Problem 1

a_n: the number of things that have label n.

$G(x) = \sum_n x^n . a_n$ is the g.f. of a_n

X: the label uniform random draw of a thing.

We have a total of N things

the p.g.f of X is:

$$F_X(S) = \mathbb{E}[S^X] = \sum_{x=0}^{n} S^x . \underbrace{\mathbb{P}(X=x)}_{\frac{a_x}{N} = p_x}$$

$$F_X'(S) = \sum_{x=0}^{n} x . S^{x-1} . p_x$$

Note that: $\mathbb{E}[X] = \sum_{x=0}^{n} x . p_x = F_X'(1)$ ✓

$$Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$F_X''(S) = \sum_{x=0}^{n} x . (x-1) . S^{x-2} . p_x$$

$$F_X''(1) = \sum_{x=0}^{n} x^2 . p_x - \sum_{x=0}^{n} x . p_x$$

Thus,

$$Var[X] = F_X''(1) + F_X'(1) - F_X'(1)^2$$ ✓

Problem 2.

- $(X_1, \dots, X_n)$ be a collection of integer-valued R.V.s with joint distribution:

$$\mathbb{P}(X_1 = k_1, \dots, X_n = k_n) = P_{k_1, \dots, k_n}$$

And for each $1 \leqslant i \leqslant n$ define the marginal distribution of X_i to be:

$$q_k^{(i)} = \mathbb{P}(X_i = k).$$

a) Prove: $\mathbb{E}[f(X_i)] = \sum_k f(k) \cdot q_k^{(i)}$

Let $f(X_i) = Y$.

$$\mathbb{E}[Y] = \sum_y y \cdot \mathbb{P}(Y=y) =$$

$$\mathbb{E}[f(X_i)=Y] = \sum_y y \sum_{k: f(x_i)=y} \mathbb{P}(X_i = k)$$

$$\mathbb{E}[X_i] = \sum_k f(k) \cdot q_k^{(i)}.$$

b) Prove expectation of sum equals the sum of expectations.

Let $X = X_i$ and $Y = X_j$ for any $1 \leqslant i, j < n$.

$$\mathbb{E}[X+Y] = \sum_x \sum_y (x+y) \cdot \mathbb{P}(X=x, Y=y)$$

$$= \sum_x \sum_y x \cdot \mathbb{P}(X=x, Y=y) + \sum \sum y \, \mathbb{P}(X=x, Y=y)$$

(extend to all $X_i, \dots X_n$ by induction) $= \sum_x x \underbrace{\sum_y \mathbb{P}(X=x, Y=y)}_{\mathbb{P}(X=x)} + \sum_y y \underbrace{\sum_x \mathbb{P}(X=x, Y=y)}_{\mathbb{P}(Y=y)}$

$$\mathbb{E}[X] + \mathbb{E}[Y] \Leftarrow = \sum_x x \, \mathbb{P}(X=x) + \sum_y y \, \mathbb{P}(Y=y)$$

Problem 3.

π is a permutation on $\underline{n}$ letters
For $n = 5$, for example:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$$

or
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix}$$

where the inversions are defined such that a pair (i, j) with $1 \leq i \leq j \leq n$ such that $\pi(i) > \pi(j)$.

If we fix k such that $\pi(l) = n$, then there are no inversions in which l is the second member of the pair and $n - l$ inversions have k as the first member of the pair.
So if we remove n from our list of letters, we get a permutation of $n-1$ with $n - l$ fewer inversions.

$$I(n, k) = \sum_{l : 1 \leq l \leq n} I(n-1, k-(n-l))$$

Then:

$$G_n(x) = \sum_{k}^{n} I(n, k) \cdot x^k$$

but
$$G_{n-1}(x) = \sum_{k}^{n-1} I(n-1, k) \cdot x^k$$

$$I(n, k) = I(n-1, k) + \sum_{l : 1 \leq l < n} I(n-1, k-(n-l))$$

then :

$$G_n(x) = (1 + x + x^2 + \dots + x^{n-1}) \cdot G_{n-1}(x)$$

$$G_n(x) = \prod_{j=1}^{n} \frac{1 - x^j}{1 - x}$$

b) $\mathbb{E}[N] = G_n'(x) \Big|_{x=1}$

$=$

d) $\theta_{ij} = 1$ if $\pi(i) > \pi(j)$

given the permutations are randomly uniform:

$$\mathbb{P}(\pi(i) > \pi(j)) = \frac{1}{2}$$

Thus,

$$\mathbb{E}[X] = \sum_{i<j} \frac{1}{2} = \frac{1}{2}\binom{n}{2}$$ ✓

Problem 4.

Die A: $\{1, 3, 4, 5, 6, 8\}$

" B: $\{1, 2, 2, 3, 3, 4\}$

a) Is $\mathbb{P}(A+B=n) = \mathbb{P}(X+Y=n)$

for X and Y R.V.s with the roll of two indep. dice?

$$G_X(S) = \sum_{k=1}^{6} \frac{1}{6} \cdot S^k = \frac{1}{6} \cdot \sum_{k=1}^{6} S^k$$

and $G_Y(S) = G_X(S)$.

We showed that $G_{X+Y}(S) = G_X(S) \cdot G_Y(S)$ (indep)

$$= \frac{1}{6^2} \cdot \left(\sum_{k=1}^{6} S^k\right)^2$$

Now the generating functions for A and B are:

$$G_A(S) = \frac{1}{6}(S + S^3 + S^4 + S^5 + S^6 + S^8) \leftarrow P_A$$

$$G_B(S) = \frac{1}{6} \cdot (S + S^2 + S^2 + S^3 + S^3 + S^4) \leftarrow P_B$$

$$G_{A+B}(S) = \frac{1}{6^2} \cdot (P_A \cdot P_B)$$

$$P_A \cdot P_B = S^2 + 2S^3 + 2\cdot S^4 + S^5 + S^4 + 2S^5 \cancel{+2S}^6 + S^7 + S^5 + 2S^6 + 2S^7 +$$
$$S^8 + S^6 + 2S^7 + 2S^8 + S^9 + S^7 + 2S^8 + 2S^9 + S^{10} + S^9 + 2S^{10} + 2S^{11} +$$
$$S^{12}$$

$$= S^2 + 2S^3 + 3S^4 + 4S^5 + 5S^6 + 6S^7 + 5S^8 + 4S^9 + 3S^{10} + 2S^{11} + S^{12}$$

Note that $\left(\sum_{k=1}^{6} S^k\right)^2 = P_A \cdot P_B$

Thus, $G_{A+B}(S) = G_{X+Y}(S)$ and the sum of two weird dice has the same distribution of two regular dice.