Edge Fault Tolerance in Graphs

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A graph or multigraph G^* is k-edge fault-tolerant with respect to a graph G, denoted k-EFT(G), if every graph obtained by removing any k edges from G^* contains G. We observe that for k sufficiently large a k-EFT(G) graph must be a multigraph, and we present some basic conditions that such multigraphs must meet. We then study the problem of constructing k-EFT(G) graphs that are optimal in that they contain the minimum number of edges among all k-EFT(G) graphs. Families of optimal k-EFT(G) graphs, where G is the n-node path or cycle, are presented for all k and n. We also give an optimal 1-EFT design for the n-dimensional hypercube.

1. INTRODUCTION AND NOTATION

Motivated by the study of computers and communication networks that tolerate failure of their components, various graph theoretic models of fault tolerance have been proposed. They typically involve two separate graphs: a primary graph G^* and a secondary graph G that must be embedded in G^*, even when "faults" that remove nodes or edges from G^* are present. In the widely used model of [6], for instance, G^* is said to be *k-fault-tolerant* or *k-FT* with respect to G if the removal of a set F of k for fewer nodes from G^* results in a graph $G^* - F$ (with F standing for faults) that contains a subgraph isomorphic to G. A class of problems of practical interest is to construct a *k-FT* primary graph G^* with respect to useful secondary graphs (trees, cycles, hypercubes, and the like) where G^* is optimal in some sense, such as having the smallest number of nodes or edges among all *k-FT* primary graphs of interest [1, 6-9]. As nodes often represent expensive components (processors) and edges represent less expensive interconnections (wires), most attention has been devoted to node faults, i.e., the removal of nodes (and their incident edges), rather than edge faults where only edges are removed.

We treat node and edge fault tolerance as intrinsic properties of a single graph by, in effect, equating G with a zero-fault-tolerant primary graph G^*. This suggests that fault tolerance is a more fundamental feature of a graph than has hitherto been thought. After introducing suitable notation, we analyze edge fault tolerance in detail; a companion paper examines node fault tolerance [4]. We show that multigraphs are an inherent feature of edge fault-tolerant graph families. For several important graph types, including paths, cycles, and hypercubes, we show how to construct k-edge fault-tolerant families that are optimal in that they contain the minimum number of edges for a given k. In general, we follow the terminology and notation of [3].

First, we review the concept of node fault tolerance. Let G be a graph with p nodes and q edges. A $(p + k)$-node graph G^* is said to be *k-node fault-tolerant*, or k-NFT, with respect to G, if every graph obtained by removing any k nodes from G^* contains G. For

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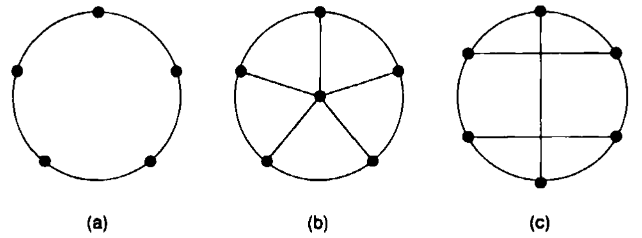

Fig. 1. (a) The cycle C_5; (b) a nonoptimal 1-NFT(C_5); (c) an optimal 1-NFT(C_5).

brevity, we will refer to G^* as a k-NFT(G) graph. If G^* has the smallest number of edges, i.e., $q + nft_k(G)$ edges, among all $(p + k)$-node graphs that are k-NFT with respect to G, then G^* is *optimally k-NFT* with respect to G, and the number $nft_k(G)$ is called the *k-node fault tolerance* of G. Figure 1(b) shows a wheel, which is a 1-NFT version of the 5-node cycle C_5 of Figure 1(a). The extra node in the center serves as a spare and obviously can replace any node appearing in the original C_5. This graph is however, not optimally 1-NFT with respect to C_5. As can readily be verified [6], the triangular prism of Figure 1(c) is an optimal 1-NFT(C_5) graph, that requires one less edge than the wheel and implies that $nft_1(C_5) = 4$.

Node fault tolerance of the above kind is usually referred to simply as "fault tolerance" in the computer literature. We now introduce the corresponding concepts for edge fault tolerance. A p-node graph G^* is said to be *k-edge fault-tolerant*, or k-EFT, with respect to G, if every graph obtained by removing any k edges from G^* contains G. If G^* has the smallest number $q + eft_k(G)$ of edges among all p-node graphs that are k-EFT with respect to G, then G^* is called *optimally k-EFT* with respect to G, and $eft_k(G)$ is the *k-edge fault tolerance* of G. We will refer to G^* as a k-EFT(G) graph, or simply as a k-EFT(G). Figure 2 shows 1-, 2-, and 3-EFT versions of C_5, which we will later see to be optimal. Unlike the corresponding process for node fault tolerance, the addition of edges in this manner will produce a multigraph such as that of Figure 2(c) when k becomes sufficiently large. (This may place a practical limit on the largest acceptable value of k.) We will assume that all the preceding definitions apply to multigraphs as well as to graphs.

2. PRELIMINARY RESULTS

It is useful to be able to express fault-tolerant multigraphs as combinations of several graphs that share the same set of nodes. The standard graph operations such as union, join, and composition [3] do not include an operator for combining multigraphs. Let G_1 and G_2 be two multigraphs defined on the same set V of labeled nodes, but with disjoint edge sets. The *merger* of G_1 and G_2, denoted $G_1 \circ G_2$, is the graph on V whose edge set is the union of the edges of G_1 and G_2. For example, the 3-EFT(C_5) multigraph of Figure 2(c) can be expressed as $K_5 \circ C_5$. As in [5], we denote the k-fold power graph $G \circ G \circ \cdots \circ G$ with respect to merging by ${}_kG$. The shorthand "network" notation for multigraphs given in Figure 3 is also useful [3, p. 53]. A multiedge consisting of j simple edges between the same two nodes will be called *multiedge of order j*, or simply a *j-edge*, and will be represented by a single heavy line labeled with j. A simplified version of the multigraph of Figure 2(c) using this notation appears in Figure 3(c). If every j-edge of a multigraph G is replaced by a simple edge, i.e., a 1-edge, then the resulting simple (ordinary) graph is called the *underlying graph UG* of G. It will be seen that families of fault-tolerant graphs often have a common underlying graph. For example, the 2-EFT(C_5) graph K_5 of Figure 2(b) is the underlying graph of the 3-EFT(C_5) multigraph in Figure 3(c).

The following theorem is of fundamental importance in constructing graphs or multigraphs of specified edge fault tolerance.

Theorem 1. If (multi) graphs G_1 and G_2 are k_1-EFT and k_2-EFT, respectively, with respect to G, then the mer-

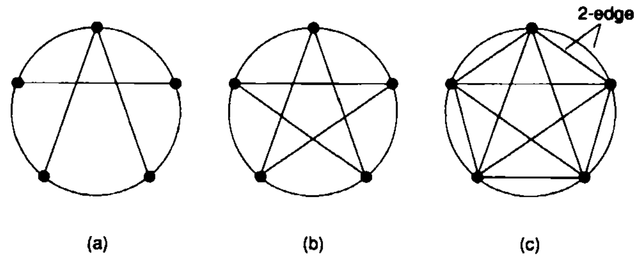

Fig. 2. Optimal k-EFT(C_5) for (a) $k = 1$, (b) $k = 2$, and (c) $k = 3$.

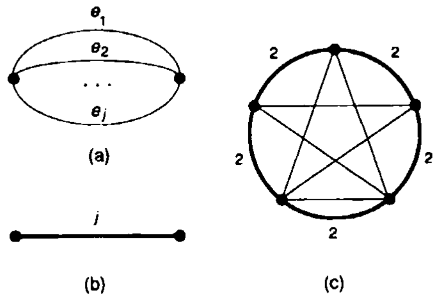

Fig. 3. (a) A j-edge of a multigraph; (b) simplified notation for the j-edge; (c) application to the 3-EFT(C_5) multigraph of Figure 2(c).

ger graph $G_{12} = G_1 \circ G_2$ is $(k_1 + k_2 + 1)$-EFT with respect to G.

Proof. Consider a fault F in G_{12} produced by removing some set of $k_1 + k_2 + 1$ edges to form the faulty graph $G_{12} - F$. We must show that $G_{12} - F$ contains a copy of G. If at most k_1 edges of F lie in G_1, then we can find G in G_1; the fault is clearly tolerated. If $k_1 + 1$ or more edges of F lie in G_1, then at most k_2 edges of F are in G_2. This means that G_2 contains a copy of G because it is k_2-EFT with respect to G. Hence, in all cases, $G_{12} - F$ contains G, and the theorem follows. ∎

Corollary 1. The k-th order merger power graph ${}_kG$ is $(k - 1)$-EFT with respect to G, for all $k \geq 1$. ∎

In general, even though G_1 and G_2 are optimally edge fault-tolerant, their merger $G_1 \circ G_2$ need not be. For example, since C_5, like every graph, is trivially and optimally 0-EFT with respect to itself, Corollary 1 implies that ${}_2C_5 = C_5 \circ C_5$ is a 1-EFT(C_5) multigraph. However, ${}_2C_5$ is not optimal because it contains 10 edges, one more than the (optimal) 1-EFT(C_5) appearing in Figure 2(a).

As k increases, G^* often acquires enough edges to contain the complete graph K_p on p nodes as a subgraph. For example, the optimal 2-EFT(C_5) graph of Figure 2(b) is also K_5, and it can be shown that every optimal k-EFT(C_5), for $k \geq 2$ has K_5 as its underlying graph. [A nonoptimal k-EFT(C_5) such as ${}_{k+1}C_5$ need not, however, contain K_5.] The edge fault tolerance of K_p is easily characterized.

Theorem 2. The complete graph K_p on p nodes has a unique and optimal k-EFT realization ${}_{k+1}K_p$, for all $k \geq 0$. ∎

The next theorem is useful in determining the optimality of an edge fault-tolerant (multi) graph. If G is a multigraph with underlying graph UG, we associate two types of degrees with the each node x of G. The degree $deg(x)$ of x is the number of edges incident with x in G; this is the usual definition. The *underlying degree* of x, denoted $undeg(x)$, is the number of edges incident with x in UG. In other words, $undeg(x)$ is the number of multiedges incident with x in G. In a simple graph, $undeg(x) = deg(x)$, and each is the number of nodes adjacent to x. In a multigraph, we can have $undeg(x) < deg(x)$, in which case only $undeg(x)$ represents the number of nodes adjacent to x. For any node x in the multigraph of Figure 2(c), $undeg(x) = 4$, but $deg(x) = 6$.

Theorem 3. Let G^* be a k-EFT(G). If the minimum degree of G is d, then every node x^* in G^* must have degree d^* satisfying

$$d^* \geq d + k. \tag{1}$$

If the minimum underlying degree of G is m, then x^* must have underlying degree $m^* \geq m$ such that for every set E of $n = m^* - m + 1$ multiedges incident with x^*:

$$\sum_{i=1}^{n} j_i > k, \tag{2}$$

where $j_1, j_2, \ldots j_n$ are the orders of the multiedges in E.

Proof. The proof follows directly from the fact that if it is to replace x, then node x^* must have at least m multiedges and at least d edges incident with it, even after it loses any k of its original edges. ∎

Theorem 3 provides bounds on the degrees and underlying degrees of the nodes in G^* that facilitate optimality tests. Consider the three edge fault-tolerant versions of C_5 in Figure 2. Since C_5, like every cycle, is regular of degree 2, each of the five nodes in a 1-EFT(C_5) graph must have degree 3 or more to satisfy constraint (1). One node must have degree of at least 4 to make the total number of edges an integer. Now, the graph of Figure 2(a) has one node of degree 4 and four nodes of degree 3, so if it is 1-EFT(C_5), it must also be optimal. In the case of the 3-EFT(C_5) graph C_5^* of Figure 2(c), each node x must have degree 5 or more by (1). Because there are only five nodes in C_5^*, node x can have an underlying degree of at most 4. Hence, a multiedge of order 2 or more must be incident with x. If x has four multiedges with orders 1, 1, 1, and 2, then its three 1-edges violate constraint (2). [Removal of these three edges would reduce $undeg(x)$ to 1.] Hence, x's four multiedges must have orders of at least 1, 1, 2, and 2; these minimum values are met exactly by every set of four nodes in C_5^*. We conclude that C_5^*'s nodes have the minimum possible degrees and underlying degrees consistent with being 3-EFT with respect to C_5; consequently C_5^* is the unique optimal 3-EFT(C_5) multigraph.

3. PATHS

We next consider optimal k-EFT graphs for the n-node path P_n. The corresponding k-NFT graphs are identified in [6]. As we will see, the edge fault tolerance properties of paths and cycles are closely linked; the latter are examined in the next section.

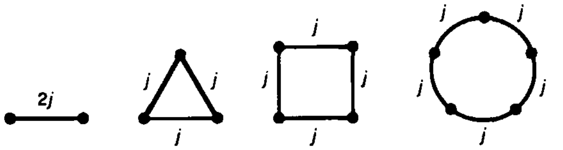

Fig. 4. Optimal k-EFT(P_n)'s for all odd $k = 2j - 1$ with $2 \le n \le 5$.

It is obvious that C_n is a 1-EFT(P_n) graph and $eft_1(P_n) = 1$, because removal of one edge from C_n leaves P_n. However, a k-EFT(P_n) need not contain C_n. In fact, by Corollary 1, the multigraph ${}_{k+1}P_n$ is a k-EFT(P_n) that does not contain any cycles. For the trivial case P_2 where the path is a single edge, ${}_{k+1}P_2$ is an optimal k-EFT(P_2) for all $k \ge 0$. For $n \ge 3$, the multigraph ${}_{k+1}P_n$ is far from optimally k-EFT. If k is odd, specifically, if $k = 2j - 1$ for all integers $j \ge 1$, then jC_5 is an optimal $(2j - 1)$-EFT(P_n) that is formed simply by replacing each edge of C_n by a j-edge. Figure 4 shows optimal $(2j - 1)$-EFT(P_n)'s for small values of n. Each node in these multigraphs has degree $k + 1 = 2j$, so the multigraphs are optimal by Theorem 3. To see that jC_n is $(2j - 1)$-EFT with respect to P_n, for $n \ge 3$, note that jC_n contains P_n as long as no more than one complete j-edge is removed from jC_n by a $(2j - 1)$-edge fault F. If F removes a j-edge, then it can remove at most $j - 1$ edges from any other j-edge in jC_n, since $k = 2j - 1$. Consequently, jC_n will always contain a copy of P_n after the removal of any $2j - 1$ edges.

With even $k = 2j$ and $n \ge 3$, an optimal k-EFT(P_n) is somewhat more difficult to construct. We present a general way of doing so by adding a minimum number of edges to jC_n, which is an optimal $(j - 1)$-EFT(P_n) multigraph, in order to increase its edge fault tolerance by one. This method is illustrated by Figure 5 for the two basic cases, even $n = 2m$ and odd $n = 2m - 1$, for $m \ge 2$. With the n nodes labeled $x_0, x_1, \ldots x_{n-1}$ as shown, we add to jC_n the set S of $\lfloor n/2 \rfloor$ edges (x_i, x_{n-i}) for $0 < i < \lfloor n/2 \rfloor$; note that when n is even, $\lfloor n/2 \rfloor = \lceil n/2 \rceil = n/2 = m$. The set S consists of all the horizontal edges in the figure. When n is even, we also insert the vertical edge $(x_0, x_{n/2})$. Now every node has its degree increased by one. When n is odd, we have three remaining nodes that we connect via the two new edges $(x_0, x_{\lfloor n/2 \rfloor})$ and $(x_0, x_{\lceil n/2 \rceil})$ drawn as diagonals in Figure 5(b). In this case, x_0 has its degree increased by two; all other nodes have their degrees increased by one.

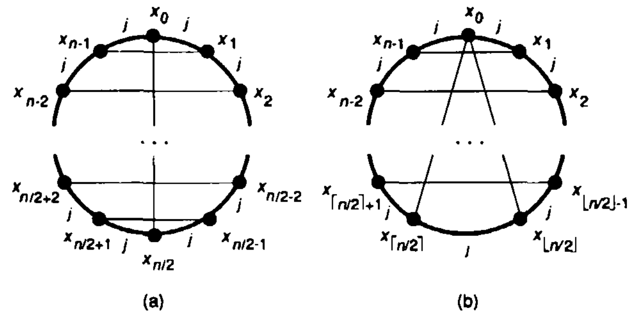

Fig. 5. Optimal $2j$-EFT(P_n) multigraphs: (a) $G_j(2m)$ for n even; (b) $G_j(2m - 1)$ for n odd.

Next we prove that the (multi) graphs $G_j(2m)$ and $G_j(2m - 1)$ of Figure 5 are $2j$-EFT by demonstrating that after any $2j - 1$ edges are removed the remaining graph contains the cycle C_n, i.e., the reduced graph is hamiltonian. It follows that the path P_n remains after any additional edge is removed. As pointed out above, the deletion of $2j - 1$ edges can only remove one complete j-edge (heavy line) from jC_n. Consequently, we only need show that $G_j(2m)$ and $G_j(2m - 1)$ are hamiltonian after the removal of any j-edge. That this true may be seen from Figure 6. For $G_j(2m)$, Figure 6(a) and (b) show by means of hatched lines two possible C_n's. Since every j-edge of $G_j(2m)$ is excluded from one of these two cases, we can conclude that $G_j(2m)$ is an optimal k-EFT(P_n) for $n = 2m$. Figure 6(c) shows a similar case for $G_j(2m - 1)$; again, we have a copy of C_n, this time excluding $m = \lfloor n/2 \rfloor$ of the original j-edges. If this graph is reflected about the vertical axis, we obtain a C_n that excludes all the remaining j-edges, except the bottom one, namely, $(x_{\lfloor n/2 \rfloor}, x_{\lceil n/2 \rceil})$. Now to exclude the latter, we always can find a hamiltonian cycle like that shown in Figure 6(d). The exact form of

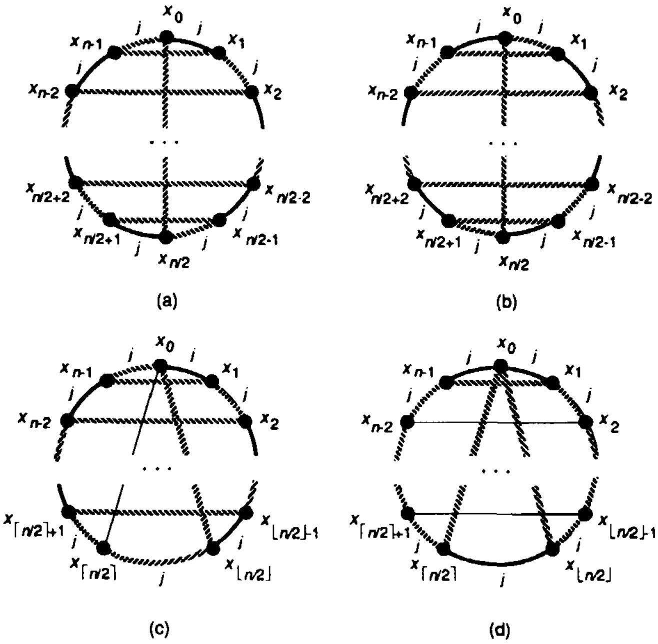

Fig. 6. Hamiltonian cycles (hatched lines) in (a, b) $G_j(2m)$ and (c, d) $G_j(2m - 1)$.

these cycles in $G_j(2m - 1)$ varies slightly with the value of n. Thus, $G_j(2m - 1)$ is an optimal k-EFT(P_n) for $n = 2m - 1$. We note in passing that the underlying graphs of $G_j(2m)$ and $G_j(2m - 1)$ are hamiltonian-connected graphs of n nodes with the minimum number of edges, a fact that has been observed in other contexts [6]. The next theorem summarizes the results covered by the above discussion.

Theorem 4. The edge fault tolerance of paths is as follows:

(*a*) For $n = 2$ and $k \geq 0$, ${}_{k+1}P_2$ is an optimal k-EFT(P_n) and $eft_k(P_2) = k$.

(*b*) For $n \geq 3$ and odd $k = 2j - 1$, jC_n is an optimal $(2j - 1)$-EFT(P_n) and $eft_{2j-1}(P_n) = (j - 1)n + 1$.

(*c*) For $n \geq 3$ and even $k = 2j$, $G_j(2m)$ and $G_j(2m - 1)$ defined by Figure 5(a) and 5(b) are optimal $2j$-EFT(P_n)'s for $n = 2m$ and $n = 2m - 1$, respectively. In this case,

$$eft_{2j}(P_n) = (j - 1)n + \lceil n/2 \rceil + 1. \qquad \blacksquare$$

Figure 7(a) and (c) illustrate 3-EFT(P_5) = ${}_2C_5$ and 4-EFT(P_6) = $G_2(6)$ constructed according to the above procedure. These optimal k-EFT(P_n)'s are by no means unique. It is easily verified that the graph K_5 of Figure 7(b) is also optimally 3-EFT with respect of P_5. Since it is not a proper multigraph, it cannot be isomorphic to ${}_2C_5$. Similarly, Figure 7(d) gives an optimal 4-EFT(P_6) multigraph that is not isomorphic to the multigraph $G_2(6)$ of Figure 7(c).

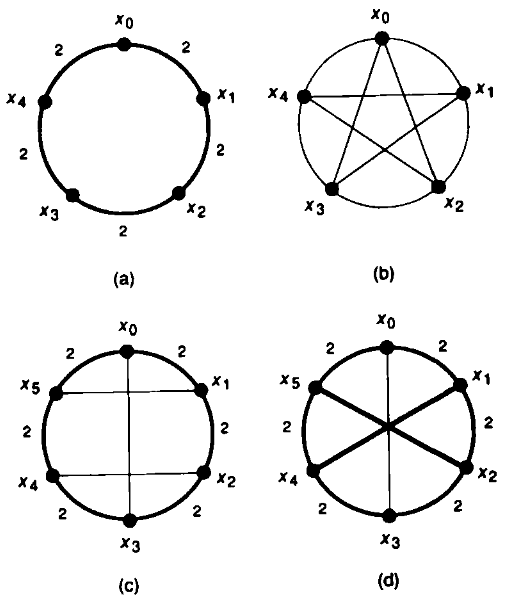

Fig. 7. Optimal k-EFT(P_n)'s for (a, b) $k = 3$ and $n = 5$ and (c, d) $k = 4$ and $n = 6$.

4. CYCLES

We turn next to the construction of optimal edge fault-tolerant implementations of the n-node cycle C_n. Certain cases have been studied in [8, 9], where a k-EFT(C_n) graph is called k-edge hamiltonian and is not allowed to be a multigraph. Obviously, a (multi) graph that is k-EFT with respect to C_n is $(k + 1)$-EFT with respect to P_n. However, as observed earlier, a k-EFT(P_n) may not even be 1-EFT with respect to any C_n. We begin with two easily characterized special cases, $k = 1$ and $k = n - 3$; note that Theorems 5–7 below cover all cases.

Theorem 5. The edge fault tolerance of cycles for $k = 1$ and $k = n - 3$ is as follows:

(*a*) The graphs $G_1(2m)$ and $G_1(2m - 1)$ defined by Figure 5(a) and (b) are optimal 1-EFT(C_n) graphs for $n = 2m$ and $n = 2m - 1$, respectively, and $eft_1(C_n) = \lceil n/2 \rceil$.

(*b*) K_n is the optimal $(n - 3)$-EFT(C_n) graph, and $eft_{n-3}(C_n) = n(n - 3)/2$.

Proof. Part (*a*) is an immediate consequence of the properties of the optimal k-EFT(P_n) graphs discussed in the preceding section. Since every node of K_n has degree $n - 1 = k + 2$, Theorem 3 implies that if K_n is $(n - 3)$-EFT with respect to C_n it is also optimal. To determine its edge fault tolerance, consider the effect of a fault F that removes $n - 3$ edges from K_n. Let x_1 and x_2 be any two nodes of the faulty graph $K_n - F$, and let F remove the edge (x_1, x_2) between them, along with $n - 4$ other edges. The effect of F is to reduce the combined node degrees of x_1 and x_2 from $2(n - 1)$ to no less than $2(n - 1) - 2 - (n - 4) = n$. Hence,

$$deg(x_1) + deg(x_2) \geq n. \tag{3}$$

By a well-known theorem due to Ore [3, p. 68], since all pairs of nonadjacent nodes of $K_n - F$ satisfy (3), $K_n - F$ must be hamiltonian. Part (*b*) then follows. $\blacksquare$

An alternative 1-EFT(C_n) design appears in [8]. Next, we consider the optimal EFT cycles lying between the limiting cases covered by Theorem 5, all of which are proper graphs. The jth *power* graph of C_n is denoted by C_n^j and is formed by adding edges to each node x_i in C_n joining it to all nodes at distance d or less from x_i, for $2 \leq d \leq j$. As before, let the nodes of C_n be labeled $x_0, x_1, \ldots, x_{n-1}$.

Theorem 6. The edge fault tolerance of cycles for k between 1 and $n - 3$ is as follows:

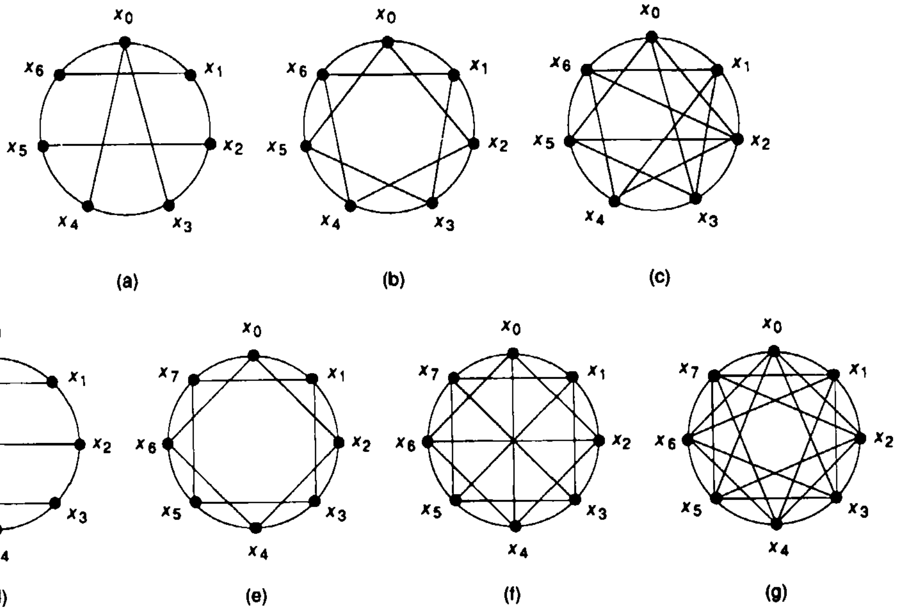

Fig. 8. Optimal k-EFT(C_n)'s for (a–c) $n = 7$ and $k = 1, 2$, and 3 and (d–g) $n = 8$ and $k = 1, 2, 3$, and 4.

(*a*) For even $k = 2j \geq 0$, C_n^{j+1} is an optimal k-EFT(C_n) graph, and $eft_k(C_n) = jn$.

(*b*) For odd $k = 2j + 1 \geq 1$, $C_n^{j+1} \circ D_n$ is an optimal k-EFT(C_n) graph, where D_n is the graph consisting of $\lceil n/2 \rceil$ "diameter" edges $(x_0, x_{\lfloor n/2 \rfloor})$, $(x_1, x_{\lfloor n/2 \rfloor + 1})$, . . . , $(x_{\lceil n/2 \rceil - 1}, x_{2\lceil n/2 \rceil - 1})$ and, when n is odd only, the edge $(x_{\lfloor n/2 \rfloor - 1}, x_{n-1})$. In this case, $eft_k(C_n) = jn + \lceil n/2 \rceil$. ■

It takes two papers [8, 9] to prove this highly nontrivial result! Optimality is immediate by Theorem 3. However, proving the k-EFT property is very difficult; complete proofs can be found for k odd and k even in [8] and [9], respectively. The graphs C_n^{j+1} and $C_n^{j+1} \circ D_n$ were identified earlier [2] as maximally connected graphs with a minimum number of edges. Figure 8 illustrates Theorem 6 for $n = 7$ and 8. Figure 8(b), for example, shows an optimal 2-EFT(C_7) = C_7^2. By adding $\lceil 7/2 \rceil = 4$ diameters to C_7^2, as in Figure 8(c), we obtain an optimal 3-EFT(C_7) graph. Figure 8(g) shows an optimal 4-EFT(C_8) = C_8^3. Figures 9(a–d) give optimal k-EFT(C_6) graphs for $k = 0, 1, 2$, and 3.

We can directly extend the preceding results to obtain C_n^*, an optimal k-EFT(C_n), when k is $n - 2$ or more, in which case C_n^* is a proper multigraph. All that is necessary is to merge one of the optimal graphs defined by Theorems 5 and 6 with an appropriate power of the complete graph K_n. For example, K_6 is an optimal 3-EFT(C_6) graph, as shown in Figure 9(d). Merging the 0-EFT(C_6) graph C_6 of Figure 9(a) with K_6 yields $K_6 \circ C_6$ [Fig. 9(e)], which is an optimal 4-EFT(C_6). Merging the 1-EFT(C_6) graph $C_6 \circ D_6$ of Figure 9(b) with K_6 produces $K_6 \circ C_6 \circ D_6$ [Fig. 9(f)], which is an optimal 5-EFT(C_6), and so on. When k reaches $2n - 5$, we obtain $_2K_6$ [Fig. 9(h)], which is a 7-EFT(C_6). Next, we have the 8-EFT(C_6) = $_2K_6 \circ C_6$, etc. These results are collected and generalized in the following theorem.

Theorem 7. The edge fault tolerance of cycles for $k > n - 3$ is as follows: For $n > 3$, the merger $_{\lfloor k/(n-2) \rfloor}K_n \circ C_{n,j}$ is an optimal k-EFT(C_n) multigraph, where $C_{n,j}$ is the optimal j-EFT(C_n) graph defined by Theorems 5(a) and 6 with $j = k \bmod (n - 2)$, which is the residue of k divided by $n - 2$. The corresponding fault tolerance number is

$$eft_k(C_n) = \binom{n(n-1)}{2}^{\lfloor k/(n-2) \rfloor} + (j - 1)n + \alpha \lceil n/2 \rceil,$$

where $\alpha = 0$ if $j = k \bmod (n - 2)$ is even and $\alpha = 1$ if this j is odd. ■

Proof. First, we show that $G^* = {}_{\lfloor k/(n-2) \rfloor}K_n \circ C_{n,j}$ tolerates k edge faults. By Theorems 1 and 5(b), $_{\lfloor k/(n-2) \rfloor}K_n$ is $((n - 2)\lfloor k/(n - 2) \rfloor - 1)$-EFT with respect to C_n. The graph $C_{n,j}$ has edge fault tolerance $j = k \bmod (n - 2)$. Hence, Theorem 1 implies that the fault tolerance of G^* with respect to C_n is $(n - 2)\lfloor k/(n - 2) \rfloor - 1 + (k \bmod (n - 2)) + 1 = k$, because $\lfloor k/(n - 2) \rfloor$ and $k \bmod (n - 2)$ are the quotient and remainder, respectively, of $k/(n - 2)$.

To see the optimality of our construction, consider

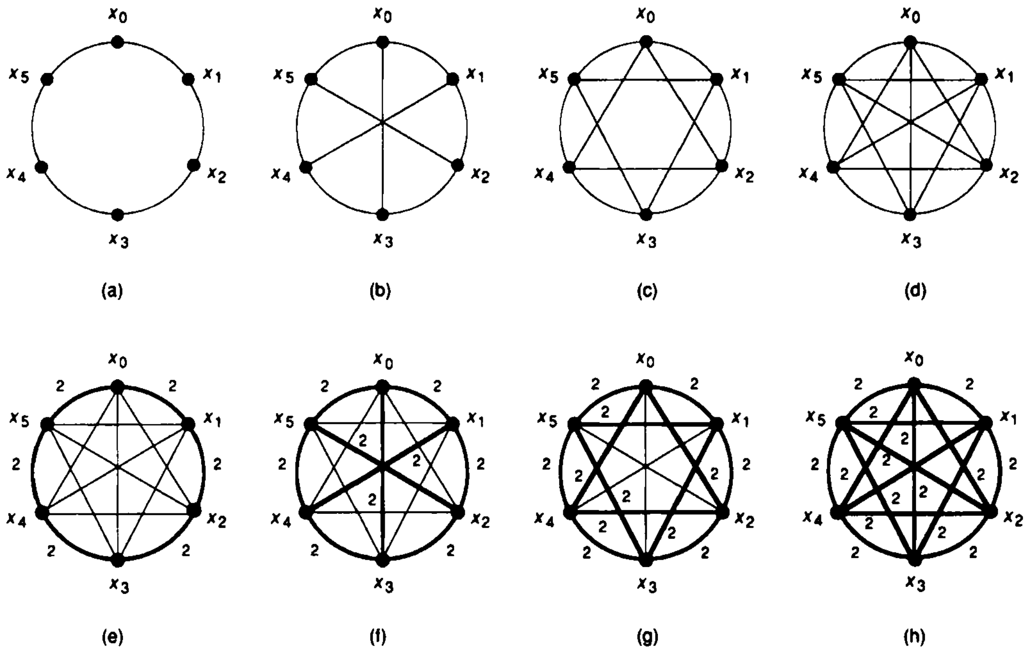

Fig. 9. Optimal k-EFT(C_6)'s for (a–d) k = 0, 1, 2, and 3 and (e–h) k = 4, 5, 6, and 7.

what happens to G^* as k increases beyond $n - 3$. The underlying degree of each node x_i remains constant at the maximum value $n - 1$, but the orders of the edges incident with x_i increase, as can be seen from Figure 9. Suppose that a fault F removes k edges from G^*. The underlying degree $undeg(x_i)$ of each node x_i in $G^* - F$ must remain at least two. Hence, the sum of the orders of any $n - 2$ of the $n - 1$ multiedges incident with x_i cannot be k or less; otherwise, F could reduce $undeg(x_i)$ to one, obviously making $G^* - F$ non-Hamiltonian. Let t_{ih} denote the order of the multiedge linking x_i to x_h in G^*. For any fixed value of $j \neq i$, the sum of the orders of all $n - 2$ multiedges linking x_i to all x_h, except the selected x_j, must satisfy

$$\sum_{\text{All } h \neq j} t_{ih} \geq k + 1. \qquad (4)$$

If we now sum the $n - 1$ possible inequalities of the form (4) for every distinct $j \neq i$, we obtain

$$(n - 2) \sum_{\text{All } h} t_{ih} \geq (n - 1)(k + 1). \qquad (5)$$

Hence,

$$\sum_{\text{All } h} t_{ih} \geq \left\lceil \frac{(n - 1)(k + 1)}{n - 2} \right\rceil,$$

where the left-hand side is the sum of the orders of all edges incident with any node x_i of G^*. The total number q of edges in G^* must then satisfy

$$q \geq \frac{n}{2} \left\lceil \frac{(n - 1)(k + 1)}{n - 2} \right\rceil. \qquad (6)$$

It can readily be seen that inequality (6) is met exactly by $G^* = {}_{\lfloor k/(n-2) \rfloor}K_n \circ C_{n,j}$. For example, in the case of Figure 9(f), $n = 6$ and $k = 5$, so

$$\frac{n}{2} \left\lceil \frac{(n - 1)(k + 1)}{n - 2} \right\rceil = 3 \left\lceil \frac{5 \times 6}{4} \right\rceil = 24.$$

Hence, G^* is indeed an optimal k-EFT(C_n) as claimed. ■

5. OTHER GRAPHS

Many other graph types pose interesting and potentially useful edge-fault-tolerance problems. We consider two representative examples of interest in the design of multiprocessing computers, meshes and hypercubes, restricting our attention to the 1-EFT case.

The *n-dimensional mesh* $M(m_1, m_2, \ldots, m_n)$ may be defined as the graph product $P_{m_1} \times P_{m_2} \times \cdots \times P_{m_n}$ of n paths [3]. It has $m_1 m_2 \cdots m_n$ nodes connected in the gridlike configuration illustrated in Figure 10(a) for the most common, 2-dimensional, case. As noted in Section 3, we can make P_m 1-EFT by adding an edge between its first and last nodes, thus extending it to C_m. This suggests that $G^* = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_n}$ might be an optimal 1-EFT version of the mesh M, where $m_i \geq 3$ for all i. Clearly, G^* can be formed by adding m_i spare edges to each dimension i of mesh M

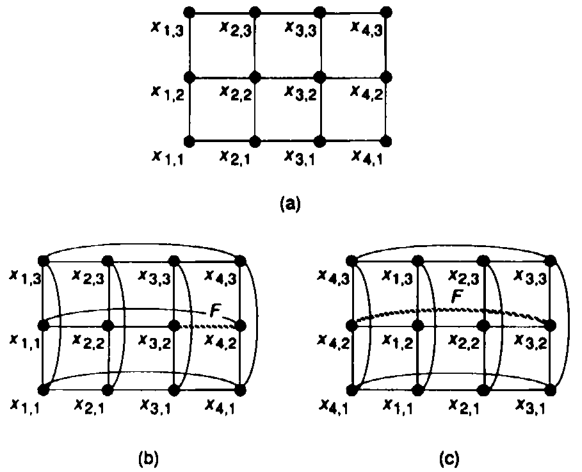

Fig. 10. (a) The 2-dimensional mesh $M(3, 4)$; (b) a 1-EFT($M(3, 4)$) with faulty (hatched) edge $F = (x_{3,2}, x_{4,2})$; (c) reconfiguration around F.

that link its peripheral nodes [see Fig. 10(b)]. Suppose that an edge F in dimension i is removed from G^*. We can rotate G^* in this dimension to move F to the spare position, as illustrated for $F = (x_{3,2}, x_{4,2})$ in Figure 10(c), thus obtaining a subgraph isomorphic to M that does not contain F. Hence, G^* is 1-EFT. We conjecture that 1-EFT meshes constructed in this way are also optimal, but this appears to be difficult to prove.

The n-dimensional mesh with $m_i = 2$ for all i is, of course, the n-dimensional hypercube Q_n. In the 2-dimensional case, $Q_2 = C_4$, and so by Theorem 5(b), 1-EFT(Q_2) $= K_4$, and $eft(Q_2) = 2$. The next theorem generalizes this result to n-dimensional hypercubes. It is convenient (and customary) to label the 2^n nodes of Q_n using binary numbers of the form $b_1b_2 \cdots b_n$, so that the node to which node $b_1b_2 \cdots b_i \cdots b_n$ is connected along dimension i has the label $b_1b_2 \cdots b_i' \cdots b_n$, where $b_i' = 0$ if and only if $b_i = 1$.

Let A_n denote a 2^n-node graph labeled in the same manner as Q_n but with only those edges joining each

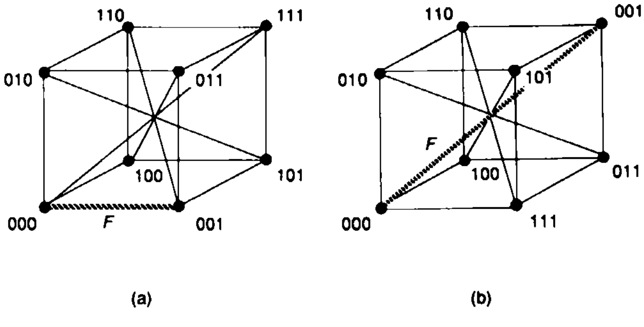

Fig. 11. (a) Optimal 1-EFT(Q_3) for the 3-dimensional hypercube Q_3; (b) recovery from a fault affecting the hatched edge $F = (000, 001)$.

node $b_1b_2 \cdots b_n$ to its "antipodal" counterpart $b_1'b_2' \cdots b_n'$. Thus, node $0 \cdots 00$ is connected to $1 \cdots 11$, node $0 \cdots 01$ is connected to $1 \cdots 10$, and so on.

Theorem 8. For the n-dimensional hypercube Q_n with $n \geq 2$, the graph $Q_n \circ A_n$ is an optimal 1-EFT(Q_n), and $eft_1(Q_n) = 2^{n-1}$.

Proof. An edge $F = (b_1b_2 \cdots b_i \cdots b_n, b_1b_2 \cdots b_i' \cdots b_n)$ along dimension i of Q_n links two copies of Q_{n-1}. When a fault removes F, it can be replaced by the antipodal edge $F' = (b_1b_2 \cdots b_i \cdots b_n, b_1'b_2' \cdots b_i' \cdots b_n')$ supplied by A_n. In fact, all 2^{n-1} original edges lying in dimension i are replaced by the corresponding edges from A_n. These edges link the two copies of Q_{n-1} originally linked by F and form a copy of Q_n. Optimality follows from Theorem 3. ■

Figure 11(a) shows $Q_3 \circ A_3$ constructed according to Theorem 8. Recovery from a fault affecting the hatched edge $F = (000,001)$, or any of the four edges lying in dimension 3, is depicted in Figure 11(b). This can be visualized as a rotation of the rightmost face of the Q_3 in the figure by 180°.

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