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Dedicated to the memory of A.P. Favorskii

Issues of Dynamic Graph Theory

A. A. Kochkarov[a,b], R. A. Kochkarov[b], and G. G. Malinetskii[c]

[a] *RTI ul. 8 Marta 10, str. 1, Moscow, 127083 Russia*
[b] *Financial University under the Government of the Russian Federation, Leningradskii pr. 49, Moscow, 125993 Russia*
[c] *Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Miusskaya pl. 4, Moscow, 125047 Russia*
e-mail: Akochkar@gmail.com, Rasul_kochkarov@mail.ru, Gmalin@keldysh.ru

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Abstract—The notion of dynamic graphs is introduced and some properties of these graphs are examined. Engineering applications and main directions of development of dynamic graph theory are described. Conditions of the diameter conservation of dynamic graph trajectories are obtained.

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1. INTRODUCTION

Nowadays, the computational mathematics techniques applied for the analysis of such classical mathematical models as partial differential equations are rapidly developing. This is needed to improve the accurateness of computations using new generations of computers.

However, another phenomenon—the occurrence of new types of mathematical models and related computational problems—is no less important. In turn, this is a consequence of the expansions of the application of mathematical methods in various branches of science and of changes in the viewpoint of earlier studied objects.

Both interrelated processes can be clearly seen in the works of Prof. Favorskii—a prominent expert in computational mathematics and mathematical modeling; this paper is dedicated to his memory.

He belonged to the scientific school of Tikhonov and Samarskii, which studied dynamic systems

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, \dots, x_N), \\ &\vdots \\ \frac{dx_N}{dt} &= f_N(x_1, \dots, x_N), \quad 0 \le t < 0, \end{aligned} \tag{1}$$

or mappings

$$\begin{aligned} x_1^{n+1} &= g_1(x_1^n, \dots, x_N^n), \\ &\vdots \\ x_N^{n+1} &= g_N(x_1^n, \dots, x_N^n), \quad n = 1, 2, \dots, \end{aligned} \tag{2}$$

which provide best approximations of fluid dynamics and magnetic hydrodynamics equations. These issues were successfully studied by Favorskii and his associates [1]. Depending on the statement of the problem, available computational resources, and requirements for the approximation (the interpretation of "best" approximations), we obtain different objects (1) or (2) for the same partial differential equations.

However, exactly the same models (1) and (2) occur in quite different fields. In these fields, the object under study is associated with a graph with N vertices, where the state of each vertex is described by a dynamic systems the arguments of which are its own state $x_p(t)$ (or x_p^n in discrete case) and the states of their neighbors. First, such objects related to directed graphs were used in cognitive modeling (see [2]). In such models, the graph vertices correspond to factors and edges correspond to cause-and-effect relations. Similar mathematical objects occur in flow models in biology [2]. Such models also make it possible to estimate the response of a system as a whole to local forces. Similar objects were considered by Favorskii and his associates in the simulation of the reaction of blood vascular system to various drugs (the influence on the properties of edges of the corresponding graph) (see [1]).

Here, we face a new element—the structure of dependence of the functions $\{f_i\}$ or $\{g_i\}$ on their arguments—and specific problems. For example, the inverse problem in the theory of neural networks is to find a dependence on the arguments (the structure of connections) and functions $\{f_i\}$ or $\{g_i\}$ such that a given set of singular points $\{\mathbf{y}_i\}$ in the phase space correspond to the attractors of dynamic systems (1) or (2). If one of these attractors is reached, the image $\mathbf{y}_p$ is recognized. In this case, new criteria that differentiate "good" solutions from the "bad" ones appear. Typically, one of them is the simplicity of the (algorithmic) construction of the functions $\{f_i\}$ or $\{g_i\}$ allowing for the automation of this procedure.

Even though the class of problems and applications in which dynamic systems (1) or (2) are used is very wide, the dimensionality N of the phase space was typically assumed to be invariable. We dealt with a time-independent graph G that determined the functional dependences in these dynamic systems. However, recently new problems have arisen in which the very dimensionality of the phase space depends on time.

Here are some examples of such problems. In the numerical simulation of non-steady-state flows on adaptive grids in Lagrangian coordinates, some cells should be eliminated from time to time (e.g., if their size became too small in the course of computations or some angles in them became too small) or some cells should be added. In this case, the graph $G(t)$ or G_l $(l = 1, 2, \ldots)$ of the functional dependence becomes time-dependent. As in the case of neural networks, a criterion of finding a good solution is its simplicity, which makes it possible to automate the algorithm for eliminating and adding cells.

Presently, considerable attention is paid to the dynamics of self-developing systems, in particular, social networks. Not only the dynamics of data propagation over the network with a given graph G, but also changes of the graph are of great interest. Moreover, this graph itself has many interesting properties (the so-called scale-free networks, in which the number of vertices with a given number of incident edges is a power function of the number of edges). Hence, an inverse problem arises—construct a model that has the same growth dynamics $G(t)$ as that observed in the network.

The third class of problems related to multiagent systems was mentioned as early as in the 1950s in connection with space research. These are teams of mobile robots executing a common task; in this case, the graph of connections between them is time-dependent.

2. DEFINITION OF A DYNAMIC GRAPH

The concept of dynamic networks [2] is widely used for the study of complex networks with a variable structure of various natures. These are social networks [4], communication networks [3, 6], cooperative networks [7], structures of stock markets [8], and structures of mutual obligations in the interbank system [9]. Despite the accumulated empirical data about dynamic networks, the dynamic network analysis or network science has not yet been formed. To form such a branch of application science, a theoretical basis is required. The dynamic graph theory can provide such a basis. The main subject in this theory is the dynamic graph, which provides a model of a dynamic network.

Considered as a model of a dynamic network, a dynamic graph Γ is a sequence of classical graphs G_l that have no parallel edges and loops; transitions between these graphs are described by various graph-theoretic operations $\varphi(G_l) = G_{l+1}$ (removal or addition of an edge [10], removal or addition of a node [10], replacement of a node with a seed [11], priority addition of vertices and nodes [12], etc.). The index l corresponds to a kind of "topological time" at which the graph structure is modified. The operations of the removal or addition of an edge and the removal or addition of a node will be called *simple* or *basic* operations. Any other operation that can be described by a sequence of simple operations will be called *complex*. In the general case, a dynamic graph is a sequence of finite unweighted (not necessarily connected) graphs $G_1, G_2, \ldots, G_l, \ldots, G_L, \ldots$ in which the next graph G_{l+1} is obtained by applying an operation $\varphi(G_l) = G_{l+1}$. This operation may be simple or complex. To construct a trajectory of the dynamic graph, several (a finite set) alternating operations $\Phi = \{\varphi^t\}$ may be used. In such an operation, a rule for selecting graph elements (edges, nodes, subgraphs) to which this operation is applied may be specified. The sequence of graphs $G_l = (V_l, E_l)$, $(l = 1, 2, \ldots, L, \ldots)$ constituting the dynamic graph will be called the trajectory of the dynamic graph Γ.

The following obvious propositions demonstrate the applications of the concepts introduced above.

Proposition 1. *If $\varphi(V_l) \neq V_l$, $l = 1, 2, \ldots, L, \ldots$, then the trajectory of the dynamic graph Γ is infinite.*

Proposition 2. *In the set of all complete n-vertex simple chains* $\{P_n\}$ $(n \geq 2)$*, there exists a dynamic graph* Γ *with an infinite trajectory* $P_2, P_3, \ldots, P_l, \ldots, P_L, \ldots$ *for which the operation* $\varphi(P_l) = P_{l+1}$ $(l = 2, 3, \ldots, L, \ldots)$ *is defined as the addition of a vertex that is adjacent to one of the leaf nodes of* P_l.

Proposition 3. *In the set of all complete n-vertex simple cycles* $\{C_n\}$ $(n \geq 3)$*, there exists a dynamic graph* Γ *with an infinite trajectory* $C_3, C_4, \ldots, C_l, \ldots, C_L, \ldots$ *for which the operation* $\varphi(C_l) = C_{l+1}$ $(l = 3, 4, \ldots, L, \ldots)$ *is defined as the removal of an edge and the subsequent addition of a vertex that is adjacent to both leaf nodes of* P_l.

Proposition 4. *In the set of all complete n-vertex graphs* $\{K_n\}$*, there exists a dynamic graph* Γ *with an infinite trajectory* $K_1, K_2, \ldots, K_l, \ldots, K_L, \ldots$ *for which the operation* $\varphi(K_l) = K_{l+1}$ $(l = 1, 2, \ldots, L, \ldots)$ *is defined as the addition of a vertex and the* $(l + 1)$ *edges that are incident upon it.*

Proposition 5. *If the transition operations* $\varphi(G_l) = G_{l+1}$ *of a dynamic graph do not use the simple operation of vertex addition, then the dynamic graph is finite.*

A specific case of dynamic graph is a fractal graph [11]. The definition of the *fractal (prefractal) graph* is based on the complex operation of *replacing a vertex with a seed*. This operation is defined as follows. For the vertex $\tilde{v} \in V$ selected for replacement in the given graph $G = (V, E)$, a set of its adjacent vertices $\tilde{V} = \{\tilde{v}_j\} \subseteq \mathrm{V}$ $(j = 1, 2, \ldots, |\tilde{V}|)$ is selected. Then, $\tilde{v}$ and all the edges that are incident upon it are removed, and each vertex $\tilde{v}_j \in \tilde{V}$ $(j = 1, 2, \ldots, |\tilde{V}|)$ is connected by an edge with a vertex of a seed $H = (W, Q)$. The vertices are connected arbitrarily or according to a certain rule.

By replacing each time in the graph G_l $(l = 1, 2, \ldots, L - 1)$ each vertex by a seed H, we obtain the trajectory of a prefractal graph. At $l = 1$, the prefractal graph is the seed $G_1 = H$. A fractal graph Γ is defined by an infinite trajectory. Every fractal graph is infinite (see [13]) because its transition operation $\varphi(G_l) = G_{l+1}$ increases the number of vertices V_l and the number of edges E_l in the trajectory.

The figure shows the trajectory of the prefractal graph $G_3 = (V_3, E_3)$ generated by the set of seeds $\{H_1, H_2, H_3\}$ with ordered growth (see [11]), where H_t is the complete graph with $(t + 2)$ vertices, $t = 1, 2, 3$.

There are other versions of constructing fractal graphs in which the seed replaces edges rather than vertices (see [14, 15]).

3. ON HEREDITARY PROPERTIES OF DYNAMIC GRAPHS

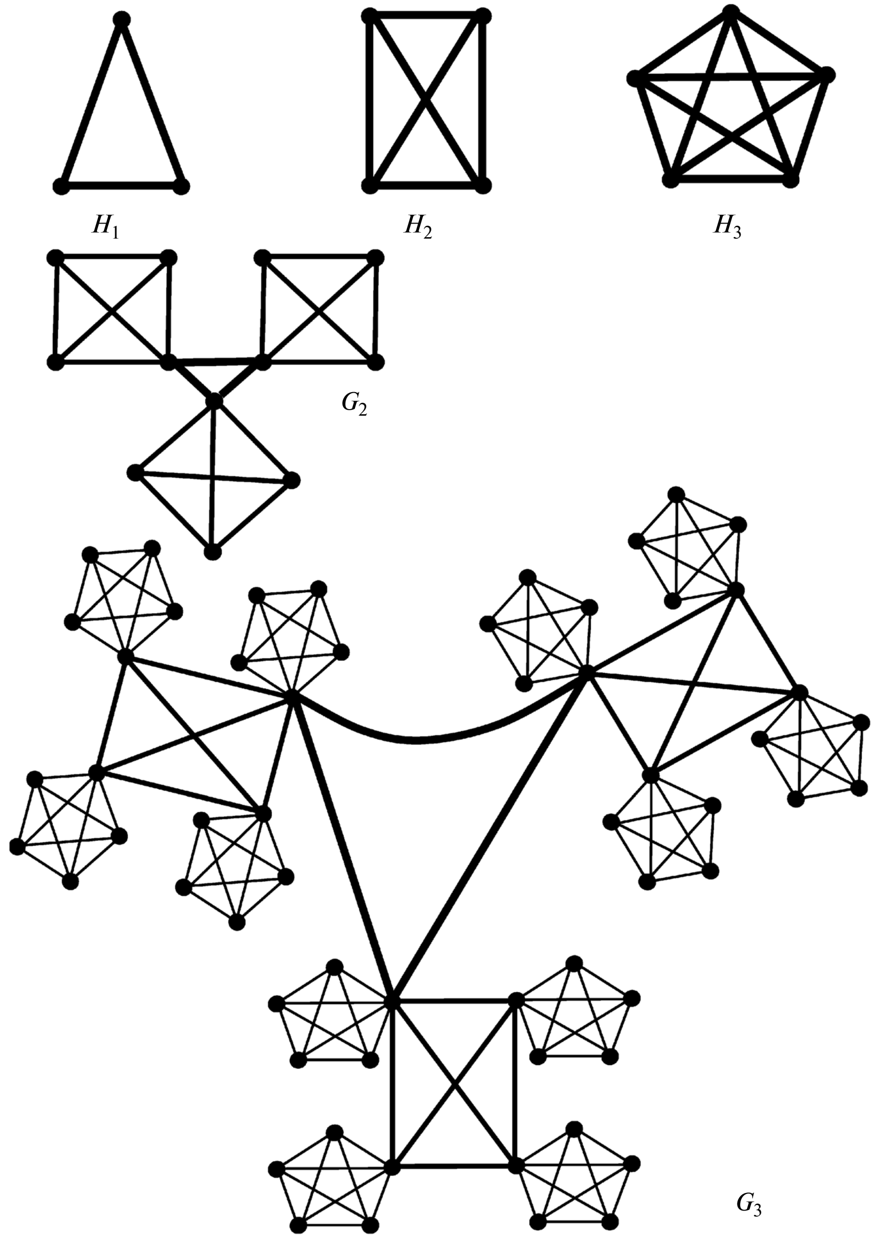
In the classical graph theory, the key optimization problem is to find a subgraph (or spanning tree) possessing given characteristics (e.g., to find a spanning tree of minimum weight). In dynamic graph theory, the key problem is to establish a relationship between the solutions of an optimization problem on different classical (time-independent) graphs that compose the dynamic graph. If the solutions on different graphs are comparable in terms of given criteria, we can speak of the *inheritance in the class of dynamic graphs* with common transition rules in the sequences of underlying graphs. A logical continuation of this problem is establishing a formal connection between a hereditary property and transition operations in the trajectory that form the dynamic graph. If such a connection were established, we could speak of programmable self-organization (see [16]), i.e., of obtaining guaranteed inherited structure properties and characteristics of dynamic graphs.

Recall that *the eccentricity* (see [10]) of an arbitrary fixed vertex of a graph is defined as the maximum distance from it to all other vertices. The greatest eccentricity is by definition equal to the graph diameter. The graph vertices whose eccentricities are equal to the graph diameter are called *peripheral* (see [10]).

Lemma. *For a dynamic graph* Γ *in which the transition operation* $\varphi(G_l) = G_{l+1}$ $(l = 1, 2, \ldots, L, \ldots)$ *in the trajectory is defined as the connection of one vertex to any nonperipheral vertex of* G_l *(see* [10]*), the diameter* $d(G_l) = d(G_1)$ *(see* [10]*) does not change if* G_1 *contains at least one nonperipheral vertex.*

Corollary. For a dynamic graph Γ in which the transition operation $\varphi(D_l) = D_{l+1}$ $(l = 1, 2, \ldots, L, \ldots)$ in the trajectory is defined as the connection of one vertex to any nonleaf vertex of the tree D_l (see [8]), the diameter $d(D_l) = d(D_1)$ does not change.

Remark. The graph G_1 in the trajectory of the dynamic graph Γ can consist of only peripheral vertices. An example is a complete graph, in which each vertex is connected to all other vertices. In this case, the operation in Lemma 4 is inapplicable.



Trajectory of the prefractal graph generated by the set of complete seeds with ordered growth.

Theorem. *For a dynamic graph Γ in which the transition operation $\varphi(G_l) = G_{l+1}$ $(l = 1, 2, \ldots, L, \ldots)$ in the trajectory is defined as the connection of a new vertex to any number of nonperipheral vertices of G_l, the diameter $d(G_l) \le d(G_1)$ does not increase if G_1 contains at least one nonperipheral vertex.*

This theorem can be considered as an abstract explanation of the well-known Milgram's six handshakes hypothesis [16]. According to this hypothesis, any two persons on the Earth are connected just via six people that know each other. This hypothesis was experimentally confirmed for various social networks (see [17, 18]), i.e., for different communities of people. The structure of connections in such communities can be represented by a graph, and the change of its structure can be described by simple graph-theoretic operations. Thus, a model of social network growth can be described by a dynamic graph. Presently, there is no rigorous analytical justification of the six handshakes hypothesis. From the viewpoint of graph theory, the graph of a social network is a graph in which ô considerable part of distances between its vertices is equal to six. The six handshakes property of social networks is preserved in the course of its formation and evolution. For that reason, the dynamic graph that models a social network also preserves this property in its trajectory. In an idealized and simplified representation, the dynamic graph of a social network should preserve its diameter in a neighborhood of six. In an even greater simplification, the dynamic graph should preserve a constant diameter in its trajectory. The theorem suggests a simple mechanism that guarantees the conservation of the initial diameter, but it does not explain while Milgram's hypothesis is based on exactly six handshakes.

4. SOME ENGINEERING APPLICATIONS OF DYNAMIC GRAPH THEORY

The principles and methods of dynamic graph theory are especially useful for the design of command and data interaction between mobile subscribers in network systems (see [19]). Network systems are engineering systems based on networks. In this sense, network systems are in a greater degree an engineering concept than a rigorous mathematical one.

The history of wireless network development shows that the field of application of this class of telecommunication technologies grows. Presently, wireless networks are superior to wired networks in terms of security, cost, reliability, functionality, and convenience of use. Furthermore, the range of tasks related to new applications of wireless technologies and networks still grows. There are two main fields of application of wireless networks—telecommunications and monitoring. In "large" systems, wireless networks can perform both data transfer and monitoring functions.

Networks with mobile subscribers (sensors) are of special interest. It is extremely important to ensure high-quality communication is such networks. This will improve the connectedness and increase the data transfer rate between mobile agents and reduce the expenses for the ground network segment due to routing and retranslation between mobile nodes. The complexity of this problem grows with the increasing number of network subscribers. It is clear that the efficiency of network operation can be improved using coordinated actions of the subscribers. In the recent decade, a new concept of the organization of team operations has been under development. However, the vast majority of the existing networking algorithms are actually specific engineering solutions.

The appearing dynamic graph theory can provide a theoretical basis for the design of algorithms of command and information interaction of mobile subscribers in network systems. The topology of a network consisting of mobile subscribers cannot be strictly fixed. Moreover, this topology must change due to a variety of reasons, e.g., due to the increase in the number of subscribers in the network. Since data transfer in a network depends on the length of the chain of subscribers, it is reasonable to control the network diameter as new subscribers join the network. This can be done using an algorithm that in the trivial case satisfies the requirements of the lemma.

5. PROOF OF THE LEMMA AND THE THEOREM

Proof of the lemma. 1. Consider a simple n-vertex chain $C_n = (V^{C_n}, E^{C_n})$. It is clear that the length of the only diametral chain in the graph C_n is $n - 1$. Both peripheral vertices of C_n coincide with its leaf vertices. Therefore, the addition of a vertex to any peripheral vertex of C_n yields a graph C_{n+1}, which also consists of a single diametral chain of length n. It is known that the graph diameter is given by the rule $d(G) = \max_{v\in V} \varepsilon(v)$ (see [10]); in turn, the eccentricity is $\varepsilon(v) = \max_{u\in V} \rho(v,u)$ (see [10]), where $\rho(v,u)$ is the distance between the vertices $v \in V$ and $u \in V$ of the graph $G = (V, E)$. It is clear that the eccentricity of any nonperipheral vertex $w \in V^{C_n}$ satisfies the inequality $\varepsilon(w) \le n - 2$.

Let the graph $G_2 = (V_2, E_2) = \varphi(C_n)$, where φ is the transition operation defined in the condition of the lemma. Then, the eccentricity of a vertex $v' \in V_2$ that is adjacent to any nonleaf vertex of C_n and such that $G_2 - v' = C_n$, satisfies the inequality $\varepsilon(v') \le n - 1$. Hence, the diameter is given by $d(G_2) = d(\varphi(C_n)) = d(C_n) = n - 1$.

2. The assertion justified in the first part of this proof remains valid for any diametral chain of the arbitrary graph G_1 if the added vertex is adjacent to none of the peripheral vertices of G_1. This makes it possible to preserve the diameter of all the graphs G_l for $l = 1, 2, \ldots, L, \ldots$ in the trajectory of the dynamic graph Γ. This completes the proof of the lemma.

Proof of the corollary. In any tree D, only leaf vertices can be peripheral because the eccentricity of a leaf vertex is always greater than the eccentricity of the vertices adjacent to it (see [10]). Therefore, for a dynamic tree, the application of the transition operation $\varphi(D_l) = D_{l+1}$, $l = 1, 2, \ldots, L, \ldots$, satisfying the condition of the lemma to nonleaf vertices preserves the diameter on the entire trajectory, which completes the proof of the corollary.

Proof of the theorem. 1. As before, consider a simple n-vertex chain $C_n = (V^{C_n}, E^{C_n})$, which coincides with its only diametral chain of length $n - 1$.

Let $v' \in V^{C_n}$ and $v'' \in V^{C_n}$ be the peripheral vertices of C_n, which also are the only leaf vertices of C_n. Let $u' \in V^{C_n}$ and $u'' \in V^{C_n}$ be two nonadjacent nonperipheral vertices of C_n. The distance between them obviously satisfies the inequality $\rho(u', u'') \geq 2$. Add a new vertex w to C_n in such a way that it is adjacent to u' and u'' in the new graph $C_n + w$. Since the vertices u' and u'' are the endpoints of the chain (u', w, u'') in $C_n + w$, its length, due to the inequality $\rho(u', u'') \geq 2$ is now $\rho(u', u'') = 2$. Therefore, the diametral chain in $C_n + w$ is not longer than the diametral chain in C_n, i.e., $d(C_n + w) \leq d(C_n)$. By reasoning in a similar way, we can show that $d(C_n + w) = d(C_n)$ if the vertex is added to C_n in such a way that it is adjacent to two adjacent nonperipheral vertices. Both versions of adding a vertex to C_n do not increase its diameter.

2. Let the graph $C_n + w$ be obtained from C_n by adding a vertex w in such a way that w is adjacent to m arbitrary nonperipheral vertices of C_n and $m \leq n - 2$. According to the first part of the proof of this theorem, the addition of w does not increase the eccentricities of the vertices of $C_n \subseteq C_n + w$ independently of the mutual arrangement of $u_1, u_2, \ldots, u_m$, i.e., on the distances between them; furthermore, $\varepsilon(w) \leq n - 2$. Therefore, $d(C_n + w) \leq d(C_n)$.

3. Consider a dynamic graph Γ in which the transition operation $\varphi(G_l) = G_{l+1}$ $(l = 1, 2, \ldots, L, \ldots)$ is defined according to the condition of the theorem. For any diametral chain in G_l, item 2 of the proof of the present theorem holds. Therefore, $d(G_l) \geq d(G_{l+1})$ and, hence, $d(G_l) \geq d(G_1)$. This completes the proof of the theorem.

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