Problem 1

Calculate and compare the expected value and standard deviation of price at time $t(P_t)$, given each of the 3 types of price returns, assuming $r_t \sim N(0, \sigma^2)$. Simulate each return equation using r and show the mean and standard deviation match your expectations.

Analysis

Firstly, give the 3 types of prices returns, I calculated the mean and standard deviation as expectations of each of them manually.

$$E(r_t) = 0$$

$$std(r_t) = \sigma$$

For Classical Brownian Motion:

$$P_t = P_{t-1} + r_t$$

Mean:

$$E(P_t) = E(P_{t-1} + r_t)$$
$$E(P_t) = E(P_{t-1}) + E(r_t)$$
$$As\ E(P_{t-1})\ is\ known\ and\ E(r_t) = 0$$
$$E(P_t) = P_{t-1}$$

Standard deviation:

$$std(P_t) = std(P_{t-1} + r_t)$$
$$std(P_t) = std(P_{t-1}) + std(r_t)$$
$$As\ std(P_{t-1}) = 0$$

$$E(P_t) = \sigma$$

For Arithmetic Return System:

$$P_t = P_{t-1}(1 + r_t)$$

Mean:

$$E(P_t) = E[P_{t-1}(1 + r_t)]$$
$$E(P_t) = P_{t-1}E(1 + r_t)$$
$$E(P_t) = P_{t-1}[1 + E(r_t)]$$
$$As\ E(r_t) = 0$$
$$E(P_t) = P_{t-1}$$

Standard Deviation

$$std(P_t) = std[P_{t-1}(1 + r_t)]$$

$$std(P_t) = std(P_{t-1} + P_{t-1}r_t)$$
$$std(P_t) = std(P_{t-1}r_t)$$
$$std(P_t) = P_{t-1}\sigma$$

For Geometric Brownian Motion:

$$P_t = P_{t-1}e^{r_t}$$

Mean:

$$E(P_t) = E(P_{t-1}e^{r_t})$$
$$E(P_t) = P_{t-1}E(e^{r_t})$$
$$As\ e^{r_t}\ is\ lognormal$$
$$E(P_t) = P_{t-1}e^{\frac{\sigma^2}{2}}$$

Standard Deviation:

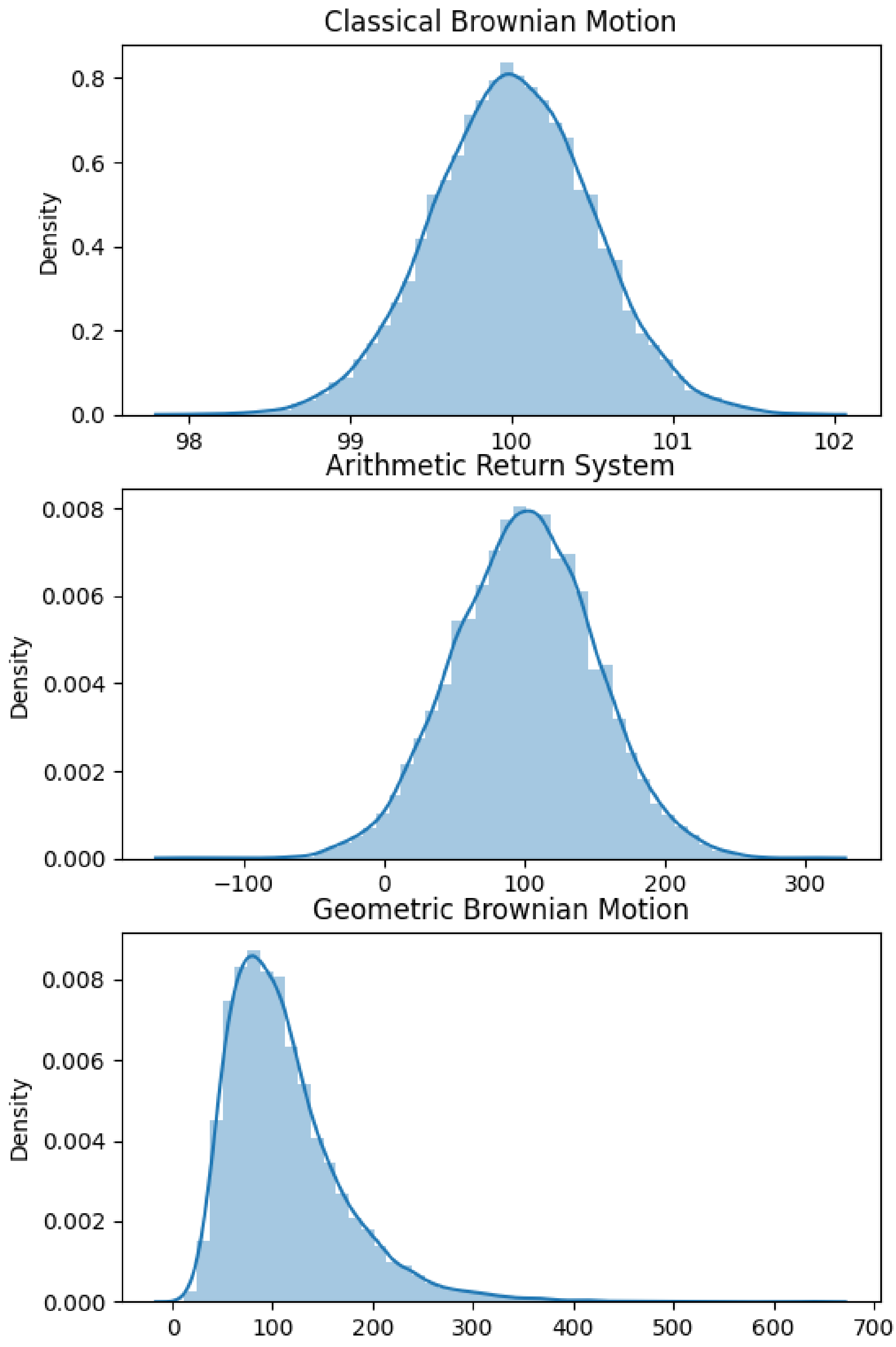
$$std(P_t) = std(P_{t-1}e^{r_t})$$
$$std(P_t) = P_{t-1}std(e^{r_t})$$
$$std(P_t) = P_{t-1}\left[\left(e^{\sigma^2} - 1\right)e^{\sigma^2}\right]^{\frac{1}{2}}$$

I set $P_{t-1} = 100$, $std(r_t) = 0.5$, and simulated each process 10000 times to get the sample means and standard deviations and calculated the expected values of each. The results are as follows:

	Mean		Standard Deviation	
	Expected Mean	Simulated Mean	Expected Standard Deviation	Simulated Standard Deviation
Classical Brownian	100	99.9975	0.5	0.4996
Arithmetic Return	100	99.5737	50	50.3797
Geometric Brownian	113.3148	113.0685	60.3900	60.3873

As can be seen from the table that the results of simulation match my expected values, the differences are very small.

And I plotted the distribution of the simulated P_t using these 3 methods.


Classical Brownian Motion
Density
0.0
0.2
0.4
0.6
0.8
98
99
100
101
102
Arithmetic Return System
Density
0.000
0.002
0.004
0.006
0.008
−100
0
100
200
300
Geometric Brownian Motion
Density
0.000
0.002
0.004
0.006
0.008
0
100
200
300
400
500
600
700

Problem 2

Implement a function similar to the "return_calculate()" in this week's code. Allow the user to specify the method of return calculation.

Use DailyPrices.csv. Calculate the arithmetic returns for all prices.

Remove the mean from the series so that the mean(META)=0

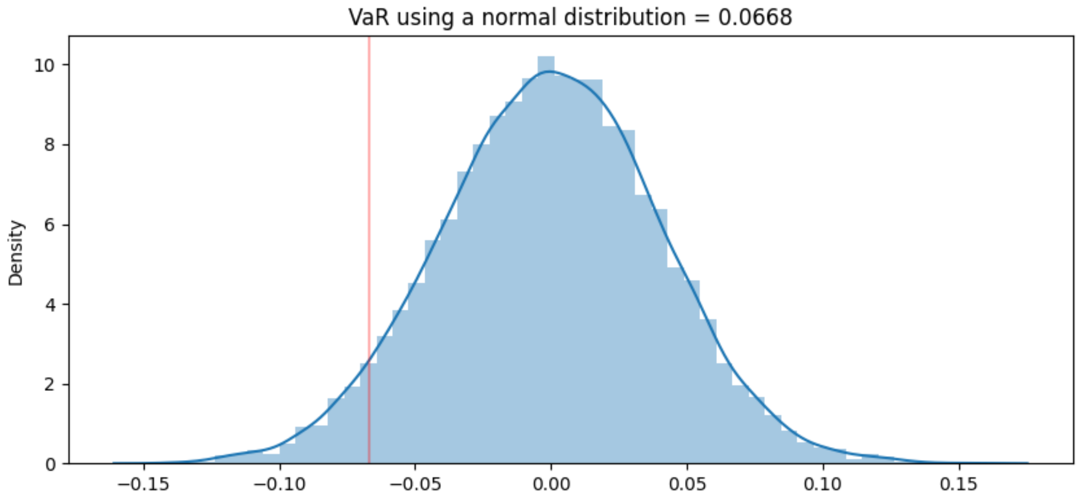
Calculate VaR
1. Using a normal distribution.
2. Using a normal distribution with an Exponentially Weighted variance ($\lambda = 0.94$)
3. Using a MLE fitted T distribution.
4. Using a fitted AR(1) model.
5. Using a Historic Simulation.

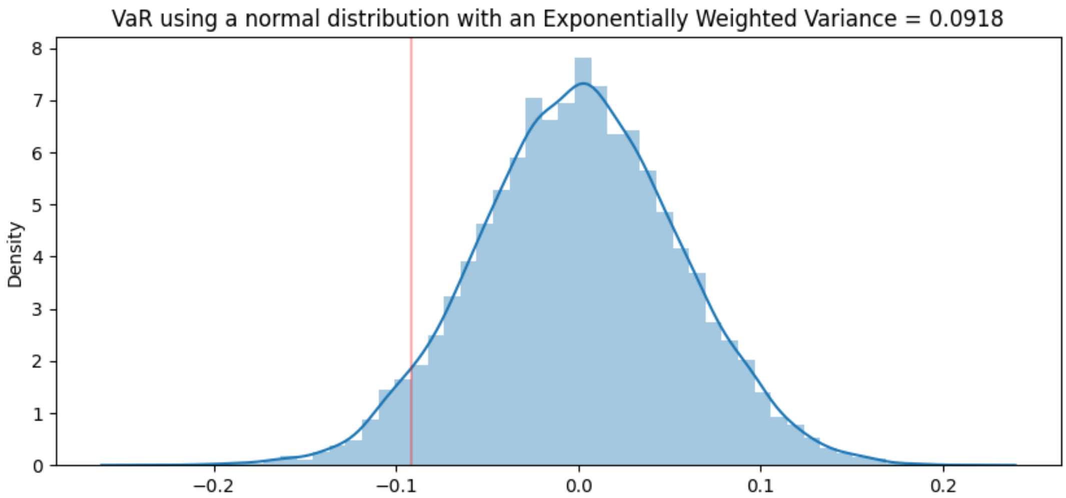
Compare the 5 values.

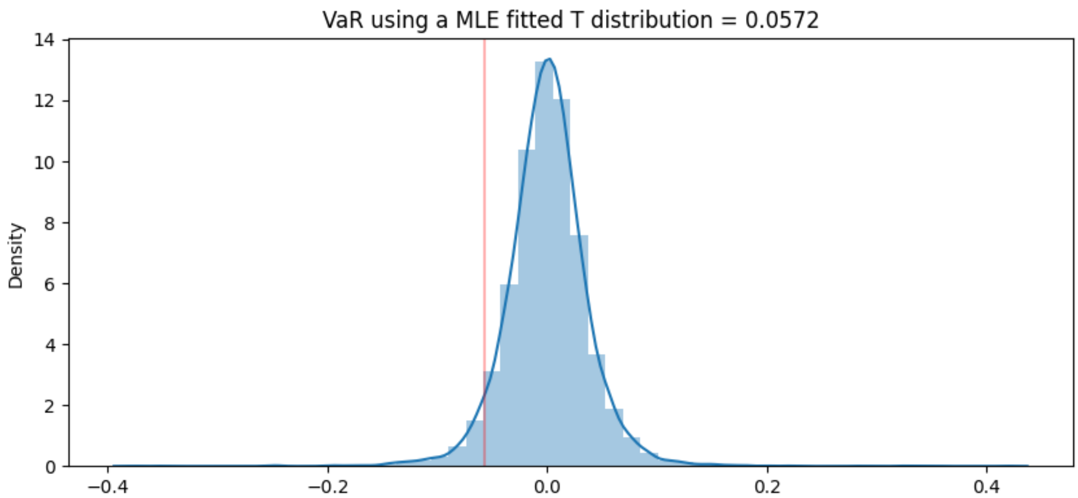
Analysis

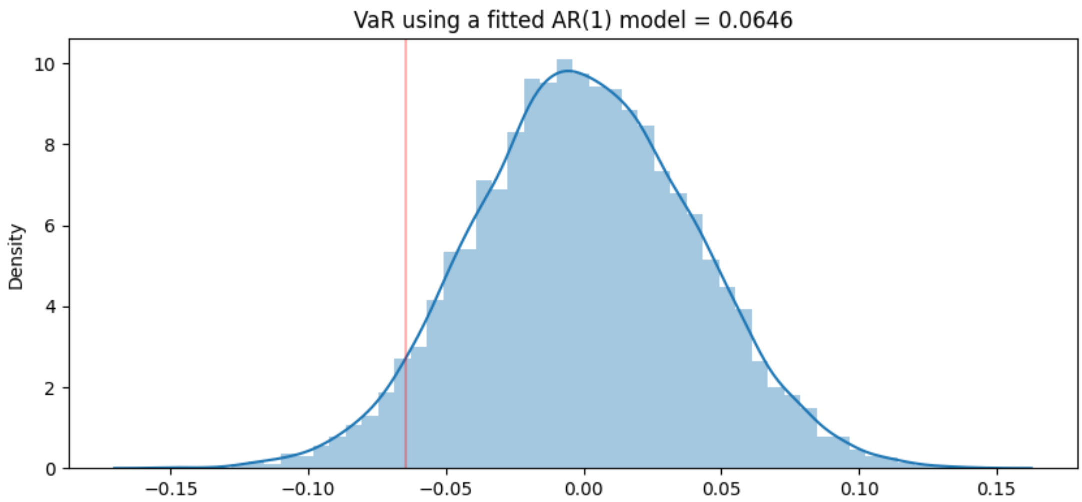
Firstly, I created a function called "return_calculate()" to calculate returns of all stocks in DailyPrices.csv, at the same time allow users to choose which method to use to calculate returns.

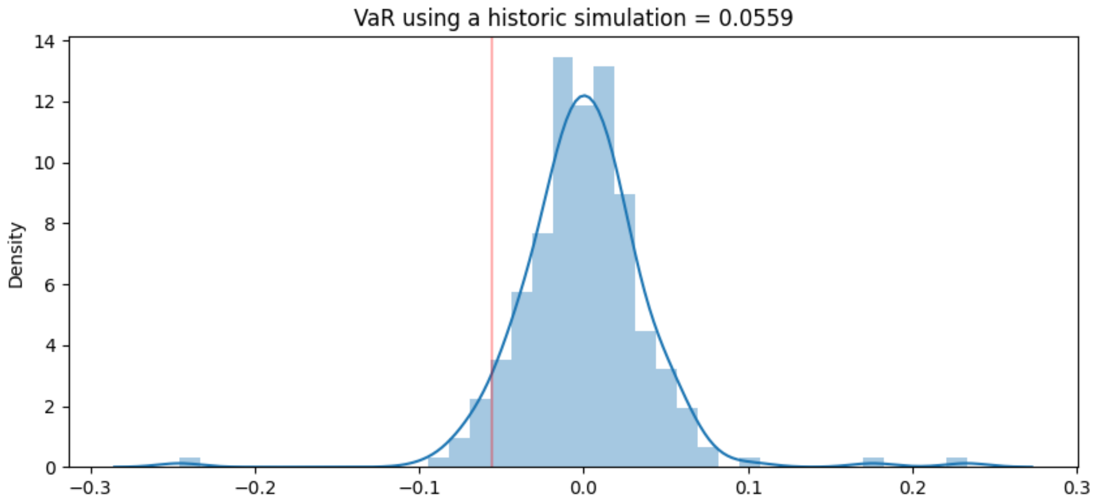
Then I extracted the return data of META and removed its mean by subtracting its mean from every return data. And, I calculated the VaR in return using the above 5 ways. The results are as follows:


VaR using a normal distribution = 0.0668
Density
0
2
4
6
8
10
−0.15
−0.10
−0.05
0.00
0.05
0.10
0.15


VaR using a normal distribution with an Exponentially Weighted Variance = 0.0918
Density
0
1
2
3
4
5
6
7
8
−0.2
−0.1
0.0
0.1
0.2


VaR using a MLE fitted T distribution = 0.0572
Density
0
2
4
6
8
10
12
14
−0.4
−0.2
0.0
0.2
0.4





Then, I transferred the results to VaR in dollar, to measure the minimum we can expect to lose in money on a 5% bad day.

Method	VaR in dollar
Normal distribution	11.6660
Normal distribution with an EWV	16.3619
MLE fitted T distribution	10.1234
Fitted AR(1) model	11.3426
Historic simulation	9.9498

As can be seen from the above table, VaR calculated by historic simulation is the lowest, 9.9498, and VaR calculated by normal distribution with an EWV is the highest one, 16.3619, which is almost the double of historical one. And it can be seen that using MLE fitted T distribution to simulate the returns, the obtained data is most similar to the historical data, which is 10.1234. And using normal distribution and fitted AR(1) model to calculate VaRs, the results are familiar. So, the simulation using MLE fitted T distribution can best describe the distribution of historical data. Simulations using normal distribution and fitted AR(1) model are also close to the distribution of historical data. The simulation using normal distribution with an EWV is not that same as the the distribution of historical data.

Problem 3

Using Portfolio.csv and DailyPrices.csv. Assume the expected return on all stocks is 0.

This file contains the stock holdings of 3 portfolios. You own each of these portfolios. Using an exponentially weighted covariance with lambda = 0.94, calculate the VaR of each portfolio as well as your total VaR (VaR of the total holdings). Express VaR as a $.
Discuss your methods and your results.

Choose a different model for returns and calculate VaR again. Why did you choose that model? How did the model change affect the results?

Analysis

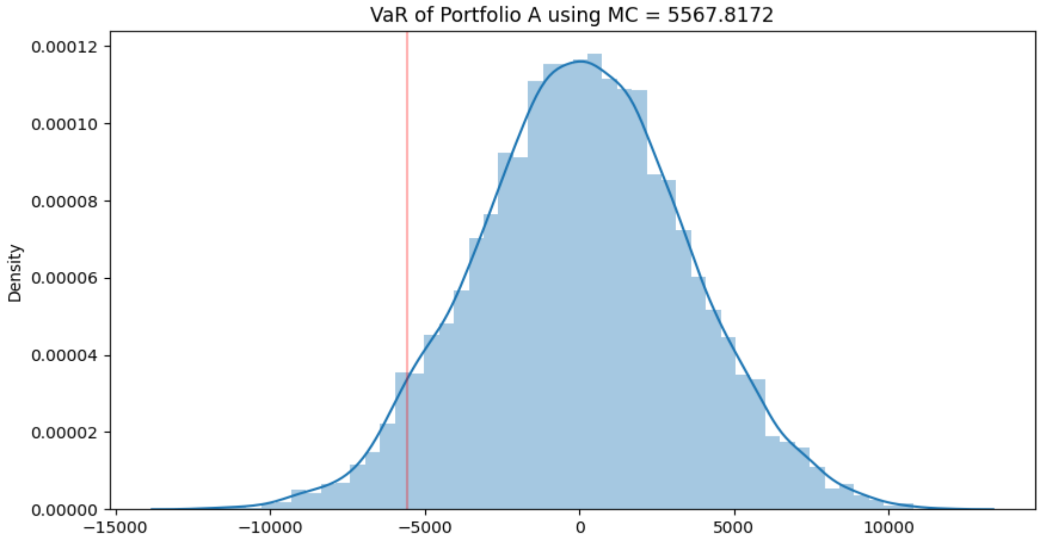
Firstly, I calculated Delta Normal VaR of each portfolio by assuming payoffs are linear and returns are distributed multivariate normal. Because I have no other information to calculate the $\delta = \frac{dA_i}{dP_i}$, I assumed that all $\delta = 1$.

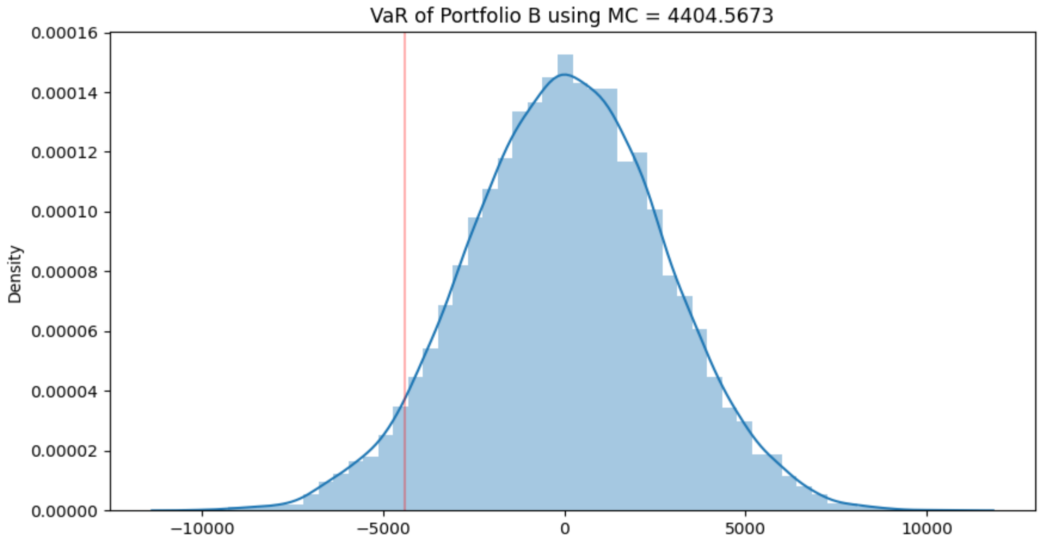
$$VaR(\alpha) = -PV * F_X^{-1}(\alpha) * (\nabla R^T \Sigma \nabla R)^{\frac{1}{2}}$$

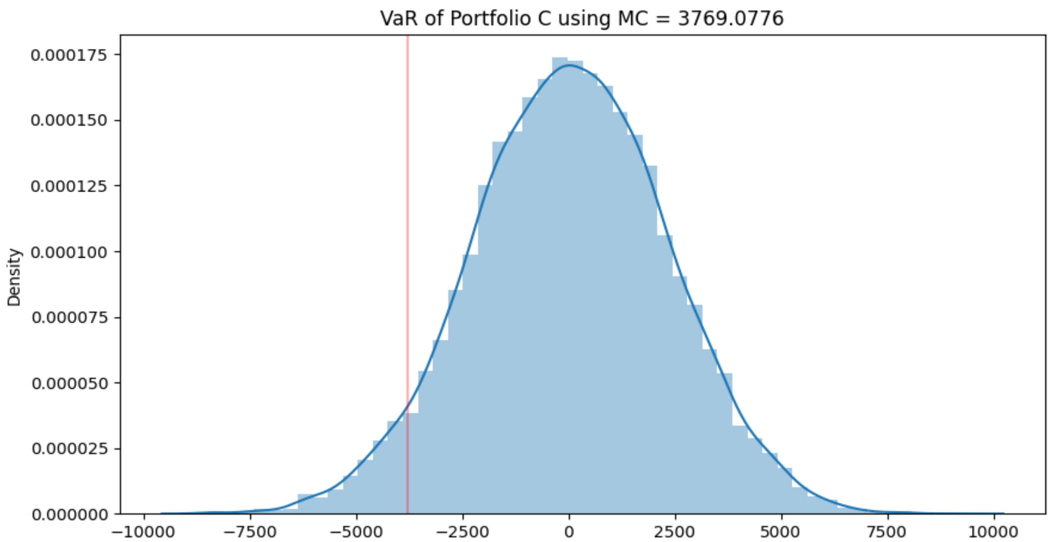
$PV = Portfolio\ Value$
$F_X^{-1}(\alpha) = The\ quantile\ function\ for\ standard\ normal$
$\nabla R = Gradient\ of\ the\ portfolio\ return\ respect\ to\ underlying\ price\ returns$
Σ = Exponentially weighted covariance matrix

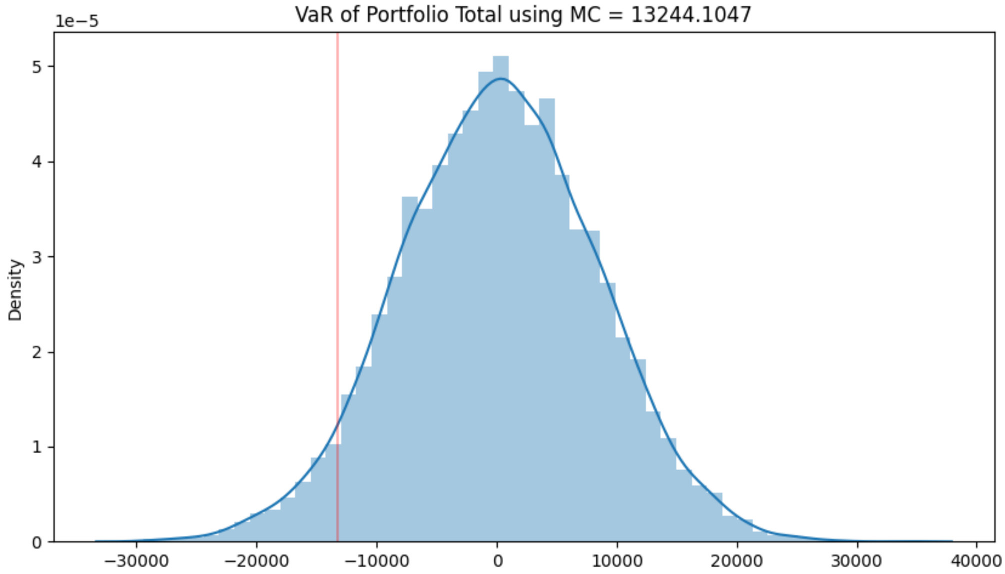
Then, I relaxed linearity assumption, and calculated Normal Monte Carlo Var. The results are as follows.

	PV($)	VaR_Delta($)	VaR_MC($)
A	299950.0590	5670.2029	5567.8172
B	294385.5908	4494.5984	4404.5673
C	270042.8305	3786.589	3769.0776
TOTAL	864378.4804	13577.0754	13244.1047


VaR of Portfolio A using MC = 5567.8172
Density
0.00000
0.00002
0.00004
0.00006
0.00008
0.00010
0.00012
−15000
−10000
−5000
0
5000
10000


VaR of Portfolio B using MC = 4404.5673
Density
0.00000
0.00002
0.00004
0.00006
0.00008
0.00010
0.00012
0.00014
0.00016
−10000
−5000
0
5000
10000


VaR of Portfolio C using MC = 3769.0776
Density
0.000000
0.000025
0.000050
0.000075
0.000100
0.000125
0.000150
0.000175
−10000
−7500
−5000
−2500
0
2500
5000
7500
10000

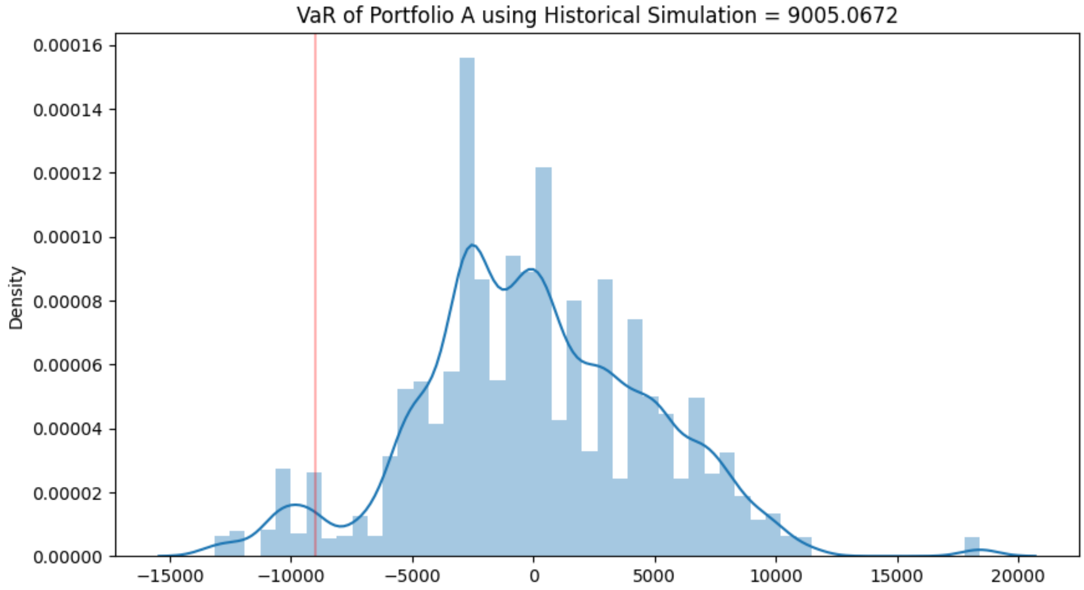


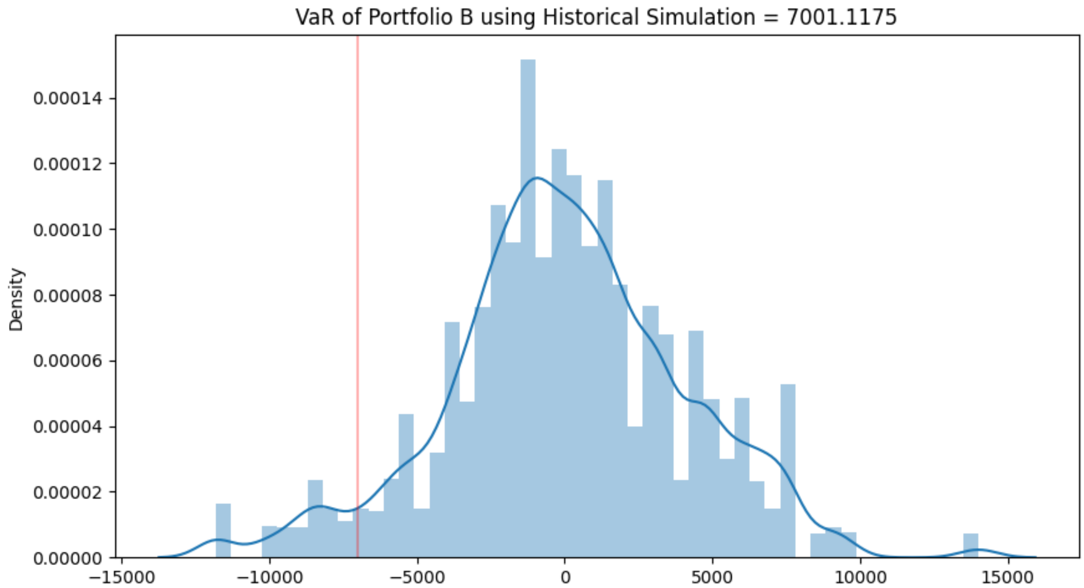
As can be seen from the above table and charts. The results of Delta Normal VaR and Normal Monte Carlo Var are pretty similar.

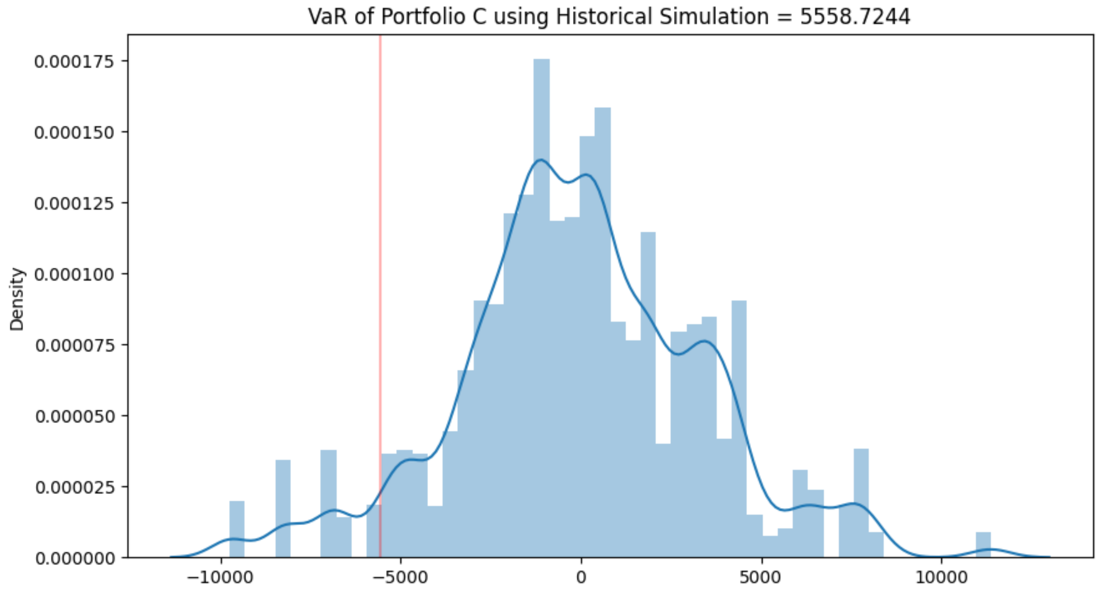
Then I used normaltest to test whether the historic data are in normal distribution. And I found that:
The proportion of Portfolio A that does not fit a normal distribution is 0.7143.
The proportion of Portfolio B that does not fit a normal distribution is 0.7812.
The proportion of Portfolio C that does not fit a normal distribution is 0.8125.
The proportion of Portfolio TOTAL that does not fit a normal distribution is 0.7677.

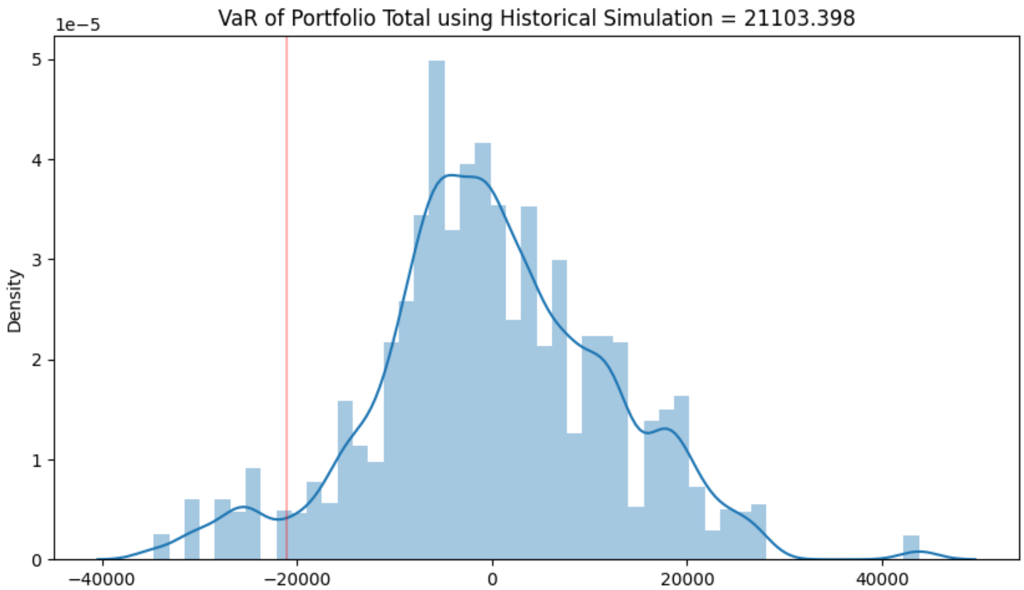
The above test results showed that the assumptions of normality cannot hold in historic data. So, I chose to use historic data to calculate Historical VaR, in which I relaxed both the linearity and normality assumption. I added the Historical VaRs to the above table to compare.

	PV($)	VaR_Delta($)	VaR_MC($)	His_VaR
A	299950.0590	5670.2029	5567.8172	9005.0672
B	294385.5908	4494.5984	4404.5673	7001.1175
C	270042.8305	3786.589	3769.0776	5558.7244
TOTAL	864378.4804	13577.0754	13244.1047	21103.398


VaR of Portfolio A using Historical Simulation = 9005.0672
Density
0.00016
0.00014
0.00012
0.00010
0.00008
0.00006
0.00004
0.00002
0.00000
−15000
−10000
−5000
0
5000
10000
15000
20000


VaR of Portfolio B using Historical Simulation = 7001.1175
Density
0.00014
0.00012
0.00010
0.00008
0.00006
0.00004
0.00002
0.00000
−15000
−10000
−5000
0
5000
10000
15000





It can be seen from the above table that Historical VaRs are much larger than Delta Normal VaR and Normal Monte Carlo Var. This means the actual money loss may much larger than the estimated money loss under normality.