AE 352 Final Project: Quadcopter

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https://github.com/mugreenstein/AE-352-Final-Project

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I. Nomenclature

Θ	=	roll angle (rotors 1 and 3)
Φ	=	pitch angle (rotors 2 and 4)
Ψ	=	yaw angle (rotors 1–4)
r_{rotor}, r_r	=	distance from center of mass to each rotor
x_i	=	x–distance from center of mass to rotor i, $i \in \{1, 2, 3, 4\}$
y_i	=	y–distance from center of mass to rotor i
z_i	=	z–distance from center of mass to rotor i
x_{com}, x_c	=	x–position of center of mass in ground frame
y_{com}, y_c	=	y–position of center of mass in ground frame
z_{com}, z_c	=	z–position of center of mass in ground frame
V_{drone}	=	generalized drone velocity (Lagrangian)
V_i	=	generalized rotor velocity for rotor i
V_1	=	rotor 1 velocity magnitude
V_2	=	rotor 2 velocity magnitude
V_3	=	rotor 3 velocity magnitude
V_4	=	rotor 4 velocity magnitude
$\dot{x}_i$	=	x–component velocity of rotor i
$\dot{y}_i$	=	y–component velocity of rotor i
$\dot{z}_i$	=	z–component velocity of rotor i
$\dot{\theta}$	=	angular velocity of roll
$\dot{\phi}$	=	angular velocity of pitch
$\dot{\psi}$	=	angular velocity of yaw
M	=	mass of the drone
I_{body}	=	moment of inertia of drone body
ω_1	=	angular velocity of rotors 1 and 3
ω_2	=	angular velocity of rotors 2 and 4
q	=	generalized coordinate
τ_{thrust}	=	torque generated by rotor thrust (roll, pitch, yaw)
F_{thrust}	=	thrust force produced by each rotor

II. Introduction and Summary of Problem

Unmanned Aerial Vehicles (UAVs) are emerging vehicles in the aerospace industry. They are highly useful because their small size and maneuverability allow them to operate in environments where larger, less agile vehicles cannot. Their versatility stems from their ability to operate in environments that would otherwise be challenging for traditional aircraft. This includes confined spaces, low altitude, and missions that require rapid directional changes. This makes drones, specifically quadcopters, perfect for aerial imaging, autonomous navigation, and emergency response missions, among many other tasks. We are analyzing a specific UAV, a quadcopter; it is a drone with 4 axisymmetric rotors capable of hovering and vertical take-off and landing (VTOL). The independent rotor speeds regulate the quadcopter's attitude, including its roll, pitch, and yaw, and therefore its overall motion. Our goal is to create a dynamic and mathematical model that will be able to describe the equations of motion of the quadcopter through the changing revolutions of the four rotor blades.

Quadcopters generate forces and torques by independently adjusting their rotor speeds. Accurate dynamic modeling is therefore essential for predicting their motion, which is governed by coupled translational and rotational dynamics. Representing this behavior requires accounting for all six degrees of freedom.

To make sure our model works as intended and is realistic, we have defined four engineering requirements to which we must adhere. These requirements help with the accuracy of predicting and controlling the drone's dynamics. They are listed in order as follows:

- **ER1:** The drone model shall have 6 degrees of freedom (center of mass motion plus attitude) and account for gravity and lift generated from 4 rotors.
- **ER2:** The drone model shall be able to modify its state and attitude solely through increasing or decreasing the number of revolutions per second of its 4 rotor blades, as determined by the motor torques.
- **ER3:** The model must be able to complete the following performance requirements without exhausting its power supply:
  - Hover 1 m above ground for 2 minutes.
  - Fly in a circle of radius 2 m, at an altitude of 1 m above ground, at a speed of 0.5 m/s for at least 1 minute.
  - Launch from the ground and ascend vertically until 1 m above ground. Move in a straight line 1 m above ground at an average speed of 1 m/s for 5 m. Stop (hover), yaw 90° to the left, and move in another straight line for 5 m. Stop (hover), land vertically with a speed of no more than 1 cm/s.
- **ER4:** The model, as determined through its parameters such as mass, power consumption, engine torques, and rotor blade size shall be realistic and implementable, preferably through the use of off-the-shelf parts. In particular, the mass of the drone shall be between 0.1 kg and 10 kg.

To properly complete the mission, our model must fulfill each one of these requirements. Now that we have a clear statement of our goals we can jump into the mathematical modeling of the system. There are several parts that are needed in order to create a dynamic model. These include:

- Initial conditions - this includes position, orientation, linear and angular velocity, and the individual rotor speeds
- Inertial Parameters - this includes mass, frame dimensions, and the full inertia tensor of the body

- Force/Torque from the controller - this relates the rotor's angular velocities to thrust and reaction torque
- Inertia tensor from controller - this allows rotational dynamics to be integrated alongside the translational equations

With all of these elements assembled, the resulting model will serve as the foundation for analyzing the quadcopter's performance. We can evaluate whether or not the drone will be able to complete its assigned performance goals using the equations of motion that were derived. We aim to understand the dynamics and behavior of a drone that we designed to fit the parameters listed.

NOTE: Additional derivations A, larger figures B, and hand-written C work can be found in the **Appendix** VII. Extra work is stored in the Appendix in order not to exceed the page limit.

III. Mathematical Derivation

A. Lagrangian Energy Formulation

This model for translational and rotational kinetic energy forms the foundation for the equations of motion used throughout this project. Therefore, we derived equations of motion dependent on pitch, roll, and yaw to fit any situation or motion needed for the drone. For the mission of this project, we constrained the final equations of motion to fit the requested motions (i.e., hovering, etc).

$$T = \tfrac{1}{2}Mv_{\text{drone}}^2 + \tfrac{1}{2}\boldsymbol{\omega}_{\text{body}}^\top I_{\text{body}}\boldsymbol{\omega}_{\text{body}} + \sum_i \left(\tfrac{1}{2}m_{r,i}v_{r,i}^2 + \tfrac{1}{2}\boldsymbol{\omega}_{r,i}^\top I_{r,i}\boldsymbol{\omega}_{r,i}\right), \tag{1}$$

$$V = Mgz_{\text{com}} + \sum_i m_{r,i}gz_{r,i}, \tag{2}$$

$$L = T - V. \tag{3}$$

$$\begin{aligned} \dot{v}_i^2 = \;& \dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2 + 2r_r\Big[\dot{\theta}\big(-\dot{x}_c\sin\theta + \dot{z}_c\cos\theta\big) + \dot{\phi}\big(-\dot{y}_c\sin\phi + \dot{z}_c\cos\phi\big) + \dot{\psi}\big(\dot{x}_c\cos\phi - \dot{y}_c\sin\psi\big)\Big] \\ & + r_r^2(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2) + 2r_r^2\Big(-\dot{\theta}\dot{\psi}\sin\theta\cos\psi + \dot{\phi}\dot{\psi}\sin\phi\sin\psi + \dot{\theta}\dot{\phi}\cos\theta\cos\phi\Big). \end{aligned} \tag{4}$$

In order to see the setup and thought process of the translational and rotational kinetic energies and velocities, see Appendix A.A.

B. Motor Thrust and Torque Mapping

There is a specific angular velocity for the rotors required to balance our drone. We use $\omega_{hover} = 400$. This value is obtained by combining typical thrust characteristics of comparable propellers, the drone's mass, and a simple vertical force balance. This $omega_{hover}$ is treated as our baseline [1].

Now that we have the ω to maintain equilibrium in the vertical direction, our thrust force must be equal to gravity. We found our equations for thrust and K_T from Gibiansky [1]. The thrust equation that they derived was for one rotor; however, because we have 4 rotors, we must account for that by finding the total thrust, thus multiplying ω by 4.

$$F_g = T = M_{total} * g$$

$$T = 4\omega_{hover}^2 K_T$$

$$K_T = \frac{M_{total} * g}{4 * \omega_{hover}^2} \tag{5}$$

The k drag coefficient can be shown as: $k_{drag} = 1 * 10^{-6}$ [2].

Each rotor produces a thrust and a reaction torque (from air resistance of the props):

$$F_D = \tfrac{1}{2}\rho C_D A v^2 \;\longrightarrow\; \tau_D = \tfrac{1}{2}R\rho C_D A v^2 \;\longrightarrow\; \tau_{d,i} = k_{\text{drag}}\,\omega_i^2$$

$$\textbf{Roll Torque:} \qquad \tau_\phi = Lk_T(\omega_1^2 - \omega_3^2) \tag{6}$$

$$\textbf{Pitch Torque:} \qquad \tau_\theta = Lk_T(\omega_2^2 - \omega_4^2) \tag{7}$$

$$\textbf{Yaw Torque:} \qquad \tau_\psi = k_{\text{drag}}(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2) \tag{8}$$

To see the vector form of the thrust and torques as well as the allocation matrix and motor speed equations see Appendix Section A.B.

C. Euler Angle Kinematics

The drone attitude is represented using a 3–2–1 (yaw–roll–pitch) Euler angle sequence. The relationship between the body-frame angular rates and the Euler–angle rates is

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{T}^{-1}(\phi, \theta) \begin{bmatrix} p \\ q \\ r \end{bmatrix},$$

where the inverse transformation matrix is defined in Appendix A.C.

$$\mathbf{T}^{-1}(\phi, \theta) = \begin{bmatrix} 1 & \sin\phi\,\tan\theta & \cos\phi\,\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \dfrac{\sin\phi}{\cos\theta} & \dfrac{\cos\phi}{\cos\theta} \end{bmatrix}.$$

This yields the Euler–angle kinematic equations used in the simulation:

$$\dot{\phi} = p + q\,\sin\phi\,\tan\theta + r\,\cos\phi\,\tan\theta,$$

$$\dot{\theta} = q\,\cos\phi - r\,\sin\phi,$$

$$\dot{\psi} = q\,\sin\phi\,\sec\theta + r\,\cos\phi\,\sec\theta.$$

The full derivation of these relations and the full rotation matrix is provided in Appendix A.C.

D. Simplified Translational Equations of Motion

We begin by doing simple $F = MA$ EOMs in the Newtonian form. Body forces are added, which in this case is only mg, as air resistance on the body of the drone is neglected. Thrust forces are added, and are represented as $T * Angle$, and are explained below.

$$m\ddot{x} \approx T\sin\theta \approx T\,\theta,$$

$$m\ddot{y} \approx -T\sin\phi \approx -T\,\phi,$$

$$m\ddot{z} \approx -mg + T.$$

Using the small-angle approximation, we multiply the body torque matrix (Eq: 6 7, 8) by the rotation matrix Eq: 32 to get an approximate force from propeller torques, and combine these with the original body forces we found above.

For the simplified equations of motion that are used for this project's performance goals, we use

$$m\ddot{x} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + RT_B + F_D \tag{9}$$

where R is the rotation matrix shown in the appendix in equation 32, T_b is the torques in the body frame seen in Eq: 6,7, and 8, And F_d is the drag force, represented as the air resistance of the propellers, which adds an additional thrust-like force, and is a nonzero matrix. F_d is represented in Eq: III.B.

E. Rotational Equations of Motion

Rotational Equations of motion can be constrained to the Torques from roll, pitch, and yaw. These equations of motion are constrained to the specific, simple motion that the performance goals require in this project (simple hover, circle hover, L-straight-and-level flight). Originally, a Lagrange Energy model was derived and explained (see Appendix Section A.D and A.E), which allows the model to execute any type of motion or torque the drone needs to make (in terms of pitch, roll, and yaw).

The derivations for the torque-based generalized force about the x, y, and z axes are shown below, which are only applicable to the rotors affected by the specific angle change (note that not all rotors are affected by every change; it is case-dependent).

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \tau_{\text{thrust}}(\theta) = \frac{d}{dt}\left(\frac{\partial V_2}{\partial \dot{\theta}} + \frac{\partial V_4}{\partial \dot{\theta}}\right) - \left(\frac{\partial V_2}{\partial \theta} + \frac{\partial V_4}{\partial \theta}\right).$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = \tau_{\text{thrust}}(\phi) = \frac{d}{dt}\left(\frac{\partial V_1}{\partial \dot{\phi}} + \frac{\partial V_3}{\partial \dot{\phi}}\right) - \left(\frac{\partial V_1}{\partial \phi} + \frac{\partial V_3}{\partial \phi}\right).$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) - \frac{\partial \mathcal{L}}{\partial \psi} = \tau_{\text{thrust}}(\psi) = \frac{d}{dt}\left(\frac{\partial V_1}{\partial \dot{\psi}} + \frac{\partial V_3}{\partial \dot{\psi}} + \frac{\partial V_2}{\partial \dot{\psi}} + \frac{\partial V_4}{\partial \dot{\psi}}\right) - \left(\frac{\partial V_1}{\partial \psi} + \frac{\partial V_3}{\partial \psi} + \frac{\partial V_2}{\partial \psi} + \frac{\partial V_4}{\partial \psi}\right).$$

Therefore, we utilized our previously derived EOM's from our Lagrange Model and applied them (see Appendix

Section A.E) to this simplified version of rotational equations of motion as shown below:

$$\dot{\omega} = \begin{bmatrix} \tau_\phi I_{xx}^{-1} \\ \tau_\theta I_{yy}^{-1} \\ \tau_\psi I_{zz}^{-1} \end{bmatrix} - \begin{bmatrix} \frac{I_{yy} - I_{zz}}{I_{xx}} \omega_y \omega_z \\ \frac{I_{zz} - I_{xx}}{I_{yy}} \omega_x \omega_z \\ \frac{I_{xx} - I_{yy}}{I_{zz}} \omega_x \omega_y \end{bmatrix} \tag{10}$$

F. Drone Properties and Inertia

The total mass for the drone, m_{tot}, is 0.943 kg. Our motor's mass is represented as $m_{motor} = 0.052$ and each propeller as $m_{prop} = 0.01$ kg [3][4]. With these known masses, we can compute the masses of each rotor (rotor defined as the assembly on the end of each of the four "legs", and consisting of a motor and propeller) and the mass of the drone's body (CoM) specifically.

$$m_{\text{rotor}} = m_{\text{motor}} + m_{\text{propeller}} \; \Big| \; m_{\text{drone}} = m_{\text{total}} - 4\, m_{\text{rotor}}$$

Additionally, the length of each of the four arms is: $L = 0.225$ [5] With equations for the mass of the drone and the rotors, we can now model the moment of inertia in each direction. The moment of inertia for each rotor can be seen as

$$I_{rotor} = (m_{motor} + m_{prop}) * L^2$$

The expressions for I_{xx}, I_{yy}, and I_{zz} follow from the symmetry of the quadcopter and the placement of its four rotor assemblies. Each rotor is modeled as a point mass located a distance L from the center of mass, which contributes an inertia I_{rotor} about any axis perpendicular to the arm.

For roll (I_{xx}) and pitch (I_{yy}), only the two rotors whose arms lie perpendicular to the corresponding axis contribute. Because each of those two masses provides the same inertia I_{rotor}, the total becomes $2I_{\text{rotor}}$ for both axes. This matches the physical intuition that rotating about the roll or pitch axis moves only two of the four rotors through circular arcs.

Yaw rotation (I_{zz}), however, involves all four rotors sweeping around the vertical axis. Since each rotor contributes equally to resistance about this axis, the yaw inertia is the sum of all four contributions, giving $4I_{\text{rotor}}$. This higher inertia reflects the fact that the drone must rotate the entire cross-shaped mass distribution during a yaw maneuver.

With this equation for each propeller, we can model the moment of inertia in every direction.

$$I_{xx} = 2\, I_{\text{rotor}} \; \Big| \; I_{yy} = 2\, I_{\text{rotor}} \; \Big| \; I_{zz} = 4\, I_{\text{rotor}}$$

The radius of the prop is 0.1 m [3]. The area of the prop can be found by:

$$A_{disk} = \pi * R_{rotor}^2$$

G. Battery and Power Equations

For our chosen power solution, the battery voltage = 11.1 V, while battery capacity = 1.5 Ah [6]. To find the total energy stored in the battery, we have to use the equation $E = VQ$. This means we multiply the battery voltage by the charge represented by the capacity. Since the capacity is given in ampere-hours (Ah), we must convert it to Coulombs by multiplying by 3600 (because 1Ah=3600C). Once the capacity is converted to Coulombs, multiplying by the voltage gives the battery energy in Joules.

$$E_{bat} = V_{bat} * Q_{bat} * 3600 = 11.1 \times 1.5 \times 3600 = 59,940 \text{ J}$$

We can now find the power coefficient using K_T

$$CP_{rotor} = \frac{K_T^{1.5}}{\sqrt{2 * \rho * A_{disk}}} \tag{11}$$

The rotor power can be defined as $P_{rotor} = CP_{rotor} * \omega^3$, while the total power is 4 times this, or the summation of the previous equation. $P_{\text{total}} = \sum_{i=1}^{4} P_{\text{rotor}}(\omega_i)$

H. Simplifying Modeling Assumptions

- **Identical rotors and motors.** All four propellers/motors are assumed identical in mass, thrust and drag coefficients, and geometry.
- **Symmetric + square geometry layout.** The quadcopter is modeled with four equally spaced rotors at the corners of a square frame, enabling a diagonal inertia tensor.
- **Rigid-body with point-mass rotor approximation.**The frame is rigid, and rotors are treated as point masses at arm ends for inertia calculations.
- **Thrust and drag torque ω^2.** Rotor thrust and reaction torque scale with the the rotor's angular velocity squared; any perturbations are neglected
- **Hover / low-speed + still air assumption.** Ambient air is assumed stationary (no wind or free-stream velocity), and the model is derived assuming low translational velocities (ex. hover).
- **Negligible motor electrical losses.** Motor resistance and no-load current are neglected in power calculations, simplifying to an ideal motor model.
- **No aerodynamic drag on the body.** Global drag (on the frame) is neglected, drag on propellers is considered by F_d
- **Small-angle / small-disturbance assumption for control.** The control model assumes small roll/pitch/yaw angles and small deviations, enabling linear control approximations, and the small angle approximation to be used in the derivation.
- **Underactuation / rotor-speed control only.** The only control inputs are rotor angular velocities, no angular adjustment, or additional control surfaces.

IV. Model Design

Table 2 Selected Quadcopter Components and Specifications

Component	Component Name (Hyperlinked)	Weight (kg)	Relevant Specs
Frame	F450 Drone Frame (Hawk's Work)	0.280	450 mm wheelbase; landing skid gear; max takeoff weight 1.8 kg
Motors (x4)	2212 Brushless Motor (Hawk's Work)	0.052 each	11.1 V; 0.600–0.680 kg thrust at 100% throttle
Propellers (x4)	HQProp 8x5 Bi-Blade	0.01 each	8" diameter; 5" pitch; polycarbonate material
Battery	11.1 V 1.5 Ah Li-Po Battery	0.127	11.1 V; 1.5 Ah capacity; meets motor voltage requirement
ESC	Holybro Tekko32 35A 4-in-1 ESC	0.017	35 A max current; BLHeli32 firmware; 3–6s LiPo compatible
Flight Controller	Holybro Pixhawk 6C Mini	0.0392	High-performance autopilot; PX4-compatible flight controller
Telemetry Radio	Holybro SiK Telemetry Radio	0.0235	USB/PC connectivity; wireless command and telemetry link

The calculated weight of the drone is **0.735 kg**. The final weight calculation will take into account items that are not shown in the parts list, including, but not limited to, wiring, screws, imperfections in hardware, etc. The final weight number we came up with is **0.943 kg**, which is well within the required range. The final price of the drone comes out to **$345**.

V. Model Performance

To evaluate the quadcopter's performance, we must subject it to 3 tests and ensure that all of these missions can be completed without exhausting the power supply. The quadcopter must:

- Hover 1 m above ground for 2 minutes.
- Fly in a circle of radius 2 m, at an altitude of 1 m above ground, at a speed of 0.5 m/s for at least 1 minute.
- Launch from the ground and ascend vertically until 1 m above ground. Move in a straight line 1 m above ground at an average speed of 1 m/s for 5 m. Stop (hover), yaw 90° to the left, and move in another straight line for 5 m. Stop (hover), land vertically with a speed of no more than 1 cm/s.

Note: All figure labels are hyperlinked to view the fully enlarged version of the image, which can be found within the Appendix VII.

A. Mission 1: Hover

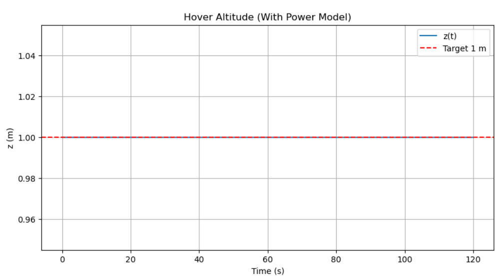

Fig. 1 Hover Altitude vs Time

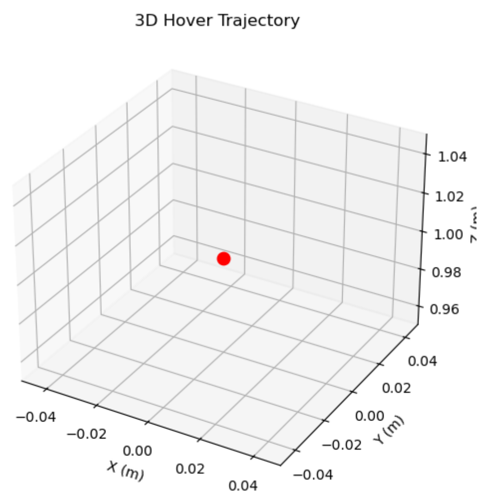

Fig. 2 3D Hover Trajectory

The first mission requires the quadcopter to maintain a hover at an altitude of 1m for a duration of 2 minutes. As shown in figure 1, the vehicle successfully maintains a constant altitude near 1m with no deviation. The corresponding three-dimensional trajectory in figure 2 shows that it is stable in its position without deviating horizontally. The Euler angles in figure 3 confirm the attitude remains constant throughout the hover

The power consumption profile during hover is presented in Figure 4, which shows that the power draw remains steady, drawing under 51 W. The cumulative energy usage compared with total battery capacity is shown in Figure 5, and the remaining battery energy after mission completion is shown in Figure 6. The quadcopter retains more than 50 kJ of energy after the 2-minute hover, indicating that the mission is completed well within the available power margin. Thus, the quadcopter meets all the performance and power requirements for mission 1.

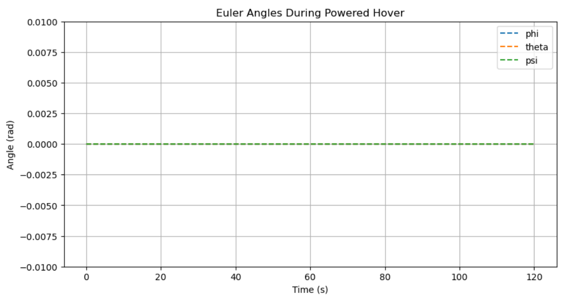

Fig. 3 Euler Angles

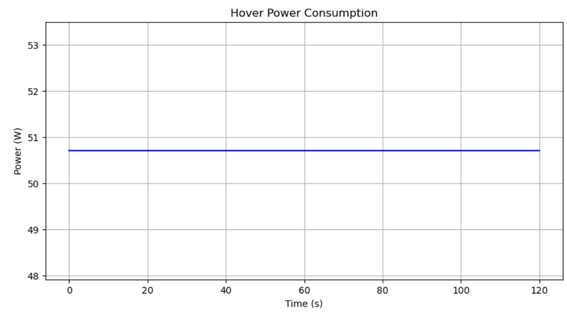

Fig. 4 Hover Power

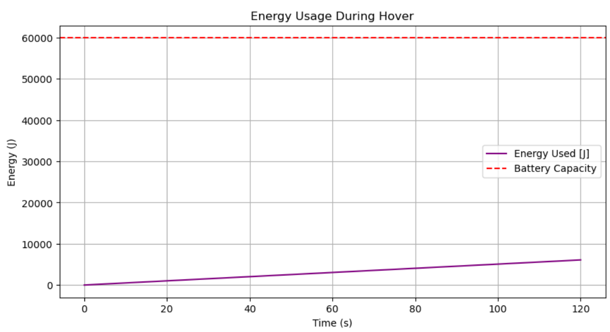

Fig. 5 Energy vs battery

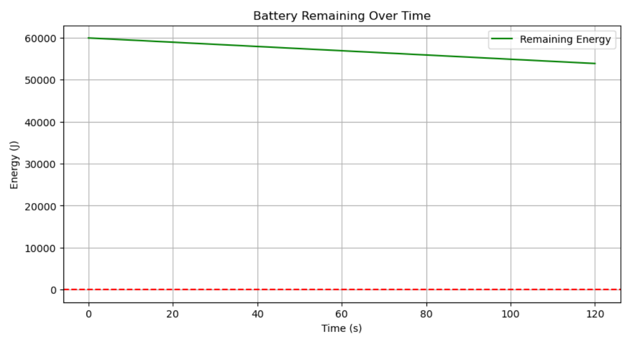

Fig. 6 Remaining Battery

B. Mission 2: Circular flight

The second mission requires the quadcopter to fly in a circle of radius 2 m at altitude of 1 m, with a speed of 0.5 m/s for 1 minute. The X-Y trajectory in figure 7 shows the circular flight path with a radius of 2 m, while figure 8 shows the altitude profile. The altitude briefly rises to 1.23 m before settling to 1 m. This overshoot results from the transient response of the control system because the controller initially commands a large thrust, but due to rotor lag, the quadcopter temporarily increases altitude before it stabilizes at the correct height.

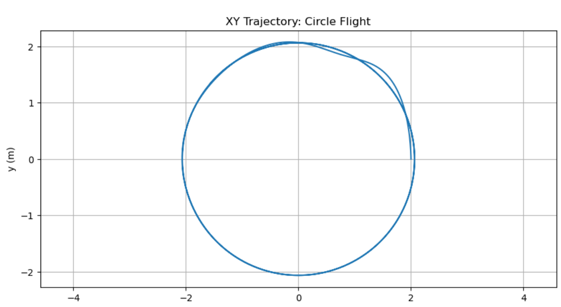

Fig. 7 XY Trajectory

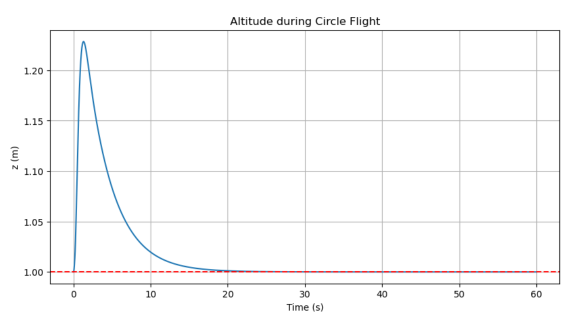

Fig. 8 Altitude

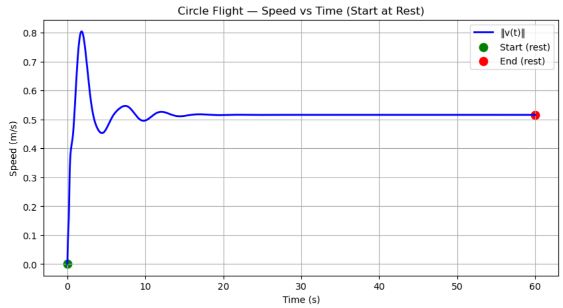

Fig. 9 Velocity vs Time

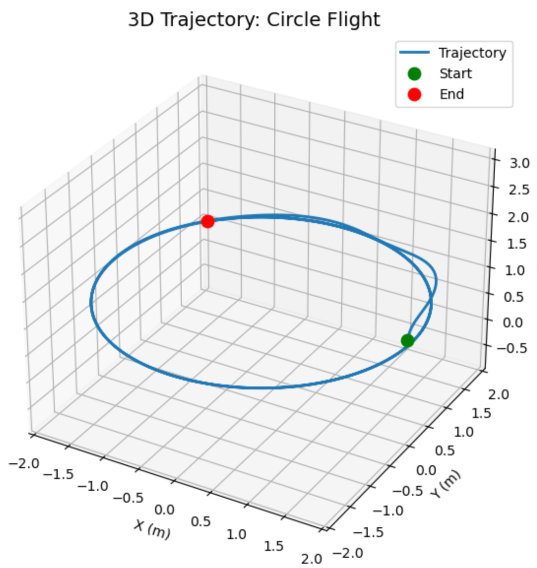

Fig. 10 3D Trajectory

The attitude behavior, shown in figure 11 indicates stable roll, pitch, and yaw throughout, with the yaw angle dictating the circular flight. The speed profile in figure 9 confirms that although the quadcopter initially spikes to a speed of 0.8 m/s during the transient, it quickly stabilizes just slightly above 0.5 m/s. The full three-dimensional trajectory in figure 10 further confirms the completion of the mission.

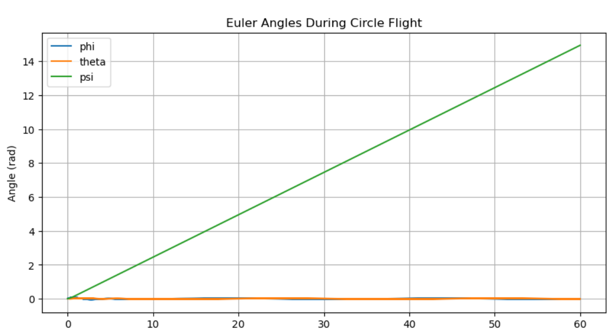

Fig. 11 Euler Angles

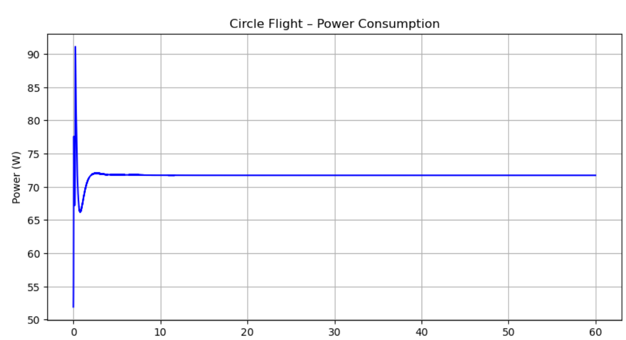

Fig. 12 Circle Power

Power consumption during circular flight is shown in Figure 12, where the power draw initially spikes as the maneuver begins, then settles to approximately 72 W. The cumulative energy and remaining battery levels shown in Figure 13 and Figure 14 indicate that more than 55 kJ of energy remains after one minute of circular flight. Therefore, the quadcopter completes Mission 2 with sufficient power and accurate trajectory tracking.

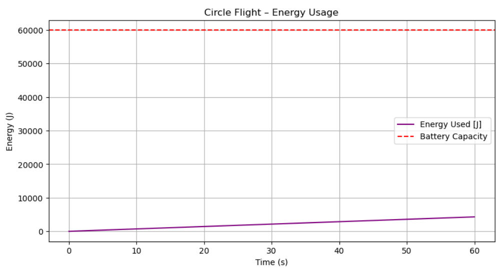

Fig. 13 Energy vs battery

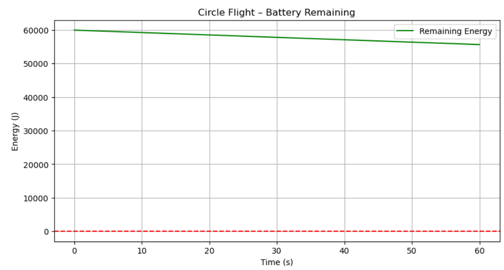

Fig. 14 Remaining Battery

C. Mission 3: L-Shaped Flight

The third mission involves several sequential maneuvers: vertical ascent to 1 m, horizontal translation at 1 m altitude over two legs, a 90° yaw rotation between legs, and a controlled landing with a descent rate not exceeding 1 cm/s.

The altitude profile in Figure 15 shows a smooth ascent from the ground to 1 m. The X–Y trajectory in Figure 16 confirms that the quadcopter first travels 5 m in a straight line, executes the required 90° yaw rotation (verified by the yaw angle reaching $\pi/2$ in Figure 18), and then proceeds along the second 5 m straight path. The complete three-dimensional trajectory of this mission is shown in Figure 17, illustrating accurate execution of all phases. The final descent is shown in Figure 19, where the vertical velocity remains below the required threshold of 1 cm/s, confirming that the quadcopter meets the soft-landing specification.

Battery performance for this mission is shown in Figures 20, 21, 22 where the total energy consumption remains comfortably below the available capacity. The quadcopter completes the entire sequence with substantial battery margin, satisfying both performance and power requirements.

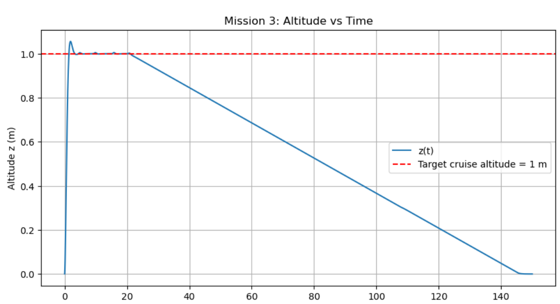

Fig. 15 Altitude vs Time

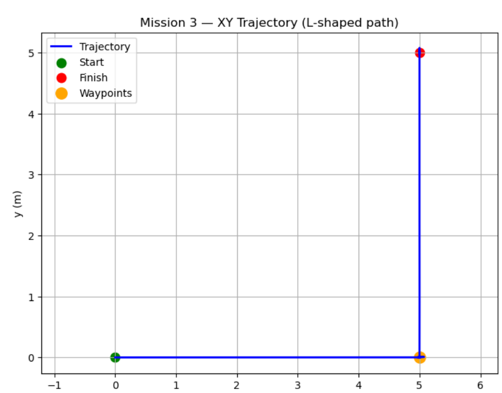

Fig. 16 XY Position

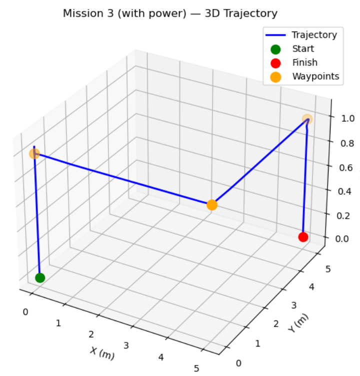

Fig. 17 3D Trajectory

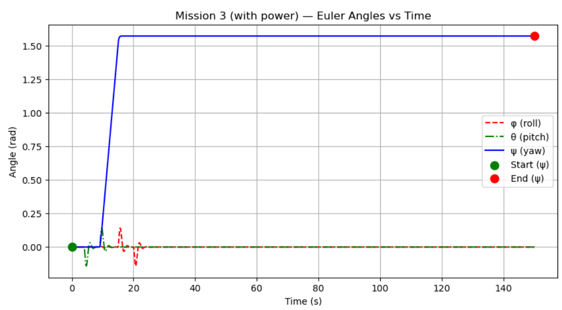

Fig. 18 Euler Angles vs Time

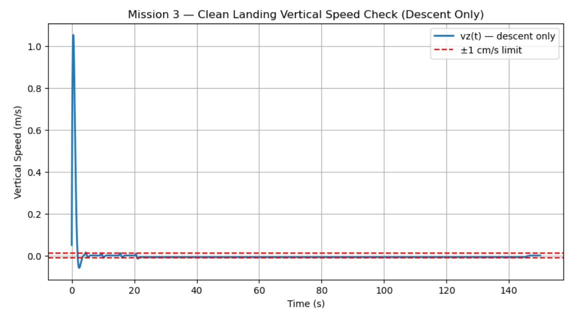

Fig. 19 Vertical Descent Speed Check

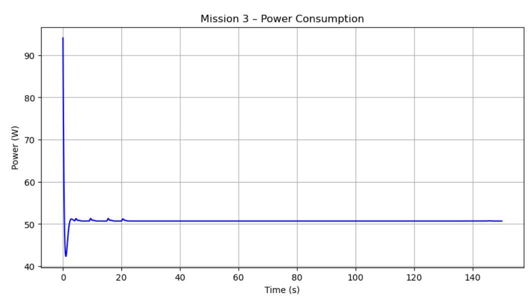

Fig. 20 Circle Power

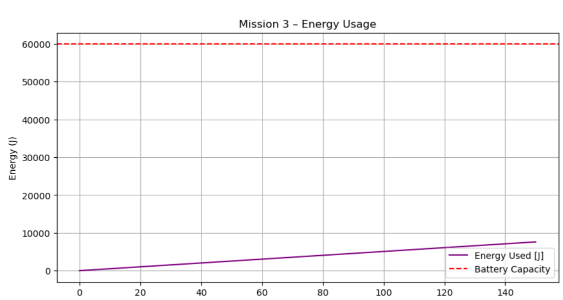

Fig. 21 Energy vs Battery

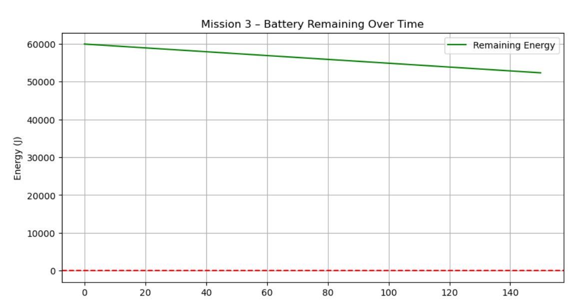

Fig. 22 Remaining Battery

VI. Contributions

A. Matthew Greenstein:

As the implementation and software lead, I was the responsible engineer for the code model (Github), and produced a replicable Python model with plots demonstrating complete control of the drone through the different performance goals V. Building on Angel's EOMs, I also added equations myself for other engineering requirements, including battery equations III.G, thrust and torque equations III.F, and propeller drag III.B. In addition, I learned how to apply a simple controller for the drone and how to best structure all of these functions in Python.

When it became time to begin working on the project, I refined the structure and presentation of our deliverables. The math section layout was most difficult, but we were able to organize it in a way that made sense for the order in which we solved the problem, while keeping supporting derivation work in an Appendix. Additionally, grammar, spelling, and formatting were a large part of my workload as we iterated through different versions of this project.

B. Angel De La Cruz:

Design Engineer and Mathematical Engineer. I did the math derivations by hand as seen on. Additionally, I did all the modeling for the Lagrange derivations and did all derivations and modeling of drone motion by hand, as seen on appendix for the hand derivationsC. I took charge of the complex math derivations involving Lagrange and all the work of Lagrange equations of motion.

C. Sebastian White:

As the lead engineer, I was responsible for coordinating the project and schedule and writing the report. Initially I set early deadlines for the equations of motion so that we could have our code ready to be implemented and run. With these completed, hand-derived EOM's seen in appendix section C, I then converted them into Overleaf format and fully wrote them out along with the steps seen in the derivation appendix section A. Once these were fully transcribed, Matthew was able to implement the code and provide data and plots to track our mission goals. Then, I took these plots and input them into the document, formatted them, wrote explanations, and demonstrated how we achieved each mission goal.

With those stages of the project then complete, we began to refine the project, which involved rewriting and shortening the math sections to fit the page requirements, which presented the need for an appendix section, which I helped format as well, adding in the larger images and the hand derivations.

D. Julian Chawla:

As the quality control engineer, I was in charge of making sure the report turned in matched the project description, and overall was of high quality. Early on, I helped the lead engineer write the introduction of the paper, along with all of the hand-derived equations that needed to be typed out. Once we were done with writing out the math derivations, I went through all of our sources and added them to the bibliography and cited them throughout the report where they were needed. Towards the end of the project, I enforced the page limits and did multiple final report checks for grammatical and flow errors.

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VII. Appendix

A. Additional Derivation Steps

A. Setup of Pre-Lagrange Equations

The first step to defining our equations of motion is to set up coordinates, x,y,z, then we must take derivatives to find the $\dot{x}$, $\dot{y}$, and $\dot{z}$

$$V_i \overset{=}{} mgh \tag{12}$$

$$a_i = \frac{1}{4}T + \tau \tag{13}$$

$$v_i^2 = \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \tag{14}$$

$$\dot{x}_i = \dot{x}_{com} + r_{rotor}\big(- \dot{\theta}\sin\theta + \dot{\psi}\cos\psi\big) \tag{15}$$

The equation $\dot{x}_i = -\dot{x}_j$ represents the opposite motion each rotor will experience. Since rotors 1 and 3 are on a fixed axis opposite each other, rotor 3 will experience the negative motion of rotor 1 to balance the drone. The same condition applies between rotors 2 and 4.

$$\dot{x}_i = \dot{x}_{com} + r_{rotor}(-\dot{\theta}\sin\theta + \dot{\psi}\cos\psi) \tag{16}$$

$$\dot{y}_i = \dot{y}_{com} + r_{rotor}(-\dot{\phi}\sin\phi + \dot{\psi}\cos\psi) \tag{17}$$

$$\dot{z}_i = \dot{z}_{com} + r_{rotor}(-\dot{\theta}\sin\theta + \dot{\phi}\cos\phi) \tag{18}$$

Now that we have equations for $\dot{x}$,$\dot{y}$, and $\dot{z}$ we can now square each of these equations.

$$\dot{x}_i^2 = (\dot{x}_{com} + r_{rotor}(-\dot{\theta}\sin\theta + \dot{\psi}\cos\psi))^2 \tag{19}$$

$$\dot{x}_i^2 = \dot{x}_{com}[\dot{x}_{com} + 2r_{rotor}(-\dot{\theta}\sin\theta + \dot{\psi}\cos\psi)] + r_{rotor}^2(\dot{\theta}^2\sin^2\theta + \dot{\psi}^2\cos^2\psi) - 2r_{rotor}^2(\dot{\theta}\dot{\psi}\sin\theta\cos\psi) \tag{20}$$

$$\dot{y}_i^2 = \dot{y}_{com}[\dot{y}_{com} + 2r_{rotor}(-\dot{\phi}\sin\phi + \dot{\psi}\sin\psi)] + r_{rotor}^2(\dot{\phi}^2 sin^2\phi^2 + \dot{\psi}^2 sin^2\psi) + 2r_{rotor}^2(\dot{\phi}^2\dot{\psi}^2 sin\theta sin\phi) \tag{21}$$

$$\dot{z}_i^2 = \dot{z}_{com}[\dot{z}_{com} + 2r_{rotor}(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi)] + r_{rotor}^2(\dot{\theta}^2 cos^2\theta + \dot{\phi}^2 cos^2\phi) + 2r_{rotor}^2(\dot{\theta}\dot{\phi} cos\theta cos\phi) \tag{22}$$

With these squared values, we can then use them to find a generalized velocity equation $\dot{v}_i^2 = \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2$, when we add everything together and plug it into our current $\dot{v}_i^2$ equation, we get:

$$\begin{aligned} \dot{v}_i^2 = {} & \dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2 + 2r_r\Big[\dot{\theta}\big(- \dot{x}_c\sin\theta + \dot{z}_c\cos\theta) + \dot{\phi}\big(- \dot{y}_c\sin\phi + \dot{z}_c\cos\phi\big) + \dot{\psi}\big(\dot{x}_c\cos\phi - \dot{y}_c\sin\psi\big)\Big] \\ & + r_r^2(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2) + 2r_r^2\Big(- \dot{\theta}\dot{\psi}\sin\theta\cos\psi + \dot{\phi}\dot{\psi}\sin\phi\sin\psi + \dot{\theta}\dot{\phi}\cos\theta\cos\phi\Big). \end{aligned} \tag{23}$$

With this generalized equation for velocity, we must then constrain each of the rotors in order to determine the individual velocities for each rotor. In the first case, for rotor 1, it is constrained with $\phi = 0$ because there is no roll component, since rotor 1 only affects roll and yaw.

$$v_1^2 = \dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2 + 2r_r[\dot{\theta}(-\dot{x}_c \sin\theta + \dot{z}_c cos\theta + \dot{\psi}(\dot{x}_c cos\psi - \dot{y}_c sin\psi)] + r_r^2(\dot{\theta}^2 + \dot{\psi}^2) + 2r_r^2(-\dot{\theta}\dot{\psi} sin\theta cos\psi) \quad (24)$$

In the case of rotor 2, because there are only pitch and yaw components, we can consider $\psi = 0$. In order to keep the angles the same, we must also adjust for the negative angles, which is where the $\frac{\pi}{2} - \psi$ is implemented.

$$v_2^2 = \dot{x}_c^2+\dot{y}_c^2+\dot{z}_c^2+2r_r[\dot{\phi}(-\dot{y}_c \sin\phi+\dot{z}_c cos\phi)+\dot{\psi}(-\dot{x}_c cos(\frac{\pi}{2} - \psi)+\dot{y}_c sin(\frac{\pi}{2}-\psi))]+r_r^2(\dot{\phi}^2+\dot{\psi}^2)+2r_r^2(-\dot{\phi}\dot{\psi} sin(\phi) sin(\frac{\pi}{2}-\psi) \quad (25)$$

In rotor 3, there is no pitch component, so we can take $\phi = 0$. We also have to account for the change in angle, so instead of just using ψ, we must use $\frac{3\pi}{2} - \psi$

$$v_3^2 = \dot{x}_c^2+\dot{y}_c^2+\dot{z}_c^2+2r_r[\dot{\theta}(-\dot{x}_c \sin(2\pi - \theta)+\dot{z}_c cos(2\pi - \theta)+\dot{\psi}(-\dot{x}_c cos\frac{3\pi}{2} - \psi-\dot{y}_c sin\frac{3\pi}{2} - \psi)]+r_r^2(\dot{\theta}^2+\dot{\psi}^2)+2r_r^2(-\dot{\theta}\dot{\psi} sin\theta cos\psi) \quad (26)$$

$$\begin{aligned} v_4^2 = {} & \dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2 + 2r_r[\dot{\phi}\big(\dot{y}_c \sin(2\pi - \phi)\big) + \dot{z}_c \cos(2\pi - \phi) + \dot{\psi}\big(\dot{x}_c \cos(2\pi - \psi) + \dot{y}_c \sin(2\pi - \psi)\big)] \\ & + r_r^2\Big(\dot{\phi}^2 + \dot{\psi}^2\Big) + 2r_r^2\big(\dot{\phi}\dot{\psi} \ \sin(2\pi - \phi) \ \sin(2\pi - \psi)\big) . \end{aligned} \quad (27)$$

We form the Lagrangian using translational and rotational kinetic energies of the rigid body and each rotor, plus gravitational potential energy. Define generalized coordinates as $q = (\theta, \phi, \psi)$ and generalized velocities as $\dot{q}$.

B. Torque, Thrust, and Motor Speed Equations

Define the vector of desired thrust and torques

$$u = \begin{bmatrix} T \\ \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} \qquad W = \begin{bmatrix} \omega_1^2 \\ \omega_2^2 \\ \omega_3^2 \\ \omega_4^2 \end{bmatrix}$$

The relation between them is

$$u = AW, \quad (28)$$

where the allocation matrix is

$$A = \begin{bmatrix} k_T & k_T & k_T & k_T \\ k_T L & 0 & -k_T L & 0 \\ 0 & k_T L & 0 & -k_T L \\ k_{\text{drag}} & -k_{\text{drag}} & k_{\text{drag}} & -k_{\text{drag}} \end{bmatrix}. \tag{29}$$

To obtain the required motor speeds, we solve

$$W = A^{-1} u, \tag{30}$$

and then recover the motor speeds via

$$\omega_i = \sqrt{W_i}. \tag{31}$$

This provides the four rotor speeds required to achieve a specified total thrust and body torques.

C. Euler-Angles Derivation

We describe the drone's attitude using a 3–2–1 (yaw–roll–pitch) Euler angle sequence:

$$\psi \text{ (yaw)}, \quad \theta \text{ (roll)}, \quad \phi \text{ (pitch)}.$$

The rotation from the inertial frame to the body frame is written as

$$\mathbf{R} = \mathbf{R}_x(\phi)\, \mathbf{R}_y(\theta)\, \mathbf{R}_z(\psi),$$

$$R = \begin{bmatrix} \cos\phi\, \cos\psi - \cos\theta\, \sin\phi\, \sin\psi & -\cos\psi\, \sin\phi - \cos\phi\, \cos\theta\, \sin\psi & \sin\theta\, \sin\psi \\ \cos\theta\, \cos\psi\, \sin\phi + \cos\phi\, \sin\psi & \cos\phi\, \cos\theta\, \cos\psi - \sin\phi\, \sin\psi & -\cos\psi\, \sin\theta \\ \sin\phi\, \sin\theta & \cos\phi\, \sin\theta & \cos\theta \end{bmatrix} \tag{32}$$

where $\mathbf{R}_x$, $\mathbf{R}_y$, and $\mathbf{R}_z$ are the standard rotation matrices about the x, y, and z axes.

The body–frame angular velocity vector is

$$\omega_b = \begin{bmatrix} p \\ q \\ r \end{bmatrix},$$

where p, q, and r are the rotation rates about the body x, y, and z axes, respectively. On the other hand, the same motion can be described as the sum of the three incremental rotations associated with the Euler angles:

$$\boldsymbol{\omega} = \dot{\phi}\, \mathbf{e}'_x + \dot{\theta}\, \mathbf{e}''_y + \dot{\psi}\, \mathbf{e}'''_z.$$

Here $\mathbf{e}'_x$, $\mathbf{e}''_y$, and $\mathbf{e}'''_z$ are the intermediate axes after successive rotations by ϕ, θ, and ψ. Expressing these intermediate axes in the body frame and collecting terms yields a linear relationship between the body rates and the Euler–angle rates:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{T}(\phi, \theta) \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix},$$

with the transformation matrix

$$\mathbf{T}(\phi, \theta) = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\,\sin\phi \\ 0 & -\sin\phi & \cos\theta\,\cos\phi \end{bmatrix}.$$

To recover the Euler–angle rates from the body rates, we invert this 3×3 matrix:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{T}^{-1}(\phi, \theta) \begin{bmatrix} p \\ q \\ r \end{bmatrix},$$

where

$$\mathbf{T}^{-1}(\phi, \theta) = \begin{bmatrix} 1 & \sin\phi\,\tan\theta & \cos\phi\,\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \dfrac{\sin\phi}{\cos\theta} & \dfrac{\cos\phi}{\cos\theta} \end{bmatrix}.$$

Reading off the components gives the Euler–angle kinematic equations used in the code:

$$\dot{\phi} = p + q\,\sin\phi\,\tan\theta + r\,\cos\phi\,\tan\theta,$$

$$\dot{\theta} = q\,\cos\phi - r\,\sin\phi,$$

$$\dot{\psi} = q\,\sin\phi\,\sec\theta + r\,\cos\phi\,\sec\theta.$$

These relations convert the body angular rates (p, q, r) produced by the rotational dynamics into Euler–angle rates $(\dot{\phi}, \dot{\theta}, \dot{\psi})$, which can then be integrated in time to update the drone's attitude.

D. Lagrange Definition

$$T = \tfrac{1}{2}Mv_{\text{drone}}^2 + \tfrac{1}{2}\boldsymbol{\omega}_{\text{body}}^\top I_{\text{body}}\boldsymbol{\omega}_{\text{body}} + \sum_i \left(\tfrac{1}{2}m_{r,i}v_{r,i}^2 + \tfrac{1}{2}\boldsymbol{\omega}_{r,i}^\top I_{r,i}\boldsymbol{\omega}_{r,i}\right), \tag{33}$$

$$V = Mgz_{\text{com}} + \sum_i m_{r,i}gz_{r,i}, \tag{34}$$

$$L = T - V. \tag{35}$$

Here each $v_{\text{com}}, v_{r,i}, \boldsymbol{\omega}_{\text{body}}, \boldsymbol{\omega}_{r,i}$ are functions of the generalized coordinates q and $\dot{q}$ (explicit derivations omitted for brevity).

E. Equations of motion components

Lagrangian component with respect to θ

$$\begin{aligned}\frac{d}{dt}(\frac{\partial v_1}{\partial \dot{\theta}}) - \frac{\partial v_1}{\partial \theta} &= \tfrac{1}{2}m\left[2r_r(\ddot{x}_c\sin(\theta) - \ddot{\theta}\cos(\theta))\dot{x}_c - \ddot{z}_c\cos(\theta) + \dot{\theta}\sin(\theta)\dot{z}_c + r_r^2(2\ddot{\theta})\right] \\ &\quad + \tfrac{1}{2}m\left[2r_r^2\Big(-\ddot{\psi}\sin(\theta)\cos(\psi) + \big(-\dot{\theta}\cos(\theta)\cos(\psi) + \dot{\psi}\sin(\theta)\sin(\psi)\big)\dot{\psi}\Big)\right] \\ &\quad - \tfrac{1}{2}m\left[2r_r\dot{\theta}\big(-\dot{x}_c\cos(\theta) - \dot{z}_c\sin(\theta)\big)\right]\end{aligned} \tag{36}$$

$$\frac{d}{dt}(\frac{\partial v_2}{\partial \dot{\theta}}) - \frac{\partial v_2}{\partial \theta} = 0 \tag{37}$$

$$\begin{aligned}\frac{d}{dt}(\frac{\partial v_3}{\partial \dot{\theta}}) - \frac{\partial v_3}{\partial \theta} &= \tfrac{1}{2}m\left[2r_r(\ddot{x}_c\sin(2\pi-\theta) - \ddot{\theta}\cos(2\pi-\theta))\dot{x}_c - \ddot{z}_c\cos(2\pi-\theta) + \dot{\theta}\sin(2\pi-\theta)\dot{z}_c + r_r^2(2\ddot{\theta})\right] \\ &\quad + \tfrac{1}{2}m\left[2r_r^2\Big(-\ddot{\psi}\sin(2\pi-\theta)\cos\Big(\tfrac{3\pi}{2}-\psi\Big) + \big(-\dot{\theta}\cos(2\pi-\theta)\cos\Big(\tfrac{3\pi}{2}-\psi\Big) + \dot{\psi}\sin\Big(\tfrac{3\pi}{2}-\psi\Big)\sin(2\pi-\theta)\big)\dot{\psi}\Big)\right] \\ &\quad - \tfrac{1}{2}m\left[2r_r\dot{\theta}\big(-\dot{x}_c\cos(2\pi-\theta) - \dot{z}_c\sin(2\pi-\theta)\big)\right]\end{aligned} \tag{38}$$

$$\frac{d}{dt}(\frac{\partial v_4}{\partial \dot{\theta}}) - \frac{\partial v_4}{\partial \theta} = 0 \tag{39}$$

Lagrangian components with respect to ϕ

$$\frac{d}{dt}(\frac{\partial v_1}{\partial \dot{\phi}}) - \frac{\partial v_1}{\partial \phi} = 0 \tag{40}$$

$$\begin{aligned}
\frac{d}{dt}(\frac{\partial v_2}{\partial \dot{\phi}}) - \frac{\partial v_2}{\partial \phi} &= \tfrac{1}{2}m\left[2r_r(-\ddot{y}_c\sin(\phi) - \dot{\phi}\cos(\phi)\dot{y}_c - \ddot{z}_c\cos(\phi) - \dot{\phi}\sin(\phi)\dot{z}_c + r_r^2(2\ddot{\phi})\right] \\
&+ \tfrac{1}{2}m\left[2r_r^2\Big(-\ddot{\psi}\sin(\phi)\sin(\frac{\pi}{2}-\psi) + \dot{\psi}(-\dot{\phi}\cos(\phi)\sin(\frac{\pi}{2}-\psi)) - \dot{\psi}\cos(\frac{\pi}{2}-\psi)\sin(\psi)\Big)\right] \\
&- \tfrac{1}{2}m\left[2r_r\dot{\phi}\big(-\dot{y}_c\cos(\phi) - \dot{z}_c\sin(\phi)\big)\right]
\end{aligned} \tag{41}$$

$$\frac{d}{dt}(\frac{\partial v_3}{\partial \dot{\phi}}) - \frac{\partial v_3}{\partial \phi} = 0 \tag{42}$$

$$\begin{aligned}
\frac{d}{dt}(\frac{\partial v_4}{\partial \dot{\phi}}) - \frac{\partial v_4}{\partial \phi} &= \tfrac{1}{2}m\left[2r_r(-\ddot{y}_c\sin(2\pi-\phi) - \dot{\phi}\cos(2\pi-\phi)\dot{y}_c - \ddot{z}_c\cos(2\pi-\phi) - \dot{\phi}\sin(2\pi-\phi)\dot{z}_c + r_r^2(2\ddot{\phi})\right] \\
&+ \tfrac{1}{2}m\left[2r_r^2\Big(-\ddot{\psi}\sin(2\pi-\phi)\sin(2\pi-\psi) + \dot{\psi}(-\dot{\phi}\cos(2\pi-\phi)\sin(2\pi-\psi)) - \dot{\psi}\cos(2\pi-\psi)\sin(2\pi-\phi)\Big)\right] \\
&- \tfrac{1}{2}m\left[2r_r\dot{\phi}\big(-\dot{y}_c\cos(2\pi-\phi) - \dot{z}_c\sin(2\pi-\phi)\big)\right]
\end{aligned} \tag{43}$$

Lagrangian components with respect to ψ

$$\begin{aligned}
\frac{d}{dt}(\frac{\partial v_1}{\partial \dot{\psi}}) - \frac{\partial v_1}{\partial \psi} &= \tfrac{1}{2}m\left[2r_r(\ddot{x}_c\cos(\psi) - \dot{\psi}\cos(\psi)\dot{x}_c) - \ddot{y}_c\sin(\psi) - \dot{\psi}\sin(\psi)\dot{y}_c + r_r^2(2\ddot{\psi})\right] \\
&+ \tfrac{1}{2}m\left[2r_r^2\Big(-\ddot{\theta}\sin(\theta)\cos(\psi) + \dot{\theta}\big(\dot{\theta}\cos(\theta)\cos(\psi) - \dot{\psi}\sin(\psi)\sin(\theta)\big)\Big)\right] \\
&- \tfrac{1}{2}m\left[2r_r\dot{\psi}\big(-\dot{x}_c\sin(\psi) - \dot{y}_c\cos(\psi)\big) + 2r_r^2(\dot{\theta}\dot{\psi}sin(\theta)sin(\psi))\right]
\end{aligned} \tag{44}$$

$$\frac{d}{dt}(\frac{\partial v_2}{\partial \dot{\psi}}) - \frac{\partial v_2}{\partial \psi} = \tfrac{1}{2}m\bigg[2r_r(-\ddot{x}_c\cos(\frac{\pi}{2}-\psi) + \dot{\psi}\cos(\frac{\pi}{2}-\psi)\dot{x}_c + \ddot{y}_c\sin(\frac{\pi}{2}-\psi) - \dot{\psi}\cos(\frac{\pi}{2}-\psi)\dot{y}_c\big) + r_r^2(2\ddot{\psi})\bigg]$$

$$+ \tfrac{1}{2}m\bigg[2r_r^2\Big(-\ddot{\phi}\sin(\phi)\sin(\frac{\pi}{2}-\psi) + \dot{\phi}(\dot{\phi}\cos(\phi)\sin(\frac{\pi}{2}-\psi)) - \dot{\psi}\cos(\frac{\pi}{2}-\psi)\sin(\phi)\Big)\bigg]$$

$$- \tfrac{1}{2}m\bigg[2r_r\dot{\psi}\big(-\dot{x}_c\sin(\frac{\pi}{2}-\psi) + \dot{y}_c\cos(\frac{\pi}{2}-\psi)\big) + 2r_r^2(\dot{\phi}\dot{\psi}cos(\frac{\pi}{2}-\psi))\bigg] \tag{45}$$

$$\frac{d}{dt}(\frac{\partial v_3}{\partial \dot{\psi}}) - \frac{\partial v_3}{\partial \psi} = \tfrac{1}{2}m\bigg[2r_r(\ddot{x}_c\cos(\frac{3\pi}{2}-\psi) - \dot{\psi}\sin(\frac{3\pi}{2}-\psi)\dot{x}_c) - \ddot{y}_c\sin(\frac{3\pi}{2}-\psi) - \dot{\psi}\cos(\frac{3\pi}{2}-\psi)\dot{y}_c + r_r^2(2\ddot{\psi})\bigg]$$

$$+ \tfrac{1}{2}m\bigg[2r_r^2\Big(-\ddot{\theta}\sin(\frac{3\pi}{2}-\theta)\cos(\frac{3\pi}{2}-\theta) + \dot{\theta}\big(-\dot{\theta}\cos(\frac{3\pi}{2}-\theta)\cos(\frac{3\pi}{2}-\psi) - +\dot{\psi}\sin(\frac{3\pi}{2}-\psi)\sin(2\pi theta)\big)\Big)\bigg]$$

$$- \tfrac{1}{2}m\bigg[2r_r\dot{\psi}\big(-\dot{x}_c\sin(\frac{3\pi}{2}-\psi) - \dot{y}_c\cos(\frac{3\pi}{2}-\psi)\big) + 2r_r^2(\dot{\theta}\dot{\psi}sin(2\pi-\theta)sin(\frac{3\pi}{2}-\psi))\bigg] \tag{46}$$

$$\frac{d}{dt}(\frac{\partial v_4}{\partial \dot{\psi}}) - \frac{\partial v_4}{\partial \psi} = \tfrac{1}{2}m\bigg[2r_r(-\ddot{x}_c\cos(2\pi-\psi) + \dot{\psi}\sin(2\pi-\psi)\dot{x}_c + \ddot{y}_c\sin(2\pi-\psi) - \dot{y}_c\cos(2\pi-\psi)\big) + r_r^2(2\ddot{\psi})\bigg]$$

$$+ \tfrac{1}{2}m\bigg[2r_r^2\Big(\ddot{\phi}\sin(2\pi-\phi)\sin(2\pi-\psi) + \dot{\phi}(-\dot{\phi}\cos(2\pi-\phi)\sin(2\pi-\psi)) - \dot{\psi}\cos(2\pi-\psi)\sin(2\pi-\phi)\Big)\bigg]$$

$$- \tfrac{1}{2}m\bigg[2r_r\dot{\psi}\big(-\dot{x}_c\sin(2\pi-\psi) + \dot{y}_c\cos(2\pi-\psi)\big) + 2r_r^2(-\dot{\phi}\dot{\psi}cos(2\pi-\phi)cos(2\pi-\psi))\bigg] \tag{47}$$

B. Full-Size Figures

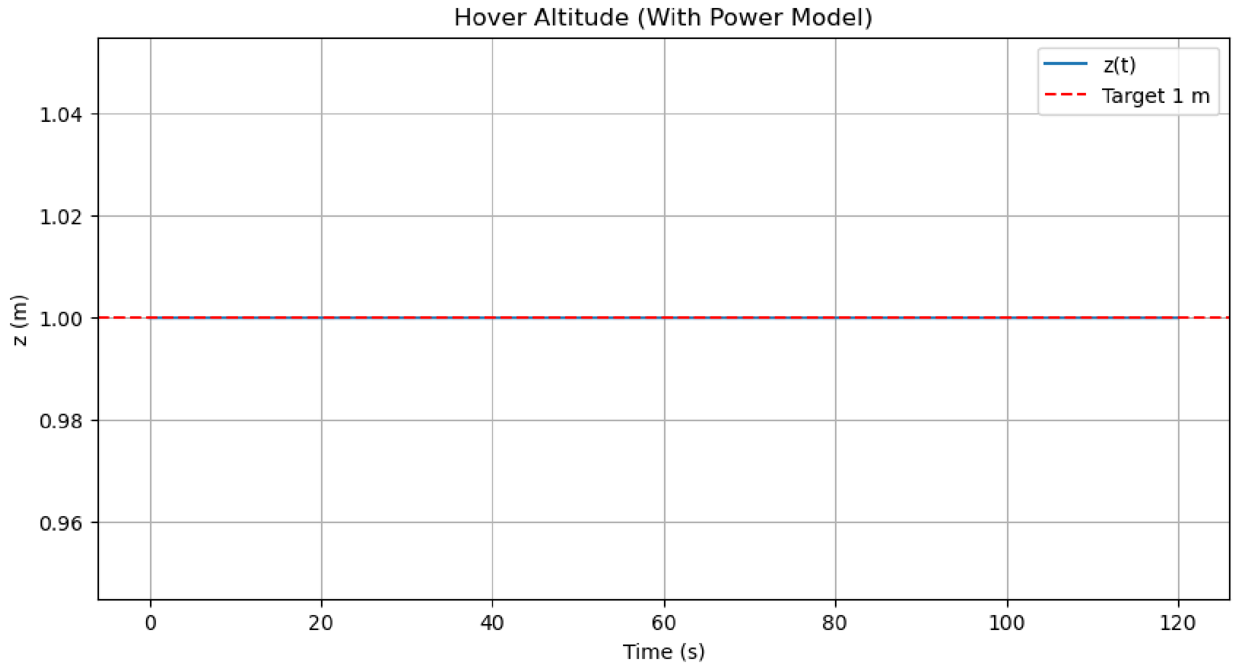


Fig. 23 hover-altitude-vs-time

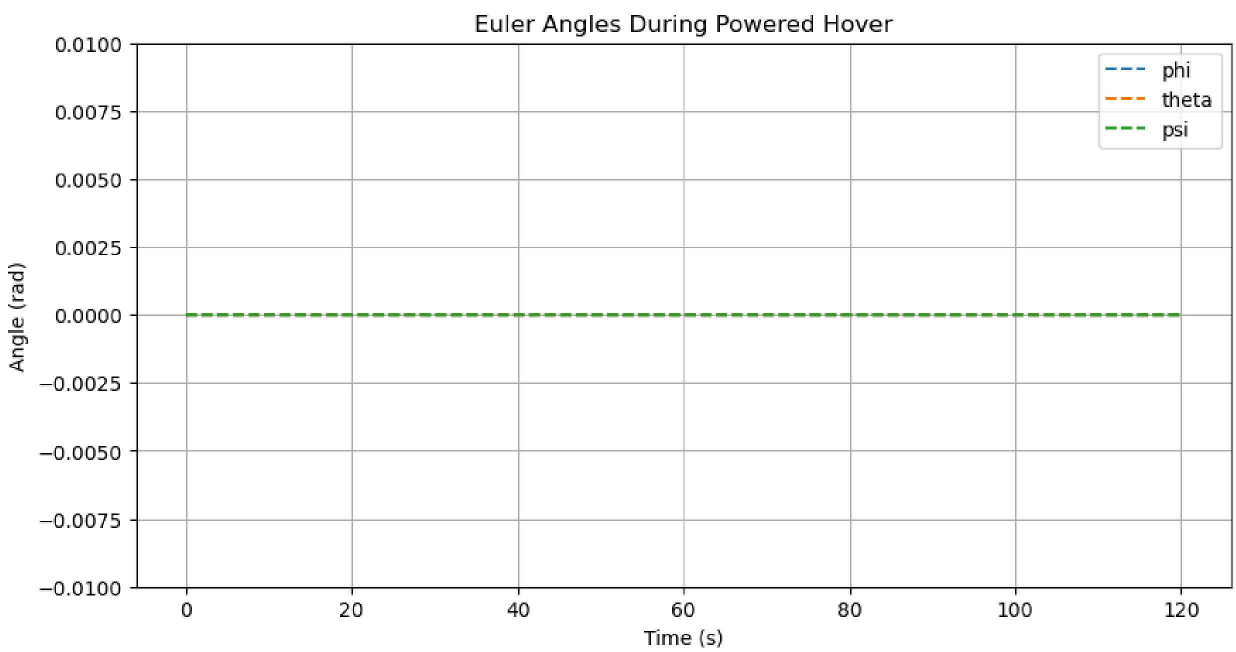


Fig. 25 Euler Angles

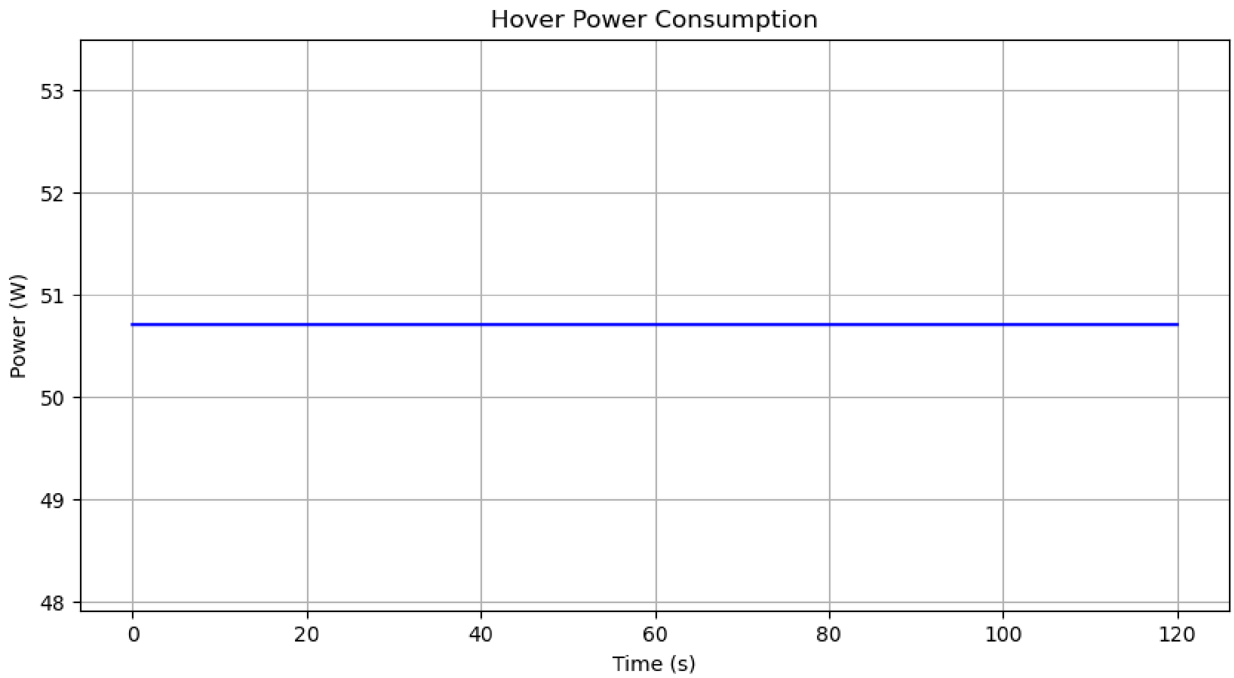


Fig. 26 Hover Power

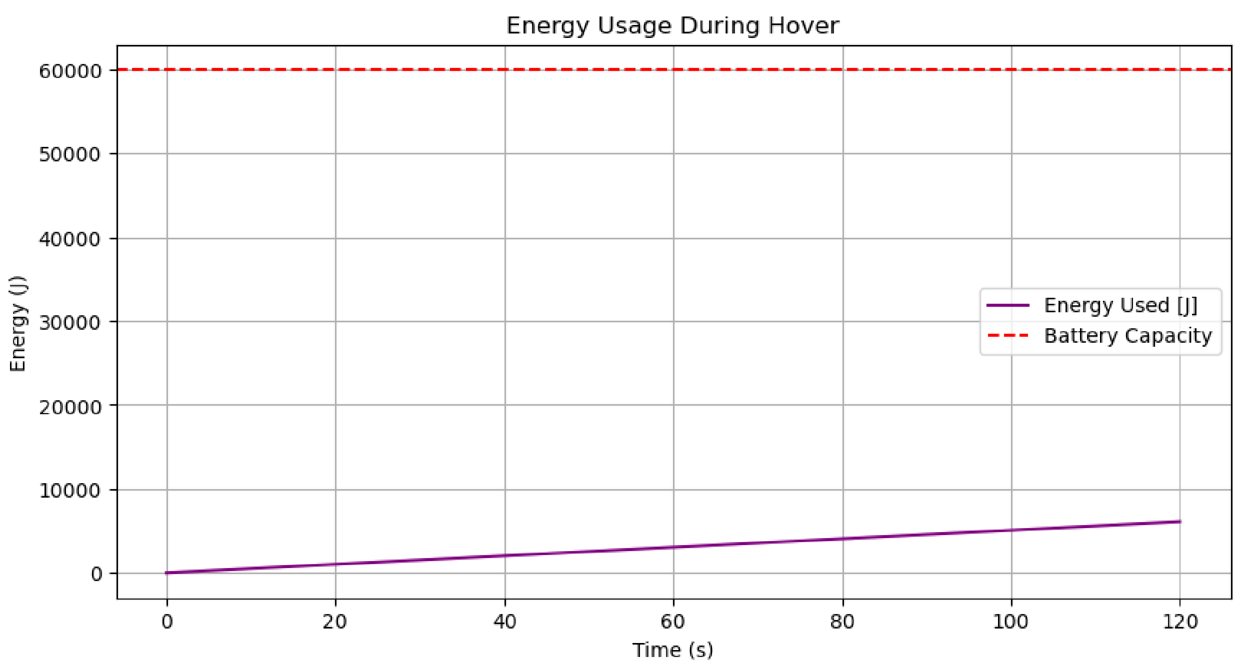


Fig. 27 Energy vs battery

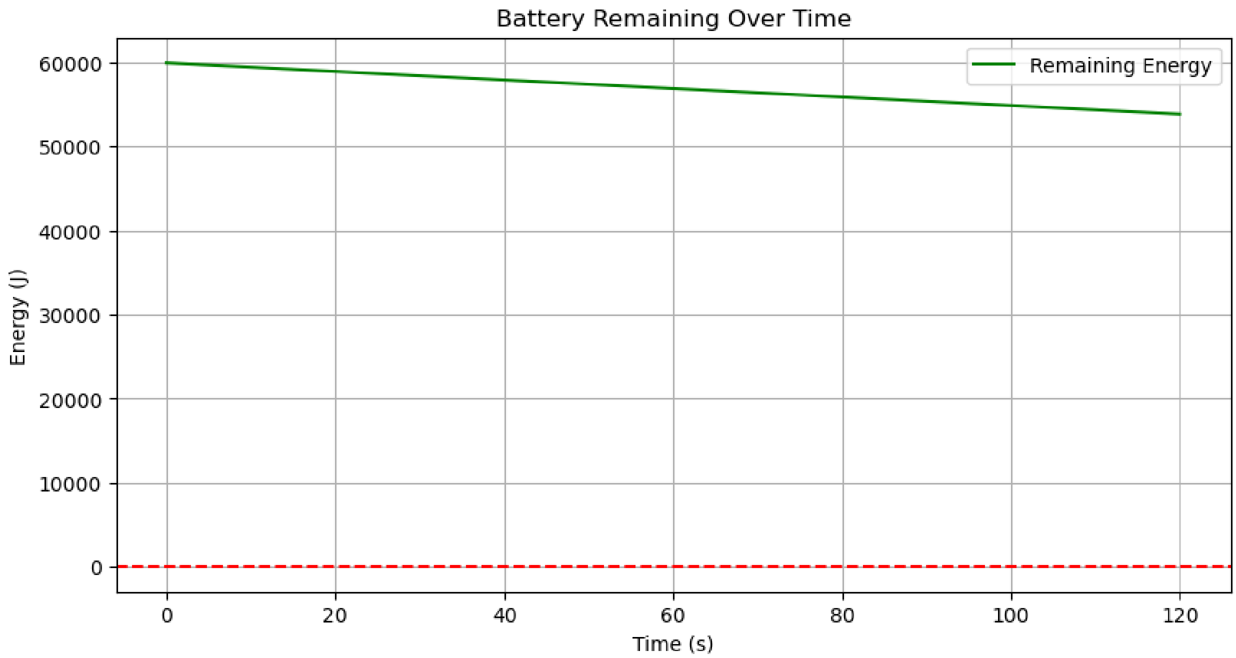


Fig. 28 Remaining Battery

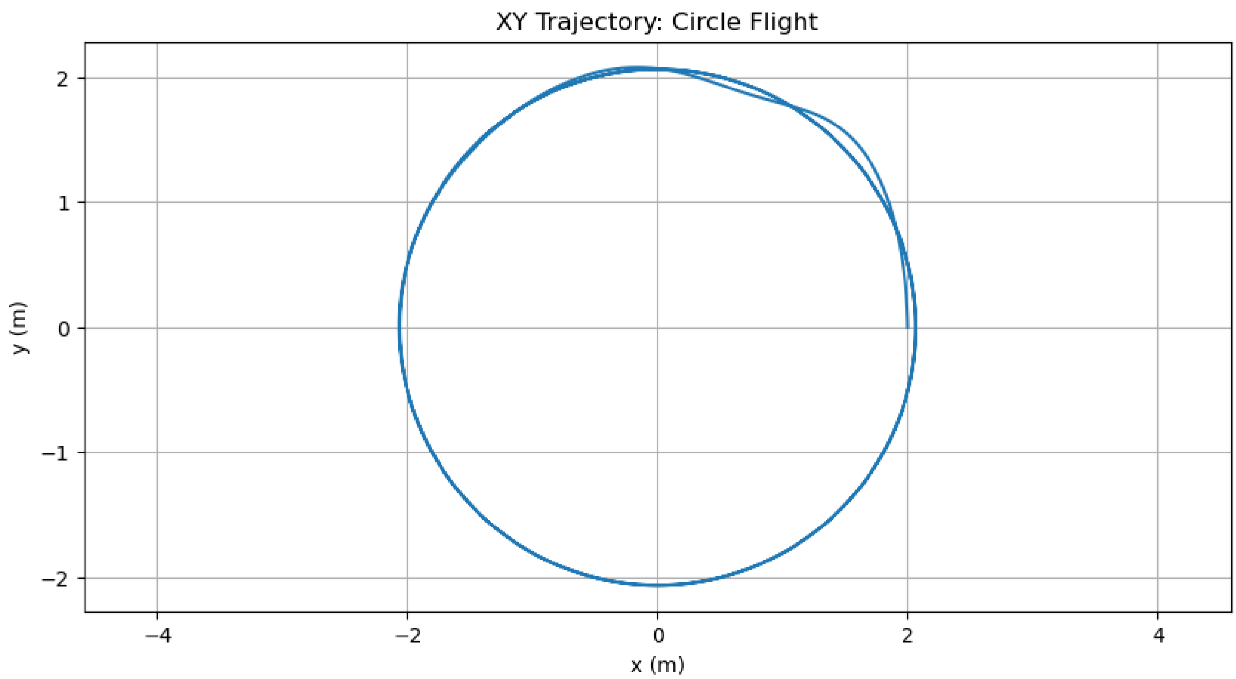


Fig. 29 XY Trajectory

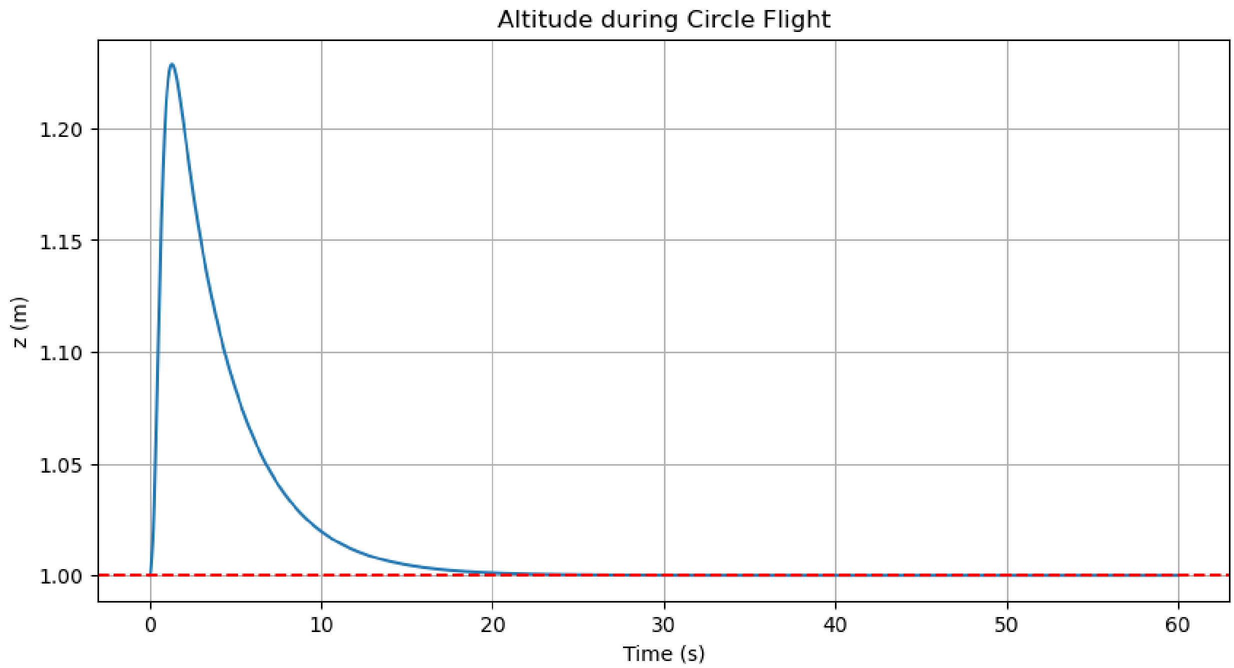


Fig. 30 XY Trajectory

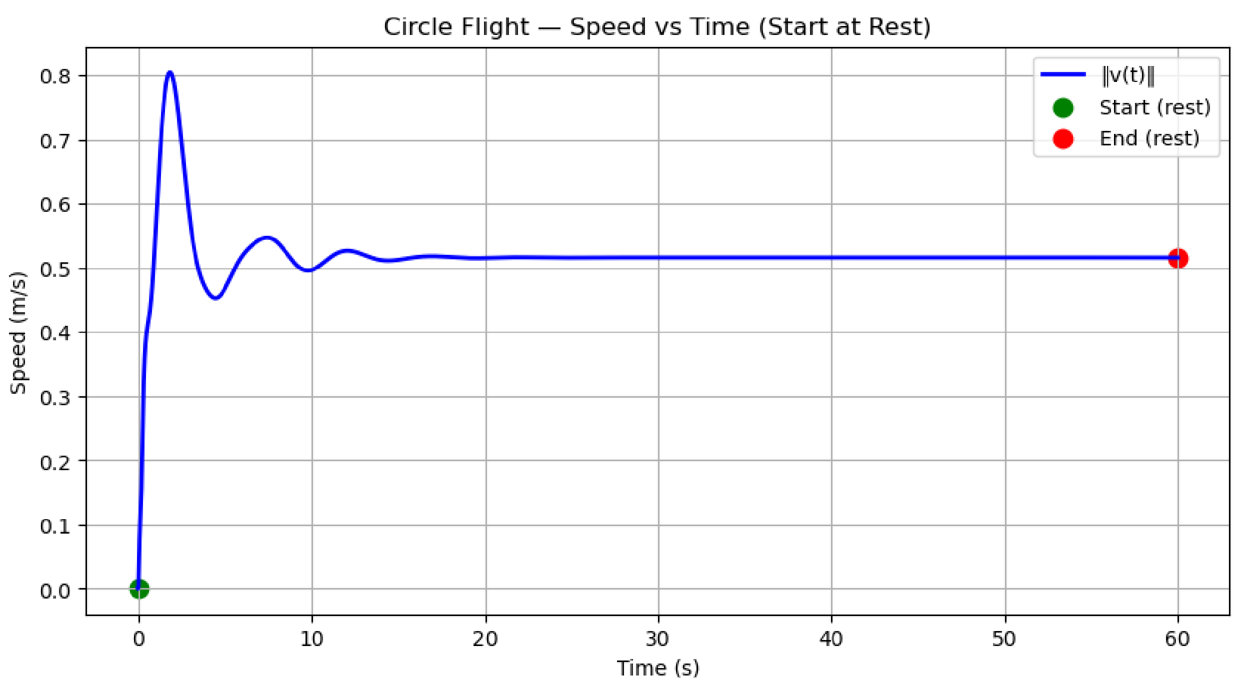


Fig. 31 Velocity vs Time

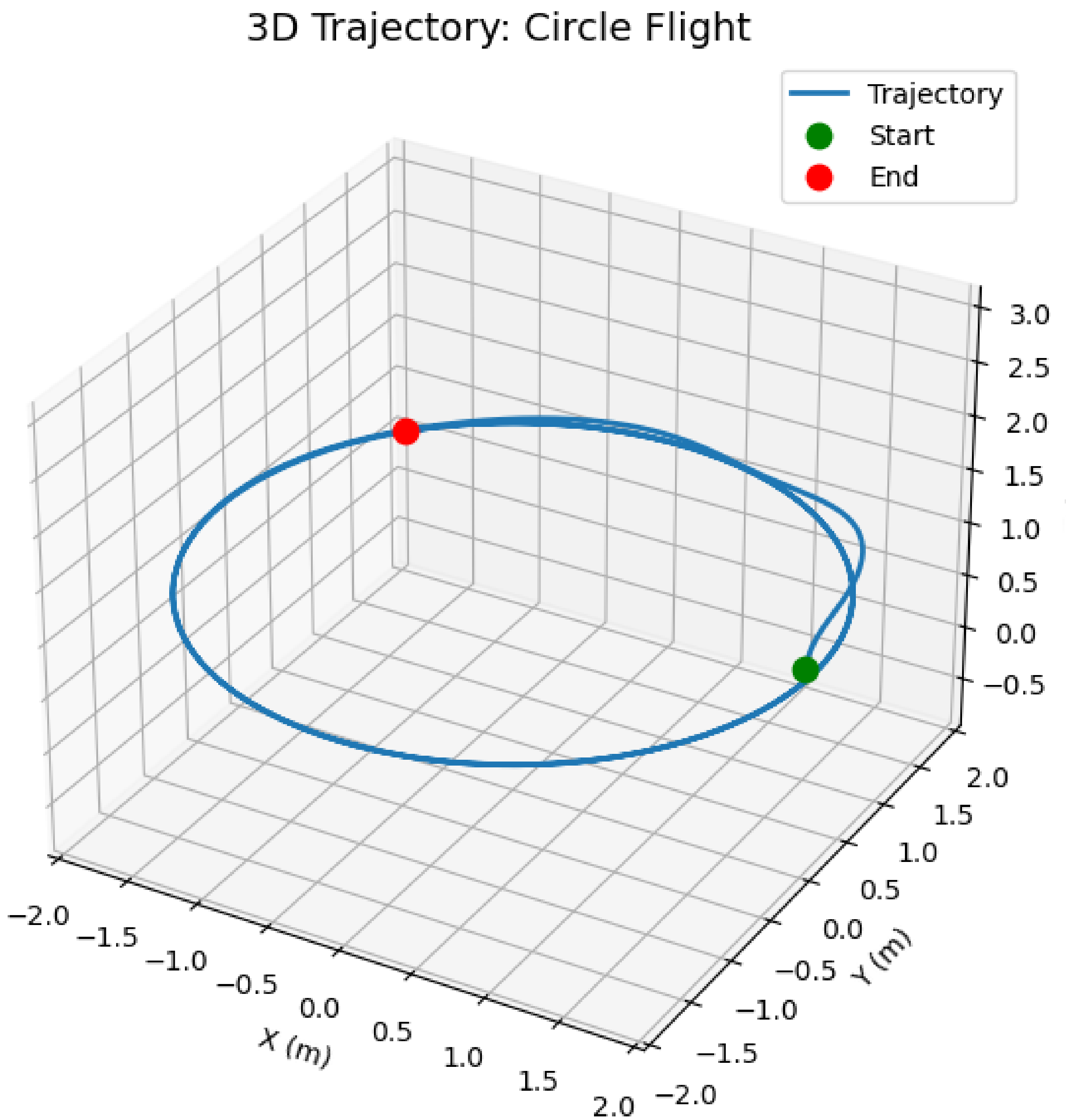


Fig. 32 3D Trajectory

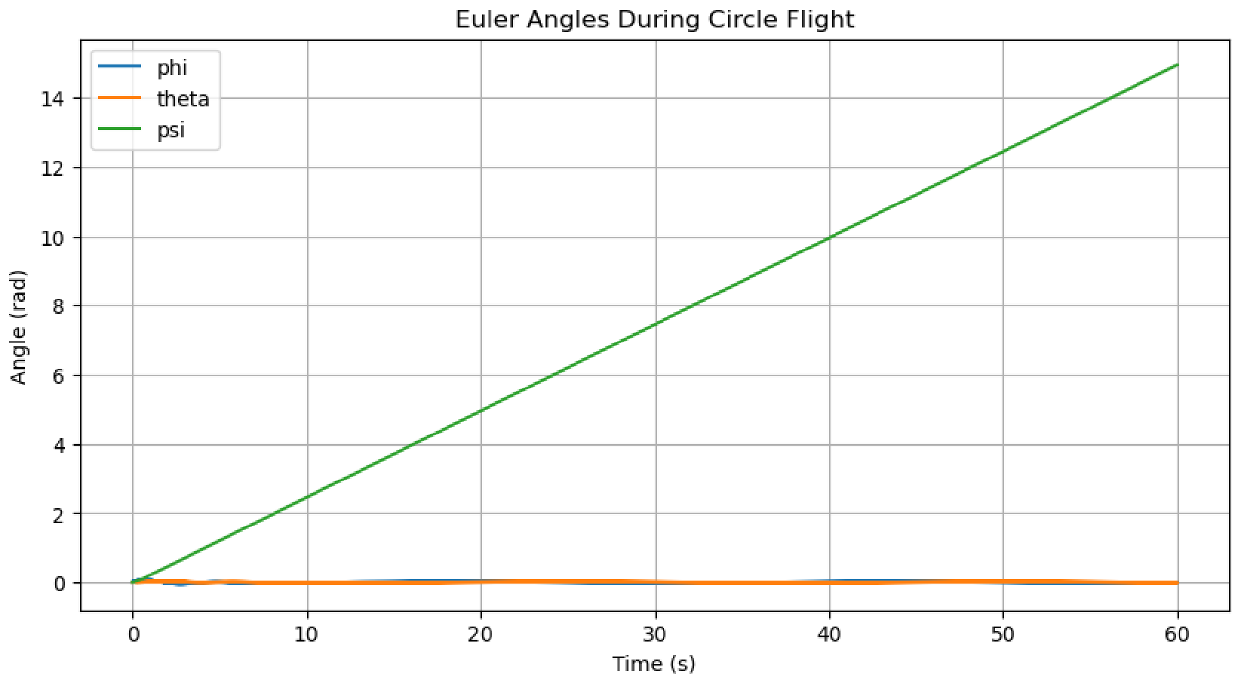


Fig. 33 Euler Angles

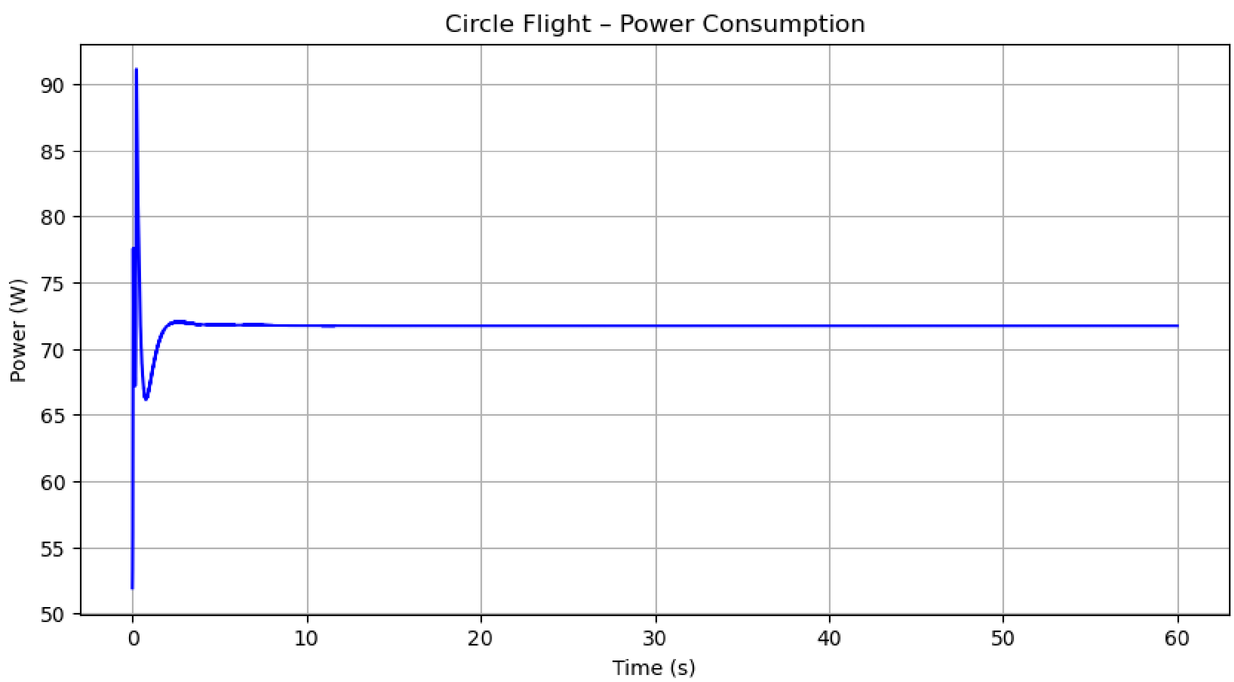


Fig. 34 Circle Power

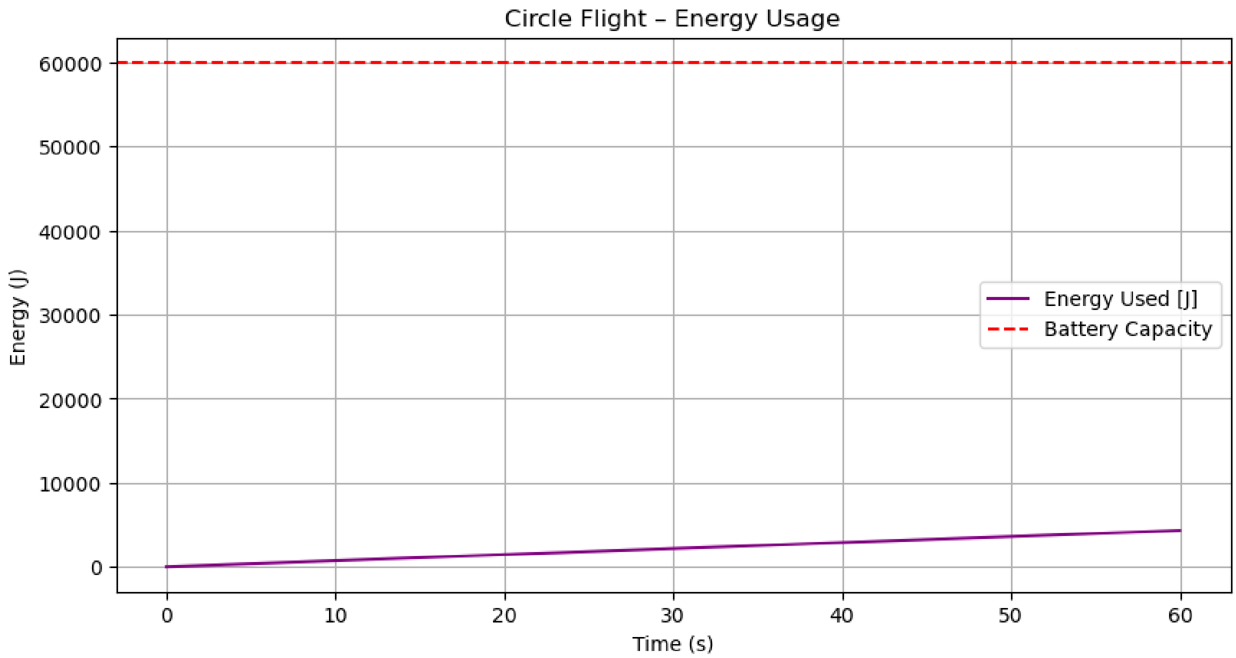


Fig. 35 Energy vs Battery

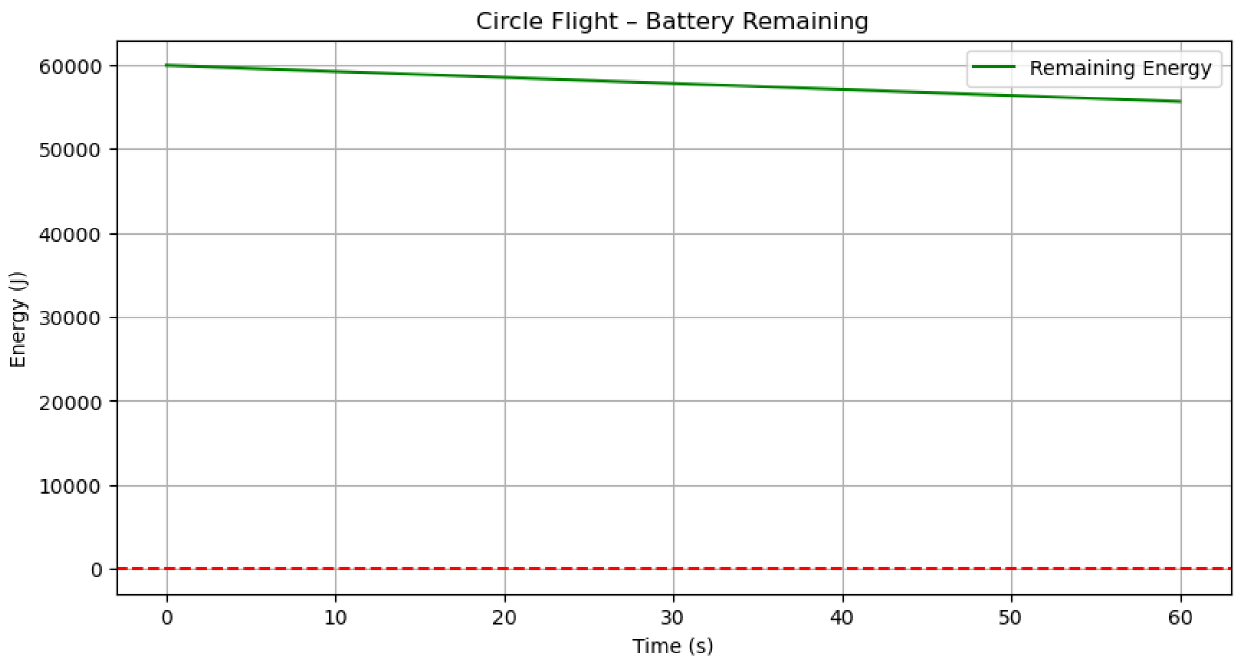


Fig. 36 Remaining Battery

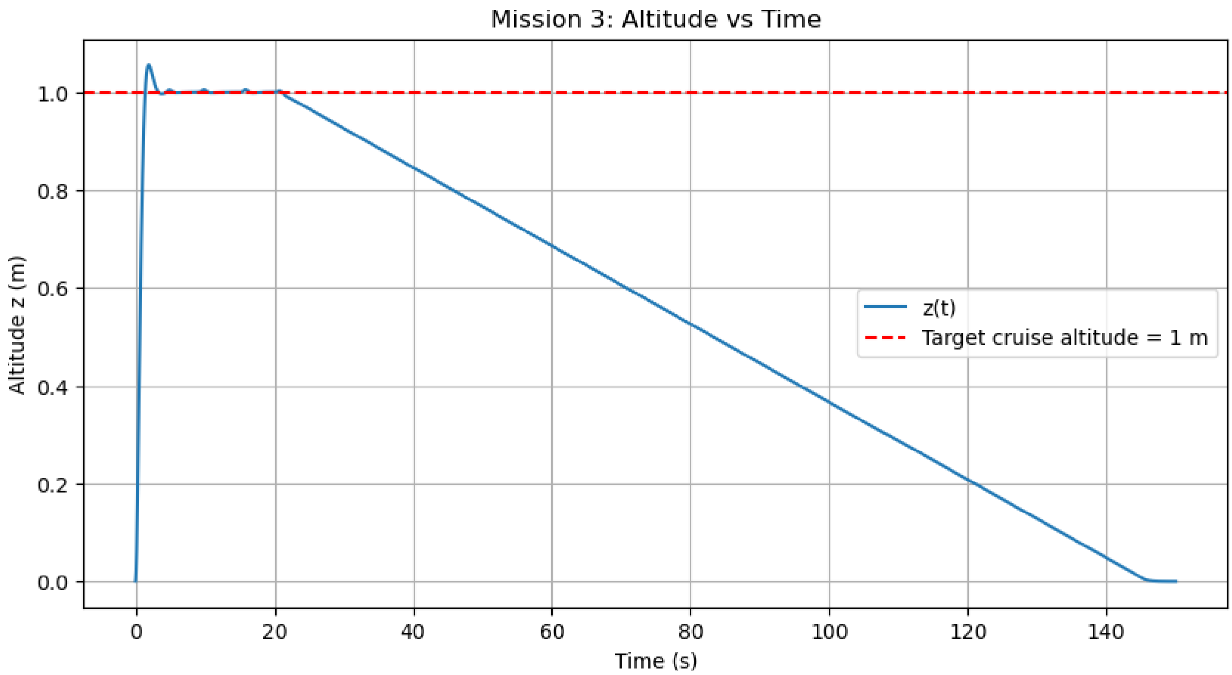


Fig. 37 Altitude vs Time

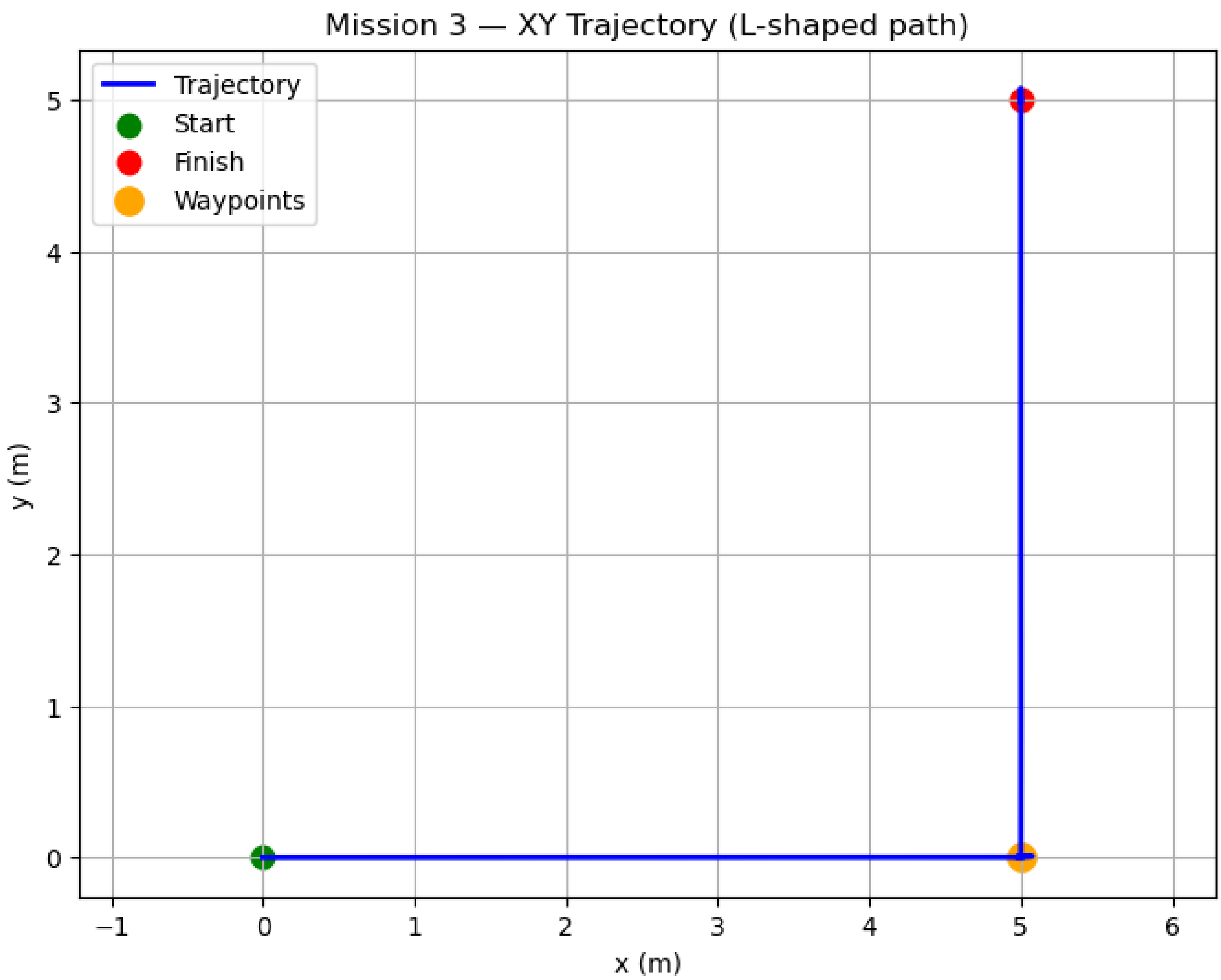


Fig. 38 XY position

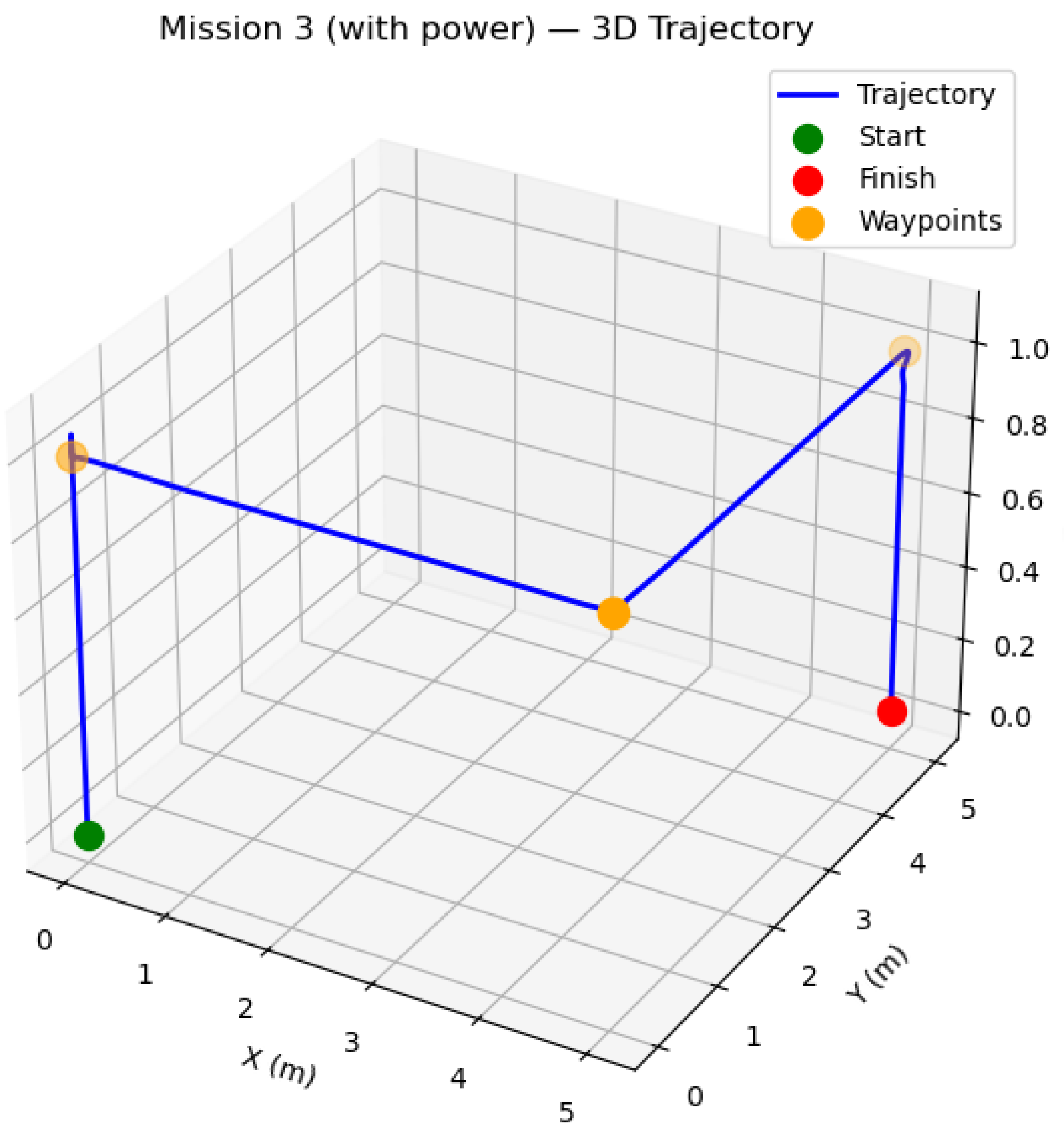


Fig. 39 3D Trajectory

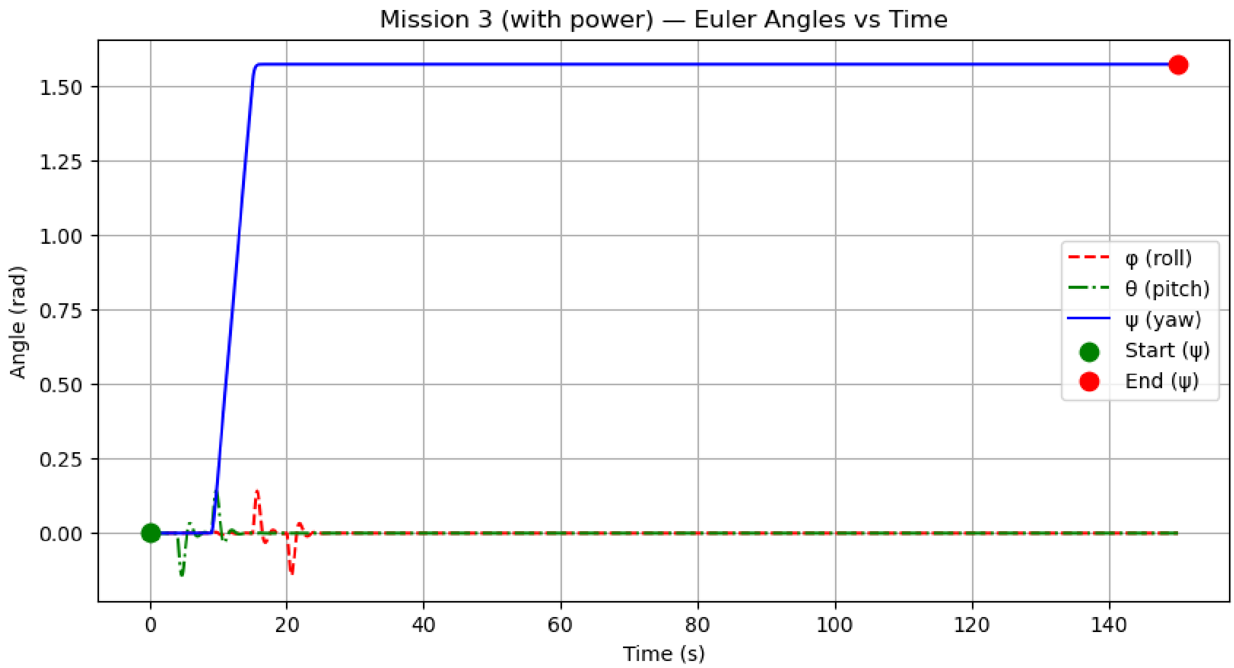


Fig. 40 3D Trajectory

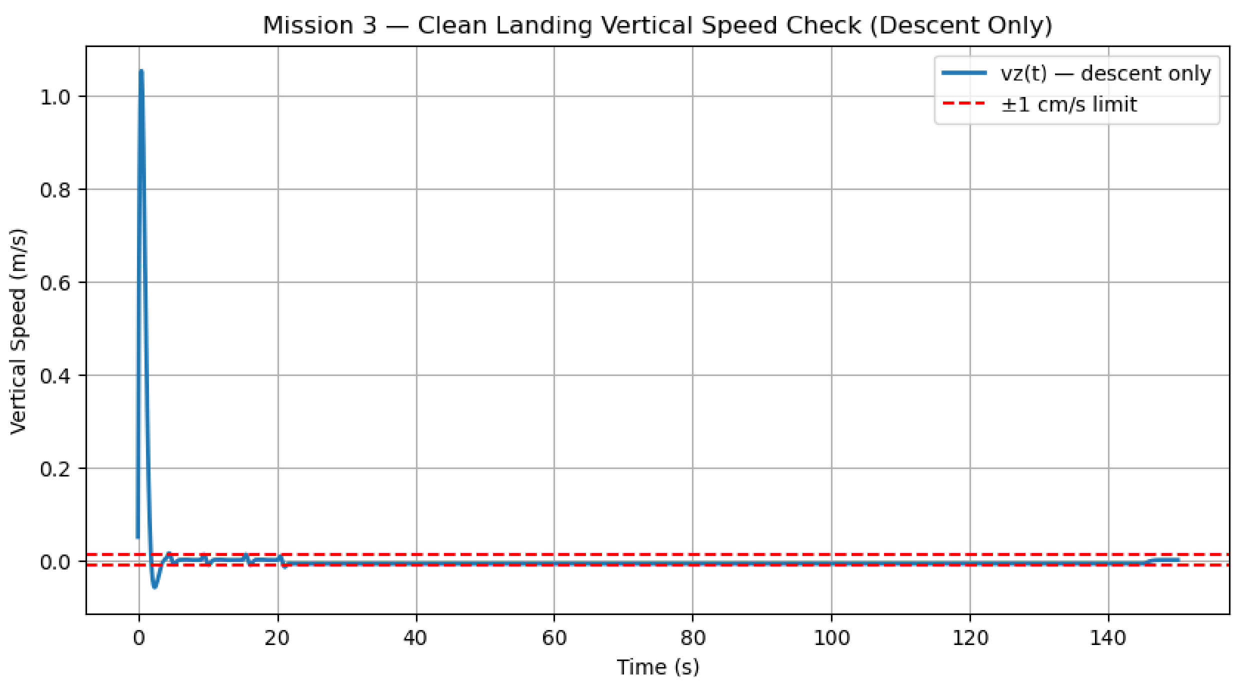


Fig. 41 Vertical Descent Speed Check

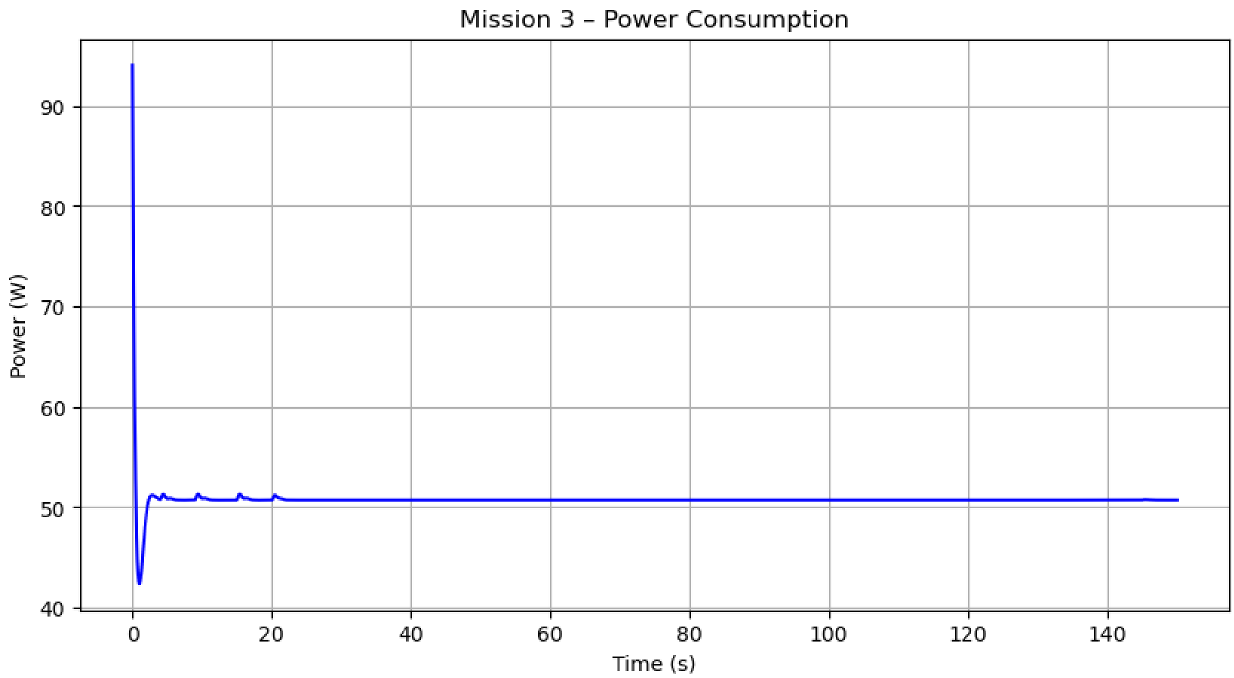


Fig. 42 Circle Power

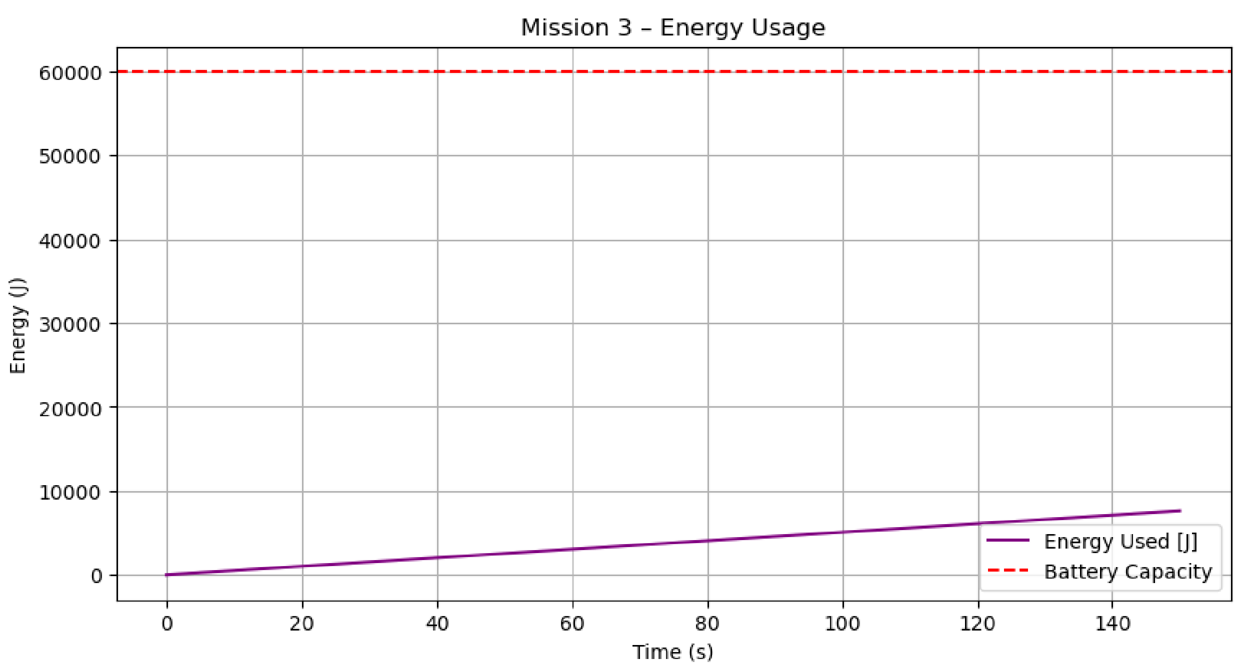


Fig. 43 Energy vs Battery

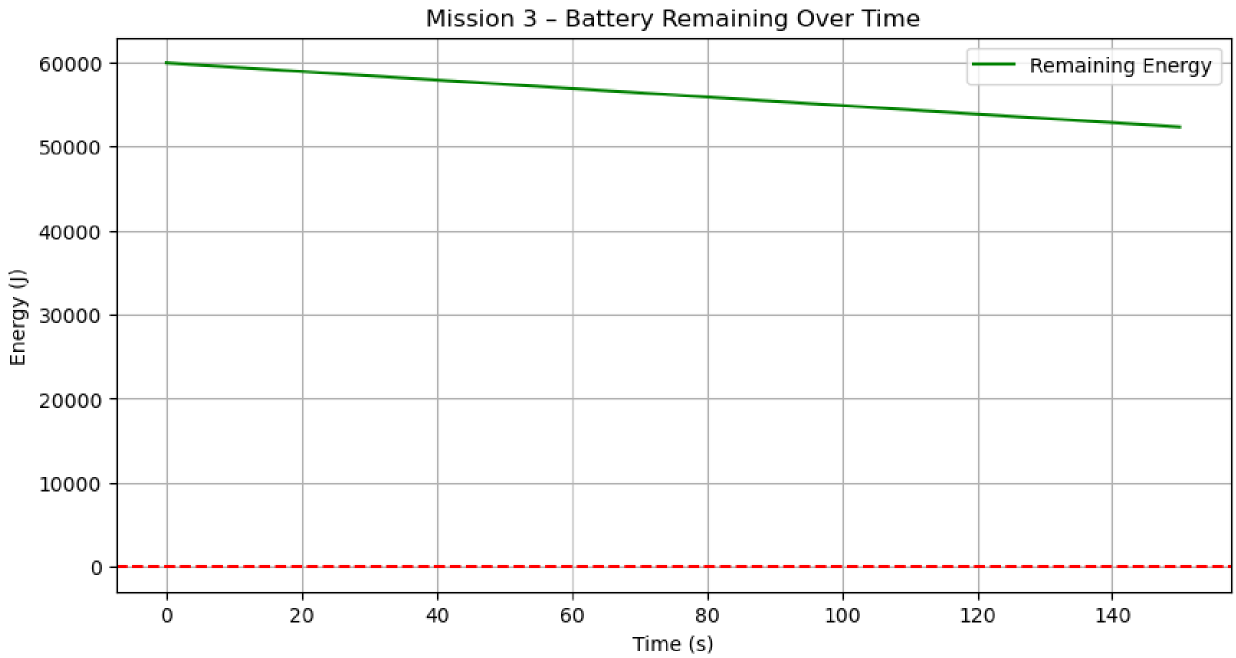
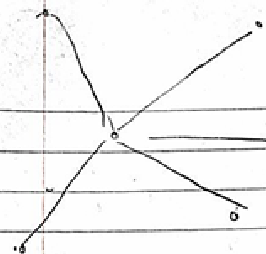


Fig. 44 Remaining Battery

C. Hand Derivations



$\rightarrow$ $y = x$

$x = y$

$z = z$

displacements, yes, good!

$$v_i^2 = \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2$$

Translational motion for rotors:

$$\left[\begin{aligned} x_i &= r_{com\,x} + r_{rotor}(\cos\theta_i + \sin\phi_i) \\ y_i &= r_{com\,y} + r_{rotor}(\cos\phi_i + \cos\psi_i) \\ z_i &= r_{com\,z} + r_{rotor}(\sin\theta_i + \sin\phi_i) \end{aligned}\right.$$

4 of these (1/rotor)

rotor angular speeds (opposite direction on opposite sides cancel out).

$$\left.\begin{aligned} x_{com} &= X \\ y_{com} &= Y \\ z_{com} &= Z \end{aligned}\right]$$ COM translational motion

$$T = \frac{1}{2} m (v_1^2 + v_2^2 + v_3^2 + v_4^2) + \frac{1}{2} m v_{com}^2 + \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3(-\omega_1)^2 + I_4(-\omega_2)^2) + \frac{1}{2} I_{com}\omega_{com}^2$$

$$T = \frac{1}{2} m (v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_{com}^2) + \frac{1}{2}[I_1\omega_1^2 + I_2\omega_2^2 + I_3(-\omega_1)^2 + I_4(-\omega_2)^2 + I_{com}\omega_{com}^2)$$

$V = mgh$

$a_i = \frac{1}{4}$ F_{thrust} $+ T_i$

$v_i^2 = \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2$

$\dot{x}_i = \dot{x}_{com} - \dot{\theta}\sin\theta \cdot r_{rotor} + \dot{\phi}\cos\phi \, r_{rotor}$

$\dot{x}_i = \dot{x}_{com} + r_{rotor}(-\dot{\theta}\sin\theta + \dot{\phi}\cos\phi)$

$\hookrightarrow$ equal + opposite θ, ϕ, ψ

$\dot{y}_i = -\dot{x}_j \rightarrow [\text{rotor } 1,3] \rightarrow [\text{rotor } 2,4]$

$\dot{y} = \dot{y}_{com} + r_{rotor}(-\dot{\phi}\sin\phi - \dot{\psi}\sin\psi)$

$\dot{z} = \dot{z}_{com} + r_{rotor}(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi)$

$\dot{x}_i^2 = (\dot{x}_{com} - r_{rotor}\dot{\theta}\sin\theta + r_{rotor}\dot{\phi}\cos\phi)$
$(\dot{x}_{com} - r_{rotor}\dot{\theta}\sin\theta + r_{rotor}\dot{\phi}\cos\phi)$

$= \dot{x}_{com}^2 - r_{rotor}\dot{\theta}\sin\theta\,\dot{x}_{com} + r_{rotor}\dot{\phi}\cos\phi\,\dot{x}_{com}$

$- r_{rotor}\dot{\theta}\sin\theta\,\dot{x}_{com} + (r_{rotor}\dot{\theta}\sin\theta)^2 - r_{rotor}^2(\dot{\theta}\dot{\phi}\sin\theta\cos\phi)$

$+ r_{rotor}\dot{\phi}\cos\phi\,\dot{x}_{com} - r_{rotor}^2\dot{\theta}\dot{\phi}\sin\theta\cos\phi$

$+ (r_{rotor}\dot{\phi}\cos\phi)^2$

$= \dot{x}_{com}^2 - 2r_{rotor}\dot{\theta}\sin\theta\,\dot{x}_{com} + 2r_{rotor}\dot{\phi}\cos\phi\,\dot{x}_{com}$
$+ (r_{rotor}\dot{\theta}\sin\theta)^2 + (r_{rotor}\dot{\phi}\cos\phi)^2$
$- 2r_{rotor}^2(\dot{\theta}\dot{\phi}\sin\theta\cos\phi)$

$$\boxed{\begin{aligned}\dot{x}_i^2 &= \dot{x}_{com}(\dot{x}_{com} - 2r_{rotor}\dot{\theta}\sin\theta + 2r_{rotor}\dot{\phi}\cos\phi) \\ &+ r_{rotor}^2(\dot{\theta}^2\sin^2\theta + \dot{\phi}^2\cos^2\phi) \\ &- 2r_{rotor}^2(\dot{\theta}\dot{\phi}\sin\theta\cos\phi)\end{aligned}}$$

$$\dot{y}_i^2 = (\dot{y}_{com} - r_{motor}\dot{\phi}\sin\phi - r_{motor}\dot{\psi}\sin\psi)$$
$$(\dot{y}_{com} - r_{motor}\dot{\phi}\sin\phi - r_{motor}\dot{\psi}\sin\psi)$$

$$\boxed{\begin{aligned}\dot{y}_i^2 &= \dot{y}_{com}[\dot{y}_{com} - 2r_{motor}(\dot{\phi}\sin\phi + \dot{\psi}\sin\psi)] \\ &+ r_{motor}^2(\dot{\phi}^2\sin^2\phi + \dot{\psi}^2\sin^2\psi) \\ &+ 2r_{motor}^2(\dot{\phi}\dot{\psi}\sin\phi\sin\psi)\end{aligned}}$$

$$\boxed{\begin{aligned}\dot{z}_i^2 &= \dot{z}_{com}[\dot{z}_{com} + 2r_{motor}(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi)] \\ &+ r_{motor}^2(\dot{\theta}^2\cos^2\theta + \dot{\phi}^2\cos^2\phi) \\ &+ 2r_{motor}^2(\dot{\theta}\dot{\phi}\cos\theta\cos\phi)\end{aligned}}$$

$$\dot{v}_i^2 = \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2$$

$$\dot{x}_{com}[\dot{x}_{com} + 2r_{motor}(-\dot{\theta}\sin\theta + \dot{\psi}\cos\psi)]$$
$$+ \dot{y}_{com}[\dot{y}_{com} - 2r_{motor}(\dot{\phi}\sin\phi + \dot{\psi}\sin\psi)]$$
$$+ \dot{z}_{com}[\dot{z}_{com} + 2r_{motor}(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi)]$$

$$\dot{x}_{com}^2 + \dot{y}_{com}^2 + \dot{z}_{com}^2 + 2r_{motor}[-\dot{x}_c\dot{\theta}\sin\theta + \dot{x}_c\dot{\psi}\cos\psi$$
$$+ \dot{y}_c(-\dot{\phi}\sin\phi) - \dot{y}_c(\dot{\psi}\sin\psi) + \dot{z}_c\dot{\theta}\cos\theta + \dot{z}_c\dot{\phi}\cos\phi]$$

$$\dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2 + 2r_{motor}[\dot{\theta}(-\dot{x}_c\sin\theta + \dot{z}_c\cos\theta)$$
$$+ \dot{\phi}(-\dot{y}_c\sin\phi + \dot{z}_c\cos\phi) + \dot{\psi}(\dot{x}_c\cos\psi - \dot{y}_c\sin\psi)] \quad (1)$$

$r_{rotor}^2 [\dot{\theta}^2 (1) + \dot{\phi}^2 (1) + \dot{\psi}^2 (1)]$

(3) $2 r_{rotor}^2 (-\dot{\theta}\dot{\psi} \sin\theta \cos\psi + \dot{\phi}\dot{\psi} \sin\phi \sin\psi + \dot{\theta}\dot{\phi} \cos\theta \cos\phi)$

(1) + (2) + (3)

$$\boxed{\begin{aligned} v_i^2 = {} & \dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2 + 2 r_{rotor} [\dot{\theta}(-\dot{x}_c \sin\theta + \dot{z}_c \cos\theta) \\ & + \dot{\phi}(-\dot{y}_c \sin\phi + \dot{z}_c \cos\phi) + \dot{\psi}(\dot{x}_c \cos\psi + \dot{y}_c \sin\psi)] \\ & + r_{rotor}^2 (\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2) + 2 r_{rotor}^2 (-\dot{\theta}\dot{\psi} \sin\theta \cos\psi \\ & + \dot{\phi}\dot{\psi} \sin\phi \sin\psi + \dot{\theta}\dot{\phi} \cos\theta \cos\phi) \end{aligned}}$$

$v_1 \Rightarrow v_3$ $\quad \underbrace{\dot{x}_c^2, \dot{y}_c^2 + \dot{z}_c^2}_{\text{Same}}$

$v_2 \Rightarrow v_4$

$\theta = -\theta$

$\psi = -\psi$ $\Rightarrow$

$\phi = \phi$

pitch

vs 1-3 $\quad \theta, -\theta \Rightarrow v_1$

roll 2-4 $\Rightarrow v_2, v_4 \neq 0$ $\quad v_1 = v_3 = 0$

yaw 1-2-3-4 $\quad v_1, v_2, v_3, v_4 \neq 0$

$$\boxed{\begin{aligned} v_1^2 = {} & \dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2 + 2 r_{rotor} [\dot{\theta}(-\dot{x}_c \sin\theta + \dot{z}_c \cos\theta) \\ & + \dot{\psi}(\dot{x}_c \cos\psi - \dot{y}_c \sin\psi)] + r_{rotor}^2 (\dot{\theta}^2 + \dot{\psi}^2) \\ & + 2 r_{rotor}^2 (-\dot{\theta}\dot{\psi} \sin\theta \cos\psi) \end{aligned}}$$

$$v_2^2 = \dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2 + 2r_{rotor}[\dot{\phi}(-\dot{y}_c \sin\phi + \dot{z}_c \cos\phi)$$
$$+\dot{\psi}(-\dot{y}_c \cos(\tfrac{\pi}{2}-\psi) + \dot{x}_c \sin(\tfrac{\pi}{2}-\psi)]$$
$$+r_{rotor}^2(\dot{\phi}^2 + \dot{\psi}^2) + 2r_{rotor}^2(-\dot{\phi}\dot{\psi} \sin\phi \sin(\tfrac{\pi}{2}-\psi))$$

$$v_3^2 = \dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2 + 2r_{rotor}[\dot{\theta}(\dot{x}_c \sin(2\pi-\theta) - \dot{z}_c \cos(2\pi-\theta))$$
$$+\dot{\psi}(-\dot{x}_c \cos(\tfrac{3\pi}{2}-\psi) + \dot{y}_c \sin(\tfrac{3\pi}{2}-\psi)]$$
$$+r_{rotor}^2(\dot{\theta}^2 + \dot{\psi}^2) + 2r_{rotor}^2(-\dot{\theta}\dot{\psi} \sin(2\pi-\theta)\cos(\tfrac{3\pi}{2}-\psi))$$

$$v_4^2 = \dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2 + 2r_{rotor}[\dot{\phi}(\dot{y}_c \sin(2\pi-\phi)$$
$$+\dot{z}_c \cos(2\pi-\phi)) + \dot{\psi}(-\dot{x}_c \cos(2\pi-\psi)$$
$$+\dot{y}_c \sin(2\pi-\psi)] + r_{rotor}^2(\dot{\phi}^2 + \dot{\psi}^2)$$
$$+2r_{rotor}^2(\dot{\phi}\dot{\psi} \sin(2\pi-\phi)\sin(2\pi-\psi))$$

$$\vec{v}_{com}^2 = \dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2$$

$$T = \frac{1}{2} m(v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_{com}^2)$$
$$+ \frac{1}{2}[I_1(\omega_1^2) + I_2(\omega_2^2) + I_3\omega_3^2 + I_4\omega_4^2]$$
$$+ \frac{1}{2} I_{com} \cdot \omega_{com}^2$$

$$U = Mgz$$

$$Q = \frac{1}{4} \text{Thrust} + \tau$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}}\right) - \frac{dL}{d\theta} = 0$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\phi}}\right) - \frac{dL}{d\phi} = 0.25 \ldots$$

$$+\frac{d}{dt}\left(\frac{dL}{d\dot{\psi}}\right) - \frac{dL}{d\psi} = 0.25 \ldots$$

$$+\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) - \frac{dL}{dx} = 0 \quad (1)$$

$$+\frac{d}{dt}\left(\frac{dL}{d\dot{y}}\right) - \frac{dL}{dy} = 0$$

$$+\frac{d}{dt}\left(\frac{dL}{d\dot{z}}\right) - \frac{dL}{dz} = mgh$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\omega}_1}\right) - \frac{dL}{d\omega_1} = \tau_1 \quad (2)$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\omega}_2}\right) - \frac{dL}{d\omega_2} = \tau_2 \quad (3)$$

$$\frac{dL}{d\dot{\theta}} \rightarrow \frac{1}{2} m\left[v_1^2 + v_2^2 + v_3^2 + v_4^2\right]$$

$$\frac{dv_1}{d\dot{\theta}} = 0 + 0 + 0 + 2r\,r_{obj}\,(-\dot{y}_c \sin\theta + \dot{z}_c \cos\theta)$$
$$+ r\,r_{obj}^2\,(2\dot{\theta}) + 2r\,r_{obj}^2\,(-\dot{\psi} \sin\phi \cos\phi)$$

$$\frac{dv_2}{d\dot{\theta}} = \frac{dv'}{d\dot{\theta}} = 0$$

$$\frac{dv_3^2}{d\dot\theta} = 2\,11\,\text{ctr}\left[\dot x_c \sin(2\Pi-\theta) - \dot z_c \cos(2\Pi-\theta)\right]$$
$$+\,11\,\text{ctr}^2(2\dot\theta) + 2\,11\,\text{ctr}^2\left(-\dot\varphi \sin(2\Pi-\theta)\cos\left(\tfrac{3\Pi}{2}-\varphi\right)\right)$$

v_3

$\sin(2\Pi-\theta)$

$\cos\left(\frac{3\Pi}{2}-\varphi\right)$

$$\frac{d}{dt}\left(\frac{dv_3^2}{d\dot\theta}\right) = 2\,11\,\text{ctr}\left[\ddot x_c \sin(2\Pi-\theta) - \dot\theta\cos(2\Pi-\theta)\,\dot x_c - \left[\ddot z_c\cos(2\Pi-\theta) + \dot\theta\sin(2\Pi-\theta)\cdot\dot z_c\right]\right]$$ ①

$$\boxed{+\,11\,\text{ctr}^2(2\ddot\theta)} + 2\,11\,\text{ctr}^2\left[-\left(\ddot\varphi\sin(2\Pi-\theta)\cos\left(\tfrac{3\Pi}{2}-\varphi\right)\right.\right.$$
$$\left.\left.-\dot\theta\cos(2\Pi-\theta)\cos\left(\tfrac{3\Pi}{2}-\varphi\right) + \dot\varphi\sin\left(\tfrac{3\Pi}{2}-\varphi\right)\sin(2\Pi-\theta)\right)\cdot\dot\varphi\right]$$

v_1

① $$\frac{d}{dt}\left(\frac{dv_1^2}{d\dot\theta}\right) = 2\,11\,\text{ctr}\left[-\left(\ddot x_c\sin\theta + \dot\theta\cos\theta\,\dot x_c\right) + \left(\ddot z_c\cos\theta - \dot\theta\sin\theta\,\dot z_c\right)\right]$$

② $$\boxed{+\,11\,\text{ctr}^2(2\ddot\theta)} + 2\,11\,\text{ctr}^2\left[-\left(\ddot\varphi\sin\theta\cos\varphi\right.\right.$$

③ $$\left.\left.+\left(\dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi\right)\cdot\dot\varphi\right)\right]$$

$\dot\theta$

$$\frac{1}{2}m\left(A + B\right) = \frac{d}{dt}\left(\frac{dL}{d\dot\theta}\right)$$

v_1 A $$\frac{dv_1}{d\theta} = 2\,11\,\text{ctr}\left[\dot\theta\left(-\dot x_c\cos\theta - \dot z_c\sin\theta\right)\right]$$

$+$

v_3 B $$\frac{dv_3}{d\theta} = 2\,11\,\text{ctr}\left[\dot\theta\left(-\dot x_c\cos(2\Pi-\theta) - \dot z_c\sin(2\Pi-\theta)\right)\right]$$

$$\frac{dL}{d\theta} = \frac{1}{2}m\left(\frac{A}{B}\right)$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot\phi}\right)-\frac{dL}{d\phi}=F_{Hsl} \times roll$$

$$\hookrightarrow \frac{dV_1}{d\dot\phi}=\frac{dV_3}{d\dot\phi}=0$$

ϕ

$$\frac{dV_2}{d\dot\phi}=2 r_{rotor}[-\dot y \sin\phi+\dot z\cos\phi]$$

$$+r_{rotor}^2(2\dot\phi)+2r_{rotor}^2(-\dot\psi\sin\phi\sin(\tfrac{H}{2}-\phi)$$

$$\frac{d}{dt}\left(\frac{dV_2}{d\dot\phi}\right)=2r_{rotor}[-(\ddot y\sin\phi+\dot\phi\cos\phi\,\dot y)+\ddot z\cos\phi$$

$$-\dot\phi\sin\phi\,\dot z]$$

$$+r_{rotor}^2(2\ddot\phi)+2r_{rotor}^2(-(\ddot\psi\sin\phi\sin(\tfrac{H}{2}-\phi)$$

$$+\dot\psi(\dot\phi\cos\phi\sin(\tfrac{H}{2}-\phi)-\dot\psi\cos(\tfrac{H}{2}-\phi)\sin\phi))]$$

$$\frac{dV_4}{d\dot\phi}=2r_{rotor}[\dot y\sin(2H-\phi)+\dot z\cos(2H-\phi)]$$

$$+r_{rotor}^2(2\dot\phi)+2r_{rotor}^2(\dot\psi\sin(2H-\phi)\sin(2H-\psi))$$

$\boxed{\frac{1}{4}m}$ $$\frac{d}{dt}\left(\frac{dV_4}{d\dot\phi}\right)=2r_{rotor}[\ddot y\sin(2H-\phi)-\dot\phi\cos(2H-\phi)\dot y$$

$$-(\ddot z\cos(2H-\phi)+\dot\phi\sin(2H-\phi)\dot z)]$$

$$+r_{rotor}^2(2\ddot\phi)+2r_{rotor}^2(\ddot\psi\sin(2H-\phi)\sin(2H-\phi)$$

$$+\dot\psi(-\dot\phi\cos(2H-\phi)\sin(2H-\phi))-\dot\psi\cos(2H-\phi)\sin(2H-\phi)$$

$\boxed{\frac{1}{4}m}$

$$\frac{dV_4}{d\phi}=2r_{rotor}[\dot\phi(-\dot y\cos(2H-\phi))+\dot\phi\dot z\sin(2H-\phi))]$$

$$\frac{dV_2}{d\phi}=2r_{rotor}[\dot\phi(-\dot y\cos\phi-\dot z\sin\phi)]$$

$$\partial \omega_1 \quad \ldots \quad [\; -\dot{x}\cos(3\pi/2-\phi) + \dot{y}\sin(3\pi/2-\phi) \ldots$$

$$\frac{d}{d\theta}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = r F_{thrust} \times y_{aw} \quad \ldots \left(\frac{\pi}{2} - \phi\right))$$

$$\frac{\partial \omega_1}{\partial \dot{\phi}} = 2 r_{rotor}\left[\dot{x}\cos\phi - \dot{y}\sin\phi\right]$$

$$+ r_{rotor}^2 (2\dot{\phi}) + 2 r_{rotor}^2 (-\dot{\theta}\sin\theta\cos\phi)$$

$$\frac{d}{dt}\left(\frac{\partial \omega_1}{\partial \dot{\phi}}\right) = 2 r_{rotor}\left[\ddot{x}\cos\phi - \dot{\phi}\sin\phi\,\dot{x} - (\ddot{y}\sin\phi + \dot{\phi}\cos\phi\,\dot{y})\right]$$

$$+ 2 r_{rotor}^2 (2\ddot{\phi}) + 2 r_{rotor}^2 (-(\ddot{\theta}\sin\theta\cos\phi)$$

$$+ \dot{\theta}[\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\phi\sin\theta])]$$

$$\frac{\partial \omega_2}{\partial \dot{\phi}} = 2 r_{rotor}\left[-\dot{x}\cos\left(\frac{\pi}{2} - \phi\right) + \dot{y}\sin\left(\frac{\pi}{2} - \phi\right)\right]$$

$$+ r_{rotor}^2 (2\dot{\phi}) + 2 r_{rotor}^2 \left[-\dot{\phi}\sin\phi\sin\left(\frac{\pi}{2} - \phi\right)\right]$$

$$\frac{\partial \omega_2}{dt} = 2 r_{rotor}\left[-(\ddot{x}\cos\left(\frac{\pi}{2} - \phi\right) + \dot{\phi}\sin\left(\frac{\pi}{2} - \phi\right)\dot{x})\right.$$

$$\left. + \ddot{y}\sin\left(\frac{\pi}{2} - \phi\right) - \dot{\phi}\cos\left(\frac{\pi}{2} - \phi\right)\dot{y}\right]$$

$$r_{rotor}^2 (2\ddot{\phi}) + 2 r_{rotor}^2 \left[-(\ddot{\phi}\sin\phi\sin\left(\frac{\pi}{2} - \phi\right)\right.$$

$$\left. + \dot{\phi}(\dot{\phi}\cos\phi\sin\left(\frac{\pi}{2} - \phi\right) - \dot{\phi}\cos\left(\frac{\pi}{2} - \phi\right)\sin\phi)\right]$$

$$\frac{\partial v_3}{\partial \dot{\psi}} = 2 l_{rotor} \left[-\dot{x} \cos\left(\tfrac{3\pi}{2} - \psi\right) + \dot{y} \sin\left(\tfrac{3\pi}{2} - \psi\right) \right] + l_{rotor}^2 (2\dot{\psi}) + 2 l_{rotor}^2 \left[-\dot{\theta} \sin(2\pi - \theta) \cos\left(\tfrac{3\pi}{2} - \psi\right) \right]$$

$$\frac{d}{dt}(\;) = 2 l_{rotor} \left[-\left(\ddot{x} \cos\left(\tfrac{3\pi}{2} - \psi\right) + \dot{\psi} \sin\left(\tfrac{3\pi}{2} - \psi\right) \dot{x} \right) + \ddot{y} \sin\left(\tfrac{3\pi}{2} - \psi\right) - \dot{\psi} \cos\left(\tfrac{3\pi}{2} - \psi\right) \dot{y} \right]$$
$$+ l_{rotor}^2 (2\ddot{\psi})$$
$$+ 2 l_{rotor}^2 \Big(- \Big(\ddot{\theta} \sin(2\pi - \theta) \cos\left(\tfrac{3\pi}{2} - \psi\right) + \dot{\theta} \Big[-\dot{\theta} \cos(2\pi - \theta) \cos\left(\tfrac{3\pi}{2} - \psi\right) + \dot{\psi} \sin\left(\tfrac{3\pi}{2} - \psi\right) \sin(2\pi - \theta) \Big]$$

$\dot{\psi}$

$$\frac{d v_4}{\partial \dot{\psi}} = 2 l_{rotor} \left[-\dot{x} \cos(2\pi - \psi) + \dot{y} \sin(2\pi - \psi) \right] + l_{rotor}^2 (2\dot{\psi}) + 2 l_{rotor}^2 \left(\dot{\phi} \sin(2\pi - \phi) \sin(2\pi - \psi) \right)$$

$$\frac{d}{dt}(\;) = 2 l_{rotor} \left[-\left(\ddot{x} \cos(2\pi - \psi) + \dot{\psi} \sin(2\pi - \psi) \dot{x} \right) + \ddot{y} \sin(2\pi - \psi) + \dot{y} \left(-\dot{\psi} \cos(2\pi - \psi) \right) \right]$$
$$+ l_{rotor}^2 (2\ddot{\psi})$$
$$+ 2 l_{rotor}^2 \Big[\ddot{\phi} \sin(2\pi - \phi) \sin(2\pi - \psi) + \dot{\phi} \left(-\dot{\phi} \cos(2\pi - \phi) \sin(2\pi - \psi) - \dot{\psi} \cos(2\pi - \psi) \sin(2\pi - \phi) \right) \Big]$$

$$\frac{d v_1}{\partial \dot{\psi}} = 2 l_{rotor} \left(\dot{\psi} \left(-\dot{x} \sin\psi - \dot{y} \cos\psi \right) \right) + 2 l_{rotor}^2 \left(\dot{\theta} \dot{\psi} \sin\theta \sin\psi \right)$$

$$\frac{dV_2}{d\dot\psi} = (\dot\phi\,(-\dot\alpha \sin(2\pi-\phi) + \dot y_1 \cos(\tfrac{\pi}{2}-\psi))$$

$2r_{rotor}$

$$+ 2r_{rotor}^2\,(\cdot\dot\phi\,\dot\psi \cos(\tfrac{\pi}{2}-\psi))$$

$\boxed{\frac{1}{2}m}$

$$\frac{dV_3}{d\psi} = 2r_{rotor}(\dot\phi\,(-\dot x_c \sin(\tfrac{3\pi}{2}-\psi) - \dot y \cos(\tfrac{3\pi}{2}-\phi))$$

$$+ 2r_{rotor}^2(-\dot\theta\,\dot\psi \sin(2\pi-\theta)\sin(\tfrac{3\pi}{2}-\psi))$$

$$\frac{dV_1}{d\psi} = 2r_{rotor}(\dot\psi(-\dot x \sin(2\pi-\psi) - \dot y \cos(2\pi-\psi))$$

$$+2r_{rotor}^2(-\dot\phi\,\dot\psi \sin(2\pi-\phi)\cos(2\pi-\phi))$$

$\dot\psi$

ω_1, ω_2

$$\cancel{\frac{d}{dt}\left(\frac{dL}{d\dot\psi}\right)} = \frac{dL}{d\omega} \qquad \frac{1}{2}m(v_1^2+v_2^2+v_3^2+v^4+v_{\dots}^2)$$

$\dot\omega_1$

$$\frac{dL}{d\omega_1} = \frac{1}{2}(2I_1\omega_1 + 2I_3\omega_1) = \tau_1$$

$$\frac{dL}{d\omega_1} \boxed{= -(I_1\omega_1 + I_3\omega_1) = \tau_1}$$

$\dot\omega_2$

$$\boxed{\frac{dL}{d\omega_2} = -(I_2\omega_2 + I_4\omega_2)} = \tau_2$$

$\dot\omega_{com}$

$$\frac{dL}{d\omega_{com}} = -I_{com}\,\omega_{com}$$

$\dots \ldots -\alpha) + \dot{y} \sin(3\frac{\psi}{2} - \alpha)]$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) = \frac{1}{2} m d (2\dot{x}) \cdot 4$$

$$= 4md(\ddot{x})$$

$\ddot{x}$ $\boxed{4md\ddot{x} = 0}$

$\ddot{y}$ $\boxed{\frac{d}{dt}\frac{dL}{dt} = 4md\ddot{y} = 0}$

$\ddot{z}$ $\boxed{\frac{d}{dt}\left(\frac{dL}{\dot{z}}\right) = 4md\ddot{z} + mg = \Sigma ft \ (\text{Thrust})}$

$$\boxed{q := [x, y, z, \theta, \phi, \psi, \omega_1, \omega_2, \omega_{com}]}$$

~~Foo~~ roll ϕ

Pitch $\emptyset$

Yaw ψ

$\frac{1}{2} m [\quad]$

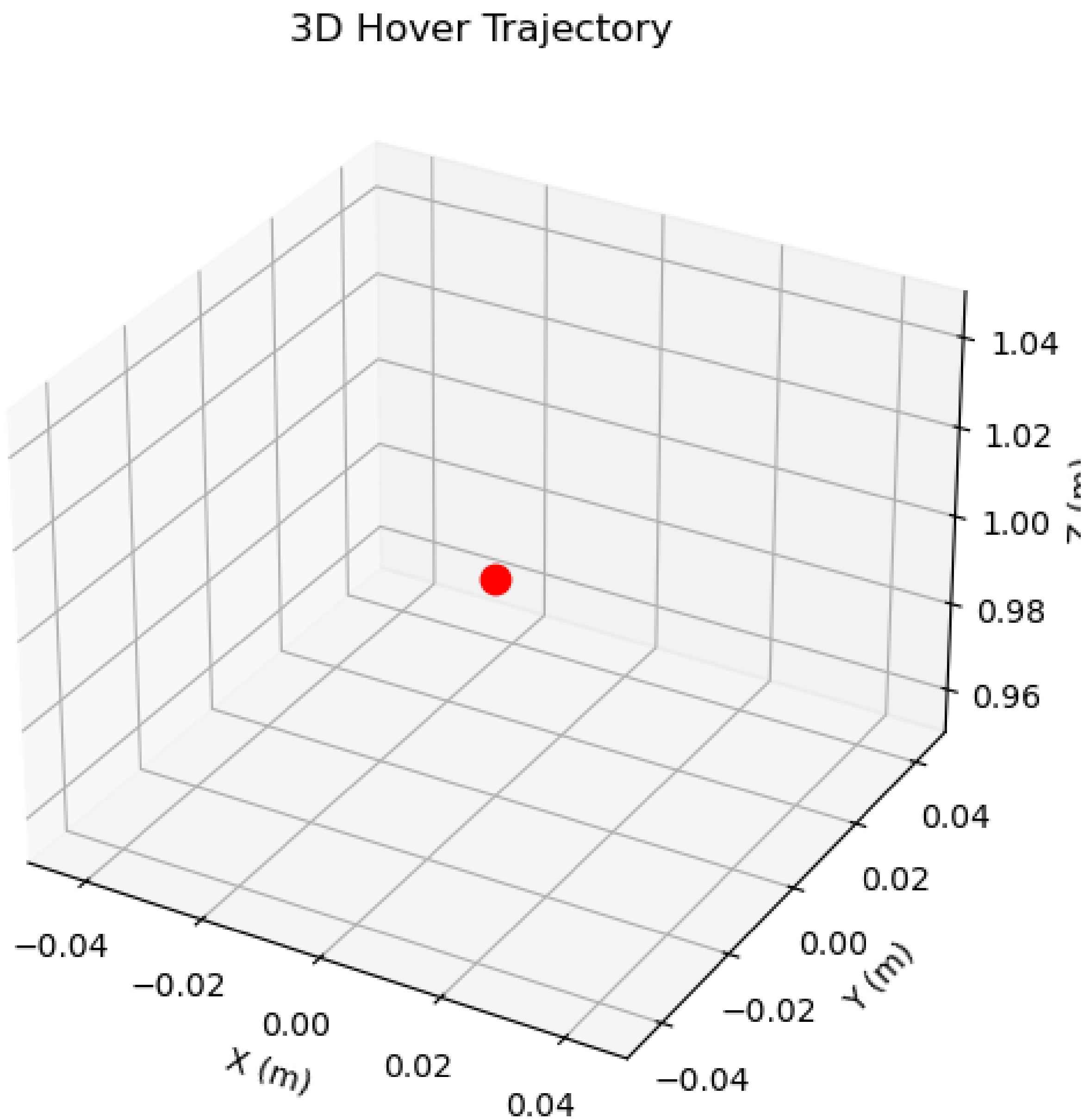


Fig. 24 3D Hover Trajectory