

This vector equation is equivalent to a system of three linear equations in three unknowns. If you row reduce the augmented matrix for this system, you find that

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

The system is consistent if and only if there is no pivot in the fourth column. That is, $h - 5$ must be 0. So \mathbf{y} is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if $h = 5$.

Remember: The presence of a free variable in a system does not guarantee that the system is consistent.

3. Since the vectors \mathbf{u} and \mathbf{v} are in $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, there exist scalars c_1, c_2, c_3 and d_1, d_2, d_3 such that

$$\mathbf{u} = c_1 \mathbf{w}_1 + c_2 \mathbf{w}_2 + c_3 \mathbf{w}_3 \quad \text{and} \quad \mathbf{v} = d_1 \mathbf{w}_1 + d_2 \mathbf{w}_2 + d_3 \mathbf{w}_3.$$

Notice

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= c_1 \mathbf{w}_1 + c_2 \mathbf{w}_2 + c_3 \mathbf{w}_3 + d_1 \mathbf{w}_1 + d_2 \mathbf{w}_2 + d_3 \mathbf{w}_3 \\ &= (c_1 + d_1) \mathbf{w}_1 + (c_2 + d_2) \mathbf{w}_2 + (c_3 + d_3) \mathbf{w}_3 \end{aligned}$$

Since $c_1 + d_1, c_2 + d_2$, and $c_3 + d_3$ are also scalars, the vector $\mathbf{u} + \mathbf{v}$ is in $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector. The following definition permits us to rephrase some of the concepts of Section 1.3 in new ways.

DEFINITION

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if \mathbf{x} is in \mathbb{R}^n , then the **product of A and \mathbf{x}** , denoted by $A\mathbf{x}$, is **the linear combination of the columns of A using the corresponding entries in \mathbf{x} as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n$$

Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in \mathbf{x} .

EXAMPLE 1

$$\begin{aligned} \text{a. } \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} &= 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{aligned}$$

$$\text{b. } \begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix} + \begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix} \quad \blacksquare$$

EXAMPLE 2 For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in \mathbb{R}^m , write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.

SOLUTION Place $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the columns of a matrix A and place the weights 3, -5 , and 7 into a vector \mathbf{x} . That is,

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathbf{x} \quad \blacksquare$$

Section 1.3 showed how to write a system of linear equations as a vector equation involving a linear combination of vectors. For example, the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned} \quad (1)$$

is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad (2)$$

As in Example 2, the linear combination on the left side is a matrix times a vector, so that (2) becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad (3)$$

Equation (3) has the form $A\mathbf{x} = \mathbf{b}$. Such an equation is called a **matrix equation**, to distinguish it from a vector equation such as is shown in (2).

Notice how the matrix in (3) is just the matrix of coefficients of the system (1). Similar calculations show that any system of linear equations, or any vector equation such as (2), can be written as an equivalent matrix equation in the form $A\mathbf{x} = \mathbf{b}$. This simple observation will be used repeatedly throughout the text.

Here is the formal result.

THEOREM 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if \mathbf{b} is in \mathbb{R}^m , the matrix equation

$$A\mathbf{x} = \mathbf{b} \quad (4)$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \quad (5)$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}] \quad (6)$$

Theorem 3 provides a powerful tool for gaining insight into problems in linear algebra, because a system of linear equations may now be viewed in three different but equivalent ways: as a matrix equation, as a vector equation, or as a system of linear equations. Whenever you construct a mathematical model of a problem in real life, you are free to choose whichever viewpoint is most natural. Then you may switch from one formulation of a problem to another whenever it is convenient. In any case, the matrix equation (4), the vector equation (5), and the system of equations are all solved in the same way—by row reducing the augmented matrix (6). Other methods of solution will be discussed later.

Existence of Solutions

The definition of $A\mathbf{x}$ leads directly to the following useful fact.

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if \mathbf{b} is a linear combination of the columns of A .

Section 1.3 considered the existence question, “Is \mathbf{b} in $\text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$?” Equivalently, “Is $A\mathbf{x} = \mathbf{b}$ consistent?” A harder existence problem is to determine whether the equation $A\mathbf{x} = \mathbf{b}$ is consistent for *all* possible \mathbf{b} .

EXAMPLE 3 Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3 ?

SOLUTION Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array} \right] &\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{array} \right] \end{aligned}$$

The third entry in column 4 equals $b_1 - \frac{1}{2}b_2 + b_3$. The equation $A\mathbf{x} = \mathbf{b}$ is *not* consistent for every \mathbf{b} because some choices of \mathbf{b} can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero. ■

The reduced matrix in Example 3 provides a description of all \mathbf{b} for which the equation $A\mathbf{x} = \mathbf{b}$ is consistent: The entries in \mathbf{b} must satisfy

$$b_1 - \frac{1}{2}b_2 + b_3 = 0$$

This is the equation of a plane through the origin in \mathbb{R}^3 . The plane is the set of all linear combinations of the three columns of A . See Figure 1.

The equation $A\mathbf{x} = \mathbf{b}$ in Example 3 fails to be consistent for all \mathbf{b} because the echelon form of A has a row of zeros. If A had a pivot in all three rows, we would not care about the calculations in the augmented column because in this case an echelon form of the augmented matrix could not have a row such as $[0 \ 0 \ 0 \ 1]$.

In the next theorem, the sentence “The columns of A span \mathbb{R}^m ” means that *every* \mathbf{b} in \mathbb{R}^m is a linear combination of the columns of A . In general, a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^m **spans** (or **generates**) \mathbb{R}^m if every vector in \mathbb{R}^m is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$ —that is, if $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = \mathbb{R}^m$.

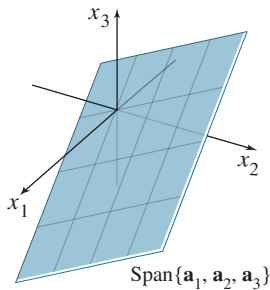


FIGURE 1

The columns of $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$ span a plane through $\mathbf{0}$.

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A , either they are all true statements or they are all false.

- For each \mathbf{b} in \mathbb{R}^m , the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
- Each \mathbf{b} in \mathbb{R}^m is a linear combination of the columns of A .
- The columns of A span \mathbb{R}^m .
- A has a pivot position in every row.

Theorem 4 is one of the most useful theorems in this chapter. Statements (a), (b), and (c) are equivalent because of the definition of $A\mathbf{x}$ and what it means for a set of vectors to span \mathbb{R}^m . The discussion after Example 3 suggests why (a) and (d) are equivalent; a proof is given at the end of the section. The exercises will provide examples of how Theorem 4 is used.

Warning: Theorem 4 is about a *coefficient matrix*, not an augmented matrix. If an augmented matrix $[A \ \mathbf{b}]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ may or may not be consistent.

Computation of $A\mathbf{x}$

The calculations in Example 1 were based on the definition of the product of a matrix A and a vector \mathbf{x} . The following simple example will lead to a more efficient method for calculating the entries in $A\mathbf{x}$ when working problems by hand.

EXAMPLE 4 Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

SOLUTION From the definition,

$$\begin{aligned} \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix} \end{aligned} \quad (7)$$

The first entry in the product $A\mathbf{x}$ is a sum of products (sometimes called a *dot product*), using the first row of A and the entries in \mathbf{x} . That is,

$$\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \end{bmatrix}$$

This matrix shows how to compute the first entry in $A\mathbf{x}$ directly, without writing down all the calculations shown in (7). Similarly, the second entry in $A\mathbf{x}$ can be calculated at once by multiplying the entries in the second row of A by the corresponding entries in \mathbf{x} and then summing the resulting products:

$$\begin{bmatrix} -1 & 5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + 5x_2 - 3x_3 \end{bmatrix}$$

Likewise, the third entry in $A\mathbf{x}$ can be calculated from the third row of A and the entries in \mathbf{x} . ■

Row-Vector Rule for Computing $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the i th entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector \mathbf{x} .

EXAMPLE 5

$$\begin{aligned} \text{a. } & \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 3 + (-1) \cdot 7 \\ 0 \cdot 4 + (-5) \cdot 3 + 3 \cdot 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ \text{b. } & \begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \cdot 4 + (-3) \cdot 7 \\ 8 \cdot 4 + 0 \cdot 7 \\ (-5) \cdot 4 + 2 \cdot 7 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix} \\ \text{c. } & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \cdot r + 0 \cdot s + 0 \cdot t \\ 0 \cdot r + 1 \cdot s + 0 \cdot t \\ 0 \cdot r + 0 \cdot s + 1 \cdot t \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix} \quad \blacksquare \end{aligned}$$

By definition, the matrix in Example 5(c) with 1's on the diagonal and 0's elsewhere is called an **identity matrix** and is denoted by I . The calculation in part (c) shows that $I\mathbf{x} = \mathbf{x}$ for every \mathbf{x} in \mathbb{R}^3 . There is an analogous $n \times n$ identity matrix, sometimes written as I_n . As in part (c), $I_n\mathbf{x} = \mathbf{x}$ for every \mathbf{x} in \mathbb{R}^n .

Properties of the Matrix–Vector Product $A\mathbf{x}$

The facts in the next theorem are important and will be used throughout the text. The proof relies on the definition of $A\mathbf{x}$ and the algebraic properties of \mathbb{R}^n .

THEOREM 5

If A is an $m \times n$ matrix, \mathbf{u} and \mathbf{v} are vectors in \mathbb{R}^n , and c is a scalar, then:

- $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;
- $A(c\mathbf{u}) = c(A\mathbf{u})$.

PROOF For simplicity, take $n = 3$, $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$, and \mathbf{u}, \mathbf{v} in \mathbb{R}^3 . (The proof of the general case is similar.) For $i = 1, 2, 3$, let u_i and v_i be the i th entries in \mathbf{u} and \mathbf{v} , respectively. To prove statement (a), compute $A(\mathbf{u} + \mathbf{v})$ as a linear combination of the columns of A using the entries in $\mathbf{u} + \mathbf{v}$ as weights.

$$\begin{aligned} A(\mathbf{u} + \mathbf{v}) &= [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3] \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \\ &= (u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + (u_3 + v_3)\mathbf{a}_3 \\ &= (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3) \\ &= A\mathbf{u} + A\mathbf{v} \end{aligned}$$

Entries in $\mathbf{u} + \mathbf{v}$

Columns of A

To prove statement (b), compute $A(c\mathbf{u})$ as a linear combination of the columns of A using the entries in $c\mathbf{u}$ as weights.

$$\begin{aligned} A(c\mathbf{u}) &= [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3] \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = (cu_1)\mathbf{a}_1 + (cu_2)\mathbf{a}_2 + (cu_3)\mathbf{a}_3 \\ &= c(u_1\mathbf{a}_1) + c(u_2\mathbf{a}_2) + c(u_3\mathbf{a}_3) \\ &= c(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) \\ &= c(A\mathbf{u}) \quad \blacksquare \end{aligned}$$

Numerical Note

To optimize a computer algorithm to compute $A\mathbf{x}$, the sequence of calculations should involve data stored in contiguous memory locations. The most widely used professional algorithms for matrix computations are written in Fortran, a language that stores a matrix as a set of columns. Such algorithms compute $A\mathbf{x}$ as a linear combination of the columns of A . In contrast, if a program is written in the popular language C, which stores matrices by rows, $A\mathbf{x}$ should be computed via the alternative rule that uses the rows of A .

PROOF OF THEOREM 4 As was pointed out after Theorem 4, statements (a), (b), and (c) are logically equivalent. So, it suffices to show (for an arbitrary matrix A) that (a) and (d) are either both true or both false. This will tie all four statements together.

Let U be an echelon form of A . Given \mathbf{b} in \mathbb{R}^m , we can row reduce the augmented matrix $[A \ \mathbf{b}]$ to an augmented matrix $[U \ \mathbf{d}]$ for some \mathbf{d} in \mathbb{R}^m :

$$[A \ \mathbf{b}] \sim \cdots \sim [U \ \mathbf{d}]$$

If statement (d) is true, then each row of U contains a pivot position and there can be no pivot in the augmented column. So $A\mathbf{x} = \mathbf{b}$ has a solution for any \mathbf{b} , and (a) is true. If (d) is false, the last row of U is all zeros. Let \mathbf{d} be any vector with a 1 in its last entry. Then $[U \ \mathbf{d}]$ represents an *inconsistent* system. Since row operations are reversible, $[U \ \mathbf{d}]$ can be transformed into the form $[A \ \mathbf{b}]$. The new system $A\mathbf{x} = \mathbf{b}$ is also inconsistent, and (a) is false. ■

Practice Problems

- Let $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$. It can be shown that \mathbf{p} is a solution of $A\mathbf{x} = \mathbf{b}$. Use this fact to exhibit \mathbf{b} as a specific linear combination of the columns of A .
- Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$. Verify Theorem 5(a) in this case by computing $A(\mathbf{u} + \mathbf{v})$ and $A\mathbf{u} + A\mathbf{v}$.
- Construct a 3×3 matrix A and vectors \mathbf{b} and \mathbf{c} in \mathbb{R}^3 so that $A\mathbf{x} = \mathbf{b}$ has a solution, but $A\mathbf{x} = \mathbf{c}$ does not.

1.4 Exercises

Compute the products in Exercises 1–4 using (a) the definition, as in Example 1, and (b) the row–vector rule for computing $A\mathbf{x}$. If a product is undefined, explain why.

$$1. \begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$$

$$2. \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$3. \begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$4. \begin{bmatrix} 8 & 3 & 1 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

In Exercises 5–8, use the definition of $A\mathbf{x}$ to write the matrix equation as a vector equation, or vice versa.

$$5. \begin{bmatrix} 7 & 2 & -9 & 3 \\ -4 & -5 & 7 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ -9 \\ 1 \\ -8 \end{bmatrix} = \begin{bmatrix} -9 \\ 44 \end{bmatrix}$$

$$6. \begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$$

$$7. x_1 \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

$$8. x_1 \begin{bmatrix} 4 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$$

In Exercises 9 and 10, write the system first as a vector equation and then as a matrix equation.

$$9. 4x_1 + x_2 - 7x_3 = 8$$

$$10. 8x_1 - x_2 = 4$$

$$x_2 + 6x_3 = 0$$

$$5x_1 + 4x_2 = 1$$

$$x_1 - 3x_2 = 2$$

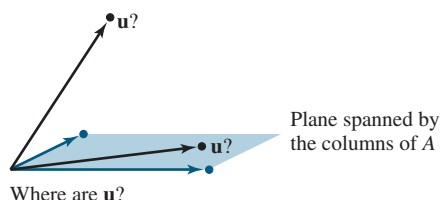
Given A and \mathbf{b} in Exercises 11 and 12, write the augmented matrix for the linear system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$. Then solve the system and write the solution as a vector.

$$11. A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}$$

$$12. A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$13. \text{ Let } \mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \text{ and } A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}. \text{ Is } \mathbf{u} \text{ in the plane in } \mathbb{R}^3$$

spanned by the columns of A ? (See the figure.) Why or why not?



$$14. \text{ Let } \mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \text{ and } A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}. \text{ Is } \mathbf{u} \text{ in the subset of } \mathbb{R}^3 \text{ spanned by the columns of } A? \text{ Why or why not?}$$

$$15. \text{ Let } A = \begin{bmatrix} 3 & -4 \\ -6 & 8 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \text{ Show that the equation } A\mathbf{x} = \mathbf{b} \text{ does not have a solution for all possible } \mathbf{b}, \text{ and describe the set of all } \mathbf{b} \text{ for which } A\mathbf{x} = \mathbf{b} \text{ does have a solution.}$$

$$16. \text{ Repeat Exercise 15: } A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Exercises 17–20 refer to the matrices A and B below. Make appropriate calculations that justify your answers and mention an appropriate theorem.

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$$

17. How many rows of A contain a pivot position? Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for each \mathbf{b} in \mathbb{R}^4 ?

18. Do the columns of B span \mathbb{R}^4 ? Does the equation $B\mathbf{x} = \mathbf{y}$ have a solution for each \mathbf{y} in \mathbb{R}^4 ?

19. Can each vector in \mathbb{R}^4 be written as a linear combination of the columns of the matrix A above? Do the columns of A span \mathbb{R}^4 ?

20. Can every vector in \mathbb{R}^4 be written as a linear combination of the columns of the matrix B above? Do the columns of B span \mathbb{R}^3 ?

$$21. \text{ Let } \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span \mathbb{R}^4 ? Why or why not?

$$22. \text{ Let } \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix}.$$

Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span \mathbb{R}^3 ? Why or why not?

In Exercises 23–34, mark each statement True or False (T/F). Justify each answer.

23. (T/F) The equation $A\mathbf{x} = \mathbf{b}$ is referred to as a *vector equation*.

24. (T/F) Every matrix equation $A\mathbf{x} = \mathbf{b}$ corresponds to a vector equation with the same solution set.

25. (T/F) If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then \mathbf{b} is not in the set spanned by the columns of A .

26. (T/F) A vector \mathbf{b} is a linear combination of the columns of a matrix A if and only if the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

27. (T/F) The equation $A\mathbf{x} = \mathbf{b}$ is consistent if the augmented matrix $[A \ \mathbf{b}]$ has a pivot position in every row.

28. (T/F) If A is an $m \times n$ matrix whose columns do not span \mathbb{R}^m , then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some \mathbf{b} in \mathbb{R}^m .

29. (T/F) The first entry in the product $A\mathbf{x}$ is a sum of products.

30. (T/F) Any linear combination vectors can always be written in the form $A\mathbf{x}$ for a suitable matrix A and vector \mathbf{x} .

31. (T/F) If the columns of an $m \times n$ matrix A span \mathbb{R}^m , then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each \mathbf{b} in \mathbb{R}^m .

32. (T/F) The solution set of a linear system whose augmented matrix is $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$ is the same as the solution set of $A\mathbf{x} = \mathbf{b}$, if $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$.

33. (T/F) If A is an $m \times n$ matrix and if the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some \mathbf{b} in \mathbb{R}^m , then A cannot have a pivot position in every row.

34. (T/F) If the augmented matrix $[A \ \mathbf{b}]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.

35. Note that $\begin{bmatrix} 3 & -4 & 2 \\ 6 & -3 & 4 \\ -8 & 9 & -5 \end{bmatrix} \begin{bmatrix} -4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -9 \\ 8 \end{bmatrix}$. Use this fact

(and no row operations) to find scalars c_1, c_2, c_3 such that

$$\begin{bmatrix} -2 \\ -9 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 6 \\ -8 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ -3 \\ 9 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 4 \\ -5 \end{bmatrix}.$$

36. Let $\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$.

It can be shown that $3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = \mathbf{0}$. Use this fact (and no row operations) to find x_1 and x_2 that satisfy the equation

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}.$$

37. Let $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$, and \mathbf{v} represent vectors in \mathbb{R}^5 , and let x_1, x_2 , and x_3 denote scalars. Write the following vector equation as a matrix equation. Identify any symbols you choose to use.

$$x_1\mathbf{q}_1 + x_2\mathbf{q}_2 + x_3\mathbf{q}_3 = \mathbf{v}$$

38. Rewrite the (numerical) matrix equation below in symbolic form as a vector equation, using symbols $\mathbf{v}_1, \mathbf{v}_2, \dots$ for the vectors and c_1, c_2, \dots for scalars. Define what each symbol represents, using the data given in the matrix equation.

$$\begin{bmatrix} -3 & 5 & -4 & 9 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

39. Construct a 3×3 matrix, not in echelon form, whose columns span \mathbb{R}^3 . Show that the matrix you construct has the desired property.

40. Construct a 3×3 matrix, not in echelon form, whose columns do *not* span \mathbb{R}^3 . Show that the matrix you construct has the desired property.

41. Let A be a 3×2 matrix. Explain why the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all \mathbf{b} in \mathbb{R}^3 . Generalize your argument to the case of an arbitrary A with more rows than columns.

42. Could a set of three vectors in \mathbb{R}^4 span all of \mathbb{R}^4 ? Explain. What about n vectors in \mathbb{R}^m when n is less than m ?

43. Suppose A is a 4×3 matrix and \mathbf{b} is a vector in \mathbb{R}^4 with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. What can you say about the reduced echelon form of A ? Justify your answer.

44. Suppose A is a 3×3 matrix and \mathbf{b} is a vector in \mathbb{R}^3 with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Explain why the columns of A must span \mathbb{R}^3 .

45. Let A be a 3×4 matrix, let \mathbf{y}_1 and \mathbf{y}_2 be vectors in \mathbb{R}^3 , and let $\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2$. Suppose $\mathbf{y}_1 = A\mathbf{x}_1$ and $\mathbf{y}_2 = A\mathbf{x}_2$ for some vectors \mathbf{x}_1 and \mathbf{x}_2 in \mathbb{R}^4 . What fact allows you to conclude that the system $A\mathbf{x} = \mathbf{w}$ is consistent? (Note: \mathbf{x}_1 and \mathbf{x}_2 denote vectors, not scalar entries in vectors.)

46. Let A be a 5×3 matrix, let \mathbf{y} be a vector in \mathbb{R}^3 , and let \mathbf{z} be a vector in \mathbb{R}^5 . Suppose $A\mathbf{y} = \mathbf{z}$. What fact allows you to conclude that the system $A\mathbf{x} = 4\mathbf{z}$ is consistent?

T In Exercises 47–50, determine if the columns of the matrix span \mathbb{R}^4 .

47. $\begin{bmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{bmatrix}$ 48. $\begin{bmatrix} 5 & -7 & -4 & 9 \\ 6 & -8 & -7 & 5 \\ 4 & -4 & -9 & -9 \\ -9 & 11 & 16 & 7 \end{bmatrix}$

49. $\begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ -9 & 4 & -8 & 7 & -3 \\ -6 & 11 & -7 & 3 & -9 \\ 4 & -6 & 10 & -5 & 12 \end{bmatrix}$

50. $\begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ -7 & -8 & 5 & 6 & -9 \\ 11 & 7 & -7 & -9 & -6 \\ -3 & 4 & 1 & 8 & 7 \end{bmatrix}$

T 51. Find a column of the matrix in Exercise 49 that can be deleted and yet have the remaining matrix columns still span \mathbb{R}^4 .

T 52. Find a column of the matrix in Exercise 50 that can be deleted and yet have the remaining matrix columns still span \mathbb{R}^4 . Can you delete more than one column?

STUDY GUIDE offers additional resources for mastering the concept of span.

Solutions to Practice Problems

1. The matrix equation

$$\begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$$

is equivalent to the vector equation

$$3 \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} - 2 \begin{bmatrix} 5 \\ 1 \\ -8 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 9 \\ -1 \end{bmatrix} - 4 \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix},$$

which expresses \mathbf{b} as a linear combination of the columns of A .

$$\begin{aligned} 2. \quad \mathbf{u} + \mathbf{v} &= \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\ A(\mathbf{u} + \mathbf{v}) &= \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 + 20 \\ 3 + 4 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \end{bmatrix} \\ A\mathbf{u} + A\mathbf{v} &= \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 11 \end{bmatrix} + \begin{bmatrix} 19 \\ -4 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \end{bmatrix} \end{aligned}$$

Remark: There are, in fact, infinitely many correct solutions to Practice Problem 3. When creating matrices to satisfy specified criteria, it is often useful to create matrices that are straightforward, such as those already in reduced echelon form. Here is one possible solution:

3. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \text{ and } \mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Notice the reduced echelon form of the augmented matrix corresponding to $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which corresponds to a consistent system, and hence $A\mathbf{x} = \mathbf{b}$ has solutions. The reduced echelon form of the augmented matrix corresponding to $A\mathbf{x} = \mathbf{c}$ is

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which corresponds to an inconsistent system, and hence $A\mathbf{x} = \mathbf{c}$ does not have any solutions.

1.5 Solution Sets of Linear Systems

Solution sets of linear systems are important objects of study in linear algebra. They will appear later in several different contexts. This section uses vector notation to give explicit and geometric descriptions of such solution sets.