

2. Applying the definition of linearly dependent to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ implies that there exist scalars c_1, c_2 , and c_3 , not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

Adding $0\mathbf{v}_4 = \mathbf{0}$ to both sides of this equation results in

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0}.$$

Since c_1, c_2, c_3 and 0 are not *all* zero, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ satisfies the definition of a linearly dependent set.

1.8 Introduction to Linear Transformations

The difference between a matrix equation $A\mathbf{x} = \mathbf{b}$ and the associated vector equation $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$ is merely a matter of notation. However, a matrix equation $A\mathbf{x} = \mathbf{b}$ can arise in linear algebra (and in applications such as computer graphics and signal processing) in a way that is not directly connected with linear combinations of vectors. This happens when we think of the matrix A as an object that “acts” on a vector \mathbf{x} by multiplication to produce a new vector called $A\mathbf{x}$.

For instance, the equations

$$\begin{array}{ccccccc} \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} & = & \begin{bmatrix} 5 \\ 8 \end{bmatrix} & \text{and} & \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} & = & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \uparrow & \uparrow & & \uparrow & & \uparrow & \uparrow & & \uparrow \\ A & \mathbf{x} & & \mathbf{b} & & A & \mathbf{u} & & \mathbf{0} \end{array}$$

say that multiplication by A transforms \mathbf{x} into \mathbf{b} and transforms \mathbf{u} into the zero vector. See Figure 1.

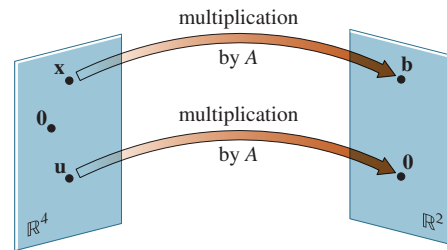


FIGURE 1 Transforming vectors via matrix multiplication.

From this new point of view, solving the equation $A\mathbf{x} = \mathbf{b}$ amounts to finding all vectors \mathbf{x} in \mathbb{R}^4 that are transformed into the vector \mathbf{b} in \mathbb{R}^2 under the “action” of multiplication by A .

The correspondence from \mathbf{x} to $A\mathbf{x}$ is a *function* from one set of vectors to another. This concept generalizes the common notion of a function as a rule that transforms one real number into another.

A **transformation** (or **function** or **mapping**) T from \mathbb{R}^n to \mathbb{R}^m is a rule that assigns to each vector \mathbf{x} in \mathbb{R}^n a vector $T(\mathbf{x})$ in \mathbb{R}^m . The set \mathbb{R}^n is called the **domain** of T , and \mathbb{R}^m

is called the **codomain** of T . The notation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ indicates that the domain of T is \mathbb{R}^n and the codomain is \mathbb{R}^m . For \mathbf{x} in \mathbb{R}^n , the vector $T(\mathbf{x})$ in \mathbb{R}^m is called the **image** of \mathbf{x} (under the action of T). The set of all images $T(\mathbf{x})$ is called the **range** of T . See Figure 2.

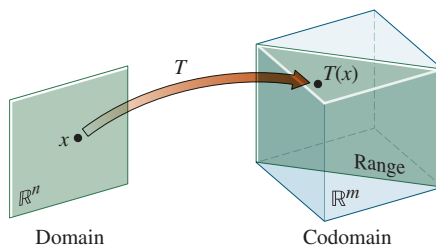


FIGURE 2 Domain, codomain, and range of $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

The new terminology in this section is important because a dynamic view of matrix–vector multiplication is the key to understanding several ideas in linear algebra and to building mathematical models of physical systems that evolve over time. Such *dynamical systems* will be discussed in Sections 1.10, 4.8, and throughout Chapter 5.

Matrix Transformations

The rest of this section focuses on mappings associated with matrix multiplication. For each \mathbf{x} in \mathbb{R}^n , $T(\mathbf{x})$ is computed as $A\mathbf{x}$, where A is an $m \times n$ matrix. For simplicity, we sometimes denote such a *matrix transformation* by $\mathbf{x} \mapsto A\mathbf{x}$. Observe that the domain of T is \mathbb{R}^n when A has n columns and the codomain of T is \mathbb{R}^m when each column of A has m entries. The range of T is the set of all linear combinations of the columns of A , because each image $T(\mathbf{x})$ is of the form $A\mathbf{x}$.

EXAMPLE 1 Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

- Find $T(\mathbf{u})$, the image of \mathbf{u} under the transformation T .
- Find an \mathbf{x} in \mathbb{R}^2 whose image under T is \mathbf{b} .
- Is there more than one \mathbf{x} whose image under T is \mathbf{b} ?
- Determine if \mathbf{c} is in the range of the transformation T .

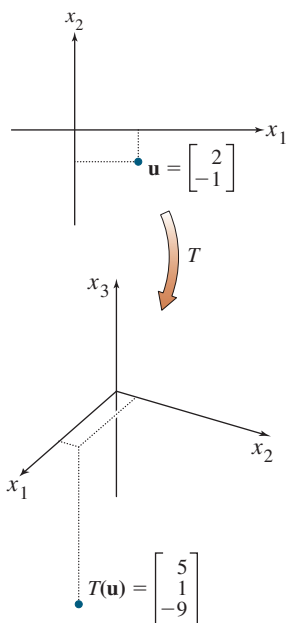
SOLUTION

- Compute

$$T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

- Solve $T(\mathbf{x}) = \mathbf{b}$ for \mathbf{x} . That is, solve $A\mathbf{x} = \mathbf{b}$, or

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \quad (1)$$



Using the method discussed in Section 1.4, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \quad (2)$$

Hence $x_1 = 1.5$, $x_2 = -.5$, and $\mathbf{x} = \begin{bmatrix} 1.5 \\ -.5 \end{bmatrix}$. The image of this \mathbf{x} under T is the given vector \mathbf{b} .

- c. Any \mathbf{x} whose image under T is \mathbf{b} must satisfy equation (1). From (2), it is clear that equation (1) has a unique solution. So there is exactly one \mathbf{x} whose image is \mathbf{b} .
- d. The vector \mathbf{c} is in the range of T if \mathbf{c} is the image of some \mathbf{x} in \mathbb{R}^2 , that is, if $\mathbf{c} = T(\mathbf{x})$ for some \mathbf{x} . This is just another way of asking if the system $A\mathbf{x} = \mathbf{c}$ is consistent. To find the answer, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

The third equation, $0 = -35$, shows that the system is inconsistent. So \mathbf{c} is *not* in the range of T . ■

The question in Example 1(c) is a *uniqueness* problem for a system of linear equations, translated here into the language of matrix transformations: Is \mathbf{b} the image of a *unique* \mathbf{x} in \mathbb{R}^n ? Similarly, Example 1(d) is an *existence* problem: Does there *exist* an \mathbf{x} whose image is \mathbf{c} ?

The next two matrix transformations can be viewed geometrically. They reinforce the dynamic view of a matrix as something that transforms vectors into other vectors. Section 2.7 contains other interesting examples connected with computer graphics.

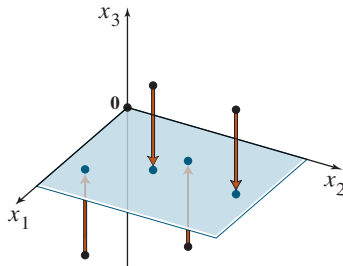


FIGURE 3

A projection transformation.

EXAMPLE 2 If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ projects points in \mathbb{R}^3 onto the x_1x_2 -plane because

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

See Figure 3. ■



sheep



sheared sheep

EXAMPLE 3 Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. The transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$T(\mathbf{x}) = A\mathbf{x}$ is called a **shear transformation**. It can be shown that if T acts on each point in the 2×2 square shown in Figure 4, then the set of images forms the sheared parallelogram. The key idea is to show that T maps line segments onto line segments (as shown in Exercise 35) and then to check that the corners of the square map onto the vertices of the parallelogram. For instance, the image of the point $\mathbf{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is

$T(\mathbf{u}) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, and the image of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$. T deforms the square as if the top of the square were pushed to the right while the base is held fixed. Shear transformations appear in physics, geology, and crystallography. ■

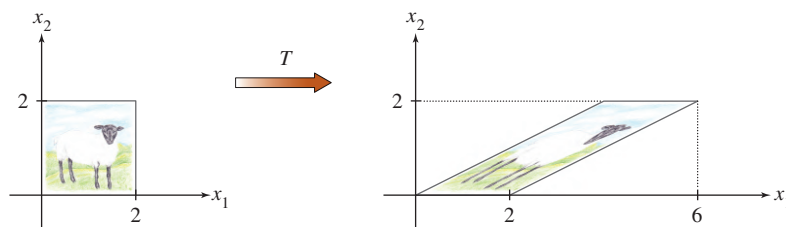


FIGURE 4 A shear transformation.

Linear Transformations

Theorem 5 in Section 1.4 shows that if A is $m \times n$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ has the properties

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} \quad \text{and} \quad A(c\mathbf{u}) = cA\mathbf{u}$$

for all \mathbf{u}, \mathbf{v} in \mathbb{R}^n and all scalars c . These properties, written in function notation, identify the most important class of transformations in linear algebra.

DEFINITION

A transformation (or mapping) T is **linear** if

- (i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u}, \mathbf{v} in the domain of T ;
- (ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all \mathbf{u} in the domain of T .

Every matrix transformation is a linear transformation. Important examples of linear transformations that are not matrix transformations will be discussed in Chapters 4 and 5.

Linear transformations *preserve the operations of vector addition and scalar multiplication*. Property (i) says that the result $T(\mathbf{u} + \mathbf{v})$ of first adding \mathbf{u} and \mathbf{v} in \mathbb{R}^n and then applying T is the same as first applying T to \mathbf{u} and to \mathbf{v} and then adding $T(\mathbf{u})$ and $T(\mathbf{v})$ in \mathbb{R}^m . These two properties lead easily to the following useful facts.

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0} \tag{3}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}) \tag{4}$$

for all vectors \mathbf{u}, \mathbf{v} in the domain of T and all scalars c, d .

Property (3) follows from condition (ii) in the definition, because $T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}$. Property (4) requires both (i) and (ii):

$$T(c\mathbf{u} + d\mathbf{v}) = T(c\mathbf{u}) + T(d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

Observe that if a transformation satisfies (4) for all \mathbf{u}, \mathbf{v} and c, d , it must be linear. (Set $c = d = 1$ for preservation of addition, and set $d = 0$ for preservation of scalar multiplication.) Repeated application of (4) produces a useful generalization:

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) \tag{5}$$

In engineering and physics, (5) is referred to as a *superposition principle*. Think of $\mathbf{v}_1, \dots, \mathbf{v}_p$ as signals that go into a system and $T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)$ as the responses of that system to the signals. The system satisfies the superposition principle if whenever an input is expressed as a linear combination of such signals, the system's response is *the same* linear combination of the responses to the individual signals. We will return to this idea in Chapter 4.

EXAMPLE 4 Given a scalar r , define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\mathbf{x}) = r\mathbf{x}$. T is called a **contraction** when $0 \leq r \leq 1$ and a **dilation** when $r > 1$. Let $r = 3$, and show that T is a linear transformation.

SOLUTION Let \mathbf{u}, \mathbf{v} be in \mathbb{R}^2 and let c, d be scalars. Then

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= 3(c\mathbf{u} + d\mathbf{v}) && \text{Definition of } T \\ &= 3c\mathbf{u} + 3d\mathbf{v} \\ &= c(3\mathbf{u}) + d(3\mathbf{v}) && \left. \begin{array}{l} \\ \end{array} \right\} \text{Vector arithmetic} \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$

Thus T is a linear transformation because it satisfies (4). See Figure 5. ■

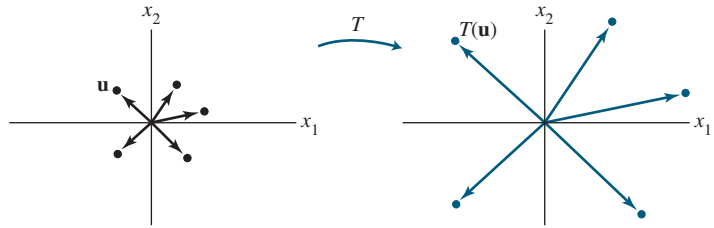


FIGURE 5 A dilation transformation.

EXAMPLE 5 Define a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

Find the images under T of $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

SOLUTION

$$\begin{aligned} T(\mathbf{u}) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, & T(\mathbf{v}) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \\ T(\mathbf{u} + \mathbf{v}) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix} \end{aligned}$$

Note that $T(\mathbf{u} + \mathbf{v})$ is obviously equal to $T(\mathbf{u}) + T(\mathbf{v})$. It appears from Figure 6 that T rotates \mathbf{u} , \mathbf{v} , and $\mathbf{u} + \mathbf{v}$ counterclockwise about the origin through 90° . In fact, T transforms the entire parallelogram determined by \mathbf{u} and \mathbf{v} into the one determined by $T(\mathbf{u})$ and $T(\mathbf{v})$. (See Exercise 36.) ■

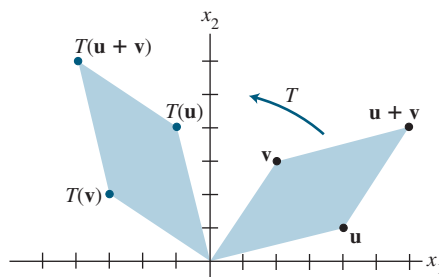


FIGURE 6 A rotation transformation.

The final example is not geometrical; instead, it shows how a linear mapping can transform one type of data into another.

EXAMPLE 6 A company manufactures two products, B and C. Using data from Example 7 in Section 1.3, we construct a “unit cost” matrix, $U = [\mathbf{b} \ \mathbf{c}]$, whose columns describe the “costs per dollar of output” for the products:

$$U = \begin{array}{cc} \begin{array}{c} \text{Product} \\ \mathbf{B} \quad \mathbf{C} \end{array} & \\ \begin{bmatrix} .45 & .40 \\ .25 & .30 \\ .15 & .15 \end{bmatrix} & \begin{array}{c} \text{Materials} \\ \text{Labor} \\ \text{Overhead} \end{array} \end{array}$$

Let $\mathbf{x} = (x_1, x_2)$ be a “production” vector, corresponding to x_1 dollars of product B and x_2 dollars of product C, and define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by

$$T(\mathbf{x}) = U\mathbf{x} = x_1 \begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix} + x_2 \begin{bmatrix} .40 \\ .30 \\ .15 \end{bmatrix} = \begin{bmatrix} \text{Total cost of materials} \\ \text{Total cost of labor} \\ \text{Total cost of overhead} \end{bmatrix}$$

The mapping T transforms a list of production quantities (measured in dollars) into a list of total costs. The linearity of this mapping is reflected in two ways:

1. If production is increased by a factor of, say, 4, from \mathbf{x} to $4\mathbf{x}$, then the costs will increase by the same factor, from $T(\mathbf{x})$ to $4T(\mathbf{x})$.
2. If \mathbf{x} and \mathbf{y} are production vectors, then the total cost vector associated with the combined production $\mathbf{x} + \mathbf{y}$ is precisely the sum of the cost vectors $T(\mathbf{x})$ and $T(\mathbf{y})$. ■

Practice Problems

1. Suppose $T : \mathbb{R}^5 \rightarrow \mathbb{R}^2$ and $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A and for each \mathbf{x} in \mathbb{R}^5 . How many rows and columns does A have?
2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Give a geometric description of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.
3. The line segment from $\mathbf{0}$ to a vector \mathbf{u} is the set of points of the form $t\mathbf{u}$, where $0 \leq t \leq 1$. Show that a linear transformation T maps this segment into the segment between $\mathbf{0}$ and $T(\mathbf{u})$.

1.8 Exercises

1. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

Find the images under T of $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

2. Let $A = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Define $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Find $T(\mathbf{u})$ and $T(\mathbf{v})$.

In Exercises 3–6, with T defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector \mathbf{x} whose image under T is \mathbf{b} , and determine whether \mathbf{x} is unique.

3. $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$

7. Let A be a 4×6 matrix. What must a and b be in order to define $T : \mathbb{R}^a \rightarrow \mathbb{R}^b$ by $T(\mathbf{x}) = A\mathbf{x}$?

8. How many rows and columns must a matrix A have in order to define a mapping from \mathbb{R}^3 into \mathbb{R}^6 by the rule $T(\mathbf{x}) = A\mathbf{x}$?

For Exercises 9 and 10, find all \mathbf{x} in \mathbb{R}^4 that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the given matrix A .

9. $A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$

11. Let $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and let A be the matrix in Exercise 9. Is \mathbf{b} in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

12. Let $\mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$, and let A be the matrix in Exercise 10. Is \mathbf{b} in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

In Exercises 13–16, use a rectangular coordinate system to plot $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, and their images under the given transformation T . (Make a separate and reasonably large sketch for each exercise.) Describe geometrically what T does to each vector \mathbf{x} in \mathbb{R}^2 .

13. $T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

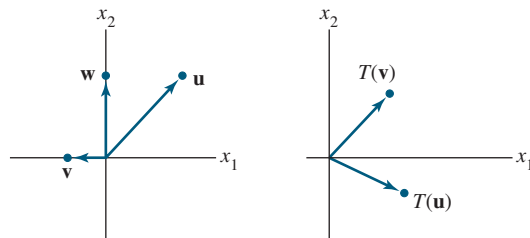
14. $T(\mathbf{x}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

15. $T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

16. $T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

17. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and maps $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 1 \\ -5 \end{bmatrix}$. Use the fact that T is linear to find the images under T of $5\mathbf{u}$, $4\mathbf{v}$, and $5\mathbf{u} + 4\mathbf{v}$.

18. The figure shows vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} , along with the images $T(\mathbf{u})$ and $T(\mathbf{v})$ under the action of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Copy this figure carefully, and draw the image $T(\mathbf{w})$ as accurately as possible. [Hint: First, write \mathbf{w} as a linear combination of \mathbf{u} and \mathbf{v} .]



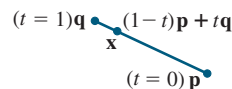
19. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, and let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation that maps \mathbf{e}_1 into \mathbf{y}_1 and maps \mathbf{e}_2 into \mathbf{y}_2 . Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

20. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -9 \end{bmatrix}$, and let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation that maps \mathbf{x} into $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$. Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for each \mathbf{x} .

In Exercises 21–30, mark each statement True or False (T/F). Justify each answer.

21. (T/F) A linear transformation is a special type of function.
22. (T/F) Every matrix transformation is a linear transformation.
23. (T/F) If A is a 3×5 matrix and T is a transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the domain of T is \mathbb{R}^3 .
24. (T/F) The codomain of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of all linear combinations of the columns of A .
25. (T/F) If A is an $m \times n$ matrix, then the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is \mathbb{R}^m .
26. (T/F) If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation and if \mathbf{c} is in \mathbb{R}^m , then a uniqueness question is “Is \mathbf{c} in the range of T ?”
27. (T/F) Every linear transformation is a matrix transformation.
28. (T/F) A linear transformation preserves the operations of vector addition and scalar multiplication.
29. (T/F) A transformation T is linear if and only if $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$ for all \mathbf{v}_1 and \mathbf{v}_2 in the domain of T and for all scalars c_1 and c_2 .
30. (T/F) The superposition principle is a physical description of a linear transformation.
31. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation that reflects each point through the x_1 -axis. (See Practice Problem 2.) Make two sketches similar to Figure 6 that illustrate properties (i) and (ii) of a linear transformation.
32. Suppose vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ span \mathbb{R}^n , and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation. Suppose $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, \dots, p$. Show that T is the zero transformation. That is, show that if \mathbf{x} is any vector in \mathbb{R}^n , then $T(\mathbf{x}) = \mathbf{0}$.
33. Given $\mathbf{v} \neq \mathbf{0}$ and \mathbf{p} in \mathbb{R}^n , the line through \mathbf{p} in the direction of \mathbf{v} has the parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$. Show that a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ maps this line onto another line or onto a single point (a *degenerate line*).
34. Let \mathbf{u} and \mathbf{v} be linearly independent vectors in \mathbb{R}^3 , and let P be the plane through \mathbf{u} , \mathbf{v} , and $\mathbf{0}$. The parametric equation of P is $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (with s, t in \mathbb{R}). Show that a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ maps P onto a plane through $\mathbf{0}$, or onto a line through $\mathbf{0}$, or onto just the origin in \mathbb{R}^3 . What must be true about $T(\mathbf{u})$ and $T(\mathbf{v})$ in order for the image of the plane P to be a plane?
35. a. Show that the line through vectors \mathbf{p} and \mathbf{q} in \mathbb{R}^n may be written in the parametric form $\mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$. (Refer to the figure with Exercises 25 and 26 in Section 1.5.)
b. The line segment from \mathbf{p} to \mathbf{q} is the set of points of the form $(1-t)\mathbf{p} + t\mathbf{q}$ for $0 \leq t \leq 1$ (as shown in the figure

below). Show that a linear transformation T maps this line segment onto a line segment or onto a single point.



36. Let \mathbf{u} and \mathbf{v} be vectors in \mathbb{R}^n . It can be shown that the set P of all points in the parallelogram determined by \mathbf{u} and \mathbf{v} has the form $a\mathbf{u} + b\mathbf{v}$, for $0 \leq a \leq 1, 0 \leq b \leq 1$. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Explain why the image of a point in P under the transformation T lies in the parallelogram determined by $T(\mathbf{u})$ and $T(\mathbf{v})$.
37. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = mx + b$.
a. Show that f is a linear transformation when $b = 0$.
b. Find a property of a linear transformation that is violated when $b \neq 0$.
c. Why is f called a linear function?
38. An *affine transformation* $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ has the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, with A an $m \times n$ matrix and \mathbf{b} in \mathbb{R}^m . Show that T is *not* a linear transformation when $\mathbf{b} \neq \mathbf{0}$. (Affine transformations are important in computer graphics.)
39. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set in \mathbb{R}^n . Explain why the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

In Exercises 40–44, column vectors are written as rows, such as $\mathbf{x} = (x_1, x_2)$, and $T(\mathbf{x})$ is written as $T(x_1, x_2)$.

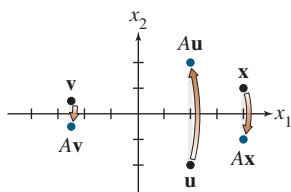
40. Show that the transformation T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.
41. Show that the transformation T defined by $T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$ is not linear.
42. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Show that if T maps two linearly independent vectors onto a linearly dependent set, then the equation $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution. [Hint: Suppose \mathbf{u} and \mathbf{v} in \mathbb{R}^n are linearly independent and yet $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent. Then $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$ for some weights c_1 and c_2 , not both zero. Use this equation.]
43. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the transformation that reflects each vector $\mathbf{x} = (x_1, x_2, x_3)$ through the plane $x_3 = 0$ onto $T(\mathbf{x}) = (x_1, x_2, -x_3)$. Show that T is a linear transformation. [See Example 4 for ideas.]
44. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the transformation that projects each vector $\mathbf{x} = (x_1, x_2, x_3)$ onto the plane $x_2 = 0$, so $T(\mathbf{x}) = (x_1, 0, x_3)$. Show that T is a linear transformation.

T In Exercises 45 and 46, the given matrix determines a linear transformation T . Find all \mathbf{x} such that $T(\mathbf{x}) = \mathbf{0}$.

45. $\begin{bmatrix} 4 & -2 & 5 & -5 \\ -9 & 7 & -8 & 0 \\ -6 & 4 & 5 & 3 \\ 5 & -3 & 8 & -4 \end{bmatrix}$ 46. $\begin{bmatrix} -9 & -4 & -9 & 4 \\ 5 & -8 & -7 & 6 \\ 7 & 11 & 16 & -9 \\ 9 & -7 & -4 & 5 \end{bmatrix}$

T 47. Let $\mathbf{b} = \begin{bmatrix} 7 \\ 5 \\ 9 \\ 7 \end{bmatrix}$ and let A be the matrix in Exercise 45. Is \mathbf{b}

STUDY GUIDE offers additional resources for mastering linear transformations.



The transformation $\mathbf{x} \mapsto A\mathbf{x}$.

in the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$? If so, find an \mathbf{x} whose image under the transformation is \mathbf{b} .

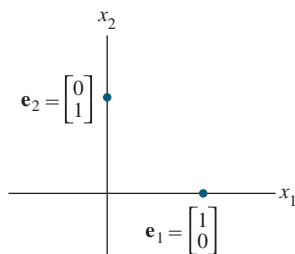
T 48. Let $\mathbf{b} = \begin{bmatrix} -7 \\ -7 \\ 13 \\ -5 \end{bmatrix}$ and let A be the matrix in Exercise 46. Is \mathbf{b} in the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$? If so, find an \mathbf{x} whose image under the transformation is \mathbf{b} .

Solutions to Practice Problems

1. A must have five columns for $A\mathbf{x}$ to be defined. A must have two rows for the codomain of T to be \mathbb{R}^2 .
2. Plot some random points (vectors) on graph paper to see what happens. A point such as $(4, 1)$ maps into $(4, -1)$. The transformation $\mathbf{x} \mapsto A\mathbf{x}$ reflects points through the x -axis (or x_1 -axis).
3. Let $\mathbf{x} = t\mathbf{u}$ for some t such that $0 \leq t \leq 1$. Since T is linear, $T(t\mathbf{u}) = tT(\mathbf{u})$, which is a point on the line segment between $\mathbf{0}$ and $T(\mathbf{u})$.

1.9 The Matrix of a Linear Transformation

Whenever a linear transformation T arises geometrically or is described in words, we usually want a “formula” for $T(\mathbf{x})$. The discussion that follows shows that every linear transformation from \mathbb{R}^n to \mathbb{R}^m is actually a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ and that important properties of T are intimately related to familiar properties of A . The key to finding A is to observe that T is completely determined by what it does to the columns of the $n \times n$ identity matrix I_n .



EXAMPLE 1 The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Suppose T is a linear transformation from \mathbb{R}^2 into \mathbb{R}^3 such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

With no additional information, find a formula for the image of an arbitrary \mathbf{x} in \mathbb{R}^2 .

SOLUTION Write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 \quad (1)$$

Since T is a linear transformation,

$$T(\mathbf{x}) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) \quad (2)$$

$$= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 + 0 \end{bmatrix}$$