# UNIT 9 DIFFERENTIAL EQUATIONS

This unit tells us, how to:

- define the differential equation, its order, degree, general and particular solutions, and its identification as linear and nonlinear ordinary differential equations.
- demonstrate the concept in forming a differential equation.
- solve the first order linear and nonlinear ordinary differential equations by separable variable form, and homogeneous form and then how to reduce differential equations in the standard form of homogeneous.
- solve the real-life problems related to differential equations.
- define the orthogonal trajectories and then how to show the orthogonal trajectories of the two families of curves.



## 9.1 Introduction

The laws of the universe are written in the language of Mathematics. Algebra is sufficient to solve many static problems, but **the most interesting natural phenomena involve change and are describe only by equations that relates rates at which quantities change.**

Suppose the solution of problems concerning the motion of objects, the flow of charged particles, heat transport, etc often involves discussion of relations of the form

$$\frac{dx}{dt} = f(x,t) \quad \text{or} \quad \frac{dq}{dt} = g(q,t)$$

In the first equation, x might represent distance. For this case, $\frac{dx}{dt}$ is the rate of change of distance with respect to time t that is speed. In the second equation, q might be a charge and $\frac{dq}{dt}$ is the rate of flow of charge that is current. These are examples of **different equations**, so called because these are equations involving the derivatives of various quantities. Such equations arise out of situations in which change is occurring.

In engineering, differential equations are most commonly used to model dynamic systems. These are the systems which change with time. Examples include an electronic circuit with time-dependent currents and voltages, a chemical production line in which pressure, tank levels, flow rates, etc, vary with time.

There is a wide variety of differential equations which occur in engineering applications, and consequently there is a wide variety of solution techniques available.

### i) *Definition of ordinary differential equation, order, degree, general and particular solutions*

**Definition 9.1.1: [Differential Equation]:** A differential equation is an equation that involves the derivatives of an unknown function (dependent variable) of one or more variables (independent variables).

If the unknown function depends on only one variable, then the derivative is an ordinary derivative, and the equation is then called an **ordinary differential equation.**

If the unknown function depends on more than one variable, then the derivative is partial derivative, and the equation is then called **partial differential equation.**

The following differential equations are the examples of ordinary differential equations with their corresponding unknown functions:

$$\frac{dy}{dx}=xy, \quad y(x)=? \qquad (1)$$

$$\frac{dy}{dx}=x+y, \quad y(x)=? \qquad (2)$$

$$\frac{dy}{dx}=\frac{x+y}{x-y}, \quad y(x)=? \qquad (3)$$

$$\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+y=3, \quad y(x)=? \qquad (4)$$

$$\left(\frac{d^3y}{dx^3}\right)^2-\frac{d^2y}{dx^2}+x^2\frac{dy}{dx}+y=3, \quad y(x)=? \qquad (5)$$

**Definition 9.1.2: [Order of a Differential Equation]:** The order of a differential equation is the order of the highest-order derivative occurring in the equation.

**Definition 9.1.3:[Degree of a Differential Equation]:** The degree of a differential equation is the power of the highest-order derivative occurring in the equation.

**Example 9.1.1:** Determine the order and degree of the following ordinary differential equations:

a. $\dfrac{dy}{dx}=\dfrac{x+y}{x-y}$

b. $\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+y=3$

c. $\left(\frac{d^3y}{dx^3}\right)^2 - \frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 3$

**Solution:** Differential equation (a) is an ordinary differential equation of order 1 and degree 1, since the highest ordinary derivative is of order 1 and the exponent of the highest ordinary derivative is 1. Differential equation (b) is an ordinary differential equation of order 2 and degree 1, while Differential equation (c) is an ordinary differential equation of order 3 and degree 2.

**Solution of a Differential Equation:** A solution of an equation in a single variable is a **number** which satisfies the equation. In similar fashion, solutions of the differential equations are **functions**, rather than numbers, which satisfy the differential equation. The variables which appear in equations are called **"unknowns."** Exactly, the only dependent variable in differential equations is referred to as **"unknown."**

For illustration, a solution of the differential equation dy/dx=1 is an expression of the unknown dependent variable y in terms of the independent variable x.

**Definition: 9.1.4:[Solution of a Differential Equation]:** A solution of an ordinary differential equation is any function $y = f(x)$ or $f(x, y)$ which when substituted in the differential equation, reduces the differential equation to an identity' that is, it satisfies the equation.

**Example 9.1.2:** Show that $y = x + A$ is a solution of the first order differential equation $dy/dx = 1$.

**Solution:** The given function y=x+A and its derivative dy/dx=1 is used in the differential equation dy/dx=1 to obtain:

$$\frac{dy}{dx} = 1$$

$$1 = 1, \textit{ identity left side} = \textit{right side}$$

This shows that $y = x + A$ is a solution of the ordinary differential equation $dy/dx = 1$.

The solution that depends on an arbitrary constant quantity is called the **general solution** of a differential equation. For example, by choosing different values of an arbitrary constant quantity, different solutions of a differential equation are obtained. This means that the general solution represents a family of curves (many solution curves) for any choice of **arbitrary constant quantity.**

Thus, the solution $y=x+A$ of a differential equation $dy/dx=1$ is declared the general solution. The general solution $y=x+A$ represents a family of curves (parallel lines) for any choice of arbitrary constant A.

If we give definite value to arbitrary constant quantity in the general solution, the solution so obtained is called a **particular solution.** For example, to determine a particular value of A for particular solution (line), we need to be given more information in the form of **initial condition.** For example, if we are given x=0, y=1, then from $y=x+A$, we have the definite value of arbitrary constant A:

$$y=x+A \quad \Rightarrow \quad 1=0+A \quad \Rightarrow \quad A=1$$

The definite value of A is used in the general solution $y=x+A$ to obtain the **particular solution** (line) $y=x+A=x+1$ which additionally satisfies the initial condition y(0)=1.

**Definition 9.1.5: [General and Particular Solution]:** The solution of a differential equation when depends on a single arbitrary constant quantity, is then called the **general solution** of the first order different equation. If we give particular steps for value to a single arbitrary constant quantity, then the solution to obtain is called the **particular solution** or **specific solution** or **exact solution** or **actual solution** of first order differential equation.

**Graphically,**

- the general solution of a first order deferential equation represents a family of curves for any choice of arbitrary constant quantity.
- The particular solution of a first order differential equation is a particular curve chosen from a family of curves (general solution) for a particular value of a constant quantity.

**Example 9.1.3:[General & Particular Solution]:** Graphically, show that $y=x+A$ is a general solution of the first order differential equation $dy/dx=1$. Find a particular solution, when $x=0$ and $y=1$.

**Solution:** The general solution $y=x+A$ of a first order differential equation $dy/dx=1$, represents a family of parallel straight lines for different values of arbitrary constant quantities $A=0, 1. 2,...$

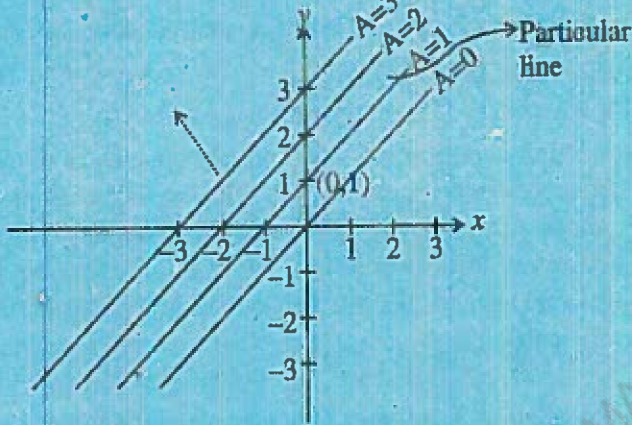


Fig. 9.1

The particular value for the particular line that passes through a point P(0, 1) can be found from the general solution y=x+A by putting x=0, y=1:

$y = x + A \Rightarrow 1 = 0 + A \Rightarrow A = 1$

Use this particular value of A=1 in general solution y=x+A to obtain a particular solution (line) $y = x + 1$.

If we are to determine the solutions of a differential equation subject to conditions on the unknown function and its derivatives specified for one value of the independent variable, the conditions are then called **initial conditions** and the related differential equation is called an **initial value problem** "IVP".

Thus, the problem of example 10.1.3 is the initial value problem that leads the notation:

$$\frac{dy}{dx} = 1, \quad y(0) = 1, \quad IVP \qquad (6)$$

**Example 9.1.4:** Determine a particular solution for the first order differential equation $ds/dt = -32\,ft/\sec$ that satisfies the initial condition s=0, when t=0.

**Solution:** This information develops the initial value problem

$ds/dt = -32, \; s(0) = 0$

for which the solution is the unknown function s(t) that can be found by integrating directly the first order differential equation with respect to t:

$$\frac{ds}{dt}=-32$$

$$\int\frac{ds}{dt}=\int-32dt+A$$

$$s(t)=-32t+A$$

The general solution $s(t)=-32t-A$ at a point P(0, 0) is giving $A=0$. Use this $A=0$ in general solution to obtain the particular solution $s(t)=-32t$.

Another classification of an ordinary differential equation is to determine whether it is linear or non-linear.

**Definition 9.1.6: [Linear Differential Equation]:** A differential equation is said to be **linear**, if the dependent variable and its derivatives occur to the **first power** only and if there are no products involving the dependent variable and/or its derivatives. There should be no non-linear functions of the dependent variable, such as Sine, Exponential, etc. A differential equation which is not linear is said to be non-linear. The linearity of a differential equation is not affected by the presence of non-linear terms involving the independent variable.

The $n^{th}$ order differential equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+.....+a_1(x)\frac{dy}{dx}+a_o(x)y=f(x), \qquad (7)$$

is linear in y (dependent variable) and its all derivatives $dy/dx, d^2y/dx^2, \ldots, d^ny/dx^n$. Here $f(x), a_n(x), \ldots\ldots, a_1(x), a_o(x)$ are functions of $x$ (or real constant quantities) and $a_n(x)$ is not zero.

**Example 9.1.5:[Linear & Non-linear]:** The ordinary differential equations are

$$\frac{dv}{dt}=-32, \qquad \frac{d^2s}{dt^2}=-32, \qquad \frac{dy}{dx}=x+1, \qquad \frac{d^2y}{dx^2}+2x\frac{dy}{dx}+y=3$$

are linear differential equations, while the differential equation

$$\frac{d^2y}{dx^2}+4y\left(\frac{dy}{dx}\right)^2+2y=\cos x$$

is non-linear differential equation, since there is a non-linear term which is the product of y and its derivatives dy/dx whose exponent is 2.

Sometimes the given differential equation, say,

$$y\frac{dy}{dx}=y^2x+xy$$

is not linear, but can be reduced into linear form (7):

$$y\frac{dy}{dx}=y^2x+xy, \quad \text{non-linear}$$

$$\frac{dy}{dx}=\frac{y^2x}{y}+\frac{xy}{y}, \quad \text{divide out by } y$$

$$\frac{dy}{dx}=xy+x$$

$$\frac{dy}{dx}-xy=x, \quad \text{linear}$$

The differential equations

$$\frac{dy}{dx}=\frac{x+y}{x-y}, \quad \left(\frac{d^3y}{dx^3}\right)^2-\frac{d^2y}{dx^2}+x^2\frac{dy}{dx}+y=3$$

are non-linear ordinary differential equations. It is not possible to reduce them into linear form (7).

## 9.2 Formation of Differential Equation

The steps for forming the differential equations are the following:

1. Discover the differential equation that describes a specified physical situation.
2. Find either exactly or approximately, the appropriate solution of that equation.
3. Interpret the solution that is found.

### i) *The concept of formation of a differential equation*

The mathematical model of a real-life problem likes,

1. the rate at which the distance travels by you is 30 mph (have you seen on the right hand side of the roads). Find the total distance travelled by you at a time $t$ hours.

   The rate at which the distance travelled, is modeled by a first order differential equation:

   $$\frac{dS}{dt}=30, \quad \text{the per hour speed}$$

   Here $S(t)$ is the unknown distance travelled by you w.r.t $t$ number of hours, and the rate at which the distance travelled is the first derivative of $S(t)$ with respect to $t$.

Integrating with respect to t to obtain S(t)

$$\int \frac{dS}{dt} = 30$$
$$S(t) = 30t + A$$

the distance traveled by you with respect to t number of hours and the constant quantity A is the fixed distance in this situation.

2. The rate at which the animal population is growing at a constant rate 4%. The habitat will support no more than 10,000 animals. There are 3000 animals present now. Find an equation that gives the animal population y w.r.t x number of years.

The rate at which the animal population grows, is modeled by a first order differential equation:

$$\frac{dP}{dx} \propto (N-P)$$
$$\frac{dP}{dx} = k(N-P), \textit{ k is the constant of proportionality}$$

Here P(x) is the unknown animal population w.r.t x number of years and the rate at which the animal population grows, is the first derivative of P(x) with respect to x:

$$\frac{dP}{dx} = k(N-P)$$
$$= 0.04(10,000-P)$$
$$\frac{dP}{(10,000-P)} = 0.04dx$$

Here k =0.04 is the constant growth, N=10,000, is the total size of the animal population in that habitat.

Integrating with respect to x to obtain P(x)

$$\int \frac{dP}{(10,000-P)} = \int (0.04)dx + A$$

the total population and A, the fixed population that depends on P=3000 when x=0. This problem is the IVP problem with the initial condition P(0)=3000.

## Exercise 9.1

1. Find the order and degree of each the following ordinary differential equations:

a. $\frac{dy}{dx} = x^2 + y$    b. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 11y = 3x$

c. $\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)^3-y=0$

Also indicate the linear and non-linear in the above differential equations.

2. In each case, show that the indicated function is a solution of the differential equation:

a. $y=e^x+e^{2x}$, $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$

b. $y=x-x\ln x$, $x\frac{dy}{dx}+x-y=0$

c. $y=(x+c)e^{-x}$, $\frac{dy}{dx}+y=e^{-x}$

3. For each of the following equations, determine whether or not it becomes linear when divided by dx or dy:

a. $(x+y)dy=(x-y)dx$ b. $ady+by\sin xdx=0$

c. $3ydx+2xdy=0$ d. $e^xdy+xy^{\frac{1}{3}}dx=0$

4. In each case, find the particular solution, when the initial condition and the general solution of the differential equation are the following:

a. $xy=c$, $y(2)=1$ b. $y=x-x\ln x+c$, $y(1)=2$

c. $\sin(xy)+y=c$, $y(\pi/4)=1$ d. $\frac{y^2}{x}=\frac{x^2}{2}+c$, $y(1)=1$

5. Solve the following initial value problems:

a. $\frac{dy}{dx}=\cos x$, $y(0)=1$ b. $\frac{dy}{dx}=x^2$, $y(0)=1$

c. $\frac{dy}{dx}=2xy^2$, $y(3)=-1$ d. $\frac{dy}{dx}+y=y^3$, $y(0)=1/2$

e. $y\frac{dy}{dx}+xy^2-x=0$, $y(0)=-1$ f. $2\frac{dy}{dx}=4xe^{-x}$, $y(0)=42$

## 9.3 Solving differential equations

If the solution of a first order differential equation is not possible by direct integration, then, the integral process (in case of difficulties) for obtaining the solution of a differential equation indicates the actual concept of a differential equation.

### *i) Solutions of first order and first degree differential equations*

We examine techniques for solving first order differential equations. For this unit, the recommended techniques for solving the differential equations are the separation of variables, reducible to separation form, homogeneous and equations reducible to homogeneous form.

- ***Separable differential equation***

If the solution of a differential equation is not possible by direct integration, then the integral technique called **separation of variables** will be used for solving the differential equation. Separation of variables is a technique commonly used to solve first order differential equations. It is so called because we try to rearrange the equation to be solved in such a way that all terms involving the dependent variable (y say) appear on one side of the equation, and all terms involving the independent variable (x, say) appear on the other side. It is not possible to rearrange all first order differential equations in this way so this technique is not always appropriate. Further, it is not always possible to perform the integration even if the variables are separable.

In general, a differential equation of the form

$$\frac{dy}{dx}=\frac{f(x)}{g(y)},\ g(y)\neq 0 \tag{8}$$

that by shifting $x$ on one side and $y$ on the other side

$$g(y)dy=f(x)dx \tag{9}$$

is giving a **separable differential equation.** The solution to separable differential equation (9) can be found by integrating left hand side w.r.t y and right hand side w.r.t $x$.

**Example 9.3.1:[General Solution]:** Find the general solution of the linear differential equation $\dfrac{dy}{dx}=y$.

**Solution:** The solution of the given linear differential equation is not possible by direct integration. The separable form of the given first order differential equation is obtained by shifting y on the left and x on the right:

$$\frac{1}{y}dy = dx$$

that by integration w.r.t x

$$\int \frac{1}{y}dy = \int dx$$

$$\ln y = x + c$$

$$y = e^{x+c}$$

$$= e^x e^c = c_1 e^x, \quad c_1 = e^c \qquad (10)$$

is giving the general solution of the first order differential equation. This general solution represents a family of exponential functions as shown in the figure (9.2).

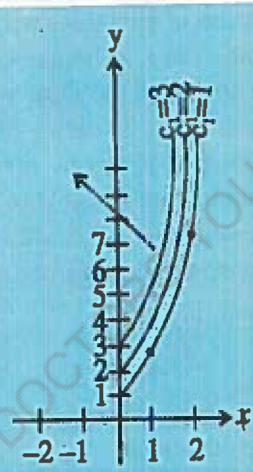


Fig. 9.2

- ***Differential equations reducible to separable form***

If the solution of the differential equation is not possible by separable form, then the given differential equation can be reduced in separable form by substitution. This substitution changes the dependent variable form $y$ to a new variable, say, u and keeps $x$ as the independent variable.

**Example 9.3.2:[Reducible to Separable Form]:** Find the general solution of the non-linear differential equation $\frac{dy}{dx} = (x+y)^2$.

**Solution:** The given non-linear differential equation is not separable differential equation, but can be reduced into separable form by substitution,

$$x+y=u(x)$$

that on differentiation w.r.t x is giving:

$$\frac{d}{dx}(x+y)=\frac{du}{dx}\Rightarrow 1+\frac{dy}{dx}=\frac{du}{dx}\Rightarrow\frac{dy}{dx}=\frac{du}{dx}-1$$

Use x+y=u and dy/dx=(du/dx)-1 in the given differential equation to obtain separable differential equation in variable u and its derivative du/dx:

$$\frac{du}{dx}-1=u^2$$

$$\frac{du}{dx}=1+u^2$$

$$\frac{du}{1+u^2}=dx \qquad (11)$$

Integrating equation (11) to obtain the general solution of ordinary differential equation (11)

$$\int\frac{du}{1+u^2}=\int dx$$

$$\tan^{-1}u=x+c$$

$$u=\tan(x+c)$$

that by back substitution of $u=x+y$ is giving

$$x+y=\tan(x+c)$$

$$y=-x+\tan(x+c)$$

the general solution of the given ordinary differential equation that depends on a single arbitrary constant c.

If the differential equation is not reducible to separable form by substitution, then the differential equation might be a homogeneous differential equation. This can be solved by the procedure developed is as under:

- ***Homogeneous differential equation***

The homogeneous differential equations are related to homogeneous functions that are already discussed in detail in Unit-11.

**Definition 9.3.1:[Homogeneous Function]:** A function f(x,y) is homogeneous function of degree n in variables x and y if and only if for all values of the variables x, and y and for every positive value of $\lambda$, the identity is true:

$$f(\lambda x,\lambda y)=\lambda^n f(x,y),\quad n=1,2,3,... \qquad (12)$$

For illustration, the function $f(x,y)=x^2+y^2$ is homogeneous function of degree 2, since the identity (12) is true:

$$f(\lambda x,\lambda y)=(\lambda x)^2+(\lambda y)^2=\lambda^2(x^2+y^2)=\lambda^2 f(x,y),\ x=\lambda x, y=\lambda y$$

The identity (12) is not true for a function $f(x,y)=x^2+y^2+1$, since the function is not homogeneous.

**Definition 9.3.2:[Homogeneous Differential Equation]:** The differential equation

$$\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)} \qquad (13)$$

is called a homogenous differential equation, if it defines a homogenous function of degree zero.

The homogeneous differential equation (13) can be reduced to separable form by introducing a new variable:

$$u(x)=\frac{y}{x}\ or\ y=ux \quad \text{and} \quad \frac{dy}{dx}=\frac{d}{dx}(ux)=u+x\frac{du}{dx} \qquad (14)$$

The substitution of (14) in equation (13) automatically converts the homogeneous differential equation in separable differential equation.

**Example 9.3.3:[Homogeneous Differential Equation]:** Find the general solution of the homogeneous differential equation:

$$\frac{dy}{dx}=\frac{y-x}{y+x}$$

**Solution:** The given differential equation defines a homogeneous function of degree zero, when the function on the right of the given differential equation defines a homogeneous function of degree zero:

$$\frac{dy}{dx}=\frac{y-x}{y+x}$$
$$=\frac{\lambda y-\lambda x}{\lambda y+\lambda x}$$

$$=\frac{\lambda(y-x)}{\lambda(y+x)}$$
$$=\frac{\lambda}{\lambda}\left[\frac{y-x}{y+x}\right]$$
$$=\lambda^{1-1}\left[\frac{y-x}{y+x}\right]=\lambda^{0}\left[\frac{y-x}{y+x}\right]=\left[\frac{y-x}{y+x}\right],\ HDE$$

The given homogeneous differential equation is used for the assumptions

$y=ux,\ \frac{dy}{dx}=u+x\frac{du}{dx}$ to obtain a separable differential equation of the form:

$$u+x\frac{du}{dx}=\frac{ux-x}{ux+x}=\frac{u-1}{u+1}$$
$$x\frac{du}{dx}=\frac{u-1}{u+1}-u$$
$$=\frac{u-1-u^2-u}{u+1}=\frac{-(u^2+1)}{u+1}$$
$$-\frac{(u+1)}{u^2+1}du=\frac{dx}{x},\ SDE \qquad (15)$$

Integrating SDE (15) to obtain the general solution of the SDE (15):

$$-\int\frac{(u+1)du}{u^2+1}=\int\frac{dx}{x}$$
$$-\int\frac{2(u+1)du}{2(u^2+1)}=\int\frac{dx}{x},\quad \textit{Multiply and divide out by } 2$$
$$-\frac{1}{2}\int\frac{2udu}{u^2+1}-\int\frac{du}{u^2+1}=\ln x+c,$$
$$-\frac{1}{2}\ln\left(u^2+1\right)-\tan^{-1}u=\ln x+\ln c$$
$$-\ln\sqrt{u^2+1}-\tan^{-1}u=\ln cx$$
$$-\tan^{-1}u=\ln\sqrt{u^2+1}+\ln cx$$
$$=\ln cx\sqrt{u^2+1}$$
$$\tan^{-1}u=-\ln cx\sqrt{u^2+1} \qquad (16)$$

The back substitution $u = y/x$ is used in equation (16) to obtain the general solution of the given homogeneous differential equation:

$$\tan^{-1}\frac{y}{x} = -\ln cx\sqrt{\frac{y^2}{x^2}+1}$$

$$\tan^{-1}\frac{y}{x} = -\ln c\sqrt{y^2+x^2} \Rightarrow \frac{y}{x} = \tan\left[-\ln c\sqrt{y^2+x^2}\right]$$

- ***Differential equations reducible to homogeneous differential equations***

**Example 9.3.4:[Reducible to Homogeneous]:** Find the general solution of the differential equation $x\frac{dy}{dx} = x+y$.

**Solution:** The given differential equation is not in the standard form of homogeneous differential equation, but it can be reduced in the standard form of homogenous differential equation by the following procedure:

Divide out by x to obtain the standard form of homogeneous differential equation:

$$\frac{dy}{dx} = \frac{x+y}{x}, \; HDE \qquad (17)$$

Homogeneous differential equation (17) is used for the assumptions $y = ux$, $dy/dx = u + xdu/dx$ to obtain a separable differential equation of the form:

$$u + x\frac{du}{dx} = \frac{x+xu}{x}$$

$$= 1+u$$

$$x\frac{du}{dx} = 1 \Rightarrow du = \frac{dx}{x}, \; SDE \qquad (18)$$

Integrating the SDE (18) to obtain the general solution of the SDE (18):

$$\int du = \int \frac{dx}{x}$$

$$u = \ln x + \ln c = \ln cx$$

that by back substitution $u = \frac{y}{x}$ is giving

$$\frac{y}{x} = \ln cx$$

$$y = x \ln cx$$

the general solution of the given homogeneous differential equation that depends on a single arbitrary constant c.

If the differential equation is not homogeneous differential equation, then it might be a nonhomogeneous differential equation. The nonhomogeneous differential equation is beyond of this curriculum.

## *ii) Real-life problems related to differential equations*

**Natural Growth and Decay:** The differential equation

$$\frac{dy}{dt} \alpha\, y \Rightarrow \frac{dy}{dt} = k\,y, \quad k \text{ is constant of proportionality} \qquad (19)$$

is a mathematical model, serves as remarkable wide range of natural phenomena. Any quantity involving whose time rate of change is proportional to its current value. For example, in problems to determine rates of decay, heating and cooling, compound interest, evaporation, mixture, rumor and others. A positive value of k indicates growth, while a negative value of k indicates decay.

In problems, such as space restrictions (a limited amount of food or maximum size of population) tend to inhibit growth of populations as time goes on, the rate of growth of population is proportional to how close the population is to that maximum size N:

$$\frac{dy}{dt} = k(N - y) \qquad (20)$$

The constant k is called the growth rate constant, while N represents the maximum size of the population.

**Example 9.3.5:[Real-Life Problem; IVP]:** A certain bacteria grows at a rate that is proportional to the number present at a particular time. If the number of bacterial at a time t=0 is $N_o$ and at time $t=1$ hour, the number of bacteria is $5N_o/2$. Determine the time necessary for the number of bacteria to be quadruple.

**Solution:** If N(t) is the unknown number of bacteria w.r.t time t hours, then, the rate at which bacterial grows, is represented by:

$$\frac{dN}{dt}\alpha N \Rightarrow \frac{dN}{dt} = kN \qquad (21)$$

Reduce the differential equation to separable form

$$\frac{dN}{N} = k\,dt$$

that on integration is giving the general solution of (21):

$$\int \frac{dN}{N} = \int k\,dt$$

$$\ln N = kt + c \qquad (22)$$

The initial condition $N(0) = N_o$ is used in equation (22) to obtain c:

$$\ln N_o = 0 + c \Rightarrow c = \ln N_o \qquad (23)$$

Use c in equation (22) to obtain a particular solution:

$$\ln N = kt + \ln N_o \Rightarrow \ln\frac{N}{N_o} = kt \Rightarrow \frac{N}{N_o} = e^{kt} \Rightarrow N = N_o e^{kt} \qquad (24)$$

The condition $N(1) = 5\,N_o/2$ is used in equation (24) to obtain the value of k:

$$5\,N_o/2 = N_o e^{k} \Rightarrow e^{k} = 5/2 \Rightarrow k = \ln(5/2) = 0.9163 \qquad (25)$$

Use the value of k in equation (24) to obtain a particular solution (specific number of bacteria):

$$N = N_o e^{0.9163t} \qquad (26)$$

The condition $N = 4N_o$ (when the bacterial have quadrupled) is used in equation (26) to obtain the time

$$4N_o = N_o e^{0.9163t}$$

$$4 = e^{0.9163t} \Rightarrow 0.9163t = \ln 4 \Rightarrow t = \ln 4/0.9163 = 1.51\ hr$$

at which the bacteria is four times of the original number of bacteria.

**Example 9.3.6:[Maximum Production]:** The rate at which a new worker in a certain factory produces items, is given by

$$\frac{dy}{dx} = k(N-y) = 0.2(125-y), \quad k = 0.2,\ N = 125$$

Where $y$ is the number of items produced by the worker per day, $x$ is the number of days worked and the maximum production per day is 125 items. Assume that the worker produced 20 items the first day on the job x = 0. How many items will be produced by the worker on the job of x = 10 (days)?

**Solution:** If y(x) is the unknown number of items produced by the worker w.r.t x number of days, then the rate at which the items produced, was approximated by:

$$\frac{dy}{dx}=k(N-y)=0.2(125-y),\quad k=0.2,\ N=125$$

Reduce the given differential equation to separable form

$$\frac{dy}{(125-y)}=0.2\,dx$$

that on integration is giving a general solution (general items production):

$$\int\frac{dy}{125-y}=\int 0.2\,dx$$

$$\frac{\ln(125-y)}{-1}=0.2x+c$$

$$\ln(125-y)=-0.2x-c$$

$$(125-y)=e^{-0.2x-c}$$

$$=e^{-0.2x}e^{-c}$$

$$=c_1e^{-0.2x},\qquad c_1=e^{-c}$$

$$y=125-c_1^{-0.2x}\qquad(27)$$

The initial condition y = 20 for x = 0 is used in equation (27) to obtain the particular value of constant c:

$$y=125-c_1e^{-0.2x}$$

$$20=125-c.e^{0}\ \Rightarrow\ c_1=125-20=105$$

$c_1=105$ is used in equation (27) to obtain a particular solution y(x) (the specific number of items produced by the worker in *x* number of days):

$$y(x)=125-105e^{-0.2x}\qquad(28)$$

The items produced by the worker on the job of *x* = 10 days is obtained by putting *x* = 10 in equation (28):

$$y(10)=125-105e^{-2}=110.825\ \text{items}$$

## 9.4 Orthogonal Trajectories

Our experience with first order differential equations has taught us that such equations often have general solutions containing a single arbitrary constant. Each such solution defines a corresponding set of integral curves. A nonempty set of plane curves defined by a differential equation involving just one parameter (single arbitrary constant) is commonly called a one-parameter family of curves. Of special importance in certain applications are those one-parameter families of curves which are orthogonal trajectories of one another.

**Definition 9.4.1:[Orthogonal Trajectories]:** The curves of a family $F(x, y, c_1)$ are said to be orthogonal trajectories of curves of a family $G(x, y, c_2)$, if and only if each curve of either family is intersected by at least one curve of the other family and at every point of intersection of a curve of F with a curve of G, the two curves are perpendicular.

### i) *Orthogonal trajectories of the given family of curves*

The two families of curves $F(x, y, c_1)$ and $G(x, y, c_2)$ are perpendicular at a point of intersection, if and only if their tangents are perpendicular at the point of intersection. If their tangent lines, say, $L_1$ and $L_2$, are perpendicular, then the product of their slopes equals -1:

$m_1 m_2 = -1,$ *$m_1$ and $m_2$ are the slopes of the two tangent lines $L_1$ and $L_2$*

$$m_1 = -\frac{1}{m_2} \Rightarrow \left(\frac{dy}{dx}\right)_G = -\frac{1}{\left(\frac{dy}{dx}\right)_F} \qquad (29)$$

This is called the **differential equation of orthogonal trajectories**. If one family of curves F is given, then the other family of curves G can be found by solving the differential equation of orthogonal trajectories (29).

**Example 9.4.1:[Orthogonal Trajectories]:** Determine the orthogonal trajectories of the family of curves (circles) $x^2 + y^2 = c$.

**Solution:** To determine the orthogonal trajectories of the circles, we need to determine the slope (derivative) of the family of circles

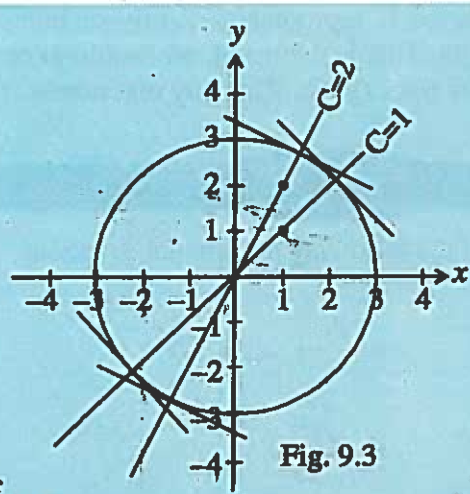

Fig. 9.3

$x^2+y^2=c$

with respect to x

$$2x+2y\frac{dy}{dx}=0$$

$$\left(\frac{dy}{dx}\right)_F=-\frac{x}{y} \qquad (30)$$

The differential equation of the orthogonal trajectories (29) with the slope of the given orthogonal trajectories (30) is used to obtain the other family of curves G of orthogonal trajectories:

$$\left(\frac{dy}{dx}\right)_G=\frac{-1}{\left(\frac{dy}{dx}\right)_F}$$

$$\frac{dy}{dx}=\frac{-1}{-\frac{x}{y}}=\frac{y}{x}$$

$$\frac{dy}{y}=\frac{dx}{x},\ SDE$$

$$\int\frac{dy}{y}=\int\frac{dx}{x}$$

$$\ln y=\ln x+\ln C'$$

$$y=Cx \qquad (31)$$

Thus, the family of curves G represents a family of homogeneous straight lines that pass through the origin. This is the result, we would expect, since the radii of a circle are the homogeneous lines (y=Cx, C is any real number) perpendicular to the lines tangent to a circle.

## Exercise 9.2

1. Find general solution of the following differential equations:

a. $\dfrac{2x\dfrac{dy}{dx}-2y}{x^2}=0$  b. $\dfrac{dy}{x}+ydx=2dx$

c. $\left(\dfrac{dy}{dx}\right)^2=1-y^2$  d. $e^x\dfrac{dy}{dx}+y^2=0$

e. $\sqrt{1-x^2}\,dy=\sqrt{1-y^2}\,dx$  f. $\operatorname{cosec}^2 x dy+\sec y dx=0$

2. Reduce the following differential equations in separable form and then solve:

a. $y'=(y+x)^2$  b. $y'=\tan(x+y)-1$

c. $y'=(x+e^y-1)e^{-y}$  d. $y\dfrac{dy}{dx}+xy^2-x=0$

e. $\dfrac{dy}{dx}=\dfrac{y}{x}+\dfrac{x^2}{y^2}$  f. $(y-3)dy=(x^3+1)dx$

3. Solve the following homogeneous differential equations:

a. $\dfrac{dy}{dx}=\dfrac{x+y}{x-y}$  b. $\dfrac{dy}{dx}=\dfrac{xy-y^2}{x^2}$  c. $\dfrac{dy}{dx}=\dfrac{x^2+3y^2}{2xy}$

d. $\dfrac{dy}{dx}=\dfrac{\sqrt{x^2-y^2}+y}{x}$  e. $\dfrac{dy}{dx}=\dfrac{xy+y^2}{x^2+xy+y^2}$

4. Reduce the following differential equations in the standard form of homogeneous form and then solve:

a. $x\dfrac{dy}{dx}=y+\sqrt{x^2+y^2},\ y(4)=3$  b. $(x^4+y^4)dx=2x^3ydy,\ y(1)=0$

5. The slope of a family of curves at a point P(x,y) is $\dfrac{y-1}{1-x}$. Determine the equation of the curve that passes through the point P(4, –3).

6. Find the solution curve of the differential equation $xyy' = 3y^2 + x^2$ which passes through the point P(-1, 2).

7. Determine the particular solution $y = f(t)$ of the homogeneous differential equation $t^2 y' = y^2 + 2ty$ with initial condition $y(1) = 2$.

8. A particle moves along the x-axis so that its velocity at any point is equal to half its abscissa minus three times the time. At a time $t = 2$, $x = -4$, determine the motion of a particle along the x-axis.

9. The rate of consumption of oil (billions of barrels) is given by $\frac{dx}{dt} = 1.2e^{0.04t}$. Where $t = 0$ correspond to 1990. Find the total amount of oil used from 1990 to year 1995. At this rate, how much oil will be used in $8(t = 8)$ years?

10. The rate of infection of a disease (in people per month) is given by:
$$\frac{dI}{dt} = \frac{100t}{t^2 + 1}$$
Where $t$ is the time in months since the disease broke out. Find the total number of infected people over the first four months of the disease.

11. The rate of reaction to a drug is given by
$$\frac{dR}{dx} = 2x^2 e^{-x}$$
Where $x$ is the number of hours since the drug was administered. Find the total reaction to the drug form $x = 1$ to $x = 6$.

12. The rate of increase of the number of cellular phone subscribers (in millions) since services began, was given by:
$$\frac{ds}{dx} = 0.38x + 0.04$$
Where x is the number of years since 1998, when the services started. There were 0.25 million subscribers in year 1998 (x=0). Find a function that gives the number of subscribers for the year 2004.

13. Determine the equations of the orthogonal trajectories of the following families of curves:

| | | |
|---|---|---|
| a. $y = cx^3$ | b. $xy = c$ | c. $y = cxe^x$ |
| d. $y^2 = x^2 + c$ | e. $y = c\sin 2x$ | f. $e^x \cos y = c$ |
| g. $y = \sqrt{x + c}$ | h. $y = x^2 + c$ | i. $e^x \sin y = c$ |
| j. $\cos x \cosh y = c$ | k. $e^x(x\cos y - y\sin y) = c$ | |

# Glossary

- **Differential Equation:** A differential equation is an equation that involves the derivatives of an unknown function (dependent variable) of one or more variables (independent variables).
- **Order of a Differential Equation:** The order of a differential equation is the order of the highest-order derivative which appears in the equation.
- **Degree of a Differential Equation:** The degree of a differential equation is the power of the highest-order derivative which appears in the equation.
- **Solution of a Differential Equation:** A solution of an ordinary differential in one dependent variable y on an interval I is a function $y(x)$ which, when substituted for the dependent variable y and its derivatives $y'$, $y''$, ..., reduces the differential equation to an identity in the independent variable x over interval I.
- **Linear Differential Equation:** A differential equation is linear in a set of one or more of its dependent variables if and only if each term of the equation which contains a variable of the set or any of their derivatives is of the first degree in those variables and their derivatives.
- **Homogeneous Function:** A function f(x,y) is homogeneous function of degree n in variables x and y if and only if for all values of the variables x, and y and for every positive value of $\lambda$, the identity is true:

  $$f(\lambda x, \lambda y) = \lambda^n f(x, y), \quad n = 1, 2, 3, ...$$
- **Homogeneous Differential Equation:** The differential equation

  $$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

  is called a homogenous differential equation, if it defines a homogenous function of degree zero.