ECSE 443 – ASSIGNMENT 2

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Chapter 3

Exercises

3.3

The least square is given below:

$$\boldsymbol{Ax} = \begin{bmatrix} 1 & e \\ 2 & e^2 \\ 3 & e^3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \boldsymbol{b}$$

3.18

a) The elementary elimination matrix that annihilates the third component of vector **a** is given below:

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

b) A householder transformation that annihilates the third component of **a** is as follows:

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.6 & -0.8 \\ 0 & -0.8 & 0.6 \end{bmatrix}$$

c) Matrix **G** below is Givens rotation that annihilates the third component of **a**:

$$\boldsymbol{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0 & -0.8 & 0.6 \end{bmatrix}$$

d) It is impossible to have annihilation give the same Householder and eliminatory matrix, as Householder transformation is always both symmetric and orthogonal but elementary elimination matrix can never be symmetric or orthogonal except in the case of the identity matrix.

e) During the annihilation of a non-zero component of a matrix, its impossible to have the same Householder transformation and a Givens rotation, because Householder transformation is always both symmetric and orthogonal but a Givens rotation is never symmetric except for identity matrix.

3.24

The add and multiply expressions of the QR factorization are as follows:

$$\sum_{k=1}^{n}[2(m-k)+\sum_{j=k}^{n}2(m-k)]$$

$$=\sum_{k=1}^{n}[2(m-k)+2(m-k)(n-k)]$$

$$=\sum_{k=1}^{n}[2mn+2m-2(m+n+1)k\ +2k^2]$$

$$=2(mn+m)n-(m+n+1)n(n+1)+\frac{n(n+1)(2n+1)}{3}$$

Keeping the highest ordered terms gives us the multiply and add operations, which is $\mathrm{n}^2\mathrm{m}-\frac{\mathrm{n}^3}{3}$.

3.25

a) $(QQ^T)^T = QQ^T\ and\ (QQ^T)^2 = Q(Q^TQ)Q^T = QQ^T$

The above equalities prove that QQ^T is symmetric and idempotent. For a vector v with columns Q, the basis for the subspace S: $(QQ^T)v = Q(Q^Tv)$ lies in S. Hence, this proves that QQ^T is an orthogonal projector onto the subspace S.

b) $(A(A^TA)^{-1}A^T)^T = A(A^TA)^{-T}A^T = A(A^TA)^{-1}A^T$

$$(A(A^TA)^{-1}A^T)^2 = A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T$$

The above inequalities show that $A(A^TA)^{-1}A^T$ is symmetric and idempotent. For an arbitrary vector v, $(A(A^TA)^{-1}A^T)v = A((A^TA)^{-1}A^Tv)$ lies in the column space of A. Hence, $A(A^TA)^{-1}A^T$ is an orthogonal projector onto the column space of A.

In linear least squares equation, Ax = b, solution of the normal equation is given by x = $(A^TA)^{-1}A^Tb$, and Ax is the projection of b onto column space of A.

c) If P is an orthogonal projector onto a subspace S, then $P^2 = P = P^T$.

$(I-P)^T = (I^T-P^T) = I-P$, showing that $I-P$ is symmetric.

$(I-P)^2 = I-2P+P^2 = (I-P)$, showing that $I-P$ is idempotent.

For any vector v, $(I-P)v = v-Pv$, lies in the orthogonal complement of S as Pv is removed. Hence, this shows that $I-P$ is an orthogonal projector onto the orthogonal complement of S.

d) For any non-zero vector v, let $P = (vv^T)/(v^Tv)$. Which proves that P is symmetric. We can also show that $P^2 = (vv^T)(vv^T)/(v^Tv)^2 = (v(v^Tv)v^T)/(v^Tv)^2 = (vv^T)/(v^Tv)$. Which proves that P is idempotent.

Also, for any vector u, $Pu = ((vv^T)/(v^Tv))u \ = \ v(v^Tu)/(v^Tv)$, which is a scalar multiple. Therefore, P is the projector onto the subspace spanned by v.

e) The modified Gram-Schmidt algorithm can be proven via induction. If the claim holds for k = 1, then for k-1 we have:

$$
\begin{aligned}
&(I - P_k)(I - P_{k-1}) \ldots . (I - P_1) \\
&= \ (I - P_k)(I - P_{k-1} - \cdots - P_1) \\
&= (I - P_{k-1} - \cdots - P_1) - P_k(I - P_{k-1} - \cdots - P_1) \\
&= I - P_k - P_{k-1} - \ldots - P_1
\end{aligned}
$$

For i≠j, $P_iP_j = (q_iq_i^T)(q_jq_j^T) = O$ because of the orthogonality of q_i and q_j. Hence the claim holds for all k.

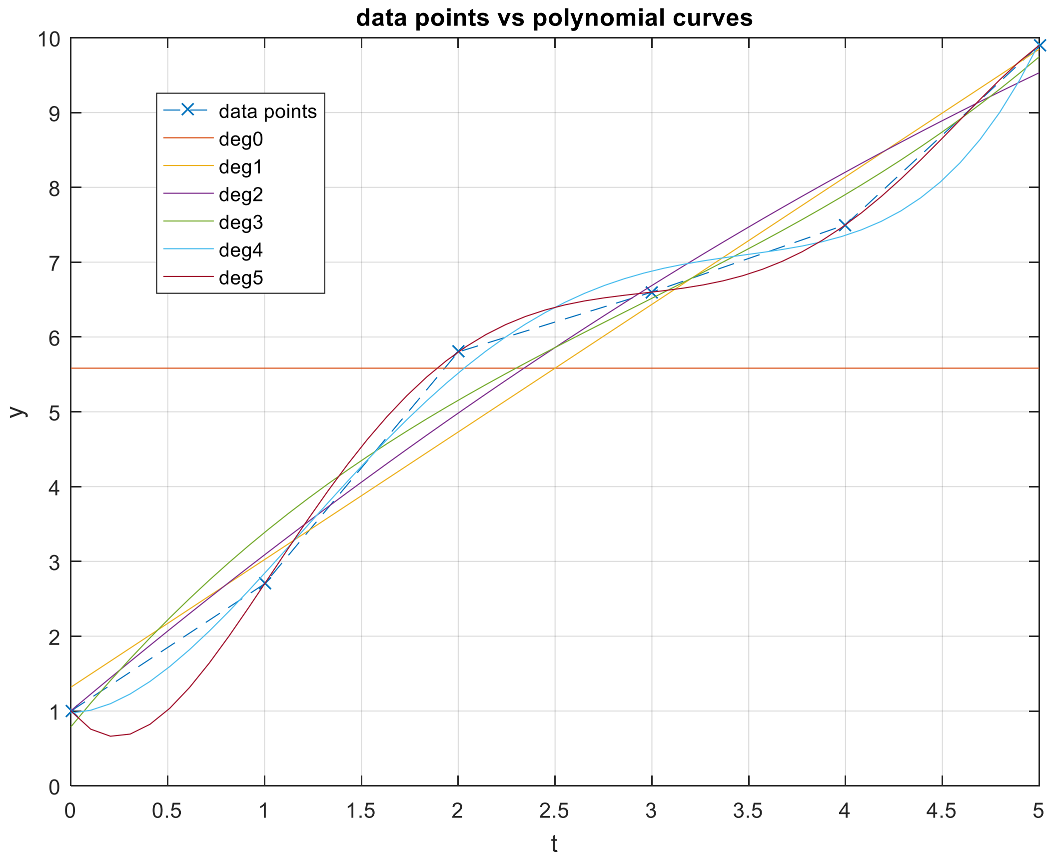
The classical Gram-Schmidt algorithm is equivalent to:

$q_k = (I - (P_1 + \cdots + P_{k-1}))a_k = a_k - P_1a_k - \cdots - P_{k-1}a_k = a_k - \ q_1q_1^Ta_k - \cdots - q_{k-1}q_{k-1}^Ta_k$.

f) The classical and the modified Gram-Schmidt algorithms (shown in (e)) are equivalent to the classical Gram-Schmidt equation, whereas the alternative algorithm (iterative refinement) is mathematically equivalent to modified Gram-Schmidt algorithm, as the projectors are idempotent.

Computer Problems

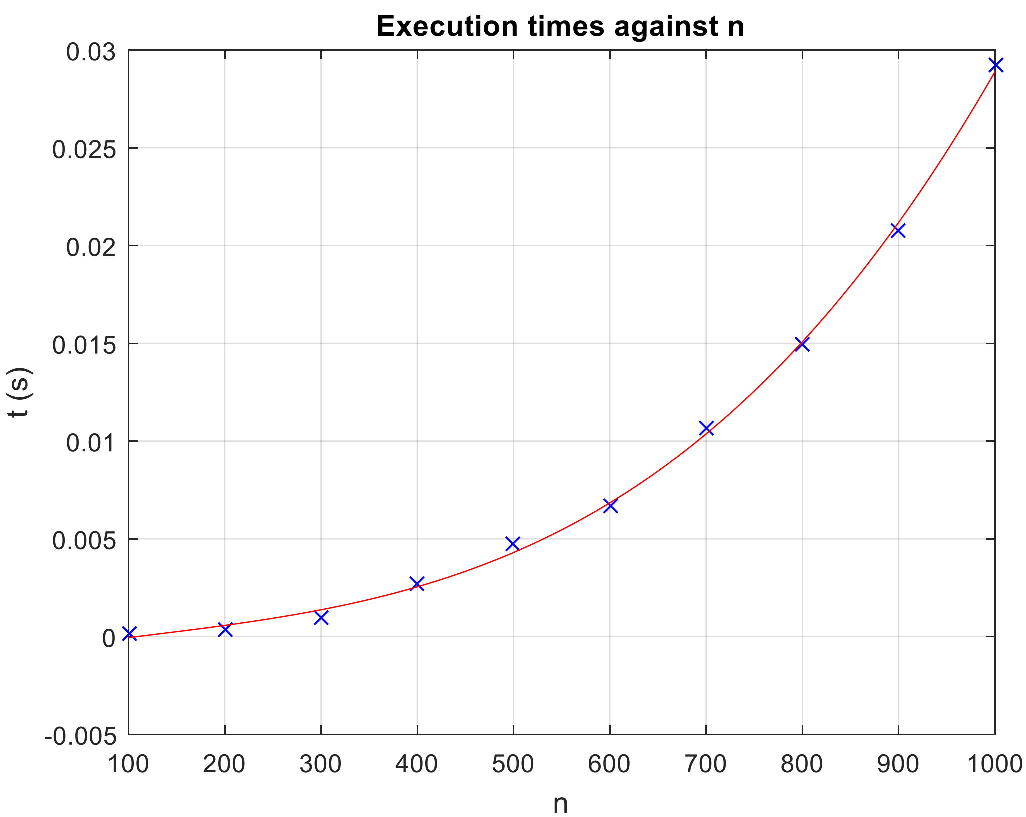
3.1



The plot above shows the data points as well as the polynomial plots of different degrees. By inspecting the figure, we can conclude that a polynomial of degree 5 gives the best approximation for the data points, as the distance between degree 5 polynomial points and data points is the least.

3.3

The graph below shows the execution time of LU factorization against varying values of n for the given range. The same MATLAB script with the polynomial function was used to estimate the execution time for a matrix order of 10,000, which was found to be 32.774 seconds.



3.4

```
x1 =

    1.0000
    1.0000


x2 =

    7.0089
   -8.3957


del1 =

    7.8862


condA =

   1.0975e+03
```

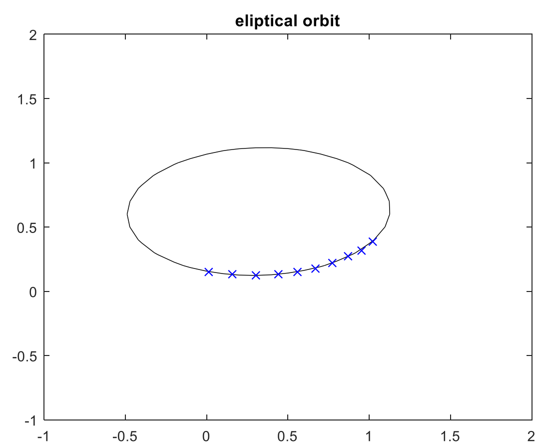
The snapshot above gives the resulting answer from MATLAB when the given equations are solved. X1 is the answer for part (a) and x2 for part (b). For comparison purposes, The del norm was computed according to the following formula:

$$delNorm = \frac{norm(x2-x1)}{norm(x1)}$$

3.5

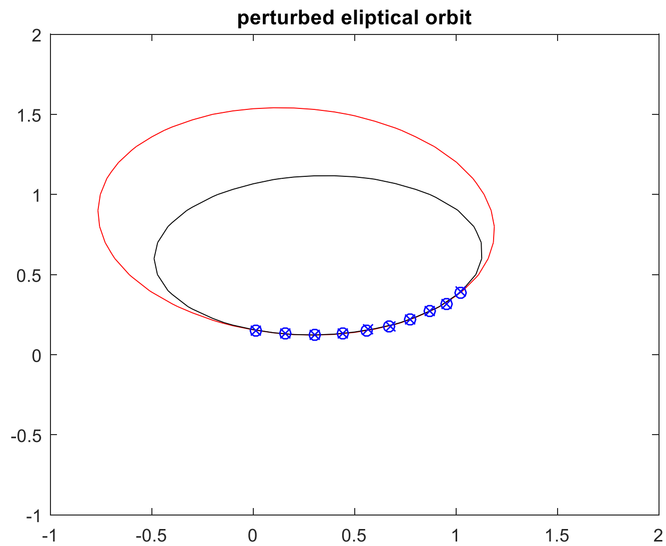
a) The computed values of the orbital parameters are as follows:
a = -2.6356, b = 0.1436, c = 0.5514, d = 3.2229 and e = -0.4329.

The plot of the resulting orbit is given below:

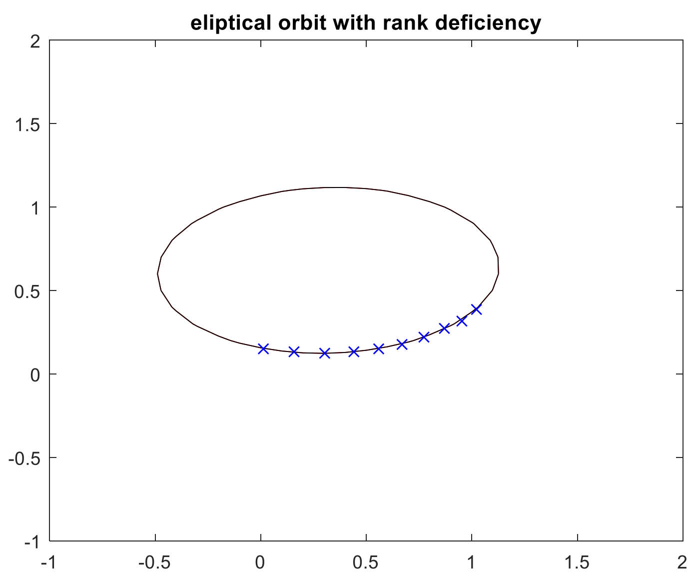


b) The new values for the orbital parameters with the perturbed data is as follows:
a = -1.8915, b = -0.2311, c = 0.6147, d = 3.1971 and e = -0.4452.

The effects of these new values on the orbit can be seen in the figure below which compares it with the previous orbit:



c) The figure obtained after solving the problem with rank deficiency is given below. As can be seen, the rank deficiency solver deals with any perturbations on the input data:



3.10

Chapter 4

Exercises

4.3

a) The characteristic polynomial of A can be calculated as follows:

$$\det(A - \lambda I) = \lambda^2 - 2\lambda - 3$$

b) The roots of the characteristic polynomial of A are found by factorizing the characteristic polynomial. The roots are: $\lambda_1 = 3$, $\lambda_2 = -1$

c) The eigenvalues of A are: $\lambda_1 = 3$, $\lambda_2 = -1$

d) The Eigenvector corresponding to $\lambda_1 = 3$ is $\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$

The Eigenvector corresponding to $\lambda_1 = -1$ is $\begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$

e) With a single iteration of power method on A, we get the following results:
$y_1 = Ax_0 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $x_1 = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}$

f) The power method will ultimately converge A to a multiple of the eigenvector corresponding the dominant eigenvalue, which is $\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$.

g) Using the vector $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the Rayleigh quotient gives an eigenvalue estimate = 3.5

h) Inverse iteration will ultimately converge to a multiple of the eigenvector corresponding to the eigenvector $\begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$

i) If the inverse iteration were used with shift σ = 2, the eigenvalue would be 3, since 2 is closer to 3 than -1.

j) If QR iteration were applied to A, it would converge to a triangular matrix since A is not symmetric.

4.9

a) Let the eigenvalue of *A* be λ with the corresponding eigenvector as x.

Hence, $(x^T Ax)^T = x^T A^T x = x^T A^T$ is real. Therefore, λ = $x^T A^T / x^T x$ is also real.

b) If A is real and symmetric, the $A^H = A$, according to part (a) above. Therefore it can be shown that all its eigenvalues are real.

4.10

Let the eigenvalue of *A* be λ with the corresponding eigenvector as x.

We know that, $x^H Ax = x^H(\lambda x) = \lambda x^H x$
Hence, λ $= x^H Ax / x^H x$

As, $x^H x$ is positive and $x^H Ax$ > 0 since A is positive definite. Therefore, λ > 0. This shows that the eigenvalues of a positive definite matrix A are all positive as well.

4.15

a) Let the eigenvalue of *A* be λ with the corresponding eigenvector as x.

$$Ax = \lambda x$$

$$A^{-1}x = \left(\frac{1}{\lambda}\right)x$$

Proving that the eigenvalue of A^{-1} are the reciprocals of the eigenvalues of *A.*

b) Part (a) also proves that the eigenvectors of A^{-1} are the same as the eigenvectors of *A.*

4.19

a) $\lambda y^T x = y^T(\lambda x) = y^T(Ax) = y^T(A^T x) = (Ay)^T x = \gamma y^T x$

This shows that (λ-γ)$y^T x = 0$

Hence, $y^T x = 0$, as λ≠ γ

b) $\lambda y^T x$ = $y^T(\lambda x) = y^T Ax = \gamma y^T x$

This shows that (λ-γ)$y^T x = 0$

Hence, $y^T x = 0$, as λ≠ γ

4.25

a) Let the eigenvalue of *A* be λ with the corresponding eigenvector as x.

$$Qx = \lambda x$$

$$||Qx||^2 = ||\lambda x||^2$$

The properties of an orthogonal matrix reserve the 2-norm, so $||x||^2 = |\lambda|.||x||^2$ implying that $|\lambda| = 1$

b) The positive square roots of the eigenvalues of $Q^T Q$ are the singular values of Q. Since, Q is orthogonal, $Q^T Q = I$ meaning that all singular values of Q are equal.

Computer Problems

4.2

```
lambda1 =

    2.1331


lambda2 =

    0.5789
    2.1331
    7.2880
```

The snapshot above is taken from MATLAB and shows the result for both parts of the question. Lambda1 is the resulting answer for part (a), whereas lambda2 shows all the eigenvalues for part (b). The inverse iteration found the correct eigenvalue nearest to 2.

4.5

Snapshot below shows the result from the implemented function, when the matrix to test are passed in the function.

```
A =

    9.0000    4.5000    3.0000
  -56.0000  -28.0000  -18.0000
   60.0000   30.0000   19.0000

>> QRiteration(A)

ans =

   -1.0000
    0.0000
    1.0000
```

4.7

```
Companion matrix
c =

     0     0     0   -24
     1     0     0    40
     0     1     0   -35
     0     0     1    13


lambda =

   0.6839 + 0.9410i
   0.6839 - 0.9410i
   1.8048 + 0.0000i
   9.8274 + 0.0000i


Proots =

   9.8274 + 0.0000i
   1.8048 + 0.0000i
   0.6839 + 0.9410i
   0.6839 - 0.9410i
```

The snapshot above shows the result from the function implemented with the given polynomial. The results show the companion matrix, the eigen values of companion matrix and the roots of the polynomial.

4.9

The curve below shows how the ratio $\frac{\sigma_{max}}{\sigma_{min}}$ changes with increasing order of matrix. It can be concluded that the ratio increases exponentially for matrix order greater than or equal to 15.