RANDOM VARIABLE (r.v)

A r.v is a function that associates a real no. with each element in the sample space

Discrete random variable: possible outcomes countable

Continous random variable: values on a continous scale

Q) Let X= no. of Heads in toss of 3 coins

X		$2^3=8$	P(x)	F(x)
0	← TTT		$\frac{1}{8}$	$\frac{1}{8}$
1	← HTT, THT, TTH		$\frac{3}{8}$	$\frac{4}{8}$
2	← HHT, THH, HTH		$\frac{3}{8}$	$\frac{7}{8}$
3	← HHH		$\frac{1}{8}$	$\frac{8}{8}$

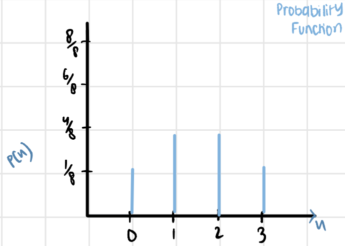
Cumulative Distribution function (CDF)

$F(x) = P(X \leq n)$

$$F(x) = \begin{cases} 0 & n < 0 & P(n<0) \\ \frac{1}{8} & 0 \leq n < \frac{1}{8} & P(n=0) \\ \frac{4}{8} & \frac{1}{8} \leq n < \frac{4}{8} & P(n=0)+P(n=1) \\ \frac{7}{8} & \frac{4}{8} \leq n < \frac{7}{8} & P(n=0)+P(n=1)+P(n=2) \\ 1 & \frac{7}{8} \leq n < 1 & P(n=0)+P(n=1)+P(n=2)+P(n=3) \end{cases}$$

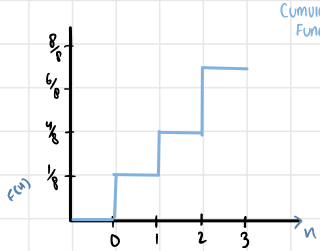
Line Graph → PMF → uses P(n)

Probability Mass Function



Line Graph → CDF → uses F(n)

Cumulative Distribution Function



Q) Let X= no. of Heads in toss of 4 coins

X		$2^4=16$	P(x)	F(x)
0	← TTTT		$\frac{1}{16}$	$\frac{1}{16}$
1	← HTTT, THTT, TTHT, TTTH		$\frac{4}{16}$	$\frac{5}{16}$
2	← HHTT, TTHH, HTHT, THTH		$\frac{4}{16}$	$\frac{9}{16}$
3	← THHH, HTHH, HHTH, HHHT		$\frac{4}{16}$	$\frac{13}{16}$
4	← HHHH		$\frac{1}{16}$	$\frac{14}{16}$

Q) find $P(n=2) = f(n=2)$ using CDF

$f(2) = F(2) - F(1)$

$= P(n \leq 2) - P(n \leq 1)$

$= 11/16 - 5/16$

$= 3/8$

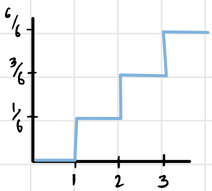
Q) PMF $f(n) = n/6$, $n = 1,2,3$

1. find CDF and sketch its graph
2. $P[1.5 < n \leq 4.5]$

n	f(n)	F(n)
1	$1/6$	$1/6$
2	$2/6$	$3/6$
3	$3/6$	$6/6$

$$F(n) = \begin{cases} 0 & n < 1 \\ 1/6 & 1 \leq n \leq 2 \\ 3/6 & 2 \leq n \leq 3 \\ 6/6 & n \geq 3 \end{cases}$$

CDF



Q1) A fair coin is tossed until H appears for the first time. Find

a) PMF: $(\frac{1}{2})^n$: $n = 1,2,3,\ldots$

b) CDF: $\sum_{n=1}^{n=n} (\frac{1}{2})^n$

c) $F(4) = \sum_{n=1}^{n=4} (\frac{1}{2})^n$

Q2) CDF: $F(n) = 1 - (\frac{1}{2})^{n+1}$ for $n = 0,1,2\ldots$

a) $P(3) = F(1) - F(2)$

b) $P[7 \leq n < 10] = F(9) - F(6)$

c) PMF: $F(n) - F(n-1)$

$= 1 - (\frac{1}{2})^{n+1} - [1 - (\frac{1}{2})^{n-1+1}]$

$= (-\frac{1}{2})^{n+1} + (\frac{1}{2})^n$

$= (\frac{1}{2})^n (\frac{1-1}{2})$

$= \frac{1}{2^{n+1}}$

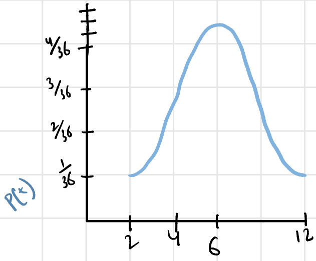
Q) if two dice are rolled once, find the PMF of the sum of points on dice, CDF and their graph

let $t = x + y$

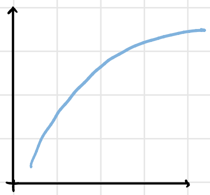
z	2	3	4	5	6	7	8	9	.....	12
$f(t) = P(t)$:	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$7/36$	$8/36$	.....	
$F(t)$:	$1/36$	$3/36$	$6/36$	.....						$36/36$

$$F(t) = \begin{cases} 0 & z < 2 \\ 1/36 & 2 \leq z \leq 3 \\ 3/36 & 3 \leq z \leq 4 \\ \vdots & \\ 36/36 & z \geq 12 \end{cases}$$

PMF



CDF



Q3) A coin is biased so that head across 3 times of tails.
If the coin is tossed 3 times, find the prob dist. for the no of heads $P[1 \leq n \leq 3]$

H	T
$\frac{3}{4}$	$\frac{1}{4}$

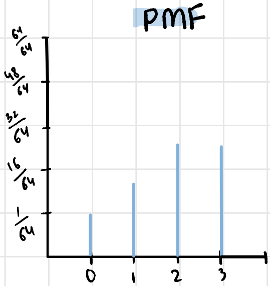
$0 \{$ T T T $\quad \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times 1$

$1 \{$ H T T $\quad \frac{3}{4} \times \frac{1}{4} \times \frac{1}{4} \times 3$
T H T
T T H

$2 \{$ H H T $\quad \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} \times 3$
H T H
T H H

$3 \{$ H H H $\quad \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times 1$

X	P(n)
0	$\frac{1}{64}$
1	$\frac{9}{64}$
2	$\frac{27}{64}$
3	$\frac{27}{64}$



4) Determine the value of c so that the f(n) becomes PMF
$f(n) = c(n^2+4)$ for $n = 0,1,2,3$

$$\sum_{n=0}^{3} c(n^2+4) = 1$$

$4c + 5c + 8c + 13c = 1$

$30c = 1$

$c = \frac{1}{30}$

* Expected Value

let suppose a coin tossed 2 times
X = Head calculate

E(X) = Mass of a row X

X	P(x)	X.P(x)	x^2.P(x)
0	$\frac{1}{4}$	0	0
1	$\frac{2}{4}$	$\frac{2}{4}$	$\frac{2}{4}$
2	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{4}{4}$
Sum	1	E(x) = 1	$E(x^2) = \frac{3}{2}$

$E(x) = \sum (x) P(x) \longrightarrow$ Expected value

$\sigma^2 x = V(x) = E(x^2) - [E(x)]^2 \longrightarrow$ Variance

$3B, 2R, 3G \Rightarrow 8$ → 2 w/o replacement

$X = B$ & $Y = R$

JOINT PROB MASS FUNCTION

x \ y	0	1	2	Total (Marginal dist g(x))
0	$\frac{{}^3C_0\,{}^2C_0\,{}^3C_2}{{}^8C_2}$	$\frac{{}^3C_0\,{}^2C_1\,{}^3C_1}{{}^8C_2}$	$\frac{{}^3C_0\,{}^2C_2\,{}^3C_0}{{}^8C_2}$	$\frac{10}{28}$
1	$\frac{{}^3C_1\,{}^2C_0\,{}^3C_1}{{}^8C_2}$	$\frac{{}^3C_1\,{}^2C_1\,{}^3C_0}{{}^8C_2}$	—	$\frac{15}{28}$
2	$\frac{{}^3C_2\,{}^2C_0\,{}^3C_0}{{}^8C_2}$	—	—	$\frac{3}{28}$
h(y) Total marginal dist of y	$\frac{13}{28}$	$\frac{12}{28}$	$\frac{1}{28}$	1

Conditional Distribution of Y

$$f(y|x) = \frac{f(x,y)}{g(x)} \quad \therefore g(x) > 0$$

Conditional Distribution of X

$$f(x|y) = \frac{f(x,y)}{h(y)} \quad \therefore h(y) > 0$$

MATHAMETICAL EXPECTATION

Mean $= E(x) = \Sigma x f(x)$

$E[\ ] = \Sigma[\ ] f(x)$

x	0	1	2	
$g(x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$	1
$x\,g(x)$	0	$\frac{15}{28}$	$\frac{6}{28}$	$\frac{21}{28} = E(x)$
$x^2 g(x)$	0	$\frac{15}{28}$	$\frac{12}{28}$	$\frac{27}{28} = E(x^2)$

	0	1	2	
$h(y)$	$\frac{13}{28}$	$\frac{12}{28}$	$\frac{1}{28}$	
$y\,h(y)$	0	$\frac{12}{28}$	$\frac{2}{28}$	$\frac{14}{28} = E(y)$
$y^2 h(y)$	0	$\frac{12}{28}$	$\frac{4}{28}$	$\frac{16}{28} = E(y^2)$

$E(x^4)$

$E(\sqrt{x})$

$E[3x] = 3E(x)$

$$\sigma^2 x = V(x) = E(x^2) - [E(x)]^2 \longrightarrow \text{Variance}$$

$$= \frac{27}{28} - \left(\frac{21}{28}\right)^2$$

a) $P[X \leq 2, Y = 1] = f(0,1) + f(1,1) + f(2,1)$

$= 6/28 + 6/28 + 3/28$

$= 12/28$

b) $P[X \geq 2, Y \leq 1] = f(2,1) + f(2,0)$

c) $P[X > Y] = f(1,0) + f(2,0)$

d) $P[X + Y = 4] = 0$

Conditional Probability

e) $P[X = 0 / Y = 1] = \frac{f(x=0, y=1)}{h(y=1)}$

$$= \frac{6/28}{12/28} = \frac{6}{12}$$

f) $P[Y = 1 / X = 0] = \frac{f(y=1, x=0)}{g(x=0)}$

$$= \frac{6/28}{10/28}$$

g) $E[XY] = \Sigma\Sigma xy f(x,y) = 6/28$

h) Covariance $(x,y) = E(XY) - E(x).E(Y)$

$= 6/28 - (21/28)(14/28)$

$= -9/56$

$$-1 \leq \text{Corelation}(x,y) = \frac{\text{Covariance}(x,y)}{\sqrt{V(x)}\sqrt{V(Y)}} \leq +1$$

$$= \frac{-9/56}{\sqrt{45/112}\,\sqrt{9/28}}$$

$= -0.441$ moderate -ve relationship

$\pm 1 \longrightarrow$ strong
$\pm 0.5 \longrightarrow$ moderate
$0 \longrightarrow$ negligible

JPMF

$f(x,y) = \frac{x+y}{30}$ for $x = 0,1,2,3$

$y = 0,1,2$

Calculated

a) $P[X \leq 2, Y=1] = 6/30$

b) $P[X > 2, Y \leq 1] = 7/30$

c) $P[X > Y] = 18/30$

d) $P[X+Y=4] = 8/30$

e) Calculate

$x \backslash y$	0	1	2	$g(x)$ (marginal Dist X)	$x\,g(x)$	$x^2 g(x)$
0	0	1/30	2/30	3/30	0	0
1	1/30	2/30	3/30	6/30	6/30	6/30
2	2/30	3/30	4/30	9/30	18/30	36/30
3	3/30	4/30	5/30	12/30	36/30	108/30
$h(y)$	6/10	10/30	14/30	1	$E(x)=$ 2	$E(x^2)=$ 5
$y\,h(y)$	0	10/30	28/30	$E(y)$: 38/30		
$y^2 h(y)$	0	10/30	56/30	$E(y^2)$: 66/30		

$E(x) = 60/30 = 2$

$E(x^2) = \frac{150}{30} = 5$

$V(x) = E(x^2) - [E(x)]^2 = 5 - (2)^2 = 1$

$V(Y) = E(Y^2) - [E(Y)]^2 = 66/30 - (38/30)^2 = 0.59$

$E(xy) = \sum\sum xy\, f(x,y) = 2.4$

$\text{Covariance}(x,y) = E(xy) - E(x).E(y) = 2.4 - 2(38/30) \rightarrow -2/15 \rightarrow -0.133$

$\text{Corelation}(x,y) = \frac{\text{Covariance}(x,y)}{\sqrt{V(x)}\,\sqrt{V(Y)}} = -0.17277$

↳ weak relation

↳ inverse relation as -ve sign

$E(xy) = \sum\sum xy\, f(x,y) = 2.4$

00	0
01	$1/30 \times 0$
02	$2/30 \times 0$
10	$1/30 \times 0$
11	$2/30 \times 1$
12	$3/30 \times 2$
20	$2/30 \times 0$
21	$3/30 \times 2$
22	$4/30 \times 4$
30	$3/30 \times 0$
31	$4/30 \times 3$
32	$5/30 \times 6$

Statistical independence

$f(x,y) = g(x)\,h(y)$

Joint Probability Distribution

1. $f(x,y) \geq 0$
2. $\sum_x \sum_y f(x,y) = 1$
3. $P(X=x, Y=y) = f(x,y)$

↳ $P[(X,Y) \in A] = \sum \sum_A f(x,y)$

Marginal Distribution

$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ ↳ for continous variable

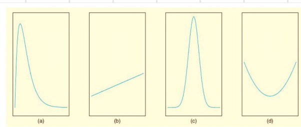
$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

Continous Probability Distribution

↳ $F(x)$ → probability density function (PDF)

↳ Areas used to represent probabilities

↳ can not be given in tabular form



• A probability density function is constructed so that the area under its curve bounded by the *x* axis is equal to 1 when computed over the range of *X*.

Probability Density Function (PDF)

1. $f(x) \geq 0$, for all $x \in R$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $P(a < X < b) = \int_a^b f(x)\,dx$

Joint Density Function (JDF)

↳ for continous variable

1. $f(x,y) \geq 0$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$
3. $P[(X,Y) \in A] = \int_A \int f(x,y)\,dx\,dy$

Cumulative Distribution Function (CDF)

↳ $F(x) = P(X \leq x)$ ↳ for continous RV

$= \int_{-\infty}^{\infty} f(t)\,dt$

↳ $P(a < x < b) = F(b) - F(a)$

$f(x) = \dfrac{dF(x)}{dx}$

PDF

$f(n)$ = measurement of error

Q1) $f(n) = \begin{cases} K(3-n^2) & ; -1 < n < 1 \\ 0 & ; \text{elsewhere} \end{cases}$

a) Determine k that renders $f(n)$ a valid PDF

$\int_{-1}^{1} K(3-n^2)\,dn = 1$ → equate to 1 as valid PDF equates to 1

$K(3n - n^3/3) = 1$

$K\left[3 - \frac{1}{3} - \left(3(-1) - \frac{(-1)^3}{3}\right)\right] = 1$

$K = \frac{3}{16}$

b) $P(n < 1/2) = \int_{-1}^{1/2} \frac{3}{16}(3-n^2)\,dn \Rightarrow \frac{99}{128}$

c) $P[\,|n| < 0.8\,] = P[n < -0.8] + P[n \geq 0.8]$

$= F(-0.8) + [1 - F(0.8)]$

$= 0.164$

$-0.8 < n < 0.8$

Q2) $f(n) = \begin{cases} 3n^{-4} & ; n > 1 \\ 0 & ; \text{elsewhere} \end{cases}$

a) Verify that $f(n)$ is a PDF

$= \int_1^{\infty} 3n^{-4}$

$= \left[-\frac{\cancel{3}}{\cancel{3}} n^{-3}\right]_1^{\infty}$

$= \left[-n^{-3}\right]_1^{\infty}$

$= \left(-\frac{1}{\infty^3}\right) - \left(-\frac{1}{1^3}\right)$

$= 0 - (-1) = 1$

b) $F(n) = CDF$

use n as upper limit

$= \int_1^{n} \frac{3}{n^4}\,dn$

$= \left[-n^{-3}\right]_1^{n}$

$= -n^{-3} + 1$

$= 1 - n^{-3}$

c) $P(n > 4)$

JPDF

1. $f(x,y) \geq 0$ for all (x,y)
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) = 1$
3. $P(X=x, Y=y) = f(x,y)$

For any region A in the xy plane, $P[(X,Y) \in A] = \iint_A f(x,y)$

Q3) $f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0<x<1 \\ & 0<y<1 \\ & \text{elsewhere} \end{cases}$

a) Verify that $f(x,y)$ is a valid JPDF

$\int_0^1\int_0^1 \frac{2}{5}(2x+3y)\,dx\,dy$

$\frac{2}{5}\int_0^1 (x^2+3xy)\Big|_0^1 dy$

$\frac{2}{5}\int_0^1 (1+3y)\,dy$

$\frac{2}{5}\left[y+\frac{3y^2}{2}\right]_0^1$

$\frac{2}{5}\left[1+\frac{3}{2}\right]$

$\frac{2}{5}\left[\frac{5}{2}\right]$

1

b) $P[0<x<\frac{1}{2}, \frac{1}{4}<y<\frac{1}{2}]$

$= \frac{2}{5}\int_{\frac{1}{4}}^{\frac{1}{2}}\int_0^{\frac{1}{2}} (2x+3y)\,dx\,dy$

$= \frac{2}{5}\int_{\frac{1}{4}}^{\frac{1}{2}} (x^2+3xy)\Big|_0^{\frac{1}{2}} dy$

$= \frac{2}{5}\int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{4}+\frac{3y}{2}\right) dy$

$= \frac{2}{5}\left(\frac{y}{4}+\frac{3y^2}{4}\right)\Big|_{\frac{1}{4}}^{\frac{1}{2}}$

$= \frac{2}{5}\left[\frac{1}{8}+\frac{3}{16}-\left(\frac{1}{16}+\frac{3}{64}\right)\right]$

$= 0.08$

c) find $g(x)$ & $h(y)$

Marginal $g(x) = \frac{2}{5}\int_0^1 (2x+3y)\,dy$

$= \frac{2}{5}\left(2xy+\frac{3y^2}{2}\right)\Big|_0^1$

$= \frac{2}{5}\left(2x+\frac{3}{2}\right)$

Marginal $f(y) = \frac{2}{5}\int_0^1 (2x+3y)\,dx$