Differential Equation:-

- Eulers Method

$$w_{i+1} = w_i + h f(t_i, w_i)$$

- Mid Point method

$$w_{i+1} = w_i + h f\left[t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right]$$

Ex 5.2

Q1) Use Euler's method to approximate the Sol for each of the following initial value Problem.

a) $y' = te^{3t} - 2y$

$0 < t \leq 1$

$y(0) = 0$

$h = 0.5$

Sol)

$F(t, w) = y' = te^{3t} - 2y$

$w = y$

$y(0) = 0$

$y_0 = w_0 = 0$

$t_0 = 0$

1st iteration $i=0$

By Euler's method

$w_1 = w_0 + hf(t_0, w_0)$

$= 0 + (0.5)f(0,0)$

$= (0.5)(0 \times e^{3\times 0} - 2\times 0)$

$= 0$

$\boxed{w_1 = 0 \quad , t_1 = 0.5}$

2nd iteration :-

$w_2 = w_1 + hf(t_1, w_1)$

$w_2 = 0 + (0.5)f(0.5, 0)$

$w_2 = 0 + 0.5(0.5 \times e^{0.5\times 3} - 2(0))$

$\boxed{w_2 = 1.12 \quad t_2 = 1}$

$y' = \cos 2t + \sin 3t$

$0 \leq t \leq 1$

$y(0) = 1$

$h = 0.25$

1st iteration

$W_1 = w_0 + h f(t_0, w_0)$

$W_1 = 1 + 0.25 \left(\cos 2(0) + \sin 3(0) \right)$

$\boxed{W_1 = 1.25}$

2nd iteration

$W_2 = 1.25 + 0.25 \left(\cos 2(0.25) + \sin(3 \times 0.25) \right)$

$\boxed{w_2 = 1.64}$

Ex # 5.4

Differential Equation

Modified Euler's Method

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right]$$

$i = 0, 1, \ldots$

$t_i = a + ih$

Heun's Method :-

$$w_{i+1} = w_i + \frac{h}{4}\left[f(t_i, w_i) + 3\left(f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\right)\right)\right)\right]$$

$t_i = a + ih$

Ex#5.4

Q1) Use the Modified Euler's Method and Heun's method to approximate the sol Initial value problem and Compare the results actual so.

(a) $y' = te^{3t} - 2y \quad 0 \le t \le 1 \quad , y(0) = 0$

$h = 0.5$

actual sol

$$y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$$

Sol)

$$w_1 = w_0 + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))]$$

$$w_1 = 0 + \frac{0.5}{2}[f0 + f(1.5, 0 + 0.5(0))]$$

$$w_1 = \frac{0.5}{2}[0.f(1.5, 0)]$$

$$w_i = 0.56 \qquad t = 0.5$$

$$w_2 = 0.56 + \frac{0.5}{2}(f(0.5, 0.56) + f(1.5, 0.56 + 0.5f(0.5, 0.56)$$

$$w_2 = 0.56 + \frac{0.5}{2}(1.12 + f(1.5, 0.56 + 0.6272)$$

i	t_i	w_0	$y(t)$
0	0	0	0
1	0.5	0.560	0.283
2	1	5.3012	3.219

Runge-kutta Method order four

$k_1 = hf(t_i, w_i)$

$k_2 = hf(t_i + h/2, w_i + 1/2 k_1)$

$k_3 = hf(t_i + h/2, w_i + 1/2 k_2)$

$k_4 = hf(t_{i+1}, w_i + k_3)$

$w_{i+1} = w_i + 1/6 (k_1 + 2k_2 + 2k_3 + k_4)$

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2) $k_1 = 0.5f(0,0) = 0$

$k_2 = 0.5f(0.5/2, 0)$

$k_2 = 0.264$

$k_3 = 0.5f(0.25, 0.132)$

$k_3 = 0.132$

$k_4 = 0.5f(0.5, 0+0.132)$

$k_4 = \cancel{1.97}\ 0.985$

$w_1 = 0 + 1/6\left(0 + 2(0.264) + 2(0.132) + 0.985 \cancel{1.97}\right)$

$w_1 = 0.296$

Sol Linear System Eq

LU decomposition Method

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0$$

$$Ax = b \quad -(1)$$
$$A = LU \quad -(2)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix} \qquad Ly = b \quad -(3)$$

$$y = Ux \quad -(4)$$

Q1)

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}}$$

$$Ax = b \quad -(1)$$

$x = 1018$

$y = -203\cancel{7}946$

$z = -0.0763$

(7) $\Rightarrow$ $A = LU$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} * \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}U_{12} & l_{11}U_{13} \\ l_{21} & l_{21}U_{12}+l_{22} & l_{21}U_{13}+l_{22}U_{23} \\ l_{31} & l_{31}U_{12}+l_{32} & l_{31}U_{13}+l_{32}U_{23}+l_{33} \end{bmatrix}$$

1st Row

$\boxed{l_{11} = 1}$

$l_{11} \cdot U_{12} = 2$ $\quad \boxed{U_{12} = 2}$

$l_{11} \cdot U_{13} = 3$ $\quad \boxed{U_{13} = 3}$

2nd Row

$\boxed{l_{21} = 2}$

$l_{21}U_{12} + l_{22} = -4$ $\quad \boxed{l_{22} = -8}$

$l_{21}U_{13} + l_{22}U_{23} = 6$ $\quad \boxed{U_{23} = 0}$

3rd Row

$l_{31} = 3$

$l_{32} = -15$

$l_{33} = -12$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) => $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ 2y_1 - 8y_2 \\ 3y_1 - 15y_2 - 12y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}$$

$\boxed{y_1 = 5}$ $\boxed{y_2 = -1}$ $\cancel{y_3 = 1/2}$

$\boxed{y_3 = 2}$

(4) => $Y = Ux$

$$\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 = 1$
$x_2 = -1$
$x_3 = 2$

Q1) Solve the following Linear System

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}}_{B}$$

(3)

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}}_{B}$$

$$\underbrace{\begin{bmatrix} 2y_1 \\ -y_1 + y_2 \\ 3y_1 + 2y_2 - y_3 \end{bmatrix}}_{L} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

$\boxed{y_1 = -1/2}$

$\boxed{y_2 = 5/2}$

$\boxed{y_3 = 7/2}$

(4) $Y = Ux$

$$\begin{bmatrix} -1/2 \\ 5/2 \\ 7/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} -1/2 \\ 5/2 \\ 7/2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_2 + 2x_3 \\ x_3 \end{bmatrix}$$

$\boxed{x_3 = 7/2}$ $\boxed{x_2 = -9/2}$

$\boxed{x_1 = 1/2}$

(Q2) Consider $A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

<u>determine Choleskey LDL^t factorization</u>

$A = LDL^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} d_1 & 0 & 0 \\ l_{21} d_1 & d_2 & 0 \\ l_{31} d_1 & l_{32} d_2 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} d_1 & d_1 l_{21} & d_1 l_{31} \\ l_{21} d_1 & l_{21} l_{21} d_2 & l_{21} l_{31} d_1 \end{bmatrix}$$

$$A = \begin{bmatrix} d_1 & d_1 l_{21} & d_1 l_{31} \\ l_{21} d_1 & l_{21}^2 d_1 + d_2 & l_{21} d_1 l_{31} + l_{31} d_2 l_{32} \\ l_{31} d_1 & l_{31} d_1 l_{21} + l_{32} d_2 & \cancel{l_{31}^2 d_1 + \ldots} \\ & & L_{31}^2 d_1 + l_{32}^2 d_2 + d_3 \end{bmatrix}$$

$\boxed{d_1 = 4}$ $\boxed{L_{21} = -1/4}$ $\boxed{L_{31} = 1/4}$

$\boxed{d_2 = 11/4}$ $\boxed{d_3 = 1.727}$

find the unknows

Jacobi Method
- Gauss Siedel Method

Solve by Jacobi Method.

Q) $10x_1 - x_2 + 2x_3 = 6$ — (1)

$-x_1 + 11x_2 - x_3 + 3x_4 = 25$ — (11)

$2x_1 - x_2 + 10x_3 - x_4 = -11$ — (111)

$3x_2 - x_3 + 8x_4 = 15$ — (4)

initially $x_1 = x_2 = x_3 = x_4 = 0$

find x_1 from eq1
x_2 from eq2 and so on

first iteration,

$$\boxed{x_1 = \frac{6 + x_2 - 2x_3}{10}}$$ $x_1 = 0.6$

$$\boxed{x_2 = \frac{25 + x_1 + x_3 - 3x_4}{11}}$$ $x_2 = 2.2727$

$$\boxed{x_3 = \frac{-11 + x_4 + x_2 - 2x_1}{10}}$$ $x_3 = -1.1$

$$\boxed{x_4 = \frac{15 + x_3 - 3x_2}{8}}$$ $x_4 = 1.875$

2nd iteration

$x_1 = 1.04727$
$x_2 = 1.7159$
$x_3 = -0.80523$
$x_4 = 0.885$

3rd iteration

$x_1 = 0.932634$
$x_2 = 2.0527$
$x_3 = -1.0479$
$x_4 = 1.130885$

Q2) Solve $Ax=b$ for the following system of Linear eq.

$$\begin{bmatrix} 1 & 1 & 5 \\ -3 & -6 & 2 \\ 10 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -21.5 \\ -61.5 \\ 27 \end{bmatrix}$$

initial value $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Perform three iteration from of Gauss Siedel Method.

$x_1 + x_2 + 5x_3 = -21.5$ — (i)

$-3x_1 - 6x_2 + 2x_3 = -61.5$ — (ii)

$10x_1 + 2x_2 - x_3 = 27$ — (iii)

~~$x_1 = -21.5 - x_2 - 5x_3$ (i)~~

~~$x_2 = \frac{-61.5 - 2x_3 + 3x_1}{-6}$ (ii)~~

~~$x_1 = -21.5$~~

find x_n with big coefficient and find its ~~value~~ equation.

1st iteration

$x_3 = \frac{-21.5 - x_1 - x_2}{5}$; $x_3 = -6.62$

$x_2 = \frac{-61.5 - 2x_3 + 3x_1}{-6}$; $x_2 = 8.9$

$x_1 = \frac{27 + x_3 - 2x_2}{10}$; $x_1 = 2.7$

2nd iteration | 3rd iteration

$x_1 = 0.258$ | $x_1 = 0.525$

$x_2 = 7.91$ | $x_2 = 7.79$

$x_3 = -5.93$ | $x_3 = -5.963$

Eigenvalues & Eigen Vector by Power Method.

Q1) $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$, $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

find eigen value & eigen vector by power method.

three iterations.

$X_1 = Ax_0$ min value as eigen in 2x2

$X_2 = Ax_1$ and max value

$x_3 = Ax_2$ as eigen in 3x3

$x_n = Ax_{n-1}$

<u>for n = 1</u>

$x_1 = Ax_0$

$$\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} -10 \\ -4 \end{bmatrix} = -4 \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$$

eigen value = -4

eigen vector = $\begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$

$x_2 = A x_1$

$$x_2 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} -10 \\ -4 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 28 \\ 10 \end{bmatrix} = 10 \begin{bmatrix} 2.8 \\ 1 \end{bmatrix}$$

eigen value $= 10$

eigen vector $= \begin{bmatrix} 2.8 \\ 1 \end{bmatrix}$

$x_3 = A x_2$

$$x_3 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 28 \\ 10 \end{bmatrix} = \begin{bmatrix} -64 \\ -22 \end{bmatrix}$$

$$x_3 = -22 \begin{bmatrix} 2.9 \\ 1 \end{bmatrix}$$

Q2) $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$ $\quad x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$X_1 = A x_0$

$X_1 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$X_1 = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 3/5 \\ 1/5 \\ 1 \end{bmatrix}$

$X_2 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$

$X_2 = \begin{bmatrix} 5 \\ 5 \\ 11 \end{bmatrix} = 11 \begin{bmatrix} 5/11 \\ 5/11 \\ 1 \end{bmatrix}$

$X_3 = \begin{bmatrix} 15 \\ 17 \\ 31 \end{bmatrix}$

Determining eigen value from an eigen vector:-

If x is an eigen vector of a matrix A, then it's corresponding eigen value is given by

$$\lambda = \frac{Ax \cdot x}{x \cdot x}$$

This quotient is called the rayleigh quotient.

Q1) $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ $x = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Use the rayleigh quotient to complete the eigen value λ of corresponding to the given eigen vector

$$\lambda = \frac{Ax \cdot x}{x \cdot x} = \frac{\begin{bmatrix} 10 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix}}{\begin{bmatrix} 5 \\ 2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix}} = \frac{58}{29} = 2$$

$$\boxed{\lambda = 2}$$

Q2) $A = \begin{bmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{bmatrix}$. $x = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$$\lambda = \frac{\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}} = \frac{18+0+2}{9+1} = 2$$

$$\boxed{\lambda = 2}$$

Q3) $\cancel{3/\lambda/ = 0}$ $\cancel{(3-\lambda)+(-4-\lambda)+(5-\lambda)=0}$

$\cancel{-4 - \lambda =}$ $\cancel{4 - 3\lambda = 0}$

$$\boxed{\lambda = \frac{3}{4}}$$

Q3) $\begin{bmatrix} 3-\lambda & 2 & -3 \\ -3 & -4-\lambda & 9 \\ -1 & -2 & 5-\lambda \end{bmatrix}$

$(3-\lambda)\begin{vmatrix} -4-\lambda & 9 \\ -2 & 5-\lambda \end{vmatrix} - 2 \begin{vmatrix} -3 & 9 \\ -1 & 5-\lambda \end{vmatrix} - 3 \begin{vmatrix} -3 & -4-\lambda \\ -1 & -2 \end{vmatrix}$

$(3-\lambda)\left((-4-\lambda)(5-\lambda) - (-18)\right) - 2\left(-3(5-\lambda)+9\right)$

$- 3\left(6 - (4+\lambda)\right)$

$$(3-\lambda)(-20+4\lambda-5\lambda+\lambda^2)+18)$$

$$-2(-15+3\lambda+9) \quad -3(2-\lambda)$$

$$(3-\lambda)(-2-\lambda+\lambda^2)+30-6\lambda-18-6+3\lambda$$

$$-6-3\lambda+3\lambda^2+2\lambda+\lambda^2-\lambda^3+3\cancel{0}-\cancel{24}-3\lambda=0$$

$$-\lambda^3+4\lambda^2-4\lambda=0$$

$$\lambda^3-4\lambda^2+4\lambda=0$$

$$\boxed{\lambda=0} \qquad \boxed{\lambda=2,} \; 2$$

$$\lambda^2-4\lambda+4=0$$

$$\lambda^2-2\lambda-2\lambda+4=0$$

$$\lambda(\lambda-2)-2(\lambda-2)$$

Optimization.

Golden Section Search Method

Golden Ratio : $d = 0.618$
$C = 1 - d = 0.382$

$x_1 = a + c(b-a)$
$x_2 = a + d(b-a)$

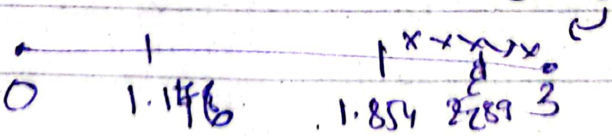
Q1) Find the minimum of $f(x) = \dfrac{x^4+1}{x^2+1}$ within the interval $[\overset{a}{0} \quad \overset{b}{3}]$, tolerance $= 0.5$

if $f(x_1) < f(x_2)$

1st iteration :-

$x_1 = 1.146 \qquad f(x_1) = 1.178$
$x_2 = 1.854 \qquad f(x_2) = 2.89$

Since $f(x_1) < f(x_2)$



$a = 0$
$b = 1.854$

if $f(x_1) < f(x_2)$ $b = x_2$

$f(x_1) > f(x_2)$ $a = x_1$

2nd iteration:-

$x_1 = 0.708$ $f(x_1) = 0.8335$

$x_2 = 1.146$ $f(x_2) =$

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