Lecture # 43

ADDITIONAL CONVERGENCE TESTS

In this section we shall develop some additional convergence tests for series with positive terms.

THEOREM 11.5.1
(The Comparison Test)

Let $\sum a_k$ and $\sum b_k$ be series with nonnegative terms and suppose that

$a_1 \le b_1, a_2 \le b_2, a_3 \le b_3, \ldots\ldots\ldots\ldots, a_k \le b_k$

(a) If the "bigger series" $\sum b_k$ converges than the "smaller series" $\sum a_k$ also converges.

(b) If the "smaller series" $\sum a_k$ diverges, then the "bigger series" $\sum b_k$ also diverges.

This test basically compares two series with each other, and if the series in which each term is bigger than that of the other series converges, then so does the smaller one.

If the series in which each term is smaller than that of the other series diverges, then so does the bigger one.

We will see more of this later. For now, we shall use the comparison test to develop some other tests that are easier to apply.

THEOREM 11.5.2
(The Ratio Test)

Let $\sum u_k$ be a series with positive terms and suppose that $\rho = \lim_{k \to \infty} \frac{u_{k+1}}{u_k}$

(a) If $\rho < 1$, the series converges.
(b) If $\rho > 1$ or $\rho = +\infty$, the series diverge.
(c) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

Use the ratio test to determine whether the following series converge or diverge.

a) $\sum_{k=1}^{\infty} \frac{1}{k!}$ b) $\sum_{k=1}^{\infty} \frac{k}{2^k}$

c) $\sum_{k=1}^{\infty} \frac{k^k}{k!}$ d) $\sum_{k=1}^{\infty} \frac{(2k)!}{4^k}$

a)

$$\rho = \lim_{k \to +\infty} \frac{u_k + 1}{u_k} = \lim_{k \to +\infty} \frac{1/(k+1)!}{1/k!}$$

$$= \lim_{k \to +\infty} \frac{k!}{(k+1)!} = \lim_{k \to +\infty} \frac{k!}{(k+1)k!}$$

$$= \lim_{k \to +\infty} \frac{1}{(k+1)} = 0 < 1$$

The series converges.

b)

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k}$$

$$= \frac{1}{2} \lim_{k \to +\infty} \frac{k+1}{k} = \frac{1}{2} < 1$$

The series converges

c)

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k}$$

$$= \lim_{k \to +\infty} \frac{(k+1)^k}{k^k} = \lim_{k \to +\infty} \left(1 + \frac{1}{k}\right)^k$$

$$= e > 1$$

The series diverges

d)

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{[2(k+1)]!}{4^{k+1}} \cdot \frac{4^k}{2k!}$$

$$= \lim_{k \to +\infty} \left(\frac{(2k+2)!}{(2k)!} \cdot \frac{1}{4} \right)$$

$$= \frac{1}{4} \lim_{k \to +\infty} (2k+2)(2k+1) = +\infty$$

The series diverges

Determine whether the series

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots\ldots\ldots+\frac{1}{2k-1}+\ldots\ldots$$

converges or diverges.

The ratio test is of no help since

$$\rho=\lim_{k\to+\infty}\frac{u_{k+1}}{u_k}=\lim_{k\to+\infty}\frac{1}{2(k+1)-1}\cdot\frac{2k-1}{1}$$

$$=\lim_{k\to+\infty}\frac{2k-1}{2k+1}=1$$

However, the integral test proves that the series diverges since

$$\int_1^{+\infty}\frac{dx}{2x-1}=\lim_{l\to+\infty}\int_1^{l}\frac{dx}{2x-1}$$

$$=\lim_{l\to+\infty}\frac{1}{2}\ln(2x-1)\Big]_1^{l}=+\infty$$

So, this series diverges.

Sometimes the following result is easier to apply than ratio test

Theorem 11.5.3
(The Root Test)
Let $\sum u_k$ be a series with positive terms and suppose that

(a) If ρ <1, the series converges.
(d) If ρ >1 or ρ =+ ∞ , the series diverge.
(e) If ρ =1, the series may converge or diverge, so that another test must be tried.

Example

$$\sum_{k=2}^{\infty}\left(\frac{4k-5}{2k+1}\right)^k$$

Use the root test to determine whether the following series converge or diverge.

Solution: The series diverges, since $\rho=\lim_{k\to\infty}\left(u_k\right)^{1/k}=\lim_{k\to\infty}\frac{4k-5}{2k+1}=2>1$

We have not really talked about the comparison test so far. To apply comparison test, we use an alternative version of this test that is easier to work with.

INFORMAL PRINCIPLE 11.6.1
Constant terms in the denominator of u_k can usually be deleted without affecting the convergence or divergence of the series.

Example
Use the above informal principle to help guess whether the following series converge or diverge

$$\sum_{k=1}^{\infty} \frac{1}{2^k + 1}$$

Solution: Deleting the constant 1 suggests that $\sum_{k=1}^{\infty} \frac{1}{2^k + 1}$ & $\sum_{k=1}^{\infty} \frac{1}{2^k}$ behaves alike.

The modified series is a convergent geometric series, so the given series is likely to converge.

INFORMAL PRINCIPLE 11.6.2
If a polynomial in k appears as a factor in the numerator or denominator of u_k, all but the highest power of k in the polynomial may usually be deleted without affecting the convergence or divergence of the series.

Example

Use the above principle to help guess whether the following series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3 + 2k}}$ converge or diverge.

Solution: Deleting the term 2k suggests that $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3 + 2k}}$ & $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$

behaves alike
Since the modified series is a convergent p-series (p=3/2) the given series is likely to converge.

The Limit Comparison Test
The following result can be used to establish convergence or divergence by examining the limit of the ratio of the general term of the series in question with general term of a series whose convergence properties are known.

THEOREM 11.6.3
(The Limit Comparison Test)

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that $\rho = \lim_{k \to \infty} \frac{a_k}{b_k}$

If ρ is finite and ρ >0, then the series both converge or both diverge.

Example

Use the limit comparison test to determine whether the following series $\sum_{k=1}^{\infty} \frac{1}{2k^2 - k}$

converge or diverge.

Solution: The given series behaves like the series $\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2}$

$a_k = \frac{1}{2k^2 - k}$ $\quad b_k = \frac{1}{2k^2}$ which is constant times a convergent p-series.
Thus the given series is likely to converge.
By applying theorem 11.6.3 with $\rho = \lim_{k \to +\infty} \frac{a_k}{b_k} = \lim_{k \to +\infty} \frac{2k^2}{2k^2 - k} = \lim_{k \to +\infty} \frac{2}{2 - 1/k} = 1$

Since ρ is finite and positive, it follows from Theorem 11.6.3 that the given series converge.

The Comparison Test
We shall now discuss some techniques for applying the comparison test (Theorem 11.5.1).
There are two basic steps required to apply the comparison test to a series $\sum u_k$ of positive terms:

- Guess whether the series $\sum u_k$ converges or diverges.

- Find a series that proves the guess to be correct. Thus if the guess is divergence, we must find a divergent series whose terms are "smaller" than the corresponding terms of $\sum u_k$, and if the guess is convergence,

we must find a convergent series whose terms are "bigger" than the corresponding terms of $\sum u_k$.

Example
$$\sum_{k=1}^{\infty} \frac{1}{k - 1/4}$$
Use the comparison test to determine whether the following series converge or diverge.

Solution:
As $\frac{1}{k - 1/4} > \frac{1}{k}$
Note that the series behaves like the divergent harmonic series, and hence is likely to diverge. Thus our goal is to find a divergent series that is "smaller" than the given series. We can do this by dropping the constant -1/4 in the denominator, thereby decreasing the size of the general term:

for k=1, 2, ……..

Thus the given series diverges by the comparison test, since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.