Lecture # 45

Taylor and Maclaurin Series

One of the early applications of calculus was the computation of approximate numerical values for functions such as sin x, ln x, and ex. One common method for obtaining such values is to approximate the function by polynomial, then use that polynomial to compute the desired numerical values.

Problem

Given a function **f** and a point **a** on the x-axis, find a polynomial of specified degree that best approximates the function **f** in the "vicinity" of the point **a**.
Suppose that we are interested in approximating a function f in the vicinity of the point a=0 by a polynomial

$$P(x) = c_0 + c_1 x + c_2 x^2 + \dots\dots\dots\dots + c_n x_n \qquad (1)$$

Because P(x) has n+1 coefficients, it seems reasonable that we should be able to impose n+1 conditions on this polynomial to achieve a good approximation to f(x). Because the point a=0 is the center of interest, our strategy will be to choose the coefficients of P(x) so that the value of P and its first n derivatives are the same as the value of f and its first n derivatives at a=0. By forcing this high degree of "match" at a=0, it is reasonable to hope that f(x) and P(x) will remain close over some interval (possibly quite small) centered at a=0. Thus, we shall assume that f can be differentiated n times at 0, and we shall try to find the coefficients in (1) such that

$$f(0) = p(0),\ f'(0) = p'(0),\ f''(0) = p''(0), \dots\dots\dots,\ f^n(0) = p^n(0) \qquad (2)$$

We have

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots\dots\dots + c_n x^n$$
$$p'(x) = c_1 x + 2c_2 x + 3c_3 x^2 + \dots\dots\dots + nc_n x^{n-1}$$
$$p''(x) = 2c_2 + 3\cdot 2c_3 x + \dots\dots\dots + n(n-1)c_n x^{n-2}$$
$$p'''(x) = 3.2c_3 + \dots\dots\dots + n(n-1)(n-2)c_n x^{n-3}$$

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$$.p^n(x) = n(n-1)(n-2)\dots\dots.(1)c_n$$

Thus, to satisfy (2) we must have

$$f(0) = p(0) = c_0$$
$$f'(0) = p'(0) = c_1$$
$$f''(0) = p''(0) = 2c_2 = 2!c_2$$
$$f'''(0) = p'''(0) = 3.2c_3 = 3!c_3$$

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$$f^n(0) = p^n(0) = n(n-1)(n-2)\dots..(1)c_n = n!c_n$$

Which yields the following values for the coefficients of P(x)?

$$c_0 = f(0), c_1 = f'(0), c_2 = \frac{f''(0)}{2!}, c_3 = \frac{f'''(0)}{3!}, ..., c_n = \frac{f^n(0)}{n!}$$

MACLAURIN POLYNOMAILS

If *f* can be differentiated *n* times at 0 , then we define the *nth* Maclaurin Polynomial for *f* to be

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n$$

This polynomial has the property that its value and the values of its first *n* derivatives match the value of *f(x)* and its first *n* derivatives when *x* = 0.

Example

Find the Maclaurin polynomials P_o, P_1, P_2, P_3, and P_n for e^x .

Solution: Let f(x) = e^x, Thus

$$f'(x) = f''(x) = f'''(x) = ... = f^n(x) = e^x$$

and

$$f(0) = f'(0) = f''(0) = f'''(0) = ... = f^n(0) = e^0 = 1$$

Therefore,

$$p_0(x) = f(0) = 1$$

$$p_1(x) = f(0) + f'(0)x = 1 + x$$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{x^2}{2!}$$

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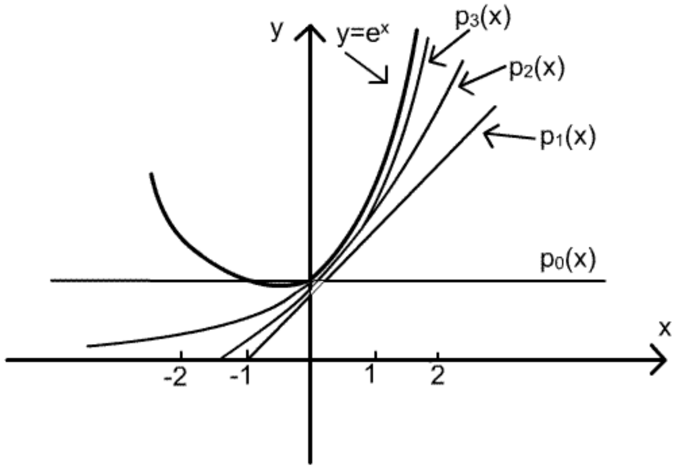
$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^n(0)}{n!}x^n$$

$$= 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}$$



Graphs of e^x and first four Maclaurin polynomials are shown here. Note that the graphs of $P_1(x)$,P_2 (x), P_3 (x) are virtually indistinguishable from the graph of e^x near the origin, so these polynomials are good approximations of e^x near the origin.
But away from origin it does not give good approximation.
To obtain polynomial approximations of f(x) that have their best accuracy near a general point x=a, it will be convenient to express polynomials in powers of x-a, so that they have the form

$$P(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots\ldots\ldots\ldots.. + cn(x-a)^n$$

DEFINITION 11.9.2

If *f* can be differentiated *n* times at 0, then we define the *nth* Taylor polynomial for *f* about *x* = *a* to be

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots\ldots.. + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor and Maclaurin Series

For a fixed value of x near a, one would expect that the approximation of f(x) by its Taylor polynomial P_n (x) about x=a should improve as n increases, since increasing n has the effect of matching higher and higher derivatives of f(x) with those of P_n (x) at x=a. Indeed, it seems plausible that one might be able to achieve any desired degree of accuracy by choosing n sufficiently large; that is, the value of $P_n(x)$ might actually converge to f(x) as $n \to \infty$

DEFINITION 11.9.3

If f has derivatives of all orders at a , then we define the Taylor Series for f about $x = a$ to be

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{2}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \ldots\ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \ldots\ldots$$

In the special case where a=0, the **Taylor series** for **f** is called the **Maclaurin series** for **f**.

Find the Maclaurin Series for

a) e^x b) sinx

a) The nth Maclaurin polynomial for e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \ldots\ldots + \frac{x^n}{n!}$$

Thus, the Maclaurin series for e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\ldots + \frac{x^k}{k!} + \ldots\ldots$$

b) Let $f(x) = \sin x$,

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

Since $f''''(x) = \sin x = f(x)$, the pattern 0,1,0,-1 will repeat over and over as we evaluate successive derivatives at 0.

Therefore , the successive Maclaurin polynomials for sinx are

$p_0(x) = 0$

$p_1(x) = 0 + x$

$p_2(x) = 0 + x + 0$

$p_3(x) = 0 + x + 0 - x^3/3!$

$p_4(x) = 0 + x + 0 - x^3/3! + 0$

$p_5(x) = 0 + x + 0 - x^3/3! + 0 + x^5/5!$

$p_6(x) = 0 + x + 0 - x^3/3! + 0 + x^5/5! + 0$

$p_7(x) = 0 + x + 0 - x^3/3! + 0 + x^5/5! + 0 - x^7/7!$

Because of the zero terms, each even-numbered Maclaurin polynomial [after $p_0(x)$] is the same as the odd-number Maclaurin polynomial; that is

$$p_{2n+1}(x) = p_{2n+2}(x)$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots. + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(n = 0,1,2,3,.......)

Thus, the Maclaurin series for sinx is

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \ldots..$$