Ch5: PAIRS OF RVs

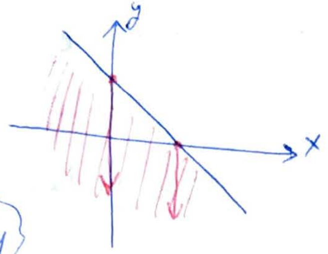
ex $X(\eta) = \begin{bmatrix} H(\eta) \\ W(\eta) \end{bmatrix}$, event $B = \{H(\eta)<183 \cap W(\eta)<80\}$

$= \{H(\eta)<183,\ W(\eta)<80\}$

★ Graphical Interpretation:

ex1 RVs X & Y

$A = \{X+Y \leq 1\}$



① find where equality holds

② fix 1 variable, use the inequality

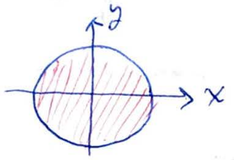
→ ① $x+y=1 \longrightarrow y=-x+1$

→ ② @ $x=0 \longrightarrow y \leq 1$

@ $x=1 \longrightarrow y \leq 0$

---

ex2 $B = \{X^2+Y^2 \leq 1\}$



→ ① $x^2+y^2=1 \longrightarrow y=\sqrt{1-x^2}$

→ ② @ $x=0 \longrightarrow y^2 \leq 1$

$-1 \leq y \leq 1$

★ PMF:

(joint)

$$\boxed{P_{XY}(x_j, y_h) = P[\{X=x_j\} \cap \{Y=y_h\}] = P[X=x_j,\ Y=y_h]}$$

→ $P[B] = \sum_{(x_j,y_h)\in B} P_{XY}(x_j, y_h)$

→ $\sum_{\text{all } j} \sum_{\text{all } h} P_{XY}(x_j, y_h) = 1$

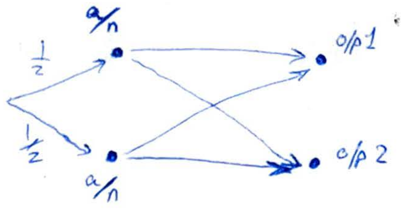
(Marginal)

$$\boxed{P_X(x_j) = P[X=x_j] = \sum_{h=1}^{\infty} \overset{\text{all } y}{P[X=x_j,\ Y=y_h]}}$$

$$\boxed{P_Y(y_h) = P[Y=y_h] = \sum_{j=1}^{\infty} \overset{\text{all } x}{P[X=x_j,\ Y=y_h]}}$$

@ i/p

ex Packet Switch



X: # of packets arriving @ o/p 1

Y: # of packets arriving @ o/p 2

$S_{XY} = \{(0,0), (0,1), (1,0), (1,1), (2,0), (0,2)\}$

no packets arrive @ any o/p

2 packets arrive @ o/p 1

$P_{XY}(\overset{\text{o/p1}}{0}, \overset{\text{o/p2}}{0}) = P[\{\overset{\text{i/p1}}{\text{none}}, \overset{\text{i/p2}}{\text{none}}\}] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\cap$

$P_{XY}(0,1) = P[\{(n, a2), (a2, n)\}] = \frac{1}{2}(\frac{1}{2}\cdot\frac{1}{2}) + \frac{1}{2}(\frac{1}{2}\cdot\frac{1}{2}) = \frac{1}{4}$

none @ i/p1 & arrive @ i/p2 & go to o/p 2

$P_{XY}(1,0) = P[\{(a1, n), (n, a1)\}] = (\frac{1}{2}\cdot\frac{1}{2})\cdot\frac{1}{2} + \frac{1}{2}(\frac{1}{2}\cdot\frac{1}{2}) = \frac{1}{4}$

$P_{XY}(1,1) = P[\{(a1, a2), (a2, a1)\}] = (\frac{1}{2}\cdot\frac{1}{2})(\frac{1}{2}\cdot\frac{1}{2}) + (\frac{1}{2}\cdot\frac{1}{2})(\frac{1}{2}\cdot\frac{1}{2}) = \frac{1}{8}$

$P_{XY}(2,0) = P[(a1, a1)] = (\frac{1}{2}\cdot\frac{1}{2})\cdot(\frac{1}{2}\cdot\frac{1}{2}) = \frac{1}{16}$

$P_{XY}(0,2) = P[(a2, a2)] = (\frac{1}{2}\cdot\frac{1}{2})\cdot(\frac{1}{2}\cdot\frac{1}{2}) = \frac{1}{16}$

→ $\sum_{x_j, y_h} P_{XY}(x_j, y_h) = 1$

$p(1|a) = \frac{1}{2}$

$p(2|a) = \frac{1}{2}$

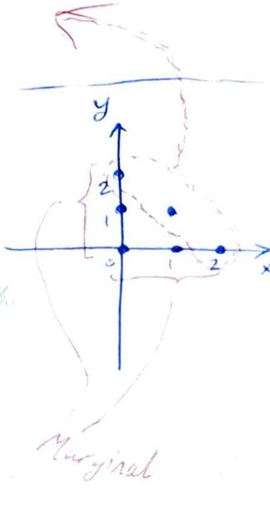
$p(a2) = p(a \cap 2)$

$= p(2|a)\,p(a)$

✱ $B = \{X+Y=2\}$

$\therefore P[B] = P_{XY}(1,1) + P_{XY}(2,0) + P_{XY}(0,2)$

$= \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4}$



✱ Marginal

$P_X(0) = P_{XY}(0,0) + P_{XY}(0,1) + P_{XY}(0,2)$

$= 9/16$

$P_X(1) = P_{XY}(1,0) + P_{XY}(1,1)$

$= 3/8$

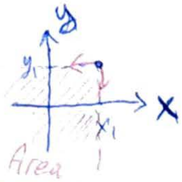
$P_X(2) = P_{XY}(2,0)$

$= 1/16$

$\therefore \sum_{\text{all } x} P_X(x) = 1$

* Joint CDF

$$F_{XY}(x_1, y_1) = P[X \le x_1, Y \le y_1]$$

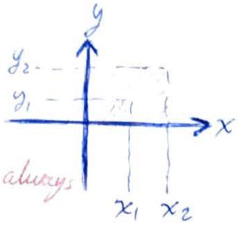


⇒ Properties of joint CDF:

① $F_{XY}(x_1, y_1) \le F_{XY}(x_2, y_2)$

if $x_1 \le x_2$
$y_1 \le y_2$

cdf is always +ve



② $F_{XY}(-\infty, y_1) = F_{XY}(x_1, -\infty) = 0$

$\downarrow$ $x < 0$ (-ve) $\qquad \downarrow$ $y < 0$ (-ve)

③ $F_{XY}(\infty, \infty) = 1$

④ Marginal CDF :

$$\boxed{\begin{aligned} F_X(x) &= F_{XY}(x, +\infty) \\ &= P[X \le x, Y < \infty] \end{aligned}}$$

$$\boxed{\begin{aligned} F_Y(y) &= F_{XY}(+\infty, y) \\ &= P[X < \infty, Y \le y] \end{aligned}}$$

---

ex magical loaded coins
X & Y
not independent

		Y	
		H	T
		0	1
X	H 0	1/3	1/6
	T 1	1/6	1/3

$$P_X(0) = P_{XY}(0,0) + P_{XY}(0,1) \pm \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

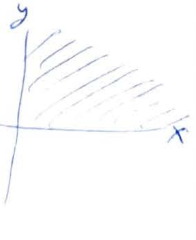
$$P_Y(0) = P_{XY}(0,0) + P_{XY}(1,0) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

→ From marginal pmfs, i feel they are independent, but from joint pmf, i know they are dependent

---

ex $F_{XY}(x,y) = \begin{cases} (1-e^{-\alpha x})(1-e^{-\beta y}) & x \ge 0, \ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$

↳ Marginal CDF



$F_X(x) = F_{XY}(x, \infty) = 1 - e^{-\alpha x} \qquad x \ge 0$

$F_Y(y) = F_{XY}(\infty, y) = 1 - e^{-\beta y} \qquad y \ge 0$

$$\boxed{F_{XY}(x,y) = F_X(x)\, F_Y(y)}$$

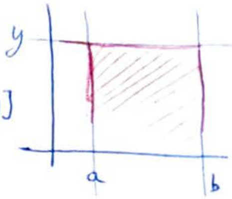
↳ X & Y independent

---

* $P[a \le X \le b, Y \le y]$

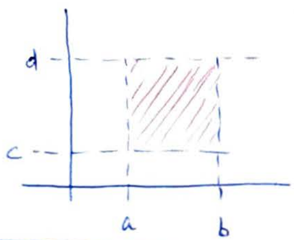
$= P[X \le b, Y \le y] - P[X \le a, Y \le y]$

$= F_{XY}(b, y) - F_{XY}(a, y)$



* $P[a \le X \le b, c \le Y \le d]$

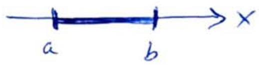
$= F_{XY}(b,d) - F_{XY}(a,d)$
$- [F_{XY}(b,c) - F_{XY}(a,c)]$



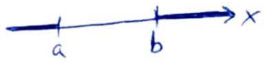
* 1D

$P[X \le a] = F_X(a)$

$P[X \ge b] = 1 - F_X(b)$



$P[a \le X \le b] = F_X(b) - F_X(a)$

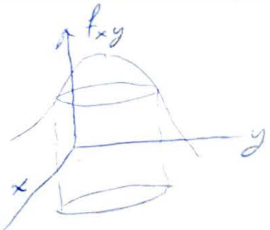


$P[X \le a, X \ge b] = 1 - [F_X(b) - F_X(a)]$

---

* Joint PDF

$$\boxed{f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \, \partial y}}$$



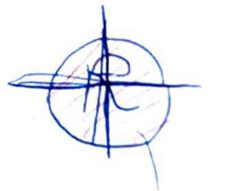
$$\boxed{F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(x', y')\, dx'\, dy'}$$

↳ $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$$

Marginal : $$\boxed{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy}$$
$$\boxed{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx}$$

ex $P[X^2 + Y^2 \le 1]$

$$= \iint_R f_{XY}(x,y)\, dx\, dy$$

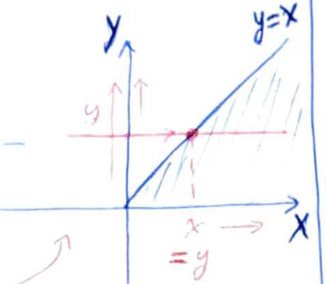


ex $f_{XY}(x,y) = \begin{cases} c\,e^{-x}e^{-y} & 0 \leq y < x < \infty \\ 0 & \text{otherwise} \end{cases}$

$\boxed{1}$ $c = ?$

$\boxed{2}$ $f_X(x) = ?$, $f_Y(y) = ?$

$\boxed{3}$ $P[X+Y \leq 1] = ?$



$\boxed{1}$ parametrization of the region for $\iint$

a) integration by x then by y

$y \leq x < \infty$

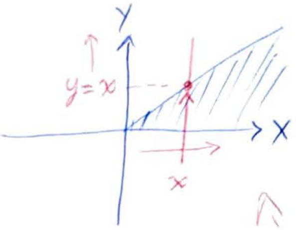
$0 \leq y < \infty$

OR

b) integration by y then by x

$0 \leq y \leq x$

$0 \leq x < \infty$

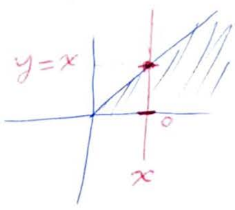


$$\Rightarrow a) \int_0^\infty \int_y^\infty c\,e^{-x}e^{-y}\,dx\,dy = c\int_0^\infty e^{-y}\left[\int_y^\infty e^{-x}dx\right]dy$$

$$= c\int_0^\infty e^{-2y}\,dy = \frac{c}{2} = 1$$

$\therefore \boxed{c = 2}$

$\boxed{2}$ $f_X(x) = ?$

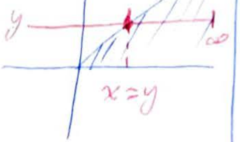


$f_X(x) = 0 \qquad x < 0$

$$f_X(x) = \int_0^x c\,e^{-x}e^{-y}\,dy$$

$$= c\,e^{-x}\int_0^x e^{-y}\,dy = 2e^{-x}(1-e^{-x}) \quad x \geq 0$$

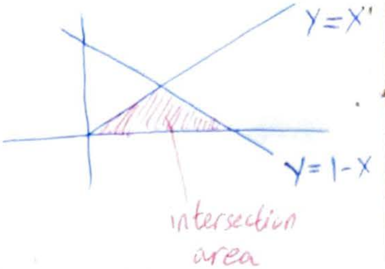
$$f_Y(y) = \int_y^\infty 2e^{-x}e^{-y}\,dx$$



$$= 2e^{-2y} \qquad y \geq 0$$

$\boxed{3}$

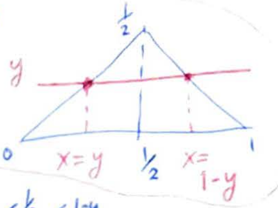
$\boxed{3}$ $P[X+Y \leq 1] = ?$



① Integration by x then by y

$y \leq x \leq \frac{1}{2}$, $\frac{1}{2} \leq x \leq 1-y$

$0 \leq y \leq \frac{1}{2}$ $\quad 0 \leq y \leq \frac{1}{2}$



$$\therefore P = \int_0^{\frac{1}{2}}\int_y^{\frac{1}{2}} c\,e^{-x}e^{-y}\,dx\,dy + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1-y} c\,e^{-x}e^{-y}\,dx\,dy$$

$$= \int_0^{1/2} 2e^{-y}\left[e^{-y} - \cancel{e^{-1/2}}\right]dy + \int_0^{1/2} 2e^{-y}\left[\cancel{e^{-1/2}} - e^{y-1}\right]dy$$

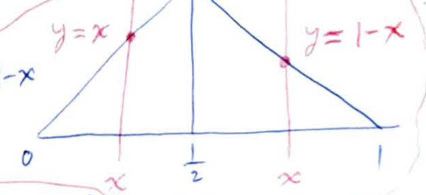
$$= \int_0^{\frac{1}{2}} 2e^{-2y}\,dy - \int_0^{1/2} 2e^{-1}\,dy$$

$$= 2\left[-e^{-2y}\right]_0^{1/2} - 2e^{-1}\left[\tfrac{1}{2}\right] = 1 - 3e^{-1}$$

② Integration by y then by x

$0 \leq y \leq x$, $0 \leq y \leq 1-x$

$0 \leq x \leq \frac{1}{2}$ $\quad \frac{1}{2} \leq x \leq 1$



$$\therefore P = \int_0^{\frac{1}{2}}\int_0^x 2e^{-x}e^{-y}\,dy\,dx + \int_{\frac{1}{2}}^1\int_0^{1-x} 2e^{-x}e^{-y}\,dy\,dx$$

$$= \int_0^{1/2} 2e^{-x}\left[-e^{-y}\right]_0^x dx + \int_{\frac{1}{2}}^1 2e^{-x}\left[-e^{-y}\right]_0^{1-x} dx$$

$$= \int_0^{\frac{1}{2}} 2e^{-x}\left[-e^{-x}+1\right]dx + \int_{\frac{1}{2}}^1 2e^{-x}\left[-e^{x-1}+1\right]dx$$

$$f_{XY}(x,y) \neq f_X(x)\,f_Y(y)$$

$\therefore X$ & Y are not independent

☆ Independence:

Continuous

$$\boxed{\begin{aligned} f_{XY}(x,y) &= f_X(x)\, f_Y(y) \\ F_{XY}(x,y) &= F_X(x)\, F_Y(y) \end{aligned}}$$

discrete

$$\boxed{P_{XY}(x,y) = P_X(x)\, P_Y(y)}$$

⟷ if X & Y are independent

A: $X^2+Y^2 \leq 1$
↳ in this event, X & Y not indep.

$$E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

$$\boxed{E[X+Y] = E[X] + E[Y]}$$

Proof: $E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f_{XY}(x,y)\, dx\, dy$

$$= \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f_{XY}(x,y)\, dy}_{f_X(x)}\, dx + \int_{-\infty}^{\infty} y \underbrace{\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx}_{f_Y(y)}\, dy$$

$$= E[X] + E[Y]$$

$$\boxed{E[XY] = E[X]\, E[Y]}$$ if X, Y independent

Proof: $E[XY] = \iint xy\, f_{XY}(x,y)\, dx\, dx$

$$= \int x\, f_X(x)\, dx \int y\, f_Y(y)\, dy$$

But this does not necessarily mean X & Y are independent

☆ Correlation of X, Y = E[XY]

↳ if $E[XY]=0 \longrightarrow \therefore$ X, Y orthogonal

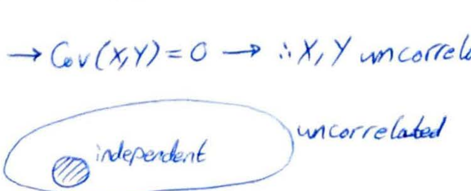
☆ Covariance of X, Y = Cov(X,Y)

$$= E\left((X-E(X))(Y-E(Y))\right)$$

↳ if $Cov(X,Y) = 0 \longrightarrow \therefore$ X, Y uncorrelated

↳ $Cov(X,X) = E\left((X-E(X))^2\right) = Var(X)$

↳ if X, Y independent → $Cov(X,Y)=0$ → ∴ X, Y uncorrelated



Proof: $Cov(X,Y) = E\left[(X-E(X))(Y-E(Y))\right]$

$$= E\left[(X-E(X))\right] \cdot E\left[(Y-E(Y))\right]$$

↓ Const.

$$\therefore E[X-E(X)] = E[X] - E[E(X)] = E[X] - E[X] = 0$$

$\therefore Cov(X,Y) = 0$
if X, Y independent

☆ Correlation Coeff $\boxed{\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X\, \sigma_Y}}$

Proof:

$$E\left[\left(\frac{X-E(X)}{\sigma_X} \pm \frac{Y-E(Y)}{\sigma_Y}\right)^2\right] \geq 0$$

$$E\left[\left(\frac{X-E(X)}{\sigma_X}\right)^2\right] + E\left[\left(\frac{Y-E(Y)}{\sigma_Y}\right)^2\right] \pm 2E\left[\frac{(X-E(X))(Y-E(Y))}{\sigma_X\, \sigma_Y}\right] \geq 0$$

$$\frac{1}{\sigma_X^2} E\left[(X-E(X))^2\right] = \frac{1}{\sigma_X^2}\sigma_X^2 = 1$$

$$1 + 1 \pm \frac{2}{\sigma_X\, \sigma_Y} Cov(X,Y) \geq 0$$

$$\therefore \boxed{|\rho_{XY}| \leq 1} \longrightarrow \text{fully correlated}$$

---

ex1 $Y = aX + b$

→ $E[Y] = aE[X] + b$

→ $\therefore Cov(X,Y) = E\left[(Y-E(Y))(X-E(X))\right]$

$$= E\left[(aX+b-aE(X)-b)(X-E(X))\right]$$

$$= aE\left((X-E(X))^2\right) = a\sigma_X^2$$

→ $\sigma_Y^2 = E\left((Y-E(Y))^2\right) = a^2\sigma_X^2$

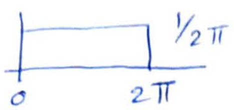
$\therefore \sigma_Y = |a|\,\sigma_X$

$$\therefore \rho_{XY} = \frac{(a\sigma_X^2)}{\sigma_X(|a|\sigma_X)} = \frac{a}{|a|} = \begin{cases} 1 & a>0 \\ -1 & a<0 \end{cases}$$

If Y & X are linearly related, then the
are 100% correlated (& Vice Versa)

ex2 $X = \cos\theta$
$Y = \sin\theta$

θ Uniform $[0, 2\pi[$



$$E(X) = E(\cos\theta) = \int_0^{2\pi} \cos\theta \left(\frac{1}{2\pi}\right) d\theta = 0$$

$$E(Y) = E(\sin\theta) = \int_0^{2\pi} \sin\theta \left(\frac{1}{2\pi}\right) d\theta = 0$$

$$E(XY) = E(\cos\theta \sin\theta) = \int_0^{2\pi} \underbrace{\cos\theta \sin\theta}_{\frac{1}{2}\sin 2\theta} \frac{d\theta}{2\pi} = 0$$

$$\therefore \operatorname{Cov}(XY) = E(XY) - E(X)E(Y) = 0$$

$\therefore$ X & Y are uncorrelated, but they are dependent (Sin & Cos are dependent)

$$\Rightarrow \operatorname{Cov}(X,Y) = E[(X-E(x))(Y-E(Y))]$$

$$= E[XY + \underbrace{E(x)E(Y)} - X\underbrace{E(Y)} - Y\underbrace{E(x)}]$$

constants

$$= E[XY] + E(x)E(Y) - E(Y)E(x) - E(x)E(Y)$$

$$\therefore \boxed{\operatorname{Cov}(x,Y) = E(xY) - E(x)E(Y)}$$

---

* Conditional Probability: Ⓐ X,Y Discrete

$$\boxed{P_Y(y|x) = \frac{P_{XY}(x,y)}{P_X(x)}}$$

$$P[Y \text{ in } A] = \sum_x P[Y \text{ in } A \mid X=x]\, P_X(x) = \sum_{\text{all } x} \sum_{y\in A} P_{XY}(x,y)$$

$$= \sum_x \left[\sum_{y\in A} P_Y[y \mid X=x]\right] P_X(x)$$

Ⓑ X Discrete, Y Continuous

$$F_Y(y|x) = P(Y \leq y \mid X=x)$$ Conditional cdf

$$f_Y(y|x) \neq \frac{d}{dy} F_Y(y|x)$$ Conditional pdf

$$\boxed{P[Y \text{ in } A \mid X=x] = \int_{y\in A} f_Y(y \mid X=x)\, dy}$$

Ⓒ X, Y Continuous

$$F_Y(y|x) = \lim_{h\to 0} P[Y \leq y \mid x \leq X \leq x+h]$$

$$f_Y(y|x) = \frac{d}{dy} F_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$\boxed{P[Y \text{ in } A \mid x] = \int_{y\in A} f_Y(y|x)\, dy}$$

$$P[Y \text{ in } A] = \int_{y\in A} \left(\int_{-\infty}^{\infty} \underbrace{f_{XY}(x,y)}_{f_Y(y|x)\, f_X(x)} dx \right) dy$$

$$= \int_{-\infty}^{\infty} f_X(x) \left(\int_{y\in A} f_Y(y|x)\, dy \right) dx$$

$$\therefore \boxed{P[Y \text{ in } A] = \int_{-\infty}^{\infty} P[Y \text{ in } A \mid X=x]\, f_X(x)\, dx}$$

Theorem of total Probability

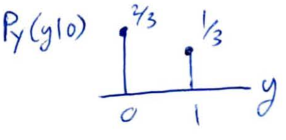
---

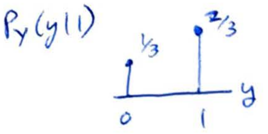
ex1 Magical Coins

Y \ X	0	1
0	1/3	1/6
1	1/6	1/3

$$P[y \mid X=0] = \frac{P_{XY}(0,y)}{P_X(0)} = 2P_{XY}(0,y) = \begin{cases} 2/3 & y=0 \\ 1/3 & y=1 \end{cases}$$

$$P[y \mid X=1] = \frac{P_{XY}(1,y)}{P_X(1)} = 2P_{XY}(1,y) = \begin{cases} 1/3 & y=0 \\ 2/3 & y=1 \end{cases}$$





$$E[P_Y(y|0)] = \left(0 \times \tfrac{2}{3}\right) + \left(1 \times \tfrac{1}{3}\right) = \tfrac{1}{3}$$

$$P[Y > \tfrac{1}{2} \mid X=0] = \tfrac{1}{3}$$

ex2

$P[X=+1] = 1/3$

$P[X=-1] = 2/3$, $N \sim N(0,1)$

$$f_N(n) = \frac{1}{\sqrt{2\pi}} e^{-n^2/2}$$

ⓐ $f_Y(y \mid X=+1) = ?$

$$F_Y(y \mid X=+1) = P[Y \leq y \mid X=+1]$$
$$= P[N+1 \leq y]$$
$$= P[N \leq y-1] = F_N[y-1]$$

$$\therefore f_Y(y \mid X=+1) = f_N(y-1) = \frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2}$$

Similarity

$$f_Y(y \mid X=-1) = f_N(y+1) = \frac{1}{\sqrt{2\pi}} e^{-(y+1)^2/2}$$

ⓑ $$P[X=+1 \mid Y>0] = \frac{P[X=+1 \cap Y>0]}{P[Y>0]}$$
$$= \frac{P[Y>0 \mid X=+1]\, P[X=+1]}{P[Y>0]}$$

$$P[Y>0] = P[Y>0 \mid X=+1]\, P[X=+1] \rightarrow \frac{1}{3}$$
$$+ P[Y>0 \mid X=-1]\, P[X=-1] \rightarrow 2/3$$

$\int_0^\infty f_Y(y \mid x=+1)\, dy$ $\quad \int_0^\infty f_Y(y \mid X=-1)\, dy$

$$\therefore P[X=+1 \mid Y>0] = \frac{\frac{1}{3}(1-Q(1))}{\frac{1}{3}(1+Q(1))} = 0.726$$

⇓

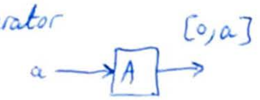
Prob the i/p is +1 given I saw +1 @ the o/p

↳ Prob of error $= 1 - P[X=+1 \mid Y>0]$

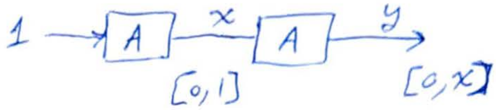
$$\boxed{E[Y \mid X] = \int_{-\infty}^{\infty} y\, f_Y(y \mid x)\, dy} \rightarrow \text{fn of } x$$

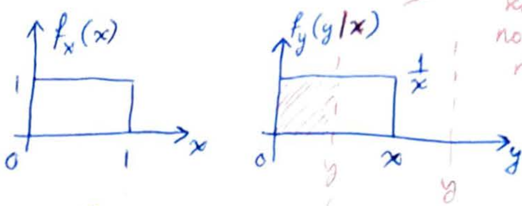
$$\boxed{\begin{gathered} E[E[Y \mid X]] = E[Y] \\ E[E[g(Y) \mid X]] = E[g(Y)] \end{gathered}}$$

ex3 Uniform Random Number Generator



Now, with 2 Generators





when x is known, it's no longer random

$$F_Y(y) = P[Y \leq y]$$
$$= \int_0^1 P[Y \leq y \mid X=x]\, \underset{1}{\cancel{f_X}}(x)\, dx$$

y → dependent var.

x → independent var.

$$P[Y \leq y \mid X=x] = \begin{cases} 1 & y > x \\ y/x & 0 < y < x \\ 0 & y < 0 \end{cases}$$

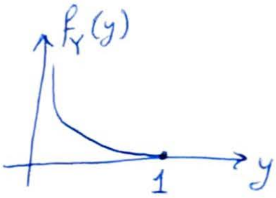
$$\therefore F_Y(y) = \int_0^y (1)\, dx + \int_y^1 \frac{y}{x}\, dx$$
$$= y + y\,[\text{Log}\, x]_y^1$$

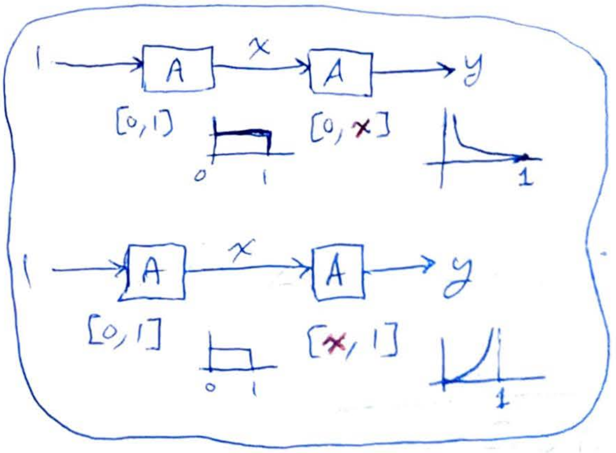
$$\boxed{F_Y(y) = y - y \,\text{Log}\, y} \quad 0<y<1$$

$Log = ln = log_e$

$Log = log_{10}$

$$\boxed{\begin{aligned} \therefore f_Y(y) &= 1 - \text{Log}\, y - y\left(\tfrac{1}{y}\right) \\ &= -\text{Log}\, y \end{aligned}}$$





ex4 Customers arriving to Bank

Rate = $\beta \frac{\text{customers}}{\text{min}}$ (Poisson) $\lambda = \beta t$

Service time for 1 certain customer is RV & follows exp. distribution

$$f_T(t) = \alpha e^{-\alpha t} \quad t \geq 0, \ \alpha > 0$$

➡ Prob[k customers arrive within service time of a customer] =?

*2 sources of randomness:

① # of customers → $P[N=k] = e^{-\beta t}\frac{(\beta t)^k}{k!}$ $k=0,1,2,..$

② Service time → $f_T(t) = \alpha e^{-\alpha t}$

So, Use Cond. Prob. (I want to fix "t")

$$P[N(t)=k] = \int_0^\infty P[N(t)=k \mid T(t)=t]\, f_T(t)\, dt$$

Theorem of Total Prob.

$$= \int_0^\infty e^{-\beta t}\frac{(\beta t)^k}{k!}\alpha e^{-\alpha t} dt$$

$$= \frac{\alpha \beta^k}{k!}\int_0^\infty t^k e^{-(\alpha+\beta)t} dt$$

let $r = (\alpha+\beta)t$

$$= \frac{\alpha\beta^k}{(\alpha+\beta)^{k+1} k!}\underbrace{\int_0^\infty r^k e^{-r} dr}_{k!}$$

Gamma Function

$$= \frac{\alpha}{\alpha+\beta}\left(\frac{\beta}{\alpha+\beta}\right)^k \quad k=0,1,2,.....$$

$$E[N] = \sum_{k=0}^\infty k \frac{\alpha}{\alpha+\beta}\left(\frac{\beta}{\alpha+\beta}\right)^k$$

(OR)

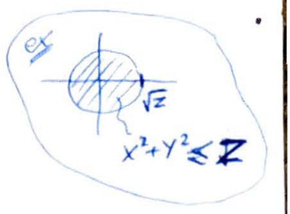
$$E[N(t)=k \mid T=t] = \lambda = \beta t$$

$$E[E[N(t)=k \mid T=t]] = \beta E[t] = \frac{\beta}{\alpha}$$

★ $Z = g(X,Y)$, $f_Z(z) = ?$

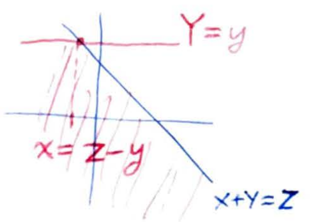
$$\boxed{F_Z(z) = P[Z \leq z] = P[g(X,Y) \leq z] = \iint_{R_Z} f_{XY}(x,y)\, dx\, dy}$$

$$\boxed{f_Z(z) = \frac{d}{dz}F_Z(z)}$$



ex $Z = X+Y \longrightarrow f_Z(z) = ?$ from $f_{XY}(x,y)$

$$F_Z(z) = P[X+Y \leq z] = \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{XY}(x,y)\, dx\, dy$$



$$f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty}\left[\frac{d}{dz}\int_{-\infty}^{z-y} f_{XY}(x,y)\, dx\right] dy$$

$-\infty < x \leq z-y$

$-\infty < y < \infty$

$$= \int_{-\infty}^{\infty} f_{XY}(z-y, y)\, dy$$

OR

$$= \int_{-\infty}^{\infty} f_{XY}(x, z-x)\, dx$$

$$\frac{d}{dz}\int_a^z f(x)\, dx = f(z)$$

$$\frac{d}{dz}\int_{-\infty}^{g(z)} f(x)\, dx = g'(z)\, f(g(z))$$

If X & Y are independent → ∴ $f_{XY}(x,y) = f_X(x) f_Y(y)$

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx = f_X * f_Y$$

pdf of X+Y is a convolution if X,Y independent

★ X, Y are jointly Gaussian

- $X \sim N(m_1, \sigma_1)$, ρ_{XY} = Correlation Coeff
  $Y \sim N(m_2, \sigma_2)$

$$f_X(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$\boxed{f_{XY}(x,y) = \frac{\exp\left[\frac{-1}{2(1-\rho_{XY}^2)}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2\rho_{XY}\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right) + \left(\frac{y-m_2}{\sigma_2}\right)^2\right]\right]}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{XY}^2}}}$$

$$X|y \sim N\left(m_1 + \rho_{XY}\frac{\sigma_1}{\sigma_2}(y-m_2) \,,\; \sigma_1^2(1-\rho_{XY}^2)\right)$$

↳ if $\rho_{XY}=0$ "uncorrelated" → $\therefore X|y = X$

↳ For Gaussian Only

Independent ⇌ Un Correlated

---

ex ↳ $m_1 = m_2 = 1$, $\rho = \frac{1}{2}$, $Z = X + Y$
$\sigma_1 = \sigma_2 = 1$

↳ $$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\}$$

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\, dx$$

$$= \frac{1}{\pi\sqrt{3}} \int_{-\infty}^{\infty} \exp\left[\frac{-2}{3}(x^2 - xz + z^2)\right] dx$$

$$x^2 - 2x\frac{z}{2} + z^2 \left(+\frac{z^2}{4} - \frac{z^2}{4}\right)$$

$$= \frac{1}{\pi\sqrt{3}} \int_{-\infty}^{\infty} \exp\left[\frac{-2}{3}\left(\left(x-\frac{z}{2}\right)^2 + \frac{3z^2}{4}\right)\right] dx$$

$$= e^{-z^2/2} \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\, 3/4}}\, e^{-\frac{(x-z/2)^2}{2(3/4)}}\, dx}_{N\left(\frac{z}{2}, \frac{3}{4}\right)}$$

$= 1$

$$\therefore f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$\sim N(0,1)$

if $X \xrightarrow{pdf} N(0,1)$

$\therefore 2X \longrightarrow N(0,4)$

8

ex $Z = \frac{X}{Y}$

X exponential with $\lambda = 1$
Y " " "
} independent
$x \geq 0$
$y \geq 0$

↳ [illegible] Assume $Y = y$

$\therefore Z = \frac{X}{y}$

$$Z = \alpha X$$
$$f_Z(z) = \frac{f_X\left(\frac{z}{\alpha}\right)}{|\alpha|}$$

$$\therefore f_Z(z|y) = \frac{f_X(yz)}{\frac{1}{|y|}}$$

$$= |y|\, f_X(yz)$$

$$= y\, e^{-yz}$$

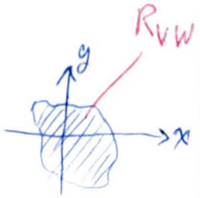
$$\therefore f_Z = \int_{-\infty}^{\infty} \underbrace{f_Z(z|y)\, f_Y(y)}_{f_{YZ}(y,z)}\, dy$$

$$= \int_{-\infty}^{\infty} y e^{-yz} e^{-y}\, dy = \frac{1}{(z+1)^2}$$

$$f_X(x) \xrightarrow{Y=g(X)} f_Y(y)$$

$$f_{XY}(x,y) \xrightarrow{Z=g(X,Y)} f_Z(z)$$

$$f_{XY}(x,y) \xrightarrow{\substack{V=g_1(X,Y)\\ W=g_2(X,Y)}} f_{VW}(v,w)$$

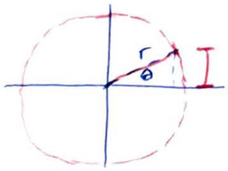


$$F_{VW}(v,w) = P[V \leq v,\ W \leq w]$$
$$= P[g_1(X,Y) \leq v,\ g_2(X,Y) \leq w]$$
$$= \iint^{v\ w} f_{XY}(x,y)\,dx\,dy$$

ex $\left.\begin{matrix} X \sim N(0,1) \\ Y \sim N(0,1) \end{matrix}\right\}$ Independent

$$\therefore f_{XY}(x,y) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,\frac{e^{-y^2/2}}{\sqrt{2\pi}} = \frac{e^{-(x^2+y^2)/2}}{2\pi}$$

$$r = \sqrt{x^2+y^2}$$
$$\theta = \tan^{-1}\frac{y}{x}$$
$$f_{R\theta}(r,\theta) = ?$$



$$F_{R\theta}(r,\theta) = P[R \leq r,\ \theta \leq \theta]$$
$$= P\left[\sqrt{x^2+y^2} \leq r,\ \tan^{-1}\frac{y}{x} \leq \theta\right]$$
$$= \underset{\text{region } \Omega}{\iint} \frac{1}{2\pi} e^{-(x^2+y^2)/2}\,dx\,dy \rightarrow \text{change to polar coordinates}$$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ dxdy &= r\,dr\,d\theta \end{aligned}}$$

$$= \int_0^\theta\int_0^r \frac{e^{-r^2/2}}{2\pi}\,r\,dr\,d\theta$$

$$= \underbrace{\int_0^\theta \frac{d\theta}{2\pi}}_{\theta/2\pi}\ \underbrace{\int_0^r r e^{-r^2/2}\,dr}_{-e^{-r^2/2}\big|_0^r}$$

$$\therefore F_{R\theta}(r,\theta) = \frac{\theta}{2\pi}\left(1 - e^{-r^2/2}\right)$$

$$\therefore f_{R\theta}(r,\theta) = \frac{\partial}{\partial\theta\partial r} F_{R\theta}(r,\theta) = \underbrace{\frac{1}{2\pi}}_{f(\theta)}\ \underbrace{r e^{-r^2/2}}_{f(r)} \qquad \begin{matrix} r \geq 0 \\ 0 \leq \theta \leq 2\pi \end{matrix}$$

θ & R are independent

4

★ Linear relation between X, Y & V, W

$$\begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ c & e \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\boxed{f_{VW}(v,w) = \frac{f_{XY}(x,y)}{\left|\frac{\partial(v,w)}{\partial(x,y)}\right|}}$$

$$= \det\begin{bmatrix} \partial v/\partial x & \partial v/\partial y \\ \partial w/\partial x & \partial w/\partial y \end{bmatrix}$$
$$= \frac{\partial v}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial y}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & e \end{bmatrix}^{-1} \begin{bmatrix} V \\ W \end{bmatrix}$$

$$\frac{1}{ae-bc}\begin{bmatrix} e & -b \\ -c & a \end{bmatrix}$$

ex $\begin{bmatrix} V \\ W \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$

X, Y jointly Gaussian

$$f_{XY}(x,y) = \frac{1}{2\pi(1-\rho^2)} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right)$$

$$\downarrow$$

$$f_{VW} = ?$$

$$V = \frac{X+Y}{\sqrt{2}}, \quad W = \frac{-X+Y}{\sqrt{2}},$$

$$\det\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = 1$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} V \\ W \end{bmatrix} \Rightarrow \begin{matrix} X = \frac{V-W}{\sqrt{2}} \\ Y = \frac{V+W}{\sqrt{2}} \end{matrix}$$

$$\therefore f_{VW}(v,w) = \underbrace{\frac{e^{-v^2/2(1+\rho)}}{\sqrt{2\pi(1+\rho)}}}_{\substack{f_V \\ \sim N(0,(1+\rho))}}\ \underbrace{\frac{e^{-w^2/2(1-\rho)}}{\sqrt{2\pi(1-\rho)}}}_{\substack{f_W \\ \sim N(0,(1-\rho))}}$$

Orthogonalization

X & Y not independent $\xrightarrow{\text{linear transf.}}$ V & W independent