EE 210: HW 05 Solutions

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Short review of Fourier series

A Fourier series represents a periodic function, such as a square wave, as the sum of sinusoids. The sinusoids being summed have frequencies which are integer multiples of $f_s = 1/T_s$ where f_s and T_s are, respectively, the *fundamental frequency* and *fundamental period* of the signal. x_k, a complex-valued number, is the "weight", or contribution to the Fourier sum, of a sinusoid oscillating at $k f_s$[1]. Plotting x_k against k gives the frequency response of $x(t)$ at these discrete frequencies, termed *harmonics*. This representation reveals the frequency components of the signal.

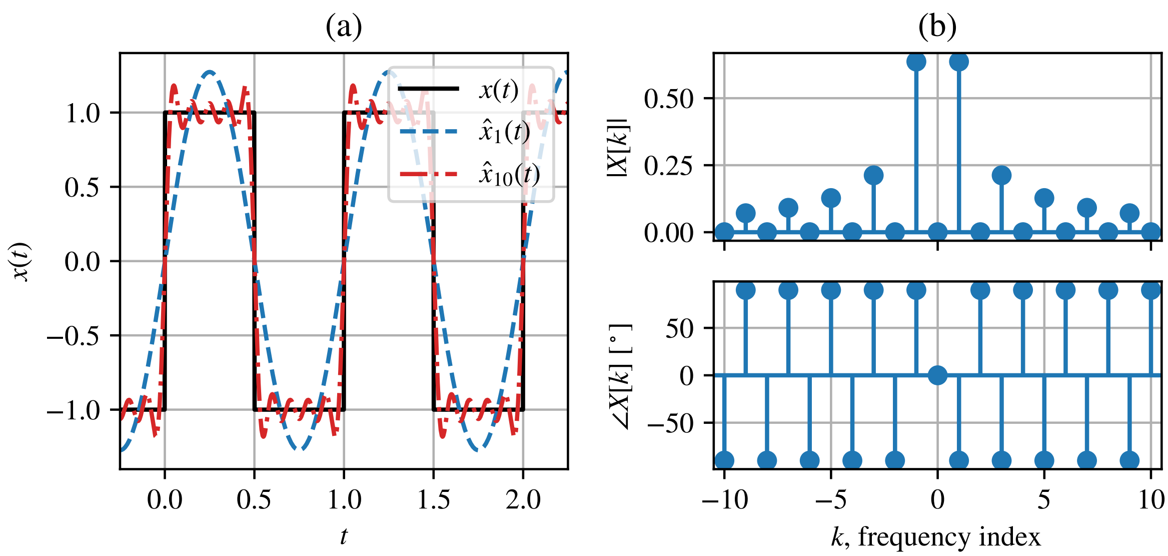

Figure 1: Fourier series expansion of a 1 Hz square wave.

Fig. 1 shows $x(t)$, a 1 Hz square wave, and its associated Fourier series representation. Fig. 1(a) shows approximations of $x(t)$ with a single sinusoid and with a sum of ten sinusoids. Each additional sinusoid contributes higher-frequency components, and the approximation approaches the original function as the number of summed terms increases. Fig. 1(b) plots the frequency response of the Fourier series coefficients. Notice that the discrete points trace out a sinc function, which we know to be the frequency response of a single rectangular pulse. From the magnitude response, we see that there is no DC component (bias) — $|X[0]| = 0$—and that the square wave mostly contains low-frequency components. This is justified by noting that a square wave is almost entirely constant-valued over a period, with very short rising and falling edges.

[1] x_k, being complex, contains amplitude *and* phase information for each sinusoid.

1 Fourier series of rectangular-pulse train

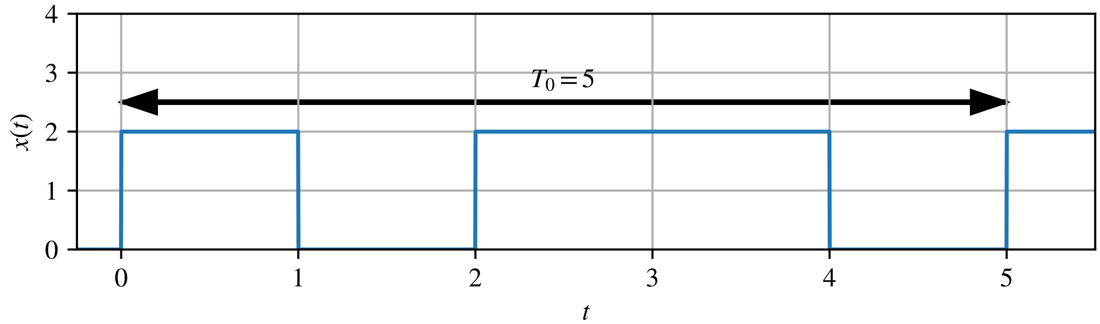

Figure 2: $x(t)$ for problem 1.

The definitions provided in the homework assignment are reiterated here. For a T_s-periodic function, $x(t)$, integrable over a period,

$$x_k = \frac{1}{T_s} \int_{T_s} x(t) e^{-j2\pi(k/T_s)t} \, \mathrm{d}t, \qquad k = (-\infty, \infty), \text{ an integer} \tag{1a}$$

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{-j2\pi(k/T_s)t} \tag{1b}$$

Comparing the equations to the definition of the Fourier transform of an *aperiodic* function, $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, \mathrm{d}t$, we can consider Eq. (1a) to be the Fourier transform, sampled at $f = k/T_s$ and scaled by $1/T_s$, of a single period of $x(t)$:

$$x_k = \frac{1}{T_s} X\left(\frac{k}{T_s}\right) \tag{2}$$

This will allow us to use a standard table of Fourier transforms to calculate Fourier series representations of functions.

Fig. 2 plots the given $x(t)$ and marks off the fundamental period, $T_0 = 5$. Before applying Eq. 2, we write a single period of $x(t)$ explicitly:

$$x(t) = 2 \sqcap (t - 0.5) + 2 \sqcap \left(\frac{t-3}{2}\right)$$

We use the transform pairs

$$\mathcal{F}\left\{A \sqcap \left(\frac{t}{2t_0}\right)\right\} = 2At_0 \operatorname{sinc}\left(2t_0 f\right) \tag{3a}$$

$$\mathcal{F}\left\{x(t - t_0)\right\} = X(f) e^{-j2\pi f t_0} \tag{3b}$$

to obtain

$$
\begin{aligned}
X(f) &= 4\,\mathrm{sinc}\,(2f)\,e^{-j2\pi f(0.5)} + 8\,\mathrm{sinc}\,(4f)\,e^{-j2\pi f(3)} \\
x_k &= \frac{1}{T_s} X\left(\frac{k}{T_s}\right) \\
&= \boxed{\frac{4}{5}\,\mathrm{sinc}\left(2\frac{k}{T_s}\right) e^{-j\pi f} + \frac{8}{5}\,\mathrm{sinc}\left(4\frac{k}{T_s}\right) e^{-j6\pi f}}
\end{aligned}
$$

$X(f)$, the frequency response for a single cycle of $x(t)$, is plotted in Fig. 3. The magnitude and phase of the Fourier series coefficients x_k are plotted in Fig. 4. Note that k, the frequency index, is discrete, and that x_k samples and scales $X(f)$.

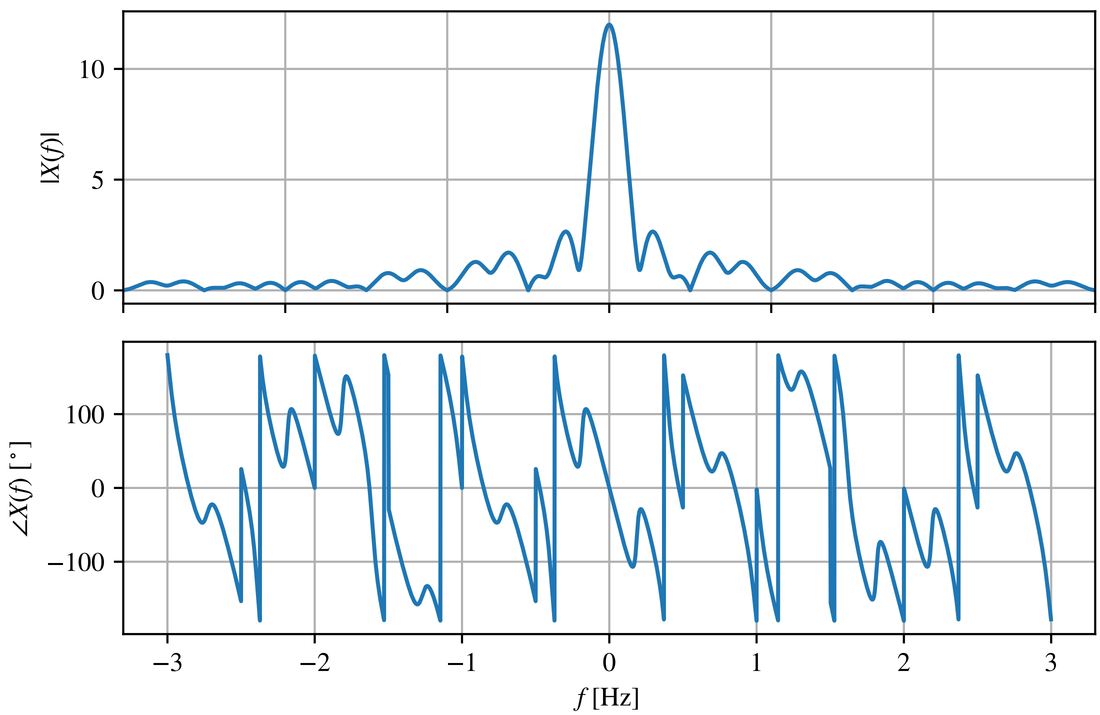

Figure 3: Fourier transform for a single cycle of $x(t)$ (Problem 1).

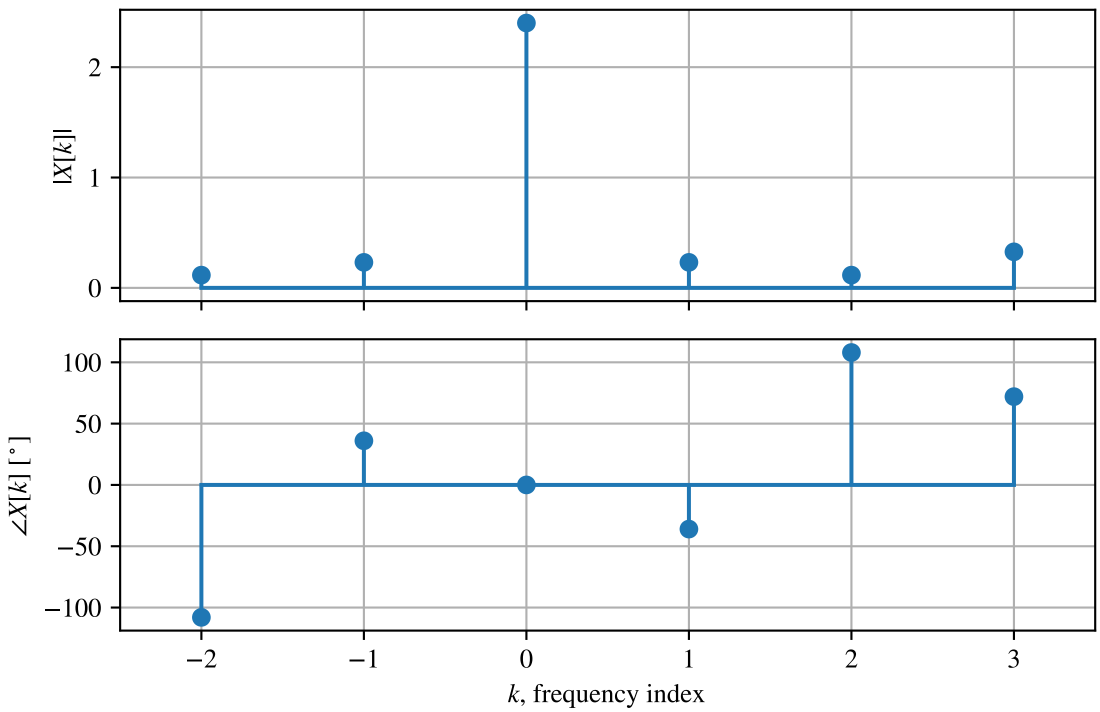

Figure 4: Fourier series coefficients' magnitude and phase for $x(t)$ (Problem 1).

2 Fourier series of triangular-pulse train

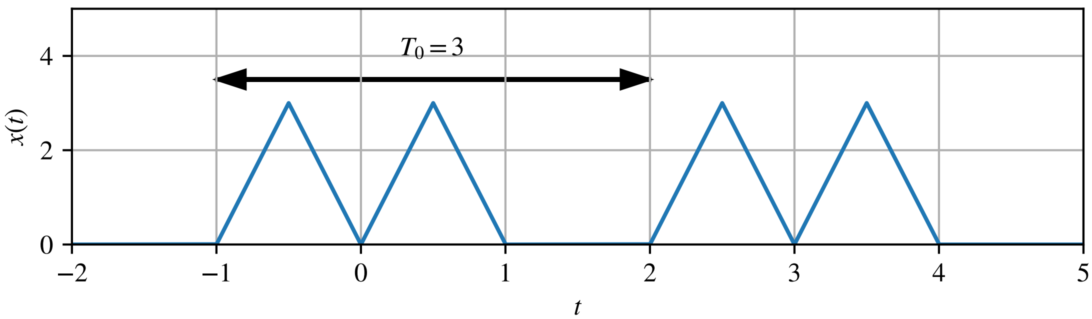

Figure 5: $x(t)$ for problem 2.

Fig. 5 plots the given $x(t)$ and marks off the fundamental period, $T_0 = 3$. Before applying Eq. 2, we need to write $x(t)$ explicitly. The *triangle function* is defined in Eq. (4) and plotted in Fig. 6.

$$\triangle\left(\frac{t}{2t_0}\right) = \begin{cases} \frac{1}{t_0}t + 1 & \left(-t_0 \leq t \leq 0\right) \\ -\frac{1}{t_0}t + 1 & \left(0 \leq t \leq t_0\right) \end{cases} \tag{4}$$

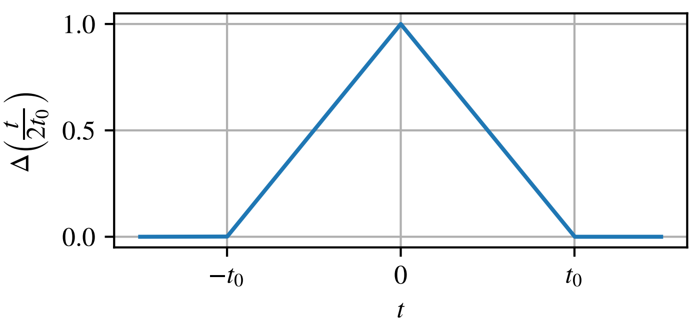

Figure 6: A general triangle function.

From Eq. (4) and Fig. 5, we obtain the following equation for one period of $x(t)$, taken to be over $-1 \leq t < 2$:

$$x(t) = 3\left(\triangle(t + 0.5) + \triangle(t - 0.5)\right)$$

Here, we have $t_0 = 0.5$. Stated without proof[2], the Fourier transform of Eq. (4) is

$$\mathcal{F}\left\{\triangle\left(\frac{t}{2t_0}\right)\right\} = At_0 \operatorname{sinc}^2\left(\pi t_0 f\right) \tag{5}$$

[2] A triangle function can be obtained by convolution of a rectangle function with itself. This corresponds to multiplication of the rectangles' Fourier transforms, which results in the sinc^2 term.

Transforming each term of $x(t)$ and applying Eq. (2) yields

$$\begin{aligned}
X(f) &= (3)(0.5)\,\mathrm{sinc}^2(\pi f)\,e^{-j2\pi f(-0.5)} + 3\,\mathrm{sinc}^2(\pi f)\,e^{-j2\pi f(0.5)} \\
X(k) &= \frac{1}{3}\cdot 3\cdot 0.5\,\mathrm{sinc}^2\left(\pi\frac{k}{T_s}\right)\left[e^{j\pi(k/T_s)} + e^{-j\pi(k/T_s)}\right] \\
&= \boxed{\frac{1}{2}\,\mathrm{sinc}^2\left(\pi\frac{k}{T_s}\right)\left[e^{j\pi(k/T_s)} + e^{-j\pi(k/T_s)}\right]}
\end{aligned}$$

$X(f)$, the frequency response for a single cycle of $x(t)$, is plotted in Fig. 7. The magnitude and phase of the Fourier series coefficients x_k are plotted in Fig. 8.

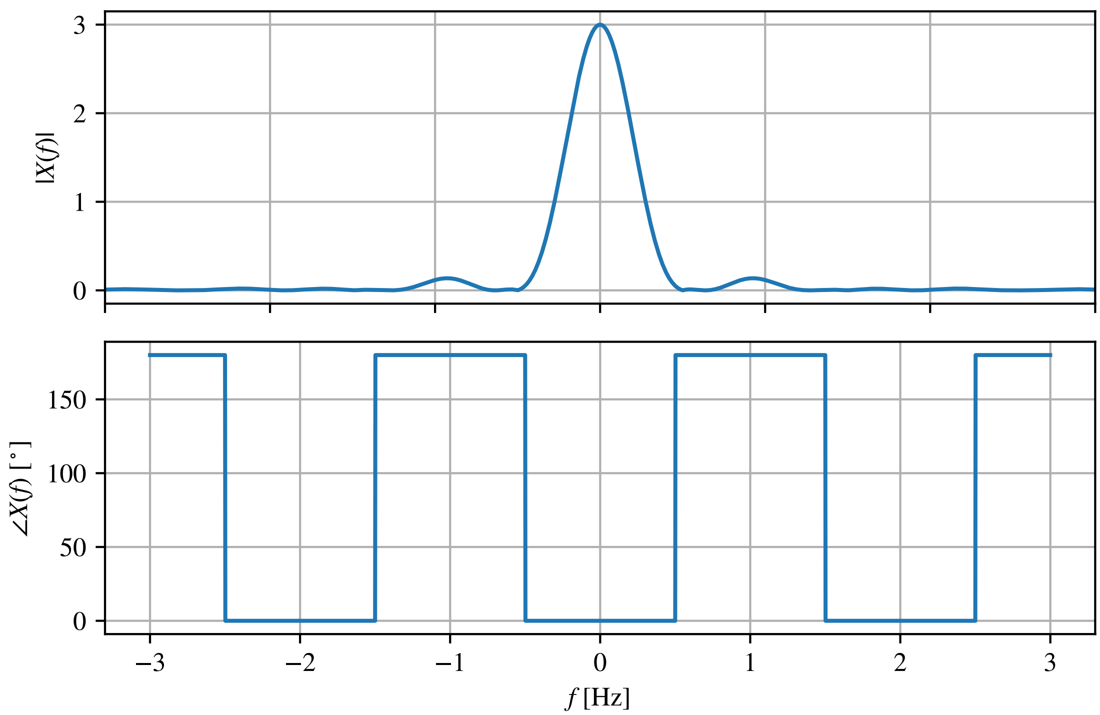

Figure 7: Fourier transform for a single cycle of $x(t)$ (Problem 2).

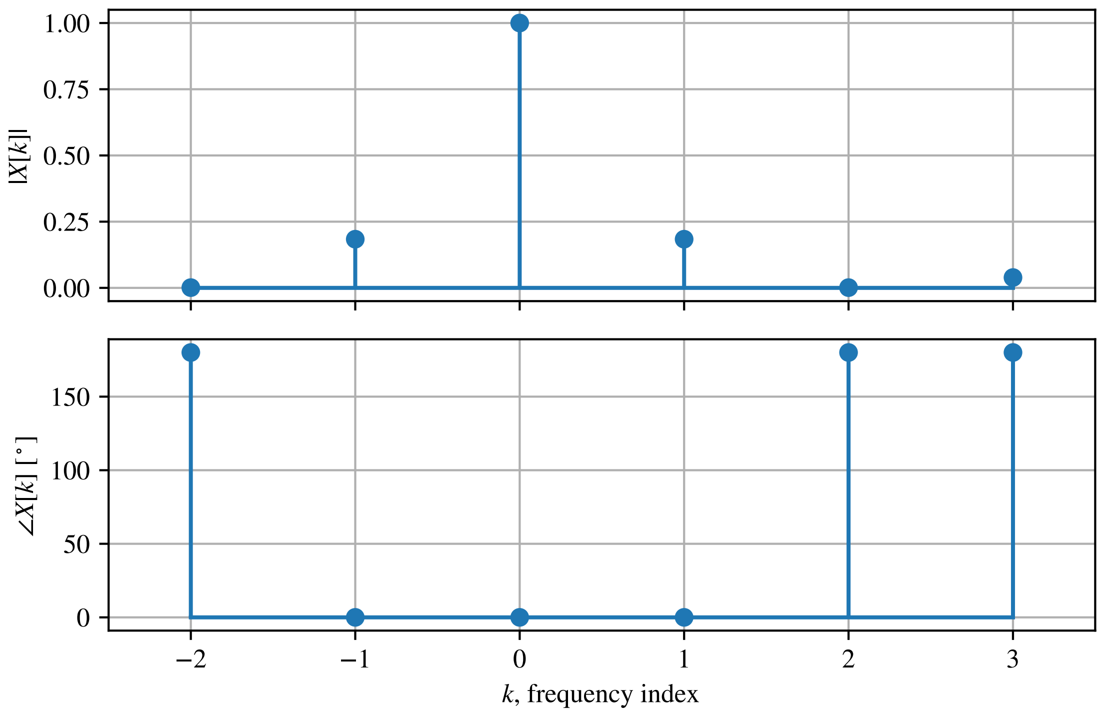

Figure 8: Fourier series coefficients' magnitude and phase for $x(t)$ (Problem 2).