By using transfer function in z domain, **the same result will be achieved more easily**.

EXAMPLE 6.14

Find the difference equation that corresponds to the cascaded sections presented in Figure 6.6, repeated here from Figure 4.19.

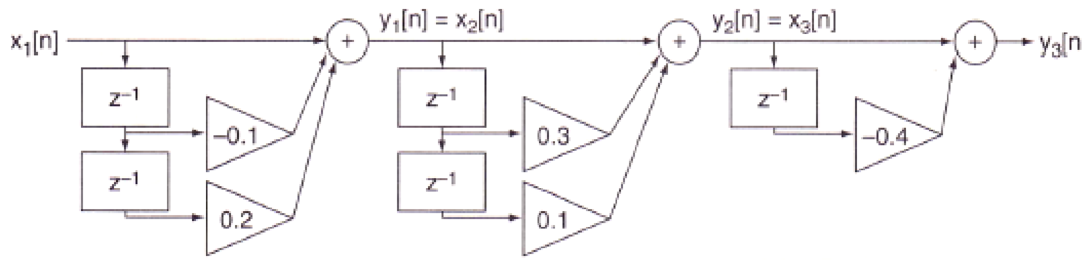


FIGURE 6.6
Difference equation diagram for Example 6.14.

In Example 4.7, the same cascaded system was investigated. The difference equations for each stage were identified as:

$$y_1[n] = x_1[n] - 0.1x_1[n-1] + 0.2x_1[n-2]$$

$$y_2[n] = x_2[n] + 0.3x_2[n-1] + 0.1x_2[n-2]$$

$$y_3[n] = x_3[n] - 0.4x_3[n-1]$$

These equations were combined to produce the difference equation for the cascaded filter. In this example, the same result will be achieved more easily with transfer functions. The transfer functions for the three filters are:

$$H_1(z) = 1 - 0.1z^{-1} + 0.2z^{-2}$$

$$H_2(z) = 1 + 0.3z^{-1} + 0.1z^{-2}$$

$$H_3(z) = 1 - 0.4z^{-1}$$

The overall transfer function is the product

$$\begin{aligned} H(z) &= H_1(z)H_2(z)H_3(z) \\ &= 1 - 0.2z^{-1} + 0.19z^{-2} - 0.058z^{-3} - 0.008z^{-5} \end{aligned}$$

As in Example 4.7, the difference equation is

$$y_3[n] = x_1[n] - 0.2x_1[n-1] + 0.19x_1[n-2] - 0.058x_1[n-3] - 0.008x_1[n-5]$$

Ex 4.7

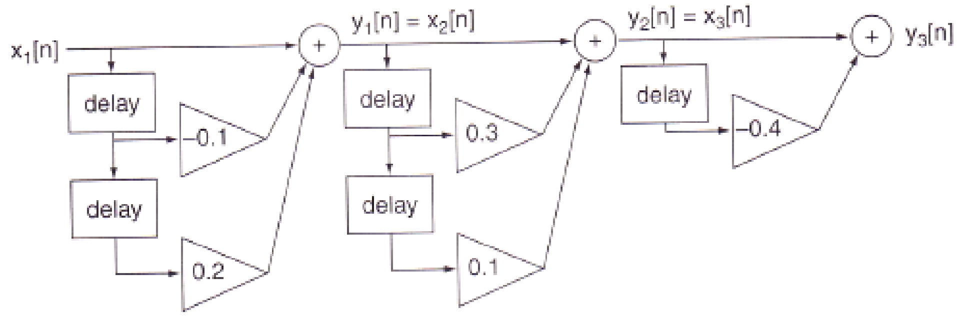


FIGURE 4.19
Difference equation diagram for Example 4.7.

The first stage produces the difference equation

$$y_1[n] = x_1[n] - 0.1x_1[n-1] + 0.2x_1[n-2]$$

The second stage produces the difference equation

$$y_2[n] = x_2[n] + 0.3x_2[n-1] + 0.1x_2[n-2]$$

The third stage produces the difference equation

$$y_3[n] = x_3[n] - 0.4x_3[n-1]$$

Setting the output of the first stage equal to the input of the second stage, or $x_2[n] = y_1[n]$, and setting the output of the second stage equal to the input of the third stage, or $x_3[n] = y_2[n]$, gives the overall output of the cascaded system $y_3[n]$. Beginning with the difference equation for the third stage and substituting the difference equation for the second stage gives:

$$\begin{aligned} y_3[n] &= x_3[n] - 0.4x_3[n-1] = y_2[n] - 0.4y_2[n-1] \\ &= (x_2[n] + 0.3x_2[n-1] + 0.1x_2[n-2]) - 0.4(x_2[n-1] + 0.3x_2[n-2] + 0.1x_2[n-3]) \\ &= x_2[n] - 0.1x_2[n-1] - 0.02x_2[n-2] - 0.04x_2[n-3] \end{aligned}$$

Substituting the difference equation for the first stage into this expression gives:

$$\begin{aligned} y_3[n] &= x_2[n] - 0.1x_2[n-1] - 0.02x_2[n-2] - 0.04x_2[n-3] \\ &= y_1[n] - 0.1y_1[n-1] - 0.02y_1[n-2] - 0.04y_1[n-3] \\ &= (x_1[n] - 0.1x_1[n-1] + 0.2x_1[n-2]) \\ &\quad - 0.1(x_1[n-1] - 0.1x_1[n-2] + 0.2x_1[n-3]) \\ &\quad - 0.02(x_1[n-2] - 0.1x_1[n-3] + 0.2x_1[n-4]) \\ &\quad - 0.04(x_1[n-3] - 0.1x_1[n-4] + 0.2x_1[n-5]) \\ &= x_1[n] - 0.2x_1[n-1] + 0.19x_1[n-2] - 0.058x_1[n-3] - 0.008x_1[n-5] \end{aligned}$$

which is the difference equation for the filter as a whole. This difference equation is much easier to obtain using z transforms, as shown in Example 6.14.

EXAMPLE 6.15

Find the transfer function of the transposed direct form 2 realization of the filter depicted in Figure 6.7.

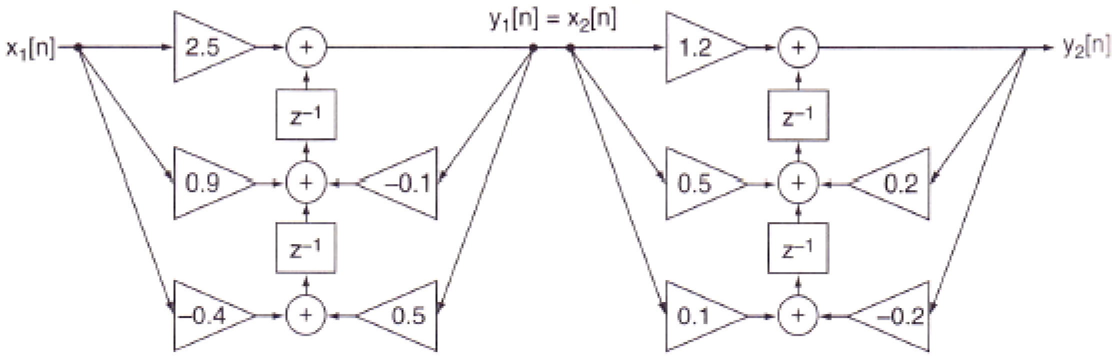


FIGURE 6.7

Difference equation diagram for Example 6.15.

The figure shows a cascade combination of two second order filters. Using the methods of Section 4.6.2.2, the difference equations for the two filters are:

$$y_1[n] = -0.1y_1[n-1] + 0.5y_1[n-2] + 2.5x_1[n] + 0.9x_1[n-1] - 0.4x_1[n-2]$$

$$y_2[n] = 0.2y_2[n-1] - 0.2y_2[n-2] + 1.2x_2[n] + 0.5x_2[n-1] + 0.1x_2[n-2]$$

The difference equation for the filter is most easily found by multiplying the transfer functions for each section:

$$\begin{aligned} H(z) &= H_1(z)H_2(z) \\ &= \left(\frac{2.5 + 0.9z^{-1} - 0.4z^{-2}}{1 + 0.1z^{-1} - 0.5z^{-2}}\right)\left(\frac{1.2 + 0.5z^{-1} + 0.1z^{-2}}{1 - 0.2z^{-1} + 0.2z^{-2}}\right) \\ &= \frac{3 + 2.33z^{-1} + 0.22z^{-2} - 0.11z^{-3} - 0.04z^{-4}}{1 - 0.1z^{-1} + 0.32z^{-2} + 0.12z^{-3} - 0.1z^{-4}} \end{aligned}$$

Finding inverse z transform

1. Simple inverse z transforms by using Table
2. Inverse z transforms by long division
3. Inverse z transforms by partial fraction expansion

<u>Standard form</u>

$$H(z) = \frac{b_0\left(1 + \frac{b_1}{b_0}z^{-1} + \ldots + \frac{b_M}{b_0}z^{-M}\right)}{a_0\left(1 + \frac{a_1}{a_0}z^{-1} + \ldots + \frac{a_N}{a_0}z^{-N}\right)}$$

$$= \frac{K\left(z^N + \frac{b_1}{b_0}z^{N-1} + \ldots + \frac{b_M}{b_0}z^{N-M}\right)}{\left(z^N + \frac{a_1}{a_0}z^{N-1} + \ldots + \frac{a_N}{a_0}\right)} \qquad \textbf{(6.5)}$$

- All exponents of z in the z transform be positive
- Coefficient of the highest power term in both the numerator and the denominator be one

- **Proper rational function** – the degree of the numerator is l**ess than or equal to** the degree of the denominator.
- **Strictly proper rational function** – degree of the numerator is less than the degree of the denominator
- **Improper rational function** – the degree of the numerator is **greater than** the degree of denominator.

EXAMPLE 6.17

Express the following transfer function in standard form:

$$H(z) = \frac{z^{-1}}{4 - 2.5z^{-1} + z^{-2}}$$

The first step in the conversion to standard form is to make the exponents of all delay terms positive. If the term with the most negative exponent is z^{-N}, then every term in the transfer function should be multiplied by z^N to make all exponents positive. In this example, the largest delay is represented by the z^{-2} term in the denominator. Once each term has been multiplied by z^2, the transfer function becomes

$$H(z) = \frac{z}{4z^2 - 2.5z + 1}$$

The second step in the conversion is to ensure that the highest power denominator coefficient is one. For this reason, each term in the transfer function is divided by four to give the final standard form:

$$H(z) = \frac{0.25z}{z^2 - 0.625z + 0.25}$$

EXAMPLE 6.18

Convert the proper transfer function

$$H(z) = \frac{z^2 + 0.1z + 0.3}{z^2 - 0.5z + 0.9}$$

into a strictly proper expression.

Using long division:

$$\begin{array}{r} 1 \\ z^2 - 0.5z + 0.9 \overline{\big) z^2 + 0.1z + 0.3} \\ \underline{z^2 - 0.5z + 0.9} \\ 0.6z - 0.6 \end{array}$$

Thus,

$$H(z) = 1 + \frac{0.6z - 0.6}{z^2 - 0.5z + 0.9}$$

The rational part of the transfer function is strictly proper.

1. Simple inverse z transforms by using Table 6.1

EXAMPLE 6.19

Find the signal $x[n]$ that corresponds to the z transform $X(z) = \dfrac{z}{z - 0.8}$.

Table 6.1 provides the inverse transform:

$$x[n] = \mathbf{Z}^{-1}\{X(z)\} = (0.8)^n u[n]$$

EXAMPLE 6.21

A system's transfer function is

$$H(z) = \frac{z^{-2}}{1 + 0.25z^{-1}}$$

a. Find the difference equation for this system.
b. Find the impulse response for this system.

a. The difference equation for the system is $y[n] + 0.25\, y[n-1] = x[n-2]$.

b. The impulse response for the system is the inverse z transform of its transfer function. The transfer function is manipulated to isolate a transform found in Table 6.1.

$$H(z) = \frac{z^{-2}}{1 + 0.25z^{-1}} = z^{-2}\frac{1}{1 + 0.25z^{-1}} = z^{-2}\frac{z}{z + 0.25}$$

This expression shows that the inverse z transform of $H(z)$ will be delayed by two steps from the inverse transform of the function $z/(z + 0.25)$. The inverse z transform of this function, from Table 6.1, is $(-0.25)^n u[n]$, so the impulse response $h[n]$, after imposing the two-step delay, must be

$$h[n] = (-0.25)^{n-2} u[n-2] \qquad \textbf{(6.6)}$$

ii) Another way of finding the impulse response is

$$\begin{aligned}
H(z) &= \frac{z^{-2}}{1+0.25z^{-1}} \\
&= \frac{z^2}{z^2}\cdot\frac{z^{-2}}{1+0.25z^{-1}} \\
&= \frac{1}{z^2+0.25z} \\
&= \frac{1}{z(z+0.25)} \\
&= \frac{4}{z} + \frac{-4}{z+0.25}
\end{aligned}$$

$$Z^{-1}\{H(z)\} = h[n]$$
$$= 4\delta[n-1] - 4(-0.25)^{n-1} u[n-1]$$

iii) Another impulse response can be written as

$$y[n] = -0.25y[n-1] + x[n-2]$$
$$\Rightarrow h[n] = -0.25h[n-1] + \delta[n-2]$$

EXAMPLE 6.22

The input to a digital filter is $x[n] = u[n]$. The output from the filter is $y[n] = (0.89)^n u[n]$.

a. Find the transfer function of the filter.
b. Find the impulse response of the filter.

a. From Table 6.1, $X(z) = \dfrac{z}{z-1}$ and $Y(z) = \dfrac{z}{z-0.89}$, so the transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z-0.89}\,\frac{z-1}{z} = \frac{z-1}{z-0.89}$$

b. Taking an inverse z transform of the transfer function is the only easy way to find the impulse response from the information given. First $H(z)$ is converted to strictly proper form using long division, and then it is rewritten to isolate a transform found in Table 6.1:

$$H(z) = \frac{z-1}{z-0.89} = 1 - \frac{0.11}{z-0.89} = 1 - z^{-1}\frac{0.11z}{z-0.89}$$

From the table, the inverse transform of 1 is $\delta[n]$ and the inverse transform of $\dfrac{0.11z}{z-0.89}$ is $0.11(0.89)^n u[n]$. The factor z^{-1} causes a one-step delay, so the impulse response is

$$h[n] = \mathbf{Z}^{-1}\{H(z)\} = \delta[n] - 0.11(0.89)^{n-1}u[n-1]$$

EXAMPLE 6.23

Find the time domain signal $x[n]$ that corresponds to the z transform

$$X(z) = \frac{5}{z^2 + 0.2z}$$

This z transform is in standard form. One solution is to factor $X(z)$ and isolate a transform from Table 6.1:

$$X(z) = \frac{5}{z(z + 0.2)} = z^{-2}\left(\frac{5z}{z + 0.2}\right)$$

The factor $5z/(z + 0.2)$ gives an inverse transform $5(-0.2)^n u[n]$. After the shift by two steps due to the factor z^{-2} term, the inverse transform is

$$x[n] = 5(-0.2)^{n-2}u[n-2] \tag{6.7}$$

2. Inverse z transforms by long division

Advantage:

- It is relatively straightforward
- It can be applied to any rational function

Disadvantage:

- In general, it is **<u>not</u>** easy to find a closed form.

EXAMPLE 6.24

Find the inverse z transform of

$$H(z) = \frac{z^2 - 0.1z}{z^2 + 0.4z + 0.8}$$

Using long division,

$$\begin{array}{r l}
 & 1 - 0.5z^{-1} - 0.6z^{-2} + 0.64z^{-3} - \ldots \\
\cline{2-2}
z^2 + 0.4z + 0.8 \,\big)\!\! & z^2 - 0.1z \\
 & \underline{z^2 + 0.4z + 0.8} \\
 & \quad -0.5z - 0.8 \\
 & \quad \underline{-0.5z - 0.2 - 0.4z^{-1}} \\
 & \qquad\qquad -0.6 + 0.4z^{-1} \\
 & \qquad\qquad \underline{-0.6 - 0.24z^{-1} - 0.48z^{-2}} \\
 & \qquad\qquad\qquad 0.64z^{-1} + 0.48z^{-2} \\
 & \qquad\qquad\qquad \underline{0.64z^{-1} + 0.256z^{-2} + 0.512z^{-3}} \\
 & \qquad\qquad\qquad\qquad\quad 0.224z^{-2} - 0.512z^{-3} \\
 & \qquad\qquad\qquad\qquad\qquad \ldots
\end{array}$$

So,

$$H(z) = 1 - 0.5z^{-1} - 0.6z^{-2} + 0.64z^{-3} - \ldots$$

Inverse transforming term by term gives the impulse response

$$h[n] = \delta[n] - 0.5\delta[n-1] - 0.6\delta[n-2] + 0.64\delta[n-3] - \ldots$$

EXAMPLE 6.25

Using long division, find the time domain signal $x[n]$ that corresponds to the z transform of Example 6.23, that is,

$$X(z) = \frac{5}{z^2 + 0.2z}$$

Long division gives:

$$
\begin{array}{r l}
 & 5z^{-2} - z^{-3} + 0.2z^{-4} - 0.04z^{-5} + \ldots \\
\cline{2-2}
z^2 + 0.2z & \big)\, 5 \\
 & \underline{5 + z^{-1}} \\
 & \quad -z^{-1} \\
 & \quad \underline{-z^{-1} - 0.2z^{-2}} \\
 & \qquad\qquad 0.2z^{-2} \\
 & \qquad\qquad \underline{0.2z^{-2} + 0.04z^{-3}} \\
 & \qquad\qquad\qquad\quad -0.04z^{-3} \\
 & \qquad\qquad\qquad\quad \underline{-0.04z^{-3} - 0.008z^{-4}} \\
 & \qquad\qquad\qquad\qquad\qquad\quad 0.008z^{-4} \\
 & \qquad\qquad\qquad\qquad\qquad\quad \ldots
\end{array}
$$

This means that $X(z) = 5z^{-2} - z^{-3} + 0.2z^{-4} - 0.04z^{-5} + \ldots$ It is now easy to find $x[n]$ by taking an inverse z transform of each term:

$$x[n] = 5\delta[n-2] - \delta[n-3] + 0.2\delta[n-4] - 0.04\delta[n-5] + \ldots$$

In this particular case, it so happens that the pattern that defines $x[n]$ can be identified, so a solution in closed form can be found:

$$x[n] = \sum_{k=0}^{\infty} 5(-0.2)^k \delta[n - 2 - k] \tag{6.8}$$

3. Inverse z transforms by partial fraction expansion

$$X(z) = \mathbf{Z}\{u[n-1]\} = z^{-1}\frac{z}{z-1} = \frac{1}{z-1}$$

The z transform of the impulse response is the transfer function of the system. From Table 6.1, the transfer function is

$$H(z) = \frac{z}{z+0.25}$$

Then, according to Equation (6.4),

$$Y(z) = H(z)X(z) = \frac{z}{(z+0.25)(z-1)} \tag{6.9}$$

This z transform is strictly proper, but is not one of the basic transforms from Table 6.1. The purpose of partial fraction expansion is to decompose it into terms that are in Table 6.1, one for each factor in the denominator. The partial fraction expansion has the form

$$Y(z) = \frac{z}{(z+0.25)(z-1)} = \frac{A}{z+0.25} + \frac{B}{z-1}$$

The coefficients A and B may be found by what is sometimes called the **cover-up method**.[1] The method requires the roots of the denominator, -0.25 and 1 in this case. The value for

$$A = \frac{-0.25}{-0.25-1} = 0.2$$

and

$$B = \frac{1}{1+0.25} = 0.8$$

Therefore,

$$Y(z) = \frac{0.2}{z+0.25} + \frac{0.8}{z-1} \tag{6.10}$$

As a check, the terms of $Y(z)$ may be recombined with a common denominator to verify that the transfer function matches Equation (6.9). To use the basic transforms in Table 6.1, $Y(z)$ can be rewritten as

$$Y(z) = z^{-1}\left(\frac{0.2z}{z+0.25} + \frac{0.8z}{z-1}\right)$$

This equation is mathematically identical with Equation (6.10). The portion within the brackets has the inverse transform $0.2(-0.25)^n u[n] + 0.8u[n]$. The z^{-1} term outside the brackets simply indicates a time shift by one step, producing the final inverse transform

$$y[n] = 0.2(-0.25)^{n-1}u[n-1] + 0.8u[n-1]$$

EXAMPLE 6.26

Find the inverse z transform of

$$Y(z) = \frac{0.5}{z(z-1)(z-0.6)}$$

The denominator is already factored into simple factors. The partial fraction expansion of $Y(z)$ has three terms, one for each of the roots in the denominator:

$$Y(z) = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-0.6}$$

Covering up the z term in the denominator and evaluating $Y(z)$ at $z = 0$,

$$A = \frac{0.5}{(0-1)(0-0.6)} = \frac{5}{6}$$

Covering up the $(z-1)$ term in the denominator and evaluating at $z = 1$,

$$B = \frac{0.5}{(1)(1-0.6)} = \frac{5}{4}$$

Covering up the $(z-0.6)$ term and evaluating at $z = 0.6$,

$$C = \frac{0.5}{(0.6)(0.6-1)} = -\frac{25}{12}$$

Hence,

$$Y(z) = \frac{\frac{5}{6}}{z} + \frac{\frac{5}{4}}{z-1} + \frac{-\frac{25}{12}}{z-0.6} = z^{-1}\left(\frac{5}{6} + \frac{\frac{5}{4}z}{z-1} + \frac{-\frac{25}{12}z}{z-0.6}\right)$$

The inverse z transform is, using Table 6.1,

$$y[n] = \frac{5}{6}\delta[n-1] + \frac{5}{4}u[n-1] - \frac{25}{12}(0.6)^{n-1}u[n-1]$$

Partial Fraction Expansion of Repeated Roots by Differentiation

Singly Repeated Roots

Consider the case in which one of the roots is repeated:

$$F(s)=\frac{N(s)}{(s+a)^2 D'(s)}$$

In this fraction the denominator polynomial has a repeated root at s=-a. The remainder of the denominator polynomial is called D'(s); it has no roots as s=-a. The numerator polynomial is N(s). If we expand this fraction we get

$$F(s)=\frac{N(s)}{(s+a)^2 D'(s)}=\frac{A_1}{(s+a)^2}+\frac{A_2}{(s+a)}+\frac{N'(s)}{D'(s)}$$

The term N'(s)/D'(s) represents the expansion of all of the terms except those with roots at s=-a.

We can find A_1 by multiplying by $(s+a)^2$ and setting s=-a (i.e., the cover-up method).

$$(s+a)^2 F(s)=A_1+(s+a)A_2+(s+a)^2\frac{N'(s)}{D'(s)}$$

$$A_1=(s+a)^2 F(s)\Big|_{s=-a}$$

To find A_2 we note that the we can get rid of the A_1 term by differentiating the result above

$$\frac{d}{ds}\left\{(s+a)^2 F(s)\right\}=A_2+\left[2(s+a)\frac{N'(s)}{D'(s)}+(s+a)^2\frac{d}{ds}\frac{N'(s)}{D'(s)}\right]$$

Now we can solve for A_2 by setting s=-a.

$$\left[\frac{d}{ds}\left\{(s+a)^2 F(s)\right\}\right]\Bigg|_{s=-a}=A_2$$

Extension to multiply repeated roots.

Now consider the case of multiply repeated roots (n>2).

$$F(s)=\frac{N(s)}{(s+a)^n D'(s)}$$

The resulting partial fraction expansion is

$$F(s)=\frac{N(s)}{(s+a)^n D'(s)}=\frac{A_1}{(s+a)^n}+\frac{A_2}{(s+a)^{n-1}}+\cdots+\frac{A_n}{(s+a)}+\frac{N'(s)}{D'(s)}$$

As before we can easily find A_1 and A_2

$$A_1=(s+a)^n F(s)\Big|_{s=-a}$$

$$A_2=\left[\frac{d}{ds}\left\{(s+a)^n F(s)\right\}\right]\Bigg|_{s=-a}$$

If we continue to differentiate we can find the other coefficients in a similar manner

$$A_k=\frac{1}{k!}\left[\frac{d^k}{ds^k}\left\{(s+a)^n F(s)\right\}\right]\Bigg|_{s=-a}$$

http://www.swarthmore.edu/NatSci/echeeve1/Ref/LPSA/BackGround/PartialFraction/RootsRepeat.html

EXAMPLE 6.27

Find the impulse response for the system

$$H(z) = \frac{z^{-2}}{1 + 0.25z^{-1}}$$

The impulse response for this transfer function was computed in Example 6.21. In this example, partial fraction expansion is used to accomplish the same goal. Changing to standard form, the transfer function becomes:

$$H(z) = \frac{z^{-2}}{1 + 0.25z^{-1}} = \frac{1}{z^2 + 0.25z}$$

Its partial fraction expansion is

$$H(z) = \frac{1}{z(z + 0.25)} = \frac{A}{z} + \frac{B}{z + 0.25} = \frac{4}{z} + \frac{-4}{z + 0.25} = z^{-1}\left(4 - \frac{4z}{z + 0.25}\right)$$

The portion within the brackets gives the inverse transform $4\delta[n] - 4(-0.25)^n u[n]$, so the final inverse transform is

$$h[n] = 4\delta[n-1] - 4(-0.25)^{n-1}u[n-1] \tag{6.11}$$

While Equations (6.6) and (6.11) appear very different, they are identical functions for all values of n.

EXAMPLE 6.28

Find the time domain signal $x[n]$ that corresponds to the z transform

$$X(z) = \frac{5}{z^2 + 0.2z}$$

This example is a repetition of Examples 6.23 and 6.25. This time, partial fraction expansion will be used to solve the problem. The denominator of $X(z)$ can be factored to give

$$X(z) = \frac{5}{z(z + 0.2)} \tag{6.12}$$

The partial fraction expansion is

$$X(z) = \frac{A}{z} + \frac{B}{z + 0.2} = \frac{25}{z} + \frac{-25}{z + 0.2} = z^{-1}\left(25 - 25\frac{z}{z + 0.2}\right)$$

Thus, the final inverse transform is

$$x[n] = 25\delta[n-1] - 25(-0.2)^{n-1}u[n-1] \tag{6.13}$$

Though this function looks completely different from Equations (6.7) and (6.8), it produces exactly the same values as those shown in Figure 6.9.

Transfer function and stability

- Poles are the values of z that make the denominator of a transfer function zero.

- Zeros are the values of z that make the numerator of a transfer function zero.

- **Of the two, poles have the biggest effect on the behavior of a digital filter.**

- In general, the numerator and denominator polynomials of a transfer function in standard form can always be factored.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-m}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

$$H(z) = \frac{K(z - z_1)(z - z_2)\dots(z - z_M)}{(z - p_1)(z - p_2)\dots(z - p_N)}$$

z_j : zeros of the filter
p_j : poles of the filter
K : Gain of the filter

EXAMPLE 6.31

Find and plot the poles and zeros for the transfer function

$$H(z) = \frac{1}{0.8 + 0.23z^{-1} + 0.15z^{-2}}$$

To find the poles and zeros, convert $H(z)$ to standard form:

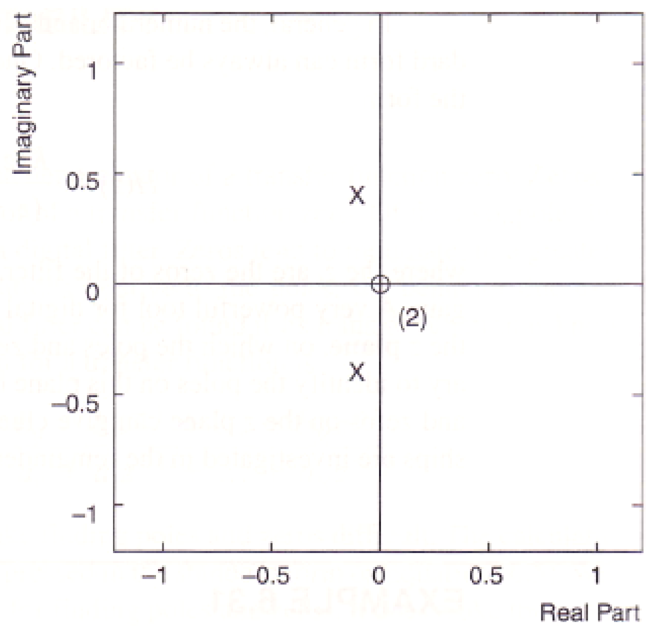
$$H(z) = \frac{z^2}{0.8z^2 + 0.23z + 0.15} = \frac{1.25z^2}{z^2 + 0.2875z + 0.1875}$$

The zeros are located where $z^2 = 0$. In other words, there are two zeros, both located at $z = 0$. The pole locations can be found using the quadratic formula:

$$z = \frac{-0.2875 \pm \sqrt{0.2875^2 - 4(1)(0.1875)}}{2(1)}$$

$$= \frac{-0.2875 \pm j0.8169}{2} = -0.1438 \pm j0.4085$$

FIGURE 6.12
Poles and zeros for Example 6.31.



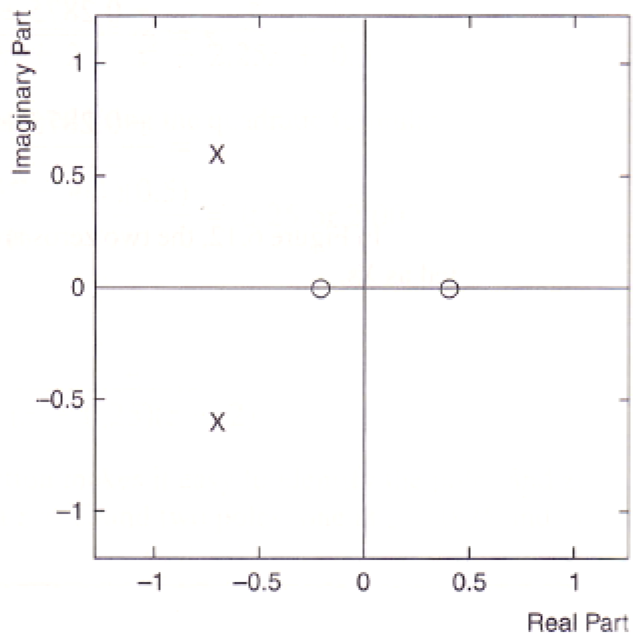
EXAMPLE 6.32

A digital filter has zeros at -0.2 and 0.4, poles at $-0.7 \pm j0.6$, and a gain of 0.5.

a. Draw a pole-zero plot for the filter.
b. Find the transfer function of the filter.

a. The pole-zero plot is shown in Figure 6.13.
b. Each zero produces a factor in the numerator of the transfer function, and each pole produces a factor in the denominator. The transfer function is

FIGURE 6.13
Pole-zero plot for Example 6.32.



$$H(z) = \frac{K(z - z_1)(z - z_2) \ldots (z - z_M)}{(z - p_1)(z - p_2) \ldots (z - p_N)}$$

$$= \frac{0.5(z - (-0.2))(z - 0.4)}{(z - (-0.7 + j0.6))(z - (-0.7 - j0.6))}$$

Simplifying, the transfer function becomes

$$H(z) = \frac{0.5(z + 0.2)(z - 0.4)}{(z + 0.7)^2 + 0.6^2} = \frac{0.5(z^2 - 0.2z - 0.08)}{z^2 + 1.4z + 0.85}$$

$$= \frac{0.5 - 0.1z^{-1} - 0.04z^{-2}}{1 + 1.4z^{-1} + 0.85z^{-2}}$$

Stability

- All useful filters are stable.
- Important aspect of filter design is to guarantee stability.
- A filter is **stable** if all its poles are inside the unit circle.
- A filter with poles on the unit circle is said to be **marginally stable**.
- A filter with poles outside the unit circle is **unstable**.

EXAMPLE 6.33

The transfer function for a digital filter is

$$H(z) = \frac{1 - z^{-2}}{1 + 0.7z^{-1} + 0.9z^{-2}}$$

Is the filter stable?

In standard form,

$$H(z) = \frac{z^2 - 1}{z^2 + 0.7z + 0.9}$$

The zeros are located where $z^2 - 1 = 0$, or $z = \pm 1$. The poles are located at

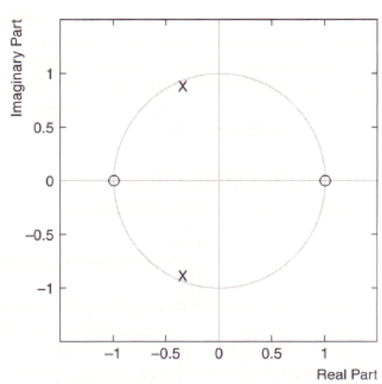
$$z = \frac{-0.7 \pm \sqrt{0.7^2 - 4(1)(0.9)}}{2(1)} = \frac{-0.7 \pm j\sqrt{3.11}}{2} = -0.35 \pm j0.8818$$

For these poles, the distance from the center of the unit circle is

$$|z| = \sqrt{(-0.35)^2 + (0.8818)^2} = 0.9487$$

Because the distance is less than one, as illustrated in Figure 6.16, both poles are within the unit circle and the system is stable.

FIGURE 6.16
Pole-zero plot for Example 6.33.



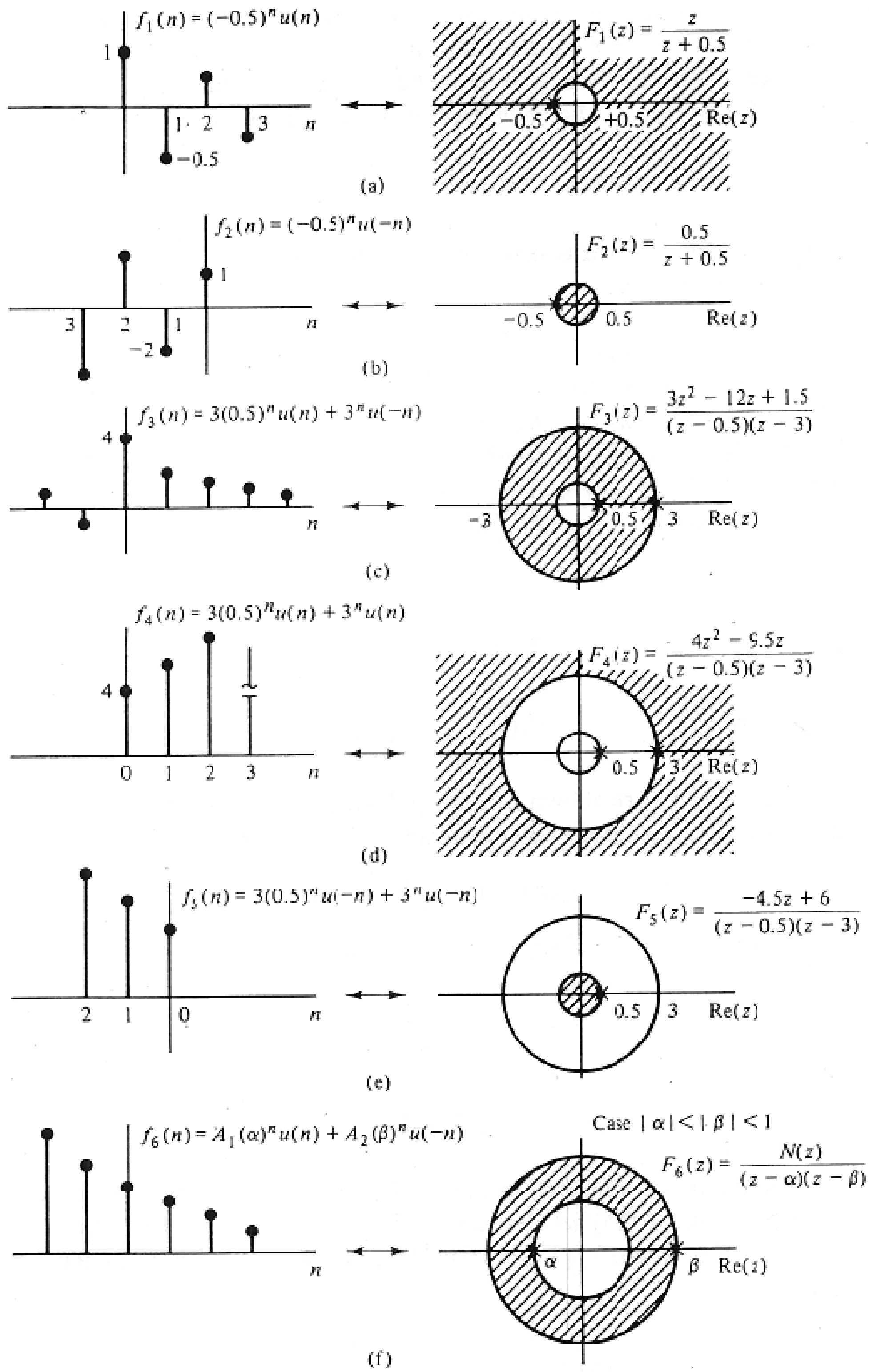


Figure 7-1 The discrete time functions of Example 7-1 and their two-sided Z transforms.

EXAMPLE 6.34

The difference equation for a filter is

$$y[n] + 0.8y[n-1] - 0.9y[n-2] = x[n-2]$$

Is the filter stable?

Poles are found most easily from the transfer function,

$$H(z) = \frac{z^{-2}}{1 + 0.8z^{-1} - 0.9z^{-2}} = \frac{1}{z^2 + 0.8z - 0.9}$$

The quadratic formula gives the pole locations as

$$z = \frac{-0.8 \pm \sqrt{0.8^2 - 4(1)(-0.9)}}{2(1)} = \frac{-0.8 \pm 2.059}{2} = 0.630 \text{ and } -1.430$$

The poles in this case are purely real, without any imaginary component. Clearly the pole at $z = -1.430$ lies outside the unit circle, so the system is unstable.

EXAMPLE 6.35

A filter has the transfer function $H(z) = \dfrac{2}{1 + 0.4z^{-1}}$.

a. Find the pole-zero plot for the filter. Is the filter stable?
b. Find the impulse response for the filter.
c. Find the step response for the filter.

a. To find the poles and zeros, $H(z)$ is first expressed in standard form:

$$H(z) = \frac{2}{1 + 0.4z^{-1}} = \frac{2z}{z + 0.4}$$

There is a single zero at $z = 0$ and a single pole at $z = -0.4$. These points are plotted in Figure 6.18. Since the pole is within the unit circle, the filter is stable.

b. The impulse input has a z transform of $X(z) = 1$. The output $Y(z)$ is

$$Y(z) = H(z)X(z) = \frac{2z}{z + 0.4}$$

The inverse z transform, from Table 6.1, is

$$y[n] = h[n] = 2(-0.4)^n u[n]$$

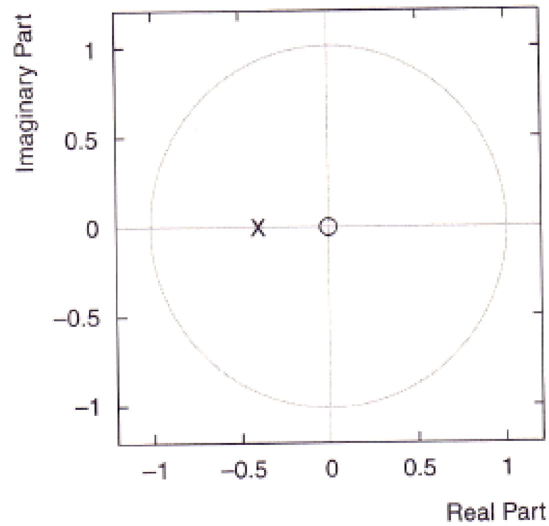


FIGURE 6.18
Pole-zero plot for Example 6.35.

This impulse response is plotted in Figure 6.19.

c. The step input has a z transform of $X(z) = z/(z-1)$. The output is

$$Y(z) = H(z)X(z) = \frac{2z}{z+0.4}\frac{z}{z-1}$$

$$= 2z^2\left(\frac{1}{(z+0.4)(z-1)}\right) = 2z^2\left(\frac{A}{z+0.4} + \frac{B}{z-1}\right)$$

$$= 2z^2\left(\frac{-\frac{5}{7}}{z+0.4} + \frac{\frac{5}{7}}{z-1}\right) = z\left(\frac{-\frac{10}{7}z}{z+0.4} + \frac{\frac{10}{7}z}{z-1}\right)$$

Thus, the step response is

$$y[n] = s[n] = -\frac{10}{7}(-0.4)^{n+1}u[n+1] + \frac{10}{7}u[n+1]$$

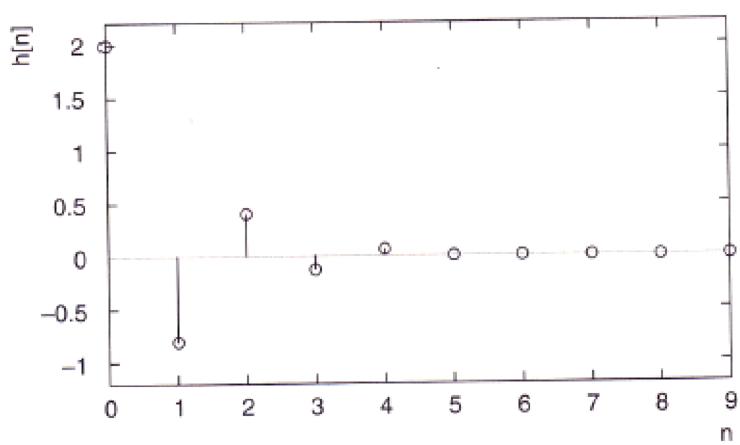


FIGURE 6.19
Impulse response for Example 6.35.

EXAMPLE 6.36

The transfer function of a filter (similar to the one in Example 6.35) is:

$$H(z) = \frac{2}{1 - 0.4z^{-1}}$$

a. Determine the difference equation of the filter.
b. Find the pole-zero plot and evaluate stability.
c. Find and plot the impulse response.

a. The difference equation is $y[n] - 0.4y[n-1] = 2x[n]$.
b. The poles and zeros are found from

$$H(z) = \frac{2}{1 - 0.4z^{-1}} = \frac{2z}{z - 0.4}$$

There is a single zero at $z = 0$ and a single pole at $z = 0.4$, plotted in Figure 6.21. The pole is within the unit circle, so the filter is stable.

c. The impulse response, $h[n] = 2(0.4)^n u[n]$, is the inverse z transform of the transfer function $H(z)$ and is plotted in Figure 6.22.

FIGURE 6.21
Pole-zero plot for Example 6.36.

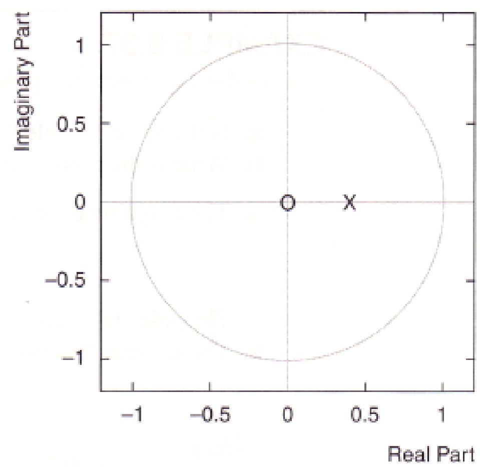


FIGURE 6.22
Impulse response for Example 6.36.