CH6: Random Vectors

$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \in \mathbb{R}^n$ (RVs) $\rightsquigarrow$ joint pmf $\boxed{p_{X_1, X_2, \ldots X_n}(x_1, x_2, \ldots x_n) = P[X_1 = x_1, \ldots X_n = x_n]}$

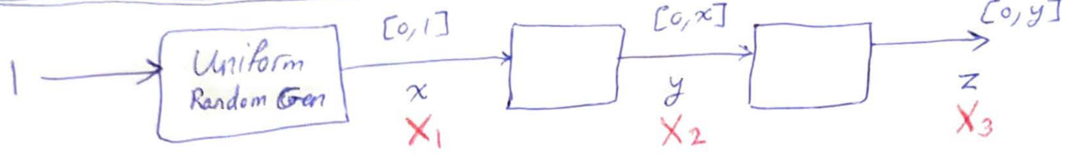
$p_{X_1, X_2}(x_1, x_2) = P[X_1 = x_1, X_2 = x_2]$

$$\boxed{P_{\underline{X}}(\underline{x}) = P[\underline{X} = \underline{x}]}$$

joint cdf

$$\boxed{F_{\underline{X}}(\underline{x}) = P[X_1 \leq x_1, \ldots X_n \leq x_n]} \rightsquigarrow \text{joint pdf } \boxed{f_{\underline{X}}(\underline{x}) = \frac{\partial^n F_{\underline{X}}(\underline{x})}{\partial x_1 \ldots \partial x_n}}$$

ex



$f_Z(z) = ?$

$$f_{X_1 X_2 X_3}(x_1, x_2, x_3) = f_{X_3}(x_3 | x_2, x_1)\, f_{X_2}(x_2 | x_1)\, f_{X_1}(x_1)$$

$$\therefore f_{XYZ}(x, y, z) = \underbrace{f_Z(z | y, x)}_{1/y}\ \underbrace{f_Y(y | x)}_{1/x}\ \underbrace{f_X(x)}_{1} = \frac{1}{xy}$$

$0 < z < y \qquad 0 < y < x \qquad 0 < x < 1$

int. by x

$$f_{YZ}(y, z) = \int f_{XYZ}(x, y, z)\, dx$$

$$= \int_y^1 \frac{1}{xy}\, dx = \frac{1}{y}\left[\ln(x)\right]_y^1 = -\frac{1}{y} \ln y \qquad 0 < z < y < 1$$

$0 < x < 1$ & $y < x$

int. by y

$$f_Z(z) = \int f_{YZ}(y, z)\, dy$$

$$= \int_z^1 -\frac{1}{y} \ln(y)\, dy = \int_z^1 -\ln(y)\, d(\ln(y)) = -\frac{1}{2}\left[(\ln(y))^2\right]_z^1 = \frac{1}{2}(\ln(z))^2 \qquad 0 < z < 1$$

$0 < y < 1$ & $z < y$

★ Independent RVs: $\boxed{f_{\underline{X}}(\underline{x}) = f_{X_1}(x_1)\, f_{X_2}(x_2) \ldots f_{X_n}(x_n)}$

★ Independent Identical Distribution (iid): $X_1, X_2, X_3 \ldots X_n$ are ① independent & ② have same distribution.

$\left.\begin{array}{l} f_{X_i}(x_i) \\ F_{X_i}(x_i) \end{array}\right\}$ marginal $\qquad f_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots x_n) = f_{X_1}(x_1)\, f_{X_2}(x_2) \ldots f_{X_n}(x_n)$

RV① $Z = Max(X_1, X_2, X_3, \ldots X_n)$

RV② $W = Min(X_1, X_2, X_3, \ldots X_n)$

$\Downarrow$

① $F_Z(z) = P[Z \leq z]$

$= P[\mathrm{Max}[X_1 \dots X_n] \leq z]$

$= P[X_1 \leq z, X_2 \leq z, \dots X_n \leq z]$ independent

$= P[X_1 \leq z] \cdot P[X_2 \leq z] \cdot \dots P[X_n \leq z]$

$= F_{X_1}(z) \cdot F_{X_2}(z) \cdot \dots F_{X_n}(z)$ identical

$\therefore \boxed{F_Z(z) = [F_X(z)]^n}$

$\xrightarrow{d}$ $\therefore \boxed{f_Z(z) = n\,[F_X(z)]^{n-1} f_X(z)}$

② $F_W(\omega) = P[W \leq \omega]$

$1 - F_W(\omega) = P[\mathrm{Min}[X_1 \dots X_n] \geq \omega]$

$= P[X_1 \geq \omega] \cdot P[X_2 \geq \omega] \dots P[X_n \geq \omega]$

$\therefore \boxed{1 - F_W(\omega) = (1 - F_X(\omega))^n}$

$\xrightarrow{d}$ $\boxed{+f_W(\omega) = n\,(1 - F_X(\omega))^n (+f_X(\omega))}$

★ **Mean Vector:** $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}_{n\times 1} \longrightarrow \boxed{\underline{m_X} = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{bmatrix} = E(\underline{X})}$

★ **Correlation Matrix:** $\underset{n\times n}{R_X} = E(\underset{n\times 1}{\underline{X}}\ \underset{1\times n}{\underline{X}^T}) = \begin{bmatrix} E(X_1^2) & E(X_1 X_2) & \cdots & E(X_1 X_n) \\ E(X_2 X_1) & E(X_2^2) & & \\ \vdots & & \ddots & \\ E(X_n X_1) & & & E(X_n^2) \end{bmatrix}$ → Symmetric Matrix

★ **Covariance Matrix:** $\boxed{K_X = E\left((\underline{X} - \underline{m_X})(\underline{X} - \underline{m_X})^T\right)} = E(\underline{X}\,\underline{X}^T) - m_X m_X^T - \cancel{m_X m_X^T} + \cancel{m_X m_X^T}$

$\sigma^2 = E(X^2) - m_X^2$

$K_X = R_X - m_X m_X^T = \begin{bmatrix} E(X_1 - m_1)^2 & E((X_1 - m_1)(X_2 - m_2)) & \cdots & E((X_1 - m_1)(X_n - m_n)) \\ E((X_1 - m_1)(X_2 - m_2)) & E(X_2 - m_2)^2 & & \\ \vdots & & \ddots & \\ E((X_1 - m_1)(X_n - m_n)) & & & \sigma_n^2 \end{bmatrix}$ → Symmetric Matrix

↳ If $\underline{X}$ uncorrelated, K_X will be diagonal matrix (all elements = 0 except for the diagonal)

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* $\underset{n\times 1}{\underline{Y}} = \underset{n\times n}{A}\ \underset{n\times 1}{\underline{X}} \longrightarrow$

$y_1 = a_{11} X_1 + a_{12} X_2 + \dots + a_{1n} X_n$

$y_2 = a_{21} X_1 + a_{22} X_2 + \dots + a_{2n} X_n$

$\vdots$

$E[Y_1] = a_{11} E[X_1] + a_{12} E[X_2] + \dots + a_{1n} E[X_n]$

$Y = aX$

$m_Y = a\,m_X$

$\sigma_Y^2 = a^2 \sigma_X^2$

⇒ $\underline{E[Y]} = A\ \underline{E[X]}$

$\boxed{\underline{m_Y} = A\ \underline{m_X}}$

⇒ $K_Y = E[(\underline{Y} - \underline{m_Y})(\underline{Y} - \underline{m_Y})^T]$

$= E[A(\underline{X} - \underline{m_X})(\underline{X} - \underline{m_X})^T A^T]$

$= A\ E[(\underline{X} - \underline{m_X})(\underline{X} - \underline{m_X})^T]\ A^T$

$[A(\underline{X} - \underline{m_X})]^T = (\underline{X} - \underline{m_X})^T A^T$

$\therefore \boxed{K_Y = A\, K_X\, A^T}$

ex $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\underline{m_x} = 0$, $K_x = \begin{bmatrix} 1 & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\underline{Y} = A\,\underline{X}$$

$$\underline{m_Y} = A\,\underline{m_x} = 0$$

$$K_Y = A K_x A^T = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C}$$

$$C = \begin{bmatrix} \frac{1}{\sqrt{2}} - \frac{1}{2} & \frac{1}{\sqrt{2}} + \frac{1}{2} & 0 \\ \frac{1}{\sqrt{2}} - \frac{1}{2} & \frac{-1}{\sqrt{2}} - \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore K_Y = \begin{bmatrix} 1-\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 1+\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Transformation from X (uncorrelated RV) to Y (correlated RV)

★ $\underline{X}$ jointly Gaussian

$$f_{\underline{X}}(\underline{x}) = \frac{\exp\left\{-\frac{1}{2}\underbrace{(\underline{x}-\underline{m_x})^T}_{1\times n}\underbrace{K^{-1}}_{n\times n}\underbrace{(\underline{x}-\underline{m_x})}_{n\times 1}\right\}}{(2\pi)^{n/2}\,(\det K)^{1/2}}$$

$1\times 1 \rightarrow$ scalar

$\underline{m_x}$ & $K \rightarrow$ parameters

Assume $\underline{X}$ is uncorrelated → so independent also

$$K = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \cdots & 0 \\ \vdots & \sigma_2^2 & & & \\ & & \ddots & & \\ 0 & & & & \sigma_n^2 \end{bmatrix} \Rightarrow \therefore K^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & & \cdots & 0 \\ \vdots & \frac{1}{\sigma_2^2} & & \vdots \\ & & \ddots & \\ 0 & \cdots & & \frac{1}{\sigma_n^2} \end{bmatrix}$$

General:
Independence $\Rightarrow$ Uncorrelated
$\Leftarrow$ not necessarily

Gaussian:
Independence $\Longleftrightarrow$ Uncorrelated

$$\therefore (\det K)^{\frac{1}{2}} = \sigma_1 \sigma_2 \sigma_3 \cdots \sigma_n$$

$$\therefore (\underline{x}-\underline{m_x})^T K^{-1} (\underline{x}-m_x) = (\underline{x}-\underline{m_x})^T \begin{bmatrix} \frac{x_1 - m_{x_1}}{\sigma_1^2} \\ \vdots \\ \frac{x_n - m_{x_n}}{\sigma_n^2} \end{bmatrix} = \sum_{i=1}^{n} \frac{(x_i - m_i)^2}{\sigma_i^2}$$

$$\therefore e^{\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-m_i)^2}{\sigma_i^2}} = e^{(x_1-m_1)^2/2\sigma_1^2} \cdots\cdots e^{(x_n-m_n)^2/2\sigma_n^2} = \prod_{i=1}^{n} e^{(x_i-m_i)^2/2\sigma_i^2}$$

$$\therefore f_{\underline{X}}(\underline{x}) = \frac{\prod_{i=1}^{n} e^{(x_i-m_i)^2/2\sigma_i^2}}{(2\pi)^{n/2}\sqrt{\sigma_1\sigma_2\cdots\sigma_n}} = f_{X_1}(x_1)\, f_{X_2}(x_2)\, f_{X_3}(x_3) \cdots\cdots f_{X_n}(x_n)$$

∴ if X_is are uncorrelated, they are also independent

CH9: Random Processes

- Random Process $\Rightarrow X(t, \eta)$ $\eta \in S$, $-\infty < t < \infty$ $\rightsquigarrow X(t_m, \eta)$: RV

  (t: function of time; η: Randomness)

	Cont. time	Discrete time
Cont R.P.		
Discrete R.P.		

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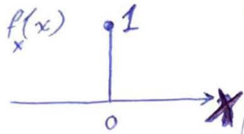
ex η uniform $[-1, 1]$

$$\boxed{X(t, \eta) = \eta \cos(2\pi t)}$$

→ for $\cos(2\pi t_0) = 0$

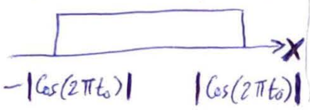
$X(t_0, \eta) = 0$ "not random

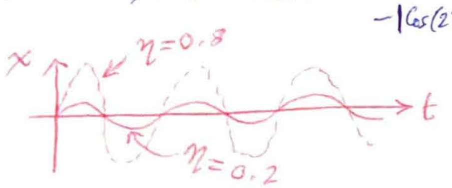


∴ @certain t values, X is not random.

→ for $\cos(2\pi t_0) \neq 0$ $f_X(X(t_0, \eta)) = \frac{1}{2|\cos(2\pi t_0)|}$

∴ @other t's, X is uniform

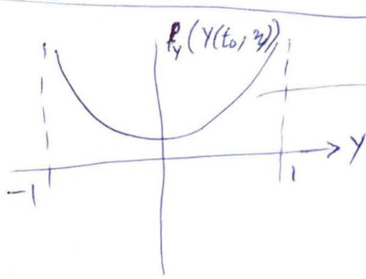




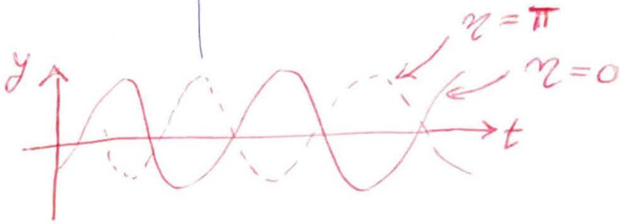
η Uniform $[0, 2\pi]$

$$\boxed{Y(t, \eta) = \cos(2\pi t + \eta)}$$

from before



$$\frac{1}{\pi\sqrt{1 - y^2}}$$



★ $$\boxed{F_{X(t_1), X(t_2), \ldots X(t_k)}(x_1, x_2, \ldots x_k) = P[X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots X(t_k) \leq x_k]}$$ for all ks & ts

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★ Mean fn: $\boxed{m_X(t) = E(X(t))}$ → deterministic fn, not random (no η)

★ Auto correlation:

$$\boxed{R_X(t_1, t_2) = E[X(t_1) X(t_2)]} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_{X(t_1)X(t_2)}(x, y)\, dx\, dy$$

★ Auto covariance:

$$\boxed{C_X(t_1, t_2) = E[(X(t_1) - E(X(t_1)))(X(t_2) - E(X(t_2)))]}$$
$$\boxed{= R_X(t_1, t_2) - m_X(t_1) m_X(t_2)}$$

★ Cross correlation:

$$\boxed{R_{XY}(t_1, t_2) = E[X(t_1) Y(t_2)]}$$

if $= 0$ for all t_1 & t_2 → ∴ X & Y orthogonal

★ Cross Covariance:

$$\boxed{C_{XY}(t_1, t_2) = E[(X(t_1) - E(X(t_1)))(Y(t_2) - E(Y(t_2)))]}$$

if $= 0$ for all t_1 & t_2 → ∴ X & Y uncorrelated

Note: $R_X(t, t) = E[X^2(t)]$

$C_X(t, t) = Var[X(t)]$

Correlation Coeff

$$\rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sqrt{C_X(t_1, t_1)}\sqrt{C_X(t_2, t_2)}}$$

ex $X(t) = \cos(2\pi t + \theta)$, θ uniform $[0, 2\pi]$

Random process (θ is random)

$$m_X = E[X(t)] = \int_0^{2\pi} \cos(2\pi t + \theta)\left(\frac{1}{2\pi}\right) d\theta = 0$$

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \underbrace{m_X(t_1)}_{0}\,\underbrace{m_X(t_2)}_{0}$$

$2\cos X \cos Y = \cos(X-Y) + \cos(X+Y)$

$$= \int_0^{2\pi} \cos(2\pi t_1 + \theta)\cos(2\pi t_2 + \theta)\left(\frac{1}{2\pi}\right) d\theta$$

$$= \frac{1}{2}\underbrace{\int_0^{2\pi} \cos(2\pi(t_1+t_2) + 2\theta)\left(\frac{1}{2\pi}\right) d\theta}_{0} + \frac{1}{2}\int_0^{2\pi} \cos(2\pi(t_1 - t_2))\left(\frac{1}{2\pi}\right) d\theta$$

$$= \frac{1}{2}(2\pi)\cos(2\pi(t_1 - t_2))\left(\frac{1}{2\pi}\right) = \frac{1}{2}\cos(2\pi(t_1 - t_2))$$

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for $Y = \sin(2\pi t + \theta)$, θ uniform $[0, 2\pi]$

$$m_Y = 0$$

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \underbrace{m_X(t_1)}_{0}\,\underbrace{m_X(t_2)}_{0}$$

$$= \int_0^{2\pi} \cos(2\pi t_1 + \theta)\sin(2\pi t_2 + \theta)\left(\frac{1}{2\pi}\right) d\theta$$

$$= \overset{\text{(the term with } 2\theta)}{0} + \frac{-1}{2}\sin(2\pi(t_1 - t_2))\int_0^{2\pi} \frac{1}{2\pi} d\theta$$

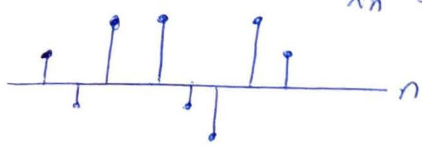
$$= -\frac{1}{2}\sin(2\pi(t_1 - t_2)) = R_{XY}(t_1, t_2)$$

at $t_1 = t_2$ or $|t_1 - t_2| =$ integer $\longrightarrow R_{XY} = 0$: X & Y are orthogonal & uncorrelated.

$C_{XY} = 0$

A Discrete time Random Processes = Random Sequences

X_n or $X[n]$



(class I) $\underline{iid}$ = independent identical distribution

$$\boxed{\begin{aligned} F_{X(t_1), X(t_2), \ldots X(t_k)}(x_1, x_2, \ldots x_k) &= F_{X(t_1)}(x_1) \cdot F_{X(t_2)}(x_2) \cdot F_{X(t_3)}(x_3) \ldots F_{X(t_k)}(x_k) \\ &= F_X(x_1)\, F_X(x_2)\, F_X(x_3) \ldots F_X(x_k) \end{aligned}}$$

X identical & independent RV @ any time (could be cont. or discrete)

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ex Binomial $P[I_n = 0] = 1 - p$ at any time n

$P[I_n = 1] = p$

$\therefore E[I_n] = 1 \times p + (1-p) \times 0 = p$, $VAR[I_n] = p(1-p)$

$$P[I_0 = 1, I_1 = 0, I_2 = 1] = \underbrace{P[I_0 = 1] \cdot P[I_1 = 0] \cdot P[I_2 = 1]}_{\text{independent}} = p(1-p)p$$

$D_n = 2I_n - 1$ $\quad P[D_n = 1] = p$

$P[D_n = -1] = 1 - p$

$\hookrightarrow E[D_n] = 2E[I_n] - 1 = 2p - 1$

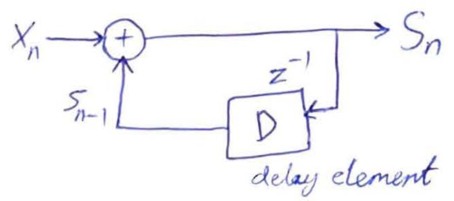
$Var[D_n] = 4\,Var[I_n] = 4p(1-p)$

I_n & D_n are discrete Random Processes & discrete time

(class II) Sum processes

$$\boxed{S_n = S_{n-1} + X_n}$$ where $S_0 = 0$ & X_n is iid

digital filter implementation



$S_1 = X_1$

$S_2 = S_1 + X_2 = X_1 + X_2$

$S_3 = S_2 + X_3 = X_1 + X_2 + X_3$

$E[S_n] = E[X_1] + \ldots + E[X_n] = n\,E[X_1]$ → mean (didribution) changes with time $\neq$ iid

Sum processes are generated by iid, but are not iid in themselves

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ex. Binomial Counting Process

$S_n = I_1 + I_2 + \ldots + I_n$ → Each I_n is either 0 or 1 → S_n can be $0, 1, 2, \ldots, n$

$S_n = \{0, 1, \ldots, n\}$

$\hookrightarrow P[S_n = j] = \binom{n}{j} p^j (1-p)^{n-j}$ $\quad j = 0, 1, \ldots, n$ → # of successes

$E[S_n] = n\,E[I_n] = np$

$Var[I_n] = E[I_n^2] - (E[I_n])^2 = p(1-p)$

$Var[S_n] = n\,Var[I_n] = np(1-p)$

Proof: $Y = X_1 + X_2 + \ldots + X_n \rightarrow \mathrm{I}$

$E[Y] = E[X_1] + E[X_2] + \ldots + E[X_n] \rightarrow \mathrm{II}$

I−II & square $\quad (Y - E[Y])^2 = ((X_1 - E[X_1]) + (X_2 - E[X_2]) + \ldots + (X_n - E[X_n]))^2$

$\therefore (Y - E[Y])^2 = (X_1 - E[X_1])^2 + \ldots + (X_n - E[X_n])^2$

$$\underbrace{+\,2(X_1 - E[X_1])(X_2 - E[X_2]) + \ldots}_{\text{cross terms}}$$

Take E

$E(Y - E[Y])^2 = E(X_1 - E[X_1])^2 + \ldots + E(X_n - E[X_n])^2$

$+\,2\underbrace{E((X_1 - E[X_1])(X_2 - E[X_2]))}_{0} + \ldots$

$X_1, X_2, \ldots X_n$ are uncorrelated since they are independent

$$\boxed{\sigma_Y^2 = \sum_{i=1}^{n} \sigma_X^2}$$ if X_is are uncorrelated

ex Random Walk

$S_n = D_1 + D_2 + \cdots + D_n$ $\quad$ $S_n = \{-n, -n+2, -n+4, \cdots, n\}$

all -1s $\quad$ all +1s

if one -1 flips to +1, you increase by 2

$P[S_n = 2k-n] = \binom{n}{k} p^k (1-p)^{n-k}$ $\quad$ $k = 0, 1, \cdots n$

of +1s (successes)

$E[S_n] = n E[D_n] = n(2p-1)$

$Var[S_n] = n(4p(1-p))$

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* Independent Increment Property:

$n_0 < n_1 < n_2 \cdots$

(1)
$$\left.\begin{aligned} S_{n_0} &= X_{n_0} + X_{n_0-1} + \cdots + X_1 \\ S_{n_1} - S_{n_0} &= X_{n_1} + X_{n_1-1} + \cdots + X_{n_0+1} \\ S_{n_2} - S_{n_1} &= X_{n_2} + \cdots + X_{n_1+1} \end{aligned}\right\}$$

Non-overlapping intervals $\Downarrow$ independent

(2) $\boxed{S_{n_2} - S_{n_1} = S_{n_2-n_1}}$ $\longrightarrow$ $X_{n_2} + X_{n_2-1} + \cdots + X_{n_1+1} = X_{n_2-n_1} + X_{n_2-n_1-1} + \cdots + X_1$

$n_1 < n_2 < n_3$

$P[S_{n_1} = y_1, S_{n_2} = y_2, S_{n_3} = y_3] \neq P[S_{n_1} = y_1] \cdot P[S_{n_2} = y_2] \cdot P[S_{n_3} = y_3]$ $\quad$ S_ns are not independent

$= P[S_{n_1} = y_1, S_{n_2} - S_{n_1} = y_2 - y_1, S_{n_3} - S_{n_2} = y_3 - y_1]$ $\quad$ non-overlapping intervals $\Rightarrow$ independent

$= P[S_{n_1} = y_1] \cdot P[S_{n_2} - S_{n_1} = y_2 - y_1] \cdot P[S_{n_3} - S_{n_2} = y_3 - y_1]$

$= P[S_{n_1} = y_1] \cdot P[S_{n_2-n_1} = y_2 - y_1] \cdot P[S_{n_3-n_2} = y_3 - y_1]$

Now we only need to find $P[S_m]$ and substitute $m \leftarrow n_1,\ n_2 - n_1,\ n_3 - n_1$.

Similarily, for Continuous:

$$f_{S_{n_1}, S_{n_2}, \cdots S_{n_k}}(y_1, y_2, \cdots y_k) = f_{S_{n_1}}(y_1)\, f_{S_{n_2-n_1}}(y_2 - y_1) \cdots f_{S_{n_k - n_{k-1}}}(y_k - y_{k-1})$$ $\longrightarrow$ we only need $f_{S_m}(y)$

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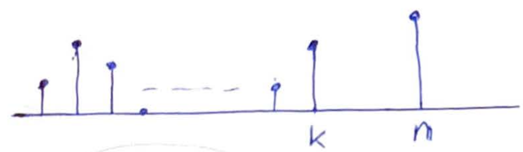
ex $S_n = X_1 + X_2 + \cdots + X_n$ $\quad$ where $X_i \sim N(0, \sigma^2)$ & X_is are iid

I want the joint pdf of 2 time instances:

$$f_{S_{n_1}, S_{n_2}}(y_1, y_2) = f_{S_{n_1}}(y_1)\, f_{S_{n_2-n_1}}(y_2 - y_1)$$

$$
\left.
\begin{aligned}
&E(S_{n_1}) = n_1 E[X_i] = 0 \\
&E(S_{n_2-n_1}) = (n_2-n_1)\, E[X_i] = 0 \\
&Var(S_{n_1}) = n_1 \sigma_x^2 \\
&Var(S_{n_2-n_1}) = (n_2-n_1)\, \sigma_x^2
\end{aligned}
\right\}
\quad \therefore f_{S_{n_1}, S_{n_2}}(y_1, y_2) = \frac{1}{\sqrt{2\pi n_1 \sigma^2}}\, e^{\frac{-y_1^2}{2 n_1 \sigma^2}} \cdot \frac{1}{\sqrt{2\pi (n_2-n_1)\sigma^2}}\, e^{\frac{-(y_2-y_1)^2}{2(n_2-n_1)\sigma^2}}
$$

☆ Covariance between 2 instances in a sum process:



$m = E[X]$

$$C_S(k,n) = E\Big[\big(S_n - \underbrace{E(S_n)}_{nm}\big)\big(S_k - \underbrace{E(S_k)}_{km}\big)\Big]$$

$\neq E(S_n - E(S_n)) \cdot E(S_k - E(S_k))$ as S_n & S_k are not independent

$$\cancel{= E[(S_{n-k} - (n-k)m)(S_k - km)]}$$

$$= E[(S_n + S_k - S_k - nm + km - km)(S_k - km)]$$

$$= \underbrace{E[(S_n - S_k - (n-k)m)(S_k - km)]}_{0} + E[(S_k - km)(S_k - km)]$$

0 as $(S_n - S_k)$ & (S_k) independent

$$= E((S_k - km)^2) = k\sigma^2$$

$\sigma^2 = Var[X_i]$

$$\therefore \boxed{C_S(k,n) = k\sigma^2 = \min(k,n) \cdot \sigma^2}$$

B Continuous time Random Process

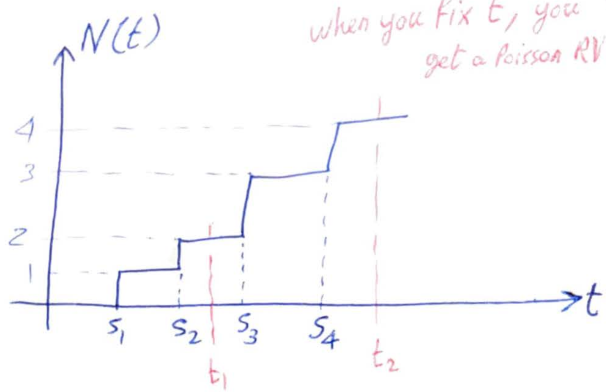
① Poisson Random Process:

$$\boxed{P[N=k] = e^{-\alpha}\frac{\alpha^k}{k!}}$$

$k = 0, 1, \ldots$

$\alpha = \lambda t$ (α → mean, λ → rate)

when you fix t, you get a Poisson RV



* $N(t_1)$ & $N(t_2)$ are dependent.
  ex: if $N(t_1) = 2$, $\therefore N(t_2) \geq 2$

* $N(t_1)$ & $N(t_2) - N(t_1)$ are independent
  "Independent Increment Property"

→ Continuous time → t can take any value
→ Discrete RV → N = integer only

a) $P[N(t_1)=i, N(t_2)=j] \neq P[N(t_1)=i] \cdot P[N(t_2)=j]$

$$= P[N(t_1)=i,\ N(t_2)-N(t_1)=j-i]$$

$$= P[N(t_1)=i] \cdot P[N(t_2)-N(t_1)=j-i]$$

$P[N(t_2)-N(t_1)=k] = P[N(t_2-t_1)=k]$

$$\therefore \boxed{P[N(t_1)=i, N(t_2)=j] = P[N(t_1)=i] \cdot P[N(t_2-t_1)=j-i]}$$

$$= e^{-\lambda t_1}\frac{(\lambda t_1)^i}{i!} \cdot e^{-\lambda(t_2-t_1)}\frac{(\lambda(t_2-t_1))^{j-i}}{(j-i)!}$$

b) $C_N(t_1, t_2) = E\left[(N(t_1) - \underbrace{E(N(t_1))}_{\lambda t_1})\,(N(t_2) - \underbrace{E(N(t_2))}_{\lambda t_2})\right]$

$+N(t_1) - N(t_1) + \lambda t_1 - \lambda t_1$

$$= \cancel{E\left[(N(t_2-t_1) - \lambda(t_2-t_1))(N(t_1)-\lambda t_1)\right]} + E\left[(N(t_1)-\lambda t_1)^2\right]$$

0 "$N(t_2-t_1)$ & $N(t_1)$ are independent

$$= E\left[(N(t_1)-\lambda t_1)^2\right]$$

$$\therefore \boxed{C_N(t_1,t_2) = \lambda t_1 = \lambda \cdot \min(t_1, t_2)}$$

Remember: In Poisson: $E[N] = Var[N] = \alpha$

---

ex: Calls with $\lambda = 15$ calls/min

P[I get 3 calls in 1st 10 secs & 2 calls in last 15 secs] = ?

$$P[N(10) = 3,\ N(60) - N(45) = 2]$$ → non-overlapping intervals

$$= P[N(10)=3] \cdot P[N(60)-N(45)=2]$$

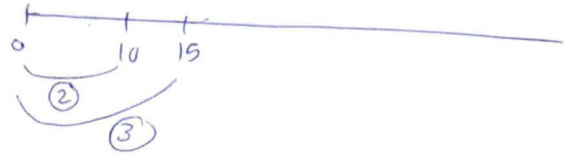
$$= P[N(10)=3] \cdot P[N(15)=2] = e^{-2.5}\frac{(2.5)^3}{3!} \cdot e^{-3.75}\frac{(3.75)^2}{2!} = 0.035 = 3.5\%$$

$\alpha = 10\lambda = 2.5$, $\alpha = 15\lambda = 3.75$

$\hookrightarrow P[N(10)=2\ ,\ N(15)=3]$

$= P[N(10)=2,\ N(15-10)=3-2]$

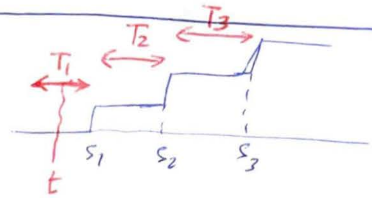
$= P[N(10)=2] \cdot P[N(5)=1] = \ldots$



$\star$ $P(S_1 > t) = P(N(t)=0) = e^{-\lambda t}$

$\therefore 1 - P[S \le t] = e^{-\lambda t}$

$\therefore F_{S_1}(t) = 1 - e^{-\lambda t} \rightsquigarrow \therefore \boxed{f_{S_1} = \lambda e^{-\lambda t}} \quad t \geq 0$



$\boxed{T_i = S_i - S_{i-1}} \to$ T_i s are all exponential & independent $\to$ iid

* each S_m $\longrightarrow$ is an m-erlang distribution
$(S_m - 0)$

Conditional Probability

ex: Prof. left his office for 1 hr & came back saw 1 student waiting.
P[this student came in the 1^{st} $\frac{1}{2}$ hr] = ?

$\hookrightarrow P[N(\tfrac{1}{2})=1 \mid N(1)=1] = \dfrac{P[N(\frac{1}{2})=1\ ;\ N(1)=1]}{P[N(1)=1]}$

$= \dfrac{P[N(\frac{1}{2})=1,\ N(1)-N(\frac{1}{2})=1-1]}{P[N(1)=1]}$

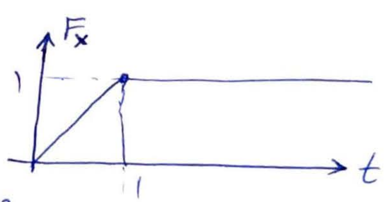
$= \dfrac{P[N(\frac{1}{2})=1] \cdot P[N(\frac{1}{2})=0]}{P[N(1)=1]} = \dfrac{\frac{\lambda}{2} e^{-\lambda/2} e^{-\lambda/2}}{\lambda e^{-\lambda}} = \dfrac{1}{2}$

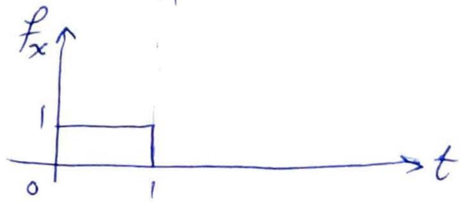
arrival time

$\hookrightarrow P[X \le x]$

$\therefore P[N(x)=1 \mid N(1)=1] = \dfrac{P[N(x)=1] \cdot P[N(1-x)=0]}{P[N(1)=1]} = x$

$\therefore$ Cond. Prob of a Poisson event (arrival) given another Poisson event (arrival), gives a Uniform Distribution
&
These Cond. Probabilities are independent.





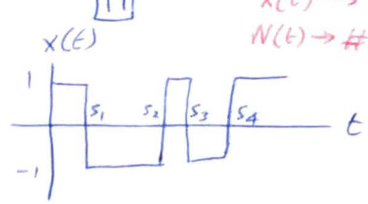
* If # of arrivals in $[0,t]$ is k, then the individual arrival times are distributed independently & uniformly in the interval.

② Random Telegraph Signal

→ output flips the sign whenever a Poisson event happens

$X(t) \longrightarrow$ output

$N(t) \rightarrow$ # of +1 s



Assume $P[X(0)=1] = P[X(0)=-1] = \frac{1}{2}$ "symmetric"

$\therefore P[X(t)] = ?$

$$P[X(t)=1] = \underbrace{P[X(t)=1 \mid X(0)=1]}_{P[N(t)=\text{even}]} P[X(0)=1] + \underbrace{P[X(t)=1 \mid X(0)=-1]}_{P[N(t)=\text{odd}]} P[X(0)=-1]$$

$\because e^{\lambda t} = 1 + \lambda t + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \cdots$

$e^{-\lambda t} = 1 - \lambda t + \frac{(\lambda t)^2}{2!} - \frac{(\lambda t)^3}{3!} + \cdots$

even terms $P[N(t)=\text{even}] = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{2k}}{2k!} = e^{-\lambda t}\left(\frac{e^{\lambda t}+e^{-\lambda t}}{2}\right)$

odd terms $P[N(t)=\text{odd}] = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k+1}}{2k+1!} = e^{-\lambda t}\left(\frac{e^{\lambda t}-e^{-\lambda t}}{2}\right)$

$$\therefore P[X(t)=1] = e^{-\lambda t}\left(\frac{e^{\lambda t}+e^{-\lambda t}}{2}\right)\cdot\left(\frac{1}{2}\right) + e^{-\lambda t}\left(\frac{e^{\lambda t}-e^{-\lambda t}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{1}{4} e^{-\lambda t}\left(e^{\lambda t} + e^{-\lambda t} + e^{\lambda t} - e^{-\lambda t}\right) = \frac{1}{2}$$

$\therefore P[X(0)=\pm 1] = \frac{1}{2}$

$\&\ P[X(t)=\pm 1] = \frac{1}{2}$

↪ $m_X(t) = (1)\times\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right) = 0$

↪ $Var[X(t)] = E[X^2(t)] - m_X^2 = (1)^2\left(\frac{1}{2}\right) + (-1)^2\left(\frac{1}{2}\right) = 1$

↪ Covariance

$$C_X(t_1,t_2) = E[X(t_1)X(t_2)] - \underbrace{m_X(t_1)}_{0}\underbrace{m_X(t_2)}_{0}$$

$x_1=1,\ x_2=1$
$x_1=-1,\ x_2=-1$

$$= (1)\cdot \underbrace{P[X(t_1)=X(t_2)]}_{P[N=\text{even}]} + (-1)\cdot \underbrace{P[X(t_1)\neq X(t_2)]}_{P[N=\text{odd}]}$$

$x_1=1,\ x_2=-1$
$x_1=-1,\ x_2=1$

$$= (1)\left(\frac{1}{2}\left(1+e^{-2\lambda|t_2-t_1|}\right)\right) + (-1)\left(\frac{1}{2}\left(1 - e^{-2\lambda|t_2-t_1|}\right)\right)$$

$$\therefore C_X(t_1,t_2) = e^{-2\lambda|t_2-t_1|}$$

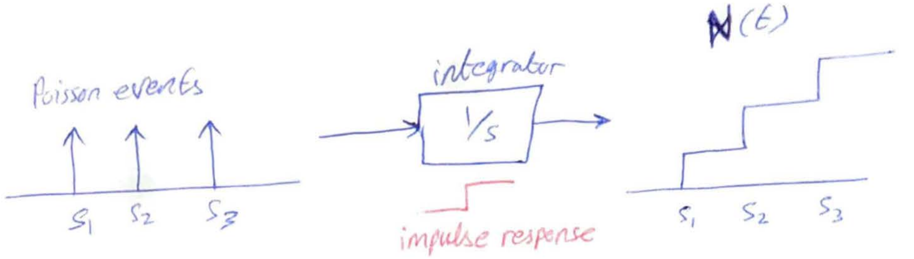
↪ ~ 0 as $t_2 - t_1 \ggg$

C Continuous time Continuous process

Shot Noise Process:

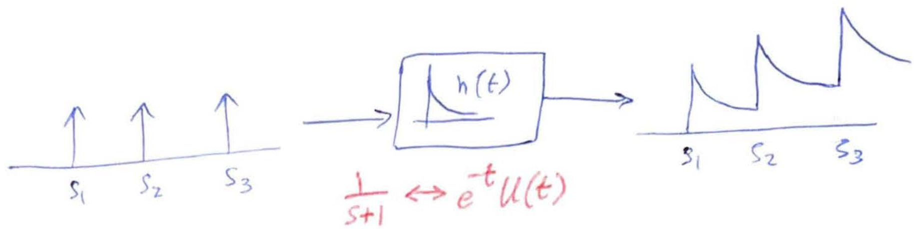
① Poisson

$$\boxed{N(t) = \sum_{i=1}^{\infty} u(t - S_i)}$$



② Shot Noise Process (General)

$$\boxed{X(t) = \sum_{i=1}^{\infty} h(t - S_i)}$$



2 sources of randomness:
① # of poisson events by t
② when they arrive
⇓
use Conditional Prob.

$$\Rightarrow E[X(t)] = E\Big[\underbrace{E(X(t) \mid N(t) = k)}\Big]$$

$$E\left[\sum_{i=1}^{k} h(t - S_i)\right] = \sum_{i=1}^{k} \underbrace{E[h(t - S_i)]}$$

$$E[h(t - S_i)] = \int_0^t h(t - S_i)\, \frac{1}{t}\, ds$$

$\to f_S(s) \Rightarrow S_i$ s are independent & uniformly distributed on interval $[0, t]$, given the impulses happened

$$= \frac{1}{t} \int_t^0 h(u)\, (-du)$$

$$= \frac{1}{t} \int_0^t h(u)\, du$$

$t - S = u$
$-ds = -du$
$s = 0: \; u = t$
$s = t: \; u = 0$

$$\therefore E[X(t)] = E\left[\sum_{i=1}^{k} E[h(t - S_i)]\right]$$

$$= E\left[k \cdot \frac{1}{t} \int_0^t h(u)\, du\right] \quad \longrightarrow \text{only } k \text{ is random, not } t$$

$$= \left(\frac{1}{t} \int_0^t h(u)\, du\right) \cdot E[k]$$

$$= \frac{1}{t} \int_0^t h(u)\, du \cdot E[N(t)] \quad \longrightarrow \alpha = \lambda t$$

$$\therefore \boxed{E[X(t)] = \lambda \int_0^t h(u)\, du}$$

ex/ Random Walk

$$S_n = D_1 + D_2 + \cdots + D_n$$

$P[D_n = \pm 1] = \frac{1}{2}$

$\therefore E[D_n] = 0$

$\therefore Var[D_n] = 1$

$$X(t) = \lim_{\delta \to 0} h\, S_n$$

$$X_\delta(t) = h\left(D_1 + D_2 + \cdots + D_{[\frac{t}{\delta}]}\right)$$

$$E[X_\delta(t)] = 0$$

$$Var[X_\delta(t)] = h^2\, Var[S_n] = h^2 \cdot \left[\tfrac{t}{\delta}\right] \cdot Var[D_n] = h^2 \cdot \frac{t}{\delta}$$

$$\boxed{\frac{h^2}{\delta} = \alpha}$$

as $\delta \to 0$, $h \to 0$ "time points come closer to each other"

$\therefore X_\delta(t) \longrightarrow X(t)$

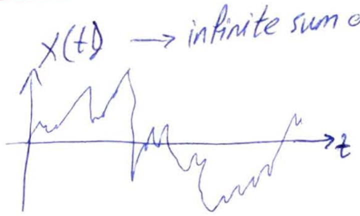
★ Weiner Process

$$X(t) = \lim_{\delta \to 0} h\, S_n$$

$$E[X(t)] = 0$$

$$Var[X(t)] = h^2 \frac{t}{\delta} = \alpha t$$

$X(t) \longrightarrow$ infinite sum of RVs



@ $t=0$: $E[X(t)] = 0$

$Var[X(t)] = 0$

$X(t) = 0 \rightarrow$ Not Random

@ $t\uparrow$: Randomness of $X(t)$ linearly increases

By central limit theorem, pdf of $X(t)$ approaches Gaussian distribution $(m=0,\ \sigma^2 = \alpha t)$

$$\boxed{f_{X(t)}(x) = \frac{1}{\sqrt{2\pi \alpha t}}\, e^{-x^2/2\alpha t}}$$

Also, inherets the independent increments property

$$f_{X(t_1)X(t_2)X(t_3)\cdots X(t_k)}(x_1, x_2, x_3, \ldots, x_k) = f_{X(t_1)}(x_1)\, f_{X(t_2-t_1)}(x_2 - x_1) \cdots f_{X(t_k - t_{k-1})}(x_k - x_{k-1})$$

$$= \frac{1}{\sqrt{2\pi\alpha t_1}}\, e^{-x_1^2/2\alpha t_1} \cdot \frac{1}{\sqrt{2\pi\alpha(t_2 - t_1)}}\, e^{-\frac{(x_2-x_1)^2}{2\alpha(t_2-t_1)}} \cdots$$