EE 210: HW 10 Solutions

Alvin Maningding

2020 November 20

A short review of the DFT (Discrete Fourier Transform

By now, you have seen a few different kinds of "transform".

1. The continuous-time Fourier transform, $X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi f t)\,\mathrm{d}t$, gives the continuous ($f$ or ω) frequency content—the "frequency response"—of an aperiodic signal $x(t)$. Taking the scaled Fourier transform of one period of a T_0−periodic signal $x(t)$ and sampling it at multiples of its fundamental frequency, $\frac{1}{T_0}X\left(\frac{k}{T_0}\right)$, gives its discrete-frequency *Fourier series* representation.

2. The discrete-time Fourier transform (DTFT), $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\exp(-j\Omega n)$, gives the continuous (Ω) frequency response of a discrete-time sequence $x[n]$, where n is an integer.

3. The Z-transform, $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ is a generalization of the DTFT, also for discrete-time sequences $x[n]$. $z = e^{j\Omega}$ is a continuous, complex frequency variable. When the region of convergence of a signal's Z-transform contains the unit circle ($|z| = 1$), the DTFT of the signal exists.

We see that there are transforms for both continuous- and discrete-time signals. Why introduce another? Notice that all of the transforms listed use a continuous frequency variable. In the same way that an analog signal cannot be stored in a computer—it must be sampled—a continuous frequency response cannot be directly stored either.

We introduce another transform, the (confusingly-named) **discrete Fourier transform (DFT)**, that uses a discrete frequency index k and operates on an N-point signal (i.e. n ranges from $0, 1, 2, \ldots, N-1$). It is defined by the equations

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi\frac{k}{N}n} \tag{1a}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{k}{N}n} \tag{1b}$$

These two equations form a transform pair: $\mathcal{F}\{x[n]\} = X[k]$. $X[k]$ is the new function of frequency; like $X(f)$ and $X(e^{j\Omega})$, $X[k]$ is *complex-valued* and conveys both magnitude and phase information. It does not matter whether the N-point signal was aperiodic to begin with, or a periodic signal that has been truncated. The DFT views both the time-domain signal, $x[n]$ and the frequency response it outputs, $X[k]$, as N-periodic.

The summations in Eqs. (1a) and (1b) become tedious for large N. These equations can be written compactly as matrix operations. First, define $w = e^{-\frac{j2\pi}{N}}$, which is constant for a given N, and define $\boldsymbol{W}_N$

as an $N \times N$ matrix consisting of entries $\left[w^{mn}\right]$. m and n correspond to the rows and columns of $\boldsymbol{W}_N$, respectively, and range from $0 \leq m, n \leq N-1$. For $N = 4$, for example,

$$\begin{aligned}\boldsymbol{W}_4 &= \begin{bmatrix} w^{0\cdot 0} & w^{0\cdot 1} & w^{0\cdot 2} & w^{0\cdot 3} \\ w^{1\cdot 0} & w^{1\cdot 1} & w^{1\cdot 2} & w^{1\cdot 3} \\ w^{2\cdot 0} & w^{2\cdot 1} & w^{2\cdot 2} & w^{2\cdot 3} \\ w^{3\cdot 0} & w^{3\cdot 1} & w^{3\cdot 2} & w^{3\cdot 3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w^1 & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\end{aligned}$$

Then, Eqs. (1a) and (1b) can be rewritten as

$$\boldsymbol{X} = \boldsymbol{W}_N \boldsymbol{x} \tag{2a}$$

$$\boldsymbol{x} = \frac{1}{N} \boldsymbol{W}_N^{-1} \boldsymbol{X} \tag{2b}$$

where $\boldsymbol{x} = \begin{bmatrix} x[0] & x[1] & \cdots x[N-1] \end{bmatrix}^T$ and $\boldsymbol{X} = \begin{bmatrix} X[0] & X[1] & \cdots X[N-1] \end{bmatrix}^T$ are $N \times 1$ column vectors. Boldface font is used here to denote vectors and matrices. Notice that $\boldsymbol{W}_N$ depends only on N, not $x[n]$.

Now compare Eq. (1a) to the DTFT. Comparing their exponential terms, $e^{-j2\pi(k/N)n}$ and $e^{-j\Omega n}$, we see that they are related by $\Omega = 2\pi\frac{k}{N}$. Since Ω is 2π-periodic, this relation means that the DFT effectively *samples* $X(e^{j\Omega})$ at N points. Let $x[n] = \left(\frac{1}{2}\right)^n \mathbf{u}[n]$, a damped cosine, as shown in Fig. 1.

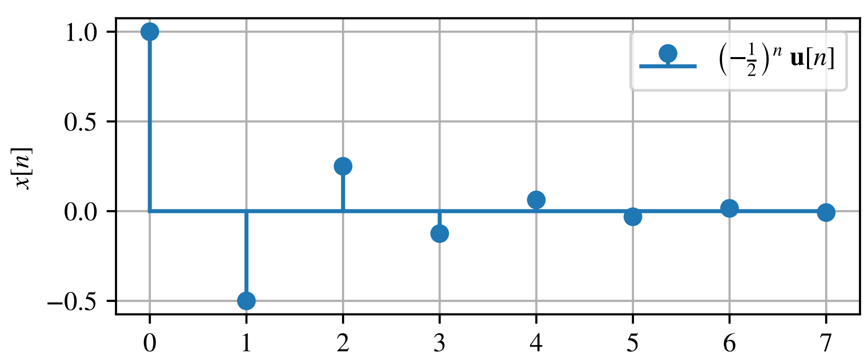
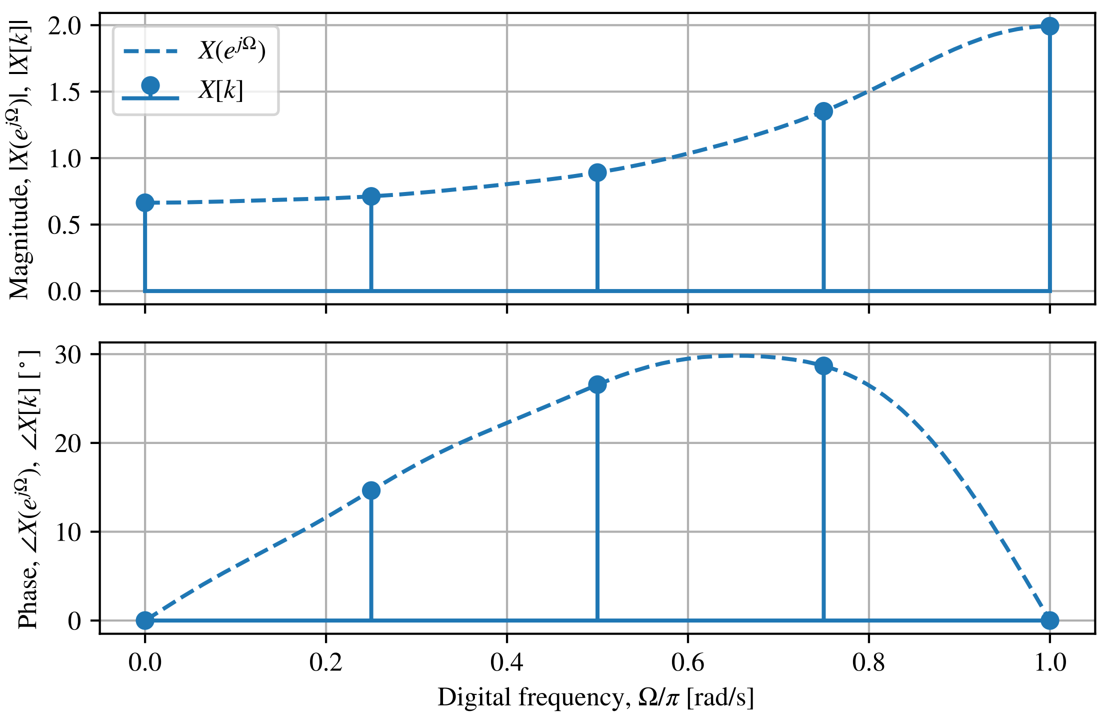


Figure 1

Its DTFT, $X(e^{j\Omega})$, and DFT, $X[k]$, are shown in Fig. 2. Notice that each point of the DFT's magnitude and phase exactly correspond with the same signal's DTFT. They are made to overlap by positioning the ks along the horizontal axis at discrete frequencies $\Omega_k = 2\pi k/N$.

In practice, however, the DFT is obtained through Fast Fourier Transform (FFT) algorithms rather than calculating and sampling the DTFT or performing matrix multiplication. You can verify your answers in this assignment by using the **fft()** MATLAB function.



$X(e^{j\Omega})$
$X[k]$
Magnitude, $|X(e^{j\Omega})|$, $|X[k]|$
2.0
1.5
1.0
0.5
0.0
Phase, $\angle X(e^{j\Omega})$, $\angle X[k]$ [$^\circ$]
30
20
10
0
0.0
0.2
0.4
0.6
0.8
1.0
Digital frequency, Ω/π [rad/s]

Figure 2

Solutions

1. (a) The given signal is reproduced in Fig. 3.

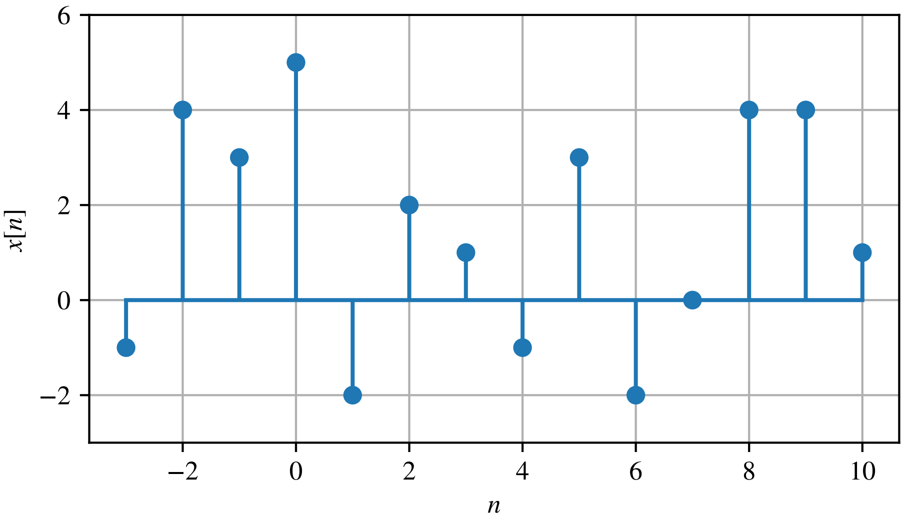


Figure 3

We only need to calculate the DFT over $0 \leq n \leq 7$, however. This gives $N = 8$ and $k = [0, N-1] = 0, 1, 2, \ldots, 7$. Use Eq. (1a) directly:

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{7} x[n] e^{j2\pi \frac{k}{8} n} \\
&= x[0]e^{-j2\pi \frac{k}{8}(0)} + x[1]e^{-j2\pi \frac{k}{8}(1)} + \cdots + x[7]e^{-j2\pi \frac{k}{8}(7)} \\
&= 5e^{-j2\pi \frac{k}{8}(0)} - 2e^{-j2\pi \frac{k}{8}(1)} + 2e^{-j2\pi \frac{k}{8}(2)} + 1e^{-j2\pi \frac{k}{8}(3)} - 1e^{-j2\pi \frac{k}{8}(4)} + 3e^{-j2\pi \frac{k}{8}(5)} - 2e^{-j2\pi \frac{k}{8}(6)} + 0e^{-j2\pi \frac{k}{8}(7)} \\
&= 5 - 2e^{-j\pi \frac{k}{4}} + 2e^{-j\pi \frac{k}{2}} + e^{-j\pi \frac{3k}{4}} - e^{-j\pi k} + 3e^{-j\pi \frac{5k}{4}} - 2e^{-j\pi \frac{3k}{2}}
\end{aligned}
$$

Evaluating this result for each k gives the results in Table 1. The magnitude and phase of $X[k]$ are plotted in Fig. 4.

(b) Recall that the complex exponential, e^{jx}, is 2π-periodic like $\sin(x)$ and $\cos(x)$. For a given N, the exponential term $e^{j2\pi \frac{k}{N} n}$ in Eqs. (1a) and (1b) and $X[k]$ are therefore N-periodic, with a period of $N = 8$. Evaluating the DFT for $k > 7$ simply cycles through $X[k]$ again[1].

[1]Specifically, we are evaluating $X[k \pmod N]$.

k	$X[k]$
0	$6 + j0$
1	$1.7574 - j1.1716$
2	$4 + j0$
3	$10.2426 + j6.8284$
4	$2 + j0$
5	$10.2426 - j6.8284$
6	$4 + j0$
7	$1.7574 + j1.1716$

Table 1

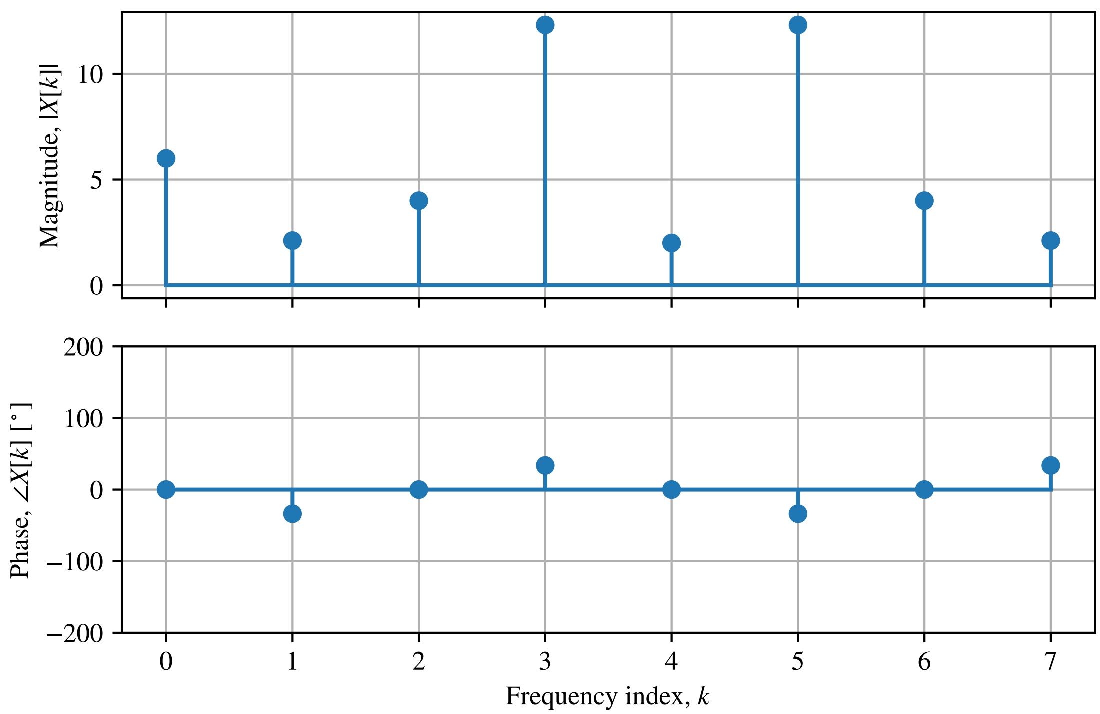


Figure 4

2. The given signal $x[n] = e^{-0.5n}\left(\mathbf{u}[n] - \mathbf{u}[n-4]\right)$, shown in Fig. 5, is non-zero for $0 \leq n \leq 3$ (notice that $n = 4$ is excluded).

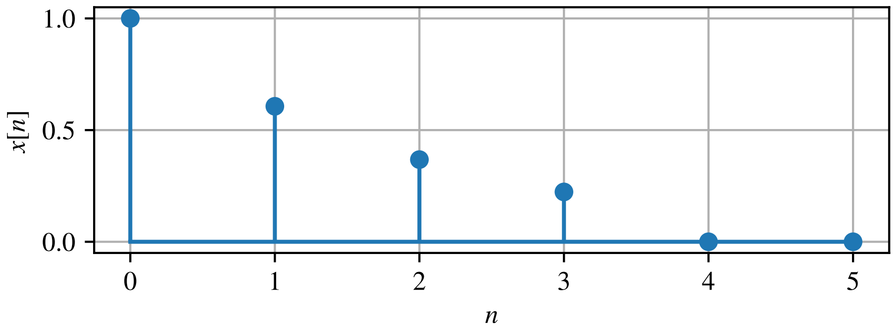


Figure 5

The DTFT is shown, with the DFT (for $k = 0, 1, 2, 3$) superimposed, in Fig. 6.

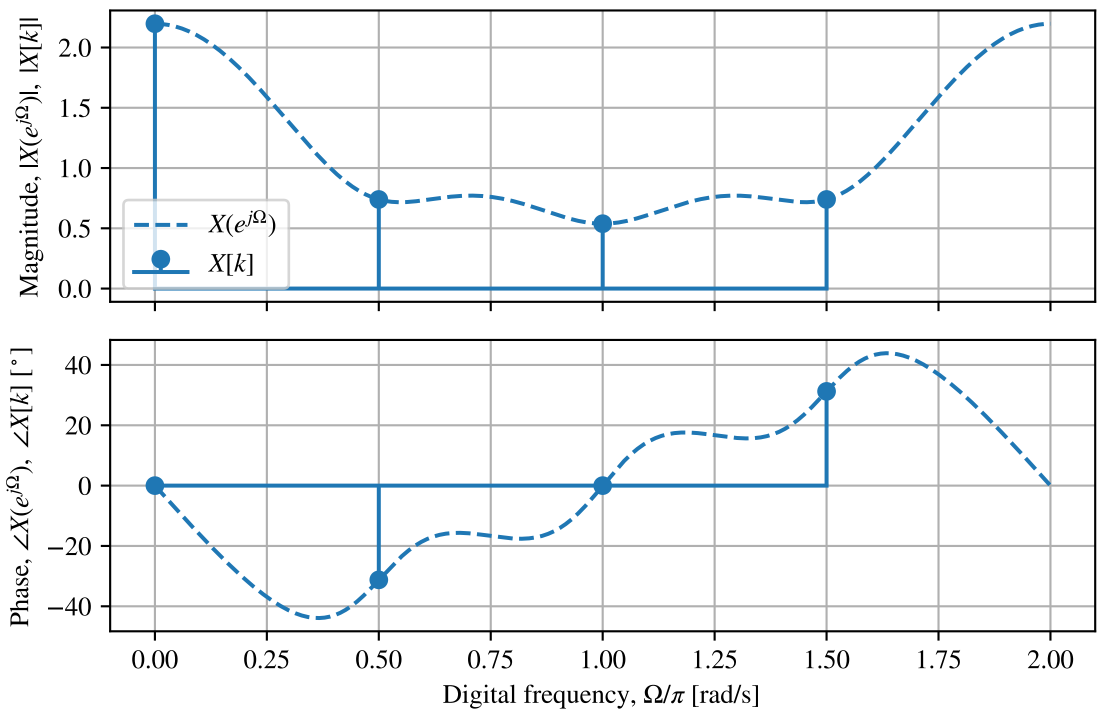


Figure 6

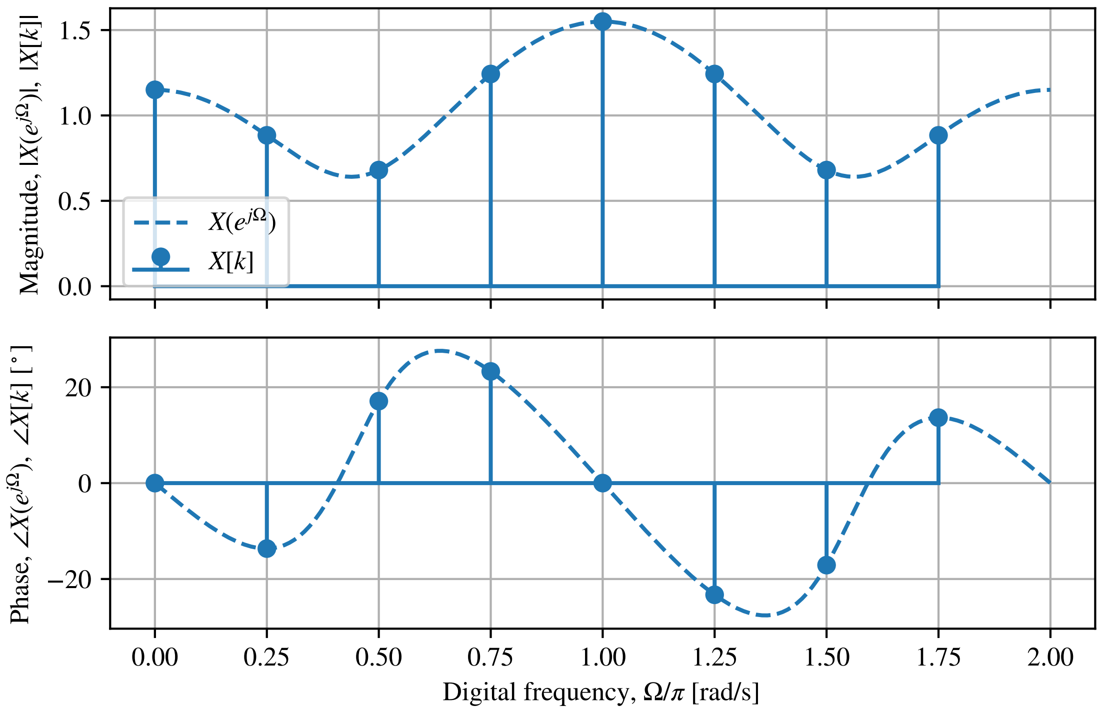


Figure 7

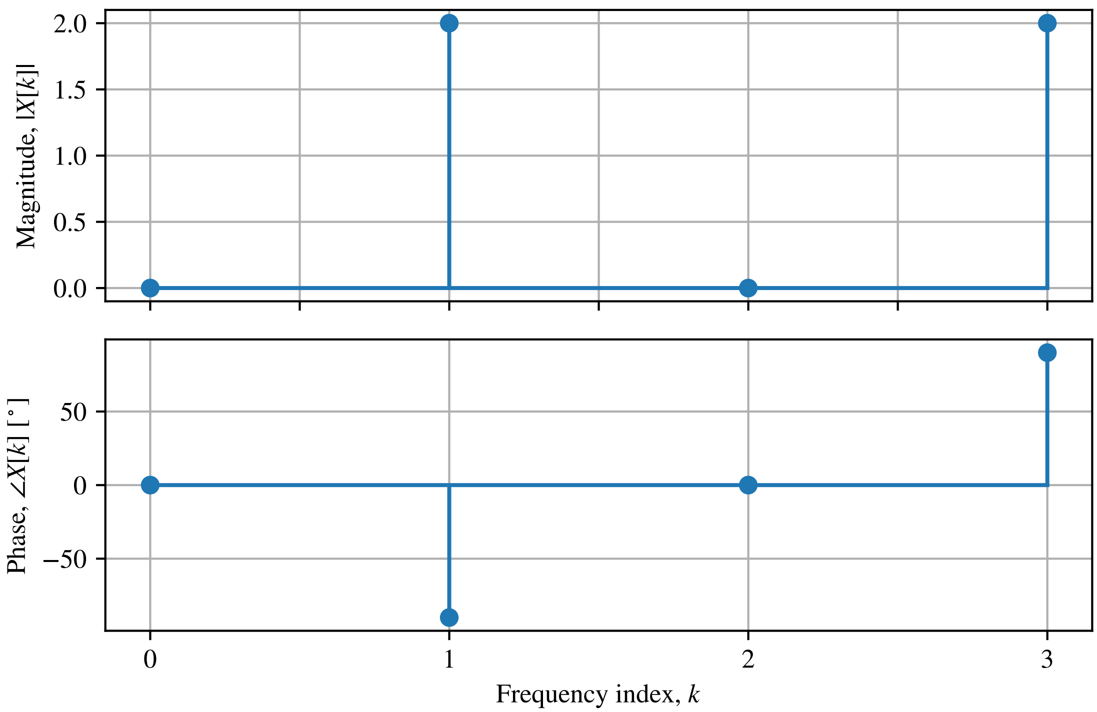
3. The given DTFT can be sampled at $N = 8$ points to obtain an equivalent DFT representation. Since Ω is 2π-periodic and k is N-periodic, k corresponds to increments of $2\pi/N = \pi/4$ rad/s from 0 to 2π. To obtain magnitude and phase, evaluate the given $X(e^{j\Omega})$ at $\Omega_k = \frac{2\pi k}{8} = \frac{\pi k}{4}$ to obtain $X[k]$, where $0 \leq k \leq 7$. Fig. 7 shows both the given DTFT and calculated DFT.

4. Evaluating $x[n] = \sin(n\pi/2)$ for $0 \leq n \leq 3$ gives $\begin{bmatrix}\mathbf{0} & 1 & 0 & -1\end{bmatrix}$ ($x[0]$ in bold).

   As an example of the matrix form of the DFT [Eq. (2a)], we let $N = 4$ and $\boldsymbol{x} = \begin{bmatrix}0 & 1 & 0 & -1\end{bmatrix}^T$. Then,

$$\begin{aligned}
\boldsymbol{X} &= \boldsymbol{W}_4 \boldsymbol{x} \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -j2 \\ 0 \\ j2 \end{bmatrix}
\end{aligned}$$

so $X[k] = \begin{bmatrix}\mathbf{0} & -j2 & 0 & j2\end{bmatrix}$. Its magnitude and phase are plotted in Fig. 8.



2.0
1.5
1.0
0.5
0.0
Magnitude, |X[k]|
50
0
−50
Phase, ∠X[k] [°]
0
1
2
3
Frequency index, k

Figure 8

5. As before, substitute $0 \le n \le 3$ into the given $h[n]$ to obtain $\begin{bmatrix} \mathbf{1} & -0.95 & 0.9025 & -0.8574 \end{bmatrix}$. Using Eqs. (1a) or (2a) give $H[k]$, plotted in Fig. 9.

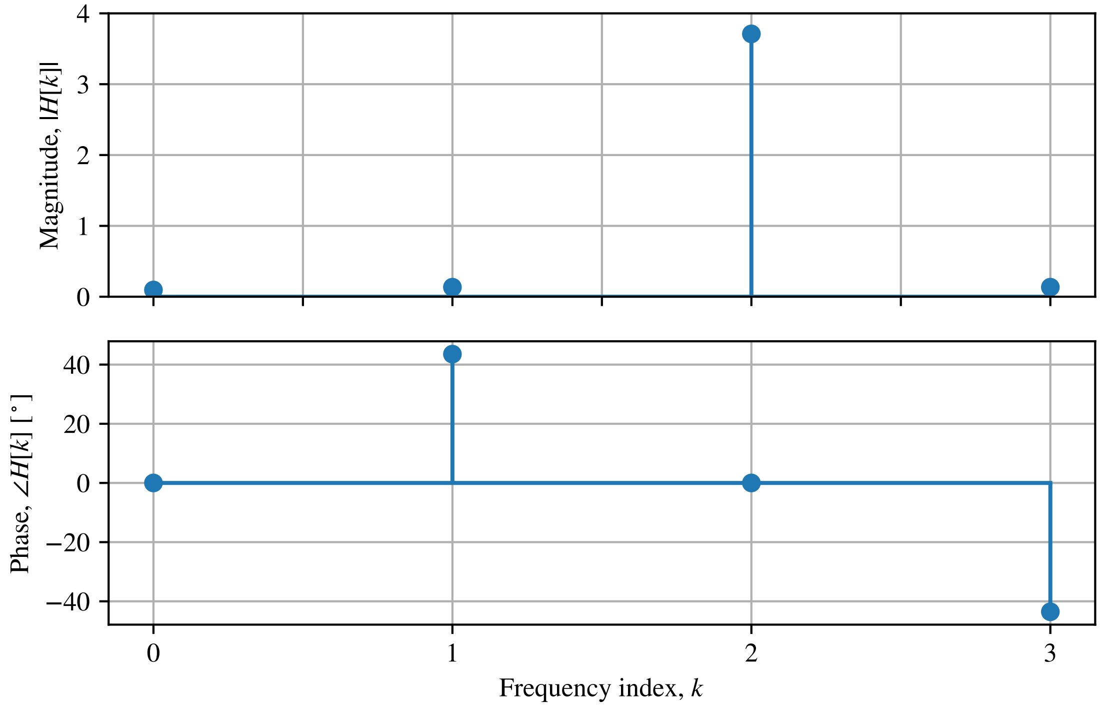


Figure 9

6. In this problem, we consider a sequence $x_1[n] = \begin{bmatrix} \mathbf{4} & 3 & 2 & 1 \end{bmatrix}$ and periodic extensions of it—$x_2[n]$ repeats it twice and $x_3[n]$ repeats it three times, as shown in Fig. 10. Note that the values of the sequences are given, but not their indices n. When n is not provided, a sequence is taken to be from $0 \le n \le N-1$.

   The DFTs of $x_1[n]$, $x_2[n]$, and $x_3[n]$ are shown in Figs. 11, 12, and 13. As shown, periodic extension of a signal (or calculation of the DFT over more than one period of a periodic signal) scales the magnitude response by the number of periods and produces zero-valued magnitude and phase components between the original points.

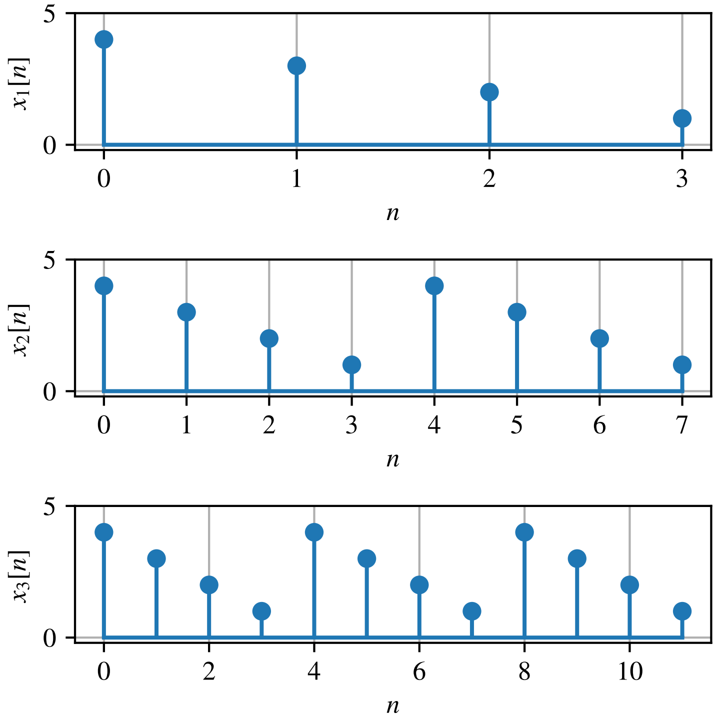


Figure 10

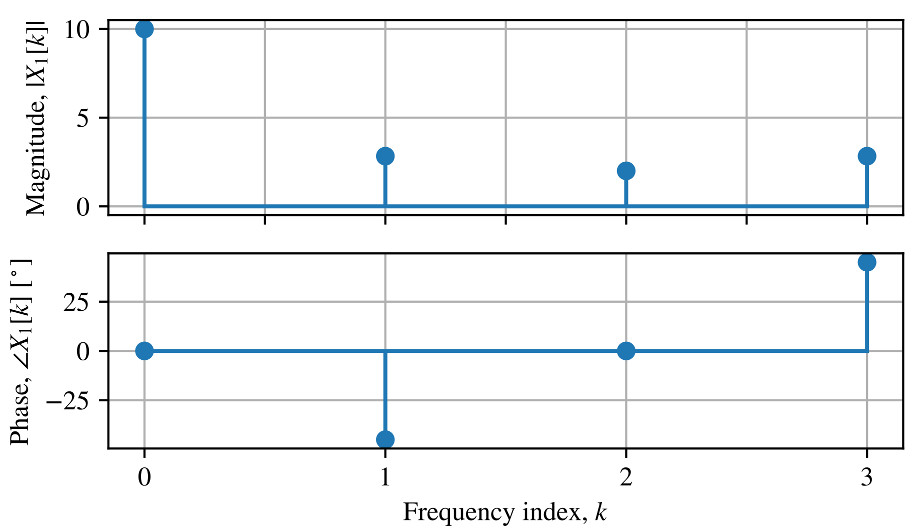


Figure 11

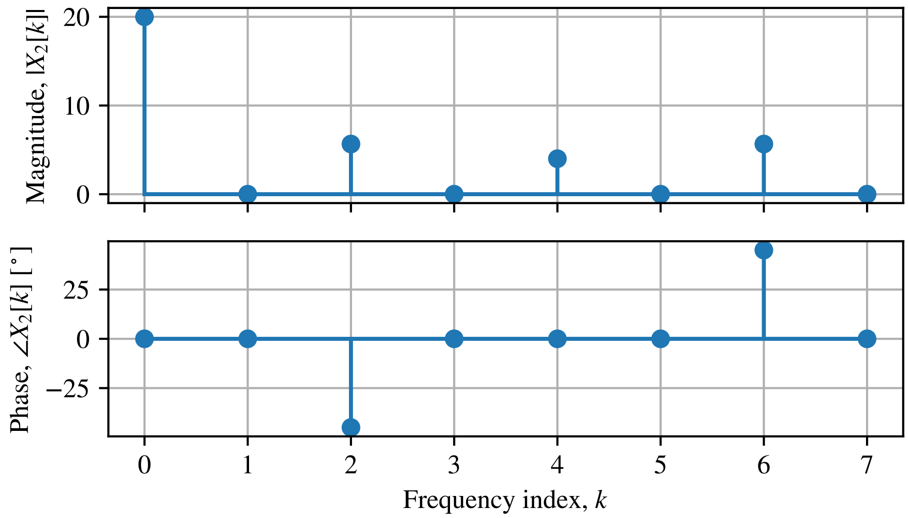


Figure 12

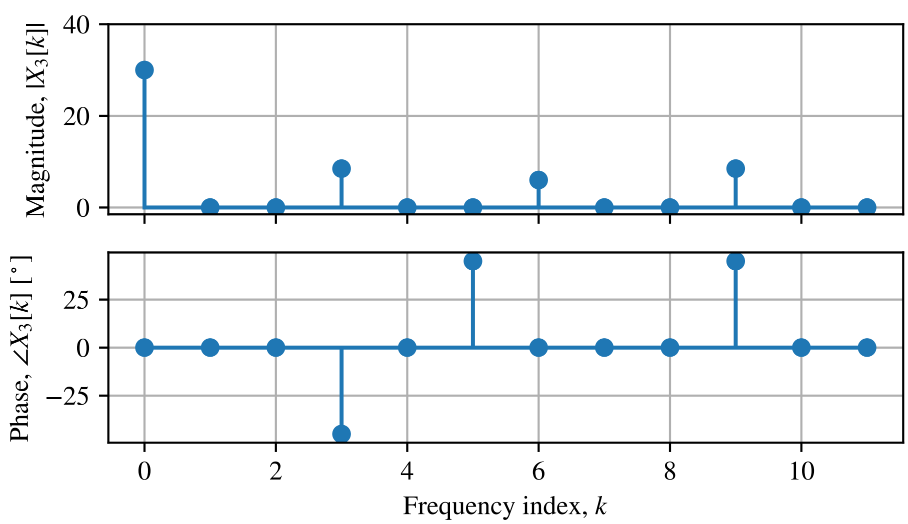


Figure 13

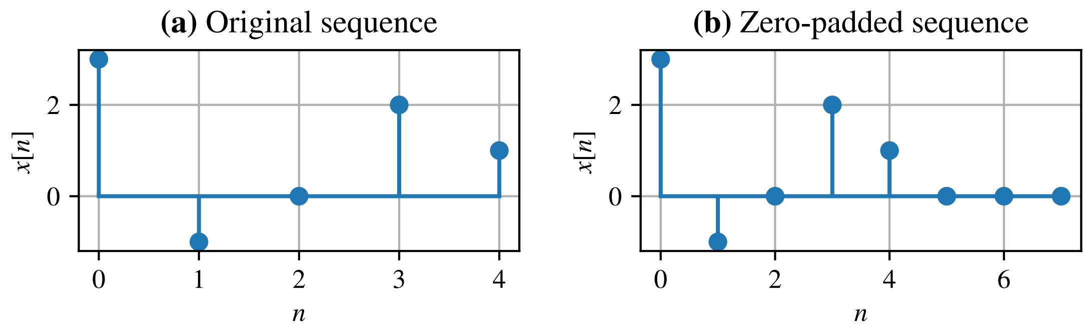


Figure 14: The original and zero-padded sequences for Problem 11.7.

7. (a) Fig. 14(a) shows the original sequence $x[n] = \begin{bmatrix} 3 & -1 & 0 & 2 & 1 \end{bmatrix}$. Its DFT ($k = 0, 1, \ldots, 4$) and DTFT are plotted in Fig. 15.

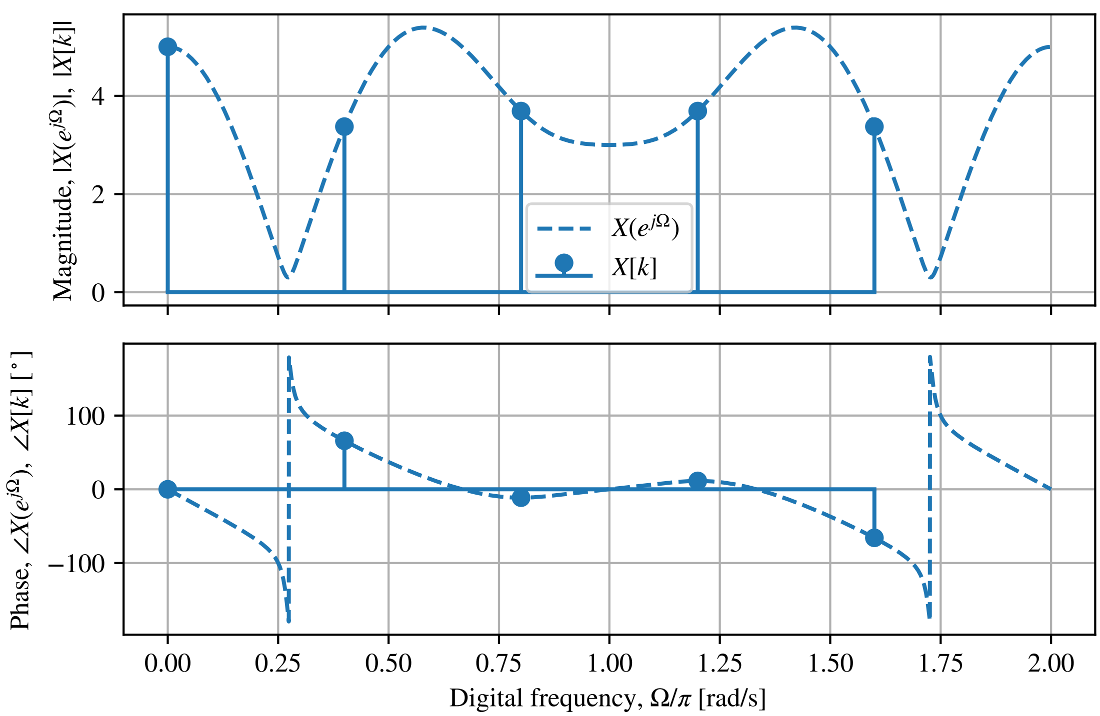


Figure 15

(b) Fig. 14(b) shows the “zero-padded" sequence $x[n] = \begin{bmatrix} 3 & -1 & 0 & 2 & 1 & 0 & 0 & 0 \end{bmatrix}$, which is now an $N = 8$-point signal. Its DFT ($k = 0, 1, \ldots, 7$) and DTFT are plotted in Fig. 16.

Zero-padding does not change the information contained in $x[n]$, so its DTFT does not change. However, zero-padding increases the signal length N, so that $X[k]$ samples $X(e^{j\Omega}$ at smaller intervals. This increases the resolution of the DFT without additional computation.

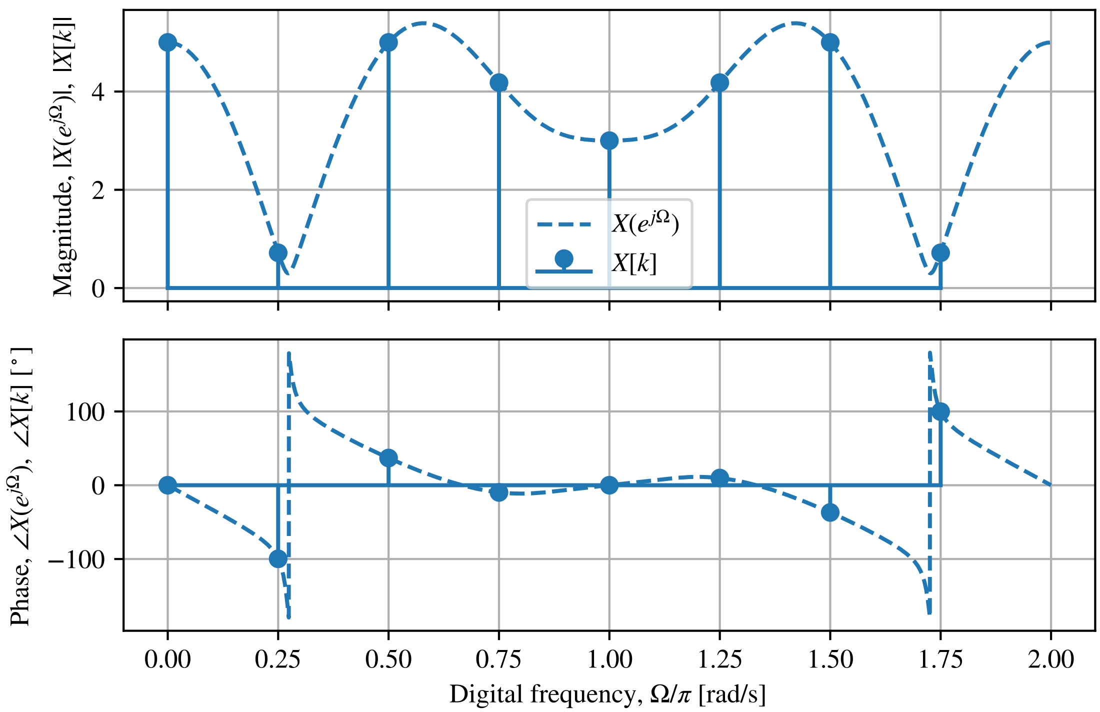

Magnitude, $|X(e^{j\Omega})|$, $|X[k]|$
4
2
0
$X(e^{j\Omega})$
$X[k]$
Phase, $\angle X(e^{j\Omega})$, $\angle X[k]$ [°]
100
0
−100
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
Digital frequency, Ω/π [rad/s]

Figure 16

12. (a) Recall that the analog frequency in Hz, f, of a signal, and the frequency index k are related as

$$f = \frac{1}{N} k f_s \tag{3}$$

The resolution of a DFT is the smallest change in f that a DFT can represent, i.e. the change in f from k to $k+1$. To calculate a DFT's resolution, Δf, let f_k be the analog frequency represented by some k, and let f_{k+1} be the analog frequency represented by the next larger $k+1$. Substituting into Eq. (3) and taking their difference,

$$\begin{aligned}\Delta f &= f_{k+1} - f_k \\ &= \left(\frac{1}{N}(k+1)f_s\right) - \left(\frac{1}{N}k f_s\right) \\ &= \frac{1}{N} f_s \left((k+1) - k\right) \\ &= \frac{1}{N} f_s\end{aligned}$$

For $N = 512$ and $f_s = 16$ kHz, the DFT resolution is $\boxed{31.25 \text{ Hz}}$.

(b) The analog frequencies corresponding to the given ks are shown in Table 2.

k	f [kHz]
0	0
127	3.969
255	7.969
511	15.969

Table 2

NOTE: When a signal is sampled, the frequency information retained is located between zero and $f_s/2$ Hz, half the sampling rate. This corresponds to $0 \leq k \leq N/2$.

Consider the DTFT (dashed line) and DFT (stemmed circles) magnitude response plots, shown in Fig. 17, of an 4th-order Butterworth low-pass filter with cutoff frequency $\Omega_c = 0.5\pi$ rad/s. The DFT is obtained by sampling $H(e^{j\Omega})$ at $N = 11$ points. The plot at the left has a frequency range $-\pi < \Omega < \pi$, corresponding to discrete frequencies $-\frac{N-1}{2} \leq k \leq \frac{N-1}{2}$ Notice that both the DTFT and DFT magnitudes in this range are even-symmetric about zero, with the *low* frequencies in the center; this is how you have plotted DTFTs in class before. However, it is conventional for DFTs to be plotted for $0 \leq k \leq N-1$, corresponding to $0 \leq \Omega < 2\pi$, as shown in the plot on the right. This is exactly the same low-pass response, but in the right half of the plot where $N/2 \leq k < N$ (corresponding to $\pi \leq \Omega < 2\pi$), you are actually seeing the *left* half of the next period/cycle of $X[k]$ or $X(e^{j\Omega})$, since they are both periodic.

The meaningful content of a DFT is therefore located in the range $0 \leq k \leq N/2$, in the same way that the symmetry of the DTFT allows the entire frequency response of a signal to be found by calculating $X(e^{j\Omega})$ over $0 \leq \Omega < \pi$. While you can substitute $k > N/2$ into the equation $f = k f_s/N$ and obtain an analog frequency greater than half the sampling rate, as in this particular problem, that frequency should be considered to belong to the left half of the *next* period.

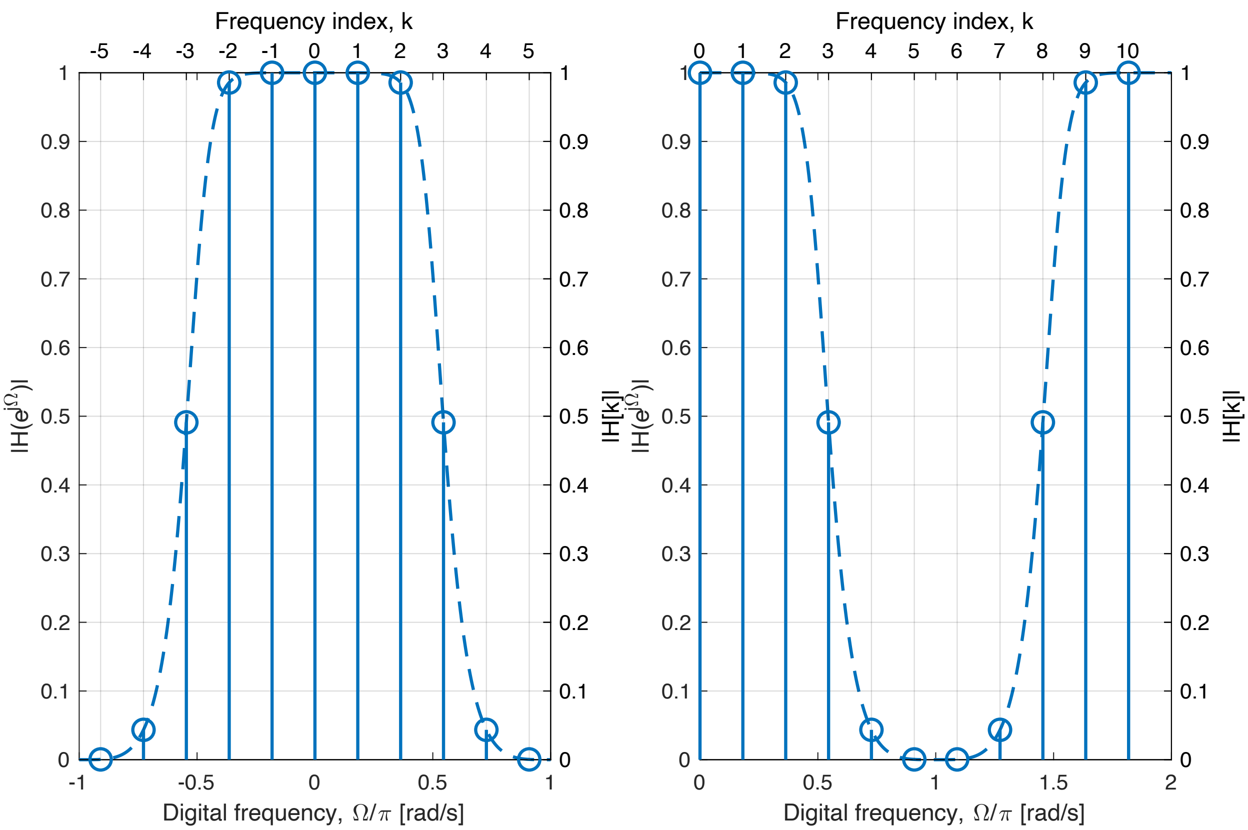


Figure 17: (left) The magnitude response of an 4th-order Butterworth low-pass filter ($\Omega_c = 0.5\pi$ rad/s) in the range $-\pi < \Omega < \pi$, or $-\frac{N-1}{2} \leq k \leq \frac{N-1}{2}$, where $N = 11$. (right) The same magnitude response plotted over the range $0 < \Omega < 2\pi$, or $0 \leq k \leq N-1$.

13. The sampled sine wave $x[n]$ is shown in Fig. 18 for $N = 32$.

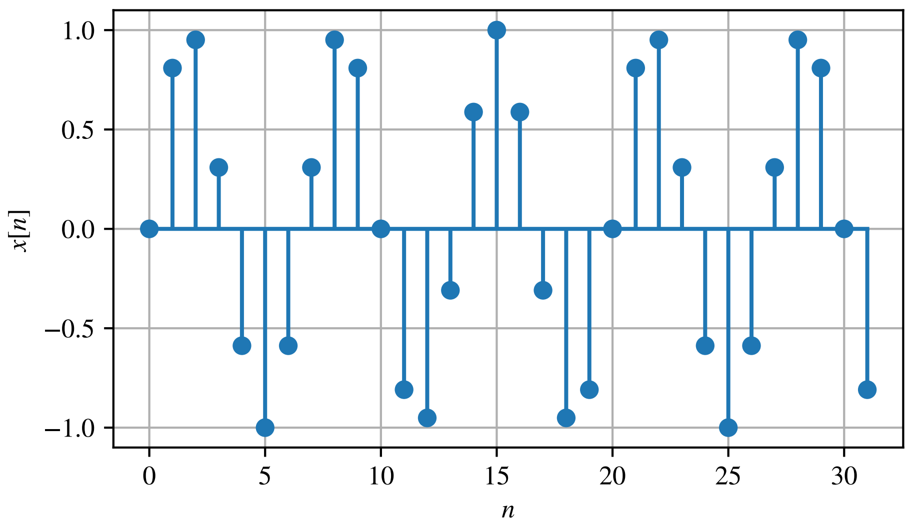


Figure 18

We know that the signal's fundamental analog frequency is 6 kHz. We need to determine the value(s) of k corresponding to that frequency.

(a) $\boldsymbol{N = 32}$: The magnitude response of $x[n]$ calculated over 32 samples is shown in Fig. 19.

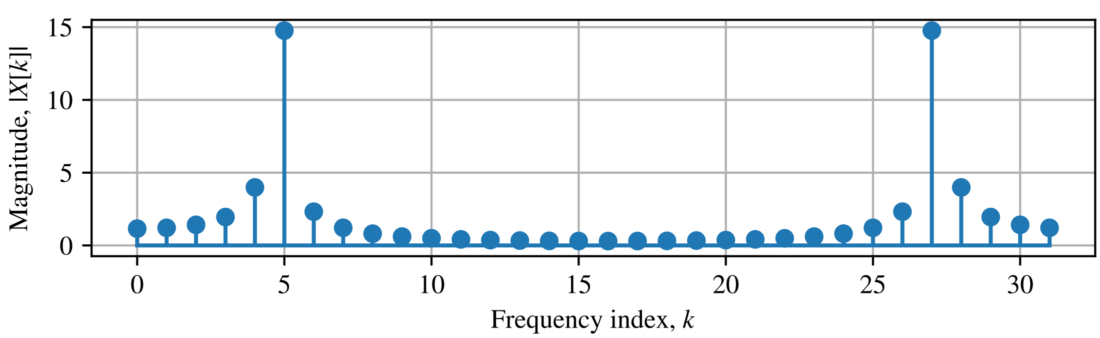


Figure 19

We know that the peak location must correspond to 6 kHz.

$$
\begin{aligned}
f &= \frac{1}{N} k f_s \\
k &= N \frac{f}{f_s} \\
&= (32) \frac{6 \text{ kHz}}{40 \text{ kHz}} = 4.8 \\
&\approx \boxed{5}
\end{aligned}
$$

This agrees with Fig. 19, though note that solving for k may not always result in an integer (this depends on f and f_s), so it must be rounded.

(b) $\boldsymbol{N = 64}$: The magnitude response of $x[n]$ calculated over 64 samples is shown in Fig. 20.

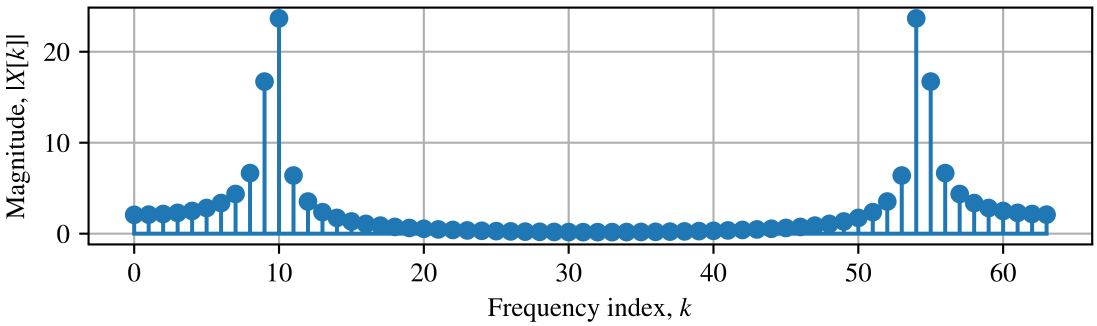


Figure 20

$$
\begin{aligned}
k &= N\frac{f}{f_s} \\
&= (128)\frac{6\ \text{kHz}}{40\ \text{kHz}} = 9.6 \\
&\approx \boxed{10}
\end{aligned}
$$

(c) $\boldsymbol{N = 128}$: The magnitude response of $x[n]$ calculated over 128 samples is shown in Fig. 21.

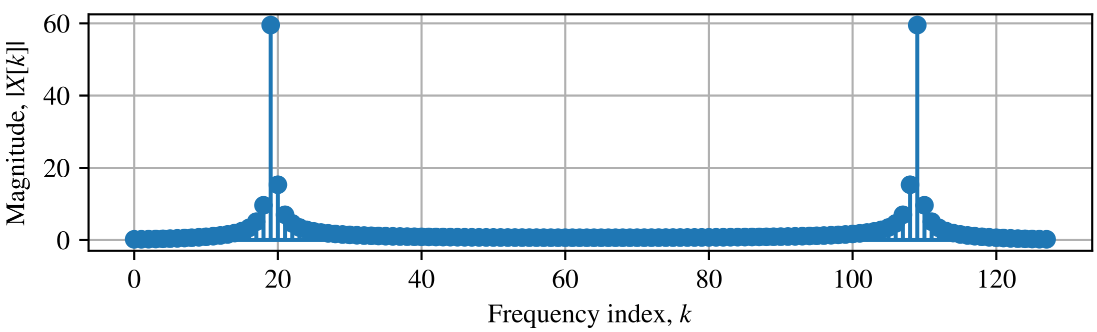


Figure 21

$$
\begin{aligned}
k &= N\frac{f}{f_s} \\
&= (128)\frac{6\ \text{kHz}}{40\ \text{kHz}} = 19.2 \\
&\approx \boxed{19}
\end{aligned}
$$