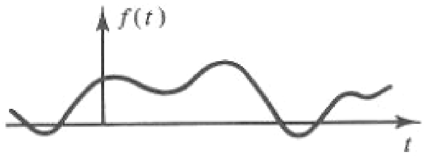
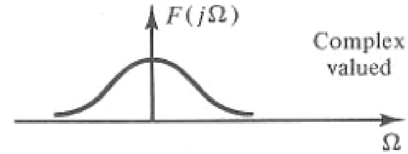
NONPERIODIC CONTINUOUS-TIME



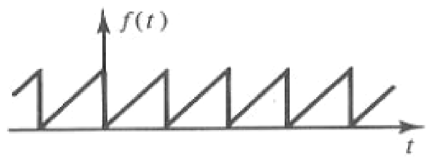
Fourier transform



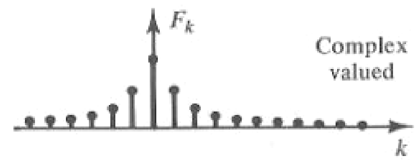
Complex valued

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\Omega)\,e^{j\Omega t}\,d\Omega \qquad F(j\Omega) = \int_{-\infty}^{\infty} f(t)\,e^{-j\Omega t}\,dt$$

PERIODIC CONTINUOUS-TIME



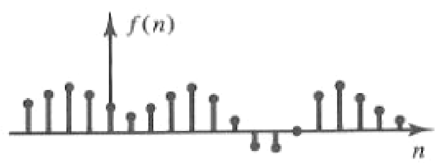
Fourier series



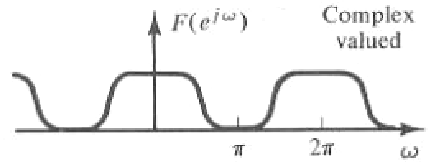
Complex valued

$$f(t) = \sum_{n=-\infty}^{\infty} F_k\,e^{jk\Omega_0 t} \qquad \Omega_0 = 2\pi/T \qquad F_k = \frac{1}{T}\int_{0}^{T} f(t)\,e^{-jk\Omega_0 t}\,dt$$

NONPERIODIC DISCRETE-TIME



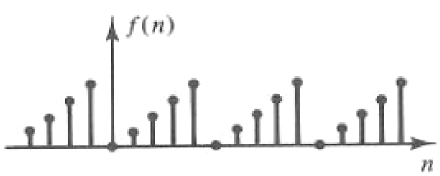
Fourier transform



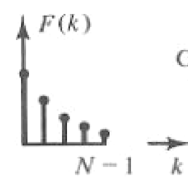
Complex valued

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega})\,e^{j\omega n}\,d\omega \qquad F(e^{j\omega}) = \sum_{n=-\infty}^{\infty} f(n)\,e^{-j\omega n}$$

PERIODIC DISCRETE-TIME



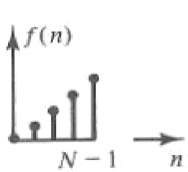
Fourier series



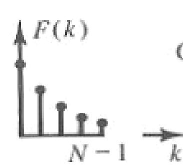
Complex valued

$$f(n) = \frac{1}{N}\sum_{k=0}^{N-1} F(k)\,e^{\left(j\frac{2\pi}{N}nk\right)} \qquad F(k) = \sum_{n=0}^{N-1} f(n)\,e^{\left(-j\frac{2\pi}{N}nk\right)}$$

FIXED LENGTH DISCRETE-TIME

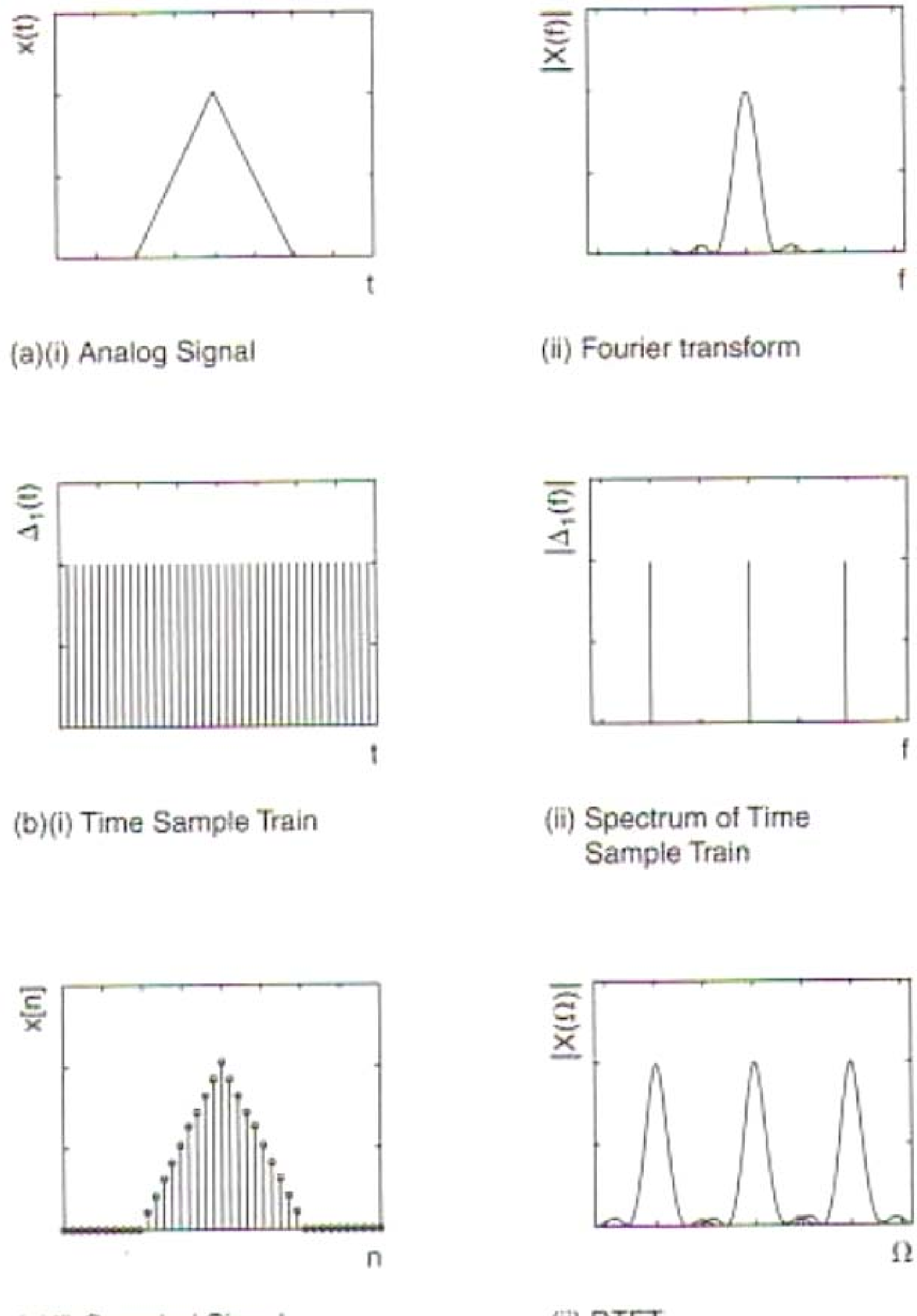


Discrete Fourier transform



Complex valued

$$f(n) = \frac{1}{N}\sum_{k=0}^{N-1} F(k)\,e^{\left(j\frac{2\pi}{N}nk\right)} \qquad F(k) = \sum_{n=0}^{N-1} f(n)\,e^{\left(-j\frac{2\pi}{N}nk\right)}$$



Examples of sampling process

Fourier series representation of an impulse train is denoted by

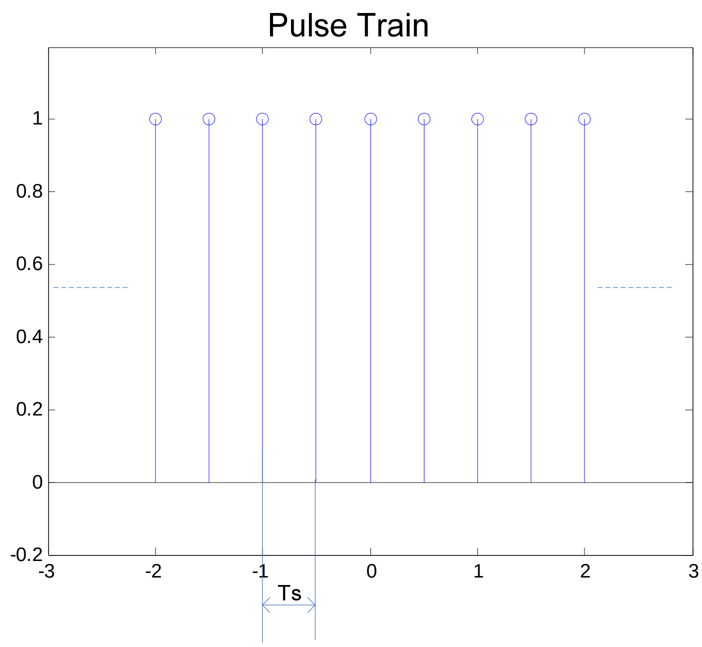


Figure 1: Pulse Train with 2 samples/second

The Fig. 1 shows the pulse train with $T_s = 0.5 \;\; or \;\; \left(f_s = \frac{1}{T_s} = 2 \right)$

Mathematical representation of the pulse train signal is

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \tag{1}$$

Since Fig. 1 is periodic signal, the Fourier series representation for the case $n = 0$ is

$$x_n = \underbrace{\frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} x(t) e^{-\frac{j2\pi nt}{T_s}}}_{\text{For one period}} dt = \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t) e^{-\frac{j2\pi nt}{T_s}} dt = \frac{1}{T_s} \cdot e^{-\frac{j2\pi nt}{T_s}} \Big|_{t=0} = \frac{1}{T_s} e^{-\frac{j2\pi n0}{T_s}} = \frac{1}{T_s} \tag{2}$$

where x_n is the Fourier coefficient.

To find the $x(t)$ again, plug in x_n to the inverse Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \cdot e^{\frac{j2\pi nt}{T_s}} = \sum_{n=-\infty}^{\infty} \frac{1}{T_s} \cdot e^{\frac{j2\pi nt}{T_s}} = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{\frac{j2\pi nt}{T_s}} \tag{3}$$

To find out the frequency response of the signal, the **Fourier transform** has to be applied

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)\cdot e^{-j2\pi ft}dt \\
&= \int_{t=-\infty}^{\infty} \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{\frac{j2\pi nt}{T_s}} \cdot e^{-j2\pi ft}dt \\
&= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\int_{t=-\infty}^{\infty} e^{\frac{j2\pi nt}{T_s}} \cdot e^{-j2\pi ft}dt \\
&= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\int_{t=-\infty}^{\infty} e^{-j2\pi\left(f-\frac{n}{T_s}\right)t}dt \\
&= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\int_{t=-\infty}^{\infty} e^{-j2\pi\left(f-\frac{n}{T_s}\right)t}dt \\
&= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_s}\right) \\
&= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta\left(f-nf_s\right)
\end{aligned}
\tag{4}
$$

The mathematical fact for the equation (4) (5)

$$F\left\{\delta(t)\right\} = \int_{t=-\infty}^{\infty}\delta(t)e^{-j2\pi ft}dt = e^{-j2e\pi ft}\Big|_{t=0} = e^{-j2\pi f0} = 1$$

$$\delta(t) = F^{-1}\left\{F\left\{\delta(t)\right\}\right\} = F^{-1}\left\{1\right\} = \int_{f=-\infty}^{\infty} 1\cdot e^{j2\pi ft}df = \int_{f=-\infty}^{\infty} e^{j2\pi ft}df$$

$$F\left\{\delta(t-t_0)\right\} = \int_{t=-\infty}^{\infty}\delta(t-t_0)e^{-j2\pi ft}dt = e^{-j2\pi ft}\Big|_{t=t_0} = e^{-j2\pi ft_0}$$

$$\delta(t-t_0) = F^{-1}\left\{F\left\{\delta(t-t_0)\right\}\right\} = F^{-1}\left\{e^{-j2\pi ft_0}\right\} = \int_{f=-\infty}^{\infty} e^{-j2\pi ft_0}\cdot e^{j2\pi ft}df = \int_{f=-\infty}^{\infty} e^{j2\pi f(t-t_0)}df$$

$$F^{-1}\left\{\delta(f)\right\} = \int_{f=-\infty}^{\infty}\delta(f)e^{j2\pi ft}df = e^{j2\pi ft}\Big|_{f=0} = e^{j2\pi 0t} = 1$$

$$\delta(f) = F\left\{F^{-1}\left\{\delta(f)\right\}\right\} = F\left\{1\right\} = \int_{t=-\infty}^{\infty} 1\cdot e^{-j2\pi ft}dt$$

$$F^{-1}\left\{\delta(f-f_0)\right\} = \int_{f=-\infty}^{\infty}\delta(f-f_0)e^{j2\pi ft}df = e^{j2\pi ft}\Big|_{f=f_0} = e^{je\pi f_0t}$$

$$\delta(f-f_0) = F\left\{F^{-1}\left\{\delta(f-f_0)\right\}\right\} = F\left\{e^{j2\pi f_0t}\right\} = \int_{t=-\infty}^{\infty} e^{j2\pi f_0t}\cdot e^{-j2\pi ft}dt = \int_{t=-\infty}^{\infty} e^{-j2\pi(f-f_0)t}dt$$

The **sampling theory**:

If the impulse train, $x(t)$, is multiplied with any continuous signal, $s(t)$, then the frequency representation of the multiplication in the time domain is the convolution in the frequency domain.

The mathematical fact for the convolution:

$$\begin{aligned} y(t) &= x(t) * s(t) \\ &= \int_{\tau=-\infty}^{\infty} x(\tau)s(t-\tau)\,d\tau = \int_{\tau=-\infty}^{\infty} s(\tau)x(t-\tau)\,d\tau \\ Y(f) &= X(f) * S(f) \\ &= \int_{\tau=-\infty}^{\infty} X(\tau)S(f-\tau)\,d\tau \end{aligned} \tag{6}$$

Let impulse train be $x(t)$, the continuous signal be $s(t)$, and the output signal be $y(t)$. And if two signals, $x(t)$ and $s(t)$, are multiplied to get the output $y(t)$, then

$$\begin{aligned} & y(t) = x(t)s(t) \\ & \Updownarrow \\ & Y(f) = X(f) * S(f) \\ & \quad = \left[\frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_s} \right) \right] * S(f) \\ & \quad = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \underbrace{\left[\int_{\tau=-\infty}^{\infty} \delta\left(\tau - \frac{n}{T_s} \right) S(f-\tau)\,d\tau \right]}_{\text{convolution of two signals}} \\ & \quad = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} S\left(f - \frac{n}{T_s} \right) \\ & \quad = \underline{\underline{\frac{1}{T_s} \sum_{n=-\infty}^{\infty} S(f - nf_s)}} \end{aligned} \tag{7}$$

The equation (7) shows that the frequency representation, $S(f)$, of the signal, $s(t)$, repeats every $f = nf_s$.