EE 210

HW#: 08

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Assigned question #s: **10**

HW 08

3.6

Professor mentioned that we don't need to do the power expansion

ⓐ $X(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \cdot \frac{z}{z}$

$= \frac{z}{z+\frac{1}{2}} \qquad |z| > \frac{1}{2}$

$\boxed{x[n] = (-\frac{1}{2})^n u[n]}$

For Fourier Transform to exist: $\sum |x[n]|\,|e^{-j\Omega n}| < \infty$

(The summation must be finite)

$\because x[n] = (a)^n u[n]$ Where $|a| = |-\frac{1}{2}| < 1$

$\therefore \sum_{-\infty}^{\infty} |x[n]|\,|e^{-j\Omega n}| < \infty$

$\therefore$ $\boxed{\text{Fourier Transform exists}}$

ⓑ $X(z) = \frac{1}{1+\frac{1}{2}z^{-1}}$

$= \frac{z}{z+\frac{1}{2}} \qquad |z| < \frac{1}{2}$

$\boxed{x[n] = -(-\frac{1}{2})^n u[-n-1]}$

$\because x[n] = -a^n u[-n-1]$ where $|a| = |-\frac{1}{2}| < 1$

$\therefore \sum_{-\infty}^{\infty} |x[n]|\,|e^{-j\Omega n}| < \infty$

$\therefore$ $\boxed{\text{Fourier Transform exists}}$

ⓒ $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} \cdot \dfrac{z^2}{z^2}$ $\qquad |z| > \frac{1}{2}$

$$= \frac{z\left(z-\frac{1}{2}\right)}{z^2+\frac{3}{4}z+\frac{1}{8}}$$

$$= \frac{z(z-1/2)}{(z+1/2)(z+1/4)} = z\left(\frac{A}{z+\frac{1}{2}} + \frac{B}{z+\frac{1}{4}}\right)$$

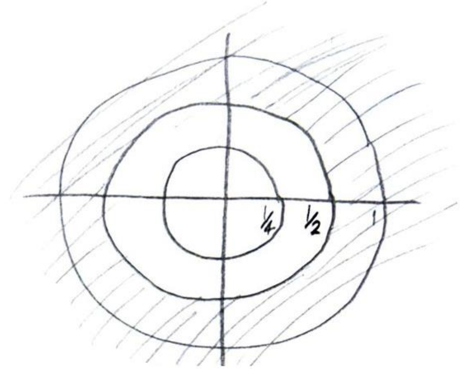
$$\left(z-\tfrac{1}{2}\right) = A\left(z+\tfrac{1}{4}\right) + B\left(z+\tfrac{1}{2}\right)$$

@ $z = -\frac{1}{2} \rightsquigarrow A = 4$

@ $z = -\frac{1}{4} \rightsquigarrow B = -3$

$\therefore X(z) = z\left(\dfrac{4}{z+\frac{1}{2}} - \dfrac{3}{z+\frac{1}{4}}\right) \qquad |z| > \frac{1}{2}$

$$\boxed{x[n] = 4\left(\tfrac{-1}{2}\right)^n u[n] - 3\left(\tfrac{-1}{4}\right)^n u[n]}$$



$\because x[n] = 4\,a^n u[n] - 3\,b^n u[n]$ where $|a|$ & $|b| < 1$

$\therefore \sum_{-\infty}^{\infty} |x[n]|\,|e^{-j\Omega n}| < \infty$

$\therefore$ $\boxed{\text{Fourier Transform exists}}$

3.7.

ⓐ $x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n]$

$\hookrightarrow X(z) = \frac{-z}{z-1} + \frac{z}{z-\frac{1}{2}}$

$\rightarrow |z| > |\frac{1}{2}|$

$\rightarrow |z| < |1|$

$\Big\} \quad \frac{1}{2} < |z| < 1$

$$X(z) = \frac{-\cancel{z^2} - \frac{1}{2}z + \cancel{z^2} - z}{(z-1)(z-\frac{1}{2})}$$

$$= \frac{-\frac{3}{2}z}{(z-1)(z-\frac{1}{2})}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{\cancel{-}\frac{1}{\cancel{2}}}{(\cancel{z-\frac{1}{2}})(z+1)} \cdot \frac{(z-1)(\cancel{z-\frac{1}{2}})}{\cancel{-}\frac{3}{\cancel{2}}z}$$

$$\boxed{H(z) = \frac{1}{3}\,\frac{(z-1)}{z(z+1)}}$$

$\rightsquigarrow$ Causal system

$\therefore$ ROC: $|z| > |1|$

ⓑ $$Y(z) = \frac{-\frac{1}{2}}{(z-\frac{1}{2})(z+1)}$$

Different possible ROCs for $Y(z)$:

$\boxed{1}$ $|z| > |\frac{1}{2}|$, $|z| > |1|$ $\rightsquigarrow$ $\boxed{|z| > |1|}$ if it is causal

$\boxed{2}$ $|z| < |\frac{1}{2}|$, $|z| < |1|$ $\rightsquigarrow$ $\boxed{|z| < |\frac{1}{2}|}$

$\boxed{3}$ $|z| > |\frac{1}{2}|$, $|z| < |1|$ $\rightsquigarrow$ $\boxed{\frac{1}{2} < |z| < 1}$

Not causal

$\boxed{4}$ $|z| \cancel{<} |\frac{1}{2}|$, $|z| > |1|$ $\rightarrow$ Not possible

* In $Y(z)$, the $z = \frac{1}{2}$ pole comes from $X(z)$ where $|z| > |\frac{1}{2}|$

and the $z = -1$ pole comes from $H(z)$ where $|z| > |1|$

$\therefore$ ROC of $Y(z)$ is $\boxed{|z| > 1}$ "causal"

ⓒ $Y(z) = \frac{-1/2}{(z-\frac{1}{2})(z+1)} = \frac{A}{(z-1/2)} + \frac{B}{(z+1)}$ ➡ $-\frac{1}{2} = A(z+1) + B(z-\frac{1}{2})$

@ $z=-1 \rightsquigarrow B = 1/3$

@ $z=\frac{1}{2} \rightsquigarrow A = -1/3$

$$Y(z) = \frac{-1/3 \,.Z.Z^{-1}}{z - 1/2} + \frac{1/3 \,.Z.Z^{-1}}{Z+1}$$

→ For caussal system: $y[n] = \left(-\frac{1}{3}\left(\frac{1}{2}\right)^n u[n]\right) * \delta(n-1) + \left(\frac{1}{3}(-1)^n u[n]\right) * \delta(n-1)$

$$\boxed{y[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1] + \frac{1}{3}(-1)^{n-1} u[n-1]}$$

ROC $\boxed{|z|>1}$

3.8

ⓐ

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4} z^{-1}} \begin{matrix} \cdot z \\ \cdot z \end{matrix}$$

$$= \frac{z - 1}{z + 3/4} = \frac{z}{z + 3/4} - \frac{1}{z + 3/4} \cdot z \cdot z^{-1}$$

$$\therefore h[n] = \left(\frac{-3}{4}\right)^n u[n] - \left(\left(\frac{-3}{4}\right)^n u[n]\right) * \delta(n-1)$$

$$\boxed{\therefore h[n] = \left(-\frac{3}{4}\right)^n u[n] - \left(-\frac{3}{4}\right)^{n-1} u[n-1]}$$

ROC $|z| > \frac{3}{4}$

Causal system

ⓑ

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + u[-n-1]$$

$$X(z) = \frac{z}{z - \frac{1}{3}} - \frac{z}{z - 1} \qquad \frac{1}{3} < |z| < 1$$

$$= \frac{z^2 - z - z^2 + \frac{1}{3} z}{(z - \frac{1}{3})(z - 1)} = -\frac{2/3}{(z - \frac{1}{3})(z - 1)}$$

$$\therefore Y(z) = H(z) . X(z) = \frac{\cancel{z - 1}}{z + 3/4} \cdot \frac{-2/3}{(z - \frac{1}{3})\cancel{(z - 1)}}$$

$$= \frac{-2/3}{(z + 3/4)(z - 1/3)} = \frac{A}{z + 3/4} + \frac{B}{z - 1/3}$$

$$\rightsquigarrow -\frac{2}{3} = A\left(z - \frac{1}{3}\right) + B\left(z + \frac{3}{4}\right)$$

$$@ z = -\frac{3}{4} \longrightarrow \therefore A = \frac{8}{13}$$

$$@ z = \frac{1}{3} \longrightarrow \therefore B = -8/13$$

$$\therefore Y(z) = \frac{8/13 \cdot z z^{-1}}{z + 3/4} - \frac{8/13 \cdot z z^{-1}}{z - 1/3}$$

ROC $|z| > 3/4$

$|z| > |-3/4|$

$|z| > |1/3|$

Causal system

$$\therefore y[n] = \frac{8}{13}\left(\left(-3/4\right)^n u[n]\right) * \delta[n-1] - \frac{8}{13}\left(\left(\frac{1}{3}\right)^n u[n]\right) * \delta[n-1]$$

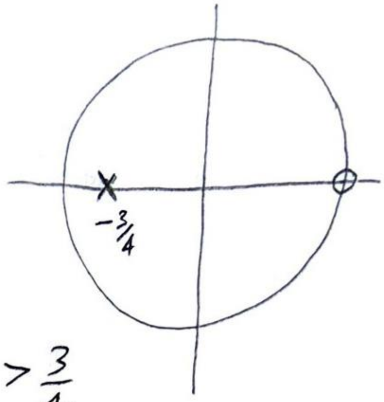
$$\boxed{\therefore y[n] = \frac{8}{13}\left(-3/4\right)^{n-1} u[n-1] - \frac{8}{13}\left(\frac{1}{3}\right)^{n-1} u[n-1]}$$

ⓒ For $H(z)$, there is a pole at $z = -\frac{3}{4}$
which is inside the unit circle in the complex plane
So it is $\boxed{\text{Stable}}$

↳ it is summable $\sum_{-\infty}^{\infty} |h[n]| < \infty$

OR $|H(z)|_{z=1} < \infty$

Causal $\Downarrow$
ROC: $|z| > \frac{3}{4}$

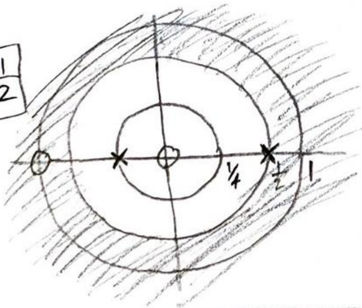


3.9

ⓐ $H(z) = \dfrac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})} \begin{matrix} \cdot z^2 \\ \cdot z^2 \end{matrix} = \dfrac{z(z+1)}{(z-\frac{1}{2})(z+\frac{1}{4})}$

$|z| > |\frac{+1}{2}|$ $\quad$ $|z| > |\frac{-1}{4}|$ $\quad$ Causal system

ROC : $\boxed{|z| > \frac{1}{2}}$



ⓑ The system is stable because $H(z)$ has 2 poles $z = \frac{1}{2}$ and $z = -1/4$ and both are inside the unit circle of the ~~Z-~~plane.

ⓒ $y[n] = -\frac{1}{3}\left(\frac{-1}{4}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1]$

$Y(z) = -\frac{1}{3}\dfrac{z}{z+\frac{1}{4}} + \frac{4}{3}\dfrac{z}{z-2}$ $\rightarrow$ ROC: $\frac{1}{4} < |z| < 2$

$= \dfrac{-\frac{1}{3}z^2 + \frac{2}{3}z + \frac{4}{3}z^2 + \frac{1}{3}z}{(z+\frac{1}{4})(z-2)}$

$= \dfrac{z(z+1)}{(z+\frac{1}{4})(z-2)}$

$\therefore X(z) = \dfrac{Y(z)}{H(z)} = \dfrac{z(z+1)}{(z+\frac{1}{4})(z-2)} \cdot \dfrac{(z-\frac{1}{2})\cancel{(z+\frac{1}{4})}}{\cancel{z}\cancel{(z+1)}}$

$\therefore \boxed{X(z) = \dfrac{z-1/2}{z-2}}$ $\rightarrow |z| < |2|$

ⓓ $$H(z) = \frac{z(z+1)}{(z-\frac{1}{2})(z+\frac{1}{4})} = z\left(\frac{A}{z-\frac{1}{2}} + \frac{B}{z+\frac{1}{4}}\right)$$

$$z+1 = A(z+\tfrac{1}{4}) + B(z-\tfrac{1}{2})$$

@ $z = -\frac{1}{4}$: $B = -1$

@ $z = \frac{1}{2}$: $A = 2$

$$\therefore H(z) = z\left(\frac{2}{z-\frac{1}{2}} - \frac{1}{z+\frac{1}{4}}\right)$$

Causal system

$$\boxed{h[n] = 2\left(\tfrac{1}{2}\right)^n u[n] - \left(-\tfrac{1}{4}\right)^n u[n]}$$

3.10

ⓓ $x[n] = \left(\frac{1}{4}\right)^{n+4} u[n-1] - \left(e^{j\pi/3}\right)^{n} u[n-1]$

$$= \left(\frac{1}{4}\right)^{n-1+5} u[n-1] - \left(e^{j\pi/3}\right)^{n-1+1} u[n-1]$$

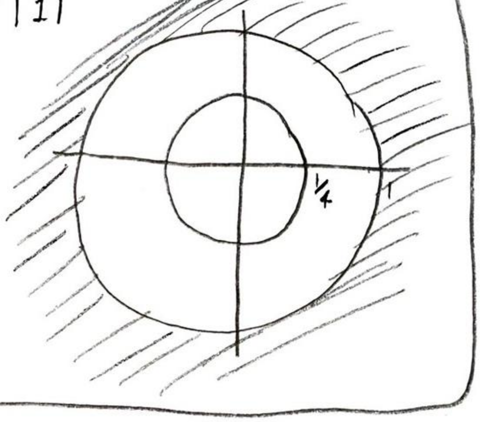
$$= \left(\frac{1}{4}\right)^{5} \left[\left(\frac{1}{4}\right)^{n} u[n]\right] * \delta(n-1) - e^{j\pi/3} \left[\left(e^{j\pi/3}\right)^{n} u[n]\right] * \delta(n-1)$$

$\Downarrow$ ROC: $|z| > \left|\frac{1}{4}\right|$

$\Downarrow$ ROC: $|z| > \left|e^{j\pi/3}\right|$

$|z| > |1|$

∴ ROC: $|z| > |1|$



$*$ The Fourier Transform does converge because

$$\sum_{n=-\infty}^{\infty} |x[n]| \, |e^{-j\Omega n}| < \infty \quad \text{"finite"}$$

In both terms of $x[n]$, they are in the form of $a^n u[n]$ where $|a| < 1$

3.11

ⓑ $X(z) = \dfrac{(z-1)^2}{(z-\frac{1}{2})}$

$$= \frac{z^2-2z+1}{z-\frac{1}{2}}$$

$$= \underbrace{z}\,\frac{z}{z-\frac{1}{2}} - \frac{2z+1}{z-\frac{1}{2}}$$

This will cause a convolution by $\delta[n+1]$ which will shift the function towards the future

This is a z-transform of a NON-CAUSAL sequence, because the degree of the numerator is higher than the degree of the denominator,

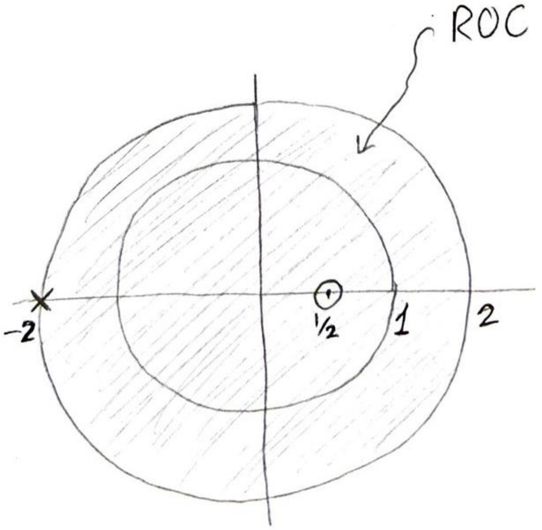
which will result in a shift towards future times in time-domain.

3.12

ⓐ $X_1(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+2z^{-1}} \quad \dfrac{\cdot z}{\cdot z}$

$$= \frac{z-\frac{1}{2}}{z+2}$$

ROC: $|z| < 2$



3.16

ⓐ

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + (2)^n u[-n-1]$$

$$X(z) = \frac{z}{z-\frac{1}{3}} - \frac{z}{z-2} \qquad \text{ROC: } \frac{1}{3} < |z| < 2$$

$$= \frac{\cancel{z^2} - 2z - \cancel{z^2} + \frac{1}{3}z}{(z-\frac{1}{3})(z-2)}$$

$$= -\frac{5}{3}\,\frac{z}{(z-\frac{1}{3})(z-2)}$$

$$y[n] = 5\left(\frac{1}{3}\right)^n u[n] - 5\left(\frac{2}{3}\right)^n u[n]$$

$$Y(z) = 5\,\frac{z}{z-\frac{1}{3}} - 5\,\frac{z}{z-\frac{2}{3}} \qquad \text{ROC: } |z| > \frac{2}{3}$$

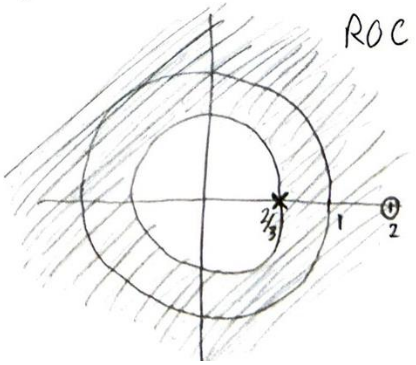
$$= 5\,\frac{\cancel{z^2} - \frac{2}{3}z - \cancel{z^2} + \frac{1}{3}z}{(z-\frac{1}{3})(z-\frac{2}{3})}$$

$$= -\frac{5}{3}\,\frac{z}{(z-\frac{1}{3})(z-\frac{2}{3})}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \cancel{-\frac{5}{3}}\,\frac{\cancel{z}}{\cancel{(z-\frac{1}{3})}(z-\frac{2}{3})} \cdot \cancel{-\frac{3}{5}}\,\frac{\cancel{(z-\frac{1}{3})}(z-2)}{\cancel{z}}$$

$$\therefore \boxed{H(z) = \frac{z-2}{z-\frac{2}{3}}}$$

$$\text{ROC: } |z| > \frac{2}{3}$$



ⓑ $$H(z) = \frac{z}{z-\frac{2}{3}} - \frac{2\,.\,z z^{-1}}{z-\frac{2}{3}}$$

Causal

$$h[n] = \left(\tfrac{2}{3}\right)^n u[n] - 2\left(\left(\left(\tfrac{2}{3}\right)^n u[n]\right) * \delta(n-1)\right)$$

$$\boxed{h[n] = \left(\tfrac{2}{3}\right)^n u[n] - 2\left(\tfrac{2}{3}\right)^{n-1} u[n-1]}$$

ⓒ $$\frac{Y(z)}{X(z)} = \frac{z-2}{z-\frac{2}{3}} \rightsquigarrow Y(z)\left(z-\tfrac{2}{3}\right) = X(z)\,(z-2)$$

$$y[n+1] - \tfrac{2}{3}y[n] = x[n+1] - 2x[n]$$

∴ Difference Equation: $\boxed{y[n] - \tfrac{2}{3}y[n-1] = x[n] - 2x[n-1]}$

ⓓ System is stable → because all the poles of $H(z)$ is within the unit circle in the complex plane. $(z = 2/3)$

System is causal → because ROC of $H(z)$ is outwards of the circle of the furthest pole from the origin $\left(|z| > \frac{2}{3}\right)$

3.17

$$y[n] - \frac{5}{2} y[n-1] + y[n-2] = x[n] - x[n-1]$$

Z transform

$$Y(z) - \frac{5}{2} Y(z) z^{-1} + Y(z) z^{-2} = X(z) - X(z) z^{-1}$$

$$Y(z) . \left[1 - \frac{5}{2} z^{-1} + z^{-2}\right] = X(z).(1 - z^{-1})$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}} = \frac{z(z-1)}{z^2-\frac{5}{2}z+1}$$

$$\therefore \boxed{H(z) = \frac{z(z-1)}{(z-2)(z-\frac{1}{2})}} = z \frac{A}{z-2} + \frac{B}{z-\frac{1}{2}}$$

$$(z-1) = A(z-\tfrac{1}{2}) + B(z-2)$$

$@z=\frac{1}{2} \longrightarrow \therefore B = \frac{1}{3}$

$@z=2 \longrightarrow \therefore A = \frac{2}{3}$

$$\therefore H(z) = \frac{2}{3}\frac{z}{z-2} + \frac{1}{3}\frac{z}{z-\frac{1}{2}}$$

⇒ There are 3 possible ROCs:

① $|z| > |2| , |z| > |\frac{1}{2}| \rightsquigarrow \underline{|z| > 2}$ "causal"

$\therefore h[n] = \frac{2}{3}(2)^n u[n] + \frac{1}{3}(\frac{1}{2})^n u[n] \longrightarrow h[0] = \frac{2}{3}(2)^0 + \frac{1}{3}(\frac{1}{2})^0 = 1$

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② $|z| < |2| , |z| > |\frac{1}{2}| \rightsquigarrow \underline{\frac{1}{2} < |z| < 2}$ "non-causal"

$\therefore h[n] = -\frac{2}{3}(2)^n u[-n-1] + \frac{1}{3}(\frac{1}{2})^n u[n] \longrightarrow h[0] = 0 + \frac{1}{3}(\frac{1}{2})^0 = \frac{1}{3}$

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③ $|z| < |2| , |z| < |\frac{1}{2}| \rightsquigarrow |z| < \frac{1}{2}$ "non-causal"

$\therefore h[n] = -\frac{2}{3}(2)^n u[-n-1] - \frac{1}{3}(\frac{1}{2})^n u[-n-1] \longrightarrow h[0] = 0 + 0 = 0$

$\therefore$ The possible values of $h[0] = \{0, \frac{1}{3}, 1\}$

3.19

ⓒ

$$X(z) = \frac{1}{(1-\frac{1}{5}z^{-1})(1+3z^{-1})} = \frac{z^2}{(z-\frac{1}{5})(z+3)}$$

$|z| > \frac{1}{5}$
$|z| < 3$

$$H(z) = \frac{1+3z^{-1}}{1+\frac{1}{3}z^{-1}} = \frac{z+3}{z+\frac{1}{3}} \qquad |z| > \frac{1}{3}$$

$$\therefore Y(z) = H(z).X(z) = \frac{\cancel{z+3}}{z+\frac{1}{3}} \cdot \frac{z^2}{(z-\frac{1}{5})\cancel{(z+3)}}$$

$$\therefore Y(z) = \frac{z^2}{(z+\frac{1}{3})(z-\frac{1}{5})}$$

$|z| > \frac{1}{5}$
$|z| > \frac{1}{3}$

$\therefore$ ROC: $|z| > \frac{1}{3}$