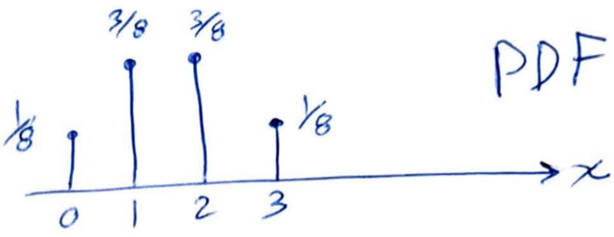
CH4 Continuous RV

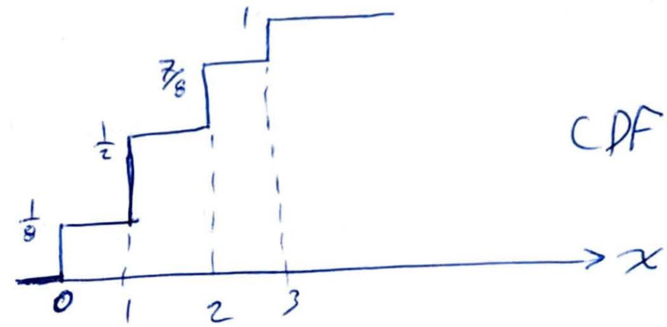
ex Toss Coin 3 times

\# of Heads $S_X = \{0,1,2,3\}$



PMF: $\boxed{P_X(x) = P[X=x]}$

CDF: $\boxed{F_X(x) = P[X \leq x]}$



for $x<0$: $F_X(x) = P[X \leq \cancel{x}] = 0$

$x=0$: $F_X(x) = P[X \leq x] = \frac{1}{8}$

$0<x<1$: $F_X(x) = P[X \leq x] = \frac{1}{8}$

$x=1$: $F_X(x) = P[\cancel{X}=1] + P[X=0] = \frac{4}{8}$

⋮

Discontinuous

$\lim_{x\to a^-} F(x) \neq \lim_{x\to a^+} F(x)$

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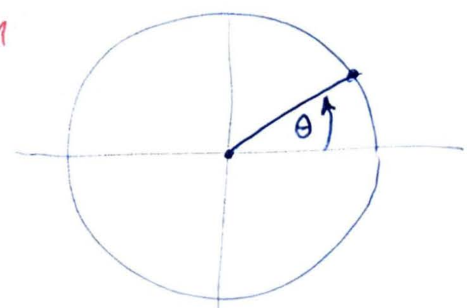
ex $0 < \theta \leq 2\pi$ $P[\theta] =$ Uniform

find CDF of X where $X = \frac{\theta}{2\pi}$?



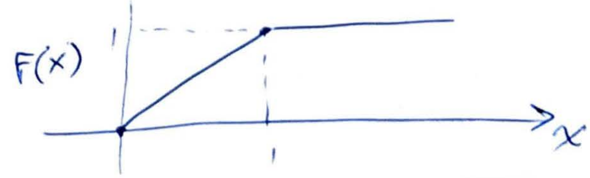
$0 < X \leq 1$

$x<0$: $F_X(x) = P[X \leq x] = 0$

$0<x\leq 1$: $F_X(x) = P[X \leq x]$

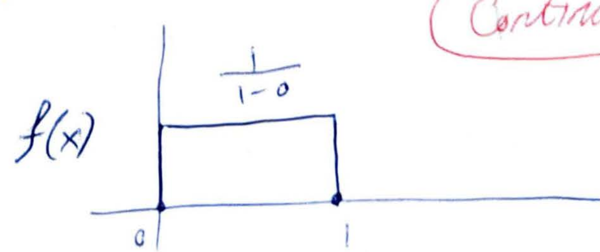
$= P[\frac{\theta}{2\pi} \leq x] = P[\theta \leq 2\pi x] = \frac{\theta}{2\pi} = \frac{2\pi x}{2\pi} = x$

$x > 1$: $F_X(x) = 1$



$\frac{d}{dx}$

Continuous



P(x)

ex// Waiting time of customer X → 0 if taxi is there
→ Uniformly distributed Random Length of time $[0, 1[$ in hrs if taxi is not there

Prob of taxi there $= p$

Find CDF of X?

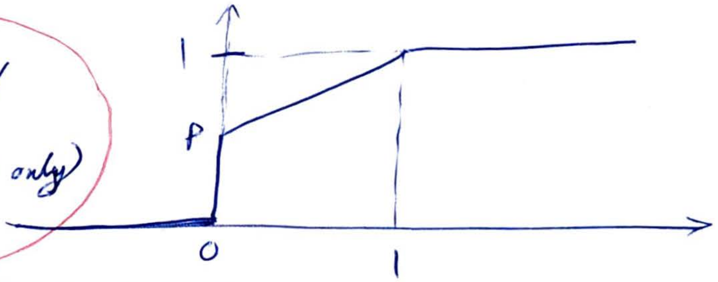
$$F_X(x) = P[X \leq x] \qquad 0 \leq x < 1$$

$$= P[X \leq x \mid \text{taxi is there}]\, P[\text{taxi is there}] + P[X \leq x \mid \text{taxi not there}]\, P[\text{taxi not there}]$$

$$= \underbrace{P\left[\begin{matrix}\text{waiting time}\\ \text{is less than } x\end{matrix} \middle| \begin{matrix}\text{taxi is}\\ \text{there}\end{matrix}\right]}_{1} \cdot \underbrace{p}_{} + \underbrace{P\left[\begin{matrix}\text{waiting time}\\ \text{less than } x\end{matrix} \middle| \begin{matrix}\text{taxi not}\\ \text{there}\end{matrix}\right]}_{x} \cdot (1-p)$$

$$\therefore F_X(x) = p + (1-p)x$$

Mixed RV
(discontinuous @ 0 only)



CDF properties

a) $\lim_{x \to -\infty} F_X(x) = 0$

b) $\lim_{x \to +\infty} F_X(x) = 1$

c) if $\alpha < \beta$, $F(\alpha) \leq F(\beta)$ → $F(x)$ monotonically increasing

d) $P[\alpha < X \leq \beta] = F(\beta) - F(\alpha^+)$

e) $P[X = \alpha] = F_X(\alpha^+) - F_X(\alpha^-)$

$$P[\alpha \leq X \leq \beta] = P[X=\alpha] + P[\alpha < X \leq \beta]$$
$$= F_X(\alpha^+) - F_X(\alpha^-) + (F(\beta) - F(\alpha^+))$$
$$= F_X(\beta) - F_X(\alpha^-)$$

pdf

$$f_X(x) = \frac{dF_X(x)}{dx} \simeq \frac{F_X(x+h) - F_X(x)}{h} = \frac{P[x < X < x+h]}{h}$$

a) $f_X(x) \geq 0$

b) $P[a < X < b] = F_X(b) - F_X(a) = \int_a^b f_X(x)\,dx$ "area"

c) $\int_{-\infty}^{\infty} f_X(x)\,dx = F_X(+\infty) - F_X(-\infty) = 1$ "normalization eqn"

$$F_X(x) = \int_{-\infty}^{x} f(x)\,dx$$

ex) pdf of speech signal : $f_X(x) = c e^{-\alpha |x|}$

$-\infty < x < +\infty$
$\alpha > 0$

a) find c?

$\int_{-\infty}^{\infty} f_X(x) = 1 \rightarrow 2\int_0^{\infty} c e^{-\alpha x} = 1$

$\int_0^{\infty} e^{-\alpha x}\,dx = \frac{1}{\alpha}$ $\quad \alpha > 0$

$2\frac{c}{\alpha} = 1 \quad \therefore \boxed{c = \frac{\alpha}{2}}$

b) $P[|X| < \nu] = ?$

$P[|X| < \nu] = P[-\nu < X < \nu] = \int_{-\nu}^{\nu} f_X(x)\,dx$ even function

$= 2\int_0^{\nu} \frac{\alpha}{2} e^{-\alpha x}\,dx$

$= \alpha\left[\frac{-1}{\alpha} e^{-\alpha x}\right]_0^{\nu} = \alpha\left[\frac{-1}{\alpha} e^{-\alpha \nu} - \left(\frac{-1}{\alpha} e^{0}\right)\right] = 1 - e^{-\alpha \nu}$

☆ Conditional Pdf

pmf Discrete	CDF	pdf Continuous
$P_X(x) = P[X=x]$	$F_X(x) = P[X \leq x]$	
$P_X(x\|A) = P[X=x\|A]$	$F_X(x\|A) = P[X \leq x \| A]$	
	$= \dfrac{P[X \leq x \cap A]}{P[A]} \xrightarrow{d}$	$f_X(x\|A) = \dfrac{dF(x\|A)}{dx}$
$P_X(x) = \sum_{i=1}^{n} P_X(x\|B_i)\, p(B_i)$	$F_X(x) = \sum_{i=1}^{n} F_X(x\|B_i)\, p(B_i) \xrightarrow{d}$	

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ex

$$F_X(x \mid X>t) = P[X \leq x \mid X>t]$$

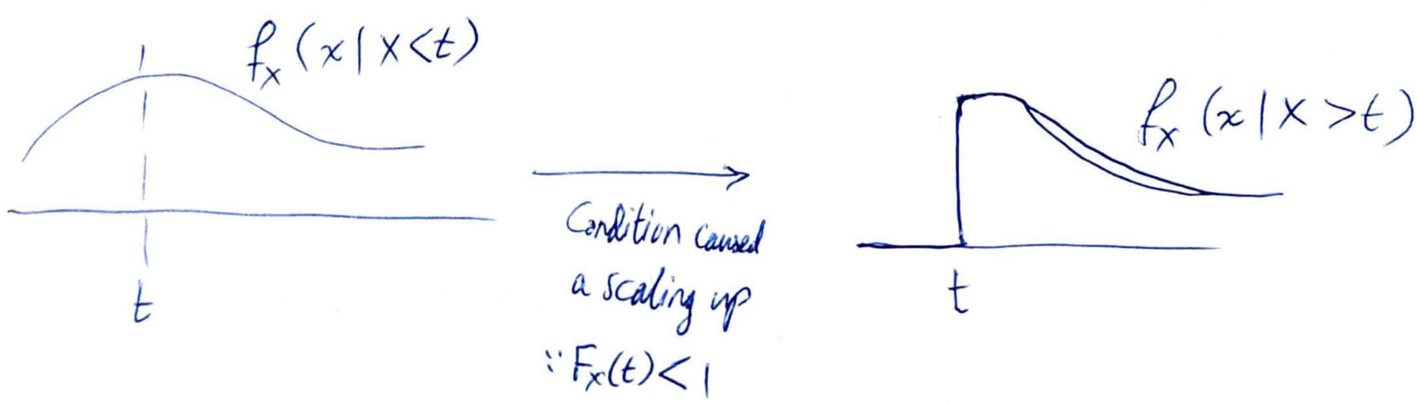
$$= \frac{P[X \leq x \cap X>t]}{P[X>t]}$$

$$= \begin{cases} \dfrac{P[t < X \leq x]}{1-P[X \leq t]} & x>t \\ 0 & x<t \end{cases}$$

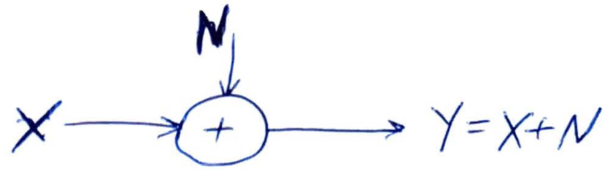
$$= \begin{cases} \dfrac{F_X(x) - F_X(t)}{1-F_X(t)} & x>t \\ 0 & x<t \end{cases}$$

Constant

$$\therefore f_X(x \mid X>t) = \frac{d}{dx} F_X(x \mid X>t) = \begin{cases} \dfrac{f_X(x)}{1-F_X(t)} & x>t \\ 0 & x<t \end{cases}$$



ex. Channel adds White Noise AWGN



$P[X=+v]=p$

$P[X=-v]=1-p$

pdf of noise $f_N(n)=\dfrac{e^{-n^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$ $\quad -\infty<n<+\infty$ Gaussian

$f_Y(y)=?$

always start with CDF since it is directly related to PMF probability

$F_Y(y)=F_Y(Y=y\mid X=v)\,P[X=v]+F_Y(Y=y\mid X=-v)\,P[X=-v]$

$=P(Y\leq y\mid X=v)\,P[X=v]+P(Y\leq y\mid X=-v)\cdot P[X=-v]$

$y=x+n$

$n=y-x=y-v$

$=P(N\leq y-v)\;p+P(N\leq y+v)\;(1-p)$

$\therefore f_Y(y)=f_N(y-v)\,p+f_N(y+v)\,(1-p)$

☆ Expectation:

$$\boxed{E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx}$$

proof

$$\int_{-\infty}^{\infty} (m-x)\, f_X(x)\, dx = \int_{-\infty}^{m} \underbrace{(m-x)}_{u} f_X(x)dx + \int_{m}^{\infty} \underbrace{(m-x)}_{-u} f_X(x)dx$$

$x=m \to u=0$ $\quad$ $x=m \to u=0$

$x=-\infty \to u=\infty$ $\quad$ $x=\infty \to u=\infty$

$$= \int_{\infty}^{0} (u)\, f_X(m-u)(-du) + \int_{0}^{\infty} (-u)\, f_X(m+u)\, du$$

$$= \int_{0}^{\infty} u\, [f_X(m-u) - f_X(m+u)]\, du = 0$$

assuming m is the point of symmetry

$$\therefore m \underbrace{\int_{-\infty}^{\infty} f_X(x)\, dx}_{1} - \underbrace{\int_{-\infty}^{\infty} x\, f_X(x)\, dx}_{E[X]} = 0$$

$\therefore E[X] = m$

☆ Variance:

$$\boxed{\begin{aligned} VAR[X] &= E[X - E[X]]^2 \\ &= \int_{-\infty}^{\infty} (x-m)^2 f_X(x)\, dx \\ &= E[X^2] - m^2 \end{aligned}}$$

☆ n^{th} order moment

$$\boxed{E[X^n]}$$

$$\Rightarrow \boxed{E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx}$$

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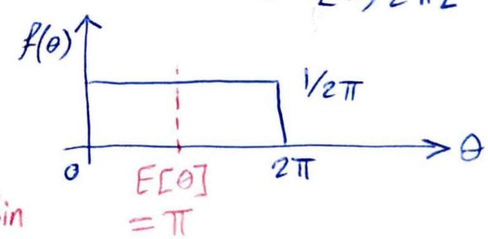
ex Sinusoid RV with Uniform Random Phase

$$Y = a \cos(\omega t + \theta)$$

θ = Uniform RV $[0, 2\pi[$



$$E[Y] = E[a \cos(\omega t + \theta)]$$

$$= \int_{0}^{2\pi} \frac{a}{2\pi} \cos(\omega t + \theta)\, d\theta = 0$$

average of Cos & Sin over 2π is 0

$$E[Y^2] = \int_{0}^{2\pi} a^2 \cos^2(\omega t + \theta) \underbrace{\left(\frac{1}{2\pi}\right)}_{f_X(x)} d\theta$$

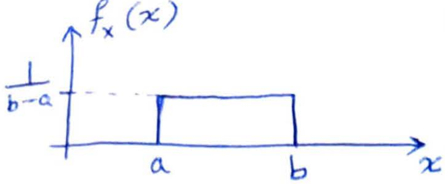
$$\cos 2\theta = 2\cos^2\theta - 1$$

$$= \frac{a^2}{4\pi}\left[\int_{0}^{2\pi} d\theta + \int_{0}^{2\pi} \cos(2\omega t + 2\theta)\, d\theta\right]$$

$$= \frac{a^2}{4\pi}\left[2\pi + \left[\frac{1}{2}\sin(2\omega t + 2\theta)\right]_{0}^{2\pi}\right] = \frac{a^2}{2}$$

$\sin(2\omega t + 4\pi) - \sin(2\omega t) = 0$

① Uniform RV



$$\boxed{f_X(x) = \frac{1}{b-a}} \quad a \le x \le b$$

2 parameters a & b

$$\boxed{E[X] = \frac{a+b}{2}}$$

$$Var[X] = E(X-m)^2$$

$$= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\, dx$$

Let $\left(x - \frac{a+b}{2}\right) = y$

$\therefore x = y + \frac{a+b}{2}$

$x = a \longrightarrow y = \frac{a-b}{2}$

$x = b \longrightarrow y = \frac{b-a}{2}$

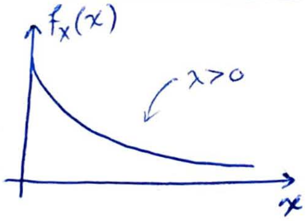
$dx = dy$

$$= \int_{-\frac{(b-a)}{2}}^{\frac{(b-a)}{2}} y^2 \frac{1}{b-a}\, dy$$

$$= \frac{1}{b-a}\left[\frac{1}{3}y^3\right]_{\frac{a-b}{2}}^{\frac{b-a}{2}}$$

$$\therefore \boxed{Var[X] = \frac{(b-a)^2}{12}}$$

② Exponential RV



$$\boxed{f_X(x) = \lambda e^{-\lambda x}} \quad x \ge 0$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

$$= \int_0^x \lambda e^{-\lambda x} dx = \lambda\left[\frac{1}{-\lambda} e^{-\lambda x}\right]_0^x$$

$$\therefore \boxed{F_X(x) = 1 - e^{-\lambda x}}$$

$$E[X] = \int_0^\infty x \lambda e^{-\lambda x} dx$$

Integration by parts

$$\boxed{\int u\, dv = uv - \int v\, du}$$

$$= \int_0^\infty x\, \lambda\left(\frac{-1}{\lambda} d e^{-\lambda x}\right)$$

$$= \left[-x e^{-\lambda x}\right]_0^\infty - \left[\int_0^\infty -e^{-\lambda x} dx\right]$$

$$\therefore = \left[\frac{-1}{\lambda} e^{-\lambda x}\right]_0^\infty$$

$$\therefore \boxed{E[X] = \frac{1}{\lambda}}$$

③ Pareto RV

⑦

$$\boxed{P[X > x] = \begin{cases} 1 & x < x_m \\ \frac{x_m^\alpha}{x^\alpha} & x \ge x_m \end{cases}} \qquad \alpha > 0$$

2 parameters x_m & α

$$F_X[x] = P[X \le x] = 1 - P[X > x]$$

$$\boxed{F_X(x) = \begin{cases} 0 & x < x_m \\ 1 - \frac{x_m^\alpha}{x^\alpha} & x \ge x_m \end{cases}}$$

$$\boxed{f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 0 & x < x_m \\ \frac{\alpha x_m^\alpha}{x^{\alpha+1}} & x \ge x_m \end{cases}}$$

$$\Rightarrow E[X] = \int_{x_m}^{\infty} x f_X(x) dx = \int_{x_m}^{\infty} \frac{\alpha x_m^\alpha}{x^\alpha} dx$$

$$= \alpha x_m^\alpha \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_{x_m}^{\infty}$$

$$\therefore \boxed{E[X] = \frac{\alpha}{\alpha - 1} x_m} \qquad \alpha > 1$$

$$E[X^2] = \frac{\alpha}{\alpha-2} x_m^2 \qquad \alpha > 2$$

$$\therefore \boxed{Var[X] = \frac{\alpha x_m^2}{\alpha - 2} - \left(\frac{\alpha x_m}{\alpha - 1}\right)^2} \qquad \alpha > 2$$

Exponential RV is the only memoryless Continuous RV

$$\boxed{P[X > t+h \mid X > t] = P[X > h]}$$

Proof

$$F_X(x) = 1 - e^{-\lambda x} \qquad x > 0$$

RHS

$$\therefore P[X > h] = 1 - P[X \le h]$$

$$= 1 - F_X(h) = e^{-\lambda h}$$

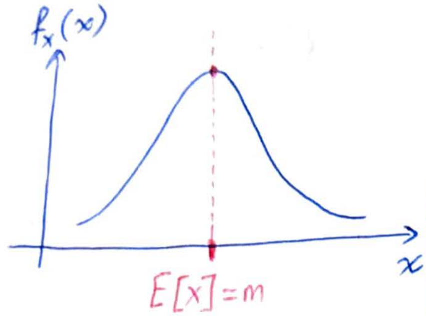
LHS

$$\therefore P[X > t+h \mid X > t] = \frac{P[X > t+h \cap X > t]}{P[X > t]}$$

$$= \frac{P[X > t+h]}{P[X > t]} = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = e^{-\lambda h}$$

4 Gaussian RV

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-m)^2}{2\sigma^2}}}$$



$-\infty < x < \infty$

2 parameters: m & σ (m → mean, σ → standard dev.)

* To get Var[X]

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-m)^2}{2\sigma^2}} dx = 1$$

$$\int_{-\infty}^{\infty} e^{\frac{-(x-m)^2}{2\sigma^2}} dx = \sqrt{2\pi}\,\sigma$$

$\frac{d}{d\sigma}$ LHS $= \frac{d}{d\sigma}$ RHS

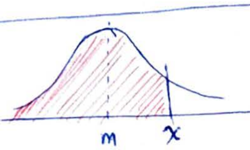
$$\int_{-\infty}^{\infty} \frac{d}{d\sigma}\left(\frac{-(x-m)^2}{2\sigma^2}\right) e^{\frac{-(x-m)^2}{2\sigma^2}} dx = \sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} (x-m)^2 \sigma^{-3} e^{\frac{-(x-m)^2}{2\sigma^2}} dx = \sqrt{2\pi}$$

$\times \frac{\sigma^2}{2\pi}$ both sides

$$\int_{-\infty}^{\infty} (x-m^2) \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-m)^2}{2\sigma^2}} dx = \boxed{\sigma^2 = Var[X]}$$

* $P[X \le x] = F_X(x)$



$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-m)^2}{2\sigma^2}} dx$$

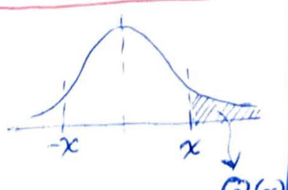
let $t = \frac{x-m}{\sigma}$, $dt = dx/\sigma$

$$= \int_{-\infty}^{\frac{x-m}{\sigma}} \frac{1}{\sqrt{2\pi\cancel{\sigma^2}}} e^{-t^2/2} \sigma\, dt$$

$$\therefore \boxed{F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right) = \int_{-\infty}^{\frac{x-m}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt}$$

$$\boxed{Q\left(\frac{x-m}{\sigma}\right) = 1 - \Phi\left(\frac{x-m}{\sigma}\right) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt}$$

- $Q(0) = \frac{1}{2}$
- $Q(-x) + Q(x) = 1$
- $Q(x) = \frac{1}{2}\,\text{erfc}\left(\frac{x}{\sqrt{2}}\right)$



ex $m = 2, \sigma = 5$: $F_X(10) = 1 - Q\left(\frac{10-2}{5}\right) = 1 - Q(1.6)$ (at $x = 10$)

Gaussian Notation

$$X \sim N(m, \sigma^2)$$

↳ Normalized Gaussian: $X \sim N(0,1)$

ex o/p of Comm. Channel $Y = \alpha V + N$

(Y → o/p, α → Gain, V → transmitted Voltage, N → noise $N(0,4)$: $m = 0$, $\sigma = 2$mV)

$\alpha = 0.01 = -40$dB ($20\log(0.01)$)

What should V be for $\underbrace{P[Y<0]}_{\text{bit error rate}} = 10^{-6}$? (Where V is a +ve value)

$$P[Y<0] = P[\alpha V + N < 0]$$
$$= P[N < -\alpha V]$$

$$P[N < -\alpha V] = F_N(n)\big|_{n = -\alpha V}$$
$$= F_N(-\alpha V)$$
$$= 1 - Q\left(\frac{-\alpha V - 0}{\sigma}\right)$$

$Q(x) = 1 - Q(-x)$

$$= Q\left(\frac{+\alpha V}{\sigma}\right) = 10^{-6}$$

from table: $\frac{\alpha V}{\sigma} = 4.753$ ($\alpha = 0.01$, $\sigma = 2$mV) $\therefore \boxed{V = 950\text{mV}}$

(5) Gamma RV

$$\boxed{f_X(x) = \frac{\lambda (\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}}$$

$\alpha > 0$
$\lambda > 0$
$\boxed{x > 0}$

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$ "Gamma function"

* $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
* $\Gamma(z+1) = z\,\Gamma(z)$, * $\Gamma(m+1) = m!$ if m = integer

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$\boxed{A}$ for $\alpha = 1 \rightarrow f_X(x) = \lambda e^{-\lambda x}$ $\quad x > 0$

→ Exponential RV

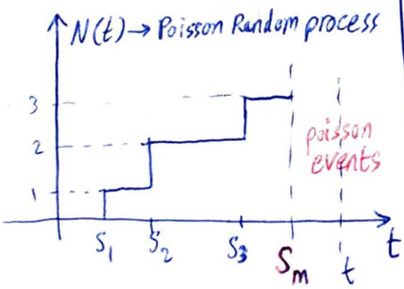
$\boxed{B}$ for $\alpha = m$ (integer) $\rightarrow f_X(x) = \frac{\lambda(\lambda x)^{m-1} e^{-\lambda x}}{(m-1)!}$ $\quad \lambda > 0$, $\alpha > 0$, $x > 0$

→ m-Erlang RV

* m-Erlang RV is related to Poisson RV.

$\alpha = \lambda t$

mean ↙ α, rate ↙ λ (like in Possion), → time t

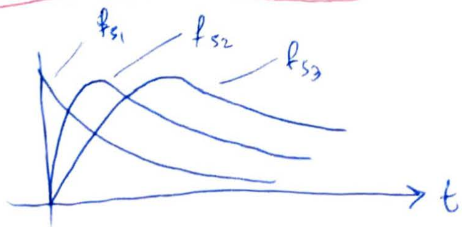


$$F_{S_m}(t) = P[S_m \leq t]$$
$$= P[N(t) \geq m]$$
$$= 1 - P[N(t) < m] = 1 - \sum_{k=0}^{m-1} e^{-\alpha} \frac{\alpha^k}{k!}$$

$$\therefore \boxed{F_{S_m}(t) = 1 - \sum_{k=0}^{m-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!}}$$

$$\therefore \boxed{f_{S_m}(t) = \frac{d F_{S_m}(t)}{dt} = \lambda e^{-\lambda t} \frac{(\lambda t)^{m-1}}{(m-1)!}}$$

* m-Erlang is used for time intervals of Poisson events



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ex Rate of failure of a part in a factory $= 1 \frac{\text{failure}}{\text{month}}$ on average.

We bought 2 spare, So we have 3 parts. But this part is out of order.

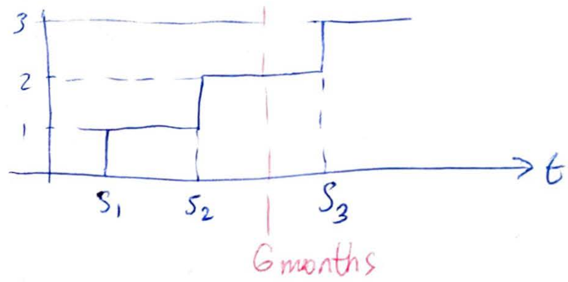
➡ P[factory operates for 6 months] = ?

$\hookrightarrow P[S_3 > 6]$ → 3rd failure happens after 6 months

$= P[N(6\text{ months}) \leq 2]$ → 2 parts fail in 6 months

$= P[N(6)=0] + P[N(6)=1] + P[N(6)=2]$

$$P[N(t)=k] = e^{-\alpha} \frac{\alpha^k}{k!} \qquad \alpha = \lambda t$$



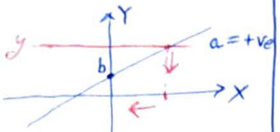
$$P[N(6) \leq 2] = e^{-6} + e^{-6}\frac{6}{1!} + e^{-6}\frac{6^2}{2!}$$
$$= 0.062$$

Functions of a RV

$Y = g(X)$

$f_Y(y) \leftarrow f_X(x)$

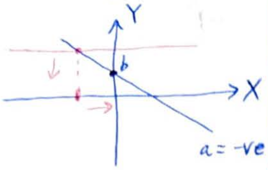
A Linear Relationship



$$\boxed{Y = aX + b}$$

$$F_Y(y) = P[Y \leq y] = P[aX + b \leq y]$$

$$= \begin{cases} P\left[X \leq \frac{y-b}{a}\right] & a > 0 \\ P\left[X \geq \frac{y-b}{a}\right] & a < 0 \end{cases}$$



$$\therefore \boxed{F_Y(y) = \begin{cases} F_X\left(\frac{y-b}{a}\right) & a > 0 \\ 1 - F_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases}}$$

Differentiate

$$\frac{d}{dx} f(g(x)) = g'(x) \cdot f'(g(x))$$

$$\therefore f_Y(y) = \begin{cases} \frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a > 0 \\ \frac{-1}{a} f_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases}$$

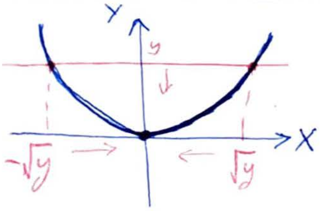
$$\frac{dF}{dy} = \frac{dF}{du} \frac{du}{dy}$$

where $u = \frac{y-b}{a}$

$$\therefore \boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

B Squared Relationship

$$\boxed{Y = X^2}$$



$$F_Y(y) = P[Y \leq y]$$

$$\therefore \boxed{F_Y(y) = \begin{cases} 0 & y < 0 \\ P[-\sqrt{y} \leq x \leq \sqrt{y}] = F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y > 0 \end{cases}}$$

$$\boxed{f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})}$$

---

ex. X is Gaussian & goes into LINEAR operator

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

$$\boxed{Y = aX + b}$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{|a|} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-\left(\frac{y-b}{a} - m\right)^2}{2\sigma^2}\right]$$

$$= \frac{1}{\sqrt{2\pi(a\sigma)^2}} \exp\left[-\frac{(y - (am + b))^2}{2(a\sigma)^2}\right]$$

$m_y = a m_x + b$

$\sigma_y = a \sigma_x$

$f_Y(y) \rightarrow$ also Gaussian

---

ex. X is Gaussian & goes into square-law system

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

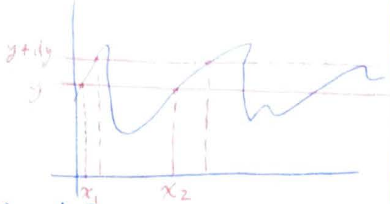
$$Y = X^2$$

$$f_Y(y) = \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)$$

$$= \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi\sigma^2}} \left(e^{\frac{-(\sqrt{y}-m)^2}{2\sigma^2}} + e^{\frac{-(-\sqrt{y}-m)^2}{2\sigma^2}}\right)$$

Not Gaussian

C General $Y=g(X)$



$P[y \leq Y \leq y+dy] \approx f_Y(y)\,dy$

$= P[x_1 \leq X \leq x_1+dx] + P[x_2 \leq X \leq x_2+dx] + \ldots\ldots$

$= f_X(x_1)\,dx_1 + f_X(x_2)\,dx_2 + \ldots\ldots$

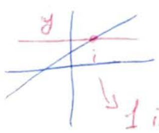
if no intersection, $\therefore f_Y(y) = 0$

$$\therefore f_Y(y) = \frac{f_X(x_1)}{\left|\frac{dy}{dx}\right|_{x_1}} + \frac{f_X(x_2)}{\left|\frac{dy}{dx}\right|_{x_2}} + \ldots\ldots$$

point of intersection

* Slope can be -ve, but the Prob. is always +ve, so we take absolute value

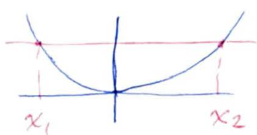
ex linear $Y = aX+b$



1 intersection $x_1 = \frac{y-b}{a}$

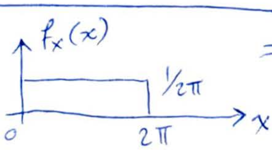
$$\therefore f_Y(y) = \frac{f_X\left(\frac{y-b}{a}\right)}{|a|}$$

ex Squared $Y = X^2$

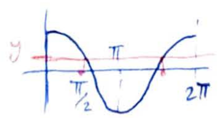


$$\therefore f_Y(y) = \frac{f_X(-\sqrt{y})}{|2x_1|} + \frac{f_X(\sqrt{y})}{|2x_2|} \rightarrow \therefore f_Y(y) = \frac{f_X(-\sqrt{y})}{2\sqrt{y}} + \frac{f_X(\sqrt{y})}{2\sqrt{y}}$$

ex X is Uniform RV, $Y = \cos(x)$



$\left|\frac{dy}{dx}\right|_{x_1 = \cos^{-1}y} = -\sin(\cos^{-1}y) = -\sqrt{1-y^2}$



$x_1 = \cos^{-1}y$

$x_2 = 2\pi - \cos^{-1}y$

$\left|\frac{dy}{dx}\right|_{x_2 = 2\pi - \cos^{-1}y} = -\sin(2\pi - \cos^{-1}y)$

$= \sin(\cos^{-1}y - 2\pi)$

$= \sin(\cos^{-1}y)$

$= +\sqrt{1-y^2}$

$$\therefore f_Y(y) = \frac{f(x_1)}{\sqrt{1-y^2}} + \frac{f(x_2)}{\sqrt{1-y^2}} \quad \frac{1}{2\pi}$$

$$= \frac{1}{\pi\sqrt{1-y^2}} \qquad |y|<1$$

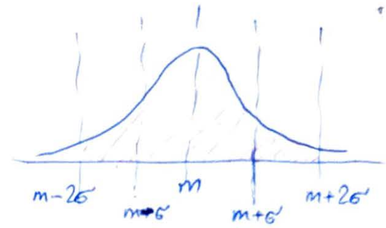
$\sin^2\theta + \cos^2\theta = 1$

$\cos^{-1}y = \theta$

$\cos\theta = y$

$\therefore \sin(\cos^{-1}y) = \sin\theta = \sqrt{1-y^2}$

Prob. of confidence (for Gaussian)



* $P[|x-m| > 2\sigma]$

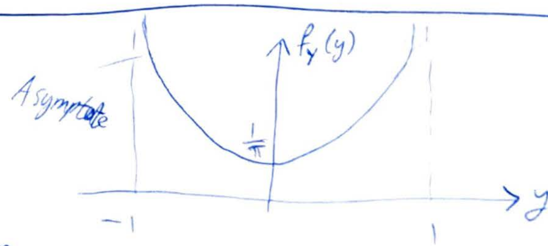
$= 2P[X-m > 2\sigma]$

$= 2Q\left(\frac{(m+2\sigma)-m}{\sigma}\right)$

$= 2Q(2) = 0.0456 \longrightarrow$ Prob of Error

* $\therefore$ Prob. of confidence $= 1 - 0.0456 = 95.44\%$

$\Longrightarrow$



$$\int_{-\infty}^{\infty} \frac{dy}{\pi\sqrt{1-y^2}} = 1$$

1 Markov Inequality

* If X non negative $\boxed{P[X \geq a] \leq \frac{E[X]}{a}}$ $a>0$

Proof

$$E[X] = \int_{-\infty}^{\infty} t f(t)\,dt = \int_{0}^{\infty} t f(t)\,dt$$ X non-negative

$$= \int_0^a t f(t)\,dt + \int_a^{\infty} t f(t)\,dt$$

$$\therefore E[X] \geq \int_a^{\infty} t f(t)\,dt \geq a \int_a^{\infty} f(t)\,dt$$

here t is between a & ∞

2 Chebychev's Inequality

* X is RV, $E[X]=m$, $Var[X]=\sigma^2$ $\boxed{P[|X-m| \geq a] \leq \frac{\sigma^2}{a^2}}$

Proof

$$P[D \geq a^2] \leq \frac{E[D]}{a^2} \quad \text{where } D=(X-m)^2$$

Markov

$$P[D \geq a^2] = P[(X-m)^2 \geq a^2] = P[X-m \geq a]$$

$$\therefore P[X-m \geq a] \leq \frac{E[(X-m)^2]}{a^2} = \frac{\sigma^2}{a^2}$$

ex. X Bernoulli RV

$$P[X = \pm \nu] = \frac{1}{2}$$

$$E[X] = 0$$

$$Var[X] = E[X^2] - E[X]^2$$

$$= \nu^2 \tfrac{1}{2} + (-\nu)^2 \tfrac{1}{2} = \nu^2 = \sigma^2$$

⇒ From Chebychev: $P[|X-m| \geq \sigma] \leq \frac{\sigma^2}{\sigma^2} = 1$

$$P[|X-m| \geq \sigma] = P[|X| \geq \nu] = 1$$

Bernoulli is the only RV to satisfy this equality

Transform Methods

* Characteristic Function of X:

$$\boxed{\phi_X(\omega) = E[e^{j\omega X}]}$$

$j=\sqrt{-1}$, $\omega \in \mathbb{R}$

⇒ For Continuous

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(\omega) e^{-j\omega x}\,d\omega$$

⇒ For Discrete

$$\phi_X(\omega) = \sum_x e^{j\omega x} p_X(x)$$

* $\boxed{E[X^n] = \frac{1}{j^n} \left.\frac{d^n \phi_X(\omega)}{d\omega^n}\right|_{\omega=0}}$

Proof

$$e^{j\omega x} = 1 + j\omega x + \frac{(j\omega x)^2}{2!} + \frac{(j\omega x)^3}{3!} + \cdots$$

$$\therefore \phi_X(\omega) = \underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx}_{1} + \underbrace{\int_{-\infty}^{\infty} (j\omega x) f_X(x)\,dx}_{j\omega E[X]} + \cdots$$

$$\therefore \phi_X(0) = 1$$

$$\left.\frac{d\phi_X(\omega)}{d\omega}\right|_{\omega=0} = jE[X] \quad \cdots$$

Probability Generating Function (PGF)

* Z discrete non-negative RV

$$G_N(z) = E[z^n] = \sum_{n=0}^{\infty} z^n p_N(n) = p_N(0) + z\,p_N(1) + z^2 p_N(2) + \cdots$$

poisson, exponential, geometric → non-negative

Z-transform

$$\boxed{p_N(k) = \frac{1}{k!} \left.\frac{d^k G_N(z)}{dz^k}\right|_{z=0}}$$

→ used to find nth moment.

$\hookrightarrow G_N(0) = p_N(0)$

$$\left.\frac{dG_N(z)}{dz}\right|_{z=0} = p_N(1) \quad , \quad \left.\frac{d^2 G_N(z)}{dz^2}\right|_{z=0} = p_N(2)$$

$\hookrightarrow G_N(1) = 1$

$$\left.\frac{dG_N(z)}{dz}\right|_{z=1} = \sum_n n\, z^{n-1} p_N(n)$$

$$= \sum_n n\, p_N(n) = E[N]$$

$$\left.\frac{d^2 G_N(z)}{dz}\right|_{z=1} = \sum_n n(n-1)\, z^{n-2} p_N(n)$$

$$= \sum_n n^2 p_N(n) - n\, p_N(n)$$

$$= E[N^2] - E[N]$$

---

* For X continuous non-negative RV

$$\boxed{X^*(s) = E[e^{-sX}] = \int_0^\infty e^{-sx} f_X(x)\, dx}$$

Laplace transform

$$\boxed{E[X^n] = (-1)^n \left.\frac{d^n X^*(s)}{ds^n}\right|_{s=0}}$$

$$f_X(x) = \mathcal{L}^{-1}(X^*(s))$$

Ex 1 Gaussian

Ch. Fn

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{\sigma^2}}$$

$$X \sim N(m, \sigma^2)$$

Normalization

$$Z \sim N(0, 1)$$

let $Z = \frac{X-m}{\sigma}$

$$E[Z] = E\left[\frac{X-m}{\sigma}\right] = \frac{1}{\sigma}(E[X]-m) = 0$$

$$Var[Z] = Var\left[\frac{X-m}{\sigma}\right] = \frac{1}{\sigma^2} Var[X-m] = \frac{1}{\sigma^2} Var[\cancel{x}] = 1$$

$$dz = \frac{1}{\sigma} dx$$

$$\phi_X(\omega) = E[e^{j\omega x}]$$

$$= \int_{-\infty}^{\infty} e^{j\omega x}\left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}\right] dx \rightarrow \text{Hard}$$

$$\phi_Z(\omega) = \int_{-\infty}^{\infty} e^{j\omega z}\left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-z^2/2} \sigma\, dz\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{j\omega z - z^2/2} dz$$

let $j\omega = s$

$\therefore sz - \frac{z^2}{2}$

$= \frac{2sz - z^2}{2} + s^2 - s^2$

$= \frac{-(s^2+z^2-2sz)+s^2}{2}$

$= \frac{-(s-z)^2+s^2}{2}$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} e^{s^2/2} \int_{-\infty}^{\infty} e^{-\frac{(z-s)^2}{2}} dz}_{=1 \quad N(s,1)}$$

$$= e^{s^2/2}$$

$$\therefore \boxed{\phi_Z(\omega) = e^{-\omega^2/2}} \rightarrow \text{Characteristic fn for } Z$$

$$\phi_X(\omega) = E\left[e^{j\omega(\sigma z+m)}\right]$$

$$= e^{j\omega m} E\left[e^{j\omega\sigma z}\right] = e^{j\omega m} \underbrace{E\left[e^{j(\omega\sigma)z}\right]}_{\phi_Z(\omega\sigma)}$$

$$\boxed{\therefore \phi_X(\omega) = e^{j\omega m} e^{-\frac{(\sigma\omega)^2}{2}}} \rightarrow \text{Characteristic fn for } X$$

$\Rightarrow$ Check: ① $\phi_X(0) = 1$

② $$\left.\frac{d\,\phi_X(\omega)}{d\omega}\right|_{\omega=0} = \left.(jm\, e^{j\omega m}) e^{-\frac{\sigma^2\omega^2}{2}} + e^{j\omega m}\left(-\sigma^2\omega e^{-\frac{\sigma^2\omega^2}{2}}\right)\right|_{\omega=0} = jm$$

Ex 2 Poisson

z-transform

$$P_N(n) = e^{-\alpha}\frac{\alpha^n}{n!} \qquad n = 0, 1, 2, \ldots$$

$$G_N(z) = E[z^n]$$

$$= \sum_{n=0}^{\infty} z^n e^{-\alpha} \frac{\alpha^n}{n!}$$

$$= e^{-\alpha} \sum_{n=0}^{\infty} \frac{(\alpha z)^n}{n!} \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$= e^{\alpha(z-1)}$$

$$G_N(1) = 1$$

$$G_N'(z) = \alpha e^{\alpha(z-1)} \rightarrow \therefore G_N'(1) = \alpha = E[N]$$

$$G_N''(z) = \alpha^2 e^{\alpha(z-1)} \rightarrow \therefore G_N''(1) = \alpha^2$$

$$= E[N^2] - E[N]$$

$$\therefore E[N^2] = \alpha^2 + \alpha$$

$$\therefore Var[N] = E[N^2] - (E[N])^2 = \alpha$$

Ex 3 Exponential

s-place

$$f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0, \quad \lambda > 0$$

$$X^*(s) = E[e^{-sX}]$$

$$= \int_0^{\infty} e^{-sx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} e^{-x(s+\lambda)} dx$$

$$= \lambda \frac{1}{s+\lambda} \qquad \int_0^{\infty} e^{-mx} dx = \frac{1}{m}$$

$$E[X] = (-1)^1 \left.\frac{d X^*(s)}{ds}\right|_{s=0}$$

$$= -1 \left.\left(\lambda(-1)(s+\lambda)^{-2}\right)\right|_{s=0} = \frac{1}{\lambda}$$