Lec09

Recursive difference equations

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b[k]\, x[n-k]$$

$$= -a_1 y[n-1] - a_2 y[n-2] - \ldots - a_N y[n-N]$$

$$+ b_0 x[n] + b_1 x[n\text{-}1] + b_2 x[n\text{-}2] + \ldots + b_M x[n\text{-}M]$$

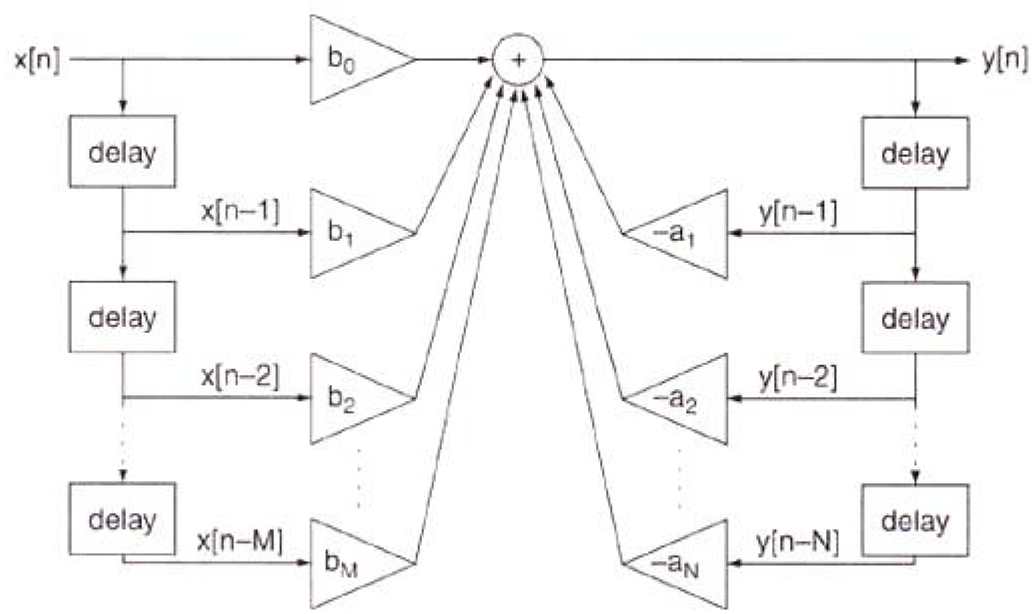


FIGURE 4.20
Recursive difference equation diagram, direct form 1 realization.

The following steps develop a **direct form II realization**.

$$\sum_{p=0}^{N} a_p y\left[n-p\right] = \sum_{k=0}^{M} b_p x\left[n-k\right] \tag{1}$$

Let’s define a signal $w[n]$ that satisfies the following relationship:

$$\sum_{p=0}^{N} a_p w\left[n-p\right] = x[n] \tag{2}$$

Substituting equation (2) to the (1) gives

$$\sum_{p=0}^{N} a_p y\left[n-p\right] = \sum_{k=0}^{M} b_k \sum_{p=0}^{N} a_p w\left[n-k-p\right]$$

$$= \sum_{p=0}^{N} a_p \left(\sum_{k=0}^{M} b_k w\left[n-k-p\right] \right)$$

$$\Rightarrow y\left[n-p\right] = \sum_{k=0}^{M} b_k w\left[n-k-p\right]$$

$$\Rightarrow y\left[n\right] = \sum_{k=0}^{M} b_k w\left[n-k\right]$$

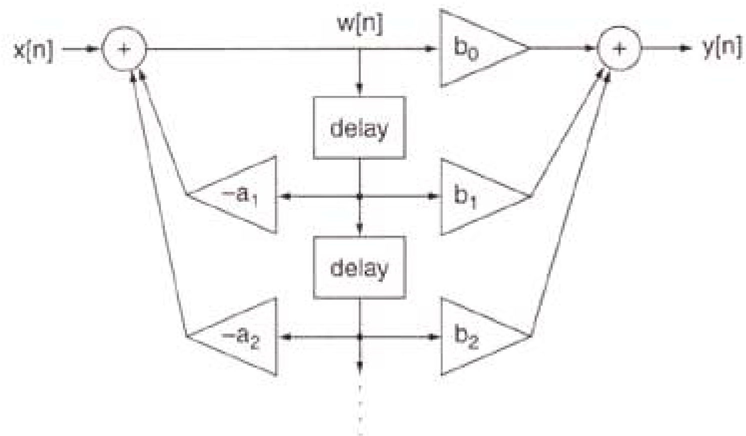
From (2), if we assume that $a_0 = 1$, then

$$\sum_{p=0}^{N} a_p w\left[n-p\right] = x[n]$$

$$\Rightarrow w[n] + \sum_{p=1}^{N} a_p w\left[n-p\right] = x[n]$$

$$\Rightarrow w[n] = x[n] - \sum_{p=1}^{N} a_p w\left[n-p\right]$$

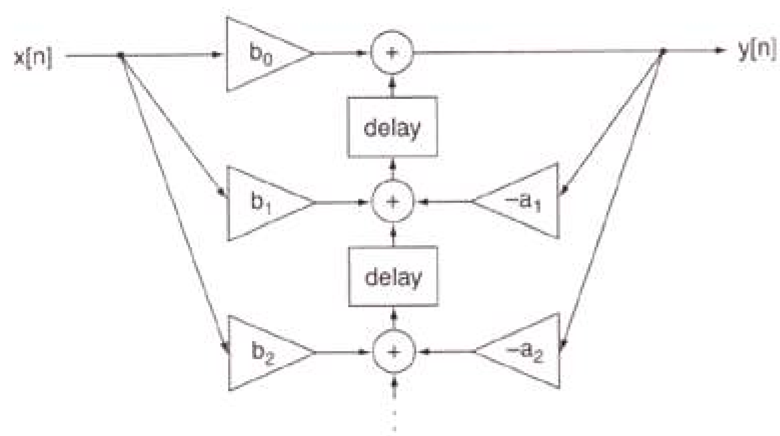
FIGURE 4.23
Recursive difference equation diagram, direct form 2 realization.



$$w[n] = x[n] - \sum_{k=1}^{N} a_k w[n-k] \tag{4.6}$$

$$y[n] = \sum_{k=0}^{N} b_k w[n-k] \tag{4.7}$$

FIGURE 4.24
Transpose of direct form 2 realization.



Ch6 Z Transform

- **The purpose of Z transform**
  - To make descriptions of digital signal and systems more compact
  - To make calculations with digital signals easier

Definitions of Z transform

$$Z\{x[n]\} = X(Z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$: Frequency domain containing complex numbers

$$Z^{-1}\{X(Z)\} = x[n]$$

- The Z transform may not be defined for all values of z in the z domain.
- The values of z for which it is defined is called region of convergence (ROC)

EXAMPLE 6.1

Find the z transform $X(z)$ of the signal $x[n] = \delta[n]$.

This signal is nonzero at only a single place, $n = 0$. Thus,

$$\mathbf{Z}\{x[n]\} = X(z) = \sum_{n=0}^{\infty}\delta[n]z^{-n} = \delta[0] = 1$$

This z transform is defined for all values of z, so its region of convergence includes all z.

EXAMPLE 6.2

Find the z transform of $x[n] = \delta[n-1]$.

The signal is nonzero only at $n = 1$. Therefore,

$$\mathbf{Z}\{x[n]\} = X(z) = \sum_{n=0}^{\infty} \delta[n-1]z^{-n} = \delta[0]z^{-1} = z^{-1}$$

which is defined as long as $z \neq 0$, so its region of convergence is all z except $z = 0$.

EXAMPLE 6.4

A signal $x[n]$ is depicted in Figure 6.1. Find the z transform of the signal.

The signal may be described as

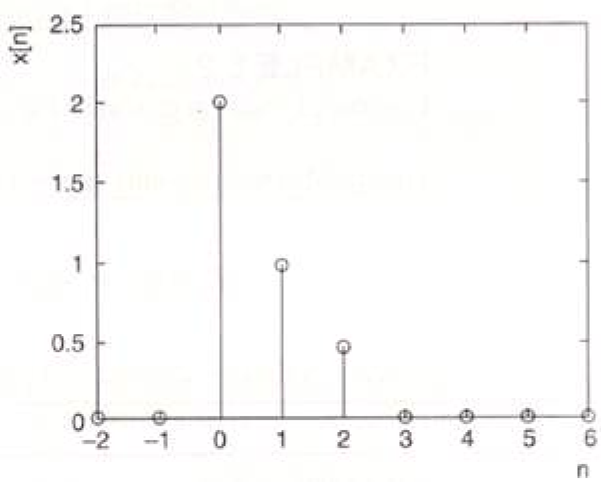
$$x[n] = 2\delta[n] + \delta[n-1] + 0.5\delta[n-2]$$

It has only three nonzero elements, so the z transform contains the same number of terms. The z transform is

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} = 2 + z^{-1} + 0.5z^{-2}$$

which is defined as long as $z \neq 0$.

FIGURE 6.1
Signal for Example 6.4.



Ex]

Find the Z transform including the region of convergence of

$$x[n] = \begin{cases} a^n & n \ge 0 \\ \\ 0 & n < 0 \end{cases}$$

$$x[n] = a^n u[n]$$

By definition, the Z transform of $x[n]$ becomes

$$X(Z) = Z\{a^n u[n]\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

By the geometric progression formula

$$X(Z) = Z\{a^n u[n]\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$X(Z) = \left(az^{-1}\right)^0 + \left(az^{-1}\right)^1 + \left(az^{-1}\right)^2 + \left(az^{-1}\right)^3 + \ldots + \left(az^{-1}\right)^\infty$$

$$X(Z) = \quad 1 + \left(az^{-1}\right)^1 + \left(az^{-1}\right)^2 + \left(az^{-1}\right)^3 + \ldots + \left(az^{-1}\right)^\infty \quad \text{-- (1)}$$

$$az^{-1}X(Z) = \quad \left(az^{-1}\right)^1 + \left(az^{-1}\right)^2 + \left(az^{-1}\right)^3 + \ldots + \left(az^{-1}\right)^\infty \quad \text{-- (2)}$$

Subtracting (2) from (1), then

$$\begin{array}{rl} & X(Z) = \quad 1 + \cancel{\left(az^{-1}\right)^1} + \cancel{\left(az^{-1}\right)^2} + \cancel{\left(az^{-1}\right)^3} + \ldots + \cancel{\left(az^{-1}\right)^\infty} \quad \text{-- (1)} \\ - & az^{-1}X(Z) = \quad \cancel{\left(az^{-1}\right)^1} + \cancel{\left(az^{-1}\right)^2} + \cancel{\left(az^{-1}\right)^3} + \ldots + \cancel{\left(az^{-1}\right)^\infty} \quad \text{-- (2)} \\ \hline & X(Z) - az^{-1}X(Z) = 1 \quad \text{-- (3)} \end{array}$$

The equation (3) is

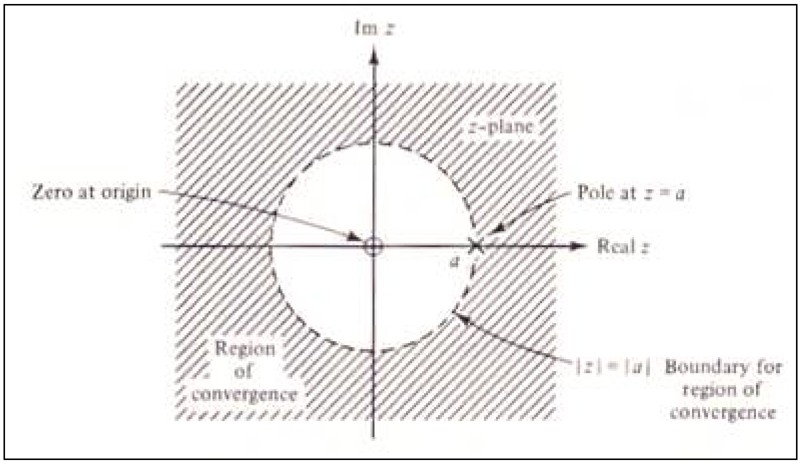
$$X(Z) - az^{-1}X(Z) = 1$$

$$\Rightarrow X(Z)\left[1 - az^{-1}\right] = 1$$

$$\Rightarrow X(Z) = \frac{1}{1 - az^{-1}}$$
$$= \frac{z}{z - a}$$

This result converges if

$$\left.\begin{array}{l} \left|az^{-1}\right| < 1 \\ \Rightarrow |z| > |a| \end{array}\right. \quad \text{ROC}$$



- Values of z for which X(z)=0 are called **zeros** of X(z)
- Values for z for which X(z)$\rightarrow \infty$ are called **poles** of X(z)

EXAMPLE 6.3

Find $X(z)$ if $x[n] = u[n]$.

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} u[n]z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + \ldots$$

This is a geometric series of the form $a + ar + ar^2 + \ldots$ with initial term a equal to one and multiplier r equal to z^{-1}. As shown in Appendix A.16, the sum of an infinite geometric series is given by

$$S_\infty = \frac{a}{1 - r}$$

as long as $|r| < 1$. Therefore,

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

provided $|z^{-1}| < 1$. That is, the region of convergence for this z transform is $|z| > 1$.

EXAMPLE 6.5

Find the z transform of the signal $x[n] = (-0.5)^n u[n]$.

Since $u[n] = 1$ for $n \geq 0$,

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} (-0.5)^n z^{-n} = \sum_{n=0}^{\infty} (-0.5z^{-1})^n$$

$$= 1 - 0.5z^{-1} + 0.25z^{-2} - 0.125z^{-3} + \ldots$$

As in Example 6.3, this is an infinite geometric series. In this series, $a = 1$ and $r = -0.5z^{-1}$, so its sum is

$$X(z) = \frac{1}{1 + 0.5z^{-1}} = \frac{z}{z + 0.5}$$

The region of convergence for this z transform is $|-0.5z^{-1}| < 1$, or $|z| > 0.5$.