Ex) A signal, $y(t)$, is shown below and it is going to be sampled based on the given sampling rate

$$x_1(t) = \cos(10\pi t) \qquad x_2(t) = 2\cos(16\pi t)$$
$$y(t) = x_1(t) + x_2(t)$$

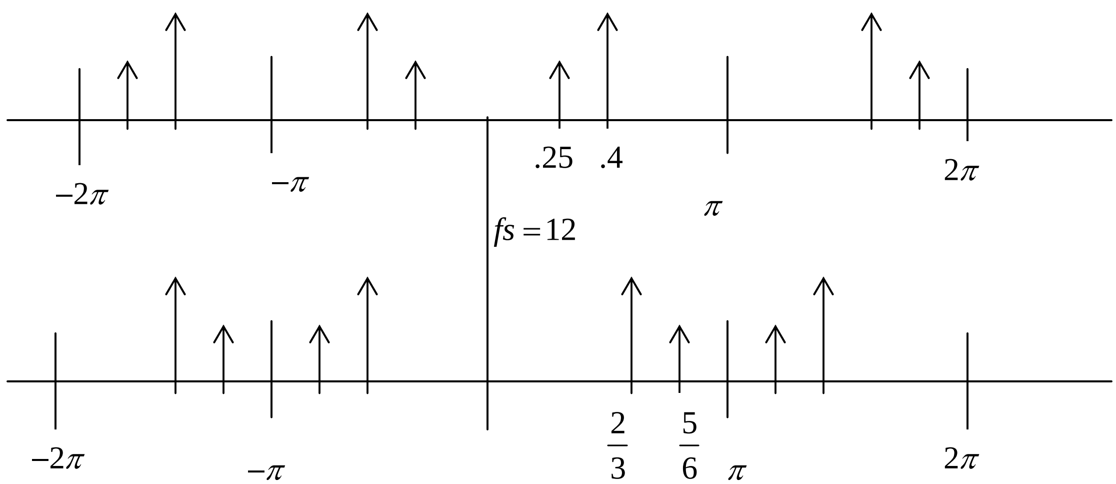
a) Given the two sampling rates, $f_s = 40$ and $f_s = 12$ samples/sec, plot the magnitude spectrum from -2π to 2π, your answer must be in terms of radians on the x axis.

b) When you sample with the $f_s = 12$, what is the new frequency corresponding frequency of $x_1(t)$ and $x_2(t)$.

c) What can you observe if you use $f_s = 12$.

$f_1 = 5,\ f_2 = 8$

$fs = 40$



b)

$$x_1 = 5Hz$$
$$x_2 = \frac{2}{3} \text{x} 6 = 4Hz$$

c)

The aliasing happens. The x_1 signal is kept but the x_2 signal is aliased and it is shown as $4Hz$ signal instead of $8Hz$ signal

Quantization

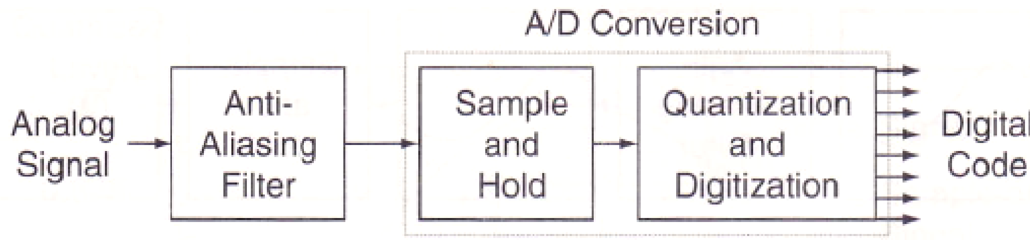
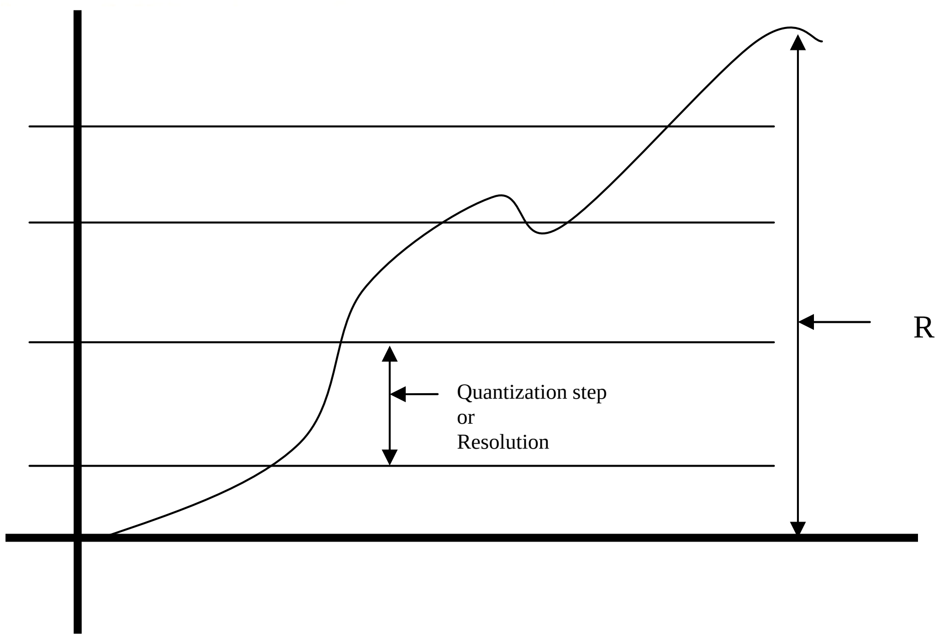


FIGURE 2.19
Analog-to-digital conversion.



Quantization step: The gap between the levels

$$Q = \frac{R}{2^N}$$

$R:$ is the full scale analog range.
$N:$ is the number of bits used for the quantization.

Quantization error = Quantized value –Actual value.

- Unipolar: Analog data varies zero to positive maximum.
- Bipolar: Analog data varies between negative minimum to positive maximum.

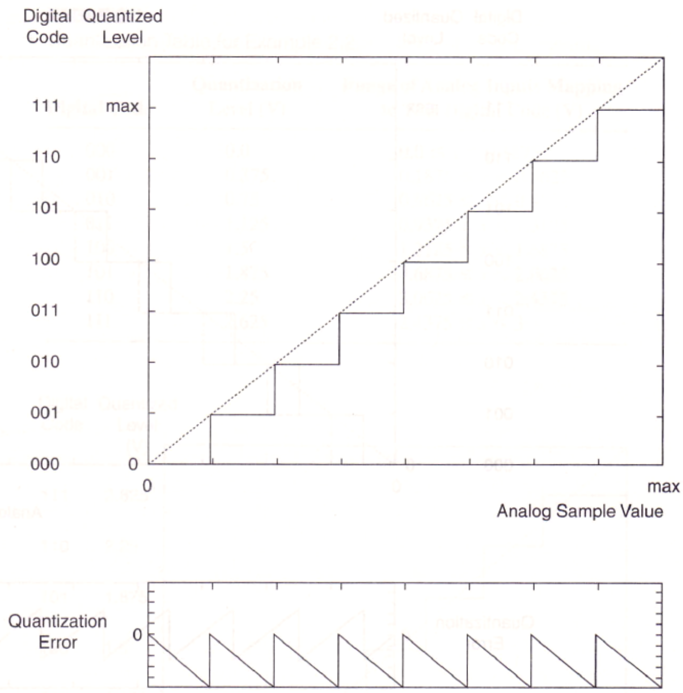


FIGURE 2.15
Quantization of unipolar data (maximum error = full step).

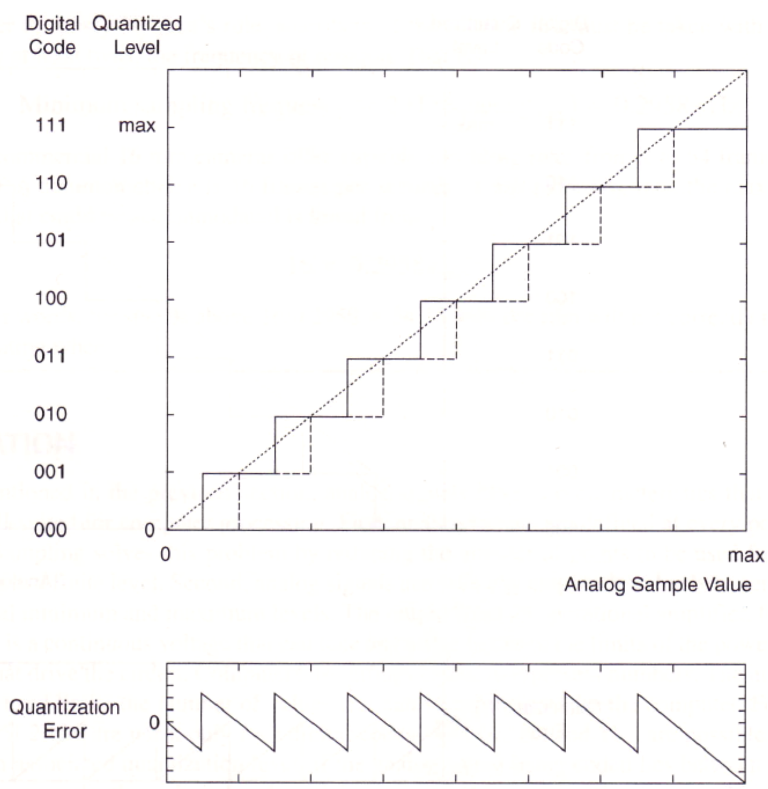


FIGURE 2.16
Quantization of unipolar data (maximum error = half step).

EXAMPLE 2.2

Analog pressures are recorded, using a pressure transducer, as voltages between 0 and 3 V. The signal must be quantized using a 3-bit digital code. Indicate how the analog voltages will be converted to digital values.

Since the range of the signal is 3 V, the quantization step size is

$$Q = \frac{3\text{ V}}{2^3} = 0.375\text{ V}$$

Quantization Table for Example 2.2

Digital Code	Quantization Level (V)	Range of Analog Inputs Mapping to This Digital Code (V)
000	0.0	$0.0 \leq x < 0.1875$
001	0.375	$0.1875 \leq x < 0.5625$
010	0.75	$0.5625 \leq x < 0.9375$
011	1.125	$0.9375 \leq x < 1.3125$
100	1.5	$1.3125 \leq x < 1.6875$
101	1.875	$1.6875 \leq x < 2.0625$
110	2.25	$2.0625 \leq x < 2.4375$
111	2.625	$2.4375 \leq x \leq 3$

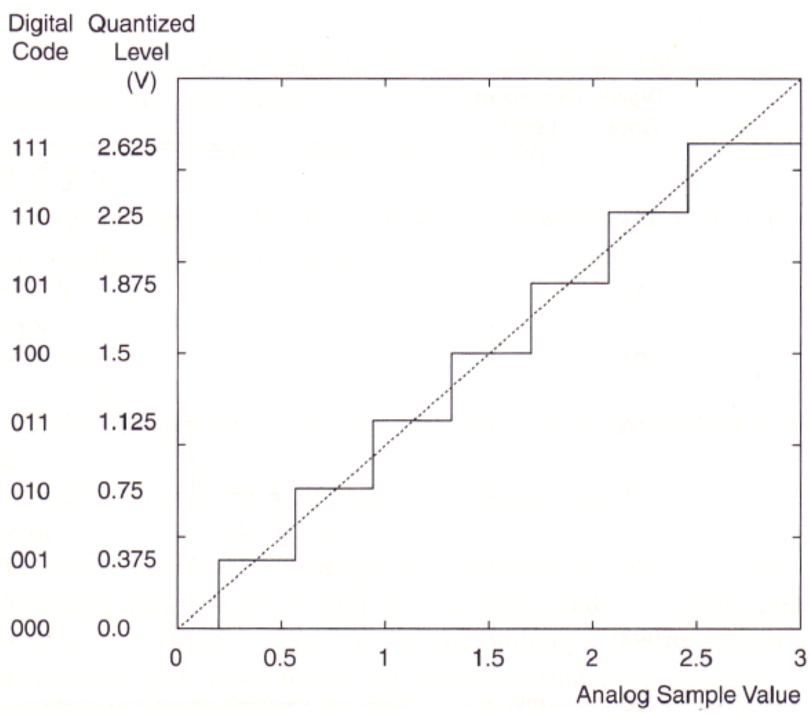


FIGURE 2.17
Quantization diagram for Example 2.2.

EXAMPLE 2.3

An analog voltage between -5 V and 5 V must be quantized using 3 bits. Quantize each of the following samples, and record the quantization error for each:

a. -3.4 V
b. 0.0 V
c. 0.625 V

Using the quantization step size $10\text{V}/2^3 = 1.25$ V, the quantization table in Table 2.2 can be constructed.

a. The analog sample -3.4 V produces the digital code 101, with a quantization error of $-3.75 - (-3.4) = -0.35$ V.
b. The analog sample 0.0 V codes as 000, with zero quantization error.
c. The analog sample 0.625 V generates the digital code 001, with a quantization error of $1.25 - 0.625 = 0.625$ V. For the midrange, this represents a worst-case quantization error, equal to half a quantization step.

TABLE 2.2
Quantization Table for Example 2.3

Digital Code	Quantization Level (V)	Range of Analog Inputs Mapping to This Digital Code (V)
100	-5.0	$-5.0 \le x < -4.375$
101	-3.75	$-4.375 \le x < -3.125$
110	-2.5	$-3.125 \le x < -1.875$
111	-1.25	$-1.875 \le x < -0.625$
000	0.0	$-0.625 \le x < 0.625$
001	1.25	$0.625 \le x < 1.875$
010	2.5	$1.875 \le x < 3.125$
011	3.75	$3.125 \le x \le 5.0$

- **Quantization error** are determined by the size of the **quantization step**

- **Dynamic range**: # of levels it can distinguish in noise

$$dB = 20 \cdot \log_{10} |X|$$

of **distinct levels** that can be identified without error is

$$\frac{R}{Q} = 2^N$$

Dynamic range of quantizer in dB is

$$20 \log_{10} \frac{R}{Q} = 20 \log_{10} 2^N = N\left(20 \log_{10} 2\right) = N \cdot \left(6.02\right) dB$$

- **Dynamic range** improves as the number of bits N increases.

- **Dynamic range** is connected to the concept of **signal to noise ratio**

$$\text{SNR} = 10 \log\left(\frac{\text{Signal power}}{\text{Noise power}}\right)$$

$$\text{SNR} = 20 \log\left(\frac{\text{Signal amplitude}}{\text{Noise amplitude}}\right)$$

Decibels

- Decibel is a comparison of two quantities expressed on a logarithmic scale.
- The power can be expressed in decibel (dB) as

$$10 \cdot \log\left(\frac{P(watts)}{1(watt)}\right) \text{ dB or dBW}$$

as the reference is 1 watt

- Power in dB or dBW can be converted to watts as

$$\mathrm{P}(\mathrm{dB}) = 10 \cdot \log\left(\frac{\mathrm{P}(\mathrm{watt})}{1(\mathrm{watt})}\right)$$

Example: Convert 5W into dB.

$$5\mathrm{W}(\mathrm{dB}) = 10 \cdot \log(5) = 6.9897 \text{ dB}$$

We can convert from dBs to watts as

$$\mathrm{P}(\mathrm{watt}) = \log^{-1}\left(\frac{\mathrm{dB}}{10}\right) = 10^{\left(\frac{\mathrm{dB}}{10}\right)}$$

Example: Convert 12 dB into watts.

$$\mathrm{P}(\mathrm{watts}) = \log^{-1}\left(\frac{\mathrm{P}(\mathrm{dB})}{10}\right) = \log^{-1}\left(\frac{\mathrm{P}(12)}{10}\right) = 15.8489 \text{ W}$$

- Power is proportional to the square of the voltage $\left(\mathrm{P} = \frac{\mathrm{V}^2}{\mathrm{R}}\right)$

$$10\log\left(\mathrm{V}^2\right) = 2(10)\log(\mathrm{V}) = 20 \log(\mathrm{V})$$

$$\text{Voltage} = \log^{-1}\left(\frac{25}{20}\right) = 17.78 \text{ V}$$

dBm:

a. The dBm is used extensively in communication systems, specifically in cases where we are dealing with **lower powers**. The spectrum analyzers represent a signal in the frequency domain. The horizontal axis represents frequency while the vertical axis represents power in dBm. The dBs and dBm are used in cascaded systems and path loss equations.

b. The letter that follows dB is used as a reference. **The letter "m" refers to milli-watt.** The given power in watts is being compared with 1 milli-watt. We must know how to **convert watts into dBm and dBm into watts.**

$$P(\text{dBm}) = 10\log\left(\frac{P(\text{watt})}{1\text{mW}}\right)$$

Example: Convert 0.52 watts into dBm.

$$P(\text{dBm}) = 10\log\left(\frac{P(0.52)}{1\text{mW}}\right) = 27.16 \text{ dBm}$$

$$\frac{P(\text{dBm})}{10} = \log\left(\frac{P(\text{watt})}{1\text{mW}}\right)$$

$$\left(\frac{P(\text{watt})}{1\text{mW}}\right) = \log^{-1}\left(\frac{P(\text{dBm})}{10}\right)$$

$$P(\text{watt}) = (1\text{mW})\left(\log^{-1}\left(\frac{P(\text{dBm})}{10}\right)\right)$$

Example: Convert 15 dBm into watts.

$$P(\text{watt}) = (1\text{mW})\left(\log^{-1}\left(\frac{P(15)}{10}\right)\right) = 31.6227\,\text{mW}$$

Example: Convert 1 mW into dBm.

$$P(\text{dBm}) = 10\log\left(\frac{P(1\text{mW})}{1\text{mW}}\right) = 0\,\text{dBm}$$

Why did you get 0 dBm? Explain!

$$2^N = \frac{R}{Q}$$

$$\log_2\left(2^N\right) = \log_2\left(\frac{R}{Q}\right)$$

$$N = \log_2\left(\frac{R}{Q}\right) = \frac{\log\left(\frac{R}{Q}\right)}{\log 2}$$

EXAMPLE 2.4

An analog signal whose range lies between 0 and 5 V must be quantized, with midrange quantization errors no bigger than 6×10^{-5} V. How many bits are required to meet this requirement?

If the maximum allowable quantization error is 6×10^{-5}, then the quantization step must be no greater than 12×10^{-5}. For an analog signal with a 5 V range, the number of quantization bits would then be

$$N = \log_2\left(\frac{R}{Q}\right) = \log_2\left(\frac{5}{12 \times 10^{-5}}\right) = 15.35$$

Thus, a 16-bit quantizer is adequate.

Bit rate: A measure of the rate at which bits are generated.

Bit rate = $N \cdot f_s$ where f_s: Sampling rate.

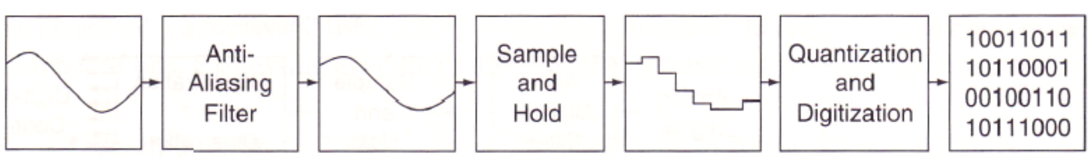
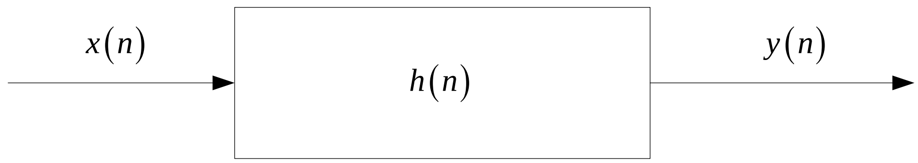


FIGURE 2.20
A/D.

Digital convolution



$$
\begin{aligned}
y(n) &= x(n) * h(n) \\
&= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \\
\\
&= h(n) * x(n) \\
&= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)
\end{aligned}
$$

The frequency response of $y(n)$, DTFT (Discrete Time Fourier Transform), is defined as

$$
\begin{aligned}
F\left[y(n)\right] &= F\left[x(n) * h(n)\right] \\
&= \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n} \qquad \text{where } \Omega = \frac{2\pi f}{f_s} \\
&= \sum_{n=-\infty}^{\infty} \left[x(n) * h(n)\right] e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, h(n-k)\right] e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, h(n-k)\right] e^{-j\Omega(n-k+k)} \\
&= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, h(n-k)\right] e^{-j\Omega(n-k)} e^{-j\Omega k} \\
&= \underbrace{\left[\sum_{n=-\infty}^{\infty} h(n-k)\, e^{-j\Omega(n-k)}\right]}_{H(\Omega)} \cdot \underbrace{\left[\sum_{k=-\infty}^{\infty} x(k)\, e^{-j\Omega k}\right]}_{X(\Omega)} \\
&= H(\Omega) \cdot X(\Omega) \\
&= Y(\Omega)
\end{aligned}
$$

And this shows that the convolution in the time domain is the multiplication in the frequency domain in the DTFT (Discrete time Fourier transform).

It is also true as proven in the continuous time signal case; the convolution in the frequency domain is the multiplication in the time domain.

$$
\begin{aligned}
F^{-1}\left[Y(\Omega)\right] = F^{-1}\left[X(\Omega) * H(\Omega)\right] \\
&= F^{-1}\left[\int_{\tau=-\infty}^{\infty} X(\tau) H(\Omega-\tau)\, d\tau\right] \\
&= \frac{1}{2\pi}\int_{\Omega=-\infty}^{\infty}\left[\int_{\tau=-\infty}^{\infty} X(\tau) H(\Omega-\tau)\, d\tau\right] e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi}\int_{\Omega=-\infty}^{\infty}\left[\int_{\tau=-\infty}^{\infty} X(\tau) H(\Omega-\tau)\, d\tau\right] e^{j(\Omega-\tau+\tau)n} d\Omega \\
&= \frac{1}{2\pi}\int_{\Omega=-\infty}^{\infty}\left[\int_{\tau=-\infty}^{\infty} X(\tau) H(\Omega-\tau)\, d\tau\right] e^{j(\Omega-\tau)n} e^{j\tau n} d\Omega \\
&= \frac{1}{2\pi}\left[\int_{\Omega=-\infty}^{\infty} H(\Omega-\tau) e^{j(\Omega-\tau)n} d\Omega\right]\left[\int_{\tau=-\infty}^{\infty} X(\tau) e^{j\tau n} d\tau\right] \\
&= \frac{1}{2\pi}\underbrace{\left[\int_{\Omega=-\infty}^{\infty} H(\Omega-\tau) e^{j(\Omega-\tau)n} d\Omega\right]}_{h(n)}\underbrace{\left[\int_{\tau=-\infty}^{\infty} X(\tau) e^{j\tau n} d\tau\right]}_{x(n)} \\
&= h(n)\cdot x(n) \\
&= y(n)
\end{aligned}
$$

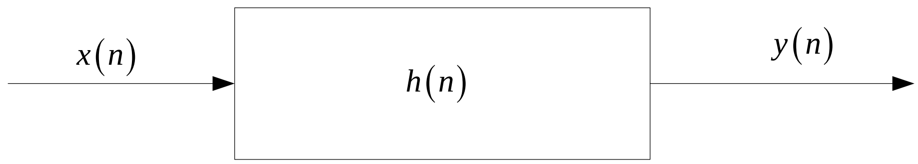
Just like the continuous signal case, the convolution of the digital signal has following properties

$$x(n) * h(n) \Leftrightarrow X(\Omega)\cdot H(\Omega)$$

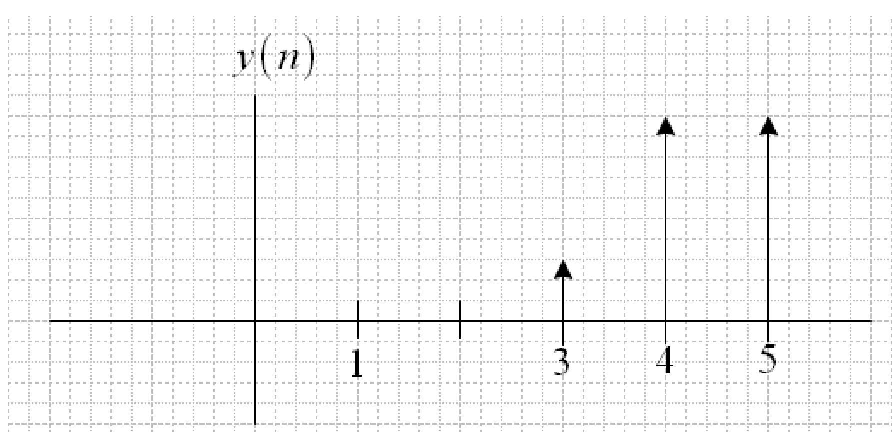
$$X(\Omega) * H(\Omega) \Leftrightarrow x(n)\cdot h(n)$$

Ex)
An input and impulse response are shown below, find the output of the system.

$$h[n] = \delta(n-1) + 2\delta(n-2)$$
$$x[n] = \delta(n-2) + 2\delta(n-3)$$



$$\begin{aligned}
y(n) &= h(n) * x(n) \\
&= \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \\
&= \sum_{k=-\infty}^{\infty} \left[\delta(k-1) + 2\delta(k-2)\right]\left[\delta(n-k-2) + 2\delta(n-k-3)\right] \\
&= \sum_{k=-\infty}^{\infty} \left[\delta(k-1)\right]\left[\delta(n-k-2) + 2\delta(n-k-3)\right] + \left[2\delta(k-2)\right]\left[\delta(n-k-2) + 2\delta(n-k-3)\right] \\
&= \left[\delta(n-k-2) + 2\delta(n-k-3)\right]\Big|_{k=1} + \left[2\delta(n-k-2) + 4\delta(n-k-3)\right]\Big|_{k=2} \\
&= \left[\delta(n-1-2) + 2\delta(n-1-3)\right] + \left[2\delta(n-2-2) + 4\delta(n-2-3)\right] \\
&= \left[\delta(n-3) + 2\delta(n-4)\right] + \left[2\delta(n-4) + 4\delta(n-5)\right] \\
&= \delta(n-3) + 4\delta(n-4) + 4\delta(n-5)
\end{aligned}$$



Convolution chart for lecture