Lec 15

Conditioning by a Random Variable

Conditional PMF

For any event $[Y = y]$ such that $\left[P_Y[y] > 0\right]$, the conditional PMF of $[X]$ given $Y = y$ is

- $$P_{X|Y}[x \mid y] = P\left[X = x \mid Y = y\right]$$

- $$\begin{cases} P_{X,Y}[x, y] = P_{X|Y}[x \mid y] \cdot P_Y[y] = P_{Y|X}[y \mid x] \cdot P_X[x] \\ P_{X|Y}[x \mid y] = \dfrac{P_{X,Y}[x, y]}{P_Y[y]} \end{cases}$$

Conditional PDF

For y such that $f_Y(y) > 0$, the conditional PDF of X given $(Y = y)$ is

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

$$f_{X,Y}(x, y) = f_{X|Y}(x \mid y) \cdot f_Y(y) = f_{Y|X}(y \mid x) \cdot f_X(x)$$

Ex. 4.19 Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & o.w \end{cases}$$

Find the conditional pdf $f_{Y|X}(y|x)$ and $f_{X|Y}(x|y)$.

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

a. $$\begin{cases} \Rightarrow f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \\ \qquad\qquad = \int_0^x 2dy \\ \qquad\qquad = 2x \qquad\qquad 0 \le x \le 1 \end{cases}$$

b. $$\begin{cases} f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} \\ \qquad\qquad = \dfrac{2}{2x} = \dfrac{1}{x} \qquad\qquad 0 \le y \le x \end{cases}$$

c. $$\begin{cases} \Rightarrow f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \\ \qquad\qquad = \int_y^1 2dx \\ \qquad\qquad = 2(1-y) \qquad\qquad 0 \le y \le 1 \end{cases}$$

d. $$\begin{cases} f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} \\ \qquad\qquad = \dfrac{2}{2(1-y)} = \dfrac{1}{(1-y)} \qquad\qquad y \le x \le 1 \end{cases}$$

Conditional Expected Value of a Function

For continuous r.v $[X \text{ and } Y]$ and any y such that $f_Y(y) > 0$, the conditional expected value of $g(X,Y)$ given $Y = y$ is

$$E\left(g(X,Y)\middle|Y=y\right) = \int_{-\infty}^{\infty} g(x,y)\, f_{X|Y}(x|y)\,dx$$

$$E\left(X\middle|Y=y\right) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$$

The conditional expected value $E(X|Y)$ is a function of random variable Y s.t. if $Y = y$ then $E(X|Y) = E(X|Y=y)$

Ex 4.20) The conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{1-y} & y \le x \le 1 \\ 0 & o.w \end{cases}$$

Find the conditional expected values $E(X|Y=y)$ and $E(X|Y)$.

$$E(X|Y=y) = \begin{cases} \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx \\ = \int_y^1 x \cdot \dfrac{1}{1-y}\,dx \\ = \dfrac{x^2}{2(1-y)}\Big|_{x=y}^{x=1} \\ = \dfrac{1-y^2}{2(1-y)} = \dfrac{1+y}{2} \end{cases}$$

$$E(X|Y) = \frac{1+Y}{2}$$

Quiz 4.9) A) The probability model for random variable A is

$$P_A[a] = \begin{cases} 0.4 & a = 0 \\ 0.6 & a = 2 \\ 0 & o.w \end{cases}$$

The conditional probability model for random variable B given A is

$$P_{B|A}[b|0] = \begin{cases} 0.8 & b = 0 \\ 0.2 & b = 1 \\ 0 & o.w \end{cases} \qquad P_{B|A}[b|2] = \begin{cases} 0.5 & b = 0 \\ 0.5 & b = 1 \\ 0 & o.w \end{cases}$$

1) What is the probability model for $[A \text{ and } B]$? Write the joint PMF $P_{A,B}[a,b]$ as a table.
2) The conditional expected value $E[B|A=2]$
3) If $[B=0]$, what is the conditional PMF $P_{A|B}[a|0]$
4) If $[B=0]$, what is the conditional variance $Var[A|B=0]$

1)

$P_{A,B}[a,b]$	$b=0$	$b=1$
$a=0$	$P[A=0,B=0]=P_{B\|A}[0\|0]P_A[a]=0.8\cdot 0.4$	$0.2\cdot 0.4$
$a=2$	$0.5\cdot 0.6$	$0.5\cdot 0.6$

2) $E[B|A=2] = \sum_{b=0}^{1} b\cdot P_{B|A}[b|2] = 0\cdot(0.5)+1\cdot(0.5) = 0.5$

3) $$P_{A|B}[a|0] = \frac{P_{A,B}[a,0]}{P_B[0]} = \begin{cases} \dfrac{0.8\cdot 0.4}{0.32+0.3} & a=0 \\ \dfrac{0.5\cdot 0.6}{0.32+0.3} & a=2 \\ 0 & o.w \end{cases}$$

4) $$\begin{cases} Var\left[A\middle|B=0\right]=E\left[A^2\middle|B=0\right]-\left(E\left[A\middle|B=0\right]\right)^2 \\ \\ \begin{cases} E\left[A^2\middle|B=0\right]=\sum\limits_{a=0,2}a^2P_{A|B}\left[a\middle|0\right]=0^2\cdot\dfrac{0.32}{0.62}+2^2\cdot\dfrac{0.30}{0.62}=\dfrac{1.2}{0.62} \\ E\left[A\middle|B=0\right]=\sum\limits_{a=0,2}a\cdot P_{A|B}\left[a\middle|0\right]=0\cdot\dfrac{0.32}{0.62}+2\cdot\dfrac{0.30}{0.62}=\dfrac{0.6}{0.62} \end{cases} \\ \\ Var\left[A\middle|B=0\right]=E\left[A^2\middle|B=0\right]-\left(E\left[A\middle|B=0\right]\right)^2=\dfrac{1.2}{0.62}-\left(\dfrac{0.6}{0.62}\right)^2 \end{cases}$$

B) The PDF of random variable X and the conditional PDF of random variable Y given X are

$$f_X(x)=\begin{cases}3x^2 & 0\le x\le 1\\ 0 & o.w\end{cases}\qquad f_{Y|X}(y|x)=\begin{cases}\dfrac{2y}{x^2} & 0\le y\le x,\ \ 0\le x\le 1\\ 0 & o.w\end{cases}$$

1) Find $f_{X,Y}(x,y)$
2) If $X=\frac{1}{2}$ find the conditional PDF $f_{Y|X}\left(y|\tfrac{1}{2}\right)$
3) If $Y=\frac{1}{2}$ find the conditional PDF $f_{X|Y}\left(x|\tfrac{1}{2}\right)$
4) If $Y=\frac{1}{2}$ find the conditional variance $Var\left(X\middle|Y=\frac{1}{2}\right)$

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1) $f_{X,Y}(x,y)=f_{Y|X}(y|x)\,f_X(x)=\dfrac{2y}{x^2}\cdot 3x^2=\begin{cases}6y & 0\le y\le x,\ \ 0\le x\le 1\\ 0 & o.w\end{cases}$

2) $f_{Y|X}\left(y|\tfrac{1}{2}\right)=\begin{cases}\dfrac{2y}{x^2}=\dfrac{2y}{\left(\frac{1}{2}\right)^2}=8y & 0\le y\le \dfrac{1}{2}\\ 0 & o.w\end{cases}$

3) $$\begin{cases}f_{X|Y}\left(x|\tfrac{1}{2}\right)=\dfrac{f_{X,Y}\left(x,\frac{1}{2}\right)}{f_Y\left(\frac{1}{2}\right)}\\ f_Y\left(\tfrac{1}{2}\right)=\int_{-\infty}^{\infty}f_{X,Y}\left(x,\tfrac{1}{2}\right)dx=\int_{\frac{1}{2}}^{1}6\cdot\dfrac{1}{2}dx=\dfrac{3}{2}\\ \qquad\text{for } \dfrac{1}{2}\le x\le 1\\ f_{X|Y}\left(x|\tfrac{1}{2}\right)=\dfrac{f_{X,Y}\left(x,\frac{1}{2}\right)}{f_Y\left(\frac{1}{2}\right)}=\dfrac{6\,\frac{1}{2}}{\frac{3}{2}}=2\end{cases}$$

4) If $Y=\dfrac{1}{2}$, the conditional PDF of X is uniform for $\dfrac{1}{2}\le x\le 1$. As we saw from theorem 3.6 of textbook page 114, the variance is $\left[\dfrac{1}{12}(b-a)^2\right]$.

$$\frac{(b-a)^2}{12}=\frac{\left(1-\frac{1}{2}\right)^2}{12}=\frac{1}{48}$$

4.10 Independent Random Variables

In page 21in Ch1, it is defined that events $[A \text{ and } B]$ are independent **if and only if**

$$P[AB] = P[A] \cdot P[B].$$

And when events $[A \text{ and } B]$ have nonzero probabilities, the following formulas are equivalent to the definitions of independent events:

$$\begin{cases} P[A \mid B] = P[A] \\ P[B \mid A] = P[B] \end{cases}$$

The above theorem can be applied to the random variable case with PMF and PDF. Random variables $[A \text{ and } B]$ are independent if and only if

Discrete: $P_{X,Y}[x, y] = P_X[x] P_Y[y]$

Continuous: $f_{X,Y}(x, y) = f_X(x) f_Y(y)$

As we can see from the independent definition of conditional probability theorem, it also implies that if X and Y are independent discrete/continuous random variables, then

$$\begin{cases} P_{X|Y}[x \mid y] = P_X[x] \\ P_{Y|X}[y \mid x] = P_Y[y] \end{cases} \qquad \begin{cases} f_{X|Y}(x \mid y) = f_X(x) \\ f_{Y|X}(y \mid x) = f_Y(y) \end{cases}$$

Ex 4.23) Are $[X \text{ and } Y]$ independent?

$$f_{X,Y}(x,y) = \begin{cases} 4xy & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & o.w \end{cases}$$

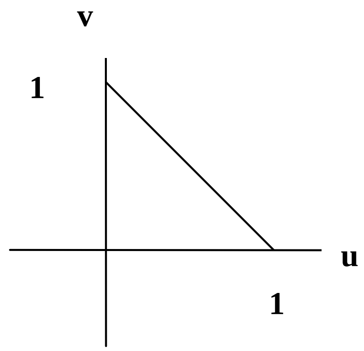
If it is true$\left[f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \right]$, then it is independent. So we need to find out the marginal probability of $[X \text{ and } Y]$

$$f_X(x) = \begin{cases} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \\ = \int_0^1 4xy dy \\ = \frac{4}{2} xy^2 \mid_0^1 \\ = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & o.w \end{cases} \end{cases} \qquad f_Y(y) = \begin{cases} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx \\ = \int_0^1 4xy dx \\ = \frac{4}{2} x^2 y \mid_0^1 \\ = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & o.w \end{cases} \end{cases}$$

$$\begin{cases} f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y) \\ \\ 2x \cdot 2y = 4xy \end{cases} \quad \text{so it is independent}$$

Ex 4.24) Are U and V independent?

$$f_{U,V}(u,v)=\begin{cases} 24uv & 0\le u,\ 0\le v,\ u+v\le 1 \\ 0 & o.w \end{cases}$$



We need to test $f_{U,V}(u,v)= f_U(u)\, f_V(v)$

$$f_U(u)=\begin{cases} \int_{-\infty}^{\infty} f_{U,V}(u,v)dv \\ =\int_0^{1-u} 24uv\cdot dv \\ =\frac{24}{2}uv^2 \Big|_0^{1-u} \\ =\begin{cases} 12u(1-u)^2 & 0\le u\le 1 \\ 0 & o.w \end{cases} \end{cases} \qquad f_V(v)=\begin{cases} \int_{-\infty}^{\infty} f_{U,V}(u,v)du \\ =\int_0^{1-v} 24uvdu \\ =\frac{24}{2}u^2v \Big|_0^{1-v} \\ =\begin{cases} 12v(1-v)^2 & 0\le v\le 1 \\ 0 & o.w \end{cases} \end{cases}$$

$$\begin{aligned} f_{U,V}(u,v) &\ne f_U(u)\cdot f_V(v) \\ 24uv &\ne \left[12u(1-u)^2\right]\cdot\left[12v(1-v)^2\right] \end{aligned}$$

It is **not** independent

For **independent random variables** X and Y,

a. $E\left(g(X)h(Y)\right)=E\left(g(X)\right)E\left(h(Y)\right)$

b. $r_{X,Y}=E(XY)=E(X)E(Y)$

c. $Cov(X,Y)=\rho_{X,Y}=0$ independent r.v $\overset{\rightarrow}{\not\leftarrow}$ uncorrelated.

d. $Var(X+Y)=Var(X)+Var(Y)$

e. $E(X\,|\,Y=y)=E(X)\ for\ all\ y\in S_Y$

f. $E(Y\,|\,X=x)=E(Y)\ for\ all\ x\in S_X$

Independent random variables are uncorrelated, but the reverse is not true all the time.

Ex 4.25) Are X and Y independent? Are X and Y uncorrelated?

$P_{X,Y}[x,y]$	$y=-1$	$y=0$	$y=1$
$x=-1$	0	0.25	0
$x=1$	0.25	0.25	0.25

$$\begin{cases} P_X[-1]=0.25 \\ P_X[1]=0.75 \end{cases} \qquad \begin{cases} P_Y[-1]=0.25 \\ P_Y[0]=0.5 \\ P_Y[1]=0.25 \end{cases}$$

$$\begin{cases} P_{X,Y}[-1,-1]=0 \\ \\ P_X[-1]P_Y[-1]=[0.25]\cdot[0.25] \end{cases} \qquad \text{Not independent}$$

$$\begin{cases} E[X]=\sum_{x=-1,1} x\cdot P_X[x]=(-1)\cdot[0.25]+1\cdot[0.75]=0.5 \\ E[Y]=\sum_{y=-1,0,1} y\cdot P_Y[y]=(-1)\cdot[0.25]+0+1\cdot[0.25]=0 \\ \\ E[XY]=\begin{cases} \sum_{x=-1,1}\sum_{y=-1,0,1} xyP_{X,Y}[x,y] \\ =[-1\cdot-1\cdot 0]+[-1\cdot 0\cdot 0.25]+[-1\cdot 1\cdot 0]+[1\cdot-1\cdot 0.25]+[1\cdot 0\cdot 0.25]+[1\cdot 1\cdot 0.25] \\ =0+0+0-0.25+0+0.25 \\ =0 \end{cases} \end{cases}$$

$$Cov[X,Y]=\begin{cases} E[XY]-E[X]E[Y] \\ =0-0.5\cdot 0 \\ =0 \end{cases} \qquad \Rightarrow \rho_{X,Y}=0$$

This is **uncorrelated**.

Quiz 4.10) Random variables X_1 and X_2 are ***independent and identically distributed*** **(iid) with PDF**

$$f_X(x) = \begin{cases} 1-\dfrac{x}{2} & 0 \le x \le 2 \\ 0 & o.w \end{cases}$$

a) What is the joint PDF?

Since X_1 and X_2 are independent,

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} f_{X_1}(x_1)\cdot f_{X_2}(x_2) \\ = \begin{cases} \left(1-\dfrac{x_1}{2}\right)\cdot\left(1-\dfrac{x_2}{2}\right) & 0 \le x_1 \le 2,\ 0 \le x_2 \le 2 \\ 0 & o.w \end{cases} \end{cases}$$

b) Find the CDF of $Z = \max(X_1, X_2)$

$$P(Z \le z) = \begin{cases} P(X_1 \le z, X_2 \le z) \\ = P(X_1 \le z)P(X_2 \le z) \\ = (F_X(z))^2 \end{cases}$$

$$F_X(x) = \begin{cases} \int_{-\infty}^{\infty} f_X(x)\,dx \\ = \int_{-\infty}^{x} f_X(x)\,dx \\ = \int_0^x \left(1-\dfrac{x}{2}\right)dx \\ = \left(x-\dfrac{x^2}{4}\right)_0^x \\ = x-\dfrac{x^2}{4} \end{cases} \quad \Rightarrow \quad F_Z(z) = \begin{cases} 0 & z<0 \\ \left(z-\dfrac{z^2}{4}\right)^2 & 0 \le z < 2 \\ 1 & 2<z \end{cases}$$

4.11 Bivariate Gaussian Random Variables

The Bivariate Gaussian distribution is a probability model for $[X \text{ and } Y]$ with the property that X **and** Y **are each Gaussian random variables.**

Random Variables X and Y have a **Bivariate Gaussian PDF** with parameters $\mu_1,\ \sigma_1,\ \mu_2,\ \sigma_2,$ and ρ

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\left(\frac{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)} \right) \right]$$

where μ_1 and μ_2 can be any real numbers, $\sigma_1 > 0$, $\sigma_2 > 0$,and $-1 < \rho < 1$

Let $\tilde{\mu}_2(x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1),\ \ \tilde{\sigma}_2 = \sigma_2\sqrt{1-\rho^2}$ to examine properties of

$$f_{X,Y}(x,y) = \left(\frac{1}{\sigma_1\sqrt{2\pi}} \mathrm{e}^{\frac{-(x-\mu_1)^2}{2\sigma_1^2}} \right) \cdot \left(\frac{1}{\tilde{\sigma}_2\sqrt{2\pi}} \mathrm{e}^{\frac{-(y-\tilde{\mu}_2(x))^2}{2\tilde{\sigma}_2^2}} \right) \qquad (4.11\text{-}1)$$

The equation (4.11-1) becomes the products of two Gaussian PDFs.

If X and Y are the Bivariate Gaussian random variables in the equation (4.11-1)

- X is the Gaussian $(\mu_1,\ \sigma_1)$ random variable
- Y is the Gaussian $(\mu_2,\ \sigma_2)$ random variable
- $f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}} \mathrm{e}^{\frac{-(x-\mu_1)^2}{2\sigma_1^2}}$
- $f_Y(y) = \frac{1}{\sigma_2\sqrt{2\pi}} \mathrm{e}^{\frac{-(y-\mu_2)^2}{2\sigma_2^2}}$

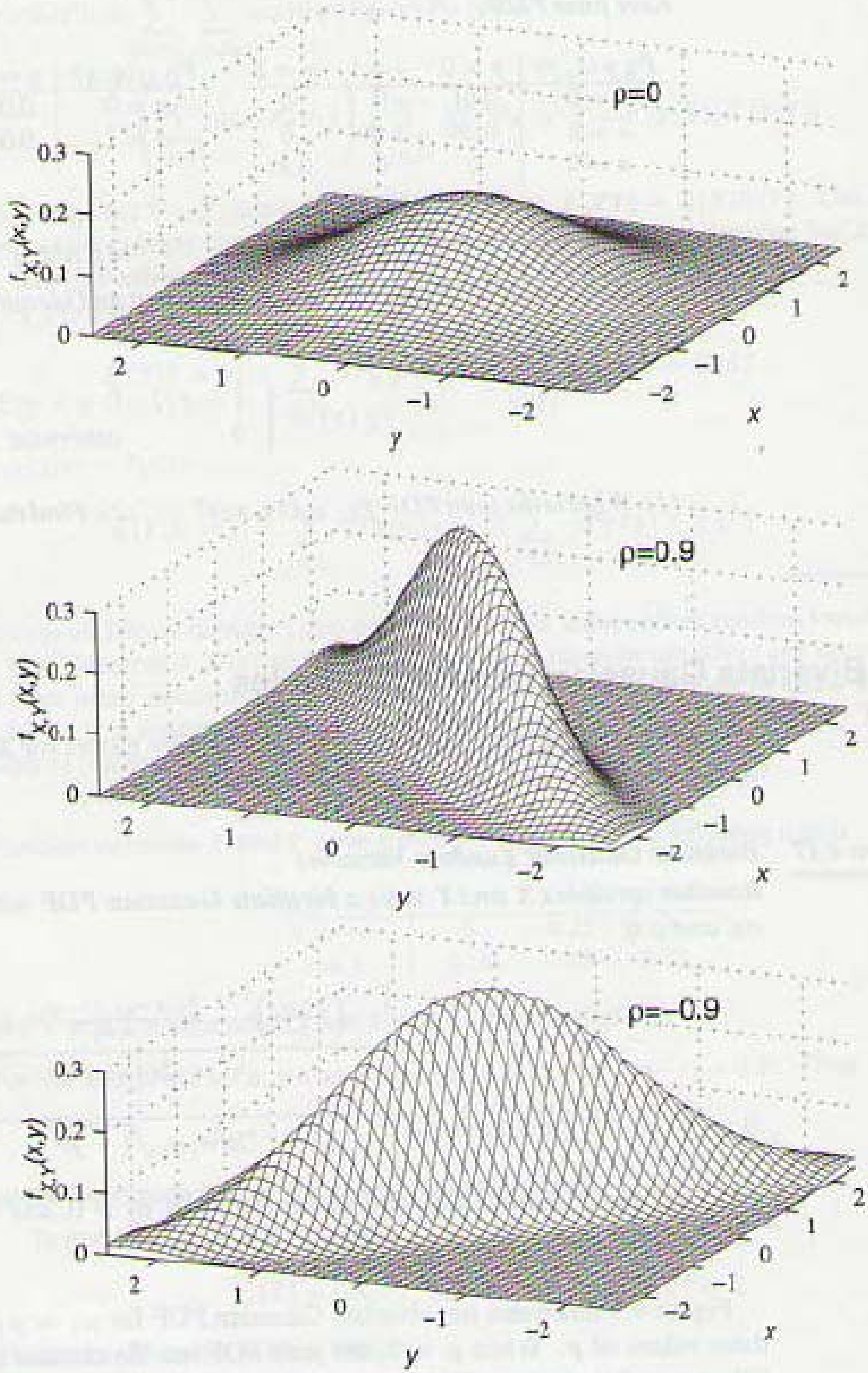


Figure 4.5 The Joint Gaussian PDF $f_{X,Y}(x, y)$ for $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$, and three values of ρ.

If X and Y are the Bivariate Gaussian random variables in the equation (4.11-1), the conditional PDF of Y given Y is

$$f_{Y|X}(y|x) = \frac{1}{\tilde{\sigma}_2\sqrt{2\pi}} \mathrm{e}^{\frac{-(y-\tilde{\mu}_2(x))^2}{2\tilde{\sigma}_2^2}}$$

where, given $X = x$, the conditional expected value and variance of Y are

$$\tilde{\mu}_2(x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1), \quad \tilde{\sigma}_2{}^2 = \sigma_2{}^2\sqrt{1-\rho^2}$$

If X and Y are the Bivariate Gaussian random variables in the equation (4.11-1), the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{1}{\tilde{\sigma}_1\sqrt{2\pi}} \mathrm{e}^{\frac{-(x-\tilde{\mu}_1(y))^2}{2\tilde{\sigma}_1^2}}$$

where, given $Y = y$, the conditional expected value and variance of X are

$$\tilde{\mu}_1(x) = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2), \quad \tilde{\sigma}_1{}^2 = \sigma_1{}^2\sqrt{1-\rho^2}$$

Bivariate Gaussian random variables X and Y have correlation coefficient

$$\rho_{X,Y} = \rho$$

Bivariate Gaussian random variables X and Y are **uncorrelated if and only if they are independent**.

Quiz 4.11 Let X and Y be jointly Gaussian (0,1) random variables with correlation coefficient, $\rho = \frac{1}{2}$.

(1) What is the joint PDF of X and Y?

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[\frac{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)} \right]$$

$$= \frac{1}{2\pi\sqrt{1-\left(\frac{1}{2}\right)^2}} \exp\left[2\left(\frac{x^2 - xy + y^2}{3} \right) \right]$$

where μ_1 and μ_2 can be any real numbers, $\sigma_1 > 0$,$\sigma_2 > 0$,and $-1 < \rho < 1$

$$\mu_1 = \mu_2 = 0,\ \sigma_1 = \sigma_2 = 1$$

```
clc; clear all;
x = -5:0.1:5;
y = -5:0.1:5;

mu1 = 0; mu2 = 0;
sig1 = 1; sig2 = 1;
ro = -0.90;%correlation cofficient

[x, y ] = meshgrid(x,y);



f = exp(-((x-mu1/sig1).^2 - (2*ro*(x-mu1).*(y-mu2))./(sig1*sig2) + ...
    (y-mu2/sig2).^2)./(2*(1-ro^2)))./(2*pi*sig1*sig2*(1-ro^2)^0.5);

meshc(x,y,f)

figure;
surf(x,y,f)
% figure;
% meshz(x,y,f)



whos
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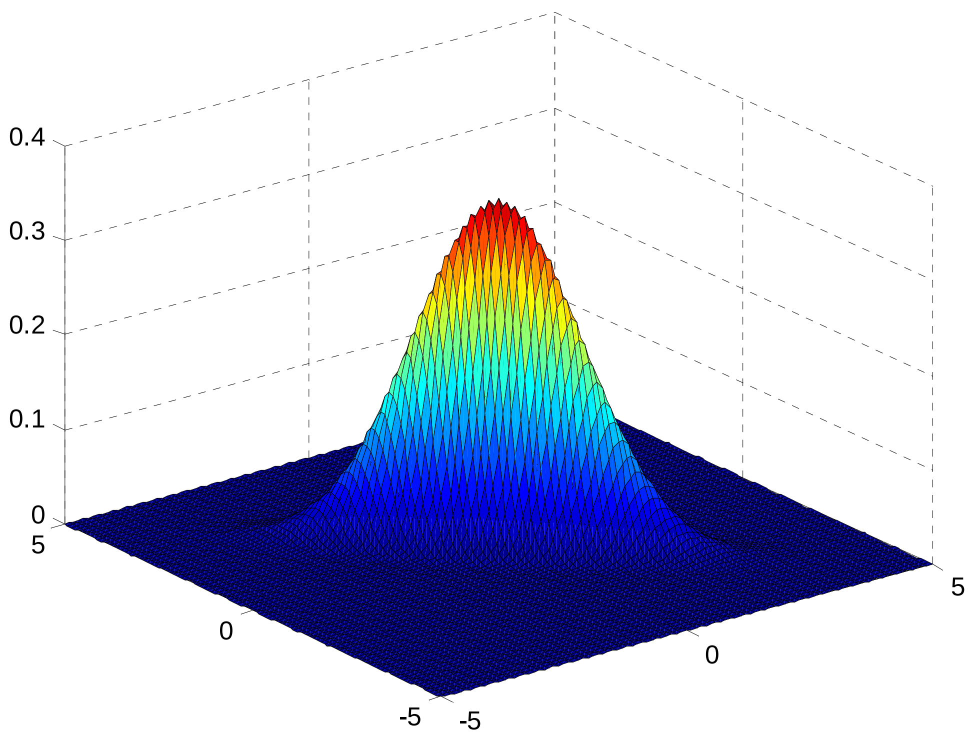

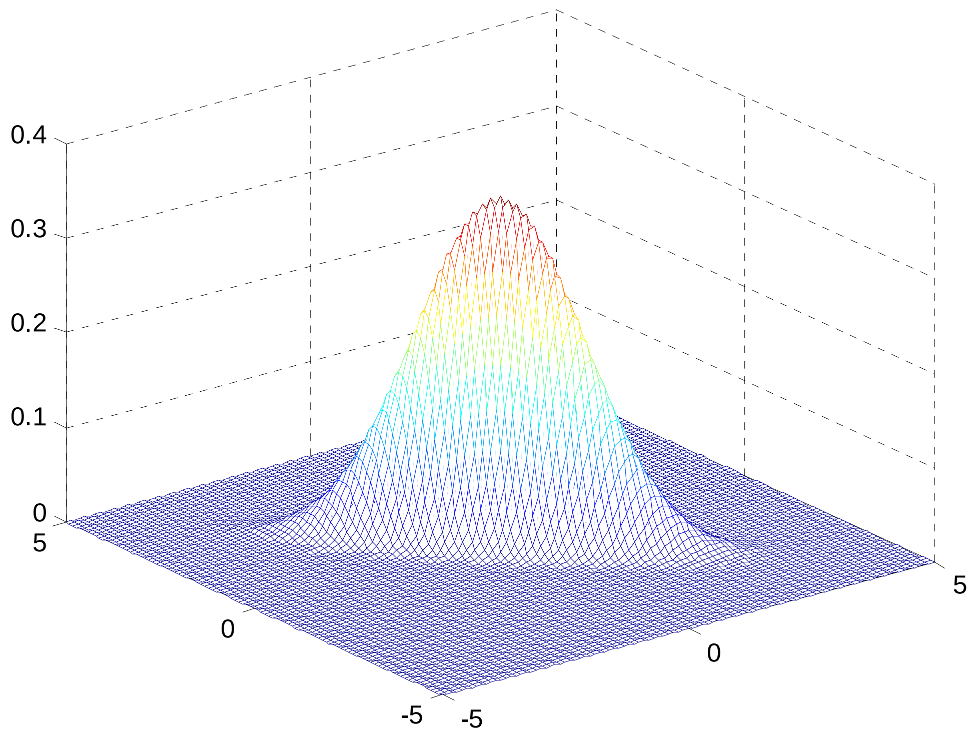

0.4
0.3
0.2
0.1
0
5
0
-5
-5
0
5