CH3 Discrete RVs

Ex: Toss Coin 3 times

η : HHH, HHT, HTH, HTT, THH, THT, TTH, TTT $\rightarrow S$

$X(\eta)$: 3 2 2 1 2 1 1 0

$S_X = \{0, 1, 2, 3\} \rightarrow$ Range of X

$$P[X=3] = P[\{\eta : X(\eta)=3\}] = P[\{HHH\}] = \frac{1}{8}$$

event $A = \{\eta : \overset{RV}{X(\eta)} \in B\}$

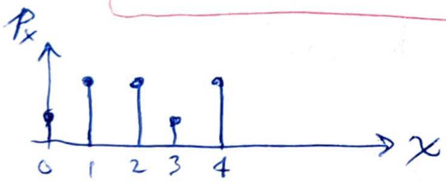
$$P[A] = P[\{\eta : X(\eta) \in B\}] = P[X \in B]$$

☆ Discrete RV

$S_X = \{x_1, x_2, x_3, \dots\}$ → Range of RV X

$\Rightarrow$ Prob. mass function (PMF)

$$p_X(x) = P[X=x] = P[\{\eta : X(\eta) = x\}]$$



* Properties of PMF:

① $p_X(x) \geq 0$

② $\sum_{x \in S_X} p_X(x) = 1$

③ $P[X \text{ in } B] = \sum_{x \in S_X} p_X(x)$ $\quad B \subset S_X$

Expected Value (Mean)

$$m_X = E[X] = \sum_{x \in S_X} x \, p_X(x)$$

$E[|X|] = \sum_{x \in S_X} |x| \, p(x) < \infty$

$$E[g(x)] = \sum_{x \in S_X} g(x) \, p(x)$$

$Z = a\, g(x) + b\, h(x) + c$

$$E[Z] = a E[g(x)] + b E[h(x)] + c$$

* $E[X+Y] = E[X] + E[Y]$

* $E[aX] = a E[X]$

* $E[X+c] = E[X] + c$

B Variance

$$\sigma^2 = VAR[X] = E[(X - m_X)^2] = \sum (x-m_X)^2 p_X(x)$$
$$= E[X^2] - m_X^2$$

n^{th} moment $= E[X^n]$

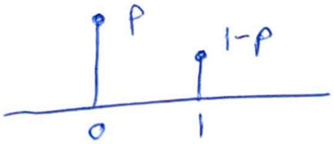
* $Var[cX] = c^2 Var[X]$

* $std(x) = \sigma = \sqrt{Var[x]}$

* $Var[X+c] = Var[x]$

① Bernoulli RV

pass
fail



$$E[X] = (0 \times p) + (1 \times (1-p))$$

$$\boxed{m_X = 1-p}$$

$$Var[X] = E[X^2] - m_X^2$$

$$= (0^2 \times p + 1^2 \times (1-p)) - (1-p)^2$$

$$= (1-p) - (1+p^2-2p) = p - p^2$$

$$\boxed{Var[X] = p(1-p)}$$

② Binomial RV

k successes in n experiments

pass $\rightarrow p$
fail $\rightarrow 1-p$

$$\boxed{P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}} \quad k = 0, 1, 2, \ldots n$$

$$E[X=k] = \sum_{k=0}^{n} k \, P_X(k)$$

$$= \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$\frac{n \quad (n-1)!}{k \quad (k-1)! \, ((n-1)-(k-1))!} = \frac{n}{k} \binom{n-1}{k-1}$$

$$= n \sum_{k=1}^{k=n} \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

let $j = k-1$
$k = j+1$

$$= n \sum_{j=0}^{n-1} \binom{n-1}{j} p^j \cdot p^1 (1-p)^{n-j-1}$$

$$= np \underbrace{\sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{(n-1)-j}}_{P(S)=1}$$

$$\boxed{E[k] = np}$$

$$Var[X] = E[X^2] - (np)^2$$

$$E[X^2] = \sum_{k=0}^{n} k^2 P_X(k)$$

$$= n \sum_{j=0}^{n-1} (j+1) \binom{n-1}{j} p^j \cdot p^1 (1-p)^{n-j-1}$$

$$= np \Big[\underbrace{\sum_{j=0}^{n-1} j \binom{n-1}{j} p^j (1-p)^{n-j-1}}_{\rightarrow E(X) = (n-1)p}$$

$$+ \underbrace{\sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-j-1}}_{\rightarrow 1} \Big]$$

$$= np[(n-1)p + 1]$$

$$\boxed{\begin{aligned} \therefore Var[X] &= np[(n-1)p+1] - n^2p^2 \\ &= np(1-p) \end{aligned}}$$

③ Geometric RV

1^{st} success in k^{th} experiment

pass $\rightarrow p$
fail $\rightarrow 1-p$

$$\boxed{P_X(k) = (1-p)^{k-1} p} \quad k = 1, 2, \ldots\ldots$$

$$E_X[X] = \sum_{k \in S_X} k \, P_X(k)$$

$$= \sum_{k=1}^{\infty} k (1-p)^{k-1} p$$

$$= p \frac{1}{(1-(1-p))^2}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

d

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{-1}{(1-x)^2}(-1)$$

$$\therefore \boxed{E[X] = \frac{1}{p}}$$

$$Var[X] = E[X^2] - (E[X])^2$$

$$E[X^2] = \sum_{k \in S_X} k^2 P_X(k)$$

$$= \sum_{1}^{\infty} k^2 (1-p)^{k-1} p$$

d

$$\sum_{n=1}^{\infty} n(n-1) x^{n-2} = \frac{-2}{(1-x)^3}(-1)$$

$\cdot x$ both sides

$$\sum_{1}^{\infty} n^2 x^{n-1} - \sum_{1}^{\infty} n x^{n-1} = \frac{2x}{(1-x)^3}$$

$$\sum_{1}^{\infty} n x^{n-1} \rightarrow \frac{1}{(1-x)^2}$$

$$\therefore \sum_{1}^{\infty} n^2 x^{n-1} = \frac{2x + (1-x)}{(1-x)^3}$$

$$\therefore E[X^2] = \cancel{\quad} = \frac{2(1-p)+p}{p^3}$$

$$\boxed{\therefore Var[X] = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}}$$

Conditional PMF

$$\boxed{P_X(x|C) = P[X=x|C] = \frac{P[X=x \cap C]}{P[C]}}$$

Theorem of Total Prob

$$P(A) = P(A|B_1)P(B_1) + \ldots\ldots + P(A|B_n)P(B_n)$$

$A = \{X=x\}$

$$\boxed{P_X(x) = \sum_{i=1}^{n} P_X(x|B_i)\, P(B_i)}$$

$$\boxed{E[X|C] = \sum_x x\, p_X(x|C)}$$

$$E[X] = \sum_x x\, p(x)$$

$$= \sum_x x \cancel{\sum_i} \sum_i p(x|B_i)\, p(B_i)$$

$$= \sum_i p(B_i) \sum_x x\, p(x|B_i)$$

$$\therefore \boxed{E[X] = \sum_i E(x|B_i)\, p(B_i)}$$

Also

$$E[g(x)] = \sum_i E[g(x)|B_i]\, p(B_i)$$

$$E[X^2] = \sum_i E[x^2|B_i]\, p(B_i)$$

Going back to Geometric RV:

$$P[M \geq k+j \,|\, M > j]$$

k & j are +ve integers

subset

$$= \frac{P[M \geq k+j \cap M > j]}{P[M > j]}$$

$$= \frac{P[M \geq k+j]}{P[M > j]} = \frac{(1-p)^{k+j-1}}{(1-p)^{j}} = (1-p)^{k-1}$$

$$= P[M \geq k]$$

$$\boxed{P[M \geq k+j \,|\, M > j] = P[M \geq k]}$$

④ Poisson RV

$$\boxed{P[X=k] = p_X(k) = e^{-\alpha}\frac{\alpha^k}{k!}}$$

$k = 0, 1, 2, \ldots$

$$\boxed{E[X] = \alpha}$$

Taylor's series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\frac{d}{dx}: \quad e^x = \sum_{k=0}^{\infty} k \frac{x^{k-1}}{k!}$$

$$x e^x = \sum_{k=0}^{\infty} k \frac{x^k}{k!}$$

$$E[X] = \sum_{k=0}^{\infty} k\, p_X(k)$$

$$= \sum k e^{-\alpha} \frac{\alpha^k}{k!}$$

$$= e^{-\alpha} \sum k \frac{\alpha^k}{k!}$$

$$= e^{-\alpha} \alpha e^{\alpha}$$

$$= \alpha$$

$$\boxed{VAR[X] = \alpha}$$

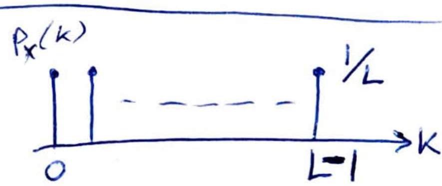
Binomial → Poisson

Binomial: $k = 0, 1, 2, \ldots n$, $\binom{n}{k} p^k (1-p)^{n-k}$

$\left.\begin{matrix} n \to \infty \\ p \to 0 \end{matrix}\right\} \alpha = np$

Poisson: $k = 0, \ldots \infty$, $e^{-\alpha}\frac{\alpha^k}{k!}$

⑤ Uniform RV



$$E[X] = \sum_{k=0}^{L-1} k \cdot \frac{1}{L}$$

$$= \frac{1}{L} \sum_{k=0}^{L-1} k = \frac{1}{L} \frac{L(L-1)}{2}$$

$$\therefore \boxed{E[X] = \frac{L-1}{2}}$$

$$\boxed{VAR[X] = \frac{L^2-1}{12}}$$

⑥ Zipf RV

A large body of text, words arranged from most frequent to less frequent (ranking)

$$S_X = \{1, 2, \ldots\ldots L\}$$ → rank of words

$$\boxed{P_X(k) = \frac{1}{C_L}\frac{1}{k}} \qquad k = 1, 2 \ldots\ldots L$$

$$\sum_{k=1}^{L} P_X(k) = 1$$

$$\sum_{k=1}^{L} \frac{1}{C_L}\frac{1}{k} = 1$$

$$\therefore C_L = \sum_{k=1}^{L} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{L}$$ Normalization factor

$$E[X] = \sum_{k=1}^{L} k\, P_X(k)$$

$$= \frac{1}{C_L} \sum_{k=1}^{L} \frac{k}{k} = \frac{L}{C_L}$$

$$\boxed{Var[X] = \frac{L(L+1)}{2C_L} - \frac{L^2}{C_L^2}}$$

Zipf

$$P[X > m] = 1 - P[X \le m]$$

$$= 1 - \sum_{k=1}^{m} P_X(k)$$

$$= 1 - \sum \frac{1}{C_L}\frac{1}{k} = 1 - \frac{C_m}{C_L}$$

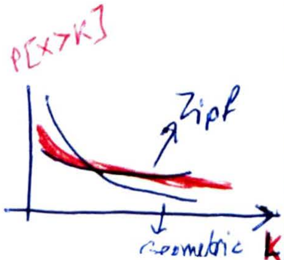
Geometric

$$P[Y > m] = (1-p)^m$$

at $L = 100$

$$E[X] = E[Y]$$

$$\frac{100}{C_L} = \frac{1}{p}$$



Examples

RV

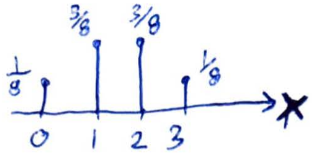
ex Coin tossed 3 times.

X : number of Hs counted

Y : number of $$ player takes — \$1 if $X=2$; \$8 if $X=3$; \$0 otherwise

$S_X = \{0, 1, 2, 3\}$

$S_Y = \{0, 1, 8\}$

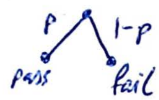


$P[Y = \$8] = \frac{1}{8}$

$P[Y = 0] = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

Geo RV

ex Message transmission



X : # of times message is sent ~~till~~ received correctly

(Consecutive transmissions are independent)

a) $P[X=1] = p$

$P[X=2] = (1-p)\,p$

$\boxed{P[X=k] = (1-p)^k\, p}$

b) $P[X = \text{even}] = P[X=2] + P[X=4] + P[X=6] + \dots$

$= (1-p)p + (1-p)^3 p + \dots$

$= p(1-p)\left[\sum_{k=0}^{\infty} (1-p)^{2k}\right]$

$= p(1-p)\left[\sum_{k=0}^{\infty} \left((1-p)^2\right)^k\right]$

$\boxed{\sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \quad |q|<1}$

$= p(1-p)\left(\frac{1}{1-(1-p)^2}\right)$

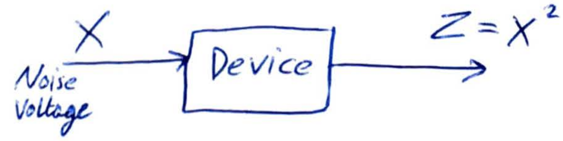
$= \frac{p(1-p)}{\cancel{1}-\cancel{1}-p^2+2\cancel{p}}$

$= \frac{1-p}{2-p}$

c) $P[X > k] = (1-p)^k$

Mean

ex Square law device



Assume X : $S_X = \{-3, -1, 1, 3\}$

$P_X(x) = \frac{1}{4}, \quad x \in S_X$

① Z : $S_Z = \{9, 1\}$

$P_Z(9) = \frac{1}{2}$

$P_Z(1) = \frac{1}{2}$

$E[Z] = \sum_{z \in S_Z} z\, P_Z(z)$

$= (9 \times \frac{1}{2}) + (1 \times \frac{1}{2}) = 5$

② $E[Z] = E[X^2]$

$= \sum_{x \in S_X} x^2\, P_X(x)$

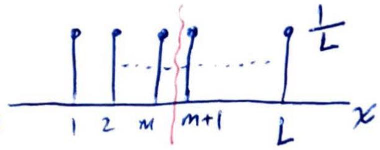
$= ((-3)^2 \times \frac{1}{4}) + ((-1)^2 \times \frac{1}{4}) + ((1)^2 \times \frac{1}{4}) + ((3)^2 \times \frac{1}{4})$

$= 5$

Cond

ex Uniformly distributed RV

X : time required to transmit a message



⇒ Suppose message has already been transmitting for m time units, Find Prob that the remaining time is j time units

$P[X = x \mid x > m] = \frac{P[X = x \cap x > m]}{P[x > m]}$ $\rightarrow \frac{1}{L}$, $\rightarrow (L-m) \times \frac{1}{L}$

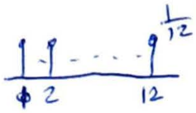
$= \begin{cases} 0 & x \le m \\ \frac{1/L}{(L-m) \times 1/L} & x > m \end{cases}$

[redacted]

$\boxed{P[X = m+j \mid m+j > m] = \frac{1}{L-m}}$

(ex) Random Clock P. 112

hours $X: S_X = \{1, 2, 3, \dots, 12\}$



$B = \{1, 2, 3, 4\}$

$$P[X = x \mid B] = \frac{P[X = x \cap B]}{P[B]}$$

$P[X = x \cap B] = 0$ if $x > 4$, $= \frac{1}{12}$ if $x \leq 4$

$P[B] \rightarrow \frac{4}{12}$

$$= \begin{cases} 0 & x > 4 \\ \frac{1}{4} & x \leq 4 \end{cases}$$

~~min $M: S_M = \{1, 2, \dots, 60\}$ 1/60 60~~

~~$D = \{1 < x \leq 11\}$ minutes~~

(ex) Lifetime of memory element is given by Geometric RV & is an integer (sampled per hr)

$P_{X_1}(x) = r(1-r)^{x-1}$ $\rightarrow \alpha$ $\qquad x = 1, 2, \dots\dots$

$P_{X_2}(x) = s(1-s)^{x-1}$ $\rightarrow 1-\alpha$

Prob of choosing an element that has a certain lifetime

$$P_X(x) = P_X(x \mid r \text{ is chosen})\, P(r \text{ is chosen}) + P_X(x \mid s \text{ is chosen})\, P(s \text{ is chosen})$$

$$= [r(1-r)^{x-1}]\alpha + [s(1-s)^{x-1}](1-\alpha)$$

Poisson

(ex) Arrivals @ packet Multiplexer with average rate $\lambda = 4$ arrivals/min

a) $P[N > 4 \text{ in } 10 \text{ sec}]$

$\hookrightarrow$ in 10 sec $\rightsquigarrow \alpha_1 = \frac{4}{60} \times 10 = \frac{2}{3}$ arrivals/10 secs

$P[N > 4] = 1 - P[N \leq 4]$

$= 1 - [P[0] + \dots P[4]]$

$$= 1 - e^{-2/3}\left[1 + \frac{(2/3)}{1!} + \frac{(2/3)^2}{2!} + \frac{(2/3)^3}{3!} + \frac{(2/3)^4}{4!}\right]$$

b) $P[N < 5 \text{ in } 2 \text{ min}]$

$\hookrightarrow \alpha_2 = 4 \times 2 = 8$

$\therefore P[N < 5] = P[0] + \dots P[4]$

(ex) Data Rate = 1 Gbps

Prob of error of 1 bit $= p = 10^{-9}$

$P[5 \text{ or more errors in } 1 \text{ sec}] = ?$

① Exact "Binomial" $\qquad n = 10^9$

$P[5 \text{ or more errors in } 1 \text{ sec}]$

$= 1 - [P[0] + \dots + P[4]]$

$= 3.659818 \times 10^{-3}$

② Approximate "Poisson" $\qquad n \sim \infty$, $\alpha = np$

$P[5 \text{ or more errors in } 1 \text{ sec}]$

$$= 1 - e^{-1}\left[1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}\right]$$

$= 3.659846 \times 10^{-3}$