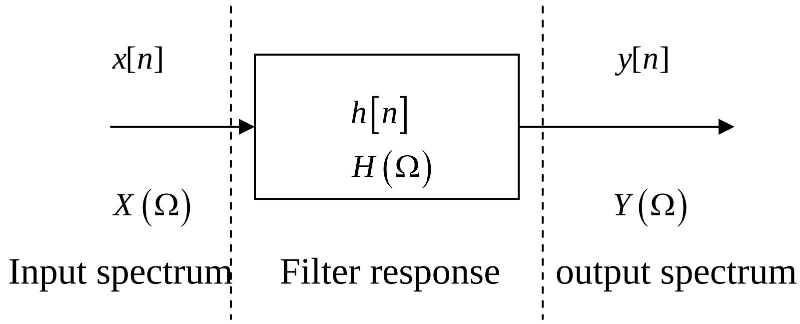
CH 8 Digital Signal Spectra

Spectrum of a signal is a detailed description of the frequency components the signal contains.

- The smooth transition in a signal comes from its low frequency elements.
- Sharp edges and rapid changes come from its high frequency elements.



Spectrum:

- Magnitude spectrum: related to the size or amplitude of the components at each frequency.
- Phase spectrum: gives the phase relationships between the components at different frequencies.

All digital signals have spectra but different tools must be used to compute them: depending on whether the signal is **non-periodic** or **periodic**.

Non-periodic digital signal

Non-periodic signals are those that do not repeat at regular intervals.

Tools to calculate frequency response: **DTFT** (Discrete Time Fourier Transform)

The DTFT for non-periodic signal gives the signal's spectrum as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$= \left| X(\Omega) \right| e^{j\theta(\Omega)}$$

- Continuous in frequency
- Periodic with period 2π
- Magnitude spectrum shows $\left| X(\Omega) \right|$ vs Ω

  in dB, $20\log_{10}\left(\left| X(\Omega) \right|\right)$, or in linear gains.
- Phase spectrum shows $\theta(\Omega)$ vs Ω

  in degrees or radians.

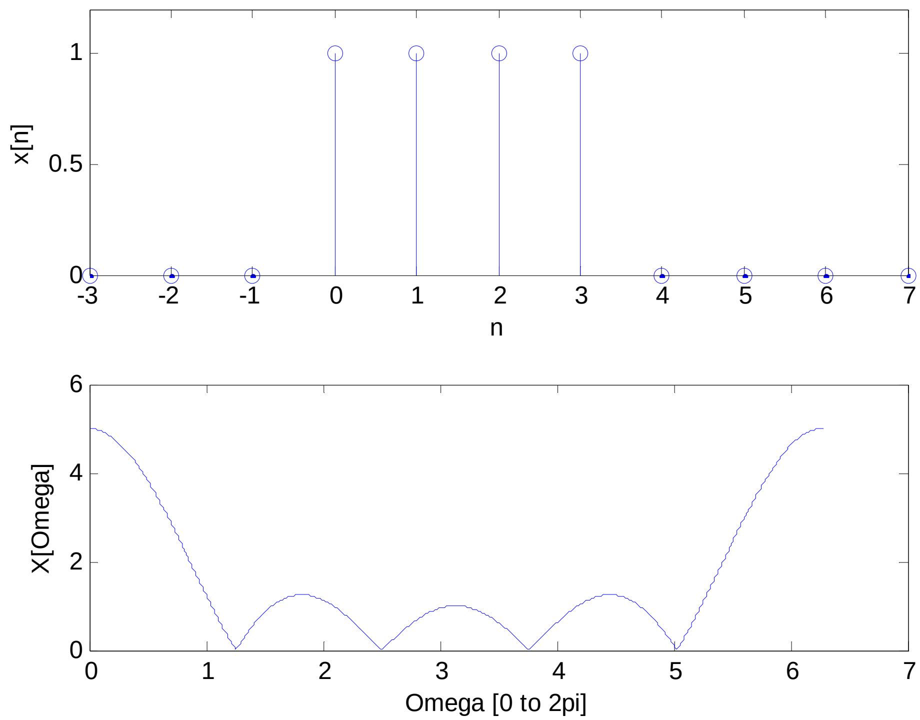
$f = \Omega \dfrac{fs}{2\pi}$ where f is analog frequency.

<u>If fs is not known, then it is **not** possible to find the analog frequency f.</u>

<u>Ex 8.1</u> Find the magnitude and phase spectra for the rectangular pulse $x[n] = u[n] - u[n-4]$ as functions of Ω. Plot linear gains, and phases in radians.

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = \sum_{n=0}^{3} e^{-j\Omega n} = 1 + e^{-j\Omega} + e^{-j2\Omega} + e^{-j3\Omega}$$

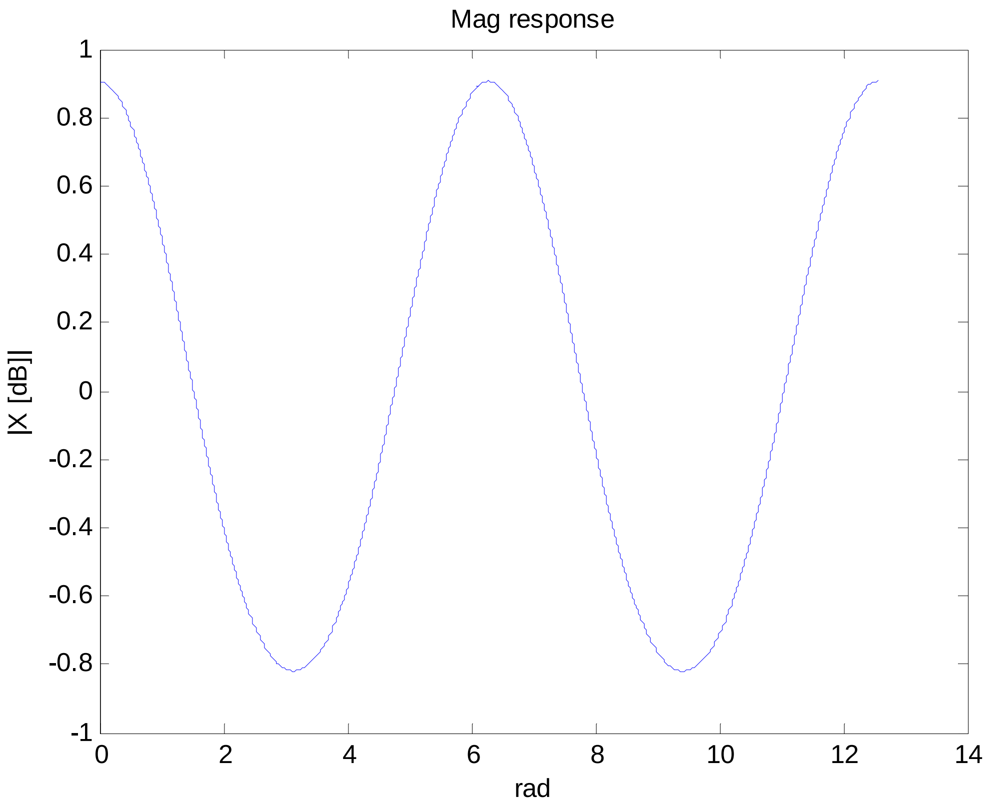



```
clc
clear all

n = [-3 -2 -1 0 1 2 3 4 5 6 7];
x = [ 0  0  0 1 1 1 1 0 0 0 0];

subplot(2,1,1), stem(n,x)
xlabel('n');
ylabel('x[n]');
axis([-3 7 0 1.2])

OM = 0:0.001:(2*pi);
X = 1 + exp(-j*OM) + exp(-j*2*OM) + exp(-j*3*OM) + exp(-j*4*OM);

subplot(2,1,2), plot(OM,abs(X));
xlabel('Omega [0 to 2pi]');
ylabel('X[Omega]');
```

Ex 8.2) Find the magnitude and phase spectra for the signal $x[n] = (0.1)^n u[n]$.

Sampled at 15 kHz, plot the magnitude spectrum in dB and the phase spectrum in degrees, both against frequency in Hz.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega} \approx 1 + (0.1)e^{-j\Omega} + (0.01)e^{-j2\Omega}$$



Periodic digital signal

Periodic signals are those that repeat at regular intervals for all time.

Digital period of the signal: The number of samples that occur in each interval.

DTFT is not an appropriate tool for calculating the spectrum: the infinite sum that is part of the DTFT would give an infinite result.

Tool: **DFS** (Discrete Fourier series)

- According to Fourier theory, every periodic signal can be expressed as the sum of sines and cosines or the sum of complex exponentials.

The Fourier series representation for a periodic digital signal $x[n]$ with period N is

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} c_k \cdot e^{j2\pi \frac{k}{N}n} \qquad k = 0 \ldots N\text{-}1$$

$$or \text{ any } N \text{ consecutive values}$$

Fourier coefficients c_k

$$c_k = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N}n} \qquad n = 0 \ldots N\text{-}1$$

Only N samples of signal need to be used to find c_k for all k

$$c_{k+N}=\sum_{n=0}^{N-1}x[n]e^{-j2\pi\frac{(k+N)}{N}n}$$

$$=\sum_{n=0}^{N-1}x[n]e^{-j2\pi\frac{k}{N}n}e^{-j2\pi\frac{N}{N}n}$$

$$=\sum_{n=0}^{N-1}x[n]e^{-j2\pi\frac{k}{N}n}\underbrace{\cancel{e^{-j2\pi\frac{N}{N}n}}}_{=1}$$

$$=\sum_{n=0}^{N-1}x[n]e^{-j2\pi\frac{k}{N}n}$$

$$c_k=\left|c_k\right|e^{j\phi_k}$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}\underbrace{\left(\left|c_k\right|e^{j\phi_k}\right)}_{c_k}e^{j2\pi\frac{k}{N}n}$$

$$=\sum_{k=0}^{N-1}\frac{\left|c_k\right|}{N}e^{j\left(2\pi\frac{k}{N}n+\phi_k\right)}$$

- Magnitude information is carried by $\frac{\left|c_k\right|}{N}$
- Phase information is carried by ϕ_k

Comparing frequencies shows that the index k is proportional to frequency f.

$$2\pi \frac{k}{N} n = \Omega n$$

$$2\pi \frac{k}{N} n = 2\pi \frac{f}{fs} n$$

$$f = \frac{k}{N} fs$$

where the index k takes the values 0 to N-1.

- The **DC component** of the signal $\frac{|c_0|}{N}$ is given by $k = 0$: it is the average value of the signal. (since the DC term is divided with $\frac{1}{N}$)
- The other frequencies, given by $k > 0$, are called **harmonics of the periodic** signal.
- $k = 1$ gives the signal's first harmonic or fundamental frequency, $\left(f = \frac{fs}{N} \right)$.
- Its reciprocal is the time in seconds to complete one full cycle of the signal.
- The harmonics fall at frequencies that are integer multiples of the fundamental.
- The magnitude spectrum for a periodic digital signal is obtained by plotting $\left(\frac{|c_k|}{N} vs.\ k \right)$.
- The phase spectrum is obtained by plotting ϕ_k vs. k
- Both magnitude and phase spectra are periodic with period N.
- **Magnitude spectrum is always even and the phase spectrum is always odd.**

$$\left|c_{-k}\right| = \left|c_k\right|$$

$$\phi_k = -\phi_k$$

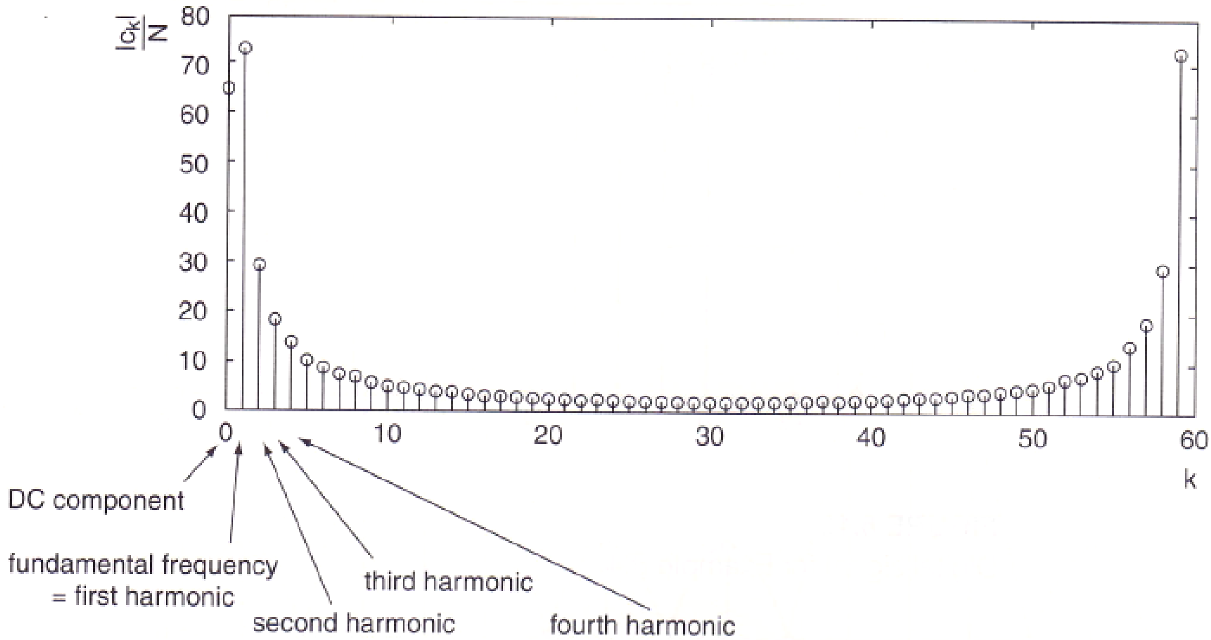


FIGURE 8.12
Magnitude spectrum for a periodic signal.

Comparing DTFTs and DFSs for digital signal

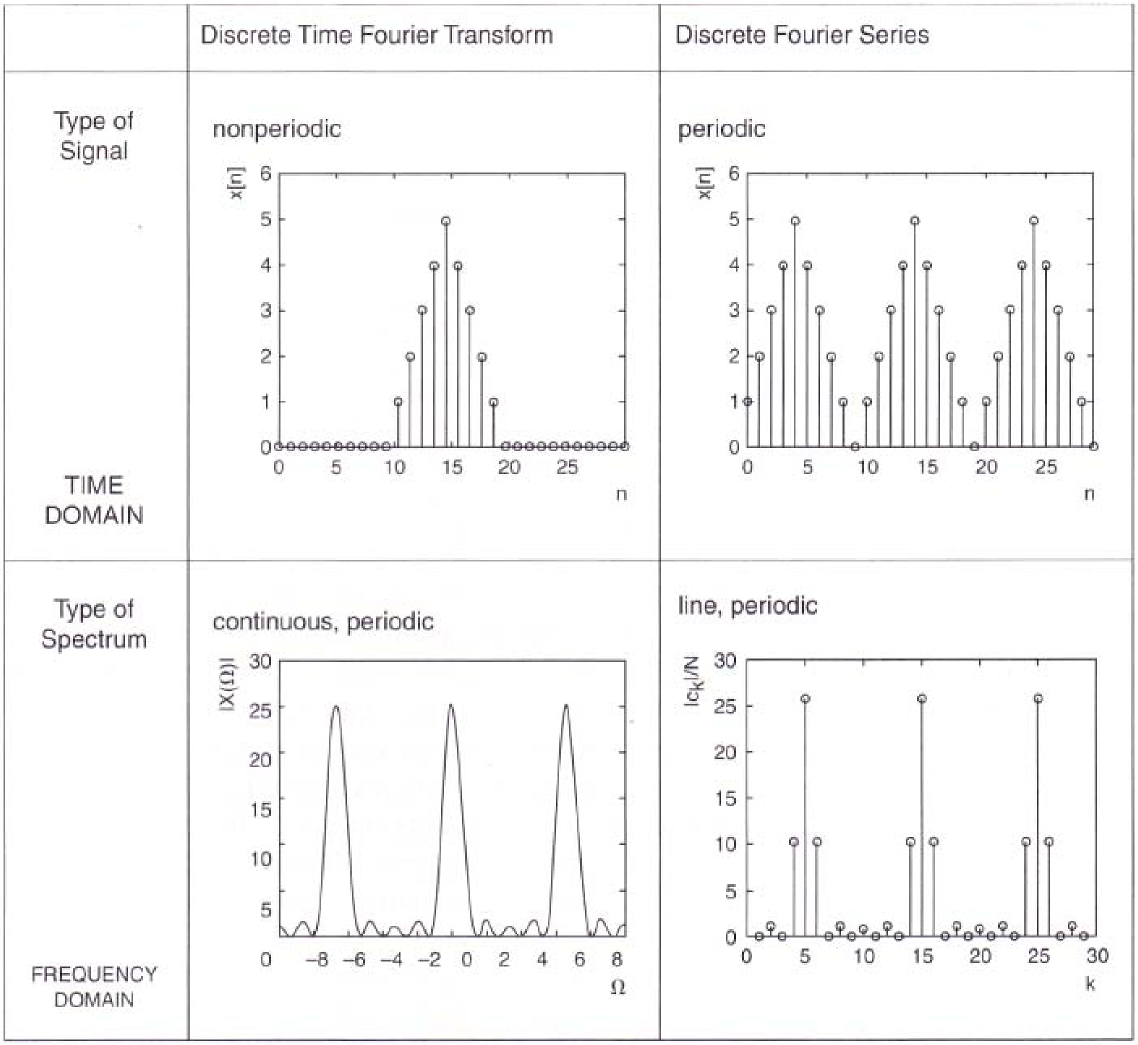
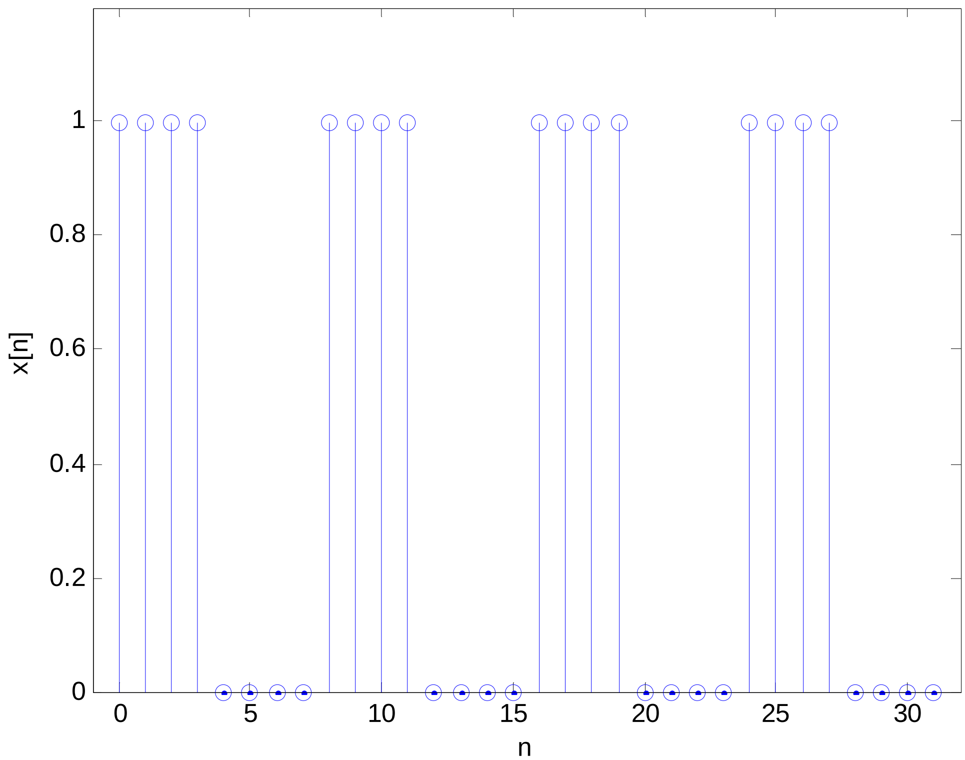


FIGURE 8.11
Comparing DTFTs and DFSs for digital signals.

The difference between the DTFT and DFS spectrums

- DTFT produces continuous spectra
- DFS produces only N points of a spectrum, covering the finite list of frequencies
    - Magnitude and phase spectra for periodic signals are line spectra, vertical lines spaced at equal intervals.

Ex 8.4) Find the magnitude and phase spectra for the periodic square wave signal.

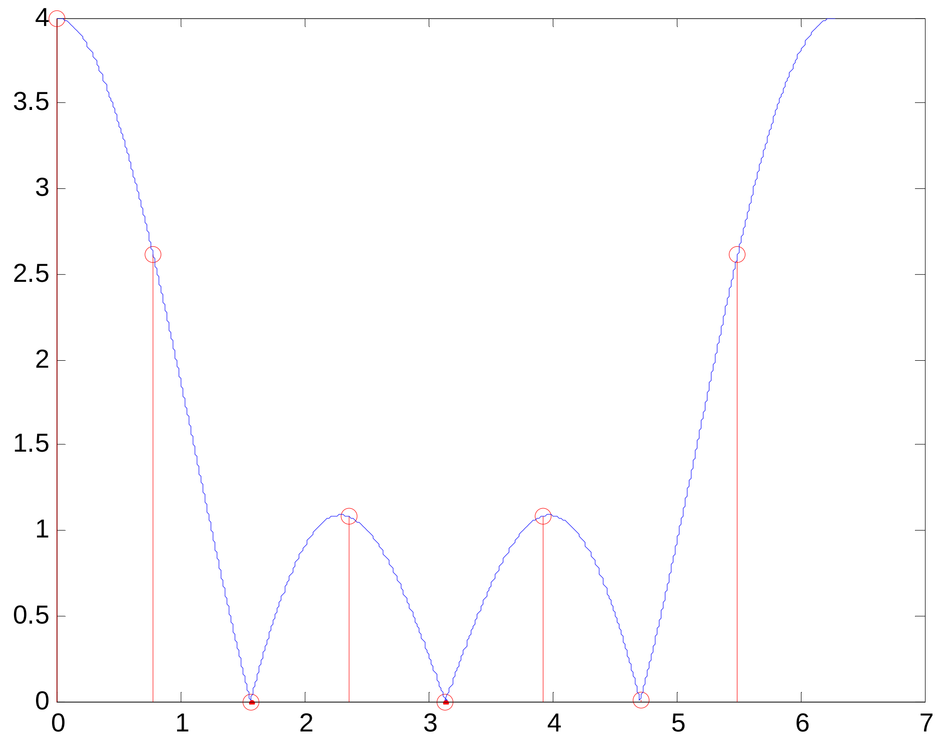


Since the digital signal is periodic, the discrete Fourier series is the tool of choice for finding the magnitude and phase spectra. The period is $N = 8$.

$$c_k = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n} = \sum_{n=0}^{7} x[n] e^{-j2\pi \frac{k}{8} n}$$

$$= x[0] + x[1] e^{-j2\pi \frac{k}{8} n} + x[2] e^{-j2\pi \frac{k}{8} 2} + x[3] e^{-j2\pi \frac{k}{8} 3} + x[4] e^{-j2\pi \frac{k}{8} 4} + x[5] e^{-j2\pi \frac{k}{8} 5}$$
$$+ x[6] e^{-j2\pi \frac{k}{8} 6} + x[7] e^{-j2\pi \frac{k}{8} 7}$$

$$= 1 + e^{-j\pi \frac{k}{4}} + e^{-j\pi \frac{k}{2}} + e^{-j\pi \frac{3k}{4}}$$



DFTF vs DFS

```
clc; clear all;


for k = 0:7

    C(k+1) = 1 + exp(-j*pi*k/4) + exp(-j*pi*k/2) + exp(-j*pi*k*3/4);

end

OM = 0:0.001:2*pi;
len_OM = length(OM);
n = 0:3;
for F = 0:len_OM-1
   C_DTFT(F+1) = exp(-j*OM(F+1)*n)*ones(4,1);
end
k = 0:7;
stem(k*2*pi/8,abs(C),'r')
hold on;
plot(OM,abs(C_DTFT))
hold off;
```