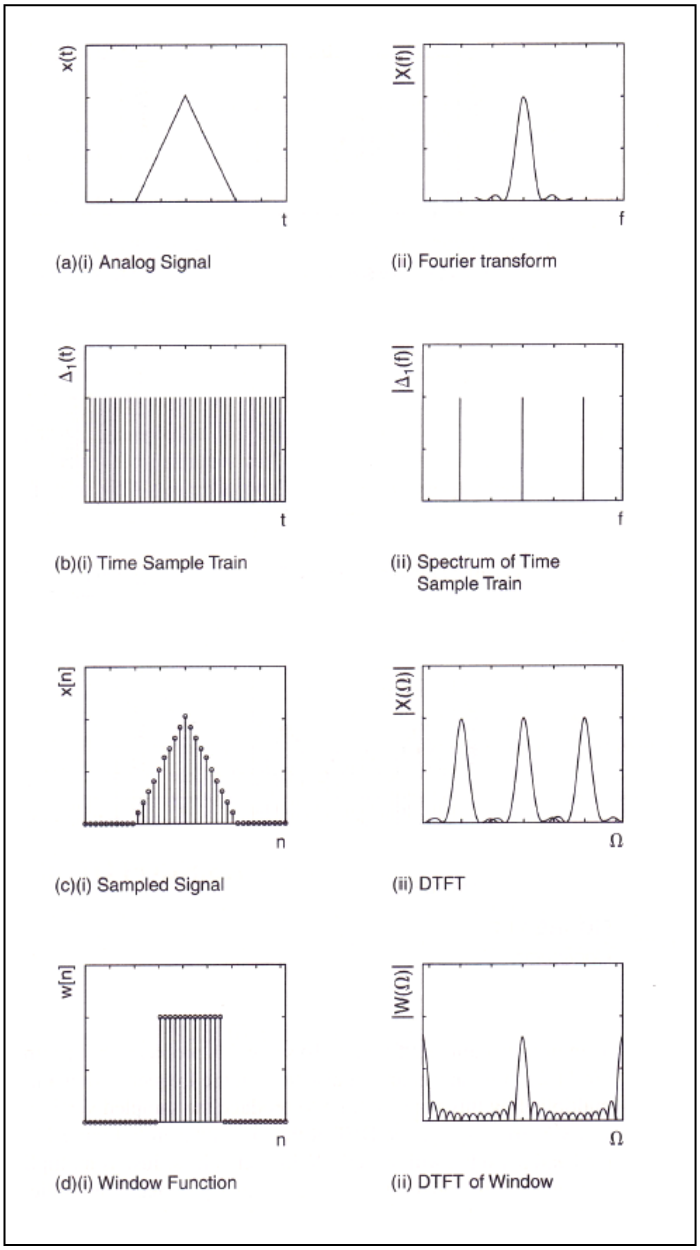
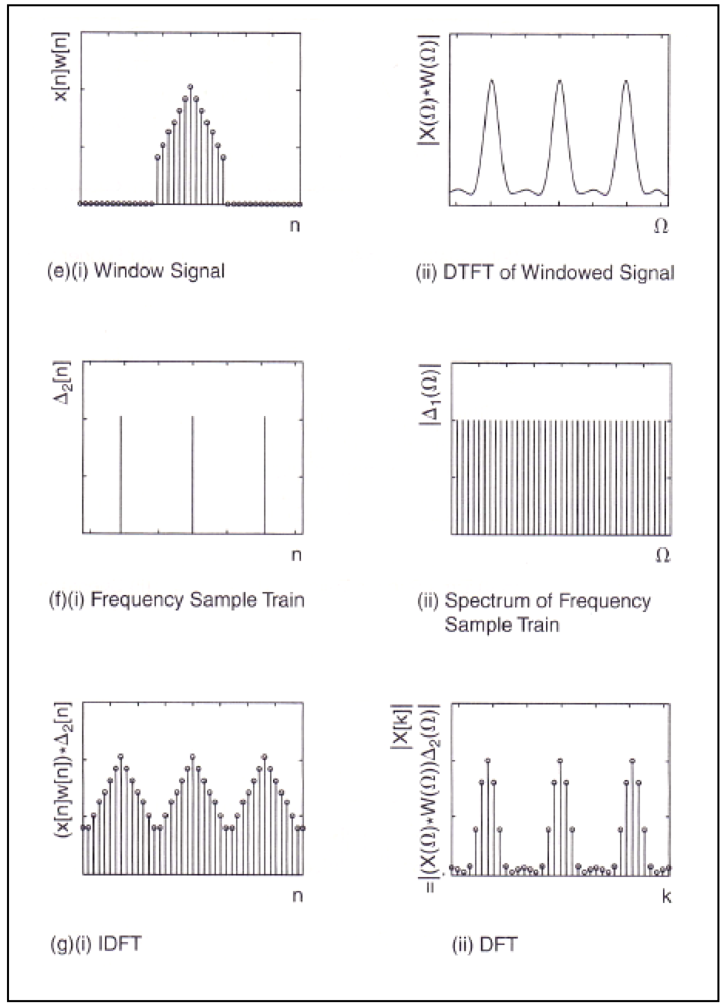

x(t)
t
(a)(i) Analog Signal
|X(f)|
f
(ii) Fourier transform
$\Delta_1(t)$
t
(b)(i) Time Sample Train
$|\Delta_1(f)|$
f
(ii) Spectrum of Time Sample Train
x[n]
n
(c)(i) Sampled Signal
$|X(\Omega)|$
Ω
(ii) DTFT
w[n]
n
(d)(i) Window Function
$|W(\Omega)|$
Ω
(ii) DTFT of Window


x[n]w[n]
n
(e)(i) Window Signal
$|X(\Omega)*W(\Omega)|$
Ω
(ii) DTFT of Windowed Signal
$\Delta_2[n]$
n
(f)(i) Frequency Sample Train
$|\Delta_1(\Omega)|$
Ω
(ii) Spectrum of Frequency Sample Train
$(x[n]w[n])*\Delta_2[n]$
n
(g)(i) IDFT
$|X[k]| = |(X(\Omega)*W(\Omega))\Delta_2(\Omega)|$
k
(ii) DFT

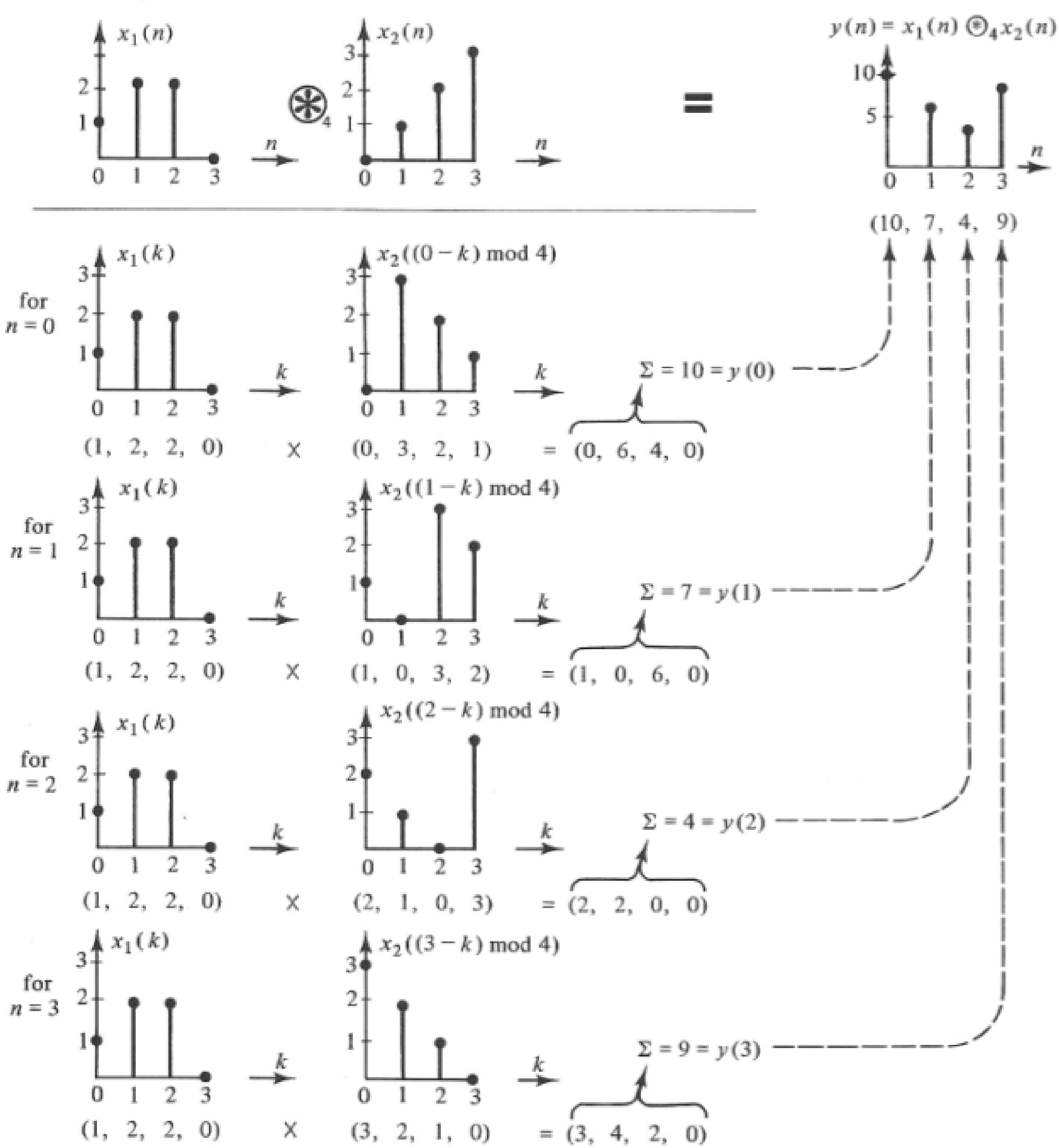
Circular convolution

The N-point circular convolution of two signals $x_1[n]$ and $x_2[n]$ denoted by $x_1[n] \otimes_N x_2[n]$ is defined by the following:

$$
\begin{aligned}
x_1[n] \otimes_N x_2[n] &= \sum_{k=0}^{N-1} \left(x_1 \left[n-k \right] \bmod N \right) \left(x_2 \left[k \right] \right) \\
&= \sum_{k=0}^{N-1} \left(x_1 \left[k \right] \right) \left(x_2 \left[n-k \right] \bmod N \right)
\end{aligned}
\tag{1}
$$

where $\left(x_1 \left[n-k \right] \bmod N \right)$ is the reflected and circularly translated version of $x_1[n]$.

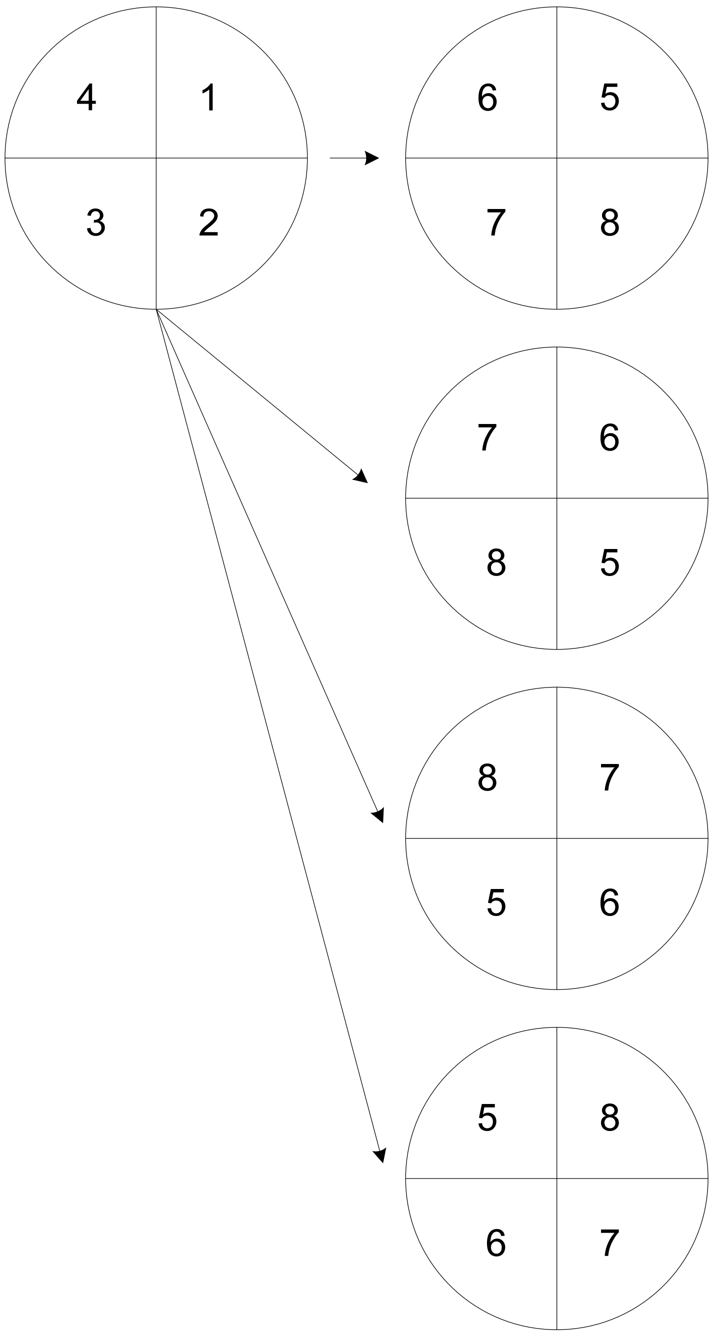
Ex 1)



$x_1(n)$
$x_2(n)$
$y(n) = x_1(n) \circledast_4 x_2(n)$
(10, 7, 4, 9)
for $n = 0$
$x_1(k)$
$x_2((0-k) \bmod 4)$
$\Sigma = 10 = y(0)$
(1, 2, 2, 0) × (0, 3, 2, 1) = (0, 6, 4, 0)
for $n = 1$
$x_1(k)$
$x_2((1-k) \bmod 4)$
$\Sigma = 7 = y(1)$
(1, 2, 2, 0) × (1, 0, 3, 2) = (1, 0, 6, 0)
for $n = 2$
$x_1(k)$
$x_2((2-k) \bmod 4)$
$\Sigma = 4 = y(2)$
(1, 2, 2, 0) × (2, 1, 0, 3) = (2, 2, 0, 0)
for $n = 3$
$x_1(k)$
$x_2((3-k) \bmod 4)$
$\Sigma = 9 = y(3)$
(1, 2, 2, 0) × (3, 2, 1, 0) = (3, 4, 2, 0)

Ex 2). Show the circular convolution of the two signals $x_1[n]=[1\ 2\ 3\ 4]$ and $x_2[n]=[5\ 6\ 7\ 8]$.

$x_1[n]=[1\ 2\ 3\ 4]$



$1\cdot 5+2\cdot 8+3\cdot 7+4\cdot 6=66$

$1\cdot 6+2\cdot 5+3\cdot 8+4\cdot 7=68$

$1\cdot 7+2\cdot 6+3\cdot 5+4\cdot 8=66$

$1\cdot 8+2\cdot 7+3\cdot 6+4\cdot 5=60$

$$y(n)=x_1(n)\otimes_4 x_2(n)$$
$$=[68\ \ 66\ \ 66\ \ 60]$$

The Fast Fourier Transform (FFT)

- The FFT is an algorithm or a procedure with which the **discrete Fourier transform** can be computed using far fewer calculations.
    - Decimation in time algorithm
    - Decimation in frequency algorithm

- DFT requires N^2 complex multiply and $(N-1)N$ add operations
- FFT needs only approximately $\frac{N}{2}\log_2 N$ operations.

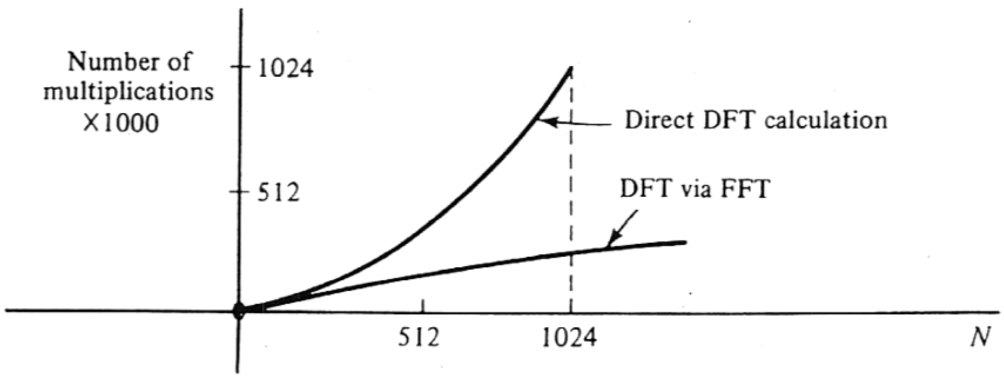


Figure 9-11 Computation for direct DFT and DFT via FFT.

Number of stages, ν	Number of points, N	Number of complex multiplications using direct calculation, N^2	Number of complex multiplications using Cooley–Tukey FFT algorithm, $(N/2)\log_2 N$	Times faster than direct evaluation, $R = N^2/((N/2)\log_2 N)$
2	4	16	4	4
3	8	64	12	5.333
4	16	256	32	8
5	32	1,024	80	12.8
6	64	4,096	192	21.33
7	128	16,384	448	36.57
8	256	65,536	1024	64
9	512	262,144	2304	113.77
10	1024	1,048,576	5120	204.8

Radix 2 decimation in time FFT

- In the following presentation, the number of points is assumed as a power of 2, that is $N = 2^v$.

- The decimation-in-time approach is one of breaking the (N)-point transform into two $(N/2)$ point transforms, then breaking each $(N/2)$ point transform into two $(N/4)$ point transforms, and continuing this process until two-point transforms are obtained.

- Given a sequence

$$\left[x(0)\, x(1)\, x(2) \cdots x\left(\frac{N}{2}-1\right) \cdots\cdots x(N-1) \right] \tag{2}$$

Even indexed sequence is

$$\left[x(0)\, x(2)\, x(4) \cdots x(N-2) \right] \tag{3}$$

Odd indexed sequence is

$$\left[x(1)\, x(3)\, x(5) \cdots x(N-1) \right] \tag{4}$$

Breaking the sum into two parts, one for the even and one for the odd indexed values, gives

$$X(k) = \sum_{\underbrace{n=0}_{n \text{ even}}}^{N-2} x(n) e^{\frac{-j2\pi kn}{N}} + \sum_{\underbrace{n=1}_{n \text{ odd}}}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}} \tag{5}$$

Changing the even and odd terms, we can represent as follows

$$\begin{aligned} X(k) &= \sum_{\underbrace{n=0}_{n \text{ even}}}^{N-2} x(n) e^{\frac{-j2\pi kn}{N}} + \sum_{\underbrace{n=1}_{n \text{ odd}}}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}} \\ &= \sum_{n=0}^{\frac{N}{2}-1} x(2n) e^{\frac{-j2\pi k(2n)}{N}} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) e^{\frac{-j2\pi k(2n+1)}{N}} \end{aligned} \tag{6}$$

Letting

- Even samples in the first group: $y(n) = x(2n)$
- Odd samples in the second group $z(n) = x(2n+1)$

Rewriting the equation again, we can see

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{k}{N} n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) e^{-j2\pi \frac{k}{N}(2n)} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) e^{-j2\pi \frac{k}{N}(2n+1)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} y(n) e^{-j4\pi \frac{k}{N} n} + \sum_{n=0}^{\frac{N}{2}-1} z(n) e^{-j4\pi \frac{k}{N} n} \left(e^{-j2\pi \frac{k}{N}} \right)$$

$$= \sum_{n=0}^{\frac{N}{2}-1} y(n) e^{-j4\pi \frac{k}{N} n} + \left(e^{-j2\pi \frac{k}{N}} \right) \sum_{n=0}^{\frac{N}{2}-1} z(n) e^{-j4\pi \frac{k}{N} n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} y(n) e^{-j2\pi \frac{k}{N/2} n} + \left(e^{-j2\pi \frac{k}{N}} \right) \sum_{n=0}^{\frac{N}{2}-1} z(n) e^{-j2\pi \frac{k}{N/2} n} \qquad (7)$$

The first half of the calculation is for $k = 0, \ldots, \left(\frac{N}{2} - 1 \right)$

$$X[k] = \underbrace{Y[k]}_{\text{DFT of even samples}} + \underbrace{e^{-j2\pi \frac{k}{N}}}_{\text{FFT parlance}} \underbrace{Z[k]}_{\text{DFT of odd samples}} \qquad (8)$$

The second half of the calculation is for $k = 0, \ldots, \left(\frac{N}{2} - 1 \right)$

$$X\left[\frac{N}{2} + k\right] = Y\left[\frac{N}{2} + k\right] + e^{-j2\pi \frac{\left(\frac{N}{2}+k\right)}{N}} Z\left[\frac{N}{2} + k\right] \qquad (9)$$

Since $Y[k]$ is the DFT of an $\frac{N}{2}$ point signal, it is periodic with period $\frac{N}{2}$,

so is $Y[k] = Y\left[k + \frac{N}{2}\right]$. Similarly, $Z[k] = Z\left[k + \frac{N}{2}\right]$ is and

$$
\begin{aligned}
e^{-j2\pi\frac{\left(\frac{N}{2}+k\right)}{N}} &= e^{-j\cancel{2}\pi\frac{\frac{\cancel{N}}{\cancel{2}}}{\cancel{N}}} e^{-j2\pi\frac{k}{N}} \\
&= \underbrace{\left(e^{-j\pi}\right)}_{=-1} e^{-j2\pi\frac{k}{N}} \\
&= -e^{-j2\pi\frac{k}{N}}
\end{aligned}
\tag{10}
$$

So,

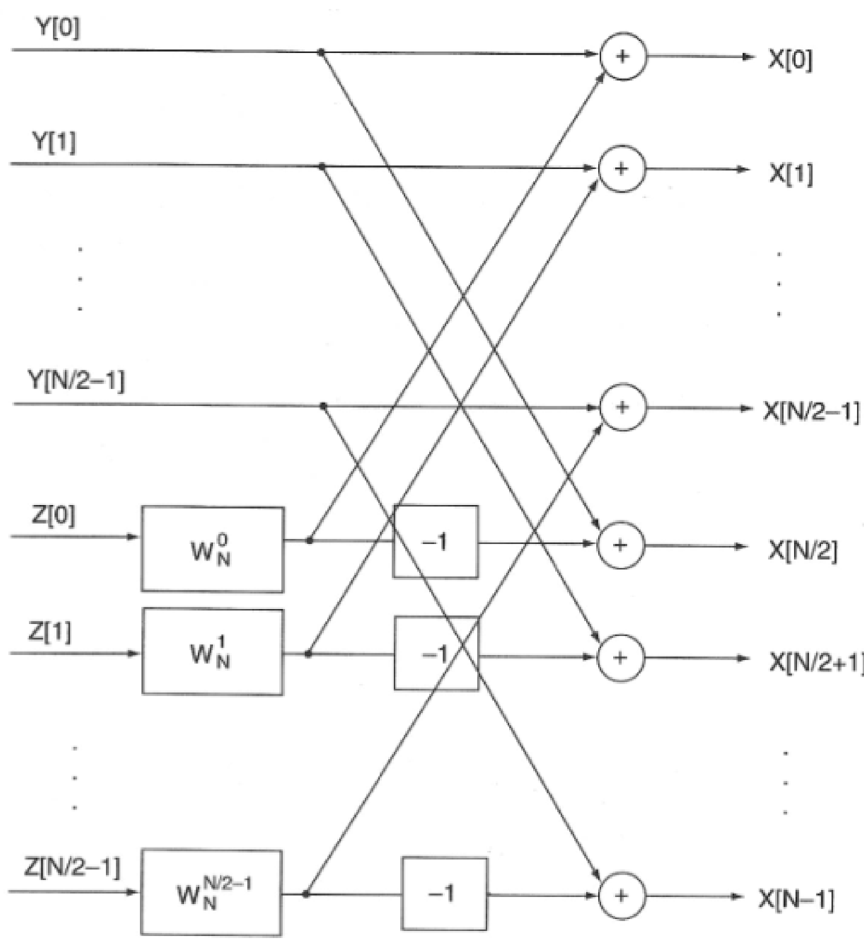
$$
X\left[\frac{N}{2}+k\right] = Y\left[k\right] - e^{-j2\pi\frac{k}{N}} Z\left[k\right] \tag{11}
$$

Rewriting eq. (9) and (11), we can see the Fig 11.39

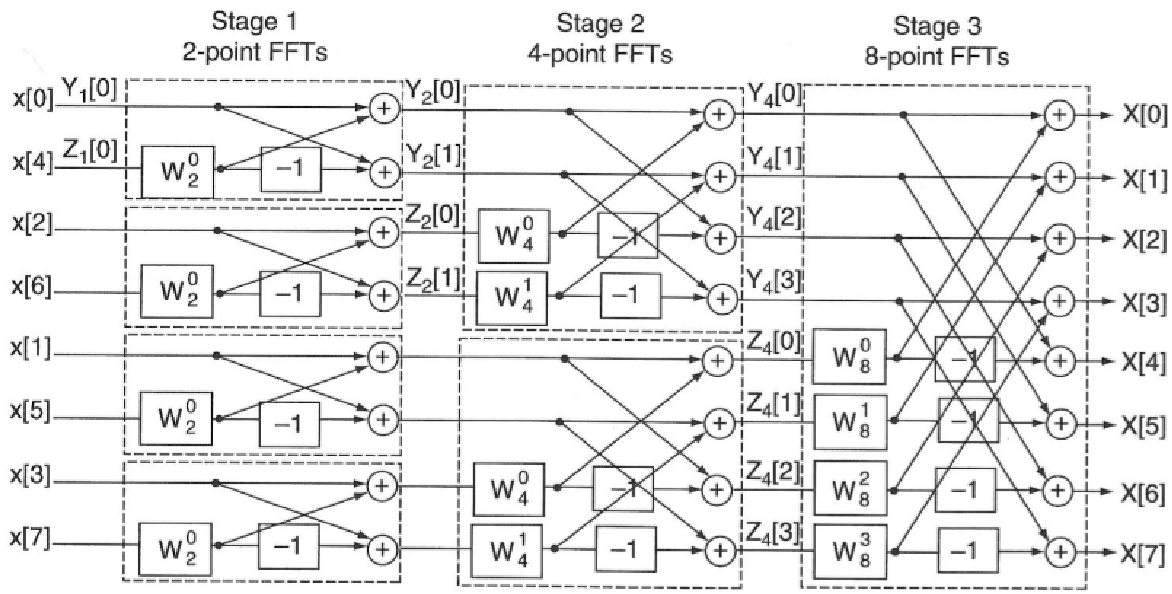
$$X[k]=Y[k]+e^{-j2\pi\frac{k}{N}}Z[k] \tag{12}$$

$$X\left[\frac{N}{2}+k\right]=Y\left[k\right]-e^{-j2\pi\frac{k}{N}}Z\left[k\right] \tag{13}$$

FIGURE 11.39
One stage of FFT.*



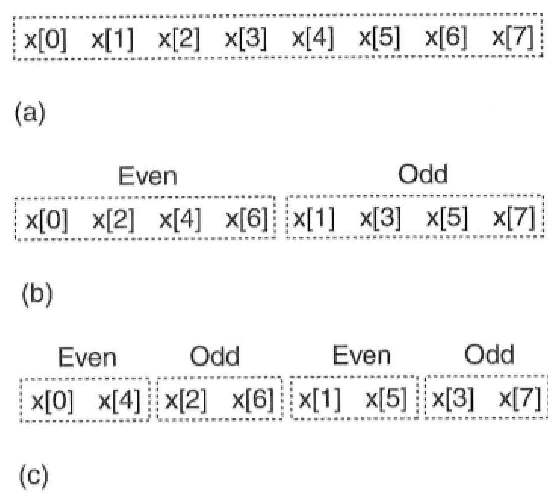
$^* W_N^k = e^{-j\frac{2\pi k}{N}}$



$* W_N^k = e^{-j\frac{2\pi k}{N}}$

FIGURE 11.40
All three stages of 8-point FFT.*

FIGURE 11.41
Regrouping into even and odd sequences.



At this point we need to compare the total # of calculation between DFT and FFT.

$$X(k)=\sum_{n=0}^{N-1}x(n)e^{\frac{-j2\pi nk}{N}} \tag{14}$$

As we can see the DFT equation, the total number of complex multiplication calculation is N^2. However the total number of complex multiplications required to evaluate the N point transform with this first decimation becomes

$$\eta_1=\left(\frac{N}{2}\right)^2+\left(\frac{N}{2}\right)^2+N \tag{15}$$

- The first term in the sum is the number of complex multiplication of for the direct calculation of the $\frac{N}{2}$ point DFT of the even indexed sequence.
- The second term is the number of complex multiplications for the direct calculation of the $\frac{N}{2}$ point DFT of the odd indexed sequence.
- The third term is the number of complex multiplications required for the combining algebra.

Each of the $\frac{N}{2}$ point sequences can be decimated further into two sequences of length $\frac{N}{4}$.

The number of complex multiplications after the second decimation is

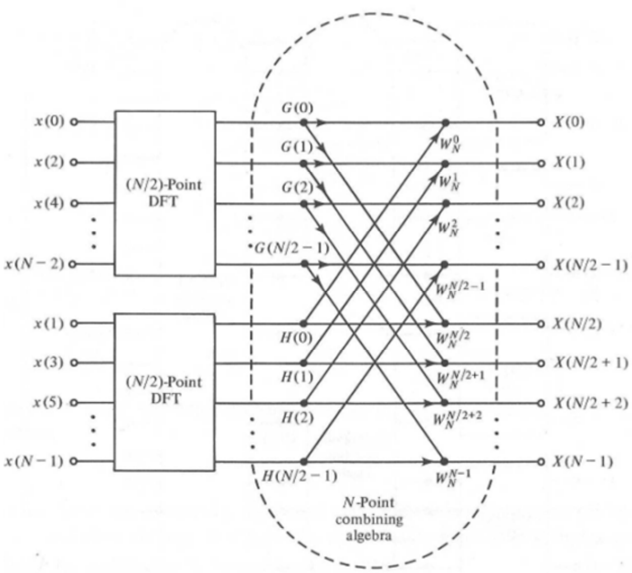
$$\begin{aligned}\eta_2&=4\left(\frac{N}{4}\right)^2+2\left(\frac{N}{2}\right)+N\\&=\frac{N^2}{4}+2N\end{aligned} \tag{16}$$

- It has been conventional to count the W^0 as complex multiplications, even though there is no multiplications.
- The approximate number of complex multiplications for the total decomposition becomes

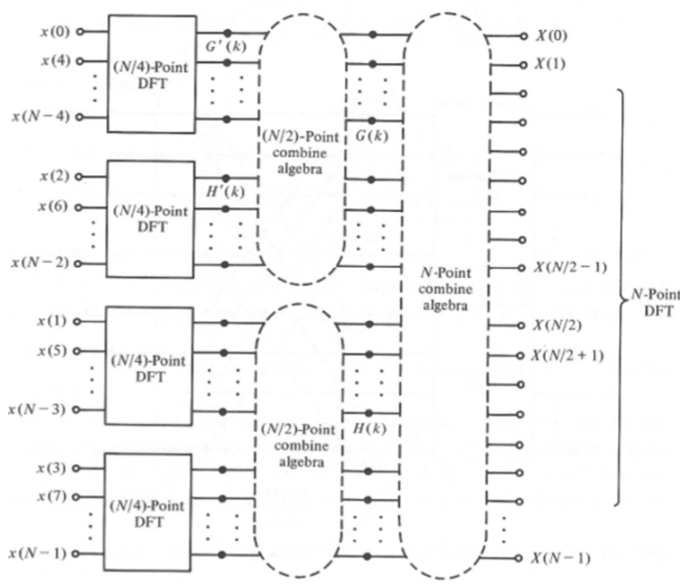
$$\begin{aligned}\eta &= \text{Number of complex multiplications} \\ &= \frac{N}{2}\log_2 N\end{aligned} \tag{17}$$

- Number of complex additions required for calculating the DIT FFT is

$$\begin{aligned}\eta &= \text{Number of complex additions} \\ &= N\ \log_2 N\end{aligned} \tag{18}$$



First stage of the decomposition for the decimation in time FFT

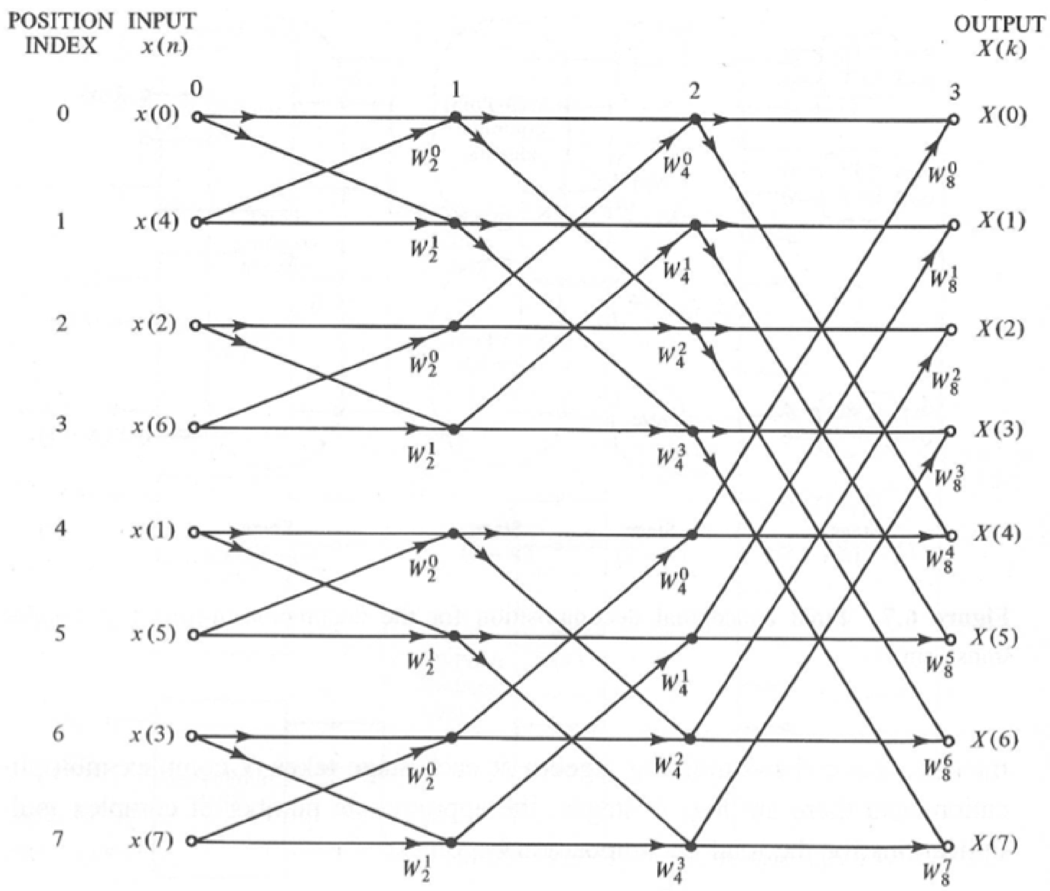


Second stage of the decomposition for the decimation in time FFT

The input data appear in what is called "bit reversed" order, for an example for

$N = 8$ case

Position		Binary equivalent		Bit Reversed		Sequence index
6	$\rightarrow$	110	$\rightarrow$	011	$\rightarrow$	3
2	$\rightarrow$	010	$\rightarrow$	010	$\rightarrow$	2



The flow graph for an eight-point decimation in time FFT