EE 210: Quiz 02 Solution

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1 Just the answers

Time interval	$y(t)$
$4 \le t < 5$	$\int_{2}^{t-2}(2)(1)\mathrm{d}\tau + \int_{t-2}^{t-1}(-t+\tau+4)(1)\ \mathrm{d}\tau$
$5 \le t < 6$	$\int_{2}^{t-2}(2)(1)\mathrm{d}\tau + \int_{t-2}^{4}(-t+\tau+4)(1)\ \mathrm{d}\tau$
$6 \le t < 8$	$\int_{t-4}^{4}(2)(1)\mathrm{d}\tau$

Table 1: The convolution result $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\ \mathrm{d}\tau$

2 Full solutions

In this section, I solve the given problem completely – once with $x(t)$ flipped and shifted, then with $h(t)$. These are not part of the quiz, but you may wish to use these as examples when studying for the exams.

2.1 Convolution review

The *convolution* operation is defined in Eqs. (1) and (2).

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\ \mathrm{d}\tau \tag{1}$$

$$= h(t) * t(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\ \mathrm{d}\tau \tag{2}$$

The two equations are equivalent because convolution is commutative. In either case, both x and h are functions of a single dummy variable, τ, with one function flipped and shifted. To see this, rewrite $h(t-\tau)$ as $h(-(\tau-t))$. Although t is the original independent variable of the two functions, it is simply a constant in these equations that determines how far to the left or right $x(t-\tau)$ or $h(t-\tau)$ are shifted, as shown in Figs. 1 and 3. When the shifted function overlaps the other function, Eqs (1) and (2) indicate that you integrate the product of the two functions over the region of overlap. Because the convolution involves a definite integral with bounds that are either constant or functions of t, τ does not appear in the result (hence the term *dummy variable*); the convolution result is only a function of t.

2.2 Flipping $x(t)$

In the quiz, you were required to use Eq. (2), though it is generally preferable to flip the simpler of the two functions. First, let's write $x(t) = x(\tau)$ as a piecewise-defined function:

$$x(\tau) = \begin{cases} -\tau + 4 & 1 \leq \tau \leq 2 \\ 2 & 2 \leq \tau \leq 4 \\ 0 & \text{elsewhere} \end{cases} \tag{3}$$

From here, we obtain $x(t-\tau)$. To start, Fig. 1 shows a visual process by which $x(t)$ is transformed into $x(t-\tau)$.

1. First, let $x(t) = x(\tau)$.
2. Then, flip $x(\tau)$ to form $x(-\tau)$, which can be viewed as $x(t-\tau)$ with $t = 0$.
3. Finally, substitute t back into $x(-\tau)$ to get $x(t-\tau)$.

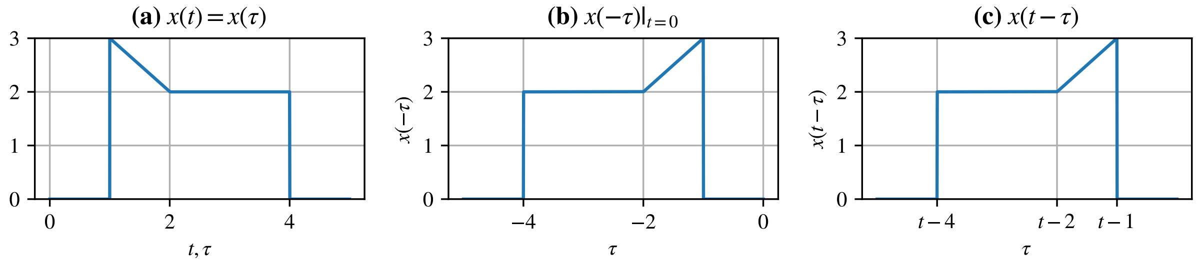

Figure 1: $x(t)$ is flipped and shifted.

Notice that I labelled all the discontinuities of $x(t-\tau)$; this is very important, as discontinuities in either $x(t-\tau)$ or $h(\tau)$ separate the regions of integration. Also note that when working with τ, t is just a constant that sets the position of $x(t-\tau)$. Note that while we can sketch $x(t-\tau)$, we don't have a piecewise definition of it like Eq. (2.2), so we don't yet know what to substitute into Eq. (2). This is not a problem if the flipped/shifted function is a rectangular function like $h(t)$ [Fig. 3] that is constant, but $x(t-\tau)$ is not constant between $t-2$ and $t-1$.

To find $x(t-\tau)$ explicitly, substitute $\tau = t - \tau$ into $x(\tau)$. Solve the inequality for each interval for τ.

$$\begin{aligned} x(t-\tau) &= \begin{cases} -(t-\tau) + 4 & 1 \leq t-\tau \leq 2 \\ 2 & 2 \leq t-\tau \leq 4 \\ 0 & \text{elsewhere} \end{cases} \\ &= \begin{cases} 2 & t-4 \leq \tau \leq t-2 \\ -t+\tau+4 & t-2 \leq \tau \leq t-1 \\ 0 & \text{elsewhere} \end{cases} \end{aligned} \tag{4}$$

This agrees with Fig. 1.

Fig. 2 shows the four regions of integration over $3 \leq t \leq 8$, with the overlapping area shaded in blue. Table 2 shows the result of convolution in each time interval. $y(t)$ is plotted in Fig. 5 after repeating the problem with $h(t)$ flipped.

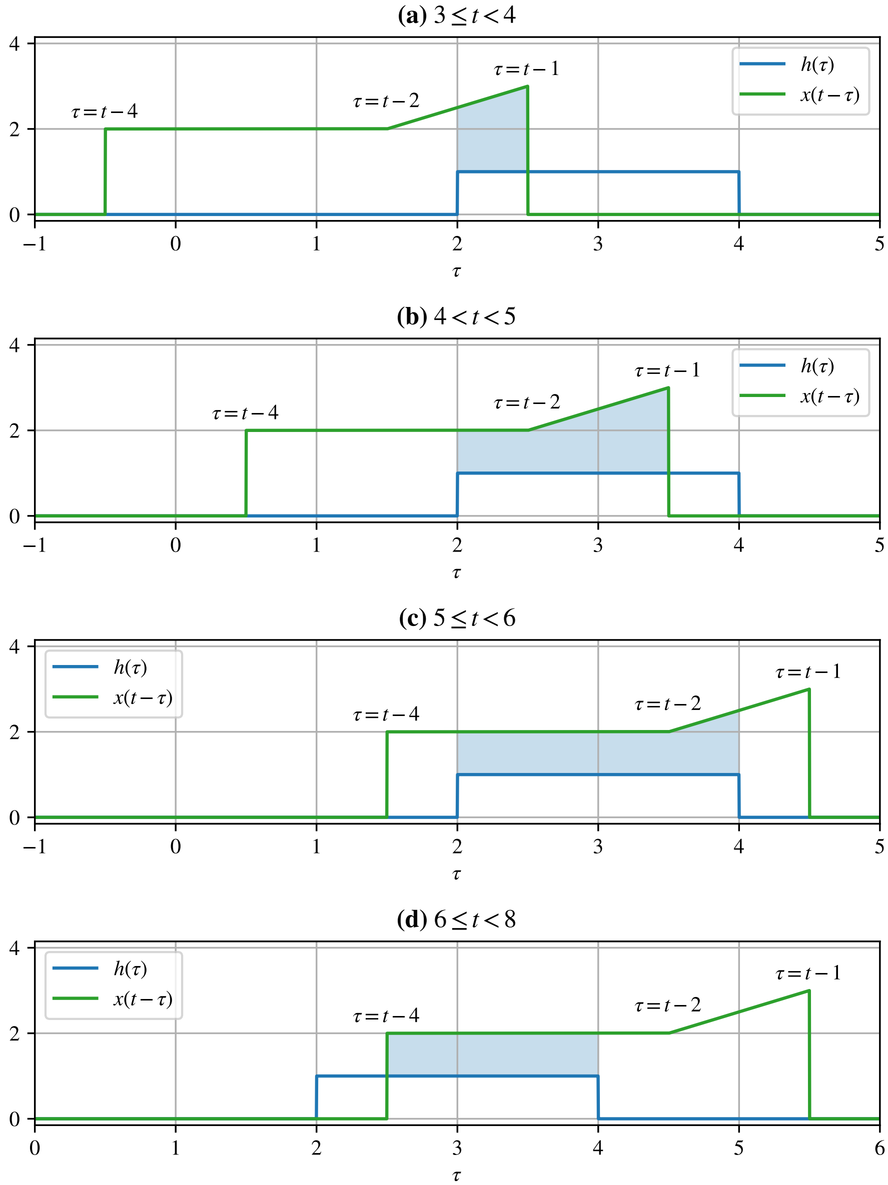

Figure 2: Regions of integration for Eq. (2).

Time interval	$\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau$	$y(t)$
$3 \le t < 4$	$\int_{2}^{t-1}(1)(-t+\tau+4)\,\mathrm{d}\tau$	$-\frac{1}{2}t^2+6t-13.5$
$4 \le t < 5$	$\int_{2}^{t-2}(1)(2)\,\mathrm{d}\tau+\int_{t-2}^{t-1}(1)(-t+\tau+4)\,\mathrm{d}\tau$	$2t-5.5$
$5 \le t < 6$	$\int_{2}^{t-2}(1)(2)\,\mathrm{d}\tau+\int_{t-2}^{4}(1)(-t+\tau+4)\,\mathrm{d}\tau$	$\frac{1}{2}t^2-6t+22$
$6 \le t < 8$	$\int_{t-4}^{4}(1)(2)\,\mathrm{d}\tau$	$-2t+16$
elsewhere	0	0

Table 2: The integrals for each time interval shown in Fig. 2.

2.3 Flipping $h(t)$

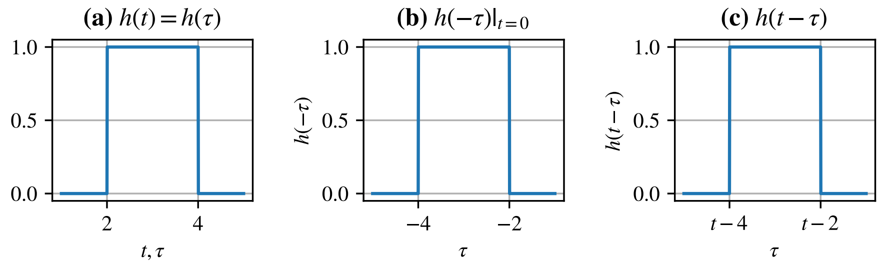

Figure 3: $h(t)$ is flipped and shifted.

Now we apply Eq. 1, choosing $h(t)$ to flip and shift as shown in Fig. 3. Applying the same procedure as in the preceding section, we obtain the regions of integration in Fig. 3. $y(t)$ is plotted in Fig. 5.

Time interval	$\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau$	$y(t)$
$3 \le t < 4$	$\int_{1}^{t-2}(1)(-\tau+4)\,\mathrm{d}\tau$	$-\frac{1}{2}t^2 + 6t - 13.5$
$4 \le t < 5$	$\int_{1}^{2}(1)(-\tau+4)\,\mathrm{d}\tau + \int_{2}^{t-2}(1)(2)\,\mathrm{d}\tau$	$2t - 5.5$
$5 \le t < 6$	$\int_{t-4}^{2}(1)(-\tau+4)\,\mathrm{d}\tau + \int_{2}^{t-2}(1)(2)\,\mathrm{d}\tau$	$\frac{1}{2}t^2 - 6t + 22$
$6 \le t < 8$	$\int_{t-4}^{4}(1)(2)\,\mathrm{d}\tau$	$-2t + 16$
elsewhere	0	0

Table 3: The integrals for each time interval shown in Fig. 4.

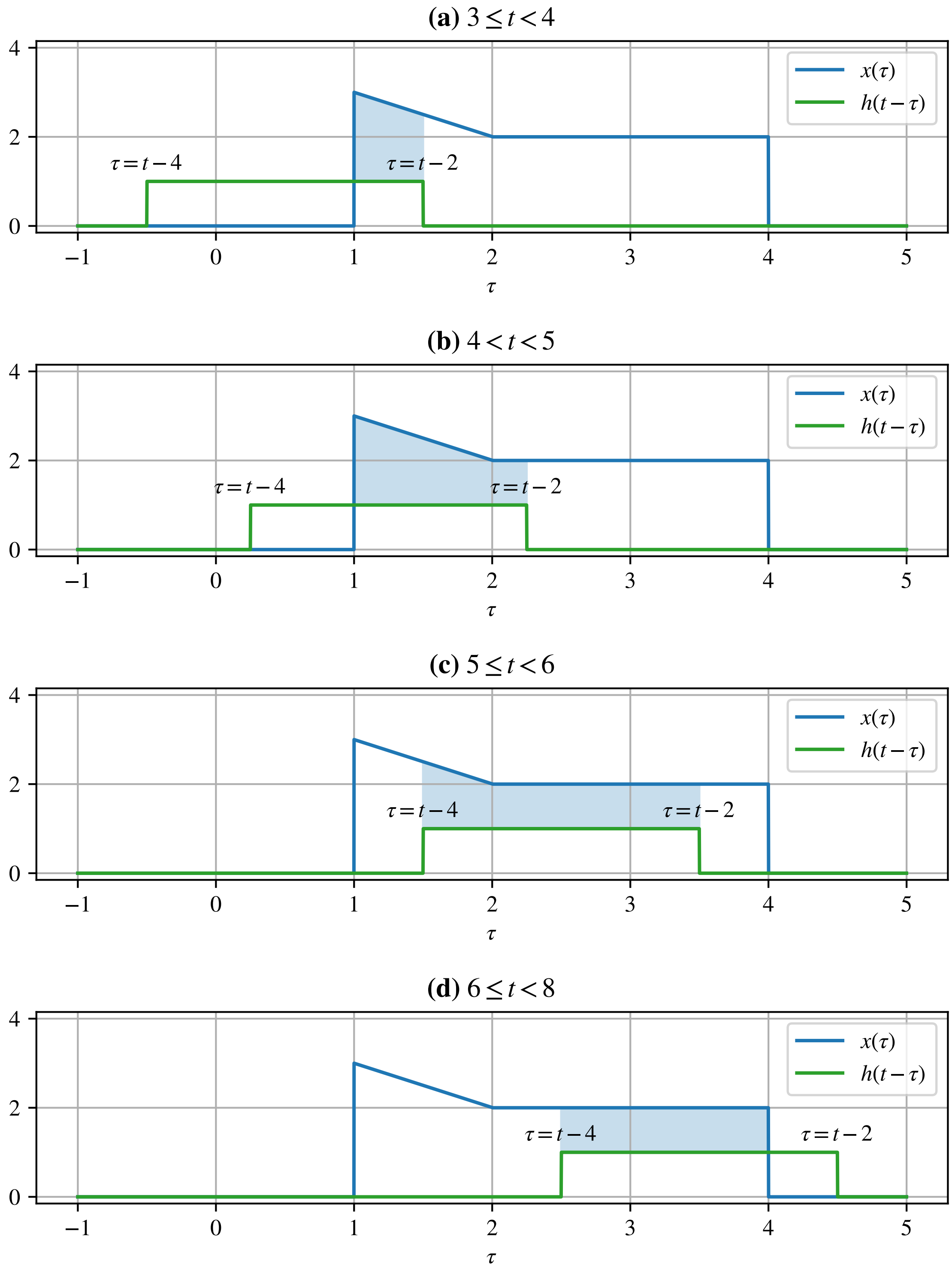

Figure 4: Regions of integration for Eq. (1).

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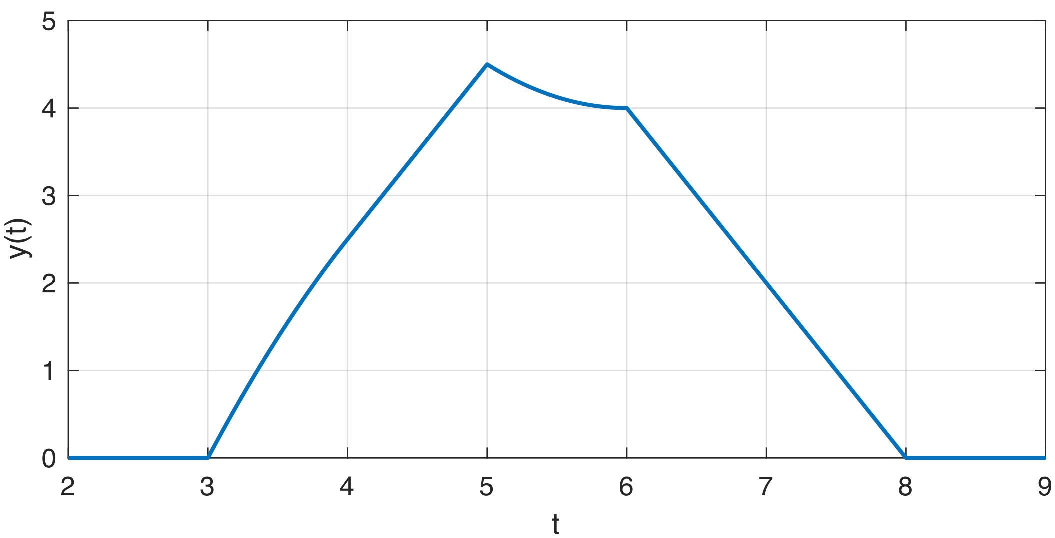

Figure 5: The final convolution result, $y(t)$.