Fall 2020

EE 210

HW#: 09

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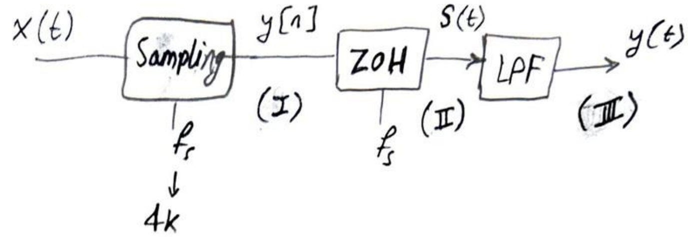
Assigned question #s: **5**

HW09

(2.4)

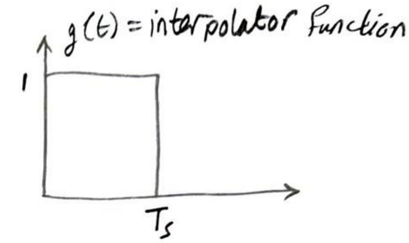
(I) $Y(f)_{\Omega} = \frac{1}{T_s} \sum_{K=-\infty}^{\infty} X(f + k f_s)$

(II) $S(f) = G(f) . Y(\Omega)$



a) $g(t) = \text{rect}\left(\frac{t}{T_s}\right) * \delta(t - T_s/2)$

$\therefore \boxed{G(f) = T_s \, \text{Sinc}(T_s f) . \, e^{-j\pi f T_s}}$



$$y[n] = 2 \cos(0.3\pi n + \pi/4)$$
$$= 2 \cos(2\pi(600) n T_s + \pi/4)$$
$$= 2 \cos\left(2\pi(600)\left(nT_s + \frac{1}{4800}\right)\right)$$

$\boxed{\Omega = 2\pi f T_s}$

$0.3\pi = 2\pi f \, T_s \rightsquigarrow \therefore f = 600$

$\rightsquigarrow x(t) = 2 \cos\left(2\pi(600)\left(t + \frac{1}{4800}\right)\right)$

$\therefore X(f) = 2\left[\frac{1}{2}\left(\delta(f-600) + \delta(f+600)\right) e^{j2\pi \frac{1}{4800} f}\right]$

$$\boxed{\therefore Y(f) = f_s \sum_{K=-\infty}^{\infty} \left(\delta(f - 600 + kf_s) + \delta(f + 600 + kf_s)\right) e^{j2\pi \frac{1}{4800}(f + kf_s)}}$$

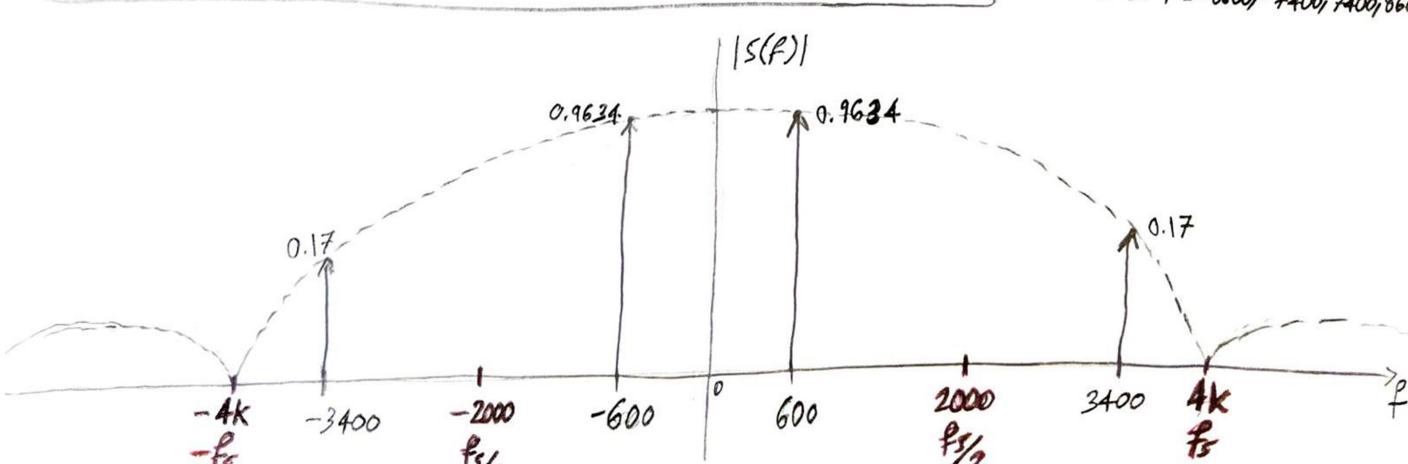
$$\boxed{\begin{aligned} \therefore S(f) &= G(f) . Y(f) \\ &= \text{Sinc}(f/f_s) \, e^{-j\pi f/f_s} . \sum_{K=-\infty}^{\infty} \left(\delta(f - 600 \pm kf_s) + \delta(f + 600 \pm kf_s)\right) e^{j\frac{\pi(f \pm kf_s)}{2400}} \end{aligned}}$$

K=0: $f = -600, 600$

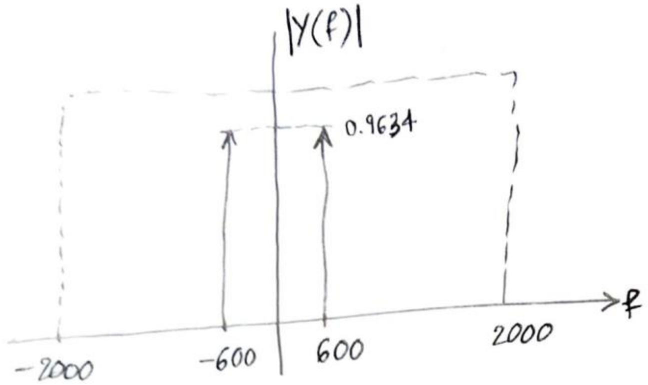
K=1: $f = -4600, -3400, 3400, 4600$

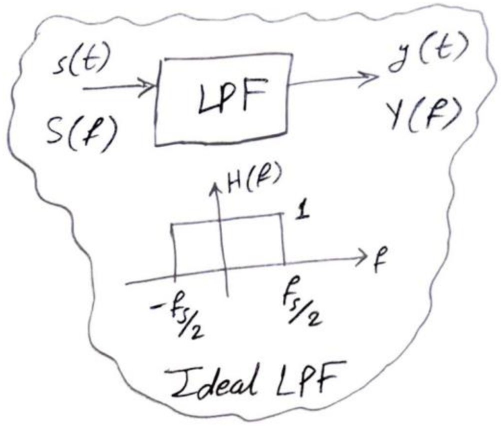
K=2: $f = -8600, -7400, 7400, 8600$

$$\boxed{\therefore |S(f)| = |\text{Sinc}(f/f_s)| . \left| \sum_{K=-\infty}^{\infty} \left(\delta(f - 600 + kf_s) + \delta(f + 600 + kf_s)\right)\right|}$$



b)





$$Y(f) = \mathrm{Sinc}\left(\frac{600}{4000}\right) \cdot \left[\delta(f-600) + \delta(f+600)\right] \cdot e^{-j\pi \frac{f}{4000}} \cdot e^{j\pi\frac{f}{2400}} \qquad (k=0)$$

$$= 0.9634\left(\delta(f-600) + \delta(f+600)\right) e^{j\pi\frac{f}{6000}}$$

$$= 1.9268\left(\tfrac{1}{2}\left(\delta(f-600) + \delta(f+600)\right)\right) e^{j2\pi\frac{f}{12000}}$$

$$\boxed{\begin{aligned} \therefore y(t) &= 1.9268 \cos\left(2\pi(600)\left(t + \frac{1}{12000}\right)\right) \\ &= 1.9268 \cos\left(1200\pi t + \frac{\pi}{10}\right) \end{aligned}}$$

2.5 $x(t) = 2\cos(2\pi(250)t) - 3\sin(2\pi(500)t) + \cos(2\pi(750)t)$ $\boxed{f_s = 2k}$

$$X(f) = (\delta(f-250)+\delta(f+250)) + \frac{3}{2}j(\delta(f-500)-\delta(f+500)) + \frac{1}{2}(\delta(f-750)+\delta(f+750))$$

a) After Sampling

$$x_s[n] = 2\cos(2\pi(250)nT_s) - 3\sin(2\pi(500)nT_s) + \cos(2\pi(750)nT_s)$$

$$X_s(f) = \frac{1}{T_s}\sum_{K=-\infty}^{\infty} X(f+kf_s)$$

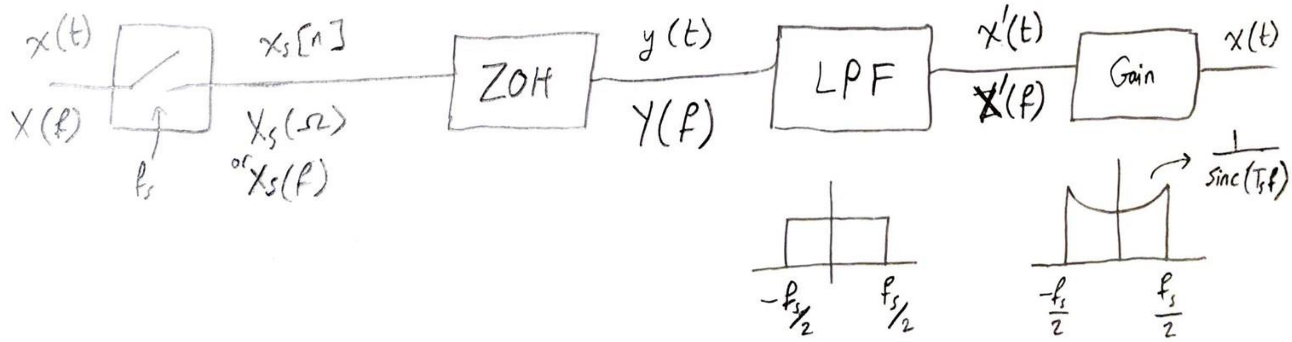
After ZOH (reconstruction)

$$Y(f) = G(f)\cdot X_s(\Omega) \qquad \text{where } \Omega = \frac{2\pi f}{f_s}$$

$$\boxed{Y(f) = \left(T_s \operatorname{Sinc}(T_s f)\cdot e^{-j\pi f T_s}\right)\cdot\left(\frac{1}{T_s}\sum_{K=-\infty}^{\infty} X(f+kf_s)\right)}$$

$$\boxed{\begin{aligned} y(t) &= \sum_{n=-\infty}^{\infty} g(t-nT_s)\, x_s[n] \\ &= \sum_{n=-\infty}^{\infty} \operatorname{rect}\left(\frac{t-nT_s-T_s/2}{T_s}\right)\cdot\left[2\cos(500\pi nT_s) - 3\sin(1000\pi nT_s) + \cos(1500\pi nT_s)\right]\end{aligned}}$$

b) To recontruct the signal $x(t)$ from its samples, we need to pass $y(t)$ through a LPF & multiply the spectrum by a gain to normalize the "$\operatorname{Sinc}(T_s f)$" factor.



↳ After LPF

$\because$ There's no aliasing ($f_s > 2f_{max}$)

$\therefore$ We can take the $k=0$ terms in $Y(f)$

$$X'(f) = \text{Sinc}(T_s f)\, e^{-j\pi f T_s} \cdot X(f)$$

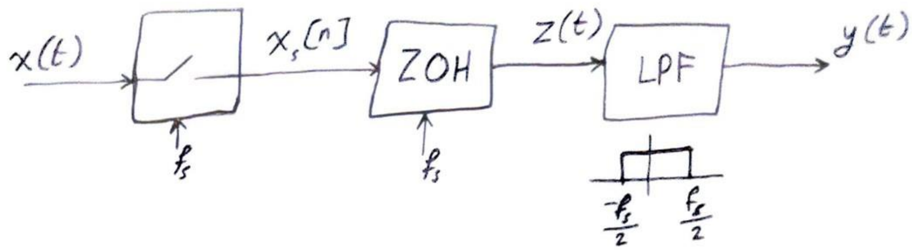
$$\boxed{X'(f) = e^{-j\pi f T_s}\,\text{Sinc}(T_s f) \cdot \left[\delta(f-250) + \delta(f+250) + \tfrac{3}{2}j\,\delta(f-500) - \tfrac{3}{2}j\,\delta(f+500) + \tfrac{1}{2}\delta(f-750) + \tfrac{1}{2}\delta(f+750) \right]}$$

$$x'(t) = \frac{1}{T_s}\,\text{rect}\left(\frac{t - T_s/2}{T_s}\right) * \left(2\cos(500\pi t) - 3\sin(1000\pi t) + \cos(1500\pi t)\right)$$

$$\boxed{x'(t) = 2\,\text{Sinc}\left(\tfrac{250}{2000}\right)\cdot\left(\cos\left(500\pi\left(t - \tfrac{1}{4000}\right)\right)\right) - 3\,\text{sinc}\left(\tfrac{500}{2000}\right)\cdot\left(\cos\left(1000\pi\left(t-\tfrac{1}{4000}\right)\right)\right) + \text{Sinc}\left(\tfrac{750}{2000}\right)\cdot\left(\cos\left(1500\pi\left(t - \tfrac{1}{4000}\right)\right)\right)}$$

To reconstruct the exact $x(t)$,
$X'(f)$ must be multiplied by $e^{j\pi f T_s}\dfrac{1}{\text{Sinc}(T_s f)}$ after LPF

2.6



$f_s = 5000$ Hz

a) $x(t) = e^{j2000\pi t} = e^{j2\pi(1000)t}$

$X(f) = \delta(f-1000)$

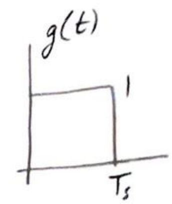
After Sampling:

$x_s[n] = e^{j2\pi(1000)nT_s} = e^{j2\pi(\frac{1}{5})n}$

$X_s(\Omega) = 2\pi \sum_{K=-\infty}^{\infty} \delta(\Omega + 2\pi K - 2\pi(\frac{1}{5}))$

OR

$X_s(f) = f_s \sum_{K=-\infty}^{\infty} \delta(f-1000+kf_s)$



After ZOH:

$z(t) = \sum_{n=-\infty}^{\infty} g(t-nT_s)\, x_s[n]$

$= \sum_{n=-\infty}^{\infty} rect\left(\frac{t-nT_s-T_s/2}{T_s}\right) . e^{j\frac{2}{5}\pi n}$

$Z(f) = G(f) . X_s(\Omega)$

$= \cancel{T_s}\, Sinc(T_s f)\, e^{-j\pi f T_s} . \cancel{f_s} \sum_{K=-\infty}^{\infty} \delta(f-1000+kf_s)$

$= Sinc(T_s f)\, e^{-j\pi f T_s} \sum_{K=-\infty}^{\infty} \delta(f-1000+kf_s)$

After LPF: (no aliasing because $f_s > 2f_{max}$) take $K=0$

$Y(f) = Sinc(T_s(1000)) . e^{-j\pi(1000)T_s} . \delta(f-1000)$

$\therefore \boxed{y(t) = Sinc(\frac{1}{5})\, e^{-j\pi\frac{1}{5}}}$

b) $x(t) = \cos(2000\pi t + 0.15\pi)$

$= \cos\left(2000\pi\left(t + \frac{0.15}{2000}\right)\right)$

$X(f) = \frac{1}{2}\left[\delta(f-1000) + \delta(f+1000)\right] \cdot e^{j2\pi\left(\frac{0.15}{2000}\right)f}$

After Sampling

$x_s[n] = \cos(2000\pi n T_s + 0.15\pi)$

$X_s(f) = f_s \sum_{k=-\infty}^{\infty} X(f + kf_s)$

After ZOH

$z(t) = \sum_{n=-\infty}^{\infty} g(t-nT_s) \cdot x_s[n] = \sum_{n=-\infty}^{\infty} rect\left(\frac{t-nT_s-T_s/2}{T_s}\right) \cdot \cos(2000\pi n T_s + 0.15\pi)$

$Z(f) = G(f) \cdot X_s(f)$

$= \cancel{T_s}\, \text{Sinc}(T_s f)\, e^{-j\pi f T_s} \cdot \frac{\cancel{f_s}}{2} \sum_{k=-\infty}^{\infty} \left[\delta(f+kf_s-1000) + \delta(f+kf_s+1000)\right] e^{j2\pi\left(\frac{0.15}{2000}\right)(f+kf_s)}$

After LPF (no aliasing, $f_s > 2f_{max}$) $\rightarrow$ take $k=0$

$Y(f) = \text{Sinc}(T_s f)\, e^{-j\pi f T_s} \cdot \left(\frac{1}{2}\right)\left(\delta(f-1000) + \delta(f+1000)\right) e^{j\pi\left(\frac{0.15}{1000}\right)f}$

$= \frac{1}{2}\,\text{Sinc}\left(\frac{1}{5}\right)\left(\delta(f-1000) + \delta(f+1000)\right) e^{-j2\pi f\left(\frac{1}{40000}\right)}$

$$\boxed{\begin{aligned} \therefore y(t) &= \text{Sinc}\left(\tfrac{1}{5}\right) \cos\left(2\pi(1000)\left(t - \tfrac{1}{40000}\right)\right) \\ &= \text{Sinc}\left(\tfrac{1}{5}\right) \cos\left(2000\pi t - \tfrac{\pi}{20}\right) \end{aligned}}$$

d) $x(t) = 2\sin(2\pi(2500)t)$

$X(f) = -j\left(\delta(f-2500) - \delta(f+2500)\right)$

After Sampling

$x_s[n] = 2\sin(2\pi(2500)\,n\,T_s) = 2\sin(\pi n)$

$X_s(f) = f_s \sum_{K=-\infty}^{\infty} X(f+kf_s)$

After ZOH

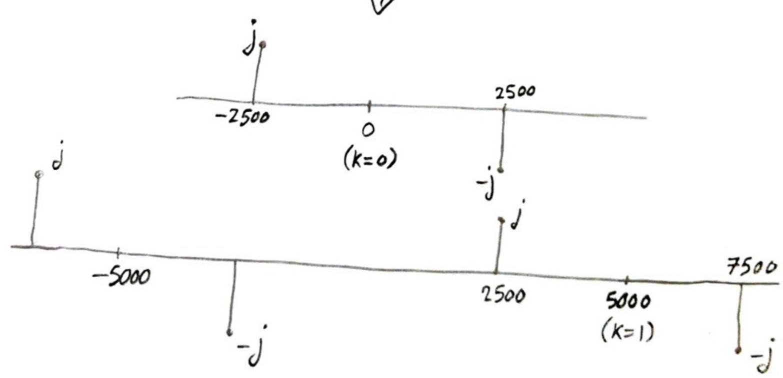
$Z(t) = \sum_{n=-\infty}^{\infty} g(t-nT_s)\,.\,x_s[n] = \sum_{n=-\infty}^{\infty} rect\left(\frac{t-nT_s-T_s/2}{T_s}\right).\left(2\sin(\pi n)\right)$

$Z(f) = G(f).\,X_s(f)$

$= T_s\,\text{Sinc}(T_s f)\,e^{-j\pi f T_s}\,.\,\cancel{f_s} \sum_{K=-\infty}^{\infty} \left(-j\,\delta(f-2500+kf_s) + j\,\delta(f+2500+kf_s)\right)$

After LPF

$\boxed{\begin{array}{l} Y(f) = 0 \\ y(t) = 0 \end{array}}$



Here, $f_{max} = f_s/2 = 2500\text{ Hz}$

So, after sampling, the spectrum cancel out

e) $x(t) = \cos(2\pi(1000)t + 0.1\pi) - \cos(2\pi(2750)t)$

$$= \cos\left(2\pi(1000)\left(t + \frac{0.1}{2000}\right)\right) - \cos(2\pi(2750)t)$$

$$X(f) = \frac{1}{2}\left(\delta(f-1000) + \delta(f+1000)\right) e^{j2\pi f\left(\frac{0.1}{2000}\right)} - \frac{1}{2}\left(\delta(f-2750) + \delta(f+2750)\right)$$

After sampling

$$x_s[n] = \cos(2000\pi n T_s + 0.1\pi) - \cos(5500\pi n T_s)$$

$$X_s(f) = f_s \sum_{K=-\infty}^{\infty} X(f + kf_s)$$

After ZOH

$$Z(t) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t - nT_s - T_s/2}{T_s}\right) \cdot \left(\cos(2000\pi n T_s + 0.1\pi) - \cos(5500\pi n T_s)\right)$$

$$Z(f) = G(f) \cdot X_s(f)$$

$$= T_s \,\text{Sinc}(T_s f)\, e^{-j\pi f T_s} \cdot f_s \sum_{K=-\infty}^{\infty} \left[\frac{1}{2}\left(\delta(f-1000+kf_s) + \delta(f+1000+kf_s)\right) e^{j2\pi\frac{0.1}{2000}(f+kf_s)} - \frac{1}{2}\left(\delta(f-2750+kf_s) + \delta(f+2750+kf_s)\right)\right]$$

$K=-1: \; f = -7750, -2250$

$K=0: \; f = -2750, 2750$

$K=1: \; f = 2250, 7750$

Here there is aliasing because $f_{max} > \frac{f_s}{2}$ ($f_s < 2f_{max}$)

So, we will have aliases at $f = -2250$ Hz & 2250 Hz.

After LPF

$$Y(f) = \frac{1}{2}\,\text{Sinc}\left(\frac{1}{5}\right) e^{-j\pi\frac{f}{5000}}\left(\delta(f-1000) + \delta(f+1000)\right) e^{j2\pi\frac{0.1}{2000}f}$$

$$- \frac{1}{2}\,\text{Sinc}\left(\frac{9}{20}\right) e^{-j\pi\frac{f}{5000}}\left(\delta(f-2250) + \delta(f+2250)\right)$$

$\frac{2250}{5000}$

$$Y(f) = \frac{1}{2}\,\text{Sinc}\left(\frac{1}{5}\right)\Big(\delta(f-1000) + \delta(f+1000)\Big)\, e^{-j2\pi \frac{1}{20000} f}$$

$$- \frac{1}{2}\,\text{Sinc}\left(\frac{9}{20}\right)\Big(\delta(f-2250) + \delta(f+2250)\Big)\, e^{-j2\pi \frac{1}{10000} f}$$

$$\therefore y(t) = \text{Sinc}\left(\frac{1}{5}\right) \cdot \text{Cos}\left(2\pi(1000)\left(t - \frac{1}{20\,000}\right)\right)$$

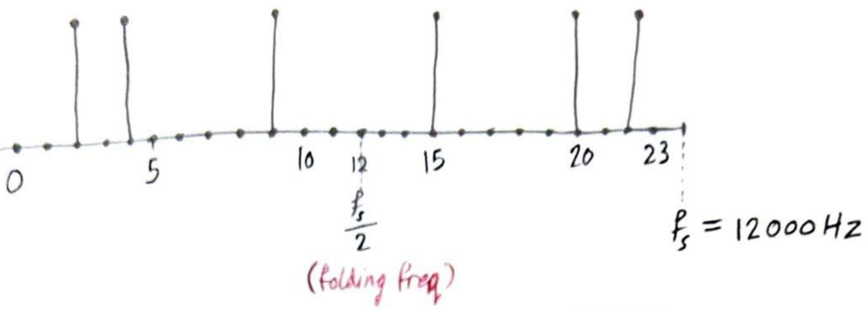
$$- \text{Sine}\left(\frac{9}{20}\right) \cdot \text{Cos}\left(2\pi(2250)\left(t - \frac{1}{10\,000}\right)\right)$$

$$\boxed{y(t) = \text{Sinc}\left(\frac{1}{5}\right) \cdot \text{Cos}\left(2000\pi t - \frac{\pi}{10}\right) - \text{Sinc}\left(\frac{9}{20}\right) \cdot \text{Cos}\left(4500\pi t - \frac{9}{20}\pi\right)}$$

The frequency here is different from the input because of aliasing

$f_s < 2\,f_{max}$

8.15



a) Period of digital signal in number of samples $\rightarrow$ $\boxed{N = 24 \text{ samples}}$

b) Frequencies in analog signal : $\boxed{f = 1k,\ 2k,\ 4.5k\ Hz}$
(Before sampling)

$\underbrace{\frac{2}{24}f_s,\ \frac{4}{24}f_s,\ \frac{9}{24}f_s}$, $\underbrace{\frac{15}{24}f_s,\ \frac{20}{24}f_s,\ \frac{22}{24}f_s}$

same

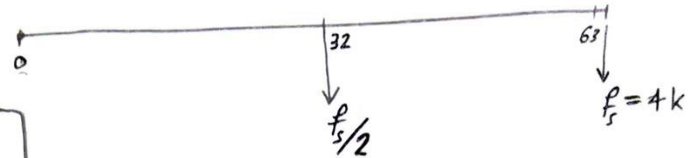
c) Digital frequency $= \Omega = \frac{2\pi f}{f_s}$

Within 1 period of the spectrum $(k = 0 \rightarrow 23)$: $\Omega = \frac{1}{6}\pi,\ \frac{1}{3}\pi,\ \frac{3}{4}\pi,\ \frac{5}{4}\pi,\ \frac{5}{3}\pi,\ \frac{11}{6}\pi$

$\frac{5}{4}\pi \downarrow -\frac{3}{4}\pi \quad \frac{5}{3}\pi \downarrow -\frac{1}{3}\pi \quad \frac{11}{6}\pi \downarrow -\frac{1}{6}\pi$

The original components of the digital signal are between 0 and $f_s/2$
$\hookrightarrow \Omega = \frac{\pi}{6},\ \frac{\pi}{3},\ \frac{3\pi}{4}$ rad

8.22 $N = 64$, $f_s = 4kHz$

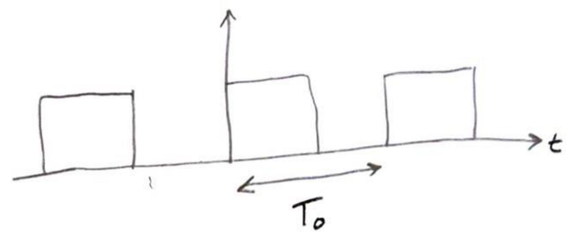


a) fundamental frequency = $\boxed{f_{fund} = \frac{1}{64} \times 4k = 62.5\,Hz}$

b) The spectrum of square wave contains components at f_0, $3f_0$, $5f_0$, -----
(The odd harmonics).

So, Period of square wave $= \frac{1}{f_{fundamental}}$

$$\boxed{T_0 = 0.016\,s = 16\,ms}$$



c) Since spectrum is Zero at $k = 0$

$\boxed{\therefore \text{Average of square wave} = 0}$

d) $f_{cutoff} = 500\ Hz = \frac{8}{64}(4000) = \frac{8}{64} f_s \longrightarrow$ at $k = 8$

$\therefore$ The filtered output will contain: $(k = 0 \rightarrow 7)$

$$f_{out} = \frac{1}{64} f_s\ ,\ \frac{3}{64} f_s\ ,\ \frac{5}{64} f_s\ ,\ \frac{7}{64} f_s$$

$$\boxed{f_{out} = 62.5,\ 187.5,\ 312.5,\ 437.5\ Hz}$$