Lec 14

For two random variables X and Y

$$E(X+Y)=E(X)+E(Y)$$

- Joint PMF, $P_{X,Y}[x,y]$ or PDF, $f_{X,Y}(x,y)$ is **not** required for $E(X+Y)=E(X)+E(Y)$
- However, the $\left[\text{variance of}\,(X+Y)\right]$ depends on the entire joint PMF, PDF or joint CDF

Variance of sum of two random variables X and Y

$$Var(X+Y)=\begin{cases} =E\left[\left([X+Y]-[\mu_X+\mu_Y]\right)^2\right] \\ =E\left[\left((X-\mu_X)+(Y-\mu_Y)\right)^2\right] \\ =E\left[(X-\mu_X)^2+(Y-\mu_Y)^2+2\cdot(X-\mu_X)\cdot(Y-\mu_Y)\right] \\ =E\left[(X-\mu_X)^2\right]+E\left[(Y-\mu_Y)^2\right]+E\left[2\cdot(X-\mu_X)\cdot(Y-\mu_Y)\right] \\ =Var[X]+Var[Y]+2\underbrace{E\left[(X-\mu_X)\cdot(Y-\mu_Y)\right]}_{\text{Covariance}} \end{cases}$$

Covariance

$$Cov[X,Y]=\sigma_{X,Y}=E\left[(X-\mu_X)\cdot(Y-\mu_Y)\right]$$

Correlation of two random variables X and Y

$$r_{X,Y}=E[X,Y]$$

Summary of above relationships

$$Cov(X,Y) = \begin{cases} E\left(\left(X-\mu_X\right)\left(Y-\mu_Y\right)\right) \\ = E\left(XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y\right) \\ = E\left(XY\right) - \mu_X\mu_Y \\ = r_{X,Y} - \mu_X\mu_Y \end{cases}$$

$$Var(X+Y) = \begin{cases} Var(X) + Var(Y) + 2E\left[\left(X-\mu_X\right)\cdot\left(Y-\mu_Y\right)\right] \\ = Var(X) + Var(Y) + 2Cov(X,Y) \end{cases}$$

$$If\ \ X = Y,\ \begin{cases} Cov(X,X) = \begin{cases} E\left(\left(X-\mu_X\right)\left(X-\mu_X\right)\right) \\ = E\left(\left(X-\mu_X\right)^2\right) \\ = Var(X) \end{cases} \\ \\ r_{X,X} = E\left(X^2\right) \end{cases}$$

Orthogonal random variables

Random variable X and Y are orthogonal
if $r_{X,Y} = 0$ (correlation is zero)

Uncorrelated random variables

Random variable X and Y are uncorrelated if
$Cov(X,Y) = 0$

Correlation Coefficient

The correlation coefficient of two random variables X and Y is

$$\begin{aligned}\rho_{X,Y} &= \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}} \\ &= \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} \qquad\qquad -1 \le \rho_{X,Y} \le 1 \\ &= \frac{E(XY)-\mu_X\mu_Y}{\sigma_X \cdot \sigma_Y}\end{aligned}$$

- The correlation is 1 in the case of an increasing linear relationship.

- The correlation is $[-1]$ in the case of a decreasing linear relationship.

- The correlation is some value in between in all other cases. Indicating the degree of linear dependence between the variables

- The closer the coefficient is to either −1 or 1, the stronger the correlation between the variables. (*Wikipedia*)

Theorem 4.17 Let $\sigma_X{}^2$ and $\sigma_Y{}^2$ denote the variance of X and Y for a constant a, let $W = X - aY$. Find $Var(W)$.

$$
\begin{aligned}
Var(W) &= E\left((W-\mu_W)^2\right) \\
&= E\left(W^2 - 2W\mu_W + \mu_W{}^2\right) \\
&= E\left(W^2\right) - E\left(2W\mu_W\right) + E\left(\mu_W{}^2\right) \\
&= E\left(W^2\right) - \mu_W{}^2
\end{aligned}
$$

$$
\begin{aligned}
W^2 &= X^2 + (aY)^2 - 2aXY \\
\mu_W &= \mu_X - a\mu_Y, \\
\mu_W{}^2 &= \mu_X{}^2 + (a\mu_Y)^2 - 2a\mu_X\mu_Y
\end{aligned}
$$

$$
\begin{aligned}
Var(W) &= \underbrace{E\left(X^2\right) - \mu_X{}^2}_{Var(X)} + \underbrace{E\left((aY)^2\right) - (a\mu_Y)^2}_{a^2Var(Y)} - \underbrace{\left(E(2aXY) - 2a\mu_X\mu_Y\right)}_{2aCov(XY)} \\
&= Var(X) + a^2Var(Y) - 2aCov(XY)
\end{aligned}
$$

Since $Var(W) \geq 0$ for any value of a

$$
\begin{aligned}
& Var(X) + a^2Var(Y) - 2aCov(XY) \geq 0 \\
\Leftrightarrow\ & Var(X) + a^2Var(Y) \geq 2aCov(XY)
\end{aligned}
$$

If $a = \dfrac{\sigma_X}{\sigma_Y}$, then

$$
\begin{aligned}
& Var(X) + a^2Var(Y) \geq 2aCov(XY) \\
&= \sigma_X{}^2 + \left(\frac{\sigma_X}{\sigma_Y}\right)^2 \sigma_Y{}^2 \geq 2\left(\frac{\sigma_X}{\sigma_Y}\right)Cov(XY) \\
&\Rightarrow \sigma_X{}^2\left(\frac{\sigma_Y}{\sigma_X}\right) + \left(\frac{\sigma_X}{\sigma_Y}\right)^2\left(\frac{\sigma_Y}{\sigma_X}\right)\sigma_Y{}^2 \geq 2Cov(XY) \\
&\Rightarrow \sigma_X\sigma_Y + \sigma_X\sigma_Y \geq 2Cov(XY) \\
&\Rightarrow 2\sigma_X\sigma_Y \geq 2Cov(XY) \\
&\Rightarrow \sigma_X\sigma_Y \geq Cov(XY)
\end{aligned}
$$

which implies that

$$
\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} \leq 1
$$

Example 4.12 For the integrated circuits tests in Example 4.1, we found in Example 4.3 that the probability model for X and Y is given by the following matrix.

$P_{XY}(x,y)$,	$y=0$	$y=1$	$y=2$	$P_X(x)$
$x=0$	0.01	0	0	0.01
$x=1$	0.09	0.09	0	0.18
$x=2$	0	0	0.81	0.81
$P_Y(y)$	0.10	0.09	0.81	

Find $r_{X,Y}$ and $Cov[X,Y]$

$$E[XY] = \sum_{x=0}^{2}\sum_{y=0}^{2} xyP_{XY}(x,y) = (1)(1)0.09 + (2)(2)0.81 = 3.33$$

$$E[X] = (1)(0.18) + (2)(0.81) = 1.80$$

$$E[Y] = (1)(0.09) + (2)(0.81) = 1.71$$

$$Cov[X, Y] = 3.33 - (1.80)(1.71) = 0.252$$

4.7.7) For a random variables X , let $Y = aX + b$. Show that if $a > 0$ then $\rho_{X,Y} = 1$ and $a < 0$ then $\rho_{X,Y} = -1$.

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$$

$$\begin{cases} E(X) = \mu_X \\ E(Y) = \begin{cases} E(aX+b) \\ = a\mu_X + b \end{cases} \end{cases}$$

$$Cov(X,Y) = \begin{cases} Cov(X, aX+b) \\ = E\left(\left(X-\mu_X\right)\left(\left(aX+b\right)-\left(a\mu_X+b\right)\right)\right) \\ = a\cdot E\left(\left(X-\mu_X\right)\left(X-\mu_X\right)\right) \\ = a\cdot E\left(\left(X-\mu_X\right)^2\right) \\ = a\cdot Var(X) \end{cases}$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}} = \begin{cases} \dfrac{a\cdot Var(X)}{\sqrt{Var(X)}\sqrt{a^2\cdot Var(X)}} \\ = \dfrac{a\cdot Var(X)}{|a|\cdot Var(X)} \\ = \dfrac{a}{|a|} \end{cases}$$

Quiz 4.7) Random variable L and T have joint **PMF**

$P_{L,T}[l,t]$	t=40	t=60	
l=1	0.15	0.1	**0.25**
l=2	0.3	0.2	**0.5**
l=3	0.15	0.1	**0.25**
	0.6	**0.4**	

Find the following

a) $E[L]$ and $Var[L]$

b) $E[T]$ and $Var[T]$

c) The correlation $\{r_{L,T} = E[L,T]\}$

d) The covariance $\{Cov[L,T] = E[(L-\mu_L)(T-\mu_T)]\}$

e) The correlation coefficient $\left\{\rho_{L,T} = \dfrac{Cov[L,T]}{\sqrt{Var[L]\cdot Var[T]}} = \dfrac{Cov[L,T]}{\sigma_L \cdot \sigma_T}\right\}$

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a) $$\begin{cases} E[L] = \sum_{l=1}^{3} l \cdot P_L[l] = 1\cdot(0.25)+2\cdot(0.5)+3\cdot(0.25) = 2 \\ Var[L] = E\left[(L-\mu_L)^2\right] = E\left[L^2\right] - {\mu_L}^2 = \left[1\cdot(0.25)+2^2\cdot(0.5)+3^2\cdot(0.25)\right] - 2^2 = 0.5 \end{cases}$$

b)

$$\begin{cases} E[T] = \sum_{t=40,60} t \cdot P_L[t] = 40\cdot(0.6)+60\cdot(0.4) = 48 \\ Var[T] = E\left[(T-\mu_T)^2\right] = E\left[T^2\right] - {\mu_T}^2 = \left[40^2\cdot(0.6)+60^2\cdot(0.4)\right] - 48^2 = 2400 - 48^2 = 96 \end{cases}$$

c)

$$r_{L,T} = E[L,T] = \sum_{t=40,60}\sum_{l=1,2,3} l\cdot t\cdot P_{L,T}[l,t] = (40\cdot 1)\cdot(0.15)+(40\cdot 2)\cdot(0.3)+(40\cdot 3)\cdot(0.15)$$

$$+(60\cdot 1)\cdot(0.1)+(40\cdot 2)\cdot(0.2)+(40\cdot 3)\cdot(0.1) = 48+48 = 96$$

d) $Cov[L,T] = E\left[(L-\mu_L)(T-\mu_T)\right] = r_{L,T} - \mu_L\cdot\mu_T = 96 - (2\cdot 48) = 0$

e) $\rho_{L,T} = \dfrac{Cov[L,T]}{\sqrt{Var[L]\cdot Var[T]}} = \dfrac{0}{\sigma_L\cdot\sigma_T} = 0$

B) The joint probability density function of random variables X and Y is

$$f_{X,Y}(x,y)=\begin{cases} xy & 0\le x\le 1,\ 0\le y\le 2 \\ 0 & o,w \end{cases}$$

Find the following quantities.

a) $E(X)$ and $Var(X)$

b) $E(Y)$ and $Var(Y)$

c) The correlation $r_{X,Y}=E(X,Y)$

d) The covariance $Cov(X,Y)=E\big((X-\mu_X)(Y-\mu_Y)\big)$

e) The correlation coefficient

$$f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy=\int_0^2 x\cdot y dy=\frac{1}{2}xy^2\,|_0^2=2x \qquad 0\le x\le 1$$

$$f_Y(y)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx=\int_0^1 x,y dx=\frac{1}{2}x^2\cdot y\,|_0^1=\frac{1}{2}y \qquad 0\le y\le 2$$

a) $$\begin{cases} E(X)=\int_{-\infty}^{\infty} x\cdot f_X(x)dx=\int_0^1 x\cdot 2\cdot x dx=\int_0^1 2\cdot x^2 dx=\frac{2}{3} \\ E(X^2)=\int_{-\infty}^{\infty} x^2\cdot f_X(x)dx=\int_0^1 x^2\cdot 2\cdot x dx=\int_0^1 \frac{2}{4}\cdot x^3 dx=\frac{2}{4}=\frac{1}{2} \\ Var(X)=E(X^2)-E(X)^2=\frac{1}{2}-\left(\frac{2}{3}\right)^2=\frac{1}{18} \end{cases}$$

b) $$\begin{cases} E(Y)=\int_0^2 y\cdot f_Y(y)dy=\frac{4}{3} \\ E(Y^2)=\int_{-\infty}^{\infty} y^2\cdot f_Y(y)dy=2 \\ Var(Y)=E(Y^2)-E(Y)^2=\frac{2}{9} \end{cases}$$

c) $r_{X,Y}=E(X,Y)=\int_0^1\int_0^2 (x\cdot y)\cdot f_{X,Y}(x,y)dxdy=\int_0^1\int_0^2 (x\cdot y)\cdot(x\cdot y)dxdy=\frac{8}{9}$

d) $Cov(X,Y)=E\big((X-\mu_X)(Y-\mu_Y)\big)=r_{X,Y}-\mu_X\mu_Y=\frac{8}{9}-\frac{8}{9}=0$

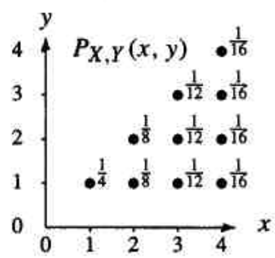
e) $\rho_{X,Y}=0$

Conditional Joint PMF

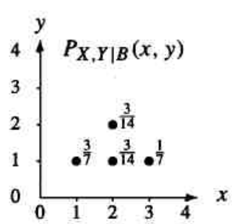
For discrete r.v X and Y and an event, B with $P[B] > 0$, the conditional joint PMF of X and Y given B is

$$P_{X,Y|B}[x,y] = \begin{cases} P[X = x, Y = y | B] \\ = \begin{cases} \dfrac{P_{X,Y}[x,y]}{P[B]} & (x,y) \in B \\ 0 & o.w \end{cases} \end{cases}$$

Example 4.13



Random variables X and Y have the joint PMF $P_{X,Y}(x, y)$ as shown. Let B denote the event $X + Y \leq 4$. Find the conditional PMF of X and Y given B.



Event $B = \{(1, 1), (2, 1), (2, 2), (3, 1)\}$ consists of all points (x, y) such that $x + y \leq 4$. By adding up the probabilities of all outcomes in B, we find

$$P[B] = P_{X,Y}(1, 1) + P_{X,Y}(2, 1) + P_{X,Y}(2, 2) + P_{X,Y}(3, 1) = \frac{7}{12}.$$

The conditional PMF $P_{X,Y|B}(x, y)$ is shown on the left.

Conditional Joint PDF

Given an event B with $P[B] > 0$, the conditional joint probability density function of X and Y is

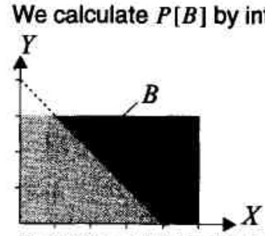
$$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P(B)} & (x,y) \in B \\ 0 & o.w \end{cases}$$

Example 4.14 X and Y are random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \le x \le 5, 0 \le y \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

Find the conditional PDF of X and Y given the event $B = \{X + Y \ge 4\}$.

We calculate $P[B]$ by integrating $f_{X,Y}(x,y)$ over the region B.



$$P[B] = \int_0^3 \int_{4-y}^5 \frac{1}{15}\, dx\, dy$$
$$= \frac{1}{15} \int_0^3 (1+y)\, dy$$
$$= 1/2.$$

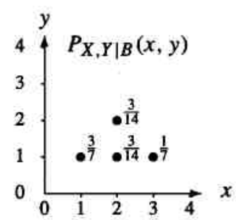
Definition 4.10 leads to the conditional joint PDF

$$f_{X,Y|B}(x,y) = \begin{cases} 2/15 & 0 \le x \le 5, 0 \le y \le 3, x + y \ge 4, \\ 0 & \text{otherwise.} \end{cases}$$

Conditional Expected Value

For r.v $[X \text{ and } Y]$ and an event, B of nonzero probability, the conditional expected value of $[W = g(X,Y)]$ given B is

Discrete case: $E[W|B] = \sum_{x \in S_X} \sum_{y \in S_Y} g[x,y] P_{X,Y|B}[x,y]$



Event $B = \{(1,1), (2,1), (2,2), (3,1)\}$ consists of all points (x, y) such that $x + y \leq 4$. By adding up the probabilities of all outcomes in B, we find

$$P[B] = P_{X,Y}(1,1) + P_{X,Y}(2,1) + P_{X,Y}(2,2) + P_{X,Y}(3,1) = \frac{7}{12}.$$

The conditional PMF $P_{X,Y|B}(x, y)$ is shown on the left.

Example 4.15 Continuing Example 4.13, find the conditional expected value and the conditional variance of $W = X + Y$ given the event $B = \{X + Y \leq 4\}$.

We recall from Example 4.13 that $P_{X,Y|B}(x, y)$ has four points with nonzero probability: $(1,1)$, $(1,2)$, $(1,3)$, and $(2,2)$. Their probabilities are 3/7, 3/14, 1/7, and 3/14, respectively. Therefore,

$$E[W|B] = \sum_{x,y} (x+y) P_{X,Y|B}(x,y) \tag{4.88}$$

$$= 2\frac{3}{7} + 3\frac{3}{14} + 4\frac{1}{7} + 4\frac{3}{14} = \frac{41}{14}. \tag{4.89}$$

Similarly,

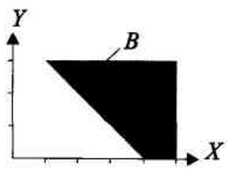
$$E\left[W^2|B\right] = \sum_{x,y} (x+y)^2 P_{X,Y|B}(x,y) \tag{4.90}$$

$$= 2^2\frac{3}{7} + 3^2\frac{3}{14} + 4^2\frac{1}{7} + 4^2\frac{3}{14} = \frac{131}{14}. \tag{4.91}$$

The conditional variance is $\text{Var}[W|B] = E[W^2|B] - (E[W|B])^2 = (131/14) - (41/14)^2 = 153/196$.

Continuous case: $E\left(W \middle| B\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(x, y\right) f_{X,Y|B}\left(x, y\right) dxdy$

Example 4.16 Continuing Example 4.14, find the conditional expected value of $W = XY$ given the event $B = \{X + Y \geq 4\}$.



For the event B shown in the adjacent graph, Example 4.14 showed that the conditional PDF of X, Y given B is

$$f_{X,Y|B}(x, y) = \begin{cases} 2/15 & 0 \leq x \leq 5, 0 \leq y \leq 3, (x, y) \in B, \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 4.20,

$$\begin{aligned} E[XY|B] &= \int_0^3 \int_{4-y}^5 \frac{2}{15} xy\, dx\, dy \\ &= \frac{1}{15} \int_0^3 \left(x^2 \Big|_{4-y}^5 \right) y\, dy \\ &= \frac{1}{15} \int_0^3 \left(9y + 8y^2 - y^3\right) dy = \frac{123}{20}. \end{aligned}$$

Conditional Variance

The conditional variance of the random variable $W = g(X,Y)$ is

$$Var(W|B) = \begin{cases} E\left(\left(W - \mu_{W|B}\right)^2 |B\right) \\ = E\left(W^2 |B\right) - \left(\mu_{W|B}\right)^2 \end{cases}$$