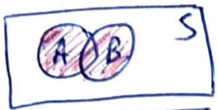
CH2 { PROBABILITY }

* Events:
- Null (impossible) $\rightarrow A = \emptyset$, $P(A) = 0$
- Certain $\rightarrow A = S$, $P(A) = 1$

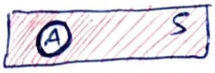
* Sets:

① Union $A \cup B$ — OR



② Intersect $A \cap B$ — AND

③ Complement A^c



④ A Difference B $A - B$

➜ Mutually Exclusive $\equiv$ $A \cap B = \emptyset$

➜ A & A^c are mutually exclusive
$A \cap A^c = \emptyset$.

➜ $A \cup A^c = S$.

➜ if $A \subset B$ & $B \subset A$, $\therefore A = B$

* Axioms

① $P(A) \geq 0 \longrightarrow +ve$

② $P(S) = 1$

Finite ③ if A & B mutually exclusive, ($A \cap B = \emptyset$)
$P(A \cup B) = P(A) + P(B)$

infinite ③ if $A_i \cap A_j = \emptyset$ for any $i \neq j$, $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

* Corollary

① $P(A^c) = 1 - P(A)$

② $P(A) \leq 1$

③ $P(\emptyset) = 0$

④ if $A_1, \dots, A_n$ are mutually exclusive,
$\therefore P[\bigcup_{i=1}^{n} A_i] = \sum_{i=1}^{n} P(A_i)$ for $n \geq 2$

Proof

a) $n = 2 \longrightarrow$ Axiom ③

b) assume it is true for $n = k$: $P[\bigcup_{i=1}^{k} A_i] = \sum_{i=1}^{k} P(A_i)$

c) $n = k+1$: $P(\underbrace{A_1 \cup \dots \cup A_k}_{M} \cup A_{k+1})$

$= P((A_1 \cup \dots \cup A_k) \cup A_{k+1})$

$= P(\bigcup_{i=1}^{k} A_i) + P(A_{k+1})$

$= \sum_{i=1}^{k+1} P(A_i)$

A) Commutative property: $A \cup B = B \cup A$ ($\cap$)

B) Associative property: $A \cup (B \cup C) = (A \cup B) \cup C$ ($\cap$)

C) Distributive property: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ ($\cap$, $\cup$)

D) De Morgan rule:
$(A \cup B)^c = A^c \cap B^c$
$(A \cap B)^c = A^c \cup B^c$

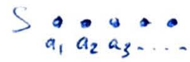
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) \leq P(A) + P(B)$

I Discrete S:

① $P[\{a_i\}] = \frac{1}{n}$

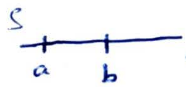
② $P[\{a_1, \dots, a_k\}] = \frac{k}{n}$



II Continuous S:

① $P[\{a\}] = 0$

② $P[[a,b]] = b - a$



⑤ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

⑥ $P[\bigcup_{i=1}^{n} A_i] = \sum_{j=1}^{n} P(A_j) - \sum_{j<k} P(A_j \cap A_k) + \dots + (-1)^{n+1} P(A_1 \cap \dots \cap A_n)$

$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - [P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_3)] + (-1)^4 P(A_1 \cap A_2 \cap A_3)$

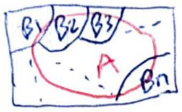
⑦ $A \subset B \rightsquigarrow \therefore P(A) \leq P(B)$

☆ Conditional Probability

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}} \qquad P(B) > 0$$

* if $A = B$
OR
* if $B \subset A$ $\Big\} \therefore P(A|B) = 1$

☆ Theorem on Total Probability



$B_1 \cup B_2 \cdots \cup B_n = S$ & $B_i \cap B_j = \emptyset$

① B_is partitioned S ② B_is are mutually exc.

$$P(A) = P(A \cap S)$$
$$= P[(A \cap B_1) \cup (A \cap B_2) \cup \cdots (A \cap B_n)]$$
$$= P(A \cap B_1) + P(A \cap B_2) + \cdots P(A \cap B_n)$$

$$\therefore \boxed{P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_n)P(B_n)}$$

☆ Baye's Rule

$$P(B_j|A) = \frac{P(B_j \cap A)}{P(A)}$$
$$= \frac{P(A \cap B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

Independance:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$$
$$P(A \cap B) = P(A)P(B)$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = P(B)$$

↳ For 3 events, independance must satisfy 4 conditions:

$$P(A \cap B) = P(A)P(B)$$
$$P(A \cap C) = P(A)P(C)$$
$$P(B \cap C) = P(B)P(C)$$
$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

↳ for n events $\longrightarrow$ $2^n - n - 1$ conditions

1 ☆ Binomial Prob. law

only 2 outcomes

$k = 0, 1, 2, \ldots n$

Done n times

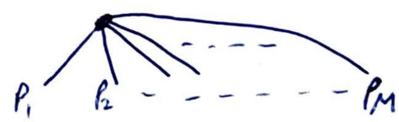
1. $p(0 \text{ success}) = p(\text{all fail}) = (1-p)^n$
2. $p(0 \text{ fail}) = p(\text{all success}) = p^n$
3. $p(k \text{ success}) = \binom{n}{k} p^k (1-p)^{n-k}$

$$\# \text{ of } k \text{ successes} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$n! = n \times n-1 \times \cdots \times 2 \times 1$

$0! = 1$

2 ☆ Multinomial Prob. law



M outcomes

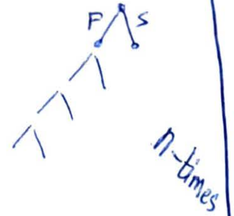
Done n times

$$P[(k_1, k_2, \ldots k_M)] = \frac{n!}{k_1! k_2! \cdots k_M!} p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots p_M^{k_M}$$

\# of times each outcome occurs in the n times

★ Geometric Prob. law

$$\boxed{P[1^{st}\text{ success is in } m^{th}\text{ experiment}] = p(1-p)^{m-1} = P[m]}$$



$P[1^{st}$ success happens after K^{th} experiment$]$

$$= p[m=K+1] + p[m=K+2] + \dots$$

$$= \underbrace{p(1-p)^K}_{\text{Common factor}} \left[1 + (1-p) + (1-p)^2 + \dots\right]$$

$$= p(1-p)^K \sum_{r=0}^{\infty} (1-p)^r$$

$$= p(1-p)^K \frac{1}{1-(1-p)}$$

$$= (1-p)^K$$

$$= P(1^{st}\ K \text{ are all failures})$$

Geometric Progression

$$\boxed{\sum_{r=0}^{\infty} q^r = \frac{1}{1-q}}$$

$|q|<1$

Binomial (Bernoulli)
Multinomial
Geometric

} All are for Sequences of Independent experiments

➔ Sequence of Dependent experiments:

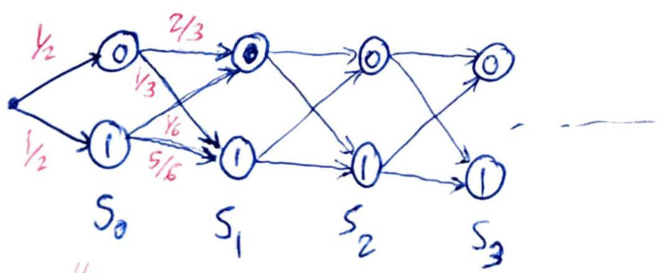
Flip Coin

Urn 0 < 1 ① ball, 2 ⓪ balls

Urn 1 < 5 ① balls, 1 ⓪ balls

If ① ball is chosen, next choise will be from Urn ①.

⇒ With replacement



$$P[\text{a certain sequence } S_0 S_1 S_2] = P[(S_0 \cap S_1) \cap S_2]$$

$$= P(\underbrace{S_2 \mid S_0 \cap S_1}_{= S_2 \mid S_1}) \, P(S_0 \cap S_1)$$

$\to$ since prob. of S_2 is only effected by S_1

$$= P(S_2 \mid S_1)\, P(S_1 \mid S_0)\, P(S_0)$$

$$\boxed{P(S_0 \cap S_1 \cap S_2 \cap \dots S_n) = P(S_n \mid S_{n-1})\, P(S_{n-1} \mid S_{n-2}) \dots P(S_1 \mid S_0)\, P(S_0)}$$

Examples

① Discrete S

ex: Toss coin 3 times

[1] Sample of observing outcome for all 3 tosses

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

↳ $n = 2^3 = 8$

$P\{1 \text{ outcome}\} = \frac{1}{8}$

$P\{2\,Hs\} = P\{HHT, HTH, THH\} = \frac{3}{8}$

[2] Sample of observing number of heads

$S_x = \{0H, 1H, 2H, 3H\}$

$= \{\{TTT\}, \{HTT, THT, TTH\}, \{HHT, THH, HTH\}, \{HHH\}\}$

② Continuous S

ex: lifetime of a computer chip is measured

↳ Prob. the chip's lifetime exceeds t decreases exponentially with rate α

$\boxed{P[(t,\infty)] = e^{-\alpha t}}$ where $\alpha > 0$

Axioms

① $P[A] = P[(t,\infty)] = e^{-\alpha t} > 0$

② $P[S] = P[[0,\infty[] = e^{-\alpha(0)} - \lim_{t\to\infty} e^{-\alpha t} = 1 - 0$

③ $B = [r, s[$ & $C = [s, \infty[$

B & C are mutually exclusive

$P(B \cup C) = P([r,\infty[)$ LHS

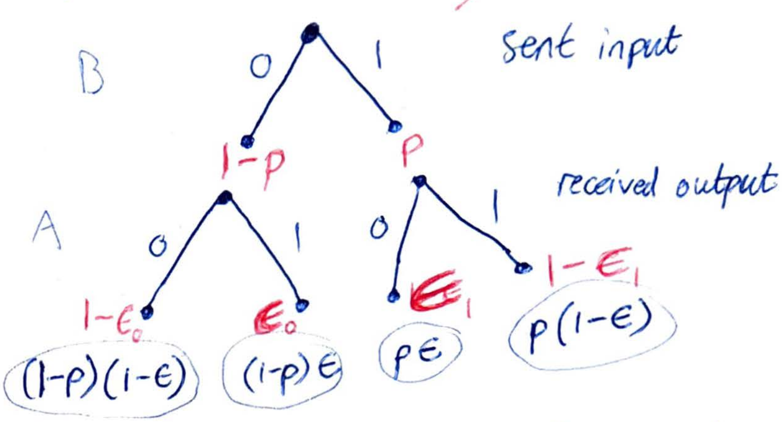
$= e^{-\alpha r}$

$P(B) + P(C) = (e^{-\alpha r} - \cancel{e^{-\alpha s}}) + \cancel{e^{-\alpha s}} = e^{-\alpha r}$ RHS

∴ LHS = RHS

★ Binary Communication System

1 branch is an intersection



B_0 = 0 transmitted , A = Error event

B_1 = 1 transmitted

① $P(A) = P(A|B_0)P(B_0) + P(A|B_1)P(B_1)$

$= P(A\cap B_0) + P(A \cap B_1)$ $\boxed{\begin{matrix} B_0 \cap B_1 = \emptyset \\ B_0 \cup B_1 = S \end{matrix}}$

$= (1-p)\epsilon_0 + p\epsilon_1$

probability of error

② $P(B_0|A) = \frac{P(B_0 \cap A)}{P(A)} = \frac{P(A|B_0)P(B_0)}{P(A)}$

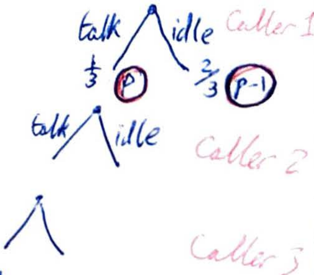
$= \frac{\epsilon_0(1-p)}{\epsilon_0(1-p) + p\epsilon_1}$

probability of 0 transmitted, if an Error happened

★ Call Center: 8 callers ⟺ 6 channels

$P(\text{1 caller talks}) = \frac{1}{3}$

$P(\text{caller doesn't talk}) = \frac{2}{3}$



$P(\text{overflow occurs})$

$= P(K > 6)$ → more than 6 talk

$= P(K=7) + P(K=8)$

$= \binom{8}{7}p^7(1-p)^1 + \binom{8}{8}p^8(1-p)^0$

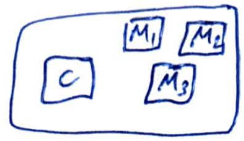
$= \binom{8}{7}\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right) + \binom{8}{8}\left(\frac{1}{3}\right)^8 = 0.00259$

Binomial

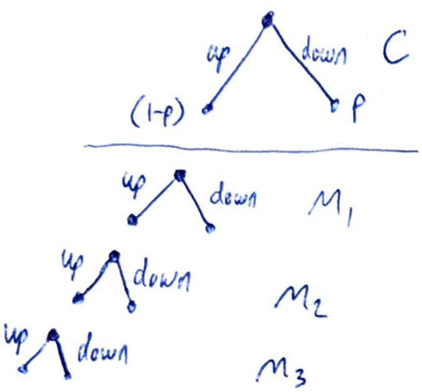
System is up if:
1 Controller & at least 2 Memories are up

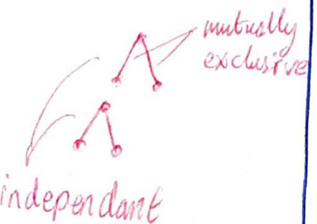
$P(\text{system is up}) = ?$

$P(\overset{1}{C} \text{ up} \cap \overset{2}{M_i}\text{s up})$



$P(C \text{ fails}) = p$
$P(M_i \text{ fails}) = a$





$P(1\,C \text{ up}) = 1-p$

$P(2\,M_s \text{ up}) = P(M_1 \cap M_2 \cap M_3^c)$
$\quad + P(M_1 \cap M_2^c \cap M_3)$
$\quad + P(M_1^c \cap M_2 \cap M_3) + P(M_1 \cap M_2 + M_3)$

mutually exclusive

$P(K \geq 2) = P(K=2) + P(K=3)$
$= \binom{3}{2}(1-a)^2(a)^{3-2} + \binom{3}{3}(1-a)^3(1)$
$= 3a(1-a)^2 + (1-a)^3$

independant

$P(\text{system up}) = P(C \text{ up}) \cdot P(K \geq 2)$
$(1-p) \cdot (3a(1-a)^2 + (1-a)^3)$

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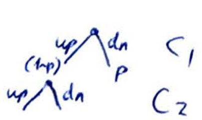
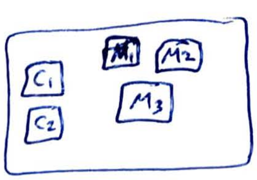
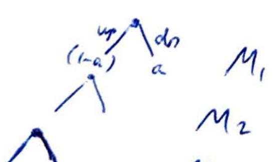
* If I add one more controller







a) Controller
$P(K \geq 1) = P(K=1) + P(K=2)$
$= \binom{2}{1}(1-p)^1(p)^{2-1} + \binom{2}{2}(1-p)^2(1)$
$= 2p(1-p) + (1-p)^2$

b) Memory
$P(K \geq 2) =$ "As before"

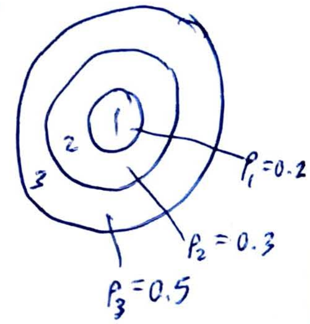
$\therefore P(\text{System up}) = \underset{\text{Controller}}{P(K \geq 1)} \cdot \underset{\text{Memory}}{P(K \geq 2)}$
$= (2p(1-p) + (1-p)^2)(3a(1-a)^2 + (1-a)^3)$

if $a = 10\% = 0.1$
$p = 20\% = 0.2$

$\therefore P(\text{system up}) = 0.93312$

Multinomial

Darts thrown 9 times $(n=9)$.
What's the prob. that 3 hit region 1, 3 hit region 2 & 3 hit region 3 ?



$P[(K_1=3, K_2=3, K_3=3)]$
$= \frac{9!}{3!\,3!\,3!}(0.2)^3(0.3)^3(0.5)^3$