Stationary Random Process

stationary
⇓
* mean, Variance = Constant
* joint pdf & pmfs don't change with time shifts

* Stationarity:

$$\boxed{F_{X(t_1)X(t_2)\dots X(t_k)}(x_1, x_2, \dots x_k) = F_{X(t_1+\tau)X(t_2+\tau)\dots X(t_k+\tau)}(x_1, x_2 \dots x_k)}$$

for all k
all $t_1, t_2, \dots t_k$
all τ

↳ If true, ∴ We have a stationary Random Process

---

ex// for $k=1$:

$$F_{X(t_1)}(x_1) = F_{X(t_1+\tau)}(x_1)$$

→ pdf don't depend on time (pdf is the same for all t)

$$= F_X(x_1)$$

$$\left.\begin{aligned} \therefore E[X(t)] &= m_X \\ Var[X(t)] &= \sigma_X^2 \\ \text{all moments} &= \text{Constant} \end{aligned}\right\} \text{Constants w.r.t } t$$

---

for $k=2$:

$$F_{X(t_1)X(t_2)}(x_1, x_2) = F_{X(t_1+\tau)X(t_2+\tau)}(x_1, x_2)$$

if $\tau = -t_1$

$$\therefore F_{X(t_1)X(t_2)}(x_1, x_2) = F_{X(0)X(t_2-t_1)}(x_1, x_2)$$

→ joint pdf only depends on the time difference (in Stationary RP)

$$\left.\begin{aligned} \therefore \overbrace{R_X(t_1, t_2)}^{E[X(t_1)X(t_2)]} &= R_X(t_2-t_1) \\ C_X(t_1, t_2) &= C_X(t_2-t_1) = \end{aligned}\right\} C_X(0, t_2-t_1)$$

$$E((X(t_1)-m_X(t_1))(X(t_2)-m_X(t_2))) = f(t_2-t_1)$$

implied by Stationarity

iid sequence $X(t) \rightarrow$ Is it stationary?

↳ identical independent distribution at all times

$\Rightarrow$ LHS of Stationarity definition $=$ RHS

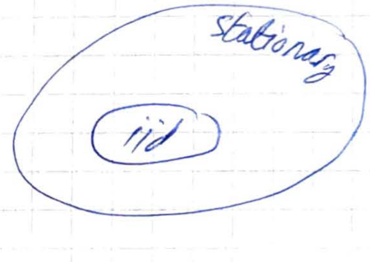
$$F_{X(t_1)X(t_2)\ldots X(t_k)}(x_1, x_2, \ldots x_k) \stackrel{?}{=} F_{X(t_1+\tau)X(t_2+\tau)\ldots X(t_k+\tau)}(x_1, x_2, \ldots x_k)$$

$$= F_{X(t_1)}(x_1)\, F_{X(t_2)}(x_2) \ldots F_{X(t_k)}(x_k) \quad\Big|\quad = F_{X(t_1+\tau)}(x_1) \ldots F_{X(t_k+\tau)}(x_k)$$

$$= F_X(x_1)\, F_X(x_2) \ldots F_X(x_k) \quad\Big|\quad = F_X(x_1) \ldots F_X(x_k)$$

$=$

$\therefore$ iid are stationary



$\rightarrow$ In iid $\rightarrow C_X(t_1, t_2) = C_X(t_2 - t_1) = 0$

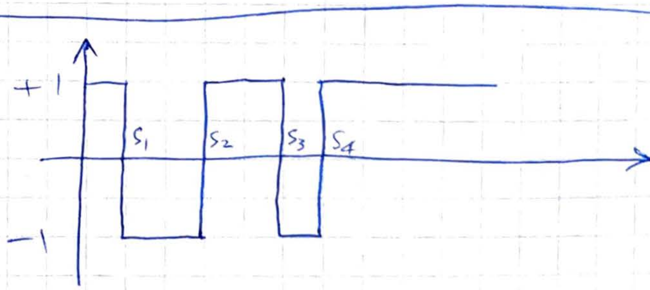
---

ex 2 Sum Process $S_n = X_1 + X_2 + \ldots + X_n$ (X_i's are iid)

↳ Is not Stationary as $E[\underset{n+\tau}{S_n}] = \underset{n+\tau}{n} E[X]$ (mean increases with time)

---

ex 3 Random Telegraph signal



$$P[X(t_2) = X(t_1)] = \frac{1}{2}\left(1 + e^{-2\alpha|t_2 - t_1|}\right)$$

→ more probable that $X(t_2) = X(t_1)$ (so they are not independent)

$$P[X(t_2) \neq X(t_1)] = \frac{1}{2}\left(1 - e^{-2\alpha|t_2 - t_1|}\right)$$

$$P[X(t) = 1] = P[X(t) = 1 \mid X(0) = 1]\, P[X(0) = 1] + P[X(t) = 1 \mid X(0) = -1]\, P[X(0) = -1]$$

$$= \frac{1}{2}\left(1 + e^{-2\alpha t}\right) \cdot \left(\frac{1}{2}\right) + \frac{1}{2}\left(1 - e^{-2\alpha t}\right)\left(\frac{1}{2}\right) = \frac{1}{2}$$

$\therefore$ If You start symmetric, You stay symmetric

if $P[X(0) = \pm 1] = \frac{1}{2} \Rightarrow \therefore P[X(t) = \pm 1] = \frac{1}{2}$

Is it stationary

LHS

$$P_{X(t_1)X(t_2)\ldots X(t_k)}(x_1, x_2, \ldots x_k)$$

$$= P[X(t_1)=a_1, X(t_2)=a_2, \ldots X(t_k)=a_k]$$

$$= P[\underbrace{X(t_k)=a_k}_{A} \mid \underbrace{X(t_1)=a_1 \ldots X(t_{k-1})=a_{k-1}}_{B}] . \underbrace{P[X(t_1)=a_1 \ldots X(t_{k-1})=a_{k-1}]}_{B}$$

RHS

$$P_{X(t_1+\tau)X(t_2+\tau)\ldots X(t_k+\tau)}(x_1, x_2, \ldots x_k)$$

$$= P[X(t_1+\tau)=a_1, \ldots X(t_k+\tau)=a_k]$$

$$= P[X(t_k)=a_k \mid X(t_{k-1})=a_{k-1}] . P[X(t_1)=a_1 \ldots X(t_{k-1})=a_{k-1}]$$

t_{k-1} t_k only the event # of events between the 2

$$= P[X(t_k)=a_k \mid X(t_{k-1})=a_{k-1}] . P[X(t_{k-1})=a_{k-1} \mid X(t_{k-2})=a_{k-2}] . \ldots . P[X(t_1)=a_1]$$

$$P[X(t_k)=\pm 1 \mid X(t_{k-1})=1] = \frac{1}{2}\left(1 \pm e^{-2\alpha|t_k-t_{k-1}|}\right)$$

general for symmetric or unsymmetric cases

all the terms depend on time differences, so a shift doesn't affect it.

Except $P[X(t_1)=a_1] \rightarrow$ is it $= P[X(t_1+\tau)=a_1]$?

Yes for symmetric $P[X(0)=\pm 1] = \frac{1}{2}$

If Start Symmetric $\Rightarrow$ Symmetric @ all times $\Rightarrow$ Stationary

---

$$P[X(t)=1] = P[X(t+\tau)=1] = \frac{1}{2}$$

for the symmetric case

For asymmetric $P[X(0)=1]=p$

$P[X(0)=-1]=1-p$

$$P[X(t)=1] = \underbrace{P[X(t)=1 \mid X(0)=1]}_{\frac{1}{2}(1+e^{-2\alpha t})} \underbrace{P[X(0)=1]}_{p} + \underbrace{P[X(t)=1 \mid X(0)=-1]}_{\frac{1}{2}(1-e^{-2\alpha t})} \underbrace{P[X(0)=-1]}_{1-p}$$

$$P[X(t)=1] = \frac{p}{2}(1+e^{-2\alpha t}) + \frac{1-p}{2}(1-e^{-2\alpha t})$$

→ Not Stationary if didn't start symmetric

→ Gets closer to stationary as $t\uparrow\uparrow$

lec 26

Strong Sense Stationary (SSS)

$$F_{X(t_1)X(t_2)\cdots X(t_k)}^{(x_1, x_2, \cdots x_k)} = F_{X(t_1+\tau)X(t_2+\tau)\cdots X(t_k+\tau)}^{(x_1, x_2, \cdots x_k)}$$

$\Downarrow$ implies

$\Uparrow$ doesn't necessarily imply

$$m_X(t) = m_X$$
$$R_X(t_1, t_2) = R_X(t_2 - t_1)$$
$$C_X(t_1, t_2) = C_X(t_2 - t_1)$$

Wide Sense Stationary = "Stationary" in most electrical applications

$$m_X(t) = m_X$$
$$R_X(t_1, t_2) = R_X(t_2 - t_1)$$
$$C_X(t_1, t_2) = C_X(\underbrace{t_2 - t_1}_{\tau}) = R_X(t_2 - t_1) - \underbrace{E(X(t_1))E(X(t_1))}_{m_X^2}$$

$$R_X(t_1, t_2) = E(X(t_1)\, X(t_2)) = E(X(t)\, X(t+\tau))$$

$$\therefore \boxed{R_X(\tau) = E(X(t)\, X(t+\tau))}$$

value of t doesn't really matter (due to stationarity)

① $R_X(0) = E(X^2(t))$

$C_X(0) = E(X^2(t)) - m_X^2 = Var(X(t))$ → proved $Var(X(t))$ is also constant in WSS

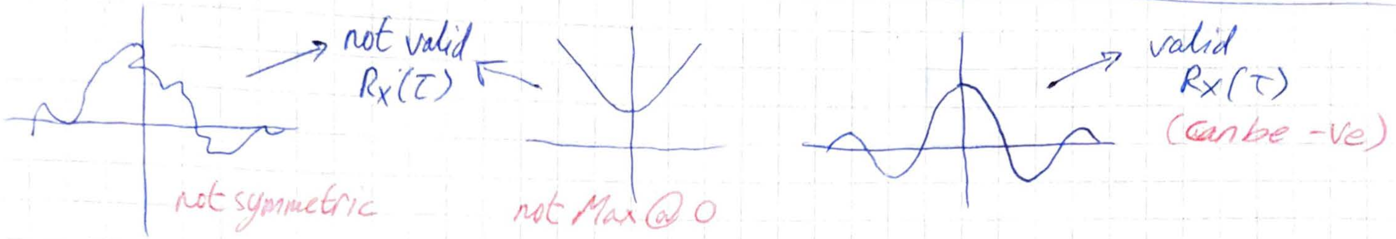
② $R_X(-\tau) = R_X(\tau)$ symmetric fn

$$R_X(-\tau) = E(X(t)\, X(t-\tau))$$
$$= E(\underbrace{X(t-\tau)}_{t}\, \underbrace{X(t)}_{t+\tau}) = R_X(\tau)$$

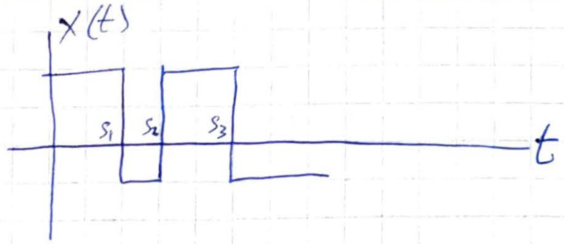
③ $|R_X(\tau)| \leq R_X(0)$

$R_X(\tau)$ is a measure of similarity of X fn at 2 time points

($R_X(0) = $ Max as we compare @ same ~~time~~ time point)



ex/ Random Telegraph Signal



→ for symmetric:

$$P[x(0)=\pm 1] = \frac{1}{2} \longrightarrow P[x(t)=\pm 1] = \frac{1}{2} \underset{\text{Proven}}{\Longrightarrow} SSS \xrightarrow[\text{also}]{\text{so}} WSS$$

$$R_X(t_1,t_2) = E(x(t_1)\,x(t_2))$$

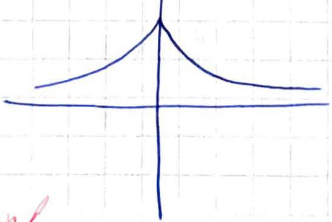
this product can only be +1 or −1

$$= 1 \cdot P[x(t_1)=x(t_2)] + (-1) \cdot P[x(t_1) \neq x(t_2)]$$

$$= \frac{1}{2}\left(1+e^{-2\alpha|t_2-t_1|}\right) - \frac{1}{2}\left(1-e^{-2\alpha|t_2-t_1|}\right)$$

$$= e^{-2\alpha|t_2-t_1|}$$

$$\therefore \boxed{R_X(\tau) = e^{-2\alpha|\tau|}}$$



only depends on the time difference
(then I know my solution is correct)

ex $S_n = X_1 + \dots + X_n$, X_i are iid,

$P[X_i = 0] = P[X_i = 1] = \frac{1}{2}$

find $P[S_2 = 0, S_3 = 1, S_5 = 1]$?

$= P[S_2 = 0] \cdot P[S_{3-2} = 1-0] \cdot P[S_{5-3} = 1-\phi]$

$= P[S_2 = 0] \cdot P[S_1 = 1] \cdot P[S_2 = 0]$

$= \underbrace{P[X_2 + X_1 = 0]}_{P[X_2=0] \cdot P[X_1=0]} \cdot P[X_1 = \phi] \cdot P[X_2 + X_1 = 0] = (\frac{1}{2} \cdot \frac{1}{2}) \cdot (\frac{1}{2}) \cdot (\frac{1}{2} \cdot \frac{1}{2})$

$= \frac{1}{32}$

---

ex $E[X(t)] = 0 \qquad t < 0$

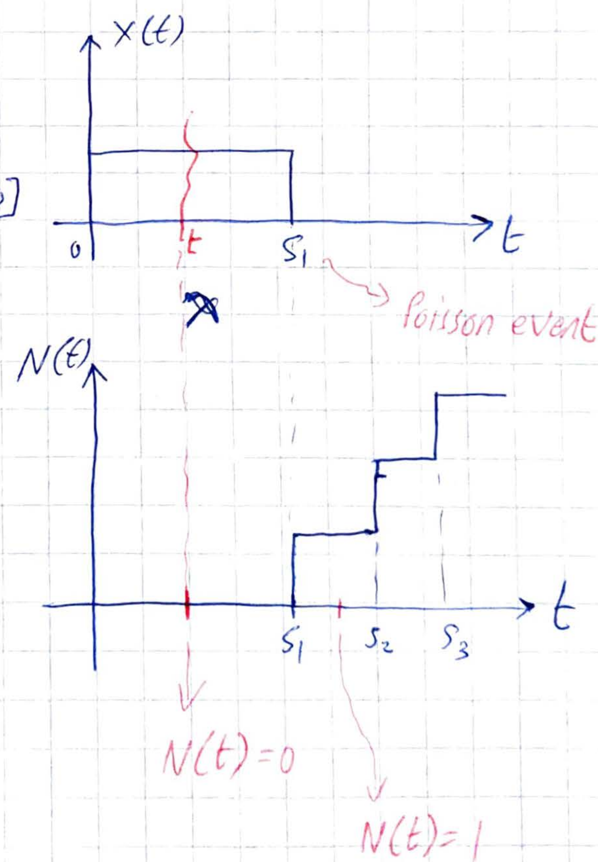
$E[X(t)] = (1) \cdot P[X(t) = 1] + (0) \cdot P[X(t) = 0]$

$= P[X(t) = 1]$

$= P[N(t) = 0]$

equivalent events

$= e^{-\lambda t}$



ex// $X(t)$ = Stationary , $y(t) = X(t) - aX(t - T)$ $\boxed{B}$

$E[X(t)] = 0$

$R_X(\tau) = e^{-|\tau|}$

(a)

$R_Y(\tau) = E[y(t)\, y(t+\tau)]$

$= E[(X(t) - aX(t-T))(X(t+\tau) - aX(t-T+\tau))]$

$= E[X(t)X(t+\tau)] - a\,E[X(t)X(t+\tau-T)]$

time differences

$- a\,E[X(t-T)X(t+\tau)] + a^2 E[X(t-T)X(t+\tau-T)]$

$= R_X(\tau) - a\,R_X(\tau - T) - a\,R_X(\tau + T) + a^2 R_X(\tau)$

$\therefore R_Y(\tau) = (1+a^2)R_X(\tau) - a(R_X(\tau - T) + R_X(\tau + T))$

Y is a Stationary variable, since R_Y only depends on τ (no t)

(b)

Choose a to minimize $Var[y(t)]$

$Var[y(t)] = C_y(0) = R_y(0) - \underbrace{m_y^2}_{0}$

same

$= (1+a^2)\,R_X(0) - a\,(R_X(-T) + R_X(T))$

$= (1+a^2)\,R_X(0) - 2a\,R_X(T)$

$\frac{d}{dy} Var[y(t)] = 2a\,R_X(0) - 2\,R_X(T) = 0$

$\therefore a = \frac{R_X(T)}{R_X(0)} = \frac{e^{-T}}{1} = e^{-T}$

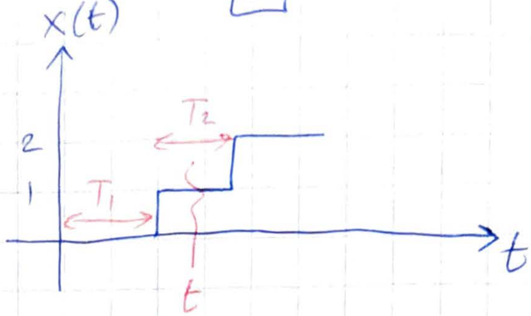
$\boxed{C}$

~~ex~~ Poisson like Random process

T_1 & T_2 uniform $[0, T]$

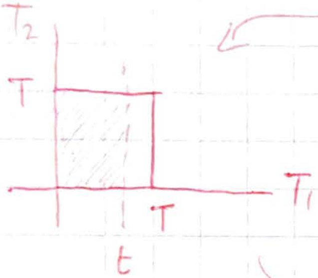
(not exponential

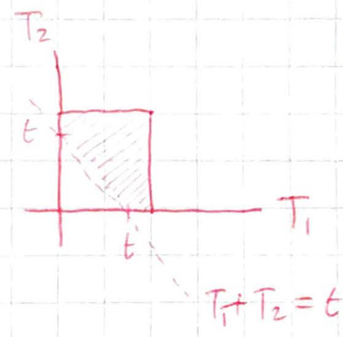
are also independent

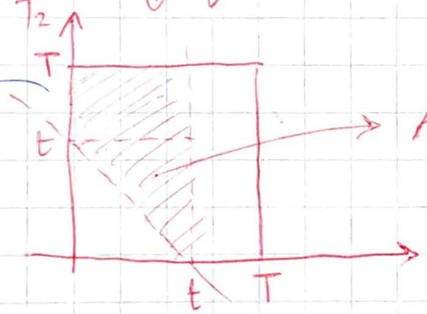


$$P[X(t) = 1] = P[T_1 < t < T_1 + T_2]$$

$$= P[t > T_1 \; , \; t < T_1 + T_2]$$







$$\text{Area} = \tfrac{1}{2}\, t . t + (T-t)\cancel{t}$$

We need to integrate the pdf in this area

(But since this is Uniform, so we only take the Area)

$$\therefore \; P[X(t)=1] = \frac{tT - t^2/2}{T^2} = \frac{t}{T} - \frac{t^2}{2T^2}$$