* Mutually exclusive: $P(A \cup B) = P(A) + P(B)$ $\quad \cap = \emptyset$

* $P[\cup A_i] = \sum P(A_i) - \sum P(A_i \cap A_k) + (-1)^{n+1} P(A_1 \cap \dots \cap A_n)$

* Cond. Prob. $P(A|B) = \frac{P(A \cap B)}{P(B)}$

* Th. on T. Prob. $P(A) = P(A|B_1)P(B_1) + \dots P(A|B_n)P(B_n)$

* Baye's Rule $P(B_j|A) = \frac{P(B_j \cap A)}{P(A)} = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$

* Independence: $P(A \cap B) = P(A)P(B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$

→ For n independent events: $2^n - n - 1$ conditions

① Binomial Prob. law:

fail, pass; 1-P, P $\quad K = 0, 1, \dots, n$

1. $p(0 \text{ success}) = (1-p)^n$
2. $p(0 \text{ fail}) = p^n = p(\text{all success})$
3. $p(k \text{ success}) = \binom{n}{k} p^k (1-p)^{n-k}$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

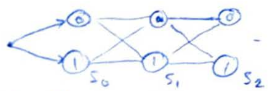
② Multinomial Prob. law: $P_1, P_2, \dots, P_M$ — M outcomes done n times

$P[(k_1, k_2, \dots, k_M)] = \frac{n!}{k_1! k_2! \dots k_M!} P_1^{k_1} P_2^{k_2} \dots P_M^{k_M}$

③ Geo. Prob. law: fail, pass; 1-P, P $\quad n = \infty$

1. $P[1^{st} \text{ success in } m^{th} \text{ exp}] = p[m] = (1-p)^{m-1} p$ $\quad P[X=m]$
2. $P[1^{st} \text{ pass is after } k] = p[1^{st}\ k \text{ all fail}] = (1-p^k)$ $\quad P[X>k]$

* Seq. of dependent experiments $\quad \sum_{r=0}^{\infty} q^r = \frac{1}{1-q}$



$P(S_0 S_1 S_2) = P((S_0 \cap S_1) \cap S_2) = P(S_2 | S_0 \cap S_1) P(S_0 \cap S_1)$

$= P(S_2|S_1) P(S_1|S_0) P(S_0)$

* Discrete RV: $\sum_{S_x} P_x(x) = 1$ $\quad P[X \in B] = \sum_{x \in B} P_x(x)$

* Expectation: $E[X] = \sum_{x \in S_x} x P_x(x)$ $\quad E[aX] = aE[X]$, $E[X+b] = E[X] + b$

$E[g(x)] = \sum_{all} g(x) P_x(x)$ $\quad E[X+Y] = E[X] + E[Y]$

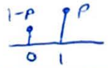
* Variance: $\sigma^2 = Var(x) = E[(x-m_x)^2]$ $\quad Var(aX) = a^2 Var(X)$

$= \sum (x-m_x)^2 P_x(x)$ $\quad Var(X+b) = Var(X)$

$= E[x^2] - m_x^2$ $\quad std(x) = \sigma_x$

* n^{th} moment $= E[x^n]$

1. Bernoulli RV $\quad E[X] = p$, $Var[X] = p(1-p)$



2. Binomial RV — k successes in n times; pass = p, fail = 1 - p

$P_x(k) = \binom{n}{k} p^k (1-p)^{n-k}$ $\quad k = 0, 1, \dots, n$

$E[X=k] = \sum_{k=0}^{n} k P_x(k) = np$ $\quad Var[X] = np(1-p)$

3. Geometric RV — 1^{st} success in k^{th} experiment; pass = p, fail = 1 - p

$P_x(k) = (1-p)^{k-1} p$ $\quad k = 1, 2, 3, \dots$

$E[X] = \sum_{k=1}^{\infty} k P_x(k) = \frac{1}{p}$

$Var[X] = \frac{2-p}{p^2} - \frac{1}{p^2}$

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$\frac{d}{dx}$: $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$

$\frac{d}{dx}$: $\sum_{n=1}^{\infty} n(n-1) x^{n-2} = \frac{2}{(1-x)^3}$

★ Cond PMF

$P_x(x|C) = P[X=x|C]$

$= \frac{P[X=x \cap C]}{P[C]}$

$P_x(x) = \sum_{i=1}^{n} P_x(x|B_i) P(B_i)$ — Th. on T. Prob.

$E(X|C) = \sum x P_x(x|C)$ $\quad E[X] = \sum_{i}^{n} E(X|B_i) P(B_i)$

* In Geo RV: "memoryless" $\quad P[M \geq k+j \mid M > j] = P[M \geq k]$

$E[g(x)] = \sum E(g(x)|B_i) P(B_i)$

4. Poisson RV

$P[N=k] = P_x(k) = e^{-\alpha} \frac{\alpha^k}{k!}$ $\quad k = 0, 1, 2, \dots$

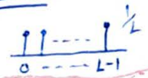
$E[X] = Var[X] = \alpha$

Binomial → Poisson: $n \to \infty$, $p \to 0$ $\} \alpha = np$

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

5. Uniform RV

$P_x(x) = \frac{1}{L}$



$E[X] = \frac{1}{L} \sum_{k=0}^{L-1} k = \frac{1}{L} \frac{L(L-1)}{2} = \frac{L-1}{2}$

$Var[X] = \frac{L^2 - 1}{12}$

6. Zipf RV $\quad S_x = \{1, 2, \dots, L\}$ most → least

$P_x(k) = \frac{1}{C_L} \frac{1}{k}$ $\quad k = 1, 2, \dots, L$

$\sum_{k=1}^{L} P_x(k) = 1 \longrightarrow C_L = \sum_{k=1}^{L} \frac{1}{k}$

$E[X] = \frac{L}{C_L}$

$Var[X] = \frac{L(L+1)}{2C_L} - \frac{L^2}{C_L^2}$

★ Continuous RV $\quad F_x(x) = P[X \leq x]$ CDF

→ $P[\alpha < X \leq \beta] = F(\beta) - F(\alpha^+)$

→ $P[X = \alpha] = F(\alpha^+) - F(\alpha^-)$

→ $P[\alpha \leq X \leq \beta] = P[X = \alpha] + P[\alpha < X \leq \beta]$

$f_x(x) = \frac{dF_x(x)}{dx} = \frac{P[x \leq X \leq x+h]}{h}$ PDF

* $f_x(x) \geq 0$ $\quad F_x(x) = \int_{-\infty}^{x} f_x(x) dx$

* $\int_{-\infty}^{\infty} f_x(x) dx = F(+\infty) - F(-\infty) = 1$

* $P[a < X < b] = \int_a^b f_x(x) dx = F(b) - F(a)$

$F_x(x|A) = P[X \leq x \mid A]$

$= \frac{P[X \leq x \cap A]}{P[A]} \xrightarrow{d} f_x(x|A)$

$F_x(x) = \sum_{i=1}^{n} F_x(x|B_i) p(B_i) \xrightarrow{d} f_x(x)$

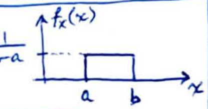
$E[X] = \int_{-\infty}^{\infty} x f_x(x) dx$ $\quad Var[X] = E((x-m)^2)$

$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x) dx$ $\quad = \int_{-\infty}^{\infty} (x-m)^2 f_x(x) dx$

$= E[x^2] - m^2$

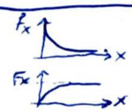
1. Uniform RV

$f_x(x) = \frac{1}{b-a}$ $\quad a \leq x \leq b$



$E[X] = \frac{a+b}{2}$ $\quad Var[X] = \frac{(b-a)^2}{12}$

2. Exponential RV



$f_x(x) = \lambda e^{-\lambda x}$ $\quad x \geq 0$

$F_x(x) = 1 - e^{-\lambda x}$

$E[X] = \frac{1}{\lambda}$

Int. by parts: $\int u\, dv = uv - \int v\, du$

3. Pareto RV $\quad \alpha$, x_m

$F_x(x) = \begin{cases} 0 & x < x_m \\ 1 - \frac{x_m^{\alpha}}{x^{\alpha}} & x \geq x_m \end{cases}$ $\quad f_x(x) = \begin{cases} 0 & x < x_m \\ \frac{\alpha x_m^{\alpha}}{x^{\alpha+1}} & x \geq x_m \end{cases}$

$E[X] = \frac{\alpha}{\alpha - 1} x_m$ $\quad \alpha > 1$ $\quad Var[X] = \frac{\alpha x_m^2}{\alpha - 2} - E[X]^2$ $\quad \alpha > 2$

Exp RV is memoryless → $P[X > t+h \mid X > t] = P[X > h]$

$\sum_{j=0}^{n} q^j = \frac{1 - q^{n+1}}{1-q}$ $\quad \frac{dF}{dy} = \frac{dF}{du} \frac{du}{dy}$

Cos 0 = 1
Cos 30 = $\sqrt{3}/2$
Cos 45 = $\sqrt{2}/2$
Cos 60 = $1/2$
Cos 90 = 0

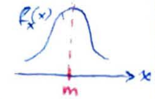
$\mathcal{L}(e^{-ax}) = \frac{1}{s+a}$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\int_0^{\infty} e^{-mx} dx = \frac{1}{m} e^{-mx} \Big|^{\infty}$

4. Gaussian RV $\quad X \sim N(m, \sigma^2)$

$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$



$F_x(x) = \Phi\left(\frac{x-m}{\sigma}\right) = \int_{-\infty}^{\frac{x-m}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$

$Q\left(\frac{x-m}{\sigma}\right) = 1 - \Phi\left(\frac{x-m}{\sigma}\right) = P[X > x]$

↳ $Q(0) = \frac{1}{2}$

↳ $Q(-x) + Q(x) = 1$

↳ $Q(x) = \frac{1}{2} \text{erfc}$

5. Gamma RV $\quad f_x(x) = \frac{\lambda (\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$

$\alpha > 0$, $\lambda > 0$, $x > 0$

$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx$ $\quad$ * $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, * $\Gamma(z+1) = z\Gamma(z)$, * $\Gamma(m+1) = m!$ int.

A) $\alpha = 1 \longrightarrow$ Exponential RV

B) $\alpha = m \longrightarrow$ m-Erlang RV ➡ Poisson events

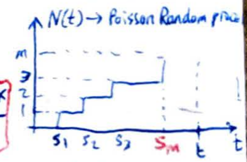
$f_x(t) = \frac{\lambda (\lambda x)^{m-1} e^{-\lambda x}}{(m-1)!}$ $\quad \alpha = \lambda t$ — mean, rate "poisson", time

$F_{S_m}(t) = P[S_m \leq t]$

$= P[N(t) \geq m]$

$= 1 - P[N(t) < m]$

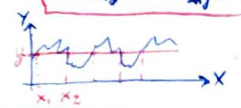
$F_{S_m}(t) = 1 - \sum_{k=0}^{m-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!}$



ex $P[S_3 > 6] = P[N(6 \text{ months}) \leq 2]$

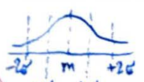
Functions of a RV

1. linear: $Y = aX + b \longrightarrow f_Y = \frac{1}{|a|} f_x\left(\frac{y-b}{a}\right)$
2. Square: $Y = X^2 \longrightarrow f_Y = \frac{1}{2\sqrt{y}} f_x(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_x(-\sqrt{y})$
3. General: $Y = g(x)$



$f_Y(y) = \frac{f_x(x_1)}{\left|\frac{dy}{dx}\right|_{x_1}} + \frac{f_x(x_2)}{\left|\frac{dy}{dx}\right|_{x_2}}$

* For Gaussian:



Prob of error

$= P[|x-m| > 2\sigma]$

$= 2P[x-m > 2\sigma]$

$= 2Q\left(\frac{(m+2\sigma)-m}{\sigma}\right)$

Prob of Confidence $= 1 -$ Prob of error

$\sin^2\theta + \cos^2\theta = 1$

$\cos^{-1} y = \theta$

$y = \cos\theta$

$\sin \cos^{-1} y = \sin\theta$

$= \sqrt{1 - y^2}$

[1] Markov Inequality

$$\boxed{P[X \geq a] \leq \frac{E[X]}{a}}$$ $X \geq 0$, $a > 0$

[2] Chebychev Inequality

$$\boxed{P[|X-m| \geq a] \leq \frac{\sigma^2}{a^2}}$$ X RV, m, σ^2

★ Transform Methods

charac. fn of X: $\varnothing_X(\omega) = E[e^{j\omega x}]$

→ Continuous:

$$\varnothing_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varnothing_X(\omega)\, e^{-j\omega x}\, dx$$

→ Discrete:

$$\varnothing_X(\omega) = \sum_x e^{j\omega x} P_X(x)$$

$$\boxed{E[X^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \varnothing_X(\omega) \Big|_{\omega=0}}$$ $\varnothing_X(0) = 1$

★ Prob. Generating fn (PGF)

z-transform

✱ Z discrete non-neg. RV

$$\boxed{G_N(z) = E[z^k] = \sum_{k=0}^{\infty} z^k P_N(k)}$$ $G_N(0) = P_N(0)$, $G_N(1) = 1$

$$\boxed{P_N(k) = \frac{1}{k!} \frac{d^k}{dz^k} G_N(z) \Big|_{z=0}}$$

↳ $\frac{d}{dz} G_N(z) \Big|_{z=1} = E[N]$

↳ $\frac{d^2}{dz^2} G_N(z) \Big|_{z=1} = E[N^2] - E[N]$

✱ X continuous non-neg RV — Laplace

$$\boxed{X^*(s) = E[e^{-sx}] = \int_0^{\infty} e^{-sx} f_X(x)\, dx}$$

$$\boxed{E[X^n] = (-1)^n \frac{d^n}{ds^n} X^*(s) \Big|_{s=0}}$$ $f_X(x) = \mathcal{L}^{-1}(X^*(s))$

RV Pairs

✱ PMF:

→ joint: $\boxed{P_{XY}(x_j, y_h) = P[X = x_j, Y = y_h]}$

$P[B] = \sum_{x,y \in B} P_{XY}(x,y)$

$\sum_{all} \sum_{all} P_{XY}(x,y) = 1$

→ marginal: $P_X(x_j) = \sum_{all\ y} P_{XY}(x_j, y)$

$P_Y(y_h) = \sum_{all\ x} P_{XY}(x, y_h)$

✱ CDF:

→ joint

$$\boxed{F_{XY}(x_1,y_1) = P[X \leq x_1, Y \leq y_1]}$$

→ marginal

$$F_X(x) = F_{XY}(x, +\infty) = P[X \leq x, Y < \infty]$$

$$F_Y(y) = F_{XY}(+\infty, y) = P[X \leq \infty, Y \leq y]$$

✱ PDF

→ joint

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

$$F_{XY}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(x,y)\, dx\, dy$$

→ marginal

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$$

✱ Independence:

Cont: $f_{XY}(x,y) = f_X(x) f_Y(y)$

$F_{XY}(x,y) = F_X(x) F_Y(y)$

Disc: $P_{XY}(x,y) = P_X(x) P_Y(y)$

$E[X+Y] = E[X] + E[Y]$

$E[XY] = E[X]E[Y]$ if X, Y independent

$$E[g(x)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x) f_{XY}(x,y)\, dx\, dy$$

✱ Correlation of X, Y $= E[XY]$ → = 0 if X, Y orthogonal

✱ $\mathrm{Cov}(X,Y) = E((X - E(X))(Y - E(Y))) = E(XY) - E(X)E(Y)$ → = 0 if X, Y uncorrelated

↳ $\mathrm{Cov}(X,X) = E((X - E(X))^2) = \mathrm{Var}(X)$

↳ If X, Y independent, ∴ $\mathrm{Cov}(X,Y) = 0$ "uncorr."

✱ Corr. Coeff.: $\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$ $|\rho| \leq 1$

✱ Cond. Prob.

(A) X, Y discrete

$$P_Y(y|x) = \frac{P_{XY}(x,y)}{P_X(x)}$$

$$P[Y \text{ in } A] = \sum_x P[Y \text{ in } A \mid X = x]\, P_X(x) = \sum_x \left(\sum_{y \in A} P_Y(y \mid X = x)\right) P_X(x) = \sum_{all\ x} \sum_{y \in A} P_{XY}(x,y)$$

(B) X Discrete, Y continuous

$$F_Y(y|x) = P(Y \leq y \mid X = x)$$

$$f_Y(y|x) = \frac{d}{dy} F_Y(y|x)$$

$$\boxed{P[Y \text{ in } A \mid X = x] = \int_{y \in A} f_Y(y \mid X = x)\, dy}$$

(C) X, Y Continuous

$$F_Y(y|x) = \lim_{h \to 0} P[Y \leq y \mid x \leq X \leq x + h]$$

$$f_Y(y|x) = \frac{d}{dy} F_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$\boxed{P[Y \text{ in } A \mid x] = \int_{y \in A} f_Y(y|x)\, dy}$$

$$P[Y \text{ in } A] = \int_{y \in A} \int_{-\infty}^{\infty} \underbrace{f_{XY}(x,y)}_{f_Y(y|x) f_X(x)}\, dx\, dy = \int_{-\infty}^{\infty} f_X(x) \int_{y \in A} f_Y(y|x)\, dy\, dx$$

$$\boxed{P[Y \text{ in } A] = \int_{-\infty}^{\infty} P[Y \text{ in } A \mid X = x]\, f_X(x)\, dx}$$

$$\boxed{E[Y|X] = \int_{-\infty}^{\infty} y\, f_Y(y|x)\, dy}$$ → fn. of x

$E[E[Y|X]] = E[Y]$

$E[E[g(Y)|X]] = E[g(Y)]$

$F_Y(y) = P[Y \leq y] = \int_0 P[Y \leq y \mid X = x]\, f(x)\, dx$

★ $Z = g(x,y)$

$$F_Z(z) = P[Z \leq z] = P[g(x,y) \leq z] = \iint_R f_{XY}(x,y)\, dx\, dy$$

$$f_Z(z) \neq \frac{d}{dz} F_Z(z)$$

e.g. $x^2 + y^2 \leq z$, $\sqrt{z}$

ex: $Z = X + Y$

$\frac{d}{dz} \int_{-\infty}^{g(z)} f(x)\, dx = g'(z) f(g(z))$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z-y, y)\, dy$$

if X, Y independent → $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx = f_X(x) * f_Y(y)$

ex: $Z = \frac{X}{Y}$ → $f_Z(z|y) = \frac{f_X(yz)}{|\frac{1}{y}|}$

$$f_Z = \int_{-\infty}^{\infty} \underbrace{f_Z(z|y) f_Y(y)}_{f_{YZ}(y,z)}\, dy$$

✱ X, Y jointly Gaussian

$X \sim N(m_1, \sigma_1)$, $Y \sim N(m_2, \sigma_2)$

$$f_{XY} = \frac{\exp\left[\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right) + \left(\frac{y-m_2}{\sigma_2}\right)^2\right]\right]}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}}$$

$$X|y \sim N\left(m_1 + \rho \frac{\sigma_1}{\sigma_2}(y - m_2),\ \sigma_1^2(1-\rho^2)\right)$$

★ $F_{VW}(v,w) = P(V \leq v, W \leq w) = P[g_1(x,y) \leq v, g_2(x,y) \leq w] = \iint_R^{vw} f_{XY}(x,y)\, dx\, dy$

★ Linear

$$\begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ c & e \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$f_{VW}(v,w) = \frac{f_{XY}(x,y)}{\left|\frac{\partial(v,w)}{\partial(x,y)}\right|}$$

$$= \det\begin{bmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = \frac{\partial v}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial x}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{ae - bc} \begin{bmatrix} e & -b \\ -c & a \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix}$$

✱ Trig:

$\cos^2\theta + \sin^2\theta = 1$

$\sin 2\theta = 2 \sin\theta \cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

$\sec = \frac{1}{\cos}$, $\operatorname{cosec} = \frac{1}{\sin}$

$\sin X \sin Y = \frac{1}{2}(\cos(X-Y) - \cos(X+Y))$

$\cos X \cos Y = \frac{1}{2}(\cos(X-Y) + \cos(X+Y))$

$\cos X \sin Y = \frac{1}{2}(\sin(X+Y) - \sin(X-Y))$

✱ Derivatives:

$\frac{d}{dx} F(x)^n = n\, F(x)^{n-1} F'(x)$

$\frac{d}{dx} FG = F\,dG + G\,dF$

$\frac{d}{dx} e^F = F' e^F$

$\frac{d}{dx} \ln F = \frac{F'}{F}$

$d \sin x \longrightarrow \cos x$

$d \cos x \longrightarrow -\sin x$

$d \tan x \longrightarrow \sec^2 x$

$d \sec x \longrightarrow \sec x \tan x$

$d \operatorname{cosec} x \longrightarrow -\operatorname{cosec} x \cot x$

$d \cot x \longrightarrow -\operatorname{cosec}^2 x$

$\int$

✱ Integrals:

$\int dx = x$	$\int x^n dx = \frac{1}{n+1} x^{n+1}$
$\int \frac{1}{x} dx = \ln x$	$\int x^{-n} dx = \frac{1}{-n+1} x^{-n+1}$
$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln\lvert ax+b\rvert$	$\int e^{nx} dx = \frac{1}{n} e^{nx}$

$\int_{-\infty}^{\infty}$ even function $= 2\int_0^{\infty}$ even function, $\int_{-\infty}^{\infty}$ odd function $= 0$

$$\sum_{j=0}^{n} \binom{n}{j} a^j = (1+a)^n$$