Decimation in Time (DIT) processing

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-\frac{j2\pi kn}{N}}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[2n] \cdot e^{-\frac{j2\pi k(2n)}{N}} + \sum_{n=0}^{\frac{N}{2}-1} x[2n+1] \cdot e^{-\frac{j2\pi k(2n+1)}{N}}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[2n] \cdot e^{-\frac{j2\pi kn}{(N/2)}} + \sum_{n=0}^{\frac{N}{2}-1} x[2n+1] \cdot e^{-\frac{j2\pi kn}{(N/2)}} \cdot e^{-\frac{j2\pi k}{N}}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[2n] \cdot e^{-\frac{j2\pi kn}{N/2}} + \left(e^{-\frac{j2\pi k}{N}} \right) \cdot \sum_{n=0}^{\frac{N}{2}-1} x[2n+1] \cdot e^{-\frac{j2\pi kn}{N/2}}$$

$$Let\ \ y[n] = \left[x[0] \ \ x[2] \right] \quad \& \quad z[n] = \left[x[1] \ \ x[3] \right]$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \underbrace{y[n]}_{even\ samples} \cdot e^{-\frac{j2\pi kn}{N/2}} + \left(e^{-\frac{j2\pi k}{N}} \right) \cdot \sum_{n=0}^{\frac{N}{2}-1} \underbrace{z[n]}_{odd\ samples} \cdot e^{-\frac{j2\pi kn}{N/2}}$$

$$X[k]=\sum_{n=0}^{\frac{N}{2}-1}\underbrace{y[n]}_{even\ samples}\cdot e^{-\frac{j2\pi kn}{N/2}}+\left(e^{-\frac{j2\pi k}{N}}\right)\cdot\sum_{n=0}^{\frac{N}{2}-1}\underbrace{z[n]}_{odd\ samples}\cdot e^{-\frac{j2\pi kn}{N/2}}\qquad\leftarrow[k=0\ \&\ 1]$$

$$=\left[\sum_{n=0}^{\frac{N}{2}-1}y[n]\cdot e^{-\frac{j2\pi kn}{N/2}}\right]+\left[\left(e^{-\frac{j2\pi k}{N}}\right)\cdot\sum_{n=0}^{\frac{N}{2}-1}z[n]\cdot e^{-\frac{j2\pi kn}{N/2}}\right]$$

$$=Y[k]\ +\ \left(e^{-\frac{j2\pi k}{N}}\right)\cdot Z[k]$$

$$X\left[k+\frac{N}{2}\right]=\sum_{n=0}^{\frac{N}{2}-1}\underbrace{y[n]}_{even\ samples}\cdot e^{-\frac{j2\pi\left(k+\frac{N}{2}\right)n}{N/2}}\ +\ \left(e^{-\frac{j2\pi\left(k+\frac{N}{2}\right)}{N}}\right)\cdot\sum_{n=0}^{\frac{N}{2}-1}\underbrace{z[n]}_{odd\ samples}\cdot e^{-\frac{j2\pi\left(k+\frac{N}{2}\right)n}{N/2}}$$

$$=\sum_{n=0}^{\frac{N}{2}-1}y[n]\cdot e^{-\frac{j2\pi kn}{N/2}}\cdot e^{-\frac{j2\pi\left(\frac{N}{2}\right)}{N/2}}\ +\ \left(e^{-\frac{j2\pi k}{N}}\cdot e^{-\frac{j2\pi\left(\frac{N}{2}\right)}{N}}\right)\cdot\sum_{n=0}^{\frac{N}{2}-1}z[n]\cdot e^{-\frac{j2\pi kn}{N/2}}\cdot e^{-\frac{j2\pi\left(\frac{N}{2}\right)}{N/2}}$$

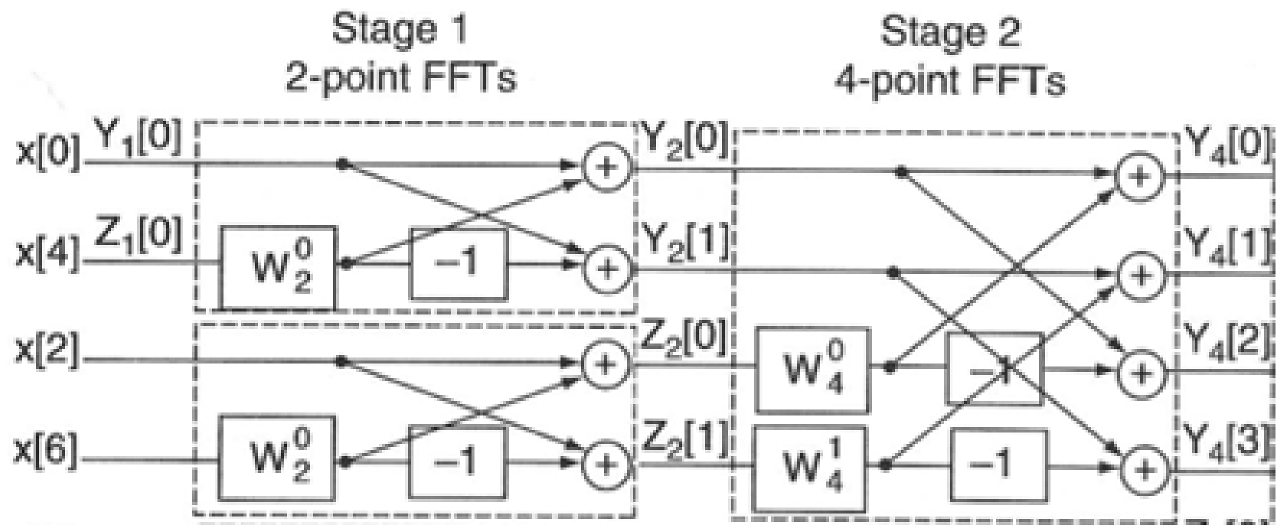
$$=\sum_{n=0}^{\frac{N}{2}-1}y[n]\cdot e^{-\frac{j2\pi kn}{N/2}}\cdot e^{-\ j2\pi}\ +\ \left(e^{-\frac{j2\pi k}{N}}\cdot e^{-\ j\pi}\right)\cdot\sum_{n=0}^{\frac{N}{2}-1}z[n]\cdot e^{-\frac{j2\pi kn}{N/2}}\cdot e^{-\ j2\pi}$$

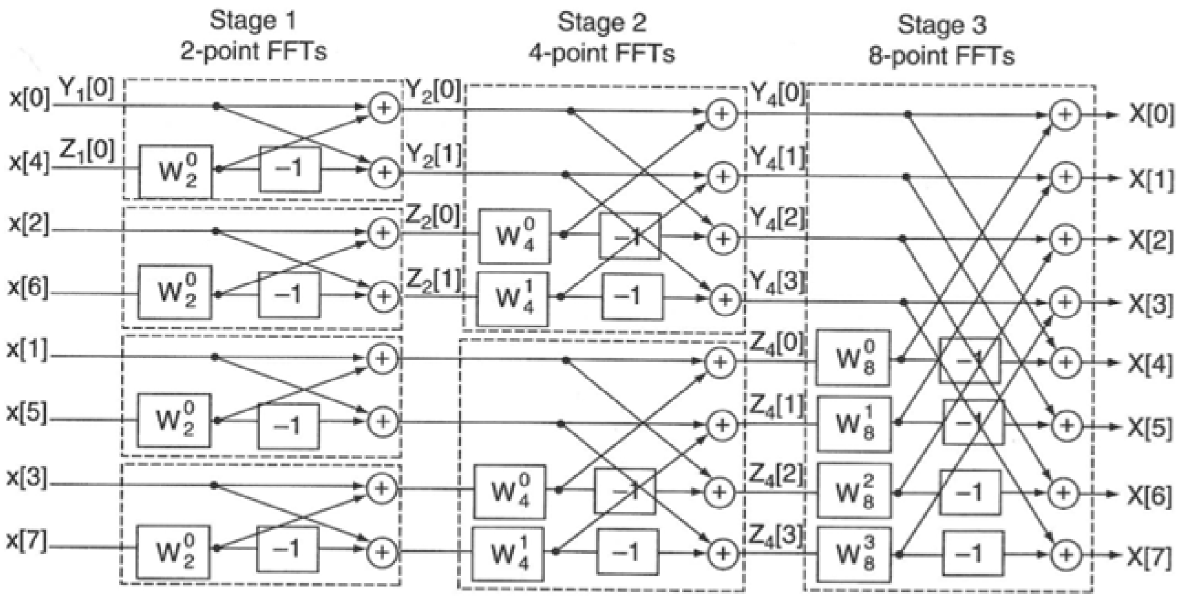
$$=\sum_{n=0}^{\frac{N}{2}-1}y[n]\cdot e^{-\frac{j2\pi kn}{N/2}}\cdot(1)\ +\ \left(e^{-\frac{j2\pi k}{N}}\cdot(-j)\right)\cdot\sum_{n=0}^{\frac{N}{2}-1}z[n]\cdot e^{-\frac{j2\pi kn}{N/2}}\cdot(1)$$

$$=Y[k]-j\left(e^{-\frac{j2\pi k}{N}}\right)\cdot Z[k]$$

$Assume\ x[n]=[1\ \ 2\ \ 3\ \ 4]\ then\ y[n]=[1\ \ 3]\ \ and\ \ z[n]=[2\ \ 4]$

$$\left[\begin{aligned}
&X[k]=Y[k]\ +\ \left(e^{-\frac{j2\pi k}{N}}\right)\cdot Z[k]\\
&X[k=0]=Y[0]\ +\ \left(e^{-\frac{j2\pi\cdot 0}{N}}\right)\cdot Z[0]\\
&\qquad\qquad=\sum_{n=0}^{\frac{N}{2}-1} y[n]\cdot e^{-\frac{j2\pi 0\cdot n}{N/2}}+\left(e^{-\frac{j2\pi\cdot 0}{N}}\right)\cdot\sum_{n=0}^{\frac{N}{2}-1} z[n]\cdot e^{-\frac{j2\pi 0\cdot n}{N/2}}\\
&\qquad\qquad=3+6=10
\end{aligned}\right.$$





$* \, W_N^k = e^{-j\frac{2\pi k}{N}}$

The Fast Fourier Transform (FFT)

- The **FFT** is an algorithm or a procedure with which the **discrete Fourier transform** can be computed using far fewer calculations.
  - Decimation in time algorithm
  - Decimation in frequency algorithm

- DFT requires $\left[N^2\right]$ complex multiply and $\left[(N-1)\cdot N\right]$ add operations
- FFT needs only approximately $\left[\frac{N}{2}\log_2 N\right]$ operations.

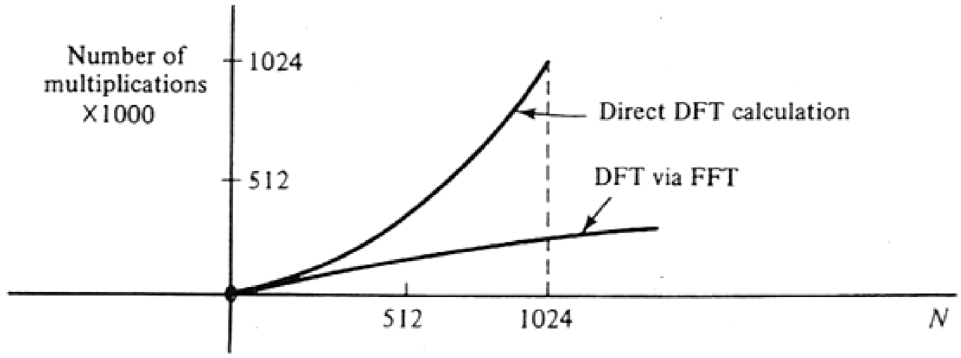


Figure 9-11 Computation for direct DFT and DFT via FFT.

Number of stages, ν	Number of points, N	Number of complex multiplications using direct calculation, N^2	Number of complex multiplications using Cooley–Tukey FFT algorithm, $(N/2)\log_2 N$	Times faster than direct evaluation, $R = N^2/((N/2)\log_2 N)$
2	4	16	4	4
3	8	64	12	5.333
4	16	256	32	8
5	32	1,024	80	12.8
6	64	4,096	192	21.33
7	128	16,384	448	36.57
8	256	65,536	1024	64
9	512	262,144	2304	113.77
10	1024	1,048,576	5120	204.8