Chapter 6

FFT
Decimation in Frequency

Spring

2009

Decimation in Frequency

The discrete Fourier transform can be found using

$$X(k)=\sum_{n=0}^{N-1} x(n)W_N^{kn} \qquad k=0,1,2,\ldots\ldots,N-1$$

Where N = 2, 4, 8, 16,... and

$$W_N = e^{-j2\pi/N}$$

$X(k)$ can be expressed as

$$X(k)=\sum_{n=0}^{N/2-1} x(n)W_N^{kn} + \sum_{n=N/2}^{N-1} x(n)W_N^{kn}$$

$$X(k)=\sum_{n=0}^{N/2-1} x(n)W_N^{kn} + \sum_{n=0}^{N/2-1} x(n+N/2)W_N^{k(n+N/2)}$$

$$X(k)=\sum_{n=0}^{N/2-1} x(n)W_N^{kn} + W_N^{kN/2}\sum_{n=0}^{N/2-1} x(n+N/2)W_N^{kn}$$

Decimation in Frequency

But

$$W_N^{(N/2)k} = e^{2\pi k(N/2)/N} = e^{\pi k} = (-1)^k$$

Then

$$X(k) = \sum_{n=0}^{N/2-1} x(n)W_N^{kn} + (-1)^k \sum_{n=0}^{N/2-1} x(n+N/2)W_N^{kn}$$

If $k = 2m$ or an even number

$$X(2m) = \sum_{n=0}^{N/2-1} \big(x(n) + x(n+N/2)\big)W_N^{2mn}$$

$$X(2m) = \sum_{n=0}^{N/2-1} a(n)W_{N/2}^{mn}$$

Decimation in Frequency

Noting That

$$W_N^{2mn} = e^{-j2mn\times 2\pi/N} = e^{-j2\pi mn/(N/2)} = W_{N/2}^{mn}$$

$$a(n) = x(n) + x(n+N/2)$$

Then *X(2m)* is *N/2*-point DFT for *a(n)*

If $k = 2m+1$ (odd number) and using the same method

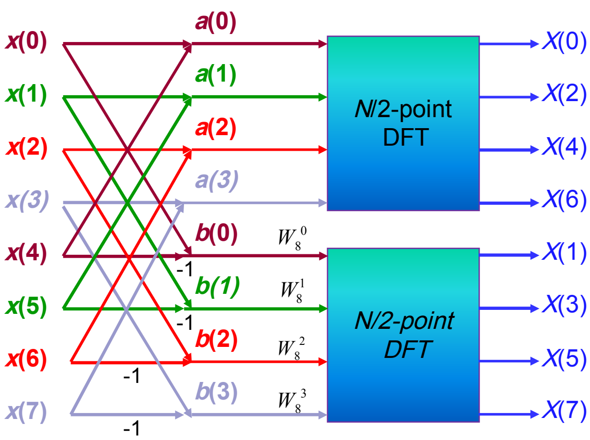
$$X(2m+1) = \sum_{n=0}^{N/2-1} \big(x(n) - x(n+N/2)\big)W_N^{n}W_N^{2mn}$$

$$X(2m+1) = \sum_{n=0}^{N/2-1} \big(b(n)W_N^{n}\big)W_{N/2}^{mn}$$

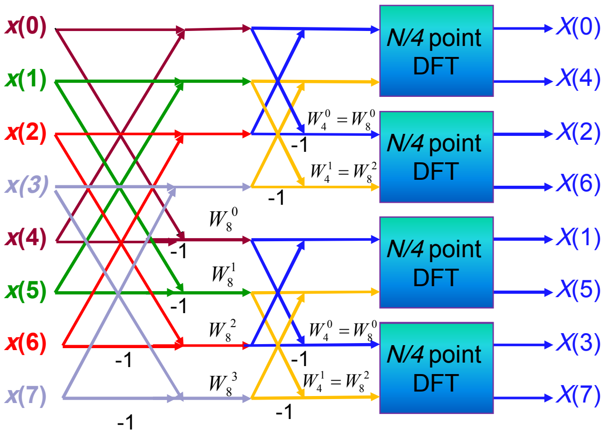
$$b(n) = x(n) - x(n+N/2)$$

X(2m+1) is *N/2*-point DFT for $b(n)W_N^n$

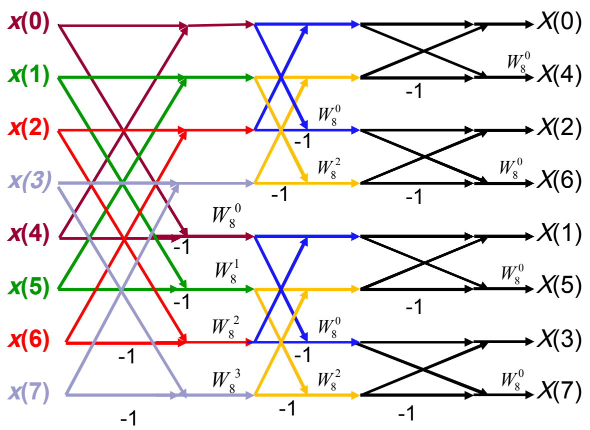
Decimation in Frequency



Decimation in Frequency



Decimation in Frequency



Decimation in Frequency

Using the previous algorithm , the complex multiplications needed is only 12. While using the normal DFT would require 64 complex multiplications

In general

Complex multiplication of DFT is: N^2

Complex multiplication of FFT is $(N/2)\log_2(N)$

If $N = 1024$

Complex multiplication of DFT is: 1,048,576

Complex multiplication of FFT is: 5,120

Decimation in Frequency

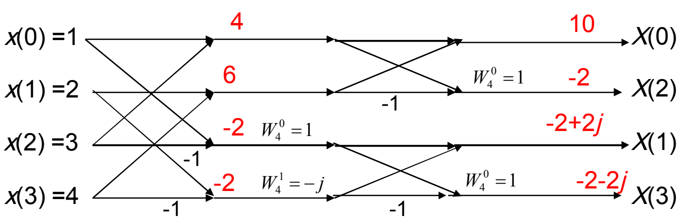
Index mapping for Fast Fourier Transform

Input Data index *n*	Index Bits	Reversal Bits	Output data index *k*
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Decimation in Frequency

Example Given a sequence *x(n)* where *x*(0) = 1, *x*(1) = 2, *x*(2) = 3, *x*(3) = 4 and *x*(n) = 0 elsewhere ,find DFT for the first four points

solution



Inverse Fourier Transform

The inverse discrete Fourier transform can be found using

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad n = 0,1,2,\ldots\ldots, N-1$$

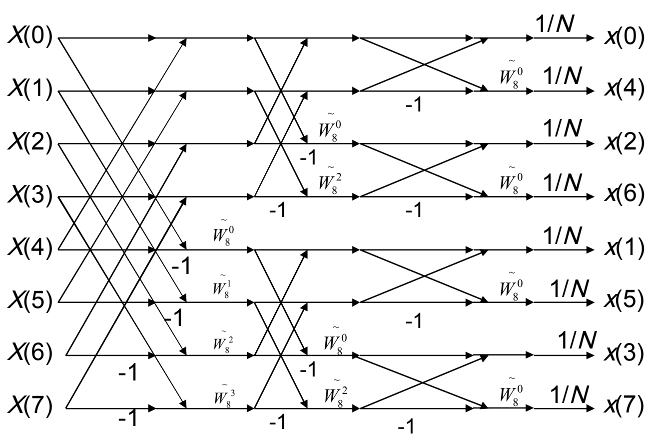
Which can be expressed as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) \tilde{W}_N^{kn} \qquad n = 0,1,2,\ldots\ldots, N-1$$

where $\tilde{W}_N^{kn} = W_N^{-kn}$ is called the twiddled factor

We can see that the difference between the inverse discrete Fourier and forward Fourier transform is the twiddled factor and the division by 1/N

Inverse Fourier Transform



Inverse Fourier Transform

Example Given a sequence *X(n)* given in the previous example. Find the IFFT using decimation in frequency method

solution