1. Two signals, $x_1(t)$ and $x_2(t)$, are defined as follows,

$$x_1(t)=\begin{cases}1 & -1\le t\le 0\\ 2 & 0\le t\le 1\\ 0 & \text{elsewhere}\end{cases}, \qquad x_2(t)=\begin{cases}2 & -2\le t\le 1\\ 1 & 1\le t\le 2\\ 0 & \text{elsewhere}\end{cases}$$

and are shown in Figure 1. Perform the convolution $y(t)=x_1(t)*x_2(t)$. Plot the result $y(t)$ in the time domain and, using MATLAB, in the frequency domain ($\mathcal{F}\{y(t)\}$) (both magnitude & phase plots).

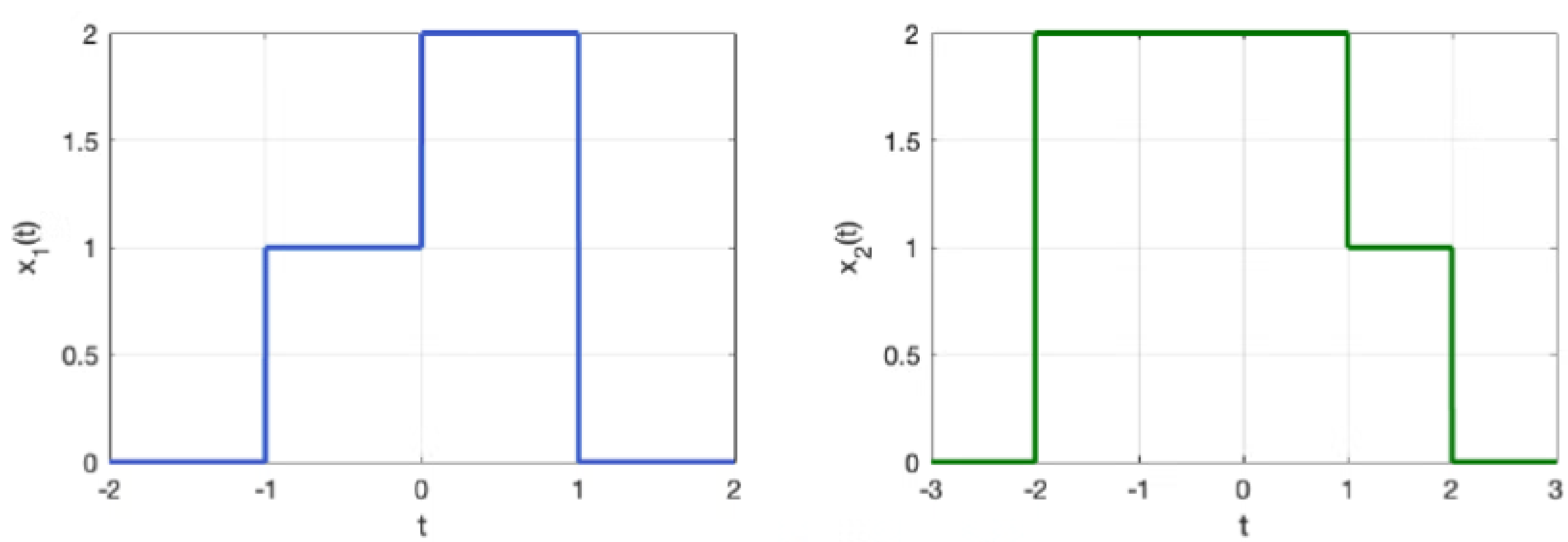
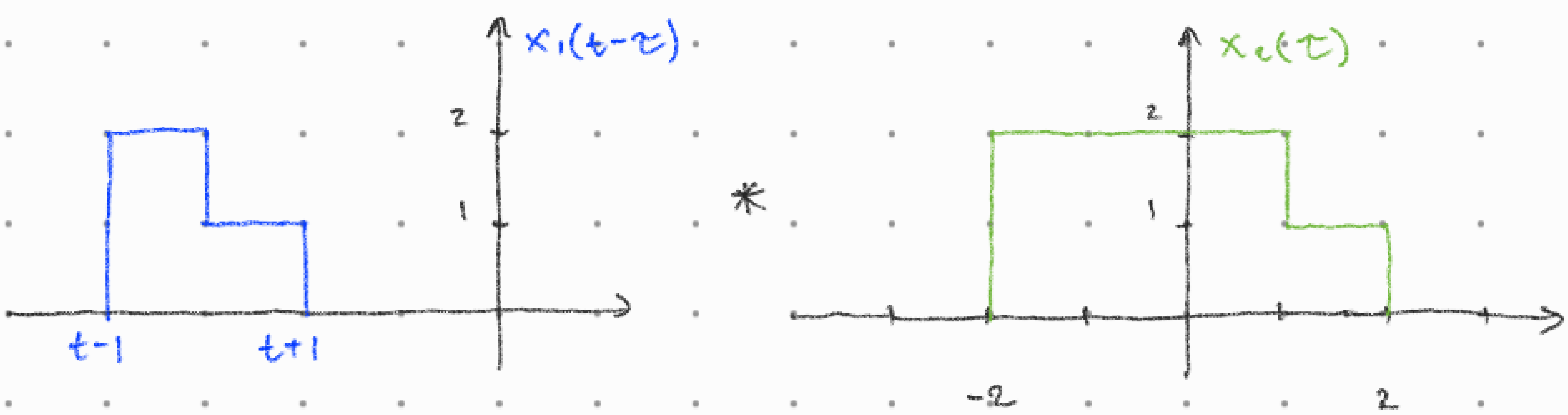
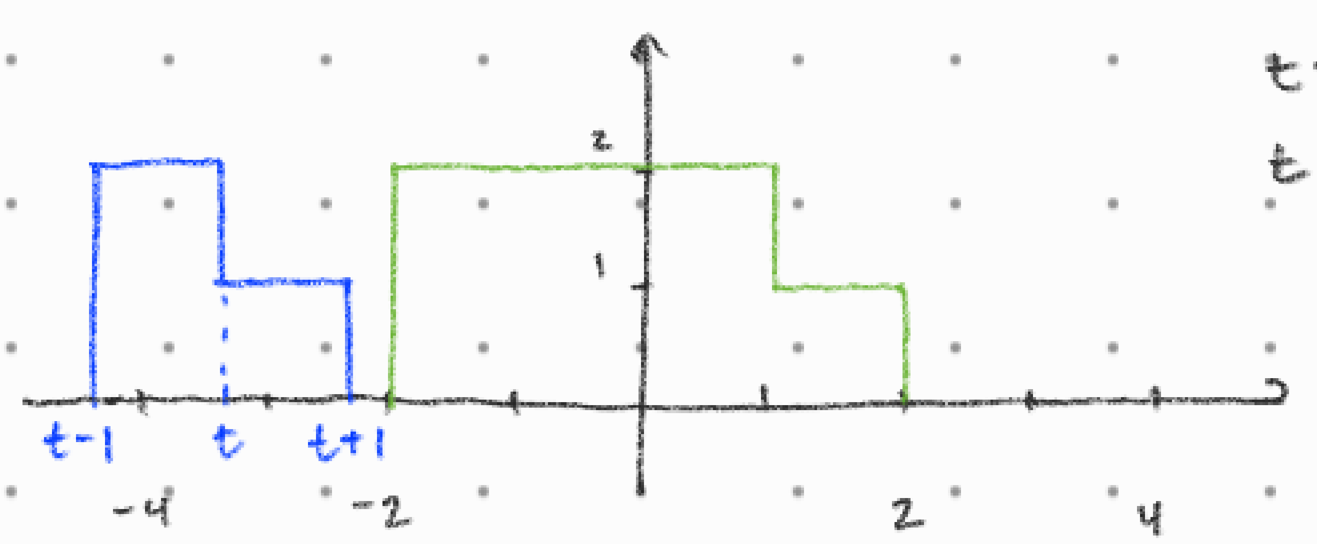


Figure 1: $x_1(t)$ and $x_2(t)$ for Problem 1.



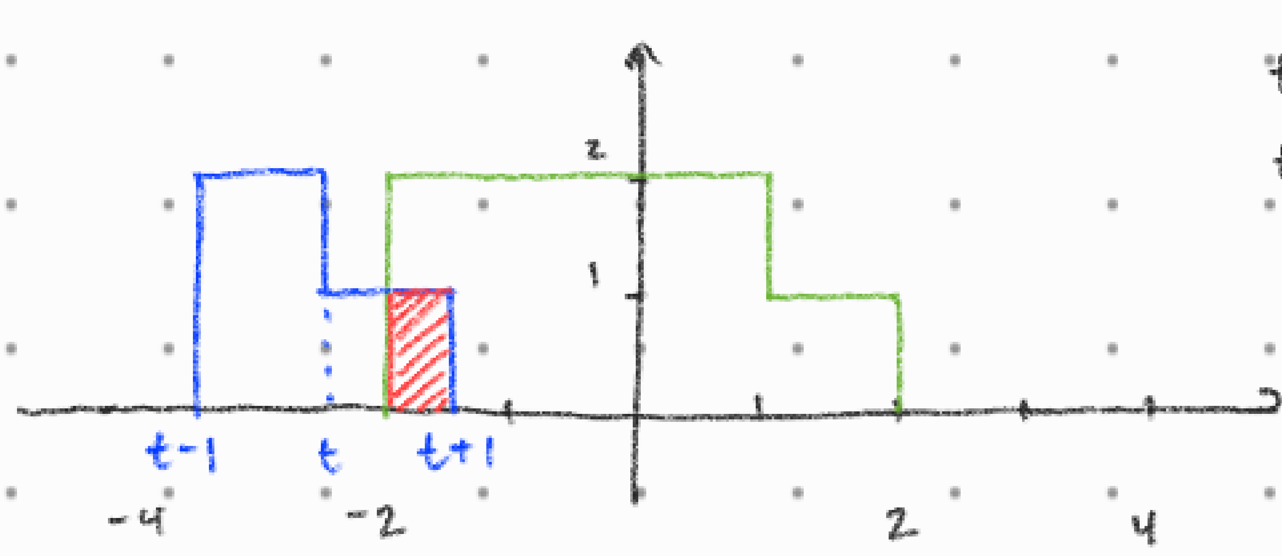


$t+1<-2$

$t<-3$

$y(t)=0$

no overlap.



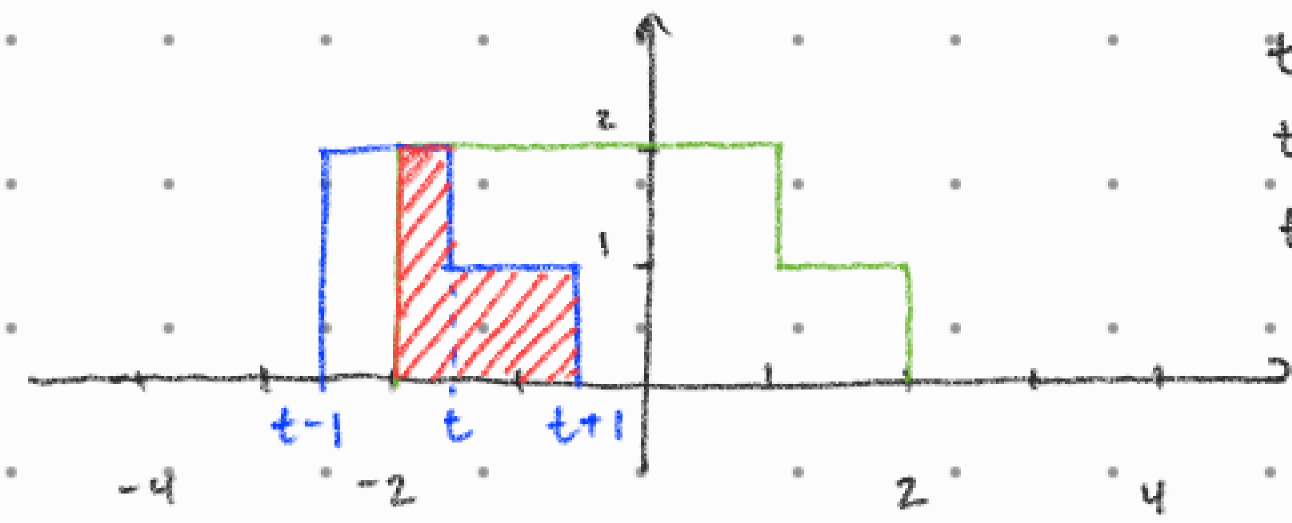
$t>-3$

$t<-2$

$$y(t)=\int_{\tau=-2}^{t+1}(2)(1)\,d\tau$$

$$=2[t+1-(-2)]$$

$$=2t+6$$



$t > -2$

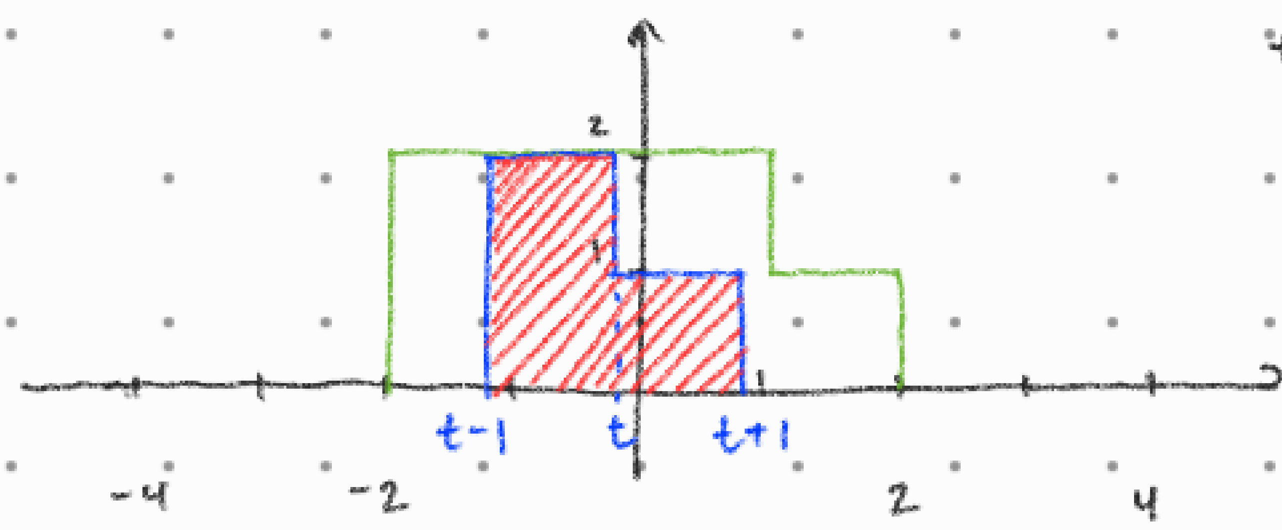
$t-1 < -2$

$t < -1$

$$y(t) = \int_{\tau=t}^{t+1} (2)(1)\, d\tau + \int_{\tau=-2}^{t} (2)(2)\, d\tau$$

$$= 2[\cancel{t}+1-\cancel{t}] + 4[t-(-2)]$$

$$\boxed{= 4t+10}$$



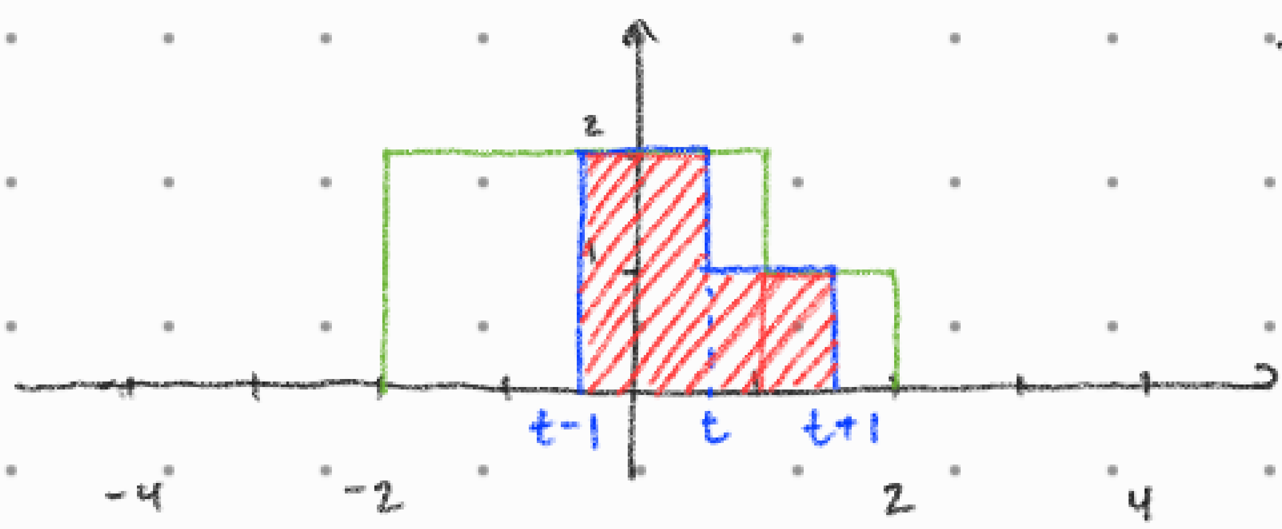
$t > -1$

$t < 0$

$$y(t) = \int_{\tau=t}^{t+1} (2)(1)\, d\tau + \int_{\tau=t-1}^{t} (2)(2)\, d\tau$$

$$= 2[\cancel{t}+1-\cancel{t}] + 4[\cancel{t}-\cancel{t}+1]$$

$$\boxed{= 6}$$



$t > 0$

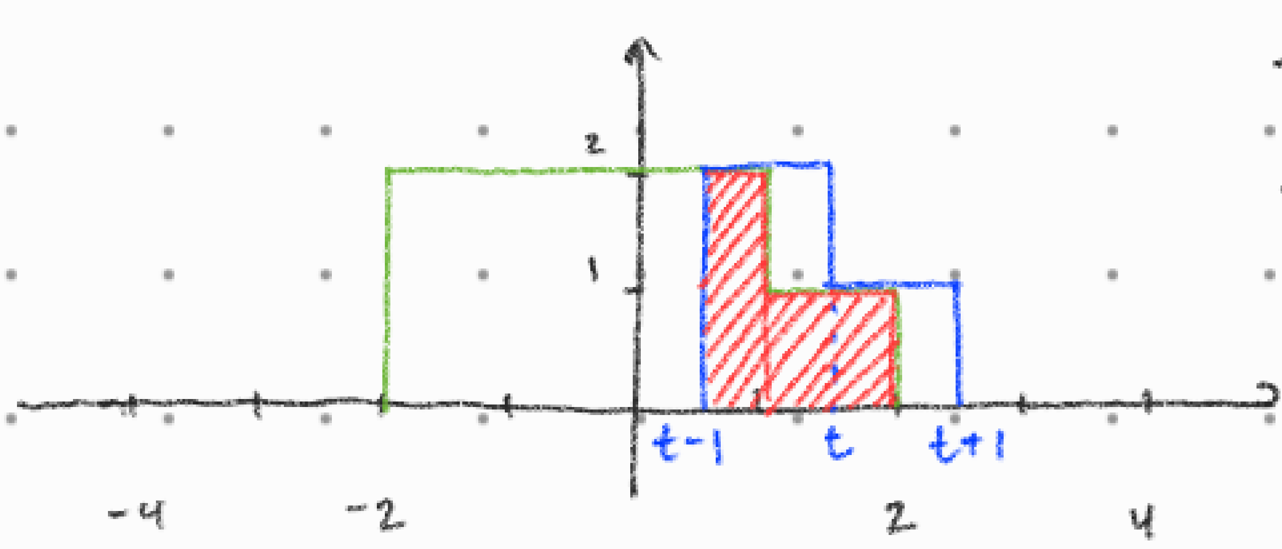
$t < 1$

$$y(t) = \int_{\tau=1}^{t+1} (1)(1)\, d\tau + \int_{\tau=t}^{1} (1)(2)\, d\tau + \int_{\tau=t-1}^{t} (2)(2)\, d\tau$$

$$= [t+\cancel{1}-\cancel{1}] + 2[1-t] + 4[\cancel{t}-\cancel{t}+1]$$

$$= t + 2 - 2t + 4$$

$$\boxed{= -t+6}$$



$t > 1$

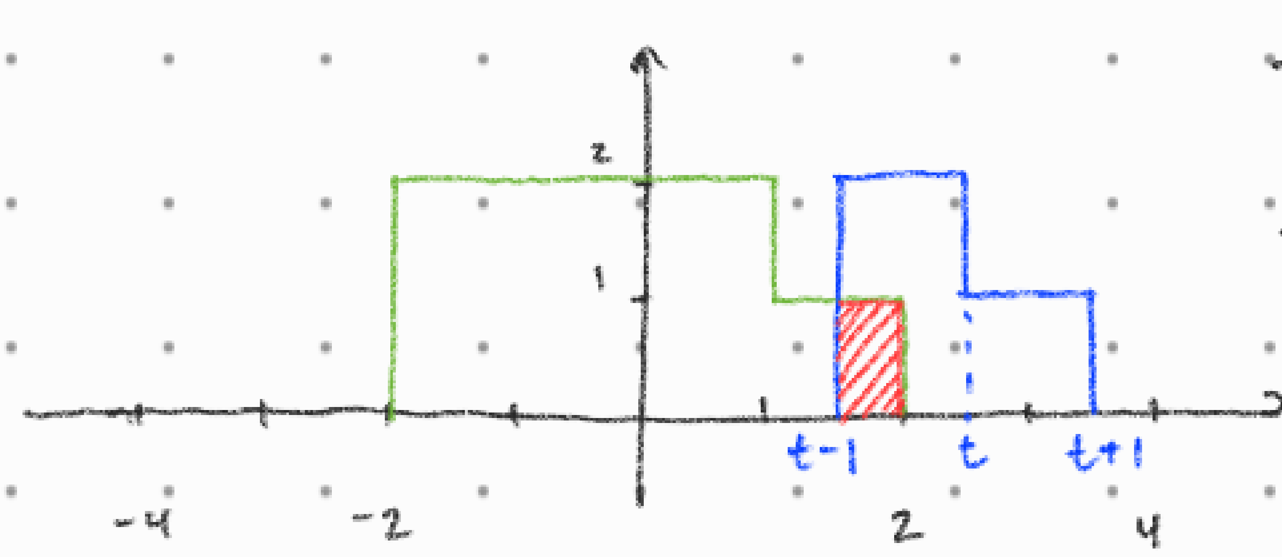
$t < 2$

$$y(t) = \int_{\tau=t}^{2} (1)(1)\, d\tau + \int_{\tau=1}^{t} (2)(1)\, d\tau + \int_{\tau=t-1}^{1} (2)(2)\, d\tau$$

$$= [2-t] + 2[t-1] + 4[1-t+1]$$

$$= \cancel{2} - t + 2t - \cancel{2} + 8 - 4t$$

$$\boxed{= -3t+8}$$



$t > 2$

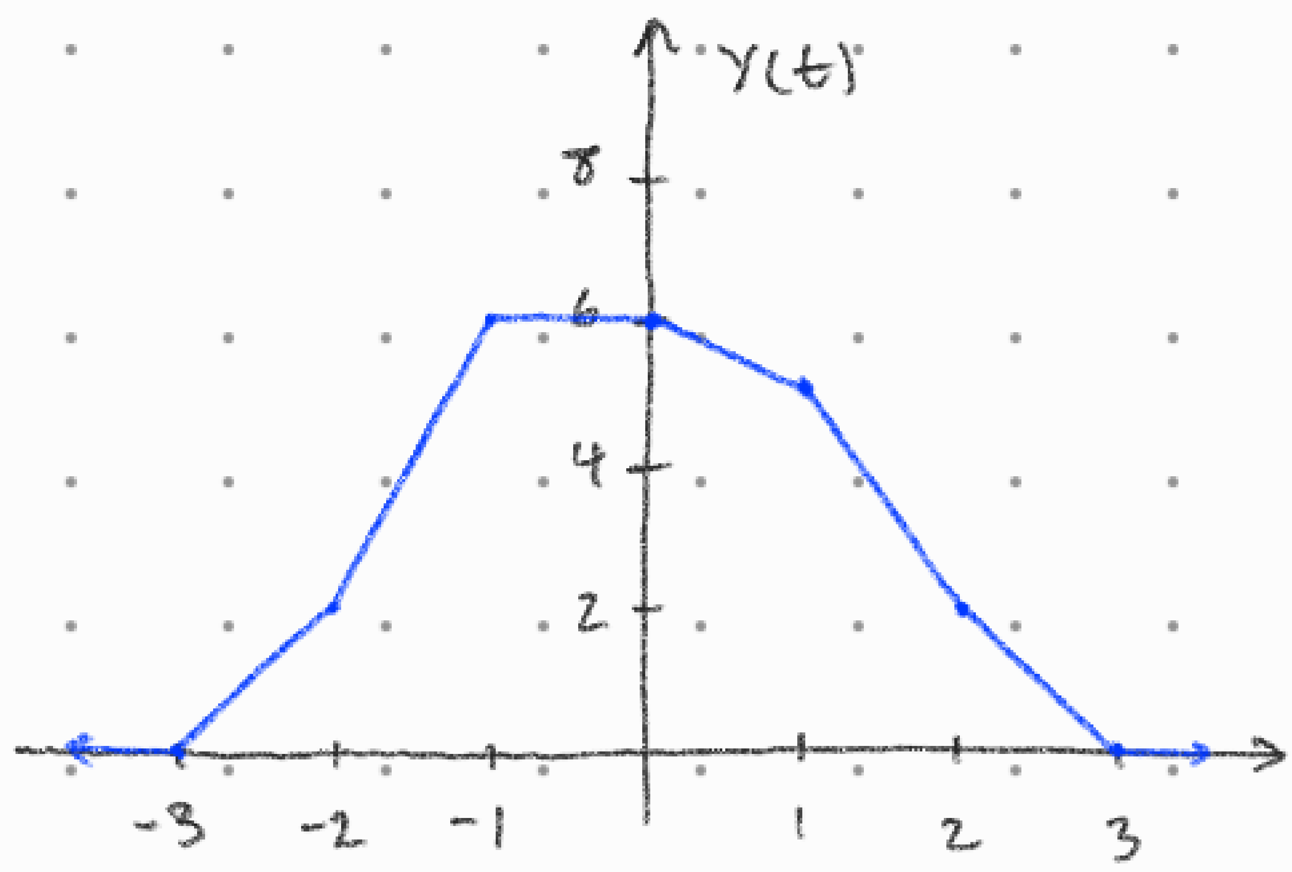
$t-1 < 2$

$t < 3$

$$y(t) = \int_{\tau=t-1}^{2} (2)(1)\, d\tau$$

$$= 2[2-t+1]$$

$$\boxed{= -2t+6}$$



$t < -3$	$y(t) = 0$
$-3 < t < -2$	$= 2t + 6$
$-2 < t < -1$	$= 4t + 10$
$-1 < t < 0$	$= 6$
$0 < t < 1$	$= -t + 6$
$1 < t < 2$	$= -3t + 8$
$2 < t < 3$	$= -2t + 6$

2. Two signals, $x_3(t)$ and $x_4(t)$, are defined as follows,

$$x_3(t)=\begin{cases} t^2 & -2\le t\le 0 \\ 0 & \text{elsewhere} \end{cases}, \qquad x_4(t)=\begin{cases} -3t+15 & 4\le t\le 5 \\ 0 & \text{elsewhere} \end{cases}$$

and are shown in Figure 2. Perform the convolution $x_3(t)*x_4(t)$ and plot the result in the time domain only.

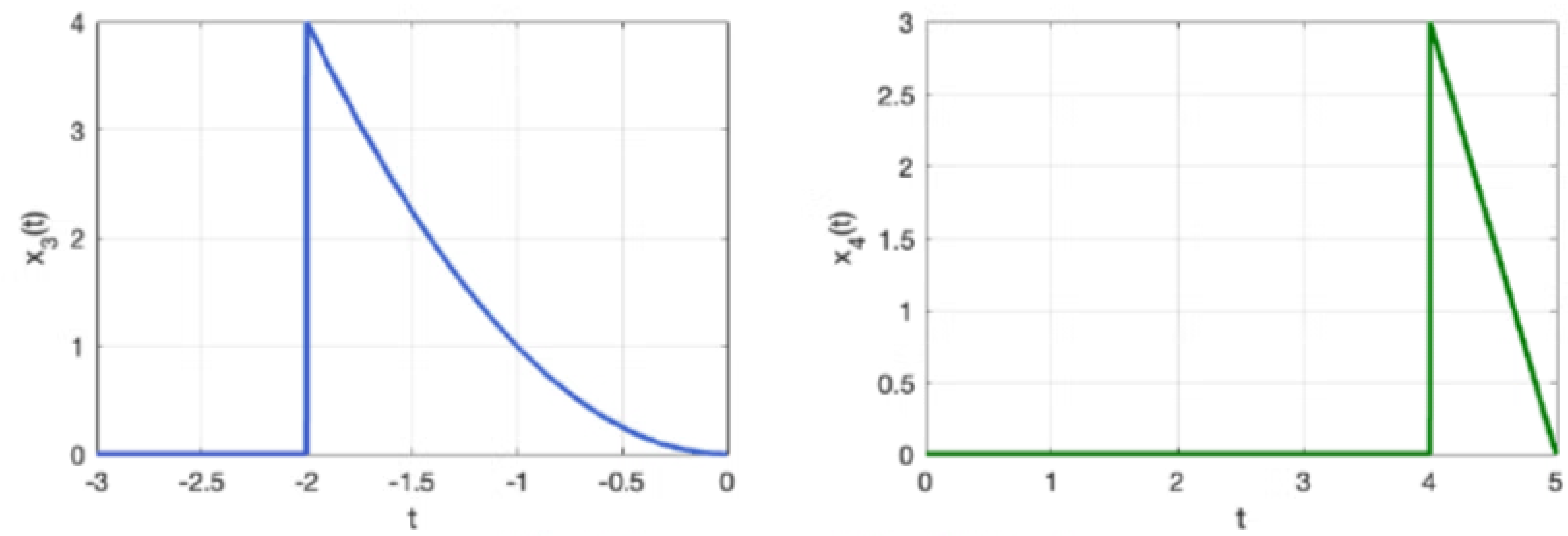
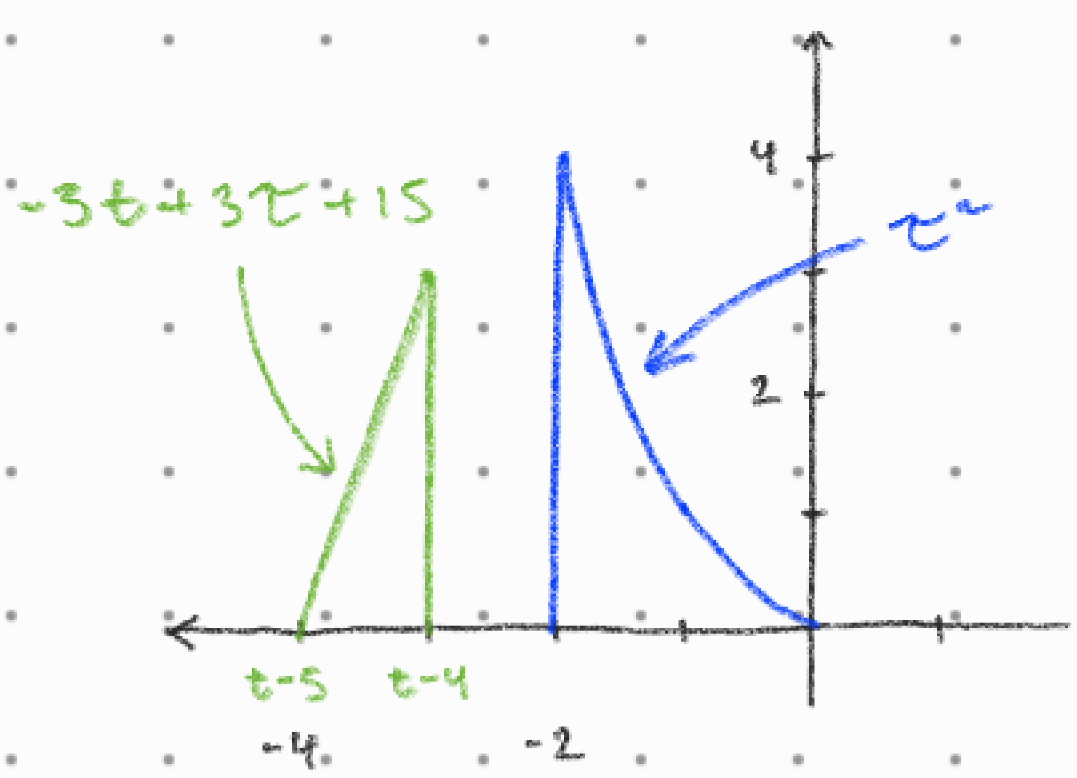


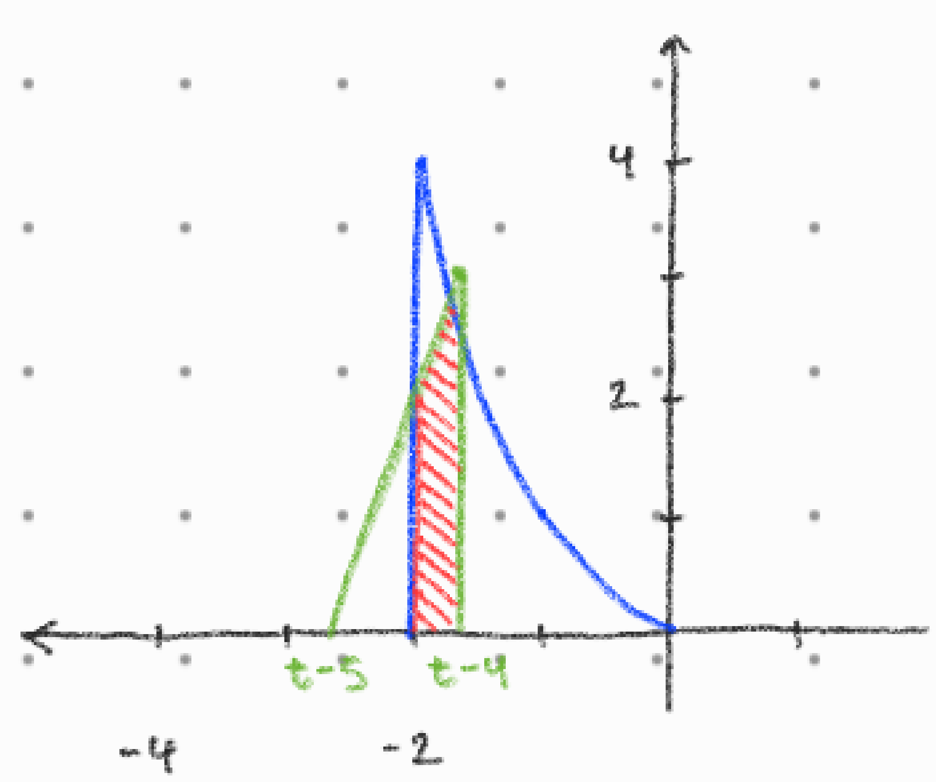
Figure 2: $x_3(t)$ and $x_4(t)$ for Problem 2.

$x_3(t-\tau) * x_4(\tau)$



$t-4<-2$ $\qquad$ $y(t)=0$

$t<2$ $\qquad$ no overlap



$t>2$

$t-5<-2$

$t<3$

$$y(t)=\int_{\tau=-2}^{t-4}(\tau^2)(-3t+3\tau+15)\,d\tau$$

$$=\int_{-2}^{t-4}(-3t\tau^2+3\tau^3+15\tau^2)\,d\tau$$

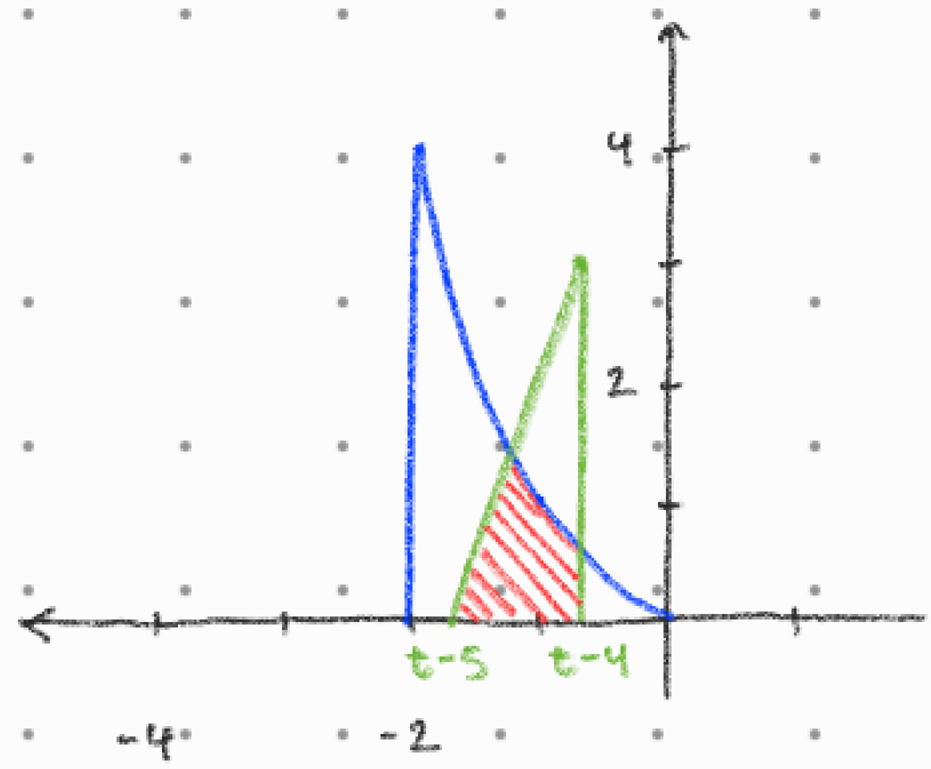
$$=\Big[\underset{①}{-t\cdot\tau^3}+\underset{②}{\tfrac{3}{4}\cdot\tau^4}+\underset{③}{\tfrac{15}{3}\cdot\tau^3}\Big]_{-2}^{t-4}$$

$$y(t)=-\tfrac{1}{4}t^4+5t^3-24t^2+44t-28$$

①: $-t[t-2]^3 \longrightarrow -t[t^3-6t^2+12t-8]$
$= -t^4+6t^3-12t^2+8t$

②: $\frac{3}{4}[t-2]^4 \longrightarrow \frac{3}{4}[t^4-8t^3+24t^2-32t+16]$
$=\frac{3}{4}t^4-6t^3+18t^2-24t+12$

③: $\frac{15}{3}[t-2]^3 \longrightarrow 5[t^3-6t^2+12t-8]$
$=5t^3-30t^2+60t-40$



$t > 3$

$t - 4 < 0$

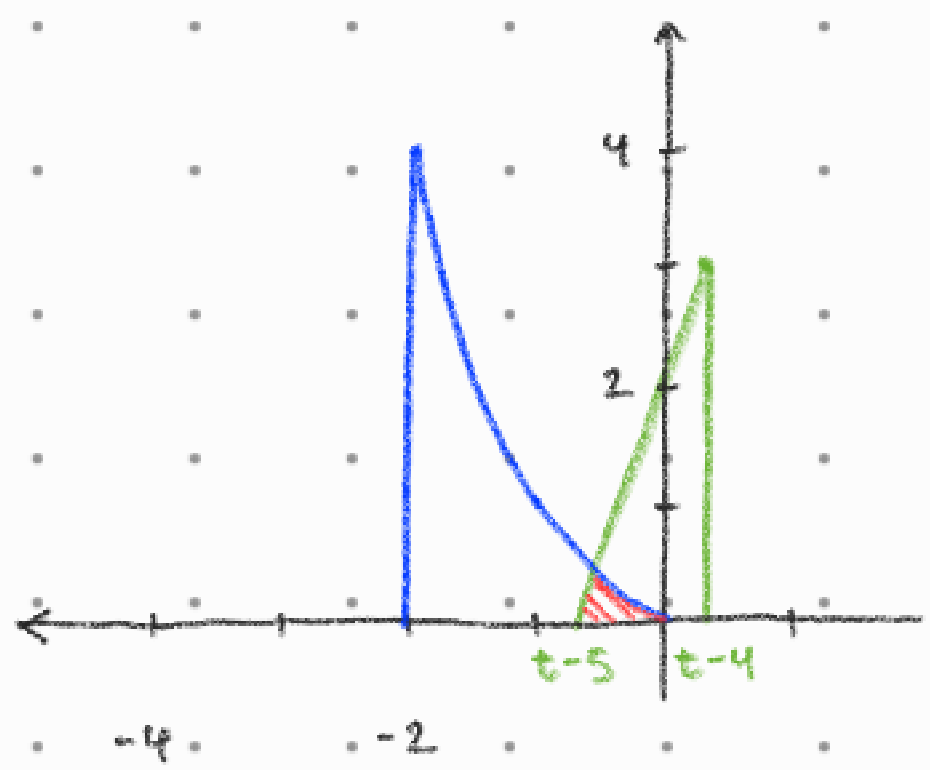
$t < 4$

$$Y(t) = \int_{\tau = t-5}^{t-4} (\tau^2)(-3t + 3\tau + 15)\, d\tau$$

$$= \left[-t\cdot\tau^3 + \frac{3}{4}\cdot\tau^4 + \frac{15}{3}\cdot\tau^3 \right]_{t-5}^{t-4}$$

$$= -t + \frac{3}{4} + 5$$

$$= -t + \frac{23}{4}$$



$t > 4$

$t - 5 < 0$

$t < 5$

$$Y(t) = \int_{\tau = t-5}^{0} (\tau^2)(-3t + 3\tau + 15)\, d\tau$$

$$= \left[\underbrace{-t\cdot\tau^3}_{1} + \underbrace{\frac{3}{4}\cdot\tau^4}_{2} + \underbrace{\frac{15}{3}\cdot\tau^3}_{3} \right]_{t-5}^{0}$$

$$Y(t) = 1.75t^4 - 35t^3 + 262.5t^2 - 875t + 1093.75$$

1: $-t[-t+5]^3 \longrightarrow -t[-t^3 + 15t^2 - 75t + 125]$

$= t^4 - 15t^3 + 75t^2 - 125t$

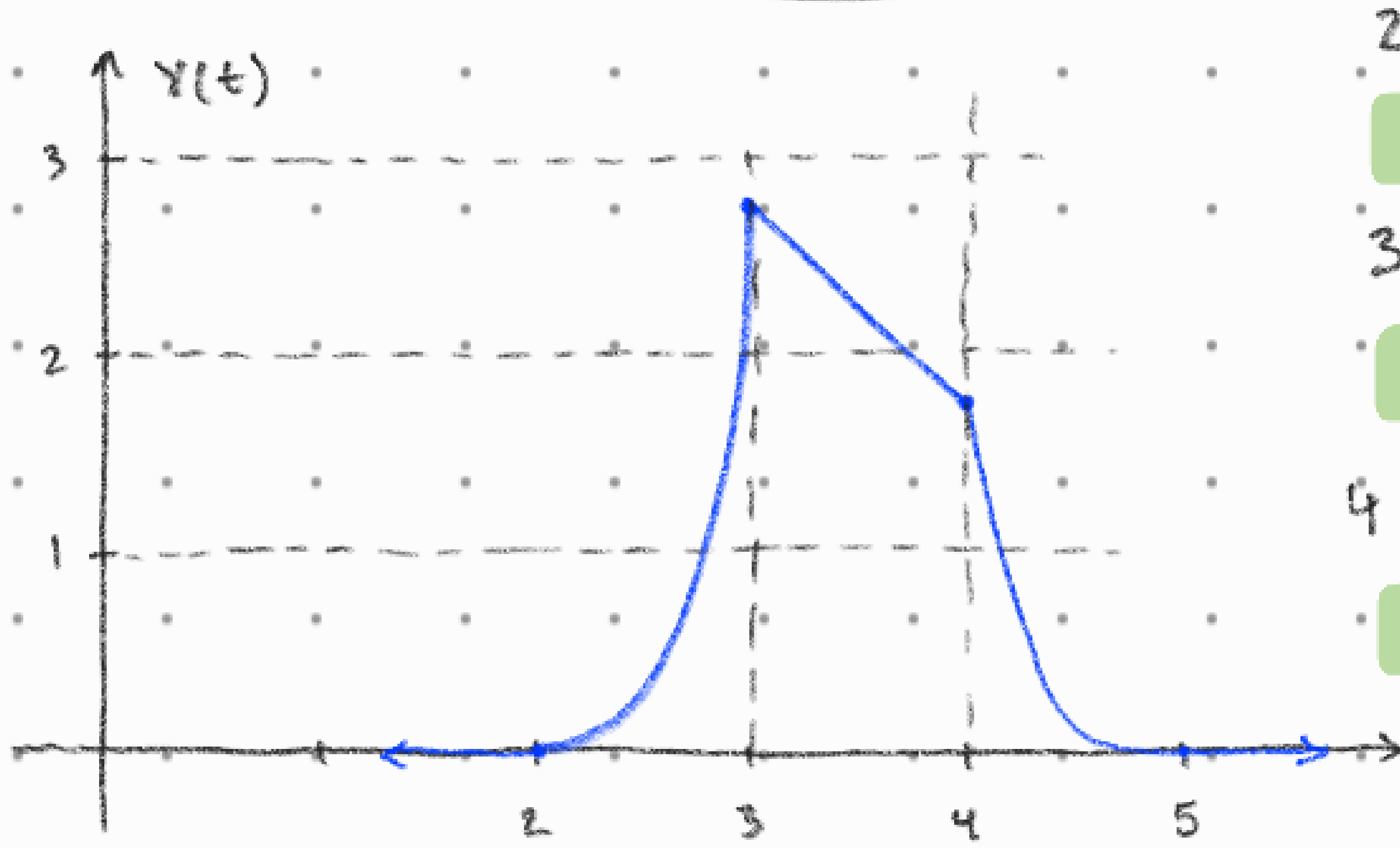
2: $\frac{3}{4}[-t+5]^4 \longrightarrow \frac{3}{4}[t^4 - 20t^3 + 150t^2 - 500t + 625]$

$= \frac{3}{4}t^4 - 15t^3 + 112.5t^2 - 375t + 468.75$

3: $5[-t+5]^3 \longrightarrow 5[-t^3 + 15t^2 - 75t + 125]$

$= -5t^3 + 75t^2 - 375t + 625$

t^4	t^3	t^2	t	1
1	-15	75	-125	0
$\frac{3}{4}$	-15	112.5	-375	468.75
0	-5	75	-375	625



$2 < t < 3$:

$= -\frac{1}{4}t^4 + 5t^3 - 24t^2 + 44t - 28$

$3 < t < 4$:

$= -t + \frac{23}{4}$

$4 < t < 5$:

$= 1.75t^4 - 35t^3 + 262.5t^2 - 875t + 1093.75$

3. If a function is given as

$$y(t) = 3\Delta\left(\frac{t-5}{4}\right)$$

Note: As you may know that $\left[3\Delta\left(\frac{t-5}{4}\right)\right]$ *means just like the rect function. Amplitude: 3, centered at: 5, width: 4*

Find a function $h(t)$ and plot it so the following equation holds

a) $y(t) = 1 \cdot \Pi\left(\frac{t-1}{2}\right) * h(t)$

b) $y(t) = 2 \cdot \Pi\left(\frac{t-7}{2}\right) * h(t)$

c) $y(t) = -1 \cdot \Pi\left(\frac{t+5}{2}\right) * h(t)$

a.) $h(t) \longrightarrow$ centered: $c + 1 = 5$

$c = 4$

width: 2

Amplitude: $(2)(1)(A) = 3$

$A = \frac{3}{2}$

$h(t) = \frac{3}{2}\Pi\left(\frac{t-4}{2}\right)$

b.) $h(t) \longrightarrow$ centered: $c + 7 = 5$

$c = -2$

width: 2

Amplitude: $(2)(2)(A) = 3$

$A = \frac{3}{4}$

$h(t) = \frac{3}{4}\Pi\left(\frac{t+2}{2}\right)$

c.) $h(t) \longrightarrow$ centered: $c - 5 = 5$

$c = 10$

width: 2

Amplitude: $(2)(-1)(A) = 3$

$A = -\frac{3}{2}$

$h(t) = -\frac{3}{2}\Pi\left(\frac{t-10}{2}\right)$

4. Find convolution of two functions

$$y(t) = 1 \cdot \Pi\left(\frac{t-1}{2}\right) * \left[2 \cdot \Pi\left(\frac{t-2}{2}\right) + 2 \cdot \Pi(t-5) + 2 \cdot \delta(t-7)\right]$$

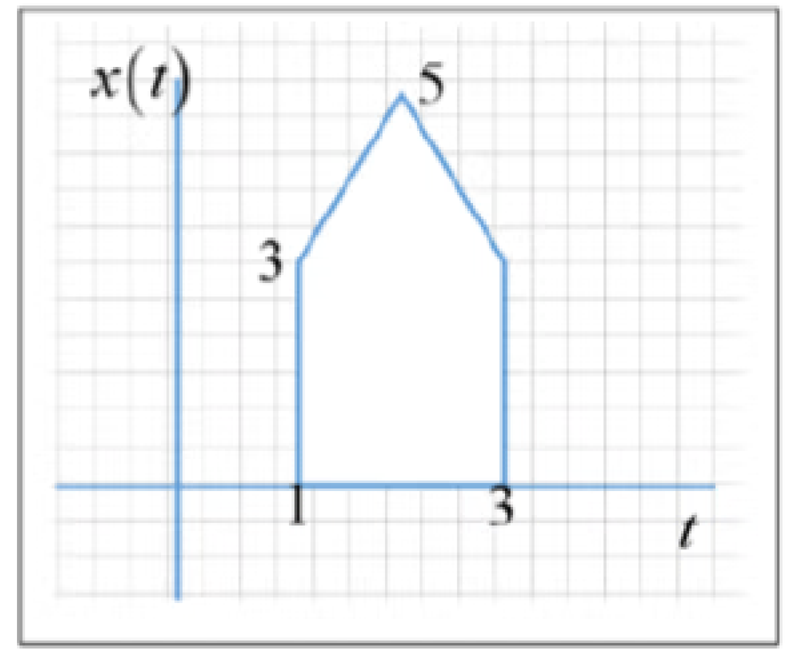
Yes there is no mistake; you see a Dirac delta function on the right functions

a) Find $y(t)$

b) Plot $y(t)$

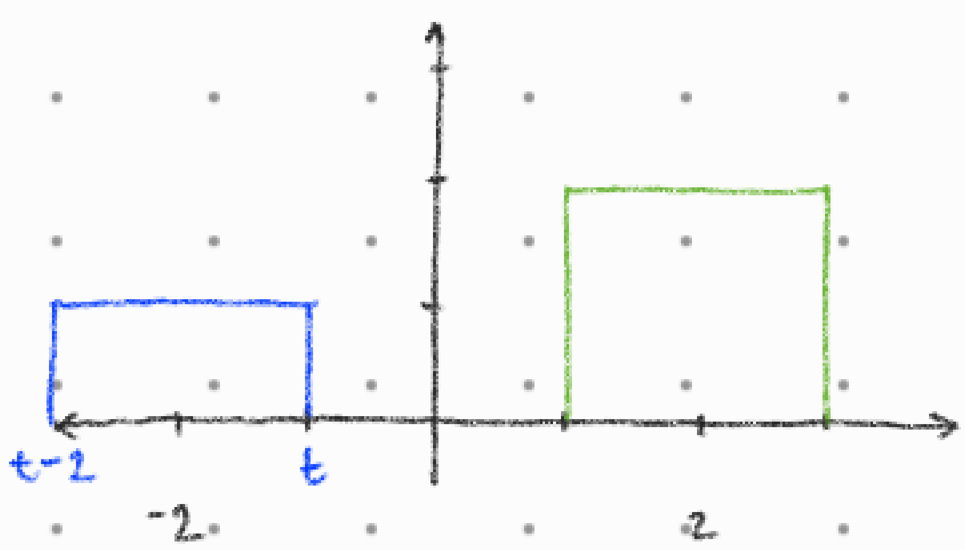
c) Find frequency response of $y(t)$

d) Plot magnitude and phase of part (c)



a.) $y(t) = \underbrace{\Pi\left(\frac{t-1}{2}\right) * 2 \cdot \Pi\left(\frac{t-2}{2}\right)}_{①} + \underbrace{\Pi\left(\frac{t-1}{2}\right) * 2 \cdot \Pi\left(\frac{t-5}{1}\right)}_{②} + \underbrace{\Pi\left(\frac{t-1}{2}\right) * 2 \cdot \delta(t-7)}_{③}$

①:



$t < 1$: $y(t) = 0$

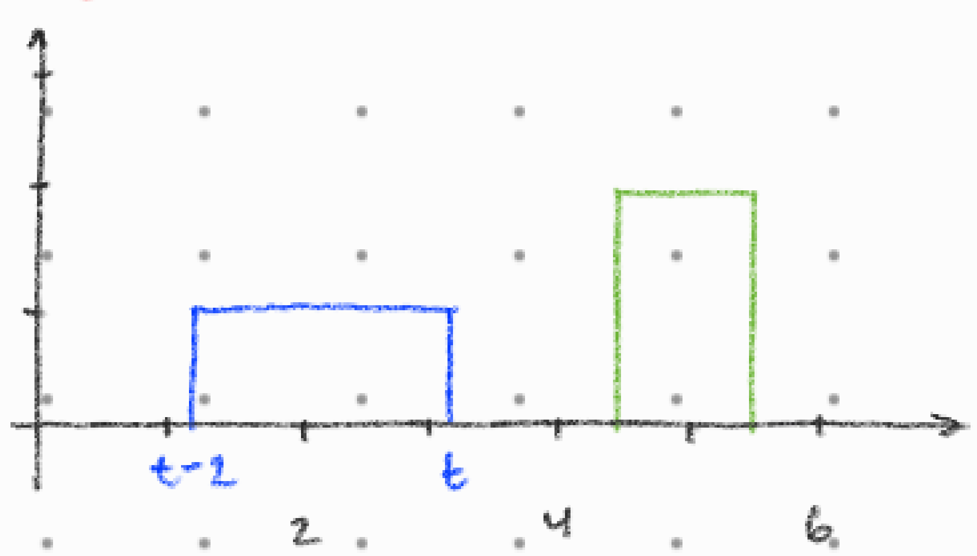
$1 < t < 3$: $y(t) = \int_1^t (2)(1)\, d\tau$

$= 2t - 2$

$3 < t < 5$: $y(t) = \int_{t-2}^3 (2)(1)\, d\tau$

$= 10 - 2t$

②:



$t < 4.5$: $y(t) = 0$

$4.5 < t < 5.5$: $y(t) = \int_{4.5}^t (2)(1)\, d\tau$

$= 2t - 9$

$5.5 < t < 6.5$: $y(t) = \int_{4.5}^{5.5} (2)(1)\, d\tau$

$= 2$

$6.5 < t < 7.5$: $y(t) = \int_{t-2}^{5.5} (2)(1)\, d\tau$

$= 15 - 2\tau$

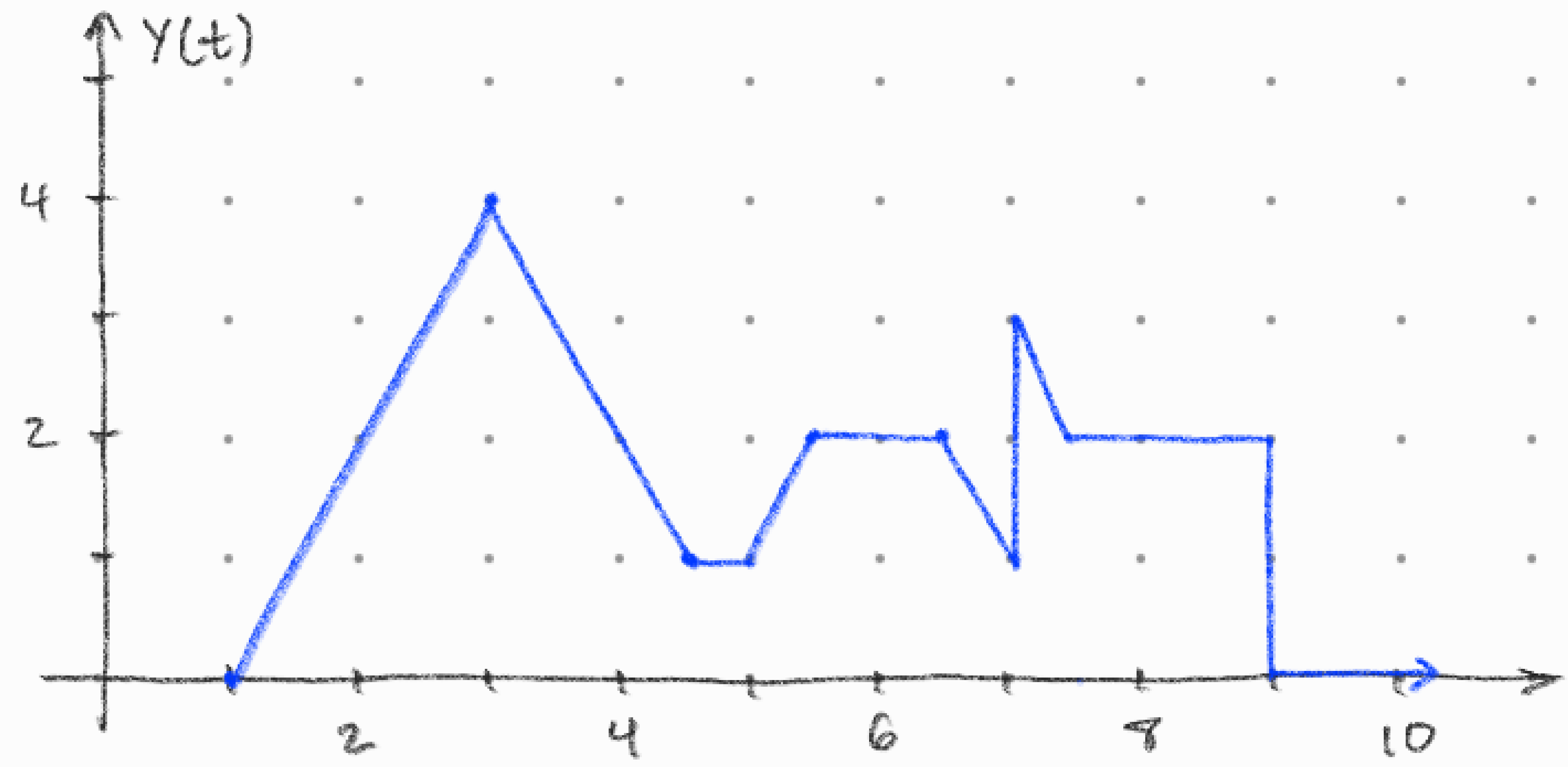
③: $\Pi\left(\frac{t-1}{2}\right) * 2 \cdot \delta(t-7)$

$= 2 \cdot \Pi\left(\frac{t-8}{2}\right) \longrightarrow$

$t < 7$: $y(t) = 0$

$7 < t < 9$: $y(t) = 2$

$t > 9$: $y(t) = 0$



$t < 1$: $Y(t) = 0$

$1 < t < 3$: $= 2t - 2$

$3 < t < 4.5$: $= 10 - 2t$

$4.5 < t < 5$: $= (\cancel{2t} - 9) + (10 - \cancel{2t})$

$= 1$

$5 < t < 5.5$: $= 2t - 9$

$5.5 < t < 6.5$: $= 2$

$6.5 < t < 7$: $= 15 - 2t$

$7 < t < 7.5$: $= (15 - 2t) + (2)$

$= 17 - 2t$

$7.5 < t \leq 9$: $= 2$

5 In class, we proved that convolution in the time domain corresponds to multiplication in the frequency domain. That is,

$$\mathrm{F}\{x(t) * h(t)\} = F\left\{\int_{\tau=-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\,d\tau\right\} = X(f)H(f)$$

where $\mathcal{F}\{x(t)\} = X(f)$ denotes the Fourier transform of $x(t)$. **Now show** that convolution in the frequency domain corresponds to multiplication in the time domain:

$$\mathcal{F}^{-1}\{X(f) * H(f)\} = x(t)h(t)$$

where $\mathcal{F}^{-1}\{X(f)\}$ denotes the inverse Fourier transform of $X(f)$.

$$F^{-1}\{X(f) * H(f)\} = F^{-1}\left[\int_{-\infty}^{\infty} X(\tau)\cdot H(f-\tau)\,d\tau\right]$$

$$Y(t) = \int_{f=-\infty}^{\infty}\left[\int_{\tau=-\infty}^{\infty} X(\tau)\cdot H(f-\tau)\,d\tau\right]\cdot e^{j2\pi f t}\,df$$

$$= \int_{f=-\infty}^{\infty} \left[\int_{\tau=-\infty}^{\infty} X(\tau)\cdot H(f-\tau)\, d\tau \right] \cdot e^{j2\pi[f-\tau+\tau]t}\, df$$

$$= \underbrace{\int_{\tau=-\infty}^{\infty} X(\tau)\cdot e^{j2\pi f t}\, d\tau}_{x(t)} \cdot \underbrace{\int_{f=-\infty}^{\infty} H(f-\tau)\cdot e^{j2\pi(f-\tau)}\, df}_{h(t)}$$

$y(t) = x(t)\cdot h(t).$

6 Convolution

a. $x(t)$ is shown in Figure 1. Perform the convolution $y(t) = x(t) * x(t)$; that is, $x(t)$ convolved with itself. And plot $Y(f)$ in magnitude and phase where $[-10 \le f \le 10]$

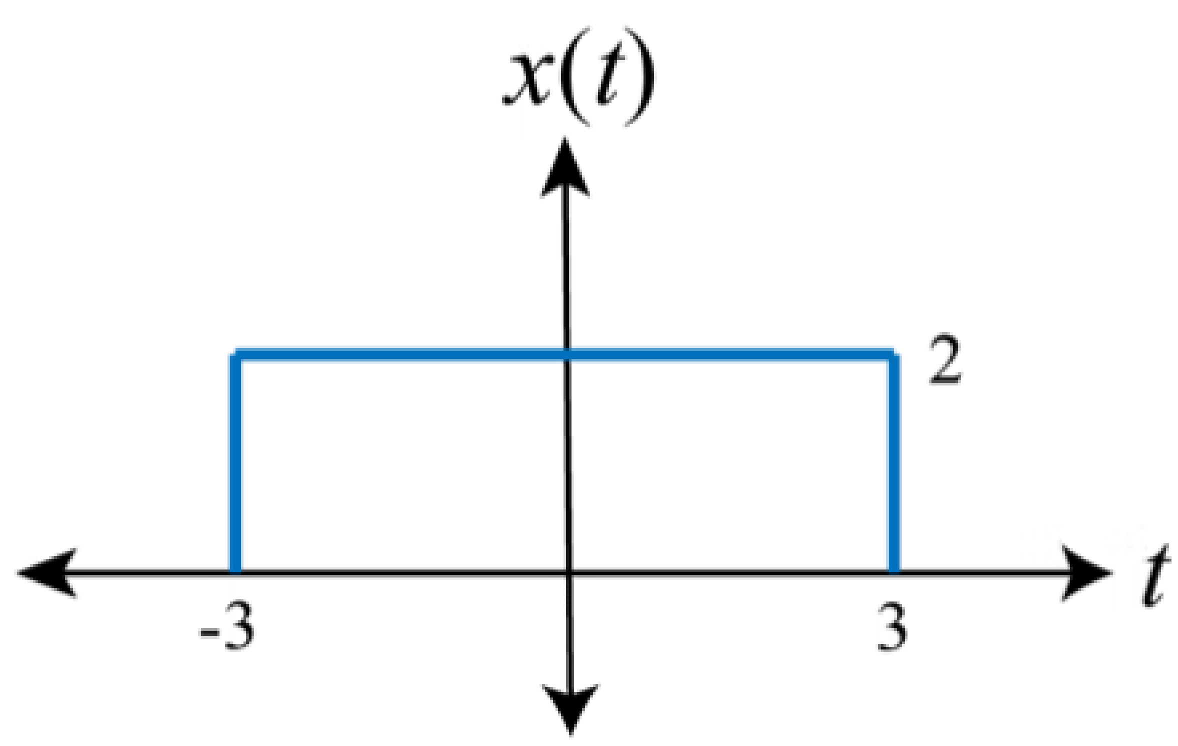


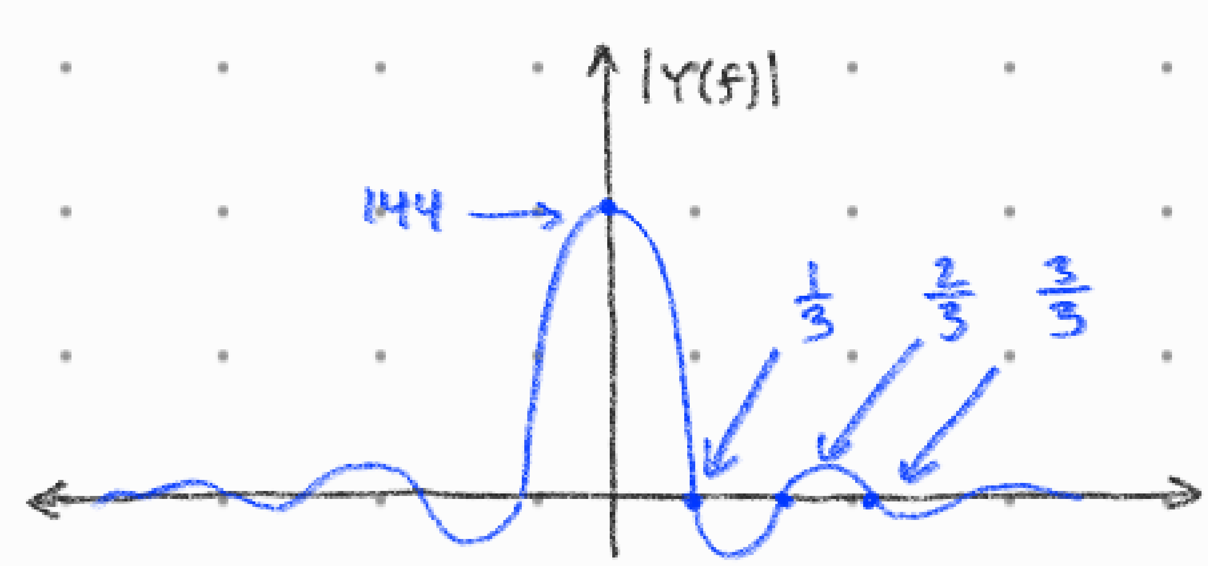
Figure 1: x(t) for problem 2(a).

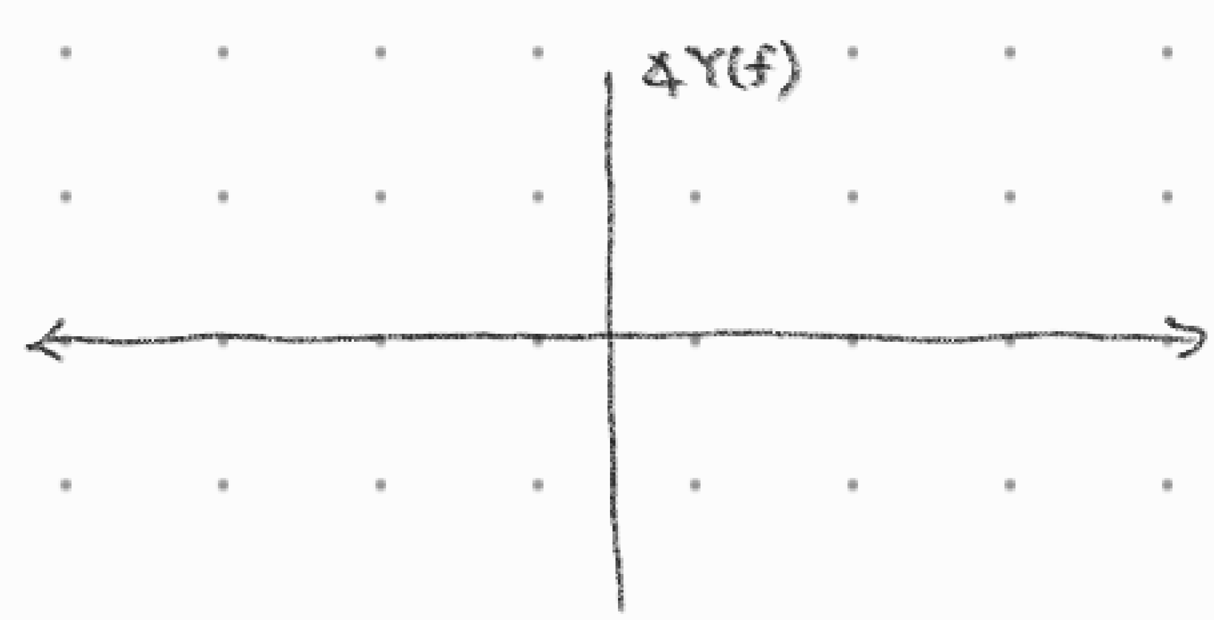
a.) $y(t) = x(t) * x(t)$

$= (2)(2)(6)\cdot \Lambda\left(\frac{t}{6}\right)$

$Y(f) = 24(6)\ \mathrm{sinc}^2(6f)$

$Y(f) = 144\ \mathrm{sinc}^2(6f)$





b. $x_1(t)$ and $x_2(t)$ are shown in Figure 2. Perform the convolution $y(t) = x_1(t) * x_2(t)$. And plot $Y(f)$ in magnitude and phase where $[-10 \le f \le 10]$

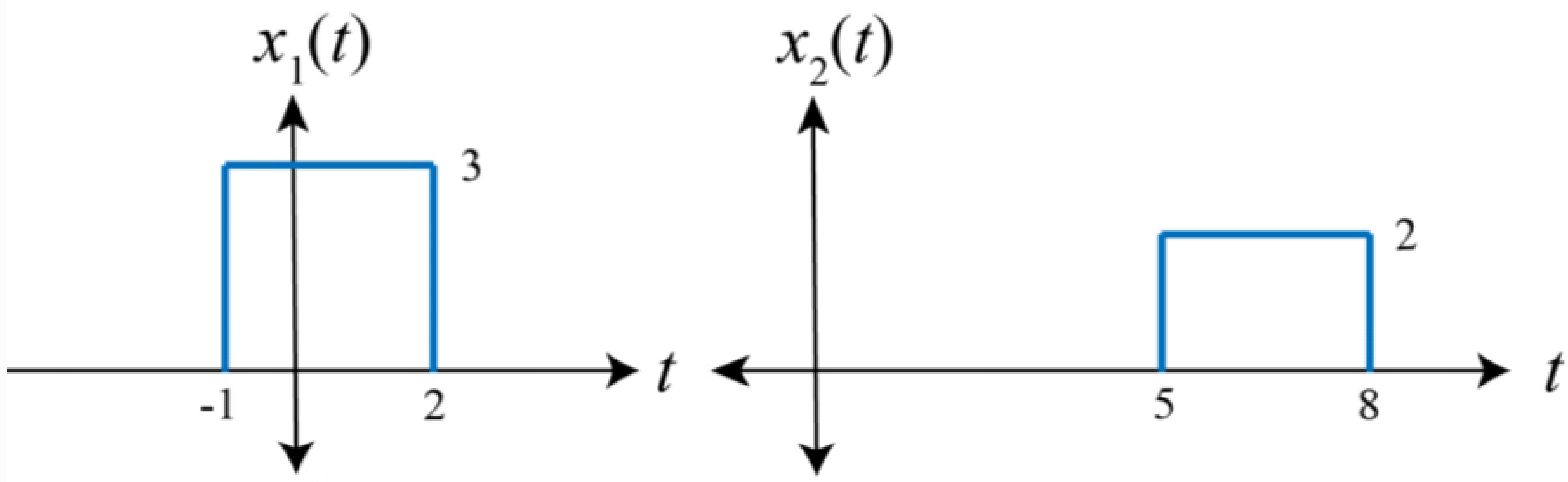


Figure 2: $x_1(t)$ and $x_2(t)$ for problem 2(b).