Ch 02 Analog-to-digital and digital-to-analog conversion

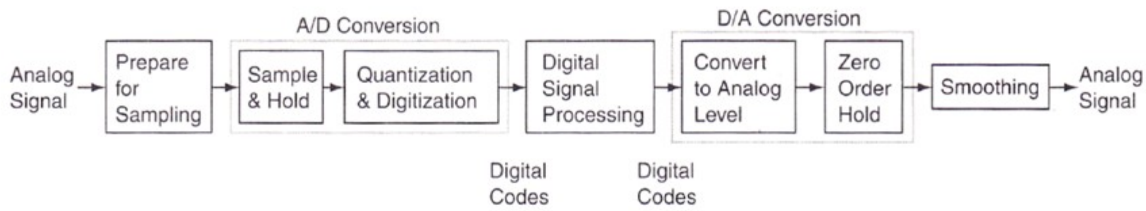


Figure 1: A/D and D/A conversions

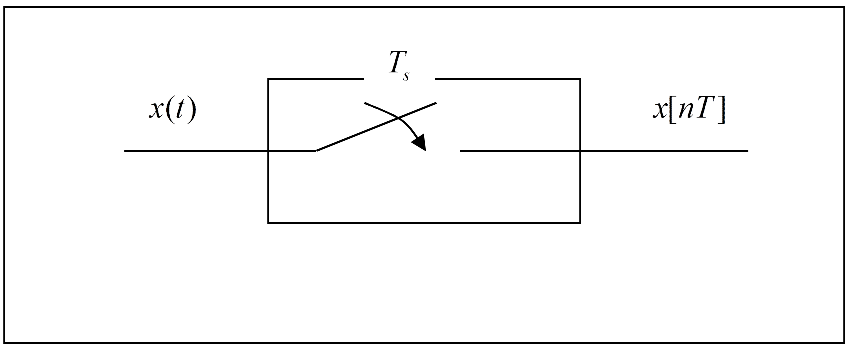
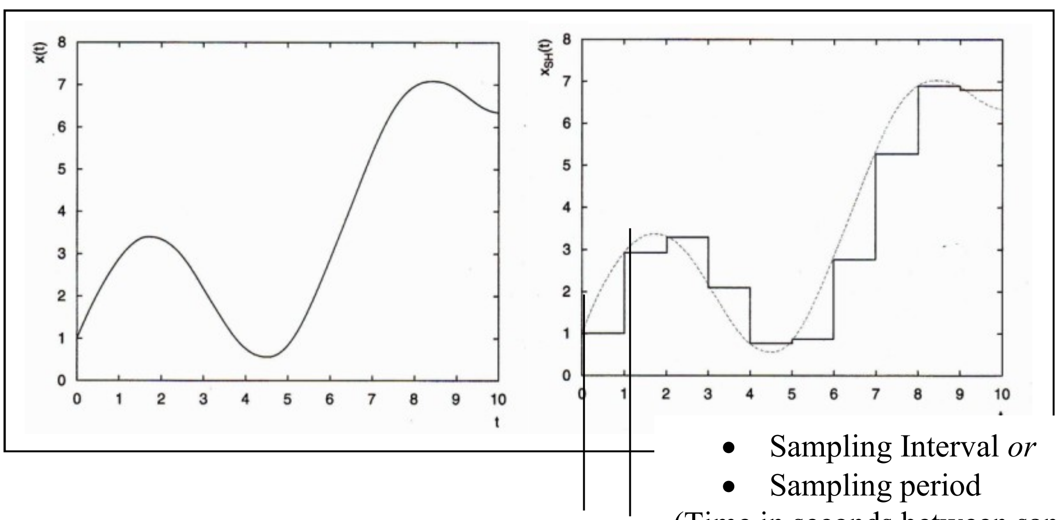


Figure 2: Ideal sampler



- Sampling Interval *or*
- Sampling period

(Time in seconds between samples)

Figure3: Sampling interval representation

$$\text{Sampling frequency} = \frac{1}{\text{Sampling interval}}$$
$$f_s = \frac{1}{T_s} \tag{1}$$

Sampling theorem: Due to **Nyquest**, an analog signal can be perfectly re-created from its sample values, provided the sampling interval is chosen correctly.

- According to **Nyquest theory**, a signal with maximum frequency of $W\ Hz$ must be sampled **at least 2**W to make it possible to reconstruct the original signal from the samples.

- **Nyquest sampling rate**: Minimum sampling frequency
  - The higher frequency signal content means a higher sampling frequency.

- **Nyquest frequency**: Half of sampling rate of a system.

- **Nyquest range**: Range of frequencies between zero and the Nyquest frequency

<u>Ex)</u>

A signal containing frequencies up to 20 kHz must be sampled a minimum of 40,000 samples per second.

The continuous time **Fourier transform** pair is defined as

- Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

- Inverse Fourier Transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

Euler's Identity

$$e^{jx} = \cos(x) + j\sin(x)$$

$$\cos(x) = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$$

$$\sin(x) = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right) = -\frac{j}{2}\left(e^{jx} - e^{-jx}\right) = \frac{j}{2}\left(e^{-jx} - e^{jx}\right)$$

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \quad \text{or} \quad \operatorname{sinc}(x) = \frac{\sin(x)}{x}$$

$$a = e^{\ln(a)}$$

and that

$$e^a e^b = e^{a+b}$$

both valid for any complex numbers *a* and *b*.

Therefore, one can write:

$$z = |z| e^{i\phi} = e^{\ln|z|} e^{i\phi} = e^{\ln|z| + i\phi}$$

for any $z \neq 0$. Taking the logarithm of both sides shows that:

$$\ln z = \ln|z| + i\phi \ .$$

and in fact this can be used as the definition for the complex logarithm. multi-valued.

Finally, the other exponential law

$$(e^a)^k = e^{ak} \ ,$$

http://en.wikipedia.org/wiki/Euler%27s_formula

Relationship to trigonometry

Euler's formula provides a powerful connection between analysis and trigonometry, and provides an interp weighted sums of the exponential function:

$$\cos x = \mathrm{Re}\{e^{ix}\} = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin x = \mathrm{Im}\{e^{ix}\} = \frac{e^{ix} - e^{-ix}}{2i}\ .$$

The two equations above can be derived by adding or subtracting Euler's formulas:

$$e^{ix} = \cos x + i \sin x$$
$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i\sin x$$

and solving for either cosine or sine.

These formulas can even serve as the definition of the trigonometric functions for complex arguments x. F

$$\cos(iy) = \frac{e^{-y} + e^{y}}{2} = \cosh(y)$$
$$\sin(iy) = \frac{e^{-y} - e^{y}}{2i} = -\frac{1}{i}\frac{e^{y} - e^{-y}}{2} = i\sinh(y)\ .$$

Complex exponentials can simplify trigonometry, because they are easier to manipulate than their sinusc convert sinusoids into equivalent expressions in terms of exponentials. After the manipulations, the simpl

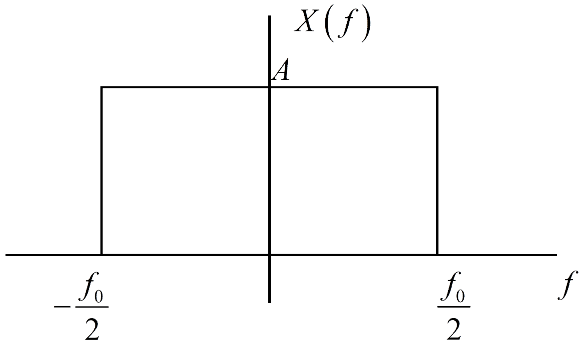
$$\begin{aligned}
\cos x \cdot \cos y &= \frac{(e^{ix} + e^{-ix})}{2} \cdot \frac{(e^{iy} + e^{-iy})}{2} \\
&= \frac{1}{2} \cdot \frac{e^{i(x+y)} + e^{i(x-y)} + e^{i(-x+y)} + e^{i(-x-y)}}{2} \\
&= \frac{1}{2}\left[\underbrace{\frac{e^{i(x+y)} + e^{-i(x+y)}}{2}}_{\cos(x+y)} + \underbrace{\frac{e^{i(x-y)} + e^{-i(x-y)}}{2}}_{\cos(x-y)} \right]\ .
\end{aligned}$$

Another technique is to represent the sinusoids in terms of the real part of a more complex expression, a expression. For example:

$$\begin{aligned}
\cos(nx) &= \mathrm{Re}\{\ e^{inx}\ \} = \mathrm{Re}\{\ e^{i(n-1)x} \cdot e^{ix}\ \} \\
&= \mathrm{Re}\{\ e^{i(n-1)x} \cdot (e^{ix} + e^{-ix} - e^{-ix})\ \} \\
&= \mathrm{Re}\{\ e^{i(n-1)x} \cdot \underbrace{(e^{ix} + e^{-ix})}_{2\cos(x)} - e^{i(n-2)x}\ \} \\
&= \cos[(n-1)x] \cdot 2\cos(x) - \cos[(n-2)x]\ .
\end{aligned}$$

This formula is used for recursive generation of cos(nx) for integer values of n and arbitrary x (in radians).

Ex) Finding the **inverse Fourier transform** of the given functions



The representation of above plot is written as

$$X(f) = A \cdot \Pi\left(\frac{f}{f_0}\right)$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j\omega t} df \qquad \text{where } \omega = 2\pi f$$

$$= \int_{-\frac{f_0}{2}}^{\frac{f_0}{2}} A \cdot e^{j2\pi f t} df$$

$$= \frac{A}{j2\pi t} e^{j2\pi f t} \Big|_{-\frac{f_0}{2}}^{\frac{f_0}{2}}$$

$$= \frac{A}{j2\pi t}\left(e^{j2\pi \frac{f_0}{2} t} - e^{j2\pi\left(-\frac{f_0}{2}\right)t}\right)$$

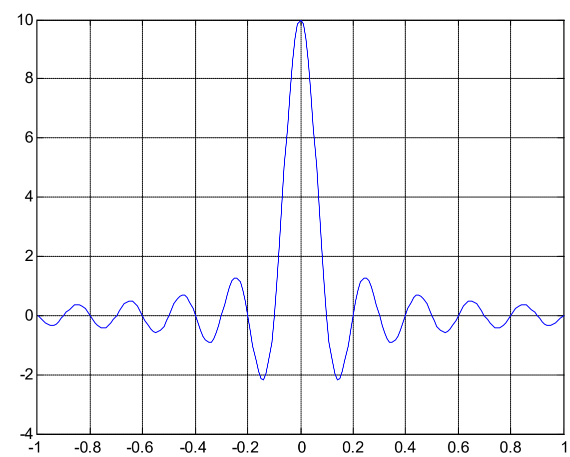
$$= \frac{A}{\pi t}\left(\frac{e^{j2\pi \frac{f_0}{2} t} - e^{j2\pi\left(-\frac{f_0}{2}\right)t}}{2j}\right)$$

$$= \frac{A}{\pi t}\sin(\pi f_0 t)$$

$$= \frac{A}{\pi t}\frac{\pi f_0 t}{\pi f_0 t}\sin(\pi f_0 t)$$

$$= \frac{A}{\cancel{\pi t}}\frac{\cancel{\pi t} f_0}{1}\frac{\sin(\pi f_0 t)}{\pi f_0 t}$$

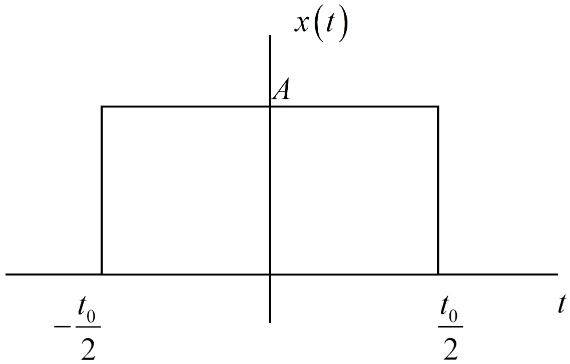
$$= \underline{\underline{(Af_0)\,\mathrm{sinc}(f_0 t)}}$$




```
clear all;

A = 1; %Amplitude
f0 = 10; %frequency
t = -1:0.01:1; % time
x = (A*f0)*sinc(f0*t);
plot(t,x); %plot
grid;
```

Ex1) The rectangular signal in the continuous time domain is shown in the above figure. Find the Fourier transform of the given function.



The representation for this signal is written by

$$x(t) = A \cdot \Pi\left(\frac{t}{t_0}\right)$$

$$\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \\
&= \int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} A \cdot e^{-j2\pi ft} dt \\
&= \frac{A}{-j2\pi f} e^{-j2\pi ft} \Big|_{-\frac{t_0}{2}}^{\frac{t_0}{2}} \\
&= \frac{A}{-j2\pi f}\left(e^{-j2\pi f \frac{t_0}{2}} - e^{j2\pi f \frac{t_0}{2}} \right) \\
&= \frac{A}{\pi f}\left(\frac{e^{j2\pi f \frac{t_0}{2}} - e^{-j2\pi f \frac{t_0}{2}}}{2j} \right) \\
&= \frac{A}{\pi f} \sin(\pi f t_0) \\
&= \frac{A}{\cancel{\pi f}} \frac{\cancel{\pi f}\, t_0}{\pi f t_0} \sin(\pi f t_0) \\
&= \frac{A}{\cancel{\pi f}} \frac{\cancel{\pi t}\, t_0}{1} \left[\frac{\sin(\pi f t_0)}{\pi f t_0} \right] \\
&= \underline{\underline{(A \cdot t_0) \operatorname{sinc}(f t_0)}}
\end{aligned}$$

Continuous time unit impulse: The Dirac function

- In continuous time, a unit impulse $\delta(t)$ is defined as a function that is zero for all $t \neq 0$, and yet its integral is nonzero. In particular $\delta(t)$

$$\delta(t) = 0 \qquad for\ \ all\ \ \ t \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t)\,dt = \int_{0-}^{0+} \delta(t)\,dt = 1$$

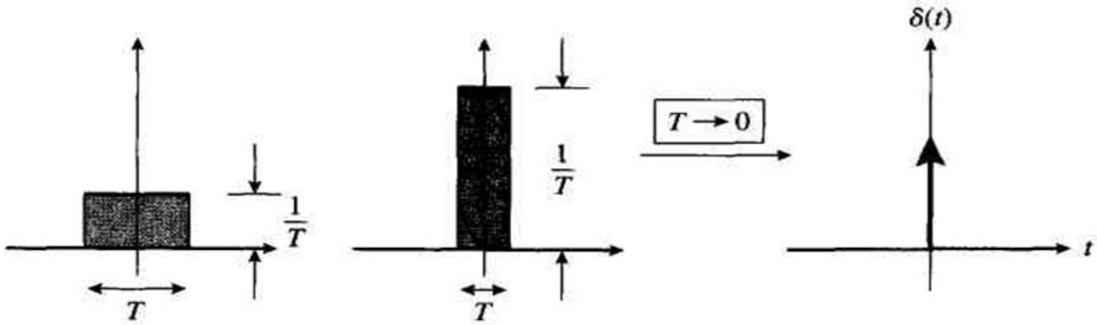


Figure: Dirac delta function

- It can be viewed as the limit of a sequence of rectangular signal of width T and height 1/T as T→0. Its significance is the fact that for any signal $x(t)$, continuous at time *t*, we can write

$$\begin{aligned}\int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)\,d\tau &= \int_{0-}^{0+} x(t-\tau)\delta(\tau)\,d\tau \\ &= x(t)\int_{0-}^{0+} \delta(\tau)\,d\tau \\ &= x(t)\end{aligned}$$

- The property is called **sifting** theory

$$\begin{aligned}x(t) &= \int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\end{aligned}$$

Some of the properties

1. $\delta(t) = 0$ for all $t \neq 0$ and $\delta(0) = \infty$
2. $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$
3. For any $\phi(t)$ continuous at t_0

$$\int_{-\infty}^{\infty} \phi(t)\delta(t - t_0)\,dt = \phi(t_0)$$

4. For any $\phi(t)$ continuous at t_0

$$\int_{-\infty}^{\infty} \phi(t + t_0)\delta(t)\,dt = \phi(t_0)$$

5. For all $a \neq 0$

$$\delta(at) = \delta\left(\frac{\overset{\text{Center}}{t}}{\underbrace{\left(\frac{1}{a}\right)}_{\text{width}}}\right) = \frac{1}{|a|}\delta(t)$$

6. The convolution

$$x(t) * \delta(t) = x(t)$$
$$x(t) * \delta(t - t_0) = x(t - t_0)$$

7. The unit step signal is the integral of the impulse signal, and the impulse signal is the generalized derivative of the unit step signal

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$
$$\delta(t) = \frac{d}{dt}u(t)$$

8. The generalized derivatives of $\delta(t)$

$$\int_{-\infty}^{\infty} \delta^n(t)\phi(t)\,dt = (-1)^n \frac{d^n}{dt^n}\phi(t)\Big|_{t=0}$$
$$\int_{-\infty}^{\infty} \delta^n(t)\phi(t - t_0)\,dt = (-1)^n \frac{d^n}{dt^n}\phi(t)\Big|_{t=t_0}$$

9. The convolution of any signal with n^{th} derivative of $x(t)$

$$x(t) * \delta'(t) = x'(t)$$
$$x(t) * \delta^n(t) = x^n(t)$$

10. The convolution of any signal $x(t)$ with the unit step signal is

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

11. For even values of n $\delta^{(n)}(t)$ are even; for odd value of n, it is odd. In particular $\delta(t)$ is even and $\delta'(t)$ is odd

Convolution in continuous time domain

The general description of LTI systems tells us that the general output y(t) is obtained from the general input x(t) by the convolution operation with impulse response h(t).

Let $x(t), h(t)$ and $y(t)$ be the input, impulse response, and output of linear time invariant system.

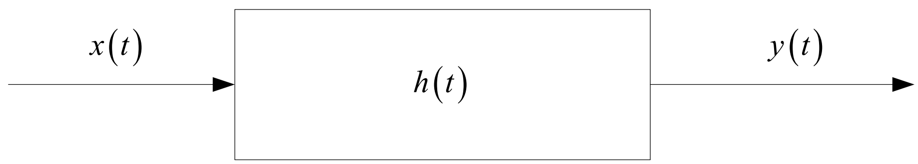


Figure: LTI system

Then the output is defined as

$$
\begin{aligned}
y(t) = x(t) * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\
= h(t) * x(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau
\end{aligned}
$$

Now we want to find the frequency response of the output of the system which is

$$
\begin{aligned}
F\left[y(t)\right] &= F\left[x(t) * h(t)\right] \\
&= F\left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\right] \\
&= \int_{t=-\infty}^{\infty}\left[\int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\right] e^{-j2\pi f t}\, dt \\
&= \int_{t=-\infty}^{\infty}\left[\int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\right] e^{-j2\pi f(t-\tau+\tau)}\, dt \\
&= \int_{t=-\infty}^{\infty}\left[\int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\right]\left(e^{-j2\pi f(\tau)}\right)\cdot\left(e^{-j2\pi f(t-\tau)}\right) dt \\
&= \int_{t=-\infty}^{\infty} h(t-\tau)\, e^{-j2\pi f(t-\tau)}\, dt \left[\int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j2\pi f(\tau)}\, d\tau\right] \\
&= \underbrace{\int_{t=-\infty}^{\infty} h(t-\tau)\, e^{-j2\pi f(t-\tau)}\, dt}_{H(f)} \underbrace{\int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j2\pi f(\tau)}\, d\tau}_{X(f)} \\
&= H(f)\, X(f) \\
&= Y(f)
\end{aligned}
$$

And we want to verify that what if the convolution of two functions in the frequency domain is given

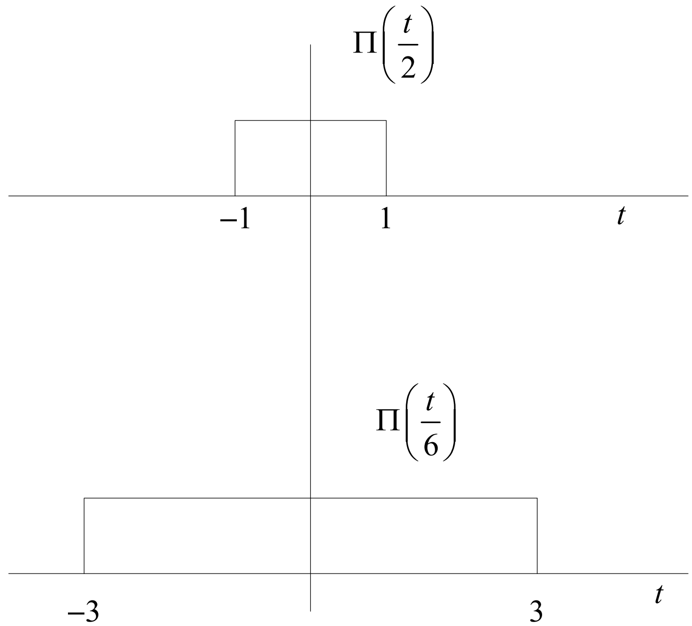
$$
\begin{aligned}
F^{-1}\left[Y(f)\right] &= F^{-1}\left[X(f)*H(f)\right] \\
&= F^{-1}\left[\int_{-\infty}^{\infty} X(\tau)H(f-\tau)\,d\tau\right] \\
&= \int_{f=-\infty}^{\infty}\left[\int_{\tau=-\infty}^{\infty} X(\tau)H(f-\tau)\,d\tau\right]e^{j2\pi ft}\,df \\
&= \int_{f=-\infty}^{\infty}\left[\int_{\tau=-\infty}^{\infty} X(\tau)H(f-\tau)\,d\tau\right]e^{j2\pi(f-\tau+\tau)t}\,df \\
&= \int_{f=-\infty}^{\infty}\left[\int_{\tau=-\infty}^{\infty} X(\tau)H(f-\tau)\,d\tau\right]e^{j2\pi(\tau)t}e^{j2\pi(f-\tau)t}\,df \\
&= \int_{f=-\infty}^{\infty} H(f-\tau)e^{j2\pi(f-\tau)t}\,df\left[\int_{\tau=-\infty}^{\infty} X(\tau)e^{j2\pi(\tau)t}\,d\tau\right] \\
&= \underbrace{\int_{t=-\infty}^{\infty} H(f-\tau)e^{j2\pi(f-\tau)t}\,df}_{h(t)}\underbrace{\int_{\tau=-\infty}^{\infty} X(\tau)e^{j2\pi(\tau)t}\,d\tau}_{x(t)} \\
&= h(t)\cdot x(t) \\
&= y(t)
\end{aligned}
$$

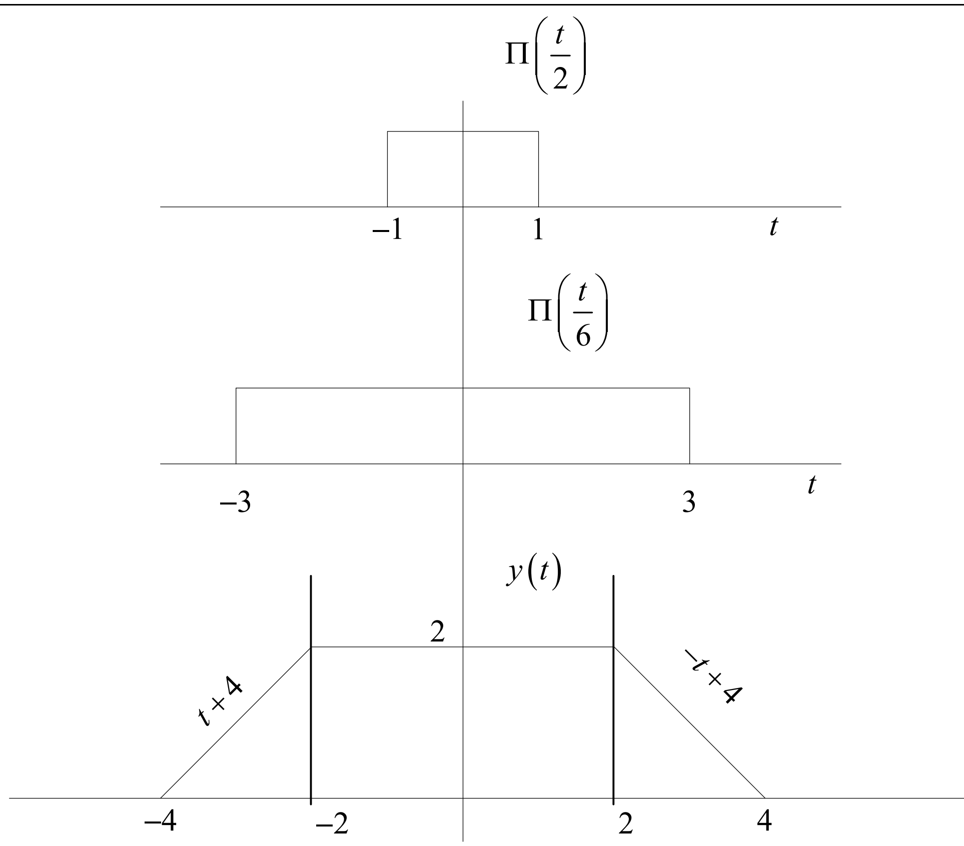
In summary, we can see that multiplication in one domain is convolution in the transformed domain and vice versa.

$$
\boxed{\begin{array}{l} x(t)*h(t) \Leftrightarrow X(f)\cdot H(f) \\ \\ X(f)*H(f) \Leftrightarrow x(t)\cdot h(t) \end{array}}
$$

<u>Ex)</u> Find the convolution of the following

$$y(t) = \Pi\left(\frac{t}{2}\right) * \Pi\left(\frac{t}{6}\right)$$





$t < -4$	$y(t) = 0$
$-4 \le t < -2$	$y(t) = \int_{-3}^{t+1} 1 \cdot d\tau = t + 4$
$-2 \le t < 2$	$y(t) = \int_{t-1}^{t+1} 1 \cdot d\tau = 2$
$2 \le t < 4$	$y(t) = \int_{t-1}^{4} 1 \cdot d\tau = -t + 4$