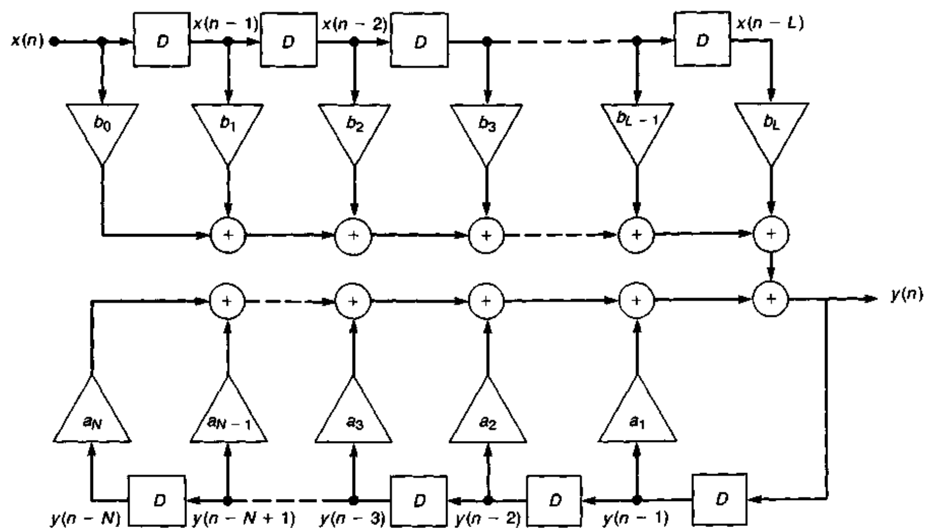
SYSTEM DIAGRAMS OR REALIZATIONS



$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{L} b_k x[n-k]$$

For any causal, stable filter with real coefficients, all of the poles of the system transfer function $H(z)$ must lie inside the unit circle. The zeros may be located anywhere. Both poles and zeros may be real or complex, but because of the real coefficients complex poles or zeros must occur in conjugate pairs.

Thus, **for a stable, causal system, all poles have magnitudes less than 1** and <u>the ROC of the transfer function $H(z)$ includes the unit circle</u>.

A linear time invariant system with system function $H(z)$ is BIBO stable if and only if the region of convergence for $H(z)$ contains the unit circle.

For a stable, anti-causal system, all poles have magnitudes greater than 1 and the ROC of the transfer function $H(z)$ includes the unit circle.

Ex)

This example illustrates the importance of the region of convergence.

Given the transform

$$Y(z) = 2 + \frac{3z}{z + \frac{1}{2}} + \frac{4z}{z - 3}$$

Find the inverse transform $y(n)$ for the region of convergence (ROC) specified.

a) $|z| < \frac{1}{2}$

$b)$ $\frac{1}{2} < |z| < 3$

$c)$ $|z| > 3$

Solution: For all three parts the inverse transform of 2 is simply $2\delta[n]$. We need to concern ourselves only with the relation of the two poles ($z = -\frac{1}{2}$ and $z = 3$) to the specified ROCs.

a) The ROC is inside both poles and consequently we have two left-sided sequences.

$$y[n] = 2\delta[n] - 3\left(-\frac{1}{2}\right)^n u[-n-1] - 4(3)^n u[-n-1]$$

b) The ROC of $\frac{1}{2} < |z| < 3$ is inside the pole at $z = 3$ and outside the one at $z = -\frac{1}{2}$.

Thus, we have a two-sided sequence, namely

$$y[n] = 2\delta[n] + 3\left(-\frac{1}{2}\right)^n u[n] - 4(3)^n u[-n-1]$$

c) This ROC is outside of both poles so we have a familiar right-sided sequence, in this case

$$y[n] = 2\delta[n] + 3\left(-\frac{1}{2}\right)^n u[n] + 4(3)^n u[n]$$

Ex)

Find the inverse transform of $X(z) = \dfrac{z}{3z^2 - 4z + 1}$ for the regions of convergence shown

$a)\quad |z| > 1$

$b)\quad |z| < \dfrac{1}{3}$

$c)\quad \dfrac{1}{3} < |z| < 1$

Define $F(z) = \dfrac{X(z)}{z}$; therefore

$$F(z) = \frac{X(z)}{z} = \frac{1}{3z^2 - 4z + 1} = \frac{1}{3\left(z^2 - \frac{4}{3}z + \frac{1}{3}\right)} = \frac{1}{3(z-1)\left(z - \frac{1}{3}\right)}$$

$$= \frac{A}{(z-1)} + \frac{B}{\left(z - \frac{1}{3}\right)}$$

$$= \frac{\frac{1}{2}}{(z-1)} + \frac{-\frac{1}{2}}{\left(z - \frac{1}{3}\right)}$$

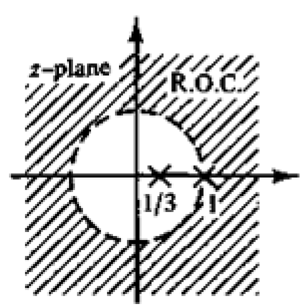
Multiplying z to the right terms, then

$$X(z) = \frac{\frac{1}{2}z}{(z-1)} + \frac{-\frac{1}{2}z}{\left(z - \frac{1}{3}\right)}$$

(a).

For the part (a), we see that the region of convergence is outside the largest pole signifying right-hand sequences for each pole. Therefore, $x[n]$ becomes

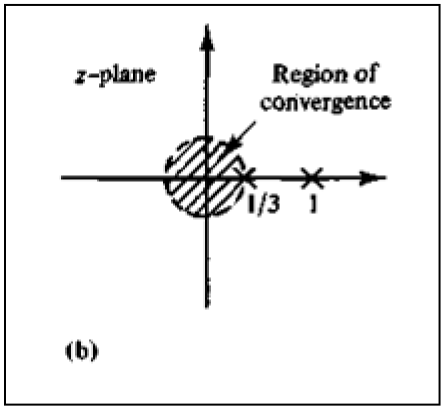
$$x[n] = \frac{1}{2}u[n] - \frac{1}{2}\left(\frac{1}{3}\right)^n u[n]$$



(a)

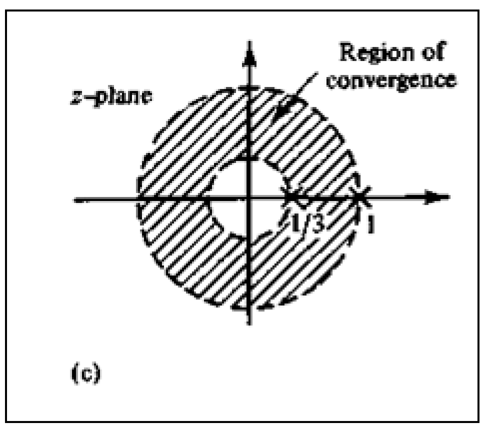
(b). $|z| < \frac{1}{3}$

$$x[n] = -\frac{1}{2}u[-n-1] + \frac{1}{2}\left(\frac{1}{3}\right)^n u[-n-1]$$



(c). $\frac{1}{3} < |z| < 1$

$$x[n] = -\frac{1}{2}u[-n-1] - \frac{1}{2}\left(\frac{1}{3}\right)^n u[n]$$



Ex)
Let an LTI system have impulse response $h[n] = 0.5^{|n|}$ and input $x[n] = u[n]$.

Find system function $H(z)$, $X(z)$, and $y[n]$.

$$h[n] = 0.5^{|n|}$$
$$= (0.5)^n u[n] + (0.5)^{-n} u[-n-1]$$

$$H(z) = \frac{z}{z-0.5} - \frac{z}{z-2} = \frac{-1.5z}{(z-0.5)(z-2)} \qquad 0.5 < |z| < 2$$

$$X(z) = \frac{z}{z-1} \qquad |z| > 1$$

$$Y(z) = H(z)X(z)$$

$$= \frac{-1.5z}{(z-0.5)(z-2)} \cdot \frac{z}{z-1} \qquad 1 < |z| < 2$$

By partial fraction expansion, $Y(z)$ can be

$$Y(z) = 3\frac{z}{z-1} - \frac{z}{z-0.5} - 2\frac{z}{z-2} \qquad 1 < |z| < 2$$

So the output signal is

$$y[n] = 3u[n] - 0.5^n u[n] + 2 \cdot 2^n u[-n-1]$$

For a stable transfer function in the form of

$$H(z) = \frac{1}{z^2 + \alpha z + \beta}$$

then the difference equation is written as

$$y[n] + \alpha y[n-1] + \beta y[n-2] = x[n-2]$$

For a unit step input, $x[n-2] = 1$ at steady state. For large n,

$y[n] \approx y[n-1] \approx y[n-2] \approx y_{ss}$, so that

$$y_{ss}(1 + \beta + \alpha) = 1$$

$$y_{ss} = \frac{1}{1 + \beta + \alpha}$$

For a unit step input, the output of a stable system is guaranteed to approach a constant value at steady state.

<u>Ex 6.40)</u>

Compute the steady state outputs for the step response.

$$H(z) = \frac{z^{-2}}{1 + 0.2z^{-1} + 0.001z^{-2}}$$

$$y[n] + 0.2y[n-1] + 0.001y[n-2] = x[n-2]$$
$$y_{ss}(1 + 0.2 + 0.001) = 1$$

$$y_{ss} = \frac{1}{1 + 0.2 + 0.001} = 0.826$$

CH 7 Fourier Transform and Filter Shape

$$X(\Omega) = F\{x[n]\}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\left(\cos(\Omega n) - j\sin(\Omega n)\right)$$

- Magnitude
- Phase

Ex) Find the discrete Fourier transform of

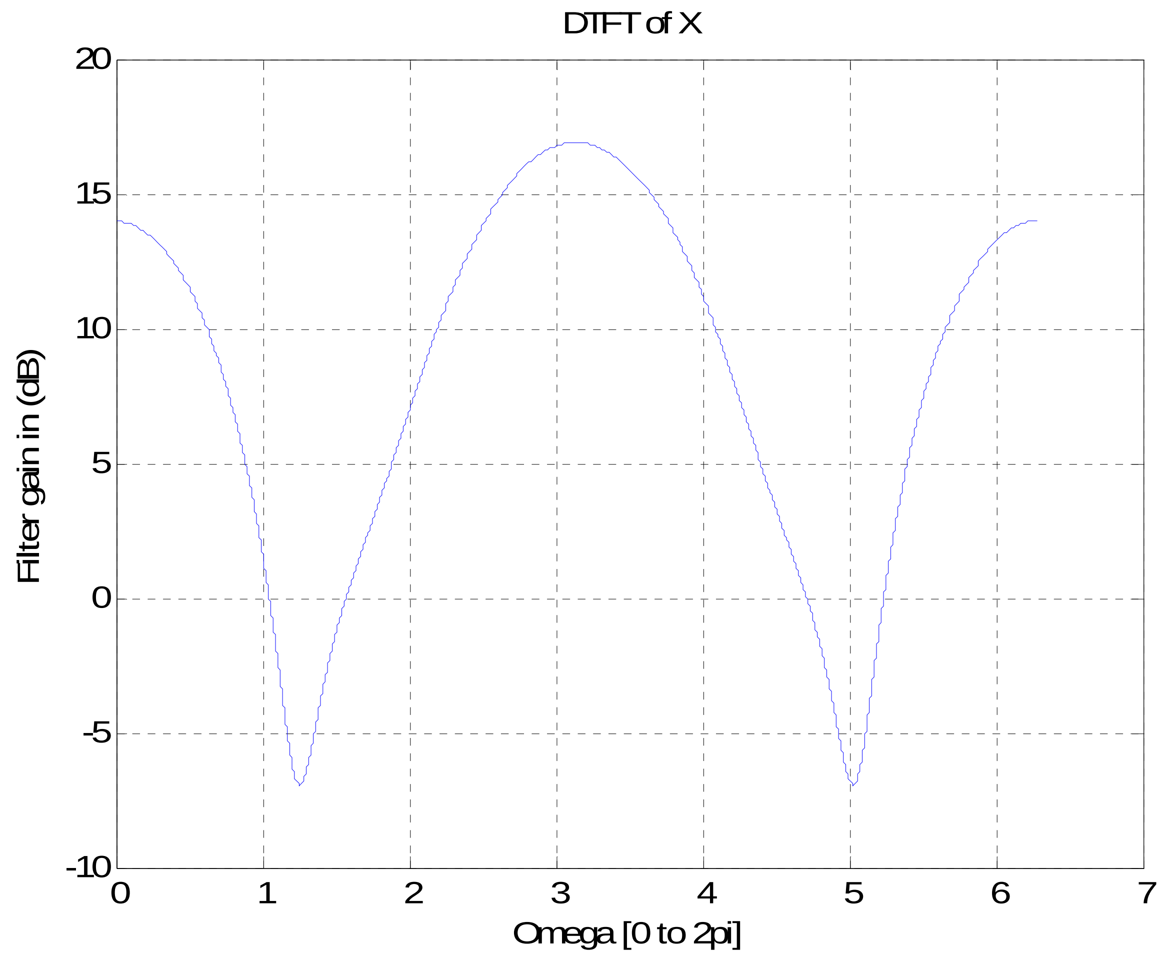
$$x[n] = 2\delta[n] - \delta[n-1] + 3\delta[n-2] + \delta[n-4]$$

$$X(\Omega) = F\{x[n]\}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(2\delta[n] - \delta[n-1] + 3\delta[n-2] + \delta[n-4]\right) e^{-j\Omega n}$$

$$= 2 - e^{-j1\Omega} + 3e^{-j2\Omega} + e^{-j4\Omega}$$




```
clc; clear all;

OM      = 0:0.001:(2*pi);
X       = 2 - exp(-j*OM) + 3*exp(-j*2*OM)+ exp(-j*4*OM);
X_dB    = 20*log10(abs(X));
plot(OM,X_dB)

xlabel('Omega [0 to 2pi]');
ylabel('Filter gain in (dB)');
title('DTFT of X');

grid;
```

Two properties of DTFT

- Effects of time delay
- Periodicity

Effects of time delay

Suppose that the DTFT of a signal $x[n]$ exists and is called $X(\Omega)$. The discrete time FT of a signal $x[n-n_0]$ will then be

$$F\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0]e^{-j\Omega n}$$

Letting $m = n - n_0$, or $(n = m + n_0)$, then the new equation is

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n-n_0]e^{-j\Omega n} &= \sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega(m+n_0)} \\ &= \sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega m}e^{-j\Omega n_0} \\ &= e^{-j\Omega n_0}\sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega m} \\ &= e^{-j\Omega n_0}X(\Omega)\end{aligned}$$

The delay of n_0 in the time domain introduces a complex exponential $e^{-j\Omega n_0}$ in the frequency domain.

Periodicity

Consider $X(\Omega-2\pi)$:

$$\begin{aligned} X(\Omega-2\pi) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega-2\pi)n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}\underbrace{e^{j2\pi n}}_{=1} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \\ &= X(\Omega) \end{aligned}$$

Since $X(\Omega-2\pi)=X(\Omega)$,the DTFT is periodic with period 2π .

The general difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

By taking the DTFT of each term,

$$a_0Y(\Omega)+a_1Y(\Omega)e^{-j\Omega}+a_2Y(\Omega)e^{-j2\Omega}+\ldots+a_NY(\Omega)e^{-jN\Omega}$$

$$=b_0X(\Omega)+b_1X(\Omega)e^{-j\Omega}+b_2X(\Omega)e^{-j2\Omega}+\ldots+b_NX(\Omega)e^{-jM\Omega}$$

$$Y(\Omega)\left(a_0+a_1e^{-j\Omega}+\ldots+a_Ne^{-jN\Omega}\right)=X(\Omega)\left(b_0+b_1e^{-j\Omega}+\ldots+b_Me^{-jM\Omega}\right)$$

$$\frac{Y(\Omega)}{X(\Omega)}=H(\Omega)=\frac{b_0+b_1e^{-j\Omega}+\ldots+b_Me^{-jM\Omega}}{a_0+a_1e^{-j\Omega}+\ldots+a_Ne^{-jN\Omega}}=$$

where $\mathbf{H}(\boldsymbol{\Omega})$ **is referred as the frequency response of the filter**.

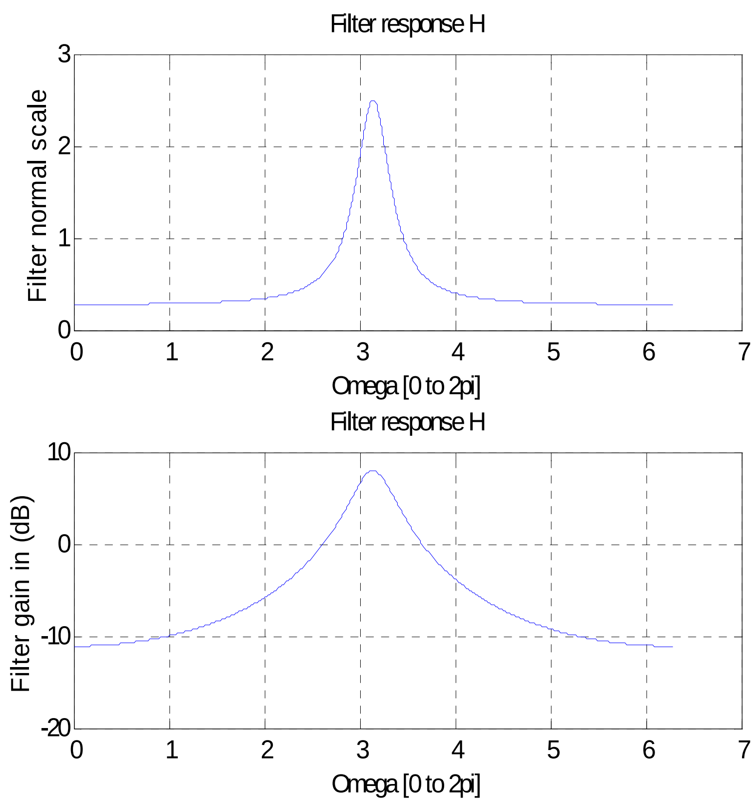
Ex 7.3) Find an expression for the frequency response of the filter described by the difference equation.

$$y[n] = -0.85y[n-1] + 0.5x[n]$$

$$Y(\Omega) + 0.85Y(\Omega)e^{-j\Omega} = 0.5X(\Omega)$$

$$Y(\Omega)\left(1 + 0.85e^{-j\Omega}\right) = 0.5X(\Omega)$$

$$\frac{Y(\Omega)}{X(\Omega)} = H(\Omega) = \frac{0.5}{1 + 0.85e^{-j\Omega}}$$



Assume that an output of a filter is $y[n]$ with an input $x[n]$, then

$$y[n] = h[n] * x[n]$$

$$\Rightarrow Y(\Omega) = H(\Omega)\, X(\Omega)$$

Since the frequency response of each is complex numbers, they can be expressed in polar form

$$Y(\Omega) = \left|Y(\Omega)\right| e^{j\theta_Y(\Omega)}$$
$$X(\Omega) = \left|X(\Omega)\right| e^{j\theta_X(\Omega)}$$
$$H(\Omega) = \left|H(\Omega)\right| e^{j\theta_H(\Omega)}$$

where $|\cdot|$ **is the gain or magnitude and** $\theta(\Omega)$ **is the phase**

So the output can be written with the polar forms

$$\begin{aligned} Y(\Omega) &= \left|Y(\Omega)\right| e^{j\theta(\Omega)} \\ &= X(\Omega) H(\Omega) \\ &= \left(\left|X(\Omega)\right| e^{j\theta_X(\Omega)}\right)\left(\left|H(\Omega)\right| e^{j\theta^H(\Omega)}\right) \\ &= \left|X(\Omega)\right|\left|H(\Omega)\right| e^{j(\theta_X(\Omega)+\theta_H(\Omega))} \end{aligned}$$

$$\left|Y(\Omega)\right| = \left|X(\Omega)\right|\left|H(\Omega)\right|$$

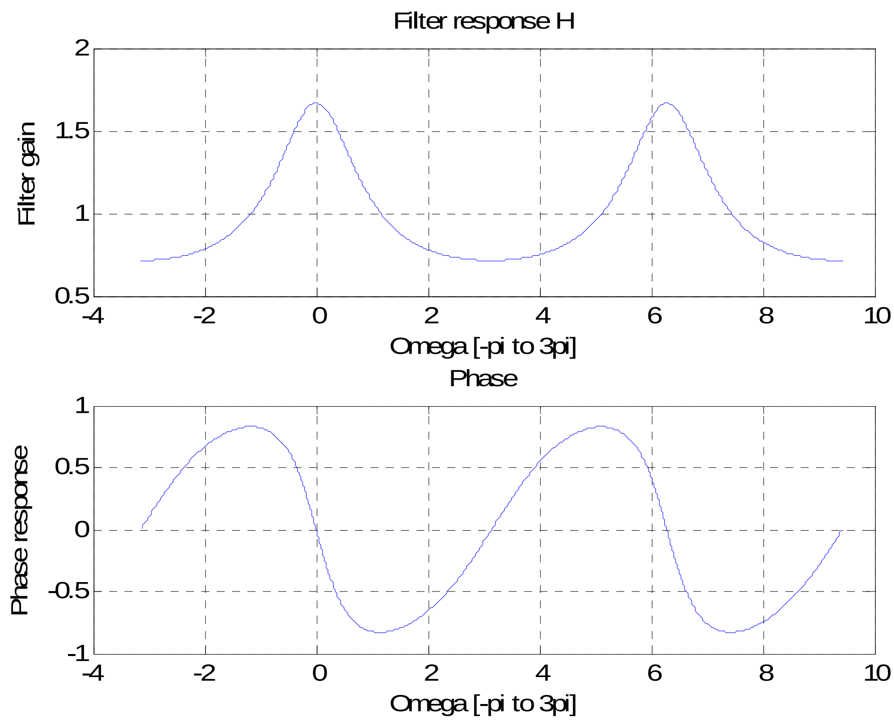
$$\theta_Y(\Omega) = \theta_H + \theta_X$$

The gain of the output is multiplication and the phase is addition

Ex 7.9)

The frequency response of the system is given. Plot the magnitude and phase response of the given system from - π to 3π.

$$H(\Omega) = \frac{1}{1-0.4e^{-j\Omega}}$$




```
clc; clear all;

OM      = -pi:0.001:(3*pi);
H       = 1./(1-0.4*exp(-j*OM));
H_dB    = 20*log10(abs(H));

subplot(2,1,1),plot(OM,H);
xlabel('Omega [-pi to 3pi]');
ylabel('Filter gain');
title('Filter response H');
grid;

% figure;
PH      = -atan((sin(OM))./(1-0.4*cos(OM)));

subplot(2,1,2),plot(OM,PH)
xlabel('Omega [-pi to 3pi]');
ylabel('Phase response');
title('Phase');
grid;
```

The above representation is

$$H(\Omega)=\frac{1}{1-0.4e^{-j\Omega}}$$

$$=\frac{1}{1-0.4\left(\cos(\Omega)-j\sin(\Omega)\right)}$$

$$=\frac{1}{\left[1-0.4\cos(\Omega)\right]+0.4j\sin(\Omega)}$$

$$=\frac{\left(\left[1-0.4\cos(\Omega)\right]-0.4j\sin(\Omega)\right)}{\left(\left[1-0.4\cos(\Omega)\right]+0.4j\sin(\Omega)\right)\cdot\left(\left[1-0.4\cos(\Omega)\right]-0.4j\sin(\Omega)\right)}$$

$$=\frac{\left(\left[1-0.4\cos(\Omega)\right]-0.4j\sin(\Omega)\right)}{\left(\left[1-0.4\cos(\Omega)\right]^2+\left[0.4\sin(\Omega)\right]^2\right)}$$

$$\left|H(\Omega)\right|=\frac{1}{\sqrt{\left[1-0.4\cos(\Omega)\right]^2+\left[0.4\sin(\Omega)\right]^2}}$$

$$\theta_H=-\operatorname{atan}\left(\frac{0.4\sin(\Omega)}{1-0.4\cos(\Omega)}\right)$$

Ex 7.15

The difference equation describing a digital filter is shown below. Find the frequency response of gain and phase.

$$y[n]=1.5y[n-1]-0.85y[n-2]+x[n]$$

$$Y(\Omega)\left(1-1.5e^{-j\Omega}+0.85e^{-j2\Omega}\right)=X(\Omega)$$

or

$$Y(z)\left(1-1.5z^{-1}+0.85z^{-2}\right)=X(z) \qquad \text{where} \quad z=e^{j\Omega}$$

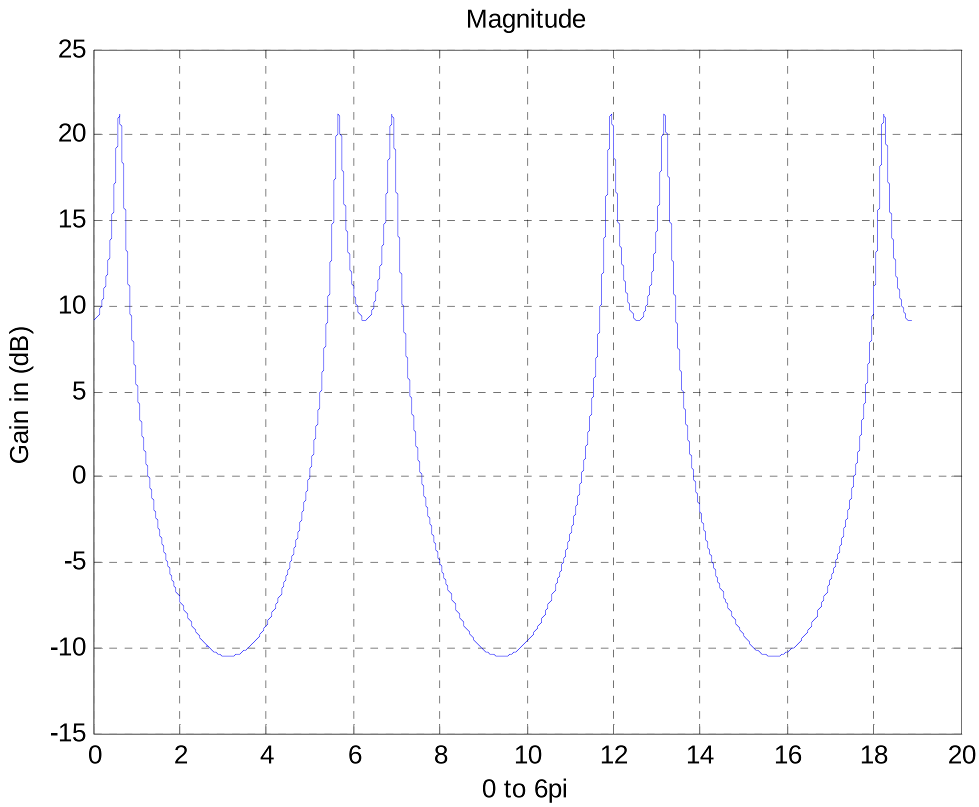
$$H(z)=H(\Omega)=\frac{Y(z)}{X(z)}=\frac{Y(\Omega)}{X(\Omega)}$$

$$=\frac{1}{1-1.5z^{-1}+0.85z^{-2}}$$

$$=\frac{1}{1-1.5e^{-j\Omega}+0.85e^{-j2\Omega}}$$

$$=\frac{1}{1-1.5\left(\cos(\Omega)-j\sin(\Omega)\right)+0.85\left(\cos(2\Omega)-j\sin(2\Omega)\right)}$$

$$=\frac{1}{\left[1-1.5\cos(\Omega)+0.85\cos(2\Omega)\right]+j\left[1.5\left(\sin(\Omega)\right)-0.85\left(\sin(2\Omega)\right)\right]}$$




```
clc; clear all;

OM      = 0:0.001:6*pi;
H       = 1./(1-1.5*exp(-j*OM)+0.85*exp(-j*2*OM));

H_dB    = 20*log10(H);

plot(OM, H_dB);
grid;
xlabel('0 to 6pi');
ylabel('Gain in (dB)');
title('Magnitude');

figure;
PH      = -atan((1.5*sin(OM)-0.85*sin(2*OM))./(1-
1.5*cos(OM)+0.85*cos(2*OM)));
plot(OM,PH)
```

Filter shape from poles and zeros

$$H(z) = \frac{1}{z-p}$$

The frequency response for this filter is

$$H(\Omega) = \frac{1}{e^{j\Omega} - p}$$

and the magnitude response, or filter shape is

$$|H(\Omega)| = \left|\frac{1}{e^{j\Omega} - p}\right| = \frac{1}{|e^{j\Omega} - p|} = \frac{1}{\cos(\Omega) + j\sin(\Omega) - p}$$

Assume $p = a + j\beta$

$$|H(\Omega)| = \left|\frac{1}{e^{j\Omega} - p}\right| = \frac{1}{|e^{j\Omega} - p|} = \frac{1}{|\cos(\Omega) + j\sin(\Omega) - (a + j\beta)|}$$

The magnitude is finding the distance of two points,

$(\cos(\Omega), \sin(\Omega))$ and (a, β)

$$|\cos(\Omega) + j\sin(\Omega) - (a + j\beta)| = \sqrt{[\cos(\Omega) - a]^2 + [\sin(\Omega) + \beta]^2}$$

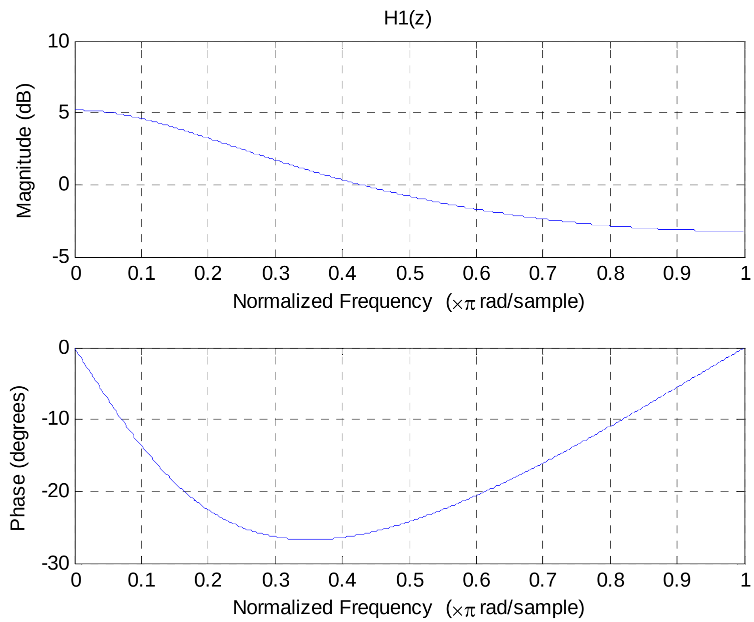
$$|H(\Omega)| = \left|\frac{1}{e^{j\Omega} - p}\right| = \frac{1}{|e^{j\Omega} - p|} = \frac{1}{\text{distance from } (e^{j\Omega} \text{ to } p)}$$

In general

$$\left|H(\Omega)\right| = \left|\frac{K(z-z_0)}{(z-p_0)(z-p_1)}\right| = \frac{\mathrm{K}(\text{distance to zeros})}{(\text{distance to } p_0)(\text{distance to } p_1)}$$

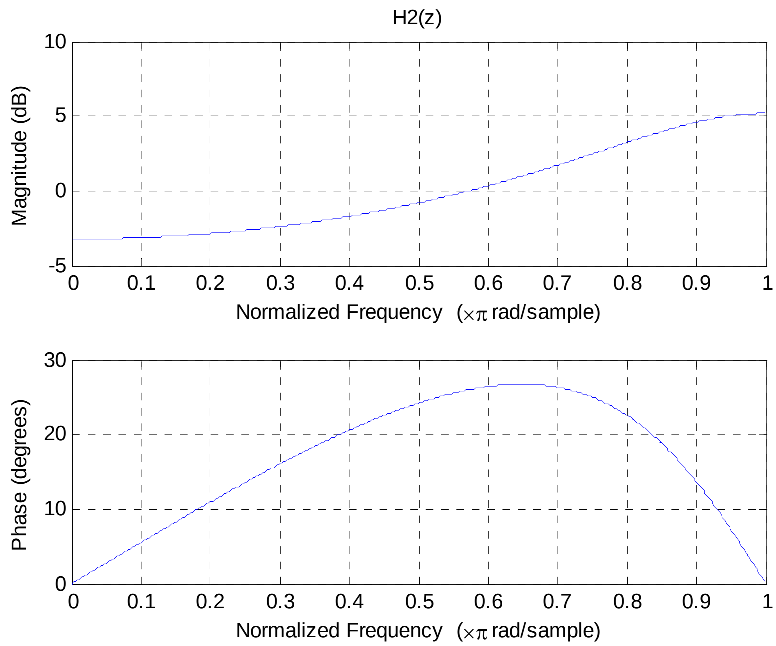
Ex

$$H_1(z) = \frac{1}{1-0.45z^{-1}} \quad \text{vs} \quad H_2(z) = \frac{1}{1+0.45z^{-1}}$$



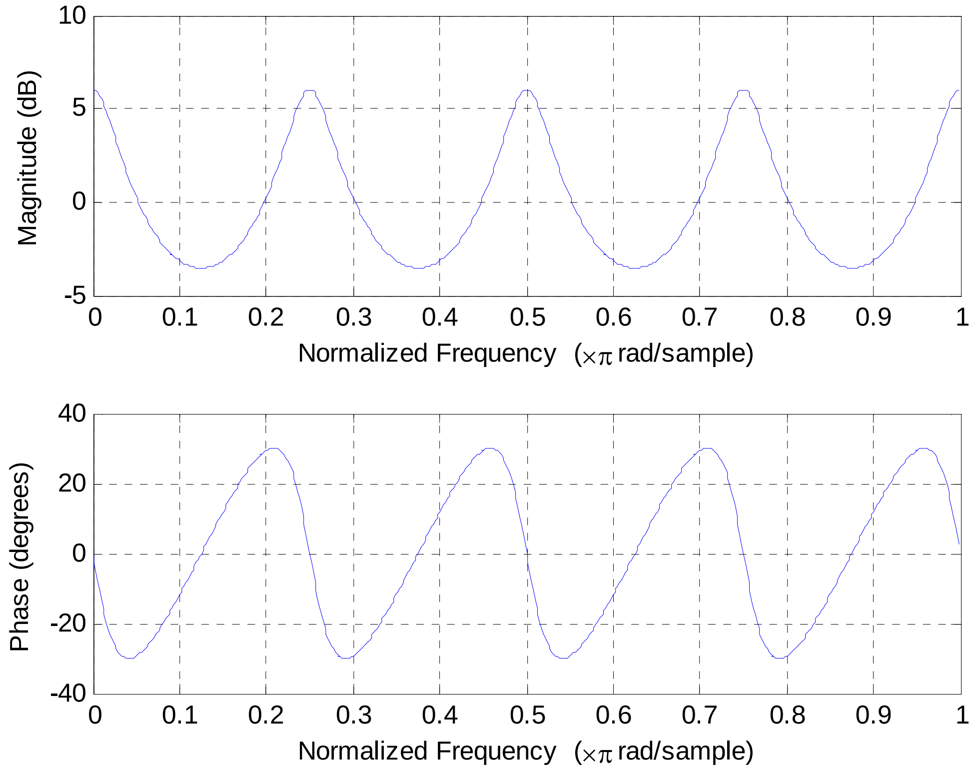

```
freqz(1,[1 -0.45])
title('H1(z)')

figure(2)
freqz(1,[1 0.45])
title('H2(z)')
```

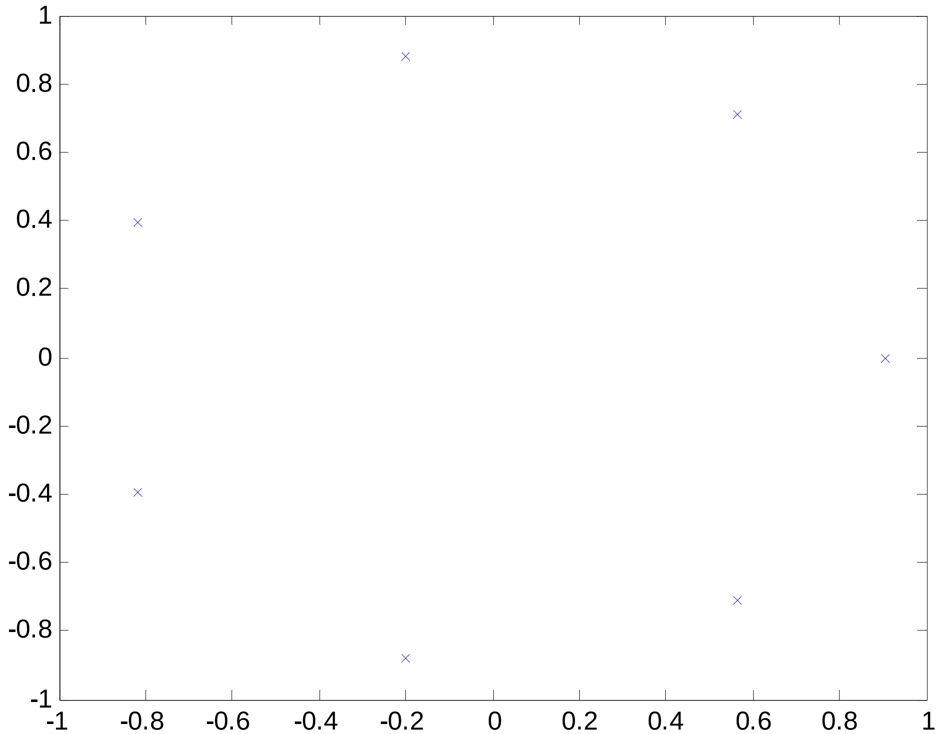


Ex7.16)

$$H(z)=\frac{1}{1-0.5z^{-8}}$$




```
freqz(1, [1 zeros(1,7) -0.5])
```


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1