Ch 10 Infinite Impulse Response Filters (IIR)

This chapter introduces a second major class of digital filters: **infinite impulse response, or IIR filters.**

The chapter:

1. reviews the difference equation and transfer function for a recursive filter
2. defines simple low pass analog filter types
3. introduces the bilinear transformation as a means of converting from an analog to a digital filter
4. describes how to choose IIR filter order given filter specifications presents low pass Butterworth filter designs
5. presents low pass Chehyshev Type 1 filter designs
6. describes the impulse invariants design method
7. discusses methods for creating band pass, high pass, and band stop filters from low pass filters
8. highlights practical considerations

The general difference equation with $a_0 = 1$ is

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\begin{aligned} y[n] &= -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \\ &= \underbrace{-a_1[y-1] - a_2 y[n-2] - \ldots a_N y[n-N]}_{\text{past outputs}} + \underbrace{b_0 x[n]}_{\text{current input}} + \underbrace{b_1 x[n-1] + \ldots b_M x[n-M]}_{\text{past inputs}} \end{aligned}$$

The transfer function for a general recursive filter is

$$H(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\ldots b_M z^{-M}}{1+a_1z^{-1}+a_2z^{-2}+\ldots a_N z^{-N}}$$

Assuming $N > M$

$$H(z)=\frac{b_0z^N+b_1z^{N-1}+b_2z^{N-2}+\ldots+b_M z^{N-M}}{z^N+a_1z^{N-1}+a_2z^{N-2}+\ldots+a_N}$$

- Recursive filters **cannot be guaranteed to be stabl**e because of poles.
- It is **not easy to achieve the linear phase** that was guaranteed for nonrecursive filters.
- Recursive filters **require far fewer coefficients than nonrecursive filters** to achieve similar performance specifications.

The design approach for IIR filters:

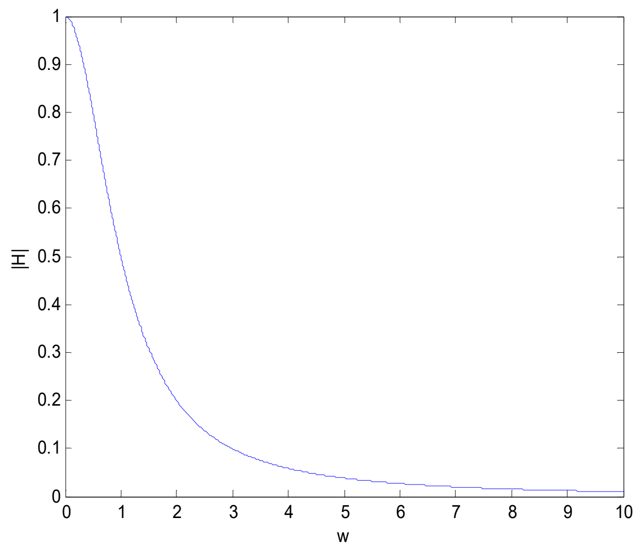
- **Choose a prototype analog filter that has the desired characteristics and then convert it to a digital filter.**

In continuous domain, the transfer function for a simple **low pass analog filter** is

$$H(s)=\frac{1}{s+1}$$

$$H(\omega)=\frac{1}{j\omega+1}$$

$$|H(\omega)|=\left|\frac{1}{j\omega+1}\right|=\frac{1}{\sqrt{\omega^2+1}}$$



If $\omega = 1$, then $H(\omega) = \frac{1}{\sqrt{2}} \rightarrow 20\log\frac{1}{\sqrt{2}} = \text{-}3dB$

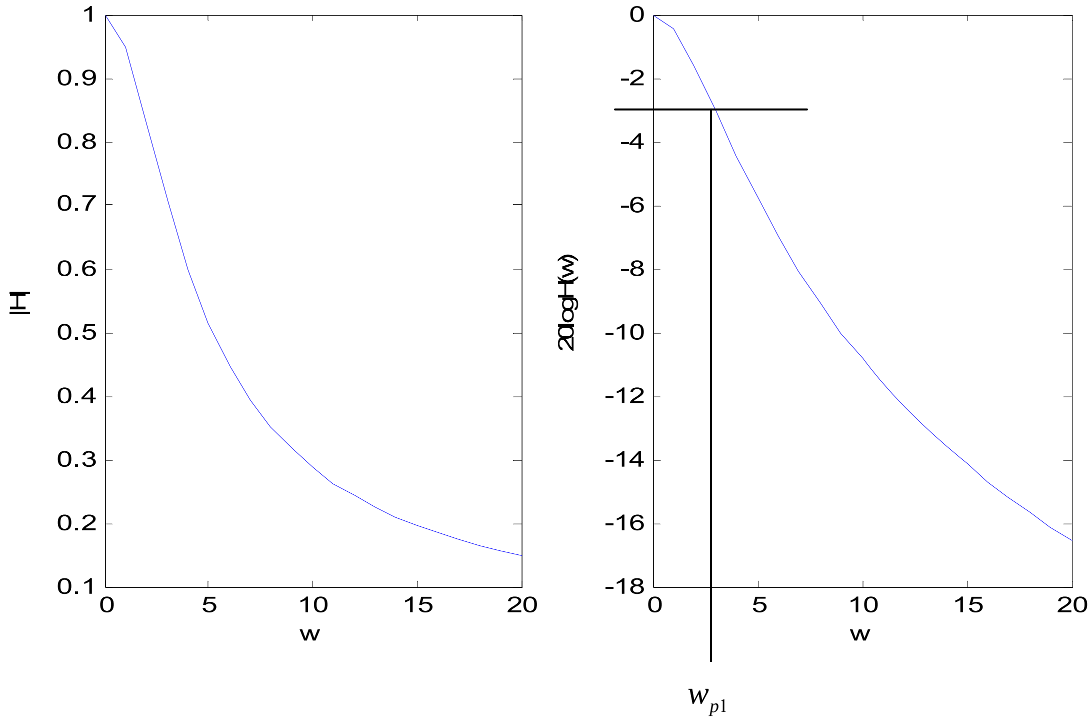
More general transfer function to change the frequency at $\text{-}3dB$ is

$$H(s) = \frac{\omega_{p1}}{s + \omega_{p1}}$$

$$H(\omega) = \frac{\omega_{p1}}{j\omega + \omega_{p1}} = \frac{1}{j\left(\frac{\omega}{\omega_{p1}}\right) + 1}$$

$$|H(\omega)| = \left|\frac{\omega_{p1}}{j\omega + w_{p1}}\right| = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_{p1}}\right)^2 + 1}}$$

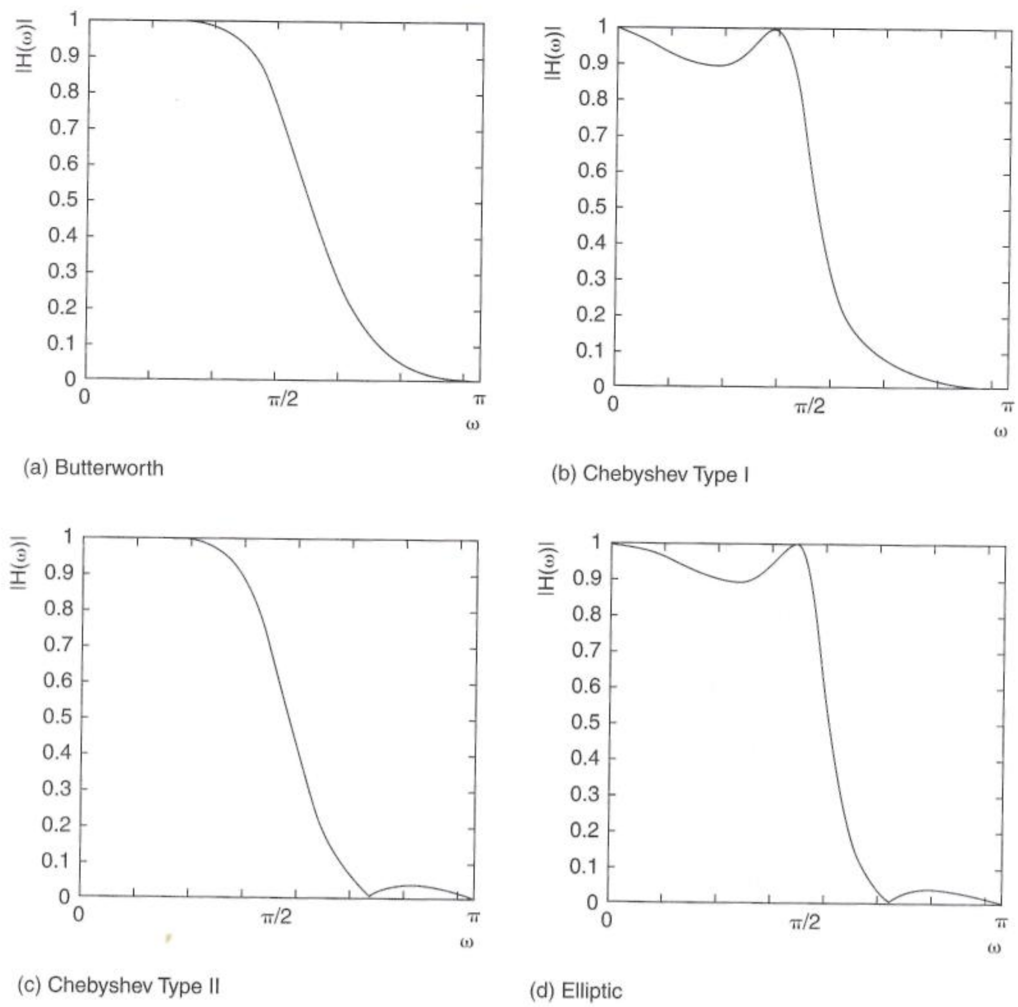
- This is still low pass filter, but the $\text{-}3dB$ frequency occurs when $\omega = \omega_{p1}$ rad/sec or $\left[f = \frac{\omega_{p1}}{2\pi}\ Hz \right]$



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
|H|
0
5
10
15
20
w
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
20log H(w)
0
5
10
15
20
w
w_{p1}

Common analog filter types:

- **Butterworth**: Monotonic through the pass and stop band.
- **Chebyshev Type I**: Monotonic through the stop band
- **Chebyshev Type II**: Monotonic in the pass band
- **Elliptic**: Ripple in both pass and stop ban.



(a) Butterworth (b) Chebyshev Type I (c) Chebyshev Type II (d) Elliptic

Bilinear transformation

- Provides a means of converting between an analog filter and a digital filter

$$s \Leftrightarrow 2\,fs\left(\frac{z-1}{z+1}\right)$$

fs : sampling frequency

$z \rightarrow e^{j\Omega}$

$$s \Leftrightarrow 2\,fs\frac{e^{j\Omega}-1}{e^{j\Omega}+1} = 2\,fs\frac{\cancel{e^{j\frac{\Omega}{2}}}\left(e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}\right)}{\cancel{e^{j\frac{\Omega}{2}}}\left(e^{j\frac{\Omega}{2}}+e^{-j\frac{\Omega}{2}}\right)} = 2\,fs\frac{e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}}{e^{j\frac{\Omega}{2}}+e^{-j\frac{\Omega}{2}}}$$

$$= 2\cdot fs\cdot\frac{\frac{2j}{2j}e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}}{\frac{2}{2}e^{j\frac{\Omega}{2}}+e^{-j\frac{\Omega}{2}}} = 2\cdot fs\cdot\frac{\left(\frac{2je^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}}{2j}\right)}{\left(\frac{2e^{j\frac{\Omega}{2}}+e^{-j\frac{\Omega}{2}}}{2}\right)} = \cancel{2}\cdot fs\cdot\frac{(2j)\sin\left(\frac{\Omega}{2}\right)}{\cancel{2}\cos\left(\frac{\Omega}{2}\right)} = (2j)\cdot fs\cdot\tan\left(\frac{\Omega}{2}\right)$$

$$\cancel{j}\omega = \cancel{j}2\cdot fs\cdot\tan\left(\frac{\Omega}{2}\right) \Rightarrow \underline{\omega = 2\cdot fs\cdot\tan\left(\frac{\Omega}{2}\right)} \quad \leftarrow \textbf{prewarping equation}$$

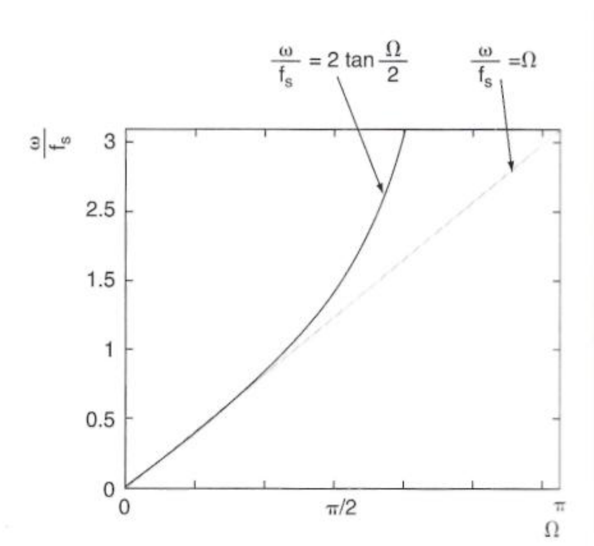
$$\omega = 2\cdot fs\cdot\tan\left(\frac{\Omega}{2}\right) \Leftrightarrow \frac{\omega}{2\,fs} = \tan\left(\frac{\Omega}{2}\right) \Rightarrow \tan^{-1}\left(\frac{\omega}{2\,fs}\right) = \frac{\Omega}{2} \Rightarrow \left[\Omega = 2\tan^{-1}\left(\frac{\omega}{2\,fs}\right)\right]$$

Ω: range of this frequency is 0 to π

So the digital frequency, Ω, maps to 0 to ∞ rad/sec for the analog frequency ω.

$$\omega = 2\pi f$$
$$\Omega = 2\pi \frac{f}{fs}$$

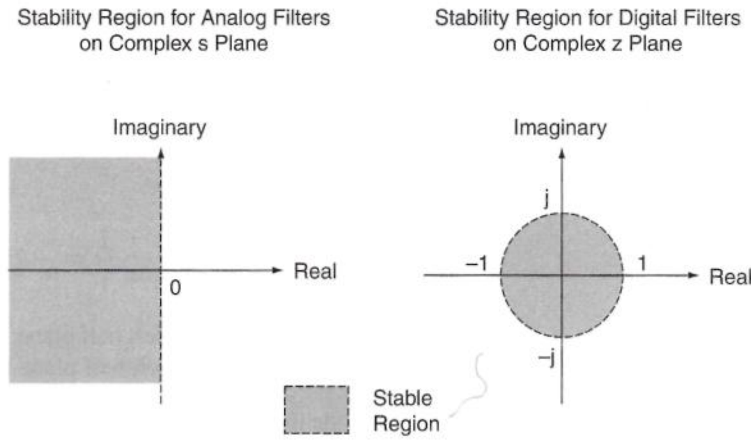
$$\omega = \cancel{2\pi}\, \cancel{f} \Leftrightarrow \Omega = \cancel{2\pi} \frac{\cancel{f}}{fs} \rightarrow \left[\Omega = \frac{\omega}{fs} \right]$$



For small values of Ω, $\tan\left(\frac{\Omega}{2}\right) = \frac{\Omega}{2}$

$$\omega = 2 \cdot fs \cdot \tan\left(\frac{\Omega}{2}\right) = 2 \cdot fs \cdot \frac{\Omega}{2} = 2 \cdot fs \cdot \frac{\left(2\pi \frac{f}{fs}\right)}{2} = 2\,\cancel{fs}\, \frac{\cancel{2}\pi \frac{f}{\cancel{fs}}}{\cancel{2}} = 2\pi f$$

Stability regions for analog and digital filters



EXAMPLE 10.2

This example provides some sample calculations to show how points are mapped from the *z* domain complex plane to the *s* domain complex plane by the bilinear transformation.

a. $z = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$ (on unit circle)

Performing the bilinear transformation:

$$s = 2f_S\frac{z-1}{z+1} = 2f_S\frac{\frac{1+j}{\sqrt{2}} - 1}{\frac{1+j}{\sqrt{2}} + 1} = 2f_S\frac{1+j-\sqrt{2}}{1+j+\sqrt{2}}$$

$$= 2f_S\left(\frac{1-\sqrt{2}+j}{1+\sqrt{2}+j}\right)\left(\frac{1+\sqrt{2}-j}{1+\sqrt{2}-j}\right) = 2f_S\frac{j2\sqrt{2}}{4+2\sqrt{2}} = jf_S\frac{2\sqrt{2}}{2+\sqrt{2}}$$

This complex number is purely imaginary, so it lies on the imaginary axis: A point on the unit circle in the *z* domain maps to a point on the imaginary axis in the *s* domain.

b. $z = 0$ (inside the unit circle)

Using the bilinear transformation:

$$s = 2f_S \frac{z-1}{z+1} = -2f_S$$

This complex number lies in the left half plane: A point inside the unit circle in the z domain maps to a point in the left half plane in the s domain.

c. $z = 2 + j$ (outside the unit circle)

Transforming:

$$s = 2f_S \frac{z-1}{z+1} = 2f_S \frac{2+j-1}{2+j+1} = 2f_S \frac{1+j}{3+j}$$

$$= 2f_S \frac{1+j3-j}{3+j3-j} = 2f_S \frac{4+j2}{10} = \frac{4f_S}{5} + j\frac{2f_S}{5}$$

This complex number lies in the right half plane: A point outside the unit circle in the z domain maps to a point in the right half plane in the s domain.

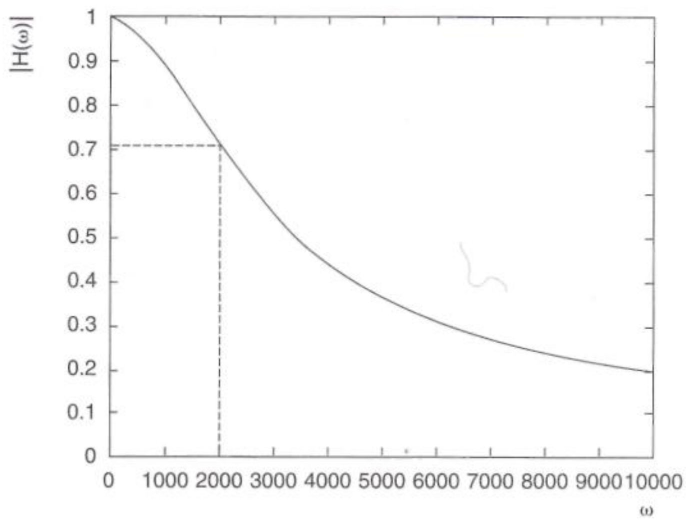
EXAMPLE 10.3

The transfer function for a first order analog low pass filter is described by Equation. The filter has a -3 dB frequency of 2000 rad/sec, or $2000/2\pi$ = 318.31 Hz.

Find the transfer function $H(z)$ for a digital filter that matches this analog filter, for a 1500 Hz sampling rate.

The original transfer function $H(s)$ in the analog domain is

$$H(s) = \frac{2000}{s+2000} \Rightarrow |H(w)| = \frac{1}{\sqrt{\left(\frac{w}{2000}\right)^2 + 1}}$$



Magnitude response for the analog filter

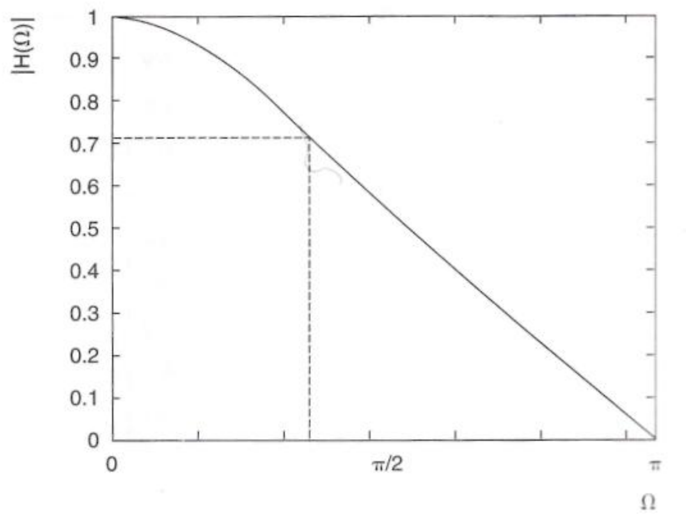
The magnitude response for the analog filter is shown in the above Figure.

A digital filter must be designed to match this analog filter. The sampling rate is 1500 Hz, and the bilinear transformation requires that each *s* in the analog transfer function of $\left[H(s)=\dfrac{2000}{s+2000}\right]$ be replaced by the ratio $\left[s \Leftrightarrow 2\,fs\dfrac{z-1}{z+1}\right]$. The transfer function in the digital domain becomes:

$$
\begin{aligned}
H(z) &= \frac{2000}{3000\dfrac{z-1}{z+1}+2000} \\
&= \frac{2000(z+1)}{3000(z-1)+2000(z+1)} = \frac{2000(z+1)}{5000z-1000} \\
&= \frac{0.4(z+1)}{z-0.2} = \frac{0.4(1+z^{-1})}{1-0.2z^{-1}}
\end{aligned}
$$

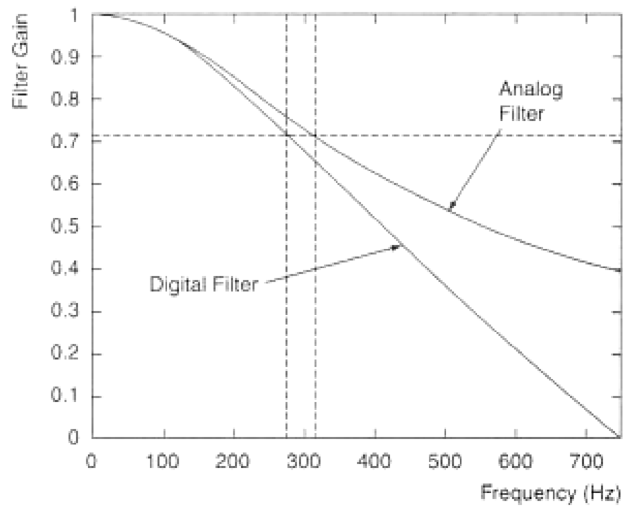
The frequency response is

$$H(\Omega)=\frac{0.4\left(1+e^{-j\Omega}\right)}{1-0.2e^{-j\Omega}}$$



Magnitude response for the digital filter

- The Fig. 10-9 compares the analog and digital filter shape for this example
- The gains for both are plotted against frequency in Hz.
- The plot is produced from those of $|H(\Omega)|$ vs. Ω and $|H(\omega)|$ vs. ω.
- Using the relationship $\left[\Omega = \frac{2\pi f}{f_s}\right]$ and $[\omega = 2\pi f]$ to convert Ω and ω to frequencies in Hz.
- **Notice that the roll-off for the digital filter is steeper than that for the analog filter.**
- **The bandwidth of the digital filter does not agree with that of the analog filter.**
- **This error occurs because of the warping effect of the bilinear transformation.**
- Fortunately, warping errors can be overcome by prewarping the -3dB frequency using $\left[\omega = 2\, fs \tan\left(\frac{\Omega}{2}\right)\right]$.
- When this step is added to the design process, the correct bandwidth is obtained.



Effect of frequency warping

EXAMPLE 10.4

A simple analog low pass filter described by the analog transfer function in Equation (10.1) must be converted into a digital filter with a −3 dB frequency of 318.3 Hz. The sampling rate is 1500 Hz. Find expressions for the transfer function and the frequency response of the digital filter.

The same goal was stated, but not achieved, in Example 10.3. In this example, steps are taken to ensure the goal is reached. With a −3 dB frequency of $f_{p1} = 318.3$ Hz and a sampling frequency of 1500 Hz, the digital cut-off frequency is

$$\Omega_{p1} = 2\pi\frac{f_{p1}}{f_S} = 2\pi\frac{318.3}{1500} = 0.4244\pi \text{ radians}$$

To allow for warping by the bilinear transformation, prewarp the analog frequency using Equation (10.5):

$$\omega_{p1} = 2f_S\tan\left(\frac{\Omega_{p1}}{2}\right) = 2f_S\tan\left(\frac{0.4244\pi}{2}\right) = 2360.4\text{rad/sec}$$

This frequency will be used in place of the unwarped $\omega_{p1} = 2\pi(318.31) = 2000$ rad/sec. The transfer function for the prototype analog filter is then

$$H(s) = \frac{\omega_{p1}}{s + \omega_{p1}} = \frac{2360.4}{s + 2360.4} \qquad \textbf{(10.7)}$$

Converting to a digital filter using the bilinear transformation $s = 3000\,(z - 1)/(z + 1)$ from Equation (10.4) gives the transfer function

$$H(z) = \frac{0.4403(1 + z^{-1})}{1 - 0.1193z^{-1}}$$

The frequency response for this prewarped digital filter is

$$H(\Omega) = \frac{0.4403(+ e^{-j\Omega})}{1 - 0.1193e^{-j\Omega}}$$

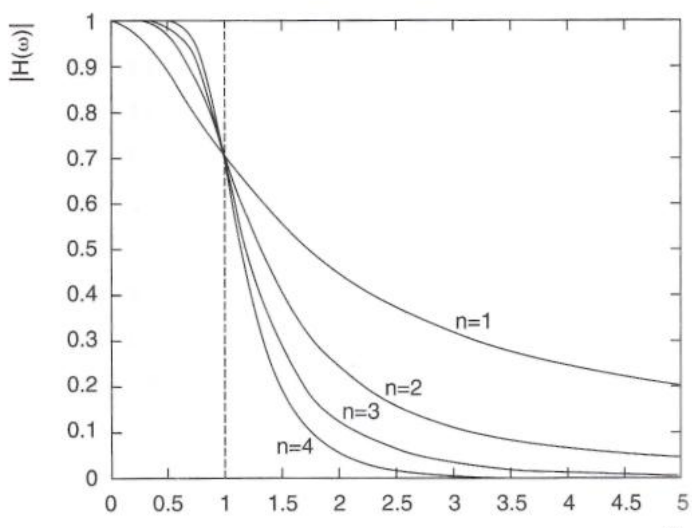
Butterworth Filter Design

- Butterworth filters are the simplest filters in the IIR family.
- This filter doesn't offer linear phase characteristic:
- The n^{th} order analog Butterworth filter magnitude response is

$$\left|H(w)\right| = \frac{1}{\sqrt{\left(\frac{w}{w_{p1}}\right)^{2n} + 1}}$$

- If $n = 1$, first order, magnitude response of Butterworth filter is

$$H(s) = \frac{w_{p1}}{s + w_{p1}} \Rightarrow \left|H(w)\right| = \frac{1}{\sqrt{\left(\frac{w}{w_{p1}}\right)^{2} + 1}}$$



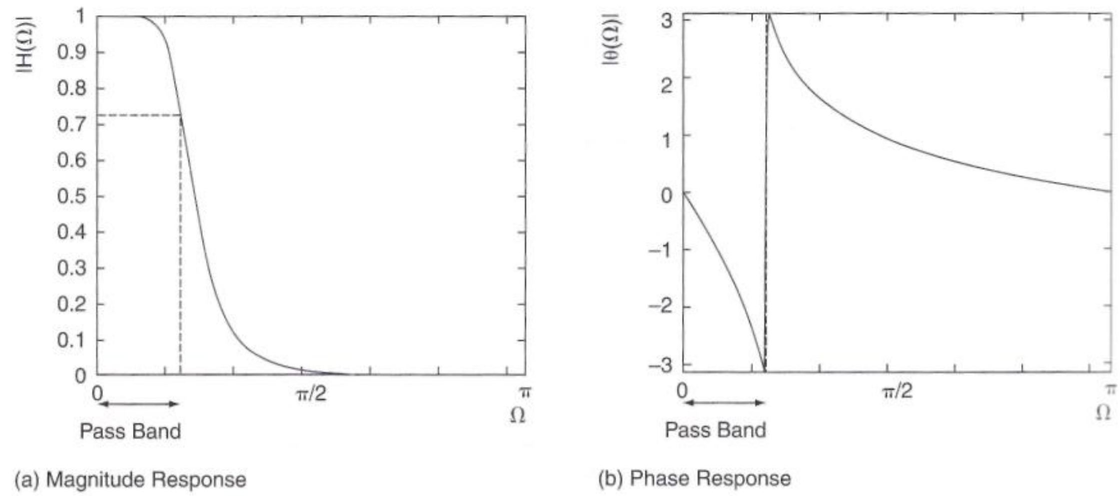
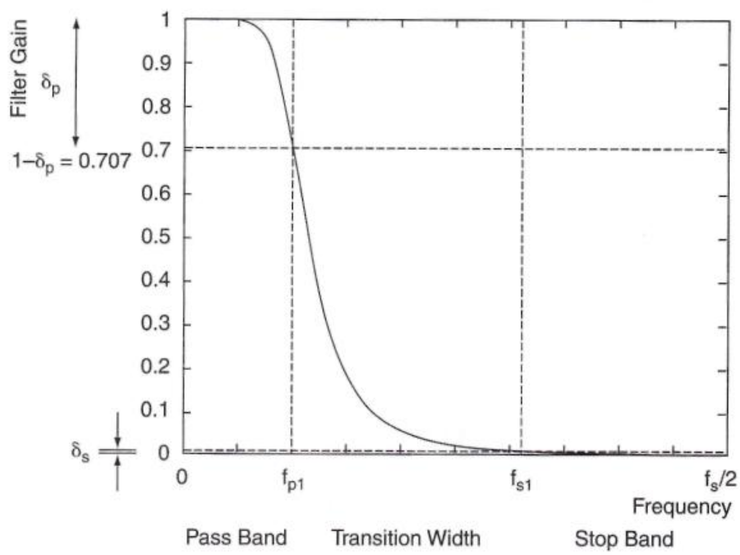
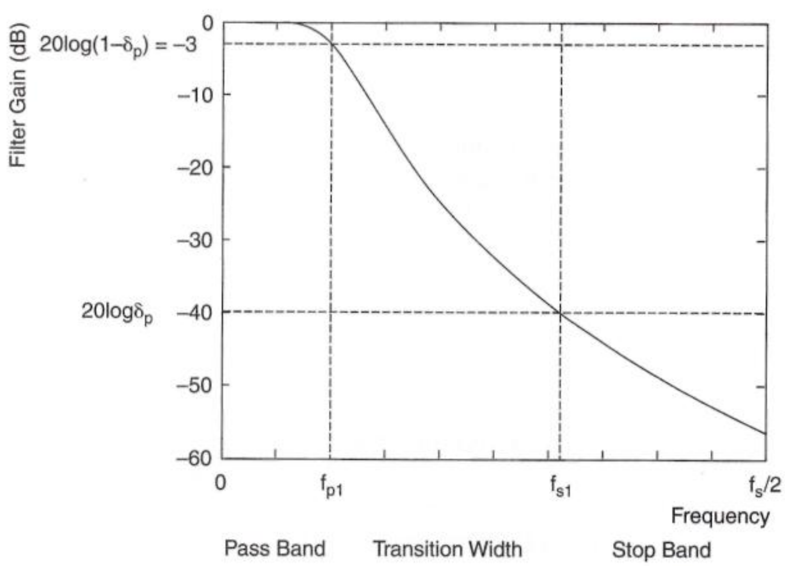


FIGURE 10.14

Frequency response of fourth order Butterworth filter.



(a) Linear Gain Versus Frequency



(b) Logarithmic Gain Versus Frequency

FIGURE 10.13
Butterworth filter parameters.

TABLE 10.2

Design Steps for Low Pass Butterworth Filter Design

1. Identify the desired pass band edge frequency f_{p1} Hz, the desired stop band edge frequency f_{s1} Hz, and the desired stop band attenuation $-20\log\delta_s$ dB (or the desired stop band gain $20\log\delta_S$ dB) for the digital filter. The pass band edge frequency must correspond to a gain of -3 dB.
2. Convert the desired edge frequencies in Hz into digital frequencies in radians, using $\Omega = 2\pi f/f_S$, to obtain Ω_{p1} and Ω_{s1}.
3. Calculate prewarped analog frequencies to avoid distortions due to bilinear transformation. Use $\omega = 2f_S \tan(\Omega/2)$ to obtain ω_{p1} and ω_{s1}, in radians per second.
4. Determine the gain δ_s at the edge of the stop band from the specified attenuation $-20\log\delta_s$ (or gain $20\log\delta_s$).
5. Calculate the filter order required using

$$n \geq \frac{\log\left(\frac{1}{\delta_s^2} - 1\right)}{2\log\left(\frac{\omega_{s1}}{\omega_{p1}}\right)}$$

An integer value for n must be chosen.

6. Substitute ω_{p1} into the nth order analog Butterworth transfer function $H(s)$, and perform a bilinear transformation on $H(s)$ to obtain the nth order digital transfer function $H(z)$.[1] The difference equation needed for filter implementation can be found directly from the transfer function $H(z)$. Filter shape $|H(\Omega)|$ can be obtained by substituting $\omega = 2f_S \tan(\Omega/2)$ into

$$|H(\omega)| = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_{p1}}\right)^{2n} + 1}}$$

Ex 6)

A low pass filter with a Butterworth characteristic is to be designed to have a $-3dB$ frequency of 1200Hz. The gain must drop to -25dB by 1500 Hz. The sampling rate is 8000Hz. Choose a suitable order for the filter and plot the filter shape.

These specs are fairly demanding since the gain must drop by 25-3=22 dB in only 300 Hz transition band, rather a small proportion of the Nyquist frequency of $\frac{fs}{2}$=4000 Hz. The required order will be relatively high. Following the design steps in Table 10.2.:

Step1: The analog edge frequencies are $f_{p1} = 1200Hz$ and $f_{s1} = 1500Hz$. The gain at the edge of the stop band is $20\log\delta_s = -25dB$

Step2: The digital edge frequencies are

$$\Omega_{p1} = 2\pi\frac{f_{p1}}{fs} = 2\pi\frac{1200}{8000} = 0.3\pi \text{ radians}$$

$$\Omega_{s1} = 2\pi\frac{f_{s1}}{f_S} = 2\pi\frac{1500}{8000} = 0.375\pi \text{ radians}$$

Step 3: The prewarped analog edge frequencies are:

$$\omega_{p1} = 2f_S\tan\frac{\Omega_{p1}}{2} = 8152.4 \text{ rad/sec}$$

$$\omega_{s1} = 2f_S\tan\frac{\Omega_{s1}}{2} = 10690.9 \text{ rad/sec}$$

Step 4: Since, from Step 1, $20\log\delta_s = -25$, then $\log\delta_s = -25/20$, and $\delta_s = 10^{-25/20} = 0.0562$.

Step 5: The required order of the filter is

$$n \geq \frac{\log\left(\frac{1}{\delta_s^2} - 1\right)}{2\log\left(\frac{\omega_{s1}}{\omega_{p1}}\right)} = \frac{\log\left(\frac{1}{(0.0562)^2} - 1\right)}{2\log\left(\frac{10690.9}{8152.4}\right)} = 10.6$$

A Butterworth filter of order 11 will slightly exceed the specifications given.

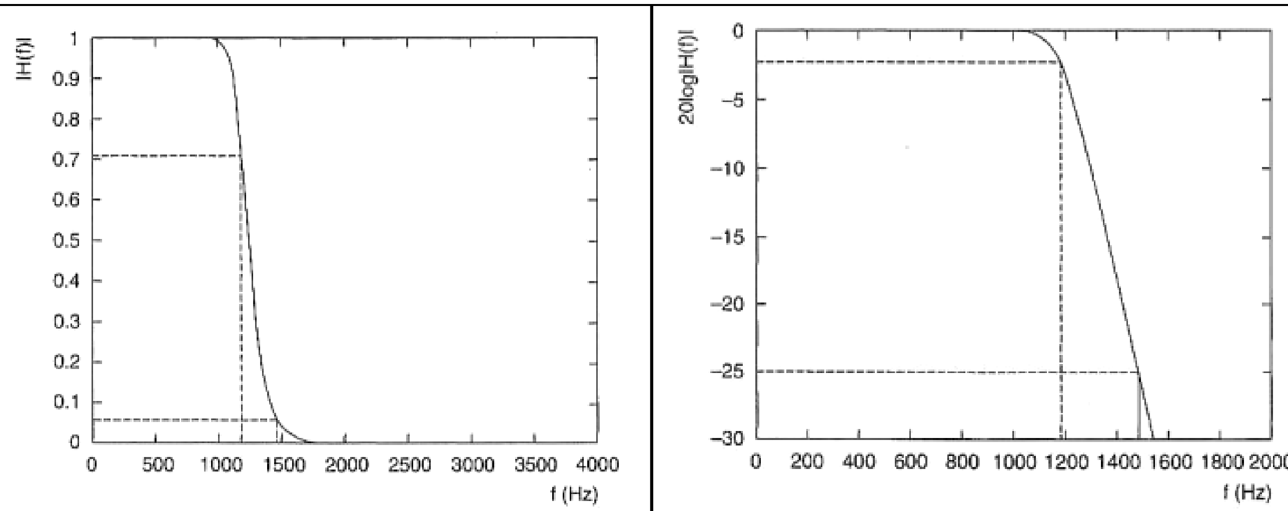
Step 6: Filter design software is best used to produce the digital transfer function $H(z)$ for this 11th order Butterworth filter. The filter shape for the analog prototype is:

$$|H(\Omega)| = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_{p1}}\right)^{2n} + 1}} = \frac{1}{\sqrt{\left(\frac{\omega}{8152.4}\right)^{22} + 1}}$$

Substituting $\omega = 2f_S \tan(\Omega/2)$ into this analog prototype, the filter shape for the digital filter is:

$$|H(\Omega)| = \frac{1}{\sqrt{\left(\frac{2f_S \tan\left(\frac{\Omega}{2}\right)}{\omega_{p1}}\right)^{2n} + 1}} = \frac{1}{\sqrt{\left(\frac{16000\tan\left(\frac{\Omega}{2}\right)}{8152.4}\right)^{22} + 1}}$$

The shape of the filter is presented using linear magnitudes in Figure 10.15(a). Instead of plotting the filter shape against Ω from 0 to π, the shape is plotted against frequencies from 0 to $f_S/2$, as discussed in Section 7.3.3. Figure 10.15(b), which plots the magnitudes in dB, demonstrates that the specifications have been satisfied. The cut-off point is in the expected location, and a gain of -25 dB is reached just below 1500 Hz.



(a) Linear Magnitude Response

(b) Logarithmic Magnitude Response

EXAMPLE 10.7

A low pass Butterworth filter has a gain of −3 dB at the edge of its pass band, at 1 kHz, and a stop band attenuation at 12 kHz of 30 dB. Find the difference equation and plot the frequency response for the filter, if the sampling rate is 25 kHz.

Following the design steps in Table 10.2:

Step 1: The analog edge frequencies are:

$$f_{p1} = 1000 \text{ Hz}$$

$$f_{s1} = 12000 \text{ Hz}$$

The attenuation at the edge of the stop band is $-20\log\delta_s = 30$ dB.

Step 2: The digital edge frequencies are:

$$\Omega_{p1} = 2\pi\frac{f_{p1}}{f_S} = 2\pi\frac{1000}{25000} = 0.08\pi \text{ radians}$$

$$\Omega_{s1} = 2\pi\frac{f_{s1}}{f_S} = 2\pi\frac{12000}{25000} = 0.96\pi \text{ radians}$$

Step 3: The prewarped analog edge frequencies are:

$$\Omega_{p1} = 2f_S\tan\frac{\Omega_{p1}}{2} = 6316.5 \text{ rad/sec}$$

$$\Omega_{s1} = 2f_S\tan\frac{\Omega_{s1}}{2} = 794727.2 \text{ rad/sec}$$

Step 4: Since, from Step 1, $-20\log\delta_s = 30$, then $\log\delta_s = -30/20$, and $\delta_s = 10^{-30/20} = 0.03162$.

Step 5: The required order of the filter is

$$n \geq \frac{\log\left(\frac{1}{\delta_s^2} - 1\right)}{2\log\left(\frac{\omega_{s1}}{\omega_{p1}}\right)} = \frac{\log\left(\frac{1}{(0.03162)^2} - 1\right)}{2\log\left(\frac{794727.2}{6316.5}\right)} = 0.714$$

Therefore, a first order Butterworth filter is adequate to meet the specifications.

Step 6: The transfer function for the first order analog Butterworth filter, as noted early in this section, is

$$H(s) = \frac{\omega_{p1}}{s + \omega_{p1}} = \frac{6316.5}{s + 6316.5}$$

The transfer function for the digital filter is found using the bilinear transformation defined in Equation (10.4).

$$H(z) = \frac{6316.5}{50000\frac{z-1}{z+1} + 6316.5} = \frac{6316.5(z+1)}{50000(z-1) + 6316.5(z+1)}$$

$$= \frac{0.1122(1+z^{-1})}{1 - 0.7757z^{-1}}$$

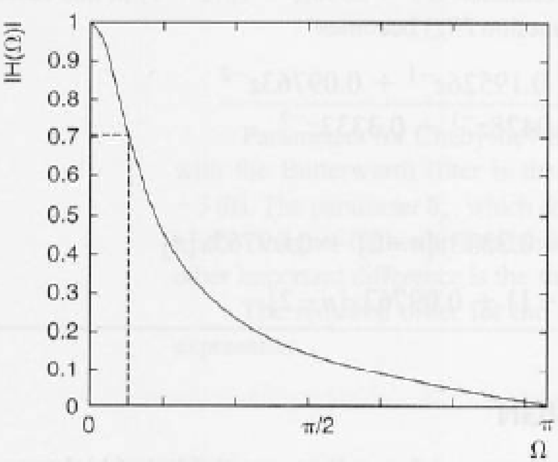
Therefore, the difference equation is

$$y[n] = 0.7757\,y[n-1] + 0.1122\,x[n] + 0.1122\,x[n-1]$$

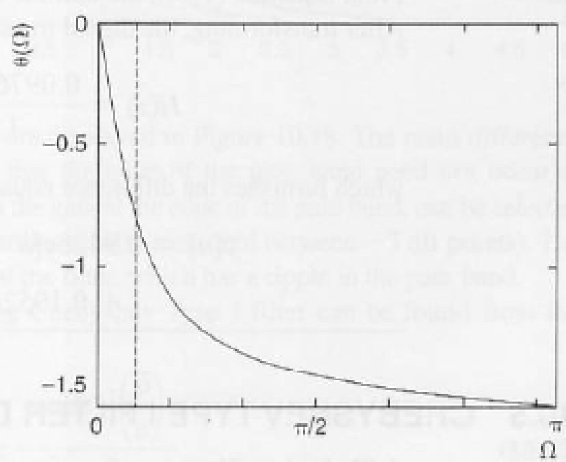
Replacing z in $H(z)$ by $e^{j\Omega}$, the frequency response

$$H(\Omega) = \frac{0.1122(1+e^{-j\Omega})}{1 - 0.7757e^{-j\Omega}}$$

is obtained, from which the magnitude and phase can be computed for any value of Ω. The magnitude and phase responses are plotted in Figure 10.16. Note that the phase response in the pass band, where the filter gains are over 0.707, is not linear.



(a) Magnitude Response



(b) Phase Response

FIGURE 10.16
Magnitude and phase response for Example 10.7.

EXAMPLE 10.8

The transfer function for a second order low pass analog Butterworth filter with −3 dB frequency ω_{p1} rad/sec is

$$H(s) = \frac{\omega_{p1}^2}{s^2 + \sqrt{2}\omega_{p1}s + \omega_{p1}^2}$$

Design a second order low pass digital Butterworth filter with a bandwidth of 500 Hz for a sampling frequency of 4 kHz.

For a low pass filter, the bandwidth is identical with the −3 dB frequency, 500 Hz. The digital frequency that corresponds to this frequency is

$$\Omega_{p1} = 2\pi\frac{f_{p1}}{f_S} = 2\pi\frac{500}{4000} = 0.25\pi \text{ radians}$$

Prewarping to prepare for the bilinear transformation gives:

$$\omega_{p1} = 2f_S \tan\frac{\Omega_{p1}}{2} = 3313.7 \text{ rad/sec}$$

Substituting this value into the analog transfer function gives

$$H(s) = \frac{10980607.7}{s^2 + 4686.3s + 10980607.7}$$

From Equation (10.4), the bilinear transformation is $s = 8000(z-1)/(z+1)$ in this case. After transforming, the digital transfer function $H(z)$ becomes

$$H(z) = \frac{0.09763 + 0.19526z^{-1} + 0.09763z^{-2}}{1 - 0.9428z^{-1} + 0.3333z^{-2}}$$

which furnishes the difference equation

$$y[n] = 0.9428y[n-1] - 0.3333y[n-2] + 0.09763x[n] + 0.19526x[n-1] + 0.09763x[n-2]$$

10.5 Chebyshev Type I Filter Design

A Chebyshev filter can often accomplish the same job as a Butterworth filter of higher order. at the cost of slightly greater complexity. The Chebyshev transfer function is rather complicated, but the expression for filter shape is quite straightforward.

For an n^{th} order analog Chebyshev Type I filter, the filter shape is defined as:

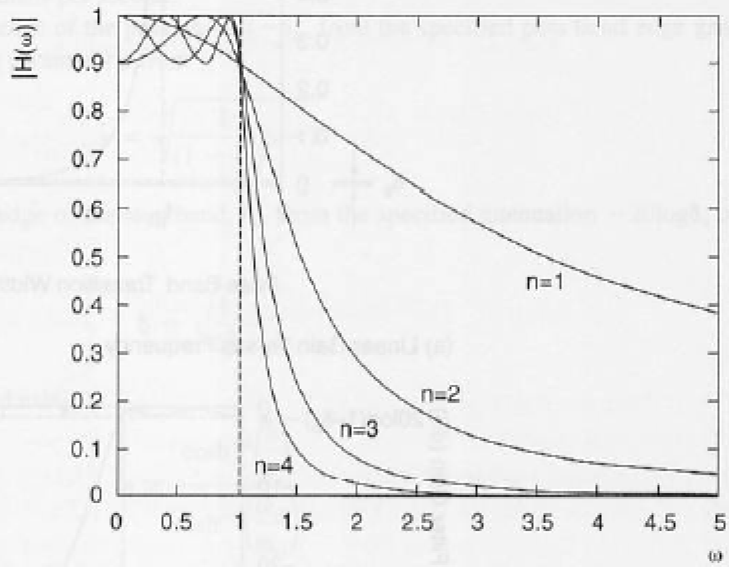
$$|H(\omega)| = \frac{1}{\sqrt{1 + \varepsilon^2 C_n^2\left(\frac{\omega}{\omega_{p1}}\right)}}$$

where ε is a parameter that depends on pass band ripple, and the function $C_n(x)$ is defined as:

$$C_n(x) = \begin{cases} \cos(n\cos^{-1}(x)) & \text{for } |x| \le 1 \\ \cosh(n\cosh^{-1}(x)) & \text{for } |x| > 1 \end{cases}$$

where $\cosh(x)$ refers to the hyperbolic cosine function. The function $C_n(x)$ is known as the Chebyshev polynomial because it can be expanded as a polynomial. Chebyshev filter shapes for several values of n are shown in Figure 10.17. A comparison with Figure 10.12 shows that, as long as the order is greater than one, the Chebyshev filters offer steeper roll-offs than the Butterworth filters.

FIGURE 10.17
Analog Chebyshev Type I filter shapes, order n.



Parameters for Chebyshev filters are displayed in Figure 10.18. The main difference with the Butterworth filter is the fact that the edge of the pass band need not occur at −3 dB. The parameter δ_p, which defines the gain at the edge of the pass band, can be selected by the designer (although bandwidth is still normally measured between −3 dB points). The other important difference is the shape of the filter, which has a ripple in the pass band.

The required order for the analog Chebyshev Type I filter can be found from the expression

$$n \ge \frac{\cosh^{-1}\left(\dfrac{\delta}{\varepsilon}\right)}{\cosh^{-1}\left(\dfrac{\omega_{s1}}{\omega_{p1}}\right)} \tag{10.12}$$

which is derived in Appendix M. The parameters in Equation (10.12) are described in Table 10.3, which contains the steps necessary to design a low pass Chebyshev Type I filter. As in Section 10.4, an nth order filter will have a transfer function of the form

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}$$

with $2n + 1$ coefficients, n zeros, and n poles.

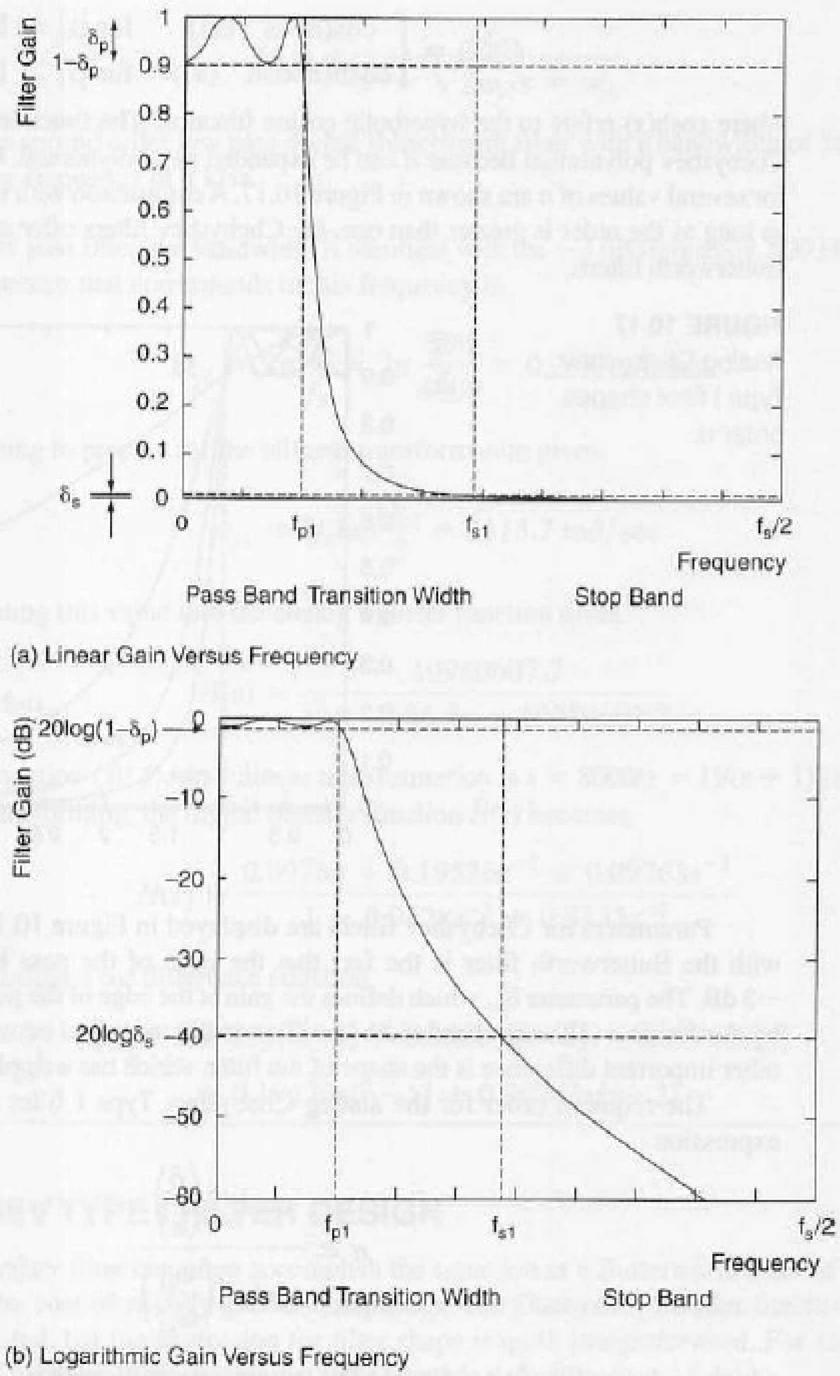


FIGURE 10.18
Chebyshev filter parameters.

TABLE 10.3
Design Steps for Low Pass Chebyshev Type I Filter Design

1. Identify the desired pass band edge and stop band edge frequencies, f_{p1} Hz and f_{s1} Hz, as well as the desired gain at the edge of the pass band $20\log(1-\delta_p)$ and the desired stop band attenuation $-20\log\delta_s$ (or the desired stop band gain $20\log\delta_s$).
2. Convert the desired edge frequencies to digital frequencies in radians, using $\Omega = 2\pi f/f_S$, to obtain Ω_{p1} and Ω_{s1}.
3. Prewarp the digital frequencies to avoid errors due to bilinear transformation. Use $\omega = 2f_S \tan(\Omega/2)$ to obtain ω_{p1} and ω_{s1}, in radians per second.
4. Determine the gain at the edge of the pass band, $1-\delta_p$, from the specified pass band edge gain $20\log(1-\delta_p)$. Compute the parameter ε from

$$\varepsilon = \sqrt{\frac{1}{(1-\delta_p)^2} - 1}$$

5. Determine the gain at the edge of the stop band, δ_s, from the specified attenuation $-20\log\delta_s$ or gain $20\log\delta_s$ dB. Compute

$$\delta = \sqrt{\frac{1}{\delta_s^2} - 1}$$

6. Calculate the order required using

$$n \geq \frac{\cosh^{-1}\left(\frac{\delta}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\omega_{s1}}{\omega_{p1}}\right)}$$

An integer value for n must be chosen.

7. Substitute ω_{p1} and δ_p into the nth order analog Chebyshev Type I transfer function $H(s)$, and perform a bilinear transformation on $H(s)$ to obtain the nth order digital transfer function $H(z)$.[2] The difference equation needed for filter implementation can be found directly from the transfer function $H(z)$. Filter shape $|H(\Omega)|$ can be obtained by substituting $\omega = 2f_S \tan(\Omega/2)$ into

$$|H(\omega)| = \frac{1}{\sqrt{1 + \varepsilon^2 C_n^2\left(\frac{\omega}{\omega_{p1}}\right)}}$$

EXAMPLE 10.9

An IIR filter with a Chebyshev Type 1 characteristic must be designed for a system with a 20 kHz sampling rate. The maximum pass band gain is 0 dB. The pass band edge occurs at 5 kHz, with a gain of −1 dB. The stop band edge begins at 7.5 kHz, with a gain of −32 dB.

[2] Because the general form of the nth order analog Chebyshev Type I transfer function $H(s)$ is rather complicated, this step is usually performed with the assistance of filter design software.

Following the design steps in Table 10.3:

Step 1: The analog edge frequencies are:

$$f_{p1} = 5000 \text{ Hz}$$

$$f_{s1} = 7500 \text{ Hz}$$

The gain at the edge of the pass band is $20\log(1-\delta_p) = -1$ dB. The gain at the edge of the stop band is $20\log\delta_s = -32$ dB.

Step 2: The digital edge frequencies are:

$$\Omega_{p1} = 2\pi\frac{f_{p1}}{f_S} = 2\pi\frac{5000}{20000} = 0.5\pi \text{ radians}$$

$$\Omega_{s1} = 2\pi\frac{f_{s1}}{f_S} = 2\pi\frac{7500}{20000} = 0.75\pi \text{ radians}$$

Step 3: The prewarped analog edge frequencies are:

$$\omega_{p1} = 2f_S \tan\frac{\Omega_{p1}}{2} = 40000 \text{ rad/sec}$$

$$\omega_{s1} = 2f_S \tan\frac{\Omega_{s1}}{2} = 96568.5 \text{rad/sec}$$

Step 4: Since, from Step 1, $20\log(1-\delta_p) = -1$, then $\log(1-\delta_p) = -1/20$, and $1-\delta_p = 10^{-1/20} = 0.89125$. Therefore,

$$\varepsilon = \sqrt{\frac{1}{(1-\delta_p)^2} - 1} = \sqrt{\frac{1}{0.89125^2} - 1} = 0.5088$$

Step 5: Since, from Step 1, $20\log\delta_s = -32$, then $\log\delta_s = -32/20$, and $\delta_s = 10^{-32/20} = 0.0251$. Therefore,

$$\delta = \sqrt{\frac{1}{\delta_s^2} - 1} = \sqrt{\frac{1}{0.0251^2} - 1} = 39.8$$

Step 6: The required order of the filter is:

$$n \geq \frac{\cosh^{-1}\left(\frac{\delta}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\omega_{s1}}{\omega_{p1}}\right)} = \frac{\cosh^{-1}\left(\frac{39.8}{0.5088}\right)}{\cosh^{-1}\left(\frac{96568.5}{40000}\right)} = 3.31$$

To meet the specifications, a filter of order 4 must be chosen.

Step 7: Filter design software can be used to find the transfer function $H(z)$. The filter shape for the analog prototype is

$$|H(\omega)| = \frac{1}{\sqrt{1 + \varepsilon^2 C_n^2\left(\frac{\omega}{\omega_{p1}}\right)}} = \frac{1}{\sqrt{1 + 0.2589 C_4^2\left(\frac{\omega}{40000}\right)}}$$

The filter shape for the digital filter is found by substituting $\omega = 2f_S \tan(\Omega/2)$ into the analog prototype:

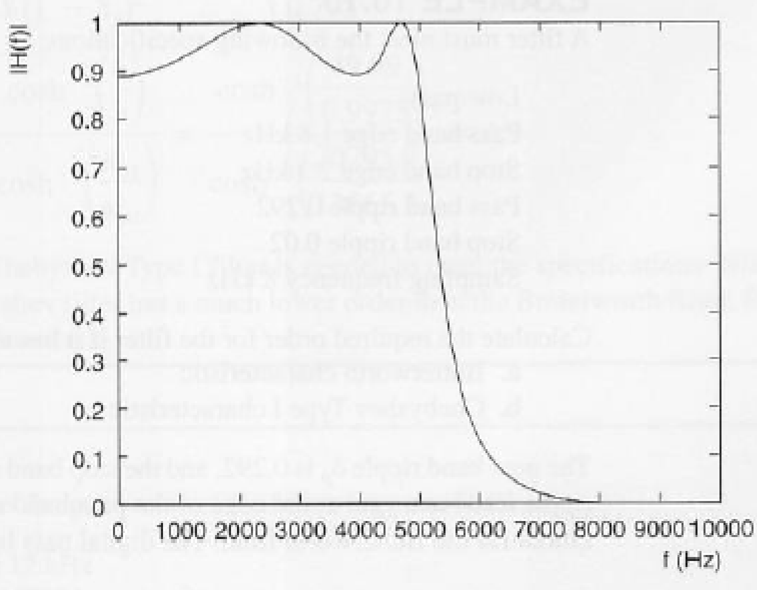
$$|H(\Omega)| = \frac{1}{\sqrt{1 + 0.2589 C_4^2\left(\frac{40000\tan\left(\frac{\Omega}{2}\right)}{40000}\right)}} = \frac{1}{\sqrt{1 + 0.2589 C_4^2\left(\tan\left(\frac{\Omega}{2}\right)\right)}}$$

where

$$C_4(x) = \begin{cases} \cos(4\cos^{-1}(x)) & |x| \le 1 \\ \cosh(4\cosh^{-1}(x)) & |x| > 1 \end{cases}$$

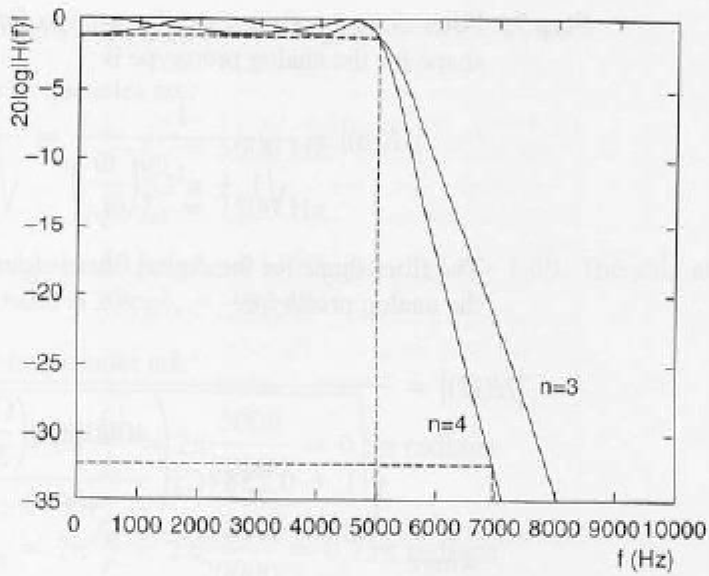
The shape of the filter is plotted in Figure 10.19, where frequencies from 0 to $f_S/2$ Hz have been used along the frequency axis in place of 0 to π rads, following the discussion in Section 7.3.3. Notice the characteristic shape of the Chebyshev Type I filter in Figure 10.19(a). The logarithmic plot in Figure 10.19(b) shows that the specifications have been satisfied. A gain of -1 dB is reached at 5 kHz, but, be-

FIGURE 10.19
Filter shape for Example 10.9.



(a) Linear Filter Shape

FIGURE 10.19
Continued



(b) Logarithmic Filter Shape

cause the order was rounded up to 4 from 3.13, the gain drops to −32 dB below 7000 Hz instead of the specified 7500 Hz. Unfortunately, order 3, also presented in Figure 10.19(b), is too low to meet the requirements.

EXAMPLE 10.10

A filter must meet the following specifications:

Low pass
Pass band edge 1.8 kHz
Stop band edge 2.3 kHz
Pass band ripple 0.292
Stop band ripple 0.02
Sampling frequency 8 kHz

Calculate the required order for the filter if it has a

a. Butterworth characteristic
b. Chebyshev Type I characteristic

The pass band ripple δ_p is 0.292, and the stop band ripple δ_s is 0.02. Note that the pass band ripple leads to a gain at the edge of the pass band of $1 - 0.292 = 0.708$, or −3 dB, as required for the Butterworth filter. The digital pass band and stop band edge frequencies are

$$\Omega_{p1} = 2\pi \frac{f_{p1}}{f_S} = 2\pi \frac{1800}{8000} = 0.45\pi \text{ radians}$$

$$\Omega_{s1} = 2\pi\frac{f_{s1}}{f_S} = 2\pi\frac{2300}{8000} = 0.575\pi \text{ radians}$$

Prewarping these yields the frequencies to be used for the design of the analog prototype:

$$\omega_{p1} = 2f_S\tan\frac{\Omega_{p1}}{2} = 13665.3 \text{ rad/sec}$$

$$\omega_{s1} = 2f_S\tan\frac{\Omega_{s1}}{2} = 20295.9 \text{ rad/sec}$$

a. For the Butterworth version of the filter,

$$n \geq \frac{\log\left(\frac{1}{\delta^2} - 1\right)}{2\log\left(\frac{\omega_{s1}}{\omega_{p1}}\right)} = \frac{\log\left(\frac{1}{0.02^2} - 1\right)}{2\log\left(\frac{20295.9}{13665.3}\right)} = 9.9$$

A tenth order Butterworth filter is needed to meet the specifications.

b. For the Chebyshev Type I filter,

$$\delta = \sqrt{\frac{1}{\delta_s^2} - 1} = \sqrt{\frac{1}{0.02^2} - 1} = 49.99$$

$$\varepsilon = \sqrt{\frac{1}{(1-\delta_p)^2} - 1} = \sqrt{\frac{1}{0.708^2} - 1} = 0.9975$$

$$n \geq \frac{\cosh^{-1}\left(\frac{\delta}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\omega_{s1}}{\omega_{p1}}\right)} = \frac{\cosh^{-1}\left(\frac{49.99}{0.9975}\right)}{\cosh^{-1}\left(\frac{20295.9}{13665.3}\right)} = 4.9$$

A fifth order Chebyshev Type I filter is needed to meet the specifications. Note that the Chebyshev filter has a much lower order than the Butterworth filter, for the same requirements.

EXAMPLE 10.11

A low pass filter must be designed to meet the following specifications:

Pass band edge at 12 kHz
Transition width 4 kHz
Attenuation at pass band edge 0.06 dB
Stop band attenuation 44 dB
Sampling frequency 44 kHz

Determine how many coefficients in the transfer function $H(z)$ are needed to meet the specifications for each of the following filter types:

a. FIR

b. IIR, Chebyshev Type I characteristic

a. According to Table 9.3, to achieve a stop band attenuation of 44 dB, this filter requires a Hanning window. This window also provides the required attenuation at the edge of the pass band. The number of terms required for the specified transition width and sampling frequency is

$$N = 3.32\frac{f_S}{\text{T.W.}} = 3.32\frac{44}{4} = 36.5$$

Thus, $N = 37$ terms are needed. As indicated in Section 9.6.2, a complete design for an FIR filter with N terms has an impulse response with N samples and a difference equation with N coefficients. Therefore, this filter design produces 37 impulse response values, and also a difference equation with 37 coefficients.

b. As shown in Figure 10.18, the gain at the edge of the pass band, -0.06 dB, is given by $20\log(1-\delta_p)$ dB. Thus, $\log(1-\delta_p) = -0.003$, $(1-\delta_p) = 10^{-0.003}$, and $\delta_p = 0.00688$. A stop band attenuation of 44 dB gives a stop band gain of -44 dB, which equals $20\log\delta_s$ dB. Solving $20\log\delta_s = -44$ gives $\delta_s = 0.00631$. Thus, for the Chebyshev filter,

$$\varepsilon = \sqrt{\frac{1}{(1-\delta_p)^2} - 1} = \sqrt{\frac{1}{(1-0.00688)^2} - 1} = 0.1179$$

$$\delta = \sqrt{\frac{1}{\delta_s^2} - 1} = \sqrt{\frac{1}{0.00631^2} - 1} = 158.48$$

The stop band edge is found by adding the transition width to the pass band edge, giving 16 kHz. Thus the digital frequencies at the edges of the pass band and stop band are

$$\Omega_{p1} = 2\pi\frac{f_{p1}}{f_S} = 2\pi\frac{12000}{44000} = 0.5455\pi \text{ radians}$$

$$\Omega_{s1} = 2\pi\frac{f_{s1}}{f_S} = 2\pi\frac{16000}{44000} = 0.7273\pi \text{ radians}$$

Prewarping these yields the frequencies to be used in the design of the analog prototype:

$$\omega_{p1} = 2f_S\tan\frac{\Omega_{p1}}{2} = 101572 \text{ rad/sec}$$

$$\omega_{s1} = 2f_S\tan\frac{\Omega_{s1}}{2} = 192715 \text{ rad/sec}$$

The order of the filter, by Equation (10.12), must be

$$n \geq \frac{\cosh^{-1}\left(\dfrac{\delta}{\varepsilon}\right)}{\cosh^{-1}\left(\dfrac{\omega_{s1}}{\omega_{p1}}\right)} = \frac{\cosh^{-1}\left(\dfrac{158.48}{0.1179}\right)}{\cosh^{-1}\left(\dfrac{192715}{101572}\right)} = 6.3$$

Rounding up, the required order for the Chebyshev filter is $n = 7$. The transfer function for this filter has $2n + 1$ coefficients, so it requires 15 coefficients in all. The Chebyshev Type I filter is able to meet the filter specifications with fewer than half of the coefficients required by the FIR filter.

10.6 IMPULSE INVARIANCE IIR FILTER DESIGN

- Every analog filter has an impulse response $h(t)$, just as every digital filter has an impulse response $h[n]$.
- The impulse invariance method of IIR filter design chooses a digital impulse response $h[n]$ that is a sampled version of the impulse response $h(t)$ of the analog filter that meets the required specifications, as illustrated in Figure 10.20. That is,

$$h[n] = h[nT] \tag{10.13}$$

where T is the sampling interval used.

- The idea is that a sampled impulse response produced in this manner will give a filter shape that is close to that of the original analog filter.
- The impulse response $h(t)$ for an analog filter with the transfer function $H(s)$ can be found using methods outside the scope of this text (Ambardar, 1999).
- For the simple case of a first order Butterworth filter, for example, the transfer function is. from Equation (10.1),

$$H(s) = \frac{\omega_{p1}}{s + \omega_{p1}}$$

where ω_{p1} is the -3 dB frequency in radians per second.

- This transfer function gives the impulse response

$$h(t) = \omega_{p1} e^{-\omega_{p1} t} u(t)$$

where $u(t)$ is an analog step function, zero for $t < 0$ and one for $t \geq 0$. When this analog impulse response is sampled every T seconds, the sampled times are given by $t = nT$, and, using Equation (10.13), the digital impulse response becomes

$$h[n] = h(nT) = \omega_{p1} e^{-\omega_{p1} nT} u(nT) = \omega_{p1} e^{-\omega_{p1} Tn} u[n]$$

since a sampled analog step function $u(nT)$ is identical with a digital step function $u[n]$. From Table 6.1, the z transform of $\beta^n u[n]$ is $z/(z - \beta)$. Therefore, the transfer function for

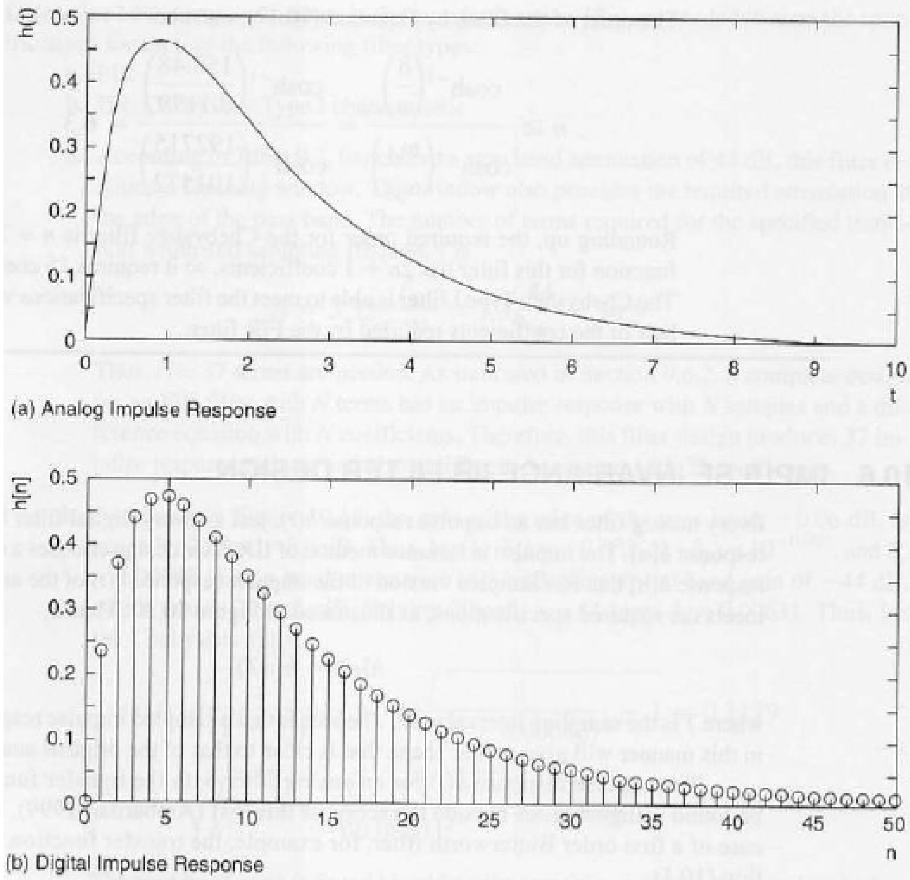


FIGURE 10.20
Impulse invariance.

the digital filter obtained through impulse invariance from a first order analog Butterworth filter may be found with $\beta = e^{-\omega_{p1} T}$ as

$$H(z) = \frac{\omega_{p1} z}{z - e^{-\omega_{p1} T}} = \frac{\omega_{p1}}{1 - e^{-\omega_{p1} T} z^{-1}} \qquad \textbf{(10.14)}$$

This transfer function leads to the difference equation

$$y[n] - e^{-\omega_{p1} T} y[n-1] = \omega_{p1} x[n]$$

The digital filter described by Equation (10.14) has the frequency response

$$H(\Omega) = \frac{\omega_{p1}}{1 - e^{-\omega_{p1} T} e^{-j\Omega}} \qquad \textbf{(10.15)}$$

Through the impulse invariance design method, the digital filter should have the same filter shape as the analog filter that provides the source impulse response $h(t)$. According to Equation (10.9), the shape of the first order analog Butterworth filter is given by

$$|H(\omega)| = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_{p1}}\right)^2 + 1}}$$

Provided that the sampling interval T used to implement the impulse-invariant digital filter is small, to minimize aliasing, then the shapes of the digital and analog filters are very close, as Example 10.12 shows.

EXAMPLE 10.12

Use the impulse invariance method to design a first order Butterworth filter with a cut-off frequency of 750 Hz.

From Equation (10.8), the first order Butterworth transfer function $H(s)$ for a 750 Hz cut-off frequency is

$$H(s) = \frac{\omega_{p1}}{s + \omega_{p1}} = \frac{2\pi(750)}{s + 2\pi(750)} = \frac{1500\pi}{s + 1500\pi}$$

Note that prewarping is not required because the bilinear transformation is not used. The filter shape for this analog filter, from Equation (10.9), is

$$|H(\omega)| = \frac{1}{\sqrt{\left(\frac{\omega}{1500\pi}\right)^2 + 1}}$$

This shape is to be duplicated by a digital filter designed with the impulse invariance method. The transfer function of the digital filter, from Equation (10.14), is

$$H(z) = \frac{\omega_{p1}}{1 - e^{-\omega_{p1}T}z^{-1}} = \frac{1500\pi}{1 - e^{-1500\pi T}z^{-1}}$$

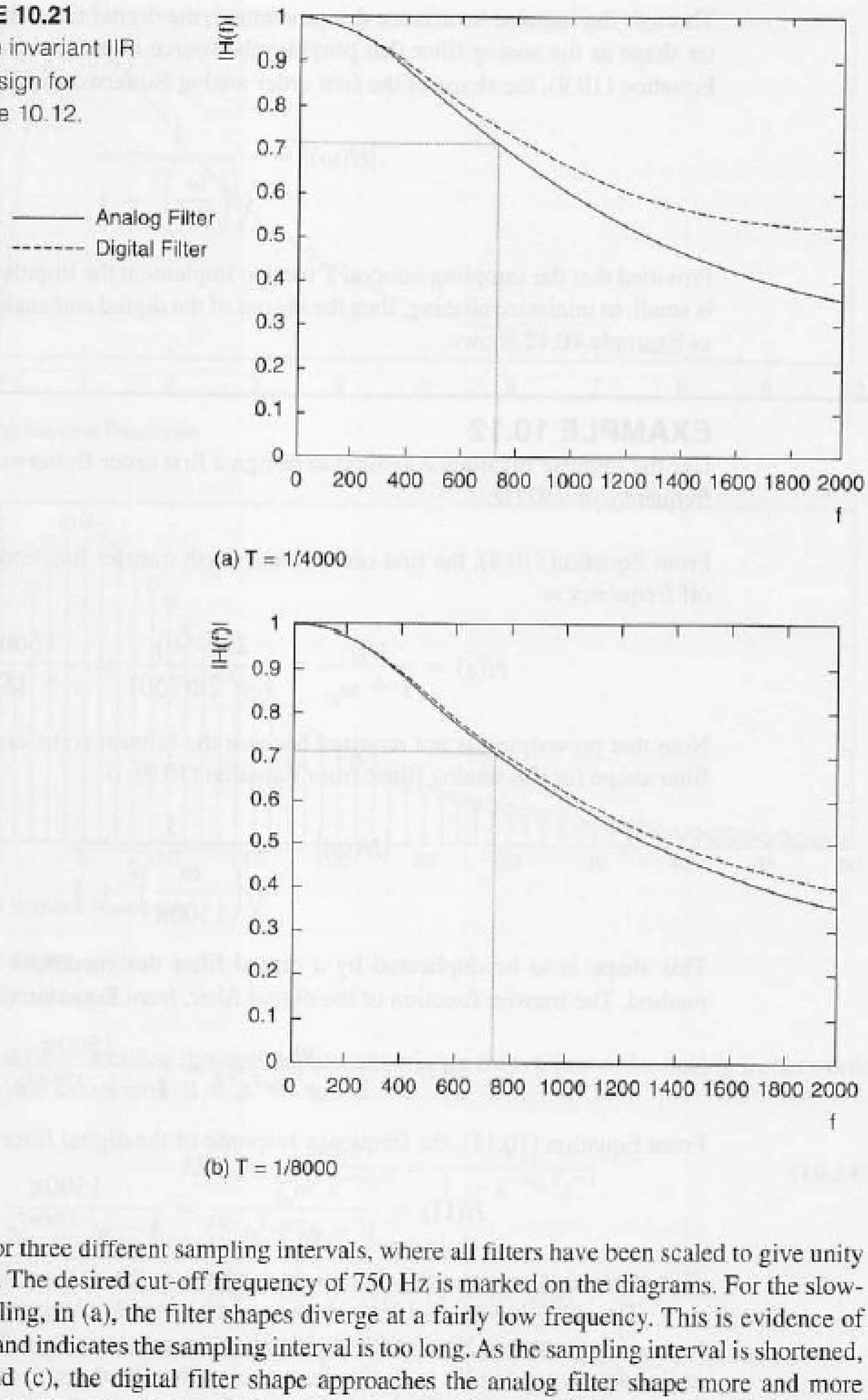
From Equation (10.15), the frequency response of the digital filter is

$$H(\Omega) = \frac{\omega_{p1}}{1 - e^{-\omega_{p1}T}e^{-j\Omega}} = \frac{1500\pi}{1 - e^{-1500\pi T}e^{-j\Omega}}$$

This expression provides the magnitude response, or filter shape, for a sampling interval T.

The analog and digital filter shapes are most readily compared if both are presented in terms of frequency in Hz. For the analog filter, the expression $\omega = 2\pi f$ may be used to relate analog frequency f in Hz to analog frequency ω in radians per second. For the digital filter, the expression $\Omega = 2\pi f/f_S = 2\pi fT$ relates analog frequency f to digital frequency Ω in radians, for a given sampling interval T. Figure 10.21 compares the analog and digital filter

FIGURE 10.21
Impulse invariant IIR filter design for Example 10.12.

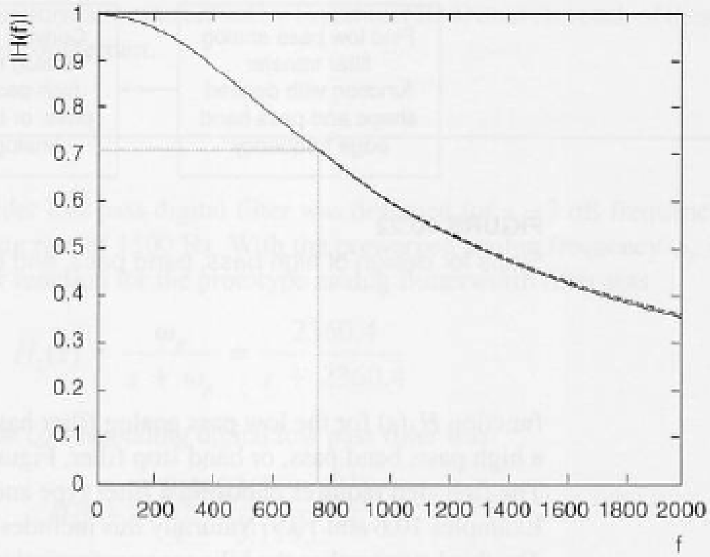


(a) T = 1/4000

(b) T = 1/8000

shapes for three different sampling intervals, where all filters have been scaled to give unity DC gain. The desired cut-off frequency of 750 Hz is marked on the diagrams. For the slowest sampling, in (a), the filter shapes diverge at a fairly low frequency. This is evidence of aliasing and indicates the sampling interval is too long. As the sampling interval is shortened, in (b) and (c), the digital filter shape approaches the analog filter shape more and more closely. The closest match is obtained when the sampling rate $1/T$ is many times higher than the desired cut-off frequency.

FIGURE 10.21
Continued



(c) T = 1/16000

10.7 "BEST FIT" FILTER DESIGN

As the examples have shown, design of recursive filters involves manipulating rather complicated equations. For this reason, software support is nearly always needed to complete these designs. One way to automate the design process is to create a program that implements the equations of the previous sections. When the equations become excessively complex, or when no convenient analog prototype exists for a desired filter, approaching the correct filter through iteration can become the best choice for design. Programs that use this method begin by guessing a transfer function, then calculate the filter shape and compare it mathematically to the filter shape. When mismatches occur, the filter coefficients are adjusted and the filter shape is recalculated. The process continues until a transfer function is found that produces the required filter shape.

This "best fit" method is used in many commercial filter design packages, even in cases when other techniques, like those presented in the previous sections, could be used. When filter coefficients do not need to be updated on the fly, as is required by some adaptive systems, speedy coefficient calculation becomes less of an issue, and the extra time needed to go through many iterative cycles is unimportant.

10.8 BAND PASS, HIGH PASS, AND BAND STOP IIR FILTERS

The order for band pass and high pass filters is chosen by working from a low pass prototype. This prototype must have the same shape as the desired filter. Once the transfer

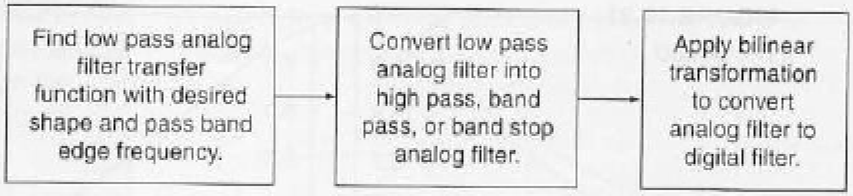


FIGURE 10.22
Steps for design of high pass, band pass, and band stop filters.

function $H_L(s)$ for the low pass analog filter has been obtained, it can be transformed into a high pass, band pass, or band stop filter. Figure 10.22 suggests the steps to be followed. The first step requires choosing a filter type and calculating a minimum filter order, as in Examples 10.6 and 10.9. Naturally this includes prewarping the digital filter requirements. The third step applies the bilinear transformation, as in Example 10.4. The conversions required in the second step can be accomplished by one of a group of simple tricks, described in the following paragraph.

To convert a low pass filter with cut-off frequency ω_p rad/sec, such as

$$H_L(s) = \frac{\omega_p}{s + \omega_p}$$

into a high pass filter with cut-off frequency ω_p' rad/sec, every s must be replaced by $\omega_p\omega_p'/s$, that is,

$$H_H(s) = H_L\left(\frac{\omega_p\omega_p'}{s}\right) \tag{10.16}$$

where $H_L(s)$ is the transfer function of the low pass analog filter and $H_H(s)$ is the transfer function of the high pass analog filter. The transfer function of a low pass filter with cut-off frequency ω_p rad/sec may be converted into the transfer function of a band pass filter with lower cut-off frequency ω_l and upper cut-off frequency ω_u with the equation

$$H_{BP}(s) = H_L\left(\omega_p\frac{s^2 + \omega_l\omega_u}{s(\omega_u - \omega_l)}\right) \tag{10.17}$$

where $H_{BP}(s)$ is the transfer function of the band pass analog filter. A band stop analog filter can be constructed from the low pass analog filter using

$$H_{BS}(s) = H_L\left(\omega_p\frac{s(\omega_u - \omega_l)}{s^2 + \omega_l\omega_u}\right) \tag{10.18}$$

where $H_{BS}(s)$ is the transfer function of the band stop analog filter. Each cut-off frequency ω_p, ω_p', ω_l, and ω_u must be prewarped using Equation (10.5) to avoid distortion of the filter shape. Note that the band pass and band stop filters have twice the order of their low pass prototypes. The bilinear transformation described by Equation (10.4) converts each of these analog filters into its digital counterpart.

EXAMPLE 10.13

In Example 10.4, a first order low pass digital filter was designed for a -3 dB frequency of 318.3 Hz with a sampling rate of 1500 Hz. With the prewarped analog frequency $\omega_p = 2360.4$ rad/sec, the transfer function for the prototype analog Butterworth filter was

$$H_L(s) = \frac{\omega_p}{s + \omega_p} = \frac{2360.4}{s + 2360.4}$$

The transfer function for the corresponding digital low pass filter was

$$H_L(z) = \frac{0.4403(1 + z^{-1})}{1 - 0.1193z^{-1}}$$

Using the first order Butterworth prototype $H_L(s)$ and a sampling frequency of 1500 Hz, determine a transfer function $H(z)$ for each of the following digital filters:

a. A high pass filter with a -3 dB frequency of 318.3 Hz.
b. A band pass filter with -3 dB frequencies of 318.3 and 636.6 Hz.
c. A band stop filter with -3 dB frequencies of 318.3 and 636.6 Hz.

a. With a 1500 Hz sampling rate, an analog frequency of 318.3 Hz corresponds to a digital frequency of $\Omega_p' = 2\pi f/f_S = 2\pi(318.3/1500) = 0.4244\pi$ rads. Prewarping gives $\omega_p' = 2f_S \tan(\Omega_p'/2) = 2360.4$ rad/sec, which happens to be the same as the cut-off for the low pass filter. The analog transfer function for the high pass filter is obtained using Equation (10.16):

$$H_H(s) = H_L\left(\frac{\omega_p\omega_p'}{s}\right) = H_L\left(\frac{(2360.4)(2360.4)}{s}\right)$$

$$= \frac{2360.4}{\left(\frac{(2360.4)(2360.4)}{s}\right) + 2360.4} = \frac{s}{s + 2360.4}$$

The digital counterpart is given by the bilinear transformation of Equation (10.4):

$$H_H(z) = \frac{3000\frac{z-1}{z+1}}{3000\frac{z-1}{z+1} + 2360.4} = \frac{0.5597(1 - z^{-1})}{1 - 0.1193z^{-1}}$$

The magnitude responses for this high pass filter and the original low pass filter $H_L(z)$ are plotted in Figure 10.23. Because the cut-off frequencies are the same for the two filters, their responses are symmetrical across 318.3 Hz.

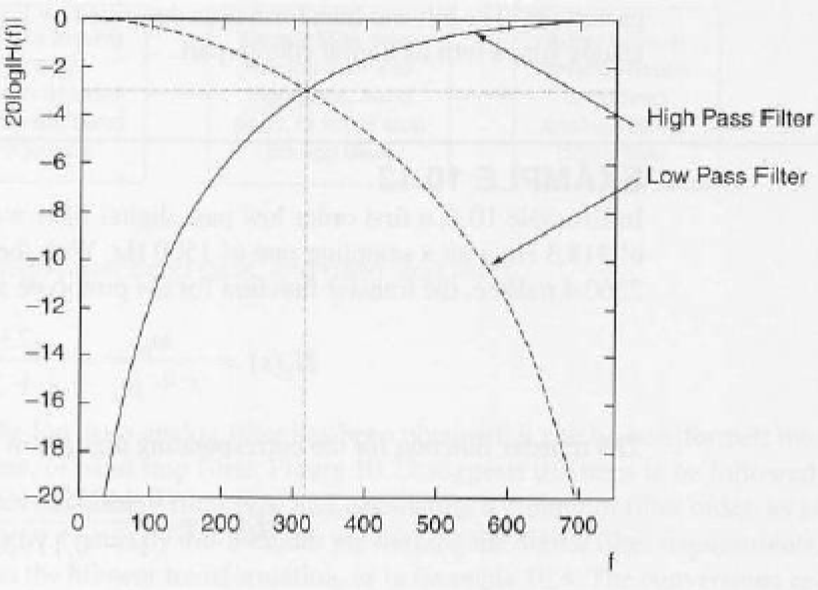


FIGURE 10.23 High pass filter from low pass prototype for Example 10.13.

b. As in (a), the prewarped lower cut-off frequency is $\omega_l = 2360.4$ rad/sec. The upper cut-off frequency will be $\Omega_u = 2\pi(636.6/1500) = 0.8488\pi$ rads, or $\omega_u = 2f_S \tan(0.8488\pi/2) = 12392.9$ rad/sec after prewarping. From Equation (10.17),

$$H_{BP}(s) = H_L\left(\omega_P \frac{s^2 + \omega_l\omega_u}{s(\omega_u - \omega_l)}\right)$$

$$= H_L\left(2360.4\frac{s^2 + (2360.4)(12392.9)}{10032.5s}\right)$$

$$= \frac{2360.4}{2360.4\dfrac{s^2 + (2360.4)(12392.9)}{10031.5s} + 2360.4}$$

$$= \frac{10032.5s}{s^2 + 10032.5s + 29252201.2}$$

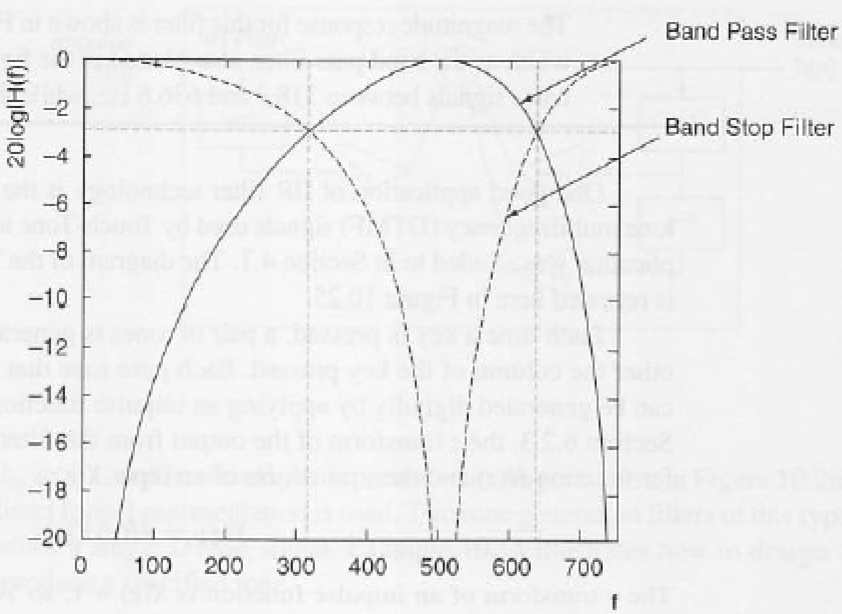
The bilinear transformation gives the digital transfer function:

$$H_{BP}(z) = \frac{10032.5\left(3000\dfrac{z-1}{z+1}\right)}{\left(3000\dfrac{z-1}{z+1}\right)^2 + 10032.5\left(3000\dfrac{z-1}{z+1}\right) + 29252201.2}$$

$$= \frac{0.4404(1 - z^{-2})}{1 + 0.5926z^{-1} + 0.1193z^{-2}}$$

The magnitude response for this filter is shown in Figure 10.24.

FIGURE 10.24 Band pass and band stop filters from low pass prototype for Example 10.13.



c. As in (b), the lower prewarped cut-off frequency is $\omega_l = 2360.4$ rad/sec and the upper prewarped cut-off frequency $\omega_u = 12392.9$ rad/sec. According to Equation (10.18),

$$H_{BS}(s) = H_L\left(\omega_p \frac{s(\omega_u - \omega_l)}{s^2 + \omega_l \omega_u}\right)$$

$$= H_L\left(2360.4 \frac{10032.5s}{s^2 + (2360.4)(12392.9)}\right)$$

$$= \frac{2360.4}{2360.4 \dfrac{10032.5s}{s^2 + (2360.4)(12392.9)} + 2360.4}$$

$$= \frac{s^2 + 29252201.2}{s^2 + 10032.5s + 29252201.2}$$

The transfer function for the digital filter is found using the bilinear transformation:

$$H_{BS}(z) = \frac{\left(3000\dfrac{z-1}{z+1}\right)^2 + 29252201.2}{\left(3000\dfrac{z-1}{z+1}\right)^2 + 10032.5\left(3000\dfrac{z-1}{z+1}\right) + 29252201.2}$$

$$= \frac{0.5597 + 0.5926z^{-1} + 0.5597z^{-2}}{1 + 0.5926z^{-1} + 0.1193z^{-2}}$$

The magnitude response for this filter is shown in Figure 10.24. It has the same bandwidth as the band pass filter, also plotted in the figure, but the band stop filter attenuates signals between 318.3 and 636.6 Hz, while the band pass filter passes them.

One good application of IIR filter technology is the generation and recovery of dual tone multifrequency (DTMF) signals used by Touch-Tone telephones (Mock, 1985). This application was alluded to in Section 4.1. The diagram of the Touch-Tone keypad in Figure 4.1 is repeated here in Figure 10.25.

Each time a key is pressed, a pair of tones is generated, one to code the row and the other the column of the key pressed. Each pure tone that forms part of a DTMF tone pair can be generated digitally by applying an impulse function to an IIR filter. As described in Section 6.2.3, the z transform of the output from the filter will be the product of the transfer function $H(z)$ and the z transform of an input $X(z)$:

$$Y(z) = H(z)X(z)$$

The z transform of an impulse function is $X(z) = 1$, so $Y(z) = H(z)$ in this case. This also means that $Y(\Omega) = H(\Omega)$. Thus, the spectrum of the output signal will be identical with the

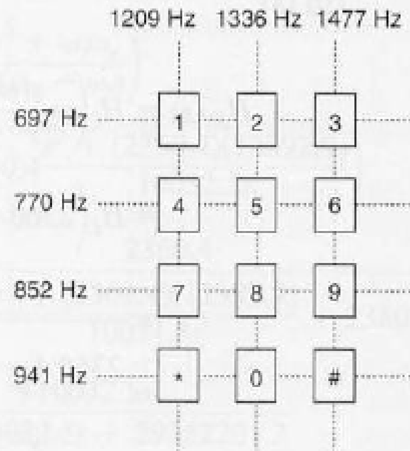


FIGURE 10.25
Touch-Tone keypad.

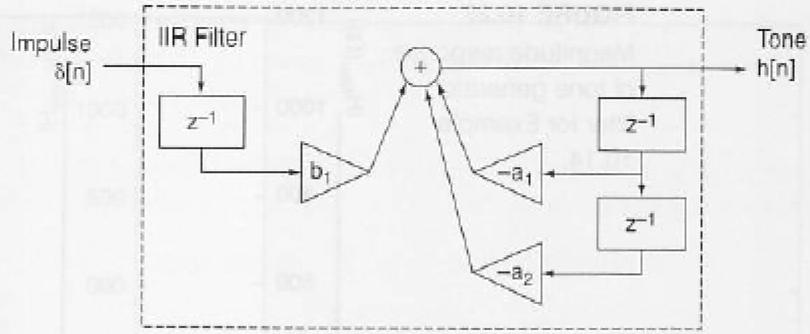
frequency response of the filter. A single pure tone is produced by a sinusoidal signal, so a filter that is capable of creating a pure tone will have a transfer function that is identical with the z transform of a sine signal:

$$H(z) = \frac{z\sin\Omega_0}{z^2 - 2z\cos\Omega_0 + 1} \quad \textbf{(10.19)}$$

from Table 6.1. Ω_0 is the digital frequency for the desired tone. The difference equation for this tone generator will be

$$y[n] = 2\cos\Omega_0 y[n-1] - y[n-2] + \sin\Omega_0 x[n-1] \quad \textbf{(10.20)}$$

FIGURE 10.26
Tone generation filter.



with $a_1 = -2\cos\Omega_0$, $a_2 = 1$, and $b_1 = \sin\Omega_0$. It is presented in diagram form in Figure 10.26. For simplicity, a direct form 1 representation is used. Two tone generation filters of this type are needed to produce a single DTMF signal. Example 10.14 illustrates how to design a tone generator to produce a specified tone.