TABLE 6-1 A TABLE OF Z TRANSFORMS

$f(n)$	$F(z)$
a^n or $a^n u(n)$	$\frac{z}{z-a}$, $\lvert z \rvert > \lvert a \rvert$
$\delta(n)$	1, for all z
$u(n)$ or 1	$\frac{z}{z-1}$, $\lvert z \rvert > 1$
na^{n-1}	$\frac{z}{(z-a)^2}$, $\lvert z \rvert > \lvert a \rvert$
n	$\frac{z}{(z-1)^2}$, $\lvert z \rvert > 1$
$n(n-1)a^{n-2}$	$\frac{2z}{(z-a)^3}$, $\lvert z \rvert > \lvert a \rvert$
$n(n-1)$	$\frac{2z}{(z-1)^3}$, $\lvert z \rvert > 1$
n^2	$\frac{z^2+z}{(z-1)^3}$, $\lvert z \rvert > 1$
$n(n-1)(n-2)a^{n-3}$	$\frac{6z}{(z-a)^4}$, $\lvert z \rvert > \lvert a \rvert$
$n(n-1)(n-2)$	$\frac{6z}{(z-1)^4}$, $\lvert z \rvert > 1$
n^3	$\frac{*}{(z-1)^4}$, $\lvert z \rvert > 1$
$(n)_{p-1}a^{n-p+1}$	$\frac{(p-1)!\,z}{(z-a)^p}$, $\lvert z \rvert > \lvert a \rvert$
e^{jnw}	*,
Cos nw	$\frac{z^2 - z\,\mathrm{Cos}w}{(z-1\angle w)(z-1\angle -w)}$
Sin nw	$\frac{z\,\mathrm{Sin}w}{(z-1\angle w)(z-1\angle -w)}$

TABLE 6-2 SOME ONE-SIDED Z TRANSFORM THEOREMS

Given $f(n) \leftrightarrow F(z)$, $|z| > \rho_1$, $g(n) \leftrightarrow G(z)$, $|z| > \rho_2$

Function	Transform	Theorem's name
$a^n f(n)$	$F\left(\frac{z}{a}\right)$	Transform-scaling
$f(n-1)u(n-1)$	$z^{-1}F(z)$	Shifting for a causal function
$f(n-k)u(n-k)$	$z^{-k}F(z)$	
$f(n-1)$	$z^{-1}F(z) + f(-1)$	
$f(n-2)$	$z^{-2}F(z) + z^{-1}f(-1) + f(-2)$	Shifting with initial conditions
$f(n-k)$	$z^{-k}F(z) + \sum_{p=1}^{k} z^{-k+p} f(-p)$	
$f(n)u(n) * g(n)u(n) = \left[\sum_{0}^{n} f(k)g(n-k)\right] \cdot u(n)$	$F(z)G(z)$	Convolution
$f(0) = \lim_{z\to\infty} F(z)$		Initial value

<u>Some of the proof:</u>

- $$Z\left\{a^n f[n]\right\} \to \sum_{n=0}^{\infty} a^n f[n] z^{-n} = \sum_{n=0}^{\infty} f[n]\left(\frac{z}{a}\right)^{-n} = F\left(\frac{z}{a}\right)$$

- $$f[n-1]u[n-1] \to \sum_{n=1}^{\infty} f[n-1] z^{-n}$$
  Let $k = n\text{-}1$
  $$\to \sum_{k=0}^{\infty} f[k] z^{-k-1} = z^{-1}F(z)$$

- $$Z\left\{f(n-1)u[n]\right\} = \sum_{n=0}^{\infty} f(n-1) z^{-n}$$
  Let $n-1 = k$ then $n = k+1$
  $$\sum_{n=0}^{\infty} f(n-1) z^{-1} = \sum_{k=-1}^{\infty} f(k) z^{-(k+1)} = f(-1) + \sum_{k=0}^{\infty} f(k) z^{-k} z^{-1} = f(-1) + z^{-1}\sum_{k=0}^{\infty} f(k) z^{-k}$$

➢

$$f(n)*g(n)=\sum_{k=-\infty}^{\infty} f(k)g(n-k)$$

$$\begin{aligned}
Z\{f(n)*g(n)\} &= \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} f(k)g(n-k)\right]z^{-n} \\
&= \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} f(k)g(n-k)\right]z^{-(n-k+k)} \\
&= \sum_{n=-\infty}^{\infty} g(n-k)z^{-(n-k)}\sum_{k=-\infty}^{\infty} f(k)z^{-k} \\
&= \underbrace{\sum_{n=-\infty}^{\infty} g(n-k)z^{-(n-k)}}_{G(z)}\underbrace{\sum_{k=-\infty}^{\infty} f(k)z^{-k}}_{F(z)} \\
&= G(z)F(z)
\end{aligned}$$

EXAMPLE 6.6

Find the z transform of the signal $x[n] = 2u[n-2]$.

Since $\mathbf{Z}\{u[n]\} = \dfrac{z}{z-1}$,

$$\mathbf{Z}\{u[n-2]\} = z^{-2}\frac{z}{z-1} = \frac{1}{z(z-1)}$$

Therefore,

$$X(z) = \frac{2}{z(z-1)}$$

Ex] Find the z transform including region of convergence of

$$x[n] = -b^n u[-n-1]$$

$$X(z) = Z\{-b^n u[-n-1]\} = \sum_{n=-\infty}^{-1} -b^n z^{-n} = -\sum_{n=-\infty}^{-1}\left(\frac{b}{z}\right)^n = -\sum_{n=1}^{\infty}\left(\frac{b}{z}\right)^{-n} = -\sum_{n=1}^{\infty}\left(\frac{z}{b}\right)^n$$

$$X(Z) = -\left(\frac{b}{z}\right)^{-1} - \left(\frac{b}{z}\right)^{-2} - \left(\frac{b}{z}\right)^{-3} - \ldots - \left(\frac{b}{z}\right)^{-\infty}$$

$$X(Z) = \quad -\left(\frac{b}{z}\right)^{-1} - \left(\frac{b}{z}\right)^{-2} - \left(\frac{b}{z}\right)^{-3} - \ldots - \left(\frac{b}{z}\right)^{-\infty} \quad \text{-- (4)}$$

$$\left(\frac{b}{z}\right)^{-1} X(Z) = \quad -\left(\frac{b}{z}\right)^{-2} - \left(\frac{b}{z}\right)^{-3} - \ldots - \left(\frac{b}{z}\right)^{-\infty} \quad \text{-- (5)}$$

Subtracting (5) from (4), then

$$-\left|\begin{array}{ll} X(Z) = & -\left(\frac{b}{z}\right)^{-1} - \cancel{\left(\frac{b}{z}\right)^{-2}} - \cancel{\left(\frac{b}{z}\right)^{-3}} - \ldots - \cancel{\left(\frac{b}{z}\right)^{-\infty}} \quad \text{-- (4)} \\ \left(\frac{b}{z}\right)^{-1} X(Z) = & \qquad\qquad - \cancel{\left(\frac{b}{z}\right)^{-2}} - \cancel{\left(\frac{b}{z}\right)^{-3}} - \ldots - \cancel{\left(\frac{b}{z}\right)^{-\infty}} \quad \text{-- (5)} \end{array}\right.$$

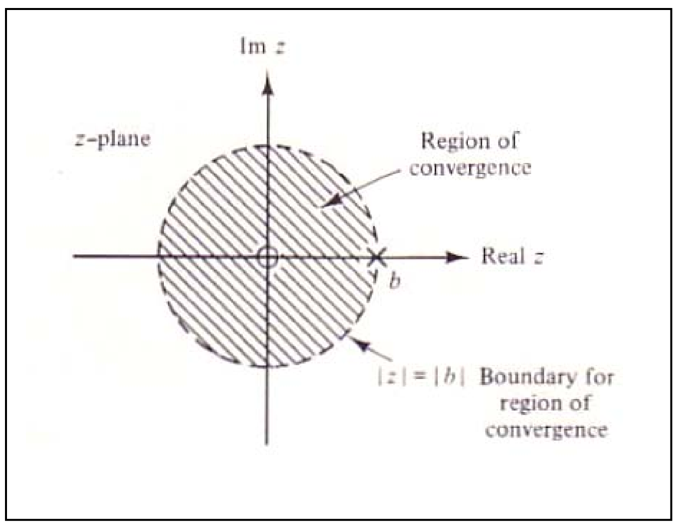
$$X(Z) - \left(\frac{b}{z}\right)^{-1} X(Z) = -\left(\frac{b}{z}\right)^{-1}$$

$$X(Z)\left(1 - \left(\frac{b}{z}\right)^{-1}\right) = -\left(\frac{b}{z}\right)^{-1}$$

$$X(Z) = \frac{-\left(\frac{b}{z}\right)^{-1}}{1 - \left(\frac{b}{z}\right)^{-1}} = \frac{-\frac{z}{b}}{1 - \frac{z}{b}} = \frac{\frac{z}{b}}{\frac{z}{b} - 1} = \frac{z}{z-b}$$

$$\left|\frac{z}{b}\right| < 1 \quad \text{ROC}$$

$$|z| < |b|$$



Ex] Find the z transform and region of convergence of y[n]

$$y[n] = a^n u[n] - b^n u[-n-1]$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$Y(z) = \frac{z}{z-a} + \frac{z}{z-b} \qquad \text{with ROC } \{|z| > |a|\} \cap \{|z| < |b|\}$$

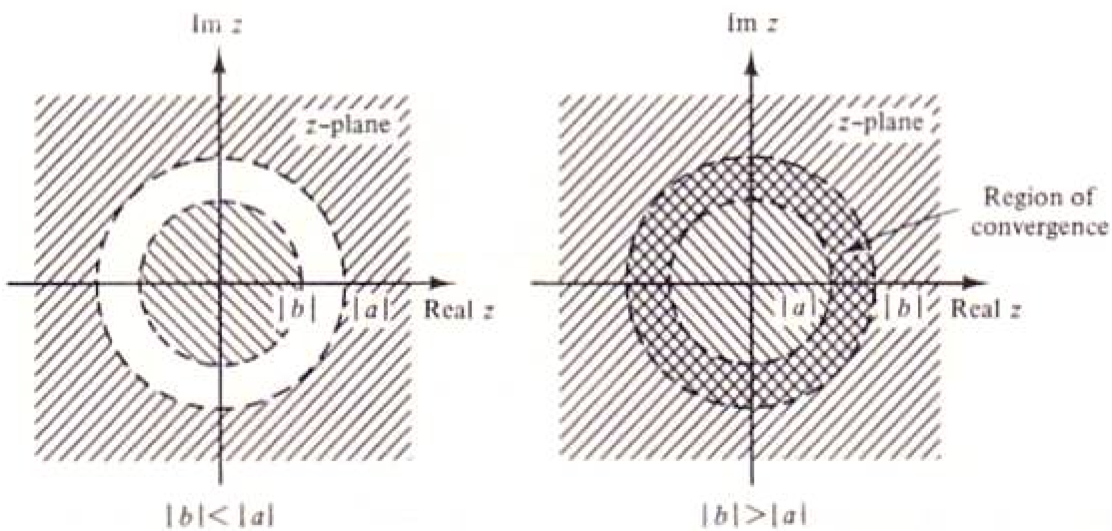


TABLE 6.1
Basic z Transforms

Signal $x[n]$	z Transform $X(z)$	Region of Convergence
$\delta[n]$	1	all z
$u[n]$	$\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
$\beta^n u[n]$	$\dfrac{z}{z-\beta}$	$\lvert z\rvert > \lvert\beta\rvert$
$nu[n]$	$\dfrac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$\cos(n\Omega)u[n]$	$\dfrac{z^2 - z\cos\Omega}{z^2 - 2z\cos\Omega + 1}$	$\lvert z\rvert > 1$
$\sin(n\Omega)u[n]$	$\dfrac{z\sin\Omega}{z^2 - 2z\cos\Omega + 1}$	$\lvert z\rvert > 1$
$\beta^n\cos(n\Omega)u[n]$	$\dfrac{z^2 - \beta z\cos\Omega}{z^2 - 2\beta z\cos\Omega + \beta^2}$	$\lvert z\rvert > \lvert\beta\rvert$
$\beta^n\sin(n\Omega)u[n]$	$\dfrac{\beta z\sin\Omega}{z^2 - 2\beta z\cos\Omega + \beta^2}$	$\lvert z\rvert > \lvert\beta\rvert$

The **z transform of time shifted version** of signal $x\left[n-1\right]$ is

$$X\left(z\right)=\sum_{n=0}^{\infty}x[n-1]z^{-n} \quad \text{let } m=n\text{-}1, \quad n=m+1$$

$$=\sum_{m=-1}^{\infty}x[m]z^{-(m+1)}=\sum_{m=0}^{\infty}x[m]z^{-m}z^{-1}=z^{-1}\sum_{m=0}^{\infty}x[m]z^{-m}=z^{-1}X\left(z\right)$$

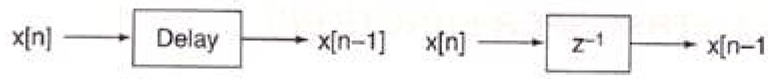
The z transform of time shifted version of signal $x\left[n-k\right]$ is

$$X\left(z\right)=\sum_{n=0}^{\infty}x[n-k]z^{-n} \quad \text{let } m=n\text{-}k, \quad n=m+k$$

$$=z^{-k}X\left(z\right)$$

$$x[n] \rightarrow X(z)$$
$$x[n-1] \rightarrow Z^{-1}X(z)$$
$$x[n-k] \rightarrow Z^{-k}X(z)$$
$$x[n+k] \rightarrow Z^{k}X(z)$$

FIGURE 6.2
Delays in z domain.



EXAMPLE 6.6

Find the z transform of the signal $x[n] = 2u[n-2]$.

Since $\mathbf{Z}\{u[n]\} = \dfrac{z}{z-1}$,

$$\mathbf{Z}\{u[n-2]\} = z^{-2}\frac{z}{z-1} = \frac{1}{z(z-1)}$$

Therefore,

$$X(z) = \frac{2}{z(z-1)}$$

EXAMPLE 6.7

Reexpress the nonrecursive difference equation diagram of Figure 4.15 using the z^{-1} notation.

The general form for a nonrecursive difference equation is

$$y[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] + \ldots + b_Mx[n-M]$$

The corresponding diagram is shown in Figure 6.3.

FIGURE 6.3
Difference equation diagram using z^{-1} notation for Example 6.7.

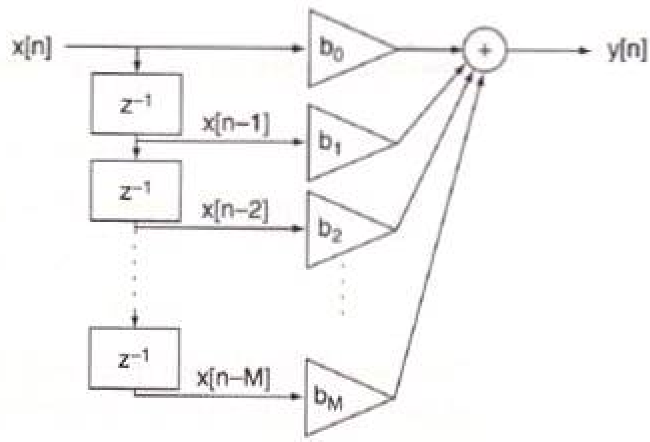
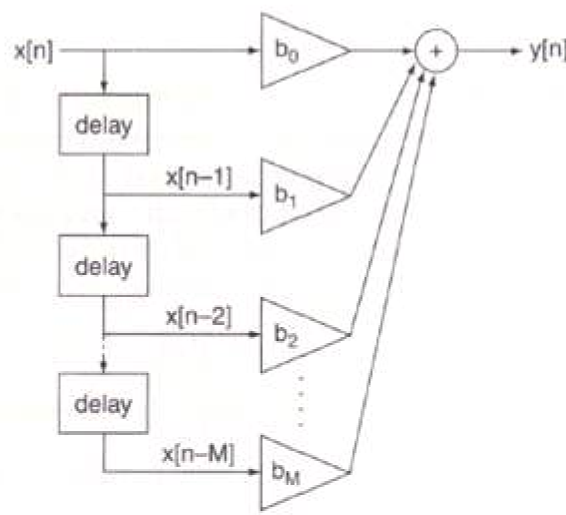


FIGURE 4.15
Nonrecursive difference equation diagram.



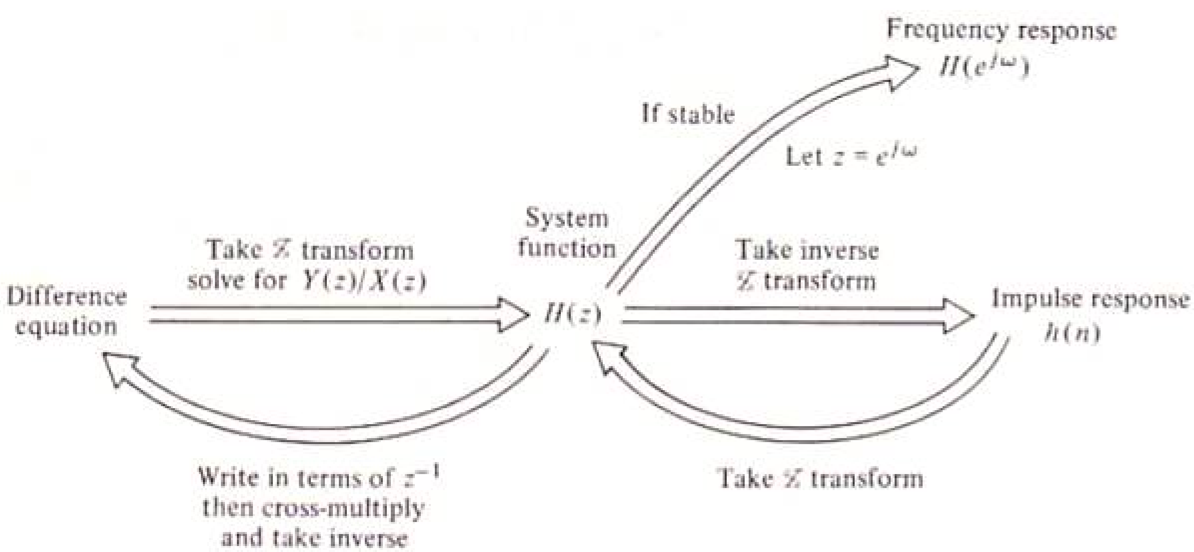
Transfer function and difference equation

$$H(z) = \frac{output}{input} = \frac{Y(z)}{X(z)}$$

The **transfer function** is a way of summarizing all information about digital systems behavior

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$



EXAMPLE 6.8

Find the transfer function of the system described by the difference equation.

$$2y[n] + y[n-1] + 0.9y[n-2] = x[n-1] + x[n-4]$$

Taking z transforms term by term:

$$2Y(z) + z^{-1}Y(z) + 0.9z^{-2}Y(z) = z^{-1}X(z) + z^{-4}X(z)$$

where $Y(z)$ is the z transform of the filter output $y[n]$, and $X(z)$ is the z transform of the filter input $x[n]$. Factoring out $Y(z)$ on the left and $X(z)$ on the right:

$$(2 + z^{-1} + 0.9z^{-2})Y(z) = (z^{-1} + z^{-4})X(z)$$

Solving this for $\frac{Y(z)}{X(z)}$ gives the system transfer function

$$H(z) = \frac{\text{output}}{\text{input}} = \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-4}}{2 + z^{-1} + 0.9z^{-2}}$$

EXAMPLE 6.9

Find the transfer function for the system with the difference equation

$$y[n] - 0.2y[n-1] = x[n] + 0.8x[n-1]$$

Virtually by inspection, the transfer function is

$$H(z) = \frac{1 + 0.8z^{-1}}{1 - 0.2z^{-1}}$$

EXAMPLE 6.10

Find the transfer function for the difference equation

$$y[n] = 0.75x[n] - 0.3x[n-2] - 0.01x[n-3]$$

The transfer function for this nonrecursive difference equation is

$$H(z) = 0.75 - 0.3z^{-2} - 0.01z^{-3}$$

EXAMPLE 6.11

Find the difference equation that corresponds to the transfer function

$$H(z) = \frac{1 + 0.5z^{-1}}{1 - 0.5z^{-1}}$$

Since the transfer function is the ratio of $Y(z)$ to $X(z)$,

$$\frac{Y(z)}{X(z)} = \frac{1 + 0.5z^{-1}}{1 - 0.5z^{-1}}$$

Cross-multiplication gives

$$Y(z)(1 - 0.5z^{-1}) = X(z)(1 + 0.5z^{-1})$$

or,

$$Y(z) - 0.5z^{-1}Y(z) = X(z) + 0.5z^{-1}X(z)$$

Inversely transforming term by term yields the difference equation

$$y[n] - 0.5y[n-1] = x[n] + 0.5x[n-1]$$

Transfer functions and impulse response

FIGURE 6.4
Finding filter outputs in time and z domains.

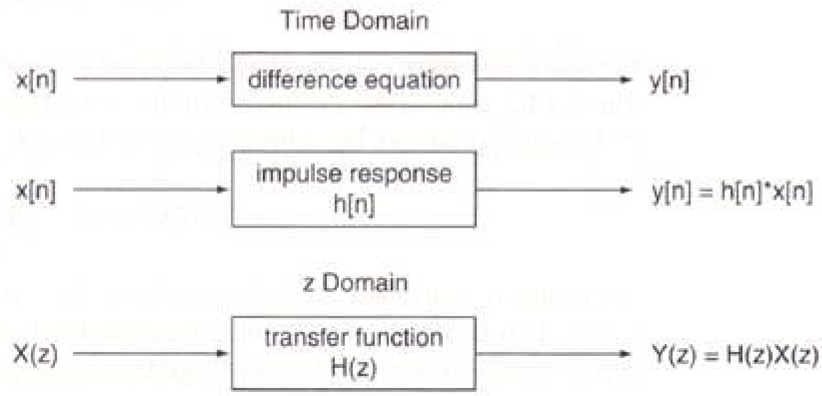
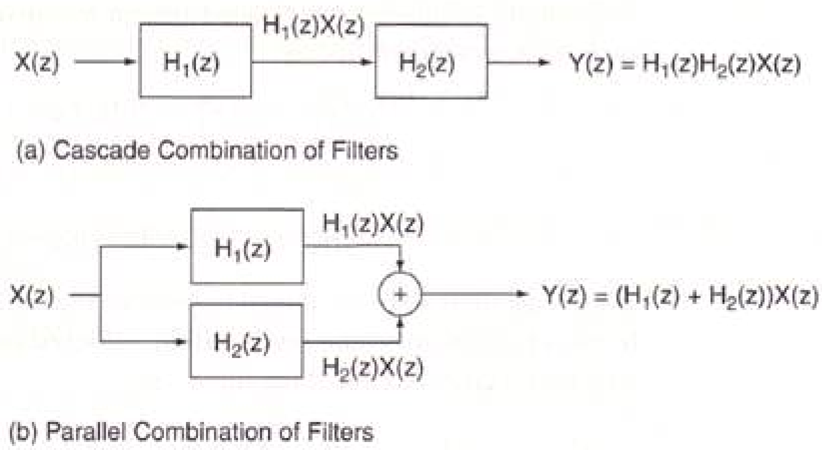
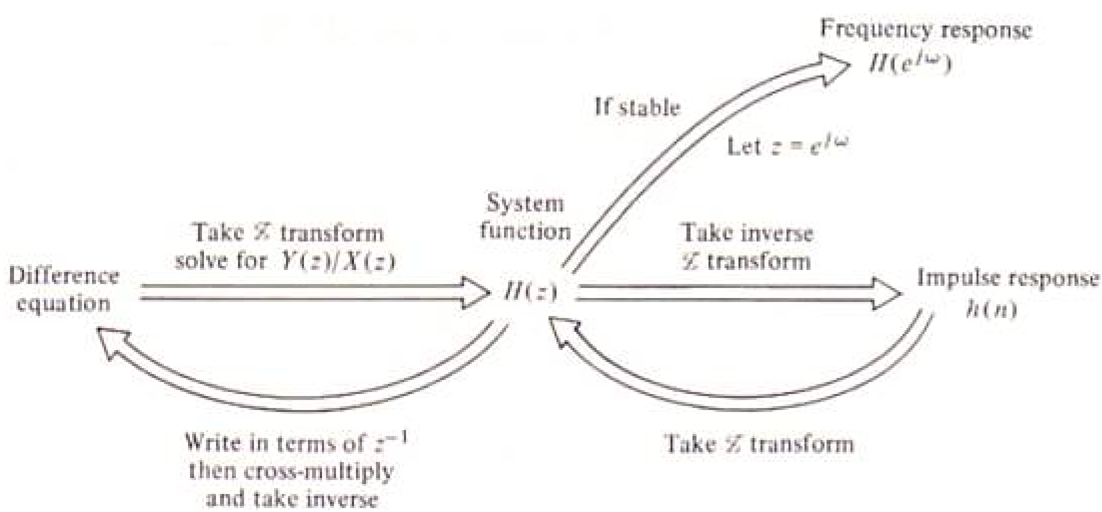


FIGURE 6.5
Cascade and parallel combinations of filters.





EXAMPLE 6.13

The impulse response for a digital filter is

$$h[n] = \delta[n] + 0.4\delta[n-1] + 0.2\delta[n-2] + 0.05\delta[n-3]$$

Find the transfer function of the filter.

The transfer function for the filter is nothing more than the z transform of the impulse response:

$$H(z) = 1 + 0.4z^{-1} + 0.2z^{-2} + 0.05z^{-3}$$

Note that this transfer function leads to the difference equation

$$y[n] = x[n] + 0.4x[n-1] + 0.2x[n-2] + 0.05x[n-3]$$