Signal reconstruction: The zero order hold (ZOH)

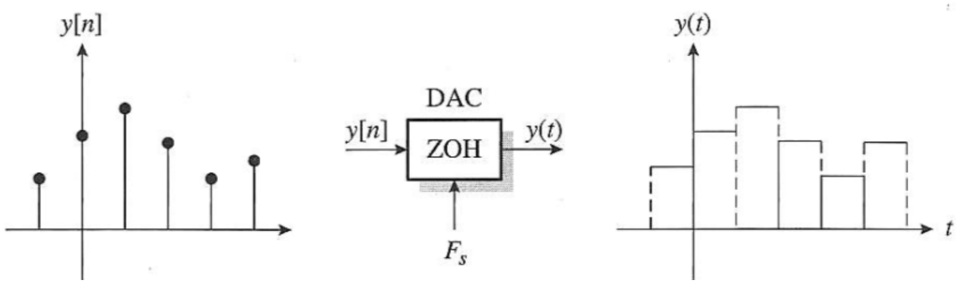
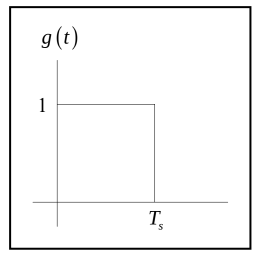


Figure 1: Signal reconstruction by the zero order holders (ZOH)

Reconstructor is a system in which the input $y[n]$ is a numerical sequence and the output $y(t)$ is a continuous time signal as shown in the Fig.1.

- The Reconstructor has to supply an interpolation function.
- If we assume linearity, this function can be expressed as



$$y(t) = \sum_{n=-\infty}^{\infty} g(t - nT_s)\, y[n] \tag{1}$$

where the continuous time signal $g(t)$ is called the **interpolation function**, provides the interpolation between the samples.

- In the case of the zero order hold, this interpolation is simply a constant.

Now we want to see the frequency response of the equation (1), then

$$\begin{aligned} F\left[y(t)\right] &= F\left[\sum_{n=-\infty}^{\infty} g(t - nT_s)\, y[n]\right] \\ Y(f) &= F\left[\sum_{n=-\infty}^{\infty} g(t - nT_s)\, y[n]\right] \end{aligned} \tag{2}$$

We need to know that $y(t)$ and $g(t - nT_s)$ are continuous signal and $y[n]$ is discrete signal.
The frequency response of interpolation signal can be defined as in the equation (3)

$$g(t - nT_s) = g(t) * \delta(t - nT_s)$$

$$\begin{aligned} F\left[g(t - nT_s)\right] &= F\left[g(t) * \delta(t - nT_s)\right] \\ &= F\left[g(t)\right] \cdot F\left[\delta(t - nT_s)\right] \\ &= G(f) \cdot e^{-j2\pi f n T_s} \end{aligned} \tag{3}$$

Plugging in (3) into (2)

$$\begin{aligned} Y(f) &= \sum_{n=-\infty}^{\infty} \left(G(f) e^{-j2\pi f n T_s} \right) \cdot y[n] \\ &= G(f) \left[\sum_{n=-\infty}^{\infty} e^{-j2\pi f n T_s} y[n] \right] \\ &= G(f) \underbrace{\left[\sum_{n=-\infty}^{\infty} y[n] e^{-j2\pi f n T_s} \right]}_{Y(\Omega) = \{DTFT\ of\ y[n]\}} \end{aligned} \tag{4}$$

You can notice that in the equation (4), the $Y(\Omega)$ is DTFT of $y[n]$

$$F\left[y[n] \right] = \sum_{n=-\infty}^{\infty} y[n] e^{-j\Omega n} \qquad \left\{ \text{where } \Omega = \frac{2\pi f}{f_s} = 2\pi f T_s \right\} \tag{5}$$

So, the equation (4) can be rewritten as

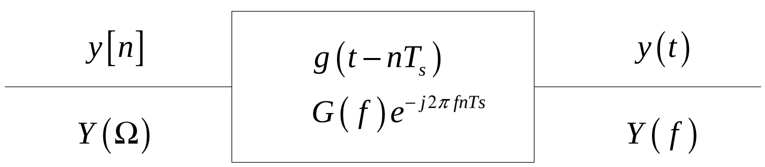


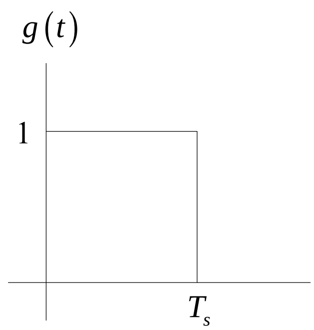
Figure 2: Reconstructor function

$$Y(f) = G(f) Y(\Omega) \tag{6}$$

We can notice that the Reconstructor function acts like a transfer function, determined by interpolating function as you can see in the Fig. 2.

- In the ideal case, $G(f)$ is the frequency response of an ideal filter.
- In practice, it is common to use a simple circuit called the zero order hold (ZOH), which performs interpolation by a zero degree polynomial: **that is constant**.
- The continuous $y(t)$ is piecewise constant within the sampling interval.

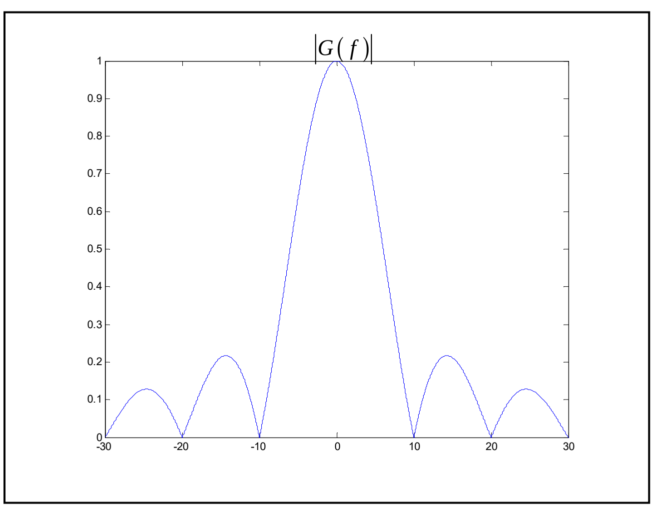
The interpolating function and magnitude of its Fourier transform are shown below



The frequency response of $g(t)$ is found in the (7)

$$
\begin{aligned}
G(f) &= \int_0^{T_s} 1 \cdot e^{-j2\pi ft} dt \\
&= \frac{-1}{j2\pi f} e^{-j2\pi ft} \Big|_0^{T_s} \\
&= \frac{1}{j2\pi f} \left(1 - e^{-j2\pi fT_s}\right) \\
&= \frac{1}{\pi f} e^{-j\pi fT_s} \left(\frac{e^{j\pi fT_s} - e^{-j\pi fT_s}}{2j} \right) \\
&= \frac{1}{\pi f} e^{-j\pi fT_s} \sin(\pi fT_s) \\
&= e^{-j\pi fT_s} \cdot T_s \cdot \frac{\sin(\pi fT_s)}{\pi fT_s} \qquad \left\{ \mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \right\} \\
&= \left[e^{-j\pi fT_s} \right] \cdot \left[T_s \cdot \mathrm{sinc}(fT_s) \right]
\end{aligned}
\tag{7}
$$

The **magnitude** response of (7) is




```
clear all;

f = -30:0.001:30;
Ts = 10;
fs = 1/Ts;

G = abs(sinc(f*fs));
plot(f,G)
```

Problem:

- The zero order hold is not ideal, and it creates some artifacts.

Ex.) Let's take a look at the case when the input sequence is a sinusoid

$$y[n] = \cos(w_0 n) \quad \text{with } 0 \le w_0 \le \pi \tag{8}$$

Note: The frequency response of

$$\begin{aligned} y(t) &= \cos(2\pi f t) \\ F\{y(t)\} &= F\{\cos(2\pi f_0 t)\} \\ &= \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] \end{aligned} \tag{9}$$

And one of the properties $\delta(t)$ shows that

$$\delta(at) = \frac{1}{|a|}\delta(t) \tag{10}$$

The frequency response of $y[n]$ is $Y(\Omega)$ with $0 \le \Omega_0 \le \pi$ being the digital frequency of the sinusoid. The DTFT, $Y(\Omega)$, in the interval $-\pi \le \Omega \le \pi$ has two delta functions at $-\Omega_0$ and $+\Omega_0$. The expression is

$$Y(\Omega) = \sum_{k=-\infty}^{\infty}\left[\pi\delta(\Omega - \Omega_0 + k2\pi) + \pi\delta(\Omega + \Omega_0 + k2\pi)\right] \tag{11}$$

The Fourier transform of the reconstructed signal is obtained by rescaling the frequency axis from w to f and using the equation (10)

$$\begin{aligned} Y(\Omega)\Big|_{\Omega=\frac{2\pi f}{f_s}} &= \sum_{k=-\infty}^{\infty}\left[\pi\cdot\delta\left(\frac{2\pi f}{f_s} - \frac{2\pi f_0}{f_s} + k2\pi\right) + \pi\delta\left(\frac{2\pi f}{f_s} + \frac{2\pi f_0}{f_s} + k2\pi\right)\right] \\ &= \sum_{k=-\infty}^{\infty}\left[\pi\cdot\delta\left(\frac{2\pi}{f_s}(f - f_0) + \frac{f_s}{2\pi}k2\pi\right) + \pi\delta\left(\frac{2\pi}{f_s}(f + f_0) + \frac{f_s}{2\pi}k2\pi\right)\right] \\ &= \sum_{k=-\infty}^{\infty}\left[\frac{f_s}{2\pi}\cdot\pi\cdot\delta\left((f - f_0) + kf_s\right) + \frac{f_s}{2\pi}\pi\delta\left((f + f_0) + kf_s\right)\right] \\ &= \sum_{k=-\infty}^{\infty}\left[\frac{f_s}{2}\delta\left((f - f_0) + kf_s\right) + \frac{f_s}{2}\delta\left((f + f_0) + kf_s\right)\right] \end{aligned} \tag{12}$$

Multiplying the result by $\left[G(f)=e^{-j\pi f T_s}\cdot T_s \mathrm{sinc}(f\cdot T_s)\right]$ and finding the result after ZOH,

The magnitude response is

$$\begin{aligned}
|Y(f)| &= \left|e^{-j\pi f T_s}\cdot T_s \mathrm{sinc}(fT_s)\right|\left|\sum_{k=-\infty}^{\infty}\left[\frac{f_s}{2}\delta\left((f-f_0)+kf_s\right)+\frac{f_s}{2}\delta\left((f+f_0)+kf_s\right)\right]\right| \\
&= \left|\frac{1}{f_s}\mathrm{sinc}(fT_s)\right|\left|\sum_{k=-\infty}^{\infty}\left[\frac{f_s}{2}\delta\left((f-f_0)+kf_s\right)+\frac{f_s}{2}\delta\left((f+f_0)+kf_s\right)\right]\right| \qquad (13) \\
&= \left|\mathrm{sinc}(fT_s)\right|\left|\frac{1}{2}\sum_{k=-\infty}^{\infty}\left[\delta\left((f-f_0)+kf_s\right)+\delta\left((f+f_0)+kf_s\right)\right]\right|
\end{aligned}$$

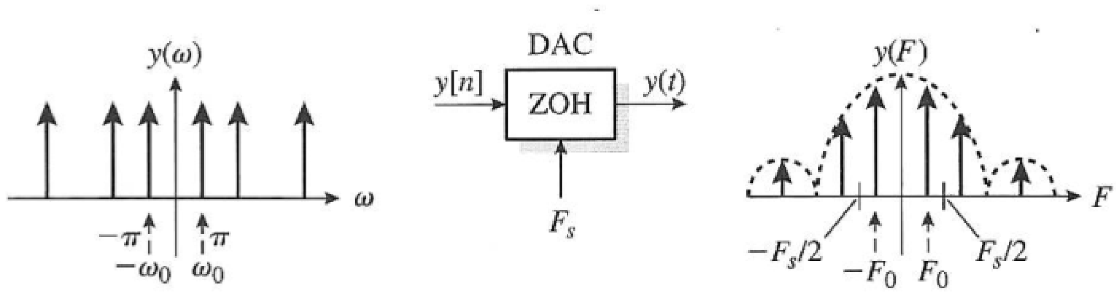


Figure 3

The example was shown with following specifications

```
clc; clear all;

f = 5% Frequency of input
fs = 10; % Sampling rate
n = -100:100;

y = cos(2*pi*f/fs*n);% y[n]

OM = -8*pi:0.01:8*pi;% range of OM for DTFT
len_OM = length(OM);

for indx = 1:len_OM
   Y(indx)   = exp(-j*OM(indx)*n)*y'; %DTFT values
end

plot(OM,abs(Y))% Magnitude response of DTFT
G = sin(pi*OM/fs)./(pi*OM/fs);% Generation of sinc function

figure;
plot(OM,abs(G))% Magnitude response of sinc function

figure;
out = abs(G).*Y;
plot(OM,out)
```

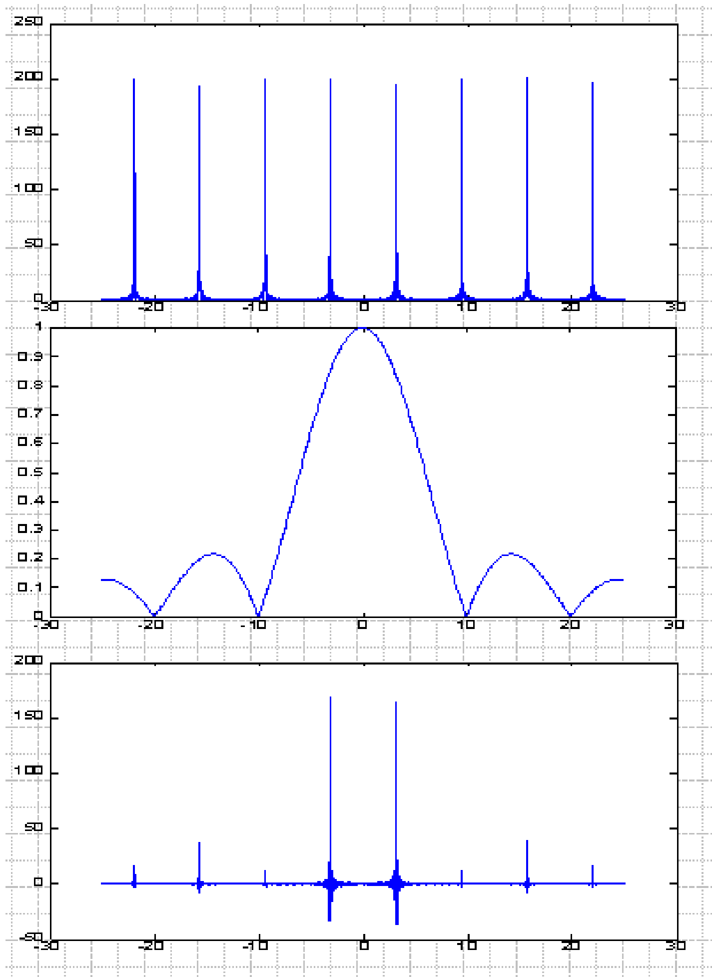


Figure4: Frequency response of input with cosine function, ZOH, and reconstructed function

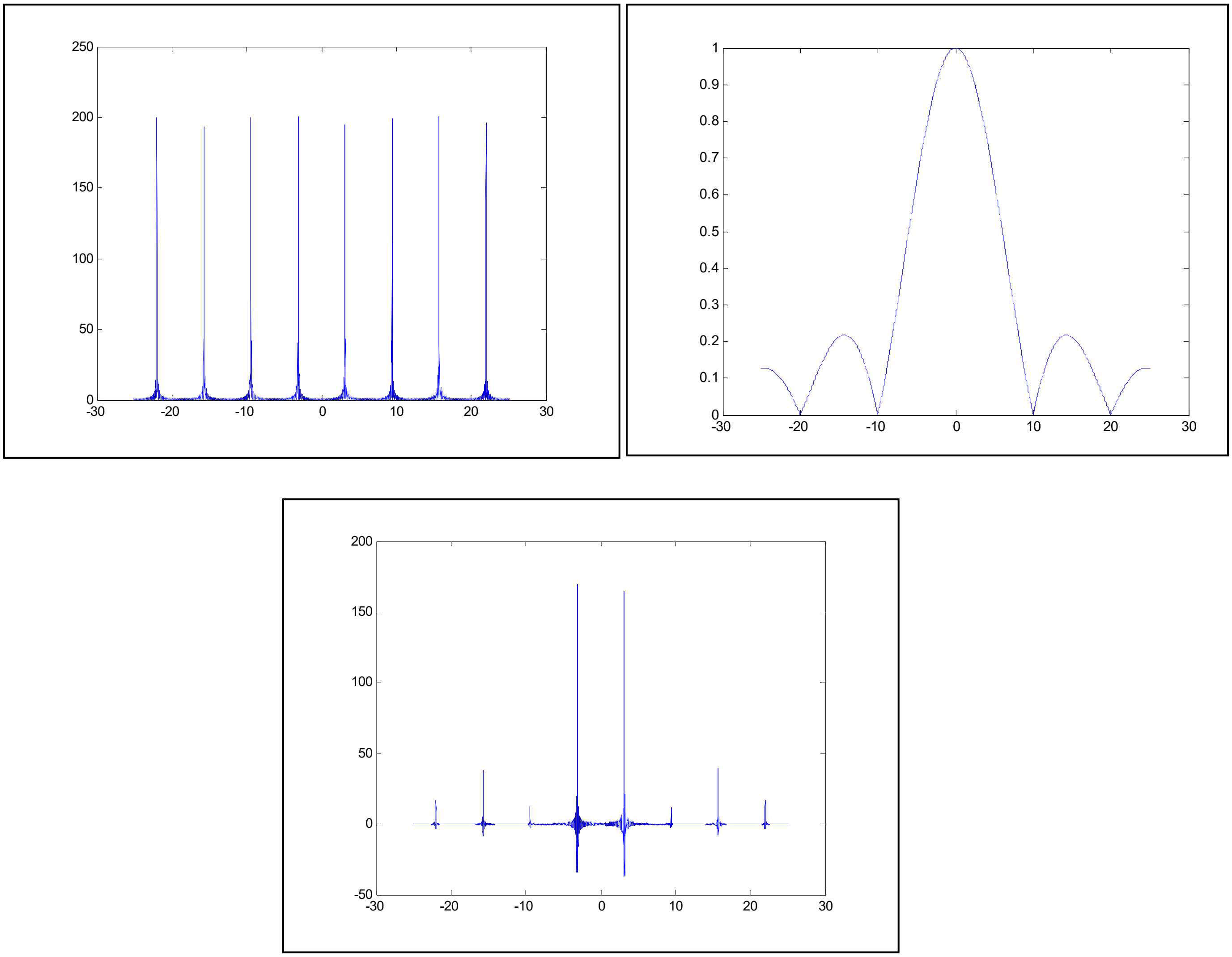


Figure 5: Plot by plot of each function

Overall frequency response of a digital filter.

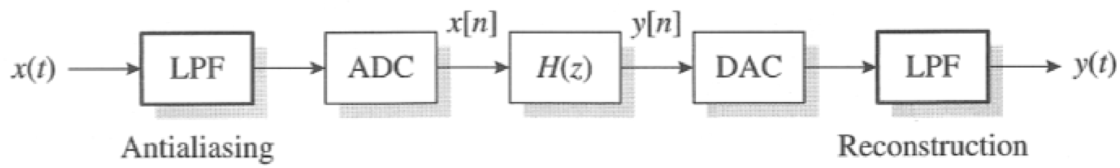


Figure 6: Block diagram of digital filter

Let's determine the overall frequency response of the digital filter and all its components.

$$Y(f) = W_R(f) \cdot e^{-j\pi f T_s} \cdot \text{sinc}(fT_s) \cdot H(\Omega) \cdot W_A(f) \cdot X(f) \tag{14}$$

where

W_R : Frequency response of the **reconstruction filter**

W_A : Frequency response of the **antialiasing filter**

Ideally, both these filters are assumed to eliminate the frequency component of the signal above the sampling frequency: that is

$$W_A(f) = W_R(f) = 0 \qquad \text{when} \quad |f| > \frac{fs}{2} \tag{15}$$

If we assume the filters to be ideal, and the input signal already bandlimited within half the sampling frequency, then the overall frequency response of the digital, with the ZOH as a reconstructor, becomes

$$\begin{aligned} Y(f) &= \left[G(f) H\left(\frac{2\pi f}{fs}\right) \right] X(f), \quad -\frac{fs}{2} < f < \frac{fs}{2} \\ &= \left[\left(e^{-j\pi \frac{f}{fs}} \text{sinc}\left(\frac{f}{fs}\right) \right) H\left(\frac{2\pi f}{fs}\right) \right] X(f) \end{aligned}$$

The equation shows that there are distortions

- There is time delay term
- A magnitude of sinc function attenuates the frequency components around $fs/2$ as shown below.

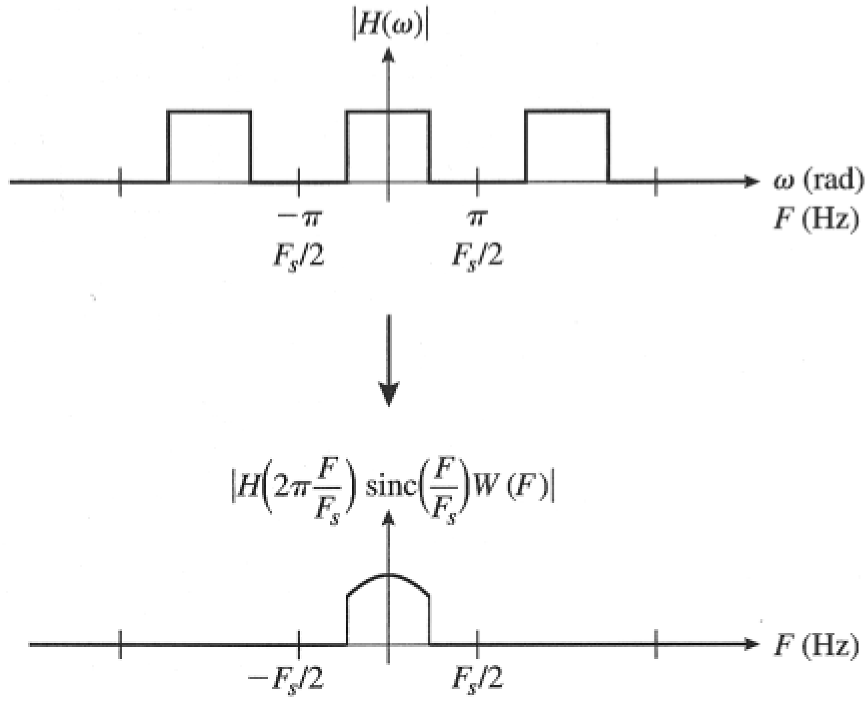


Figure 7: Magnitude of the overall frequency response