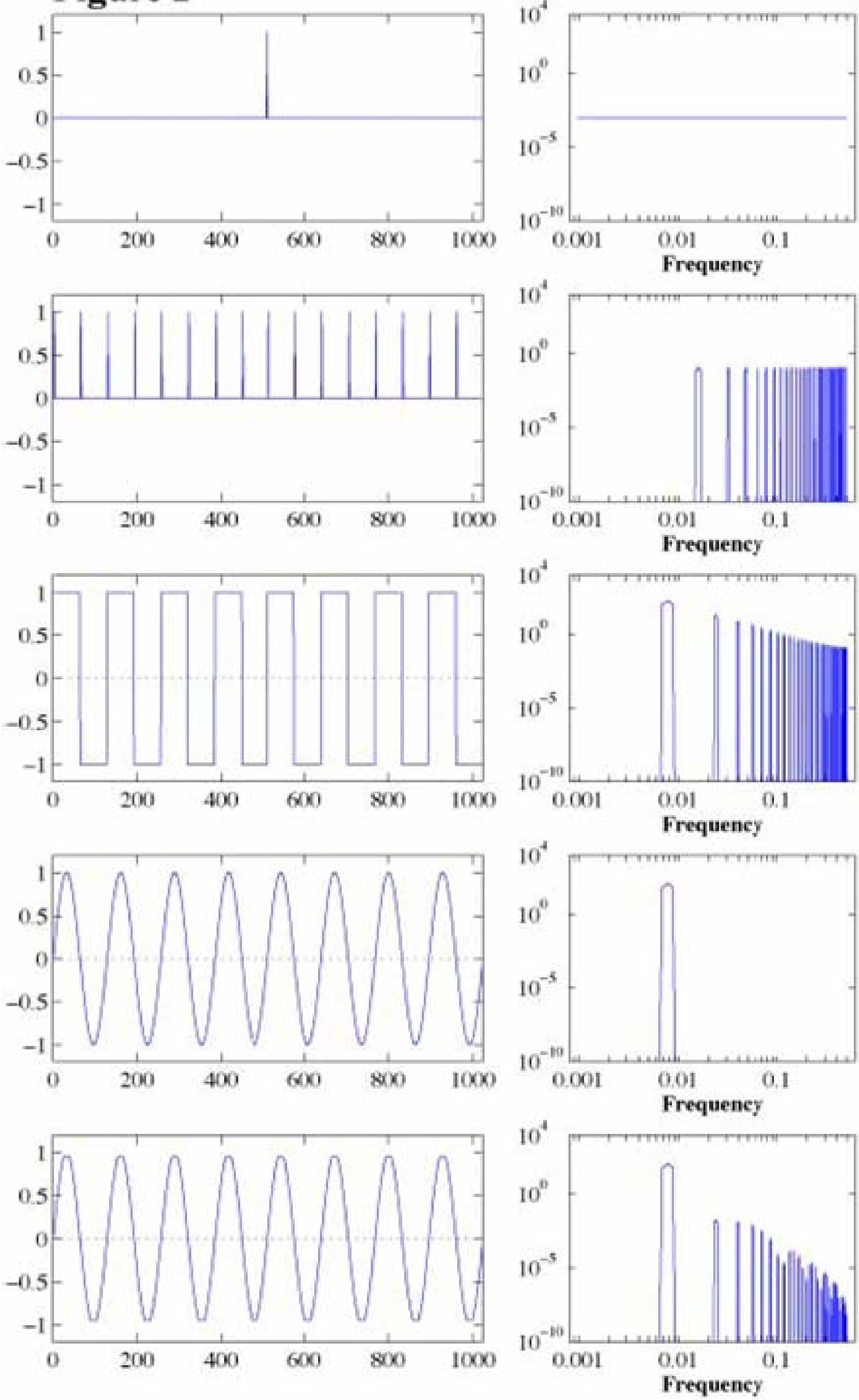
Power Spectral Density

Figure 2



Power spectral density

- Amount of *power* per unit (*density*) of frequency (*spectral*) as a function of the frequency.
- The power spectral density (PSD), which describes how the power of a signal or time series is distributed with **frequency**

The **PSD is the Fourier transform of the autocorrelation function, *R*(τ)**, of the signal if the signal can be treated as a **stationary random** process.

An interpretation of the PSD

A stationary sequence is a random sequence such that the joint PDF (probability density function) of the sequence is invariant over time.

Given stationary process $X(t)$, consider the finite support segment,

$$X_T(t) = \begin{cases} X(t) & -T \le t \le T \\ 0 & o.w \end{cases}$$

The Fourier transform of the above function is

$$FT\{X_T(t)\} = \int_{-T}^{T} X(t) e^{-j\omega t} dt$$

The magnitude squared of this random variable is

$$\left| FT\{X_T(t)\} \right|^2 = \int_{-T}^{T}\int_{-T}^{T} X(t_1)\left[X(t_2)\right]^* e^{-j\omega t_1}\left[e^{-j\omega t_2}\right]^* dt_1 dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} X(t_1)\left[X(t_2)\right]^* e^{-j\omega(t_1-t_2)} dt_1 dt_2$$

Dividing by 2T and taking the expected values, we get

$$\frac{1}{2T}E\left[\left|FT\left\{X_T(t)\right\}\right|^2\right]=\frac{1}{2T}E\left[\int_{-T}^{T}\int_{-T}^{T}X(t_1)\left[X(t_2)\right]^* e^{-j\omega(t_1-t_2)}dt_1dt_2\right]$$

$$=\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}E\left[X(t_1)\left[X(t_2)\right]^*\right]e^{-j\omega(t_1-t_2)}dt_1dt_2$$

$$=\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}R_X(t_1-t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2$$

$$=\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}R_X(\tau)e^{-j\omega(\tau)}dt_1dt_2$$

$$=\int_{-T}^{T}\left[1-\frac{|\tau|}{2T}\right]R_X(\tau)e^{-j\omega\tau}d\tau$$

In the limit $T\to\infty$ the above equation becomes, Fourier transform of correlation function.

$$S_X(\omega)=\lim_{T\to\infty}\frac{1}{2T}E\left[\left|FT\left\{X_T(t)\right\}\right|^2\right]$$

So the Power spectral density is real and non-negative and it is related to average power at frequency ω.

Let $R_X(\tau)$ be an autocorrelation function. Then we define the power spectral density $S_X(\omega)$ to be its Fourier transform

$$S_X(\omega)=\int_{-\infty}^{\infty}R_X(\tau)e^{-j\omega\tau}d\tau$$

The inverse Fourier transform of the above function, $S_X(\omega)$ is

$$R_X(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_X(\omega)e^{j\omega\tau}d\omega$$

<u>CH6 Sums of Random Variables</u>

$$W_n = X_1 + X_2 + X_3 + ... + X_n$$

For any set of random variables $X_1,...,X_n$, the expected values of $W_n = X_1 + X_2 + X_3 + ... + X_n$ is

$$\begin{aligned}E[W_n] &= E[X_1 + X_2 + X_3 + ... + X_n] \\ &= E[X_1] + E[X_2] + E[X_3] + ... + E[X_n]\end{aligned}$$

$$\begin{aligned}Var[W_n] &= E\left[\left(\sum_{i=1}^{n} X_i - \mu_i\right)^2\right] \\ &= E\left[\left(\sum_{i=1}^{n} X_i - \mu_i\right)\left(\sum_{j=1}^{n} X_j - \mu_i\right)\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} Cov\left(X_i X_j\right) \\ &= \sum_{i=1}^{n} Var\left(X_i\right) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} Cov\left(X_i X_j\right)\end{aligned}$$

If $X_1,...,X_n$ are uncorrelated, then the $Var[W_n]$ becomes

$$\begin{aligned}Var[W_n] &= \sum_{i=1}^{n} Var\left(X_i\right) \\ &= Var[X_1] + Var[X_2] + ... + Var[X_n]\end{aligned}$$

Ex 6.1 $X_1, X_2, X_3....$ is a sequence of random variables with expected values $E[X_i] = 0$ and covariance, $Cov\left[X_i, X_j\right] = 0.8^{|\text{i-j}|}$. Find the expected value and variance of a random variable Y_i defined as the sum of three consecutive values of the random sequence

$$Y_i = X_i + X_{i-1} + X_{i-2}$$

$$\begin{aligned}
E\left[Y_i\right] &= E\left[X_i + X_{i-1} + X_{i-2}\right] \\
&= E\left[X_i\right] + E\left[X_{i-1}\right] + E\left[X_{i-2}\right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
Var\left[Y_i\right] &= Var\left[X_i + X_{i-1} + X_{i-2}\right] \\
&= \sum_{i=1}^{n} Var\left(X_i\right) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} Cov\left(X_i X_j\right) \\
&= \sum_{i=1}^{3} Var\left(X_i\right) + 2\sum_{i=1}^{2}\sum_{j=i+1}^{3} Cov\left(X_i X_j\right) \\
&= Var\left(X_i\right) + Var\left(X_{i-1}\right) + Var\left(X_{i-2}\right) \\
&\quad + Cov\left(X_i, X_{i-1}\right) + Cov\left(X_i, X_{i-2}\right) + Cov\left(X_{i-1}, X_{i-2}\right) \\
&= 3\left(0.8\right)^0 + 2\left(0.8\right)^1 + 2\left(0.8\right)^2 + 2\left(0.8\right)^1 \\
&= 7.48
\end{aligned}$$

Moment Generating Functions

The moment generating function generates the moments of the probability distribution.

For a random variable X, the moment generating function (MGF) of X is defined as

$$\phi_X(s) = E\left(e^{sX}\right)$$

- $$\begin{aligned}\phi_X(s) &= E\left(e^{sX}\right) \\ &= \int_{-\infty}^{\infty} e^{sx} \cdot f_X(x)\,dx\end{aligned}$$ for continuous random variable X

- $$\begin{aligned}\phi_X(s) &= E\left(e^{sX}\right) \\ &= \sum_{x_i \in S_x}^{\infty} e^{sx_i} \cdot P_X\left[x_i\right]\end{aligned}$$ for discrete random variable X

- The set of values of s for which $\phi_X(s)$ exists is called region of convergence.
- The MGF and PMF or PDF form a transform pair, the MGF is also a complete probability model of random variable. (given the MGF, it is possible to compute the PDF or PMF)

The definition of the MGF implies that

$$\phi_X(s=0) = E\left(e^0\right) = E(1) = 1$$

- **The derivatives of $\phi_X(s)$ evaluated at $s=0$ are the moment of X .**

A random variable X with MGF $\phi_X(s)$ has n^{th} moment

$$E\left(X^n\right)=\left.\frac{d^n\phi_X(s)}{ds^n}\right|_{s=0}$$

Proof:

$$\begin{aligned}\left.\frac{d\phi_X(s)}{ds}\right|_{s=0} &= \frac{d}{ds}\left(\int_{-\infty}^{\infty} e^{sx} f_X(x)dx\right)\\ &= \left.\left(\int_{-\infty}^{\infty} xe^{sx} f_X(x)dx\right)\right|_{s=0}\\ &= \int_{-\infty}^{\infty} x f_X(x)dx\\ &= E(X)\end{aligned}$$

Similarly, the n^{th} moment of $\phi_X(s)$ is $E\left(X^n\right)$

Ex6.5 X is an exponential random variable with MGF $\left[\phi_X(s) = \frac{\lambda}{\lambda - s}\right]$. What is the first and second moments of X? Write a general expression for the n^{th} moment.

As you know the 1st moment is the expected value, $E(X)$.

$$E(X) = \frac{d}{ds}(\phi_X(s)) = \frac{d}{ds}\left(\frac{\lambda}{\lambda - s}\right)$$

$$= \left.\frac{\lambda}{(\lambda - s)^2}\right|_{s=0} = \frac{1}{\lambda}$$

$$E\left(X^2\right) = \frac{d^2}{ds^2}(\phi_X(s)) = \left.\frac{2\lambda}{(\lambda - s)^3}\right|_{s=0} = \frac{2}{\lambda^2}$$

$$E\left(X^n\right) = \frac{d^n}{ds^n}(\phi_X(s)) = \left.\frac{n!\lambda}{(\lambda - s)^{n+1}}\right|_{s=0} = \frac{n!}{\lambda^n}$$

The MGF of $Y = aX + b$ is $\phi_Y(s) = e^{sb} \cdot \phi_X(as)$

$$\phi_Y(s) = E\left(e^{ys}\right) = E\left(e^{(ax+b)s}\right) = E\left(e^{axs}e^{bs}\right) = e^{bs}E\left(e^{axs}\right) = e^{bs}\phi_X(as)$$

Quiz 6.3 Random variable K has PMF

$$P_K[k] = \begin{cases} 0.2 & k = 0,1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

Use the MGF $\phi_K(s)$ to find the first, second, third, fourth moments of K.

The MGF of K is

$$\begin{aligned} \phi_K(s) &= E\left[e^{ks}\right] \\ &= \sum_{k=0,1,2,3,4} e^{ks}P_K[k] \\ &= \sum_{k=0,1,2,3,4} e^{ks}P_K[k] = \sum_{k=0,1,2,3,4} e^{ks}(0.2) \\ &= (0.2)\left(1 + e^{1s} + e^{2s} + e^{3s} + e^{4s}\right) \end{aligned}$$

We can find the moments by taking derivatives. The 1st derivative of $\phi_K(s)$ is

$$\begin{aligned} \frac{d}{ds}\phi_K(s) &= \frac{d}{ds}E\left[e^{ks}\right] \\ &= (0.2)\left(e^{1s} + 2e^{2s} + 3e^{3s} + 4e^{4s}\right)\Big|_{s=0} \\ &= (0.2)(1+2+3+4) = 2 \end{aligned}$$

$$\begin{aligned} \frac{d^2}{ds^2}\phi_K(s) &= \frac{d^2}{ds^2}E\left[e^{ks}\right] \\ &= (0.2)\left(e^{1s} + 4e^{2s} + 9e^{3s} + 16e^{4s}\right)\Big|_{s=0} \\ &= (0.2)(1+4+9+16) = 6 \end{aligned}$$

$$\frac{d^3}{ds^3}\phi_K(s) = \frac{d^3}{ds^3}E\left[e^{ks}\right] = 20$$

$$\frac{d^4}{ds^4}\phi_K(s) = \frac{d^4}{ds^4}E\left[e^{ks}\right] = 70.8$$

MGF of the sum of independent random variables

For a set of independent random variables $X_1,...,X_n$, the moment generating function of $W = X_1 + X_2 + X_3 ... + X_n$ is

$$\begin{aligned}\phi_W(s) &= E\left(e^{sX_1} \cdot e^{sX_2} \cdot e^{sX_3} \dots \cdot e^{sX_n}\right) \\ &= E\left(e^{sX_1}\right)E\left(e^{sX_2}\right)\dots E\left(e^{sX_n}\right)\end{aligned}$$

When $X_1,...,X_n$ are iid, each with MFG $\phi_W(s) = \left[\phi_X(s)\right]^n$

Ex6.6 J and K are independent random variables with probability mass function.

$$P_J(j) = \begin{cases} 0.2 & j=1 \\ 0.6 & j=2 \\ 0.2 & j=3 \\ 0 & o.w \end{cases} \qquad P_K(k) = \begin{cases} 0.5 & k=-1 \\ 0.5 & j=1 \\ 0 & o.w \end{cases}$$

Find the MGF of $M = J + K$ what are $E\left[M^3\right]$?

J and K have moment generating functions

$$\phi_J(s) = 0.2e^s + 0.6e^{2s} + 0.2e^{3s}$$
$$\phi_K(s) = 0.5e^{-s} + 0.5e^s$$

$$\begin{aligned}\phi_M(s) = \phi_K(s)\phi_J(s) &= \left(0.2e^s + 0.6e^{2s} + 0.2e^{3s}\right)\left(0.5e^{-s} + 0.5e^s\right) \\ &= 0.1 + 0.3e^s + 0.2e^{2s} + 0.3e^{3s} + 0.1e^{4s}\end{aligned}$$

To find $E\left[M^3\right]$, the 3rd moment of M, $\phi_M(s)$ has to be differentiate three times and evaluated s=0.

$$E\left[M^3\right] = \frac{d^3}{ds^3}\phi_M(s)\Big|_{s=0}$$

$$= 0.3e^s + 0.2\left(2^3\right)e^{2s} + 3\left(3^3\right)e^{3s} + 0.1\left(4^3\right)e^{4s}\Big|_{s=0} = 16.4$$

Central Limit Theorem

The **central limit theorem (CLT)** states that the sum of a large number of independent and identically-distributed random variables will be approximately normally distributed (i.e., following a Gaussian distribution, or bell-shaped curve) if the random variables have a finite variance. *[wikipedia]*

Central limit theorem explains why so many practical phenomena produce data that can be modeled as Gaussian random variables.

Given $X_1, X_2, \ldots,$ a sequence of iid random variables with expected value μ_X and variance $\sigma_X{}^2$, the CDF of

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu_X}{\sqrt{n\sigma_X{}^2}}$$

has the property

$$\lim_{n\to\infty} F_{Z_n}(z) = \Phi(z).$$

Central limit theorem approximation

Let $W_n = X_1 + \ldots + X_n$ be the sum of n iid random variables, each with $E[X] = \mu_X$ and $Var[X] = \sigma_X{}^2$. The central limit theorem approximation to the CDF of W_n is

$$F_{W_n}(w) \approx \Phi\left(\frac{w - n\mu_x}{\sqrt{n\sigma_X{}^2}}\right)$$

Quiz6.6 The random variable X millisecond is the total access time (waiting time + reading time) to get one block of information from a computer disk. X is uniformly distributed between 0 and 12 milliseconds. Before performing a certain task, the computer must access 12 different blocks of information from the disk. (Access times for different blocks are independent of one another) The total access time for all the information is a random variable A milliseconds.

1) What is $E(X)$, the expected value of the access time?
2) What is $Var(X)$, the variance of the access time?
3) What is $E(A)$, the expected value of the total access time?
4) What is σ_A, the standard deviation of the total access time.
5) Use the central limit theorem to estimate $P(A > 75ms)$, the probability that the total access time exceeds 75 ms.
6) Use the central limit theorem to estimate $P(A < 48ms)$, the probability that the total access time is less than 48 ms.

1). $E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx$

$$= \int_0^{12} x \frac{1}{12}\,dx$$
$$= 6 \;\; ms$$

2). $Var(X) = E(X^2) - E(X)^2$

$$E(X^2) = \int_0^{12} x^2 f_X(x)\,dx$$
$$= \int_0^{12} x^2 \frac{1}{12}\,dx$$
$$= 48$$
$$Var(X) = 48 - (6)^2$$
$$= 12$$

3). $A = X_1 + X_2 + X_3 + \ldots + X_{12}$

$$\begin{aligned} E(A) &= E(X_1 + X_2 + X_3 + \ldots + X_{12}) \\ &= E(X_1) + E(X_2) + E(X_3) + \ldots + E(X_{12}) \\ &= 12E(X) \\ &= 12 \cdot 6 \\ &= 72 \end{aligned}$$

4). $$\begin{aligned} Var(A) &= Var(X_1 + X_2 + X_3 + \ldots + X_{12}) \\ &= Var(X_1) + Var(X_2) + Var(X_3) + \ldots + Var(X_{12}) \\ &= 12 \cdot 12 \\ &= 144 \end{aligned}$$

$$\sigma_A = \sqrt{\sigma_A^{\ 2}} = 12$$

5). $P(A > 75ms)$

$$\begin{aligned} P(A > 75) &= 1 - P(A \le 75) \\ &= 1 - \Phi\left(\frac{A - \mu_A}{\sigma_A}\right) \\ &\approx 1 - \Phi\left(\frac{75 - 72}{12}\right) \\ &= 1 - \Phi\left(\frac{75 - 72}{12}\right) \end{aligned}$$

6). $P(A < 48ms)$

$$\begin{aligned} P(A < 48) &\approx \Phi\left(\frac{A - \mu_A}{\sigma_A}\right) \\ &= \Phi\left(\frac{48 - 72}{12}\right) \\ &= \Phi(-2) \\ &= 1 - \Phi(2) \\ &= 1 - 0.9773 \end{aligned}$$