Stochastic Control Study Notes

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Contents

Chapter 1

Some Preliminaries

1.1 Martingale Representation Theorem

Theorem 1.1.1 (Martingale Representation). *Suppose that* $M = \left\{(M_t^{(1)}, .., M_t^{(d)})\right\}_{t \in \mathbb{T}}$ *is a* d*-dimensional cadlag local martingale on* $(\Sigma, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$*. Suppose that the bracket* $\langle M^{(i)}, M^{(j)} \rangle$ *is absolutely continuous w.r.t time* t $\mathbb{P}$*-a.s. Then there exists an extension* $(\tilde{\Sigma}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t\in\mathbb{T}}, \tilde{\mathbb{P}})$ *of* $(\Sigma, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ *in which is defined a Brownian motion* $W = \left\{(W_t^{(1)}, .., W_t^{(d)})\right\}_{t\in\mathbb{T}}$ *and a measurable and adapted matrix-valued process* $\alpha = \left\{(\alpha_t^{(i,k)})_{i,k=1}^d\right\}_{t\in\mathbb{T}}$ *such that each component is in* L_2^{loc} *and*

$$M_t = M_0 + \int_0^t \alpha_s \, dW_s \quad \tilde{\mathbb{P}}\text{-a.s.} \tag{1.1}$$

PROOF. Because the bracket $\langle M^{(i)}, M^{(j)} \rangle$ are absolutely continuous, then it is differentiable a.e. and is an integral.

$$\left\langle M^{(i)}, M^{(j)} \right\rangle_t = \int_0^t z_s^{(i,j)} \, ds \tag{1.2}$$

Form the matrix-valued process $Z = \left\{(z_s^{(i,j)})_{i,j=1}^d\right\}_{t\in T}$. Obviously, Z is symmetric and for any $y = (y_1, ..., y_d)^T \in \mathbb{R}^d$,

$$y^T Z_t y = \frac{d}{dt} \left\langle \sum_{i=1}^d M^{(i)} \right\rangle_t \geq 0 \tag{1.3}$$

as the bracket is non-decreasing, making Z positive-semidefinite. Then, there is an orthogonal matrix-valued process $Q = \left\{(q_t^{(i,k)})_{i,k=1}^d\right\}_{t\in\mathbb{T}}$ and diagonal matrix-valued process $\Lambda = \left\{\operatorname{diag}(\lambda_t^{(i)})_{i=1}^d\right\}_{t\in\mathbb{T}}$ (in which the diagonal is nonnegative) such that $\Lambda = Q^{-1} Z Q$.

Define a d-dimensional process N

$$N_t = \int_0^t Q_s \, dM_s \tag{1.4}$$

N is a local martingale because M is and $\int_0^t Q_s^T Q_s \, d\langle M\rangle_s = \int_0^t I \, d\langle M\rangle_s = \langle M\rangle_s < \infty$.

If we denote the i-th column of Q as $Q^{(i)}$, then we also have

$$
\begin{aligned}
\left\langle N^{(i)}, N^{(j)} \right\rangle_t &= \left\langle \sum_{k=1}^{d} \int_0^t q_s^{(i,k)} \, dM_s^{(k)}, \sum_{l=1}^{d} \int_0^t q_s^{(j,l)} \, dM_s^{(l)} \right\rangle_t \\
&= \sum_{k=1}^{d} \sum_{l=1}^{d} \int_0^t q_s^{(i,k)} q_s^{(j,l)} \, d\left\langle M^{(k)}, M^{(l)} \right\rangle_s \\
&= \sum_{k=1}^{d} \sum_{l=1}^{d} \int_0^t q_s^{(i,k)} q_s^{(l,j)} \, d\left\langle M^{(k)}, M^{(l)} \right\rangle_s \\
&= \sum_{k=1}^{d} \sum_{l=1}^{d} \int_0^t q_s^{(i,k)} z_s^{(k,l)} q_s^{(l,j)} \, ds \\
&= \int_0^t \left(Q_s^{(i)}\right)^T Z_s Q_s^{(j)} \, ds \\
&= \int_0^t \delta_{i,j} \lambda_s^{(i)} \, ds = \delta_{i,j} \int_0^t \lambda_s^{(i)} \, ds
\end{aligned}
\tag{1.5}
$$

Extend $(\Sigma, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ to $(\tilde{\Sigma}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t\in\mathbb{T}}, \tilde{\mathbb{P}})$ so that a Brownian motion $B = \left\{(B_t^{(1)}, .., B_t^{(d)})\right\}_{t\in\mathbb{T}}$ independent of N can be constructed.

Define

$$
W_t^{(i)} = \int_0^t 1_{\lambda_s^{(i)}>0} \frac{1}{\sqrt{\lambda_s^{(i)}}} \, dN_s^{(i)} + \int_0^t 1_{\lambda_s^{(i)}=0} \, dB_s^{(i)} \tag{1.6}
$$

As B and N are independent, we may compute

$$
\begin{aligned}
\left\langle W^{(i)}, W^{(j)} \right\rangle_t &= \int_0^t 1_{\lambda_s^{(i)}>0} 1_{\lambda_s^{(j)}>0} \frac{1}{\sqrt{\lambda_s^{(i)} \lambda_s^{(j)}}} \, d\left\langle N^{(i)}, N^{(j)} \right\rangle_s + \int_0^t 1_{\lambda_s^{(i)}=0} 1_{\lambda_s^{(j)}=0} \, d\left\langle B^{(i)}, B^{(j)} \right\rangle_s \\
&= \delta_{i,j} \int_0^t 1_{\lambda_s^{(i)}>0} \frac{1}{\lambda_s^{(i)}} \, d\left\langle N^{(i)} \right\rangle_s + \delta_{i,j} \int_0^t 1_{\lambda_s^{(i)}=0} \, d\left\langle B^{(i)} \right\rangle_s \\
&= \delta_{i,j} \int_0^t 1_{\lambda_s^{(i)}>0} \frac{1}{\lambda_s^{(i)}} \lambda_s^{(i)} \, ds + \delta_{i,j} \int_0^t 1_{\lambda_s^{(i)}=0} \, ds \\
&= \delta_{i,j} \cdot t
\end{aligned}
\tag{1.7}
$$

$\boxed{\text{Q.E.D}}$

Then W is also a Brownian motion on $(\tilde{\Sigma}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t\in\mathbb{T}}, \tilde{\mathbb{P}})$.

Define $X = Q\sqrt{\Lambda}$. Then

$$\begin{aligned}
\int_0^t X_s\,dW_s &= \int_0^t Q_s\sqrt{\Lambda_s}\,\mathrm{diag}\left(1_{\left\{\lambda_s^{(i)}>0\right\}}\frac{1}{\sqrt{\lambda_s^{(i)}}}\right)_{i=1}^d dN_s \\
&\quad+\int_0^t Q_s\sqrt{\Lambda_s}\,\mathrm{diag}\left(1_{\left\{\lambda_s^{(i)}=0\right\}}\right)_{i=1}^d dB_s \\
&= \int_0^t Q_s\,\mathrm{diag}\left(1_{\left\{\lambda_s^{(i)}>0\right\}}\right)_{i=1}^d dN_s + 0 \\
&= \int_0^t Q_s\,dN_s - \int_0^t Q_s\,\mathrm{diag}\left(1_{\left\{\lambda_s^{(i)}=0\right\}}\right)_{i=1}^d dN_s
\end{aligned} \tag{1.8}$$

But

$$\begin{aligned}
\left\langle \int_0^t Q_s\,\mathrm{diag}\left(1_{\left\{\lambda_s^{(i)}=0\right\}}\right)_{i=1}^d dN_s \right\rangle_t &= \int_0^t \mathrm{diag}\left(1_{\left\{\lambda_s^{(i)}=0\right\}}\right)_{i=1}^d Q_s^T Q_s \mathrm{diag}\left(1_{\left\{\lambda_s^{(i)}=0\right\}}\right)_{i=1}^d d\langle N\rangle_s \\
&= \int_0^t \mathrm{diag}\left(1_{\left\{\lambda_s^{(i)}=0\right\}}\right)_{i=1}^d d\langle N\rangle_s \\
&= \int_0^t \mathrm{diag}\left(1_{\left\{\lambda_s^{(i)}=0\right\}}\right)_{i=1}^d (\delta_{i,j}\lambda_s^{(i)})_{i,j=1}^d\,ds \\
&= 0
\end{aligned} \tag{1.9}$$

Because $\int_0^t Q_s\,\mathrm{diag}\left(1_{\left\{\lambda_s^{(i)}=0\right\}}\right)_{i=1}^d dN_s$ has 0 quadratic variation and is a local martingale, then it must be indistinguishable from 0 (?).

Because of this, we may easily show

$$\begin{aligned}
\int_0^t X_s\,dW_s &= \int_0^t Q_s\,dN_s \\
&= \int_0^t Q_s Q_s\,dM_s \\
&= \int_0^t I\,dM_s \\
&= M_t
\end{aligned} \tag{1.10}$$

1.2 Girsanov's Theorem

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1.3 Stochastic Differential Equations

1.3.1 Strong Solutions of SDEs

Let us fix a filtered probability space $(\Sigma, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ satisfying the usual conditions and a d-dimensional Brownian motion W with respect to $\mathbb{F}$. In this case, $\mathbb{T} = [0, T]$ where T is any positive numbet but cannot be ∞.

Consider a stochastic differential equation

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t \tag{1.11}$$

which is really a shortened form of the integral equation

$$X_t - X_s = \int_s^t b(u, X_u)\, du + \int_s^t \sigma(u, X_u)\, dW_u \tag{1.12}$$

Here X is an $\mathbb{R}^n$-valued process, so the drift b is n-dimensional and the diffusion σ is matrix-valued with dimension $(n \times d)$.

Definition 1.3.1 (Strong Solution of SDE)**.** A strong solution of SDE (1.11) starting from s is a progressively measurable process X s.t.

$$\int_s^t |b(u, X_u)|\, du + \int_s^t \mathrm{tr}(\sigma(u, X_u)^T \sigma(u, X_u))\, du < \infty \tag{1.13}$$

and satisfying the equation (1.12) for any $t \geq s$, $t \in \mathbb{T}$ almost surely.

Basically, for strong solutions the filtered probability space and the Brownian motion have been fixed from the start.

Let us denote the initial condition $X_s = \xi$ where ξ has finite p-moment for some $p \geq 2$.

Now, let us assume that the drift and diffusion are Lipschitz and have linear growth.

Assumption 1.3.2. There exists a constant $K < \infty$ and a real-valued process $\kappa = (\kappa_t)_{t\in\mathbb{T}}$ such that for all $t \in \mathbb{T}$, $\omega \in \Omega$, $x, y \in \mathbb{R}^n$,

$$|b(t, x, \omega) - b(t, y, \omega)| + |\sigma(t, x, \omega) - \sigma(t, y, \omega)| \leq K|x - y| \tag{1.14}$$

$$|b(t, x, \omega)| + |\sigma(t, x, \omega)| \leq |\kappa(\omega)| + K|x| \tag{1.15}$$

$$\mathbb{E}\left[\int_0^t |\kappa_u|^2\, du\right] + \mathbb{E}\left[\int_0^t |\kappa_u|^p\, du\right] < \infty \tag{1.16}$$

Theorem 1.3.1 (Existence of Solution to SDE)**.** *Under Assumption 1.3.2, there exists for all $s \in \mathbb{T}$, a strong solution to the SDE* (1.12) *with initial value $X_s = \xi$.*

As with proving solution existence for deterministic ODEs, we introduce the following Picard iteration $X_t^{(0)} = \xi$ and

$$X_t^{(n)} = \xi + \int_s^t b(u, X_u^{(n-1)})du + \int_s^t \sigma(u, X_u^{(n-1)})dW_u \tag{1.17}$$

It turns out we may bound the expectation of these iterations as follows.

Lemma 1.3.2. *Assume that the initial condition $X_0 = \xi$ is of finite p-moment for some $p \geq 2$. Then there exists a constant C depending only on K and T such that the sequence of processes obtained from Picard's iteration in* (1.17) *can be bounded as*

$$\mathbb{E}\left[|X_t^{(n)}|^p\right] \leq C(1 + \mathbb{E}\left[|\xi|^p\right]) \exp(C(t - s)) \quad \forall t \in \mathbb{T}, t \geq s, n \in \mathbb{N} \tag{1.18}$$

PROOF. We may compute, using triangle inequality for the p-norm,

$$\left(\mathbb{E}\left[|X_t^{(n+1)}|^p\right]\right)^{\frac{1}{p}} \leq \left(\mathbb{E}\left[|\xi|^p\right]\right)^{\frac{1}{p}} + \left(\mathbb{E}\left[\left|\int_s^t b(u, X_u^{(n)})\, du\right|^p\right]\right)^{\frac{1}{p}} + \left(\mathbb{E}\left[\left|\int_s^t \sigma(u, X_u^{(n)})\, dW_u\right|^p\right]\right)^{\frac{1}{p}}$$

From Hölder's inequality for q such that $1/p + 1/q = 1$,

$$\begin{aligned}
\left(\mathbb{E}\left[\left|\int_s^t b(u, X_u^{(n)})\, du\right|^p\right]\right)^{\frac{1}{p}} &\leq \left(\mathbb{E}\left[\left|(t-s)^{\frac{1}{q}}\left(\int_s^t |b(u, X_u^{(n)})|^p\, du\right)^{\frac{1}{p}}\right|^p\right]\right)^{\frac{1}{p}} \\
&= (t-s)^{\frac{1}{q}}\mathbb{E}\left[\int_s^t |b(u, X_u^{(n)})|^p\, du\right]^{\frac{1}{p}} \\
&\leq (t-s)^{\frac{1}{q}}\mathbb{E}\left[\int_s^t (|\kappa_u| + K|X_u^{(n)}|)^p\, du\right]^{\frac{1}{p}} \\
&\leq T^{\frac{1}{q}}\left\{\mathbb{E}\left[\int_s^t |\kappa_u|^p\, du\right]^{\frac{1}{p}} + K\mathbb{E}\left[\int_s^t |X_u^{(n)}|^p\, du\right]^{\frac{1}{p}}\right\}
\end{aligned} \tag{1.19}$$

Because we have assumed $p \geq 2$, then $p/2 \geq 1$. From Hölder's inequality for r such that $1/(p/2) + 1/r = 1$ and Ito's isometry,

$$\begin{aligned}
\left(\mathbb{E}\left[\left|\int_s^t \sigma(u, X_u^{(n)})\, dW_u\right|^p\right]\right)^{\frac{1}{p}} &\leq \left(\mathbb{E}\left[\left|(t-s)^{\frac{1}{r}}\left(\int_s^t |\sigma(u, X_u^{(n)})|^{p/2}\, dW_u\right)^{\frac{1}{p/2}}\right|^p\right]\right)^{\frac{1}{p}} \\
&= (t-s)^{\frac{1}{r}}\mathbb{E}\left[\left(\int_s^t |\sigma(u, X_u^{(n)})|^{p/2}\, dW_u\right)^2\right]^{\frac{1}{p}} \\
&= (t-s)^{\frac{1}{r}}\mathbb{E}\left[\int_s^t |\sigma(u, X_u^{(n)})|^p\, du\right]^{\frac{1}{p}} \\
&\leq T^{\frac{1}{r}}\left\{\mathbb{E}\left[\int_s^t |\kappa_u|^p\, du\right]^{\frac{1}{p}} + K\mathbb{E}\left[\int_s^t |X_u^{(n)}|^p\, du\right]^{\frac{1}{p}}\right\}
\end{aligned} \tag{1.20}$$

From Power Mean Inequality,

$$\begin{aligned}
&\left(\mathbb{E}\left[|\xi|^p\right]\right)^{\frac{1}{p}} + \mathbb{E}\left[\int_s^t |\kappa_u|^p\, du\right]^{\frac{1}{p}} + \mathbb{E}\left[\int_s^t |X_u^{(n)}|^p\, du\right]^{\frac{1}{p}} \\
\leq\ & 3^{1-\frac{1}{p}}\left(\mathbb{E}\left[|\xi|^p\right] + \mathbb{E}\left[\int_s^t |\kappa_u|^p\, du\right] + \mathbb{E}\left[\int_s^t |X_u^{(n)}|^p\, du\right]\right)^{\frac{1}{p}}
\end{aligned} \tag{1.21}$$

Then there must be a constant $C \geq \mathbb{E}\left[\int_0^T |\kappa_u|^p\, du\right]$ such that

$$\left(\mathbb{E}\left[|X_t^{(n+1)}|^p\right]\right)^{\frac{1}{p}} \leq \left\{C\mathbb{E}\left[|\xi|^p\right] + C\left(1 + \mathbb{E}\left[\int_s^t |X_u^{(n)}|^p\, du\right]\right)\right\}^{\frac{1}{p}} \tag{1.22}$$

or

$$\mathbb{E}\left[|X_t^{(n+1)}|^p\right] \leq C\mathbb{E}\left[|\xi|^p\right] + C\left(1 + \mathbb{E}\left[\int_s^t |X_{u_1}^{(n)}|^p \, du_1\right]\right) \tag{1.23}$$

We now have an iterable inequality. Iterating this inequality backward to $\mathbb{E}\left[|X_s^{(0)}|^p\right] = \mathbb{E}\left[|\xi|^p\right]$, we may obtain

$$\begin{aligned}\mathbb{E}\left[|X_t^{(n+1)}|^p\right] &\leq (C\mathbb{E}\left[|\xi|^p\right] + C)\left(\sum_{j=0}^n \frac{C^j(t-s)^j}{j!}\right) + C^{n+1}\int_s^t \int_u^{s_1} \dots \int_u^{s_n} \mathbb{E}\left[|\xi|^p\right] du_{n+1}\dots du_2 du_1 \\ &\leq (C\mathbb{E}\left[|\xi|^p\right] + C)\exp(C(t-s)) \\ &= C(1 + \mathbb{E}\left[|\xi|^p\right])\exp(C(t-s))\end{aligned}$$

$\boxed{\text{Q.E.D}}$

Now we may begin proving the solution existence.

Let us follow a Picard's iteration as we have described before.

Now we may denote

$$X_t^{(n+1)} - X_t^{(n)} = D_t^{(n)} + M_t^{(n)} \tag{1.24}$$

where

$$\begin{aligned} D_t^{(n)} &= \int_s^t b(u, X_u^{(n)}) - b(u, X_u^{(n-1)}) \, du \\ M_t^{(n)} &= \int_s^t \sigma(u, X_u^{(n)}) - \sigma(u, X_u^{(n-1)}) \, dW_u \end{aligned}$$

From Power Mean inequality,

$$|X_u^{(n+1)} - X_u^{(n)}|^p \leq 2^{p-1}(D_u^{(n)p} + M_u^{(n)p}) \tag{1.25}$$

Observe that using the bound on lemma 1.3.2, we can obtain

$$\begin{aligned}\mathbb{E}\left[\int_s^T |\sigma(u, X_u^{(n)}) - \sigma(u, X_u^{(n-1)})|^p \, du\right] &= \int_s^T \mathbb{E}\left[|\sigma(u, X_u^{(n)}) - \sigma(u, X_u^{(n-1)})|^p\right] du \\ &\leq \int_s^T K\mathbb{E}\left[|X_u^{(n)} - X_u^{(n-1)}|^p\right] ds \\ &\leq \int_s^T 2^p K C_2(1 + \mathbb{E}\left[|\xi|^p\right])\exp(C_2(u-s)) \, du \\ &< \infty\end{aligned}$$

Then $\sigma(s, X_s^{(n)}) - \sigma(s, X_s^{(n-1)})$ is an $\mathcal{H}^2$ process, implying that $(M_s^{(n)})_{0\leq s\leq T}$ is a martingale. Thus we may use Doob's L^p inequality and Itô isometry consecutively as follows.

$$\begin{aligned}\mathbb{E}\left[\sup_{s\leq t\leq T} |M_s^{(n)}|^p\right] &\leq C_p\mathbb{E}\left[|M_T^{(n)}|^p\right] \\ &= C_p\int_s^t \mathbb{E}\left[\left|\sigma(u, X_u^{(n)}) - \sigma(u, X_u^{(n-1)})\right|^p\right] du \\ &\leq C_p K^p \int_s^t \mathbb{E}\left[|X_u^{(n)} - X_u^{(n-1)}|^p\right] du \end{aligned} \tag{1.26}$$

Using Hölder's inequality, we may obtain

$$\begin{aligned}
|D_t^{(n)}|^p &\leq K^p (t-s)^{\frac{p}{q}} \int_s^t \left| X_u^{(n)} - X_u^{(n-1)} \right|^2 du \\
\Rightarrow \sup_{s \leq t \leq T} |D_t^{(n)}|^p &\leq K^p T^{\frac{p}{q}} \int_s^T \left| X_u^{(n)} - X_u^{(n-1)} \right|^p du
\end{aligned} \tag{1.27}$$

Using equation (1.24), power-mean inequality and the inequalities (1.25), (1.26), (1.27),

$$\begin{aligned}
\mathbb{E}\left[\sup_{s \leq t \leq T} \left| X_s^{(n+1)} - X_s^{(n)} \right|^p \right] &= \mathbb{E}\left[\sup_{s \leq t \leq T} \left| M_t^{(n)} + D_t^{(n)} \right|^p \right] \\
&\leq 2^{p-1} \mathbb{E}\left[\sup_{s \leq t \leq T} \left\{ \left| M_t^{(n)} \right|^p + \left| D_t^{(n)} \right|^p \right\} \right] \\
&\leq 2^{p-1} (C_p K^p + K^p T^{\frac{p}{q}}) \int_s^T \mathbb{E}\left[\left| X_u^{(n)} - X_u^{(n-1)} \right|^p \right] du \\
&\leq L_{p,T} \int_s^T \mathbb{E}\left[\sup_{s \leq u \leq T} \left| X_u^{(n)} - X_u^{(n-1)} \right|^p \right] du
\end{aligned} \tag{1.28}$$

Iterating this inequality (denoting $L_{p,t}$ as L), we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s \leq t \leq T} \left| X_s^{(n+1)} - X_s^{(n)} \right|^p \right] &\leq L \int_s^T \frac{L^{n-1}(u-s)^{n-1}}{(n-1)!} \mathbb{E}\left[\left| X_u^{(1)} - X_u^{(0)} \right|^p \right] du \\
&\leq L \int_s^T \frac{L^{n-1}(u-s)^{n-1}}{(n-1)!} \sup_{s \leq u \leq T} \mathbb{E}\left[\left| X_u^{(1)} - X_u^{(0)} \right|^p \right] du \\
&= \frac{L^n (T-s)^n}{n!} \sup_{s \leq u \leq T} \mathbb{E}\left[\left| X_u^{(1)} - X_u^{(0)} \right|^p \right] \\
&\leq C^* \frac{L^n T^n}{n!} \leq \infty
\end{aligned} \tag{1.29}$$

where $C^* = \sup_{s \leq u \leq T} \mathbb{E}\left[\left| X_u^{(1)} - X_u^{(0)} \right|^p \right] < \infty$ because of lemma 1.3.2.

Using Markov's inequality,

$$\mathbb{P}\left(\sup_{s \leq u \leq T} \left| X_u^{(n+1)} - X_u^{(n)} \right| > \frac{1}{2^{n+1}} \right) \leq 2^p C^* \frac{2^{np} L^n T^n}{n!} \tag{1.30}$$

If we sum the right-hand part of inequality (1.30) across all n, the summation is absolutely convergent. Thus, using Borel-Cantelli lemma, for each ω in a set $\Omega^* \subseteq \Omega$ of probability 1, there exists $N(\omega) \in \mathbb{N}$ such that

$$\begin{aligned}
\sup_{s \leq u \leq T} \left| X_u^{(n+1)}(\omega) - X_u^{(n)}(\omega) \right| &\leq \frac{1}{2^{n+1}} \quad \forall n \geq N(\omega) \\
\Rightarrow \sup_{s \leq u \leq T} \left| X_u^{(n+k)}(\omega) - X_u^{(n)}(\omega) \right| &\leq \frac{1}{2^n} \quad \forall n \geq N(\omega)
\end{aligned} \tag{1.31}$$

Then $X_u^{(n)}$ converges uniformly for each $u \in [s,T]$ almost surely. Denote the limit as X_u. Because of uniform convergence and the almost-sure continuity of each $X_u^{(n)}$, it should follow that X_u is also continuous almost-surely.

From Jensen's inequality, Fatou's lemma, and lemma 1.3.2

$$\begin{aligned}\mathbb{E}\left[|X_u|^2\right] &\leq \mathbb{E}\left[|X_u|^p\right]^{\frac{2}{p}} \\ &\leq \lim_{n\to\infty} \mathbb{E}\left[|X_u^{(n)}|^2\right]^{\frac{2}{p}} \\ &\leq \left[C(1+\mathbb{E}\left[|\xi|^2\right])\exp(C(u-s))\right]^{\frac{2}{p}} \end{aligned} \tag{1.32}$$

Then the process $(X_u)_{s\leq u\leq T}$ is also an $\mathcal{H}^2$ process. Thus, by Jensen's inequality and Itô isometry

$$\begin{aligned}\mathbb{E}\left[\left|\int_s^t \sigma(u,X_u^{(n)})-\sigma(u,X_u)\,dW_u\right|^p\right] &\leq K^p\mathbb{E}\left[\left|\int_s^t |X_u-X_u^{(n)}|\,dW_u\right|^p\right] \\ &\leq K^p\mathbb{E}\left[\left|\int_s^t |X_u-X_u^{(n)}|\,dW_u\right|^2\right]^{\frac{2}{p}} \\ &= K^p\left\{\int_s^t \mathbb{E}\left[|X_u-X_u^{(n)}|^2\right]\,du\right\}^{\frac{2}{p}} \end{aligned} \tag{1.33}$$

Because $X_u^{(n)}$ converges uniformly to X_u on $[s,t]$ almost surely and the bound on $\mathbb{E}\left[|X_u^{(n)}|^2\right]$ and $\mathbb{E}\left[|X_u|^2\right]$ in lemma 1.3.2 and inequality (1.32), by dominated convergence theorem, the right-hand side of (1.33) converges to 0.

Thus,

$$\int_s^t \sigma(u,X_u^{(n)})\,dW_u \xrightarrow{L^p(\Omega)} \int_s^t \sigma(u,X_u)\,dW_u \tag{1.34}$$

Similarly,

$$\int_s^t b(u,X_u^{(n)})\,du \xrightarrow{L^p(\Omega)} \int_s^t b(u,X_u)\,du \tag{1.35}$$

Because of this $L^p(\Omega)$ convergence, there is a subsequence $(X_t^{(n_k)})_{k=1}^{\infty}$ such that the convergence in (1.34) and (1.35) becomes almost-sure convergence. But because $X_t^{(n)}$ converges almost surely to X_t, it should follow that

$$X_t = X_s + \int_s^t b(u,X_u)\,du + \int_s^t \sigma(u,X_u)\,dW_u \quad a.s. \tag{1.36}$$

Because $\mathbb{Q}$ is countable, there exists a set $\Omega_1 \in \mathcal{F}$ of probability 1 such that

$$X_t = X_s + \int_s^t b(u,X_u)\,du + \int_s^t \sigma(u,X_u)\,dW_u \quad \forall t\in[s,T]\cap\mathbb{Q}\,a.s. \tag{1.37}$$

Because integration (ordinary Lebesgue or Itô) is continuous, equation (1.37) should also be satisfied for all $t \in [s, T]$ almost surely.

We may summarise the above results as follows:

Theorem 1.3.3 (Existence)**.** *Let b and σ satisfy 1.3.2. Let us fix a Brownian motion $(W_t)_{s \le t \le T}$. Then there is a unique solution $(X_t)_{s \le t \le T}$ to the SDE* (1.12) *with initial condition $X_s = \xi$ of finite p-moment. Furthermore, X_t has finite p-moment for any $t \in [s, T]$ and that there exists a constant C, depending only on K (the Lipschitz constant) and T such that*

$$\mathbb{E}\left[\sup_{s \le u \le T} |X_u|^p\right] \le C(1 + \mathbb{E}\left[|\xi|^p\right]) \exp(C(T - s)) \quad \forall t \in [s, T] \tag{1.38}$$

Chapter 2

Stochastic Optimization Problem

2.1 Example: Portfolio Allocation

Let S^0 and S be the price process of bonds/saving accounts and stock/risky assets, respectively. Let β and α be the shares process of bonds/saving accounts and stock/risky assets, respectively. Then the total wealth accumulated by the portfolio at time t is

$$X_t = \beta_t S_t^0 + \alpha_t S_t \tag{2.1}$$

By self-financing principle, there are no cash flow in or out of the system, so the changes in wealth are due to the changes in bond or stock prices.

$$dX_t = \beta_t \, dS_t^0 + \alpha_t \, dS_t \tag{2.2}$$

By substituting $\beta_t = (X_t - \alpha_t S_t)/S_t^0$,

$$dX_t = \frac{(X_t - \alpha_t S_t)}{S_t^0} \, dS_t^0 + \alpha_t \, dS_t \tag{2.3}$$

Here in stochastic optimization problem:

1. $X_t(\omega)$ is the state of the system.
2. α is the control; we can "control" α to optimize X as long as α is NOT forward-looking (i.e. it can only look in the past, in other words, adapted).

What is the performance criterion for this problem?

Let $U(x)$ be the utilization function of how much "satisfied" (or "happy") the portfolio's owner in holding x amount of wealth. Obviously, U must be nondecreasing (larger wealth don't decrease "satisfaction"). Furthermore, U is generally concave (adding more wealth do not increase "satisfaction" as much if you already own a huge amount of wealth).

On a finite time-horizon $[0, T]$, one can define a performance criterion as below.

$$\sup_{\alpha} \mathbb{E}\left[X_T\right] \tag{2.4}$$

Chapter 3

Dynamic Programming for Solving Stochastic Control Problems

3.1 Controlled SDE

Consider an SDE with control

$$dX_t = b(X_t, \alpha_t)\,dt + \sigma(X_t, \alpha_t)\,dW_t \tag{3.1}$$

Assume X, W, b, σ are respectively $\mathbb{R}^n$, $\mathbb{R}^d$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times d}$-valued, and the control α must be progressively-measurable and take values in $A \subseteq \mathbb{R}^m$. Furthermore, assume that the drift and diffusion terms are globally Lipschitz w.r.t. the first entry (excluding the control input).

3.1.1 Finite Horizon Time

First, let us fix a finite horizon time $\mathbb{T} = [0, T]$ with $T < \infty$. Denote

$$\mathcal{A} = \left\{\alpha = (\alpha_t)_{t\in\mathbb{T}} : \alpha \text{ is prog. measr. } \& \ \mathbb{E}\left[\int_0^T |b(0,\alpha_s)|^2 + |\sigma(0,\alpha_s)|^2\,ds\right] < \infty\right\} \tag{3.2}$$

The Lipschitz condition ensures that b and σ have linear growth which is integrable

$$|b(X_u, \alpha_u)| \leq K|X_u| + |b(0, \alpha_u)| \tag{3.3}$$
$$|\sigma(X_u, \alpha_u)| \leq K|X_u| + |\sigma(0, \alpha_u)| \tag{3.4}$$

because $\mathbb{E}\left[\int_0^T |b(0,\alpha_u)|^2 + |\sigma(0,\alpha_u)|^2\,du\right] < \infty$.

Due to this, on our Picard construction with $p = 2$, we may bound the 2-moments estimate to the solution of (3.1) as

$$\mathbb{E}\left[\sup_{t\leq s\leq T} |X_s^{t,x}|^2\right] < \infty \tag{3.5}$$

$$\lim_{h\downarrow 0} \mathbb{E}\left[\sup_{t\leq s\leq t+h} |X_s^{t,x} - x|^2\right] = 0 \tag{3.6}$$

Assuming a control α, we define the (expected) gain function beginning from $X_t = x$ to the terminal time T as

$$J(t, x, \alpha) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)\, ds + g(X_T^{t,x})\right] \tag{3.7}$$

Here, the integral inside the expectation can be interpreted as the continuous gain obtained while g is the terminal gain. Note that we assume $\alpha \in \mathcal{A}(t, x)$, that is the controls in $\mathcal{A}$ such that the expectation of the integral part is well-defined.

$$\mathbb{E}\left[\int_t^T |f(s, X_s \alpha_s)|\, ds\right] < \infty \tag{3.8}$$

We also assume that the terminal function g satisfies the (**Hg**)condition, that is

Assumption 3.1.1 ((**Hg**))**.**

1. g is lower-bounded i.e. $\exists\, c_0 > 0$ s.t. $|g(x)| \geq c_0\, \forall x \in \mathbb{R}^n$
2. g has quadratic growth i.e. $\exists\, C$ s.t. $|g(x)| \leq C(1 + |x|^2)\, \forall x \in \mathbb{R}^n$

We wish to maximise this expected gain, and as such, we define the associated value function

$$v(t, x) = \sup_{\alpha \in \mathcal{A}(t,x)} J(t, x, \alpha) \tag{3.9}$$

We say that

1. A control $\alpha \in \mathcal{A}(t, x)$ is optimal iff $v(t, x) = J(t, x, \alpha)$
2. A control α is Markovian iff $\alpha_s = a(s, X_s^{t,x})$ for some measurable function a i.e. α only depends on the present state.

3.1.2 Infinite Horizon Time

Now let us fix $\mathbb{T} = [0, \infty]$. We may define the gain and value function similarly as in the finite-time version but with no terminal gain term (because of the infinite time). Another difference is because of the infinite time horizon, we modify the sets of admissible controls as

$$\mathcal{A}_0 = \left\{\alpha = (\alpha_t)_{t \in \mathbb{T}} : \alpha \text{ is prog. measr. } \& \ \mathbb{E}\left[\int_0^T |b(0, \alpha_s)|^2 + |\sigma(0, \alpha_s)|^2\, ds\right] < \infty\, \forall 0 < T < \infty\right\} \tag{3.10}$$

The gain function is then defined as

$$J(t, x, \alpha) = \mathbb{E}\left[\int_0^\infty e^{-\beta s} f(X_s, \alpha_s)\, ds\right] \tag{3.11}$$

with f no longer depends on time (instead, we introduce an additional discount factor $e^{-\beta s}$). For the expectation (and the integral inside the expectation) above to be well-defined, we further restrict the admissible controls to $\mathcal{A}(x) \subseteq \mathcal{A}_0$ containing controls α satisfying

$$\mathbb{E}\left[\int_0^\infty e^{-\beta s} |f(X_s, \alpha_s)|\, ds\right] < \infty \tag{3.12}$$

with the assumption that $\mathcal{A}(x)$ is nonempty for any x.

The value function is

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(x)} J(t,x,\alpha) \tag{3.13}$$

3.2 Dynamic Programming Principle (DPP)

Theorem 3.2.1 (Dynamic Programming Principle: Finite Horizon Time)**.** *Let* $(t,x) \in [0,T] \times \mathbb{R}^n$ *and* $\mathcal{T}_{t,T}$ *the set of stopping times valued in* $[t,T]$*. Then*

$$\begin{aligned} v(t,x) &= \sup_{\alpha \in \mathcal{A}(t,x)} \sup_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\, ds + v(\theta, X_\theta^{t,x})\right] \\ &= \sup_{\alpha \in \mathcal{A}(t,x)} \inf_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\, ds + v(\theta, X_\theta^{t,x})\right] \end{aligned} \tag{3.14}$$

Interpretation: ?

The theorem above is a stronger version of the usual DPP, which is

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\, ds + v(\theta, X_\theta^{t,x})\right] \quad \forall \theta \in \mathcal{T}_{t,T} \tag{3.15}$$

(this explains why the sup sup expression is the same as the sup inf one)

The interpretation of the (usual) DPP is that the optimal value function (starting from x at time t) is the same as if we optimize for the value function starting from $X_\theta^{t,x}$ at time θ to T first, then optimize along the remaining previous time horizon $[t,\theta]$.

PROOF. Let $\omega \in \Omega$ such that the initial condition $X_t(\omega) = x$ is satisfied. By the uniqueness of the path, for every $t \le s \le \theta(\omega)$,

$$\begin{aligned} X_s^{t,x}(\omega) &= X_s(\omega) \\ &= X_s^{\theta(\omega), X_{\theta(\omega)}^{t,x}}(\omega) \end{aligned} \tag{3.16}$$

Then we may calculate

$$
\begin{aligned}
J(t,x,\alpha) &= \mathbb{E}\left[\int_t^T f(s,X_s^{t,x},\alpha_s)\,ds + g(X_T^{t,x})\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_t^T f(s,X_s^{t,x},\alpha_s)\,ds + g(X_T^{t,x})\right]\Big|\mathcal{F}_\theta\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\,ds + \int_\theta^T f(s,X_s^{t,x},\alpha_s)\,ds + g(X_T^{t,x})\right]\Big|\mathcal{F}_\theta\right] \\
&= \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\,ds + \mathbb{E}\left[\int_\theta^T f(s,X_s^{\theta,X_\theta^{t,x}},\alpha_s)\,ds + g(X_T^{\theta,X_\theta^{t,x}})\Big|\mathcal{F}_\theta\right]\right] \\
&= \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\,ds + J(\theta,X_\theta^{t,x},\alpha)\right] \\
&\leq \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\,ds + v(\theta,X_\theta^{t,x})\right]
\end{aligned}
\tag{3.17}
$$

Taking the infimum first w.r.t $\mathcal{T}_{t,T}$ and the supremum w.r.t $\mathcal{A}(t,x)$, we obtain

$$
v(t,x) \leq \sup_{\alpha\in\mathcal{A}(t,x)} \inf_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\,ds + v(\theta,X_\theta^{t,x})\right] \tag{3.18}
$$

If we can prove

$$
v(t,x) \geq \sup_{\alpha\in\mathcal{A}(t,x)} \sup_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\,ds + v(\theta,X_\theta^{t,x})\right] \tag{3.19}
$$

then we are finished. Fix any arbitrary $\alpha\in\mathcal{A}(t,x)$ and $\theta\in\mathcal{T}_{t,T}$. From the definition of value function, for any $\varepsilon>0$ and $\omega\in\Omega$, there exists a control $\alpha^{\varepsilon,\omega}\in\mathcal{A}(\theta(\omega),X_{\theta(\omega)}^{t,x}(\omega))$ s.t.

$$
v(\theta(\omega),X_{\theta(\omega)}^{t,x}(\omega)) - \varepsilon \leq J(\theta(\omega),X_{\theta(\omega)}^{t,x}(\omega),\alpha^{\varepsilon,\omega}) \tag{3.20}
$$

Define a process $\hat{\alpha}$ as

$$
\hat{\alpha}_s(\omega) = \begin{cases} \alpha_s(\omega), & s\in[t,\theta(\omega)] \\ \alpha_s^{\varepsilon,\omega}(\omega), & s\in[\theta(\omega),T] \end{cases} \tag{3.21}
$$

It can be shown that $\hat{\alpha}$ is prog. measr. (**prove it!**). Because α and $\alpha^{\varepsilon,\omega}$ are in $\mathcal{A}(t,x)$ and $\mathcal{A}(\theta(\omega),X_\theta^{t,x})$ respectively, it can be shown that $\hat{\alpha}\in\mathcal{A}(t,x)$. Thus,

$$
\begin{aligned}
v(t,x) &\geq J(t,x,\hat{\alpha}) \\
&= \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\hat{\alpha}_s)\,ds + J(\theta,X_\theta^{t,x},\hat{\alpha})\right] \\
&= \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\,ds + J(\theta,X_\theta^{t,x},\alpha^{\varepsilon,\omega})\right] \\
&\geq \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\,ds + v(\theta,X_\theta^{t,x})\right] - \varepsilon
\end{aligned}
\tag{3.22}
$$

Because α, θ, and ε are arbitrary, then

$$
v(t,x) \geq \sup_{\alpha\in\mathcal{A}(t,x)} \sup_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\,ds + v(\theta,X_\theta^{t,x})\right] \tag{3.23}
$$

concluding our proof. $\boxed{\text{Q.E.D}}$

3.3 Hamilton-Jacobi-Bellman Equation

This is an "infinitesimal" version of DPP; that is, what happens when we send $\theta \downarrow t$.

Let us consider $X_s^{t,x}$ be the solution to 3.1 with constant control $\alpha = a$. From DPP, for any stopping times $\tau \in \mathcal{T}_{t,T}$,

$$v(t,x) \geq \mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x}, a)\, ds + v(\tau, X_\tau^{t,x})\right] \tag{3.24}$$

Assuming v is smooth enough (have (locally)-bounded second derivatives?), we may apply Ito's lemma to v.

$$\begin{aligned} v(t+h, X_{t+h}^{t,x}) = v(t,x) &+ \int_t^{t+h} \frac{\partial v}{\partial t}(s, X_s^{t,x})\, ds + \int_t^{t+h} \frac{\partial v}{\partial x}(s, X_s^{t,x})^T\, dX_s^{t,x} \\ &+ \frac{1}{2}\mathrm{tr}\left(\int_t^{t+h} \frac{\partial^2 v}{\partial x^2}(s, X_s^{t,x})\, d\left\langle X_s^{t,x}\right\rangle\right) \end{aligned} \tag{3.25}$$

From 3.1, we have $d\left\langle X_s^{t,x}\right\rangle = \sigma\sigma^T\, ds$, so we have

$$v(t+h, X_{t+h}^{t,x}) = v(t,x) + \int_t^{t+h} \left[\frac{\partial v}{\partial t}(s, X_s^{t,x}) + \mathcal{L}^a v\right] ds + \int_t^{t+h} \frac{\partial v}{\partial x}(s, X_s^{t,x})^T \sigma(X_s^{t,x}, a)\, dW_s \tag{3.26}$$

where

$$\mathcal{L}^a v = (D_x v)^T b(x,a) + \frac{1}{2}\mathrm{tr}(\sigma(x,a)\sigma(x,a)^T D_x^2 v) \tag{3.27}$$

If the last integrand in 3.26 is globally-bounded, then it is a martingale and thus having expectation 0. If it is only locally-bounded (e.g. continuous), we have to try bounding our X process.

<u>How to "bound" X</u>

Take stopping time τ_h as the first time the diffusion X exits the ball

$$B_h = \{(s,y) : |s-t| < h \text{ and } |y-x| < \varepsilon\}$$

i.e. $\tau_h = T \wedge \inf\{u > t : (u, X_u^{t,x}) \notin B_h\}$. It is clear from the continuity of X that for all $\omega \in \Omega$ a.s., there exists a $h(\omega)$ such that $\tau_h(\omega) = t + h \quad \forall\, 0 < h < h(\omega)$.

Taking the expectation to 3.26 stopped by τ_h, we have

$$\mathbb{E}\left[v(\tau_h, X_{\tau_h}^{t,x})\right] = v(t,x) + \mathbb{E}\left[\int_t^{\tau_h} \left[\frac{\partial v}{\partial t}(s, X_s^{t,x}) + \mathcal{L}^a v\right] ds\right] \tag{3.28}$$

We have ensured that the expectation of the stochastic integral is 0 due to our stopping times forcing a bound on X, thus forcing a bound on the dW integrands, making it a martingale with expectation 0.

End of how to "bound" X

Substituting into 3.24 with $\tau := \tau_h$, we obtain

$$0 \geq \mathbb{E}\left[\int_t^{\tau_h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^{t,x}) + f(s, X_s^{t,x}, a)\, ds\right] \tag{3.29}$$

Dividing by h on both sides,

$$0 \geq \mathbb{E}\left[\frac{1}{h}\int_t^{\tau_h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^{t,x}) + f(s, X_s^{t,x}, a)\, ds\right] \tag{3.30}$$

Remember that for all $\omega \in \Omega$ a.s., there exists a $h(\omega)$ such that $\tau_h(\omega) = t+h \quad \forall\, 0 < h < h(\omega)$. Because of this, from Mean-Value Theorem and the continuity of the integrand,

$$\lim_{h\downarrow 0} \frac{1}{h}\int_t^{\tau_h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^{t,x}) + f(s, X_s^{t,x}, a)\, ds = \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(t,x) + f(t,x,a) \quad a.s. \tag{3.31}$$

On the event that $\tau_h = t + h$, by our definition of τ_h and Mean Value Theorem,

$$\begin{aligned}
\left|\frac{1}{h}\int_t^{\tau_h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^{t,x}) + f(s, X_s^{t,x}, a)\, ds\right| &= \left|\left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s^*, X_{s^*}^{t,x}) + f(s^*, X_{s^*}^{t,x}, a)\right| \\
&\quad \exists s^* \in [t, t+h] \\
&\leq \sup_{(u,y)\in\overline{B_h}} \left|\left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(u,y) + f(u,y,a)\right| \\
&= \text{ a finite deterministic constant}
\end{aligned} \tag{3.32}$$

On the other hand, if $\tau < t+h$, we can rewrite $\frac{1}{h}$ as $\frac{\tau_h}{h} \cdot \frac{1}{\tau_h}$, and with the fact that $0 < \frac{\tau_h}{h} < 1$, we may also bound the expression as in (3.32). Due to this, we may apply the Dominated Convergence Theorem to imply

$$\begin{aligned}
0 &\geq \lim_{h\downarrow 0} \mathbb{E}\left[\frac{1}{h}\int_t^{\tau_h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^{t,x}) + f(s, X_s^{t,x}, a)\, ds\right] \\
&= \mathbb{E}\left[\lim_{h\downarrow 0} \frac{1}{h}\int_t^{\tau_h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^{t,x}) + f(s, X_s^{t,x}, a)\, ds\right] \\
&= \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(t,x) + f(t,x,a)
\end{aligned} \tag{3.33}$$

Taking the supremum on $a \in A$ (the permissible set of realized values for the control α), we have

$$\sup_{a\in A} \left[\mathcal{L}^a v(t,x) + f(t,x,a)\right] \leq -\frac{\partial v}{\partial t}(t,x) \tag{3.34}$$

However, if the controlled system has an optimal control (say, α^*) with some nice properties (e.g. continuous almost-surely), then

$$v(t,x) = \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{*t,x}, \alpha_s^*)\, ds + v(t+h, X_{t+h}^{*t,x})\right] \tag{3.35}$$

and we can repeat our arguments above to obtain

$$\mathcal{L}^{\alpha_t^*} v(t,x) - f(t,x,\alpha_t^*) = -\frac{\partial v}{\partial t}(t,x) \tag{3.36}$$

Combining (3.34) and (3.36), we conclude

$$\sup_{a\in A}\left[\mathcal{L}^a v(t,x) + f(t,x,a)\right] = -\frac{\partial v}{\partial t}(t,x) \tag{3.37}$$

or

$$-\frac{\partial v}{\partial t}(t,x) - \sup_{a\in A}\left[\mathcal{L}^a v(t,x) + f(t,x,a)\right] = 0 \tag{3.38}$$

with terminal condition $v(T,x) = g(x)$.

Infinite-time Horizon

Following the same steps as before, we may obtain

$$\beta v(x) - \sup_{a\in A}\left[\mathcal{L}^a v(x) + f(x,a)\right] = 0 \tag{3.39}$$

3.4 Verification Theorem

We have shown that given a (sufficiently-nice) value function v with an optimal control, then it must satisfy the HJB equation. But how about the other way around, if a function satisfies the HJB? Does this function equals the value function? How about its optimal control?

The answer is yes (assuming some nice properties). This result is called the Verification Theorem.

Theorem 3.4.1 (Verification Theorem)**.** *Let w be a function in $C^{1,2}([0,T)\times\mathbb{R}^n)\cap C^0([0,T]\times\mathbb{R}^n)$ (the function is continuous but its derivatives may "explode" at T). Assume w has a quadratic growth rate, that is there exists a constant C such that*

$$|w(t,x)| \le C(1+|x|^2) \quad \forall (t,x) \in [0,T]\times\mathbb{R}^n \tag{3.40}$$

1. *If w satisfies*

$$-\frac{\partial w}{\partial t}(t,x) - \sup_{a\in A}\left[\mathcal{L}^a w(t,x) + f(t,x,a)\right] \ge 0 \quad \forall (t,x)\in[0,T)\times\mathbb{R}^n \tag{3.41}$$

$$w(T,x) \ge g(x) \quad \forall x\in\mathbb{R}^n \tag{3.42}$$

   then w dominates the value function v i.e. $w(t,x) \ge v(t,x)$ for $(t,x)\in[0,T]\times\mathbb{R}^n$.

2. *Suppose further that the inequality above becomes equality and there exists a measurable function* $\hat{\alpha} : [0,T) \times \mathbb{R}^n \to A$ *such that*

$$-\frac{\partial w}{\partial t}(t,x) - \left[\mathcal{L}^{\hat{\alpha}(t,x)} w(t,x) + f(t,x,\hat{\alpha}(t,x))\right] = 0 \tag{3.43}$$

   Assume further that the controlled SDE in (3.1) *with control* $\alpha_s = \hat{\alpha}(s, X_s^{t,x})$ *has a unique solution* $\hat{X}^{t,x}$ *and the process* $\hat{\alpha}(., X_.^{t,x})$ *lies in* $\mathcal{A}(t,x)$ *for any* $t \geq T$.

   Then $w = v$ *on* $[0,T] \times \mathbb{R}^n$ *and* $\hat{\alpha}(., X_.^{t,x})$ *is an optimal Markovian control.*

PROOF.

1. Let $(t,x) \in [0,T) \times \mathbb{R}^n$, $\alpha \in \mathcal{A}(t,x)$, and $s \in [t,T)$. Applying Ito's formula to w, we have

$$\begin{aligned} w(s, X_s^{t,x}) = w(t,x) &+ \int_0^s \left(D_t w + \mathcal{L}^{\alpha_u} w\right)(u, X_u^{t,x})\, du \\ &+ \int_t^s D_x w(u, X_u^{t,x})^T \sigma(X_u^{t,x}, \alpha_u)\, dW_u \end{aligned} \tag{3.44}$$

   We wish to take the expectation of , but the last (stochastic) integral may not be a true martingale (only a local martingale). To work around this, we introduce a localizing sequence (τ_n) with

$$\tau_n = \inf\left\{ s \geq t : \int_t^s \left| D_x w(u, X_u^{t,x})^T \sigma(X_u^{t,x}, \alpha_u)\right|^2 du \geq n \right\} \tag{3.45}$$

   Note that $\tau_n \uparrow \infty$. The sequence is obviously increasing, and if it is bounded, then the integral is blowing up at a certain point, an impossibility because the integrand is continuous.

   By our construction of the localizing sequence, the stochastic integral is a martingale if stopped by our stopping times, so

$$\mathbb{E}\left[w(s \wedge \tau_n, X_{s\wedge\tau_n}^{t,x})\right] = w(t,x) + \mathbb{E}\left[\int_t^{s\wedge\tau_n} \left(D_t w + \mathcal{L}^{\alpha_u} w\right)(u, X_u^{t,x})\, du\right] \tag{3.46}$$

   From (3.41)

$$\frac{\partial w}{\partial t}(u, X_u^{t,x}) + \mathcal{L}^{\alpha_u} w(u, X_u^{t,x}) + f(u, X_u^{t,x}, \alpha_u) \leq 0 \tag{3.47}$$

   Substituting (3.47) to (3.46), we obtain

$$\mathbb{E}\left[w(s \wedge \tau_n, X_{s\wedge\tau_n}^{t,x})\right] \leq w(t,x) - \mathbb{E}\left[\int_t^{s\wedge\tau_n} f(u, X_u^{t,x}, \alpha_u)\, du\right] \tag{3.48}$$

   It is easy to see that

$$\left|\int_t^{s\wedge\tau_n} f(u, X_u^{t,x}, \alpha_u)\, du\right| \leq \int_t^T |f(u, X_u^{t,x}, \alpha_u)|\, du \tag{3.49}$$

and the right-hand side has a finite expectation by the definition of $\mathcal{A}(t,x)$, so that we may apply DCT (Dominated Convergence Theorem) to obtain

$$\mathbb{E}\left[\int_t^{s\wedge\tau_n} f(u, X_u^{t,x}, \alpha_u)\,du\right] \to \mathbb{E}\left[\int_t^{s} f(u, X_u^{t,x}, \alpha_u)\,du\right] \tag{3.50}$$

Also note that by our condition on w,

$$\begin{aligned}\left|w(s\wedge\tau_n, X_{s\wedge\tau_n}^{t,x})\right| &\le C(1+\left|X_{s\wedge\tau_n}^{t,x}\right|^2)\\ &\le C(1+\sup_{s\in[t,T]}\left|X_s^{t,x}\right|^2)\end{aligned} \tag{3.51}$$

Recall that because we have initial condition $X_t = x$ is a constant (which obviously has a finite 2-moment). Also recall that our drift b and diffusion σ are globally-Lipschitz on the first term and have linear growth. Due to this, $\sup_{s\in[t,T]}\left|X_s^{t,x}\right|^2$ has a finite 1-moment, so we may also apply DCT to get

$$\mathbb{E}\left[w(s\wedge\tau_n, X_{s\wedge\tau_n}^{t,x})\right] \to \mathbb{E}\left[w(s, X_s^{t,x})\right] \tag{3.52}$$

,

which implies

$$\mathbb{E}\left[w(s, X_s^{t,x})\right] \le w(t,x) - \mathbb{E}\left[\int_t^{s} f(u, X_u^{t,x}, \alpha_u)\,du\right] \tag{3.53}$$

By the same argument as in (3.51), $w(s, X_s^{t,x})$ has an integrable dominant which does not depend on s, so that we may send $s \to T$, and by the continuity of w and (3.42),

$$\begin{aligned}\mathbb{E}\left[g(X_T^{t,x})\right] &\le \mathbb{E}\left[w(T, X_T^{t,x})\right]\\ &\le w(t,x) - \mathbb{E}\left[\int_t^{T} f(u, X_u^{t,x}, \alpha_u)\,du\right]\\ \Rightarrow w(t,x) &\ge J(t,x,\alpha)\end{aligned} \tag{3.54}$$

Because α is arbitrary (in $\mathcal{A}(t,x)$), we conclude that $w \ge v$.

2. By our assumption that the process $\left(\hat{\alpha}(s, X_s^{t,x})\right)$ lies in $\mathcal{A}(t,x)$, we may repeat the arguments above to obtain similar result (this time with equality).

$$w(t,x) = J(t,x,(\hat{\alpha}(\cdot, X_\cdot^{t,x}))) \le v(t,x) \tag{3.55}$$

Because $w \ge v$ and $w \le v$, we must have $w = v$, also concluding that $\left(\hat{\alpha}(s, X_s^{t,x})\right)$ is an optimal Markovian control.

$\boxed{\text{Q.E.D}}$

Chapter 4

Optimal Execution with Continuous Trading

Let's say that you are a trader with a task to execute an order of $\mathfrak{N}$ stock shares. You may trade them all at once, but a sudden, large transaction may swing the prices to your disadvantage. However, if you trade them gradually over a period of time, you may be exposed to uncertainties in the price movement. Because of this, you have to optimise how fast (or slow) you trade, and the theory of stochastic control we've touched before comes in handy.

4.1 The Model

In this model we consider a finite-time horizon. Denote

1. $\nu = (\nu_t)_{t\in\mathbb{T}}$: the rate at which the agent is trading (either liquidating or acquiring) shares. This is the agent's control.
2. $S^\nu = (S^\nu_t)_{t\in\mathbb{T}}$: the midprice (the average between the best ask and bid price) of shares. This may depend on the agent's trading rate.
3. $Q^\nu = (Q^\nu_t)_{t\in\mathbb{T}}$: the number of shares held by the agent; the inventory process. This depends on the trading rate process ν.
4. $\hat{S}^\nu = (\hat{S}^\nu_t)_{t\in\mathbb{T}}$: the execution price process. This depends on the midprice and the agent's trading rate.
5. $X^\nu = (X^\nu_t)_{t\in\mathbb{T}}$: the agent's cash process. This depends on the trading rate ν.

We formulate the following SDE for the midprice.

$$dS_t = \pm g(\nu_t)\,dt + \sigma\,dW_t, \quad S^\nu_0 = S, \quad g : \mathbb{R}^+ \to \mathbb{R}^+ \tag{4.1}$$

Equation (4.1) may be interpreted as follows: aside from randomness, the agent's trading rate may influence the midprice permanently with the effect of g. The $\pm$ sign represents the acquirement and liquidation action respectively (if the agent is acquiring a large number of shares, the sudden increase of demand may permanently increase the price, and similarly the other way around). This midprice may also be taken as the fundamental price; the price which best reflects the current information. In this case, the permanent effect reflects the agent's informed decision making; when the value of the firm changes, the agent will update their trading rate accordingly.

The execution price (the price that the trader receives) is modelled as

$$\hat{S}_t^\nu = S_t^\nu \pm \left(\frac{1}{2}\Delta + f(\nu_t)\right), \quad \hat{S}_0^\nu = \hat{S}, \quad f : \mathbb{R}^+ \to \mathbb{R}^+ \tag{4.2}$$

Equation (4.2) may be interpreted as follows: the price that the trader receives is the midprice with additional terms $\frac{1}{2}\Delta$, which is half of the bid-ask spread, and the temporary effect on the price f caused by the agent's trading rate. The spread $\Delta \geq 0$ is assumed to be constant. The $\pm$ sign represents the acquirement and liquidation action respectively. The additional term f swings the execution price to the trader's disadvantage (making it worse than the bid/ask price) as a large market order (MO) will "walk the book".

The inventory process will have to satisfy

$$dQ_t^\nu = \pm \nu_t \, dt \tag{4.3}$$

Again, the $\pm$ sign represents the acquirement and liquidation action respectively.

4.2 Complete Liquidation with only Temporary Impact

In this section, we will discuss the model with liquidation as the action to be performed. The agent has to liquidate all $\mathfrak{N}$ shares at the end of the period. Moreover, the agent's trading rate only affects the execution (temporary) price and not the midprice ($g = 0$). Assume that the temporary effect is linear $f(\nu_t) = k\nu_t$.

The revenue at the end of period is

$$\int_0^T \hat{S}_u^\nu \nu_t \, du = \int_0^T (S_u^\nu - k\nu_t)\nu_t \, du \tag{4.4}$$

Naturally, the agent's value function is

$$H(t, S, q) = \sup_{\nu \in \mathcal{A}} \mathbb{E}\left[\int_t^T (S_u^\nu - k\nu_t)\nu_t \, du \Big| S_t = s, Q_t = q\right] \tag{4.5}$$

Here we have an additional term q (we've only discussed value functions in the form $v(t, x)$), but that does not matter. We only need to add the $D_q H dQ_t = -D_q H \nu_t dt$ term, and because Q satisfy equation (4.3), it has 0 quadratic variation.

Our HJB equation for this model is

$$\begin{aligned}
D_t H + \sup_{\nu^* \in A} \left[\mathcal{L}^{\nu^*} H - \nu^* D_q H + (S - k\nu^*)\nu^*\right] &= 0 \\
\Rightarrow D_t H + \sup_{\nu^* \in A} \left[\frac{1}{2}\sigma^2 D_{SS} H - \nu^* D_q H + (S - k\nu^*)\nu^*\right] &= 0 \\
\Rightarrow D_t H + \frac{1}{2}\sigma^2 D_{SS} H + \sup_{\nu^* \in A} \left[(S - k\nu^*)\nu^* - \nu^* D_q H\right] &= 0
\end{aligned} \tag{4.6}$$

Note that $(S - k\nu^*)\nu^* - \nu^* D_q H$ is a quadratic equation in ν^* with maximum global on $\nu^* = \dfrac{S - D_q H}{2k}$. Thus, our equation becomes

$$D_t H + \frac{1}{2}\sigma^2 D_{SS} H + \frac{1}{4k}\left(S - D_q H\right)^2 = 0 \tag{4.7}$$

The agent is required to sell all of their inventory at time T, so we require

$$\lim_{t \to T} H(t, S, q) = -\infty \quad \forall q > 0 \tag{4.8}$$

$$\lim_{t \to T} H(t, S, 0) = 0 \tag{4.9}$$

4.2.1 Solving the PDE (4.7)-(4.9)

References

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