Multivariate Calculus:

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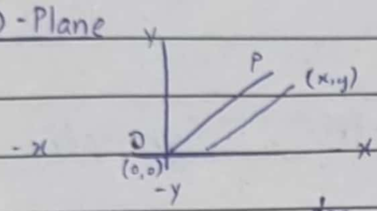
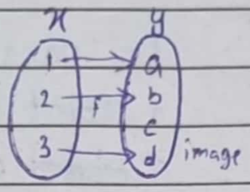
2D - Plane



$\overline{OP} = P(x,y) - O(0,0)$ Position formula.

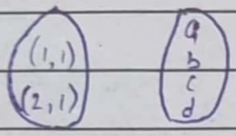
$\overline{P_1P_2} = P_2 - P_1 = (x_2 - x_1,\ y_2 - y_1)$

Funtion: 2 variable : $\underset{\text{dep}}{y} = f(\underset{\text{indep}}{x})$.



$\therefore$ image should be unique for every x.

Function 2 variable : $z = f(x,y)$



Domain: an ordered pair.

$V = \pi r^2 h$ (π constant)

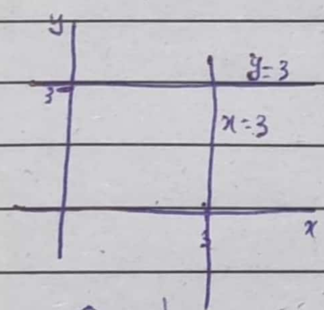
$\therefore V = f(r,h)$

A function f of two variables is a rule that assigns to each ordered pair of real number (x,y) in a set D a unique real number denoted by $f(x,y)$. The set D is the domain of f & its range is the set values that f takes on, that is $\{f(x,y) \mid (x,y) \in D\}$.

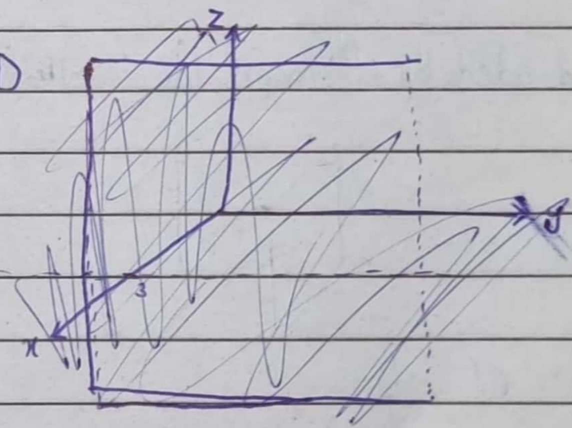
Example: $f(3,2)$

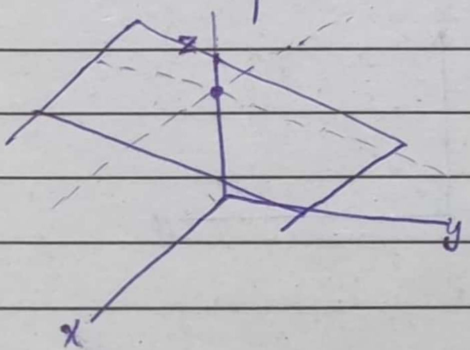
$f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$, $D = (x,y) \mid x+y+1 \geq 0,\ x \neq 1$

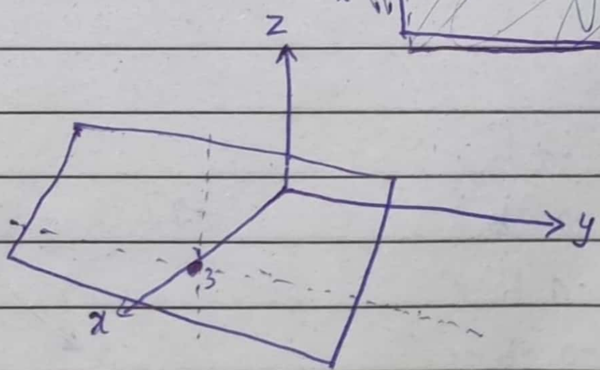
2-D



3-D





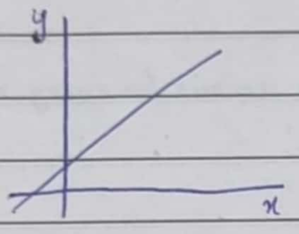


MIGHTY PAPER PRODUCT

CamScanner

Line & Plane in Space

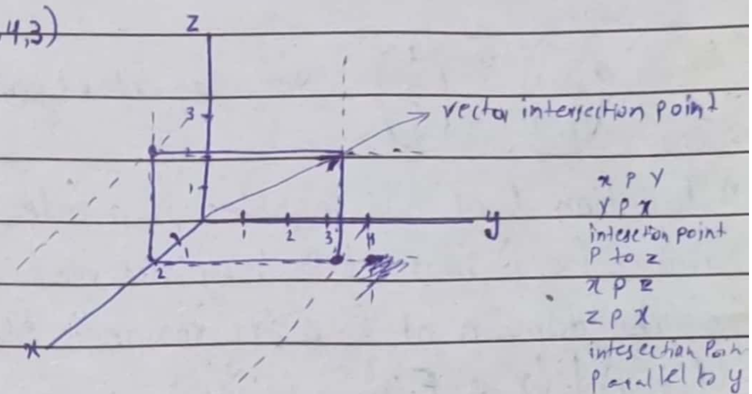
In 2-D : $y = mx + c$



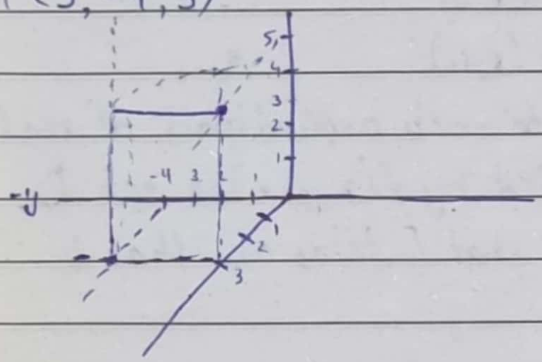
In 3-D : Parametric , Symmetric

For parametric equation we should know a point on our line i.e (x_0, y_0, z_0) and a ~~short~~ parallel vector

p(2,4,3)



P(3,-4,5)

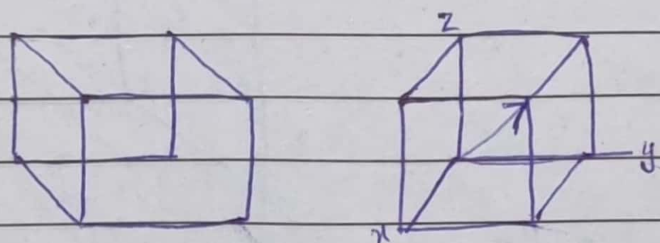


Distance formula: $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

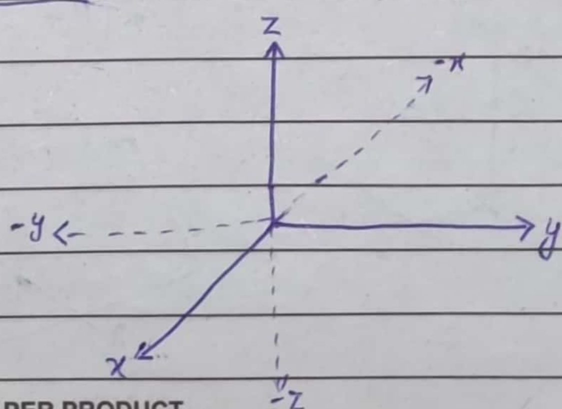
Mid-Point Formula

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

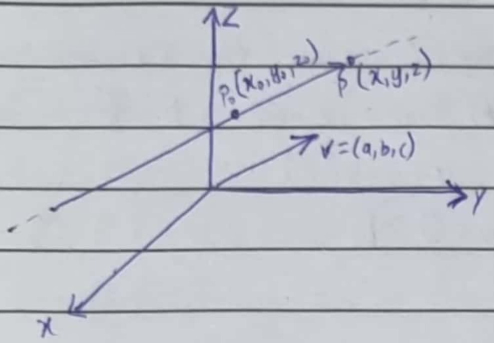
Vector divided by its magnitude is called unit vector.



Octants of 3-D



1st: a, b, c	5th: -a, b, c
2nd: a, b, -c	6th: -a, -b, c
3rd: a, -b, c	7th: -a, b, -c
4th: a, -b, -c	8th: -a, -b, -c



Scalar multiple

$\overrightarrow{P_0P} = t\,v$ ↳ Parameter

$tv = \langle a,b,c \rangle$

$P - P_0$

$\langle x-x_0, y-y_0, z-z_0 \rangle$

$x-x_0 = at$, $y-y_0 = bt$, $z-z_0 = ct$

Example: $P_0(-2,1,3)$ $v=(1,2,-2)$

$P_0P = tv$

$\langle x-x_0, y-y_0, z-z_0 \rangle = t\langle 1,2,-2 \rangle$

$x+2=t$, $y-1=2t$, $z-3=-2t$

Parametric

They all are equal

$t = \dfrac{x-x_0}{a}$, $t = \dfrac{y-y_0}{b}$, $t = \dfrac{z-z_0}{c}$

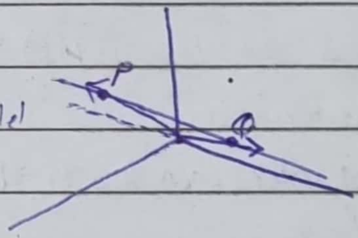
Symmetric Equation of Line.

Example: 2) Two points are given $P(-3,3,-2)$ & $Q(2,-1,4)$

So, we need to parallel vector.

$\overrightarrow{PQ} = Q - P$; this will give a vector on given line / collinear / parallel

$\langle \overset{a}{5i}, \overset{b}{-4j}, \overset{c}{6k} \rangle = V$



$x+3=5t$, $y-3=-4t$, $z+2=6t$

$\boxed{\dfrac{x+3}{5} = \dfrac{y-3}{-4} = \dfrac{z+2}{6}}$ Symmetric

Q) At what point does L intersect the xy plane?

$z=0$; $z-z_0=ct$

$0-(-2)=6t$

$\boxed{t=\frac{1}{3}}$

$x-x_0=at$ $y-y_0=bt$

$\boxed{x=-4/3}$ $\boxed{y=5/3}$

- Two vectors are said to be Parallel if their cross product is a zero vector.
- And their dot product is $\boxed{a \cdot b = |a||b|}$ equal to their magnitude.

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xample: 2 (a) Find parametric & Symmetric equations
oints $P(-3,3,-2)$ & $Q(2,-1,4)$

Sol: $\overrightarrow{PQ} = Q - P = \langle 5,-4,6 \rangle$

PE = $x = -3+5t$, $y = 3-4t$, $z = -2+6t$

SE = $\frac{x+3}{5} = \frac{y-3}{-4} = \frac{z+2}{6}$

(b) At what point does L intersact the xy-plane.

$z = 0$; $z = -2+6(t) \Rightarrow 0 = -2+6t \Rightarrow t = \frac{1}{3}$

Put $t \Rightarrow x = -3+5(\frac{1}{3})$, $y = 3-4(\frac{1}{3})$

$\boxed{\langle -\frac{4}{3}, \frac{5}{3}, 0 \rangle}$

Example: 3 (Q) $L_1: x = 1+2t$, $y = 2-3t$, $z = 2+t$

$L_2: x = 3-4t$, $y = 1+4t$, $z = -3+4t$

Sol (a)

$v_1: \langle 2,-3,1 \rangle \neq v_2: \langle -4,4,4 \rangle$

$\therefore$ Not Parallel.

(b) $\overbrace{1+2t_1 = 3-4t_2}^{1}$, $\overbrace{2-3t_1 = 1+4t_2}^{2}$, $\overbrace{2+t_1 = -3+4t_2}^{3}$

Add First 2 equa: $3 - t_1 = 4+0 = \boxed{t_1 = -1}$

Sub value t_1: $1+2(-1) = 3-4t_2 \Rightarrow \boxed{t_2 = 1}$

Sub t_1 & t_2 in eqa 3: $2+(-1) = -3+4(1) \Rightarrow 1 = 1$

$\therefore$ Line intersects.

To Find point of intersection sub $t_1 \rightarrow l_1$ & $t_2 \rightarrow l_2$

We get : $(-1,5,1)$

Distance b/w point & plane

$$D = \frac{|\overrightarrow{P_0P_1} \cdot n|}{|n|}$$

(Q) Point $P(-2,1,3)$ & plane: $2x-3y+z=1$

$$D = \frac{|2(-2)-3(1)+1(3)-1|}{\sqrt{2^2+(-3)^2+1^2}} \Rightarrow \boxed{\frac{5}{\sqrt{14}}}$$

Assignment # 01

M. Anas
CS221004

Date: ______

Q.1) $P(-1,3,-7)$, $Q(-3,9,-21)$

(i)

$Q - P = \langle -3, 9, -21\rangle - \langle -1, 3, -7\rangle$

Parallel Vector $= \langle \overset{a}{-2i}, \overset{b}{6j}, \overset{c}{-14} \rangle$

Parametric: $x - x_0 = at$, $y - y_0 = bt$, $z - z_0 = ct$

$= x + 3 = -2t$, $y - 9 = 6t$, $z + 21 = -14t$

Symmetric: $\dfrac{x+3}{-2} = \dfrac{y-9}{6} = \dfrac{z+21}{-14}$

(ii) $4x + 2y - 3z = 5$, $x - 3y + 2z = 5$

$z = 0$

$4x + 2y = 5$, $x - 3y = 5$

$P_0 = \left(\underset{x_0}{\frac{25}{14}}, \underset{y_0}{-\frac{15}{14}}, \underset{z_0}{0} \right)$

$n_1 : \langle 4, 2, -3 \rangle$ $\quad$ $n_2 : \langle 1, -3, 2 \rangle$

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 4 & 2 & -3 \\ 1 & -3 & 2 \end{vmatrix}$$

$= (4+9)i - (8+3)j + (-12-2)k$

$= 13i - 11j - 14k$

Parametric Equation:

$x = \frac{25}{14} + 13t$

$y = -\frac{15}{14} - 11t$

$z = 0 - 14t$

(Q.02) (i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2 - 3y^2}{x^2 + y^2} \Rightarrow \dfrac{2(0) - 3(0)}{0 + 0} = \dfrac{0}{0}$ indeterminant form

$x = 0$: $\dfrac{2(0) - 3y^2}{(0) + y^2} \to \dfrac{-3\cancel{y^2}}{\cancel{y^2}} \Rightarrow \boxed{-3}$

$\therefore$ Limit doesn't exist

$y = 0$: $\dfrac{2x^2 - 3(0)}{x^2 + (0)} \to \dfrac{2\cancel{x^2}}{\cancel{x^2}} = \boxed{2}$

$\therefore$

(ii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2} \Rightarrow \dfrac{(0)(0)}{0 + 0} = \dfrac{0}{0}$ indeterminant

$x = 0$: $\boxed{0}$ $\quad$ $y = 0$: $\boxed{0}$ $\quad$ $y = x$: $\boxed{\frac{1}{2}}$

$\therefore$ Limit doesn't exist.

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Q.2) (iii) $\lim\limits_{(x,y)\to(1,\pi)} \dfrac{y\cos y + \sin(xy)}{x} \Rightarrow \dfrac{\pi\cos(y) + \sin(1\pi)}{1}$

$\pi(-1) + 0 \Rightarrow \boxed{-\pi}$

$\therefore$ Limit exist

(iv) $\lim\limits_{(x,y)\to(0,0)} \dfrac{3y}{x}$

$\dfrac{3(0)}{0} \Rightarrow \dfrac{0}{0}$

$x=0$: $\dfrac{3y}{0} \Rightarrow \boxed{\infty}$

$y=0$: $\dfrac{3(0)}{x} \Rightarrow \dfrac{3}{x} \Rightarrow \dfrac{3}{0} \Rightarrow \boxed{\infty}$

$y=x$: $\dfrac{3(x)}{x} = \boxed{3}$

$\therefore$ Limit doesn't exist.

Q.2) (b) $x^2 + y^2 = z - 30$ at point $(1, 5, -3)$

$x^2 + y^2 - z + 30 = 0$

$\nabla f = \dfrac{\partial f}{\partial x}\hat{i} + \dfrac{\partial f}{\partial y}\hat{j} + \dfrac{\partial f}{\partial R}\hat{k}$

$\nabla F = 2x\hat{i} + 2y\hat{j} - 1\hat{k}$

$\nabla f_p = 2(1)\hat{i} + 2(5)\hat{j} - 1(1)\hat{k}$

$= 2\hat{i} + 10\hat{j} - 1\hat{k}$

$\boxed{2(x-1) + 10(y-5) + 1(z+3) = 0}$

$2x - 2 + 10y - 50 + 1z + 3 = 0$

$\boxed{2x + 10y + z = 49}$

$x = x_0 + at$ $\quad$ $x = 1 + 2t$

$y = y_0 + bt$ $\quad$ $y = 5 + 10t$

$z = z_0 + ct$ $\quad$ $z = -3 + 1t$

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Q.3)(i) $F(x,y) = Cos(xy)e^{xy}$

Apply Product Rule

$f_x = Cos(xy) \cdot y e^{xy} + e^{xy} \cdot -y Sin(xy)$

$\boxed{f_x = y Cos(xy) \cdot e^{xy} - y Sin(xy) e^{xy}}$

$f_y = Cos(xy) \cdot x e^{xy} + e^{xy} - x Sin(xy)$

$\boxed{f_y = x Cos(xy) \cdot e^{xy} - x Sin(xy) \cdot e^{xy}}$

$f_{xx} = y(Cos(xy) \cdot y e^{xy} + e^{xy} \cdot -y Sin(xy)) \rightarrow$

$\quad -y(S(xy) \cdot y x^{xy} + e^{xy} \cdot y Cos(xy))$

$\quad = y^2 Cos(xy) \cdot e^{xy} - y^2 Sin(xy) \cdot e^{xy} \rightarrow$

$\quad -y^2 Sin(xy) \cdot e^{xy} - y^2 Cos(xy) \cdot e^{xy}.$

$\boxed{f_{xx} = -2y^2 Sin(xy) \cdot e^{xy}}$

$f_{yy} = x^2 Cos(xy) \cdot e^{xy} - x^2 Sin(xy) \cdot e^{xy}$

$\quad -x^2 Sin(xy) \cdot e^{xy} - x^2 Cos(xy) \cdot e^{xy}$

$\boxed{f_{yy} = -2x^2 Sin(xy) \cdot e^{xy}}$

Q.3)(ii) $f(x,y) = e^{xy} sin(2y)$ at point $(0, \frac{\pi}{3})$

vector $u = -4i + 4j$

$\nabla F = \frac{\partial}{\partial x}(e^{xy} sin(2y))\hat{i} + \frac{\partial}{\partial y}(e^{xy} sin(2y))\hat{j}$

$\nabla F = (y Sin(2y) \cdot e^{xy})\hat{i} + (e^{xy} \cdot 2Cos(2y) + x e^{xy} \cdot Sin(2y))\hat{j}$

$\nabla F = (y Sin(2y) \cdot e^{xy})\hat{i} + (2Cos(2y) \cdot e^{xy} + x Sin(2y) \cdot e^{xy})\hat{j}$

Gradient at given point

$\nabla f_p = (\frac{\pi}{3} Sin(2(\frac{\pi}{3})) \cdot e^0) i + (2 Cos(2(\frac{\pi}{3})) + 0)\hat{j}$

$v_1 \leftarrow \nabla f_p = 0.91\,\hat{i} - \hat{j}$

$v_2 \leftarrow \hat{u} = \frac{u}{|u|} = \frac{-4i+4j}{\sqrt{(-4)^2+(4)^2}} \Rightarrow \frac{-4}{4\sqrt{2}}\hat{i} + \frac{4}{4\sqrt{2}}\hat{j}$

$V_1 \cdot V_2 = \langle 0.91, -1 \rangle \cdot \langle \frac{-4}{4\sqrt{2}}, \frac{4}{4\sqrt{2}} \rangle$

$= -0.64 - \frac{4}{4\sqrt{2}}$

$\boxed{\text{Directional Derivative} = -1.4} \approx -1.348$

$f_{xy} = [y(Cos(xy) \cdot e^{xy})' + (Cos(xy) \cdot e^{xy})(y)'] -$

$\quad [(y)(Sin(xy) \cdot e^{xy})' + (Sin(xy) \cdot e^{xy})(y)']$

$= [xy\,\cancel{Cos(xy)} \cdot e^{xy} - xy\,Sin(xy) \cdot e^{xy} + Cos(xy) \cdot e^{xy}]$

$- [xy\,Sin(xy) e^{xy} + \cancel{xy\,Cos(xy)} \cdot e^{xy} + Sin(xy) \cdot e^{xy}]$

f_{xy}:

$\boxed{-2xy\,Sin(xy) \cdot e^{xy} + Cos(xy) \cdot e^{xy} - Sin(xy) \cdot e^{xy}}$

Equation of Plane:

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The standard form of the equation of a plane containing the point $P_0(x_0, y_0, z_0)$ and having the normal vector $n = (a, b, c)$ is: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Example 5: $P_0(3, -3, 2)$, $n = (4, 2, 3)$. Sol: $4(x-3) + 2(y+3) + 3(z-2) = 0$

$$4x + 2y + 3z = 12$$

For x-intercept: $(y, z) = 0$
" y " : $(x, z) = 0$

Example 6: Find an equation of the plane containing points $P(3, -1, 1)$, $Q(1, 4, 2)$ & $R(0, 1, 4)$

$\vec{PQ} = \langle -2, 5, 1 \rangle$ $\vec{PR} = \langle -3, 2, 3 \rangle$ $n = PQ \times PR$:

$$n = \begin{vmatrix} i & j & k \\ -2 & 5 & 1 \\ -3 & 2 & 3 \end{vmatrix} = 13i + 3j + 11k$$

$13(x-3) + 3(y+1) + 11(z-1) = 0$

$13x + 3y + 11z = 47$

Angle between two planes: $\cos\theta = \dfrac{n_1 \cdot n_2}{|n_1||n_2|}$

•) Parallel
•) Point

$\boxed{\text{Pg\#33}}$

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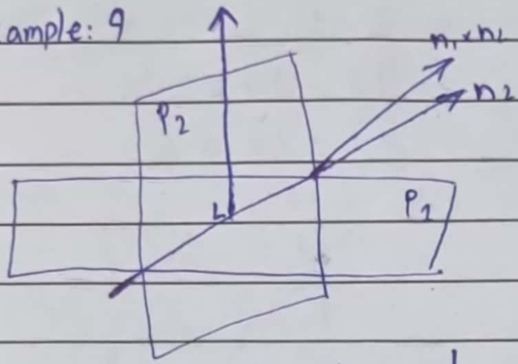
Example: 9



if 2 planes intersects on a line then ~~there non~~ the cross product of there normal vectors will be parallel to the line of intersection of planes.

$$v = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 2 & 3 & -1 \end{vmatrix} = -5i + 7j + 11k$$

$$i(1-6) - j(-3-4) + k(9-(-2))$$

$x^2 + y^2 = r^2$ eq.a of circle

$z = x^2$ Parabola.

$3x - y + 2z = 1$

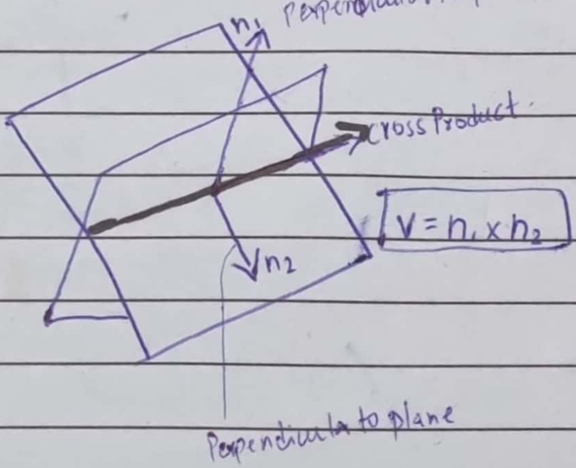
$2x + 3y - z = 4$

$z = 0$

$3x - y = 1$

$2x + 3y = 4$

$P_0 = \left(\underset{x_0}{\frac{7}{11}}, \underset{y_0}{\frac{10}{11}}, \underset{z_0}{0}\right)$



$\boxed{v = n_1 \times n_2}$

Cross Product are perpendicular.

$n_1 = \langle 3, -1, 2 \rangle$

$n_2 = \langle 2, 3, -1 \rangle$

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 2 & 3 & -1 \end{vmatrix}$$

$$i(1-6) - j(-3-4) + k(9-(-2))$$

$\boxed{V = -5i + 7j + 11k}$ line kae parallel

Points ka perpendicular

Parametric Equa:

$x = \frac{7}{11} - 5t$

$y = \frac{10}{11} + 7t$

$z = 0 + 11t$

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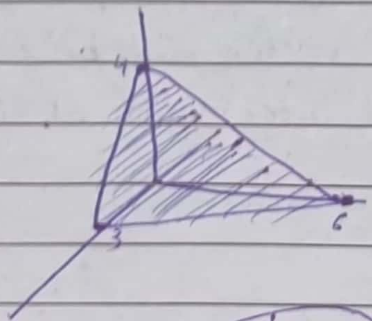
Tracing:

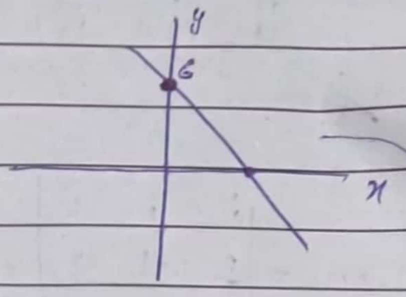
$4x + 2y + 3z = 12$

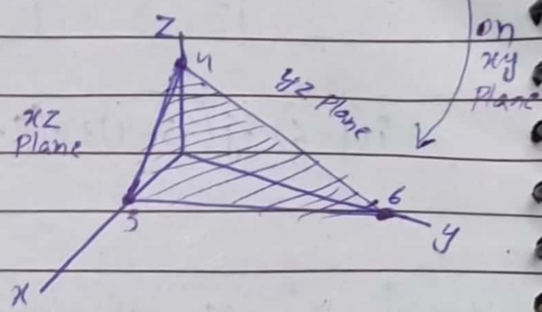


$z = 0$ xy Plane

$4x + 2y = 12$

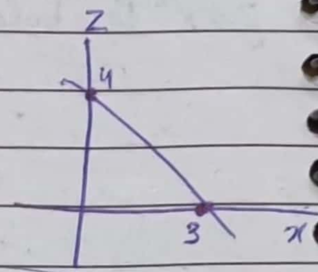
$y = -2x + 6$





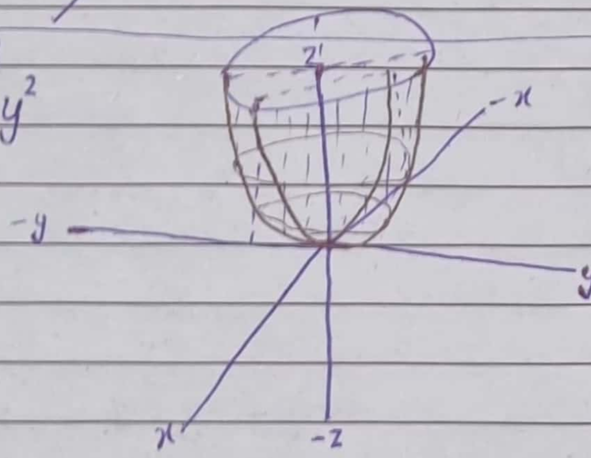
$y = 0$

$4x + 3z = 12$



Paraboloied

Ex: $z = x^2 + y^2$



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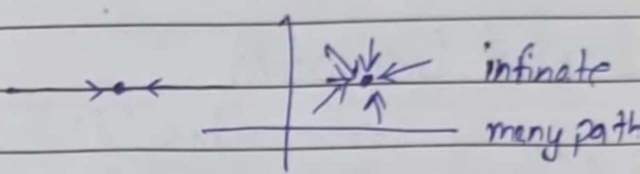
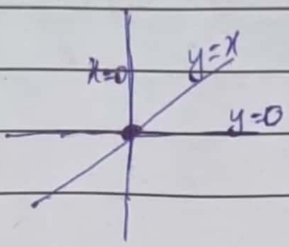
$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{0}{0}$$ indeterminant/Undefined.



Q) $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \frac{0}{0}$$



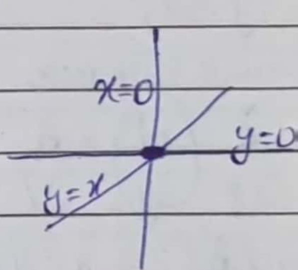
$x=0$: $\frac{0-y^2}{0+y^2} = \boxed{-1}$

$y=0$: $\frac{x^2-0}{x^2+0} = \boxed{1}$

∴ Limit doesn't exist.

Q) $$\lim_{(x,y)\to(1,0)} \frac{x^2-y^2}{x^2+y^2} = \frac{1-0}{1+0} = 1$$ Limit exist.

Q) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{0}{0}$$ indeterminant



$x=0$: $\frac{0(y)}{0+y^2} = \boxed{0}$

$y=0$: $\frac{x(0)}{x^2+0} = \boxed{0}$

$y=x$: $\frac{x(x)}{x^2+x} = \boxed{\frac{1}{2}}$ Limit doesn't exi

Q) $$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2}$$ Numerator's power is greater. Very fast towards zero, so might limit exist.

$\frac{2(0)(0)}{0+0} = \frac{0}{0}$ indeterminant

$y=x$: $\frac{2x^2(x)}{x^2+x^2} \Rightarrow \frac{2x^3}{2x^2} = x \Rightarrow (0) = \boxed{0}$

$x=0$: $\boxed{0}$ $y=0$: $\boxed{0}$

$y=x^2$: $\frac{2x^2(x^2)}{x^2+x^4} = \boxed{0}$

Q.) $$\lim_{(x,y)\to(1,\pi)} \frac{y\cos y + \sin(xy)}{x}$$

$\frac{\pi\cos\pi}{1} + \sin(1(\pi))$ $\to 0$

$\frac{\pi(-1)}{1} + \sin\pi = -\pi$

Date: ________

b) Find the equa of tangent plane & normal line to $x^2+y^2=z-30$ at point $(1,5,-3)$

Partial Derivatives:

$\cdot)\ f(x,y) = x\,y$ → Constant treate

$f_x \leftarrow \frac{\partial f}{\partial x}$ $\quad \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy) = y = \frac{\partial}{\partial x}(x) = y(1)$

$\boxed{f_x = y}$

$\cdot)\ f(x,y) = e^{xy}$

$\frac{\partial}{\partial x}(e^{xy}) = ye^{xy} \qquad \therefore e^{ax} = ae^{ax}$

$\frac{\partial f}{\partial x} = f_x = ye^{xy}$

$f_y = xe^{xy}$

derivate with respect of x, y constant

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(xe^{xy}) \qquad xe^{ax}$

$f_{xy}: \boxed{\frac{\partial^2 f}{\partial x \partial y} = x(ye^{xy}) + e^{xy}(1)}$

$f_y = xe^{xy}$

with respect to y:

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(xe^{xy})$

$x\frac{\partial}{\partial y}(e^{xy})$

$x(ye^{xy}) = xye^{xy}$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = xye^{xy}$

Partial Derivatives: Date: ______

Q) $f(x,y) = e^x \sin(xy)$

Find f_x

$\frac{\partial f}{\partial x}$

applying $\frac{\partial}{\partial x}$ on both sides

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\underset{1}{e^x}\ \underset{2}{\sin(xy)}\right]$$

$$f_x = e^x(\cos(xy)y) + \sin(xy)e^x$$

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left[ye^x\cos(xy) + e^x\sin(xy)\right]$$

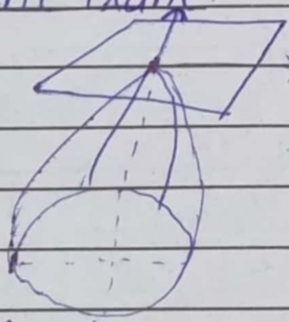
Q) $f(x,y) = ye^x$

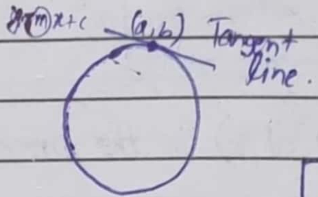
find f_{xy}

$\frac{\partial}{\partial x}$: f_x: ye^x

$\boxed{f_{xy}: e^x}$

Tangent Plane:





$$\boxed{\begin{aligned}&\text{Tangent Line: } f'(x_1)(x-x_1)\\ &\text{Normal Line: } -\tfrac{1}{f'(x_1)}(x-x_1)\end{aligned}}$$

Equation of plane:

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

① Point (Given)

② Gradient at point (Partial Derivative)

Q) Find the Equa of tangent plane to the surface $z = x^2+y^2$ at point $p(-2,1,$

$f = x^2+y^2-z$ we need to take derivative.

$$\nabla f = \frac{\partial}{\partial x}(f)\,\hat{i} + \frac{\partial}{\partial y}(f)\,\hat{j} + \frac{\partial}{\partial z}(f)\,\hat{k}$$

$$\nabla f = 2x\hat{i} + 2y\hat{j} + (-1)\hat{k}$$

$$\nabla f_p = 2(-2)\hat{i} + 2(1)\hat{j} + (-1)\hat{k}$$

$-4i + 2j - k$

$\langle -4, 2, -1 \rangle$

Date: ____________

$V = \langle \overset{a}{-4}, \overset{b}{2}, \overset{c}{-1} \rangle$ Perpendicular Vector.

$P_0 = \langle \underset{x_0}{-2}, \underset{y_0}{1}, \underset{z_0}{5} \rangle$

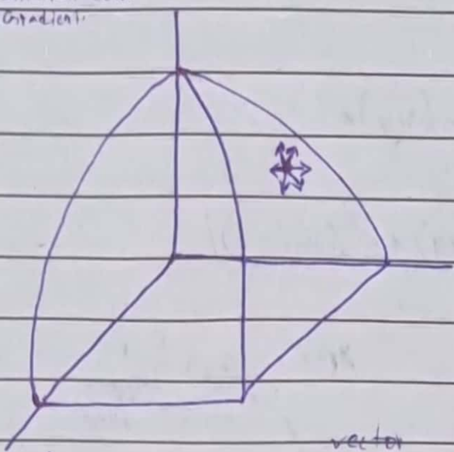
$$a(x-x_0)+b(y-y_0)+c(z-z_0)$$
$$-4(x+2)+2(y-1)-1(z-5)=0$$

Directional Derivatives:

we need
1) direction/vector
2) Gradient.

Gradient

$$\nabla F = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$



Steapest Decent.

Unit Vectors

Vector divide by its magnitude.

Q) $\underset{\text{function}}{f = 4xz^3 - 3x^2y^2z}$ at point $(2,-1,2)$ in the direction $\underset{\text{vector}}{2\hat{i}-3\hat{j}+6\hat{k}}$.

Step 1: Find the Gradient at P:

$$\nabla f = \frac{\partial}{\partial x}[4xz^3-3x^2y^2z]\hat{i} + \frac{\partial}{\partial y}[4xz^3-3x^2y^2z]\hat{j} + \frac{\partial}{\partial z}[4xz^3-3x^2y^2z]\hat{k}$$

$$\nabla f = (4z^3 - 6xy^2z)\hat{i} + [0-6x^2yz]\hat{j} + [12xz^2 - 3x^2y^2]\hat{k}.$$

Step 2 Gradient at point

$$\nabla f_p: (4(2)^3 - 6(2)(-1)^2(2))\hat{i} + [-6(2)^2(-1)(2)]\hat{j} + (12(2)(2)^2 - 3(2)^2(-1)^2]\hat{k}$$

$$\boxed{\nabla f_p = 8\hat{i} + 48\hat{j} + 84\hat{k}} - V_1$$

$$\hat{u} = \frac{u}{|u|} = \frac{2\hat{i}-3\hat{j}+6k}{\sqrt{2^2+(-3)^2+(6)^2}}$$

$$\hat{u} = \frac{2}{\sqrt{49}}\hat{i} - \frac{3}{\sqrt{49}}\hat{j} + \frac{6}{\sqrt{49}}k$$

$$\boxed{\hat{u} = \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}} - V_2$$

$f_p \cdot \hat{u}$

$$V_1 \cdot V_2 = \langle 8, 48, 84 \rangle \cdot \langle 2/7, -3/7, 6/7 \rangle$$

$$\frac{16}{7} - \frac{144}{7} + \frac{504}{7}$$

$$\boxed{\frac{376}{7}}$$

MIGHTY PAPER PRODUCT

CamScanner

Date: ____________

Example 1: Find the directional derivative of $f(x,y) = 4 - 2x^2 - y^2$ at the point (1,1) in the direction of unit vector u that makes an angle of $\pi/3$ radians with positive x-axis.

$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \Rightarrow -4x - 2y\hat{j}$

$\nabla f_P = -4(1)\hat{i} - 2(1)\hat{j}$

$\boxed{\nabla f_P = -4\hat{i} - 2\hat{j}}$

$u = \cos\left(\frac{\pi}{3}\right)\hat{i} + \sin\left(\frac{\pi}{3}\right)\hat{j}$

$\boxed{u = \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}}$

$|u| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$ $\therefore$ Unit vector is

Dot Product

$\nabla f_P \cdot u = (-4\hat{i} - 2\hat{j}) \cdot \left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right)$

$= \boxed{-2 - \sqrt{3}}$ Directional Derivative.

Partial Derivative:

Q) $f(x,y) = \frac{x^2}{y+1}$

$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2}{y+1}\right)$

$\frac{1}{y+1}(x^2) \Rightarrow \boxed{\frac{2x}{y+1}}$

$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2}{y+1}\right)$

$= x^2 \frac{\partial}{\partial y}\left(\frac{1}{y+1}\right)$

$\boxed{f_y = \frac{-x^2}{(y+1)^2}}$

Q) $g(x,y) = e^{x+3y}\sin(xy)$

$f_x = \frac{\partial f}{\partial x} \Rightarrow \frac{\partial}{\partial x}(e^{x+3y})\sin(xy)$

$\frac{\partial}{\partial x}\left[e^x . e^{3y}\sin(xy)\right]$

$e^{3y}\left[\frac{\partial}{\partial x} e^x . \sin(xy)\right] \rightarrow$ Product Rule $\therefore \frac{\partial}{\partial x}[\sin(xy)] \Rightarrow \cos(xy) . \frac{\partial}{\partial x}(xy) \Rightarrow y\cos(xy)$

$f_x = e^{3y}\left[\sin(xy) . e^x + e^x . y\cos(xy)\right]$

$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial^2 g}{\partial x^2}$

Q) $f(x,y) = e^x\sin(xy)$, Find f_{xx}

$f_x = e^x . \cos(xy) . y + e^x\sin(xy)$

$f_{xx} = y\left(e^x\cos(xy) - ye^x\sin(xy)\right) + \left(e^x\sin(xy) + ye^x\sin(xy)\right)$

$f_{xy} = e^x\left(\cos(xy) - xy\sin(xy)\right) + e^x\cos(xy) . x$

$a^x = a^x \ln a$
$\ln x = \frac{1}{x}$
$\log_a x = \frac{1}{(\ln a) x}$

$\cancel{u'v - u(x)v'(x)}$

$\frac{u'v - uv'}{v^2}$

$c + = c$
$i + = 3$
$\boxed{i = i + 3}$

Directional Derivative:

Date:

Example: 2) $f(x,y) = e^x \cos(2y)$, Point$(0, \frac{\pi}{4})$, $v = 2\hat{i} + 3\hat{j}$

$$\nabla F = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \Rightarrow [e^x \cdot -0 \cdot \sin(2y) + e^x \cdot \cos(2y)]\hat{i} + [e^x \cdot -2\sin(2y) + 0]\hat{j}$$

$$\nabla f = e^x \cdot \cos(2y)\,\hat{i} - 2e^x \cdot \sin(2y)\hat{j}$$

$$\nabla f_p = e^0 \cdot \cos(2(\tfrac{\pi}{4}))\hat{i} - 2e^0 \cdot \sin(2\tfrac{\pi}{4})\hat{j}$$

$$\boxed{\nabla f_p = 0\,\hat{i} - 2\hat{j}} - v_1$$

$$u = \frac{2\hat{i} + 3\hat{j}}{\sqrt{2^2 + 3^2}} \Rightarrow \boxed{\frac{2}{\sqrt{13}}\hat{i} + \frac{3}{\sqrt{13}}\hat{j}} - v_2$$

$$v_1 \cdot v_2 = 0 \times \frac{2}{\sqrt{13}} + (-2)\left(\frac{3}{\sqrt{13}}\right)$$

$$= \boxed{-\frac{6\sqrt{13}}{13}} \rightarrow \text{Directional Derivative.}$$

Q) Gradient of $f(x,y) = x\sin y + y\ln x$, point (e, π)

$$\nabla f = [(x \cdot 0 \sin y + 1 \cdot \sin y) + (y(\tfrac{1}{x}) + 0)]\hat{i} + [(x \cdot \cos y + 0) + (y \cdot 0 + 1(\ln x))]\hat{j}$$

$$= [\sin(y) + \tfrac{y}{x}]\hat{i} + [x\cos y + \ln x]\hat{j}$$

$$\nabla F_p = [\sin(\pi) + \tfrac{\pi}{e}]\hat{i} + [e\cos(\pi) + \ln e]\hat{j}$$

$$= [\tfrac{\pi}{e}]\hat{i} + [1 - e]\hat{j}$$

Tangent Plane & Normal Line:

$4x^2 + y^2 + 4z^2 = 16$ point $(1, 2, \sqrt{2})$

$$8x\hat{i} + 2y\hat{j} + 8z\hat{k} = 0$$

$$8\hat{i} + 4\hat{j} + 8\sqrt{2}\,k$$

Equ of tangent Plane

$$8(x-1) + 4(y-2) + 8\sqrt{2}(z - \sqrt{2}) = 0$$

Normal line:

$$\frac{x-1}{8} = \frac{y-2}{4} = \frac{z-\sqrt{2}}{8\sqrt{2}}$$

•) Agr limits variable hongi tou x ya y ki form ma hongi.
•) Agr limits constent hogi tou ham change kerkty.

if limit is constent so we transform without any change.

Integrations

Date:

Q) $\int_0^1\int_0^x \frac{\sin x}{x}\, dy\, dx$

$\int_0^1 \frac{\sin x}{x}[dy]_0^x\, dx$

$\int_0^1 \frac{\sin x}{x}[y]_0^x\, dx$

$\int_0^1 \frac{\sin x}{x}[x-0]\, dx$

$\int_0^1 \sin x\, dx$

$-\cos x|_0^1$

$(-\cos(1)+\cos(0))$

$-\cos(1)+\cos(0)$

$\boxed{1-\cos(1)}$

Q) $\int_{-1}^1\int_0^1 xy\, dx\, dy$

$\int_0^1\int_{-1}^1 xy\, dy\, dx$

$\int_0^1 x\left(\frac{y^2}{2}\right)_0^1 dx$

$\int_0^1 \frac{x}{2}\, dx$

Q) $\int_0^2\int_1^4 y\sqrt{x}\, dy\, dx$

$\int_0^2 \sqrt{x}\left(\frac{y^2}{2}\right)_1^4 dx$

$\int_0^2 \sqrt{x}\left(\frac{4^2}{2}-\frac{1^2}{2}\right) dx$

$\int_0^2 \sqrt{x}\left(15/2\right) dx$

$\frac{15}{2}\left[\frac{2x^{3/2}}{3}\right]\Big|_0^2$

$\boxed{\frac{5(2)^{3/2}}{3}}$

Q) $\int_0^{\pi/2}\int_0^{\ln 2} e^{-x}\sin y\, dx\, dy$

$\int_0^{\pi/2} \sin y\left(-e^{-x}|_0^{\ln 2}\right) dy$

$\int_0^{\pi/2} \sin y\left(-e^{\ln 2}-(-e^0)\right) dy$

$e^{\ln y}=y$

$e^{-\ln 2}=e^{\ln 2^{-1}}$

$2^{-1}=1/2$

$\int_0^{\pi/2} \sin y\left(-1/2+1\right) dy$

Q) $\int_0^{1/2}\int_0^{\sqrt{1-x}} 2xy\, dy\, dx$

$\int_0^{1/2} \cancel{2}x \times \frac{y^2}{\cancel{2}}\Big|_0^{\sqrt{1-x}} dx$

$\int_0^{1/2} x\left((\sqrt{1-x})^2-0\right) dx$

$\int x(1-x)\, dx$

$\frac{x^2}{2}-\frac{x^3}{3}$

$\frac{(1/2)^2}{2}-\frac{(1/2)^3}{3}$

$\frac{1/4}{2}-\frac{1/8}{3}$

$1/2 - 3/8$

$\frac{8-6}{16}$

$\boxed{1/4}$

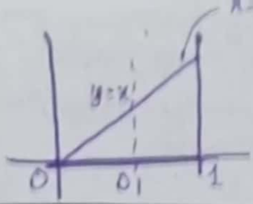
LIATE

Q) $\int_0^1\int_0^1 \frac{x}{1+xy}\, dy\, dx$

$\int_0^1 \ln(1+xy)|_0^1\, dx$

$\int_0^1 (\ln(1+x)-\ln(1))\, dx$

$\int_0^1 \ln(1+x)\, dx$



Date:

Q) $\int_0^1\int_y^1 e^{x^2}\,dx\,dy$

$\int_0^1\int_0^x e^{x^2}\,dy\,dx$

$\int_0^1 y e^{x^2}\Big|_0^x dx$

$\int_0^1 e^{x^2}|x-0|\,dx$

$\int_0^1 x e^{x^2}\,dx$

Let $u=x^2,\ du=2x\,dx$

$\frac{du}{2}=x\,dx$

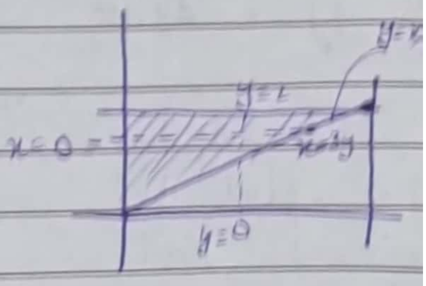
$\frac{1}{2}\int e^u\,du=\frac{e^u}{2}$

$\frac{e^x}{2}\Big|_0^1$

$\frac{e^1-e^0}{2}$

$\boxed{\frac{1}{2}(e-1)}$

Q) $\int_0^6\int_{\frac{x}{3}}^2 x\sqrt{y^3+1}\,dy\,dx$



$\int_0^2\int_0^{3y} x\sqrt{y^3+1}\,dx\,dy$

$\int_0^2 \frac{x^2}{2}\sqrt{y^3+1}\Big|_0^{3y} dy$

$\int_0^2 \sqrt{y^3+1}\left|\frac{(3y)^2}{2}-0\right|dy$

$\int_0^2 \frac{9y^2}{2}\sqrt{y^3+1}\,dy$

$u=y^3+1,\ du=3y^2\,dy$

$\frac{3}{2}\int\sqrt{u}\,du$

$\frac{3}{2}\ \frac{u}{3/2}\Rightarrow u^{3/2}$

$(y^3+1)^{3/2}\Big|_0^2$

$27-1$

$\boxed{26}$

Green's Theorem:

Date: ____________

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

(1) $\oint_C \overbrace{(x^2+y^3)\,dx}^{M\,dx} + \overbrace{3xy^2\,dy}^{N\,dy}$

$\frac{\partial N}{\partial x} = 3y^2$, $\frac{\partial M}{\partial y} = 3y^2$

$\iint_R (3y^2 - 3y^2)\,dA = 0$

(1) $\oint_C \overbrace{(e^x+6xy)\,dx}^{M\,dx} + \overbrace{(8x^2+\sin y^2)\,dy}^{N\,dy}$

$\frac{\partial N}{\partial x} = 16x$, $\frac{\partial M}{\partial y} = 6x$

$\iint_R (16x-6x)\,dA \Rightarrow \iint_R (10x)\,dA$

$x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$

$10\iint_R x\,dA \Rightarrow 10\iint_R r\cos\theta \cdot r\,dr\,d\theta$

$10\int_0^{\pi/2}\int_1^3 r^2\cos\theta\,dr\,d\theta$

$10\int_0^{\pi/2} \cos\theta \left(\frac{r^3}{3}\right)\Big|_1^3 d\theta$

$\frac{260}{3}\int_0^{\pi/2} \cos\theta\,d\theta$

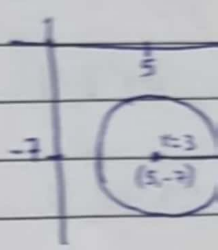
$\frac{260}{3} \sin\theta \Big|_0^{\frac{\pi}{2}}$

$\frac{260}{3}\sin\left(\frac{\pi}{2}\right)$

$\boxed{\frac{260}{3}}$

(1) $\oint_C \overbrace{(7y - e^{\sin x})\,dx}^{M\,dx} + \overbrace{[15x - \sin(y^3+8y)]\,dy}^{N\,dy}$

$\frac{\partial N}{\partial x} = 15$, $\frac{\partial M}{\partial y} = 7$



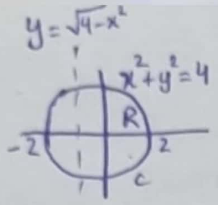
$\iint_R (15-7)\,dA$

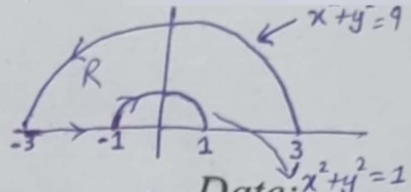
$8\iint_R dA = 8(\pi r^2)$

$8(\pi(3)^2)$

$\boxed{72\pi}$

Date:





Q) $\oint_c \underbrace{(y^2+\tan x)dx}_{M\,dx} + \underbrace{(x^3+2xy-\sqrt{y})dy}_{N\,dy}$

where C is circle $x^2+y^2=4$

$\frac{\partial M}{\partial y} = 2y$

$\frac{\partial N}{\partial x} = 3x^2+2y$

$\iint_R (3x^2+2y-2y)\, dA$

$\iint_R 3x^2\, dA$

$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 3x^2\, dy\, dx$

$3\int_{-2}^{2} x^2 y \Big|_L^U dx$

$3\int_{-2}^{2} x^2\left(\sqrt{4-x^2} - (-\sqrt{4-x^2})\right)dx$

$3\int_{-2}^{2} x^2 \cdot 2\sqrt{4-x^2}\, dx$

$6\int_{-2}^{2} x^2 \cdot \sqrt{4-x^2}\, dx$

Even Function

$2\times 6\int_0^2 x^2\sqrt{4-x^2}\, dx$

$\boxed{12\pi}$

↳ Using Polar Coordinate.

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\, dr\, d\theta$

$\int_0^{2\pi}\int_0^2 3r^2\cos^2\theta \cdot r\, dr\, d\theta$

$\int_0^{2\pi}\int_0^2 3r^3\cos^2\theta .\, dr\, d\theta$

$\cos^2\theta \int_0^{2\pi} \frac{3r^4}{4}\Big|_0 d\theta$

$\int_0^{2\pi} 12 \cos^2\theta\, d\theta$

Calculator.

$\boxed{12\pi}$

Q) $\oint_c \underbrace{(e^x+y^2)dx}_{M\,dx} + \underbrace{(x^2+3xy)dy}_{N\,dy}$

C is +ve Circle (semicircle) $x^2+y^2=1$ & $x^2+y^2=9$

$\frac{\partial N}{\partial x} = 2x+3y$

$\frac{\partial M}{\partial y} = 2y$

$\iint_R (2x+3y-2y)\, dA$

$\iint_R (2x+y)\, dA$

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\, dr\, d\theta$

$\int_0^{\pi}\int_1^3 (2r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$

$\int_0^{\pi}\int_1^3 2r^2\cos\theta + r^2\sin\theta)\, dr\, d\theta$

$\int_0^{\pi} (2\cos\theta+\sin\theta)\frac{r^3}{3}\Big|_1^3 d\theta$

$\int_0^{\pi} \left(\frac{3^3}{3} - \frac{1^3}{3}\right)(2\cos\theta+\sin\theta)\, d\theta$

$\frac{26}{3}\int_0^{\pi} 2\cos\theta + \sin\theta \,.\, d\theta$

$\frac{26}{3}(2\sin\theta - \cos\theta)\Big|_0^{\pi}$

$\boxed{\frac{52}{3}}$

Critical Point

Date:

Q) $f(x,y) = x^3 + y^3 - 3axy$

$\frac{\partial f}{\partial x} = 3x^2 - 3ay = 0$

$\frac{\partial f}{\partial y} = 3y^2 - 3ax = 0$

$x^2 = ay$ — (1)

$y^2 = ax$ — (2)

$\left(\frac{x^2}{a}\right)^2 = ax$

$x^4 = a^3 x$

$x^4 - a^3 x = 0$

$x(x^3 - a^3) = 0$

$\boxed{x=0}$, $x^3 - a^3 = 0$ $\quad \therefore \sqrt[3]{x^3} = \sqrt[3]{a^3}$

$\boxed{x=a}$

When $x=0$

$y=0$ $(0,0)$

When $x=a$

$y=a$ (a,a)

Q) Examine $f(x,y) = 2x^2 - 4x + xy^2 - 1$ for relative Extrema.

$f_x = 4x + y^2 - 4 = 0$

$f_y = 2xy = 0$

$\hookrightarrow x=0, y=0$

$y=0$

$4x + (0)^2 - 4 = 0$

$4x - 4 = 0$

$x = 1$ $(1,0)$

$x=0$

$4(0) + y^2 - 4 = 0$

$y = \pm 2$

$(0,2)$, $(0,-2)$

$f_{xx} = 4 > 0 \rightarrow$ at every point.

$f_{yy} = 2x$

$f_{xy} = 2y$

$f_{xx} f_{yy} - f_{xy}^2 > 0$

at $(1,0)$

$4(2x) - (2y)^2$

$8(1) - 0$

$8 > 0$

$(1,0)$ is relative minima.

at $(0,2)$

$4(2x) - (2y)^2$

$0 - 16 < 0$

$(0,2)$ is Saddle point. $(0,-2)$ is saddle point

Q) Examine $f(x,y) = 2x^2 - 4xy + 4xy^2 - 1$ for relative Extrema.

$f_x = 4x - 4y + 4y^2 = 0$

$f_y = -4x + 8xy = 0$

$4x(-1 + 2y) = 0$

$x = 0$, $y = \frac{1}{2}$

when $x=0$

$-4y + 4y^2 = 0$

$y(-1 + y) = 0$

$y = 0$ or $y = 1$

$(0,0)$, $(0,1)$

when $y = \frac{1}{2}$

$4x - 4(\frac{1}{2}) + 4(\frac{1}{2})^2 = 0$

$4x - 2 + 1 = 0$

$x = \frac{1}{4}$

$(\frac{1}{4}, \frac{1}{2})$

$f_{xx} = 4 > 0$

$f_{yy} = 8x$

$f_{xy} = -4 + 8y$

at $(\frac{1}{4}, \frac{1}{2})$

$4(8(\frac{1}{4})) - (-4 + 8(\frac{1}{2}))^2$

$8 > 0 \longrightarrow$

$f_{xx} f_{yy} - f_{xy}^2 > 0$

at $(0,0)$

$4(8x) - (-4 + 8y)^2$

$-16 < 0$

$(0,0)$ is saddle point

$(0,1)$ is saddle point

$(\frac{1}{4}, \frac{1}{2})$ is relative minima

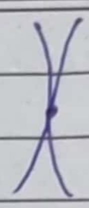
Date: ________

$\boxed{\nabla f = \lambda \nabla g}$ Constrained optimization $\rightarrow$ restriction

method of Lagrange Multipliers.

$f(x,y)$

$g(x,y)$



Q) $f(x,y) = x^{2/3} y^{1/3} \rightarrow (2.52)^{\frac{2}{3}}(1.26)^{\frac{1}{3}} = \boxed{2}$

$g(x,y) = x + y \leq 3.78$

$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}$

$\nabla f = \frac{2}{3} x^{-\frac{1}{3}} y^{\frac{1}{3}} \hat{i} + \frac{1}{3} x^{\frac{2}{3}} y^{-\frac{2}{3}} \hat{j}$

$\nabla g = \frac{\partial}{\partial x}(x+y-3.78)\hat{i} + \frac{\partial}{\partial y}(x+y-3.78)\hat{j}$

$\nabla g = 1\hat{i} + 1\hat{j}$

Using $\nabla f = \lambda \Delta g$

$\frac{2}{3} x^{-1/3} y^{1/3} \hat{i} + \frac{1}{3} x^{2/3} y^{-2/3} \hat{j} = \lambda(1\hat{i} + 1\hat{j})$

$\frac{2}{3} x^{-\frac{1}{3}} y^{\frac{1}{3}} = \lambda, \quad \frac{1}{3} x^{\frac{2}{3}} y^{-\frac{2}{3}} = \lambda$

$\boxed{\lambda = 0.53}$

$\frac{2}{3} x^{-\frac{1}{3}} y^{\frac{1}{3}} = \frac{1}{3} x^{\frac{2}{3}} y^{-\frac{2}{3}}$

$\frac{2x^{-\frac{1}{3}}}{x^{\frac{2}{3}}} = \frac{y^{-\frac{2}{3}}}{y^{\frac{1}{3}}}$

$2x^{-1} = y^{-1}$

$2y - x = 0$

$x = 2y$

$g(x,y) = x + y = 3.78$

$= 2y + y = 3.78$

$= \boxed{y = 1.26}$

$x = 2(1.26)$

~~$x = 1.26$~~

$\boxed{x = 2.52}$

Q) $f(x,y,z) = 2x - y + 3z$

$g = x^2 + y^2 + z^2 = 25$

$h: x + y - z = 0$

$\nabla f = \nabla g \lambda + \nabla h \mu$

$2i - j + 3k = (2x\hat{i} + 2y\hat{j} + 2z\hat{k})\lambda + (\hat{i} + j - k)\mu$

$2 = 2x\lambda + \mu$ — ① $\rightarrow \mu = 2 - 2x\lambda$

$-1 = 2y\lambda + \mu$ — ② $\rightarrow \mu = -1 - 2y\lambda$

$3 = 2z\lambda - \mu$ — ③ $\rightarrow \mu = 2z\lambda - 3$

① & ②

$2 - 2x\lambda = -1 - 2y\lambda$

$3 = -2y\lambda + 2x\lambda$

$2\lambda(-y + x) = 3$

$\lambda = \frac{3}{2(-y+x)}$

① & ③

$2 - 2x\lambda = 2z\lambda - 3$

$5 = 2z\lambda + 2x\lambda$

$5 = 2\lambda(z + x)$

$\lambda = \frac{5}{2(z+x)}$

$6(z + x) = 10(-y + x)$

Date: ______

Q) Find the extrema of $f(x,y,z) = x^2+y^2+z^2$ subject to $z^2 = x^2+y^2$ and $x+y-z+1=0$, using lagrange multiplier.

$\nabla f = \nabla g \lambda + \nabla h \mu$

$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda(-2x\hat{i} - 2y\hat{j} + 2z\hat{k}) + \mu(i+j-k)$

$2x = -2x\lambda + \mu$ — (1) $\rightarrow 2x + 2x\lambda = \mu$

$2y = -2y\lambda + \mu$ — (2) $\rightarrow 2y + 2y\lambda = \mu$

$2z = 2z\lambda - \mu$ — (3)

$2x + 2x\lambda = 2y + 2y\lambda$

$2x(1+\lambda) = 2y(1+\lambda)$

$2x(1+\lambda) - 2y(1+\lambda) = 0$

$(x-y)(1+\lambda) = 0$

$x - y = 0$

$x = y$

$z^2 = x^2 + y^2$

$\therefore x = y$

$z^2 = y^2 + y^2$

$z = \sqrt{2y^2}$

$z = \pm\sqrt{2}\, y$

$x + y - z + 1 = 0$

$y + y - \sqrt{2}y + 1 = 0$ | $y + y + \sqrt{2}y + 1 = 0$

$(2-\sqrt{2})y + 1 = 0$ | $(2+\sqrt{2})y + 1 = 0$

$\boxed{y = -\frac{1}{2-\sqrt{2}}}$ | $\boxed{y = -\frac{1}{2+\sqrt{2}}}$

$z = \sqrt{2}\left(\frac{-1}{2-\sqrt{2}}\right)$ | $z = -\sqrt{2}\left(\frac{-1}{2+\sqrt{2}}\right)$

$\boxed{z = \frac{-\sqrt{2}}{2-\sqrt{2}}}$ | $\boxed{z = \frac{+\sqrt{2}}{2+\sqrt{2}}}$

$\boxed{x = \frac{-1}{2-\sqrt{2}}}$ | $\boxed{x = \frac{-1}{2+\sqrt{2}}}$

$$\int_{-a}^{a} F(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad \therefore \text{ if even function.}$$

Date: ______

Q.) Find the volume of the region enclosed by surfaces.

$$z = x^2 + 3y^2 \qquad z = 8 - x^2 - y^2$$

$$\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$\iint z\,dy\,dx \Rightarrow \iint [8-x^2-y^2-(x^2+3y^2)]\,dy\,dx$$

$$\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{+\sqrt{\frac{4-x^2}{2}}} (8-2x^2-4y^2)\,dy\,dx.$$

$\therefore$ Even function

$$2\int_{-2}^{2}\int_{0}^{\sqrt{\frac{4-x^2}{2}}} (8-2x^2-4y^2)\,dy\,dx$$

$$2\int_{-2}^{2} \left. 8y - 2x^2y - \frac{4y^3}{3} \right|_{0}^{\sqrt{\frac{4-x^2}{2}}}$$

$$4\int_{-2}^{2} \left. 4y - x^2y - \frac{2y^3}{3} \right|$$

$$4\int_{-2}^{2} 4\left(\sqrt{\frac{4-x^2}{2}}\right) - x^2\left(\frac{4-x^2}{2}\right) - \frac{2}{3}\left(\frac{4-x^2}{2}\right)^{3/2}$$

$$4\int_{-2}^{2} \frac{\sqrt{4-x^2}}{\sqrt{2}}(4-x^2) - \frac{2}{3}\left[\frac{(4-x^2)^{3/2}}{2^{3/2}}\right]$$

$$4\int_{-2}^{2} \left[\frac{1}{2}(4-x^2)\right]^{3/2}\left[\frac{1}{\sqrt{2}} - \frac{2}{3\times 2^{3/2}}\right]dx$$

$$4\int_{-2}^{2} (4-x^2)^{3/2}\left[\frac{1}{\sqrt{2}} - \frac{2}{3\times 2^{3/2}}\right]dx \Rightarrow 4\left(\frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}}\right)\int_{-2}^{2}(4-x^2)^{3/2}\,dx$$

$$\frac{8}{3\sqrt{2}}\int_{-2}^{2}(4-x^2)^{3/2}\,dx \qquad \text{Let } x = 2\sin\theta,\ \frac{dx}{d\theta} = 2\cos\theta \Rightarrow dx = 2\cos\theta\,d\theta.$$

Limits: $-2 = 2\sin\theta$, $\theta = \sin^{-1}(-1)$, $\boxed{\theta = -\frac{\pi}{2}}$ $\boxed{\theta = \frac{\pi}{2}}$

$$\frac{8}{3\sqrt{2}}\int [4-4\sin^2\theta]^{3/2}\, 2\cos\theta\,d\theta \Rightarrow \frac{8}{3\sqrt{2}}\int 4^{3/2}(\cos^3\theta)\cos\theta\,d\theta = \frac{64}{3\sqrt{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\,d\theta$$

$$\frac{64}{3\sqrt{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\,d\theta$$

$$\int \cos^2\theta.\cos^2\theta\,d\theta$$

$$\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)$$

$$\frac{1}{4}\left[1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right]$$

$$\int \frac{1}{4} + \cos 2\theta + \frac{1}{4} + \frac{\cos 4\theta}{4}$$

$$\theta + 8\frac{\sin 2\theta}{2} + \frac{1}{4}\theta + \frac{\sin 4\theta}{16}$$

$$\frac{64}{3\sqrt{2}}\left[\theta + \frac{\sin 2\theta}{2} + \frac{\theta}{4} + \frac{\sin 4\theta}{16}\right]_{-\pi/2}^{\pi/2}$$

$$\frac{64}{3\sqrt{2}}\left[\frac{5\theta}{4} + \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{16}\right]_{-\pi/2}^{\pi/2}$$

MIGHTY PAPER PRODUCT

Date:

Line Integral:

Q) Compute the line integral $\int_C (4x3 + 2y)\,dx$, where C is the segment from $P(4,2,0)$ to $Q(8,0,4)$

$PQ = \langle 8-4, 0-2, 4-0 \rangle$

$= \langle \underset{a}{4}, \underset{b}{-2}, \underset{c}{4} \rangle$

$x = x_0 + at$, $y = y_0 + bt$, $3 = 3_0 + ct$.

$x = 4 + 4t$, $y = 2 - 2t$, $3 = 0 + 4t$

$0 \leq t \leq 1$ $\quad \frac{dx}{dt} = 0 + 4 \Rightarrow dx = 4dt$

$\int (4x3 + 2y) \cdot dx$

$\int_0^1 [4(4+4t)(4t) + 2(2-2t)] \cdot 4dt$

$4\int_0^1 (64t + 64t^2 + 4 - 4t)dt \Rightarrow 4\int_0^1 (64t^2 + 60t + 4)\,dt$

$4\left[\frac{64t^3}{3} + \frac{60t^2}{2} + 4t\right)\Big|_0^1$

$4\left[\frac{64(1)^3}{3} + \frac{60(1)}{2} + 4(1)\right] \Rightarrow \frac{664}{3} \Rightarrow \boxed{221.33}$

MIGHTY PAPER PRODUCT

Conversation Vector Field:

Date: ___________

- if $\nabla \times \vec{f} = 0$ vector field is conservative

Q) $\vec{f} = \langle 1, y, z^2 \rangle \quad \nabla = \langle \partial/\partial x, \partial/\partial y, \frac{\partial}{\partial z} \rangle$

$$\nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & y & z^2 \end{vmatrix}$$

$$\hat{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(1)\right] + k\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(1)\right]$$

$(0-0)\hat{i} - (0-0)\hat{j} + (0-0)k = 0$ $\therefore$ Vector field is conservative.

Lagrang Multiplier: $f: xyz$, $g: x+y+z=30$, $h = x+y-z=0$

$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k} \qquad \nabla g = i+j+k, \ \nabla h = i+j-k$

$yz = \lambda + \mu$ — ①
$xz = \lambda + \mu$ — ②
$xy = \lambda - \mu$ — ③

① & ② $yz - \cancel{\lambda} = xz - \cancel{\lambda}$
$yz = xz$
$\boxed{x = y}$

② & ③ $xz - \lambda = \lambda + xy$
$2\lambda = xy - xz$

$x+y-z=0$
$y+y-z=0$
$2y-z=0$
$z=2y$
$z=2\left(\frac{15}{2}\right)$
$\boxed{z=15}$

$x+y+z=30$
$y+y+z=30$
$2y+2y=30$
$y=\frac{30}{4}$
$\boxed{y=\frac{15}{2}}$

$F = \frac{15}{2} \times \frac{15}{2} \times 15$
$= \boxed{843.75}$

$\left(\frac{15}{2}, \frac{15}{2}, 15\right)$

Assignment

Date: ____________

Q1) (i) $\int_0^4\int_{y/2}^{2} e^{x^2}\,dx\,dy$

$\int_0^4\int_{y/2}^{2} e^{x^2}\,dx\,dy$

Change of order

$\int_0^2\int_0^{2x} e^{x^2}\,dy\,dx$

$\int_0^{2x} e^{x^2}\,dy$

$ye^{x^2}\Big|_0^{2}$

$(2x)e^{x^2}-0(e^{x^2})$

$\int_0^2 2xe^{x^2}\,dx$

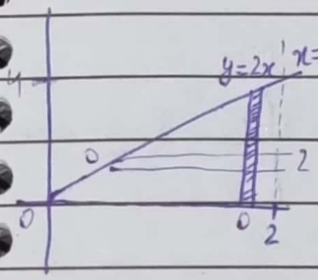
$u=x^2$, $du=2x\cdot dx$

$\int_0^2 e^u\cdot du$

$e^u\Big|_0^2 \Rightarrow e^{x^2}\Big|_0^2$

$e^{(2)^2}-e^{(0)^2}$

$\boxed{e^4-1}$



(ii) $\int_0^2\int_{y^2}^4 y\sin x^2\,dx\,dy$

$\int_0^4\int_0^{\sqrt{x}} y\sin x^2\,dy\,dx$

$\sin x^2\int_0^{\sqrt{x}} y\,dy$

$\frac{y^2}{2}\sin x^2\Big|_0^{\sqrt{x}}$

$\int_0^4 \frac{x}{2}\sin x^2\,dx$

$\frac{1}{4}\int_0^4 2x\sin x^2\,dx$

$u=x^2$, $\frac{du}{dx}=2x \Rightarrow dx=\frac{du}{2x}$

$\frac{1}{4}\int_0^4 \sin u\,du$

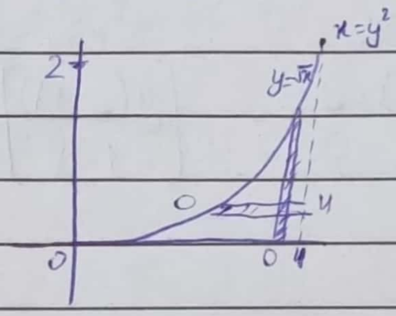
$-\frac{1}{4}\cos u\Big|_0^4$

$-\frac{1}{4}\cos x^2\Big|_0^4$

$-\frac{1}{4}\cos(4)^2+\frac{1}{4}\cos(0)^2$

$-\frac{1}{4}\cos 16+\frac{1}{4}$

$\boxed{\frac{1}{4}(-\cos(16)+1)}$



(iii) $\int_0^4\int_{\sqrt{y}}^2 y\sin x^5\,dx\,dy$

$\int_0^2\int_0^{x^2} y\sin x^5\,dy\,dx$

$\sin x^5\int_0^{x^2} y\cdot dy \Rightarrow \frac{y^2}{2}\cdot\sin x^5\Big|_0^{x^2}$

$\frac{x^4}{2}\cdot\sin x^5$

$\int_0^2 \frac{5\times x^4\cdot\sin x^5}{5\times 2}\,dx$

$u=x^5$, $du=5x^4\cdot dx$

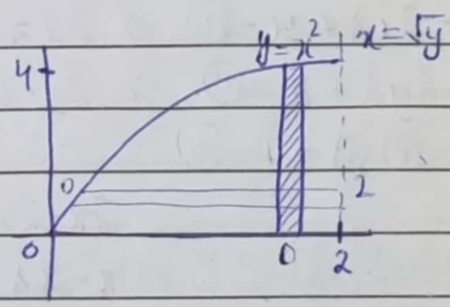
$\frac{1}{10}\int_0^2 \sin u\cdot du \Rightarrow \frac{1}{10}\cdot-\cos u\Big|_0^2$

$-\frac{1}{10}\cos x^5\Big|_0^2$

$-\frac{1}{10}\cos(2)^5+\frac{1}{10}\cos(0)^5$

$-\frac{1}{10}\cos(32)+\frac{1}{10}$

$\boxed{\frac{1}{10}(-\cos(32)+1)}$



Date: ________

Q.2)(a)

$f(x,y) = y^2 - e^{x^2}$

$F_x = 0 - 2xe^{x^2}$

$F_y = 2y$

$2y = 0$ | $-2xe^{x^2} = 0$

$y = 0$ | $x = 0$

$(0,0)$

$F_{xx} = -2e^{x^2} + 4x^2 \cdot e^{x^2}$

$F_{yy} = 2$

$F_{xy} = 0$

$F_{xx} \cdot F_{yy} - F_{xy}^2$

$(-2e^{x^2} + 4x^2 \cdot e^{x^2}) \cdot 2 - 0$

$(-2) \cdot 2 - 0$

-4

$\therefore -4 < 0$, so $(0,0)$ is saddle point.

Q.2)(b) $f = x^2 + y^2 + z^2$, $g = (x,y) = x^2 + 3y^2 - 5z^2 = 0$, $h = x + y - z = 2$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda(2x\hat{i} + 6y\hat{j} - 10z\hat{k}) + \mu(\hat{i} + \hat{j} - \hat{k})$

$2x = 2x\lambda + \mu$ — (I) $\rightarrow 2x = 2x\lambda = \mu$

$2y = 6y\lambda + \mu$ — (II) $\rightarrow 2y - 6y\lambda = \mu$

$2z = -10z\lambda - \mu$ — (III)

$2x - 2x\lambda = 2y - 6y\lambda$

$x - x\lambda - y + 3y\lambda = 0$

$x(1-\lambda) - y(1-3\lambda) = 0$

$x = \dfrac{y(1-3\lambda)}{1-\lambda}$

$x + y - z - 2 = 0$ — (IV)

$\dfrac{y(1-3\lambda)}{1-\lambda} + y - 2 = z$

$x^2 + 3y^2 - 5z^2 = 0$ — (V)

$\left[\dfrac{y(1-3\lambda)}{1-\lambda}\right]^2 + 3y^2 - 5\left[\dfrac{y(1-3\lambda)}{1-\lambda} + y - 2\right.$

Date: ____________

(Q.4)(a) $\int_C (4xz + 2y)\,dx$, From $\overset{P}{(4,0,2)}$ to $\overset{Q}{(2,1,0)}$

$PQ = \langle 2-4,\ 1-0,\ 0-2 \rangle$; $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$

$= \langle \underset{a}{-2}, \underset{b}{1}, \underset{c}{-2} \rangle$; $x = 4 - 2t$, $y = 0 + t$, $z = 2 - 2t$

$2 \leq x \leq 4$

$x = 4 - 2t$	$x = 4 - 2t$
$(2) = 4 - 2t$	$(4) = 4 - 2t$
$t = 1$	$t = 0$

$0 \leq t \leq 1$

$\frac{dx}{dt} = -2$, so $dx = -2\,dt$

$$\int_C 4xz + 2y$$

$$-2\int_0^1 4(4-2t)(2-2t) + 2(t) \cdot dt$$

$$(16-8t)(2-2t) + 2t \cdot dt$$

$$32 - 32t - 16t + 16t^2 + 2t$$

$$-2\int_0^1 16t^2 - 46t + 32 \cdot dt$$

$$-2\left[\frac{16t^3}{3} - \frac{46t^2}{2} + 32t\right]\Big|_0^1$$

$$-2\left[\frac{16}{3} - \frac{46}{2} + 32\right]$$

$$-2\left[\frac{43}{3}\right]$$

$$\boxed{-\frac{86}{3}}$$

(Q.4)(b) $F = \langle xe^z, xyz, -4y^2 \rangle$, conservative?

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^z & xyz & -4y^2 \end{vmatrix}$$

$$i\left[\frac{\partial}{\partial y}(-4y^2) - \frac{\partial}{\partial z}(xyz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-4y^2) - \frac{\partial}{\partial z}(xe^z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial}\right.$$

$$[-8y - xy]\hat{i} - (0 - xe^z)\hat{j} + (yz - 0)\hat{k}$$

$$(-8y - xy)\hat{i} - (-xe^z)\hat{j} + (yz)\hat{k}$$

$\therefore -8y \neq xy$, $xe^z \neq 0$, $yz \neq 0$

so vector field isn't conservative

Date:

Q.3) $\displaystyle\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$

$\displaystyle\int_{x^2+3y^2}^{8-x^2-y^2} dx$

$z\Big|_L^u$

$(8-x^2-y^2)-(x^2+3y^2)$

$8-x^2-y^2-x^2-3y^2$

$\displaystyle\iint 8-2x^2-4y^2\,dy\,dx$

↳ Even Function

$\displaystyle 2\int 8y-2x^2y-\frac{4y^3}{3}\Big|_0^u$

$\displaystyle 2\int_{-2}^{2} 8\left(\sqrt{\frac{4-x^2}{2}}\right)-2x^2\left(\sqrt{\frac{4-x^2}{2}}\right)-\frac{4\left(\sqrt{\frac{4-x^2}{2}}\right)^3}{3}$

$\displaystyle 4\left[4\sqrt{\frac{4-x^2}{2}}-x^2\sqrt{\frac{4-x^2}{2}}-\frac{2\left(\sqrt{\frac{4-x^2}{2}}\right)^3}{3}\right]$

$\displaystyle 4\left[\sqrt{\frac{4-x^2}{2}}(4-x^2)-\frac{2}{3}\left(\sqrt{\frac{4-x^2}{2}}\right)^{3/2}\right]$

$\displaystyle 4\left[\frac{(4-x^2)^{1/2}}{\sqrt{2}}(4-x^2)-\frac{2}{3}\left(\frac{(4-x^2)^{3/2}}{2^{3/2}}\right)\right]$

$\displaystyle 4\left[(4-x^2)^{3/2}\left(\frac{1}{\sqrt{2}}-\frac{2}{3\times2^{3/2}}\right)\right.$

$4\times0.5\,[4-x^2]^{3/2}$

$\displaystyle 1.89\int_{-2}^{2}(4-x^2)^{3/2}\,dx$

Even Function

$\displaystyle 3.78\int_0^2(4-x^2)^{3/2}\,dx$

Calaulator : $\boxed{8\sqrt{2}\pi}$

MIGHTY PAPER PRODUCT

CamScanner