CS6015 Linear Algebra and Random Processes

Muhammad Arsath K F(CS16S035) and Srihari Maruthachalam (CS16S024)

Programming Assignment-I

Group I

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1 Projection of Matrix

1.1 Column space and Row space of Matrix

Given: $A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$ and $A^T = \begin{bmatrix} 3 & 4 \\ 6 & 8 \\ 6 & 8 \end{bmatrix}$

Column space of $A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and Row space of $A = \begin{bmatrix} 3 & 6 & 6 \end{bmatrix}$

$P_C = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}$ and $P_R = \begin{bmatrix} 0.11 & 0.22 & 0.22 \\ 0.22 & 0.44 & 0.44 \\ 0.22 & 0.44 & 0.44 \end{bmatrix}$

1.2 Projection of Matrix

Projection matrix: $(P_C)_{2\times 2} = A_{2\times 3}\ (A^T_{3\times 2} A_{2\times 3})^{-1}_{3\times 3}\ A^T_{3\times 2}$ and $(P_R)_{3\times 3} = A^T_{3\times 2}\ (A_{2\times 3} A^T_{3\times 2})^{-1}_{2\times 2}\ A_{2\times 3}$

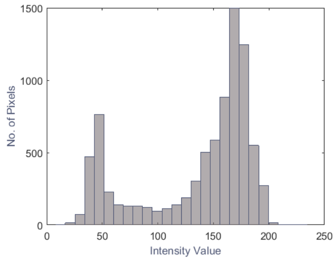
$$
\begin{aligned}
B &= P_C A P_R \\
&= (A(A^T A)^{-1} A^T) A (A^T (AA^T)^{-1} A) \\
&= A \underbrace{(A^T A)^{-1}} \underbrace{A^T A} (A^T (AA^T)^{-1} A) \\
&= \underbrace{AA^T} \underbrace{(AA^T)^{-1}} A \\
&= A
\end{aligned}
$$

Inference

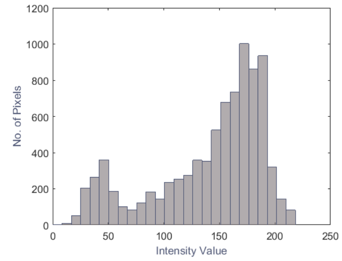
- Given matrix A is of rank 1.
- We used Reduced Row Echelon Form ("rref") to get linearly independent row space and column space of the matrix.
- Another interesting observation is that after projecting given matrix to column space and row space, we got the same matrix. The detailed theoretical proof is shown above.
- When we apply projection on transpose of the given matrix $P_R * A^T * P_C$, we get A^T.
- When we apply column space projection on the matrix P_C * A, we got A. Similarly A * P_R gives A.

2 Reconstruction of Images from Eigen Bases

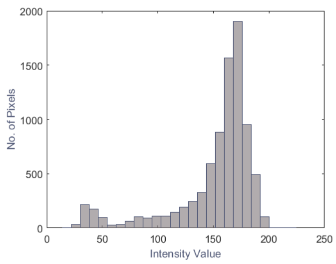
2.1 Pixel Intensity Histograms



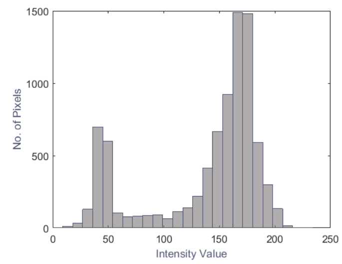
(a) face1.pgm



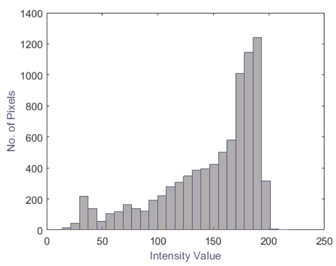
(b) face2.pgm



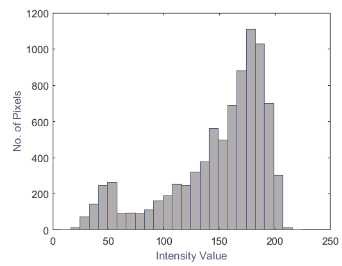
(c) face3.pgm



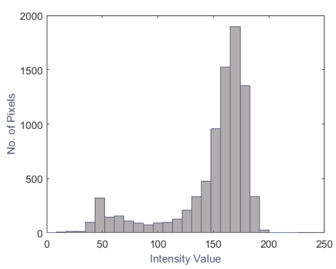
(d) face4.pgm



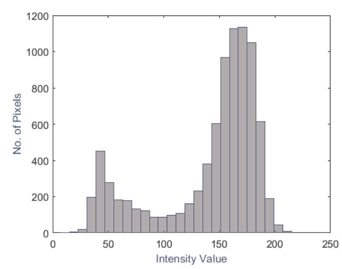
(e) face5.pgm



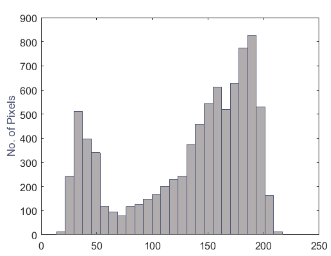
(f) face6.pgm



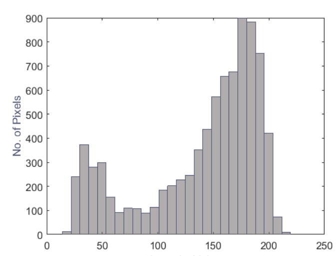
(g) face7.pgm



(h) face8.pgm



(i) face9.pgm



(j) face10.pgm

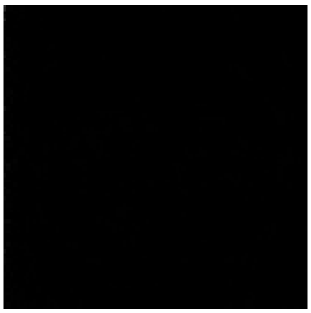
Figure 1: Histograms of pixel intensities of given images

Intuition and Inference

- Given images are grey-scale and presence of white and black shades are predominant. From that inference, we expected an “U” shaped histogram(or bi modal histogram) for pixel intensities.

2.2 Construction of images using Eigen bases

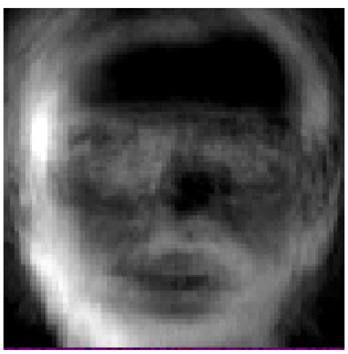
We tried to plot the images from the given set of Eigen bases. The following plotted images are in the order as they are given.



(a) Combining all 8464 Eigen bases

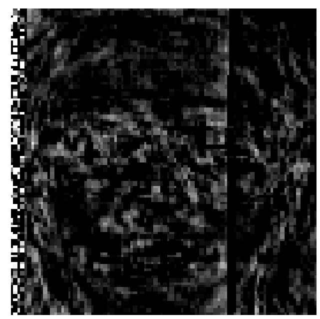
(b) First Eigen basis scaled by 10000

(c) First 2 Eigen bases scaled by 6000



(d) First 5 Eigen bases scaled by 4000

(e) First 100 Eigen bases scaled by 800



(f) First 6000 Eigen bases scaled by 180

Figure 2: Images from given Eigen bases

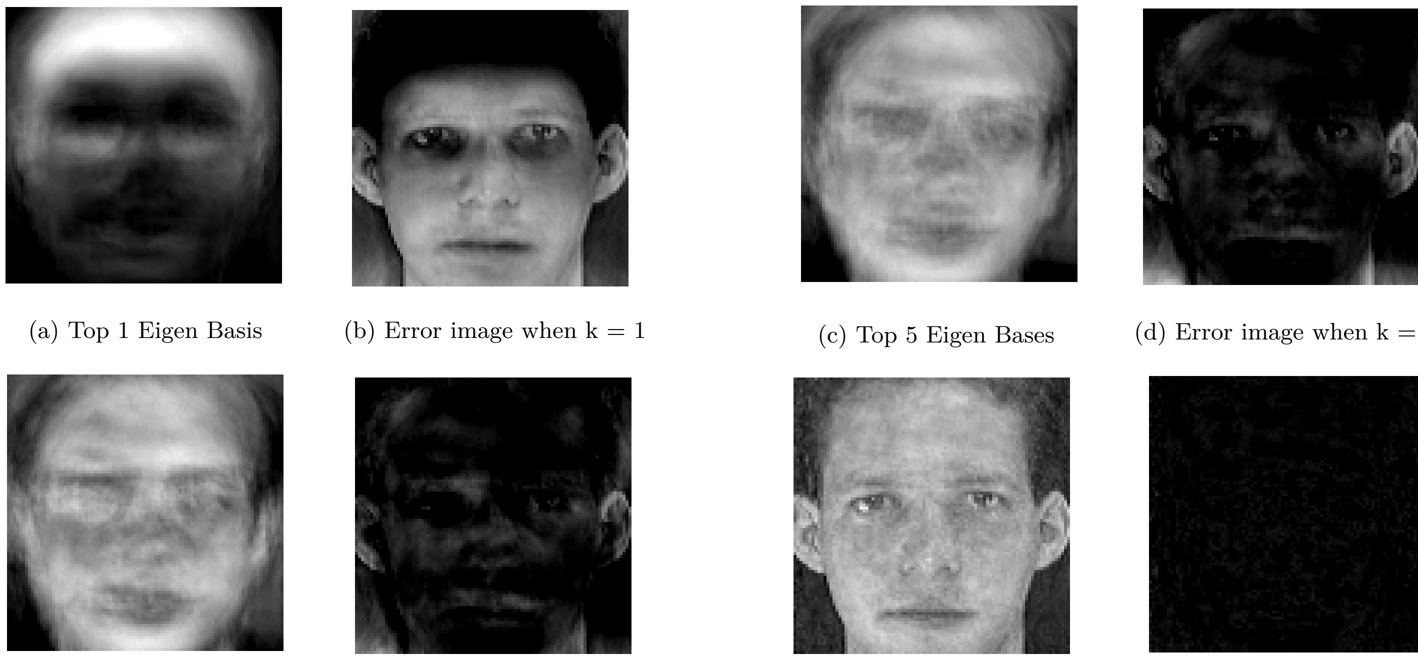
Inference

- If we include all the Eigen bases, we didn’t get any information in the plotted images.
- For first Eigen basis, we can see a blurred human face.
- As we increase the Eigen bases, we could see the evolution of human face.
- But after including most of the Eigen Bases, human face got collapsed. This is because Eigen bases are extracted from various postures of the human face.

2.3 Reconstruction of Given Images

2.3.1 Evolution of an Image with Eigen Bases

We tried to plot the images as we increase the Eigen Bases to show the evolution of the human face. The order of the Eigen Bases is based on the value of the coefficient of the Eigen bases.



(a) Top 1 Eigen Basis
(b) Error image when k = 1
(c) Top 5 Eigen Bases
(d) Error image when k = 5
(e) Top 10 Eigen Bases
(f) Error image when k = 10
(g) Top 1000 Eigen Bases
(h) Error image when k = 1000

Figure 3: Evolution of face1.pgm with Top K Eigen Bases

2.3.2 Reconstruction of Images with Eigen Bases

We reconstructed the given images with the Eigen bases using coefficients. We increased Eigen bases to obtain relative error(Frobenius norm) less than 1%. The Frobenius norm is a matrix norm of an m x n matrix A is defined as the square root of the sum of the absolute squares of its elements,

$$||A||_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}$$

The formula of relative error is,

$$\frac{||A - \hat{A}||_F}{||A||_F}$$

where,
A is original image, and
$\hat{A}$ is the reconstructed image.

(a) K=5738
Error=0.00999645

(b) K=5679
Error=0.00999989

(c) K=5612
Error=0.00999847

(d) K=5717
Error=0.00999992

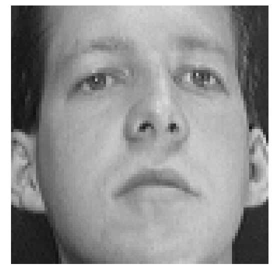
(e) K=5632
Error=0.00999683

(f) K=5613
Error=0.00999763

(g) K=5628
Error=0.00999658

(h) K=5700
Error=0.00999701

(i) K=5736
Error=0.00999736



(j) K=5689
Error=0.00999717

Figure 4: Reconstruction of given images with error $< 1\%$

Intuition and Inference

- We projected each image into the Eigen bases and obtained the corresponding coefficients. We sorted the absolute value of coefficients to fetch the Eigen bases contributing to the image.
- We increased the Eigen bases until we get the relative error is less than 1%.
- For similar posture of human faces, the similar set of Eigen basis responded with its coefficient.
- Face 1 and Face 4 gives similar pixel density distribution and Face 8 and Face 10 gives similar pixel density distribution as they have similar posture.

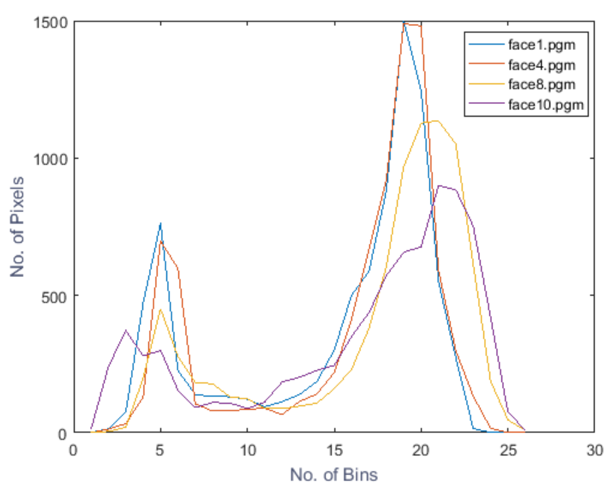


Figure 5: Pixel density distribution of Face 1 & Face 4 and Face 8 & Face 10

2.4 Intuition for Computational Optimization

We know that computing outer product of vector sized 8464 x 1 is a very intensive computational task and the machine will take a lot of time to compute. So we came up with an efficient computational paradigm to solve this problem.

We know that projection of the image vector into the Eigen basis is obtained by,

Obtained projection vector = $\frac{Basis*Basis^T}{Basis^T*Basis} * Image\ vector$

Now, we want to find the coefficient of the Basis. In order to do that, it is enough to get the scaling value of only one component of the Basis and Image vector.

$\hat{x} = \frac{First\ Component\ of\ Obtained\ projection\ vector}{First\ Component\ of\ Eigen\ vector}$

So we just found the first component of the Obtained projection vector.

$$\underbrace{\begin{bmatrix} . & . & . \\ & & \\ & & \end{bmatrix}}_{Basis*Basis^T} * \underbrace{\begin{bmatrix} . \\ . \\ . \end{bmatrix}}_{\text{Image vector}} = \underbrace{\begin{bmatrix} \square \\ . \\ . \end{bmatrix}}_{\text{Obtained projection vector}}$$

Now, it is enough to calculate the first row of the $Basis * Basis^T$ matrix. In order to do that the first component of the Basis vector should be multiplied with the $Basis^T$ vector.

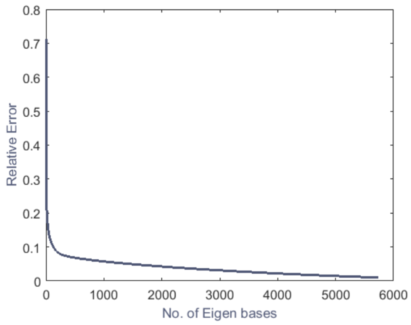
$$\text{The first component of the Basis vector } * \underbrace{\begin{bmatrix} . & . & . \end{bmatrix}}_{\text{Basis}^T} * \underbrace{\begin{bmatrix} . \\ . \\ . \end{bmatrix}}_{\text{Image vector}} = \underbrace{\square}_{\text{First component of obtained projection vector}}$$

Now we obtain the coefficient for the Basis.

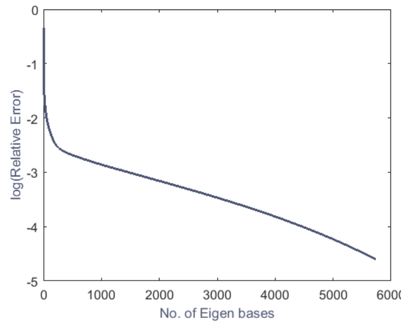
- Earlier, computation of outer product took **1.5 hours per image**. But, after the optimization, the running time was **15.589356 seconds per ten images**.

2.5 Analysis of errors and coefficients of reconstruction

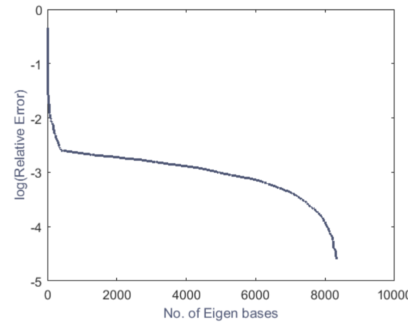
2.5.1 Analysis of errors during reconstruction



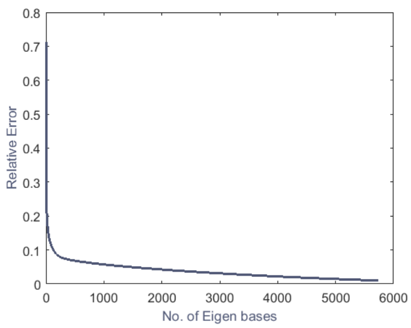
(a) Relative Error for face1.pgm Top K Eigen Bases



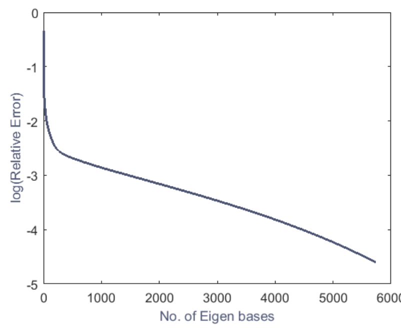
(b) Logarithmic Relative Error for face1.pgm Top K Eigen Bases



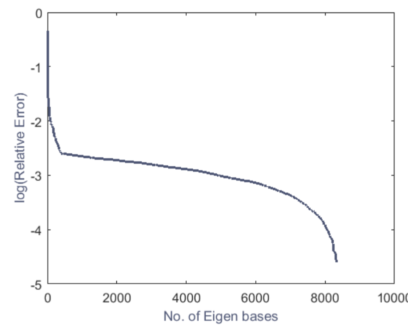
(c) Logarithmic Relative Error for face1.pgm Given K Eigen Bases



(d) Relative Error for face5.pgm Top K Eigen Bases



(e) Logarithmic Relative Error for face5.pgm Top K Eigen Bases



(f) Logarithmic Relative Error for face5.pgm Given K Eigen Bases

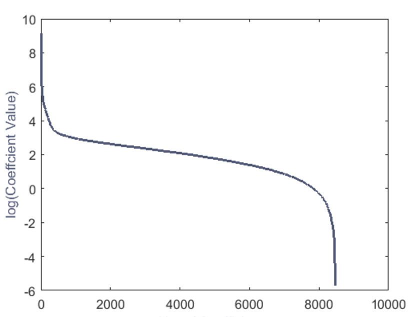
Figure 6: Error plots during image reconstruction

Inference

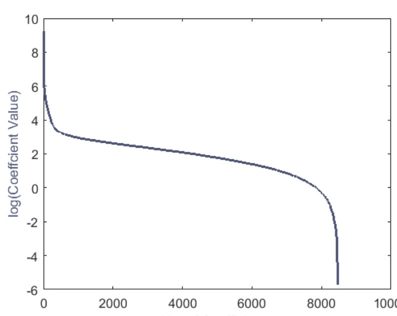
- Error curve is smooth if Eigen bases are chosen as in the increasing order of absolute coefficients.
- Error curve is flickering if Eigen bases are chosen as in the given order.

2.5.2 Analysis of coefficients during reconstruction

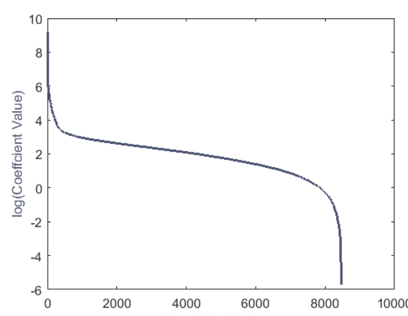
Sorted coefficients obtained are plotted against its Eigen bases.



(a) Logarithmic coefficients for face1.pgm



(b) Logarithmic coefficients for face5.pgm



(c) Logarithmic coefficients for face10.pgm

Figure 7: Sorted coefficients obtained during image reconstruction

Inference

- We can see that first few sorted coefficients are very large.
- As we increase Eigen bases, the value of coefficients is almost same.
- Last few coefficients are not contributing to the image.

3 References

- Gilbert Strang, “Linear algebra and applications”, CENGAGE LEARNING, 2007.
- Edgar Goodaire, “Linear Algebra: Pure and Applied”, World Scientific Publishing Co Pte Ltd, 2013.
- Linear Algebra videos by Strang
- https://www.mathworks.com/