MACT 4212: Stochastic Processes

Assignment 2

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Question 1

Consider the following Markov chain with state space $\{1, 2, 3\}$ and transition matrix and Find:

$$P = \begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.5 & 0 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}$$

1. $P[X_2 = 2|X_0 = 1]$.

$$P^2 = P * P = \begin{pmatrix} 0.44 & 0.16 & 0.4 \\ 0.225 & 0.525 & 0.25 \\ 0.3 & 0.325 & 0.375 \end{pmatrix}$$

$P[X_2 = 2|X_0 = 1] = 0.16$

$P[X_2 = 2|X_0 = 1]$

$\qquad = P[X_2 = 2|X_1 = 1]P[X_1 = 1|X_0 = 1]$

$\qquad + P[X_2 = 2|X_1 = 2]P[X_1 = 2|X_0 = 1]$

$\qquad + P[X_2 = 2|X_1 = 3]P[X_1 = 3|X_0 = 1] = 0.8 * 0.2 + 0 * 0.8 + 0.25 * 0$

$\qquad = 0.16 + 0 + 0 = 0.16$

2. $P[X_3 = 2|X_0 = 1, X_2 = 1]$.

$P[X_3 = 2|X_0 = 1, X_2 = 1] = P[X_3 = 2|X_2 = 1] = 0.8$

3. $P[X_1 = 2|X_2 = 1]$.

$$P[X_1 = 2|X_2 = 1] = \frac{P[X_1 = 2, X_2 = 1]}{P[X_2 = 1]}$$

$$P[X_1 = 2, X_2 = 1] = P[X_2 = 1|X_1 = 2]P[X_1 = 2]$$

$$\pi^T * P^2 = [0.3055 \quad 0.352 \quad 0.3425]$$

$$P[X_2 = 1] = 0.3055$$

$$\pi^T * P = [0.315 \quad 0.285 \quad 0.4]$$

$$P[X_1 = 2] = 0.285$$

$$P[X_1 = 2, X_2 = 1] = P[X_2 = 1|X_1 = 2]P[X_1 = 2] = 0.5 * 0.285 = 0.1425$$

$$P[X_1 = 2|X_2 = 1] = \frac{0.1425}{0.3055} = 0.4664484452 \approx 0.466$$

4. $P[X_2 = 3]$ given that $\pi^T = [0.2 \quad 0.3 \quad 0.5]$

$$P^2 = P * P = \begin{pmatrix} 0.44 & 0.16 & 0.4 \\ 0.225 & 0.525 & 0.25 \\ 0.3 & 0.325 & 0.375 \end{pmatrix}$$

$$\pi^T * P^2 = [0.3055 \quad 0.352 \quad 0.3425]$$

$$P[X_2 = 3] = 0.3425$$

Question 2

Three white and three black balls are distributed equally into two urns. We say that the system is in state i, $i = 0,1,2,3$, if the first urn contains 3 white balls. At each step, we draw one ball from each urn and exchange them. Let X_n denote the state of the system after the n-th step. Find the transition probability matrix.

This is a Markov chain where the system's state is specified by the number of white balls in the first urn.

There are four states:
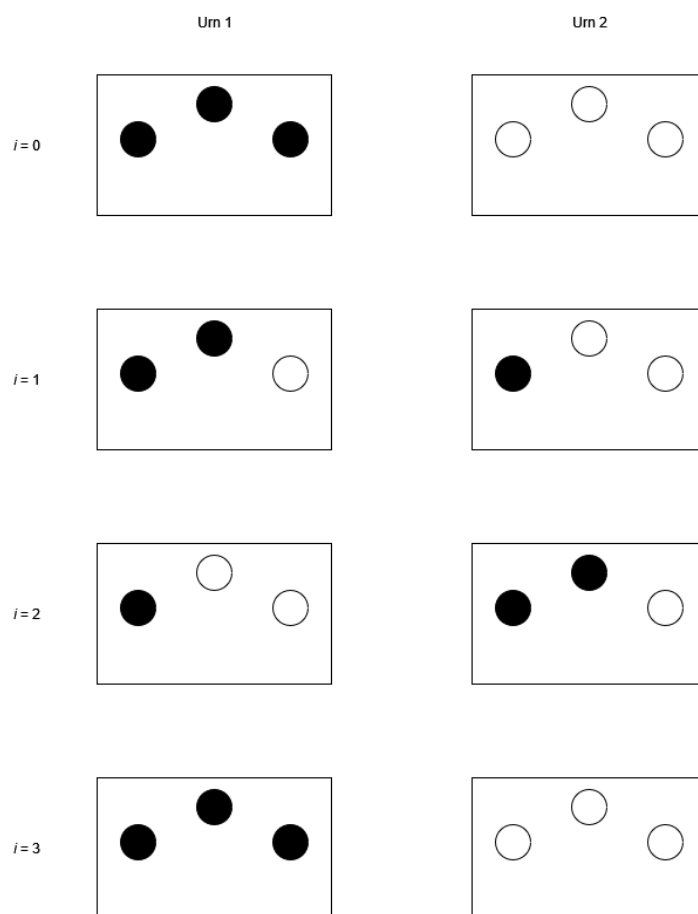
The diagram above shows both urns with the one on the left being the first urn in which we count the balls.

According to the problem statement, we can say that this is a Discrete-Time Markov Chain (DTMC). The probability of the next state is dependent only on the current state. If we are in a state, it does not matter how we arrived at that state.

For instance,

$$P[X_{n+1} = 2|X_n = 2] = \frac{4}{9}$$

Since we need to choose either of the white balls with a probability of 2/3 in the first urn and the white ball in the second urn with a probability of 1/3 or either of the black balls in the second urn with a probability of 2/3 and the black ball in the first urn with a probability of 1/3. Events are independent. It is impossible for the system to remain in state 3. If the system is in state 3, it is certainly going to transition to state 2.

$P[X_{n+1} = 0|X_n = 0] = 0$

$P[X_{n+1} = 1|X_n = 0] = 1$

$P[X_{n+1} = 2|X_n = 0] = 0$

$P[X_{n+1} = 3|X_n = 0] = 0$

$P[X_{n+1} = 0|X_n = 1] = \frac{1}{9}$

$P[X_{n+1} = 1|X_n = 1] = \frac{4}{9}$

$P[X_{n+1} = 2|X_n = 1] = \frac{4}{9}$

$P[X_{n+1} = 3|X_n = 1] = 0$

$P[X_{n+1} = 0|X_n = 2] = 0$

$P[X_{n+1} = 1|X_n = 2] = \frac{4}{9}$

$$P[X_{n+1} = 2 | X_n = 2] = \frac{4}{9}$$

$$P[X_{n+1} = 3 | X_n = 2] = \frac{1}{9}$$

$$P[X_{n+1} = 0 | X_n = 3] = 0$$

$$P[X_{n+1} = 1 | X_n = 3] = 0$$
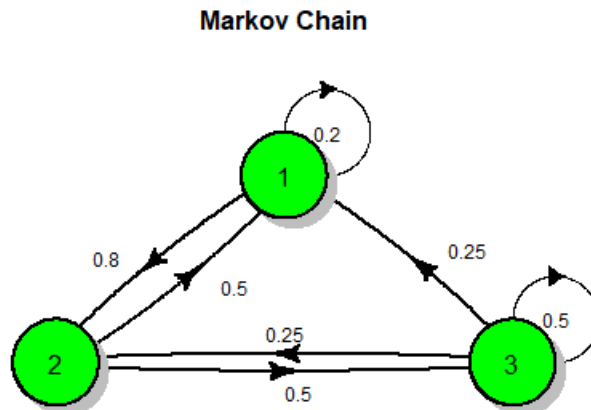
$$P[X_{n+1} = 2 | X_n = 3] = 1$$

$$P[X_{n+1} = 3 | X_n = 3] = 0$$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \dfrac{1}{9} & \dfrac{4}{9} & \dfrac{4}{9} & 0 \\ 0 & \dfrac{4}{9} & \dfrac{4}{9} & \dfrac{1}{9} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Question 3: R application

Open the R code attached, modify and run the code to answer the following questions

1. Draw the transition diagram in question 1 using the R code provided. Label the states as 1, 2, and 3.

Markov Chain

2. Find $P(X_4 = 2|X_0 = 1)$ using the R code.

$P(X_4 = 2|X_0 = 1) = 0.2844$

Assignment 2 Question 3

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```r
install.packages("markovchain", repos = "https://cran.r-project.org")
```

```
## Installing package into 'C:/Users/muhammadazzazy/AppData/Local/R/win-library/4.4'
## (as 'lib' is unspecified)
```

```
## package 'markovchain' successfully unpacked and MD5 sums checked
##
## The downloaded binary packages are in
##   C:\Users\muhammadazzazy\AppData\Local\Temp\RtmpUV0DDZ\downloaded_packages
```

```r
install.packages("diagram", repos = "https://cran.r-project.org")
```

```
## Installing package into 'C:/Users/muhammadazzazy/AppData/Local/R/win-library/4.4'
## (as 'lib' is unspecified)
```

```
## package 'diagram' successfully unpacked and MD5 sums checked
##
## The downloaded binary packages are in
##   C:\Users\muhammadazzazy\AppData\Local\Temp\RtmpUV0DDZ\downloaded_packages
```

```r
library(markovchain)
```

```
## Package:  markovchain
## Version:  0.9.5
## Date:     2023-09-24 09:20:02 UTC
## BugReport: https://github.com/spedygiorgio/markovchain/issues
```

```r
library(diagram)
```

```
## Loading required package: shape
```

```r
###### Simulate discrete-time Markov chain ######################
# Simulates n steps of a Markov chain
# markov(init,mat,n,states)
# Generates X0, ..., Xn for a Markov chain with initial
#  distribution init and transition matrix mat
# Labels can be a character vector of states; default is 1, .... k

markov <- function(init,mat,n,labels) {
  if (missing(labels)) labels <- 1:length(init)
  simlist <- numeric(n+1) #creating a vector of zeros

  states <- 1:length(init)
  simlist[1] <- sample(states,1,prob=init)
  for (i in 2:(n+1))
  { simlist[i] <- sample(states,1,prob=mat[simlist[i-1],]) }
```

```
    labels[simlist]
}
# define a transition matrix
tmQ1 <- matrix(c(0.2,0.8,0,0.5,0,0.5,0.25,0.25,0.5),nrow = 3, byrow = TRUE)
# create the DTMC
dtmcQ1 <- new("markovchain",transitionMatrix=tmQ1, states=c("1","2","3"),
            name="MarkovChain")
dtmcQ1
```

```
## MarkovChain
##  A  3 - dimensional discrete Markov Chain defined by the following states:
##  1, 2, 3
##  The transition matrix  (by rows)  is defined as follows:
##      1    2   3
## 1 0.20 0.80 0.0
## 2 0.50 0.00 0.5
## 3 0.25 0.25 0.5
```
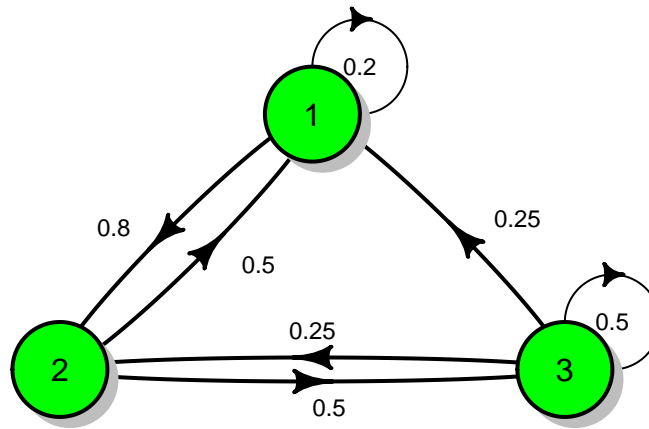
```
plot(dtmcQ1)
```
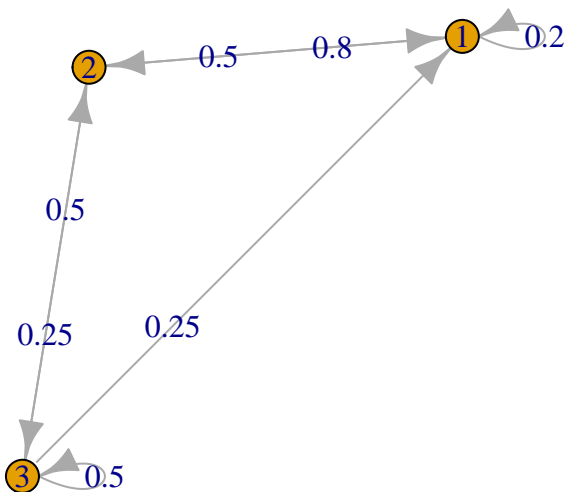


```
stateNames <- c("1","2","3")
row.names(tmQ1) <- stateNames; colnames(tmQ1) <- stateNames
plotmat(t(tmQ1), pos = c(1, 2), box.type = "circle", main = "Markov Chain",
        box.col = "green", box.size=0.047, cex.txt=0.75, dtext=0.75)
```

Markov Chain


```r
# It is possible to simulate states distribution after n-steps
init<-c(1,0,0)
steps<-4
finalState<-init*dtmcQ1^steps #using power operator
finalState
```

```
##           1      2     3
## [1,] 0.3496 0.2844 0.366
```

```r
finalState[2]
```

```
## [1] 0.2844
```