Computer Science BSc Analyisis-2 Practices

Semester 1 (Autumn) in the academic year 2024/2025. Edited by Csörgő István.

The abbreviation „AN-2" denotes the following reference:

Analysis 2 lecture schemes, written by István Csörgő

The abbreviation „AN-3" denotes the following reference:

Analysis 3 lecture schemes, written by István Csörgő

They can be found in the Digital Library of the Faculty of Informatics or they can be opened from CANVAS

Remember that $\mathbb{N} = \{1, 2, 3, \ldots\}$.

week 1 (9-13 of September, 2024):

1. *(to Lesson 1 in „AN-2")*

   Discuss the continuity of the following functions (at which points of the domain it is continuous, at which points is it not, the type of the discontinuities, e.t.c.):

   a)

$$f(x) := \begin{cases} \dfrac{x^2 - 5x + 6}{x^2 - 7x + 10} & \text{if } x \in \mathbb{R},\ x \neq 2,\ x \neq 5 \\ 0 & \text{if } \quad x \in \{2; 5\} \end{cases}$$

   b)

$$f(x) := \begin{cases} \dfrac{3 - \sqrt{x}}{9 - x} & \text{if } x \geq 0,\ x \neq 9 \\ 0 & \text{if } \quad x = 9 \end{cases}$$

   c)

$$f(x) := \begin{cases} \dfrac{x - 7}{|x - 7|} & \text{if } x \neq 7 \\ 5 & \text{if } x = 7 \end{cases}$$

   d)

$$f(x) := \begin{cases} 2x + 1 & \text{if } \quad x \leq -1 \\ 3x & \text{if } -1 < x < 1 \\ 2x - 1 & \text{if } \quad x \geq 1 \end{cases}$$

e)

$$f(x) := \begin{cases} \dfrac{1}{x+1} & \text{if} \quad x < -1 \\ x & \text{if} \quad -1 \leq x < 0 \\ \ln(x+1) & \text{if} \quad x \geq 0 \end{cases}$$

2. Determine $f(0)$, such that the extended function $f : \mathbb{R} \to \mathbb{R}$ will be continuous:

$$a) \quad f(x) = \frac{(1+x)^n - 1}{x} \quad (x \neq 0, \ n \in \mathbb{N}) \qquad\qquad b) \quad f(x) = \frac{1 - \cos x}{x^2} \quad (x \neq 0)$$

Homework to this topic: „AN-2" p. 8, ex. 1., 2., 3.

3. *(to Lesson 2 in „AN-2")*

   Prove that the given equations have roots in the given intervals.

   a) $x^3 - 3x + 1 = 0$ in the interval $(0,\,1)$. Compute the first 3 terms of the sequence that approximates the root. Estimate the error of approximation with this 3-rd term.

   b) $x^2 = \sqrt{x+1}$ in the interval $(1,\,2)$

   c) $\cos x = x$ in the interval $(0,\,\frac{\pi}{2})$

   d) $e^x = 2 - x$ in the interval $\mathbb{R}$

   e) $x^5 - x^2 + 2x + 3 = 0$ in the interval $\mathbb{R}$

   Homework to this topic: „AN-2" p. 15, ex. 2

week 2 (16-20 of September, 2024):

4. *(to Lesson 3 in „AN-2")*

   Determine $f'(a)$ by definition:

$$a) \quad f(x) = \sqrt{x}, \ a = 3 \qquad\qquad b) \quad f(x) = x^2 + 2x - 1, \ a = 1$$

$$c) \quad f(x) = \frac{x+2}{x^2 - 9}, \ a = -1$$

5. *(to Lesson 3 in „AN-2")*

   Discuss the differentiability of the following functions (in c) α is a real parameter):

   $a)$ $f(x) = x \cdot |x|, \ (x \in \mathbb{R})$ $\qquad$ $b)$ $f(x) := \begin{cases} 1-x & \text{if } x<0 \\ e^{-x} & \text{if } x \geq 0 \end{cases}$

   $c)$ $f(x) := \begin{cases} \alpha x + x^2 & \text{if } x<0 \\ x - x^2 & \text{if } x \geq 0 \end{cases}$

6. *(to Lesson 3 in „AN-2")*

   Differentiate the following functions (determine the derivatives):

   $a)$ $f(x) = 4x^5 - 3x^4 + 2x^3 - 7x^2 + 6x + 7$ $\qquad$ $b)$ $f(x) = x^2 \cdot \sqrt[3]{x}$

   $c)$ $f(x) = \sqrt{x \cdot \sqrt[3]{x}}$ $\qquad$ $d)$ $f(x) = e^x \cdot \sin x$ $\qquad$ $e)$ $f(x) = (x^3 + \ln x) \cdot \cos x$

   $f)$ $f(x) = \dfrac{2x^2+3x+1}{x^3+x^2+x+1}$ $\qquad$ $g)$ $f(x) = \sin(x^3 + \ln x)$ $\qquad$ $h)$ $f(x) = e^{\sin^3 x}$

   $i)$ $f(x) = \dfrac{1}{\sqrt[3]{x+\sqrt{x}}}$ $\qquad$ $j)$ $f(x) = x^x$ $\qquad$ $k)$ $f(x) = (\sin x)^{\cos\sqrt{x}}$

   Homework to this topic: „AN-2" p. 21, ex. 1., 2.

week 3 (23-27 of September, 2024):

7. *(to Lesson 3 in „AN-2")*

   Differentiate the following functions (determine the derivatives):

   $a)$ $f(x) = \arcsin\left(2x^2 - \sqrt{x}\right)$ $\qquad$ $b)$ $f(x) = \dfrac{\arcsin x}{\arctan x}$

8. *(to Lesson 3 in „AN-2")*

   Determine the equation of tangent line to the following curves at the given points (in a) and in b) only the first coordinates are given):

   $a)$ $y = \dfrac{x}{x^2-2}, \qquad x_0 = 2$ $\qquad$ $b)$ $y = e^x + e^{2x}, \qquad x_0 = 0$

   $c)$ $x^2 y = 2y + x^{x+1}, \qquad P_0 = (1, -1)$

   Homework to this topic: „AN-2" p. 21, ex. 3.

9. *(to Lesson 4 in „AN-2")*

Discuss the monotonicity, the local extreme values and the global (=absolute) extreme values of the following $\mathbb{R} \to \mathbb{R}$ type functions:

$$a) \quad f(x) = 1 - 4x - x^2 \qquad b) \quad f(x) = \frac{x}{x^2 - 6x - 16}$$

$$c) \quad f(x) = e^{x^2 - 4x} \qquad d) \quad f(x) = x \cdot \ln x$$

$$e) \quad f(x) = x^3 - 3x^2 + 3x + 2 \qquad f) \quad f(x) = x^2 \cdot e^{-x}$$

$$g) \quad f(x) = x - \ln(1 + x)$$

Homework to this topic: „AN-2" p. 27, ex. 1.

week 4 (30 of September - 4 of October, 2024):

10. *(to Lesson 4 in „AN-2")*

Determine the global extreme values of the following functions:

$$a) \; f(x) = \frac{x}{1 + x^2} \quad (x \in \mathbb{R})$$

$$b) \; f(x) = \sin^4 x + \cos^4 x \quad (-2\pi/3 \le x \le \pi/3)$$

$$c) \; f(x) = x - \ln(1 + x) \quad (x > -1)$$

$$d) \; f(x) = 2x^3 + 3x^2 - 12x + 1 \quad (-10 \le x \le 12)$$

Homework to this topic: „AN-2" p. 27, ex. 3.

11. *(to Lesson 4 in „AN-2")*

(a) Find the rectangle of largest area having a given perimeter $p > 0$.

(b) Find the point on the line $6x + y = 9$ that is closest to the point $(-3; 1)$.

(c) Determine the equation of the line trough the point $(3; 5)$ which cuts off the smallest area from the first quadrant of the coordinate system.

(d) We divide a 10 m long rope into two parts. We form a square from the one part and we form an equilateral triangle from the other part. When will the sum of the areas of the two plane figures be the smallest?

(e) The hypotenuse of a right triangle is 1 unit. Denote by x and y the legs of the triangle. When will $x + 2y$ be the largest?

(f) Find the dimensions of the cone of maximum volume that can be inscribed in a sphere of radius $R > 0$.

Homework to this topic: „AN-2" p. 27, ex. 2.

week 5 (7-11 of October, 2024):

12. *(to Lesson 5.1 in „AN-2")*
Use L'Hospital's Rule for determination of the following limits:

$$a)\ \lim_{x\to 0}\frac{a^x - a^{\sin x}}{x^2}\quad (a>0)\qquad b)\ \lim_{x\to 0}\frac{\tan x - x}{x-\sin x}\qquad c)\ \lim_{x\to 1}\ln x\cdot\ln(1-x)$$

$$d)\ \lim_{x\to 1}\frac{x^2-1}{2x^2-x-1}\qquad e)\ \lim_{x\to+\infty}(x\cdot e^{1/x}-x)\qquad f)\ \lim_{x\to-\infty}x^2\cdot e^{-x}$$

$$g)\ \lim_{x\to 0}e^{-x}\cdot\ln x\qquad h)\ \lim_{x\to+\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$$

Homework to this topic: „AN-2" p. 31, ex. 1.

13. *(to Lesson 5.2 in „AN-2")* Use Taylor-polynomials for writing the polynomial $f(x)=(1+2x)^3$ by the powers of

$$a)\ x+1\qquad b)\ x-\frac{1}{2}\qquad c)\ x+\frac{1}{2}$$

14. Let $f(x)=\ln(1+x)\quad (x>-1)$.
    (a) Find the second Taylor-polynomial $T_2(x)$ centered at 0.
    (b) Estimate the error of approximation $f(x)\approx T_2(x)$ for $x>0$ and for $-1<x<0$.
    (c) Approximate $\ln 2$ by means of $T_2(1)$, and estimate the error.

15. Estimate the error of the approximation

$$\tan x\approx x+\frac{x^3}{3}\qquad (|x|\le 10^{-1})$$

Homework to this topic: „AN-2" p. 32, ex. 3.

week 6 (14-18 of October, 2024):

16. *(to Lessons 4. and 5. in „AN-2")*
Discuss and sketch the graph of f if

$$a)\ f(x)=2-2x^2-x^3\quad (x\in\mathbb{R})\qquad b)\ f(x)=\frac{1}{x(x-3)^2}\quad (x\in\mathbb{R}\setminus\{0;3\})$$

$$c)\ f(x)=\frac{5x}{(x+2)^2}\quad (-2\ne x\in\mathbb{R})\qquad d)\ f(x)=5x^3-4x^4\quad (x\in\mathbb{R})$$

$$e)\ f(x)=\frac{x^2-1}{x^2-5x+6}\quad (x\in\mathbb{R}\setminus\{2;3\})$$

Homework to this topic: „AN-2" p. 32, ex. 2.

week 7 (21-25 of October) and week 8 (4-8 of November, 2024):

17. *(to Lessons 6.1, 6.2 in „AN-2")*

Find the integrals

$$a) \int 6x^2 - 8x + 3 \; dx \qquad b) \int 2x + \frac{5}{\sqrt{1-x^2}} \; dx \qquad c) \int \frac{x^2}{x^2+1} \; dx$$

$$d) \int \frac{\cos^2 x - 5}{1+\cos(2x)} \; dx \qquad e) \int \frac{1}{\sqrt{x}} \; dx \qquad f) \int x \cdot \sqrt{x} \; dx$$

$$g) \int \frac{(x+1)^2}{\sqrt{x}} \; dx \qquad h) \int \sin^2 x \; dx \qquad i) \int (3x+2)^4 \; dx$$

$$j) \int \frac{2}{3+2x^2} \; dx \qquad k) \int \frac{1}{\sqrt{1-2x^2}} \; dx \qquad l) \int 5^{2-3x} \; dx$$

$$m) \int \sin x \cdot \cos x \; dx \qquad n) \int x^2 \cdot (2x^3+4) \; dx \qquad o) \int x^2 \cdot \sqrt{6x^3+4} \; dx$$

$$p) \int e^x \cdot (1-e^x)^3 \; dx \qquad q) \int \frac{x}{x^2+3} \; dx \qquad r) \int \sin^3 x \; dx$$

$$s) \int \sin^2 x \cdot \cos^3 x \; dx \qquad t) \int \frac{1}{x \cdot \ln x} \; dx \qquad u) \int \sin^2 x \cdot \cos^4 x \; dx$$

Homework to this topic: „AN-2" p. 37, ex. 1a, 1b

18. *(to Lesson 6.3 in „AN-2")*

Find the integrals of the following rational functions

$$a) \int \frac{1}{(x-2)(x+4)} \; dx \qquad b) \int \frac{4x^2+13x-9}{x^3+2x^2-3x} \; dx \qquad c) \int \frac{1}{x^3+2x^2+x} \; dx$$

Homework to this topic: „AN-2" p. 37, ex. 1c, 1e *week 9 (11-15 of November, 2024):*

19. *(to Lesson 7 in „AN-2")*

Find the integrals

$$a)\ \int x \cdot e^{2x}\ dx \qquad b)\ \int x^2 \cdot e^{2x}\ dx \qquad c)\ \int x^2 \cdot \sin 5x\ dx$$

$$d)\ \int \ln x\ dx \qquad e)\ \int (x^3 + 2x + 2) \cdot \ln x\ dx$$

Homework to this topic: „AN-2" p. 39, ex. 1., 2.

20. *(to Lesson 10 in „AN-2")*

Find the integrals

$$a)\ \int\limits_0^{\pi/3} \sin(8x)\ dx \qquad b)\ \int\limits_0^1 \frac{1}{3x-5}\ dx \qquad c)\ \int\limits_1^e \frac{\ln^2(x)}{x}\ dx$$

21. *(to Lesson 10. in „AN-2")*

Find the integrals

$$a)\ \int\limits_0^1 x \cdot e^{-x^2}\ dx \qquad b)\ \int\limits_{1/2}^1 \frac{\sqrt{1-x^2}}{x^2}\ dx \qquad c)\ \int\limits_e^{e^2} x \cdot \ln x\ dx \qquad d)\ \int\limits_0^{\pi} x \cdot \cos x\ dx$$

Homework to this topic: „AN-2" p. 54, ex. 1., 2., 3.

week 10 (18-22 of November, 2024):

22. *(to Lesson 11 in „AN-2")*

Find the improper integrals

$$a)\ \int\limits_0^{+\infty} \frac{1}{1+x^2}\ dx \qquad b)\ \int\limits_1^2 \frac{1}{\sqrt{x-1}}\ dx \qquad c)\ \int\limits_0^{+\infty} x \cdot e^{-2x}\ dx$$

Homework to this topic: „AN-2" p. 57, ex. 1., 2.

23. *(to Lesson 12 in „AN-2")*

Application of the integrals

(a) Determine the area bounded by the curves $y = x^2$ and $y = 1 - x^2$.

(b) Compute the arc length of the graph of the function $f(x) = \sqrt{4 - x^2}$ bounded by the points $(0, 2)$ and $(2, 0)$.

(c) Revolve the curve

$$y = e^{2x}, \qquad 0 \leq x \leq 2$$

about the x-axis. Determine the volume of this solid of revolution.

Homework to this topic: „AN-2" p. 59, ex. 1., 2., 3.

week 11 (25-29 of November, 2024):

Partial derivatives *(to Lesson 6.1 in „AN-3")*

24. Find the partial derivatives of f, if

a) $f(x, y) = \ln(xy^2) - x^3y^2 \cos(x^2 + y^2) \quad ((x, y) \in \mathbb{R}^2)$

b) $f(x, y) = \operatorname{arctg}\left(\frac{y}{x}\right) \quad ((x, y) \in \mathbb{R}^2)$

Homework to this topic: „AN-3" p. 33, ex. 2.

Local extreme values *(to Lessons 7.1, 8.1, 8.2, 9.1 in „AN-3")*:

25. Find the local extreme values and their places of f, if

a) $f(x, y) = x^2 - 4xy + y^3 + 4y \quad ((x, y) \in \mathbb{R}^2)$

b) $f(x, y) = x^4 - 4xy + y^4 \quad ((x, y) \in \mathbb{R}^2)$

Homework to this topic: „AN-3" p. 47, ex. 1.

Absolute extreme values on compact sets:

Find the absolute extreme values and their places of f on the set H, if

$$f(x, y) = 2xy - 3y, \qquad H = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2,\ 0 \leq y \leq y^2\}$$

Homework to this topic: „AN-3" p. 44, ex. 1.

week 12 (2-6 of December, 2024):

Double and triple integral over intervals and over normal regions
(to Lessons 10 and 11 in „AN-3")

26. Integrate the function $f(x, y) = xy^2 + 3x^2y$ over the interval (rectangle) whose vertices are:

$$A(1; -1), \quad B(4; -1)), \quad C(4; 2), \quad D(1; 2)$$

27. Integrate the function $f(x, y, z) = xy + xz$ over the interval (cuboid):

$$[0; 2] \times [1; 2] \times [1; 3]$$

Homework to this topic: „AN-3" p. 54, ex. 1., 2., 3., 4.

28. Integrate the function $f(x, y) = xy$ over the triangle whose vertices are:

$$A(1; 1), \quad B(4; 5)), \quad C(4; 2)$$

29. Integrate the function $f(x, y, z) = 2xy$ over the region determined by

$$(x,\, y,\, z) \in \mathbb{R}^3, \quad x \geq 0,\ y \geq 0,\ z \geq 0,\ x + y + z \leq 1$$

Homework to this topic: „AN-3" p. 58, ex. 1., 2., 3., 4., 5.

week 13 (9-13 of December, 2024):

Consultations