Theorem 7.8. *Let $\mathscr{D}$ be a skew-adjoint $q \times q$ matrix differential operator and $\Theta = \frac{1}{2}\int \{\theta \wedge \mathscr{D}\theta\}\, dx$ the corresponding functional bi-vector. Then $\mathscr{D}$ is Hamiltonian if and only if*

$$\operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}(\Theta) = 0. \tag{7.18}$$

Example 7.9. Let us return one final time to the Hamiltonian operator $\mathscr{E}$ associated with the Korteweg–de Vries equation. According to (7.15) and (7.18) we need only check the vanishing of

$$\begin{aligned}\operatorname{pr} \mathbf{v}_{\mathscr{E}\theta} \int \{\tfrac{1}{2}\theta \wedge \theta_{xxx} + \tfrac{1}{3}u\theta \wedge \theta_x\}\, dx &= \tfrac{1}{3}\int \{\mathscr{E}(\theta) \wedge \theta \wedge \theta_x\}\, dx \\ &= \tfrac{1}{3}\int (\theta_{xxx} \wedge \theta \wedge \theta_x + \tfrac{2}{3}u\theta_x \wedge \theta \wedge \theta_x \\ &\qquad + \tfrac{1}{3}u_x\theta \wedge \theta \wedge \theta_x)\, dx \\ &= 0,\end{aligned}$$

by our earlier computation. Thus we have proved, in a completely elementary fashion, the fact that $\mathscr{E}$ is Hamiltonian.

Example 7.10. Consider the Euler equatons of inviscid ideal fluid flow

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p, \qquad \nabla \cdot \mathbf{u} = 0.$$

(See Example 2.45 for the notation.) As these stand, they cannot take the form of a Hamiltonian system since, in particular, we have no equation governing the temporal evolution of the pressure p. The easiest way to circumvent this difficulty is to rewrite the equations in terms of the *vorticity* $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. Taking the curl of the first set of equations, we find the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \boldsymbol{\omega}, \tag{7.19}$$

which we will put into Hamiltonian form

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \mathscr{D} \frac{\delta \mathscr{H}}{\delta \boldsymbol{\omega}} \tag{7.20}$$

for a suitable Hamiltonian operator $\mathscr{D}$. The Hamiltonian functional is the energy

$$\mathscr{H} = \int \tfrac{1}{2}|\mathbf{u}|^2\, dx,$$

but for (7.20) we need to compute its variational derivative with respect to $\boldsymbol{\omega}$, not $\mathbf{u}$! This is done (formally) by introducing the vector stream function

$\boldsymbol{\psi}$ satisfying $\nabla \times \boldsymbol{\psi} = \mathbf{u}$, $\nabla \cdot \boldsymbol{\psi} = 0$. Let $\boldsymbol{\eta}(\mathbf{x})$ have compact support, with $\nabla \times \boldsymbol{\eta} = \boldsymbol{\zeta}$. Then

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathscr{H}[\boldsymbol{\omega} + \varepsilon\boldsymbol{\zeta}] = \int \mathbf{u}\cdot\boldsymbol{\eta}\, d\mathbf{x} = \int (\nabla \times \boldsymbol{\psi})\cdot\boldsymbol{\eta}\, d\mathbf{x} = \int \boldsymbol{\psi}\cdot(\nabla \times \boldsymbol{\eta})\, d\mathbf{x}$$
$$= \int \boldsymbol{\psi}\cdot\boldsymbol{\zeta}\, d\mathbf{x},$$

hence $\delta\mathscr{H}/\delta\boldsymbol{\omega} = \boldsymbol{\psi}$.

In the two-dimensional case, $\mathbf{u} = (u, v)$ depends on (x, y, t) and there is a single vorticity $\omega = v_x - u_y$. The vorticity equation is then

$$\frac{\partial \omega}{\partial t} = -u\omega_x - v\omega_y = \omega_x \psi_y - \omega_y \psi_x, \tag{7.21}$$

where ψ is the stream function, with $\psi_x = v$, $\psi_y = -u$. If we set

$$\mathscr{D} = \omega_x D_y - \omega_y D_x,$$

then we see that (7.21) is of the form (7.20) using the energy as the Hamiltonian functional.

To prove that $\mathscr{D}$ is a Hamiltonian operator, note first that

$$\mathscr{D}^* = -D_y\cdot\omega_x + D_x\cdot\omega_y = -\mathscr{D},$$

so $\mathscr{D}$ is skew-adjoint. The Jacobi identity is proved by checking (7.18):

$$0 = \operatorname{pr}\mathbf{v}_{\mathscr{D}\theta} \int \{\omega_x\theta \wedge \theta_y - \omega_y\theta \wedge \theta_x\}\, dx\, dy$$
$$= \int \{D_x(\omega_x\theta_y - \omega_y\theta_x) \wedge \theta \wedge \theta_y - D_y(\omega_x\theta_y - \omega_y\theta_x) \wedge \theta \wedge \theta_x\}\, dx\, dy$$
$$= \int \{\omega_x(\theta_{xy} \wedge \theta \wedge \theta_y - \theta_{yy} \wedge \theta \wedge \theta_x)$$
$$+ \omega_y(\theta_{xy} \wedge \theta \wedge \theta_x - \theta_{xx} \wedge \theta \wedge \theta_y)\}\, dx\, dy.$$

Integrating the second and fourth terms by parts, we find this equals

$$\int \{\omega_x(\theta_{xy} \wedge \theta \wedge \theta_y + \theta_y \wedge \theta \wedge \theta_{xy}) + \omega_{xy}\theta_y \wedge \theta \wedge \theta_x$$
$$+ \omega_y(\theta_{xy} \wedge \theta \wedge \theta_x + \theta_x \wedge \theta \wedge \theta_{xy}) + \omega_{xy}\theta_x \wedge \theta \wedge \theta_y\}\, dx\, dy = 0,$$

since the wedge product is alternating. Thus $\mathscr{D}$ verifies the conditions of Theorem 7.8 and defines a true Poisson bracket, relative to which the two-dimensional Euler equations are Hamiltonian. The three-dimensional version is left to the reader; see Exercise 7.5.

7.2. Symmetries and Conservation Laws

In outline, the correspondence between Hamiltonian symmetry groups and conservation laws for systems of evolution equations in Hamiltonian form proceeds exactly as in the finite-dimensional case discussed in Section 6.3. First, we need to investigate the "distinguished functionals" arising from degeneracies of the Poisson bracket itself; these will provide conservation laws for *any* system having the given Hamiltonian structure. Further conservation laws, particular to the symmetry properties of the individual Hamiltonian functionals, can then be deduced from generalized symmetries which are themselves Hamiltonian.

Distinguished Functionals

Definition 7.11. Let $\mathscr{D}$ be a $q \times q$ Hamiltonian differential operator. A *distinguished functional* for $\mathscr{D}$ is a functional $\mathscr{C} \in \mathscr{F}$ satisfying $\mathscr{D}\delta\mathscr{C} = 0$ for all x, u.

In other words, the Hamiltonian system corresponding to a distinguished functional is completely trivial: $u_t = 0$. From (7.4) we conclude that a functional $\mathscr{C}$ is distinguished if and only if its Poisson bracket with every other functional is trivial:

$$\{\mathscr{C}, \mathscr{H}\} = 0 \quad \text{for all} \quad \mathscr{H} \in \mathscr{F}.$$

This immediately implies the conservative nature of such functionals.

Proposition 7.12. *Let $\mathscr{D}$ be a Hamiltonian operator. If $\mathscr{C}$ is a distinguished functional for $\mathscr{D}$, then $\mathscr{C}$ determines a conservation law for every Hamiltonian system $u_t = \mathscr{D}\delta\mathscr{H}$ relative to $\mathscr{D}$.*

Example 7.13. For the first Hamiltonian operator $\mathscr{D} = D_x$ of the Korteweg–de Vries equation, a distinguished functional must satisfy $D_x\delta\mathscr{C} = 0$, or, equivalently, $\delta\mathscr{C}$ is constant. Every such functional is a constant multiple of the mass $\mathscr{M}[u] = \int u\,dx$. Thus, according to Proposition 7.12, the L^1 solutions to any evolution equation of the form $u_t = D_x\delta\mathscr{H}$ automatically satisfy the constraint of mass conservation $\int u\,dx = \text{constant}$. (Actually, this can be generalized, see Exercise 7.8.) The second Hamiltonian operator $\mathscr{E}$, on the other hand, has no nontrivial distinguished functionals, and thus might be regarded as "symplectic".

Lie Brackets

As in the finite-dimensional set-up, the main result required for establishing a Noether-type theorem relating symmetry groups and conservation laws is the correspondence between the Poisson bracket of functionals and the commutator of their corresponding Hamiltonian vector fields.

Proposition 7.14. *Let $\{\cdot, \cdot\}$ be a Poisson bracket determined by a differential operator $\mathscr{D}$. Let $\mathscr{P}, \mathscr{Q} \in \mathscr{F}$ be functionals, with corresponding Hamiltonian vector fields $\hat{\mathbf{v}}_{\mathscr{P}}, \hat{\mathbf{v}}_{\mathscr{Q}}$. Then the Hamiltonian vector field corresponding to the Poisson bracket $\{\mathscr{P}, \mathscr{Q}\}$ is the Lie bracket of the two vector fields*;

$$\hat{\mathbf{v}}_{\{\mathscr{P},\mathscr{Q}\}} = -[\hat{\mathbf{v}}_{\mathscr{P}}, \hat{\mathbf{v}}_{\mathscr{Q}}] = [\hat{\mathbf{v}}_{\mathscr{Q}}, \hat{\mathbf{v}}_{\mathscr{P}}]. \tag{7.22}$$

(*The Lie bracket is that given by Definition* 5.14.)

Proof. Let $\mathscr{R}$ be an arbitrary functional. Applying the prolongation of $\hat{\mathbf{v}}_{\{\mathscr{P},\mathscr{Q}\}}$ to $\mathscr{R}$, and using (7.4) and the Jacobi identity, we find

$$\begin{aligned}
\operatorname{pr} \hat{\mathbf{v}}_{\{\mathscr{P},\mathscr{Q}\}}(\mathscr{R}) &= \{\mathscr{R}, \{\mathscr{P}, \mathscr{Q}\}\} \\
&= \{\{\mathscr{R}, \mathscr{P}\}, \mathscr{Q}\} - \{\{\mathscr{R}, \mathscr{Q}\}, \mathscr{P}\} \\
&= \operatorname{pr} \hat{\mathbf{v}}_{\mathscr{Q}}(\{\mathscr{R}, \mathscr{P}\}) - \operatorname{pr} \hat{\mathbf{v}}_{\mathscr{P}}(\{\mathscr{R}, \mathscr{Q}\}) \\
&= (\operatorname{pr} \hat{\mathbf{v}}_{\mathscr{Q}} \cdot \operatorname{pr} \hat{\mathbf{v}}_{\mathscr{P}} - \operatorname{pr} \hat{\mathbf{v}}_{\mathscr{P}} \cdot \operatorname{pr} \hat{\mathbf{v}}_{\mathscr{Q}})\mathscr{R}.
\end{aligned}$$

By (5.21), this latter expression is the prolongation of the Lie bracket between $\hat{\mathbf{v}}_{\mathscr{P}}, \hat{\mathbf{v}}_{\mathscr{Q}}$, so

$$\operatorname{pr} \hat{\mathbf{v}}_{\{\mathscr{P},\mathscr{Q}\}}(\mathscr{R}) = -\operatorname{pr}([\hat{\mathbf{v}}_{\mathscr{P}}, \hat{\mathbf{v}}_{\mathscr{Q}}])\mathscr{R}$$

for every $\mathscr{R} \in \mathscr{F}$. Exercise 5.52 says that this can happen only if the two generalized vector fields are equal, which proves (7.22). □

Conservation Laws

As we remarked in Chapter 4, any conservation law of a system of evolution equations takes the form

$$D_t T + \operatorname{Div} X = 0,$$

in which Div denotes spatial divergence and the conserved density $T(x, t, u^{(n)})$ can be assumed without loss of generality to depend only on x-derivatives of u. Equivalently, for $\Omega \subset X$, the functional

$$\mathscr{T}[t; u] = \int_\Omega T(x, t, u^{(n)})\, dx$$

is a constant, independent of t, for all solutions u such that $T(x, t, u^{(n)}) \to 0$ as $x \to \partial\Omega$.

Note that if $T(x, t, u^{(n)})$ is any such differential function, and u is a solution to the evolutionary system $u_t = P[u]$, then

$$D_t T = \partial_t T + \operatorname{pr} \mathbf{v}_P(T),$$

where $\partial_t = \partial/\partial t$ denotes the partial t-derivative. Thus T is the density for a conservation law of the system if and only if its associated functional $\mathscr{T}$ satisfies

$$\partial \mathscr{T}/\partial t + \operatorname{pr} \mathbf{v}_P(\mathscr{T}) = 0. \tag{7.23}$$

In the case our system is of Hamiltonian form, the bracket relation (7.4) immediately leads to the Noether relation between Hamiltonian symmetries and conservation laws.

Theorem 7.15. *Let $u_t = \mathscr{D}\delta\mathscr{H}$ be a Hamiltonian system of evolution equations. A Hamiltonian vector field $\hat{\mathbf{v}}_{\mathscr{P}}$ with characteristic $\mathscr{D}\delta\mathscr{P}$, $\mathscr{P} \in \mathscr{F}$, determines a generalized symmetry group of the system if and only if there is an equivalent functional $\tilde{\mathscr{P}} = \mathscr{P} - \mathscr{C}$, differing only from $\mathscr{P}$ by a time-dependent distinguished functional $\mathscr{C}[t; u]$, such that $\tilde{\mathscr{P}}$ determines a conservation law.*

Proof. By a time-dependent distinguished functional we mean, in analogy with Chapter 6, a functional $\mathscr{C}[t; u] = \int C(t, x, u^{(n)})\, dx$, with C depending on t, x, u and x-derivatives of u, and with the property that for each fixed t_0, $\mathscr{C}[t_0; u]$ is a distinguished functional: $\mathscr{D}\delta\mathscr{C} = 0$. Now, according to Proposition 5.19, $\hat{\mathbf{v}}_{\mathscr{P}}$ is a symmetry of the Hamiltonian system if and only if

$$\partial \hat{\mathbf{v}}_{\mathscr{P}}/\partial t + [\hat{\mathbf{v}}_{\mathscr{H}}, \hat{\mathbf{v}}_{\mathscr{P}}] = 0, \tag{7.24}$$

$\hat{\mathbf{v}}_{\mathscr{H}}$ being the associated Hamiltonian vector field. Since $\mathscr{D}$ does not explicitly depend on t, $\partial \hat{\mathbf{v}}_{\mathscr{P}}/\partial t$ is the Hamiltonian vector field corresponding to the functional $\partial\mathscr{P}/\partial t$, while by the previous proposition, $[\hat{\mathbf{v}}_{\mathscr{H}}, \hat{\mathbf{v}}_{\mathscr{P}}]$ is the Hamiltonian vector field for the Poisson bracket of $\mathscr{P}$ and $\mathscr{H}$. Thus (7.24) says that the Hamiltonian vector field for the combined functional $\partial_t\mathscr{P} + \{\mathscr{P}, \mathscr{H}\}$ is zero, and hence

$$\frac{\partial \mathscr{P}}{\partial t} + \{\mathscr{P}, \mathscr{H}\} = \tilde{\mathscr{C}}$$

for some time-dependent distinguished functional

$$\tilde{\mathscr{C}}[t; u] = \int \tilde{C}(t, x, u^{(n)})\, dx.$$

Now set

$$\mathscr{C}[t; u] = \int_{t_0}^{t} \tilde{\mathscr{C}}[s; u]\, ds \equiv \int \left(\int_{t_0}^{t} \tilde{C}(s, x, u^{(n)})\, ds \right) dx,$$

and let $\tilde{\mathscr{P}} = \mathscr{P} - \mathscr{C}$. Then

$$\frac{\partial \tilde{\mathscr{P}}}{\partial t} = \frac{\partial \mathscr{P}}{\partial t} - \tilde{\mathscr{C}},$$

while by the definition of distinguished functional,

$$\{\mathscr{P}, \mathscr{H}\} = \{\tilde{\mathscr{P}}, \mathscr{H}\}.$$

Thus $\tilde{\mathscr{P}}$ satisfies the condition (7.23) that it be conserved, and the theorem is proved. □

Example 7.16. Consider the Korteweg–de Vries equation

$$u_t = u_{xxx} + uu_x, \tag{7.25}$$

whose two Hamiltonian structures were discussed in Example 7.6. Let's investigate which of the classical symmetry groups of Example 2.44 are Hamiltonian and hence lead to conservation laws. The symmetries are generated by

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = t\partial_x - \partial_u, \qquad \mathbf{v}_4 = x\partial_x + 3t\partial_t - 2u\partial_u,$$

cf. (2.68) with x replaced by $-x$, with corresponding characteristics

$$Q_1 = u_x, \ Q_2 = u_{xxx} + uu_x, \ Q_3 = 1 + tu_x, \ Q_4 = 2u + xu_x + 3t(u_{xxx} + uu_x),$$

(up to sign).

For the first Hamiltonian operator $\mathscr{D} = D_x$, there is one independent nontrivial distinguished functional, the mass $\mathscr{P}_0 = \mathscr{M} = \int u\,dx$, which is therefore conserved. Of the above four characteristics, the first three are Hamiltonian

$$Q_i = D_x \delta \mathscr{P}_i, \qquad i = 1, 2, 3, \tag{7.26}$$

with conserved functionals

$$\mathscr{P}_1 = \int \tfrac{1}{2}u^2\,dx, \qquad \mathscr{P}_2 = \int (\tfrac{1}{6}u^3 - \tfrac{1}{2}u_x^2)\,dx, \qquad \mathscr{P}_3 = \int (xu + \tfrac{1}{2}tu^2)\,dx.$$

Note that $\mathscr{P}_2$ is just the Hamiltonian functional for (7.25) with $\mathscr{D}$ as the Hamiltonian operator. Invariance of $\mathscr{P}_3$, when combined with that of $\mathscr{P}_1$, proves that the first moment of u is a linear function of t,

$$\int xu\,dx = at + b, \qquad a, b \text{ constant},$$

with $a = -\int \frac{1}{2}u^2\,dx$, for any solution u decaying sufficiently rapidly as $|x| \to \infty$, or any periodic solution, where the integral is now over one period. The fourth characteristic Q_4 is not of the form (7.26), and hence does not lead to a conservation law.

What about the second Hamiltonian operator $\mathscr{E} = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x$? Here there are no longer any distinguished functionals. In this case Q_1, Q_2 and Q_4 (but not Q_3) are Hamiltonian,

$$Q_i = \mathscr{E}\delta\tilde{\mathscr{P}}_i, \qquad i = 1, 2, 4, \tag{7.27}$$

where

$$\tilde{\mathscr{P}}_1 = 3\mathscr{P}_0 = 3\int u\,dx, \qquad \tilde{\mathscr{P}}_2 = \mathscr{P}_1 = \int \tfrac{1}{2}u^2\,dx,$$

$$\tilde{\mathscr{P}}_4 = 3\mathscr{P}_3 = 3\int (xu + \tfrac{1}{2}tu^2)\,dx,$$

are the corresponding conservation laws. In this case, nothing new is obtained. Note that the other conservation law $\mathscr{P}_2$ did *not* arise from one of the geometrical symmetries. According to Theorem 7.15, however, there *is* a Hamiltonian symmetry which gives rise to it, namely $\hat{\mathbf{v}}_{\mathscr{P}_2}$. The characteristic of this generalized symmetry is

$$Q_5 = \mathscr{E}\delta\mathscr{P}_2 = (D_x^3 + \tfrac{2}{3}uD_x + \tfrac{1}{3}u_x)(u_{xx} + \tfrac{1}{2}u^2)$$
$$= u_{xxxxx} + \tfrac{5}{3}uu_{xxx} + \tfrac{10}{3}u_xu_{xx} + \tfrac{5}{6}u^2u_x.$$

We have thus rediscovered the fifth order generalized symmetry of Section 5.2! Pressing on, we note that Q_5 happens to satisfy the Hamiltonian condition (7.26) for $\mathscr{D}$ with the functional

$$\mathscr{P}_5 = \int (\tfrac{1}{2}u_{xx}^2 - \tfrac{5}{6}uu_x^2 + \tfrac{5}{72}u^4)\,dx$$

providing a further conservation law for the Korteweg–de Vries equation. By now, the signs of a recursive procedure of generating conservation laws and corresponding Hamiltonian symmetries of the Korteweg–de Vries equation are starting to appear. We take the new conservation law $\mathscr{P}_5$, determine its Hamiltonian vector field relative to the operator $\mathscr{E}$, which, by Theorem 7.15 is necessarily a symmetry, and then try to find a further functional $\mathscr{P}_6$ for which it is the Hamiltonian vector field relative to the other Hamiltonian operator $\mathscr{D}$, and so on. The rigorous implementation of this recursion scheme for general equations with two Hamiltonian structures will be the subject of Section 7.3.

Example 7.17. The two-dimensional Euler equations were cast into Hamiltonian form in Example 7.10. Let us investigate what type of conservation laws arise as a result. First we need to look at the distinguished functionals for the Hamiltonian operator $\mathscr{D} = \omega_xD_y - \omega_yD_x$. A straightforward computation proves that a differential function $P[\omega]$ lies in the kernel of $\mathscr{D}$, so $\mathscr{D}P = 0$, if and only if $P = P(\omega)$ is a function of ω (but not x or y, nor any derivatives of ω). We conclude that the functionals

$$\mathscr{C}[\omega] = \int C(\omega)\,dx\,dy,$$

where $C(\omega)$ is *any* smooth function of the vorticity ω, are all distinguished and hence are conserved for solutions of the Euler equations. These are the well-known "area integrals" and reflect the point-wise conservation of vorticity for two-dimensional incompressible fluid flow.

Conservation laws arising from the Euclidean symmetries of the Euler equations found in Example 2.45 are constructed next. Note first that we need to find the "ω-characteristic" of each of the symmetry generators, i.e. rewrite it as the prolongation of an evolutionary vector field in the form $\mathbf{v} = Q(x, y, t, u, v, p, \omega, \ldots)\partial_\omega$. If $\mathbf{v}$ is a Hamiltonian vector field, we may then deduce the existence of a conservation law $\mathscr{P}[\omega]$ with $\mathscr{D}\delta\mathscr{P} = Q$. For instance, the translational symmetry

$$\mathbf{v}_\alpha = \alpha\partial_x + \alpha_t\partial_u - \alpha_{tt}x\partial_p, \qquad \alpha = \alpha(t),$$

has evolutionary form

$$\tilde{\mathbf{v}}_\alpha = (\alpha_t - \alpha u_x)\partial_u - \alpha v_x\partial_v - (\alpha_{tt}x + \alpha p_x)\partial_p.$$

Prolonging $\tilde{\mathbf{v}}_\alpha$, we see that its ω-coefficient is

$$Q = -\alpha\omega_x = -\mathscr{D}(\alpha y) = -\mathscr{D}\delta\mathscr{P}_\alpha,$$

where

$$\mathscr{P}_\alpha = \int \alpha(t)y\omega\, dx\, dy = \int \alpha(t)u\, dx\, dy$$

(integrating by parts) is the associated conserved functional. Similarly, the translational symmetries $\mathbf{v}_\beta$ lead to conservation laws

$$\mathscr{P}_\beta = -\int \beta(t)x\omega\, dx\, dy = \int \beta(t)v\, dx\, dy,$$

where $\beta(t)$ is also an arbitrary function of t. The fact that $\mathscr{P}_\alpha$ and $\mathscr{P}_\beta$ are conservation laws for any functions $\alpha(t)$ and $\beta(t)$ appears to be paradoxical, but this is resolved by looking at the boundary contributions. In vector form, if $\boldsymbol{\alpha}(t) = (\alpha(t), \beta(t))$, then we have the divergence identity

$$D_t(\boldsymbol{\alpha}\cdot\mathbf{u}) + \operatorname{Div}[(\boldsymbol{\alpha}\cdot\mathbf{u} - \boldsymbol{\alpha}_t\cdot\mathbf{x})\mathbf{u} + p\boldsymbol{\alpha}] = 0$$

valid for all solutions $\mathbf{u} = (u, v)$ of the Euler equations. This integrates to the generalized momentum relations

$$\frac{d}{dt}\int_\Omega (\boldsymbol{\alpha}(t)\cdot\mathbf{u})\, d\mathbf{x} = -\int_{\partial\Omega} [(\boldsymbol{\alpha}\cdot\mathbf{u} - \boldsymbol{\alpha}_t\cdot\mathbf{x})\mathbf{u} + p\boldsymbol{\alpha}]\cdot dS$$

valid over an arbitrary subdomain Ω. It is in this sense that the above functionals $\mathscr{P}_\alpha$, $\mathscr{P}_\beta$ are "conserved".

The rotational symmetry

$$y\partial_x - x\partial_y + v\partial_u - u\partial_v$$

has ω-evolutionary form

$$(y\omega_x - x\omega_y)\partial_\omega,$$

which is Hamiltonian. We find

$$(y\omega_x - x\omega_y) = \mathscr{D}(\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2) = \mathscr{D}\delta\mathscr{T},$$

where

$$\mathscr{T} = \int \tfrac{1}{2}(x^2 + y^2)\omega\, dx\, dy = \int (yu - xv)\, dx\, dy$$

is the conserved angular momentum of the fluid. For the two-dimensional Euler equations, then, there are three infinite families of conservation laws—two coming from the generalized translational symmetries and one, the area integrals, reflecting the degeneracy of the underlying Poisson bracket—together with the individual conservation laws of angular momentum and energy. The three-dimensional case is markedly different—see Exercise 7.5.

7.3. Bi-Hamiltonian Systems

In this final section, we discuss the remarkable properties of systems of evolution equations which, like the Korteweg–de Vries equation, can be written in Hamiltonian form in not just one but *two* different ways. We will thus be interested in systems of the form

$$\frac{\partial u}{\partial t} = K_1[u] = \mathscr{D}\delta\mathscr{H}_1 = \mathscr{E}\delta\mathscr{H}_0 \tag{7.28}$$

in which both $\mathscr{D}$ and $\mathscr{E}$ are Hamiltonian operators, and $\mathscr{H}_0[u]$ and $\mathscr{H}_1[u]$ appropriate Hamiltonian functionals. Subject to a compatibility condition between the two Poisson structures determined by $\mathscr{D}$ and $\mathscr{E}$, we will be able to recursively construct an infinite hierarchy of symmetries and conservation laws for the system in the following manner.

According to Theorem 7.15, if $\mathscr{P}[u]$ is any conserved functional for (7.28), then *both* of the Hamiltonian vector fields $\mathbf{v}_{\mathscr{D}\delta\mathscr{P}}$ and $\mathbf{v}_{\mathscr{E}\delta\mathscr{P}}$ are symmetries. In particular, since both $\mathscr{H}_0$ and $\mathscr{H}_1$ are conserved, not only is the original vector field $\mathbf{v}_{K_1} = \mathbf{v}_{\mathscr{D}\delta\mathscr{H}_1} = \mathbf{v}_{\mathscr{E}\delta\mathscr{H}_0}$ a symmetry of (7.28), but so are the two additional vector fields $\mathbf{v}_{\mathscr{D}\delta\mathscr{H}_0}$ and $\mathbf{v}_{\mathscr{E}\delta\mathscr{H}_1}$. The recursion algorithm proceeds on the assumption that one of these new symmetries, say $\mathbf{v}_{\mathscr{E}\delta\mathscr{H}_1}$, is a Hamiltonian vector field for the other Hamiltonian structure, so

$$\mathscr{E}\delta\mathscr{H}_1 = \mathscr{D}\delta\mathscr{H}_2$$

for some functional $\mathscr{H}_2$. Again, by Theorem 7.15, $\mathscr{H}_2$ (or some equivalent functional) is conserved, and so we obtain yet a further symmetry, this time with characteristic $\mathscr{E}\delta\mathscr{H}_2$. At this stage, the recursion pattern is clear. At the n-th stage we determine a new functional $\mathscr{H}_{n+1}$ satisfying the recursion relation

$$K_n \equiv \mathscr{D}\delta\mathscr{H}_n = \mathscr{E}\delta\mathscr{H}_{n-1}. \tag{7.29}$$

This will provide both a further conservation law for the original system (7.28) plus a further symmetry $\mathbf{v}_{n+1}$ with characteristic $K_{n+1} = \mathscr{E}\delta\mathscr{H}_n$. Note that if we define the operator $\mathscr{R} = \mathscr{E}\cdot\mathscr{D}^{-1}$ then, formally,

$$K_{n+1} = \mathscr{R}K_n,$$

and, as will be the case, we suspect that $\mathscr{R}$ will define a recursion operator for our system.

Example 7.18. Consider the Korteweg–de Vries equation, which was shown to have two Hamiltonian structures in Example 7.6, with

$$\mathscr{D} = D_x, \qquad \mathscr{E} = D_x^3 + \tfrac{2}{3}uD_x + \tfrac{1}{3}u_x.$$

The operator connecting our hierarchy of Hamiltonian symmetries is thus

$$\mathscr{R} = \mathscr{E}\cdot\mathscr{D}^{-1} = D_x^2 + \tfrac{2}{3}u + \tfrac{1}{3}u_x D_x^{-1},$$

which is nothing but the Lenard recursion operator of Section 5.2! Thus our results on bi-Hamiltonian systems will provide ready-made proofs of the existence of infinitely many conservation laws and symmetries for the Korteweg–de Vries equation.

To proceed rigorously, however, we need to ensure that the two Hamiltonian structures be "compatible" in the following precise sense:

Definition 7.19. A pair of skew-adjoint $q \times q$ matrix differential operators $\mathscr{D}$ and $\mathscr{E}$ is said to form a *Hamiltonian pair* if every linear combination $a\mathscr{D} + b\mathscr{E}$, $a, b \in \mathbb{R}$, is a Hamiltonian operator. A system of evolution equations is a *bi-Hamiltonian system* if it can be written in the form (7.28) where $\mathscr{D}$, $\mathscr{E}$ form a Hamiltonian pair.

Lemma 7.20. *If $\mathscr{D}$, $\mathscr{E}$ are skew-adjoint operators, then they form a Hamiltonian pair if and only if $\mathscr{D}$, $\mathscr{E}$ and $\mathscr{D} + \mathscr{E}$ are all Hamiltonian operators.*

PROOF. Note that the Jacobi identity, in any of its forms (7.3), (7.11) or (7.18) is a *quadratic* expression in $\mathscr{D}$. The lemma is then an easy consequence of the fact that any quadratic polynomial vanishing at three distinct points must vanish identically. More specifically, given $P, Q, R \in \mathscr{A}^q$, let $\mathscr{J}(\mathscr{D}, \mathscr{D}; P, Q, R)$ denote the left-hand side of (7.11). The corresponding symmetric bilinear form is

$$\mathscr{J}(\mathscr{D}, \mathscr{E}; P, Q, R) =$$

$$\tfrac{1}{2}\int [P\cdot \operatorname{pr} \mathbf{v}_{\mathscr{D}R}(\mathscr{E})Q + R\cdot \operatorname{pr} \mathbf{v}_{\mathscr{D}Q}(\mathscr{E})P + Q\cdot \operatorname{pr} \mathbf{v}_{\mathscr{D}P}(\mathscr{E})R$$

$$+ P\cdot \operatorname{pr} \mathbf{v}_{\mathscr{E}R}(\mathscr{D})Q + R\cdot \operatorname{pr} \mathbf{v}_{\mathscr{E}Q}(\mathscr{D})P + Q\cdot \operatorname{pr} \mathbf{v}_{\mathscr{E}P}(\mathscr{D})R]\, dx. \tag{7.30}$$

If $\mathscr{D}$, $\mathscr{E}$ and $\mathscr{D} + \mathscr{E}$ are Hamiltonian, then

$$\mathscr{J}(\mathscr{D}, \mathscr{D}; P, Q, R) = \mathscr{J}(\mathscr{E}, \mathscr{E}; P, Q, R) = 0$$

and

$$\begin{aligned}\mathscr{J}(\mathscr{D} + \mathscr{E}, \mathscr{D} + \mathscr{E}; P, Q, R) &= \mathscr{J}(\mathscr{D}, \mathscr{D}; P, Q, R) + 2\mathscr{J}(\mathscr{D}, \mathscr{E}; P, Q, R) \\ &\quad + \mathscr{J}(\mathscr{E}, \mathscr{E}; P, Q, R) \\ &= 0,\end{aligned}$$