# 9 Introduction to renormalization theory and renormalization group (RG)

In Chapter 8, we have shown that a straightforward construction of a local, relativistic quantum field theory (QFT) leads to ultraviolet (UV) divergences, and that a QFT has to be regularized by modifying its short-distance or large-energy–momentum structure. Since such a modification is somewhat arbitrary, it is necessary to prove that the resulting large scale predictions are, at least to a large extent, short-distance insensitive.

Such a proof relies on the renormalization theory, as initiated in Refs. [24, 25, 54], and the corresponding RG [55] (for early works, see Refs. [56]). Therefore, we describe here the essential steps of a proof of the *perturbative renormalizability* of the scalar $\phi^4$ QFT in dimension 4. All the basic difficulties of renormalization theory are already present in this simple example, and it will eventually become apparent how to extend the method, based on power counting, to other theories (however, the preservation of symmetries requires a specific discussion).

We have followed the elegant presentation of Callan [57, 58], which makes it possible to prove renormalizability, and RG equations (in Callan–Symanzik's form [60]) simultaneously. This presentation is especially suited to our general purpose, since a large part of this work is devoted to applications of RG. Moreover, it already emphasizes, at the technical level, the direct relation between renormalizability and the existence of an RG. Although this may not always be obvious, the background of the discussion is effective field theory (EFT), and emergent renormalizable theory (see Sections 8.8.1 and 8.9).

As a technical tool, we define the initial unrenormalized theory by *momentum cut-off regularization*, although this leads to a derivation slightly more complicated than dimensional regularization. However, dimensional regularization has no direct physical interpretation and, moreover, already performs a partial renormalization, since it cancels what in momentum regularization are power-law divergences. This may lead to erroneous conclusions from the physics viewpoint. By contrast, the inverse of the momentum cut-off is an artificial replacement for a true initial short-distance scale below which the *local EFT* is no longer meaningful.

Renormalization involves a renormalization of the field necessary to generate a limiting distribution, something expected when one sums over an infinite number of random variables (*cf.* the central limit theorem of probabilities). However, we have already pointed out that it also requires *a tuning of the parameters of the initial action as a function of the regularization parameter*, something non-physical in the framework of particle physics and which, even in statistical physics, is peculiar. *The interpretation of QFTs as effective low energy or large distance theories* (EFTs), and the RG formulated in terms of initial (bare) parameters, provide a framework for discussing this issue.

The proof of renormalizability given here applies only to massive theories and, thus, in Section 9.9, we discuss the existence of a massless theory. A different, homogeneous form of RG equations follows. In Section 9.10.1, we discuss the covariance of RG functions.

Another, non-perturbative RG approach, which is outlined in Appendix A9.1, deals with a general EFT. It has been employed to give an alternative proof of renormalizability [59]. It relies on a partial integration of large momentum modes in a form proposed by Wegner and Wilson [61–63], and reduces to the QFT bare RG in the perturbative regime.



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It is described in the spirit of this work in Chapter 16 of Ref. [64].

Appendix A9.2 contains a few remarks about divergences in super-renormalizable theories and the technique of normal-ordering.

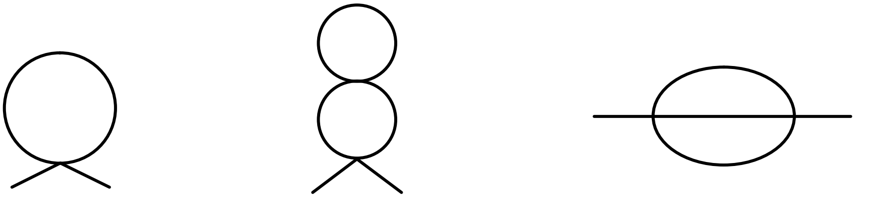


**Fig. 9.1** Contributions to the $\phi$-field two-point vertex function: $l = 0$, $n = 2$, $\delta = 2$

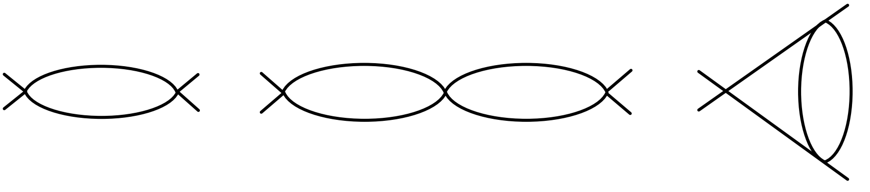


**Fig. 9.2** Contributions to the $\phi$-field four-point vertex function: $l = 0$, $n = 4$, $\delta = 0$

## 9.1 Power counting. Dimensional analysis

We discuss the divergences of the correlation functions of a relativistic QFT corresponding to the classical, non-regularized, Euclidean action for a scalar field $\phi(x)$ in four dimensions (the dimension relevant to particle physics),

$$\mathcal{S}(\phi) = \int \mathrm{d}^4 x \left[ \tfrac{1}{2} \left( \nabla \phi(x) \right)^2 + \tfrac{1}{2} r \phi^2(x) + \frac{1}{4!} g \phi^4(x) \right], \tag{9.1}$$

where $\nabla \equiv \{\partial/\partial x_1, \ldots, \partial/\partial x_4\}$, and $r$ and $g \geq 0$ are two constant parameters.

When the coupling constant $g$ vanishes, the coefficient $r$ of $\phi^2$ is the physical mass squared but, for $g > 0$, the existence of a necessary phase transition requires $r < 0$.

Renormalization theory is mainly formulated in terms of *vertex (one-particle irreducible or 1PI) functions*,

$$\Gamma^{(n)}(x_1, \ldots, x_n) = \langle \phi(x_1) \cdots \phi(x_n) \rangle_{1\mathrm{PI}} ,$$

(in the notations of equation (8.1)) which have simpler properties than correlation functions. These are generated by the functional $\Gamma(\varphi)$, Legendre transform of the generating functional of connected correlation functions (Section 7.7),

$$\Gamma(\varphi) = \sum_{n=0} \frac{1}{n!} \int \Gamma^{(n)}(x_1, \ldots, x_n) \prod_{i=1}^{n} \mathrm{d}^4 x_i \, \varphi(x_i). \tag{9.2}$$

*Power counting in four dimensions.* In Section 8.3, we have shown that the $\phi^4$ vertex has UV dimension 0 (which in a scalar QFT is the same as mass dimension) and the action is renormalizable in the sense of power counting: the superficial degree of divergence of vertex functions is independent of the order in perturbation theory.

The degree of divergence $\delta$ of the vertex function $\Gamma^{(n)}$ is $\delta = 4 - n$ (equations (8.9) and (8.14)). In Figs. 9.1 and 9.2, the superficially divergent one- and two-loop Feynman diagrams corresponding to the two-point and four-point functions are displayed.

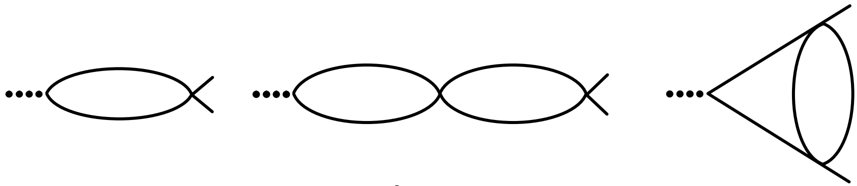


**Fig. 9.3** Contributions to the $\langle\phi^2\phi\phi\rangle$ vertex function: $l = 1$, $n = 2$, $\delta = 0$

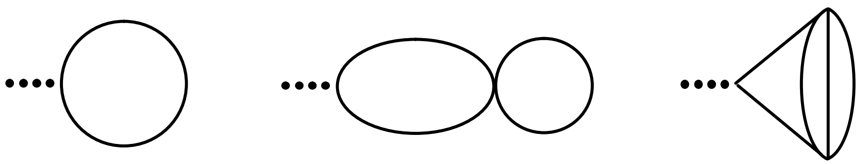


**Fig. 9.4** Contributions to the constant $\langle\phi^2\rangle$ expectation value: $l = 1$, $n = 0$, $\delta = 2$

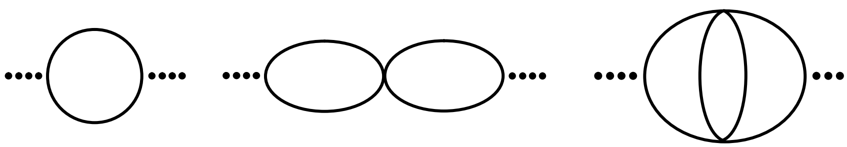


**Fig. 9.5** Contributions to the $\langle\phi^2\phi^2\rangle$ correlation function: $l = 2$, $n = 0$, $\delta = 0$

$\phi^2(x)$ *insertions.* We also need vertex functions with $\phi^2(x)$ insertions. The degree $\delta$ of divergence of the vertex function with $l$ insertions of $\frac{1}{2}\phi^2$,

$$\Gamma^{(l,n)}(y_1,\ldots,y_l;x_1,\ldots,x_n) = 2^{-l}\left\langle\phi^2(y_1)\cdots\phi^2(y_l)\phi(x_1)\cdots\phi(x_n)\right\rangle_{1\mathrm{PI}},$$

is $\delta = 4 - n - 2l$ (equation (8.43)).

In Figs. 9.3–9.5, the first few superficially divergent diagrams corresponding to the new divergent functions are displayed. Diagrams with $n = 0$ never arise as subdiagrams, since they have no external lines, and can be discussed separately.

## 9.2 Regularization. Bare and renormalized QFT

To define a finite perturbation theory, we replace the Euclidean action $\mathcal{S}(\phi)$ by a regularized action $\mathcal{S}_\Lambda(\phi)$, using one of the regularization schemes described in Section 8.4.2. We refer explicitly to a cut-off, because it is in the spirit of this work. Some arguments would have to be slightly modified in the case of dimensional regularization.

*Cut-off regularization.* We introduce the regularized action (the subscript $\Lambda$ emphasizes that we are using a momentum-cut-off scheme),

$$\mathcal{S}_\Lambda(\phi) = \int \mathrm{d}^4x \left[\tfrac{1}{2}\left(\nabla_\Lambda\phi(x)\right)^2 + \tfrac{1}{2}r\phi^2(x) + \frac{1}{4!}g\phi^4(x)\right], \tag{9.3}$$

where (a form of Pauli–Villars's regularization),

$$\nabla_\Lambda \equiv \nabla\prod_{i=1}^{s}\left(1 - \nabla^2/M_i^2\right)^{1/2}, \tag{9.4}$$

and the masses $M_i > 0$ are of the order of the cut-off $\Lambda$.

To generate finite $\phi$ and $\phi^2$ vertex functions in four dimensions, $s = 2$ suffices (the fields contributing to the field integral then become continuous). The regularization yields propagators, in the Fourier representation, of the form

$$[\tilde{\Delta}(p)]^{-1} = p^2 \prod_{i=1}^{2} (1 + p^2/M_i^2) + r\,.$$

*Dimensional analysis.* In what follows, mass dimensional analysis, which in a scalar QFT is equivalent to UV dimension, is useful: the action is dimensionless, space has dimension $-1$, by convention. Then, the field $\phi$ has mass dimension 1, as well as the cut-off $\Lambda$. The parameter $r$ has dimension 2 and the coupling $g$ is dimensionless.

Connected correlation functions satisfy

$$W^{(n)}(x_1/\lambda, \ldots, x_n/\lambda; \lambda^2 r, g, \lambda\Lambda) = \lambda^n W^{(n)}(x_1, \ldots, x_n; r, g, \Lambda). \tag{9.5}$$

After Fourier transformation and factorization of the $\delta$-function of momentum conservation, one obtains

$$\widetilde{W}^{(n)}(\lambda p_1, \ldots, \lambda p_n; \lambda^2 r, g, \lambda\Lambda) = \lambda^{4-3n} \widetilde{W}^{(n)}(p_1, \ldots, p_n; r, g, \Lambda). \tag{9.6}$$

From the expansion (9.2), one infers the scaling of the vertex function $\Gamma^{(n)}(x_1, \ldots, x_n)$, and its Fourier transform $\tilde{\Gamma}^{(n)}(p_1, \ldots, p_n)$ (after factorization of a $\delta$-function):

$$\begin{aligned} \Gamma^{(n)}(x_1/\lambda, \ldots, x_n/\lambda; \lambda^2 r, g, \lambda\Lambda) &= \lambda^{3n} \Gamma^{(n)}(x_1, \ldots, x_n; r, g, \Lambda), \\ \tilde{\Gamma}^{(n)}(\lambda p_1, \ldots, \lambda p_n; \lambda^2 r, g, \lambda\Lambda) &= \lambda^{4-n} \tilde{\Gamma}^{(n)}(p_1, \ldots, p_n; r, g, \Lambda). \end{aligned}$$

The function $\tilde{\Gamma}^{(n)}$ has a mass dimension $(4 - n)$, which coincides with its UV dimension in the sense of power counting.

The argument generalizes to

$$\tilde{\Gamma}^{(l,n)}(\lambda q_i, \lambda p_j; \lambda^2 r, g, \lambda\Lambda) = \lambda^{4-n-2l} \tilde{\Gamma}^{(l,n)}(q_i, p_j; r, g, \Lambda). \tag{9.7}$$

### *9.2.1 Bare and renormalized action: Counter-terms*

Before we discuss renormalization, a few definitions are required. The action (9.3) is often called the *bare action.* The parameters $r$ and $g$ that appear in the action are then called *bare parameters*; the correlation functions of the *bare* field $\phi$ are *bare correlation functions.* From the physics viewpoint, the bare action is obtained from the *effective action at the cut-off scale* by performing a Gaussian renormalization (equation (8.52)) and neglecting all non-renormalizable interactions.

To the bare parameters $r$ and $g$ correspond two *renormalized parameters* (or effective parameters at the physical scale) $m_\mathrm{r}$ and $g_\mathrm{r}$: $m_\mathrm{r}$ characterizes the imaginary-time decay of correlation functions and is proportional to the physical mass of the theory; $g_\mathrm{r}$ is dimensionless and becomes the expansion parameter of the renormalized theory.

We define a renormalized field $\phi_\mathrm{r}$ obtained from the bare field by a rescaling,

$$\phi(x) = Z^{1/2} \phi_\mathrm{r}(x). \tag{9.8}$$

The correlation functions of the field $\phi_\mathrm{r}$ are the *renormalized correlation functions.*

We also define the *renormalized action* $\mathcal{S}_{\rm r}$, which is the initial action expressed in terms of the renormalized field $\phi_{\rm r}$ and renormalized parameters, which we write as

$$\mathcal{S}_\Lambda(\phi) \equiv \mathcal{S}_{\rm r}(\phi_{\rm r}) = \mathcal{S}_{{\rm r},0}(\phi_{\rm r}) + \mathcal{S}_{\rm C.T.}(\phi_{\rm r}), \tag{9.9}$$

where $\mathcal{S}_{{\rm r},0}(\phi_{\rm r})$ is the *tree order action* (which, for $\Lambda \to \infty$, reduces to the classical action):

$$\mathcal{S}_{{\rm r},0}(\phi_{\rm r}) = \int {\rm d}^4 x \left[\tfrac{1}{2}\big(\nabla_\Lambda \phi_{\rm r}(x)\big)^2 + \tfrac{1}{2} m_{\rm r}^2 \phi_{\rm r}^2(x) + \frac{1}{4!} g_{\rm r} \phi_{\rm r}^4(x)\right], \tag{9.10}$$

$\mathcal{S}_{\rm C.T.}(\phi_{\rm r})$ is the sum of *counter-terms*:

$$\mathcal{S}_{\rm C.T.}(\phi_{\rm r}) = \int {\rm d}^4 x \left[\tfrac{1}{2}(Z-1)\big(\nabla \phi_{\rm r}(x)\big)^2 + \tfrac{1}{2} Z \delta r \phi_{\rm r}^2(x) + \frac{1}{4!} g_{\rm r}(Z_g - 1)\phi_{\rm r}^4(x)\right], \tag{9.11}$$

and $\delta r$ and $Z_g$ are two additional renormalization constants. The identity between the renormalized action (9.9) and the regularized action (9.3) is expressed by relation (9.8) and the relations between renormalized and bare quantities:

$$g = g_{\rm r} Z_g / Z^2, \quad r = m_{\rm r}^2/Z + \delta r\,, \tag{9.12}$$

where $Z$ is the field renormalization constant, $Z_g/Z^2$ is the coupling constant renormalization constant and $\delta r$ characterizes the mass renormalization.

*Perturbative renormalization theorem.* In this chapter, we derive (non-rigorously) the following theorem: the renormalization constants $\delta r$, $Z_g$, and $Z$, calculated as power series in $g_{\rm r}$ of the form,

$$\begin{cases} \delta r = \Lambda^2 \left[a_1(\Lambda) g_{\rm r} + a_2(\Lambda) g_{\rm r}^2 + O(g_{\rm r}^3)\right] \\ Z_g = 1 + b_1(\Lambda) g_{\rm r} + b_2(\Lambda) g_{\rm r}^2 + O(g_{\rm r}^3) \\ Z = 1 + c_1(\Lambda) g_{\rm r} + c_2(\Lambda) g_{\rm r}^2 + O(g_{\rm r}^3), \end{cases} \tag{9.13}$$

can be chosen as functions of $g_{\rm r}$, in such a way that the renormalized vertex functions, order by order in $g_{\rm r}$, have a finite limit for $\Lambda \to \infty$ (a mathematical limit that is not necessarily physical).

*Perturbative and loop expansions.* The action (9.1) can be rewritten as

$$\mathcal{S}(\phi) = \frac{1}{g}\int {\rm d}^4 x \left[\tfrac{1}{2}\big(\sqrt{g}\,\nabla\phi(x)\big)^2 + \tfrac{1}{2} r\big(\sqrt{g}\phi(x)\big)^2 + \frac{1}{4!}\big(\sqrt{g}\phi(x)\big)^4\right]. \tag{9.14}$$

The loop expansion in the $\phi^4$ QFT is thus an expansion in powers of $g$ at $\phi\sqrt{g}$ fixed.

Because the renormalization constants are series in $g_{\rm r}$, the expansion in powers of $g_{\rm r}$ at $(\sqrt{g_{\rm r}}\phi_{\rm r})$ fixed is no longer a loop expansion in the diagrammatic sense, in contrast with the expansion in powers of $g$ (equation (9.14)). At order $g_{\rm r}^{L-1}$, contribute $L$ loop diagrams and diagrams with less than $L$ loops multiplied by contributions coming from renormalization constants. Nevertheless, in the next section, when no confusion is possible, we call the expansion in powers of $g_{\rm r}$ at $(\sqrt{g_{\rm r}}\phi_{\rm r})$ fixed, loop expansion.

### 9.2.2 *Bare and renormalized correlation functions*

The relation between bare and renormalized fields $\phi = Z^{1/2}\phi_{\rm r}$ directly implies the relations between connected renormalized and bare correlation functions,

$$W_{\rm r}^{(n)}(x_1, \ldots, x_n) = Z^{-n/2} W^{(n)}(x_1, \ldots, x_n),$$

which can be summarized by

$$\mathcal{W}(J/\sqrt{Z}) = \mathcal{W}_{\rm r}(J), \tag{9.15}$$

in which $\mathcal{W}(J)$ and $\mathcal{W}_{\rm r}(J)$ are, respectively, the generating functionals of bare and renormalized connected correlation functions.

After Legendre transformation, one verifies that the corresponding generating functionals of vertex functions are related by ($J$ and $\varphi$ are dual)

$$\Gamma_{\mathrm{r}}(\varphi) = \Gamma(\varphi\sqrt{Z}), \tag{9.16}$$

a relation that, for the renormalized and bare vertex functions, translates into

$$\Gamma_{\mathrm{r}}^{(n)}(x_1, \dots, x_n) = Z^{n/2}\Gamma^{(n)}(x_1, \dots, x_n). \tag{9.17}$$

*Operator $\phi^2$ insertions.* We also need the bare and renormalized $\phi^2$ insertions. Therefore, we introduce a source $J_2(x)$ for $\phi^2(x)$ and add to the bare action (9.1) the corresponding source term:

$$\mathcal{S}(\phi, J_2) = \mathcal{S}(\phi) + \tfrac{1}{2}\int \mathrm{d}^4x\, J_2(x)\phi^2(x). \tag{9.18}$$

Functional differentiations with respect to $J_2(x)$ of the field integral

$$\mathcal{Z}(J, J_2) = \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}(\phi, J_2) + \int \mathrm{d}^4x\, J(x)\phi(x)\right], \tag{9.19}$$

then generate insertions of the operator $-\frac{1}{2}\phi^2(x)$ (Section 8.6). Similarly, the field integral

$$\mathcal{Z}_{\mathrm{r}}(J, J_2) = \int [\mathrm{d}\phi_{\mathrm{r}}] \exp\left[-\mathcal{S}_{\mathrm{r}}(\phi_{\mathrm{r}}, J_2) + \int \mathrm{d}^4x\, J(x)\phi_{\mathrm{r}}(x)\right], \tag{9.20}$$

where

$$\mathcal{S}_{\mathrm{r}}(\phi_{\mathrm{r}}, J_2) = \mathcal{S}_{\mathrm{r}}(\phi_{\mathrm{r}}) + \tfrac{1}{2}Z_2\int J_2(x)\phi_{\mathrm{r}}^2(x)\mathrm{d}^4x\,, \tag{9.21}$$

in which $Z_2$ is a new renormalization constant, generates the renormalized correlation functions with $-\frac{1}{2}\phi^2$ insertions (we temporarily normalize $\mathcal{Z}(J, J_2)$ to $\mathcal{Z}(0, J_2) = 1$ in order to eliminate the pure $\phi^2(x)$ correlation functions).

The relation between renormalized and bare functionals is then

$$\mathcal{W}_{\mathrm{r}}(J, J_2) = \mathcal{W}(J/\sqrt{Z}, J_2Z_2/Z). \tag{9.22}$$

After Legendre transformation, the relation implies

$$\Gamma_{\mathrm{r}}(\varphi, J_2) = \Gamma(\varphi\sqrt{Z}, J_2Z_2/Z), \tag{9.23}$$

or, in terms of vertex functions,

$$\Gamma_{\mathrm{r}}^{(l,n)}(y_1, \dots, y_l; x_1, \dots, x_n) = Z^{(n/2)-l}Z_2^l\Gamma^{(l,n)}(y_1, \dots, y_l; x_1, \dots, x_n). \tag{9.24}$$

If the source $J_2$ is a constant, then it simply generates a shift of the bare parameter $r$, and the action (9.18) can be rewritten as

$$\mathcal{S}(\phi, J_2) = \int \mathrm{d}^4x \left[\tfrac{1}{2}\big(\nabla\phi(x)\big)^2 + \tfrac{1}{2}(r + J_2)\phi^2(x) + \tfrac{1}{4!}g\phi^4(x)\right]. \tag{9.25}$$

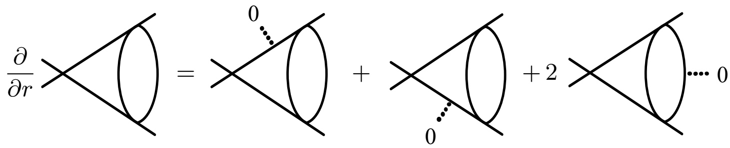



**Fig. 9.6** A contribution to $\partial\Gamma^{(4)}/\partial r$ and the corresponding contributions to $\Gamma^{(1,4)}$

In this limit, differentiation with respect to $J_2$ is equivalent to differentiation with respect to $r$. Moreover, if $J_2(x)$ is a constant, its Fourier transform is proportional to $\delta^{(d)}(p)$, which means that it generates vertex functions with insertions of the Fourier transform of $\phi^2(x)$ at zero momentum,

$$\left.\frac{\partial}{\partial r}\right|_g \tilde{\Gamma}^{(l,n)}(q_1,\ldots,q_l;p_1,\ldots,p_n)=\tilde{\Gamma}^{(l+1,n)}(0,q_1,\ldots,q_l;p_1,\ldots,p_n). \qquad (9.26)$$

Equation (9.26) has a diagrammatic interpretation: the diagrams contributing to the right-hand side are obtained from the diagrams contributing to $\Gamma^{(l,n)}$ by doubling a propagator in all possible ways (up to a sign). In Fig. 9.6, we illustrate the relation between $\Gamma^{(4)}$ and $\Gamma^{(1,4)}$ with a two-loop order diagram.

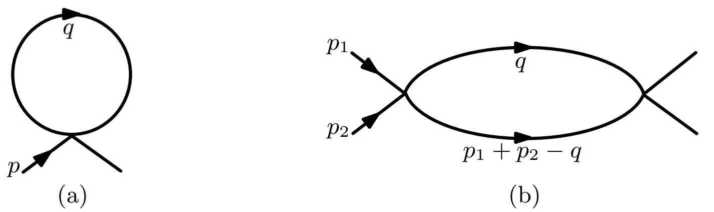



**Fig. 9.7** One-loop divergent diagrams in the $\phi^4$ QFT

## 9.3 One-loop divergences

In Section 8.5, we have calculated the one-loop divergences in the $\phi^3_6$ (the subscript refers to the space dimension) QFT. We examine here the one-loop divergences of the $\phi^4_4$ QFT. We expand the generating functional of renormalized vertex functions $\Gamma_{\rm r}(\varphi)$ at one-loop order (really in powers of $g_{\rm r}$ at $\varphi\sqrt{g_{\rm r}}$ fixed).

At tree order, the counter-terms do not contribute and thus (equation (9.10)),

$$\begin{aligned}\Gamma_{\rm r}(\varphi)=\Gamma_{{\rm r},0}(\varphi)&=\lim_{\Lambda\to\infty}\mathcal{S}_{{\rm r},0}(\varphi)\\&=\int {\rm d}^4x\left[\tfrac12\big(\nabla\varphi(x)\big)^2+\tfrac12 m_{\rm r}^2\varphi^2(x)+\frac{1}{4!}g_{\rm r}\varphi^4(x)\right].\end{aligned} \qquad (9.27)$$

It follows that,

$$\tilde{\Gamma}^{(2)}_{\rm r}(p)=p^2+m_{\rm r}^2,\quad \tilde{\Gamma}^{(4)}_{\rm r}(p_1,\ldots,p_4)=g_{\rm r}\,,\quad \tilde{\Gamma}^{(n)}_{\rm r}=0\,,\ \text{for } n>4\,. \qquad (9.28)$$

At one-loop order contribute the one-loop terms generated by the tree order action and the counter-terms at leading order. The former is given by equation (7.93):

$$\begin{aligned}\Gamma_1(\varphi)&=\tfrac12\,{\rm tr}\ln\left[1+\left(m_{\rm r}^2-\nabla_\Lambda^2\right)^{-1}g_{\rm r}\varphi^2/2\right]\\&=\tfrac14\,{\rm tr}\left(m_{\rm r}^2-\nabla_\Lambda^2\right)^{-1}g_{\rm r}\varphi^2-\tfrac{1}{16}\,{\rm tr}\left[\left(m_{\rm r}^2-\nabla_\Lambda^2\right)^{-1}g_{\rm r}\varphi^2\right]^2+O\left(\varphi^6\right).\end{aligned}$$

The first two terms in the expansion in powers of $\varphi^2$ correspond to the two divergent diagrams displayed in Fig. 9.7.

Using a specific regularization of the form (9.4) with $s=2$, $M_1=M_2=\Lambda$ (to simplify the propagator, to $m_{\mathrm{r}}^2$ have been added terms of order $m_{\mathrm{r}}^4/\Lambda^2$), one finds:
(i) for $n=2$, the coefficient (a) of $\frac{1}{2}g_{\mathrm{r}}\varphi^2$,

$$\frac{1}{2}\,(\mathrm{a}) \;=\frac{1}{2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+m_{\mathrm{r}}^2)\,(1+p^2/\Lambda^2)^2}=\frac{1}{16\pi^2}\left(\frac{\Lambda^2}{2}-m_{\mathrm{r}}^2\ln\frac{\Lambda}{m_{\mathrm{r}}}\right)+O\left(\Lambda^0\right);$$

(ii) for $n=4$, the coefficient (b) of $g_{\mathrm{r}}^2\varphi^4/4!$,

$$-\frac{3}{2}(\mathrm{b})=-\frac{3}{2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+m_{\mathrm{r}}^2)_\Lambda\left[(q-p_1-p_2)^2+m_{\mathrm{r}}^2\right]_\Lambda}\sim-\frac{3}{16\pi^2}\ln\frac{\Lambda}{m_{\mathrm{r}}}\,.$$

The divergent part $\Gamma_1^{\mathrm{div.}}(\varphi)$ of $\Gamma_1(\varphi)$ generated by the tree action (9.10) is thus

$$\Gamma_1^{\mathrm{div.}}=\frac{1}{16\pi^2}\int\mathrm{d}^4x\left[\frac{1}{2}\left(\frac{\Lambda^2}{2}-m_{\mathrm{r}}^2\ln\frac{\Lambda}{m_{\mathrm{r}}}\right)g_{\mathrm{r}}\varphi^2(x)-\frac{3}{4!}\ln\frac{\Lambda}{m_{\mathrm{r}}}g_{\mathrm{r}}^2\varphi^4(x)\right].\tag{9.29}$$

Note the absence of a term proportional to $\int\mathrm{d}^4x(\nabla\varphi_{\mathrm{r}})^2$ in the $\phi^4$ QFT at one-loop order, as a consequence at one-loop order of the $\phi\to-\phi$ symmetry of the action.

We now consider the modified action

$$\mathcal{S}_{\mathrm{r},1}(\phi_{\mathrm{r}})=\mathcal{S}_{\mathrm{r},0}(\phi_{\mathrm{r}})-\Gamma_1^{\mathrm{div}}(\phi_{\mathrm{r}}).$$

At this order, $\Gamma_1^{\mathrm{div}}(\phi_{\mathrm{r}})$ contributes additively to $\Gamma_{\mathrm{r}}(\varphi)$. Its addition cancels the divergences in the $\Lambda\to\infty$ limit, and the $\phi^4$ QFT is renormalized at one-loop order,

$$\Gamma_{\mathrm{r}}(\varphi)=\Gamma_{\mathrm{r},0}(\varphi)+\Gamma_{\mathrm{r},1}(\varphi),\quad\text{with}\quad\Gamma_{\mathrm{r},1}(\varphi)\underset{\Lambda\to\infty}{=}\Gamma_1(\varphi)-\Gamma_1^{\mathrm{div}}(\varphi).$$

Identifying $\mathcal{S}_{\mathrm{r},1}(\phi_{\mathrm{r}})$ with the action (9.9), we infer the divergent part of the counter-terms expanded at one-loop order and, in the parametrization (9.13), $a_1$, $b_1$, and $c_1$.

The condition of finiteness of vertex functions determines these coefficients only up to arbitrary finite constants. The difference between the vertex functions corresponding to two different choices is of the form of the tree order functions, that is, a constant for the four-point function in Fourier space and a first degree polynomial in $p^2$, $p$ being the momentum, for the two-point function. In this chapter, it is convenient to impose a set of *renormalization conditions* to the renormalized vertex functions $\Gamma_{\mathrm{r}}^{(n)}$. In the Fourier representation, we impose

$$\begin{gathered}\tilde{\Gamma}_{\mathrm{r}}^{(2)}(p=0)=m_{\mathrm{r}}^2\,,\\ \frac{\partial}{\partial p^2}\tilde{\Gamma}_{\mathrm{r}}^{(2)}\;(p)|_{p=0}=1\,,\\ \tilde{\Gamma}_{\mathrm{r}}^{(4)}(0,0,0,0)=g_{\mathrm{r}}\,,\end{gathered}\tag{9.30}$$

conditions consistent with the tree approximation (9.28). They completely determine the three renormalization constants. At one-loop order, one obtains

$$\begin{cases}a_1(\Lambda)=-\dfrac{1}{2}\displaystyle\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+m_{\mathrm{r}}^2)_\Lambda}=-\frac{1}{16\pi^2}\left(\tfrac{1}{2}\Lambda^2-m_{\mathrm{r}}^2\ln(\Lambda/m_{\mathrm{r}})-\tfrac{1}{2}m_{\mathrm{r}}^2\right),\\ b_1(\Lambda)=\dfrac{3}{2}\displaystyle\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+m_{\mathrm{r}}^2)_\Lambda^2}=\frac{1}{16\pi^2}\left(3\ln(\Lambda/m_{\mathrm{r}})-\tfrac{17}{4}\right),\\ c_1(\Lambda)=0\,.\end{cases}\tag{9.31}$$

*Operator $\phi^2$ insertions.* One obtains the one-loop contribution to the superficially divergent $\langle\phi^2\phi\phi\rangle$ vertex (1PI) function by adding $J_2(x)$ to $\frac{1}{2}g_r\varphi_r^2$ in the tr ln, and identifying the $J_2\varphi^2$ term:

$$\tilde{\Gamma}^{(1,2)}(q;p_1,p_2) = 1 - \frac{1}{2}\frac{g_r}{(2\pi)^4}\int \frac{\mathrm{d}^4k}{(k^2+m_r^2)_\Lambda[(k+q)^2+m_r^2]_\Lambda} + O(g_r^2). \tag{9.32}$$

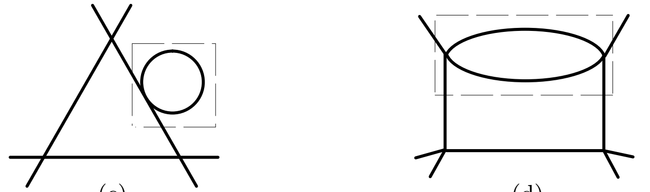
The additional renormalization condition,

$$\tilde{\Gamma}_r^{(1,2)}(q=0;p_1=p_2=0)=1\,, \tag{9.33}$$

which again is consistent with the tree approximation, determines the new renormalization constant $Z_2$ (equation (9.24)). At one-loop order,

$$Z_2 - 1 = \frac{1}{2}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+m_r^2)_\Lambda^2}g_r + O(g_r^2) = \frac{1}{16\pi^2}\left(\ln(\Lambda/m_r) - \tfrac{17}{12}\right)g_r + O(g_r^2)\,. \tag{9.34}$$

*The fine-tuning problem.* In the conventional interpretation of the renormalization procedure, one adjusts the parameters of the initial (or bare) action as functions of the cut-off and physical parameters, and then takes the infinite cut-off limit. This tuning procedure is quite strange, since the bare parameters are singular in this limit and become meaningless. Although we will follow this formal procedure, we will give it a different interpretation. First, we will implicitly assume that the cut-off is large but not infinite (we simply neglect the contributions that formally vanish for infinite cut-off in the perturbative expansion). Second, for the fine tuning of the $\phi^2$ coefficient, we have a physics interpretation in the framework of statistical physics: this parameter plays the role of the temperature and we know that the correlation length can become large only near the critical temperature. However, for particle physics, the origin of such tuning, which demands fixing values of parameters with unreasonable precision, is unknown. Finally, the tuning of the coupling constant renormalization is examined in Section 9.11. Its interpretation involves 'bare' (and perturbative) RG equations.



(c) (d)

**Fig. 9.8** The six-point function: divergent subdiagrams

(c') (d')

**Fig. 9.9** Two-loop contributions from one-loop counter-terms

## 9.4 Divergences beyond one-loop: Skeleton diagrams

Power counting shows that, to all orders in the loop expansion, the two-point function has superficial quadratic divergences and the four-point function logarithmic divergences. However, at higher orders a new difficulty arises: superficially convergent diagrams may have divergent subdiagrams. For example, the six-point function at one-loop order is finite but, at two-loop order, the diagrams displayed in Fig. 9.8 contribute.

Inside the dashed boxes, one recognizes divergent subdiagrams. However, they can be identified with one-loop divergences of the two-point function (c) and the four-point function (d), for which counter-terms have already been provided. Indeed, at this order a diagram (c′) appears, in which the one-loop counter-term for the two-point function is inserted on a propagator, and another one (d′), in which the vertex of the tree order action is replaced by the one-loop counter-term of the four-point function (Fig. 9.9).

This property is generally true: counter-terms that render the divergent functions finite, at higher orders also cancel the divergence of subdiagrams of superficially convergent functions.

The proof of this property is based on the introduction of *skeleton diagrams*: a skeleton diagram is a superficially convergent without divergent subdiagrams.

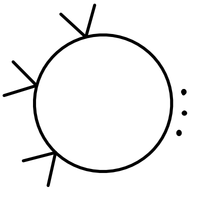


**Fig. 9.10** One-loop skeleton diagrams

For example, the one-loop diagrams of the $2n$-point functions, for $n > 2$, are all skeleton diagrams (see Fig. 9.10).

An arbitrary superficially convergent diagram can then be obtained from a skeleton diagram by replacing all vertices by $\Gamma^{(4)}$ and all propagators by $(\Gamma^{(2)})^{-1}$ and expanding in powers of the coupling constant $g_{\mathrm{r}}$.

For example, the diagrams (c) and (d) are generated by the expansion of the *dressed skeleton diagram* of Fig. 9.11.

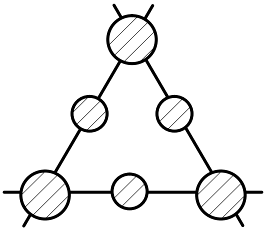


**Fig. 9.11** Dressed skeleton diagram

An important property is the following: if in a dressed skeleton diagram, $\Gamma^{(4)}$ and $\Gamma^{(2)}$ are replaced by the renormalized functions $\Gamma_{\mathrm{r}}^{(4)}$ and $\Gamma_{\mathrm{r}}^{(2)}$, the dressed skeleton diagram is finite.

This is a direct consequence of the bounds on the large momentum behaviour [65]:

$$\left.\begin{aligned} \left|\tilde{\Gamma}_{\mathrm{r}}^{(2)}(\lambda p)\right| &\le \lambda^2 \times \text{ power of } \ln\lambda, \\ \left|\tilde{\Gamma}_{\mathrm{r}}^{(4)}(\lambda p_i)\right| &\le \text{ power of } \ln\lambda, \\ \left|\tilde{\Gamma}_{\mathrm{r}}^{(1,2)}(\lambda q;\lambda p_1,\lambda p_2)\right| &\le \text{ power of } \ln\lambda, \end{aligned}\right\} \text{at any finite order for } \lambda\to\infty. \tag{9.35}$$

A few comments about them can be found at the end of Section 9.6. These bounds for the large momentum behaviour of the various renormalized functions, which are valid for arbitrary momenta, differ from the tree order behaviour only by powers of logarithms (at any finite order in the loop expansion). Therefore, power counting arguments still apply and superficially convergent diagrams are thus convergent.

Similar estimates exist for superficially convergent functions, but are then valid only for generic momenta (see Sections 11.3 and 11.4).

The bounds (9.35) together with the skeleton expansion completely reduce the problem of renormalization of superficially convergent vertex functions to the renormalization of the divergent vertex functions.

The argument also applies to the vertex functions $\Gamma^{(l,n)}$ with $l$ $\phi^2$ insertion. For example, the diagram of Fig. 9.12, which contributes to the superficially convergent function $\Gamma^{(1,4)}$, has divergent subdiagrams and is generated from a skeleton diagram by replacing propagators by $(\Gamma^{(2)})^{-1}$, vertices by $\Gamma^{(4)}$, and the $\phi^2\phi\phi$ vertex by $\Gamma^{(1,2)}$, as shown in Fig. 9.13.

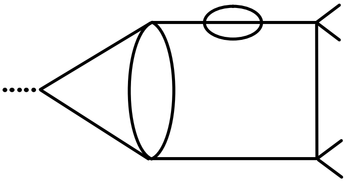


**Fig. 9.12** Divergent contribution to $\Gamma^{(1,4)}$

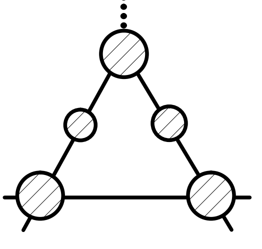


**Fig. 9.13** Dressed skeleton diagram with $\phi^2$ insertion

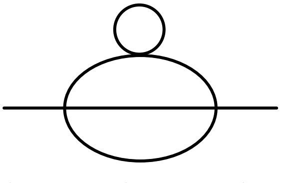


**Fig. 9.14** Three-loop contribution to the two-point function

*Superficially divergent functions.* In the next section, we thus examine the renormalization of $\Gamma^{(2)}$, $\Gamma^{(4)}$ and $\Gamma^{(1,2)}$. The diagrams contributing to these functions are superficially divergent, but also have divergent subdiagrams corresponding to the divergence of the same functions at lower orders. Fig. 9.14 provides an example.

*Overlapping divergences.* However, a new problem arises, the problem of *overlapping divergences.* For example, let us examine the diagram of Fig. 9.15.

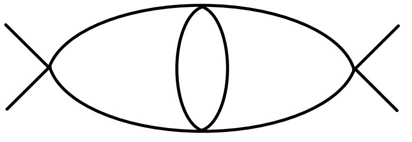


**Fig. 9.15** Three-loop contribution to the four-point function

Figure 9.16 displays the three divergent subdiagrams. These subdiagrams have a common part. Therefore, the concept of insertion of divergent diagrams of lower order is no longer well-defined. This is the problem of the so-called *overlapping divergences.*

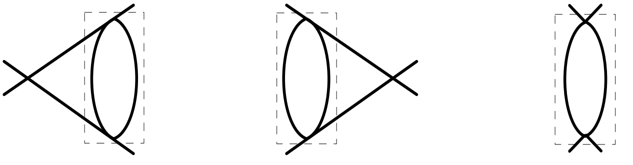


**Fig. 9.16** Overlapping divergent subdiagrams in the diagram 9.15

In Section 9.5, we develop a specific technique, based on differentiating diagrams with respect to the mass, to deal with this problem.

## 9.5 Callan–Symanzik equations

The starting point of the analysis is equation (9.26), which shows that a differentiation with respect to the bare mass improves the large momentum behaviour of Feynman diagrams. This provides a method to relate superficially divergent vertex functions to functions which have a skeleton expansion. Furthermore, at a given number of loops, a function which has a skeleton expansion is only expressed in terms of divergent functions which have at least *one loop less.* We see here a mechanism to prove renormalizability by induction. However, we want to insert into the skeleton expansion renormalized vertex functions. This introduces some additional difficulties that will become apparent once we have transformed equation (9.26) into an equation for the renormalized vertex functions.

We first introduce a notation and apply the chain rule to transform differentiation with respect to $m_{\mathrm{r}}$ at $g, \Lambda$ fixed, into differentiation at $g_{\mathrm{r}}, \Lambda$ fixed:

$$\mathrm{D}_{\mathrm{r}} \equiv m_{\mathrm{r}} \left.\frac{\partial}{\partial m_{\mathrm{r}}}\right|_{g,\Lambda} = m_{\mathrm{r}} \left.\frac{\partial}{\partial m_{\mathrm{r}}}\right|_{g_{\mathrm{r}},\Lambda} + \beta(g_{\mathrm{r}}, m_{\mathrm{r}}/\Lambda) \left.\frac{\partial}{\partial g_{\mathrm{r}}}\right|_{m_{\mathrm{r}},\Lambda} \tag{9.36}$$

with the definition (we take dimensional analysis immediately into account),

$$\mathrm{D}_{\mathrm{r}}\, g_{\mathrm{r}} = \beta\left(g_{\mathrm{r}}, m_{\mathrm{r}}/\Lambda\right). \tag{9.37}$$

Similarly, we define the functions $\eta$, $\eta_2$ and $\sigma$ as

$$\mathrm{D_r}\,\ln Z\left(g_\mathrm{r}, m_\mathrm{r}/\Lambda\right) = \eta\left(g_\mathrm{r}, m_\mathrm{r}/\Lambda\right), \tag{9.38}$$

$$\mathrm{D_r}\,\ln\left(Z_2/Z\right) = \eta_2\left(g_\mathrm{r}, m_\mathrm{r}/\Lambda\right), \tag{9.39}$$

$$Z Z_2^{-1}\left(\mathrm{D_r}\, m^2\right) = m_\mathrm{r}^2 \sigma\left(g_\mathrm{r}, m_\mathrm{r}/\Lambda\right). \tag{9.40}$$

In terms of the differential operator

$$\mathrm{D_{CS}} = \mathrm{D_r} - \tfrac{1}{2} n\,\eta\left(g_\mathrm{r}, m_\mathrm{r}\ \Lambda\right) - l\,\eta_2\left(g_\mathrm{r}, m_\mathrm{r}\ \Lambda\right),$$

one finds the equation,

$$\begin{aligned}&\mathrm{D_{CS}}\tilde{\Gamma}_\mathrm{r}^{(l,n)}(q_1,\ldots,q_l;p_1,\ldots,p_n)\\ &\quad= m_\mathrm{r}^2\sigma\left(g_\mathrm{r}, m_\mathrm{r}/\Lambda\right)\tilde{\Gamma}_\mathrm{r}^{(l+1,n)}(0,q_1,\ldots,q_l;p_1,\ldots,p_n).\end{aligned} \tag{9.41}$$

In the infinite cut-off limit, equation (9.41) becomes an equation for the renormalized vertex functions called Callan–Symanzik (CS) equation [60], which has also played an important role in the calculations of universal quantities in continuous phase transitions (Chapters 15–32).

To prove renormalizability, we prove inductively on the number of loops both the existence of the CS equation and the finiteness of vertex functions.

*Renormalization conditions.* The CS equation in the form (9.41) expresses only that we have rescaled the vertex functions and made an arbitrary change of parametrization. To be able to prove that the renormalized vertex functions have a finite $\Lambda \to \infty$ limit, it is necessary to determine the renormalization constants and, therefore, to impose on equation (9.41) the consequences of the renormalization conditions (9.30) and (9.33).

(i) $n = 2, l = 0$:
at zero momentum, equation (9.41) yields

$$\left(m_\mathrm{r}\frac{\partial}{\partial m_\mathrm{r}} - \eta(g_\mathrm{r}, \Lambda/m_\mathrm{r})\right) m_\mathrm{r}^2 = m_\mathrm{r}^2\sigma(g_\mathrm{r}, \Lambda/m_\mathrm{r}),$$

and thus, the relation

$$\sigma = 2 - \eta\,. \tag{9.42}$$

If one then differentiates the same equation with respect to a momentum squared, one finds

$$-\eta = m_\mathrm{r}^2(2-\eta)\frac{\partial}{\partial p^2}\tilde{\Gamma}_\mathrm{r}^{(1,2)}(0;p,-p)\Big|_{p^2=0}. \tag{9.43}$$

(ii) $n = 4, l = 0$:
at zero momentum, equation (9.41) yields

$$\beta - 2g_\mathrm{r}\eta = m_\mathrm{r}^2(2-\eta)\tilde{\Gamma}_\mathrm{r}^{(1,4)}(0;0,0,0,0). \tag{9.44}$$

(iii) $n = 2, l = 1$:
again, the zero momentum limit yields

$$-\eta - \eta_2 = m_\mathrm{r}^2(2-\eta)\tilde{\Gamma}_\mathrm{r}^{(2,2)}(0,0;0,0). \tag{9.45}$$

We have related all the coefficients of the partial differential equation (9.41) to values of vertex functions at zero momentum. From these relations it follows that, if we can show that the renormalized vertex functions have an infinite cut-off limit, the functions $\beta$, $\eta$, and $\eta_2$ will also have a limit.

Also note that, if we know the coefficients of the CS equations, we can calculate the renormalization constants from the set of equations (9.37–9.40).

*Leading order contributions.*
(i) $\tilde{\Gamma}^{(1,2)}(0;p,-p)$ at order $g$ does not depend on $p$ (equation (9.32)). The equation (9.34) then implies

$$\tilde{\Gamma}_{\rm r}^{(1,2)}(0;p,-p) = 1 + O\left(g_{\rm r}^2\right).$$

Therefore, the expansion of $\eta(g_{\rm r})$, which can be calculated from equation (9.43), begins at order $g_{\rm r}^2$.
(ii) The first diagram contributing to $\tilde{\Gamma}^{(1,4)}$ is of order $g_{\rm r}^2$ (see Fig. 9.17 (a)). It then follows from equation (9.44) and the preceding remark that the function $\beta(g_{\rm r})$ has also an expansion that begins at order $g_{\rm r}^2$. Thus, the operator $\beta_2\partial/\partial g_{\rm r}$, which appears in the CS equation is of order $g_{\rm r}$.
(iii) The function $\tilde{\Gamma}^{(2,2)}$ has a first contribution of order $g_{\rm r}$ (see Fig. 9.17 (b)). Equation (9.45) then shows that $\eta_2$ also begins at order $g_{\rm r}$.

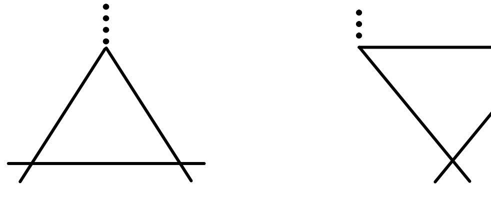


**Fig. 9.17** Contributions (a) to $\tilde{\Gamma}^{(1,4)}$, and (b) to $\tilde{\Gamma}^{(2,2)}$

*Cluster properties and analyticity at low momentum.* In Section 8.5, we have used the regularity of the one-loop diagrams near zero momentum to show that the divergent contributions are polynomials in the momentum variables. This is more generally true: in a massive theory, connected and vertex (or 1PI) functions are analytic functions around zero momentum, as can be seen on the expression of regularized Feynman diagrams. This property, which will again be needed in the inductive proof, implies cluster properties: connected correlation functions decrease exponentially in space for large separations of the arguments (for details, see Section A7.3).

## 9.6 Inductive proof of renormalizability

In Section 9.3, we have constructed a theory finite at one-loop order. We now assume that the vertex functions defined by equations (9.24) and renormalization conditions (9.30, 9.33) have a finite limit up to loop order $L$, at $m_{\rm r}$ and $g_{\rm r}$ fixed, for $\Lambda \to \infty$.

This means that $\tilde{\Gamma}_{\rm r}^{(2)}$, $\tilde{\Gamma}_{\rm r}^{(4)}$, $\tilde{\Gamma}_{\rm r}^{(1,2)}$ have a limit for $\Lambda \to \infty$ up to order $g_{\rm r}^L$, $g_{\rm r}^{L+1}$ and $g_{\rm r}^L$, respectively.

As we have shown in Section 9.5, from equations (9.24), (9.30), and (9.33) follow the CS equations (9.41) and the relations (9.42–9.45).

We now use the CS equation (9.41) in the form

$$m_{\mathrm{r}} \frac{\partial}{\partial m_{\mathrm{r}}} \tilde{\Gamma}_{\mathrm{r}}^{(l,n)} = \left( -\beta_2 \frac{\partial}{\partial g_{\mathrm{r}}} + \frac{n}{2}\eta + l\eta_2 \right) \tilde{\Gamma}_{\mathrm{r}}^{(l,n)} + m_{\mathrm{r}}^2 (2-\eta) \tilde{\Gamma}_{\mathrm{r}}^{(l+1,n)}, \tag{9.46}$$

and show that the right-hand side is finite at loop order $(L+1)$.

We note that $\Gamma^{(l,n)}$ in the right-hand side is only needed at loop order $L$, because its coefficient is of order $g_{\mathrm{r}}$. For $\Gamma^{(l+1,n)}$ two cases arise: either it has a skeleton expansion and is, therefore, finite at loop order $(L+1)$, or the CS equation has to be iterated. However, before discussing vertex functions, we examine the coefficient functions.

### *9.6.1 Coefficients of the CS equation*

(i) Because $\partial\tilde{\Gamma}^{(1,2)}/\partial p^2$ is of order $g_{\mathrm{r}}^2$, equation (9.43) then implies that $\eta$ is finite up to order $g_{\mathrm{r}}^L$.
(ii) The function $\tilde{\Gamma}^{(1,4)}$ is superficially convergent. Therefore, it has a skeleton expansion. The first dressed skeleton diagram contributing to $\tilde{\Gamma}^{(1,4)}$ is shown in Fig. 9.18.

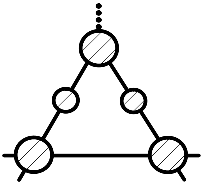


**Fig. 9.18** Skeleton diagram contributing to $\Gamma^{(1,4)}$

If the functions $\tilde{\Gamma}_{\mathrm{r}}^{(2)}$, $\tilde{\Gamma}_{\mathrm{r}}^{(4)}$ and $\tilde{\Gamma}_{\mathrm{r}}^{(1,2)}$ are finite up to $L$ loops, $\tilde{\Gamma}_{\mathrm{r}}^{(1,4)}$ is finite up to loop order $(L+1)$, which means up to order $g_{\mathrm{r}}^{L+2}$. Equation (9.44) then shows that the combination $\beta - 2g_{\mathrm{r}}\eta$ is finite up to order $g_{\mathrm{r}}^{L+2}$. Since $\eta$ is finite up to order $g_{\mathrm{r}}^L$, this implies that $\beta$ is finite up to order $g_{\mathrm{r}}^{L+1}$.
(iii) The function $\tilde{\Gamma}^{(2,2)}$ is also superficially convergent. It has a skeleton expansion. The first dressed skeleton is shown in Fig. 9.19.

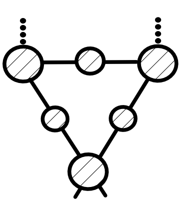


**Fig. 9.19** Skeleton diagram contributing to $\Gamma^{(2,2)}$

Therefore, the function $\tilde{\Gamma}_{\mathrm{r}}^{(2,2)}$ is also finite up to loop order $(L+1)$, which means up to order $g_{\mathrm{r}}^{L+1}$. Since $\tilde{\Gamma}^{(2,2)}$ is of order $g_{\mathrm{r}}$, at order $g_{\mathrm{r}}^{L+1}$ the sum $\eta+\eta_2$ calculated from equation (9.45) involves only $\eta$ at order $g_{\mathrm{r}}^L$ and $\tilde{\Gamma}_{\mathrm{r}}^{(2,2)}$ at order $g_{\mathrm{r}}^{L+1}$, and is thus finite. The function $\eta_2$ is then finite up to order $g_{\mathrm{r}}^L$.

We now prove that the functions $\tilde{\Gamma}_{\mathrm{r}}^{(2)}$, $\tilde{\Gamma}_{\mathrm{r}}^{(4)}$, and $\tilde{\Gamma}_{\mathrm{r}}^{(1,2)}$ are, with the induction assumptions, finite up to loop order $(L+1)$.

*9.6.2 The $\langle\phi\phi\phi\phi\rangle$ vertex function ($l = 0$, $n = 4$)*

We consider the coefficient of order $g_\mathrm{r}^{L+2}$ in equation (9.46):

$$\left[m_\mathrm{r}\frac{\partial}{\partial m_\mathrm{r}}\tilde{\Gamma}_\mathrm{r}^{(4)}\right]_{L+2} = \left[\left(-\beta_2\frac{\partial}{\partial g_\mathrm{r}} + 2\eta\right)\tilde{\Gamma}_\mathrm{r}^{(4)}\right]_{L+2} + m_\mathrm{r}^2\left[(2-\eta)\tilde{\Gamma}_\mathrm{r}^{(1,4)}\right]_{L+2}.$$

Since $\beta_2\partial/\partial g_\mathrm{r}$ is of order $g_\mathrm{r}$ and $\eta$ of order $g_\mathrm{r}^2$ for $g_\mathrm{r}$ small, in the first term of the right-hand side we need $\tilde{\Gamma}_\mathrm{r}^{(4)}$ only up to order $g_\mathrm{r}^{L+1}$, which is finite by assumption. We now separate in $\tilde{\Gamma}_\mathrm{r}^{(4)}$ the leading term $g_\mathrm{r}$ and a remainder of order $g_\mathrm{r}^2$. For the terms of order $g_\mathrm{r}^2$ and higher, we need $\eta$ only up to order $g_\mathrm{r}^L$ and $\beta$ up to order $g_\mathrm{r}^{L+1}$, which are finite. The leading term in $\tilde{\Gamma}_\mathrm{r}^{(4)}$ then involves the combination

$$\left[\left(-\beta_2\frac{\partial}{\partial g_\mathrm{r}} + 2\eta\right)g_\mathrm{r}\right]_{L+2} = \left[-\beta_2 + 2\eta g_\mathrm{r}\right]_{L+2},$$

and we have shown above that $\beta - 2g_\mathrm{r}\eta$ is finite up to order $g_\mathrm{r}^{L+2}$.

Finally, $\tilde{\Gamma}_\mathrm{r}^{(1,4)}$ is finite up to order $g_\mathrm{r}^{L+2}$. In addition, its expansion in powers of $g_\mathrm{r}$ begins only at order $g_\mathrm{r}^2$. Therefore, the factor $(2-\eta)$ is only needed up to order $g_\mathrm{r}^L$. The conclusion is that the left-hand side is finite at loop order $(L+1)$.

*Perturbative integration of CS equations.* In what follows, we denote by $\mathbf{p}$ the set of all four momenta $(p_1, p_2, p_3, p_4)$.

By integrating equation (9.46), we now want to show that $\tilde{\Gamma}_\mathrm{r}^{(4)}$ itself is finite at $(L+1)$ loop order. The function $\tilde{\Gamma}_\mathrm{r}^{(4)}$ is dimensionless, that is, invariant in a dilatation

$$(\mathbf{p}, m_\mathrm{r}, \Lambda) \mapsto (\rho\mathbf{p}, \rho m_\mathrm{r}, \rho\Lambda).$$

Therefore, we set

$$m_\mathrm{r}\frac{\partial}{\partial m_\mathrm{r}}\tilde{\Gamma}_\mathrm{r}^{(4)} = f^{(4)}\left(\mathbf{p}/m_\mathrm{r}, m_\mathrm{r}/\Lambda, g_\mathrm{r}\right), \tag{9.47}$$

with

$$\lim_{\Lambda\to\infty} f^{(4)}\left(\mathbf{p}/m_\mathrm{r}, m_\mathrm{r}/\Lambda, g_\mathrm{r}\right) < \infty\,, \text{ and } \quad f^{(4)}\left(0, m_\mathrm{r}/\Lambda, g_\mathrm{r}\right) = m_\mathrm{r}\frac{\partial}{\partial m_\mathrm{r}}g_\mathrm{r} = 0\,.$$

We integrate equation (9.47) between $m_\mathrm{r}$ and $\Lambda$:

$$\tilde{\Gamma}_\mathrm{r}^{(4)}\left(\mathbf{p}/m_\mathrm{r}, m_\mathrm{r}/\Lambda, g_\mathrm{r}\right) = \tilde{\Gamma}_\mathrm{r}^{(4)}\left(\mathbf{p}/\Lambda, 1, g_\mathrm{r}\right) - \int_{m_\mathrm{r}}^{\Lambda}\frac{\mathrm{d}\rho}{\rho}f^{(4)}\left(\mathbf{p}/\rho, \rho/\Lambda, g_\mathrm{r}\right).$$

Renormalization conditions (9.30) together with the regularity at low momentum imply

$$\lim_{\Lambda\to\infty}\tilde{\Gamma}_\mathrm{r}^{(4)}\left(\mathbf{p}/\Lambda, 1, g_\mathrm{r}\right) = \tilde{\Gamma}_\mathrm{r}^{(4)}\left(0, 1, g_\mathrm{r}\right) = g_\mathrm{r}\,.$$

The integral has a large cut-off limit if we restrict the domain of integration to $[m_\mathrm{r}, \mu]$, $m_\mathrm{r} \ll \mu \ll \Lambda$:

$$\lim_{\substack{\Lambda\to\infty\\ \mu\ll\Lambda}}\int_{m_\mathrm{r}}^{\mu}\frac{\mathrm{d}\rho}{\rho}f^{(4)}\left(\mathbf{p}/\rho, \rho/\Lambda, g_\mathrm{r}\right) < \infty\,.$$

We still have to examine the values of $\rho$ of order $\Lambda$. Then $\Lambda/\rho$ is of order 1, while $\mathbf{p}/\rho$ is small. The integral depends on the small momentum behaviour of the four-point vertex function. Again, we use the property that, in a massive theory (the mass is $\rho \sim \Lambda$), the vertex functions are analytic around $\mathbf{p} = 0$. Since, in addition, $f^{(4)}$ vanishes at $\mathbf{p} = 0$,

$$\left| f^{(4)}\left(\mathbf{p}/\rho, \rho/\Lambda, g_\mathrm{r}\right) \right| < \frac{C\left(\mathbf{p}, g_\mathrm{r}\right)}{\rho^2}\,, \quad \text{for} \quad \rho \gg |\mathbf{p}|\,,$$

the remaining integral is then bounded by

$$\int_\mu^\Lambda \frac{\mathrm{d}\rho}{\rho} \left| f^{(4)}\left(\mathbf{p}/\rho, \rho/\Lambda, g_\mathrm{r}\right) \right| < \tfrac{1}{2} C\left(\mathbf{p}, g_\mathrm{r}\right) \left( \frac{1}{\mu^2} - \frac{1}{\Lambda^2} \right).$$

We conclude that $\tilde{\Gamma}_\mathrm{r}^{(4)}$ has a large cut-off limit at $(L+1)$ loop order given by

$$\tilde{\Gamma}_\mathrm{r}^{(4)}\left(\mathbf{p}, m_\mathrm{r}, g_\mathrm{r}\right) = g_\mathrm{r} - \int_1^\infty \frac{\mathrm{d}\rho}{\rho} f^{(4)}\left(\mathbf{p}/\rho m_\mathrm{r}, 0, g_\mathrm{r}\right), \tag{9.48}$$

both sides being expanded up to order $g_\mathrm{r}^{L+2}$.

### *9.6.3 The $\langle \phi^2 \phi \phi \rangle$ vertex function ($l = 1$, $n = 2$)*

We now repeat the argument for $\tilde{\Gamma}_\mathrm{r}^{(1,2)}$. We consider the term of order $g_\mathrm{r}^{L+1}$ in equation (9.46),

$$m_\mathrm{r} \frac{\partial}{\partial m_\mathrm{r}} \left[ \tilde{\Gamma}_\mathrm{r}^{(1,2)} \right]_{L+1} = -\left[ \beta \frac{\partial}{\partial g_\mathrm{r}} \tilde{\Gamma}_\mathrm{r}^{(1,2)} \right]_{L+1} + \left[ \left(\eta + \eta_2\right) \tilde{\Gamma}_\mathrm{r}^{(1,2)} \right]_{L+1} + m_\mathrm{r}^2 \left[ \left(2 - \eta\right) \tilde{\Gamma}_\mathrm{r}^{(2,2)} \right]_{L+1}.$$

The first term in the right-hand side involves $\tilde{\Gamma}_\mathrm{r}^{(1,2)}$ up to order $L$, since $\beta \partial/\partial g_\mathrm{r}$ is of order $g_\mathrm{r}$, and $\beta$ up to order $(L+1)$. Both are finite. The second term again involves $\tilde{\Gamma}_\mathrm{r}^{(1,2)}$ up to order $L$, since $(\eta + \eta_2)$ is of order $g_\mathrm{r}$ and $(\eta + \eta_2)$ at order $(L+1)$, which is finite (although $\eta$ and $\eta_2$ separately are not). The last term involves $\tilde{\Gamma}_\mathrm{r}^{(2,2)}$ up to order $(L+1)$ and $\eta$ up to order $L$, since $\tilde{\Gamma}_\mathrm{r}^{(2,2)}$ is of order $g_\mathrm{r}$. We conclude that $m_\mathrm{r} \partial \tilde{\Gamma}_\mathrm{r}^{(1,2)} / \partial m_\mathrm{r}$ is finite at $(L+1)$ loop order.

The function $\tilde{\Gamma}_\mathrm{r}^{(1,2)}$ is also dimensionless. Its value at zero momentum is fixed by the renormalization condition (9.33), therefore,

$$m_\mathrm{r} \frac{\partial}{\partial m_\mathrm{r}} \tilde{\Gamma}_\mathrm{r}^{(1,2)}\left(0; 0, 0\right) = 0\,.$$

The analysis is then the same as for $\tilde{\Gamma}_\mathrm{r}^{(4)}$, and we conclude that $\tilde{\Gamma}_\mathrm{r}^{(1,2)}$ has a finite limit at infinite cut-off at loop order $(L+1)$. We can now use equation (9.43) and argument (i): Since $\tilde{\Gamma}^{(1,2)}$ is finite up to order $g_\mathrm{r}^{L+1}$, $\eta$ is finite up to the same order $g_\mathrm{r}^{L+1}$.

*9.6.4 The $\langle\phi\phi\rangle$ vertex function ($l = 0$, $n = 2$)*

The term of order $g_{\rm r}^{L+1}$ in equation (9.46) is

$$m_{\rm r}\frac{\partial}{\partial m_{\rm r}}\left[\tilde{\Gamma}_{\rm r}^{(2)}\right]_{L+1} = \left[\left(-\beta\frac{\partial}{\partial g_{\rm r}}+\eta\right)\tilde{\Gamma}_{\rm r}^{(2)}\right]_{L+1} + m_{\rm r}^2\left[(2-\eta)\,\tilde{\Gamma}_{\rm r}^{(1,2)}\right]_{L+1}. \tag{9.49}$$

In the first term of the right-hand side, we need $\tilde{\Gamma}_{\rm r}^{(2)}$ only up to order $g_{\rm r}^{L}$, since it is multiplied by terms of order $g_{\rm r}$. We have also shown previously that $\eta$ is finite up to order $g_{\rm r}^{L+1}$ and $\beta$ is finite at this order by argument (ii). For the second term, we have just shown that the two factors are finite up to order $g_{\rm r}^{L+1}$. We now consider the quantity $\tilde{\Gamma}_{\rm r}^{(2)}(p) - m_{\rm r}^2 - p^2$. Renormalization conditions imply that it vanishes as $(p^2)^2$ for $|\mathbf{p}|$ small. It has mass dimension 2. We set

$$m_{\rm r}\frac{\partial}{\partial m_{\rm r}}\left[\tilde{\Gamma}_{\rm r}^{(2)}(p) - m_{\rm r}^2 - p^2\right] = m_{\rm r}^2 f^{(2)}(p/m_{\rm r}, m_{\rm r}/\Lambda, g_{\rm r}), \tag{9.50}$$

with

$$\begin{cases} \lim\limits_{\Lambda\to\infty} f^{(2)}(p/m_{\rm r}, m_{\rm r}/\Lambda, g_{\rm r}) < \infty\,, \\ f^{(2)}(p/m_{\rm r}, m_{\rm r}/\Lambda, g_{\rm r}) = O(p^4)\ \text{for}\ |p|\to 0\,. \end{cases}$$

The integration of equation (9.50) then yields

$$\begin{aligned}\tilde{\Gamma}_{\rm r}^{(2)}(p, m_{\rm r}, g_{\rm r}, \Lambda) - m_{\rm r}^2 - p^2 &= \Lambda^2\left[\tilde{\Gamma}_{\rm r}^{(2)}(p/\Lambda, 1, g_{\rm r}, 1) - 1 - p^2/\Lambda^2\right] \\ &\quad - \int_{m_{\rm r}}^{\Lambda}\rho\,\mathrm{d}\rho\, f^{(2)}(p/\rho, \rho/\Lambda, g_{\rm r}).\end{aligned} \tag{9.51}$$

The integrated term in the right-hand side decreases like $(p^2)^2/\Lambda^4$ for $\Lambda$ large and, therefore, goes to zero.

For the same reason, $f^{(2)}(p/\rho, \rho/\Lambda, g_{\rm r})$ is of order $1/\rho^4$, for $\rho \sim \Lambda$, and, therefore, the infinite cut-off limit can be taken in the integral.

This concludes the induction. The advantage of the method is that, simultaneously, we have proved renormalizability and derived the CS equations,

$$\begin{aligned}&\left(m_{\rm r}\frac{\partial}{\partial m_{\rm r}} + \beta(g_{\rm r})\frac{\partial}{\partial g_{\rm r}} - \frac{\eta}{2}\eta(g_{\rm r}) - l\eta_2(g_{\rm r})\right)\tilde{\Gamma}_{\rm r}^{(l,n)}(q_j; p_i) \\ &\qquad = \big(2-\eta(g_{\rm r})\big)m_{\rm r}^2\tilde{\Gamma}_{\rm r}^{(l+1,n)}(0, q_j; p_i).\end{aligned} \tag{9.52}$$

*9.6.5 The large momentum behaviour of superficially divergent vertex functions*

We infer from this derivation that we can also use induction to estimate the large momentum behaviour of vertex functions. If we assume the bounds (9.35) at loop order $L$, then $\tilde{\Gamma}^{(1,4)}$ and $\tilde{\Gamma}^{(2,2)}$ are given by convergent integrals at loop order $(L+1)$. It is not too difficult to bound their large momentum behaviour by powers of logarithms.

Let us now again consider the example of the four-point function. Once we have established the representation (9.48), the induction hypothesis tells us that the function $f^{(4)}$ in the right-hand side is bounded by (we omit now the cut-off dependence)

$$\left|f^{(4)}(\lambda\mathbf{p}/m_{\rm r}, g_{\rm r})\right| < \text{const.}\ (\ln\lambda)^{k(L)} \quad \text{for}\ \lambda\to\infty\,.$$

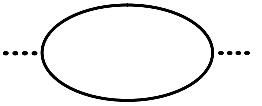


**Fig. 9.20** One-loop contribution to $\Gamma^{2,2)}$

We then decompose the integral over $\rho$ into the sum of two terms,

$$\int_1^{\infty} \frac{\mathrm{d}\rho}{\rho} f^{(4)}\left(\lambda \mathbf{p}/\rho m_{\mathrm{r}}, g_{\mathrm{r}}\right) = \int_{\lambda}^{\infty} \frac{\mathrm{d}\rho}{\rho} f^{(4)}\left(\lambda \mathbf{p}/\rho m_{\mathrm{r}}, g_{\mathrm{r}}\right) + \int_1^{\lambda} \frac{\mathrm{d}\rho}{\rho} f^{(4)}\left(\lambda \mathbf{p}/\rho m_{\mathrm{r}}, g_{\mathrm{r}}\right).$$

The first integral can be bounded by a constant. The second integral can be bounded using the large-momentum behaviour of $f^{(4)}$,

$$\int_1^{\lambda} \frac{\mathrm{d}\rho}{\rho} \left| f^{(4)}\left(\lambda \mathbf{p}/\rho m_{\mathrm{r}}, g_{\mathrm{r}}\right)\right| < \text{const.} \int_1^{\lambda} \frac{\mathrm{d}\rho}{\rho} \left[\ln\left(\lambda/\rho\right)\right]^{k(L)} \sim \left(\ln \lambda\right)^{k(L)+1}.$$

The argument is the same for $\tilde{\Gamma}^{(1,2)}$. This last bound can then be used to bound $\tilde{\Gamma}^{(2)}$.

## 9.7 The $\langle \phi^2 \phi^2 \rangle$ vertex function

We have seen that the superficial degree of divergence of the vertex function $\tilde{\Gamma}^{(2,0)}$ is 0. Actually, even in free QFT ($g_{\mathrm{r}} = 0$), $\tilde{\Gamma}^{(2,0)}$ is divergent (see Fig. 9.20). An additional renormalization is needed. We impose the renormalization condition

$$\tilde{\Gamma}_{\mathrm{r}}^{(2,0)}(p = 0, m_{\mathrm{r}}, g_{\mathrm{r}}) = 0\,, \tag{9.53}$$

which is obviously consistent with the tree level approximation. We expect the relation between bare and renormalized vertex function to be

$$\tilde{\Gamma}_{\mathrm{r}}^{(2,0)}(p) = \left( Z_2/\, Z\right)^2 \left[\tilde{\Gamma}^{(2,0)}(p) - \tilde{\Gamma}^{(2,0)}(0)\right]. \tag{9.54}$$

Differentiating with respect to $m_{\mathrm{r}}$ at $g$ and $\Lambda$ fixed, we then obtain

$$\begin{aligned}\left[m_{\mathrm{r}} \frac{\partial}{\partial m_{\mathrm{r}}} + \beta\left(g_{\mathrm{r}}\right) \frac{\partial}{\partial g_{\mathrm{r}}} - 2\eta_2\left(g_{\mathrm{r}}\right)\right] \tilde{\Gamma}_{\mathrm{r}}^{(2,0)}\left(p; m_{\mathrm{r}}, g_{\mathrm{r}}\right) \\ = m_{\mathrm{r}}^2 \left[2 - \eta\left(g_{\mathrm{r}}\right)\right] \left[\tilde{\Gamma}_{\mathrm{r}}^{(3,0)}\left(p, -p, 0; m_{\mathrm{r}}, g_{\mathrm{r}}\right) - \tilde{\Gamma}_{\mathrm{r}}^{(3,0)}\left(0,0,0; m_{\mathrm{r}}, g_{\mathrm{r}}\right)\right].\end{aligned} \tag{9.55}$$

The derivation of the equation follows the same lines as in previous cases. It uses the properties that $\tilde{\Gamma}^{(3,0)}$ has a skeleton expansion and that the $L$ loop order of $\tilde{\Gamma}^{(2,0)}$ is of order $g_{\mathrm{r}}^{L-1}$. Finally, in the integration with respect to $m_{\mathrm{r}}$, one takes into account the renormalization condition (9.53) and notes that $\tilde{\Gamma}^{(2,0)}$ is dimensionless in mass units.

It is possible then to summarize all CS equations by

$$\begin{aligned}\left[m_{\mathrm{r}} \frac{\partial}{\partial m_{\mathrm{r}}} + \beta\left(g_{\mathrm{r}}\right) \frac{\partial}{\partial g_{\mathrm{r}}} - \frac{\eta}{2}\eta\left(g_{\mathrm{r}}\right) - l\eta_2\left(g_{\mathrm{r}}\right)\right] \tilde{\Gamma}_{\mathrm{r}}^{(l,n)}\left(q_j; p_i; m_{\mathrm{r}}, g_{\mathrm{r}}\right) \\ = m_{\mathrm{r}}^2 \left(2 - \eta\left(g_{\mathrm{r}}\right)\right) \tilde{\Gamma}_{\mathrm{r}}^{(l+1,n)}\left(0, q_j; p_i; m_{\mathrm{r}}, g_{\mathrm{r}}\right) + \delta_{n0}\delta_{l2} B\left(g_{\mathrm{r}}\right),\end{aligned} \tag{9.56}$$

with the notation

$$-m_{\mathrm{r}}^2 \left(2 - \eta\left(g_{\mathrm{r}}\right)\right) \tilde{\Gamma}_{\mathrm{r}}^{(3,0)}\left(0,0,0; m_{\mathrm{r}}, g_{\mathrm{r}}\right) = B\left(g_{\mathrm{r}}\right), \tag{9.57}$$

since the left-hand side is dimensionless.

## 9.8 The renormalized action: General construction

The proof of the renormalizability, *in the sense of power counting*, of arbitrary massive local QFTs is a rather simple generalization of the proof we have given previously for the $\phi^4$ QFT in four dimensions. Therefore, we summarize only the main steps of the construction of the renormalized theory.

We consider an arbitrary QFT renormalizable by power counting. We perform a loop expansion of the generating functional of vertex functions,

$$\Gamma(\varphi) = \mathcal{S}_{\mathrm{r},0}(\varphi) + \sum_{l=1}^{\infty} \Gamma_l(\varphi).$$

After addition to the tree order action $\mathcal{S}_{\mathrm{r},0}(\phi)$ of all the counter-terms needed to render the theory finite up to loop-order $L$, for $\Lambda \to \infty$, the functionals $\Gamma_l(\varphi)$ have a finite limit for $l \leq L$, and the diagrams contributing to $\Gamma_{L+1}(\varphi)$ have no divergent subdiagrams. Therefore, the divergent part $\Gamma^{\text{div.}}_{L+1}(\varphi)$ of $\Gamma_{L+1}(\varphi)$ is a general local functional linear combination of all vertices of non-positive UV dimensions (except if symmetries forbid some terms) [54]. By adding the counter-terms $-\Gamma^{\text{div}}_{L+1}(\varphi)$ to the renormalized action, one renders the theory finite at loop order $(L+1)$. The counter-terms are defined only up to an arbitrary finite part, a linear combination of the same vertices that appear in $\Gamma^{\text{div.}}_{L+1}$. *The resulting renormalized action is a general local functional of the fields, a linear combination of all vertices of non-positive UV dimension.*

This statement summarizes the result derived in Section 9.6 in the case of the $\phi^4$ QFT in four dimensions. The divergent part of $\Gamma(\varphi)$ at loop order $L$, after renormalization up to order $(L-1)$, has the form

$$\Gamma^{\text{div.}}_L(\varphi) = -\int \mathrm{d}^4x \left[\tfrac{1}{2}\delta m_L^2 \varphi^2(x) + \tfrac{1}{2}\delta Z_L \big(\nabla\varphi(x)\big)^2 + \tfrac{1}{4!} g_\mathrm{r} \delta Z_{g,L} \varphi^4(x)\right], \tag{9.58}$$

because the vertex $\phi^2$ has dimension $-2$, and the vertices $(\nabla\phi)^2$ and $\phi^4$ dimension 0. No odd power of $\phi$ appears because the tree order action is symmetric in $\phi \mapsto -\phi$.

Note that, quite generally, if some monomials allowed by power counting and symmetries are absent from the initial (bare) action, they will be generated as counter-terms. Note also that algorithms have been developed to generate directly renormalized Feynman diagrams [66, 67, 68].

## 9.9 The massless theory

We have derived the renormalizability of the $\phi^4$ QFT by a method that applies only to massive QFTs, because the mass insertion operation plays an essential role in decreasing the degree of divergence of Feynman diagrams. To derive a renormalized massless $\phi^4_4$ QFT, we rescale the mass, at fixed cut-off and momenta [69],

$$m_\mathrm{r} \mapsto m_\mathrm{r}/\rho\,, \quad \rho \to \infty\,.$$

At fixed cut-off, vertex functions have a limit, as will be discussed extensively in Chapters 14–17, in dimensions larger than or equal to the dimension in which the theory is exactly renormalizable, provided this dimension is larger than 2 (because the propagator is $1/p^2$). In addition, the set of arguments of the vertex functions in the momentum representation must be non-exceptional, that is, all non-trivial subsets of momenta should have a non-vanishing sum.

These conditions are met in the $\phi^4_4$ QFT (at non-exceptional momenta) and we now examine what happens when the cut-off is removed. We combine CS equations with Weinberg's theorem [65].

### *9.9.1 Large momentum behaviour and massless theory*

We start from the CS equations

$$\left[m_\mathrm{r}\frac{\partial}{\partial m_\mathrm{r}}+\beta(g_\mathrm{r})\frac{\partial}{\partial g_\mathrm{r}}-\frac{n}{2}\eta(g_\mathrm{r})\right]\tilde{\Gamma}_\mathrm{r}^{(n)}(p_i)=m_\mathrm{r}^2(2-\eta)\tilde{\Gamma}_\mathrm{r}^{(1,n)}(0;p_i).$$

Weinberg's theorem states that if we scale all momenta $p_i \mapsto \rho p_i$, the large $\rho$ behaviour at *non-exceptional momenta*, at any finite order of perturbation theory, is given by the mass dimension up to powers of logarithms. Thus,

$$\begin{aligned}\tilde{\Gamma}_\mathrm{r}^{(l,n)}(\rho p_i)&\underset{\rho\to\infty}{\sim}\rho^{4-n-2l}\times\text{power of }\ln\rho\,,\\ \tilde{\Gamma}_\mathrm{r}^{(l+1,n)}(0;\rho p_i)&\underset{\rho\to\infty}{\sim}\frac{1}{\rho^2}\rho^{4-n-2l}\times\text{power of }\ln\rho\,.\end{aligned}$$

Therefore, if in the asymptotic expansion of $\tilde{\Gamma}_\mathrm{r}^{(n)}(\rho p_i)$ for $\rho$ large, we neglect all terms subleading by a power $\rho^{-2}$ up to powers of $\ln\rho$, we obtain a vertex function $\tilde{\Gamma}_\mathrm{r,as.}^{(n)}$ that satisfies the homogeneous RG equation,

$$\left(m_\mathrm{r}\frac{\partial}{\partial m_\mathrm{r}}+\beta(g_\mathrm{r})\frac{\partial}{\partial g_\mathrm{r}}-\frac{n}{2}\eta(g_\mathrm{r})\right)\tilde{\Gamma}_\mathrm{r,as.}^{(n)}(p_i)=0\,. \tag{9.59}$$

However, we know from dimensional analysis that

$$\tilde{\Gamma}_\mathrm{r}^{(n)}(p_i,m_\mathrm{r})=m_\mathrm{r}^{4-n}\tilde{\Gamma}_\mathrm{r}^{(n)}(p_i/m_\mathrm{r},1). \tag{9.60}$$

Therefore, scaling all momenta by a factor $\rho$ is equivalent, up to a global factor, to scaling the mass by a factor $\rho^{-1}$. The solutions of equation (9.59) are thus the vertex functions of the massless $\phi^4$ QFT.

*The critical bare mass.* When the physical mass vanishes, the parameter $r$ in expression (9.1) is fixed to a critical value $r=r_c$ and begins with a term of order $g_\mathrm{r}$. The first of equations (9.31) then shows that $r$ is negative for $g_\mathrm{r}$ small. This is the sign one expects from the point of view of phase transitions: the existence of a phase transition requires a double-well potential. This justifies our notation $r$ for the coefficient of $\phi^2$ in the bare action, instead of the traditional $m^2$.

*Perturbative solution of the homogeneous CS equations.* It is actually interesting to study the structure of vertex functions implied by equation (9.59). For illustration purpose, we consider here only the two-point function $\tilde{\Gamma}_\mathrm{r,as.}^{(2)}(p)$. We introduce the function $\zeta(g_\mathrm{r})$ defined by

$$\ln\zeta(g_\mathrm{r})=\int_0^{g_\mathrm{r}}\frac{\mathrm{d}g'\,\eta(g')}{\beta(g')}.$$

Since both $\eta$ and $\beta$ are of order $g_\mathrm{r}^2$, the function has a perturbative expansion.

We then set

$$\tilde{\Gamma}_\mathrm{r,as.}^{(2)}(p)=p^2\zeta(g_\mathrm{r})A(g_\mathrm{r},p/m_\mathrm{r}).$$

The function $A$ satisfies

$$\left(m_\mathrm{r}\frac{\partial}{\partial m_\mathrm{r}}+\beta(g_\mathrm{r})\frac{\partial}{\partial g_\mathrm{r}}\right)A(g_\mathrm{r},p/m_\mathrm{r})=0\,. \tag{9.61}$$

We expand $A$ and $\beta(g_{\mathrm{r}})$ in powers of $g_{\mathrm{r}}$, setting

$$A(g_{\mathrm{r}}, p/m_{\mathrm{r}}) = 1 + \sum_{1}^{\infty} g_{\mathrm{r}}^{n} a_n(p/m_{\mathrm{r}}), \quad \beta(g_{\mathrm{r}}) = \sum_{2}^{\infty} \beta_n g_{\mathrm{r}}^{n} . \tag{9.62}$$

Introducing these expansions into equation (9.61), we find for $n \geq 2$,

$$-z a_n'(z) + \sum_{m=1}^{n-1} m a_m(z) \beta_{n-m+1} = 0 . \tag{9.63}$$

The equation for $n = 1$ is special ($C_1$ is a constant),

$$-z a_1'(z) = 0 \quad \Rightarrow \quad a_1(z) = C_1 .$$

The solution of the equation for $n = 2$ is given by ($C_2$ is another constant)

$$-z a_2'(z) + C_1 \beta_2 = 0 \Rightarrow \ a_2(z) = C_1 \beta_2 \ln z + C_2 .$$

From this example, the general structure of $a_n(z)$ can be guessed: it is a polynomial of degree $(n-1)$ in $\ln z$ of the form

$$a_n(z) = P_{n-1}(\ln z), \quad \text{with} \quad P_{n-1}'(x) = \sum_{m=1}^{n-1} m P_{m-1}(x) \beta_{n-m+1} . \tag{9.64}$$

The new information specific to the order $n$ is characterized by two constants, $\beta_n$, which enters in the coefficient of $\ln z$, and $C_n$, which is the integration constant (to which one should add the coefficients of $\eta(g_{\mathrm{r}})$, which appear in the function $\zeta(g_r)$). Moreover, the term of highest degree in $P_n$ is entirely determined by one-loop results, the next term by one- and two-loop results and so on. Finally, $\tilde{\Gamma}^{(2)}_{\mathrm{r,as.}}(p)$ is entirely determined by the functions $\beta(g_{\mathrm{r}})$ and $\eta(g_{\mathrm{r}})$ and, for example, $\tilde{\Gamma}^{(2)}_{\mathrm{r,as.}}(1, g_{\mathrm{r}})/m_{\mathrm{r}}^2$ which is a third function of $g_{\mathrm{r}}$.

It also follows from these equations that $\tilde{\Gamma}^{(2)}_{\mathrm{r,as.}}(p)$ has a limit for $p = 0$:

$$\tilde{\Gamma}^{(2)}_{\mathrm{r,as.}}(p^2 = 0) = 0 , \tag{9.65}$$

confirming, as expected, that the theory is massless. However, its derivative $\partial \tilde{\Gamma}^{(2)}_{\mathrm{r,as.}}/\partial p^2$ has no zero-momentum limit. It is easy to verify that no other vertex function has a zero-momentum limit either.

We have constructed a massless theory by scaling a massive theory and shown that the corresponding vertex functions satisfy a homogeneous CS equation, also called an RG equation. We now show that such an equation can be derived more directly if one assumes the existence of a renormalized massless QFT.

### *9.9.2 RG equations in a massless QFT*

*Renormalization conditions.* We have shown that the renormalized massless $\phi^4$ QFT exists in four dimensions. To determine renormalization constants by renormalization conditions, we have to impose them at non-exceptional momenta; in particular, we cannot use zero momentum except for the two-point function as we have indicated previously. Therefore, we introduce an arbitrary mass scale $\mu$ (the physical mass scale) and impose

$$\tilde{\Gamma}^{(2)}_{\mathrm{r}}(p^2 = 0) = 0 , \tag{9.66}$$

$$\frac{\partial}{\partial p^2} \tilde{\Gamma}^{(2)}_{\mathrm{r}}(p^2 = \mu^2) = 1 , \tag{9.67}$$

$$\tilde{\Gamma}^{(4)}_{\mathrm{r}}(p_i = \mu \theta_i) = g_{\mathrm{r}} , \tag{9.68}$$

in which the $\theta_i$ form a set of arbitrary non-exceptional numerical vectors.

The bare vertex functions in a massless theory depend only on the cut-off and momenta, since the bare mass parameter is fixed by imposing that the renormalized mass $m_\mathrm{r}$ vanishes. The renormalized vertex functions depend on the arbitrary scale $\mu$ and momenta. They are related for $\Lambda$ large by

$$\tilde{\Gamma}_\mathrm{r}^{(n)}(p_i;\mu,g_\mathrm{r}) = Z^{n/2}(\Lambda/\mu,g_\mathrm{r})\,\tilde{\Gamma}^{(n)}(p_i;\Lambda,g)\,. \tag{9.69}$$

*RG equations.* The bare theory does not depend on the parameter $\mu$, which has just been introduced to fix the normalization scale. Therefore,

$$\mu\frac{\partial}{\partial\mu}\tilde{\Gamma}^{(n)}(p_i;\Lambda,g)\bigg|_{g,\Lambda} = 0\,.$$

Then if one differentiates equation (9.69) with respect to $\mu$ at $\Lambda$ and $g$ fixed, using chain rule, one finds [70–72]

$$\left[\mu\frac{\partial}{\partial\mu}+\tilde{\beta}(g_\mathrm{r})\frac{\partial}{\partial g_\mathrm{r}}-\frac{n}{2}\tilde{\eta}(g_\mathrm{r})\right]\tilde{\Gamma}_\mathrm{r}^{(n)}(p_i;\mu,g_\mathrm{r}) = 0\,, \tag{9.70}$$

with the definitions

$$\tilde{\beta}(g_\mathrm{r}) = \mu\frac{\partial}{\partial\mu}g_\mathrm{r}\bigg|_{g,\Lambda},\quad \tilde{\eta}(g_\mathrm{r}) = \mu\frac{\partial}{\partial\mu}\ln Z\bigg|_{g,\Lambda}. \tag{9.71}$$

A priori, the functions $\tilde{\beta}$ and $\tilde{\eta}$ could also depend of the dimensionless ratio $\Lambda/\mu$ but, since they can also be expressed directly in terms of the renormalized vertex functions, they must have a large cut-off limit.

Equation (9.70) is analogous to equation (9.59). It differs only in the definition of $g_\mathrm{r}$ and a finite field renormalization. Both sets of equations will be essential tools for the analysis of the large momentum behaviour of vertex functions. Moreover, equation (9.70) is used in Chapter 15 to discuss the small momentum behaviour in massless theories.

$\phi^2$ *insertions.* In the massless theory, one can also define renormalized vertex functions with $\phi^2$ insertions (see Section 9.7 and equations (9.54) and (11.6)), given by

$$\begin{aligned}&\tilde{\Gamma}_\mathrm{r}^{(l,n)}(q_1,\ldots,q_l;p_1,\ldots,p_n;\mu,g_\mathrm{r})\\ &\quad = Z^{(n/2)-l}Z_2^l\left[\tilde{\Gamma}^{(l,n)}(q_1,\ldots,q_l;p_1,\ldots,p_n;\Lambda,g)-\delta_{l2}\delta_{n0}C(g)\right].\end{aligned} \tag{9.72}$$

The $\phi^2$ renormalization constant $Z_2$ can be fixed by a renormalization condition of the form

$$\tilde{\Gamma}_\mathrm{r}^{(1,2)}(q;p_1,p_2)\Big|_{p_1^2=p_2^2=q^2=\mu^2} = 1\,. \tag{9.73}$$

The additional renormalization constant $C(g)$ needed for the $\langle\phi^2\phi^2\rangle$ correlation function can be fixed by the renormalization condition (see equation (9.53)),

$$\tilde{\Gamma}_\mathrm{r}^{(2,0)}(p^2=\mu^2) = 0\,. \tag{9.74}$$

From equation (9.72), RG equations can be derived by differentiating with respect to $\mu$. Introducing the RG function

$$\tilde{\eta}_2(g_\mathrm{r}) = \mu\frac{\partial}{\partial\mu}\ln Z_2\bigg|_{g,\Lambda}, \tag{9.75}$$

one obtains (see equation (9.56)),

$$\begin{aligned}&\left[\mu\frac{\partial}{\partial\mu}+\tilde{\beta}(g_\mathrm{r})\frac{\partial}{\partial g_\mathrm{r}}-\frac{n}{2}\tilde{\eta}(g_\mathrm{r})-l\tilde{\eta}_2(g_\mathrm{r})\right]\tilde{\Gamma}_\mathrm{r}^{(l,n)}(q_1,\ldots,q_l;p_1,\ldots,p_n;\mu,g_\mathrm{r})\\ &\quad=\delta_{n0}\delta_{l2}C_\mathrm{r}(g_\mathrm{r})\,.\end{aligned} \tag{9.76}$$

## 9.10 Homogeneous RG equations: Massive QFT

The renormalized vertex functions $\tilde{\Gamma}_{\mathrm{r}}^{(l,n)}$ of the massless theory have been defined and shown to satisfy RG equations. Since a constant source term for $\phi^2$ at zero momentum generates a mass shift (equation (9.25)), one could think about expressing the vertex functions of a massive theory in terms of the vertex functions with $\phi^2$ insertions of the massless theory [70–72]. However, an immediate problem arises: one verifies that $\phi^2$ insertions at zero momentum in a massless theory are infrared (IR) divergent.

The problem can be solved by the following method: one introduces a space-dependent source $J_2(x)$ for $\phi^2(x)$ insertions [74]. One performs a partial summation of the two-point function, and then takes the constant source limit. After summation, the propagator becomes massive, and the limit is no longer IR divergent.

We consider the renormalized action with a source $J_2(x)$ for renormalized $\phi^2$ insertions,

$$\mathcal{S}_{\mathrm{r}}(\phi_{\mathrm{r}}, J_2) = \int \mathrm{d}^4 x \left[\tfrac{1}{2} Z \left(\nabla\phi_{\mathrm{r}}(x)\right)^2 + \tfrac{1}{2}\left(\delta m^2 + Z_2 J_2(x)\right)\phi_{\mathrm{r}}^2(x) + \frac{1}{4!} g_{\mathrm{r}} Z_g \phi_{\mathrm{r}}^4(x)\right], \quad (9.77)$$

where the renormalization constants are those of the massless theory. In the Fourier representation, a vertex function in the presence of the source $J_2(x)$, expressed in terms of its Fourier transform $\tilde{J}_2(q)$, has the expansion,

$$\tilde{G}_{\mathrm{r}}^{(n)}(p_1, p_2, \ldots, p_n; J_2) = \sum_{l=0} \frac{1}{2^l}\frac{1}{l!} \int \mathrm{d}^4 q_1 \mathrm{d}^4 q_2 \cdots \mathrm{d}^4 q_l\, \tilde{J}_2(q_1)\tilde{J}_2(q_2)\cdots\tilde{J}_2(q_l)$$
$$\times (2\pi)^4 \delta^{(4)}(q_1 + \cdots + q_l + p_1 + \cdots + p_n)\tilde{\Gamma}_{\mathrm{r}}^{(l,n)}(q_1, q_2, \ldots, q_l; p_1, p_2, \ldots, p_n), \quad (9.78)$$

in which the vertex function $\tilde{G}_{\mathrm{r}}^{(n)}$ differs from $\tilde{\Gamma}_{\mathrm{r}}^{(n)}$ because the $\delta$-function of momentum conservation cannot be factorized when $J_2$ is space dependent.

Applying the differential operator

$$\mathrm{D} \equiv \mu\frac{\partial}{\partial\mu} + \tilde{\beta}(g_{\mathrm{r}})\frac{\partial}{\partial g_{\mathrm{r}}}, \quad (9.79)$$

to equation (9.78), using the RG equations (9.76), and noting that

$$\int \mathrm{d}^4 q\, \tilde{J}_2(q)\frac{\delta}{\delta\tilde{J}_2(q)} = l\,, \quad (9.80)$$

one obtains

$$\left[\mu\frac{\partial}{\partial\mu} + \tilde{\beta}(g_{\mathrm{r}})\frac{\partial}{\partial g_{\mathrm{r}}} - \frac{n}{2}\tilde{\eta}(g_{\mathrm{r}}) - \tilde{\eta}_2(g_{\mathrm{r}}) \int \mathrm{d}^4 q\, \tilde{J}_2(q)\frac{\delta}{\delta\tilde{J}_2(q)}\right]$$
$$\times \tilde{G}_{\mathrm{r}}^{(n)}(J_2; p_1, \ldots, p_n) = 0\,. \quad (9.81)$$

After some partial summation, one can set $\tilde{J}_2(q) = \tau_{\mathrm{r}}$ constant and the equation becomes [70–74]

$$\left[\mu\frac{\partial}{\partial\mu} + \tilde{\beta}(g_{\mathrm{r}})\frac{\partial}{\partial g_{\mathrm{r}}} - \frac{n}{2}\tilde{\eta}(g_{\mathrm{r}}) - \tilde{\eta}_2(g_{\mathrm{r}})\tau_{\mathrm{r}}\frac{\partial}{\partial\tau_{\mathrm{r}}}\right]\tilde{\Gamma}_{\mathrm{r}}^{(n)}(p_1, \ldots, p_n; \mu, g_{\mathrm{r}}, \tau_{\mathrm{r}}) = 0\,. \quad (9.82)$$

Therefore, we have derived a new RG equation for a massive theory, which differs from the original CS equation in two respects:

(i) Vertex functions now depend on two mass parameters while we know that only one is required. However, in this parametrization the massless limit of correlation functions is directly obtained by setting $\tau_{\mathrm{r}} = 0$ and no asymptotic expansion at large momenta is needed.

(ii) In contrast to the CS equation, the RG equation (9.82) is homogeneous. It is thus easier to solve.

### 9.10.1 Covariance of RG functions

In the example of the massless $\phi^4$ theory, we have introduced two different renormalization schemes. It is interesting to understand how RG functions of different schemes are related. Theories may differ by a redefinition of the coupling constant and a finite field renormalization. We denote by $g_{\mathrm{r}}$ and $\tilde{g}_{\mathrm{r}}$ the coupling constants in two schemes. Comparing two RG equations, one infers

$$\beta(g_{\mathrm{r}})\frac{\partial}{\partial g_{\mathrm{r}}} = \tilde{\beta}(\tilde{g}_{\mathrm{r}})\frac{\partial}{\partial \tilde{g}_{\mathrm{r}}} \ \Rightarrow \ \beta(g_{\mathrm{r}})\frac{\partial \tilde{g}_{\mathrm{r}}}{\partial g_{\mathrm{r}}} = \tilde{\beta}(\tilde{g}_{\mathrm{r}}). \tag{9.83}$$

In a $\phi^4$-like QFT, the function $\beta(g_{\mathrm{r}})$ has the expansion,

$$\beta(g_{\mathrm{r}}) = \beta_2 g_{\mathrm{r}}^2 + \beta_3 g_{\mathrm{r}}^3 + O\left(g_{\mathrm{r}}^4\right). \tag{9.84}$$

Expanding $\tilde{g}_{\mathrm{r}}$ in terms of $g_{\mathrm{r}}$,

$$\tilde{g}_{\mathrm{r}} = g_{\mathrm{r}} + \gamma_2 g_{\mathrm{r}}^2 + O\left(g_{\mathrm{r}}^3\right), \tag{9.85}$$

and using equation (9.84), after a short calculation one finds

$$\tilde{\beta}(\tilde{g}_{\mathrm{r}}) = \beta_2 {\tilde{g}_{\mathrm{r}}}^2 + \beta_3 {\tilde{g}_{\mathrm{r}}}^3 + O\left({\tilde{g}_{\mathrm{r}}}^4\right). \tag{9.86}$$

The first two terms in the expansion of the $\beta$-function are universal and all others are formally arbitrary.

*One should not conclude from this observation that the physical consequences inferred from the RG $\beta$-function for finite coupling are arbitrary.* Only regular mappings $g_{\mathrm{r}} \mapsto \tilde{g}_{\mathrm{r}}$ are allowed. In particular, this implies that $\partial\tilde{g}_{\mathrm{r}}/\partial g_{\mathrm{r}}$ in equation (9.83) must be strictly positive. Therefore, the *sign and the existence of zeros* of the $\beta$-function are properties of the QFT. Equation (9.83) also shows that the slope of the $\beta$-function at a zero is scheme independent.

Of course, we do not know in general which renormalization scheme leads to regular functions of the coupling constant. Our intuition is that 'natural' definitions as induced by momentum or minimal subtraction have the most chance to satisfy this criterion.

In the same way, if we call $\zeta(\tilde{g}_{\mathrm{r}})$ the additional finite field renormalization, we find

$$\tilde{\eta}(\tilde{g}_{\mathrm{r}}) = \eta(g_{\mathrm{r}}) + \tilde{\beta}(\tilde{g}_{\mathrm{r}})\frac{\partial}{\partial \tilde{g}_{\mathrm{r}}}\ln\zeta(\tilde{g}_{\mathrm{r}}). \tag{9.87}$$

Since the field renormalization appears at order $g_{\mathrm{r}}^2$, $\ln\zeta(\tilde{g}_{\mathrm{r}})$ is of order ${\tilde{g}_{\mathrm{r}}}^2$ and, therefore, the modification to $\tilde{\eta}$ of order ${\tilde{g}_{\mathrm{r}}}^3$. The coefficient of order $g_{\mathrm{r}}^2$ is universal. The value of $\eta$ at a zero of $\beta(g_{\mathrm{r}})$ is also universal. Similar arguments apply to $\eta_2$: its first coefficient is universal as well as its value at a zero of $\beta(g_{\mathrm{r}})$.

Finally, these arguments generalize to the relation between bare (Section 9.11) and renormalized RG functions.

*Several coupling constants.* We shall meet actions depending on several fields and coupling constants $g_i$. In a change of parametrization of the coupling space, the RG $\beta$-functions transform like

$$\tilde{\beta}_i\left(\tilde{g}\right) = \sum_j \frac{\partial \tilde{g}_i}{\partial g_j}\beta_j(g), \tag{9.88}$$

where the mapping should satisfy $\det(T_{ij} \equiv \partial\tilde{g}_i/\partial g_j) > 0$. Thus the existence of zeros of the $\beta$-functions is universal and, since at a zero $g_i^*$,

$$\frac{\partial\tilde{\beta}_i}{\partial\tilde{g}_j} = \sum_{k,l} T_{ik}\frac{\partial\beta_k}{\partial g_l}T_{lj}^{-1},$$

the eigenvalues of $\partial\beta_i/\partial g_j$ are scheme independent.

## 9.11 EFT and RG

In Sections 8.8.1 and 8.9, we have explained that the initial bare QFT is an *effective* microscopic theory (EFT) in which non-renormalizable interactions have been neglected. It is an approximation to a more complete *finite physical theory*, and designed to describe only its large distance scale, low energy properties. The cut-off $\Lambda$, introduced to render the QFT finite, represents the mass scale at which new physics has to be taken into account and at which the EFT is no longer relevant. In the framework of EFTs, the restriction to renormalizable QFTs is not a new law of nature but only a consequence of the weakness of the possible non-renormalizable interactions due to the large ratio between microscopic and physical scales.

The renormalizability of a QFT implies that the large distance, small physical mass behaviour of correlation functions is *universal*, that is, independent from the arbitrary cut-off implementation, order by order in a perturbative expansion, because the renormalized and bare correlation functions are asymptotically proportional.

The renormalized functions, when they exist beyond perturbation theory, contain the whole information about the asymptotic large distance properties.

One may then wonder why one should consider bare correlation functions. The main reason is that renormalization theory implies *tuning the parameters of the Lagrangian as a function of the cut-off*, something that is obviously non-physical in particle physics. The tuning of the $\phi^2$ coefficient in the $\phi^4$ QFT (it corresponds to adjusting the temperature close to the critical temperature in critical phenomena, see Chapter 15) seems unavoidable, and requires an explanation outside of the $\phi^4$ theory. However, one wants to avoid at least the tuning of the $\phi^4$ interaction, which, even for critical phenomena, corresponds to a non-generic situation.

A further reason is that the *renormalized theory may not exist beyond perturbation theory*, equation (9.90) being then valid only for $\Lambda$ large but finite (see Section 9.12).

Finally, one may want to study corrections to the leading large distance behaviour.

*Super-renormalizable interactions.* Renormalization theory implicitly assumes that the scale in super-renormalizable interactions is the physical scale, for example, the physical mass, instead of the cut-off scale. This is a highly non-generic situation.

### *9.11.1 Bare (or microscopic) and renormalized vertex functions*

Rescaling the parameter $r$, we rewrite the bare (effective microscopic) action (9.3) as

$$\mathcal{S}_\Lambda(\phi) = \int \mathrm{d}^4x \left[\tfrac{1}{2}\big(\nabla_\Lambda \phi(x)\big)^2 + \tfrac{1}{2}\Lambda^2 r\phi^2(x) + \frac{1}{4!}g\phi^4(x)\right], \tag{9.89}$$

where parameters $g$ and $r$ are dimensionless and the cut-off $\Lambda$ is the only mass parameter.

For generic values of $r$, the physical mass $m$ is of order $\Lambda$ and all correlations vanish at large distance. For a particular negative value $r = r_c$ (the critical temperature in the sense of critical phenomena, see Section 15.3), which must exist beyond perturbation theory, the physical mass $m_\mathrm{r}$ vanishes and for $r < r_c$ the $\mathbb{Z}_2$ reflection symmetry is spontaneously broken (see Section 16.5.4). A small physical mass in the unbroken phase implies $1 \gg r - r_c \geq 0$, a condition that implies a *fine tuning* of the parameter $r$.

Studying the low energy, low mass situation is equivalent to studying the large cut-off limit. Thus, we can use the results of the renormalization theory. For simplicity, specializing to $r = r_c$, the value at which the physical mass $m_\mathrm{r}$ vanishes, we introduce rescaled vertex functions, defined by the renormalization conditions (9.67) and (9.68), adapted to a massless theory at a physical scale $\mu \ll \Lambda$, and functions of a renormalized coupling constant $g_\mathrm{r}$.

These vertex functions are related to the microscopic (bare) ones by equations of the form (9.69), which we write here slightly differently as

$$\tilde{\Gamma}_{\mathrm{r}}^{(n)}(p_i; g_{\mathrm{r}}, \mu, \Lambda) = Z^{n/2}(g, \Lambda/\mu)\tilde{\Gamma}^{(n)}(p_i; g, \Lambda). \tag{9.90}$$

Renormalization theory tells us that the functions $\tilde{\Gamma}_{\mathrm{r}}^{(n)}$ in equation (9.90) have, at $p_i$, $g_{\mathrm{r}}$, and $\mu$ fixed, and *order by order in the perturbative expansion*, finite limits for $\Lambda \to \infty$. Moreover, the renormalized functions $\tilde{\Gamma}_{\mathrm{r}}^{(n)}$ do not depend on the specific regularization procedure and, given the normalization conditions (9.68), are *universal*. More precisely, the corrections at loop order $L$ are of the order $(\ln \Lambda)^L/\Lambda^2$. We assume here that the sum of all corrections still vanishes for $\Lambda \to \infty$, something that has not been proved here.

### *9.11.2 Bare or asymptotic microscopic RG equations*

Differentiating equation (9.90) with respect to $\Lambda$ at $g_{\mathrm{r}}$ and $\mu$ fixed, one derives the new identity

$$\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}},\mu \text{ fixed}} Z^{n/2}(g, \Lambda/\mu)\tilde{\Gamma}^{(n)}(p_i; g, \Lambda) = o\left(\Lambda^{-1}\right). \tag{9.91}$$

Using chain rule, one infers from equation (9.91),

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g, \Lambda/\mu)\frac{\partial}{\partial g} - \frac{n}{2}\eta(g, \Lambda/\mu)\right]\tilde{\Gamma}^{(n)}(p_i; g, \Lambda) = 0, \tag{9.92}$$

where corrections vanishing for $\Lambda$ large have been neglected (see Chapter 17). The functions $\beta$ and $\eta$ are given by

$$\beta(g, \Lambda/\mu) = \Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}},\mu} g, \quad \eta(g, \Lambda/\mu) = -\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}},\mu} \ln Z(g, \Lambda/\mu). \tag{9.93}$$

Being dimensionless, they may depend only on the dimensionless quantities $g$ and $\Lambda/\mu$. However, the functions $\beta$ and $\eta$ can also be directly calculated from equation (9.92) in terms of functions $\tilde{\Gamma}^{(n)}$ which do not depend on $\mu$. Therefore, the functions $\beta$ and $\eta$ cannot depend on the ratio $\Lambda/\mu$. This simplifies equation (9.92), which becomes [72]

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\eta(g)\right)\tilde{\Gamma}^{(n)}(p_i; g, \Lambda) = 0. \tag{9.94}$$

Equation (9.94) is satisfied, when the *cut-off is large with respect to the physical scale, but not infinite*, by the bare vertex functions of the $\phi^4$ QFT. It is a direct consequence of the existence of a perturbative renormalized theory. Its solution also implies the existence of a renormalized theory, but in a slightly different form and with a new interpretation.

*Massive theory.* RG equations can also be proved for $0 < (r - r_c) \ll 1$. They take a form analogous to equations (9.82), $\tau_{\mathrm{r}}$ being replaced by $\tau = r - r_c$ [72].

*Leading corrections for* $\Lambda$ *large.* Finally, we characterize the terms neglected in equation (9.94) more precisely. In a series expansion in powers of $g$,

$$\tilde{\Gamma}^{(n)}(p_i; g, \Lambda) = \sum_{\nu} \tilde{\Gamma}_{\nu}^{(n)}(p_i, \Lambda)g^{\nu}, \tag{9.95}$$

the coefficients $\tilde{\Gamma}_{\nu}^{(n)}$ have an asymptotic expansion for all $|p_i|/\Lambda$ small of the form

$$\tilde{\Gamma}_{\nu}^{(n)}(p_i, \Lambda) = \sum_{\ell=0}^{L(n,\nu)} \left[(\ln \Lambda)^{\ell} A_{\ell\nu}^{(1)}(p_i) + \frac{1}{\Lambda^2}(\ln \Lambda)^{\ell} A_{\ell\nu}^{(2)}(p_i) + \cdots\right], \tag{9.96}$$

with $L(n, \nu) = \nu - 1$ for $n = 4$ and $L(n, \nu) = \nu - n/2$ for $n \neq 4$.

The RG equation (9.94) is exact for the sum of the perturbative contributions which do not vanish for $\Lambda$ large, as can be verified by expanding equation (9.94) in powers of $g$.

*Bare and renormalized RG equations.* Formally, the bare and renormalized RG equations differ only by their parametrization. Even the first terms of the small coupling expansion are the same (Section 9.10.1). However, the bare RG equations combined with the Gaussian renormalization, describing the RG flow from the microscopic scale to the physical scale, determine some properties of the renormalized parameters. By contrast, the renormalized equations describe only the flow between different physical scales. Still, the two approaches are equivalent to study the leading IR behaviour of a massless (or critical) or near massless QFT, but the latter is more practical.

## 9.12 Solution of bare RG equations: The triviality issue

The RG equations (9.94) (as well as equations (9.70)) can be solved by the method of characteristics. Introducing a dilatation parameter $\lambda$, one looks for functions $g(\lambda)$ and $Z(\lambda)$ such that

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \Lambda\lambda\big)\right] = 0\,. \tag{9.97}$$

Differentiating explicitly with respect to $\lambda$, one finds that equations (9.97) and (9.94) are compatible provided that

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda) = \beta\big(g(\lambda)\big), \tag{9.98}$$

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln Z(\lambda) = \eta\big(g(\lambda)\big), \tag{9.99}$$

or after integration with the boundary conditions $g(1) = g$ and $Z(1) = 1$,

$$\ln \lambda = \int_g^{g(\lambda)} \frac{\mathrm{d}g'}{\beta(g')} \tag{9.100}$$

$$\ln Z(\lambda) = \int_1^{\lambda} \frac{\mathrm{d}\sigma}{\sigma}\eta\big(g(\sigma)\big) = \int_g^{g(\lambda)} \mathrm{d}g' \frac{\eta(g')}{\beta(g')}. \tag{9.101}$$

Equation (9.97) implies

$$\tilde{\Gamma}^{(n)}(p_i; g, \Lambda) = Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \Lambda\lambda\big),$$

and after the rescaling $\Lambda\lambda \mapsto \Lambda$, since $\tilde{\Gamma}^{(n)}$ has mass dimension $(4 - n)$,

$$\tilde{\Gamma}^{(n)}(\lambda p_i; g, \Lambda) = \lambda^{4-n} Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \Lambda\big). \tag{9.102}$$

The equation shows that the function $g(\lambda)$ is the effective coupling at the scale $\mu = \lambda\Lambda$ and we are interested in the limit $\lambda \to 0$. Equation (9.94) is the RG equation in differential form. Equations (9.102), (9.100), and (9.101) are the integrated RG equations.

### *9.12.1 RG functions at leading order*

After a short calculation, one obtains for the massless propagator

$$\Delta(x) = \frac{1}{4\pi^2}\frac{1}{x^2}. \tag{9.103}$$

The one-loop contribution to the two-point function is a constant which is cancelled by a shift of $r_c$ and thus $\eta(g) = O(g^2)$. The one-loop diagram contributing to the four-point function (first diagram in Fig. 9.2) can be written as

$$B_4(p) = \int \mathrm{d}^4 x \; \mathrm{e}^{ipx}\, \Delta^2(x).$$

The integral diverges at $x = 0$ and, at this leading order, we simply cut the integral at $|x| = 1/\Lambda$ and, after having also cut the integral in the IR at $|x| = 1/\mu$ for $|x|$ large, we set $p = 0$. The divergent part (consistent with equation (9.29)) is then

$$B_4(p) \sim \frac{1}{8\pi^2}\ln(\Lambda/\mu) \;\Rightarrow\; \tilde{\Gamma}^{(4)}(p_i; g, \Lambda) = g - \frac{3}{16\pi^2}[\ln(\Lambda/\mu) + O(1)]g^2 + O(g^3). \quad (9.104)$$

Substituting the expansion into equation (9.94) with $n = 4$, one infers (as expected from the discussion of Section 9.10.1)

$$\beta(g) = \frac{3}{16\pi^2}g^2 + O(g^3). \quad (9.105)$$

The two-loop diagram contributing to the two-point function (third diagram in Fig. 9.1) can be written as

$$\Omega_4(p) = \int \mathrm{d}^4x\; \mathrm{e}^{ipx}\, \Delta^3(x).$$

Again we introduce an IR cut-off $\mu$ and a UV cut-off $\Lambda$. We can then expand in powers of the momentum $p$. The relevant contribution is of order $p^2$ and given by

$$[\Omega_4(p)]_{\text{div.}} \sim -\frac{1}{8}p^2 \int \mathrm{d}^4x\, x^2 \Delta^3(x) \sim -\frac{1}{4}\frac{1}{(8\pi^2)^2}p^2 \ln(\Lambda/\mu). \quad (9.106)$$

Inserting the result into equation (9.94) with $n = 2$, one finds

$$\eta(g) = \frac{1}{(8\pi^2)^2}\frac{g^2}{24} + O(g^3). \quad (9.107)$$

Since the $\beta$-function is of order $g^2$, for $g(\lambda) \to 0$, $Z(\lambda)$ has a constant limit (for details see Section 17.2).

### *9.12.2 The triviality issue*

For $g$ small, the $\beta$-function is positive. If it remains positive for all $g > 0$ (in agreement with empirical evidence), then equation (9.98) implies that $g(\lambda)$ is an increasing function of $\lambda$. For physical scales $\mu$ such that $\mu/\Lambda = \lambda \to 0$, in equation (9.100) $\ln\lambda \to -\infty$ and the integral in the right-hand side must diverge. Therefore, the effective coupling $g(\lambda)$ must go to zero. The integral is then dominated by the $g^2$ term of $\beta(g)$ (equation (9.105)). One concludes

$$g(\lambda = \mu/\Lambda) \sim \frac{16\pi^2}{3}\frac{1}{\ln(\Lambda/\mu)}\,. \quad (9.108)$$

The effective coupling $g(\mu/\Lambda)$ can be identified with the renormalized coupling $g_{\mathrm{r}}$. This leads to the *triviality* issue: in the infinite cut-off limit, the *renormalized coupling vanishes*: it is impossible to define a *renormalized* $\phi^4$ QFT in four dimensions for non-zero coupling [73]. However, in the logic of *EFTs*, the problem is reformulated differently. The QFT has only a limited energy or momentum range of validity, where it is consistent (in the sense of satisfying all usual physical requirements). This range is small compared to the cut-off, which represents the scale of some new physics and can no longer be assumed to be infinite.

RG arguments then imply that the consistent physical range decreases when the renormalized or effective charge increases, as indicated by equation (9.108). Note that if $g$ is generic (not too small) and $\Lambda/\mu$ very large, then the renormalized coupling constant $g_r$ becomes essentially independent of the initial coupling constant $g$.

Finally, in the $O(N)$ symmetric $(\boldsymbol{\phi}^2)^2$ field theory, for $N$ large, one finds that the renormalized theory has a non-physical pole, dubbed the 'Landau ghost' (Section 18.5).

# A9 Functional RG equations. Super-renormalizable QFTs. Normal order

## A9.1 Large-momentum mode integration and functional RG equations

Here, we briefly describe a direct constructiom of a general functional RG, closer to Wilson's ideas [62] (in a form also developed by Wegner [61]), which was proposed by Polchinski [59] as an alternative method to prove renormalizability. The physical motivation for such a method will become more apparent when we discuss RG and critical phenomena starting in Chapter 15. The method is based on a recursive integration over the large momentum modes of the field. The procedure leads to an exact RG in the space of all local interactions: one expresses the equivalence between a variation of the cut-off and a modification of the coefficients of the interaction terms. The equivalence takes the form of functional RG equations for an effective Hamiltonian, valid for large distance or low momentum (in the cut-off scale). For a simple and much more detailed introduction see, for example, Chapter 16 of Ref. [64].

*A9.1.1 A basic equivalence*

From the arguments of Section 8.4.3, based on a Gaussian integration, it follows that the two actions (translation invariance is assumed)

$$\mathcal{S}(\phi) = \frac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\phi(x)\Delta^{-1}(x-y)\phi(y) + V(\phi), \quad \text{and} \tag{A9.1}$$

$$\begin{aligned}\mathcal{S}(\phi_1,\phi_2) = \tfrac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\left[\phi_1(x)\Delta_1^{-1}(x-y)\phi_1(y) + \phi_2(x)\Delta_2^{-1}(x-y)\phi_2(x)\right]\\ + V(\phi_1+\phi_2),\end{aligned}$$

with $\Delta = \Delta_1 + \Delta_2$, generate the same perturbation theory.

Here, $\Delta^{-1}$ means inverse of $\Delta$ in the sense of operators (as well as $\Delta_1^{-1}$, $\Delta_2^{-1}$):

$$\int \mathrm{d}^d z\,\Delta^{-1}(x-z)\Delta(z-y) = \delta^{(d)}(x-y).$$

We take for $\Delta$, $\Delta_1$ and $\Delta_2$ massless propagators of the general form

$$\Delta(x-y) = \frac{1}{(2\pi^d)}\int \mathrm{d}^d k\;\mathrm{e}^{i(x-y)}\,\tilde{\Delta}(k),\ \text{with}\ \tilde{\Delta}(k) = \frac{C(k^2/\Lambda^2)}{k^2}, \tag{A9.2}$$

in which the function $C(t)$ is smooth, go to 1 for $t$ small and decreases faster than any power for $t$ large (see Section A8.3), and $\Lambda$ is a large momentum cut-off.

We now use this equivalence in the limit in which the propagator $\Delta_2$ goes to 0. Then only small values of $\phi_2$ contribute to the partition function. Expanding the interaction for $\phi_2$ small,

$$V(\phi_1+\phi_2) = V(\phi_1) + \int \mathrm{d}^d x \frac{\delta V(\phi_1)}{\delta\phi(x)}\phi_2(x) + \tfrac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y \frac{\delta^2 V}{\delta\phi(x)\delta\phi(y)}\phi_2(x)\phi_2(y) + \cdots\ ,$$

we integrate over $\phi_2$ to obtain the leading order correction

$$\begin{aligned}&\int [\mathrm{d}\phi_2]\exp\left\{-\left[\tfrac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\phi_2(x)\Delta_2^{-1}(x-y)\phi_2(y) + V(\phi+\phi_2) - V(\phi)\right]\right\}\\ &\quad\sim 1 + \tfrac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\Delta_2(x-y)\left[\frac{\delta V}{\delta\phi(x)}\frac{\delta V}{\delta\phi(y)} - \frac{\delta^2 V}{\delta\phi(x)\delta\phi(y)}\right] + \cdots\ .\end{aligned} \tag{A9.3}$$

Taking the logarithm of expression ($A9.3$), we rewrite the partition function as a field integral over a field $\phi$ (the index 1 is no longer useful) with the effective action

$$\begin{aligned}\mathcal{S}'(\phi) &= \tfrac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\phi(x)\Delta_1^{-1}(x-y)\phi(y) + V(\phi)\\ &\quad + \tfrac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\Delta_2(x-y)\left[\frac{\delta^2 V}{\delta\phi(x)\delta\phi(y)} - \frac{\delta V}{\delta\phi(x)}\frac{\delta V}{\delta\phi(y)}\right] + \cdots\,. \end{aligned} \quad (A9.4)$$

We have thus established that the actions ($A9.1$) and ($A9.4$) lead to the same partition function, up to a trivial change of normalization. Note that the same identity can be proven by playing with integrations by parts as in the case of the quantum equation of motion (7.33).

We will now show that this equivalence can be used to partially integrate out the large momenta in a field theory with a cut-off $\Lambda$.

### *A9.1.2 Large-momentum mode partial integration and RG equations*

We now choose

$$\tilde{\Delta}(k) = \frac{C(k^2/\Lambda^2)}{k^2}\,, \quad \tilde{\Delta}_1(k) = \frac{C(k^2/\Lambda^2(1+\sigma)^2)}{k^2}\,, \quad (A9.5)$$

where the propagator $\Delta_1$ is obtained from $\Delta$ by a rescaling of the cut-off $\Lambda$.

For $\sigma \to 0$, at leading order $\Delta_2$ is then given by

$$\tilde{\Delta}_2(k) \sim \frac{2\sigma}{\Lambda^2}C'(k^2/\Lambda^2) \equiv \sigma\tilde{D}(k). \quad (A9.6)$$

We note that the propagator $\tilde{\Delta}_2$ *has no pole at* $k=0$. Moreover, if we choose a function $C(t)$ which is very close to 1 for $t$ small

$$|C(t)-1|\,t^{-p} \to 0 \quad \forall p > 0\,,$$

then $\tilde{\Delta}_2$ is only large for $k$ of order $\Lambda$. The integration over $\phi_2$ thus corresponds to an integration over the large momentum modes of the field $\phi$.

The equivalence between actions ($A9.1$) and ($A9.4$), which is the starting point of an RG, can be written as ($D(x)$ is the Fourier transform of $\tilde{D}(k)$)

$$\Lambda\frac{\mathrm{d}}{\mathrm{d}^d\Lambda}V(\phi,\Lambda) = \frac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\,D(x-y)\left[\frac{\delta^2 V}{\delta\phi(x)\delta\phi(y)} - \frac{\delta V}{\delta\phi(x)}\frac{\delta V}{\delta\phi(y)}\right], \quad (A9.7)$$

or, after Fourier transformation ($\tilde{\phi}(p)$ is the Fourier transform $\phi(x)$),

$$\Lambda\frac{\mathrm{d}}{\mathrm{d}\Lambda}V(\phi,\Lambda) = \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\tilde{D}(k)\left[\frac{\delta^2 V}{\delta\tilde{\phi}(k)\delta\tilde{\phi}(-k)} - \frac{\delta V}{\delta\tilde{\phi}(k)}\frac{\delta V}{\delta\tilde{\phi}(-k)}\right]. \quad (A9.8)$$

The equation can also be derived from the quantum field equations and, therefore, partial integration does not imply a loss of information, by contrast with the lattice equation.

To study the existence of fixed points, we start with a given interaction $V_0(\phi)$ at a scale $\Lambda_0$ and use equation ($A9.7$) to calculate the effective interaction $V(\phi,\Lambda)$ at a scale $\Lambda \ll \Lambda_0$. A fixed point is defined by the property that $V(\phi,\Lambda)$, after a suitable rescaling of $\phi$, goes to a limit.

We denote by $\tilde{V}^{(n)}(p_1,p_2,\ldots,p_n)$ the coefficients of $V(\phi,\Lambda)$ in an expansion in powers of $\tilde{\phi}(p)$. Equation ($A9.8$) can then be written in component form as

$$\Lambda\frac{\mathrm{d}}{\mathrm{d}\Lambda}\tilde{V}^{(n)}(p_1,p_2,\ldots,p_n)=\frac{1}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\tilde{D}(k)\tilde{V}^{(n+2)}(p_1,p_2,\ldots,p_n,k,-k)$$
$$-\frac{1}{2}\sum_I D(p_0)\tilde{V}^{(l+1)}(p_{i_1},\ldots,p_{i_l},p_0)\tilde{V}^{(n-l+1)}(p_{i_{l+1}},\ldots,p_{i_n},-p_0), \qquad (A9.9)$$

in which the momentum $p_0$ is determined by momentum conservation, and the set $I \equiv \{i_1,i_2,\ldots,i_l\}$ runs over all distinct subsets of $\{1,2,\ldots,n\}$.

Equation ($A9.9$) shows that, even if we start with a pure $g\phi^4$ interaction, at scale $\Lambda$ we obtain a general local interaction because all functions $\tilde{V}^{(n)}$ are coupled. However, in the spirit of the perturbative methods used so far, it is possible to solve equation ($A9.7$) as an expansion in the coupling constant $g$ with the ansatz that the terms of $V(\phi,\Lambda)$ quadratic and quartic in $\phi$ are of order $g$ and the general term of degree $2n$ is of order $g^{n-1}$.

*Correlation functions.* To generate correlation functions, one has to add a source to the interaction $V(\phi)$:

$$V(\phi)\mapsto V(\phi)-\int \mathrm{d}^d x\, J(x)\phi(x).$$

However, equation ($A9.4$) then shows that $\mathcal{S}'(\phi)$ becomes in general a complicated functional of the source $J(x)$. A solution to this problem is the following: one takes a source whose Fourier transform $\tilde{J}(k)$ vanishes for $k^2 \geq \Lambda^2$, together with a propagator $\Delta_2$ which propagates only momenta such that $k^2 \geq \Lambda^2$. This implies that $C'(t)$ vanishes identically for $t \leq 1$ (unfortunately, such cut-off functions are inconvenient for practical calculations). Then $\int \mathrm{d}^d x\, J(x)\phi_2(x)$ does not contribute in integral ($A9.3$) and $\mathcal{S}'(\phi)-\mathcal{S}(\phi)$ does not depend on $J(x)$.

However, we note that then the RG transformation is such that the correlation functions corresponding to the action $\mathcal{S}(\phi)$ and $\mathcal{S}'(\phi)$ are only identical when all momenta are smaller than the cut-off. The differences between correlation functions are smooth functions of momenta and thus decay at large distances in space faster than any power.

## A9.2 The $\phi^4$ QFT in three dimensions: Divergences

Although much of this work is focused on strictly renormalizable QFTs, we make a few remarks here about UV divergences in super-renormalizable scalar QFTs in three and two dimensions. Note that, below four dimensions, the perturbative expansion of the *massless theory is IR divergent* (see Section 10.5.6).

We first take the example of the $\phi^4$ QFT in three dimensions. As explained in Chapter 8, the $\phi^4$ QFT then has only three superficially divergent diagrams, all contributing to the two-point function, which are displayed in Fig. 9.21.

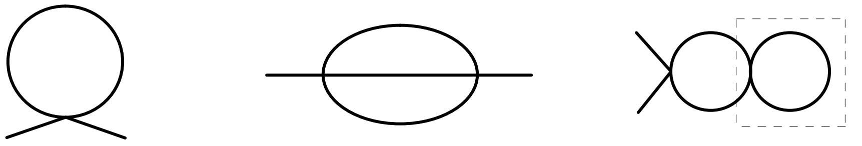



**Fig. 9.21** The three divergent diagrams in the $\phi_3^4$ QFT

The two first diagrams are given by

$$(a) = \frac{1}{(2\pi)^3} \int \frac{d^3 q}{(q^2 + m^2)_\Lambda}.$$
$$(b) = \frac{1}{(2\pi)^6} \int \frac{d^3 q_1 d^3 q_2}{(q_1^2 + m^2)_\Lambda (q_2^2 + m^2)_\Lambda \left[(p - q_1 - q_2)^2 + m^2\right]_\Lambda}.$$

By adding counter-terms of the form

$$\tfrac{1}{2} \int d^3 x \left[a_1(\Lambda) g + a_2(\Lambda) g^2\right] \phi^2(x),$$

we can render the first two diagrams finite.

The diagram (c) is also superficially divergent, but it is clear that the counter-term which renders the diagram (a) finite, also renormalizes the diagram (c). Thus, the counter-terms that renormalize the diagrams (a) and (b) render the whole theory finite.

Actually the divergence (a), which corresponds to a self-contraction of the vertex $\phi^4$, can be eliminated *a priori* by replacing the vertex $\phi^4$ by a *normal-ordered vertex* $:\phi^4:$,

$$:\phi^4:(x) = \phi^4(x) - 6\phi^2(x) \left\langle \phi^2(x) \right\rangle + 3 \left( \left\langle \phi^2(x) \right\rangle \right)^2, \tag{A9.10}$$

in which the expectation value $\left\langle \phi^2(x) \right\rangle$ is calculated in a free QFT with a mass $\mu$ which may or may not be equal to the renormalized mass $m$,

$$\left\langle \phi^2(x) \right\rangle_\mu = \frac{1}{(2\pi)^3} \int \frac{d^3 q}{(q^2 + \mu^2)_\Lambda}. \tag{A9.11}$$

The denomination normal order comes from the operator terminology. The quantity $:\phi^4(x):$ is such that

$$\begin{gathered} \left\langle :\phi^4(x): \right\rangle = 0 \\ \left\langle :\phi^4(x): \phi(y_1)\phi(y_2) \right\rangle = 0\,, \end{gathered} \tag{A9.12}$$

in which again the expectation values are calculated with the action $\mathcal{S}_\mu(\phi)$,

$$\mathcal{S}_\mu(\phi) = \frac{1}{2} \int d^3 x \left[ \left(\nabla\phi(x)\right)^2 + \mu^2 \phi^2(x) \right]. \tag{A9.13}$$

The expectation values calculated with another mass are then finite. Finally, we still have to add a counter-term for diagram (b).

### A9.3 Super-renormalizable scalar QFTs in two dimensions: Normal order

An action $\mathcal{S}(\phi)$ of the form

$$\mathcal{S}(\phi) = \int \mathrm{d}^2 x \left[\tfrac{1}{2}(\nabla\phi(x))^2 + \tfrac{1}{2}m^2\phi^2(x) + V\big(\phi(x)\big)\right], \tag{A9.14}$$

in which $V(\phi)$ is an arbitrary regular function of $\phi$, is super-renormalizable. If $V(\phi)$ is a polynomial, only a finite number of diagrams are superficially divergent. However, it is a peculiarity of dimension 2 that the field is dimensionless and, therefore, the interaction $V(\phi)$ may have an infinite series expansion in powers of $\phi$. Although the theory is super-renormalizable, one finds an infinite number of superficially divergent diagrams. One then notes that all divergences come from self-contractions of the vertex (see Fig. 9.22). Therefore, the operation $: V(\phi) :$ removes all divergences.

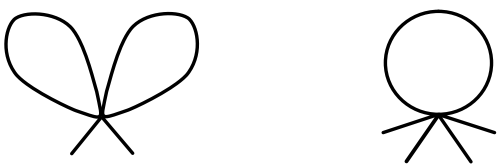


**Fig. 9.22** Divergent contributions

To obtain an explicit expression for $: V(\phi) :$, we first consider the special interaction

$$V(\phi) = \mathrm{e}^{\lambda\phi}\,. \tag{A9.15}$$

The expectation value of $V(\phi)$ in the presence of a source term can then be calculated explicitly:

$$\begin{aligned}&\int [\mathrm{d}\phi] \exp\left\{-\int \mathrm{d}^2 y \left[\tfrac{1}{2}\left((\nabla\phi(y))^2 + \mu^2\phi^2(y)\right) - \left(J(y) + \lambda\delta^{(2)}(x-y)\right)\phi(y)\right]\right\} \\ &\quad = \exp\left[\tfrac{1}{2}\int \mathrm{d}^2 y\, \mathrm{d}^2 y' J(y)\Delta(y,y')J(y') + \lambda\int \mathrm{d}^2 y\, \Delta(x,y)J(y) + \tfrac{1}{2}\lambda^2\Delta(x,x)\right],\end{aligned} \tag{A9.16}$$

in which $\Delta(x,y)$ is the free propagator with mass $\mu$. The normal ordering operation has to suppress the term coming from self-contractions, which is proportional to $\Delta(x,x)$. It is thus clear that the normal ordered interaction is

$$: \exp[\lambda\phi(x)] : = \exp\left[\lambda\phi(x) - \lambda^2\left\langle\phi^2(x)\right\rangle/2\right], \tag{A9.17}$$

in which we have used

$$\left\langle\phi^2(x)\right\rangle = \Delta(x,x). \tag{A9.18}$$

We then express an arbitrary interaction term as a Laplace transform,

$$V\big(\phi(x)\big) = \int \mathrm{d}\rho(\lambda)\, \mathrm{e}^{\lambda\phi(x)}\,. \tag{A9.19}$$

The normal ordering is a linear operation. Thus,

$$: V(\phi) : = \int \mathrm{d}\rho(\lambda) : \mathrm{e}^{\lambda\phi(x)} : \ .$$

We then use the result ($A9.17$) for the exponential interaction and obtain

$$: V(\phi) : = \left\{ \exp \left[ -\tfrac{1}{2} \left\langle \phi^2 \right\rangle (\partial / \partial \phi)^2 \right] \right\} V(\phi), \qquad (A9.20)$$

or more explicitly,

$$: V(\phi) : = V(\phi) + \left[ \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!} \left\langle \phi^2 \right\rangle^n \left( \frac{\partial}{\partial \phi} \right)^{2n} \right] V(\phi).$$

The existence of the Laplace transform of a given interaction is irrelevant in this argument, since the final identity is purely algebraic.