7 Perturbative quantum field theory (QFT): Algebraic methods

In Chapter 6, we have introduced the quantum theory of the relativistic scalar field, natural relativistic extension of the non-relativistic quantum statistical physics of the Bose gas (Section 4.4.2), in the formalism of Section 4.4.7.

In this chapter, we begin a more systematic study of the algebraic properties of perturbation theory in the example of a *local*, *relativistic* QFT. We discuss only scalar fields and postpone the study of relativistic fermions to Chapter 12, because this requires first describing the representations of the spin group. However, most algebraic results can be easily generalized.

We consider the Euclidean formulation of QFT based on the density matrix at thermal equilibrium, but we mainly investigate the simpler zero-temperature limit where all d coordinates, Euclidean time and space, can be treated symmetrically.

Our discussion is based on field integrals (for an early textbook see, *e.g.* Ref. [29]). The field integral defines a functional measure to which correspond expectation values of product of fields called *correlation functions*. These functions are analytic continuations to imaginary (Euclidean) time of the vacuum expectation values of time-ordered products of field operators, which *we shall also call correlation functions*, although the functional measure then is complex. As we have already explained in Sections 2.4.2 and 2.5, in the case of the quantum particle, they have also an interpretation as correlation functions in some models of classical statistical physics, in continuum formulations, or at equal time of finite temperature quantum field theory. Finally, they naturally appear in non-relativistic quantum statistical physics in various limits, high or critical temperatures.

We first calculate a Gaussian integral, which includes the example of the free field theory. Adding a source term to the action, we obtain a generating functional of Gaussian correlation functions. The field integral, corresponding to a general action with an interaction polynomial in the field, can then be expressed as an infinite sum of Gaussian expectation values, which can be calculated, for example, with the help of Wick's theorem. Each contribution has a graphical representation in terms of *Feynman diagrams*.

We show, because this may no longer be obvious, that the properties of the perturbative expansion are consistent with the usual manipulations performed on integrals, like integration by parts and change of variables. We define the functional δ-function.

Quite generally, the field integral, corresponding to an action to which a term linear in the field coupled to an external source (or field) J has been added, defines a generating functional $\mathcal{Z}(J)$ of field correlation functions (the partition function in an external field) [9, 30]. The functional $\mathcal{W}(J) = \ln \mathcal{Z}(J)$ (analogous to the free energy of statistical physics) is then the generating functional of *connected correlation functions*, to which contribute only connected Feynman diagrams. In a local field theory (the action is the integral of a function of the field and its derivatives at the same point), connected correlation functions, as a consequence of locality, have *cluster properties*.

The Legendre transform $\Gamma(\varphi)$ of $\mathcal{W}(J)$ is the generating functional of *vertex functions* (sometimes called effective potential and analogous to the thermodynamic potential of statistical physics) [31]. Only one-line irreducible Feynman diagrams (diagrams that cannot be disconnected by cutting only one line) contribute to vertex functions.

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These diagrams are also called one-particle irreducible (or *1PI, an acronym we often use in this work*) in particle physics.

Vertex functions play a basic role in the renormalization of local quantum field theories. In Section 7.11, we give a quantum interpretation to $\Gamma(\varphi)$. We also relate it to the partition function at fixed field time average. This relation explains why $\Gamma(\varphi)$ also appears in the discussion of symmetry breaking.

We explain how to calculate these functionals in a reorganized perturbative expansion, called *loop expansion* [32].

In the appendix, we calculate the generating functional of two-loop Feynman diagrams, introduce the background field method, and discuss some properties of Feynman diagrams relevant for cluster properties.

Warning. Many quantities that we meet in the chapter may be infinite in the case of relativistic quantum field theories in their straightforward definition, a problem that is carefully studied in Chapters 8 and 9. Therefore, the theories have to be *regularized*, for example, by modifying the propagator at short distance, or using a space-time lattice. In this chapter, we implicitly assume such a regularization (see Chapter 8).

7.1 Generating functionals of correlation functions

Correlation functions. We consider Euclidean, local action $\mathcal{S}(\phi)$ for a neutral scalar field ϕ in d-Euclidean dimensions, of the type introduced in Section 6.1, and define the n-point ϕ-field correlation function (a generalized moment) by the field integral,

$$Z^{(n)}(x_1,\ldots,x_n) \equiv \langle \phi(x_1)\cdots\phi(x_n)\rangle \equiv \frac{1}{\mathcal{Z}_0}\int [\mathrm{d}\phi]\,\phi(x_1)\cdots\phi(x_n)\,\mathrm{e}^{-\mathcal{S}(\phi)}, \tag{7.1}$$

where $\mathcal{Z}_0$ is the partition function,

$$\mathcal{Z}_0 = \int [\mathrm{d}\phi]\,\mathrm{e}^{-\mathcal{S}(\phi)}\,.$$

The field integral

$$\mathcal{Z}(J) = \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}(\phi) + \int \mathrm{d}^d x\, J(x)\phi(x)\right], \tag{7.2}$$

where $J(x)$ is an external field or source, is a generating functional of correlation functions (see also Section 2.5).

Indeed, expanded in powers of J, it yields

$$\mathcal{Z}(J) = \mathcal{Z}_0 \sum_{n=0}^{\infty} \frac{1}{n!}\int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_n\, Z^{(n)}(x_1,\ldots,x_n) J(x_1)\cdots J(x_n). \tag{7.3}$$

We now introduce functional differentiation, a differentiation completely defined by

$$\frac{\delta J(x)}{\delta J(y)} = \delta^{(d)}(x-y), \tag{7.4}$$

where $\delta^{(d)}(x)$ is the d-dimensional Dirac function. Then,

$$\frac{\delta}{\delta J(y)}\,\mathrm{e}^{J\cdot\phi} = \phi(y)\,\mathrm{e}^{J\cdot\phi}\,. \tag{7.5}$$

Applying this identity to the field integral (7.2), one verifies that ϕ-field correlation functions can also be obtained from the *generating functional* (7.2) by functional differentiation, as

$$Z^{(n)}(x_1,\ldots,x_n) = \frac{1}{\mathcal{Z}_0}\left[\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\mathcal{Z}(J)\right]\Bigg|_{J=0}. \tag{7.6}$$

7.2 Perturbative expansion. Wick's theorem and Feynman diagrams

We assume *translation invariance* in this chapter.

To generate a perturbative expansion, we write $\mathcal{S}(\phi)$ as the sum of a quadratic term and an interaction, of the form,

$$\mathcal{S}(\phi) = \tfrac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, \phi(x) K(x-y)\phi(y) + \mathcal{V}_{\mathrm{I}}(\phi), \tag{7.7}$$

where $\mathcal{V}_{\mathrm{I}}(\phi)$ is a *local* polynomial, and where the symmetric, positive kernel K is chosen to render all terms of the perturbative expansion in powers of $\mathcal{V}_{\mathrm{I}}(\phi)$ finite (a regularization, see, *e.g.*, Section 8.4.2). For a field with mass m, the kernel K can be chosen as a local operator of the form ($\nabla \equiv (\partial/\partial x_1, \dots, \partial/\partial x_d)$),

$$K(x-y) = \kappa(-\nabla^2)\, \delta^{(d)}(x-y), \tag{7.8}$$

where κ is an analytic function on the real axis, positive for $z>0$, of the form

$$\kappa(z) = m^2 + z + O(z^2).$$

In what follows, we use the symbolic notation

$$(\phi K \phi) \equiv \int \mathrm{d}^d x\, \mathrm{d}^d y\, \phi(x) K(x-y)\phi(y), \quad J\cdot\phi \equiv \int \mathrm{d}^d x\, J(x)\phi(x). \tag{7.9}$$

7.2.1 Gaussian integral and free field theory

We consider the general Gaussian field integral,

$$\mathcal{Z}_{\mathrm{G}}(J) = \int [\mathrm{d}\phi] \exp\left[-\tfrac{1}{2}(\phi K\phi) + J\cdot\phi\right]. \tag{7.10}$$

In expression (7.10), a normalization is implied; we choose $\mathcal{Z}_{\mathrm{G}}(0) = 1$. The functional $\mathcal{Z}_{\mathrm{G}}(J)$ is the generating functional of ϕ-field correlation functions corresponding to the weight $\mathrm{e}^{-(\phi K\phi)/2}$. The general expression (7.6) yields the Gaussian correlation functions,

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle_0 = \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\mathcal{Z}_{\mathrm{G}}(J)\bigg|_{J=0},$$

where $\langle\bullet\rangle_0$ means Gaussian expectation value.

The kernel Δ, inverse of K, is defined by

$$\int \mathrm{d}^d z\, \Delta(x-z)K(z-y) = \delta^{(d)}(x-y). \tag{7.11}$$

To calculate the field integral (7.10), we shift $\phi(x)$ by $\int \mathrm{d}^d y\, \Delta(x-y)J(y)$, and find after integration,

$$\mathcal{Z}_{\mathrm{G}}(J) = \exp\left[\tfrac{1}{2}(J\Delta J)\right], \tag{7.12}$$

with the notation

$$(J\Delta J) = \int \mathrm{d}^d x\, \mathrm{d}^d y\, J(x)\Delta(x-y)J(y).$$

Symmetry implies $\langle\phi(x)\rangle_0 = 0$. Then, differentiating expression (7.12), one infers,

$$\langle \phi(x_1)\phi(x_2)\rangle_0 = \frac{\delta^2\mathcal{Z}_{\mathrm{G}}(J)}{\delta J(x_1)\delta J(x_2)}\bigg|_{J\equiv 0} = \Delta(x_1-x_2). \tag{7.13}$$

An example of a Gaussian theory is provided by a free scalar field theory, where $\kappa(z) = m^2 + z$. The kernel Δ then is given by

$$\Delta(x) = \frac{1}{(2\pi)^d}\int \mathrm{d}^d p\, \frac{\mathrm{e}^{ip\cdot x}}{p^2+m^2}. \tag{7.14}$$

7.2.2 Perturbative expansion: A compact expression

In Section 2.6, we show how to calculate a path integral for a Hamiltonian of the form $\frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + V_{\mathrm{I}}(q)$ as an expansion in powers of $V_{\mathrm{I}}(q)$, for any function $V_{\mathrm{I}}(q)$ expandable in powers of q. The result is based on the calculation of a reference Gaussian integral (in Chapter 2, the harmonic oscillator). Here, we apply a similar method to field integrals.

Furthermore, although most of the results derived in this chapter are illustrated with examples corresponding to an action of the form (6.45) (but suitably regularized, see Section 8.4.2), the results apply to any boson field theory, and this explains the choice of the abstract notation (7.9).

We consider a general Euclidean action of the form (7.7). We use the compact notation,

$$D_J \equiv \delta/\delta J\,, \tag{7.15}$$

each time $\delta/\delta J$ appears as an argument. Then, it follows from the property (7.5) that we can express the field integral

$$\mathcal{Z}(J) = \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}(\phi) + J\cdot\phi\right], \tag{7.16}$$

in terms of $\mathcal{Z}_{\mathrm{G}}(J)$ (equation (7.10)) as

$$\mathcal{Z}(J) = \exp\left[-\mathcal{V}_{\mathrm{I}}\left(D_J\right)\right]\mathcal{Z}_{\mathrm{G}}(J) = \exp\left[-\mathcal{V}_{\mathrm{I}}\left(D_J\right)\right]\exp\left(\tfrac{1}{2}J\Delta J\right). \tag{7.17}$$

The expression (7.17) expresses, in a compact algebraic form, the result of the calculations of all Gaussian expectation values obtained by expanding the field integral (7.1), in the example of equation (7.7), as a formal series in powers of the interaction potential $\mathcal{V}_{\mathrm{I}}(\phi)$. The expansion, then combined with the identity (7.6), generates the perturbative expansion of all ϕ-field correlation functions [34–37].

7.2.3 Wick's theorem

The direct expansion of the field integral (7.1), in the example of equation (7.7), in powers of $\mathcal{V}_{\mathrm{I}}(\phi)$ reduces all calculations to Gaussian expectation values of products of fields. From the expression (7.17), and using the arguments of Section 1.1, one obtains a straightforward generalization of equations (1.9–1.14) or (2.61), which expresses Wick's theorem [1] in field theory,

$$\begin{aligned}\left\langle \prod_1^{2s} \phi(z_i)\right\rangle_0 &= \left[\prod_{i=1}^{2s} \frac{\delta}{\delta J(z_i)} \exp\left(\tfrac{1}{2}J\Delta J\right)\right]\Bigg|_{J\equiv 0} \\ &= \sum_{\substack{\text{all possible pairings}\\ \text{of }\{1,2,\ldots,2s\}}} \Delta(z_{i_1}-z_{i_2})\cdots\Delta(z_{i_{2s-1}}-z_{i_{2s}}).\end{aligned} \tag{7.18}$$

Perturbation theory involves, as a basic ingredient, the two-point function Δ of the Gaussian theory (equation (7.13)), which we call the *propagator*.

Graphically, each term in the sum can be represented by a set of contractions corresponding to a particular pairing. For example, for $s = 2$, one finds

$$\begin{aligned}&\langle\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle_0 = \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4) \;+\; 2 \text{ terms} \\ &\quad = \Delta(z_1-z_2)\Delta(z_3-z_4) + \Delta(z_1-z_3)\Delta(z_2-z_4) + \Delta(z_1-z_4)\Delta(z_2-z_3).\end{aligned}$$

7.2.4 Feynman diagrams

When the interaction terms are local, that is, integrals of polynomials of the field $\phi(x)$ and its derivatives, any perturbative contribution to the n-point correlation function is a Gaussian expectation value of the form (for simplicity, we omit derivative couplings, because they leave the argument unchanged)

$$\left\langle \phi(x_1)\cdots\phi(x_n)\int \mathrm{d}^d y_1\,\phi^{p_1}(y_1)\int \mathrm{d}^d y_2\,\phi^{p_2}(y_2)\cdots\int \mathrm{d}^d y_k\,\phi^{p_k}(y_k)\right\rangle_0 .$$

Therefore, it is a sum of products of propagators integrated over the points corresponding to interaction vertices. It is then possible to give a graphical representation of each product [37]: a propagator is represented by a line joining the two points which appear as arguments; moreover, any point that is common to more than one line corresponds to an argument that has to be integrated over.

Ultraviolet divergences. Since interacting theories with a propagator of the form (7.14) have large momentum (ultraviolet) or short distance divergences, in what follows we assume that the field theory has been *regularized* by any one of the methods that ensures the convergence of all terms in the perturbative expansion (see Sections 8.4.2, 8.7, and 10.1).

7.3 Connected correlation functions: Generating functional

We now define another generating functional, analogous to the free energy of statistical physics,

$$\mathcal{W}(J) = \ln \mathcal{Z}(J), \tag{7.19}$$

the generating functional of *connected correlation functions.*

We introduce the notation

$$\mathcal{W}(J) = \sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}^d x_1\cdots \mathrm{d}^d x_n\, W^{(n)}(x_1,x_2,\ldots,x_n)J(x_1)J(x_2)\cdots J(x_n). \tag{7.20}$$

The connected diagrams that contribute to the partition function $\mathcal{Z}_0$ (also called vacuum diagrams in particle physics), contribute only to $\mathcal{W}(J=0)$.

Then,

$$W^{(n)}(x_1,\ldots,x_n) = \left[\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\mathcal{W}(J)\right]\bigg|_{J=0}. \tag{7.21}$$

The functions $W^{(n)}$ are more directly related to physical observables.

The three first relations, as implied by the definition (7.19), between complete correlation functions $Z^{(n)}$ and connected functions $W^{(n)}$ are

$$\begin{aligned} Z^{(1)}(x) &= W^{(1)}(x),\\ Z^{(2)}(x_1,x_2) &= W^{(2)}(x_1,x_2) + W^{(1)}(x_1)W^{(1)}(x_2),\\ Z^{(3)}(x_1,x_2,x_3) &= W^{(3)}(x_1,x_2,x_3) + W^{(1)}(x_1)W^{(2)}(x_2,x_3) + W^{(1)}(x_2)W^{(2)}(x_3,x_1)\\ &\quad + W^{(1)}(x_3)W^{(2)}(x_1,x_2) + W^{(1)}(x_1)W^{(1)}(x_2)W^{(1)}(x_3). \end{aligned}$$

Note that the right-hand side involves all possible products of connected functions with coefficient 1.

7.3.1 An alternative proof of connectivity

In Section 1.2.1, we give a combinatorial proof in a sense of a perturbative diagrammatic expansion: only connected Feynman diagrams contribute to $\mathcal{W}(J)$. As a consequence, the functions $W^{(n)}$, unlike the functions $Z^{(n)}$, contain no contributions that can be factorized into products of the form

$$F_1(x_1,\ldots,x_p)F_2(x_{p+1},\ldots,x_n),$$

where the two disjoint sets of arguments are not empty.

This property is the starting point of an alternative global algebraic proof of connectivity which relies, in particular, on linearity and locality: a linear combination of connected functions is still connected.

In Section 7.2.1, we have calculated the generating functional $\mathcal{Z}_{\mathrm{G}}(J)$ for a Gaussian theory, and found (equation (7.12)),

$$\mathcal{Z}_{\mathrm{G}}(J)=\exp\left[\tfrac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\, J(x)\Delta(x-y)J(y)\right], \tag{7.22}$$

where $\Delta(x-y)$ is the propagator.

Expanding in powers of J, we immediately verify that, except for the two-point correlation function, all correlation functions are disconnected. By contrast, the expansion of the functional

$$\mathcal{W}_{\mathrm{G}}(J)=\ln\mathcal{Z}_{\mathrm{G}}(J)=\frac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\, J(x)\Delta(x-y)J(y),$$

only involves the connected contribution. All functions $W^{(n)}$ vanish, except $W^{(2)}=\Delta$, which is connected.

We now assume that, for some action $\mathcal{S}(\phi)$, we have proved that $\mathcal{W}(J)$ generates connected correlation functions, and we add a local perturbation to the action:

$$\mathcal{S}_\varepsilon(\phi)=\mathcal{S}(\phi)+\varepsilon\int \mathrm{d}^d x\,\phi^N(x),$$

(derivative couplings leave the argument unchanged). Then, expanding to first order in ε, we obtain,

$$\begin{aligned}\mathrm{e}^{\mathcal{W}_\varepsilon(J)}&=\int[\mathrm{d}\phi]\exp\left[-\mathcal{S}_\varepsilon(\phi)+\int \mathrm{d}^d x\,\phi(x)J(x)\right]\\&=\left[1-\varepsilon\int \mathrm{d}^d x\left(\frac{\delta}{\delta J(x)}\right)^N\right]\exp\mathcal{W}(J)+O(\varepsilon^2),\end{aligned}$$

and, therefore,

$$\mathcal{W}_\varepsilon(J)=\mathcal{W}(J)-\varepsilon\,\mathrm{e}^{-\mathcal{W}(J)}\int \mathrm{d}^d x\left(\frac{\delta}{\delta J(x)}\right)^N \mathrm{e}^{\mathcal{W}(J)}+O\left(\varepsilon^2\right).$$

The contribution of order ε is a linear combination of products of derivatives of $\mathcal{W}(J)$ with respect to a source at a unique point x. For example, for $N = 3$,

$$\mathcal{W}_\varepsilon(J) - \mathcal{W}(J) = -\varepsilon \int \mathrm{d}^d x \left[\frac{\delta^3 \mathcal{W}(J)}{[\delta J(x)]^3} + 3 \frac{\delta^2 \mathcal{W}(J)}{[\delta J(x)]^2} \frac{\delta \mathcal{W}(J)}{\delta J(x)} + \left(\frac{\delta \mathcal{W}(J)}{\delta J(x)} \right)^3 \right] + O\left(\varepsilon^2\right).$$

Since $\mathcal{W}(J)$ contains only connected contributions, all terms are products of connected contributions linked to the same point, x. Therefore, if $\mathcal{W}(J)$ is connected, all terms of order ε are also connected.

The argument immediately generalizes to any local polynomial $\lambda V_{\mathrm{I}}(\phi)$ (λ is a parameter). Since $\mathcal{W}(J)$ is connected for a general Gaussian theory, it follows, after integration over the parameter λ, that it remains connected for any interaction.

The functions $W^{(n)}$ are thus indeed connected correlation functions. Occasionally, we will emphasize this character by using the notation

$$W^{(n)}(x_1, \ldots, x_n) = \langle \phi(x_1) \cdots \phi(x_n) \rangle_{\mathrm{c}} , \tag{7.23}$$

where the symbol $\langle \bullet \rangle_{\mathrm{c}}$ means a connected contribution to the correlation function.

7.3.2 *Inversion*

The relation between action and generating functional, which has the form of a Laplace transformation, can formally be inverted:

$$\mathrm{e}^{-\mathcal{S}(\phi)} = \int [\mathrm{d}J] \exp\left[\mathcal{W}(J) - \int \mathrm{d}^d x\, J(x)\phi(x) \right], \tag{7.24}$$

where one integrates over imaginary sources $J(x)$ [38]. A truncated loop expansion (see Section 7.9) of the field integral then yields approximate non-linear equations for correlation functions. It is actually convenient to introduce, in the right-hand side, the generating functional of vertex functions defined in Section 7.7.

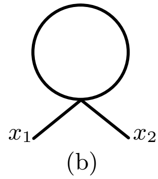


Fig. 7.1 Two-point function at order g

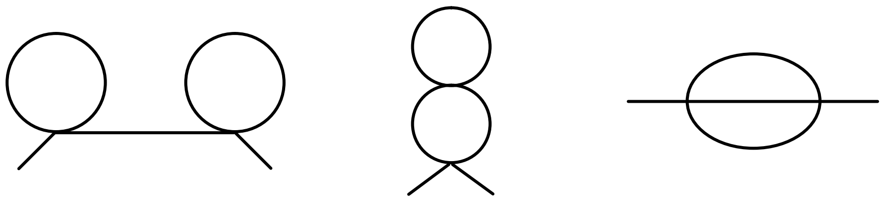
7.4 The example of the ϕ^4 QFT

We illustrate the previous discussion with the second order expansion of the important quartic example,

$$\mathcal{V}_{\mathrm{I}}(\phi) = \frac{g}{4!} \int \mathrm{d}^d x\, \phi^4(x).$$

The two-point function. The two-point function to order g^2 has the expansion

$$\langle \phi(x_1)\phi(x_2) \rangle = \text{(a)} - \tfrac{1}{2} g\, \text{(b)} + \tfrac{1}{4} g^2\, \text{(c)} + \tfrac{1}{4} g^2\, \text{(d)} + \tfrac{1}{6} g^2\, \text{(e)} + O\left(g^3\right).$$



(c) (d) (e)

Fig. 7.2 Contributions of order g^2 to the two-point function

Note that, three additional contributions, which factorize into

$$\langle\phi(x_1)\phi(x_2)\rangle_0 \left\langle\phi^4(y)\right\rangle_0\,,\ \left\langle\phi(x_1)\phi(x_2)\phi^4(y_1)\right\rangle_0 \left\langle\phi^4(y_2)\right\rangle_0\,,\ \text{and}$$
$$\langle\phi(x_1)\phi(x_2)\rangle_0 \left\langle\phi^4(y_1)\phi^4(y_2)\right\rangle_0\,,$$

cancel in the division by $\mathcal{Z}(J=0)$. The diagrams contributing to $\mathcal{Z}(0)$ (the partition function) are called *vacuum diagrams* in particle terminology.

Then,

(a) is the propagator: x_1 ——— x_2;

(b) is the Feynman diagram that appears at order g and is displayed in Fig. 7.1; (c), (d), (e) are the three diagrams displayed in Fig. 7.2.

Let us explain, for example, in detail the weight 1/6 in front of diagram (e). Expanding the exponential at second order, we have to calculate the Gaussian expectation value of

$$\frac{g^2}{2!\,(4!)^2}\int \mathrm{d}^d y_1 \int \mathrm{d}^d y_2 \left\langle\phi(x_1)\phi(x_2)\phi^4(y_1)\phi^4(y_2)\right\rangle_0 .$$

We apply Wick's theorem. First, $\phi(x_1)$ can be associated with any ϕ field of the interaction terms; there are eight choices, and one interaction term is distinguished. Then, $\phi(x_2)$ must be contracted with a field of the remaining interaction term: four choices. Finally, the three remaining fields of the first interaction term can be paired with any permutation of the fields of the second one: 3! equivalent possibilities. Multiplying all factors, one finds

$$\frac{1}{2}\frac{1}{(4!)^2}\times 8\times 4\times 3! = \frac{1}{6}\,.$$

Note also that the factor 1/6 multiplying the diagram can be shown to have an interpretation as 1/3!, the combinatorial factor in the denominator reflecting the symmetry under permutation of the three lines joining the two vertices. There exist systematic expressions giving the weight factor of Feynman diagrams in terms of the symmetry group of the graph.

A useful practical remark is the following: the sum of all weight factors at a given order can be calculated from the 'zero-dimensional' field theory obtained by suppressing the arguments of the field and all derivatives and integration in the action, because the propagator can then be normalized to 1. For example, in the case of the ϕ^4 field considered here, the action becomes

$$\mathcal{S}(\phi) = \tfrac{1}{2}\phi^2 + \tfrac{1}{4!}g\phi^4,$$

and the two-point function is given by

$$Z^{(2)} = \frac{\int \mathrm{d}\phi\,\phi^2 \exp\left[-\mathcal{S}(\phi)\right]}{\int \mathrm{d}\phi\,\exp\left[-\mathcal{S}(\phi)\right]} = 1 - \frac{g}{2} + \frac{2}{3}g^2 + O\left(g^3\right),$$

in which the expressions correspond to ordinary one-variable integrals.

The sum rules are satisfied. For example, at order g^2,

$$\frac{2}{3} = \frac{1}{4} + \frac{1}{4} + \frac{1}{6}\,.$$

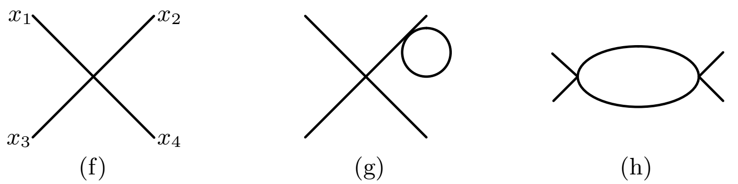


Fig. 7.3 New Feynman diagrams contributing to the four-point function

The four-point function. The expansion of the four-point function to order g^2 is

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= [\,(\mathrm{a})_{12}\ (\mathrm{a})_{34}\ +2\,\text{terms}\,] - \tfrac{1}{2}g\,[\,(\mathrm{a})_{12}\ (\mathrm{b})_{34} + 5\,\text{terms}\,]\\ &- g\,(\mathrm{f}) + g^2\left\{(\mathrm{a})_{12}\left[\tfrac{1}{4}\big((\mathrm{c})_{34} + (\mathrm{d})_{34}\big) + \tfrac{1}{6}(\mathrm{e})_{34}\right] + \ 5\ \text{terms}\right\}\\ &+ \tfrac{1}{4}g^2\,[\,(\mathrm{b})_{12}\ (\mathrm{b})_{34}\ +2\,\text{terms}\,] + \tfrac{1}{2}g^2\,[(\mathrm{g}) + \ 3\ \text{terms}\,] + \tfrac{1}{2}g^2\,[(\mathrm{h}) + \ 2\ \text{terms}\,] + O\left(g^3\right).\end{aligned}$$

The notation $(\mathrm{a})_{12}$, for example, means diagram (a) contributing to the two-point function with arguments x_1 and x_2. Finally, the additional terms are obtained by exchanging the external arguments to restore the permutation symmetry of the four-point function.

Again, as for the two-point function, we have omitted disconnected diagrams in which one factor has no external arguments. As one can check directly here, and as the general arguments in Section 7.3 will confirm, these diagrams are cancelled by the perturbative expansion of $\mathcal{Z}(J=0)$.

The graphs that involve only contributions to the two-point functions ((a), (b), (c), (d), (e)) are disconnected, that is, factorize into a product of functions depending on disjoint subsets of variables. The origin of this phenomenon has already been indicated in Section 1.2.1, and further discussed in Section 7.3. Finally, the new *connected diagrams*, (f), (g), (h), are displayed in Fig. 7.3.

A final remark: local interaction terms may also involve derivatives of the field $\phi(x)$. Then in expression (7.18) derivatives of the propagator appear. The representation in terms of the Feynman diagrams given previously is no longer faithful, since it does not indicate the positions of the derivatives. A more faithful representation can be obtained by splitting points at the vertices and putting arrows on lines.

7.5 Algebraic properties of field integrals. Quantum field equations

Field integrals: Perturbative definition. Rigorously proving the existence of field integrals is not a simple mathematical problem and is solved only in particular examples. However, some difficulties in the very definition of field integrals simply reflect the problem of ambiguities or divergences in perturbation theory, as the discussion of Section 3.3 has already shown. Since straightforward perturbation theory in QFT exhibits divergences, we assume that the QFT has been regularized (see Chapters 8–10), and the perturbation theory defined. This implies, in particular, that the fields contributing to the field integral are regular enough (at least continuous).

Assuming a suitable regularization, here we want to prove that algebraic properties of perturbation theory derived by field integral techniques are consistent with results obtained by working directly with the perturbative expansion [39, 40]. This is sometimes an issue, especially in the case of *dimensional continuation*.

The proofs rely on simple but powerful techniques, which, simultaneously, lead to a derivation of important identities satisfied by correlation functions, like *Schwinger–Dyson equations* (a form of the quantum field equations expressed in terms of correlation functions). Finally, the formal manipulations involved in these derivations, should familiarize the reader with some algebraic properties of the perturbative expansion.

Therefore, we want to show that some properties derived from the field integral (7.16) with action (7.7),

$$\mathcal{Z}(J) = \int [\mathrm{d}\phi] \exp\left[-\tfrac{1}{2}(\phi K \phi) - \mathcal{V}_{\mathrm{I}}(\phi) + J \cdot \phi\right], \tag{7.25}$$

are consistent with properties derived from the corresponding perturbative expression (7.17) (using the notation (7.15)),

$$\mathcal{Z}(J) = \exp\left[-\mathcal{V}_{\mathrm{I}}(D_J)\right] \exp\left[\tfrac{1}{2}(J\Delta J)\right]. \tag{7.26}$$

We assume that all terms in the expansion of this expression in powers of $\mathcal{V}_{\mathrm{I}}$ exist and are finite (which, as we shall see later, implies some conditions for the kernel K and the interaction term $\mathcal{V}_{\mathrm{I}}(\phi)$). From the field integral representation (7.16), we now derive identities satisfied by the generating functional $\mathcal{Z}(J)$, and then prove algebraically that these identities also follow from the perturbative definition (7.17).

7.5.1 Integration by parts and quantum field equations

The integral of a total derivative vanishes. This simple property implies

$$\int [\mathrm{d}\phi] \frac{\delta}{\delta\phi(x)} \exp\bigl(-\mathcal{S}(\phi) + J \cdot \phi\bigr) = 0\,, \tag{7.27}$$

and, therefore,

$$\int [\mathrm{d}\phi] \left[J(x) - \frac{\delta \mathcal{S}}{\delta\phi(x)}\right] \exp\bigl(-\mathcal{S}(\phi) + J \cdot \phi\bigr) = 0\,. \tag{7.28}$$

Using the identity (7.5), one can transform the equation into

$$\left[\frac{\delta \mathcal{S}(D_J)}{\delta\phi(x)} - J(x)\right] \mathcal{Z}(J) = 0\,. \tag{7.29}$$

Equation (7.29) is a compact form of quantum field or Schwinger–Dyson equations [36, 30]. It is equivalent to an infinite set of relations between correlation functions obtained by expanding in powers of the source $J(x)$. These equations, in turn, can be solved perturbatively, and, as we show, determine correlation functions completely.

Solution of the quantum field equations. Conversely, the solution of equation (7.29) is the field integral (7.16). To prove this statement, we express $\mathcal{Z}(J)$, the solution of equation (7.29), as a generalized Laplace transform,

$$\mathcal{Z}(J) = \int [\mathrm{d}\phi] \exp\left[-\Sigma(\phi) + J \cdot \phi\right]. \tag{7.30}$$

Equation (7.29) implies

$$\int [\mathrm{d}\phi] \exp\left[-\Sigma(\phi) + J \cdot \phi\right] \left[\frac{\delta \mathcal{S}(\phi)}{\delta\phi(x)} - J(x)\right] = 0\,. \tag{7.31}$$

Using equation (7.28), but with $\mathcal{S}$ replaced by Σ, we transform the equation into

$$\int [\mathrm{d}\phi] \exp\left[-\Sigma(\phi) + J\cdot\phi\right] \frac{\delta}{\delta\phi(x)} \left[\mathcal{S}(\phi) - \Sigma(\phi)\right] = 0\,. \tag{7.32}$$

Therefore, $\mathcal{S}(\phi) - \Sigma(\phi)$ is independent of ϕ and the difference only affects the normalization of the field integral.

Schwinger–Dyson equations: An example. We consider the Euclidean action (omitting, for notational simplicity, the required regularization)

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \left[\frac{1}{2}\left[\nabla_x \phi(x)\right]^2 + \frac{1}{2} m^2 \phi^2(x) + \frac{g}{4!}\phi^4(x)\right].$$

Equation (7.29), in this example, reads

$$\left[\left(-\nabla_x^2 + m^2\right)\frac{\delta}{\delta J(x)} + \frac{g}{3!}\left(\frac{\delta}{\delta J(x)}\right)^3 - J(x)\right]\mathcal{Z}(J) = 0\,.$$

Explicit equations can be obtained by expanding in powers of $J(x)$. For example, differentiating once with respect to $J(y)$ and setting $J = 0$, one finds

$$\left(-\nabla_x^2 + m^2\right)\langle\phi(x)\phi(y)\rangle + \frac{g}{3!}\left\langle\phi(x)^3\phi(y)\right\rangle = \delta^{(d)}(x-y).$$

Differentiating thrice and setting $J = 0$, one obtains

$$\begin{aligned}&\left(-\nabla_x^2 + m^2\right)\langle\phi(x)\phi(y_1)\phi(y_2)\phi(y_3)\rangle + \frac{g}{3!}\left\langle\phi(x)^3\phi(y_1)\phi(y_2)\phi(y_3)\right\rangle \\ &\quad = \delta^{(d)}(x-y_1)\langle\phi(y_2)\phi(y_3)\rangle + \text{ 2 terms obtained by permutation of 123}\,.\end{aligned}$$

More generally, these equations relate the $2n$-, $(2n{+}2)$- and $(2n{+}4)$-point functions. They can be solved by expanding all functions in powers of g, and this leads to perturbation theory. One can also try solving these equations in a non-perturbative, but approximate, way by truncating, in some form, the infinite set.

7.5.2 *Direct algebraic proof of the quantum field equations*

We now show with purely algebraic transformations that $\mathcal{Z}(J)$ defined by equation (7.17) indeed satisfies equation (7.29) which, for the explicit form (7.7) of $\mathcal{S}(\phi)$, reads

$$\left[\int \mathrm{d}^d y\, K(x-y)\frac{\delta}{\delta J(y)} + \frac{\delta \mathcal{V}_{\mathrm{I}}}{\delta\phi(x)}(D_J) - J(x)\right]\mathcal{Z}(J) = 0\,. \tag{7.33}$$

We want thus to prove algebraically

$$\left[\int \mathrm{d}^d y\, K(x-y)\frac{\delta}{\delta J(y)} + \frac{\delta \mathcal{V}_{\mathrm{I}}(D_J)}{\delta\phi(x)} - J(x)\right]\exp\left[-\mathcal{V}_{\mathrm{I}}(D_J)\right]\exp\left[\tfrac{1}{2}(J\Delta J)\right] = 0\,. \tag{7.34}$$

We first observe that the identity is true in the Gaussian theory, since, as a consequence of the definition (7.11) of Δ, one infers

$$\left[\int \mathrm{d}^d y\, K(x-y)\frac{\delta}{\delta J(y)} - J(x)\right]\exp\left[\tfrac{1}{2}(J\Delta J)\right] = 0\,. \tag{7.35}$$

We then act with the operator $\exp(-\mathcal{V}_{\mathrm{I}}(D_J))$ on the left of equation (7.35).

(i) The operator commutes with KD_J.

(ii) We use the commutation relation

$$[F(D_J), J(x)] = \frac{\delta F(D_J)}{\delta\phi(x)}, \tag{7.36}$$

which can easily be verified by expanding F in powers of D_J. Applied to the functional

$$F(\phi) = \exp\left[-\mathcal{V}_{\mathrm{I}}(\phi)\right],$$

the commutation relation implies

$$\exp\bigl(-\mathcal{V}_{\mathrm{I}}(D_J)\bigr)J(x) = J(x)\exp\bigl(-\mathcal{V}_{\mathrm{I}}(D_J)\bigr) - \frac{\delta\mathcal{V}_{\mathrm{I}}(D_J)}{\delta\phi(x)}\exp\bigl(-\mathcal{V}_{\mathrm{I}}(D_J)\bigr). \tag{7.37}$$

This completes the proof.

7.5.3 The infinitesimal change of variables

Various identities satisfied by correlation functions in the case of field theories possessing symmetries can be proved by the method of infinitesimal change of variables.

We change variables $\phi(x) \mapsto \chi(x)$ in a field integral, setting

$$\phi(x) = \chi(x) + \varepsilon F(x;\chi), \tag{7.38}$$

in which ε is an infinitesimal parameter, and $F(x;\chi)$ a general functional of χ,

$$F(x;\chi) = \sum_1^\infty \frac{1}{n!}\int \mathrm{d}^d y_1 \cdots \mathrm{d}^d y_n\, \chi(y_1)\cdots\chi(y_n) F^{(n)}(x; y_1,\ldots,y_n). \tag{7.39}$$

The variation of the action $\mathcal{S}(\phi)$ is

$$\mathcal{S}(\phi) - \mathcal{S}(\chi) = \varepsilon\int \mathrm{d}^d x \frac{\delta\mathcal{S}}{\delta\chi(x)} F(x;\chi) + O\left(\varepsilon^2\right). \tag{7.40}$$

The change of variables (7.38) in the field integral generates the Jacobian

$$\mathcal{J} = \det\frac{\delta\phi(x)}{\delta\chi(y)} = \det\left[\delta^{(d)}(x-y) + \varepsilon\frac{\delta F(x;\chi)}{\delta\chi(y)}\right]. \tag{7.41}$$

As a consequence of identity ($A2.4$),

$$\det\left(1+\varepsilon M\right) = 1 + \varepsilon\ \mathrm{tr}\, M + O\left(\varepsilon^2\right),$$

one obtains

$$\mathcal{J} = 1 + \varepsilon\int \mathrm{d}^d x \frac{\delta F(x;\chi)}{\delta\chi(x)} + O\left(\varepsilon^2\right). \tag{7.42}$$

It follows that

$$\mathcal{Z}(J)=\int[\mathrm{d}\chi]\left(1+\varepsilon\int \mathrm{d}^d x\frac{\delta F(x;\chi)}{\delta\chi(x)}\right)\left[1-\varepsilon\int \mathrm{d}^d x\frac{\delta\mathcal{S}}{\delta\chi(x)}F(x;\chi)\right.$$
$$\left.+\varepsilon\int \mathrm{d}^d x\,J(x)F(x;\chi)\right]\exp\left[-\mathcal{S}(\chi)+J\cdot\chi\right]+O\left(\varepsilon^2\right). \quad (7.43)$$

The term of order ε^0 is $\mathcal{Z}(J)$ itself. The terms of order ε thus must cancel. Collecting them, replacing χ by D_J when appropriate, one obtains the identity

$$\int \mathrm{d}^d x\left[F(x;D_J)\frac{\delta\mathcal{S}(D_J)}{\delta\chi(x)}-\frac{\delta F(x;D_J)}{\delta\chi(x)}-J(x)F(x;D_J)\right]\mathcal{Z}(J)=0\,. \quad (7.44)$$

Algebraic proof. The algebraic proof of the identity relies on acting with the differential operator $\int \mathrm{d}^d x\,F(x;D_J)$ on the field equation (7.29),

$$\left[\frac{\delta\mathcal{S}(D_J)}{\delta\phi(x)}-J(x)\right]\mathcal{Z}(J)=0\,.$$

Equation (7.44) then follows immediately from the commutation relation (7.36) used in the form

$$[F(y;D_J),J(x)]=\frac{\delta F(y;D_J)}{\delta\chi(x)}.$$

Note that we have dealt, in this section, with an infinitesimal change of variables at first order in ε, although the algebraic proof can be extended to all orders in ε.

7.5.4 The choice of the Gaussian measure

We want to show that, provided the sums of some geometric series exist, a part of the quadratic term K can be treated as a perturbation.

We thus decompose the kernel K in expression (7.7) into a sum of two terms:

$$K=K_1+K_2\,,\quad \text{with}\quad K_1>0\,.$$

We want to prove algebraically that

$$\exp\left[-\mathcal{V}_{\mathrm{I}}(D_J)-\tfrac{1}{2}(D_J K_2 D_J)\right]\exp\left[\tfrac{1}{2}(JK_1^{-1}J)\right]$$
$$=\mathcal{N}(K_1,K_2)\exp\left[-\mathcal{V}_{\mathrm{I}}(D_J)\right]\exp\left[\tfrac{1}{2}(J\left(K_1+K_2\right)^{-1}J)\right], \quad (7.45)$$

where $\mathcal{N}$ is independent of the source J. Since the operator $\exp\left[-\mathcal{V}_{\mathrm{I}}(\delta/\delta J)\right]$ can be factorized in both sides of equation (7.45), it is sufficient to prove the identity for $\mathcal{V}_{\mathrm{I}}=0$, that is, to calculate the functional

$$\mathcal{Z}_{\mathrm{G}}(J)=\exp\left[-\tfrac{1}{2}(D_J K_2 D_J)\right]\exp\left[\tfrac{1}{2}(JK_1^{-1}J)\right]. \quad (7.46)$$

We act with $K_1 D_J$ on $\mathcal{Z}_{\mathrm{G}}(J)$:

$$\int \mathrm{d}^d y\,K_1(x-y)\frac{\delta}{\delta J(y)}\mathcal{Z}_{\mathrm{G}}(J)=\exp\left[-\tfrac{1}{2}(D_J K_2 D_J)\right]J(x)\exp\left[\tfrac{1}{2}(JK_1^{-1}J)\right]. \quad (7.47)$$

In the right-hand side, we then commute J to bring it to the left (equation (7.36)):

$$\exp\left[-\tfrac{1}{2}(D_J K_2 D_J)\right] J(x) = J(x)\exp\left[-\tfrac{1}{2}(D_J K_2 D_J)\right] - (K_2 D_J)(x)\exp\left[-\tfrac{1}{2}(D_J K_2 D_J)\right].$$

We conclude that $\mathcal{Z}_{\mathrm{G}}$ satisfies the equation

$$\left[\int \mathrm{d}^d y\,(K_1+K_2)(x-y)\frac{\delta}{\delta J(y)} - J(x)\right]\mathcal{Z}_{\mathrm{G}}(J) = 0\,. \tag{7.48}$$

Integrating the equation, one finds

$$\mathcal{Z}_{\mathrm{G}}(J) = \mathcal{N}(K_1,K_2)\exp\left[\tfrac{1}{2}(J\,(K_1+K_2)^{-1}\,J)\right],$$

proving identity (7.45). After some additional algebra, one verifies that $\mathcal{N}^2 = \det(1 + K_2 K_1^{-1})$.

In the same way, one can show that a part of the source term can be treated as an interaction, without changing $\mathcal{Z}(J)$. The result follows from the identity

$$\exp\left[\int \mathrm{d}^d x\, L(x)\frac{\delta}{\delta J(x)}\right] F(J) = F(J+L).$$

All these identities make sense only if both sides exist.

7.5.5 The functional Dirac δ-function

We consider, here, a field $\boldsymbol{\phi}(x)$ with N components $\phi_i(x)$, $i=1,\ldots,N$, satisfying a constraint of the form

$$F(x,\boldsymbol{\phi}) = 0\,, \tag{7.49}$$

where F is expandable in powers of $\boldsymbol{\phi}$. We assume that it is possible to solve the constraint and calculate one component, for example $\phi_N(x)$, as a formal power series in the remaining components.

We then define the functional Dirac δ-function by

$$\delta(F) \equiv \int [\mathrm{d}\lambda(x)]\exp\left[\int \mathrm{d}^d x\,\lambda(x)F(x,\boldsymbol{\phi})\right], \tag{7.50}$$

where the λ-integration runs along the imaginary axis. This is a generalized Fourier representation. For reasons explained before, we have omitted any normalization factor.

We now show that representation (7.50) has the properties expected from a δ-function. We consider a field integral in which the integration is restricted to fields $\boldsymbol{\phi}$ satisfying the constraint $F(x,\boldsymbol{\phi})=0$:

$$\mathcal{Z}_F(\mathbf{J}) = \int \prod_{i=1}^{N}[\mathrm{d}\phi_i(x)]\,\delta(F)\exp\left(-\mathcal{S}(\boldsymbol{\phi}) + \mathbf{J}\cdot\boldsymbol{\phi}\right). \tag{7.51}$$

We use the representation (7.50) to write $\mathcal{Z}_F$ as

$$\mathcal{Z}_F(\mathbf{J}) = \int [\mathrm{d}\lambda]\prod_{i=1}^{N}[\mathrm{d}\phi_i]\exp\left[-\mathcal{S}(\boldsymbol{\phi}) + \lambda\cdot F(\boldsymbol{\phi}) + \mathbf{J}\cdot\boldsymbol{\phi}\right]. \tag{7.52}$$

We now assume that the quadratic part in the fields $(\boldsymbol{\phi}, \lambda)$ in the total action is not singular in such a way that, after adding a source term for $\lambda(x)$, we can use the equivalent of equation (7.17) to define, algebraically, integral (7.52) (it is sufficient, in particular, for the quadratic part of $\mathcal{S}(\boldsymbol{\phi})$ to be non-singular). Then, if we add a term proportional to $F(x, \boldsymbol{\phi})$ to $\mathcal{S}(\boldsymbol{\phi})$,

$$\mathcal{S}(\boldsymbol{\phi}) \mapsto \mathcal{S}(\boldsymbol{\phi}) + \int \mu(x) F(x, \boldsymbol{\phi}) \mathrm{d}^d x \,, \tag{7.53}$$

we can cancel the change by the change of variables $\lambda(x) \mapsto \lambda(x) + \mu(x)$. We have actually proven invariance by change of variables only for the infinitesimal change of variables, but for translations the property can be generalized to finite changes.

Finally, note that with the field $\lambda(x)$ is associated a simple field equation. From the identity

$$\int \prod_i \left[\mathrm{d}\phi_i\right] \left[\mathrm{d}\lambda\right] \frac{\delta}{\delta \lambda(x)} \exp\left[\lambda \cdot F + \mathbf{J} \cdot \boldsymbol{\phi} - \mathcal{S}(\boldsymbol{\phi})\right] = 0 \,,$$

one derives

$$F(x; D_J) \mathcal{Z}_F(\mathbf{J}) = 0 \,. \tag{7.54}$$

This equation expresses the constraint $F = 0$ on the generating functional of correlation functions $\mathcal{Z}_F(J)$.

Examples. In Section 19.9, a functional δ-function in the form (7.50) is used to express the non-linear σ model in terms of an N-vector field $\boldsymbol{\phi}$ satisfying an $O(N)$ invariant constraint $\boldsymbol{\phi}^2(x) = 1$.

In Chapter 21, one covariant gauge corresponds to constraining the gauge field A_μ to satisfy the condition $\sum_\mu \partial_\mu A_\mu = 0$. Again, the representation (7.50) is useful. The equation (7.54) then becomes $\sum_\mu \partial^x_\mu \delta \mathcal{Z}/\delta J_\mu(x) = 0$, where J_μ is the gauge field source.

7.6 Connected correlation functions. Cluster properties

We now consider *local* Euclidean actions $\mathcal{S}(\phi)$ functions of a scalar field $\phi(x)$ of the form

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \, \mathcal{L}(\phi; x), \tag{7.55}$$

where the Euclidean Lagrangian density $\mathcal{L}(\phi; x)$ is a function of the field $\phi(x)$ and its derivatives and does not depend on space explicitly, but only through ϕ (as in the example (6.1) after continuation to Euclidean time).

We have already observed in quantum mechanics that the partition function $\operatorname{tr} \mathrm{e}^{-\beta H}$ in the large volume limit, that is, for $\beta \to \infty$, has an exponential behaviour; more precisely (equation (2.1)),

$$\lim_{\beta \to \infty} -\frac{1}{\beta} \ln \operatorname{tr} \mathrm{e}^{-\beta H} = E_0 \,,$$

in which E_0 is the ground state energy. Moreover, the convergence towards the limit is exponential when the ground state is isolated (equation (2.2)).

Here, we generalize the quantum mechanics result to local field theories. The discussion that follows is somewhat intuitive and tries only to motivate results that can be proven rigorously. Some peculiarities found in massless theories or due to ground state degeneracy are ignored here and discussed later, starting with Chapter 13.

The functional $\mathcal{Z}(J)$ is a partition function in the presence of an external source $J(x)$. We take for source $J(x)$ the sum of two terms $J_1(x)$ and $J_2(x)$, in which $J_1(x)$ and $J_2(x)$ have disjoint supports consisting of two domains Ω_1 and Ω_2 of large volumes V_1 and V_2:

$$J(x) = J_1(x) + J_2(x), \quad \begin{cases} J_1(x) = 0 & \text{for } x \notin \Omega_1, \\ J_2(x) = 0 & \text{for } x \notin \Omega_2 \,, \end{cases} \quad \Omega_1 \cap \Omega_2 = \emptyset \,. \tag{7.56}$$

We also assume that $J_1(x)$ and $J_2(x)$ fluctuate around an arbitrarily small but non-vanishing constant. Then the *locality* of $\mathcal{S}(\phi)$ implies the decomposition

$$\begin{aligned} \mathcal{S}(\phi) - \int \mathrm{d}^d x \, J(x)\phi(x) &= \int_{x \in \Omega_1} \mathrm{d}^d x \, [\mathcal{L}(\phi; x) - J_1(x)\phi(x)] \\ &+ \int_{x \in \Omega_2} \mathrm{d}^d x \, [\mathcal{L}(\phi; x) - J_2(x)\phi(x)] + \int_{x \notin \Omega_1 \cup \Omega_2} \mathrm{d}^d x \, \mathcal{L}(\phi; x) \\ &+ \text{ contributions from boundaries} \,. \end{aligned} \tag{7.57}$$

We then write $\mathcal{Z}(J)$ as

$$\mathcal{Z}(J) = \mathcal{Z}_1(J_1)\mathcal{Z}_2(J_2)\mathcal{Z}_{12}(J_1, J_2), \tag{7.58}$$

with the definitions,

$$\begin{aligned} \mathcal{Z}_1(J_1) &= \int_{x \in \Omega_1} [\mathrm{d}\phi(x)] \exp\left[-\mathcal{S}(\phi) + \int \mathrm{d}^d x \, J_1(x)\phi(x)\right], \\ \mathcal{Z}_2(J_2) &= \int_{x \in \Omega_2} [\mathrm{d}\phi(x)] \exp\left[-\mathcal{S}(\phi) + \int \mathrm{d}^d x \, J_2(x)\phi(x)\right]. \end{aligned}$$

Both functionals $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are normalized to 1 for $J_1 = 0$ or $J_2 = 0$, respectively. The functional $\mathcal{Z}_{12}$ is defined by equation (7.58). Its dependence in J_1 and J_2 comes entirely from the existence of boundary terms in equation (7.57). When we scale up Ω_1 and Ω_2, these boundary terms grow like surfaces while the first two terms in equation (7.57) grow like the volumes V_1 and V_2. Therefore, $\ln \mathcal{Z}_{12}(J_1, J_2)$ becomes asymptotically negligible compared to $\ln \mathcal{Z}_1(J_1)$ and $\ln \mathcal{Z}_2(J_2)$ when V_1 and $V_2 \to \infty$.

To express this property, it is natural to introduce the functional $\mathcal{W}(J) = \ln \mathcal{Z}(J)$ (equation (7.19)), which then satisfies

$$\mathcal{W}(J_1 + J_2) = \mathcal{W}_1(J_1) + \mathcal{W}_2(J_2) + \text{negligible} \,. \tag{7.59}$$

In particular, if $\mathcal{S}(\phi) - J\phi$ is translation invariant (which implies that J is a constant), $\mathcal{W}(J)$ is extensive, that is, proportional to the total volume. This property generalizes property (2.1).

Cluster properties. After the infinite volume limit has been taken, one can expand $\mathcal{W}(J_1 + J_2)$ in powers of J_1 and J_2:

$$\begin{aligned} \mathcal{W}(J_1 + J_2) = \sum_{0 \le p \le n}^{\infty} \frac{1}{p!(n-p)!} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_p \, \mathrm{d}^d y_{p+1} \cdots \mathrm{d}^d y_n \\ \times W^{(n)}(x_1, \ldots, x_p, y_{p+1}, \ldots, y_n) J_1(x_1) \cdots J_1(x_p) J_2(y_{p+1}) \cdots J_2(y_n), \end{aligned} \tag{7.60}$$

with $x_i \in \Omega_1, \; y_j \in \Omega_2$.

The property (7.59) implies that all terms with $p \neq 0$ or $p \neq n$ are negligible for V_1 and V_2 large. Considering expression (7.60), we note that this implies that the functions $W^{(n)}$ must decrease rapidly enough when the two non-empty sets of points $\{x_1, \ldots, x_p\}$ and $\{y_{p+1}, \ldots, y_n\}$ have large separations. More precisely,

$$W^{(n)}(x_1, \ldots, x_p, y_{p+1}, \ldots, y_n) \to 0 \quad \text{for} \quad \min_{\substack{i=1\cdots p \\ j=p+1\cdots n}} |x_i - y_j| \to \infty\,. \tag{7.61}$$

This property, which we describe here only qualitatively, is called the *cluster property*, and is a *characteristic property of the connected correlation functions* generated by the functional $\mathcal{W}(J)$ (see Section A7.3 for details).

Feynman diagrams. We have seen that a Feynman diagram that is disconnected in the sense of graphs can be factorized into a product of the form

$$F_1(x_1, \ldots, x_p)F_2(y_1, \ldots, y_q).$$

In a translation invariant theory, when the two set of points $\{x_i\}$ and the set $\{y_i\}$ are not empty, we can separate the two sets in a way which leaves such a product invariant: a disconnected diagram cannot satisfy the cluster property. We recover the property that the Feynman diagrams which contribute to the perturbative expansion of $\mathcal{W}(J)$ are all connected. Furthermore, it can be verified that, in a field theory containing only massive fields, connected Feynman diagrams decrease exponentially, when points are separated, with a minimal rate which is the inverse of the smallest mass in the theory. This property is a consequence of the exponential decrease of the propagator (Section A7.3).

7.7 Legendre transformation. Vertex functions

We introduce a new generating functional, $\Gamma(\varphi)$, Legendre transform (see Section 1.8) of the generating functional of connected correlation functions $\mathcal{W}(J)$, called the generating functional of *vertex functions* also called proper vertices or 1PI correlation functions for reasons that are explained in Section 7.10.

Legendre transformation. In the context of statistical physics and phase transitions, it is natural to consider the thermodynamic potential, Legendre transform of the *free energy* $\mathcal{W}(J)$ and generator of vertex functions. Here, vertex functions are introduced, because they have special properties from the point of view of perturbation theory, as we demonstrate in Sections 7.9 and 7.10. The generating functional $\Gamma(\varphi)$ of vertex functions, in which $\varphi(x)$ is a classical field argument of Γ, is related to $\mathcal{W}(J)$ by

$$\Gamma(\varphi) + \mathcal{W}(J) - \int \mathrm{d}^d x\, J(x)\varphi(x) = 0\,, \tag{7.62}$$

with

$$\varphi(x) = \frac{\delta \mathcal{W}}{\delta J(x)}. \tag{7.63}$$

We have described some properties of the Legendre transformation in Section 1.8 and shown that it is involutive. In particular,

$$J(x) = \frac{\delta \Gamma}{\delta \varphi(x)}. \tag{7.64}$$

Moreover, if $\mathcal{W}(J)$ depends on a parameter v, then,

$$\frac{\partial \mathcal{W}}{\partial v} + \frac{\partial \Gamma}{\partial v} = 0\,. \tag{7.65}$$

This identity, derived here for one external parameter, obviously applies also for an external field or source. It will be used frequently.

Expansion of $\Gamma(\varphi)$. If we set $J = 0$ in equation (7.63), we obtain

$$\varphi(x)|_{J=0} = \left.\frac{\delta \mathcal{W}}{\delta J(x)}\right|_{J=0} \equiv W^{(1)}(x) = \langle \phi(x) \rangle\,,$$

that is, that $\varphi(x)$ for a vanishing source is the expectation value of the field ϕ. Moreover, equation (7.64) then implies that the expectation value of $\phi(x)$ is an extremum of $\Gamma(\varphi)$.

Inverting the relation (7.63), we can expand the source $J(x)$ as a series of powers of

$$\xi(x) = \varphi(x) - W^{(1)}(x) = \varphi(x) - \langle \phi(x) \rangle\,. \tag{7.66}$$

The classical field $\xi(x)$ is related to the correlation functions of the field

$$\Xi(x) = \phi(x) - \langle \phi(x) \rangle\,, \tag{7.67}$$

which has a vanishing expectation value, $\langle \Xi(x) \rangle = 0$.

The first terms of the expansion of equation (7.63) are

$$\xi(x) = \int \mathrm{d}^d x_1\, W^{(2)}(x, x_1) J(x_1) + \frac{1}{2!} \int \mathrm{d}^d x_1\, \mathrm{d}^d x_2\, W^{(3)}(x, x_1, x_2) J(x_1) J(x_2) + \cdots\,. \tag{7.68}$$

We introduce the inverse $S(x, y)$ of the connected two-point function:

$$\int \mathrm{d}^d z\, S(x, z) W^{(2)}(z, y) = \delta^{(d)}(x - y). \tag{7.69}$$

We can then write the expansion of $J(x)$ as

$$\begin{aligned} J(x) = \int \mathrm{d}^d x_1\, S(x, x_1) \xi(x_1) - \frac{1}{2!} \int \mathrm{d}^d y_1\, \mathrm{d}^d y_2\, \mathrm{d}^d y_3\, \mathrm{d}^d x_1\, \mathrm{d}^d x_2\, S(x, y_3) \\ \times S(x_1, y_1) S(x_2, y_2) W^{(3)}(y_1, y_2, y_3) \xi(x_1) \xi(x_2) + \cdots\,. \end{aligned} \tag{7.70}$$

The expansion can be expressed in terms of so-called amputated correlation functions

$$W^{(n)}_{\text{amp.}}(x_1, \ldots, x_n) = \int \left[\prod_{i=1}^{n} \mathrm{d}^d y_i\, S(x_i, y_i)\right] W^{(n)}(y_1, \ldots, y_n). \tag{7.71}$$

In terms of Feynman diagrams, this means that propagators and contributions to the two-point function on external lines are cancelled.

One can use equation (7.64) to calculate $\Gamma(\varphi)$. We set

$$\Gamma(\varphi) = \sum_{1}^{\infty} \frac{1}{n!} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_n\, \Gamma^{(n)}(x_1, \ldots, x_n) \xi(x_1) \cdots \xi(x_n), \tag{7.72}$$

where the $\Gamma^{(n)}$ are associated with the field Ξ (defined by equation (7.67)).

One finds

$$
\begin{aligned}
&\Gamma^{(1)}(x) = 0\,,\\
&\Gamma^{(2)}(x_1,x_2) = S(x_1,x_2) = [W^{(2)}]^{-1}(x_1,x_2),\\
&\Gamma^{(3)}(x_1,x_2,x_3) = -W^{(3)}_{\text{amp.}}(x_1,x_2,x_3),\\
&\Gamma^{(4)}(x_1,x_2,x_3,x_4) = -W^{(4)}_{\text{amp.}}(x_1,x_2,x_3,x_4)\\
&\quad + \int \mathrm{d}^d y\,\mathrm{d}^d z\, W^{(3)}_{\text{amp.}}(x_1,x_2,y)W^{(2)}(y,z)W^{(3)}_{\text{amp.}}(z,x_3,x_4) + 2 \text{ terms}\\
&\quad \ldots
\end{aligned}
$$

The inverse relations are even more useful, because, as we show in Section 7.10, $\Gamma(\varphi)$ has simpler properties than $\mathcal{W}(J)$. For example,

$$
\begin{aligned}
&W^{(2)}(x_1,x_2) = [\Gamma^{(2)}]^{-1}(x_1,x_2),\\
&W^{(3)}_{\text{amp.}}(x_1,x_2,x_3) = -\Gamma^{(3)}(x_1,x_2,x_3),\\
&W^{(4)}_{\text{amp.}}(x_1,x_2,x_3,x_4) = -\Gamma^{(4)}(x_1,x_2,x_3,x_4),\\
&\quad + \int \mathrm{d}^d y\,\mathrm{d}^d z\,\Gamma^{(3)}(x_1,x_2,y)W^{(2)}(y,z)\Gamma^{(3)}(z,x_3,x_4) + 2 \text{ terms}\,,\\
&\ldots
\end{aligned}
$$

Fig. 7.4 Graphical representation of the two-point function and the n-point vertex function

With the graphical definitions of Fig.7.4, we can give a graphical representation of the first equations. The correlation functions $W^{(3)}$ and $W^{(4)}$ are then be represented as shown in Figs. 7.5 and 7.6, respectively.

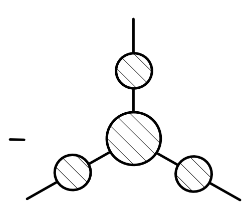


Fig. 7.5 The connected three-point function $W^{(3)}$

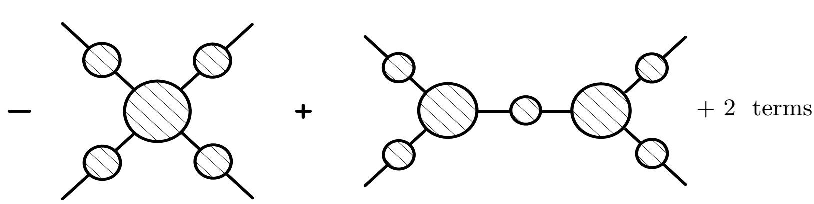


Fig. 7.6 The connected four-point function $W^{(4)}$

Mass operator. It follows from the set of relations between connected correlation functions and vertex functions that Feynman diagrams that contribute to vertex functions appear in the expansion of connected functions with the opposite sign except in the case of the two-point function. Indeed, if we set (in the notation (7.7))

$$\Gamma^{(2)}(x,y) = K(x-y) + \Sigma(x,y),$$

where K is the two-point function $\Gamma^{(2)}$ in the free (Gaussian) theory, then $\Sigma(x,y)$, called the mass operator, contains all the perturbative corrections. The connected two-point function $W^{(2)}(x,y)$ then takes the form of a geometric series of the form,

$$W^{(2)}(x,y) = \Delta(x-y) - \int \mathrm{d}^d z_1\, \mathrm{d}^d z_2\, \Delta(x-z_1)\Sigma(z_1,z_2)\Delta(z_2-y) + \cdots\ ,$$

where Δ is the propagator (7.11). In this case, K, the first term in perturbation theory contributing to $\Gamma^{(2)}$, is special from the point of view of Feynman graph expansion, since it has the same sign as the propagator in $W^{(2)}$.

Convexity. The connected two-point function can be written as

$$W^{(2)}(x,y) = \langle [\phi(x) - \langle\phi(x)\rangle]\,[\phi(y) - \langle\phi(y)\rangle]\rangle = \langle \Xi(x)\Xi(y)\rangle\,.$$

Setting $(J\cdot\Xi) \equiv \int \mathrm{d}^d x\, J(x)\Xi(x)$, we immediately obtain

$$\int \mathrm{d}^d x\, \mathrm{d}^d y\, J(x)J(y)W^{(2)}(x,y) = \left\langle (J\cdot\Xi)^2\right\rangle \geq 0\,.$$

It follows that the two-point function $W^{(2)}(x,y)$ is the kernel of a positive operator. Since $\Gamma^{(2)}(x,y)$ is the inverse of $W^{(2)}(x,y)$ in the sense of kernels, it is also the kernel of a positive operator,

$$\int \mathrm{d}^d x\, \mathrm{d}^d y\, \varphi(x)\Gamma^{(2)}(x,y)\varphi(y) \geq 0\,. \tag{7.73}$$

For the same reasons, $\delta^2\mathcal{W}(J)/\delta J(x)\delta J(y)$ and $\delta^2\Gamma(\varphi)/\delta\varphi(x)\delta\varphi(y)$, which are the two-point functions in an external source, are positive operators. In particular, in the case of constant fields J or φ, $\mathcal{W}(J)$ and $\Gamma(\varphi)$, both divided by the total space volume, are convex functions of the sources. We shall recall this property when we examine the physics of spontaneous symmetry breaking and meet functions $\Gamma(\varphi)$ that do not seem obviously convex (see Section 7.11).

7.8 Momentum representation

In this work, we mainly study theories invariant under space translations. Correlation functions then depend only on differences of space arguments. It is thus natural to introduce their Fourier representation. To establish a consistent set of conventions, we start from the generating functional of vertex functions (in d space dimensions)

$$\Gamma(\varphi) = \sum_n \frac{1}{n!}\int \prod_{i=1}^{n} \mathrm{d}^d x_i\, \varphi(x_i)\Gamma^{(n)}(x_1,\ldots,x_n).$$

We introduce the Fourier components of the field,

$$\varphi(x) = \int \mathrm{e}^{ipx}\, \widetilde{\varphi}(p) \mathrm{d}^d p\,. \tag{7.74}$$

We define the vertex function $\widetilde{\Gamma}^{(n)}(p_1, \ldots, p_n)$ in the Fourier (or momentum) representation in terms of the coefficient of $\widetilde{\varphi}(p_1) \cdots \widetilde{\varphi}(p_n)$ in $\Gamma(\varphi)$, assuming translation invariance, which implies total momentum conservation,

$$(2\pi)^d \delta^{(d)}\,(p_1 + \cdots + p_n)\, \widetilde{\Gamma}^{(n)}(p_1, \ldots, p_n) = \int \Bigl(\prod_k \mathrm{d}^d x_k\, \mathrm{e}^{ix_k p_k} \Bigr) \Gamma^{(n)}(x_1, \ldots, x_n). \tag{7.75}$$

In the same way, we introduce the Fourier components $\tilde{J}(p)$ of the source,

$$J(x) = \int \mathrm{e}^{ipx}\, \tilde{J}(p) \mathrm{d}^d p\,.$$

We insert this representation into the generating functional $\mathcal{W}(J)$ of connected correlation functions and define $\widetilde{W}^{(n)}(p_1, \ldots, p_n)$ in terms of the coefficient of $\tilde{J}(p_1) \cdots \tilde{J}(p_n)$:

$$(2\pi)^d \delta^{(d)}\,(p_1 + \cdots + p_n)\, \widetilde{W}^{(n)}(p_1, \ldots, p_n) = \int \Bigl(\prod_k \mathrm{d}^d x_k\, \mathrm{e}^{ix_k p_k} \Bigr) W^{(n)}(x_1, \ldots, x_n). \tag{7.76}$$

Inverting the Fourier transformation, we find for the various two-point functions,

$$\begin{aligned} \Gamma^{(2)}(x, y) &= \frac{1}{(2\pi)^d} \int \mathrm{d}^d p\; \mathrm{e}^{ip(x-y)}\, \widetilde{\Gamma}^{(2)}(-p, p), \\ W^{(2)}(x, y) &= \frac{1}{(2\pi)^d} \int \mathrm{d}^d p\; \mathrm{e}^{ip(x-y)}\, \widetilde{W}^{(2)}(-p, p). \end{aligned}$$

In what follows, we denote simply by $\widetilde{\Gamma}^{(2)}(p)$ and $\widetilde{W}^{(2)}(p)$, the functions

$$\widetilde{\Gamma}^{(2)}(p) = \widetilde{\Gamma}^{(2)}(-p, p)\,, \quad \widetilde{W}^{(2)}(p) = \widetilde{W}^{(2)}(-p, p).$$

The Legendre transformation takes the simple form

$$\widetilde{\Gamma}^{(2)}(p) \widetilde{W}^{(2)}(p) = 1\,. \tag{7.77}$$

The explicit expressions of Section 7.7 (*e.g.* equation (7.71)) then show that amputation and Legendre transformation become in the Fourier representation purely algebraic operations in the sense that no momentum integration is involved, because the two-point function is proportional to $\delta^{(d)}(p_1 + p_2)$. For example:

$$\widetilde{W}^{(n)}_{\text{amp.}}(p_1, \ldots, p_n) = \widetilde{W}(p_1, \ldots, p_n) \prod_{i=1}^{n} \left[\widetilde{W}^{(2)}(p_i) \right]^{-1},$$

and also

$$\begin{aligned} \widetilde{\Gamma}^{(3)}(p_1, p_2, p_3) &= -\widetilde{W}^{(3)}_{\text{amp.}}(p_1, p_2, p_3), \\ \widetilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4) &= -\widetilde{W}^{(4)}_{\text{amp.}}(p_1, p_2, p_3, p_4) + \Bigl[\widetilde{W}^{(3)}_{\text{amp.}}(p_1, p_2, -p_1 - p_2) \widetilde{W}^{(2)}(p_1 + p_2) \\ &\quad \times \widetilde{W}^{(3)}_{\text{amp.}}(p_3, p_4, -p_3 - p_4) + \text{cyclic permutation of } \{p_2, p_3, p_4\} \Bigr], \\ &\cdots\,. \end{aligned}$$

This remark will be used in the discussion of divergences in perturbation theory.

7.9 Loop or semi-classical expansion

It is sometimes useful to reorganize perturbation theory by grouping some classes of Feynman diagrams. An important example is provided by the *loop expansion* [32]. We set $\hbar$ in front of the classical action and the source term and consider the generating functional

$$\mathcal{Z}(J) = \mathcal{N} \int [\mathrm{d}\phi] \exp\left[-\frac{1}{\hbar}\big(\mathcal{S}(\phi) - J\cdot\phi\big)\right], \tag{7.78}$$

($\mathcal{Z}(0) = 1$), in the symbolic notation of Section 7.2.2. For $\hbar \to 0$, the field integral can be calculated by the *steepest descent method* (see Section 1.3). The successive corrections to the leading order approximation generate an expansion in powers of $\hbar$. We show below that this expansion gathers Feynman diagrams according to the number of loops.

Because it is an expansion in powers of $\hbar$, the expansion is also called *semi-classical*, although classical field equations correspond to a true classical limit only for massless theories like quantum electrodynamics.

7.9.1 Loop expansion at leading order

The saddle point equation is

$$\frac{\delta \mathcal{S}}{\delta\phi(x)}\left[\phi_c(J)\right] = J(x). \tag{7.79}$$

We assume in the following calculations that the field ϕ has been defined in such a way that $\phi_c(J=0) = 0$ and that $S(\phi = 0) = 0$.

Substituting the solution $\phi_c(J)$ into the classical action, one obtains $\mathcal{Z}(J)$ at leading order,

$$\ln \mathcal{Z}(J) \sim \ln \mathcal{Z}_0(J) \equiv \frac{1}{\hbar}\left[-\mathcal{S}(\phi_c) + J\cdot\phi_c\right]. \tag{7.80}$$

When $\hbar$ is explicit, it is convenient to define $\mathcal{W}(J)$ by

$$\mathcal{W}(J) = \hbar \ln \mathcal{Z}(J). \tag{7.81}$$

Then, at leading order,

$$\mathcal{W}(J) = \mathcal{W}_0(J) \equiv -\mathcal{S}(\phi_c) + J\cdot\phi_c\,. \tag{7.82}$$

Together, the two equations (7.79) and (7.82) imply that the two functionals $\mathcal{S}(\phi)$ and $\mathcal{W}_0(J)$ *are related by a Legendre transformation* (see also Section 1.8).

Perturbation theory. Ordinary perturbation theory is recovered by expanding the solution $\phi_c(J)$ in powers of J.

To show this, we decompose $\mathcal{S}(\phi)$ into a sum of a quadratic part and interaction terms (expression (7.7)). Moreover, we assume

$$\mathcal{V}_\mathrm{I}(\phi) = O\left(\phi^3\right) \quad \text{for } \phi \to 0\,.$$

Equation (7.79) then takes the symbolic form

$$K\phi_c + \frac{\delta\mathcal{V}_\mathrm{I}(\phi_c)}{\delta\phi} = J\,. \tag{7.83}$$

It can be solved iteratively as ($K\Delta = 1$)

$$\phi_c = \Delta J - \Delta\frac{\delta\mathcal{V}_\mathrm{I}(\phi_c)}{\delta\phi} = \Delta J - \Delta\frac{\delta\mathcal{V}_\mathrm{I}(\Delta J)}{\delta\phi} + \cdots\,. \tag{7.84}$$

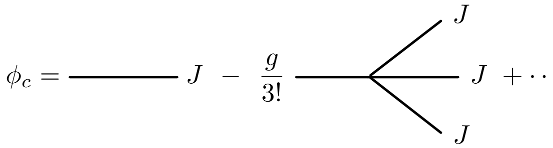


Fig. 7.7 Feynman-diagram expansion of ϕ_c in the ϕ^4 field theory

If, for example,

$$\mathcal{V}_{\mathrm{I}}(\phi) = \frac{g}{4!}\int \mathrm{d}^d x\, \phi^4(x),$$

ϕ_c has the Feynman diagram expansion displayed in Fig. 7.7. We observe that only trees [33], that is, diagrams without loops, are generated. Substituting the expansion into equation (7.82), we note that the perturbative expansion of $\mathcal{W}_0(J)$ in powers of J also contains only connected tree Feynman diagrams. The functional $\mathcal{W}_0(J)$ is the generating functional of connected tree diagrams.

Vertex functions. Since the action $\mathcal{S}(\phi)$ and $\mathcal{W}_0(J)$, the generating functional of connected tree diagrams are related by a Legendre transformation, we conclude immediately

$$\Gamma_0(\varphi) = \mathcal{S}(\varphi). \tag{7.85}$$

At leading order, $\Gamma(\varphi)$, the generating functional of vertex functions, is identical to the classical action. From the point of view of Feynman diagrams, it reduces to the vertices of the field theory.

7.9.2 Order $\hbar$ or one-loop contributions

The Gaussian integral obtained by expanding around the saddle point generates the order $\hbar$ corrections (Appendix B of Ref. [41] and Ref. [42]). We set

$$\phi(x) = \phi_c(J,x) + \sqrt{\hbar}\chi(x), \tag{7.86}$$

$$S^{(2)}(x_1,x_2;\phi) = \frac{\delta^2\mathcal{S}}{\delta\phi(x_1)\delta\phi(x_2)}\,. \tag{7.87}$$

Expanding the action in powers of $\hbar$, one finds

$$\mathcal{S}(\phi) - J\cdot\phi = \mathcal{S}(\phi_c) - J\cdot\phi_c + \frac{\hbar}{2}\int \mathrm{d}^d x_1\,\mathrm{d}^d x_2\, S^{(2)}(x_1,x_2;\phi_c)\chi(x_1)\chi(x_2) + O(\hbar^{3/2}).$$

The field integral at this order becomes

$$\mathcal{Z}(J) \sim \mathcal{Z}_0(J)\int [\mathrm{d}\chi]\exp\left[-\frac{1}{2}\int \mathrm{d}^d x_1\,\mathrm{d}^d x_2\, S^{(2)}(x_1,x_2;\phi_c)\chi(x_1)\chi(x_2)\right],$$

and, therefore,

$$\mathcal{Z}(J) \propto \mathcal{Z}_0(J)\left[\det S^{(2)}(x_1,x_2;\phi_c)/\det S^{(2)}(x_1,x_2;0)\right]^{-1/2}, \tag{7.88}$$

where the normalization follows from the conditions $\phi_c(0)=0$ and $\mathcal{Z}(0)=1$.

At this order, the connected generating functional $\mathcal{W}(J)$ is then ($\ln\det = \mathrm{tr}\ln$)

$$\mathcal{W}(J) = \mathcal{W}_0(J) + \hbar\mathcal{W}_1(J) + O\left(\hbar^2\right), \tag{7.89}$$

with

$$\mathcal{W}_1(J) = -\tfrac{1}{2}\left[\mathrm{tr}\ln S^{(2)}(x_1,x_2;\phi_c) - \mathrm{tr}\ln S^{(2)}(x_1,x_2;0)\right]. \tag{7.90}$$

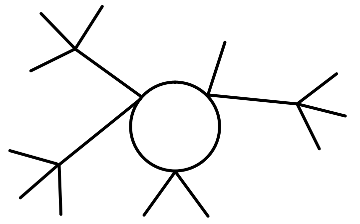


Fig. 7.8 One-loop connected contribution in the ϕ^4 field theory

We again illustrate the result with the example of the ϕ^4 interaction, where

$$S^{(2)}(x_1, x_2; \phi) = K(x_1 - x_2) + \tfrac{1}{2} g\phi^2(x_1)\delta^{(d)}(x_1 - x_2).$$

Then,

$$\operatorname{tr}\ln S^{(2)}(x_1, x_2; \phi_c) - \operatorname{tr}\ln S^{(2)}(x_1, x_2; 0) = \operatorname{tr}\ln\left[\delta^{(d)}(x_1 - x_2) + \tfrac{1}{2} g\Delta(x_1 - x_2)\phi_c^2(x_2)\right].$$

The expansion of $\mathcal{W}_1(J)$ in powers of ϕ_c takes the form

$$\begin{aligned}\mathcal{W}_1(J) = -\frac{1}{2}\bigg[&\frac{g}{2}\int \mathrm{d}^d x_1\, \Delta(x_1 - x_1)\phi_c^2(x_1)\\ &- \frac{g^2}{8}\int \mathrm{d}^d x_1\, \mathrm{d}^d x_2\, \Delta(x_1 - x_2)\phi_c^2(x_2)\Delta(x_2 - x_1)\phi_c^2(x_1) + \cdots\bigg].\end{aligned}$$

Note that the trace operation has generated a set of *one-loop Feynman diagrams*.

To recover perturbation theory, one still has to expand $\phi_c(J)$ in powers of J. A typical contribution to $\mathcal{W}_1(J)$ has the representation displayed in Fig. 7.8.

At order $\hbar$, the generating functional $\Gamma(\varphi)$ can be obtained from the relation

$$\frac{\partial \Gamma}{\partial \hbar} + \frac{\partial \mathcal{W}}{\partial \hbar} = 0\,,$$

for $\hbar \to 0$. Thus,

$$\Gamma(\varphi) = \mathcal{S}(\varphi) - \hbar\mathcal{W}_1(J) = \mathcal{S}(\varphi) + \tfrac{1}{2}\hbar\left[\operatorname{tr}\ln S^{(2)}(x_1, x_2; \phi_c) - \operatorname{tr}\ln S^{(2)}(x_1, x_2; 0)\right]. \quad (7.91)$$

At this order, we can replace $\phi_c(J)$ by φ. Setting

$$\Gamma(\varphi) = \mathcal{S}(\varphi) + \hbar\Gamma_1(\varphi) + O\left(\hbar^2\right), \quad (7.92)$$

we obtain the order $\hbar$ correction,

$$\Gamma_1(\varphi) = \tfrac{1}{2}\operatorname{tr}\left[\ln S^{(2)}(x_1, x_2; \varphi) - \ln S^{(2)}(x_1, x_2; 0)\right]. \quad (7.93)$$

If $\mathcal{S}(\phi)$ has the decomposition (7.7), and $\mathcal{V}_{\mathrm{I}}(\phi)$ the local form,

$$\mathcal{V}_{\mathrm{I}}(\phi) = \int \mathrm{d}^d x\, V_{\mathrm{I}}\big(\phi(x)\big),$$

the expansion of $\Gamma_1(\varphi)$ in powers of V_{I} takes the form,

$$\begin{aligned}\Gamma_1(\varphi) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}&\int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_n V_{\mathrm{I}}''(\varphi(x_1))\Delta(x_1 - x_2)\\ &\times V_{\mathrm{I}}''(\varphi(x_2))\Delta(x_2 - x_3)\cdots V_{\mathrm{I}}''(\varphi(x_n))\Delta(x_n - x_1).\end{aligned} \quad (7.94)$$

In terms of Feynman diagrams, it is a sum of one-loop diagrams (see Fig. 7.9).

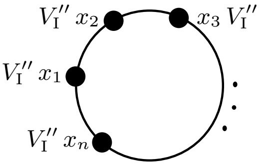


Fig. 7.9 Vertex functions: One-loop diagrams

The remarkable property of the Feynman graph expansion of the functional $\Gamma_1(\varphi)$ at this order is that all diagrams are one-line irreducible (1PI in the terminology of particle physics): they cannot be disconnected by cutting only one line. The functional $\Gamma_1(\varphi)$ is *the generating functional of vertex functions, which sum the one-line or 1PI one-loop Feynman diagrams.*

7.9.3 Loop expansion at higher orders

We now expand the field integral

$$\mathcal{Z}(J) = \int [\mathrm{d}\phi] \exp\left[-\frac{1}{\hbar}\big(\mathcal{S}(\phi) - J\cdot\phi\big)\right], \tag{7.95}$$

with $\mathcal{S}(\phi)$ of the form (7.7), to all orders in perturbation theory. The propagator is $\hbar\Delta$. Any vertex generated by $\mathcal{V}_{\mathrm{I}}(\phi)$ is multiplied by $1/\hbar$. In the same way, at the end of all external lines is attached a factor J which also yields a factor $1/\hbar$. Denoting by I the number of internal lines of a Feynman diagram (propagators which join two vertices), by E the number of external lines (propagator joining a vertex to a source J), by V the number of vertices, we find that the power of $\hbar$ that multiplies a *connected* diagram (a contribution to $\mathcal{W}(J)$) is

$$\hbar^{I+E-(V+E)+1} = \hbar^{I-V+1},$$

the last factor $\hbar$ coming from our normalization of $\mathcal{W}(J)$ (equation (7.81)).

We prove in the next section that the same result is obtained for 1PI Feynman diagrams (*i.e.* contributions to $\Gamma(\varphi)$), because the factor $\hbar$ coming from the source cancels the factor coming from the external propagator.

We now show that the power of $\hbar$ that we have found counts the number of loops of a diagram [43]. The number of loops is defined in the following way: if, by cutting a line of a connected diagram γ we obtain a new connected diagram γ', then,

$$\#\ \text{loops}\ \gamma = \#\ \text{loops}\ \gamma' + 1\,.$$

From this definition follows a relation between the number of loops L, the number of internal lines I and the number of vertices V:

$$L = I - V + 1\,. \tag{7.96}$$

Indeed, we note that each time we can remove an internal line without disconnecting the diagram, we decrease I by 1 and L by 1. Eventually, we obtain a tree diagram, that is, a diagram in which no internal line can be cut without disconnecting the diagram. We then have to show that

$$I - V + 1 = 0\,.$$

From a tree diagram, we can remove systematically a vertex at the boundary with the line connecting it to the diagram until we obtain the simplest diagram, composed of a line joining two vertices, which satisfies the equation.

We have thus shown that the expansion in powers of $\hbar$ reorganizes perturbation theory according to the number of loops of Feynman diagrams.

The number of loops is also the number of independent internal intensities in the corresponding electric circuit, the current being conserved at each vertex, the intensities flowing into the diagram being fixed.

Indeed, the number L of independent intensities is equal to the total number of intensities I minus the number of conservation equations $(V-1)$ (because one equation gives the total conservation of the current) and thus equation (7.96) is again satisfied. We will eventually use this remark to relate the number of loops to the number of independent momentum integration variables.

Higher order calculations. We now indicate how successive terms in the loop expansion can be calculated by applying the steepest descent method to the field integral (7.95).

The saddle point $\phi_c(J)$ is the solution of the equation

$$\frac{\delta \mathcal{S}}{\delta\phi(x)}(\phi_c) = J(x). \tag{7.97}$$

We change variables in the field integral (7.78), $\phi \mapsto \chi$, setting

$$\phi(x) = \phi_c(x) + \sqrt{\hbar}\chi(x). \tag{7.98}$$

We expand $\mathcal{S}(\phi)$ in powers of χ,

$$\mathcal{S}(\phi_c + \chi) = \mathcal{S}(\phi_c) + \sum_{n=2}^{\infty} \frac{\hbar^{n/2}}{n!} \int \left(\prod_{i=1}^{n} \mathrm{d}^d x_i\, \chi(x_i)\right) S^{(n)}(x_1, x_2, \ldots, x_n; \phi_c). \tag{7.99}$$

where we have introduced the notation

$$\left.\frac{\delta^n \mathcal{S}(\phi)}{\delta\phi(x_1)\cdots\delta\phi(x_n)}\right|_{\phi=\phi_c} = S^{(n)}(x_1, \ldots, x_n; \phi_c). \tag{7.100}$$

The expansion in powers of $\hbar$ and the integration over χ generate vacuum Feynman diagrams with a ϕ_c-dependent propagator Δ solution of,

$$\int S^{(2)}(x, y; \phi_c)\Delta(y, z; \phi_c)\mathrm{d}^d y = \delta^{(d)}(x - z), \tag{7.101}$$

and ϕ_c-dependent vertices $S^{(n)}$. At a finite order in $\hbar$, only a finite number of vertices contribute.

In this expansion, $\mathcal{W}(J)$ is the sum of all connected vacuum diagrams. To calculate $\Gamma(\varphi)$ we have to express ϕ_c and J in terms of φ. At leading order, $\phi_c = \varphi + O(\hbar)$. Since the Legendre transformation removes only the 1PI diagrams, we conclude that $\Gamma(\varphi) - \mathcal{S}(\varphi)$ is given by the sum of the 1PI vacuum diagrams, in which ϕ_c has been replaced by φ. We verify this property at the two-loop order in Section A7.1.

7.10 Vertex functions: One-line irreducibility

We have shown that the two first orders of the loop expansion of $\Gamma(\varphi)$ contain only 1PI Feynman diagrams. We now suspect that the coefficient of $\hbar^L$ is the generating functional of all L-loop 1PI diagrams. By realizing that vertex functions and connected correlation functions are affected by global combinatorial factors identical, respectively, to those of vertices in the action and connected tree diagrams, it is possible to give a general proof that $\Gamma(\varphi)$ is the generating functional of 1PI Feynman diagrams: one takes $\Gamma(\phi)$ as the action, one calculates at leading order in $\hbar$, and recovers $\mathcal{W}(J)$ as the generating functional of the corresponding connected tree diagrams. The result then follows from the considerations of the beginning of the section (equation (7.82)) (for a discussion, see Ref. [44]). However, we give here a more powerful and completely algebraic proof.

To prove that vertex functions are given in perturbation theory by a sum of 1PI Feynman diagrams, we directly use the definition and prove that by cutting one line in all possible ways in all diagrams contributing to $\Gamma(\varphi)$, the diagrams remain connected.

We consider the modified action,

$$\mathcal{S}_\varepsilon(\phi) = \mathcal{S}(\phi) + \frac{\varepsilon}{2}\left(\int \mathrm{d}^d x\, \phi(x)\right)^2 = \frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, \phi(x)\phi(y)\,[K(x-y)+\varepsilon] + \mathcal{V}_\mathrm{I}(\phi), \tag{7.102}$$

where we have introduced a parameter ε in which we will expand at first order. The corresponding propagator $\Delta_\varepsilon(x-y)$ given by

$$\int \Delta_\varepsilon(x-z)\,[K(z-y)+\varepsilon]\,\mathrm{d}^d z = \delta^{(d)}(x-y),$$

can be written as

$$\Delta_\varepsilon(x,y) = \Delta(x-y) - \varepsilon\eta(x)\eta(y) + O\left(\varepsilon^2\right), \tag{7.103}$$

with the definition

$$\eta(x) = \int \Delta(x-z)\mathrm{d}^d z\,.$$

If we now expand in ε a Feynman diagram with the new propagator $\Delta_\varepsilon(x-y)$, we obtain, at first order, a sum of terms which consist of all possible ways in which a propagator $\Delta(x-y)$ has been replaced by the product $-\eta(x)\eta(y)$. Since, in this product, the dependence in x and y is factorized, this means topologically that, in the Feynman diagram, the corresponding line has been cut. A necessary and sufficient condition for a diagram to be 1PI is that all terms at order ε are connected.

Higher orders in ε make it possible to study irreducibility with respect to cutting two, three, or n lines.

The partition function $\mathcal{Z}_\varepsilon(J)$, at first order in ε, is given by

$$\mathcal{Z}_\varepsilon(J) = \int [\mathrm{d}\phi]\left(1 - \frac{\varepsilon}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, \phi(x)\phi(y)\right)\exp\left[-\mathcal{S}(\phi) + J\cdot\phi\right] + O\left(\varepsilon^2\right) \tag{7.104}$$

and, therefore,

$$\mathcal{Z}_\varepsilon(J) = \left[1 - \frac{\varepsilon}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, \frac{\delta^2}{\delta J(x)\delta J(y)} + O\left(\varepsilon^2\right)\right]\mathcal{Z}(J). \tag{7.105}$$

One infers for the generating functional $\mathcal{W}_\varepsilon(J) = \ln \mathcal{Z}_\varepsilon(J)$,

$$\mathcal{W}_\varepsilon(J) = \mathcal{W}(J) - \frac{\varepsilon}{2}\left\{\left[\int \mathrm{d}^d x \, \frac{\delta \mathcal{W}}{\delta J(x)}\right]^2 + \int \mathrm{d}^d x \, \mathrm{d}^d y \, \frac{\delta^2 \mathcal{W}}{\delta J(y) \delta J(x)}\right\} + O\left(\varepsilon^2\right). \quad (7.106)$$

It contains at order ε a contribution of the form

$$\left[\int \mathrm{d}^d x \frac{\delta \mathcal{W}}{\delta J(x)}\right]^2,$$

which is disconnected, as expected.

In the Legendre transformation, we use the relation (7.65) in the form

$$\frac{\partial \Gamma}{\partial \varepsilon} = -\frac{\partial \mathcal{W}}{\partial \varepsilon},$$

for $\varepsilon = 0$. Therefore,

$$\Gamma_\varepsilon(\varphi) = \Gamma(\varphi) + \frac{\varepsilon}{2}\left\{\left[\int \mathrm{d}^d x \, \varphi(x)\right]^2 + \int \mathrm{d}^d x \, \mathrm{d}^d y \left[\frac{\delta^2 \Gamma}{\delta \varphi(x) \delta \varphi(y)}\right]^{-1}\right\} + O\left(\varepsilon^2\right). \quad (7.107)$$

The first order in ε contains two terms, the term $\frac{1}{2}\varepsilon\left[\int \mathrm{d}^d x \, \varphi(x)\right]^2$, which we have added explicitly to the action, and a second term which is the connected two-point function in the presence of an external field. The disconnected terms have been removed by the Legendre transformation. Since the variation of $\Gamma(\varphi)$ is connected, $\Gamma(\varphi)$ is indeed one line or, in the terminology of particle physics, one particle irreducible. In the chapters that follow, we use the term of vertex functions but recall occasionally that the functions are 1PI. Finally, note that in Section 7.7 we have shown that

$$W^{(n)}_{\text{amp.}} = -\Gamma^{(n)} + \text{ reducible terms} \quad (n > 2),$$

because the difference contains only lower correlation functions related by propagators. The diagrams contributing to $\Gamma^{(n)}$ thus differ from the 1PI amputated diagrams of $W^{(n)}$ only by a sign. In the same way, the diagrams contributing to $\Gamma^{(2)}$ beyond tree level are, up to a change of sign, the amputated diagrams of the mass operator.

7.11 Statistical and quantum interpretation of the vertex functional

The functional $\mathcal{Z}(J)$ can be considered as the classical partition function in an external field (or source) $J(x)$, and then $\mathcal{W}(J)$ is proportional to the free energy. We now provide a statistical interpretation to $\Gamma(\varphi)$.

7.11.1 Interpretation and variational principle

Statistical physics interpretation. We consider the free energy $\mathcal{W}(J)$ given by

$$\mathrm{e}^{\mathcal{W}(J)} = \int [\mathrm{d}\phi] \, \mathrm{e}^{-\mathcal{S}(\phi) + J \cdot \phi}.$$

We introduce a field $\varphi(x)$ and constrain the source $J(x)$ to satisfy

$$\varphi(x) = \frac{\delta \mathcal{W}(J)}{\delta J(x)} = \langle \phi(x) \rangle_J ,$$

where, by $\langle \bullet \rangle_J$, we denote the expectation value with the weight $\mathrm{e}^{-\mathcal{S}(\phi) + J \cdot \phi}$.

The exponential function being convex, we infer the inequality,

$$\ln \left\langle \mathrm{e}^{-J \cdot \phi} \right\rangle_J = \mathcal{W}(0) - \mathcal{W}(J) \geq \langle -J \cdot \phi \rangle_J = -J \cdot \varphi ,$$

or introducing the Legendre transform $\Gamma(\varphi)$ of $\mathcal{W}(J)$,

$$\mathcal{W}(0) \geq -\Gamma(\varphi). \quad (7.108)$$

The inequality is the starting point of a variational principle. Moreover, $\mathcal{W}(0)$ is obtained in the limit $J = 0$ by taking the solution of

$$\frac{\delta\Gamma(\varphi)}{\delta\varphi(x)} = 0\,, \tag{7.109}$$

which minimizes $\Gamma(\varphi)$. In the framework of statistical physics, the Legendre transform $\Gamma(\varphi)$ is the *thermodynamic potential*, a quantity which plays a central role in the discussion of critical phenomena.

Finally, note that we have uncovered a general property of the Legendre transformation and, therefore, if we introduce a source J for any function of $\phi(x)$, a similar arguments will apply.

Quantum interpretation [45]. We now distinguish in the d-dimensional space $\mathbb{R}^d$ Euclidean time direction and denote by t, x the time and space arguments, respectively. We explicitly assume time translation invariance: the quantum Hamiltonian $\hat{H}(\phi)$ corresponding to the action $\mathcal{S}(\phi)$ is time independent. We assume that t varies in a finite interval $[0, \beta]$ and impose periodic boundary conditions on the fields in the time direction. Moreover, we restrict the analysis to time-independent sources

$$J(t, x) = J(x),$$

and, therefore, also to functionals $\Gamma(\varphi)$ where the field φ is time independent, $\varphi(t, x) = \varphi(x)$. As a consequence, the functional $\mathcal{Z}(J)$ becomes, from the point of view of quantum statistical physics, the partition function at temperature $T = 1/\beta$:

$$\mathcal{Z} = \operatorname{tr} \mathrm{e}^{-\beta\hat{H}}\,.$$

The inequality (7.108) becomes

$$\ln \operatorname{tr} \mathrm{e}^{-\beta\hat{H}} \geq -\Gamma(\varphi), \tag{7.110}$$

where φ is the thermal expectation value corresponding to the quantum Hamiltonian

$$\hat{H}(\phi, J) = \hat{H}(\phi) - \int \mathrm{d}^{d-1}x\, J(x)\hat{\phi}(x), \tag{7.111}$$

$\hat{\phi}(x)$ being the field operator.

In the large β limit, the partition function is dominated by the ground state energy E_0 of $\hat{H}$, and, thus,

$$\mathcal{W}(0) \underset{\beta\to\infty}{\sim} -\beta E_0\,. \tag{7.112}$$

In this limit, the inequality (7.110) becomes

$$E_0 \leq \frac{1}{\beta}\Gamma(\varphi)\,, \tag{7.113}$$

where again E_0 is obtained by looking for the solution of equation (7.109) that minimizes $\Gamma(\varphi)$.

7.11.2 Vertex functional and free energy at fixed field time average

We now present a related, but slightly different, interpretation. We calculate the partition function with the same periodic boundary condition in time but restricted to fields satisfying the constraint

$$\frac{1}{\beta}\int_0^\beta \mathrm{d}t\, \phi(t,x) = \varphi(x), \tag{7.114}$$

Time translation invariance then implies $\varphi(x) = \langle \phi(t,x) \rangle$. We denote by $-\beta\mathcal{G}(\varphi)$ the corresponding free energy

$$\mathrm{e}^{-\beta\mathcal{G}(\varphi)} = \int [\mathrm{d}\phi(t,x)] \exp[-\mathcal{S}(\phi)]. \tag{7.115}$$

We have written the free energy in the form $-\beta\mathcal{G}(\varphi)$, because we anticipate that, in the large β limit, the free energy is proportional to β.

Then, the free energy corresponding to the sum over all field configurations in the presence of a time-independent source $J(x)$ is given by

$$\mathrm{e}^{\mathcal{W}(J)} = \int [\mathrm{d}\varphi(x)] \exp\left[-\beta\mathcal{G}(\varphi) + \beta \int \mathrm{d}^d x\, J(x)\varphi(x)\right]. \tag{7.116}$$

In the $\beta \to \infty$ limit, the field integral can be calculated by the steepest descent method. The saddle point equation is

$$J(x) = \frac{\delta \mathcal{G}}{\delta \varphi(x)}\,. \tag{7.117}$$

When the equation has several solutions, one has to take the solution that corresponds to a local maximum and yields the largest contribution to the free energy. Then,

$$\mathcal{W}(J) \sim -\beta\mathcal{G}(\varphi) + \beta \int \mathrm{d}^d x\, J(x)\varphi(x). \tag{7.118}$$

After Legendre transformation, one finds

$$\beta\mathcal{G}(\varphi) = \Gamma(\varphi),$$

where again $\Gamma(\varphi)$ is the vertex functional restricted to time-independent fields. However, note that $\mathcal{G}(\varphi)$ has in general no reasons to be convex. One may find field configurations such that the operator

$$\frac{\delta^2 \mathcal{G}(\varphi)}{\delta\varphi(x)\delta\varphi(y)}$$

is not positive. On the other hand, because $\Gamma(\varphi)$ is the result of a steepest descent calculation, it may coincide with $\beta\mathcal{G}(\varphi)$ only in regions of field space where the operator is positive. In general, in perturbation theory, one calculates a quantity which, restricted to time-independent fields, coincides with $\mathcal{G}$ rather than Γ. This explains an apparent paradox: in the several phase region, one often pretends to be discussing the minima of $\Gamma(\varphi)$, that is, the minima of a quantity which has convexity properties and can have only one minimum. In fact, one discusses the properties of $\mathcal{G}$.

A7 Additional results and methods

A7.1 Generating functional at two loops

We illustrate the discussion of Section 7.9.3 by an explicit two-loop calculation. Expanding the interaction terms $S^{(3)}$ and $S^{(4)}$ (equation (7.100)), integrating term by term over χ (equation (7.98)) and choosing a simple normalization of the field integral, we expand $\mathcal{W}(J) = \hbar \ln \mathcal{Z}(J)$ as

$$\mathcal{W}(J) = -\mathcal{S}(\phi_c) + J\cdot\phi_c - \tfrac{1}{2}\hbar \operatorname{tr}\ln S^{(2)}(x_1,x_2;\phi_c) + \hbar^2 \mathcal{W}_2(J) + O\left(\hbar^3\right), \qquad (A7.1)$$

where the two-loop contribution is given by

$$\begin{aligned}\mathcal{W}_2(J) = &-\frac{1}{8}\int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_4\, S^{(4)}(x_1,x_2,x_3,x_4)\Delta(x_1,x_2;\phi_c)\Delta(x_3,x_4;\phi_c)\\ &+\int \mathrm{d}^d x_1 \cdots \mathrm{d}^d y_3\, S^{(3)}(x_1,x_2,x_3;\phi_c) S^{(3)}(y_1,y_2,y_3;\phi_c)\left[\frac{1}{8}\Delta(x_1,x_2;\phi_c)\right.\\ &\left.\times\Delta(y_1,y_2;\phi_c)\Delta(x_3,y_3;\phi_c) + \frac{1}{12}\Delta(x_1,y_1;\phi_c)\Delta(x_2,y_2;\phi_c)\Delta(x_3,y_3;\phi_c)\right]. \qquad (A7.2)\end{aligned}$$

We now perform the Legendre transformation (7.62). We need $\varphi(x)$ only up to order $\hbar$,

$$\varphi(x) = \phi_c(J;x) - \frac{\hbar}{2}\frac{\delta}{\delta J(x)} \operatorname{tr}\ln S^{(2)}(x_1,x_2;\phi_c) + O\left(\hbar^2\right), \qquad (A7.3)$$

because expression $\Gamma(\varphi) - J\cdot\varphi$ is stationary in φ. Using $\delta \operatorname{tr}\ln X = \operatorname{tr}\delta X X^{-1}$, valid for any matrix or operator X, and applying the chain rule, we rewrite the order $\hbar$ correction as

$$\frac{\delta}{\delta J(x)} \operatorname{tr}\ln S^{(2)}(x_1,x_2;\phi_c) = \int \mathrm{d}^d y\, \mathrm{d}^d z_1\, \mathrm{d}^d z_2 \frac{\delta\phi_c(y)}{\delta J(x)} S^{(3)}(y,z_1,z_2;\phi_c)\Delta(z_2,z_1,\phi_c).$$

Using equations (7.97), (A7.3), and the definition (7.101), we express ϕ_c in terms of φ. At order $\hbar$,

$$\varphi(x) = \phi_c(J,x) - \frac{\hbar}{2}\int \mathrm{d}^d y\, \mathrm{d}^d y_1\, \mathrm{d}^d y_2\, S^{(3)}(y,y_1,y_2;\varphi)\Delta(x,y;\varphi)\Delta(y_1,y_2;\varphi). \qquad (A7.4)$$

We still need $J(x)$ at order $\hbar$. Using equations (7.97) and (A7.4), we find that

$$J(x) = \frac{\delta\mathcal{S}(\varphi)}{\delta\phi(x)} + \frac{\hbar}{2}\int \mathrm{d}^d y_1\, \mathrm{d}^d y_2\, S^{(3)}(x,y_1,y_2;\varphi)\Delta(y_1,y_2;\varphi) + O\left(\hbar^2\right). \qquad (A7.5)$$

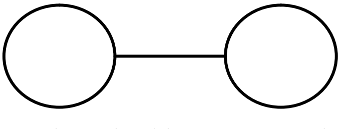


Fig. 7.10 The reducible part at two-loop order

Equation (7.62) then yields $\Gamma(\varphi)$ at two-loop order. As expected the reducible part in expression (A7.2) (Fig. 7.10) cancels, and we obtain

$$\Gamma(\varphi) = \mathcal{S}(\varphi) + \tfrac{1}{2}\hbar \operatorname{tr}\ln S^{(2)}(x_1, x_2; \varphi) + \hbar^2 \Gamma_2(\varphi) + O\left(\hbar^3\right), \tag{A7.6}$$

where the two-loop contribution is

$$\begin{aligned}\Gamma_2(\varphi) = \frac{1}{8}\int & \mathrm{d}^d x_1\, \mathrm{d}^d x_2\, \mathrm{d}^d x_3\, \mathrm{d}^d x_4\, \Delta(x_1 - x_2; \varphi) S^{(4)}(x_1, x_2, x_3, x_4; \varphi)\Delta(x_3 - x_4; \varphi) \\ & - \frac{1}{12}\int \mathrm{d}^d x_1\, \mathrm{d}^d x_2\, \mathrm{d}^d x_3\, \mathrm{d}^d y_1\, \mathrm{d}^d y_2\, \mathrm{d}^d y_3\, S^{(3)}(x_1, x_2, x_3; \varphi)\Delta(x_1 - y_1; \varphi)\Delta(x_2 - y_2; \varphi) \\ & \times \Delta(x_3 - y_3; \varphi) S^{(3)}(y_1, y_2, y_3; \varphi). \end{aligned} \tag{A7.7}$$

Fig. 7.11 gives a diagrammatic representation of the two-loop contributions.

Fig. 7.11 The two-loop contributions to $\Gamma(\varphi)$

A7.2 The background field method

In most of our work, we regard correlation functions of the field as fundamental physical objects. However, in some cases all or some local functionals of the field, which have a non-trivial linear part in ϕ, are equivalent. Important examples are:

(i) Some models are defined on Riemannian manifolds and fields correspond to a particular choice of coordinates on the manifold. For some issues, only quantities intrinsic to the manifold are physical.

(ii) In gauge theories, only gauge independent quantities are physical. Change of gauges correspond to field redefinitions.

(iii) In particle physics, normalized S-matrix elements are invariant under a change of field variables (see Section 6.5.4).

Here, we introduce a method, the background field method [46], which has among its main merits, the power to make a more direct calculation possible of quantities that are rather insensitive to a change of field variables.

The generating functional of vertex functions $\Gamma(\varphi)$, corresponding to an action $\mathcal{S}(\phi)$, can be derived from

$$\mathrm{e}^{-\Gamma(\varphi)+J\cdot\varphi} = \int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{S}(\phi)+J\cdot\phi}, \tag{A7.8}$$

or, using

$$J(x) = \frac{\delta\Gamma}{\delta\varphi(x)},$$

from

$$\mathrm{e}^{-\Gamma(\varphi)} = \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}(\phi) + \int \mathrm{d}^d x\, (\phi(x) - \varphi(x))\, \frac{\delta\Gamma}{\delta\varphi(x)}\right], \tag{A7.9}$$

or, equivalently, translating $\phi(x)$, $\phi \mapsto \varphi + \chi$:

$$\mathrm{e}^{-\Gamma(\varphi)} = \int [\mathrm{d}\chi] \exp\left[-\mathcal{S}(\chi + \varphi) + \int \mathrm{d}^d x\, \chi(x) \frac{\delta\Gamma}{\delta\varphi(x)}\right]. \tag{A7.10}$$

We now assume that the equation

$$\frac{\delta\Gamma}{\delta\varphi(x)} = 0 \tag{A7.11}$$

has a non-trivial solution $\varphi_c(x)$, which, at leading order in perturbation theory, is a solution $\varphi_c^{(0)}(x)$ of the classical equation of motion:

$$\frac{\delta\mathcal{S}}{\delta\varphi(x)}\left[\varphi_c^{(0)}\right] = 0\,. \tag{A7.12}$$

Then equation ($A7.10$) becomes (see also the discussion at the end of Section 7.9.3)

$$\mathrm{e}^{-\Gamma(\varphi_c)} = \int [\mathrm{d}\chi]\, \mathrm{e}^{-\mathcal{S}(\chi+\varphi_c)}\,. \tag{A7.13}$$

The quantity $\Gamma(\varphi_c)$ is clearly independent of the representation of the field ϕ, and contains, therefore, only physical information in the sense defined at the beginning of the section. We introduce Γ_r, the renormalized generating functional of vertex functions (see Chapter 9), given by

$$\Gamma_\mathrm{r}(\varphi_c) = -\ln \int [\mathrm{d}\chi] \exp\left[-\mathcal{S}_0(\chi+\varphi_c) + \text{counter-terms}\right],$$

in which $\mathcal{S}_0(\phi)$ is the tree order action. The solution φ_c of

$$\frac{\delta\Gamma_\mathrm{r}}{\delta\varphi_c(x)} = 0$$

is expanded around the solution of $\varphi_c^{(0)}(x)$ of

$$\frac{\delta\mathcal{S}_0}{\delta\varphi_c(x)} = 0\,.$$

It can be inferred from the discussion of Section 6.5.3 that, in real time, the background field method yields the S-matrix. We shall provide other examples of calculations involving the background field method in the coming chapters.

A7.3 Connected Feynman diagrams: Cluster properties

We briefly describe the cluster properties of connected Feynman diagrams contributing to Euclidean correlation functions, in a massive field theory. We restrict ourselves to a theory with one massive scalar field, but the generalization is straightforward. We then discuss the influence of threshold effects for real-time diagrams at large time separation.

A7.3.1 Decay of connected Feynman diagrams in Euclidean space

The propagator can be written (equation (7.14)) as

$$\Delta(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d p \, \frac{\mathrm{e}^{ip\cdot x}}{p^2 + m^2} \,. \tag{A7.14}$$

We assume in the following that m is the physical mass, and this requires a mass renormalization of Feynman diagrams (see Chapter 9).

To determine the behaviour of Δ for $|x|$ large, we rewrite the integral as

$$\Delta(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d p \int_0^{+\infty} \mathrm{d}t \, \mathrm{e}^{ip\cdot x} \, \mathrm{e}^{-t(p^2+m^2)} \,. \tag{A7.15}$$

We then perform the Gaussian integration over the momentum p:

$$\Delta(x) = \frac{\pi^{d/2}}{(2\pi)^d} \int \frac{\mathrm{d}t}{t^{d/2}} \exp\left(-tm^2 - \frac{1}{4t} x^2\right). \tag{A7.16}$$

The behaviour of Δ for large separation is given by the method of steepest descent. The saddle point is

$$t = \frac{|x|}{2m} \,. \tag{A7.17}$$

Finally, the Gaussian integration over fluctuations around the saddle point yields

$$\Delta(x) \sim \frac{1}{2m} \left(\frac{m}{2\pi\,|x|}\right)^{(d-1)/2} \mathrm{e}^{-m|x|} \,. \tag{A7.18}$$

The large distance decay of the propagator is governed by the inverse of the particle mass m (for $\hbar = c = 1$), which, in the theory of phase transitions, is also the *correlation length*. This asymptotic estimate remains valid even when the propagator is replaced by a momentum-regularized propagator, as one can verify more easily using the regularization scheme of Section 8.4.3.

Using this asymptotic estimate, one can derive the following property: if in a connected diagram we separate two sets of points by a distance l, then at large l the diagram decreases as $\exp(-nml)$. In this expression, n is the smallest number of lines it is necessary to cut in order to disconnect the diagram, the two sets of points being attached to different connected components (see Fig. 7.12).

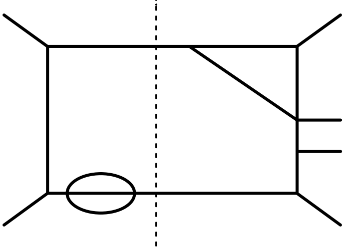


Fig. 7.12 Example with $n = 2$

By contrast, in a massless theory ($m = 0$) the decay is only algebraic when the propagator exists (this implies, in perturbation theory, $d > 2$).

A7.3.2 Threshold effects

The precise large distance behaviour of diagrams is related to the strength of the leading singularity in momentum space (or Fourier representation). For example, if, in momentum space, a contribution to the two-point function has the algebraic singularity,

$$\tilde{K}^{(2)}(p) \propto (m^2 + p^2)^{-\alpha},$$

a generalization of the preceding calculation yields the large distance behaviour

$$K^{(2)}(r) \propto r^{\alpha - d/2 - 1/2}\, \mathrm{e}^{-mr}\,.$$

If we now consider an 1PI diagram with n internal lines, it yields, for r large, the contribution,

$$K^{(2)}(r) \propto r^{-n(d-1)/2}\, \mathrm{e}^{-nmr}\,.$$

This, in turn, corresponds to a singularity

$$\tilde{K}^{(2)}(p) \propto (m^2 + p^2)^{-\alpha}, \quad \alpha = \tfrac{1}{2}[(n-1)d - n - 1].$$

In particular, for $d > 1$, the singularity softens when n increases. The two-particle threshold yields the strongest singularity $(p^2 + 4m^2)^{(d-3)/2}$. The nature of the singularity is important for the large-time behaviour of correlation functions in real time: the leading large-time behaviour is then related to the leading singularity in the energy variable (a property of the Fourier transformation). Therefore, if we consider the two-point function, its large-time behaviour is given by the one-particle pole, then the next to leading term is related to the two-particle threshold and so on.