alternating functions

$$\omega: TM|_x \times \cdots \times TM|_x \to \mathbb{R}.$$

Specifically, if we denote the evaluation of ω on the tangent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in TM|_x$ by $\langle \omega; \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle$, then the basic requirements are that for all tangent vectors at x,

$$\langle \omega; \mathbf{v}_1, \ldots, c\mathbf{v}_i + c'\mathbf{v}_i', \ldots, \mathbf{v}_k \rangle = c\langle \omega; \mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_k \rangle + c'\langle \omega; \mathbf{v}_1, \ldots, \mathbf{v}_i', \ldots, \mathbf{v}_k \rangle$$

for $c, c' \in \mathbb{R}$, $1 \leqslant i \leqslant k$, and

$$\langle \omega; \mathbf{v}_{\pi 1}, \ldots, \mathbf{v}_{\pi k} \rangle = (-1)^{\pi} \langle \omega; \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle$$

for every permutation π of the integers $\{1, \ldots, k\}$, with $(-1)^{\pi}$ denoting the sign of π. The space $\bigwedge_k T^*M|_x$ is, in fact, a vector space under the obvious operations of addition and scalar multiplication. A 0-form at x is, by convention, just a real number, while the space $T^*M|_x = \bigwedge_1 T^*M|_x$ of one-forms, called the *cotangent space* to M at x, is the space of linear functions on $TM|_x$, i.e. the dual vector space to the tangent space at x. A *smooth differential* k-*form* ω on M (or k-*form* for short) is a collection of smoothly varying alternating k-linear maps $\omega|_x \in \bigwedge_k T^*M|_x$ for each $x \in M$, where we require that for all smooth vector fields $\mathbf{v}_1, \ldots, \mathbf{v}_k$,

$$\langle \omega; \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle (x) \equiv \langle \omega|_x; \mathbf{v}_1|_x, \ldots, \mathbf{v}_k|_x \rangle$$

is a smooth, real-valued function of x. In particular, a 0-form is just a smooth real-valued function $f: M \to \mathbb{R}$.

If $(x^1, \ldots, x^m)$ are local coordinates, then $TM|_x$ has basis $\{\partial/\partial x^1, \ldots, \partial/\partial x^m\}$. The dual cotangent space has a dual basis, which is traditionally denoted $\{dx^1, \ldots, dx^m\}$; thus $\langle dx^i; \partial/\partial x^j \rangle = \delta^i_j$ for all i, j, where δ^i_j is 1 for $i = j$ and 0 otherwise. A differential one-form ω thereby has the local coordinate expression

$$\omega = h_1(x)\, dx^1 + \cdots + h_m(x)\, dx^m,$$

where each coefficient function $h_j(x)$ is smooth. Note that for any vector field $\mathbf{v} = \sum \xi^i(x) \partial/\partial x^i$,

$$\langle \omega; \mathbf{v} \rangle = \sum_{i=1}^{m} h_i(x) \xi^i(x)$$

is a smooth function. Of particular importance are the one-forms given by the differentials of real-valued functions:

$$df = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i} dx^i, \quad \text{with} \quad \langle df; \mathbf{v} \rangle = \mathbf{v}(f).$$

To proceed to higher order differential forms, we note that, given a collection of differential one-forms $\omega_1, \ldots, \omega_k$, we can form a differential k-form

$\omega_1 \wedge \cdots \wedge \omega_k$, called the *wedge product*, using the determinantal formula

$$\langle \omega_1 \wedge \cdots \wedge \omega_k; \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle = \det(\langle \omega_i; \mathbf{v}_j \rangle), \tag{1.49}$$

the right-hand side being the determinant of a $k \times k$ matrix with indicated (i, j) entry. Note that, by the usual properties of determinants, the wedge product itself is both multi-linear and alternating

$$\omega_1 \wedge \cdots \wedge (c\omega_i + c'\omega_i') \wedge \cdots \wedge \omega_k = c(\omega_1 \wedge \cdots \wedge \omega_i \wedge \cdots \wedge \omega_k) + c'(\omega_1 \wedge \cdots \wedge \omega_i' \wedge \cdots \wedge \omega_k),$$

$$\omega_{\pi 1} \wedge \cdots \wedge \omega_{\pi k} = (-1)^\pi \omega_1 \wedge \cdots \wedge \omega_k.$$

It is not hard to see that in local coordinates, $\bigwedge_k T^*M|_x$ is spanned by the basis k-forms

$$dx^I \equiv dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where I ranges over all strictly increasing multi-indices $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m$. Thus $\bigwedge_k T^*M|_x$ has dimension $\binom{m}{k}$; in particular, $\bigwedge_k T^*M|_x \simeq \{0\}$ if $k > m$. Any smooth differential k-form on M has the local coordinate expression

$$\omega = \sum_I \alpha_I(x)\, dx^I,$$

where, for each strictly increasing multi-index I, the coefficient α_I is a smooth, real-valued function. For example, a two-form in $\mathbb{R}^3$ takes the form

$$\omega = \alpha(x, y, z)\, dy \wedge dz + \beta(x, y, z)\, dz \wedge dx + \gamma(x, y, z)\, dx \wedge dy, \tag{1.50}$$

using the basis $dy \wedge dz$, $dz \wedge dx = -dx \wedge dz$, and $dx \wedge dy$, attuned to the notation for surface integrals. We have

$$\langle \omega; \xi\partial_x + \eta\partial_y + \zeta\partial_z, \hat{\xi}\partial_x + \hat{\eta}\partial_y + \hat{\zeta}\partial_z \rangle = \alpha(\eta\hat{\zeta} - \hat{\eta}\zeta) + \beta(\zeta\hat{\xi} - \hat{\zeta}\xi) + \gamma(\xi\hat{\eta} - \hat{\xi}\eta).$$

If

$$\omega = \omega_1 \wedge \cdots \wedge \omega_k, \qquad \theta = \theta_1 \wedge \cdots \wedge \theta_l,$$

are "decomposible" forms, their *wedge product* is the form

$$\omega \wedge \theta = \omega_1 \wedge \cdots \wedge \omega_k \wedge \theta_1 \wedge \cdots \wedge \theta_l,$$

with the definition extending bilinearly to more general types of forms:

$$(c\omega + c'\omega') \wedge \theta = c(\omega \wedge \theta) + c'(\omega' \wedge \theta),$$

$$\omega \wedge (c\theta + c'\theta') = c(\omega \wedge \theta) + c'(\omega \wedge \theta'),$$

for $c, c' \in \mathbb{R}$. This wedge product is associative:

$$\omega \wedge (\theta \wedge \zeta) = (\omega \wedge \theta) \wedge \zeta,$$

and *anti*-commutative,

$$\omega \wedge \theta = (-1)^{kl}\theta \wedge \omega$$

for ω a k-form and θ an l-form. For example, the wedge product of (1.50) with a one-form $\theta = \lambda\, dx + \mu\, dy + \nu\, dz$ is the three-form

$$\omega \wedge \theta = (\alpha\lambda + \beta\mu + \gamma\nu)\, dx \wedge dy \wedge dz.$$

Pull-Back and Change of Coordinates

If $F: M \to N$ is a smooth map between manifolds, its differential dF maps tangent vectors on M to tangent vectors on N. There is thus an induced linear map F^*, called the *pull-back* or *codifferential* of F, which takes differential k-forms on N back to differential k-forms on M,

$$F^*: \textstyle\bigwedge_k T^*N|_{F(x)} \to \bigwedge_k T^*M|_x.$$

It is defined so that if $\omega \in \bigwedge_k T^*N|_{F(x)}$,

$$\langle F^*(\omega); \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle = \langle \omega; dF(\mathbf{v}_1), \ldots, dF(\mathbf{v}_k) \rangle$$

for any set of tangent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in TM|_x$. In contrast to the differential, the pull-back *does* take smooth differential forms on N back to smooth differential forms on M. If $x = (x^1, \ldots, x^m)$ are local cordinates on M and $y = (y^1, \ldots, y^n)$ coordinates on N, then

$$F^*(dy^i) = \sum_{j=1}^{m} \frac{\partial y^i}{\partial x^j} \cdot dx^j, \quad \text{where} \quad y = F(x),$$

gives the action of F^* on the basis one-forms. We conclude that in general

$$F^*\left(\sum_I \alpha_I(y)\, dy^I \right) = \sum_{I,J} \alpha_I(F(x)) \frac{\partial y^I}{\partial x^J} dx^J, \tag{1.51}$$

where $\partial y^I/\partial x^J$ stands for the Jacobian determinant $\det(\partial y^{i_\kappa}/\partial x^{j_\nu})$ corresponding to the increasing multi-indices $I = (i_1, \ldots, i_k)$, $J = (j_1, \ldots, j_k)$. In particular, if $y = F(x)$ determines a change of coordinates on M, then (1.51) provides the corresponding change of coordinates for differential k-forms on M. Note also that the pull-back preserves the algebraic operation of wedge product:

$$F^*(\omega \wedge \theta) = F^*(\omega) \wedge F^*(\theta).$$

Interior Products

If ω is a differential k-form and $\mathbf{v}$ a smooth vector field, then we can form a $(k-1)$-form $\mathbf{v} \,\lrcorner\, \omega$, called the *interior product* of $\mathbf{v}$ with ω, defined so that

$$\langle \mathbf{v} \,\lrcorner\, \omega; \mathbf{v}_1, \ldots, \mathbf{v}_{k-1} \rangle = \langle \omega; \mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_{k-1} \rangle$$

for every set of vector fields $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$. The interior product is clearly bilinear in both its arguments, so it suffices to determine it for basis elements:

$$\frac{\partial}{\partial x^i} \lrcorner (dx^{j_1} \wedge \cdots \wedge dx^{j_k})$$

$$= \begin{cases} (-1)^{\kappa-1}\, dx^{j_1} \wedge \cdots \wedge dx^{j_{\kappa-1}} \wedge dx^{j_{\kappa+1}} \wedge \cdots \wedge dx^{j_k}, & i = j_\kappa, \\ 0, & i \neq j_\kappa \quad \text{for all } \kappa. \end{cases}$$

For example,

$$\partial_x \lrcorner\, dx \wedge dy = dy, \qquad \partial_x \lrcorner\, dz \wedge dx = -dz, \qquad \partial_x \lrcorner\, dy \wedge dz = 0,$$

so that if ω is as in (1.50),

$$(\xi\partial_x + \eta\partial_y + \zeta\partial_z) \lrcorner\, \omega = (\zeta\beta - \eta\gamma)\, dx + (\xi\gamma - \zeta\alpha)\, dy + (\eta\alpha - \xi\beta)\, dz.$$

Note that the interior product acts as an *anti-derivation* on forms, meaning that

$$\mathbf{v} \lrcorner (\omega \wedge \theta) = (\mathbf{v} \lrcorner\, \omega) \wedge \theta + (-1)^k \omega \wedge (\mathbf{v} \lrcorner\, \theta) \tag{1.52}$$

whenever ω is a k-form, θ an l-form.

The Differential

Besides the purely algebraic operations of wedge and interior products, there are two important differential operations. The first of these generalizes the concept of the differential of a smooth function (or 0-form) to an arbitrary differential k-form. In local coordinates, if $\omega = \sum \alpha_I(x)\, dx^I$ is a smooth differential k-form on a manifold M, its *differential* or *exterior derivative* is the $(k + 1)$-form

$$d\omega = \sum_I d\alpha_I \wedge dx^I = \sum_{I,j} \frac{\partial \alpha_I}{\partial x^j} dx^j \wedge dx^I. \tag{1.53}$$

Proposition 1.60. *The differential d, taking k-forms to $(k + 1)$-forms, has the following properties*:

(a) Linearity

$$d(c\omega + c'\omega') = c\, d\omega + c'\, d\omega' \quad \textit{for } c, c' \textit{ constant},$$

(b) Anti-derivation

$$d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^k \omega \wedge (d\theta), \quad \textit{for } \omega \textit{ a } k\textit{-form}, \theta \textit{ an } l\textit{-form}. \tag{1.54}$$

(c) Closure

$$d(d\omega) \equiv 0. \tag{1.55}$$

(d) Commutation with Pull-Back

$$F^*(d\omega) = d(F^*\omega), \tag{1.56}$$

for $F: M \to N$ smooth, ω a k-form on N.

The proofs of these properties are reasonably straightforward. Linearity is obvious and the anti-derivational property is an easy consequence of Leibniz' rule. To check closure, we need only prove $d(df) = 0$ for f a smooth function since we can then use properties (a) and (b) to extend this to the general case (1.53). However,

$$d(df) = \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x^i \, \partial x^j} dx^i \wedge dx^j = \sum_{i<j} \left(\frac{\partial^2 f}{\partial x^i \, \partial x^j} - \frac{\partial^2 f}{\partial x^j \, \partial x^i} \right) dx^i \wedge dx^j$$

by the alternating property of the wedge product, so closure just reduces to the equality of mixed partial derivatives. In fact, properties (a), (b), and (c) together with the action of d on functions serve to *uniquely* characterize the differential and so d is in fact independent of the choice of local coordinates. Finally, the proof of (1.56) needs only be done in the case of functions: $F^*(df) = dF^*(f)$, where $F^*(f) = f \circ F$, and then it reduces to the ordinary chain rule. □

If $M = \mathbb{R}^3$, then the differential of a one-form,

$$\begin{aligned} d(\lambda \, dx + \mu \, dy + \nu \, dz) = (\nu_y - \mu_z) \, dy \wedge dz + (\lambda_z - \nu_x) \, dz \wedge dx \\ + (\mu_x - \lambda_y) \, dx \wedge dy, \end{aligned}$$

can be identified with curl of its coefficients: $\nabla \times \boldsymbol{\lambda} \equiv \nabla \times (\lambda, \mu, \nu)$. Similarly, the differential of a two-form

$$d(\alpha \, dy \wedge dz + \beta \, dz \wedge dx + \gamma \, dx \wedge dy) = (\alpha_x + \beta_y + \gamma_z) \, dx \wedge dy \wedge dz$$

can be identified with the divergence $\nabla \cdot \boldsymbol{\alpha} \equiv \nabla \cdot (\alpha, \beta, \gamma)$. The closure property (1.55) of d therefore translates into the familiar vector calculus identities

$$\nabla \times (\nabla f) = 0, \qquad \nabla \cdot (\nabla \times \boldsymbol{\lambda}) = 0.$$

The reader may find it instructive to see which vector calculus identities are implied by the anti-derivational property (1.54) in this case.

The de Rham Complex

Given a manifold M, we let $\bigwedge_k = \bigwedge_k(M)$ denote the space of all smooth differential k-forms on M. The differential d, mapping k-forms to $(k + 1)$-forms, serves to define a "complex"

$$0 \to \mathbb{R} \to \textstyle\bigwedge_0 \xrightarrow{d} \bigwedge_1 \xrightarrow{d} \bigwedge_2 \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge_{m-1} \xrightarrow{d} \bigwedge_m \to 0$$

called the *de Rham complex* of M. In general, a *complex* is defined as a sequence of vector spaces, and linear maps between successive spaces, with the property that the composition of any pair of successive maps is identically 0. In the present case, this last requirement is a restatement of the closure property $d \circ d = 0$, (1.55), of the differential. The initial map $\mathbb{R} \to \bigwedge_0$ takes a constant $c \in \mathbb{R}$ to the constant function (0-form) $f(x) \equiv c$ on M. Note that $dc = 0$ for any constant c.

The definition of a complex requires that the kernel of one of the linear maps contains the image of the preceding map. The complex is *exact* if this containment is, in fact, equality. In the case of the de Rham complex, exactness means that a *closed* differential k-form ω, meaning that $d\omega \equiv 0$, is necessarily an *exact* differential k-form, meaning that there exists a $(k-1)$-form θ with $\omega = d\theta$. (For $k = 0$, it says that a smooth function f is closed, $df = 0$, if and only if it is constant.) Clearly, any exact form is closed, but the converse need not hold. A simple example is when $M = \mathbb{R}^2\backslash\{0\}$, on which $\omega = (x^2 + y^2)^{-1}(y\,dx - x\,dy)$ is easily seen to be closed, but is not the differential of any smooth, single-valued function defined on all of M. Thus the de Rham complex is *not* in general exact. Remarkably, the extent to which it fails to be exact measures purely topological information about the manifold M. This result, the celebrated de Rham theorem, lies beyond the scope of this book and we refer the interested reader to the books of Warner, [1], and Bott and Tu, [1], for a development of this beautiful theory.

On the local side, for special types of domains in Euclidean space $\mathbb{R}^m$, there is only trivial topology and we *do* have exactness of the de Rham complex. This result, known as the *Poincaré lemma*, will hold for *star-shaped* domains $M \subset \mathbb{R}^m$, where "star-shaped" means that whenever $x \in M$, so is the entire line segment joining x to the origin: $\{\lambda x : 0 \leqslant \lambda \leqslant 1\} \subset M$.

Theorem 1.61. *Let $M \subset \mathbb{R}^m$ be a star-shaped domain. Then the de Rham complex over M is exact.*

Example 1.62. For $M \subset \mathbb{R}^m$, any m-form ω is uniquely determined by a single smooth function f, with $\omega = f(x)\,dx^1 \wedge \cdots \wedge dx^m$ relative to the standard volume form. (This does depend on our choice of coordinates.) Similarly, an $(m-1)$-form ξ is determined by an m-tuple of smooth functions $p = (p_1, \ldots, p_m)$, so that

$$\xi = \sum_{j=1}^{m} (-1)^{j-1} p_j(x)\, dx^{\hat{j}},$$

where

$$dx^{\hat{j}} \equiv dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^m.$$

The differential $\omega = d\xi$ is then determined as the usual divergence of p:

$$f(x) = \operatorname{div} p(x) = \sum_{j=1}^{m} \partial p_j / \partial x^j.$$

Note that any m-form on $\mathbb{R}^m$ is always closed, $d\omega = 0$, as there are no nonzero $(m+1)$-forms. Exactness of the de Rham complex at the $\bigwedge_m$-stage, then, says that any function f defined on a star-shaped subdomain of $\mathbb{R}^m$ can always be written as a divergence: $f = \operatorname{div} p$ for some p. Similarly, an $(m-1)$-form η is determined by $m(m-1)/2$ functions $q_{jk}(x)$, j, $k = 1, \ldots, m$, with $q_{jk} = -q_{kj}$, so that

$$\eta = \sum_{\substack{j,k=1 \\ j<k}}^{m} (-1)^{j+k-1} q_{jk}(x)\, dx^{\widehat{jk}},$$

where

$$dx^{\hat{jk}} \equiv dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \cdots \wedge dx^m.$$

Note that $d\eta = \xi$ is equivalent to the condition that p be a "generalized curl" of q:

$$p_j(x) = \sum_{k=1}^{m} \partial q_{jk}/\partial x^k, \qquad j = 1, \ldots, m.$$

Exactness of the de Rham complex at this stage then says that any vector field $p(x)$, defined over a star-shaped domain in $\mathbb{R}^m$, which is divergence-free: $\operatorname{div} p \equiv 0$, is necessarily the generalized curl of some such q.

Lie Derivatives

Let $\mathbf{v}$ be a vector field on a manifold M. We are often interested in how certain geometric objects on M, such as functions, differential forms and other vector fields, vary under the flow $\exp(\varepsilon\mathbf{v})$ induced by $\mathbf{v}$. The *Lie derivative* of such an object will in effect tell us its infinitesimal change when acted on by the flow. (Our standard integration procedures will tell us how to reconstruct the variation under the flow from this infinitesimal version.) For instance, the behaviour of a function under the flow induced by a vector field $\mathbf{v}$ has already been established so $\mathbf{v}(f)$, cf. (1.17), will be the "Lie derivative" of the function f with respect to $\mathbf{v}$.

More generally, let σ be a differential form or vector field defined over M. Given a point $x \in M$, after "time" ε it has moved to $\exp(\varepsilon\mathbf{v})x$ and the goal is to compare the value of σ at $\exp(\varepsilon\mathbf{v})x$ with its original value at x. However, $\sigma|_{\exp(\varepsilon\mathbf{v})x}$ and $\sigma|_x$, as they stand are, strictly speaking, incomparable as they belong to different vector spaces, e.g. $TM|_{\exp(\varepsilon\mathbf{v})x}$ and $TM|_x$ in the case of a vector field. To effect any comparison, we need to "transport" $\sigma|_{\exp(\varepsilon\mathbf{v})x}$ back to x in some natural way, and then make our comparison. For vector fields, this natural transport is the inverse differential

$$\phi_\varepsilon^* \equiv d\exp(-\varepsilon\mathbf{v}) \colon TM|_{\exp(\varepsilon\mathbf{v})x} \to TM|_x,$$

whereas for differential forms we use the pull-back map

$$\phi_\varepsilon^* \equiv \exp(\varepsilon\mathbf{v})^* \colon \textstyle\bigwedge_k T^*M|_{\exp(\varepsilon\mathbf{v})x} \to \bigwedge_k T^*M|_x.$$

This allows us to make the general definition of a Lie derivative.

Definition 1.63. Let $\mathbf{v}$ be a vector field on M and σ a vector field or differential form defined on M. The *Lie derivative* of σ with respect to $\mathbf{v}$ is the object whose value at $x \in M$ is

$$\mathbf{v}(\sigma)|_x = \lim_{\varepsilon \to 0} \frac{\phi_\varepsilon^*(\sigma|_{\exp(\varepsilon\mathbf{v})x}) - \sigma|_x}{\varepsilon} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi_\varepsilon^*(\sigma|_{\exp(\varepsilon\mathbf{v})x}). \tag{1.57}$$

(Note that $\mathbf{v}(\sigma)$ is an object of the same type as σ.)

In the case that σ is a vector field, its Lie derivative is a by now familiar object—the Lie bracket!

Proposition 1.64. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be smooth vector fields on* M. *The Lie derivative of* $\mathbf{w}$ *with respect to* $\mathbf{v}$ *coincides with the Lie bracket of* $\mathbf{v}$ *and* $\mathbf{w}$:

$$\mathbf{v}(\mathbf{w}) = [\mathbf{v}, \mathbf{w}]. \tag{1.58}$$

PROOF. Let $(x^1, \ldots, x^m)$ be local coordinates, with $\mathbf{v} = \sum \xi^i(x)\partial/\partial x^i$, $\mathbf{w} = \sum \eta^i(x)\partial/\partial x^i$. Expanding in powers of ε, we see that

$$\mathbf{w}|_{\exp(\varepsilon\mathbf{v})x} = \sum_{i=1}^{m} [\eta^i(x) + \varepsilon\mathbf{v}(\eta^i) + O(\varepsilon^2)]\frac{\partial}{\partial x^i},$$

hence, using (1.23) and (1.19),

$$d\exp(-\varepsilon\mathbf{v})[\mathbf{w}|_{\exp(\varepsilon\mathbf{v})x}] = \sum_{i=1}^{m} \{\eta^i(x) + \varepsilon[\mathbf{v}(\eta^i) - \mathbf{w}(\xi^i)] + O(\varepsilon^2)\}\frac{\partial}{\partial x^i}.$$

Substituting into the definition (1.57), we deduce (1.58) from (1.28). □

Turning to differential forms, we find that the Lie derivative can be most easily reconstructed from its basic properties:

(a) *Linearity*

$$\mathbf{v}(c\omega + c'\omega') = c\mathbf{v}(\omega) + c'\mathbf{v}(\omega'), \qquad c, c' \text{ constant}, \tag{1.59}$$

(b) *Derivation*

$$\mathbf{v}(\omega \wedge \theta) = \mathbf{v}(\omega) \wedge \theta + \omega \wedge \mathbf{v}(\theta), \tag{1.60}$$

(c) *Commutation with the Differential*

$$\mathbf{v}(d\omega) = d\mathbf{v}(\omega). \tag{1.61}$$

The commutation property is proved using the analogous property of pullbacks (1.56). The derivational property is proved just like Leibniz' rule. In fact, a Leibniz-type argument extends to Lie derivatives of more general bilinear combinations of geometric objects. Thus we have the useful formula

$$\mathbf{v}(\mathbf{w} \lrcorner \omega) = [\mathbf{v}, \mathbf{w}] \lrcorner \omega + \mathbf{w} \lrcorner \mathbf{v}(\omega), \tag{1.62}$$

for vector fields $\mathbf{v}$ and $\mathbf{w}$ and ω a differential form. (See Exercise 1.35.)

In local coordinates, the Lie derivative of a differential form is determined as follows. If

$$\mathbf{v} = \sum_{i=1}^{m} \xi^i(x)\frac{\partial}{\partial x^i},$$

then

$$\mathbf{v}(dx^i) = d\mathbf{v}(x^i) = d\xi^i = \sum_{j=1}^{m} \frac{\partial \xi^i}{\partial x^j}dx^j.$$

Therefore, we have the general formula

$$\mathbf{v}\left(\sum_I \alpha_I(x)\, dx^I\right) = \sum_I \left\{\mathbf{v}(\alpha_I)\, dx^I + \sum_{\kappa=1}^{k} \alpha_I\, dx^{i_1} \wedge \cdots \wedge d\xi^{i_\kappa} \wedge \cdots \wedge dx^{i_k}\right\}. \tag{1.63}$$

For example, if $M = \mathbb{R}^2$ and

$$\mathbf{v} = \xi(x, y)\partial_x + \eta(x, y)\partial_y,$$

then the Lie derivative of a two-form is

$$\begin{aligned}\mathbf{v}(\gamma(x, y)\, dx \wedge dy) &= \mathbf{v}(\gamma)\, dx \wedge dy + \gamma\, d\xi \wedge dy + \gamma\, dx \wedge d\eta \\ &= \{\xi\gamma_x + \eta\gamma_y + \gamma\xi_x + \gamma\eta_y\}\, dx \wedge dy.\end{aligned}$$

For instance, the Lie derivative of $dx \wedge dy$ with respect to $\mathbf{v} = -y\partial_x + x\partial_y$, the generator of the rotation group, is identically 0 and reflects the fact that rotations in $\mathbb{R}^2$ preserve area. (See Exercise 1.36.) Note that the three properties (1.59–1.61) along with its action on smooth functions serve to define the Lie derivative operation uniquely.

Proposition 1.65. *A differential k-form on M is invariant under the flow of a vector field* $\mathbf{v}$:

$$\omega|_{\exp(\varepsilon\mathbf{v})x} = \exp(-\varepsilon\mathbf{v})^*(\omega|_x),$$

if and only if $\mathbf{v}(\omega) = 0$ *everywhere. (A similar result holds for vector fields.)*

PROOF. Applying $\phi_\varepsilon^* = \exp(\varepsilon\mathbf{v})^*$ to (1.57) and using the basic group property of the flow, we find

$$\exp(\varepsilon\mathbf{v})^*(\mathbf{v}(\omega)|_{\exp(\varepsilon\mathbf{v})x}) = \frac{d}{d\varepsilon}\{\exp(\varepsilon\mathbf{v})^*(\omega|_{\exp(\varepsilon\mathbf{v})x})\} \tag{1.64}$$

for all ε where defined. From this the proposition is easily deduced. □

The most important formula for our purposes is one that relates the Lie derivative and the differential.

Proposition 1.66. *Let* ω *be a differential form and* $\mathbf{v}$ *be a vector field on M. Then*

$$\mathbf{v}(\omega) = d(\mathbf{v} \lrcorner\, \omega) + \mathbf{v} \lrcorner\, (d\omega). \tag{1.65}$$

PROOF. Define the operator $\mathscr{L}_\mathbf{v}(\omega)$ by the right-hand side of (1.65). Since the Lie derivative is uniquely determined by its action on functions and the properties (1.59–1.61), it suffices to check that $\mathscr{L}_\mathbf{v}$ enjoys the same properties. First

$$\mathscr{L}_\mathbf{v}(f) = \mathbf{v} \lrcorner\, df = \langle df; \mathbf{v}\rangle = \mathbf{v}(f),$$

so the action on functions is the same. Linearity of $\mathscr{L}_{\mathbf{v}}$ is clear, while the closure property (1.55) of d immediately proves the commutation property:

$$\mathscr{L}_{\mathbf{v}}(d\omega) = d(\mathbf{v} \,\lrcorner\, d\omega) = d\mathscr{L}_{\mathbf{v}}(\omega).$$

Finally, if ω is a k-form and θ an l-form, we use (1.52), (1.54) to prove that

$$\begin{aligned}
&\mathscr{L}_{\mathbf{v}}(\omega \wedge \theta) \\
&\quad = d[(\mathbf{v} \,\lrcorner\, \omega) \wedge \theta + (-1)^k \omega \wedge (\mathbf{v} \,\lrcorner\, \theta)] + \mathbf{v} \,\lrcorner\, [(d\omega) \wedge \theta + (-1)^k \omega \wedge (d\theta)] \\
&\quad = d(\mathbf{v} \,\lrcorner\, \omega) \wedge \theta + (-1)^{k-1}(\mathbf{v} \,\lrcorner\, \omega) \wedge d\theta + (-1)^k (d\omega) \wedge (\mathbf{v} \,\lrcorner\, \theta) \\
&\qquad + (-1)^{2k} \omega \wedge d(\mathbf{v} \,\lrcorner\, \theta) + (\mathbf{v} \,\lrcorner\, d\omega) \wedge \theta + (-1)^{k+1}(d\omega) \wedge (\mathbf{v} \,\lrcorner\, \theta) \\
&\qquad + (-1)^k (\mathbf{v} \,\lrcorner\, \omega) \wedge (d\theta) + (-1)^{2k} \omega \wedge (\mathbf{v} \,\lrcorner\, d\theta) \\
&\quad = \mathscr{L}_{\mathbf{v}}(\omega) \wedge \theta + \omega \wedge \mathscr{L}_{\mathbf{v}}(\theta),
\end{aligned}$$

the remaining terms cancelling. □

Homotopy Operators

The key to the proof of the exactness of the de Rham complex (or any other complex for that matter) lies in the construction of suitable *homotopy operators*. By definition, these are linear operators $h\colon \bigwedge_k \to \bigwedge_{k-1}$, taking differential k-forms to $(k-1)$-forms, and satisfying the basic identity

$$\omega = dh(\omega) + h(d\omega) \tag{1.66}$$

for all k-forms ω. (The case $k = 0$ is slightly different, as explained below.) The discovery of such a set of operators immediately implies exactness of the complex. For if ω is closed, $d\omega = 0$, then (1.66) reduces to $\omega = d\theta$ where $\theta = h(\omega)$, so ω is exact. Thus we need only concentrate on finding these homotopy operators.

Let us look back at the Lie derivative formula (1.65). If we could treat the Lie derivative as an ordinary derivative, then we could integrate both sides of (1.65) and deduce the homotopy formula (1.66). More rigorously, we can integrate the Lie derivative formula (1.64) with respect to ε; using (1.65) and (1.56), we find

$$\begin{aligned}
\exp(\varepsilon\mathbf{v})^*[\omega|_{\exp(\varepsilon\mathbf{v})x}] - \omega|_x &= \int_0^\varepsilon \exp(\tilde{\varepsilon}\mathbf{v})^*[\mathbf{v}(\omega)|_{\exp(\tilde{\varepsilon}\mathbf{v})x}]\, d\tilde{\varepsilon} \\
&= \int_0^\varepsilon \{d[\exp(\tilde{\varepsilon}\mathbf{v})^*(\mathbf{v} \,\lrcorner\, \omega|_{\exp(\tilde{\varepsilon}\mathbf{v})x})] \\
&\qquad + \exp(\tilde{\varepsilon}\mathbf{v})^*[\mathbf{v} \,\lrcorner\, d\omega|_{\exp(\tilde{\varepsilon}\mathbf{v})x}]\}\, d\tilde{\varepsilon}.
\end{aligned}$$

If we define the operator

$$h_{\mathbf{v}}^{\varepsilon}(\omega)|_x \equiv \int_0^\varepsilon \exp(\tilde{\varepsilon}\mathbf{v})^*[\mathbf{v} \,\lrcorner\, \omega|_{\exp(\tilde{\varepsilon}\mathbf{v})x}]\, d\tilde{\varepsilon},$$