Example 5.57. Consider the functional

$$\mathscr{L}[u] = \int_a^b \tfrac{1}{2}u_x^2\,dx, \qquad x, u \in \mathbb{R}.$$

The generalized symmetry $\mathbf{v} = -u_x\partial_u$ is easily seen to be a variational symmetry of $\mathscr{L}$:

$$\operatorname{pr}\mathbf{v}(\tfrac{1}{2}u_x^2) = -u_x u_{xx} = -D_x(\tfrac{1}{2}u_x^2).$$

Indeed $\mathbf{v}$ is just the evolutionary form of the translation field $\tilde{\mathbf{v}} = \partial_x$, and generates the one-parameter group

$$\exp(\varepsilon\mathbf{v})f(x) = f(x-\varepsilon).$$

If $[c, d] \subset (a, b)$ is any subinterval, the boundary contribution in the proof of (5.88) is

$$\mathscr{B}(x, u^{(1)}) = -\tfrac{1}{2}u_x^2\Big|_{x=c}^{d} = \tfrac{1}{2}[f'(c)^2 - f'(d)^2];$$

indeed (5.88) in this case reads

$$\int_c^d \tfrac{1}{2}[f'(x-\varepsilon)]^2\,dx$$

$$= \int_c^d \tfrac{1}{2}[f'(x)]^2\,dx + \int_0^\varepsilon \tfrac{1}{2}\{[f'(c-\tilde{\varepsilon})]^2 - [f'(d-\tilde{\varepsilon})]^2\}\,d\tilde{\varepsilon}.$$

Note especially that we cannot dispense with the boundary contribution in general since the only solution vanishing on the boundary is the trivial solution $u \equiv 0$.

Noether's Theorem

As the reader may have already noticed, in the case that the system of differential equations Δ is the Euler–Lagrange equations for some variational problem, the condition (5.82) for Q to be the characteristic of a conservation law and the condition (5.87) for $\mathbf{v}_Q$ to generate a variational symmetry group coincide. Thus, using Theorem 4.26, we immediately deduce the general form of Noether's theorem.

Theorem 5.58. *A generalized vector field* $\mathbf{v}$ *determines a variational symmetry group of the functional* $\mathscr{L}[u] = \int L\,dx$ *if and only if its characteristic* $Q \in \mathscr{A}^q$ *is the characteristic of a conservation law* $\operatorname{Div} P = 0$ *for the corresponding Euler–Lagrange equations* $\mathsf{E}(L) = 0$. *In particular, if* $\mathscr{L}$ *is a nondegenerate variational problem, there is a one-to-one correspondence between equivalence classes of nontrivial conservation laws of the Euler–Lagrange equations and equivalence classes of variational symmetries of the functional.*

Note that two variational symmetries are equivalent provided they differ by a trivial symmetry, meaning one whose characteristic vanishes on all solutions of the Euler–Lagrange equations. (However, it is *not* true that a symmetry which happens to be equivalent to a variational symmetry is necessarily variational—see Exercise 5.32.)

Example 5.59. As a first illustration of this result, consider the Kepler problem $\ddot{x} + \mu r^{-3}x = 0$, $\ddot{y} + \mu r^{-3}y = 0$, $\ddot{z} + \mu r^{-3}z = 0$, $r^2 = x^2 + y^2 + z^2$, for a mass moving in a gravitational potential due to a fixed mass at the origin. The associated Lagrangian is $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \mu r^{-1}$. We've already seen in Example 4.31 how the conservation laws of energy and angular momenta arise from the variational symmetry groups of time translations and rotations in $\mathbb{R}^3$. Owing to the Newtonian nature of the force field, there are three additional "hidden" generalized variational symmetries of this system, leading to three further independent conservation laws. One such infinitesimal generator is the vector field

$$\mathbf{v}_x = (y\dot{y} + z\dot{z})\partial_x + (\dot{x}y - 2x\dot{y})\partial_y + (\dot{x}z - 2x\dot{z})\partial_z,$$

the other two being obtained by permuting the variables x, y, z. To prove that $\mathbf{v}_x$ is indeed a variational symmetry, we compute

$$\begin{aligned}\mathrm{pr}^{(1)}\,\mathbf{v}_x = \mathbf{v}_x &+ (y\ddot{y} + z\ddot{z} + \dot{y}^2 + \dot{z}^2)\partial_{\dot{x}} + (\ddot{x}y - 2x\ddot{y} - \dot{x}\dot{y})\partial_{\dot{y}}\\ &+ (\ddot{x}z - 2x\ddot{z} - \dot{x}\dot{z})\partial_{\dot{z}},\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{pr}^{(1)}\,\mathbf{v}_x(L) &= (y\dot{y} + z\dot{z})\ddot{x} + (\dot{x}y - 2x\dot{y})\ddot{y} + (\dot{x}z - 2x\dot{z})\ddot{z}\\ &\quad + \mu r^{-3}[(y^2 + z^2)\dot{x} - xy\dot{y} - xz\dot{z}]\\ &= D_t[\dot{x}(y\dot{y} + z\dot{z}) - x(\dot{y}^2 + \dot{z}^2) + \mu r^{-1}x],\end{aligned}$$

verifying (5.85). The corresponding conservation laws are found from the characteristic form (5.81), or, more simply, by noting that $\mathrm{pr}^{(1)}\,\mathbf{v}_x(L)$ itself vanishes on solutions of the Euler–Lagrange equations, so

$$R_x \equiv \dot{x}(y\dot{y} + z\dot{z}) - x(\dot{y}^2 + \dot{z}^2) + \mu r^{-1}x$$

is a first integral of the Kepler problem. Coupled with the other conservation laws R_y and R_z obtained by permuting the variables, we deduce the constancy of the *Runge–Lenz vector*, which can be written as

$$\mathbf{R} \equiv (R_x, R_y, R_z) = \dot{\mathbf{x}} \times \mathbf{A} - \mu\mathbf{x}/|\mathbf{x}| = \dot{\mathbf{x}} \times (\mathbf{x} \times \dot{\mathbf{x}}) - \mu\mathbf{x}/|\mathbf{x}|,$$

where $\mathbf{x} = (x, y, z)$ is the position vector and $\mathbf{A} = \mathbf{x} \times \dot{\mathbf{x}}$ the angular momentum. Physically, $\mathbf{R}$ points along the major axis of the conic section determined by the planetary orbit, its magnitude determining the eccentricity. (See Thirring, [1; p. 147].)

Example 5.60. The sine–Gordon equation $u_{xt} = \sin u$ is the Euler–Lagrange equation for the functional

$$\mathscr{L}[u] = \iint (\tfrac{1}{2}u_x u_t - \cos u)\, dx\, dt.$$

The generalized vector field $\mathbf{v}_1$ with characteristic $Q_1 = u_{xxx} + \frac{1}{2}u_x^3$ is a variational symmetry of $\mathscr{L}$. This can be seen directly, or, slightly easier, by using Proposition 5.55. Note that

$$\mathsf{D}_{Q_1} = D_x^3 + \tfrac{3}{2}u_x^2 D_x, \qquad \mathsf{D}_{Q_1}^* = -D_x^3 - \tfrac{3}{2}u_x^2 D_x - 3u_x u_{xx}.$$

A short calculation shows that

$$\begin{aligned} \mathrm{pr}\, \mathbf{v}_{Q_1}[u_{xt} - \sin u] &= u_{xxxxt} + \tfrac{3}{2}u_x^2 u_{xxt} + 3u_x u_{xx} u_{xt} - (u_{xxx} + \tfrac{1}{2}u_x^3)\cos u \\ &= -\mathsf{D}_{Q_1}^*[u_{xt} - \sin u], \end{aligned}$$

verifying (5.87). The associated conservation law has characteristic form

$$\begin{aligned} &D_t(-\tfrac{1}{2}u_{xx}^2 + \tfrac{1}{8}u_x^4) + D_x(u_{xx}u_{xt} - u_{xx}\sin u + \tfrac{1}{2}u_x^2 \cos u) \\ &\quad = (u_{xxx} + \tfrac{1}{2}u_x^3)(u_{xt} - \sin u). \end{aligned}$$

In particular, the conserved density determines a functional

$$\mathscr{T}_1[u] = \int_{-\infty}^{\infty} (\tfrac{1}{8}u_x^4 - \tfrac{1}{2}u_{xx}^2)\, dx,$$

whose value is independent of t whenever $u(x, t)$ is a solution whose derivatives decay rapidly as $|x| \to \infty$.

An even more tedious computation shows that

$$\mathbf{v}_{Q_2} = (u_{xxxxx} + \tfrac{5}{2}u_x^2 u_{xxx} + \tfrac{5}{2}u_x u_{xx}^2 + \tfrac{3}{8}u_x^5)\partial_u$$

is also a variational symmetry, with associated conservation law

$$\mathscr{T}_2 = \int_{-\infty}^{\infty} (\tfrac{1}{2}u_{xxx}^2 - \tfrac{5}{4}u_x^2 u_{xx}^2 + \tfrac{1}{16}u_x^6)\, dx.$$

(See Exercises 5.12 and 5.31 for further results on this equation.)

Self-adjoint Linear Systems

Consider a homogeneous system of linear differential equations $\Delta[u] = 0$ determined by a $q \times q$ matrix of differential operators

$$\Delta_{\mu\nu} = \sum_J a_{\mu\nu}^J(x) D_J, \qquad \mu, \nu = 1, \ldots, q,$$

whose coefficients depend only on x. As is well known, this system is the Euler–Lagrange equations for a variational problem if and only if Δ is self-adjoint: $\Delta^* = \Delta$. In this case, we can take the functional simply to be

$$\mathscr{L}[u] = \tfrac{1}{2} \int u \cdot \Delta[u] \, dx. \tag{5.89}$$

(See also Theorem 5.92.)

Any conservation law for the given self-adjoint linear system can, without loss of generality, be taken in characteristic form $\operatorname{Div} P = Q \cdot \Delta$. By Noether's theorem, the characteristic Q determines a variational symmetry of the corresponding quadratic variational problem. Here we investigate in some detail the cases of *linear* conservation laws, where P is linear in u and its derivatives, and hence Q depends only on x, and quadratic conservation laws, with P being quadratic and Q linear in u and its derivatives. The former case will lead to "reciprocity" relations relating pairs of solutions of the system; the latter will be closely tied to our theory of recursion operators for linear systems developed in the preceding section.

For a linear conservation law, note that $\mathbf{v}_q = \sum q_\alpha(x) \partial_{u^\alpha}$ generates a symmetry group of a linear system if and only if $q(x)$ is a solution itself: $\Delta[q] = 0$. (The group transformations are just $u \mapsto u + \varepsilon q$, reflecting the linearity of Δ.) Also note that the Fréchet derivative in this case is automatically 0, so (5.87) is verified and $\mathbf{v}_q$ is always a variational symmetry. Noether's theorem allows us to conclude the existence of a linear conservation law

$$\operatorname{Div} \hat{P}[u] = q(x) \cdot \Delta[u] \tag{5.90}$$

for any solution $q(x)$ of Δ. Alternatively, we can derive (5.90) directly using our basic integration by parts procedure:

Proposition 5.61. *Let $\Delta[u] = 0$ be a self-adjoint linear system. Then, for any functions $u(x)$, $v(x)$, we have the reciprocity relation*

$$v \cdot \Delta[u] - u \cdot \Delta[v] = \operatorname{Div} P[u, v], \tag{5.91}$$

where $P \in \mathscr{A}^p$ is some bilinear expression involving u and v and their derivatives.

The general formula for P in terms of Δ is quite complicated. In the second order case, however, we can derive a relatively simple expression. It is not difficult to see that any self-adjoint second order matrix differential operator can be written in the particular form

$$\Delta_{\mu\nu} = \sum_{i,j=1}^{p} D_i \cdot a^{ij}_{\mu\nu}(x) D_j + \sum_{i=1}^{p} (b^i_{\mu\nu}(x) \cdot D_i + D_i \cdot b^i_{\mu\nu}(x)) + c_{\mu\nu}(x),$$

$$\mu, \nu = 1, \ldots, q,$$

where the coefficients satisfy

$$a^{ij}_{\mu\nu} = a^{ji}_{\nu\mu}, \qquad b^i_{\mu\nu} = -b^i_{\nu\mu}, \qquad c_{\mu\nu} = c_{\nu\mu}.$$

The corresponding variational problem can either be taken in the form (5.89), or, by a simple integration by parts, in first order form

$$\mathscr{L}[u] = \frac{1}{2} \int \sum_{\mu,\nu=1}^{q} \left\{ -\sum_{i,j=1}^{p} a_{\mu\nu}^{ij} u_i^\mu u_j^\nu + \sum_{i=1}^{p} b_{\mu\nu}^i (u^\mu u_i^\nu - u_i^\mu u^\nu) + c_{\mu\nu} u^\mu u^\nu \right\} dx. \tag{5.92}$$

If we define the $q \times q$ matrix differential operators $\mathscr{D}^i$, $i = 1, \ldots, p$, with entries

$$\mathscr{D}_{\mu\nu}^i = \sum_{j=1}^{p} a_{\mu\nu}^{ij}(x) D_j + b_{\mu\nu}^i(x),$$

then the reciprocity relation (5.91) holds with

$$P_i = v \cdot \mathscr{D}^i[u] - u \cdot \mathscr{D}^i[v], \qquad i = 1, \ldots, p.$$

Equivalently, we have the integral form

$$\int_{\partial\Omega} (v \cdot \mathscr{D}[u] - u \cdot \mathscr{D}[v]) \cdot dS = \int_{\Omega} (v \cdot \Delta[u] - u \cdot \Delta[v]) \, dx, \tag{5.93}$$

where $v \cdot \mathscr{D}[u] \equiv (v \cdot \mathscr{D}^1[u], \ldots, v \cdot \mathscr{D}^p[u])$.

For instance, in the case of Laplace's equation, (5.93) is the familiar form of Green's formula since $\mathscr{D}[u] = \nabla u$. For Navier's equations

$$\mu \Delta u + (\mu + \lambda) \nabla(\nabla \cdot u) = 0$$

of linear isotropic elasticity, (5.93) is equivalent to the standard Betti reciprocal theorem

$$\int_{\partial\Omega} (u \cdot \sigma[v] - v \cdot \sigma[u]) \, dS = \int_{\Omega} \{u \cdot [\mu \Delta v + (\mu + \lambda) \nabla(\nabla \cdot v)]$$
$$- v[\mu \Delta u + (\mu + \lambda) \nabla(\nabla \cdot u)]\} \, dx,$$

in which

$$\sigma[u] = \mu(\nabla u + \nabla u^T) + \lambda(\nabla \cdot u) \mathrm{I}$$

is the stress tensor associated with the displacement u. (Here (5.92), which is

$$\mathscr{L}[u] = -\frac{1}{2} \int \{\mu \|\nabla u\|^2 + (\mu + \lambda)(\nabla \cdot u)^2\} \, dx$$

is not exactly the same as the usual variational principle derived from the stored energy function, but differs from it only by the null Lagrangian

$$N = \sum_{\substack{i \neq j \\ \alpha \neq \beta}} \mu \frac{\partial(u^\alpha, u^\beta)}{\partial(x^i, x^j)}.$$

Turning to the quadratic conservation laws, the characteristic Q is a linear function of u and its derivatives, hence $Q(x, u^{(m)}) = \mathscr{D}[u]$ for some $q \times q$ matrix of differential operators $\mathscr{D}$ whose coefficients depend only on x.

Noether's theorem implies that Q is the characteristic of a variational symmetry, and hence a symmetry of the Euler–Lagrange equations themselves. Proposition 5.22 implies that $\mathscr{D}$ is a recursion operator for the linear system, so $\Delta\mathscr{D} = \tilde{\mathscr{D}}\Delta$ for some differential operator $\tilde{\mathscr{D}}$. Not every recursion operator gives rise to a variational symmetry, however, but it is easy to characterize those which do.

Proposition 5.62. *A q-tuple $Q = \mathscr{D}[u]$ of linear functions in u and its derivatives forms the characteristic of a conservation law for the linear system $\Delta[u] = 0$ if and only if the product differential operator $\mathscr{D}^* \cdot \Delta$ is skew-adjoint.*

This is an immediate consequence of (5.82) using the fact that the Fréchet derivative of a linear q-tuple $\Delta[u]$ is the same as the differential operator Δ which determines it. In particular, if Δ is self-adjoint, this condition takes the form

$$\Delta \cdot \mathscr{D} = -\mathscr{D}^* \cdot \Delta, \tag{5.94}$$

meaning that the operator $\tilde{\mathscr{D}}$ appearing in the recursion condition $\Delta\mathscr{D} = \tilde{\mathscr{D}}\Delta$ must agree with $-\mathscr{D}^*$. Note that in this case, any *odd* power $\mathscr{D}^{2k+1}$ of $\mathscr{D}$ also satisfies (5.94). We conclude that a self-adjoint linear system with one quadratic conservation law always has an *infinite* hierarchy of such laws

$$\operatorname{Div} P^{(k)} = \mathscr{D}^{2k+1}[u] \cdot \Delta[u]$$

depending on higher and higher order derivatives of u.

Under our nondegeneracy hypothesis, a symmetry operator $\mathscr{D}$ determines a trivial conservation law if and only if it is a multiple of Δ, i.e. $\mathscr{D} = \mathscr{E} \cdot \Delta$ for some differential operator $\mathscr{E}$. The question of how many nontrivial quadratic conservation laws of a given order there are, then, is related to the (complicated) question of how many inequivalent symmetry operators of a given order there are, which was considered in Section 5.2. Note further that if $\mathscr{D}$ is any linear recursion operator, so $\Delta\mathscr{D} = \tilde{\mathscr{D}}\Delta$ for some operator $\tilde{\mathscr{D}}$, then we can always "skew-symmetrize" $\mathscr{D}$ to produce a new recursion operator $\hat{\mathscr{D}} = \frac{1}{2}(\mathscr{D} - \tilde{\mathscr{D}}^*)$ which does satisfy (5.94) and hence does determine a conservation law. To see this, it suffices to take the adjoint of the symmetry condition,

$$\Delta\tilde{\mathscr{D}}^* = (\tilde{\mathscr{D}}\Delta)^* = (\Delta\mathscr{D})^* = \mathscr{D}^*\Delta,$$

using the self-adjointness of Δ, hence

$$\Delta\hat{\mathscr{D}} = \tfrac{1}{2}(\Delta\mathscr{D} - \Delta\tilde{\mathscr{D}}^*) = \tfrac{1}{2}(\tilde{\mathscr{D}}\Delta - \mathscr{D}^*\Delta) = -\hat{\mathscr{D}}^*\Delta.$$

In particular, since for any symmetry operator $\mathscr{D}$, the operator $\tilde{\mathscr{D}}$ has the same leading order terms, we see that there is a one-to-one correspondence between quadratic conservation laws and the skew-adjoint leading terms of recursion operators. For scalar equations, the leading order terms must be of odd order, and for every such term we get a conservation law. If $\mathscr{D}_1, \ldots, \mathscr{D}_k$ are linear first order variational symmetry operators, then the skew-symmetrized product

$$\tfrac{1}{2}[\mathscr{D}_1\mathscr{D}_2\cdots\mathscr{D}_k + (-1)^{k-1}\mathscr{D}_k\mathscr{D}_{k-1}\cdots\mathscr{D}_1] \tag{5.95}$$

gives a k-th (or lower) order variational operator. (For scalar equations, we need only take k odd, and then the operator is k-th order.) In many examples, it appears that every quadratic conservation law can be generated in this way.

Example 5.63. Here we conclude our investigation into the symmetries and conservation laws of the two-dimensional wave equation $u_{tt} = u_{xx} + u_{yy}$ of Examples 2.43, 4.36 and 5.23. Here $\Delta = D_t^2 - D_x^2 - D_y^2$. Of the recursion operators listed in (5.30), the first six, corresponding to translations and rotations, all commute with Δ and are all skew-adjoint and hence all satisfy (5.94). The resulting conservation laws were determined in Example 4.36. For the dilatational operator $\mathscr{D}$, we find $\Delta\mathscr{D} = (\mathscr{D} + 2)\Delta$, but $\mathscr{D}^* = -\mathscr{D} - 3$. Thus $\mathscr{D}$ does not determine a conservation law; however, the modified dilatation operator $\mathscr{M} = \mathscr{D} + \frac{1}{2}$ does satisfy (5.94):

$$\Delta\mathscr{M} = (\mathscr{M} + 2)\Delta, \quad \text{and} \quad \mathscr{M}^* = \mathscr{D}^* + \tfrac{1}{2} = -\mathscr{D} - \tfrac{5}{2} = -\mathscr{M} - 2.$$

See Example 4.36 for the conservation law. Finally, each inversional operator also determines a conservation law, since, for $\mathscr{I}_x$, say,

$$\Delta\mathscr{I}_x = (\mathscr{I}_x + 4x)\Delta, \quad \text{and} \quad \mathscr{I}_x^* = -\mathscr{I}_x - 4x.$$

The corresponding conservation laws were found in Example 4.36.

Higher order quadratic conservation laws are found by looking at "skew-symmetrized" odd order products (5.95) of these basic recursion operators, e.g. $\frac{1}{2}[\mathscr{R}_{xy}\mathscr{M}\mathscr{I}_x + \mathscr{I}_x\mathscr{M}\mathscr{R}_{xy}]$. A partial listing of some of the second order conservation laws and their corresponding symmetry operators is given in the following table. (See also Example 5.65.)

Recursion Operator	Characteristic	Conserved Density
D_x^3	u_{xxx}	$u_{xx}u_{xt}$
$D_x^2 D_t$	u_{xxt}	$\frac{1}{2}(u_{xt}^2 + u_{xx}^2 + u_{xy}^2)$
D_t^3	u_{ttt}	$\frac{1}{2}(u_{tt}^2 + u_{xt}^2 + u_{yt}^2)$
$D_x\mathscr{R}_{xy}D_x$	$-yu_{xxx} + xu_{xyy} + u_{xy}$	$u_{xt}(xu_{xy} - yu_{xx})$
$D_x\mathscr{R}_{xy}D_y - \frac{1}{2}D_x^2 - \frac{1}{2}D_y^2$	$-yu_{xxy} + xu_{xyy} - \frac{1}{2}u_{xx} + \frac{1}{2}u_{yy}$	$u_{xx}(yu_{yt} + \frac{1}{2}u_t) - u_{yy}(xu_{xt} + \frac{1}{2}u_t)$
$D_x\mathscr{R}_{xt}D_x$	$xu_{xxt} + tu_{xxx} + u_{xt}$	$x\hat{T} + tu_{xx}u_{xt}$
$D_x\mathscr{M}D_x$	$xu_{xxx} + yu_{xxy} + tu_{xxt} + \frac{3}{2}u_{xx}$	$T^* + t\hat{T}$
$\mathscr{R}_{xt}D_t\mathscr{R}_{xt}$	$x^2u_{xtt} + 2xtu_{xxt} + t^2u_{xxx} + xu_{tt} +$ $+ 2tu_{xt} + xu_{xx}$	$(x^2 + t^2)\hat{T} + \frac{1}{2}u_y^2 +$ $+ t(2xu_{xt}u_{tt} - u_yu_{yt})$
$D_x\mathscr{I}_tD_x$	$(x^2 + y^2 + t^2)u_{xxt} + 2xtu_{xxx} +$ $2ytu_{xxy} + 2xu_{xt} + 3tu_{xx}$	$(x^2 + y^2 + t^2)\hat{T} + u_x^2 + 2tT^*$

where

$$\hat{T} = \tfrac{1}{2}(u_{xt}^2 + u_{xx}^2 + u_{xy}^2), \qquad T^* = xu_{xx}u_{xt} + yu_{xx}u_{yt} + \tfrac{1}{2}u_{xx}u_t$$

Action of Symmetries on Conservation Laws

A second method of generating conservation laws is to apply known symmetry group generators to known conservation laws. Unfortunately, the method is not guaranteed to produce nontrivial laws, but we can determine precisely when it does.

Proposition 5.64. *Let Δ be totally nondegenerate and* $\operatorname{Div} P = 0$ *a conservation law. If $\mathbf{v}_R$ is an evolutionary symmetry of Δ, then the induced p-tuple $\tilde{P} = \operatorname{pr} \mathbf{v}_R(P)$, with entries $\tilde{P}_i = \operatorname{pr} \mathbf{v}_R(P_i)$, is also a conservation law:* $\operatorname{Div} \tilde{P} = 0$. *Moreover, if $\Delta = E(L)$ is a system of Euler–Lagrange equations, P has characteristic Q corresponding to the variational symmetry $\mathbf{v}_Q$, and $\mathbf{v}_R$ is a variational symmetry, then $\tilde{P}$ has characteristic $\tilde{Q}$ corresponding to the Lie bracket $\mathbf{v}_{\tilde{Q}} = [\mathbf{v}_R, \mathbf{v}_Q]$ of the two symmetries.*

PROOF. We assume that the conservation law is in characteristic form (5.81). (Note that if P_0 is a trivial conservation law, so is $\operatorname{pr} \mathbf{v}_R(P_0)$, so this first step is justified.) Applying $\operatorname{pr} \mathbf{v}_R$, we find

$$\operatorname{Div}[\operatorname{pr} \mathbf{v}_R(P)] = \operatorname{pr} \mathbf{v}_R(Q) \cdot \Delta + Q \cdot \operatorname{pr} \mathbf{v}_R(\Delta) \tag{5.96}$$

using (5.19). Since $\operatorname{pr} \mathbf{v}_R(\Delta) = 0$ for solutions of Δ, the right-hand side of (5.96) vanishes on solutions, proving the first part of the theorem. If $\Delta = \mathsf{E}(L)$ and $\mathbf{v}_R$ is variational, then we can use Proposition 5.55 to rewrite the second term in (5.96) and integrate by parts,

$$Q \cdot \operatorname{pr} \mathbf{v}_R(\Delta) = -Q \cdot \mathsf{D}_R^*(\Delta) = -\mathsf{D}_R(Q) \cdot \Delta - \operatorname{Div} B = -\operatorname{pr} \mathbf{v}_Q(R) \cdot \Delta - \operatorname{Div} B$$

for some p-tuple B which depends linearly on Δ and its total derivatives, and hence forms a trivial conservation law of the first kind. Thus, by (5.22),

$$\operatorname{Div}[\operatorname{pr} \mathbf{v}_R(P) + B] = \{\operatorname{pr} \mathbf{v}_R(Q) - \operatorname{pr} \mathbf{v}_Q(R)\} \cdot \Delta = \tilde{Q} \cdot \Delta$$

is the characteristic form of our conservation law and the proof is complete. □

This result is most useful in the case of self-adjoint linear systems. Indeed, if $P \in \mathscr{A}^p$ determines a quadratic conservation law corresponding to the linear characteristic $Q = \mathscr{D}[u]$, and $\mathbf{v}_R$ is a linear symmetry, so $R = \mathscr{E}[u]$ for some differential operator $\mathscr{E}$ satisfying $\Delta \cdot \mathscr{E} = \tilde{\mathscr{E}} \cdot \Delta$, then $\operatorname{pr} \mathbf{v}_R(P)$ yields a conservation law with characteristic $\tilde{Q} = (\mathscr{D} \cdot \mathscr{E} + \tilde{\mathscr{E}}^* \cdot \mathscr{D})[u]$. In particular, if $\mathbf{v}_R$ is a variational symmetry, then $\tilde{Q}$ has characteristic corresponding to the commutator operator $[\mathscr{D}, \mathscr{E}] = \mathscr{D} \cdot \mathscr{E} - \mathscr{E} \cdot \mathscr{D}$.

Example 5.65. For the two-dimensional wave equation, the conserved densities in the table of Example 5.63 are most easily computed using this method. For example, the conservation law with characteristic u_{xxt} can be constructed either by applying the prolongation of the symmetry $\mathbf{v} = \frac{1}{2} u_{xx} \partial_u$ to the energy conservation law with characteristic u_t, or the prolongation of

$\mathbf{w} = \frac{1}{2}u_{xt}\partial_u$ to the momentum conservation law with characteristic u_x. (In the former case, $\Delta D_x^2 = D_x^2\Delta$, so the new characteristic is indeed $\operatorname{pr}\mathbf{v}(u_t) + \frac{1}{2}(D_x^2)^* u_t = u_{xxt}$.) In the first case, the new density is

$$\operatorname{pr}\mathbf{v}[\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_y^2] = \tfrac{1}{2}(u_t u_{xxt} + u_x u_{xxx} + u_y u_{xxy}) \equiv T,$$

while in the second it is

$$\operatorname{pr}\mathbf{w}[u_x u_t] = \tfrac{1}{2}(u_t u_{xxt} + u_x u_{xtt}) \equiv \tilde{T}.$$

Since both these densities have the same characteristics, they should be equivalent:

$$T = \tilde{T} + D_x R + D_y S$$

on solutions of the wave equation. In other words, we have the freedom to (a) substitute for derivatives according to the equation and its prolongation, and (b) integrate by parts with respect to x and y (but *not* t); thus $u_t u_{xxt}$ is equivalent to $-u_{xt}^2$, but *not* $u_{xx}u_{tt}$. The reader can verify that T and $\tilde{T}$ are both equivalent to the second order density listed in the above-mentioned table.

As a second example, the conservation law corresponding to the operator $D_x \mathscr{R}_{xt} D_x$ is found by applying the symmetry $\hat{\mathbf{v}} = \frac{1}{2}\mathscr{R}_{xt}D_x[u]\partial_u = \frac{1}{2}(xu_{xt} + tu_{xx})\partial_u$ to the conservation law with characteristic u_x. We find

$$\begin{aligned}\operatorname{pr}\hat{\mathbf{v}}[u_x u_t] &= \tfrac{1}{2}\{D_x(xu_{xt} + tu_{xx})u_t + u_x D_t(xu_{xt} + tu_{xx})\}\\ &= \tfrac{1}{2}(xu_{xxt} + u_{xt} + tu_{xxx})u_t + \tfrac{1}{2}u_x(xu_{xtt} + tu_{xxt} + u_{xx}).\end{aligned}$$

Both lower order terms $u_t u_{xt}$ and $u_x u_{xx}$ are x-derivatives, hence this density is equivalent to the one in the table by a similar integration by parts.

Abnormal Systems and Noether's Second Theorem

The connection between variational symmetries and conservation laws for systems which fail to be totally nondegenerate is less transparent. Although the basic integration by parts formula (4.39) still yields a variational symmetry for each conservation law and vice versa, there is now no guarantee that nontrivial symmetries will give rise to nontrivial conservation laws or the reverse. In the case of analytic systems, we saw that there are two basic types of abnormality possible. Over-determined systems are less well understood in this regard, and the precise relationship between their symmetries and conservation laws has yet to be fully sorted out. Under-determined systems, however, fall under the ambit of Noether's second theorem which is concerned with systems posessing infinite-dimensional groups of variational symmetries. The resulting dependencies among the Euler–Lagrange equations can be re-interpreted as trivial conservation laws determined by nontrivial variational symmetry groups, so the nice one-to-one correspondence of Theorem 5.58 breaks down in the under-determined case.

Theorem 5.66. *The variational problem $\mathscr{L}[u] = \int L\,dx$ admits an infinite-dimensional group of variational symmetries whose characteristics $Q[u; h]$ depend on an arbitrary function $h(x)$ (and its derivatives) if and only if there exist differential operators $\mathscr{D}_1, \ldots, \mathscr{D}_q$, not all zero, such that*

$$\mathscr{D}_1 \mathsf{E}_1(L) + \cdots + \mathscr{D}_q \mathsf{E}_q(L) \equiv 0 \tag{5.97}$$

for all x, u.

PROOF. Assume first that the Euler–Lagrange equations for $\mathscr{L}$ are underdetermined, so there is a relation of the form (5.97) among them. Let $h(x)$ be arbitrary. Then an easy integration by parts shows that

$$\begin{aligned} 0 &= h(x)\{\mathscr{D}_1 \mathsf{E}_1(L) + \cdots + \mathscr{D}_q \mathsf{E}_q(L)\} \\ &= \mathscr{D}_1^*[h]\,\mathsf{E}_1(L) + \cdots + \mathscr{D}_q^*[h]\,\mathsf{E}_q(L) - \operatorname{Div} P \end{aligned} \tag{5.98}$$

for some p-tuple $P \in \mathscr{A}^p$ depending linearly on $\mathsf{E}(L)$ and its derivatives. If we set $Q_\nu = \mathscr{D}_\nu^*[h]$, $\nu = 1, \ldots, q$, then the above identity is in the form of a conservation law in characteristic form, where Q is the characteristic and $P = P[u; h] \in \mathscr{A}^p$ the conservation law, which is actually trivial (of the first kind). Now we can clearly use (5.98) to prove that for any function $h(x)$, $\mathbf{v}_{Q[u;h]}$ determines a variational symmetry of the functional $\mathscr{L}[u]$.

The proof of the converse is straightforward if $Q_\nu[u; h] = \tilde{\mathscr{D}}_\nu[h]$ are all linear in h and its derivatives, $\tilde{\mathscr{D}}_\nu$ being differential operators whose coefficients can depend on u. Starting with the condition (5.85) that $\mathbf{v}_Q$ be a variational symmetry, we integrate by parts to obtain the corresponding conservation law

$$\operatorname{Div} P = Q \cdot \mathsf{E}(L) = \tilde{\mathscr{D}}_1[h]\,\mathsf{E}_1(L) + \cdots + \tilde{\mathscr{D}}_q[h]\,\mathsf{E}_q(L).$$

Further integration by parts, effectively reversing the arguments in (5.98), leads to an identity of the form

$$\operatorname{Div} \tilde{P} = h(x)\{\tilde{\mathscr{D}}_1^* \mathsf{E}_1(L) + \cdots + \tilde{\mathscr{D}}_q^* \mathsf{E}_q(L)\}, \tag{5.99}$$

which holds for an arbitrary function $h(x)$. The proof is completed using the following "formal" version of the du Bois–Reymond lemma of the variational calculus.

Lemma 5.67. *Let $R(x, u^{(n)})$ be a differential function and suppose for every smooth function $h(x)$ there exists $P[u] = P_h[u] \in \mathscr{A}^p$ such that*

$$h(x)R(x, u^{(n)}) = \operatorname{Div} P(x, u^{(m)}).$$

Then $R(x, u^{(n)}) = r(x)$ is a function of x alone.

PROOF. Assume R depends on the n-th and lower order derivatives of u and that $\partial R(x_0, u_0^{(n)})/\partial u_J^\alpha \neq 0$ for some $\#J = n \geq 0$, $(x_0, u_0^{(n)}) \in M^{(n)}$. Choose $h(x)$ such that $\partial_J h(x_0) \neq 0$, but all other derivatives of h of order $\leq n$ vanish at x_0.