for any rotation θ. But κ is just the curvature of the curve determined by the graph of $u = f(x)$, so we've just reproved the fact that the curvature of a curve is invariant under rotations. (This is a special case of the theory of differential invariants—see Section 2.5 for further results of this type.)

Example 2.38. Consider the special case $p = 2$, $q = 1$ in the prolongation formula, so we are looking at a partial differential equation involving a function $u = f(x, t)$. A general vector field on $X \times U \simeq \mathbb{R}^2 \times \mathbb{R}$ takes the form

$$\mathbf{v} = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \phi(x, t, u)\frac{\partial}{\partial u}.$$

The first prolongation of $\mathbf{v}$ is the vector field

$$\mathrm{pr}^{(1)}\,\mathbf{v} = \mathbf{v} + \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t},$$

where, using (2.39),

$$\begin{aligned} \phi^x &= D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt} \\ &= D_x\phi - u_x D_x\xi - u_t D_x\tau \\ &= \phi_x + (\phi_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \\ \phi^t &= D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt} \\ &= D_t\phi - u_x D_t\xi - u_t D_t\tau \\ &= \phi_t - \xi_t u_x + (\phi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2, \end{aligned} \tag{2.45}$$

the subscripts on ϕ, ξ, τ denoting partial derivatives. Similarly,

$$\mathrm{pr}^{(2)}\,\mathbf{v} = \mathrm{pr}^{(1)}\,\mathbf{v} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}},$$

where, for instance,

$$\begin{aligned} \phi^{xx} &= D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt} \\ &= D_x^2\phi - u_x D_x^2\xi - u_t D_x^2\tau - 2u_{xx}D_x\xi - 2u_{xt}D_x\tau \\ &= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t \\ &\quad - \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_t + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} \\ &\quad - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}. \end{aligned} \tag{2.46}$$

These formulae will be used in the following section to compute symmetry groups of some well-known evolution equations.

Properties of Prolonged Vector Fields

Theorem 2.39. *Suppose* $\mathbf{v}$ *and* $\mathbf{w}$ *are smooth vector fields on* $M \subset X \times U$. *Then their prolongations have the properties*

$$\mathrm{pr}^{(n)}(c\mathbf{v} + c'\mathbf{w}) = c \cdot \mathrm{pr}^{(n)}\,\mathbf{v} + c' \cdot \mathrm{pr}^{(n)}\,\mathbf{w},$$

for c, c' *constant, and*

$$\mathrm{pr}^{(n)}[\mathbf{v}, \mathbf{w}] = [\mathrm{pr}^{(n)}\,\mathbf{v}, \mathrm{pr}^{(n)}\,\mathbf{w}]. \tag{2.47}$$

Proof. The linearity is left to the reader. The Lie bracket property can be proved by direct computation using (2.38), (2.39). However, it is easier to proceed as follows. Note first that if g, h are group elements of some transformation group, then

$$\mathrm{pr}^{(n)}(g \cdot h) = \mathrm{pr}^{(n)}\,g \cdot \mathrm{pr}^{(n)}\,h,$$

and, if we use the fact that M is a subset of some Euclidean space,

$$\mathrm{pr}^{(n)}(g + h) = \mathrm{pr}^{(n)}\,g + \mathrm{pr}^{(n)}\,h,$$

where $(g + h) \cdot x = g \cdot x + h \cdot x$ by definition. Let $\mathbb{1}$ denote the identity map of M, so $\mathbb{1}^{(n)} = \mathrm{pr}^{(n)}\,\mathbb{1}$ is the identity map of $M^{(n)}$. Using the characterization of the Lie bracket in Theorem 1.33,

$$\begin{aligned}
&[\mathrm{pr}^{(n)}\,\mathbf{v}, \mathrm{pr}^{(n)}\,\mathbf{w}] \\
&\quad = \lim_{\varepsilon \to 0+} \frac{\mathrm{pr}^{(n)}\exp(-\sqrt{\varepsilon}\mathbf{w})\exp(-\sqrt{\varepsilon}\mathbf{v})\exp(\sqrt{\varepsilon}\mathbf{w})\exp(\sqrt{\varepsilon}\mathbf{v}) - \mathbb{1}^{(n)}}{\varepsilon} \\
&\quad = \mathrm{pr}^{(n)}\left\{\lim_{\varepsilon \to 0+} \frac{\exp(-\sqrt{\varepsilon}\mathbf{w})\exp(-\sqrt{\varepsilon}\mathbf{v})\exp(\sqrt{\varepsilon}\mathbf{w})\exp(\sqrt{\varepsilon}\mathbf{v}) - \mathbb{1}}{\varepsilon}\right\} \\
&\quad = \mathrm{pr}^{(n)}[\mathbf{v}, \mathbf{w}].
\end{aligned}$$

□

Corollary 2.40. *Let* Δ *be a system of differential equations of maximal rank defined over* $M \subset X \times U$. *The set of all infinitesimal symmetries of this system forms a Lie algebra of vector fields on* M. *Moreover, if this Lie algebra is finite-dimensional, the (connected component of the) symmetry group of the system is a local Lie group of transformations acting on* M.

Characteristics of Symmetries

Finally, we note an equivalent, computationally useful way of writing down the general prolongation formula (2.39). Given $\mathbf{v}$ as above, set

$$Q_\alpha(x, u^{(1)}) = \phi_\alpha(x, u) - \sum_{i=1}^{p} \xi^i(x, u)u_i^\alpha, \qquad \alpha = 1, \ldots, q; \tag{2.48}$$

the q-tuple $Q(x, u^{(1)}) = (Q_1, \ldots, Q_q)$ is referred to as the *characteristic* of the vector field $\mathbf{v}$. With this definition, (2.39) takes the form

$$\phi_\alpha^J = D_J Q_\alpha + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha. \tag{2.49}$$

Substituting into (2.38) and rearranging terms, we find

$$\mathrm{pr}^{(n)}\,\mathbf{v} = \sum_{\alpha=1}^{q} \sum_{J} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha} + \sum_{i=1}^{p} \xi^i \left\{ \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J} u_{J,i}^\alpha \frac{\partial}{\partial u_J^\alpha} \right\}.$$

The terms in brackets we recognize to be just the total derivative operators, as given by (2.36), hence

$$\mathrm{pr}^{(n)}\,\mathbf{v} = \mathrm{pr}^{(n)}\,\mathbf{v}_Q + \sum_{i=1}^{p} \xi^i D_i, \tag{2.50}$$

where, by definition

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha(x, u^{(1)}) \frac{\partial}{\partial u^\alpha}, \qquad \mathrm{pr}^{(n)}\,\mathbf{v}_Q = \sum_{\alpha=1}^{q} \sum_{J} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha}. \tag{2.51}$$

In all the above formulae, the summations extend over all multi-indices J of order $0 \leqslant \#J \leqslant n$. Of course, the two terms on the right-hand side of (2.50) are just formal algebraic expressions since they each involve $(n+1)$-st order derivative of the u's. Only when they are combined together do the terms involving the $(n+1)$-st order derivatives cancel and we have a genuine vector field on the jet space $M^{(n)}$. The importance of (2.50) will become manifest once we discuss generalized symmetries in Chapter 5.

2.4. Calculation of Symmetry Groups

Theorem 2.31, when coupled with the prolongation formulae (2.38), (2.39) provides an effective computational procedure for finding the most general (connected) symmetry group of almost any system of partial differential equations of interest. In this procedure, one lets the coefficients $\xi^i(x, u)$, $\phi_\alpha(x, u)$ of the infinitesimal generator $\mathbf{v}$ of a hypothetical one-parameter symmetry group of the system be unknown functions of x and u. The coefficients ϕ_α^J of the prolonged infinitesimal generator $\mathrm{pr}^{(n)}\,\mathbf{v}$ will be certain explicit expressions involving the partial derivatives of the coefficients ξ^i and ϕ_α with respect to both x and u. The infinitesimal criterion of invariance (2.25) will thus involve x, u and the derivatives of u with respect to x, as well as $\xi^i(x, u)$, $\phi_\alpha(x, u)$ and their partial derivatives with respect to x and u. After eliminating any dependencies among the derivatives of the u's caused by the system itself (since (2.25) need only hold on solutions of the system), we can then equate the coefficients of the remaining unconstrained partial derivatives of u to zero. This will result in a large number of elementary partial differential equations for the coefficient functions ξ^i, ϕ_α of the infinitesimal generator, called the *determining equations* for the symmetry group of the given system.

In most instances, these determining equations can be solved by elementary methods, and the general solution will determine the most general infinitesimal symmetry of the system. Corollary 2.40 assures us that the resulting system of infinitesimal generators forms a Lie algebra of symmetries; the general symmetry group itself can then be found by exponentiating the given vector fields. The process will become clearer in the following examples.

Example 2.41. *The Heat Equation.* Consider the equation for the conduction of heat in a one-dimensional rod

$$u_t = u_{xx}, \tag{2.52}$$

the thermal diffusivity having been normalized to unity. Here there are two independent variables x and t, and one dependent variable u, so $p = 2$ and $q = 1$ in our notation. The heat equation is of second order, $n = 2$, and can be identified with the linear subvariety in $X \times U^{(2)}$ determined by the vanishing of $\Delta(x, t, u^{(2)}) = u_t - u_{xx}$.

Let

$$\mathbf{v} = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \phi(x, t, u)\frac{\partial}{\partial u} \tag{2.53}$$

be a vector field on $X \times U$. We wish to determine all possible coefficient functions ξ, τ and ϕ so that the corresponding one-parameter group $\exp(\varepsilon\mathbf{v})$ is a symmetry group of the heat equation. According to Theorem 2.31, we need to know the second prolongation

$$\mathrm{pr}^{(2)}\,\mathbf{v} = \mathbf{v} + \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}$$

of $\mathbf{v}$, whose coefficients were calculated in Example 2.38. Applying $\mathrm{pr}^{(2)}\,\mathbf{v}$ to (2.52), we find the infinitesimal criterion (2.25) to be

$$\phi^t = \phi^{xx}, \tag{2.54}$$

which must be satisfied whenever $u_t = u_{xx}$. Substituting the general formulae (2.45), (2.46) into (2.54), replacing u_t by u_{xx} whenever it occurs, and equating the coefficients of the various monomials in the first and second order partial derivatives of u, we find the determining equations for the symmetry group of the heat equation to be the following:

Monomial	Coefficient	
$u_x u_{xt}$	$0 = -2\tau_u$	(a)
u_{xt}	$0 = -2\tau_x$	(b)
u_{xx}^2	$-\tau_u = -\tau_u$	(c)
$u_x^2 u_{xx}$	$0 = -\tau_{uu}$	(d)
$u_x u_{xx}$	$-\xi_u = -2\tau_{xu} - 3\xi_u$	(e)
u_{xx}	$\phi_u - \tau_t = -\tau_{xx} + \phi_u - 2\xi_x$	(f)
u_x^3	$0 = -\xi_{uu}$	(g)
u_x^2	$0 = \phi_{uu} - 2\xi_{xu}$	(h)
u_x	$-\xi_t = 2\phi_{xu} - \xi_{xx}$	(j)
1	$\phi_t = \phi_{xx}$	(k)

(As usual, subscripts indicate derivatives.) The solution of the determining equations is elementary. First, (a) and (b) require that τ be just a function of t. Then (e) shows that ξ doesn't depend on u, and (f) requires $\tau_t = 2\xi_x$, so $\xi(x, t) = \frac{1}{2}\tau_t x + \sigma(t)$, where σ is some function of t only. Next, by (h), ϕ is linear in u, so

$$\phi(x, t, u) = \beta(x, t)u + \alpha(x, t)$$

for certain functions α and β. According to (j), $\xi_t = -2\beta_x$, so β is at most quadratic in x, with

$$\beta = -\tfrac{1}{8}\tau_{tt}x^2 - \tfrac{1}{2}\sigma_t x + \rho(t).$$

Finally, the last equation (k) requires that both α and β be solutions of the heat equation,

$$\alpha_t = \alpha_{xx}, \qquad \beta_t = \beta_{xx}.$$

Using the previous form of β, we find

$$\tau_{ttt} = 0, \qquad \sigma_{tt} = 0, \qquad \rho_t = -\tfrac{1}{4}\tau_{tt}.$$

Thus τ is quadratic in t, σ is linear in t, and we can read off the formulae for ξ and ϕ directly from those of ρ, σ and τ. Since we have now satisfied all the determining equations, we conclude that the most general infinitesimal symmetry of the heat equation has coefficient functions of the form

$$\begin{aligned}
\xi &= c_1 + c_4 x + 2c_5 t + 4c_6 xt,\\
\tau &= c_2 + 2c_4 t + 4c_6 t^2,\\
\phi &= (c_3 - c_5 x - 2c_6 t - c_6 x^2)u + \alpha(x, t),
\end{aligned}$$

where $c_1, \ldots, c_6$ are arbitrary constants and $\alpha(x, t)$ an arbitrary solution of the heat equation. Thus the Lie algebra of infinitesimal symmetries of the heat equation is spanned by the six vector fields

$$\begin{aligned}
\mathbf{v}_1 &= \partial_x,\\
\mathbf{v}_2 &= \partial_t,\\
\mathbf{v}_3 &= u\partial_u,\\
\mathbf{v}_4 &= x\partial_x + 2t\partial_t,\\
\mathbf{v}_5 &= 2t\partial_x - xu\partial_u,\\
\mathbf{v}_6 &= 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u,
\end{aligned} \tag{2.55}$$

and the infinite-dimensional subalgebra

$$\mathbf{v}_\alpha = \alpha(x, t)\partial_u,$$

where α is an arbitrary solution of the heat equation. The commutation relations between these vector fields is given by the following table, the entry in row i and column j representing $[\mathbf{v}_i, \mathbf{v}_j]$:

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_\alpha$
$\mathbf{v}_1$	0	0	0	$\mathbf{v}_1$	$-\mathbf{v}_3$	$2\mathbf{v}_5$	$\mathbf{v}_{\alpha_x}$
$\mathbf{v}_2$	0	0	0	$2\mathbf{v}_2$	$2\mathbf{v}_1$	$4\mathbf{v}_4 - 2\mathbf{v}_3$	$\mathbf{v}_{\alpha_t}$
$\mathbf{v}_3$	0	0	0	0	0	0	$-\mathbf{v}_\alpha$
$\mathbf{v}_4$	$-\mathbf{v}_1$	$-2\mathbf{v}_2$	0	0	$\mathbf{v}_5$	$2\mathbf{v}_6$	$\mathbf{v}_{\alpha'}$
$\mathbf{v}_5$	$\mathbf{v}_3$	$-2\mathbf{v}_1$	0	$-\mathbf{v}_5$	0	0	$\mathbf{v}_{\alpha''}$
$\mathbf{v}_6$	$-2\mathbf{v}_5$	$2\mathbf{v}_3 - 4\mathbf{v}_4$	0	$-2\mathbf{v}_6$	0	0	$\mathbf{v}_{\alpha'''}$
$\mathbf{v}_\alpha$	$-\mathbf{v}_{\alpha_x}$	$-\mathbf{v}_{\alpha_t}$	$\mathbf{v}_\alpha$	$-\mathbf{v}_{\alpha'}$	$-\mathbf{v}_{\alpha''}$	$-\mathbf{v}_{\alpha'''}$	0

where

$$\alpha' = x\alpha_x + 2t\alpha_t, \qquad \alpha'' = 2t\alpha_x + x\alpha,$$
$$\alpha''' = 4tx\alpha_x + 4t^2\alpha_t + (x^2 + 2t)\alpha.$$

Note that since Corollary 2.40 assures us that the totality of infinitesimal symmetries must be a Lie algebra, we can conclude that if $\alpha(x, t)$ is any solution of the heat equation, so are α_x, α_t, and α', α'' and α''' as given above.

The one-parameter groups G_i generated by the $\mathbf{v}_i$ are given in the following table. The entries give the transformed point $\exp(\varepsilon\mathbf{v}_i)(x, t, u) = (\tilde{x}, \tilde{t}, \tilde{u})$:

$$\begin{aligned}
G_1&: \quad (x + \varepsilon, t, u),\\
G_2&: \quad (x, t + \varepsilon, u),\\
G_3&: \quad (x, t, e^{\varepsilon}u),\\
G_4&: \quad (e^{\varepsilon}x, e^{2\varepsilon}t, u),\\
G_5&: \quad (x + 2\varepsilon t, t, u\cdot\exp(-\varepsilon x - \varepsilon^2 t)),\\
G_6&: \quad \left(\frac{x}{1 - 4\varepsilon t}, \frac{t}{1 - 4\varepsilon t}, u\sqrt{1 - 4\varepsilon t}\exp\left\{\frac{-\varepsilon x^2}{1 - 4\varepsilon t}\right\}\right),\\
G_\alpha&: \quad (x, t, u + \varepsilon\alpha(x, t)).
\end{aligned} \tag{2.56}$$

Since each group G_i is a symmetry group, (2.14) implies that if $u = f(x, t)$ is a solution of the heat equation, so are the functions

$$\begin{aligned}
u^{(1)} &= f(x - \varepsilon, t),\\
u^{(2)} &= f(x, t - \varepsilon),\\
u^{(3)} &= e^{\varepsilon}f(x, t),\\
u^{(4)} &= f(e^{-\varepsilon}x, e^{-2\varepsilon}t),\\
u^{(5)} &= e^{-\varepsilon x + \varepsilon^2 t}f(x - 2\varepsilon t, t),\\
u^{(6)} &= \frac{1}{\sqrt{1 + 4\varepsilon t}}\exp\left\{\frac{-\varepsilon x^2}{1 + 4\varepsilon t}\right\}f\left(\frac{x}{1 + 4\varepsilon t}, \frac{t}{1 + 4\varepsilon t}\right),\\
u^{(\alpha)} &= f(x, t) + \varepsilon\alpha(x, t),
\end{aligned}$$

where ε is any real number and $\alpha(x, t)$ any other solution to the heat equation. (See Example 2.22 for a detailed discussion of how these expressions are derived from the group transformations.)

The symmetry groups G_3 and G_α thus reflect the linearity of the heat equation; we can add solutions and multiply them by constants. The groups G_1 and G_2 demonstrate the time- and space-invariance of the equation, reflecting the fact that the heat equation has constant coefficients. The well-known scaling symmetry turns up in G_4, while G_5 represents a kind of Galilean boost to a moving coordinate frame. The last group G_6 is a genuinely local group of transformations. Its appearance is far from obvious from basic physical principles, but it has the following nice consequence. If we let $u = c$ be just a constant solution, then we immediately conclude that the function

$$u = \frac{c}{\sqrt{1 + 4\varepsilon t}} \exp\left\{\frac{-\varepsilon x^2}{1 + 4\varepsilon t}\right\}$$

is a solution. In particular, if we set $c = \sqrt{\varepsilon/\pi}$ we obtain the fundamental solution to the heat equation at the point $(x_0, t_0) = (0, -1/4\varepsilon)$. To obtain the fundamental solution

$$u = \frac{1}{\sqrt{4\pi t}} \exp\left\{\frac{-x^2}{4t}\right\}$$

we need to translate this solution in t using the group G_2 (with ε replaced by $-1/4\varepsilon$).

The most general one-parameter group of symmetries is obtained by considering a general linear combination $c_1\mathbf{v}_1 + \cdots + c_6\mathbf{v}_6 + \mathbf{v}_\alpha$ of the given vector fields; the explicit formulae for the group transformations are very complicated. Alternatively, we can use (1.40), and represent an arbitrary group transformation g as the composition of transformations in the various one-parameter subgroups $G_1, \ldots, G_6, G_\alpha$. In particular, if g is near the identity, it can be represented uniquely in the form

$$g = \exp(\mathbf{v}_\alpha)\cdot\exp(\varepsilon_6\mathbf{v}_6)\cdot\ldots\cdot\exp(\varepsilon_1\mathbf{v}_1).$$

Thus the most general solution obtainable from a given solution $u = f(x, t)$ by group transformations is of the form

$$u = \frac{1}{\sqrt{1 + 4\varepsilon_6 t}} \exp\left\{\varepsilon_3 - \frac{\varepsilon_5 x + \varepsilon_6 x^2 - \varepsilon_5^2 t}{1 + 4\varepsilon_6 t}\right\}$$
$$\times f\left(\frac{e^{-\varepsilon_4}(x - 2\varepsilon_5 t)}{1 + 4\varepsilon_6 t} - \varepsilon_1, \frac{e^{-2\varepsilon_4} t}{1 + 4\varepsilon_6 t} - \varepsilon_2\right) + \alpha(x, t),$$

where $\varepsilon_1, \ldots, \varepsilon_6$ are real constants and α an arbitrary solution to the heat equation.

Example 2.42. *Burgers' Equation.* A nonlinear equation closely allied with the heat equation is Burgers' equation, which, for symmetry group purposes, is convenient to take in "potential form"

$$u_t = u_{xx} + u_x^2. \tag{2.57}$$

Note that if we differentiate this with respect to x and substitute $v = u_x$, we derive the more usual form

$$v_t = v_{xx} + 2vv_x \tag{2.58}$$

of Burgers' equation; it represents the simplest wave equation combining both dissipative and nonlinear effects, and therefore appears in a wide variety of physical applications.

The symmetry group of (2.57) will again the generated by vector fields of the form (2.53). Applying the second prolongation $\mathrm{pr}^{(2)}\,\mathbf{v}$ to (2.57), we find that ξ, τ, ϕ must satisfy the symmetry conditions

$$\phi^t = \phi^{xx} + 2u_x\phi^x, \tag{2.59}$$

where the coefficients ϕ^t, ϕ^x, ϕ^{xx} of $\mathrm{pr}^{(2)}\,\mathbf{v}$ were determined in Example 2.38, and we are allowed to substitute $u_{xx} + u_x^2$ for u_t, wherever it occurs in (2.59). We could, at this juncture, write out (2.59) in full detail and equate coefficients of the various first and second order derivatives of u to get the full determining equations, as was done in the previous example. In practice, however, it is far more expedient to tackle the solution of the symmetry equations in stages, first extracting information from the higher order derivatives appearing in them, and then using this information to simplify the prolongation formulae at the lower order stages. Working this way, "from the top down", is extremely efficient, and, even more to the point, well-nigh the only course available for higher order systems of equations, for which the full system of determining equations would take many pages to write down in full detail.

In the present case, using (2.45), (2.46) and keeping in mind that u_t has been replaced by $u_{xx} + u_x^2$, we find that the coefficients of $u_x u_{xt}$ and u_{xt} require that $\tau_u = \tau_x = 0$, so τ is a function of t only. (Note that this already simplifies the formulae for ϕ^x and ϕ^{xx} quite a bit.) The coefficient of $u_x u_{xx}$ implies that ξ doesn't depend on u, while from that of u_{xx} we find that $\tau_t = 2\xi_x$, so $\xi(x, t) = \frac{1}{2}\tau_t x + \sigma(t)$. The coefficient of u_x^2 is

$$\phi_u - \tau_t = \phi_{uu} + 2\phi_u - 2\xi_x,$$

hence

$$\phi = \alpha(x, t)e^{-u} + \beta(x, t).$$

The coefficient of u_x requires

$$\xi_t = -2\phi_{xu} - 2\phi_x = -2\beta_x,$$

hence $\beta = -\frac{1}{8}\tau_{tt}x^2 - \frac{1}{2}\sigma_t x + \rho(t)$. The remaining terms not involving any derivatives of u are just

$$\phi_t = \phi_{xx},$$

This implies that

$$\begin{aligned}
\xi &= c_1 + c_4 x + 2c_5 t + 4c_6 xt,\\
\tau &= c_2 + 2c_4 t + 4c_6 t^2,\\
\phi &= \alpha(x, t)e^{-u} + c_3 - c_5 x - 2c_6 t - c_6 x^2,
\end{aligned}$$

where $c_1, \ldots, c_6$ are arbitrary constants and $\alpha(x, t)$ is an arbitrary solution to the *heat equation*: $\alpha_t = \alpha_{xx}$. The symmetry algebra is thus generated by

$$\begin{aligned}
\mathbf{v}_1 &= \partial_x,\\
\mathbf{v}_2 &= \partial_t,\\
\mathbf{v}_3 &= \partial_u,\\
\mathbf{v}_4 &= x\partial_x + 2t\partial_t,\\
\mathbf{v}_5 &= 2t\partial_x - x\partial_u,\\
\mathbf{v}_6 &= 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)\partial_u,
\end{aligned} \tag{2.60}$$

and

$$\mathbf{v}_\alpha = \alpha(x, t)e^{-u}\partial_u,$$

where α is any solution to the heat equation.

Note the remarkable similarity between the symmetry algebra for Burgers' equation and that derived previously for the heat equation! Indeed, if we replace u by $w = e^u$, then $\mathbf{v}_1, \ldots, \mathbf{v}_\alpha$ are changed over to the corresponding vector fields (2.55) with w replacing u. Indeed, if we set $w = e^u$ in Burgers' equation, we find

$$w_t = u_t e^u, \qquad w_{xx} = (u_{xx} + u_x^2)e^u,$$

hence w satisfies the heat equation

$$w_t = w_{xx}!$$

We have rediscovered the famous Hopf–Cole transformation reducing solutions of Burgers' equation to positive solutions of the heat equation. (For the usual form (2.58) of Burgers' equation, this takes the form

$$v = (\log w)_x = w_x/w.$$

It is much more difficult to deduce this transformation from the symmetry properties of (2.58), which, as the reader may check, has only a five-parameter symmetry group.) Since we've reduced (2.57) to the heat equation, there is no further need to discuss symmetry properties here.

Example 2.43. *The Wave Equation.* Consider the wave equation

$$u_{tt} - u_{xx} - u_{yy} = 0 \tag{2.61}$$

in two spatial dimensions. A typical vector field on the space of independent and dependent variables takes the form

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial \tau} + \phi \frac{\partial}{\partial u},$$

where ξ, η, τ, ϕ depend on x, y, t, u. In this example, it is easier to work with the infinitesimal criterion of invariance in the form (2.26), which, in the present case, takes the form

$$\phi^{tt} - \phi^{xx} - \phi^{yy} = Q \cdot (u_{tt} - u_{xx} - u_{yy}) \tag{2.62}$$

in which $Q(x, y, t, u^{(2)})$ can depend on up to second order derivatives of u. The coefficient functions ϕ^{tt}, ϕ^{xx}, ϕ^{yy} of $\mathrm{pr}^{(2)}\,\mathbf{v}$ are determined by expressions similar to those in (2.46) but with extra terms involving the y-derivatives thrown in; for example,

$$\begin{aligned}
\phi^{tt} &= D_t^2(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xtt} + \eta u_{ytt} + \tau u_{ttt} \\
&= D_t^2 \phi - u_x D_t^2 \xi - u_y D_t^2 \eta - u_t D_t^2 \tau - 2u_{xt} D_t \xi - 2u_{yt} D_t \eta - 2u_{tt} D_t \tau,
\end{aligned}$$

etc.

To solve (2.62), we look first at the terms involving the mixed second order partial derivatives of u, namely u_{xy}, u_{xt} and u_{yt}, each of which occurs linearly on the left-hand side. This requires that ξ, η and τ do not depend on u, and, moreover

$$\xi_y + \eta_x = 0, \qquad \xi_t - \tau_x = 0, \qquad \eta_t - \tau_y = 0. \tag{2.63}$$

The coefficients of the remaining second order derivatives of u yield the relations

$$\phi_u - 2\tau_t = \phi_u - 2\xi_x = \phi_u - 2\eta_y = Q,$$

hence

$$\tau_t = \xi_x = \eta_y. \tag{2.64}$$

The equations (2.63), (2.64) are the equations for an infinitesimal conformal transformation on $\mathbb{R}^3$ with Lorentz metric $dt^2 - dx^2 - dy^2$, cf. Exercise 1.30. It is not difficult to show that ξ, η, τ are quadratic polynomials of x, y, t of the form

$$\begin{aligned}
\xi &= c_1 + c_4 x - c_5 y + c_6 t + c_8(x^2 - y^2 + t^2) + 2c_9 xy + 2c_{10} xt, \\
\eta &= c_2 + c_5 x + c_4 y + c_7 t + 2c_8 xy + c_9(-x^2 + y^2 + t^2) + 2c_{10} yt, \\
\tau &= c_3 + c_6 x + c_7 y + c_4 t + 2c_8 xt + 2c_9 yt + c_{10}(x^2 + y^2 + t^2),
\end{aligned}$$