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The renormalization group method in statistical hydrodynamics

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This paper gives a first principles formulation of a renormalization group (RG) method appropriate to study of turbulence in incompressible fluids governed by Navier–Stokes equations. The present method is a momentum-shell RG of Kadanoff–Wilson type based upon the Martin–Siggia–Rose (MSR) field-theory formulation of stochastic dynamics. A simple set of diagrammatic rules are developed which are exact within perturbation theory (unlike the well-known Ma–Mazenko prescriptions). It is also shown that the claim of Yakhot and Orszag (1986) is false that higher-order terms are irrelevant in the ϵ expansion RG for randomly forced Navier–Stokes (RFNS) with power-law force spectrum $\hat{F}(k)=D_0k^{-d+(4-\epsilon)}$. In fact, as a consequence of Galilei covariance, there are an infinite number of higher-order nonlinear terms marginal by power counting in the RG analysis of the power-law RFNS, even when $\epsilon \ll 4$. The difficulty does not occur in the Forster–Nelson–Stephen (FNS) RG analysis of thermal fluctuations in an equilibrium NS fluid, which justifies a linear regression law for $d>2$. On the other hand, the problem occurs also at the nontrivial fixed point in the FNS Model A, or its Burgers analog, when $d<2$. The marginal terms can still be present at the strong-coupling fixed point in true NS turbulence. If so, infinitely many fixed points may exist in turbulence and be associated to a somewhat surprising phenomenon: nonuniversality of the inertial-range scaling laws depending upon the dissipation-range dynamics.

I. INTRODUCTION

In this work we will consider the renormalization-group (RG) analysis of several problems in hydrodynamic statistics. These include incompressible fluid turbulence at high Reynolds number, as well as some model problems in which the Navier–Stokes equations are supplemented with stochastic force terms. To be precise, these latter models are defined by a dynamical equation of the form

$$\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla p+\nu_0\Delta\mathbf{v}+\mathbf{f}, \tag{1}$$

in which $\mathbf{f}$ is a Gaussian random force with mean zero and covariance

$$\langle f_i(\mathbf{r},t)f_j(\mathbf{r}',t')\rangle=2P_{ij}(\nabla_\mathbf{r})F(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{2}$$

In this expression $P_{ij}(\nabla_\mathbf{r})$ is the projection onto solenoidal vector fields, required to maintain the incompressibility constraint

$$\nabla\cdot\mathbf{v}=0. \tag{3}$$

We refer to this general class of models as "randomly forced Navier–Stokes" (RFNS). The Fourier transform of F or *force spectrum*, $\hat{F}(\mathbf{k})$, is necessarily non-negative.

A case which has been much studied in the literature[1–4] is the power-law force spectrum,

$$\hat{F}(\mathbf{k})=D_0k^{-d+(4-\epsilon)}, \tag{4}$$

where the exponent is written in the given form for later convenience. The choice $\epsilon=2-d$ was studied by Forster, Nelson, and Stephens (FNS) in Ref. 1 as "Model A." If the noise strength is chosen by the fluctuation-dissipation relation $D_0=(\nu_0/\rho)k_BT$, then the force in Model A represents molecular noise and the equation is a realistic model of fluctuations in an equilibrium fluid at absolute temperature T. The general case for arbitrary ϵ was first proposed for study by DeDominicis and Martin (DM) in Ref. 2, who realized that the model has a weak-coupling fixed point for small ϵ in any dimension d, which may be studied perturbatively. For $\epsilon=4$ the constant D_0 has units of energy dissipation per mass, and a Kolmogorov-style dimensional argument gives a 5/3-energy spectral law. More generally, as pointed out in Ref. 3, a dimensional argument gives the energy spectrum

$$E(k)\sim D_0^{2/3}k^{1-2\epsilon/3}. \tag{5}$$

This was also "derived" with RG methods by DM in Ref. 2, but that argument used RG only to justify the existence of the limit $\nu_0\rightarrow0$, whereas the result Eq. (5) requires also that the limit $L\rightarrow+\infty$ exist, with L a large-length scale cutoff for the force F. As noted by DM themselves, such a cutoff is probably required when $\epsilon>3$ and Eq. (5) then receives an L-dependent correction. More recently, Yakhot and Orszag (YO) have considered the power-law RFNS system in connection with an ambitious "correspondence principle" which they have proposed between the RFNS with $\epsilon=4$ and true turbulence in a long inertial range.[4] The latter can be considered for convenience to be also a model of the RFNS class, but with a force spectrum $\hat{F}(\mathbf{k})\equiv0$ when $k>1/L$. More appropriately, "true turbulence" is taken to mean stationary random flow at high Reynolds number produced by driving at long wavelengths from a mean shear, forced flow past a boundary, etc., or even the transient state in free decay. YO attempted to study true turbulence by assuming the correspondence principle and then extrapolating to $\epsilon=4$ the results of a perturbative RG study of the power-law RFNS model for small ϵ. On this basis they predicted various scaling laws, such as the Kolmogorov energy spectrum, including some constant factors such as the Kolmogorov constant, the Obukhov–Corrsin constant and the turbulent Prandtl number. By an associated technique, they attempted to derive some familiar turbulence models, such as the Smagorinsky subgrid eddy viscosity, and two-equation $K-\epsilon$ models of the large scales, including the phenomenological constants.

The new contributions of our work here are as follows: In Sec. II we formulate a Wilson-type RG method for problems of hydrodynamic statistics using MSR field theory. We emphasize how RG gives a controlled approximation in favorable cases, using crucially the "irrelevance" of neglected terms. In Sec. III, we show that, contrary to the claim of YO, the higher-order nonlinear terms generated in their RG analysis are not irrelevant but marginal by power-counting. Because of this, neglected terms have not been shown by their analysis to be small *even for small* ϵ. The problem is traced to the fact that the RG is in a class of Galilei covariant theories, and it is shown not to occur in the FNS analysis of fluctuation dynamics for equilibrium NS fluids. In Sec. IV, we shall examine in detail the possibilities of using the ϵ-expansion RG for turbulence modeling. We show that a "good RG" procedure can be devised which avoids the previous problems in the YO method for $\epsilon\ll 4$. On this basis we attempt an explanation of why the YO theory has been reasonably successful in making predictions of turbulence parameters at moderate Reynolds number. (However, any sound and consistent version of the YO analysis would require considerable reformulation, e.g., use of a Lagrangian representation when $\epsilon>3$.) Finally, we examine the extent to which the YO predictions require the use of RG-type methods or get any justification from them. In Sec. V we shall present our own view on the use of the RG method in turbulence theory. We argue that its basic use is to show how scaling laws can emerge from precise limit hypotheses rather than from *ad hoc* phenomenological models. We also show that the appearance of infinitely many marginal variables may be an unavoidable fact for the strong-coupling case of true turbulence and connected with a phenomenon counter to the traditional Kolmogorov picture: nonuniversal scaling in the inertial range depending upon the precise dynamics in the dissipation range.

To keep the discussion clear, we will use the following terminology to describe the RFNS models introduced previously:

$$\text{Model P:}\quad F(k)=D_0\cdot k^{-d+(4-\epsilon)}, \tag{6}$$

which is the power-law force spectrum case, and

$$\text{Model T:}\quad F(k)=0 \quad \text{for } k>1/L, \tag{7}$$

which is true turbulence, or at least a rather realistic model of turbulence. Model P was studied by DM and YO, and it corresponds to Model A of FNS when $\epsilon=2-d$.

II. RENORMALIZATION GROUP AND HYDRODYNAMIC STATISTICS

A. Why use renormalization group?

We recall in this section the rationale of the RG procedure, in the context of statistical hydrodynamics. There are actually two somewhat related, but distinct, RG methods—the original field-theoretic RG and the Wilson–Kadanoff RG. The latter was first applied to hydrodynamic problems by FNS in Ref. 1, by Fournier and Frisch,[3] and later by YO.[4] The field-theoretic RG was used in this class of problems first by DM,[2] and later by others. A particularly clear formulation is given in a recent paper of Teodorovich.[5] The field-theoretic RG has mostly been employed as a perturbative method (although it is not restricted to weak-coupling problems in principle.) The Wilson–Kadanoff approach has explicitly a nonperturbative basis and range of application. The connection between them is that the field-theoretic RG flow can be interpreted as the Wilson–Kadanoff map—which acts in an infinite-dimensional space of effective theories—when it is restricted to a low-dimensional "inertial manifold" parametrized by canonical field theories with a finite number of interaction terms. The connection between the "old" and "new" RG methods is further discussed by DiCastro and Jona-Lasinio in Ref. 6.

Here we shall mainly discuss the Wilson method, which is reviewed for general applications in Refs. 7 and 8. We have recently given an extensive discussion for turbulence in the context of shell models,[9] so that we may be relatively brief here. Wilson's basic idea was to study large-scale effective theories by a partial elimination of short-wavelength degrees of freedom with a subsequent rescaling of parameters. The rescaling is always chosen to keep the high-wave number cutoff Λ fixed. The rationale of this two-step procedure is that the change in effective theories under change in scale can be visualized, through the rescaling, as a "dynamical flow," in a space of theories with fixed cutoff Λ. Even if it is possible in principle to calculate one step in this flow, very complicated, nontrivial behavior can emerge asymptotically in the "long-time limit," just as for any dynamical flow. Therefore, it is best to examine the global character of "flow" to understand the large-scale physics. In particular, scaling behavior at large lengths can be identified with "fixed points" of the flow and universality classes with their "basins of attraction." The rescaling also plays an important role in control of errors, as we discuss further below. As emphasized by Jona-Lasinio,[6] the rescaling factors are entirely analogous to the normalization factor $1/\sqrt{N}$ in a central limit theorem, chosen just so that a limit exists. However, it is important to keep in mind that they are only a clever mathematical device to obtain a limit and that real interest is in the effective theories without rescaling, so that, for comparison of the RG results with the physics one must "undo" the rescalings.

B. The RG transformation on subgrid dynamics

For static statistical problems, the RG acts in a space of probability distributions of low-wave number degrees of freedom or effective distributions. In dynamical problems the RG acts in a space of effective dynamics of the low-wave number variables, or "subgrid models" in the language of turbulence theory. These are stochastic Langevin-type dynamics, or stochastic processes, characterized by their distributions on the space of *histories*. They are best described in terms of a path-integral representation of the probability generating functionals, using a so-called MSR action.[10–12] For our models this takes the form

$$Z[\boldsymbol{\eta},\hat{\boldsymbol{\eta}}]=\int \mathscr{D}\mathbf{v}\mathscr{D}\hat{\mathbf{v}}e^{S[\mathbf{v},\hat{\mathbf{v}}]+i\langle\boldsymbol{\eta},\mathbf{v}\rangle+i\langle\hat{\boldsymbol{\eta}},\hat{\mathbf{v}}\rangle}, \tag{8}$$

with

$$S[\mathbf{v},\hat{\mathbf{v}}]=-i\int dt\int d^d\mathbf{r}\ \hat{\mathbf{v}}\cdot[\partial_t\mathbf{v}-\nu_0\Delta\mathbf{v}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}]$$
$$-\int dt\int d^d\mathbf{r}\int d^d\mathbf{r}'\ \hat{\mathbf{v}}(\mathbf{r},t)F(\mathbf{r}-\mathbf{r}')\hat{\mathbf{v}}(\mathbf{r}',t). \tag{9}$$

This formula is well defined if the fields are Fourier truncated at wave numbers Λ and the time integrals are approximated by a discretization. (Observe the regularizing role of the forcing term associated to F.) Functional differentiation of $Z[\boldsymbol{\eta},\hat{\boldsymbol{\eta}}]$ with respect to $\boldsymbol{\eta}$, $\hat{\boldsymbol{\eta}}$ yields the statistical correlation and response functions.

To define the RG transformation, we make a decomposition of the velocity field $\mathbf{v}$ and its "response field" $\hat{\mathbf{v}}$ into low-wave number and high-wave number components, as

$$\mathbf{v}(\mathbf{r},t)=\mathbf{v}^<(\mathbf{r},t)+\mathbf{v}^>(\mathbf{r},t),\quad \hat{\mathbf{v}}(\mathbf{r},t)=\hat{\mathbf{v}}^<(\mathbf{r},t)+\hat{\mathbf{v}}^>(\mathbf{r},t), \tag{10}$$

where "$<$" indicates a low-pass filtered field containing only modes with wave numbers $<e^{-r}\Lambda$ and "$>$" represents a high-pass filtered field containing wave numbers in the complementary range $[e^{-r}\Lambda,\Lambda]$. Here $r>0$ is a disposable parameter, although a judicious choice may turn out to be important for the analysis (as we see later). Thereafter, the effective process of the $\mathbf{v}^<$, $\hat{\mathbf{v}}^<$ modes is defined by simply integrating out the $\mathbf{v}^>$, $\hat{\mathbf{v}}^>$ variables in the above formula, which gives a new formula of the type

$$Z[\boldsymbol{\eta}^<,\hat{\boldsymbol{\eta}}^<]=\int \mathscr{D}\mathbf{v}^<\mathscr{D}\hat{\mathbf{v}}^<e^{S^{\text{eff}}[\mathbf{v}^<,\hat{\mathbf{v}}^<]+i\langle\boldsymbol{\eta}^<,\mathbf{v}^<\rangle+i\langle\hat{\boldsymbol{\eta}}^<,\hat{\mathbf{v}}^<\rangle}, \tag{11}$$

but where now $S^{\text{eff}}[\mathbf{v}^<,\hat{\mathbf{v}}^<]$ is no longer given by expression Eq. (9), but contains infinitely many higher-order nonlinear terms than just the cubic nonlinearity in Eq. (9). The second step of the RG transformation is a rescaling operation

$$\mathbf{v}^<(\mathbf{r},t)\to e^{xr}\cdot\mathbf{v}(e^{-r}\mathbf{r},e^{-zr}t), \tag{12}$$

and

$$\hat{\mathbf{v}}^<(\mathbf{r},t)\to e^{\hat{x}r}\cdot\hat{\mathbf{v}}(e^{-r}\mathbf{r},e^{-zr}t), \tag{13}$$

which redefines variables $\mathbf{v}$, $\hat{\mathbf{v}}$ whose distribution of histories is now given by the action

$$S'[\mathbf{v},\hat{\mathbf{v}}]=S^{\text{eff}}[e^{xr}\mathbf{v}(e^{-r}\cdot,e^{-zr}\cdot),e^{\hat{x}r}\hat{\mathbf{v}}(e^{-r}\cdot,e^{-zr}\cdot)]. \tag{14}$$

The transformation $\mathscr{R}:S\to S'$ gives the RG flow map in the space of theories. Note that the rescaling of space was just that necessary to bring back the UV cutoff wave number to Λ. The other scaling exponents, $x,\hat{x},z$, must be chosen according to principles that depend upon the application at hand. The single constraint that is imposed in general is

$$x+\hat{x}=-d, \tag{15}$$

which ensures that the $\hat{\mathbf{v}}\partial_t\mathbf{v}$ term in the action is invariant. This is natural, since the MSR action of any dynamics first order in time should contain such a term. We discuss the other constraints on the rescaling factors for our models in the following section.

The RG procedure we have defined is an exact one, which involves no approximations and is not limited to a weak-coupling regime. In Refs. 1 and 4 a different dynamic RG procedure was employed, which is due to Mazenko and Ma.[13] This approach involves approximations of unknown validity and is strictly perturbative, limited to weak coupling. The Mazenko–Ma elimination step involves decoupling the dynamics into separate equations for $\mathbf{v}^<$ and $\mathbf{v}^>$. The equation for $\mathbf{v}^>$ is solved perturbatively in the nonlinearity in terms of $\mathbf{v}^<$ and $\mathbf{f}^>$. This solution is then used to eliminate $\mathbf{v}^>$ everywhere in the equation for $\mathbf{v}^<$ and subsequently this equation is averaged over the known statistics of $\mathbf{f}^>$, *assuming independence from* $\mathbf{v}^<$, to give the effective dynamics of the variables $\mathbf{v}^<$. However, this is an uncontrolled approximation, since the $\mathbf{v}^<$ variables get a statistical dependence on the forces $\mathbf{f}^>$ through their coupling to the $\mathbf{v}^>$ variables and a conditional average over the $\mathbf{f}^>$ forces with $\mathbf{v}^<$ fixed will change the distribution of the forces $\mathbf{f}^>$ in an unknown way. Nevertheless, we will simply remark here that the exact RG procedure we have described and the Mazenko–Ma approximate procedure lead to essentially identical results in our models at the second order in perturbation theory. The elimination step in our RG method can be easily performed perturbatively and exactly the terms appear in our MSR effective action that would be associated to the effective dynamics in the Mazenko–Ma procedure, except that there are additional terms proportional to $(\hat{\mathbf{v}}^<)^2$ and $(\hat{\mathbf{v}}^<)^2(\mathbf{v}^<)^2$. These represent noise terms in the effective dynamics of $\mathbf{v}^<$, the second one "multiplicative" with a strength proportional to $(\mathbf{v}^<)^2$. Nevertheless, when the rescaling factors are defined for our models it turns out that the multiplicative noise is irrelevant by power counting and does not substantially change the analysis. Therefore, the results of the Mazenko–Ma procedure are essentially recovered, at least at second-order in perturbation theory.

A general action in the space we consider has the form

$$S[\mathbf{v},\hat{\mathbf{v}}]=-i\int d^{d+1}x\ \hat{\mathbf{v}}(x)(\partial_t\mathbf{v}(x)-\mathbf{K}[x;\mathbf{v}])$$
$$+\sum_{p\geqslant 2}\frac{(-i)^p}{p!}\int d^{d+1}x_1\cdots\int d^{d+1}x_p$$
$$\times D_{(p)}^{i_1\cdots i_p}[x_1,\ldots,x_p;\mathbf{v}]\hat{v}_{i_1}(x_1)\cdots\hat{v}_{i_p}(x_p), \tag{16}$$

where $x=(\mathbf{r},t)$ is a space–time point and we have separated the action into parts linear and nonlinear in $\hat{\mathbf{v}}$, with $\mathbf{K}[\mathbf{v}]$ and $\mathbf{D}_{(p)}[\mathbf{v}]$ arbitrary functionals of $\mathbf{v}$. [These functionals, must, however, be "nonanticipating" or "causal," i.e., they can depend only upon the values of $\mathbf{v}(x)$ in the "past" of their largest time argument.] One can introduce, depending upon a history $\mathbf{v}$, a zero-mean random force $\mathbf{f}'[\mathbf{v}]$ by its generating functional

$$\int \mathscr{D}\mathbf{f}'[\mathbf{v}]e^{-i\langle\hat{\mathbf{v}},\mathbf{f}'[\mathbf{v}]\rangle}$$

$$=\exp\left\{\sum_{p\geq 2}\frac{(-i)^p}{p!}\int d^{d+1}x_1\cdots\int d^{d+1}x_p \times D_{(p)}^{i_1\cdots i_p}[x_1,\ldots,x_p;\mathbf{v}]\hat{v}_{i_1}(x_1)\cdots\hat{v}_{i_p}(x_p)\right\}, \tag{17}$$

which is non-Gaussian if $\mathbf{D}_{(p)}\neq 0$ for any $p\geqslant 3$. By integrating over the response variable $\hat{\mathbf{v}}$ it is easy to see that the generating functional for this action has the form

$$Z[\boldsymbol{\eta},\hat{\boldsymbol{\eta}}]=\int\mathscr{D}\mathbf{v}\int\mathscr{D}\mathbf{f}'[\mathbf{v}]\Delta[\partial_t\mathbf{v}-\mathbf{K}[\mathbf{v}]-\hat{\boldsymbol{\eta}}-\mathbf{f}'[\mathbf{v}]]e^{i\langle\boldsymbol{\eta},\mathbf{v}\rangle}, \tag{18}$$

where $\Delta[\cdots]$ denotes a delta functional for the path integration. Therefore, the probability measure corresponding to this generating functional is supported on the solutions of the "generalized Langevin dynamics"

$$\partial_t\mathbf{v}(x)=\mathbf{K}[x;\mathbf{v}]+\mathbf{f}[x;\mathbf{v}], \tag{19}$$

where $\mathbf{f}\equiv\hat{\boldsymbol{\eta}}+\mathbf{f}'$ now represents the total force. However, this equation is somewhat symbolic, since the dynamical field $\mathbf{K}$ and the distribution of the random noise $\mathbf{f}'$ may depend upon the entire past history of the velocity field.

C. RG Classification of the interactions

It is often convenient to use a compact representation for the action,

$$S[\Phi]=\tfrac{1}{2}\Phi(1)i\sigma^{(2)}\partial_{t_1}\Phi(1)+\sum_{k\geq 2}\frac{1}{k!}\gamma_k(12\cdots k)\Phi(1)\cdots\Phi(k), \tag{20}$$

where a "doublet field" has been defined as $\Phi=(\mathbf{v},i\hat{\mathbf{v}})^{\perp}$, and (a) indicates indices $(\mathbf{r}_a,t_a,\epsilon_a)$ with a summation convention understood for repeated indices. An infinite number of "coupling constants" g_0^α are thereby defined, which may, for example, be taken to be Fourier coefficients of the interaction potentials γ_k. The first step in the RG transformation defines a new set of "effective couplings"

$$g_{\text{eff}}^\alpha=G^\alpha[g_0], \tag{21}$$

which are some complicated nonlinear functions of the original g_0's. It is convenient to separate out the linear part, which is just the term g_0^α which comes from replacing all the Φ's in the original interaction term by $\Phi^<$'s. Therefore, generally

$$g_{\text{eff}}^\alpha=g_0^\alpha+N^\alpha[g_0], \tag{22}$$

where N^α denotes the nonlinear part. The second step of the RG transformation consists of the rescaling, which gives

$$g_1^\alpha\equiv\mathscr{R}^\alpha[g_0]=e^{y^\alpha r}(g_0^\alpha+N^\alpha[g_0]). \tag{23}$$

The exponent y^α will generally be some linear combination of $x,\hat{x},z,d$, and y. There is an important distinction between couplings (and associated interaction terms in the action), depending upon the sign of y^α. If $y^\alpha>0$, then the variable is said to be "relevant by power counting." Likewise, if $y^\alpha<0$, then the variable is "irrelevant by power counting" and if $y^\alpha=0$ it is "marginal by power counting." Obviously variables with $y^\alpha<0$ will tend to get damped out under RG iteration, so the epithet "irrelevant" seems appropriate. This idea will be justified in detail in the the following section.

The qualification "by power counting" is added because the above distinctions only strictly apply at the Gaussian fixed point where all nonlinear couplings vanish. In general, at any fixed point of the RG transformation, defined by

$$g_*^\alpha=\mathscr{R}^\alpha[g_*], \tag{24}$$

one can define a "linearized transformation" corresponding to small departures $\bar{g}^\alpha=g^\alpha-g_*^\alpha$ from the fixed point, as

$$\bar{g}_1^\alpha=\mathscr{T}^{\alpha\beta}\bar{g}_0^\beta, \tag{25}$$

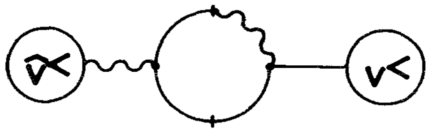
with

$$\mathscr{T}^{\alpha\beta}=\frac{\partial\mathscr{R}^\alpha}{\partial g^\beta}[g_*]. \tag{26}$$

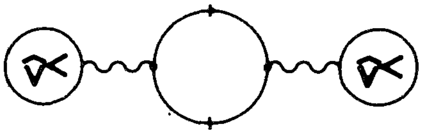
The classification into "relevant," "irrelevant," and "marginal" variables must be made separately at each fixed point g_* and corresponds to the classification into eigenvectors of $\mathscr{T}^{\alpha\beta}$ with eigenvalues $e^{y^\alpha r}$ and $y^\alpha>0$, $y^\alpha<0$, or $y^\alpha=0$, respectively.

D. Diagrammatic representation of the perturbative RG

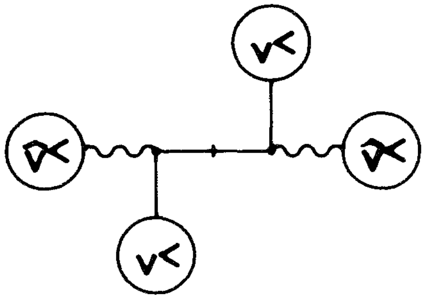
There is an intuitively appealing diagrammatic representation of the terms appearing in the perturbative evaluation of the elimination integral in our RG. All the terms are obtained by calculating "vacuum graphs" with usual Wyld–Feynman rules for the averages over $\hat{\mathbf{v}}^>$, $\mathbf{v}^>$. These correspond to all graphs with arbitrarily many insertions of $\hat{\mathbf{v}}^<$, $\mathbf{v}^<$ and with all internal lines carrying a "hard" wave number, i.e., in the interval $[e^{-r}\Lambda,\Lambda]$, which we indicate with a slash mark. At second-order order in the coupling they are the following:



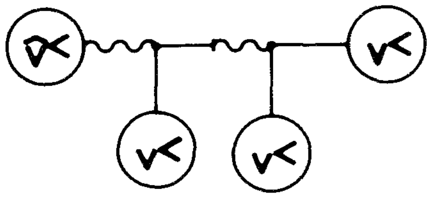
which gives the linear damping on $\mathbf{v}^<$ due to the eliminated high-wave number modes,



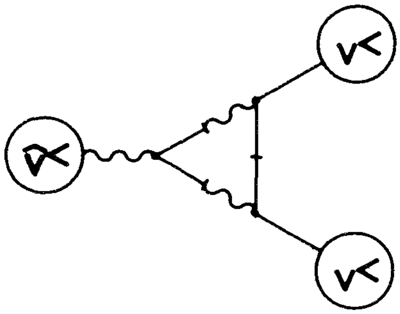
and



which represent the generated noise terms, and



which is the lowest order cubic nonlinearity $\sim \hat{\mathbf{v}}^{<}(\mathbf{v}^{<})^3$ in the effective action. Note that at third order there will also be loop contributions to the original nonlinearity $\sim \hat{\mathbf{v}}^{<}(\mathbf{v}^{<})^2$, i.e., "vertex corrections," as



It is the new cubic term $\sim(\mathbf{v}^{<})^3$ and the higher-order nonlinear terms generated in the effective dynamics by the elimination procedure which are the subject of the following section.

III. GALILEI COVARIANCE AND MARGINAL TERMS

A. Prescriptions for the RG scaling factors

We are now going to finish the definition of our RG transformation for the RFNS models. For Model P we will choose scaling exponent $\hat{x}$ so that D_0 remains invariant. It is easy to check that this requires that

$$\hat{x}=-\tfrac{1}{2}(z+d+y). \tag{27}$$

Recalling that $x+\hat{x}=-d$ was also imposed gives

$$x=\tfrac{1}{2}(z-d+y). \tag{28}$$

The rationale of these prescriptions is that one wishes to consider the RG as acting in a space with fixed external forcing. In the case of Model T the prescription to leave the force covariance invariant, i.e., to leave the term $\hat{\mathbf{v}}F\hat{\mathbf{v}}$ in the action invariant, seems appropriate. In fact, it corresponds to keeping the mean dissipation fixed, since they are related by

$$\bar{\epsilon}=F(0) \tag{29}$$

(see Novikov[14]). It turns out that this is accomplished by the same rescalings as above with $y=d$, or $\epsilon=4$. Note *that* a power-law force covariance transforms under the space rescaling as $F\rightarrow e^{-(d-y)r}F$, so that for Model P the prescription to keep D_0 fixed does *not* coincide with the prescription to keep mean-dissipation $\bar{\epsilon}$ fixed, unless $\epsilon=4$. [In the latter case, however, the mean dissipation is a logarithmically divergent function of the uv cutoff by Eq. (29): $\bar{\epsilon}=(\text{const.})\log\Lambda$.] The choice to keep D_0 fixed was used also by Yakhot and Orszag in their analysis of Model P[4] and by FNS in their analysis of Model A.[1]

To fix all three scaling exponents, x, $\hat{x}$, and z, one additional condition is required. It seems natural to impose also that the strength of the cubic term $\lambda_0\hat{\mathbf{v}}(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ remain fixed at $\lambda_0\equiv 1$, which corresponds to having the RG act in a space of theories covariant under the usual Galilei transformation

$$\mathbf{v}'(\mathbf{r},t)=\mathbf{v}(\mathbf{r}-\mathbf{u}t,t)+\mathbf{u}. \tag{30}$$

For invariance of the action, this must be supplemented with a transformation law for the response field

$$\hat{\mathbf{v}}'(\mathbf{r},t)=\hat{\mathbf{v}}(\mathbf{r}-\mathbf{u}t,t). \tag{31}$$

Imposing the condition $\lambda\equiv 1$, yields the final relation

$$\hat{x}+2x=-(z+d-1). \tag{32}$$

Solving now for the rescaling factors in Model P gives

$$z=2-\frac{\epsilon}{3}, \tag{33}$$

$$x=-1+\frac{\epsilon}{3}, \tag{34}$$

$$\hat{x}=-(d-1)-\frac{\epsilon}{3}. \tag{35}$$

For Model T we make the same choice with $\epsilon=4$. Note that our prescriptions are the same as those of YO in Ref. 4. However, as we shall emphasize below, the RG for Model A of FNS was *not* constructed to preserve Galilei symmetry.

B. Fixed points and Lagrangian histories

There is an important point that must be stressed here. The exponents x and z are scaling exponents of the velocity field $\mathbf{v}$ and of time t, respectively, and so have physical significance. Yet we seem to have obtained precise values for these quantities simply by making definitions! For example, for $\epsilon=4$, we get Kolmogorov values of the velocity exponent $x=1/3$ and dynamic scaling exponent $z=2/3$. Of course, it is not possible to get physical results by simply making definitions. The solution to this puzzle is that the previous values only correspond to physical scaling behavior *if the RG we have defined in fact has a fixed point*. This is a nontrivial fact and so far we have done nothing to demonstrate it. Following earlier work of DM,[2] YO in Ref. 4 define a dimensionless effective coupling (which they denote $\bar{\lambda}_0$)

$$g_0^2\equiv\frac{D_0\lambda_0^2}{\nu_0^3\Lambda^{\epsilon}}, \tag{36}$$

whose RG recursion at second-order is of the form

$$g_1 = e^{\epsilon r} g_0 - A_{(r)} \cdot g_0^3. \tag{37}$$

This recursion has a fixed point $g_* = O(\epsilon^{1/2})$. [YO considered the limit $r \to 0$, but, since $A_{(r)} \propto r$, the result is the same.] To really establish the existence of a fixed point requires careful consideration of higher-order nonlinearities and, in fact, a non-perturbative analysis. We will argue below perturbatively in ϵ that there is, indeed, a fixed point for $\epsilon \ll 4$.

However, we do *not* expect the RG we have defined above to have a fixed point when $\epsilon > 3$. The reason is that we have formulated our RG in terms of Eulerian velocity histories, whereas $z = 2 - (\epsilon/3)$ is the dynamic scaling exponent z_L that would be expected only for a Lagrangian time correlation when $\epsilon > 3$. Since the energy will be mostly contained in the infrared for $\epsilon > 3$, there should be a random sweeping effect and the Eulerian dynamical scaling exponent will presumably "stick" at $z_E = 1$ for all $\epsilon > 3$ (see Tennekes[15]). Therefore, at the very least, we believe our RG must be reformulated in terms of Lagrangian histories if it is to have a nontrivial fixed point when $\epsilon > 3$. This will be discussed further in Sec. IV B.

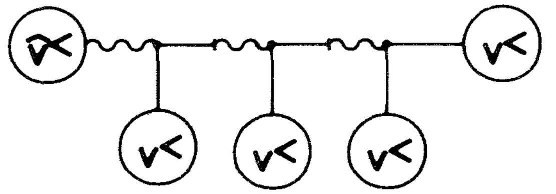
C. Power-counting analysis of higher nonlinearities

Our RG is now completely defined for the RFNS models, and we can turn to the problem of power counting. We are going to show now that the claim of YO in Sec. 2.5 of Ref. 4 is incorrect that the quartic nonlinearity $\sim \hat{\mathbf{v}}\mathbf{v}^3$ generated at $O(g^2)$ is irrelevant in Model P for $\epsilon < 4$. Their claim is based upon a simple algebraic mistake and, in fact, the term is marginal by power counting. This can be checked in detail for the stated prescriptions, but it is most illuminating to see that it results from the requirement of Galilei covariance alone. Written out in detail, the cubic term has the explicit analytic form

$$\int d^{d+1}x \; \hat{v}_l^{<}(x)[G_0 P^{>}((\mathbf{v}^{<} \cdot \nabla) v_m^{<})](x) \nabla_m v_l^{<}(x), \tag{38}$$

where $G_0 = (\partial_t + \bar{\mathbf{v}} \cdot \nabla - \nu_0 \Delta)^{-1}$ is the Galilei covariant form of the bare response operator (with $\bar{\mathbf{v}}$ the space average of $\mathbf{v}$) and $P^{>}$ is the spectral projection onto the eliminated wave number range $[e^{-r}\Lambda, \Lambda]$. (Its physical interpretation will be commented upon in Sec. IV B.) The projection operator is dimensionless, while the response operator scales like a time. Note, however, that by Galilei covariance ∂_t and $\mathbf{v}^{<} \cdot \nabla$ must rescale in the same way. Therefore, $\mathbf{v}^{<} \cdot \nabla$ must scale as an inverse time, while the bare response function G_0 must scale as a time, *so that their rescaling factors cancel identically*. The terms which are left are exactly the same as those which appear in the original cubic nonlinearity, so that the total rescaling factor is exactly 1. Therefore, the term is marginal by power counting.

It is not hard to see that there will be infinitely many such marginal terms. In fact, a new such term can be obtained by replacing any $\mathbf{v}$ with a $G_0 P^{>}[(\mathbf{v} \cdot \nabla)\mathbf{v}]$, which scales in the same way. For example, at third order there is the following quintic nonlinearity:



In general, there will be terms at order $O(g^n)$ which go as $\sim \hat{v} G_0^{n-1} \nabla^n v^{n+1}$ and which are all marginal by power counting.

D. The problem of marginal variables

This has a destructive effect on the claims of YO. In fact, it cannot be concluded from their analysis that the new cubic term $\sim v^3$ in the "fixed-point dynamics" has a coefficient $O(\epsilon)$. It is true that if this term is initially absent, then it is first generated at order $O(g^2)$, which is $O(\epsilon)$ when $g = O(\epsilon^{1/2})$. However, if the recursion is iterated $O(1/\epsilon)$ number of times, then the coefficient, call it $g^{(3)}$, can become $O(1)$! The fixed point is obtained only by iterating infinitely often, so that the problem is real. It must be appreciated that control of the bounds on neglected terms depends crucially on the "irrelevancy" of those terms, as strongly emphasized by Wilson in Sec. 5.1 of Ref. 7 and Sec. V of Ref. 8. On the other hand, "marginal" variables are the worst nightmare of any RG analysis and call for a more complicated higher-order analysis. This can be appreciated by considering the following formal solution for the RG recursion Eq. (23):

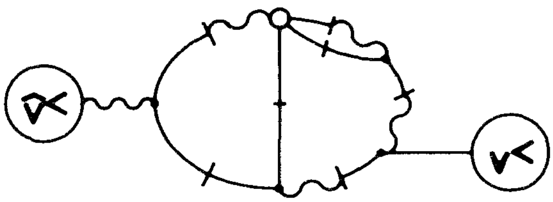
$$g_k^{\alpha} = e^{y^{\alpha} \cdot kr} g_0^{\alpha} + \sum_{l=0}^{k-1} e^{y^{\alpha}(k-l)r} N^{\alpha}[g_l], \tag{39}$$

which is equivalent to Wilson's Eq. (V.10) in Ref. 8. The interest of this formal solution is that, when $y^{\alpha} < 0$, the terms in the sum are all weighted by exponentially small damping factors and the sum is dominated by the terms with $l \approx k$. Therefore, it can be deduced from this equation by induction that an irrelevant variable g^{α}, if small initially, will remain small forever. This is how the bounds on neglected "irrelevant" variables are obtained. (Notice also that under RG iteration an irrelevant variable forgets its initial value and becomes "slaved" to the slower variables.) On the contrary, if $y^{\alpha} = 0$ and the variable is marginal, then nothing can be inferred from the Eq. (39) and, even if all the terms in the sum are $O(\epsilon)$, the output may be $O(1)$ for $k \geq 1/\epsilon$. The difficult problem of marginal variables is discussed extensively by Wilson in Sec. V of Ref. 8. In perturbation theory a marginal variable may "act" as relevant or irrelevant depending upon loop corrections. For example, if the RG recursion for a marginal variable is of the simple form

$$g_1^{\alpha} = g_0^{\alpha} + C \cdot (g_0^{\alpha})^2 + O(g_0^3), \tag{40}$$

then its behavior is determined by the sign of C.

We are not here going to attempt such a higher-order analysis for the RG procedure used by YO in Ref. 4. In the following section we will discuss a possible solution to the problem when $\epsilon \ll 4$. Here we just wish to stress that the original YO analysis *is not even controlled for $\epsilon \ll 4$ and higher-order nonlinearities in the fixed point dynamics could* be $O(1)$. Therefore, there is no basis to YO's claim that the fixed point model is just NS dynamics with a "renormalized" eddy viscosity even at $\epsilon \ll 4$, let alone for $\epsilon=4$. (It might be suggested that the YO limit procedure $r \to 0$ solves the problem, since then $P^{>} \to 0$ and the indicated marginal terms vanish. However, this is obviously wrong. The problem appears for any small $r>0$ and the RG is not even defined for $r=0$ since no variables are then eliminated! The limit $r \to 0$ is too singular to control and estimates of error bounds must be obtained at finite r.) Observe, incidentally, that the neglected marginal variables feed back into the recursion for the basic coupling variable g at higher orders; e.g., the following graph indicates an $O(g^{(3)}g^4)$ contribution to the eddy damping:



The four-point vertex in this graph represents the $\hat{\mathbf{v}}\mathbf{v}^3$-interaction term proportional to $g^{(3)}$. Because of such feedback, the recursion for g itself will change at order $O(g^4)$.

E. The RG analysis of equilibrium fluids

Now we will explain why these problems do not appear in the corresponding RG analysis of Model A by FNS in Ref. 1. As we have noted, that corresponds to our Model P with $\epsilon=2-d$ and with the fluctuation–dissipation relation (FDR) $D_0=(\nu_0/\rho)k_BT$, so that it describes the thermal fluctuations in a body of fluid at perfect rest. The main difference is that FNS defined their RG so that ν_0, as well as D_0, was fixed and the FDR maintained. This does *not* correspond to the prescription with $\lambda \equiv 1$, preserving Galilei covariance. In fact, it is easy to see from the Eq. (3.23) in Ref. 1 of FNS that $\lambda_k \to 0$ in their model when $d>2$. The non-Galilei covariance of the RG prescription of FNS is not a problem but corresponds to the physics of their model. In fact, FNS are considering not the entire fluid velocity but only its small thermal fluctuations around the mean value $\mathbf{v}' \equiv \mathbf{v}-\bar{\mathbf{v}}$ (where Galilei covariance *may* be used to set $\bar{\mathbf{v}}=0$.) The dynamics of the velocity fluctuations on large length scales and long time scales for $d>2$ is shown by FNS to be driven to a Gaussian fixed point, described by the following linear Langevin equation:

$$\partial_t \mathbf{v}' = \nu_0 \Delta \mathbf{v}' + \mathbf{f}, \tag{41}$$

with the FDR for the noise strength. This is nothing more than Onsager's linear regression law for the fluctuations.[16] It should be mentioned that FNS also derive a less trivial result for Model A, namely, they derive the "long-time tail" for the velocity correlation by an application of Wegner's theory of corrections to scaling (Sec. B of Ref. 1). All these results are obtained by FNS in a controlled approximation. In fact, as correctly discussed by FNS in Sec. F of Ref. 1, the higher-order terms which appear in the elimination step (which are exactly the same as those we discussed above) *are* irrelevant for their RG when $d>2$.

Even the argument of FNS is not a proof, of course, and it is only given in Ref. 1 at the level of a perturbation analysis. However, it is probably possible to give nonperturbative estimates for Model A. The difficulty with the FNS perturbation analysis is not that the coupling may be large. In fact, the dimensionless effective coupling $g_0^2=D_0\lambda_0^2/\nu_0^3\Lambda^\epsilon$ is quite small in almost any equilibrium fluid, when Λ is chosen to be a "semimacroscopic" wave number. Note particularly that $\epsilon<0$, when $d>2$. For example, for water in $d=3$ at 20 °C, $g_0^2=(2.50\times10^{-6}\text{ cm})\times\Lambda$ with $\lambda_0=1$. (Data is taken from Table VI, in Ref. 17). Therefore, g_0 starts out already small and is driven to zero under RG recursion. The primary problem with a purely perturbative analysis is the same here as for purely static equilibrium problems, and was discussed by Wilson in Ref. 18. The difficulty is that a term with a small coefficient but nonlinear in a stochastic variable, like $\lambda_0(\mathbf{v}\cdot\nabla)\mathbf{v}$, need not be small relative to a lower-order term with $O(1)$ coefficient, like $\partial_t\mathbf{v}$, if there are large fluctuations in the variable $\mathbf{v}$. In Ref. 18, Wilson estimated the size of this effect for critical spin systems by using equilibrium fluctuation theory. Essentially, he applied the Boltzmann principle, which estimates the probability of a fluctuation in terms of the entropy, or free energy, and which gives an exponentially small estimate for the probability of large fluctuations. A rigorous application of this idea requires a large-deviations type estimate of the probability of large-spin values: for a pedagogical discussion, see the author's papers[19] which use an elementary version of this argument devised by Gawedzki and Kupiainen. In the case of Model A of FNS similar estimates are required, but now they must be given for the probabilities of *velocity histories* with large amplitudes, not just static values. In this case, the required estimates should be provided by a version of Onsager's principle (see Refs. 16 and 20), which estimates the probability of large fluctuations of histories in terms of the dissipation required to produce them.

The situation is rather different for Model A when $d<2$. In this case, the one-loop RG recursion equation for the effective coupling g has a nontrivial fixed point of order $O(\epsilon^{1/2})$. However, at this fixed point, Galilei covariance is preserved [note that $d\lambda(l)/dl=0$ from the FNS Eq. (3.23) when $z=2-\frac{1}{2}\epsilon$, its fixed point value for $d<2$]. Therefore, the same difficulty with marginal variables occurs here as for the Model P of YO in general. This point is somewhat academic, since a solenoidal vector field does not exist for $d<2$ and the model does not make sense other than as an analytically continued perturbation series. On the other hand, the Burgers model studied by FNS in Sec. III C is well defined for any $d \geqslant 1$ and it has the same RG structure as Model A. Therefore, the difficulties are actually present in that case. This contradicts a statement by Kardar, Parisi, and Zhang[21] that all higher-order terms are irrelevant for this model, or rather is equivalent version, the "KPZ equation," which they proposed as a universal model of interfacial growth.

F. Renormalization and Galilei covariance

It should be clear by now that the Galilei symmetry in the RG used by us, as well as by YO, for Model P is the crux of the problem. It is physically unavoidable, since the random velocity field in turbulence is described by the full NS equations (or a suitable subgrid dynamics at lower wave numbers), which is Galilei covariant. Unfortunately, Galilei covariance is not very restrictive and allows a large number of possible terms. For contrast, recall that Lorentz covariance requires that any interaction terms be local in space–time, and this is an extremely important fact in the renormalization theory of relativistic fields. Notice that the marginal interaction terms which we have discussed are nonlocal both in space and in time, but are fully consistent with Galilei covariance.

Up until this point we have not mentioned the "old" field-theoretic RG, but similar problems also occur there. For example, if Λ is an arbitrarily selected wave-number scale, then the term such as Eq. (38)

$$\int d^{d+1}x\ \hat{v}_l^<(x)[G_0P^>((\mathbf{v}^<\cdot\nabla)v_m^<)](x)\nabla_m v_l^<(x), \tag{42}$$

may be added to the continuum MSR action and is marginal by the power-counting analysis of DM in Ref. 2 even for $\epsilon \ll 4$. DM did not consider such a term in their discussion of higher-order terms, because they restricted their consideration only to space–time local interactions. However, as we have seen, there is no reason to restrict the interactions to space–time local ones in a Galilei covariant theory, so that they cannot be ignored. It is of interest for our later considerations to remark that such terms are probably marginal, not only by power counting, *but to all orders in perturbation theory*. The reason is that, within the field-theoretic renormalization, all counterterms are local. (The old field-theoretic renormalization and RG are carefully discussed in the book of Collins.[22]) Therefore, nonlocal variables, such as those above, should not mix under renormalization except with local variables and their scaling dimensions will get no loop corrections. This argument for lack of loop corrections was already made in the context of our Model P in the interesting paper of Adzhemyan *et al.*[23] If correct, the term like that in Eq. (42) will be *exactly marginal* and not merely marginal by power counting. This argument assumes certain facts, such as that the usual perturbative BPHZ renormalization can be carried through for MSR field theories. So far as we are aware, this has never been established. However, the result itself seems plausible, since "anomalous dimensions" in the field-theory arise from short-distance divergences in the products of local variables at coinciding points. It is difficult to see how such a nonlocal variable as Eq. (42) could have any short-distance divergences. This is rather interesting, because variables which are exactly marginal are associated to degeneracy in the fixed point, i.e., a direction in the space of theories along which there is a line of fixed points. This point will be discussed further in the final section of this paper.

IV. ϵ-EXPANSION RG AND TURBULENCE MODELING

A. An improved RG strategy for the ϵ expansion

Before discussing the behavior of the higher-order nonlinearities under RG iteration, let us first consider what may be inferred about their magnitude from a dimensional analysis alone. Suppose that the effective theory, or subgrid dynamics, obtained by integrating out wave numbers $\geqslant\Lambda$ (and without any rescalings at all) is

$$(\partial_t+\mathbf{v}^<\cdot\nabla)\mathbf{v}^< = -\nabla p^< + \nu(\Lambda)\Delta\mathbf{v}^< + \mathbf{f}^< + \mu(\Lambda)\times[G_0P^>((\mathbf{v}^<\cdot\nabla)v_m^<)]\nabla_m\mathbf{v}^< + \cdots . \tag{43}$$

Here $\nu(\Lambda)$ is the "effective" or "eddy" viscosity at scale Λ, while the force $\mathbf{f}^<$ has a strength $\sim D(\Lambda)=D_0$. Additional terms, including nonlocal contributions to eddy damping, etc., are represented by the dots $(\cdots)$. For $\epsilon\ll 4$ it seems reasonable to assume that the limit $L\to+\infty$ may be taken, and then the natural length, time, and velocity scales are

$$l_0 \sim \frac{1}{\Lambda}, \tag{44}$$

$$t_0 \sim \frac{1}{\nu(\Lambda)\Lambda^2}, \tag{45}$$

$$v_0 \sim \left(\frac{D_0\Lambda^{2-\epsilon}}{\nu(\Lambda)}\right)^{1/2}. \tag{46}$$

Using these combinations to introduce rescaled *dimensionless* variables, $\tilde{\mathbf{r}}$, $\tilde{t}$, $\tilde{\mathbf{v}}$, one finds that the dynamical equation takes the form

$$(\partial_t+g(\Lambda)\tilde{\mathbf{v}}^<\cdot\tilde{\nabla})\tilde{\mathbf{v}}^< = -g(\Lambda)\tilde{\nabla}\tilde{p}^< + \tilde{\Delta}\tilde{\mathbf{v}}^< + \tilde{\mathbf{f}}^< + \mu(\Lambda)g^2(\Lambda)\times[\tilde{G}_0P^>((\tilde{\mathbf{v}}^<\cdot\tilde{\nabla})\tilde{v}_m^<)]\tilde{\nabla}_m\tilde{\mathbf{v}}^< + \cdots , \tag{47}$$

with the dimensionless coupling $g^2(\Lambda)\equiv[D_0/\nu^3(\Lambda)\Lambda^\epsilon]$ as in Eq. (36). The rescaling has effectively set $D_0\equiv 1, \nu(\Lambda)\equiv 1$. For $\epsilon\ll 4$, $g(\Lambda)$ is expected to approach a "fixed point" value $g_*=O(\epsilon^{1/2})$ as $\Lambda\to+\infty$. Therefore, it is possible to make here a perturbation expansion in the parameter $g(\Lambda)$, and in this expansion the higher-order terms appear with larger powers of $g(\Lambda)$. *However, it cannot be inferred from this argument that the higher-order terms are negligible in the "fixed point" dynamics.* In fact, $\mu_* g_*^2 = O(1)$ if $\mu_*=O(1/\epsilon)$. To establish that the cubic term is negligible it must be shown that $\mu_*=O(1)$. RG is a tool precisely designed to investigate the asymptotics of the parameters like $\mu(\Lambda)$ and to establish suitable bounds on their limiting values. It is important to keep in mind, in any case, that the strength of the cubic term for the ordinary dimensional variables is—at best—$O(1)$ and that RG must be used nontrivially to establish even that. For $\epsilon\approx 4$ there is no reason whatsoever to believe that the higher-order terms are negligible compared to the quadratic nonlinearity.

The bounds on coefficients of higher-order nonlinearities are usually obtained in RG by exploiting their "irrelevance" but we have found in Model P that there are many such terms which are "marginal." To see how this difficulty may be at least partially solved, consider the marginal quartic term in the action, i.e.,

$$g_0^{(3)}\int d^{d+1}x\ \hat{v}_l(x)[G_0\bar{P}((\mathbf{v}\cdot\nabla)v_m)](x)\nabla_m v_l(x), \quad (48)$$

Here, $\bar{P}$ is defined to be the spectral projection operator onto wave numbers in the range $[\Lambda, e^r\Lambda]$. This is the additional nonlinear term which would be produced at $O(g^2)$ starting from pure NS, after performing the specified rescalings in the definition of the RG. As we have noted, the overall scaling factor of this term is 1, but the projection $P^>$ is transformed into $\bar{P}$ under the space rescaling. Of interest is how this term iterates under RG, especially the *linear* part in $g^{(3)}$, which comes from replacing all of the $\mathbf{v}$ variables in the term by $\mathbf{v}^<$ variables. Doing so, and performing the specified rescalings, gives a contribution, proportional to $g_0^{(3)}$, to the interaction term

$$g^{(3,1)}\int d^{d+1}x\ \hat{v}_l(x)[G_0\bar{P}^{(1)}((\mathbf{v}\cdot\nabla)v_m)](x)\nabla_m v_l(x), \quad (49)$$

in which $\bar{P}^{(1)}$ is now the projection onto wave numbers in the range $[e^r\Lambda, e^{2r}\Lambda]$. In other words, the recursion relation for the coupling coefficient $g^{(3,1)}$ of the previous term is of the form

$$g_1^{(3,1)}=g_0^{(3)}+N^{(3,1)}[g_0], \quad (50)$$

which contains no term linear in $g^{(3,1)}$ itself, while the coupling $g^{(3)}$ has a recursion which contains no linear part at all

$$g_1^{(3)}=N^{(3)}[g_0]. \quad (51)$$

More generally, if we define interaction terms

$$g^{(3,s)}\int d^{d+1}x\ \hat{v}_l(x)[G_0\bar{P}^{(s)}((\mathbf{v}\cdot\nabla)v_m)](x)\nabla_m v_l(x), \quad (52)$$

where $\bar{P}^{(s)}$ is the projector onto wave number range $[e^{sr}\Lambda, e^{(s+1)r}\Lambda]$, then the coupling $g^{(3,s)}$ has an RG recursion of the form

$$g_1^{(3,s)}=g_0^{(3,s-1)}+N^{(3,s)}[g_0] \quad (53)$$

for $s\geqslant 1$. [We set $g^{(3,0)}=g^{(3)}$.] This RG recursion is rather curious, since we see that the coupling $g^{(3,s)}$ does not feed back into own recursion at a linear level, but instead into the recursion for $g^{(3,s+1)}$. Therefore, although these variables all have rescaling factors equal to one, their recursion is not that of a marginal variable in the classical sense.[24] To understand the precise behavior of these couplings under the RG flow requires to examine the coupled system of difference equations Eqs. (51)–(53).

However, a better solution suggests itself. The most natural thing to do is to consider the RG transformation with $r\geqslant\ln 2$. The point is that, when $r\geqslant\ln 2$, then

$$G_0\bar{P}[(\mathbf{v}^<\cdot\nabla)\mathbf{v}^<]\equiv 0, \quad (54)$$

since the sum of two wave vectors $<e^{-r}\Lambda$ is always $<\Lambda$ when $r\geqslant\ln 2$. For similar reasons

$$G_0\bar{P}^{(s)}[(\mathbf{v}\cdot\nabla)\mathbf{v}]\equiv 0, \quad (55)$$

when $s\geqslant 1$. Therefore, choosing $r\geqslant\ln 2$ accomplishes two things: all the additional quartic terms proportional to $g^{(3,s)}$ above vanish *identically* for $s\geqslant 1$ and the RG map generates no term linearly proportional to $g^{(3)}$ at all. Therefore, the recursion of $g^{(3)}$ is still given by Eq. (51) above and it feeds into the recursion of no other variable at linear level. Thus the RG analysis is actually made simpler here by choosing r to be sufficiently large. Observe that the recursion for $g^{(3)}$ is now the same as

$$g_1^{(3)}=e^{yr}g_0^{(3)}+N^{(3)}[g_0], \quad (56)$$

with $y=-\infty$, so that $g^{(3)}$ is formally acting like a "super-irrelevant" variable! The nonlinear part $N^{(3)}[g]$ at lowest order is just the quadratic contribution from g, i.e.,

$$g_k^{(3)}=g_{k-1}^2+K^{(3)}[g_{k-1}], \quad (57)$$

where $K^{(3)}[g_{k-1}]$ contains the contributions of cubic and higher order in all the couplings g^α. Therefore, we can see that, at lowest order, $g^{(3)}$ is completely "slaved" to the variable g and only depends upon the value of g at the previous step. From this it can be concluded that the variable $g^{(3)}$ indeed will remain $O(\epsilon)$ under iteration, since under infinite recursion it will approach a fixed point value $g_*^{(3)} = g_*^2 + K^{(3)}[g_*]$.

We expect that by means of this "improved" RG method, the basic results of YO for $\epsilon\ll 4$ can be recovered in a controlled way. We emphasize that we have *not* proved this. The previous argument shows how the additional quartic term in the action can be controlled at the lowest order in perturbation theory. We have not shown that it suffices to consider just the lowest order in perturbation theory, nor considered all other possible marginal variables which might appear at higher orders. Ultimately, a nonperturbative analysis would have to made, particularly to judge the importance of large fluctuations in $\mathbf{v}$. We suspect that everything is okay for $\epsilon\ll 4$, since the noise then grows weaker for $k\to 0$, but an analysis is required.

What we want to emphasize is that *RG is not a "magic formula."* There is nothing inherent in the idea of successive integration, or looking for "fixed points" in lowest-order recursion formulas, etc., which guarantees that the results will have some special validity. Always some analysis of the higher-order terms and sources of error must be made. Otherwise, the method is just an uncontrolled approximation, no better than naive use of low Reynolds number expansions or *ad hoc* closures.

B. Why the YO predictions are not justified by RG

Let us recall what are the main predictions of the YO theory. Their results fall into three general categories: (i) The first is a set of inertial-range scaling laws along with constants of proportionality. These include the eddy viscosity law

$$\nu(k)=0.49\bar{\epsilon}^{1/3}k^{-5/3}, \quad (58)$$

the Kolmogorov energy spectrum

$$E(k)=1.61\bar{\epsilon}^{2/3}k^{-5/3}; \quad (59)$$

the Obukhov–Corrsin spectrum for passive scalar concentration

$$\Phi(k)=1.16\,\frac{\chi}{\epsilon^{1/3}}\,k^{-5/3} \tag{60}$$

(where χ is the scalar dissipation), and some other numerical constants such as turbulent Prandtl number $P_t=0.7179$ and inertial-range velocity skewness $\bar{S}=-0.59$. (ii) A second result of the YO work is a derivation of the simple Smagorinsky subgrid model, with eddy viscosity formula

$$\nu(\Delta)=0.0062\Delta^2\|\sigma^<\|, \tag{61}$$

where Δ is the grid mesh size and

$$\sigma_{ij}^<=\left(\frac{\partial v_i^<}{\partial x_j}+\frac{\partial v_j^<}{\partial x_i}\right)$$

is the strain field of the explicit velocity modes. (iii) A final set of results are simple two-equation models of K-ϵ type for the large scales

$$\partial_t K+V_i\nabla_i K=P-\epsilon+\nabla_i(\alpha\nu\nabla_i K) \tag{62}$$

and

$$\partial_t\epsilon+V_i\nabla_i\epsilon=1.42\,\frac{\epsilon}{K}\,P-1.68\,\frac{\epsilon^2}{K}+\nabla_i(\alpha\nu\nabla_i\epsilon), \tag{63}$$

where

$$\nu=0.085\,\frac{K^2}{\epsilon}, \tag{64}$$

is the eddy-viscosity formula,

$$P=-(\nabla_i V_j)R_{ij}, \tag{65}$$

is the turbulence production due to the Reynolds stress $R_{ij}=\langle v_i' v_j'\rangle$ and large-scale strain field $\nabla_i V_j+\nabla_j V_i$, and $\alpha=1.39$ is another Prandtl-type constant. In reporting these results we have incorporated revisions of the original YO work due to Smith and Reynolds[25] and Yakhot and Smith.[26]

There are, at the outset, two serious sources of doubt about the YO theory. First, the "correspondence principle" is a drastic assumption, especially for higher-order statistical characteristics of turbulence (although it may be reasonably accurate at the level of energetics, where dimensional reasoning is rather successful). Second, as noted by DM, there is an important qualitative change in Model P at $\epsilon=3$, since the (dimensionally determined) energy spectrum there undergoes a transition from being UV dominated for $\epsilon<3$ to ir dominated for $\epsilon>3$. Hence, the intermittency corrections normally associated to the "energy cascade" are likely to appear when $\epsilon>3$, whereas the YO methods predict no corrections to naive dimensional analysis. Therefore, the scaling predictions of YO for Model P itself at $\epsilon=4$—let alone for true turbulence—are likely to be wrong. It might be objected that for equilibrium critical systems in $d=3$ setting $\epsilon=1$ yields remarkably accurate predictions in the corresponding ϵ expansion with $\epsilon=4-d$. However, the ϵ expansion of YO for Model P is crucially different from the the ϵ expansion in critical phenomena and we believe that any analogy between them is misleading. In the case of critical systems there is no analogous "transition" phenomenon between $\epsilon=0$ and $\epsilon=1$, and corrections to "mean field" appear already at $\epsilon=0$. From these considerations it is clear that, even if one assumes that the results of YO for Model P can be justified by the analyses of the previous section for $\epsilon\ll 4$, there still remains the question whether the extrapolation to $\epsilon=4$ may be made and to what extent the reasonable success of YO predictions for turbulence parameters may be understood on the basis of RG. We shall now discuss each of the YO predictions (i), (ii), (iii) in turn to see to what extent these results may be justified by RG analysis.

First, as to (i), it should be recalled that the power-laws derived by YO are *not* a true result of RG but simply of dimensional analysis, since YO have assumed—without any justification—that the limit $L\to+\infty$ exists. The forcing constant D_0, which has units of mean dissipation $\bar{\epsilon}$ for $\epsilon=4$, is therefore left as the only dimensional constant. In this way, YO enforce Kolmogorov scaling and eliminate the possibility of obtaining any intermittency corrections by fiat. Therefore, the only issue to understand is whether their RG analysis justifies the constants of proportionality in these formulas. An immediate source of uneasiness is that RG in no other application predicts absolute constants, but, at most, ratios of amplitudes. A constant, such as a fixed point value g_*, in general depends upon the precise RG transformation used. [For example, at second order in perturbation theory, the fixed point in Model P depends upon the choice of r, as $g_*^2=(e^{\epsilon r}-1)/A_{(r)}$.] Only scaling exponents, or ratios of certain amplitudes, are generally physical and independent of the RG procedure. This is a serious concern but we ignore it here.

Let us attempt an explanation for the "good" numbers of YO cited above. We will assume that the YO predictions for Model P with $\epsilon\ll 4$ are correct. This now seems reasonable. Second, we will assume that for $\epsilon\ll 4$ Lagrangian and Eulerian time statistics are nearly identical. This seems also likely, since, for $\epsilon<3$ most of the energy is in the uv modes and there will be little sweeping effect. Put another way, the sweeping frequency is always $\sim v_0 k$ but the internal dynamical frequency (from dimensional analysis) is $\sim D_0^{1/3}k^z$, with $z=2-(\epsilon/3)$, and the later dominates at high k for $\epsilon<3$. Third, we will assume that in Model P the Lagrangian statistics all extend continuously from $\epsilon\ll 4$ to $\epsilon=4$. This is probably not precisely true. In particular, it rules out any intermittency type corrections to the energy law, or other high-order statistics. Since it seems likely that an "energy cascade" and associated intermittency begins at $\epsilon=3$, it is unlikely this third assumption is strictly true. Nevertheless, the corrections at the level of low-order statistics, like the energy law, are certainly small so that the errors from this third assumption are likely to be small at that level. Fourth, and finally, we assume YO's "correspondence principle" holds. Again, this does not seem very reasonable for high-order statistics, but should be satisfactory at the level of energetics. Therefore, we conclude that the numbers obtained by YO above should be a reasonable approximation (although the predicted "constants" may not in reality even be Reynolds-number independent, because of intermittency corrections.)

We should comment here on the necessity of a Lagrangian interpretation of the YO theory. This point was already stressed by Kraichnan long ago.[27] In fact, it is one of the most unconventional aspects of the YO theory that it predicts Kolmogorov dynamical scaling exponent $z=2/3$ for *Eulerian* velocity correlations. The issue has been discussed by Nelkin and Tabor in Ref. 28. There is a bare possibility that "random sweeping" effects are less than expected if kinetic energy fluctuations have a spectrum given by naive dimensional analysis $\sim k^{-7/3}$. This is, of course, the prediction of YO theory, which always gives the scaling of naive dimensional analysis. If instead the kinetic energy fluctuations have spectrum given by a Gaussian ansatz, then the spectrum is $\sim k^{-5/3}$ and the "sweeping effect" is significant. Van Atta and Wyngard[29] have given detailed experimental vindication of the latter spectrum. An opposite conclusion on "random sweeping" has been reached on the basis of a numerical simulation by Panda *et al.*[30] However, it was pointed out by Chen and Kraichnan[31] that at the low Reynolds numbers of that simulation ($\mathrm{Re}_\lambda \leqslant 64$) the Eulerian sweeping frequency and the internal Lagrangian frequency in the putative inertial range are of the same order of magnitude. Therefore, this simulation is not a meaningful test. The preponderance of evidence at the moment seems to be that there is a random sweeping effect and that the opposite prediction of YO is wrong.

Let us now consider the YO result (ii) concerning the Smagorinsky model. It seems reasonable to assume that the fixed-point dynamics of Model P for $\epsilon \ll 4$ is well described by NS with a simple renormalized viscosity. However, the higher-order correction terms are proportional to some powers of ϵ and not zero. Therefore, setting $\epsilon=4$, there is no reason to believe that such additional nonlinearities are negligible or insignificant to the physics. The cubic nonlinearity, for example, represents the rate of change of a mode $v_l^{<}(\mathbf{k},t)$ due to the retarded interaction of three other explicit modes induced by feedback or "backscatter" from an excited subgrid mode. The dynamical process involved is one in which the subgrid mode $v_m^{>}(\mathbf{p},t)$ at time t—produced in the past by interaction of two explicit modes—in conjunction with a third explicit mode $v_n^{<}(\mathbf{q},t)$ contributes as a wave vector triad to the instantaneous evolution of the supergrid mode $v_l^{<}(\mathbf{k},t)$. Zhou *et al.*[32] have incorporated the additional cubic nonlinearity in numerical simulations to test its effect and they find that it gives an additional source of energy dissipation, in some sense analogous to (and replacing) the cusp-up effect in the eddy viscosity at the cutoff scale in simple closures like the test-field model. This phenomenon was already observed long ago by Rose for dissipation of scalar concentration.[33] Another source of doubt is the "correspondence principle." Therefore, we do not see how the YO analysis in any way justifies a simple subgrid model by RG methodology. It is an uncontrolled, *ad hoc* approximation.

Finally, let us consider the results (iii) of YO on the $K-\epsilon$ models. These model equations are derived by YO using a method that is described most clearly by Smith and Reynolds in Ref. 25. However, it must be stressed that the YO "RNG" method for deriving these model equations is just a "cookbook procedure" and *it has nothing to do with renormalization group methods at all!* The only thing the YO RNG method seems to have in common with traditional RG is the idea to remove degrees of freedom successively. A major part of ordinary RG, the rescaling, which is vital to error estimation, is dropped entirely. Most importantly, there is no discussion at all of how the given RNG rules will help to give a controlled approximation or how errors will be bounded. Unless some rationale is supplied, it is again an ad hoc, uncontrolled approximation. It is not supported by any RG methodology.

In conclusion, we see little hope that ϵ-expansion RG methods can be used to derive any of the YO predictions analytically, with a clear understanding of their range of validity. Only the constants of proportionality in (i) might have any possible justification from RG. Even here additional assumptions—with unknown validity—are required to argue for the approximate correctness of the results. In short, the YO theory is not an RG analysis at all!

Not only is the YO method not sufficient to justify their conclusions but also it is probably not necessary. In fact, in a very important work, Kraichnan[34] has shown how to recover several of the main predictions of the YO theory—the eddy viscosity coefficient, the Kolmogorov constant, the Obhukov–Corrsin constant, and the turbulent Prandtl number—by methods which do not involve RG strategies at all. Kraichnan's method, the distant interaction algorithm (DSTA), has greatly clarified the origin of the good numbers of the YO theory. In fact, by making *explicitly* the same assumptions made implicitly by YO, Kraichnan recovers equivalent results in a direct manner. It is important to emphasize that Kraichnan does not attempt to *justify* the approximations, but simply to clarify the essential ingredients of the YO analysis which give rise to the numbers. The RNG procedure of successive elimination of scales is totally unnecessary. Instead, Kraichnan gets the results by making the approximation—also used by YO without discussion—that the eddy damping on a band of wave numbers just $<k$ by those $>k$ is the same as the damping on the zero wave number modes by those with wave number $>\beta k$, for some $\beta>1$. Kraichnan uses an exact eddy viscosity formula for NS equations,[35] evaluated within this "distant interaction approximation," to derive the turbulent energy balance. Just as YO, Kraichnan assumes a model in which the internal driving forces are represented by a Langevin white-noise force. His final results are quite close to those of YO: with $\beta=1$, he gets 0.43 for the eddy-viscosity coefficient, 1.56 for the Kolmogorov constant, 0.96 for the Obukhov–Corrsin constant, and 0.61 for the turbulent Prandtl number. The discrepancies do not appear significant. Kraichnan's derivation clarifies the physical origin of the numbers, whereas the RNG analysis is complicated but does nothing to help justify the results. Further work along these lines would be very desirable.

D. How can RG be useful in turbulence theory?

In our judgement the YO theory has created some misconceptions about the role of RG in turbulence theory. The most serious of these is the idea that RG should be useful for the purpose of justifying phenomenological turbulence models of K-ϵ type or subgrid models for large-eddy simulation (LES). Although RG theory makes some use of the idea of "subgrid models" conceptually, in fact the interest is very different from the concerns of practical subgrid modeling. The problem is that RG was designed for an entirely different purpose: to derive asymptotic scaling laws, including the "anomalous" corrections to naive dimensional analysis. See our discussion in Ref. 9. There is no reason to believe—from any RG methodology—that the fixed point dynamics in true turbulence will be a simple object. The fixed point model, if it is accurately determined, must allow the computation of all intermittency corrections to scaling. Clearly, any subgrid model accurate to capture *perfectly* all of the statistical effects of eliminated modes must be extremely complex. It will doubtless contain many higher-order nonlinearities, non-Markovian and nonlocal interactions, etc. The ideal of subgrid modeling is to find a dynamics, simple enough to simulate at reasonable cost, but accurate enough to capture as much of the physics of eliminated modes as possible. Unfortunately, RG does not suggest any asymptotic simplifications at all. It is probably far better to base modeling efforts on sound phenomenological constraints, such as Galilei covariance, and dimensional analysis. Similar remarks apply to large-scale model equations of K-ϵ type and Reynolds stress modeling. More reasonable progress should be expected by imposing *a priori* constraints such as realizability inequalities, Galilei covariance, and "material frame indifference" conditions,[36] or by using bald but explicit approximations based on reverted expansion techniques,[35] two-scale approximations,[37] etc. Turbulence modeling remains an art and is not yet an exact science.

Where RG seems to be useful in turbulence theory, as we have explained in Ref. 9, is in deriving anomalous scaling laws based upon precise limit assumptions. This is not surprising: it is exactly how RG was used in the theory of critical systems. In our paper[38] we showed how RG may be used in this way to derive "multifractal scaling" for NS turbulence, as well as relations between inertial- and dissipation-range scaling exponents. Just to recapitulate those results, we recall that the operator-product expansion was used to relate the Reynolds-number scaling of "pth-order flatnesses"

$$F_p \equiv \frac{\langle (\partial u^{<}/\partial x)^p\rangle}{[\langle (\partial u^{<}/\partial x)^2\rangle]^{p/2}} \sim (\mathrm{Re})^{\xi_p}, \tag{66}$$

for $\mathrm{Re} \gg 1$, with the inertial-range scaling of pth moments of velocity differences

$$\langle [u(\mathbf{r}+l\cdot\hat{\mathbf{e}}_x)-u(\mathbf{r})]^p\rangle \sim l^{\zeta_p}, \tag{67}$$

for $L \gg l \gg \eta$. In the first equation $\mathbf{v}^{<}$ represents a "coarse-grained" velocity, with wave numbers $>k_\eta=(\bar{\epsilon}/\nu^3)^{1/4}$ filtered out. In the usual language of field theory, the blowup of the flatness F_p with Reynolds number is an "ultraviolet divergence" for $k_\eta \to +\infty$. It was also shown in Ref. 38 that the following relation between scaling exponents should hold

$$\xi_p = \frac{3\zeta_p}{4} - \frac{3p\zeta_2}{8}. \tag{68}$$

Hopefully RG will play a useful role in numerically evaluating scaling corrections, but that requires the discovery of suitable approximation methods. At the moment RG can only be used to indicate some general qualitative features of turbulence scaling behaviors. However, as discussed in the following section, some of the indicated features are quite surprising and counter to conventional beliefs.

V. ARE THERE INFINITELY MANY FIXED POINTS IN TURBULENCE?

A. True turbulence and renormalization group

We have already discussed the RG prescription for Model T, which, unlike Model P, is a realistic model of turbulence. In fact, the same RG prescription as for Model T is appropriate for true turbulence without any artificial stochastic forces at all, e.g., decaying turbulence. Note that our RG has the property that the instantaneous energy flux

$$\Pi(k,t) \equiv \frac{1}{|\Omega|}\int_\Omega d^d\mathbf{r}\, \mathbf{v}^{<k}(\mathbf{r},t)\cdot[(\mathbf{v}(\mathbf{r},t)\cdot\boldsymbol{\nabla})\mathbf{v}(\mathbf{r},t)], \tag{69}$$

transforms as

$$\Pi'(k,t)=\Pi(e^{-r}k,e^{zr}t)+\cdots, \tag{70}$$

i.e., it is marginal by power counting. This is analogous to the requirement in the forced case that the mean energy injection rate by the force, $\bar{\epsilon}=F(0)$, be invariant. Hence, the RG we have defined for Model T should also be the relevant RG for many situations of true turbulence.

The stationary PDF of the velocity field should satisfy a renormalization-group equation, which, we have explained in Ref. 9, is an expression of the independence of the PDF from the molecular viscosity. Let $P_{\mathbf{v};\Lambda}[\mathbf{u};L,\{\lambda\}]$ be the single-time distribution of all the $\mathbf{v}$ modes of wave number $\leqslant\Lambda$, i.e., of the low-pass filtered field $\mathbf{v}^{<}$, in the turbulent state with integral scale L. The (infinite) set of dimensionless parameters $\{\lambda\}$ are those needed in addition to the length-scale L and a velocity-scale v_0 to fully specify the statistics of the large length-scale modes. (For example, they may be nondimensionalized higher-order cumulants of the low-wave number field in the case of decaying turbulence.) The natural velocity-scale v_0 defined in terms of the mean dissipation is

$$v_0=(\bar{\epsilon}L)^{1/3}, \tag{71}$$

which is supposed to stay constant in the limit $\nu_0\to 0$. This is certainly true for Model T, if any steady-state exists at all, and it is also supposed to be true instantaneously for decaying turbulence in the quasistationary regime. Then the RG invariance condition is that

$$P_{\mathbf{v}',\Lambda}[\mathbf{u};\bar{\epsilon},L,\{\lambda\}]=P_{\mathbf{v};\Lambda}[\mathbf{u};\bar{\epsilon},e^{-r}L,\{\lambda\}], \tag{72}$$

where $\mathbf{v}'(\mathbf{r})\equiv e^{-r/3}\mathbf{v}(e^r\mathbf{r})$. This equation is expected to hold when Λ is in the inertial range of wave numbers for finite viscosity ν_0, and with any $\Lambda\leqslant+\infty$ in the limit $\nu_0\to 0$. It should be emphasized that such an invariance condition is not proved, but only conjectured, although *it is* basic to the whole RG approach. Assuming that the $\nu_0 \to 0$ limit exists and has been taken, one can see by a dimensional analysis that any joint velocity PDF has the form

$$P_{\mathbf{v}(\mathbf{r}_1)\cdots\mathbf{v}(\mathbf{r}_p),\Lambda}(\mathbf{u}_1\cdots\mathbf{u}_p;\bar{\epsilon},L,\{\lambda\})$$
$$=(\bar{\epsilon}L)^{-p/3}F\left(\frac{\mathbf{u}_1}{(\bar{\epsilon}L)^{1/3}},\ldots,\frac{\mathbf{u}_p}{(\bar{\epsilon}L)^{1/3}};\right.$$
$$\left.\times\frac{\mathbf{r}_1}{L},\ldots,\frac{\mathbf{r}_p}{L};\Lambda L,\{\lambda\}\right). \tag{73}$$

It may be noted that for decaying, homogeneous turbulence, the zero-viscosity limit of velocity distributions has been rigorously proved to exist in three dimensions for any finite time; see Theorem VIII.3.1 in Ref. 39. From the definition of $\mathbf{v}'$,

$$P_{\mathbf{v}'(\mathbf{r}_1)\cdots\mathbf{v}'(\mathbf{r}_p),\Lambda}(\mathbf{u}_1\cdots\mathbf{u}_p;\bar{\epsilon},L,\{\lambda\})$$
$$=e^{pr/3}P_{\mathbf{v}(e^r\mathbf{r}_1)\cdots\mathbf{v}(e^r\mathbf{r}_p),e^{-r}\Lambda}(e^{r/3}\mathbf{u}_1\cdots e^{r/3}\mathbf{u}_p;\bar{\epsilon},L,\{\lambda\}). \tag{74}$$

It then follows easily from Eq. (73) that the RG equation Eq. (72) holds. This is essentially the "fixed point" condition (although that is only precisely a correct description for the limit $L\to+\infty$, if this limit exists).

Although the RG equation Eq. (72) holds for all wave numbers smaller than the Kolmogorov wave number $k_\eta=(\bar{\epsilon}/\nu^3)^{1/4}$ under the stated assumptions, it is additionally expected that the PDFs have no functional dependence on the parameters $\{\lambda\}$ in the inertial range of wave numbers $1/L \ll k \ll k_\eta$. This is Kolmogorov's *universality hypothesis*, which is supposed to hold for some appropriately selected high-wave number variables. In the language of RG theory, their inertial-range PDF is an *ultraviolet fixed point* which is universally approached at high-wave numbers for a large domain of low-wave number statistics specified by $\{\lambda\}$. Observe that universality requires that the inertial-range modes be statistically independent of the low-wave-number modes at scale $1/L$. In fact, because the low-wave-number modes may be imagined to simply passively convect the high-wave numbers, their values may be averaged over any arbitrary distribution without affecting either $\bar{\epsilon}$ or L. (This assumption is clearly true for the zero-wave-number mode, where it follows from Galilei covariance.) However, if the PDF of the inertial-range modes $\mathbf{v}_I$ conditioned on the energy-range modes $\mathbf{v}_E$, $P[\mathbf{v}_I|\mathbf{v}_E]$, were dependent on $\mathbf{v}_E$ other than simply through $\bar{\epsilon},L$, then averaging with respect to an arbitrary distribution $P[\mathbf{v}_E;\{\lambda\}]$ would give a dependence of $P[\mathbf{v}_I]$ on other parameters $\{\lambda\}$ in addition to $\bar{\epsilon}$ and L. This would contradict the universality hypothesis.

With our earlier definition of the RG, the fixed-point (FP) condition in the strict sense—that is, Eq. (72) in the limit $L\to+\infty$—can hold only for single-time statistics. On the contrary, for multitime statistics the RG would have to be defined in terms of *Lagrangian histories* $\mathbf{V}(\mathbf{r},t)$ for a FP equation to hold for the distribution in path–space with $\mathbf{V}'(\mathbf{r},t)\equiv e^{-r/3}\mathbf{V}(e^r\mathbf{r},e^{2r/3}t)$. This can be argued just as for the RFNS models, since $z=\frac{2}{3}$ is the scaling exponent expected of a Lagrangian time correlation. The deeper reason for the need of Lagrangian representation lies in the presumed short-distance universality of the velocity PDF. As emphasized by Kraichnan in a fundamental work[40] (Secs. 3 and 7), the velocity histories in a high wave-number band can only be independent of the statistics of modes at the scale L in a Lagrangian representation. On the other hand, the Eulerian multitime correlations will have, even at arbitrarily high wave number, dependence on the statistics at scale L, e.g., the rms sweeping velocity $v_0\sim(\bar{\epsilon}L)^{1/3}$, and their $L\to+\infty$ limits will not exist.

We do not discuss in detail here the formalism for defining a Lagrangian-history RG, but we note simply that it can be done along the previous lines using Kraichnan's two-time velocity field, or, generalized Lagrangian representation (see Sec. 7 of Ref. 40 and Sec. 2 of Ref. 41). Recall that $\mathbf{v}(\mathbf{r},s|t)$ was defined by Kraichnan as the velocity of the fluid particle at time t which was at point $\mathbf{r}$ at time s. It is easy to write down an MSR action for this field, of the form

$$S[\mathbf{v},\hat{\mathbf{v}}]=-i\int d^d\mathbf{r}\int dt\ \hat{\mathbf{v}}(\mathbf{r},t)\cdot[\partial_t\mathbf{v}(\mathbf{r},t)-\nu_0\Delta\mathbf{v}(\mathbf{r},t)+(\mathbf{v}(\mathbf{r},t)\cdot\nabla)\mathbf{v}(\mathbf{r},t)]-\frac{1}{2}\eta\int d^d\mathbf{r}\int dt|\hat{\mathbf{v}}(\mathbf{r},t)|^2$$
$$-i\int d^d\mathbf{r}\int ds\int dt\ \hat{\mathbf{v}}(\mathbf{r},s|t)\cdot[\partial_s\mathbf{v}(\mathbf{r},s|t)+(\mathbf{v}^S(\mathbf{r},s|s)\cdot\nabla)\mathbf{v}(\mathbf{r},s|t)]-\frac{1}{2}\eta\int d^d\mathbf{r}\int ds\int dt|\hat{\mathbf{v}}(\mathbf{r},s|t)|^2. \tag{75}$$

In the functional integral expression for the generating functional $Z[\boldsymbol{\eta},\hat{\boldsymbol{\eta}}]$ the integration is over all the fields $\mathbf{v}(\mathbf{r},t)$, $\hat{\mathbf{v}}(\mathbf{r},t)$, $\mathbf{v}(\mathbf{r},s|t)$, $\hat{\mathbf{v}}(\mathbf{r},s|t)$ with the constraint $\mathbf{v}(\mathbf{r},t|t)=\mathbf{v}(\mathbf{r},t)$. To regularize the integrals a space–time white noise with strength $\eta\to 0$ was added to the dynamics. Our other notations follow Kraichnan's. As usual in the two-time formalism, the pure Lagrangian histories are obtained as a subset of the "two-time histories," $\mathbf{V}(\mathbf{r},t)=\mathbf{v}(\mathbf{r},0|t)$.

B. Non-universal scaling in the inertial range

For Model T and for true turbulence, unlike Model P for $\epsilon\ll 4$, the fixed point will be a strong-coupling theory, very far from the simple Gaussian fixed point

$$S[\Phi]=\tfrac{1}{2}\Phi(1)(i\sigma^{(2)}\partial_{t_1}\delta(1,2)$$
$$+(1-\sigma^{(3)})F(1,2))\Phi(2) \tag{76}$$

(which describes a Brownian velocity field, obeying the linear stochastic equation $\partial_t \mathbf{v}=\mathbf{f}$ with white-noise force $\mathbf{f}$). The short-distance statistics in turbulence show, of course, strong departures from Gaussianity. The perturbative RG which we described diagrammatically in Sec. II D is no longer applicable, and there is at the present time no honest approximation method for dealing with the true, strong-coupling fixed point in turbulence. The power-counting analysis of nonlinear terms which we made in Sec. III C does not indicate any longer marginality of the variables at the strong-coupling fixed point. All that can be inferred from that discussion is that there are infinitely many variables, like that in Eq. (48), for which the rescaling factor $e^{y^{\alpha}r}$ is unity, and whose RG recursion is of the form

$$g_1^{\alpha}=g_0^{\alpha}+N^{\alpha}[g_0]. \tag{77}$$

To study issues of relevance, irrelevance and marginality at the strong-coupling fixed point g_* one must determine the eigenvalues and associated eigenvectors of the full linear map

$$\mathcal{T}^{\alpha\beta}=e^{y^{\alpha}r}\left(\delta^{\alpha\beta}+\frac{\partial N^{\alpha}}{\partial g^{\beta}}[g_*]\right), \tag{78}$$

obtained by linearizing around the fixed point. However, it should be noted that the idea we used in Sec. IV A to define an RG for which the variables like that in Eq. (48) are "super-irrelevant," no longer works at a strong-coupling fixed point. In fact, the linear map which must be analyzed for such marginal-by-power-counting variables with that definition of the RG is (for $g^{(\alpha)}$ such a variable)

$$\mathcal{T}^{\alpha\beta}=\frac{\partial N^{\alpha}}{\partial g^{\beta}}[g_*], \tag{79}$$

and the question of relevance, irrelevance, or marginality depends upon an eigenvalue analysis of the linear map. All that can be said is that, since the rescaling factor of infinitely many variables is 1, the linear map in the infinite-dimensional space is likely to have an nonempty eigenspace (perhaps even infinite dimensional) associated to the eigenvalue 1.

The question whether there are variables O at the fixed point which are *exactly* marginal, i.e., $O'\equiv O$, is of considerable interest. If we consider the fixed-point distribution P_* (on histories or on instantaneous configurations), then adding the marginal variable as a "perturbation" produces a new fixed point

$$(P_* e^{O})'=P_* e^{O}. \tag{80}$$

(This might not be true if the marginal variable are "redundant," i.e., obtained from the original fixed point action just by an infinitesimal change of variables: see Ref. 42.) Such exactly marginal variables, not merely marginal to lowest order, are encountered in some equilibrium lattice models, e.g., the Baxter eight-vertex model where it is associated to a line of fixed points.[43] In that case a very unusual phenomenon occurs, the existence of critical exponents which vary continuously with a parameter (describing the location along the line of fixed points). It is interesting to consider whether such a phenomenon might occur in turbulence.

In fact, we have already pointed out elsewhere the possibility of such a continuous variation of scaling exponents in turbulence, associated to a dependence of the inertial-range behavior on the detailed dynamics in the dissipation range. Specifically, at the very end of Ref. 9 we noted the possible existence of infinitely many fixed points and in the conclusion of Ref. 44 we attempted to explain the inertial-range nonuniversal scaling in terms of a dependence of scaling exponents upon distinct dissipative regularizations of the inviscid Euler equations, e.g., hyperviscosity versus ordinary viscosity.

The argument in Ref. 44 was based upon the analysis of energy transfer for Euler equations, where it was observed that local energy cascade breaks down in the vicinity of "singular structures" associated to negative Hölder singularities of the velocity field. As a consequence of this, energy transfer may be effected directly between the inertial and dissipation range in one local turnover time, without proceeding through a sequence of cascade steps. Observe that this is only a breakdown of local transfer in the *ultraviolet* whereas the fraction of energy transfer from the energy range directly across Δ octave bands is $O(2^{-(1-h)\Delta})$ relative to the contribution from Δ successive "cascade steps." Hence, the *infrared* locality is secure as long as $h<1$ and the independence of the inertial and energy range modes seems reasonable. However, because of highly singular objects in the flow, like vortex filaments and sheets, there will be direct communication between the viscous and inertial ranges of wave number.

The relations like Eqs. (66)–(68) between inertial and dissipation-range scaling behavior derived in Ref. 38 suggest as well a nonuniversal scaling in the inertial range depending upon the dissipation-range dynamics. In fact, the high strain and vorticity structures in the dissipation range will certainly depend upon the dissipation mechanism, e.g., a hyperviscosity will tend to make the velocity field smoother than ordinary viscosity. Since the dissipation-range scaling will likely be affected by the change in singularity structure, so also will the inertial-range scaling by our argument in Ref. 38.

C. A numerical test of the conjecture

These conjectures may be more than an academic speculation. In a recent numerical study, She[45] has found indications that inertial-range scaling in the so-called "shell models" of turbulence depend continuously upon the exponent α in the dynamical dissipation term $\nu_{(\alpha)}k_n^{\alpha}u_n$. In particular, he has found that the energy spectral exponent decreases with increasing α, so that the spectrum becomes flatter and flatter as α increases. She has given independent theoretical arguments for this behavior there. (In fact, while our RG argument from Ref. 9 still applies to this model, the argument in Ref. 44 does not, since it invokes the nonlocal interactions, which are absent in the shell models.)

However, there is another possible explanation of these observations, which was pointed out to us by Kraichnan.[46] Since there is an impedance mismatch at the boundary of the inertial and dissipation ranges, which becomes greater as α increases, there should be a large reflection of energy flux back into the inertial range. This effect will lead to a "back-up" of energy in the inertial range and therefore a flatter spectrum. The phenomenon may be very similar to the "bump" observed at the high-wave-number end of the inertial-range spectrum with ordinary viscous dissipation. Actually, this explanation of the phenomenon is very similar to She's.[45] However, the key issue is whether the change in slope is just a finite Reynolds number effect—i.e., a very long, but finite, "bump"—or whether it represents a true change in the asymptotic scaling behavior over an infinitely long range.

What we want to stress here is that the RG invariance condition for PDFs, like Eq. (72), is precisely a statement of independence from molecular viscosity. It can be very efficiently checked numerically by simulating two systems with stirring length L and $2^{-1}L$ and then comparing the distributions of the variables u_n' and u_n, respectively, in the two systems. This is exactly analogous to the test for "approximate fixed points" used in MCRG studies of equilibrium critical systems.[47] If the two distributions match exactly, then this a sensitive test of absence of viscous effects. Therefore, the verification of the fixed point condition is a critical test which should be performed to determine whether the observed effects represent true asymptotic high Reynolds number behavior. Whether or not She's candidate fixed points pass this numerical test, we would like to emphasize the practical utility of the RG equation as a check whether the infinite Reynolds number limit has been achieved.

If She's examples pass this test, then it is a demonstration of existence of infinitely many fixed points in turbulence, at least for shell models. Since the shell models have only local interactions and *a priori* more reason to exhibit universality than Euler equations, it would be a strong indication that there are many fixed points for Euler equations as well. This fact would create a problem for any strategy to determine turbulence scaling behavior "by just looking for the fixed point" in a big (infinite) space of theories. In fact, such a problem occurs in Polyakov's approach to 2-D turbulence.[48] Even granting that his "conformal hypothesis" is correct, it still allows an infinite number of solutions. As he discusses himself, there is a nontrivial problem of matching these conformal solutions with the "viscous layer" in wave-number space.

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[1]D. Forster, D. Nelson, and M. Stephen, "Large-distance and long-time properties of a randomly stirred fluid," Phys. Rev. A **16**, 732 (1977).
[2]C. DeDominicis and P. C. Martin, "Energy spectra of certain randomly-stirred fluids," Phys. Rev. A **19**, 419 (1979).
[3]J. P. Fournier and U. Frisch, "Remarks on the renormalization group in statistical fluid mechanics," Phys. Rev. A **28**, 1000 (1983).
[4]V. Yakhot and S. A. Orszag, "Renormalization group analysis of turbulence, I. Basic theory," J. Sci. Comp. **1**, 3 (1986).
[5]E. V. Teodorovich, "Use of the renormalization-group method to describe intermittency and to derive the corrections to the exponents in Kolmogorov turbulence theory," Sov. Phys. JETP **75**, 472 (1992).
[6]C. Di Castro and G. Jona-Lasinio, "The renormalization group approach to critical phenomena," in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6.
[7]K. G. Wilson and J. B. Kogut, "Renormalization group and the ϵ-expansion," Phys. Rep. C **12**, 75 (1974).
[8]K. G. Wilson, "The renormalization group: critical phenomena and the Kondo problem," Rev. Mod. Phys. **47**, 773 (1975).
[9]G. L. Eyink, "Renormalization group and operator-product expansion in turbulence: shell models," Phys. Rev. E **48**, 1823 (1993).
[10]P. C. Martin, E. D. Siggia, and H. A. Rose, "Statistical dynamics of classical systems," Phys. Rev. A **8**, 423 (1973).
[11]H. K. Janssen, "On a Lagrangian for classical field dynamics and renormalization group calculations of dynamical critical properties," Z. Phys. B **23**, 377 (1976).
[12]C. DeDominicis, "Techniques de renormalisation de la théorie des champs et dynamique des phénomenès critiques," J. Phys. (Paris) C **1**, 247 (1976).
[13]S. K. Ma and G. Mazenko, "Critical dynamics of ferromagnetics in 6-ϵ dimensions: general discussion and detailed calculation," Phys. Rev. B **11**, 4077 (1975).
[14]E. A. Novikov, "Functionals and the random-force method in turbulence theory," Sov. Phys. JETP **20**, 1290 (1965).
[15]H. Tennekes, "Eulerian and Lagrangian time microscales in isotropic turbulence," J. Fluid Mech. **67**, 561 (1975).
[16]L. Onsager, "Reciprocal relations in irreversible processes, I, II," Phys. Rev. **37**, 405 (1931); **38**, 2265 (1931).
[17]S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Dover, New York, 1981).
[18]K. G. Wilson, "Renormalization group and critical phenomena. II. phase-space cell analysis of critical behavior," Phys. Rev. B **4**, 3184 (1971).
[19]G. L. Eyink, "Quantum field-theory models on fractal spacetime, I, II," Commun. Math. Phys. **125**, 613 (1989); **126**, 85 (1989).
[20]G. L. Eyink, "Dissipation and large thermodynamic fluctuations," J. Stat. Mech. **61**, 533 (1990).
[21]M. Kardar, G. Parisi, and Y.-C. Zhang, "Dynamic scaling of growing interfaces," Phys. Rev. Lett. **56**, 889 (1986).
[22]J. Collins, *Renormalization* (Cambridge University Press, Cambridge, 1984).
[23]L. D. Adzhemyan, A. N. Vasil'ev, and Yu. M. Pis'mak, "Renormalization-group approach in the theory of turbulence: The dimensions of composite operators," Teor. Mat. Fiz. **57**, 268 (1983).
[24]However, the same type of difficulties still occur. The formal solution to the recursion relations Eq. (53) is $g_k^{(3,s)}=g_0^{(3,s-k)}+\Sigma_{l=0}^{k-1}N^{(3,s-l)}[g_{k-l-1}]$ when $s>k$, and it can be $O(1)$ if $k\geqslant 1/\epsilon$.
[25]L. M. Smith and W. C. Reynolds, "On the Yakhot–Orszag renormalization group method for deriving turbulence statistics and models," Phys. Fluids A **4**, 364 (1992).
[26]V. Yakhot and L. M. Smith, "The renormalization group, the ϵ-expansion, and derivation of turbulence models," J. Sci. Comput. **7**, 35 (1992).
[27]R. H. Kraichnan, "Hydrodynamic turbulence and the renormalization group," Phys. Rev. A **25**, 3281 (1982).
[28]M. Nelkin and M. Tabor, "Time correlations and random sweeping in isotropic turbulence," Phys. Fluids A **2**, 81 (1990).
[29]C. Van Atta and J. C. Wyngarde, "On higher-order spectra of turbulence," J. Fluid Mech. **72**, 673 (1975).
[30]R. Panda, E. Clemente, V. Sonnad, S. A. Orszag, and V. Yakhot, "Turbulence in a randomly stirred fluid," Phys. Fluids A **1**, 1045 (1989).
[31]S. Chen and R. H. Kraichnan, "Sweeping decorrelation in isotropic turbulence," Phys. Fluids A **1**, 2019 (1989).
[32]Y. Zhou, G. Vahala, and M. Hossain, "Renormalization-group theory for the eddy viscosity in subgrid modeling," Phys. Rev. A **37**, 2590 (1988).
[33]H. A. Rose, "Eddy diffusivity, eddy noise and sub-grid scale modeling," J. Fluid Mech. **81**, 719 (1977).
[34]R. H. Kraichnan, "An interpretation of the Yakhot–Orszag turbulence theory," Phys. Fluids **30**, 2400 (1987).
[35]R. H. Kraichnan, "Eddy viscosity and diffusivity: exact formulas and approximations," Complex Systems **1**, 805 (1987).
[36]C. G. Speziale, "Turbulence modeling in noninertial frames of reference," Theor. Comp. Fluid Dyn. **1**, 3 (1989).
[37]A. Yoshizawa, "A statistically derived system of equations for turbulent shear flows," Phys. Fluids **28**, 59 (1985).
[38]G. L. Eyink, "Lagrangian field theory, multifractals, and universal scaling in turbulence," Phys. Lett. A **172**, 355 (1993).
[39]M. J. Vishik and A. V. Fursikov, *Mathematical Problems of Statistical Hydrodynamics* (Kluwer Academic, Dordrecht, 1988).

[40]R. H. Kraichnan, "Kolmogorov's hypotheses and Eulerian turbulence theory," Phys. Fluids **7**, 1723 (1964).
[41]R. H. Kraichnan, "Langrangian-history closure approximation for turbulence," Phys. Fluids **8**, 575 (1965).
[42]F. J. Wegner, "The critical state, general aspects," in *Phase Transitions and Critical Phenomena*, Vol. 6, edited by C. Domb and M. S. Green (Academic, New York, 1976).
[43]R. J. Baxter, "Partition function of the eight-vertex lattice model," Ann. Phys. (N.Y.) **70**, 193 (1972).
[44]G. L. Eyink, "Energy dissipation without viscosity in ideal hydrodynamics, I. Fourier analysis and local transfer," Physica D (to appear).
[45]Z.-S. She (private communication).
[46]R. H. Kraichnan (private communication).
[47]R. H. Swendson, "Monte-Carlo renormalization," in *Real-Space Renormalization*, edited by W. Burkhardt and J. M. J. van Leeuwen (Springer, New York, 1982).
[48]A. M. Polyakov, "The theory of turbulence in two dimensions," Nucl. Phys. B **396**, 367 (1993).