On the RNG theory of turbulence

S. H. Lam

Citation: Physics of Fluids A: Fluid Dynamics **4**, 1007 (1992);
View online: https://doi.org/10.1063/1.858517
View Table of Contents: http://aip.scitation.org/toc/pfa/4/5
Published by the American Institute of Physics

On the RNG theory of turbulence

S. H. Lam
Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, New Jersey 08544-5263

(Received 23 September 1991; accepted 23 January 1992)

The Yakhot and Orszag [J. Sci. Comput. **1**, 3 (1986)] RNG theory of turbulence has generated a number of scaling law constants in reasonable quantitative agreement with experiments. The theory itself is highly mathematical, and its assumptions and approximations are not easily appreciated, particularly since several rather unconventional artifices are employed. The present paper reviews the RNG theory and recasts it in more conventional terms using a distinctly different viewpoint. A new formulation based on an alternative interpretation of the origin of the random force is presented, showing that the artificially introduced ε in the original theory is an adjustable parameter, thus offering a plausible explanation for the remarkable record of quantitative success of the so-called ε-expansion procedure.

I. INTRODUCTION

The incompressible Navier–Stokes equations are

$$\nabla\cdot\mathbf{v} = 0, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu_0\nabla^2\mathbf{v}, \tag{2}$$

where ν_0 is the (constant) molecular kinematic viscosity. The problem at hand is to construct a turbulence theory for flows at large Reynolds numbers. The issues related to stability and transition of laminar flows are excluded from consideration.

The 1987 paper by Yakhot and Orszag[1] (YO) first applied the *renormalization group* (RNG) technique to high Reynolds number turbulent flows, following the basic ideas in the 1976 and 1977 works of Forster, Nelson, and Stephens[2] (FNS) on fluctuations in a randomly stirred fluid otherwise at rest. So far, the YO theory has generated, without using "any experimentally adjustable parameters,"[1,3] a number of quantitative results in reasonable agreement with experiments. However, several unconventional artifices were used in the theoretical developments. First, following FNS, *a scale invariant* random force **f** with certain statistical properties was inserted into the Navier–Stokes equations via the *correspondence principle*, which postulated that solutions forced solely by this special **f** were "statistically equivalent...in the inertial range" to the unforced solutions for problems with nontrivial initial and/or boundary conditions. Second, an *ε-expansion procedure* was used extensively when the value of the artificially introduced expansion parameter, ε, was 4. The present paper reviews the YO derivations, examines the roles played by the correspondence principle and the ε-expansion procedure, and proposes a new formulation that not only views the correspondence principle in a new light but also can recover the YO results without the use of the ε-expansion procedure. Most importantly, the new viewpoint no longer claims the absence of adjustable parameters. Instead, ε in the YO theory is shown to be a free and unrestricted adjustable parameter, thus offering a plausible explanation for the good quality of the YO quantitative results.

Recently, Smith and Reynolds[4] (SR) carefully examined the YO theory, identified several algebraic errors, made important observations about the RNG equation for energy dissipation, and clarified the details of many derivations. The SR paper is an excellent tutorial of the YO theory because it follows precisely the original viewpoint. In contrast, *the present presentation employs a distinctly different viewpoint*, and assumes that the reader is already familiar with the YO paper. In Sec. II, the classical concept of turbulent kinematic viscosity ν_T is briefly reviewed. In Sec. III, the correspondence principle and the postulated scale invariant functional form for the correlation function of the random force are displayed as accepted premises. The strategy of filtering solutions with the user-specified artificial ultraviolet cutoff wave number Λ is explained. The scaling laws which can already be derived at this point from dimensional analysis are presented in Sec. IV. The next three sections review the YO developments: the iteration algorithm for the forced flow field in Sec. V, the derivation of $\nu_T(\Lambda)$ in Sec. VI, and the evaluation of the scaling law constants using the ε-expansion procedure in Sec. VII. The critical role played by the scale invariant assumption in the derivations is highlighted. A detailed discussion of the ε-expansion procedure itself is given in Sec. VIII. A new interpretation of the origin of the random force is adopted in Sec. IX where the YO scale invariance assumption is unequivocally abandoned. A new formulation, exploiting the adjustable parameter(s) available in the assumed random force, is then presented without the use of any conventional artifices. Additional discussions of the new viewpoint and concluding remarks are presented in Secs. X and XI.

II. THE BOUSSINESQ POSTULATE

The traditional approach to turbulence is to divide the flow variables into a mean part and a fluctuating part with zero mean:

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}', \quad p = p_0 + p'. \tag{3}$$

Substituting (3) into (1) and (2), and taking the appropriate time average, equations identical to (1) and (2) are obtained except the laminar stress term in (2) is augmented by

 0899-8213/92/051007-11$04.00

the addition of a turbulent stress, $-\langle \mathbf{v}'\mathbf{v}' \rangle$. To proceed further, the so-called "closure" problem must be addressed: how to determine this turbulent stress. An approach first advocated by Boussinesq[5] assumes the mean turbulent stress can be expressed approximately in the same form as the laminar stress with a turbulent kinematic viscosity, ν_T:

$$\text{turbulent stress} \approx \rho \nu_T [\nabla \mathbf{v}_0 + (\nabla \mathbf{v}_0)^T] . \tag{4}$$

If this *Boussinesq postulate* is accepted, the closure problem is reduced to the determination of a single scalar. In principle, once a theory for ν_T is developed, the problem for the mean flow is solved. Since ν_T is obviously dependent on the characteristics of the fluctuating part of the solution, the two parts are coupled, and both must be given appropriate theoretical attention.

Experimentally, the energy spectrum of the small eddies of high Reynolds number flows is found to follow the universal theoretical scaling law derived by Kolmogorov[6] using dimensional analysis. *This observation is fully exploited by the RNG theory.*

III. THE RNG THEORY OF TURBULENCE

The YO RNG theory of turbulence begins by postulating the correspondence principle which inserts a divergence-free isotropic random force into (2):

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu_0 \nabla^2 \mathbf{v} + \mathbf{f} , \tag{5}$$

where $\mathbf{f}$ is "chosen to generate the velocity field described by the (Kolmogorov) spectrum...in the limit of large wave number k." The correlation function of the random force $\mathbf{f}$ chosen by YO is

$$\begin{aligned} &\langle \hat{f}_i(\mathbf{k},\omega)\hat{f}_j(\mathbf{k}',\omega') \rangle \\ &\quad = 2D_0^*(\varepsilon,k)k^{-d}(2\pi)^{d+1}(\delta_{i,j} - k_i k_j/k^2) \\ &\qquad \times \delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') , \end{aligned} \tag{6a}$$

where $\hat{\mathbf{f}}_i$ is the space-time Fourier transform of $\mathbf{f}$, $\mathbf{k}$ is wave number, ω is frequency, d is the dimension of the physical space, and $D_0^*(\varepsilon,k)$, which has the dimension of velocity squared per unit time, is expressed as a product of two factors according to their k dependence:

$$D_0^*(\varepsilon,k) \equiv D_0(\varepsilon)(k)^{4-\varepsilon},$$

$$(\text{dimension} = \text{velocity squared/time}) . \tag{6b}$$

Here $D_0(\varepsilon)$ and k (the magnitude of $\mathbf{k}$) are both dimensional, and ε is a dimensionless parameter. *The YO theory assumed $D_0(\varepsilon)$ to be independent of any characteristic length scale,* especially the user-specified ultraviolet cutoff wave number Λ. As will be shown in Sec. IV, this *scale invariance* assumption on $\mathbf{f}$ leads to $\varepsilon = 4$ for turbulence theory. In the following derivations, ε will be left as a symbolic parameter (an exception will be noted) in anticipation of the application of the YO ε-expansion procedure.

In the present paper, $D_0(\varepsilon)$ instead of D_0 will be used throughout to emphasize its dependence on ε, particularly its dimension. As mentioned earlier, a more general functional form for $D_0^*(\varepsilon,k)$ will be considered in Sec. IX when $D_0(\varepsilon)$ is allowed to depend on Λ. Until then, the present presentation simply accepts the correspondence principle and the scale invariance assumption without further comment.

The logic of RNG theory can be described as follows. In turbulent flows, the energy spectrum of the small turbulent eddies naturally decays exponentially for $k \gtrsim \Lambda_0$, where Λ_0 is a large ν_0-dependent "dissipation" *ultraviolet cutoff* wave number. So, if all the $k > \Lambda_0$ Fourier components of an exact solution were removed, the filtered solution in physical space-time would be little affected and would still satisfy the same original Navier–Stokes equations approximately. In the so-called inertial range $\Lambda_0 \gtrsim k \gg 1/L$, the magnitude of the energy spectrum becomes significant and is expected to follow the Kolmogorov scaling law—provided the Reynolds number is sufficiently large. If a narrow band $\Delta\Lambda$ at the ultraviolet edge $\Lambda_0 \gtrsim k \gtrsim \Lambda_0 - \Delta\Lambda$ were filtered out, the filtered solution would now be significantly affected, and would no longer satisfy the original Navier–Stokes equations. The filtering process generates new terms. The RNG theory, obviously prompted by the Boussinesq postulate, uses a correction $\Delta\nu$ to ν_0 to absorb the major effects of these new terms at low k and ω. By recursively filtering the solution, the value of the user-specified ultraviolet cutoff wave number Λ can be moved downward to some lower value in the inertial range, while the value of $\nu_T(\Lambda)$ increases by accumulating the corrections from ν_0 to some much larger value. The residual of the new terms not absorbed by the $\Delta\nu$ correction also accumulates. Note that the random force $\mathbf{f}$, assumed by YO to be scale invariant, was not even mentioned in this exposition.

In spite of the fact that ε was required to be 4, YO applied the so-called ε-expansion procedure in the derivation of ν_T, and argued that the residual new terms (beyond those already accounted for by ν_T) in the modified Navier–Stokes equations for the filtered solution could be ignored because they were "higher order" in ε. The "lowest-order" theory,[1,3] when supplemented by an appropriate closure assumption, yielded a number of scaling law constants in reasonable agreement with experiments. Since both the insertion of the random force and the ε-expansion procedure are unconventional, they will be examined in some detail.

IV. DIMENSIONAL ANALYSIS

In general, the RNG theory of turbulence contains five dimensional input parameters: a characteristic physical length scale L, a characteristic velocity U, the molecular kinematic viscosity ν_0, the dimensional $D_0(\varepsilon)$ factor in the random force, and the artificial ultraviolet cutoff wave number Λ. Under the assumptions that the Reynolds number is sufficiently large ($UL/\nu_0 \gg 1$), the artificial ultraviolet cutoff is sufficiently high ($\Lambda L \gg 1$), and that the characteristic time scale of the problem is sufficiently long in comparison to the time scales of the small eddies ($L/U \gg [D_0^*(\varepsilon,\Lambda)]^{1/3}(\Lambda)^{-2/3}$), the dynamics of the small turbulent eddies is expected to behave in some universal manner; its weak dependence on U, L, and ν_0 can be ignored altogether. Under these conditions, $D_0(\varepsilon)$ and Λ are the only dimensional input parameters available. The following re-

sults can be straightforwardly obtained using only dimensional analysis.

The only possible form for ν_T, the main dimensional output parameter, is the following:

$$\nu_T = \Pi_1 [D_0^*(\varepsilon,\Lambda)]^{1/3}(\Lambda)^{-4/3} = \Pi_1 [D_0(\varepsilon)]^{1/3}(\Lambda)^{-\varepsilon/3}, \tag{7}$$

where Π_1 is a dimensionless constant.

Let $E(\Lambda)$ be the value of the energy spectrum at the user-specified ultraviolet cutoff $\Lambda \gg k_* = O(1/L)$ where k_* is the lower edge of the inertial range. The only possible form for $E(\Lambda)$ is

$$E(\Lambda) = \Pi_2 [D_0^*(\varepsilon,\Lambda)]^{2/3}(\Lambda)^{-5/3} = \Pi_2 [D_0(\varepsilon)]^{2/3}(\Lambda)^{1-2\varepsilon/3}, \tag{8}$$

where Π_2 is a dimensionless constant. Here Λ_0 is assumed to be infinite.

Let $\mathcal{E}$ be *tentatively* identified with some relevant energy dissipation rate (with dimension velocity squared per unit time) of the *unfiltered* flow field. The only possible form for $\mathcal{E}$, assumed to be independent of k, is

$$\mathcal{E} = \Pi_3 D_0^*(\varepsilon,\Lambda) = \Pi_3 D_0(\varepsilon)(\Lambda)^{4-\varepsilon}, \tag{9}$$

where Π_3 is a dimensionless constant. Since $\mathcal{E}$ must also be independent of Λ, (9) clearly indicates that $D_0(\varepsilon)$ is Λ dependent when $\varepsilon \neq 4$. It is useful to remark here that the innocent looking (9) is at the heart of the matter being addressed in this paper.

The immediate task of a turbulence theory is to determine the values of ε, Π_1, Π_2, and Π_3. The YO theory concluded from (8) that $\varepsilon = 4$ was needed to recover the Kolmogorov energy by spectrum. Actually, this conclusion is solely the consequence of assuming $D_0(\varepsilon)$ to be independent of Λ, and will be challenged. In what follows, ε will be kept symbolically in (9) *without committing ε to any specific value.*

The scaling laws (7)–(9) all have ε-dependent exponents, and all involve $D_0(\varepsilon)$. For reasons which will become clear later, it is preferable to use $\mathcal{E}$ instead. Using (9) to eliminate $D_0(\varepsilon)$ in favor of $\mathcal{E}$ from (7) and (8), one obtains the following:

$$\nu_T = \pi_1(\mathcal{E})^{1/3}(\Lambda)^{-4/3}, \tag{10}$$

$$E(\Lambda) = \pi_2(\mathcal{E})^{2/3}(\Lambda)^{-5/3}, \tag{11}$$

where

$$\pi_1 = \Pi_1/\Pi_3^{1/3}, \tag{12a}$$

$$\pi_2 = \Pi_2/\Pi_3^{2/3} = \text{denoted as } C_K \text{ in YO}. \tag{12b}$$

Equation (11) is the Kolmogorov $E(k)$ in the inertial range, and it has been straightforwardly derived here by eliminating $D_0(\varepsilon)$ in favor of $\mathcal{E}$ using (9).

Many additional useful relations can be derived from (10) and (11). Eliminating $\mathcal{E}$ between them, one obtains the following:

$$\pi_3 = [E(\Lambda)]^{-1}(\Lambda)(\nu_T)^2 = \Pi_1^2/\Pi_2 = \pi_1^2/\pi_2. \tag{13}$$

Eliminating Λ, one obtains

$$\pi_4 = [E(\Lambda)]^{-4/5}(\mathcal{E})^{1/5}(\nu_T) = \Pi_3^{1/5}\Pi_1/\Pi_2^{4/5} = \pi_1/(\pi_2)^{4/5}. \tag{14}$$

The kinetic energy $K(\Lambda)$ contained in all the eddies with $k > \Lambda$ in the inertial range is readily computed by integrating $E(k)$ as given by (11) from Λ to infinity while holding $\mathcal{E}$ constant:

$$K(\Lambda) = \int_\Lambda^\infty E(k)dk = \tfrac{3}{2}\Lambda E(\Lambda). \tag{15}$$

Other interesting relations which appear in YO are

$$\pi_5 = [K(\Lambda)]^{-2}(\mathcal{E})(\nu_T) = \frac{4}{9}\frac{\Pi_3\Pi_1}{\Pi_2^2} = \frac{4}{9}\frac{\pi_1}{(\pi_2)^2} = \text{denoted as } C_\nu \text{ in YO}, \tag{16}$$

which is of particular interest in K-$\mathcal{E}$ modeling, and

$$\pi_6 \equiv (\mathcal{E})(\Lambda)^{-3}[E(\Lambda)]^{-1}(\nu_T)^{-1} = \Pi_3/(\Pi_1\Pi_2) = 1/(\pi_1\pi_2), \tag{17}$$

$$\pi_7 \equiv (\mathcal{E})(\Lambda)^{-2}[E(\Lambda)]^{-2}(\nu_T) = \Pi_1\Pi_3/(\Pi_2)^2 = \pi_1/(\pi_2)^2, \tag{18}$$

$$\pi_8 \equiv (\Lambda)^{-2}[K(\Lambda)](\nu_T)^{-2} = \frac{3}{2}\frac{\Pi_2}{(\Pi_1)^2} = \frac{3}{2}\frac{\pi_2}{(\pi_1)^2}. \tag{19}$$

Only two of the above scaling law constants are independent. Most importantly, the dimensional analysis above does not care whether a random force has in fact been inserted or not, nor does it have any interest in the details of the RNG and/or any associated solution algorithms. The critical physical assumption is that the only relevant dimensional parameters are Λ and $\mathcal{E}$.

Note that once $\mathcal{E}$ is used in place of $D_0(\varepsilon)$, the scaling laws no longer have ε-dependent exponents.

V. THE ITERATION ALGORITHM

Equation (5) can be written in the following form:

$$\frac{\partial \mathbf{v}}{\partial t} - \nu_T\nabla^2\mathbf{v} = \mathbf{f} + (\nu_0 - \nu_T)\nabla^2\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v} - \frac{1}{\rho}\nabla p, \tag{20}$$

or

$$\mathcal{L}(\mathbf{v};\nu_T) = \mathbf{f} + \lambda_0\mathbf{g}(\mathbf{v};\nu_T;p), \tag{21}$$

where $\mathcal{L}$ is the linear Stokes operator containing the turbulent kinematic viscosity, and $\mathbf{g}$ contains all the remaining terms on the right-hand side of (20):

$$\mathcal{L}(\mathbf{v};\nu_T) \equiv \frac{\partial \mathbf{v}}{\partial t} - \nu_T\nabla^2\mathbf{v}, \tag{22}$$

$$\mathbf{g}(\mathbf{v};\nu_T) \equiv (\nu_0 - \nu_T)\nabla^2\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v} - (1/\rho)\nabla p. \tag{23}$$

The parameter λ_0 in (21) is simply a tag (see the Appendix) and has the numerical value of 1. In view of (1), the vector $\mathbf{g}$ is required to be divergence free. A Poisson equation for the

scalar pressure p is obtained by taking the divergence of (23).

The YO theory formally used the following iteration algorithm:

$$\mathscr{L}[\mathbf{v}(n+1);\nu_T] = \mathbf{f} + \lambda_0 \mathbf{g}[\mathbf{v}(n);\nu_T;p(n)], \quad (24)$$

$$\nabla^2 p(n+1) = -\rho[\nabla\mathbf{v}(n+1):\nabla\mathbf{v}(n+1)], \quad (25)$$

where n is the iteration index and

$$\nabla\mathbf{v}:\nabla\mathbf{v} \equiv \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}. \quad (26)$$

The iteration began with $\mathbf{g}[\mathbf{v}(0);\nu_T;p(0)] = 0$, and was performed totally in Fourier space. The term containing the turbulent kinematic viscosity ν_T that appears on both sides of (24) is responsible for extracting out of $\mathbf{g}$ those turbulent stress terms which can be emulated by the Boussinesq postulate and putting them into $\mathscr{L}$.

Most importantly, YO tacitly assumed the iterants were totally forced by $\mathbf{f}$, and that initial and boundary conditions, *which are expected to be long wavelength entities*, made no separate contributions to the iterants, either directly or indirectly, in spite of the nonlinearity of $\mathbf{g}$. The iterants are expressed as power series of λ_0:

$$\mathbf{v}(n) = \mathbf{V}_0(n) + \mathbf{V}_1(n)\lambda_0^1 + \cdots + \mathbf{V}_m(n)\lambda_0^m + \cdots, \quad n \geqslant 1, \quad (27a)$$

$$p(n) = P_0(n) + P_1(n)\lambda_0^1 + \cdots + P_m(n)\lambda_0^m + \cdots, \quad n \geqslant 1, \quad (27b)$$

where $\mathbf{V}_m(n)$ and $P_m(n)$ are independent of n when $m < n$. The iterant $\mathbf{v}(1)$ is the Stokes solution of (24) forced by $\mathbf{f}$ alone, while the iterant $\mathbf{v}(2)$ is forced by $\mathbf{f} + \lambda_0\mathbf{g}[\mathbf{v}(1);\nu_T;p(1)]$. The YO theory used $\mathbf{v}(3)$, forced by $\mathbf{f} + \lambda_0\mathbf{g}[\mathbf{v}(2);\nu_T;p(2)]$, and included terms in $\lambda_0\mathbf{g}^{\text{fast}}$ only to $O(\lambda_0^2)$, the minimum necessary to obtain the viscosity correction. Note that $\mathbf{f}$ indeed played a dominant role in calculating the flow field iterants. Since the truncated power series representation of the iterants only makes sense for small λ_0, the adequacy of the leading iterants for $\lambda_0 = 1$ is at best uncertain; in fact, it will be found wanting in Sec. VII later.

VI. THE DETERMINATION OF ν_T

The RNG theory divides the flow variables into two components in analogy with (3):

$$\mathbf{v} = \mathbf{v}^< + \mathbf{v}^>, \quad p = p^< + p^>, \quad (28)$$

where $\mathbf{v}^>$ is the component of $\mathbf{v}$ whose Fourier transform has wave number in the range $\Lambda - \Delta\Lambda \leqslant k \leqslant \Lambda$, and $\mathbf{v}^<$ is the filtered solution. The pressure p is divided consistently [see (29c) and (29d) later]. In contrast to (3), both $\mathbf{v}^<$ and $p^<$ are time dependent. The user-specified ultraviolet cuttoff Λ is initially set at Λ_0 and $\Delta\Lambda$ is left arbitrary at this point.

Using (28), the original Navier–Stokes equations (1) and (5) can be rewritten as follows:

$$\nabla\cdot\mathbf{v}^< = 0, \quad (29a)$$

$$\mathscr{L}(\mathbf{v}^<;\nu_T) = \mathbf{f} + \mathbf{g}^{\text{fast}} + \mathbf{g}^{\text{slow}}, \quad (29b)$$

where

$$\mathbf{g}^{\text{fast}} \equiv -(1/\rho)\nabla p^> + \nabla\cdot(\mathbf{v}^<\mathbf{v}^< - \mathbf{v}\mathbf{v}) - \mathscr{L}(\mathbf{v}^>;\nu_0) + (\nu_0 - \nu_T)\nabla^2\mathbf{v}^<, \quad (29c)$$

$$\mathbf{g}^{\text{slow}} \equiv -(1/\rho)\nabla p^< - \nabla\cdot(\mathbf{v}^<\mathbf{v}^<), \quad (29d)$$

$$\nabla\cdot\mathbf{v}^> = 0. \quad (29e)$$

Poisson's equation for $p^<$ and $p^>$ are obtained by requiring that $\mathbf{g}^{\text{fast}}$ and $\mathbf{g}^{\text{slow}}$ be divergence free. Equations (29) are exact. It is seen that filtering[7] has introduced a new term $\mathbf{g}^{\text{fast}}$ into the Navier–Stokes equation for the filtered solution $\mathbf{v}^<$, and this new term not only depends on both $\mathbf{v}^>$ and $\mathbf{v}^<$, but also contains the yet unknown turbulent kinematic viscosity ν_T.

Clearly, it is highly desirable to choose $\nu_T[\Lambda;D_0(\varepsilon),\varepsilon]$ in such a way that $\mathbf{g}^{\text{fast}}$ is rendered unimportant in some sense. The RNG theory determines $\nu_T[\Lambda;D_0(\varepsilon),\varepsilon]$ by requiring that the value of $\mathbf{g}^{\text{fast}}$, when time averaged *over the fast time scales* associated with the band of fast eddies (i.e., those with frequency above $\nu_T\Lambda^2$), be as small as possible for low $\mathbf{k}$ and ω. The residual value of $\mathbf{g}^{\text{fast}}$ is not expected to be small for $k \approx \Lambda$, and its role there will be carefully scrutinized later in Sec. IX.

The derivation proceeds by first assuming that $\nu_T[\Lambda;D_0(\varepsilon),\varepsilon]$ exists for a certain Λ. Then the value of Λ is perturbed by a small amount $\Delta\Lambda$, causing a certain change, $\Delta\nu_T$. Working in the Fourier transform space and using the iteration algorithm outlined in Sec. V for $k \approx \Lambda$ *only*, one sums the coefficients of all terms $k^2\mathbf{v}^<\Delta\Lambda$ in the Fourier transform of $\mathbf{g}^{\text{fast}}$ and time averages it over the fast time scale. Equating the result to $\Delta\nu_T$, dividing by $\Delta\Lambda$, and taking the limit $\Delta\Lambda \to 0$, one obtains a differential equation for $\nu_T(\Lambda)$:

$$\frac{d\nu_T}{d\Lambda} = -\frac{\nu_T}{\Lambda} A_d \bar{\lambda}^2 F(\bar{\lambda}^2), \quad (30)$$

where

$$A_d(\varepsilon) = (d^2 - d - \varepsilon)/2d(d+2), \quad (31)$$

d is the dimension of the physical space,

$$\bar{\lambda}^2 \equiv [D_0(\varepsilon)S_d/(2\pi)^d](1/\nu_T^3\Lambda^\varepsilon), \quad (32)$$

$$F(\bar{\lambda}^2) = 1 + C_1(\varepsilon,d)(\bar{\lambda}^2)^1 + \cdots + C_n(\varepsilon,d)(\bar{\lambda}^2)^n + \cdots, \quad (33)$$

$S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of a d-dimensional unit sphere, and $C_n(\varepsilon,d)$, $n \geqslant 1$, are dimensionless numbers which depend on terms in the forced flow field iterants of order higher than $O(\lambda_0^2)$ and have never been calculated. It is easy to show that with the appropriate nondimensionalizations, the role of λ_0 is played by $\bar{\lambda}$. The form of (33) was first given by Yakhot and Smith[8] (YS). In order to follow the YO derivation, *it is essential that $D_0(\varepsilon)$ be kept in (32) and is not eliminated in favor of ε by (9)*, and that ε in (31) and (32) is *not* replaced by 4.

The initial condition for (30) is $\nu_T(\Lambda = \Lambda_0) = \nu_0$. Yakhot and Orszag integrated (30) analytically—assuming $D_0(\varepsilon)$ to be independent of Λ—using the leading approximation $F(\bar{\lambda}^2) \approx 1$, and showed that $\bar{\lambda}^2$ quickly approached a *fixed point* value $\bar{\lambda}_*^2$ independent of ν_0 as Λ moves downward in the inertial range. It is easy to show that the same

conclusion holds for arbitrary $F(\bar{\lambda}^2)$ provided[2] the derivative of $A_d(\varepsilon)\bar{\lambda}^2 F(\bar{\lambda}^2)$ with respect to $\bar{\lambda}^2$ is positive at $\bar{\lambda}_*^2$. Taking the logarithmic derivative of (32) with respect to Λ and using (30), one obtains the following equation for $\bar{\lambda}$:

$$\frac{d\bar{\lambda}}{d\Lambda}=\frac{\bar{\lambda}}{2\Lambda}\left(3A_d\bar{\lambda}^2F(\bar{\lambda}^2)-\varepsilon+\frac{\Lambda}{D_0(\varepsilon)}\frac{dD_0(\varepsilon)}{d\Lambda}\right). \tag{34a}$$

The fixed point $\bar{\lambda}_*^2$ of (34a) is given by setting the right-hand side of (34a) to zero. Imposing the YO scale invariance assumption [i.e., $D_0(\varepsilon)$ is independent of Λ], one obtains

$$3A_d(\varepsilon)\bar{\lambda}_*^2F(\bar{\lambda}_*^2)=\varepsilon. \tag{34b}$$

The scaling law for ν_T is then obtained simply by setting $\bar{\lambda}^2=\bar{\lambda}_*^2$ in (32). It is critical to the YO theory that ε appears symbolically on the right-hand side of (34b). Note that if (9) had been used to eliminate $D_0(\varepsilon)$ from (34a) in favor of $\mathcal{E}$ and Λ, the right-hand side of (34b) would simply be 4. The following development shall follow YO and adopt (34b) until Sec. IX when the scale invariance assumption is challenged.

A relation between Π_1 and $\bar{\lambda}_*^2$ is found by comparing (7) with (32):

$$\Pi_1=[S_d/(2\pi)^d]^{1/3}(\bar{\lambda}_*^2)^{-1/3}. \tag{35}$$

It is useful to rewrite (34b) as

$$\bar{\lambda}_*^2=\varepsilon/3F_1^*, \tag{36}$$

where

$$F_1^*=F_1^*(\varepsilon,\bar{\lambda}_*^2)\equiv A_d(\varepsilon)F(\bar{\lambda}_*^2). \tag{37}$$

It is emphasized again that (34b) and (36), which are later pivotal for YO to conclude $\bar{\lambda}_*^2=O(\varepsilon)$, are valid only if $D_0(\varepsilon)$ is assumed to be independent of Λ.

The energy spectrum $E(k)$ can be straightforwardly computed from the flow field iterants. It is easy to show that Π_2 can always be expressed as

$$\Pi_2=[S_d/(2\pi)^d]^{2/3}(\bar{\lambda}_*^2)^{1/3}F_2^*, \tag{38}$$

where F_2^* is some power series of $\bar{\lambda}_*^2$:

$$F_2^*=F_2^*(\varepsilon,\bar{\lambda}_*^2)\equiv A_d'(\varepsilon)[1+C_1'(\varepsilon,d)(\bar{\lambda}_*^2)^1+\cdots+C_n'(\varepsilon,d)(\bar{\lambda}_*^2)^n+\cdots], \tag{39}$$

and the leading-order term $A_d'(\varepsilon)$ may depend on d and ε. The notations have been chosen to emphasize the similarity between (33), (37), and (39). Up to this moment, none of the coefficients, $C_n(\varepsilon,d)$ or $C_n'(\varepsilon,d)$, has ever been calculated.

The definition of $\mathcal{E}$, tentatively identified as an energy dissipation rate, is not self-contained in the RNG theory. Without committing $\mathcal{E}$ to a specific definition, one can express Π_3 as

$$\Pi_3=[S_d/(2\pi)^d]F_3^*, \tag{40}$$

where

$$F_3^*=\pi_6F_2^*, \tag{41}$$

using (17), (35), and (38).

VII. EVALUATION OF SCALING LAW CONSTANTS

In Sec. IV, π_3 through π_8 were expressed in terms of π_1 and π_2. Theoretically, any two constants can be used as the basic set. Since π_3 and π_6 are most simply related to F_n^*, they are chosen to be the basic set:

$$\pi_3=(3/\varepsilon)(F_1^*/F_2^*), \tag{42}$$

$$\pi_6=(F_3^*/F_2^*). \tag{43}$$

The other six constants can then be expressed as follows:

$$\pi_1=(\pi_3)^{1/3}/(\pi_6)^{1/3}, \tag{44}$$

$$\pi_2=1/[(\pi_3)^{1/3}(\pi_6)^{2/3}], \tag{45}$$

$$\pi_4=(\pi_3)^{3/5}(\pi_6)^{1/5}, \tag{46}$$

$$\pi_5=4\pi_3\pi_6/9, \tag{47}$$

$$\pi_7=\pi_3\pi_6, \tag{48}$$

$$\pi_8=3/(2\pi_3). \tag{49}$$

Finally, the obstacle of evaluating the power series F_n^* when only its first term is available must be confronted. This is overcome by the so-called ε-expansion procedure which will be discussed fully in Sec. VIII. Applying the ε-expansion procedure to F_1^*, YO obtained $F_1^*\approx A_3(0)\times 1=0.20$ "to lowest order in ε" in the small ε limit. Since according to (37) F_1^* equals $A_3(4)F(\bar{\lambda}_*^2)$, this result implies $F(\bar{\lambda}_*^2)\approx A_3(0)/A_3(4)=3$, indicating that the leading term of (33) alone is not good enough. Evaluating $E(k)$ using the leading order $\mathbf{v}^>(1)$, YO obtained $F_2^*\approx A_3'(0)\times 1\approx 1$, also to lowest order in ε. Together $\pi_3\approx 3A_3(0)/4=0.150$ is obtained by formally neglecting terms of $O(\bar{\lambda}_*^2)$ and $O(\varepsilon)$ *simultaneously*.

In order to evaluate the second independent constant, it is *now* necessary to define $\mathcal{E}$. This is the closure step for the RNG theory. A reasonable choice is

$$\mathcal{E}\equiv\lim_{\Lambda\to\infty}[\nu_T(\Lambda)\langle\nabla\mathbf{v}^<:\nabla\mathbf{v}^<+(\nabla\mathbf{v}^<)^T:\nabla\mathbf{v}^<\rangle], \tag{50a}$$

which can be evaluated by (assuming isotropic small scale eddies)

$$\mathcal{E}=\lim_{\Lambda\to\infty}\left(2\nu_T(\Lambda)\int_0^\Lambda k^2E(k)dk\right), \tag{50b}$$

where the limiting process $\Lambda\to\infty$, which was tacitly assumed in YO but perhaps inadequately stressed, has been added to make sure that $\mathcal{E}$ does not depend on Λ. Note that (50b) is consistent with (9)–(11).

Using (10), (11), (35), and (38) in (50b), one obtains, in the limit $\Lambda\gg k_*$:

$$\mathcal{E}=(3\pi_1\pi_2/2)\mathcal{E}, \tag{51a}$$

which yields $\pi_1\pi_2=2/3$ or $\pi_6=3/2$ valid for any ε. Instead, YO chose to equate $\mathcal{E}$ to Kraichnan's expression for transport power[9]—this is later rederived within the RNG framework using the ε-expansion procedure by Dannevik *et al.*[3]—to obtain

$$\mathcal{E}=(2/1.594)[S_d/(2\pi)^d]D_0,\quad D_0=D_0(4), \tag{51b}$$

which yields $\pi_6\approx 2/1.594=1.255$. In contrast to all other YO derivations, (51b) explicitly commits itself to $\varepsilon=4$. To

be consistent with (9), ε should be left as a symbolic parameter in (51b). This innocent point is *crucial*, and shall be further discussed in Sec. X.

Note that π_6 is determined solely by the closure relation assumed for ε. Strictly speaking, the YO value of 1.594 used in (51b) is in error, and the correct value should be 1.576 which, together with $\pi_3 = 0.1500$, yields $\pi_6 = 1.269$ and $\pi_7 = 0.1904$; the latter value agrees with several non-RNG turbulent theories.[10] Using these values one obtains the following results:

YO results	variation #1	variation #2	
$\pi_3 = 3A_3(0)/4$	$\pi_3 = 3A_3(0)/4$	$\pi_3 = 3A_3(0.923)/4$	
$= 0.150$	$= 0.150$	$= 0.1269$	
$\pi_6 = 1.269$	$\pi_6 = 1.500$	$\pi_6 = 1.500$	
$\pi_1 \approx 0.491$	0.464	0.439	(52a)
$\pi_2 \approx 1.605$	1.436	1.518	(52b)
$\pi_3 \approx 0.150$	0.150	0.1269	(52c)
$\pi_4 \approx 0.336$	0.347	0.314	(52d)
$\pi_5 \approx 0.0846$	0.100	0.0846	(52e)
$\pi_6 \approx 1.269$	1.500	1.500	(52f)
$\pi_7 \approx 0.1904$	0.225	0.1904	(52g)
$\pi_8 \approx 10.000$	10.000	11.820	(52h)
$\bar{\lambda}_*^2 \approx 6.667$	6.667	7.880	(52i)

Experimentally, the value of π_2 ranges between 1.3 and 2.3, and $\pi_5 \approx 0.09$. Since only two rows of (52) are independent, the values in the first column of (52b) and (52e) pave the way for the long list of evidence cited[1,3] in support of the ε-expansion procedure. Note that data under variation #1 obtained using $\pi_6 = 1.500$ did quite well also. The data under variation #2 is obtained using $\varepsilon = 0.923$ in $A_3(\varepsilon)$ along with $\pi_6 = 1.500$; its rationale will be explained later in Sec. IX. It can easily be verified that if $A_3(4)$ is used instead in the evaluation of π_3, none of the scaling law constants with either π_6 is unacceptable.

Other constants such as the turbulent Prandtl number $P_t = 0.7179$ for heat transfer at high Reynolds number and the skewness factor $\overline{S_3} = 0.4878$ were derived by YO with the assistance of the ε-expansion procedure. A benign algebraic error in the computation for $\overline{S_3}$ was identified and corrected by SR, resulting in a slightly different numerical value, $\overline{S_3} = 0.59$. In all cases, RNG predictions without the ε-expansion procedure were always unacceptable. And in nearly all cases they became quite acceptable after the procedure was applied.

The lone exception is its prediction on the decay of homogeneous isotropic turbulence at high Reynolds numbers. The following theoretical model is used:

$$\frac{dK(0)}{dt} = -\varepsilon, \tag{53a}$$

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 2}\frac{\varepsilon^2}{K(0)}, \tag{53b}$$

where $K(0)$ is the kinetic energy of all the eddies $(0 < k < \infty)$ and $C_{\varepsilon 2}$ is a dimensionless constant to be determined. Equations (53a) and (53b) can be integrated analytically to yield:

$$K(0) \propto [1/(C_{\varepsilon 2} - 1)t]^{1/(C_{\varepsilon 2}-1)} \quad (C_{\varepsilon 2} > 1 \text{ assumed}). \tag{54}$$

The original YO paper, without calling for additional closure assumptions, derived $C_{\varepsilon 2} = 1.7512$ in good agreement with available experimental values ($C_{\varepsilon 2} \approx 1.9$) and data from direct numerical simulations ($C_{\varepsilon 2} \approx 1.7$). However, this derivation is disputed by SR who obtained $C_{\varepsilon 2} = 5.65$ by carefully repeating the YO procedures, and attributed the discrepancy to several algebraic errors. Hence the original YO theory failed for this problem.

Reynolds[11] gave a thorough discussion of this problem in 1976, and showed that, at this "two-equation" level, $C_{\varepsilon 2}$ can be determined by assuming a $E(k)$ profile. Using

$$E(k) = Ak^n, \quad \text{for } k \ll k_*, \tag{55}$$

and (11) for $k > k_*$, Reynolds obtained $C_{\varepsilon 2} = 11/6 = 1.833$ with $n = 2$ *and* the assumption that A is "permanent," i.e., time independent. Physically, it is clear that $C_{\varepsilon 2}$ is affected by the dynamics of the low k modes; additional closure assumptions must be introduced to generate a theory for this problem. In other words, RNG methodology *alone* should not be capable of producing a theory. But any RNG-based theory, when properly augmented by additional closure assumptions, will be hard pressed to compete with the simplicity of the Reynolds formulation mentioned above.

VIII. THE ε-EXPANSION PROCEDURE

Theoretically, so long as $C_n(\varepsilon,d)$ of the power series $F(\bar{\lambda}^2)$ are not available, no further progress toward the determination of $\bar{\lambda}_*^2$ can be made. Without $\bar{\lambda}_*^2$, the values of $F_2^*(\bar{\lambda}_*^2)$ and $F_3^*(\bar{\lambda}_*^2)$ cannot be determined even if $C_n'(\varepsilon,d)$ were known. Nevertheless, the YO theory managed to proceed and obtain good answers without knowledge of any of them by using the radial ε-expansion procedure: *the lowest-order result of a small ε expansion in RNG turbulence theory can be used* for $\varepsilon = 4$. The following conjecture was

suggested: "the ε expansion is an asymptotic series with sum close to the value computed to lowest order $\varepsilon \to 0$."[3]

By treating ε as a small number, the formal conclusion that $\bar{\lambda}_*^2 = O(\varepsilon)$ is easily obtained from (36) which was derived assuming $D_0(\varepsilon)$ to be independent of Λ [see (58a) later]. Two simplifications are immediately available. First, the higher-order terms in relevant power series of $\bar{\lambda}_*^2(\varepsilon)$ now no longer matter since only the leading-order term in the small ε limit is needed. Second, the magnitude of $\mathbf{g}^{\text{fast}}$ in (29b) is formally estimated to be $O(\varepsilon^{3/2})$ and can be classified as a higher-order term and therefore ignored. In other words, the Boussinesq postulate is "derived" as a leading-order result of the small ε expansion. In addition to the above, the small ε expansion is formally applied to the final results obtained, and only the leading-order term is retained. Since ε is known to be 4 and $\bar{\lambda}_*^2$ (4) is eventually found to be 6.667, these steps are unorthodox indeed. The remarkable record of success of this artifice has been a puzzle to researchers in turbulence theory ever since the publication of the YO paper.

In spite of its pervasive use and its crucial role in obtaining acceptable results, the ε-expansion procedure was not clearly defined in YO. The principal rationalization was that the small ε expansion was needed to be consistent with the expansion in powers of $\bar{\lambda}_*^2$ because $\bar{\lambda}_*^2 = O(\varepsilon)$. The speculation was that by simultaneously neglecting terms of order $O(\bar{\lambda}_*^2)$ and $O(\varepsilon)$ the surviving lone leading term somehow managed to give accurate answers. By carefully examining the YO derivations, one finds that the procedure is used only on expressions involved with a power series of $\bar{\lambda}_*^2$, and *exponents are always exempted*. Most importantly, *the ε dependence of $D_0(\varepsilon)$ is totally ignored*, and (51b) which is valid only for $\varepsilon = 4$ is used for $\mathcal{E}$. For example, the small ε expansion is used in (31), (33), (37), (39), and the left side of (34b), while $\varepsilon = 4$ is used in (6b), (7), (8), (9), and (32), and the right side of (34b), (36), and (42). In other words, the small ε expansion is only applied to F_1^* and F_2^*, or, to $A_d(\varepsilon)$, $A_d'(\varepsilon)$, $C_n(\varepsilon,d)$, and $C_n'(\varepsilon,d)$ in the derivations presented here. Since it is *not* applied uniformly whenever ε appears, the ε-expansion procedure cannot be considered as a formal leading-order extrapolation from a physical flow being stirred by a random force with small ε to $\varepsilon = 4$ (see Sec. XI for additional comments). Rather, it can only be viewed as an *ad hoc* artifice which, when selectively applied according to the above rules, has been successful in generating good answers.

In principle, the evaluation of additional coefficients of all the relevant power series is straightforward. In other words, the RNG theory is capable of determining the scaling law constants without the ε-expansion procedure, provided that the leading terms of the power series for the forced YO flow field iterants are adequate approximations. A simple example which cautions against such optimistic expectations is given in the Appendix.

All RNG presentations readily conceded the *ad hoc* nature[1,3] of the ε-expansion procedure. At the present time, its justification rests primarily on its remarkable and surprising record of empirical success. *This ability to consistently generate reasonable results is unlikely to be fortuitous*, and deserves a better explanation.

IX. THE ORIGIN OF THE RANDOM FORCE

Neither FNS nor YO adequately explained the origin of the extra random force $\mathbf{f}$, although Yakhot[12] had written in 1981 that it could be a manifestation of pumping occurring in the long wavelength regime. Yakhot and Orszag clearly assumed $D_0(\varepsilon)$ to be Λ independent, and this scale invariance assumption played the pivotal role in the derivation of $\bar{\lambda}_*^2 = O(\varepsilon)$, the basis of the ε-expansion procedure. In this section, we shall reexamine the origin of the random force $\mathbf{f}$ and the functional form assumed for its correlation function.

Once the flow variables are divided according to (28) and ν_T is introduced to emulate the turbulent stress, the modified Navier–Stokes equations (29) for $\mathbf{v}^<$ has a new body force $\mathbf{g}^{\text{fast}}$. The RNG theory more or less ignored $\mathbf{g}^{\text{fast}}$—YO considered it "marginal" and "irrelevant"—but firmly insisted on inserting the extra $\mathbf{f}$. When viewed in this light, a totally new interpretation of $\mathbf{f}$ emerges. Instead of being an *extra* random force inserted into the original Navier–Stokes equations for $\mathbf{v}$ via the correspondence principle, $\mathbf{f}$ *is simply the RNG guess of what* $\mathbf{g}^{\text{fast}}$ *should be for* $k \approx \Lambda$ *in the equation for* $\mathbf{v}^<$. It is quite an intelligent guess: $\mathbf{f}$ is allowed to depend only on $\mathcal{E}$, the dimensional energy dissipation rate of the physical flow field that is needed in the Kolmogorov law. Instead of using the exact $\mathbf{g}^{\text{fast}}$ (even if available), which would have produced a $\mathbf{v}^<$ with a mathematically sharp cutoff for its energy spectrum, the RNG theory used $\mathbf{f}$ to extend the inertial range of $\mathbf{v}^<$ with guaranteed Kolmogorov scaling for $k \approx \Lambda$ and *beyond*. In other words—according to the new viewpoint—the RNG theory simply made a guess for the magnitude of $\mathbf{g}^{\text{fast}}$ for $k \approx \Lambda$, *extrapolated* it to all $k > \Lambda$ with the guaranteed scaling, and used it as a surrogate for $\mathbf{g}^{\text{fast}}$ to generate the leading iterants for $\mathbf{v}^<$. The new expectation, understandably without proof, is that the $\mathbf{v}^<$ so generated is a good approximation to the original $\mathbf{v}$ *but only for* $k \approx \Lambda$. Formally, there is no $\mathbf{f}$ in (5), and the $\mathbf{g}^{\text{fast}}$ in (29c) should be interpreted as $(\mathbf{g}^{\text{fast}} - \mathbf{f})$ in this new formulation.

It is instructive to rewrite the original $\mathbf{g}^{\text{fast}}$ as given by (29c) in the following form:

$$\mathbf{g}^{\text{fast}} = \nabla\cdot\{(\mathbf{v}^<\mathbf{v}^< - \mathbf{v}\mathbf{v}) - \nu_T[\nabla\mathbf{v}^< + (\nabla\mathbf{v}^<)^T]\} - [\mathscr{L}(\mathbf{v}^>;\nu_0) + (1/\rho)\nabla p^>] + (\nu_0\nabla^2\mathbf{v}^<) . \quad (56)$$

The first term represents the Boussinesq postulate at work, the second term the direct effects of the band of small eddies, and the last term is negligible since $\nu_0 \ll \nu_T$. By construction, $\mathbf{g}^{\text{fast}}$ *clearly depends on* Λ, *and is small for low* k. In contrast, the YO $\mathbf{f}$ is by assumption independent of Λ, and not small for low k. In order for $\mathbf{f}$ to be a surrogate for $\mathbf{g}^{\text{fast}}$ (in some approximate sense), some reassessment is clearly needed.

Using only dimensional arguments, the dimensional factor $D_0^*(\varepsilon,k)$ in the correlation function of $\mathbf{f}$ can be expressed, *without loss of generality*, as follows:

$$D_0^*(\varepsilon,k) = D_0^*(\varepsilon,k;\Lambda) = \mathcal{E}h(\eta) , \quad (57a)$$

where $h(\eta)$ is a dimensionless function of η:

$$\eta \equiv k/\Lambda . \tag{57b}$$

The YO choice for $h(\eta)$ can be found as follows. Since $\mathcal{E}$ is independent of either k or Λ, $D_0(\varepsilon)$ must depend on Λ as given by (9):

$$D_0(\varepsilon) = \mathcal{E}/\Pi_3 \Lambda^{4-\varepsilon} . \tag{58a}$$

Using (58a) in (6b), one obtains

$$D_0^*(\varepsilon,k;\Lambda) = (\mathcal{E}/\Pi_3)(k/\Lambda)^{4-\varepsilon}, \tag{58b}$$

which contains two dimensionless parameters, Π_3 and ε. Comparing (58b) with (57a), one obtains

$$h(\eta) = (1/\Pi_3)\eta^{4-\varepsilon}, \quad \text{for all } \eta \geqslant 0 . \tag{58c}$$

The YO restriction $\varepsilon = 4$ is seen to be based solely on the philosophical insistence that $h(\eta)$ or $D_0(\varepsilon)$ or $D_0^*(\varepsilon,k)$ must not contain Λ. As indicated earlier, the Kolmogorov scaling is separately assured for arbitrary ε so long as (9) is accepted, and Π_3 (or π_6) is determined by the closure relation between $\mathcal{E}$ and the filtered flow field [e.g., (50a) and (50b) or others].

With the new interpretation of the origin of **f**, this restriction on ε no longer applies. *The Λ dependence of $D_0(\varepsilon)$, which was ignored by (51b) in the YO derivation, must now be properly acknowledged.* Of special interest is that when $dD_0(\varepsilon)/d\Lambda$ is evaluated using (9) or (58a) and used in (34a), the ε appearing symbolically on the right-hand sides of (34b) and (36) is replaced by 4. As a consequence, the relation $\bar{\lambda}_*^2 = O(\varepsilon)$, once the cornerstone of the YO rationale for the ε-expansion procedure, is now simply not true.

The following choice for $h(\eta)$ is equivalent to (58c) but is physically more appealing [see the discussions following (62) later]:

$$h(\eta) \approx 1/\Pi_3, \quad \text{for } \eta > 1 , \tag{59a}$$

$$h(\eta) \approx (1/\Pi_3)\eta^{4-\varepsilon_0}, \quad \text{for } \eta \approx 1 , \tag{59b}$$

where ε_0 is a free and unrestricted dimensionless parameter. It can easily be shown that (59a) expedites the recovery of the Kolmogorov $E(k)$ for the leading $\mathbf{v}^>$ iterants for $k > \Lambda$, and that (59b) provides all the information necessary [i.e., coefficients of the Taylor series of $h(\eta)$ about $\eta = 1$] to perform the needed RNG computations in a finite thin spherical shell, $\eta \approx 1$, in **k** space. *No precise information on $h(\eta)$ for $\eta \ll 1$ is required except that it be sufficiently small.* To develop the "new" theory, one needs only to use (57) and (59)—but one must *never* use (51b) under any circumstances—early to remove all presence of $D_0(\varepsilon)$ in favor of $\mathcal{E}$. By repeating all the YO derivations, one observes that ε_0 now can be treated as an ordinary parameter needing no special considerations such as exemptions extended to exponents. The value of ε_0 is determined empirically, and $\varepsilon = 0$ together with $\pi_6 = 1.269$ recovers the variation #1 data in (52) previously obtained using the ε-expansion procedure. If $\pi_6 = 1.500$ is adopted instead, then $\varepsilon_0 = 0.923$ would be a better choice; together they yield the very respectable variation #2 in (52). In other words, ε_0 (or ε) has been shown to be an adjustable parameter.

When **f** was considered an extra inserted random force in the YO theory, the expectation of accuracy of the flow field iterants using (27) was poor because $\bar{\lambda}_*^2$ was not a small number. *The new viewpoint changes this expectation.* While the assumption $\mathbf{g}^{\text{slow}}$ and $\mathbf{g}^{\text{fast}}$ are small in comparison to **f** is clearly untenable for $k \ll \Lambda$, it is quite plausible for $k \approx 6\Lambda$: $\mathbf{g}^{\text{slow}}$ is insignificant there when both **f** and $\mathbf{g}^{\text{fast}}$ are absent, and the residual of $\mathbf{g}^{\text{fast}}$ after its surrogate **f** is removed should be quite benign and harmless. Hence the $\mathbf{v}^>$ iterants, which are not computed from their own equations but are extrapolated across $k \approx \Lambda$ from the $\mathbf{v}^<$ iterants computed from (27) under the assumption that **f** is dominant and initial and boundary conditions can be ignored, could be quite good. Some necessary attributes of good $\mathbf{v}^>$ iterants are the ability to reproduce the Kolmogorov energy spectrum for $k \geqslant \Lambda$ and to generate acceptable scaling law constants. The YO $\mathbf{v}^>$ iterants have these attributes, albeit acquired in the name of the correspondence principle and the ε-expansion procedure. Ironically, the $\mathbf{v}^<$ iterants computed from (27) for $k \ll \Lambda$ are totally unreliable; the RNG theory tacitly recommends $\mathbf{v}^<$ to be computed from (29a) and (29b) by ignoring the residual $\mathbf{g}^{\text{fast}}$.

According to the new viewpoint, the turbulent stress in the $\mathbf{v}^<$ equations is being emulated by the Boussinesq postulate for low k, and by the random force for $k \approx \Lambda$. This viewpoint is consistent with the casual treatment of the residual $\mathbf{g}^{\text{fast}}$ in RNG theory: it received little attention. In principle, terms of the form $k^2\mathbf{v}^<\Delta\Lambda$ may also be extracted from the residual $\mathbf{g}^{\text{fast}}$ and converted into $\nu_\tau \nabla^4 \mathbf{v}$ to reduce its magnitude further. The RNG derivation of ν_τ would be quite a tedious task. However, its general form can readily be deduced from dimensional analysis:

$$\nu_\tau = \Pi_4(\mathcal{E})^{1/3}(\Lambda)^{-10/3} \left(\text{dimension} = \frac{\text{velocity}}{\text{wave number cubed}}\right), \tag{60}$$

where the dimensionless constant Π_4 is expected to depend on Π_3 and ε_0.

X. DISCUSSIONS

In the present paper, the strategy of filtering solutions and modifying the kinematic viscosity to absorb the main effects of the new terms is emphasized, while the roles played by the λ_0 tag in the flow field iteration, the massive amount of detailed calculations and manipulations in Fourier space, the ε-expansion procedure, and even the concept of renormalization group transformations, are downgraded. By identifying the random force with the filtering and extrapolation processes, the basic YO premise that its correlation functions contains no characteristic length scale is challenged and abandoned. Once (9) or (57) is accepted, a totally new viewpoint emerges. A new formulation based on this alternative rationale for the correspondence principle is presented, and the ε-expansion procedure is no longer needed. In addition, the flow field iteration algorithm is seen in a different light, and a plausible explanation is offered for the remarkable record of success of the YO theory. The new derivation proceeds identically as before except that $D_0(\varepsilon)$, its Λ dependence openly acknowledged, is eliminated in the

very beginning from the theory in favor of $\mathcal{E}$ using (9) or (58a). The entire mathematical burden of the new viewpoint is now carried by the unproven but plausible assumption that the leading $\mathbf{v}^>$ iterants are good enough in some pragmatic sense. In fact, the iteration algorithm used for $\mathbf{v}^<$ need not be convergent; an iterant is theoretically useful whenever the assumption that $(\mathbf{g}^{\text{fast}} - \mathbf{f})$ is unimportant for $k \approx \Lambda$ becomes plausible (see the Appendix).

The RNG theory itself does not provide the needed closure relation between $\mathcal{E}$ and the flow field. Theoretically, $\mathcal{E}$, which is a property of the unfiltered solution, must be independent of k and Λ. The new viewpoint prefers the definition given in (50a) and (50b) because it requires $\Lambda \ll k_*$, allowing it to be evaluated from either the unfiltered or the filtered solution. Since (50a) and (50b) are valid for any ε and consistent with (9), the Λ dependence of $D_0(\varepsilon)$ is tacitly conceded. The YO derivation, however, chose to adopt (51b) and explicitly committed itself to $\varepsilon = 4$ based on the scale invariance requirement. According to the new viewpoint, (51b) should be replaced by

$$\mathcal{E} = (2/1.594)\,[S_d/(2\pi)^d]\,D_0(\varepsilon)\Lambda^{4-\varepsilon} \tag{61}$$

so that it is consistent with (9) for any value of ε. If the YO theory had used (61) instead of (51b) in its derivations, the ε-expansion procedure would have no role to play at all.

If $\mathbf{f}$ and $\mathbf{g}^{\text{fast}}$ are both neglected and $\nu_T(\Lambda)$ is taken from (10), the $E(k)$ of a large-eddy simulation generated from (29) in response to initial and boundary conditions would have a natural ultraviolet cutoff at $k = \Lambda_B$ given approximately by

$$\Lambda_B = \gamma\left(\mathcal{E}/\nu_T^3\right)^{1/4} = \gamma\Lambda/(\pi_1)^{3/4}, \tag{62}$$

where the commonly accepted value for γ is 0.2. Using π_1 from (52a), it is seen that in general $\Lambda_B < \Lambda$. Hence, the mission assigned to $\mathbf{f}$, surrogate for $\mathbf{g}^{\text{fast}}$, is to extend $E(k)$ into the $\Lambda_B \leqslant k \leqslant \Lambda$ region with as little side effect as possible. This observation suggests that $\mathbf{f}$ should have a presence mainly inside this wave-number region. On the $k < \Lambda_B$ side, $\mathbf{f}$ should be small to avoid any nonlinear coupling with initial and boundary conditions; on the $k > \Lambda$ side, any reasonable $\mathbf{f}$ there should have negligible influence on $\mathbf{v}^>$ near $k \approx \Lambda$ since little reverse energy cascading is expected. The form assumed by $h(\eta)$ for $\eta > 1$ in (59a) therefore should not matter.

In the new formulation, $\mathbf{f}$ has been called a surrogate for, and not an approximation to, $\mathbf{g}^{\text{fast}}$. Ideally, $\mathbf{f}$ should be a good approximation to $\mathbf{g}^{\text{fast}}$ for the calculation of the $\mathbf{v}^<$ iterants. However, $\mathbf{g}^{\text{fast}}$ must be quite difficult to approximate since it is solely responsible for recovering the mathematically sharp $E(k)$ discontinuity of the $\mathbf{v}^<$ generated. The new formulation bypasses this difficulty by explicitly abandoning the goal of recovering the (totally uninteresting) $E(k)$ discontinuity, and uses a surrogate $\mathbf{f}$ charged with the more pragmatic mission: to recover the Kolmogorov $E(k)$ smoothly across $k \approx \Lambda$. The relationship between $\mathbf{f}$ and $\mathbf{g}^{\text{fast}}$ is left vague and imprecise: $\mathbf{f}$ is to be chosen such that $(\mathbf{g}^{\text{fast}} - \mathbf{f})$ is unimportant for $k \approx \Lambda$ in some approximate sense. Two independently adjustable parameters, Π_3 and ε (or ε_0), are made available; the Kolmogorov scaling law is separately assured by choosing $\mathcal{E}$, an energy dissipation rate of the flow field, to carry the dimension of $D_0^*(\varepsilon,k)$. Hence, the YO claim that no experimentally adjustable parameters were used is explicitly abandoned. Moreover, whenever good quantitative answers for (52) are generated—by choosing an appropriate closure relation [e.g., either (50b) or (61)] and adjusting the value of ε—*empirical legitimacy* is acquired by the $\mathbf{v}^>$ iterants used in the calculations, making plausible the assumption that the leading term alone in (33) and (39) is adequate. The good quality of results obtained by the YO theory beyond those listed in (52) is then no longer a surprise; each successful extension is a testimonial to the internal consistency of RNG methodology in exploiting the empirically proven $\mathbf{v}^>$ iterants.

The grid size L_{grid} of large-eddy simulation is readily associated with the RNG Λ. Yakhot and Orszag suggested using $\Lambda = \pi/L_{\text{grid}}$ in (10) to obtain a grid size dependent ν_T, and showed that the Smagorinsky[13] eddy viscosity formula can be recovered, including its constant in agreement with commonly accepted values. It is important to note that the simulation is expected to be time dependent (the time averaging used in the RNG theory is only over the fast time scales), and that the $E(k)$ so generated must have a substantial inertial range (i.e., $\Lambda \gg k_*$) as mentioned earlier. This is in contrast to the traditional approach of calculating strictly steady mean-flow solutions under the Boussinesq postulate.

Many interesting problems in science and engineering involve a large number of physical processes with vastly disparate time scales, and often their governing equations can be expressed in the same generic form as (29b): the left-hand side is a timelike linear operator, the right-hand side consists of the sum of two terms, $\mathbf{g}^{\text{fast}}$ and $\mathbf{g}^{\text{slow}}$, which evolve with different time scales. For deterministic problems in which $\mathbf{g}^{\text{fast}}$ is expected to decay rapidly (e.g., for most problems in chemical kinetics), the simplified models valid in the slow time scale can be obtained by simply dropping $\mathbf{g}^{\text{fast}}$ after applying the *computational singular perturbation* (CSP)[14,15] "refinement" algorithm to reduce its magnitude below some user-specified threshold. For problems in which $\mathbf{g}^{\text{fast}}$ is mainly oscillatory [e.g., Wentzel–Kramers–Brillouin (WKB)-type problems], the CSP refinement algorithm does not work. The problem of turbulence is of the latter type. The RNG theory in essence takes full advantage of available physical insights on the problem, and uses plausible assumptions to supplement mathematics to reduce the magnitude of $\mathbf{g}^{\text{fast}}$ and to rationalize its eventual neglect. This philosophy may indeed be what is needed to deal with this class of difficult problems, and may serve as a model in the CSP search for refinement algorithms to deal with nondeterministic, nonlinear WKB-type problems.

XI. CONCLUDING REMARKS

The YO theory of turbulence has three major features: the boldness of inserting an extra random force, the unconventional nature of the ε-expansion procedure, and the surprisingly good quality of its quantitative results without the use of adjustable parameters. The latter has been a source of

amazement to all, particularly in view of the fact that the forced flow field iterants used were unlikely to be accurate, and the leading-order results obtained before application of the ε-expansion procedure were unacceptably poor.

The new formulation presented here uses a distinctly different new viewpoint. No extra scale invariant random force needs to be inserted into the Navier–Stokes equations; $\mathbf{f}$ is a guess of $\mathbf{g}^{\text{fast}}$ which is self-generated by the filtering and extrapolating processes and is clearly Λ dependent. The scale invariance assumption on $\mathbf{f}$ is *unequivocally abandoned*. Since $D_0(\varepsilon)$ can now depend on Λ, the relation $\bar{\lambda}_*^2 = O(\varepsilon)$ is no longer true and the ε-expansion procedure loses its principal rationale. Moreover, pessimistic expectation of accuracy of the YO flow field iterants is replaced by the sanguine view that the leading $\mathbf{v}^>$ iterants could be quite good: they not only automatically follow the Kolmogorov scaling law but also possess adjustable free parameter(s). The factor $D_0^*(\varepsilon,k)$ in the correlation function of $\mathbf{f}$ becomes $D_0^*(\varepsilon,k;\Lambda)$, which is allowed to depend on Λ through the undetermined function $h(\eta)$. Any reasonable $h(\eta)$ containing two or more free parameters can be used in the same spirit as any assumed velocity profile is used in a conventional boundary-layer momentum-integral analysis. Hence, agreement with any two independent experiment scaling law constants can always be achieved by adjusting these free parameters. If (58c) is adopted for $h(\eta)$ and Π_3 (or its equivalent, π_6) is determined by a closure assumption [(50a), (50b), or (61)], then ε is available as an adjustable parameter to match any single desired scaling law constant in (52). No unconventional artifices are needed.

According to this new viewpoint, the critical issue in the correlation function of $\mathbf{f}$ is *not* the value of ε adopted, but the assumed closure relation between D_0^* and the relevant dimensional parameter(s) of the physical problem. For example, in model A of FNS on fluctuations in a fluid in thermal equilibrium stirred by random thermal noise, the critical physical assumption is that the relevant parameter is $\nu_0 k_B T/\rho$, where k_B is the Boltzmann constant, ρ is mass density, and T is absolute temperature. By dimensional analysis this assumption leads to $\varepsilon = -1$, and no other value is acceptable. If somehow it could physically be argued that the relevant parameter is $(\nu_0)^3(L)^{-\varepsilon}$, where L is some characteristic length such as the mean-free path of the fluid under study, then any reasonable ε would be dimensionally acceptable *including* $\varepsilon = 4$. If for this hypothetical case, $\mathbf{f}$ is in addition also physically required to be scale invariant (i.e., not allowed to depend on L), then $\varepsilon = 0$ must be imposed. In turbulence theory, the critical physical assumption is that Λ and $\mathcal{E}$, the ultraviolet cutoff and the energy dissipation rate, are the relevant parameters. By dimensional analysis $D_0(\varepsilon)$ must depend on Λ in accordance with (58a), and any reasonable ε is dimensionally acceptable, *including* $\varepsilon = -1$. According to this new viewpoint, the nearly obligatory paragraph found in most RNG turbulence papers—classifying problems according to the value of ε in (6b) while D_0 is tacitly held fixed—is misleading, and is responsible for much of the confusion.

Teodorovich[16] pointed out as early as 1987 that $\varepsilon = 4$ was not needed to recover the Kolmogorov scaling law provided one was willing to abandon the scale invariance assumption. He adopted (58c) for all $\eta \geqslant 0$, suggested $\varepsilon \to 0^-$ to suppress the so-called infrared divergences of certain integrals, and obtained results identical to YO without the ε-expansion procedure. Probably because no justification for the abandonment of the scale invariance assumption was provided, this work received little attention. Recently, McComb and Watt[17] developed an alternative RNG theory without an inserted random force, and, in terms of the terminology here, attempted to evaluate $\mathbf{g}^{\text{fast}}$ approximately. Nakano[18] developed another alternative RNG theory with an inserted but self-consistent Λ-dependent random force. Both theories required the removal of the $\Lambda_0 \geqslant k \geqslant \Lambda$ components as a finite block and both contained a free dimensionless parameter. Reasonable quantitative results were obtained with reasonable choices of the free parameters. The relative merits of these competing viewpoints are not yet clear, and additional research is needed.

The physical dimension d has always been left arbitrary in RNG theory. In principle, predictions for two-dimensional turbulence[9] can simply be obtained by setting $d = 2$. However, according to the new viewpoint here, no obvious theoretical reason exists to suggest that ε should be independent of d. Since the d dependence of ε is not empirically available, RNG predictions for two-dimensional turbulence are not possible at the present time unless ε is assumed to be independent of d. While laboratory two-dimensional turbulence data is difficult to obtain, data from two- and three-dimensional large-eddy simulations can be used to test this assumption.

The RNG methodology has been successfully used to exploit the empirically proven $\mathbf{v}^>$ iterants beyond the scaling law constants in (52). The turbulent Prandtl number, the skewness factor, etc., are valuable and impressive extensions. However, in spite of its remarkable record of success, cautions and sound physical reasonings must accompany the selection and execution of RNG extensions. Its failure for the decay of homogeneous isotropic turbulence is a reminder that the methodology, even with the new viewpoint, has limitations.

ACKNOWLEDGMENTS

The author acknowledges numerous stimulating discussions with Garry Brown and Victor Yakhot in the course of this research. In particular, the author is indebted to Dr. Yakhot for his patience and for providing the English translation of Ref. 16.

This work is supported by the NASA Langley Aerothermodynamics Branch, Space System Division.

APPENDIX: EXPANDING ITERANTS USING THE λ_0 TAG

In addition to assuming the convergence of the flow field iteration algorithm (24) and (25), the YO theory further assumes a truncated n-term power series is adequate to represent the nth iterant using the artifice of the λ_0 tag. The following simple algebraic problem provides some idea of the validity and accuracy of these assumptions.

Suppose it is desired to find a root of the following algebraic equation:

$$\lambda_0 x^2 - 2x + f = 0\,, \tag{A1}$$

where $\lambda_0 = 1$ is a tag and $f = 1$. The following iteration algorithm is proposed:

$$\mathcal{L}x(n+1) = f + \lambda_0 g[x(n)]\,, \tag{A2}$$

where

$$\mathcal{L} \equiv 2\,, \tag{A3a}$$

$$g(x) \equiv x^2\,. \tag{A3b}$$

The similarity between (A2) and (21) should be obvious.

The truncated power series of the iterants are

$$x(1) = \tfrac{1}{2} + \{g[x(0)]/2\}\lambda_0\,, \tag{A4a}$$

$$x(2) \approx \tfrac{1}{2} + \tfrac{1}{8}\lambda_0 + \{g[x(0)]/4\}\lambda_0^2 + O(\lambda_0^3)\,, \tag{A4b}$$

$$x(3) \approx \tfrac{1}{2} + \tfrac{1}{8}\lambda_0 + \tfrac{1}{16}\lambda_0^2 + \{\tfrac{1}{128} + g[x(0)]/8\}\lambda_0^3 + O(\lambda_0^4)\,, \tag{A4c}$$

$$x(4) \approx \tfrac{1}{2} + \tfrac{1}{8}\lambda_0 + \tfrac{1}{16}\lambda_0^2 + \tfrac{5}{128}\lambda_0^3 + \{\tfrac{3}{256} + g[x(0)]/16\}\lambda_0^4 + O(\lambda_0^5)\,. \tag{A4d}$$

Note that the first n terms of the nth iterant have "converged," and that the initial iterant $x(0)$ appears only in the last term which, formally being of order $O(\lambda_0^n)$, is ignored by the truncated representation. *Iterants represented by such truncated power series do not take advantage of a good initial guess of the initial iterant.* If the fixed point value x^* were somehow available to be used as $x(0)$, (A4) would yield the same answers when terms of order $O(\lambda_0^n)$ are ignored, and would reap no benefits at all.

These truncated power series (A4) are certainly accurate and useful for sufficiently small λ_0. But how good are they when λ_0 is of order unity?

For $\lambda = 1$, the only fixed point value is $x^* = 1$. Neglecting terms of order $O(\lambda_0^n)$ as is done in the RNG flow field iteration, (A4) yields $x(1)/x^* = 0.5$, $x(2)/x^* \approx 0.625$, $x(3)/x^* \approx 0.6953$, and $x(4)/x^* \approx 0.7383$ for any initial iterant $x(0)$. For $\lambda_0 = -1$, the only "reachable" fixed point value is $x^* = 0.4142$. Neglecting terms of order $O(\lambda_0^n)$, (A4) yields $x(1)/x^* = 1.207$, $x(2)/x^* \approx 0.905$, $x(3)/x^* \approx 1.056$, and $x(4)/x^* \approx 0.962$ for any initial iterant $x(0)$.

The difference in performance between the two cases is clearly not because of the magnitude of λ_0, but the relative magnitude of $g(x^*)$ in comparison to f. In the first case, $g(1)/f = 1$, the performance is poor; the truncated power series for $x(3)$ is not close to the correct answer. In the second case, $g(0.4142)/f = 0.1716$ is a moderately small number, and the performance is much better; the truncated power series for $x(3)$ is quite reasonable. The other fixed point, $x^* = -2.4142$, has $g(-2.4142)/f = 5.828$ and is simply not reachable by this iteration scheme.

Intuitively, the iteration algorithm should perform best when the magnitude of $|\lambda_0 g(x^*)/f|$ is as small as possible. In the YO theory, the relative magnitude of the tagged term in (21) can be estimated by $\bar{\lambda}_* \approx \sqrt{6.667} = 2.582$, which is not a small number. Thus $\mathbf{v}^>(3)$ or its truncated power series representation in the YO formulation is not expected to be very accurate.

If $\lambda_0 = 1$ and $f = 0$, the fixed point of (A1) is $x^* = 2$. The iteration algorithm (A2) for this case is theoretically divergent. However, if an inspired guess for $g[x(0)]$ is used, the leading iterant $x(1)$ could be very good.

[1] V. Yakhot and S. A. Orszag, "Renormalization group analysis of turbulence. I. Basic Theory," J. Sci. Comput. **1**, 3 (1986).

[2] D. Forster, D. Nelson, and M. Stephen, "Large-distance and long-time properties of a randomly stirred fluid," Phys. Rev. A **16**, 732 (1977); see also, "Longtime tails and the large-eddy behavior of a randomly stirred fluid," Phys. Rev. Lett. **36**, 867 (1976).

[3] W. P. Dannevik, V. Yakhot, and S. A. Orszag, "Analytical theories of turbulence and the ε-expansion," Phys. Fluids **30**, 2021 (1987).

[4] L. M. Smith and W. C. Reynolds, "On the Yakhot–Orszag renormalization group method for deriving turbulence statistics and models," Phys Fluids A **4**, 364 (1992).

[5] J. Boussinesq, "Théorie de l'écoulement tourbillant," Mém. Prés. Acad. Sci. (Paris) **XXIII**, 46 (1877).

[6] S. K. Friedlander and L. Topper, *Turbulence, Classic Papers on Statistical Turbulence* (Interscience, New York, 1961).

[7] A. Leonard, "Energy cascade in large-eddy simulations of turbulent fluid flows," Adv. Geophys. A **18**, 237 (1974).

[8] V. Yakhot and L. M. Smith, "The renormalization group, the ε-expansion and derivation of turbulent models," to appear in J. Sci. Comput. (1992).

[9] R. H. Kraichnan, "Inertial-range transfer in two- and three-dimensional turbulence," J. Fluid Mech. **47**, 525 (1971).

[10] D. C. Leslie, *Developments in the Theory of Turbulence* (Clarendon, Oxford, 1972).

[11] W. C. Reynolds, "Computation of turbulent flows," Annu. Rev. Fluid Mech. **8**, 183 (1976).

[12] V. Yakhot, "Universal stirring force in the theory of isotropic turbulence," Phys. Rev. A **24**, 3268 (1981).

[13] J. Smagorinsky, "General circulation experiments with the primitive equations. I. The basic experiment," Mon. Weather Rev. **91**, 99 (1963).

[14] S. H. Lam, "Singular perturbation for stiff equations using numerical methods," in *Recent Advances in the Aerospace Sciences*, edited by C. Casci (Plenum, New York, 1985), pp. 3–20.

[15] S. H. Lam and D. A. Goussis, "Conventional asymptotics and computational singular perturbation for simplified kinetics modelling," in *Reduced Kinetic Mechanisms and Asymptotic Approximations for Methane Air Flames*, Lecture Notes in Physics, edited by M. Smooke (Springer-Verlag, Berlin, 1991), pp. 227–242.

[16] E. V. Teodorovich, "On calculation of turbulent viscosity," Mech. Fluids Gas **4**, 29 (1987) (in Russian).

[17] W. D. McComb and A. G. Watt, "Conditional averaging procedure for the elimination of the small-scale modes from incompressible fluid turbulence at high Reynolds numbers," Phys. Rev. Lett. **65**, 3281 (1990).

[18] T. Nakano (private communication, 1991).