Definition 1.14. A *group* is a set G together with a group operation, usually called multiplication, such that for any two elements g and h of G, the product $g \cdot h$ is again an element of G. The group operation is required to satisfy the following axioms:

(1) *Associativity.* If g, h and k are elements of G, then

$$g \cdot (h \cdot k) = (g \cdot h) \cdot k.$$

(2) *Identity Element.* There is a distinguished element e of G, called the identity element, which has the property that

$$e \cdot g = g = g \cdot e$$

for all g in G.

(3) *Inverses.* For each g in G there is an inverse, denoted g^{-1}, with the property

$$g \cdot g^{-1} = e = g^{-1} \cdot g.$$

Before proceeding to Lie groups, we discuss a couple of elementary examples of groups which give some idea of the features which distinguish Lie groups from more general types of groups.

Example 1.15. (a) Let $G = \mathbb{Z}$, the set of integers, with addition being the group operation. Clearly associativity is satisfied, the identity element is 0 and the "inverse" of an integer x is $-x$.

(b) Similarly $G = \mathbb{R}$, the set of real numbers, is also a group with addition serving as the group operation. Again 0 is the identity, and $-x$ the inverse of the real number x. In both of these cases the group operation is commutative: $g \cdot h = h \cdot g$ for $g, h \in G$. Such groups are called *abelian*; they form only a small subclass of the full range of possibilities for groups.

(c) Let $G = \mathrm{GL}(n, \mathbb{Q})$, the set of invertible $n \times n$ matrices with rational numbers for entries. The group operation is given by matrix multiplication. The identity element is, of course, the identity matrix I, the inverse of a matrix A is the ordinary matrix inverse, which again has rational entries.

(d) Similarly, $\mathrm{GL}(n, \mathbb{R})$, the set of all invertible $n \times n$ matrices with real entries is a group under matrix multiplication, the identity and inverse being the same as in the previous example. For brevity, we will usually denote the *general linear group* $\mathrm{GL}(n, \mathbb{R})$ by just $\mathrm{GL}(n)$.

The distinguishing feature of a Lie group is that it also carries the structure of a smooth manifold, so the group elements can be continuously varied. Thus, in each of the above pairs of examples of groups, the second case is a Lie group since it is also a smooth manifold. For $\mathbb{R}$, the manifold structure is clear. As for the general linear group, it can be identified with the open subset

$$\mathrm{GL}(n) = \{X : \det X \neq 0\}$$

of the space $\mathbf{M}_{n\times n}$ of all $n \times n$ matrices. But $\mathbf{M}_{n\times n}$ is isomorphic to $\mathbb{R}^{n^2}$, the coordinates being the matrix entries x_{ij} of X. Thus $\mathrm{GL}(n)$ is also an n^2-dimensional manifold. In both cases the group operation is smooth (indeed analytic). This leads to the general definition of a Lie group.

Definition 1.16. An *r-parameter Lie group* is a group G which also carries the structure of an r-dimensional smooth manifold in such a way that both the group operation

$$m: G \times G \to G, \qquad m(g, h) = g \cdot h, \qquad g, h \in G,$$

and the inversion

$$i: G \to G, \qquad i(g) = g^{-1}, \qquad g \in G,$$

are smooth maps between manifolds.

Example 1.17. Here we discuss a couple of examples of Lie groups besides the two already presented.

(a) Let $G = \mathbb{R}^r$, with the obvious manifold structure, and let the group operation be vector addition $(x, y) \mapsto x + y$. The "inverse" of a vector x is the vector $-x$. Both operations are clearly smooth, so $\mathbb{R}^r$ is an example of an r-parameter abelian Lie group.

(b) Let $G = \mathrm{SO}(2)$, the group of rotations in the plane. In other words

$$G = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : 0 \leqslant \theta < 2\pi \right\},$$

where θ denotes the angle of rotation. Note that we can identify G with the unit circle

$$S^1 = \{(\cos\theta, \sin\theta): 0 \leqslant \theta < 2\pi\}$$

is $\mathbb{R}^2$, which serves to define the manifold structure on $\mathrm{SO}(2)$. If we include reflections we obtain the orthogonal group

$$\mathrm{O}(2) = \{X \in \mathrm{GL}(2): X^T X = \mathrm{I}\}.$$

It has the manifold structure of two disconnected copies of S^1.

(c) More generally, we can consider the group of *orthogonal* $n \times n$ *matrices*

$$\mathrm{O}(n) = \{X \in \mathrm{GL}(n): X^T X = \mathrm{I}\}.$$

Thus $\mathrm{O}(n)$ is the subset of $\mathbb{R}^{n^2}$ defined by the n^2 equations

$$X^T X - \mathrm{I} = 0,$$

involving the matrix entries x_{ij} of X. It can be shown that precisely $\frac{1}{2}n(n+1)$ of these equations, corresponding to the matrix entries on or above the diagonal, are independent and satisfy the maximal rank condition everywhere on $\mathrm{O}(n)$. Thus, by Theorem 1.13, $\mathrm{O}(n)$ is a regular submanifold of

GL(n) of dimension $\frac{1}{2}n(n-1)$. Moreover, matrix multiplication and inversion remain smooth maps when restricted to O(n), hence O(n) is a Lie group in its own right. The *special orthogonal group*

$$\mathrm{SO}(n) = \{X \in \mathrm{O}(n)\colon \det X = +1\},$$

being the connected component of the identity of the orthogonal group, is also an $\frac{1}{2}n(n-1)$-parameter Lie group for the same reasons. (A simpler proof of these facts will appear shortly.)

(d) The group T(n) of upper triangular matrices with all 1's on the main diagonal is an $\frac{1}{2}n(n-1)$-parameter Lie group. As a manifold T(n) can be identified with the Euclidean space $\mathbb{R}^{n(n-1)/2}$ since each matrix is uniquely determined by its entries above the diagonal. For instance, in the case of T(3), we identify the matrix

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{T}(3)$$

with the vector (x, y, z) in $\mathbb{R}^3$. However, except in the special case of T(2), T(n) is *not* isomorphic to the abelian Lie group $\mathbb{R}^{n(n-1)/2}$. In the case of T(3), the group operation is given by

$$(x, y, z)\cdot(x', y', z') = (x + x', y + y', z + z' + xy'),$$

using the above identification. This is not the same as vector addition—in particular, it is not commutative. Thus a fixed manifold may be given the structure of a Lie group in more than one way.

A Lie group *homomorphism* is a smooth map $\phi\colon G \to H$ between two Lie groups which respects the group operations:

$$\phi(g\cdot\tilde{g}) = \phi(g)\cdot\phi(\tilde{g}), \qquad g, \tilde{g} \in G.$$

If ϕ has a smooth inverse, it determines an *isomorphism* between G and H. In practice, we will not distinguish between isomorphic Lie groups. For example, the Lie group $\mathbb{R}^+$ consisting of all positive real numbers, with ordinary multiplication being the group operation, is isomorphic to the additive Lie group $\mathbb{R}$. The exponential function $\phi\colon \mathbb{R} \to \mathbb{R}^+$, $\phi(t) = e^t$, provides the isomorphism. For all practical purposes, $\mathbb{R}$ and $\mathbb{R}^+$ are the same Lie group. (In fact, up to isomorphism there are only two connected one-parameter Lie groups: $\mathbb{R}$ and SO(2).)

If G and H are r- and s-parameter Lie groups, then their Cartesian product $G \times H$ is an $(r + s)$-parameter Lie group with group operation

$$(g, h)\cdot(\tilde{g}, \tilde{h}) = (g\cdot\tilde{g}, h\cdot\tilde{h}), \qquad g, \tilde{g} \in G, \qquad h, \tilde{h} \in H,$$

which is easily seen to be a smooth map in the product manifold structure. Thus, for example, the tori T^r are all Lie groups, being r-fold Cartesian products of the Lie group $S^1 \simeq \mathrm{SO}(2)$. The group law on T^2, for instance, is given in terms of the angular coordinates (θ, ρ) by addition modulo integer multiples of 2π:

$$(\theta, \rho)\cdot(\theta', \rho') = (\theta + \theta', \rho + \rho') \bmod 2\pi.$$

Note that each torus T^r is a connected, compact, abelian r-parameter Lie group, and, in fact, is the only such Lie group up to isomorphism.

Our blanket assumption on manifolds that they be connected also carries over to the Lie groups we consider in this book. Thus *unless explicitly stated otherwise, all Lie groups are assumed to be connected.* For instance, the orthogonal groups $\mathrm{O}(n)$ are not connected, while the special orthogonal groups $\mathrm{SO}(n)$ are connected Lie groups. By restricting our attention to connected Lie groups, we are consciously excluding discrete symmetries, such as reflections, from consideration and concentrating on symmetries, like rotations, which can be continuously connected to the identity element in the group. There are, of course, important applications of discrete groups to differential equations, but these lie outside the scope of this book. (Technically speaking, without the assumption of connectivity, both Examples 1.15(a) and 1.15(c) are Lie groups, being totally disconnected zero-dimensional manifolds. However, none of the infinitesimal techniques vital to Lie group theory have any relevance there, and so we are justified in excluding them.) The general linear group $\mathrm{GL}(n)$ can be shown to consist of two connected components: $\mathrm{GL}^+(n) = \{X\colon \det X > 0\}$, which is itself a Lie group, and $\mathrm{GL}^-(n) = \{X\colon \det X < 0\}$. More generally, if G is any (not necessarily connected) Lie group, the connected component of the identity G^+ will always be a Lie group of the same dimension, and we will always concentrate on this part of G, the other components of G being obtained from G^+ via a discrete subgroup of elements.

Lie Subgroups

Often Lie groups arise as subgroups of certain larger Lie groups; for example, the orthogonal groups are subgroups of the general linear groups of all invertible matrices. In general we will be interested only in subgroups of Lie groups which can be considered as Lie groups in their own right. The proper definition of a Lie subgroup is modelled on that of an (immersed) submanifold.

Definition 1.18. A *Lie subgroup H* of a Lie group G is given by a submanifold $\phi\colon \tilde{H} \to G$, where $\tilde{H}$ itself is a Lie group, $H = \phi(\tilde{H})$ is the image of ϕ, and ϕ is a Lie group homomorphism.

For example, if ω is any real number, the submanifold

$$H_\omega = \{(t, \omega t) \bmod 2\pi\colon t \in \mathbb{R}\} \subset T^2$$

is easily seen to be a one-parameter Lie subgroup of the toroidal group T^2. If ω is rational, then H_ω is isomorphic to the circle group SO(2), and forms a closed, regular subgroup of T^2, while if ω is irrational, then H_ω is isomorphic to the Lie group $\mathbb{R}$, and is dense in T^2. Thus Lie subgroups of Lie groups do not have to be regular submanifolds. However, for many applications there is one very simple method of testing whether a subgroup is a regular Lie subgroup.

Theorem 1.19. *Suppose G is a Lie group. If H is a closed subgroup of G, then H is a regular submanifold of G and hence a Lie group in its own right. Conversely, any regular Lie subgroup of G is a closed subgroup.*

Note that we need only check that H is a subgroup of G and is closed as a subset of G in order to conclude that H is a regular Lie subgroup. This circumvents the problem of actually proving that H is a submanifold. In particular, if H is a subgroup defined by the vanishing of a number of (continuous) real-valued functions

$$H = \{g \colon F_i(g) = 0, i = 1, \dots, n\},$$

then H is automatically a Lie subgroup of G; we do not need to check the maximal rank conditions on the F_i! (Of course, to find the dimension of H we need to determine how many of the F_i are independent.) Thus, for example, the orthogonal group $\mathrm{O}(n)$ is a Lie group, being given by the n^2 equations

$$A^T A = \mathrm{I}, \qquad A \in \mathrm{GL}(n).$$

Another important example is the *special linear group*

$$\mathrm{SL}(n) \equiv \mathrm{SL}(n, \mathbb{R}) \equiv \{A \in \mathrm{GL}(n) \colon \det A = 1\},$$

which is an $(n^2 - 1)$-dimensional subgroup, given by the vanishing of a single function $\det A - 1$.

Local Lie Groups

Often we are not interested in the full Lie group, but only in group elements close to the identity element. In this case we can dispense with the abstract manifold theory and define a local Lie group solely in terms of local coordinate expressions for the group operations.

Definition 1.20. An *r-parameter local Lie group* consists of connected open subsets $V_0 \subset V \subset \mathbb{R}^r$ containing the origin 0, and smooth maps

$$m \colon V \times V \to \mathbb{R}^r,$$

defining the group operation, and

$$i \colon V_0 \to V,$$

defining the group inversion, with the following properties.

(a) *Associativity.* If $x, y, z \in V$, and also $m(x, y)$ and $m(y, z)$ are in V, then

$$m(x, m(y, z)) = m(m(x, y), z).$$

(b) *Identity Element.* For all x in V, $m(0, x) = x = m(x, 0)$.
(c) *Inverses.* For each x in V_0, $m(x, i(x)) = 0 = m(i(x), x)$.

If we write $x \cdot y$ for $m(x, y)$, and x^{-1} for $i(x)$, then the above axioms translate into the usual group axioms, except that they are not necessarily defined everywhere. Thus $x \cdot y$ makes sense only for x and y sufficiently near 0. Associativity says that $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ whenever both sides of this equation are defined. The identity element of the group is the origin 0. Finally, x^{-1} again is defined only for x sufficiently near 0, and $x \cdot x^{-1} = 0 = x^{-1} \cdot x$ for such x's.

Example 1.21. Here we present a nontrivial example of a local (but not global) one-parameter Lie group. Let $V = \{x: |x| < 1\} \subset \mathbb{R}$ with group multiplication

$$m(x, y) = \frac{2xy - x - y}{xy - 1}, \qquad x, y \in V.$$

A straightforward computation verifies the associativity and identity laws for m. The inverse map is $i(x) = x/(2x - 1)$, defined for $x \in V_0 = \{x: |x| < \frac{1}{2}\}$. Thus m defines a local one-parameter Lie group.

One easy method of constructing local Lie groups is to take a global Lie group G and use a coordinate chart containing the identity element. Less trivial is the fact that (locally) every local Lie group arises in this fashion. In other words, every local Lie group is locally isomorphic to a neighbourhood of the identity of some global Lie group G.

Theorem 1.22. *Let $V_0 \subset V \subset \mathbb{R}^r$ be a local Lie group with multiplication $m(x, y)$ and inversion $i(x)$. Then there exists a global Lie group G and a coordinate chart $\chi: U^* \to V^*$, where U^* contains the identity element, such that $V^* \subset V_0$, $\chi(e) = 0$, and*

$$\chi(g \cdot h) = m(\chi(g), \chi(h))$$

whenever $g, h \in U^*$, and

$$\chi(g^{-1}) = i(\chi(g))$$

whenever $g \in U^*$. Moreover, there is a unique connected, simply-connected Lie group G^* having the above properties. If G is any other such Lie group, there exists a covering map $\pi: G^* \to G$ which is a group homomorphism, whereby G^* and G are locally isomorphic Lie groups. (G^* is called the simply-connected covering group *of G.)*

Example 1.23. The only connected, simply-connected one-parameter Lie group is $\mathbb{R}$, so the local Lie group of Example 1.21 must coincide with some coordinate chart containing 0 in $\mathbb{R}$. Indeed, if we let $\chi: U^* \to V^* \subset \mathbb{R}$, where

$$\chi(t) = t/(t-1), \qquad t \in U^* = \{t < 1\},$$

then we easily see that

$$\chi(t+s) = m(\chi(t), \chi(s)) = \frac{2\chi(t)\chi(s) - \chi(t) - \chi(s)}{\chi(t)\chi(s) - 1},$$

$$\chi(-t) = i(\chi(t)) = \frac{\chi(t)}{2\chi(t) - 1},$$

where defined, so χ satisfies the requirements of the theorem.

Once we know that such a global Lie group exists, we can essentially reconstruct it from knowledge of just the neighbourhood of the identity determining the local Lie group.

Proposition 1.24. *Let G be a connected Lie group and $U \subset G$ a neighbourhood of the identity. Also, let $U^k \equiv \{g_1 \cdot g_2 \cdot \ldots \cdot g_k : g_i \in U\}$ be the set of k-fold products of elements of U. Then*

$$G = \bigcup_{k=1}^{\infty} U^k.$$

In other words, every group element $g \in G$ can be written as a finite product of elements of U.

As shown in Exercise 1.26 this follows directly from the connectedness of G. A similar result holds for connected local Lie groups as well.

Local Transformation Groups

In practice, Lie groups arise most naturally not as abstract, self-contained entities, but rather concretely as groups of transformations on some manifold M. For instance, the group SO(2) arises as the group of rotations in the plane $M = \mathbb{R}^2$, while GL(n) appears as the group of invertible linear transformations on $\mathbb{R}^n$. In general a Lie group G will be realized as a group of transformations of some manifold M if to each group element $g \in G$ there is associated a map from M to itself. It is important not to restrict our attention solely to linear transformations. Moreover, the group may act only locally, meaning that the group transformations may not be defined for all elements of the group nor for all points on the manifold.

Definition 1.25. Let M be a smooth manifold. A *local group of transformations* acting on M is given by a (local) Lie group G, an open subset $\mathscr{U}$, with

$$\{e\} \times M \subset \mathscr{U} \subset G \times M,$$

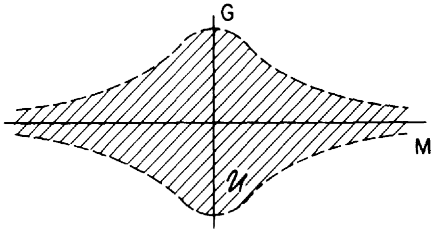

Figure 3. Domain for a local transformation group.

which is the domain of definition of the group action, and a smooth map $\Psi: \mathscr{U} \to M$ with the following properties:

(a) If $(h, x) \in \mathscr{U}$, $(g, \Psi(h, x)) \in \mathscr{U}$, and also $(g \cdot h, x) \in \mathscr{U}$, then

$$\Psi(g, \Psi(h, x)) = \Psi(g \cdot h, x).$$

(b) For all $x \in M$,

$$\Psi(e, x) = x.$$

(c) If $(g, x) \in \mathscr{U}$, then $(g^{-1}, \Psi(g, x)) \in \mathscr{U}$ and

$$\Psi(g^{-1}, \Psi(g, x)) = x.$$

(Note that except for the assumption of the form of the domain $\mathscr{U}$, part (c) follows directly from parts (a) and (b).)

For brevity, we will denote $\Psi(g, x)$ by $g \cdot x$, and the conditions of this definition take the simpler form:

$$g \cdot (h \cdot x) = (g \cdot h) \cdot x, \qquad g, h \in G, \quad x \in M, \tag{1.1}$$

whenever both sides of this equation make sense,

$$e \cdot x = x \quad \text{for all} \quad x \in M, \tag{1.2}$$

and

$$g^{-1} \cdot (g \cdot x) = x, \qquad g \in G, \quad x \in M, \tag{1.3}$$

provided $g \cdot x$ is defined. As a consequence of (1.3), we see that each group transformation is a diffeomorphism where it is defined.

Note that for each x in M, the group elements g such that $g \cdot x$ is defined form a local Lie group

$$G_x \equiv \{g \in G : (g, x) \in \mathscr{U}\}.$$

Conversely, for any $g \in G$, there is an open submanifold

$$M_g \equiv \{x \in M : (g, x) \in \mathscr{U}\}$$

of M where the transformation given by g is defined. In certain cases, the only group element which acts on all of M might be the identity element. At the other extreme, a *global* group of transformations is one in which we can take $\mathscr{U} = G \times M$. In this case, $g \cdot x$ is defined for every $g \in G$ and every $x \in M$. Thus (1.1), (1.2), (1.3) hold for all $g, h \in G$, and all $x \in M$, and there is no need to worry about precise domains of definition.

A group of transformations G acting on M is called *connected* if the following requirements hold:

(a) G is a connected Lie group and M is a connected manifold;
(b) $\mathscr{U} \subset G \times M$ is a connected open set; and
(c) for each $x \in M$, the local group G_x is connected.

As with manifolds and Lie groups, we make the blanket restriction that *unless explicitly stated otherwise, all local groups of transformations are assumed to be connected, in the above restricted sense.* These connectivity requirements help us avoid several technical complications when we come to discuss infinitesimal methods and invariants. They can always be realized by suitably shrinking the domain of definition $\mathscr{U}$ of the group action.

Orbits

An *orbit* of a local transformation group is a minimal nonempty group-invariant subset of the manifold M. In other words, $\mathcal{O} \subset M$ is an orbit provided it satisfies the conditions

(a) If $x \in \mathcal{O}$, $g \in G$ and $g \cdot x$ is defined, then $g \cdot x \in \mathcal{O}$.
(b) If $\tilde{\mathcal{O}} \subset \mathcal{O}$, and $\tilde{\mathcal{O}}$ satisfies part (a) then either $\tilde{\mathcal{O}} = \mathcal{O}$, or $\tilde{\mathcal{O}}$ is empty.

In the case of a global transformation group, for each $x \in M$ the orbit through x has the explicit definition $\mathcal{O}_x = \{g \cdot x \colon g \in G\}$. For local transformation groups, we must look at products of group elements acting on x:

$$\mathcal{O}_x = \{g_1 \cdot g_2 \cdot \ldots \cdot g_k \cdot x \colon k \geqslant 1, g_i \in G \text{ and } g_1 \cdot g_2 \cdot \ldots \cdot g_k \cdot x \text{ is defined}\}.$$

As we will see, the orbits of a Lie group of transformations are in fact submanifolds of M, but they may be of varying dimensions, or may not be regular. We distinguish two important subclasses of group actions.

Definition 1.26. Let G be a local group of transformations acting on M.

(a) The group G acts *semi-regularly* if all the orbits $\mathcal{O}$ are of the same dimension as submanifolds of M.
(b) The group G acts *regularly* if the action is semi-regular, and, in addition, for each point $x \in M$ there exist arbitrarily small neighbourhoods U of x with the property that each orbit of G intersects U in a pathwise connected subset.

Note that in particular, if G acts regularly on M then each orbit of G is a regular submanifold of M. However, the regularity condition on the group action is much stronger than this last statement, as Exercise 1.8 will bear out.

A group action is called *transitive* if there is only one orbit, namely the manifold M itself. Clearly any transitive group of transformations acts regularly. In most of our applications, the most interesting group actions will *not* be transitive.

Example 1.27. *Examples of Transformation Groups.*

(a) The group of *translations* in $\mathbb{R}^m$: Let $a \neq 0$ be a fixed vector in $\mathbb{R}^m$, and let $G = \mathbb{R}$. Define

$$\Psi(\varepsilon, x) = x + \varepsilon a, \qquad x \in \mathbb{R}^m, \quad \varepsilon \in \mathbb{R}.$$

This is readily seen to give a global group action. The orbits are straight lines parallel to a, so the action is regular with one-dimensional orbits.

(b) Groups of *scale transformations*: Let $G = \mathbb{R}^+$ be the multiplication group. Fix real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$, not all zero. Then $\mathbb{R}^+$ acts on $\mathbb{R}^m$ by the scaling transformations

$$\Psi(\lambda, x) = (\lambda^{\alpha_1} x^1, \ldots, \lambda^{\alpha_m} x^m), \qquad \lambda \in \mathbb{R}^+, \quad x = (x^1, \ldots, x^m) \in \mathbb{R}^m.$$

The orbits of this action are all one-dimensional regular submanifolds of $\mathbb{R}^m$, except for the singular orbit consisting of just the origin $\{0\}$. For instance, in the special case of $\mathbb{R}^2$ with $\Psi(\lambda, (x, y)) = (\lambda x, \lambda^2 y)$, the orbits are halves of the parabolas $y = kx^2$ (corresponding to either $x > 0$ or $x < 0$), the positive and negative y-axes, and the origin. In general, this scaling group action is regular on the open subset $\mathbb{R}^m \backslash \{0\}$. These group actions arise in the dimensional analysis of partial differential equations and historically provided the main impetus behind the development of the general theory of group-invariant solutions of differential equations.

(c) An action similar to the following comes up in the study of the heat equation. Let $M = \mathbb{R}^2$, $G = \mathbb{R}$ and consider the map

$$\Psi(\varepsilon, (x, y)) = \left(\frac{x}{1 - \varepsilon x}, \frac{y}{1 - \varepsilon x}\right),$$

which is defined on

$$\mathscr{U} = \left\{(\varepsilon, (x, y)) \colon \varepsilon < \frac{1}{x} \text{ for } x > 0, \text{ or } \varepsilon > \frac{1}{x}, \text{ for } x < 0\right\} \subset \mathbb{R} \times \mathbb{R}^2.$$

To show that this is indeed a group action, we must check condition (a) of Definition 1.25:

$$\begin{aligned}
\Psi(\delta, \Psi(\varepsilon, (x, y)) &= \Psi\left(\delta, \left(\frac{x}{1 - \varepsilon x}, \frac{y}{1 - \varepsilon x}\right)\right) \\
&= \left(\frac{x/(1 - \varepsilon x)}{1 - \delta x/(1 - \varepsilon x)}, \frac{y/(1 - \varepsilon x)}{1 - \delta x/(1 - \varepsilon x)}\right) \\
&= \left(\frac{x}{1 - (\delta + \varepsilon)x}, \frac{y}{1 - (\delta + \varepsilon)x}\right) \\
&= \Psi(\delta + \varepsilon, (x, y))
\end{aligned}$$