where $c_1, \dots, c_{10}$ are constants. For instance, we find

$$\xi_{xxx} = \eta_{xxy} = -\xi_{xyy} = -\tau_{yyt} = -\eta_{ytt} = -\xi_{xtt} = -\tau_{xxt} = -\xi_{xxx},$$

hence all these third order derivatives vanish; similar arguments prove that *all* third order derivatives of ξ, η and τ are zero, and the structure of the resulting quadratic polynomials follows easily from (2.63), (2.64).

Next the coefficient of u_x^2 (or u_y^2 or u_t^2) in (2.62) says $\phi_{uu} = 0$, so

$$\phi(x, y, t, u) = \beta(x, y, t)u + \alpha(x, y, t).$$

Finally, the coefficients of the linear terms in the first order derivatives of u, and the terms without u in them at all yield the relations

$$\begin{aligned} 2\beta_x &= \xi_{xx} + \xi_{yy} - \xi_{tt}, \\ 2\beta_y &= \eta_{xx} + \eta_{yy} - \eta_{tt}, \\ 2\beta_t &= \tau_{tt} - \tau_{xx} - \tau_{yy}, \\ \alpha_{tt} &- \alpha_{xx} - \alpha_{yy} = 0. \end{aligned}$$

Thus α is any solution of the wave equation, and

$$\beta = c_{11} - c_8 x - c_9 y - c_{10} t.$$

This gives the most general solution of the determining equations of the symmetry group of the wave equation. We have thus reproved the well-known result that the infinitesimal symmetry group of the wave equation is spanned by the ten vector fields

$$\begin{array}{ll} \partial_x, \quad \partial_y, \quad \partial_t, & \text{translations,} \\ \mathbf{r}_{xy} = -y\partial_x + x\partial_y, \quad \mathbf{r}_{xt} = t\partial_x + x\partial_t, \quad \mathbf{r}_{yt} = t\partial_y + y\partial_t, & \text{hyperbolic rotations,} \\ \mathbf{d} = x\partial_x + y\partial_y + t\partial_t, & \text{dilatation,} \\ \left.\begin{array}{l} \mathbf{i}_x = (x^2 - y^2 + t^2)\partial_x + 2xy\partial_y + 2xt\partial_t - xu\partial_u, \\ \mathbf{i}_y = 2xy\partial_x + (y^2 - x^2 + t^2)\partial_y + 2yt\partial_t - yu\partial_u, \\ \mathbf{i}_t = 2xt\partial_x + 2yt\partial_y + (x^2 + y^2 + t^2)\partial_t - tu\partial_u, \end{array}\right\} & \text{inversions,} \end{array} \tag{2.65}$$

which generate the conformal algebra for $\mathbb{R}^3$ with the given Lorentz metric, and the additional vector fields

$$u\partial_u, \qquad \mathbf{v}_\alpha = \alpha(x, y, t)\partial_u,$$

for α an arbitrary solution of the wave equation, reflecting the linearity of the equation.

The corresponding group transformations for the translations and dilatation are easily found. Of the rotations, owing to the indefinite character of the underlying metric $dt^2 - dx^2 - dy^2$, only the rotations in the (x, y)-plane are true rotations; the other two are "hyperbolic rotations". For example $\mathbf{r}_{xt}$

generates the group

$$(x, y, t) \mapsto (x \cosh \varepsilon + t \sinh \varepsilon, y, x \sinh \varepsilon + t \cosh \varepsilon).$$

The inversional groups can be constructed, as in Exercise 1.30, from the primary inversion

$$I(x, y, t) = \left(\frac{x}{t^2 - x^2 - y^2}, \frac{y}{t^2 - x^2 - y^2}, \frac{t}{t^2 - x^2 - y^2}\right),$$

which is defined provided (x, y, t) does not lie in the light cone $t^2 = x^2 + y^2$. We find that the group generated by $\mathbf{i}_x$ say, is given by first inverting, then translating the x-direction, and then re-inverting:

$$\exp(\varepsilon \mathbf{i}_x) = I \circ \exp(\varepsilon \partial_x) \circ I.$$

The general formula is

$$(x, y, t) \mapsto \left(\frac{x + \varepsilon(t^2 - x^2 - y^2)}{1 - 2\varepsilon x - \varepsilon^2(t^2 - x^2 - y^2)}, \frac{y}{1 - 2\varepsilon x - \varepsilon^2(t^2 - x^2 - y^2)}, \right.$$
$$\left. \frac{t}{1 - 2\varepsilon x - \varepsilon^2(t^2 - x^2 - y^2)}\right),$$

which is well defined even for (x, y, t) in the light cone (which is an invariant subvariety). The corresponding transformation of u under $\exp(\varepsilon \mathbf{i}_x)$ is then

$$u \mapsto \sqrt{1 - 2\varepsilon x - \varepsilon^2(t^2 - x^2 - y^2)}\, u.$$

We conclude that if $u = f(x, y, t)$ is a solution to the wave equation, so is

$$\tilde{u} = \frac{1}{\sqrt{1 + 2\varepsilon x - \varepsilon^2(t^2 - x^2 - y^2)}} f\left(\frac{x - \varepsilon(t^2 - x^2 - y^2)}{1 + 2\varepsilon x - \varepsilon^2(t^2 - x^2 - y^2)}, \right.$$
$$\left. \frac{y}{1 + 2\varepsilon x - \varepsilon^2(t^2 - x^2 - y^2)}, \frac{t}{1 + 2\varepsilon x - \varepsilon^2(t^2 - x^2 - y^2)}\right).$$

Example 2.44. *The Korteweg-de Vries Equation.* As a higher order example, we consider the Korteweg-de Vries equation

$$u_t + u_{xxx} + uu_x = 0, \tag{2.66}$$

which arises in the theory of long waves in shallow water and other physical systems in which both nonlinear and dispersive effects are relevant. A vector field $\mathbf{v} = \xi \partial_x + \tau \partial_t + \phi \partial_u$ generates a one-parameter symmetry group if and only if

$$\phi^t + \phi^{xxx} + u\phi^x + u_x \phi = 0 \tag{2.67}$$

whenever u satisfies (2.66). Here ϕ^t and ϕ^x, the coefficients of the first prolongation of $\mathbf{v}$, are determined by the explicit prolongation formulae (2.45); the coefficient of $\partial/\partial u_{xxx}$ in $\mathrm{pr}^{(3)}\, \mathbf{v}$ is

$$\phi^{xxx} = D_x^3 \phi - u_x D_x^3 \xi - u_t D_x^3 \tau - 3u_{xx} D_x^2 \xi - 3u_{xt} D_x^2 \tau - 3u_{xxx} D_x \xi - 3u_{xxt} D_x \tau.$$

Substituting into (2.67) and replacing u_t by $-u_{xxx} - uu_x$ wherever it occurs, we obtain the determining equations for the symmetry group. To analyze these, we work our way down the order of the derivatives which appear. The coefficient of u_{xxt} is $D_x\tau = 0$, hence τ depends only on t. The coefficient of u_{xx}^2 shows that $\xi_u = 0$. From the coefficient of u_{xxx}, we find $\tau_t = 3\xi_x$ (the ϕ_u-terms cancelling), hence $\xi = \frac{1}{3}\tau_t x + \sigma(t)$. Now the coefficient of u_{xx} reveals that $\phi_{uu} = 0 = \phi_{xu}$, so ϕ is linear in u, the coefficient of u being a function of t alone. The remaining terms in (2.67) are those involving u_x, which give

$$-\xi_t - u(\phi_u - \tau_t) + u(\phi_u - \xi_x) + \phi = 0,$$

and those without any derivatives of u,

$$\phi_t + \phi_{xxx} + u\phi_x = 0.$$

These all have the general solution

$$\begin{aligned} \xi &= c_1 + c_3 t + c_4 x, \\ \tau &= c_2 + 3c_4 t, \\ \phi &= c_3 - 2c_4 u, \end{aligned}$$

where c_1, c_2, c_3, c_4 are arbitrary constants. Therefore the symmetry algebra of the Korteweg-de Vries equation is spanned by the four vector fields

$$\begin{aligned} \mathbf{v}_1 &= \partial_x, && \text{space translation,} \\ \mathbf{v}_2 &= \partial_t, && \text{time translation,} \\ \mathbf{v}_3 &= t\partial_x + \partial_u, && \text{Galilean boost,} \\ \mathbf{v}_4 &= x\partial_x + 3t\partial_t - 2u\partial_u, && \text{scaling.} \end{aligned} \tag{2.68}$$

Their commutator table is

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	0	0	0	$\mathbf{v}_1$
$\mathbf{v}_2$	0	0	$\mathbf{v}_1$	$3\mathbf{v}_2$
$\mathbf{v}_3$	0	$-\mathbf{v}_1$	0	$-2\mathbf{v}_3$
$\mathbf{v}_4$	$-\mathbf{v}_1$	$-3\mathbf{v}_2$	$2\mathbf{v}_3$	0

Exponentiation shows that if $u = f(x, t)$ is a solution of the Korteweg-de Vries equation, so are

$$\begin{aligned} u^{(1)} &= f(x - \varepsilon, t), \\ u^{(2)} &= f(x, t - \varepsilon), \\ u^{(3)} &= f(x - \varepsilon t, t) + \varepsilon, \\ u^{(4)} &= e^{-2\varepsilon} f(e^{-\varepsilon} x, e^{-3\varepsilon} t). \end{aligned} \qquad \varepsilon \in \mathbb{R}.$$

These can easily be checked by inspection. (For the reader familiar with the many remarkable "soliton" properties of the Korteweg-de Vries equation, this list of symmetries may seem disappointingly small. Further symmetry properties, reflecting the existence of infinitely many conservation laws and, presumably, the linearization of the inverse scattering method, cf. Newell, [1], will require our development of the theory of generalized symmetries in Chapters 5 and 7.)

Example 2.45. *The Euler Equations.* As a last illustration of the basic method of computing symmetry groups, we consider the system of Euler equations for the motion of an inviscid, incompressible ideal fluid in a three-dimensional domain. Here there are four independent variables, $\mathbf{x} = (x, y, z)$ being spatial coordinates and t the time, together with four dependent variables, the velocity field $\mathbf{u} = (u, v, w)$ and the pressure p. (The density ρ is normalized to be 1.) In vector notation, the system has the form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p, \qquad \nabla\cdot\mathbf{u} = 0, \tag{2.69}$$

in which the components of the nonlinear terms $\mathbf{u}\cdot\nabla\mathbf{u}$ are

$$(uu_x + vu_y + wu_z,\ uv_x + vv_y + wv_z,\ uw_x + vw_y + ww_z).$$

An infinitesimal symmetry of the Euler equations will be a vector field of the form

$$\mathbf{v} = \xi\partial_x + \eta\partial_y + \zeta\partial_z + \tau\partial_t + \phi\partial_u + \psi\partial_v + \chi\partial_w + \pi\partial_p,$$

where $\xi, \eta, \ldots, \pi$ are functions of $\mathbf{x}$, t, $\mathbf{u}$ and p. Applying the first prolongation $\mathrm{pr}^{(1)}\,\mathbf{v}$ to the Euler equations (2.69), we find the following system of symmetry equations

$$\phi^t + u\phi^x + v\phi^y + w\phi^z + u_x\phi + u_y\psi + u_z\chi = -\pi^x, \tag{2.70a}$$

$$\psi^t + u\psi^x + v\psi^y + w\psi^z + v_x\phi + v_y\psi + v_z\chi = -\pi^y, \tag{2.70b}$$

$$\chi^t + u\chi^x + v\chi^y + w\chi^z + w_x\phi + w_y\psi + w_z\chi = -\pi^z, \tag{2.70c}$$

$$\phi^x + \psi^y + \chi^z = 0, \tag{2.70d}$$

which must be satisfied whenever $\mathbf{u}$ and p satisfy (2.69). Here ϕ^t, ψ^x, etc. are the coefficients of the first order derivatives $\partial/\partial u_t$, $\partial/\partial v_x$, etc. appearing in $\mathrm{pr}^{(1)}\,\mathbf{v}$; typical expressions for these functions follow from the prolongation formula (2.43), so

$$\phi^t = D_t\phi - u_xD_t\xi - u_yD_t\eta - u_zD_t\zeta - u_tD_t\tau,$$

$$\psi^x = D_x\psi - v_xD_x\xi - v_yD_x\eta - v_zD_x\zeta - v_tD_x\tau,$$

and so on.

Since (2.70) need only hold on solutions of (2.69), we can substitute for p_x, p_y, p_z and w_z wherever they occur in (2.70) using their expressions from the four equations in (2.69). We may then equate all the coefficients of the remaining first order derivatives of $\mathbf{u}$, p in (2.70) and solve the resulting system of determining equations for $\xi, \eta, \ldots, \pi$.

As a first step, let us show that the symmetry is necessarily projectable, meaning that ξ, η, ζ and τ only depend on $\mathbf{x}$ and t. The coefficient of p_t in (2.70a) is

$$\phi_p - \xi_p u_x - \eta_p u_y - \zeta_p u_z - \tau_p u_t = D_x \tau = \tau_x + \tau_u u_x + \tau_v v_x + \tau_w w_x + \tau_p p_x,$$

Therefore $\tau_v = \tau_w = 0$, and, by consideration of the same coefficient in (2.70b), $\tau_u = 0$ also. Furthermore, if we substitute for p_x according to (2.69), we find

$$\phi_p = \tau_x, \qquad \psi_p = \tau_y, \qquad \chi_p = \tau_z, \tag{2.71}$$

$$\xi_p = u\tau_p, \qquad \eta_p = v\tau_p, \qquad \zeta_p = w\tau_p, \tag{2.72}$$

where the equations for ψ_p and χ_p come from similar considerations in (2.70b, c). Next consider the quadratic monomial $v_t v_x$ in (2.70a). This can also arise from the monomials $p_y v_x$, $p_y v_t$ and p_y^2, all of which only appear in π^x. The resulting coefficient is $0 = -\eta_v$. Similarly, the coefficient of $v_t w_x$ in (2.70a) proves that $\eta_w = 0$. Further analysis of quadratic terms in (2.70a, b, c) proves that ξ, η, ζ are independent of u, v, w. Then differentiating (2.72) with respect to u, v and w we find $\tau_p = 0$, hence $\xi_p = \eta_p = \zeta_p = 0$ and the symmetry is projectable.

The next step is to look at the coefficients of u_t, v_t and w_t in (2.70d), keeping in mind that these can also arise from ∇p upon substitution. This implies that

$$\phi_p + \tau_x = \psi_p + \tau_y = \chi_p + \tau_z = 0.$$

Comparison with (2.71) proves that τ depends on t alone, and ϕ, ψ, χ are independent of the pressure. Consider next the coefficients of v_t and v_x in (2.70a), which are

$$\phi_v = -\eta_x, \qquad \phi_v = -\eta_x - \pi_v.$$

Thus $\pi_v = 0$, and, by similar considerations, π does not depend on u or w either. From the coefficients of u_t and w_t we also find that

$$\phi_u = \tau_t - \xi_x + \pi_p, \qquad \phi_w = -\zeta_x,$$

and so on. These all imply that ϕ, ψ, χ have the general form

$$\phi = (\tau_t - \xi_x + \pi_p)u - \eta_x v - \zeta_x w + \hat{\phi},$$

$$\psi = -\xi_y u + (\tau_t - \eta_y + \pi_p)v - \zeta_y w + \hat{\psi},$$

$$\chi = -\xi_z u - \eta_z v + (\tau_t - \zeta_z + \pi_p)w + \hat{\chi},$$

where $\hat{\phi}$, $\hat{\psi}$ and $\hat{\chi}$ depend only on $\mathbf{x}$ and t. The coefficients of the spatial derivatives of $\mathbf{u}$ in (2.70a, b, c) then require

$$\hat{\phi} = \xi_t, \qquad \hat{\psi} = \eta_t, \qquad \hat{\chi} = \zeta_t,$$
$$\xi_x = \eta_y = \zeta_z = \tau_t + \tfrac{1}{2}\pi_p,$$
$$\xi_y + \eta_x = \xi_z + \zeta_x = \eta_z + \zeta_y = 0.$$

In particular, the spatial component $\xi\partial_x + \eta\partial_y + \zeta\partial_z$ of $\mathbf{v}$ generates a (time-dependent) conformal symmetry group of $\mathbb{R}^3$ with the Euclidean metric. The remaining terms in (2.70) involve no derivatives of $\mathbf{u}$ or p. These require ξ, η, ζ to be linear in x, y, z and, furthermore,

$$\xi_{yt} = \xi_{zt} = \eta_{xt} = \eta_{zt} = \zeta_{xt} = \zeta_{yt} = 0,$$
$$\xi_{xt} = \eta_{yt} = \zeta_{zt} = \tau_{tt},$$
$$\xi_{tt} = -\pi_x, \qquad \eta_{tt} = -\pi_y, \qquad \zeta_{tt} = -\pi_z.$$

Therefore

$$\begin{aligned}
\xi &= \delta_t x + c_1 y - c_2 z + \alpha,\\
\eta &= -c_1 x + \delta_t y + c_3 z + \beta,\\
\zeta &= c_2 x - c_3 y + \delta_t z + \gamma,\\
\tau &= 2\delta + c_4 t + c_5,\\
\phi &= -(\delta_t + c_4)u + c_1 v - c_2 w + \alpha_t,\\
\psi &= -c_1 u - (\delta_t + c_4)v + c_3 w + \beta_t,\\
\chi &= c_2 u - c_3 v - (\delta_t + c_4)w + \gamma_t,\\
\pi &= -2(\delta_t + c_4)p - \tfrac{1}{2}\delta_{tt}(x^2 + y^2 + z^2) - \alpha_{tt}x - \beta_{tt}y - \gamma_{tt}z + \theta,
\end{aligned}$$

in which α, β, γ, δ and θ are functions of t, and c_1, c_2, c_3, c_4, c_5 constants. Finally, the divergence-free condition (2.70d) imposes the further restriction that $\delta_{tt} = 0$, so $\delta = c_6 t + c_7$.

We have thus shown that the symmetry group of the Euler equations in three dimensions is generated by the vector fields

$$\begin{aligned}
&\left.\begin{aligned}
\mathbf{v}_\alpha &= \alpha\partial_x + \alpha_t\partial_u - \alpha_{tt}x\partial_p,\\
\mathbf{v}_\beta &= \beta\partial_y + \beta_t\partial_v - \beta_{tt}y\partial_p,\\
\mathbf{v}_\gamma &= \gamma\partial_z + \gamma_t\partial_w - \gamma_{tt}z\partial_p,
\end{aligned}\right\} && \text{(moving coordinates)}\\
&\mathbf{v}_0 = \partial_t, && \text{(time translation)}\\
&\left.\begin{aligned}
\mathbf{d}_1 &= x\partial_x + y\partial_y + z\partial_z + t\partial_t,\\
\mathbf{d}_2 &= t\partial_t - u\partial_u - v\partial_v - w\partial_w - 2p\partial_p,
\end{aligned}\right\} && \text{(scaling)} \qquad (2.73)\\
&\left.\begin{aligned}
\mathbf{r}_{xy} &= y\partial_x - x\partial_y + v\partial_u - u\partial_v,\\
\mathbf{r}_{zx} &= x\partial_z - z\partial_x + u\partial_w - w\partial_u,\\
\mathbf{r}_{yz} &= z\partial_y - y\partial_z + w\partial_v - v\partial_w,
\end{aligned}\right\} && \text{(rotations)}\\
&\mathbf{v}_\theta = \theta\partial_p, && \text{(pressure changes)}
\end{aligned}$$

in which α, β, γ and θ are arbitrary functions of t. The corresponding one-parameter groups of symmetries of the Euler equations are then:

(a) Transformation to an arbitrarily moving coordinate system:

$$G_{\boldsymbol{\alpha}}: (\mathbf{x}, t, \mathbf{u}, p) \mapsto (\mathbf{x} + \varepsilon\boldsymbol{\alpha}(t), t, \mathbf{u} + \varepsilon\boldsymbol{\alpha}_t, p - \varepsilon\mathbf{x}\cdot\boldsymbol{\alpha}_{tt} - \tfrac{1}{2}\varepsilon^2\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}_{tt}),$$

where $\boldsymbol{\alpha} = (\alpha, \beta, \gamma)$ and $G_{\boldsymbol{\alpha}}$ is generated by the linear combination $\mathbf{v}_{\boldsymbol{\alpha}} = \mathbf{v}_\alpha + \mathbf{v}_\beta + \mathbf{v}_\gamma$ of the first three vector fields.

(b) Time translations:

$$G_0: (\mathbf{x}, t, \mathbf{u}, p) \mapsto (\mathbf{x}, t + \varepsilon, \mathbf{u}, p).$$

(c) Scale transformations:

$$G_1: (\mathbf{x}, t, \mathbf{u}, p) \mapsto (\lambda\mathbf{x}, \lambda t, \mathbf{u}, p),$$

$$G_2: (\mathbf{x}, t, \mathbf{u}, p) \mapsto (\mathbf{x}, \lambda t, \lambda^{-1}\mathbf{u}, \lambda^{-2}p),$$

where $\lambda = e^\varepsilon$ is a multiplicative group parameter.

(d) The group

$$\mathrm{SO}(3): (\mathbf{x}, t, \mathbf{u}, p) \mapsto (R\mathbf{x}, t, R\mathbf{u}, p)$$

of simultaneous rotations in both space and the velocity field $\mathbf{u}$. Here R is an arbitrary 3×3 orthogonal matrix.

(e) Pressure changes:

$$G_p: (\mathbf{x}, t, \mathbf{u}, p) \mapsto (\mathbf{x}, t, \mathbf{u}, p + \varepsilon\theta(t)).$$

The corresponding action on solutions of the Euler equations says that if $\mathbf{u} = \mathbf{f}(\mathbf{x}, t)$, $p = g(\mathbf{x}, t)$ are solutions, so are

$$\begin{aligned}
G_{\boldsymbol{\alpha}}:&\quad \mathbf{u} = \mathbf{f}(\mathbf{x} - \varepsilon\boldsymbol{\alpha}(t), t) + \varepsilon\boldsymbol{\alpha}_t, &\quad p &= g(\mathbf{x} - \varepsilon\boldsymbol{\alpha}(t), t) - \varepsilon\mathbf{x}\cdot\boldsymbol{\alpha}_{tt} + \tfrac{1}{2}\varepsilon^2\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}_{tt},\\
G_0:&\quad \mathbf{u} = \mathbf{f}(\mathbf{x}, t - \varepsilon), &\quad p &= g(\mathbf{x}, t - \varepsilon),\\
G_1:&\quad \mathbf{u} = \mathbf{f}(\lambda\mathbf{x}, \lambda t), &\quad p &= g(\lambda\mathbf{x}, \lambda t),\\
G_2:&\quad \mathbf{u} = \lambda\mathbf{f}(\mathbf{x}, \lambda t), &\quad p &= \lambda^2 g(\mathbf{x}, \lambda t),\\
\mathrm{SO}(3):&\quad \mathbf{u} = R\mathbf{f}(R^{-1}\mathbf{x}, t), &\quad p &= g(R^{-1}\mathbf{x}, t),\\
G_p:&\quad \mathbf{u} = \mathbf{f}(\mathbf{x}, t), &\quad p &= g(\mathbf{x}, t) + \varepsilon\theta(t).
\end{aligned}$$

(In G_1 and G_2 we have replaced λ by λ^{-1}.) Note that in our change to a moving coordinate system $G_{\boldsymbol{\alpha}}$, we must adjust the pressure according to the induced acceleration $\varepsilon\boldsymbol{\alpha}_{tt}$. The final group G_p results from the fact that the pressure p is only defined up to the addition of an arbitrary function of t. This completes the list of symmetries of the Euler equations.

2.5. Integration of Ordinary Differential Equations

One of the most appealing applications of Lie group theory is to the problem of integrating ordinary differential equations. Lie's fundamental observation was that knowledge of a sufficiently large group of symmetries of a system of ordinary differential equations allows one to integrate the system by quadratures (indefinite integrals) and thereby deduce the general solution. This approach unifies and significantly extends the various special methods introduced for the integration of certain types of first order equations such as homogeneous, separable, exact and so on. Similar results hold for systems of ordinary differential equations. In this section, a comprehensive survey of these methods is presented.

First Order Equations

We begin by considering a single first order ordinary differential equation

$$\frac{du}{dx} = F(x, u). \tag{2.74}$$

It will be shown that if this equation is invariant under a one-parameter group of transformations, then it can be integrated by quadrature. If G is a one-parameter group of transformations on an open subset $M \subset X \times U \simeq \mathbb{R}^2$, let

$$\mathbf{v} = \xi(x, u)\frac{\partial}{\partial x} + \phi(x, u)\frac{\partial}{\partial u}$$

be its infinitesimal generator. The first prolongation of $\mathbf{v}$ is the vector field

$$\mathrm{pr}^{(1)}\,\mathbf{v} = \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial u} + \phi^x\frac{\partial}{\partial u_x}, \tag{2.75}$$

where

$$\phi^x = D_x\phi - u_x D_x\xi = \phi_x + (\phi_u - \xi_x)u_x - \xi_u u_x^2.$$

Thus the infinitesimal condition that G be a symmetry group of (2.74) is

$$\frac{\partial \phi}{\partial x} + \left(\frac{\partial \phi}{\partial u} - \frac{\partial \xi}{\partial x}\right)F - \frac{\partial \xi}{\partial u}F^2 = \xi\frac{\partial F}{\partial x} + \phi\frac{\partial F}{\partial u}, \tag{2.76}$$

and any solution $\xi(x, u)$, $\phi(x, u)$ of the partial differential equation (2.76) generates a one-parameter symmetry group of our ordinary differential equation. Of course, in practice finding solutions of the determining equation (2.76) is usually a much more difficult problem than solving the original ordinary differential equation. However, led on by inspired guess-work, or geometric intuition, we may be able to ascertain a particular solution of (2.76) which will allow us to integrate (2.74). Herein lies the art of Lie's method.

Once we have found a symmetry group G, there are several different methods we can employ to integrate (2.74). Suppose $\mathbf{v}$ is the infinitesimal generator of the symmetry group, and assume that $\mathbf{v}|_{(x_0, u_0)} \neq 0$. (If the vector field $\mathbf{v}$ vanishes at a point (x_0, u_0), then we will expect some kind of singularity for solutions near this point. The behaviour of solutions $u = f(x)$ near such a singularity can be deduced by extrapolation once the equation has been integrated at nearby values of x.) According to Proposition 1.29, we can introduce new coordinates

$$y = \eta(x, u), \qquad w = \zeta(x, u), \tag{2.77}$$

near (x_0, u_0) such that in the (y, w)-coordinates the symmetry vector field has the simple translational form $\mathbf{v} = \partial/\partial w$, with first prolongation

$$\mathrm{pr}^{(1)}\,\mathbf{v} = \mathbf{v} = \partial/\partial w$$

also. Thus in the new coordinate system, in order to be invariant, the differential equation must be independent of w, so (2.74) is equivalent to the elementary equation

$$\frac{dw}{dy} = H(y),$$

for some function H. This equation is trivially integrated by quadrature, with

$$w = \int H(y)\, dy + c$$

for some constant c. Re-substituting the expressions (2.77) for w and y, we obtain a solution $u = f(x)$ of our original system in implicit form.

The change of variables (2.77) is constructed using the methods for finding group invariants presented in Section 2.1. Indeed, (1.16) implies that $\mathbf{v}$ is transformed into the form $\partial/\partial w$ provided η and ζ satisfy the linear partial differential equations

$$\mathbf{v}(\eta) = \xi \frac{\partial \eta}{\partial x} + \phi \frac{\partial \eta}{\partial u} = 0, \tag{2.78a}$$

$$\mathbf{v}(\zeta) = \xi \frac{\partial \zeta}{\partial x} + \phi \frac{\partial \zeta}{\partial u} = 1. \tag{2.78b}$$

The first of these equations just says that $\eta(x, u)$ is an invariant of the group generated by $\mathbf{v}$. We can thus find η by solving the associated characteristic ordinary differential equation

$$\frac{dx}{\xi(x, u)} = \frac{du}{\phi(x, u)}. \tag{2.79}$$

Often the corresponding solution ζ of (2.78b) can be found by inspection. More systematically, we can introduce an auxiliary variable v and note that $\zeta(x, u)$ satisfies (2.78b) if and only if the function $\chi(x, u, v) = v - \zeta(x, u)$ is an invariant of the vector field $\mathbf{w} = \mathbf{v} + \partial_v = \xi\partial_x + \phi\partial_u + \partial_v$. Thus we require

$$\mathbf{w}(\chi) = \xi \frac{\partial \chi}{\partial x} + \phi \frac{\partial \chi}{\partial u} + \frac{\partial \chi}{\partial v} = 0.$$

This we can again solve by the method of characteristics,

$$\frac{dx}{\xi(x, u)} = \frac{du}{\phi(x, u)} = \frac{dv}{1}, \tag{2.80}$$

where we seek a solution of the form $v - \zeta(x, u) = k$, for k an arbitrary constant of integration.

In general, it may be just as difficult to solve (2.79) and (2.80), being again ordinary differential equations, as it was to integrate the original differential equation. In particular, if

$$\phi(x, u)/\xi(x, u) = F(x, u), \tag{2.81}$$

then we automatically have a solution of the determining equation (2.76), so such a vector field $\mathbf{v} = \xi\partial_x + \phi\partial_u$ is always a symmetry of the equation. In this case, finding the invariant $\eta(x, u)$ of the group, i.e. solving (2.79), is exactly the same problem as integrating the original equation, so the method is of no help. Only when the group of symmetries is of a reasonably simple form, so that we can explicitly solve (2.79), (2.80), do we stand any chance of making progress towards the solution of our problem.

Example 2.46. A homogeneous equation is one of the form

$$\frac{du}{dx} = F\left(\frac{u}{x}\right),$$

where F only depends on the ratio of u to x. Such an equation has the group of scaling transformations

$$G: (x, u) \mapsto (\lambda x, \lambda u), \qquad \lambda > 0,$$

as a symmetry group. This can be seen directly from the form of the first prolongation of G,

$$\mathrm{pr}^{(1)}\, G: (x, u, u_x) \mapsto (\lambda x, \lambda u, u_x),$$

which obviously leaves the equation invariant. Alternatively, we can look at the infinitesimal generator

$$\mathbf{v} = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},$$

which, according to (2.75), has first prolongation $\mathrm{pr}^{(1)}\,\mathbf{v} = \mathbf{v}$, and use the infinitesimal criterion of invariance.

New coordinates y, w satisfying (2.78) are given by

$$y = \frac{u}{x}, \qquad w = \log x.$$

Employing the chain rule, we find

$$\frac{du}{dx} = \frac{du/dy}{dx/dy} = \frac{x(1 + yw_y)}{xw_y} = \frac{1 + yw_y}{w_y},$$