

10 Dimensional continuation, regularization, minimal subtraction (MS). Renormalization group (RG) functions

In this chapter, we introduce the notions of *dimensional continuation* and *dimensional regularization*, by defining a continuation of Feynman diagrams to analytic functions of the space dimension.

Dimensional continuation, which is essential for generating Wilson–Fisher’s famous ε -expansion in the theory of critical phenomena [75], and dimensional regularization seem to have no meaning outside the perturbative expansion of quantum field theory (QFT), and thus no direct physical interpretation.

Dimensional regularization is a powerful regularization technique, which is often used, when applicable, because it leads to simpler perturbative calculations. Dimensional regularization *performs a partial renormalization*, cancelling what would show up as power-law divergences in momentum or lattice regularization. In particular, it cancels the commutator of quantum operators in local QFTs. These cancellations may be convenient, but may also, occasionally remove divergences that have an important physical meaning.

It is not applicable when some essential property of the field theory is specific to the initial dimension. For example, in even space dimensions, the relation between γ_S (identical to γ_5 in four dimensions) and the other γ matrices involving the completely antisymmetric tensor $\epsilon_{\mu_1 \dots \mu_d}$ (see Section A12) may be needed in theories violating parity symmetry.

Its use requires some care in massless theories, because its rules may lead to unwanted cancellations between ultraviolet (UV) and infrared (IR) logarithmic divergences.

Within the framework of dimensional regularization, we also introduce the concept of renormalization by *MS*. We show that the MS scheme simplifies the expressions of renormalization constants and RG functions.

We perform explicit calculations at two-loop order, first in the simple one-component ϕ^4 QFT, and then in an N -component QFT with a general four-field interaction. The results of these calculations will be directly used for lower order estimates of critical exponents (see Sections 15.2–15.5).

10.1 Dimensional continuation and dimensional regularization

We first define dimensional continuation, and then dimensional regularization, which simplifies perturbative calculations in QFT.

10.1.1 Dimensional continuation

We express the Fourier representation of a regularized scalar propagator $\tilde{\Delta}(p)$ of a massive theory, in d dimensions, as a Laplace transform (see equation (A8.23)),

$$\tilde{\Delta}(p) = \int_0^\infty dt \rho(t\Lambda^2) e^{-(p^2+m^2)t}, \quad (10.1)$$

where Λ is the cut-off and

$$\lim_{t \rightarrow \infty} \rho(t) = 1, \quad \rho(t) = O(t^\sigma), \text{ with } \sigma > d, \text{ for } t \rightarrow 0.$$

For $\rho(t) \equiv 1$, one recovers the unregularized propagator $1/(p^2 + m^2)$.

Using the representation (10.1), and the property that the momentum associated with an internal line is a linear combination of loop and external momenta, one can then express a scalar Feynman diagram γ with constant vertices, and external momenta p_α , in the form

$$I_\gamma(\{p_\alpha\}) = (2\pi)^{-Ld} \int \prod_{i=1}^I dt_i \rho(\Lambda^2 t_i) \prod_{\ell=1}^L d^d q_\ell \\ \times \exp \left[- \sum_{\ell, \ell'=1}^L q_\ell \cdot q_{\ell'} M_{\ell\ell'}(t_i) - 2 \sum_{\ell=1}^L q_\ell \cdot k_\ell(\{p_\alpha\}, t_i) - S[\{(p_\alpha\}, t_i) \right]. \quad (10.2)$$

The Gaussian integration over all loop momenta q_ℓ can then be performed. The resulting expression is

$$I_\gamma(p) = \frac{1}{(4\pi)^{Ld/2}} \int \prod_{i=1}^I dt_i \rho(\Lambda^2 t_i) (\det \mathbf{M})^{-d/2} \\ \times \exp \left[\sum_1^L k_\ell (M^{-1})_{\ell\ell'} k_{\ell'} - S(\{p_\alpha\}, t_i) \right]. \quad (10.3)$$

In the integrated expression, the dependence in the dimension d is explicit and, therefore, continuation in d is achieved. This continuation makes it possible to study properties of phase transitions near two (Chapter 19) and four dimensions (Chapter 15).

10.1.2 Dimensional regularization: Defining properties of d -dimensional integrals

After continuation, we can choose $\text{Re } d$, small enough, even negative if necessary, such that all Feynman diagrams become formally convergent in the sense of power counting and take the infinite cut-off limit. The result is a meromorphic function of d , the dimensional regularization of the Feynman diagram [76].

If $d > 2$, one can then also take the massless limit.

One can also define directly an integral in d dimensions by a set of conditions which, when d is an integer, lead to the usual integral. It should satisfy the three conditions,

- (i) $\int d^d p F(p+q) = \int d^d p F(p)$ translation
- (ii) $\int d^d p F(\lambda p) = |\lambda|^{-d} \int d^d p F(p)$ dilatation
- (iii) $\int d^d p d^{d'} q f(p) g(q) = \int d^d p f(p) \int d^{d'} q g(q)$ factorization.

These simple rules define a dimensional continuation of Feynman diagrams. It is equivalent to the previous method. Indeed, from property (iii), one derives

$$\int d^d p e^{-tp^2} = \left(\int ds e^{-ts^2} \right)^d = \left(\frac{\pi}{t} \right)^{d/2}. \quad (10.4)$$

and then one can calculate the Feynman diagram using the Laplace representation. However, one can also use the rules more directly with an unregularized propagator.

Examples.

(i) The massless propagator is given by

$$\Delta(x) = \frac{1}{(2\pi)^d} \int d^d p \frac{e^{ip \cdot x}}{p^2} = \frac{1}{(2\pi)^d} \int_0^\infty ds \int d^d p e^{-sp^2 + ip \cdot x}.$$

The Gaussian momentum integral can be performed, followed by the s integral, and the result is

$$\Delta(x) = \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}} |x|^{2-d}. \quad (10.5)$$

The pole at $d = 2$ reflects the IR divergences of the two-dimensional massless theory.

(ii) The one-loop contribution to the two-point function in the massless ($m = 0$) ϕ^3 QFT as well as the contribution to the four-point function in the ϕ^4 QFT (first diagram of Fig. 9.2) involve the function

$$\begin{aligned} I_\gamma(p) &= \int d^d x e^{ip \cdot x} \Delta^2(x) = \frac{\Gamma^2(d/2 - 1)}{16\pi^d} \int d^d x e^{ip \cdot x} |x|^{4-2d} \\ &= \frac{\Gamma^2(d/2 - 1)}{16\pi^d \Gamma(d - 2)} \int_0^\infty ds s^{d-3} \int d^d x e^{ip \cdot x - s x^2}. \end{aligned}$$

Finally, integrating over x and then s , one obtains,

$$I_\gamma(p) = \frac{1}{(4\pi)^{d/2}} \Gamma(2 - d/2) \frac{\Gamma^2((d/2) - 1)}{\Gamma(d - 2)} (p^2)^{(d/2)-2}. \quad (10.6)$$

This expression has a pole at $d = 2$ corresponding to IR (low momentum) singularities because the theory is massless and has poles at $d = 4, 6$, and so on, which clearly are consequences of the UV (large momentum) divergences of the Feynman diagram.

It is interesting to explain the interplay between dimensional continuation and cut-off regularization in this example. If we regularize the propagator, for example by the method of Pauli–Villars, the Feynman diagram I_γ becomes a regular function of d for $d > 2$ up to some even integer larger than 4.

In the neighbourhood of $d = 4$, it has the form

$$I_\gamma \sim \frac{1}{8\pi^2(4-d)} \left[(p^2)^{(d/2)-2} - \Lambda^{d-4} \right], \quad \text{for } d \rightarrow 4.$$

At $d < 4$ fixed, the limit of infinite cut-off yields the continuation of the initial diagram with a pole at $d = 4$. At cut-off fixed, the $d = 4$ limit yields a finite result in which $\ln \Lambda$ has replaced the pole at $d = 4$.

(iii) The one-loop contribution to the two-point function in a massive ϕ^4 QFT (Fig. 9.1) is proportional to

$$\begin{aligned} \Omega_d(m) &= \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m^2} = \int_0^\infty dt \frac{d^d q}{(2\pi)^d} e^{-t(q^2 + m^2)} \\ &= \frac{1}{(4\pi)^{d/2}} \int_0^\infty dt t^{-d/2} e^{-m^2 t} = \frac{1}{(4\pi)^{d/2}} \Gamma(1 - d/2) m^{d-2}. \end{aligned} \quad (10.7)$$

This expression has poles for $d = 2 + 2n$, $n \geq 0$, corresponding to expected UV divergences, but nothing equivalent to quadratic divergences. The divergence at $d = 4$ obtained after analytic continuation has the same nature as at $d = 2$.

Important remark. The expression (10.7) has the following property: for $\text{Re } d > 2$, the limit $m = 0$ vanishes. The result generalizes, leading to the peculiar result

$$\int \frac{d^d p}{p^2} = 0, \quad (10.8)$$

for this integral that exists for no value of d .

The argument generalizes, for $\text{Re } d > 2n$, to

$$\int \frac{d^d p}{p^{2n}} = 0. \quad (10.9)$$

This result confirms that dimensional regularization is a partial renormalization since all UV divergences that take the form of powers of the cut-off, in cut-off regularization, are cancelled.

Dangerous extrapolation. Let us also point out one dangerous consequence. The result (10.9) extrapolated naively to $d \rightarrow 2n$ would lead to a cancellation between a UV and IR divergence:

$$\int \frac{d^d p}{p^{2n}} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \left[\int_1^\infty p^{d-1-2n} dp + \int_0^1 p^{d-1-2n} dp \right].$$

For example, in a field theory involving massless fields having a propagator $1/p^2$, IR divergences appear in 2 dimensions. If this theory is renormalizable in 2 dimensions, it also has UV divergences. In such a case, UV and IR singularities get mixed. Therefore, to be able to identify poles coming from the large momentum region, it is necessary to introduce an IR cut-off, for example, by giving a mass to the field.

Operator commutation in local products. Dimensional regularization leads to the commutation of quantum operators in local field theories. Indeed commutators, for example, in the case of a scalar field take the form (6.7),

$$[\hat{\phi}(x), \hat{\pi}(y)] = i\hbar \delta^{(d-1)}(x-y) = i\hbar (2\pi)^{1-d} \int d^{d-1}p e^{ip \cdot (x-y)},$$

where $\hat{\pi}(x)$ is the momentum conjugate to the field $\hat{\phi}(x)$. As we have already stressed, in a local theory all fields are taken at the same point and, therefore a commutation in the product $\hat{\phi}(x)\hat{\pi}(x)$ generates a divergent contribution (for $d > 1$), formally proportional to

$$\delta^{d-1}(0) = (2\pi)^{1-d} \int d^{d-1}p.$$

In dimensional regularization, taking the limit $n = 0$ of the equation (10.9), one concludes $\int d^d p = 0$, in contrast with momentum regularization, where it is proportional to Λ^d . Therefore, the order between quantum operators becomes irrelevant, because the commutator vanishes. Dimensional regularization is thus especially useful for perturbative calculations in geometric models where the problems of quantization occur, like non-linear σ -models whose Hamiltonians have the generic form (3.24) (see Chapters 19 and 29), or gauge theories. However, the cancellations have then to be justified by other methods, like lattice regularization.

Continuation of tensor structures. So far, we have only considered diagrams corresponding to scalar fields. Any diagram which is not a scalar can be expanded on a set of fixed tensors with scalar coefficients. For example,

$$\int d^d q q_\mu q_\nu f(q^2, p^2, p \cdot q) = A(p^2) p_\mu p_\nu + B(p^2) \delta_{\mu\nu}. \quad (10.10)$$

The scalar diagrams contributing to $A(p^2)$ and $B(p^2)$ can be obtained by taking the trace and the scalar product with p_μ :

$$\begin{aligned} A(p^2) &= \frac{1}{d-1} \frac{1}{(p^2)^2} \int d^d q [d(p \cdot q)^2 - p^2 q^2] f(q^2, p^2, p \cdot q), \\ B(p^2) &= \frac{1}{d-1} \frac{1}{p^2} \int d^d q [-(p \cdot q)^2 + p^2 q^2] f(q^2, p^2, p \cdot q). \end{aligned} \quad (10.11)$$

We have reduced the problem to the calculation of integrals of the form (10.2) with additional factors polynomial in momenta. The integration over momenta can then also be performed to yield the continuation in d .

10.1.3 Dimensional regularization and UV divergences

When the dimension d approaches the dimension where UV divergences are expected, Feynman diagrams have poles as a function of the dimension. The singular contributions can be isolated by performing a Laurent expansion of the diagram. For example, the expression (10.6) is the value of a Feynman diagram of the massless ϕ^4 QFT, which is renormalizable for $d = 4$. The Laurent expansion is

$$I_\gamma = N_d \left[\frac{1}{4-d} + \frac{1}{2} - \frac{1}{2} \ln p^2 + O(d-4) \right].$$

As we have implicitly done previously, in general, we include the loop factor

$$N_d = \frac{\text{area of the sphere } S_{d-1}}{(2\pi)^d} = \frac{2}{(4\pi)^{d/2} \Gamma(d/2)}, \quad (10.12)$$

in the definition of the loop expansion parameter, because it is generated naturally by each loop integration.

As we have already shown in an example, powers of $\ln \Lambda$ (Λ being the cut-off) which appear in a cut-off regularization in the large Λ limit are replaced by powers of $1/(d-4)$.

For example, at loop order L , in a renormalizable theory like the ϕ^4 QFT, one expects multiple poles like $1/(d-4)^L$.

10.2 RG functions

We again discuss the example of the ϕ^4 QFT, first in the massive formalism of the Callan–Symanzik (CS) equation (see Chapter 9), and then in the case of the massless theory.

10.2.1 The massive ϕ^4 field theory

We consider the bare action,

$$\mathcal{S}(\phi_0) = \int d^d x \left[\frac{1}{2} (\nabla \phi_0(x))^2 + \frac{1}{2} m_0^2 \phi_0^2(x) + \frac{1}{4!} g_0 \phi_0^4(x) \right], \quad (10.13)$$

within the framework of *dimensional regularization*. Following the discussion of Section 9.2, we introduce a renormalized mass m , and a dimensionless renormalized coupling constant g , in such a way that the renormalized action can be written as

$$\mathcal{S}_r(\phi) = \int d^d x \left[\frac{1}{2} Z(g) (\nabla \phi(x))^2 + \frac{1}{2} m^2 Z_m(g) \phi^2(x) + \frac{1}{4!} m^{4-d} g Z_g(g) \phi^4(x) \right]. \quad (10.14)$$

With this parametrization, the renormalization constants $Z(g)$, $Z_m(g)$, and $Z_g(g)$, are dimensionless and, thus, depend on the only dimensionless parameter available, the coupling constant g . In particular, the mass is multiplicatively renormalizable [70, 71, 77].

Since the renormalized action is obtained from the bare action by the field rescaling $\phi_0 = \phi \sqrt{Z}$, the bare and renormalized parameters are related by

$$m_0^2 = m^2 Z_m(g) / Z(g), \quad (10.15)$$

$$g_0 = g m^{4-d} Z_g(g) / Z^2(g). \quad (10.16)$$

Note that, in the dimensional regularization scheme, renormalization constants depend on an additional hidden parameter, the dimension d .

From the relation (10.15), we can now calculate the RG β -function, the coefficient of the CS equation. Setting

$$g Z_g / Z^2 = G(g), \quad (10.17)$$

and differentiating equation (10.16) with respect to m at g_0 fixed, we find (equation (9.37))

$$0 = (4 - d) G(g) + \beta(g) \frac{\partial}{\partial g} G(g) \Rightarrow \beta(g) = - (4 - d) \left(\frac{d \ln G(g)}{dg} \right)^{-1}. \quad (10.18)$$

Using the notation of Section 9.2, we call $Z_2(g)$ the renormalization constant associated with the renormalization of ϕ^2 . From equations (9.38) and (9.39), we then derive

$$\eta(g) = \beta(g) \frac{d}{dg} \ln Z(g), \quad (10.19)$$

$$\eta_2(g) = \beta(g) \frac{d}{dg} \ln [Z_2(g) / Z(g)] \quad (10.20)$$

and, finally, from equations (10.15) and (9.40),

$$\frac{Z_m}{Z_2} \left[2 + \beta(g) \frac{d}{dg} \ln (Z_m / Z) \right] = \sigma(g). \quad (10.21)$$

This equation shows that Z_m / Z_2 is a finite function of g and, therefore, Z_m is not a new renormalization constant.

10.2.2 The massless theory

Dimensional regularization can also be used to define the massless ϕ^4 field theory. However, unlike the massive theory, due to small momentum (IR) divergences, the regularized theory only exists in an infinitesimal neighbourhood of the dimension 4. This problem is discussed at length in Chapters 15 and 16, devoted to critical phenomena. Then, since the massless theory is renormalizable, the *MS* scheme is also applicable.

The bare action can be written as

$$\mathcal{S}(\phi_0) = \int d^d x \left[\frac{1}{2} (\nabla \phi_0(x))^2 + \frac{1}{4!} g_0 \phi_0^4(x) \right]. \quad (10.22)$$

Note the absence of a bare mass term. Indeed, if the propagator is massless, no mass can be generated, because there is no dimensional parameter, besides the coupling constant: all diagrams contributing to the two-point function have a power-law behaviour given by simple dimensional considerations ($\varepsilon = 4 - d$ is infinitesimal):

$$\tilde{\Gamma}^{(2)}(p) = p^2 + \sum_{n=2} C_n(\varepsilon) g_0^n p^{2-n\varepsilon}.$$

To define a renormalized theory, it is necessary to introduce a mass scale μ , which takes care of the dimension of the ϕ^4 coupling constant. The renormalized action then takes the form

$$\mathcal{S}_r(\phi) = \int d^d x \left[\frac{1}{2} Z(g) (\nabla \phi(x))^2 + \frac{1}{4!} \mu^{4-d} g Z_g \phi^4(x) \right]. \quad (10.23)$$

To the action, correspond the relations between bare and renormalized vertex functions,

$$Z^{n/2}(g)\tilde{\Gamma}_r^{(n)}(p_i; \mu, g) = \tilde{\Gamma}^{(n)}(p_i; g_0), \quad (10.24)$$

with

$$g_0 = \mu^{4-d} g Z_g / Z^2. \quad (10.25)$$

A differentiation of equation (10.24) with respect to μ at g_0 fixed, yields RG equations that express that vertex functions depend on μ and g only through the combination $g_0(\mu, g)$:

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2} \eta(g) \right] \tilde{\Gamma}_r^{(n)}(p_i; \mu, g) = 0, \quad (10.26)$$

where β and η are given by expressions formally identical to equations (10.18) and (10.19):

$$\beta(g) = -(4-d) \left(\frac{d \ln G(g)}{dg} \right)^{-1}, \quad \eta(g) = \beta(g) \frac{d}{dg} \ln Z(g). \quad (10.27)$$

As we have discussed in Section 9.10, it is then possible to define a massive theory by adding to the renormalized action a mass term of the form

$$\mathcal{S}_r(\phi, m) = \mathcal{S}_r(\phi) + \frac{1}{2} m^2 \int d^d x Z_2(g) \phi^2(x).$$

10.3 The structure of renormalization constants

The renormalizability of the ϕ^4 field theory in four dimensions implies that the renormalized vertex functions and, therefore, also the RG functions $\beta(g)$, $\eta(g)$, $\eta_2(g)$, and $\sigma(g)$ have a finite limit when the deviation $\varepsilon = 4-d$ from the dimension 4 goes to zero. Since

$$G(g) = g + O(g^2),$$

the function $\beta(g)$ can be written as

$$\beta(g) = -\varepsilon g + \beta_2(\varepsilon) g^2 + \beta_3(\varepsilon) g^3 + \dots, \quad (10.28)$$

in which all the functions $\beta_n(\varepsilon)$ have regular Taylor series expansion at $\varepsilon = 0$:

$$\beta_n(\varepsilon) = \beta_n(0) + \varepsilon \beta'_n(0) + \dots$$

Conversely, we now determine the form of $G(g)$ from the knowledge of $\beta(g)$:

$$g \frac{G'(g)}{G(g)} = -\frac{\varepsilon g}{\beta(g)} \equiv \left[1 - \frac{1}{\varepsilon} \beta_2(\varepsilon) g - \frac{1}{\varepsilon} \beta_3(\varepsilon) g^2 - \dots \right]^{-1}.$$

Expanding the right-hand side in powers of g , we observe that, at a fixed order in g , the most singular term in ε comes from the term of order g^2 in $\beta(g)$:

$$\frac{G'(g)}{G(g)} = \frac{1}{g} + \left(\frac{\beta_2(0)}{\varepsilon} + O(1) \right) + g \left(\frac{\beta_2^2(0)}{\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right) \right) + \dots$$

Integrating the expansion term by term, we find

$$G(g) = g + \sum_{n=2} g^n \left[\left(\frac{\beta_2(0)}{\varepsilon} \right)^{n-1} + \text{less singular terms} \right].$$

The coefficient $\tilde{G}_n(\varepsilon)$ of the expansion of $G(g)$ in powers of g ,

$$G(g) = g + \sum_2^{\infty} g^n \tilde{G}_n(\varepsilon),$$

has thus a Laurent series expansion in ε , for ε small, of the form

$$\tilde{G}_n(\varepsilon) = \frac{\beta_2^{n-1}(0)}{\varepsilon^{n-1}} + \frac{G_{n,2-n}}{\varepsilon^{n-2}} + \cdots + G_{n,0} + G_{n,1}\varepsilon + \cdots.$$

The finiteness of $\eta(g)$, and $\eta_2(g)$ leads to similar conclusions for $Z(g)$ and $Z_2(g)$, which can be written as

$$\begin{aligned} Z(g) &= 1 + \sum_1^{\infty} \frac{\alpha^{(n)}(g)}{\varepsilon^n} + \text{regular terms in } \varepsilon, \\ Z_2(g) &= 1 + \sum_1^{\infty} \frac{\alpha_2^{(n)}(g)}{\varepsilon^n} + \text{regular terms in } \varepsilon, \end{aligned}$$

with $\alpha^{(n)}(g) = O(g^{n+1})$, $\alpha_2^{(n)} = O(g^n)$.

We conclude that, at order L in the loop expansion, the divergent part of $\Gamma(\varphi)$, the generating functional of vertex functions, is a polynomial of degree L in $1/\varepsilon$.

10.4 MS scheme

Although the MS idea can be used in any regularization scheme (see, for example, equation (8.34)), it is especially useful in dimensional regularization. Renormalization constants are determined in the following way: instead of imposing renormalization conditions to divergent vertex functions, one just subtracts, at each order in the loop expansion, the singular part of Laurent expansion in ε [78, 79]. We denote by $\Gamma_L^{\text{div.}}(\varphi)$ the divergent part of the generating functional of vertex functions, renormalized up to $(L-1)$ loops:

$$\Gamma_L^{\text{div.}}(\varphi) = \sum_{\ell=1}^L \frac{\gamma_{L,\ell}}{\varepsilon^\ell}. \quad (10.29)$$

We then add, as a counter-term, $-\Gamma_L^{\text{div.}}(\varphi)$ to the action.

The $\overline{\text{MS}}$ scheme. In the calculation of low order Feynman diagrams, the factor $(N_d)^L$, where L is the number of loops of the diagram and N_d the loop factor (equation (10.12)),

$$N_d = 2/(4\pi)^{d/2} \Gamma(d/2) \Rightarrow N_4 = 1/8\pi^2,$$

is generated naturally. Therefore, it is convenient to rescale the loop expansion parameter (*e.g.* by multiplying it by the factor N_4/N_d) to suppress the factor, avoiding in this way to expand it in ε [80]. Combined with the MS scheme, this is called the modified MS, or $\overline{\text{MS}}$ scheme.

Example. We again consider the ϕ^4 QFT in the scheme of Section 10.2.1. Since the field renormalization vanishes at this order, the one-loop divergent part of $\Gamma(\varphi)$ can be inferred from the functional density at constant field φ . In the Fourier representation (see equation (7.93)),

$$\gamma_1(\varphi) = \Gamma_1(\varphi)/\text{volume} = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \ln(p^2 + m^2 + \frac{1}{2} g m^\varepsilon \varphi^2). \quad (10.30)$$

Setting $m^2 + \frac{1}{2} g m^\varepsilon \varphi^2 = K$, we calculate (using the result (A10.5))

$$\gamma(K) = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \ln(p^2 + K) \Rightarrow \gamma'(K) = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + K} = \frac{1}{4} N_d \frac{\pi}{\sin(\pi d/2)} K^{d/2-1}.$$

Integrating over K , we obtain (see also equation (A8.22)),

$$\gamma_1(\varphi) = \frac{1}{2d} N_d \frac{\pi}{\sin(\pi d/2)} (m^2 + \frac{1}{2} g m^\varepsilon \varphi^2)^{d/2}. \quad (10.31)$$

From an expansion in φ and ε one infers, in the $\overline{\text{MS}}$ scheme,

$$\Gamma_1^{\text{div.}}(\varphi) = -\frac{N_4}{4\varepsilon} \left[m^2 g \int \varphi^2(x) d^d x + \frac{1}{4} g^2 m^\varepsilon \int \varphi^4(x) d^d x \right]. \quad (10.32)$$

RG functions at one-loop order. The functions Z , Z_m , Z_g at one-loop order then are

$$Z = 1 + O(g^2), \quad Z_g = 1 + \frac{3}{2} N_d \frac{g}{\varepsilon} + O(g^2), \quad Z_m = 1 + \frac{1}{2} N_d \frac{g}{\varepsilon} + O(g^2). \quad (10.33)$$

Directly calculating the vertex function $\langle \phi^2 \phi \phi \rangle$ at one-loop order, one notes that $Z_2 = Z_m$. Since the MS scheme eliminates any possible finite renormalization and Z_2/Z_m is finite, the relation remains true to all orders.

The RG functions are then

$$\beta(g) = -\varepsilon g + \frac{3}{2} N_d g^2 + O(g^3), \quad (10.34)$$

$$\eta(g) = O(g^2), \quad (10.35)$$

$$\eta_2(g) = -\frac{1}{2} g N_d + O(g^2). \quad (10.36)$$

10.4.1 RG functions in the *MS* scheme

The RG β -function. Expanding, for instance, the function (equation (10.17))

$$G(g) = g + \sum_1^\infty \frac{G_n(g)}{\varepsilon^n}, \quad G_n(g) = O(g^{n+1}), \quad (10.37)$$

one can calculate the RG function

$$\beta(g) = -\varepsilon \left[g + \sum_1^\infty \frac{G_n(g)}{\varepsilon^n} \right] \left[1 + \sum_1^\infty \frac{G'_n(g)}{\varepsilon^n} \right]^{-1}.$$

Since $G'_n(g)$ is of order g^n , one can expand the denominator as

$$\beta(g) = -\varepsilon \left[g + \sum_1^\infty \frac{G_n(g)}{\varepsilon^n} \right] \left[1 - \frac{G'_1(g)}{\varepsilon} + \frac{[G'_1(g)]^2}{\varepsilon^2} - \frac{G'_2(g)}{\varepsilon^2} + \dots \right].$$

This expression can be rewritten as

$$\beta(g) = -\varepsilon g - G_1(g) + gG'_1(g) + \sum_1^\infty \frac{b_n(g)}{\varepsilon^n}.$$

The finiteness of $\beta(g)$ then implies $b_n(g) = 0$ for all $n \geq 1$, and leads to the simple form

$$\beta(g) = -\varepsilon g + gG'_1(g) - G_1(g). \quad (10.38)$$

The functions $\beta(g)$ and $G_n(g)$, $n \geq 2$, are uniquely determined by the function $G_1(g)$, that is, the coefficients of $1/\varepsilon$ in the divergences.

The RG functions η and η_2 . General arguments show that, in the expansion of equation (10.29), the whole new L -loop information about divergences is contained in $\gamma_{L,1}(\varphi)$. All other counter-terms are determined by the counter-terms of previous orders. This implies that the RG functions $\eta(g)$, and $\eta_2(g)$ also have a simple dependence on ε . In the MS scheme, the renormalization constants Z and Z_2 have the form

$$Z(g) = 1 + \sum_1^\infty \frac{\alpha^{(n)}(g)}{\varepsilon^n}, \quad \alpha^{(n)}(g) = O(g^{n+1}), \quad (10.39)$$

$$Z_2(g) = 1 + \sum_1^\infty \frac{\alpha_2^{(n)}(g)}{\varepsilon^n}, \quad \alpha_2^{(n)}(g) = O(g^n). \quad (10.40)$$

Using relation (10.19) and the form (10.38) of $\beta(g)$, one obtains

$$\eta(g) = [-\varepsilon g + gG'_1(g) - G_1(g)] \left[\frac{1}{\varepsilon} \frac{d}{dg} \alpha^{(1)}(g) + O\left(\frac{1}{\varepsilon^2}\right) \right].$$

Since $\eta(g)$ has a finite limit, it is given by

$$\eta(g) = -g \frac{d}{dg} \alpha^{(1)}(g). \quad (10.41)$$

Similarly, for $\eta_2(g)$ we find

$$\eta_2(g) = -g \frac{d}{dg} \alpha_2^{(1)}(g). \quad (10.42)$$

Finally, the explicit dependence of the renormalization constants on ε can be obtained by calculating them from β , η , and η_2 . For example,

$$G(g) = g \exp \left(-\varepsilon \int_0^g \left[\frac{1}{-\varepsilon g' + b(g')} + \frac{1}{\varepsilon g'} \right] dg' \right), \quad (10.43)$$

in which we have set

$$\beta(g, \varepsilon) = -\varepsilon g + b(g). \quad (10.44)$$

Massive and massless scheme. Since, in the MS scheme the renormalization constants are uniquely defined, one concludes that the renormalization constants in the massless scheme of Section 10.2.2 and the massive scheme of Section 10.2.1 are identical.

10.5 RG functions at two-loop order: The ϕ^4 QFT

As an exercise, we now calculate explicitly, at two-loop order, the renormalization constants and RG functions in the QFT corresponding to the renormalized action (10.23), to which a source $\tau(x)$ for the $\phi^2(x)/2$ monomial is added,

$$\mathcal{S}_r(\phi) = \int d^d x \left[\frac{1}{2} Z (\nabla \phi(x))^2 + \frac{1}{4!} \mu^\varepsilon g Z_g \phi^4(x) + \frac{1}{2} Z_2 \tau(x) \phi^2(x) \right]. \quad (10.45)$$

We recall that, in dimensional regularization, no mass counter-term is generated in the massless theory.

We define the renormalization constants in the $\overline{\text{MS}}$ scheme (Section 10.4) where the loop factor (10.12) is factorized in each loop integral and not expanded in ε .

For technical convenience, we work with the massless theory. Note that for higher order calculations more sophisticated methods are used. For example, one considers diagrams to which all divergent subdiagrams have been subtracted. The remaining global divergence is then independent of external momenta and internal masses. To calculate the divergent part, one sets as many masses and momenta to 0 as possible, consistently in the diagrams and the subtracted subdiagrams, as long as one does not encounter IR (zero momentum) divergences [81]. An example is given below.

10.5.1 The perturbative expansion

In Section 7.4, we have already listed the diagrams contributing to $\Gamma^{(2)}$ at this order. We display them again in Fig. 10.1. The two first diagrams vanish in dimensional regularization and the renormalized two-point vertex function reduces to

$$\tilde{\Gamma}_r^{(2)}(p) = Z p^2 - \frac{1}{6} g^2 \mu^{2\varepsilon} A(p) + O(g^3), \quad (10.46)$$

where $A(p) \propto p^{2-2\varepsilon}$.

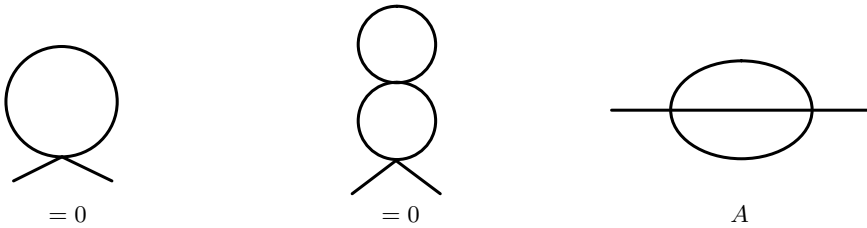
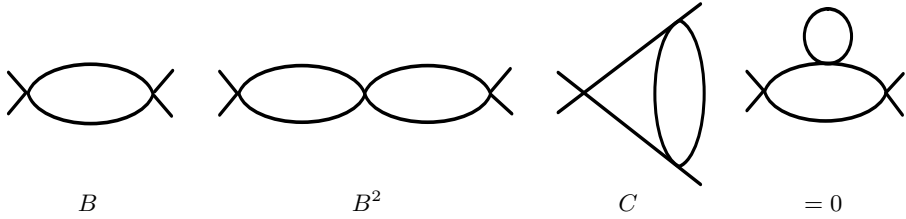


Fig. 10.1 Feynman diagrams contributing to $\Gamma^{(2)}$

The diagrams contributing to $\Gamma^{(4)}$ are displayed in Fig. 10.2. The renormalized four-point vertex function is given by (Z does not contribute at this order)

$$\begin{aligned} \tilde{\Gamma}_r^{(4)}(p_i) = & g Z_g \mu^\varepsilon - \frac{1}{2} Z_g^2 g^2 \mu^{2\varepsilon} [B(p_1 + p_2) + 2 \text{ terms}] + \frac{1}{4} g^3 \mu^{3\varepsilon} [B_d^2(p_1 + p_2) + 2 \text{ terms}] \\ & + \frac{1}{2} g^3 \mu^{3\varepsilon} [C(p_1, p_2) + 5 \text{ terms}] + O(g^4), \end{aligned} \quad (10.47)$$

the additional terms restoring the permutation symmetry of the four-point function between the four momenta (p_1, p_2, p_3, p_4) .

**Fig. 10.2** Feynman diagrams contributing to $\Gamma^{(4)}$

10.5.2 Diagrams: Divergences

First, we list a few identities useful for the calculation of renormalization constants.

Feynman's parametrization. We use the special case of identity (A10.4) several times,

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 ds \frac{s^{\alpha-1}(1-s)^{\beta-1}}{(as + b(1-s))^{\alpha+\beta}}. \quad (10.48)$$

We also need the integral (A10.5),

$$\frac{1}{(2\pi)^d} \int \frac{d^d q}{(q^2 + 1)^\nu} = \frac{N_d}{2} \frac{\Gamma(d/2)\Gamma(\nu - d/2)}{\Gamma(\nu)}. \quad (10.49)$$

The massless propagator and diagram A. The massless propagator

$$\Delta(x) = \frac{1}{(2\pi)^d} \int d^d p \frac{e^{ipx}}{p^2},$$

is given by equation (10.5),

$$\Delta(x) = \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}} |x|^{2-d}. \quad (10.50)$$

The diagram A (Fig. 10.1) can be expressed in terms of the propagator as

$$\begin{aligned} A(p) &= \int d^d x e^{ipx} \Delta^3(x) = \frac{\Gamma^3(d/2 - 1)}{64\pi^{3d/2}} \int d^d x e^{ipx} |x|^{6-3d} \\ &= \frac{N_d^2}{4} \Gamma^2(d/2) \Gamma^3(d/2 - 1) \frac{\Gamma(3-d)}{\Gamma(3d/2 - 3)} p^{2d-6} \underset{\varepsilon \rightarrow 0}{\sim} -\frac{N_d^2}{8\varepsilon} p^2. \end{aligned} \quad (10.51)$$

The bubble diagram B. The diagram B of Fig. 10.2 is given by

$$B(p) = \frac{1}{(2\pi)^d} \int \frac{d^d q}{q^2(p+q)^2}. \quad (10.52)$$

Using the identity (10.48), one can rewrite the diagram as

$$B(p) = \frac{1}{(2\pi)^d} \int_0^1 ds \int \frac{d^d q}{[(1-s)q^2 + s(p+q)^2]^2} = K_d p^{d-4},$$

with, for $\varepsilon = 4 - d \rightarrow 0$,

$$K_d = -\frac{N_d}{2} \frac{\pi}{\sin(\pi d/2)} \frac{\Gamma(d/2 - 1)\Gamma(d/2)}{\Gamma(d - 2)} = \frac{N_d}{\varepsilon} (1 + \frac{1}{2}\varepsilon) + O(\varepsilon). \quad (10.53)$$

At the two-loop order, one needs,

$$B(p) = \frac{N_d}{\varepsilon} \left(1 + \left(\frac{1}{2} - \ln p \right) \varepsilon + O(\varepsilon^2) \right) = b_1 + B_r(p), \quad (10.54)$$

where $B_r(p)$ is the finite part and b_1 the divergent part of $B(p)$ in the $\overline{\text{MS}}$ scheme,

$$b_1 = \frac{N_d}{\varepsilon}. \quad (10.55)$$

The diagram C. More work is required for the evaluation of the divergent part of the third diagram of Fig. 10.2,

$$C(p_1, p_2) = \frac{1}{(2\pi)^{2d}} \int \frac{d^d q d^d k}{(q + p_1)^2 (q - p_2)^2 (q - k)^2 k^2}.$$

However, if we subtract to C the contribution of the divergent subdiagram, $C(p_1, p_2) - b_1 B_r(p_1 + p_2)$, the divergent part becomes momentum independent. Therefore, we can specialize the calculation of $C(p_1, p_2)$ to $C(p, 0)$, which is not IR divergent [81]. Then, after integration over k ,

$$C(p, 0) = \frac{1}{(2\pi)^d} \int \frac{d^d q}{(p + q)^2 q^2} B(q).$$

We set

$$C(p, 0) = K_d I(p),$$

where

$$I(p) = \frac{1}{(2\pi)^d} \int \frac{d^d q}{(p + q)^2 (q^2)^{3-d/2}}.$$

Using the identity (10.48) again, we can rewrite the integral as

$$\begin{aligned} I(p) &= \frac{1}{(2\pi)^d} \int_0^1 ds s^{2-d/2} \int \frac{d^d q}{[(1-s)q^2 + s(p+q)^2]^{4-d/2}} \\ &= \frac{N_d}{2} \Gamma(d/2) \frac{\Gamma(4-d)}{\Gamma(3-d/2)} \int_0^1 ds (1-s)^{d/2-4} s^{d-4} p^{2d-8} \\ &= \frac{N_d}{2} \Gamma(d/2) \frac{\Gamma(4-d)}{\Gamma(3-d/2)} \frac{\Gamma(d/2-1)\Gamma(d-3)}{\Gamma(3d/2-4)} p^{2d-8} \\ &= \frac{N_d}{2\varepsilon} (1 + \varepsilon - 2\varepsilon \ln p). \end{aligned}$$

Thus,

$$C(p, 0) = \frac{N_d^2}{2\varepsilon^2} \left(1 + \frac{3}{2}\varepsilon - 2\varepsilon \ln p \right). \quad (10.56)$$

Subtracting the divergent subdiagram, one obtains

$$C(p, 0) - b_1 B_r(p) = \frac{N_d^2}{2\varepsilon^2} \left(1 + \frac{1}{2}\varepsilon \right) + O(1), \quad (10.57)$$

which is momentum independent, as expected.

10.5.3 The renormalization constants

Field renormalization. At this order, only the diagram A contributes to $\tilde{\Gamma}^{(2)}$. Therefore, expressing that $\tilde{\Gamma}^{(2)}$ is finite and using equation (10.51), one finds

$$Z = 1 - \frac{N_d^2}{48\varepsilon} g^2 + O(g^3). \quad (10.58)$$

Coupling renormalization. After setting $\mu = 1$ and $B(p) = b_1 + B_r(p)$, where $B_r(p)$ is finite (equation (10.54)), the expansion (10.47) becomes

$$\begin{aligned} \tilde{\Gamma}_r^{(4)}(p_i) = & gZ_g - \frac{3}{2}b_1g^2Z_g^2 - \frac{1}{2}Z_g^2g^2[B_r(p_1 + p_2) + 2 \text{ terms}] \\ & + \frac{1}{4}g^3[B_r^2(p_1 + p_2) + 2 \text{ terms}] + \frac{1}{2}b_1g^3[B_r(p_1 + p_2) + 2 \text{ terms}] + \frac{3}{4}b_1^2g^3 \\ & + \frac{1}{2}g^3[C(p_1, p_2) + 5 \text{ terms}]. \end{aligned}$$

Then, using equation (10.55) and expressing that $\Gamma_r^{(4)}$ is finite at one-loop order, one finds

$$Z_g = 1 + \frac{3}{2}b_1g + O(g^2) = 1 + \frac{3N_d}{2\varepsilon}g + O(g^2).$$

The perturbative expansion becomes

$$\begin{aligned} \tilde{\Gamma}_r^{(4)}(p_i) = & gZ_g - \frac{3}{2}b_1g - \frac{15}{4}b_1^2g^3 - \frac{1}{2}g^2[B_r(p_1 + p_2) + 2 \text{ terms}] \\ & + \frac{1}{4}g^3[B_r^2(p_1 + p_2) + 2 \text{ terms}] + \frac{1}{2}g^3[C(p_1, p_2) - b_1B_r(p_1 + p_2) + 5 \text{ terms}]. \end{aligned}$$

The divergent part of $[C(p_1, p_2) - b_1B_r(p_1 + p_2)]$ is momentum independent and given by equation (10.57). Then, the divergent part of $\tilde{\Gamma}^{(4)}$ reduces to

$$\tilde{\Gamma}_{\text{div}}^{(4)}(p_i) = g(Z_g - 1) - \frac{3}{2}b_1g^2 - \frac{15}{4}b_1^2g^3 + \frac{3N_d^2}{2\varepsilon^2}(1 + \frac{1}{2}\varepsilon)g^3.$$

Expressing that $\tilde{\Gamma}_{\text{div}}^{(4)}$ vanishes at two-loop order, one finds

$$Z_g = 1 + N_d \frac{3g}{2\varepsilon} + N_d^2 \left(\frac{9}{4\varepsilon^2} - \frac{3}{4\varepsilon} \right) g^2 + O(g^3). \quad (10.59)$$

10.5.4 The ϕ^2 insertion

The two-loop expansion of the renormalized $\frac{1}{2}\phi^2\phi\phi$ vertex function is given by

$$\tilde{\Gamma}_r^{(2,1)}(p_1, p_2) = Z_2 - \frac{1}{2}gZ_gZ_2B(p_1 + p_2) + \frac{1}{4}g^2B^2(p_1 + p_2) + \frac{1}{2}g^2C(p_1, p_2) + O(g^3).$$

At this order, the field renormalization is not yet needed. We can specialize again to $p_1 = p, p_2 = 0$. Introducing the function B_r , we can rewrite the expressions as,

$$\begin{aligned} \tilde{\Gamma}_r^{(2,1)}(p, 0) = & Z_2 - \frac{1}{2}gZ_2B_r(p) - \frac{1}{2}b_1Z_2g - \frac{3}{4}b_1^2g^2 - \frac{3}{4}b_1B_r(p)g^2 \\ & + \frac{1}{4}g^2(B_r^2(p) + 2b_1B_r(p) + b_1^2) + \frac{1}{2}g^2C(p, 0) + O(g^3). \end{aligned}$$

At one-loop order,

$$Z_2 = 1 + \frac{1}{2}b_1g = 1 + \frac{N_d}{2\varepsilon}g.$$

Then, the divergent part of $\tilde{\Gamma}^{(2,1)}$ is given by

$$\left[\tilde{\Gamma}^{(2,1)}(p, 0)\right]_{\text{div.}} = Z_2 - \frac{1}{2}b_1g - \frac{3}{4}b_1^2g^2 + \frac{1}{2}g^2 [C(p, 0) - b_1B_r(p)] + O(g^3).$$

Expressing that $[\tilde{\Gamma}^{(2,1)}]_{\text{div.}}$ vanishes at two-loop order, one obtains

$$Z_2 - 1 - \frac{1}{2}b_1g - \frac{3}{4}b_1^2g^2 + g^2 \frac{N_d^2}{4\varepsilon^2} (1 + \frac{1}{2}\varepsilon) = 0,$$

and thus,

$$Z_2 = 1 + N_d \frac{g}{2\varepsilon} + N_d^2 \left(\frac{1}{2\varepsilon^2} - \frac{1}{8\varepsilon} \right) g^2 + O(g^3). \quad (10.60)$$

10.5.5 RG functions

Equations (10.18–10.20) then yield the three RG functions,

$$\tilde{\beta}(\tilde{g}) = N_d\beta(\tilde{g}) = -\varepsilon\tilde{g} + \frac{3}{2}\tilde{g}^2 - \frac{17}{12}\tilde{g}^3 + O(\tilde{g}^4), \quad (10.61)$$

$$\eta(\tilde{g}) = \frac{\tilde{g}^2}{24} + O(\tilde{g}^3), \quad (10.62)$$

$$\eta_2(\tilde{g}) = -\frac{\tilde{g}}{2} + \frac{5\tilde{g}^2}{24} + O(\tilde{g}^3), \quad (10.63)$$

with the notation

$$\tilde{g} = N_d g. \quad (10.64)$$

Field renormalization: Positivity of the RG function. Note that for g small, that is, in the perturbative domain, the field renormalization (10.58) satisfies $Z < 1$. This is a general property in unitary theories implied by the spectral representation of the two-point function (see Section 6.6). This implies $\eta(g) > 0$ for g small.

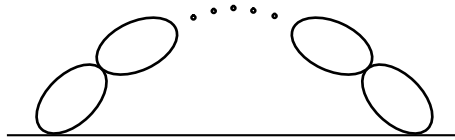


Fig. 10.3 Diagrams contributing to $\Gamma^{(2)}$

10.5.6 The massless theory at fixed dimension $d < 4$

We now show, by working out an example, that the perturbative expansion of the massless ϕ^4 QFT is IR divergent for all dimensions $d < 4$. We consider one contribution to the two-point vertex function of order g^{n+1} , proportional to the diagram (Fig. 10.3),

$$D_n(p) = \frac{1}{(2\pi)^d} \int \frac{d^d q}{(p+q)^2} B^n(q), \quad (10.65)$$

where $B(q)$ is the diagram (10.52),

$$B(q) = \frac{1}{(2\pi)^d} \int \frac{d^d k}{q^2(k+q)^2} = -\frac{N_d}{2} \frac{\pi}{\sin(\pi d/2)} \frac{\Gamma(d/2-1)\Gamma(d/2)}{\Gamma(d-2)} q^{d-4}.$$

Therefore, for

$$n(4-d) \geq d \Rightarrow n \geq d/(4-d),$$

the integral (10.65) is IR divergent. For $d = 3$, this happens for $n = 3$.

10.6 Generalization to N -component fields

We now generalize the preceding calculations to a field ϕ with N components ϕ_i and a ϕ^4 -like QFT, symmetric under a group, subgroup of $O(N)$, which admits only one quadratic invariant. In particular, this implies

$$\tilde{\Gamma}_{ij}^{(2)}(p) = \delta_{ij} \tilde{\Gamma}^{(2)}(p).$$

The renormalized action of the massless theory can be written as

$$\mathcal{S}_r(\phi) = \int d^d x \left[\frac{1}{2} Z \sum_i \nabla \phi_i(x) \nabla \phi_i(x) + \frac{1}{4!} \sum_{i,j,k,l} G_{ijkl} \phi_i(x) \phi_j(x) \phi_k(x) \phi_l(x) \right], \quad (10.66)$$

where G_{ijkl} is a tensor symmetric in its four indices. We denote by μ the renormalization scale. To determine the renormalization constants, we use again the $\overline{\text{MS}}$ scheme. We denote by g_{ijkl} the renormalized coupling constant, which is also symmetric in its four indices. At leading order,

$$G_{ijkl} = \mu^\varepsilon g_{ijkl} + O(g^2). \quad (10.67)$$

The symmetry implies a number of constraints on the tensors G_{ijkl} or g_{ijkl} . At two-loop order, one needs

$$\sum_k g_{ijkk} = \gamma_1 \delta_{ij}, \quad \gamma_1 = \frac{1}{N} \sum_{i,j} g_{iijj}, \quad (10.68)$$

$$\sum_{k,l,m} g_{iklm} g_{jklm} = \gamma_2 \delta_{ij}, \quad \gamma_2 = \frac{1}{N} \sum_{j,k,l,m} g_{jklm} g_{jklm}. \quad (10.69)$$

10.6.1 Renormalization constants

The calculation of renormalization constants reduces to the calculation of weight factors, the momentum dependence of the diagrams being the same as in Section 10.5.

The two-point function. At two-loop order, the renormalized two-point vertex function has the expansion

$$\tilde{\Gamma}_{ij}^{(2)}(p) = Z \delta_{ij} p^2 - \frac{1}{6} \sum_{k,l,m} g_{iklm} g_{jklm} \mu^{2\varepsilon} A(p) + O(g^3). \quad (10.70)$$

The field renormalization is then (equations (10.69)),

$$Z = 1 - \frac{N_d^2}{48\varepsilon} \gamma_2 + O(g^3). \quad (10.71)$$

The four-point function. The renormalized four-point vertex function $\langle \phi_i \phi_j \phi_k \phi_l \rangle_{1\text{PI}}$, at the same order is

$$\begin{aligned} \tilde{\Gamma}_{ijkl}^{(4)}(p_i) &= G_{ijkl} - \frac{1}{2} \sum_{m,n} (G_{ijmn} G_{mnkl} B(p_1 + p_2) + 2 \text{ terms}) \\ &\quad + \frac{1}{4} \sum_{m,n,p,q} (G_{ijmn} G_{mnpq} G_{pqkl} B^2(p_1 + p_2) + 2 \text{ terms}) \\ &\quad + \frac{1}{2} \sum_{m,n,p,q} (G_{klmn} G_{mpqi} G_{npqj} C(p_1, p_2) + 5 \text{ terms}) + O(G^4). \end{aligned} \quad (10.72)$$

The additional terms restore the permutation symmetry of the four-point function in its four arguments.

To evaluate renormalization constants, for notational convenience we now set $\mu = 1$. At one-loop order, the cancellation of divergence implies

$$G_{ijkl} = g_{ijkl} + \frac{1}{2}b_1 \sum_{m,n} (g_{ijmn}g_{mnkl} + 2 \text{ terms}) + O(g^2). \quad (10.73)$$

Then, expanding expression (10.72) in powers of g_{ijkl} , using the evaluations of the functions B (equation (10.54)) and C (equation (10.57)), and demanding the cancellation of the divergent part, one obtains

$$\begin{aligned} G_{ijkl} = & g_{ijkl} + \frac{N_d}{2\varepsilon} \sum_{m,n} (g_{ijmn}g_{mnkl} + 2 \text{ terms}) + \frac{N_d^2}{4\varepsilon^2} \sum_{m,n,p,q} (g_{ijmn}g_{mnpq}g_{pqkl} + 2 \text{ terms}) \\ & + \frac{N_d^2}{4\varepsilon^2} \left(1 - \frac{\varepsilon}{2}\right) \sum_{m,n,p,q} (g_{ijmn}g_{mpqk}g_{npql} + 5 \text{ terms}) + O(g^4). \end{aligned}$$

The expansion of the bare coupling constant

$$g_{0;ijkl} = \mu^{-\varepsilon} Z^{-2} G_{ijkl}, \quad (10.74)$$

follows (equation (10.71)),

$$\begin{aligned} g_{0;ijkl} &= g_{ijkl} + \frac{N_d}{2\varepsilon} \sum_{m,n} (g_{ijmn}g_{mnkl} + 2 \text{ terms}) + \frac{N_d^2}{4\varepsilon^2} \sum_{m,n,p,q} (g_{ijmn}g_{mnpq}g_{pqkl} + 2 \text{ terms}) \\ &+ \frac{N_d^2}{4\varepsilon^2} \left(1 - \frac{\varepsilon}{2}\right) \sum_{m,n,p,q} (g_{ijmn}g_{mpqk}g_{npql} + 5 \text{ terms}) + \frac{N_d^2}{24\varepsilon} \gamma_2 + O(g^4). \end{aligned} \quad (10.75)$$

The $\frac{1}{2}\phi^2$ insertion. To determine the renormalization constant Z_2 of the local polynomial $\frac{1}{2}\phi^2(x)$, we calculate the renormalized vertex function $\frac{1}{2}\langle\tilde{\phi}^2(p_1+p_2)\tilde{\phi}_i(p_1)\tilde{\phi}_j(p_2)\rangle_{1\text{PI}}$, which has the form

$$\tilde{\Gamma}_{ij}^{(1,2)}(p_1, p_2) = \delta_{ij} \tilde{\Gamma}^{(1,2)}(p_1, p_2).$$

After introduction of the ϕ^2 renormalization constant Z_2 , the renormalized function $\tilde{\Gamma}_r^{(1,2)}$ has the two-loop expansion,

$$\begin{aligned} \tilde{\Gamma}_{r;ij}^{(1,2)}(p_1, p_2) = & Z_2 \delta_{ij} - \frac{1}{2} Z_2 \sum_k G_{k k i j} B(p_1 + p_2) + \frac{1}{4} \sum_k G_{k k m n} G_{m n i j} B^2(p_1 + p_2) \\ & + \frac{1}{2} \sum_{k,p,q} G_{k p q i} G_{k p q j} C(p_1, p_2) + O(G^3). \end{aligned}$$

From equation (10.73), one infers,

$$\sum_k G_{k k i j} = \gamma_1 + \frac{1}{2} b_1 (\gamma_1^2 + 2\gamma_2) + O(g^3).$$

It follows that

$$\begin{aligned}\tilde{\Gamma}_r^{(1,2)}(p_1, p_2) = & Z_2 - \frac{1}{2}Z_2 \left[\gamma_1 + \frac{1}{2}b_1(\gamma_1^2 + \gamma_2) \right] B(p_1 + p_2) + \frac{1}{4}\gamma_1^2 B^2(p_1 + p_2) \\ & + \frac{1}{2}\gamma_2 C(p_1, p_2) + O(g^3).\end{aligned}\quad (10.76)$$

Then, setting $p_1 = p$, $p_2 = 0$, substituting $B = B_r + b_1$, one obtains

$$\begin{aligned}\tilde{\Gamma}_r^{(1,2)}(p, 0) = & Z_2 - \frac{1}{2}Z_2 \left[\gamma_1 + \frac{1}{2}b_1(\gamma_1^2 + 2\gamma_2) \right] b_1 - \frac{1}{2}Z_2 \left[\gamma_1 + \frac{1}{2}b_1(\gamma_1^2 + 2\gamma_2) \right] B_r(p) \\ & + \frac{1}{4}b_1^2\gamma_1^2 + \frac{1}{2}b_1B_r(p)\gamma_1^2 + \frac{1}{4}\gamma_1^2B_r^2(p) + \frac{1}{2}\gamma_2 C(p, 0) + O(g^3).\end{aligned}$$

The renormalization constant at one-loop order is then

$$Z_2 = 1 + \frac{1}{2}b_1\gamma_1 + O(g^2).$$

The replacement, in the one-loop term, of Z_2 by the expansion yields

$$\begin{aligned}\tilde{\Gamma}_r^{(1,2)}(p, 0) = & Z_2 - \frac{1}{2} \left[\gamma_1 + b_1(\gamma_1^2 + \gamma_2) \right] b_1 - \frac{1}{2} \left[\gamma_1 + b_1(\gamma_1^2 + \gamma_2) \right] B_r(p) \\ & + \frac{1}{4}b_1^2\gamma_1^2 + \frac{1}{2}b_1B_r(p)\gamma_1^2 + \frac{1}{4}\gamma_1^2B_r^2(p) + \frac{1}{2}\gamma_2 C(p, 0) + O(g^3).\end{aligned}$$

The divergent part of this expression takes the form,

$$\left[\tilde{\Gamma}_r^{(1,2)} \right]_{\text{div.}} = Z_2 - 1 - \frac{1}{2}b_1\gamma_1 - \frac{1}{4}(\gamma_1^2 + 2\gamma_2)b_1^2 + \frac{1}{2}\gamma_2 [C(p, 0) - b_1B_r(p)]_{\text{div.}}.$$

Using the result (10.57), one infers

$$Z_2 = 1 + \frac{N_d}{2\varepsilon}\gamma_1 + \frac{N_d^2}{4\varepsilon^2}\gamma_1^2 + \frac{N_d^2}{4\varepsilon^2} \left(1 - \frac{\varepsilon}{2} \right) \gamma_2 + O(g^3). \quad (10.77)$$

Actually, we need the renormalization constant which expresses the renormalized operator in terms of the bare fields,

$$\zeta_2 = Z_2/Z. \quad (10.78)$$

At two-loop order, one finds

$$\zeta_2 = Z_2 + \frac{N_d^2}{48\varepsilon}\gamma_2 + O(g^3). \quad (10.79)$$

10.6.2 RG equations

The relation between renormalized and bare vertex functions takes the form

$$\tilde{\Gamma}_{r;i_1i_2\dots i_n}^{(n)}(p, g, \mu) = Z^{n/2}\tilde{\Gamma}_{i_1i_2\dots i_n}^{(n)}(p, g_0, \Lambda), \quad (10.80)$$

in which g stands for g_{ijkl} and g_0 for $\mu^\varepsilon g_{0;ijkl}$.

We set

$$D \equiv \mu \frac{\partial}{\partial \mu} + \sum_{i,j,k,l} \beta_{ijkl}(g) \frac{\partial}{\partial g_{ijkl}},$$

with the definition

$$\sum_{i',j',k',l'} \beta_{i'j'k'l'} \frac{\partial g_{0;ijkl}}{\partial g_{i'j'k'l'}} = -\varepsilon g_{0;ijkl}. \quad (10.81)$$

Differentiating equation (10.80) with respect to μ at g_0 and Λ fixed, one obtains the RG equation

$$(D - \tfrac{1}{2}n\eta(g)) \tilde{\Gamma}_{r;i_1 i_2 \dots i_n}^{(n)} = 0, \quad (10.82)$$

with the definition

$$\eta(g) = \sum_{i,j,k,l} \beta_{ijkl}(g) \frac{\partial \ln Z}{\partial g_{ijkl}}. \quad (10.83)$$

Similarly, to renormalize vertex functions with $\frac{1}{2}\phi^2(x)$ insertions, we must multiply each insertion by the matrix ζ_2 . This leads to the RG equation

$$[D - l\eta_2(g) - \tfrac{1}{2}n\eta(g)] \tilde{\Gamma}_{r;j_1 j_2 \dots j_l, i_1 \dots i_n}^{(l,n)} = 0, \quad (10.84)$$

with the definition

$$\eta_2(g) = \sum_{i,j,k,l} \beta_{ijkl}(g) \frac{\partial \ln \zeta_2}{\partial g_{ijkl}}. \quad (10.85)$$

RG functions. From the expressions (10.71–10.79), one derives the expansion of the RG functions at two-loop order:

$$\begin{aligned} \beta_{ijkl} = & -\varepsilon g_{ijkl} + \frac{N_d}{2} \sum_{m,n} (g_{ijmn} g_{mnkl} + 2 \text{ terms}) \\ & - \frac{N_d^2}{4} \sum_{m,n,p,q} (g_{ijmn} g_{mpqk} g_{npql} + 5 \text{ terms}) \\ & + \frac{N_d^2}{48} \sum_{m,n,p,q} (g_{ijkm} g_{mnpq} g_{npql} + 3 \text{ terms}) + O(g^4), \end{aligned} \quad (10.86)$$

and

$$\eta = \frac{N_d^2}{24} \gamma_2 + O(g^3), \quad \eta_2 = -\frac{N_d}{2} \gamma_1 + \frac{5N_d^2}{24} \gamma_2 + O(g^3). \quad (10.87)$$

10.6.3 $O(N)$ symmetry: Fixed point and exponents

These expressions will be used in Section 16.6 in a rather general form. Here, we specialize to the $(\phi^2)^2$ QFT with $O(N)$ symmetry. We then have to substitute

$$g_{ijkl} = \frac{g}{3} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \quad (10.88)$$

The two quantities γ_1, γ_2 become

$$\gamma_1 = \frac{N+2}{3} g, \quad \gamma_2 = \frac{(N+2)}{3} g^2.$$

A short calculation in the notation of equation (10.64) leads to

$$\tilde{\beta}(\tilde{g}) = N_d \beta(\tilde{g}) = -\varepsilon \tilde{g} + \frac{1}{6} (N+8) \tilde{g}^2 - \frac{(3N+14)}{12} \tilde{g}^3 + O(\tilde{g}^4), \quad (10.89)$$

$$\eta(\tilde{g}) = \frac{(N+2)}{72} \tilde{g}^2 + O(\tilde{g}^3). \quad (10.90)$$

$$\eta_2(\tilde{g}) = -\frac{1}{6} (N+2) \tilde{g} (1 - \frac{5}{12} \tilde{g}) + O(\tilde{g}^3). \quad (10.91)$$

One notes that, for $\varepsilon = 4 - d$ positive and small, the β -function has a zero, $\tilde{g}^* \sim 6\varepsilon/(N+8)$, which corresponds in this framework to *Wilson–Fisher’s RG fixed point* in the theory of critical phenomena [75, 133] (see Chapter 15). From $\eta(\tilde{g}^*)$ and $\eta_2(\tilde{g}^*)$ one then infers two critical exponents of the N - vector model:

$$\eta = \eta(\tilde{g}^*), \quad \nu = 1/(2 + \eta_2(\tilde{g}^*)).$$

A10 Feynman parametrization

In explicit calculations of Feynman diagrams a simple identity is often useful. One starts from

$$\frac{1}{a^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty dt t^{\alpha-1} e^{-at}. \quad (\text{A10.1})$$

Therefore,

$$\prod_{i=1}^n (a_i)^{-\alpha_i} = \prod_{i=1}^n (\Gamma(\alpha_i))^{-1} \int_0^\infty \left(\prod_{i=1}^n dt t_i^{\alpha_i-1} \right) \exp \left(- \sum_{i=1}^n a_i t_i \right). \quad (\text{A10.2})$$

Then, setting

$$t_i = s u_i, \quad (\text{A10.3})$$

with

$$u_i \geq 0, \quad \sum_{i=1}^n u_i = 1,$$

one can integrate over s to obtain

$$\prod_{i=1}^n \frac{1}{a_i^{\alpha_i}} = \frac{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n)}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_{u_i \geq 0} \prod_{i=1}^n \frac{du_i u_i^{\alpha_i-1}}{\left(\sum_{j=1}^n a_j u_j \right)^{\alpha_i}} \delta \left(\sum_i u_i - 1 \right). \quad (\text{A10.4})$$

If the quantities a_1, \dots, a_n correspond to propagators, $a_i \equiv p_i^2 + m_i^2$, the integral over momenta can then be explicitly performed.

At one-loop order, only one integral is needed:

$$\frac{1}{(2\pi)^d} \int \frac{d^d p}{(p^2 + 1)^\nu} = \frac{1}{(2\pi)^d \Gamma(\nu)} \int_0^\infty t^{\nu-1} dt \int d^d p e^{-tp^2 - t} = \frac{\Gamma(\nu - d/2)}{(4\pi)^{d/2} \Gamma(\nu)}. \quad (\text{A10.5})$$

Note that the representation (10.3) leads to an expression similar to (A10.4). When $\rho(t)$ is of the form e^{-tm^2} , the argument of the exponential is linear in the variables t_i and, after the change of variables (A10.3), the integral over the homogeneous variable s can be performed.

In the calculations we have also used the integral

$$B(\alpha, \beta) \equiv \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}. \quad (\text{A10.6})$$