Solving for u_x, we are left with one final quadrature to produce the general solution to the original equation (2.111).

One interesting thing to note is that although the equation (2.113) is invariant under a reduced vector field corresponding to the symmetry $\mathbf{v}_3$ of (2.111), there is *no* corresponding symmetry of the intermediate reduced equation (2.112). Indeed,

$$\mathrm{pr}^{(3)}\,\mathbf{v}_3 = u\frac{\partial}{\partial x} - v^2\frac{\partial}{\partial v} - 3vv_x\frac{\partial}{\partial v_x} - (4vv_{xx} + 3v_x^2)\frac{\partial}{\partial v_{xx}}$$

in terms of the invariants x, $v = u_x$ of $\mathbf{v}_1$, but this vector field *cannot* be reduced to one which does not depend on u. As a consequence of this observation, we see that in the general reduction procedure, it is important to wait until we have the invariants for $\mathfrak{g}^{(k)}$ before trying to reduce the next vector field $\mathbf{v}_{k+1}$; one cannot expect $\mathbf{v}_{k+1}$ to naturally reduce relative to an earlier subalgebra $\mathfrak{g}^{(j)}$ if $j < k$!

Systems of Ordinary Differential Equations

Knowledge of a group of symmetries of a system of first order ordinary differential equations has much the same consequences as knowledge of a similar group of symmetries of a single higher order equation. If we know a one-parameter symmetry group, then we can find the solution by quadrature from the solution to a first order system with one fewer equation in it. Similarly, knowledge of an r-parameter solvable group of symmetries allows us to reduce the number of equations by r. These results clearly extend to higher order systems as well, so that invariance of an n-th order system under a one-parameter group, say, allows us to reduce the order of *one* of the equations in the system by one. However, a higher order system can always be replaced by an equivalent first order system, so we are justified in restricting our attention to the latter case.

Theorem 2.66. *Let*

$$\frac{du^\nu}{dx} = F_\nu(x, u), \qquad \nu = 1, \ldots, q, \tag{2.114}$$

be a first order system of q ordinary differential equations. Suppose G is a one-parameter group of symmetries of the system. Then there is a change of variables $(y, w) = \psi(x, u)$ under which the system takes the form

$$\frac{dw^\nu}{dy} = H_\nu(y, w^1, \ldots, w^{q-1}), \qquad \nu = 1, \ldots, q. \tag{2.115}$$

Thus the system reduces to a system of $q - 1$ ordinary differential equations for $w^1, \ldots, w^{q-1}$ together with the quadrature

$$w^q(y) = \int H_q(y, w^1(y), \ldots, w^{q-1}(y))\, dy + c.$$

Proof. Let $\mathbf{v}$ be the infinitesimal generator of G. Assuming $\mathbf{v}|_{(x,u)} \neq 0$, we can locally find new coordinates $y = \eta(x, u)$, $w^\nu = \zeta^\nu(x, u)$, $\nu = 1, \ldots, q$, such that $\mathbf{v} = \partial/\partial w^q$ in these coordinates. In fact,

$$\eta(x, u), \zeta^1(x, u), \ldots, \zeta^{q-1}(x, u),$$

will be a complete set of functionally independent invariants of G, so

$$\mathbf{v}(\eta) = \mathbf{v}(\zeta^\nu) = 0, \qquad \nu = 1, \ldots, q - 1,$$

while $\zeta^q(x, u)$ satisfies

$$\mathbf{v}(\zeta^q) = 1.$$

It is then a simple matter to check that the equivalent first order system for $w^1, \ldots, w^q$ is invariant under the translation group generated by $\mathbf{v} = \partial/\partial w^q$ if and only if the right-hand sides are all independent of w^q, i.e. it is of the form (2.115). □

Example 2.67. Consider an autonomous system of two equations

$$\frac{du}{dx} = F(u, v), \qquad \frac{dv}{dx} = H(u, v).$$

Clearly $\mathbf{v} = \partial/\partial x$ generates a one-parameter symmetry group, so we can reduce this to a single first order equation plus a quadrature. The new coordinates are $y = u$, $w = v$ and $z = x$, in which we are viewing w and z as functions of y. Then

$$\frac{du}{dx} = \frac{1}{dz/dy}, \qquad \frac{dv}{dx} = \frac{dw/dy}{dz/dy},$$

so we have the equivalent system

$$\frac{dw}{dy} = \frac{H(y, w)}{F(y, w)}, \qquad \frac{dz}{dy} = \frac{1}{F(y, w)}.$$

We thus are left with a single first order equation for $w = w(y)$; the corresponding value of $z = z(y)$ is determined by a quadrature:

$$z = \int \frac{dy}{F(y, w)} + c.$$

If we revert to our original variables x, u, v we see that we just have the equation

$$\frac{dv}{du} = \frac{H(u, v)}{F(u, v)}$$

for the phase plane trajectories of the system, the precise motion along these trajectories being then determined by quadrature:

$$x = \int \frac{du}{F(u, v(u))} + c.$$

Theorem 2.68. *Suppose $du/dx = F(x, u)$ is a system of q first order, ordinary differential equations, and suppose G is an r-parameter solvable group of symmetries, acting regularly with r-dimensional orbits. Then the solutions $u = f(x)$ can be found by quadrature from the solutions of a reduced system $dw/dy = H(y, w)$ of $q - r$ first order equations. In particular, if the original system is invariant under a q-parameter solvable group, its general solution can be found by quadratures alone.*

The proof is left to the reader.

Example 2.69. Any linear, two-dimensional system

$$u_t = \alpha(t)u + \beta(t)v,$$

$$v_t = \gamma(t)u + \delta(t)v,$$

is invariant under the one-parameter group of scale transformations $(t, u, v) \mapsto (t, \lambda u, \lambda v)$ with infinitesimal generator $\mathbf{v} = u\partial_u + v\partial_v$, and hence can be reduced to a single first order equation by the method of Theorem 2.66. We set $w = \log u$, $z = v/u$, which straightens out $\mathbf{v} = \partial_w$. These new variables satisfy the transformed system

$$w_t = \alpha(t) + \beta(t)z,$$

$$z_t = \gamma(t) + (\delta(t) - \alpha(t))z - \beta(t)z^2,$$

so if we can solve the Riccati equation for z, we can find w (and hence u and v) by quadrature.

However, if the original system possesses some additional symmetry property, it may be unwise to carry out this preliminary reduction, as the resulting Riccati equation may no longer be invariant under some "reduced" symmetry group. For example, the system

$$u_t = -u + (t + 1)v,$$

$$v_t = u - tv$$

has an additional one-parameter symmetry group with generator $\mathbf{w} = t\partial_u + \partial_v$, as the reader may verify, but the associated Riccati equation

$$z_t = 1 + (1 - t)z - (1 + t)z^2$$

has no obvious symmetry property. The problem is that the vector fields $\mathbf{v}$ and $\mathbf{w}$ generate a solvable, two-dimensional Lie group, but have the commutation relation $[\mathbf{v}, \mathbf{w}] = -\mathbf{w}$, so we should be reducing first with respect to $\mathbf{w}$. To implement the reduction procedure of Theorem 2.68, we need to first straighten out $\mathbf{w} = \partial_{\tilde{w}}$ by choosing coordinates

$$\tilde{w} = v, \qquad \tilde{z} = u - tv.$$

The scaling group still has generator $\mathbf{v} = \tilde{w}\partial_{\tilde{w}} + \tilde{z}\partial_{\tilde{z}}$ in these variables. To straighten its $\tilde{z}$-component we further set $\hat{z} = \log \tilde{z} = \log(u - tv)$, in terms of which

$$\mathbf{w} = \partial_{\tilde{w}}, \qquad \mathbf{v} = \tilde{w}\partial_{\tilde{w}} + \partial_{\hat{z}}.$$

The system now takes the form

$$\frac{d\tilde{w}}{dt} = e^{\hat{z}}, \qquad \frac{d\hat{z}}{dt} = -t - 1,$$

which, as guaranteed by Theorem 2.68, can be integrated by quadratures. We find

$$\hat{z}(t) = -\tfrac{1}{2}(t+1)^2 + \tilde{c},$$
$$\tilde{w}(t) = c \operatorname{erf}[(t+1)/\sqrt{2}] + k,$$

where $\tilde{c} = \log(c\sqrt{2/\pi})$, and

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-x^2}\, dx$$

is the standard error function. Thus the general solution to the original system is

$$u(t) = \sqrt{\frac{2}{\pi}}\, c e^{-(t+1)^2/2} + ct \operatorname{erf}\left(\frac{t+1}{\sqrt{2}}\right) + kt, \qquad v(t) = c \operatorname{erf}\left(\frac{t+1}{\sqrt{2}}\right) + k,$$

where c and k are arbitrary constants.

2.6. Nondegeneracy Conditions for Differential Equations

Often one is interested in classifying all the symmetries of a system of differential equations, and so it is important to know when the infinitesimal methods developed in Section 2.3 can construct the most general connected symmetry group of the given system. For this to be the case, it will be necessary to impose an additional nondegeneracy condition, known as "local solvability", beyond the maximal rank condition of Definition 2.30. This relatively unfamiliar and somewhat technical condition requires that the system have solutions for "arbitrary initial data". In this section we discuss this concept and some of its consequences in detail.

Local Solvability

In order to motivate the definition of local solvability, let's see why, in contrast to the case of systems of algebraic equations, the infinitesimal criterion (2.25) is *not* in general a necessary condition for a Lie group G to be a symmetry group of a system of differential equations of maximal rank. For a system of algebraic equations $F(x) = 0$, to each point x_0 on the subvariety $\mathscr{S}_F = \{x: F(x) = 0\}$ there is, tautologously, a solution to the system; namely, x_0 itself! In contrast, if $\Delta(x, u^{(n)}) = 0$ is a system of differential equations, and $(x_0, u_0^{(n)})$ a point on the corresponding subvariety $\mathscr{S}_\Delta = \{(x, u^{(n)})): \Delta(x, u^{(n)}) = 0\}$, there is in general no guarantee that there exists a solution $u = f(x)$ of the system which has these particular values for its derivatives at x_0, i.e. $u_0^{(n)} = \mathrm{pr}^{(n)} f(x_0)$. Therefore if G, a local group of transformations, is a symmetry group of the system of differential equations, in the sense that it transforms solutions to solutions, there is no assurance that G will leave the entire subvariety $\mathscr{S}_\Delta$ invariant. We can only conclude that those points $(x_0, u_0^{(n)})$ in $\mathscr{S}_\Delta$ for which there does exist such a solution are transformed into other such points in $\mathscr{S}_\Delta$ under group transformations. Therefore, to prove the necessity of the infinitesimal criterion of invariance, we need to assume that every point in $\mathscr{S}_\Delta$ has a corresponding solution.

Definition 2.70. A system of n-th order differential equations $\Delta(x, u^{(n)}) = 0$ is *locally solvable* at the point

$$(x_0, u_0^{(n)}) \in \mathscr{S}_\Delta = \{(x, u^{(n)}): \Delta(x, u^{(n)}) = 0\}$$

if there exists a smooth solution $u = f(x)$ of the system, defined for x in a neighbourhood of x_0, which has the prescribed "initial conditions" $u_0^{(n)} = \mathrm{pr}^{(n)} f(x_0)$. The system is *locally solvable* if it is locally solvable at every point of $\mathscr{S}_\Delta$. A system of differential equations is *nondegenerate* if at every point $(x_0, u_0^{(n)}) \in \mathscr{S}_\Delta$ it is both locally solvable and of maximal rank.

For a system of ordinary differential equations, this condition of local solvability coincides with the usual initial value problem, with $(x_0, u_0^{(n)})$ corresponding to the usual initial data in this case. For instance, for a single second order equation

$$u_{xx} = F(x, u, u_x), \tag{2.116}$$

the initial data for local solvability consist of four numbers $(x^0, u^0, u_x^0, u_{xx}^0)$ subject only to the condition that they satisfy the equation, i.e.

$$u_{xx}^0 = F(x^0, u^0, u_x^0).$$

We are then required to find a solution $u = f(x)$, defined for x near x^0, such that

$$u^0 = f(x^0), \qquad u_x^0 = f'(x^0), \qquad u_{xx}^0 = f''(x^0).$$

Clearly the first two of these conditions form the usual initial value problem for (2.116), and we are thus assured of the existence of a solution $u = f(x)$ satisfying these two conditions. (Indeed, we only need to assume that F is continuous.) The third condition $u_{xx}^0 = f''(x^0)$ is then given to us "for free" since f is a solution, even at x^0, so

$$f''(x^0) = F(x^0, f(x^0), f'(x^0)) = F(x^0, u^0, u_x^0) = u_{xx}^0.$$

Similar reasoning shows that nonsingular systems of ordinary differential equations are always locally solvable.

For systems of partial differential equations, the problem of local solvability is of a completely different character than the more usual existence problems, e.g. Cauchy problems or boundary value problems. In the case of local existence, the initial data is only being prescribed at a single point x_0, whereas one ordinarily requires the specification of the data along an entire submanifold of the space of independent variables. For example, in the case of the wave equation

$$u_{tt} - u_{xx} = 0,$$

the question of local solvability becomes that of determining whether for every set of initial values

$$(x^0, t^0; u^0; u_x^0, u_t^0; u_{xx}^0, u_{xt}^0, u_{tt}^0),$$

subject only to the condition $u_{tt}^0 = u_{xx}^0$, there exists a solution $u = f(x, t)$ of the wave equation in a neighbourhood of (x^0, t^0) with

$$u^0 = f(x^0, t^0), \qquad u_x^0 = \frac{\partial f}{\partial x}(x^0, t^0), \qquad u_t^0 = \frac{\partial f}{\partial t}(x^0, t^0),$$

$$u_{xx}^0 = \frac{\partial^2 f}{\partial x^2}(x^0, t^0), \qquad u_{xt}^0 = \frac{\partial^2 f}{\partial x \partial t}(x^0, t^0), \qquad u_{tt}^0 = \frac{\partial^2 f}{\partial t^2}(x^0, t^0).$$

Clearly in this case the answer is yes, since by design $u_{xx}^0 = u_{tt}^0$, so we can take f to be the polynomial solution

$$\begin{aligned} f(x, t) = u^0 &+ u_x^0(x - x^0) + u_t^0(t - t^0) + \tfrac{1}{2}u_{xx}^0[(x - x^0)^2 + (t - t^0)^2] \\ &+ u_{xt}^0(x - x^0)(t - t^0), \end{aligned}$$

hence the wave equation is locally solvable. (Note that there is no question of uniqueness for the solutions to the local existence problem—even in this simple example no such result is valid.) The reader should contrast this problem with the usual Cauchy problem, in which the initial data is specified along the entire x-axis:

$$u(x, 0) = g(x), \qquad \frac{\partial u}{\partial t}(x, 0) = h(x).$$

There are two principal reasons why a system of partial differential equations might fail to be locally solvable. The first is that the system may have integrability conditions obtained by cross-differentiating the various equations. For example, the over-determined system

$$u_x = yu, \qquad u_y = 0, \tag{2.117}$$

is not locally solvable since at any point (x_0, y_0) there is no smooth solution $u(x, y)$ with "initial conditions"

$$u^0 = u(x_0, y_0) = 1, \qquad u_x^0 = u_x(x_0, y_0) = y_0, \qquad u_y^0 = u_y(x_0, y_0) = 0,$$

values which algebraically satisfy the two equations. Indeed, cross-differentiation shows that

$$0 = u_{xy} = (yu)_y = yu_y + u,$$

hence $u(x, y) \equiv 0$ is the only solution.

Another interesting example, which actually arises from a variational problem (cf. Section 4.1), is the second order system

$$\begin{aligned} u_{xx} + v_{xy} + v_x &= 0, \\ u_{xy} + v_{yy} - u_x &= 0. \end{aligned} \tag{2.118}$$

As it stands, (2.118) is not locally solvable since there is an additional relationship among second order derivatives, namely

$$u_{xx} + v_{xy} = 0,$$

obtained by differentiating the first equation with respect to y, the second with respect to x and subtracting. This in turn implies $v_x = 0$ and $u_{xx} = 0$, so any assignation of initial values

$$(x^0, y^0; u^0, v^0; u_x^0, u_y^0, v_x^0, v_y^0; u_{xx}^0, u_{xy}^0, u_{yy}^0, v_{xx}^0, v_{xy}^0, v_{yy}^0)$$

which satisfies (2.118), but which does not have $v_x^0 = v_{xx}^0 = v_{xy}^0 = u_{xx}^0 = 0$, has no local solution pertaining to it.

The second source of systems which are not locally solvable are certain smooth, but not analytic, systems of differential equations which have no solutions. The original example of such a system was discovered by Lewy, [1], who showed that there exist smooth functions $h(x, y, z)$ such that the first order system

$$\begin{aligned} u_x - v_y + 2yu_z + 2xv_z &= h(x, y, z), \\ u_y + v_x - 2xu_z + 2yv_z &= 0, \end{aligned}$$

has no smooth (or even C^1) solutions on any open subset of $\mathbb{R}^3$. A related example is given by Nirenberg, [1; p. 8], who constructs a function $h(x, y)$ such that the homogeneous linear system

$$\begin{aligned} u_x - h(x, y)v_y &= 0, \\ v_x + h(x, y)u_y &= 0, \end{aligned} \tag{2.119}$$

has only constant solutions in a neighbourhood of the origin.

As we will see, for analytic systems, the Cauchy-Kovalevskaya theorem provides the key to the proof of local solvability. For C^∞ systems, the question is much more delicate, owing to the Lewy-type phenomena, and very few general results are known. Before investigating the analytic case in more detail, we apply the local solvability criterion to the infinitesimal condition for group invariance of a system of differential equations.

Invariance Criteria

Theorem 2.71. *Let $\Delta(x, u^{(n)}) = 0$ be a nondegenerate system of differential equations. A connected local group of transformations G acting on an open subset $M \subset X \times U$ is a symmetry group of the system if and only if*

$$\operatorname{pr}^{(n)} \mathbf{v}[\Delta_\nu(x, u^{(n)})] = 0, \quad \nu = 1, \ldots, l, \quad \textit{whenever} \quad \Delta(x, u^{(n)}) = 0, \tag{2.120}$$

for every infinitesimal generator $\mathbf{v}$ of G.

PROOF. We already know that (2.120) is sufficient for G to be a symmetry group, so we need only prove the necessity of this condition. In light of the algebraic counterpart of this result in Theorem 2.8, it suffices to prove that the subvariety $\mathscr{S}_\Delta = \{\Delta(x, u^{(n)}) = 0\}$ is an invariant subset of the prolonged group action $\operatorname{pr}^{(n)} G$ whenever G transforms solutions of the system to other solutions. Let $(x_0, u_0^{(n)}) \in \mathscr{S}_\Delta$. Using the local solvability, let $u = f(x)$ be a solution of the system defined in a neighbourhood of x_0 such that $u_0^{(n)} = \operatorname{pr}^{(n)} f(x_0)$. If g is a group element such that $\operatorname{pr}^{(n)} g \cdot (x_0, u_0^{(n)})$ is defined, then by appropriately shrinking the domain of definition of f, we can ensure that the transformed function $\tilde{f} = g \cdot f$ is a well-defined function in a neighbourhood of $\tilde{x}_0$, where $(\tilde{x}_0, \tilde{u}_0) = g \cdot (x_0, u_0)$. Since G is a symmetry group, $u = \tilde{f}(x)$ is also a solution to the system. Moreover, by the definition of the prolonged group action, (2.18),

$$\operatorname{pr}^{(n)} g \cdot (x_0, u_0^{(n)}) = (\tilde{x}_0, \operatorname{pr}^{(n)}(g \cdot f)(\tilde{x}_0)) = (\tilde{x}_0, \tilde{u}_0^{(n)}),$$

hence the transformed point $(\tilde{x}_0, \tilde{u}_0^{(n)})$ must again lie in $\mathscr{S}_\Delta$. This proves that $\mathscr{S}_\Delta$ is an invariant subset of G, and the theorem follows. □

In order to appreciate the necessity of the local solvability condition in Theorem 2.71, we discuss a couple of examples. First consider the Nirenberg system (2.119). Since the only solutions near the origin are constants, the translational group $(x, y, u, v) \mapsto (x, y + \varepsilon, u, v)$ is a symmetry group. However, the infinitesimal criterion (2.120) does not hold; applying $\operatorname{pr}^{(1)} \mathbf{v} = \partial_y$ to the first equation we get $h_y v_y$, which is not zero as an algebraic consequence of the system. This group is also, for the same reason, a symmetry group of the over-determined system (2.117). However,

$$\operatorname{pr}^{(1)} \mathbf{v}(u_x - yu) = -u,$$

which does not vanish as an algebraic consequence of (2.117), and again the infinitesimal criterion (2.120) does not apply. However, we can get (2.120) to be both necessary and sufficient for G to be a symmetry group if we only require that it hold at points of local solvability:

Theorem 2.72. *Let $\Delta(x, u^{(n)}) = 0$ be a system of differential equations of maximal rank. A local group of transformations G is a symmetry group of the system if and only if for every point $(x_0, u_0^{(n)}) \in \mathcal{S}_\Delta$ at which the system is locally solvable, we have*

$$\mathrm{pr}^{(n)} \mathbf{v}(\Delta_\nu)(x_0, u_0^{(n)}) = 0, \qquad \nu = 1, \ldots, l,$$

for all infinitesimal generators $\mathbf{v}$ of G.

The proof is immediate. □

The Cauchy-Kovalevskaya Theorem

For analytic systems of partial differential equations, the Cauchy-Kovalevskaya theorem plays a pivotal role in the existence theory. Besides being the principal general existence result for solutions of such systems, this theorem also provides the key to the general theory of characteristics, which underlies any serious investigation of the behaviour of solutions of systems of partial differential equations. As we will see, the Cauchy-Kovalevskaya theorem also gives a proof of the local solvability of most analytic systems of differential equations.

In its original form, the Cauchy-Kovalevskaya theorem treats the Cauchy problem on the initial hyperplane $\{t = t_0\}$ for a system in *Kovalevskaya form*

$$u_{nt}^\alpha \equiv \frac{\partial^n u^\alpha}{\partial t^n} = \Gamma_\alpha(y, t, \widetilde{u^{(n)}}), \qquad \alpha = 1, \ldots, q. \tag{2.121}$$

Here $(y, t) = (y^1, \ldots, y^{p-1}, t)$ are the independent variables, and $\widetilde{u^{(n)}}$ denotes all partial derivatives of u with respect to both y and t up to order n *except* the derivatives u_{nt}^β which appear on the left-hand side of (2.121). The *Cauchy data* for this system is given by

$$\frac{\partial^k u^\alpha}{\partial t^k}(y, t_0) = h_k^\alpha(y), \qquad \alpha = 1, \ldots, q, \quad k = 0, \ldots, n-1, \tag{2.122}$$

where the h_k^α are analytic functions on the hyperplane $\{t = t_0\}$ for y in a neighbourhood of a point $y_0 \in \mathbb{R}^{p-1}$.

Theorem 2.73. *Suppose the functions Γ_α in the Kovalevskaya system* (2.121) *are analytic in their arguments, and the Cauchy data $h_k^\alpha(y)$ in* (2.122) *are also analytic functions for y near y_0. Then there exists a unique analytic solution $u = f(y, t)$ for the Cauchy problem* (2.121), (2.122) *defined for (y, t) in some neighbourhood of the point (y_0, t_0).*

This theorem immediately proves the local solvability of the Kovalevskaya system (2.121).

Corollary 2.74. *If Δ is an analytic system in Kovalevskaya form* (2.121), *then Δ is locally solvable.*

PROOF. Note that for the local solvability problem for (2.121) we can prescribe the lower order t-derivatives $\widetilde{u_0^{(n)}}$ at the initial point (y_0, t_0) in an arbitrary manner, the remaining n-th order derivatives, namely $u^\alpha_{nt,0}$, are then determined by the requirement that $(y_0, t_0, u_0^{(n)})$ be a solution to Δ. Given $(y_0, t_0, u_0^{(n)})$, choose analytic functions $h^\alpha_k(y)$, $k = 0, \ldots, n-1$, $\alpha = 1, \ldots, q$, such that the y-derivatives

$$\partial_J h^\alpha_k(y_0) = \partial^i h^\alpha_k(y_0)/\partial y^{j_1} \cdots \partial y^{j_i}, \qquad 0 \leqslant i \leqslant n-k,$$

agree with the corresponding prescribed values $u^\alpha_{kt,J0}$, where $u^\alpha_{kt,J} \equiv D_J(u^\alpha_{kt})$. (Again, $h^\alpha_k(y)$ could be an appropriate Taylor polynomial.) The corresponding solution $u = f(y, t)$ to the Cauchy problem (2.121), (2.122) ensured by the Cauchy-Kovalevskaya theorem then solves the local existence problem for Δ. Indeed

$$\partial_J \partial_t^k f^\alpha(y_0, t_0) = \partial_J h^\alpha_k(y_0) = u^\alpha_{kt,J0}$$

for $0 \leqslant k \leqslant n-1$, $\#J \leqslant n-k$, while the n-th order derivatives $\partial_t^n f^\alpha(y_0, t_0)$ and $u^\alpha_{nt,0}$ agree because both satisfy the given equations (2.121) at (y_0, t_0) with the same values of $\widetilde{u_0^{(n)}}$. □

More generally, we can admit different order t-derivatives on the left-hand side, whereby a system will be in *general Kovalevskaya form* if

$$\partial^{n_\alpha} u^\alpha/\partial t^{n_\alpha} = \Gamma_\alpha(t, y, \widetilde{u^{(n)}}), \tag{2.123}$$

in which $n = \max\{n_1, \ldots, n_q\}$, and $\widetilde{u^{(n)}}$ denotes all derivatives of each u^β up to order n_β except the particular derivatives $\partial^{n_\beta} u^\beta/\partial t^{n_\beta}$ appearing on the left-hand side. The Cauchy problem (2.122) is the same except that for each α, the index k runs from 0 to $n_\alpha - 1$. All the results of this section, including Corollary 2.74, remain valid for these more general Kovalevskaya forms; the proofs are only slightly more complicated, and are left for the reader to fill in the details.

Characteristics

The range of applicability of the Cauchy-Kovalevskaya existence theorem, and hence the local solvability theorem of Corollary 2.74, can be greatly extended by allowing the possibility of transforming a given system of analytic differential equatons into a system in Kovalevskaya form (2.121) (or (2.123)) by a change of independent variables. To begin with, the system

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, q,$$