15 The renormalization group (RG) approach: The critical theory near four dimensions

In Chapter 14, we have studied Ising-like systems (and more generally ferromagnetic systems with $O(N)$ symmetry) with short range interactions, and determined the behaviour of thermodynamic functions near a second order phase transition, in the mean-field approximation, which is, as we have indicated, a *quasi-Gaussian* approximation. The mean-field approximation predicts a set of *universal* properties, that is, properties independent of the detailed structure of the microscopic Hamiltonian, the dimension of space, and, to a large extent, of the symmetry of systems. We have then systematically examined corrections to the mean-field approximation. We have found that, above four dimensions, these corrections do not change universal quantities. By contrast, below and at four dimensions, the corrections diverge at the critical temperature, and the universal predictions of the mean-field approximation can certainly not be correct. Such an analysis shows that different scales of physics do not decouple, and leads to the important question, how much do the large scale properties of the critical theory depend on the microscopic structure? In Chapter 9, we have answered the question in four dimensions. With some general assumptions about the relevance of perturbation theory, we have shown that the large scale properties can be described by a renormalized quantum field theory, which is largely universal, and depends only on a small number of effective parameters, masses and ϕ^4-like coupling constants. Moreover, below four dimensions, we have identified, in an expansion around the mean field, the most singular terms near criticality, and found indications that they can be also formally recovered from a continuum, low mass ϕ^4 field theory (Section 14.6), a result we use later in the chapter.

However, to understand universality beyond the mean-field approximation, it is necessary to call upon a new idea. Flow equations for effective Hamiltonians based on the RG concept provide the essential tools one needs.

The RG idea, as applied to critical phenomena, has been suggested by Kadanoff [128] and then implemented by Wilson [129]. We first describe the basic RG ideas in a somewhat abstract and intuitive framework. The formulation lacks precision, and the arguments are largely heuristic, but the importance of fixed points in Hamiltonian space emerges. The special role of Gaussian models and their universal properties can be related to the existence of a fixed point, the Gaussian fixed point.

The abstract RG scheme is extremely suggestive, but it requires discovering proper implementations, which is not a straightforward task. Non-perturbative implementations can be found in Refs. [61] and [62]. Numerical investigations have been initiated by Wetterich [130]. However, once the role of the ϕ^4 interaction was recognized by Wilson [120], perturbative methods started to be investigated.

In the framework of the *perturbative mean-field expansion*, this problem has strong similarities with the question of the low mass, low momentum behaviour in effective quantum field theories (QFTs), as discussed in Chapter 8. Universality within perturbation theory relies first on the recognition of the role of renormalizable field theories. Universality is then related, to a large extent, to the question of *regularization independence*, a problem tackled by using *renormalization theory* (Chapter 9). Asymptotic RG equations with a small number of parameters then follow (Section 9.11).

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Therefore, in Section 15.3, we use what we have learned both from the analysis of corrections to the mean-field approximation in Section 14.4, and of the relevant eigenoperators at the Gaussian fixed point near dimension 4 [131], and construct an asymptotic RG for an effective ϕ^4 QFT [72]. We show that the RG equations that appear as a consequence of the renormalizability of a class of local field theories provide a direct realization of the abstract RG equations we have introduced in Section 15.1. Universality in the theory of critical phenomena is thus directly related to the property that renormalizable local field theories are insensitive, to a large extent, to the short distance structure and physics can, therefore, be inferred, at leading order at large distance, from the properties of renormalized correlation functions of QFT. Conversely, from the viewpoint of phase transitions, QFTs are always close to criticality and their existence, beyond perturbation theory, relies, from the abstract RG point of view, on the presence of fixed points in Hamiltonian space.

The results confirm in a very direct manner the deep connection, first recognized by Wilson, between QFT and the theory of critical phenomena.

Finally, after dimensional continuation (Chapter 10), the solution of the RG equations in $d = 4 - \varepsilon$ dimensions makes it possible to calculate universal quantities in an ε-expansion, following Wilson and Fisher [75].

More generally, using the methods of perturbative QFT, it is possible to efficiently calculate universal quantities, and derive universal properties to all orders, for a large class of statistical models of physical interest [80, 132, 133] (see also the textbooks Refs. [134, 135], and the elementary introduction Ref. [64]).

Many systems to which the RG predictions, based on the $O(N)$ symmetric $(\boldsymbol{\phi}^2)^2$ field theory, apply are non-magnetic. In this chapter, we consider three systems, or classes of systems, which are of special interest, because they lead to precise comparisons between RG results, experiments, and simulations: $N = 0$ describes the statistical properties of polymer chains (or self-avoiding random walks) asymptotically when their length becomes large [136]; $N = 1$ and $N = 2$ also describe the liquid–vapour and the superfluid helium transitions, respectively.

15.1 RG: The general idea

We consider a general effective (because it describes only the large scale properties of some initial microscopic model) Hamiltonian, local functional of a field $\phi(x)$ (the classical spin of the lattice models of Chapter 14) in the continuum, in d dimensions, of the form,

$$\mathcal{H}(\phi) = \sum_{n=0} \frac{1}{n!} \int \mathrm{d}^d x_1 \mathrm{d}^d x_2 \cdots \mathrm{d}^d x_n \, \mathcal{H}_n(x_1, x_2, \ldots, x_n) \phi(x_1) \cdots \phi(x_n), \tag{15.1}$$

which has the properties of the thermodynamic potential (14.82) of Landau's theory (Section 14.5.4) as well as of the effective actions of Section 8.8.1. We assume that, in the domain of interest, it is a regular function of all parameters; that it is *translation invariant* and, therefore, the Fourier transforms of the functions $\mathcal{H}_n$ are products of a δ-function of momentum conservation by functions analytic at low momenta (assumption of short range forces or locality). Such a Hamiltonian depends on an infinite number of parameters or coupling constants.

To the Hamiltonian $\mathcal{H}(\phi)$ (or configuration energy), correspond connected correlation functions defined by

$$W^{(n)}(x_1, x_2, \ldots, x_n) = \left[\int [\mathrm{d}\phi] \, \phi(x_1) \cdots \phi(x_n) \, \mathrm{e}^{-\beta \mathcal{H}(\phi)} \right]_{\text{connect.}} . \tag{15.2}$$

We want to study the large distance behaviour of connected correlation functions at a second order phase transition, that is, the behaviour of $W^{(n)}(\lambda x_1, \ldots, \lambda x_n)$ (with all points distinct) when the dilatation parameter λ becomes large.

15.1.1 The RG idea: Fixed points

The RG idea is to trade the initial problem, studying the behaviour of correlation functions as a function of dilatation parameter λ acting on space variables, for the study of the flow of a scale-dependent (effective) Hamiltonian $\mathcal{H}_\lambda(\phi)$, which has the same large scale correlation functions, at fixed space variables. More precisely, we want to construct a Hamiltonian $\mathcal{H}_\lambda(\phi)$ that has correlation functions $W_\lambda^{(n)}(x_i)$ satisfying

$$W_\lambda^{(n)}(x_1, \ldots, x_n) - Z^{-n/2}(\lambda) W^{(n)}(\lambda x_1, \ldots, \lambda x_n) = R_\lambda^{(n)}(x_1, \ldots, x_n), \tag{15.3}$$

where the functions $R^{(n)}$ decrease faster than any power of λ for $\lambda \to \infty$. The mapping $\mathcal{H}(\phi) \mapsto \mathcal{H}_\lambda(\phi)$ is called an RG transformation. We define the transformation such that $\mathcal{H}_{\lambda=1}(\phi) \equiv \mathcal{H}(\phi)$.

The Hamiltonian flow can be constructed by a partial integration over short distance degrees of freedom [61, 62] (see also Ref. [64] for an elementary introduction). Then, the Hamiltonian $\mathcal{H}_\lambda(\phi)$ has a $\lambda \to \infty$ limit only for a suitable choice of the renormalization factor $Z(\lambda)$ (the simplest example is provided by the central limit theorem of probabilities [64]).

Due to the assumption of space-translation invariance, after a Fourier transformation and a factorization of the δ-function of momentum conservation, equation (15.3) becomes

$$\widetilde{W}_\lambda^{(n)}(p_1, \ldots, p_n) = Z^{-n/2}(\lambda) \lambda^{(1-n)d} \widetilde{W}^{(n)}(p_1/\lambda, \ldots, p_n/\lambda) + \widetilde{R}_\lambda^{(n)}. \tag{15.4}$$

Various RG transformations differ by the form of $R^{(n)}$, and the function $Z(\lambda)$. In explicit constructions, the $R^{(n)}$ are generated by the integration over the large momentum modes of $\phi(x)$. When both space and field are continuous variables, one can find RG transformations with $R^{(n)} \equiv 0$. The simplest such RG transformation corresponds to a rescaling of space and field, but this transformation has a fixed point only in exceptional cases (Gaussian models), and thus more general transformations are required [61, 62]. In the following equations we omit the terms $R^{(n)}$ and thus equalities between correlation functions will mean up to terms decreasing faster than any power.

The fixed point Hamiltonian. The coupling constants appearing in $\mathcal{H}_\lambda$ are all explicit functions of λ. We assume that we have found an RG transformation such that, for $\lambda \to \infty$, the Hamiltonian $\mathcal{H}_\lambda(\phi)$ has a limit $\mathcal{H}^*(\phi)$, the (infrared (IR)) fixed point Hamiltonian. If such a fixed point exists in Hamiltonian space, then the correlation functions $W_\lambda^{(n)}$ have corresponding limits $W_*^{(n)}$, and equation (15.3) becomes

$$W^{(n)}(\lambda x_1, \ldots, \lambda x_n) \underset{\lambda \to \infty}{\sim} Z^{n/2}(\lambda) W_*^{(n)}(x_1, \ldots, x_n). \tag{15.5}$$

We now introduce a second scale parameter μ, and calculate $W^{(n)}(\lambda \mu x_i)$ from equation (15.5) in two different ways. We obtain a relation involving only $W_*^{(n)}$:

$$W_*^{(n)}(\mu x_1, \ldots, \mu x_n) = Z_*^{n/2}(\mu) W_*^{(n)}(x_1, \ldots, x_n), \tag{15.6}$$

with

$$Z_*(\mu) = \lim_{\lambda \to \infty} Z(\lambda\mu)/Z(\lambda). \tag{15.7}$$

Equation (15.6), being valid for arbitrary μ, immediately implies that Z_* forms a representation of the dilatation semi-group:

$$Z_*(\lambda_1)Z_*(\lambda_2) = Z_*(\lambda_1\lambda_2). \tag{15.8}$$

Then, under reasonable assumptions,

$$Z_*(\lambda) = \lambda^{-2d_\phi}. \tag{15.9}$$

The fixed point correlation functions have a power-law behaviour characterized by a positive number d_ϕ, which can be called the IR *dimension* of the order parameter $\phi(x)$.

Returning to equation (15.7), we conclude that $Z(\lambda)$ also has asymptotically a power-law behaviour. Equation (15.5) then shows that the correlation functions $W^{(n)}$ have the scaling behaviour at large distances,

$$W^{(n)}(\lambda x_1, \ldots, \lambda x_n) \underset{\lambda\to\infty}{\sim} \lambda^{-nd_\phi} W_*^{(n)}(x_1, \ldots, x_n), \tag{15.10}$$

with a power d_ϕ that is a property of the fixed point.

The right-hand side of the equation, which determines the critical behaviour of correlation functions, depends only on the fixed point Hamiltonian. Therefore, the correlation functions corresponding to all Hamiltonians that flow after RG transformations into the same fixed point, have the same critical behaviour. This property is called *universality*. The space of Hamiltonians is thus divided into *universality classes*. Universality, beyond the mean-field approximation, relies upon the existence of IR fixed points in the space of Hamiltonians.

Applied to the two-point function, this result shows, in particular, that, if $2d_\phi < d$, the correlation length ξ diverges and, therefore, the corresponding Hamiltonians are necessarily critical. Critical Hamiltonians define in Hamiltonian space the *critical surface*, which is invariant under an RG flow. In the generic case where ξ is finite, the correlation length ξ/λ corresponding to $\mathcal{H}_\lambda$ goes to zero. The Fourier components of correlation functions become momentum independent, and thus correlation functions become δ-functions in space. This trivial fixed point corresponds to $2d_\phi = d$.

15.1.2 Hamiltonian flows. Scaling operators

We now assume that the dilatation parameter can be varied continuously (on the lattice this can only be implemented approximately) and perform an additional small dilatation which leads from the scale λ to the scale $\lambda(1 + \mathrm{d}\lambda/\lambda)$.

We perform an RG transformation, the consequence of equation (15.5), for the Hamiltonian itself. We assume that the transformation corresponds to a stationary Markov process (Section 2.1), where $\ln\lambda$ plays the role of time. We thus can write differential equations of the form,

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{H}_\lambda = \mathcal{T}\left[\mathcal{H}_\lambda\right], \tag{15.11}$$

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln Z(\lambda) = 2 - d - \eta\left[\mathcal{H}_\lambda\right], \tag{15.12}$$

where we assume that $\mathcal{T}$ is a differentiable mapping (in some topology) of the space of Hamiltonians into itself, and that η is a real continuous function, defined on the space of Hamiltonians. Moreover, both equations do not depend on λ explicitly, but only through $\mathcal{H}_\lambda$.

Explicit examples of such equations are provided by the quadratic flow equations in Refs. [61, 62], which are also discussed in Chapter 16 of Ref. [64]. Note that we have written the right-hand side of equation (15.12) in an unnatural way for later convenience.

We assume that the RG transformation (15.11) has a fixed point, necessarily solution of the equation

$$\mathcal{T}\,[\mathcal{H}^*] = 0\,, \tag{15.13}$$

where $\mathcal{H}^*$ is the fixed point Hamiltonian, and that $\mathcal{H}_\lambda$ converges towards $\mathcal{H}^*$ for $\lambda \to \infty$.

The dimension d_ϕ of the field ϕ is then

$$d_\phi = \tfrac{1}{2}(d - 2 + \eta\,[\mathcal{H}^*]). \tag{15.14}$$

Linearized flow equations. To study the Hamiltonian flow near the fixed point, and characterize the fixed point stability, we linearize the RG equation (15.11) near the fixed point [131]. Setting $\mathcal{H}_\lambda = \mathcal{H}^* + \Delta\mathcal{H}_\lambda$, we obtain

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta\mathcal{H}_\lambda = L^*(\Delta\mathcal{H}_\lambda), \tag{15.15}$$

where L^* is a linear operator, independent of λ, acting in Hamiltonian space.

We assume that L^* has a discrete set of eigenvalues l_i corresponding to a set of local eigenoperators $\mathcal{O}_i$. Then, $\Delta\mathcal{H}_\lambda$ can be expanded on the $\mathcal{O}_i$'s as

$$\Delta\mathcal{H}_\lambda = \sum_i h_i(\lambda)\mathcal{O}_i\,, \tag{15.16}$$

and the transformation (15.15) becomes

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} h_i(\lambda) = l_i h_i(\lambda). \tag{15.17}$$

The integration then yields

$$h_i(\lambda) = \lambda^{l_i} h_i(1)\,. \tag{15.18}$$

15.1.3 Classification of eigenvectors or scaling fields

The eigenvectors $\mathcal{O}_i$ can be classified into four families depending on the values of the eigenvalues l_i [131].

(i) Eigenvalues with a positive real part: the corresponding eigenoperators are called *relevant*. If $\mathcal{H}_\lambda$ has a component on one of these operators, this component grows with λ, and $\mathcal{H}_\lambda$ moves away from the neighbourhood of $\mathcal{H}^*$. Operators that generate a finite correlation length (and thus $T \neq T_c$) are clearly relevant since a dilatation further decreases the effective correlation length. In Section 15.6, we calculate the corresponding eigenvalue for the ϕ^4 field theory.

(ii) Eigenvalues for which $\mathrm{Re}(l_i) = 0$. Then, two situations can arise: either $\mathrm{Im}(l_i)$ does not vanish and the corresponding component has a periodic behaviour (no example is met in this work) or $l_i = 0$. Eigenoperators corresponding to a vanishing eigenvalue are called *marginal*. In Section 15.5, we show that in the ϕ^4 field theory the operator $\phi^4(x)$ is marginal in four dimensions. To solve the RG equation (15.11), and determine the behaviour of the corresponding component, it is necessary to expand beyond the linear approximation. Generically, one finds

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} h_i(\lambda) \sim B h_i^2(\lambda). \tag{15.19}$$

Depending on the sign of the constant B and the initial sign of h_i, the fixed point then is marginally unstable or stable. In the latter case, the solution takes for λ large the form

$$h_i(\lambda) \sim -1/(B \ln \lambda)\,. \tag{15.20}$$

A marginally stable operator generally leads to a logarithmic approach to a fixed point.

An exceptional example is provided by the XY model in two dimensions ($O(2)$ symmetric non-linear σ-model), which, instead of an isolated fixed point, has a line of fixed points. The operator that corresponds to a motion along the line is obviously marginal (see Chapter 31).

(iii) Eigenvalues with negative real parts. The corresponding operators are called *irrelevant*. The effective components on these operators go to zero for large dilatations.

All these eigenvalues, which are characteristic of the fixed point, may appear in the asymptotic expansion at large distances of the correlation functions corresponding to some critical or near-critical Hamiltonian.

(iv) Finally, some operators do not affect physics. An example is provided by the operator associated to a constant multiplicative renormalization of the dynamical fields $\phi(x)$. These operators are called *redundant*. In QFT, quantum equations of motion correspond to redundant operators with vanishing eigenvalue.

Classification of fixed points. Fixed points can be classified according to their local stability properties, that is, to the number of relevant operators. This number is also the number of parameters it is necessary to fix to impose to a general Hamiltonian to be on the surface which flows into the fixed point. For a non-trivial fixed point corresponding to critical Hamiltonians, it is the co-dimension of the critical surface.

The critical domain. In the mean-field approximation, we have derived universal properties not only for the critical theory, but also for temperatures close to T_c. They can be found also in this RG framework. Indeed, let us add to a critical Hamiltonian a term proportional to a relevant operator with a very small coefficient. For small dilatations, the RG flow is hardly affected. After some large dilatation, the flow will start deviating substantially from the flow of the critical Hamiltonian. But, by then, the components of the Hamiltonian on all irrelevant operators are already small. In the case in which relevant operators induce a finite correlation length, the maximal dilatation is of the order of the ratio between the correlation length and the microscopic scale.

This argument shows that the behaviour of correlation functions as a function of amplitudes of relevant operators is universal in the limit of asymptotically small amplitudes. We call the domain of parameters in which we expect universality the *critical domain*. Note that this strategy is reminiscent of renormalization theory (Chapter 9), where it corresponds to fine tuning.

15.1.4 Explicit RG equations for correlation functions

Equation (15.3) can be rewritten (neglecting R_n) as

$$Z^{n/2}(\lambda) W_\lambda^{(n)}(x_1/\lambda, \ldots, x_n/\lambda) = W^{(n)}(x_1, \ldots, x_n). \tag{15.21}$$

We assume that the Hamiltonian $\mathcal{H}_\lambda$ has been parametrized in terms of constants $h_i(\lambda)$,

$$\mathcal{H}_\lambda = \sum_i h_i(\lambda) \mathcal{O}_i\,.$$

We can then change notation and set

$$W_\lambda^{(n)}(x_1,\dots,x_n) \equiv W^{(n)}\big(\{h(\lambda)\};x_1,\dots,x_n\big),$$

where $\{h(\lambda)\}$ stands for the set of all $h_i(\lambda)$. We then differentiate equation (15.21) with respect to λ. The right-hand side does not depend on λ and, therefore,

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big[Z^{n/2}(\lambda)W^{(n)}\big(\{h(\lambda)\};x_1/\lambda,\dots,x_n/\lambda\big)\Big]=0\,. \tag{15.22}$$

We introduce the differential operator

$$\mathrm{D}_{\mathrm{RG}} \equiv -\sum_\ell x_\ell\frac{\partial}{\partial x_\ell}-\sum_i \beta_i(\{h\})\frac{\partial}{\partial h_i}+\frac{n}{2}\big(2-d-\eta(\{h\})\big),$$

with

$$\beta_i(\{h\})=-\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}h_i(\lambda),\quad 2-d-\eta(\{h\})=\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln Z(\lambda), \tag{15.23}$$

where we have used properties of equations (15.11) and (15.12), which imply that β_i et η do not depend on λ explicitly, but only through the $h_i(\lambda)$. Equation (15.22) becomes

$$\mathrm{D}_{\mathrm{RG}}W^{(n)}\big(\{h(\lambda)\};x_1/\lambda,\dots,x_n/\lambda\big)=0\,.$$

Finally, since λ plays no explicit role anymore, we set $\lambda=1$. The equation becomes a partial differential equation for correlation functions, the form of RG equations we use most frequently,

$$\left[-\sum_\ell x_\ell\frac{\partial}{\partial x_\ell}-\sum_i\beta_i(\{h\})\frac{\partial}{\partial h_i}+\frac{n}{2}\big(2-d-\eta(\{h\})\big)\right]W^{(n)}(\{h\};x_1,\dots,x_n)=0\,. \tag{15.24}$$

With this notation, a fixed point is defined by the common solution to all equations

$$\beta_i(h^*)=0\,,\quad \forall i\,,$$

and equation (15.24) then implies the scaling behaviour derived more directly.

15.2 The Gaussian fixed point

In a rather general framework, we have shown that universality emerges as a consequence of the existence of RG fixed points in Hamiltonian space. Therefore, to be able to quantitatively describe the critical behaviour, one has to construct RG flows explicitly and to find their fixed points. A global analysis has never been performed. In general, one is able only to exhibit a few fixed points and study their local stability.

However, the subspace of quadratic Hamiltonians can be explored completely (free field theories in the QFT terminology), which corresponds to Gaussian Boltzmann weights.

Gaussian distributions play a special role, since they appear as asymptotic distributions in the case of a large number of weakly coupled stochastic degrees of freedom. Moreover, as we have explained in Section 14.7, the weakly perturbed Gaussian (or quasi-Gaussian) model reproduces all results of the mean-field approximation, which makes the study of Gaussian models especially interesting.

We consider a general quadratic Hamiltonian in the continuum, invariant under space translations and, for simplicity, rotations. For short range interactions, the class of interactions considered in Section 14.4, in terms of the Fourier components $\tilde{\phi}(q)$ of the field $\phi(x)$, the Hamiltonian has the form

$$\mathcal{H}_{\mathrm{G}}(\phi) = \tfrac{1}{2} \int \mathrm{d}^d q \, \tilde{\phi}(q) U(q) \tilde{\phi}(-q) \,, \tag{15.25}$$

with

$$U(q) = \sum_{r \geq 0} \tilde{u}_r^{(2)} q^{2r}, \tag{15.26}$$

where U is a positive function with a convergent q^2 expansion. This form is equivalent to the derivative expansion

$$\mathcal{H}_{\mathrm{G}}(\phi) = \tfrac{1}{2} \int \mathrm{d}^d x \, \phi(x) U(-\nabla^2) \phi(x). \tag{15.27}$$

The simple RG transformations $\phi \mapsto \phi \sqrt{Z(\lambda)}$, $x \mapsto \lambda x$, which leads to the Hamiltonian

$$\mathcal{H}_{\mathrm{G},\lambda}(\phi) = \tfrac{1}{2} Z(\lambda) \int \lambda^d \mathrm{d}^d x \, U(-\nabla^2/\lambda^2) \phi(x), \tag{15.28}$$

implies the relation (15.3), with $R^{(2)} = 0$:

$$W_\lambda^{(2)}(x) = Z^{-1}(\lambda) W^{(2)}(\lambda x).$$

In the Hamiltonian, the coefficient of the term with $2r$ derivatives becomes

$$u_r^{(2)} \mapsto u_r^{(2)}(\lambda) = Z(\lambda) \lambda^{d-2r} u_r^{(2)}(1). \tag{15.29}$$

For $\lambda \to \infty$, the leading terms are those with the smallest number of derivatives. For $u_0^{(2)} \neq 0$, the Hamiltonian is non-critical. If we choose for renormalization factor $Z(\lambda) = \lambda^{-d}$, we obtain the trivial fixed point

$$\mathcal{H}_{\mathrm{G}}^*(\phi) = \tfrac{1}{2} u_0^{(2)} \int \mathrm{d}^d x \, \phi^2(x).$$

The two-point correlation function has a δ-function limit ($d_\phi = d/2$), with a vanishing correlation length.

Critical Hamiltonian. For $u_0^{(2)} = 0$, the Hamiltonian is critical. If we choose

$$Z(\lambda) = \lambda^{-(d-2)} \;\Rightarrow\; d_\phi = \tfrac{1}{2}(d-2), \tag{15.30}$$

and thus (see equation (8.52))

$$x \mapsto \lambda x, \quad \phi \mapsto \phi \lambda^{-(d-2)/2}, \tag{15.31}$$

$$u_r^{(2)}(\lambda) = \lambda^{2-2r} u_r^{(2)}(1), \tag{15.32}$$

all $u_r^{(2)}$ with $r > 1$ vanish in the $\lambda \to \infty$ limit.

In the limit, we find the *Gaussian fixed point*,

$$\mathcal{H}_{\mathrm{G}}^*(\phi) = \tfrac{1}{2} u_1^{(2)} \int \mathrm{d}^d x \big(\nabla_x \phi(x)\big)^2, \tag{15.33}$$

which generates the asymptotic behaviour (14.68) of the two-point function at T_c in the mean-field approximation. Finally, if we take $u_0^{(2)} \propto T - T_c > 0$, but small enough such that when $\lambda^2 u_0^{(2)} = O(1)$, $u_r^{(2)}(\lambda)$ for all $r > 1$ is small, we recover the universal two-point function of Ornstein–Zernicke form of the mean-field approximation (14.70), which is valid in the critical domain above T_c, and implies $\nu = \frac{1}{2}$.

15.2.1 Eigenoperators

We now study the RG behaviour of perturbations to the Gaussian fixed point (15.33), within the subspace of *even* Hamiltonians, that is, having an Ising-like $\mathbb{Z}_2$ symmetry. We consider perturbations $\Delta\mathcal{H}$ of the form of local operators, which can thus be expanded both in powers of ϕ and derivatives:

$$\Delta\mathcal{H}(\phi) = \sum_{n=2}^{\infty}\sum_{r=0}^{\infty}\sum_{\alpha}\mathcal{O}_{\alpha,r,n}(\phi),$$

where the operators $\mathcal{O}_{\alpha,r,n}(\phi)$ are space integrals of monomials $O_{\alpha,r,n}(\phi,x)$ in $\phi(x)$ and its derivatives (see equation (8.8)) taken at the same point, of degree n (n even) in ϕ, and with exactly r derivatives (r is also even):

$$\mathcal{O}_{\alpha,r,n}(\phi) = \int \mathrm{d}^d x\, O_{\alpha,r,n}(\phi,x).$$

The index α emphasizes that a pair of values n, r correspond, in general, to several independent homogeneous polynomials.

The same RG transformations then yield (see equation (8.54))

$$[\mathcal{O}_{\alpha,r,n}(\phi)]_\lambda = Z^{n/2}(\lambda)\lambda^{d-r}\mathcal{O}_{\alpha,r,n}(\phi) = \lambda^{d-n(d-2)/2-r}\mathcal{O}_{\alpha,r,n}(\phi). \tag{15.34}$$

We see that these operators are eigenoperators with eigenvalues $l_{n,r} = d - n(d-2)/2 - r$.

Another way to express the same property is to take a fixed basis, and expand

$$\Delta\mathcal{H}(\phi) = \sum_{n=2}^{\infty}\sum_{r=0}^{\infty}\sum_{\alpha}h_{\alpha,r,n}\mathcal{O}_{\alpha,r,n}(\phi).$$

Then,

$$h_{\alpha,r,n}(\lambda) = \lambda^{d-n(d-2)/2-r}h_{\alpha,r,n}(1)$$

or, in differential form,

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}h_{\alpha,r,n}(\lambda) = L^* h_{\alpha,r,n}(\lambda) = l_{r,n}h_{\alpha,r,n}(\lambda), \tag{15.35}$$

which provides an example of equation (15.17), with (*cf.*, equations (8.57) or (8.15))

$$l_{r,n} = d - \tfrac{1}{2}n(d-2) - r\,. \tag{15.36}$$

We can now classify all even operators:

(i) For $n = 2$, $r = 0$, then $l_{0,2} = 2$, and the operator is relevant, corresponding to a deviation from the critical temperature.

(ii) For $n = 2$, $r = 2$, then $l_{2,2} = 0$, and the operator is redundant, corresponding to a simple renormalization of the dynamical field.

(iii) Above dimension 4, all other operators are irrelevant: on the critical surface, the Gaussian fixed point is stable. At dimension 4, $\int \phi^4(x)\mathrm{d}^4x$ ($n = 4, r = 0$ and thus $l_{0,4} = 4 - d$) is marginal, and logarithmic corrections are expected.

Below dimension 4, $\int \phi^4(x)\mathrm{d}^d x$ is relevant and when the dimension decreases additional operators become relevant too. The Gaussian fixed point is IR unstable.

This analysis generalizes our analysis of corrections to the mean-field approximation. It is also equivalent to power counting in QFT as discussed in Section 8.8.1. The monomials that appear in the renormalizable field theory correspond exactly to the set of relevant or marginal operators.

Finally, among the odd operators, which break Ising symmetry, one is always relevant $n = 1, r = 0$, and thus $l_{0,1} = d/2 + 1$, and corresponds to adding a linear coupling to a magnetic field.

15.2.2 Beyond the Gaussian fixed point

The Gaussian fixed point is unstable for $d < 4$. In Section 10.1, we have shown that it is possible to define a dimensional continuation of the perturbative expansion of QFT to real or complex values of the dimension of space d. In this framework, we can consider a critical Hamiltonian in the neighbourhood of dimension 4, that is, for $\varepsilon = 4 - d$ small. If the dimensions of eigenoperators are continuous functions of the dimension of space, for $\varepsilon > 0$ and small, only the operator ϕ^4 will be relevant, with a small dimension. If initially the coefficient g of ϕ^4 is small, and if ε is small enough, there will be a range of dilatations large enough to render the amplitudes of all irrelevant operators negligible, but small enough for the coefficient $g(\lambda)$ to remain small because its evolution is slow. In the leading approximation, the flow of the Hamiltonian will then be governed by the flow of $g(\lambda)$, which eventually must become positive for the transition to be second order.

With these assumptions, the flow of $g(\lambda)$ beyond the linear approximation depends only on $g(\lambda)$ itself. It is plausible that the flow equation has an expansion of the form

$$\lambda \frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda} \equiv -\beta\big(g(\lambda)\big) = (4-d)g(\lambda) - \beta_2 g^2(\lambda) + O(g^3(\lambda)). \tag{15.37}$$

In what follows, we assume that the dilatation $\lambda = 1$ corresponds to a situation where the expansion already makes sense, and thus $g(1)$ is small and positive.

The coefficient β_2 depends on the dimension of space, but at leading order can be replaced by its value at $d = 4$.

The direction of the flow depends on the sign of the function $\beta(g)$. We immediately note that the sign of β_2 plays a crucial role. Let us examine the different cases:

(i) $\beta_2 < 0$. Then, for $d < 4$, the first two terms have the same sign and $g(\lambda)$ increases until the expansion becomes meaningless, and nothing more can be concluded. By contrast, for $d > 4$, one finds a non-trivial repulsive fixed point,

$$g^* = \varepsilon/\beta_2 + O(\varepsilon^2). \tag{15.38}$$

If initially $g(1) < g^*$, $g(\lambda)$ converges towards the Gaussian fixed point. If $g(1) = g^*$, $g(\lambda) = g^*$ for all λ. If $g(1) > g^*$ g(λ) increases, and again one cannot conclude.

(ii) $\beta_2 = 0$. In this exceptional case, the RG flow is governed by the term of order g^3.

(iii) $\beta_2 > 0$. In this case, for $d > 4$, the Gaussian fixed point is stable. By contrast, for $d < 4$, the Gaussian fixed point is unstable, as we already know, but for ε small, another fixed point of the form (15.38) appears, $g^* \sim \varepsilon/\beta_2$, which is stable. Indeed, if initially $g(1) < g^*$, then $g(\lambda)$ increases, and if $g(1) > g^*$, $g(\lambda)$ decreases. This is an especially interesting situation and, as we show in Section 15.4, it is realized in the $O(N)$ symmetric spin model, as first noticed by Wilson and Fisher [75]. Then, universality is predicted, and all universal quantities can be calculated in a power series in the deviation ε from dimension 4.

15.3 Critical behaviour: The effective ϕ^4 field theory

The main difficulty with the general RG approach is that it requires an explicit construction of RG transformations for Hamiltonians, which have a chance to lead to fixed points. Although the general idea is to integrate over large momentum modes of the dynamical variables, its practical implementation is not straightforward [61, 62]. Here, we use a method based on a perturbative RG. The limitation of the method is that it is applicable only when there exists a fixed point that, in a sense which will become slowly clearer, is close to the Gaussian fixed point, in the spirit of the discussion of Section 15.2.2. Moreover, we write RG equations directly for correlation or vertex functions.

In Section 14.7, we have argued that, for an Ising-like system with short range ferromagnetic interactions, in the critical domain and for $d \leq 4$, the sum of the most divergent contributions, order by order in a mean-field expansion, can be reproduced by an effective local field theory whose action is given by the first relevant terms of Landau–Ginzburg–Wilson's Hamiltonian [120] (after some renormalization of the field),

$$\mathcal{H}(\phi) \equiv \beta H = \int \mathrm{d}^d x \left[\tfrac{1}{2} \big(\nabla\phi(x)\big)^2 + \tfrac{1}{2} v\phi^2(x) + \tfrac{1}{4!} g\phi^4(x) \right], \tag{15.39}$$

where v and $g > 0$ are *regular* functions of the temperature for T close to T_c.

Consistently, the analysis of the stability of the Gaussian fixed point has shown that, in four dimensions, the ϕ^4 interaction becomes marginal while all other interactions remain irrelevant. If the dimensions of operators are *continuous* functions of the space dimension, the Hamiltonian (15.39) should contain all relevant operators, at least in some neighbourhood of dimension 4.

The Hamiltonian (15.39) generates a perturbative expansion of QFT type, which can be described in terms of Feynman diagrams. These have to be calculated with a cut-off of order unity, a reflection of the initial microscopic structure. Since the field theory with cut-off provides itself an example of a microscopic theory, we will call it microscopic theory. In the following calculations we choose below a Pauli–Villars's type regularization, which amounts to adding a number of irrelevant terms to the Hamiltonian, which, therefore, are not expected to modify the large scale properties. More general regularizations are possible, but they should satisfy some general conditions. For example, the propagator can be modified (as in momentum regularization), but the Fourier transform of the inverse propagator must remain a regular function of momentum (the forces are short range).

In most of the chapter, we consider a one-component field and a system with Ising-like symmetry. In Section 15.7, we generalize to the N-vector model with $O(N)$ symmetry. Several-component models with more than one ϕ^4 coupling constant are briefly discussed in Section 16.6.

In Section 14.7, we have already noted that a convenient way to study large scale properties (or low-momentum behaviour) is to rescale all space or momentum variables and measure distances in units of the correlation length or, at the critical temperature, in some arbitrary unit much larger than the lattice spacing, and corresponding to the typical distances at which correlations are measured. After such a rescaling, the momentum cut-off becomes a large momentum Λ, analogous to the cut-off used to regularize QFT.

Therefore, we perform such a rescaling here, together with the corresponding RG transformation (15.30) on the field $\phi(x)$, associated to the Gaussian fixed point:

$$x \mapsto \Lambda x\,, \quad \phi(x) \mapsto \Lambda^{1-d/2}\phi(x). \tag{15.40}$$

This transformation is similar to the transformation (8.52) and can be called a Gaussian renormalization. After the rescaling, all quantities have a dimension in units of Λ, the field ϕ having its Gaussian dimension $(d/2-1)$. The regularized Hamiltonian (15.39) can then be written as (see also Section 8.9)

$$\mathcal{H}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2} \big(\nabla_\Lambda\phi(x)\big)^2 + \tfrac{1}{2} r\phi^2(x) + \tfrac{1}{4!} g\Lambda^{4-d}\phi^4(x) \right], \tag{15.41}$$

with $r = v\Lambda^2$ and

$$\nabla_\Lambda = \nabla \left(1 - \alpha_1 \nabla^2/\Lambda^2 + \alpha_2 \nabla^4/\Lambda^4 + \cdots\right).$$

For a value $r = r_c(g, \Lambda)$, which corresponds to the critical temperature T_c, the correlation length ξ diverges. In terms of the scale Λ, the critical domain is then defined by

$$\begin{aligned} \text{physical mass } &= \xi^{-1} \ll \Lambda \;\Rightarrow\; |r - r_c| \ll \Lambda^2, \\ \text{distances } &\gg 1/\Lambda \quad \text{or momenta} \ll \Lambda, \\ \text{magnetization } M &\equiv \langle \phi(x) \rangle \ll \Lambda^{(d/2)-1}. \end{aligned} \tag{15.42}$$

These conditions are met if Λ is identified with the cut-off of a usual QFT. However, an inspection of the action (15.41) also shows that, in contrast with what is assumed in conventional QFT, the dependence on Λ of the ϕ^4 coupling constant is fixed. For $d = 4$, we have already discussed this issue in Section 9.12. For $d < 4$, in the usual formulation of QFT, the quantity $g\Lambda^{4-d}$ is fixed, a situation which, from the viewpoint of phase transitions, corresponds to a highly non-generic small coupling (and assumes some *fine tuning*).

15.4 RG equations near four dimensions

The Hamiltonian (15.41) can now be studied by field theoretical methods. By direct calculation, we have shown in Section 10.5.6 that the perturbative expansion for the critical (or massless) theory has IR divergences for any dimension smaller than 4. If we define, by dimensional continuation, a critical theory in $d = 4 - \varepsilon$ dimension, even for ε arbitrarily small but fixed, there always exists an order $\ell \sim 4/\varepsilon$ in the perturbative expansion, at which IR divergences appear. Therefore, the idea, originally due to Wilson and Fisher [75], is to perform a double series expansion in powers of the coupling constant g and ε, with g and ε considered of the same order. Order by order in the expansion, for $r = r_c$ (*i.e.*, $T = T_c$), the critical behaviour differs from the mean-field behaviour only by powers of logarithm, and one can construct a perturbative critical theory.

To study the large cut-off limit, we then use methods developed for the construction of the renormalized massless ϕ^4 field theory in Section 9.9.2. We introduce rescaled vertex functions, called renormalized, defined by renormalization conditions at a new scale $\mu \ll \Lambda$, and functions of a renormalized coupling constant $g_{\rm r}$ (the effective coupling at scale μ). In the Fourier representation, they can be defined by

$$\begin{aligned} \tilde{\Gamma}^{(2)}_{\rm r}\ (p; g_{\rm r}, \mu, \Lambda)|_{p^2=0} &= 0\,, \\ \frac{\partial}{\partial p^2} \tilde{\Gamma}^{(2)}_{\rm r}\ (p; g_{\rm r}, \mu, \Lambda)|_{p^2=\mu^2} &= 1\,, \\ \tilde{\Gamma}^{(4)}_{\rm r}(p_i = \mu\theta_i; g_{\rm r}, \mu, \Lambda) &= \mu^{\varepsilon} g_{\rm r}\,, \end{aligned} \tag{15.43}$$

in which θ_i is a non-vanishing numerical vector. The renormalized and microscopic vertex functions are related by the equations

$$\tilde{\Gamma}^{(n)}_{\rm r}(p_i; g_{\rm r}, \mu, \Lambda) = Z^{n/2}(g, \Lambda/\mu) \tilde{\Gamma}^{(n)}(p_i; g, \Lambda), \tag{15.44}$$

where the factor Z has to be understood as the ratio between the renormalization (15.3) and the Gaussian renormalization (15.30). Renormalization theory (more precisely a slightly extended version adapted to the ε-expansion) implies that, within the double g, ε expansion, the functions $\tilde{\Gamma}^{(n)}_{\rm r}(p_i; g_{\rm r}, \mu, \Lambda)$ of equation (15.44) have, at p_i, $g_{\rm r}$, and μ fixed large cut-off limits, which are the renormalized correlation functions $\tilde{\Gamma}^{(n)}_{\rm r}(p_i; g_{\rm r}, \mu)$. A detailed analysis actually shows that

$$\tilde{\Gamma}^{(n)}_{\rm r}(p_i; g_{\rm r}, \mu, \Lambda) = \tilde{\Gamma}^{(n)}_{\rm r}(p_i; g_{\rm r}, \mu) + O(\Lambda^{-2} (\ln \Lambda)^L), \tag{15.45}$$

in which the power L increases with the order in g and ε (see equation (9.96)).

Moreover, the renormalized functions $\tilde{\Gamma}_{\rm r}^{(n)}$ do not depend on the specific cut-off procedure and given the normalization conditions (15.43), are, therefore, *universal*. Since the renormalized functions $\tilde{\Gamma}_{\rm r}^{(n)}$ and the initial ones $\tilde{\Gamma}^{(n)}$ are asymptotically proportional, both functions have the same small momentum or large distance behaviour. The renormalized functions thus contain the whole information about the asymptotic universal critical behaviour. Therefore, we could consider only the renormalized field theory, which indeed is the only one really useful for explicit leading order calculations (see Section 16.9). However, universality is not limited to the asymptotic critical behaviour; leading corrections have also some interesting universal properties. Moreover, renormalized quantities are not directly obtained in non-perturbative calculations and may not exist for all values of $g_{\rm r}$ (see Section 9.12). For these reasons, it is useful to study the implications of equation (15.44) also, directly in the microscopic theory.

RG equations in the microscopic parameters. Since the RG equations have been derived in Section 9.11.2, we do not repeat the derivation here. In the first part of the study, we neglect corrections subleading by powers of Λ (the assumption of perturbative RG). We return to this point in Chapter 17.

The RG equation is given by equation (9.94) and reads [72]

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}-\frac{n}{2}\eta(g)\right]\tilde{\Gamma}^{(n)}(p_i;g,\Lambda)=0\,. \tag{15.46}$$

Equation (15.46) is satisfied, when the cut-off is large, by the microscopic vertex functions of statistical mechanics, which are also the bare vertex functions of QFT. It is a direct consequence of the existence of a renormalized theory. We also note that equation (15.46) is a simplified form of equation (15.24), when only one operator is retained.

The form of the contributions that have been neglected has been given in equation (9.96). The RG equation (15.46) is exact for the sum of the perturbative contributions that do not vanish for Λ large, as can be verified by expanding equation (15.46) in powers of g and ε.

The RG functions β and η can be calculated in perturbation theory. At one-loop order $\eta(g)$ vanishes. The four-point vertex function has the one-loop expansion for Λ large ($\varepsilon=4-d$) (see equation (9.104)),

$$\tilde{\Gamma}^{(4)}(p_i)=\Lambda^{\varepsilon}\left[g-\frac{3g^2}{16\pi^2}\big(\ln\Lambda+O(1)\big)\right]+O(g^3,g^2\varepsilon). \tag{15.47}$$

Taking into account $\eta(g)=O(g^2)$ and inserting the expansion into equation (15.46), one obtains

$$\beta(g,\varepsilon)=-\varepsilon g+\frac{3g^2}{16\pi^2}+O(g^3,g^2\varepsilon). \tag{15.48}$$

Similarly, the two-point function at two loops is (equation (9.106)),

$$\tilde{\Gamma}^{(2)}(p)=p^2+\frac{1}{24}\frac{1}{(8\pi^2)^2}g^2p^2\big(\ln\Lambda+O(1)\big)+O(g^3,g^2\varepsilon).$$

It follows (see equation (9.107)),

$$\eta(g)=\frac{1}{6}\frac{1}{(4\pi)^4}g^2+O(g^3,g^2\varepsilon). \tag{15.49}$$

15.5 Solution of the RG equations: The ε-expansion

Equation (15.46) can be solved by the method of characteristics (see Section 9.12 for details). One introduces a dilatation parameter λ and looks for functions $g(\lambda)$ and $Z(\lambda)$ such that

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[Z^{-n/2}(\lambda) \tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \lambda\Lambda\big) \right] = 0 \,. \tag{15.50}$$

Differentiating explicitly with respect to λ, one finds that equation (15.50) is compatible with equation (15.46) provided that (with $g(1) = g$, $Z(1) = 1$)

$$\ln \lambda = \int_g^{g(\lambda)} \frac{\mathrm{d}g'}{\beta(g')}, \quad \ln Z(\lambda) = \int_1^{\lambda} \frac{\mathrm{d}\sigma}{\sigma} \eta\big(g(\sigma)\big) = \int_g^{g(\lambda)} \frac{\mathrm{d}g' \eta(g')}{\beta(g')}. \tag{15.51}$$

Equation (15.50) implies

$$\tilde{\Gamma}^{(n)}(p_i; g, \Lambda) = Z^{-n/2}(\lambda) \tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \lambda\Lambda\big).$$

It is actually convenient to rescale Λ by a factor $1/\lambda$ and write the equation as

$$\tilde{\Gamma}^{(n)}(p_i; g, \Lambda/\lambda) = Z^{-n/2}(\lambda) \tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \Lambda\big). \tag{15.52}$$

Equations (9.98), (9.99), and (15.52) implement approximately (because terms subleading by powers of Λ have been neglected) the general RG ideas as presented in Section 15.1: $g(\lambda)$, the effective coupling constant at scale λ, characterizes the Hamiltonian $\mathcal{H}_\lambda$, equation (9.98) is the equivalent of equation (15.11) (up to the change $\lambda \mapsto 1/\lambda$); equations (15.12) and (9.99) differ by the definition of $Z(\lambda)$, because the Gaussian renormalization (15.30) has already been performed.

Equation (15.52) shows that it is equivalent to increase Λ or to decrease λ. Therefore, to investigate the large Λ limit, we study the behaviour of the effective coupling constant $g(\lambda)$ when λ goes to zero. In what follows, we explicitly assume that the RG functions $\beta(g)$ and $\eta(g)$ are regular functions of g for $g \geq 0$.

IR fixed points. The first equation (15.51) shows that $g(\lambda)$ increases if the function β is negative, or decreases in the opposite case. Fixed points correspond to zeros of the β-function which, therefore, play an essential role in the analysis of the critical behaviour. Those where the β-function has a negative slope are IR repulsive: the effective coupling moves away from such zeros, except if the initial coupling has an exact fixed point value. Conversely, those where the slope is positive are IR attractive.

We now assume that initially g is in the range where the behaviour of the β-function is still governed by the explicit expression (15.48).

Then, above four dimensions, that is $\varepsilon < 0$, $g(\lambda)$ decreases, approaching the origin $g = 0$. We recover that the Gaussian fixed point is IR stable.

Below four dimensions, if initially g is small enough, $g(\lambda)$ first increases, a behaviour reflecting the instability of the Gaussian fixed point.

However, for ε infinitesimal, $\beta(g)$ has another zero,

$$\beta(g^*) = 0 \,, \quad \text{for} \quad g^* = 16\pi^2 \varepsilon/3 + O(\varepsilon^2), \tag{15.53}$$

with a positive slope,

$$\omega \equiv \beta'(g^*) = \varepsilon + O(\varepsilon^2) > 0 \,. \tag{15.54}$$

This agrees with one of the possibilities envisaged in Section 15.2.2. Then, the first equation (15.51) shows that $g(\lambda)$ has g^* as an asymptotic limit. Linearizing the equation near the fixed point, one finds

$$\int_g^{g(\lambda)} \frac{\mathrm{d}g'}{\omega(g'-g^*)} \sim \ln\lambda\,, \tag{15.55}$$

or in integrated form,

$$|g(\lambda)-g^*| \underset{\lambda\to 0}{\propto} \lambda^{\omega}. \tag{15.56}$$

Below dimension 4, at least for ε infinitesimal, this non-Gaussian fixed point is IR stable. In dimension 4, this IR fixed point merges with the Gaussian fixed point, and the eigenvalue ω vanishes, indicating the appearance of the marginal operator already identified in the analysis of the Gaussian fixed point in Section 15.2.

The behaviour of correlation functions. We now assume that $\tilde{\Gamma}^{(n)}(g^*)$ and $\eta(g^*)$ are finite, conditions that are satisfied within the framework of the ε-expansion. The second equation (15.51) determines the behaviour of $Z(\lambda)$ for λ small. The integral in the left-hand side is dominated by small values of σ. It follows that

$$\ln Z(\lambda) \underset{\lambda\to 0}{\sim} \eta \ln\lambda\,, \tag{15.57}$$

where we have set

$$\eta = \eta(g^*).$$

Equation (15.52) then determines the behaviour of $\tilde{\Gamma}^{(n)}(p_i; g, \Lambda)$ for Λ large:

$$\tilde{\Gamma}^{(n)}(p_i; g, \Lambda/\lambda) \propto \lambda^{-n\eta/2}\tilde{\Gamma}^{(n)}(p_i; g^*, \Lambda). \tag{15.58}$$

Moreover, simple dimensional considerations (which are directly related to the Gaussian renormalization) imply that

$$\tilde{\Gamma}^{(n)}(p_i; g, \Lambda/\lambda) = \lambda^{-d+(n/2)(d-2)}\tilde{\Gamma}^{(n)}(\lambda p_i; g, \Lambda). \tag{15.59}$$

Combining this equation with equation (15.58), one obtains

$$\tilde{\Gamma}^{(n)}(\lambda p_i; g, \Lambda) \underset{\lambda\to 0}{\propto} \lambda^{d-(n/2)(d-2+\eta)}\tilde{\Gamma}^{(n)}(p_i; g^*, \Lambda). \tag{15.60}$$

After Legendre and Fourier transformations, taking into account the δ-function of momentum conservation, one infers for the connected correlation functions,

$$\tilde{W}^{(n)}(\lambda p_i; g, \Lambda) \underset{\lambda\to 0}{\propto} \lambda^{d-(n/2)(d+2-\eta)}\tilde{W}^{(n)}(p_i; g^*, \Lambda), \tag{15.61}$$

$$W^{(n)}(x_i/\lambda; g, \Lambda) \underset{\lambda\to 0}{\propto} \lambda^{(n/2)(d-2+\eta)}W^{(n)}(x_i; g^*, \Lambda). \tag{15.62}$$

These equations show that the critical vertex and connected correlation functions have a power-law behaviour for small momenta, independent of the initial value of the ϕ^4 coupling constant g, at least if g initially is small enough for perturbation theory to be applicable, or if the β-function has no additional zero.

For $n = 2$, equation (15.62) implies,

$$W^{(2)}(x) \underset{|x|\to\infty}{\propto} 1/|x|^{d-2+\eta} \tag{15.63}$$

Therefore, the definition (15.57) coincides with the usual definition of the critical exponent η. The spectral representation of the two-point function (Section 6.6) implies $\eta > 0$. Since in perturbation theory, the first contribution to the field renormalization $Z(g, \Lambda/\mu)$ arises at order g^2, $\eta(g)$ is of order g^2, and η of order ε^2.

A two-loop calculation (equation (15.49)) yields

$$\eta = \frac{\varepsilon^2}{54} + O(\varepsilon^3). \tag{15.64}$$

A semi-quantitative prediction is that η is numerically small in three dimensions.

Finally, equation (15.60) has the following interpretation: the field $\phi(x)$, which had at the Gaussian fixed point the Gaussian dimension $(d-2)/2$, has now acquired the 'anomalous' or IR dimension (equation (15.14))

$$d_\phi = \tfrac{1}{2}(d - 2 + \eta).$$

Universality. These results call for a few comments. Within the framework of the ε-expansion, we have shown that all correlation functions have, for $d < 4$, a large distance behaviour different from the one predicted by the mean-field approximation. In addition, the critical behaviour does not depend on the initial value of the ϕ^4 coupling constant g. At least for ε small, we can hope that the analysis of leading IR singularities of Section 14.4 remains valid, and thus it does not depend on any other coupling either (this point will be further discussed in Section 17.4). Therefore, the critical behaviour is *universal*, although less universal than in the mean-field approximation, since it depends on a small number of general properties of the system under consideration, like symmetry or dimension of space.

Moreover, the correlation functions obtained by neglecting, in perturbation theory and within the ε-expansion, power-law corrections when the cut-off is large, and which satisfy exactly RG equations (15.46), define implicitly a one parameter family of critical Hamiltonians, which correspond to an RG trajectory which goes from the neighbourhood of the Gaussian fixed point $g = 0$, which is IR unstable below four dimensions, to a non-trivial stable fixed point g^*.

Finally, the consistency of this analysis, based on the ε-expansion, relies on the following observation. The IR divergences found in the perturbative expansion at fixed dimension result from an expansion around an IR repulsive fixed point. The ε-expansion makes an exchange of limits, and thus a perturbative expansion around an attractive IR fixed point possible.

15.6 Critical correlation functions with $\phi^2(x)$ insertions

In Chapter 16, we study the whole critical domain (15.42).

Following a method explained in Section 9.10, correlation functions for $T \neq T_c$ are expanded in powers of *critical* correlation functions with $\phi^2(x)$ insertions.

Here, the monomial $\phi^2(x)$ has a direct physics interpretation: it is the most singular contribution (*i.e.* the most relevant) to the energy density (15.39).

Therefore, we now discuss the large distance properties of the critical mixed correlation functions of the order parameter ϕ and the energy density $\frac{1}{2}\phi^2$. We denote by $\tilde{\Gamma}^{(l,n)}(q_1, \ldots, q_l; p_1, \ldots, p_n; g, \Lambda)$ the Fourier transforms of vertex functions of $n\,\phi(x)$ fields and $l\,\frac{1}{2}\phi^2(x)$ monomials, (with $(l+n) \geq 2$).

15.6.1 RG equations

Renormalization theory implies that one can define renormalized vertex functions $\tilde{\Gamma}_{\rm r}^{(l,n)}$ related to the microscopic functions by (see equations (9.72) or (11.6))

$$\tilde{\Gamma}_{\rm r}^{(l,n)}(q_i;p_j;g_{\rm r},\mu) = \lim_{\substack{\Lambda\to\infty \\ \text{at } \mu,g_{\rm r} \text{ fixed}}} Z^{n/2-l}Z_2^l\left[\tilde{\Gamma}^{(l,n)}(q_i;p_j;g,\Lambda)-\delta_{n0}\delta_{l2}\Lambda^{-\varepsilon}C(g,\Lambda/\mu)\right], \quad (15.65)$$

where $Z_2(g,\Lambda/\mu)$ and $C(g,\Lambda/\mu)$ are two new renormalization constants.

They can be fixed by imposing, in addition to the conditions (15.43), the conditions

$$\tilde{\Gamma}_{\rm r}^{(1,2)}(q;p_1,p_2;g_{\rm r},\mu)\Big|_{p_1^2=p_2^2=\mu^2,\ p_1\cdot p_2=-\frac{1}{3}\mu^2} = 1\,,\ \ \tilde{\Gamma}_{\rm r}^{(2,0)}(q,-q;g_{\rm r},\mu)\Big|_{q^2=\frac{4}{3}\mu^2} = 0\,. \quad (15.66)$$

Differentiating with respect to Λ at $g_{\rm r}$ and μ fixed, as in Section 15.4, and using chain rule, one obtains the RG equations [72],

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}-\tfrac{1}{2}n\eta(g)-l\eta_2(g)\right]\tilde{\Gamma}^{(l,n)} = \delta_{n0}\delta_{l2}\Lambda^{-\varepsilon}B(g). \quad (15.67)$$

In addition to β and η, two new RG functions, $\eta_2(g)$ and $B(g)$, appear:

$$\eta_2(g) = -\Lambda\left.\frac{\partial}{\partial\Lambda}\right|_{g_{\rm r},\mu}\ln\left[Z_2(g,\Lambda/\mu)\,/Z(g,\Lambda/\mu)\right], \quad (15.68)$$

$$B(g) = \left[\Lambda\left.\frac{\partial}{\partial\Lambda}\right|_{g_{\rm r},\mu}-2\eta_2(g)-\varepsilon\right]C(g,\Lambda/\mu). \quad (15.69)$$

Again, these two RG functions, because they are calculable in terms of correlation functions which do not depend on μ, cannot depend on the ratio Λ/μ.

Note that for $n=0$, $l=2$, the RG equation (15.67) is not homogeneous. This is a consequence of the non-multiplicative character of the renormalization in this case. Multiple insertions of operators of higher dimension like $\phi^4(x)$ lead to even more complicated RG equations. This question has been discussed in Chapter 11, and we want only to warn the reader here against a too naive application of RG ideas.

In the homogeneous case, equation (15.67) can be solved by the method of characteristics explained in Section 15.5, in exactly the same way as equation (15.46). A new scale-dependent function $\zeta_2(\lambda)$ is associated with the RG function $\eta_2(g)$. We set

$$\lambda\frac{\rm d}{{\rm d}\lambda}\left[Z^{-n/2}(\lambda)\zeta_2^{-l}(\lambda)\tilde{\Gamma}^{(l,n)}(q_i;p_j;g(\lambda),\lambda\Lambda)\right] = 0\,,$$

and $\zeta_2(\lambda)$ satisfies

$$\lambda\frac{\rm d}{{\rm d}\lambda}\ln\zeta_2(\lambda) = \eta_2[g(\lambda)].$$

It follows from the arguments of Section 15.5 that the critical behaviour of $\tilde{\Gamma}^{(l,n)}$ is governed by the IR fixed point g^*.

To relate the RG function $\eta_2(g)$ to standard exponents, we introduce the function

$$\nu(g) = \left[\eta_2(g)+2\right]^{-1}. \quad (15.70)$$

For λ small, ζ_2 behaves like

$$\ln \zeta_2 \sim (1/\nu - 2) \ln \lambda\,, \quad \text{with} \quad \nu = \nu(g^*).$$

Combining again the solution of equation (15.67) with simple dimensional analysis, one obtains the behaviour

$$\tilde{\Gamma}^{(l,n)}(\lambda q_i; \lambda p_j; g, \Lambda) \underset{\lambda \to 0}{\propto} \lambda^{d-n(d-2+\eta)/2-l/\nu} \tilde{\Gamma}^{(l,n)}(q_i; p_j; g^*, \Lambda). \tag{15.71}$$

For connected correlation functions, this translates into the relations

$$\tilde{W}^{(l,n)}(\lambda q_i; \lambda p_j; g, \Lambda) \underset{\lambda \to 0}{\propto} \lambda^{d-(n/2)(d+2-\eta)-l/\nu} \tilde{W}^{(l,n)}(q_i; p_j; g^*, \Lambda), \tag{15.72}$$

$$W^{(l,n)}(y_i/\lambda; x_j/\lambda; g, \Lambda) \underset{\lambda \to 0}{\propto} \lambda^{(n/2)(d-2+\eta)+l(d-1/\nu)} W^{(l,n)}(y_i; x_j; g^*, \Lambda). \tag{15.73}$$

Using equations (15.68) and (15.70), it is straightforward to calculate $\nu(g)$ at one-loop order. One finds

$$2\nu(g) = 1 + \frac{g}{32\pi^2} + O(g^2), \tag{15.74}$$

and then, setting $g = g^*$ (equation (15.53)), the exponent ν at first order in ε:

$$2\nu = 1 + \tfrac{1}{6}\varepsilon + O(\varepsilon^2).$$

The $\langle \phi^2 \phi^2 \rangle$ correlation function. The ϕ^2 (energy density) two-point function $\tilde{\Gamma}^{(2,0)}$ satisfies the inhomogeneous RG equation (15.67). To solve it, we first look for a particular solution of the form $\Lambda^{-\varepsilon} C_2(g)$. Then,

$$\beta(g) C_2'(g) - [\varepsilon + 2\eta_2(g)]\, C_2(g) = B(g). \tag{15.75}$$

The solution is uniquely determined by imposing its *regularity* at $g = g^*$.

The general solution of equation (15.67) then is the sum of this particular solution and the general solution of the homogeneous equation, which has a behaviour given by equation (15.71):

$$\tilde{\Gamma}^{(2,0)}(q; g, \Lambda) - \Lambda^{-\varepsilon} C_2(g) \underset{q \to 0}{\sim} K \Lambda^{2/\nu - 4} q^{d-2/\nu}. \tag{15.76}$$

Note that the regular contribution depends on g but not the constant K.

After Fourier transformation, the singular term has the scaling behaviour (15.73) and the constant gives a δ-function.

The dimension of the ϕ^2 operator. The scaling behaviour (15.73) determines the dimension of the operator ϕ^2 at the IR fixed point. Quite generally, the connected n-point correlation function with insertions of l scaling fields $\mathcal{O}_i$ of dimensions d_i has, in real space, the scaling behaviour

$$\left\langle \prod_i \mathcal{O}_i(y_i/\lambda) \prod_j \phi(x_j/\lambda) \right\rangle_{\lambda \to 0} \propto \lambda^D, \quad \text{with } D = n d_\phi + \sum_i d_i\,.$$

Equation (15.73) then implies $d_{\phi^2} = d - 1/\nu$.

Remark. Real physics corresponds to integer values of ε, $\varepsilon = 1, 2$. Although one can prove the validity of RG results only within the framework of the ε-expansion, we will eventually assume that their validity extends beyond an infinitesimal neighbourhood of dimension 4. The comparison between RG results derived from the ε expansion and experimental, or numerical, data thus provides a crucial test of the theory (see Section 16.9).

The fixed point Hamiltonian. Field theory considerations indicate that the explicit determination of a fixed point Hamiltonian $\mathcal{H}^*$ is in general not easy, except for the Gaussian fixed point. Since fixed point correlation functions have been calculated, they define implicitly a class of fixed point Hamiltonians. It has also been proposed to define $\mathcal{H}^*$ as a limit of a sequence of Hamiltonians $\mathcal{H}_k$ whose correlation functions $W_k^{(n)}$ converge towards fixed point correlation functions $W_*^{(n)}$ in the following way:

$$\left| W_k^{(n)}(\lambda q_i) - W_*^{(n)}(\lambda q_i) \right| = O(\lambda^{2k}).$$

In Sections 15.3–15.5, we have constructed $\mathcal{H}_1$.

15.6.2 Renormalization theory and critical phenomena

Let us comment about the significance of these results for QFT in the context of renormalization theory. From the preceding analysis, we infer that the existence of a renormalizable field theory, beyond perturbation theory, relies on two properties:

(i) The existence of a continuous phase transition.

(ii) The existence of an IR RG fixed point that ensures that the large distance physics is short-distance insensitive, and can be described by a renormalized QFT.

However, an inspection of the action (15.39) shows that, in contrast with conventional renormalization theory, the ϕ^4 coupling constant has a dependence in Λ given *a priori*. For $d < 4$, the ϕ^4 coupling is very large in terms of the scale relevant for the critical domain. By contrast, in QFT, in the usual formulation of renormalization theory, the *microscopic* coupling constant is considered as an adjustable parameter: a super-renormalizable QFT corresponds to a Hamiltonian maintained close to the Gaussian fixed point by adjusting the coefficients of all relevant and marginal operators. The resulting theory is thus less generic.

These coefficients then introduce a new scale μ^{-1} in the theory, much larger than the microscopic scale, in such a way that it is possible to define, in addition to the universal large distance physics, a short distance or large momentum physics with $\mu \ll |p_i| \ll \Lambda$.

By contrast, if one assumes that parameters in particle physics theories are *generic*, that is, that coupling constants at the cut-off scale (which can be identified with the scale of some new physics) are numbers of order 1, we are driven back into the statistical framework. This is also the viewpoint of effective field theories (Section 8.8).

Finally, we have performed the RG analysis of the ϕ^4 QFT in four dimensions, in the context of particle physics, in Section 9.12. In the domain of attraction of the IR fixed point $g = 0$, the renormalized coupling constant of the ϕ^4 field theory goes to 0 logarithmically when the cut-off becomes infinite, leading to the so-called *triviality problem*.

15.7 The $O(N)$-symmetric $(\phi^2)^2$ field theory

We now briefly indicate how the considerations of the chapter generalize to models with an $O(N)$ symmetry (see Section (14.75)). We consider a field $\boldsymbol{\phi}$ with N components ϕ_i, and the $O(N)$ symmetric Hamiltonian

$$\mathcal{H}(\boldsymbol{\phi}) = \int \left[\tfrac{1}{2} \big(\nabla \boldsymbol{\phi}(x) \big)^2 + \tfrac{1}{2} r \boldsymbol{\phi}^2(x) + \tfrac{1}{4!} g \Lambda^{4-d} \big(\boldsymbol{\phi}^2(x) \big)^2 \right] \mathrm{d}^d x \,. \tag{15.77}$$

The vertex functions have now group indices, and satisfy WT identities of the kind described in Section 13.2. As a consequence, for example,

$$\begin{aligned} \tilde{\Gamma}^{(2)}_{ij}(p) &= \delta_{ij} \tilde{\Gamma}^{(2)}(p) \\ \tilde{\Gamma}^{(4)}_{ijkl}(p_1, p_2, p_3, p_4) &= \delta_{ij}\delta_{kl} \tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4) + \delta_{ik}\delta_{jl} \tilde{\Gamma}^{(4)}(p_1, p_3, p_2, p_4) \\ &\quad + \delta_{il}\delta_{kj} \tilde{\Gamma}^{(4)}(p_1, p_4, p_3, p_2), \end{aligned}$$

where $\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4)$ is symmetric in the exchanges $p_1 \leftrightarrow p_2$ and $p_3 \leftrightarrow p_4$.

With these definitions, the two first renormalization conditions (15.43) are unchanged, and the third one becomes

$$\tilde{\Gamma}^{(4)}_{\mathrm{r}}(p_i = \mu \theta_i; g_{\mathrm{r}}, \mu, \Lambda) = \tfrac{1}{3} \mu^{\varepsilon} g_{\mathrm{r}} \,. \tag{15.78}$$

The derivation of the RG equations is identical, and one finds,

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2} \eta(g) \right] \tilde{\Gamma}^{(n)}_{i_1, \ldots, i_n}(p_1, \ldots, p_n; g, \Lambda) = 0 \,. \tag{15.79}$$

The RG equations with $\frac{1}{2}\boldsymbol{\phi}^2$ insertions are identical to equations (15.67). The RG functions at leading order are (equations (10.89–10.91)),

$$\beta(g) = -\varepsilon g + \frac{N+8}{48\pi^2} g^2 + O\left(g^3\right), \tag{15.80}$$

$$\eta(g) = \frac{(N+2)}{72(8\pi^2)^2} g^2 + O\left(g^3\right), \tag{15.81}$$

$$\eta_2(g) = -\frac{1}{48\pi^2}(N+2) g + O\left(g^2\right). \tag{15.82}$$

One finds again an IR fixed point $g^* \sim 48\pi^2 \varepsilon/(N+8)$, with $\omega = \beta'(g^*) \sim \varepsilon > 0$. All the scaling relations follow, but with different exponents,

$$\eta = \frac{N+2}{2(N+8)^2} \varepsilon^2 + O(\varepsilon^3), \quad \eta_2 = -\frac{N+2}{N+8} \varepsilon + O(\varepsilon^2) \;\Rightarrow \nu = \frac{1}{2} + \frac{(N+2}{4(N+8)} \varepsilon + O(\varepsilon^2).$$

15.8 Statistical properties of long self-repelling chains

Remarkably enough, the statistical properties of long polymer chains

can be derived from the critical behaviour of the $O(N)$ symmetric $(\boldsymbol{\phi}^2)^2$ field theory in the 'non-physical' $N = 0$ limit [136]. (For a comprehensive experimental and theoretical discussion of polymer physics, see Ref. [137].) To describe the asymptotic statistical properties of long polymers, we consider a model of chains in the continuum with repulsive contact self-interaction. We will follow the derivation of Ref. [138] (see also Ref. [139]).

We denote by u the position along the chain, $\mathbf{r}(u)$ the position in space of the point on the chain of parameter u. We characterize the chain by the probability distribution for a chain $\mathbf{r}(u)$ of total length s,

$$[\mathrm{d}\rho(\mathbf{r}(u))] = [\mathrm{d}\mathbf{r}(u)] \exp\left\{ -\left[\frac{1}{4} \int_0^s \dot{\mathbf{r}}^2(u)\mathrm{d}u + \frac{g}{6} \int_0^s \mathrm{d}u_1 \mathrm{d}u_2\, \delta^{(d)}\big(\mathbf{r}(u_1) - \mathbf{r}(u_2)\big) \right] \right\}. \quad (15.83)$$

The special limit $g = 0$ corresponds to a Brownian chain, or a Gaussian random walk (see also Section 34.5). The self-avoiding random walk on a lattice is a discretized form of the model, and provides a regularization: at a microscopic scale Λ^{-1}, the chain becomes much stiffer than what is implied by expression (15.83). Finally, note that any short-range potential would yield, at leading order for long chains, the same results as the δ-function interaction, as one can verify by repeating, in the more general case, the derivation of an equivalent ϕ^4 like field theory, and using the analysis of corrections to scaling as in the case of critical phenomena.

15.8.1 A generating function

Various characteristic properties of the chain can be derived from the two-point function

$$G^{(2)}(\mathbf{k}, s) = \left\langle \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}(s)-\mathbf{r}(0))} \right\rangle, \quad (15.84)$$

where brackets mean average with respect to the distribution (15.83). Indeed, the expansion of $G^{(2)}(\mathbf{k},s)$ in powers of $\mathbf{k}$ yields the successive moments of the distribution of the origin-to-end positions:

$$G^{(2)}(\mathbf{k}, s) = 1 - \frac{1}{2!}\frac{\mathbf{k}^2}{d} \left\langle (\mathbf{r}(s) - \mathbf{r}(0))^2 \right\rangle + \frac{(\mathbf{k}^2)^2}{4!} \frac{1}{d(d+2)} \left\langle (\mathbf{r}(s) - \mathbf{r}(0))^4 \right\rangle + \cdots, \quad (15.85)$$

where the rotation symmetry of the distribution (15.83) has been used.

Actually, it is easier to calculate the Laplace transform of $G^{(2)}(\mathbf{k}, s)$ with respect to s,

$$\tilde{Z}^{(2)}(\mathbf{k}, t) = \int_0^\infty \mathrm{e}^{-st}\, G^{(2)}(\mathbf{k}, s)\mathrm{d}s\,. \quad (15.86)$$

The function $G^{(2)}(\mathbf{k}, s)$ is then recovered by inverting the Laplace transformation,

$$G^{(2)}(\mathbf{k}, s) = \frac{1}{2i\pi} \oint_C \mathrm{e}^{st}\, \tilde{Z}^{(2)}(\mathbf{k}, t)\mathrm{d}t\,, \quad (15.87)$$

in which the contour C is parallel to the imaginary t axis at the right of all singularities of $\tilde{Z}^{(2)}$. Since only the expansion of $G^{(2)}$ in powers of $\mathbf{k}$ (equation (15.85)) is useful, $\mathbf{k}$ has to be considered as infinitesimal. Thus, only the location of singularities in the limit $\mathbf{k} \to 0$ plays a role.

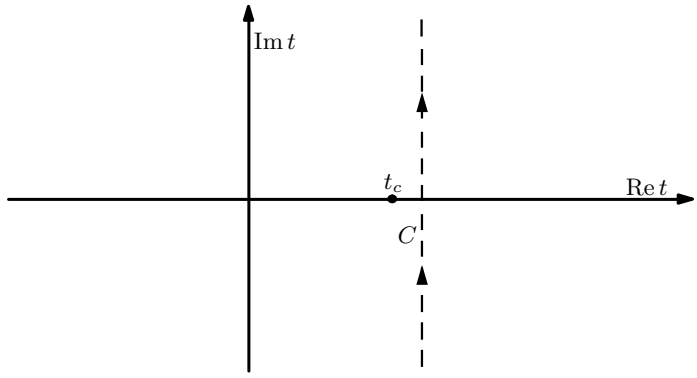

Fig. 15.1 Contour C in the t plane

We denote by t_c the first singularity met when the contour is displaced to the left (Fig. 15.1). On C, the real part $\mathrm{Re}(t)$ of t thus satisfies

$$\mathrm{Re}\,(t) > \mathrm{Re}(t_c).$$

One verifies that the behaviour of $\tilde{Z}^{(2)}$ near the singularity governs the large positive s behaviour of the integral (15.87), and thus of the moments of $\mathbf{r}(s) - \mathbf{r}(0)$.

15.8.2 Some exact results. Flory's approximation

The Gaussian random walk. In expression (15.84), we can substitute

$$\mathbf{r}(s) - \mathbf{r}(0) = \int_0^s \dot{\mathbf{r}}(u)\mathrm{d}u\,.$$

For vanishing self-repulsion, one obtains

$$G^{(2)}(\mathbf{k}, s) = \int [\mathrm{d}\mathbf{r}(u)] \exp\left\{ - \int_0^s \mathrm{d}u \left[\tfrac{1}{4}\dot{\mathbf{r}}^2(u) - i\mathbf{k}\cdot\dot{\mathbf{r}}(u) \right] \right\}.$$

Shifting $\dot{\mathbf{r}}$, one finds

$$G^{(2)}(\mathbf{k}, s) = \mathrm{e}^{-s\mathbf{k}^2}, \tag{15.88}$$

which implies the well-known scaling laws of the Gaussian random walk:

$$\left\langle (\mathbf{r}(s) - \mathbf{r}(0))^{2n} \right\rangle \sim s^n. \tag{15.89}$$

The Laplace transform (15.86) takes the free field form,

$$\tilde{Z}^{(2)}(\mathbf{k}, t) = (t + \mathbf{k}^2)^{-1}. \tag{15.90}$$

The one-dimensional chain. If the chain self-repulsion plays a role, it must increase the average spatial extension of the chain for s large, compared to the Gaussian case. This is obvious in the one-dimensional example. The chain is then completely stretched:

$$r(s) = \pm s\,, \tag{15.91}$$

and, therefore,

$$G^{(2)}(k, s) = \cos ks\,, \quad \tilde{Z}^{(2)}(k, t) = \frac{t}{t^2 + k^2}\,. \tag{15.92}$$

The one-dimensional example shows that the self-repulsion can change the statistical properties of the chain. It also provides an upper bound on the moments. RG arguments will show that

$$\left\langle (\mathbf{r}(s)-\mathbf{r}(0))^{2n} \right\rangle \sim s^{2\nu n}, \quad \text{for } s \to \infty\,,$$

in which the exponent ν is bounded from below by $1/2$, the value for the Gaussian chain, and from above by 1, the value for the stretched chain.

The upper-critical dimension. Note that the self-avoiding condition becomes weaker when the dimension d of embedding space increases. Actually, a simple argument suggests, as in the case of the critical behaviour, the existence of an upper-critical dimension. For a very long chain, one can consider the effect of self-avoiding as the influence of one chain onto another. A Brownian chain has Hausdorff dimension 2. Therefore, above dimension 4 (2 +2) two chains no longer see each other, self-avoiding should no longer play a role, and the chain should have the statistical properties of a Brownian chain.

Beyond the Gaussian model: Flory's approximation. Before solving the problem by more systematic methods, we first describe a simple approximate solution based on an energy balance argument. For s large, the Gaussian term should scale as the interaction:

$$\left\langle \int_0^s \dot{\mathbf{r}}^2(u)\mathrm{d}u \right\rangle \sim s^{2\nu-1}, \tag{15.93}$$

$$\int \mathrm{d}u_1 \mathrm{d}u_2\, \delta^{(d)}(\mathbf{r}(u_1) - \mathbf{r}(u_2)) \sim s^{2-d\nu}. \tag{15.94}$$

Equating the two exponents, one obtains the equation,

$$2\nu - 1 = 2 - d\nu \;\Rightarrow\; \nu = \frac{3}{d+2}, \tag{15.95}$$

valid for $1 \leq d \leq 4$. For $d = 1$, Flory's approximation yields the exact value $\nu = 1$. It predicts that, for $d \geq 4$, the interaction can never balance the Gaussian term, since ν is bounded by $1/2$. More surprisingly, for $d = 2$ it yields $\nu = 3/4$ which is also known, from conformal field theory arguments, to be the exact value.

Flory's approximation relies on intuitive arguments and it is difficult to see how it can be improved. Actually, the values it yields for the exponent ν are so good that one may wonder whether the result is not exact. We discuss generic dimensions in the following sections, and show that it is only an approximation. For this purpose, it is convenient to completely reformulate the model based on the distribution (15.83).

15.8.3 Equivalence with the $(\phi^2)^2$ field theory in the $N = 0$ limit

We first transform the interaction term in equation (15.83) by expressing it as a Gaussian integral over an auxiliary field $\sigma(\mathbf{r})$. For a given chain $\mathbf{r}(u)$, the following identity holds:

$$\int [\mathrm{d}\sigma(\mathbf{r})] \exp\left[\frac{3}{2g}\int \mathrm{d}^d r\, \sigma^2(r) - \int \mathrm{d}u\, \sigma\big(\mathbf{r}(u)\big)\right] = \exp\left[-\frac{g}{6}\int \mathrm{d}u_1 \mathrm{d}u_2\, \delta^{(d)}(\mathbf{r}(u_1) - \mathbf{r}(u_2))\right], \tag{15.96}$$

where the σ-field is imaginary, and we have used

$$\int \mathrm{d}u\, \sigma(\mathbf{r}(u)) = \int \mathrm{d}u \int \mathrm{d}^d r\, \sigma(r)\delta^{(d)}(\mathbf{r}(u) - \mathbf{r}) = \int \mathrm{d}^d r\, \sigma(r) \int \mathrm{d}u\, \delta^{(d)}(\mathbf{r}(u) - \mathbf{r}).$$

Substituting the field integral into expression (15.83), we identify the $\mathbf{r}(u)$ integration measure with the integrand in the path integral representation of the statistical operator in Euclidean time u of a d-dimensional quantum system with potential $\sigma(\mathbf{r})$ (see Chapter 2). The two-point function $\tilde{Z}^{(2)}$ can thus be written as, in bra–ket quantum notation,

$$\tilde{Z}^{(2)}(\mathbf{k},t) = \int [\mathrm{d}\sigma(\mathbf{r})] \exp\left[\frac{3}{2g}\int \mathrm{d}^d r\, \sigma^2(r)\right] \int_0^\infty \mathrm{e}^{-ts}\,\mathrm{d}s \times \int \mathrm{d}^d r\, \mathrm{d}^d r'\, \mathrm{e}^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} \left\langle \mathbf{r}' \left| \mathrm{e}^{-sH} \right| \mathbf{r} \right\rangle, \tag{15.97}$$

in which H is the quantum Hamiltonian,

$$H = -\nabla_{\mathbf{r}}^2 + \sigma(\mathbf{r}). \tag{15.98}$$

After integration over s, the expression becomes

$$\tilde{Z}^{(2)}(\mathbf{k},t) = \int [\mathrm{d}\sigma(\mathbf{r})] \exp\left[\frac{3}{2g}\int \mathrm{d}^d r\, \sigma^2(r)\right] \int \mathrm{d}^d r\, \mathrm{d}^d r'\, \mathrm{e}^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} \times \left\langle \mathbf{r}' \left| (-\nabla^2 + t + \sigma)^{-1} \right| \mathbf{r} \right\rangle. \tag{15.99}$$

We now apply a variant of the so-called *replica trick* (see also Section A14.5). We introduce an N-component scalar field $\boldsymbol{\phi}$. Then,

$$\begin{aligned} &\lim_{N\to 0} \int [\mathrm{d}\boldsymbol{\phi}(\mathbf{r})]\, \phi_1(\mathbf{r})\phi_1(\mathbf{r}') \exp\left\{-\frac{1}{2}\int \mathrm{d}^d r \left[\big(\nabla\boldsymbol{\phi}(r)\big)^2 + \big(t+\sigma(r)\big)\boldsymbol{\phi}^2(r)\right]\right\} \\ &\quad \propto \lim_{N\to 0} [\det(-\nabla^2 + t + \sigma)]^{-N/2} \left\langle \mathbf{r}' \left| (-\nabla^2 + t + \sigma)^{-1} \right| \mathbf{r} \right\rangle \\ &\quad = \left\langle \mathbf{r}' \left| (-\nabla^2 + t + \sigma)^{-1} \right| \mathbf{r} \right\rangle. \end{aligned} \tag{15.100}$$

Substituting the identity into expression (15.99), one obtains

$$\tilde{Z}^{(2)}(\mathbf{k},t) = \int [\mathrm{d}\boldsymbol{\phi}]\, \tilde{\phi}_1(\mathbf{k})\tilde{\phi}_1(-\mathbf{k}) \int [\mathrm{d}\sigma(r)] \times \exp\left[\frac{1}{2}\int \mathrm{d}^d r \left(3\sigma^2(\mathbf{r})/g - \sigma(\mathbf{r})\boldsymbol{\phi}^2(\mathbf{r}) - \big(\nabla\boldsymbol{\phi}(\mathbf{r})\big)^2 - t\boldsymbol{\phi}^2(\mathbf{r})\right)\right]. \tag{15.101}$$

Then, the Gaussian integration over σ can be performed, and yields

$$\begin{aligned} \tilde{Z}^{(2)}(\mathbf{k},t) &= \int [\mathrm{d}\boldsymbol{\phi}]\, \tilde{\phi}_1(\mathbf{k})\tilde{\phi}_1(-\mathbf{k}) \exp\left[-\mathcal{S}(\boldsymbol{\phi})\right], \\ \mathcal{S}(\boldsymbol{\phi}) &= \int \mathrm{d}^d r \left[\frac{1}{2}(\nabla\boldsymbol{\phi})^2 + \frac{1}{2}t\boldsymbol{\phi}^2 + \frac{g}{4!}(\boldsymbol{\phi}^2)^2\right]. \end{aligned} \tag{15.102}$$

This is a most remarkable result: the statistical properties of polymers are related to the properties of the $O(N)$ symmetric $(\boldsymbol{\phi}^2)^2$ field theory in the non-physical $N = 0$ limit, as first pointed out by de Gennes.

15.8.4 RG approach to self-avoiding walk (SAW) and statistical properties of polymers

The large scale statistical properties of the chain are thus related to the singularities of correlation functions of the $(\phi^2)^2$ field theory in the massless or critical limit. The short distance stiffness of the chain provides an ultraviolet (UV) cut-off Λ for the theory. Near t_c, in which t_c is the critical temperature or critical bare mass, the two-point function $\tilde{Z}^{(2)}$ has the scaling behaviour:

$$\tilde{Z}^{(2)}(\mathbf{k}, t) \sim (t - t_c)^{-\gamma} f\left[\mathbf{k}(t - t_c)^{-\nu}\right], \tag{15.103}$$

in which ν and γ are the $N = 0$ limits of the critical exponents of the $(\phi^2)^2$ field theory.

Inserting the form (15.103) into the integral (15.87), one obtains the scaling behaviour of $G^{(2)}(s, k)$ for s large:

$$G^{(2)}(s, \mathbf{k}) \sim \mathrm{e}^{st_c}\, s^{\gamma-1} g(|\mathbf{k}|\, s^{\nu}). \tag{15.104}$$

Expanding in powers of $\mathbf{k}^2$, one finds a first term

$$\langle 1 \rangle \sim \mathrm{e}^{st_c}\, s^{\gamma-1} g(0),$$

instead of 1, which shows that the distribution is incorrectly normalized. The first term is actually the ratio of the number of configurations of the SAW to the number of configurations of the Gaussian random walk. Therefore, the properly normalized average $\langle \exp\left[i\mathbf{k}(r(s) - r(0))\right]\rangle$ has a scaling behaviour entirely characterized by the exponent ν. For example,

$$\left\langle (\mathbf{r}(s) - \mathbf{r}(0))^2 \right\rangle \sim s^{2\nu}. \tag{15.105}$$

The exponent can be obtained from the ε-expansion:

$$\nu = \tfrac{1}{2}\left(1 + \tfrac{1}{8}\varepsilon + \tfrac{15}{256}\varepsilon^2\right) + O(\varepsilon^3). \tag{15.106}$$

By contrast, Flory's formula (15.95) predicts at the same order:

$$\nu(\text{Flory}) = \tfrac{1}{2}\left(1 + \tfrac{1}{6}\varepsilon + \tfrac{1}{36}\varepsilon^2\right) + O(\varepsilon^3). \tag{15.107}$$

Comparing the two expressions, we note immediately that Flory's formula is not exact in general, though, as mentioned before, it correctly predicts the upper-critical dimension $d = 4$, and the values of ν for $d = 1$ and 2.

For $d = 3$, RG calculations yield (see Section 41.5)

$$\nu = 0.5874 \pm 0.0003\,,$$

result that can be compared with $\nu_{\text{Flory}} = 0.6000$. The two values are close, but significantly different. Actually, one can estimate the difference approximately. If one simply takes into account the orders ε and ε^2, and the property that Flory's approximation is exact for $\varepsilon = 0, 2, 3$, one can write

$$\nu_{\text{RG}}^{-1} - \nu_{\text{Flory}}^{-1} = \tfrac{1}{12}\varepsilon(1 - \tfrac{1}{2}\varepsilon)(1 - \tfrac{1}{3}\varepsilon)(1 + \tfrac{19}{96}\varepsilon)^{-1} + O(\varepsilon^3)).$$

For $\varepsilon = 1$, one finds successively $\nu = 0.590$ at order ε, and $\nu = 0.592$ at order ε^2, results consistent with the RG value obtained by summing all terms (Tables 41.3 and 41.4). Experimental results, $\nu = 0.586 \pm 0.004$, and Monte Carlo simulations, $\nu = 0.58760$ (Table 41.5), agree with the RG prediction, and also exclude the value predicted by Flory's approximation.

Moreover, with the use of RG arguments, one can derive a number of universal properties and calculate many other universal quantities, which cannot be inferred from Flory's argument.

15.9 Liquid–vapour phase transition and ϕ^4 field theory

A priori, it is not obvious that the liquid–vapour phase transition in classical fluids belongs to the same universality class as the Ising model, with its $\mathbb{Z}_2$ symmetry, and can be described by a ϕ^4 field theory. In particular, no apparent discrete symmetry is broken. However, there exist several methods to relate the liquid–vapour transition and the $N = 1$ ϕ^4 theory: one can either start from a model which is physically not very realistic, the lattice gas model, but rigorously equivalent to the Ising model, or from a true gas model in the continuum, and by a number of approximations for which it is difficult to provide a rigorous justification, directly obtaining a ϕ^4 field theory.

The lattice gas model [140]. One considers particles living on a lattice, such that at each site i of a d-dimensional lattice (i is the position on the lattice) the particle occupation number n_i can only be 0 or 1. The lattice is a schematic way to represent a hard-core type interaction. A longer range attractive potential is provided by a nearest neighbour (n.n.) interaction which favours the occupation of neighbour sites. Finally, one adds a chemical potential μ in the form of a linear coupling to the number of particles. The configuration energy is given by

$$-\beta\mathcal{H}(n_i) = J\sum_{\text{n.n.}} n_i n_j - \mu\sum_i n_i\,. \tag{15.108}$$

This model is directly related to the Ising model by the transformation

$$n_i \mapsto S_i = 2n_i - 1\,, \tag{15.109}$$

where S_i is an Ising spin. In d dimensions, one finds

$$-\beta\mathcal{H}(S_i) = \tfrac{1}{4}J\sum_{\text{n.n.}} S_i S_j + \tfrac{1}{2}(dJ-\mu)\sum_i S_i + \text{ const.}\,. \tag{15.110}$$

One recognizes the energy of the Ising model with a nearest-neighbour interaction and a magnetic field. Since we have shown that the Ising model leads to the ϕ^4 field theory, the same applies to the lattice gas model, and the relation between the two models makes it possible to translate the RG results into the language of classical fluids.

15.9.1 The classical gas in the continuum: Field integrals

Since the introduction of a lattice is somewhat artificial, we now consider a classical gas in the continuum. For a class of models, using a sequel of exact identities one can derive a field integral representation of the partition function (in the grand canonical formalism) [141]. Then, a few additional steps, which are quite instructive but harder to justify rigorously, lead to the ϕ^4 field theory.

We consider a real classical gas in the continuum, and assume that the potential between particles is the sum of a hard-core (h.c) potential, with a core of size a, and a short-range attractive two-body potential V such that,

$$V(\mathbf{r}) = 0 \text{ for } |\mathbf{r}| < a\,, \text{ and } V(\mathbf{r}) = V(r=|\mathbf{r}|) \le 0\,, \quad \forall\ \mathbf{r}\,,$$

but with a range much larger than a. The canonical partition function for n particles of mass m in a d-dimensional volume Ω at temperature $1/\beta$ then has the form

$$\mathcal{Z}(n,\beta,\Omega) = \left(\frac{2\pi m}{\beta h^2}\right)^{nd/2} \int_{\substack{\mathbf{r}_i\in\Omega \\ |\mathbf{r}_i-\mathbf{r}_j|>2a}} \prod_{i=1}^{n} \mathrm{d}^d r_i \exp\Big[-\beta\sum_{i<j} V(\mathbf{r}_i-\mathbf{r}_j)\Big]. \tag{15.111}$$

Since the potential depends only on the distance r and is short range, its Fourier transform

$$\tilde{V}(\mathbf{k}) = \int \mathrm{d}^d r \; \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} \, V(\mathbf{r}),$$

is an analytic function of k^2 in a strip, and can be expanded for k^2 small as

$$\tilde{V}(\mathbf{k}) = -v(1 - \sigma^2 \mathbf{k}^2) + O(k^4), \quad v > 0\,, \tag{15.112}$$

where $\sigma^2 = \langle \mathbf{r}^2 \rangle \gg a^2$ is the second moment of the distribution $-V$. We assume that the longer range of V reflects into the condition $\sigma \gg a$.

We introduce the local gas density

$$\rho(\mathbf{r}) = \sum_{i=1}^{n} \delta^{(d)}(\mathbf{r} - \mathbf{r}_i), \tag{15.113}$$

where $\mathbf{r}_i$ is the position of a particle in the gas, and $\int \mathrm{d}^d r \, \rho(\mathbf{r}) = n$ is the number of particles. The gas potential energy can then be written as

$$\sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j) = \frac{1}{2} \int \mathrm{d}^d r \, \mathrm{d}^d r' V(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}) \rho(\mathbf{r}').$$

We now impose the relation (15.113) by a field integral in the spirit of the method of Section 14.4.2, and insert the identity

$$\int [\mathrm{d}\rho(\mathbf{r})] \int [\mathrm{d}\phi(\mathbf{r})] \exp\left[\int \mathrm{d}^d r \, \phi(\mathbf{r}) \rho(\mathbf{r}) - \sum_{i=1}^{n} \phi(\mathbf{r}_i) \right] = \text{const.}$$

into the expression (15.111) (note that the integral over ϕ represents a functional δ function, and thus the ϕ-integration contour is parallel to the imaginary axis). After these transformations, the partition function becomes

$$\mathcal{Z}(n, \beta, \Omega) = \mathrm{e}^{n\beta V(0)/2} \int [\mathrm{d}\phi(r) \mathrm{d}\rho(r)] \exp\left[-\frac{1}{2}\beta \int \mathrm{d}^d r \, \mathrm{d}^d r' \rho(\mathbf{r}) V(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') \right.$$
$$\left. + \int \mathrm{d}^d r \, \rho(\mathbf{r}) \phi(\mathbf{r}) \right] \mathcal{Z}_{\text{h.c.}}(n, \beta, \Omega, \phi), \tag{15.114}$$

where $\mathcal{Z}_{\text{h.c.}}(n, \beta, \Omega, \phi)$ is the canonical partition function of particles interacting through a h.c. two-body potential in a random external (imaginary) potential $\phi(\mathbf{r})$:

$$\mathcal{Z}_{\text{h.c.}}(n, \beta, \Omega, \phi) = \left(\frac{2\pi m}{\beta h^2} \right)^{nd/2} \int_{\substack{\mathbf{r}_i \in \Omega \\ |\mathbf{r}_i - \mathbf{r}_j| > 2a}} \prod_{i=1}^{n} \mathrm{d}^d r_i \exp\left(-\sum_i \phi(\mathbf{r}_i) \right). \tag{15.115}$$

We can transform equation (15.114) into an identity between grand canonical partition functions setting

$$\mathcal{Z}(\mu, \beta, \Omega) = \sum_n \frac{z^n}{n!} \mathcal{Z}(n, \beta, \Omega), \tag{15.116}$$

where z is the gas fugacity. We also set

$$z = \mathrm{e}^{\beta\mu}\,, \quad \rho_0 = (2\pi)^{d/2}/\lambda^d\,, \tag{15.117}$$

where μ is a chemical potential and λ the thermal wavelength,

$$\lambda = \hbar\sqrt{\beta/m}\,. \tag{15.118}$$

The normalization density ρ_0, which can be changed by shifting the chemical potential, thus increases with the temperature.

Expressing n as the space integral of the local density ρ, and using the integral representation (15.114), we obtain the field integral

$$\mathcal{Z}(\mu,\beta,\Omega) = \int [\mathrm{d}\phi(r)\mathrm{d}\rho(r)] \exp\left[-\mathcal{S}(\rho,\phi)\right], \tag{15.119}$$

with the action

$$\begin{aligned}\mathcal{S}(\rho,\phi) = \frac{1}{2}\beta \int \mathrm{d}^d r\, \mathrm{d}^d r' \rho(\mathbf{r}) V(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') \\ - \int \mathrm{d}^d r\, \rho(\mathbf{r})\big(\phi(\mathbf{r}) + \beta\mu\big) - \ln \mathcal{Z}_{\mathrm{h.c.}}(\Omega,\phi),\end{aligned} \tag{15.120}$$

and

$$\mathcal{Z}_{\mathrm{h.c.}}(\Omega,\phi) = \sum_n \frac{\rho_0^n}{n!} \int_{\substack{\mathbf{r}_i \in \Omega \\ |\mathbf{r}_i - \mathbf{r}_j| > 2a}} \prod_{i=1}^n \mathrm{d}^d r_i \exp\left[-\sum_i \phi(\mathbf{r}_i)\right]. \tag{15.121}$$

Note that, if we replace the chemical potential μ by an external potential $\mu(\mathbf{r})$, we transform the partition function (15.121) into the generating functional of density correlation functions.

The integral over the density field ρ is Gaussian and can thus be calculated explicitly. The integration is particularly useful in the case of long range potentials, like the Coulomb potential, which have a Fourier transform $\tilde{V}$ singular at $\mathbf{k} = 0$, a situation we face in Chapter 31. Here, by contrast, we consider only short range forces and the integration then slightly obscures the physical meaning of the dynamic variables, ϕ having the nature of an imaginary external potential. Instead, we introduce the integral over ϕ, even though we cannot calculate it exactly.

Finally, we note that the expression (15.119) can easily be generalized to many-body forces, leading to an action which is a polynomial of higher degree in the local density ρ.

15.9.2 Phase transition

The partition function of the hard-core potential in presence of a given external one-body potential cannot be calculated exactly. If the hard core is completely neglected, the partition function is given by the field integral,

$$\begin{aligned}\mathcal{Z}(\mu,\beta,\Omega) &= \int [\mathrm{d}\phi(r)\mathrm{d}\rho(r)] \exp\left[-\mathcal{S}(\rho,\phi)\right] \\ \mathcal{S}(\rho,\phi) &= \frac{1}{2}\beta \int \mathrm{d}^d r\, \mathrm{d}^d r' \rho(\mathbf{r}) V(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') \\ &\quad - \int \mathrm{d}^d r (\rho(\mathbf{r})\big(\phi(\mathbf{r}) + \beta\mu\big) + \rho_0\, \mathrm{e}^{-\phi(\mathbf{r})}).\end{aligned} \tag{15.122}$$

In a calculation by the steepest descent method, we can, at leading order, replace the field ϕ by the solution of the corresponding saddle point equation

$$\rho(\mathbf{r}) = \rho_0\, \mathrm{e}^{-\phi(\mathbf{r})} \;\Rightarrow\; \phi(\mathbf{r}) = -\ln\big(\rho(\mathbf{r})/\rho_0\big).$$

The action becomes

$$\mathcal{S}(\rho) = \frac{1}{2}\beta \int \mathrm{d}^d r\, \mathrm{d}^d r' \rho(\mathbf{r}) V(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') + \int \mathrm{d}^d r\, \rho(\mathbf{r}) \left[\ln\big(\rho(\mathbf{r})/\rho_0\big) - 1 - \beta\mu\right]. \quad (15.123)$$

We now look for a homogeneous saddle point ρ. With the notation (15.112), we obtain

$$-\beta v\rho + \ln(\rho/\rho_0) = \beta\mu\,. \quad (15.124)$$

Using the equation to eliminate the chemical potential, we obtain the action density

$$\mathcal{S}/\Omega = -\rho + \tfrac{1}{2}\beta v\rho^2 = -\beta p\,,$$

where Ω is the volume and p the pressure. This yields a reasonable equation of state for small densities but, for large densities, the pressure is negative and unbounded. This reflects the collapse of the system in the case of a purely attractive potential.

In the presence of the hard core, the action has to be understood with an implicit short distance cut-off a. Moreover, the hard core for large densities, $\rho = O(a^{-d})$, becomes important, its contributions to $\mathcal{S}(\rho)$ can no longer be neglected and ensure the stability of the system. More precisely, the effect of the hard core can be studied by calculating the ϕ potential. Since we are interested in phenomena associated with distances much larger than the size a of the hard core, we can make a local expansion of $\ln \mathcal{Z}_{\mathrm{h.c.}}(\phi)$ governed by the length scale a, in the successive terms of the sum (15.121). Using the leading order equation (15.124), we see that the effect is an addition of local ρ contributions to the action, which stabilize the ρ potential. For dimensional reasons, since ρ is an inverse volume, the combinations that appear are $a^d\rho$ and $a\nabla$ in the case of derivatives, justifying the idea that the hard core is important for large or fast-varying densities. Neglecting derivatives, one obtains an expression of the form

$$\mathcal{S}(\rho) = \frac{1}{2}\beta \int \mathrm{d}^d r\, \mathrm{d}^d r' \rho(\mathbf{r}) V(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') - \int \mathrm{d}^d r \left[\rho(\mathbf{r})\big(\phi(\mathbf{r}) + \beta\mu\big) + w\big(\phi(\mathbf{r})\big)\right], \quad (15.125)$$

where $w(\phi)$ is the hard-core free energy in the thermodynamic limit, as a function of a chemical potential $-\phi/\beta$:

$$\exp\left[\Omega w(\phi)\right] = 1 + \sum_{n=1} \frac{\rho_0^n\, \mathrm{e}^{-n\phi}}{n!} \int_{\substack{\mathbf{r}_i \in \Omega \\ |\mathbf{r}_i - \mathbf{r}_j| > 2a}} \prod_{i=1}^{n} \mathrm{d}^d r_i\,.$$

Again, we integrate over ϕ by the steepest descent method and replace ϕ by the solution of the saddle point equation:

$$\rho(\mathbf{r}) + w'\big(\phi(\mathbf{r})\big) = 0\,. \quad (15.126)$$

We introduce $U(\rho)$, the Legendre transform of $w(\phi)$ defined by equation (15.126) and

$$U(\rho) = -w(\phi) - \rho\phi\,, \quad (15.127)$$

in such a way that the approximated action now reads

$$\mathcal{S}(\rho) = \frac{\beta}{2} \int \mathrm{d}^d r\, \mathrm{d}^d r' \rho(\mathbf{r}) V(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') + \int \mathrm{d}^d r \left[U\big(\rho(\mathbf{r})\big) - \beta\mu\rho(\mathbf{r})\right]. \quad (15.128)$$

We then need some information about the hard-core free energy in the thermodynamic limit. The sum cannot be calculated for generic dimensions, but is easy in one dimension, because one can order positions on the line. Summation relies on introducing the Fourier transform of the θ functions. Setting

$$\mathrm{e}^{\Omega F(z)} \underset{\Omega\to\infty}{=} 1 + \sum_{n=1} \frac{z^n}{n!} \int_{\substack{r_i\in\Omega \\ |r_i-r_j|>2a}} \prod_{i=1}^{n} \mathrm{d}r_i \,,$$

one finds that F is solution of the transcendental equation,

$$F(z) = z\,\mathrm{e}^{-aF(z)} \,.$$

Of particular interest is the behaviour of the function for $z \to +\infty$. One finds

$$aF = \ln z - \ln\ln z + \cdots \,.$$

The leading term is generic and reflects the situation of close packing and thus maximal density. Indeed, for Ω large but finite, the partition sum is really a polynomial of degree of the order Ω/a^d. For z large, the polynomial is dominated by the terms of highest degree. Therefore,

$$F(z) \underset{z\to\infty}{\propto} a^{-d} \ln z \,.$$

This implies that the function $U(\rho)$ diverges for the close packing density, as expected (we disregard here possible phenomena which could occur for densities very close to close packing of the nature of a liquid–solid transition). It is straightforward to show that for $d = 1$ $U(\rho)$ is convex, a property which generalizes to generic dimensions. Indeed, $U''(\rho) = 1/w''(\phi)$ and $w''(\phi)$ is positive, because it has the form $\langle n^2\rangle - (\langle n\rangle)^2$.

With these properties, we can now discuss the action (15.128) in the tree approximation, following the lines of Section 14.4.3, where the mean-field approximation was considered. We look for a homogeneous saddle point ρ solution of

$$\mathcal{S}'(\rho)/\Omega = -\beta v\rho + U'(\rho) - \beta\mu = 0 \,. \tag{15.129}$$

Depending on the number of solutions of the equation, one finds one phase, or two phases separated by a transition. This is determined by the second derivative

$$\mathcal{S}''(\rho)/\Omega = U''(\rho) - \beta v \,.$$

The function $U''(\rho)$ is positive, diverges as $1/\rho$ for $\rho \to 0$, and is expected to diverge at the close packing density. Therefore, it has an absolute minimum at an intermediate density. For βv smaller than the minimum of U'', the system has only one phase, the high temperature phase. For β larger than the minimum the equation has one or three solutions depending on the chemical potential. However, in general, the corresponding transition is first order because the third derivative of the action does not vanish. A second order transition is characterized by the vanishing of both the first and third derivative of the thermodynamic potential. The vanishing of the third derivative yields the condition

$$U'''(\rho) = 0 \,.$$

This equation determines a critical value ρ_c of order a^{-d} of the density and therefore, from equation (15.129), a relation between chemical potential and temperature. The vanishing of the second derivative then determines the critical temperature $T_c = 1/\beta_c = v/U''(\rho_c)$.

Near β_c, the action for constant fields has the typical form of the ϕ^4 field theory, up to irrelevant higher order corrections,

$$\mathcal{S}(\rho)/\Omega = \tfrac{1}{2}s_2(\beta)(\rho-\rho_c)^2 + \tfrac{1}{4!}s_4(\rho-\rho_c)^4 + O\big((\rho-\rho_c)^5\big),$$

with

$$s_2(\beta) = U''(\rho_c) - v\beta = v(\beta_c - \beta)\,.$$

Finally, for what concerns long distance properties the potential energy can be approximated by its two first terms in the derivative expansion (equation (15.112)):

$$\tfrac{1}{2}\beta\int \mathrm{d}^d r\,\mathrm{d}^d r'\rho(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') = \tfrac{1}{2}\beta v\int \mathrm{d}^d r\left[-\rho^2(\mathbf{r}) + \sigma^2\big(\nabla\rho(\mathbf{r})\big)^2\right].$$

The universal properties of the liquid–vapour transition near the special second-order transition point can thus be described by the one-component ϕ^4 field theory, and the transition belongs to the universality class of the Ising model although, here, the action has no reflection symmetry.

By varying both the temperature and a second thermodynamic variable like the chemical potential, one has been able to cancel both the $(\rho-\rho_c)^2$ and the $(\rho-\rho_c)^3$ term in the expansion of the action for $\rho-\rho_c$ small, and reach a situation of second order phase transition. Then, the odd term in $(\rho-\rho_c)$ of lowest dimension is $(\rho-\rho_c)^5$ which has dimension 5 for $d=4$ and is, therefore, irrelevant at the phase transition. This explains that the ϕ^4 field theory can correctly describe the liquid–vapour phase transition. Let us, nevertheless, mention a few specific features of this transition with respect to magnetic transitions:

(i) The complicated, although regular, relations between the thermodynamic variables to the more natural, from the symmetry point of view, magnetic field, and temperature variables of the magnetic systems. This introduces additional parameters in the description of experimental results.

(ii) The appearance of corrections to scaling due to operators of odd canonical dimensions like ϕ^5.

15.10 Superfluid transition

In this section, we briefly justify that the $(\boldsymbol{\phi}^2)^2$ field theory with $O(2)$ symmetry describes the superfluid transition. Our discussion is based on the formalism introduced in Section 4.4.5.

15.10.1 The Bose partition function

In Section 4.4.5, we have shown that the partition function for a Bose fluid at temperature $1/\beta$, for particles of mass m, with two-body interactions, in the grand canonical formulation, can be expressed as a field integral of the form (equations (4.61, 4.62)),

$$\mathcal{Z}(\beta) = \int [\mathrm{d}\bar\varphi(t,x)\mathrm{d}\varphi(t,x)]\exp[-\mathcal{S}(\varphi,\bar\varphi)],$$

with the periodic boundary conditions $\bar\varphi(-\beta/2,x) = \bar\varphi(\beta/2,x)$, $\varphi(-\beta/2,x) = \varphi(\beta/2,x)$, and the non-relativistic Euclidean action

$$\mathcal{S}(\bar\varphi,\varphi) = \int \mathrm{d}t\,\mathrm{d}^d x\,\bar\varphi(x,t)\left(\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\nabla^2 - \mu\right)\varphi(x,t) + \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^d x\,\mathrm{d}^d y\,\bar\varphi(x,t)\varphi(x,t)V_2(x,y)\bar\varphi(y,t)\varphi(y,t), \qquad (15.130)$$

where μ is the chemical potential and V_2 the pair potential.

In the case of short range forces, because we are interested in large distance phenomena, we can approximate the potential by a δ-function pseudo-potential, and the action becomes local (Section 4.4.7):

$$\mathcal{S}(\bar\varphi,\varphi)=\int \mathrm{d}t\,\mathrm{d}^d x\left[\bar\varphi(x,t)\left(\frac{\partial}{\partial t}-\frac{\hbar^2}{2m}\nabla^2-\mu\right)\varphi(x,t)+\frac{G}{2}\big(\bar\varphi(x,t)\varphi(x,t)\big)^2\right]. \quad (15.131)$$

We recall that the strength G of the interaction, which can be parametrized by $ma^{d-2}/\hbar^2$, where a is a length proportional to the scattering length in $d=3$, must be positive for stability, and this corresponds to a repulsive interaction between bosons.

The model has an obvious $U(1)\equiv O(2)$ symmetry,

$$\varphi\mapsto\varphi\,\mathrm{e}^{i\theta},\quad \bar\varphi\mapsto\bar\varphi\,\mathrm{e}^{-i\theta},$$

corresponding to particle number conservation.

The partition function is obtained by integrating over the complex field φ with periodic boundary conditions in the Euclidean time direction.

The action density for constant fields is

$$\mathcal{E}(\bar\varphi,\varphi)=-\mu\varphi\bar\varphi+\tfrac{1}{2}G(\varphi\bar\varphi)^2.$$

It predicts, in the tree approximation, a second-order phase transition at vanishing chemical potential μ. For $\mu>0$, the field acquires a non-vanishing expectation value,

$$|\varphi|^2=\mu/G\,.$$

In the Gaussian approximation, in the disordered phase, the non-relativistic two-point function $\tilde\Delta(\omega,\mathbf{k})$ in the Fourier representation is

$$\tilde\Delta(\omega,\mathbf{k})=\frac{1}{i\omega+k^2/2m-\mu}\,.$$

As a consequence of the periodic boundary conditions, the field can be expanded on a basis of periodic functions of period β as

$$\varphi(x,t)=\sum_{\nu\in\mathbb{Z}}\mathrm{e}^{2i\pi\nu t/\beta}\,\varphi_\nu(x),\quad \bar\varphi(x,t)=\sum_{\nu\in\mathbb{Z}}\mathrm{e}^{-2i\pi\nu t/\beta}\,\bar\varphi_\nu(x),$$

and thus ω is quantized: $\omega=\omega_\nu\equiv 2\pi\nu/\beta$. Near the transition point $\mu=0$, the large distance behaviour of the equal-time two-point function,

$$\Delta(x,0)=\frac{1}{(2\pi)^d}\frac{1}{\beta}\sum_{\nu\in\mathbb{Z}}\int\frac{\mathrm{d}^d k\;\mathrm{e}^{ikx}}{i\omega_\nu+k^2/2m}\,, \quad (15.132)$$

is dominated by the zero mode $\nu=0$, the contributions of other modes decreasing exponentially,

$$\Delta(x,0)\underset{|x|\to\infty}{\sim}\frac{1}{(2\pi)^d}\frac{1}{\beta}\int\frac{\mathrm{d}^d k\;\mathrm{e}^{ikx}}{k^2/2m}\propto|x|^{2-d}.$$

One finds the usual behaviour of the propagator in a massless field theory in d isotropic dimensions.

15.10.2 Critical properties: Beyond the Gaussian approximation

The critical behaviour of correlation functions, in the Gaussian approximation, is entirely obtained from the contribution of the zero mode. In a perturbative expansion, at each order, the leading singularities are again generated by the zero-mode contributions. This argument indicates that the critical behaviour, beyond the Gaussian approximation, can be derived from a simplified field theory. At leading order, we simply replace in the action the field $\varphi(x,t)$ by its zero mode $\varphi_0(x)$. A more detailed discussion in Chapter 32 confirms the analysis. In fact, the quantity $\sqrt{m/\beta}$ acts as an effective momentum cut-off, or the thermal wavelength

$$\lambda = \hbar\sqrt{\beta/m}\,, \tag{15.133}$$

as a short-distance cut-off. Note that the δ-function approximation for the potential makes sense if λ is much larger than the range of the potential, that is, if the temperature is low enough.

At leading order, the effective Hamiltonian for the zero mode then reads

$$\mathcal{H}(\bar\varphi_0,\varphi_0) = \beta\int \mathrm{d}^d x\left[\bar\varphi_0(x)\left(-\frac{\hbar^2}{2m}\nabla^2-\mu\right)\varphi_0(x)+\frac{G}{2}\big(\bar\varphi_0(x)\varphi_0(x)\big)^2\right], \tag{15.134}$$

which is also the classical approximation. An integration over the non-zero, non-critical modes simply generates a renormalization of the parameters of the effective long distance action. We recognize the Euclidean action of the $O(2)$ symmetric $(\boldsymbol{\phi}^2)^2$ field theory,

$$\mathcal{S}(\boldsymbol{\phi}) = \int\left[\tfrac{1}{2}\big(\nabla\boldsymbol{\phi}(x)\big)^2+\tfrac{1}{2}r\boldsymbol{\phi}^2(x)+\tfrac{1}{4!}g\Lambda^{4-d}\big(\boldsymbol{\phi}^2(x)\big)^2\right]\mathrm{d}^d x\,,$$

where $\boldsymbol{\phi}$ is a two-component vector, in a different normalization and parametrization, the parameters and dynamical variables having also a different interpretation. To identify the two actions, one can set

$$\phi_1(x)+i\phi_2(x) = \zeta\varphi_0(\theta x),$$

with

$$\theta = \lambda\Lambda,\quad \zeta^2 = \Lambda^{d-2}\lambda^d.$$

Comparing the two expressions, one obtains the relations between parameters (a is the scattering length)

$$g = 12\lambda^{-d}\beta G \propto (a/\lambda)^{d-2}\,,\quad r = -2\mu\beta\Lambda^2.$$

This field theory can then be studied by RG methods as in Section 15.7.

Note that from the derivation of the field integral representation in Section 4.4.5, it follows that the field $\boldsymbol{\phi}$ is not a direct observable. Only correlations of $\boldsymbol{\phi}^2$, which is the density field, are physical. Two quantities which can be measured directly are the specific heat and the density near criticality. For instance, the density ρ is given by the derivative of the free energy per unit volume with respect to $\beta\mu$:

$$\rho = \Omega^{-1}\frac{\partial\mathcal{W}}{\partial(\beta\mu)} = -2\Lambda^{2-d}(m/\beta\hbar^2)^{d/2}\Omega'^{-1}\frac{\partial\mathcal{W}}{\partial r}\,,$$

where Ω is the volume in the initial variables and Ω' in those of the $(\boldsymbol{\phi}^2)^2$ field theory. They depend on the two critical exponents ν and α, which are related. A measurement of both quantities yields two different determinations of the exponent ν [143].

In this system, two parameters can be tuned to reach criticality, temperature, and chemical potential in the theoretical model, in actual experiments temperature and pressure. By varying the pressure, one can check universality and get some handle on corrections to scaling.

The zero-temperature limit. A special situation corresponds to zero temperature. Then, the Fourier modes are continuous, and the complete non-relativistic theory becomes relevant. Power counting of the type encountered in critical dynamics, with time identified to the square of a space variable, shows that the theory is exactly renormalizable in $d = 2$ space dimensions, where logarithmic corrections to the mean-field approximation are expected.

Quantum and classical fluids: The density field. We have shown that critical properties can be obtained from the classical approximation. However, we have studied a region of parameter where the zero mode is critical. By contrast, in the case of classical fluids, the field φ_0 is not critical; the thermal wave length is small, and the δ-function approximation is no longer justified. It is the density $\rho(x) = \bar{\varphi}_0(x)\varphi_0(x)$ that is the order parameter and has critical fluctuations. The field φ_0 can be integrated out to generate an effective field theory for the density field ρ of the form discussed in Section 15.9, and which eventually leads to a $N = 1$, ϕ^4 field theory in the critical domain.

15.10.3 Superfluid transition and Bose–Einstein condensation

In the case of a Bose gas, the approximation of two-body interaction is natural because the system is dilute. In the case of a Bose fluid, this is no longer the case. However, additional higher powers of $\bar{\varphi}\varphi$ in the action are irrelevant, and thus do not modify the critical behaviour. Only in the exceptional case where the renormalized $(\bar{\varphi}\varphi)^2$ interaction is small, three-body interactions, which generate a $(\bar{\varphi}\varphi)^3$ term, may become important and lead to a tricritical behaviour [144].

By contrast, in dilute Bose gases, of the kind studied in Bose–Einstein condensation experiments, three-body interactions are totally negligible compared to two-body interactions, even though the latter are also weak. In such systems, at the transition temperature, correlation functions depend only on the RG invariant length scale ξ, which satisfies [145]

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g}\right)\xi(\Lambda, g) = 0\,,$$

and thus is proportional to

$$\xi \propto \Lambda^{-1} g^{1/(d-4)} \exp\left[\int_0^g \mathrm{d}g' \left(\frac{1}{\beta(g')} - \frac{1}{(d-4)g'}\right)\right].$$

We note that, for generic values of the ϕ^4 coupling, that is, $g = O(1)$, typical for a fluid, the length scale ξ is of the order of the microscopic scale, and long distance physics corresponds to distances much larger than $1/\Lambda$ and thus than ξ. This physics is entirely described by the QFT of the IR fixed point.

However, if as in the case of the dilute Bose gas g is very small, then ξ is much larger than the microscopic length scale. It becomes a *crossover length* between two universal behaviours. For distances much larger than ξ, physics can still be derived from the IR fixed point, and is analogous to the superfluid transition. By contrast, for distances much larger than $1/\Lambda$ but much smaller than ξ, physics is dominated by the Gaussian (UV) fixed point, and, therefore, one observes a behaviour close to the Bose–Einstein condensation of the free Bose gas.