7

Renormalization group

As we saw in Chapter 3, the renormalization procedure has considerable arbitrariness: the counterterm for a graph must cancel its divergence but may contain any amount of finite part. A rule for choosing the value of the counterterm we called a renormalization prescription. In one-loop order it was clear from the examples that a change in renormalization prescription can be cancelled by a change in the finite, renormalized couplings corresponding to each divergence. Thus a change in renormalization prescription does not change the theory but only the parametrization by renormalized coupling and mass. What is not so easy is to see that this property is true to all orders. This we will show in Section 7.1. The invariance of the theory under such transformations is called renormalization-group (RG) invariance.

A particularly useful type of change of renormalization prescription is to change the renormalization mass μ. Infinitesimal changes are conveniently described by a differential equation, called the renormalization-group equation, which is derived in Section 7.3. This leads to the concept of the effective momentum-dependent coupling. This concept is very useful in calculations of high-energy behavior, as explained in Section 7.4. The coefficients in the renormalization-group equation are called the renormalization-group coefficients and are important properties of a theory. Various developments of the formalism occupy the remaining sections.

The renormalization group was first discussed by Stueckelberg & Petermann (1953) and by Gell-Mann & Low (1954). Very similar ideas are applied in statistical physics (Wilson & Kogut (1974)). Many important recent applications arise because of the asymptotic freedom of QCD.

Results of calculations of renormalization-group coefficients can be found in many places:

(1) Gross (1976) lists many one-loop results for theories with scalars and fermions and up to two loops for gauge theories with only fermions.
(2) Cheng, Eichten & Li (1974) give the β-function for a general renormalizable theory to one-loop order.

(3) Tarasov, Vladimirov & Zharkov (1980) compute renormalization-group coefficients to three-loop order in gauge theories with fermions using minimal subtraction.
(4) Vladimirov, Kazakov & Tarasov (1979) compute to four-loop order in ϕ^4 theory.
(5) Chetyrkin, Kataev & Tkachov (1981) and Chetyrkin & Tkachov (1981) compute the anomalous dimension in ϕ^4 theory at five-loop order.
(6) Tkachov (1981) summarizes the methods used for the above calculations.
(7) Caswell & Zanon (1981) perform calculations in supersymmetric theories at three-loop order.

7.1 Change of renormalization prescription

7.1.1 *Change of parametrization*

The techniques we will describe are valid for any theory. However, to be specific, we will mainly work with the theory we have been using as a source of examples, the ϕ^3 theory in six space-time dimensions. There are three alternative, but equivalent, forms in which to write the Lagrangian. First of all, we can write it in terms of the bare field ϕ_0:

$$\mathscr{L} = (\partial\phi_0)^2/2 - m_0^2\phi_0^2/2 - g_0\phi_0^3/6. \tag{7.1.1a}$$

(As before, we ignore the term linear in ϕ.) The importance of this form is that the bare field ϕ_0 is invariant under change of renormalization prescription: its normalization is determined, because it satisfies canonical equal-time commutation relations.

When we renormalize the theory, we obtain finite Green's functions of the renormalized field $\phi = Z^{-1/2}\phi_0$. In terms of the renormalized field, the Lagrangian is

$$\begin{aligned}\mathscr{L} &= Z(\partial\phi)^2/2 - m_0^2 Z\phi^2/2 - g_0 Z^{3/2}\phi^3/6 \\ &\equiv Z(\partial\phi)^2/2 - m_B^2\phi^2/2 - g_B\phi^3/6.\end{aligned} \tag{7.1.1b}$$

This is the second of the three forms.

In the perturbative theory of renormalization, we wrote the Lagrangian as the sum of a free Lagrangian, a basic interaction Lagrangian, and a counterterm Lagrangian:

$$\begin{aligned}\mathscr{L} &= \mathscr{L}_0 + \mathscr{L}_b + \mathscr{L}_{ct} \\ &= \mathscr{L}_{basic} + \mathscr{L}_{ct}.\end{aligned} \tag{7.1.1c}$$

This is the third form of the Lagrangian. Here, we have chosen to define a

basic Lagrangian

$$\mathscr{L}_{\text{basic}} = (\partial\phi)^2/2 - m^2\phi^2/2 - \mu^{3-d/2} g\phi^3/6, \tag{7.1.2}$$

where m and g are the renormalized mass and coupling. Since we will mostly use minimal subtraction, it is sensible to define g to be dimensionless, and therefore to introduce the unit of mass μ. The counterterm Lagrangian is

$$\mathscr{L}_{\text{ct}} = \delta Z(\partial\phi)^2/2 - \delta m^2\phi^2/2 - \delta g\phi^3/6, \tag{7.1.3}$$

and the counterterms δZ, δm^2, and δg are computed as definite functions of g, m and μ with the aid of some renormalization prescription.

To be concrete, let us use minimal subtraction, so that

$$\delta Z = \sum_{j=1}^{\infty} (6-d)^{-j} c_j(g, m, \mu), \tag{7.1.4a}$$

$$\delta m^2 = m^2 \sum_{j=1}^{\infty} (6-d)^{-j} b_j(g, m, \mu), \tag{7.1.4b}$$

$$\delta g = \mu^{3-d/2} \sum_{j=1}^{\infty} (6-d)^{-j} a_j(g, m, \mu). \tag{7.1.4c}$$

We saw, in Section 5.8, that in fact the coefficients a_i, b_i, c_i are independent of m and μ; they are functions of the dimensionless coupling g only. However we will not use this fact at the moment.

The three forms of the Lagrangian listed in (7.1.1) are equivalent – if we use any of them in the functional integral, then the same Green's functions will result. The coefficients Z, m_0^2, and g_0 will be singular when d approaches 6 with g, m, and μ fixed. The singularities will be just such as to give finite Green's functions of ϕ at $d = 6$.

The parametrization of the Green's functions by g, m, and μ is rather arbitrary. Suppose that we change variables to g', m', and μ', which are some given functions of g, m, and μ. These functions may even depend on the regulator, d, provided that the change of variable remains non-singular at $d = 6$. Then we get the same theory, for the collection of Green's functions G_N is unchanged. It is just the numerical values of the renormalized mass and coupling and of the unit of mass that have changed.

We may even change the scale of the renormalized field by writing $\phi' = \zeta\phi$, where ζ is finite. The Green's functions are now different:

$$G'_N = \zeta^N G_N.$$

But observe that the value of ζ is irrelevant for a physical observable like the S-matrix. For example, consider an S-matrix element involving N particles. It is obtained from G_N by (1) dividing out an external propagator $G_2(p_i)$ for

each external line and (2) multiplying by $\hat{z}^{1/2}$ for each external line, where $i\hat{z}$ is the residue of the pole of $G_2(p)$. Finally we let the momenta p_i go on-shell. Thus

$$S = \lim_{p^2 \to m_{ph}^2} \left\{ G_N \hat{z}^{N/2} \Big/ \prod_{i=1}^{N} G_2 \right\}$$
$$= \lim \{ G'_N (\zeta \hat{z}^{1/2})^N / \prod G'_2 \}. \tag{7.1.5}$$

But

$$G_2 = \zeta^{-2} G'_2, \tag{7.1.6}$$

so the particle pole is at the same position in G'_2 as in G_2 and the new residue is

$$\hat{z}' = \zeta^2 \hat{z}. \tag{7.1.7}$$

Hence

$$S = \lim \{ G'_N (\hat{z}'^{1/2})^N / \prod G'_2 \}. \tag{7.1.8}$$

and the S-matrix is invariant, as claimed.

7.1.2 *Renormalization-prescription dependence*

In the bare Lagrangian (7.1.1*a*), there are only two parameters. So there should be only a two-parameter collection of physical theories obtained from it. As we have just seen, the freedom to vary the scale of the renormalized field ϕ in the second form of $\mathscr{L}$, viz. (7.1.1*b*), does not introduce a third real parameter into the physics.

Unfortunately, we appear to have introduced a large and indefinite number of parameters by having to choose one out of the infinitely many possible renormalization prescriptions. One might suppose that in different renormalization prescriptions, the singular behavior of m_0 and g_0 as $d \to 6$ could be different in such a way that one picks up different phases of the theory. In fact, this is not so. We will show that a change of renormalization prescription is one of the reparametrizations discussed in the previous subsection 7.1.1. This is the property we have defined as renormalization-group invariance.

Even within a single renormalization prescription, we have introduced a third parameter, the unit of mass, μ. Notice that the basic Lagrangian does not depend on μ and g separately, but only on the combination $\mu^{3-d/2} g$; but notice also that this property is not true for, say, the renormalized Green's functions at *one-loop* order. However, a change of μ is in effect a change in renormalization prescription. Indeed, we could include in our definition of the renormalization prescription the requirement that μ and m have a fixed

ratio; we would still have two free parameters. A change in this ratio is then a change in renormalization prescription. Our proof of renormalization-group invariance will, in fact, only explicitly cover the case of a change in μ. The more general case will be essentially the same.

We will prove that if we change μ to μ', then the physics is unchanged, provided that we choose suitable new values, g' and m', for the renormalized coupling and mass. That is, an S-matrix element $S(g, m, \mu)$ satisfies

$$S(g, m, \mu) = S(g', m', \mu').$$

The bare mass and coupling m_0 and g_0 are similarly invariant.

The new renormalized field ϕ' with the new values (g', m', μ') of the parameters is not the same as with the old values but is related by

$$\phi' = \zeta\phi,$$

where ζ is a finite function of g, m, μ and μ'. Hence the renormalized Green's functions satisfy

$$\begin{aligned} G_N(p_1, \ldots, p_N; g, m, \mu) &= \zeta^{-N} G_N(p_1, \ldots, p_N; g', m', \mu') \\ &= \zeta^{-N} G'_N. \end{aligned} \tag{7.1.9}$$

It will be convenient, in our proof, to compute Green's functions of the original field ϕ, but with the new value μ' of the unit of mass. Now, in terms of the new field ϕ', the Lagrangian is

$$\mathscr{L} = Z'\partial\phi'^2/2 - m_B'^2\phi'^2/2 - g'_B\phi'^3/6. \tag{7.1.10}$$

Here we write

$$Z' = Z(g', m'/\mu', d), \tag{7.1.11a}$$

$$m_B'^2 = m_B^2(g', m', \mu', d), \tag{7.1.11b}$$

$$g'_B = g_B(g', \mu', d). \tag{7.1.11c}$$

These are the same functions of the new renormalized parameters m', g', and μ' as the original bare parameters were of the old renormalized parameters m, g, and μ. To get the Green's functions of the original field ϕ but with the new value of the unit of mass, we substitute $\zeta\phi$ for ϕ' to obtain

$$\begin{aligned} \mathscr{L} = {} & Z(g', m'/\mu', d)\zeta^2\partial\phi^2/2 - m_B^2(g', m'/\mu', d)\zeta^2\phi^2/2 \\ & - g_B(g', m', \mu', d)\zeta^3\phi^3/6. \end{aligned} \tag{7.1.12}$$

7.1.3 *Low-order examples*

Let us remind ourselves how the changes in g and m are obtained at one-loop order. We must start at tree approximation, where we ignore all counterterms. In order that the two formulae (7.1.1*b*) and (7.1.12) for $\mathscr{L}$ be

the same in tree approximation when we change μ to μ', we must write

$$\begin{aligned} g' &= g_{\text{new}} = (\mu/\mu')^{3-d/2} g, \\ m' &= m, \\ \zeta &= 1. \text{ (All tree approximation.)} \end{aligned} \tag{7.1.13}$$

Note that at $d = 6$ we have $g' = g$ to lowest order. We distinguish g' and g_{new}: g_{new} is defined to be exactly the value of g' in tree approximation. In higher-order calculations, g' gets corrections, but g_{new} will be defined to be $(\mu/\mu')^{3-d/2} g$ always.

To treat higher-order corrections, we write the Lagrangian (7.1.12) as

$$\begin{aligned} \mathscr{L} = {} & (\partial\phi)^2/2 - m^2\phi^2/2 - \mu'^{3-d/2}[g(\mu/\mu')^{3-d/2}]\phi^3/6 \\ & + \delta' Z(\partial\phi)^2/2 - \delta' m^2\phi^2/2 - \delta' g\phi^3/6. \end{aligned} \tag{7.1.14}$$

Our strategy will be to express the counterterms in (7.1.14) as minimal subtraction counterterms plus some new finite pieces. The finite pieces will accomplish the change of parametrization.

First we consider the one-loop graph for the self-energy, Fig. 3.1.1. The unrenormalized value is

$$\Gamma_1 = \frac{ig^2\Gamma(2-d/2)}{128\pi^3} \int_0^1 dx\,[m^2 - p^2 x(1-x)]^{d/2-2} (4\pi\mu^2)^{3-d/2}. \tag{7.1.15}$$

This is invariant under the transformation $(\mu,\ m,\ g) \to (\mu',\ m,\ g(\mu/\mu')^{3-d/2})$. If we use unit of mass μ, then the counterterm is

$$\begin{aligned} C(\Gamma_1) &= -\operatorname{pole}(\Gamma_1) \\ &= \frac{-ig^2}{64\pi^3(d-6)}(m^2 - \tfrac{1}{6}p^2). \end{aligned} \tag{7.1.16}$$

Next we use μ' and $g_{\text{new}} = (\mu/\mu')^{3-d/2} g$ instead of μ and g. The counterterm changes to

$$\begin{aligned} C'(\Gamma_1) &= \frac{-ig_{\text{new}}^2}{64\pi^3(d-6)}(m^2 - \tfrac{1}{6}p^2) \\ &= \frac{-ig^2}{64\pi^3(d-6)}(m^2 - \tfrac{1}{6}p^2)\left(\frac{\mu}{\mu'}\right)^{6-d}. \end{aligned} \tag{7.1.17}$$

Notice that we define $C'(\Gamma_1)$ to be the negative of the pole part of Γ_1, with the d-dependence of g_{new} ignored. That is, we consider the function $\Gamma_1 = \Gamma_1(p, g_{\text{new}}, m, \mu')$ and extract its pole at $d = 6$ with g_{new} fixed. This prescription ensures that we may later replace g_{new} by its value at $d = 6$ without changing the renormalized value of the graph.

Since both $C(\Gamma_1)$ and $C'(\Gamma_1)$ cancel the divergence, their difference is finite. We may therefore obtain the same value for the graph plus

counterterm by letting the counterterm coefficients in (7.1.14), namely $\delta' Z$ and $\delta' m^2$, each be a sum of two terms:

$$\delta' Z = \frac{g_{\text{new}}^2}{384\pi^3(d-6)} + \frac{g_{\text{new}}^2}{384\pi^3}\left[\frac{(\mu'/\mu)^{6-d}-1}{d-6}\right], \tag{7.1.18}$$

$$\delta' m^2 = \frac{m^2 g_{\text{new}}^2}{64\pi^3(d-6)} + \frac{m^2 g_{\text{new}}^2}{64\pi^3}\left[\frac{(\mu'/\mu)^{6-d}-1}{d-6}\right]. \tag{7.1.19}$$

The first term in each equation is the minimal subtraction counterterm for the new coupling, while the second term is finite as $d \to 6$. Using the formula (7.1.12) for the Lagrangian, we may regroup these terms to give

$$\begin{aligned}\zeta^2 &= 1 + \frac{g_{\text{new}}^2}{384\pi^3}\left[\frac{(\mu'/\mu)^{6-d}-1}{d-6}\right] \\ &= 1 + \frac{g^2}{384\pi^3}\left[\frac{1-(\mu/\mu')^{6-d}}{d-6}\right],\end{aligned} \tag{7.1.20}$$

$$m'^2\zeta^2 = m^2 + \frac{m^2 g^2}{64\pi^3}\left[\frac{1-(\mu/\mu')^{6-d}}{d-6}\right]. \tag{7.1.21}$$

Hence we obtain the value of m':

$$\begin{aligned}m'^2 &= m^2\left\{1 + \frac{5g^2}{384\pi^3}\left[\frac{1-(\mu/\mu')^{6-d}}{d-6}\right] + O(g^4)\right\} \\ &\to m^2\left[1 + \frac{5g^2}{384\pi^3}\ln(\mu/\mu') + O(g^4)\right] \quad \text{as } d \to 6.\end{aligned} \tag{7.1.22}$$

Also at $d = 6$ we have

$$\zeta(d=6) = 1 + \frac{g^2}{768\pi^3}\ln(\mu/\mu') + O(g^4). \tag{7.1.23}$$

We can apply the same procedure to the vertex graph, Fig. 3.6.1, whose value is the factor in curly brackets in (5.3.5). The counterterm $\delta' g$ is written as

$$\frac{g_{\text{new}}^3 \mu'^{3-d/2}}{64\pi^3(d-6)} + \frac{g_{\text{new}}^3 \mu'^{3-d/2}}{64\pi^3}\left[\frac{(\mu'/\mu)^{6-d}-1}{d-6}\right], \tag{7.1.24}$$

with the result that

$$\begin{aligned}\zeta^3 g' \mu'^{3-d/2} &= g_{\text{new}}\mu'^{3-d/2}\left\{1 + \frac{g_{\text{new}}^2}{64\pi^3}\left[\frac{(\mu'/\mu)^{6-d}-1}{d-6}\right]\right\} \\ &= g\mu^{3-d/2}\left\{1 + \frac{g^2}{64\pi^3}\left[\frac{1-(\mu/\mu')^{6-d}}{d-6}\right]\right\}.\end{aligned} \tag{7.1.25}$$

It follows that

$$g' = g(\mu/\mu')^{3-d/2}\left\{1 + \frac{3g^2}{256\pi^3}\left[\frac{1-(\mu/\mu')^{6-d}}{d-6}\right] + O(g^4)\right\}$$

$$\to g\left[1 + \frac{3g^2}{256\pi^3}\ln(\mu/\mu') + O(g^4)\right] \quad \text{as } d \to 6. \tag{7.1.26}$$

Our strategy for understanding the effect of a change in renormalization prescription is to absorb the difference into a finite counterterm. The counterterm will itself generate divergent counterterms when we insert it into a bigger graph. Finally we reorganize the Lagrangian by putting all the finite counterterms into the basic Lagrangian. Then we see that the change in renormalization prescription is exactly compensated by a change in the parameters of the theory.

What happens when we go to higher order? An example is given by the two-loop self-energy graph of Fig. 7.1.1(*a*). Graphs (*b*), (*c*), and (*d*) renormalize its subdivergences and its overall divergence. We write the renormalized graph with the original value of μ as

$$\Sigma_1(p, g, m, \mu) = \Sigma_a + 2\Sigma_b + \Sigma_d. \tag{7.1.27}$$

(We used the fact that $\Sigma_b = \Sigma_c$.) The unrenormalized graph is unchanged if we replace μ by μ' and g by g_{new}. But the vertex counterterm is treated exactly as at (7.1.24), so that:

$$\Sigma_b(p, g, m, \mu) = \Sigma_b(p, g_{\text{new}}, m, \mu') + [\Sigma_b(p, g, m, \mu) - \Sigma_b(p, g_{\text{new}}, m, \mu')]. \tag{7.1.28}$$

The first term contains the counterterm for one of the subgraphs, computed by minimal subtraction with unit of mass μ'. The second term exactly compensates the difference. It has the counterterm replaced by the finite

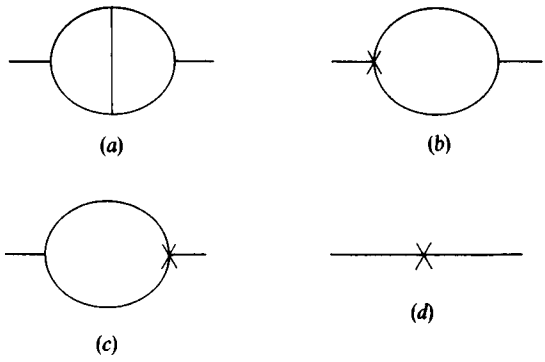

Fig. 7.1.1. Self-energy graph with counterterm graphs.

part in (7.1.24), viz.

$$\Delta g = \frac{g_{\text{new}}^3 \mu'^{3-d/2}}{64\pi^3}\left[\frac{(\mu'/\mu)^{6-d}-1}{d-6}\right]. \tag{7.1.29}$$

We now write

$$\Sigma_1 = \Sigma_a(p, g_{\text{new}}, m, \mu') + 2\Sigma_b(p, g_{\text{new}}, m, \mu') + \Sigma_d(p, g_{\text{new}}, m, \mu') + \left\{2\Gamma_1 \frac{\Delta g}{g_{\text{new}}\mu'^{3-d/2}} + \Sigma_d(p, g, m, \mu) - \Sigma_d(p, g_{\text{new}}, m, \mu')\right\}. \tag{7.1.30}$$

Here we wrote out the first three terms as the minimal renormalization of Σ_1 with unit of mass μ'. The remainder is finite, since Σ_1 is finite. The term $2\Gamma_1 \Delta g/(g_{\text{new}}\mu'^{3-d/2})$ is the one-loop self-energy graph with one of its couplings replaced by Δg. It has a divergence which can be cancelled by a minimal counterterm:

$$C = \frac{-ig_{\text{new}}\Delta g}{64\pi^3(d-6)}(m^2 - \tfrac{1}{6}p^2). \tag{7.1.31}$$

Hence the term in curly brackets is

$$\left[\frac{2\Gamma_1 \Delta g}{g_{\text{new}}\mu'^{3-d/2}} + 2C\right] + [\Sigma_d(p, g, m, \mu) - \Sigma_d(p, g_{\text{new}}, m, \mu') - 2C]. \tag{7.1.32}$$

The second term is finite, since there are no remaining divergences. It is of the form

$$\mathrm{i}(-\Delta m^2 + \Delta Z p^2),$$

and so gives rise to another finite contribution to m'^2 and to ζ.

7.2 Proof of RG invariance

To show to all orders of perturbation theory that g', m' and ζ can be chosen so that (7.1.9) holds, we generalize from our treatment of the examples. We write

$$\begin{aligned} \zeta^3 g' \mu'^{3-d/2} &= g\mu^{3-d/2} + \Delta g, \\ \zeta^2 &= 1 + \Delta\zeta^2, \\ \zeta^2 m'^2 &= m^2 + \Delta m^2. \end{aligned} \tag{7.2.1}$$

The original version of the theory has counterterms computed with unit of mass μ:

$$\mathscr{L} = \mathscr{L}_{\text{basic}} + \mathscr{L}_{\text{ct}}(\mu). \tag{7.2.2}$$

We now change to unit of mass μ' and wish to show that identically the

same Green's functions and total Lagrangian are obtained if we make changes of the form (7.2.1).

The Lagrangian is written in the form

$$\mathscr{L} = \mathscr{L}_{\text{basic}} + \mathscr{L}_{\text{c}} + \mathscr{L}'_{\text{ct}}, \tag{7.2.3}$$

with the basic Lagrangian the same as before, but written as

$$\mathscr{L}_{\text{basic}} = (\partial\phi)^2/2 - m^2\phi^2/2 - g_{\text{new}}\mu'^{3-d/2}\phi^3/6. \tag{7.2.4}$$

(As before, we define $g_{\text{new}} = (\mu/\mu')^{3-d/2}g$.) The term $\mathscr{L}_{\text{c}}$ is a 'compensating Lagrangian' of the form

$$\mathscr{L}_{\text{c}} = \Delta\zeta^2\partial\phi^2/2 - \Delta m^2\phi^2/2 - \Delta g\phi^3/6. \tag{7.2.5}$$

The counterterm Lagrangian $\mathscr{L}'_{\text{ct}}$ in (7.2.3) is computed using minimal subtraction with unit of mass μ'.

We may later reorganize (7.2.3) so that the basic Lagrangian is taken as $\mathscr{L}_{\text{basic}} + \mathscr{L}_{\text{c}}$. We may drop the d-dependence of the finite counterterms Δg, $\Delta\zeta^2$, and Δm^2, since renormalized Green's functions are finite functions of renormalized quantities. Finally we may rescale the fields to give (7.1.10).

But a proof is most easily given with the form (7.2.3). Each of the finite counterterms is computed as a sum of terms, one for each 1PI graph contributing to the relevant Green's function:

$$\Delta g = \sum_{\Gamma} \Delta_{\Gamma}(g), \text{ etc.} \tag{7.2.6}$$

Particular cases are given by the examples in Section 7.1. Thus

$$\Delta_{3.1.1}\zeta^2 = \frac{g_{\text{new}}^2}{384\pi^3}\left[\frac{(\mu'/\mu)^{6-d}-1}{d-6}\right],$$

$$\Delta_{3.1.1}m^2 = \frac{m^2 g_{\text{new}}^2}{64\pi^3}\left[\frac{(\mu'/\mu)^{6-d}-1}{d-6}\right],$$

$$\Delta_{3.6.1}g = \mu'^{3-d/2}\frac{g_{\text{new}}^3}{64\pi^3}\left[\frac{(\mu'/\mu)^{6-d}-1}{d-6}\right],$$

$$\Delta_{7.1.1}(\zeta^2 p^2 - m^2) = -\mathrm{i}[\Sigma_d(p, g, m, \mu) - \Sigma_d(p, g_{\text{new}}, m, \mu') - 2C],$$

where the label on Δ indicates the number of the figure depicting the basic graph.

The general proof is by induction on the size of a graph. Consider a graph G contributing to some Green's function. Using the basic interaction $-g\mu^{3-d/2}\phi^3/6$ we renormalize it with unit of mass μ using the method of Section 5.11. The renormalized value of G is then

$$R(G) = G + \sum_{\gamma \subseteq G} C_{\gamma}(G). \tag{7.2.7}$$

Here the sum is over subgraphs of G that consist of one or more disjoint 1PI graphs, and $C_\gamma(G)$ denotes the replacement of each of these 1PI graphs by a counterterm with unit of mass μ. Similarly we can renormalize G with a different unit of mass μ' but with the same basic interaction $-g_{\text{new}}\mu'^{3-d/2}\phi^3/6$ to get

$$R'(G) = G + \sum_{\gamma \subseteq G} C'_\gamma(G). \tag{7.2.8}$$

Here, we use the prime to denote use of the unit of mass μ' instead of μ.

We will now derive a series of new basic graphs containing *finite* counterterms. These counterterms will be used to generate the compensating Lagrangian $\mathscr{L}_c$ in (7.2.3). The new graphs will need renormalization, and we will arrange them so that when they are added to $R'(G)$, we get back the original value $R(G)$. The aim will be to have a finite counterterm Δ_γ for every 1PI graph γ that is a vertex or self-energy graph. We will arrange the Δ_γ's so that

$$R(G) = R'\left[G + \sum_{\gamma \subsetneq G} \Delta_\gamma(G) + \Delta(G) \right]. \tag{7.2.9}$$

Here the sum over γ is over products of 1PI graphs. The overall counterterm for a graph is computed using minimal subtraction, but with the d-dependence of g_{new} and of the finite counterterms Δ_γ ignored (see our remarks below (7.1.17)). That is, the counterterms in R' are a series of poles at $d = 6$ with their coefficients being power series in g_{new}. In the case of a vertex subgraph, there is also the usual factor $\mu'^{3-d/2}$. The finite subtraction $\Delta(G)$ for the complete graph in (7.2.9) is only non-zero if G is a 1PI vertex or a self-energy graph.

We will prove the following relation between counterterms for a 1PI graph

$$C(\gamma) = C'\left(\gamma + \sum_{\delta \subsetneq \gamma} \Delta_\delta(\gamma) \right) + \Delta(\gamma). \tag{7.2.10}$$

The above equations (7.2.9) and (7.2.10) are trivially true for tree graphs, where no counterterms are needed, for we can set $\Delta(\text{tree graph}) = 0$. So let us assume they are true for all graphs smaller than a given graph G, with all the Δ_γ's finite. There are two cases: (*a*) G is not overall divergent; then we must prove (7.2.9) with no counterterm $\Delta(G)$. (*b*) G is 1PI and overall divergent; then we use (7.2.10) with γ replaced by G to define $\Delta(G)$. We must prove $\Delta(G)$ finite and prove (7.2.9). We must also prove $\Delta(G)$ is polynomial in the external momenta of G, with degree equal to the degree of divergence

of G; we will assume inductively that this is true with G replaced by any smaller graph.

Consider the terms in (7.2.7). For each we identify a contribution in (7.2.9). The term G is the same as G in $R'(G)$. Next let γ be a 1PI subgraph of G (G itself being excluded). Decompose $C_\gamma(G)$ by (7.2.10):

(1) The $C'(\gamma)$ term occurs in $R'(G)$ as $C'_\gamma(G)$.
(2) The term $\Delta(\gamma)$ occurs as $\Delta_\gamma(G)$.
(3) The term $C'_\gamma(\Delta_\delta(\gamma))$ occurs as a counterterm in the renormalization $R'(\Delta_\delta(G))$.

If G is not overall divergent, these exhaust all of the terms in $R(G)$ and on the right-hand side of (7.2.9). But if G is 1PI and overall divergent then there remains $C(G)$ in (7.2.7) and the terms

$$C'\left[G + \sum_{\delta \subsetneq G} \Delta_\delta(G)\right] + \Delta(G)$$

in (7.2.9). We are therefore forced to define $\Delta(G)$ by (7.2.10), and it remains to prove $\Delta(G)$ finite. This is now easy, since

$$R(G), R'(G), \text{ and } R'\left[\sum_{\delta \subsetneq G} \Delta_\delta(G)\right] \tag{7.2.11}$$

are all finite, while we have

$$R'(\Delta(G)) = \Delta(G).$$

Moreover all the terms in (7.2.10), except possibly $\Delta(G)$, are ordinary minimal-subtraction counterterms. So they are polynomial in the external momenta of G, with degree equal to the degree of divergence of G. So $\Delta(G)$ is polynomial, of the same degree. (Note that the replacement of a subgraph γ by Δ_γ does not change the overall degree of divergence of any graph Γ satisfying $\gamma \subseteq \Gamma \subseteq G$. This is because of our inductive assumption on the polynomial degree of Δ_γ.)

The theorem is also true if R and R' stand not for renormalization with different units of mass, but for any two renormalization prescriptions. It is important that the d-dependence of Δ_γ is taken outside of the extraction of pole parts when computing a counterterm like $C_\Gamma(\Delta_\gamma)$. This can be seen from the example of Fig. 7.1.1, at (7.1.31). The primed counterterms must be a particular function of g_{new}, μ', m', and d. The fact that g' itself is a function of other variables is ignored.

Equation (7.2.10) expresses the counterterm $C(\gamma)$ for a graph γ (with unit of mass μ) in terms of renormalization counterterms with unit of mass μ' and a finite counterterm $\Delta(\gamma)$. We can use the finite counterterms to generate

the compensating Lagrangian $\mathscr{L}_c$ in (7.2.3), and the primed counterterms to generate $\mathscr{L}'_{ct}$. It should be evident that the new Lagrangian is the same as the original one (with unit of mass μ), considered as a function of ϕ and $\partial\phi$. If we set $\phi_0 = Z^{1/2}\phi$, with $Z = Z(g, m, \mu)$, then we can deduce that the bare parameters m_0 and g_0 are renormalization-group invariant:

$$m_0(g, m, \mu) = m_0(g', m', \mu'),$$
$$g_0(g, m, \mu) = g_0(g', m', \mu'). \tag{7.2.12}$$

7.3 Renormalization-group equation

We saw in Sections 7.1 and 7.2 that a change in the unit of mass μ accompanied by suitable changes in coupling and mass does not change the theory, while the Green's functions satisfy

$$G_N(x_1, \ldots, x_N; g, m, \mu) = \zeta^{-N} G_N(x_1, \ldots, x_N; g', m', \mu'). \tag{7.3.1}$$

We wish to compute g', m'^2 and ζ as functions of g, m^2, μ, and μ'. If the ratio μ/μ' is large, it is not sufficient to use lowest-order perturbation theory, since, for example, in (7.1.26) the coefficient of g^2 may be large.

An important device is to consider a large change in μ as being made up of a sequence of very small changes, so that g', m'^2, and ζ are obtained as solutions of differential equations. This is the subject to which we now turn.

The consequence of our work in Sections 7.1 and 7.2 is that for a given physical theory, we have for each value of μ a definite value of the coupling $g(\mu)$ and mass $m(\mu)$. These are called the effective (or running) coupling and mass. We will derive differential equations for $g(\mu)$ and $m(\mu)$.

The easiest way to derive the results is to look at the Green's functions and the Lagrangian expressed in terms of the bare field ϕ_0. The important point is that the Green's functions of ϕ_0 are invariant under our change of parametrization $(\mu, g, m) \to (\mu', g', m')$. (This is because the mass and coupling in the bare Lagrangian are invariant.)

7.3.1 *Renormalization-group coefficients*

Physical quantities like the S-matrix are invariant under the change of variable $(\mu, g(\mu), m(\mu)) \to (\mu', g(\mu'), m(\mu'))$. This invariance is conveniently expressed by considering a small change in μ, accompanied by the corresponding changes in g and m. We write the result as

$$\mu \mathrm{d}S/\mathrm{d}\mu = 0. \tag{7.3.2}$$

The total derivative with respect to μ can be written as

$$\mu \mathrm{d}/\mathrm{d}\mu = \mu\partial/\partial\mu + \beta\partial/\partial g - \gamma_m m^2 \partial/\partial m^2. \tag{7.3.3}$$

On the right-hand side the partial derivatives with respect to μ, g or m are taken with the other two fixed, and the coefficients β and γ_m give the variations of $g(\mu)$ and $m^2(\mu)$ when μ is varied. The sign of the $m^2\partial/\partial m^2$ term is the usual convention (Weinberg (1973)). We have

$$\beta = \mu \mathrm{d}g(\mu)/\mathrm{d}\mu, \tag{7.3.4a}$$

$$\gamma_m = -m^{-2}\mu \mathrm{d}m^2(\mu)/\mathrm{d}\mu. \tag{7.3.4b}$$

The coefficients β and γ_m are called renormalization-group coefficients. As we will see they are easy to calculate in terms of the counterterms, as functions of g, m, μ, and d. If we use minimal subtraction, they have no mass dependence. This means that (7.3.4) can be readily solved as differential equations for $g(\mu)$ and $m(\mu)$. Indeed this is the easiest way in practice to compute the effective coupling and mass.

To compute β and γ_m it is convenient to consider the Lagrangian expressed in terms of the bare field ϕ_0 – see (7.1.1*a*). We saw in Section 7.2 that m_0^2 and g_0 are renormalization-group invariant:

$$\begin{aligned} \mu \mathrm{d}g_0/\mathrm{d}\mu &= 0, \\ \mu \mathrm{d}m_0^2/\mathrm{d}\mu &= 0. \end{aligned} \tag{7.3.5}$$

Suppose we have computed g_0 and m_0 to some order in g. Then (7.3.5) can be solved to give β and γ_m.

An example of such a calculation comes from our results on ϕ^3 theory, where

$$\begin{aligned} g_0 &= \mu^{3-d/2} g\left[1 + \frac{3g^2}{256\pi^3(d-6)} + O(g^4)\right], \\ m_0^2 &= m^2\left[1 + \frac{5g^2}{384\pi^3(d-6)} + O(g^4)\right]. \end{aligned} \tag{7.3.6}$$

Thus

$$0 = \mu\frac{\mathrm{d}g_0}{\mathrm{d}\mu} = \mu^{3-d/2}\left\{g(3-d/2) - \frac{3g^3}{512\pi^3} + O(g^5) + \beta(g)\left[1 + \frac{9g^2}{256\pi^3(d-6)} + O(g^4)\right]\right\},$$

so

$$\begin{aligned} \beta(g) &= (d/2-3)g - \frac{3g^3}{256\pi^3} + O(g^5) \\ &\to -\frac{3g^3}{256\pi^3} + O(g^5) \text{ at } d = 6. \end{aligned} \tag{7.3.7}$$

Similarly

$$0 = \mu \frac{\mathrm{d}m_0^2}{\mathrm{d}\mu}$$

$$= -m^2\gamma_m[1 + O(g^2)] + m^2\beta(g)\left[\frac{5g}{192\pi^3(d-6)} + O(g^3)\right],$$

so that

$$\gamma_m = \frac{5}{6}\frac{g^2}{64\pi^3} + O(g^4). \tag{7.3.8}$$

Observe that the $(d/2 - 3)g$ term in β is important in these derivations, even though the term disappears at $d = 6$. Observe also that, even though the coefficients in g_0 and m_0 diverge at $d = 6$, it is crucial to expand strictly in powers of g. A phenomenon true to all orders is that β and γ_m are independent of m and μ, provided that we renormalize by minimal subtraction.

The general calculation of β and γ_m can be organized with the aid of (7.1.4) for the counterterms. Since we use minimal subtraction, the m- and μ-dependence of the bare parameters is simple:

$$g_0 = \mu^{3-d/2}\bar{g}_0(g, d),$$
$$m_0^2 = m^2 Z_m(g, d). \tag{7.3.9}$$

Then

$$\beta(g, d) = (d/2 - 3)\bar{g}_0 \Big/ \frac{\partial \bar{g}_0}{\partial g},$$

$$\gamma_m(g) = \beta(g, d)\frac{\partial \ln Z_m}{\partial g}. \tag{7.3.10}$$

The expressions (7.3.9) are to be expanded in powers of g with the aid of (7.1.4). Now

$$g_0 = (\mu^{3-d/2}g + \delta g)Z^{-3/2}$$
$$= \mu^{3-d/2}\left[g + \frac{a_1(g) - \frac{3}{2}gc_1(g)}{6-d} + \text{higher poles}\right],$$

$$Z_m = (1 + \delta m^2/m^2)Z^{-1}$$
$$= \left[1 + \frac{b_1(g) - c_1(g)}{6-d} + \text{higher poles}\right], \tag{7.3.11}$$

where we have picked out the single poles in the series expansion of the counterterms (7.1.4). (These are all that will be relevant.) Then (7.3.10)

becomes

$$\beta(g,d)=\left(\frac{d}{2}-3\right)\frac{\left[g+\dfrac{a_1(g)-\frac{3}{2}gc_1(g)}{6-d}+\text{higher poles}\right]}{\left[1+\dfrac{a_1'(g)-\frac{3}{2}gc_1'(g)-\frac{3}{2}c_1(g)}{6-d}+\text{higher poles}\right]}$$

$$=(d/2-3)g+\tfrac{1}{2}\left(1-g\frac{\partial}{\partial g}\right)\left[\tfrac{3}{2}gc_1(g)-a_1(g)\right]+\text{poles (that cancel)}$$

$$\equiv(d/2-3)g+\bar{\beta}(g), \tag{7.3.12a}$$

$$\gamma_m(g)=\left[(d/2-3)g+\bar{\beta}(g)\right]\frac{\partial}{\partial g}\left[\frac{b_1(g)-c_1(g)}{6-d}+\text{higher poles}\right]$$

$$=\tfrac{1}{2}g\frac{\partial}{\partial g}\left[c_1(g)-b_1(g)\right]. \tag{7.3.12b}$$

These manipulations are made by expanding in powers of g.

In the last line of each of (7.3.12) we have used the fact that although pole terms are in principle present, they must cancel in order that β and γ_m be finite as $d\to 6$. Notice that β and γ_m are independent of m and μ, and that the only d-dependence is the $(d/2-3)g$ term in β. Only the single-pole terms are needed for the calculation. There is a series of relations between these and the higher poles that ensures that the poles cancel in (7.3.12); we will investigate these later.

7.3.2 *RG equation*

The RG coefficients β and γ_m are computed from two out of three combinations of the counterterms. The differential equations (7.3.4) then enable g' and m' in (7.3.1) to be computed. To complete the calculation we need ζ. This is related to the wave-function renormalization. It is easiest to obtain by observing that the bare Green's functions $G_N^{(0)}=Z^{N/2}G_N$ are renormalization-group invariant, so that

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_N=\mu\frac{\mathrm{d}}{\mathrm{d}\mu}(G_N^{(0)}Z^{-N/2})$$

$$=-\frac{N}{2}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}(\ln Z)G_N. \tag{7.3.13}$$

Let us define the finite coefficient

$$\gamma=\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z; \tag{7.3.14}$$

then G_N satisfies the following renormalization group equation (e.g. Weinberg (1973)):

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu}+\frac{N}{2}\gamma\right]G_N=\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}-\gamma_m m^2\frac{\partial}{\partial m^2}+\frac{N}{2}\gamma\right]G_N=0. \quad (7.3.15)$$

In the minimal subtraction scheme for ϕ^3 we find

$$\begin{aligned}\gamma&=[(d/2-3)g+\bar{\beta}(g)]\frac{\partial}{\partial g}\left[\frac{c_1(g)}{6-d}+\text{higher poles}\right]\\&=-\tfrac{1}{2}g\frac{\mathrm{d}}{\mathrm{d}g}c_1(g)\\&=\frac{g^2}{384\pi^3}+O(g^4).\end{aligned} \quad (7.3.16)$$

7.3.3 *Solution*

We wish to solve the RG equations to find $g(\mu')$, $m(\mu')$ and $\zeta(\mu',\mu)$ in (7.3.1), given that $g(\mu)=g$ and $m(\mu)=m$. The RG equation tells us that

$$\begin{aligned}\mu'\frac{\mathrm{d}}{\mathrm{d}\mu'}\ln\zeta&=\frac{1}{N}\mu'\frac{\mathrm{d}}{\mathrm{d}\mu'}\ln[G_N(\mu')/G_N(\mu)]\\&=-\tfrac{1}{2}\gamma[g(\mu')].\end{aligned} \quad (7.3.17)$$

So we must solve this equation together with (7.3.4) for g and m. The boundary conditions are

$$\begin{aligned}g(\mu)&=g,\\m(\mu)&=m,\\\zeta(\mu,\mu)&=1.\end{aligned} \quad (7.3.18)$$

Explicit solutions can be written:

$$\begin{aligned}\ln(\mu'/\mu)&=\int_{g(\mu)}^{g(\mu')}\frac{\mathrm{d}g'}{\beta(g')},\\m^2(\mu')&=m^2(\mu)\exp\left\{-\int_\mu^{\mu'}\frac{\mathrm{d}\bar{\mu}}{\bar{\mu}}\gamma_m(g(\bar{\mu}))\right\}\\&=m^2(\mu)\exp\left\{-\int_{g(\mu)}^{g(\mu')}\mathrm{d}g'\frac{\gamma_m(g')}{\beta(g')}\right\},\\\zeta(\mu',\mu)&=\exp\left\{-\tfrac{1}{2}\int_\mu^{\mu'}\frac{\mathrm{d}\bar{\mu}}{\bar{\mu}}\gamma(g(\bar{\mu}))\right\}\\&=\exp\left\{-\tfrac{1}{2}\int_{g(\mu)}^{g(\mu')}\mathrm{d}g'\frac{\gamma(g')}{\beta(g')}\right\}.\end{aligned} \quad (7.3.19)$$

Approximations can be made by taking a finite number of terms in the perturbation series for β, γ_m and γ. For example:

$$\ln(\mu'/\mu) = -\frac{256\pi^3}{3}\int_{g(\mu)}^{g(\mu')}\frac{dg}{g^3}[1+O(g^2)]$$
$$= \frac{128\pi^3}{3}\left[\frac{1}{g(\mu')^2}-\frac{1}{g(\mu)^2}\right]+O\left[\ln\left(\frac{g(\mu')}{g(\mu)}\right)\right]. \quad (7.3.20)$$

This is accurate if $g(\mu')$ and $g(\mu)$ are small. Notice that $g(\mu')\to 0$ as $\mu'\to\infty$. This is the property called asymptotic freedom. It is determined by the negative sign of the first term in β at $d=6$.

The full solution to the RG equation is

$$G_N(x;g,m,\mu) = \exp\left[-\frac{N}{2}\int_{\mu'}^{\mu}\frac{d\bar{\mu}}{\bar{\mu}}\gamma(g(\bar{\mu}))\right]G_N(x;g(\mu'),m(\mu'),\mu'). \quad (7.3.21)$$

7.4 Large-momentum behavior of Green's functions

The most important application of the renormalization group is to compute large-momentum behavior. In this section we treat the simplest case, that of a Green's function $\tilde{G}_N(p_1,\ldots,p_N)$ all of whose external momenta are made large. (Notice that we have used our standard notation, where the tilde indicates Fourier transformation into momentum space.)

Let us suppose initially that all the Lorentz invariants formed from the momenta are non-zero. Then we scale all the momenta by a factor κ: $p_i\to\kappa p_i$, and let κ get large. Thus all the Lorentz invariants $p_i\cdot p_j$ are scaled by a factor κ^2 and become large. Under these conditions, Weinberg's theorem tells us that at least in a renormalizable theory all internal lines of graphs for $\tilde{G}_N$ carry large momenta, and that graphs for $\tilde{G}_N$ have the asymptotic behavior

$$\kappa^{\dim \tilde{G}}(\text{logarithms of } \kappa). \quad (7.4.1)$$

Since all propagator denominators are large, we should be able to neglect masses and make only an error a power of κ smaller than the leading behavior (7.4.1).

For example consider the propagator in ϕ^3 theory at $d=6$. The tree graph goes like i/p^2 at large p^2, while the one-loop correction is

$$\frac{i}{p^2}\frac{g^2}{128\pi^3}\left\{\tfrac{1}{6}(\gamma-1)+\int_0^1 dx\, x(1-x)\ln\left[\frac{-p^2x(1-x)}{4\pi\mu^2}\right]\right.$$
$$\left.+O\left[\frac{m^2}{p^2}\ln\left(\frac{m^2}{p^2}\right)\right]\right\}. \quad (7.4.2)$$

(We used (3.6.10) for the self-energy graph to derive this equation.)

Now there is a term proportional to $\ln(-p^2/\mu^2)/p^2$ that gets large relative to the tree graph if p^2 is large enough. Thus the perturbation series has large coefficients and is not directly useful. However we may use the renormalization group to set $\mu^2 = O(p^2)$. This makes the coefficient small again. So we use the following strategy to compute $\tilde{G}_N(\kappa p)$ at large κ:

(1) Set $\mu' = \kappa\mu$, and use the solution of the RG equation to write

$$\tilde{G}_N(\kappa p, g, m, \mu) = \zeta(\kappa\mu, \mu)^{-N}\tilde{G}_N(\kappa p, g(\kappa\mu), m(\kappa\mu), \kappa\mu). \tag{7.4.3}$$

(2) Neglect m (if $m(\kappa\mu)$ does not get too large). Then use dimensional analysis to give

$$\begin{aligned}\tilde{G}_N(\kappa p, g, m, \mu) &\sim \zeta^{-N}\tilde{G}_N(\kappa p, g(\kappa\mu), 0, \kappa\mu) \\ &= \kappa^{\dim(\tilde{G}_N)}\zeta(\kappa\mu, \mu)^{-N}\tilde{G}_N(p, g(\kappa\mu), 0, \mu).\end{aligned} \tag{7.4.4}$$

(3) Large κ-dependent coefficients, as in (7.4.2), are now removed, so if $g(\kappa\mu)$ is small, a low-order calculation suffices.

This procedure makes it evident that the coupling that is relevant is the effective coupling at the scale of the momenta involved.

It should be noted that we have related the large-momentum behavior of $\tilde{G}_N$ to the finite-momentum behavior of the zero-mass theory. It is therefore crucial that the zero-mass limit exists. However this limit does not always exist: if we use mass-shell or zero-momentum subtractions, then we see from, for example, (3.4.7), that the same self-energy as considered in (7.4.2) diverges as $m \to 0$. Now Weinberg's theorem tells us that, in the dominant momentum region for a graph without counterterms, all lines are far off-shell; hence masses can be neglected. So the problem must be that with mass-shell or zero-momentum renormalization prescriptions, the counterterms diverge as $m \to 0$. This is easily checked from our explicit calculations (see (5.10.2)).

We can now see the practical importance of the theorem whose proof was summarized at the end of Section 5.8, that the counterterms may be *chosen* polynomial in masses. It ensures that the zero-mass limit may be taken directly and used to compute large momentum behavior. The minimal subtraction scheme is one way of ensuring that counterterms are polynomial in mass.

If one uses, say, zero-momentum subtractions, large-momentum behavior may be computed by changing renormalization prescription to, say, minimal subtraction. Another approach is to observe that the logarithms of p^2 break a possible symmetry of the theory under scaling transformations. The consequences of this point of view were worked out by Callan (1970)

and Symanzik (1970b). They derived the Ward identity for scaling transformations. It is called the Callan–Symanzik equation and looks similar to the RG equation. This equation may also be used to discuss high-energy behavior.

7.4.1 *Generalizations*

The behavior of $\tilde{G}_N$ when all momenta are scaled by a large factor κ is not normally experimentally relevant, for all the external momenta are then far off-shell. In coordinate space the corresponding region is the short-distance limit of $G_N(x_1, \ldots x_N)$, where every $x_j - x_k$ is made small: $x_j - x_k \to (x_j - x_k)/\kappa$.

This means that we should be able to use RG methods to discuss the renormalization of the theory, for renormalization is a purely short-distance phenomenon. We will work out the details in Section 7.10. On the other hand, physical experiments involve long distances. To get results for high-energy experiments we need the so-called factorization theorems. The simplest of these is the operator-product expansion which we will treat in Chapter 10. These theorems typically give a cross-section as a product of a factor which can be computed by pure short-distance methods and of simple factors related to wave-functions of the incoming and/or outgoing particles.

We could also use RG methods to discuss the infra-red limit $\kappa \to 0$. This is only useful if masses can be neglected. Certainly this is true in a purely massless theory, as we will see in Section 7.5.4, and then IR behavior is computable if and only if the theory is not asymptotically free. But in a massive theory, it is not useful to take μ much less than a typical mass, for one obtains logarithms of m/μ, and these prevent a simple use of perturbative methods when $\mu \ll m$.

7.5 Varieties of high- and low-energy behavior

7.5.1 *Asymptotic freedom*

In solving the RG equation to obtain high-energy behavior, we find two cases according to whether β is positive or negative. In this section we discuss the asymptotically free case, when β is negative. Suppose the effective coupling $g(\mu)$ is small for one value of μ, so that β is well approximated by its first term. Then the evolution equation (7.3.4*a*) shows that g becomes even smaller at larger values of μ and in fact goes to zero as $\mu \to \infty$. Thus perturbation theory is reliable for computing high-energy

behavior. From (7.3.7) we see that ϕ^3 in six dimensions is asymptotically free.

It is instructive to compute the behavior of g from the first two terms in β. Let us define

$$\beta = -A_1 g^3 - A_2 g^5 + O(g^7). \tag{7.5.1}$$

We have the equation for the evolution of the effective coupling:

$$\mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = \beta(g). \tag{7.5.2}$$

Equations of the same form as (7.5.1) and (7.5.2) hold in any renormalizable theory, for example in QCD, though A_1 and A_2 are not necessarily positive in the general case.

The solution of (7.5.2) is

$$\begin{aligned} \ln \mu &= \text{constant} + \int^{g(\mu)} \mathrm{d}g'/\beta(g') \\ &= \text{constant} + \int^{g(\mu)} \mathrm{d}g' \left[\frac{-1}{A_1 g'^3} + \frac{A_2}{A_1^2 g'} + O(g') \right], \\ &= \text{constant} + \frac{1}{2A_1 g(\mu)^2} + \frac{A_2}{A_1^2} \ln [g(\mu)] + O(g^2). \end{aligned} \tag{7.5.3}$$

The constant can be computed from a knowledge of $g(\mu)$ at one value of μ. It is conventional (Buras, Floratos, Ross & Sachrajda (1977)) to write the constant in the form $\ln \Lambda + \frac{1}{2} A_2 A_1^{-2} \ln(A_1)$, where Λ is a parameter with the dimensions of mass. Then

$$\ln(\mu^2/\Lambda^2) = \frac{1}{A_1 g(\mu)^2} + \frac{A_2}{A_1^2} \ln [A_1 g(\mu)^2] + O(g(\mu)^2), \tag{7.5.4}$$

so that

$$g(\mu)^2 = \frac{1}{A_1 \ln(\mu^2/\Lambda^2)} - \frac{A_2 \ln[\ln(\mu^2/\Lambda^2)]}{A_1^3 \ln^2(\mu^2/\Lambda^2)} + O\left\{ \frac{\ln^2[\ln(\mu/\Lambda)]}{\ln^3(\mu/\Lambda)} \right\}. \tag{7.5.5}$$

A specification of $g(\mu)$ at one value of μ is exactly equivalent to a specification of Λ. The precise choice of the scale of Λ is that in (7.5.5) the omitted terms are of the order shown rather than of order $1/\ln^2(\mu/\Lambda)$. The expansion (7.5.5) is much used in QCD. A higher-order calculation of g can be made from the following form of the solution

$$\begin{aligned} \ln(\mu^2/\Lambda^2) = &\frac{1}{A_1 g(\mu)^2} + \frac{A_2}{A_1^2} \ln [A_1 g(\mu)^2] \\ &+ 2 \int_0^{g(\mu)} \mathrm{d}g' \left[\frac{1}{\beta(g')} + \frac{1}{A_1 g'^3} - \frac{A_2}{A_1^2 g'} \right]. \end{aligned} \tag{7.5.6}$$

It is necessary to go to two-loop order to obtain both the terms on the right of (7.5.6) that diverge as $g \to 0$.

The values of $m(\mu)$ and ζ may be similarly calculated. For example, if

$$\gamma = C_1 g^2 + \cdots, \tag{7.5.7}$$

then

$$\begin{aligned}\zeta(\mu', \mu) &= \exp\left[\frac{1}{2}\int_{g(\mu)}^{g(\mu')} \frac{dg}{g}\frac{C_1}{A_1} + O(g^2)\right] \\ &= \exp\left[\frac{C_1}{2A_1}\ln\frac{g(\mu')}{g(\mu)} + O(g^2)\right] \\ &= \left[\frac{g(\mu')}{g(\mu)}\right]^{C_1/2A_1}[1 + O(g^2)] \\ &\propto [\ln(\mu'/\mu)]^{-C_1/4A_1}, \quad \text{as } \mu' \to \infty. \end{aligned} \tag{7.5.8}$$

7.5.2 *Maximum accuracy in an asymptotically free theory*

The results above enable calculations of Green's functions to be made at high energy. By taking more and more terms in the series, we may improve the predictions. However, in general, perturbation series are asymptotic, not convergent. A trivial example is the ordinary integral

$$I(g, m) = m(2\pi)^{-1/2}\int_{-\infty}^{\infty} dz \exp(-m^2 z^2/2 - gz^4/4!)$$

This can be considered to be a functional integral in Euclidean ϕ^4 field theory at zero space-time dimension with the normalization chosen to give $I = 1$ when $g = 0$. The perturbation expansion is

$$\begin{aligned} I &\sim m(2\pi)^{-1/2}\sum_{N=0}^{\infty}\int_{-\infty}^{\infty} dz \exp(-m^2 z^2/2)\left(\frac{-gz^4}{4!}\right)^N \frac{1}{N!} \\ &= \sum_{N=0}^{\infty}\left(\frac{-g}{6m^4}\right)^N \pi^{-1/2}\frac{\Gamma(2N + 1/2)}{\Gamma(N+1)} \\ &\equiv \sum_{N=0}^{\infty}\left(\frac{g}{m^4}\right)^N I_N. \end{aligned} \tag{7.5.9}$$

Now

$$I_N = N^N\left(\frac{-2}{3e}\right)^N (\pi N)^{-1/2}[1 + O(1/N)], \quad \text{as } N \to \infty, \tag{7.5.10}$$

so the series is divergent. The divergence is associated with the fact that the defining integral diverges if g is negative, so that I is not analytic at $g = 0$. The corresponding property of ϕ^4 theory is that the Hamiltonian is unbounded below (i.e., the vacuum does not exist) if g is negative.

What we can say is that we can approximate I by truncating the series:

$$I = \sum_{N=0}^{N_m} (g/m^4)^N I_N + O[(g/m^4)^{N+1}]. \tag{7.5.11}$$

The error is estimated by the first term omitted, i.e.,

$$O[I_{N_m+1}(g/m^4)^{N_m+1}].$$

These results are standard in the theory of asymptotic expansions for simple integrals. All experience, together with rigorous theorems for quantum mechanics and super-renormalizable field theories (Glimm & Jaffe (1981) and references therein), indicate that this behavior is typical for functional integrals in a non-trivial dimension (i.e., $d > 0$).

Next, let us suppose we wish to compute some quantity in an asymptotically free theory by truncating its perturbation expansion

$$\sum_{N=0}^{N_m} g(\mu)^{2N} I_N.$$

We assume that the quantity depends on some momentum p, and that we set the unit of mass μ to be of order p. A case would be the propagator with $\mu = O(|p|)$. Suppose that the coefficients in the expansion behave like

$$I_N \sim N^N b^N N^a d[1 + O(1/N)], \tag{7.5.12}$$

for large N. What is the best accuracy with which we can calculate the quantity? This is given by the minimum error, i.e., the minimum of $I_N g^{2N}$ as N varies. The result is that the minimum possible error in a perturbative calculation is of order

$$\text{constant}\,|p|^{-2A_1/eb}(\ln|p|)^a.$$

This means that beyond a certain level, power-law corrections to the asymptotic behavior computed in perturbation theory are meaningless since they are smaller than the irreducible error in using perturbation theory. Power-law corrections are those that are a power of p^2 smaller than the leading term.

7.5.3 *Fixed point theories*

In four dimensions, the only theories that are asymptotically free are non-abelian gauge theories with a small enough number of matter fields – see Coleman & Gross (1973) and Gross (1976). Other theories, like ϕ^4 and QED, have an effective coupling that increases with energy. Thus, in such theories it is impossible to compute the true high-energy behavior by perturbation theory. (Note however that the coupling in QED is α/π

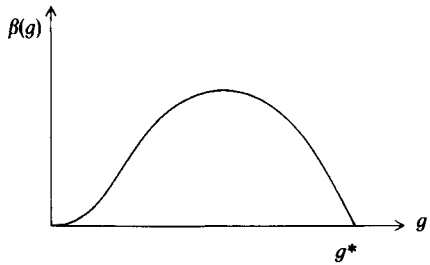

Fig. 7.5.1. $\beta(g)$ in a non-asymptotically free theory with a fixed point at $g = g^*$.

$\sim 1/430$. This is very small, so the non-perturbative region in QED does not occur until very many orders of magnitude beyond experimentally accessible energies.)

An interesting possibility is that $\beta(g)$ has the form shown in Fig. 7.5.1, with a zero at $g = g^*$. Then $g(\mu)$ approaches the 'fixed point' g^* as $\mu \to \infty$. At large momentum Green's functions behave like

$$\tilde{G}_N(\kappa p_1, \ldots, \kappa p_N, g, m, \mu) \sim \text{const}\ \kappa^{\dim \tilde{G}_N + N\gamma(g^*)/2} \tilde{G}_N(p_1, \ldots, p_N, g^*, 0, \mu). \tag{7.5.13}$$

This behavior is as if ϕ had an extra term $\gamma(g^*)/2$ in its dimension. Consequently, the function $\gamma(g)/2$ is called the anomalous dimension of the field ϕ.

7.5.4 *Low-energy behavior of massless theory*

If $m = 0$, then the renormalization group can be used to compute infra-red behavior. The calculability is the opposite of that for the UV behavior. Consider first an asymptotically free theory. There, the effective coupling $g(\mu)$ goes to zero when μ goes to infinity, so that short-distance behavior is computable perturbatively, as we saw in Section 7.5.1. But, when μ is small, $g(\mu)$ is large, so the infra-red behavior cannot be computed reliably by perturbation theory. (This is the case for strong interactions, according to QCD.)

Let us now consider a non-asymptotically free theory. For large μ, the effective coupling is large, so the short-distance behavior is not perturbatively computable. (For example, a perturbative calculation in low order of the position of the fixed point, g^*, in Fig. 7.5.1 and of the value of $\beta(g^*)$ is subject to large errors from higher-order uncalculated corrections.) But when μ goes to zero, so does the effective coupling. We are assuming the absence of a mass term for the field, so there are no large logarithms of m/μ as μ goes to zero. Hence, we can compute IR behavior in such a theory, just

as we computed UV behavior in an asymptotically free theory. We will now do this.

Now, for almost any graph there are large logarithms of p^2/μ^2 as $p^2 \to 0$ in a massless theory, just as in the ultra-violet. In the case of the propagator, these logarithms mean that the propagator's singularity is not a pole, at least order-by-order.

To investigate this singularity let us again write

$$\gamma(g) = C_1 g^2, \quad \beta(g) = -A_1 g^3, \tag{7.5.14}$$

using the same notation as before, but now with $A_1 < 0$. We assume that g is below the first non-zero fixed point g^*, if there is one. The propagator is

$$\begin{aligned}\tilde{G}_2(\kappa p; g(\mu), \mu) &= \kappa^{-2} \tilde{G}_2(p; g(\kappa\mu), \mu) \exp \int_{g(\mu)}^{g(\kappa\mu)} \mathrm{d}g\, \gamma(g)/\beta(g) \\ &= \mathrm{i}/(\kappa^2 p^2)[1 + O(g(\kappa\mu)^2)] \exp\left\{ - \int \mathrm{d}g (C_1/A_1 g)[1 + O(g^2)] \right\} \\ &\sim \mathrm{i}/(\kappa^2 p^2) \cdot \text{constant} \cdot [\ln(1/\kappa)]^{C_1/2A_1},\end{aligned} \tag{7.5.15}$$

as $\kappa \to 0$. Hence if C_1 is non-zero, then $p^2 \tilde{G}_2(p)$ does not have a finite non-zero limit as $p \to 0$; the singularity of $\tilde{G}_2(p^2)$ at $p^2 = 0$ does not correspond to a *simple* single-particle pole. The massless particle that gives rise to the singularity cannot be treated as an ordinary particle, because its long-range interactions are too strong. Positivity of the metric of the state vectors constrains C_1 to be positive, and we assumed a theory with A_1 negative, so κ^2 times the right-hand side of (7.5.15) goes to zero. (The positivity argument is the one given in the textbooks (e.g., Bjorken & Drell (1966)) that the residue of the pole in a propagator is less than unity if the propagator is of the canonical field. Application of this argument in the theory with an ultra-violet cut-off shows that the divergence of the self-energy must be such that C_1 is positive.)

Notice that if $C_1 = 0$, then the propagator does have a finite residue at $p = 0$:

$$-\mathrm{i}\kappa^2 p^2 \tilde{G}_2(\kappa p, g(\mu), \mu) \to \exp\left\{ - \int_0^{g(\mu)} \mathrm{d}g\, \gamma(g)/\beta(g) \right\}. \tag{7.5.16}$$

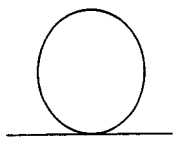

Fig. 7.5.2. Lowest-order self-energy graph in ϕ^4 theory.

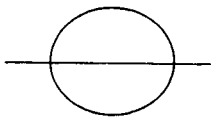

Fig. 7.5.3. Lowest-order self-energy graph that contributes to the anomalous dimension in ϕ^4 theory.

Observe that only the first term in γ is relevant to the finiteness of the limit, and that the limit does not exist order-by-order. A case where $C_1 = 0$ is the ϕ^4 theory, because the one-loop graph Fig. 7.5.2 is independent of p. The two-loop graph Fig. 7.5.3 provides the lowest-order term in the anomalous dimension.

7.6 Leading logarithms, etc.

7.6.1 *Renormalization-group logarithms*

We saw in Section 7.4 how to compute the large momentum behavior of a Green's function by approximating it by a Green's function with $m = 0$. Then we used the renormalization group to reorganize the perturbation series into a form with small coefficients. It is of interest to examine how the complete result can be obtained by a systematic resummation of the perturbation expansion.

For concreteness, let us examine the propagator $\tilde{G}_2(p, g, \mu)$ in massless ϕ^3 theory in six dimensions. We write its perturbation expansion as

$$\tilde{G}_2 = (\mathrm{i}/p^2) \sum_{n=0}^{\infty} g^{2n} T_n(-p^2/\mu^2), \tag{7.6.1}$$

where the lowest coefficient is $T_0 = 1$. We will prove that each T_n is a polynomial in $\ln \mu$ (and hence in $\ln(-p^2/\mu^2)$) of degree at most n, with n being the number of loops. To do this we will regard the RG equation (7.3.15) not as an equation to give the variation of $\tilde{G}_2$ when μ is changed with g set equal to the effective coupling $g(\mu)$ (thus keeping the theory fixed), but as an equation for the μ-dependence of $\tilde{G}_2$ with g fixed. Picking out the order g^{2n} term gives

$$\frac{\partial}{\partial \ln \mu} T_n = \left\{ \left[-\gamma(g) - \beta(g) \frac{\partial}{\partial g} \right] \sum_{n'=0}^{n-1} T_{n'} g^{2n'} \right\}_{\text{coefficient of } g^{2n}}. \tag{7.6.2}$$

Since γ is $O(g^2)$ and β is $O(g^3)$, this equation determines T_n in terms of lower-order $T_{n'}$'s:

$$T_n = \text{constant} + \left\{ \sum_{n'<n} - \left[\gamma(g) + \beta(g) \frac{\partial}{\partial g} \right] g^{2n'} \int_0^{\ln \mu} \mathrm{d} \ln \mu' \, T_{n'}(-p^2/\mu'^2) \right\}_{\text{coefficient of } g^{2n}}. \tag{7.6.3}$$

Iteration of this procedure another $n-1$ times gives T_n in terms of T_0 and n constants of integration. Evidently T_n is a polynomial of degree n in $\ln \mu$ as

claimed:

$$T_n = \sum_{l=0}^{n} T_{n,n-l}[\ln(-p^2/\mu^2)]^l. \tag{7.6.4}$$

All but the constant term are determined in terms of lower-order coefficients.

A convenient way of organizing the series is to define for each term

$$L = -\text{number of logarithms} + \text{number of powers of } g^2 = -l + n. \tag{7.6.5}$$

This is non-negative. The sum of the terms with $L=0$ (viz., $T_{n0}[\ln(-p^2/\mu^2)]^n$) gives what is called the leading logarithm approximation to $\tilde{G}_2$. Application of $\mu\partial/\partial\mu$, $\beta\partial/\partial g$ or γ to $\tilde{G}_2$ (with one-loop values for β and γ) increases L by 1. All the non-leading logarithms give even higher values of L. So the leading logarithm series exactly satisfies the one-loop approximation to the RG equation. Hence we may sum the leading logarithm series by solving this approximation to the RG equation

$$\begin{aligned}\tilde{G}_2(p^2;g,\mu)_{L=0} &= \left\{\exp\left[\int_{\mu}^{\sqrt{-p^2}} \frac{d\mu'}{\mu'}\gamma(g(\mu'))\right]\tilde{G}_2(p^2;g(\sqrt{-p^2}),\sqrt{-p^2})\right\}_{L=0} \\ &= [1 + A_1 g^2 \ln(-p^2/\mu^2)]^{C_1/2A_1} i/p^2. \end{aligned} \tag{7.6.6}$$

Here we used the same notation as in Section 7.5.1. This equation reproduces the approximation derived at (7.5.8).

Another way of treating both the leading and non-leading logarithms is to use the RG equation (7.6.2) to give a recursion relation for the $T_{n,L}$'s:

$$\begin{aligned}&\sum_{L=0}^{n-1} 2(L-n)T_{n,L}[\ln(-p^2/\mu^2)]^{n-L-1} \\ &\quad = [2A_1(n-1) - C_1]\sum_{L=0}^{n-1} T_{n-1,L}[\ln(-p^2/\mu^2)]^{n-1-L} \\ &\qquad + [2A_2(n-2) - C_2]\sum_{L=0}^{n-2} T_{n-2,L}[\ln(-p^2/\mu^2)]^{n-2-L} + \cdots, \end{aligned} \tag{7.6.7}$$

where $\gamma = C_1 g^2 + C_2 g^4 + \cdots$, and $\beta = -A_1 g^3 - A_2 g^5 + \cdots$.

The leading logarithm part of this equation is

$$-2nT_{n,0} = [2A_1(n-1) - C_1]T_{n-1,0}. \tag{7.6.8}$$

This equation determines the leading logarithm series in terms of $T_{00} = 1$, and of A_1 and C_1; this series sums to (7.6.6).

Equation (7.6.7) also determines the non-leading logarithms. For example the next-to-leading terms are

$$-2(n-1)T_{n,1} = [2A_1(n-1) - C_1]T_{n-1,1} + [2A_2(n-2) - C_2]T_{n-2,0}. \tag{7.6.9}$$

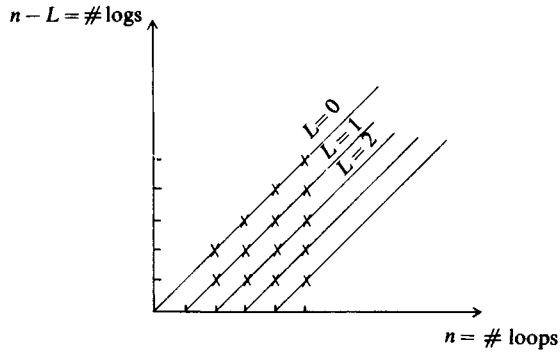

Fig. 7.6.1. Illustrating the leading logarithms and non-leading logarithms of a Green's function.

Again a convergent series results. Its sum is equally accurate as the result of using the two-loop approximation to β and γ in the solution (7.3.21). There μ is set equal to $(-p^2)^{1/2}$ and we take the one-loop approximation to $\tilde{G}_2(p; g(\sqrt{-p^2}), m=0, \sqrt{-p^2})$, i.e., $(i/p^2)(1+g^2 T_{11})$.

The series for larger L may be similarly determined. In Fig. 7.6.1 we illustrate the structure of the calculations. The diagonal lines are lines of constant L, and the recursion relation (7.6.7) determines a coefficient $T_{n,L}$ in terms of lower-order terms on its diagonal and on the higher diagonals.

Suppose we have computed perturbation theory to $n-1$ loops for $\tilde{G}_2$ and wish to compute the n-loop term. In this term the coefficients of all but $\ln(-p^2/\mu^2)$ and the constant are fixed by the lower-order calculations. Thus the new information is in the nth order coefficient C_n for γ and in the terms with one and no logarithms, i.e., in $T_{n,n-1}$ and $T_{n,n}$. The $(\ln)^0$ term in (7.6.7) is

$$-2T_{n,n-1} = -C_n + \sum_{j=1}^{n-1} [2A_{n-j}j - C_{n-j}] T_{jj}. \tag{7.6.10}$$

This shows that knowing the coefficient, $T_{n,n-1}$, of the singly logarithmic term in T_n is equivalent to knowing the n-loop coefficient C_n in $\gamma(g)$.

Exactly the same procedure may be applied to any Green's function $\tilde{G}_N$. The only difference is that there are several external momenta. It is enough to consider the connected graphs. Let us write

$$\tilde{G}_N(\text{connected}) = (p^2)^{\dim(G)/2} g^{N-2} \sum_{n=0}^{\infty} \sum_{L=0}^{n} G_{n,L}^{(N)} \ln^{n-L}(-p^2/\mu^2) g^{2n}, \tag{7.6.11}$$

where p is one of the external momenta and we have factored out g^{N-2}, which is the power of g appearing in the tree approximation. The

coefficients $G_{n,L}^{(N)}$ are now functions of the dimensionless ratios of the Lorentz invariants formed from the external momenta.

The leading logarithm series and all the non-leading series are convergent, so they can be summed. The $n!$ behavior of large orders only appears when we consider the single log and constant terms, and thus in the sum over L.

7.6.2 *Non-renormalization-group logarithms*

In all of the above cases there was one logarithm of the large momentum per loop. There are more complicated situations where not all invariants get large. A simple standard example is the form-factor of the electron in QED with a massive photon (Fig. 7.6.2). Here $q^2 = (p_1 - p_2)^2$ gets large but p_1^2 and p_2^2 are fixed. It turns out (Sudakov (1956), Jackiw (1968)) that there are *two* logarithms per loop. These must be in the coefficients $G_{n,L}^{(N)}$ in (7.6.11), since the power of the logarithms is too high for them to be the explicit logarithms in (7.6.11).

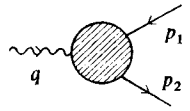

Fig. 7.6.2. The electron's form factor in QED.

One would like to find the large-momentum behavior in such situations. A much-used technique is to sum the leading logarithms, which are often relatively easy to compute. For the on-shell form-factor this gives a convergent series which sums to (Jackiw (1968))

$$F \sim \exp\left[-(e^2/16\pi^2)\ln^2 q^2\right]. \tag{7.6.12}$$

(See also Mueller (1981). For the simple cases that we considered earlier, with all momenta large, the leading logarithm approximation is justified by renormalization-group methods, as we have seen. For cases like the present one of the form-factor, it may be a bad approximation (Collins & Soper (1981, 1982b)). However methods are available to obtain large-momentum behavior in some of these situations. See Mueller (1979, 1981) and Collins (1980) for the electron form-factor and Collins & Soper (1981) for cases in strong interactions.

7.6.3 *Landau ghost*

The leading logarithmic approximation (7.6.6) for the propagator has a

singularity when

$$p^2 = -\mu^2 \exp[-1/(A_1 g^2)]. \tag{7.6.13}$$

This singularity, if present in the true propagator, would signal a state of this value of mass squared. Since the residue has the opposite sign to that for a normal propagator pole, this would be a state with unphysical properties. It is called the Landau ghost (Landau & Pomeranchuk (1955)). In a non-asymptotically free theory like QED, it occurs at very large energies and in an asymptotically free theory like QCD, it occurs at low energies. In either case it occurs where perturbation theory is inapplicable and so where the leading logarithmic approximation is a bad approximation.

7.7 Other theories

We restricted our attention in deriving the renormalization group equation to a theory with one field, one coupling, and one mass parameter. However we may treat, by exactly the same method, a theory with several fields $\phi_1, \ldots, \phi_A$ (each may be Bose or Fermi), several couplings $g_1, \ldots, g_B$, and several masses. A change in the unit of mass μ is compensated by a change in each of the parameters and in the scale of the fields. The main problem is a proliferation of indices. It is easiest to treat couplings and masses on the same footing. So we have a collection $g_1, \ldots, g_C$ of renormalized parameters, with C being the total number of couplings and masses. Then we must write

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_j(\mu) = \beta_j(\mathbf{g}), \tag{7.7.1}$$

each function β_j being, *a priori*, a function of all the parameters. For the case of a theory with a single coupling, and a single mass, would have $(g_1, g_2) = (g, m^2)$, and $\beta_1 = \beta(g)$ and $\beta_2 = -\gamma_m(g)m^2$.

Given a function f of the renormalized parameters and of μ, we have

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} f(\mathbf{g}, \mu) = \left(\mu \frac{\partial}{\partial \mu} + \sum_j \beta_j \frac{\partial}{\partial g_j} \right) f. \tag{7.7.2}$$

The RG coefficients can be determined by noting that the bare couplings $g_{j(0)}$ are invariant:

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_{j(0)}(\mathbf{g}, \mu, d) = 0. \tag{7.7.3}$$

These form C equations for C unknowns.

The RG equations for Green's functions are complicated by the possibility that fields of the same quantum numbers may mix under

renormalization. Writing

$$\phi_{(0)i} = \sum_j \zeta_{ij}(\mathbf{g}, d)\phi_j,$$

we find, for example, that 'kinetic energy' terms in $\mathscr{L}$ are of the form

$$\tfrac{1}{2}\sum_i \partial\phi_{(0)i}^2 = \tfrac{1}{2}\sum_{j,l} \partial\phi_j \partial\phi_l \sum_i \zeta_{ij}\zeta_{il} = \tfrac{1}{2}\sum_{j,l} \partial\phi_j \partial\phi_l Z_{jl}.$$

Hence, we have a matrix counterterm for the field-strength renormalization

$$Z_{ij}(\mathbf{g}, d) = \sum_l \zeta_{li}\zeta_{lj} = (\zeta^{\mathrm{T}}\zeta)_{ij}, \tag{7.7.4}$$

where $^{\mathrm{T}}$ denotes transpose. (Note that ζ has a different meaning here than in Sections 7.1 and 7.2.)

If we define a matrix anomalous dimension by

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\phi_i = -\tfrac{1}{2}\sum_j \gamma_{ij}\phi_j, \tag{7.7.5}$$

then invariance of the bare fields gives

$$0 = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\phi_{(0)i} = \left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\zeta - \tfrac{1}{2}\zeta\gamma\right)\phi, \tag{7.7.6a}$$

i.e.,

$$\tfrac{1}{2}\zeta\gamma = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\zeta,$$

or

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\mathbf{Z} = \tfrac{1}{2}\{\mathbf{Z}, \boldsymbol{\gamma}\}. \tag{7.7.6b}$$

If $\mathbf{Z}$ is diagonal (as is the case in most theories we consider): $Z_{ij} = \delta_{ij}Z_i$, then (7.7.6) reduces to an anomalous dimension for each field

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\phi_i = -\tfrac{1}{2}\gamma_i\phi_i, \tag{7.7.7}$$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_i = \gamma_i.$$

In the case of a diagonal $\mathbf{Z}$, the renormalization group equation for an N-point Green's function is

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_N \equiv \left[\mu\frac{\partial}{\partial\mu} + \sum_{j=1}^{B}\beta_j(\mathbf{g})\frac{\partial}{\partial g_i}\right]G_N$$

$$= -\sum_{\alpha=1}^{N}\tfrac{1}{2}\gamma_{i_\alpha}(\mathbf{g})G_N. \tag{7.7.8}$$

Here γ_{i_α} is the anomalous dimension function for the αth external field of G_N. If the renormalization matrix $\mathbf{Z}$ is not diagonal, then we have a similar, but more complicated, set of equations for the Green's functions.

The equations for the evolution of the couplings are coupled, so their solution is in general complicated. Considerable simplification can be achieved by using our knowledge of the dependence of the counterterms on massive couplings. Let the couplings $g_1, \ldots, g_A$ be dimensionless and let the corresponding bare couplings be

$$\mu^{(4-d)\rho_i} g_{(0)i}(\mathbf{g}, d),$$

where the $g_{(0)i}$ depend only on the renormalized dimensionless couplings and on the UV cut-off. For the sake of definiteness, we assume that the physical dimension of space-time in the theory is $d = 4$. The wave-function renormalizations Z_i also only depend on the dimensionless couplings and on d. Let the other parameters (masses and super-renormalizable couplings) be denoted by f_s, and let the dimension of f_s be $(4-d)\tau_s + \sigma_s$. If f_s is the mass of a fermion, then $\sigma_s = 1, \tau_s = 0$. If it is a boson mass squared, then $\sigma_s = 2$ and $\tau_s = 0$, while for a super-renormalizable coupling $\tau_s \neq 0$ and $\sigma_s > 0$. Then (by Section 5.8) the bare quantity corresponding to f_s has the form

$$f_{(0)s} = \sum_{d_X = \sigma_s} \mu^{(4-d)\tau_s} X F_{sX}(\mathbf{g}, d). \tag{7.7.9}$$

Here X is any product of the dimensional couplings with dimension (at $d = 4$) equal to the dimension σ_s of f_s.

Requiring invariance of the bare couplings gives

$$0 = (4-d)\rho_i g_{(0)i} + \sum_{j=1}^{A} \beta_j \frac{\partial}{\partial g_j} g_{(0)i}, \tag{7.7.10a}$$

$$\gamma_j = \sum_{j=1}^{A} \beta_j \frac{\partial}{\partial g_j} \ln Z_j, \tag{7.7.10b}$$

$$0 = (4-d)\tau_s \sum_X X F_{sX} + \sum_{X,j} X \beta_j \frac{\partial}{\partial g_j} F_{sX} + \sum_{t,X} \beta_t \frac{\partial X}{\partial f_t} F_{s,X}. \tag{7.7.10c}$$

There is then a triangular structure to the evolution equation: the evolution of a coupling depends only on couplings of the same and lower dimensions. In these equations, the index j runs over the values 1 to A, i.e., over those values that correspond to the dimensionless couplings, while the indices s and t run over the labels for the dimensionful couplings.

If we use minimal subtraction, the calculation of the coefficients is rather easy. Let $\hat{G}_i(g)$, $\hat{Z}_i(g)$, and $\hat{F}_{sX}(g)$ be the coefficients of single poles in $g_{(0)i}$, Z_i and F_{sX}. Then we have

$$\begin{aligned}\beta_j(\mathbf{g},d) &= (d-4)\rho_j g_j + \bar{\beta}_j(\mathbf{g}),\\ \beta_s(\mathbf{g},\mathbf{f},d) &= (d-4)\tau_s f_s + \bar{\beta}_s(\mathbf{g},\mathbf{f}),\end{aligned} \tag{7.7.11}$$

with γ_j, $\bar{\beta}_j$ and $\bar{\beta}_s$ satisfying

$$\begin{aligned}\bar{\beta}_i &= \rho_i \hat{G}_i - \sum_{j=1}^{A} \rho_j g_j \frac{\partial}{\partial g_j}\hat{G}_i,\\ \gamma_i &= \sum_{j=1}^{A} \rho_j g_j \frac{\partial}{\partial g_j}\hat{Z}_i,\\ \bar{\beta}_s &= \sum_X \left\{\left(\tau_s - \sum_t \tau_t f_t \frac{\partial}{\partial f_t}\right)(X\hat{F}_{sX}) - X\sum_{j=1}^{A}\rho_j g_j \frac{\partial}{\partial g_j}\hat{F}_{sX}\right\}.\end{aligned} \tag{7.7.12}$$

7.8 Other renormalization prescriptions

It was only for the sake of simplicity that we restricted our attention to the minimal subtraction procedure. The proof in Section 7.2 in fact shows that any change in renormalization prescription can be compensated by a change in renormalized parameters and a change in the scale of the renormalized field. Let us examine what happens in more general schemes. It is sufficient to restrict our attention to a theory with a single coupling and mass, like ϕ^3 theory in six dimensions.

If we choose a renormalization scheme with an extra mass μ, which might be a renormalization point, then renormalization-group coefficients can still be defined and computed by (7.3.4), (7.3.5), and (7.3.14). What we lose in general are:

(1) the simple formulae (7.3.12),
(2) the lack of dependence of β, γ_m and γ on the masses.

In order to discuss UV limits, it is sensible to choose a scheme in which the limit $m \to 0$ exists. This means that β, γ_m, γ are finite, order-by-order, as $m \to 0$.

Now, different renormalization schemes are related by finite renormalizations of the parameters. So we may relate the RG coefficients in different schemes by looking at the theory in the physical space-time dimension and then computing $\mu \mathrm{d}/\mathrm{d}\mu$ in one scheme in terms of $\mu \mathrm{d}/\mathrm{d}\mu$ in another scheme with the aid of the chain rule.

Suppose we have a second scheme in which the new mass, coupling and

field are m', g' and ϕ':

$$\left.\begin{aligned} g' &= g'(g, m^2/\mu^2), \\ m'^2 &= m^2 z_m(g, m^2/\mu^2), \\ \phi' &= \phi\zeta(g, m^2/\mu^2). \end{aligned}\right\} \tag{7.8.1}$$

Then the Green's functions in the new scheme are

$$G'_N(p; g', m'^2, \mu) = \zeta^N G_N(p; g, m^2, \mu). \tag{7.8.2}$$

The renormalization-group coefficients in the new scheme are:

$$\beta'(g', m'^2/\mu^2) \equiv \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g' = \left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} - \gamma_m m^2\frac{\partial}{\partial m^2}\right)g'(g, m^2/\mu^2), \tag{7.8.3}$$

$$\begin{aligned}\gamma'_m(g', m'^2/\mu^2) &\equiv -\frac{\mu}{m'^2}\frac{\mathrm{d}}{\mathrm{d}\mu}m'^2 \\ &= \gamma_m - \left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} - \gamma_m m^2\frac{\partial}{\partial m^2}\right)\ln\left[z_m(g, m^2/\mu^2)\right].\end{aligned} \tag{7.8.4}$$

Our definition of the total derivative $\mathrm{d}/\mathrm{d}\mu$ is as the derivative with respect to μ when the bare coupling g_0 and bare mass m_0 are held fixed. Therefore, it is the same in both schemes. Notice that there are two steps in computing β' or γ'_m: First, compute the right-hand side expressed in terms of g, m, and μ; second, change variables to the new coupling and mass.

The anomalous dimension of ϕ' is obtained as

$$\begin{aligned}\gamma'(g', m'^2/\mu^2) &= -\frac{2}{N}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln G'_N \\ &= -2\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln\zeta + \gamma \\ &= \gamma - 2\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} - \gamma_m m^2\frac{\partial}{\partial m^2}\right)\ln\zeta(g, m^2/\mu^2).\end{aligned} \tag{7.8.5}$$

A considerable simplification occurs in relating mass-independent schemes. Then g', ζ and z_m are functions of g alone, so that the renormalization-group coefficients in the new scheme satisfy

$$\left.\begin{aligned} \beta'(g') &= \beta(g)\frac{\partial}{\partial g}g'(g), \\ \gamma'_m(g') &= \gamma_m - \beta\frac{\partial}{\partial g}\ln z_m(g), \\ \gamma'(g') &= \gamma - 2\beta\frac{\partial}{\partial g}\ln\zeta(g). \end{aligned}\right\} \tag{7.8.6}$$

In this case, let g', z_m and ζ have perturbation expansions

$$\left.\begin{aligned} g' &= g + a_1 g^3 + a_2 g^5 + \cdots, \\ z_m &= 1 + b_1 g^2 + b_2 g^4 + \cdots, \\ \zeta &= 1 + c_1 g^2 + c_2 g^4 + \cdots, \end{aligned}\right\} \tag{7.8.7}$$

and let the expansions of β, γ_m, and γ be

$$\left.\begin{aligned} \beta(g) &= -A_1 g^3 - A_2 g^5 - A_3 g^7 - \cdots, \\ \gamma_m(g) &= B_1 g^2 + B_2 g^4 + \cdots, \\ \gamma(g) &= C_1 g^2 + C_2 g^4 + \cdots \end{aligned}\right\} \tag{7.8.8}$$

The expansions of β', γ'_m, and γ' are written similarly with all quantities primed. Then we can express them in terms of g by using (7.8.7). For example:

$$\begin{aligned} \beta'(g'(g)) &= -A'_1 g^3 (1 + a_1 g^2 + a_2 g^4 + \cdots)^3 - A'_2 g^5 (1 + a_1 g^2 + \cdots)^5 - A'_3 g^7 + \cdots \\ &= -A'_1 g^3 - g^5 (A'_2 + 3a_1 A'_1) - g^7 (A'_3 + 5A'_2 a_1 + 3A'_1 a_2 + 3A'_1 a_1^2) + \cdots . \end{aligned} \tag{7.8.9}$$

This must agree with the perturbation expansion of the right-hand side of (7.8.6)

$$\beta \partial g'/\partial g = -A_1 g^3 - g^5 (A_2 + 3a_1 A_1) - g^7 (A_3 + 3A_2 a_1 + 5A_1 a_2) + \cdots . \tag{7.8.10}$$

From these equations we see that the first two coefficients in β do not change when the renormalization prescription is changed, i.e., $A_1 = A'_1$, $A_2 = A'_2$. By generalizing the above equations to all orders we also see that, by adjusting the terms in the expansion of $g'(g)$, we may choose the terms beyond the second in β' to be whatever we want. In similar fashion we see that only the $O(g^2)$ terms in γ_m and γ are invariant.

Note that if the one-loop term in γ or γ_m is zero then the whole of γ (or γ_m respectively) may be made zero by a choice of renormalization prescription. This privilege does not extend to β: if the first non-vanishing term in β is at n-loop order ($n > 1$) then that term is RG invariant (but not the $(n+1)$-loop term).

In a theory with more than one dimensionless coupling we may try to apply the same methods. This is left as an exercise. It will be found that only the first term in each β is now invariant, except in the case that the one-loop β-function does not mix the different couplings.

The invariance of these coefficients only applies within mass-independent renormalization prescriptions. If one were to use, say, on-shell subtractions, then the parameter μ would not appear, so all derivatives with respect to μ are zero. Then we have $\beta = \gamma = \gamma_m = 0$. (The asymptotic

behavior that we extract by varying μ can no longer be computed by renormalization-group methods, if we stay within this renormalization prescription. In this case the Callan–Symanzik equation must be used instead – see Callan (1970) and Symanzik (1970b).)

7.9 Dimensional transmutation

Consider a renormalizable field theory with one dimensionless coupling g and no masses. A physically important case is QCD with several flavors of massless quark; with two or three flavors this is an approximation to actual strong interactions.

Since the basic theory has no masses we must use a renormalization prescription with an arbitrary renormalization mass μ. Although the theory apparently has two parameters, g and μ, we saw that this is not so: a change in μ can be compensated by a change in g. In fact, as Coleman & Weinberg (1973) pointed out, the theory really has no parameters at all. The point is simple but somewhat elusive, so we explain it at length.

A physically measurable quantity must be renormalization-group invariant. For example, let $M(g,\mu)$ be a particle mass. By dimensional analysis, it is μ times a function of g alone. So

$$0 = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} M = M + \beta \frac{\partial}{\partial g} M. \tag{7.9.1}$$

Hence

$$\begin{aligned} M &= \mu \cdot \text{constant} \cdot \exp\left[-\int^{g} \frac{\mathrm{d}g'}{\beta(g')} \right] \\ &= \mu C \exp\left\{ -\frac{1}{2A_1 g^2} - \frac{A_2}{A_1^2} \ln(g) - \int_0^{g} \mathrm{d}g' \frac{[A_1^2 g'^3 + \beta(g')(A_1 - A_2 g'^2)]}{A_1^2 g'^3 \beta(g')} \right\} \\ &= \mu C \exp\left\{ -\frac{1}{2A_1 g^2} - \frac{A_2}{A_1^2} \ln(g) + O(g^2) \right\}. \end{aligned} \tag{7.9.2}$$

Here C is a constant and we have written $\beta(g) = -A_1 g^3 - A_2 g^5 + \cdots$, as usual.

Note that the Green's functions are not renormalization-group invariant: to measure a Green's function, one must define the field operators. This definition has an arbitrariness, which is the freedom to vary its scale.

The formula (7.9.2) has a number of consequences:

(1) Non-zero particle masses cannot be computed in ordinary perturbation theory (in a theory with no mass in the Lagrangian). For to avoid large logarithms one must set μ to be of order M, where M is the particle mass

being computed. Then (7.9.2) tells us that $g(M)$ is not a free parameter; it is a number of order unity.

(2) In a non-asymptotically free theory ($A_1 < 0$), suppose we have a small value for $g(\mu)$. Then $\mu \ll M$, where M is the value of the mass of any given massive particle. Perturbation theory is therefore only useful for Green's functions when the external momenta are much below the threshold for producing any of the massive particles.

(3) In an asymptotically free theory ($A_1 > 0$), we have $\mu \gg M$ whenever $g(\mu)$ is small. Perturbation theory is useful in such a theory only when momenta are much bigger than particle masses.

(4) Since the g-dependence of (7.9.2) is universal, i.e., the same for all particles, ratios of particle masses are pure numbers independent of g and μ.

Let us emphasize once more that these results are true when there are no explicit mass terms in the Lagrangian.

The observation of Coleman & Weinberg (1973) comes from asking what can be measured in the theory. Suppose we start with $\mu = \mu_1$ and $g = g_1$ and ask how the theory changes when we work with the theory with a different value of g, $g = g_2$. (We suppose g_1 and g_2 are between $g = 0$ and $g = g^*$, the first fixed point of β.) Each version of the theory has an effective coupling satisfying

$$g_{\text{vers }1}(\mu_1) = g_1, \quad g_{\text{vers }2}(\mu_1) = g_2.$$

Now evolve $g(\mu)$ in the second version to the value of μ where

$$g_{\text{vers }2}(\mu_2) = g_1.$$

Then the second version of the theory is just the first version with all momenta scaled by a factor μ_2/μ_1. For example let σ be a cross-section depending on momenta $p_1, \ldots, p_N$. Then RG invariance and dimensional analysis give us

$$\begin{aligned}\sigma(p_1, \ldots, p_N; g_2, \mu_1) &= \sigma(p_1, \ldots, p_N; g_1, \mu_2) \\ &= \left(\frac{\mu_2}{\mu_1}\right)^{\dim \sigma} \sigma\left(\frac{\mu_1}{\mu_2} p_1, \ldots, \frac{\mu_1}{\mu_2} p_N; g_1, \mu_1\right). \qquad (7.9.3)\end{aligned}$$

The last factor is the cross-section in version 1 of the theory, with its momenta scaled.

We see that changing the dimensionless coupling in a massless theory does not basically change the theory, but only its mass scale. This is called dimensional transmutation.

There are many ways of specifying the scale of the theory: in QCD one might give the proton mass. For perturbative purposes it is better to use

something that can be directly used in perturbation theory, for example the value of μ at which $g(\mu)$ has some given value (e.g., 0.1) in one's chosen renormalization prescription. One standard way is to notice that for large μ, $g(\mu)$ has its asymptotic behavior given by

$$g^2(\mu) = \frac{1}{A_1 \ln(\mu^2/\mu_0^2)} - \frac{A_2 \ln(\ln(\mu^2/\mu_0^2))}{A_1^3 \ln^2(\mu^2/\mu_0^2)} + \frac{\text{constant}}{\ln^2(\mu/\mu_0)} + O\left[\frac{\ln^2(\ln(\mu/\mu_0))}{\ln^3(\mu/\mu_0)}\right]. \tag{7.9.4}$$

Here μ_0 is a reference value of μ. If μ_0 is changed then the series is reorganized; only the first two terms are unchanged. As is conventional (Buras, Floratos, Ross & Sachrajda (1977)), we define the scale Λ of strong interactions as the value of μ_0 for which the $1/\ln^2(\mu^2/\mu_0^2)$ term is zero. This gives (7.5.5).

If we change from, say, minimal subtraction to momentum-space subtraction, then the theory is unchanged provided the coupling is adjusted. This may be done in perturbation theory. For example, we might find that g in the MS scheme and in the momentum-space subtraction scheme are related by

$$g_{\text{MS}} = g_{\text{mom}} + a_1 g_{\text{mom}}^3 \cdots. \tag{7.9.5}$$

Now let g_{MS} be given by (7.5.5) with $\Lambda = \Lambda_{\text{MS}}$, and let g_{mom} be given by (7.5.5) with $\Lambda = \Lambda_{\text{mom}}$. (We already know that A_1 and A_2 are the same in both schemes.) Substituting these expansions into (7.5.5) and requiring consistency gives

$$\Lambda_{\text{MS}} = \Lambda_{\text{mom}} \exp(a_1/A_1). \tag{7.9.6}$$

Notice that both A_1 and a_1 are obtained from one-loop calculations and that there are no higher-order corrections whatever (Celmaster & Gonsalves (1979)).

An amusing consequence is obtained by substituting (7.5.5) for g in (7.9.2). Since M is independent of μ we may let $\mu \to \infty$. The higher-order terms all go away and leave

$$M = C\Lambda(A_1)^{A_2/2A_1^2}. \tag{7.9.7}$$

This equation is not very useful for performing perturbative calculations.

If the theory is a complete theory of physics, then measurements of σ and the p's in (7.9.3) will be in terms of a standard of mass. This we may take to be the mass M of some particle (say, the proton). Let us now change the theory by changing the coupling from g_1 to g_2, just as we did earlier. Then the standard of mass is multiplied by μ_2/μ_1. So if we do experiments in

version 2 with numerical values of momenta equal to those in version 1, the momenta are actually increased by a factor μ_2/μ_1. Therefore, (7.9.3) tells us that σ gets multiplied by a factor $(\mu_2/\mu_1)^{\dim\sigma}$. But its unit of measurement increases by the same factor, so the numerical value is unchanged. In this sense massless theories with different values of the coupling (or different values of Λ) are indistinguishable. This is perhaps the most important result of Coleman & Weinberg (1973).

However, there are many experiments that claim to *measure* Λ. There are even some that give (without qualification) a single measured value of g. How can this be? The second problem is easy to dispose of. What is being measured is the effective coupling g in some renormalization scheme with μ set to a value of the order of the energy of the experiment (typically in e^+e^--annihilation). Strictly one should specify not only the value of g but also the scheme and the value of μ. Now the experiments are at around 10 to 30 GeV, and Λ is at most a few 100 MeV. The variation of g over this range and the variations between the usual renormalization schemes are often no more than the size of experimental errors. So it is possible to talk loosely.

However, we just asserted that massless QCD with different values of Λ is the same theory. The sense of a measurement of Λ is that we measure the numerical value of the ratio of Λ (defined by (7.5.5)) to a standard of mass. For the purposes of the argument, we may regard the standard as being that the nucleon mass is 939 MeV. In terms of dimensionless quantities the measurement is of the constant C in (7.9.7) when M is the nucleon mass. (In the MS scheme, we find that C is between about 5 and 20.) The non-zero masses of the quarks make a relatively small perturbation of the above argument.

Notice that if we play God and double the size of Λ, then the size of the standard mass also doubles, so that numerical results of experiments are unchanged.

In QED the situation is different. The electron has a mass, and its Coulomb field is classical at large distances. A mass-shell renormalization scheme is natural. Since there is a very important mass-scale, an unqualified statement of a measurement of the QED coupling, viz., $e = (4\pi/137)^{1/2}$, makes good sense. QED with a different value of e is a different theory, unlike QCD in the absence of quark masses.

7.10 Choice of cut-off procedure

It is very convenient to use dimensional continuation as an ultra-violet cut-off in perturbation theory. However, there is no known construction of a

complete theory in an arbitrary complex dimension, so one must beware of assigning too much physical significance to use of dimensional continuation. This is especially true when we use minimal subtraction, which is a procedure that exploits the form of the cut-off dependence of the theory. However, the renormalized theory with the cut-off removed does not depend on the form of the cut-off. We saw this in our one-loop calculations. In general the fact is easiest to see by using BPHZ renormalization, in which an integrand is constructed that gives a manifestly convergent integral. The only freedom left is a change of renormalization prescription, otherwise known as a change of parametrization.

In this section we will examine the renormalization-group properties when a different UV cut-off is used. For definiteness we cut off the theory by using a lattice, with spacing a. We consider any theory with a single dimensionless coupling g and a single mass m. It is, of course, possible to generalize to any cut-off procedure and to any theory. In general we will need a renormalization mass μ, in order that we can take the massless limit. The bare coupling g_0, bare mass m_0, and the field-strength renormalization Z are written as functions of the finite parameters g, m and μ, and of the cut-off a. Then the renormalized Green's functions are written in terms of the bare Green's functions

$$G_N(x_1,\ldots,x_N;g,m,\mu,a) = Z^{-N/2}(g,m,\mu,a)G_N^{(0)}(x_1,\ldots,x_N;g_0,m_0,a), \quad (7.10.1)$$

and for them the limit $a\to 0$ exists.

The renormalization-group structure is essentially unchanged. Let us again choose a mass-independent renormalization prescription, so that g_0, Z, and m_0 have the forms:

$$\left.\begin{aligned} g_0 &= g_0(g,\mu a),\\ Z &= Z(g,\mu a),\\ m_0^2 &= m^2 Z_m(g,\mu a) + a^{-2}Y(g,\mu a). \end{aligned}\right\} \quad (7.10.2)$$

The massless theory has $m=0$, and, as before, g_0 and Z are independent of mass. But now the cut-off parameter is dimensional, so g_0 and Z have explicit dependence on μ as shown. But the dimension of g_0 is fixed at zero, so the d-dependent power of μ is not used.

The m^2-dependence of the bare mass squared is again linear. But it is no longer true that $m_0=0$ when $m=0$. In the case of dimensional regularization the only remaining dimensional parameter is μ, and it is not possible (Collins (1974)) to generate by minimal subtraction a counterterm $\mu^2 Y(g,d)$. But with a lattice cut-off a term $a^{-2}Y$ is both possible and needed, as we will now verify by computing a low-order graph.

7.10.1 *Example*: ϕ^4 *self-energy*

The simplest example that shows the existence of the Y-term in (7.10.2) is the self-energy graph of Fig. 7.10.1, not in the ϕ^3 theory that we have been using, but in the ϕ^4 theory in four space-time dimensions (with $m = 0$). The Lagrangian is (2.3.1). With dimensional regularization the value of the graph is

$$\tfrac{1}{2}g_0(2\pi)^{-d}\int \mathrm{d}^d k/k^2 = 0,$$

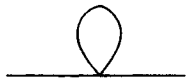

Fig. 7.10.1. Lowest-order self-energy graph in ϕ^4 theory.

but with a lattice cut-off we find

$$-\mathrm{i}g_0(32\pi^4)^{-1}\int_{|k^\mu|<\pi/a} \mathrm{d}^3k\,\mathrm{d}\omega\, D(\omega,k;a). \tag{7.10.3}$$

Here the Euclidean lattice propagator is $1/(\omega^2 + k^2)$ if ω and $|k|$ are much smaller than $1/a$. For general values of k^μ, it is

$$a^2\Big/\Big\{4\sum_{\mu=1}^{4}\sin^2(k^\mu a/2)\Big\},$$

which is positive definite, so that the integral (7.10.3) is non-zero and diverges to a number of order $1/a^2$ as $a \to 0$.

A similar divergence occurs in the self-energy of a scalar field in any theory.

7.10.2 *RG coefficients*

We now continue our general discussion of the renormalization group when a lattice cut-off is used. As in the treatment using dimensional regularization we define a renormalization-group operator

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} = \mu\frac{\partial}{\partial\mu} + \bar{\beta}(g,\mu a)\frac{\partial}{\partial g} - \bar{\gamma}_m(g,\mu a)m^2\frac{\partial}{\partial m^2}. \tag{7.10.4}$$

We have changed our notation slightly, and used an overbar to indicate renormalization-group coefficients in the cut-off theory. These coefficients $\bar{\beta}$ and $\bar{\gamma}_m$ have finite limits, $\beta(g)$ and $\gamma_m(g)$, as $a \to 0$. In our later work it will be rather important to distinguish the coefficients before and after the

cut-off is removed. The coefficients can be computed from

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g_0 = 0 = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}m_0^2,$$

with (7.10.4) used for $\mu\mathrm{d}/\mathrm{d}\mu$. We also have the anomalous dimension $\bar{\gamma} = \mu\mathrm{d}\ln Z/\mathrm{d}\mu$, just as with dimensional regularization. This all results in

$$\bar{\beta} = \frac{-(\mu a)\dfrac{\partial}{\partial(\mu a)}g_0(g,\mu a)}{\dfrac{\partial}{\partial g}g_0(g,\mu a)},$$

$$\bar{\gamma}_m = \left[\mu a\frac{\partial}{\partial(\mu a)} + \bar{\beta}\frac{\partial}{\partial g}\right]\ln Z_m(g,\mu a), \tag{7.10.5}$$

$$\bar{\gamma} = \left[\mu a\frac{\partial}{\partial(\mu a)} + \bar{\beta}\frac{\partial}{\partial g}\right]\ln Z(g,\mu a).$$

In addition there is the constraint

$$0 = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}(a^{-2}Y) = \frac{1}{a^2}\left[\mu a\frac{\partial}{\partial(\mu a)} + \bar{\beta}\frac{\partial}{\partial g}\right]Y(g,\mu a). \tag{7.10.6}$$

The information on the divergences is all contained in the finite functions $\bar{\beta}$, $\bar{\gamma}_m$ and $\bar{\gamma}$. If desired, we can use minimal subtraction with the form

$$g_0 = g + g^3G_{11}\ln(a\mu) + g^5[G_{22}\ln^2(a\mu) + G_{21}\ln(a\mu)] + \cdots,$$

so that $\bar{\beta} = -g^3G_{11} - g^5G_{21} - \cdots$ is a function of g alone. In order that $\bar{\beta}$ be finite as $a \to 0$ all the logarithms of $a\mu$ must cancel in $\bar{\beta}$. This implies a set of relations for the counterterms, the first of which is $2G_{22} = 3G_{11}^2$. An analogous set of relations occurs when we use dimensional regularization, as can be seen from (7.3.12). These we will discuss further in Section 7.11. Note that for minimal subtraction with the lattice cut-off we have $\beta = \bar{\beta}$, $\gamma_m = \bar{\gamma}_m$, etc.

7.10.3 *Computation of g_0 and Z; asymptotically free case*

If we were to compute the exact theory, rather than a perturbative approximation, we would need to know how $g_0(g,\mu a)$ depends on a as $a \to 0$ with g and μ fixed. A low-order calculation is not sufficient, for g_0 has large logarithms in its perturbative expansion. Provided the theory is asymptotically free, we can remedy this by using the renormalization group to improve the calculation, just as we did for the large-momentum behavior of

Green's functions. The starting point is the equation

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_0 \equiv \mu \frac{\partial}{\partial \mu} g_0 + \bar{\beta} \frac{\partial}{\partial g} g_0 = 0, \tag{7.10.7}$$

which is in effect the renormalization-group equation for g_0. We may solve it just as for the Green's functions.

Ultimately, we will let a approach zero while holding g and μ fixed. But first let us keep a non-zero. Then we can define an effective coupling $\bar{g}(\mu)$ by

$$\mu' \mathrm{d}\bar{g}(\mu')/\mathrm{d}\mu' = \bar{\beta}(\bar{g}(\mu')), \tag{7.10.8a}$$

with the boundary condition

$$\bar{g}(\mu) = g. \tag{7.10.8b}$$

We will also need the effective coupling at $a = 0$. For the moment, let us denote it by the symbol $\hat{g}(\mu')$. It satisfies

$$\mu' \mathrm{d}\hat{g}(\mu')/\mathrm{d}\mu' = \beta(\hat{g}(\mu')) = \bar{\beta}(\hat{g}(\mu'); \mu' a = 0),$$
$$\hat{g}(\mu) = g.$$

Implicitly there is dependence of $\hat{g}$ on μ and g, and of $\bar{g}$ on μ, g and a:

$$\hat{g} = \hat{g}(\mu'; \mu, g), \quad \bar{g} = \bar{g}(\mu'; \mu, a, g).$$

Of course, $\bar{g}(\mu') \to \hat{g}(\mu')$ as $a \to 0$.

We can solve the renormalization-group equation (7.10.7) for g_0 to find

$$g_0 = g_0(g, a\mu) = g_0(\bar{g}(1/a), 1).$$

Now, when a is small, it might appear that we can replace $\bar{g}(1/a)$ by $\hat{g}(1/a)$, and that g_0 is well approximated by the first term in its perturbation expansion (since $\bar{g}(1/a)$ is small). That is,

$$g_0 = \hat{g}(1/a) + \text{negligible error}.$$

These suppositions are actually false, for two reasons. First, $\bar{\beta}(g, a\mu)$ in general depends on $a\mu$, so we cannot just replace $\bar{g}$ at $\mu = 1/a$ by $\hat{g}(1/a)$ computed in the $a = 0$ theory. Secondly, we cannot simply drop the higher-order terms in g_0, since the dependence of renormalized Green's functions on g_0 is singular. Thus small errors in g_0 may give rise to large errors in a Green's function computed as a function of bare quantities.

To derive the correct formula we must examine the a-dependence of $\bar{\beta}$ more closely. So we write the perturbation expansion of g_0 in the form:

$$g_0 = g + g^3[G_{11} \ln(a\mu) + G_{10} + \bar{G}_1(a\mu)]$$
$$+ g^5[G_{22} \ln^2(a\mu) + G_{21} \ln(a\mu) + G_{20} + \bar{G}_2(a\mu)] + \cdots. \tag{7.10.9}$$

Here we have not specified the renormalization prescription, so in addition to the logarithms we need finite functions $\bar{G}_1(a\mu)$, etc. We have explicit constant terms G_{i0}, so we define $\bar{G}_i(a\mu)$ to be zero at $a\mu = 0$. Once divergences and subdivergences have been subtracted from Feynman graphs, the remainders converge with power-law convergence in momentum. This is a consequence of our treatment of Weinberg's theorem, and is further treated in Weinberg (1960). Therefore we can say

$$\bar{G}_i(a\mu) = O((a\mu)^{c_i})$$

as $a\mu \to 0$, for some positive number c_i. In general, $\bar{G}_i$ equals $a\mu$ times logarithms of $a\mu$, so we can safely set $c_i = 1/2$.

First we compute the $\bar{\beta}$-function:

$$\bar{\beta} = -\mu \frac{\partial}{\partial \mu} g_0 \Big/ \frac{\partial g_0}{\partial g}$$

$$= -\frac{\left\{ g^3 \left[G_{11} + \frac{\partial}{\partial \ln(a\mu)} \bar{G}_1 \right] + g^5 \left[2G_{22} \ln(a\mu) + G_{21} + \frac{\partial}{\partial \ln(a\mu)} \bar{G}_2 \right] + \cdots \right\}}{\{1 + 3g^2 [G_{11} \ln(a\mu) + G_{10} + \bar{G}_1] + \cdots\}}$$

$$= -g^3 \left[G_{11} + \mu a \frac{\partial}{\partial(\mu a)} \bar{G}_1 \right]$$

$$- g^5 \left\{ G_{21} - 3G_{10}G_{11} + \frac{\partial}{\partial \ln(\mu a)} [\bar{G}_2 - \tfrac{3}{2}\bar{G}_1^2 - 3G_{11}\bar{G}_1 \ln(a\mu) - 3G_{10}\bar{G}_1] \right\} + \cdots. \quad (7.10.10)$$

The relation $2G_{22} = 3G_{11}^2$ must hold in order that $\bar{\beta}$ is finite as $a \to 0$. The limit $a \to 0$ gives

$$\beta(g) = \bar{\beta}(g, 0) = -g^3 G_{11} - g^5 (G_{21} - 3G_{10}G_{11}) + \cdots, \quad (7.10.11)$$

so that with our usual notation $A_1 = G_{11}$ and $A_2 = G_{21} - 3G_{11}G_{10}$. Since $\bar{G}_1(a\mu)$ and $\bar{G}_2(a\mu)$ go to zero like a power of $a\mu$ (times logarithms) when $a\mu \to 0$, their logarithmic derivatives $a\mu \partial \bar{G}_j / \partial(a\mu)$ also go to zero like a power.

The first step in our calculation of g_0 is to observe that the RG invariance of g_0 implies that

$$g_0(g, \mu a) = g_0(\bar{g}(\mu'), \mu' a)$$
$$= g_0(\bar{g}(1/a), 1). \quad (7.10.12)$$

The next step is to examine the size of the error that is made in replacing $\bar{g}(1/a)$ by the effective coupling $\hat{g}(1/a)$ in the $a = 0$ theory. Finally, we will find the accuracy to which $g_0(\bar{g}(1/a), 1)$ must be computed in order to obtain

the correct renormalized Green's functions at $a = 0$.

The difference between the two effective couplings $\hat{g}(1/a)$ and $\bar{g}(1/a)$ will turn out to be of order $\hat{g}(1/a)^3$ when a is small. So let us define the fractional error $\hat{g}^2\Delta$ by

$$\bar{g}(\mu';\mu,a,g) = \hat{g}(\mu';\mu,g)[1 + \hat{g}(\mu';\mu,g)^2\Delta(\mu';\mu,a,g)]. \quad (7.10.13)$$

We will now show that Δ is finite when $\mu' = 1/a$ and $a \to 0$.

From (7.10.13) and the definitions of $\bar{\beta}$ and β we find that

$$\mu'\frac{\partial}{\partial\mu'}\Delta(\mu') = \hat{g}^{-3}[\bar{\beta}((1+\hat{g}^2\Delta)\hat{g}, \mu' a) - (1 + 3\hat{g}^2\Delta)\beta(\hat{g})]$$

$$= -\mu' a\frac{\partial}{\partial(\mu' a)}\bar{G}_1 + O(\hat{g}^2(\mu' a)^{1/2}) + O(\hat{g}^4\Delta). \quad (7.10.14)$$

Now $\hat{g}^2(\mu') \sim 1/A_1 \ln(\mu'/\Lambda)$ as $\mu' \to \infty$, so this equation tells us that $\Delta(\mu', \mu, a, g)$ is finite when $a \to 0$ and $\mu \le \mu' \le 1/a$. In fact it implies that

$$\Delta(\mu') = -\bar{G}_1(a\mu') + O(1/\ln(1/a)). \quad (7.10.15)$$

We now compute g_0. It is convenient to write a formula for its square:

$$\begin{aligned} g_0(g, \mu a)^2 &= g_0(\bar{g}(1/a), 1)^2 \\ &= \{\hat{g}(1/a)(1 + \hat{g}^2\Delta) + \hat{g}^3(1 + \hat{g}^2\Delta)^3[G_{10} + \bar{G}_1(1)] + O(\hat{g}^5)\}^2 \\ &= \frac{1}{[A_1 \ln(1/a^2\Lambda^2)]} - \frac{A_2 \ln[\ln(1/a^2\Lambda^2)]}{[A_1^3 \ln^2(1/a^2\Lambda^2)]} \\ &\quad + \frac{2G_{10}}{[A_1^2 \ln^2(1/a^2\Lambda^2)]} + O\{\ln^2[\ln(1/a)]/\ln^3(1/a)\}. \end{aligned} \quad (7.10.16)$$

Here we used the formula for $\hat{g}(\mu)$ in terms of μ and Λ – (7.5.5). The formula for g_0 in terms of $1/a$ and Λ is the same as (7.5.5) except for an additional $1/\ln^2$ term. Observe that it was essential to keep the a-dependence in $\bar{\beta}(g, a\mu)$; the $-\bar{G}_1(1)$ term in (7.10.15) canceled the $\bar{G}_1(1)$ in the two-loop coefficient in g_0.

Finally we express (7.10.16) in terms of g, a and μ:

$$\begin{aligned} g_0(g, a\mu)^2 &= \frac{1}{A_1 \ln(1/a^2\mu^2)} - \frac{A_2 \ln(\ln(1/a^2\mu^2))}{A_1^3 \ln^2(1/a^2\mu^2)} \\ &\quad + \frac{1}{A_1^2 \ln^2(1/a^2\mu^2)}\left[2G_{10} - \frac{1}{g^2} - \frac{A_2}{A_1}\ln(A_1 g^2) - 2A_1 f(g)\right] \\ &\quad + O\{\ln^2[\ln(1/a\mu)]/\ln^3(1/a\mu)\}, \end{aligned} \quad (7.10.17)$$

where

$$f(g) = \int_0^g dg'[1/\beta(g') + 1/(A_1 g'^3) - A_2/(A_1^2 g')]. \quad (7.10.18)$$

Similar formulae hold for Z, for Z_m, and for Y. Thus

$$Z = [A_1 g^2 \ln(1/a^2\mu^2)]^{-C_1/2A_1} \times \\ \times \exp\left\{\int_0^g dg' \left[\frac{\gamma(g')}{\beta(g')} + \frac{C_1}{A_1 g'}\right]\right\}\left\{1 + O\left[\frac{\ln(\ln(a\mu))}{\ln(a\mu)}\right]\right\}, \tag{7.10.19}$$

where $\gamma = C_1 g^2 + O(g^4)$.

7.10.4 *Accuracy needed for* g_0

Let us now suppose we compute the renormalized Green's functions:

$$G_N(x_1, \ldots, x_N; g, m, \mu; a) = Z^{-N/2} G_{(0)N}(x_1, \ldots, x_N; g_0, m_0; a). \tag{7.10.20}$$

We must now let a approach zero, and ask how accurately we need to compute g_0 and Z. In (7.10.17) and (7.10.19) we gave formulae for g_0 and Z, with explicit estimates for the errors coming from uncalculated corrections. These equations tell us the value of $g_0(g, a\mu)$ when we let $a \to 0$ while keeping g and μ fixed. Since the bare Green's functions have singular dependence on g_0, the uncalculated corrections might affect the values of the renormalized Green's functions. In fact these terms do not affect the renormalized Green's functions in the continuum limit, as we will now show.

The key observation is that the renormalized Green's functions are finite functions of the renormalized parameters. Thus we do not need to hold the renormalized coupling and mass fixed while taking the continuum limit $a \to 0$. We may in fact let them vary continuously, provided only that their values at $a = 0$ are the same as before. Now examine (7.10.17). It is evident that we may absorb the whole of the correction term into just such a variation of g. In fact the necessary change in g is of order $\ln^2(\ln(1/a))/\ln(1/a)$ as $a \to 0$. So we may choose the bare coupling to be

$$g_0^2 = \frac{1}{A_1 \ln(1/a^2\mu^2)} - \frac{A_2 \ln[\ln(1/a^2\mu^2)]}{A_1^3 \ln^2(1/a^2\mu^2)} \\ + \frac{1}{A_1^2 \ln^2(1/a^2\mu^2)}\left[2G_{10} - \frac{1}{g^2} - \frac{A_2}{A_1}\ln(A_1 g^2) - 2A_1 f(g)\right]. \tag{7.10.21a}$$

Hence in (7.10.16), we can also drop the $O\{\ln^2[\ln(1/a^2)]/\ln^3(1/a)\}$ terms. So we have the following formula for g_0 in terms of a Λ alone:

$$g_0^2 = \frac{1}{A_1 \ln(1/a^2\Lambda^2)} - \frac{A_2 \ln[\ln(1/a^2\Lambda^2)]}{A_1^3 \ln^2(1/a^2\Lambda^2)} + \frac{2G_{10}}{A_1^2 \ln^2(1/a^2\Lambda^2)}. \tag{7.10.21b}$$

In the case of the wave-function renormalization Z, the uncalculated corrections can be absorbed into a factor ζ^N multiplying the Green's function G_N. This factor must approach unity in the continuum limit. Hence we may use

$$Z = \left[A_1 g^2 \ln\left(\frac{1}{a^2 M^2}\right)\right]^{-C_1/2A_1} \exp\left\{ \int_0^g dg' \left[\frac{\gamma(g')}{\beta(g')} + \frac{C_1}{A_1 g'} \right] \right\}, \tag{7.10.22}$$

where M is an arbitrary mass that is irrelevant when $a \to 0$. Notice that for the coupling we had a form (7.10.21*b*) that had dependence on Λ, but not on μ or on g. This was because g_0 is renormalization-group invariant: we may take μ arbitrarily large without affecting g_0, provided that we also set g equal to the effective coupling at μ. When μ is very big, g is very small, and the higher-order corrections contained in $f(g)$ go to zero. But Z is not invariant; it must depend on g. What we can say is that any dependence on a of the form

$$Z = \text{finite} \cdot [\ln(1/a)]^{-C_1/2A_1}$$

will produce finite Green's functions.

Notice that if the one-loop divergence in Z vanishes, then we may let Z be finite:

$$Z = \exp\left[\int_0^g dg' \gamma(g')/\beta(g') \right].$$

There will in general be divergences in the self-energy graphs in higher orders. What we have proved is that they must sum to something finite.

In the case of g_0, any a-dependence of the form

$$g_0^2 = 1/A_1 \ln(1/a^2) - A_2 \ln[\ln(1/a)]/[A_1^3 \ln^2(1/a^2)] + \text{finite}/\ln^2 a$$

will give finite renormalized Green's functions. Only knowledge of A_2 and A_1 is necessary for this. They are obtained from one- and two-loop calculations. The coefficient of the $1/\ln^2 a$ determines the value of g.

The formula (7.10.21*b*) shows the fundamental significance of the Λ-parameter. In a renormalizable field theory, there are divergences, so one cannot simply specify a single number as the bare coupling constant. Rather, one must construct the theory as the continuum limit of some lattice theory, with g_0 depending on the lattice spacing, a. Equation (7.10.21*b*) gives g_0 as a definite numerical function of a.

Unfortunately, there is a certain arbitrariness in precisely how one constructs a lattice approximation to a continuum theory. This arbitrariness is physically irrelevant (although some particular approximation may

be superior when it is used for a numerical calculation). So (7.10.21*b*) is also important because it expresses the bare coupling in terms of quantities (Λ, A_1, and A_2), which have direct meaning in the continuum theory, and in terms of one number G_{10}, which depends on the lattice approximation, but which can actually be computed analytically (Hasenfratz & Hasenfratz (1980)). The result of such a lattice calculation is normally given as the ratio of a Λ_{lattice} to the value of Λ in some standard continuum renormalization scheme. The definition of Λ_{lattice} is the value for which

$$g_0^2 = 1/[A_1 \ln(1/a^2\Lambda_{\text{lattice}}^2)] - A_2 \ln[\ln(1/a^2\Lambda_{\text{lattice}}^2)]/[A_1^3 \ln^2(1/a^2\Lambda_{\text{lattice}}^2)] \tag{7.10.23}$$

gives the same continuum limit as (7.10.21). It is easily checked that this is

$$\Lambda_{\text{lattice}} = \Lambda \exp(G_{10}/A_1). \tag{7.10.24}$$

Despite the fundamental significance of Λ, there is a convention dependence in its definition. In specifying a theory by its value of Λ, one must specify these conventions. This is analogous in its effect to the need for specifying a system of units in electromagnetism. The main convention is that of the renormalization prescription. The other convention is the one implicit in the choice of the constant in (7.5.3). It is sensible to follow the usual convention, to avoid confusion.

We have seen that higher-order corrections (beyond two loops) do not enter into our formula for g_0 in terms of Λ. This is in contrast to (7.10.21*a*), which expresses g_0 in terms of g and μ. So it is sensible to treat Λ as a fundamental parameter of the theory – say in strong interactions. But practical considerations intervene if one tries to measure Λ. A typical measurement consists of measuring a quantity for which a useful perturbation expansion exists (for example, a jet cross-section in $e^+ - e^-$ annihilation). The experiment therefore measures the effective coupling $g(\mu)$ at some value of μ which is of the order of the energy of the experiment. There are errors in this measurement caused by uncalculated higher-order terms in the theoretical calculation of the cross-section, not to mention non-perturbative corrections. We can then deduce Λ from (7.5.6), with further errors due to corrections in β.

Since g is more directly related to the size of the cross-section, it is perhaps correct to argue that experiments should quote their results as a value of g. But to give the value of Λ is equally valid. However, a small fractional error in g corresponds to a much larger fractional error in Λ. This can be seen from (7.5.6). If we change g and Λ while holding μ fixed, then

$$|\mathrm{d}\Lambda/\Lambda| = |\mathrm{d}g/g|[2A_1 g^2]^{-1}[1 + O(g^2)].$$

If one could do a real calculation of the mass of, say, the proton in QCD, then it is the error in the value of Λ that would determine the error in the mass. As we saw in Section 7.9, when we discussed dimensional transmutation, the mass is proportional to Λ.

7.10.5 m_0^2

Unfortunately m_0^2 has a $1/a^2$ term, but the variation of m_0 with m^2 depends on a $[\ln(1/a)]^{-b_{11}/A_1}$ term. So we need the coefficient of $1/a^2$ to very high accuracy. Any slight error (say of order $1/a$) will be equivalent to making the renormalized mass diverge like $1/a$ as $a \to 0$. The resulting need to be very accurate in m_0 leads many people to consider scalar field theories unnatural.

In the case of fermion theories there is a symmetry under $\psi \to \gamma_5 \psi$ when $m = 0$, so the Y term is absent and we have

$$m_0 = mZ_m \simeq m \cdot \text{constant}\,[\ln(1/a)]^{-B_1/2A_1}.$$

7.10.6 *Non-asymptotically free case*

The values of g_0, etc., as $a \to 0$ are not perturbatively computable unless the theory is asymptotically free. However, if we suppose that β in a non-asymptotically free theory has a fixed point, then we may write

$$\begin{aligned} g_0(g, a\mu) &= g_0(\bar{g}(1/a), 1) \\ &\to g_0(g^*, 1) \text{ as } a \to 0. \end{aligned} \tag{7.10.25}$$

Note that $g_0(\bar{g}(1/a), 1)$ is a finite function of $\bar{g}$, so the limit exists. However the same value is obtained for g_0 at $a = 0$ for any value of $g(\mu)$. So the way in which the limit is approached determines the value of g.

An example is easily constructed. Suppose we have a theory in which

$$\bar{\beta}(g) = \sin^2(g^2)/(2g), \tag{7.10.26}$$

and

$$g_0(g, 1) = g. \tag{7.10.27}$$

Then the effective coupling has the form

$$g = [\arctan(1/\ln(\Lambda/\mu))]^{1/2}.$$

There is a fixed point $g^* = \pi^{1/2}$. We therefore find that the bare coupling as $a \to 0$ must be

$$\begin{aligned} g_0(g, a\mu) &= [\arctan(1/\ln(a\Lambda))]^{1/2} \\ &= \pi^{1/2} - \frac{1}{2\pi^{1/2}\ln(1/a\Lambda)} + O(1/\ln^2(a\Lambda)). \end{aligned} \tag{7.10.28}$$

It is necessary to know how $g_0(g,a\mu)$ approaches its limit $g_0(g^*,1)$ in order to determine the value of Λ.

7.11 Computing renormalization factors using dimensional regularization

In the previous section, Section 7.10, we computed how the bare coupling g_0 should behave as a function of the lattice spacing a. In this section we present the corresponding argument using dimensional continuation as the cut-off. We do this by treating the defining equation (7.3.10) of β as a differential equation to compute g_0. Our argument will be valid in any asymptotically free theory, like ϕ^3 theory in six dimensions or QCD in four dimensions. If we let d_0 be the physical space-time dimension, then we regularize by going to a lower dimension $d = d_0 - \varepsilon$.

First we compute the relations between lower and higher poles in the renormalization. Now we write

$$g_0 = \mu^{\varepsilon/2}\left[g + \sum_{j=1}^{\infty} d_j(g)\,\varepsilon^{-j} \right], \tag{7.11.1}$$

and we have the definition of $\bar{\beta}$:

$$\varepsilon g_0/2 + \bar{\beta}(g,d)\partial g_0/\partial g = 0. \tag{7.11.2}$$

Let us expand (7.11.2) in powers of ε. The terms proportional to ε and ε^0 give us:

$$\bar{\beta} = -\varepsilon g/2 + \beta(g) = -\varepsilon g/2 + \tfrac{1}{2}(g\partial/\partial g - 1)d_1(g).$$

We have changed notation from our original definitions to correspond to the definitions that we used in Section 7.10 for the lattice cut-off. There we defined $\bar{\beta}$ to be RG coefficient in the cut-off theory, while we defined β as the limit of $\bar{\beta}$ as the cut-off is removed.

Now, the coefficient of the pole ε^{-j} in (7.11.2) is

$$\tfrac{1}{2}(1 - g\partial/\partial g)d_{j+1}(g) + \beta(g)\partial d_j/\partial g = 0. \tag{7.11.3}$$

This is a differential equation which, when solved using the boundary condition $d_j(0) = 0$, gives all the higher coefficients $d_j(g)$ in terms of the single pole $d_1(g)$.

Similar relations ('t Hooft (1973)) hold for all renormalization counterterms. The structure is similar to the leading logarithm expansion. They show that in each order of perturbation theory the only new information in the counterterm in a given order of perturbation theory is in the single pole.

A convenient way of solving these relations is to work out the solution of the differential equation (7.11.2). This gives

$$\ln\left[g_0(g,\mu,d)\mu^{-\varepsilon/2}\right] = \int_0^g \mathrm{d}g' \left[\frac{1}{g' - 2\beta(g')/\varepsilon} - \frac{1}{g'} \right] + \ln g, \tag{7.11.4}$$

i.e.,

$$g_0 = \mu^{\varepsilon/2} g \exp\left\{\int_0^g \mathrm{d}g' \frac{2\beta(g')}{g'^2\varepsilon} \frac{1}{[1-2\beta(g')/(g'\varepsilon)]}\right\}. \tag{7.11.5}$$

The boundary condition $g_0/g \to \mu^{\varepsilon/2}$ as $g \to 0$ has been used.

We now ask how g_0 must behave as $\varepsilon \equiv d_0 - d \to 0$, with g (and μ) fixed. If the theory is not asymptotically free (so that in $\beta(g) = -A_1 g^3 + \cdots$, A_1 is negative), then the integrand has a pole at

$$g'^2 = -(d_0 - d)/2A_1 + O(d - d_0)^2.$$

The solution (7.11.4) only unambiguously exists if g^2 is less than this value, which is zero when $d = d_0$. To get to the $d = d_0$ theory with g non-zero we must continue g_0 so that the integration avoids the pole. The result is that g_0 has an imaginary part. This, among other things, suggests that the theory is unphysical (see Wilson (1973), Gross (1976)). Recently, evidence has accumulated that the lattice ϕ^4 theory does not have a non-trivial continuum limit – see Symanzik (1982) for a review.

If the theory is asymptotically free then we may continue (7.11.4) to $d = d_0$, i.e., $\varepsilon = 0$. The integrand becomes singular when $\varepsilon = 0$, and we examine the singularity by expanding in powers of g':

$$\begin{aligned}\ln g_0 = \ln g + \frac{\varepsilon}{2}\ln\mu \\ -\frac{\varepsilon}{2}\int_0^g \mathrm{d}g' \left\{\frac{1}{[-\varepsilon g'/2 + \beta(g')]} - \frac{1}{[-\varepsilon g'/2 - A_1 g'^3]} \right. \\ \left. - \frac{A_2 g'^5}{[-\varepsilon g'/2 - A_1 g'^3]^2}\right\} \\ -\frac{\varepsilon}{2}\int_0^g \mathrm{d}g' \left\{\frac{1}{[-\varepsilon g'/2 - A_1 g'^3]} + \frac{A_2 g'^5}{[-\varepsilon g'/2 - A_1 g'^3]^2} + \frac{2}{\varepsilon g'}\right\}.\end{aligned} \tag{7.11.6}$$

In the first integral we may set $\varepsilon = 0$ and have errors that are $o(1)$. So

$$\begin{aligned}\ln g_0 = \tfrac{1}{2}\ln\left(\frac{\varepsilon}{2A_1}\right) \\ + \frac{\varepsilon}{2}\left[\ln\mu - \frac{1}{2A_1 g^2} - \frac{A_2}{2A_1^2}\ln\left(\frac{2A_1 g^2}{\varepsilon}\right) + \frac{A_2}{2A_1^2} - f(g)\right] + o(\varepsilon),\end{aligned} \tag{7.11.7}$$

where $f(g)$ is defined by (7.10.18). Thus

$$g_0^2 = \Lambda^\varepsilon\left(\frac{\varepsilon}{2A_1}\right)\left\{1 + \frac{\varepsilon A_2}{2A_1^2}\left[1 - \ln\left(\frac{2}{\varepsilon}\right)\right] + o(\varepsilon)\right\}. \tag{7.11.8}$$

7.12 Renormalization group for composite operators

We have seen how a change in renormalization prescription for the interactions of a theory can be compensated by a change in the values of the renormalized parameters. The same property holds for the composite operators we defined in Chapter 6.

For example, consider the renormalized $[\phi^2]$ operator in ϕ^3 theory at $d = 6$. In Section 6.2 we calculated it in the one-loop approximation:

$$\langle 0|T\phi(x)\phi(y)[\phi^2](z)/2|0\rangle = \\ = \text{tree graph} + \{\text{one-loop graph} + \text{counterterm graphs}\} + \cdots \quad (7.12.1)$$

A change in renormalization prescription amounts to a finite change in the counterterm graphs. Since the counterterms are of the form

$$\tfrac{1}{2}[\phi^2] = \tfrac{1}{2}\phi^2 + \tfrac{1}{2}\delta Z_a \phi^2 + \delta Z_b m^2 \phi + \delta Z_c \Box\phi + \text{higher order}, \quad (7.12.2)$$

we have

$$\tfrac{1}{2}[\phi^2]_{\text{newRP}} = \tfrac{1}{2}[\phi^2]_{\text{oldRP}} + \tfrac{1}{2}a[\phi^2] + b\phi + c\Box\phi. \quad (7.12.3)$$

Here a, b, and c are finite quantities of the same order in the coupling as the one-loop counterterms. The equation (7.12.3) is, so far, only derived at the one-loop order – so the finite counterterms are to be used with their operators inserted in tree graphs.

Let us examine the situation we expect to all orders. We will use minimal subtraction. Then the renormalization in the notation of (6.2.12) is

$$\tfrac{1}{2}[\phi^2] = \tfrac{1}{2}Z_a Z^{-1}\phi_0^2 + \mu^{d/2-3}Z_b Z^{-1/2} m^2 \phi_0 + \mu^{d/2-3} Z_c Z^{-1/2}\Box\phi_0. \quad (7.12.4)$$

Now the bare field is independent of μ, so we may write

$$\begin{aligned} \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\tfrac{1}{2}[\phi^2] &= Z_a Z^{-1}\tfrac{1}{2}\phi_0^2 \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln(Z_a/Z) \\ &\quad + \mu^{d/2-3}Z_b Z^{-1/2} m^2 \phi_0 \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln(\mu^{d/2-3}Z_b Z^{-1/2} m^2) \\ &\quad + \mu^{d/2-3}Z_c Z^{-1/2}\Box\phi_0 \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln(\mu^{d/2-3}Z_c Z^{-1/2}) \\ &= \tfrac{1}{2}[\phi^2]\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln(Z_a/Z) \\ &\quad + \mu^{d/2-3}Z_b m^2 \phi \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln(\mu^{d/2-3}Z_b Z^{1/2} Z_a^{-1} m^2) \\ &\quad + \mu^{d/2-3}Z_c \Box\phi \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln(\mu^{d/2-3}Z_c Z^{1/2} Z_a^{-1}), \end{aligned} \quad (7.12.5)$$

which has the form

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\tfrac{1}{2}[\phi^2] = \gamma_a\tfrac{1}{2}[\phi^2] + \gamma_b m^2\mu^{d/2-3}\phi + \gamma_c\mu^{d/2-3}\Box\phi. \tag{7.12.6}$$

We can formulate this as a matrix equation:

$$\Phi \equiv \begin{pmatrix} \frac{1}{2}[\phi^2] \\ \phi \\ \Box\phi \end{pmatrix} = \begin{pmatrix} Z_a Z^{-1} & m^2 Z_b Z^{-1/2}\mu^{d/2-3} & Z_c Z^{-1/2}\mu^{d/2-3} \\ 0 & Z^{-1/2} & 0 \\ 0 & 0 & Z^{-1/2} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\phi_0^2 \\ \phi_0 \\ \Box\phi_0 \end{pmatrix} \equiv M\Phi_0, \tag{7.12.7}$$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\Phi = \mu\frac{\mathrm{d}M}{\mathrm{d}\mu}M^{-1}\Phi = \gamma_\Phi\Phi. \tag{7.12.8}$$

The coefficients γ_a, γ_b, and γ_c are finite at $d = 6$. From our calculations in Section 6.2, we have

$$\left.\begin{aligned} \gamma_a &= \frac{5g^2}{384\pi^3} = \frac{5}{6}\lambda, \\ \gamma_b &= \frac{g}{64\pi^3} = \lambda/g, \\ \gamma_c &= \frac{g}{384\pi^3} = \frac{1}{6}\lambda/g, \end{aligned}\right\} \tag{7.12.9}$$

where $\lambda = g^2/64\pi^3$. Thus

$$\gamma_\Phi = \begin{pmatrix} \frac{5}{6}\lambda & \frac{\lambda m^2\mu^{d/2-3}}{g} & \frac{\lambda\mu^{d/2-3}}{6g} \\ 0 & \frac{1}{6}\lambda & 0 \\ 0 & 0 & \frac{1}{6}g \end{pmatrix} + \text{higher order}. \tag{7.12.10}$$

Observe that γ_a, γ_b, and γ_c are all independent of μ and m. This follows from the same arguments that we used to prove the same property for γ_m and γ.

From the RG equation (7.12.7) we prove renormalization-group equations for Green's functions of the composite operators. For example:

$$\begin{aligned} \left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} - \gamma_m m^2\frac{\partial}{\partial m^2}\right)&\langle 0|T\phi(x)\phi(y)\tfrac{1}{2}[\phi^2(z)]|0\rangle \\ &= (\gamma_a - \gamma)\langle 0|T\phi\phi\tfrac{1}{2}[\phi^2]|0\rangle \\ &\quad + (\gamma_b m^2 + \gamma_c\Box)\langle 0|T\phi(x)\phi(y)\phi(z)|0\rangle, \end{aligned} \tag{7.12.11}$$

where we have used $\mu\mathrm{d}\phi/\mathrm{d}\mu = -\gamma\phi/2$, and we have set $d = 6$, thus eliminating the $\mu^{d/2-3}$ factors.

We must prove (7.12.8) both to all orders for the $[\phi^2]$ operator and in its generalizations to deal with any operators. Since bare operators are automatically RG invariant, the only question is whether the anomalous dimensions are finite. This is handled by a simple generalization of the proof given in Section 7.2 for the ordinary Green's functions. We will not spell out the details – for that is just a mathematical exercise.