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Peter J. Olver

Applications of Lie Groups to Differential Equations

Second Edition

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Preface to the First Edition

This book is devoted to explaining a wide range of applications of continuous symmetry groups to physically important systems of differential equations. Emphasis is placed on significant applications of group-theoretic methods, organized so that the applied reader can readily learn the basic computational techniques required for genuine physical problems. The first chapter collects together (but does not prove) those aspects of Lie group theory which are of importance to differential equations. Applications covered in the body of the book include calculation of symmetry groups of differential equations, integration of ordinary differential equations, including special techniques for Euler–Lagrange equations or Hamiltonian systems, differential invariants and construction of equations with prescribed symmetry groups, group-invariant solutions of partial differential equations, dimensional analysis, and the connections between conservation laws and symmetry groups. Generalizations of the basic symmetry group concept, and applications to conservation laws, integrability conditions, completely integrable systems and soliton equations, and bi-Hamiltonian systems are covered in detail. The exposition is reasonably self-contained, and supplemented by numerous examples of direct physical importance, chosen from classical mechanics, fluid mechanics, elasticity and other applied areas. Besides the basic theory of manifolds, Lie groups and algebras, transformation groups and differential forms, the book delves into the more theoretical subjects of prolongation theory and differential equations, the Cauchy–Kovalevskaya theorem, characteristics and integrability of differential equations, extended jet spaces over manifolds, quotient manifolds, adjoint and co-adjoint representations of Lie groups, the calculus of variations and the inverse problem of characterizing those systems which are Euler–Lagrange equations of some variational problem, differential operators, higher Euler operators and the variational complex, and the general theory of Poisson structures, both for finite-dimensional Hamiltonian systems as well as systems of evolution equations, all of which have direct bearing on the symmetry analysis of differential equations. It is hoped that after reading this book, the reader will, with a minimum of difficulty, be able to readily apply these important group-theoretic methods to the systems of differential equations he or she is interested in, and make new and interesting deductions concerning them. If so, the book can be said to have served its purpose.

A preliminary version of this book first appeared as a set of lecture notes, distributed by the Mathematical Institute of Oxford University, for a graduate seminar held in Trinity term, 1979. It is my pleasure to thank the staff of Springer-Verlag for their encouragement for me to turn these notes into book form, and for their patience during the process of revision that turned out to be far more extensive than I originally anticipated.

Preface to the Second Edition

For the second edition, I have corrected a number of misprints and inadvertent mathematical errors that found their way into the original version. More substantial changes are the inclusion of a simpler proof of Theorem 4.26 due to Alonso, [1], and the omission of the false (at least in the form stated in the first edition) Theorem 5.22 on the commutativity of generalized symmetries. Also, I have corrected some of the exercises and added several new ones. Hopefully this now eliminates all of the major (and almost all of the minor) mistakes. The one substantial addition to the second edition is a short presentation of the calculus of pseudo-differential operators and their use in Shabat's theory of formal symmetries, which provides a powerful, algorithmic method for determining the integrability of evolution equations.

The years since the appearance of the original edition of the book have witnessed a remarkable explosion of research, both pure and applied, into symmetry group methods in differential equations, proceeding at a pace well beyond my expectations. Innumerable papers, as well as several substantial textbooks devoted to the subject of symmetry and differential equations, have appeared in the literature. The former are too numerous to try to list here, although I have added a few of the more notable contributions to the list of references and have correspondingly updated the historical notes at the end of each chapter. Of the latter, I recommend the books of Bluman and Kumei, [2], and Stephani [3], on symmetry methods, and Zharinov, [1], on the geometrical theory of differential equations. There has also been a lot of activity in the development of computer algebra (symbolic manipulation) computer programs to (partially) automate the determination of symmetry groups of differential equations. A good survey of the available codes, as of 1991, including a discussion of their strengths and weaknesses, can be found in the paper of Champagne, Hereman, and Winternitz, [1].

I would like to acknowledge, with gratitude, Ian Anderson, Ken Driessel, Darryl Holm, Niky Kamran, John Maddocks, Jerry Marsden, Sascha Mikhailov, and Alexei Shabat, who offered valuable comments and suggestions for improving the first edition. Finally, I should reiterate my thankfulness and love to my wife, Cheri, and children, Pari, Sheehan, and Noreen, for their continued, all-important love and support!

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