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Renormalization in Quantum Field Theory and the Riemann–Hilbert Problem II: The β-Function, Diffeomorphisms and the Renormalization Group⋆

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Abstract: We showed in Part I that the Hopf algebra $\mathcal{H}$ of Feynman graphs in a given QFT is the algebra of coordinates on a complex infinite dimensional Lie group G and that the renormalized theory is obtained from the unrenormalized one by evaluating at $\varepsilon = 0$ the holomorphic part $\gamma_+(\varepsilon)$ of the Riemann–Hilbert decomposition $\gamma_-(\varepsilon)^{-1}\gamma_+(\varepsilon)$ of the loop $\gamma(\varepsilon) \in G$ provided by dimensional regularization. We show in this paper that the group G acts naturally on the complex space X of dimensionless coupling constants of the theory. More precisely, the formula $g_0 = gZ_1Z_3^{-3/2}$ for the effective coupling constant, when viewed as a formal power series, does define a Hopf algebra homomorphism between the Hopf algebra of coordinates on the group of formal diffeomorphisms to the Hopf algebra $\mathcal{H}$. This allows first of all to read off directly, without using the group G, the bare coupling constant and the renormalized one from the Riemann–Hilbert decomposition of the unrenormalized effective coupling constant viewed as a loop of formal diffeomorphisms. This shows that renormalization is intimately related with the theory of non-linear complex bundles on the Riemann sphere of the dimensional regularization parameter ε. It also allows to lift both the renormalization group and the β-function as the asymptotic scaling in the group G. This exploits the full power of the Riemann–Hilbert decomposition together with the invariance of $\gamma_-(\varepsilon)$ under a change of unit of mass. This not only gives a conceptual proof of the existence of the renormalization group but also delivers a scattering formula in the group G for the full higher pole structure of minimal subtracted counterterms in terms of the residue.

1. Introduction

We showed in Part I of this paper [1] that perturbative renormalization is a special case of a general mathematical procedure of extraction of finite values based on the Riemann–Hilbert problem. More specifically we associated to any given renormalizable quantum

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field theory an (infinite dimensional) complex Lie group G. We then showed that passing from the unrenormalized theory to the renormalized one was exactly the replacement of the loop $d \to \gamma(d) \in G$ of elements of G obtained from dimensional regularization (for $d \neq D =$ dimension of space-time) by the value $\gamma_+(D)$ of its Birkhoff decomposition, $\gamma(d) = \gamma_-(d)^{-1}\,\gamma_+(d)$.

The original loop $d \to \gamma(d)$ not only depends upon the parameters of the theory but also on the additional "unit of mass" μ required by dimensional analysis. We shall show in this paper that the mathematical concepts developed in Part I provide very powerful tools to lift the usual concepts of the β-function and renormalization group from the space of coupling constants of the theory to the complex Lie group G.

We first observe, taking φ_6^3 as an illustrative example to fix ideas and notations, that even though the loop $\gamma(d)$ does depend on the additional parameter μ,

$$\mu \to \gamma_\mu(d), \tag{1}$$

the negative part γ_{μ^-} in the Birkhoff decomposition,

$$\gamma_\mu(d) = \gamma_{\mu^-}(d)^{-1}\,\gamma_{\mu^+}(d) \tag{2}$$

is actually independent of μ,

$$\frac{\partial}{\partial\mu}\,\gamma_{\mu^-}(d) = 0. \tag{3}$$

This is a restatement of a well known fact and follows immediately from dimensional analysis. Moreover, by construction, the Lie group G turns out to be graded, with grading,

$$\theta_t \in \operatorname{Aut} G\;, \quad t \in \mathbb{R}, \tag{4}$$

inherited from the grading of the Hopf algebra $\mathcal{H}$ of Feynman graphs given by the loop number,

$$L(\Gamma) = \text{loop number of } \Gamma \tag{5}$$

for any 1PI graph Γ.

The straightforward equality,

$$\gamma_{e^t\mu}(d) = \theta_{t\varepsilon}(\gamma_\mu(d)) \qquad \forall\, t \in \mathbb{R},\ \varepsilon = D - d \tag{6}$$

shows that the loops γ_μ associated to the unrenormalized theory satisfy the striking property that the negative part of their Birkhoff decomposition is unaltered by the operation,

$$\gamma(\varepsilon) \to \theta_{t\varepsilon}(\gamma(\varepsilon)). \tag{7}$$

In other words, if we replace $\gamma(\varepsilon)$ by $\theta_{t\varepsilon}(\gamma(\varepsilon))$ we do not change the negative part of its Birkhoff decomposition. We settle now for the variable,

$$\varepsilon = D - d \in \mathbb{C}\backslash\{0\}. \tag{8}$$

Our first result (Sect. 2) is a complete characterization of the loops $\gamma(\varepsilon) \in G$ fulfilling the above striking invariance. This characterization only involves the negative part $\gamma_-(\varepsilon)$ of their Birkhoff decomposition which by hypothesis fulfills,

$$\gamma_-(\varepsilon)\,\theta_{t\varepsilon}(\gamma_-(\varepsilon)^{-1}) \text{ is convergent for } \varepsilon \to 0. \tag{9}$$

It is easy to see that the limit of (9) for $\varepsilon \to 0$ defines a one parameter subgroup,

$$F_t \in G, \; t \in \mathbb{R} \tag{10}$$

and that the generator $\beta = \left(\frac{\partial}{\partial t} F_t\right)_{t=0}$ of this one parameter group is related to the *residue* of γ,

$$\operatorname*{Res}_{\varepsilon=0} \gamma = -\left(\frac{\partial}{\partial u} \gamma_- \left(\frac{1}{u}\right)\right)_{u=0}, \tag{11}$$

by the simple equation,

$$\beta = Y \operatorname{Res} \gamma, \tag{12}$$

where $Y = \left(\frac{\partial}{\partial t} \theta_t\right)_{t=0}$ is the grading.

This is straightforward, but our result is the following formula (14) which gives $\gamma_-(\varepsilon)$ in closed form as a function of β. We shall for convenience introduce an additional generator in the Lie algebra of G (i.e. primitive elements of $\mathcal{H}^*$) such that,

$$[Z_0, X] = Y(X) \qquad \forall\, X \in \operatorname{Lie} G. \tag{13}$$

The scattering formula for $\gamma_-(\varepsilon)$ is then,

$$\gamma_-(\varepsilon) = \lim_{t \to \infty} e^{-t\left(\frac{\beta}{\varepsilon} + Z_0\right)} \, e^{t Z_0}. \tag{14}$$

Both factors in the right-hand side belong to the semi-direct product,

$$\widetilde{G} = G \underset{\theta}{\rtimes} \mathbb{R} \tag{15}$$

of the group G by the grading, but of course the ratio (14) belongs to the group G.

This shows (Sect. 3) that the higher pole structure of the divergences is uniquely determined by the residue and gives a strong form of the t'Hooft relations, which will come as an immediate corollary.

In Sect. 4 we show, specializing to the massless case, that the formula for the bare coupling constant,

$$g_0 = g \, Z_1 \, Z_3^{-3/2}, \tag{16}$$

where both $g \, Z_1 = g + \delta g$ and the field strength renormalization constant Z_3 are thought of as power series (in g) of elements of the Hopf algebra $\mathcal{H}$, does define a Hopf algebra homomorphism,

$$\mathcal{H}_{CM} \xrightarrow{g_0} \mathcal{H}, \tag{17}$$

from the Hopf algebra $\mathcal{H}_{CM}$ of coordinates on the group of formal diffeomorphisms of $\mathbb{C}$ such that,

$$\varphi(0) = 0, \; \varphi'(0) = \mathrm{id} \tag{18}$$

to the Hopf algebra $\mathcal{H}$ of the massless theory. We had already constructed in [2] a Hopf algebra homomorphism from $\mathcal{H}_{\mathrm{CM}}$ to the Hopf algebra of rooted trees, but the physical significance of this construction was unclear.

The homomorphism (17) is quite different in that for instance the transposed group homomorphism,

$$G \xrightarrow{\rho} \operatorname{Diff}(\mathbb{C}) \tag{19}$$

lands in the subgroup of *odd* diffeomorphisms,

$$\varphi(-z) = -\varphi(z) \qquad \forall\, z. \tag{20}$$

Moreover its physical significance will be transparent. We shall show in particular that the image by ρ of $\beta = Y\,\mathrm{Res}\,\gamma$ is the usual β-function of the coupling constant g.

We discovered the homomorphism (17) by lengthy concrete computations. We have chosen to include them in an appendix besides our conceptual proof given in Sect. 4. The main reason for this choice is that the explicit computation allows to validate the concrete ways of handling the coproduct, coassociativity, symmetry factors... that underly the theory.

As a corollary of the construction of ρ we get an *action* by (formal) diffeomorphisms of the group G on the space X of (dimensionless) coupling constants of the theory. We can then in particular formulate the Birkhoff decomposition *directly* in the group,

$$\mathrm{Diff}\,(X) \tag{21}$$

of formal diffeomorphisms of the space of coupling constants.

The unrenormalized theory delivers a loop

$$\delta(\varepsilon) \in \mathrm{Diff}\,(X), \quad \varepsilon \neq 0 \tag{22}$$

whose value at ε is simply the unrenormalized effective coupling constant.

The Birkhoff decomposition,

$$\delta(\varepsilon) = \delta_+(\varepsilon)\,\delta_-(\varepsilon)^{-1} \tag{23}$$

of this loop gives directly then,

$$\delta_-(\varepsilon) = \text{ bare coupling constant} \tag{24}$$

and,

$$\delta_+(D) = \text{renormalized effective coupling constant.} \tag{25}$$

This result now, in its statement, no longer depends upon our group G or the Hopf algebra $\mathcal{H}$. But of course the proof makes heavy use of the above ingredients.

Now the Birkhoff decomposition of a loop,

$$\delta(\varepsilon) \in \mathrm{Diff}\,(X), \tag{26}$$

admits a beautiful geometric interpretation. If we let X be a complex manifold and pass from formal diffeomorphisms to actual ones, the data (26) is the initial data to perform, by the clutching operation, the construction of a complex bundle,

$$P = (S^+ \times X) \cup_\delta (S^- \times X) \tag{27}$$

over the sphere $S = P_1(\mathbb{C}) = S^+ \cup S^-$, and with fiber X,

$$X \longrightarrow P \xrightarrow{\pi} S. \tag{28}$$

The meaning of the Birkhoff decomposition (23),

$$\delta(\varepsilon) = \delta_+\,(\varepsilon)\,\delta_-(\varepsilon)^{-1}$$

is then exactly captured by an isomorphism of the bundle P with the trivial bundle,

$$S \times X. \tag{29}$$

2. Asymptotic Scaling in Graded Complex Lie Groups

We shall first prove the formula (14) of the introduction in the general context of graded Hopf algebras and then apply it to the Birkhoff decomposition of the loop associated in Part I to the unrenormalized theory.

We let $\mathcal{H}$ be a connected commutative graded Hopf algebra (connected means that $\mathcal{H}^{(0)} = \mathbb{C}$) and let θ_t, $t \in \mathbb{R}$ be the one parameter group of automorphisms of $\mathcal{H}$ associated with the grading so that for $x \in \mathcal{H}$ of degree n,

$$\theta_t(x) = e^{tn}\, x \qquad \forall\, t \in \mathbb{R}. \tag{1}$$

By construction θ_t is a Hopf algebra automorphism,

$$\theta_t \in \mathrm{Aut}(\mathcal{H}). \tag{2}$$

We also let $Y = \left(\frac{\partial}{\partial t}\, \theta_t\right)_{t=0}$ be the generator which is a derivation of $\mathcal{H}$.

We let G be the group of characters of $\mathcal{H}$,

$$vp : \mathcal{H} \to \mathbb{C}, \tag{3}$$

i.e. of homomorphisms from the algebra $\mathcal{H}$ to $\mathbb{C}$. The product in G is given by,

$$(\varphi_1\, \varphi_2)(x) = \langle \varphi_1 \otimes \varphi_2,\ \Delta x\rangle, \tag{4}$$

where Δ is the coproduct in $\mathcal{H}$. The augmentation $\overline{e}$ of $\mathcal{H}$ is the unit of G and the inverse of $\varphi \in G$ is given by,

$$\langle \varphi^{-1}, x\rangle = \langle \varphi, Sx\rangle, \tag{5}$$

where S is the antipode in $\mathcal{H}$.

We let L be the Lie algebra of derivations,

$$\delta : \mathcal{H} \to \mathbb{C}, \tag{6}$$

i.e. of linear maps on $\mathcal{H}$ such that

$$\delta(xy) = \delta(x)\, \overline{e}(y) + \overline{e}(x)\, \delta(y) \qquad \forall\, x, y \in \mathcal{H}. \tag{7}$$

Even if $\mathcal{H}$ is of finite type so that $\mathcal{H}^{(n)}$ is finite dimensional for any $n \in \mathbb{N}$, there are more elements in L than in the Lie algebra P of primitive elements,

$$\Delta Z = Z \otimes 1 + 1 \otimes Z \tag{8}$$

in the graded dual Hopf algebra $\mathcal{H}^*_{\mathrm{gr}}$ of $\mathcal{H}$. But one passes from P to L by completion relative to the I-adic topology, I being the augmentation ideal of $\mathcal{H}^*_{\mathrm{gr}}$.

The linear dual $\mathcal{H}^*$ is in general an algebra (with product given by (4)) but not a Hopf algebra since the coproduct is not necessarily well defined. It is however well defined for characters φ or derivations δ which satisfy respectively $\Delta\varphi = \varphi \otimes \varphi$, and $\Delta\delta = \delta \otimes 1 + 1 \otimes \delta$.

For $\delta \in L$ the expression,

$$\varphi = \exp\, \delta \tag{9}$$

makes sense in the algebra $\mathcal{H}^*$ since when evaluated on $x \in \mathcal{H}$ one has $\langle x, \delta^n\rangle = 0$ for n large enough (since $\langle x, \delta^n\rangle = \langle \Delta^{(n-1)} x, \delta \otimes \cdots \otimes \delta\rangle$ vanishes for $n > \deg x$). Moreover φ is a group-like element of $\mathcal{H}^*$, i.e. a character of $\mathcal{H}$. Thus $\varphi \in G$.

The one parameter group $\theta_t \in \mathrm{Aut}\,(\mathcal{H})$ acts by automorphisms on the group G,

$$\langle \theta_t(\varphi), x\rangle = \langle \varphi, \theta_t(x)\rangle \qquad \forall\, x \in \mathcal{H} \tag{10}$$

and the derivation Y of $\mathcal{H}$ acts on L by

$$\langle Y(\delta), x\rangle = \langle \delta, Y(x)\rangle, \tag{11}$$

and defines a derivation of the Lie algebra L where we recall that the Lie bracket in L is given by,

$$\langle [\delta_1, \delta_2], x\rangle = \langle \delta_1 \otimes \delta_2 - \delta_2 \otimes \delta_1,\ \Delta x\rangle \qquad \forall\, x \in \mathcal{H}. \tag{12}$$

Let us now consider a map,

$$\varepsilon \in \mathbb{C}\backslash\{0\} \to \varphi_\varepsilon \in G \tag{13}$$

such that for any $x \in \mathcal{H}$, $\overline{e}(x) = 0$ one has,

$$\varepsilon \to \langle \varphi_\varepsilon, x\rangle \text{ is a polynomial in } \frac{1}{\varepsilon} \text{ without constant term.} \tag{14}$$

Thus $\varepsilon \to \varphi_\varepsilon$ extends to a map from $P_1(\mathbb{C})\backslash\{0\}$ to G, such that

$$\varphi_\infty = 1. \tag{15}$$

For such a map we define its *residue* as the derivative at ∞, i.e. as,

$$\mathrm{Res}\,\varphi = \lim_{\varepsilon\to\infty} \varepsilon(\varphi_\varepsilon - 1). \tag{16}$$

By construction $\mathrm{Res}\,\varphi \in L$ is a derivation $\mathcal{H} \to \mathbb{C}$. When evaluated on $x \in \mathcal{H}$, $\mathrm{Res}\,\varphi$ is just the residue at $\varepsilon = 0$ of the function $\varepsilon \to \langle \varphi_\varepsilon, x\rangle$.

We shall now assume that for any $t \in \mathbb{R}$ the following limit exists for any $x \in \mathcal{H}$,

$$\lim_{\varepsilon\to 0} \langle \varphi_\varepsilon^{-1}\, \theta_{t\varepsilon}(\varphi_\varepsilon), x\rangle. \tag{17}$$

Using (10), (4) and (5) we have,

$$\langle \varphi_\varepsilon^{-1}\, \theta_{t\varepsilon}(\varphi_\varepsilon), x\rangle = \langle \varphi_\varepsilon \otimes \varphi_\varepsilon, (S \otimes \theta_{t\varepsilon})\, \Delta x\rangle, \tag{18}$$

so that with $\Delta x = \sum x_{(1)} \otimes x_{(2)}$ we get a sum of terms $\langle \varphi_\varepsilon, S\, x_{(1)}\rangle\, \langle \varphi_\varepsilon, \theta_{t\varepsilon}(x_{(2)})\rangle = P_1\left(\frac{1}{\varepsilon}\right) e^{kt\varepsilon}\, P_2\left(\frac{1}{\varepsilon}\right)$. Thus (17) just means that the sum of these terms is holomorphic at $\varepsilon = 0$. It is clear that the value at $\varepsilon = 0$ is then a polynomial in t,

$$\langle F_t, x\rangle = \lim_{\varepsilon\to 0} \langle \varphi_\varepsilon^{-1}\, \theta_{t\varepsilon}(\varphi_\varepsilon), x\rangle. \tag{19}$$

Let us check that $t \to F_t \in G$ is a one parameter group,

$$F_{t_1+t_2} = F_{t_1}\, F_{t_2} \qquad \forall\, t_i \in \mathbb{R}. \tag{20}$$

The group G is a topological group for the topology of simple convergence, i.e.,

$$\varphi_n \to \varphi \quad \text{iff} \quad \langle \varphi_n, x\rangle \to \langle \varphi, x\rangle \qquad \forall\, x \in \mathcal{H}. \tag{21}$$

Moreover, using (10) one checks that

$$\theta_{t_1\varepsilon}(\varphi_\varepsilon^{-1}\, \theta_{t_2\varepsilon}(\varphi_\varepsilon)) \to F_{t_2} \qquad \text{when } \varepsilon \to 0. \tag{22}$$

We then have $F_{t_1+t_2} = \lim_{\varepsilon\to 0} \varphi_\varepsilon^{-1}\, \theta_{(t_1+t_2)\varepsilon}\,(\varphi_\varepsilon) = \lim_{\varepsilon\to 0} \varphi_\varepsilon^{-1}\, \theta_{t_1\varepsilon}(\varphi_\varepsilon)\, \theta_{t_1\varepsilon}(\varphi_\varepsilon^{-1}\, \theta_{t_2\varepsilon}(\varphi_\varepsilon)) = F_{t_1}\, F_{t_2}$.

This proves (20) and we let,

$$\beta = \left(\frac{\partial}{\partial t}\, F_t\right)_{t=0} \tag{23}$$

which defines an element of L such that,

$$F_t = \exp(t\beta) \qquad \forall\, t \in \mathbb{R}. \tag{24}$$

As above, we view $\mathcal{H}^*$ as an algebra on which Y acts as a derivation by (11). Let us prove,

Lemma 1. *Let $\varepsilon \to \varphi_\varepsilon \in G$ satisfy* (17) *with $\varphi_\varepsilon = 1 + \sum_{n=1}^{\infty} \frac{d_n}{\varepsilon^n}$, $d_n \in \mathcal{H}^*$. One then has*

$$Y(d_1) = \beta \qquad Y\, d_{n+1} = d_n\, \beta \qquad \forall\, n \geq 1.$$

Proof. Let $x \in \mathcal{H}$ and let us show that

$$\langle \beta, x\rangle = \lim_{\varepsilon\to 0}\, \varepsilon \langle \varphi_\varepsilon \otimes \varphi_\varepsilon, (S \otimes Y)\, \Delta(x)\rangle. \tag{25}$$

Using (18) we know by hypothesis that,

$$\langle \varphi_\varepsilon \otimes \varphi_\varepsilon, (S \otimes \theta_{t\varepsilon})\, \Delta(x)\rangle \to \langle F_t, x\rangle, \tag{26}$$

where the convergence holds in the space of holomorphic functions of t in say $|t| \leq 1$ so that the derivatives of both sides at $t = 0$ are also convergent, thus yielding (25).

Now the function $\varepsilon \to \varepsilon\, \langle \varphi_\varepsilon \otimes \varphi_\varepsilon, (S \otimes Y)\, \Delta x\rangle$ is holomorphic for $\varepsilon \in \mathbb{C}\backslash\{0\}$ and also at $\varepsilon = \infty \in P_1(\mathbb{C})$ since $\varphi_\infty = 1$. Moreover by (25) it is also holomorphic at $\varepsilon = 0$ and is thus a constant, which gives,

$$\langle \varphi_\varepsilon \otimes \varphi_\varepsilon, (S \otimes Y)\, \Delta(x)\rangle = \frac{1}{\varepsilon}\, \langle \beta, x\rangle. \tag{27}$$

Using the product in $\mathcal{H}^*$ this means that

$$\varphi_\varepsilon^{-1}\, Y(\varphi_\varepsilon) = \frac{1}{\varepsilon}\, \beta, \tag{28}$$

and multiplying by φ_ε on the left, that,

$$Y(\varphi_\varepsilon) = \frac{1}{\varepsilon}\, \varphi_\varepsilon\, \beta. \tag{29}$$

One has $Y(\varphi_\varepsilon) = \sum_{n=1}^{\infty} \frac{Y(d_n)}{\varepsilon^n}$ and $\frac{1}{\varepsilon}\, \varphi_\varepsilon\, \beta = \frac{1}{\varepsilon}\, \beta + \sum_{n=1}^{\infty} \frac{1}{\varepsilon^{n+1}}\, d_n\, \beta$. Thus (29) gives the lemma. $\square$

In particular we get $Y(d_1) = \beta$ and since d_1 is the residue, $\operatorname{Res}\varphi$, this gives,

$$\beta = Y\,(\operatorname{Res}\varphi), \tag{30}$$

which shows that β is uniquely determined by the residue of φ_ε.

We shall now write a formula for φ_ε in terms of β. This is made possible by Lemma 1 which shows that β uniquely determines φ_ε. What is not transparent from Lemma 1 is that for $\beta \in L$ the elements $\varphi_\varepsilon \in \mathcal{H}^*$ are group-like, so that $\varphi_\varepsilon \in G$. In order to obtain a nice formula we take the semi direct product of G by $\mathbb{R}$ acting on G by the grading θ_t,

$$\widetilde{G} = G \underset{\theta}{\rtimes} \mathbb{R}, \tag{31}$$

and similarly we let $\widetilde{L}$ be the Lie algebra

$$\widetilde{L} = L \oplus \mathbb{C}\, Z_0, \tag{32}$$

where the Lie bracket is given by

$$[Z_0, \alpha] = Y(\alpha) \qquad \forall\, \alpha \in L \tag{33}$$

and extends the Lie bracket of L.

We view $\widetilde{L}$ as the Lie algebra of $\widetilde{G}$ in a way which will become clear in the proof of the following,

Theorem 2. *Let $\varepsilon \to \varphi_\varepsilon \in G$ satisfy* (17) *as above. Then with $\beta = Y(\operatorname{Res}\varphi)$ one has,*

$$\varphi_\varepsilon = \lim_{t\to\infty} e^{-tZ_0}\, e^{t\left(\frac{\beta}{\varepsilon}+Z_0\right)}.$$

The limit holds in the topology of simple convergence in G. Both terms e^{-tZ_0} and $e^{t\left(\frac{\beta}{\varepsilon}+Z_0\right)}$ belong to $\widetilde{G}$ but their product belongs to G.

Proof. We endow $\mathcal{H}^*$ with the topology of simple convergence on $\mathcal{H}$ and let θ_t act by automorphisms of the topological algebra $\mathcal{H}^*$ by (10). Let us first show, with,

$$\varphi_\varepsilon = 1 + \sum_{n=1}^{\infty} \frac{d_n}{\varepsilon^n}, \quad d_n \in \mathcal{H}^*, \tag{34}$$

that the following holds,

$$d_n = \int_{s_1 \geq s_2 \geq \cdots \geq s_n \geq 0} \theta_{-s_1}(\beta)\,\theta_{-s_2}(\beta)\ldots\theta_{-s_n}(\beta)\,\Pi\, ds_i. \tag{35}$$

For $n = 1$, this just means that,

$$d_1 = \int_0^\infty \theta_{-s}(\beta)\, ds, \tag{36}$$

which follows from (30) and the equality

$$Y^{-1}(x) = \int_0^\infty \theta_{-s}(x)\, ds \qquad \forall\, x \in \mathcal{H},\ \bar{e}(x) = 0. \tag{37}$$

We see from (37) that for $\alpha, \alpha' \in \mathcal{H}^*$ such that

$$Y(\alpha) = \alpha', \ \langle \alpha, 1 \rangle = \langle \alpha', 1 \rangle = 0 \tag{38}$$

one has,

$$\alpha = \int_0^\infty \theta_{-s}(\alpha')\, ds. \tag{39}$$

Combining this equality with Lemma 1 and the fact that $\theta_s \in \mathrm{Aut}\, \mathcal{H}^*$ is an automorphism, gives an inductive proof of (35). The meaning of this formula should be clear; we pair both sides with $x \in \mathcal{H}$, and let

$$\Delta^{(n-1)}\, x = \sum x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(n)}. \tag{40}$$

Then the right-hand side of (35) is just,

$$\int_{s_1 \geq \cdots \geq s_n \geq 0} \langle \beta \otimes \cdots \otimes \beta \ , \ \theta_{-s_1}(x_{(1)}) \otimes \theta_{-s_2}(x_{(2)}) \cdots \otimes \theta_{-s_n}(x_{(n)}) \rangle \, \Pi \, ds_i \,, \tag{41}$$

and the convergence of the multiple integral is exponential since,

$$\langle \beta, \theta_{-s}(x_{(i)}) \rangle = O\,(e^{-s}) \qquad \text{for} \quad s \to +\infty. \tag{42}$$

We see moreover that if x is homogeneous of degree $\deg(x)$, and if $n > \deg(x)$, at least one of the $x_{(i)}$ has degree 0 so that $\langle \beta, \theta_{-s}(x_{(i)}) \rangle = 0$ and (41) gives 0. This shows that the pairing of φ_ε with $x \in \mathcal{H}$ only involves finitely many non-zero terms in the formula,

$$\langle \varphi_\varepsilon, x \rangle = \overline{e}(x) + \sum_{n=1}^{\infty} \frac{1}{\varepsilon^n} \, \langle d_n, x \rangle. \tag{43}$$

With all convergence problems out of the way we can now proceed to prove the formula of Theorem 2 without care for convergence.

Let us first recall the expansional formula [3],

$$e^{(A+B)} = \sum_{n=0}^{\infty} \int_{\sum u_j = 1,\, u_j \geq 0} e^{u_0 A}\, B e^{u_1 A} \ldots B e^{u_n A} \, \Pi \, du_j \tag{44}$$

(cf. [3] for the exact range of validity of (44)).

We apply this with $A = t Z_0$, $B = t\beta$, $t > 0$ and get,

$$e^{t(\beta + Z_0)} = \sum_{n=0}^{\infty} \int_{\sum v_j = t,\, v_j \geq 0} e^{v_0 Z_0}\, \beta e^{v_1 Z_0}\, \beta \ldots \beta e^{v_n Z_0} \, \Pi \, dv_j. \tag{45}$$

Thus, with $s_1 = t - v_0$, $s_1 - s_2 = v_1, \ldots, s_{n-1} - s_n = v_{n-1}$, $s_n = v_n$ and replacing β by $\frac{1}{\varepsilon}\,\beta$, we obtain,

$$e^{t(\beta/\varepsilon + Z_0)} = \sum_{n=0}^{\infty} \frac{1}{\varepsilon^n} \int_{t \geq s_1 \geq s_2 \geq \cdots \geq s_n \geq 0} e^{t Z_0}\, \theta_{-s_1}(\beta) \ldots \theta_{-s_n}(\beta) \, \Pi \, ds_i. \tag{46}$$

Multiplying by e^{-tZ_0} on the left and using (41) thus gives,

$$\varphi_\varepsilon = \lim_{t \to \infty} e^{-tZ_0}\, e^{t(\beta/\varepsilon + Z_0)}. \tag{47}$$

□

It is obvious conversely that this formula defines a family $\varepsilon \to \varphi_\varepsilon$ of group-like elements of $\mathcal{H}^*$ associated to any preassigned element $\beta \in L$.

Corollary 3. *For any $\beta \in L$ there exists a (unique) map $\varepsilon \to \varphi_\varepsilon \in G$ satisfying* (17) *and* (34).

3. The Renormalization Group Flow

Let us now apply the above results to the group G associated in Part I to the Hopf algebra $\mathcal{H}$ of 1PI Feynman graphs of a quantum field theory. We choose φ_6^3 for simplicity. As explained in Part I the group G is a semi-direct product,

$$G = G_0 \rtimes G_c \tag{1}$$

of an abelian group G_0 by the group G_c associated to the Hopf subalgebra $\mathcal{H}_c$ constructed on 1PI graphs with two or three external legs and fixed external structure. Passing from G_c to G is a trivial step and we shall thus concentrate on the group G_c. The unrenormalized theory delivers, using dimensional regularization with the unit of mass μ, a loop,

$$\varepsilon \to \gamma_\mu(\varepsilon) \in G_c, \tag{2}$$

and we first need to see the exact μ dependence of this loop. We consider the grading of $\mathcal{H}_c$ and G_c given by the loop number of a graph,

$$L(\Gamma) = I - V + 1, \tag{3}$$

where I is the number of internal lines and V the number of vertices.

One has,

$$\gamma_{e^t\mu}(\varepsilon) = \theta_{t\varepsilon}(\gamma_\mu(\varepsilon)) \qquad \forall\, t \in \mathbb{R}. \tag{4}$$

Let us check this using the formulas of Sect. 3 of Part I. For $N = 2$ external legs the dimension B of $\langle \sigma, U_\Gamma \rangle$ is equal to 0 by (12) of loc.cit. Thus the μ dependence is given by

$$\mu^{\frac{\varepsilon}{2} V_3}, \tag{5}$$

where V_3 is the number of 3-point vertices of Γ. One checks that $\frac{1}{2}\, V_3 = L$ as required. Similarly if $N = 3$ the dimension B of $\langle \sigma, U_\Gamma \rangle$ is equal to $\left(1 - \frac{3}{2}\right)\, d + 3$, $d = 6 - \varepsilon$ by (12) of loc.cit. so that the μ-dependence is,

$$\mu^{\frac{\varepsilon}{2} V_3}\, \mu^{-\varepsilon/2}. \tag{6}$$

But this time, $V_3 = 2L + 1$ and we get

$$\mu^{\varepsilon L} \tag{7}$$

as required.

We now reformulate a well known result, the fact that counterterms, once appropriately normalized, are independent of m^2 and μ^2.

Lemma 4. *Let $\gamma_\mu = (\gamma_{\mu^-})^{-1}\,(\gamma_{\mu^+})$ be the Birkhoff decomposition of γ_μ. Then γ_{μ^-} is independent of μ.*

As in Part I we perform the Birkhoff decomposition with respect to a small circle C with center $D = 6$ and radius < 1.

Proof. The proof of the lemma follows immediately from [4]. Indeed the dependence in m^2 has in the minimal subtraction scheme the same origin as the dependence in p^2 and we have chosen the external structure of graphs (Eq. (41) of Part I) so that no m^2 dependence is left [1]. But then, since μ^2 is a dimensionful parameter, it cannot be involved any longer. $\square$

Corollary 5. *Let $\varphi_\varepsilon = (\gamma_{\mu^-})^{-1}(\varepsilon)$, then for any $t \in \mathbb{R}$ the following limit exists in G_c:*

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon^{-1}\, \theta_{t\varepsilon}(\varphi_\varepsilon).$$

In other words $\varepsilon \to \varphi_\varepsilon \in G_c$ fulfills condition (17) of Sect. 2.

Proof. The product $\varphi_\varepsilon^{-1}\, \gamma_\mu(\varepsilon)$ is holomorphic at $\varepsilon = 0$ for any value of μ. Thus by (4), for any $t \in \mathbb{R}$, both $\varphi_\varepsilon^{-1}\, \gamma_\mu(\varepsilon)$ and $\varphi_\varepsilon^{-1}\, \theta_{\varepsilon t}(\gamma_\mu(\varepsilon))$ are holomorphic at $\varepsilon = 0$. The same holds for $\theta_{-\varepsilon t}\,(\varphi_\varepsilon^{-1})\, \gamma_\mu(\varepsilon)$ and hence for the ratio

$$\varphi_\varepsilon^{-1}\, \gamma_\mu(\varepsilon)\, (\theta_{-\varepsilon t}(\varphi_\varepsilon^{-1})\, \gamma_\mu(\varepsilon))^{-1} = \varphi_\varepsilon^{-1}\, \theta_{-\varepsilon t}(\varphi_\varepsilon). \qquad \square$$

We let $\gamma_-(\varepsilon) = \varphi_\varepsilon^{-1}$ and translate the results of Sect. 2.

Corollary 6. *Let $F_t = \lim_{\varepsilon \to 0} \gamma_-(\varepsilon)\, \theta_{t\varepsilon}(\gamma_-\,(\varepsilon)^{-1})$. Then F_t is a one parameter subgroup of G_c and $F_t = \exp(t\beta)$, where $\beta = Y\,\mathrm{Res}\,\varphi_\varepsilon$ is the grading operator Y applied to the residue of the loop $\gamma(\varepsilon)$.*

In general, given a loop $\varepsilon \to \gamma(\varepsilon) \in G$ it is natural to define its *residue* at $\varepsilon = 0$ by first performing the Birkhoff decomposition on a small circle C around $\varepsilon = 0$ and then taking

$$\mathrm{Res}_{\varepsilon=0}\, \gamma = \frac{\partial}{\partial u}\, (\varphi_{1/u})_{u=0}, \tag{8}$$

where $\varphi_\varepsilon = \gamma_-(\varepsilon)^{-1}$ and $\gamma(\varepsilon) = \gamma_-(\varepsilon)^{-1}\, \gamma_+(\varepsilon)$ is the Birkhoff decomposition.

As shown in Sect. 2, the residue or equivalently $\beta = Y\,\mathrm{Res}$ uniquely determines $\varphi_\varepsilon = \gamma_-(\varepsilon)^{-1}$ and we thus get, from Theorem 2,

Corollary 7. *The negative part $\gamma_-(\varepsilon)$ of the Birkhoff decomposition of $\gamma_\mu(\varepsilon)$ is independent of μ and given by,*

$$\gamma_-(\varepsilon) = \lim_{t \to \infty} e^{-t\left(\frac{\beta}{\varepsilon} + Z_0\right)}\, e^{tZ_0}.$$

As above we adjoined the primitive element Z_0 to implement the grading Y (cf. Sect. 2). Our choice of the letter β is of course not innocent and we shall see in Sect. 5 the relation with the β-function.

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[1] This can be easily achieved by maintaining non-vanishing fixed external momenta. γ_{μ^-} is independent on such external structures by construction [1].

4. The Action of G_c on the Coupling Constants

We shall show in this section that the formula for the bare coupling constant g_0 in terms of 1PI graphs, i.e. the generating function,

$$g_0 = (g\, Z_1)\, (Z_3)^{-3/2}, \tag{1}$$

where we consider the right-hand side as a formal power series with values in $\mathcal{H}_c$ given explicitly by, (with $\ell = L(\Gamma)$ the loop number of the graphs),

$$g_0 = \left(x + \sum x^{2l+1} \frac{\Gamma}{S(\Gamma)} \right) \left(1 - \sum x^{2l} \frac{\Gamma}{S(\Gamma)} \right)^{-3/2} \tag{2}$$

does define a Hopf algebra homomorphism,

$$\Phi : \mathcal{H}_{\mathrm{CM}} \to \mathcal{H} \tag{3}$$

from the Hopf algebra $\mathcal{H}_{\mathrm{CM}}$ of coordinates on the group of formal diffeomorphisms of $\mathbb{C}$ with

$$\varphi(0) = 0, \;\; \varphi'(0) = 1, \tag{4}$$

to the Hopf algebra $\mathcal{H}_c$ of 1PI graphs.

This result is only valid if we perform on $\mathcal{H}_c$ the simplification that pertains to the massless case, $m = 0$, but because of the m-independence of the counterterms all the corollaries will be valid in general. The desired simplification comes because in the case $m = 0$ there is no need to indicate by a cross left on an internal line the removal of a self energy subgraph. Indeed and with the notations of Part I we can first of all ignore all the $_{(0)}$ since $m^2 = 0$; moreover the $_{k}\;{}_{(1)}$ yield a k^2 term which exactly cancels out with the additional propagator when we remove the subgraph and replace it by $_{(1)}$. This shows that we can simply ignore all these crosses and write coproducts in the simplest possible way. To get familiar with this coproduct and with the meaning of the Hopf algebra morphism (3) we urge the reader to begin by the concrete computation done in the appendix, which checks its validity up to order six in the coupling constant.

Let us now be more explicit on the meaning of formula (2). We first expand g_0 as a power series in x and get a series of the form,

$$g_0 = x + \sum_2^{\infty} \alpha_n\, x^n, \tag{5}$$

where the even coefficients α_{2n} are zero and the coefficients α_{2n+1} are finite linear combinations of products of graphs, so that,

$$\alpha_{2n+1} \in \mathcal{H} \qquad \forall n \geq 1. \tag{6}$$

We let $\mathcal{H}_{\mathrm{CM}}$ be the Hopf algebra of the group of formal diffeomorphisms such that (4) holds. We take the generators a_n of $\mathcal{H}_{\mathrm{CM}}$ given by the equality

$$\varphi(x) = x + \sum_{n \geq 2} a_n(\varphi)\, x^n, \tag{7}$$

and define the coproduct in $\mathcal{H}_{\mathrm{CM}}$ by the equality

$$\langle \Delta a_n,\ \varphi_1 \otimes \varphi_2 \rangle = a_n(\varphi_2 \circ \varphi_1). \tag{8}$$

We then define uniquely the algebra homomorphism

$$\Phi : \mathcal{H}_{\mathrm{CM}} \to \mathcal{H}$$

by the condition,

$$\Phi(a_n) = \alpha_n. \tag{9}$$

By construction Φ is a morphism of algebras. We shall show that it is comultiplicative, i.e.

$$(\Phi \otimes \Phi)\, \Delta\, x = \Delta\, \Phi(x) \qquad \forall\, x \in \mathcal{H}_{\mathrm{CM}} \tag{10}$$

and comes from a group morphism,

$$\rho : G_c \to G_2, \tag{11}$$

where G_2 is the group of characters of $\mathcal{H}_{\mathrm{CM}}$ which is by construction the opposite of the group of formal diffeomorphisms. In fact we shall first describe the corresponding Lie algebra morphism, ρ.

Let us first recall from Part I that a 1PI graph Γ defines a primitive element (Γ) of $\mathcal{H}_{\mathrm{gr}}^*$ which only pairs nontrivially with the monomial Γ of $\mathcal{H}$ and satisfies $\langle (\Gamma),\ \Gamma \rangle = 1$. We take the following natural basis $\underline{\Gamma} = S(\Gamma)(\Gamma)$ for the Lie algebra of primitive elements of $\mathcal{H}_{\mathrm{gr}}^*$, labelled by 1PI graphs with two or three external legs. By Part I, Theorem 2, their Lie bracket is given by

$$[\underline{\Gamma}, \underline{\Gamma'}] = \sum_{v'} \underline{\Gamma' \circ^{v'} \Gamma} - \sum_{v} \underline{\Gamma \circ^{v} \Gamma'}, \tag{12}$$

where $\Gamma' \circ_{v'} \Gamma$ is the graph obtained by grafting Γ at the vertex v' of Γ'. (Our basis differs from the one used in loc. cit. by an overall – sign, but the present choice will be more convenient.) In our context of the simplified Hopf algebra the places where a given graph Γ can be inserted in another graph Γ' are no longer always labelled by vertices of Γ'. They are when Γ is a vertex graph but when Γ is a self energy graph such places are just labelled by the internal lines of Γ', as we could discard the use of external structures and two-point vertices for self energy graphs.

We also let Z'_n be the natural basis of the Lie algebra of primitive elements of $\mathcal{H}_{\mathrm{CM}}^*$ which corresponds to the vector fields $x^{n+1}\,\frac{\partial}{\partial x}$. More precisely, Z'_n is given as the linear form on $\mathcal{H}_{\mathrm{CM}}$ which only pairs with the monomial a_{n+1},

$$\langle Z'_n,\ a_{n+1} \rangle\ = 1, \tag{13}$$

and the Lie bracket is given by,

$$[Z'_n, Z'_m] = (m - n)\, Z'_{n+m}. \tag{14}$$

We then first prove,

Lemma 8. *Let $\rho_\Gamma = \frac{3}{2}$ for 2-point graphs and $\rho_\Gamma = 1$ for 3-point graphs. The equality $\rho(\underline{\Gamma}) = \rho_\Gamma\, Z'_{2\ell}$, where $\ell = L(\Gamma)$ is the loop number, defines a Lie algebra homomorphism.*

Proof. We just need to show that ρ preserves the Lie bracket. Let us first assume that Γ_1, Γ_2 are vertex graphs and let V_i be the vertex number of Γ_i. One has,

$$V = 2L + 1 \tag{15}$$

for any vertex graph Γ. Thus the Lie bracket $\rho\,[\underline{\Gamma}_1, \underline{\Gamma}_2]$ provides $V_2 - V_1 = 2(L_2 - L_1)$ vertex graph contributions all equal to $Z'_{2(L_1+L_2)}$ so that

$$\rho\,([\underline{\Gamma}_1, \underline{\Gamma}_2]) = 2(L_2 - L_1)\, Z'_{2(L_1+L_2)} \tag{16}$$

which is exactly $[\rho\,(\underline{\Gamma}_1), \rho\,(\underline{\Gamma}_2)]$ by (14).

Let then Γ_1 and Γ_2 be 2-point graphs. For any such graph one has

$$I = 3L - 1, \tag{17}$$

where I is the number of internal lines of Γ. Thus $\rho\,([\underline{\Gamma}_1, \underline{\Gamma}_2])$ gives $I_2 - I_1 = 3(L_2 - L_1)$ 2-point graph contributions, each equal to $\frac{3}{2}\, Z'_{2(L_1+L_2)}$. Thus,

$$\rho\,([\underline{\Gamma}_1, \underline{\Gamma}_2]) = \frac{3}{2}\, 3(L_2 - L_1)\, Z'_{2(L_1+L_2)}, \tag{18}$$

but the right-hand side is $\rho_{\Gamma_1}\, \rho_{\Gamma_2}\, 2(L_2 - L_1)\, Z'_{2(L_1+L_2)}$ so that,

$$\rho\,([\underline{\Gamma}_1, \underline{\Gamma}_2]) = [\rho\,(\underline{\Gamma}_1), \rho\,(\underline{\Gamma}_2)], \tag{19}$$

as required.

Finally if say Γ_1 is a 3-point graph and Γ_2 a 2-point graph, we get from $[\underline{\Gamma}_1, \underline{\Gamma}_2]$ a set of V_2 2-point graphs minus I_1 3-point graphs which gives,

$$\left(\frac{3}{2}\, V_2 - I_1\right) Z'_{2(L_1+L_2)}. \tag{20}$$

One has $V_2 = 2\, L_2$, $I_1 = 3\, L_1$ so that $\frac{3}{2}\, V_2 - I_1 = 3(L_2 - L_1) = \rho_{\Gamma_1}\, \rho_{\Gamma_2}\, 2(L_2 - L_1)$ which gives (19) as required. $\square$

We now have the Lie algebra morphism ρ and the algebra morphism Φ. To ρ corresponds a morphism of groups,

$$\rho : G_c \to G_2, \tag{21}$$

and we just need to check that the algebra morphism Φ is the transposed of ρ on the coordinate algebras,

$$\Phi(a) = a \circ \rho \qquad \forall\, a \in \mathcal{H}_{\mathrm{CM}}. \tag{22}$$

To prove (22) it is enough to show that Φ is equivariant with respect to the action of the Lie algebra L of primitive elements of $\mathcal{H}^*_{\mathrm{gr}}$. More precisely, given a primitive element

$$Z \in \mathcal{H}^*_{\mathrm{gr}}\,,\ \ \Delta Z = Z \otimes 1 + 1 \otimes Z, \tag{23}$$

we let ∂_Z be the derivation of the algebra $\mathcal{H}$ given by,

$$\partial_Z(y) = \langle Z \otimes \mathrm{id}, \Delta y\rangle \in \mathcal{H} \qquad \forall\, y \in \mathcal{H}. \tag{24}$$

What we need to check is the following:

Lemma 9. *For any $a \in \mathcal{H}_{\mathrm{CM}}$, $Z \in L$ one has $\partial_Z\, \Phi(a) = \Phi(\partial_{(\rho Z)}(a))$.*

Proof. It is enough to check the equality when Z is of the form $\underline{\Gamma} = S(\Gamma)(\Gamma)$ with the above notations. Thus we let Γ be a 1PI graph and let ∂_Γ be the corresponding derivation of $\mathcal{H}$ given by (24) with $Z = S(\Gamma)(\Gamma)$. Now by definition of the primitive element (Γ) one has (cf. (48) Sect. 2 of Part I)

$$\langle (\Gamma) \otimes \mathrm{id}, \Delta\Gamma' \rangle = \sum n(\Gamma, \Gamma''; \Gamma')\, \Gamma'', \tag{25}$$

where the integer $n(\Gamma, \Gamma''; \Gamma')$ is the number of subgraphs of Γ' which are isomorphic to Γ while $\Gamma'/\Gamma \cong \Gamma''$. By Theorem 2 of Part I we have

$$S(\Gamma)\, S(\Gamma'')\, n(\Gamma, \Gamma''; \Gamma') = i(\Gamma, \Gamma''; \Gamma')\, S(\Gamma'), \tag{26}$$

where $i(\Gamma, \Gamma''; \Gamma')$ is the number of times Γ' appears in $\Gamma'' \circ \Gamma$. We thus get

$$\partial_\Gamma \frac{\Gamma'}{S(\Gamma')} = \sum i(\Gamma, \Gamma''; \Gamma') \frac{\Gamma''}{S(\Gamma'')}, \tag{27}$$

which shows that ∂_Γ admits a very simple definition in the generators $\frac{\Gamma'}{S(\Gamma')}$ of $\mathcal{H}$.

The derivation $\partial_{(\rho Z)}$ of $\mathcal{H}_{\mathrm{CM}}$ is also very easy to compute. One has by construction (Lemma 8),

$$\rho(Z) = \rho_\Gamma\, Z'_{2\ell}, \qquad \ell = L(\Gamma), \tag{28}$$

and the derivation d_k of $\mathcal{H}_{\mathrm{CM}}$ associated to the primitive element Z'_k of $\mathcal{H}^*_{\mathrm{CM}}$ is simply given, in the basis $a_n \in \mathcal{H}_{\mathrm{CM}}$ by

$$d_k(a_n) = (n-k)\, a_{n-k}. \tag{29}$$

We thus get

$$\partial_{(\rho Z)} = \rho_\Gamma\, d_{2\ell}\ , \quad \ell = L(\Gamma). \tag{30}$$

Now by construction both $\partial_Z\, \Phi$ and $\Phi \circ \partial_{(\rho Z)}$ are derivations from the algebra $\mathcal{H}_{\mathrm{CM}}$ to $\mathcal{H}$ viewed as a bimodule over $\mathcal{H}_{\mathrm{CM}}$, i.e. satisfy

$$\delta(ab) = \delta(a)\, \Phi(b) + \Phi(a)\, \delta(b). \tag{31}$$

Thus, to prove the lemma we just need to check the equality

$$\partial_\Gamma\, \Phi(a_n) = \rho_\Gamma\, \Phi(d_{2\ell}(a_n)), \qquad \ell = L(\Gamma), \tag{32}$$

or equivalently using the generating function

$$g_0 = x + \sum \Phi(a_n)\, x^n, \tag{33}$$

that

$$\partial_\Gamma\, g_0 = \rho_\Gamma\, x^{2\ell+1} \frac{\partial}{\partial x}\, g_0. \tag{34}$$

Now by construction of Φ we have

$$g_0 = (x\, Z_1)(Z_3)^{-3/2}, \quad Z_3 = 1 - \delta Z, \tag{35}$$

where

$$Z_1 = 1 + \sum x^{2l} \frac{\Gamma}{S(\Gamma)},\ \ \delta Z = \sum x^{2l} \frac{\Gamma}{S(\Gamma)}. \tag{36}$$

Thus, since both ∂_Γ and $\frac{\partial}{\partial x}$ are derivations we can eliminate the denominators in (34) and rewrite the desired equality, after multiplying both sides by $(1-\delta Z)^{5/2}$ as

$$\left(x \frac{\partial}{\partial \Gamma} Z_1\right)(1-\delta Z) + \frac{3}{2}\left(\frac{\partial}{\partial \Gamma}\delta Z\right)(x\, Z_1)$$
$$= \rho_\Gamma \left(x^{2\ell+1}\left(\frac{\partial}{\partial x}(x\, Z_1)\right)(1-\delta Z) + \frac{3}{2}(x\, Z_1)\, x^{2\ell+1}\frac{\partial}{\partial x}\delta Z\right). \quad (37)$$

Both sides of this formula are bilinear expressions in the 1PI graphs. We first need to compute $\frac{\partial}{\partial \Gamma} Z_1$ and $\frac{\partial}{\partial \Gamma}\delta Z$. One has

$$\frac{\partial}{\partial \Gamma} Z_1 = \sum x^{2l+2l'} c(\Gamma, \Gamma')\frac{\Gamma'}{S(\Gamma')} \quad (38)$$

and

$$\frac{\partial}{\partial \Gamma}\delta Z = \sum x^{2l+2l'} c(\Gamma, \Gamma')\frac{\Gamma'}{S(\Gamma')}, \quad (39)$$

where $\ell = L(\Gamma)$, $\ell' = L(\Gamma')$ are the loop numbers and the integral coefficient $c(\Gamma, \Gamma')$ is given by

$$c(\Gamma, \Gamma') = V' \text{ if } \rho_\Gamma = 1 \quad \text{and} \quad c(\Gamma, \Gamma') = I' \text{ if } \rho_\Gamma = 3/2 \quad (40)$$

(where V' and I' are respectively the number of vertices and of internal lines of Γ').

To prove (38) and (39) we use (27) and we get in both cases expressions like (38), (39) with

$$c(\Gamma, \Gamma') = \sum_{\Gamma''} i(\Gamma, \Gamma'; \Gamma''). \quad (41)$$

But this is exactly the number of ways we can insert Γ inside Γ' and is thus the same as (40). Let now Γ_1 be a 3-point graph and Γ_2 a 2-point graph. The coefficient of the bilinear term,

$$\frac{\Gamma_1}{S(\Gamma_1)}\,\frac{\Gamma_2}{S(\Gamma_2)}, \quad (42)$$

in the left-hand side of (37) is given by

$$\left(\frac{3}{2}\, c(\Gamma, \Gamma_2) - c(\Gamma, \Gamma_1)\right) x^{2\ell+2\ell_1+2\ell_2+1}. \quad (43)$$

Its coefficient in the right-hand side of (37) is coming from the terms,

$$x^{2\ell+1}\frac{\partial}{\partial x}\left(x^{2\ell_1+1}\frac{\Gamma_1}{S(\Gamma_1)}\right)\left(\frac{-\Gamma_2}{S(\Gamma_2)}\right) x^{2\ell_2} +$$
$$\frac{3}{2}\, x^{2\ell_1+1}\frac{\Gamma_1}{S(\Gamma_1)}\, x^{2\ell+1}\frac{\partial}{\partial x}\left(x^{2\ell_2}\frac{\Gamma_2}{S(\Gamma_2)}\right) \quad (44)$$

which gives

$$(3\ell_2 - 2\ell_1 - 1)\, x^{1+2\ell+2\ell_1+2\ell_2}. \quad (45)$$

We thus only need to check the equality

$$\frac{3}{2}\, c(\Gamma, \Gamma_2) - c(\Gamma, \Gamma_1) = (3\ell_2 - 2\ell_1 - 1)\, \rho_\Gamma. \tag{46}$$

Note that in general, for any graph with N external legs we have

$$V = 2(L-1) + N, \;\; I = 3(L-1) + N. \tag{47}$$

Let us first take for Γ a 3-point graph so that $\rho_\Gamma = 1$. Then the left-hand side of (46) gives $\frac{3}{2}\, V_2 - V_1 = \frac{3}{2}\,(2\ell_2) - (2(\ell_1 - 1) + 3) = 3\ell_2 - 2\ell_1 - 1$.

Let then Γ be a 2-point graph, i.e. $\rho_\Gamma = \frac{3}{2}$. Then the left-hand side of (46) gives $\frac{3}{2}\, I_2 - I_1 = \frac{3}{2}\,(3\ell_2 - 1) - 3\ell_1 = \rho_\Gamma\,(3\ell_2 - 2\ell_1 - 1)$, which gives the desired equality.

Finally we also need to check the scalar terms and the terms linear in Γ_1 or in Γ_2. The only scalar terms in the left hand side of (37) are coming from $x\, \frac{\partial}{\partial \Gamma}\, Z_1 + \frac{3}{2}\, x\, \frac{\partial}{\partial \Gamma}\, \delta Z$ and this gives,

$$x^{2\ell+1}\, \rho_\Gamma. \tag{48}$$

The only scalar term in the right-hand side of (37) comes from $x^{2\ell+1}$ thus they fulfill (37).

The terms linear in Γ_1 in the left-hand side of (37) come only from $x\, \frac{\partial}{\partial \Gamma}\, Z_1$ if Γ is a 3-point graph and the coefficient of $\Gamma_1 / S(\Gamma_1)$ is thus,

$$c(\Gamma, \Gamma_1)\, x^{1+2\ell_1+2\ell}. \tag{49}$$

In the right-hand side of (37) we just get

$$(2\ell_1 + 1)\, x^{1+2\ell_1+2\ell}. \tag{50}$$

We thus need to check that $c(\Gamma, \Gamma_1) = 2\ell_1 + 1$ which follows from (40) and (47) since $V_1 = 2\ell_1 + 1$.

Similarly, if Γ is a 2-point graph, the left side of (37) only contributes by $x\, \frac{\partial}{\partial \Gamma}\, Z_1 + \frac{3}{2}\, x^{2\ell+1}\, Z_1$, so that the coefficient of $\Gamma_1 / S(\Gamma_1)$ is

$$\left(c(\Gamma, \Gamma_1) + \frac{3}{2} \right) x^{1+2\ell+2\ell_1}. \tag{51}$$

In the right-hand side of (37) we get just as above

$$(2\ell_1 + 1)\, x^{1+2\ell+2\ell_1}, \tag{52}$$

multiplied by $\rho_\Gamma = 3/2$.

Now here, since Γ is a 2-point graph, we have $c(\Gamma, \Gamma_1) = I_1 = 3(\ell_1 - 1) + 3 = 3\ell_1$ so that

$$c(\Gamma, \Gamma_1) + \frac{3}{2} = \frac{3}{2}\,(2\ell_1 + 1) = \rho_\Gamma (2\ell_1 + 1)$$

as required.

The check for terms linear in Γ_2 is similar. $\square$

We can now state the main result of this section:

Theorem 10. *The map $\Phi = \mathcal{H}_{\mathrm{CM}} \to \mathcal{H}$ given by the effective coupling is a Hopf algebra homomorphism. The transposed Lie group morphism is $\rho : G_c \to G_2$.*

The proof follows from Lemma 9 which shows that the map from G_c to G_2 given by the transpose of the algebra morphism Φ is the Lie group morphism ρ. By construction the morphism Φ is compatible with the grading Θ of $\mathcal{H}$ and α of $\mathcal{H}_{CM}$ given by $\deg(a_n) = n-1$ (cf. [5]), one has indeed,

$$\Phi \circ \alpha_t = \Theta_{2t} \circ \Phi, \ \forall t \in \mathbb{R}. \tag{51}$$

Finally we remark that our proof of Theorem 10 is similar to the proof of the equality

$$F_{\phi_1 \phi_2} = F_{\phi_2} \circ F_{\phi_1} \tag{52}$$

for the Butcher series used in the numerical integration of differential equations, but that the presence of the Z_3 factor makes it much more involved in our case.

5. The β-Function and the Birkhoff Decomposition of the Unrenormalized Effective Coupling in the Diffeomorphism Group

Let us first recall our notations from Part I concerning the effective action. We work in the Euclidean signature of space time and in order to minimize the number of minus signs we write the functional integrals in the form,

$$\mathcal{N} \int e^{S(\varphi)} \, P(\varphi) \, [D\varphi] \tag{1}$$

so that the Euclidean action is[2]

$$S(\varphi) = -\frac{1}{2} \, (\partial_\mu \, \varphi)^2 - \frac{1}{2} \, m^2 \, \varphi^2 + \frac{g}{6} \, \varphi^3. \tag{2}$$

The effective action, which when used at tree level in (1) gives the same answer as the full computation using (2), is then given in dimension $d = 6 - \varepsilon$ by

$$\begin{aligned} S_{\mathrm{eff}}(\varphi) = S(\varphi) + \sum_{\mathrm{1PI}} \frac{(\mu^{\varepsilon/2} \, g)^{n-2}}{n!} \int \frac{\Gamma}{S(\Gamma)} \, (p_1, \ldots, p_n)(3) \\ \times \varphi(p_1) \ldots \varphi(p_n) \, \Pi \, dp_i, \end{aligned} \tag{3}$$

where, as in Part I, we do not consider tree graphs as 1PI and the integral is performed on the hyperplane $\sum p_i = 0$. To be more precise one should view the right-hand side of (3) as a formal power series with values in the Hopf algebra $\mathcal{H}$. The theory provides us with a loop

$$\gamma_\mu(\varepsilon) = \gamma_-(\varepsilon)^{-1} \, \gamma_{\mu_+}(\varepsilon) \tag{4}$$

of characters of $\mathcal{H}$. When we evaluate $\gamma_\mu(\varepsilon)$ (resp. γ_-, γ_{μ^+}) on the right-hand side of (3) we get respectively the unrenormalized effective action, the bare action and the renormalized effective action (in the MS scheme).

Our notation is hiding the g-dependence of $\gamma_\mu(\varepsilon)$, but this dependence is entirely governed by the grading. Indeed with $t = \log(g)$ one has, with obvious notations,

$$\gamma_{\mu,g}(\varepsilon) = \Theta_{2t}(\gamma_{\mu,1}(\varepsilon)). \tag{5}$$

Since Θ_t is an automorphism the same equality holds for both γ_{μ_+} and γ_-.

As in Sect. 4 we restrict ourselves to the massless case and let $\gamma_\mu(\varepsilon) = \gamma_-(\varepsilon)^{-1} \, \gamma_{\mu_+}(\varepsilon)$ be the Birkhoff decomposition of $\gamma_\mu(\varepsilon) = \gamma_{\mu,1}(\varepsilon)$.

[2] We know of course that the usual sign convention is better to display the positivity of the action functional.

Lemma 11. *Let $\rho : G_c \to G_2$ be given by Theorem 10. Then $\rho(\gamma_\mu(\varepsilon))(g)$ is the unrenormalized effective coupling constant, $\rho(\gamma_{\mu_+}(0))(g)$ is the renormalized effective coupling constant and $\rho(\gamma_-(\varepsilon))(g)$ is the bare coupling constant g_0.*

This follows from (3) and Theorem 10. It is now straightforward to translate the results of the previous sections in terms of diffeomorphisms. The only subtle point to remember is that the group G_2 is the opposite of the group of diffeomorphisms so that if we view ρ as a map to diffeomorphisms it is an antihomomorphism,

$$\rho(\gamma_1\,\gamma_2) = \rho(\gamma_2) \circ \rho(\gamma_1). \tag{6}$$

Theorem 12. *The renormalization group flow is the image $\rho(F_t)$ by $\rho : G_c \to \mathrm{Diff}$ of the one parameter group $F_t \in G_c$.*

Proof. The *bare coupling constant* g_0 governs the bare action,

$$S_{\mathrm{bare}}(\varphi_0) = -\frac{1}{2}\,(\partial_\mu\,\varphi_0)^2 + \mu^{\varepsilon/2}\,\frac{g_0}{6}\,\varphi_0^3 \tag{7}$$

in terms of the bare field φ_0.

Now when we replace μ by

$$\mu' = e^t\,\mu \tag{8}$$

we can keep the bare action, and hence the physical theory, unchanged provided we replace the renormalized coupling constant g by g', where

$$g_0(\varepsilon, g') = e^{-\varepsilon\,\frac{t}{2}}\,g_0(\varepsilon, g). \tag{9}$$

By construction we have

$$g' = \psi_\varepsilon^{-1}(e^{-\varepsilon\frac{t}{2}}\,\psi_\varepsilon(g)), \tag{10}$$

where ψ_ε is the formal diffeomorphism given by

$$\psi_\varepsilon = \rho\,(\gamma_-(\varepsilon)). \tag{11}$$

Now the behaviour for $\varepsilon \to 0$ of g' given by (10) is the same as for

$$\psi_\varepsilon^{-1}\,\alpha_{\varepsilon t/2}(\psi_\varepsilon), \tag{12}$$

where α_s is the grading of Diff given as above by

$$\alpha_s(\psi)(x) = e^{-s}\,\psi(e^s x). \tag{13}$$

Thus, since the map ρ preserves the grading,

$$\rho\,(\theta_t(\gamma)) = \alpha_{t/2}\,\rho(\gamma) \tag{14}$$

(by (51) of Sect. 4), we see by Corollary 6 of Sect. 3 that

$$g' \to \rho(F_t)g \quad \text{when } \varepsilon \to 0. \tag{15}$$

□

As a corollary we get of course,

Corollary 13. *The image by ρ of $\beta \in L$ is the β-function of the theory.*

In fact all the results of Sect. 3 now translate to the group G_2. We get the formula for the bare coupling constant in terms of the β-function, namely,

$$\psi_\varepsilon = \lim_{t \to \infty} e^{-tZ_0} \, e^{t\left(\frac{\beta}{\varepsilon} + Z_0\right)}, \tag{16}$$

where $Z_0 = x \, \frac{\partial}{\partial x}$ is the generator of scaling. But we can also express the main result of Part I independently of the group G or of its Hopf algebra $\mathcal{H}$. Indeed the group homomorphism $\rho : G \to G_2$ maps the Birkhoff decomposition of $\gamma_\mu(\varepsilon)$ to the Birkhoff decomposition of $\rho(\gamma_\mu(\varepsilon))$. But we saw above that $\rho\,(\gamma_\mu(\varepsilon))$ is just the *unrenormalized* effective coupling constant. We can thus state

Theorem 14. *Let the unrenormalized effective coupling constant $g_{\rm eff}(\varepsilon)$ viewed as a formal power series in g be considered as a loop of formal diffeomorphisms and let $g_{\rm eff}(\varepsilon) = g_{{\rm eff}_+}(\varepsilon)\,(g_{{\rm eff}_-})^{-1}(\varepsilon)$ be its Birkhoff decomposition in the group of formal diffeomorphisms. Then the loop $g_{{\rm eff}_-}(\varepsilon)$ is the bare coupling constant and $g_{{\rm eff}_+}(0)$ is the renormalized effective coupling.*

Note that G_2 is naturally isomorphic to the opposite group of Diff so we used the opposite order in the Birkhoff decomposition.

This result is very striking since it no longer involves the Hopf algebra $\mathcal{H}$ or the group G but only the idea of thinking of the effective coupling constant as a formal diffeomorphism. The proof is immediate, by combining Lemma 11, Theorem 10 of Sect. 4 with Theorem 4 of Part I.

Now in the same way as the Riemann–Hilbert problem and the Birkhoff decomposition for the group $G = \mathrm{GL}(n, \mathbb{C})$ are intimately related to the classification of holomorphic n-dimensional vector bundles on $P_1(\mathbb{C}) = C_+ \cup C_-$, the Birkhoff decomposition for the group $G_2 = \mathrm{Diff}^0$ is related to the classification of one dimensional complex (non linear) bundles

$$P = (C_+ \times X) \cup_{g_{\rm eff}} (C_- \times X). \tag{17}$$

Here X stands for a formal one dimensional fiber and $C_\pm$ are, as in Part I, the components of the complement in $P_1(\mathbb{C})$ of a small circle around D. The total space P should be thought of as a 2-dimensional complex manifold which blends together the $\varepsilon = D - d$ and the coupling constant of the theory.

6. Conclusions

We showed in this paper that the group G of characters of the Hopf algebra $\mathcal{H}$ of Feynman graphs plays a key role in the geometric understanding of the basic ideas of renormalization including the renormalization group and the β-function.

We showed in particular that the group G acts naturally on the complex space X of dimensionless coupling constants of the theory. Thus, elements of G are a refined form of diffeomorphisms of X and as such should be called diffeographisms.

The action of these diffeographisms on the space of coupling constants allowed us first of all to read off directly the bare coupling constant and the renormalized one from the Riemann–Hilbert decomposition of the unrenormalized effective coupling constant viewed as a loop of formal diffeomorphisms. This showed that renormalization is intimately related with the theory of non-linear complex bundles on the Riemann sphere of the dimensional regularization parameter ε. It also allowed us to lift both the renormalization group and the β-function as the asymptotic scaling in the group of diffeographisms. This used the full power of the Riemann–Hilbert decomposition together with the invariance of $\gamma_-(\varepsilon)$ under a change of unit of mass. This gave us a completely streamlined proof of the existence of the renormalization group and more importantly a closed formula of scattering nature, delivering the full higher pole structure of minimal subtracted counterterms in terms of the residue.

In the light of the predominant role of the residue in NCG we expect this type of formula to help us to decipher the message on space-time geometry buried in the need for renormalization.

Moreover, thanks to [6] the previous results no longer depend upon dimensional regularization but can be formulated in any regularization or renormalization scheme. Also, we could discard a detailed discussion of anomalous dimensions, since it is an easy corollary [7] of the knowledge of the β-function.

For reasons of simplicity our analysis was limited to the case of one coupling constant. The generalization to a higher dimensional space X of coupling constants is expected to involve the same ingredients as those which appear in higher dimensional diffeomorphism groups and Gelfand-Fuchs cohomology [5]. We left aside the detailed study of the Lie algebra of diffeographisms and its many similarities with the Lie algebra of formal vector fields. This, together with the interplay between Hopf algebras, rational homotopy theory, BRST cohomology, rooted trees and shuffle identities will be topics of future joint work.

7. Appendix: Up to Three Loops

We now want to check the Hopf algebra homomorphism $\mathcal{H}_{CM} \to \mathcal{H}$ up to three loops as an example. We regard g_0 as a series in a variable x (which can be thought of as a physical coupling) up to order x^6, making use of $g_0 = x Z_1 Z_3^{-3/2}$ and the expression of the Z-factors in terms of the 1PI Feynman graph. The challenge is then to confirm that the coordinates δ_n on G_2, implicitly defined by [5]

$$\log\left[g_0(x)'\right]^{(n)}$$

commute with the Hopf algebra homomorphism: calculating the coproduct Δ_{CM} of δ_n and expressing the result in Feynman graphs must equal the application of the coproduct Δ applied to δ_n expressed in Feynman graphs.

By (2) of Sect. 4 we write $g_0 = x Z_1 Z_3^{-3/2}$,

$$Z_1 = 1 + \sum_{k=1}^{\infty} z_{1,2k} x^{2k},$$

$$Z_3 = 1 - \sum_{k=1}^{\infty} z_{3,2k} x^{2k},$$

and

$$Z_g = Z_1 Z_3^{-3/2},\ z_{i,2k} \in \mathcal{H}_c,\ i = 1, 3,$$

as formal series in x^2. Using

$$\log\left(\frac{\partial}{\partial x} x Z_g\right) = \sum_{k=1}^{\infty} \frac{\delta_{2k}}{(2k)!} x^{2k},$$

which defines δ_{2k} as the previous generators $a_n(\phi)$ of coordinates of G_2, we find

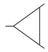
$$\frac{1}{2!}\delta_2 \equiv \tilde{\delta_2} = 3z_{1,2} + \frac{9}{2}z_{3,2}, \tag{1}$$

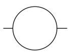
$$\frac{1}{4!}\delta_4 \equiv \tilde{\delta_4} = 5[z_{1,4} + \frac{3}{2}z_{3,4}] - \frac{9}{2}z_{1,2}^2 - 6z_{1,2}z_{3,2} - \frac{3}{4}z_{3,2}^2, \tag{2}$$

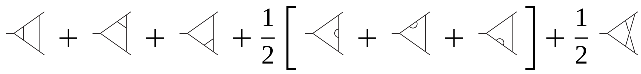
$$\frac{1}{6!}\delta_6 \equiv \tilde{\delta_6} = 9z_{1,2}^3 + 18z_{1,2}^2 z_{3,2} - 5[3z_{1,2}z_{1,4} + \frac{3}{2}z_{3,2}z_{3,4}] \tag{3}$$
$$+ 12[z_{1,2}z_{3,2}^2 - z_{1,2}z_{3,4} - z_{1,4}z_{3,2}] + 7[z_{1,6} + \frac{1}{2}z_{3,2}^3 + \frac{3}{2}z_{3,6}].$$

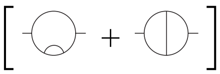
The algebra homomorphism $\mathcal{H}_c \to \mathcal{H}$ of Sect. 4 is effected by expressing the $z_{i,2k}$ in Feynman graphs, with 1PI graphs with three external legs contributing to Z_1, and 1PI graphs with two external legs, self-energies, contributing to Z_3.

Explicitly, we have

$z_{1,2} =$  ,

$z_{3,2} = \frac{1}{2}$  ,

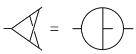
$z_{1,4} =$  ,

$z_{3,4} = \frac{1}{2}$  .

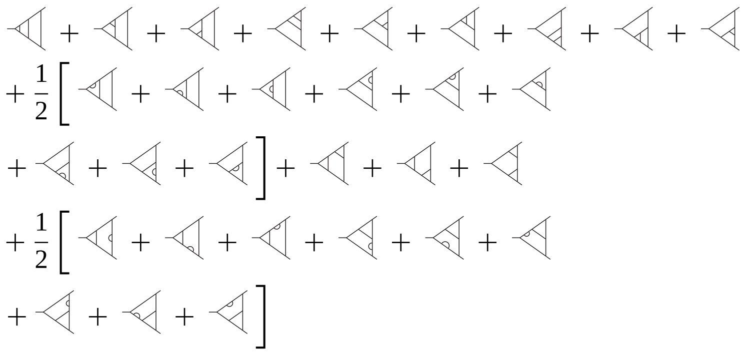
The symmetry factor

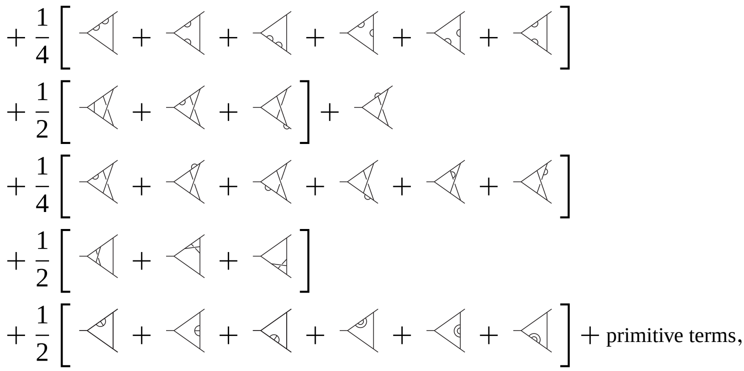
$2 = S\Big($  $\Big)$

is most obvious if we redraw

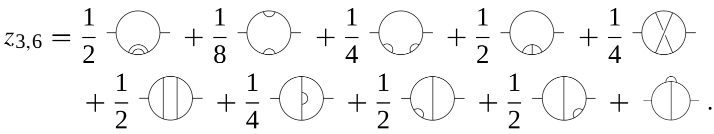
 .

Further, we have

$z_{1,6} =$ 



and



Here, primitive terms refer to 1PI three-loop vertex graphs without subdivergences. They fulfill all desired identities below trivially, and are thus not explicitly given. On the level of diffeomorphisms, we have the coproducts

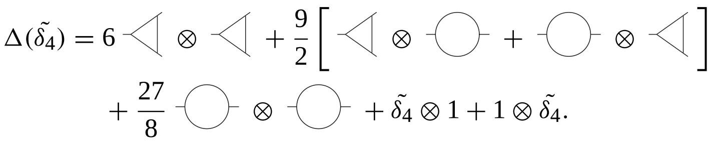
$$\Delta_{CM}[\delta_4] = \delta_4 \otimes 1 + 1 \otimes \delta_4 + 4\delta_2 \otimes \delta_2, \tag{4}$$

$$\begin{aligned} \Delta_{CM}[\delta_6] = {} & \delta_6 \otimes 1 + 1 \otimes \delta_6 + 20\delta_2 \otimes \delta_4 + 6\delta_4 \otimes \delta_2 \\ & + 28\delta_2^2 \otimes \delta_2, \end{aligned} \tag{5}$$

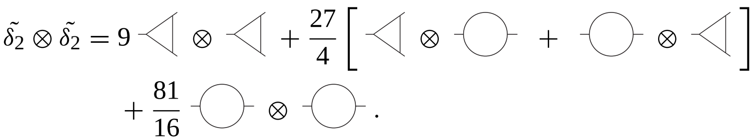
where we skip odd gradings.

We have to check that the coproduct Δ of Feynman graphs reproduces these results.

Applying Δ to the rhs of (2) gives, using the expressions for $z_{i,k}$ in terms of Feynman graphs,



This has to be compared with $\tilde{\delta_4} \otimes 1 + 1 \otimes \tilde{\delta_4} + \frac{2!2!}{4!} 4\tilde{\delta_2} \otimes \tilde{\delta_2}$, which matches perfectly, as

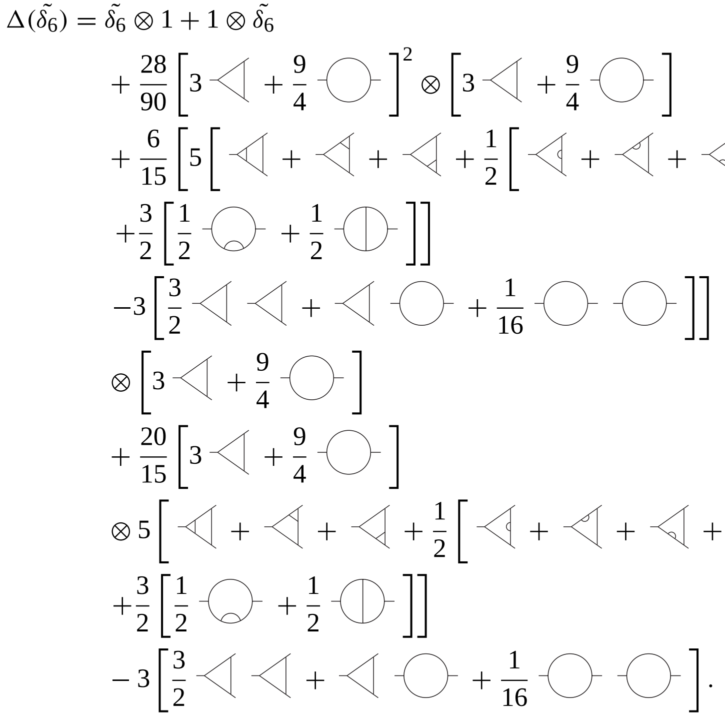


After this warming up, let us do the check at order g^6, which will be much more demanding, as the coproduct will be noncocommutative now.

We need $\Delta_{CM}(\delta_6)$ in (5) to be equivalent to

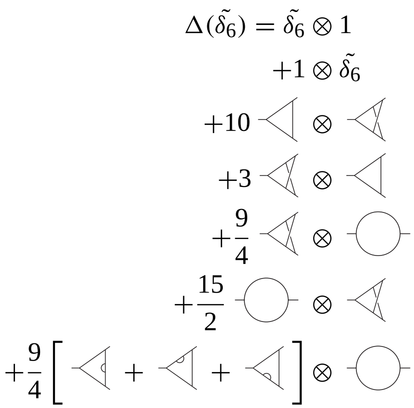
$$\Delta_{CM}(\tilde{\delta_6}) = \tilde{\delta_6} \otimes 1 + 1 \otimes \tilde{\delta_6} + 20\frac{2!4!}{6!}\tilde{\delta_2} \otimes \tilde{\delta_4} + 6\frac{2!4!}{6!}\tilde{\delta_4} \otimes \tilde{\delta_2}$$
$$+ 28\frac{2!2!2!}{6!}\tilde{\delta_2}^2 \otimes \tilde{\delta_2}. \tag{6}$$

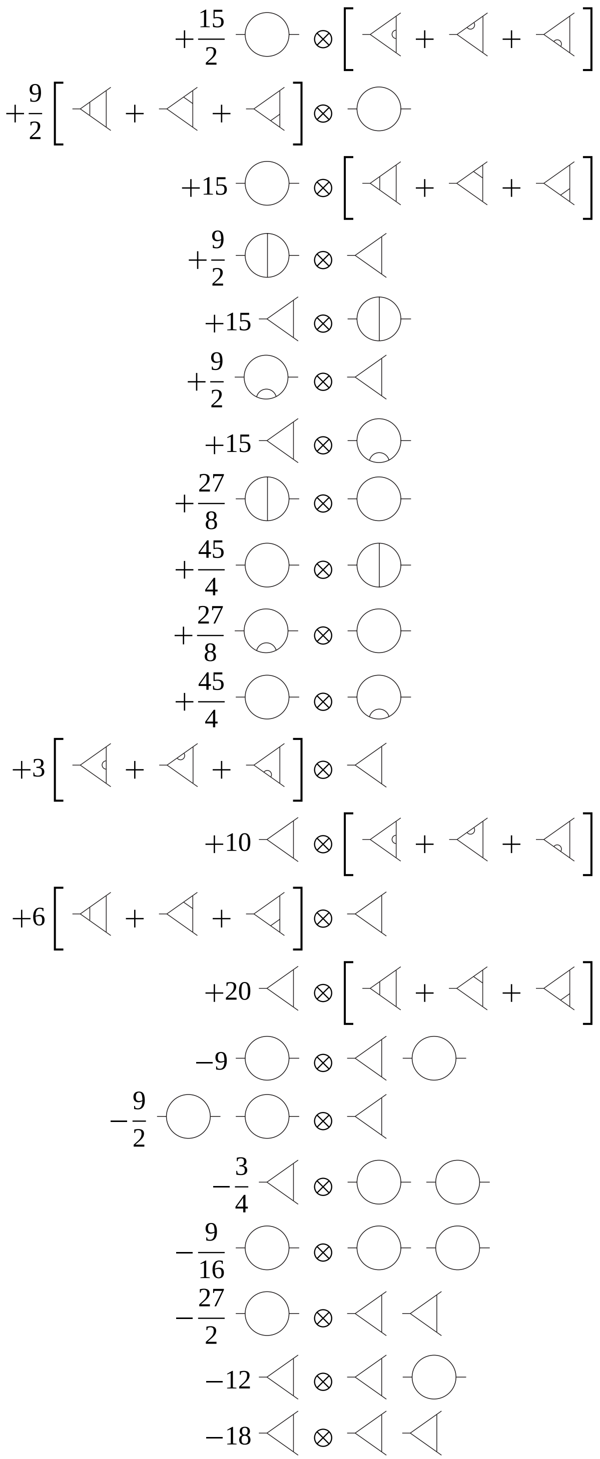
Applying the Hopf algebra homomorphism to Feynman graphs on both sides of the tensor product delivers



(7)

Multiplying this out, we find the following result:

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+3 [+ +] ⊗
+10 ⊗ [+ +]
+6 [+ +] ⊗
+20 ⊗ [+ +]
−9 ⊗
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−12 ⊗
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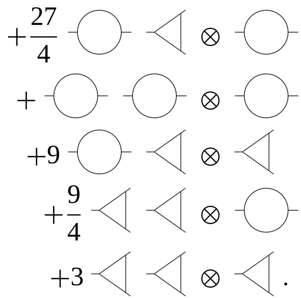
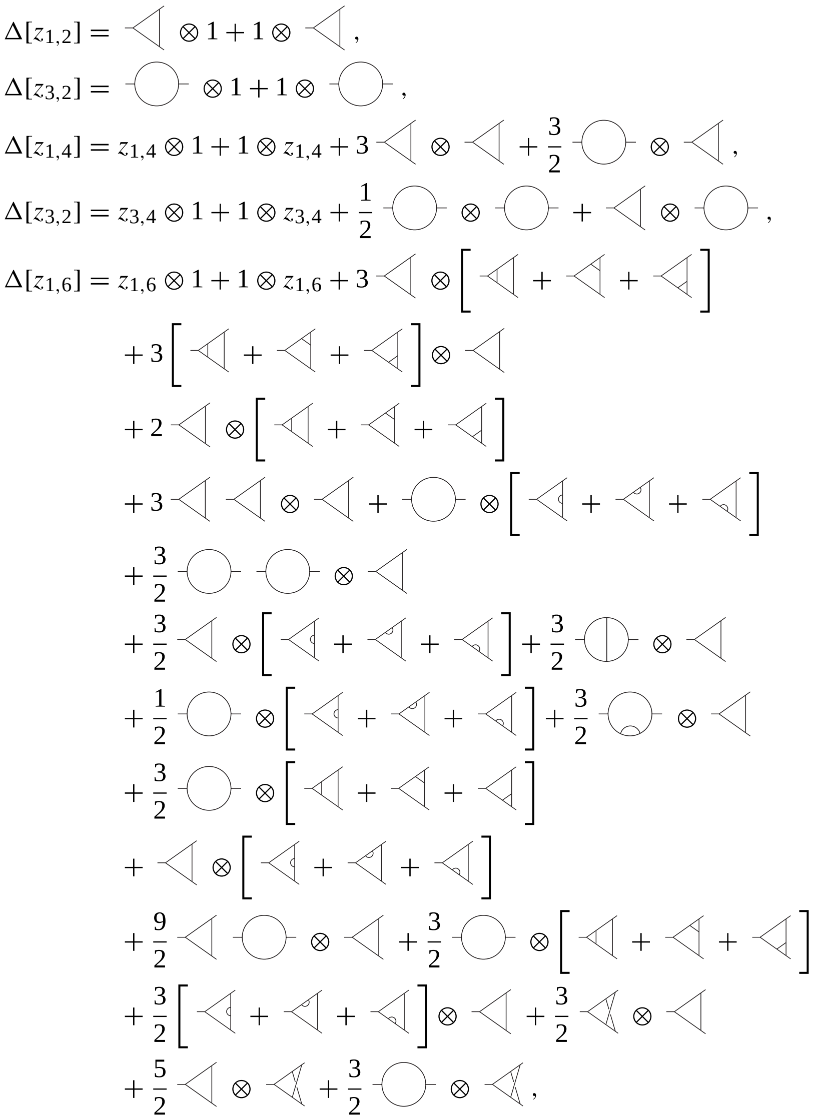


(8)

Now we have to compare with $\Delta(\tilde{\delta_6})$, so we first apply the homomorphism to graphs and use the coproduct Δ on them. For this, we need



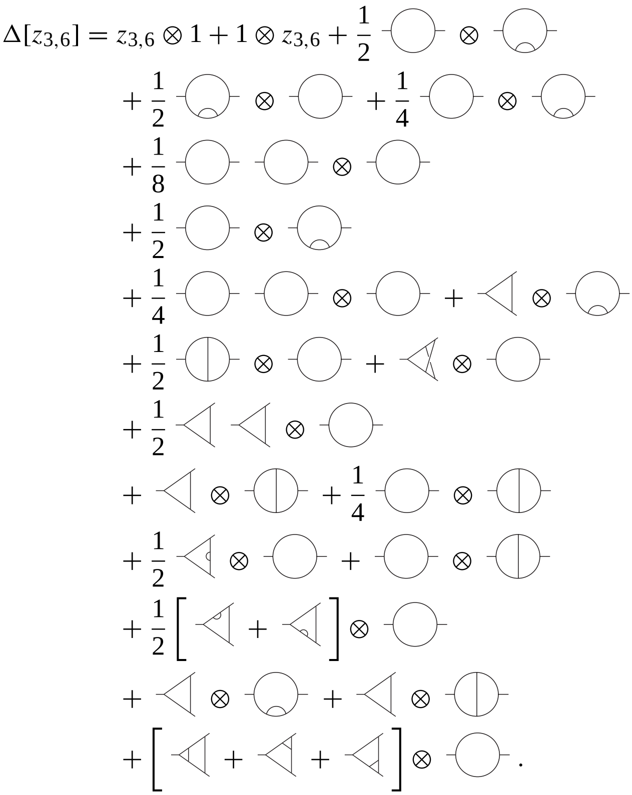
(9)

(10)

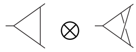
(11)

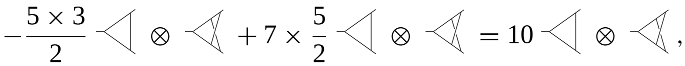
(12)

(13)



(14)

It is now only a matter of using the rhs of (3) for $\tilde{\delta_6}$ to confirm that we reproduce the result (8). For example, for the contribution to  in $\Delta(\tilde{\delta_6})$ we find



as desired. Similarly, one checks all of the 32 tensorproducts of (8).

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