A straight-forward calculation shows that

$$\mathsf{E}_\alpha(h\cdot R)(x_0, u_0^{(n)}) = (-1)^n \partial_J h(x_0)\cdot \partial R(x_0, u_0^{(n)})/\partial u_J^\alpha \neq 0.$$

Theorem 4.7 implies that $h\cdot R$ is not a total divergence, contradicting our assumption. An easy induction now proves that R can only depend on x. □

An easy corollary of this result will be quite important for subsequent developments.

Corollary 5.68. *Let $P \in \mathscr{A}^r$ be an r-tuple of differential functions. Then $\int_\Omega P\cdot Q\,dx = 0$ for all $Q \in \mathscr{A}^r$, all $\Omega \subset X$, if and only if $P \equiv 0$ for all x, u.*

Proof. Using the lemma component-wise, we conclude that $P = p(x)$ depends on x alone. Further, given $1 \leqslant \nu \leqslant r$, choose $Q_\mu[u] = \delta_\mu^\nu u^\alpha$ for any $1 \leqslant \alpha \leqslant q$. Then $\mathsf{E}_\alpha(p\cdot Q) = p_\nu(x) \equiv 0$ by Theorem 4.7, hence $P \equiv 0$ for all x, u. □

Returning to (5.99), we see that

$$\tilde{\mathscr{D}}_1^* \mathsf{E}_1(L) + \cdots + \tilde{\mathscr{D}}_q^* \mathsf{E}_q(L) = r(x)$$

is a function of x alone. If $r \equiv 0$ we're done; otherwise we divide by $r(x)$ and differentiate once more (with respect to any variable x^i) to produce an identity of the required form (5.97). □

More generally, if $h(x)$ appears nonlinearly in $Q[u; h]$, we can nevertheless reduce to the previous case using the following:

Lemma 5.69. *Suppose $Q[u; h]$ is the characteristic of a variational symmetry of $\mathscr{L}$ depending on an arbitrary function $h(x)$. Let $\mathscr{D}_Q = \mathscr{D}_{Q[u;h]}$ denote the Fréchet derivative of Q with respect to h, with entries*

$$\mathscr{D}_Q^\nu = \sum (\partial Q_\nu/\partial h_J)\cdot D_J, \qquad \nu = 1, \ldots, q, \quad (h_J = \partial_J h).$$

Then $Q' = \mathscr{D}_Q[k]$ is the characteristic of a variational symmetry depending linearly on the arbitrary function $k(x)$.

Proof. By assumption, for any function $h(x)$, there exists a p-tuple $B_h[u] \in \mathscr{A}^p$ such that

$$\operatorname{pr} \mathbf{v}_{Q[u;h]}(L) = \operatorname{Div} B_h.$$

If we replace h by $h + \varepsilon k$ in this identity and differentiate with respect to ε at $\varepsilon = 0$, we obtain

$$\operatorname{pr} \mathbf{v}_{Q'}(L) = \operatorname{Div} B', \qquad B' = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} B_{h+\varepsilon k},$$

which proves the lemma. □

In Theorem 5.66, each of the nontrivial symmetries $Q[u; h]$ (linear in h) gives rise to a trivial conservation law with Q as the characteristic. This remark has a converse also, that says that if a system of Euler–Lagrange equations has a trivial conservation law, which corresponds to a nontrivial variational symmetry, then it is necessarily under-determined, and hence admits an entire infinite-dimensional family of such symmetries depending on an arbitrary function. (See Exercise 5.34.) In applications, these are the "gauge symmetries" of the theory. (In relativity, cf. Goldberg, [1], these "trivial" conservation laws are among the most important identities of the subject. Here, perhaps, our choice of terminology is slightly misleading.)

Example 5.70. *Parametric Variational Problems.* Consider a first order variational problem of the form

$$\mathscr{L}[u, v] = \int L(x, u, v, u_x, v_x)\, dx,$$

with $x \in \mathbb{R}$. Consider the infinite-dimensional symmetry group consisting of arbitrary coordinate changes $x \mapsto \psi(x)$ in the independent variable. Its infinitesimal generators are of the form $\mathbf{v}_h = h(x)\partial_x$ for h an arbitrary function of x. The infinitesimal criterion (4.15) says that this is a variational symmetry group provided

$$h(x)L_x + h'(x)\{-u_x L_{u_x} - v_x L_{v_x} + L\} = 0,$$

subscripts denoting derivatives of L. (Generalizing to divergence symmetries doesn't add anything here.) As both h and h' are arbitrary, L must be independent of x, and of the form $L = u_x \tilde{L}(u, v, v_x/u_x)$. We conclude that we are necessarily dealing with a parametric variational problem

$$\mathscr{L}[u] = \int \tilde{L}\left(u, v, \frac{v_x}{u_x}\right) u_x\, dx = \int \tilde{L}\left(u, v, \frac{dv}{du}\right) du,$$

in which we can treat v, say, as a function of u only.

Noether's second theorem says that there is a dependency between the two original Euler–Lagrange equations

$$\mathsf{E}_u(L) = u_x \tilde{L}_u - D_x\left(\tilde{L} - \frac{v_x}{u_x}\tilde{L}_{v_u}\right) = 0, \qquad \mathsf{E}_v(L) = u_x \tilde{L}_v - D_x \tilde{L}_{v_u} = 0.$$

The evolutionary form of $\mathbf{v}_h$ is $-h(x)(u_x\partial_u + v_x\partial_v)$, so according to (5.97), (5.98) we have the identity

$$u_x \mathsf{E}_u(L) + v_x \mathsf{E}_v(L) = 0.$$

This argument clearly extends to both higher order and higher dimensional problems.

Example 5.71. Consider the variational problem

$$\mathscr{L}[u] = \tfrac{1}{2} \iint (u_x + v_y)^2\, dx\, dy,$$

whose Euler–Lagrange equations,

$$-\mathsf{E}_u(L) = u_{xx} + v_{xy} = 0, \qquad -\mathsf{E}_v(L) = u_{xy} + v_{yy} = 0,$$

were seen to be under-determined in Section 2.6, with $D_y \mathsf{E}_u(L) - D_x \mathsf{E}_v(L) \equiv 0$. The proof of Theorem 5.66 provides the corresponding infinite-dimensional symmetry group, generated by $\mathbf{v}_h = -h_y \partial_u + h_x \partial_v$ for arbitrary $h(x, y)$, with group transformations

$$\exp(\varepsilon \mathbf{v}_h)(u, v) = (u - \varepsilon h_y, v + \varepsilon h_x)$$

obviously leaving $\mathscr{L}$ invariant. Although these groups are certainly nontrivial, the corresponding conservation laws are trivial. For instance, if $h(x, y) = -y$, so $\mathbf{v}_h = \partial_u$ we get the trivial law with components $(u_x + v_y, 0)$, i.e.

$$D_x(u_x + v_y) = u_{xx} + v_{xy}.$$

Admittedly this doesn't look trivial, but if we add in the obviously trivial law (of the first kind) $(y(u_{xy} + v_{yy}), -y(u_{xx} + v_{xy}))$ we obtain an equivalent trivial conservation law of the second kind, since

$$\begin{aligned} (u_x + v_y) + y(u_{xy} + v_{yy}) &= D_y(y(u_x + v_y)), \\ -y(u_{xx} + v_{xy}) &= -D_x(y(u_x + v_y)). \end{aligned}$$

The lesson is that for abnormal systems one must exercise even more care in distinguishing trivial laws from nontrivial laws; here even the characteristics no longer are a foolproof indicator of triviality.

Formal Symmetries and Conservation Laws

There is an intimate connection between formal symmetries and conservation laws of evolution equations that does not appear to bear any obvious relationship to Noether's Theorem. (Indeed, a single evolution equation can never be the Euler–Lagrange equation for a variational problem, a fact that follows immediately from the Helmholtz conditions of Theorem 5.92.) The main observation is that the coefficient of D_x^{-1} in a formal symmetry of the appropriate rank and order will provide a conserved density of the evolution equation.

Definition 5.72. The *residue* of a pseudo-differential operator $\mathscr{D}$ is the coefficient of D_x^{-1}:

$$\operatorname{Res} \sum_{i=-\infty}^{n} P_i D_x^i = P_{-1}. \tag{5.100}$$

Proposition 5.73. *For any pair of pseudo-differential operators, the residue of their commutator is a total x-derivative*: $\operatorname{Res}[\mathscr{D}, \mathscr{E}] = D_x P$ *for some differential function P.*

PROOF. By linearity, it suffices to prove the theorem when $\mathcal{D} = QD_x^n$ and $\mathcal{E} = RD_x^m$ are monomials, with $n \geqslant m$. It is easy to see that, in this case, the residue of their commutator vanishes unless $n \geqslant -m-1 \geqslant 0$, in which case

$$\begin{aligned}
\operatorname{Res}&[QD_x^n, RD_x^m] \\
&= \binom{n}{-m-1}(PD_x^{n-m+1}Q + (-1)^{n-m+1}QD_x^{n-m+1}P) \\
&= \binom{n}{-m-1}D_x(PD_x^{n-m}Q - (D_xP)D_x^{n-m-1}Q + \cdots \pm (D_x^{n-m}P)Q). \qquad \square
\end{aligned}$$

We first show how, given a formal symmetry, one constructs a sequence of conservation laws. Recall that we can, without loss of generality, assume that the formal symmetry is given as a first order pseudo-differential operator, cf. Theorem 5.46.

Theorem 5.74. *If $\mathcal{D}$ is a first order formal symmetry of the n-th order evolution equation $u_t = K$ of rank $k \geqslant n+2$, then the residues of the first $k-n-2$ powers of $\mathcal{D}$,*

$$T_j = \operatorname{Res} \mathcal{D}^j, \quad j = 1, \ldots, k-n-2, \tag{5.101}$$

are conserved densities.

PROOF. Note first that according to Lemma 5.45, each power $\mathcal{R}^j$ is a formal symmetry of rank k also and, therefore, satisfies a formal symmetry condition

$$(\mathcal{D}^j)_t + [\mathcal{D}^j, \mathsf{D}_K] = \mathcal{E}_j, \tag{5.102}$$

where $\mathcal{E}_j$ is a pseudo-differential operator of order at most $n+j-k$. Provided $n+j-k < -1$, the coefficient of D_x^{-1} in $\mathcal{E}_j$ is zero; hence, according to Proposition 5.73, the residue of (5.102) is of the form

$$D_t T_j + D_x X_j = 0,$$

where $D_x X_j$ denotes the residue of the commutator in (5.102). This produces the required conservation law. $\square$

An important point is that there is no guarantee that these conservation laws are nontrivial, and, in the case of Burgers' equation, they are *all* trivial, in accordance with the analysis in Example 5.50.

Example 5.75. Consider the recursion operator $\mathcal{R} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1}$ for the Korteweg–de Vries equation (5.45). As a consequence of Theorem 5.74, the coefficient $\frac{1}{3}u_x$ of D_x^{-1} is a conserved density, but is trivial, being an x-derivative. To get nontrivial conservation laws, we must work with the square root of the recursion operator,

$$\begin{aligned}
\sqrt{\mathcal{R}} = D_x &+ \tfrac{1}{3}uD_x^{-1} - \tfrac{1}{18}u^2 D_x^{-3} + \tfrac{1}{9}uu_x D_x^{-4} \\
&+ (-\tfrac{1}{9}uu_{xx} - \tfrac{1}{18}u_x^2 + \tfrac{1}{54}u^3)D_x^{-5} + \cdots,
\end{aligned}$$

which, according to Lemma 5.45, is also a formal symmetry of order ∞. The residue of $\sqrt{\mathscr{R}}$ provides the first conserved density, u, of the Korteweg–de Vries equation. Moreover, all the powers $\mathscr{R}^{m/2}$ of this operator are also formal symmetries of order ∞, and hence their residues provide an infinite sequence of conserved densities. For example,

$$\mathscr{R}^{3/2} = D_x^3 + uD_x + u_x + (\tfrac{1}{3}u_{xx} + \tfrac{1}{6}u^2)D_x^{-1} - (\tfrac{1}{18}u_x^2 + \tfrac{1}{54}u^3)D_x^{-2} + \cdots$$

gives the conserved density $u_{xx} + \frac{1}{2}u^2$, which is equivalent to the nontrivial density $\frac{1}{2}u^2$;

$$\begin{aligned}\mathscr{R}^2 = D_x^4 &+ \tfrac{4}{3}uD_x^2 + 2u_xD_x + (\tfrac{4}{3}u_{xx} + \tfrac{4}{9}u^2) \\ &+ (\tfrac{1}{3}u_{xxx} + \tfrac{4}{9}uu_x)D_x^{-1} + \tfrac{1}{9}u_x^2D_x^{-2} + \cdots\end{aligned}$$

gives the trivial conserved density $u_{xxx} + \frac{4}{3}uu_x$;

$$\begin{aligned}\mathscr{R}^{5/2} &= D_x^5 + \tfrac{5}{3}uD_x^3 + \tfrac{10}{3}u_xD_x^2 + (\tfrac{10}{3}u_{xx} + \tfrac{5}{6}u^2)D_x + (\tfrac{5}{3}u_{xxx} + \tfrac{5}{3}uu_x) \\ &\quad + (\tfrac{1}{3}u_{xxxx} + \tfrac{5}{9}uu_{xx} + \tfrac{5}{18}u_x^2 + \tfrac{5}{54}u^3)D_x^{-1} + \cdots\end{aligned}$$

gives the conserved density $u_{xxxx} + \frac{5}{3}uu_{xx} + \frac{5}{6}u_x^2 + \frac{5}{18}u^3$, which is equivalent to a multiple of the next nontrivial conserved density $u_x^2 - \frac{1}{3}u^3$ for the Korteweg–de Vries equation. It can be shown that every integral power of $\mathscr{R}$ has a trivial density as its residue, whereas the half integral powers provide the well-known infinite sequence of (inequivalent) conservation laws of the Korteweg–de Vries equation.

With a little more work, one can produce two further conservation laws from a first order formal symmetry. If

$$\mathscr{D} = Q_1D_x + Q_0 + Q_{-1}D_x^{-1} + \cdots$$

is a formal symmetry of rank at least $n + 1$, then

$$T_{-1} = \frac{1}{Q_1}, \qquad T_0 = \frac{Q_1}{Q_0}, \tag{5.103}$$

are conserved densities; if the formal symmetry has rank n, just T_{-1} is conserved. Therefore, any formal symmetry of rank $k \geqslant n$ of an n-th order evolution equation ($n \geqslant 2$) provides $k - n$ *canonical conserved densities* T_j, $j = -1, \ldots, k - n + 2$. A converse to this result can also be proven: If an evolution equation has $k - n$ canonical conserved densities, then it has a formal symmetry of rank k; see Mikhailov, Shabat and Yamilov, [1]. Indeed, in their approach, the analysis is based primarily on the conditions imposed by the existence of canonical conservation laws rather than the more direct formal symmetry condition. This strategy results in some simplifications in the required calculations.

Just as there is a formal analogue of a symmetry obtained by taking the Fréchet derivative of the basic infinitesimal symmetry condition, there is also the concept of a formal conservation law. Let $T[u]$ be a conserved density which depends only on x, u and x-derivatives of u (so we are excluding explicitly time-dependent conservation laws), and set $R = \mathsf{E}(T)$. We say that the conservation law determined by T has *order m* if R depends on at most m-th order derivatives; note that this definition of order does not change if we replace T by any equivalent conserved density $T + D_x Q$. Applying the Euler operator to the conservation law

$$0 = D_t T + D_x X = \mathrm{pr}\,\mathbf{v}_Q(T) + D_x X,$$

and using (5.86) and Theorem 4.7, we find the equivalent condition

$$\mathrm{pr}\,\mathbf{v}_Q(R) + \mathsf{D}_K^* R = 0. \tag{5.104}$$

We now "linearize" (5.104) by taking its Fréchet derivative, leading via (5.60) to a differential operator identity of the form

$$(\mathsf{D}_R)_t + \mathsf{D}_R \cdot \mathsf{D}_K + \mathsf{D}_K^* \cdot \mathsf{D}_R + \mathscr{E} = 0, \tag{5.105}$$

where $\mathscr{E}$ is a differential operator of order at most n, which depends on the second derivatives of K with respect to the derivatives of u, but plays no role in the subsequent discussion.

Now suppose that the order m of the conservation law is much larger than the order n of the evolution equation, $m \gg n$. Then $\mathscr{E}$ is of much lower order that the other three differential operators appearing in (5.105). Thus, in analogy with the definition of a formal symmetry, we make the following definition of a formal conservation law.

Definition 5.76. A *formal conservation law of rank k* of an n-th order evolution equation $u_t = K$ is a pseudo-differential operator $\mathscr{C}$ of order m which satisfies

$$\mathrm{order}\{\mathscr{C}_t + \mathscr{C} \cdot \mathsf{D}_K + \mathsf{D}_K^* \cdot \mathscr{C}\} \leqslant m + n - k. \tag{5.106}$$

Proposition 5.77. *If T is a conserved density with $R = \mathsf{E}(T)$ of order m, then D_R is a formal conservation law of rank m.*

The most important fact about formal conservation laws is that two of them can be combined to provide a formal symmetry!

Theorem 5.78. *If $\mathscr{C}_1$ and $\mathscr{C}_2$ are formal conservation laws of ranks k_1, k_2, respectively, then $\mathscr{D} = \mathscr{C}_2 \cdot \mathscr{C}_1^{-1}$ is a formal symmetry of rank $k = \min\{k_1, k_2\}$.*

PROOF. Let m_i denote the order of $\mathscr{C}_i$, $i = 1, 2$. We have

$$(\mathscr{C}_i)_t + \mathscr{C}_i \cdot \mathsf{D}_K + \mathsf{D}_K^* \cdot \mathscr{C}_i = \mathscr{B}_i, \qquad i = 1, 2,$$

where $\mathscr{B}_i$ is a pseudo-differential operator of order at most $n + m_i - k_i$.

Evaluating the (1, 1)-Lie derivative $\mathbf{v}_K[\![\mathscr{D}]\!]$, we find, on solutions,

$$\begin{aligned}(\mathscr{C}_2\cdot\mathscr{C}_1^{-1})_t + [\mathsf{D}_K, \mathscr{C}_2\cdot\mathscr{C}_1^{-1}] &= (\mathscr{C}_2)_t\cdot\mathscr{C}_1^{-1} - \mathscr{C}_2\cdot\mathscr{C}_1^{-1}\cdot(\mathscr{C}_1)_t\cdot\mathscr{C}_1^{-1}\\ &\quad + \mathsf{D}_K\cdot\mathscr{C}_2\cdot\mathscr{C}_1^{-1} - \mathscr{C}_2\cdot\mathscr{C}_1^{-1}\cdot\mathsf{D}_K\\ &= \{(\mathscr{C}_2)_t + \mathscr{C}_2\cdot\mathsf{D}_K + \mathsf{D}_K^*\cdot\mathscr{C}_2\}\cdot\mathscr{C}_1^{-1}\\ &\quad - \mathscr{C}_2\cdot\mathscr{C}_1^{-1}\cdot\{(\mathscr{C}_1)_t + \mathscr{C}_1\cdot\mathsf{D}_K + \mathsf{D}_K^*\cdot\mathscr{C}_1\}\cdot\mathscr{C}_1^{-1}\\ &= \mathscr{B}_2\cdot\mathscr{C}_1^{-1} - \mathscr{C}_2\cdot\mathscr{C}_1^{-1}\cdot\mathscr{B}_1\cdot\mathscr{C}_1^{-1}.\end{aligned}$$

Now, $\mathscr{D} = \mathscr{C}_2\cdot\mathscr{C}_1^{-1}$ has order $m_2 - m_1$, whereas the two operators in the final equality have respective orders at most $n + m_2 - m_1 - k_2$, $n + m_2 - m_1 - k_1$. Therefore, order $\mathbf{v}_K[\![\mathscr{D}]\!] \leqslant n + m_2 - m_1 - k$, which proves that $\mathscr{D}$ is a formal symmetry of rank k (at least).

Corollary 5.79. *An evolution equation having two (ordinary) conservation laws of orders k_1, k_2 has a formal symmetry of rank $k \geqslant \min\{k_1, k_2\}$.*

Consequently, assuming our earlier conjecture on the integrability of equations having high order formal symmetries, we deduce that any evolution equation with two conservation laws of sufficiently high order is necessarily integrable. For example, any second order equation possessing two fifth or higher order conservation laws is integrable and equivalent, via a contact transformation, to one of the equations listed in Theorem 5.48. (Actually, since the first two equations in the list do not have any higher order conservation laws, the equation must be equivalent to either the third or fourth equation in that list.)

5.4. The Variational Complex

As mentioned in the introduction to this chapter, the variational complex draws its inspiration from three principal results that have formed the basis of much of our work on symmetries, conservation laws, differential operators and so on. First is the characterization of the kernel of the Euler operator as the space of total divergences given in Theorem 4.7; second is the characterization of all null divergences as total curls given in Theorem 4.24; third is the characterization of Euler–Lagrange equations by the self-adjointness of the their Fréchet derivatives—see the proof of Lemma 5.54. The two latter results especially are not easy to prove, as the reader may have discovered, but, when restated in a more natural differential form language, can be recovered through the construction of suitable homotopy operators similar to those used in the proof of the Poincaré lemma in Section 1.5. (The reader is well advised to become thoroughly familiar with the concepts of ordinary differential forms on manifolds, as developed in Section 1.5, before attempting to explore the more complicated types of forms to be treated here.) Although in this book we will only require the above three special instances of the full variational complex, we have chosen to include it in its entirety because (a) the proofs are not any more difficult in the general case, and (b) a familiarity with this complex will provide the reader with an excellent preparation for further reading and research into recent work on the geometric theory of the calculus of variations on manifolds.

The variational complex naturally splits in two halves. In the first half, the relevant differential forms are expressions involving the differentials dx^i of the independent variables, but whose coefficients are now differential functions. Replacing the ordinary differential d is now a "total" differential D which uses total instead of partial derivatives. Although this is the easier of the two halves to define, the proof of exactness is by far the more complicated and requires the machinery of "higher Euler operators" developed at the end of this section. The result on null divergences appears at the next-to-last stage of this half of the complex. In second half of the variational complex, the role of functions is taken by the functionals of the variational calculus, with "functional forms" being defined analogously. The differential now is similar to the variational derivative of a functional, and is hence called the variational differential. Although the objects in this half are less familiar, the proof of exactness relies on a relatively simple extension of the de Rham homotopy operator. Included here is the solution to Helmholtz' inverse problem of the calculus of variations. The Euler operator itself provides the link between the two halves, the characterization of null Lagrangians providing the remaining step in the full exactness of the variational complex.

The D-Complex

The first half of the variational complex is obtained by adapting the de Rham complex to the space of differential functions defined over $M \subset X \times U$. A *total differential r-form* will take the form

$$\omega = \sum_J P_J[u]\, dx^J$$

in which the coefficients $P_J \in \mathscr{A}$ are now differential functions, and $dx^J = dx^{j_1} \wedge \cdots \wedge dx^{j_r}$, $1 \leqslant j_1 < \cdots < j_r \leqslant p$ form the standard basis of $\bigwedge_r T^*X$. If we replace u by some function $u = f(x)$, then we recover an ordinary differential r-form on the space X. We differentiate ω treating the u's as functions of the x's, leading to the *total differential*

$$\mathsf{D}\omega = \sum_{i=1}^{p} \sum_J D_i P_J\, dx^i \wedge dx^J. \tag{5.107}$$

For example, if $p = 2$, then

$$\omega = yu_x\, dx + uu_{xy}\, dy$$

is a total one-form, with total differential

$$\mathsf{D}\omega = [D_x(uu_{xy}) - D_y(yu_x)]\,dx \wedge dy = [uu_{xxy} + u_x u_{xy} - u_x - yu_{xy}]\,dx \wedge dy.$$

Since when we specialize $u = f(x)$, the total differential agrees with the exterior derivative, it is easy to see that D defines a complex (called the "big D-complex") on the spaces of total differential forms, meaning that $\mathsf{D}(\mathsf{D}\omega) = 0$ for any form ω. Over suitable subdomains $M \subset X \times U$ this complex is exact. The precise requirement on M is that it be *totally star-shaped*, meaning that it be (a) *vertically star-shaped*, so each vertical slice $M_x = \{u: (x, u) \in M\}$ is a star-shaped subdomain of U, and (b) the base horizontal slice $\Omega = \{x: (x, 0) \in M\}$ is a star-shaped subdomain of X.

Theorem 5.80. *Let M be totally star-shaped. Then the D-complex*

$$0 \to \mathbb{R} \to \textstyle\bigwedge_0 \xrightarrow{\mathsf{D}} \bigwedge_1 \xrightarrow{\mathsf{D}} \cdots \xrightarrow{\mathsf{D}} \bigwedge_{p-1} \xrightarrow{\mathsf{D}} \bigwedge_p$$

is exact, where $\bigwedge_r$ denotes the space of total r-forms. In other words, if $\omega \in \bigwedge_r$ for $0 < r < p$, then ω is D-closed: $\mathsf{D}\omega = 0$, if and only if ω is D-exact: $\omega = \mathsf{D}\eta$ for some total $(r-1)$-form η, while if $\omega \in \bigwedge_0$, so ω is just a differential function, then $\mathsf{D}\omega = 0$ if and only if ω is constant.

Example 5.81. Exactness of the D-complex at the $\bigwedge_{p-1}$-stage is easily seen to be equivalent to the characterization of null divergences given in Theorem 4.24. Indeed, using the notation of Example 1.62, any $(p-1)$-form $\omega = \sum (-1)^{j-1} P_j\, dx^{\hat{j}}$ can be identified with its coefficients $P = (P_1, \ldots, P_p) \in \mathscr{A}^p$. We have $\mathsf{D}\omega = (\mathrm{Div}\, P)\, dx^1 \wedge \cdots \wedge dx^p$, so ω is D-closed if and only if P is a null divergence. On the other hand, a $(p-2)$-form takes the form $\eta = \sum (-1)^{j+k-1} Q_{jk}\, dx^{\widehat{jk}}$ where $Q_{jk} = -Q_{kj}$, and $\mathsf{D}\eta = \omega$ if and only if $P_j = \sum D_k Q_{jk}$. (Explicit formulae for the Q's in terms of the P's will be found in the course of the proof of Theorem 5.80.)

If we specialize $u = f(x)$ everywhere, then the D-complex reduces to the ordinary de Rham complex, which by the Poincaré lemma (Theorem 1.61) is exact. However, this does *not* prove the exactness of the D-complex! To see why not, let $\omega[u]$ be a total r-form depending on u and its derivatives and $\tilde{\omega}_f(x) = \omega[f(x)]$ the corresponding r-form on $\Omega \subset X$ once we substitute $f(x)$ for u everywhere. We have $\mathsf{D}\omega = 0$ if and only if $d\tilde{\omega}_f = 0$ for each f, and hence $\tilde{\omega}_f = d\tilde{\eta}_f$ for some $(r-1)$-form $\tilde{\eta}_f(x)$. What is *not* clear from the Poincaré homotopy formula (1.69) for $\tilde{\eta}_f$ is that there is a total $(r-1)$-form $\eta[u]$, depending just on u and its derivatives, which specializes to the given $\tilde{\eta}_f$ in every case: $\eta[f(x)] = \tilde{\eta}_f(x)$ for all f, the reason being that (1.69) is not a local map.

Indeed, the de Rham complex is exact at the $\bigwedge_{p-1} T^*\Omega \xrightarrow{d} \bigwedge_p T^*\Omega \to 0$ stage, but this is most definitely not true for the D-complex. Every total p-form $\omega = L[u]\, dx^1 \wedge \cdots \wedge dx^p$ is trivially D-closed, but it is D-exact, $\omega = \mathsf{D}\eta$, if and only if L is a total divergence, $L = \mathrm{Div}\, P$, and, as we know, not every differential function is a total divergence. The proof of Theorem 5.80 will therefore require new methods, in particular a new "total homotopy operator". The proof will be deferred until the end of this section.

The next step in the variational complex is to continue the D-complex beyond the $\bigwedge_p$-stage. This is something we essentially already know how to do, since by Theorem 4.7, $\omega = L\,dx^1 \wedge \cdots \wedge dx^p$ is D-exact, meaning $L = \operatorname{Div} P$ for some $P \in \mathscr{A}^p$, if and only if $\mathsf{E}(L) = 0$, where E is the Euler operator. Thus $\mathsf{D}\colon \bigwedge_{p-1} \to \bigwedge_p$ should be followed by the Euler operator or variational derivative expressed, perhaps, in a more intrinsic way. This will be implemented, and the variational complex continued even further, through the introduction of "functional forms" and the "variational differential", which in a sense accomplish for the dependent variables what the D-complex did for the independent variables.

Vertical Forms

The total r-forms concentrated on the "horizontal" variables x in $M \subset X \times U$ in that only the differentials dx^i appeared. Vertical forms are constructed by similarly concentrating on the "vertical" variables, which consist of the u's and all their derivatives.† Specially, a *vertical k-form* is a *finite* sum

$$\hat{\omega} = \sum P_{\mathbf{J}}^{\boldsymbol{\alpha}}[u]\, du_{J_1}^{\alpha_1} \wedge \cdots \wedge du_{J_k}^{\alpha_k}, \tag{5.108}$$

in which the coefficients $P_{\mathbf{J}}^{\boldsymbol{\alpha}} \in \mathscr{A}$ are differential functions. Since only the differentials du_J^α appear in these forms, the analogue of the differential of the ordinary de Rham complex is the *vertical differential*

$$\hat{d}\hat{\omega} = \sum \frac{\partial P_{\mathbf{J}}^{\boldsymbol{\alpha}}}{\partial u_K^\beta}\, du_K^\beta \wedge du_{J_1}^{\alpha_1} \wedge \cdots \wedge du_{J_k}^{\alpha_k}. \tag{5.109}$$

For example, if $p = q = 1$, a typical vertical two-form might be $\hat{\omega} = xu_{xx}\, du \wedge du_x$. Its vertical differential is then $\hat{d}\hat{\omega} = x\, du \wedge du_x \wedge du_{xx}$, the independent variable x only appearing parametrically.

Since any given vertical form $\hat{\omega}$ can depend on only finitely many of the variables u_J^α, and hence lives on a finite jet space $M^{(n)}$, the vertical differential $\hat{d}\hat{\omega}$ is in reality the same as the de Rham differential in these variables, the remaining independent variables playing the role of parameters. Thus the vertical differential is readily seen to have the usual bilinearity, antiderivation and closure properties of the ordinary differential:

$$\begin{aligned} \hat{d}(c\hat{\omega} + c'\hat{\omega}') &= c\,\hat{d}\hat{\omega} + c'\,\hat{d}\hat{\omega}', \\ \hat{d}(\hat{\omega} \wedge \hat{\eta}) &= (\hat{d}\hat{\omega}) \wedge \hat{\eta} + (-1)^k \hat{\omega} \wedge \hat{d}\hat{\eta}, \\ \hat{d}(\hat{d}\hat{\omega}) &= 0, \end{aligned}$$

† One can, of course, construct "hybrid" forms in both sets of variables, leading to the important "variational bicomplex". However, this would take us too far afield.