If $\mathbf{v}$ is a vector field on M, its prolongation $\mathrm{pr}^{(n)}\,\mathbf{v}$ is the vector field on $M_*^{(n)}$ which generates the prolongation $\mathrm{pr}^{(n)}\,[\exp(\varepsilon\mathbf{v})]$ of the one-parameter group generated by $\mathbf{v}$. Since this agrees with the usual prolongation on any coordinate chart $\tilde{M} \subset M$, we immediately conclude that the formula for $\mathrm{pr}^{(n)}\,\mathbf{v}$ is the same as that given in Theorem 2.36 on the subspace $\tilde{M}^{(n)} \subset \tilde{M}_*^{(n)}$. (Notice that by this remark, we conclude the invariance of (2.38) under arbitrary changes of independent and dependent variables!)

A locally solvable system of differential equations $\mathscr{S}_\Delta^* \subset M_*^{(n)}$ is invariant under the group action of G if and only if $\mathrm{pr}^{(n)}\,G$ preserves $\mathscr{S}_\Delta^*$, i.e. $\mathrm{pr}^{(n)}\,g[\mathscr{S}_\Delta^*] \subset \mathscr{S}_\Delta^*$. The corresponding infinitesimal criterion is that $\mathrm{pr}^{(n)}\,\mathbf{v}$ is tangent to $\mathscr{S}_\Delta^*$ whenever $\mathbf{v}$ is an infinitesimal generator of G. In local coordinates on $\tilde{M} \subset M$, this reduces to our usual infinitesimal criterion of invariance (2.25), which is both necessary and sufficient provided $\mathscr{S}_\Delta = \mathscr{S}_\Delta^* \cap \tilde{M}^{(n)}$ is both locally solvable and of maximal rank, and the complete subvariety $\mathscr{S}_\Delta^* \subset \tilde{M}_*^{(n)}$ is just the closure of $\mathscr{S}_\Delta$. (Otherwise we would have to check invariance in other coordinate systems.) Thus the theory of symmetry groups of systems of differential equations on extended jet bundles does not differ in any essential aspect from our previous theory of symmetry groups of differential equations, and, in fact, reduces to it as soon as local coordinates are introduced on M.

The Invariant Jet Space

The real key that unlocks the geometrical insight behind the construction of group-invariant solutions is the determination of the structure of that subset of the jet space traced out by their prolongations. Suppose G acts on a smooth manifold M, on which some system of differential equations $\mathscr{S}_\Delta^*$ is defined. The G-invariant solutions to the system will be certain p-dimensional submanifolds $\Gamma \subset M$, corresponding to the graphs of functions in local coordinate charts, which are locally invariant under the action of G. In general, these G-invariant submanifolds will not fill up the entire jet space $M_*^{(n)}$, but only a certain subspace $I_*^{(n)} = I_*^{(n)}(G)$ called the *invariant space* of G. It is defined as

$$I_*^{(n)}|_{z_0} \equiv \{z_0^{(n)} \in M_*^{(n)}|_{z_0}\colon \text{there exists a locally } G\text{-invariant } p\text{-dimensional submanifold } \Gamma \text{ passing through } z_0 \text{ with prolongation } z_0^{(n)} = \mathrm{pr}^{(n)}\,\Gamma|_{z_0}\}.$$

In most cases of practical interest, M is an open subset of some fixed Euclidean space, with ordinary jet space $M^{(n)} \subset M_*^{(n)}$. There is a corresponding invariant space $I^{(n)} = I_*^{(n)} \cap M^{(n)}$, which is determined by the prolongations of G-invariant functions $u = f(x)$:

$$I^{(n)}|_{x_0} = \{(x_0, u_0^{(n)}) \in M^{(n)}\colon \text{there exists a locally } G\text{-invariant function defined in a neighbourhood of } x_0 \text{ such that } u_0^{(n)} = \mathrm{pr}^{(n)}\,f(x_0)\}.$$

For practical purposes, the space $I^{(n)}$ is the easiest to work with, while in the theoretical proofs, its extension $I_*^{(n)}$ comes into the forefront.

Example 3.32. Consider the case $p = 2$, $q = 1$, so X has coordinates (x, t) and U the single dependent variable u. Let G be the translation group $(x, t; u) \mapsto (x + \varepsilon, t; u)$, with infinitesimal generator $\partial/\partial x$. A function $u = f(x, t)$ is G-invariant if and only if f is independent of x. Thus

$$I^{(1)} = \{(x, t; u; u_x, u_t) \colon u_x = 0\},$$

since at each point $u_x = \partial f/\partial x$ vanishes, while $u_t = \partial f/\partial t$ can be specified arbitrarily. Similarly,

$$I^{(2)} = \{(x, t; u; u_x, u_t; u_{xx}, u_{xt}, u_{tt}) \colon u_x = u_{xx} = u_{xt} = 0\},$$

and so on. As a useful exercise, at this point the reader should determine $I^{(1)}$ and $I^{(2)}$ in the case $G = \mathrm{SO}(2)$ is the rotation group with infinitesimal generator $-t\partial_x + x\partial_t$.

In Theorem 3.38 we will give an explicit characterization of the invariant space. However, we can already prove most of the important properties of this space even without the explicit formulae.

Proposition 3.33. *Let M be a smooth manifold, and G a local group of transformations acting on M. Then the invariant jet space $I_*^{(n)} \subset M_*^{(n)}$ corresponding to G is invariant under the action of* $\mathrm{pr}^{(n)} G$ *on $M_*^{(n)}$:*

$$\mathrm{pr}^{(n)} g[I_*^{(n)}] \subset I_*^{(n)}, \qquad g \in G.$$

PROOF. Let $z_0^{(n)}$ be a point in $I_*^{(n)}|_{z_0}$, so that by definition there exists a locally G-invariant p-dimensional submanifold Γ passing through z_0 with $\mathrm{pr}^{(n)} \Gamma|_{z_0} = z_0^{(n)}$. If g is any element of G such that $\tilde{z}_0 = g \cdot z_0$ is defined, then the transformed submanifold $\tilde{\Gamma} = g \cdot \Gamma = \{g \cdot z \colon z \in \Gamma, g \cdot z \text{ is defined}\}$ is also locally G-invariant. (Why?) Thus, by (3.31),

$$\mathrm{pr}^{(n)} g(z_0^{(n)}) = \mathrm{pr}^{(n)} g \cdot [\mathrm{pr}^{(n)} \Gamma|_{z_0}] = \mathrm{pr}^{(n)} \tilde{\Gamma}|_{\tilde{z}_0},$$

which, being the prolongation of a locally G-invariant submanifold, lies in $I_*^{(n)}|_{\tilde{z}_0}$. This completes the proof. (The same proof clearly works for the ordinary invariant space $I^{(n)} \subset M^{(n)}$.) $\square$

Connection with the Quotient Manifold

Since the invariant jet space $I_*^{(n)}$ for a group action is itself invariant under the prolonged group action $\mathrm{pr}^{(n)} G$, we can define a quotient space $I_*^{(n)}/\mathrm{pr}^{(n)} G$ by contracting the orbits of $\mathrm{pr}^{(n)} G$ in $I_*^{(n)}$ to points. In the case G acts regularly on the underlying manifold M, this "prolonged quotient manifold" can be identified with the n-jet space of the corresponding quotient manifold M/G.

This result, which becomes elementary to both state and prove in the language of extended jet bundles (but is considerably more complicated if we stick to ordinary jet spaces, as will be seen subsequently) immediately leads to the reduced system of differential equations for G-invariant solutions:

Proposition 3.34. *Let G be a local group of transformations acting regularly on the $(p+q)$-dimensional manifold M with s-dimensional orbits, $s \leqslant p$, and let M/G be the corresponding $(p+q-s)$-dimensional quotient manifold. Let $M_*^{(n)}$ be the extended n-jet space generated by p-dimensional submanifolds of M, and $I_*^{(n)} \subset M_*^{(n)}$ the corresponding invariant space generated by the G-invariant p-dimensional submanifolds. Then there is a natural projection $\pi^{(n)}: I_*^{(n)} \to (M/G)_*^{(n)}$ onto the extended n-jet space corresponding to $(p-s)$-dimensional submanifolds of M/G with the following properties*:

(a) *If $z \in M$ has image $\pi(z) = w \in M/G$, where $\pi: M \to M/G$ is the natural projection, then*

$$\pi^{(n)}: I_*^{(n)}|_z \to (M/G)_*^{(n)}|_w$$

is a diffeomorphism.

(b) *If $\Gamma \subset M$ is any G-invariant p-dimensional submanifold, with image $\Gamma/G = \pi[\Gamma] \subset M/G$, then*

$$\pi^{(n)}[\mathrm{pr}^{(n)}\,\Gamma|_z] = \mathrm{pr}^{(n)}\,(\Gamma/G)|_w \tag{3.32}$$

for any $z \in \Gamma$ with image $w = \pi(z) \in \Gamma/G$.

(c) *Two points $z^{(n)}$ and $\tilde{z}^{(n)}$ in $I_*^{(n)}$ have the same image in $(M/G)_*^{(n)}$ under $\pi^{(n)}$ if and only if they lie in the same orbit of $\mathrm{pr}^{(n)}\,G$. Thus*

$$I_*^{(n)}/\mathrm{pr}^{(n)}\,G \simeq (M/G)_*^{(n)}$$

with $\pi^{(n)}$ coinciding with the natural projection.

Proof. Almost all of these properties follow directly from the correspondence between G-invariant p-dimensional submanifolds of M and general $(p-s)$-dimensional submanifolds of M/G described in Proposition 3.21, and the following lemma.

Lemma 3.35. *Let Γ and $\tilde{\Gamma}$ be locally G-invariant submanifolds of M with images Γ/G and $\tilde{\Gamma}/G$ in M/G. Then Γ and $\tilde{\Gamma}$ have n-th order contact at $z_0 \in M$ if and only if Γ/G and $\tilde{\Gamma}/G$ have n-th order contact at $w_0 = \pi(z_0) \in M/G$.*

Proof. Choose flat local coordinates $(t, y, v) = (t^1, \ldots, t^s, y^1, \ldots, y^{p-s}, v^1, \ldots, v^q)$ near $z_0 = (t_0, y_0, v_0)$, the orbits of G being the slices $\{y = c, v = \tilde{c}\}$, such that Γ and $\tilde{\Gamma}$ are the graphs of functions $v = f(y, t)$, $v = \tilde{f}(y, t)$ respectively. The G-invariance of Γ and $\tilde{\Gamma}$ implies that f and $\tilde{f}$ are independent of t, and, moreover, in the corresponding local coordinates (y, v) on M/G, Γ/G and $\tilde{\Gamma}/G$ have the same respective formulae $v = f(y)$, $v = \tilde{f}(y)$. The lemma is thus trivial: n-th order contact of Γ and $\tilde{\Gamma}$ means that the n-th order derivatives of f and $\tilde{f}$ with respect to both y and t agree at y_0, t_0. But the t-derivatives are all identically zero, so this is clearly equivalent to the requirement that just the n-th order derivatives of f and $\tilde{f}$ with respect to y agree at y_0, which is the same as Γ/G and $\tilde{\Gamma}/G$ having n-th order contact. □

To prove Proposition 3.34, we define the map $\pi^{(n)}$ using (3.32), the lemma assuring us that it is well defined. Part (a) follows from the correspondence between G-invariant submanifolds of M and their images in M/G. To prove part (c), let Γ and $\tilde{\Gamma}$ be locally G-invariant submanifolds representing $z^{(n)} = \mathrm{pr}^{(n)}\,\Gamma|_z$ and $\tilde{z}^{(n)} = \mathrm{pr}^{(n)}\,\tilde{\Gamma}|_{\tilde{z}}$. The images $\pi^{(n)}(z^{(n)}) = \mathrm{pr}^{(n)}(\Gamma/G)|_{\pi(z)}$ and $\pi^{(n)}(\tilde{z}^{(n)}) = \mathrm{pr}^{(n)}(\tilde{\Gamma}/G)_{\pi(\tilde{z})}$ are the same if and only if Γ/G and $\tilde{\Gamma}/G$ have n-th order contact at $w = \pi(z) = \pi(\tilde{z})$. We conclude that z and $\tilde{z}$ lie in the same orbit of G in M, so by Proposition 1.24 there exist elements $g_1, \ldots, g_k \in G$ such that $\tilde{z} = g_1 \cdot g_2 \cdot \ldots \cdot g_k \cdot z$. Let $\Gamma^* = g_1 \cdot g_2 \cdot \ldots \cdot g_k \cdot \Gamma$. Then Γ^* passes through $\tilde{z}$, and has the same projection $\Gamma^*/G = \Gamma/G$ as Γ. By Lemma 3.35, Γ^* and $\tilde{\Gamma}$ have n-th order contact at $\tilde{z}$. Therefore

$$\begin{aligned}\tilde{z}^{(n)} &= \mathrm{pr}^{(n)}\,\tilde{\Gamma}|_{\tilde{z}} = \mathrm{pr}^{(n)}\,\Gamma^*|_{\tilde{z}} = \mathrm{pr}^{(n)}[g_1 \cdot g_2 \cdot \ldots \cdot g_k \cdot \Gamma]|_{\tilde{z}} \\ &= \mathrm{pr}^{(n)}\,g_1 \cdot \mathrm{pr}^{(n)}\,g_2 \cdot \ldots \cdot \mathrm{pr}^{(n)}\,g_k(\mathrm{pr}^{(n)}\,\Gamma|_z) = \mathrm{pr}^{(n)}\,g_1 \cdot \ldots \cdot \mathrm{pr}^{(n)}\,g_k(z^{(n)}),\end{aligned}$$

so $\tilde{z}^{(n)}$ and $z^{(n)}$ lie in the same orbit of $\mathrm{pr}^{(n)}\,G$. □

The Reduced Equation

Let $\mathscr{S}_\Delta^* \subset M_*^{(n)}$ correspond to a system Δ of partial differential equations admitting the symmetry group G. If Γ is the graph of a G-invariant solution to Δ, then not only is its prolongation $\mathrm{pr}^{(n)}\,\Gamma$ a submanifold of $\mathscr{S}_\Delta^*$, it also necessarily lies in the invariant space $I_*^{(n)}$. This suggests that the determination of such solutions is accomplished through the analysis of the intersection $\mathscr{S}_\Delta^* \cap I_*^{(n)}$ of these two subvarieties, whose invariance under $\mathrm{pr}^{(n)}\,G$ is guaranteed by Proposition 3.33. Using the projection of Proposition 3.34, we will then arrive at the *reduced system* $\mathscr{S}_{\Delta/G}^* = \pi^{(n)}[\mathscr{S}_\Delta^* \cap I_*^{(n)}]$. It is now easy to state and prove the fundamental theorem on the construction of group-invariant solutions.

Theorem 3.36. *Let G be a symmetry group of a system of differential equations $\mathscr{S}_\Delta^* \subset M_*^{(n)}$. A p-dimensional submanifold $\Gamma \subset M$ is a G-invariant solution if and only if the corresponding $(p - s)$-dimensional submanifold $\Gamma/G \subset M/G$ is a solution to the reduced system $\mathscr{S}_{\Delta/G}^* = \pi^{(n)}[\mathscr{S}_\Delta^* \cap I_*^{(n)}] \subset (M/G)_*^{(n)}$.*

PROOF. If Γ is such a solution, its prolongation $\mathrm{pr}^{(n)}\,\Gamma$ lies in the intersection $\mathscr{S}_\Delta^* \cap I_*^{(n)}$. Then by Proposition 3.34, $\pi^{(n)}[\mathrm{pr}^{(n)}\,\Gamma] = \mathrm{pr}^{(n)}(\Gamma/G)$ lies in $\mathscr{S}_{\Delta/G}^*$, hence Γ/G is a solution to the reduced system. To prove the converse, note first that by Proposition 3.33, $\mathscr{S}_\Delta^* \cap I_*^{(n)}$ is $\mathrm{pr}^{(n)}\, G$-invariant, hence if $z^{(n)} \in I_*^{(n)}$ has projection $\pi^{(n)}(z^{(n)}) \in \mathscr{S}_{\Delta/G}^*$, then $z^{(n)} \in \mathscr{S}_\Delta^* \cap I_*^{(n)}$. (Indeed, we can find $\tilde{z}^{(n)} \in \mathscr{S}_\Delta^* \cap I_*^{(n)}$ lying in the same orbit of $\mathrm{pr}^{(n)}\, G$.) Therefore, if Γ/G is a solution to the reduced system, and $\Gamma = \pi^{-1}(\Gamma/G)$ the corresponding G-invariant submanifold of M, then by (3.32) $\mathrm{pr}^{(n)}\, \Gamma \subset \mathscr{S}_\Delta^* \cap I_*^{(n)}$ since $\pi^{(n)}[\mathrm{pr}^{(n)}\, \Gamma] = \mathrm{pr}^{(n)}(\Gamma/G) \subset \mathscr{S}_{\Delta/G}^*$. Thus Γ is a solution. $\square$

Local Coordinates

Theorem 3.36 does provide the rigorous justification of the general method for constructing group-invariant solutions. Its almost trivial proof is a good illustration of the power of mathematical abstraction for simplifying and simultaneously generalizing seemingly complicated constructions. On the other hand, from a more practical standpoint its slick presentation is rather disconcerting, so we need to bring the abstract jet space constructions back down to earth, which means re-introducing local coordinates. We thus let $(x, u) = (x^1, \ldots, x^p, u^1, \ldots, u^q)$ be local coordinates on M, which we can now regard as an open subset of the Euclidean space $X \times U$, with jet space $M^{(n)} \subset M_*^{(n)}$.

If G is a local group of transformations acting on M, the invariant space $I^{(n)} \subset M^{(n)}$ differs from the extended invariant space $I_*^{(n)} \subset M_*^{(n)}$ just by the images of nontransversal G-invariant submanifolds Γ. In particular, $I^{(0)} \subset M$ consists of all points $z_0 = (x_0, u_0)$ such that there is at least one locally G-invariant function $u = f(x)$ whose graph passes through z_0. Note that while $I_*^{(0)} = M$, provided only that s, the dimension of the orbits of G, does not exceed p, the same cannot be said of $I^{(0)}$. For example, in the case $G = \mathrm{SO}(2)$ acting as the group of rotations on $X \times U \simeq \mathbb{R}^2$, the locally G-invariant functions are $u = \pm\sqrt{c^2 - x^2}$, whose graphs are circular arcs. No such graphs pass through the points on the x-axis, so $I^{(0)} = \{(x, u)\colon u \neq 0\}$ is *strictly* contained in $M = X \times U$. In general, outside $I^{(0)}$ there are no G-invariant functions at all, so we may as well restrict attention to $I^{(0)}$ itself and assume from now on that $M = I^{(0)}$, meaning that through each point of M there passes the graph of some G-invariant function $u = f(x)$. There is a simple explicit characterization of this requirement in the case of a regular group action.

Proposition 3.37. *Let G act regularly on $M \subset X \times U$. Then $z_0 \in M$ lies in $I^{(0)}$ if and only if the orbit of G through z_0 is transverse to the vertical space U_{z_0}, in which case G is said to act* transversally *at z_0.*

Proof. The necessity of transversality of G at z_0 is clear, since if Γ is locally G-invariant, Γ contains a relatively open subset $W \cap \mathcal{O}$ of the orbit $\mathcal{O}$ passing through z_0, so transversality of Γ implies transversality of $\mathcal{O}$. To prove sufficiency, note that by definition, G acts transversally at z_0 if and only if

$$\mathfrak{g}|_{z_0} \cap TU_{z_0}|_{z_0} = \{0\}, \tag{3.33}$$

since $\mathfrak{g}|_{z_0}$ is the tangent space to the orbit $\mathcal{O}$ through z_0. Let $w_0 = \pi(z_0)$ be the image point in M/G. Theorem 3.18 implies that $d\pi[TU_{z_0}|_{z_0}] \equiv TU^*|_{w_0}$ is a q-dimensional subspace of $T(M/G)|_{w_0}$. Let $\tilde{\Gamma}$ be any $(p-s)$-dimensional submanifold transverse to this subspace, meaning $T\tilde{\Gamma}|_{w_0} \cap TU^*|_{w_0} = \{0\}$. Then $\Gamma = \pi^{-1}(\tilde{\Gamma})$ is easily seen to be a G-invariant p-dimensional submanifold of M passing through z_0, transverse to U_{z_0}, and hence $z_0 \in I^{(0)}$. □

In local coordinates, if $\mathfrak{g}$ is spanned by the vector fields

$$\mathbf{v}_k = \sum_i \xi_k^i(x, u)\partial_{x^i} + \sum_\alpha \phi_\alpha^k(x, u)\partial_{u^\alpha}, \qquad k = 1, \dots, r,$$

then (3.33) is equivalent to the condition that the rank of the $p \times r$ matrix with entries $\xi_k^i(x, u)$ be exactly s, the dimension of the orbit, at $z_0 = (x_0, u_0)$:

$$\operatorname{rank}(\xi_k^i(x_0, u_0)) = \operatorname{rank}(\xi_k^i(x_0, u_0), \phi_\alpha^k(x_0, u_0)) = s. \tag{3.34}$$

For example, the action of SO(2) on $\mathbb{R}^2$ is generated by $-u\partial_x + x\partial_u$. We have $s = \operatorname{rank}(-u, x) = 1$ provided $(x, u) \neq (0, 0)$, while $\operatorname{rank}(-u) = 1$ except when $u = 0$. Thus SO(2) acts transversally everywhere except on the x-axis, which agrees with our earlier computation.

This clears up the connection between transversality and the existence of G-invariant functions, at least on the local level. The question of existence of *globally* defined G-invariant functions is considerably more delicate, and does not follow even if the local transversality condition holds everywhere. See Exercise 3.15 for an example.

From now on, we assume G acts transversally everywhere, so $I^{(0)} = M$. Then the invariant space $I^{(n)}$ can be described explicitly using the infinitesimal generators of G.

Theorem 3.38. *Let G act regularly and transversally on $M \subset X \times U$. Let*

$$\mathbf{v}_k = \sum_{i=1}^p \xi_k^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \phi_\alpha^k(x, u)\frac{\partial}{\partial u^\alpha}, \qquad k = 1, \dots, r,$$

be a basis for the infinitesimal generators. Then the n-th invariant space $I^{(n)} \subset M^{(n)}$ is determined by the equations

$$I^{(n)} = \{(x, u^{(n)}): D_J Q_\alpha^k(x, u^{(n)}) = 0, k = 1, \dots, r, \alpha = 1, \dots, q, \#J \leq n-1\},$$

where $Q_\alpha^k = \phi_\alpha^k - \sum_i \xi_k^i u_i^\alpha$ are the characteristics of the vector fields $\mathbf{v}_k$. (See (2.48).)

Proof. We outline the proof for $n = 1$, whereby $I^{(1)}$ is the common vanishing set of the characteristics $Q_\alpha^k(x, u^{(1)})$, leaving the extension to general n to the reader. If $u = f(x)$ is a G-invariant function, then, for $\alpha = 1, \ldots, q$,

$$0 = \mathbf{v}_k(u^\alpha - f^\alpha(x)) = \phi_\alpha^k - \sum_{i=1}^{p} \xi_k^i \frac{\partial f^\alpha}{\partial x^i}$$

must vanish whenever $u = f(x)$. The right-hand side is thus $Q_\alpha^k(x, \operatorname{pr}^{(1)} f(x))$. Since every point in $I^{(1)}$ is determined by the first prolongation of such a G-invariant function, we conclude that $I^{(1)}$ is contained in the set where $Q_\alpha^k = 0$ for all α, k. This part holds for any group action whatsoever.

To prove the converse, we have to use the restrictions on the group action. The easiest way to proceed is to introduce flat local coordinates $(y, v) = (y^1, \ldots, y^p, v^1, \ldots, v^q)$ where the orbits of G are the slices $\{y^1 = c_1, \ldots, y^{p-s} = c_{p-s}, v^1 = \hat{c}_1, \ldots, v^q = \hat{c}_q\}$, and hence at each point (y_0, v_0), the space of infinitesimal generators $\mathfrak{g}|_{(y_0, v_0)}$ is spanned by the tangent vectors $\partial/\partial y^{p-s+1}, \ldots, \partial/\partial y^p$. In this case, the vanishing of all the characteristics is equivalent to the conditions $\partial v^\alpha/\partial y^k = 0$ for all $\alpha = 1, \ldots, q$, $k = p - s + 1, \ldots, p$. To any point $(y_0, v_0^{(1)}) \in M^{(1)}$ satisfying these equations, it is easy to associate a function $v = h(y)$ with $v_0^{(1)} = \operatorname{pr}^{(1)} h(y_0)$. (For instance, h can be constant!) Therefore the reverse inclusion is valid, proving the theorem.

There are, however, two technical points to be dealt with in this coordinate change. The first is that changing coordinates does not alter the characteristics, or, more precisely, does not change their common vanishing set. This follows from the general formula of Exercise 3.21 for the behaviour of characteristics under a change of variables. The other point is that the graph of $u = f(x)$, when re-expressed in the (y, v)-coordinates, may fail to be transverse to the vertical v-space, and hence not be the graph of a well-defined function $v = h(y)$. This, however, is easily rectified by "skewing" the v coordinates through a linear change of variables $\tilde{y} = y + Lv$, $\tilde{v} = v$, for some constant $p \times q$ matrix L. The $(\tilde{y}, \tilde{v})$-coordinates are still flat, and L can always be determined so that the graph of $u = f(x)$ is once again transverse to the vertical space. □

Example 3.39. In the case of the rotation group SO(2), the infinitesimal generator is $\mathbf{v} = -u\partial_x + x\partial_u$, with characteristic $Q = x + uu_x$. The first invariant space at (x, u) with $u \neq 0$ is thus $I^{(1)} = \{(x, u, u_x): x + uu_x = 0\}$. Note that even if $u = 0$, Q does not vanish unless $x = 0$, so $I^{(1)}$ is still described by the vanishing of the characteristic except at the origin $x = u = 0$. There, however, $I^{(1)}$ is still empty, but $Q \equiv 0$ for all u_x, so the regularity of the group action is essential for the validity of Theorem 3.38. Higher order invariant spaces are constructed by differentiating, so for $u \neq 0$,

$$I^{(2)} = \{(x, u, u_x, u_{xx}): x + uu_x = 0,\ 1 + uu_{xx} + u_x^2 = 0\},$$

and so on.

To proceed to the quotient manifold, we make the further assumption that there exist $p + q - s$ globally defined, functionally independent invariants for

G on M, which we partition into new independent variables $y^i = \eta^i(x, u)$, $i = 1, \ldots, p - s$, and new dependent variables $v^\alpha = \zeta^\alpha(x, u)$, $\alpha = 1, \ldots, q$. (This can always be arranged by shrinking the domain M still further.) These provide global coordinates on the quotient manifold M/G, which we can therefore regard as an open subset of the $(p + q - s)$-dimensional Euclidean space $Y \times V$.

As we saw in Proposition 3.34, the projection $\pi^{(n)}$ provides a diffeomorphism between the full invariant space $I_*^{(n)}|_{z_0}$ at a point $z_0 \in M$ and the full extended jet space $(M/G)_*^{(n)}|_{w_0}$ at the image point $w_0 = \pi(z_0) \in M/G$. However, with the introduction of local coordinates on both M and M/G, we must impose transversality requirements on the relevant submanifolds to ensure that they locally look like the graphs of smooth functions. As a result, the basic correspondence between the invariant space $I^{(n)}|_{z_0}$ and the usual jet space $(M/G)^{(n)}|_{w_0}$ loses much of the innate simplicity of the extended version in Proposition 3.34.

Example 3.24 illustrated how graphs of smooth G-invariant functions might project down to nontransverse submanifolds of M/G, or, vice versa, smooth functions on M/G might correspond to nontransverse G-invariant submanifolds of M. To maintain the basic correspondence, then, we must avoid these pathological cases and concentrate on those G-invariant functions on M which correspond to smooth functions on M/G and conversely. More specifically, the invariant space $I^{(n)}|_{z_0}$ traced out by the prolongations of G-invariant functions $u = f(x)$ differs from the extended invariant space $I_*^{(n)}|_{z_0}$ only by those points in the "vertical" subvariety $\mathscr{V}^{(n)}|_{z_0} = M_*^{(n)}|_{z_0} \setminus M^{(n)}|_{z_0}$. Let $(\widetilde{\mathscr{V}/G})^{(n)}|_{w_0} \equiv \pi^{(n)}[\mathscr{V}^{(n)}|_{z_0} \cap I_*^{(n)}|_{z_0}]$ be its image in $(M/G)_*^{(n)}$; a point therein represents the prolongation of a submanifold of M/G passing through w_0 which does *not* correspond to a graph of a smooth G-invariant function passing through z_0. The remainder of the jet space,

$$(\widetilde{M/G})^{(n)}|_{w_0} = (M/G)^{(n)}|_{w_0} \setminus (\widetilde{\mathscr{V}/G})^{(n)}|_{w_0}$$

represents the prolongations of "nice" functions $v = h(y)$ which correspond to locally G-invariant functions $u = f(x)$ near z_0.

Conversely, the usual jet space $(M/G)^{(n)}|_{w_0}$ differs from the extended jet space $(M/G)_*^{(n)}|_{w_0}$ by the vertical subvariety $(\mathscr{V}/G)^{(n)}|_{w_0} \equiv (M/G)_*^{(n)}|_{w_0} \setminus (M/G)^{(n)}|_{w_0}$. Let $\tilde{\mathscr{V}}^{(n)}|_{z_0}$ be its pre-image in $I_*^{(n)}|_{z_0}$, so $\pi^{(n)}[\tilde{\mathscr{V}}^{(n)}|_{z_0}] = (\mathscr{V}/G)^{(n)}|_{w_0}$. A point thereof represents the prolongation of a G-invariant submanifold passing through z_0 which does *not* correspond to the graph of a smooth function $v = h(y)$ on M/G. The remainder of the invariant space

$$\tilde{I}^{(n)}|_{z_0} = I^{(n)}|_{z_0} \setminus \tilde{\mathscr{V}}^{(n)}|_{z_0}$$

contains the prolongations of all "nice" G-invariant functions $u = f(x)$ which correspond to explicit functions $v = h(y)$ on M/G. It is on these "nice" prolongation spaces that a correspondence similar to that of Proposition 3.34 holds.

Proposition 3.40. *The projection* $\pi^{(n)}\colon I_*^{(n)}|_{z_0} \to (M/G)_*^{(n)}|_{w_0}$ *induces a diffeomorphism* $\tilde{\pi}^{(n)}\colon \tilde{I}^{(n)}|_{z_0} \to (\widetilde{M/G})^{(n)}|_{w_0}$ *for* $w_0 = \pi(z_0)$.

So much for geometry; how does this all work out explicitly in the given local coordinates? Functional independence of the invariants η, ζ requires that the Jacobian matrix

$$\mathrm{J} \equiv \begin{pmatrix} \partial\eta^i/\partial x^j & \partial\eta^i/\partial u^\beta \\ \partial\zeta^\alpha/\partial x^j & \partial\zeta^\alpha/\partial u^\beta \end{pmatrix}$$

have rank $p + q - s$ everywhere. The transversality condition (3.34), when coupled with the infinitesimal criterion for invariance of η^i, ζ^α, requires that the last q columns of J have rank q everywhere:

$$\operatorname{rank}(\partial\eta^i/\partial u^\beta, \partial\zeta^\alpha/\partial u^\beta)^T = q. \tag{3.35}$$

By the implicit function theorem, we can then locally solve for all q dependent variables $u^1, \ldots, u^q$ along with $p - s$ of the independent variables, say $\tilde{x} = (x^{i_1}, \ldots, x^{i_{p-s}})$, in terms of $y = (y^1, \ldots, y^{p-s})$, $v = (v^1, \ldots, v^q)$ and the remaining s independent variables $\hat{x} = (x^{j_1}, \ldots, x^{j_s})$:

$$\tilde{x} = \gamma(\hat{x}, y, v), \qquad u = \delta(\hat{x}, y, v). \tag{3.36}$$

For each fixed value y_0, v_0 of the reduced variables, (3.36) determines an orbit of G in M parametrized by the "parametric variables" $\hat{x}$.

If $v = h(y)$ is a function whose graph lies in M/G, then the corresponding G-invariant p-dimensional submanifold of M is determined by the equations

$$\zeta(x, u) = h[\eta(x, u)], \tag{3.37}$$

obtained by replacing y and v by their expressions as invariants on M. This submanifold of M will be the graph of a function $u = f(x)$ if and only if we can solve (3.37) for u as a function of x, which requires that the $q \times q$ matrix $\partial\zeta/\partial u - (\partial h/\partial y)\cdot(\partial\eta/\partial u)$ be nonsingular. (As in Section 3.1, the derivative symbols denote Jacobian matrices.) Since we can identify h with v, it makes sense to write *this* transversality condition as

$$\det\left(\frac{\partial\zeta}{\partial u} - \frac{\partial v}{\partial y}\frac{\partial\eta}{\partial u}\right) \neq 0. \tag{3.38}$$

The contrary case when this determinant vanishes will correspond to G-invariant submanifolds of M which are *not* transverse to the vertical space U_z, and, hence, determine the singular subvariety $(\widetilde{\mathscr{V}/G})^{(n)}|_w$ which we must avoid!

From (3.37), we can differentiate to find the expressions for the derivatives of u with respect to x in terms of those of v with respect to y. By the chain rule,

$$\frac{\partial\zeta}{\partial x} + \frac{\partial\zeta}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\left(\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial u}\frac{\partial u}{\partial x}\right), \tag{3.39}$$

each derivative again representing a Jacobian matrix of the appropriate size. This can be rewritten in the form

$$\left(\frac{\partial\zeta}{\partial u}-\frac{\partial v}{\partial y}\frac{\partial\eta}{\partial u}\right)\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\frac{\partial\eta}{\partial x}-\frac{\partial\zeta}{\partial x},\tag{3.40}$$

whereby our transversality condition (3.38) permits us to solve explicitly for $\partial u/\partial x$,

$$\frac{\partial u}{\partial x}=\left(\frac{\partial\zeta}{\partial u}-\frac{\partial v}{\partial y}\frac{\partial\eta}{\partial u}\right)^{-1}\left(\frac{\partial v}{\partial y}\frac{\partial\eta}{\partial x}-\frac{\partial\zeta}{\partial x}\right),$$

as a function of x, u and $\partial v/\partial y$. The first $p+q$ variables can in turn be replaced by their expressions (3.36) leading to the formula

$$\frac{\partial u}{\partial x}=\delta_1\left(\hat{x},y,v,\frac{\partial v}{\partial y}\right),$$

for the first order x-derivatives of a G-invariant function $u=f(x)$ in terms of the first order derivatives of its representative $v=h(y)$.

Higher order derivatives are treated by further differentiating (3.39). If we introduce the *total Jacobian matrices* $\mathsf{D}_x\eta$, $\mathsf{D}_x\zeta$ with entries $D_i\eta^j$, $D_i\zeta^\alpha$ respectively, then (3.39) has the simpler form

$$\mathsf{D}_x\zeta=\frac{\partial v}{\partial y}\cdot\mathsf{D}_x\eta.$$

Differentiating with respect to x, we find, with self-evident notation,

$$\mathsf{D}_x^2\zeta=\frac{\partial v}{\partial y}\mathsf{D}_x^2\eta+\frac{\partial^2 v}{\partial y^2}(\mathsf{D}_x\eta)^2,\tag{3.41}$$

where

$$\mathsf{D}_x^2\zeta=\frac{\partial^2\zeta}{\partial x^2}+2\frac{\partial^2\zeta}{\partial x\partial u}\frac{\partial u}{\partial x}+\frac{\partial^2\zeta}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial\zeta}{\partial u}\frac{\partial^2 u}{\partial x^2}.$$

(We leave it to the reader to fill in the appropriate indices.) If we group the terms involving the second order derivatives of u together, we get an expression of the form

$$\left(\frac{\partial\zeta}{\partial u}-\frac{\partial v}{\partial y}\frac{\partial\eta}{\partial u}\right)\frac{\partial^2 u}{\partial x^2}=\tilde{\delta}_2\left(x,u,\frac{\partial u}{\partial x},\frac{\partial v}{\partial y},\frac{\partial^2 v}{\partial y^2}\right).$$

Again, (3.38) allows us to invert the matrix on the left-hand side, leading to an expression for $\partial^2u/\partial x^2$ in terms of x, u, $\partial u/\partial x$ and v, $\partial v/\partial y$. The first collection of variables can be replaced by their appropriate expressions in terms of y, v, $\partial v/\partial y$ *and* the parametric variables $\hat{x}$, so

$$\frac{\partial^2 u}{\partial x^2}=\delta_2(\hat{x},y,v^{(2)}),$$