# 11

# Coordinate space

In the previous chapters we set up renormalization theory in momentum space. In this chapter, we will give a treatment in coordinate space. Now, the utility of a momentum-space description, such as we gave in the earlier chapters, comes from the translation invariance of a problem. However, the momentum-space formulation rather obscures the fact that UV divergences arise from purely short-distance phenomena. Thus a coordinate-space treatment is useful from a fundamental point of view. There are also a number of situations, essentially external field problems, where a coordinate-space treatment is the most appropriate from a more practical point of view. A particular advantage is that the coordinate-space method makes it easy to see that the counterterms are the same as with no external field.

An important case, which we will treat in detail in this chapter, is that of thermal field theory at temperature $T$ (Fetter & Walecka (1971), Bernard (1974), and references therein). There one works with imaginary time using periodic boundary conditions (period $1/T$).

It is first necessary to work out the short-distance singularities of the free propagator. This we will do in Section 11.1. A number of forms of the propagator will be given, whose usefulness will become apparent when we treat some examples in Section 11.2. The reader should probably skip to this section first and refer back to Section 11.1 as the need for various properties of the propagator arises. We will explicitly show that the particular counterterms that we compute are independent of temperature. Our arguments will be of a form that will readily generalize not only to higher orders but also to other situations.

In Section 11.3 we will show to all orders of perturbation theory that counterterms at finite temperature are independent of $T$. We will do this by constructing counterterms in a theory in flat space-time in such a way as to make manifest the fact that only the short-distance singularities of the propagator affect the counterterms.

Another case, which we will treat in Section 11.4, is flat space-time with, say, an external electromagnetic field. If the field is weak it can be treated as a perturbation, i.e., as part of the interaction. But if the field is strong, one must put it in the free Lagrangian. Thus the free electron propagator satisfies

$$(\mathrm{i}\partial\!\!\!/ + \mathrm{e}A\!\!\!/ - M)S_{\mathrm{F}}(x, y; eA) = \mathrm{i}\delta(x - y), \tag{11.0.1}$$

which cannot in general be solved by expanding in a series in the field (Schwinger (1951)).

Another common external field problem is that of quantizing a quantum field theory in a curved space-time, where there is no remnant of a global Poincaré symmetry. Momentum space is then an inappropriate tool. One wishes to show that the UV counterterms can be kept the same as in flat space-time. We will not discuss this particular case here. But an extension of the techniques described should enable a fairly simple treatment to be given.

## 11.1 Short-distance singularities of free propagator

### 11.1.1 *Zero temperature*

The UV counterterm of a 1PI graph is ultimately determined by the short-distance singularities of the free propagators that make up the graph. So we need to obtain these singularities; and we will do this in this section.

It is sufficient to consider the scalar propagator

$$S_{\mathrm{F}}(x^2; d, m) = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{i}e^{\mathrm{i}q \cdot x}}{(q^2 - m^2 + \mathrm{i}\varepsilon)}. \tag{11.1.1}$$

Propagators for fields with spin can be obtained by differentiating with respect to $x$. The propagator satisfies the equation

$$(\Box + m^2)S_{\mathrm{F}} = -\mathrm{i}\delta^{(d)}(x), \tag{11.1.2}$$

together with appropriate boundary conditions. When $x$ is non-zero this equation reduces to

$$(4z\partial^2/\partial z^2 + 2d\partial/\partial z - m^2)S_{\mathrm{F}} = 0, \tag{11.1.3}$$

where $z = -x^2$. Thus $S_{\mathrm{F}} = z^{1/2 - d/4} w(mz^{1/2})$, where $w$ satisfies the modified Bessel equation of order $\nu = d/2 - 1$. The particular solution we need is determined by requiring that the $\delta$-function in (11.1.2) be obtained at $x^2 = 0$ and that $S_{\mathrm{F}} \to 0$ as $x^2 \to -\infty$.

A standard method of solving (11.1.3) is to expand in powers of $z$:

$$\begin{aligned} S_F &= z^a \sum_{n=0}^{\infty} (zm^2)^n w_n \\ &= (-x^2)^a \sum (-m^2 x^2)^n w_n. \end{aligned} \tag{11.1.4}$$

There are two independent solutions. One is analytic at $z = 0$, i.e., it has $a = 0$; the other solution is singular, with $a = 1 - d/2$. $S_F$ is a linear combination of these solutions. The quickest way to compute the singularity is to observe that, if $m = 0$, then (11.1.1) gives

$$S_F(m=0) = \Gamma(d/2 - 1)/[4\pi^{d/2}(-x^2)^{d/2-1}]. \tag{11.1.5}$$

This normalizes the coefficient of the singular solution. The normalization of the regular solution is obtained from the properties of Bessel functions, by requiring that $S_F \to 0$ as $x^2 \to -\infty$. Then we find that

$$\begin{aligned} S_F = {} & \frac{1}{4\pi^{d/2}(-x^2)^{d/2-1}} \sum_{n=0}^{\infty} \left(\frac{m^2 x^2}{4}\right)^n \frac{\Gamma(d/2-1-n)}{n!} \\ & + m^{d-2}(4\pi)^{-d/2} \sum_{n=0}^{\infty} (m^2x^2/4)^n \Gamma(1-n-d/2)/n! \\ = {} & S_{\mathrm{Fsing}} + S_{\mathrm{Fana}}. \end{aligned} \tag{11.1.6}$$

Here, we have used the series expansion of the Bessel functions. The coefficient of the regular solution can be obtained quickly by explicitly computing, from (11.1.1), that $S_F = \Gamma(1 - d/2)m^{d-2}/(4\pi)^{d/2}$ at $x = 0$, if $d$ is less than 2.

For the purposes of renormalization we will need the singularities of $S_F$. These are of two types: the singularities of $S_{\mathrm{Fsing}}$ as $x \to 0$, and the singularities of $S_{\mathrm{Fana}}$ as $d$ approaches an even integer. The fact that most of the $\Gamma$-functions have poles when $d$ is an even integer $d = 2\omega$, reflects the fact that the ansatz (11.1.4) is then incorrect for the singular solution. However, the correct result is obtained by expanding in powers of $d - 2\omega$ and then letting $d \to 2\omega$. The limits $x \to 0$ and $d \to 2\omega$ are non-uniform. Application to graphs like the $\phi^3$ self-energy need $x \neq 0$, but tadpole graphs (e.g., Fig. 11.1.1 in $\phi^4$ theory) have a propagator with $x = 0$ for all values of $d$. So

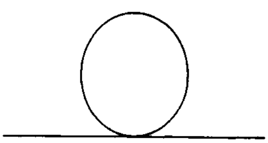

Fig. 11.1.1. Graph with a tadpole.

it is convenient to extract all the singular behavior of $S_F$ close to $d = 2\omega$ and $x = 0$ by writing

$$
\begin{aligned}
S_F &= \bar{S}_{\text{Fsing}} + \bar{S}_{\text{Fana}} \\
\bar{S}_{\text{Fsing}} &= \frac{1}{4\pi^{\omega}(-x^2)^{\omega-1}} \left\{ \sum_{n=0}^{\omega-2} \left(\frac{m^2 x^2}{4}\right)^n \frac{\Gamma(d/2-1-n)}{n!(-\pi x^2)^{d/2-\omega}} \right. \\
&\quad \left. + \sum_{n=\omega-1}^{\infty} \left(\frac{m^2 x^2}{4}\right)^n \frac{1}{n!} \left[ \frac{\Gamma(d/2-1-n)}{(-\pi x^2)^{d/2-\omega}} + \frac{(-1)^{n+\omega}\mu^{d-2\omega}}{(d/2-\omega)(n+1-\omega)!} \right] \right\}, \\
\bar{S}_{\text{Fana}} &= \frac{m^{2\omega-2}}{(4\pi)^{\omega}} \sum_{n=0}^{\infty} \left(\frac{m^2 x^2}{4}\right)^n \frac{1}{n!} \times \\
&\quad \times \left\{ \Gamma(1-n-d/2)\left(\frac{m^2}{4\pi}\right)^{d/2-\omega} - \frac{(-1)^{n+\omega}\mu^{d-2\omega}}{(d/2-\omega)(\omega+n-1)!} \right\}.
\end{aligned}
\tag{11.1.7}
$$

Neither term is a solution of the equation for $S_F$. The analytic part now has a finite limit as $d \to 2\omega$, with no singular behavior at $x = 0$. The singular term also has a finite limit as $d \to 2\omega$ if $x \neq 0$. It contains the $x \to 0$ singularities. But if we need $S_F(x = 0)$ then we first take $d < 2$, then $x \to 0$, and finally $d \to 2\omega$. Then the only singular contribution is from the $n = \omega - 1$ term in $\bar{S}_{\text{Fsing}}$:

$$
\bar{S}_{\text{Fsing}}(x=0) = \frac{(-1)^{\omega} m^{2\omega-2}\mu^{d-2\omega}}{(d/2-\omega)(4\pi)^{\omega}(\omega-1)!}. \tag{11.1.8}
$$

This result will be needed in evaluating tadpole graphs.

### 11.1.2 *Non-zero temperature*

Thermal Green's functions at inverse temperature $\beta = 1/T$ are obtained by Wick-rotating time $t \to -i\tau$ and then by imposing periodic boundary conditions (Fetter & Walecka (1971)). Thus the free propagator $S_F(x - y; \beta)$ satisfies

$$
(-\partial^2/\partial\tau^2 - \vec{\nabla}^2 + m^2) S_F(\tau, \vec{x}) = \delta(\tau)\delta^{(3)}(\vec{x}), \tag{11.1.9a}
$$

$$
S_F(\tau, \vec{x}) \to 0, \quad \text{as } \vec{x} \to \infty, \tag{11.1.9b}
$$

$$
S_F(\tau + \beta, \vec{x}) = S_F(\tau, \vec{x}). \tag{11.1.9c}
$$

All integrals over space-time are restricted to the time range between 0 and $\beta$. The propagator is obtained from the zero-temperature propagator $S_F(x; \infty)$ by writing

$$
S_F(x; \beta) = \sum_{n=-\infty}^{\infty} S_F(\tau + n\beta, \vec{x}; \infty). \tag{11.1.10}
$$

(Note that in $S_F(x-y;\beta)$ both $\tau_x$ and $\tau_y$ are between 0 and $\beta$, so when we apply $-\partial^2/\partial\tau^2 - \nabla^2 + m^2$ to it the only $\delta$-function in (11.1.9$a$) arises from the $n=0$ term.)

The only singularities in $S_F(x;\beta)$ are from the $n=0$ term when $x^\mu = 0$, so we have

$$S_F(x;\beta) = \bar{S}_{\mathrm{Fsing}}(x) + \bar{S}_{\mathrm{Fana}}(x;\beta), \qquad (11.1.11)$$

where $\bar{S}_{\mathrm{Fsing}}$ is the same as in (11.1.7), with $-x^2 = \tau^2 + \vec{x}^2$. The analytic term is different. It differs from $\bar{S}_{\mathrm{Fana}}(x;\infty)$ at zero temperature by

$$\Delta S_F = \sum_{n \neq 0} S_F(\tau + n\beta, x; \infty),$$

which, in the range $0 < t < \beta$, is a solution of the homogeneous equation $(\Box + m^2)S_F = 0$. Notice that it is not a function of $x^2$ alone, i.e., it is not Poincaré invariant.

Another, direct, way of deriving the form (11.1.11) is to expand $S_F(x;\beta)$ in a power series about $x^\mu = 0$, and then to require the differential equation (11.1.9$a$) to be satisfied. The $\delta$-function on the right-hand side ensures that $S_F = S_{\mathrm{Fsing}}(x) + S_{\mathrm{Fana}}(x;\beta)$, where $S_{\mathrm{Fsing}}$ is the same singular solution as in (11.1.6). Then $S_{\mathrm{Fana}}$ is a solution of the homogeneous equation analytic at $x=0$, and for which $S_{\mathrm{Fsing}} + S_{\mathrm{Fana}}$ satisfies periodic boundary conditions. We then add and subtract the poles at $d = 2\omega$ to obtain (11.1.11). The coefficient of the pole in $S_{\mathrm{Fana}}$ is the same as at zero temperature, because it must cancel the pole in $S_{\mathrm{Fsing}}$, which is independent of temperature.

## 11.2 Construction of counterterms in low-order graphs

To explain how renormalization works in coordinate space we consider the graphs shown in Figs. 11.2.1 to 11.2.4 for $\phi^4$ theory in four space-time dimensions. These are sufficient to show the various complications that occur.

Our treatment works as well at any temperature.

The simplest example is the one-loop correction to the propagator,

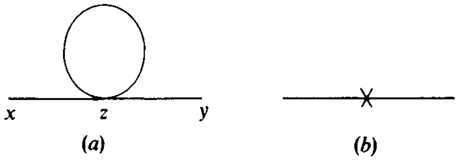


Fig. 11.2.1. Self-energy graph and counterterm.

Fig. 11.2.1. Its contribution to the full propagator is:

$$M_1 = -\frac{\mathrm{i}g}{2}\mu^{4-d}\int \mathrm{d}^d z S_F(x-z,d;1/T)S_F(z-y,d;1/T)S_F(0,d;1/T). \tag{11.2.1}$$

The functions $S_F(x-z)$ and $S_F(y-z)$ are singular at $z=x$ and at $z=y$, but the singularities are integrable, so that they do not contribute any divergence. The factor $S_F(0,d;1/T)$ is divergent, so we use (11.1.8) and (11.1.11) to write $M_1$ as

$$\begin{aligned} M_1 = &-\frac{\mathrm{i}g}{2}\int \mathrm{d}^4 z S_F(x-z)S_F(z-y)\bar{S}_{\mathrm{Fana}}(0,d=4;1/T) \\ &-\frac{\mathrm{i}g}{2}\int \mathrm{d}^d z S_F(x-z)S_F(z-y)\frac{m^2}{8\pi^2(d-4)} + O(d-4). \end{aligned} \tag{11.2.2}$$

The divergent term is evidently exactly cancelled by the mass-renormalization counterterm, Fig. 11.2.1(*b*). The result for the renormalized propagator at order $g$ is

$$-\frac{\mathrm{i}g}{2}\int \mathrm{d}^4 z S_F(x-z)S_F(z-y)\bar{S}_{\mathrm{Fana}}(0,4,1/T). \tag{11.2.3}$$

Notice that the renormalization only involved the singular term in $S_F$. Thus we have shown that the counterterm is the same at any temperature.

The unrenormalized graph of Fig. 11.2.2 for the four-point function is

$$\begin{aligned} M_2 = \tfrac{1}{2}(-\mathrm{i}g\mu^{4-d})^2 &\int \mathrm{d}^d y \mathrm{d}^d z S_F(x_1-y)S_F(x_2-y)S_F(y-z)^2 \times \\ &\times S_F(z-x_3)S_F(z-x_4). \end{aligned} \tag{11.2.4}$$

The only divergence as $d$ approaches 4 comes from the singularity of $S_F(y-z)^2$ at $y=z$; it is logarithmic. Let us write

$$M_2 = g^2\int \mathrm{d}^d y \mathrm{d}^d z g S_F(y-z)^2 f(y,z,\{x_i\}). \tag{11.2.5}$$

Since the divergence is logarithmic, it is governed by the leading behavior of

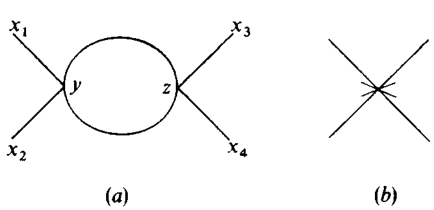


Fig. 11.2.2. Vertex graph and counterterm.

the integrand of (11.2.5) as $y \to z$. Thus we can generate a counterterm, illustrated in Fig. 11.2.2(*b*),

$$M_{2b} = -\int \mathrm{d}^d y \mathrm{d}^d z g^2 f(y, y, \{x_i\}) [\text{pole part of} \int_{z \simeq y} \mathrm{d}^d z S_{\mathrm{F1}}(y-z)^2]. \tag{11.2.6}$$

The singularity is entirely given by the first term $S_{\mathrm{F1}}$ in the expansion (11.1.7) of $\bar{S}_{\mathrm{Fsing}}$

$$S_{\mathrm{F1}} \equiv \Gamma(d/2-1)/[4\pi^{d/2}(-x^2)^{d/2-1}].$$

The pole-part factor in (11.2.6) is then

$$\begin{aligned} -\,\text{pole}\left\{\int_{z \simeq y} \mathrm{d}^d z S_{\mathrm{F1}}(y-z)^2\right\} &= -\,\text{pole}\left\{\frac{\Gamma(d/2-1)^2}{16\pi^d}\int_{z \simeq 0} \mathrm{d}^d z(-z^2)^{2-d}\right\} \\ &= \frac{\mathrm{i}}{16\pi^4}\,\text{pole}\left\{\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0 \mathrm{d}z z^{3-d}\right\} \\ &= \frac{\mathrm{i}}{8\pi^2}\frac{\mu^{d-4}}{4-d}. \end{aligned} \tag{11.2.7}$$

In the first line we shifted $z$ to $z + y$. In the second line we Wick rotated $z^0$, and the notation $\int_{z \simeq 0} \mathrm{d}z$ indicates that only the region near $z = 0$ matters. The factor $\mu^{d-4}$ in the last line is needed, as usual, to preserve explicit dimensional correctness.

We thus find that the renormalized $M_2$ is

$$\begin{aligned} M_2 = \lim_{d \to 4} \int \mathrm{d}^d z \mathrm{d}^d y \left[ g^2 S_{\mathrm{F}}(y-z)^2 + \frac{\mathrm{i} g^2 \mu^{4-d}}{8\pi^2(4-d)}\delta^{(d)}(y-z)\right] \times \\ \times f(y, z; \{x_i\}). \end{aligned} \tag{11.2.8}$$

The factor in square brackets is a well-defined distribution at $d = 4$. Again observe that we only used the singular part $\bar{S}_{\mathrm{Fsing}}$ of $S_{\mathrm{F}}$ in obtaining the counterterm, so that the counterterm is temperature-independent.

The graph Fig. 11.2.3 has a logarithmic subdivergence as well as an

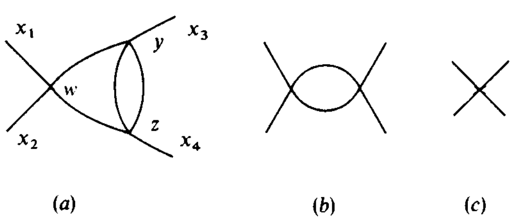


Fig. 11.2.3. Two-loop vertex graph and counterterms.

overall divergence. Its unrenormalized value is

$$M_3 = \tfrac{1}{2}(-ig\mu^{4-d})^3 \int d^d w d^d y d^d z S_F(x_1 - w) S_F(x_2 - w) S_F(x_3 - y) \times$$
$$\times S_F(x_4 - z) S_F(w - y) S_F(w - z) S_F(y - z)^2$$
$$\equiv g^3 \int d^d w d^d y d^d z f^{(3)}(w, y, z, \{x_i\}) S_F(w - y) S_F(w - z) S_F(y - z)^2. \tag{11.2.9}$$

If $w$ is not close to $y$ or $z$ there is a logarithmic divergence when $y$ approaches $z$. This is identical to the divergence in Fig. 11.2.2. There is also an overall divergence when all of the interaction points $w$, $y$, and $z$ approach each other.

We first subtract the subdivergence by the counterterm (11.2.7):

$$\bar{R}(M_3) = g^3 \int f^{(3)}(w, y, z, \{x_i\}) S_F(w - y) S_F(w - z) \times$$
$$\times \left[ S_F(y - z)^2 + \frac{i}{8\pi^2} \frac{\mu^{d-4}}{(4-d)} \delta^{(d)}(y - z) \right]. \tag{11.2.10}$$

The distribution in square brackets is singular at $y = z$. However, it is integrable; that is, it produces a finite result when $d$ goes to 4, if it is integrated with a test function continuous at $y = z$. The function $f^{(3)}(w, y, z) S_F(w - y) S_F(w - z)$ is continuous at $y = z$, unless also $w = y = z$. The remaining divergence in $\bar{R}(M_3)$ comes from the region $w \sim y \sim z$, where $f^{(3)}$ is not continuous. (There is also a singularity if $y$ and $z$ are both equal to $x_3$ or $x_4$; however, this region is again integrable.) The divergence at $w = y = z$ is logarithmic, so again it is determined entirely by the leading singular terms of the propagators. The counterterm is therefore

$$C = -\int d^d w g^3 f^{(3)}(w, w, w, \{x_i\}) \,\mathrm{pole} \left\{ \int d^d y d^d z S_{F1}(y) S_{F1}(z) \times \right.$$
$$\left. \times \left[ S_F(y - z)^2 + \frac{i}{8\pi^2} \frac{\mu^{d-4}}{(4-d)} \delta^{(d)}(y - z) \right] \right\}, \tag{11.2.11}$$

as shown in Fig. 11.2.3($c$). Recall that the factor in square brackets is singular but integrable at $y = z$. So the only divergent behavior comes when $S_F(y) S_F(z)$ is singular, i.e., at $y = z = 0$. It is easily checked that the counterterm $C$ then reduces to

$$C = \int d^d w f^{(3)}(w, w, w, \{x_i\}) \left( \frac{-g^3 \mu^{2d-8}}{(16\pi^2)^2} \right) \left[ \frac{2}{(4-d)^2} - \frac{1}{4-d} \right] \tag{11.2.12}$$

just as in momentum space (Vladimirov, Kazakov & Tarasov (1979)).

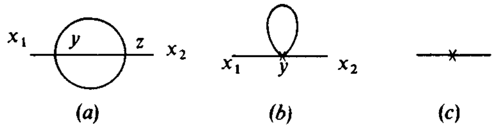


Fig. 11.2.4. Two-loop self-energy graph and counterterms.

Our final graph, Fig. 11.2.4, is a little more difficult. Its unrenormalized contribution to the propagator is

$$M_4 = \tfrac{1}{6}(-ig\mu^{4-d})^2 \int d^d y d^d z S_F(x_1 - y) S_F(y-z)^3 S_F(z - x_2)$$
$$\equiv g^2 \int d^d y d^d z f^{(4)}(y, z; \{x_i\}) S_F(y-z)^3. \tag{11.2.13}$$

The difficulty is that in momentum space there is not only the quadratic overall divergence, but also three logarithmic subdivergences. However (11.2.13) appears to have only an overall divergence, when $y$ goes to $z$; this involves all three propagators. Furthermore, the counterterm, calculated in momentum space, for a subdivergence from two of the lines gives a graph of the form of Fig. 11.2.1(*a*). There the third propagator is at $y = z$, whereas the propagators in (11.2.13) are also used at $y \neq z$.

The correct way to handle this case is to write

$$S_F = S_{F1} + S_{F2} + S_{\text{rem}}, \tag{11.2.14}$$

where $S_{F1}$ is the same leading term as before, and $S_{F2}$ is the second term in $\bar{S}_{\text{Fsing}}$:

$$S_{F2}(x) = -\frac{m^2}{16\pi^2}\left[\frac{\Gamma(d/2-2)}{(-\pi x^2)^{d/2-2}} - \frac{\mu^{d-4}}{d/2-2}\right]. \tag{11.2.15}$$

The remainder $S_{\text{rem}}(y-z)$ is finite as $d \to 4$ and/or $y \to z$. We now substitute (11.2.14) for each factor of $S_F(y-z)$ in (11.2.13) and obtain

$$M_4 = g^2\left\{\int d^d y d^d z f^{(4)}(y,z) S_{F1}(y-z)^3 + 3\int f^{(4)}(y,z) S_{F1}^2 S_{F2} + 3\int f^{(4)}(y,z) S_{F1}^2 S_{\text{rem}}\right\} + \text{finite}. \tag{11.2.16}$$

The term with all $S_{F1}$'s has the quadratic divergence, while the terms with two $S_{F1}$'s are logarithmically divergent. The other terms are finite. The term with $S_{F1}^2 S_{\text{rem}}$ has a divergence coming from the two $S_{F1}$'s, and its factor 3 reflects the fact that there are three divergent subgraphs. The graph with the counterterm for the subdivergences is Fig. 11.2.4(*b*):

$$
\begin{aligned}
M_{4b} &= \frac{3ig^2\mu^{4-d}}{8\pi^2(4-d)}\int d^d y f^{(4)}(y,y;\{x_i\})S_F(0) \\
&= \frac{3ig^2\mu^{4-d}}{8\pi^2(4-d)}\int d^d y f^{(4)}(y,y;\{x_i\})[S_{F2}(0)+S_{\text{rem}}(0)]. \qquad (11.2.17)
\end{aligned}
$$

Here we used the value we calculated at (11.2.7) for the counterterm vertex. The $S_{\text{rem}}(0)$ term here cancels the corresponding divergence in the third term in (11.2.16), while the $S_{F2}$ term combines with the second term in (11.2.16) to produce a logarithmic divergence from

$$
3g^2\int d^d y f^{(4)}(y,y)\left\{\int_{z\simeq y} d^d z S_{F1}(y-z)^2 S_{F2}(y-z)\mu^{8-2d} - \frac{\mu^{4-d}}{8\pi^2(4-d)}S_{F2}(0)\right\}. \qquad (11.2.18)
$$

Let us return to (11.2.16). The divergence in the first term is sensitive to $f^{(4)}$ and its second derivative at $y=z$:

$$
\begin{aligned}
g^2 &\int d^d z\int_{y\simeq z} d^d y S_{F1}(y-z)^3[f^{(4)}(z,z)+(y-z)^\mu(\partial f^{(4)}/\partial y^\mu)|_{y=z} \\
&\qquad + \tfrac{1}{2}(y-z)^\mu(y-z)^\nu(\partial^2 f^{(4)}/\partial y^\mu\partial y^\nu)|_{y=z}] \\
&= \text{finite} + [2ig^2\pi^{d/2}/\Gamma(d/2)]\times \\
&\quad\times\int d^d z\int_0 dy\, y^{d-1}\frac{\Gamma(d/2-1)^3}{(4\pi^{d/2})^3 y^{3d-6}}\left[f^{(4)}(z,z)-\frac{y^2}{2d}\frac{\partial^2 f^{(4)}}{\partial y^\mu\partial y_\mu}\bigg|_{y=z}\right] \\
&= \text{finite} - i\left(\frac{g}{16\pi^2}\right)^2\int d^d y\, d^d z f^{(4)}(y,z)\Box\delta^{(d)}(y-z)/(8-2d). \qquad (11.2.19)
\end{aligned}
$$

Note that the term with $f^{(4)}(z,z)$ gives a pole at $d=2$ but not at $d=4$.

## 11.3 Flat-space renormalization

In the last section we computed counterterms for some low-order graphs. Our method was not a good method for computing the finite parts. But it made very explicit the fact that the counterterms depend only on the short-distance structure of the free propagator. In particular it made it obvious that the counterterms are independent of temperature. We must now spell out how to generalize the results to an arbitrary graph in an arbitrary theory.

We define the renormalization of a graph $G$ by the same structure that we had in momentum space (i.e., by the recursion method or by the forest formula). A counterterm is needed for the overall divergence of every 1PI subgraph. The overall degree of divergence of a 1PI subgraph is obtained by first counting powers of position as all of its vertices approach each other, and by then taking the negative of the result (since it is an $x \to 0$ divergence). The integrations over relative positions of the vertices are included in the power-counting. The result coincides with the usual momentum-space definition for a 1PI subgraph. The value of $G$ is obtained by integrating over positions of its interaction vertices, and the divergence and subdivergences come from regions where some of these positions approach each other. To each region corresponds a subgraph consisting of the vertices that go to the same point, together with all the lines joining them.

It might appear that these graphs should be in one-to-one correspondence with the 1PI subgraphs that in momentum space are divergent when all their internal loop momenta are large. However this is not so, for the following reasons:

(1) There are graphs overall divergent in coordinate space that are not 1PI.
(2) There are divergent subgraphs in momentum space that are not obtained directly in coordinate space; these are when some but not all lines connecting vertices of a 1PI graph are in the subgraph.

An example of the first case is Fig. 11.3.1, where there is a logarithmically divergent graph consisting of the vertices at $w, y, z$, and of the lines joining them. In momentum space the counterterms for the self-energy subgraph cancel all the divergences. But in coordinate space $S_F(w - y)$ is singular, so we have to justify the momentum-space result. Other graphs are divergent in momentum space but do not occur directly as divergences in coordinate space. A typical example is given by the one-loop subgraphs of Fig. 11.2.4. The divergence comes from the region $y \to z$, and it appears to involve all three propagators, never just two propagators. As we saw in Section 11.2, the trick to handling this problem is to make a decomposition of the propagator, as in (11.2.14).

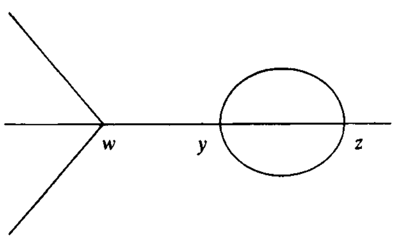


Fig. 11.3.1. The $w$–$y$–$z$ subgraph is one-particle-reducible and apparently has an overall divergence in coordinate space.

First, however, let us consider the problem of divergent 1PR graphs like the $(w, y, z)$ subgraph of Fig. 11.3.1. After subtraction of the self-energy counterterms, Fig. 11.3.1 has the form

$$\int d^d w d^d y d^d z f(w) S_F(w-y) \times \\ \times [S_F(y-z)^3 + A\delta(y-z) + B\Box\delta(y-z)] S_F(z-x_2). \qquad (11.3.1)$$

Here $f(w)$ is a function non-singular at $w = y$. The factor in square brackets is the self-energy plus counterterms. At $d = 4$, we have $S_F(w-y) \sim 1/(w-y)^2$ and $S_F(y-z)^3 \sim 1/(y-z)^6$, so it might appear that there is a logarithmic divergence when $w$ and $y$ both approach $z$. A counterterm for this divergence would be a four-point vertex and therefore allowed. However, the momentum-space result gives no such counterterm, so it is important to derive the same result directly in coordinate space.

We first integrate over $w$, and see that

$$\int d^d w f(w) S_F(w-y)$$

is finite and non-singular as a function of $y$. Then we can perform the $y$-integral. Although $S_F(y-z)$ is singular at $y = z$, it is integrated with a function with no singularity there, so the counterterms are sufficient to cancel the divergence. Then, finally, we integrate over $z$. The crucial point is that we find an order of integration with the following property: – each integral kills the singularity on precisely one propagator without introducing new singularities. If we replaced Fig. 11.3.1 by Fig. 11.2.3, say, we would have a not-quite similar integral:

$$\int d^d w d^d y d^d z \, S_F(w-y) S_F(w-z) [S_F(y-z)^2 - \text{counterterm}].$$

The integral over $w$, for example, involves two propagators $S_F(w-y)$, $S_F(w-z)$. The result becomes singular at $y = z$.

So far we have chosen to define renormalization by the forest formula or by the recursive method. We have seen that, after subtraction, only 1PI graphs have divergences. Some subgraphs that need counterterms in momentum space (like Fig. 11.2.4) do not appear explicitly as divergences in coordinate space, since we defined the subgraph corresponding to a coordinate-space divergence to include all the lines connecting its vertices. Even so, we saw in Section 11.2 that we have counterterms for such subgraphs – as in Fig. 11.2.4(*b*). To show how these counterterms arise in general and to show that the counterterms are independent of temperature, we decompose $S_F$ as

$$S_F = S_{F1} + \cdots + S_{FN} + S_{rem}. \tag{11.3.2}$$

Here $S_{F1}$ to $S_{FN}$ are the first $N$ terms in the singular part $\bar{S}_{Fsing}$ of $S_F$. We choose $N$ so that $2N$ is larger than the degree of divergence of any subgraph of the graph $G$ that is being renormalized. Thus when $S_F$ is replaced by $S_{rem}$, any (sub)graph $\gamma$ containing it becomes overall convergent.

When we substitute (11.3.2) for every propagator in a graph, we obtain a sum of graphs, in each of which every propagator is replaced by one of $S_{F1}, \ldots, S_{FN}$, or $S_{rem}$. We renormalize each of these new graphs separately. A subgraph needs a counterterm if its overall degree of divergence is a positive integer or zero, and we make counterterms for all 1PI subgraphs that are divergent.

The rules for computing the degree of divergence are essentially obvious. Any propagator with $S_F$ replaced by $S_{F1}$ contributes as in the original graph, and a replacement by $S_{FN}$ reduces the degree of divergence by $2N - 2$. The only subtlety is that replacement of $S_{F1}$ by $S_{rem}(x - y)$ is considered as contributing $-2N$ to the degree of divergence, even though it does not behave as $(x - y)^{2-d+2N}$ when $x - y \to 0$: Its analytic part behaves as $(x - y)^0$. However, the analytic part never actually contributes to any overall divergence, and the singular part of $S_{rem}$ does go like $(x - y)^{2-d+2N}$ relative to $S_{F1}$.

A typical case is the third term on the right of (11.2.16). Before subtraction of the subdivergence, this term is proportional to

$$\int f^{(4)}(y,z) S_{rem}(y - z) S_{F1}(y - z)^2 \mathrm{d}^d y \mathrm{d}^d z$$

Now $S_{F1}(y - z)^2 \sim (y - z)^{4-2d}$ as $y \to z$, while $S_{rem} \to \bar{S}_{Fana}(0) \neq 0$. So there is a logarithmic divergence at $y = z$. But after subtraction, we have:

$$\int f^{(4)}(y,z) S_{rem}(y - z) \left[ S_{F1}(y - z)^2 + \frac{\mathrm{i}\mu^{d-4}}{8\pi^2(4 - d)} \delta(y - z) \right] \mathrm{d}^d y \mathrm{d}^d z.$$

When integrated with an analytic function like $f^{(4)}(y,z)\bar{S}_{Fana}(y - z)$, the factor in square brackets gives a finite result. So any divergence comes from the singularity in $S_{rem}$. But the singularity is of order $(y - z)^{6-d}$, so no divergence occurs.

We now generalize our treatment of Fig. 11.2.4. We examine structures of the form of Fig. 11.3.2($a$). The subgraph $A$ is 1PI and overall divergent. The lines $l_1, \ldots, l_a$ join vertices of $A$. A graph that consists of $A$ and some or all of the $l_i$'s is 1PI and is overall divergent. In coordinate space, these divergences come from the region where all vertices of $A$ approach the same point; it

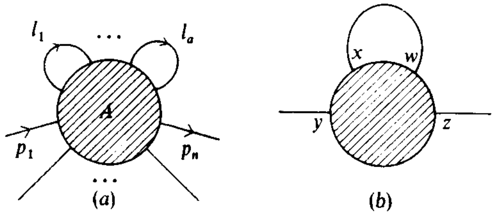


Fig. 11.3.2. General structure where subgraphs are divergent in momentum space, but appear to have no corresponding divergence in coordinate space.

appears that the lines $l_1, \ldots, l_a$ all participate in every one of the divergences.

In order to preserve the correct counterterm structure we must include counterterms for graphs of form $A$ in our definition of renormalization. This ensures that counterterms are given by terms in the action. For simplicity consider the case in $\phi^4$ theory of one line $l$ with two external lines $p_1, p_2$. Schematically we have

$$\int \mathrm{d}^d w \mathrm{d}^d x \mathrm{d}^d y \mathrm{d}^d z \, S_\mathrm{F}(x-w) A(w,x,y,z) f(y,z). \tag{11.3.3}$$

Here $S_\mathrm{F}(x-w)$ is the propagator of line $l$, while $f(y,z)$ is analytic at $y=z$ and represents the rest of the graph. $A(w,x,y,z)$ is the value of graph $A$. We subtract off all subdivergences of the graph $A \cup \{l\}$. Among these is the overall counterterm for $A$, which is of the form

$$C_A(w,x,y,z;d,g,\mu) = C(d,g,\mu)\delta^{(d)}(w-z)\delta^{(d)}(x-z)\delta^{(d)}(y-z), \tag{11.3.4}$$

since $A$ gives a logarithmic divergence. The result of replacing $A$ by its counterterm is then

$$C(d,g,\mu)\int \mathrm{d}^d z f(z,z) S_\mathrm{F}(0). \tag{11.3.5}$$

This counterterm is necessarily generated by the Feynman rules, since when $A$ occurs in a four-point Green's function without additional loops it needs an overall counterterm. We wish to show how this counterterm is needed to cancel certain parts of the divergences of (11.3.3) as $y \to z$.

Let us now substitute each propagator by (11.3.2) with $N=2$. The propagator for line $l$ may be replaced by $S_{\mathrm{F}1}$, $S_{\mathrm{F}2}$, or $S_\mathrm{rem}$. The terms with $S_{\mathrm{F}1}$ and $S_{\mathrm{F}2}$ both contribute to a divergence at $y=z$, whether in the original graph (11.3.3) or in (11.3.5). The divergence from (11.3.5) with $S_\mathrm{F} = S_{\mathrm{F}1}$ or $S_{\mathrm{F}2}$ is local and temperature-independent, so that it can be cancelled by overall wave-function and mass counterterms. The terms with $S_\mathrm{rem}$ also give divergences, but cancel in the sum of (11.3.3) and (11.3.5), just as in the simplest case of Fig. 11.2.4.

The argument presented above is considerably more cumbersome than needed for the particular case considered. However it was presented so as to emphasize its form as a special case of a general argument. This case happens to be the only one present in Green's functions. However in the vacuum bubbles (used to compute ground-state energies) and in the presence of composite operators the degree of divergence can be higher, with the consequence of needing a more general proof.

In any event, the moral is that if we choose counterterms to cancel divergent subgraphs including those of form $A$ in Fig. 11.3.2, then the counterterms need only depend on the $S_{F1}, S_{F2}, \ldots$. In particular, the value of $\bar{S}_{\mathrm{Fana}}$ is irrelevant. It is only $\bar{S}_{\mathrm{Fana}}$ that knows the boundary conditions, so only it knows the temperature. Note that the $S_{Fi}$'s are monomials in mass. Hence our counterterms are polynomials in mass parameters, as we saw by a totally different method in momentum space.

## 11.4 External fields

Consider as an example QED in the external electromagnetic field generated by a classical source $J^\mu$. We have (in covariant gauge)

$$\begin{aligned}\mathscr{L} = & -\tfrac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(\mathrm{i}\partial\!\!\!/ + eA\!\!\!/ - M)\psi - J^\mu A_\mu \\ & -\tfrac{1}{4}(Z_3 - 1)F_{\mu\nu}^2 + (Z_2 - 1)\bar{\psi}(\mathrm{i}\partial\!\!\!/ + eA\!\!\!/)\psi - \bar{\psi}\psi(Z_2 M_0 - M) \\ & -(1/2\xi)\partial\cdot A^2.\end{aligned} \tag{11.4.1}$$

To separate the classical and quantum parts of the electromagnetic field, we let $\mathscr{A}_\mu$ be a $c$-number potential that satisfies the classical Maxwell equations with source $J^\mu$. Then we replace $A_\mu$ in (11.4.1) by $A_\mu + \mathscr{A}_\mu$, with the result

$$\begin{aligned}\mathscr{L} = & -\tfrac{1}{4}F_{\mu\nu}^2 - (1/2\xi)\partial\cdot A^2 + \bar{\psi}(\mathrm{i}\partial\!\!\!/ + e\mathscr{A}\!\!\!/ - M)\psi + e\bar{\psi}A\!\!\!/\psi \\ & -\tfrac{1}{4}(Z_3 - 1)F_{\mu\nu}^2 + (Z_2 - 1)\bar{\psi}(\mathrm{i}\partial\!\!\!/ + eA\!\!\!/)\psi - (Z_2 M_0 - M)\bar{\psi}\psi \\ & +\tfrac{1}{2}\mathscr{A}_\mu J^\mu + \tfrac{1}{2}(Z_3 - 1)\mathscr{A}_\mu J^\mu + (Z_3 - 1)A_\nu J^\nu \\ & +(Z_2 - 1)\bar{\psi}e\mathscr{A}\!\!\!/\psi + \text{total derivative}.\end{aligned} \tag{11.4.2}$$

We assume the gauge condition $\partial\cdot\mathscr{A} = 0$. The total electromagnetic field is $A_\mu + \mathscr{A}_\mu$, with the classical $c$-number part satisfying

$$\begin{aligned}\partial\cdot\mathscr{A} &= 0 \\ \Box\mathscr{A}^\mu &= J^\mu.\end{aligned} \tag{11.4.3}$$

If the field $\mathscr{A}_\mu$ is large then we are not allowed to expand in powers of $\mathscr{A}_\mu$. Indeed, as Schwinger (1951) pointed out, the electron propagator $S_F(x, y; e\mathscr{A})$ in the external field is not necessarily analytic at $\mathscr{A} = 0$.

Moreover, he showed that it is the non-analytic part that is relevant for pair production in an electric field. We will therefore do perturbation theory in $e$ without assuming that $e\mathscr{A}_\mu$ is small. The lowest-order propagator (in powers of $e$) therefore satisfies the equation:

$$(\mathrm{i}\gamma^\mu \partial/\partial x^\mu + e\gamma^\mu \mathscr{A}_\mu - M) S_\mathrm{F}(x, y; e\mathscr{A}) = \mathrm{i}\delta^{(4)}(x-y)\mathbf{1}. \tag{11.4.4}$$

Since $S_\mathrm{F}$ is no longer simple in momentum space, it is not clear that the renormalizations are the same as with $\mathscr{A}_\mu = 0$. Let us work in coordinate space. We construct a power series in $x - y$ to solve (11.4.4). The solution is a series singular at $x = y$, plus a series analytic there. We only need explicitly the first few singular terms, and to prove renormalizability we decompose $S_\mathrm{F}$ as in (11.3.2). It is important not to use an infinite series for $S_\mathrm{F}$ at that stage, because the series will not converge for general values of $x$, $y$ and $\mathscr{A}$. The singular series is obtained by treating both $e$ and $M$ in (11.4.4) as perturbations and expanding in powers. This is similar to the way in which the singular part of the scalar propagator is an expansion in powers of $m^2$.

We obtain the leading power of $x - y$ by solving

$$\mathrm{i}\gamma^\mu \partial/\partial x^\mu S_{\mathrm{F}1} = \mathrm{i}\delta(x-y)\mathbf{1}. \tag{11.4.5}$$

Each non-leading term $S_{\mathrm{F}n}$ is obtained in terms of earlier terms by solving at $x \neq y$ the equation

$$\mathrm{i}\gamma^\mu \partial/\partial x^\mu S_{\mathrm{F}n} = (M - e\mathscr{A}_\mu \gamma^\mu) \sum_{j=1}^{n-1} S_{\mathrm{F}j}. \tag{11.4.6}$$

To obtain $S_{\mathrm{F}n}$ uniquely, we require it to be a singular power of $x - y$ times a function of $y$. The operation $\mathrm{i}\gamma^\mu\partial/\partial x^\mu$ makes $S_{\mathrm{F}n}$ more singular while multiplication by $M$ or by $e\gamma^\mu\mathscr{A}_\mu$ leaves the degree of singularity unchanged.

After this work we see that the renormalizations are correctly given by treating as a perturbation the term $J^\mu A_\mu$ in the Lagrangian (11.4.1) before the shift of the field to $A_\mu + \mathscr{A}_\mu$. Since $A_\mu$ is the renormalized field, this term cannot affect the divergences – it simply tells us to integrate $J^\mu(x)$ separately with each of one or more external photon fields of an ordinary Green's function.

We see that, after the shift to $A_\mu + \mathscr{A}_\mu$, the divergences are correctly treated by expanding in powers of $\mathscr{A}$ and then taking the first few terms. The non-analyticity is entirely confined to the remainder term in $S_\mathrm{F}$; this contributes to important physics, but not to the divergences.