for all smooth functions $f: M \to \mathbb{R}$. It is easy to verify that $[\mathbf{v}, \mathbf{w}]$ is indeed a vector field. In local coordinates, if

$$\mathbf{v} = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i}, \qquad \mathbf{w} = \sum_{i=1}^{m} \eta^i(x) \frac{\partial}{\partial x^i},$$

then

$$[\mathbf{v}, \mathbf{w}] = \sum_{i=1}^{m} \{\mathbf{v}(\eta^i) - \mathbf{w}(\xi^i)\} \frac{\partial}{\partial x^i} = \sum_{i=1}^{m} \sum_{j=1}^{m} \left\{ \xi^j \frac{\partial \eta^i}{\partial x^j} - \eta^j \frac{\partial \xi^i}{\partial x^j} \right\} \frac{\partial}{\partial x^i}. \tag{1.28}$$

(Note that in (1.27) the terms involving second order derivatives of f cancel.) For example, if

$$\mathbf{v} = y \frac{\partial}{\partial x}, \qquad \mathbf{w} = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y},$$

then

$$[\mathbf{v}, \mathbf{w}] = \mathbf{v}(x^2) \frac{\partial}{\partial x} + \mathbf{v}(xy) \frac{\partial}{\partial y} - \mathbf{w}(y) \frac{\partial}{\partial x} = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}.$$

Proposition 1.32. *The Lie bracket has the following properties*:

(a) Bilinearity

$$\begin{aligned} [c\mathbf{v} + c'\mathbf{v}', \mathbf{w}] &= c[\mathbf{v}, \mathbf{w}] + c'[\mathbf{v}', \mathbf{w}], \\ [\mathbf{v}, c\mathbf{w} + c'\mathbf{w}'] &= c[\mathbf{v}, \mathbf{w}] + c'[\mathbf{v}, \mathbf{w}'], \end{aligned} \tag{1.29}$$

where c, c' are constants.

(b) Skew-Symmetry

$$[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}]. \tag{1.30}$$

(c) Jacobi Identity

$$[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] = 0. \tag{1.31}$$

The proof is left to the reader. (*Hint*: Use (1.27) as your definition—trying to verify the Jacobi identity using the local coordinate formula (1.28) is horrible.)

The first definition (1.27) of the Lie bracket ensures that it is coordinate-free. (This can also be checked from the local coordinate formula (1.28), but is a fairly tedious computation.) More generally, if $F: M \to N$ is any smooth map, and $\mathbf{v}$ and $\mathbf{w}$ are vector fields on M such that $dF(\mathbf{v})$, $dF(\mathbf{w})$ are F-related to well-defined vector fields on N, then their Lie brackets are also F-related:

$$dF([\mathbf{v}, \mathbf{w}]) = [dF(\mathbf{v}), dF(\mathbf{w})]. \tag{1.32}$$

To prove this, given $f: N \to \mathbb{R}$, if $y = F(x) \in N$, then by (1.24),

$$\begin{aligned} dF([\mathbf{v}, \mathbf{w}])f(y) &= [\mathbf{v}, \mathbf{w}]\{f(F(x))\} = \mathbf{v}(\mathbf{w}\{f(F(x))\}) - \mathbf{w}(\mathbf{v}\{f(F(x))\}) \\ &= \mathbf{v}\{dF(\mathbf{w})f(F(x))\} - \mathbf{w}\{dF(\mathbf{v})f(F(x))\} \\ &= dF(\mathbf{v})dF(\mathbf{w})f(y) - dF(\mathbf{w})dF(\mathbf{v})f(y) \\ &= [dF(\mathbf{v}), dF(\mathbf{w})]f(y), \end{aligned}$$

as required.

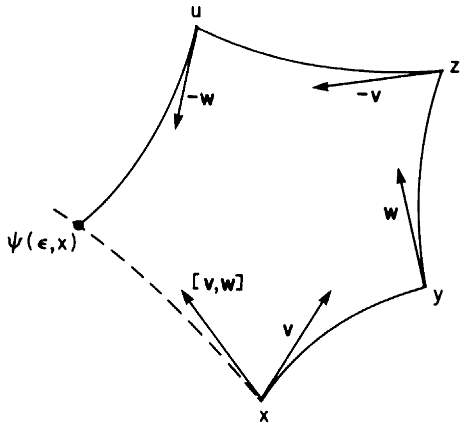

Figure 5. Commutator construction of the Lie bracket.

There is a more geometric characterization of the Lie bracket of two vector fields as the "infinitesimal commutator" of the two one-parameter groups $\exp(\varepsilon\mathbf{v})$ and $\exp(\varepsilon\mathbf{w})$.

Theorem 1.33. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be smooth vector fields on a manifold* M. *For each* $x \in M$, *the commutator*

$$\psi(\varepsilon, x) = \exp(-\sqrt{\varepsilon}\mathbf{w})\exp(-\sqrt{\varepsilon}\mathbf{v})\exp(\sqrt{\varepsilon}\mathbf{w})\exp(\sqrt{\varepsilon}\mathbf{v})x$$

defines a smooth curve for sufficiently small $\varepsilon \geqslant 0$. *The Lie bracket* $[\mathbf{v}, \mathbf{w}]|_x$ *is the tangent vector to this curve at the end-point* $\psi(0, x) = x$:

$$[\mathbf{v}, \mathbf{w}]|_x = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0+} \psi(\varepsilon, x). \tag{1.33}$$

Proof. Let $x = (x^1, \ldots, x^m)$ be local coordinates, so that

$$\mathbf{v} = \sum_{i=1}^m \xi^i(x)\frac{\partial}{\partial x^i}, \qquad \mathbf{w} = \sum_{i=1}^m \eta^i(x)\frac{\partial}{\partial x^i}.$$

Set $y = \exp(\sqrt{\varepsilon}\mathbf{v})x$, $z = \exp(\sqrt{\varepsilon}\mathbf{w})y$, $u = \exp(-\sqrt{\varepsilon}\mathbf{v})z$, so that $\psi(\varepsilon, x) = \exp(-\sqrt{\varepsilon}\mathbf{w})u$. Then we use the Taylor series expansions (1.18), (1.19) for the action of the flow generated by a vector field repeatedly:

$$\begin{aligned}
\psi(\varepsilon, x) &= u - \sqrt{\varepsilon}\eta(u) + \tfrac{1}{2}\varepsilon\mathbf{w}(\eta)(u) + O(\varepsilon^{3/2}) \\
&= z - \sqrt{\varepsilon}\{\eta(z) + \xi(z)\} + \varepsilon\{\tfrac{1}{2}\mathbf{w}(\eta)(z) + \mathbf{v}(\eta)(z) + \tfrac{1}{2}\mathbf{v}(\xi)(z)\} + O(\varepsilon^{3/2}) \\
&= y - \sqrt{\varepsilon}\xi(y) + \varepsilon\{\mathbf{v}(\eta)(y) - \mathbf{w}(\xi)(y) + \tfrac{1}{2}\mathbf{v}(\xi)(y)\} + O(\varepsilon^{3/2}) \\
&= x + \varepsilon\{\mathbf{v}(\eta)(x) - \mathbf{w}(\xi)(x)\} + O(\varepsilon^{3/2}).
\end{aligned}$$

Therefore

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0+} \psi(\varepsilon, x) = \{\mathbf{v}(\eta) - \mathbf{w}(\xi)\}(x),$$

and (1.33) is proven. □

As another illustration of the connection of the Lie bracket with the commutator, we show that the flows generated by two vector fields commute if and only if their Lie bracket vanishes everywhere.

Theorem 1.34. *Let* $\mathbf{v}$, $\mathbf{w}$ *be vector fields on* M. *Then*

$$\exp(\varepsilon\mathbf{v})\exp(\theta\mathbf{w})x = \exp(\theta\mathbf{w})\exp(\varepsilon\mathbf{v})x \tag{1.34}$$

for all $x \in M$ *and* ε, $\theta \in V$, *where* $V \subset R^2$ *is a connected open subset containing* $(0,0)$ *such that both sides of* (1.34) *are defined at all points therein, if and only if*

$$[\mathbf{v}, \mathbf{w}] = 0$$

everywhere.

Proof. Theorem 1.33 immediately shows that if the flows commute, i.e. if (1.34) holds, then the Lie bracket vanishes. Conversely, suppose $[\mathbf{v}, \mathbf{w}] = 0$, and let $x \in M$. If both $\mathbf{v}$ and $\mathbf{w}$ vanish at x, then flows of both vector fields leave x fixed, and hence they obviously commute at x. Otherwise, at least one vector field is not zero at x, say $\mathbf{v}|_x \neq 0$. Using Proposition 1.29, we can choose local coordinates $y = (y^1, \ldots, y^m)$ near x so that $\mathbf{v} = \partial/\partial y^1$ everywhere in these coordinates. Then if $\mathbf{w} = \sum \eta^i(y)\partial/\partial y^i$,

$$0 = [\mathbf{v}, \mathbf{w}] = \sum_{i=1}^{m} \frac{\partial\eta^i}{\partial y^1}\frac{\partial}{\partial y^i}.$$

Therefore each η^i is independent of y^1. The flow generated by $\mathbf{v}$ in these coordinates is just

$$\exp(\varepsilon\mathbf{v})(y^1, \ldots, y^m) = (y^1 + \varepsilon, y^2, \ldots, y^m).$$

The flow generated by $\mathbf{w}$ is a solution of the system of ordinary differential equations

$$\frac{dy^i}{d\theta} = \eta^i(y^2, \ldots, y^m), \qquad i = 1, \ldots, m.$$

Consider the functions

$$y(\theta, \varepsilon) = \exp(\theta\mathbf{w})\exp(\varepsilon\mathbf{v})y = \exp(\theta\mathbf{w})(y^1 + \varepsilon, y^2, \ldots, y^m)$$

and

$$\begin{aligned}\tilde{y}(\theta, \varepsilon) &= \exp(\varepsilon\mathbf{v})\exp(\theta\mathbf{w})y = \exp(\varepsilon\mathbf{v})y(\theta, 0)\\ &= (y^1(\theta, 0) + \varepsilon, y^2(\theta, 0), \ldots, y^m(\theta, 0)).\end{aligned}$$

Since y^1 does not appear on the right-hand side of the differential equations for the flow of $\mathbf{w}$, as functions of θ both y and $\tilde{y}$ are solutions, and both have the same initial conditions

$$y(0, \varepsilon) = (y^1 + \varepsilon, y^2, \ldots, y^m) = \tilde{y}(0, \varepsilon).$$

By uniqueness, $y(\theta, \varepsilon) = \tilde{y}(\theta, \varepsilon)$, which proves (1.34) for θ, ε sufficiently small.

To prove (1.34) in general, consider the following two subsets of the (θ, ε) plane: first V is the connected component of

$$\hat{V} = \{(\theta, \varepsilon)\text{: both sides of (1.34) are defined at } (\theta, \varepsilon)\}$$

containing the origin; second $U = \hat{U} \cap V$, where

$$\hat{U} = \{(\theta, \varepsilon)\text{: both sides of (1.34) are defined and equal at } (\theta, \varepsilon)\}.$$

By what we have just shown, U is open. On the other hand, by continuity, if (1.34) holds at $(\theta_i, \varepsilon_i) \in U$, and $(\theta_i, \varepsilon_i) \to (\theta^*, \varepsilon^*) \in V$, then (1.34) holds at $(\theta^*, \varepsilon^*)$. Thus U is both open and closed as a subset of V, so by connectivity $U = V$. *Warning*: It is not, in general, true that $\hat{U} = \hat{V}$! □

Tangent Spaces and Vector Fields on Submanifolds

Suppose $N \subset M$ is a submanifold of M parametrized by the immersion ϕ: $\tilde{N} \to M$. The tangent space to N at $y \in N$ is, by definition, the image of the tangent space to $\tilde{N}$ at the corresponding point $\tilde{y}$:

$$TN|_y = d\phi(T\tilde{N}|_{\tilde{y}}), \qquad y = \phi(\tilde{y}) \in N.$$

Note that $TN|_y$ is a subspace of $TM|_y$ of the same dimension as N. There is an analogous characterization of the tangent space to an implicitly defined submanifold:

Proposition 1.35. *Let $F: M \to \mathbb{R}^n$, $n \leqslant m$, be of maximal rank on $N = \{x: F(x) = 0\}$, so $N \subset M$ is an implicitly defined, regular $(m - n)$-dimensional submanifold. Given $y \in N$, the tangent space to N at y is precisely the kernel of the differential of F at y:*

$$TN|_y = \{\mathbf{v} \in TM|_y\colon dF(\mathbf{v}) = 0\}.$$

PROOF. If $\phi(\varepsilon)$ parametrizes a smooth curve $C \subset N$ passing through $y = \phi(\varepsilon_0)$, then $F(\phi(\varepsilon)) = 0$ for all ε. Differentiating with respect to ε, we see that

$$0 = \frac{d}{d\varepsilon} F(\phi(\varepsilon)) = dF(\dot{\phi}(\varepsilon)),$$

hence the tangent vector $\dot{\phi}$ to C is in the kernel of dF. The converse follows by a dimension count, using the fact that dF has rank n at y. □

Example 1.36. Consider the sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$. At each point $p = (x, y, z)$ on the sphere, the tangent space $TS^2|_p$ is given as the kernel of the differential of the defining function $F(x, y, z) = x^2 + y^2 + z^2 - 1$ at p. Thus

$$TS^2|_{(x,y,z)} = \left\{ a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \colon 2ax + 2by + 2cz = 0 \right\}.$$

Identifying $T\mathbb{R}^3|_p$ with $\mathbb{R}^3$, so that $\mathbf{v}|_p = a\partial_x + b\partial_y + c\partial_z$ becomes the vector (a, b, c), we see that $TS^2|_p$ consists of all vectors $\mathbf{v}|_p$ in $\mathbb{R}^3$ which are orthogonal to the radial vector $p = (x, y, z)$. Thus the tangent space $TS^2|_p$ agrees with the usual geometric tangent plane to S^2 at the point p. (The same argument generalizes to any implicitly defined surface $S = \{F(x, y, z) = 0\}$ in $\mathbb{R}^3$, where dF corresponds to the normal vector ∇F.)

Let N be a submanifold of M. If $\mathbf{v}$ is a vector field on M, then $\mathbf{v}$ restricts to a vector field on N if and only if $\mathbf{v}$ is everywhere tangent to N, meaning that $\mathbf{v}|_y \in TN|_y$ for each $y \in N$. In this case, using the definition of $TN|_y$, we immediately deduce the existence of a corresponding vector field $\tilde{\mathbf{v}}$ on the parametrization space $\tilde{N}$ satisfying $d\phi(\tilde{\mathbf{v}}) = \mathbf{v}$ on N.

Lemma 1.37. *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are tangent to a submanifold* N, *then so is* $[\mathbf{v}, \mathbf{w}]$.

PROOF. Let $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{w}}$ be the corresponding vector fields on $\tilde{N}$. Then by (1.32),

$$d\phi[\tilde{\mathbf{v}}, \tilde{\mathbf{w}}] = [d\phi(\tilde{\mathbf{v}}), d\phi(\tilde{\mathbf{w}})] = [\mathbf{v}, \mathbf{w}].$$

at each point of N. But this says that $[\mathbf{v}, \mathbf{w}]|_y \in TN|_y = d\phi(T\tilde{N}|_{\tilde{y}})$ for each $y \in N$. □

For example, in the case of the sphere S^2, since $z\partial_x - x\partial_z$ and $z\partial_y - y\partial_z$ are both tangent to S^2, so is $[z\partial_x - x\partial_z, z\partial_y - y\partial_z] = y\partial_x - x\partial_y$.

Frobenius' Theorem

We have already seen how each vector field $\mathbf{v}$ on a manifold M determines an integral curve through each point of M, such that $\mathbf{v}$ is tangent to the curve everywhere. Frobenius' theorem deals with the more general case of determining "integral submanifolds" of systems of vector fields, with the property that each vector field is tangent to the submanifold at each point.

Definition 1.38. Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be vector fields on a smooth manifold M. An *integral submanifold* of $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a submanifold $N \subset M$ whose tangent space $TN|_y$ is spanned by the vectors $\{\mathbf{v}_1|_y, \ldots, \mathbf{v}_r|_y\}$ for each $y \in N$. The system of vector fields $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is *integrable* if through every point $x_0 \in M$ there passes an integral submanifold.

Note that if N is an integral submanifold of $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$, then the dimension of the subspace of $TM|_y$ spanned by $\{\mathbf{v}_1|_y, \ldots, \mathbf{v}_r|_y\}$, which by definition is $TN|_y$, is equal to the dimension of N at each point $y \in N$. This does *not* exclude the possibility that the dimension of the subspace of $TM|_x$ spanned by $\{\mathbf{v}_1|_x, \ldots, \mathbf{v}_r|_x\}$ varies as x ranges over the entire manifold M; this just means that the given set of vector fields can have integral submanifolds of different dimensions.

Lemma 1.37 immediately gives necessary conditions that a system of vector fields be integrable. Namely, if N is an integral submanifold, then each vector field in the collection must be tangent to N at each point. Thus the Lie bracket of any pair of vector fields in the collection must again be tangent to N, and hence in the span of the set of vector fields at each point.

Definition 1.39. A system of vector fields $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ on M is *in involution* if there exist smooth real-valued functions $h_{ij}^k(x)$, $x \in M$, $i, j, k = 1, \ldots, r$, such that for each $i, j = 1, \ldots, r$,

$$[\mathbf{v}_i, \mathbf{v}_j] = \sum_{k=1}^{r} h_{ij}^k \cdot \mathbf{v}_k.$$

Frobenius' theorem, as generalized by Hermann to the case when the integral submanifolds have varying dimensions, states that this necessary condition is also sufficient:

Theorem 1.40. *Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be smooth vector fields on M. Then the system $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is integrable if and only if it is in involution.*

This theorem is *not* true as stated if the system is generated by infinitely many vector fields; see Exercise 1.13. There is, however, a useful generalization provided we make an additional restriction on the system. Let $\mathscr{H}$ be a collection of vector fields which forms a vector space. We say $\mathscr{H}$ is *in involution* if $[\mathbf{v}, \mathbf{w}] \in \mathscr{H}$ whenever $\mathbf{v}$ and $\mathbf{w}$ are in $\mathscr{H}$. In the above finite-dimensional case, $\mathscr{H}$ can be taken to be the set of linear combinations $\sum f_i(x)\mathbf{v}_i$ of the "basis" vector fields $\mathbf{v}_i$, with the f_i being arbitrary smooth real-valued functions on M (in which case $\mathscr{H}$ is called *finitely generated*). Let $\mathscr{H}|_x$ be the subspace of $TM|_x$ spanned by $\mathbf{v}|_x$ for all $\mathbf{v} \in \mathscr{H}$. An *integral submanifold* of $\mathscr{H}$ is a connected submanifold $N \subset M$ such that $TN|_y = \mathscr{H}|_y$ for all $y \in N$. We say that $\mathscr{H}$ is *rank-invariant* if for any vector field $\mathbf{v} \in \mathscr{H}$, the dimension of the subspace $\mathscr{H}|_{\exp(\varepsilon\mathbf{v})x}$ along the flow generated by $\mathbf{v}$ is a constant, independent of ε. (It can, of course, depend on the initial point x.) Note that since the integral curve $\exp(\varepsilon\mathbf{v})x$ of $\mathbf{v}$ emanating from a point x should be contained in an integral submanifold N, rank-invariance is certainly a necessary condition for complete integrability. Rank-invariance follows automatically if $\mathscr{H}$ is finitely generated, or consists of analytic vector fields on an analytic manifold.

Theorem 1.41. *Let $\mathscr{H}$ be a system of vector fields on a manifold M. Then $\mathscr{H}$ is integrable if and only if it is in involution and rank-invariant.*

In essence, the proof proceeds by direct construction of the integral submanifolds. If $x \in N$, then we can realize the integral submanifold through x by examining successive integral curves starting at x:

$$N = \{\exp(\mathbf{v}_1)\exp(\mathbf{v}_2)\cdots\exp(\mathbf{v}_k)x \colon k \geqslant 1, \mathbf{v}_i \in \mathscr{H}\}.$$

The rank invariance will imply that $\mathscr{H}|_y$ for any $y \in N$ has the correct dimension. The details of the proof that N is a submanifold can be found in Hermann, [2]. Borrowing terminology from the more usual constant-rank case, we call the collection of all maximal integral submanifolds of an integrable system of vector fields a *foliation* of the manifold M; the integral submanifolds themselves are also referred to as *leaves* of the foliation.

Example 1.42. Consider the vector fields

$$\mathbf{v} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \qquad \mathbf{w} = 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (z^2 + 1 - x^2 - y^2)\frac{\partial}{\partial z}$$

on $\mathbb{R}^3$. An easy computation proves that $[\mathbf{v}, \mathbf{w}] = 0$, so by Frobenius' theorem $\{\mathbf{v}, \mathbf{w}\}$ is integrable. Given (x, y, z), the subspace of $T\mathbb{R}^3|_{(x,y,z)}$ spanned by $\mathbf{v}|_{(x,y,z)}$ and $\mathbf{w}|_{(x,y,z)}$ is two-dimensional, except on the z-axis $\{x = y = 0\}$ and the circle $\{x^2 + y^2 = 1, z = 0\}$, where it is one-dimensional. It is not difficult to check that both the circle and the z-axis are one-dimensional integral submanifolds of $\{\mathbf{v}, \mathbf{w}\}$. All other integral submanifolds are two-dimensional tori

$$\zeta(x, y, z) = (x^2 + y^2)^{-1/2}(x^2 + y^2 + z^2 + 1) = c,$$

defined for $c > 2$. Indeed,

$$d\zeta(\mathbf{v}) = \mathbf{v}(\zeta) = 0, \qquad d\zeta(\mathbf{w}) = \mathbf{w}(\zeta) = 0,$$

everywhere, so by Proposition 1.35, both $\mathbf{v}$ and $\mathbf{w}$ are tangent to each level set of ζ where $\nabla\zeta \neq 0$. (See Section 2.1 for some general techniques for constructing integral submanifolds.)

An integrable system of vector fields $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is called *semi-regular* if the dimension of the subspace of $TM|_x$ spanned by $\{\mathbf{v}_1|_x, \ldots, \mathbf{v}_r|_x\}$ does not vary from point to point. In this case all the integral submanifolds have the same dimension. In analogy with the concept of a regular group action, we say that an integrable system of vector fields is *regular* if it is semi-regular, and, in addition, each point x in M has arbitrarily small neighbourhoods U with the property that each maximal integral submanifold intersects U in a pathwise connected subset. Although semi-regularity is a local property, which can be deduced using coordinates, regularity depends on the global structure of the system and is extremely difficult to check without explicitly finding the integral submanifolds. Any semi-regular system can be made regular, however, by restriction to a suitably small open subset of M. For example, the system in Example 1.42 is regular on the open subset $\mathbb{R}^3 \setminus (\{x = y = 0\} \cup \{x^2 + y^2 = 1, z = 0\})$ obtained by deleting the z-axis and the unit circle from $\mathbb{R}^3$.

For semi-regular systems of vector fields, Frobenius' theorem actually gives a means of "flattening out" the integral submanifolds by appropriate choice of local coordinates, just as we did for the integral curves of a single vector field in Proposition 1.29.

Theorem 1.43. *Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be an integrable system of vector fields such that the dimension of the span of $\{\mathbf{v}_1|_x, \ldots, \mathbf{v}_r|_x\}$ in $TM|_x$ is a constant s, independent of $x \in M$. Then for each $x_0 \in M$ there exist* flat local coordinates *$y = (y^1, \ldots, y^m)$ near x_0 such that the integral submanifolds intersect the given coordinate chart in the "slices" $\{y: y^1 = c_1, \ldots, y^{m-s} = c_{m-s}\}$, where $c_1, \ldots, c_{m-s}$ are arbitrary constants. If, in addition, the system is regular, then the coordinate chart can be chosen so that each integral submanifold intersects it in at most one such slice.*

For the system in Example 1.42, near any point (x_0, y_0, z_0) with $z_0 \neq 0$ and not on the z-axis, flat local coordinates are given by $\tilde{x} = x$, $\tilde{y} = y$, $\tilde{z} = \zeta(x, y, z)$. The tangent space to the plane $\{\tilde{z} = \text{constant}\}$ is spanned by the vector fields

$$\frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial x} - \frac{x(x^2 + y^2 - z^2 - 1)}{2z(x^2 + y^2)} \frac{\partial}{\partial z},$$

$$\frac{\partial}{\partial \tilde{y}} = \frac{\partial}{\partial y} - \frac{y(x^2 + y^2 - z^2 - 1)}{2z(x^2 + y^2)} \frac{\partial}{\partial z}.$$

Note that $\{\partial/\partial\tilde{x}, \partial/\partial\tilde{y}\}$ and $\{\mathbf{v}, \mathbf{w}\}$ both span the same subspace of $T\mathbb{R}^3$ at each point (x, y, z) with $z(x^2 + y^2) \neq 0$, so we have indeed locally "flattened out" the tori of Example 1.42. A more physically interesting set of flat local coordinates for $\{\mathbf{v}, \mathbf{w}\}$ are provided by the toroidal coordinates (θ, ψ, η), defined by

$$x = \frac{\sinh \eta \cos \psi}{\cosh \eta - \cos \theta}, \qquad y = \frac{\sinh \eta \sin \psi}{\cosh \eta - \cos \theta}, \qquad z = \frac{\sin \theta}{\cosh \eta - \cos \theta},$$

which arise in the theory of separation of variables for Laplace's equation, cf. Moon and Spencer, [1]. The reader can check that the level surfaces $\{\eta = c\}$ are precisely the integral tori for the system $\{\mathbf{v}, \mathbf{w}\}$; in fact $\mathbf{v} = \partial_\psi$, $\mathbf{w} = -2\partial_\theta$ under the change of coordinates!

1.4. Lie Algebras

If G is a Lie group, then there are certain distinguished vector fields on G characterized by their invariance (in a sense to be defined shortly) under the group multiplication. As we shall see, these invariant vector fields form a finite-dimensional vector space, called the Lie algebra of G, which is in a precise sense the "infinitesimal generator" of G. In fact almost all the information in the group G is contained in its Lie algebra. This fundamental observation is the cornerstone of Lie group theory; for example, it enables us to replace complicated nonlinear conditions of invariance under a group action by relatively simple linear infinitesimal conditions. The power of this method cannot be overestimated—indeed almost the entire range of applications of Lie groups to differential equations ultimately rests on this one construction!

We begin with the global Lie group picture, addressing the analogous construction for local Lie groups subsequently. Let G be a Lie group. For any group element $g \in G$, the *right multiplication map*

$$R_g: G \to G$$

defined by

$$R_g(h) = h \cdot g$$

is a diffeomorphism, with inverse

$$R_{g^{-1}} = (R_g)^{-1}.$$

A vector field $\mathbf{v}$ on G is called *right-invariant* if

$$dR_g(\mathbf{v}|_h) = \mathbf{v}|_{R_g(h)} = \mathbf{v}|_{hg}$$

for all g and h in G. Note that if $\mathbf{v}$ and $\mathbf{w}$ are right-invariant, so is any linear combination $a\mathbf{v} + b\mathbf{w}$, $a, b \in \mathbb{R}$; hence the set of all right-invariant vector fields forms a vector space.

Definition 1.44. The *Lie algebra* of a Lie group G, traditionally denoted by the corresponding lowercase German letter $\mathfrak{g}$, is the vector space of all right-invariant vector fields on G.

Note that any right-invariant vector field is uniquely determined by its value at the identity because

$$\mathbf{v}|_g = dR_g(\mathbf{v}|_e), \tag{1.35}$$

since $R_g(e) = g$. Conversely, any tangent vector to G at e uniquely determines a right-invariant vector field on G by formula (1.35). Indeed,

$$dR_g(\mathbf{v}|_h) = dR_g(dR_h(\mathbf{v}|_e)) = d(R_g \circ R_h)(\mathbf{v}|_e) = dR_{hg}(\mathbf{v}|_e) = \mathbf{v}|_{hg},$$

proving the right-invariance of $\mathbf{v}$. Therefore we can identify the Lie algebra $\mathfrak{g}$ of G with the tangent space to G at the identity element

$$\mathfrak{g} \simeq TG|_e. \tag{1.36}$$

In particular, $\mathfrak{g}$ is a finite-dimensional vector space of the same dimension as the underlying Lie group.

In addition to its vector space structure, such a Lie algebra is further equipped with a skew-symmetric bilinear operation, namely the Lie bracket. Indeed, if $\mathbf{v}$ and $\mathbf{w}$ are right-invariant vector fields on G, so is their Lie bracket $[\mathbf{v}, \mathbf{w}]$, since by (1.32)

$$dR_g[\mathbf{v}, \mathbf{w}] = [dR_g(\mathbf{v}), dR_g(\mathbf{w})] = [\mathbf{v}, \mathbf{w}].$$

This motivates the general definition of a Lie algebra.

Definition 1.45. A *Lie algebra* is a vector space $\mathfrak{g}$ together with a bilinear operation

$$[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the *Lie bracket* for $\mathfrak{g}$, satisfying the axioms

(a) *Bilinearity*

$$[c\mathbf{v} + c'\mathbf{v}', \mathbf{w}] = c[\mathbf{v}, \mathbf{w}] + c'[\mathbf{v}', \mathbf{w}], \quad [\mathbf{v}, c\mathbf{w} + c'\mathbf{w}'] = c[\mathbf{v}, \mathbf{w}] + c'[\mathbf{v}, \mathbf{w}'],$$

for constants $c, c' \in \mathbb{R}$,

(b) *Skew-Symmetry*

$$[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}],$$

(c) *Jacobi Identity*

$$[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] = 0,$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{v}', \mathbf{w}, \mathbf{w}'$ in $\mathfrak{g}$.

In this book most Lie algebras will be finite-dimensional vector spaces. (An interesting infinite-dimensional Lie algebra is given by the space of all smoth vector fields on a manifold M. However, infinite-dimensional algebras are considerably more difficult to work with.) We begin with some easy examples of Lie algebras.

Example 1.46. If $G = \mathbb{R}$, then there is, up to constant multiple, a single right-invariant vector field, namely $\partial_x = \partial/\partial x$. In fact, given $x, y \in \mathbb{R}$,

$$R_y(x) = x + y,$$

hence

$$dR_y(\partial_x) = \partial_x.$$

Similarly, if $G = \mathbb{R}^+$, then the single independent right-invariant vector field is $x\partial_x$. Finally, for SO(2) the vector field ∂_θ is again the unique independent right-invariant one. Note that the Lie algebras of $\mathbb{R}$, $\mathbb{R}^+$ and SO(2) are all the same, being one-dimensional vector spaces with trivial Lie brackets ($[\mathbf{v}, \mathbf{w}] = 0$ for all $\mathbf{v}, \mathbf{w}$). This shouldn't be surprising, as the reader can easily check from