

Action principle for OPE

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Abstract

We formulate an “action principle” for the operator product expansion (OPE) describing how a given OPE coefficient changes under a deformation induced by a marginal or relevant operator. Our action principle involves no ad-hoc regulator or renormalization and applies to general (Euclidean) quantum field theories. It implies a natural definition of the renormalization group flow for the OPE coefficients and of coupling constants. When applied to the case of conformal theories, the action principle gives a system of coupled dynamical equations for the conformal data. The last result has also recently been derived (without considering tensor structures) independently by Behan (arXiv:1709.03967) using a different argument. Our results were previously announced and outlined at the meetings “In memoriam Rudolf Haag” in September 2016 and the “Wolfhart Zimmermann memorial symposium” in May 2017.

1 Introduction

One possible viewpoint of quantum field theory (QFT) is that the operator product expansion [41, 42] (OPE) defines a theory, just as the equations of motion define a classical field theory.

Informally, the OPE states that

$$\mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_n}(x_n) = \sum_B \mathcal{C}_{A_1 \dots A_n}^B(x_1, \dots, x_n) \mathcal{O}_B(x_n), \quad (1.1)$$

which is understood in the sense of an insertion into a correlation function, and where $\{\mathcal{O}_A\}$ is the set of all composite operators of the theory; for details see sec. 2.2. Here we present a “variational-” or “action principle” for the OPE which states how the coefficients in this expansion change under a change in a coupling parameter g of the theory. If the OPE coefficients define the theory, then such a formula, relating the derivative w.r.t. g

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of a coefficient with other coefficients at the same value of g , should clearly exist, and it should depend only on the OPE coefficients, the kind of deformation (given by a relevant or marginal operator), but not on extraneous structures such as arbitrary regulators etc. However, it is not clear a priori exactly what form it should take.

The path integral suggests a formal action principle, but such naive formulas require renormalization, while we are looking for an intrinsic formula not requiring such extraneous procedures. Such a formula was derived only relatively recently in [19] building on earlier work [22, 25, 29]. It gives an expression for the derivative of an 2-point OPE coefficient w.r.t g in terms of 2- and 3-point OPEs involving the relevant or marginal operator \mathcal{V} conjugate to g . To close the system, these are supplemented by analogous formulas for the derivative of an n -point OPE coefficient w.r.t g in terms of 2- and $(n+1)$ -point OPEs. By contrast to similar hierarchies of functionals such as the Dyson-Schwinger equations, our equations have the desired property of being manifestly finite, i.e. are not in need of additional renormalisation or regularization. For antecedents of our equations see [16] which is somewhat similar in spirit but different in detail and works for relevant perturbations and OPE-coefficients of operators of sufficiently low dimensions.

The purpose of the present paper is to briefly explain the relationship of the action principle to the renormalisation group, and secondly, to apply it in the case of conformal QFTs (CFTs). In those theories, the n -point OPE coefficients and correlation functions can, in principle, be written in terms of the conformal data, i.e. the dimensions Δ_i of the conformal primaries and the structure constants λ_{ijk}^α , where $\mathcal{O}_i, \mathcal{O}_j, \dots$ are conformal primaries and α is a label for independent tensor structures needed if the operators are not scalar. Our action principle is shown to imply the following dynamical system

$$\begin{aligned} A_i \frac{d}{dg} \Delta_i &= \sum_\alpha \mathcal{D}_i^\alpha \lambda_{\mathcal{V}ii}^\alpha \\ \frac{d}{dg} \lambda_{jkl}^\mu &= \sum_m \sum_{\alpha\beta} A_m \left({}^a \mathcal{T}_{jklm}^{\alpha\beta\mu} \lambda_{\mathcal{V}jm}^\alpha \lambda_{klm}^\beta + {}^b \mathcal{T}_{jklm}^{\alpha\beta\mu} \lambda_{jkm}^\alpha \lambda_{\mathcal{V}lm}^\beta + {}^c \mathcal{T}_{jklm}^{\alpha\beta\mu} \lambda_{\mathcal{V}km}^\alpha \lambda_{jlm}^\beta \right), \end{aligned} \quad (1.2)$$

where $A_i = \lambda_{ii1}$ is a normalization factor for the 2-point function¹, and the constants \mathcal{T}, \mathcal{D} are of an entirely representation theoretic nature, i.e. in principle determined by the dimensions Δ_i and the tensor structures of the primary fields \mathcal{O}_i . We give formulas for them in terms of the spinning conformal blocks in general dimensions. Thus, to apply the dynamical system, one has to know these quantities as well as the possible tensor structures. Their determination and classification is a rather complicated problem in practice that is drawing considerable attention in the conformal bootstrap [9] community, see e.g. [3, 4, 39, 30, 31]. In $d = 2$, there are explicit expressions [35], which lead to explicit formulas in terms of hypergeometric functions also given below.

If all the coefficients \mathcal{T}, \mathcal{D} could be found (with their dependence on the dimensions Δ_i), then the dynamical system could be used in combination with a Newton iteration to find the flows $\Delta_i(g), \lambda_{ijk}(g)$, starting from a reference CFT. A particularly interesting case is when the starting point $g = 0$ corresponds to a Gaussian free field, as is the case in the $\mathcal{N} = 2, 4$ super CFTs in $d = 4$ based on gauge theories. Furthermore, this strategy

¹This could be set to 1 at the expense of a change in \mathcal{T} .

may open up a new avenue in the long-standing problem of mathematically establishing the existence of non-trivial QFTs in $d = 4$ dimensions. To accomplish this, one would need perhaps not completely explicit knowledge of the coefficients \mathcal{T}, \mathcal{D} , but at least sufficient control for large dimensions and spins. Furthermore, one would have to check the axioms for the resulting OPE coefficients (most notably associativity) which are not immediately evident from (1.2). In the context of perturbation theory, we have given in [19] an argument how the action principle can be used to derive associativity order-by-order in perturbation theory, but the argument is rather complicated. Nevertheless, it offers hope that this can be done.

The system (1.2) without tensor structures has also recently been derived independently by [2] using a somewhat different argument (with an explicit computation of the constants \mathcal{T} for $d = 1$). Our results were announced and outlined before at the meetings “In memoriam Rudolf Haag” in Hamburg in September 2016 [27] as well as at the “Wolfhart Zimmermann memorial symposium” in Munich in May 2017 [26]. This work is dedicated to the memory of Wolfhart Zimmermann whose work on the OPE [42] has been a major inspiration for us to further study this structure in QFT.

2 General structure of QFT

2.1 Euclidean QFT

Although this paper is not about axiomatic quantum field theory, to set the stage, we first recall the basic properties of correlation functions in Euclidean QFT, called the “Osterwalder-Schrader (OS)-axioms”, see e.g. [15]. The basic idea is to formulate general properties of Euclidean Green’s functions, perhaps constructed by a properly defined path integral,

$$\langle \mathcal{O}_{A_1}(x_1) \dots \mathcal{O}_{A_N}(x_N) \rangle = \int_{\phi} \mathcal{O}_{A_1}(x_1) \dots \mathcal{O}_{A_N}(x_N) \exp(-S(\phi)), \quad (2.3)$$

where S is an action such as e.g. the ϕ_d^4 -model

$$S = \int (|\partial\phi|^2 + m^2\phi^2 + g\phi^4) d^d x. \quad (2.4)$$

and where the composite fields are expressions of the form $\mathcal{O}_A = \partial^{a_1}\phi \dots \partial^{a_r}\phi$, where ∂^{a_i} is a multi-derivative. Of course, the precise definition and renormalization of such a path integral is a very complicated matter which has been successfully accomplished only so far for $d = 2, 3$ in a full, non-perturbative manner. In $d = 4$, only perturbative constructions exist, and in that case the OS-axioms are satisfied in the sense of formal series in g —to the extent that they make sense for formal series.

It is not the case that all reasonable quantum field theories will arise from a classical path integral—many counterexamples exist e.g. in $d = 2$, and presumably also in higher dimensions. Thus, in general, \mathcal{O}_A should not be thought of as somehow – i.e. modulo renormalization – corresponding to a polynomial of a “basic field” and its derivatives. Rather, A is an abstract label that incorporates the type of field and its tensor- or spinor

structure. The set of labels will carry some extra structure e.g. related to hermitian conjugation, Bose/Fermi alternative, etc., as will become clear when we need it. In the above example $A = \{a_1, \dots, a_r\}$ is basically a multi-index.

The expected properties of the correlation functions $\langle \mathcal{O}_{A_1}(x_1) \dots \mathcal{O}_{A_N}(x_N) \rangle$ are as follows:

- e1) Each $\mathcal{S}_{A_1 \dots A_n}(x_1, \dots, x_n) \equiv \langle \mathcal{O}_{A_1}(x_1) \dots \mathcal{O}_{A_N}(x_N) \rangle$ is real analytic for non-coinciding points, i.e. on the configuration space $M_n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}^d, x_i \neq x_j \ \forall i \neq j\}$.
- e2) (Identity) There is a neutral element $A = 1$ in our index set (corresponding intuitively to the identity operator $\mathcal{O}_1 = 1$) characterized by the following condition. If for $1 \leq k < n$ we have $A_k = 1$, then

$$\mathcal{S}_{A_1 \dots A_n}(x_1, \dots, x_n) = \mathcal{S}_{A_1 \dots \hat{A}_k \dots A_n}(x_1, \dots, \hat{x}_k, \dots, x_n), \quad (2.5)$$

where a hat means that the index/point is omitted.

- e3) (Bose/Fermi alternative): There exists an assignment $A \rightarrow F_A \in \{0, 1\}$ such that, for any $1 \leq i < j \leq n$, we have

$$\mathcal{S}_{\dots A_i \dots A_j \dots}(\dots, x_i, \dots, x_j, \dots) = (-1)^{F_{A_i} F_{A_j}} \mathcal{S}_{\dots A_j \dots A_i \dots}(\dots, x_j, \dots, x_i, \dots), \quad (2.6)$$

and such that $F_B = \sum F_{A_i} \bmod 2$. We think of a field \mathcal{O}_A as a Bose field if $F_A = 0$ and as a Fermi field if $F_A = 1$.

- e4) (Star operation) This axiom states that there is an involutive $*$ -operation $A \mapsto A^*$ on the set of labels A such that

$$\overline{\mathcal{S}_{A_1 \dots A_n}(x_1, \dots, x_n)} = \mathcal{S}_{A_n^* \dots A_1^*}(x_n, \dots, x_1), \quad (2.7)$$

where overbar means component-wise complex conjugation.

- e5) (Euclidean group action). Each composite field \mathcal{O}_A carries a finite-dimensional representation D of $\mathrm{SO}(d)$ (or its double cover if spinor fields are present). For any element $(r, a) \in E(d) = \mathrm{SO}(d) \ltimes \mathbb{R}^d$ of the Euclidean group, or its cover $\mathrm{Spin}(d) \ltimes \mathbb{R}^d$ in the case of fermionic fields,

$$\mathcal{S}_{A_1 \dots A_n}(rx_1 + a, \dots, rx_n + a) = \bigotimes_{j=1}^N D_j(r) \mathcal{S}_{A_1 \dots A_n}(x_1, \dots, x_n). \quad (2.8)$$

- e6) (OS reflection positivity) This axiom is the replacement of the positive definite Hilbert space condition in Lorentzian QFT. Its precise form can be found in [15].

Although the above properties allow for a reconstruction of a corresponding Lorentzian QFT by the famous OS-reconstruction theorem, they somehow do not tell us directly how the composite fields are related algebraically, and, similarly, they do not capture “short

distance factorization” properties that one typically expects in QFT. From a pragmatic viewpoint, the (related) essential shortcoming of the OS-axioms is that we cannot meaningfully say how the QFT changes under a variation of a coupling parameter such as g in ϕ_4^4 -theory. From the formal path integral, one is tempted to write:

$$\partial_g \langle \mathcal{O}_{A_1}(x_1) \dots \mathcal{O}_{A_N}(x_N) \rangle = \int d^4y \langle \mathcal{O}_{A_1}(x_1) \dots \mathcal{O}_{A_N}(x_N) \mathcal{V}(y) \rangle \quad (2.9)$$

where $\mathcal{V} = -\phi^4$ in this case. The problem is that the integral makes no sense due to the non-integrable short distance divergences at $y = x_i$, for which there is nothing intrinsic to the OS-axioms that would tell us how to regularize it. To capture these aspects, it seems inevitable to introduce the operator product expansion (OPE).

2.2 Axioms for OPE

In this paper, we take the viewpoint that a QFT is *defined* by the OPE. Thus, the burden of a construction of a QFT is (at least) to give *all* OPE coefficients $\{\mathcal{C}_{A_1 \dots A_n}^B(x_1, \dots, x_n; g)\}$ for arbitrary composite field labels $\{A_i\}$ and arbitrary n . Informally, the coefficients fulfill (1.1) in the sense of an insertion into a correlation function. The precise meaning of this relation is explained in 10) below. We have included a dependence on a parameter g in the OPE coefficients, which is thought of as representing a coupling constant. One may be interested in the theory at only one value of this parameter, or one may be interested in the whole family of theories labelled by g . Obviously, there is no need to restrict to a one-parameter family, and in principle, the parameters could belong to a manifold or even more general mathematical structures such as orbifolds.

To get a bigger conceptual picture, we first give a list of axioms that one would like the OPE coefficients in a reasonable Euclidean quantum field theory in $d > 1$ dimensions to fulfill, see [23, 24] for details.

1. (Analyticity) For each choice of A_1, \dots, A_n, B , $(x_1, \dots, x_n) \mapsto \mathcal{C}_{A_1 \dots A_n}^B(x_1, \dots, x_n)$ is a complex valued real analytic function on the “configuration manifold” $M_n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}^d, x_i \neq x_j \forall i \neq j\}$.
2. (Identity) There is a neutral element $A = 1$ in our index set (corresponding intuitively to the identity operator $\mathcal{O}_1 = 1$) characterized by the following condition. If for $1 \leq k < n$ we have $A_k = 1$, then

$$\mathcal{C}_{A_1 \dots A_n}^B(x_1, \dots, x_n) = \mathcal{C}_{A_1 \dots \hat{A}_k \dots A_n}^B(x_1, \dots, \hat{x}_k, \dots, x_n), \quad (2.10)$$

where a hat means that the index/point is omitted.

3. (Bose/Fermi alternative) There exists an assignment $A \rightarrow F_A \in \{0, 1\}$ such that, for any $1 \leq i < j < n$, we have

$$\mathcal{C}_{\dots A_i \dots A_j \dots}^B(\dots, x_i, \dots, x_j, \dots) = (-1)^{F_{A_i} F_{A_j}} \mathcal{C}_{\dots A_j \dots A_i \dots}^B(\dots, x_j, \dots, x_i, \dots), \quad (2.11)$$

and such that $F_B = \sum F_{A_i} \bmod 2$ for every non-zero OPE coefficient $\mathcal{C}_{A_1 \dots A_n}^B(x_1, \dots, x_n)$. We think of a field \mathcal{O}_A as a Bose field if $F_A = 0$ and as a Fermi field if $F_A = 1$.

4. (Base point) A fairly obvious axiom specifying how the coefficients change under a change of the base point (in our conventions, this is always the last point, x_n for a product of n operators).
5. (Dimension and scaling) There exists a assignment of “dimensions” $A \mapsto \Delta_A \in \mathbb{R}_+$ such that the identity operator has dimension $\Delta_1 = 0$ and such that, for any $\delta > 0$, we have²

$$\lim_{\lambda \rightarrow 0^+} \lambda^{\Delta_1 + \dots + \Delta_n - \Delta_B + \delta} \mathcal{C}_{A_1 \dots A_n}^B(\lambda x_1, \dots, \lambda x_n) = 0 . \quad (2.12)$$

The axiom connects the abstract notion of dimension with the short-distance behavior of the coefficients. We assume that the theory is “rational” in the sense that there is a finite number of fields—i.e. labels A —having dimension less than some given number.³

6. (Star operation) This axiom states that there is an involutive $*$ -operation $A \mapsto A^*$ on the set of labels A , and that the OPE coefficients of the starred operators are equal to the hermitian conjugates of the un-starred operators under a change of base point $x_1 \leftrightarrow x_n$.
7. (Descendants) This axiom states first that there is an operation $A \mapsto \partial A$ on the set of labels which we think of in the example of the ϕ_4^4 -model as corresponding to the partial derivative of the composite field and applying the “Leibniz rule”. The non-trivial part of this axiom is that $\mathcal{O}_{\partial A}$ should behave in the OPE just as $\partial \mathcal{O}_A$, i.e.

$$\mathcal{C}_{A_1 \dots \partial A_k \dots A_n}^B(x_1, \dots, x_n) = \frac{\partial}{\partial x_k} \mathcal{C}_{A_1 \dots A_n}^B(x_1, \dots, x_n) . \quad (2.13)$$

It is also natural to demand that the dimension of an operator to be increased by one under this operation, $\Delta_{\partial A} = \Delta_A + 1$.

8. (Associativity) To formulate the associativity of the OPE in its most basic form, some “rationality” assumption is needed stating that the index set should be countable. The strongest possible form of the associativity condition is perhaps the following. Let $1 < M < N$.

$$\mathcal{C}_{A_1 \dots A_N}^B(x_1, \dots, x_N) = \sum_C \mathcal{C}_{A_1 \dots A_M}^C(x_1, \dots, x_M) \mathcal{C}_{CA_{M+1} \dots A_N}^B(x_M, \dots, x_N) \quad (2.14)$$

holds on the domain defined by $\xi \equiv \frac{\max_{1 \leq i \leq M} |x_i - x_M|}{\min_{M < j \leq N} |x_j - x_M|} < 1$, the the sum over C is required to be absolutely convergent. We can, and should, also impose a more general condition for convergence of “nested” expansions on corresponding domains, see [23] for details.

²Note that this axioms is consistent with a scaling behavior of the form $\mathcal{C}_{AB}^C(x_1, x_2) \sim p(\log|x_{12}|) |x_{12}|^{-\Delta_A - \Delta_B + \Delta_C}$ for any polynomial p , as indeed found in perturbation theory where the $\Delta_A = [A]$ coincide with the engineering dimensions and where the degree of p increases with the loop order.

³A reasonable strengthened version of this would be that the “partition function” $\sum_A q^{\Delta_A}$ converges for sufficiently small $q > 0$.

For $N = 3$ the ratio $\xi = \frac{|x_1 - x_2|}{|x_2 - x_3|} < 1$ geometrically signifies to what extent the triangle of points x_1, x_2, x_3 is degenerate. Weaker versions of the axiom would only require that the OPE coefficient factorizes approximately if a subset of points (here x_1, \dots, x_M) get closer to each other than to the remaining points, i.e. in the limit as $\xi \rightarrow 0$.

9. (Euclidean invariance) An obvious axiom expressing that the OPE coefficients are covariant under the action of the Euclidean group in d dimensions, analogous to that for the correlation functions.
10. (Convergence): Let $1 < M < N$.

$$\langle \mathcal{O}_{A_1}(x_1) \dots \mathcal{O}_{A_N}(x_N) \rangle = \sum_C \mathcal{C}_{A_1 \dots A_M}^C(x_1, \dots, x_M) \langle \mathcal{O}_C(x_M) \dots \mathcal{O}_{A_N}(x_N) \rangle \quad (2.15)$$

holds on the domain defined by $\xi \equiv \frac{\max_{1 \leq i \leq M} |x_i - x_M|}{\min_{M < j \leq N} |x_j - x_M|} < 1$, where the sum over C is required to be absolutely convergent for fixed $\{x_i\}$ inside the domain. This axiom is very similar looking to the associativity condition and becomes identical if we have $\mathcal{C}_{A_1 \dots A_N}^1(x_1, \dots, x_N) = \langle \mathcal{O}_{A_1}(x_1) \dots \mathcal{O}_{A_N}(x_N) \rangle$. One can always apply a field-redefinition (see below) to satisfy this condition, essentially by imposing $\langle \mathcal{O}_A(x) \rangle = 0$, i.e. a vanishing 1-point function, for the redefined fields. This condition is not always natural⁴, but commonly imposed in CFTs (see below).

It is natural to consider two quantum field theories to be *equivalent* if their OPE coefficients differ by a “field-redefinition”. A field redefinition is abstractly a linear map Z from the space of fields to itself satisfying certain properties. Since we are working throughout with a fixed basis of fields \mathcal{O}_A labelled by the indices A , we may equivalently say that a field redefinition corresponds to a “matrix”, Z_A^B , the basic properties of which are dictated by the above axioms: The identity axiom 2) implies $Z_1^B = 1$ if $B = 1$ and $Z_1^B = 0$ otherwise. The Bose-Fermi alternative 3) implies that $Z_A^B = 0$ unless $F_A + F_B = 0 \pmod{2}$. The dimension and scaling axiom 5) implies that $Z_A^B = 0$ unless $\Delta_A \geq \Delta_B$. The star operation axiom 6) implies that $\overline{Z_A^B} = Z_{A^*}^{B^*}$. The Euclidean invariance axiom 9) implies that, viewed as a tensor over \mathbb{R}^d , Z_A^B should be invariant under the group action of the Euclidean group in d dimensions.

Definition 1. Given a matrix Z_A^B satisfying these properties, we say that two systems $\mathcal{C}_{A_1 \dots A_n}^{A_{\mathcal{R}}}$ and $\widehat{\mathcal{C}}_{A_1 \dots A_n}^{A_{\mathcal{R}}}$ are equivalent under Z_A^B if there holds

$$Z_{B_{\mathcal{R}}}^{A_{\mathcal{R}}} \widehat{\mathcal{C}}_{A_1 \dots A_n}^{B_{\mathcal{R}}}(x_1, \dots, x_n) = \mathcal{C}_{B_1 \dots B_n}^{A_{\mathcal{R}}}(x_1, \dots, x_n) Z_{A_1}^{B_1} \dots Z_{A_n}^{B_n} \quad (2.16)$$

for all n , all indices, and all $(x_i) \in M_n$. Note that the implicit sums over B_1, B_2, \dots etc. are finite because $Z_A^B = 0$ unless $\Delta_A \geq \Delta_B$, so there are no convergence issues.

⁴Such a normalization is not always natural because it might have non-analytic/non-smooth behavior e.g. near phase transition points in coupling constant space, or in the presence of boundaries (Casimir effect).

The transformation formula given in the definition corresponds to a redefinition

$$\hat{\mathcal{O}}_A = \sum_{B:\Delta_A \geq \Delta_B} Z_A^B \mathcal{O}_B \quad (2.17)$$

of the composite fields.

In the context of renormalized perturbation theory, the freedom to redefine fields is in one-to-one correspondence with the freedom of choosing different “renormalization conditions” [43]. There, it is also natural to view the OPE coefficients as functions of the coupling constant(s), (in our case g) and to allow also a diffeomorphism acting on g , as well as a dependence of Z_A^B on g . The renormalization group then also has a natural formulation in terms of field-redefinitions, see section 3.2.

2.3 Do the axioms hold?

Of course, it remains to be seen in models to what extent we can actually fulfill this “wish list.” There are by now many models with strictly relevant interaction for which the OS-axioms and further results about clustering, Borel summability of the perturbation series, etc. have been shown. Apart from many CFT models in $d = 2$, these include in particular the $P(\phi)_2$ -models in 2 dimensions–scalar fields with stable polynomial interaction–the ϕ_3^4 -model as well as QED_3 . For a summary of these developments up to the time of printing see the classic book of Glimm-Jaffe [15] and references therein. In a noteworthy development, Dimock [5, 6, 7] has recently picked up an approach of Balaban and reconsidered some of these models and methods. Comprehensive references to Balaban’s work can also be found there. Balaban’s method is in principle also applicable to treat the non-perturbative UV renormalization of models with marginal interaction such as YM_4 , but it is not completely clear to the author what has and what has not been achieved in his certainly very impressive series of papers in this direction. The full construction of YM_4 without infra-red cutoffs is a famous open problem. For the GN_2 -model the situation is better and an effective action has been constructed non-perturbatively by [14], and also by [10]. These results basically cover the full OS-axioms.

For models with marginal interactions, there are also many very good results in perturbation theory establishing the OS-axioms and many much more detailed properties in the sense of formal series. The most interesting theories are perhaps the YM_4 models, which have by now been treated in full mathematical detail in [11, 8] based on the method of RG-flow equations [37, 40] and the BRST-BV technique. Related prior works for the massless ϕ_4^4 -model are [17, 18], where several important new techniques for massless theories were introduced in this context.

Concerning the OPE, there are considerably fewer results. In the case of CFTs, Lüscher and Mack [33, 34] have demonstrated the OPE of two operators inside a 3-point function, but not the general version of the OPE and associativity given above. For a convincing theoretical physics style argument that this is the case, see [36]. In the perturbative setting, convergence of the OPE (with “smeared” spectator fields) for non-conformally invariant models was first shown by [20] in the massive ϕ_4^4 -model to arbitrary but fixed order in perturbation theory. Later, this was generalized to the massless case

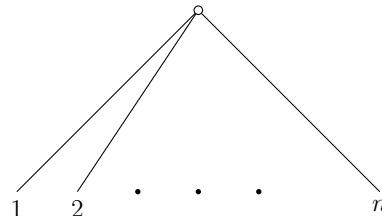
[21], with good error bounds for the remainder in a finite OPE, and also to perturbative YM₄-theory [12]. The OPE and associativity condition in the form given above was established in the massive and massless ϕ_4^4 model by [19] to arbitrary but finite orders in perturbation theory. This version of the OPE has also been established for $d = 2$ CFTs within the Vertex Operator Algebra framework [28].

3 Action principle: How the OPE changes under deformations

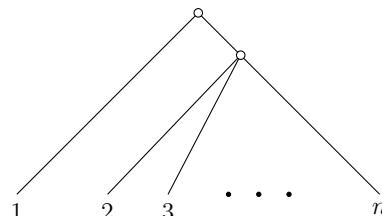
3.1 General QFTs

We will now present the “action principle” to construct the OPE coefficients, derived in [19] in the context of perturbative ϕ_4^4 -theory. The derivation [19] of the action principle generalizes straightforwardly to any theory with power counting renormalizable interaction, such as the GN₂-model, and with modifications due to local gauge symmetry, also to YM₄-theory [12]. It describes how an OPE coefficient, which is itself a complicated function not only of the points x_i but also of the coupling constant g , changes when we vary g .

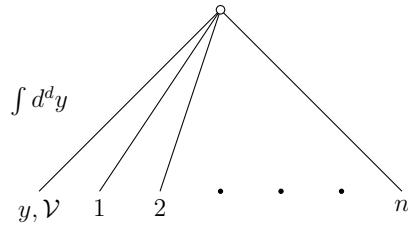
To write it down, we first use a graphical notation. We draw an OPE coefficient $\mathcal{C}_{A_1 \dots A_n}^B(x_1, \dots, x_n)$ as



Next, we draw a concatenation of OPE coefficients $\mathcal{C}_{A_1 C}^B(x_1, x_n) \mathcal{C}_{A_2 \dots A_n}^C(x_2, \dots, x_n)$ as



We also write



to mean that

- \mathcal{V} denotes the “deformation”, given e.g. by $-\phi^4$ in the ϕ_4^4 -model⁵ or by $-(\bar{\psi}\psi)^2$ in the GN_2 -model,
- $\int d^d y = \text{integral over } \{|y - x_n| < L\}$.
- L is length scale that is part of the definition of the theory; the formula is valid for points such that $|x_i - x_j| < L$.

The “action principle” for OPE coefficients is:

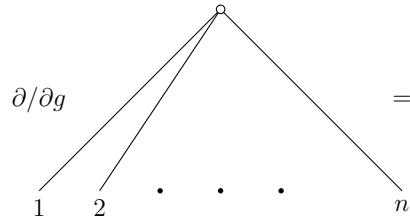


Figure 1: Functional equation, left side. The tree represents a coefficient $\mathcal{C}_{A_1 \dots A_n}^B(x_1, \dots, x_n)$

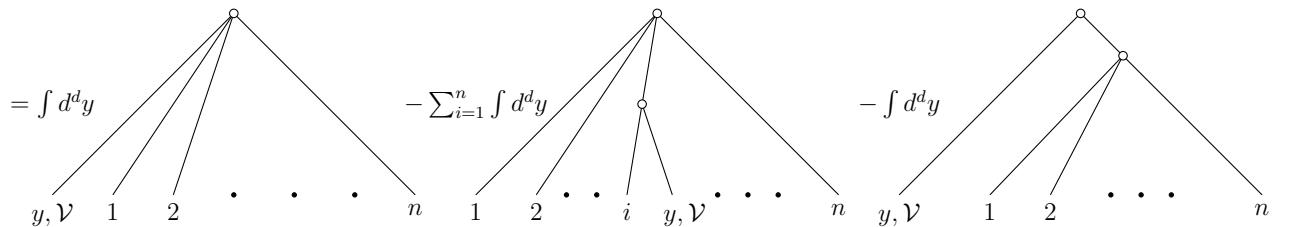


Figure 2: Functional equation, right side. The composite trees represent concatenations of coefficients, e.g. the rightmost tree means $\sum_C \mathcal{C}_{A_1 \dots A_n}^C(x_1, \dots, x_n) \mathcal{C}_{\mathcal{V}C}^B(y, x_n)$.

⁵Since this model is expected to exist only in the sense of formal series in g , our action principle is only expected to hold in that sense in this model.

In symbols, the action principle reads as follows for $n = 2$ points:

$$\begin{aligned} \partial_g \mathcal{C}_{A_1 A_2}^B(x_1, x_2) &= \int_{\epsilon < |y - x_i| \leq L} d^d y \mathcal{C}_{VA_1 A_2}^B(y, x_1, x_2) \\ &- \sum_{\Delta_C \leq \Delta_1} \int_{\epsilon < |y - x_1| < L} d^d y \mathcal{C}_{VA_1}^C(y, x_1) \mathcal{C}_{CA_2}^B(x_1, x_2) - \sum_{\Delta_C \leq \Delta_2} \int_{\epsilon < |y - x_2| < L} d^d y \mathcal{C}_{VA_2}^C(y, x_2) \mathcal{C}_{A_1 C}^B(x_1, x_2) \\ &- \sum_{\Delta_C < \Delta_B} \int_{\epsilon < |y - x_2| < L} d^d y \mathcal{C}_{A_1 A_2}^C(x_1, x_2) \mathcal{C}_{VC}^B(y, x_2), \end{aligned} \tag{3.18}$$

where the limit $\epsilon \rightarrow 0^+$ is understood in the end. The terminology “action principle” is due to the fact that a derivative with respect to g corresponds to an “insertion” of the interaction operator \mathcal{V} in the first term on the right side.

The action principle (3.18) – together with the hierarchy of similar relations with more points $\{x_i\}$ – has the following features:

1. It *only* involves the OPE-coefficients, but no correlation functions.
2. The integral over the “insertion point”, y , is shown to be *absolutely convergent*, and in particular free of any of the seemingly unavoidable UV-divergences in QFT, in the sense that the limit $\epsilon \rightarrow 0^+$ can be taken without further regulators. Here the idea is that, due to associativity, the terms in the second line of (3.18) should *precisely cancel the UV divergences* of the first line that would appear in the limit $\epsilon \rightarrow 0^+$. Similarly, the third line cancels any UV divergences that would appear when $L \rightarrow \infty$, again due to associativity.
3. The formula makes no reference to perturbation theory. Thus, given any theory with given OPE coefficients at some value of the coupling, we can chose a relevant or marginal interaction \mathcal{V} and define the deformed theory abstractly as a solution to our action principle.
4. For n points and more couplings, the generalization is given below in (3.18).

A perturbative version of the action principle was proved in [19, 22] for the ϕ_4^4 -model, starting from a definition of the renormalized OPE coefficients using flow equations. The rigorous statement is that if we make a power series expansion in g of the OPE coefficients, schematically $\mathcal{C} = \sum g^r \mathcal{C}_r$, around the free Gaussian fixed point $g = 0$, then the action principle holds to all orders in g . In other words, it was proved that:

Theorem 1. *Let $\mathcal{V} = -\phi^4$ and $d = 4$. Then to any order r in perturbation theory there holds the recursion relation (3.18) (and its obvious variants for n point OPE coefficients) where the limit $\epsilon \rightarrow 0$ is understood, and where $\Delta_A = [A]$ in the formula is given by the engineering dimension of \mathcal{O}_A at the Gaussian fixed point ($g = 0$). When the terms on the right side are written under a single integral sign, then the integral is absolutely convergent for $\epsilon \rightarrow 0$, i.e. all non-integrable singularities of the total integrand cancel out between the different terms.*

We view this as convincing evidence for the general validity of the action principle (3.18) even beyond perturbation theory (i.e. formal series), in models where a non-perturbative theory actually exists. The ϕ_4^4 -model is not believed to exist non-perturbatively except for the Gaussian fixed point ($g = 0$), but we expect the action principle to be true in models where it does, such as in the GN₂-model, or the exactly marginal flows described below. At any rate, *we will from now on assume the validity of the action principle at the non-perturbative level.*

In case we have more marginal or relevant couplings denoted by g^a corresponding to marginal or relevant operators \mathcal{O}_a in the action, then a generalized action principle can be derived by the same method as in the case of one marginal coupling. The formula is now

$$\begin{aligned} \frac{\partial}{\partial g^a} \mathcal{C}_{A_1 \dots A_n}^B(x_1, \dots, x_n) &= \int_{\epsilon < |y - x_i| \leq L} d^d y \mathcal{C}_{aA_1 \dots A_n}^B(y, x_1, \dots, x_n) \\ &- \sum_{i=1}^n \sum_{\Delta_C \leq \Delta_i} \int_{\epsilon < |y - x_i| < L} d^d y \mathcal{C}_{aA_1}^C(y, x_i) \mathcal{C}_{A_1 \dots C \dots A_i}^B(x_1, \dots, x_n) \\ &- \sum_{\Delta_C < \Delta_B} \int_{\epsilon < |y - x_n| < L} d^d y \mathcal{C}_{A_1 \dots A_n}^C(x_1, \dots, x_n) \mathcal{C}_{aC}^B(y, x_n), \end{aligned} \quad (3.19)$$

where the limit $\epsilon \rightarrow 0^+$ is understood in the end, and where we have generalized formula (3.18) to n points.

3.2 Action principle and renormalization group for OPE

3.2.1 Geometry of field redefinitions

As we have emphasized, we are free in general to make g -dependent redefinitions of the fields $\hat{\mathcal{O}}_A = \sum Z_A^B(g) \mathcal{O}_B$, where for massless theories we should demand on dimensional grounds that Z_A^B has non-vanishing entries for $\Delta_A = \Delta_B$. For the so redefined fields, we get new OPE coefficients $\hat{\mathcal{C}}_{A_1 \dots A_n}^B$ as defined in Definition 1.

The re-defined coefficients will satisfy a modified action principle. The relation with the action principle before the redefinition is best explained in geometric terms. Let us denote by $g^a = (g^1, \dots, g^n)$ the parameters of the theory which we loosely think of as associated with terms of the form $\sum g^a \mathcal{O}_a$ in the action—if the theory has one. Each operator \mathcal{O}_a is marginal or relevant, i.e. $\Delta_a \leq d$. Let

$$\mathcal{A} \equiv \mathcal{A}_{Ba}^A(g) dg^a \quad (3.20)$$

be a “connection” in field space, non-zero only for $\Delta_A \leq \Delta_B$. A curve $c^a(\tau)$ in coupling constant space is called geodesic if its tangent $\dot{c}^a(\tau)$ is parallel transported to itself under the connection \mathcal{A} ; in equations

$$\ddot{c}^a = -\mathcal{A}_{bc}^a(c) \dot{c}^b \dot{c}^c. \quad (3.21)$$

If we consider curves starting at $g = 0$, then they are uniquely given (locally) once we give $\hat{g}^a = \dot{c}^a(0)$. Geodesic normal coordinates around $g^a = 0$ are defined by assigning to \hat{g}^a the value $c^a(1)$ in coupling constant space (this is the “exponential map” in geometric terms, $g = \text{Exp}_0^{\mathcal{A}}\hat{g}$). Define $Z(\hat{g})$ to be the parallel transport (holonomy) along a geodesic from 0 to \hat{g} under \mathcal{A} , in formulas

$$Z(\hat{g}) = T \exp \int_0^1 \mathcal{A}_a(c(\tau)) \dot{c}^a(\tau) d\tau. \quad (3.22)$$

The components of Z are denoted by Z_B^A . They are non-zero only if $\Delta_B \leq \Delta_A$. We use this Z_B^A in order to define a new set of OPE coefficients $\hat{\mathcal{C}}$ as in Definition 1. These new coefficients are viewed as functions of \hat{g} . Then it is easily seen that the action principle for $\hat{\mathcal{C}}$ is the same as that (see (3.19)) for \mathcal{C} , except that we have to replace the hatted coefficients everywhere, and we have to make the replacement on the left side:

$$\frac{\partial}{\partial g^a} \mathcal{C}_{A_1 \dots A_n}^B \rightarrow \frac{\partial}{\partial \hat{g}^a} \hat{\mathcal{C}}_{A_1 \dots A_n}^B + \sum_i \mathcal{A}_{A_i a}^{C_i} \hat{\mathcal{C}}_{A_1 \dots C_i \dots A_n}^B - \mathcal{A}_C^B \hat{\mathcal{C}}_{A_1 \dots A_n}^{C}$$

$$(3.23)$$

where the hatted quantities and \mathcal{A} on the right side are now viewed as functions of \hat{g} . So, geometrically, the partial derivative $\frac{\partial}{\partial g^a}$ (parallel transport with trivial connection) is replaced by parallel transport along the geodesic tangent to $\frac{\partial}{\partial \hat{g}^a}$ with the connection \mathcal{A} . In practice, we will use field redefinition to remove such unwanted terms involving \mathcal{A} below.

In case we have a perturbatively defined theory, then all statements must be understood in the sense of formal series in g^a .

3.2.2 Renormalization group (RG) flow

The action principle is closely related to a version of the RG-flow. The discussion is again completely general, but to be concrete, we can think of a model like the GN₂-model or the ϕ_4^4 model. The OPE coefficients are defined as formal series in that case g and the action principle is understood order-by-order in powers of g . To avoid a cumbersome discussion of relevant interactions, we assume that there is only one marginal coupling constant, and that the renormalized mass is taken to be $m = 0$ (at each order in perturbation theory). Now we ask how the OPE-coefficients change if we change L to some (larger) value $\hat{L} = e^t L$, where $t > 0$. Let the new OPE coefficients (determined again by the action principle (3.18) and its obvious generalization to n points) be denoted by $\hat{\mathcal{C}}_{A_1 \dots A_n}^B$. Let us define the RG “time” as usual by

$$t = \log \frac{\hat{L}}{L}, \quad (3.24)$$

assuming that L, \hat{L} are so large that $L, \hat{L} > |x_{ij}|$ for any pair of points considered in an OPE coefficient.

The answer to this question is closely related to the geometric definition of the field redefinitions just explained. More precisely, the following proposition expresses the renormalization group:

Proposition 1. *In a massless perturbative theory with only one marginal interaction \mathcal{V} , there exist formal power series $f(t, g) = g + \sum_{r \geq 2} f_r(t)g^r$ and $Z_A^B(g, t) = \delta_A^B + \sum_{r \geq 1} Z_{rA}^B(t)g^r$ such that the OPE-coefficients $\hat{\mathcal{C}}_{A_1 \dots A_n}^{A_R}(g)$ for IR-cutoff \hat{L} are equivalent to the OPE-coefficients $\mathcal{C}_{A_1 \dots A_n}^{A_R}(f(t, g))$ for IR-cutoff L under the field redefinition*

$$\hat{\mathcal{O}}_A = \sum_B Z_A^B(g, t)\mathcal{O}_B , \quad (3.25)$$

where $Z_A^B(g, t)$ vanishes if $[B] \neq [A]$. (Here $[A]$ is the engineering dimension of a field \mathcal{O}_A).

Proof. The proof of this proposition for massless ϕ^4 -theory follows directly from the arguments given in sec. V of [19]. Apply proposition 1 of that reference twice, for L and \hat{L} . For a small mass m , it follows from the formulas given in the paper that Z, f are invariant if we rescale simultaneously L, \hat{L}, m . Therefore, in the limit as $m \rightarrow 0$, which is argued in the paper to exist, Z, f can only depend on the ratio L/\hat{L} , hence on t , as well as g . There is nothing special about the argument for this particular theory in the sense that the proof generalizes as long as we only have one marginal operator. For several marginal operators one obtains a flow in an n -dimensional space of couplings. \square

We stress that each coefficient $f_r(t)$ in the formal series for $f(g, t)$ has a well-defined, finite value. Similar remarks apply to the formal series $Z_A^B(g, t)$. Then, by applying the previous proposition twice for t and t' it therefore follows that the “cocycle” identities

$$Z(t + t', g) = Z(t', g(t, g))Z(t, g) , \quad f(t + t', g) = f(t', f(t, g)) \quad (3.26)$$

hold in the sense of formal series. It is standard that the cocycle conditions imply the existence of a formal series $\beta(g)$ and a matrix-valued formal series $\Gamma_A^B(g)$ in g with vanishing entries for $[A] \neq [B]$ satisfying

$$\partial_t f(t, g) = \beta(f(t, g)) , \quad \partial_t Z(t, g) = \Gamma(f(t, g))Z(t, g) . \quad (3.27)$$

For the GN₂ model, the first terms in the formal series $\beta(g) = \sum \beta_r g^r$ are found to be $\beta_0 = \beta_1 = 0$ and

$$\beta_2 = -(N - 1)/\pi , \quad \beta_3 = (N - 1)/(2\pi^2) . \quad (3.28)$$

These results, which we do not derive here, are equivalent to the usual “1-loop” and “2-loop” calculations of the beta function in the GN-model.

In a theory with vanishing beta coefficients to all orders, the cocycle relation becomes the 1-parameter group condition $Z(t + t', g) = Z(t', g)Z(t, g)$. Thus we can write $Z(t, g) = \exp t\Gamma(g)$. In a theory satisfying the OS-positivity condition, one can show that Γ is semi-simple, i.e. diagonalizable with real eigenvalues (no non-trivial Jordan-blocks). The eigenvalues $\gamma_A(g)$ correspond to the anomalous dimensions. More properly, we should say that the true dimensions are the engineering dimensions⁶ $[A]$ plus the anomalous

⁶Engineering dimensions are defined only at the Gaussian fixed point (free field). In a scalar field theory in $d = 4$, the engineering dimension of \mathcal{O}_A is the number of ϕ -factors plus the number of derivatives.

dimensions, $\Delta_A(g) = [A] + \gamma_A(g)$. Indeed, in a basis of fields in which $\Gamma(g)$ is diagonal, our action principle and version of the renormalization group gives

$$\mathcal{C}_{A_1 \dots A_n}^B(\lambda x_1, \dots, \lambda x_n) = \lambda^{-\Delta_{A_1} - \dots - \Delta_{A_n} + \Delta_B} \mathcal{C}_{A_1 \dots A_n}^B(x_1, \dots, x_n) \quad (3.29)$$

in the sense of formal series. This can easily be derived using that the action principle is trivially invariant under a simultaneous rescaling of $L \rightarrow \lambda L$ and $x_i \rightarrow \lambda x_i$ at each perturbation order, up to the power counting factor $\lambda^{-[A_1] - \dots - [A_n] + [B]}$. Thus, the action principle is scale-covariant. Since a multiplicative change of L corresponds to a multiplicative change of the fields by $Z(\log \lambda, g) = \lambda^{\Gamma(g)}$ with eigenvalues $\lambda^{\gamma_A(g)}$, the result follows.

The RG flow is also nicely intertwined with the geometry underlying the field redefinitions. Suppose we subject the OPE coefficients to a field redefinition given by a matrix $\zeta_B^A(g)$ and the marginal or relevant coupling parameters to the corresponding change $g^a \mapsto \hat{g}^a(g) \equiv \phi(g)$ induced by the exponential map (see above). Then under this change, the vector field β^a and the matrix of anomalous dimensions Γ_B^A change according to

$$\Gamma(g) \rightarrow \widehat{\Gamma}(g) = \zeta(g)^{-1} \Gamma(\phi(g)) \zeta(g) - \zeta(g)^{-1} \widehat{\beta} \cdot d\zeta(g), \quad \beta(g) \rightarrow \widehat{\beta}(g) = \phi_* \beta(g), \quad (3.30)$$

where ϕ_* denotes the pull-back of a vector field. In perturbation theory, these statements are understood in the sense of formal series in g . It is a simple exercise in manipulating formal series to show that, if β_2 is non-vanishing, there always exist formal series ϕ and ζ_A^B such that $\widehat{\beta}_r = 0$ for all $r \geq 4$ and $\widehat{\Gamma}_{rA}^B = 0$ for all $r \geq 2$. This corresponds to the statements in the physics literature that “only β_2, β_3 and Γ_1 are universal”.

3.3 Action principle for CFTs

3.3.1 Conformal blocks, correlation functions, and tensor structures

Conformal field theories are (Euclidean, in this paper) QFTs whose correlation functions are invariant not only under the Euclidean group but even under the conformal group, see e.g. [38] for a recent, hands-on, review. More precisely, consider the group of all globally defined orientation preserving conformal transformations g of \mathbb{R}^d (here $d \geq 2$),

$$g^* \delta_{\mu\nu} = \Omega(x)^2 \delta_{\mu\nu}, \quad (3.31)$$

where $\Omega(x)$ is called the conformal factor and $\delta_{\mu\nu}$ is the metric on Euclidean space \mathbb{R}^d . As is well-known, the conformal group is isomorphic to a covering group⁷ of the identity component of $\mathrm{SO}(d+1, 1)$. It obviously contains the (cover of the) Euclidean group $E(d)$, and in particular the rotation subgroup $\mathrm{SO}(d)$ or $\mathrm{Spin}(d)$, for which the conformal factor is $\Omega = 1$. The most important new transformations are the dilations $d(\lambda) : x \mapsto \lambda x$, which have conformal factor $\Omega = \lambda$.

To make these axioms, one assumes that within the set $\{\mathcal{O}_A\}$ of local fields (or rather, mathematically speaking in the abstract set of labels A) there are distinguished so-called

⁷Here we will only be concerned with a sufficiently small neighbourhood of the identity of this group, so the covering is in this sense unnecessary.

“primary” fields. These fields are labelled in the following by a lower case Roman index from the middle of the alphabet, i.e. the set of primary fields is denoted by $\{\mathcal{O}_j\}$. Each such primary field may have a tensor- or spinor index structure (subsumed in the index j) – alternatively speaking, they take values in some finite-dimensional irreducible representation D_j of $\text{SO}(d)$, or its cover $\text{Spin}(d)$ if we want to treat fermions, on a *finite dimensional* real or complex vector space⁸ V_j . Furthermore, it is assumed that any other local field \mathcal{O}_A can be obtained as primary field can be obtained by a linear combination of derivatives of primary fields. Such fields are called “descendants”.

The covariance axiom under the Euclidean group is now replaced by the following axiom:

e5') Correlation functions of primary fields satisfy

$$\langle \mathcal{O}_{i_1}(gx_1) \dots \mathcal{O}_{i_N}(gx_N) \rangle = \prod_{j=1}^N \Omega(x_j)^{\Delta_j} \bigotimes_{j=1}^N D_j(R(x_j, g)) \langle \mathcal{O}_{i_1}(x_1) \dots \mathcal{O}_{i_N}(x_N) \rangle, \quad (3.32)$$

where gx is the usual action – if defined – of a group element $g \in \text{SO}(d+1, 1)$ on a point $x \in \mathbb{R}^d$. The relation is required to hold for g in a neighborhood $U \subset \text{SO}(d+1, 1)$ of the identity such that for any smooth path g_t in U connecting $g \in U$ with the identity, none of the points $g_t x_i$ goes through infinity. Furthermore, $R(x, g)$ is a $\text{SO}(d)$ element obtained by decomposing the Jacobian of the transformation g as $\partial(gx)/\partial x = \Omega(x, g)R(x, g)$, and Δ_j is the dimension of the primary field. Since any other field can be obtained from primary fields by applying derivatives, we have a corresponding (more complicated) transformation formula also for the descendant fields.

As is well known, the axiom gives drastic simplifications of the correlation functions at the 2- and 3-point level. Let us assume to simplify notations that all fields are hermitian (i.e. $i^* = i$). It is a consequence of OS-positivity that, at the 2-point level, we can choose the primary fields such that

$$\langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \rangle = A_i \delta_{ij} t_{ij}(x_1, x_2) |x_{12}|^{-2\Delta_i}, \quad (3.33)$$

where $x_{12} \equiv x_1 - x_2$, where $A_i \in \mathbb{C}$ and where t_{ij} is an “invariant tensor structure”⁹. In general, an N -point invariant tensor structure is a map from M_N to the tensor product representation space $\otimes_k V_{i_k}$ (with each V_i a complex vector space carrying a *finite dimensional, irreducible* representation D_i of $\text{SO}(d)$) such that for all $g \in \text{SO}(d+1, 1)$ in some open neighborhood of the identity – depending on the points x_i as in e5’) – we have

$$t_{i_1 \dots i_N}(gx_1, \dots, gx_N) = \bigotimes_{j=1}^N D_j(R(x_j, g)) t_{i_1 \dots i_N}(x_1, \dots, x_N). \quad (3.34)$$

⁸Such representations are in correspondence with a Young-tableau indicating the symmetry properties of the tensor, or by a set of spin labels $(s_1, \dots, s_{\lfloor d/2 \rfloor})$.

⁹We could absorb A_i into the tensor structure or a field redefinition. However, it is more convenient for us not to do this here, see footnote 15. If we want, we can always set A_i to 1 by a field redefinition in the very end.

For $N = 2$ points, we can construct the tensor structures as follows¹⁰. First, we note that 2 points x_1, x_2 can be transformed by a conformal transformation to two fixed points, say $0, e$ with $|e| = 1$ on a fixed line in \mathbb{R}^d . The remaining conformal transformations leaving these two points fixed form a subgroup of $\mathrm{SO}(d+1, 1)$ isomorphic to¹¹ $\mathrm{E}(d-1) = \mathrm{SO}(d-1) \rtimes \mathbb{R}^{d-1}$. Thus, the transformation formula tells us that $t_{ij}(0, e)$ lies in the space of invariant tensors under the group action (3.34), namely

$$t_{ij}(0, e) \in (V_i \otimes V_j)^{\mathrm{E}(d-1)} \quad (3.35)$$

where the superscript denotes the set of tensors invariant under $D_i(R(0, g)) \otimes D_j(R(e, g))$ with g ranging over a sufficiently small neighborhood of the unit element in the group $\mathrm{E}(d-1)$. This space can be seen to be 1-dimensional if $V_i \cong V_j$ as a representation. So, in the 2-point function, only one tensor structure can appear for fixed i , which in the following we fix once and for all¹².

As an example, consider a totally symmetric tensor operator $\mathcal{O}^{\mu_1 \dots \mu_r}$ of rank r , i.e. $V_i = V_j = \mathrm{S}^r \mathbb{R}^d$, which is actually a reducible representation of $\mathrm{SO}(d)$. The elements in the space (3.35) are linear combinations of

$$\begin{aligned} t^{\mu_1 \dots \mu_r, \nu_1 \dots \nu_r}(0, e) &= \sum_{P, Q \in S_r} (\delta^{\mu_{P(1)} \nu_{Q(1)}} - 2e^{\mu_{P(1)}} e^{\nu_{Q(1)}}) \dots (\delta^{\mu_{P(p)} \nu_{Q(p)}} - 2e^{\mu_{P(p)}} e^{\nu_{Q(p)}}) \\ &\quad \times \delta^{\mu_{P(p+1)} \mu_{P(p+2)}} \dots \delta^{\mu_{P(r-1)} \mu_{P(r)}} \delta^{\nu_{Q(p+1)} \nu_{Q(p+2)}} \dots \delta^{\nu_{Q(r-1)} \nu_{Q(r)}}, \end{aligned} \quad (3.36)$$

for $1 \leq p \leq r$, and the form for general $x_1 \neq x_2$ is then found using the transformation law (3.34). An irreducible V_i is obtained by considering only tensors in $\mathrm{S}^r \mathbb{R}^d$ which are trace-free in any pair of indices from either $\{\mu_1, \dots, \mu_r\}$ or $\{\nu_1, \dots, \nu_r\}$. The corresponding 2-point tensor structure is then precisely the up to scaling unique totally trace-free linear combination of (3.36). The 2-point structure $t^{\mu_1 \dots \mu_r, \nu_1 \dots \nu_r}(x_1, x_2)$ is obtained via (3.34) choosing any group element $g \in \mathrm{SO}(d+1, 1)$ such that $g(0) = x_1, g(e) = x_2$.

For $r = 2$ (which in the case $\Delta_i = d$ leads to the 2-point function of the stress tensor), this gives for instance

$$t^{\mu_1 \mu_2, \nu_1 \nu_2}(x_1, x_2) = \left(\delta^{\mu_1(\nu_1)} - 2 \frac{x_{12}^{\mu_1} x_{12}^{(\nu_1)}}{|x_{12}|^2} \right) \left(\delta^{\nu_2(\mu_2)} - 2 \frac{x_{12}^{\nu_2} x_{12}^{(\mu_2)}}{|x_{12}|^2} \right) - \frac{1}{d} \delta^{\mu_1 \mu_2} \delta^{\nu_1 \nu_2}, \quad (3.37)$$

in agreement with standard formulas.

Similarly, for $N = 3$ points, we can transform an arbitrary configuration x_1, x_2, x_3 to three distinguished points on a line, say $0, e, \infty$, where ∞ denotes the point at infinity on the compactified line $\mathbb{R}e$. The fixed point group consists of those conformal transformations in the subgroup $\mathrm{SO}(d-1)$ leaving the line $e\mathbb{R}$ fixed. Therefore, we similarly have

$$t_{ijk}(0, e, \infty) \in (V_i \otimes V_j \otimes V_k)^{\mathrm{SO}(d-1)}. \quad (3.38)$$

¹⁰For a more detailed exposition of the following “colinear frame”-type arguments, see e.g. the recent paper [32] and references therein.

¹¹At the level of Lie-algebras, if we denote the generators $\mathfrak{so}(d, 1)$ by their standard notation $M_{\mu\nu}, P_\mu, K_\mu, D$ and if $e = (1, 0, \dots, 0)$, then $\mathfrak{e}(d-1)$ is generated by $M_{ij}, 2K_i - M_{ii}$, where $i = 2, \dots, d$.

¹²In particular, when \mathcal{O}_i is scalar, we take $t_{ii} = 1$.

For $N = 4$ points, we can transform an arbitrary configuration x_1, x_2, x_3, x_4 to four distinguished points on a fixed plane line, say $0, e, \infty, x$, where x denotes the point in some fixed plane containing e . Such a plane is left invariant by a corresponding subgroup $\text{SO}(d - 2)$. Therefore, we similarly have

$$t_{ijkl}(x, 0, e, \infty) \in (V_i \otimes V_j \otimes V_k \otimes V_l)^{\text{SO}(d-2)}. \quad (3.39)$$

Thus, an invariant 4-point tensor structure gives rise to a function of x valued in the invariant tensors. In fact, since we need precisely two real numbers to say where we are on our real plane, we may say that an invariant 4-point tensor structure gives rise to a function from a distinguished 2-plane in \mathbb{R}^d to in the invariant tensors. The converse to these statements also holds true, i.e. we may reconstruct invariant tensor structures for 2, 3 or 4 points by the invariants described above.

The 3-point function of primary fields can be written in terms of such 3-point structures, namely

$$\begin{aligned} \langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \mathcal{O}_k(x_3) \rangle &= |x_{12}|^{-\Delta_i - \Delta_j + \Delta_k} |x_{23}|^{-\Delta_j - \Delta_k + \Delta_i} |x_{13}|^{-\Delta_k - \Delta_i + \Delta_j} \\ &\times \sum_{\alpha} \lambda_{ijk}^{\alpha} t_{ijk}^{\alpha}(x_1, x_2, x_3), \end{aligned} \quad (3.40)$$

where $\lambda_{ijk}^{\alpha} \in \mathbb{C}$ are called the structure constants, and where t_{ijk}^{α} is some chosen basis of 3-point tensor structures from $(V_i \otimes V_j \otimes V_k)^{\text{SO}(d-1)}$. Thus, up to 3 points, all we need to know are the dimensions $\Delta_i \in \mathbb{R}_+$ and the structure constants $\lambda_{ijk}^{\alpha} \in \mathbb{C}$. In fact, in view of the OPE axiom, all correlation functions are determined by these “conformal data”.

Using the form of the 3-point function, one finds that OPE of two primary fields also has a particularly simple form in any CFT. It is given by

$$\mathcal{O}_i(x_1) \mathcal{O}_j(x_2) = \sum_k \sum_{\alpha} \lambda_{ijk}^{\alpha} \mathcal{P}_{ijk}^{\alpha}(x_{12}, \partial_2) \mathcal{O}_k(x_2), \quad (3.41)$$

where the tensor operators $\mathcal{P}_{ijk}^{\alpha}(x_{12}, \partial_2)$ are fixed pseudo-differential operators (depending only on the tensor structures associated with i, j, k and α) that can be found explicitly in principle. When formally expanded out in ∂_2 , they generate an infinite series of contributions from descendant fields. The main difference of the OPE in CFTs compared to the case of general QFTs as described above is thus that the OPE coefficients for primaries and descendants are not independent (we also get the OPE of two arbitrary descendants). Applying two successive OPEs $(ij)(kl)$ to the 4-point function gives

$$\langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \mathcal{O}_k(x_3) \mathcal{O}_l(x_4) \rangle = \sum_m \sum_{\alpha, \beta} \lambda_{ijm}^{\alpha} \lambda_{klm}^{\beta} A_m \mathcal{G}_{ijkl,m}^{\alpha\beta}(x_1, x_2, x_3, x_4), \quad (3.42)$$

(at least) in the domain $\{|x_{12}|, |x_{34}| < |x_{24}|\}$, where the “conformal blocks” are defined as

$$\mathcal{G}_{ijkl,m}^{\alpha\beta}(x_1, x_2, x_3, x_4) \equiv \mathcal{P}_{ijm}^{\alpha}(x_{12}, \partial_2) \mathcal{P}_{klm}^{\beta}(x_{34}, \partial_4) \langle \mathcal{O}_m(x_2) \mathcal{O}_m(x_4) \rangle / A_m. \quad (3.43)$$

These expressions are in principle completely fixed by the representation theory of the conformal group. It is customary to factor out the trivial dependencies of the blocks on

the points implied by conformal invariance and factor out the tensor 4-point structures. This amounts to writing

$$\begin{aligned} \mathcal{G}_{ijkl,m}^{\alpha\beta}(x_1, x_2, x_3, x_4) &= \left(\frac{|x_{14}|}{|x_{24}|}\right)^{\Delta_j - \Delta_i} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_k - \Delta_l} \frac{1}{|x_{12}|^{\Delta_i + \Delta_j} |x_{34}|^{\Delta_k + \Delta_l}} \\ &\times \sum_{\gamma} t_{ijkl}^{\gamma}(x_1, x_2, x_3, x_4) \mathcal{G}_{ijkl,m}^{\alpha\beta\gamma}(u, v). \end{aligned} \quad (3.44)$$

Here u, v are the anharmonic ratios,

$$u = \frac{|x_{12}|^2 |x_{34}|^2}{|x_{13}|^2 |x_{24}|^2}, \quad v = \frac{|x_{14}|^2 |x_{23}|^2}{|x_{13}|^2 |x_{24}|^2}, \quad (3.45)$$

and the quantities $\mathcal{G}_{ijkl,m}^{\alpha\beta\gamma}(u, v)$ are sometimes referred to as the “spinning blocks”. They are, too, in principle completely determined by the representation theory of the conformal group once we give the tensor structures and dimensions associated with i, j, k, l, m and once we fix bases of 3- and 4-point tensor structures $t_{ijm}^{\alpha}, t_{klm}^{\beta}, t_{ijkl}^{\gamma}$. But in practice their determination for general d is a very complicated problem which we do not address here. For the sake of concreteness, let us just quote the well-known result for $d = 2$:

Example: In $d = 2$, the conformal group is $\text{SO}(3, 1)$. The spin corresponds to a representation of $\text{SO}(2)$ (for bosons). The representation spaces V_j are all one-dimensional and the representations are labelled by an integer $s \in \mathbb{Z}$ (for fermions, $s \in \mathbb{Z}/2$). It is convenient to define h, \bar{h} through the relations $h + \bar{h} = \Delta, h - \bar{h} = s$, where the overbar in \bar{h} does not mean any kind of conjugation. By the general arguments above, there is only one independent invariant 3-point and 4-point tensor structure, so the labels α, β, \dots are not needed. Identifying points $x_i \in \mathbb{R}^2$ with complex numbers and using the notation $h_{ij} = h_i - h_j$ etc., we may write the 4-point tensor structure as

$$t_{ijkl}(x_1, x_2, x_3, x_4) = \left(\frac{x_{24}\bar{x}_{14}}{x_{14}\bar{x}_{24}}\right)^{s_i/2 - s_j/2} \left(\frac{x_{14}\bar{x}_{13}}{x_{13}\bar{x}_{14}}\right)^{s_k/2 - s_l/2} \left(\frac{\bar{x}_{12}}{x_{12}}\right)^{s_i/2 + s_j/2} \left(\frac{\bar{x}_{34}}{x_{34}}\right)^{s_k/2 + s_l/2} \quad (3.46)$$

and we may write the spinning conformal block as [35]

$$\mathcal{G}_{ijkl,m}(u, v) = {}_2F_1(h_m - h_{ij}, h_m + h_{kl}, 2h_m; z) \cdot {}_2F_1(\bar{h}_m - \bar{h}_{ij}, \bar{h}_m + \bar{h}_{kl}, 2\bar{h}_m; \bar{z}) z^{h_m} \bar{z}^{\bar{h}_m} \quad (3.47)$$

Here, $z \in \mathbb{C}$ is defined implicitly by $u = \bar{z}z, v = (1 - \bar{z})(1 - z)$.

3.3.2 Action principle

Equipped with this information, we now want to investigate what form our action principle takes in the context of CFTs. In general, we must assume that \mathcal{V} is some operator of dimension $\Delta_{\mathcal{V}} \leq d$. However, even if our starting point $g = 0$ corresponds to a conformally invariant theory, there is no reason that the flow determined by the action principle will remain conformal. This obviously is not the case when $\Delta_{\mathcal{V}} < d$, i.e. for relevant flows. For marginal flows, $\Delta_{\mathcal{V}} = d$, this is not necessarily the case either, because the

interaction can break conformal symmetry (as for the GN_2 -model or the ϕ_4^4 -model) in the sense that Δ_V will start differing from d as we flow. Thus, we must assume a *strictly marginal* perturbation which by definition corresponds to a 1-parameter family of CFTs parameterized by g . Such a family is characterized by a 1-parameter family of conformal data $\{\Delta_i(g), \lambda_{ijk}^\alpha(g)\}$. It is clear that our action principle will give an ordinary differential equation for the 1-parameter family of conformal data when applied to $N = 3$ points and three primary operators with labels $A = i, B = j, C = k$.

We now would like to determine what this differential equation is. First, we would like to write down the 3-point OPE coefficients \mathcal{C}_{ijk}^l associated with four primaries. From the OPE $(ijk)l$ in a 4-point function and the fact that the 2-point functions of descendants fall off faster than those of their corresponding primaries, one finds

$$\lim_{x_4 \rightarrow \infty} |x_4|^{2\Delta_l} \langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \mathcal{O}_k(x_3) \mathcal{O}_l(x_4) \rangle / A_l = \mathcal{C}_{ijk}^l(x_1, x_2, x_3) t_{ll}(x_3, \infty), \quad (3.48)$$

(no summation over l) where t_{ij} is the tensor structure appearing in the 2-point function $\langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \rangle$. Since this tensor structure is invertible, which follows from OS-positivity, we can obviously reconstruct \mathcal{C}_{ijk}^l from the 4-point function which in turn can be reconstructed from the conformal data and spinning conformal blocks. Similarly, we have

$$\lim_{x_3 \rightarrow \infty} |x_3|^{2\Delta_l} \langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \mathcal{O}_l(x_3) \rangle / A_l = \mathcal{C}_{ij}^l(x_1, x_2) t_{ll}(x_2, \infty), \quad (3.49)$$

so \mathcal{C}_{ij}^l again can be reconstructed from the conformal data.

We need to write down the 2-point and 3-point structure constants in some more detail. For simplicity, we assume for the moment that a normalization (field redefinition) of the operators has been chosen in which $A_i = 1$ – these factors can easily be reinstated and will be in the final formulas. For the 2-point OPE coefficient we then get putting $x_1 = x, x_2 = 0$:

$$\mathcal{C}_{ij}^l(x, 0) t_{ll}(0, \infty) = \sum_\alpha \frac{t_{ijl}^\alpha(x, 0, \infty) \lambda_{ijl}^\alpha}{|x|^{\Delta_i + \Delta_j - \Delta_l}}. \quad (3.50)$$

For the 3-point OPE coefficient, we get the relevant information from the 4-point function, but the formulas are a bit more complicated, because the expansion formulas for the 4-point function in terms of spinning conformal blocks do not converge everywhere. Putting $x_1 = x, x_2 = 0, x_3 = e$, we are going find separate expressions for $\mathcal{C}_{ijk}^l(x, 0, e)$ in the following domains

- a) $\{|x_{12}| < \min(|x_{13}|, |x_{23}|)\}$, equivalent to $\{|x| < \min(|x - e|, 1)\}$.
 - b) $\{|x_{23}| < \min(|x_{12}|, |x_{13}|)\}$, equivalent to $\{1 < \min(|x - e|, |x|)\}$.
 - c) $\{|x_{13}| < \min(|x_{23}|, |x_{12}|)\}$, equivalent to $\{|x - e| < \min(|x|, 1)\}$.
- a) In this domain (3.42) is valid, so we can write

$$\begin{aligned} \mathcal{C}_{ijk}^l(x, 0, e) t_{ll}(0, \infty) &= \sum_m \sum_{\alpha, \beta, \gamma} \frac{1}{|x|^{\Delta_i + \Delta_j} |x - e|^{\Delta_k - \Delta_l}} \mathcal{G}_{ijkl, m}^{\alpha\beta\gamma} \left(\frac{|x|^2}{|x - e|^2}, \frac{1}{|x - e|^2} \right) \\ &\quad \times t_{ijkl}^\gamma(x, 0, e, \infty). \end{aligned} \quad (3.51)$$

b) In this domain (3.42) is valid after permuting $(ijk) \rightarrow (jki)$ as well as $(123) \rightarrow (231)$, so we can write (assuming for simplicity that all primary fields are bosonic)

$$\begin{aligned} \mathcal{C}_{ijk}^l(x, 0, e) t_{ll}(0, \infty) &= \sum_m \sum_{\alpha, \beta, \gamma} \frac{1}{|x|^{\Delta_i - \Delta_l}} \mathcal{G}_{jkl, m}^{\alpha\beta\gamma} \left(\frac{1}{|x|^2}, \frac{|x - e|^2}{|x|^2} \right) \\ &\quad \times t_{jkl}^\gamma(0, e, x, \infty) \lambda_{jkm}^\alpha \lambda_{ilm}^\beta. \end{aligned} \quad (3.52)$$

c) In this domain (3.42) is valid after permuting $(ijk) \rightarrow (ikj)$ as well as $(123) \rightarrow (132)$, so we can write (assuming for simplicity that all primary fields are bosonic)

$$\begin{aligned} \mathcal{C}_{ijk}^l(x, 0, e) t_{ll}(0, \infty) &= \sum_m \sum_{\alpha, \beta, \gamma} \frac{1}{|x|^{\Delta_j - \Delta_l} |x - e|^{\Delta_i - \Delta_k}} \mathcal{G}_{ikjl, m}^{\alpha\beta\gamma} \left(\frac{|x - e|^2}{|x|^2}, \frac{1}{|x|^2} \right) \\ &\quad \times t_{ikjl}^\gamma(x, e, 0, \infty) \lambda_{ikm}^\alpha \lambda_{jlm}^\beta. \end{aligned} \quad (3.53)$$

Now let $\mathcal{O}_i = \mathcal{V}$ be our exactly marginal perturbation (a primary field of dimension d), and let $\epsilon > 0$ be small. We define the following complex numbers:

a) For domain a), we set

$$\begin{aligned} {}^a\mathcal{T}_{jklm}^{\alpha\beta\mu} &= \text{P.F.} \sum_\gamma \int_{\epsilon < |x| < \min(|x-e|, 1)} d^d x \frac{1}{|x|^{d+\Delta_j} |x - e|^{\Delta_k - \Delta_l}} \\ &\quad \mathcal{G}_{\mathcal{V}jkl, m}^{\alpha\beta\gamma} \left(\frac{|x|^2}{|x - e|^2}, \frac{1}{|x - e|^2} \right) t_{jkl}^\mu(0, e) \cdot t_{\mathcal{V}jkl}^\gamma(x, 0, e, \infty) \end{aligned} \quad (3.54)$$

where the dot \cdot is the natural hermitian inner product between tensors in $V_j \otimes V_k \otimes V_l$ and P.F. denotes the finite part of an asymptotic expansion¹³ for small ϵ .

b) For domain b), we set

$$\begin{aligned} {}^b\mathcal{T}_{jklm}^{\alpha\beta\mu} &= \text{P.F.} \sum_\gamma \int_{1 < \min(|x-e|, |x|) < 1/\epsilon} d^d x \frac{1}{|x|^{d-\Delta_l}} \\ &\quad \mathcal{G}_{jk\mathcal{V}l, m}^{\alpha\beta\gamma} \left(\frac{1}{|x|^2}, \frac{|x - e|^2}{|x|^2} \right) t_{jkl}^\mu(0, e) \cdot t_{jk\mathcal{V}l}^\gamma(0, e, x, \infty). \end{aligned} \quad (3.55)$$

c) For domain c), we set

$$\begin{aligned} {}^c\mathcal{T}_{jklm}^{\alpha\beta\mu} &= \text{P.F.} \sum_\gamma \int_{\epsilon < |x-e| < \min(|x|, 1)} d^d x \frac{1}{|x|^{\Delta_j - \Delta_l} |x - e|^{d-\Delta_k}} \\ &\quad \mathcal{G}_{\mathcal{V}kjl, m}^{\alpha\beta\gamma} \left(\frac{|x - e|^2}{|x|^2}, \frac{1}{|x|^2} \right) t_{jkl}^\mu(0, e) \cdot t_{\mathcal{V}jkl}^\gamma(x, 0, e, \infty) \end{aligned} \quad (3.56)$$

¹³More precisely, let $f(\epsilon)$ be a function having an asymptotic expansion of the form $\sum_{a_i < N} \sum_{b_j > M} A_{i,j} (\log \epsilon)^{a_i} \epsilon^{b_j}$ for small positive ϵ . Then F.P. means that we subtract all of the finitely many divergent terms in this expansion. Note that the right side indeed has such an expansion (with possibly non-integer powers) due to the OPE.

Note that the complex numbers ${}^a\mathcal{T}_{jklm}^{\alpha\beta\mu}$, ${}^b\mathcal{T}_{jklm}^{\alpha\beta\mu}$, ${}^c\mathcal{T}_{jklm}^{\alpha\beta\mu}$ are in principle determined by the representation theory of the conformal group alone, i.e. the spinning blocks and invariant tensor structures, as well as the dimensions of the primary fields. Further note that if we parameterize $x = (\Re(z), \Im(z)\mathbf{z})$ with $z \in \mathbb{C}, \mathbf{z} \in S^{d-2}$, then the integrand depends in each of the cases a), b), c) only on z, \bar{z} (noting that $|x|^2 = z\bar{z}$, $|x - e|^2 = (1-z)(1-\bar{z})$), due to the invariance properties of our tensor structures (3.39). Then the integration turns into an integration over the corresponding subset of \mathbb{C} , and we can effectively write

$$d^d x = (2i)^{-d+2} \text{vol}(S^{d-2})(z - \bar{z})^{d-2} d^2 z.$$

Furthermore, the integration domain is transformed in each case to the “fundamental¹⁴ domain”

$$\mathcal{F} = \{z \in \mathbb{C} \mid \epsilon < |z| < \min(|z - 1|, 1)\} \quad (3.57)$$

after the change of integration variables $z \rightarrow z/(z - 1)$ in case a), $z \rightarrow 1/z$ in case b) and $z \rightarrow (z - 1)/z$ in case c). The dot products involving the tensor structures can in the variables z, \bar{z} be written as sums of terms of the form $z^a \bar{z}^{\bar{a}} (1-z)^b (1-\bar{z})^{\bar{b}}$ where $a - \bar{a} \in \mathbb{Z}, a + \bar{a} \in \mathbb{R}$, so at the end of the day we need to perform in each case a), b), c) integrals of the form

$$\mathcal{T} \sim \text{sum of terms of form P.F. } \int_{\mathcal{F}} z^a \bar{z}^{\bar{a}} (1-z)^b (1-\bar{z})^{\bar{b}} \mathcal{G}(z, \bar{z}) d^2 z, \quad (3.58)$$

where \mathcal{G} is some spinning conformal block viewed as a function of the new variables z, \bar{z} , and a, b, \bar{a}, \bar{b} depend on the dimensions, tensor structures, and the spacetime dimension $d \geq 2$ (in a different way in each of the cases a), b), c)).

Now we consider the action principle (3.18) for general A_1, A_2, C in CFT, under the assumption that there is only one exactly marginal operator (generating a flow of CFTs). Note that, even if A_1, A_2, C correspond to primary fields, the sums in the formula involve primary and descendant fields. As we have already said, the role of the terms in the second line of (3.18) is to remove any UV-divergences from the integral in the first line i.e. the contributions that would diverge as $\epsilon \rightarrow 0$. The terms in the last line are automatically finite as long as L remains finite. According to our discussion of the renormalization group, we can accommodate a change in L by appropriate (diverging, as $L \rightarrow \infty$) field redefinitions. It is convenient to take $L = 1/\epsilon$, because then we can write

$$\begin{aligned} \partial_g \mathcal{C}_{AB}^C(x_1, x_2) &= \text{P.F.} \int_{\epsilon < |x_1 - y|, |x_2 - y| \leq 1/\epsilon} d^d y \mathcal{C}_{VAB}^C(y, x_1, x_2) \\ &\quad - \sum_{\Delta_D = \Delta_A} \mathcal{A}_A^D \mathcal{C}_{DB}^C(x_1, x_2) - \sum_{\Delta_D = \Delta_B} \mathcal{A}_B^D \mathcal{C}_{AD}^C(x_1, x_2) - \sum_{\Delta_D = \Delta_C} \mathcal{A}_C^D \mathcal{C}_{AB}^D(x_1, x_2), \end{aligned} \quad (3.59)$$

where P.F. denotes the finite part as $\epsilon \rightarrow 0$, whereas \mathcal{A}_A^B are certain (finite) complex constants. These constants arise from the RG-flow (since we are varying $L = 1/\epsilon$), as well as from the terms in the second and third line of the action principle (3.18). It

¹⁴It is a fundamental for the action of the permutation group S_3 acting by the conformal transformations $1/z, (z - 1)/z, z/(z - 1)$ which permute the points $0, 1, \infty$.

follows in particular that the *same* finite constants appear in all three terms. The fact that only terms with equal dimension appear in the summations can be seen from dilation covariance since both the finite part P.F. as well as the left side must satisfy the dilation covariance condition (3.29). It follows that the matrices of constants \mathcal{A}_A^B may be absorbed in a further finite field redefinition and a redefinition of the coupling constant, as described in (3.23). So we learn that, with these implicit field redefinitions,

$$\partial_g \mathcal{C}_{AB}^C(x_1, x_2) = \text{P.F.} \int_{\epsilon < |x_1 - y|, |x_2 - y| \leq 1/\epsilon} d^d y \mathcal{C}_{VAB}^C(y, x_1, x_2). \quad (3.60)$$

We now consider the action principle (3.60) for $A_1 = A_2 = i$, with i a label of a primary field, and $C = 1$. At this stage, we put $x_1 = x, x_2 = 0$, we use that $\mathcal{C}_{ij\dots k}^1 = \langle \mathcal{O}_i \mathcal{O}_j \dots \mathcal{O}_k \rangle$, we use the explicit form of the 3- and 2-point functions in the CFT in terms of the conformal data, and we may use (3.50). The resulting explicit integrals can be carried out easily when \mathcal{O}_i is scalar normalized so that $t_{ii} = 1$ in the 2-point function. The result is

$$\begin{aligned} |x|^{-2\Delta_i} \left(2A_i \log(|x|) \frac{d\Delta_i}{dg} + \frac{dA_i}{dg} \right) &= |x|^{-2\Delta_i+d} \lambda_{Vii} \text{ P.F.} \int_{\epsilon < |y|, |x-y| < 1/\epsilon} \frac{d^d y}{|x-y|^d |y|^d} \\ &= \left(\frac{4\pi^{d/2}}{\Gamma(d/2)} \log(|x|) + C \right) |x|^{-2\Delta_i} \lambda_{Vii} \end{aligned} \quad (3.61)$$

where C is a constant¹⁵ depending only on d . We conclude that, if \mathcal{O}_i is scalar, then

$$A_i \frac{d}{dg} \Delta_i = \frac{2\pi^{d/2}}{\Gamma(d/2)} \lambda_{Vii}.$$

In the general case when i is in a non-trivial representation of the group $\text{SO}(d)$ the integrals can still be carried out explicitly in principle, but we must take into account the tensor structures that might appear in the 3-point OPE coefficients \mathcal{C}_{Vii}^1 . The result is now of the form

$$A_i \frac{d}{dg} \Delta_i = \sum_{\alpha} \mathcal{D}_i^{\alpha} \lambda_{Vii}^{\alpha} \quad (3.62)$$

where \mathcal{D}_i^{α} are the constants that come up in these tensor integrals.

We now choose $A = i, B = j, C = l$, with i, j, l labels of primary fields, and $(x_1, x_2) = (0, e)$. We split the remaining integral (integration variable now called x) on the right side into the regions a), b), c) described before, and use in each region the formulas for the 3-point OPE coefficient given under a), b), c). Finally, we use the definition of the constants ${}^a\mathcal{T}_{jklm}^{\alpha\beta\mu}, {}^b\mathcal{T}_{jklm}^{\alpha\beta\mu}, {}^c\mathcal{T}_{jklm}^{\alpha\beta\mu}$. On the left side, we use our expression for the 2-point OPE coefficient $\mathcal{C}_{ij}^l(0, e)$. Taking the tensor structures $t_{jkl}^{\mu}(0, e, \infty)$ to form an orthogonal system in $(V_i \otimes V_j \otimes V_k)^{\text{SO}(d-1)}$ under the natural hermitian inner product on this finite dimensional space denoted by a dot above, and using that the tensor structure

¹⁵ The presence of this constant forbids us to set $dA_i/dg = 0$. Of course, we could achieve this by a further field redefinition if we wanted to, but this would change eq. (3.60).

$t_{ll}(0, \infty)$ appearing in the 2-point function is invertible (by OS-positivity) there results the equation¹⁶:

$$\frac{d}{dg} \lambda_{jkl}^\mu = \sum_m \sum_{\alpha\beta} \left({}^a \mathcal{T}_{jklm}^{\alpha\beta\mu} \lambda_{\nu jm}^\alpha \lambda_{klm}^\beta + {}^b \mathcal{T}_{jklm}^{\alpha\beta\mu} \lambda_{jkm}^\alpha \lambda_{\nu lm}^\beta + {}^c \mathcal{T}_{jklm}^{\alpha\beta\mu} \lambda_{\nu km}^\alpha \lambda_{jlm}^\beta \right) A_m \quad (3.63)$$

which is the formula claimed in the introduction.

Consistency with our normalization conditions on the 1-point, 2-point and 3-point function in CFT requires that if $i = j$ and $k = 1$, then we should have $A_i = \lambda_{ii1}$ which gives the missing evolution equation for A_i as this special case of (3.63). Consistency also requires that, if $i \neq j$ and $k = 1$, then we should have $\lambda_{ijk}^\alpha = 0$ along the flow, and hence $d/dg \lambda_{ijk}^\alpha = 0$. This should actually follow from our flow equation and is most easily seen for scalar operators i, j (so that there is no need for a tensor structure label α) going back to (3.60) with $A = i, B = j, C = 1$. So we only need to check in view of $\mathcal{C}_{ij\nu}^1(x_1, x_2, x_3) = \langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \mathcal{V}(x_3) \rangle$ and of (3.40) that

$$\text{P.F.} \int_{\epsilon < |x_1 - y|, |x_2 - y| < 1/\epsilon} \frac{d^d y}{|x_1 - y|^{-\Delta_j + \Delta_i + d} |x_2 - y|^{-\Delta_i + \Delta_j + d}} = 0 \quad (3.64)$$

which indeed holds as long as $x_1 - x_2 \neq 0$ and as long as

$$\Delta_i - \Delta_j \notin \mathbb{Z}, \quad (3.65)$$

i.e. if a certain non-degeneracy condition is fulfilled by the spectrum of dimensions. This condition is generically only violated if i corresponds to the identity operator 1 (having dimension 0) and j corresponds to \mathcal{V} (having by construction dimension d). However, in that case (3.64) automatically vanishes anyway. The conceptual reason for the vanishing of the integral (3.64) is of group theoretical nature as one may expect. The best way to see this is to analytically continue the integrand to imaginary $\nu = \Delta_i - \Delta_j$. Then the integral, viewed as a function of x_1, x_2 , can be seen as a sesquilinear bilinear form on the representation space of a spin-0, principal series representation of $\text{SO}(d+1, 1)$ labelled by ν . The only such form must be the scalar product itself, which corresponds to a delta distribution $\delta^d(x_1, x_2)$ in the principal series, and hence vanishes for $x_1 \neq x_2$. By analyticity, this must remain true as long as ν does not correspond to poles of (3.64), i.e. as long as the non-degeneracy conditions remains true. This line of reasoning also immediately shows that the corresponding statement is still true if i, j are not scalar but carry spin.

Perhaps a better way to state these results is that the field definition required to set $\mathcal{A} = 0$ in (3.59) is precisely that which respects our normalization condition of the 1,2,3-point functions in the presence of a non-degeneracy condition. In the absence of such a

¹⁶We note in passing that the coefficients \mathcal{T} with arbitrary indices can be related to 6j-symbols¹⁷ of the conformal group $\text{SO}(d-1)$ for the six representations ($V_\nu = \mathbb{R}, \Delta_\nu = 2$), (V_i, Δ_i) , (V_j, Δ_j) , (V_k, Δ_k) , (V_l, Δ_l) , (V_m, Δ_m) . Here (V, Δ) is the (infinite-dimensional) representation of $\text{SO}(d-1)$ described by the $\lfloor d/2 \rfloor$ spins encoded in the finite-dimensional representation V of $\text{SO}(d)$ and the dimension Δ . We shall come back to this in another paper.

condition, suitable field redefinition have to be applied, which unfortunately complicates matters considerably.

If we set $i = j = k = \mathcal{V}$ in our evolution equations (3.62), (3.63), then we find $\lambda_{\mathcal{V}\mathcal{V}\mathcal{V}} = 0$ from the first evolution equation, since we must have $\Delta_{\mathcal{V}} = d$ along our flow for an exactly marginal perturbation. Our second evolution equation then gives a constraint on the coefficients $\lambda_{\mathcal{V}\mathcal{V}m}$, and by differentiating this constraint w.r.t. g , further constraints follow which somehow encode that the theory actually *has* any exactly marginal deformations. Such constraints were also observed previously in [1].

Example: In $d = 2$, the conformal group is $\text{SO}(3, 1)$, and we can identify points $x \in \mathbb{R}^2$ with complex numbers. The tensor structures are unique, so the labels α, β, \dots are superfluous, and the equations become somewhat simpler:

$$\frac{d}{dg} \lambda_{jkl} = \sum_m ({}^a\mathcal{T}_{jklm} \lambda_{\mathcal{V}jm} \lambda_{klm} + {}^b\mathcal{T}_{jklm} \lambda_{jkm} \lambda_{\mathcal{V}lm} + {}^c\mathcal{T}_{jklm} \lambda_{\mathcal{V}km} \lambda_{jlm}) A_m \quad (3.66)$$

$$A_i \frac{d}{dg} \Delta_i = 2\pi \lambda_{\mathcal{V}ii}. \quad (3.67)$$

There remains, however, the problem of determining the coefficients \mathcal{T} . These were concretely defined in a), b), c) above, so in principle we can find them from the spinning blocks displayed in the previous section. For domain a), we find, for instance, after a change of variables $z = x/(x - 1)$:

$$\begin{aligned} {}^a\mathcal{T}_{jklm} = & \text{P.F.} \int_{\mathcal{F}} d^2 z \frac{(z - 1)^{h_{kl}-2}}{z^{1-h_j-h_m}} \frac{(\bar{z} - 1)^{\bar{h}_{kl}-2}}{\bar{z}^{1-\bar{h}_j-\bar{h}_m}} \times \\ & {}_2F_1(h_m + h_j - 1, h_m + h_{kl}, 2h_m; z) {}_2F_1(\bar{h}_m + \bar{h}_j - 1, \bar{h}_m + \bar{h}_{kl}, 2\bar{h}_m; \bar{z}) \end{aligned} \quad (3.68)$$

where P.F. denotes the finite part as $\epsilon \rightarrow 0$, and where the ϵ -dependent integration domain is $\mathcal{F} = \{z \in \mathbb{C} \mid \epsilon < |z| < \min(|z - 1|, 1)\}$. For domain b), we find after a change of variables $z = 1/x$:

$$\begin{aligned} {}^b\mathcal{T}_{jklm} = & \text{P.F.} \int_{\mathcal{F}} d^2 z \frac{(z - 1)^{-1-h_k}}{z^{h_{jk}+h_l+h_m+1}} \frac{(\bar{z} - 1)^{-1-\bar{h}_k}}{\bar{z}^{\bar{h}_{jk}+\bar{h}_l+\bar{h}_m+1}} \times \\ & {}_2F_1(h_m + 1 - h_k, h_m + h_{jl}, 2h_m, z) {}_2F_1(\bar{h}_m + 1 - \bar{h}_k, \bar{h}_m + \bar{h}_{jl}, 2\bar{h}_m, \bar{z}) \end{aligned} \quad (3.69)$$

For domain c), we find after a change of variables $z = (x - 1)/x$:

$$\begin{aligned} {}^c\mathcal{T}_{jklm} = & \text{P.F.} \int_{\mathcal{F}} d^2 z \frac{(z - 1)^{-1+h_j+h_l+h_k}}{z^{1+h_k-h_m}} \frac{(\bar{z} - 1)^{-1+\bar{h}_j+\bar{h}_l+\bar{h}_k}}{\bar{z}^{1+\bar{h}_k-\bar{h}_m}} \times \\ & {}_2F_1(h_m - 1 + h_k, h_m + h_{jl}, 2h_m, z) {}_2F_1(\bar{h}_m - 1 + \bar{h}_k, \bar{h}_m + \bar{h}_{jl}, 2\bar{h}_m, \bar{z}) \end{aligned} \quad (3.70)$$

The remaining integrals can be evaluated for instance using the Mellin-Barnes representation of the hypergeometric function. Using such a representation, we are led to integrals of the form $\int_{\mathcal{F}} z^a \bar{z}^{\bar{a}} (1 - z)^b (1 - \bar{z})^{\bar{b}} d^2 z$ or rather, their finite part as $\epsilon \rightarrow 0$. The final result is expressed in terms of a generalized hypergeometric series, which we will explore elsewhere. In $d = 1$ (“conformal quantum mechanics” which is formally outside the scope of our framework), the integrals reduce to one-dimensional ones and the formulas become much simpler. In that case, an explicit expression has independently been obtained by [2].

4 Outlook

The most interesting application of our dynamical equations (3.62), (3.63) for the CFT data of exactly marginal flows would be to the $\mathcal{N}=2,4$ super conformal Yang-Mills theories in $d = 4$. Such theories are Gaussian for $g = 0$ and so trivially soluble, providing thus in principle an initial condition for the dynamical system. Given sufficient information about the coefficients \mathcal{T} , which is in principle only kinematical input, might enable one to show by Newton iteration that the dynamical system has a non-perturbative solution for finite g , thus leading to a mathematical existence result for these theories without any kind or large N limit!

Unfortunately, as stated our results do not quite apply to this situation, because at the Gaussian point the non-degeneracy condition (3.65) $\Delta_i - \Delta_j \notin \mathbb{Z}$ for the spectrum is certainly very far from being fulfilled, as is even the case away from the Gaussian point due to primaries with protected dimension. In the absence of the non-degeneracy condition, our evolution equations for the conformal data have to be modified by finite field redefinitions when a “crossover” $\Delta_i - \Delta_j \in \mathbb{Z}$ occurs as we have explained. This appears to be a non-trivial problem¹⁸.

An independent technical complication, which to some extent has been addressed by [3, 4, 39, 30, 31], is that one must know the all the spinning conformal blocks of the conformal group $SO(5, 1)$ and all tensor structures to determine the coefficients \mathcal{T} in the dynamical system, or one must have another way to determine them – perhaps directly from the representation theory of the conformal group.

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References

- [1] V. Bashmakov, M. Bertolini and H. Raj, “On non-supersymmetric conformal manifolds: field theory and holography,” arXiv:1709.01749 [hep-th].
- [2] C. Behan, “Conformal manifolds: ODEs from OPEs,” arXiv:1709.03967 [hep-th].

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- [3] M. S. Costa, J. Penedones, D. Poland and S. Rychkov, “Spinning Conformal Blocks,” *JHEP* **1111**, 154 (2011)
- [4] M. S. Costa, T. Hansen, J. Penedones and E. Trevisani, “Radial expansion for spinning conformal blocks,” *JHEP* **1607**, 057 (2016) doi:10.1007/JHEP07(2016)057
- [5] J. Dimock, “The Renormalization Group According to Balaban III. Convergence,” *Annales Henri Poincare* **15**, no. 11, 2133 (2014)
- [6] J. Dimock, “The Renormalization Group According to Balaban - II. Large fields,” AIP, no. 9 doi:10.1063/1.4821275
- [7] J. Dimock, “The Renormalization Group According to Balaban, I. Small fields,” *Rev. Math. Phys.* **25**, no. 7, 1330010 (2013)
- [8] A. N. Efremov, R. Guida and C. Kopper, “Renormalization of SU(2) Yang-Mills theory with Flow Equations,” *J. Math. Phys.* **58**, no. 9, 093503 (2017)
- [9] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, “Solving the 3D Ising Model with the Conformal Bootstrap,” *Phys. Rev. D* **86**, 025022 (2012) doi:10.1103/PhysRevD.86.025022 [arXiv:1203.6064 [hep-th]].
- [10] J. Feldman, J. Magnen, V. Rivasseau and R. Seneor, “A Renormalizable Field Theory: The Massive Gross-Neveu Model in Two-dimensions,” *Commun. Math. Phys.* **103**, 67 (1986).
- [11] M. B. Fröb, J. Holland and S. Hollands, “All-order bounds for correlation functions of gauge-invariant operators in Yang-Mills theory,” *J. Math. Phys.* **57** (2016) 122301 doi:10.1063/1.4967747 [arXiv:1511.09425 [math-ph]].
- [12] M. B. Fröb and J. Holland, “All-order existence of and recursion relations for the operator product expansion in Yang-Mills theory,” arXiv:1603.08012 [math-ph].
- [13] A. Gadde, “In search of conformal theories,” arXiv:1702.07362 [hep-th].
- [14] K. Gawedzki and A. Kupiainen, “Gross-Neveu Model Through Convergent Perturbation Expansions,” *Commun. Math. Phys.* **102**, 1 (1985).
- [15] J. Glimm and A. Jaffe, “Quantum Physics, a Functional Integral Point of View,” *Springer 1987*
- [16] R. Guida and N. Magnoli, “All order IR finite expansion for short distance behavior of massless theories perturbed by a relevant operator,” *Nucl. Phys. B* **471**, 361 (1996) doi:10.1016/0550-3213(96)00175-7
- [17] R. Guida and Ch. Kopper, “All-order uniform momentum bounds for the massless ϕ^4 theory in four dimensional Euclidean space,” <http://arxiv.org/abs/1103.5692>
- [18] R. Guida and Ch. Kopper, *All-order uniform momentum bounds for the massless ϕ_4^4 field theory*, in preparation .

- [19] J. Holland and S. Hollands, “Associativity of the operator product expansion,” *J. Math. Phys.* **56** (2015) no.12, 122303. doi:10.1063/1.4937811
- [20] S. Hollands and C. Kopper, “The operator product expansion converges in perturbative field theory,” *Commun. Math. Phys.* **313**, 257 (2012)
- [21] J. Holland, S. Hollands and C. Kopper, “The operator product expansion converges in massless φ_4^4 - theory,” *Commun. Math. Phys.* **342** (2016) no.2, 385 doi:10.1007/s00220-015-2486-6 [arXiv:1411.1785 [hep-th]].
- [22] J. Holland and S. Hollands, “Recursive construction of operator product expansion coefficients,” *Commun. Math. Phys.* **336** (2015) no.3, 1555 doi:10.1007/s00220-014-2274-8 [arXiv:1401.3144 [math-ph]].
- [23] S. Hollands, “Quantum field theory in terms of consistency conditions. I. General framework, and perturbation theory via Hochschild cohomology,” *SIGMA* **5**, 090 (2009)
- [24] S. Hollands and R. M. Wald, “Axiomatic quantum field theory in curved spacetime,” *Commun. Math. Phys.* **293**, 85 (2010)
- [25] J. Holland and S. Hollands, “Operator product expansion algebra,” *J. Math. Phys.* **54** (2013) 072302 doi:10.1063/1.4810780 [arXiv:1205.4904 [math-ph]].
- [26] S. Hollands, talk at “Wolfhard Zimmermann memorial symposium,” Munich (May 2017), available at <http://wwwth.mpp.mpg.de/conf/zimmermann-memorial/talks/Hollands.pdf>
- [27] S. Hollands, talk at “In memoriam Rudolf Haag,” Hamburg (September 2016), available at <https://lqp2.org/node/1329>
- [28] Y.-Z. Huang and L. Kong, “Full field algebras,” *Commun. Math. Phys.* **272** (2007)
- [29] G. Keller and Ch. Kopper, “Perturbative renormalization of composite operators via flow equations. 2. Short distance expansion,” *Commun. Math. Phys.* **153** (1993) 245.
- [30] D. Karateev, P. Kravchuk and D. Simmons-Duffin, “Weight Shifting Operators and Conformal Blocks,” arXiv:1706.07813 [hep-th].
- [31] P. Kravchuk, “Casimir recursion relations for general conformal blocks,” arXiv:1709.05347 [hep-th].
- [32] P. Kravchuk and D. Simmons-Duffin, “Counting Conformal Correlators,” arXiv:1612.08987 [hep-th].
- [33] M. Lüscher, “Operator Product Expansions on the Vacuum in Conformal Quantum Field Theory in Two Space-Time Dimensions”, *Commun. Math. Phys.* **50** (1976) 23
- [34] G. Mack, “Convergence of Operator Product Expansions on the Vacuum in Conformal Invariant Quantum Field Theory”, *Commun. Math. Phys.* **53** (1977) 155.

- [35] H. Osborn, “Conformal Blocks for Arbitrary Spins in Two Dimensions,” Phys. Lett. B **718**, 169 (2012) doi:10.1016/j.physletb.2012.09.045
- [36] D. Pappadopulo, S. Rychkov, J. Espin, and R. Rattazzi, “OPE Convergence in Conformal Field Theory”, Phys. Rev. D **86** (2012) 105043
- [37] J. Polchinski, “Renormalization and Effective Lagrangians” Nucl. Phys. B, **231** (1984) 269.
- [38] S. Rychkov, “EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions,” doi:10.1007/978-3-319-43626-5 arXiv:1601.05000 [hep-th].
- [39] V. Schomerus, E. Sobko and M. Isachenkov, “Harmony of Spinning Conformal Blocks,” JHEP **1703**, 085 (2017) doi:10.1007/JHEP03(2017)085
- [40] Ch. Wetterich, “Exact evolution equation for the effective potential” Phys. Lett. B, **301** (1993) 90.
- [41] K. G. Wilson, “Non-Lagrangian models of current algebra,” Phys. Rev. **179** (1969) 1499.
- [42] W. Zimmermann: “Normal products and the short distance expansion in the perturbation theory of renormalizable interactions”, Ann. Phys. **77** (1973) 570.
- [43] W. Zimmermann, “Composite operators in the perturbation theory of renormalizable interactions,” Annals Phys. **77**, 536 (1973) [Lect. Notes Phys. **558**, 244 (2000)].