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Exponential Asymptotics, Transseries, and Generalized Borel Summation for Analytic, Nonlinear, Rank-One Systems of Ordinary Differential Equations

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1 Introduction and main results

In this paper we prove that the formal exponential series solutions (transseries) at an irregular singularity of rank one of an analytic linear or nonlinear system of ordinary differential equations (under some nondegeneracy and integrability conditions) are Borel summable, in a sense similar to that of Ecalle. The functions obtained by resummation of the transseries are precisely the solutions of the differential equation that decay in a specified sector in the complex plane. Subsequently, we find the dependence of the correspondence between the solutions of the differential equation and transseries as the ray in the complex plane changes (local Stokes phenomenon). Simple analytic identities lead to "resurgence" relations and to an averaging formula having, in addition to the properties of the medianization of Ecalle, the property of preserving exponential growth at infinity.

Consider an n-dimensional, rank-one, level-one vector differential equation in a neighborhood of an irregular singularity, say, $x = \infty$. We assume that the Stokes lines are simple. In normalized form (see [17], [16]), such an equation can be written in the form

$$\mathbf{y}' = \mathbf{f}_0(x) - \hat{\Lambda}\mathbf{y} - \frac{1}{x}\hat{B}\mathbf{y} + \mathbf{g}(x, \mathbf{y}), \qquad \mathbf{y} \in \mathbb{C}^n. \tag{1.1}$$

(The reason to separate out the second and third terms on the right-hand side of (1.1) is that they play a special role in the asymptotic behavior of the solutions.)

The functions $\xi \mapsto \mathbf{f}_0(\xi^{-1})$ and $(\xi, \mathbf{y}) \mapsto \mathbf{g}(\xi^{-1}, \mathbf{y})$ are taken to be analytic for small arguments. The normalization can be chosen so that $\mathbf{f}_0(x) = O(x^{-2})$ for large x, and, by construction, we have $\mathbf{g}(x, \mathbf{y}) = O(|\mathbf{y}|^2, x^{-2}\mathbf{y})$.

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$\hat{\Lambda}$ and $\hat{B}$ are $n \times n$ matrices with constant coefficients. We assume that $\hat{\Lambda}$ is invertible and that the ("nonresonance") condition $\arg \lambda_j \neq \arg \lambda_i$, for $j \neq i$, $\lambda \in \operatorname{spec} \hat{\Lambda}$, is satisfied.

By a change of variables we can then arrange that $\hat{\Lambda}$ is diagonal, $\hat{\Lambda} = \operatorname{diag}\{\lambda_i\}$ with $\arg \lambda_j > \arg \lambda_i$ for $j > i$ and make $\lambda_1 = 1$. The matrix $\hat{B}$ can be diagonalized at the same time [16].

To simplify the analysis we assume further that $\Re(\hat{B}_{1,1}) := \beta > 0$. Through normalization we make

$$\Re(\hat{B}_{1,1}) = \beta \in (0, 1]. \tag{1.2}$$

We are interested in the study of the solutions of (1.1) that are decaying for large x, in one of the half-planes $\Re(xe^{-i\phi}) > 0$ with $\phi \in (\arg \lambda_n - 2\pi, \arg \lambda_2)$. These solutions have the same asymptotic behavior at large x, described by a (typically divergent) power series

$$\mathbf{y}(x) \sim \hat{\mathbf{y}}_0 = \sum_{k=2}^{\infty} \frac{\tilde{\mathbf{y}}_{0,k}}{x^k} \quad (|x| \to \infty;\ \Re\left(xe^{-i\phi}\right) > \text{const} > 0). \tag{1.3}$$

For instance, all the solutions of the equation $y' + y = x^{-1}$ have the property $y(x) \sim \sum_{k=0}^{\infty} k! x^{-k-1}$ as $x \to \infty$. If $\phi \neq 0$, there is only one solution of (1.1) satisfying (1.3). A much more interesting case is when we take $\phi = 0$. Then, as it is known (and will also follow from the present paper), there is a one-dimensional manifold M^+ of solutions of (1.1) such that (1.3) holds. The manifold $\tilde{M}^+$ of all *formal* solutions which decay in the half-plane $\Re x > 0$

$$\hat{\mathbf{y}} = \hat{\mathbf{y}}_0 + \sum_{k=1}^{\infty} C^k e^{-kx} \hat{\mathbf{y}}_k \tag{1.4}$$

also has one free parameter, $C \in \mathbb{C}$. In (1.4), $\hat{\mathbf{y}}_k$, $k \geq 0$, are formal power series and $\hat{\mathbf{y}}$ is an instance of a transseries. In our example $y' + y = x^{-1}$, $\hat{\mathbf{y}} = \sum_{k=0}^{\infty} k! x^{-k-1} + Ce^{-x}$. See Section 2.6 for a heuristic construction leading to transseries solutions and for references.

The series $\hat{\mathbf{y}}_k$ satisfy the system of differential equations

$$\mathbf{y}_0' + \left(\hat{\Lambda} + \frac{1}{x}\hat{B}\right)\mathbf{y}_0 = \mathbf{f}_0(x) + \mathbf{g}(x, \mathbf{y}_0),$$

$$\mathbf{y}_k' + \left(\hat{\Lambda} + \frac{1}{x}\hat{B} - k - \boldsymbol{\partial}\mathbf{g}(x, \mathbf{y}_0)\right)\mathbf{y}_k = \sum_{|\mathbf{l}|>1} \frac{\mathbf{g}^{(\mathbf{l})}(x, \mathbf{y}_0)}{\mathbf{l}!} \sum_{\Sigma m = k} \prod_{i=1}^{n} \prod_{j=1}^{l_i} (\mathbf{y}_{m_{i,j}})_i, \tag{1.5}$$

where $\mathbf{g}^{(\mathbf{l})} := \partial^{(\mathbf{l})}\mathbf{g}/\partial\mathbf{y}^{\mathbf{l}}$, $(\boldsymbol{\partial}\mathbf{g})\mathbf{y}_k := \sum_{i=1}^{n}(\mathbf{y}_k)_i(\partial\mathbf{g}/\partial y_i)$, and $\sum_{\Sigma m=k}$ stands for the sum over all integers $m_{i,j} \geq 1$ with $1 \leq i \leq n, 1 \leq j \leq l_i$ such that $\sum_{i=1}^{n}\sum_{j=1}^{l_i} m_{i,j} = k$. Because $m_{i,j} \geq 1$, $\sum m_{i,j} = k$ (fixed) and $\mathrm{card}\{m_{i,j}\} = |\mathbf{l}|$, the sums in (1.5) contain only a *finite* number of terms. We use the convention $\prod_{i\in\emptyset} \equiv 0$. The system (1.5) is derived in Section 2.6.

Starting with $k = 1$, the equations (1.5) are linear. Note that the inhomogeneous term in these linear equations is zero for $k = 1$, and for $k > 1$ it involves only $\mathbf{y}_n$ with $n < k$.

While some connection between $\hat{\mathbf{y}}_0$ and actual solutions of (1.1) is given by (1.3), the interpretation of (1.4) is less immediate, since generically all the series involved are (factorially) divergent and "beyond all orders of each other." The interest in transseries is motivated partly by their formal simplicity compared to the vast class of differential equations that they "solve" and by the fact that they can be algorithmically found, once the equation is given. Finding the connection between formal expansions and true solutions is the object of exponential asymptotics, a field that has been growing constantly, especially after the pioneering works of M. Berry, J. Ecalle, and M. Kruskal. The formalism of generalized Borel summation and the theory of transseries, in a very comprehensive setting, were introduced by Ecalle [6], [8], [7].

For the problem (1.1)–(1.3), we prove that there is a one-to-one natural correspondence between actual solutions $\mathbf{y}$ and the transseries $\hat{\mathbf{y}}$ (1.4). We show that the general solution of (1.1), (1.3) is obtained by replacing each formal series in (1.4) by its Borel sum, which gives a one-to-one correspondence between the formal solutions (transseries) and the actual solutions of (1.1), (1.3):

$$\hat{\mathbf{y}} = \hat{\mathbf{y}}_0 + \sum_{k=1}^{\infty} C^k e^{-kx}\hat{\mathbf{y}}_k \;\longleftrightarrow\; \mathcal{L}_\phi\mathcal{B}_\phi\hat{\mathbf{y}}_0 + \sum_{k=1}^{\infty} C^k e^{-kx}\mathcal{L}_\phi\mathcal{B}_\phi\hat{\mathbf{y}}_k = \mathbf{y}. \tag{1.6}$$

The Borel summation operator, $\mathcal{LB}$, will be defined precisely. The function $\mathbf{y} \in M^+$ is convergently defined by (1.6) for large x. The left arrow in (1.6) means that $\mathcal{L}_\phi\mathcal{B}_\phi\hat{\mathbf{y}}_k(x) \sim \hat{\mathbf{y}}_k(x)$ for $x \to \infty$. The exact statement corresponding to (1.6) is given in Theorem 2.

We study in detail the features of the representation (1.6) and the properties of the objects involved. The technique that we use differs from that of [6], [8], [7] and leads to new results. In particular, we obtain, for the Borel transform of the formal series solutions of differential systems, an averaging formula having, as the medianization of Ecalle, the quality of preserving reality and of commuting with convolution, but involving a smaller number of analytic continuations, and in addition satisfying the condition of at most exponential growth at infinity.

For $m > 1$, the inverse Laplace transform of x^{-m} is

$$\mathcal{L}^{-1}x^{-m} = p^{m-1}/\Gamma(m-1) = \mathcal{B}\,x^{-m}.$$

The *Borel transform* $\mathcal{B}$ of a formal series

$$\hat{\mathbf{y}} = x^r \sum_{k=1}^{\infty} \tilde{\mathbf{y}}_k x^{-k}, \qquad r \in (0,1) \tag{1.7}$$

is by definition the formal series gotten by taking $\mathcal{L}^{-1}$ term by term:

$$\mathcal{B}\,\hat{\mathbf{y}} = \mathbf{Y} := p^{-r} \sum_{k=0}^{\infty} \frac{\tilde{\mathbf{y}}_{k+1}}{\Gamma(k-r)} p^k. \tag{1.8}$$

A priori $\mathbf{Y}$ is still a formal series. If it has a nonzero radius of convergence, then it generates an element of an analytic function which we will denote, all the same, by $\mathbf{Y}$.

A formal series $\hat{\mathbf{y}}$ is Borel summable in the classical sense along a ray Φ (the direction of which is given by the angle ϕ) if the following conditions are met:

(1) the series $\mathbf{Y}$ has a nonzero radius of convergence

(2) $\mathbf{Y}$ can be analytically continued along the ray, and

(3) the analytic continuation $\mathbf{Y}$ grows at most exponentially along the ray and is therefore Laplace transformable along Φ.

The Laplace transform along that ray of $\mathbf{Y}$, $\mathcal{L}_\phi \mathbf{Y}$, is well defined and gives the *Borel sum* of $\hat{\mathbf{y}}$. We prove that the conditions (1) through (3) are met by $\mathcal{B}\hat{\mathbf{y}}_k$, $k \geq 0$, away from the Stokes rays, i.e., if $\phi \neq \arg \lambda_i$, $\lambda_i \in \operatorname{spec} \hat{\Lambda}$.

Of all the formal solutions (1.4), only the one with $C = 0$ (formally) decays in a half-plane, if the half-plane *is not* centered on the real axis. On the other hand, $\mathcal{L}_\phi \mathcal{B}\hat{\mathbf{y}}_0$ turns out to be the only solution of (1.1), (1.3) which decays in the same half-plane centered on Φ. Borel summation associates uniquely a true solution to $\mathbf{Y}_0$.

The situation is more complicated and more interesting along Stokes rays. (We focus on one of them, $\Phi = \mathbb{R}^+$.) Condition (2) above is violated and, generically, the functions $\mathbf{Y}_k$ have an array of branch points along $\mathbb{R}^+$. If we reinterpret (2) and consider paths that avoid the singularities, then, first of all, analytic continuation is (a priori) ambiguous. What is worse, the Laplace transform of such analytic continuations of $\mathbf{Y}_0$ are, typically, *not* solutions of (1.1) (see Section A.2). However, Laplace transforms (of a one-parameter family) of suitable weighted combinations of analytic continuations of $\mathbf{Y}_0$ *are*, as we will prove, solutions of (1.1). If we require in addition that real series are Borel-summed to real-valued functions, then one of weighted average of analytic continuations appears as more natural (see also Theorem 5 below).

To define the Borel transform along the Stokes line $\mathbb{R}^+$, we construct a suitable space of analytic functions. Let $\phi_+ = \arg \lambda_2$, $\phi_- = 2\pi - \arg \lambda_n$, and

$$\mathcal{W}_1 := \{p : p \notin \mathbb{N} \cup \{0\} \text{ and } \arg p \in (-\phi_-, \phi_+)\} \tag{1.9}$$

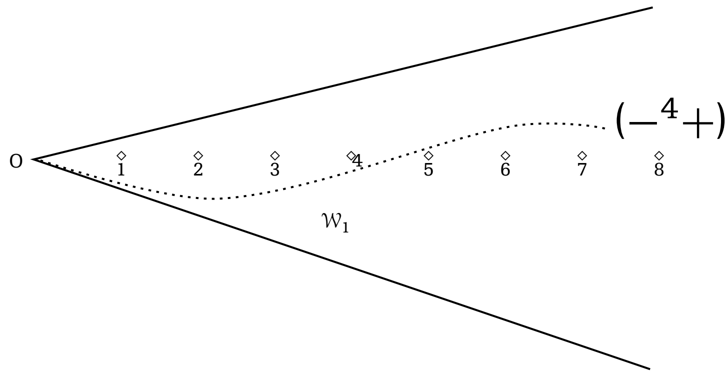

Figure 1 The region $\mathcal{W}_1$
The dotted line is one of the paths that generate $\mathcal{R}_1$.

(see Figure 1), a sector containing only the eigenvalue $\lambda_1 = 1$ and punctured at all the integers (where the functions $\mathcal{B}\hat{\mathbf{y}}_k$ are typically singular). We construct over $\mathcal{W}_1$ a surface $\mathcal{R}_1$, consisting of homotopy classes of curves starting at the origin, going only forward and crossing the real axis at most once:

$$\mathcal{R}_1 := \Big\{\gamma : (0,1) \mapsto \mathcal{W}_1 \text{ s.t. } \gamma(0_+) = 0;\ \Re(\gamma(t)) \text{ increases in } t \text{ and}$$
$$0 = \Im(\gamma(t_1)) = \Im(\gamma(t_2)) \Rightarrow t_1 = t_2\Big\} \tag{1.10}$$

modulo homotopies. Let also

$$\mathcal{D} := \mathbb{C} \setminus \cup_{i=1}^{n} \{\alpha\lambda_i : \alpha \geq 1\} \tag{1.11}$$

be the complex plane without the rays originating at the eigenvalues λ_i of $\hat{\Lambda}$.

Using notations similar to those of Ecalle, we symbolize the paths in $\mathcal{R}_1$ by a sequence of signs $\epsilon_1, \ldots, \epsilon_j, \ldots, \epsilon_n$, $\epsilon_j = +$ or $-$. For example, $----+ = -^4+$ will symbolize a path in $\mathcal{R}_1$ that crosses the real line from below through the interval $(4,5)$, and then goes only through the upper half-plane (Figure 1); "+" is a path confined to the upper half-plane, etc. The analytic continuation of a function $\mathbf{Y}$ along the path $-^4+$ will be denoted $\mathbf{Y}^{-^4+}$.

The result below gives a first characterization of the analytic properties of $\mathcal{B}\hat{\mathbf{y}}_k$. (In the following, we choose the determination of the logarithm which is real for positive argument.)

Proposition 1. (i) The function $\mathbf{Y}_0 := \mathcal{B}\hat{\mathbf{y}}_0$ is analytic in $\mathcal{D}$ and Laplace transformable along any direction in $\mathcal{D}$. In a neighborhood of $p = 1$,

$$\mathbf{Y}_0(p) = \begin{cases} S_\beta (1-p)^{\beta-1}\mathbf{A}(p) + \mathbf{B}(p) \text{ for } \beta < 1 \\ S_\beta \ln(1-p)\mathbf{A}(p) + \mathbf{B}(p) \text{ for } \beta = 1 \end{cases} \tag{1.12}$$

(see (1.2)), where $\mathbf{A}$, $\mathbf{B}$ are ($\mathbb{C}^n$-valued) analytic functions in a neighborhood of $p = 1$.

(ii) The functions $\mathbf{Y}_k := \mathcal{B}\hat{\mathbf{y}}_k$, $k = 0, 1, 2, \ldots$ are analytic in $\mathcal{R}_1$.

(iii) For small p,

$$\mathbf{Y}_0(p) = p\mathbf{A}_0(p); \qquad \mathbf{Y}_k(p) = p^{k\beta-1}\mathbf{A}_k(p), \qquad k = 1, 2, \ldots \tag{1.13}$$

where $\mathbf{A}_k$, $k \geq 0$, are analytic functions in a neighborhood of $p = 0$ in $\mathbb{C}$.

(iv) If $S_\beta = 0$, then $\mathbf{Y}_k$, $k \geq 0$, are analytic in $\mathcal{W}_1 \cup \mathbb{N}$.

(v) The analytic continuations of $\mathbf{Y}_k$ along paths in $\mathcal{R}_1$ are in $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ (their singularities along $\mathbb{R}^+$ are integrable). The analytic continuations of the $\mathbf{Y}_k$ in $\mathcal{R}_1$ can be expressed in terms of each other through "resurgence" relations of the type

$$S_\beta^k \mathbf{Y}_k = \left(\mathbf{Y}_0^- - \mathbf{Y}_0^{-^{k-1}+}\right) \circ \tau_k, \quad \text{on } (0,1) \quad (\tau_a := p \mapsto p - a) \tag{1.14}$$

relating the higher-order series in the transseries to the first series and

$$\mathbf{Y}_k^{-^m+} = \mathbf{Y}_k^+ + \sum_{j=1}^{m} \binom{k+j}{k} S_\beta^j \mathbf{Y}_{k+j}^+ \circ \tau_j. \tag{1.15}$$

□

S_β is related to the Stokes constant S by

$$S_\beta = \begin{cases} \dfrac{iS}{2\sin(\pi(1-\beta))} \text{ for } \beta \neq 1 \\ \dfrac{iS}{2\pi} \text{ for } \beta = 1. \end{cases}$$

The Borel transformability of the principal series $\hat{\mathbf{y}}_0$ has been considered for general systems of differential equations, allowing for resonances (see [1], [4]).

Let $\mathbf{Y}$ be one of the functions $\mathbf{Y}_k$ and define, on $\mathbb{R}^+ \cap \mathcal{R}_1$, the "balanced average" of $\mathbf{Y}$:

$$\mathbf{Y}^{ba} = \mathbf{Y}^+ + \sum_{k=1}^{\infty} 2^{-k} \left(\mathbf{Y}^- - \mathbf{Y}^{-^{k-1}+}\right) \mathcal{H} \circ \tau_k \tag{1.16}$$

($\mathcal{H}$ is Heaviside's function). For any value of the argument, only finitely many terms (1.16) are nonzero. Moreover, the balanced average preserves reality in the sense that if (1.1) is real and $\hat{\mathbf{y}}_0$ is real, then $\mathbf{Y}^{ba}$ is real on $\mathbb{R}^+ - \mathbb{N}$. (And in this case the formula can be symmetrized by taking 1/2 of the expression above plus 1/2 of the same expression with $+$ and $-$ interchanged.) Equation (1.16) has the main features of medianization (cf. [8]), in particular (unlike individual analytic continuations; see Appendix A.2), it commutes with convolution (cf. Theorem 5). As it will become clear, the advantage of the definition (1.16) is that $\mathbf{Y}^{ba}$ is exponentially bounded at infinity for the functions we are dealing with.

Let again $\hat{\mathbf{y}}$ be one of $\hat{\mathbf{y}}_k$ and $\mathbf{Y} = \mathcal{B}\hat{\mathbf{y}}$. We define

$$\mathcal{L}_\phi \mathcal{B}\hat{\mathbf{y}} := \mathcal{L}_\phi \mathbf{Y} = x \mapsto \int_0^{\infty e^{i\phi}} \mathbf{Y}(p) e^{-px}\, dp \quad \text{if } \Phi \neq \mathbb{R}^+$$

$$\mathcal{L}_0 \mathcal{B}\hat{\mathbf{y}} := \mathcal{L}_0 \mathbf{Y} = x \mapsto \int_0^{\infty} \mathbf{Y}^{ba}(p) e^{-px}\, dp \quad \text{if } \Phi = \mathbb{R}^+. \tag{1.17}$$

The connection between true and formal solutions of the differential equation is given in the following theorem.

Theorem 2. (i) There is a large enough b such that, for $\Re(x) > b$, the Laplace transforms $\mathcal{L}_\phi \mathbf{Y}_k$ exist for all $k \geq 0$ and $\phi \in (-\phi_-, \phi_+)$; cf. (1.9).

For $\phi \in (-\phi_-, \phi_+)$ and any C, the series

$$\mathbf{y}(x) = (\mathcal{L}_\phi \mathcal{B}\hat{\mathbf{y}}_0)(x) + \sum_{k=1}^{\infty} C^k e^{-kx} (\mathcal{L}_\phi \mathcal{B}\hat{\mathbf{y}}_k)(x) \tag{1.18}$$

is convergent for large enough x in the right half-plane. The function $\mathbf{y}$ in (1.18) is a solution of the differential equation (1.1). Furthermore, for any $k \geq 0$, we have $\mathcal{L}_\phi \mathcal{B}\hat{\mathbf{y}}_k \sim \hat{\mathbf{y}}_k$ in the right half-plane, and $\mathcal{L}_\phi \mathcal{B}\hat{\mathbf{y}}_k$ is a solution of the corresponding equation in (1.5).

(ii) Conversely, given ϕ, *any* solution of (1.1) having $\hat{\mathbf{y}}_0$ as an asymptotic series in the right half-plane can be written in the form (1.18), for a unique C.

(iii) The constant C, associated in (ii) with a given solution $\mathbf{y}$ of (1.1), depends on the angle ϕ:

$$C(\phi) = \begin{cases} C(0_+) \;\; \text{for } \phi > 0 \\ C(0_+) - \dfrac{1}{2} S_\beta \;\; \text{for } \phi = 0 \\ C(0_+) - S_\beta \;\; \text{for } \phi < 0 \end{cases} \tag{1.19}$$

(see also (1.12)). □

Note that by (1.19) the change in the correspondence (1.6) occurs when the Stokes line $\arg x = 0$ is crossed. This is a *local* manifestation of the Stokes phenomenon [15], [17], [14].

Next, we study the correspondence between the solutions of the differential equations (1.1), (1.27), their formal solutions, and the solutions of the inverse Laplace transform of these equations, which, in the transformed space, are convolution equations.

With the convolution defined as

$$f * g := p \mapsto \int_0^p f(s)g(p-s)\,ds, \tag{1.20}$$

we have, as is well known, $\mathcal{L}(f*g) = \mathcal{L}(f)\mathcal{L}(g)$, $\mathcal{L}(-pf(p)) = \mathcal{L}(f(p))'$. (See Section A.3 for a few more useful formulas.) In (1.1) we write

$$\mathbf{g}(\xi^{-1},\mathbf{y}) = \sum_{|\mathbf{l}|\geq 1} \mathbf{g}_{\mathbf{l}}(\xi)\mathbf{y}^{\mathbf{l}} = \sum_{m\geq 0;|\mathbf{l}|\geq 1} \mathbf{g}_{m,\mathbf{l}}\xi^m\mathbf{y}^{\mathbf{l}} \quad (|\xi| < \xi_0, |\mathbf{y}| < y_0), \tag{1.21}$$

where by construction $\mathbf{g}_{0,\mathbf{l}} = \mathbf{g}_{1,\mathbf{l}} = 0$ if $|\mathbf{l}| = 1$ and the notation $\mathbf{z}^{\mathbf{l}}$ means $z_1^{l_1} \cdot z_n^{l_n}$ and $|\mathbf{l}| = l_1 + \cdots + l_n$. The formal inverse Laplace transform of $\mathbf{g}(x,\mathbf{y}(x))$ is given by

$$\mathcal{L}^{-1}\sum_{|\mathbf{l}|\geq 1}\mathbf{y}(x)^{\mathbf{l}}\left(\sum_{m\geq 0}\mathbf{g}_{m,\mathbf{l}}x^{-m}\right) = \sum_{|\mathbf{l}|\geq 1}\mathbf{G}_{\mathbf{l}} * \mathbf{Y}^{*\mathbf{l}} + \sum_{|\mathbf{l}|\geq 2}\mathbf{g}_{0,\mathbf{l}}\mathbf{Y}^{*\mathbf{l}} =: \mathcal{N}(\mathbf{Y}) \tag{1.22}$$

where

$$\mathbf{G}_{\mathbf{l}}(p) = \sum_{m=1}^{\infty}\mathbf{g}_{m,\mathbf{l}}\frac{p^{m-1}}{m!} \qquad (\mathbf{G}_{1,\mathbf{l}}(0) = 0 \ \text{ if } |\mathbf{l}| = 1), \tag{1.23}$$

$$\mathbf{G}_{\mathbf{l}} * \mathbf{Y}^{*\mathbf{l}} \in \mathbb{C}^n; \quad (\mathbf{G}_{\mathbf{l}} * \mathbf{Y}^{*\mathbf{l}})_j := \left(\mathbf{G}_{\mathbf{l}}\right)_j * Y_1^{*l_1} * \cdots * Y_n^{*l_n}. \tag{1.24}$$

The inverse Laplace transform of (1.1) is the convolution equation:

$$-p\mathbf{Y}(p) = \mathbf{F}_0(p) - \hat{\Lambda}\mathbf{Y}(p) - \hat{B}\int_0^p \mathbf{Y}(s)\,ds + \mathcal{N}(\mathbf{Y})(p) \tag{1.25}$$

(see (1.22)) where, since $\mathbf{f}_0(x) = O(x^{-2})$,

$$\mathbf{F}_0(0) = 0. \tag{1.26}$$

By transforming (1.5), we get, similarly,

$$(\hat{\Lambda} - p - k)\mathbf{Y}_k(p) + \hat{B}\int_0^p \mathbf{Y}_k(s)\,ds - \sum_{j=1}^{n}\int_0^p (\mathbf{Y}_k)_j(s)\mathbf{D}_j(p-s)\,ds$$

$$= \sum_{|\mathbf{l}|>1} \mathbf{d}_{\mathbf{l}} * \sum_{\Sigma m=k} * \prod_{i=1}^{n} * \prod_{j=1}^{l_i} (\mathbf{Y}_{m_{i,j}})_i =: \mathbf{R}_k(p) \qquad (k = 1, 2, \dots) \tag{1.27}$$

with $\mathbf{d}_{\mathbf{m}} := \mathcal{L}^{-1}(\mathbf{g}^{(\mathbf{m})}(x, \mathbf{y}_0)/\mathbf{m}!)$, $\mathbf{D}_j := \mathcal{L}^{-1}(\partial\mathbf{g}(x, \mathbf{y}_0)/\partial y_j)$, and $*\prod$ standing for the convolution product.

For a given ray Φ, we consider the equations (1.25) and (1.27) in $L^1_{loc}(\Phi)$. When Φ is not a Stokes line, the description of the solutions is quite simple.

Proposition 3. (i) If Φ is a ray in $\mathcal{D}$, then equation (1.25) has a unique solution in $L^1_{loc}(\Phi)$, namely, $\mathbf{Y}_0 = \mathcal{B}\hat{\mathbf{y}}_0$.

(ii) For any ray in $\mathcal{W}_1$, the system (1.25), (1.27) has the general solution $C^k\mathbf{Y}_k = C^k\mathcal{B}\hat{\mathbf{y}}_k$, $k \ge 0$. □

The more interesting case $\Phi = \mathbb{R}^+$ is dealt with in the following theorem.

Theorem 4. (i) The general solution in $L^1_{loc}(\mathbb{R}^+)$ of the equation (1.25) can be written in the form

$$\mathbf{Y}_C(p) = \sum_{k=0}^{\infty} C^k \mathbf{Y}_k^{ba}(p-k)\mathcal{H}(x-k) \tag{1.28}$$

with $C \in \mathbb{C}$ arbitrary.

(ii) Near $p = 1$, $\mathbf{Y}_C$ is given by

$$\mathbf{Y}_C(p) = \begin{cases} S_\beta(1-p)^{\beta-1}\mathbf{A}(p) + \mathbf{B}(p) \text{ for } p < 1 \\ C(1-p)^{\beta-1}\mathbf{A}(p) + \mathbf{B}(p) \text{ for } p > 1 \end{cases} \quad (\beta < 1) \tag{1.29}$$

$$\mathbf{Y}_C(p) = \begin{cases} S_\beta \ln(1-p)\mathbf{A}(p) + \mathbf{B}(p) \text{ for } p < 1 \\ (S_\beta \ln(1-p) + C)\mathbf{A}(p) + \mathbf{B}(p) \text{ for } p > 1 \end{cases} \quad (\beta = 1)$$

where $\mathbf{A}$ and $\mathbf{B}$ extend to analytic functions in a neighborhood of $p = 1$.

(iii) With the choice $\mathbf{Y}_0 = \mathbf{Y}_0^{ba}$, the general solution of (1.27) in $L^1_{loc}(\mathbb{R}^+)$ is $C^k\mathbf{Y}_k^{ba}$, $k \in \mathbb{N}$. □

Comparing (1.29) with (1.12), we see that if $S \neq 0$ (which is the generic case), the general solution of (1.25) can be written on the interval $(0, 2)$ as a linear combination of the upper and lower analytic continuations of $\mathcal{B}\hat{\mathbf{y}}_0$:

$$\mathbf{Y}_C = \lambda_C \mathbf{Y}_0^+ + (1 - \lambda_C)\mathbf{Y}_0^-. \tag{1.30}$$

Finally, we mention the following result, which shows that the balanced average, like medianization [8], commutes with convolution.

Theorem 5. If f and g are analytic in $\mathcal{R}_1$, then $f * g$ extends analytically in $\mathcal{R}_1$ and, furthermore,

$$(f * g)^{ba} = f^{ba} * g^{ba}. \tag{1.31}$$

□

As a consequence of the linearity of the balanced averaging and its commutation with convolution, if $\tilde{\mathbf{t}}_{1,2}$ are the transseries of the solutions $\mathbf{f}_{1,2}$ of differential equations of the type considered in the present paper (cf. (1.6)), and if $\mathcal{L}\mathcal{B}\tilde{\mathbf{t}}_{1,2} = \mathbf{f}_{1,2}$, then

$$\mathcal{L}\mathcal{B}\left(a\tilde{\mathbf{t}}_1 + b\tilde{\mathbf{t}}_2\right) = a\mathbf{f}_1 + b\mathbf{f}_2. \tag{1.32}$$

Moreover, less obviously, we have for the component-wise product

$$\mathcal{L}\mathcal{B}(\tilde{\mathbf{t}}_1\tilde{\mathbf{t}}_2) = \mathbf{f}_1\mathbf{f}_2. \tag{1.33}$$

Borel summation is in fact an isomorphism between a subalgebra of transseries and a function algebra.

2 Proofs and further results

2.1 Outline of the proofs of the main results

To show the results stated in the previous section, we first obtain the general solution in L^1_{loc} of the convolution system (1.27) in $\mathcal{D}$ and then, separately, on the Stokes line $\mathbb{R}^+$. We show that along a ray in $\mathcal{D}$ the solution is unique, whereas along the ray $\mathbb{R}^+$ there is a one-parameter family of solutions of the system, branching off at $p = 1$. We show that any L^1_{loc} solution of the system is (uniformly in k) exponentially bounded at infinity and therefore is Laplace transformable; by the usual properties of the Laplace transform, these transforms solve (1.1). Conversely, any solution of (1.1) with the required asymptotic properties is inverse Laplace transformable, and therefore it has to be one of the previously obtained solutions of the equation corresponding to $k = 0$. We then study the regularity properties of the solutions of the convolution equation by local analysis.

Having the complete description of the family of L^1_{loc} solutions, we compare different ways that lead to the same solution and obtain interesting identities; the identities, together with the local properties of the solutions, are instrumental in finding the analytic properties of $\mathbf{Y}_k$ in $\mathcal{R}_1$.

Key to the main proofs. The complete connection with equation (1.16) is established in Section 2.7. For the remaining parts: Proposition 1 (i) follows from Proposition 6 and Lemma 20; (ii) and (iii) follow from Proposition 38. The proof of (1.13) is given in Remark 34, and (iv) is shown in Remark 40. Part (v) follows from Proposition 37 and Proposition 38. Theorem 2 (i) and (ii) follow from Lemma 36 and Proposition 32; (iii) is equation (2.89). Proposition 3 follows from Proposition 6 and Lemma 36. Theorem 4 follows from Proposition 26, Lemma 24, and Proposition 31. The proof of Theorem 5 starts with Proposition 41.

2.2 The convolution equation away from Stokes rays

For any star-shaped set $\mathcal{E}$ in the complex plane containing the origin (i.e., a region such that the origin can be connected with any other point in $\mathcal{E}$ by a straight line segment contained in $\mathcal{E}$), we denote by $L_{\mathrm{ray}}(\mathcal{E})$ the set of functions which are locally integrable along each ray in $\mathcal{E}$.

Proposition 6. There is a unique solution of (1.25) in $L_{\mathrm{ray}}(\mathcal{D})$ (cf. (1.11)), namely, $\mathbf{Y}_0 = \mathcal{B}\hat{\mathbf{y}}_0$. This solution is analytic in $\mathcal{D}$; it is Laplace transformable along any ray Φ contained in $\mathcal{D}$; and $\mathcal{L}_\phi \mathbf{Y}_0$ is a solution of (1.1). □

For the proof we need a few more results.

Remark 7. There is a constant $K > 0$ (independent of p and $\mathbf{l}$) such that, for all $p \in \mathbb{C}$ and all $\mathbf{l} \geq \mathbf{0}$,

$$|\mathbf{G}_{\mathbf{l}}(p)|_\wedge < K\mu^{|\mathbf{l}|}e^{\mu|p|} \tag{2.1}$$

for $\mu > \max\{\xi_0^{-1}, y_0^{-1}\}$ (cf. (1.21)). ($|\mathbf{f}|_\wedge := \max_{1\ldots n}\{|f_1|, \ldots, |f_n|\}$ is a Euclidean norm; for the definition of $\mathbf{G}$ see (1.23), (1.21), and (1.1).)

Proof. From the analyticity assumption, it follows that

$$|\mathbf{g}_{m,\mathbf{l}}|_\wedge < \text{Const } \mu^{m+|\mathbf{l}|} \tag{2.2}$$

where the constant is independent on m and $\mathbf{l}$. Then, by (1.23),

$$|\mathbf{G}_{\mathbf{l}}(p)|_\wedge < \text{Const } \mu^{|\mathbf{l}|+1}\frac{e^{\mu|p|}-1}{\mu|p|} < \text{Const } \mu^{|\mathbf{l}|+1}e^{\mu|p|}. \qquad ■$$

Consider the ray segments

$$\Phi_D = \{\alpha e^{i\phi} : 0 \le \alpha < D\} \tag{2.3}$$

and along Φ_D the L^1 norm with exponential weight

$$\|f\|_{b,\Phi} = \|f\|_b := \int_\Phi e^{-b|p|}|f(p)||dp| = \int_0^D e^{-bt}|f(te^{i\phi})|\,dt \tag{2.4}$$

and the space

$$L^1_b(\Phi_D) := \{f : \|f\|_b < \infty\}$$

(if $D < \infty$, $L^1_b(\Phi_D) = L^1_{loc}(\Phi_D)$). We mention the following elementary property.

Remark 8. The Laplace transform $\mathcal{L}$ is a continuous operator from $L^1_b(\Phi_D)$ to the space of analytic functions in the half-plane $\Re(x) > b$ with the uniform norm.

Let $\mathcal{K} \in \mathbb{C}$ be a bounded domain, $\operatorname{diam}(\mathcal{K}) = D < \infty$. On the space of continuous functions on $\mathcal{K}$, we take the uniform norm with exponential weight

$$\|f\|_u := D \sup_{p\in\mathcal{K}}\{|f(p)|e^{-b|p|}\} \tag{2.5}$$

(which is equivalent to the usual uniform norm). Let $\mathcal{O} \subset D$, $\mathcal{O} \ni 0$ be a *star-shaped*, *open set*, $\operatorname{diam}(\mathcal{O}) = D$ containing a ray segment Φ. Let $\mathcal{A}$ be the space of analytic functions f in $\mathcal{O}$ such that $f(0) = 0$, endowed with the norm (2.5).

Proposition 9. The spaces $L^1_b(\Phi_D)$ and $\mathcal{A}$ are Banach algebras with respect to the usual addition of functions and the convolution (1.20). Furthermore,

$$\begin{aligned}
&\|f * g\|_b \le \|f\|_b\|g\|_b && (f, g \in L^1_b(\Phi_D))\\
&\|f * g\|_u \le \|f\|_u\|g\|_u && (f, g \in \mathcal{A})\\
&\|f * g\|_u \le \|f\|_u\|g\|_b && (f \in C(\Phi_D), g \in L^1_b(\Phi_D))
\end{aligned} \tag{2.6}$$

($D = \infty$ is allowed in the first inequality). □

With $F(s) := f(se^{i\phi})$ and $G(s) := g(se^{i\phi})$, we have

$$\begin{aligned}
&\int_0^D dte^{-bt}\left|\int_0^t dsF(s)G(t-s)\right| \le \int_0^D dte^{-bt}\int_0^t ds|F(s)G(t-s)|\\
&= \int_0^D\int_0^{D-v} e^{-b(u+v)}|F(v)||G(u)|\,du\,dv\\
&\le \int_0^D\int_0^D e^{-b(u+v)}|F(v)||G(u)|\,du\,dv = \|f\|_b\|g\|_b.
\end{aligned} \tag{2.7}$$

On the other hand, for $f, g \in \mathcal{A}$ we have $f * g \in \mathcal{A}$. Also,

$$\|f * g\|_u = D \sup_{p\in\mathcal{O}} e^{-b|p|} \left| \int_0^p f(s)g(p-s)\, ds \right|$$
$$\leq D \sup_{p\in\mathcal{O}} \int_0^{|p|} \left| f(te^{i\arg p})e^{-bt} g(p - te^{i\arg p}) e^{-b(|p|-t)} \right| \, dt, \tag{2.8}$$

which is less than both $\|f\|_u\|g\|_u$ and $\|f\|_u\|g\|_b$.

Remark 10. For f in $\mathcal{A}$ or f in $L^1_b(\Phi_D)$,

$$\|f\|_{u,b} \to 0 \quad \text{as } b \to \infty, \tag{2.9}$$

where $\|\|_{u,b}$ is either of the $\|\|_u$ or $\|\|_b$ and $D = \infty$ is allowed in the second case.

For $\|\|_b$, equation (2.9) is an immediate consequence of the dominated convergence theorem, whereas for $\|\|_u$ it follows from the definition of $\mathcal{A}$.

Corollary 11. Let f be continuous along Φ_D, $D < \infty$, and $g \in L^1_b(\Phi_D)$. Given $\epsilon > 0$, there exists a large enough b and $K = K(\epsilon, \Phi_D)$ such that, for all k,

$$\|f * g^{*k}\|_u < K\,\epsilon^k. \qquad \square$$

By Remark 10, we can choose $b = b(\epsilon, \Phi_D)$ so large that $\|g\|_b < \epsilon$. Then, by Remark 9,

$$\left| \int_0^{pe^{i\phi}} f(pe^{i\phi} - s) g^{*k}(s)\, ds \right| \leq D^{-1} e^{b|p|} \|f\|_u \int_0^{pe^{i\phi}} e^{-b|s|} |g^{*k}(s)|| \, ds|$$
$$\leq D^{-1} e^{b|p|} \|f\|_u \|g\|_b^k < K\,\epsilon^k.$$

Remark 12. By (2.1), for any $b > \mu$ and $\Phi_D \subset \mathbb{C}$, $D \leq \infty$,

$$\|\mathbf{G_l}\|_b \leq K\mu^{|\mathbf{l}|} \int_0^\infty |dp| e^{|p|(\mu-b)} = K\frac{\mu^{|\mathbf{l}|}}{b-\mu}, \tag{2.10}$$

where, to avoid cumbersome notations, we write

$$\mathbf{f} \in L^1_b(\Phi_D) \quad \text{if and only if} \quad \|\mathbf{f}\|_b := \||\mathbf{f}|_\wedge\|_b \in L^1_b(\Phi_D) \tag{2.11}$$

(and similarly for other norms of vector functions).

Proof of Proposition 6. We first show existence and uniqueness in $L_{\text{ray}}(\mathcal{D})$, which amounts to nothing more than existence and uniqueness along each $\Phi_D \subset \mathcal{D}$. Then we show that for large enough b, there exists a unique solution of (1.25) in $L^1_b(\Phi_\infty)$. Since this solution is also in $L^1_{\text{loc}}(\Phi_\infty)$, it follows that our (unique) L^1_{loc} solution is Laplace transformable. Analyticity is proven by finding the solution as a fixed point of a contraction with respect to the uniform norm in a suitable space of analytic functions.

Proposition 13. (i) For $\Phi_D \in \mathcal{D}$ and large enough b, the operator

$$\mathcal{N}_1 := \mathbf{Y}(p) \mapsto (\hat{\Lambda} - p)^{-1}\left(\mathbf{F}_0(p) - \hat{B}\int_0^p \mathbf{Y}(s)\,ds + \mathcal{N}(\mathbf{Y})(p)\right) \tag{2.12}$$

is a contraction in a small enough neighborhood of the origin with respect to $\|\,\|_u$ if $D < \infty$ and with respect to $\|\,\|_b$ for $D \le \infty$.

(ii) For $D \le \infty$, the operator $\mathcal{N}$ given formally in (1.22) is continuous in $L^1_{\text{loc}}(\Phi_D)$. The last sum in (1.22) converges uniformly on compact subsets of Φ_D. $\mathcal{N}(L^1_{\text{loc}}(\Phi_D)) \subset AC(\Phi_D)$, the absolutely continuous functions on Φ_D. Moreover, if $\mathbf{v}_n \to \mathbf{v}$ in $\|\,\|_b$ on Φ_D, $D \le \infty$, then for $b' \ge b$ large enough, $\mathcal{N}(\mathbf{v}_n)$ exist and converge in $\|\,\|_{b'}$ to $\mathbf{v}$. □

The last statement amounts to saying that $\mathcal{N}$ is continuous in the topology of the inductive limit of the L^1_b.

Proof. Since $\hat{\Lambda}$ and $\hat{B}$ are constant matrices,

$$\|\mathcal{N}_1(\mathbf{Y})\|_{u,b} \le \text{Const}(\Phi)\left(\|\mathbf{F}_0\|_{u,b} + \|\mathbf{Y}\|_{u,b}\|1\|_b + \|\mathcal{N}(\mathbf{Y})\|_{u,b}\right). \tag{2.13}$$

As both $\|1\|_b$ and $\|\mathbf{F}_0\|_{u,b}$ are $O(b^{-1})$ for large b, the fact that $\mathcal{N}_1$ maps a small ball into itself follows from the next remark.

Remark 14. Let $\epsilon > 0$ be small enough. Then, there is a K such that for large b and all $\mathbf{v}$ such that $\|\mathbf{v}\|_{u,b} =: \delta < \epsilon$,

$$\|\mathcal{N}(\mathbf{v})\|_{u,b} \le K\left(b^{-1} + \|\mathbf{v}\|_{u,b}\right)\|\mathbf{v}\|_{u,b}. \tag{2.14}$$

By (2.2) and (2.10), for large b and some positive constants $C_1, \ldots, C_5$,

$$\|\mathcal{N}(\mathbf{v})\|_{u,b} \le C_1\left(\sum_{|\mathbf{l}|\ge 1}\|\mathbf{G}_{\mathbf{l}}\|_b\|\mathbf{v}\|_{u,b}^{|\mathbf{l}|} + \sum_{|\mathbf{l}|\ge 2}\|\mathbf{g}_{0,\mathbf{l}}\|_b\|\mathbf{v}\|_{u,b}^{|\mathbf{l}|}\right)$$
$$\le \frac{C_2}{b}\left(\sum_{|\mathbf{l}|\ge 1}\frac{\mu^{|\mathbf{l}|}}{b-\mu}\delta^{|\mathbf{l}|} + \sum_{|\mathbf{l}|\ge 2}\mu^{|\mathbf{l}|}\delta^{|\mathbf{l}|}\right) \le \left(C_2\sum_{m=1}^{\infty} + \sum_{m=2}^{\infty}\right)\mu^m\delta^m\sum_{|\mathbf{l}|=m}1$$

$$\leq \left(\frac{C_4}{b} + \mu\delta\right) \sum_{m=1}^{\infty} \mu^m \delta^m (m+4)^n \leq \left(\frac{C_4}{b} + \mu\delta\right) C_5 \delta. \tag{2.15}$$

To show that $\mathcal{N}_1$ is a contraction we need the following.

Remark 15. We have

$$\|\mathbf{h_l}\| := \|(\mathbf{f}+\mathbf{h})^{*\mathbf{l}} - \mathbf{f}^{*\mathbf{l}}\| \leq |\mathbf{l}| \left(\|\mathbf{f}\| + \|\mathbf{h}\|\right)^{|\mathbf{l}|-1} \|\mathbf{h}\| \tag{2.16}$$

where $\|\| = \|\|_u$ or $\|\|_b$.

This estimate will be useful when $\mathbf{h}$ is a "small perturbation." The proof of (2.16) is a simple induction on $\mathbf{l}$, with respect to the lexicographic ordering. For $|\mathbf{l}| = 1$, (2.16) is trivial; assume (2.16) holds for all $\mathbf{l} < \mathbf{l}_1$ and that $\mathbf{l}_1$ differs from its predecessor $\mathbf{l}_0$ at the position k (we can take $k = 1$), i.e., $(\mathbf{l}_1)_1 = 1 + (\mathbf{l}_0)_1$. We have

$$\begin{aligned}
&\|(\mathbf{f}+\mathbf{h})^{*\mathbf{l}_1} - \mathbf{f}^{*\mathbf{l}_1}\| = \|(\mathbf{f}+\mathbf{h})^{*\mathbf{l}_0} * (\mathbf{f}_1 + \mathbf{h}_1) - \mathbf{f}^{*\mathbf{l}_1}\| \\
&= \|(\mathbf{f}^{*\mathbf{l}_0} + \mathbf{h}_{\mathbf{l}_0}) * (f_1 + h_1) - \mathbf{f}^{*\mathbf{l}_1}\| = \|\mathbf{f}^{*\mathbf{l}_0} * h_1 + \mathbf{h}_{\mathbf{l}_0} * f_1 + \mathbf{h}_{\mathbf{l}_0} * h_1\| \\
&\leq \|\mathbf{f}\|^{|\mathbf{l}_0|}\|\mathbf{h}\| + \|\mathbf{h}_{\mathbf{l}_0}\|\|\mathbf{f}\| + \|\mathbf{h}_{\mathbf{l}_0}\|\|\mathbf{h}\| \\
&\leq \|\mathbf{h}\| \left(\|\mathbf{f}\|^{|\mathbf{l}_0|} + |\mathbf{l}_0|(\|\mathbf{f}\| + \|\mathbf{h}\|)^{|\mathbf{l}_0|}\right) \\
&\leq \|\mathbf{h}\|(|\mathbf{l}_0| + 1)(\|\mathbf{f}\| + \|\mathbf{h}\|)^{|\mathbf{l}_0|}.
\end{aligned} \tag{2.17}$$

Remark 16. For small δ and large enough b, $\mathcal{N}_1$ defined in a ball centered at zero, of radius δ in the norms $\|\|_{u,b}$, is contractive.

By (2.13) and (2.14), we know that the ball is mapped into itself for large b. Let $\epsilon > 0$ be small and let $\mathbf{f}, \mathbf{h}$ be such that $\|\mathbf{f}\| < \delta - \epsilon$, $\|\mathbf{h}\| < \epsilon$. Using (2.16) and the notations (1.25), (2.13), and $\|\| = \|\|_{u,b}$, we obtain, for some positive constants $C_1, \ldots, C_4$ and large b,

$$\begin{aligned}
&\|\mathcal{N}_1(\mathbf{f}+\mathbf{h}) - \mathcal{N}_1(\mathbf{f})\| \leq C_1 \left\| \left(\sum_{|\mathbf{l}|\geq 2} \mathbf{g}_{0,\mathbf{l}} \cdot + \sum_{|\mathbf{l}|\geq 1} \mathbf{G}_{\mathbf{l}} *\right) \left((\mathbf{f}+\mathbf{h})^{*\mathbf{l}} - \mathbf{f}^{*\mathbf{l}}\right) \right\| \\
&\leq C_2 \|\mathbf{h}\| \left(\sum_{|\mathbf{l}|\geq 1} \frac{\mu^{|\mathbf{l}|}}{b-\mu} |\mathbf{l}| \delta^{|\mathbf{l}|-1} + \sum_{|\mathbf{l}|\geq 2} |\mathbf{l}| \mu^{|\mathbf{l}|} \delta^{|\mathbf{l}|-1} \right) < (C_3 b^{-1} + C_4 \delta)\|\mathbf{h}\|.
\end{aligned} \tag{2.18}$$

To finish the proof of Proposition 13, take $\mathbf{v} \in \mathcal{A}$. Given $\epsilon > 0$ we can choose b large enough (by Remark 10) to make $\|\mathbf{v}\|_u < \epsilon$. Then the sum in the formal definition of $\mathcal{N}$ is convergent in $\mathcal{A}$, by (2.15). Now, if $D < \infty$, $L^1_{loc}(\Phi_D) = L^1_b(\Phi_D)$ for any $b > 0$. Suppose

$\mathbf{v}_n \to \mathbf{v}$ in $L^1_b(\Phi_D)$. If we choose ϵ small enough, b large so that $\|\mathbf{v}\|_b < \epsilon$, and finally n_0 large so that for $n > n_0$ $\|\mathbf{v}_n - \mathbf{v}\|_b < \epsilon$ (note that $\|\|_b$ decreases with respect to b), then $\|\mathbf{v}_n\|_b < 2\epsilon$, and continuity (in $L^1_b(\Phi_D)$ as well as in $L^1_{loc}(\Phi_\infty) \equiv \cup_{k\in\Phi_\infty} L^1_b(0,k)$) follows from Remark 16. Continuity with respect to the topology of the inductive limit of the L^1_b is proven in the same way. It is straightforward to show that $\mathcal{N}(L^1_{loc}(\Phi)) \subset AC(\Phi)$. ■

The fact that $\mathcal{L}_\phi \mathbf{Y}_0$ is a solution of (1.1) follows from Proposition 13, Remark 8, and the elementary properties of $\mathcal{L}$ (see also the proof of Proposition 28). Since $\mathbf{Y}_0(p)$ is analytic for small p, $(\mathcal{L}\mathbf{Y}_0)(x)$ has an asymptotic series for large x, which has to agree with $\hat{\mathbf{y}}_0$ since $\mathcal{L}\mathbf{Y}_0$ solves (1.1). This shows that $\mathbf{Y}_0 = \mathcal{B}\hat{\mathbf{y}}_0$. ■

Remark 17. For any δ, there is a constant $K_2 = K_2(\delta, |p|)$ so that for all $\mathbf{l}$, we have

$$|\mathbf{Y}_0^{*\mathbf{l}}(p)|_\wedge \le K_2 \delta^{|\mathbf{l}|}. \tag{2.19}$$

The estimates (2.19) follow immediately from analyticity and from Corollary 11.

2.3 Behavior of $\mathbf{Y}_0(p)$ near $p = 1$.

Let $\mathbf{Y}_0$ be the unique solution in $L_{ray}(\mathcal{D})$ of (1.25), and let $\epsilon > 0$ be small. Define

$$\mathbf{H}(p) := \begin{cases} \mathbf{Y}_0(p) & \text{for } p \in \mathcal{D}, |p| < 1-\epsilon \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbf{h}(1-p) := \mathbf{Y}_0(p) - \mathbf{H}(p). \tag{2.20}$$

In terms of $\mathbf{h}$, for real $z = 1 - p$, $z < \epsilon$, the equation (1.25) reads

$$-(1-z)\mathbf{h}(z) = \mathbf{F}_1(z) - \hat{\Lambda}\mathbf{h}(z) + \hat{B}\int_\epsilon^z \mathbf{h}(s)\,ds + \mathcal{N}(\mathbf{H} + \mathbf{h}) \tag{2.21}$$

where

$$\mathbf{F}_1(1-s) := \mathbf{F}_0(s) - \hat{B}\int_0^{1-\epsilon} \mathbf{H}(s)\,ds.$$

Proposition 18. For small ϵ, $\mathbf{H}^{*\mathbf{l}}(1+z)$ extends to an analytic function in the disk $\mathbb{D}_\epsilon := \{z : |z| < \epsilon\}$. Furthermore, for any δ there is an ϵ and a constant $K_1 := K_1(\delta, \epsilon)$ such that for $z \in \mathbb{D}_\epsilon$ the analytic continuation satisfies the estimate

$$|\mathbf{H}^{*\mathbf{l}}(1+z)|_\wedge < K_1 \delta^{l}. \tag{2.22}$$

□

Proof. The case $|\mathbf{l}| = 1$ is trivial: $\mathbf{H}$ itself extends as the zero analytic function. We assume by induction on $|\mathbf{l}|$ that Proposition 18 is true for all $\mathbf{l}$, $|\mathbf{l}| \le l$ and show that it then holds, e.g., for $H_1 * \mathbf{H}^{*\mathbf{l}}$, for all $\mathbf{l}$, $|\mathbf{l}| \le l$.

$\mathbf{H}$ is analytic in an ϵ-neighborhood of $[0, 1-2\epsilon]$, and therefore so is $\mathbf{H}^{*\mathbf{l}}$. Taking first $z \in \mathbb{R}^+$, $z < \epsilon$, we have

$$\int_0^{1-z} H_1(s)\mathbf{H}^{*\mathbf{l}}(1-z-s)\,ds = \int_0^{1-\epsilon} H_1(s)\mathbf{H}^{*\mathbf{l}}(1-z-s)\,ds$$
$$= \int_0^{1/2} H_1(s)\mathbf{H}^{*\mathbf{l}}(1-z-s)\,ds + \int_{1/2}^{1-\epsilon} H_1(s)\mathbf{H}^{*\mathbf{l}}(1-z-s)\,ds. \tag{2.23}$$

The integral on $[1/2, 1-\epsilon]$ is analytic for small z, since the argument of $\mathbf{H}^{*\mathbf{l}}$ varies in an ϵ-neighborhood of $[0, 1/2]$; the integral on $[0, 1/2)$ equals

$$\int_{1/2-z}^{1-z} H_1(1-z-t)\mathbf{H}^{*\mathbf{l}}(t)\,dt = \left(\int_{1/2-z}^{1/2} + \int_{1/2}^{1-\epsilon} + \int_{1-\epsilon}^{1-z}\right) H_1(1-z-t)\mathbf{H}^{*\mathbf{l}}(t)\,dt. \tag{2.24}$$

In (2.24) the integral on $[1/2-z, 1/2]$ is clearly analytic in $\mathbb{D}_\epsilon$, and the following one is the integral of an analytic function of the parameter z with respect to the absolutely continuous measure $\mathbf{H}^{*\mathbf{l}}dt$, whereas in the last integral, both $\mathbf{H}^{*\mathbf{l}}$ (by induction) and H_1 extend analytically in $\mathbb{D}_\epsilon$.

To prove now the induction step for the estimate (2.22), fix δ small and let

$$\eta < \delta;\ M_1 := \max_{|p|<1/2+\epsilon} |\mathbf{H}(p)|_\wedge; \qquad M_2(\epsilon) := \max_{0\le x\le 1-\epsilon} |\mathbf{H}(p)|_\wedge; \qquad \epsilon < \frac{\delta}{4M_1}. \tag{2.25}$$

Let $K_2 := K_2(\eta;\epsilon)$ be large enough so that (2.19) holds with η in place of δ for *real* $x \in [0, 1-\epsilon]$ and *also* in an ϵ neighborhood in $\mathbb{C}$ of the interval $[0, 1/2+2\epsilon]$. We use (2.19) to estimate the second integral in the decomposition (2.23) and the first two integrals on the right-hand side of (2.24). For the last integral in (2.24), we use the induction hypothesis. If $K_1 > 2K_2\left(2M_1 + M_2\right)$, it follows that $|\mathbf{H}^{*\mathbf{l}} * H_1|_\wedge$ is bounded by (the terms are in the order explained above)

$$M_2(\epsilon)K_2\eta^l + M_1K_2\eta^l + M_1K_2\eta^l + (2\epsilon)M_1K_1\delta^l < K_1\delta^{l+1}. \tag{2.26}$$

∎

Proposition 19. The equation (2.21) can be written as

$$-(1-z)\mathbf{h}(z) = \mathbf{F}(z) - \hat{\Lambda}\mathbf{h}(z) + \hat{B}\int_\epsilon^z \mathbf{h}(s)\,ds - \sum_{k=1}^n \int_\epsilon^z h_j(s)\mathbf{D}_j(s-z)\,ds, \tag{2.27}$$

where

$$\mathbf{F}(z) := \mathcal{N}(\mathbf{H})(1-z) + \mathbf{F}_0(z), \tag{2.28}$$

$$\mathbf{D}_j = \sum_{|\mathbf{l}|\geq 1} l_j \mathbf{G}_\mathbf{l} * \mathbf{H}^{*\bar{\mathbf{l}}^j} + \sum_{|\mathbf{l}|\geq 2} l_j \mathbf{g}_{0,\mathbf{l}} \mathbf{H}^{*\bar{\mathbf{l}}^j};\ \bar{\mathbf{l}}^j := (l_1, l_2, \ldots, (l_j - 1), \ldots, l_n) \tag{2.29}$$

(cf. also (1.24)) extend to analytic functions in $\mathbb{D}_\epsilon$ (cf. Proposition 18). Moreover, if $\mathbf{H}$ is a vector in $L^1_b(\mathbb{R}^+)$, then, for large b, $\mathbf{D}_j \in L^1_b(\mathbb{R}^+)$ and the functions $\mathbf{F}(z)$ and $\mathbf{D}_j$ extend to analytic functions in $\mathbb{D}_\epsilon$. □

Proof. Noting that $(\mathbf{Y}_0 - \mathbf{H})^{*2}(1-z) = 0$ for $\epsilon < 1/2$ and $z \in \mathbb{D}_\epsilon$, the result is easily obtained by reexpanding $\mathcal{N}(\mathbf{H}+\mathbf{h})$, since Proposition 18 guarantees the uniform convergence of the series thus obtained. The proof that $\mathbf{D}_j \in L^1_b$ for large b is very similar to the proof of (2.18). The analyticity properties follow easily from Proposition 18, since the series involved in $\mathcal{N}(\mathbf{H})$ and $\mathbf{D}_j$ converge uniformly for $|z| < \epsilon$. ■

Consider again equation (2.27). Let $\hat{\Gamma} = \hat{\Lambda} - (1-z)\hat{1}$, where $\hat{1}$ is the identity matrix. By construction, $\hat{\Gamma}$ and $\hat{B}$ are block-diagonal. Their first block is one-dimensional: $\hat{\Gamma}_{00} = z$ and $\hat{B}_{0,0} = \beta$. We write this as $\hat{\Gamma} = z \oplus \hat{\Gamma}_c(z)$ and similarly, $\hat{B} = \beta \oplus \hat{B}_c$, where $\hat{\Gamma}_c$ and $\hat{B}_c$ are $(n-1)\times(n-1)$ matrices. $\hat{\Gamma}_c(z)$ and $\hat{\Gamma}_c^{-1}(z)$ are analytic in $\mathbb{D}_\epsilon$.

Lemma 20. The function $\mathbf{Y}_0$ given in Proposition 6 can be written in the form

$$\begin{aligned}
\mathbf{Y}_0(p) &= (1-p)^{\beta-1}\mathbf{a}_1(p) + \mathbf{a}_2(p) \qquad (\beta \neq 1), \\
\mathbf{Y}_0(p) &= \ln(1-p)\mathbf{a}_1(p) + \mathbf{a}_2(p) \qquad (\beta = 1),
\end{aligned} \tag{2.30}$$

for p in the region $(\mathbb{D}_\epsilon + 1) \cap \mathcal{D}$ $(\mathbb{D}_\epsilon + 1 := \{z : 1 + z \in \mathbb{D}_\epsilon\})$, where $\mathbf{a}_1, \mathbf{a}_2$ are analytic functions in $\mathbb{D}_\epsilon + 1$ and $(\mathbf{a}_1)_j = 0$ for $j > 1$. □

Proof. Let $\mathbf{\Omega}(z) := \int_\epsilon^z \mathbf{h}(s)\,ds$. By Proposition 6, $\mathbf{\Omega}$ is analytic in $\mathbb{D}_\epsilon \cap (\mathcal{D} - 1)$. From (2.27), we obtain

$$(z \oplus \hat{\Gamma}_c(z))\mathbf{\Omega}'(z) - (\beta \oplus \hat{B}_c)\mathbf{\Omega}(z) = \mathbf{F}(z) - \sum_{j=1}^{n} \int_\epsilon^z \mathbf{D}_j(s-z) Q_j'(s)\,ds \tag{2.31}$$

or, after integration by parts in the right-hand side of (2.31) ($\mathbf{D}_j(0) = 0$; cf. (2.29)),

$$(z \oplus \hat{\Gamma}_c(z))\mathbf{\Omega}'(z) - (\beta \oplus \hat{B}_c)\mathbf{\Omega}(z) = \mathbf{F}(z) + \sum_{j=1}^{n} \int_\epsilon^z \mathbf{D}_j'(s-z) Q_j(s)\,ds. \tag{2.32}$$

With the notation $(Q_1, \mathbf{Q}_\perp) := (Q_1, Q_2, \ldots, Q_n)$, we write the system in the form

$$\begin{aligned}
&(z^{-\beta}Q_1(z))' = z^{-\beta-1}\left(F_1(z) + \sum_{j=1}^{n}\int_\epsilon^z D'_{1j}(s-z)Q_j(s)\,ds\right),\\
&(e^{\hat{C}(z)}\mathbf{Q}_\perp)' = e^{\hat{C}(z)}\hat{\Gamma}_c(z)^{-1}\left(\mathbf{F}_\perp + \sum_{j=1}^{n}\int_\epsilon^z \mathbf{D}'_\perp(s-z)Q_j(s)\,ds\right),\\
&\hat{C}(z) := -\int_0^z \hat{\Gamma}_c(s)^{-1}\hat{B}_c(s)\,ds,\\
&\mathbf{Q}(\epsilon) = 0.
\end{aligned} \tag{2.33}$$

After integration, we get

$$\begin{aligned}
&Q_1(z) = R_1(z) + J_1(\mathbf{Q}),\\
&\mathbf{Q}_\perp(z) = \mathbf{R}_\perp(z) + J_\perp(\mathbf{Q}),
\end{aligned} \tag{2.34}$$

with

$$\begin{aligned}
&J_1(\mathbf{Q}) = z^\beta \int_\epsilon^z t^{-\beta-1}\sum_{j=1}^{n}\int_\epsilon^t Q_j(s)D'_{1j}(t-s)\,ds\,dt,\\
&J_\perp(\mathbf{Q})(z) := e^{-\hat{C}(z)}\int_\epsilon^z e^{\hat{C}(t)}\hat{\Gamma}_c(t)^{-1}\left(\sum_{j=1}^{n}\int_\epsilon^z \mathbf{D}'_\perp(s-z)Q_j(s)\,ds\right)dt,\\
&\mathbf{R}_\perp(z) := e^{-\hat{C}(z)}\int_\epsilon^z e^{\hat{C}(t)}\hat{\Gamma}_c(t)^{-1}\mathbf{F}_\perp(t)\,dt,\\
&R_1(z) = z^\beta\int_\epsilon^z t^{-\beta-1}F_1(t)\,dt, \qquad (\beta \neq 1)\\
&R_1(z) = F_1(0)z\ln z + z\int_\epsilon^z \frac{F_1(s) - F_1(0)}{s}\,ds \qquad (\beta = 1).
\end{aligned} \tag{2.35}$$

Consider the following space of functions:

$$\begin{aligned}
&\mathcal{T}_\beta = \left\{\mathbf{Q} \text{ analytic in } \mathbb{D}_\epsilon \cap (\mathcal{D}-1) : \mathbf{Q} = z^\beta\mathbf{A}(z) + \mathbf{B}(z)\right\} \text{ for } \beta \neq 1 \text{ and}\\
&\mathcal{T}_1 = \left\{\mathbf{Q} \text{ analytic in } \mathbb{D}_\epsilon \cap (\mathcal{D}-1) : \mathbf{Q} = z\ln z\mathbf{A}(z) + z\mathbf{B}(z)\right\},
\end{aligned} \tag{2.36}$$

where $\mathbf{A}, \mathbf{B}$ are analytic in $\mathbb{D}_\epsilon$. (The decomposition of $\mathbf{Q}$ in (2.36) is unambiguous since z^β and $z\ln z$ are not meromorphic in $\mathbb{D}_\epsilon$.) The norm

$$\|\mathbf{Q}\| = \sup\{|\mathbf{A}(z)|_\wedge, |\mathbf{B}(z)|_\wedge : z \in \mathbb{D}_\epsilon\} \tag{2.37}$$

makes $\mathcal{T}_\beta$ a Banach space.

For $A(z)$ analytic in $\mathbb{D}_\epsilon$, the following elementary identities are useful in what follows:

$$\int_\epsilon^z A(s)s^r\,ds = \text{Const} + z^{r+1}\int_0^1 A(zt)t^r dt = \text{Const} + z^{r+1}\,\text{Analytic}(z),$$
$$\int_0^z s^r \ln s\, A(s)\,ds = z^{r+1}\ln z\int_0^1 A(zt)t^r dt + z^{r+1}\int_0^1 A(zt)t^r \ln t\,dt, \tag{2.38}$$

where the second equality is obtained by differentiating, with respect to r, the first equality.

Using (2.38), it is straightforward to check that the right-hand side of (2.34) extends to a linear inhomogeneous operator on $\mathcal{T}_\beta$ with image in $\mathcal{T}_\beta$ and that the norm of J is $O(\epsilon)$ for small ϵ. For instance, one of the terms in J for $\beta = 1$,

$$\begin{aligned} & z\int_0^z t^{-2}\int_0^t s\ln s\, A(s)D'(t-s)\,ds \\ = {} & z^2\ln z\int_0^1\int_0^1 \sigma A(z\tau\sigma)D'(z\tau - z\tau\sigma)d\sigma d\tau \\ & + z^2\int_0^1 d\tau\ln\tau\int_0^1 d\sigma\sigma(1+\ln\sigma)A(z\tau\sigma)D'(z\tau - z\tau\sigma) \end{aligned} \tag{2.39}$$

manifestly in $\mathcal{T}_\beta$ if A is analytic in $\mathbb{D}_\epsilon$. Comparing with (2.36), the extra power of z accounts for a norm $O(\epsilon)$ for this term.

Therefore, in (2.33), $(1-J)$ is invertible and the solution $\mathbf{Q} \in \mathcal{T}_\beta \subset \mathcal{L}\left(\mathcal{D}\right)$. In view of the uniqueness of $\mathbf{Y}_0$ (cf. Proposition 6), the rest of the proof of Lemma 20 is immediate. ■

2.4 The solutions of (1.25) on $[0, 1+\epsilon]$

Let $\mathbf{Y}_0$ be the solution given by Proposition 6, take ϵ small enough, and denote by $\mathcal{O}_\epsilon$ a neighborhood in $\mathbb{C}$ of width ϵ of the interval $[0, 1+\epsilon]$.

Remark 21. $\mathbf{Y}_0 \in L^1(\mathcal{O}_\epsilon)$. As $\phi \to \pm 0$, $\mathbf{Y}_0(pe^{i\phi}) \to \mathbf{Y}_0^{\pm}(p)$ in the sense of $L^1([0, 1+\epsilon])$ and also in the sense of pointwise convergence for $p \neq 1$, where

$$\mathbf{Y}_0^{\pm} := \begin{cases} \mathbf{Y}_0(p) & p < 1 \\ (1-p\pm 0i)^{\beta-1}\mathbf{a}_1(p) + \mathbf{a}_2(p) & p > 1 \end{cases} \qquad (\beta \neq 1)$$
$$\mathbf{Y}_0^{\pm} := \begin{cases} \mathbf{Y}_0(p) & p < 1 \\ \ln(1-p\pm 0i)\mathbf{a}_1(p) + \mathbf{a}_2(p) & p > 1 \end{cases} \qquad (\beta = 1). \tag{2.40}$$

Moreover, $\mathbf{Y}_0^{\pm}$ are L^1_{loc} solutions of the convolution equation (1.25) on the interval $[0, 1+\epsilon]$.

The proof is immediate from Lemma 20 and Proposition 13.

Proposition 22. For any $\lambda \in \mathbb{C}$, the combination $\mathbf{Y}_\lambda = \lambda \mathbf{Y}_0^+ + (1-\lambda)\mathbf{Y}_0^-$ is a solution of (1.25) on $[0, 1+\epsilon]$. □

Proof. For $p \in [0,1) \cup (1, 1+\epsilon]$, let $\mathbf{y}_\lambda(p) := \mathbf{Y}_\lambda - \mathbf{H}(p)$. Since $\mathbf{y}_\lambda^{*2} = 0$, the equation (1.25) is actually linear in $\mathbf{y}_\lambda$ (compare with (2.27)). ■

Note. We consider the application $\mathcal{Y} := \mathbf{y}_0 \mapsto \mathbf{Y}_\lambda$ and require that it is compatible with complex conjugation: $\mathcal{Y}(\overline{\mathbf{y}_0}) = \overline{\mathcal{Y}(\mathbf{y}_0)}$. We get $\Re\lambda = 1/2$. It is natural to choose $\lambda = 1/2$ to make the linear combination a true average. This choice corresponds, on $[0, 1+\epsilon]$, to the balanced averaging (1.16).

Remark 23. For any $\delta > 0$ there is a constant $C(\delta)$ such that, for large b,

$$\|(\mathbf{Y}_0^{ba})^{*\mathbf{l}}\|_u < C(\delta)\delta^{|\mathbf{l}|} \quad \forall \mathbf{l} \text{ with } |\mathbf{l}| > 1 \tag{2.41}$$

($\|\|_u$ is taken on the interval $[0, 1+\epsilon]$).

Without loss of generality, assume that $l_1 > 1$. Using the notation (2.29),

$$\begin{aligned} &\left\| \int_0^p (\mathbf{Y}_0)_1^{ba}(s)(\mathbf{Y}_0^{ba})^{*\bar{\mathbf{l}}^1}(p-s)\, ds \right\|_u \\ &\le \left\| \int_0^{\frac{p}{2}} (\mathbf{Y}_0^{ba})_1(s)(\mathbf{Y}_0^{ba})^{*\bar{\mathbf{l}}^1}(p-s)\, ds \right\|_{u_2} + \left\| \int_0^{\frac{p}{2}} (\mathbf{Y}_0)_1(p-s)(\mathbf{Y}_0^{ba})^{*\bar{\mathbf{l}}^1}(s)\, ds \right\|_{u_2} \end{aligned} \tag{2.42}$$

($\|\|_{u_2}$ refers to the interval $p \in [0, 1/2 + \epsilon/2]$). The first u_2 norm can be estimated directly using Corollary 11, whereas we majorize the second one by

$$\|(\mathbf{Y}_0^{ba})_1\|_b \|(\mathbf{Y}_0^{ba})^{*\bar{\mathbf{l}}^1}(x)\|_{u_2}$$

and apply Corollary 11 to it for $|\mathbf{l}| > 2$. (If $|\mathbf{l}| = 2$ simply observe that $(\mathbf{Y}_0^{ba})^{*\mathbf{l}}$ is analytic on $[0, 1/2 + \epsilon/2]$.)

Lemma 24. The set of all solutions of (1.25) in $L^1_{\mathrm{loc}}([0, 1+\epsilon])$ is parameterized by a complex constant C and is given by

$$\mathbf{Y}_0(p) = \begin{cases} \mathbf{Y}_0^{ba}(p) \text{ for } p \in [0,1) \\ \mathbf{Y}_0^{ba}(p) + C(p-1)^{\beta-1}\mathbf{A}(p) \text{ for } p \in (1, 1+\epsilon] \end{cases} \tag{2.43}$$

for $\beta \neq 1$ or, for $\beta = 1$,

$$\mathbf{Y}_0(p) = \begin{cases} \mathbf{Y}_0^{ba}(p) \text{ for } p \in [0,1) \\ \mathbf{Y}_0^{ba}(p) + C(p-1)\mathbf{A}(p) \text{ for } p \in (1, 1+\epsilon] \end{cases} \tag{2.43$'$}$$

where $\mathbf{A}$ extend analytically in a neighborhood of $p = 1$. Different values of C correspond to different solutions.

This result remains true if $\mathbf{Y}_0^{ba}$ is replaced by any other combination $\mathbf{Y}_\lambda := \lambda \mathbf{Y}_0^+ + (1-\lambda)\mathbf{Y}_0^-$, $\lambda \in \mathbb{C}$. □

Proof. We look for solutions of (1.25) in the form

$$\mathbf{Y}^{ba}(p) + \mathbf{h}(p-1). \tag{2.44}$$

By Lemma 20, $\mathbf{h}(p-1) = 0$ for $p < 1$. Note that

$$\mathcal{N}(\mathbf{Y}_0^{ba} \circ \tau_{-1} + \mathbf{h})(z) = \mathcal{N}(\mathbf{Y}_0^{ba})(1+z) + \sum_{j=1}^{n} \int_0^z h_j(s)\mathbf{D}_j(z-s)\,ds, \tag{2.45}$$

where the $\mathbf{D}_j$ are given in (2.29), and by Remark 2.41 all the infinite sums involved are uniformly convergent. For $z < \epsilon$, (1.25) translates to (compare with (2.27))

$$-(1+z)\mathbf{h}(z) = -\hat{\Lambda}\mathbf{h}(z) - \hat{B}\int_0^z \mathbf{h}(s)\,ds + \sum_{j=1}^{n} \int_0^z h_j(s)\mathbf{D}_j(z-s)\,ds. \tag{2.46}$$

Let

$$\mathbf{Q}(z) := \int_0^z \mathbf{h}(s)\,ds. \tag{2.47}$$

As we are looking for solutions $\mathbf{h} \in L^1$, we have $\mathbf{Q} \in AC[0,\epsilon]$ and $\mathbf{Q}(0) = 0$. Following the same steps as in the proof of Lemma 20, we get the system of equations

$$\begin{aligned}
&(z^{-\beta}Q_1(z))' = z^{-\beta-1}\sum_{j=1}^{n}\int_0^z D'_{1j}(z-s)Q_j(s)\,ds,\\
&(e^{\hat{C}(z)}\mathbf{Q}_\perp)' = e^{\hat{C}(z)}\hat{\Gamma}_c(z)^{-1}\sum_{j=1}^{n}\int_0^z \mathbf{D}'_\perp(z-s)Q_j(s)\,ds,\\
&\hat{C}(z) := -\int_0^z \hat{\Gamma}_c(s)^{-1}\hat{B}_c(s)\,ds,\\
&\mathbf{Q}(0) = 0,
\end{aligned} \tag{2.48}$$

which by integration gives

$$(\hat{1} + J)\mathbf{Q}(z) = C\mathbf{R}(z), \tag{2.49}$$

where $C \in \mathbb{C}$ and

$$\begin{aligned}
(J(\mathbf{Q}))_1(z) &= z^{\beta}\int_0^z t^{-\beta-1}\sum_{j=1}^{n}\int_0^t Q_j(s)D'_{1j}(t-s)\,ds\,dt,\\
J(\mathbf{Q})_{\perp}(z) &:= e^{-\hat{C}(z)}\int_0^z e^{\hat{C}(t)}\hat{\Gamma}_c(t)^{-1}\left(\sum_{j=1}^{n}\int_0^z \mathbf{D}'_{\perp}(z-s)Q_j(s)\,ds\right)dt,\\
\mathbf{R}_{\perp} &= 0,\\
R_1(z) &= z^{\beta}.
\end{aligned} \tag{2.50}$$

First we note the presence of an arbitrary constant C in (2.49). (Unlike in Lemma 20, when the initial condition given at $z = \epsilon$ was determining the integration constant, now the initial condition $\mathbf{Q}(0) = 0$ is satisfied for all C.)

For small ϵ, the norm of the operator J defined on $AC[0, \epsilon]$ is $O(\epsilon)$, as in the proof of Lemma 20. Given C, the solution of the system (2.48) is unique and can be written as

$$\mathbf{Q} = C\mathbf{Q}_0; \qquad \mathbf{Q}_0 := (\hat{1} + J)^{-1}\mathbf{R} \neq 0. \tag{2.51}$$

It remains to find the analytic structure of $\mathbf{Q}_0$. We now introduce the space

$$\mathcal{T} = \left\{\mathbf{Q} : [0, \epsilon) \mapsto \mathbb{C}^n : \mathbf{Q} = z^{\beta}\mathbf{A}(z)\right\}, \tag{2.52}$$

where $\mathbf{A}(z)$ extends to an analytic function in $\mathbb{D}_{\epsilon}$. With the norm (2.37) (with $\mathbf{B} \equiv \mathbf{0}$), $\mathcal{T}$ is a Banach space. As in the proof of Lemma 20, the operator J extends naturally to $\mathcal{T}$ where it has a norm $O(\epsilon)$ for small ϵ. It follows immediately that

$$\mathbf{Q}_0 \in \mathcal{T}. \tag{2.53}$$

The formulas (2.43) and (2.43′) follow from (2.44) and (2.47). ■

Remark 25. If $S_{\beta} \neq 0$ (cf. Lemma 20), then the *general* solution of (1.25) is given by

$$\mathbf{Y}_0(p) = (1 - \lambda)\mathbf{Y}_0^{+}(p) + \lambda\mathbf{Y}_0^{-}(p) \tag{2.54}$$

with $\lambda \in \mathbb{C}$.

Indeed, if $\mathbf{a}_1 \not\equiv 0$ (cf. Lemma 20), we get at least two distinct solutions of (2.49) (i.e., two distinct values of C) by taking different values of λ in (2.54). The remark follows from (2.53), (2.52), and Lemma 24.

2.5 The solutions of (1.25) on $[0,\infty)$

In this section we show that the leading asymptotic behavior of $\mathbf{Y}_p$, as $p \to 1_+$, determines a unique solution of (1.25) in $L^1_{loc}(\mathbb{R}^+)$. Furthermore, any L^1_{loc} solution of (1.25) is exponentially bounded at infinity and thus Laplace transformable. We also study some properties of these solutions and of their Laplace transforms.

Let $\mathbf{H}$ be a solution of (1.25) on an interval $[0, 1+\epsilon]$, which we extend to $\mathbb{R}^+$ letting $\mathbf{H}(p) = 0$ for $p > 1+\epsilon$. For a large enough b, define

$$\mathcal{S}_{\mathbf{H}} := \{f \in L^1_{loc}([0,\infty)) : f(p) = \mathbf{H}(p) \text{ on } [0, 1+\epsilon]\} \tag{2.55}$$

and

$$\mathcal{S}_0 := \{f \in L^1_{loc}([0,\infty)) : f(p) = 0 \text{ on } [0, 1+\epsilon]\}. \tag{2.56}$$

We extend $\mathbf{H}$ to $\mathbb{R}^+$ by putting $\mathbf{H}(p) = 0$ for $p > 1+\epsilon$; for $p \geq 1+\epsilon$ (1.25) reads

$$-p(\mathbf{H}+\mathbf{h}) = F_0 - \hat{\Lambda}(\mathbf{H}+\mathbf{h}) - \hat{B}\int_0^p (\mathbf{H}+\mathbf{h})(s)\,ds + \mathcal{N}(\mathbf{H}+\mathbf{h}) \tag{2.57}$$

with $\mathbf{h} \in \mathcal{S}_0$, or

$$\mathbf{h} = -\mathbf{H} + (\hat{\Lambda} - p)^{-1}\left(F_0 - \hat{B}\int_0^p (\mathbf{H}+\mathbf{h})(s)\,ds + \mathcal{N}(\mathbf{H}+\mathbf{h})\right) := \mathcal{M}(\mathbf{h}). \tag{2.58}$$

For small $\phi_0 > 0$ and $0 \leq \rho_1 < \rho_2 \leq \infty$, consider the truncated sectors

$$S^{\pm}_{(\rho_1,\rho_2)} := \{z : z = \rho e^{\pm i\phi}, \rho_1 < \rho < \rho_2;\ 0 \leq \phi < \phi_0\} \tag{2.59}$$

and the spaces of functions analytic in $S^{\pm}_{(\rho_1,\rho_2)}$ and continuous in its closure,

$$\mathcal{T}^{\pm}_{\rho_1,\rho_2} = \left\{ f : f \in C(\overline{S_{(\rho_1,\rho_2)}});\ f \text{ analytic in } S^{\pm}_{(\rho_1,\rho_2)} \right\}, \tag{2.60}$$

which are Banach spaces with respect to $\|\,\|_u$ on compact subsets of $\overline{S_{(\rho_1,\rho_2)}}$.

Proposition 26. (i) Given $\mathbf{H}$, the equation (2.58) has a unique solution in $L^1_{loc}[1+\epsilon,\infty)$. For large b, this solution is in $L^1_b([1+\epsilon,\infty))$ and thus Laplace transformable.

(ii) Let $\mathbf{Y}_0$ be the solution defined in Proposition 6. Then

$$\mathbf{Y}_0^{\pm}(p) := \lim_{\phi\to\pm 0} \mathbf{Y}_0(pe^{i\phi}) \in C(\mathbb{R}^+\backslash\{1\}) \cap L^1_{loc}(\mathbb{R}^+) \tag{2.61}$$

(and the limit exists pointwise on $\mathbb{R}^+\backslash\{1\}$ and in $L^1_{loc}(\mathbb{R}^+)$).

Furthermore, $\mathbf{Y}_0^{\pm}$ are particular solutions of (1.25) and

$$\mathbf{Y}_0^{\pm}(p) = (1-p)^{\beta-1}\mathbf{a}^{\pm}(p) \qquad (\beta \neq 1),$$
$$\mathbf{Y}_0^{\pm}(p) = \ln(1-p)\,\mathbf{a}^{\pm}(p) \qquad (\beta = 1), \tag{2.62}$$

where $\mathbf{a}^{\pm} \in \mathcal{T}^{\pm}_{0,\infty}$. □

Proof. Note first that by Proposition 13, $\mathcal{M}$ (equation (2.58)) is well defined on $\mathcal{S}_0$ (equation (2.56)). Moreover, since $\mathbf{H}$ is a solution of (1.25) on $[0,1+\epsilon)$, we have, for $\mathbf{h}_0 \in \mathcal{S}_0$, $\mathcal{M}(\mathbf{h}) = 0$ almost everywhere on $[0,1+\epsilon)$, i.e.,

$$\mathcal{M}(\mathcal{S}_0) \subset \mathcal{S}_0.$$

Remark 27. For large b, $\mathcal{M}$ is a contraction in a small neighborhood of the origin in $\|\,\|_{u,b}$.

Indeed, $\sup\{\|(\hat{\Lambda}-p)^{-1}\|_{\mathbb{C}^n\mapsto\mathbb{C}^n} : p \geq 1+\epsilon\} = O(\epsilon^{-1})$ so that

$$\|\mathcal{M}(\mathbf{h}_1) - \mathcal{M}(\mathbf{h}_2)\|_{u,b} \leq \frac{\text{Const}}{\epsilon}\|\mathcal{N}(\mathbf{f}+\mathbf{h}) - \mathcal{N}(\mathbf{f})\|_{u,b}. \tag{2.63}$$

The rest follows from (2.18), Proposition 13, and Remark 10 applied to $\mathbf{H}$.

The existence of a solution of (2.58) in $\mathcal{S}_0 \cap L^1_b([0,\infty))$ for large enough b is now immediate.

Uniqueness in L^1_{loc} is tantamount to uniqueness in $L^1([1+\epsilon,K]) = L^1_b([1+\epsilon,K]$, for all $K-1-\epsilon \in \mathbb{R}^+$. Now, assuming $\mathcal{M}$ has two fixed points in $L^1_b([1+\epsilon,K])$, by Remark 10, we can choose b large enough so that these solutions have arbitrarily small norm, in contradiction with Remark 27.

We now turn to (ii). For $p<1$, $\mathbf{Y}_0^{\pm}(p) = \mathbf{Y}_0(p)$. For $p \in (1,1+\epsilon)$, the result follows from Lemma 20. Noting (in view of the estimate (2.15)) that $\mathcal{M}(\mathcal{T}^{\pm}{}_{1+\epsilon,\infty}) \subset \mathcal{T}^{\pm}{}_{1+\epsilon,\infty}$, the rest of the proof follows from the Remark 27 and Lemma 20. ■

Proposition 28. There is a one-parameter family of solutions of equation (1.25) in $L^1_{loc}[0,\infty)$, branching off at $p=1$, and in a neighborhood of $p=1$ all solutions are of the form (2.43), (2.43′). The general solution of (1.25) is Laplace transformable for large b and the Laplace transform is a solution of the original differential equation in the half-space $\Re(x) > b$. □

Note. As of now, the correspondence (2.43), (2.43′) with the balanced average (1.16) is proven only near $p = 1$; the complete correspondence is established in Section 2.7.

Proof. Let $\mathbf{Y}$ be any solution of (1.25). By Lemma 24, Remark 10, and Proposition 26, b large implies that $\mathbf{Y} \in L^1_b([0,\infty))$ (thus $\mathcal{L}\mathbf{Y}$ exists), that $\|\mathbf{Y}\|_b$ is small, and, in particular, that the sum defining $\mathcal{N}$ in (1.22) is convergent in $L^1_b(\mathbb{R}^+)$.

By Remark 8,

$$\mathcal{L}\left(\sum_{|\mathbf{l}|\geq 1}\mathbf{G}_{\mathbf{l}} * \mathbf{Y}^{*\mathbf{l}} + \sum_{|\mathbf{l}|\geq 2}\mathbf{g}_{0,\mathbf{l}}\mathbf{Y}^{*\mathbf{l}}\right)(x)$$
$$= \sum_{|\mathbf{l}|\geq 1}(\mathcal{L}\mathbf{G}_{\mathbf{l}})(\mathcal{L}\mathbf{Y})^{\mathbf{l}}(x) + \sum_{|\mathbf{l}|\geq 2}\mathbf{g}_{0,\mathbf{l}}\left(\mathcal{L}\mathbf{Y}\right)^{\mathbf{l}} = \sum_{|\mathbf{l}|\geq 1}\mathbf{g}_{\mathbf{l}}(x)\mathbf{y}^{\mathbf{l}}(x) = \mathbf{g}(x,\mathbf{y}(x)) \tag{2.64}$$

(and $\mathbf{g}(x,\mathbf{y}(x))$ is analytic for $\Re(x) > b$). The rest is straightforward. ■

2.6 Correspondence with formal solutions

Finally, we consider formal solutions for the *large argument* of the differential equation, in the differential algebra generated by formal power series (in decreasing powers of the large variable) and (decreasing) exponentials, i.e., solutions as formal asymptotic expansions. The theory of formal solutions is classical ([9], [5], [12]; see also [8] for a vast and very interesting generalization). We only sketch the facts that are relevant to us.

The simplest formal solution of (1.1) is an asymptotic series $\hat{\mathbf{y}}_0$:

$$\hat{\mathbf{y}}_0 = \sum_{m=2}^{\infty}\frac{\mathbf{y}_{0,m}}{x^m}.$$

In view of the invertibility of $\hat{\Lambda}$, the coefficients $\{\mathbf{y}_{0,m}\}_{m\in\mathbb{N}} \subset \mathbb{C}^n$ can be determined uniquely by expanding in (1.1) in powers of $1/x$ and equating the coefficients of the x^{-m}, $m \geq 2$. The series $\hat{\mathbf{y}}_0$ is generically divergent.

Since we expect an n-*parameter* family of solutions, we look for further solutions as perturbations of $\hat{\mathbf{y}}_0$. Because of the uniqueness of $\hat{\mathbf{y}}_0$, a perturbation must be smaller than all powers of x^{-1}, i.e., "beyond all orders" of $\hat{\mathbf{y}}_0$.

Taking $\hat{\mathbf{y}} = \hat{\mathbf{y}}_0 + \hat{\mathbf{y}}_1$, we get, to the lowest order of approximation, $\hat{\mathbf{y}}_1' = -\hat{\Lambda}\hat{\mathbf{y}}_1$. The solutions to this approximate equation are linear combinations of $e^{-\lambda x}$, $\lambda \in \operatorname{spec}\hat{\Lambda}$. We only consider solutions $\hat{\mathbf{y}}_1$ that are (formally) small perturbations of $\hat{\mathbf{y}}_0$ in the half-plane $\Re(x) > 0$; this condition selects out the eigenvalue $\lambda = 1$.

Continuing the perturbative procedure until we reach a formal solution of (1.1), we end up with an exponential series

$$\hat{\mathbf{y}} = \hat{\mathbf{y}}_0 + \sum_{m=1}^{\infty} e^{-kx} \hat{\mathbf{y}}_k \tag{2.65}$$

where $\hat{\mathbf{y}}_k$ are formal power series. Substituting (2.65) into (1.1) and using the fact that $\hat{\mathbf{y}}_0$ is already a formal solution, we get, for $\hat{\mathbf{y}}_k, k \geq 1$,

$$\begin{aligned}
&\sum_{k=1}^{\infty} e^{-kx} \left[\hat{\mathbf{y}}_k' - \left(k - \hat{\Lambda} - \frac{1}{x}\hat{B} + \partial \mathbf{g}(x, \hat{\mathbf{y}}_0) \right) \hat{\mathbf{y}}_k \right] \\
&= \sum_{|\mathbf{l}|>1} \frac{\mathbf{g}^{(\mathbf{l})}(x, \hat{\mathbf{y}}_0)}{\mathbf{l}!} \left(\sum_{k=1}^{\infty} e^{-kx} \hat{\mathbf{y}}_k \right)^{\mathbf{l}} \\
&= \sum_{k=2}^{\infty} e^{-kx} \sum_{|\mathbf{l}|>1} \frac{\mathbf{g}^{(\mathbf{l})}(x, \hat{\mathbf{y}}_0)}{\mathbf{l}!} \sum_{\Sigma m=} \prod_{i=1}^{n} \prod_{j=1}^{l_i} (\hat{\mathbf{y}}_{m_{i,j}})_i .
\end{aligned} \tag{2.66}$$

Equating the coefficients of $e^{-kx}, k \geq 0$, we get the system (1.5).

By assumption, $\hat{\Lambda} - 1$ has a one-dimensional null-space. Thus, by (1.5), $\hat{\mathbf{y}}_1$ has the freedom of an arbitrary multiplicative constant. We make a definite choice of $\hat{\mathbf{y}}_1$ by requiring that the first component of the coefficient of the leading power of x is one.

Still by assumption, for $k \neq 1$, $\hat{\Lambda} - k$ is invertible, so that, taking $C = 1$, all $\hat{\mathbf{y}}_k$, $k \geq 1$, are uniquely determined. Letting C be arbitrary, we get instead $C\hat{\mathbf{y}}_1$ for $k = 1$, $C^2\hat{\mathbf{y}}_2$ for $k = 2$ (because of the condition $\sum m = 2$), etc., so that the general formal solution of the type (2.65) is

$$\hat{\mathbf{y}} = \hat{\mathbf{y}}_0 + \sum_{k=1}^{\infty} C^k e^{-kx} \hat{\mathbf{y}}_k .$$

The existence of formal exponential solutions has been considered in [9], [5], and [10], and a very comprehensive theory can be found in Ecalle [6], [8], [7].

The following proposition is a classical result and is a specialization of general theorems (see [10]).

Proposition 29. There is exactly a one-parameter family of solutions of (1.1) having the asymptotic behavior described by $\hat{\mathbf{y}}_0$ in the half-plane $\Re(x) > 0$. □

Proof. Any solution with the properties stated in Proposition 29 is inverse Laplace transformable and its inverse Laplace transform has to be one of the L^1_{loc} solutions of the convolution equation (1.25). The rest of the proof follows from Proposition 28. ■

Proposition 30. Let $\mathbf{Y}$ be any $L^1_{loc}(\mathbb{R}^+)$ solution of (1.25). For large b and some $\nu > 0$, the coefficients $\mathbf{d_m}$ in (1.27) are bounded by

$$|\mathbf{d_m}(p)|_\wedge \le e^{\mu p}\nu^{|\mathbf{m}|}. \qquad \square$$

Note that $\mathcal{L}^{-1}(\mathbf{g}^{(\mathbf{m})}(x,\mathbf{y})/\mathbf{m}!)$ is the coefficient of $\mathbf{Z}^{*\mathbf{m}}$ in the expansion of $\mathcal{N}(\mathbf{Y}+\mathbf{Z})$ in convolution powers of Z (1.22):

$$\begin{aligned}
&\left(\left(\sum_{|\mathbf{l}|\ge 2}\mathbf{g}_{0,\mathbf{l}}\cdot+\sum_{|\mathbf{l}|\ge 1}\mathbf{G}_{\mathbf{l}}*\right)(\mathbf{Y}+\mathbf{Z})^{*\mathbf{l}}\right)_{\mathbf{Z}^{*\mathbf{m}}}\\
&=\left(\left(\sum_{|\mathbf{l}|\ge 2}\mathbf{g}_{0,\mathbf{l}}\cdot+\sum_{|\mathbf{l}|\ge 1}\mathbf{G}_{\mathbf{l}}*\right)\sum_{0\le\mathbf{k}\le\mathbf{l}}\binom{\mathbf{l}}{\mathbf{k}}\mathbf{Z}^{*\mathbf{k}}\mathbf{Y}^{*(\mathbf{l}-\mathbf{k})}\right)_{\mathbf{Z}^{*\mathbf{m}}}\\
&=\left(\sum_{|\mathbf{l}|\ge 2}\mathbf{g}_{0,\mathbf{l}}\cdot+\sum_{|\mathbf{l}|\ge 1}\mathbf{G}_{\mathbf{l}}*\right)\sum_{\mathbf{l}\ge\mathbf{m}}\binom{\mathbf{l}}{\mathbf{m}}\mathbf{G}_{\mathbf{l}}*\mathbf{Y}^{*(\mathbf{l}-\mathbf{m})}
\end{aligned}\tag{2.67}$$

($\mathbf{m}$ is fixed) where $\mathbf{l}\ge\mathbf{m}$ means $l_i \ge m_i$, $i = 1\cdots n$ and $\binom{\mathbf{l}}{\mathbf{k}} := \prod_{i=1}^n\binom{l_i}{k_i}$.

Let ϵ be small and b large so that $\|\mathbf{Y}\|_b < \epsilon$. Then, for some constant K, estimate (cf. (2.1))

$$\begin{aligned}
&\left|\left(\sum_{II}\mathbf{g}_{0,\mathbf{l}}\cdot+\sum_{I}\mathbf{G}_{\mathbf{l}}*\right)\binom{\mathbf{l}}{\mathbf{m}}\mathbf{G}_{\mathbf{l}}*\mathbf{Y}^{*(\mathbf{l}-\mathbf{m})}\right|_\wedge \le \sum_I K e^{\mu|p|}(\mu\epsilon)^{|\mathbf{l}-\mathbf{m}|}\binom{\mathbf{l}}{\mathbf{m}}\\
&= \epsilon^{-|\mathbf{m}|}Ke^{\mu|p|}\prod_{i=1}^n\sum_{l_i\ge m_i}\binom{l_i}{m_i}(\mu\epsilon)^{l_i} = K\frac{e^{\mu|p|}\mu^{|\mathbf{m}|}}{(1-\epsilon\mu)^{|\mathbf{m}|+n}} < e^{\mu|p|}\nu^{|\mathbf{m}|}
\end{aligned}\tag{2.68}$$

(where $I(II) \equiv \{|\mathbf{l}| \ge 1(2); \mathbf{l}\ge\mathbf{m}\}$) for large enough ν.

For $k = 1$, $\mathbf{R}_1 = 0$ and equation (1.27) is (2.46) (with $p \leftrightarrow z$) but now on the whole line $\mathbb{R}^+$. For small z the solution is given by (2.51) (note that $\mathbf{D}_1 = \mathbf{d}_{(1,0,\dots,0)}$ and so on) and depends on the free constant C (2.51). We choose a value for C (the values of $\mathbf{Y}_1$ on $[0,\epsilon]$ are then determined), and we write the equation of $\mathbf{Y}_1$ for $p \ge \epsilon$:

$$\begin{aligned}
&(\hat\Lambda - 1 - p)\mathbf{Y}_1(p) + \hat B\int_\epsilon^p \mathbf{Y}_1(s)\,ds - \sum_{j=1}^n\int_\epsilon^p(\mathbf{Y}_1)_j(s)\mathbf{D}_j(p-s)\,ds\\
&= \mathbf{R}(p) := \int_0^\epsilon \mathbf{Y}_1(s)\,ds + \sum_{j=1}^n\int_0^\epsilon(\mathbf{Y}_1)_j(s)\mathbf{D}_j(p-s)\,ds
\end{aligned}\tag{2.69}$$

($\mathbf{R}$ only depends on the values of $\mathbf{Y}_1(p)$ on $[0,\epsilon]$). We write

$$(1 + J_1)\mathbf{Y}_1 = \hat Q_1^{-1}\mathbf{R} \tag{2.70}$$

with $Q_1 = 1 - \hat{\Lambda} + p$. The operator J_1 is defined by $(J_1\mathbf{Y}_1)(p) := 0$ for $p < \epsilon$, while, for $p > \epsilon$,

$$(J_1\mathbf{Y}_1)(p) := Q_1^{-1}\left(\hat{B}\int_\epsilon^p \mathbf{Y}_1(s)\,ds - \sum_{j=1}^{n}\int_\epsilon^p (\mathbf{Y}_1)_j(s)\mathbf{D}_j(p-s)\,ds\right).$$

By Proposition 19, (2.6), and Remark 10, noting that $\sup_{p>\epsilon}\|Q_1^{-1}\| = O(\epsilon^{-1})$, we find that $(1 + J_1)$ is invertible as an operator in L^1_b, since

$$\|J_1\|_{L^1_b \mapsto L^1_b} < \sup_{p>\epsilon}\|\hat{Q}_1^{-1}\|\left(\|\hat{B}\|\|1\|_b + n\max_{1\le j\le n}\|\mathbf{D}_j\|_b\right) \to 0 \quad \text{as } b \to \infty. \tag{2.71}$$

Given C, $\mathbf{Y}_1$ is therefore uniquely determined from (2.70) as an $L^1_b(\mathbb{R}^+)$ function.

The analytic structure of $\mathbf{Y}_1$ for small z is contained in in (2.43), (2.43′). As a result,

$$\mathcal{L}(\mathbf{Y}_1)(x) \sim C\sum_{k=0}^{\infty}\frac{\Gamma(k-\beta)}{x^{k-\beta}}\mathbf{a}_k, \tag{2.72}$$

where $\sum_{k=0}^{\infty}\mathbf{a}_k z^k$ is the series of $\mathbf{a}(z)$ near $z = 0$.

Correspondingly, we write (1.27) as

$$(1 + J_k)\mathbf{Y}_k = \hat{Q}_k^{-1}\mathbf{R}_k \tag{2.73}$$

with $\hat{Q}_k := (-\hat{\Lambda} + p + k)$ and

$$(J_k\mathbf{h})(p) := \hat{Q}_k^{-1}\left(\hat{B}\int_0^p \mathbf{h}(s)\,ds - \sum_{j=1}^{n}\int_0^p h_j(s)\mathbf{D}_j(p-s)\,ds\right), \tag{2.74}$$

$$\|J_k\|_{L^1_b \mapsto L^1_b} < \sup_{p\ge 0}\|\hat{Q}_k^{-1}\|\left(\|\hat{B}\|\|1\|_b + n\max_{1\le j\le n}\|\mathbf{D}_j\|_b\right). \tag{2.75}$$

Since $\sup_{p\ge 0}\|\hat{Q}_k^{-1}\| \to 0$ as $k \to \infty$, we have

$$\sup_{k\ge 1}\left\{\|J_k\|_{L^1_b \mapsto L^1_b}\right\} \to 0 \quad \text{as } b \to \infty. \tag{2.76}$$

We have thus proved the following result.

Proposition 31. For large b, $(1 + J_k)$, $k \ge 1$ are simultaneously invertible in L^1_b; cf. (2.76). For specified $\mathbf{Y}_0$ and C, $\mathbf{Y}_k, k \ge 1$ are uniquely determined, and moreover, for $k \ge 2$,

$$\|\mathbf{Y}_k\|_b \le \frac{\sup_{p\ge 0}\|\hat{Q}_k^{-1}\|}{1 - \sup_{k\ge 1}\|J_k\|_{L^1_b \mapsto L^1_b}}\|\mathbf{R}_k\|_b := K\|\mathbf{R}_k\|_b. \tag{2.77}$$

□

Note. As we will see later, there only is a *one-parameter* freedom in $\mathbf{Y}_k$: a change in $\mathbf{Y}_0$ can be compensated by a corresponding change in C.

Because of condition $\sum m = k$ in the definition of $\mathbf{R}_k$, we get, by an easy induction, the homogeneity relation with respect to the free constant C,

$$\mathbf{Y}_k^{[C]} = C^k \mathbf{Y}_k^{[C=1]} =: C^k \mathbf{Y}_k. \tag{2.78}$$

Proposition 32. For any $\delta > 0$, there is a large enough b, so that

$$\|\mathbf{Y}_k\|_b < \delta^k, \qquad k = 0, 1, \ldots. \tag{2.79}$$

Each $\mathbf{Y}_k$ is Laplace transformable and $\mathbf{y}_k = \mathcal{L}(\mathbf{Y}_k)$ solve (1.5). □

Proof. We first show inductively that the $\mathbf{Y}_k$ are bounded. Choose r small enough and b large so that $\|\mathbf{Y}_0\|_b < r$. Note that in the expression of $\mathbf{R}_k$, only $\mathbf{Y}_i$ with $i < k$ appear. We show by induction that $\|\mathbf{Y}_k\|_b < r$ for all k. Using (2.77), (1.27), the explanation to (1.5), and Proposition 30, we get

$$\|\mathbf{Y}_k\|_b < K\|\mathbf{R}_k\|_b \le \sum_{|\mathbf{l}|>1} \mu^{|\mathbf{l}|} r^k \sum_{\Sigma m = k} 1 \le r^k \left(\sum_{l>1} \binom{l}{k} \mu^l \right)^n \le (r(1+\mu)^n)^k < r \tag{2.80}$$

if r is small, which completes this induction step. But now if we look again at (2.80), we see that in fact $\|\mathbf{Y}_k\|_b \le (r(1+\mu)^n)^k$. Choosing r small enough (and to that end, b large enough), the first part of Proposition 32 follows. Laplace transformability as well as the fact that $\mathbf{y}_k$ solve (1.5) follow immediately from (2.79) (observe again that, given k, there are only finitely many terms in the sum in $\mathbf{R}_k$). ■

Remark 33. The series

$$\sum_{k=0}^{\infty} C^k (\mathbf{Y}_k \cdot \mathcal{H}) \circ \tau_k \tag{2.81}$$

is convergent in L_b^1 for large b, and thus the sum is Laplace transformable. By Remark 8 and Proposition 2.79,

$$\mathcal{L}\left(\sum_{k=0}^{\infty} C^k (\mathbf{Y}_k \mathcal{H}) \circ \tau_k \right)(x) = \sum_{k=0}^{\infty} e^{-kx} \mathcal{L}(\mathbf{Y}_k)(x) \tag{2.82}$$

is uniformly convergent for large x (together with its derivatives with respect to x). Thus (by its formal construction), (2.82) is a solution of (1.1).

Alternatively, we could have checked in a straightforward way that the series (2.81), truncated to order N, is a solution of the convolution equation (1.25) on the interval $p \in [0, N)$, and in view of the $L^1_b(\mathbb{R}^+)$ (or even L^1_{loc}) convergence, it has to be one of the general solutions of the convolution equation and therefore provide a solution to (1.1).

Proof of Proposition 1 (ii). We now show (1.13). This is done from the system (1.27) by induction on k. For $k = 0$ and $k = 1$, the result follows from Proposition 6 and Proposition 21. For the induction step we consider the operator J_k (2.74) on the space

$$\mathcal{T}_k = \left\{ \mathbf{Q} : [0, \epsilon) \mapsto \mathbb{C} : \mathbf{Q}(z) = z^{k\beta-1}\mathbf{A}_k(z) \right\}, \tag{2.83}$$

where $\mathbf{A}_k$ extends as an analytic function in a neighborhood $\mathbb{D}_\epsilon$ of $z = 0$. Endowed with the norm

$$\|\mathbf{Q}\|_{\mathcal{T}_k} := \sup_{z\in\mathbb{D}_\epsilon} |\mathbf{A}_k(z)|_\wedge,$$

$\mathcal{T}_k$ is a Banach space.

Remark 34. For $k \in \mathbb{N}$ the operators J_k in (2.74) extend continuously to $\mathcal{T}_k$ and their norm is $O(\epsilon)$. The functions $\mathbf{R}_k$, $k \in \mathbb{N}$ (cf. (2.73), (1.27)), belong to $\mathcal{T}_k$. Thus for $k \in \mathbb{N}$, $\mathbf{Y}_k \in \mathcal{T}_k$.

If A, B are analytic, then for $z < \epsilon$,

$$\int_0^z ds\, s^{k\beta-1}A(s)B(z-s) = z^{k\beta}\int_0^1 dt\, t^r A(zt)B(z(1-t)) \tag{2.84}$$

is in $\mathcal{T}_k$ with norm $O(\epsilon)$, and the assertion about J_k follows easily. Therefore $\mathbf{Y}_k \in \mathcal{T}_k$ if $\mathbf{R}_k \in \mathcal{T}_k$. We prove both these properties by induction, and (by the homogeneity of $\mathbf{R}_k$ and the fact that $\mathbf{R}_k$ depends only on $\mathbf{Y}_m$, $m < k$) this amounts to checking that if $\mathbf{Y}_m \in \mathcal{T}_m$ and $\mathbf{Y}_n \in \mathcal{T}_n$, then

$$\mathbf{Y}_m * \mathbf{Y}_n \in \mathcal{T}_{m+n}.$$

This follows from the identity

$$\int_0^z ds\, s^r A(s)(z-s)^q B(z-s) = z^{r+q+1}\int_0^1 dt\, t^r(1-t)^q A(zt)B(z-zt). \qquad \blacksquare$$

It is now easy to see that $\mathcal{L}_\phi\mathcal{B}\hat{\mathbf{y}}_k \sim \hat{\mathbf{y}}_k$ (cf. Theorem 2). Indeed, note that in view of Remark 34 and Proposition 32, $\mathcal{L}(\mathbf{Y}_k)$ have asymptotic power series that can be differentiated for large x in the positive half-plane. Since $\mathcal{L}(\mathbf{Y}_k)$ are true solutions of the system (1.5), their asymptotic series are formal solutions of (1.5), and by the uniqueness of the formal solution of (1.5) once C is given, the property follows.

In the next subsection, we prove that the general solution of the system (1.5) can be obtained by means of Borel transform of formal series and analytic continuation.

We define $\mathbf{Y}^+$ to be the function defined in Proposition 26, extended in $\mathcal{D} \cap \mathbb{C}^+$ by the unique solution of (1.25) $\mathbf{Y}_0$ provided by Proposition 6. (We define $\mathbf{Y}^-$ correspondingly.)

By Proposition 26 (ii), $\mathbf{Y}^{\pm}$ are solutions of (1.25) on $[0, \infty)$ (cf. (2.60)). By Lemma 24, any solution on $[0, \infty)$ can be obtained from, say, $\mathbf{Y}^+$ by choosing C and then solving uniquely (2.58) on $[1 + \epsilon, \infty)$ (Proposition 26). We now show that the solutions of (2.70), (2.73) are continuous boundary values of functions analytic in a region bounded by $\mathbb{R}^+$.

Remark 35. The function $\mathbf{D}(s)$ defined in (2.29), by substituting $\mathbf{H} = \mathbf{Y}^{\pm}$, is in $\mathcal{T}^{\pm}_{0,\infty}$ (cf. (2.60)).

By Proposition 26 (ii), it is easy to check that if $\mathbf{H}$ is any function in $\mathcal{T}^+_{0,A}$, then $\mathbf{Y}^+ * \mathbf{\Omega} \in \mathcal{T}^+_{0,A}$. Thus, with $\mathbf{H} = \mathbf{Y}^+$, all the terms in the infinite sum in (2.29) are in $\mathcal{T}^+_{0,A}$. For fixed $A > 0$, taking b large enough, the norm ρ_b of $\mathbf{Y}^+$ in L^1_b can be made arbitrarily small uniformly in all rays in $S^+_{0,A}$ ((2.60), (Proposition 26)). Then by Corollary 11 and Proposition 26 (ii), the uniform norm of each term in the series (2.29) can be estimated by Const $\rho_b^{|\mathbf{l}-1|}\nu^{|\mathbf{l}|}$, and thus the series converges uniformly in ${\mathcal{T}^+}_{0,\infty}$, for large b.

Lemma 36. (i) The system (1.27) with $\mathbf{Y}_0 = \mathbf{Y}^+$ (or $\mathbf{Y}^-$) and given C (say, $C = 1$) has a unique solution in $L^1_{\mathrm{loc}}(\mathbb{R}^+)$, namely, $\mathbf{Y}^+_k$, ($\mathbf{Y}^-_k$, resp.), $k \in \mathbb{N}$. Furthermore, for large b and all k, $\mathbf{Y}^+_k \in \mathcal{T}^+_{0,\infty}$ ($\mathbf{Y}^-_k \in \mathcal{T}^-_{0,\infty}$) (cf. (2.60)).

(ii) The general solution of the equation (1.25) in $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ can be written in either of the following forms:

$$\mathbf{Y}^+ + \sum_{k=1}^{\infty} C^k (\mathbf{Y}^+_k \cdot \mathcal{H}) \circ \tau_k$$

$$\mathbf{Y}^- + \sum_{k=1}^{\infty} C^k (\mathbf{Y}^-_k \cdot \mathcal{H}) \circ \tau_k. \tag{2.85}$$

□

Proof. (i) The first part follows from the same arguments as Proposition 31. For the last statement, it is easy to see (cf. (2.84)) that $J_k \mathcal{T}^+_{0,\infty} \subset \mathcal{T}^+_{0,\infty}$, and by Proposition 2.6 the inequalities (2.75), (2.76) hold for $\|\|_{\mathcal{T}_{0,A} \mapsto \mathcal{T}_{0,A}}$ (A arbitrary), replacing $\|\|_{L^1_b \mapsto L^1_b}$.

(ii) We already know that $\mathbf{Y}^+$ solves (1.27) for $k = 0$. For $k > 0$, by (i), $C^k \mathbf{Y}_k \in \mathcal{T}_{0,\infty}$ and so, by continuity, the boundary values of $\mathbf{Y}^+_k$ on $\mathbb{R}^+$ solve the system (1.27) on $\mathbb{R}^+$ in L^1_{loc}. The rest of (ii) follows from Lemma 24, Proposition 26, and the arbitrariness of C in (2.85) (cf. also (2.51)). ■

2.7 Analytic structure and averaging

Having the general structure of the solutions of (1.25) given in Proposition 3 and in Lemma 36, we can obtain various analytic identities. The function $\mathbf{Y}_0^{\pm} := \mathbf{Y}^{\pm}$ has been defined in the previous section.

Proposition 37. For $m \geq 0$,

$$\mathbf{Y}_m^- = \mathbf{Y}_m^+ + \sum_{k=1}^{\infty} \binom{m+k}{m} S_\beta^k (\mathbf{Y}_{m+k}^+ \cdot \mathcal{H}) \circ \tau_k. \tag{2.86}$$

□

Proof. $\mathbf{Y}_0^-(p)$ is a particular solution of (1.25). It follows from Lemma 36 that the following identity holds on $\mathbb{R}^+$:

$$\mathbf{Y}_0^- = \mathbf{Y}_0^+ + \sum_{k=1}^{\infty} S_\beta^k (\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k \tag{2.87}$$

since, by (1.29) and (1.12), (2.87) holds for $p \in (0, 2)$.

By Lemma 36 for any C_+ there is a C_- such that

$$\mathbf{Y}_0^+ + \sum_{k=1}^{\infty} C_+^k (\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k = \mathbf{Y}_0^- + \sum_{k=1}^{\infty} C_-^k (\mathbf{Y}_k^- \cdot \mathcal{H}) \circ \tau_k. \tag{2.88}$$

To find the relation C_+ and C_-, we take $p \in (1, 2)$; we get, comparing with (2.87),

$$\mathbf{Y}_0^+(p) + C_+ \mathbf{Y}_1(p-1) = \mathbf{Y}_0^-(p) + C_- \mathbf{Y}_1(p-1) \Rightarrow C_+ = C_- + S_\beta, \tag{2.89}$$

whence, for any $C \in \mathbb{C}$,

$$\mathbf{Y}_0^+ + \sum_{k=1}^{\infty} (C + S_\beta)^k (\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k = \mathbf{Y}_0^- + \sum_{k=1}^{\infty} C^k (\mathbf{Y}_k^- \cdot \mathcal{H}) \circ \tau_k. \tag{2.90}$$

Differentiating m times with respect to C and taking $C = 0$, we get

$$\sum_{k=m}^{\infty} \frac{k!}{(k-m)!} S_\beta^{k-m} (\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k = m! (\mathbf{Y}_m^- \cdot \mathcal{H}) \circ \tau_m,$$

from which we obtain (2.86) by rearranging the terms and applying τ_{-m}. ■

Proposition 38. The functions $\mathbf{Y}_k$, $k \geq 0$, are analytic in $\mathcal{R}_1$. □

Proof. Starting with (2.87), if we take $p \in (1, 2)$, we obtain

$$\mathbf{Y}_0^-(p) = \mathbf{Y}_0^+(p) + S_\beta \mathbf{Y}_1(p-1). \tag{2.91}$$

By Proposition 26 and Lemma 36, the left-hand side of (2.91) is analytic in a lower half-plane neighborhood of $(\varepsilon, 1-\varepsilon)$ (for all $\varepsilon \in (0,1)$) and continuous in the closure of such a neighborhood. The right-hand side is analytic in an upper half-plane neighborhood of $(\varepsilon, 1-\varepsilon)$ (for all $\varepsilon \in (0,1)$) and continuous in the closure of such a neighborhood. Thus, $\mathbf{Y}_0^-(p)$ can be analytically continued along a path crossing the interval $(1, 2)$ from below; i.e., $\mathbf{Y}_0^{-+}$ exists and is analytic.

Now, in (2.87), let $p \in (2, 3)$:

$$\begin{aligned} S_\beta^2 \mathbf{Y}_2(p-2) &= \mathbf{Y}_0(p)^- - \mathbf{Y}(p)^+ - S_\beta \mathbf{Y}_1(p-1)^+ \\ &= \mathbf{Y}_0(p)^- - \mathbf{Y}_0(p)^+ - \mathbf{Y}_0(p)^{-+} + \mathbf{Y}_0(p)^+ = \mathbf{Y}_0(p)^- - \mathbf{Y}_0(p)^{-+}, \end{aligned} \tag{2.92}$$

and, in general, taking $p \in (k, k+1)$, we get

$$S_\beta^k \mathbf{Y}_k(p-k) = \mathbf{Y}_0(p)^- - \mathbf{Y}_0(p)^{-^{k-1}+}. \tag{2.93}$$

Using (2.93) inductively, the same arguments that we used for $p \in (0, 1)$ show that $\mathbf{Y}_0^{-^k}(p)$ can be continued analytically in the upper half-plane.

Remark 39. The function $\mathbf{Y}_0$ is analytic in $\mathcal{R}_1$. In fact, for $p \in (j, j+1)$, $k \in \mathbb{N}$,

$$\mathbf{Y}_0^{-^j+}(p) = \mathbf{Y}_0^+(p) + \sum_{k=1}^{j} S_\beta^k \mathbf{Y}_k^+(p-k)\mathcal{H}(p-k). \tag{2.94}$$

The relation (2.94) follows from (2.93) and (2.87).

Note. Unlike (2.87), in (2.94) the sum contains a finite number of terms. For instance, we have

$$\mathbf{Y}_0^{-+}(p) = \mathbf{Y}_0^+(p) + \mathcal{H}(p-1)\mathbf{Y}_1^+(p-1). \qquad (\forall p \in \mathbb{R}^+). \tag{2.95}$$

The analyticity of $\mathbf{Y}_m$, $m \geq 1$ is shown inductively on m, using (2.86) and following exactly the same course of proof as for $k = 0$. ∎

Remark 40. If $S_\beta = 0$, then $\mathbf{Y}_k$ are analytic in $\mathcal{W}_1 \cup \mathbb{N}$.

Indeed, this follows from (2.87), (2.86), and Lemma 36 (i).

On the other hand, if $S_\beta \neq 0$, then all $\mathbf{Y}_k$ are analytic continuations of the Borel transform of $\mathbf{y}_0$ (cf. (2.92)). This is an instance of the so-called resurgence.

Moreover, we can now calculate $\mathbf{Y}_0^{ba}$. By definition (see the discussion before Remark 23), on the interval $(0,2)$,

$$\mathbf{Y}_0^{ba} = \frac{1}{2}(\mathbf{Y}_0^+ + \mathbf{Y}_0^-) = \mathbf{Y}_0^+ + \frac{1}{2} S_\beta (\mathbf{Y}_1 \mathcal{H}) \circ \tau_1. \tag{2.96}$$

Now we are looking for a solution of (1.25) which satisfies the condition (2.96). By comparing with Lemma 36, which gives the general form of the solutions of (1.25), we get, now on the whole positive axis,

$$\mathbf{Y}_0^{ba} = \mathbf{Y}_0^+ + \sum_{k=1}^{\infty} \frac{1}{2^k} S_\beta^k (\mathbf{Y}_k^+ \mathcal{H}) \circ \tau_k \qquad (\text{on } \mathbb{R}^+), \tag{2.97}$$

which we can rewrite using (2.93):

$$\mathbf{Y}_0^{ba} = \mathbf{Y}_0^+ + \sum_{k=1}^{\infty} \frac{1}{2^k} \left(\mathbf{Y}_0^{-^k} - \mathbf{Y}_0^{-^{k-1},+} \right) (\mathcal{H} \circ \tau_k). \tag{2.98}$$

Proposition 41. Let $y_1(p), y_2(p)$ be analytic in $\mathcal{R}_1$, and such that for any path $\gamma = t \mapsto t\exp(i\phi(t))$ in $\mathcal{R}_1$,

$$|y_{1,2}(\gamma(t))| < f_\gamma(t) \in L^1_{\mathrm{loc}}(\mathbb{R}^+). \tag{2.99}$$

Assume further that for some large enough b, M, and any path γ in $\mathcal{R}_1$,

$$\int_\gamma |y_{1,2}|(s) e^{-b|s|} |ds| < M. \tag{2.100}$$

Then the analytic continuation $AC_\gamma(y_1 * y_2)$ along a path γ in $\mathcal{R}_1$, of their convolution product $y_1 * y_2$ (defined for small p by (1.20)) exists, is locally integrable, and satisfies (2.99). For the same b and some γ-independent $M' > 0$,

$$\int_\gamma |y_1 * y_2|(s) e^{-b|s|} |ds| < M'. \tag{2.101}$$

$\square$

Proof. Since

$$2y_1 * y_2 = (y_1 + y_2) * (y_1 + y_2) - y_1 * y_1 - y_2 * y_2, \tag{2.102}$$

it is enough to take $y_1 = y_2 = y$. For $p \in \mathbb{R}^+ \setminus \mathbb{N}$, we write

$$y^- = y^+ + \sum_{k=1}^{\infty} (\mathcal{H} \cdot y_k^+) \circ \tau_k. \tag{2.103}$$

The functions y_k are *defined* inductively. (The superscripts "+, (−)" mean, as before, the analytic continuations in $\mathcal{R}_1$ going below (above) the real axis.) In the same way (2.93) was obtained, we get by induction

$$y_k = (y^- - y^{-^{k-1}+}) \circ \tau_{-k}, \tag{2.104}$$

where the equality holds on $\mathbb{R}^+ \setminus \mathbb{N}$ and $+, -$ mean the upper and lower continuations. For any p, only finitely many terms in the sum in (2.103) are nonzero. The sum is also convergent in $\| \|_b$ (by dominated convergence; note that, by assumption, the functions $y^{--\cdots-\pm}$ belong to the same L^1_b).

If $t \mapsto \gamma(t)$ in $\mathcal{R}_1$ is a straight line, other than $\mathbb{R}^+$, then

$$\mathcal{A}C_\gamma((y * y)) = \mathcal{A}C_\gamma(y) *_\gamma \mathcal{A}C_\gamma(y) \quad \text{if } \arg(\gamma(t)) = \text{const} \neq 0 \tag{2.105}$$

(since y is analytic along such a line). The notation $*_\gamma$ means (1.20) with $p = \gamma(t)$.

Note, however, that, suggestive as it might be, (2.105) is *incorrect* if the condition stated there is not satisfied and γ is a path that crosses the real line (see the appendix, Section A.2)!

We get from (2.105), (2.103) (see also (A.9), in the appendix)

$$\begin{aligned}(y * y)^- &= y^- * y^- = y^+ * y^+ + \sum_{k=1}^{\infty} \left(\mathcal{H} \sum_{m=0}^{k} y_m^+ * y_{k-m}^+ \right) \circ \tau_k \\ &= (y * y)^+ + \sum_{k=1}^{\infty} \left(\mathcal{H} \sum_{m=0}^{k} \left(y_m * y_{k-m}\right)^+ \right) \circ \tau_k,\end{aligned} \tag{2.106}$$

and now the analyticity of $y * y$ in $\mathcal{R}_1$ follows: on the interval $p \in (m, m+1)$, we have from (2.104)

$$\left(y * y\right)^{-^j}(p) = \left(y * y\right)^-(p) = \left(y^{*2}\right)^+(p) + \sum_{k=1}^{j} \sum_{m=0}^{k} \left(y_m * y_{k-m}\right)^+(p-k). \tag{2.107}$$

Again, formula (2.107) is useful for analytically continuing $\left(y * y\right)^{-^j}$ along a path as the one depicted in Figure 1. By dominated convergence, $(y * y)^\pm \in \mathcal{T}^\pm_{(0,\infty)}$ (2.60). By (2.104), y_m

are analytic in $\mathcal{R}_1^+ := \mathcal{R}_1 \cap \{p : \Im(p) > 0\}$ and thus by (2.105) the right-hand side of (2.107) can be continued analytically in $\mathcal{R}_1^+$. The same is then true for $(y * y)^-$. The function $(y * y)$ can be extended analytically along paths that cross the real line from below. Likewise, $(y * y)^+$ can be continued analytically in the lower half-plane so that $(y * y)$ is analytic in $\mathcal{R}_1$.

Combining (2.107), (2.105), and (2.102), we get a similar formula for the analytic continuation of the convolution product of two functions f, g satisfying the assumptions of Proposition 41:

$$(f * g)^{-^j+} = f^+ * g^+ + \sum_{k=1}^{j} \left(\mathcal{H} \sum_{m=0}^{k} f_m^+ * g_{k-m}^+ \right) \circ \tau_k. \tag{2.108}$$

Note that (2.108) corresponds to (2.103), and in those notations we have

$$\left(f * g\right)_k = \sum_{m=0}^{k} f_m * g_{k-m}. \tag{2.109}$$

Integrability as well as (2.101) follow from (2.104), (2.107), and Remark 9. ■

By (1.16) and (2.104),

$$y^{ba} = y^+ + \sum_{k=1}^{\infty} \frac{1}{2^k} (y_k^+ \mathcal{H}) \circ \tau_k$$

so that (see (A.9))

$$\begin{aligned} y^{ba} * y^{ba} &= \left(y^+ + \sum_{k=1}^{\infty} \frac{1}{2^k} (\mathcal{H} \circ \tau_k)(y_k^+ \circ \tau_k) \right)^{*2} \\ &= y^+ * y^+ + \sum_{k=1}^{\infty} \frac{1}{2^k} \mathcal{H} \circ \tau_k \sum_{m=0}^{k} (y_m^+ \circ \tau_m) * (y_{k-m}^+ \circ \tau_{k-m}) \circ \tau_k \\ &= y^+ * y^+ + \sum_{k=1}^{\infty} \frac{1}{2^k} \mathcal{H} \circ \tau_k \sum_{m=0}^{k} (y_m * y_{k-m})^+ \circ \tau_k = (y^{*2})^{ba}. \end{aligned} \tag{2.110}$$

To finish the proof of Theorem 2, note that on any finite interval the sum in (1.16) has only a finite number of terms, and by (2.110) balanced averaging commutes with any finite sum of the type

$$\sum_{k_1,\ldots,k_n} c_{k_1 \cdots k_n} f_{k_1} * \cdots * f_{k_n} \tag{2.111}$$

and then, by continuity, with any sum of the form (2.111), with a finite or infinite number of terms, provided it converges in L^1_{loc}. Averaging thus commutes with all the operations involved in the equations (2.73). By uniqueness, therefore, if $\mathbf{Y}_0 = \mathbf{Y}^{ba}$, then $\mathbf{Y}_k = \mathbf{Y}_k^{ba}$ for all k. Preservation of reality is immediate since (1.25), (1.27) are real if (1.1) is real, and therefore $\mathbf{Y}_0^{ba}$ is real-valued on $\mathbb{R}^+\backslash\mathbb{N}$ (since it is real-valued on $[0,1) \cup (1,2)$) and so are, inductively, all $\mathbf{Y}_k$.

A Appendix

A.1 Example of nontypical behavior

Consider the equation

$$f' = -f - \frac{1}{2x}f + \frac{1}{x} - \frac{1}{2x^2}. \tag{A.1}$$

The general solution of this equation is given by

$$f = \frac{1}{x} + Cx^{-1/2}e^{-x} = \int_0^\infty \left(p + \frac{C}{\sqrt{p-1}}\mathcal{H}(1-p)\right)\, dp. \tag{A.2}$$

We see that the asymptotic series of f for $x \to \infty$, $\Re(x) > 0$, is $\hat{\mathbf{y}}_0 = 1/x$. The inverse Laplace transform of f is

$$\mathcal{L}^{-1}f = p + \frac{C}{\sqrt{p-1}}\mathcal{H}(1-p). \tag{A.3}$$

(i) The Stokes constant is zero and $\mathbf{Y}_0 = \mathcal{B}(\hat{\mathbf{y}}_0) = p$ is entire.

(ii) All combinations $\lambda\mathbf{Y}_0^+ + (1-\lambda)\mathbf{Y}_0^-$ coincide. Therefore (1.30) does not hold.

Equation (A.1) is exceptional, in the sense that the properties (i), (ii) above do not withstand a small perturbation. Indeed, for the equation

$$f' = -f - \frac{1}{2x}f + \frac{1+\epsilon}{x} - \frac{1}{2x^2}, \tag{A.4}$$

we have $\mathcal{B}(\hat{\mathbf{y}}_0) = 2\epsilon + p + \epsilon(1-p)^{-1/2}$, and the inverse Laplace transform of the general solution is

$$\mathcal{L}^{-1}(f) = \begin{cases} 2\epsilon + p + \epsilon(1-p)^{-1/2} & \text{for } p < 1 \\ 2\epsilon + p + C(p-1)^{-1/2} & \text{for } p > 1. \end{cases}$$

A.2 $AC(f * g)$ versus $AC(f) * AC(g)$

Typically, the analytic continuation along curve in $\mathcal{W}_1$ which is not homotopic to a straight line will not commute with convolution. For example, in equation (A.4), $\mathcal{B}(\hat{\mathbf{y}}_0)^{-+} * \mathcal{B}(\hat{\mathbf{y}}_0)^{-+} \neq [\mathcal{B}(\hat{\mathbf{y}}_0) * \mathcal{B}(\hat{\mathbf{y}}_0)]^{-+}$, as it can be seen from Remark 42 below (or by direct calculation). This situation is generic.

Remark 42. Let y be a function satisfying the conditions stated in Proposition 41 and assume that $p = 1$ is a branch point of y. Then

$$(y * y)^{-+} \neq y^{-+} * y^{-+}. \tag{A.5}$$

Indeed, by (2.108) and (2.104),

$$\begin{aligned}(y * y)^{-+} &= y^+ * y^+ + 2[(y^+ * y_1^+)\mathcal{H}] \circ \tau_1 \neq y^{-+} * y^{-+} \\ &= [y^+ + (\mathcal{H}y_1^+) \circ \tau_1]^{*2} = y^+ * y^+ + 2[(y^+ * y_1^+)\mathcal{H}] \circ \tau_1 + [\mathcal{H}(y_1^+ * y_1^+)] \circ \tau_2,\end{aligned} \tag{A.6}$$

since, in view of (2.104), in our assumptions, $y_1 \not\equiv 0$ and thus $y_1 * y_1 \not\equiv 0$.

There is also the following intuitive reasoning leading to the same conclusion: For a generic system of the form (1.1)–(1.3), $p = 1$ is a branch point of $\mathbf{Y}_0$ and so $\mathbf{Y}_0^- \neq \mathbf{Y}_0^{-+}$. On the other hand, if AC_{-+} commuted with convolution, then $\mathcal{L}(\mathbf{Y}_0^{-+})$ would provide a solution of (1.1). By Lemma 36, $\mathcal{L}(\mathbf{Y}_0^-)$ is a different solution (since $\mathbf{Y}_0^- \neq \mathbf{Y}_0^{-+}$). As $\mathbf{Y}_0^-$ and $\mathbf{Y}_0^{-+}$ coincide up to $p = 2$, we have $\mathcal{L}(\mathbf{Y}_0^{-+}) - \mathcal{L}(\mathbf{Y}_0^-) = O(e^{-2x}x^{\text{power}})$ for $x \to +\infty$. By Theorem 2, however, no two solutions of (1.1)–(1.3) can differ by less than $e^{-x}x^{\text{power}}$ without actually being equal (also, heuristically, this can be checked using formal perturbation theory), a contradiction.

A.3 Useful formulas

We have

$$\mathcal{B}\left(\frac{1}{x^n}\right) = \frac{p^{n-1}}{\Gamma(n)} \qquad \text{or} \qquad \mathcal{L}(p^n) = \frac{\Gamma(n+1)}{x^{n+1}}, \tag{A.7}$$

$$p^q * p^r = \frac{\Gamma(q+1)\Gamma(r+1)}{\Gamma(q+r+2)} p^{q+r+1}. \tag{A.8}$$

With $f_{1,2}(p) := p \mapsto \mathcal{H}(p - k_{1,2})g_{1,2}(p - k_{1,2})$, we have

$$\Big(f_1 * f_2\Big)(p) = \mathcal{H}(p - k_1 - k_2)\Big(g_1 * g_2\Big)(p - k_1 - k_2). \tag{A.9}$$

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