Resurgence and trans-series in Quantum Field Theory: the $\mathbb{CP}^{N-1}$ model

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ABSTRACT: This work is a step towards a non-perturbative continuum definition of quantum field theory (QFT), beginning with asymptotically free two dimensional non-linear sigma-models, using recent ideas from mathematics and QFT. The ideas from mathematics are resurgence theory, the trans-series framework, and Borel-Écalle resummation. The ideas from QFT use continuity on $\mathbb{R}^1 \times \mathbb{S}_L^1$, i.e, the absence of any phase transition as $N \to \infty$ or rapid-crossovers for finite-N, and the small-L weak coupling limit to render the semi-classical sector well-defined and calculable. We classify semi-classical configurations with actions $1/N$ (kink-instantons), $2/N$ (bions and bi-kinks), in units where the 2d instanton action is normalized to one. Perturbation theory possesses the IR-renormalon ambiguity that arises due to non-Borel summability of the large-orders perturbation series (of Gevrey-1 type), for which a microscopic cancellation mechanism was unknown. This divergence must be present because the corresponding expansion is on a singular Stokes ray in the complexified coupling constant plane, and the sum exhibits the Stokes phenomenon crossing the ray. We show that there is also a non-perturbative ambiguity inherent to certain neutral topological molecules (neutral bions and bion-anti-bions) in the semiclassical expansion. We find a set of "confluence equations" that encode the *exact cancellation* of the two different type of ambiguities. There exists a resurgent behavior in the semi-classical trans-series analysis of the QFT, whereby subleading orders of exponential terms mix in a systematic way, canceling all ambiguities. We show that a new notion of "graded resurgence triangle" is necessary to capture the path integral approach to resurgence, and that graded resurgence underlies a potentially rigorous definition of general QFTs. The mass gap and the Θ angle dependence of vacuum energy are calculated from first principles, and are in accord with large-N and lattice results.

Contents

1 General idea of resurgence in QFT

> "Series don't diverge for no reason; it is not a capricious thing. The divergence
> of a series must reflect its cause."
>
> M.V. Berry, *Stokes and the Rainbow*, Newton Institute Lecture, (2003)

This work aims to give a non-perturbative continuum definition of quantum field theory, specifically here for two-dimensional non-linear sigma models, using two recent developments in mathematics and quantum field theory (QFT). The ideas from mathematics come from the beautiful and powerful notions of resurgent functions and trans-series which go beyond conventional (Poincaré) asymptotic analysis [1–10]. The new insights from QFT are the semi-classically consistent compactifications [11–13] and deformations [15, 17] of quantum field theories, to control their infrared behavior, rendering them well-defined and calculable.

Since these quantum field theories possess a non-Borel-summable asymptotic perturbative expansion around *any* background, perturbation theory on its own is ambiguous, and does not define the QFT. This is one of the major difficulties why many mathematicians would say QFT is still non-rigorous, as recently emphasized in ref. [18]. A lesser known fact is that the non-perturbative semi-classical expansion on its own is also ambiguous (in the context of QFT, see [19, 20] and the present work), and also does not define the QFT. However, there exists a mechanism to cancel the non-perturbative ambiguities of perturbation theory with the ambiguities of the semi-classical expansion within resurgence theory, and one obtains *unique*, ambiguity-free, answers for physical quantities. This is a provocative and ambitious goal which has been explored in some detail in certain quantum mechanical systems with degenerate vacua [21–37], in connection with the pioneering work of Bender and Wu [38, 39]. Here and in joint work with P. Argyres, [19, 20], we take the first steps in applying these ideas to quantum field theory, where new effects appear, such as asymptotic freedom and renormalons [40, 41].

Philosophically, the idea of resurgence is to combine perturbation theory (with small parameter λ, or $\lambda\hbar$ if one restores $\hbar$) and non-perturbative analysis (with small parameter $e^{-A/\lambda}$, with $A > 0$) into a unified "trans-series" representation of a physical observable, in which the trans-series encodes much more [and potentially all] information about the function, rather than being viewed simply as a perturbative approximation or as an asymptotic approximation:[1]

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)} \, (\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n} \, e^{-n\,A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)} \, (\lambda\hbar)^k \tag{1.1}$$

The main point of resurgence is that the perturbative and non-perturbative sectors can be related in a systematic and mutually consistent manner, and unified as a trans-series. Paraphrasing the perspective of Dingle [1] and Berry and Howls [4–6], a trans-series may be a coded version of the exact function, which requires decoding in a systematic manner. Resurgence is the statement that the coefficients $c_{(n,k)}$ in (1.1) of the series expansion in powers λ^k at large order k, for some n, are directly related to the coefficients at low order in k, for some other n. In other words, the perturbative expansion about some semiclassical configuration [a multi-instanton-anti-instanton] is directly related to the perturbative expansion about some other semiclassical configuration with different action. In section 1.6 we give some elementary but illustrative examples where such a trans-series expansion (rather than a perturbative *or* a non-perturbative expansion) is obviously necessary in order to give a complete description of the function as the phase of the expansion parameter λ is varied.

A natural and powerful approach to this kind of problem is known as *Borel-Écalle resummation*, or *generalized Borel summation* [2, 3, 7, 8, 10], a technique for extracting mathematical and physical information from a divergent series. In quantum mechanics, this method may be applied both to the divergent perturbation series representing an energy eigenvalue [22–37], as well as to the divergent asymptotic series representation of the semiclassical wavefunction [1–3, 7, 8, 10]. For certain polynomial oscillator-type potentials the relation to exact semiclassical quantization rules has been explored in detail [42–45]. As a direct physical application, resurgence yields trans-series representations of both eigenvalues and wavefunctions that, in principle, encode *all* information about the solution. Earlier, Dingle and others developed resurgent forms of WKB expansions for quantum mechanical problems and special functions. These ideas also underlie improved *hyper-asymptotic* approximations in which the remainder tails left after optimal truncation of a divergent series are repeatedly Borel resummed [1, 4–6], revealing interesting universal structures.

It is not immediately clear that we can extend these quantum mechanical results to quantum field theories with renormalization. We present here some evidence that resurgence can be applied to QFT, in the two-dimensional $\mathbb{CP}^{N-1}$ model, one of the simplest non-trivial quantum field theories which possesses features analogous to QCD such as asymptotic freedom, confinement and instantons [47]. The resurgence perspective allows us to identify certain semi-classical objects in the $\mathbb{CP}^{N-1}$ model with the elusive infrared (IR)

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[1]In general a trans-series also includes a sum over powers of logarithms [2, 3, 8, 27–31], which are associated with quasi-zero-modes. We will comment on such terms later. After (1.1), we generally set $\hbar = 1$.

renormalons, so that ambiguities in the perturbative and non-perturbative sector cancel one another. This builds on, but goes much further than, the fundamental results of Lipatov connecting instantons in QFT path integrals and the divergence of perturbation theory [48].

In fact, resurgence theory in QFTs, or in certain quantum mechanics problems with degenerate classical vacua, offers an extra feature: when we consider the effect of a topological theta angle we find that the semi-classical exponential factors may also acquire phases, and sectors with different phases cannot mix in perturbation theory, by the simple fact that perturbation theory is independent of Θ. Thus, the Θ dependence serves as a simple and useful guide that "grades" the distinct *resurgent sectors*, linking those that talk to one another and cure one anothers' ambiguities. Using the abbreviations $\frac{\tilde{\Theta}_k}{N} \equiv \frac{\Theta + 2\pi k}{N}$, $A = 4\pi$, and for the 't Hooft coupling $\lambda \equiv g^2 N$ in the bosonic $\mathbb{CP}^{N-1}$ model, the general structure that emerges out of the path integral formulation can be summarized symbolically in the following "graded resurgence triangle":

$$f_{(0,0)}$$

$$e^{-\frac{A}{\lambda} + i\frac{\tilde{\Theta}_k}{N}} f_{(1,1)} \qquad e^{-\frac{A}{\lambda} - i\frac{\tilde{\Theta}_k}{N}} f_{(1,-1)}$$

$$e^{-\frac{2A}{\lambda} + 2i\frac{\tilde{\Theta}_k}{N}} f_{(2,2)} \qquad e^{-\frac{2A}{\lambda}} f_{(2,0)} \qquad e^{-\frac{2A}{\lambda} - 2i\frac{\tilde{\Theta}_k}{N}} f_{(2,-2)}$$

$$e^{-\frac{3A}{\lambda} + 3i\frac{\tilde{\Theta}_k}{N}} f_{(3,3)} \qquad e^{-\frac{3A}{\lambda} + i\frac{\tilde{\Theta}_k}{N}} f_{(3,1)} \qquad e^{-\frac{3A}{\lambda} - i\frac{\tilde{\Theta}_k}{N}} f_{(3,-1)} \qquad e^{-\frac{3A}{\lambda} - 3i\frac{\tilde{\Theta}_k}{N}} f_{(3,-3)}$$

$$e^{-\frac{4A}{\lambda} + 4i\frac{\tilde{\Theta}_k}{N}} f_{(4,4)} \quad e^{-\frac{4A}{\lambda} + 2i\frac{\tilde{\Theta}_k}{N}} f_{(4,2)} \quad e^{-\frac{4A}{\lambda}} f_{(4,0)} \quad e^{-\frac{4A}{\lambda} - 2i\frac{\tilde{\Theta}_k}{N}} f_{(4,-2)} \quad e^{-\frac{4A}{\lambda} - 4i\frac{\tilde{\Theta}_k}{N}} f_{(4,-4)}$$

$$\iddots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \ddots \tag{1.2}$$

which represents a general expansion of some observable. The rows correspond to a given instanton number n, with associated perturbative loop expansions times an instanton prefactor $f_{(n,k)}(\lambda) \equiv (\lambda)^{-\beta_n} \sum_{k=0}^{\infty} c_{(n,k)} (\lambda)^k$, and with the topological phases specified. Only columns of this triangle with matching Θ dependence can possibly mix via resurgence.[2] For example, in the Bogomolny-Zinn-Justin (BZJ) approach to the periodic potential problem [31, 46], which has degenerate vacua and a topological theta angle, the ambiguity in the perturbative contribution $f_{(0,0)}(\lambda)$ to the ground state energy is cured by an ambiguity in the instanton-anti-instanton amplitude, at order $e^{-\frac{2A}{\lambda}} f_{(2,0)}(\lambda)$. This is in fact a general phenomenon that extends throughout the triangle: the Θ-sectors are correlated with instanton sectors, which gives another tool for probing the mixing of the different terms in the trans-series representation. Our main conjecture is that each column is a resurgent function of λ. This graded resurgence structure provides an interesting new perspective on instanton calculus and is born in a natural implementation of the theory of resurgence in the path integral formalism.

Our long-term goal in applying resurgence to QFT is rather ambitious: We aim to give a non-perturbative continuum definition of quantum field theory, and provide a mathematically rigorous foundation. We also would like that such such a definition should be

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[2]For notational simplicity we suppress log terms that generally also appear in the prefactor sums.

of practical value (not a formal tool) whose results can be compared with the numerical analysis of lattice field theory. In other words, generalizing the title of 't Hooft's seminal Erice lectures [49], we want to make sense out of general QFTs in the continuum.

We emphasize that our immediate goal is *not* to provide theorems; rather we would like to reveal *structure underlying QFT*, a framework in which we can define QFT in a self-consistent manner without running into internal inconsistencies. We hope that whatever framework emerges along these lines may form the foundation of a rigorous definition. This point of view is close in spirit to refs. [18, 50]. However, we ultimately hope that we will be able to use resurgence theory to provide exact and rigorous results for general QFTs, at least in the semi-classical domain. Our optimism stems from the work of Pham et.al. where they proved that the semi-classical expansions in certain non-trivial quantum mechanical systems are resummable to finite exact and unambiguous answers [43–46].

Similar ideas that appear in the current work and in [19, 20] also appeared in a recent talk by Kontsevich [58]. Kontsevich also examines resurgence from the path integral perspective, with the intention of establishing a non-perturbative definition of certain special QFTs and quantum mechanics, directly from the path integral. The notion of analytic continuation of paths in field space, which is also crucial in our analysis of quasi-zero mode integrals, plays an important role in his discussion. We also note that some progress has recently been made in applying the ideas of resurgence to matrix models, and certain string theories and quantum field theories (which do not require renormalization) in the recent works of Schiappa et al [51, 52], Mariño et al [53–56], and Costin et al [57]. Our works differ from the above in the sense that we study both realistic QFTs, with asymptotic freedom and renormalons, and comment upon theories with extended supersymmetry. The study of realistic QFTs requires, apart from new mathematical inputs, new physical inputs as well, rendering them calculable; this QFT program began with new compactifications [11, 12] and deformations of gauge theories [15, 17].[3]

1.1 Problems with semi-classical analysis on $\mathbb{R}^2$

To motivate our application of resurgence we recall that the $\mathbb{CP}^{N-1}$ model in two dimensions is an asymptotically free non-linear sigma model with many features in common with four dimensional Yang-Mills theory [47, 59]. Despite some progress in this class of theories, especially in the large-N limit, there are several significant long-standing open problems:

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[3]The ability to use the weak coupling limit in realistic QFTs is a newly developing program, and it is what makes the current detailed analysis possible. In non-supersymmetric theories, it is also to a certain extent an unexpected, but quite welcome aspect. The state of the art concerning the large orders in perturbation theory versus non-perturbative effects in the framework of resurgent functions and trans-series in ODEs, integral equations, quantum mechanics and a sub-class of special QFTs is explained clearly in a recent review by Mariño [56], where he asserts

"In realistic QFTs, perturbation theory is so wild that it is not feasible to pursue this program [theory of resurgence], but in some special QFTs — namely, those without renormalons, like Chern-Simons theory in 3d or $\mathcal{N} = 4$ Yang-Mills theory — there are some partial results."

We provide ample evidence in this work for $\mathbb{CP}^{N-1}$, and in [19, 20] for QCD(adj), that the resurgence formalism can in fact be applied to more realistic QFTs, even those with asymptotic freedom and renormalons.

Problem 1. Invalidity of the semi-classical dilute instanton gas approximation on $\mathbb{R}^2$. The theory on $\mathbb{R}^2$ has instanton solutions, but it does not admit a reliable semi-classical analysis because of the existence of the instanton size modulus, which implies that instantons of all sizes come with no extra action cost. Therefore, a self-consistent dilute instanton gas approximation, which relies on the assumption that the typical separation between instanton events is much larger than the instanton size, does not exist for the $\mathbb{CP}^{N-1}$ model on $\mathbb{R}^2$. This is a variant of the long-standing "infrared embarrassment" problem [59].

Problem 2. Meaning of the infra-red renormalon singularities. Another serious (and actually, we claim, related) problem is that if one works out the large orders of perturbation theory in sigma models, there are singularities (poles or branch points) on the positive real axis $\mathbb{R}^+$ of the complex Borel plane located at order N times closer to the origin than the *leading* 2d instanton-anti-instanton $[\mathcal{I}\bar{\mathcal{I}}]$ singularity. These are called *infrared (IR) renormalon* singularities, and it was previously unknown if there exists a semi-classical (or non-semiclassical) field configuration which may cure this disease of perturbation theory [49, 60, 61]. Thus, perturbation theory is ill-defined for $\mathbb{CP}^{N-1}$ on $\mathbb{R}^2$.

Problem 3. Relation between large-N results and instantons. $\mathbb{CP}^{N-1}$ in the large-N limit admits a solution with non-perturbative mass gap

$$m_g = \Lambda = \mu e^{-S_I/N} \equiv \mu e^{-4\pi/(g^2 N)}\,, \tag{1.3}$$

where μ and Λ are, respectively, the renormalization and the strong scale, and S_I is the 2d-instanton action. There is no known semi-classical (or non-semiclassical) configuration which leads to this mass gap. This has been studied, for example, in several fundamental works [62, 64–69], but so far no fully consistent semi-classical analysis exists on $\mathbb{R}^2$.

We emphasize that none of these problems is formal. They indicate that some crucial ingredients are lacking in our current understanding of QFT. We show here that it is possible to solve these problems on $\mathbb{R}^1 \times \mathbb{S}^1$, and in fact, the solutions of all three problems are deeply related. We also argue that this solution may be continuously connected to the situation on $\mathbb{R}^2$, leading to a quite radical resolution of the problems listed above. Our proposal, valid for both supersymmetric and non-supersymmetric theories, is also consistent with arguments based on mirror symmetry [72] and appropriate generalization of the quantum chiral ring relations [73] in the minimal supersymmetric case, as discussed below in section 5.1.

1.2 Good and bad for semi-classics in compactified theories

Traditionally, an infrared cut-off such as thermal compactification is used to tame the problem of the size of an instanton in asymptotically free theories [74]. However, this thermal compactification approach, as opposed to the spatial (non-thermal) compactification that we propose here, does not provide a semi-classical analysis of the confined phase. As reviewed in section 3.1, at finite temperature, the $\mathbb{CP}^{N-1}$ model has two phases (regimes): a

deconfined phase $\beta < \beta_c \approx \Lambda^{-1}$ (where β_c is the deconfinement temperature and Λ is the strong scale), and a confined phase, $\beta > \beta_c$.[4]

- The theory in the weak coupling small–$\mathbb{S}_\beta^1$ (high-temperature) regime is in a deconfined phase, governed by trivial holonomy (3.6). The instanton size problem can indeed be tamed in this phase, since

$$\rho \lesssim \beta < \beta_c \approx \Lambda^{-1} \qquad (1.4)$$

i.e., the instanton size ρ is cut-off by the box size β, *and* the box size is smaller than the strong scale. However, the information that one acquires in this regime does not apply to the confined-phase and is not continuously connected to $\mathbb{R}^2$. Despite the fact that one can make sense of a dilute instanton gas for $\beta \ll \Lambda^{-1}$ [62, 64–69], whatever is learned in this phase is not obviously directly relevant for understanding the physics below the deconfinement temperature, at large–$\mathbb{S}_\beta^1 \times \mathbb{R}$, or on $\mathbb{R}^2$.

- If the theory is in the confined phase, i.e., $\beta > \beta_c \approx \Lambda^{-1}$, it is governed by a non-trivial holonomy for the line operator at strong coupling. In this regime, there is no weakly coupled description of the long-distance physics. The dilute instanton gas approximation breaks down whenever the size modulus is of order the strong scale, $\rho \sim \Lambda^{-1}$. Therefore, the existence of a box whose size is larger than β_c is not helpful for a semi-classical dilute gas approximation. Formally, it is still true that the size modulus of the "classical instanton solution" is cut-off at the scale β. However, for

$$\Lambda^{-1} \lesssim \rho \lesssim \beta \qquad (1.5)$$

the notion of a semi-classical instanton is not meaningful, due to strong coupling. The semi-classical approximation is simply inapplicable in this regime.

In $\mathbb{CP}^{N-1}$ on $\mathbb{R}^1 \times \mathbb{S}^1$, as well as in gauge theories on $\mathbb{R}^3 \times \mathbb{S}^1$, there are three types of gauge holonomies. (In $\mathbb{CP}^{N-1}$, this is the gauge holonomy associated with the σ-model connection, defined below in (2.34), and it plays the same role as the Wilson line in gauge theory). In studying calorons, Yi and Lee [75, 76], and van Baal et al [77, 78], analyzed the difference between trivial (degenerate eigenvalue distribution) and non-trivial holonomy (maximally non-degenerate distribution), and realized the importance of non-trivial holonomy for topological configurations with fractional topological charges. However, a further refinement is still needed in order to find a quantitative semi-classical theory describing the dynamics. The types of holonomies are:

1) *weak coupling trivial holonomy*. Semi-classical analysis applies, but this regime is separated from strong coupling non-trivial holonomy by a rapid cross-over or a phase transition.

2) *Weak coupling non-trivial holonomy*. Semi-classical analysis applies, and for a large class of theories, this regime may be continuously connected to strong coupling non-trivial holonomy regime.

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[4]Strictly speaking, there is no phase transition, but a rapid crossover in the behavior of the theory at the strong scale. At $N = \infty$ thermodynamic limit, this becomes a sharp phase transition; at finite-N, the deconfined regime has $O(N)$-free energy, and the confined regime has $O(1)$ free energy.
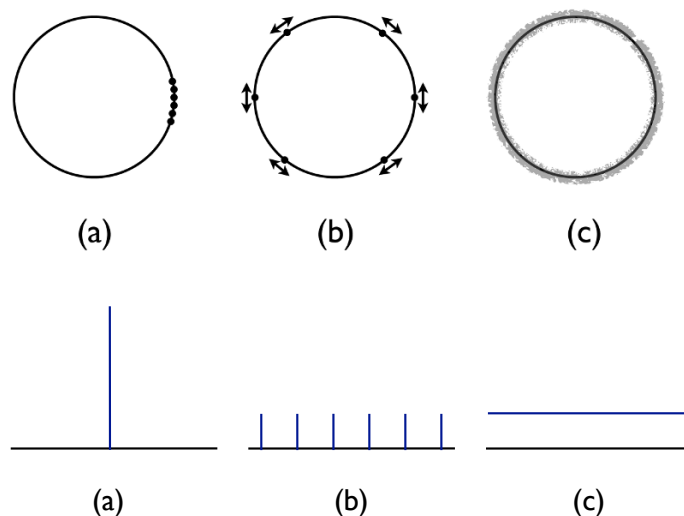
Figure 1.   Three types of gauge holonomy in non-abelian gauge theories and analogously σ-connection holonomy (defined in (2.34)) in non-linear sigma models, and their classification according to their eigenvalue distribution. a) is the weak coupling trivial holonomy. b) and c) are, respectively, the weak and strong coupling non-trivial holonomy, and they are continuously connected in the sense of gauge invariant order parameters. In b), the eigenvalues of holonomy are located at the roots of unity and their fluctuations are small. In c), the positions of the eigenvalues are uniform and randomized. In gauge theory on $\mathbb{R}^3 \times S^1$, b) is the counterpart of the weak coupling adjoint Higgs regime. There is no weak coupling or even a potential description for c). The difference between (b) and (c) in gauge theory is discussed in ref. [15].

3) *Strong coupling non-trivial holonomy.* Weak-coupling semi-classical analysis is not applicable.

The appreciation of the existence and significance of the second type of holonomy is relatively new in gauge theory [11, 15], and for $\mathbb{CP}^{N-1}$ it is presented in this work.[5]

Specifically, we propose a method to understand the dynamics of the $\mathbb{CP}^{N-1}$ theory on $\mathbb{R}^2$ using continuity and weak coupling methods. We work with a spatial (non-thermal) compactification with twisted boundary conditions. In this case, as we will show, there are $\mathbb{CP}^{N-1}$ theories without *any* phase transition (the associated free energy always remains order one, as opposed to order N^1 as in the deconfined regime of the thermal theory) and whatever is learned at weak coupling is expected to smoothly interpolate to strong coupling. We study a class of theories related to $\mathbb{CP}^{N-1}$: the bosonic model, the supersymmetric $\mathcal{N} = (2, 2)$ theory with $N_f = 1$ Dirac fermionic flavor, as well as non-supersymmetric multi-flavor theory with $N_f > 1$. Our methods apply equally well both to supersymmetric and non-supersymmetric theories, and the role that supersymmetry plays in $N_f = 1$ is in fact not particularly significant.

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[5]The associated "free-energy" in both strong and weak coupling non-trivial holonomy regime is $O(1)$, whereas the free-energy of the weak coupling trivial holonomy regime is $O(N)$. As $N \to \infty$ thermodynamic limit, there is a sharp phase transition between 1) and 3), but the limit is smooth between 2) and 3). This is what we mean by continuity on $\mathbb{R}^1 \times \mathbb{S}^1$.

1.3 Perturbation theory and spatial twisted boundary conditions

In the classical theory, we show that the twisted boundary conditions can be recast into a background field, a $U(1)^N$ σ-connection. It is realized as a background field for the line operator, defined below in (2.34). This is analogous to the background Wilson line in gauge theory on $\mathbb{R}^3 \times \mathbb{S}^1$. Determining the stability of a given background in quantum theory requires a perturbative one-loop Coleman-Weinberg type analysis at small-L, similar to gauge theory. The idea of twisted boundary conditions has recently been used to study the substructure of calorons in $\mathbb{CP}^{N-1}$ at finite temperature [79–81], with earlier relevant work by Sutcliffe [82], and we are in part inspired by these results. However, our physical interpretation is different, as we use spatially twisted boundary conditions, and we further address the crucial issue of the stability of the twisted background in the quantum theory, not addressed in earlier works.

The main result of the perturbative one-loop analysis is parallel to Yang-Mills theory with adjoint fermions, QCD(adj), on $\mathbb{R}^3 \times \mathbb{S}^1$. We find that on a temporal circle, i.e., thermal case, twisted boundary conditions are unstable for any $N_f \geq 0$ in the small-$\mathbb{S}_\beta$ regime. In contradistinction, upon spatial compactification on a cylinder $\mathbb{R}^1 \times \mathbb{S}^1_L$, $\mathbb{Z}_N$ twisted boundary conditions are *stable* for $N_f \geq 1$, and unstable for $N_f = 0$. However, even in the ($N_f = 0$) bosonic case, a $\mathbb{Z}_N$ symmetric twist can be achieved, by exploiting the fermion induced stabilization by taking fermions heavy with respect to the strong scale, but light with respect to $1/(LN)$. This theory at distances larger then m^{-1} emulates the bosonic theory and has a stable $\mathbb{Z}_N$-twisted background. We will refer to the bosonic theory obtained in this manner or obtained by adding an explicit center-stabilizing potential as "deformed-$\mathbb{CP}^{N-1}$". A stable $\mathbb{Z}_N$ twist is the first step towards a resolution of the problems listed in section 1.1.

1.4 Topological defects and molecules

The stability of the spatially-twisted-background in the weak coupling regime allows us to investigate topological defects. The leading topological defects and molecules in the $\mathbb{CP}^{N-1}$ model on $\mathbb{R}^1 \times \mathbb{S}^1_L$ are:

(i) kinks (or more rigorously kink-instantons) $\mathcal{K}_i$.

(ii) Charged bions (correlated kink-anti-kink events) $\mathcal{B}_{ij} = [\mathcal{K}_i \overline{\mathcal{K}}_j]$,

(iii) neutral bions $\mathcal{B}_{ii} = [\mathcal{K}_i \overline{\mathcal{K}}_i]$, and

(iv) neutral bion-anti-bion molecular events such as $[\mathcal{B}_{ij} \mathcal{B}_{ji}], [\mathcal{B}_{ij} \mathcal{B}_{jk} \mathcal{B}_{ki}]$ etc.

Note that the 2d instanton $\mathcal{I}$ does not appear in this list of leading topological configurations. Apart from the fact that it causes the anomaly in the classical $U(1)_A$ chiral symmetry for theories with multiple massless fermions [reducing it to a discrete chiral $\mathbb{Z}_{2N}$], the role of the 2d instanton in the semi-classical expansion is actually *insignificant* for the resolution of the problems listed in section 1.1. We emphasize this, as it is surprising and goes against the general philosophy of many works. Also note that our classification of topological defects in $\mathbb{CP}^{N-1}$ is identical to QCD(adj) compactified on $\mathbb{R}^3 \times \mathbb{S}^1$, with the role of the monopole-instantons of [19] being played by the kink-instantons in $\mathbb{CP}^{N-1}$.

Before going into a description of the topological defects, we note the important hierarchy of length (or energy) scales,

$$r_{\mathrm{k}} \ll r_{\mathrm{b}} \ll d_{\mathrm{k-k}} \ll d_{\mathrm{b-b}}, \tag{1.6}$$

where r_{k} is the size of of kink-instanton, r_{b} is the size of a bion, $d_{\mathrm{k-k}}$ and $d_{\mathrm{b-b}}$ are the typical inter-kink and inter-bion separations. This justifies the use of of a dilute gas of kinks, bions, etc. This hierarchy is also inherent to the hierarchy of successive terms in the semi-classical trans-series expansion.

The kink-instantons are self-dual configurations with topological charge $1/N$, as discussed in detail in section 4. The kink-instantons come in N types, associated with the (extended) root system of the SU(N) global symmetry of the $\mathbb{CP}^{N-1}$theory. The first $(N-1)$ types are 1d kink-instantons associated with the simple roots α_i, and the remaining one is the twisted (affine) kink-instanton, associated with the affine root $\alpha_0 = -\sum_i \alpha_i$. We give an explicit construction of this affine-kink-instanton. In a theory with massless fermions, these defects carry fermion zero modes. In the bosonic theory, they have a dependence on the topological Θ-angle. Both of these ingredients will be crucial in the construction of the trans-series and resurgent analysis.

At second order in the semi-classical expansion, we have two types of topological configurations. These are $[\mathcal{K}_i \mathcal{K}_j]$ type correlated kink-kink events, or bi-kinks, with topological charge and action $2/N$ of a 2d instanton. However, these events turn out to be not very important. This is easy to understand in the $N_f \geq 1$ theory, where $[\mathcal{K}_i \mathcal{K}_j]$ amplitudes carry twice as many fermion zero modes and so are much less important for IR physics. It is a bit more subtle to understand this in the deformed bosonic theory, but it is nevertheless true.

The more interesting events at second order are correlated kink-anti-kink instanton events $\mathcal{B}_{ij}$, which exist for all non-vanishing entries of the extended Cartan matrix $\widehat{A}_{ij} \neq 0$ of SU(N). These defects have a typical size much larger than the kink size, but exponentially smaller than the typical inter-kink separation. Therefore, these objects may be seen as molecular instanton events.

For $\widehat{A}_{ij} < 0$, there exists a *charged bion*. Charge $\mu_{\mathcal{B}_{ij}} = \alpha_i - \alpha_j$ stands for the fact that the root associated with this correlated tunneling event is non-zero. For $N_f \geq 1$ theories, the interaction between $\mathcal{K}_i$ and $\overline{\mathcal{K}}_j$ has a repulsive component due to boson exchange, in addition to an attractive fermion zero mode exchange induced interaction, leading to a bound (correlated) event, with characteristic size r_{b}. The charged bion is the counter-part of the magnetic bion in QCD(adj) on $\mathbb{R}^3 \times \mathbb{S}^1$ [12]. The $N_f = 0$ case requires more care and is discussed in section 6.5.

For $\widehat{A}_{ii} > 0$, there exists a neutral bion. These are the correlated tunneling events beginning and ending at the same position in the landscape of vacua, hence the name neutral, $\mu_{\mathcal{B}_{ii}} = \alpha_i - \alpha_i = 0$. In this case, the interaction between $\mathcal{K}_i$ and $\overline{\mathcal{K}}_i$ has two attractive components: one due to boson exchange and another due to fermion zero mode exchange. Consequently, and naively, it does not make sense to talk about such molecules, because such an object would have a size as small as the kink-instanton size, where it is meaningless to talk about a kink-anti-kink molecule. In fact, this problem in 2d field theory compactified down to quantum mechanics, reduces to a variant of a readily solved problems

in quantum mechanics. The $N_f = 0$ case reduces to a problem addressed by Bogomolny and Zinn-Justin in the context of bosonic quantum mechanics, [21–24, 27–30], while the $N_f = 1$ case reduces to supersymmetric quantum mechanics addressed by Balitsky and Yung [32], and the $N_f > 1$ case is new. The Bogomolny-Zinn-Justin (BZJ) prescription tells us how to make sense of such kink-anti-kink configuratons in quantum mechanics.

1.5 Classification of non-perturbative ambiguities and "confluence equations"

According to the BZJ-prescription, applied now to the $N_f = 0$ deformed–$\mathbb{CP}^{N-1}$model, the instanton-anti-instanton $[\mathcal{K}_i\overline{\mathcal{K}}_i]$ amplitude is two-fold ambiguous. This is the first of many non-perturbative ambiguities in non-perturbative amplitudes, which will appear repeatedly. In fact, the amplitude exhibits a jump exactly on the real positive g^2 axis. Slightly above and below the real axis $g^2 = |g^2|e^{i\theta}$, $\theta = 0\pm\epsilon$, we define the left and right bion amplitude as

$$[\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0\pm} = \text{Re}[\mathcal{K}_i\overline{\mathcal{K}}_i] \pm ig^{-r_1}e^{-2S_0}, \tag{1.7}$$

Whenever there are massless fermions in the theory, the neutral bion event $\mathcal{B}_{ii}$ is ambiguity free. For $\mathbb{CP}^{N-1}$theories with massless fermions, the first non-perturbative ambiguity appears at *fourth order* in the semi-classical expansion, and is due to the bion-anti-bion amplitude:

$$[\mathcal{B}_{ij}\overline{\mathcal{B}}_{ij}]_{\theta=0\pm} = \text{Re}[\mathcal{B}_{ij}\overline{\mathcal{B}}_{ij}] \pm ig^{-r_2}e^{-4S_0} \quad . \tag{1.8}$$

This is a new result: namely that there is an ambiguity associated with neutral topological molecules whose constituents carry a fraction of the 2d instanton action, $1/N$ or $2/N$, respectively. On the face of it, this is a disaster, rendering the semi-classical expansion meaningless. After all, we are calculating a real physical observable, say, a mass gap or vacuum energy density, in a system without an instability. What does an imaginary part mean?

In fact, what looks like a disaster turns out to be a blessing in disguise. In asymptotically-free confining field theories, perturbation theory on $\mathbb{R}^2$ is factorially divergent, a Gevrey-1 series (defined below in section 6.1). It is often non-Borel summable; that is to say, the sum is ambiguous. One can define a left and right Borel sum, $\mathbb{B}_{0,\theta=0\pm}$, the imaginary part of which is the ambiguity. The ambiguity of perturbation theory on $\mathbb{R}^2$ cannot be cured solely with instanton-anti-instanton amplitudes (which were sufficient to cure the problem in quantum mechanics [22, 27–31]). The reason is that perturbation theory on $\mathbb{R}^2$ develops singularities in the Borel plane, called "infrared renormalons", [40, 41, 49] located at approximately $1/N$ of the 2d $[\mathcal{I}\overline{\mathcal{I}}]$ singularity. But there is no known field configuration on $\mathbb{R}^2$ to cancel the non-perturbative renormalon ambiguity of the perturbation theory. However, on small $\mathbb{R}^1 \times \mathbb{S}^1$ this is precisely what our neutral molecules (1.7) and (1.8) do.

In the semi-classical domain of the $\mathbb{CP}^{N-1}$model on small $\mathbb{R} \times \mathbb{S}^1$, as in gauge theory on $\mathbb{R}^3 \times \mathbb{S}^1$, we encounter two classes of non-perturbative ambiguities:

- *ambiguities in the Borel resummation of perturbation theory* either around the perturbative vacuum or in the background of instantons or kinks.
- *Ambiguities in the definition of the non-perturbative amplitudes*(1.7), (1.8) associated with neutral topological molecules, or molecules which include neutral sub-components.
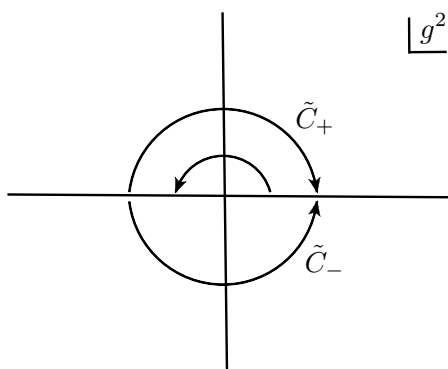
Figure 2. This figure depicts one of the main ideas of confluence equations and this work:
i) For real positive g^2, perturbation theory is non-Borel summable, i.e., ill-defined. Continue to negative g^2 where the perturbation theory becomes Borel summable. Then, continue back to $|g^2| \pm i\epsilon$, where one obtains left(right) Borel sums, $\mathbb{B}_{0,\theta=0^\pm}$. The absence of a smooth $\theta = 0$ limit means non-Borel summability, i.e., perturbation theory does not define the theory.
ii) For real positive g^2, the *neutral* bion amplitude is also ill-defined. Continue to negative g^2 where it is well-defined. Then, continue it back, via $\tilde{C}_\pm$ to $|g^2| \pm i\epsilon$. Upon continuation, one obtains a left(right) neutral bion amplitude $[\mathcal{B}_{ii}]_{\theta=0^\pm}$, with an imaginary discontinuity between the two. This means an ambiguity at $\theta = 0$. We demonstrate, analytically, the *exact cancellation* of these two ambiguities at order $e^{-2S_0} \equiv e^{-2S_I/N}$. This is the first of many such cancellation encoded into confluence equations. This means quantum field theory is non-perturbatively well-defined in continuum up to ambiguities at order e^{-4S_0}, which can further be improved systematically.
iii) The mathematical reason behind this phenomenon is that $\mathbb{R}^+$ in the complex g^2-plane is a Stokes ray. The jump in the resummed perturbation theory is the Stokes jump, which is (remarkably) mirrored by a jump in the neutral bion amplitude, to render observables meaningful even along the Stokes ray.

In order for the $\mathbb{CP}^{N-1}$ model to have a meaningful, semi-classical non-perturbative definition in the continuum, these two class of ambiguities must cancel. Denote the left (right) Borel resummation of perturbation theory, as described above, by $\mathbb{B}_{0,\theta=0^\pm}$. The fact that $\mathbb{B}_{0,\theta}$ exhibits a jump at $\theta = 0$ is the statement that the expansion is on a Stokes ray and the jump is the Stokes discontinuity. In this work, we will show explicitly the cancellation between the leading ambiguity in perturbation theory with the leading ambiguity in the non-perturbative neutral bion amplitude:

$$\operatorname{Im}\mathbb{B}_{0,\theta=0^\pm} + \operatorname{Im}[\mathcal{B}_{ii}]_{\theta=0^\pm} = 0 , \quad \text{up to } e^{-4S_0} \tag{1.9}$$

This is the first entry in a hierarchy of such cancellation equations: see section 7.4 and equations (7.26). These equations hold the key to the possibility of defining QFTs in the continuum using the resurgence framework. For this reason we give these equations a name: *perturbative-non-perturbative confluence equations*, or *confluence equations* for short. Our main conjecture is that all columns, i.e., all independent Θ sectors shown in the resurgence triangle (1.2) are resurgent functions of the parameter $\lambda\hbar \equiv \lambda$.

The first class of ambiguities is well-known in perturbation theory, both in quantum mechanics and QFT [36]. The latter class is perhaps less well known, but has been studied in the context of quantum mechanics [21–35]. Our main new result here is the connection with IR renormalons in QFT for this second class of ambiguities, in non-perturbative amplitudes for topological molecules in QFTs. This effect is primarily explored in this work and its companion on $\mathbb{R}^3 \times \mathbb{S}^1$ [19]. In the case of $\mathbb{CP}^{N-1}$, compared to Yang-Mills theories on $\mathbb{R}^3 \times \mathbb{S}^1$, the situation simplifies relatively because the small-compactified circle limit reduces directly to quantum mechanics, and certain important technical results already exist in the literature [22, 27–32].[6]

By using the resurgence formalism, we calculate the mass gap, a solely non-perturbative quantity and the Θ angle dependence of the vacuum energy density at arbitrary N. Both results are nontrivial, and are in accord with large-N and lattice results. Unlike the large-N considerations, which provide little microscopic insight, our derivation also makes explicit the microscopic origin of the mass gap and Θ-dependence in the $\mathbb{CP}^{N-1}$-model. To our knowledge, this is the first time that these non-perturbative quantities are analytically derived from first principles.

1.6 Interlude: prototypes of trans-series expansions

The cancellation of ambiguities between perturbative and non-perturbative expansions is an example of the resurgent behavior of trans-series expressions, in which the full expansion of a physical observable should be viewed as unifying the perturbative and non-perturbative parts, as in (1.1). Given the physical significance of such behavior, we sketch the main mathematical ideas of resurgence with some elementary examples. We refer the interested reader to [1–8, 10] for excellent introductory reviews. Let us state clearly at the beginning that a major advantage of this formalism is that it enables us to keep track of the relation between perturbative and non-perturbative expansions as we analytically continue in the phase of the coupling constant. The familiar relation between the single instanton sector and the large-order growth of perturbative coefficients, the Bender-Wu relations in quantum mechanics [38, 39, 83] and the Lipatov analysis in QFT [48], are the simplest examples, but they are the proverbial "tip of the iceberg". Resurgence produces a whole series of such relations between different non-perturbative sectors, and these are needed to demonstrate the full consistency of QFT.

A simple illustrative class of examples of trans-series and resurgence arises for the asymptotic behavior of functions satisfying a second-order differential equation [such as Bessel functions, Airy functions, parabolic cylinder functions, etc, or indeed for general Schrödinger equations], for which there are just *two* non-perturbative exponential terms in the trans-series expansion (1.1). Consider the following integral, related to the modified Bessel function K_0:

$$Z_1(\lambda) = \int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2\lambda} \sinh^2(\sqrt{\lambda}\, x)} \tag{1.10}$$

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[6]The same can be done for QFTs on $\mathbb{R}^4$ as well, by an unconventional compactification on $T^3 \times \mathbb{R}$. This will be explored elsewhere.

$$= \frac{1}{\sqrt{\lambda}} \, e^{\frac{1}{4\lambda}} \, K_0 \left(\frac{1}{4\lambda} \right) \tag{1.11}$$

$$\sim \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (-1)^n (2\lambda)^n \frac{\Gamma(n + \frac{1}{2})^2}{n! \, \Gamma\left(\frac{1}{2}\right)^2}, \qquad \lambda \to 0^+ \tag{1.12}$$

This is the "partition function", $Z_1(\lambda) = \operatorname{tr} e^{-V_1}$, for the 0-dimensional field theory with potential $V_1(x) = \frac{1}{2\lambda} \sinh^2(\sqrt{\lambda}\, x) = \frac{1}{2}x^2 + \frac{\lambda}{6}x^4 + \dots$. The perturbative series in (1.12) is factorially divergent (Gevrey class 1; see section 6.1), but has coefficients alternating in sign, and is Borel summable (see (1.16) below). On the other hand, for the periodic potential, $V_2(x) = \frac{1}{2\lambda} \sin^2(\sqrt{\lambda}\, x) = \frac{1}{2}x^2 - \frac{\lambda}{6}x^4 + \dots$, obtained formally by $\lambda \to -\lambda$, we have the partition function, $Z_2(\lambda) = \operatorname{tr} e^{-V_2}$, related to the modified Bessel function I_0:

$$Z_2(\lambda) = \int_0^{\pi/\sqrt{\lambda}} dx \, e^{-\frac{1}{2\lambda} \sin^2(\sqrt{\lambda}\, x)} \tag{1.13}$$

$$= \frac{\pi}{\sqrt{\lambda}} \, e^{-\frac{1}{4\lambda}} \, I_0 \left(\frac{1}{4\lambda} \right) \tag{1.14}$$

$$\sim \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (2\lambda)^n \frac{\Gamma(n + \frac{1}{2})^2}{n! \, \Gamma\left(\frac{1}{2}\right)^2}, \qquad \lambda \to 0^+ \tag{1.15}$$

Formally, it is tempting to conclude from the perturbative expansions in (1.12) and (1.15) that $Z_1(-\lambda) = Z_2(\lambda)$, but this is *not true*, as it misses important non-perturbative contributions [see (1.22) below].

The first clear sign of a problem is that the perturbative expansion in (1.15) is a *non-alternating* divergent series and is not Borel summable. There is therefore the possibility of a non-perturbative imaginary ambiguity in $Z_2(\lambda)$. However, we know that the periodic potential system is stable, so there should be no imaginary part, with $Z_2(\lambda)$ being real.

We can resolve this problem using Borel-Écalle summation. From (1.12) we see that the Borel sum of the perturbative series for $Z_1(\lambda)$ can be expressed in terms of a hypergeometric function:

$$Z_1(\lambda) = \sqrt{\frac{\pi}{2}} \, \frac{1}{2\lambda} \int_0^{\infty} dt \, e^{-\frac{t}{2\lambda}} \, {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}, 1; -t \right) \tag{1.16}$$

For $\lambda > 0$, this Borel integral is well defined, as the hypergeometric function has a cut $(-\infty, -1)$ only along the negative t axis. On the other hand, a formal Borel expression for $Z_2(\lambda)$ has a cut on the contour of integration. So we must define $Z_2(\lambda)$ by analytic continuation from $Z_1(\lambda)$. Consider rotating the phase of λ in the $Z_1(\lambda)$ Borel expression (1.16) so that $\lambda \to |\lambda| e^{i\theta}$. Then the direction of the branch cut rotates, and when θ approaches $\pm\pi$, the branch cut approaches the contour of integration, either from below or above. In this limit, when $\theta = \pm\pi$, the alternating asymptotic series in $Z_1(\lambda)$ in (1.12) becomes non-alternating. There is however an ambiguity, because we can rotate either clockwise or anti-clockwise. The difference between these two results is the difference of the two so-called "lateral Borel sums", $\mathbb{B}_\pm(\lambda)$, a measure of the ambiguity in summing the non-alternating series, and can be written as a Borel integral above and below a cut:

$$Z_1(e^{i\pi} \lambda) - Z_1(e^{-i\pi} \lambda) = \sqrt{\frac{\pi}{2}} \, \frac{1}{2\lambda} \int_1^{\infty} dt \, e^{-\frac{t}{2\lambda}} \left[{}_2F_1 \left(\frac{1}{2}, \frac{1}{2}, 1, t - i\varepsilon \right) - {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}, 1, t + i\varepsilon \right) \right]$$

$$= -(2i) \sqrt{\frac{\pi}{2}} \frac{1}{2\lambda} e^{-\frac{1}{2\lambda}} \int_0^\infty dt\, e^{-\frac{t}{2\lambda}} \, _2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, -t\right) \tag{1.17}$$

$$= -2i\, e^{-\frac{1}{2\lambda}}\, Z_1(\lambda) \tag{1.18}$$

To obtain (1.17) we used the known discontinuity property of the hypergeometric function [84]:

$$_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, t + i\varepsilon\right) - \, _2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, t - i\varepsilon\right) = 2i\, _2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, 1 - t\right) \tag{1.19}$$

Notice the amazing fact that in the discontinuity of $Z_1(\lambda)$ in (1.18) we recover an exponential factor, $e^{-2(\frac{1}{4\lambda})}$, multiplying the original function $Z_1(\lambda)$. This is not an accident — it is a sign of resurgence at work.

In fact, the modified Bessel functions $K_0(z)$ and $I_0(z)$ are related under analytic continuation of their argument by the following connection formula:

$$K_0(e^{\pm i\pi}\, |z|) = K_0(|z|) \mp i\,\pi\, I_0(|z|) \tag{1.20}$$

Therefore, we deduce that

$$Z_1(e^{\pm i\pi}\, \lambda) = \frac{\pi}{\sqrt{\lambda}}\, e^{-\frac{1}{4\lambda}}\, I_0\left(\frac{1}{4\lambda}\right) \mp \frac{i}{\sqrt{\lambda}}\, e^{-\frac{1}{4\lambda}}\, K_0\left(\frac{1}{4\lambda}\right) \tag{1.21}$$

$$= Z_2(\lambda) \mp i\, e^{-\frac{1}{2\lambda}}\, Z_1(\lambda) \tag{1.22}$$

which is consistent with (1.18).

Another way to understand this is that the naive asymptotic expansions of $K_0(z)$ and $I_0(z)$ for $z \to +\infty$ are not consistent with the connection formula (1.20), reflecting the fact that these asymptotic expansions do not fully define the functions. The asymptotic expansion, as $z \to +\infty$, of $K_0(z)$ is easily obtained from the following integral representation:

$$K_0(z) = \frac{1}{2} \int_0^\infty \frac{dt}{t}\, e^{-\frac{z}{2}\left(t + \frac{1}{t}\right)} \sim \sqrt{\frac{\pi}{2z}}\, e^{-z} \sum_{n=0}^\infty \frac{(-1)^n}{(2z)^n} \frac{\Gamma\left(n + \frac{1}{2}\right)^2}{n!\, \Gamma\left(\frac{1}{2}\right)^2} \tag{1.23}$$

The integral has saddle points at $t = \pm 1$, but only the saddle at $t = +1$ contributes. The resulting series is asymptotic but Borel summable. The other modified Bessel function, $I_0(z)$, has a quite different integral representation

$$I_0(z) = \frac{1}{2\pi i} \int_C \frac{dt}{t}\, e^{-\frac{z}{2}\left(t + \frac{1}{t}\right)} \tag{1.24}$$

where C is the contour of the anti-clockwise unit circle. Now both saddle points, at $t = \pm 1$, contribute, and the full resurgent asymptotic expansion reads:

$$I_0(z) = \sqrt{\frac{1}{2\pi z}} \left[e^z \sum_{n=0}^\infty \frac{1}{(2z)^n} \frac{\Gamma\left(n + \frac{1}{2}\right)^2}{n!\, \Gamma\left(\frac{1}{2}\right)^2} + \left\{ \begin{matrix} i \\ -i \\ 0 \end{matrix} \right\} e^{-z} \sum_{n=0}^\infty \frac{(-1)^n}{(2z)^n} \frac{\Gamma\left(n + \frac{1}{2}\right)^2}{n!\, \Gamma\left(\frac{1}{2}\right)^2} \right] \tag{1.25}$$

where the three cases are for $0 < \arg z < \pi$, $-\pi < \arg z < 0$, and $\arg z = 0$, respectively. This is a two-term trans-series expansion, with two different non-perturbative (in $z = \frac{1}{4\lambda}$)

exponential terms, $e^{\pm z}$, one associated with each saddle point. The second term is exponentially sub-leading when $\text{Re}(z) > 0$, and so is often neglected. But as the phase of z approaches $\pm \pi/2$, both terms contribute, accounting for the real oscillatory nature of the ordinary Bessel functions $J_0(z)$ and $Y_0(z)$. More importantly, the coefficients in the asymptotic expansions multiplying each exponential term are related to one another in a particular way, differing simply by a factor of $(-1)^n$. This correspondence is a direct consequence of Darboux's theorem: the high orders of the asymptotic expansion about one saddle involve high derivatives, and these are determined by the behavior of the function in the vicinity of the nearest singularity, which is the other saddle. This is also manifest in our transseries expansion: The value of the "action" $S(z) = \frac{z}{2}\left(t + \frac{1}{t}\right)$ at the two saddle points $t = \pm 1$ are $S_+ = z$ and $S_- = -z$, and the two exponential terms in the trans-series are respectively $e^{-S_+} = e^{-z}$ and $e^{-S_-} = e^z$. Dingle defines the difference of the two "actions" as the *singulant* [1, 4–6]

$$\Delta S_{+-} = S_+ - S_- \, , \tag{1.26}$$

and notes that the large order behavior of the asymptotic series is controlled by the singulant. The trans-series is roughly (ignoring inessential details)

$$I_0(z) \sim \left[e^{-S_-} \sum_{n=0}^{\infty} \frac{n!}{(\Delta S_{+-})^n} + \begin{Bmatrix} i \\ -i \\ 0 \end{Bmatrix} e^{-S_+} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(\Delta S_{+-})^n} \right] \tag{1.27}$$

This feature is generic, as was discovered by Dingle in the context of WKB analysis [1], and elaborated by Berry and Howls [4–6] for more general integrals with multiple saddles. Let $t_i, i = 1, 2, \ldots$ denote the set of saddle points with actions $S(t_i) = S_i$. Ref. [4–6] defines a singulant $S_{ij} = S_i - S_j$ for each adjacent saddle j. The leading $n!$ divergence of the asymptotic series is controlled by the saddle(s) j for which $|S_{ij}| < |S_{ij'}|$ for all j'. We refer the reader to ref. [4–6] for a discussion of the intricate topological structure that arises in an integral with multiple saddles, characterizing which saddles affect one another. This is one aspect of resurgence — the expansions around different saddles [i.e., the expansions about different instantons] are necessarily related to one another in a very precise way, as encoded in Darboux's theorem.[7]

Other important examples of resurgence, more like what we expect to encounter in quantum field theory, involve an *infinite* series of non-perturbative exponential terms. The simplest example of this type is given by Stirling's formula for the gamma function. In fact, it is easier to describe in terms of the digamma function $\psi(x) = \Gamma'(z)/\Gamma(z)$. Consider the divergent but Borel summable "perturbative" expansion for $z \to +\infty$ (so $\frac{1}{z}$ is the small perturbative parameter):

$$\psi\left(\frac{1}{2} + z\right) = \ln z - \int_0^{\infty} dt \, e^{-2zt} \left(\frac{1}{\sinh t} - \frac{1}{t} \right) \tag{1.28}$$

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[7]Clearly, the form given in (1.27) is very suggestive for quantum field theory. Indeed, we will identify a very similar structure in quantum field theory (albeit with infinitely many exponential factors, as in the sub-sequent Γ function example) in the perturbative expansion around the perturbative vacuum or different topological sectors. In particular, the singulants will be identified with neutral topological composites (like neutral bion), as opposed to single instantons.

$$\sim \ln z - 2 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!}{(2\pi z)^{2n+2}} \left(\frac{1 - 2^{2n+1}}{2^{2n+1}} \right) \zeta(2n+2) \qquad (1.29)$$

The series expansion is sufficient to satisfy the basic recurrence relation, $\psi\left(\frac{1}{2} + z\right) = \psi\left(-\frac{1}{2} + z\right) + 1/(z - 1/2)$, derived from the gamma function relation $z\,\Gamma(z) = \Gamma(z+1)$. Now imagine rotating z to $e^{\pm i\pi/2} z$, so that the series becomes non-alternating. The series representation (1.29) suggests that the difference between these two rotations might be: $\psi\left(\frac{1}{2} + iz\right) - \psi\left(\frac{1}{2} - iz\right) \sim i\pi$, coming from the log term. However, we also know the (*global*) reflection formula for the gamma function, $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, which implies that

$$\psi\left(\frac{1}{2} + iz\right) - \psi\left(\frac{1}{2} - iz\right) = i\,\pi\,\tanh(\pi\,z) = i\,\pi\left(1 + 2\sum_{k=1}^{\infty} (-1)^k\,e^{-2\pi k z}\right) \qquad (1.30)$$

We see that in addition to the expected $i\,\pi$ term, there is an infinite series of exponentially small non-perturbative terms (in $e^{-2\pi z} = e^{-2\pi/\lambda}$). These are neglected in the series representation (1.29) but are needed in order to satisfy the reflection formula. Moreover, we obtain the following expression for the real part:

$$\mathrm{Re}\,\psi\left(\frac{1}{2} + i\,z\right) \sim \ln z + 2\sum_{n=0}^{\infty} \frac{(2n+1)!}{(2\pi z)^{2n+2}} \left(\frac{1 - 2^{2n+1}}{2^{2n+1}} \right) \zeta(2n+2) - i\pi \sum_{k=1}^{\infty} (-1)^k\,e^{-2\pi k z} \quad (1.31)$$

This formula looks strange since the l.h.s. is obviously real. The formal first sum on the r.h.s. looks real, but this is deceptive. It also looks like there is an imaginary contribution on the r.h.s., and this is also deceptive. Upon Borel resummation of the formal series, the imaginary non-perturbative part on the r.h.s. cancels the imaginary part coming from the Borel summation of the non-alternating divergent series. This is how the r.h.s. is (not so obviously) real.

The reflection formula is derived from an analytic continuation of the gamma function, so it encodes *global* information about the function, more than just the basic recursion formula $z\,\Gamma(z) = \Gamma(z+1)$ (which may be viewed as the local Schwinger-Dyson equation derived from the integral representation of the gamma function) and its series solution in (1.29). In other words, a global resurgent trans-series expression for the digamma function at large $|z|$ must include both perturbative powers of $1/z$ and also exponentially small non-perturbative terms $e^{-2\pi k z}$, for all positive integer k. Furthermore, the perturbative series part is not independent of the non-perturbative exponential part, as they are connected by the reflection formula.[8] Equivalently, we can deduce the exponential terms from directed Borel integrals that go around the poles of the Borel integrand at $t = ik\pi$ in (1.28), which can in turn be identified with saddles of an integral representation, which interact with one another according to Darboux's theorem. An analogous analysis is possible for the Barnes multiple gamma functions and for the Hurwitz zeta function, the basic functions known to underly quantum field theoretic effective actions in constant gravitational and electromagnetic background fields [85].

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[8]In QFT, the counter-part of this global relation is the confluence equations discussed in section 7.4.

At first sight one might think that these examples are unrealistically special, since we know extra information about these functions [such as a differential equation or a functional relation]. However, the first example of the modified Bessel functions captures much of the physics of the perturbative and non-perturbative ambiguities arising in the quantum mechanics of periodic potentials such as the Sine-Gordon model [24, 31, 46], along the lines of the Bogomolny-Zinn-Justin analysis [21, 22, 27–30], and is central to our $\mathbb{CP}^{N-1}$ analysis in this paper (see section 7). The second example of the digamma function underlies the non-perturbative ambiguities found in quantum field theoretic effective actions in constant electromagnetic and gravitational backgrounds such as found in the analytic continuation between de Sitter and anti-de Sitter spaces [85], and in certain highly symmetric Chern-Simons models [51–56, 86].

Beyond these examples, the essence of Écalle's theory of resurgence is that by studying the analytic structure of Borel integrals in the vicinity of all the poles and cuts, one can in principle reconstruct all information about the function starting from a series representation [2, 3]. One of Écalle's main results is that the set of "analyzable functions", for which these operations apply, forms a basis of trans-series expansions which appears to be sufficient to capture all the required information for the types of functions that appear in physics and mathematics. This is a profound and provocative viewpoint, which has motivated our study here of the $\mathbb{CP}^{N-1}$ model, to see to what extent this bridge between perturbative and non-perturbative physics is actually realizable in quantum field theory.

2 $\mathbb{CP}^{N-1}$ models

We study the $\mathbb{CP}^{N-1}$ model defined on two-dimensional Euclidean space-time, $\mathbb{R}^1 \times \mathbb{S}^1$, with the topology of a cylinder, using coordinates (x_1, x_2), where x_1 is the non-compact Euclidean time direction, and x_2 is the compactified spatial dimension of length $L = 2\pi R$. The classical action of the $\mathbb{CP}^{N-1}$ model is

$$S = \frac{2}{g^2} \int d^2x \, (D_\mu n)^\dagger D_\mu n \tag{2.1}$$

where $n = (n_1, n_2, \ldots n_N)^T$ is a complex N-component column vector satisfying $n^\dagger n = 1$. The covariant derivative operator is $D_\mu = \partial_\mu + iA_\mu$, and the abelian gauge field A_μ is determined by its equation of motion to be

$$A_\mu = \frac{i}{2} \left(n^\dagger \partial_\mu n - \partial_\mu n^\dagger n \right) \tag{2.2}$$

The $\mathbb{CP}^{N-1}$ model has a local U(1) gauge redundancy,

$$n(x) \to e^{i\alpha(x)} n(x), \qquad A_\mu(x) \to A_\mu(x) - \partial_\mu \alpha(x) \tag{2.3}$$

and also a global U(N) symmetry

$$n(x) \to U \, n(x), \qquad U \in \mathrm{U}(N) \tag{2.4}$$

The n-field parametrizes the coset space

$$\mathcal{M}_{N,1} \equiv \mathbb{CP}^{N-1} = \frac{\mathrm{U}(N)}{\mathrm{U}(N-1) \times \mathrm{U}(1)} \tag{2.5}$$

and is characterized by $N^2 - 1 - (N-1)^2 = 2(N-1)$ real fields, which are massless to all orders in perturbation theory.

It will be useful to add a topological theta-term to the action:

$$S_\Theta = i\Theta\, Q \tag{2.6}$$

where Θ is an angular parameter with period 2π, and Q is the topological charge:

$$Q = -\frac{i}{2\pi} \int d^2x\, \epsilon_{\mu\nu}(D_\mu n)^\dagger D_\nu n = -\frac{i}{2\pi} \int d^2x\, \epsilon_{\mu\nu}\partial_\mu\left(n^\dagger \partial_\nu n\right) = \frac{1}{2\pi} \int d^2x\, \epsilon_{\mu\nu}\partial_\mu A_\nu \tag{2.7}$$

In bosonic and deformed $\mathbb{CP}^{N-1}$ theories, physical observables exhibit Θ dependence. The Θ term will be particularly useful in analyzing the semi-classical trans-series expansion. We will also consider the $\mathbb{CP}^{N-1}$ model with N_f Dirac fermions:

$$S_{\mathrm{fermion}} = \frac{2}{g^2} \int d^2x \left[-i\overline{\psi}\gamma_\mu D_\mu \psi + \frac{1}{4}\left((\overline{\psi}\psi)^2 + (\overline{\psi}\gamma_3\psi)^2 - (\overline{\psi}\gamma_\mu\psi)^2\right)\right] \tag{2.8}$$

where a sum over the fermion flavor index is assumed. The $N_f = 1$ theory is the $\mathcal{N} = (2,2)$ supersymmetric model [87], while for $N_f > 1$ the theory is non-supersymmetric.

All $\mathbb{CP}^{N-1}$ models are asymptotically free, because the first order β-function is independent of N_f. Whether this class of theories is confining or conformal in the IR depends on N_f. The coupling $g^2(\mu)$ is a function of energy scale μ. At one loop order, we have

$$\Lambda^{\beta_0} = \mu^{\beta_0} e^{-\frac{4\pi}{g^2(\mu)}}, \qquad \beta_0 = N \qquad \text{or} \qquad \Lambda = \mu\, e^{-S_I/\beta_0} = \mu\, e^{-\frac{4\pi}{N g^2(\mu)}}, \tag{2.9}$$

where Λ is the strong coupling scale, β_0 is the coefficient of the 1-loop beta function, and S_I is the instanton action. Λ is the natural scale of the mass gap of the theory. Note the simple but important fact that the relation between Λ and μ, via dimensional transmutation, is *not* determined by the instanton factor, but by the instanton factor divided by N. Thus, on $\mathbb{R}^2$, a physical quantity like the mass gap cannot have a semi-classical origin or description. On the other hand, in the weakly coupled semi-classical domain on $\mathbb{R}^1 \times \mathbb{S}^1$, we will show that the semi-classical expansion is in $e^{-S_I/\beta_0} \equiv e^{-\frac{4\pi}{N g^2(\mu)}}$, allowing us to identify the configurations which lead to a mass gap.

The $N_f = 1$ model has a classical $\mathrm{U}(1)_V \times \mathrm{U}(1)_A$ symmetry, acting on the 2d Dirac fermion $\psi = (\psi_+, \overline{\psi}_-)^T$ as

$$V : \psi \longrightarrow e^{i\delta}\psi, \qquad A : \psi \longrightarrow e^{i\sigma_3\beta}\psi \tag{2.10}$$

where ψ_+ and ψ_- are the left and right-mover modes. The $\mathrm{U}(1)_A$ symmetry is anomalous due to instanton effects. In the background of an instanton, chirality is violated by $2N$ units, and the associated 2d instanton amplitude is given by

$$I_{2\mathrm{d}} = e^{-S_I}(\psi_-\psi_+)^N \tag{2.11}$$

Quantum mechanically, the exact axial symmetry of the theory is $\mathbb{Z}_{2N}$. On $\mathbb{R}^2$, there is compelling evidence that the $\mathbb{Z}_{2N}$ discrete chiral symmetry is dynamically broken down to $\mathbb{Z}_2$ by a fermion-bilinear condensate

$$\langle \psi_- \psi_+ \rangle = N\Lambda\, e^{i\frac{2\pi k}{N}}, \qquad k = 0, 1, \ldots, N-1 \tag{2.12}$$

leading to N isolated vacua. The supersymmetric index for this theory is also N. We will give a new derivation of the chiral condensate and supersymmetric index in section 5.

In the multi-flavor theory $N_f > 1$ theory, the classical global symmetry is $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$. Quantum mechanically, due to instanton effects, this symmetry is reduced to

$$\left[\mathrm{U}(1)_V \times \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathbb{Z}_{2N\,N_f} \right] / \mathbb{Z}_2 \times \mathbb{Z}_{N_f} \tag{2.13}$$

The terms in the denominator are there to prevent double counting. $\mathbb{Z}_2$ is fermion number mod two, which lives both in $\mathrm{U}(1)_V$ and $\mathbb{Z}_{2N\,N_f}$. $\mathbb{Z}_{N_f}$ lives in both the diagonal subgroup of the continuous chiral symmetries as well as in $\mathbb{Z}_{2N\,N_f}$. The true discrete chiral symmetry of the multi-flavor theory is also just $\mathbb{Z}_N$, as in the case of single flavor theory. The 2d instanton amplitude in the N_f flavor theory is given by

$$I_{\mathrm{2d}} = e^{-S_I} \left[\det_{ab}(\psi_-^a \psi_+^b) \right]^N \sim e^{-S_I} \left[(\psi_-^{n_1}\psi_+^{n_1})(\psi_-^{n_2}\psi_+^{n_2}) \ldots (\psi_-^{N_f}\psi_+^{N_f}) + \text{ permutations} \right]^N \tag{2.14}$$

This is of course a singlet under continuous chiral symmetries, but reduces the discrete chiral symmetry down to $\mathbb{Z}_{2N\,N_f}$. In two-dimensions, the continuous chiral symmetry cannot be spontaneously broken, by the Coleman-Mermin-Wagner theorem [59]. However, the discrete chiral symmetry may be broken. The order parameter associated with the discrete chiral symmetry is

$$\langle \det_{ab}(\psi_-^a \psi_+^b) \rangle \sim \Lambda^{N_f} e^{i\frac{2\pi k}{N}}, \qquad k = 0, 1, \ldots, N-1 \tag{2.15}$$

leading to N isolated vacua, as in the case of the supersymmetric theory. The chiral condensates (2.12) and (2.15) on $\mathbb{R}^2$, just like the mass gap of the bosonic theory, cannot have a semi-classical description. Since the instanton size is a modulus parameter, as noted in the Introduction, there is no well-defined semi-classical dilute instanton gas description on $\mathbb{R}^2$.

2.1 Twisted boundary conditions vs. background in classical theory

Twisted boundary conditions can be used to probe more finely the structure of the $\mathbb{CP}^{N-1}$ model. We first show the equivalence at the *classical* level between the theory with twisted boundary conditions and action (2.1), and the theory with untwisted (periodic) boundary conditions in the presence of a $\mathrm{U}(1)^N$ background "gauge" field. In the next section (2.2), we show that the background field is associated with a gauge invariant line operator that we construct from the $\mathbb{CP}^{N-1}$ fields. The quantum mechanical stability of the background is discussed in section 3. As stated in the Introduction, twisted boundary conditions on their own do not necessarily imply semi-classical calculability. What is crucial is that a certain type of background must be stable, in order for a semi-classical analysis to apply usefully to $\mathbb{CP}^{N-1}$.

Consider twisted boundary conditions for the bosonic and fermionic fields in $\mathbb{CP}^{N-1}$:

$$n(x_1, x_2 + L) = \Omega_0 \, n(x_1, x_2) \,, \qquad\qquad n_i(x_1, x_2 + L) = e^{2\pi i \mu_i} \, n_i(x_1, x_2)$$
$$\psi(x_1, x_2 + L) = \pm \Omega_0 \, \psi(x_1, x_2) \,, \qquad \psi_i(x_1, x_2 + L) = \pm e^{2\pi i \mu_i} \, \psi_i(x_1, x_2) \qquad (2.16)$$

where $\Omega_0 \in \mathrm{U}(N)$ is the twist matrix

$$\Omega_0 = \begin{pmatrix} e^{2\pi i \mu_1} & 0 & \dots & 0 \\ 0 & e^{2\pi i \mu_2} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & e^{2\pi i \mu_N} \end{pmatrix}, \qquad 0 \leq \mu_1 \leq \mu_2 \leq \dots \leq \mu_N < 1 \qquad (2.17)$$

In other words, the fields are periodic only up to a global Ω_0 rotation, with $\Omega_0 \in \mathrm{U}(N)$. The twist matrix Ω_0 can always be brought to this diagonal form by a similarity transformation.

A general twisted boundary condition with non-degenerate twist explicitly breaks the global $\mathrm{U}(N)$ symmetry down to $\mathrm{U}(1)^N$. A special role will be played by the maximally symmetric twist, or $\mathbb{Z}_N$-symmetric twist:

$$(\mu_1, \mu_2, \dots, \mu_N) = \frac{1}{N}(0, 1, 2, \dots, N-1) \qquad (2.18)$$

for which the eigenvalues of Ω_0 are equidistant.

The twisted boundary condition for the theory with action (2.1) is equivalent to the theory with periodic boundary conditions and a twisted $\mathrm{U}(1)^N$ background field. To see this, define periodic fields $\widetilde{n}_j$, for $j = 1, \dots N$:

$$\widetilde{n}_j(x_1, x_2) = e^{-i\frac{2\pi \mu_j x_2}{L}} n_j(x_1, x_2) \,, \qquad\qquad \widetilde{n}_j(x_1, x_2 + L) = \widetilde{n}_j(x_1, x_2), \qquad (2.19)$$

The gauge field A_μ in (2.2) can be re-expressed in terms of the periodic fields $\widetilde{n}(x_1, x_2)$ as

$$A_\mu(n) = A_\mu(\widetilde{n}) - \frac{2\pi}{L} \delta_{\mu 2} \sum_{j=1}^{N} \widetilde{n}_j^\dagger \, \mu_j \, \widetilde{n}_j \qquad (2.20)$$

Then the bosonic action (2.1) can be expressed in terms of the periodic field as

$$S = \frac{2}{g^2} \int d^2 x \left(|D_\mu^{\mathrm{b}} \widetilde{n}_j|^2 - |\tilde{n}_j^* D_\mu^{\mathrm{b}} \tilde{n}_j|^2 \right) \,,$$
$$D_\mu^{\mathrm{b}} \cdot = \ \partial_\mu \ \cdot + i \, \delta_{\mu 2} \, \frac{2\pi}{L} \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & \mu_N \end{pmatrix} . \qquad (2.21)$$

The change in the fermonic action (2.8), under similar manipulations, is also equivalent to the replacement $D_\mu \to D_\mu^{\mathrm{b}}$. This corresponds to the action of the $\widetilde{n}$-particles coupled to a "background $\mathrm{U}(1)^N$ gauge holonomy" Ω_0. The gauge connection that we refer to here is "the sigma model connection" that will be defined in the next section. We comment that the twisted boundary conditions introduced here have some relations to, but also some significant differences from, twisted mass terms in sigma models [89, 91–96]. A closer, but also not exact, analogy is with symmetry breaking terms for studying sphalerons in sigma models [97–100].

2.2 Parametrization of $\mathbb{CP}^{N-1}$ manifold and σ-connections

The local splitting of the n field into a modulus and phase $n_i = e^{i\varphi_i}|n_i|$ provides a useful parametrization to study the dynamics of the theory. Despite the fact that the splitting is local, it will help us to build new line operators, which are useful in the study of the phases, and to make connections with the 4d gauge theory.

Definiton 1: point-wise modulus and phase splitting. The point-wise splitting amounts to rewriting the $\mathbb{CP}^{N-1}$field in *complexified hyperspherical coordinates*, involving $2(N-1)$ angular fields:

$$
\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_N \end{pmatrix} = \begin{pmatrix} e^{i\varphi_1} \cos\frac{\theta_1}{2} \\ e^{i\varphi_2} \sin\frac{\theta_1}{2} \cos\frac{\theta_2}{2} \\ e^{i\varphi_3} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} \cos\frac{\theta_3}{2} \\ \vdots \\ e^{i\varphi_N} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} \sin\frac{\theta_3}{2} \ldots \sin\frac{\theta_{N-1}}{2} \end{pmatrix} \qquad \theta_i \in [0,\pi], \qquad \varphi_i \in [0,2\pi). \quad (2.22)
$$

The angular fields $\{\theta_1,\ldots,\theta_{N-1}\}$ are independent, and one out of $\{\varphi_1,\ldots,\varphi_N\}$ can be gauged away. Hence, there are $2(N-1)$ microscopic degrees of freedom, as noted in the coset construction (2.5). We set the gauge choice to

$$
\sum_{i=1}^{N} \varphi_i = 0 \qquad (\mathrm{mod}\ 2\pi) \tag{2.23}
$$

by using the local U(1) gauge redundancy (2.3):

$$
|n_i(x_1, x_2)| \to |n_i(x_1, x_2)| \qquad \varphi_i(x_1, x_2) \to \varphi_i(x_1, x_2) + \alpha(x_1, x_2) \tag{2.24}
$$

The modulus is gauge invariant, whereas each phase transforms as a "gauge" connection. This splitting is crucial in the construction of a refined line-operator probing the structure of the theory.

Definition 2: sigma-model connection or σ-connection. The derivative of each phase, $-\partial_\mu \varphi_i, i = 1, \ldots, N$, transforms under (2.3) like a gauge connection, modulo the constraint (2.23). Therefore, and due to reasons that will follow, it is useful to define $-\partial_\mu \varphi_i \equiv \mathcal{A}_{\mu,i}$ as the "sigma-model connection", or "σ-connection" for short. Under (2.24), it rotates as

$$
-\partial_\mu \varphi_i \equiv \mathcal{A}_{\mu,i} \longrightarrow -\partial_\mu \varphi_i + \partial_\mu \alpha \equiv \mathcal{A}_{\mu,i} + \partial_\mu \alpha \tag{2.25}
$$

This N-component (minus the constraint) σ-connection is crucial to build new line operators.

The relation between various useful representations of the $\mathbb{CP}^{N-1}$fields are

$$
n(x_1, x_2) = \Omega(x_1, x_2)\mathcal{R}(x_1, x_2), \qquad \widetilde{n}(x_1, x_2) = \widetilde{\Omega}(x_1, x_2)\mathcal{R}(x_1, x_2) \tag{2.26}
$$

where $\mathcal{R} \equiv \mathrm{Diag}(|n_i|)$, with $0 \leq |n_i| \leq 1$, according to (2.22) and

$$
\Omega(x_1, x_2) = \widetilde{\Omega}(x_1, x_2)T(x_2), \qquad \text{in components},
$$

$$
\begin{pmatrix}
e^{i\varphi_1} & 0 & \dots & 0 \\
0 & e^{i\varphi_2} & \dots & 0 \\
\vdots & & & \\
0 & 0 & \dots & e^{i\varphi_N}
\end{pmatrix}
=
\begin{pmatrix}
e^{i\phi_1} & 0 & \dots & 0 \\
0 & e^{i\phi_2} & \dots & 0 \\
\vdots & & & \\
0 & 0 & \dots & e^{i\phi_N}
\end{pmatrix}
\begin{pmatrix}
e^{i\frac{2\pi\mu_1 x_2}{L}} & 0 & \dots & 0 \\
0 & e^{i\frac{2\pi\mu_2 x_2}{L}} & \dots & 0 \\
\vdots & & & \\
0 & 0 & \dots & e^{i\frac{2\pi\mu_N x_2}{L}}
\end{pmatrix}
\tag{2.27}
$$

The $\varphi_i(x_1, x_2)$ field is not periodic, as per (2.16), $\varphi_i(x_1, x_2 + L) = \varphi_i(x_1, x_2) + 2\pi\mu_i$, while the $\phi(x_1, x_2)$ fields are periodic, and $T(x_2)$ is a twist matrix depending only on the compactified coordinate x_2.

Line operators. The local (point-wise) decomposition of fields can be used to construct line operators, which are useful to probe the phases of the theory in a way which provides *more information* than the usual Wilson line associated with the auxiliary gauge field (2.2). In particular,

$$
A_\mu = -\sum_{i=1}^{N} |n_i|^2 \partial_\mu \phi_i = \sum_{i=1}^{N} |n_i|^2 \mathcal{A}_{\mu,i}
\tag{2.28}
$$

and obeys (2.3). On $\mathbb{R}^2$, one can define an open Wilson line

$$
W(\mathbf{a}, \mathbf{b}) = e^{i \int_{\mathbf{a}}^{\mathbf{b}} dx_\mu A_\mu} = e^{-i \sum_{i=1}^{N} |n_i|^2 \int_{\mathbf{a}}^{\mathbf{b}} dx_\mu \partial_\mu \phi_i} = \prod_{i=1}^{N} e^{-i|n_i|^2 \int_{\mathbf{a}}^{\mathbf{b}} dx_\mu \partial_\mu \phi_i}
\tag{2.29}
$$

which transforms covariantly. Instead of this conventional operator, we define a more refined version, because there is more information in the phases than there is in their weighted-sum (2.28).

Definition 3: σ-connection holonomy. We define a line operator associated with the σ-connection holonomy:

$$
(^L\Omega)_{ij} = (^L\Omega)_i \delta_{ij}, \qquad ^L\Omega_i(\mathbf{a}, \mathbf{b}) = e^{i \int_{\mathbf{a}}^{\mathbf{b}} dx_\mu \mathcal{A}_{\mu,i}} = e^{i(\varphi_i(\mathbf{a}) - \varphi_i(\mathbf{b}))}
\tag{2.30}
$$

as an $N \times N$ matrix. Note that the line operator, because of the way that it is constructed, can be written as a product of two-point operators at the end-points. This is because the σ-connection $\mathcal{A}_{\mu,i}$ is a total derivative. Under a U(1) gauge rotation, the line operator transform covariantly:

$$
^L\Omega(\mathbf{a}, \mathbf{b}) = \Omega(\mathbf{a})\Omega^\dagger(\mathbf{b}) \to e^{i\alpha(\mathbf{a})} \; ^L\Omega(\mathbf{a}, \mathbf{b}) \; e^{-i\alpha(\mathbf{b})}
\tag{2.31}
$$

The traced version of this line operator making a topologically non-trivial loop on the $\mathbb{S}^1$ circle will play a role parallel to the Wilson line (or Polyakov loop) in SU(N) gauge theory.

2.3 Center symmetry

When compactified on $\mathbb{R}^1 \times \mathbb{S}^1$, the $\mathbb{CP}^{N-1}$ model has a global symmetry, called *center symmetry*, which acts non-trivially on certain line operators. In order to see this, recall

that the local gauge invariance does not require the gauge rotations to be strictly periodic. Aperiodicity up to a *global* element of the center-group is both permitted and useful:

$$e^{i\alpha(x_1,x_2+L)} = e^{-i\xi}e^{i\alpha(x_1,x_2)} , \qquad e^{-i\xi} \in \mathrm{U}(1) . \tag{2.32}$$

A canonical gauge invariant order parameter on the cylinder $\mathbb{R}^1 \times \mathbb{S}^1$ is given by the Wilson loop

$$W(x_1) = \exp\left[i \int_0^L A_2(x_1, x_2)\, dx_2 \right] \tag{2.33}$$

In order to extract more information about the structure of the theory, we introduce a refined order parameter. Consider the σ-connection holonomy (2.30) which makes a circuit around the compact direction, namely

$$(^L\Omega)_j(x_1) = \exp\left[i \int_0^L dx_2\, \mathcal{A}_{2,j} \right] = \exp\left[i(\varphi_j(x_1, 0) - \varphi_j(x_1, L)) \right]$$

$$^L\Omega(x_1) = \begin{pmatrix} e^{i[\varphi_1(x_1,0)-\varphi_1(x_1,L)]} & 0 & \dots & 0 \\ 0 & e^{i[\varphi_2(x_1,0)-\varphi_2(x_1,L)]} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & e^{i[\varphi_N(x_1,0)-\varphi_N(x_1,L)]} \end{pmatrix} \tag{2.34}$$

Under an aperiodic global gauge rotation, $e^{i[\varphi_1(x_1,0)-\varphi_1(x_1,L)]} \to e^{i\xi}e^{i[\varphi_1(x_1,0)-\varphi_1(x_1,L)]}$. The constraint (2.23) implies

$$\det\left({}^L\Omega \right) = 1 \tag{2.35}$$

which is same as $e^{iN\xi} = 1$ or equivalently,

$$e^{i\xi} = e^{i\frac{2\pi k}{N}}, \qquad k = 1, \dots, N \tag{2.36}$$

Therefore, the center symmetry of the theory is $\mathbb{Z}_N$ and (2.34) is its order parameter. Under center-symmetry, it rotates by the global $\mathbb{Z}_N$- phase (2.36):

$$\mathbb{Z}_N \;:\; {}^L\Omega \longrightarrow e^{i\frac{2\pi k}{N}}\; {}^L\Omega . \tag{2.37}$$

This gauge invariant operator plays the same role as a Polyakov loop or a Wilson line does in non-abelian gauge theories.

Note that the twisted boundary conditions used in (2.16) and (2.17) can be viewed as the background field associated with the line operator (2.30). In quantum field theory, we would associate $2\pi\mu_i \leftrightarrow \langle [\varphi_i(0) - \varphi_i(L)](x_1) \rangle$ with the vev of the dynamical field, and we would interpret $[\phi_i(0) - \phi_i(L)](x_1)$ as the quantum fluctuations around the vev. In quantum mechanics, (2.17) will be the configuration that *minimizes* the one-loop potential, the background around which we can make a Born-Oppenheimer approximation:

$$^L\Omega = \underbrace{\begin{pmatrix} e^{2\pi i\mu_1} & 0 & \dots & 0 \\ 0 & e^{2\pi i\mu_2} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & e^{2\pi i\mu_N} \end{pmatrix}}_{\text{background}} \underbrace{\begin{pmatrix} e^{i[\phi_1(0)-\phi_1(L)]} & 0 & \dots & 0 \\ 0 & e^{i[\phi_2(0)-\phi_2(L)]} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & e^{i[\phi_N(0)-\phi_N(L)]} \end{pmatrix}}_{\text{fluctuations}} \tag{2.38}$$

and $[\phi_i(0) - \phi_i(L)](x_1)$ denotes the fluctuation around it.

This is an important point, distinguishing a classical analysis from a quantum one. In the quantum theory, the twisted boundary condition (which is equivalent to a background $U(1)^N$ σ-connection) is actually not a choice; rather it is determined by dynamics. In the weak coupling regime, this background is determined by a Coleman-Weinberg type one-loop analysis. As we show in the next section, not all values of μ_i are actually stable under quantum fluctuations.

3 Perturbative one-loop analysis for σ-connection holonomy

The realization of center-symmetry in the $\mathbb{CP}^{N-1}$ model in the small-$\mathbb{S}^1$ regime can be determined through a one-loop calculation. Because of asymptotic freedom, at sufficiently small $\mathbb{S}^1$ (with respect to the length scale set by the inverse of the strong scale Λ), the Kaluza-Klein modes of the theory are weakly coupled and can be integrated out perturbatively. For the general $\mathbb{CP}^{N-1}$ theory with N_f fermions, the one-loop effective potential for the gauge invariant line operator $^L\Omega$ can be obtained by standard methods [74]. The one-loop analysis picks out which "background" appearing in the covariant derivative (2.21) and (2.38) is preferred by thermal or quantum fluctuations. The use of twisted boundary conditions, and hence a twisted background, in the classical theory *does not* imply the stability of the given background in the quantum theory.

In order to check the stability of the twisted background, under thermal vs. spatial compactification, use

$$\tilde{n}(x_1, x_2 + L) = \tilde{n}(x_1, x_2), \qquad \tilde{\psi}(x_1, x_2 + L) = -\tilde{\psi}(x_1, x_2) \qquad \text{(thermal)}$$
$$\tilde{n}(x_1, x_2 + L) = \tilde{n}(x_1, x_2), \qquad \tilde{\psi}(x_1, x_2 + L) = +\tilde{\psi}(x_1, x_2) \qquad \text{(spatial)} \qquad (3.1)$$

The choice of anti-periodic (thermal) vs. periodic (spatial) boundary conditions for fermions in the path integral formalism, correspond, in the operator formalism to studying the theory by using either the thermal partition function (response to heating) or twisted (signed) partition function (response to spatial squeezing).

$$Z(\beta) \equiv \text{tr}[e^{-\beta H}] \equiv Z_{\mathcal{B}} + Z_{\mathcal{F}} \qquad \text{(thermal)} \qquad (3.2)$$
$$\widetilde{Z}(L) \equiv \text{tr}[e^{-LH}(-1)^F] \equiv Z_{\mathcal{B}} - Z_{\mathcal{F}} \qquad \text{(spatial)} \qquad (3.3)$$

where β and L are the size of the thermal and spatial $\mathbb{S}^1$, respectively. Here $(-1)^F$ is fermion number modulo two, grading the Hilbert space $\mathcal{H} = \mathcal{B} \oplus \mathcal{F}$, where states in the bosonic sub-space $\mathcal{B}$ contribute with a plus sign and states in the fermionic subspace $\mathcal{F}$ contribute with a minus sign. In the supersymmetric theory, the $N_f = 1$ case, $\widetilde{Z}(L)$ is the supersymmetric Witten index, and in the non-supersymmetric theory, with $N_f > 1$, as already stated, it probes the phase structure as a function of spatial size [11].

The main result of the one-loop analysis of the effective potential for the line operator (2.34) of the σ-connection holonomy, is

$$V_-[^L\Omega] = \frac{2}{\pi \beta^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (-1 + (-1)^n N_f)(|\text{tr}\, ^L\Omega^n| - 1) \qquad \text{(thermal)} \qquad (3.4)$$

$$V_+[{}^L\Omega] = (N_f - 1)\frac{2}{\pi L^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(|\mathrm{tr}\,{}^L\Omega^n| - 1) \qquad \text{(spatial)} \qquad (3.5)$$

This result is remarkable because it indicates a sharp quantitative difference between thermal and spatial compactification.[9,10] In particular, in the thermal case, the minimum of one-loop potential (3.4) is

$${}^L\Omega_0^{\text{thermal}} = e^{i\frac{2\pi k}{N}}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \qquad \text{(thermal)} \qquad (3.6)$$

where k labels the center-position of the lump of eigenvalues. This means that the eigenphases of the holonomy attract each other. See figure 1a. We identify this regime as the deconfined center-broken phase.

The moral behind spatial circle compactification is the absence of thermal fluctuations, and only the presence of zero temperature quantum fluctuations. A new phenomenon occurs, with far-reaching consequences. The minimum of the one-loop potential (3.5) is located at

$${}^L\Omega_0^{\text{spatial}} = \begin{pmatrix} 1 & & & \\ & e^{i\frac{2\pi}{N}} & & \\ & & \ddots & \\ & & & e^{i\frac{2\pi(N-1)}{N}} \end{pmatrix}, \qquad \text{(spatial)} \qquad (3.7)$$

This means that the eigenphases of the holonomy repel each other. See figure 1b. This simple result is the analog of the "adjoint Higgsing" at weak coupling of gauge theories, and is ultimately responsible for the rest of this paper: continuity between the small and large-$\mathbb{S}^1$ regimes in the sense of center-symmetry, the fractionalization of 2d instantons to 1d kink-instantons, realization of semi-classical renormalons, and identification of the resurgent structure in the $\mathbb{CP}^{N-1}$ model.

Note that the analog of this sharp quantitative difference between the one-loop potentials (3.4) and (3.5), based on either thermal or spatial compactification, has been obtained previously in 4d gauge theories in [102], and employed in [11, 12, 20] to build a reliable semi-classical expansion for non-perturbative effects in gauge theory. We see that a parallel structure emerges in $\mathbb{CP}^{N-1}$.

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[9]Despite its simplicity, to our knowledge, this effective potential is new in $\mathbb{CP}^{N-1}$. The reason is that the natural variable in terms of which this potential is expressed, ${}^L\Omega$, the line operator (2.34) associated with the holonomy of the σ-connection, is also new.

[10]The form of the one-loop potentials, $V_\pm$, can be deduced on physical grounds. First, it must be order N in the large-N limit. Hence, at leading order, it must be a sum over single-trace operators $\mathrm{tr}\,{}^L\Omega^n$ with order one coefficients. Second, it must be invariant under center symmetry $\mathrm{tr}\,{}^L\Omega^n \to e^{i\frac{2\pi nk}{N}}\mathrm{tr}\,{}^L\Omega^n$. Since center-symmetry is a symmetry of the compactified microscopic theory, it must also be a symmetry of the effective potential. One more piece of information is that for $N_f = 1$, the theory is supersymmetric $\mathcal{N} = (2,2)$ theory, and the potential $V_+[{}^L\Omega] = 0$ to all orders in perturbation theory. This dictates $V_+[{}^L\Omega] = (N_f - 1)\frac{1}{L^2}\sum_{n=1}^{\infty}a_n|\mathrm{tr}\,{}^L\Omega^n|$ in the spatial case (3.5).

Another reason why the spatially twisted partition function $\widetilde{Z}(L)$ is more useful than $Z(\beta)$ for semi-classical analysis is that the counter-part of the density matrix, which we may refer to as the "twisted density matrix", is not positive definite for $\widetilde{Z}(L)$. The usual density matrix that enters the study of $Z(\beta)$ is positive definite, and is ultimately responsible for Hagedorn instability. If, for example, $\rho(E) = \rho_{\mathcal{B}}(E) + \rho_{\mathcal{F}}(E) \sim e^{\beta^* E}$, at large-$E$, then $Z(\beta) = \int^{\infty} dE \; e^{(\beta^* - \beta)E}$ diverges for $\beta < \beta^*$, indicating an instability towards the deconfinement phase transitions. On the other hand, with the twisted partition function, $\widetilde{\rho}(E) = \rho_{\mathcal{B}}(E) - \rho_{\mathcal{F}}(E)$, and even if the theory has exponential growth of states in both bosonic and fermionic Hilbert spaces, the twisted partition function $\widetilde{Z}(L) = \int^{\infty} dE \; (\rho_{\mathcal{B}}(E) - \rho_{\mathcal{F}}(E)) e^{-LE}$ may be tame. The fact that the center-symmetry is unbroken in the small-$\mathbb{S}^1$ regime for periodic compactification can be traced to the non-positivity of the twisted density matrix.

3.1 Thermal compactification: center instability

For the $N_f = 0$ case, there is no distinction between the thermal and spatial compactification, and the minimum of the one-loop potential is at a center-broken configuration in the small $\mathbb{S}^1$ regime. We can verify the correctness of the one-loop potential (3.4) by independent means, using basic statistical mechanics.

The minimum of the one-loop potential (3.4) is at (3.6), $^L\Omega = \mathbf{1}$. The value of the potential at the minimum must give the leading order free energy density (or minus the pressure, P) of the hot $\mathbb{CP}^{N-1}$model:

$$\mathcal{F} = V_-[^L\Omega_0^{\text{thermal}}] = \frac{2}{\pi\beta^2}(N-1)\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{2}{\pi\beta^2}(N-1)\zeta(2) = -(2N-2)\frac{\pi}{6}T^2 \quad (3.8)$$

By comparison, a direct calculation using a non-interacting gas of bosons gives the same leading order result:

$$\mathcal{F} = (2N-2)T\int\frac{d\mathbf{p}}{2\pi}\;\log(1 - e^{-\beta|\mathbf{p}|}) = -(2N-2)\frac{\pi}{6}T^2 \quad (3.9)$$

This is just the Stefan-Boltzmann result: the number of degrees of freedom $(2N-2)$ times the Stefan-Boltzmann factor per bosonic quanta, which is $\frac{\pi}{6}T^2$.

For the thermal theory with fermions, the analysis is similar. The free energy density can be found either by evaluating the one-loop potential at its minimum $\Omega = \mathbf{1}$, or by using statistical physics,

$$\mathcal{F} = (2N-2)T\left(\int\frac{d\mathbf{p}}{2\pi}\;\log(1 - e^{-\beta|\mathbf{p}|}) - N_f\int\frac{d\mathbf{p}}{2\pi}\;\log(1 + e^{-\beta|\mathbf{p}|})\right) \quad (3.10)$$

Using $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\zeta(2)}{2}$, the free energy density can be written as

$$\mathcal{F} = -(2N-2)\frac{\pi}{6}\left(1 + \frac{N_f}{2}\right)T^2 \quad (3.11)$$

The factor of $1/2$ in front of the fermion number arises due to Fermi-Dirac statistics. This is again the expected Stefan-Boltzmann free energy of the system.

The $O(N^1)$-free energy implies that the high-temperature [i.e., small-$\mathbb{S}^1_\beta$] regime of the $\mathbb{CP}^{N-1}$ model is in the deconfined phase. At large-$\mathbb{S}^1_\beta$, the theory is expected to be in the confined phase with an $O(N^0)$ free energy. The transition must take place at a critical (inverse) temperature $\beta_c = a\Lambda^{-1}$ where a is a pure $O(1)$ number. The rapid-cross over between these two regimes at finite-N becomes a sharp phase transition at $N = \infty$. The phase transition to the deconfined phase is interesting in its own right. However, since the small and large $\mathbb{S}^1_\beta$ regimes are different phases, the information gained in the small circle regime does not help to understand the dynamics in the confined large $\mathbb{S}^1_\beta$ phase. In the large $\mathbb{S}^1_\beta$ regime, where the theory is confined, and a strong coupling center-symmetric σ-connection holonomy (See figure 1c.) is operative, one cannot use semi-classical methods due to strong coupling. What is needed is a weak-coupling semi-classical regime which is continuously connect the the strong-coupling regime. This would indeed provide a semi-classical way to study the 2d non-perturbative dynamics.

3.2 Spatial compactification: center stability

We re-write the holonomy dependent part of the one-loop potential (3.5) in the eigenvalue basis for $N_f > 1$:

$$\frac{V_+[\Omega]\pi L^2}{2(N_f - 1)} = \sum_{n=1}^\infty \frac{1}{n^2} \Big| \sum_{j=1}^N e^{i2\pi n\mu_j} \Big|, \qquad 2\pi\mu_j \equiv (\phi_j(0) - \phi_j(L)) \qquad (3.12)$$

The right hand side is non-negative definite. Because of the $1/n^2$ in the prefactor, the minimization is achieved by first minimizing $|\text{tr}\,({}^L\Omega)|$, then $|\text{tr}\,({}^L\Omega^2)|$, and all the way $|\text{tr}\,{}^L\Omega^{\lfloor \frac{N}{2} \rfloor}|$, where $\lfloor x \rfloor$ is the floor function, the largest integer not greater than x. Going all the way to $\lfloor \frac{N}{2} \rfloor$ is both necessary and sufficient to lift all possible vacuum degeneracies, and one will not gain more by going to higher orders. This procedure determines the global minimum of the potential given in (3.7).

The one-loop potential (3.5) has a non-trivial dependence on the number of flavors, in close analogy with QCD(adj) with N_f Majorana fermions in 4d [102].

- **$N_f > 1$**: the one-loop potential generates a repulsive interaction between the eigenvalues of the holonomy ${}^L\Omega$ as can be read-off easily from (3.12) and the center symmetry is preserved. Namely, the minimum of the potential is at (3.7) or

$$\text{tr}\,{}^L\Omega^n = 0, \qquad \text{for } n \neq 0 \bmod(N) \qquad (3.13)$$

- **$N_f = 1$**: The one-flavor theory is the supersymmetric $\mathcal{N} = (2,2)$ $\mathbb{CP}^{N-1}$ model. The perturbative potential vanishes at one loop order, and also to all orders due to supersymmetry. Despite this, we later demonstrate that there is a non-perturbatively induced potential stabilizing the center-symmetry. Thus, in the $N_f = 1$ case, (3.7) is also the center-symmetric vacuum of the theory.

- **$N_f = 0$**: In the purely bosonic theory, since there is no difference between the spatial and thermal compactification, the center symmetry is always broken at small $\mathbb{S}^1$. Below, we discuss how to go around this obstacle, either by using deformation or by integrating out heavy fermions.

3.3 Deformed-$\mathbb{CP}^{N-1}$ and massive fermions

Consider the theory with N_f fermions with mass m. Then, the one-loop potential with periodic boundary conditions for fermions takes the form

$$V_+[\Omega] = \frac{2}{\pi L^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[-1 + N_f(nLm)K_1(nLm) \right] (|\text{tr}\,^L\Omega^n| - 1). \qquad (3.14)$$

where $K_1(z)$ is the modified Bessel function. In the heavy and light fermion asymptotes, $K_1(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z}, z \to \infty$, and $K_1(z) \approx \frac{1}{z}, z \to 0$, so that (3.14) reduces to (3.5) with $N_f = 0$ and (3.5) with $N_f \geq 1$ respectively.

Provided $mLN \lesssim 1$, the center symmetry will be stable. This condition can be achieved at fixed-N and fixed-L, if m is taken to be sufficiently small. This can be viewed as a small mass perturbation of the theory with massless fermions.

A far more interesting (and at first glance, counter-intuitive) case arises as follows. Take $m \gg \Lambda$, so large that it is practically decoupled from the dynamics on $\mathbb{R}^2$. This theory on $\mathbb{R}^2$ emulates the pure bosonic theory, and when compactified, indeed, has a center-breaking phase transition at $L_c = a\Lambda^{-1}$, as in pure bosonic $\mathbb{CP}^{N-1}$ model. However, if one makes L sufficiently small, eventually one will achieve $mLN \lesssim 1$, so that the center symmetry will stabilize. In other words, provided the hierarchy of scales,

$$\Lambda \ll m \lesssim \frac{1}{LN} \qquad (3.15)$$

m is heavy with respect to Λ, but light with respect to $1/LN$. The corresponding small-$\mathbb{S}^1$ theory is a bosonic, center symmetric theory. Therefore, we can also consider the dynamics of bosonic center-symmetric theory at small-$\mathbb{S}^1$.

Inspired by the massive-fermion induced stabilization, and deformed Yang-Mills theory [15, 16] compactified on $\mathbb{R}^3 \times \mathbb{S}^1$, we can construct a *deformed* $\mathbb{CP}^{N-1}$ model (or d$\mathbb{CP}^{N-1}$). The deformed model at small-L, should be viewed as the analytic continuation of the confined phase of large-L undeformed $\mathbb{CP}^{N-1}$. The action of the deformed model is

$$S^{\text{d}\mathbb{CP}} = S + \Delta S, \qquad \Delta S = \frac{2}{\pi L^2} \int_{\mathbb{R}^1 \times \mathbb{S}^1} \sum_{n=1}^{\lfloor N/2 \rfloor} \frac{a_n}{n^2} (|\text{tr}\,^L\Omega^n| - 1), \qquad (3.16)$$

where $a_n = O(N^0)$ are sufficiently large pure numbers, and $\lfloor x \rfloor$ is the integer part of x. For example, one may take $a_n = 2$, twice in modulus the value of the perturbative one-loop result. The deformation respects all the symmetries of the original theory. In the small-$\mathbb{S}^1$ domain, it guarantees unbroken center-symmetry and semi-classical calculability.

3.4 Is large-N volume independence possible in a σ-model?

A class of non-abelian gauge theories, for example an SU(N) theory, when studied on toroidal compactification of $\mathbb{R}^d$, has properties independent of the compactification radius, provided center-symmetry and translation symmetry are not spontaneously broken. This property is called large-N volume independence or Eguchi-Kawai reduction [101, 102]. When center-symmetry is unbroken, then, in the reduced theory, the space-time or
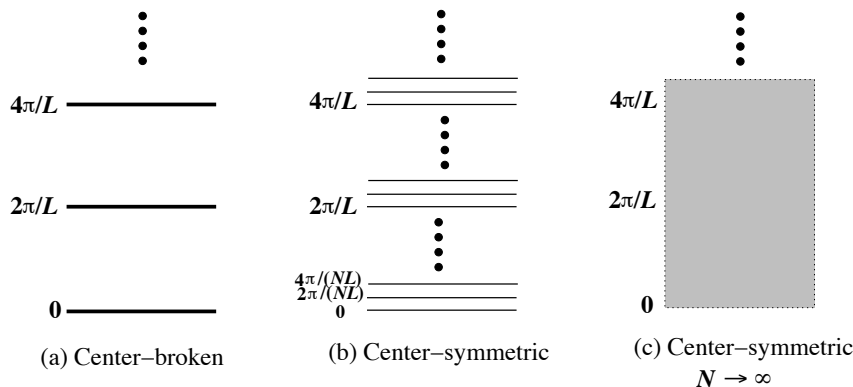
Figure 3. The perturbative spectrum of the $\mathbb{CP}^{N-1}$ theory as a function of the background σ-connection holonomy. (a) Weak-coupling trivial holonomy (as well as classical theory) gives the usual $2\pi/L$ level spacing. (b) Weak-coupling non-trivial holonomy ($\mathbb{Z}_N$ symmetric background) at finite-N produces a finer level spacing. (c) $\mathbb{Z}_N$ symmetric background at $N = \infty$ leads to a continuous spectrum. Classically, the background for the σ-connection holonomy is equivalent to twisted boundary conditions on $\mathbb{CP}^{N-1}$ fields. For $N_f \geq 1$ theories, quantum mechanically, (b) and (c) are stable upon spatial compactification and (a) is stable upon thermal compactification. To achieve (b) and (c) in the $N_f = 0$ case, we deform the $\mathbb{CP}^{N-1}$ Lagrangian appropriately. (b) admits a semi-classical analysis of the confined regime at finite N, and (c) satisfies large-N volume independence at $N = \infty$. (a) is not suitable for the semi-classical study of the confined regime/phase.

momenta are encoded into the gauge structure of the non-abelian theory. Vector models are, to our knowledge, not discussed in the volume independence context, because they do not possess a non-abelian gauge structure. In the $\mathbb{CP}^{N-1}$ model, for example, the gauge structure is U(1), and to an expert in volume independence, even talking about the possibility of having volume independence may sound absurd. However, we suggest here that the $\mathbb{CP}^{N-1}$ models should also exhibit volume independence, provided, *(i)* center-symmetry is unbroken for the toroidally compactified theory, *(ii)* translational invariance is not spontaneously broken for the theory on $\mathbb{R}^2$. We expect the latter to hold in a Lorentz invariant theory. In this work, we have showed that $\mathbb{CP}^{N-1}$ model with multiple-fermions endowed with periodic boundary conditions do not break their center-symmetries.

In this subsection, we give an intuitive plausibility argument for volume independence in $\mathbb{CP}^{N-1}$. Since there is no non-abelian gauge structure in $\mathbb{CP}^{N-1}$, there must be a new mechanism to generate space-time or momenta out of the reduced model. Indeed, there is, and this requires an interpretation of the vev that we obtained for the line operator (2.34). The key physical idea is that the vev of the order parameter (2.34), provided center-symmetry is unbroken, generates *momentum feeding* into the system in units of $\frac{2\pi\mu_k}{L} = \frac{2\pi k}{NL}$, with $k \in \mathbb{Z}$, which behave like fractional momenta with respect to the "standard" Kaluza-Klein momenta $\frac{2\pi k}{L}$, with $k \in \mathbb{Z}$.

As described in section 2.1, the spatial twist in the boundary condition can be removed in favor of a background σ-connection holonomy. The form of the Kaluza-Klein spectrum is crucially dependent on the choice of twist matrix $^L\Omega$ or equivalently, the background field

in the modified action (2.21). In the center-broken case (3.6), $^L\Omega = 1$, figure 3 a depicts the KK tower with momenta and spacing at integer multiples of $2\pi/L$. Each level has an $O(N)$ degeneracy. This is the analog of the center-symmetry broken regime in gauge theories. In this case, the critical length scale that enters the problem is L. At length scales larger than L, the non-zero frequency modes can be integrated out perturbatively.

In contrast, when a $\mathbb{Z}_N$ symmetric twist matrix $^L\Omega$, as in (3.7), is stable, we find a much finer KK spectrum with spacing of $2\pi/(NL)$ and $O(1)$ degeneracies. Expanding the periodic $\tilde{n}$-fields into their KK-modes along the compact direction,

$$\tilde{n}_j(x_1, x_2) = \sum_{k \in \mathbb{Z}} e^{\frac{i2\pi k x_2}{L}} \tilde{n}_{j,k}(x_1) \tag{3.17}$$

the quadratic terms in the bosonic action take the form

$$S^{\text{quad}} = \frac{2L}{g^2} \sum_{k \in \mathbb{Z}} \int dx_1 \left| \left(\partial_1 + i\frac{2\pi}{L}(\mu_j + k)\delta_{\mu 2} \right) \tilde{n}_{j,k}(x_1) \right|^2 \tag{3.18}$$

Therefore, a stable $\mathbb{Z}_N$-symmetric background act likes momentum quantized in units of $\frac{2\pi}{LN}$. Alternatively, the stable twisted boundary conditions shift the the phase acquired by a excitation propagating around the spatial S^1 by the amount $\frac{2\pi j}{LN}$ for the mode $n_{j,k}$.

The main observation is that the SU(N) index of the $\mathbb{CP}^{N-1}$ field and the ordinary Kaluza-Klein momentum index intertwine, and the $\mathbb{CP}^{N-1}$ field $n_{j,k}$ breaks up to into N distinct pieces with shifted offsets in the frequency quantization. The critical length scale that enters the problem is NL, and not L. The instanton and kink-instanton effects discussed in the next section, like the perturbative effects, are sensitive to length scale LN, as opposed to L.

It is important to note that in the $N \to \infty$ limit with fixed L, the perturbative spectrum of the $\mathbb{CP}^{N-1}$ model with $\mathbb{Z}_N$ symmetric twist approaches the continuous frequency spectrum of the decompactified theory theory on $\mathbb{R}^2$, as illustrated in figure 3c. This is also a property of gauge theories which satisfy volume independence in the $N = \infty$ limit. In this limit, below the energy level Λ, perturbatively, there is a continuum band of states. It seems reasonable to expect that, in analogy with gauge theory, the neutral sector observables in the $\mathbb{CP}^{N-1}$ model should exhibit volume independence. In this limit, the effects due to compactification are $1/N$ suppressed, and the leading large-N behavior of observables, including the non-perturbative mass spectrum of the theory, must be volume independent. In this regime, the description in terms of microscopic degrees of freedom is strongly coupled, this is the absence of weak coupling description of long distance physics in $\mathbb{CP}^{N-1}$. However, the macroscopic "hadrons" of the theory should have a weakly coupled description, with couplings controlled by $\frac{1}{\sqrt{N}}$.

Clearly, the relevant length scales in the problem and the approach to the $N = \infty$ limit are critically dependent on the twist-matrix, Ω. To recap, for $\mathbb{CP}^{N-1}$ on $\mathbb{R} \times \mathbb{S}^1$, with a $\mathbb{Z}_N$ symmetric twist matrix $^L\Omega$, the physically relevant length scale appearing in finite volume effects is not L, but rather NL. The theory has two distinct characteristic regimes:

$$\frac{NL\Lambda}{2\pi} \ll 1, \qquad \text{semi-classical} \Longrightarrow \text{volume dependence;} \tag{3.19}$$

$$\frac{NL\Lambda}{2\pi} \gg 1\,, \qquad \text{strongly coupled} \Longrightarrow \text{volume independence.} \qquad (3.20)$$

The techniques of this paper allow us to study the semi-classical limit at arbitrary N, including the large-N limit by analytical methods. For $N_f = 0$, we will derive the mass gap in the semi-classical domain. The result is an exact match to the well-known large-N result on $\mathbb{R}^2$. Furthermore, we will demonstrate explicitly that the positions of the expected renormalon singularities on $\mathbb{R}^2$ for $N_f \geq 1$ are the same as the positions of the bion-anti-bion singularities in the semi-classical domain. These results, together with other features to be discussed in a future publication, suggest that all qualitative aspects of the volume independence domain is captured in the semi-classical domain. The results for the mass gap and renormalon pole positions exhibit also quantitative agreement between the large-N results on $\mathbb{R}^2$ and the semi-classical results in the compactified theory, furthermore providing the microscopic mechanisms underlying the large-N results.

4 Self-dual configurations

The first part of this section reviews the standard text-book instanton construction for $\mathbb{CP}^{N-1}$ on $\mathbb{R}^2$ [103], and highlights the non-existence of a dilute 2d instanton gas approximation. Next, on top of our perturbative one-loop analysis on $\mathbb{R} \times \mathbb{S}^1$, we perform a study of leading semi-classical configurations on $\mathbb{R} \times \mathbb{S}^1$.

4.1 2d instantons in $\mathbb{CP}^{N-1}$

The 2d instanton equations can be obtained by a standard Bogomolnyi factorization of the action density:

$$(D_\mu n)^\dagger D_\mu n = |(D_\mu \pm i\epsilon_{\mu\nu}D_\nu)\,n|^2 \mp i\,\epsilon_{\mu\nu}\partial_\mu\left(n^\dagger\partial_\nu n\right) \qquad (4.1)$$

Thus, the self-dual instanton equations are

$$D_\mu n = \mp i\,\epsilon_{\mu\nu}D_\nu n \qquad (4.2)$$

For these instanton solutions, the action saturates the BPS bound:

$$S = \frac{2}{g^2}\int (D_\mu n)^\dagger D_\mu n = \frac{2}{g^2}\left|\mp i\epsilon_{\mu\nu}\int (D_\mu n)^\dagger D_\nu n\right| \geq \frac{4\pi}{g^2}|Q| \qquad (4.3)$$

where Q is the topological charge defined in (2.7). In describing instantons, it is convenient to define homogeneous coordinates for the $\mathbb{CP}^{N-1}$ fields,

$$n = \frac{v}{|v|} \quad \Rightarrow \quad A_\mu = \frac{i}{2}\left(\frac{v^\dagger\partial_\mu v - \partial_\mu v^\dagger v}{v^\dagger v}\right) \qquad (4.4)$$

where v is an N-component column vector; then the instanton equations (4.2) reduce to the Cauchy-Riemann equations $\partial_\mu v = \mp i\epsilon_{\mu\nu}\partial_\nu v$, which means that v is holomorphic (instanton) or anti-holomorphic (anti-instanton):

$$\text{instanton}: v = v(z)\,, \qquad \text{anti} - \text{instanton}: v = v(\bar{z}) \qquad (4.5)$$

Then for instanton and anti-instanton solutions we can write

$$A_\mu = \pm \frac{1}{2} \epsilon_{\mu\nu} \partial_\nu \ln v^\dagger v \qquad (4.6)$$

On $\mathbb{R}^2$, the most general instanton with charge $k \in \mathbb{N}$ is expressed in terms of a holomorphic vector v having entries that are polynomials in z, with maximal degree k, and no common roots. For example, in $\mathbb{CP}^1$ on $\mathbb{R}^2$, the single instanton can be written

$$v = \begin{pmatrix} 1 \\ (z-b)/a \end{pmatrix} \quad \Rightarrow \quad Q = \frac{1}{\pi} \int d^2x \, \frac{|a|^2}{(|a|^2 + |z-b|^2)^2} = 1 \qquad (4.7)$$

There are two complex (four real) moduli parameters entering this solution. The parameter b characterizes the location of instanton in $\mathbb{R}^2$, $\rho = |a|$ encodes the size modulus, and $\arg(a)$ is a U(1) phase of the instanton. More generally, in the $\mathbb{CP}^{N-1}$ model on $\mathbb{R}^2$, the 2d instanton has $2N$ parameters, associated with $2N$ zero modes. These are associated with the classical symmetries of the self-duality equation: two are the position of the instanton ($\mathbf{a}_I \in \mathbb{R}^2$) and arise due to translation invariance, one is the size modulus ($\rho \in \mathbb{R}^+$) and is associated with invariance under dilatations, and the remaining $2N-3$ are internal orientational modes.

$$2N \quad \longrightarrow \quad 2 + 1 + (2N-3) = (\mathbf{a}_I \in \mathbb{R}^2) + (\rho \in \mathbb{R}^+) + (\text{orientation}). \qquad (4.8)$$

This is in analogy with Yang-Mills instantons on $\mathbb{R}^4$, where the count is $4N = 4 + 1 + (4N - 5)$, with parallel physical interpretation.

For the $\mathbb{CP}^{N-1}$ theory on $\mathbb{R}^2$, the existence of the size modulus ρ implies that the instanton comes in all sizes at no cost in action. Therefore, there is no precise sense in which the typical instanton separation is much larger than the typical instanton size. This prevents, on $\mathbb{R}^2$, a meaningful semi-classical dilute instanton gas from first principles. On $\mathbb{R}^1 \times \mathbb{S}^1$, we will propose a way around this obstacle while staying continuously connected to $\mathbb{R}^2$.

4.2 Fundamental and affine kink-instantons in $\mathbb{CP}^1$

$\mathbf{CP^1}$: First, we discuss the kink-instanton events in $\mathbb{CP}^1$, and then give the generalization to $\mathbb{CP}^{N-1}$. The $\mathbb{CP}^1$ model is locally equivalent to the $O(3)$ non-linear σ-model through the identification of fields:

$$\vec{s}(x) = n_i^\dagger \vec{\sigma}_{ij} n_j, \qquad s^a(x) = n_i^\dagger \sigma_{ij}^a n_j \qquad (4.9)$$

where $\vec{\sigma}$ are the Pauli matrices. We would like to benefit from this equivalence while incorporating the $\mathbb{Z}_2$-symmetric background (3.7). It is crucial that this background must be stable in the quantum theory, and this, as explained in sections 3.2 and 3.3, can be achieved in the spatially compactified theory, either by massless fermion-induced mechanism or by integrating out massive-fermions, i.e, the deformed bosonic theory.

Let us, as before, trade the twisted boundary conditions in favor of a *stable* background field: Using (2.19) for $\mathbb{CP}^1$, we have

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} e^{i\frac{2\pi\mu_1 x_2}{L}} & \\ & e^{i\frac{2\pi\mu_2 x_2}{L}} \end{pmatrix} \begin{pmatrix} \widetilde{n}_1 \\ \widetilde{n}_2 \end{pmatrix} = \begin{pmatrix} e^{i\frac{2\pi\mu_1 x_2}{L}} & \\ & e^{i\frac{2\pi\mu_2 x_2}{L}} \end{pmatrix} \begin{pmatrix} e^{-i\phi/2} \cos\frac{\theta}{2} \\ e^{+i\phi/2} \sin\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\left(-\frac{\phi}{2}+\frac{2\pi\mu_1 x_2}{L}\right)} \cos\frac{\theta}{2} \\ e^{i\left(+\frac{\phi}{2}+\frac{2\pi\mu_2 x_2}{L}\right)} \sin\frac{\theta}{2} \end{pmatrix}, \qquad \theta \in [0,\pi], \; \phi \in [0,2\pi] \tag{4.10}$$

where $\theta(x_1, x_2)$ and $\phi(x_1, x_2)$ are periodic fields of x_2. Now, the s_a fields take the form

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \sin\theta \cos(\phi + \xi x_2) \\ \sin\theta \sin(\phi + \xi x_2) \\ \cos\theta \end{pmatrix}, \qquad \xi \equiv \frac{2\pi}{L}(\mu_2 - \mu_1) \tag{4.11}$$

The Lagrangian obtained in this manner, on $\mathbb{R} \times \mathbb{S}^1$, is given by

$$S = \frac{2}{g^2} \int_{\mathbb{R} \times \mathbb{S}^1_L} |D_\mu n_i|^2 = \frac{1}{2g^2} \int_{\mathbb{R} \times \mathbb{S}^1_L} |\partial_\mu \vec{s}|^2 = \frac{1}{2g^2} \int_{\mathbb{R} \times \mathbb{S}^1_L} (\partial_\mu \theta)^2 + \sin^2\theta(\partial_\mu \phi + \xi\delta_{\mu 2})^2 \tag{4.12}$$

which is identical to (2.21). As before, the twisted boundary conditions are undone in favor of a twisted-background $\partial_2 \phi_{\text{background}} = \xi$.

Since the fields θ, ϕ entering (2.21) are manifestly periodic, we can reduce this Lagrangian to simple quantum mechanics by truncating it to its zero Kaluza-Klein mode. (We drop the higher Kaluza-Klein modes momentarily.) The action of the associated zero mode quantum mechanics is (this does not capture all the interesting effects, which we will restore momentarily by keeping the relevant KK-modes)

$$S^{\text{zero}} = \frac{L}{2g^2} \int_{\mathbb{R}} (\partial_t \theta)^2 + \sin^2\theta(\partial_t \phi)^2 + \xi^2 \sin^2\theta \tag{4.13}$$

The equations of motions associated with this action are:[11]

$$\ddot{\theta} - \frac{1}{2}\sin 2\theta[(\dot{\phi})^2 + \xi^2] = 0$$
$$\ddot{\phi} + 2\dot{\phi}\cot\theta = 0 \tag{4.15}$$

Setting ϕ=constant upon which the second equation is satisfied, the first one reduces to the usual equation for a kink in a one-dimensional problem:

$$\ddot{\theta} - \frac{\xi^2}{2}\sin 2\theta = 0 \tag{4.16}$$

We can find the action of this configuration by using Bogomolny's method: Let $V(\theta) = (\mathcal{W}')^2$ where $\mathcal{W} = \xi\cos\theta$. Then

$$S^{\text{zero}} = \frac{L}{2g^2} \int_{\mathbb{R}} (\dot{\theta})^2 + (\mathcal{W}')^2 = \frac{L}{2g^2} \int_{\mathbb{R}} \left[(\dot{\theta} \pm \mathcal{W}')^2 \mp 2\dot{\theta}\mathcal{W}' \right] \geq \left| \frac{L}{g^2} \int d\mathcal{W} \right| \tag{4.17}$$

---

[11]Setting $\xi = 0$, this theory reduces to the quantum mechanics of a particle on a sphere S^2, and described the high-temperature deconfined regime of the thermally compactified theory at scales larger than T^{-1}. The zero mode of the $\mathbb{CP}^1$ model,

$$(\phi, \theta)(\tau) : \mathbb{R}/\mathbb{Z} \to X \qquad \text{where } \mathbb{R}/\mathbb{Z} = S^1_\beta, \; X = S^2 . \tag{4.14}$$

is just the quantum mechanics of a particle on $X = S^2$, and more generally, $X = \mathbb{CP}^{N-1}$. Resurgence in this type of quantum mechanical system is recently examined by Kontsevich from path integral point of view [58]. However, the study of resurgence in this quantum mechanics is not the relevant one for the purpose of understanding the 2d QFT in its semi-classical regime. The quantum mechanical theory relevant to $\mathbb{CP}^{N-1}$ on $\mathbb{R}^2$ is the one in which ξ is nonzero.

The kink-instanton, which we refer to as $\mathcal{K}_1$, ($\mathcal{K}_j$ in the general case), satisfies the first order differential equations $\dot{\theta} \pm \mathcal{W}' = 0$, or $\dot{\theta} \pm \xi \sin\theta = 0$. Differentiating this equation once with respect to Euclidean time, we recover (4.16). Using $\int d\mathcal{W} = 2\xi$ for the kink interpolating from $\theta = 0$ to $\theta = \pi$, we find the action of the $\mathcal{K}_1$ kink-instantanton as

$$\mathcal{K}_1 : \qquad S_1 = \frac{L}{g^2}(2\xi) = \frac{4\pi}{g^2} \times (\mu_2 - \mu_1) \equiv S_I \times (\mu_2 - \mu_1) = \frac{S_I}{2} \qquad (4.18)$$

In the last step, we used the actual value of the background μ_i. The action of the kink is *half* that of the 2d instanton. The fact that the action of the kink is determined by the separation between the eigenvalues of the $^L\Omega$ matrix (3.7) holds more generally. The kink configuration is an interpolation between Euclidean times $x_1 = \mp\infty$

$$\mathcal{K}_1 : \qquad \begin{pmatrix} \widetilde{n}_1 \\ \widetilde{n}_2 \end{pmatrix}(-\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \widetilde{n}_1 \\ \widetilde{n}_2 \end{pmatrix}(+\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (4.19)$$

and we denote its anti-kink by $\overline{\mathcal{K}}_1$.

The kink-instanton has two-zero modes associated with the two-global symmetries of the quantum mechanics:

- the position modulus $a \in \mathbb{R}$ associated with translational invariance along x_1 direction,

- Angular modulus $\phi \in \mathrm{U}(1)$ associated with the shift symmetry $\phi \to \phi + c$ of the action (4.13). Note that the one-loop potential (3.14) in terms of the holonomy Ω, given in (2.34), also respects the shift symmetry, as it depends on ϕ_i as $\phi_i(x_1, 0) - \phi_i(x_1, L)$, always as in a difference equation, despite the fact that it is not strictly derivatively coupled.

The angular modulus is reflected in the classical Euclidean equations of motions (4.15) as the choice $\phi = $constant.

Affine kink-instanton. Ordinarily, when one performs dimensional reduction of a QFT, the Kaluza-Klein modes which carry momenta in compact direction by an amount $\frac{2\pi}{L}$ decouples, as it takes a divergent amount of energy to excite the modes associated with it. However, and although not appreciated broadly, when certain conditions are satisfied, such as unbroken center-symmetry as in gauge theory or $\mathbb{CP}^{N-1}$, this argument is invalid. This is discussed in section 3.4. Figure 3b and figure 3c are manifestations of this fact. In particular, the lightest mode which may be instrumental in the low-energy dynamics may be hidden in the first KK-mode (not the zeroth KK-mode).

In finding the kink-instanton solution in (4.12), we reduced the periodic fields $\widetilde{n}_1$ and $\widetilde{n}_2$ to their zero momentum mode sector, and this leads to (4.13) and the ensuing kink-instanton solution with charge $Q = \mu_2 - \mu_1$. Instead, now, use

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} e^{i\left(-\frac{\phi}{2} + \frac{2\pi\mu_1 x_2}{L}\right)}\cos\frac{\theta}{2} \\ e^{i\left(+\frac{\phi}{2} + \frac{2\pi(-1+\mu_2)x_2}{L}\right)}\sin\frac{\theta}{2} \end{pmatrix}, \qquad (4.20)$$

which carries a single extra unit of KK-momentum in the x_2 direction. Now, substitute this into the $\mathbb{CP}^1$ action in (4.12). Then, dimensionally reduce the corresponding Lagrangian down to QM, by declaring the fields θ, ϕ entering (4.20) to be independent of the compact spatial x_2 coordinate. The resulting action is,

$$S^{\text{first}} = \frac{L}{2g^2} \int_{\mathbb{R}} (\partial_t \theta)^2 + \sin^2\theta (\partial_t\phi)^2 + (\xi')^2 \sin^2\theta, \qquad \xi' = \frac{2\pi[-1 + (\mu_2 - \mu_1)]}{L} \quad (4.21)$$

The kink (not anti-kink) solution of (4.21) interpolates from π to 0,

$$\mathcal{K}_2: \qquad \begin{pmatrix} \widetilde{n}_1 \\ \widetilde{n}_2 \end{pmatrix}(-\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} \widetilde{n}_1 \\ \widetilde{n}_2 \end{pmatrix}(+\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (4.22)$$

and has topological charge and action

$$Q = 1 - (\mu_2 - \mu_1) = \frac{1}{2}, \qquad S_2 = \frac{L}{g^2}|2\xi'| = \frac{4\pi}{g^2} \times (1 - (\mu_2 - \mu_1)) = \frac{S_I}{2} \quad (4.23)$$

It is important to note that $\mathcal{K}_2$ is not the same as $\overline{\mathcal{K}}_1$. In particular, $\mathcal{K}_2$ is not an anti-kink, as it has positive topological charge.

The simplest way to see these differences is to artificially deviate $\mu_1 - \mu_2$ from $\frac{1}{2}$. Then, the action of $\mathcal{K}_1$ and $\overline{\mathcal{K}}_1$ remain the same, and the actions of $\mathcal{K}_2$ and $\overline{\mathcal{K}}_2$ are the same, but these are not equal to each other. At $\mu_1 - \mu_2 = \frac{1}{2}$ the two types of kink configurations is distinguished by their topological charges, or equivalently, by the Θ-dependence of the associated amplitudes. These will be discussed in more detail when we discuss the role of the $\mathcal{K}_i$-events in dynamics.

To summarize, the leading self-dual kink configurations in $\mathbb{CP}^1$ are:

$$\mathcal{K}_1: \qquad [\theta: 0 \to \pi], \qquad Q = \frac{1}{2}, \qquad \overline{\mathcal{K}}_1: \qquad [\theta: \pi \to 0], \qquad Q = -\frac{1}{2}$$

$$\mathcal{K}_2: \qquad [\theta: \pi \to 0], \qquad Q = \frac{1}{2}, \qquad \overline{\mathcal{K}}_2: \qquad [\theta: 0 \to \pi], \qquad Q = -\frac{1}{2} \quad (4.24)$$

There actually exists an infinite tower of both types of kinks, with higher topological charge, and identical asymptotes. This tower may be useful to establish a duality between the particle/soliton topological defects on $\mathbb{R}^{1,1}$ and kink-instanton topological defects on $\mathbb{R} \times \mathbb{S}^1_L$ along the lines of [70]. This direction will be pursued separately.

4.3 Embedding $\mathbb{CP}^1$ kink-instantons into $\mathbb{CP}^{N-1}$

In the presence of a quantum mechanically stable twisted background or equivalently, twisted boundary conditions in the compact spatial x_2 direction, we find that a sufficiently large 2d instanton with $Q = 1$ in $\mathbb{CP}^{N-1}$ decomposes into up to N constituent kink-instantons, each of which carry $Q = \frac{1}{N}$. More precisely,

$$\begin{array}{ll} \rho < LN \ll \Lambda^{-1} & \text{small-instanton, no fractionalization} \\ LN \lesssim \rho \ll \Lambda^{-1} & \text{large-instanton, fractionalization to kinks} \end{array} \quad (4.25)$$

The observation regarding fractionalization of 2d instantons in $\mathbb{CP}^{N-1}$ recently appeared in interesting works [79–81] for the thermally compactified theory, and part of our current

analysis has been inspired by these results. However, as noted earlier, our work differs from [79–81] in the sense that we consider a zero temperature spatial compactification, and consequently, the $\mathbb{Z}_N$-symmetric background is stable against quantum fluctuations, as opposed to the thermal case in which the $\mathbb{Z}_N$-symmetric background is unstable against thermal fluctuations.

Kink-instantons in $\mathbb{CP}^{N-1}$can be constructed by embedding $\mathbb{CP}^1$ kink-instantons into $\mathbb{CP}^{N-1}$. The fundamental and affine kinks are characterized by the simple and affine roots of $\mathrm{SU}(N)$ algebra. This mimics the construction of $\mathrm{SU}(N)$ BPS monopole-instantons on $\mathbb{R}^3 \times \mathbb{S}^1$ from fundamental monopoles by the embedding of $\mathrm{SU}(2)$ monopoles, again characterized by roots of $\mathrm{SU}(N)$, including the affine root.

For $\mathbb{CP}^{N-1}$, in order to describe the kink-instantons, it is convenient to use complexified hyper-spherical coordinates (2.22), with $\phi_i \in [0, 2\pi)$ and $\theta_i \in [0, \pi]$:

$$
\begin{aligned}
\widetilde{n} &= \begin{pmatrix} e^{i\phi_1} & 0 & 0 & \ldots & 0 \\ 0 & e^{i\phi_2} & 0 & \ldots & 0 \\ 0 & 0 & e^{i\phi_3} \ldots & 0 \\ \vdots & & & \\ 0 & 0 & 0 & \ldots & e^{i\phi_N} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta_1}{2} \\ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2} \\ \vdots \\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2}\ldots\sin\frac{\theta_{N-1}}{2} \end{pmatrix} \\
n &= \begin{pmatrix} e^{i\frac{2\pi\mu_1 x_2}{L}} & 0 & \ldots & 0 \\ 0 & e^{i\frac{2\pi\mu_2 x_2}{L}} & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & e^{i\frac{2\pi\mu_N x_2}{L}} \end{pmatrix} \widetilde{n}
\end{aligned}
\tag{4.26}
$$

This representation makes it easy to see that there is a kink-configuration for each simple root of the $\mathrm{SU}(N)$ algebra.

Embedding ansatz. Perform the following truncation which reveals the existence of kink-instantons:

$$
\theta_1 = \ldots = \theta_{k-1} = \pi, \quad \theta_k = \theta_k(x_1), \quad \theta_{k+1} = 0, \quad \{\theta_{k+2}, \ldots, \theta_{N-1}\} \to \text{arbitrary} \tag{4.27}
$$

Substituting this into the action (2.21),

$$
S = \frac{L}{2g^2} \int_{\mathbb{R}} (\partial_t \theta_k)^2 + \sin^2\theta_k [\partial_t(\phi_{k+1} - \phi_k)]^2 + \left(\frac{2\pi(\mu_{k+1} - \mu_k)}{L}\right)^2 \sin^2\theta_k \tag{4.28}
$$

which can be identified with (4.13) with simple matching, $\theta_k \equiv \theta$, $(\phi_{k+1} - \phi_k) \equiv \phi$, $\frac{2\pi(\mu_{k+1} - \mu_k)}{L} \equiv \xi_k$. This is the embedding of the $\mathbb{CP}^1$ kink into $\mathbb{CP}^{N-1}$, in analogy with the embedding of the $\mathrm{SU}(2)$ monopole-instanton into $\mathrm{SU}(N)$.

The asymptotic values of the $\widetilde{n}$ configurations are

$$
\mathcal{K}_k: \quad \widetilde{n}(-\infty) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_k\,, \quad \widetilde{n}(\infty) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = e_{k+1}\,, \quad e_{N+1} \equiv e_1,\quad k = 1,\ldots,N
$$

(4.29)

The differences of the asymptotes are associated with the simple and the affine co-root of the $\mathrm{SU}(N)$ algebra and can be used to uniquely label a kink event:

$$
\mathcal{K}_k: \quad \Delta\widetilde{n} = \widetilde{n}(\infty) - \widetilde{n}(-\infty) = \begin{pmatrix} 0 \\ \vdots \\ -1 \\ +1 \\ \vdots \\ 0 \end{pmatrix} = e_{k+1} - e_k \equiv \alpha_k\,, \quad k = 1,\ldots,N
$$

(4.30)

The action of the kink-instanton $\mathcal{K}_k$ can easily be obtained by using (4.18) and (4.28)

$$
\mathcal{K}_k: \qquad S_k = \frac{L}{g^2}(2\xi) = \frac{4\pi}{g^2} \times (\mu_{k+1} - \mu_k) = \frac{S_I}{N}\,, \qquad k = 1,\ldots,N
$$

(4.31)

where in the last-step, we used the center-symmetric background (3.7). The crucial point here is that the action of the kink-instanton is $1/N$ of the action of the 2d-instanton. This will play a major role in the determination of the mass gap and the renormalon structure in the semi-classical domain.

4.4 Small thermal circle: non-fractionalization of 2d instantons

As discussed in the Introduction in section 1.2, thermal compactification does not permit one to smoothly connect weak-coupling semi-classical results to the strong-coupling phase. To further clarify this point, directly in the context of the $\mathbb{CP}^{N-1}$model, consider for example the simplest, untwisted, $Q = 1$ instanton for $\mathbb{CP}^1$, where the second component of the homogeneous $\mathbb{CP}^{N-1}$coordinate v (4.4) is a first order polynomial in $e^{-\frac{2\pi}{L}z}$ (the formulas here are for spatial compactification, which we apply later, but the formulas are not important for the point we wish to make here):

$$
v = \begin{pmatrix} 1 \\ \lambda_1 + \lambda_2\, e^{-\frac{2\pi}{L}z} \end{pmatrix}
$$
$$
v^\dagger v = 1 + |\lambda_1|^2 + |\lambda_2|^2 e^{-\frac{4\pi}{L}x_1} + 2|\lambda_1\lambda_2|e^{-\frac{2\pi}{L}x_1}\cos\left(\frac{2\pi}{L}x_2 - \arg\lambda_1 + \arg\lambda_2\right)
$$

(4.32)

The corresponding topological charge density $q(x_1, x_2) = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu$ is plotted in figure 4 for various values of the λ parameters. For large $|\lambda_1|$ the instanton looks like a single
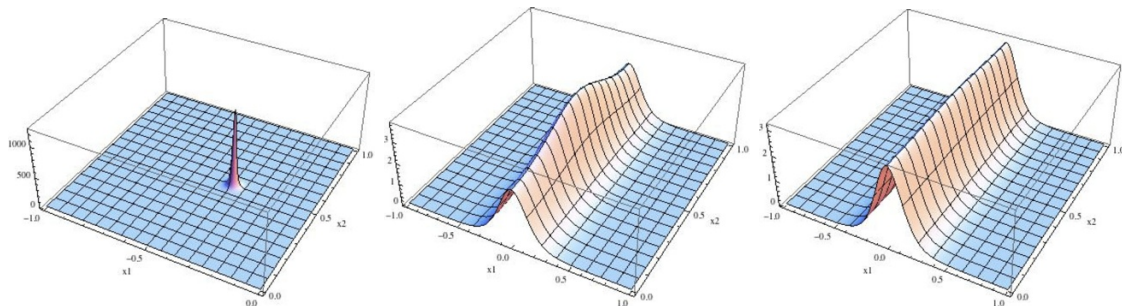
Figure 4. The topological charge density of the single untwisted $\mathbb{CP}^1$ instanton (4.32) on $\mathbb{R}^1 \times \mathbb{S}^1$ at fixed $\lambda_2 = 1$ and $L = 1$. The plots are for $\lambda_1 = 10$, $\lambda_1 = 1/10$, and $\lambda_1 = 1/100$, respectively. The fractionalization of the instanton does not occur for an untwisted background.

instanton in $\mathbb{R}^2$, while for small $|\lambda_1|$ it looks like the topological charge for a kink in the non-compact x_1 direction. In fact, when $\lambda_1 = 0$, $v^\dagger v$ is independent of the compact coordinate x_2, so $A_1 = 0$, and A_2 has a characteristic kink profile:

$$A_2 = \frac{1}{2} \partial_1 \ln v^\dagger v = \frac{\pi}{L} \left[\tanh \left(\frac{2\pi}{L} \left(x_1 - \frac{L}{2\pi} \ln |\lambda_2| \right) \right) + 1 \right] \tag{4.33}$$

$$q = \frac{\pi}{L^2} \operatorname{sech}^2 \left(\frac{2\pi}{L} \left(x_1 - \frac{L}{2\pi} \ln |\lambda_2| \right) \right) \tag{4.34}$$

This change in the form of the single instanton occurs because the compact direction sets a maximal size for an instanton, and so a small instanton looks like an instanton on $\mathbb{R}^2$, while a large instanton looks like a 1d kink. However, as noted above, in quantum theory, this effect should only be trusted when $\beta < \Lambda^{-1}$, as emphasized in (1.4). In the low-temperature confined phase, we must use a center-symmetric holonomy, as opposed to (3.6). However, in strong coupling, the eigenvalues of (2.34) are strongly fluctuating and a weak coupling analysis is not applicable.

On the other hand, we have already shown that a $\mathbb{Z}_N$-symmetric twist is stable in a compact spatial direction, at small–$\mathbb{S}_L^1$. The difference between the weak-coupling center-symmetric regime and the strong coupling center-symmetric regime is analogous to the weak coupling adjoint Higgs regime vs. the strongly coupled "unbroken" regime of gauge theories on $\mathbb{R}^3 \times \mathbb{S}_L^1$ [15]. In other words, the role of the global part of the SU(N) gauge symmetry in gauge theory on $\mathbb{R}^3 \times \mathbb{S}^1$ is played by the SU(N)-global symmetry in the $\mathbb{CP}^{N-1}$ model. In particular, in the quantum theory at large-β [low temperature] one cannot just use the weak coupling center-symmetric configuration (3.7) to reveal the kink-constituents of a 2d instanton. Doing so naively, would result in a fractionalization of instantons at a scale $\beta > \Lambda^{-1}$, however, there is no clear interpretation of what an instanton with size modulus $\rho \sim \beta > \Lambda^{-1}$ actually means. No such semi-classical configuration actually exists.

4.5 Small spatial circle: fractionalization of 2d instantons

We have already shown in section 4.2 that the theory in a $\mathbb{Z}_N$ symmetric background at weak coupling has N-types of elementary kink configuration with action $S_0 = \frac{S_I}{N}$. In
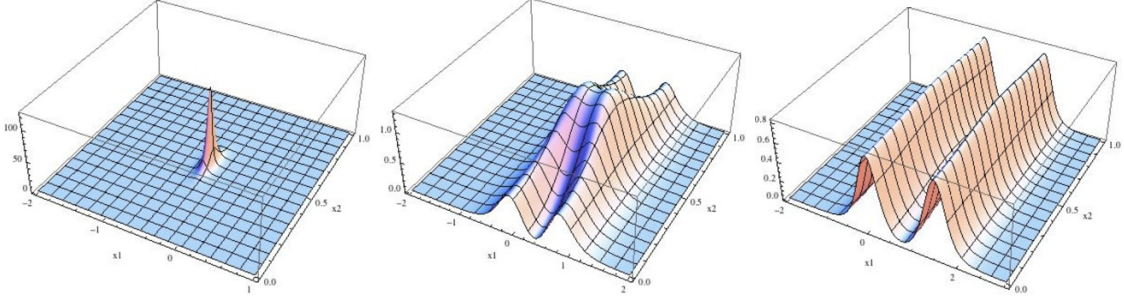
Figure 5. Small and large $Q = 1$ instantons in $\mathbb{CP}^1$ in a weak coupling center-symmetric background. Large instantons split into two $Q = \frac{1}{2}$ instantons.

this section, we re-derive this result in an alternative way. We show that a 2d instanton decomposes into N kink-instantons in the presence of a stable $\mathbb{Z}_N$ symmetric spatial twist.

Returning to the $\mathbb{CP}^1$ example, we now incorporate a *spatial* twist by multiplying the second component of the homogeneous coordinate (4.4) v_2 by a factor $e^{\frac{2\pi}{L}\mu_2 z}$, where $\mu_2 = 1/2$,

$$v_{\text{twisted}} = \begin{pmatrix} 1 \\ \left(\lambda_1 + \lambda_2\, e^{-\frac{2\pi}{L}z}\right) e^{\frac{2\pi}{L}\mu_2 z} \end{pmatrix} \tag{4.35}$$

To satisfy the twisted boundary condition (2.16) it would be enough to take a factor $e^{\frac{2\pi i}{L}\mu_2 x_2}$, but for an instanton v must be holomorphic, so we need to take $e^{\frac{2\pi}{L}\mu_2 z}$, which therefore prescribes also a certain dependence on the non-compact direction x_1. This is the essence of how the twisted spatial boundary conditions affect the structure of instantons on $\mathbb{R}^1 \times \mathbb{S}^1$.

The twisted instanton (4.35) has charge $Q = 1$, but at long distances it splits into two distinct kink-instantons, each of charge $1/2$. In general, for $\mathbb{CP}^{N-1}$ a charge $Q = 1$ decomposes into N distinct kink-instantons, each of topological charge $1/N$. To see how this works for the twisted $\mathbb{CP}^1$ instanton in (4.35), note that

$$v^\dagger_{\text{twisted}}v_{\text{twisted}} = 1 + |\lambda_1|^2 e^{\frac{2\pi}{L}x_1} + |\lambda_2|^2 e^{-\frac{2\pi}{L}x_1} + 2|\lambda_1\lambda_2|\cos\left(\frac{2\pi}{L}x_2 - \arg\lambda_1 + \arg\lambda_2\right) \tag{4.36}$$

A_1 is manifestly periodic in x_2, so it does not contribute to the topological charge. On the other hand,

$$A_2 = \frac{1}{2}\partial_1 \ln v^\dagger v = \frac{\pi}{L}\frac{|\lambda_1|^2 e^{\frac{2\pi}{L}x_1} - |\lambda_2|^2 e^{-\frac{2\pi}{L}x_1}}{1 + |\lambda_1|^2 e^{\frac{2\pi}{L}x_1} + |\lambda_2|^2 e^{-\frac{2\pi}{L}x_1} + 2|\lambda_1\lambda_2|\cos\left(\frac{2\pi}{L}x_2 - \arg\lambda_1 + \arg\lambda_2\right)} \tag{4.37}$$

Thus, $A_2 \to \pm\frac{\pi}{L}$ as $x_1 \to \pm\infty$, and so $Q = 1$. However, inspection of the form of A_2 shows that A_2 behaves like two separate kinks, each of charge $1/2$, one located at $x_1 \approx -\frac{L}{\pi}\ln\lambda_1$, and the other at $x_1 \approx \frac{L}{\pi}\ln\lambda_2$. The corresponding topological charge densities are plotted in figure 5.

Another useful way to visualize these kink-instanton constituents is using the Wilson loop (2.33). As the Euclidean time coordinate x_1 goes from $x_1 = -\infty$ to $x_1 = +\infty$, the
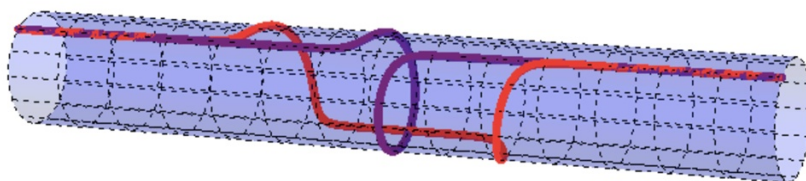
Figure 6. The Wilson loop for a small $Q = 1$ instanton is shown in purple. The large instanton splits into two separate kink-instantons. Each wraps half-way around the cylinder.
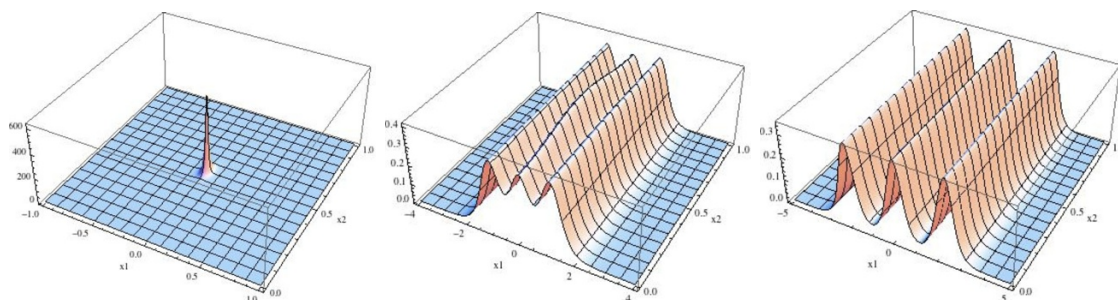
Figure 7. Same as figure 5, but now for $\mathbb{CP}^2$. Large instantons split into three $Q = \frac{1}{3}$ kink-instantons, as the scale changes.
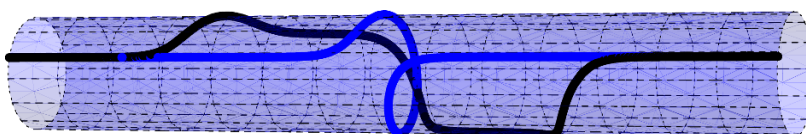
Figure 8. Same as figure 6, but now for $\mathbb{CP}^2$. The large instanton splits into three separate kink-instantons, as the scale changes.

Wilson loop winds around on a unit circle. For the twisted $Q = 1$ $\mathbb{CP}^1$ instanton in (4.35), the Wilson line $W(x_1)$ is plotted in figure 6. A small instanton on $\mathbb{R} \times \mathbb{S}_L^1$ behaves like a single instanton on $\mathbb{R}^2$, and winds fully around the circle. A sufficiently large instanton, on the other hand, decomposes into its two constituents, each of which winds half-way around the circle, but displaced from one another in the non-compact x_1 direction. The $\mathbb{CP}^2$ case ($N = 3$) is illustrated in figures 7 and 8. Again, the small instanton is essentially the same as a small instanton on $\mathbb{R}^2$, however, the large-instanton splits into three constituent kink-instantons, each of which has topological charge $Q = \frac{1}{3}$.

4.6 Matching and reinterpreting the bosonic zero modes

As reviewed in section 4.1, in the $\mathbb{CP}^{N-1}$ model on $\mathbb{R}^2$, the 2d instanton zero modes are associated with the classical symmetries of the self-duality equations: 2 are the position of the instanton ($\mathbf{a}_I \in \mathbb{R}^2$) and arise due to translation invariance, one is the size modulus

($\rho \in \mathbb{R}^+$) and is associated with invariance under dilatations, and the remaining $2N-3$ are internal orientational modes. This is equally true for the small instantons (4.25) on $\mathbb{R} \times \mathbb{S}_L^1$:

$$2N \xrightarrow{\text{short-distance}} 2 + 1 + (2N-3) = (\mathbf{a}_I \in \mathbb{R}^2) + (\rho \in \mathbb{R}^+) + (\text{orientation}). \quad (4.38)$$

As emphasized earlier, for the theory on $\mathbb{R}^2$, the existence of the size modulus ρ implies that the instanton comes in all sizes at no cost in action, and prevents a meaningful long-wavelength description of a dilute instanton gas from first principles. However, in the small $\mathbb{R}^1 \times \mathbb{S}^1$ regime of $\mathbb{CP}^{N-1}$, the instanton has a maximal size set by the eigenvlaue separation of the holonomy matrix ${}^L\Omega$. In this regime, and at long distances, the 2-d instanton is described as a composite of N separate 1d kinks. The $2N$ bosonic zero modes of the 2-d instanton on $\mathbb{R}^2$ matches the counting of the zero modes of the N kinks of the theory on $\mathbb{R} \times \mathbb{S}^1$. Each kink has two zero modes: One is the position of the kink, and arises due to translational invariance, and the other is an angular zero mode, associated with an internal symmetry. Therefore, the $2N$ collective coordinates split as

$$2N \xrightarrow{\text{long-distance}} N[1+1] = N[(a \in \mathbb{R}) + (\phi \in \mathrm{U}(1))]. \quad (4.39)$$

In particular the size modulus of the 2-d instanton is no longer present in the long distance description of the $\mathbb{CP}^{N-1}$ on small $\mathbb{R} \times \mathbb{S}^1$. This permits a meaningful dilute gas expansion.

Collective coordinates of kink-instantons. The one-loop measure for integrating over configurations of a type-j kink-instanton is

$$d\mu_{\mathrm{B}}^j d\mu_{\mathrm{F}}^j = e^{-S_j} \cdot \frac{da\, d\phi}{(2\pi)} \prod_{f=1}^{N_f} d^2\xi_f \cdot \mu^{2-N_f} \cdot J_a J_\phi (J_\xi)^{-N_f} \cdot \left[\det{}'(-D_j^2) \right]^{N_f-1}.$$

- $a \in \mathbb{R}$ is the kink-instanton position, $\phi \in \mathrm{U}(1)$ is an angle, ξ_f are the Grassmann-valued fermionic zero modes.
- μ is the (Pauli-Villars) renormalization scale. Each bosonic zero mode gives a contribution proportional to μ and each Grassmann zero modes gives a contribution proportional to $\mu^{-1/2}$, yielding μ^{2-N_f}.
- The collective coordinate Jacobians for bosonic and fermionic coordinates are: $J_a = S_j^{1/2}$, $J_\phi = LS_j^{1/2}[2\pi\overline{\alpha}_j(\varphi)]^{-1} = \frac{LNS_j^{1/2}}{2\pi}$, $J_\xi = 2S_j$.
- The primed determinant is the result of integrating over the Gaussian non-zero modes. When $N_f = 1$, the bosonic and fermionic primed determinants cancel precisely due to supersymmetry and the absence of non-compact scalars. This also helps us to deduce the result for general N_f.

The regularized (primed) determinant in the background of a type-j kink-instanton depends linearly on the renormalization scale, $\det'(-D^2) \sim \mu$. Since the the fields of a type-j kink-instanton reside entirely within a $\mathbb{CP}^1$ sub-manifold of $\mathbb{CP}^{N-1}$, the only scale which appears in the classical equations is $2\pi\overline{\alpha}_j(\varphi)/L = \frac{2\pi}{LN}$. Since the determinant is dimensionless, it must therefore have the form

$$[\det{}'(-D^2)]^{N_f-1} = C^{N_f-1} \left(\frac{\mu LN}{2\pi} \right)^{N_f-1}, \quad (4.40)$$

where C is a pure number of order one.

- The exponent of the renormalization scale μ coming from the collective coordinates μ^{2-N_f} and from integrating out non-zero modes (the primed determinant) μ^{N_f-1} combine to give, (also keeping the exponential of the kink-action)

$$\underbrace{\mu^{2-N_f}}_{\text{zero-modes}} \times \underbrace{\mu^{N_f-1}}_{\text{UV-det}'} e^{-S_0} \sim \mu e^{-S_I/N} \tag{4.41}$$

- Putting this all together, and realizing that at the center-symmetric vacuum, $S_j = S_0 = \frac{S_I}{N}$, the one-loop type-$j$ kink-instanton measure can be written as

$$d\mu_{\mathrm{B}} d\mu_{\mathrm{F}} = \frac{C^{N_f-1}}{\pi} \mu e^{-S_0} \left(\frac{LN}{2\pi}\right)^{N_f} S_0^{1-N_f} da d\phi \prod_{f=1}^{N_f} d^2\xi_f, \tag{4.42}$$

Consequently, the kink-instanton amplitude takes the form (neglecting interactions among kinks) $\mathcal{K}_j \sim \mu\ e^{-S_0}$ in the bosonic theory, and $\mathcal{K}_j \sim \mu e^{-S_0}\psi^{2N_f}$ in the theory with fermions. The crucial point is the appearance of the renormalization group invariant scale $\mu\ e^{-S_0} = \Lambda$ through the kink amplitudes, and Λ^2 through the charged and neutral bion amplitudes. The product of the amplitudes of N types of kink-instantons produces $I_{2d} \sim \prod_{j=1}^{N} \mathcal{K}_j \sim \mu^N e^{-NS_0} = \mu^{\beta_0} e^{-S_I} = \Lambda^{\beta_0}$, as expected, the 2d instanton factor, Λ^N. The kink and bion amplitudes will be crucial crucial in understanding the microscopic origins of various observables in $\mathbb{CP}^{N-1}$ models, such as the non-perturbatively induced mass gap, chiral symmetry realization and the renormalon singularity structure.

4.7   1-defects: kink-instantons

We have seen in section 4.3 that in the $\mathbb{CP}^{\mathrm{N}-1}$ model on $\mathbb{R} \times \mathbb{S}^1_L$, there are N types of elementary kink-instanton events associated with the simple and affine co-roots of the $SU(N)$ algebra. The vacuum of the theory, surprisingly, is the co-root lattice of $SU(N)$, that is the kink-instanton events are valued in the co-root lattice $\Gamma_r^\vee$.

$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i, \qquad \alpha_i \in \Gamma_r^\vee \tag{4.43}$$

The amplitudes associated with these kink-events in a $\mathbb{Z}_N$-symmetric background (3.7) are given by (ignoring interactions among the kink-instantons),

$$\mathcal{K}_j = \exp\left[-\frac{S_I}{N}\right] \qquad \text{where} \qquad S_I = \frac{4\pi}{g^2} - i\Theta, \qquad j = 1, \dots, N \tag{4.44}$$

Since Θ is an angular variable with period 2π, $\frac{\Theta}{N}$ does assume N different values. Therefore, the kink-instanton amplitude associated with each j is already a multi-branched quantity.

$$\text{fixed}-j: \qquad \mathcal{K}_j = \exp\left[-\frac{4\pi}{g^2 N} + i\frac{\Theta + 2\pi\,k}{N}\right], \qquad k = 1, \dots, N \tag{4.45}$$

Indeed, when we discuss the Θ dependence of observables, such as the vacuum energy density, mass gap or topological susceptibility, it will be crucial to note the fact that Θ dependence enters as $\frac{1}{N}$ of the 2d instanton effect and the kink-instanton contributions are multi-branched.

As noted, (4.45) does not take into account the interaction between kinks. This can be restored by writing the bosonic kink amplitudes as

$$\mathcal{K}_j = e^{-\alpha_j \cdot Y}, \qquad\qquad j = 1, \ldots N - 1,$$
$$\mathcal{K}_N = \eta e^{-\alpha_N \cdot Y}, \qquad\qquad \eta = e^{-\frac{4\pi}{g^2} + i\Theta}, \qquad\qquad (4.46)$$

where

$$\langle e^{-\alpha_j \cdot Y} \rangle = e^{-\frac{4\pi}{g^2}(\mu_{j+1} - \mu_j)} \qquad\qquad (4.47)$$

Y is an N-component complex field, with $2N$ real variables. It is related to the original variables as follows:

$$Y(x_1) = \mathrm{Re}Y(x_1) - i\,\mathrm{Im}Y(x_1) \qquad\qquad (4.48)$$

where $\mathrm{Re}Y(x_1) = \{\mathcal{A}_{2,1}, \mathcal{A}_{2,2}, \ldots, \mathcal{A}_{2,N}\}$ is the N-component sigma-model connection defined in (2.25), and $\mathrm{Im}Y$ is an N-component field which accounts for the $\{\theta_1, \ldots \theta_{N-1}\}$ induced interactions, and is non-locally dual to the θ-field. Although both real and imaginary parts of Y are N-component objects, due to the constraint $\sum_{j=1}^{N} \alpha_j \cdot Y = 0$, there are only $2(N-1)$ independent degrees of freedom entering to the N-types of kink amplitudes, which agrees with the number of degrees of freedom entering to the original n or $\tilde{n}$ fields.

4.8 Index theorem for Fredholm-type Dirac operator on $\mathbb{R} \times \mathbb{S}_L^1$

In the theory with N_f fermions, the 2d instanton has $2NN_f$ fermionic zero modes, as dictated by the Atiyah-Singer index theorem. The number of fermionic zero modes of kink-instantons is determined by an appropriate modification of the Nye-Singer index theorem, along the same lines as refs. [104, 105].

Here, we give the the index formula for topological excitations on $\mathbb{R}^1 \times S^1$ without derivation:

$$\mathrm{ind}(\slashed{D}) = \dim \ker \slashed{D} - \dim \ker \slashed{D}^\dagger$$
$$= \int \mathrm{ch}_1(F) - \frac{1}{2}\eta[D_{S^1,{}^L\Omega}] \qquad\qquad (4.49)$$

where $\mathrm{ch}_1(F)$ denotes the first Chern character, $\eta[D_{S^1,{}^L\Omega}]$ is the eta-invariant (spectral asymmetry) of the one-dimensional Dirac operator at the end of the cylinder (at $|x_1| = \infty$) coupled to the non-trivial σ-connection holonomy. The index is well-defined if and only if the operator $D_{S^1,{}^L\Omega}$ is Fredholm, i.e., $[{}^L\Omega(\infty) - 1]$ must be invertible, or non-degenerate. Typically, neither contribution to the index is actually an integer, and yet, the sum always is, as the index counts the number of fermionic zero modes associated with a kink.

The index theorem can be derived by using axial current non-conservation — an exact operator identity valid on any two-manifold — and relies on the existence of a σ-connection holonomy that satisfies the Fredholm condition. The main result is that the index associated with k_i many kinks of type α_i in $\mathbb{CP}^{N-1}$theory with N_f flavors of fermions is given by

$$\mathrm{ind}[k_1, k_2, \ldots, k_N] = N_f \sum_{i=1}^{N} 2k_i \qquad\qquad (4.50)$$

i.e., each elementary kink-instanton carries $2N_f$ fermionic zero modes. This result is identical to the index for the monopole-instantons in QCD(adj) on $\mathbb{R}^3 \times S^1$, see appendix B of [104, 105].

In the $N_f = 1$ case corresponding to compactified $\mathcal{N} = (2,2)$ theory, each kink-instanton $\mathcal{K}_j$ has two zero modes. The importance of just two zero modes this will be discussed in the next section. The kink-instanton amplitude in the N_f flavor theory is given by

$$\mathcal{K}_j = e^{-\alpha_j \cdot Y} \det_{f,f'}[(\alpha_j \cdot \psi_-^f)(\alpha_j \cdot \psi_+^{f'})] \qquad\qquad j = 1, \dots N-1,$$

$$\mathcal{K}_N = \eta e^{-\alpha_N \cdot Y} \det_{f,f'}[(\alpha_N \cdot \psi_-^f)(\alpha_N \cdot \psi_+^{f'})] \qquad\qquad \eta = e^{-\frac{4\pi}{g^2}+i\Theta}, \qquad (4.51)$$

Note that the product of the N-types of kink-instanton amplitudes gives the action and the fermion zero mode structure of the the 2d instanton amplitude, namely

$$\prod_{j=1}^{N} \mathcal{K}_j = I_{2d} \qquad\qquad (4.52)$$

The fermion-bilinear in these amplitudes transforms non-trivially under $\mathbb{Z}_{2NN_f}$, and is singlet under $\mathbb{Z}_{2N_f}$, and continuous global symmetries (2.13)

$$\mathbb{Z}_{2NN_f}: \quad \det_{f,f'}[(\alpha_j \cdot \psi_-^f)(\alpha_j \cdot \psi_+^{f'})] \longrightarrow e^{i\frac{2\pi}{N}} \det_{f,f'}[(\alpha_j \cdot \psi_-^f)(\alpha_j \cdot \psi_+^{f'})] \qquad (4.53)$$

(Recall the discussion around (2.13), the genuine discrete chiral symmetry, which may be spontaneously broken is just $[\mathbb{Z}_N]_A$ and it is broken down to $\mathbb{Z}_1$ on $\mathbb{R}^2$.) Since the kink-amplitude only respects the $\mathbb{Z}_1$ subgroup of $U(1)_A$, one may wonder if this implies that the anomaly free sub-group is $\mathbb{Z}_1$ and not $\mathbb{Z}_N$. However, this is not so in a way which differs crucially from QFT on $\mathbb{R}^3 \times S_L^1$.[12] $[\mathbb{Z}_N]_A$ is an exact symmetry of the quantum theory, which is broken spontaneously on $\mathbb{R}^2$. On small-S^1 where the low energy theory is described by a quantum mechanics, the theory develops N superselection sectors, and in fact, the existence of N-chirally broken vacua holds at any (finite) S_L^1.

5   Application: $\mathcal{N} = (2,2)$ $\mathbb{CP}^{N-1}$ theory on $\mathbb{R} \times S_L^1$

The power of the semi-classical transseries expansion, whenever applicable, transcends supersymmetry, and applies to all QFTs which admit a weak coupling limit. In this section, we study the dynamics of the supersymmetric theory by using kink-instantons (4.7),

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[12]In $\mathcal{N} = 1$ SYM and QCD(adj) on $\mathbb{R}^3 \times S_L^1$, the transformation of the fermion bilinear is as in (4.53) and $\mathbb{Z}_N$ symmetry is still exact because the "flux" part of the monopole-amplitude transforms in the opposite direction,

$$\mathbb{Z}_{2N}: \quad e^{-\alpha_j \cdot Y} \quad \longrightarrow \quad e^{-i\frac{2\pi}{N}} e^{-\alpha_j \cdot Y} \qquad (4.54)$$

rendering $e^{-\alpha_j \cdot Y}(\alpha_j \cdot \psi_-)(\alpha_j \cdot \psi_+)$ a singlet under $\mathbb{Z}_{2N}$ symmetry. This is possible because in QFT, $\sigma \equiv \mathrm{Im}Y(x)$ is the dual photon and in the absence of the monopoles, has a shift symmetry, $\sigma \to \sigma + \epsilon$. This shift symmetry intertwines and reduces to $[\mathbb{Z}_N]_A$ in the presence of monopoles. This mechanism is no longer valid on small $\mathbb{R}^1 \times S^1$, because $\mathrm{Im}Y$ is a massive field at tree level, and there is no analogous shift symmetry to undo the chiral rotation (4.53). In quantum mechanics, a new mechanism is operative. This requires a separate discussion of its own.

the index theorem (4.49) and basic supersymmetric techniques. This will help us to check the semi-classical methods applied to topological molecules with the results of the supersymmetric approach.

As discussed earlier, the one-loop potential (3.5) for the σ-connection holonomy (2.34) is zero for the supersymmetric $\mathcal{N} = (2, 2)$ $\mathbb{CP}^{N-1}$ theory with supersymmetry preserving periodic boundary conditions for fermions:

$$V_+[^L\Omega] = 0 \tag{5.1}$$

This is true to all orders in perturbation theory because of supersymmetry.

As noted in (4.51), for the $N_f = 1$ case, each kink-instanton $\mathcal{K}_j$ event carries two fermion zero modes, including the affine-one as a result of the index theorem (4.49). The N types of kink-instanton amplitudes are given by

$$\begin{aligned}
\mathcal{K}_j &= e^{-\alpha_j \cdot Y}[(\alpha_j \cdot \psi_-)(\alpha_j \cdot \psi_+)], \qquad j = 1, \ldots N - 1, \\
\mathcal{K}_N &= \eta e^{-\alpha_N \cdot Y}[(\alpha_N \cdot \psi_-)(\alpha_N \cdot \psi_+)]
\end{aligned} \tag{5.2}$$

where we dropped the prefactors for convenience. The existence of two zero modes implies that such kink-instanton amplitudes can induce a superpotential. The associated superpotential is

$$\mathcal{W} = \sum_{j=1}^{N-1} e^{-\alpha_j \cdot Y} + \eta e^{-\alpha_N \cdot Y} \tag{5.3}$$

which is called the affine Toda superpotential.

The non-perturbatively induced bosonic potential $V(Y) = \sum_{j=1}^{N} |\partial_{Y_j} \mathcal{W}|^2$ leads to a $\mathbb{Z}_N$-symmetric ground state, i.e., the same $\mathbb{Z}_N$-symmetric background (3.7) as in the $N_f > 1$ theories. In order to see this, let $V_i = \alpha_i \cdot Y$, and re-write the superpotential as

$$\mathcal{W} = \sum_{j=1}^{N-1} e^{-V_j} + \eta \prod_{j=1}^{N-1} e^{V_j} \tag{5.4}$$

The minimum can be found by extremizing the superpotential, which yields

$$e^{-V_j} = \eta \prod_{j=1}^{N-1} e^{V_j} \implies e^{-V_j} = \eta^{1/N} \tag{5.5}$$

leading to the center-symmetric background

$$\langle \mathrm{Re}(Y_{j+1} - Y_j) \rangle = \frac{4\pi}{g^2 N}, \qquad \mu_{j+1} - \mu_j = \frac{2\pi}{N} \tag{5.6}$$

given in (3.7). This provides a self-consistency condition to the use of semi-classics in supersymmetric $\mathbb{CP}^{N-1}$.

The superpotential expanded around the minimum (5.6) can be re-written as

$$\mathcal{W}(\langle Y_j \rangle + Y_j) = \eta^{1/N} \sum_{j=1}^{N} e^{-\alpha_j \cdot Y} = \eta^{1/N} \sum_{j=1}^{N} e^{-(Y_{j+1} - Y_j)} \tag{5.7}$$

where we parametrized the fluctuations around the minimum $\langle Y_j \rangle$ by Y_j. Therefore, the bosonic potential is

$$V(Y) = \sum_{j=1}^{N} \left| \partial_{Y_j} \mathcal{W} \right|^2 = \eta^{2/N} \sum_{j=1}^{N} \left| e^{-\alpha_j Y} - e^{-\alpha_{j-1} Y} \right|^2$$

$$= \eta^{2/N} \sum_{j=1}^{N} \left[2 e^{-\alpha_j(Y+Y^*)} - e^{-\alpha_j Y - \alpha_{j-1} Y^*} - e^{-\alpha_j Y^* - \alpha_{j-1} Y} \right] \qquad (5.8)$$

A few interpretational comments are in order:

- the topological configurations (4.51) with $N_f = 1$, leading to a superpotential, are 1-defects, or kink-instantons.
- The bosonic potential is induced by non-selfdual 2-defects, correlated kink-anti-kink events, called bions $\mathcal{B}_{ii} = [\mathcal{K}_i \overline{\mathcal{K}}_i]$ and $\mathcal{B}_{i,i-1} = [\mathcal{K}_i \overline{\mathcal{K}}_{i-1}]$. These 2-defects do not carry any fermionic zero modes, hence they are capable of generating a bosonic potential, as opposed to a superpotential.
- The existence of the semi-classical kink-instantons and bions relies on the $\mathbb{Z}_N$ symmetric (or more generally, non-degenerate) background such as (3.7), but not on supersymmetry. These defects generalize to arbitrary N_f. In section 6.3 we give a general argument for the construction of the two-types of bions.

5.1 Chiral ring and condensate, and mirror symmetry

In order to calculate the chiral condensate, it is useful to introduce a Veneziano-Yankielowicz (VY) Lagrange multiplier superfield S [71]. Then, the superpotential can be re-written as

$$\mathcal{W}(V_i, S) = \sum_{j=1}^{N} e^{-V_j} + S\left(\tau - \sum_{j=1}^{N} V_j\right) \qquad (5.9)$$

Integrating out S, we land on (5.4). Instead, integrating out the V_i super-fields, one obtains the VY-type superpotential, given by

$$\mathcal{W}(S) = -SN \log S - S\tau + NS \qquad (5.10)$$

It is now straightforward to obtain the chiral condensate by using the VY-superpotential. Integrate out the S-field, which amounts to $S^N = e^{-\tau}$, a quantum modified chiral ring relation, similar to its 4d counterpart [73]. In dimensionful units,

$$S^N = \Lambda^N, \qquad S = \Lambda e^{i\frac{2\pi k}{N}} \quad , \quad k = 1, \dots, N \qquad (5.11)$$

where Λ is the strong scale. The lowest component of S is the chiral condensate.

The chiral condensate can also be recovered as follows. Substituting $S^N = e^{-\tau}$ into (5.10), we obtain the superpotential in terms of the holomorphic parameter τ:

$$\mathcal{W}(\tau) = N e^{-\tau/N} \qquad (5.12)$$

and the chiral condensate reads $\langle \psi_+ \psi_- \rangle = -\partial_\tau \mathcal{W} = e^{-\tau/N}$. The supersymmetric theory, (as we will see the non-supersymmetric theories with $N_f > 1$) possess N-vacua $|\Psi_k^0\rangle$, associated with spontaneous chiral symmetry breaking:

$$\langle \Psi_k^0 | \psi_+ \psi_- | \Psi_k^0 \rangle = \Lambda e^{i\frac{2\pi k}{N}}, \qquad k = 1, \ldots, N \tag{5.13}$$

The affine Toda superpotential (5.9) has been obtained earlier, on $\mathbb{R}^2$, by using mirror symmetry in string theory, see [72]. In our framework, this follows from a simple duality in quantum mechanics on $\mathbb{R} \times \mathbb{S}_L^1$. This duality, in our formulation, amounts to re-writing of a quantum mechanical system with multiple degenerate minima, in terms of a dilute gas of kink-instantons and the two-types of bions. In string theory, the mirror of $\mathbb{CP}^{N-1}$ is obtained by using the standard $R \to 1/R$ duality and the dynamical generation of a superpotential by vortices. It would be interesting to understand the connection between these two derivations in more detail.

6 Infrared renormalons and topological molecules

6.1 Prelude: large orders in perturbation theory, Borel summation and the Stokes phenomenon

Perturbation theory in almost all interesting quantum field theories, quantum mechanics and even in ordinary integrals with multiple saddles, leads to divergent asymptotic expansions [36]. The divergence encodes physical information about the saddles of ordinary integrals, or path integrals of quantum mechanics and quantum field theory, as a consequence of Darboux's theorem [1, 4–6]. We recall a few relevant definitions and motivate (known) generalizations of those definitions by using simple quantum mechanics.

Let $P(g^2)$ denote a perturbative asymptotic series that satisfies the "Gevrey-1" condition:

$$P(g^2) = \sum_{q=0}^\infty a_q g^{2q}, \qquad \text{Gevrey} - 1: \ |a_q| \le CR^q q! \tag{6.1}$$

for some positive constants C and R [8, 10]. Known examples of perturbative series that arise in quantum mechanics and QFT satisfy the "Gevrey-1" condition [36]. We denote the Borel transform of $P(\lambda)$ by $BP(t)$ and define it as

$$BP(t) := \sum_{q=0}^\infty \frac{a_q}{q!} t^q. \tag{6.2}$$

The formal Borel transform determines "a germ of a holomorphic function" at $t = 0$, with a finite radius of convergence. Next, one analytically continues the obtained germ $BP(t)$ to the whole complex t-plane, called the Borel plane. We also assume that the analytic continuation of the Borel transform $BP(t)$ is "endlessly continuable". That roughly means that the function is represented by an analytic function with a discrete set of singularities (poles or cuts) on its Riemann surface. The Borel resummation of $P(g^2)$, when it exists, is defined as the Laplace transform of the analytic continuation of the germ:

$$\mathbb{B}(g^2) = \frac{1}{g^2} \int_0^\infty BP(t) e^{-t/g^2} dt . \tag{6.3}$$

In quantum theories with multiple-degenerate vacua, (but no instability of any kind), perturbation theory is typically a non-alternating Gevrey-1 series, hence non Borel resummable [23, 24, 31, 33, 34, 36]. Non-Borel summability means that there is no unique answer in perturbation theory; i.e., resummed perturbation theory does not produce a unique answer for a physical observable which ought to be unique, for example, the ground state energy. Of course, this is senseless. This means that perturbation theory (re-summed or otherwise) is insufficient to define the theory.

In certain cases, a perturbative sum which is not Borel summable becomes Borel summable upon continuation $g^2 \rightarrow -g^2$, see figure 2. In simple quantum mechanics, let us mention an example that is directly relevant for our purpose [24]. Perturbation theory for the periodic potential $V(x) = \frac{1}{g^2} \sin^2(gx)$ is non-Borel summable, whereas perturbation theory for $V(x) = \frac{1}{g^2} \sinh^2(gx)$ is Borel summable. [Recall and compare with the 0-dimensional partition functions discussed in section 1.6]. Both series are, of course, asymptotic and divergent. The difference between the two is that the asymptotic series which arises in the first case is *non-alternating*, whereas the series in the latter is just the *alternating* version of the former. Let us refer to the Borel resummed series for the latter, Borel resummable series, as $\mathbb{B}_0(g^2)$. Then, we can define the perturbative sum for the non-alternating series as the analytic continuation of $\mathbb{B}_0(g^2)$ in the g^2 complex plane from negative coupling, $g^2 < 0$, to the positive real axis, $g^2 > 0$. This can be done in one of the two ways as shown in figure 2. Approaching the positive real axis clock-wise (from above) and counter-clock-wise (from below).

$$\mathbb{B}_0(|g^2| \pm i\epsilon) = \mathrm{Re}\,\mathbb{B}_0(|g^2|) \pm i\,\mathrm{Im}\,\mathbb{B}_0(|g^2|) \qquad \text{where} \quad \mathrm{Im}\,\mathbb{B}_0(|g^2|) \sim e^{-2S_I} \sim e^{-2A/g^2} \quad (6.4)$$

is the ambiguous part, and is a manifestation of non-Borel-summability [compare with (1.22)].

A definition of the Borel sum equivalent to what we described above through analytic continuation in the complex g^2-plane is the *directional (sectorial) Borel sum*. Define

$$\mathcal{S}_\theta P(g^2) \equiv \mathbb{B}_\theta(g^2) = \frac{1}{g^2} \int_0^{\infty \cdot e^{i\theta}} BP(t)\, e^{-t/g^2} dt, \qquad (6.5)$$

A special case of this is the *lateral Borel sum*. The function $\mathbb{B}_{\theta\pm}(g^2)$ is associated with contours just above and just below the ray at angle θ, and is called right (left) Borel resummation. If there are no singular points in the θ direction, then the left and right Borel sums are equal, and the theory is sectorial Borel summable in the θ-direction. A theory for which there are no singularities on $\theta = 0$ is called Borel summable in physics. In many cases, there is a ray of singular points of the Borel transform $BP(t)$, as shown in figure 9. Then, the theory is non-Borel summable, but left and right Borel summable. The ambiguity described above, associated with whether we approach the real positive axis from above or below in the complex g^2-plane, in the latter language, maps to the choice of the integration contour in the Laplace-transform. The integral is, of course, dependent on the choice of the
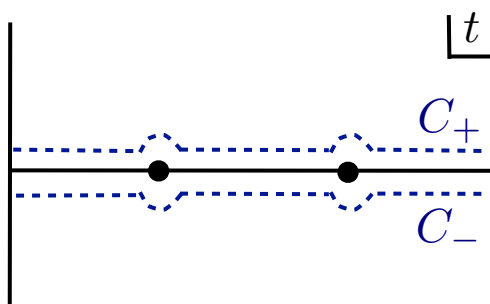
Figure 9. Lateral, or right and left, Borel sums. Dark circles are singularities (poles or branch points). Whenever a singularity exists between the right and left Borel sums, the theory is non-Borel summable. The singular direction in the t-plane corresponds to a Stokes line in the complex g^2-plane, see figure 2. The difference of the sectorial sums in passing from $\theta = 0^-$ to $\theta = 0^+$ is the Stokes "jump" across a Stokes ray.

contour, yielding (6.4). The overall procedure can be summarized through the diagram:

$$P(g^2) \xrightarrow{\text{B}-\text{transform}} BP(t) \tag{6.6}$$

$$\text{Borel}-\text{sum} \searrow \qquad \swarrow \mathcal{L}^\theta-\text{transform}$$

$$\mathcal{S}_\theta P(g^2)$$

where B- is the Borel transform operator, and $\mathcal{L}_\theta$ is Laplace transform along the ray at some angle θ.

In cases where there are singularities in the $\theta = 0$ direction, the $\mathbb{B}_{0\pm}(g^2)$ are different sums differing in their exponentially small imaginary parts, and still, both sectorial sums are associated with the asymptotic expansion (6.1). The divergence of the original series, in the original g^2 plane, just means that the perturbative expansion is taking place on a singular Stokes ray in the complexified coupling constant plane. Singular directions in the Borel t-plane correspond to Stokes lines in the g^2-plane, where the sum exhibits the Stokes phenomenon crossing the ray. The Stokes phenomenon in the g^2-plane, is mathematically, the origin of the ambiguity of the Borel sum. The understanding of the connection of the distinct sectorial sums entails the understanding of the jumps across this direction. This is achieved via the *Stokes automorphism*, or *passage automorphism* labelled by $\underline{\mathfrak{S}}_\theta$. This corresponds to composing the Laplace transform in a given direction, say θ^+, with the inverse Laplace transform in another direction, say θ^-, [88].

$$\mathcal{S}_{\theta+} = \mathcal{S}_{\theta-} \circ \underline{\mathfrak{S}}_\theta \equiv \mathcal{S}_{\theta-} \circ (\mathbf{1} - \text{Disc}_{\theta-}) , \tag{6.7}$$

where $\text{Disc}_{\theta-}$ is the full discontinuity of the Borel sum across θ. The Stokes jump is purely non-perturbative, and it is roughly

$$\text{Disc}_{\theta-} \mathbb{B} \sim e^{-t_1/g^2} + e^{-t_2/g^2} + \ldots \tag{6.8}$$

where $t_i \in e^{i\theta}\mathbb{R}^+$ is an ordered sequence of singularities along the singular direction.

In physical applications, $\mathbb{B}_0(|g^2| \pm i\epsilon) \equiv \mathcal{S}_{\pm 0} P(g^2)$ may be a physical observable in quantum mechanics, such as the energy of an eigenstate. If we take Borel-resummation literally, and if there are indeed singularities at $t_i \in \mathbb{R}^+$, then the Borel sum yields an answer which *i)* has imaginary component, *ii)* the imaginary part is two-fold ambiguous. On the other hand, we are dealing with a quantum mechanical system which does not have any instability, and the energy eigenvalues *must be* real.

In simple quantum mechanical systems, it is understood how the non-perturbative ambiguity in perturbation theory is cancelled with the ambiguity in instanton-anti-instanton events:[13] see the important collection of works [21–35]. In these theories, the non-perturbative molecular instanton-anti-instanton events are also ambiguous, and obtained through the analytic continuation and back depicted in figure 2. (The instanton and anti-instanton events are not ambiguous, only their topologically neutral composites are.) This ambiguity arises from the quasi-zero mode integrals and will be explained more generally in section 6.4. The instanton-anti-instanton molecule amplitude is

$$[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm} = \operatorname{Re}[\mathcal{I}\overline{\mathcal{I}}] + i \operatorname{Im}[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm} \tag{6.9}$$

As shown explicitly in quantum mechanics, these two classes of ambiguities, between the the re-summed perturbation theory around the perturbative vacuum and the non-perturbative amplitudes associated with neutral topological molecules, cancel precisely at the $O(e^{-2S_I})$ level, yielding our simplest "*confluence equation*", (see (7.25) and (7.26) for generalizations):

$$\operatorname{Im} \mathbb{B}_{0,\theta=0^\pm} + \operatorname{Im}[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm} = 0 , \qquad \text{up to } O(e^{-4S_I}) \tag{6.10}$$

leading to an unambiguous physical observable:

$$O(g^2) = \operatorname{Re} \mathbb{B}_0(|g^2|) + \operatorname{Re}[\mathcal{I}\overline{\mathcal{I}}] + \dots , \qquad \text{up to } O(e^{-4S_I}) \tag{6.11}$$

Resurgence is the statement that this structure repeats itself at all non-perturbative orders, and a consistent semi-classical trans-series expansion for observables in quantum mechanics can be obtained.

6.2 How to tame a theory with IR renormalons?

The ambiguity that arises in QM is related to the $q!$ factorial growth of the number of Feynman diagrams combined with the non-alternating character of the series [106]. This ambiguity is canceled by instanton-anti-instanton events. In "realistic QFTs" including asymptotically free theories such as QCD and non-linear sigma models which appear as low-energy description of certain spin systems [e.g., quantum anti-ferromagnets], however, the situation is quite different, as described by 't Hooft [49]. In particular, apart from the $q!$

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[13]It is important to note that the cancellation is with a *topologically neutral* instanton-anti-instanton event. Instantons, as they carry topological charge, cannot mix with the perturbative vacuum. It is important to realize and appreciate this difference. In particular, we will also discuss theories in which there are instantons, but the theory is Borel summable. This is due to the absence of a correlated instanton-anti-instanton event. For example, $\mathbb{CP}^{N-1}$ with extended supersymmetry, such as $\mathcal{N} = (4,4)$ is of this type.

factorial growth of the number of Feynman diagrams, in this class of theories, perturbation theory is so wild that a certain subset of diagrams also yields $q!$ growth due to integration over momenta. This situation arises in renormalizable QFTs, and these new divergences are called "renormalons" [40, 41]. There are both UV and IR renormalon singularities, associated with high and low momentum integration with respect to a given scale. In asymptotically free theories, the ambiguities that are due to IR renormalons are on the positive real axis, $t \in \mathbb{R}^+$, of the Borel plane, and furthermore are located much closer to the origin than the $[\mathcal{I}\bar{\mathcal{I}}]$ Borel pole. Until very recently [19], it was not known whether there exists a first principles non-perturbative approach to cancel these perturbative ambiguities, despite the existence of substantial literature on renormalons, see [40, 41] and references therein. We now address this problem in the context of $\mathbb{CP}^{N-1}$ model.

According to 't Hooft's analysis [49] and its generalizations to non-linear sigma models $\mathbb{CP}^{N-1}$ and $O(N)$, [40, 41, 60, 61], perturbation theory generically develops ambiguities of the form

$$\mathrm{Im}\,\mathbb{B}_0(|g^2|) \sim e^{-2nS_I/\beta_0} \sim \pi\, e^{-2nS_I/N}, \qquad n = 2, 3, \ldots, \tag{6.12}$$

which are exponentially more important than the $[\mathcal{I}\bar{\mathcal{I}}]$ singularity. We make two simple observations:

- this renormalon ambiguity is exponentially more important than the instanton-anti-instanton $[\mathcal{I}\bar{\mathcal{I}}]$ ambiguity,

$$e^{-2S_I/N} \gg e^{-2S_I} \tag{6.13}$$

The BZJ approach that suffices in quantum mechanics would only cure the $[\mathcal{I}\bar{\mathcal{I}}]$ ambiguity in QFT, not the much larger and more important renormalon ambiguity.

- On $\mathbb{R}^2$, there are no-known semi-classical (or even non-semi-classical) configurations with action $2nS_I/N, n = 2, 3, \ldots$ that can fix the IR renormalon ambiguity.

The formalism developed in this work helps us to solve this problem on $\mathbb{R} \times \mathbb{S}_L^1$ in a regime of QFT continuously connected to $\mathbb{R}^2$. Many authors previously considered some form of compact space to fix the IR-problem of the 2d instantons [64–69], however, the above mentioned problems did not find a solution there. What is new in our approach is *continuity*, an idea developed for gauge theory in [11, 12, 19, 70].

Main underlying idea of continuity. Compactify the theory on $\mathbb{R} \times \mathbb{S}_L^1$, and find the conditions under which there are no phase transitions or rapid crossovers as the $\mathbb{S}_L^1$ size is dialed to small values. Since we are reducing the theory to simple quantum mechanics and to finite volume, a sharp phase transition would only occur in the infinite-N limit. At finite-N, this phase transition would turn into a rapid cross-over, which is an equally drastic change of the long-distance theory. Our continuity argument avoids both types of drastic change in the dynamics and smoothly connects the small-L regime to the large-L regime. Since the theory is asymptotically free, at sufficiently small L, the theory is rendered weakly coupled, and since there is no phase transition or rapid cross-over, the non-perturbative phenomena must also be continuous. For example, for the mass gap, there cannot be any drastic
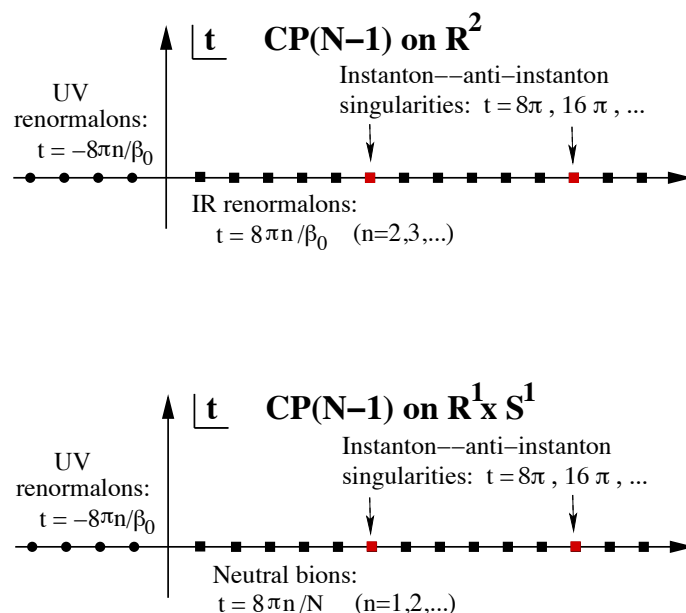
Figure 10. Upper figure: The conjectured structure of the Borel plane for $\mathbb{CP}^{N-1}$ on $\mathbb{R}^2$. Lower figure: The semi-classical singularities associated with the neutral bion molecules in $\mathbb{CP}^{N-1}$ on small $\mathbb{R} \times S^1$. For $N_f = 0$, the weak-coupling regime has an extra singularity closer to origin than the leading renormalon pole on $\mathbb{R}^2$. For $N_f \geq 1$, the location of the semi-classical and non-semi-classical renormalon singularities coincide. Although the theory moves from a weakly coupled description to a strongly coupled one, the structure of the Borel plane singularities either do not change at all or change extremely mildly. We take this as evidence that the neutral bion molecules are the semi-classical realization of renormalons. This also gives us hope that even the theory on $\mathbb{R}^2$ may potentially be solvable at arbitrary N.

changes, and indeed, we will demonstrate this non-trivial claim explicitly. Then, the non-semi-classical notion of an IR renormalon on $\mathbb{R}^2$ *must* find a semi-classical realization on $\mathbb{R} \times \mathbb{S}_L^1$. In this way, we may indeed find a semi-classical configuration continuously connected to the renormalon singularity. This is the main and simple physical idea in our formalism.

We have already shown that in the semi-classical regime, there are kink-instanton configurations $\mathcal{K}_i$ with action $\frac{S_I}{N}$. However, there is no ambiguity associated with these configurations, and their topological charge or topological Θ angle dependence suffice to distinguish them from the perturbative vacuum. Hence, the kink-instantons are *not* the realization of the IR -renormalons. Rather, the IR renormalons must be associated with certain topological configurations (or *molecules*), indistinguishable from perturbation theory.

In the next section, we give a classification of n-defects (1-defects are kinks.). We show that there is an ambiguity in the semi-classical amplitude of certain n-kinks, which is identifiable with the infrared renormalon singularities. It is not true that all n-kink configurations are ambiguous. For example, in the list of topological configurations for the bosonic $N_f = 0$ theory, only the configurations with subscript $\pm$ have ambiguous imaginary parts, and the ambiguities that arise in the Borel summation of the perturbation theory cancel with the ambiguities of these molecular events. On the other hand, in a

theory with fermions, the appearance of the first non-perturbative ambiguity is delayed by one order. A few examples of the topological configurations and the (non)existence of their ambiguities are given in the following lists:

$$N_f = 0: \quad \{\mathcal{K}_i, [\mathcal{B}_{ij}], \ [\mathcal{B}_{ii}]_{\theta=0\pm}, \ [\mathcal{B}_{ij}\mathcal{B}_{ji}]_{\theta=0\pm}, \ [\mathcal{B}_{ij}\mathcal{B}_{jk}\mathcal{B}_{ki}]_{\theta=0\pm}, \dots, [\mathcal{I}\overline{\mathcal{I}}]_{\theta=0\pm}, \dots\}$$

$$N_f \geq 1: \quad \{\mathcal{K}_i, [\mathcal{B}_{ij}], \ [\mathcal{B}_{ii}], \qquad [\mathcal{B}_{ij}\mathcal{B}_{ji}]_{\theta=0\pm}, \ [\mathcal{B}_{ij}\mathcal{B}_{jk}\mathcal{B}_{ki}]_{\theta=0\pm}, \dots, [\mathcal{I}\overline{\mathcal{I}}], \dots\} \qquad (6.14)$$

In other words, when $N_f = 0$ we first see the non-perturbative ambiguities in the neutral bion amplitude $[\mathcal{B}_{ii}]$, while for $N_f \geq 1$ the non-perturbative ambiguities first arise in the neutral correlators of two bions. The location of the ambiguities in the semi-classical molecules matches the location of the renormalon singularities on $\mathbb{R}^2$ for $N_f \geq 1$ theories, and for $N_f = 0$, the semi-classics has an extra singularity closer to the origin than the leading renormalon pole on $\mathbb{R}^2$. See figure 10.

The elegance of this analysis is that a very difficult problem in QFT, tied with the renormalon singularities, reduces to a relatively simpler problem in quantum mechanics without much change in the structure of the Borel plane singularities. The crucial physical elements permitting this analysis in QFT are continuity and compactification with spatially twisted boundary conditions.

6.3 Classification of bions and Cartan matrix

The nodes of the extended Dynkin diagram of the $SU(N)$ algebra $\widehat{A}_{N-1}$ provides a unique labeling of the kink-instanton events on $\mathbb{R} \times \mathbb{S}^1$. As described earlier, the difference of the asymptotes of the kink event $\mathcal{K}_j$ is given by

$$\Delta\widetilde{n} = \widetilde{n}(\infty) - \widetilde{n}(-\infty) = \alpha_j \in \Gamma_r^\vee \qquad (6.15)$$

Since for a given kink-instanton $\mathcal{K}_j$, $\Delta\widetilde{n} = \alpha_j$ cannot be deformed to zero, or to another $\alpha_{j'}$ where $j' \neq j$, by small perturbations, the co-root $\alpha_j \in \Gamma_r^\vee$ must be considered as a topological charge.[14]

The study of the topological molecules in the weakly coupled domain of the $\mathbb{CP}^{N-1}$ model on $\mathbb{R} \times \mathbb{S}^1$ is parallel to the study of the same class of molecules in center-symmetric QCD(adj) on $\mathbb{R}^3 \times \mathbb{S}^1$. Here, we follow ref. [20]. As in gauge theory, there exists a one-to-one mapping between the molecules at second order in the semi-classical expansion and non-vanishing entries of the extended Cartan matrix, $\widehat{A}_{ij} := (\alpha_i, \alpha_j)$, $i, j = 1, \dots, N$. The extended Cartan matrix determines the interaction between kinks of different types, and plays a crucial role in classifying the molecular kink-instanton/kink-anti-instanton events. For brevity and due to their similarity to bions in gauge theory, we also refer to these *universal* correlated events that appear in the second order in the semi-classical expansion as "bions". As in gauge theory, there are two types of bions:

---

[14]This is the counterpart of the magnetic charge of monopole-instanton. Recall that a monopole-instanton on $\mathbb{R}^3 \times S^1$ is labelled by two topological charges: the second Chern number, which is $\frac{1}{N}$, and a magnetic charge $\alpha_j \in \Gamma_r^\vee$. Reducing QFT on $\mathbb{R}^3 \times S^1$ down to quantum mechanics on $\mathbb{R} \times T^2 \times S^1$, we may define the magnetic flux change associated with the monopole events as $\Delta\Phi = \int_{T^2} B \in \Gamma_r^\vee$ in quantum mechanics as well. Identifying $\Delta\Phi$ with $\Delta\widetilde{n}$, this compactification can be used to find an exact mapping among topological configurations between $\mathbb{CP}^{N-1}$ and gauge theory. This is actually the primary reason why the classification of the topological configurations in compactified gauge theory and $\mathbb{CP}^{N-1}$ model are identical.

Definition 4: charged bions and neutral bions.

- *Charged bions:* for each pair (i,j) such that the entry of the Cartan matrix is negative $\widehat{A}_{ij} < 0$ (as a result, the bosonic interaction $V_{ij} \sim -\widehat{A}_{ij} < 0$ is repulsive at short separations), there exists a bion $[\mathcal{K}_i \overline{\mathcal{K}}_j]$, associated with the correlated tunneling-anti-tunneling event

$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i - \alpha_j \qquad \alpha_i \in \Gamma_r^\vee \tag{6.16}$$

The amplitude associated with such an event is

$$\mathcal{B}_{ij} = [\mathcal{K}_i \overline{\mathcal{K}}_j] \sim e^{-S_i(\varphi)-S_j(\varphi)} e^{i\sigma(\alpha_i-\alpha_j)}. \tag{6.17}$$

where $\sigma = \mathrm{Im}\, Y$, with Y defined in (4.48). There is no ambiguity for a charged bion, as shown in section 6.5. This is the counter-part of the magnetic bion in compactified gauge theory.

- *Neutral bions:* for each i, such that the entry of the Cartan matrix is positive $\widehat{A}_{ii} > 0$ (as a result, the bosonic interaction $V_{ii} \sim -\widehat{A}_{ii} < 0$ is attractive at short separations), there exists a bion $[\mathcal{K}_i \overline{\mathcal{K}}_i]$ with vanishing topological charge and associated with the correlated tunneling-anti-tunneling event of the same type

$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i - \alpha_i \qquad \alpha_i \in \Gamma_r^\vee \tag{6.18}$$

The real part of the amplitude for the neutral bion is unambiguous

$$\mathrm{Re}\,\mathcal{B}_{ii} = \mathrm{Re}[\mathcal{K}_i \overline{\mathcal{K}}_i] \sim e^{-2S_i(\varphi)}. \tag{6.19}$$

However, the neutral bion amplitude develops an imaginary ambiguous part, as will be discussed in section 6.6, along with its physical implications.

We define the right (and left) neutral bion amplitudes as the bion amplitude evaluated at $g^2 \pm i0$ according to the continuation shown in figure 2. These are unambiguous, but the two differ by a non-perturbative jump $e^{-2S_i(\varphi)}$ for the bosonic theory . This will be crucial in the non-perturturbative cancellation of ambiguities.

The derivations of these assertions will be given in sections 6.5 and 6.6. For the $\mathbb{CP}^{N-1}$theory, both tunneling events actually carry zero topological charge (2.7)

$$Q_T = 1/N + (-1/N) = 0, \qquad \text{for both types of bions} \tag{6.20}$$

However, the *charged* bion is still associated with a co-root charge. Correspondingly, the charged bions can be assigned a more refined topological quantum number, physically associated with where it starts and ends in the $\widetilde{n}$ landscape of classical vacua, which are different points. In contrast, the neutral bion starts at some configuration, tunnels in the α_i directions and returns back to the original point. For the theory with fermions, this first ambiguity disappears for $N_f \in \mathbb{Z}^+$ and the first ambiguity appears for a 4-defect.

6.4 Zero, quasi-zero and non-zero modes of n-defects

In the semi-classical regime, the path integral can be formally rewritten as a sum over a dilute gas of n-defects. In Euclidean space where 1-defects can be seen as Euclidean particles, n-defects should be viewed as Euclidean molecules, hence the terminology *topological molecules* that we use from time to time. In section 4.7 we discussed 1-defects and their contribution to the low-energy theory. These 1-defects are solutions to a self-duality equation. They are exact solutions to the classical equations. The n-defects are only approximate quasi-solutions.

In the background of a general n-defect, within the path-integral formulation, one needs to perform a sum over all fluctuations, meaning that one needs to obtain information about the eigen-spectrum of fluctuations. In general semi-classical analysis, the eigen-spectrum has three types of modes:

- zero-modes: zero eigenvalues of the eigenspectrum, associated with the moduli

- Quasi-zero modes: parametrically small compared to typical eigenvalue

- Non-zero modes: Modes for which the eigenspectrum is of order one in natural units.

For 1-defects, kinks, as discussed in section 4.2, there are only two moduli, the position moduli $a \in \mathbb{R}$ and the angular moduli $\phi \in \mathrm{U}(1)$; the latter is integrated trivially. Of course, there are also non-zero modes associated with the small fluctuations operator, in the background of the 1-defect. These modes and associated determinants can be dealt with in the Gaussian approximation. The quasi-zero mode does not exist for 1-defects.

For n-defects with $n \geq 2$, all three types of modes exist. The quasi-zero-modes need to be treated exactly within semi-classics. The reason is as follows: there is a characteristic length scale entering the quasi-zero mode analysis. Denote this scale by ℓ_{qzm}. As we will see, this scale is much larger than the characteristic kink size r_{k}, but much smaller than the typical inter-kink separation $d_{\mathrm{k-k}}$:

$$r_{\mathrm{k}} \ll \ell_{\mathrm{qzm}} \ll d_{\mathrm{k-k}} \tag{6.21}$$

The integral over the quasi-zero mode is mainly supported at the scale ℓ_{qzm}. Performing the quasi-zero-mode integral, one can treat the 2-defect as a point operator when considering physics at length scales larger than ℓ_{qzm}. This is the way that we construct the bion operators $\mathcal{B}_{ij}$ and other n-defect operators. The reason that semi-classical analysis in compactified $\mathbb{CP}^{N-1}$ is reliable at small-$\mathbb{S}_L^1$ is the hierarchy of length scales

$$
\begin{array}{ccccccc}
r_{\mathrm{k}} & \ll & r_{\mathrm{b}} \sim \ell_{\mathrm{qzm}} & \ll & d_{\mathrm{k-k}} & \ll & d_{\mathrm{b-b}}, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L & \ll & L \log\left(\frac{1}{g^2}\right) & \ll & L\, e^{S_0} & \ll & L\, e^{2S_0}.
\end{array}
\tag{6.22}
$$

that arises from a careful treatment of the quasi-zero modes. In this formula, r_{b} is the size of a bion, and $d_{\mathrm{b-b}}$ is the typical inter-bion separation. This physical hierarchy of length scales is also naturally built into the mathematical trans-series expansion.

6.5 2-defects: charged bions

The interaction (correlation) between $\mathcal{K}_i$ and $\overline{\mathcal{K}}_j$ has two components. One is due to the exchange of the bosonic fields and the other is due to the exchange of fermion zero modes. Following the explanation in section 5.1 of [20], and generalizing the appendix A43.2 of Zinn-Justin's book [31], we find that the interaction induced by bosonic exchange between $\mathcal{K}_i$ and $\overline{\mathcal{K}}_j$ kinks separated at a distance τ is given by

$$S_{\text{int}}(\tau) = -8\xi \frac{\alpha_i.\alpha_j}{g^2} e^{-\xi\tau}, \qquad \xi \equiv \frac{2\pi(\mu_{i+1} - \mu_i)}{L} = \frac{2\pi}{LN} \tag{6.23}$$

The fermion zero-mode exchange induces an interaction between the two kinks as well, which can be read-off from the connected correlator:

$$\left\langle \prod_{f=1}^{N_f} [\alpha_i(\psi_f)]^2 (t - \tau/2) \prod_{f=1}^{N_f} [\alpha_j(\overline{\psi}_f)]^2 (t + \tau/2) \right\rangle = \left(\frac{\alpha_i.\alpha_j}{2} \right)^{2N_f} \left(\frac{g^2}{2L} \right)^{2N_f} e^{-2N_f \xi \tau}$$

Consequently, the bion amplitude may be written as in (6.17) with coefficient involving an integral over the quasi-zero mode (separation between the two events.)

$$\mathcal{A}_{ij} = \mathcal{A}_i \mathcal{A}_j \left(\frac{\alpha_i.\alpha_j}{2} \right)^{2N_f} \left(\frac{g^2}{2L} \right)^{2N_f} 2 \int_0^\infty d\tau\, e^{-V_{\text{eff}}^{ij}(\tau)}, \tag{6.24}$$

where

$$V_{\text{eff}}^{ij}(\tau) = -8\xi \frac{\alpha_i.\alpha_j}{g^2} e^{-\xi\tau} + 2N_f \xi \tau \tag{6.25}$$

The factor of 2 in (6.24) comes from the integration over the solid angle in 1d, $\int d\Omega = 2$, as the interaction of the constituents of the bions only depends on separation. For $N \geq 3, \widehat{A}_{ij} = -\frac{1}{2}$ for non-vanishing off-diagonal entries of the extended Cartan matrix (for $N = 2$, $\widehat{A}_{ij} = -1$), the quasi-zero mode integral given in (6.24) is

$$I(g^2) = \int_0^\infty d\tau \exp\left[-\left(\frac{4\xi}{g^2} e^{-\xi\tau} + 2N_f \xi \tau \right) \right] = \left(\frac{g^2}{4\xi} \right)^{2N_f} \int_0^{\frac{4\xi}{g^2}} du\, e^{-u}\, u^{2N_f - 1}$$

$$\xrightarrow[g^2 \ll 1]{} \left(\frac{g^2}{4\xi} \right)^{2N_f} \Gamma(2N_f) = \left(\frac{g^2 N}{8\pi} \right)^{2N_f} \Gamma(2N_f) \tag{6.26}$$

The charged-bion amplitude is, therefore,

$$\mathcal{B}_{ij} = -\mathcal{A}_{ij} e^{-S_i(\varphi) - S_j(\varphi)} e^{2\pi i \sigma (\alpha_i^\vee - \alpha_j^\vee)} \tag{6.27}$$

Various comments are in order regarding (6.26): For $\mathbb{CP}^{N-1}$ and $N_f \geq 1$, for non-vanishing negative entries of the extended Cartan matrix, $\alpha_i.\alpha_j < 0$, the bosonic interaction between the constituents is repulsive, whereas the fermion zero mode induced exchange is attractive. Therefore, there is a characteristic scale dominating the integral.

$$V'_{\text{eff}}(\tau) = 0 \implies \tau^* = \frac{1}{\xi} \log\left(\frac{4\pi}{g^2 N N_f} \right), \quad r_{\text{b}} = r_{\text{k}} \log\left(\frac{4\pi}{g^2 N N_f} \right) \quad N_f \geq 1 \tag{6.28}$$

We interpret this as the size of the charged bion.

The integral (6.26) appeared in the study of molecular instantons in supersymmetric and non-supersymmetric quantum mechanics in [32]. For $N_f = 0$, where the integral at large-τ is not cut-off, this appeared already in the work of Bogomolny and Zinn-Justin [22, 27–30] in bosonic quantum mechanics. In this case, Bogomolny [22] realized that the divergence arises from the double-counting of uncorrelated instanton-anti-instanton events at large separations and upon a careful treatment of the partition function, this divergence subtracts off as $I(g^2) = \int_0^\infty d\tau \left[\exp\left[-\left(\frac{4\xi}{g^2}e^{-\xi\tau}\right)\right] - 1\right]$. Using this integral, let us evaluate the the size of the charged bion in the case $N_f = 0$. Using integration by parts, $I(g^2) = \frac{4\xi^2}{g^2}\int_0^\infty d\tau \tau \exp\left[-\left(\frac{4\xi}{g^2}e^{-\xi\tau} + \xi\tau\right)\right]$. Following (6.28), the characteristic size of the charged bion in the bosonic theory is given by

$$r_{\rm b} = r_{\rm k}\log\left(\frac{8\pi}{g^2 N}\right), \qquad N_f = 0 \tag{6.29}$$

The result of the quasi-zero mode integral can be found either by working out this integral exactly as done in bosonic quantum mechanics [22], or via an equivalent prescription: take the $N_f = \epsilon \to 0$ limit in (6.26) and subtract the pole term:

$$I(g^2) = \left(\frac{g^2 N}{8\pi}\right)^{2\epsilon}\Gamma(2\epsilon) = \frac{1}{2\epsilon} + \left(\log\left(\frac{g^2 N}{8\pi}\right) - \gamma\right) + O(\epsilon) \longrightarrow \left(\log\left(\frac{g^2 N}{8\pi}\right) - \gamma\right) \tag{6.30}$$

The two result indeed agree exactly.

The mechanism of pairing in $\mathbb{CP}^{N-1}$ with $N_f \geq 1$ is the same as in QCD(adj) with $N_f \geq 1$ [12], as well as quantum mechanics with fermions [32]; and for $N_f = 0$, the deformed bosonic $\mathbb{CP}^{N-1}$, the pairing takes place in the same way as in deformed YM [107], as well as bosonic quantum mechanics [22]. This mechanism is a universal feature of semi-classical analysis.

6.6 2-defects: neutral bions and first non-perturbative ambiguity

For each diagonal entry of the Cartan matrix, there exists a *neutral* bion $\mathcal{B}_{ii} = [\mathcal{K}_i\overline{\mathcal{K}}_i]$. In our Lie algebra convention, $\widehat{A}_{ii} = 1$, and the quasi-zero mode integral takes the form

$$\widetilde{I}(g^2) = \int_0^\infty d\tau \exp\left[-\left(-\frac{8\xi}{g^2}e^{-\xi\tau} + 2N_f\xi\tau\right)\right] \tag{6.31}$$

However, as in gauge theory, both the bosonic interaction as well as the fermion zero mode induced interaction between constituents are actually attractive, and naively, the integral is dominated at small separations with respect to $r_{\rm b}$. Consequently, a semi-classical $[\mathcal{K}_i\overline{\mathcal{K}}_i]$ configuration seems meaningless. A related issue is that all topological quantum numbers that we may associate with $\mathcal{B}_{ii}$ are actually zero. It is indistinguishable from the perturbative vacuum, and potentially may mix with the perturbative contribution to a given observable. Understanding this precise connection leads to a quantitative and rigorous theory of semi-classics.

In order to make sense out of the $\mathcal{B}_{ii}$ molecule, we apply the generalized BZJ-prescription: deform the contour of integration so that the kink anti-kink has a repulsive component, or equivalently, rotate $g^2 \to g^2 e^{i\theta}$, and take, for example, $\theta = \pi$. Then, again,

the interaction has a repulsive component. Perform the integration and then, continue back to the original g^2. The result of the BZJ-prescription is,

$$\widetilde{I}(g^2, N_f) \rightarrow I(-g^2, N_f) = \left(-\frac{g^2 N}{8\pi}\right)^{2N_f} \Gamma(2N_f) \tag{6.32}$$

For positive integer number of flavors $N_f \geq 1$, this result is unambiguous. However, for $N_f = 0$, subtracting the pole due to the uncorrelated kink-anti-kink events, we obtain

$$\widetilde{I}(g^2, N_f = 0) = \left(\log\left(-\frac{g^2 N}{8\pi}\right) - \gamma\right) = I(g^2) \pm i\pi \tag{6.33}$$

The same ambiguity is obtained by Bogomolny in bosonic quantum mechanics [22]. Thus, we learn that the kink-anti-kink amplitude in the bosonic theory is two-fold ambiguous. The left and right bion amplitude is therefore

$$
\begin{aligned}
[\mathcal{K}_i \overline{\mathcal{K}}_i]_{\theta=0^\pm} &= \mathrm{Re}\,[\mathcal{K}_i \overline{\mathcal{K}}_i] + i\,\mathrm{Im}\,[\mathcal{K}_i \overline{\mathcal{K}}_i]_{\theta=0^\pm} \\
&= \left(\log\left(\frac{g^2 N}{8\pi}\right) - \gamma\right) 2\mathcal{A}_i^2 e^{-2S_0} \pm i\pi 2\mathcal{A}_i^2 e^{-2S_0}, \qquad 2S_0 \equiv 2 \times \frac{S_I}{N} = \frac{8\pi}{\lambda}
\end{aligned}
\tag{6.34}
$$

This is the first of many non-perturbative ambiguities that we will see in the semi-classical analysis. On its own, such ambiguities are disastrous, as they would render the semi-classical expansion meaningless. However, this ambiguity, and the many other ambiguities that we will find in semi-classical configurations are actually the resolution of a long-standing puzzle, the IR-renormalons in perturbation theory around the perturbative vacuum.

Consider a typical observable in the bosonic $\mathbb{CP}^{N-1}$ theory. This observable will receive contributions to all orders in perturbation theory, and also non-perturbative contributions. Let us denote the lateral (left and right) Borel summation for perturbation theory by $\mathbb{B}_{0,\theta=0^\pm}$. Let us write $g^2 = |g^2| e^{i\theta}$, where θ is the phase of the complexified coupling. In order for QFT to make sense, these two types of ambiguities must cancel:

$$\mathrm{Im}\,\mathbb{B}_{0,\theta=0^\pm} + \mathrm{Im}\,[\mathcal{B}_{ii}]_{\theta=0^\pm} = 0, \quad \text{up to } e^{-4S_0} \tag{6.35}$$

In words, this equation means: The sum of the left (right) Borel resummation and left (right) neutral bion amplitude is unambiguous at order $e^{-2S_0} = e^{-2S_I/N}$. The limit $\theta \rightarrow 0^\pm$ is accompanied by a Stokes jump for the Borel resummation, which is mirrored with a jump in the neutral bion amplitude, such that the sum of the two gives a unique result, with a smooth limit up to ambiguities at order e^{-4S_0}. We will indeed confirm this important confluence relation by explicit computation in two different ways. As a physical effect, the neutral bion amplitude leads to a repulsion between the eigenvalues of the σ-connection holonomy, as manifest from the supersymmetric example (5.8). This is the same as the role that the neutral bion plays in non-abelian gauge theory [14, 20].

6.7 4-defects: bion-anti-bion molecules and more ambiguities

According to the general classification stated in section 5.5 of ref. [20], in theories with massless fermions, both charged and neutral bion events are unambiguous. In theories
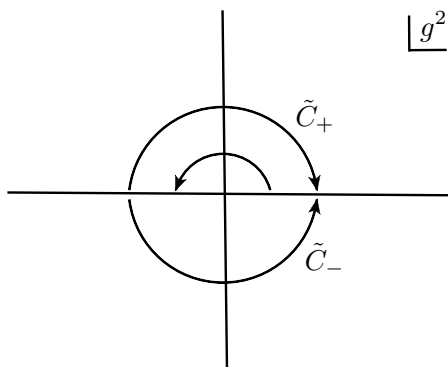
Figure 11. Defining left (right) bion-anti-bion amplitude $[\mathcal{B}_{ij}\overline{\mathcal{B}}_{ij}]_{\theta=0^\pm}$, we proceed as in the construction of left (right) Borel resummation $\mathbb{B}_{0,\theta=0^\pm}$.

with $N_f \geq 1$, the first ambiguity in semi-classical expansion arise at 4th order, as opposed to 2nd order as it was the case in bosonic theory. Below, we show that the quasi-zero mode integral does not yield an imaginary part for $[\mathcal{B}\mathcal{B}]$, but does yield an imaginary part for $[\mathcal{B}\overline{\mathcal{B}}]$. The quasi-zero mode integrals are of the form

$$I(g^2) = \int_0^\infty d\tau \, \exp\left(-V(\tau)\right) \qquad \text{for } [\mathcal{B}\mathcal{B}], \text{ and} \tag{6.36}$$

$$\widetilde{I}(g^2) = \int_0^\infty d\tau \, \exp\left(+V(\tau)\right) \qquad \text{for } [\mathcal{B}\overline{\mathcal{B}}], \tag{6.37}$$

where

$$V(\tau) = (\mu_\mathcal{B}, \mu_\mathcal{B}) \frac{8\xi}{g^2} e^{-\xi\tau} \tag{6.38}$$

and $\mu_\mathcal{B} = \alpha_i - \alpha_j \in \Gamma_r^\vee$ is the charge of the bion $\mathcal{B}_{ij}$.

This type of integral, as noted earlier, is addressed in bosonic quantum mechanics by Bogomolny [22]. Both integrals are divergent at large separation, and the latter is dominated by $\tau \to 0$ where molecular configurations are meaningless. The first of these problems is due to double-counting of the uncorrelated $[\mathcal{B}]$-$[\mathcal{B}]$ or $[\mathcal{B}]$-$[\overline{\mathcal{B}}]$ events, and is subtracted off.

The short-distance domination of $\widetilde{I}(g^2)$ can be taken care of by modifying the integration contour, or by rotating $g^2 \to -g^2$, where the bion-anti-bion interaction becomes repulsive, and continuing the integral back to positive $|g^2| + i0^\pm$. The result, as was the case with (6.34), is two-fold ambiguous:

$$[\mathcal{B}_{ij}\overline{\mathcal{B}}_{ij}]_{\theta=0^\pm} = \text{Re}\,[\mathcal{B}_{ij}\overline{\mathcal{B}}_{ij}] + i\,\text{Im}\,[\mathcal{B}_{ij}\overline{\mathcal{B}}_{ij}]_{\theta=0^\pm} \sim e^{-4S_0} \pm i\,\pi\,e^{-4S_0} \tag{6.39}$$

Consider a typical observable in $\mathbb{CP}^{N-1}$theory with $N_f \geq 1$ fermions. We expect that this observable will receive contributions to all orders in perturbation theory, as well as non-perturbative contributions. Denote the lateral Borel summation for perturbation theory by $\mathbb{B}_{0,\theta=0^\pm}$. Then write $g^2 = |g^2|e^{i\theta}$, where θ is the phase of the complexified coupling. For QFT to make sense, these two ambiguities must cancel:

$$\text{Im}\,\mathbb{B}_{0,\theta=0^\pm} + \text{Im}\,[\mathcal{B}\overline{\mathcal{B}}]_{\theta=0^\pm} = 0 \,, \quad \text{up to } e^{-6S_0} \tag{6.40}$$

This confluence relation is the counter-part of the leading ambiguity cancellation (6.35) in the $N_f = 0$ theory to $N_f \geq 1$. In the next sections, we explicitly derive (6.35).

7 Resurgence in $\mathbb{CP}^{N-1}$ QFT

7.1 Borel-Écalle summability at leading order

We derived the actions describing the low energy dynamics in eqs. (4.13) and (4.21) and described the embedding of the $\mathbb{CP}^1$ kink into $\mathbb{CP}^{N-1}$ theory in section 4.3. The Euclidean action describing this embedding is given in (4.28). Passing to a Minkowskian formulation, we write down the Hamiltonian associated with the action (4.28). It is given by

$$H_{\alpha_k}^{\text{zero}} = \frac{g^2}{2} P_\theta^2 + \frac{\xi^2}{2g^2} \sin^2 \theta + \frac{g^2}{2 \sin^2 \theta} P_\phi^2, \qquad \xi = \frac{2\pi}{N}, \qquad (\text{set } L = 1) \qquad (7.1)$$

We are interested in the ground state properties of this Hamiltonian. The field ϕ is a cyclic coordinate and P_ϕ is the associated angular momentum with eigenstates $m_\phi = 0, \pm 1, \pm 2, \ldots$. The ground state in the ϕ-sector is $m_\phi = 0$. As we will justify *a posteriori*, the energy gap of low lying modes is non-perturbative in g: it is $e^{-\frac{4\pi}{g^2 N}}$. Therefore, within the Born-Oppenheimer approximation, we drop the high ϕ-sector modes. Then, the relevant low energy Hamiltonian reduces to

$$H_{\alpha_k}^{\text{zero}} = -\frac{1}{2} \frac{d^2}{d\theta^2} + \frac{\xi^2}{4g^2} [1 - \cos(2g\theta)] \qquad (7.2)$$

and the Schrödinger equation takes the form

$$\psi'' + \left(p + \frac{\xi^2}{2g^2} \cos(2g\theta) \right) \psi = 0, \qquad p = 2E - \frac{\xi^2}{2g^2} \qquad (7.3)$$

The asymptotic perturbative expansion for the ground state energy in units of the natural frequency ξ is evaluated in ref. [24] by using the methods developed by Bender and Wu [38, 39]

$$\mathcal{E}(g^2) \equiv E_0 \xi^{-1} = \sum_{q=0}^{\infty} a_q (g^2)^q, \qquad a_q \sim -\frac{2}{\pi} \left(\frac{1}{4\xi} \right)^q q! \left(1 - \frac{5}{2q} + O(q^{-2}) \right) \qquad (7.4)$$

The series is Gevrey-1, non-alternating, and hence non-Borel summable. This is a manifestation of the fact that we are expanding the ground state energy along a Stokes ray in the complex-g^2 plane. The Borel transform (for the leading $q!$ divergence) is given by

$$B\mathcal{E}(t) = -\frac{2}{\pi} \sum_{q=0}^{\infty} \left(\frac{t}{4\xi} \right)^q = -\frac{2}{\pi} \frac{1}{1 - \frac{t}{4\xi}} \qquad (7.5)$$

and has a pole singularity on the positive real axis $\mathbb{R}^+$. The transform of the subleading term $(q-1)!$ generates a $\log(1 - \frac{t}{4\xi})$ branch point at the same position. Hence, the series
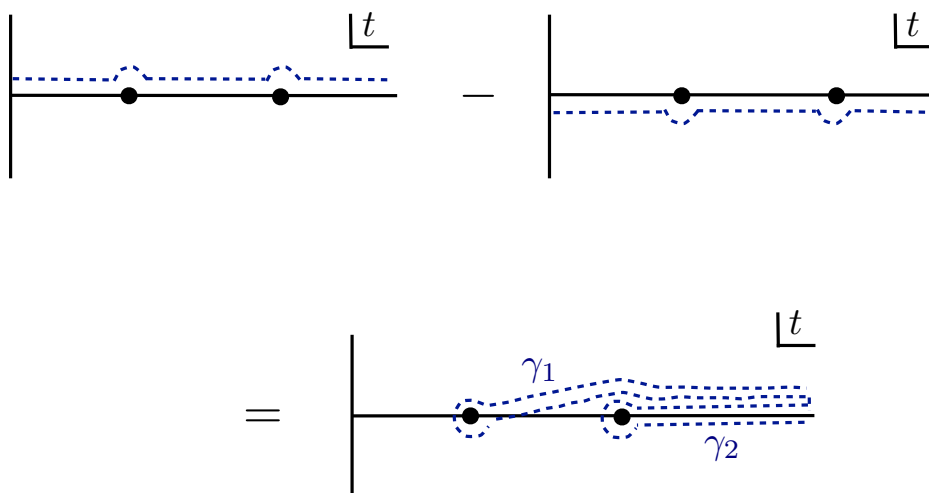
Figure 12. The difference between the left and right Borel sum defines the Stokes automorphism, as a sum over Hankel contours. This is the discontinuity in perturbation theory. It is cancelled by the discontinuity of the neutral bion and bion-anti-bion amplitudes.

is non-Borel summable. However, the series is *right and left Borel resummable*. These lateral Borel sums are

$$
\mathcal{S}_{0\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\, B\mathcal{E}(t)\, e^{-t/g^2} = \mathrm{Re}\mathcal{S}\mathcal{E}(g^2) \mp i\frac{8\xi}{g^2}e^{-\frac{4\xi}{g^2}}
$$
$$
= \mathrm{Re}\mathbb{B}_0 \mp i\frac{16\pi}{g^2 N}e^{-\frac{8\pi}{g^2 N}} \tag{7.6}
$$

The real part of the lateral Borel sum $\mathrm{Re}\mathbb{B}_0$ is unique and is given by Cauchy's principal part. The lateral resummed energy has a two-fold ambiguous imaginary part, depending on the choice of path. It is important to note that the imaginary part is *not* associated with an *instability*. The spectrum of the Hamiltonian is bounded from below, and is real. The interpretation of this result is that since the lateral resummed perturbation theory does not have a smooth limit in the $\theta \to 0^\pm$ limit, it cannot be used to define the full theory. In other words, all orders perturbation theory is not equal to full quantum field theory, or even to full quantum mechanics.

The Stokes automorphism (6.7) connecting different sectorial sums is defined as the difference between the two lateral Borel sums, and is schematically shown in figure 12. The difference

$$
\mathcal{S}_{0^+}\mathcal{E}(g^2) - \mathcal{S}_{0^-}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_+} B\mathcal{E}(t)\, e^{-t/g^2} - \frac{1}{g^2}\int_{C_-} B\mathcal{E}(t)\, e^{-t/g^2} = \sum_i \int_{\gamma_i} B\mathcal{E}(t)\, e^{-t/g^2} \tag{7.7}
$$

where $\{\gamma_1, \gamma_2, \ldots\}$ are the Hankel contours associated with singularities. In our case, the leading factorial growth of the perturbation theory yields

$$
\mathcal{S}_{0^+}\mathcal{E}(g^2) - \mathcal{S}_{0^-}\mathcal{E}(g^2) = -i\frac{32\pi}{g^2 N}e^{-\frac{8\pi}{g^2 N}} \tag{7.8}
$$

Recall from (6.34) that the non-perturbative neutral bion amplitude is also two-fold ambiguous:

$$\begin{aligned}[\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm} &= \operatorname{Re}[\mathcal{K}_i\overline{\mathcal{K}}_i] + i\operatorname{Im}[\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm} \\ &= \left(\log\left(\frac{\lambda}{8\pi}\right) - \gamma\right)\frac{16}{\lambda}e^{-8\pi/\lambda} \pm i\frac{16\pi}{\lambda}e^{-\frac{8\pi}{\lambda}}\end{aligned} \tag{7.9}$$

Both of these ambiguities, as already emphasized, are exponentially more important than the 2d instanton-anti-instanton ambiguity. Remarkably, they cancel each other *exactly*:

$$\operatorname{Im}\left[\mathcal{S}_\pm\mathcal{E}(g^2) + [\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm}\right] = 0 \qquad \text{up to } e^{-4S_0} = e^{-4S_I/\beta_0} \tag{7.10}$$

leading to a cancellation of ambiguities up to e^{-4S_I/β_0}. This is an explicit realization of how confluence equations work in the small-$\mathbb{S}^1$ regime of a non-trivial QFT, the $\mathbb{CP}^{N-1}$ model.

7.2 In the reverse direction: dispersion relations

Consider a typical observable in quantum theory, such as the energy eigenspectrum. For concreteness, let $\mathcal{E}_n(\lambda)$ be the energy of the state with quantum number n (n may be a collective quantum number; we do not make such a distinction yet) in units of natural frequency ξ as defined in (7.4). Also, assume that we do not have knowledge of the asymptotic expansion for the energy of the state n. And in fact, we will derive this asymptotic expansion by using the confluence equation (6.35), together with dispersion relations. Let

$$\mathcal{E}_n(\lambda) = a_{n,0} + a_{n,1}\lambda + a_{n,2}\lambda^2 + \ldots = \sum_{q=0}^{\infty} a_{n,q}\lambda^q \tag{7.11}$$

We recall the dispersion relation from Cauchy's theorem, and explore its consequences in connecting perturbative to non-perturbative physics [31, 38, 39, 83]. We assume that $\mathcal{E}_n(\lambda)$ is an analytic function in the cut-plane, where the function has a branch-cut along the positive λ axis. Cauchy's theorem and a contour deformation relate the discontinuity, the imaginary part and large-orders in perturbation theory. As a result of the confluence equation (6.35), we already have knowledge of the discontinuity in $\mathcal{E}_n(\lambda)$, and how it is connected to the discontinuity in the neutral bion amplitude. Using Cauchy's theorem provides a prediction for the large-order behavior of perturbation theory, which we test against the Bender-Wu method applied to the (reduced) quantum mechanics of $\mathbb{CP}^{N-1}$ theory.

For an analytic function $f(\lambda)$ in a cut-plane, Cauchy's theorem and a contour deformation implies

$$f(\lambda) = \frac{1}{2\pi i}\int_0^\infty d\lambda' \frac{\operatorname{Disc} f(\lambda')}{\lambda' - \lambda} - \frac{1}{2\pi i}\oint_{C_\infty} \frac{f(\lambda')}{\lambda' - \lambda} \tag{7.12}$$

where C_∞ is a loop at infinity, and λ is a point off the positive real axis. If $f(\lambda)$ decays sufficiently fast as $|\lambda| \to \infty$, the last term drops out. However, this is not the case for the energy spectrum, $\mathcal{E}_n(\lambda)$. Formally, as $\lambda \to \infty$ in the low-energy quantum mechanics, the potential term serves as a small perturbation. The system becomes a free

particle on a circle with circumference $\ell \sim \frac{\pi}{\sqrt{\lambda}}$. Thus, the energy levels of the theory are $E_n(\lambda) \sim \frac{1}{\ell^2} \sim \lambda$, including the ground state energy. We can build an auxiliary function $f_{\text{aux}}(\lambda)$ for which the boundary term drops out of the equation. Construct

$$f_{\text{aux}}(\lambda) = \frac{1}{\lambda^2}\left(E_n(\lambda) - a_0 - a_1\lambda\right) = \sum_{q=0}^{\infty} a_{n,q+2}\lambda^q \qquad (7.13)$$

We divided by λ^2 so that as $|\lambda| \to \infty$, $f_{\text{aux}}(\lambda) \sim 1/\lambda$ and the C_∞ term vanishes. We subtracted two terms so that we do not generate an undesired poles for $f_{\text{aux}}(\lambda)$ at the origin, see [31]. As a result, we have a more useful form of the dispersion relation:

$$f_{\text{aux}}(\lambda) = \frac{1}{2\pi i}\int_0^\infty d\lambda' \frac{\text{Disc}f_{\text{aux}}(\lambda')}{\lambda' - \lambda} \qquad (7.14)$$

Plugging (7.13) into (7.14), we find, for the ground state energy,

$$\mathcal{E}_0(\lambda) = a_{0,0} + a_{0,1}\lambda + \frac{1}{2\pi i}\sum_{q=0}^{\infty}\lambda^{q+2}\int_0^\infty d\lambda' \frac{\text{Disc}\mathcal{E}_0(\lambda')}{(\lambda')^{q+3}}\,, \qquad (7.15)$$

out of which we extract,

$$a_{0,q} = \frac{1}{2\pi i}\int_0^\infty d\lambda \frac{\text{Disc}\mathcal{E}_0(\lambda')}{\lambda^{q+1}} = \frac{1}{\pi}\int_0^\infty d\lambda \frac{\text{Im}\mathcal{E}_0(\lambda')}{\lambda^{q+1}} \qquad \text{for} \qquad q \geq 2, \qquad (7.16)$$

Now, let us investigate the implications of the confluence equation (6.35)

$$\text{Im}\,\mathcal{E}_0(\lambda') \equiv \text{Im}\,\mathbb{B}_{0,\theta=0^\pm} = -\text{Im}\,[\mathcal{B}_{ii}]_{\theta=0^\pm}\,, \qquad \text{up to } e^{-4S_0} \qquad (7.17)$$

The imaginary part of the neutral bion amplitude is given in (6.34). The pre-factor $\mathcal{A}_i = \sqrt{\frac{8}{\lambda}}$. (This is given in ref. [31], eq. (43.67), $g_{\text{there}} = \frac{\lambda}{8\pi}$ here.) Thus, after taking the pre-factors into account,

$$\text{Im}[\mathcal{B}_{ii}]_{\theta=0^\pm} = \text{Im}\left(\mathcal{K}_i\overline{\mathcal{K}}_i \times 2\,I(-g^2)\right) = \pm\frac{16\pi}{\lambda}e^{-8\pi/\lambda} \qquad (7.18)$$

Using (7.16), we obtain, at leading order, the large-order behavior of the perturbative series $P_0(\lambda)$ as

$$P_0(\lambda) \sim \sum_q a_{0,q}\lambda^q = -\sum_q \frac{2}{\pi}\left(\frac{\lambda}{8\pi}\right)^q q! = -\frac{2}{\pi}\sum_q \left(\frac{1}{4\xi}\right)^q q!\,g^{2q}, \qquad (7.19)$$

This is the expected non-alternating Gevrey-1 series. A few remarks are in order:

- the late terms obtained in reduced quantum mechanics of $\mathbb{CP}^{N-1}$, by using the cancellation of the imaginary parts of the left (right) $[\mathcal{K}_i\overline{\mathcal{K}}_i]$ amplitude and left (right) Borel-sum of perturbation theory agrees with the earlier work of Stone and Reeve [24], who applied the Bender-Wu analysis to a periodic potential.[15]

---

[15]In a beautiful paper [24], Stone and Reeve noted the ambiguity associated with the non-Borel-summability of perturbation theory for the QM periodic potential, and stressed that it is not associated with an instability. However, at that time the counterpart of the neutral bion in our confluence equation (6.35) was not yet understood.

- For $\mathbb{CP}^{N-1}$, our expansion is *not* in g^2, rather in $\frac{g^2 N}{8\pi} \equiv \frac{\lambda}{8\pi} = \frac{N}{2S_I}$ where the instanton action is $S_I = \frac{4\pi}{g^2}$. Consequently, this leads to the mechanism for the cancellation of the semi-classically realized IR-renormalon ambiguity on $\mathbb{R} \times \mathbb{S}_L^1$, which are closer to origin by a factor of N with respect to the 2d instanton-anti-instanton $[\mathcal{I}\bar{\mathcal{I}}]$ singularities.

- For $\mathbb{CP}^{N-1}$ on $\mathbb{R}^2$, the renormalon ambiguity is located at $\frac{2nS_I}{N}, n \geq 2$. On $\mathbb{R}^2$, there are no semi-classical topological configuration that these ambiguities may cancel against. The 2d instanton-anti-instanton events are located far from the origin, at $2nS_I, n \geq 1$.

- On a locally two-dimensional manifold on small $\mathbb{R} \times \mathbb{S}_L^1$, for the $N_f = 0$ theory, we find neutral bion events to cancel the ambiguities at $\frac{2nS_I}{N}, n \geq 1$. For the $N_f \geq 1$ theories, we find neutral bion events to cancel the ambiguities at $\frac{2nS_I}{N}, n \geq 2$, in exact agreement with the renormalon singularities on $\mathbb{R}^2$. According to our analysis, there is one more singularity closer to the origin in the bosonic theory.

- The all-important step is *continuity*, in the sense we made precise (absence of rapid crossovers at finite-N, or sharp phase transitions at $N = \infty$), the ability to connect the strong coupling non-trivial holonomy to the weak-coupling *non-trivial* holonomy for the σ-connection (2.34).[16]

7.3 Graded resurgence and resurgent sectors in the path integral formalism

In order to understand the general form of an observable in general QFT or QM with a topological Θ angle, we introduce the concept of "graded resurgence". The main idea of graded resurgence follows from the following simple observations:

1) perturbation theory is independent of the topological Θ-angle. Therefore, the ambiguity due to non-Borel summability of perturbation theory is also independent of Θ.

2a) The amplitude of topological configurations which carry non-vanishing topological charge *do depend* on the Θ-angle. Examples in $\mathbb{CP}^{N-1}$ are kink-instantons, 2d instantons.

2b) The amplitude of (molecular/correlated) topological configurations which carry zero topological charge *do not depend* on the Θ-angle. Examples in $\mathbb{CP}^{N-1}$ are neutral bions, bion-anti-bions.

3) Therefore, the non-Borel summability of the large orders in perturbation theory can *never* be cancelled by configurations which carry non-vanishing topological charge. Rather it can *only* be cancelled by topological configurations with zero topological charge, or equivalently, without any Θ-angle dependence.

---

[16] All earlier compactifications of this class of theories land on the weak coupling trivial holonomy in the small-$\mathbb{S}^1$ regime [62, 64–69]. This is interesting for other reasons, but this regime is unrelated to the semi-classical treatment of the confined regime due to rapid-crossover or phase transition. Our formulation of the problem provides a weak coupling regime which appears to be as close as one can get to the strong coupling regime.

This structure leads to a sectorial mechanism of cancellation, which we call "graded resurgence". To apply these ideas to $\mathbb{CP}^{N-1}$, define a "cell" $[n, m]$ as follows:

$$n = n_{\text{kink}} + n_{\text{anti-kink}}, \qquad m = n_{\text{kink}} - n_{\text{anti-kink}} \tag{7.20}$$

Here $n\frac{S_I}{N} = n\frac{4\pi}{g^2 N}$ is the action and $\frac{m}{N}$ denotes the topological charge. The $[n, m]$ sector is composed of $n_{\text{kink}} + n_{\text{anti-kink}}$ *correlated* kink-instanton events. For example, a single kink event belongs to $\mathcal{K}_j \in [1, 1]$. The proliferation of single-kink events in the Euclidean vacuum is the leading Θ dependent contribution to any observable. Neutral and charged bions belong to $\mathcal{B} \in [2, 0]$, and their proliferation generates various physical effects, such as the non-perturbative mass gap for the $N_f \geq 1$ theories.

The general form of the contribution of the events in the $[n, m]$ cell to an observable is given by

$$[n, m] \implies \mathcal{A}_{[n,m]} e^{-n\frac{4\pi}{\lambda} + im\frac{\Theta + 2\pi k}{N}} P_{[n,m]}(\lambda) \tag{7.21}$$

where $\mathcal{A}_{[n,m]}$ is the pre-factor of the associated $[n, m]$-defect amplitude, and $P_{[n,m]}(\lambda)$ denotes the formal perturbative fluctuation series around the $[n, m]$-defect. The appearance of $k = 0, \ldots, N - 1$ along with Θ is tied with the multi-branched structure of physical observables in bosonic $\mathbb{CP}^{N-1}$, either on $\mathbb{R}^2$ or $\mathbb{R} \times \mathbb{S}_L^1$.

Definition 5: graded resurgence triangle. The sectors in the $\mathbb{CP}^{N-1}$model form a structure that we refer to as the *graded resurgence triangle* (7.22), where the rows are at fixed-n (fixed-action). As one moves downward in the triangle, the action of the whole row increases by one-unit (in kink-instanton action), namely $n = 0, 1, 2, \ldots$.

$$[0, 0]$$

$$[1, 1] \qquad [1, -1]$$

$$[2, 2] \qquad [2, 0] \qquad [2, -2]$$

$$[3, 3] \qquad [3, 1] \qquad [3, -1] \qquad [3, -3]$$

$$[4, 4] \qquad [4, 2] \qquad [4, 0] \qquad [4, -2] \qquad [4, -4]$$

$$\iddots \qquad\qquad \vdots \qquad\qquad \ddots \tag{7.22}$$

The row labelled with n has $n + 1$ "cells", these are $m = n, n-2, \ldots, -n+2, -n$. Columns are fixed-$m$ (fixed topological charge) sectors. The graded structure inherent to QFTs and QM with Θ angle is shown in (7.22). For *general* QFTs, all cells in the graded resurgence triangle are ambiguous for one of the two reasons:

- *ambiguities in the Borel resummation of perturbation theory, $\mathcal{S}_\pm P_{[n,m]} \equiv \mathbb{B}_{[n,m]\pm}$ either around the perturbative vacuum or in the background of an $[n, m]$-defect.*

• *Ambiguities in the definition of the non-perturbative amplitudes*associated with neutral topological molecules, or molecules which include neutral sub-components, such as $[(\mathcal{K})^{n_{\mathrm{k}}}(\overline{\mathcal{K}})^{n_{\mathrm{ak}}}]_{\pm}$.

As discussed earlier, there is no unique meaning to perturbation theory in the $[0,0]$ cell: the corresponding series is typically non-Borel summable, but left (right) Borel summable $\mathbb{B}_{[0,0],\theta=0^{\pm}}$, and is two-fold ambiguous. This is to say, usual perturbation theory in QFT and Rayleigh-Schrödinger perturbation theory in quantum mechanics cannot be used to define the theory. Analogously, the $[1,1]$ sector, which we can write as $[\mathcal{K}_i]\mathbb{B}_{[1,0],\theta=0^{\pm}}$ is also equally ambiguous, due to the large order behavior of perturbation theory around the kink-instanton background, whereas $\mathcal{K}_i$ itself is unambiguous.

At the next level and thereafter, there is a new type of ambiguity. Consider the $[2,0]$ cell, which has elements like $[\mathcal{B}_{ij}]\mathbb{B}_{[2,0],\theta=0^{\pm}}$, as well as $[\mathcal{B}_{ii}]_{\theta=0^{\pm}}\mathbb{B}_{[2,0],\theta=0^{\pm}}$, where $\pm$ is used to indicate the presence of an ambiguity and left-right definitions of the corresponding objects. In the $[4,0]$ cell, the events which are ambiguous are $[\mathcal{B}_{ij}\mathcal{B}_{ji}]_{\theta=0^{\pm}}$.

In general QFT, our claim is that all these ambiguities are interconnected, and once we calculate a physical observable, say the vacuum energy density, mass gap or whatever we want, the ambiguities cancel to yield a unique unambiguous answer. In the previous section, we have explicitly demonstrated this mechanism, recall (6.35). This type of cancellation is at the very heart of the Borel-Écalle resummation; if it indeed continues to all non-perturbative orders then it could potentially provide a fully consistent non-perturbative continuum definition of QFT.

7.4 Confluence equations

In the graded resurgence triangle, there are certain selection rules, which dictate the possible communications and cancellations between the non-perturbative ambiguities in different cells.

Permitted (and necessary) communications. The elements of a fixed-m sector (i.e., a column) in the triangle (7.22) *can* and typically do talk with each other. These are sectors whose action differ by two units of (minimal) kink-instanton action:

$$[n,q] \Longleftrightarrow [n+2n',q] \tag{7.23}$$

Forbidden communications. Two different columns can and do contribute to a given observable. However, the ambiguities in a cell of a given column can never be cured by any cell in a different column. In this sense, the communications between

$$[n,q] \Longleftrightarrow [n',q'], \qquad q \neq q', \qquad n,n' \text{ arbitrary} \tag{7.24}$$

are forbidden. This is a simple consequence of the fact that perturbation theory does not depend on the Θ-angle.[17]

---

[17]There is actually a far more refined structure in the resurgence triangle compatible with resurgence. Each cell has a sub-structure dictated by the Lie algebra data of SU(N). Recall that kinks are associated with the co-roots $\alpha_i \in \Gamma_\vee^r$, whereas the the elements of the $[2,0]$ cell, charged and neutral bions are in

In theories such as $\mathbb{CP}^{N-1}$, typically, none of the cells exist as a self-consistent object on its own. Each cell is in need of other cells that it communicates to cure its diseases (ambiguities). In special QFTs, cells can exist self-consistently, this is synonymous with Borel summability. Later, we give evidence that extended supersymmetric theories are of this type.

For example, the ambiguity in ordinary perturbation theory, i.e., in the $[0,0]$ sector can only be cured by the ambiguity in the various neutral bion events in $[2,0]$, $[4,0]$, $[6,0]$, ... cells. We call this relation *perturbative-non-perturbative confluence equations* or *confluence equations* for short. For the $m=0$ column, the confluence equation is

$$0 = \text{Im}\Big(\mathbb{B}_{[0,0],\theta=0^\pm} + \mathbb{B}_{[2,0],\theta=0^\pm}[\mathcal{B}_{ii}]_{\theta=0^\pm} + \mathbb{B}_{[4,0],\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_{\theta=0^\pm} + \mathbb{B}_{[6,0]\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{jk}\mathcal{B}_{ki}]_{\theta=0^\pm} + \dots\Big).$$
(7.25)

Since the ambiguities of $\mathbb{B}_{[0,0]}$ form a sum of terms of the form $\{\pm ie^{-2S_0}, \pm ie^{-4S_0}, \dots, \}$ and since the imaginary (ambiguous) parts of the neutral topological molecules are of the form $\text{Im}[\mathcal{B}_{ii}]_{\theta=0^\pm} \sim ie^{-2S_0}$, $\text{Im}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_{\theta=0^\pm} \sim \pm ie^{-4S_0}$, $\text{Im}[\mathcal{B}^n]_{\theta=0^\pm} \sim \pm ie^{-2nS_0}$, (7.25) implies a hierarchy of cancellation at each order of ambiguities. These are given by

$$0 = \text{Im}\mathbb{B}_{[0,0]\pm} + \text{Re}\mathbb{B}_{[2,0]}\text{Im}[\mathcal{B}_{ii}]_\pm , \qquad (\text{up to } e^{-4S_0})$$
$$0 = \text{Im}\mathbb{B}_{[0,0]\pm} + \text{Re}\mathbb{B}_{[2,0]}\text{Im}[\mathcal{B}_{ii}]_\pm + \text{Im}\mathbb{B}_{[2,0]\pm}\text{Re}[\mathcal{B}_{ii}] + \text{Re}\mathbb{B}_{[4,0]}\text{Im}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_\pm \qquad (\text{up to } e^{-6S_0})$$
$$0 = \dots \tag{7.26}$$

where in the first relation, only the ambiguities at order e^{-2S_0} cancel, and in the second relation, the ambiguities at order e^{-2S_0} and e^{-4S_0} cancel, and so forth. Provided that these confluence equations hold, then a Θ independent contribution to a general observable will be given by

$$O(g^2) = \text{Re}\,\mathbb{B}_{[0,0]} + \text{Re}\mathbb{B}_{[2,0]}\text{Re}[\mathcal{B}_{ii}] + \text{Im}\mathbb{B}_{[2,0]\pm}\text{Im}[\mathcal{B}_{ii}]_\pm + \text{Re}\mathbb{B}_{[4,0]}\text{Re}[\mathcal{B}_{ij}\mathcal{B}_{ji}] , \quad \text{up to } O(e^{-6S_I})$$
(7.27)

which is unambiguous and unique up to order $O(e^{-6S_I})$. In this relation, $\text{Re}\,\mathbb{B}_{[2n,0]}$ is $O(1)$, the second term is $O(e^{-2S_0})$, and the third and fourth term are $O(e^{-2S_0})$.

On the other hand, the existence of the kink-instanton sector $[1,1]$, presents its own set of cancellations, leading to the confluence equations:

$$0 = \text{Im}\Big(\mathbb{B}_{[1,1],\theta=0^\pm}[\mathcal{K}_i] + \mathbb{B}_{[3,1],\theta=0^\pm}[\mathcal{K}_i\mathcal{B}_{jj}]_{\theta=0^\pm} + \mathbb{B}_{[5,1],\theta=0^\pm}[\mathcal{K}_i\mathcal{B}_{jk}\mathcal{B}_{kj}]_{\theta=0^\pm} + \dots\Big). \tag{7.28}$$

This implies, in hierarchical form

$$0 = \text{Im}\mathbb{B}_{[1,1]\pm}[\mathcal{K}_i] + \text{Re}\mathbb{B}_{[3,1]}\text{Im}[\mathcal{K}_i\mathcal{B}_{jj}]_\pm , \qquad (\text{to } e^{-5S_0})$$
$$0 = \text{Im}\mathbb{B}_{[1,1]\pm}[\mathcal{K}_i] + \text{Re}\mathbb{B}_{[3,1]}\text{Im}[\mathcal{K}_i\mathcal{B}_{jj}]_\pm + \text{Im}\mathbb{B}_{[3,1]\pm}\text{Re}[\mathcal{K}_i\mathcal{B}_{jj}] + \text{Re}\mathbb{B}_{[5,1]}\text{Im}[\mathcal{K}_i\mathcal{B}_{jk}\mathcal{B}_{kj}]_\pm \quad (\text{to } e^{-7S_0})$$
$$0 = \dots \tag{7.29}$$

One may be tempted to divide (7.28) and (7.29) by the kink amplitude $[\mathcal{K}_i]$, and write an expression virtually identical in form to (7.26). This is not quite true, because the

one-to-one correspondence with the Cartan matrix entries. For cells associated with higher action, more elaborate structure arises. We will refer to this as ramification of graded resurgence triangle. This structure will be explored in more detail in future work.

pre-factor of the $[\mathcal{K}_i \mathcal{B}_{jj}]$ amplitude is not obtained through a simple product, but rather a convolution, an integral over the quasi-zero mode. Nevertheless, it is still true that the large-order asymptotics of $P_{[0,0]}(\lambda)$ and $P_{[1,1]}(\lambda)$ have *universal* late terms that can be extracted from the dispersion relations through the formula:

$$
\begin{aligned}
\mathrm{Disc}\,\mathbb{B}_{[0,0]} &= -2\pi i \lambda^{-r_2} P_{[2,0]} e^{-2A/\lambda} + \mathcal{O}(e^{-4A/\lambda}), \\
\mathrm{Disc}\,\mathbb{B}_{[1,1]} &= -2\pi i \lambda^{-r_3+r_1} P_{[3,1]} e^{-2A/\lambda} + \mathcal{O}(e^{-4A/\lambda}).
\end{aligned}
\tag{7.30}
$$

Using (7.30) in the dispersion relation, we obtain

$$
\begin{aligned}
a_{[0,0],q} &= \sum_{q'=0}^{\infty} a_{[2,0],q'} \frac{\Gamma(q+r_2-q')}{(2A)^{q+r_2-q'}} + O\left(\left(\frac{1}{4A}\right)^q\right) \\
&= \frac{\Gamma(q+r_2-q')}{(2A)^{q+r_2}} \left[a_{[2,0],0} + \frac{2A}{(q+r_2-1)} a_{[2,0],1} + \frac{(2A)^2}{(q+r_2-1)(q+r_2-2)} a_{[2,0],2} + \ldots \right] \\
&\quad + O\left(\left(\frac{1}{4A}\right)^q\right) \\
a_{[1,1],q} &= \sum_{q'=0}^{\infty} a_{[3,1],q'} \frac{\Gamma(q+r_3-r_1-q')}{(2A)^{q+r_3-r_1-q'}} + O\left(\left(\frac{1}{4A}\right)^q\right) \\
&= \frac{\Gamma(q+r_3-r_1-q')}{(2A)^{q+r_3-r_1}} \left[a_{[3,1],0} + \frac{2A}{(q+r_3-r_1-1)} a_{[3,1],1} \right. \\
&\quad \left. + \frac{(2A)^2}{(q+r_3-r_1-1)(q+r_3-r_1-2)} a_{[3,1],2} + \ldots \right] + O\left(\left(\frac{1}{4A}\right)^q\right)
\end{aligned}
\tag{7.31}
$$

These equations describe multiple manifestations of resurgence:

- the large order behavior of $a_{[0,0],q}$ for large-q is determined by the first few $a_{[2,0],q'}$, mainly by $a_{[2,0],0}$. The subsequent terms, associated with the one-loop perturbative fluctuations around $[\mathcal{B}_{ii}]$ are suppressed by power law corrections in q, which are extra factors of $1/q$. Analogously, the large-order behavior of perturbation theory around a kink-instanton $a_{[1,1],q}$ is determined by the low orders in perturbation theory around the $[\mathcal{K}\mathcal{K}\overline{\mathcal{K}}]$ sector, mainly by $a_{[3,1],0}$.

- The one-loop fluctuations around, respectively, $[\mathcal{B}\overline{\mathcal{B}}]$ and $[\mathcal{B}\overline{\mathcal{B}}\mathcal{K}]$ saddle points determine the sub-series exponentially suppressed by a factor of 2^{-q}.

- The expansion in both cases is dictated by the nearest singularity in the Borel plane, as a consequence of Darboux's theorem [1, 4–6]. The leading large-order behaviors for the $[0,0]$ and $[1,1]$ sectors are given by

$$
\begin{aligned}
P_{[0,0]}(\lambda) &\sim \frac{a_{[2,0],0}}{(2A)^{r_2}} \sum_{q=0}^{\infty} (q+r_2-1)! \left(\frac{\lambda}{2A}\right)^q, \\
P_{[1,1]}(\lambda) &\sim \frac{a_{[3,1],0}}{(2A)^{r_3-r_1}} \sum_{q=0}^{\infty} (q+r_3-r_1-1)! \left(\frac{\lambda}{2A}\right)^q.
\end{aligned}
\tag{7.32}
$$

Despite the different backgrounds, the asymptotics of the perturbative expansions around their respective sectors have a *universal* behavior, determined by the nearest singularity in the Borel plane.

The relations (7.31) have their counterparts in matrix models and topological string theory [51–55] and 4d gauge theory compactified on $\mathbb{R}^3 \times \mathbb{S}^1$ [20].

Ignoring order one numerical factors and other (not so major) factors momentarily, (7.32) assumes the form

$$P_{[0,0]}(\lambda) \sim P_{[1,1]}(\lambda) \sim \sum_{q=0}^{\infty} \frac{q!}{\left(S_{\mathcal{K}\overline{\mathcal{K}}}\right)^q} \sim \sum_{q=0}^{\infty} \frac{q!}{(2S_{\mathcal{K}})^q} \qquad (7.33)$$

making it clear that it is the neutral bion $\mathcal{K}\overline{\mathcal{K}}$ configuration that controls the large order growth of the series $P_{[n,m]}(\lambda)$. Comparing with the ordinary integrals as discussed in section 1.6, we observe that the counterpart of the singulant (1.26) of ordinary integrals is the kink-instanton-anti-kink-instanton configuration (and not the kink-instanton configuration itself) in the compactified $\mathbb{CP}^{N-1}$ model.

7.5 Extended supersymmetric $\mathbb{CP}^{N-1}$ and Borel summability

Consider the extended supersymmetric theory, for example, $\mathcal{N} = (4,4)$ $\mathbb{CP}^{N-1}$ model, compactified on $\mathbb{R} \times \mathbb{S}^1$. In this class of theories, despite the fact that kink-instnatons are present, a superpotential is not permitted because of the number of the fermonic zero modes or large amount of supersymmetry. This is similar to 4d gauge theories with $\mathcal{N} \geq 2$ supersymmetry compactified on $\mathbb{R}^3 \times \mathbb{S}^1$. Since a superpotential is not generated, the counterpart of the neutral bion and charged bions do not exist, and [2,0]-cell is an empty set. In the extended supersymmetric cases, in fact, most cells in the resurgence triangle are empty, and the resurgence triangle (7.22) simplifies to that shown in (7.34).

$$[0,0]$$

$$[1,1] \qquad [1,-1]$$

$$[2,2] \qquad \varnothing \qquad [2,-2]$$

$$[3,3] \qquad \varnothing \qquad \varnothing \qquad [3,-3]$$

$$[4,4] \qquad \varnothing \qquad \varnothing \qquad \varnothing \qquad [4,-4]$$

$$\iddots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \ddots \qquad\qquad (7.34)$$

Since there are no neutral bion configurations, the confluence equation (7.25) and (7.28) simplify into

$$0 = \text{Im}\left(\mathbb{B}_{[0,0],\theta=0^\pm}\right), \qquad 0 = \text{Im}\left(\mathbb{B}_{[1,1],\theta=0^\pm}\right) \qquad (7.35)$$

meaning that there is no imaginary ambiguity in the Borel sum of ordinary perturbation theory, as well as in perturbation theory around the instantons. In other words, the cells $[n, \pm n]$, $n = 0, 1, 2, \ldots$ must be Borel summable, or equivalently, there are no singularities in the Borel plane along $\mathbb{R}^+$ for extended supersymmetric theories. This is the major difference between the bosonic theory and extended supersymmetric theory.

It should be noted that the existence of instantons implies that perturbation theory is a divergent asymptotic series. However, whether such a series is Borel summable (alternating, Gevrey-one) or non-Borel summable (non-alternating, Gevrey-one) is a more refined question, which is tied with the existence of singularities on the Borel complex-t plane along the $\mathbb{R}^+$ ray. These singularities, in the semi-classical regime, would be associated with neutral topological events as opposed to single instanton events. Consequently, the absence of such neutral molecules in the semi-classical regime of a given theory is the same as Borel summability.

Our argument for the Borel summability of the extended supersymmetric theory is for the semi-classical regime. In these theories, it is believed that there are no phase transition as the holomorphic parameters are varied. Therefore, if this is true, then as the theory moves from the semi-classical regime to the regime of strong coupling, the Borel summability must still hold. This implies the Borel summability of the extended supersymmetric quantum theory on $\mathbb{R}^2$.

7.6 Θ-dependence of vacuum energy density and topological susceptibility

Once the cancellation of the ambiguous imaginary parts is assured, we obtain finite and physical results for observables, such as vacuum energy density, topological susceptibility, mass gap of the theory. The result obtained in this manner is an approximation to the physical result, that can be compared with lattice gauge theory. In $N_f = 0$ deformed-$\mathbb{CP}^{N-1}$ model, we find, for example, the Θ-angle dependence of vacuum energy density as

$$\mathcal{E}(\Theta) - \mathcal{E}(0) = \text{Min}_{k=1}^{N} \left[-\frac{N}{\sqrt{\lambda}} e^{-\frac{4\pi}{\lambda}} \, \text{Re}\mathbb{B}_{[1,1]} \cos\left(\frac{\Theta + 2\pi k}{N}\right) + \ldots \right] \tag{7.36}$$

where $\text{Re}\mathbb{B}_{[1,1]}$ is the unambiguous real part of the Borel sum associated with the perturbative fluctuations in the 1-kink sector. The factor of N is present because there are N types of kink events $\mathcal{K}_j$ contributing to the vacuum energy density at leading order. There are subleading $O(e^{-\frac{8\pi}{\lambda}})$ corrections, which we ignore in the weak-coupling regime at this order.

When $LN\Lambda \gtrsim 1$, the theory moves to the strongly coupled volume independence domain, where semi-classics is no longer reliable. By continuity, and by the evidence provided by the analysis of renormalons on $\mathbb{R}^2$ versus semi-classical renormalons (bions etc.) on $\mathbb{R} \times \mathbb{S}^1$, the semi-classical regime $LN\Lambda \lesssim 1$ regime is rather similar to the strongly coupled domain. Using the dimensional transmutation (2.9), we may therefore write the topological susceptibility as

$$\frac{\partial^2 \mathcal{E}}{\partial \Theta^2}\bigg|_{\Theta=0} = a_1 \frac{\Lambda^2}{N} \tag{7.37}$$

where a_1 is a numerical factor. This result is in qualitative agreement with the large-N result, for which $a_1 = 3/\pi$. If we set this as boundary value at $N = \infty$ for the topological
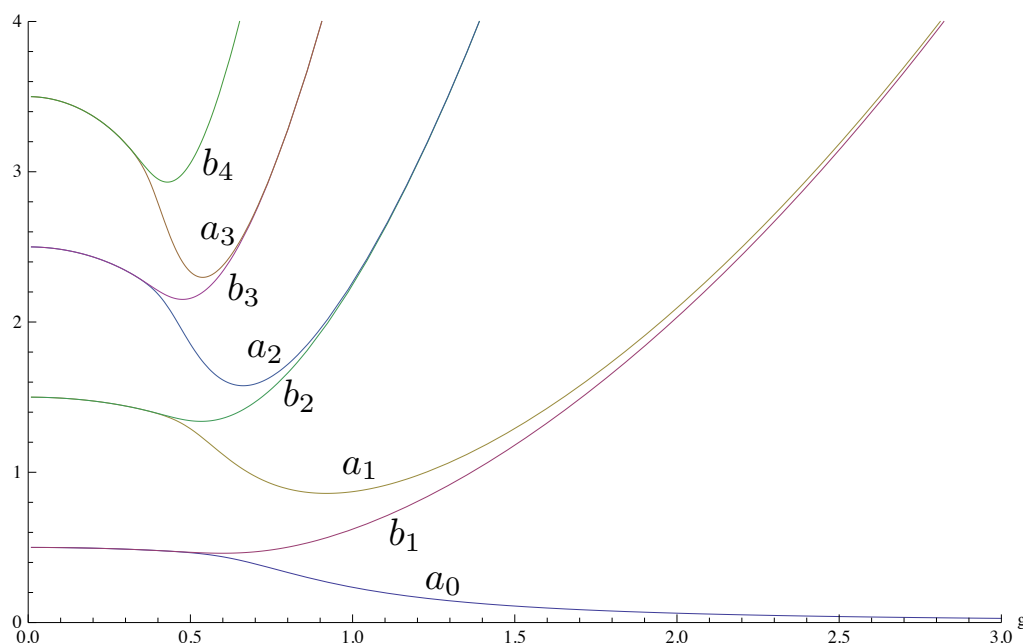
Figure 13. The energy eigenspectrum of the Mathieu equation as a function of g. For large-g^2, it describes a particle with small moment of inertia $I = 1/g^2$. The spectrum asymptotes to $E_n = \frac{g^2}{2}n^2$. The small-g^2 regime is related to the Hamiltonian for $\mathbb{CP}^1$ in the Born-Oppenheimer approximation. The curves represent a_0, b_1, a_1, b_2, a_2, ..., from (7.44), starting with the lowest curve. The mass gap $E(b_1) - E(a_0)$ is a purely non-perturbative kink-instanton effect.

susceptibility, our result provides a prediction for finite values of N, which seems to be in accord with numerical lattice simulations [108]. This result implies that in the semi-classical regime, it is the kink-instanton events that are responsible for the $O(1/N)$ topological susceptibility.

7.7 Mass gap on $\mathbb{R} \times \mathbb{S}^1$

The mass gap of the theory is the energy required to excite the system from the ground state to the first excited state. In the $g^2 \to 0$ limit, i.e., the weak coupling regime of the deformed-bosonic theory, the gap is purely non-perturbative.

The study of the spectrum in the Hamiltonian (7.2) in the Born-Oppenheimer approximation, in the case of $\mathbb{CP}^1$, reduces to the the study of the asymptotics in the Mathieu equation [84]. Let

$$\mathcal{O}(q) = -\frac{2H}{g^2} = \frac{d^2}{d\theta^2} - 2q\cos 2\theta, \qquad q = \frac{\xi^2}{4g^4} \tag{7.38}$$

denote a second order differential operator. Then, the eigenstates of the Hamiltonian[18] are the eigenfunctions of the Mathieu equation which obey

$$\mathcal{O}(q)\,\mathrm{ce}_n(\theta, q) = -a_n(q)\,\mathrm{ce}_n(\theta, q) \qquad\qquad n = 0, 1, 2, \ldots \tag{7.39}$$

---

[18]We shifted θ by $\pi/2$ to match to the standard form of the Mathieu equation, $w'' + (a - 2q\cos(2\theta))w = 0$.

$$\mathcal{O}(q)\,\mathrm{se}_n(\theta, q) = -b_n(q)\,\mathrm{se}_n(\theta, q) \qquad\qquad n = 1, 2, \ldots \qquad (7.40)$$

where

$$\mathrm{ce}_n(\theta, q) = \langle \theta | a_n(q) \rangle, \qquad \mathrm{se}_n(\theta, q) = \langle \theta | b_n(q) \rangle, \qquad (7.41)$$

are the real space wave functions.

Let P denote the parity operator acting as $\theta \to -\theta$ and T_π denote translation by π, $\theta \to \theta + \pi$. The eigenfunctions are also simultaneous eigenstates of P and T_π, transforming as

$$P\mathrm{ce}_n(\theta, q) = +\mathrm{ce}_n(\theta, q), \qquad\qquad T_\pi \mathrm{ce}_n(\theta, q) = (-1)^n \mathrm{ce}_n(\theta, q) \qquad (7.42)$$
$$P\mathrm{se}_n(\theta, q) = -\mathrm{se}_n(\theta, q), \qquad\qquad T_\pi \mathrm{se}_n(\theta, q) = (-1)^n \mathrm{se}_n(\theta, q) \qquad (7.43)$$

The energy eigenvalues of the Hamiltonian as a function of g^2 for low lying states are shown in figure 13. For any finite g^2, the eigenvalues obey

$$a_0 < b_1 < a_1 < b_2 < a_2 < \ldots \qquad (7.44)$$

The large-g^2 limit (which is not interesting for our purpose here)[19] corresponds to a particle with a small moment of inertia $I = 1/g^2 \to 0$. The Hamiltonian reduces to $H = \frac{P_\theta^2}{2I}$ and

$$H\, e^{\pm in\theta} = \frac{P_\theta^2}{2I}\, e^{\pm in\theta} = \frac{g^2}{2} n^2\, e^{in\theta} \qquad (7.45)$$

The relation between the angular momentum eigenstates $|\pm n\rangle$ and Mathieu functions in the infinite coupling limit is given by

$$|a_n(q = 0)\rangle \pm i|b_n(q = 0)\rangle = |\pm n\rangle, \qquad n = 1, 2, \ldots \qquad |a_0(q = 0)\rangle = |n = 0\rangle . \qquad (7.46)$$

Since the energy is quantized in units of g^2, the splitting between the rotational energy levels become arbitrarily large. Note that in this regime,

$$g^2 = \infty: \qquad a_0 = 0, \qquad a_n = b_n = n^2, \quad n = 1, 2, \ldots \qquad (7.47)$$

However, this $g^2 \to \infty$ limit is not relevant for the weak coupling regime of the $\mathbb{CP}^{N-1}$ models.

The small-S^1 regime of the $\mathbb{CP}^{N-1}$ model is related with the Hamiltonian (7.38) within the Born-Oppenheimer approximation, as discussed in section 7.1. In the $g^2 \to 0$ limit (or $q \to \infty$) limit, the pair of states

$$|a_n(q \to \infty)\rangle \leftrightarrow |b_{n+1}(q \to \infty)\rangle, \qquad n = 0, 1, 2, \ldots \qquad (7.48)$$

or in configuration space $\mathrm{ce}_n(\theta, q)$ and $\mathrm{se}_{n+1}(\theta, q)$ (not n, see the figure) become degenerate to all orders in perturbation theory. At $g^2 = 0$, their eigenenergy is

---

[19]This limit is sensible in purely quantum mechanical system defined by the Hamiltonian (7.38). In our case, g^2 is fixed by the asymptotic freedom of the UV theory, and the value of g^2 in the long distance Hamiltonian is $g^2(1/L)$ for the $\mathbb{CP}^1$ theory. We can take the formal large-g^2 limit, but this has nothing to do with the continuum $\mathbb{CP}^{N-1}$ model. It is, in a sense, the counter-part of the lattice strong coupling expansion.

$E(b_{n+1}) = E(a_n) \sim (n + \frac{1}{2})$, the one of simple harmonic oscillator. The splitting $E(b_{n+1}) - E(a_n)$ is purely non-perturbative and is given by (here $h = \frac{\xi}{2g^2} = \frac{2\pi}{N(2g^2)} \equiv \frac{\pi}{\lambda}$):

$$\Delta E_n = \frac{g^2}{2}(b_{n+1}(h^2) - a_n(h^2))$$

$$= \frac{g^2}{2}\left(\frac{2^{4n+5}}{n!}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} h^{n+\frac{3}{2}} e^{-4h}\left(1 - \frac{6n^2 + 14n + 7}{32h} + O\left(\frac{1}{h^2}\right)\right)\right). \quad (7.49)$$

The mass gap m_g of deformed-$\mathbb{CP}^1$ in the small-S_L^1 regime is given by

$$m_g = \Delta E_{n=0} = \frac{8\pi}{g}\left(1 - \frac{7g^2}{16\pi} + O(g^4)\right)e^{-\frac{2\pi}{g^2}} \sim e^{-S_I/2} \qquad \text{for } \mathbb{CP}^1 \qquad (7.50)$$

whereas for the $\mathbb{CP}^{N-1}$, by generalizing the deformed-$\mathbb{CP}^1$ discussion, it is given by

$$m_g = \Delta E_{n=0} = \frac{C}{\sqrt{\lambda}}\left(1 - \frac{7\lambda}{32\pi} + O(\lambda^2)\right)e^{-\frac{4\pi}{\lambda}} \sim e^{-S_I/N} \qquad \text{for } \mathbb{CP}^{N-1} \qquad (7.51)$$

Both (7.50) and (7.51) are remarkable non-perturbative consequences of the formalism, and they deserve multiple comments:

- the mass gap is zero to all orders in perturbation theory.
- The mass gap in the weak coupling regime is a purely non-perturbative factor proportional to $e^{-S_I/N}$. This is the first derivation of the all-important non-perturbative factor $e^{-S_I/N}$ from microscopic considerations. The mass gap at small-L may be considered as *the germ of the mass gap* for the theory on $\mathbb{R}^2$ [110].
- A physicist who is looking for the microscopic origin of the mass gap would be quite happy with this result, whereas a mathematician may feel disappointed. There is a series multiplying the kink-instanton, $P_{[1,1]}(\lambda)$ and it is a divergent asymptotic Gevrey-one series. So, is there a well-defined meaning to the mass gap that we obtained? Since $P_{[1,1]}(\lambda)$ is not even Borel summable, what does the germ of the mass gap that we obtained really mean?
- The rather deep and provocative answer to the problem in the previous question is that the resurgence theory cures all the ambiguities, canceling the ambiguities in the $[1,1]$ cell with the ambiguities in the $[3,1]$ sector, as we have explicitly shown for the vacuum energy. These are encoded into our confluence equations, (7.26) and (7.29).
- We claim the result is physical and meaningful

$$m_g = \Delta E_{m=0} = \frac{C}{\sqrt{\lambda}}\text{Re}\mathbb{B}_{[1,1]}e^{-\frac{4\pi}{\lambda}} \sim e^{-S_I/N} \qquad (7.52)$$

up to ambiguities of order $O(e^{-4S_I/N})$, where $\text{Re}\mathbb{B}_{[1,1]}$ is the ambiguity-free real part of the (left or right) Borel sum. Although we have not shown this fully (only a partial construction is given here), we anticipate that the mass gap, as well as other non-perturbative observables, in this theory are resurgent functions in the sense of Écalle, and resurgence theory takes care of all the ambiguities. We believe this statement can be proven along the lines of [43–46].

- The Born-Oppenheimer approximation in the weak coupling limit is justified because of the hierarchy of the energy scales,

$$\Delta E_{m=0} \ll E(a_1) - E(b_1) \sim \Delta E_\phi \qquad (7.53)$$

where ΔE_ϕ is the ϕ-sector discussed around (7.1).

- Another remarkable result is that the mass gap obtained in the semi-classical regime coincides with the result (1.3) obtained by large-N consideration, although the nature of the two semi-classical limits is completely different, weak in g^2 in terms of original degrees of freedom vs. weak in the coupling of the confined states where interactions are $1/N$. This item deserves more consideration, perhaps by incorporating the volume independence property (3.4).

The resurgent analysis of (7.38) from the Hamiltonian perspective, along with some newly developed mathematical methods, will be discussed in detail in [111].

8 Conclusion: towards a non-perturbative continuum definition of QFT

This work provides some steps towards the construction of a non-perturbative continuum definition of QFT, in particular, the two-dimensional $\mathbb{CP}^{N-1}$model. Our goal with such a construction, apart from providing a rigorous foundation to QFT, is to have a continuum formulation of practical value. We feel obliged to emphasize that this is not a formal problem. In our view, the lack of such a formulation is the root-cause underlying our rather insufficient understanding of these theories. In this work, we hope that we have made progress in this direction. There are two key elements, one from mathematics and one from physics:

- Écalle's theory of resurgent functions

- Continuity: the absence of phase transitions or rapid-crossovers upon spatial compactification of QFTs.

There are at present rigorous results using resurgent functions in quantum mechanics [43–46]. These authors, using Écalle's theory, prove that the semi-classical trans-series expansion for some of these theories is resummable to finite, *exact* and unambiguous results.

Continuity and spatial compactification provide the new physical inputs necessary to extend these QM rsults to non-trivial QFTs such as $\mathbb{CP}^{N-1}$or QCD.[20] These are used in a new way to reduce a non-trivial QFT, in its low energy limit, to quantum mechanical systems that can be studied through resurgence. Although we did this for the 2d $\mathbb{CP}^{N-1}$model, it can be done for even more interesting theories, primarily QCD(adj) and (deformed) Yang-Mills theory. A quantum mechanical version of these theories exists, in

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[20]There is an old idea of Bjorken: the femto-universe both in QCD and other theories. To move from the femto-universe to the large-scale-universe, there is either a phase transition or a rapid crossover. Our construction and idea of continuity relates the quantum mechanics in the small-circle (or torus limit) in an "as smooth as possible" manner passage to the large-volume QFT, and preserves a substantial amount of information about the QFT. This paper presents evidence for the existence of this type of passage, and another example was given in [20].

which rigorous results can be proven along the lines of [43–46]. It is apparent that there is ample opportunity here to improve significantly our understanding of QFT. In the next subsection we list some of the problems in which we feel progress can be made:

To summarize, the main results of this paper are:

- we introduced a new parametrization of the $\mathbb{CP}^{N-1}$ manifold (2.22) which makes the analysis of the QFT simpler. This parametrization immediately yields a new order parameter, that we referred to as the σ-connection holonomy (2.34), a matrix valued operator which is the counter-part of the Wilson line in non-abelian SU(N) gauge theory, and which carries more information than the regular U(1) Wilson line in the $\mathbb{CP}^{N-1}$ theory.

- The classical background of the σ-connection holonomy (2.34) is equivalent to the twisted boundary conditions for the $\mathbb{CP}^{N-1}$ fields. The quantum mechanical stability (instability) of the $\mathbb{Z}_N$-symmetric background follows from spatial (thermal) compactification. The former (but not the latter) admits a semi-classical weak coupling study of the confined phase for all $N_f \geq 1$, different from *all* earlier studies of the $\mathbb{CP}^{N-1}$ theories. In the $N_f = 0$ bosonic theory, we use a deformation to stabilize the $\mathbb{Z}_N$-symmetric background. This can also be achieved by integrating out heavy fermions.

- In the weak coupling regime ($LN\Lambda \lesssim 1$), we have shown that the leading finite action topological configurations are not 2d instantons, rather kinks with action S_I/N.

- In the same regime, we have shown that the non-perturbative ambiguities in perturbation theory are cancelled against the ambiguities in neutral bions, or bion-anti-bion configurations with action $2S_I/N, 4S_I/N, \ldots$. This is the content of our confluence equations (7.25) in QFT. This shows that the Bogomolny-Zinn-Justin mechanism of cancellation of ambiguities of perturbation theory against ambiguities in non-perturbative semiclassical configurations in quantum mechanics can be successfully applied to an asymptotically free QFT such as the $\mathbb{CP}^{N-1}$ model.

- The standard renormalon analysis on $\mathbb{R}^2$ suggests singularities in the Borel plane located at $t_n = \frac{2S_I}{\beta_0}n, n = 2, 3, \ldots$ [49]. There is no known (semi-classical) configuration that these ambiguities of perturbation theory may cancel against. The 2d instanton-anti-instanton cancels a much suppressed and unimportant ambiguity of the perturbation theory located at $t^{I\bar{I}} = n(2S_I), n = 1, 2, \ldots$ in the Borel plane. Our findings imply that the neutral bions etc. are the semi-classical realization of IR renormalons.

- The mass gap in the weak coupling regime of the bosonic theory is purely non-perturbative, proportional to $e^{-S_I/N} = e^{-4\pi/\lambda}$. This is the first derivation of the mass gap from microscopic considerations, and is sourced by the proliferation of kink-instanton events. This result, valid for arbitrary N, also matches with the large-N result. We conjecture that the mass gap is a resurgent function, at least in the small-$\mathbb{S}^1$ regime.

- The 2d-instantons, in the large-N limit, scale as $e^{-4\pi/g^2} \sim e^{-N}$. In the semi-classical domain of the theory, the kink-instantons scale as $e^{-4\pi/(g^2N)} \sim e^{-1}$ and are unsuppressed in the large-N limit. If indeed, our claim that the renormalons and our semi-classical neutral molecules being continuously connected is correct (we provided evidence that it is so), then the continuation of kinks (or charged bions) to the strong coupling domain are the resolution of the large-N vs. instanton puzzles. At least in the weak coupling domain, they are indeed the resolution of the puzzles noted in [63].

- For QFT (and QM) with degenerate vacua, and an associated topological charge and Θ-parameter, we have introduced the notion of a "graded resurgence triangle" to explain how such cancellations can be categorized according to the Θ-dependence, based on the simple observation that perturbation theory is insensitive to the Θ-parameter. Thus, non-Borel summability of the large orders of perturbation theory can never be cured by configurations with non-vanishing topological charge, but only by certain *neutral* bion molecules. In the full resurgent framework, such cancellations should proceed to all non-perturbative orders, suggesting that this approach could provide a fully consistent non-perturbative definition of QFT in the continuum.

In the small-$\mathbb{S}^1$ regime, at length scales larger than $\xi^{-1} = \frac{LN}{2\pi}$, the 2d instanton should be viewed as a composite, and kink-instantons are elementary. This, combined with large-N, matching of the mass gap between small-λ semi-classics and large-N ($1/N$ expansion), and the semi-classical description of IR renormalons suggests that the 2d-instanton on $\mathbb{R}^2$ should be viewed as an object with sub-structure. We claim that the "fractionalization scale" of a 2d instanton is

$$L^* = \min(LN, \Lambda^{-1}) \tag{8.1}$$

This seems to be the only reasonable possibility in order to merge the semi-classical domain $LN\Lambda/(2\pi) \lesssim$ with the volume independence domain $LN\Lambda/(2\pi) \gg 1$. When $LN < \Lambda^{-1}$, this is in fact true as we have shown already. If $LN \gg \Lambda^{-1}$, then, the theory is the same as the theory on $\mathbb{R}^2$ as per volume independence, for which the only relevant length scale is the strong scale Λ^{-1}. Therefore, the scale (8.1) seems to be the most reasonable guess for the fractionalization scale.

8.1 Prospects and open problems

1: Borel-Écalle summability at higher orders or proof of all-orders confluence equations. We showed the leading cancellation of ambiguities in the bosonic theory. The demonstration of the whole set of confluence equations would be a major step, plausibly equivalent to the demonstration of the existence of QFT in continuum.

2: which QFTs are Borel resummable? The general classification of which QFTs are Borel summable, and which are not, is a meaningful and important question that can partially be addressed using continuity, weak-coupling methods, and the graded resurgence triangle. For example, all extended supersymmetric theories seem to be Borel summable according to our criteria here and [20].

3: O(N) models. The techniques of this work easily generalize to the $O(N)$ model. It is often asserted, based on homotopy considerations, that there are no stable instantons for $N \geq 4$. However, our preliminary investigation shows that with judiciously chosen boundary conditions, there are kink configurations and bion configurations. The classification of these seems to be identical to $SO(N)$ gauge theory on $\mathbb{R}^3 \times S^1$ [20].

4: Grassmannian models. The techniques of this work are also suitable for Grassmannian models, with or without fermions. The $\mathbb{CP}^{N-1}$ model is a rank-one Grassmannian model and is intimately connected to $SU(N)$ gauge theory on $\mathbb{R}^4$. It would be interesting to find the theory associated with higher rank Grassmannians.

5: rigorous study of the 4d gauge theory-2d sigma model connection. The relation between 4d gauge theory and 2d $\mathbb{CP}^{N-1}$ can be made non-perturbatively rigorous by compactifying the former on asymmetric $T^2 \times S_L^1 \times \mathbb{R}$, and the latter on $S_L^1 \times \mathbb{R}$. It is transparent from our analysis and the one of the [20] that the Lie algebraic classification of the topological configurations is actually identical.

6: Three-dimensional models. The fractionalization of instantons has been studied in three-dimensional $\mathbb{CP}^{N-1}$ models [109] from a very different perspective, and it would be interesting to see if any of our ideas could be usefully applied therein.

7: Θ angle dependence. There are conjectures for the behavior of the $\mathbb{CP}^1$ model at $\Theta = \pi$. These conjectures may be tested in our framework. The Θ dependence of general $\mathbb{CP}^{N-1}$ theory can also be studied.

8: Singularities in the Borel plane vs. phase transitions. Depending on whether the background on small S^1 regime is center-asymmetric (degenerate eigenvalue for the holonomy matrix $^L\Omega$) vs. $\mathbb{Z}_N$ center-symmetric (maximally non-degenerate eigenvalue distribution) the Borel plane structure of singularities is drastically different. The singularity structure for the latter is almost identical to the theory on $\mathbb{R}^2$, perhaps only a small deformation thereof. The singularity structure for the former can be deduced from the combination of our work and Kontsevich's work on resurgence in quantum mechanics [58], see footnote 11. It is evident that the drastic changes in the singularity structure in the Borel plane is associated either with a sharp phase transition or a rapid crossover.

9: Index theorem. The index theorem (4.49) for the Fredholm-type Dirac operator on $\mathbb{R} \times \mathbb{S}_L^1$ is stated without derivation. It can be derived along the same lines as in the joint work with E. Poppitz [105] or [104]. For the $O(N)$ and Grassmannians sigma models there exists a corresponding index theorem.

10: Chiral symmetry and sectors in quantum mechanics. The low energy limit of multi-flavor $\mathbb{CP}^{N-1}$ is described by quantum mechanics systems with N_f types of fermions. This amounts to considering, in the quantum mechanical context, particles with spin (where the spin is determined by N_f) instead of spin-zero particles. There are crucial changes in the observables due to fermion zero modes. In particular, the number of sectors of associated

quantum mechanics will be related to the number of discrete chiral symmetry breaking vacua in the field theory on $\mathbb{R}^2$. This deserves a detailed study of its own.

11: Duality in quantum mechanics versus mirror symmetry. For the $\mathcal{N} = (2,2)$ theory, the dual theory obtained in quantum mechanics is equivalent to the well-known mirror symmetry dual in the stringy framework of the very same theory. This also deserves further deliberations.

12: Relation with the $\mathbb{Z}_N$ twisted mass deformed $\mathbb{CP}^{N-1}$model. A calculable deformation of the $\mathbb{CP}^{N-1}$model on $\mathbb{R}^2$ is the twisted mass deformed $\mathbb{CP}^{N-1}$model: see for example, [89–96]. We believe that this model can be studied at arbitrary size $\mathbb{S}_L^1$ by the methods of this work. It would also be useful to understand more precisely the relation between the topological configurations on $\mathbb{R}^2$ and in its compactified version.

Acknowledgments

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