INTERMEDIATE ASYMPTOTICS, SCALING LAWS AND RENORMALIZATION GROUP IN CONTINUUM MECHANICS*

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ABSTRACT. Scaling laws and self-similar solutions are very popular concepts in modern continuum mechanics. In the present paper these concepts are analyzed both from the viewpoint of intermediate asymptotics, known in classical mathematical physics and fluid mechanics, and from the viewpoint of the renormalization group technique, known in modern theoretical physics. The definition of the renormalization group is proposed, related to the intermediate asymptotics with incomplete similarity. The general presentation is illustrated by examples of essentially non-linear problems where all analytical properties of the solutions and their asymptotics are rigorously proved, as well by an example from turbulence, where the rigorous problem statement is missing.

SOMMARIO. Leggi di riscaldamento e soluzioni auto-simili sono concetti molto diffusi nella moderna meccanica del continuo. In questa memoria questi concetti vengono analizzati sia dal punto di vista degli asintoti intermedi, noti nella fisica matematica e fluido-meccanica classiche, sia dal punto di vista della tecnica del gruppo di rinormalizzazione, nota nella moderna fisica teorica. Si propone una definizione di gruppo di rinormalizzazione, collegata ad asintoti intermedi con similitudine incompleta. La presentazione generale viene illustrata da esempi di problemi essenzialmente non lineari, in cui tutte le proprietà analitiche delle soluzioni e dei loro asintoti sono dimostrati rigorosamente, nonché da un esempio di turbolenza, nel quale manca una definizione rigorosa del problema.

KEY WORDS. Scaling laws, Intermediate asymptotics, Self-similar solutions, Renormalization group, Continuum mechanics.

INTRODUCTION

Self-similar solutions of partial differential equations entered mathematical physics with the famous memoir of Fourier [1], devoted to the analytical theory of heat propagation. Obtaining such solutions was always considered to be an achievement, especially in the pre-computer era, because their construction was reduced to solving the boundary-value problems for ordinary, not partial, differential equations. The dimensions in which independent variables of partial differential equations enter the self-similar variables – like $x/t^{1/2}$ in heat conduction, where x is the space variable and t the time – were usually obtained by some simple method, e.g. by dimensional analysis, giving no particular problems to the researcher. Classical self-similarities were discussed and summarized in a remarkable review paper by Germain [2] where the general approach to problems leading to such solutions was also discussed.

The situation, however, changed drastically after the paper by Guderley [3] where the solution to the problem of very intense implosion (converging shock wave) was obtained, and the papers by von Weizsäcker [4] and Zeldovich [5] treating the plane analog of the implosion wave problem, the problem of an impulsive loading. In these problems a delicate analytical procedure was needed to obtain the dimensions n in which time entered the self-similar variable x/t^n. These dimensions appeared, generally speaking, to be certain transcendental numbers rather than simple fractions as in classical self-similarities. In fact solutions with such anomalous dimensions appeared even earlier in papers by Kolmogorov, Petrovsky and Piskunov [6], Fisher [7], and Zeldovich and Frank-Kamenetsky [8]. In these papers the wave-type solutions $f(\xi - \lambda\tau)$ of non-linear parabolic equations were considered, and the wave phase speed was to be calculated by a complicated analytical procedure. Transforming the variables $\xi = \ln x$, $\tau = \ln t$, we obtain the same problem as before for determining the dimensions of time in the self-similar variable x/t^λ. An important question arose: What is the real nature of the difference between two types of similarity solutions? To understand that, in papers [9] and [10] two special problems were considered having a parameter entering the problem formulation. For one value of this parameter a classical self-similar solution appeared. However, for all other values of the parameter anomalous dimensions appeared, which should be obtained from the solution to a nonlinear eigenvalue problem. These results allowed to understand the fundamental nature of the difference between the two types of self-similarities mentioned above. Indeed, the self-similar solutions are not only the exact solutions to

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some partial differential equations; they are also the so-called intermediate asymptotics of the non-self-similar solutions to certain more general problems, valid for times and distances from the boundaries sufficiently large to eliminate the influence of the fine details of the initial and/or boundary conditions, but small enough to keep the system distant from the ultimate equilibrium state. So, the reason for the difference is the character of these intermediate asymptotics. If these asymptotics are represented by a regular function, a self-similar solution of the first kind appears. If, however, the asymptotics are power-type (scaling), with dimensions depending on the fine analytical properties of the non-self-similar solutions, then self-similar solutions of the second kind appear. So it became clear how the transcendental dimensions appear in the solutions to the problems, which seem completely regular in their statement.

Independently, but essentially later, an activity started in theoretical physics – basically in quantum field theory and the theory of phase transitions in statistical physics – related to the so-called renormalization group. Anomalous dimensions also entered the language of physicists-theorists. The names and works of Gell-Mann and Low [11], Kadanoff [12], [13], Patashinsky and Pokrovsky [14], Wilson [15], as well as books by Bogoljubov and Shirkov [16] and Ma [17] should be mentioned here. It is essential to emphasize, however, that contrary to researchers in mechanics, physicists considered the problems where the rigorous formulation leading to correct mathematical solutions was missing.

It soon became clear that the concepts of intermediate asymptotics and of the renormalization group are closely related. This relationship was emphasized in [18], and the physicists were invited to see how the approach of intermediate asymptotics can work in the problems previously considered by the renormalization group method.

This was realized in a series of remarkable works of Goldenfeld, Oono, Martin and their students [19–22]. In particular, in these works the authors solved by the traditional method of a renormalization group several problems of continuum mechanics (filtration, elasticity, turbulence, etc.), which were previously solved by the method of intermediate asymptotics. Moreover, using the singular expansion method that was widely applied in theoretical physics, Goldenfeld and his colleagues were able to obtain some instructive and useful approximate solutions to these problems. On the other hand, by using the intermediate asymptotics method, they also obtained solutions to several problems of statistical physics, solved previously by the renormalization group approach.

These important works helped to represent in a final form the renormalization group approach from the viewpoint of intermediate asymptotics. In particular, it appeared useful to give the proper definition of the renormalization group using the concept of intermediate asymptotics. These results are presented in this paper.

1. SCALING LAWS AND SELF-SIMILAR SOLUTIONS

Scaling (power-type) laws and self-similar asymptotic solutions play an important and growing role in the continuum mechanics of this century. It is enough to mention here the well-known Taylor–von Neumann–Sedov solution to the problem of concentrated very intense explosion [23]–[25]. The relation for the radius of the shock wave r_f:

$$r_f = C\left(\frac{Et^2}{\rho_0}\right)^{1/5} \tag{1}$$

gives a typical example of a scaling law (E is the explosion energy, ρ_0 the density of ambient gas, t the time, and $C \approx 1$ is a constant). The distributions of the pressure p, density ρ and velocity u within the fire ball are represented in the form

$$p = \rho_0\left(\frac{r^2}{t^2}\right)P\left(\frac{r}{r_f}\right), \quad v = \left(\frac{r}{t}\right)V\left(\frac{r}{r_f}\right), \quad \rho = \rho_0 R\left(\frac{r}{r_f}\right) \tag{2}$$

(where r is the current radius and P, V, R are certain dimensionless functions), so that these distributions have an important property of self-similarity: at various times they can be obtained from one another by similarity transformation.

A remarkable example of a scaling law related to fracture mechanics is the conical crack of Benbow [26] formed when a punch of very small diameter is penetrated into a block of fused silica: the relation for the diameter D of the base of the conical crack has the form

$$D = \mathrm{Const}(\nu)\left(\frac{P}{K}\right)^{2/3}. \tag{3}$$

Here P is the pressing load, ν is the Poisson ratio, and K is a specific characteristic of the fracture toughness of fused silica, called the cohesion modulus.

We consider here in more detail a rather more recent example [27], [28]: axisymmetric spreading of the groundwater mound, initially concentrated near a thin well in porous stratum lying on a horizontal impermeable bed (Figure 1). The expression for the pressure at the bottom of the mound $p = \rho g h$ (where ρ is the water density, g the acceleration due to gravity, and h the local mound height) at the time t for the radius r has the form

$$p = p_0(t)\Phi\left(\frac{r}{r_f}\right), \quad p_0 = \tfrac{1}{2}\left(\frac{Q}{\kappa t}\right)^{1/2}, \quad r_f = \sqrt{8}\,(Q\kappa t)^{1/4} \tag{4}$$

where $Q = W/2\pi m$, r_f is the radius of the mound,

$$\kappa = \frac{k}{2m\mu}, \qquad \Phi = \begin{cases}(1-\zeta^2), & 0 \leqslant \zeta \leqslant 1\\ 0, & \zeta > 1\end{cases}$$

$\zeta = r/r_f$; m is the porosity of the stratum, k is its permeability, W is the total weight of the mound and μ is the

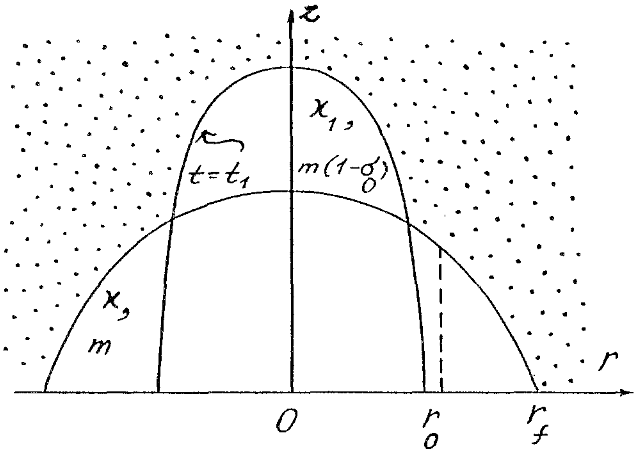

Fig. 1. Ground-water mound spreading. Leaving porous medium the ground-water partly remains in the corners and narrow channels, reducing the porosity.

dynamic viscosity of water. Here again we meet the scaling laws and self-similarity.

In obtaining the scaling laws and self-similar solutions the considerations of dimensional analysis at the beginning played an important role. The dimensional analysis is a simple sequence of rules based on the fundamental covariance principle: all physical regularities can be represented in a form that is equally valid for all observers. Let us consider a physical regularity

$$a = f(a_1, \dots, a_k, b_1, \dots, b_m) \tag{5}$$

where the arguments $a_1, \dots, a_k$ have independent dimensions. This means that by choosing appropriate fundamental units it is possible to vary arbitrarily and independently the values of $a_1, \dots, a_k$:

$$a_1' = A_1 a_1, \dots, a_k' = A_k a_k. \tag{6}$$

At the same time the dimensions of a, $b_1, \dots, b_m$ can be expressed as power monomials in dimensions of $a_1, \dots, a_k$:

$$\begin{aligned}&[b_1] = [a_1]^{p_1} \cdots [a_k]^{r_1}, \dots, [b_m] = [a_1]^{p_m} \cdots [a_k]^{r_m},\\ &[a] = [a_1]^{p} \cdots [a_k]^{r},\end{aligned} \tag{7}$$

($[\varphi]$ is the symbol of the dimension of quantity φ, introduced by J. C. Maxwell). So after transformation (6) of the parameters with independent dimensions, their values transform as

$$\begin{aligned}&b_1' = A_1^{p_1} \cdots A^{r_1} b_1, \dots, b_m' = A_1^{p_m} \cdots A_k^{r_m} b_m\\ &a' = A_1^{p} \cdots A_k^{r} a\end{aligned} \tag{8}$$

Transformations (6) and (8) form a continuous transformation group, and $A_1, \dots, A_k > 0$ are the parameters of this group. Due to the covariance principle the regularity (5) should be represented as the relation between the dimensionless invariants of this group

$$\Pi = \Phi(\Pi_1, \dots, \Pi_m) \tag{9}$$

where

$$\Pi_i = \frac{b_i}{a_1^{p_i} \cdots a_k^{r_i}}, \qquad \Pi = \frac{a}{a_1^{p} \cdots a_k^{r}}\ . \tag{10}$$

This means that the function f entering the physical regularity (5) possesses a fundamental property of 'generalized homogeneity':

$$f = a_1^{p} \cdots a_k^{r} \Phi \left(\frac{b_1}{a_1^{p_1} \cdots a_k^{r_1}} \right), \dots, \frac{b_m}{a_1^{p_m} \cdots a_k^{r_m}} \Bigg). \tag{11}$$

Reducing the number of arguments and form of the invariants sometimes allows to establish self-similarity of the solution coming from the problem statement and to obtain the form of self-similar variables.

Let us demonstrate that for the example of the ground-water mound spreading. The pressure p at the horizontal impermeable bottom of the mound satisfies the Boussinesq equation [29]:

$$\partial_t p = \kappa \left(\frac{1}{r} \right) \partial_r r\, \partial_r p^2 \tag{12}$$

whereas the initial conditions for the mound concentrated initially near the well of negligibly small radius can be written in the form:

$$p(r, 0) \equiv 0, \quad (r \neq 0); \qquad 2\pi m \int_0^\infty p(r, 0) r\, dr = W. \tag{13}$$

Therefore p depends on the quantities Q, κ, t and r having the dimensions

$$[Q] = F, \quad [\kappa] = L^4 T^{-1} F^{-1}, \quad [t] = T, \quad [r] = L. \tag{14}$$

Here F is the dimension of force, L the dimension of length, and T the dimension of time. The first three arguments (14) obviously have independent dimensions, whereas the quantities of the dimensions of length and pressure can be formed from them: $(Q\kappa t)^{1/4}$ and $(Q/\kappa t)^{1/2}$. Therefore, according to the 'Π-theorem' (9), the dimensionless quantity $\Pi = p/(Q/\kappa t)^{1/2}$ should depend on $\Pi_1 = r/r_f$ only, $\Pi = \Phi(\Pi_1)$, and $\Pi = r_f/(Q\kappa t)^{1/4} = \xi_0$ should be a constant. Substituting these expressions into partial differential equation (12) we obtain, for the function Φ, an ordinary differential equation. Moreover, from the expressions $\Pi = \Phi(\Pi_1)$ we obtain the relation

$$\int_0^{r_f} p(r, t) r\, dr = \text{Const } Q. \tag{15}$$

Due to the initial condition (13), Const = 1, so the function Φ satisfies the integral relation

$$\int_0^1 \Phi(\zeta)\zeta\, d\zeta = \tfrac{1}{4}. \tag{16}$$

Solving the ordinary differential equation obtained previcusly for Φ under condition (16), we obtain solution (4).

2. INTERMEDIATE ASYMPTOTICS

Consider now a problem, modified, seemingly only slightly. Assume that when the water goes out of a porous medium, some fixed part, σ_0, of it remains, retained by capillary forces. This means that when the water level is reduced, $\partial_t h < 0$, $\partial_t p < 0$ the effective porosity of the stratum becomes reduced, equal to $m(1-\sigma_0)$ instead of m, so that instead of κ in Equation (12) for p we get the coefficient

$$\kappa_1 = \frac{\kappa}{1-\sigma_0} > \kappa. \tag{17}$$

It is essential that the increase in κ is due to a reduction in the porosity, whereas the permeability remains the same: the water is retained in narrow corners and channels which do not contribute to permeability, only to water content.

Thus, instead of Equation (12), we obtain

$$\partial_t p = \begin{cases} \left(\frac{\kappa_1}{r}\right) \partial_r r\, \partial_r p^2, & \partial_t p \leqslant 0 \\ \left(\frac{\kappa}{r}\right) \partial_r r\, \partial_r p^2, & \partial_t p \geqslant 0. \end{cases} \tag{18}$$

If we integrate Equation (18) after multiplying it by r from $r=0$ to $r=r_f$ (where r_f is water-mound radius), and we bear in mind that if the initial mound height is monotonically decreasing with the radius, then inside a certain circle $r \leqslant r_0(t)$ the mound height and the bottom pressure decrease, whereas at $r_0(t) \leqslant r \leqslant r_f$ they increase and we obtain an integral relation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{r_f} rp(r, t)\, \mathrm{d}r = (\kappa_1 - \kappa)(r\, \partial_r p^2)_{r=r_0(t)}. \tag{19}$$

Obviously $(r\, \partial_r p^2)_{r=r_0(t)} < 0$, so that the total weight of the mound

$$2\pi m \int_0^{r_f} rp(r, t)\, \mathrm{d}r = W(t) \tag{20}$$

decreases with time. Physically it is obvious, because a certain part of the water is retained above the mound by capillary forces and does not participate in creating pressure at the mound bottom.

Let us try now to repeat the same dimensional consideration as before for solving the same initial problem (13) but for the modified equation (18). Seemingly, only one additional parameter, κ_1, appears in the problem, having the same dimensions as κ, so following the dimensional analysis the solution should essentially be of the same form as (4)

$$p = \tfrac{1}{2}\left(\frac{Q}{\kappa t}\right)^{1/2} \Phi\left(\frac{r}{r_f(t)}, \frac{\kappa_1}{\kappa}\right) \tag{21}$$

$$r_f = \xi_f\left(\frac{\kappa_1}{\kappa}\right)(Q\kappa t)^{1/4}, \quad r_0 = \xi_0\left(\frac{\kappa_1}{\kappa}\right)(Q\kappa t)^{1/4}. \tag{22}$$

However, this result for $\kappa_1 \neq \kappa$ is obviously wrong because it gives, for the mound weight, $W(t) = \text{Const}$ which, if $\kappa_1 \neq \kappa$, contradicts the non-integrable conservation law (19).

To resolve the paradox a numerical calculation was at first performed. However, in the numerical calculations we could not use the generalized function entering the initial conditions (13). Therefore, it was necessary to replace it by an ordinary function, different from zero in a finite region $0 \leqslant r \leqslant r_*$, where r_* is the initial radius of the mound. As such an initial distribution, solution (4) can be used at the moment when $r_f = r_*$:

$$\begin{aligned} p(r, 0) &= \left(\frac{4Q}{r_*^2}\right)\left(1 - \frac{r^2}{r_*^2}\right), \quad 0 < r < r_* \\ p(r, 0) &\equiv 0, \quad r > r_*. \end{aligned} \tag{23}$$

Numerical computation ([27], see also [28]) has given a seemingly unexpected result: the solution tends very quickly to a self-similar asymptotics

$$p = \left(\frac{A^2}{\kappa t^{1-2\beta}}\right) \Phi\left(\frac{r}{r_f}, \frac{\kappa_1}{\kappa}\right) \tag{24}$$

where

$$r_f = At^\beta, \qquad A = \text{Const}[Q\kappa r_*^{1/\beta - 4}]^\beta, \quad r_0 = \zeta_0 A t^\beta. \tag{25}$$

The basic parameter β depends on the ratio κ_1/κ (as well as the parameter ζ_0), but does not depend on the initial condition, in particular of Q. At $\kappa_1 = \kappa$, $\beta = \frac{1}{4}$, at $\kappa_1 > \kappa$, $\beta < \frac{1}{4}$. It is very important that this is not only a numerical result; this character of the asymptotics was rigorously proved by Hulshof and Vázquez [30].

Now it becomes clear what has happened. We are interested in fact not in the solution of a limiting initial problem described by a generalized function (13), but in the intermediate asymptotics to the initial problem (23). We call this an intermediate asymptotics because the time should be large enough to reach $r_f \gg r_*$, but small enough to have a mound weight $W(t)$ that is not too small in comparison with the initial mound weight. The solution to the initial problem (23) for the modified equation (18) depends on the parameters Q, κ, t, r, r_*, κ_1, and the dimensional analysis leads to the relation

$$p = \tfrac{1}{2}\left(\frac{Q}{\kappa t}\right)^{1/2} \Phi\left(\Pi_1, \Pi_2, \frac{\kappa_1}{\kappa}\right), \tag{26}$$

where

$$\Pi_1 = \xi = \frac{r}{(Q\kappa t)^{1/4}}, \quad \Pi_2 = \eta = \frac{r_*}{(Q\kappa t)^{1/4}}.$$

To obtain the desired asymptotics at small η we simply neglected the small parameter η, and, consequently the influence of the initial radius of the mound r_*. This is possible for $\kappa_1 = \kappa$, where the finite limit different from zero of the function $\Phi(\varepsilon, \eta, 1)$ at $\eta \to 0$ exists. However, for $\kappa_1 \neq \kappa$ such a finite limit different from zero does not exist, and simply neglecting $\Pi_2 = \eta$ in the representation of the solution (25) is incorrect. Nevertheless, for $\kappa_1 \neq \kappa$ the function $\Phi(\xi, \eta, \kappa_1/\kappa)$ in relation (26) possesses a power-type asymptotics at small η:

$$\Phi = \eta^{2\delta}\Phi_1\left(\frac{\xi}{\eta^\delta}, \frac{\kappa_1}{\kappa}\right), \quad \delta = 1 - 4\beta, \tag{27}$$

so that solution (26) assumes asymptotically at small η the form (24)–(25). As can be seen, this asymptotics 'remembers' not the initial weight of the mound, i.e. not Q, but a more complex quantity

$$Qr_*^{1/\beta-4} = \text{Const.} \tag{28}$$

However, the parameter η can be made small not only by making t large at constant r_*, but also by tending the initial radius r_* to zero at constant t. Thus, we can obtain a singular solution, corresponding to the concentrated initial amount of water, but this limiting solution at $r_* \to 0$ corresponds not to a fixed amount of ground-water concentrated initially at the axis, but to an amount of water which tends to infinity when $r_* \to 0$, so that relation (28) is valid. This solution can be obtained directly [27], [28] if we substitute the solution in the form (24)–(25) to the basic equation (18). An ordinary differential equation will be obtained, where β appears as a parameter. Generally speaking, the solution to this equation, which satisfies the necessary boundary conditions and has the required properties for an arbitrary β does not exist. However, there is an exceptional value of β for which the necessary solution does exist. So, to determine the solution directly we obtained a non-linear eigenvalue problem.

3. RENORMALIZATION GROUP

Let us now return to the general case. What can occur if, in relation (9), some dimensionless parameters $\Pi_1, \Pi_2, \ldots, \Pi_l$, corresponding to dimensional parameters $b_1, \ldots, b_l$, are small (or large)? This is always an important question in every mechanical study because in our mathematical models we seldom take into account certain factors which are considered to be non-essential.

If there exists a finite limit different from zero of the function $\Phi(\Pi_1, \ldots, \Pi_l, \Pi_{l+1}, \ldots, \Pi_m)$ at, for the sake of definiteness, $\Pi_1, \ldots, \Pi_l$ tending to zero, then for sufficiently small values of $\Pi_1, \ldots, \Pi_l$ the function Φ can be replaced with essential accuracy by a function of a lesser number of arguments

$$\Phi(0, \ldots, 0, \Pi_{l+1}, \ldots, \Pi_m) = \Phi_1(\Pi_{l+1}, \ldots, \Pi_m). \tag{29}$$

Such a reduction in the number of arguments gives further advantages in addition to those obtained by dimensional analysis. This can be interpreted from the group-theoretical viewpoint as an additional invariance with respect to a transformation group

$$\begin{aligned} &a_1' = a_1, \ldots, a_k' = a_k,\ b_1' = \lambda_1 b_1, \ldots, b_l' = \lambda_l b_l, \\ &b_{l+1}' = b_{l+1}, \ldots, b_m' = b_m,\ a' = a \end{aligned} \tag{30}$$

Here $\lambda_1, \ldots, \lambda_l$ are the parameters of the additional group (their values are restricted by the condition that the dimensionless parameters should remain small). Such was the case of a very intense explosion: in the Taylor–von Neumann–Sedov solution the size of the charge d and initial air pressure p_0 were neglected as well as corresponding dimensionless parameters

$$\Pi_d = \frac{d}{(Et^2/\rho_0)^{1/5}}, \quad \Pi_p = \frac{p_0}{\rho_0^{3/5}E^{2/5}t^{-6/5}}. \tag{31}$$

We met the same situation in the problem of ground-water mound spreading for $\kappa_1 = \kappa$, where the parameter r_* (the initial radius of the mound) as well as the corresponding dimensionless parameter $r_*/(Q\kappa t)^{1/4}$ were neglected. We refer to this case as 'the similarity of the first kind', or complete similarity. Generally speaking, however, this kind of finite limit different from zero of a function $\Phi(\Pi_1, \ldots, \Pi_l, \Pi_{l+1}, \ldots, \Pi_m)$ at $\Pi_1, \ldots, \Pi_l$ tending to zero does not exist. Therefore the dimensionless parameters $\Pi_1, \ldots, \Pi_l$, generally speaking, remain essential even although they are small, and so remain essential corresponding dimensional parameters $b_1, \ldots, b_l$. There exists, however, an important exception when such a finite limit different from zero does not exist, but the function Φ has for small (or large) $\Pi_1, \ldots, \Pi_l$ an asymptotics having the property of generalized homogeneity:

$$\Phi = \Pi_1^{\alpha_1} \cdots \Pi_l^{\alpha_l} \Phi_1 \left(\frac{\Pi_{l+1}}{(\Pi_1^{\beta_{l+1}} \cdots \Pi_l^{\delta_{l+1}})}, \ldots, \frac{\Pi_m}{(\Pi_1^{\beta_m} \cdots \Pi_l^{\delta_m})} \right) \tag{32}$$

exactly of the same form as the function f in relation (11). There is, however, one essential difference: generalized homogeneity of the function f in (11) followed from the general physical covariance principle and the constants $p, \ldots, r_m$ were obtained by dimensional analysis, whereas the generalized homogeneity of the function Φ in relation (32) is a special property of the problem under consideration, and the constants $\alpha_1, \ldots, \delta_m$ cannot be obtained from some general considerations.

Generalized homogeneity (32) is equivalent to the invariance with respect to a transformation group

$$\begin{aligned} &a_1' = a_1, \ldots, a_k' = a_k \\ &b_1' = \lambda_1 b_1, \ldots, b_l' = \lambda_l b_l \\ &b_{l+1}' = \lambda_1^{\beta_{l+1}} \cdots \lambda_l^{\delta_{l+1}} b_{l+1}, \ldots, b_m' = \lambda_1^{\beta_m} \cdots \lambda_l^{\delta_m} b_m \\ &a' = \lambda_1^{\alpha_1} \cdots \lambda_l^{\alpha_l} a \end{aligned} \tag{33}$$

($\lambda_1, \ldots, \lambda_l$ are parameters of the group). As is seen, here all the parameters $b_1, \ldots, b_l, b_{l+1}, \ldots, b_m, a$ are renormalized in a special way. This very group is the renormalization group, a concept often used in the physical literature. The numbers $\alpha_1, \ldots, \alpha_l, \beta_{l+1}, \ldots, \delta_m$ are, in the physical literature, known as *anomalous dimensions*. We refer to this case as *scaling*, or similarity of the second kind, or incomplete similarity.

Note some important special cases. First is the case when only one parameter, say Π_1 is small. The asymptotics (32) takes the form

$$\Phi = \Pi_1^{\alpha_1} \Phi_1 \left(\frac{\Pi_2}{\Pi_1^{\beta_2}}, \ldots, \frac{\Pi_m}{\Pi_1^{\beta_m}} \right). \tag{34}$$

This is the case which we met in the problem of ground-water mound spreading. The important more special case is when $\beta_2 = \cdots = \beta_m = 0$, $\alpha_1 \neq 0$. Such asymptotics appears for the initial value problem for the equation

$$\partial_t u = \begin{cases} \kappa_1 \partial_{xx}^2 u, & \partial_t u \leqslant 0 \\ \kappa \partial_{xx}^2 u, & \partial_t u \geqslant 0 \end{cases} \tag{35}$$

which describes filtration in elasto-plastic porous media [18], [31].

4. ε-EXPANSION

Consider a problem where the phenomenon depends on a parameter ε, as in our example with ground-water mound spreading, if we denote ε by

$$\varepsilon = \frac{\kappa_1}{\kappa} - 1, \tag{36}$$

and, moreover, in such way that for $\varepsilon = 0$ all degrees $\alpha_1, \ldots, \alpha_l, \beta_{l+1}, \ldots, \delta_m$ are equal to zero. Then, for small values of ε the expansion over the parameter ε can be used in the calculations of the parameters $\alpha_1, \ldots, \alpha_l$, and others. The ε-expansion is the current approach in statistical physics and quantum field theory. We will demonstrate this in the example considered above of the ground-water mound spreading.

The integral relation (19) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{r_f} rp(r,t)\mathrm{d}r = \kappa\varepsilon(r\,\partial_r p^2)_{r=r_0}. \tag{37}$$

Furthermore, using solution (4) for the case $\kappa_1 = \kappa$, we can form the expansions

$$p = \frac{A^2}{2\kappa t^{1-2\beta}} [\Phi_0(\zeta) + \varepsilon\Phi_1(\zeta) + \cdots], \quad \zeta = \frac{r}{r_f}$$
$$r_f = At^{\beta}, \quad \Phi_0(\zeta) = (1-\zeta^2), \quad r_0 = \left(\frac{1}{\sqrt{2}}\right)(Q\kappa t)^{1/4} + O(\varepsilon) \tag{38}$$

Substituting relations (38) into (37) and comparing the coefficients at various degrees of ε, we obtain

$$\tfrac{1}{4} - \beta = \frac{\varepsilon}{16} \tag{39}$$

in excellent agreement with the complete solution at small ε. This result was obtained by Chen and Goldenfeld [22] who used a different method: the exact application of the renormalization group procedure used traditionally by physicists.

5. SCALING LAWS FOR DEVELOPED TURBULENT FLOWS IN CYCLINDRICAL TUBES

From early 1930s it has been known (see, e.g., [32]) that the average velocity distribution $u(y)$ within an intermediate interval of distances y from the wall (outside the tiny viscous layer near the wall and in the close vicinity of the tube axis) with equal accuracy can be represented in two different forms:

(1) Power law, depending on Reynolds number (Re):

$$\varphi = C\eta^{\alpha}; \quad \varphi = \frac{u}{u_*}, \eta = \frac{u_* y}{\nu}, \tag{40}$$

(where $u_* = (\tau/\rho)^{1/2}$, in which τ is the wall shear stress, ρ is the fluid density, ν is the fluid kinematic viscosity; C and α are dimensionless constants known to be weakly dependent on the global flow Reynolds number $\mathrm{Re} = \bar{u}d/\nu$, with $\bar{u}$ the mean fluid velocity and d the tube diameter); and

(2) Universal logarithmic law, independent of Reynolds number (*Re*):

$$\varphi = \left(\frac{1}{\kappa}\right) \ln \eta + C_1, \tag{41}$$

where $\kappa = 0.41$–0.42 is called the von Kármán constant, and C_1 is another constant known to be equal to 5.1–5.5.

Until recently it was commonly believed that the logarithmic law (41) has some advantages: it can be derived rigorously from a seemingly plausible assumption that, at large *Re*, the average velocity gradient should be independent of molecular viscosity. On the other hand, the power law (40) was considered merely as a convenient representation of the empirical data, deprived of any theoretical basis.

More recently it was shown [33] that the power law (40) can be obtained from a different assumption – scaling, incomplete similarity – no less rigorously than the method by which the logarithmic law is usually derived from the simpler assumption of complete similarity, i.e. complete independence on the flow Reynolds number.

The mean velocity distribution is governed by the flow microstructure (vortex dissipative structure) of the turbulent flow. Recent experiments have shown this structure to be most irregular. Therefore, it seems natural to assume that the influence of molecular viscosity will not disappear, even at very large Reynolds numbers.

Therefore it seems to be not unnatural that a recent result [34] has shown an instructive coincidence with the classical experimental data of Nikuradze [35] of the scaling law (40) with the following parameters

$$\alpha = \frac{3}{2\ln \mathrm{Re}}, \quad C = \left(\frac{1}{\sqrt{3}}\right)\ln \mathrm{Re} + \tfrac{5}{2} \tag{42}$$

so that the scaling law (40) can be represented in the following quasi-universal form

$$\psi = \frac{1}{\alpha}\ln\left(\frac{2\alpha\varphi}{(\sqrt{3}+5\alpha)}\right) = \ln\eta, \quad \alpha = \frac{3}{2\ln \mathrm{Re}}. \tag{43}$$

All 16 series, containing 256 experimental points represented in the tables of paper [35] are presented in Figure 2 in the coordinates ψ, $\ln\eta$. The experimental

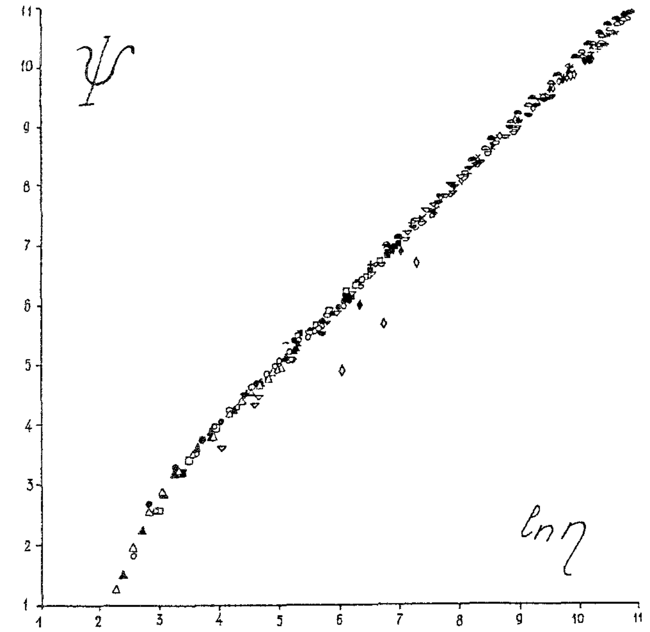

Fig. 2. Experimental points representing velocity data in reduced coordinates are situated near the bisectrix. This confirms the basic scaling law.

△ $Re = 4 \times 10^3$	▲ $Re = 6.1 \times 10^3$	○ $Re = 9.2 \times 10^3$
● $Re = 16.7 \times 10^3$	□ $Re = 23.3 \times 10^3$	■ $Re = 43.4 \times 10^3$
▽ $Re = 105 \times 10^3$	▼ $Re = 205 \times 10^3$	$Re = 396 \times 10^3$
$Re = 725 \times 10^3$	◇ $Re = 1110 \times 10^3$	◆ $Re = 1536 \times 10^3$
+ $Re = 1959 \times 10^3$	× $Re = 2350 \times 10^3$	$Re = 2790 \times 10^3$
$Re = 3240 \times 10^3$		

points lie mainly close to the bisectrix in accordance with the quasi-universal representation of the scaling law (43).

There is apparently a significant difference between the laws (40) and (41): from the scaling law (40) follows the fractality of the microstructure (vortex dissipative structure) of the developed turbulent shear flow.

We mention that relations (42) seem to be the first terms of the ε-expansion over a small parameter

$$\varepsilon = \frac{1}{\ln Re}. \tag{44}$$

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