41 Critical exponents and equation of state from series summation

Universal quantities near the phase transition of $O(N)$ symmetric vector models, can be determined, in the framework of the $(\phi^2)^2$ field theory, and the corresponding renormalization group (RG), in the form of perturbative series.

The $O(N)$ symmetric $(\phi^2)^2$ field theories do not describe the universal properties only of ferromagnetic systems. In Section 15.8 we have explained how the $N = 0$ limit is related to the statistical properties of polymers, in Section 15.9 how the Ising-like $N = 1$ model describes the physics of the liquid–vapour transition and, in Section 15.10, why the superfluid helium transition corresponds to $N = 2$.

Universal quantities have been calculated within two different schemes, the Wilson–Fisher $\varepsilon = 4 - d$ expansion [75], and perturbative expansion at fixed dimensions 2 and 3, as suggested by Parisi [146], using series reported in Refs. [418–420]. In both cases, in Sections 40.2 and 40.3, we have shown that the series are divergent (see Refs. [421] for mathematical details), and the expansion parameters are not small.

In fixed dimensions smaller than 4, the series are proven to be Borel summable [422]. For the ε expansion, there are reasons to believe that the renormalon contributions cancel, and that the property is equally true, but a proof is lacking. With this assumption, in both cases, although the series are divergent, they define unique functions.

Since the expansion parameters are not small, summation methods are then required to determine these functions. A specific summation method, based on a parametric Borel transformation and mapping has been successfully applied to the series, and has led to a precise evaluation of critical exponents and other universal quantities [423–444, 160].

In the first part of the chapter we define Borel summable series. We describe methods to sum Borel summable divergent series, as generated in quantum mechanics (QM) and in quantum field theory (QFT). In particular, we show how, in some methods based on the Borel transformation, the knowledge of the large-order behaviour of perturbation theory (see Sections 40.1.1, 40.2) can be efficiently incorporated.

41.1 Divergent series: Borel summability, Borel summation

Asymptotic series. We consider a function $f(z)$, analytic in a sector S,

$$S: \quad |\operatorname{Arg} z| \le \alpha/2\,, \quad |z| \le |z_0|. \tag{41.1}$$

We assume that, in S, f has the asymptotic (Taylor series) expansion

$$f(z) = \sum_0^\infty f_k z^k. \tag{41.2}$$

This means that the series (41.2) diverges for all $z \neq 0$, and that, in S, it satisfies a bound of the form

$$\left| f(z) - \sum_{k=0}^N f_k z^k \right| \le C_{N+1} |z|^{N+1}, \quad \forall\ N, \tag{41.3}$$

in which

$$C_N |z|^N \underset{\text{for } N\to+\infty}{\longrightarrow} \infty\,, \quad \forall\ z \neq 0\,.$$

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Although the series (41.2) diverges, it can nevertheless be used it to estimate the function $f(z)$ for $|z|$ small. At $|z|$ fixed, we can look for a minimum in the bound (41.3) when N varies. If $|z|$ is small enough, the bound first decreases with N and then, since the series is divergent, it eventually increases. A truncation of the series at the minimum yields the best possible estimate of $f(z)$, with a finite error $\varepsilon(z)$. Assuming, for definiteness, that the coefficients C_N have the form

$$C_N = M\,A^{-N}(N!)^{\beta}, \tag{41.4}$$

one can estimate $\varepsilon(z)$ explicitly. One finds

$$\varepsilon(z) = \min_{\{N\}} C_N|z|^N \sim \exp\left[-\beta(A/|z|)^{1/\beta}\right]. \tag{41.5}$$

We see that an asymptotic series does not in general define a unique function. Indeed, if we have found one function, we can add to it any function analytic in the sector (41.1) and which is smaller than $\varepsilon(z)$ in the whole sector. The new function still satisfies the condition (41.3).

41.1.1 Borel summability. Borel summation

In th case of the specific examples (41.4), a classical theorem of complex analysis states that no analytic function can satisfy a bound of the form (41.3) in a sector of angle α larger than $\pi\beta$. Therefore, for $\alpha > \pi\beta$, although the series is divergent, it defines a unique function.

In the marginal case in which the series is asymptotic only in the open interval $|\operatorname{Arg} z| \in (-\pi\beta/2, \pi\beta/2)$, additional conditions have to be imposed to prove uniqueness.

Borel transformation. We focus from now on to the case $\beta = 1$, which is typical for perturbative expansions, and $\alpha > \pi$, but the generalization to arbitrary β is simple. The unique function can be determined by introducing its Borel transform (Watson's theorem),

$$B_f(z) = \sum_{k=0}^{\infty} \frac{f_k}{k!} z^k. \tag{41.6}$$

Formally, in the sense of power series,

$$f(z) = \int_0^{\infty} \mathrm{d}s\ \mathrm{e}^{-s}\, B_f(sz). \tag{41.7}$$

The bound (41.3) and the estimate (41.4) lead to the bound on the coefficients B_k,

$$|B_k| < M\,A^{-k}. \tag{41.8}$$

Thus, $B_f(z)$ is analytic at least in a circle of radius A and uniquely defined by the series. Moreover, it can be proved that $B_f(z)$ is also analytic in a sector

$$|\operatorname{Arg} z| \in [0, \tfrac{1}{2}(\alpha - \pi)[\,, \tag{41.9}$$

and does not increase faster than an exponential in the sector, so that integral (41.7) converges for $|z|$ small enough and inside the sector

$$|\operatorname{Arg} z| < \alpha/2\,.$$

In addition, it can be shown that the integral (41.7) satisfies a bound of type (41.3). Hence, this integral representation yields the unique function that has the asymptotic expansion (41.2) in the sector S. The series (41.2) is then called *Borel summable.*

41.1.2 Large-order behaviour and Borel summability

For a large class of potentials in QM, and for a number of field theories, we know that instanton contributions for small values of the loop expansion parameter g behave like

$$Cg^{-b}\,\mathrm{e}^{-a/g}\,. \tag{41.10}$$

The corresponding contribution to the perturbative coefficients for large-order k of the loop expansion is then,

$$(C/\pi)k^{b-1}a^k k!\,. \tag{41.11}$$

Therefore, the coefficients B_k of the Borel transform $B(z)$ (equation (41.6)) behave as

$$B_k \sim (C/\pi)k^{b-1}a^k. \tag{41.12}$$

This asymptotic estimate implies that the singularity of $B(z)$ closest to the origin is located at the point $z = 1/a$, and that $B(z)$ has an algebraic singularity of the form

$$\frac{C}{\pi}\int_0 \frac{\mathrm{d}g\,\mathrm{e}^{-a/g}}{g^{b+1}}\sum_k \frac{1}{k!}\left(\frac{z}{g}\right)^k = \frac{C}{\pi}\int_0 \frac{\mathrm{d}g\,\mathrm{e}^{-(a-z)/g}}{g^{b+1}} = (C/\pi)\Gamma(b)(a-z)^{-b}.$$

Therefore, the integral (41.7) does not exist if the classical action $A = 1/a$ is positive. The perturbation series in such theories is not Borel summable. In the light of this remark, we can draw several conclusions.

(i) The field equations have no real instanton solutions. In particular, this is the case if one has expanded around the unique absolute minimum of the potential. If complex instanton solutions exist, the corresponding classical action is non-positive, and the perturbative expansion is presumably Borel summable. This is only a presumption, because various features of the perturbative expansion, invisible at large orders, could prevent Borel summability. For instance, the perturbative expansion could contain contributions all of the same sign, growing faster than any exponential of the order k, but much smaller than $k!$ (*e.g.* $\sqrt{k!}$). Then, $B(z)$ would grow too rapidly for large-argument z ($\ln B(z) \sim z^2$ in the example) and the Borel integral would not converge at infinity.

(ii) One has expanded around a relative minimum of the potential, and real instanton solutions are found, corresponding to barrier penetration: the perturbative expansion is not Borel summable.

In this case, one additional piece of information may be useful for determining the solution, if the metastable situation originates from a stable situation by analytic continuation. Then, a possible solution is to integrate in the Borel transform just above the cut, which is on the real positive axis. Therefore, from a real perturbative expansion, one derives a complex result, but this is exactly what one expects. It is easy to verify that the imaginary part is, for g small, exactly what one would have calculated directly. Actually, this is only the solution of the problem in the simplest case, when no other instanton singularities cross the contour of integration in the analytic continuation.

(iii) Real instantons connect degenerate non-continuously connected classical minima (Section 40.1.3). The theory then is not Borel summable. Integration above or below the axis yields a complex result for a real quantity. The half sum of the integral above and below is real, but even in the simple example of the quartic double well-potential, one can verify that this does not give the correct answer. In Chapter 42, in the case of several one-dimensional potentials we show that, in addition to the perturbative expansion, one has to take into account multi-instanton contributions. The corresponding problem has not been solved in field theory examples yet.

Field theory examples. Field theory examples of degenerate non-continuously connected classical minima are provided by the two-dimensional $CP(N-1)$ models (Section 39.5), and four-dimensional $SU(2)$ gauge theory (Section 39.6). In these models, real instantons connect the degenerate minima, and the corresponding classical action is positive. Therefore, the perturbative expansion is not Borel summable. Note that the instanton contribution may not necessarily dominate the large-order behaviour, because, as the example of the $\phi^4_{d=4}$ massless field theory (Section 40.3.2) illustrates, when the classical field theory is scale invariant, the perturbative expansion might be dominated by contributions unobtainable by semi-classical methods and related to ultraviolet (UV) or infrared (IR) singularities.

41.2 Borel transformation: Series summation

All summation methods rely on some additional knowledge about the analytic properties of the function that is expanded.

In the framework of the $(\boldsymbol{\phi}^2)^2$ field theory, methods based on the Borel transformation have been systematically used to sum perturbative series at fixed dimension $d < 4$, and to sum Wilson–Fisher's $\varepsilon = 4 - d$ expansions. The main motivations are:

(i) Borel summability of the perturbative expansion at the fixed dimensions 2 and 3 has been rigorously established [422]. For the ε-expansion, Borel summability is quite plausible.

(ii) The information drawn from the large-order behaviour analysis (which has been determined in all cases and compares favourably with the first available terms of the series, see Section 40.2) can easily be incorporated.

The Borel transformation reduces the problem to determining the analytic continuation of the Borel transform, defined by its Taylor series in a circle, to a neighbourhood of the real positive axis. This continuation can be performed by many methods and the optimal choice depends somewhat on the available additional information. We give here two examples that have been used.

Padé approximants. In the absence of a precise knowledge of the location of the singularities of the Borel transform in the complex plane, one can use the Padé approximation [109, 418] (the Padé–Borel method). From the series, one derives $[M, N]$ Padé approximants, which are rational functions P_M/Q_N satisfying

$$B_f(z) = \frac{P_M(z)}{Q_N(z)} + O\left(z^{N+M+1}\right), \tag{41.13}$$

where P_M and Q_N are polynomials of degrees M and N, respectively. If one knows $(K+1)$ terms of the series, one can construct all Padé approximants with $N + M \leq K$. This method is well adapted to meromorphic functions. The main shortcoming of the method is that, for a rather broad class of functions, Padé approximants are known to converge only in measure [446], and thus spurious poles may occasionally appear close to, or on the real positive axis. Even when Padé approximants converge, this property may lead to instabilities in the results when M and N vary, and make an empirical evaluation of errors for short series difficult.

Borel transformation and conformal mapping. After Borel transformation, to sum the expansion of the Borel transform efficiently, it is necessary to know, or to guess, some of its analytic properties. Since in the case of the $(\boldsymbol{\phi}^2)^2$ field theory and the perturbative expansion at fixed dimension, all known instanton actions are negative, it is plausible that all singularities of the Borel transform are located on the negative real axis.

The Borel transform is then analytic in a cut-plane, the location and nature of the singularity closest to the origin being given by the large order estimates (Section 41.1.2) [425].

For what concerns the ε-expansion, the situation is more subtle, because it is related to the four-dimensional perturbation theory, which has renormalon singularities that may even prevent Borel summability. However, one may argue, and this is supported by numerical evidence, that the ε-expansion of physical quantities is free of renormalon singularities.

In Refs. [424, 430], it has been assumed that the Borel transform is also analytic in a cut-plane. However, since in both cases this maximal analyticity is only a conjecture, the reliability of results can only be estimated by checking their stability with respect to reasonable variations of the summation method. Moreover, the comparison between the two families of results, perturbation theory at fixed dimensions $d < 4$, ε-expansion provides an internal consistency check of field theory methods.

Mapping of a cut-plane onto a disk. Analytic continuation can then be achieved by using a mapping that preserves the origin and maps the domain of analyticity onto a disk [423, 425]. In the transformed variable, the new series converges in the whole domain of analyticity. Let us explain the method with an example.

We assume that the function $B_f(z)$, Borel transform of a function f, is analytic in a cut-plane, the cut running along the real negative axis from $-\infty$ to $-1/a$. The mapping

$$z \mapsto u, \quad u(z) = \frac{\sqrt{1+az}-1}{\sqrt{1+az}+1} \ \Leftrightarrow \ z = \frac{4}{a}\frac{u}{(1-u)^2}\,, \tag{41.14}$$

maps the cut-plane onto a circle of radius 1. From the original series for the Borel transform, one derives a series in powers of the new variable u:

$$B_f(z) = \sum \frac{f_k}{k!} z^k \ \Rightarrow \ B_f[z(u)] = \sum_0^\infty B_k u^k. \tag{41.15}$$

Introducing this expansion into the Borel transformation, one obtains an expansion of $f(z)$ of the form

$$f(z) = \sum_0^\infty B_k I_k(z), \tag{41.16}$$

in which the functions $I_k(z)$ have the integral representation:

$$I_k(z) = \int_0^\infty \mathrm{e}^{-t}\,[u(zt)]^k\,\mathrm{d}t\,. \tag{41.17}$$

To determine the natural domain of convergence of the new expansion, one needs the behaviour of $I_k(z)$ for $k \to \infty$. Using for $u(z)$ the explicit expression (41.14), one can evaluate it by the steepest descent method. The saddle point equation is

$$-1 + \frac{k}{t}\frac{1}{\sqrt{1+azt}} = 0\,, \tag{41.18}$$

which, for k large, yields

$$t \sim k^{2/3}/(az)^{1/3}. \tag{41.19}$$

It follows that $I_k(z)$ behaves for k large as

$$I_k(z) \sim \exp\left[-3k^{2/3}/(az)^{1/3}\right]. \tag{41.20}$$

Three situations can then be encountered:

(i) The coefficients B_k either decrease or at least do not grow too rapidly,

$$|B_k| < M\,\mathrm{e}^{\varepsilon k^{2/3}}, \quad \forall\, \varepsilon > 0\,.$$

Then, the expansion (41.16) converges at least in the region

$$\mathrm{Re}\, z^{-1/3} > 0 \;\Rightarrow\; |\mathrm{Arg}\, z| < 3\pi/2\,. \tag{41.21}$$

This, in particular, implies that the function $f(z)$ must be analytic in the cut plane and even a part of the second sheet of the cut.

(ii) The coefficients behave like

$$\ln|B_k| \sim ck^{2/3},\; c > 0\,, \quad \text{for } k \text{ large}\,. \tag{41.22}$$

The domain of convergence is

$$\mathrm{Re}\, z^{-1/3} > \tfrac{1}{3}ca^{1/3}. \tag{41.23}$$

This condition implies analyticity in a finite domain containing a part of the second sheet since for $|z|$ small, the right-hand side is negligible.

(iii) The coefficients B_k grow faster than $\exp(ck^{2/3})$. This is quite possible, since the only constraint on the coefficients B_k is that the series (41.15) has a radius of convergence 1. For instance, the coefficients B_k could grow like $\exp(ck^{4/5})$. In such a situation, the new series is also divergent. Such a situation arises when the singularities on the boundary of the domain of analyticity are too strong. One must map a smaller fraction of the domain of analyticity onto a disk.

41.3 Summing the perturbative expansion of the $(\boldsymbol{\phi}^2)^2$ field theory

We discuss the summation of series by several variants of the method of Borel transformation and parametric mapping, whose simplest form is described in Section 41.2 [425, 430].

41.3.1 RG functions and exponents

Critical exponents and a number of other universal quantities have been calculated within the framework defined in Section 16.1 (equations (16.3, 16.4)), that is, in the massive $(\boldsymbol{\phi}^2)^2$ field theory, as perturbative series at fixed dimension. For example, in three dimensions, in the normalization

$$\tilde{g} = (N+8)g/(48\pi), \tag{41.24}$$

such that $\beta(\tilde{g}) = -\varepsilon\tilde{g} + \tilde{g}^2 + O(\tilde{g}^3)$, for $N = 1$, the RG β-function has the expansion [419],

$$\begin{aligned}\beta(\tilde{g}) = -\tilde{g} + \tilde{g}^2 - \tfrac{308}{729}\tilde{g}^3 + 0.3510695978\tilde{g}^4 - 0.3765268283\tilde{g}^5 \\ + 0.49554751\tilde{g}^6 - 0.749689\tilde{g}^7 + O\left(\tilde{g}^8\right).\end{aligned} \tag{41.25}$$

To determine exponents or other universal quantities, one must first determine the IR stable zero g^* of the function $\beta(g)$, which is given by a few terms of a divergent expansion.

Unlike the ε-expansion, there no small parameter in which to expand since g^* has a value of order 1. Already at this stage a summation method is required.

A further problem arises from the property that RG functions, unlike the universal quantities in the ε-expansion, depend explicitly on the renormalization scheme. On the other hand, because one-loop diagrams have, in three dimensions, a simple analytic expression, it has been possible to calculate quite early the RG functions of the N-vector model up to six- and partially seven-loop order [418–420].

The main drawback of this procedure is that the values of the critical exponents depend strongly on the value of $\tilde{g}^*$. Therefore, an error in the estimation of $\tilde{g}^*$ biases all exponents. A variant, which avoids this problem, has thus been used as a check. A pseudo-ε parameter has been introduced by setting (for $d = 3$)

$$\beta(\tilde{g}, \varepsilon) = \tilde{g}(1 - \varepsilon) + \beta(\tilde{g}). \tag{41.26}$$

The two functions $\beta(\tilde{g}, \varepsilon)$ and $\beta(\tilde{g})$ coincide for $\varepsilon = 1$, and the zero of $\beta(\tilde{g}, \varepsilon)$ is calculated as a power series in ε. Critical exponents are then also calculated as series in ε, and these series are summed. However, there are indications that the specific mapping $\tilde{g} \mapsto \varepsilon$ introduces singularities, because the apparent convergence is poorer. Nevertheless, all variants give consistent results, which is satisfactory. Moreover, a comparison of the different results gives useful indications about the apparent errors.

41.3.2 Equation of state

The equation of state for $N = 1$, as well a number of universal ratios of amplitude, have also been calculated [444]. Numerical results are displayed in Section 41.8.

In the framework of the massive theory, the calculation of physical quantities in the ordered phase leads to additional technical problems because the theory is parametrized in terms of the disordered phase correlation length m^{-1}, which is singular at T_c. Moreover, the normalization of correlation functions is singular at T_c (equation (16.17)).

In the example of Ising-like systems ($N = 1$), the free-energy F has the general form

$$F(M) - F(0) = \frac{m^d}{g}\varphi\big(g^{1/2}m^{1-d/2}\tilde{M}, g\big),$$

in which g has to be set to its fixed point value g^*, and $\tilde{M}$ is related to the magnetization M by the field renormalization (16.15):

$$\tilde{M} = Mm^{-\eta/2}.$$

The derivative of F with respect to M yields the equation of state under the form

$$H = g^{-1/2}m^{1+d/2-\eta/2}\mathfrak{h}\big(g^{1/2}m^{1-d/2}\tilde{M}, g\big), \quad \mathfrak{h}(z, g) = \partial\varphi(z, g)/\partial z\,. \tag{41.27}$$

For example, at one-loop order, the function $\mathfrak{h}(z, g)$ is given by

$$\begin{aligned}\mathfrak{h}(z, g) &= z + \frac{z^3}{6} + \frac{gz}{2}\frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d p}{p^2 + 1 + z^2/2} + O\,(2 \text{ loops}) \\ &= z + \frac{z^3}{6} + \frac{\pi N_d}{4\sin\pi d/2}gz\left[(1 + z^2/2)^{d/2-1} - 1 - \tfrac{1}{4}(d-2)z^2\right].\end{aligned} \tag{41.28}$$

In terms of the deviation from the critical temperature $t = m^{1/\nu}(g^*)^{1/2\beta} \sim T - T_c$, equation (41.27) takes the form

$$H = H_0 t^{\beta\delta}\mathfrak{h}(Mt^{-\beta}) \quad (H_0 \text{ being a constant}). \tag{41.29}$$

The expression is adequate for the description of the disordered phase when $Mt^{-\beta}$ is small, but all terms in the loop expansion become singular for $t \to 0$.

The ordered phase. This problem does not completely prevent calculations near the coexistence curve, that is, for $t < 0$. Since, at the fixed point g^*, all functions have simple power law singularities at T_c, it is possible to proceed by analytic continuation in the complex t-plane. The scaling variable $z = Mt^{-\beta}$ picks up a phase below T_c:

$$\text{for } t = |t|\,\mathrm{e}^{i\pi}, \quad z = |z|\,\mathrm{e}^{-i\pi\beta}. \tag{41.30}$$

The scaling variable $H(-t)^{-\beta\delta}$ then is given by

$$H(-t)^{-\beta\delta} = H_0\,\mathrm{e}^{i\pi\beta\delta}\,\mathfrak{h}(z) = H_0\,|\mathfrak{h}(z)|\,. \tag{41.31}$$

In particular, it is possible to evaluate ratios of amplitudes of singularities above and below T_c; one can calculate the complex zero of $\mathfrak{h}(z,g)$ as a power series in g and substitute it in other quantities. The result is complex, but its modulus taken at $g = g^*$ converges towards the correct result. This can be illustrated by the example of the magnetic susceptibility. From equation (41.27), one derives

$$C^+/C^- = \mathrm{e}^{-i\pi\gamma}\,\mathfrak{h}'\big(z_0(g^*),g^*\big)/\mathfrak{h}'(0,g^*) = \left|\mathfrak{h}'\big(z_0(g^*),g^*\big)\right|, \tag{41.32}$$

in which z_0 is the complex zero of $\mathfrak{h}(z,g)$. The expression can be used to obtain a series expansion for C^+/C^-.

However, this method does not make an extrapolation of the equation of state to t small possible. Following the lines of Section 16.9.3, it is natural to introduce the parametric representation (16.147) [163],

$$z = x_0^{-\beta}\theta\left(1-\theta^2\right)^{-\beta},$$

and consider the function

$$h(\theta) = \left(1-\theta^2\right)^{\beta\delta}\mathfrak{h}\big(z(\theta)\big).$$

However, in an expansion at fixed dimension, if we just replace all quantities by their perturbative expansion, the singularity of $h(\theta)$ at $\theta = 1$ (*i.e.* $t = 0$) does not cancel any more. Therefore, inspired by results coming from the ε-expansion, one also expands $h(\theta)$ in powers of θ. The method is the following: one first determines by Borel summation, as explained in Section 41.4, the first terms of the expansion of the function $\mathfrak{h}(z)$ in powers of z. As expected, the apparent precision decreases with increasing degree. One determines the corresponding coefficients of the expansion of $h(\theta)$ in powers of θ (note $z \sim \theta$). These coefficients are polynomials in $x_0^{-\beta}$. One then adjusts the arbitrary constant x_0 to minimize the last term, following the ODM method [447] (see Section A41.1). This strategy has been applied to the $N = 1$ series, which are known up to order g^5. A general representation of the equation of state has been obtained. Some amplitude ratios have been inferred (see Table 41.6). It would be interesting to apply this method to $N \neq 1$ series.

Table 41.1

Series summed by the method based on Borel transformation and mapping for the zero $\tilde{g}^*$ of the RG-β function, and the exponents γ and ν in the $\phi^4_{d=3}$ field theory, after determination of $\tilde{g}^*$.

k	2	3	4	5	6	7
$\tilde{g}^*$	1.8774	1.5135	1.4149	1.4107	1.4103	1.4105
ν	0.6338	0.6328	0.6297	0.6302	0.6302	0.6302
γ	1.2257	1.2370	1.2386	1.2398	1.2398	1.2398

41.4 Summation method: Practical implementation

The general principles and the theoretical justification of the summation method based on Borel summation and conformal mapping are explained in Section 41.2. We add here a few details about its specific implementation for the calculation of critical exponents and other universal quantities. In Table 41.1, we display three examples of transformed series, to illustrate the convergence.

The method. Several different variants based on the Borel (really Borel–Leroy) transformation and parametric conformal mapping have been implemented and tested. Let $R(z)$ be the function whose expansion has to be summed (z here represents the coupling constant $\tilde{g}$ or ε):

$$R(z) = \sum_{k=0} R_k z^k . \tag{41.33}$$

One transforms the series into

$$R(z) = \sum_{k=0}^{\infty} B_k(\rho) \int_0^{\infty} t^{\rho}\, \mathrm{e}^{-t} \left[u(zt)\right]^k \mathrm{d}t\,,\ \ \rho \geq 0\,, \quad \text{with } u(z) = \frac{\sqrt{1+az}-1}{\sqrt{1+za}+1}\,. \tag{41.34}$$

The coefficients B_k are calculated by identifying the expansion of the right-hand side of equation (41.34) in powers of z with the expansion (41.33). The constant a has been determined by the large-order behaviour analysis (equations (38.11, 38.12)). At fixed dimension (with a normalization such that $\beta(\tilde{g}) = -\varepsilon\tilde{g} + \tilde{g}^2 + O(\tilde{g}^3)$, see equation (41.24)),

$$a(d=3) = \frac{48\pi}{(N+8)A} = 0.147774232 \times \frac{9}{(N+8)}, \tag{41.35}$$

$$a(d=2, N=1) = \frac{8\pi}{3A} = 0.238659217\,, \tag{41.36}$$

and for the ε-expansion, $a = 3/(N+8)$.

The free-parameter ρ is adjusted empirically to improve the convergence of the transformed series by weakening the singularities of the Borel transform near $z = -a$. Moreover, in many cases, the conformal transformation

$$R(z) \mapsto \tilde{R}(z) = R\left[z/(1-\tau z)\right], \tag{41.37}$$

has been applied to the initial function R, where τ is left as an adjustable parameter in order to move away its closest singularities, because the location of all singularities of $R(z)$ is not known.

This transformation is necessary in the case of the ε-expansion, because the critical exponents, as functions of ε, have close singularities. It has been verified that it also improves the summation of the series at fixed dimensions.

Finally, in general, a third parameter was introduced which will not be discussed here.

Needless to say, with three parameters and short-initial series, it becomes possible to find, occasionally, some transformed series whose apparent convergence is deceptively good. Therefore, it is essential to vary the parameters in some range around the optimal values, and to examine the sensitivity of the results upon their variations. Finally, it is useful to sum independently series for exponents related by scaling relations. An underestimation of the apparent errors leads to inconsistent results. It is clear from these remarks that the quoted errors remain an educated guess, even though, compared to the central values, their determination requires much more work.

The $(\boldsymbol{\phi}^2)^2$ field theory at fixed dimensions. The RG β-function has been determined, in three dimensions, up to six-loop order while the series for the dimensions of the fields $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^2$ have been extended to seven loops. In two dimensions, the series only are known up to four loops. They have been analysed by two methods. In the first method, the series of the RG β-function has been first summed and its zero $\tilde{g}^*$ calculated ($\tilde{g} = g(N+8)/(48\pi)$ for $d=3$, $\tilde{g} = 3g/8\pi$ for $d=2$). The series of the other RG functions have then been summed for $\tilde{g} = \tilde{g}^*$.

The ε-expansion. The ε-expansion makes it possible to connect the results in three and two dimensions, a notable advantage. In particular, in the cases $N=1$ and $N=0$, it is possible to compare the ϕ^4 results with exact results coming from lattice models and to test both universality and the reliability of the summation procedure. Moreover, it is possible to constrain the three-dimensional results by imposing the exact two-dimensional values or the behaviour near two dimensions for $N>1$, the changes in the results being a check of consistency [430]. Finally, as we have already emphasized, the comparison between the different results is a check of the consistency of the field theory methods, combined with the summation procedures.

Table 41.2

Critical exponents from the summed perturbative $(\boldsymbol{\phi}^2)^2$ field theory for $N=0,1$ in dimension 2.

	γ	ν	η	β	ω
$N=0\,,\ \varepsilon^5$	1.39 ± 0.04	0.76 ± 0.03	0.21 ± 0.05	0.065 ± 0.015	1.7 ± 0.2
$N=0\,,\ \varepsilon^6$	1.333 ± 0.025	0.741 ± 0.004	0.201 ± 0.025	0.074 ± 0.010	1.90 ± 0.25
$N=0\,,$ exact	1.34375	0.75	$0.2083\cdots$	$0.0781\cdots$	?
$N=1\,,\varepsilon^5$	1.73 ± 0.06	0.99 ± 0.04	0.26 ± 0.05	0.120 ± 0.015	1.6 ± 0.2
$N=1,,\varepsilon^6$	1.68 ± 0.05	0.952 ± 0.014	0.237 ± 0.027	0.113 ± 0.015	1.71 ± 0.09
$N=1,,d=2$ fixed	1.79 ± 0.04	0.96 ± 0.04	0.18 ± 0.04	0.086 ± 0.022	1.3 ± 0.2
$N=1\,,$ exact	1.75	1.	0.25	0.125	?

Table 41.3

Estimates of critical exponents in the $O(N)$-symmetric $(\phi^2)^2_{d=3}$ field theory.

N	0	1	2	3
$\tilde{g}^*$	1.413 ± 0.006	1.411 ± 0.004	1.403 ± 0.003	1.390 ± 0.004
g^*	26.63 ± 0.11	23.64 ± 0.07	21.16 ± 0.05	19.06 ± 0.05
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
η	0.0284 ± 0.0025	0.0335 ± 0.0025	0.0354 ± 0.0025	0.0355 ± 0.0025
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
α	0.235 ± 0.003	0.109 ± 0.004	-0.011 ± 0.004	-0.122 ± 0.010
ω	0.812 ± 0.016	0.799 ± 0.011	0.789 ± 0.011	0.782 ± 0.0013
$\theta = \omega\nu$	0.478 ± 0.010	0.504 ± 0.008	0.529 ± 0.009	0.553 ± 0.012

41.5 Field theory estimates of critical exponents for the $O(N)$ model

We now display the values of critical exponents inferred from an RG analysis of the $O(N)$-symmetric $(\phi^2)^2$ field theory, based on summed perturbative expansions. Let us point out that, even if we focus here mainly on the determination of critical exponents, the advantages of field theory methods are that universality can be proven, and that all universal quantities can, in principle, be calculated. We give the examples of the equation of state, and amplitude ratios in Section 41.8.

41.5.1 Dimension 2

In Table 41.2, we display the results for $N = 0$ and $N = 1$ coming from the summed ε-expansion for $\varepsilon = 2$, including the early results at order ε^5 [424] and the more recent results at order ε^6 [160] (note that the values of γ and β are inferred by scaling), and for $N = 1$ from the perturbative expansion at fixed dimension [425]. With the latter method, the apparent errors are larger because the series are shorter. For $N = 1$, we compare the values with the Ising model exponents and, for $N = 0$, with the values proposed in Ref. [426]. The values for $N = 0$ are consistent with a convergence towards the exact values. For $N = 1$, the values of the exponents are close to the exact values but the convergence of the summed ε expansion seems to be slower. For the less precise results inferred from the fixed dimension series, the main observation is that the exponent η is underestimated.

For $N = 1$, one obtains a fixed point coupling constant $\tilde{g}^* = 1.85 \pm 0.010$. In lattice models, it is known only from high-temperature series [427–429]: $\tilde{g}^* = 1.755(2)$.

Both for $N = 0$ and $N = 1$, the identification of the correction exponent ω remains a problem. For example, only analytic corrections to scaling have been found in the Ising model, which makes the identification of the correction exponent ω difficult. However, an analysis based on conformal invariance predicts a correction exponent $\omega = 4/m$ for ϕ^{2m-2} field theories and for $m > 3$. One may conjecture that the amplitudes of the singularities involving the correction exponent ω vanish when m approaches 3 for $d = 2$, or for $m = 3$ when d approaches 2.

Table 41.4

Critical exponents in the $(\phi^2)^2_{d=3}$ field theory derived from the ε-expansion.

N	0	1	2	3
$\gamma\ (\varepsilon^5)$	1.1575 ± 0.0060	1.2355 ± 0.0050	1.311 ± 0.007	1.382 ± 0.009
$\gamma\ (\varepsilon^6)$	1.1566 ± 0.0010	1.2356 ± 0.0014	1.313 ± 0.002	1.385 ± 0.004
$\nu\ (\varepsilon^5)$	0.5875 ± 0.0025	0.6290 ± 0.0025	0.6680 ± 0.0035	0.7045 ± 0.0055
$\nu\ (\varepsilon^6)$	0.5874 ± 0.0003	0.6292 ± 0.0005	0.6690 ± 0.0010	0.7059 ± 0.0020
$\eta\ (\varepsilon^5)$	0.0300 ± 0.0050	0.0360 ± 0.0050	0.0380 ± 0.0050	0.0375 ± 0.0045
$\eta\ (\varepsilon^6)$	0.0310 ± 0.0007	0.0362 ± 0.0006	0.0380 ± 0.0006	0.0378 ± 0.0005
$\beta\ (\varepsilon^5)$	0.3025 ± 0.0025	0.3257 ± 0.0025	0.3465 ± 0.0035	0.3655 ± 0.0035
$\beta\ (\varepsilon^6)$	0.3028 ± 0.0004	0.3260 ± 0.0004	0.3472 ± 0.0007	0.3663 ± 0.0012
$\omega(\varepsilon^5)$	0.828 ± 0.023	0.814 ± 0.018	0.802 ± 0.018	0.794 ± 0.018
$\omega\ (\varepsilon^6)$	0.841 ± 0.013	0.820 ± 0.007	0.804 ± 0.003	0.795 ± 0.007
$\theta = \omega\nu\ (\varepsilon^5)$	0.486 ± 0.016	0.512 ± 0.013	0.536 ± 0.015	0.559 ± 0.017
$\theta = \omega\nu\ (\varepsilon^6)$	0.494 ± 0.009	0.516 ± 0.008	0.538 ± 0.003	0.561 ± 0.012

41.5.2 Dimension 3

Table 41.3 displays the results obtained from summed perturbation series at fixed dimension 3 [430], incorporating seven-loop terms for γ and η reported in Ref. [419]. The exponent $\theta = \omega\nu$ characterizes corrections to scaling in the temperature variable,

Table 41.4 displays the results for $\varepsilon = 1$ from the ε expansions, at order ε^5 [430] and the more recent results, which take into account the ε^6 contributions (θ is inferred from the reported values of ν and ω) [160].

Discussion. Since 1980, the fixed-dimension, three-dimensional results have only marginally changed [425, 430], and within errors. No additional progress can be expected in the near future, since a seven-loop calculation of the RG β function (a formidable task) is still lacking. By contrast, the ε-expansion predictions have steadily improved [424, 430, 160] because the order ε^5 (in two steps) and more recently, the order ε^6 have become available. It should be noted that the successive values are very close, suggesting that the errors in Ref. [430] were too conservative.

Considering the reasonable agreement between exact two-dimensional results and summed ε expansion, in Ref. [430], as an alternative method, a summation procedure has been used that automatically incorporates the $d = 2$, $\varepsilon = 2$ values. The three-dimensional results have slightly changed, but within errors.

Comparing the two sets of results coming from the perturbation series at fixed dimension, and the ε-expansion, one notes that the general agreement is excellent, especially for the exponents ν and β, even if some tension exists for η. A comparison with the most precise results provided by other methods for $N = 0$ and $N = 1$ favours the ε expansion.

41.6 Other three-dimensional theoretical estimates

The N-vector model, with nearest-neighbour interactions, has been studied on several lattices. Critical exponents are derived from an analysis of high-temperature series by different ratio methods, Padé or differential approximants (Sections 41.2 and A41.2) [431].

Table 41.5

Estimates of critical exponents of the $(\phi^2)^2_{d=3}$ field theory, by other theoretical methods.

N	0	1	2	3
γ	1.159653(1)	1.237075	1.3178(2)	1.3963(22)
ν	0.587597	0.629971	0.6717(1)	0.7116(10)
η	0.031043(3)	0.036298(2)	0.0381(2)	0.0378(3)
β	0.302919	0.326419	0.3486(1)	0.3692(5)
ω	0.900(15)	0.830(2)	0.811(10)	0.791(22)
$\theta = \omega\nu$	0.528(8)	0.523(2)	0.545(6)	0.563(13)

A number of results are obtained from computer simulations using Monte-Carlo methods [432, 433]. More recently, new results have been provided by the *conformal bootstrap method* [91, 434, 435].

Note that, when the first field theory results were published [425], they were in strong disagreement with some of high-temperature series analyses. With time, high-temperature and Monte-Carlo predictions have converged towards the field theory results, and now all available results are remarkably consistent, as Table 41.5 indicates.

However, in the $N = 0$ Monte-Carlo results [433], one notes one discrepancy for the correction exponent ω.

$N = 1$ is now dominated by the conformal bootstrap results [435] and the fixed-dimension result for ω seems now to be somewhat low.

Estimates for $N = 2$ and $N = 3$, have been obtained by using a combination of Monte-Carlo simulations and analysis of high-temperature series [432].

41.7 Critical exponents from experiments

We have discussed the N-vector model in the ferromagnetic language, even though most of our experimental knowledge comes from physical systems which are not magnetic but belong to the universality class of the N-vector model. For example, $N = 0$ describes the statistical properties of long polymer chains, that is, long not intersecting chains or self-avoiding walks (see Section 15.8). The case $N = 1$ (Ising-like systems) describes liquid–vapour transitions in classical fluids, critical binary fluids and uniaxial antiferromagnets. The helium superfluid transition corresponds to $N = 2$. Finally, for $N = 3$, the experimental information comes from ferromagnetic systems.

An early review where experiments, high-temperature series and field-theory results are compared, using the results then available, can be found in the proceedings of the Cargèse summer school 1980 [436].

Critical exponents and polymers. In the case of polymers, only the exponent ν is easily accessible. An old, not reproduced since, result is $\nu = 0.586 \pm 0.004$ [437], in excellent agreement with theory.

Ising-like systems $N = 1$. For Ising-like systems, the results are somewhat scattered and difficult to review. Earlier reviews for fluids containing, in addition, estimates of universal ratios are Refs. [438]. Ref. [439] reports the values $\alpha = 0.0.113 \pm 0.005$ and for the correction exponent θ (sometimes denoted Δ), the estimate $\theta = 0.50 \pm 0.3$.

In the more recent review of Sengers and Shanks [440], one finds $\eta = 0.032 \pm 0.013$, $\nu = 0.629 \pm 0.003$.

However, the authors also quote the more precise value $\eta = 0.030 \pm 0.0015$ [441] and $\nu = 0.632 \pm 0.002$, $\eta = 0.041 \pm 005$ [442].

The conclusion is that the exponents derived from the analysis of fluid experiments are globally totally consistent with theoretical values.

Helium superfluid transition, $N = 2$. The helium fluid makes extremely precise measurements very close to T_c possible, and this explains the precision of the determination of critical exponents. However, the order parameter is not directly accessible and, therefore, only ν and α have been determined. Reported values are [143]

$$\nu = 0.6705 \pm 0.0006\,, \quad \nu = 0.6708 \pm 0.0004\,, \quad \text{and} \quad \alpha = -0.01285 \pm 0.00038\,.$$

The agreement with RG values is quite remarkable, but the precision of ν is now a challenge to field theory.

Ferromagnetic systems, $N = 3$. The precision of experimental results is limited. One finds in the literature the estimates

$$\gamma = 1.40 \pm 0.03\,,\ \nu \in [0.700\text{–}0.725],\ \beta = 0.35 \pm 0.03\,,\ \alpha \in [-0.09\text{–}-0.012],\ \theta = 0.54 \pm 0.10\,.$$

Table 41.6

Amplitude ratios: Models and binary critical fluids ($N = 1$).

	ε-expansion	Fixed dim. $d = 3$	Lattice models	Experiment
A^+/A^-	0.527 ± 0.037	0.537 ± 0.019	$\begin{cases} 0.523 \pm 0.009 \\ 0.560 \pm 0.010 \end{cases}$	0.56 ± 0.02
C^+/C^-	4.73 ± 0.16	4.79 ± 0.10	$\begin{cases} 4.75 \pm 0.03 \\ 4.95 \pm 0.15 \end{cases}$	4.3 ± 0.3
f_1^+/f_1^-	1.91	2.04 ± 0.04	1.96 ± 0.01	1.9 ± 0.2
R_ξ^+	0.28	0.270 ± 0.001	0.266 ± 0.001	0.25– 0.32
R_c	0.0569 ± 0.0035	0.0574 ± 0.0020	0.0581 ± 0.0010	0.050 ± 0.015
$R_\xi^+ R_c^{-1/3}$	0.73	0.700 ± 0.014	0.650	0.60– 0.80
R_χ	1.648 ± 0.036	1.669 ± 0.018	1.75	1.75 ± 0.30
Q_2	1.13		1.21 ± 0.04	1.1 ± 0.3
Q_3	0.96		0.896 ± 0.005	

Table 41.7

Correction amplitude ratios for $N = 1$.

	$\varepsilon -$ expansion	Fixed dim. $d = 3$	HT series	Experiment
a_ξ^+/a_χ^+	0.56 ± 0.15	0.65 ± 0.05	0.70 ± 0.03	
a_C^+/a_ξ^+	2.03	1.45 ± 0.11		
a_C^+/a_χ^+	1.02	0.94 ± 0.10	1.96	0.87 ± 0.13
a_C^+/a_C^-	2.54	1.0 ± 0.1		0.7– 1.35
a_χ^+/a_χ^-	0.3	0.315 ± 0.013		
a_C^+/a_M		1.10 ± 0.25		1.85 ± 0.10
a_M/a_χ^+		0.90 ± 0.21		0.08– 1.4

41.8 Amplitude ratios

A classical review on amplitude ratios can be found in Ref. [443].

Amplitude ratios evaluated by field theory and RG methods (see Section 16.9.4 for definitions) are less precise than exponents, because the series are shorter. Note that a determination of the equation of state based on field theory methods (Section 41.3.2) is also available Ref. [444].

Table 41.6 contains a comparison of amplitude ratios as obtained, for $N = 1$, from field theory, lattice calculations for Ising-like models, and experiments on binary mixtures [445].

Some results are available for uniaxial magnetic systems and liquid–vapour transitions. For the latter systems, a few reported values are

$$C^+/C^- = 5. \pm 0.2 \ , \quad R_c = 0.047 \pm 0.010 \ , \quad R_\chi = 1.69 \pm 0.14 \ .$$

For A^+/A^-, results range from 0.48 to 0.53. The set of results, with indeed large errors, shows a satisfactory agreement with the field theory based RG predictions.

Finally, we give a few results concerning ratios of amplitudes of corrections to the leading scaling behaviour. If, in addition to the correlation length and the susceptibility amplitudes a_ξ and a_χ, one considers also the specific heat amplitude a_C, and the coexistence curve magnetization amplitude a_M, one can form three independent ratios. The results for $N = 1$ are displayed in Table 41.7.

Table 41.8

Amplitude ratios for $N = 2$ and $N = 3$.

	N	Field theory	HT series	Experiment
A^+/A^-	2	1.056 ± 0.004	1.08	$1.054 \pm .001$
R_ξ^+	2	0.36	0.36	
R_c	2	0.123 ± 0.003		
A^+/A^-	3	1.52 ± 0.02	1.52	1.40–1.52
R_ξ^+	3	0.42	0.42	0.45
R_c	3	0.189 ± 0.009	0.165	

A few amplitude ratios have been calculated and measured for helium ($N = 2$) and ferromagnets ($N = 3$). We give in Table 41.8 the examples of A^+/A^-, R_ξ^+ and R_c.

A review of all available data (critical exponents, amplitude ratios, and so on), clearly shows that the RG predictions are remarkably consistent with the whole experimental information available. The diversity of experimental systems is a spectacular confirmation of the concept of universality, and of the RG ideas combined with field methods.

A41 Some other summation methods

For illustration purposes, we give two examples of summation methods that do not necessarily involve a Borel transformation.

A41.1 Order-dependent mapping method (ODM)

The ODM method requires, to be applicable, some knowledge of the analyticity properties of the function itself [447]. As we have discussed, functions may be singular at an expansion point and nevertheless have a (necessarily divergent) series expansion in a sector. In the examples we have met, the function has a cut extending to the origin, and its discontinuity decreases exponentially, close to the origin. The idea of the ODM method is then to pretend that the function is analytic, in addition to its true domain of analyticity, in a small disk centred at the origin, of adjustable radius ρ, and to map this extended domain onto a circle centred at the origin, keeping the origin fixed. If the function would really be analytic in such a domain, the expansion in the transformed variable would converge in the whole domain of analyticity, and the continuation problem would be solved. Since the original series is really only asymptotic, the series in the transformed variable is also asymptotic. However, as a result of the transformation, the coefficients of the new series now depend on an adjustable parameter ρ.

For instance, let us assume that $f(z)$ is analytic in a cut-plane. We then use the mapping,

$$z = 4\rho u/(1-u)^2 . \tag{A41.1}$$

The transformed series has the form

$$f\big(z(u)\big) = \sum_0^{\infty} P_k(\rho) u^k , \tag{A41.2}$$

in which the coefficients $P_k(\rho)$ are polynomials of degree k in the parameter ρ. In more general situations, one can often use a mapping of the form

$$z = \rho\, h(u)\,, \quad h(u) = O(u)\,. \tag{A41.3}$$

The kth order approximation is obtained by truncating the series at order k and choosing ρ as one of the zeros of the polynomial $P_k(\rho)$. The zero cannot actually be chosen arbitrarily but qualitatively speaking, must be the zero of largest modulus for which the derivative $P'_k(\rho)$ is small. The idea behind the method is the following: with the original series, the best approximation is obtained by truncating the series at z fixed, at an order dependent on z such that the modulus of the last term taken into account is minimal. By introducing an additional parameter, one modifies the situation: one first chooses the order of truncation, and then tries to adjust the parameter ρ in such a way that, at z again fixed, the last term taken into account is minimal.

The kth order approximant has the form

$$\{f(z)\}_k = \sum_{l=0}^{k} P_l(\rho_k)\, [u(z)]^l\,, \quad P_k(\rho_k) = 0\,. \tag{A41.4}$$

Under some conditions, it can be shown that if the terms f_k of the original series grow, for k large, like $(k!)^\beta$, then the sequence ρ_k decreases like $1/k^\beta$. Proof of the convergence of the method in some cases can be found in Ref. [448].

The method has been successfully applied to test problems like the quartic anharmonic oscillator with a mapping

$$z = \rho u/(1-u)^{3/2},$$

the complex PT symmetric potential x^2+igx^3 [449], and to one physics relevant example, the hydrogen atom in a strong magnetic field [450].

A41.2 Linear differential approximants

Padé approximants [109] provide the simplest example of a general class of approximants, which are obtained as solutions to equations (algebraic or differential) with polynomial coefficients [451]. These polynomials are chosen to be the polynomials of the lowest degree for which the solution of the equation has the same power series expansion, up to a given order, as the function one wants to approximate. To be more concrete, we describe linear differential approximants.

Let $f(z)$ be a function for which we know a power series expansion. We can construct approximants $\bar{f}_k(z)$ to this function by looking for solutions of the differential equation

$$\sum_{n=0}^{N} P_n(z) \left(\frac{\partial}{\partial z}\right)^n \bar{f}_k(z) = R(z), \qquad (A41.5)$$

in which the polynomials $P_n(z)$ and $R(z)$ form a set of polynomials of the lowest possible total degree such that

$$f(z) - \bar{f}_k(z) = O\left(z^{k+1}\right). \qquad (A41.6)$$

In the generic situation, the degrees $[P_n]$ and $[R]$ of the polynomials P_n and R satisfy

$$\sum_{n=0}^{N} [P_n] + [R] = k+1\,. \qquad (A41.7)$$

The advantage of these kinds of approximants is that they are extremely flexible. It is possible to use much available information about the function by imposing additional constraints on the polynomials P_n and R.

Furthermore, while Padé approximants generate only approximants with poles, the more general approximants can have a large class of new singularities. The drawback of this flexibility is that this leads to approximations that are much more unstable. It is necessary to select among the large number of approximants one can construct, those for which one has reasons to believe that they are best adapted to the function one wants to approximate.

Due to the generality of the problem, a systematic study of this class of approximants is lacking. Finally, the method can be generalized to power series in several variables. One then derives partial differential equations with polynomial coefficients in all variables.