We conclude this section by specializing (5.149) to the case of total $(p-1)$-forms.

Theorem 5.104. *Let $P \in \mathscr{A}^q$ and let $L = \operatorname{Div} P$. Then*

$$P_k = \sum_{j=1}^{p} D_j Q_{jk}^* + B_k^*, \qquad k = 1, \dots, p, \tag{5.152}$$

where $B^* = B + b$ is the p-tuple determined by (5.151), and $Q_{jk}^* = Q_{jk} + q_{jk}$, where

$$Q_{jk}[u] = \int_0^1 \sum_{\alpha=1}^{q} \sum_I D_I \left\{ u^\alpha \left[\frac{\tilde{\imath}_j + 1}{\#I + 2} \mathsf{E}_\alpha^{I,j}(P_k)[\lambda u] - \frac{\tilde{\imath}_k + 1}{\#I + 2} \mathsf{E}_\alpha^{I,k}(P_j)[\lambda u] \right] \right\} d\lambda,$$

$$q_{jk}(x) = \int_0^1 \{ x^j P_k(\lambda x, 0) - x^k P_j(\lambda x, 0) \} \, d\lambda. \tag{5.153}$$

In particular, if P is a null divergence, then (5.153) shows how to write P explicitly as a "total curl" $P_k = \sum D_j Q_{jk}^*$, where $Q_{jk}^* = -Q_{kj}^*$. For example, in the case $p = 2$, we have

$$D_x P + D_y \tilde{P} = 0$$

if and only if

$$P = D_y Q, \qquad \tilde{P} = -D_x Q,$$

where

$$Q = \int_0^1 \{ \tfrac{1}{2} u \mathsf{E}^{(y)}(P) + \tfrac{1}{3} D_x[u \mathsf{E}^{(xy)}(P)] + \tfrac{2}{3} D_y[u \mathsf{E}^{(yy)}(P)] + \cdots$$
$$- \tfrac{1}{2} u \mathsf{E}^{(x)}(\tilde{P}) - \tfrac{2}{3} D_x[u \mathsf{E}^{(xx)}(\tilde{P})] - \tfrac{1}{3} D_y[u \mathsf{E}^{(xy)}(\tilde{P})] - \cdots \} \, d\lambda, \tag{5.154}$$

where $\mathsf{E}^{(x)}(\tilde{P})$, $\mathsf{E}^{(y)}(P)$, etc. are all evaluated at λu. As a specific example, let $P = u_y u_{xy} + u_x u_{yy}$, $\tilde{P} = -u_y u_{xx} - u_x u_{xy}$, which do form a null divergence. The only nonzero Euler expressions appearing in (5.154) are

$$\mathsf{E}^{(y)}(P) = -2u_{xy}, \qquad \mathsf{E}^{(xy)}(P) = u_y, \qquad \mathsf{E}^{(yy)}(P) = u_x,$$
$$\mathsf{E}^{(x)}(\tilde{P}) = 2u_{xy}, \qquad \mathsf{E}^{(xx)}(\tilde{P}) = -u_y, \qquad \mathsf{E}^{(xy)}(\tilde{P}) = -u_x.$$

Thus

$$Q = \int_0^1 \{ u(-\lambda u_{xy}) + \tfrac{1}{3} D_x[u(\lambda u_y)] + \tfrac{2}{3} D_y[u(\lambda u_x)]$$
$$- u(\lambda u_{xy}) + \tfrac{2}{3} D_x[u(\lambda u_y)] + \tfrac{1}{3} D_y[u(\lambda u_x)] \} \, d\lambda = u_x u_y$$

satisfies $P = D_y Q$, $\tilde{P} = -D_x Q$.

Notes

Generalized symmetries first made their appearance in their present form in the fundamental paper of Noether, [1], in which their role in the construction of conservation laws was clearly enunciated. Anderson and Ibragimov, [1], and Ibragimov, [1], try to make the case that they date back to the investigations of Lie and Bäcklund, hence their choice of the term "Lie–Bäcklund transformation" for these objects. As far as I can determine, Lie only allows dependence of the group transformations on derivatives of the dependent variables in his theory of contact transformations, which are much more restrictive than generalized symmetries. He further, in Lie, [1; § 1.4], proposes the problem of looking at higher order generalizations of these contact transformations. Bäcklund, [1], in response, does write down transformations depending on derivatives of the dependent variables of arbitrary order, and so in a sense anticipates the theory of generalized symmetries, but he and Lie always require that the corresponding prolongations "close off" to define genuine geometrical transformations on some jet space. Bäcklund concludes that the only such transformations are the prolongations of ordinary point transformations or of Lie's contact transformations, hence fails to make the jump to true generalized symmetries. More telling is the fact that Bäcklund requires his transformations to depend on only finitely many derivatives of the dependent variables, while for true generalized symmetries, this is only the case for the infinitesimal generators (which Bäcklund never considers); the group transformations determined by the solutions of the associated evolution equation (5.14) are truly nonlocal, and are not determined solely by the values of finitely many derivatives of the dependent variables at a single point. In particular, in the case of one dependent variable, every contact transformation is equivalent to a first order generalized symmetry, whereas for more than one dependent variable, every contact transformation is the prolongation of a point transformation.

Since their introduction by Noether, generalized symmetries have been rediscovered many times, including the papers of Johnson, [1], [2], in differential geometry, Steudel, [1], in the calculus of variations, and Anderson, Kumei and Wulfman, [1], among others. Recent applications to differential equations can be found in Anderson and Ibragimov, [1], Kosmann-Schwarzbach, [1], [2], Fokas, [3], and Ibragimov, [1]; the last reference includes an extensive discussion of those second and third order evolution equations, as well as general second order equations in two independent variables, admitting generalized symmetries. Steudel, [1], was the first to note the importance of placing a generalized vector field in its evolutionary form. Recent investigations into the symmetry properties of systems of linear equations, including those of field theory (Fushchich and Nikitin, [2], and Kalnins, Miller and Williams, [1]) and elasticity (Olver, [9], [14]) have uncovered new generalized symmetries depending on first order derivatives of the dependent variables, whose significance is not yet fully understood, although they appear to play a role in the separation of variables for such systems.

An outstanding problem in the theory of generalized symmetries is whether a system of partial differential equations can admit only a finite-dimensonal space of generalized symmetries. Exercise 5.3 in the first edition, based on a paper of Ibragimov and Shabat, [1], was not correct; see Exercise 5.16 for a candidate system. Sokolov, [1; Proposition 11], proves that any evolution equation with an infinite-dimensional algebra of generalized symmetries always has an infinite-dimensional commutative subalgebra. The Theorem of Tu presented in the first edition is not correct; however see Exercise 5.4 for what can be proven in this direction, and Exercise 5.15 for a counterexample to the theorem as stated in Tu, [1], and the first edition. A more complete discussion of the properties of the heat equation mentioned in Example 5.11 can be found in Kovalevskaya, [1], Forsyth, [1; Vol. 5, §26] and Copson, [1; §12.4, 12.5].

Various further extensions of the concept of symmetry have appeared since the first edition came out. Nonlocal symmetries and conservation laws have received a fair amount of recent attention—see Krasilshchik and Vinogradov, [1], Vladimorov and Volovich, [1], [2], and Akhatov, Gazizov and Ibragimov, [1]. Also see Bluman and Kumei, [2], for a development of the concept of a potential symmetry. (However, there is not yet, as far as I know, an adequate computational calculus for determining symmetries of nonlocal (integro-differential) equations.) Anderson, Kamran and Olver, [1], discuss internal symmetries of differential equations, including a generalization of Bäcklund's theorem showing that for most systems every internal symmetry is equivalent to a first order generalized symmetry. Fushchich and Shtelen, [1], and Baikov, Gazizov and Ibragimov, [1], discuss approximate symmetries of perturbed differential equations.

The use of recursion operators for constructing infinite families of generalized symmetries is based on the recursive construction of the higher order Korteweg–de Vries equations due to Lenard (see Gardner, Greene, Kruskal and Miura, [1]) and was first presented in the general form in Olver, [1]. The extension of the theory of recursion operators to nonlinear partial differential equations in more than two independent variables was the source of considerable difficulty in the 1980's. Zakharov and Konopelchenko, [1], proved that recursion operators naturally generalizing those discussed here do not exist. However, Fokas and Santini, [1], and Santini and Fokas, [1], finally discovered the correct form of these operators for soliton equations; see also the review paper by Fokas, [2]. The concept of a hereditary operator was introduced in Fuchssteiner, [1]. See Li Yi-Shen, [1], for examples of recursion operators which do not satisfy the hereditary property. Master symmetries were introduced by Chen, Lee and Lin, [1], and Fuchssteiner, [2]; see also Oevel, [1].

For linear partial differential equations, higher order symmetries have been directly applied to the method of separation of variables by Miller, Kalnins, Boyer, Winternitz and others using the operator-theoretic approach mentioned in the text; see Miller, [3], and the references therein. The results concerning the number of independent symmetries of a given order of Laplace's equation and the wave equation appear in Delong's thesis, [1]. Later, Shapovalov and Shirokov, [1], proved that every generalized symmetry of these two equation is a linear symmetry, which is a polynomial in the first order symmetries. However, the latter statement is not true for more general linear equations; see Exercise 5.2.

The calculus of pseudo-differential operators discussed in the section on formal symmetries originates in studies of Lax representations of soliton equations due to Gel'fand and Dikii, [3], Adler, [1], who proved Proposition 5.73, and Wilson, [1]. In Adams, Ratiu and Schmidt, [1], [2], an attempt was made to make the formal calculus rigorous, utilizing the analytical theory of pseudo-differential operators, cf. M. Taylor, [1]. Formal symmetries were developed in the work of Shabat and collaborators; see Sokolov and Shabat, [1], Mikhailov, Shabat and Yamilov, [1], [2], Sokolov, [1], and, especially, the survey paper by Mikhailov, Shabat and Sokolov, [1].

If we omit the part concerning trivial symmetries and conservations laws, the version of Noether's Theorem 5.58 stated here dates back to Bessel-Hagen, [1]. (See Exercise 5.33 for Noether's original version, which does not use divergence symmetries.) The correspondence between nontrivial conservation laws and nontrivial variational symmetry groups proved here can be found in Alonso, [1], Olver, [11], and Vinogradov, [5]. Proposition 5.56 concerning the geometric interpreation of the group transformations of a variational symmetry can be found in Edelen, [1; p. 149]. The existence of infinite families of conservation laws for self-adjoint linear systems of differential equations was the cause of some astonishment in the mid-1960's with the discovery of the "zilch tensor" and related objects by Lipkin, [1], T. A. Morgan, [1] and Kibble, [1], in their work on field theories. An explanation using generalized symmetries and the full version of Noether's theorem similar to Proposition 5.62 was soon proposed by Steudel, [3]. Proposition 5.64 discussing the action of symmetries on conservation laws also appears in Khamitova, [1]. See Abellanas and Galindo, [1], for further results on conservation laws of linear systems.

The statement and proof of Noether's Second Theorem 5.66 on infinite-dimensional symmetry groups is from Noether's paper, [1]. The connections with the abnormality of the underlying system of Euler–Lagrange equations, however, is new; see Olver, [11]. One outstanding problem here is to complete the classification of symmetries and conservation laws for over-determined systems of Euler–Lagrange equations. In particular, does there exist an over-determined system for which a trivial variational symmetry group gives rise to a nontrivial conservation law? Such a system must be quite complicated; for instance, Exercise 5.51 shows that it cannot be homogeneous in u and its derivatives. (Fokas, [2], refers to an example of Ibragimov where this occurs, but the cited paper does not contain the purported example.)

The history of the variational complex and, in particular, the inverse problem in the calculus of variations is quite interesting. Helmholtz, [1], first proposed the problem of determining which systems of differential equations are the Euler–Lagrange equations for some variational problem and found necessary conditions in the case of a single second order ordinary differential equation. Mayer, [1], generalized Helmholtz' conditions to the case of first order Lagrangians involving one independent variable and several dependent variables, and also proved they sufficed to guarantee the existence of a suitable functional. In two incisive papers on this subject Hirsch, [1], [2], extended these results to the cases of higher order Lagrangians involving either one independent and several dependent variables, or two or three independent and one dependent variable. Hirsch's papers also include further results on what order derivatives can appear in the Lagrangian, as well as the "multiplier problem": when can one multiply a differential equation by a differential function so as to make it an Euler–Lagrange equation? However, the general self-adjointness condition and the homotopy formula (5.123) were independently discovered by Volterra, [1; pp. 43, 48]; see also Vainberg, [1], for a modern version. The next major work on the inverse problem was the profound paper of Douglas, [1], in which he states and solves the problem of determining when a system of two second order ordinary differential equations is *equivalent* to the Euler–Lagrange equations for some functional depending on at most first order derivatives of the dependent variables; the complexity of his solution no doubt hindered further research in this direction. Further recent work on this more difficult version of the inverse problem—when a system of differential equations is equivalent to a system of Euler–Lagrange equations—can be found in Anderson and Duchamp, [2], and Henneaux, [1]. The general case, however, remains unsolved to this day. See also Atherton and Homsy, [1], and Anderson and Thompson, [1], for further references on the inverse problem.

In the early 1970's, the inverse problem was seen to be part of a much larger machine—the variational complex and, more generally, the variational bicomplex—which has evolved into a pre-eminent position in the geometric theory of the calculus of variations. Intimations of this machinery can be found in Dedecker's work, [1], on the applications of algebraic topology to the calculus of variations. This complex first appears explicitly in the work of Vinogradov, [1], where deep methods from algebraic topology are used to prove exactness. A closely related complex appears in contemporaneous work of Tulczyjew, [1], [2]. Further developments, including applications to conservation laws, Cartan forms and characteristic classes, are to be found in Kupershmidt, [1], Takens, [1], Anderson and Duchamp, [1], Tsujishita, [1], [2], and Anderson, [1]. (A different complex including the solution to the inverse problem can be found in Olver and Shakiban, [1], and Shakiban, [1].) The formal variational calculus methods used in the development of this complex presented in Section 5.4, and in particular the abstract definition of a functional, owes much to the work of Gel'fand and Dikii, [1], [2], on the Korteweg–de Vries equation. Further developments of this complex can be found in the comprehensive papers of Vinogradov, [2], [3], [4], and in Olver, [4]. The new proof of exactness of the variational complex presented here was discovered by I. Anderson; the homotopy operators (5.147) serve to considerably simplify the earlier computational proofs of exactness of the D-complex of Takens, [1], and Anderson and Duchamp, [1]. The higher order Euler operators themselves first appeared in work of Kruskal, Miura, Gardner and Zabusky, [1], on the Korteweg–de Vries equation, and were subsequently developed by Aldersley, [1], Galindo and Martinez-Alonso, [1], and Olver, [3]. The present presentation is due to I. Anderson.

Exercises

5.1. Prove that the full symmetry group of the Kepler problem in $\mathbb{R}^3$, including those symmetries giving the Runge–Lenz vector, is locally isomorphic to the group SO(3, 1) of "rotations" in $\mathbb{R}^4$ preserving the Lorentz metric $(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2$. (Goldstein, [1; p. 422].)

*5.2. The Schrödinger equation for a hydrogen atom is the quantized version of the Kepler problem and takes the form

$$\Delta u + |\mathbf{x}|^{-1} u = \lambda u,$$

where λ is a constant, $\mathbf{x} \in \mathbb{R}^3$ and $u \in \mathbb{R}$.

(a) Find the geometric symmetries of this equation for different values of λ.

(b) Prove that the "Runge–Lenz vector"

$$\mathbf{R}[u] = (\mathbf{x} \times \nabla) \times \nabla u - \nabla \times (\mathbf{x} \times \nabla u) + 2|\mathbf{x}|^{-1}\mathbf{x}u$$

provides three second order generalized symmetries, their characteristics being the three components of $\mathbf{R}$. Further show that these symmetries are *not* derivable from the first order symmetries of the equation coming from the evolutionary forms of the geometrical symmetries of part (a).

(Miller, [2; p. 376], Kalnins, Miller and Winternitz, [1].)

5.3. Let $u_t = K[u]$ be an n-th order evolution equation, and let $Q[u]$ be an m-th order symmetry. Prove that $(\partial K/\partial u_n)^m = c(\partial Q/\partial u_m)^n$ for some constant c.

5.4. Consider an evolution equation of the form

$$u_t = cu_n + \tilde{K}(u, u_x, \ldots, u_{n-1}),$$

where $c \neq 0$ is a constant and $u_n = D_x^n u$. (These include the Korteweg–de Vries equation and Burgers' equation.)

(a) Prove that every m-th order, x, t-independent, generalized symmetry has a characteristic of the same form: $Q[u] = \hat{c}u_m + \tilde{Q}(u, u_x, \ldots, u_{m-1})$, $\hat{c} \neq 0$. (Compare with Exercise 5.3.)

(b) Use this fact to prove that any two x, t-independent generalized symmetries $\mathbf{v}_Q$, $\mathbf{v}_R$ of such an evolution equation necessarily commute: $[\mathbf{v}_Q, \mathbf{v}_R] = 0$. (Tu, [1].)

5.5. Let Δ be a linear system of differential equations and $\mathscr{D}$ a linear recursion operator. Prove that whenever $u = f(x)$ is a solution to Δ, so is $\tilde{u} = \mathscr{D}f(x)$. How is $\exp(t\mathscr{D})u = u + t\mathscr{D}u + \cdots$ related to the flow generated by the symmetry with characteristic $\mathscr{D}u$? Does this result generalize to recursion operators for nonlinear systems?

5.6. Let $u_t = \mathscr{D}u$ be a linear evolution equation and $(\mathscr{E}u)\partial_u$ a linear symmetry. Prove that $(\mathscr{E}^*u)\partial_u$ is a linear symmetry of the adjoint equation $u_t = -\mathscr{D}^*u$.

5.7. Prove that, under appropriate existence and uniqueness assumptions, two evolutionary vector fields commute, $[\mathbf{v}_Q, \mathbf{v}_R] = 0$, if and only if their one-parameter groups $\exp(\varepsilon \mathbf{v}_Q)$, $\exp(\tilde{\varepsilon}\mathbf{v}_R)$ commute. Interpret.

5.8. An alternative approach to the definition of the flow associated with a generalized vector field $\mathbf{v} = \sum \xi^i \partial_{x^i} + \sum \phi_\alpha \partial_{u^\alpha}$ would be as follows. Consider the infinite prolongation (5.3), and write down the infinite system of ordinary differential equations

$$\frac{dx^i}{d\varepsilon} = \xi^i, \qquad \frac{du^\alpha}{d\varepsilon} = \phi_\alpha, \qquad \frac{du^\alpha_J}{d\varepsilon} = \phi^J_\alpha.$$

Define the flow of $\mathbf{v}$ on the infinite jet space to be the solution of this system with given initial values $(x, u^{(\infty)}) = (x^l, u^\alpha, u^\alpha_J)$:

$$\exp[\varepsilon \operatorname{pr} \mathbf{v}](x, u^{(\infty)}) = (x(\varepsilon), u^{(\infty)}(\varepsilon)).$$

For the "heat vector field" $\mathbf{v} = u_{xx}\partial_u$, compare this method with the evolutionary method (5.18) in the case of analytic initial data. Does this carry over to more general vector fields? (Anderson and Ibragimov, [1].)

5.9. (a) What happens if one applies the recursion operators for Burgers' equation to the symmetry with characteristic $\rho(x, t)e^{-u}$, ρ a solution to the heat equation? (See Example 5.30.)
(b) How are the recursion operators for Burgers' equation and the heat equation related through the Hopf–Cole transformation of Example 2.42?

5.10. (a) Prove that the nonlinear diffusion equation $u_t = D_x(u^{-2}u_x)$ admits the recursion operator $\mathscr{R} = D_x^2 \cdot u^{-1} D_x^{-1}$. How is this related to Exercise 2.22(d)?
(b) Prove that a general nonlinear diffusion equation $u_t = D_x(K(u)u_x)$ admits generalized symmetries if and only if K is either constant or a multiple of $(u + c)^{-2}$. (Bluman and Kumei, [1].)

5.11. The modified Korteweg–de Vries equation is $u_t = u_{xxx} + u^2 u_x$.
(a) Compute the geometrical symmetry group of this equation.
(b) Prove that the operator $\mathscr{R} = D_x^2 + \frac{2}{3}u^2 + \frac{2}{3}u_x D_x^{-1} \cdot u$ is a recursion operator. (The last summand is the operator that takes a differential function, multiplies it by u, then takes D_x^{-1} (if possible) and finally multiplies the result by $\frac{2}{3}u_x$.) What are the first few generalized symmetries of this equation?
(c) Let $v_x = u$, so v will be a solution of the "potential modified Korteweg–de Vries equation" $v_t = v_{xxx} + \frac{1}{3}v_x^3$. Find a recursion operator for this equation.
(d) Prove that if u is any solution to the modified Korteweg–de Vries equation, then its Miura transformation $w = u^2 + \sqrt{-6u_x}$ is a solution to the Korteweg–de Vries equation. How do the symmetries and recursion operators of these two equations match up? (Miura, [1], Olver, [1].)

*5.12. Prove that the operator $\mathscr{R} = D_x^2 + u_x^2 - u_x D_x^{-1} \cdot u_{xx}$ is a recursion operator for the sine–Gordon equation $u_{xt} = \sin u$. (*Hint*: In (5.42), $\tilde{\mathscr{R}} = \mathscr{R}^*$.) What are some generalized symmetries? Can you relate them to those of the potential modified Korteweg–de Vries equation in the previous example? (*Hint*: Try scaling v.) (Olver, [1].)

5.13. Discuss the conservation laws and linear recursion operators for the following linear equations:
(a) The telegraph equation $u_{tt} = u_{xx} + u$. (See Exercise 2.9.)
(b) The axially symmetric wave equation $u_{tt} - u_{xx} + (u/x) = 0$. (See Exercise 3.1.)

5.14. Find a recursion operator for the generalized Burgers' equation

$$u_t = u_{xx} + uu_x + h(x).$$

5.15. Prove that the operator $u^2 D_x^2 - uu_x D_x + uu_{xx} + u^3 u_{xxx} D_x^{-1} u^{-2}$ is a recursion operator for the alternative form $u_t = u^3 u_{xxx}$ of the Harry Dym equation. Is it hereditary? (Leo, Leo, Soliani, Solombrino, and Mancarella, [1].)

**5.16. (a) Prove that $\mathbf{v} = (u_{xxx} + 3vv_x)\partial_u + 4v_{xxx}\partial_v$ is a generalized symmetry for the evolutionary system

$$u_t = u_{xx} + \tfrac{1}{2}v^2, \qquad v_t = 2v_{xx}.$$

Find a higher order generalized symmetry and a recursion operator.
(b) Prove that the system

$$u_t = u_{xxxx} + v^2, \qquad v_t = \tfrac{1}{5}v_{xxxx}$$

has a sixth order generalized symmetry. (It seems likely that this is an example of an equation with only one generalized symmetry, but a rigorous proof of this fact seems to be quite difficult. Computer experiments of Bakirov, [1], have shown that the system has no other generalized symmetries of order $\leqslant 53$.)

5.17. Prove directly that the two types of Lie derivatives of differential operators (5.37), (5.40) are invariant under a change of variables $v = \varphi(x, u)$ (with appropriate action on the differential operator). Are changes of independent variables allowed?

5.18. Prove that the recursion operators for the physical form of Burgers' equation and the Korteweg–de Vries equation are hereditary operators. Discuss the same question for the recursion operators in Exercises 5.14–5.16. (Fuchssteiner, [1].)

5.19. Prove that a second order evolution equation of the form $u_t = \rho(x, u)u_{xx} + \sigma(x, u)u_x^2$ admits a formal symmetry of rank 3 if and only if either $\sigma = (\rho\rho_{uu}/\rho_u) - \frac{1}{2}\rho_u$ or $\rho_u = 0$. Find a recursion operator for the first type of equation. (This equation arises in the study of flow in porous media, Fokas and Yortsos, [1].)

5.20. Given a (formal) Laurent series $P[u, \xi] = \sum_{i=-\infty}^{n} P_i[u]\xi^i$ whose coefficients are differential functions, we can define an associated pseudo-differential operator $\mathcal{D} = P[u, D_x]$ by formally substituting the operator D_x for the variable ξ. Prove that the Leibniz rule for multiplying pseudo-differential operators can be restated in the following convenient form. If $P[u, \xi]$ and $Q[u, \xi]$ are Laurent series determining pseudo-differential operators $\mathcal{D} = P[u, D_x]$, $\mathcal{E} = Q[u, D_x]$, then the product operator $\mathcal{D} \cdot \mathcal{E} = R[u, D_x]$ is determined by the Laurent series

$$R[u, \xi] = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i P}{\partial \xi^i} D_x^i Q.$$

(Gel'fand and Dikii, [3].)

5.21. Prove that the coefficient of D_x^{-n} in $(D_x + u)^{-1}$ is given by $(-1)^{n-1}(D_x + u)^{n-2}u$ for $n \geqslant 2$.

5.22. Prove that any pseudo-differential operator of order $-m$, $m > 0$, has an m-th root.

*5.23. Suppose $L = D_x^2 + u$. Using the (positive) square root pseudo-differential operator $\sqrt{L} = D_x + \frac{1}{2}uD_x^{-1} + \cdots$, define the differential operators $B_m = (L^{m/2})_+$. Here the + subscript means differential operator truncation, so that if $\mathscr{D} = \sum_{i=-\infty}^n P_i D_x^i$, then $\mathscr{D} = \mathscr{D}_+ + \mathscr{D}_-$, where $\mathscr{D}_+ = \sum_{i=0}^n P_i D_x^i$ and $\mathscr{D}_- = \sum_{i<0} P_i D_x^i$.
(a) Prove that the commutator $[B_m, L]$ is zero if m is even and is a nonzero multiplication operator if m is odd. (*Hint*: Use the fact that the power $L^{m/2} = (L^{m/2})_+ + (L^{m/2})_-$ commutes with L and that $(L^{m/2})_-$ has order strictly less than 0.)
(b) Prove that the *Lax representation* $L_t = [B_m, L]$ for m odd defines an evolution equation $u_t = K_m[u]$ of order m. Show that the case $m = 3$ produces the Korteweg–de Vries equation (up to are scaling).
(c) Prove that the higher order equations are the higher order Korteweg–de Vries equations.
(d) Perform an analogous construction for the third order differential operator $L = D_x^3 + uD_x + v$. The case $B_2 = (L^{2/3})_+$ yields the Boussinesq equation, which is, like the Korteweg–de Vries equation, a model for shallow water waves—see Example 7.28.
(Gel'fand and Dikii, [1], Wilson, [1], Dickey, [1].)

*5.24. Discuss the symmetries and conservation laws of the Helmholtz equation $\Delta u + \lambda u = 0$, $x \in \mathbb{R}^3$.

**5.25. Discuss the generalized symmetries of Maxwell's equations. (See Exercise 2.16.) What about conservation laws? (Pohjanpelto, [1].)

*5.26. (a) Derive the conservation laws for the two-dimensional wave equation listed in Example 5.63. Compare the direct method with the method from Example 5.65. Continue the list to find new second order conserved densities for the wave equation.
(b) Let $x \in \mathbb{R}^p$, $t \in \mathbb{R}$, $u \in \mathbb{R}^q$, and consider an autonomous system of partial differential equations $\Delta(x, u^{(n)}) = 0$ involving u and its x and t derivatives in which t does not explicitly appear. Prove that if $T(x, t, u^{(m)})$ is a conserved density, so are the partial derivatives $\partial T/\partial t$, $\partial^2 T/\partial t^2$, etc. Use this result to check your work in part (a).

5.27. (a) Let $u_t = \mathscr{D}u$ be a linear evolution equation. Prove that $\int q(x, t)u\,dx$ is conserved if and only if $q(x, t)$ is a solution to the adjoint equation $q_t = -\mathscr{D}^*q$.
(b) Prove that if $q(x, t) = (x - 2t\partial_x)^m(1)$, then $\int q(x, t)u\,dx$ is a conservation law for the heat equation $u_t = u_{xx}$. What does this imply about the time evolution of the moment $\int x^m u(x, t)\,dx$ when u is a solution to the heat equation?
(c) Do the same as part (b) for the Fokker–Planck equation of Exercise 2.8. (Steinberg and Wolf, [1].)

5.28. Try to generalize Example 5.50 by discussing the validity of the following statement: If $u_t = P(x, u, \ldots, u_{2m})$ is an evolution equation in only one spatial variable and $\partial P/\partial u_{2m} \neq 0$, then the only nontrivial conservation laws have characteristics independent of u and its derivatives.

5.29. Hamilton's principle in geometrical optics requires the minimization of the integral $\int_a^b N(\mathbf{x})|d\mathbf{x}/dt|\,dt$ in which $\mathbf{x}(t) \in \mathbb{R}^3$, $N(\mathbf{x})$ is the optical index of the material and $|\cdot|$ is the usual length on $\mathbb{R}^3$. What are the Euler–Lagrange equations? Prove that the variational symmetries of space translations and rotations lead, respectively, to Snell's law in the form $N\mathbf{n} = \text{constant}$, where $\mathbf{n} = |d\mathbf{x}/dt|^{-1}\,d\mathbf{x}/dt$ is the unit velocity vector, and the "Optical sine theorem" $N\mathbf{n} \times \mathbf{x} = \text{constant}$. Further, prove that the time translational symmetry leads to a trivial conservation law. What does this imply about the Euler–Lagrange equations? (Baker and Tavel, [1].) (Hint: see Exercise 5.34.)

5.30. Let $p = q = 3$. Prove that any functional $\mathscr{L}[u] = \int L(\operatorname{div}\mathbf{u})\,d\mathbf{x}$ depending only on $\operatorname{div}\mathbf{u} = u_x + v_y + w_z$ admits an infinite-dimensional variational symmetry group. Discuss the consequences of Noether's second theorem in this case.

5.31. In Kumei, [1], the author finds "new" conservation laws of the sine–Gordon equation $u_{xt} = \sin u$ by starting with the elementary conservation law

$$D_t(\tfrac{1}{2}u_x^2) + D_x(\cos u) = 0$$

and applying generalized symmetries to it. For instance, the evolutionary symmetry with characteristic $Q = u_{ttt} + \frac{1}{2}u_t^3$ is shown to lead to the conservation law

$$D_t[u_x u_{xttt} + \tfrac{3}{2}u_x u_t^2 u_{xt}] - D_x[(u_{ttt} + \tfrac{1}{2}u_t^3)\sin u] = 0.$$

Prove that this conservation law is trivial! (What is its characteristic?) More generally, prove that any conservation law derived by this method from a symmetry whose characteristic does not explicitly depend on x is necessarily trivial.

5.32. Let $\mathscr{L}[u] = \int \frac{1}{2}u_x^2\,dx$. Show that the vector field $\tilde{\mathbf{v}} = u_x\partial_u$ is a variational symmetry, but the equivalent vector field (for the Euler–Lagrange equation $u_{xx} = 0$) $\tilde{\mathbf{v}} = (u_x + u_{xx})\partial_u$ is *not* a variational symmetry. Thus the equivalence relation on symmetries does *not* respect the variational property.

*5.33. *Noether's Version of Noether's Theorem.* A generalized vector field $\mathbf{v}$ will be called a *strict variational symmetry* of $\mathscr{L} = \int L\,dx$ if

$$\operatorname{pr}\mathbf{v}(L) + L\operatorname{Div}\xi = 0,$$

i.e. there is no divergence term in (5.84), as we had in our original discussion of variational symmetries in Chapter 4.

(a) Prove that for each conservation law of the Euler–Lagrange equation $\mathsf{E}(L) = 0$ there is a corresponding strict variational symmetry which gives rise to it via Noether's theorem.

(b) Prove that such a strict variational symmetry is unique up to addition of a null divergence in the sense that $\mathbf{v} = \sum \xi^i\partial_{x^i} + \sum \phi_\alpha\partial_{u^\alpha}$, and $\tilde{\mathbf{v}} = \sum \tilde{\xi}^i\partial_{x^i} + \sum \tilde{\phi}_\alpha\partial_{u^\alpha}$ are both strict variational symmetries giving rise to the same conservation law if and only if

$$\tilde{\xi}^i = \xi^i + \frac{1}{L}\sum_j D_j Q_{ij} \quad \text{where} \quad Q_{ij} = -Q_{ji}.$$

(c) What are the strict variational symmetries corresponding to the inversional symmetries of the wave equation?

(Noether, [1].)