4

Dimensional regularization

We have seen how convenient it is to regulate the UV divergences of perturbation theory by continuation in the dimension of space-time. To date, no-one has shown how to use the method in the complete theory. But in perturbation theory, as we will now demonstrate, it is consistent and well-defined. Now all results obtained by this method can be obtained by other, more physical methods (say, a lattice regulator). But frequently much more labor is involved. This is not a triviality, for in complicated situations, especially in gauge theories, it enables us to handle the technicalities of renormalization in a simple way.

The idea of dimensional continuation has been used for a long time in statistical mechanics (see, for example, Fisher & Gaunt (1964)). It became very prominent when Wilson & Fisher (1972) discovered the ε-expansion and applied it to field-theoretic methods in statistical mechanics (Wilson (1973), Mack (1972), and Wilson & Kogut (1974)). In the ε-expansion one works in $4 - \varepsilon$ spatial dimensions, and expands in powers of ε. At the same time, in a purely field-theoretic context, a need arose to find a way of regulating non-abelian gauge theories that preserved gauge invariance and Poincaré invariance. This led to dimensional regularization ('t Hooft & Veltman (1972a), Bollini & Giambiagi (1972), Cicuta & Montaldi (1972), and Ashmore (1972)). Speer & Westwater (1971) had actually discovered the method earlier, but their paper is considerably more abstract, and had not attracted much attention.

Now vector spaces either have infinite dimension or a finite integer dimension. So the concept of integration on a space of finite non-integer dimension, d, cannot be taken completely literally. Either it is a set of purely formal rules for obtaining answers or it is an operation that is not literally integration in d dimensions, but only behaves in many respects as if it were integration in d dimensions. It is not sufficient to treat it only as a set of formal rules (even though that is what it becomes in practice), because one must know that the rules are consistent with one another and with the algebraic manipulations one carries out on integrals. To show that no inconsistencies can arise, we must construct an explicit definition.

https://doi.org/10.1017/CBO9780511622656.004 Published online by Cambridge University Press

There are three issues to address: (1) uniqueness, (2) existence, and (3) properties. Uniqueness is necessary, to avoid the possibility of constructing two definitions, each definition being self-consistent but giving different results from the other definition. Existence, shown by construction of an explicit definition, is necessary to prove that no inconsistencies arise. Once having seen that integration in non-integer dimension can be defined, we cannot just assume that all properties associated with ordinary integration are true; indeed they need not be.

So we also have to prove those properties which we need and which are true. We also must prove that the results agree with ordinary integration if d is an integer.

These considerations are quite non-trivial, as can be seen by considering, for example, the anomaly in the Ward identity for the axial current $j^\mu_{(5)} = \bar{\psi}\gamma^\mu\gamma_5\psi$ in the gauge model (2.11.7). If the fermion masses are zero, then a naive application of the fermion equations of motion shows that the current is conserved: $\partial_\mu j^\mu_{(5)} = 0$. In fact, the current is not conserved, as shown by Adler (1969, 1970) and Bell & Jackiw (1969). A counterterm can be added to $j^\mu_{(5)}$ to make it conserved, but only at the expense of removing its gauge invariance.

Among the objects to be extended to d dimensions are the Dirac matrices (γ^μ and γ_5). If we assumed the obvious generalization of their anticommutation relations, then for all values of d we would have

$$\left.\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}1, \\ \{\gamma_5, \gamma^\mu\} &= 0, \\ {\gamma_5}^2 &= 1. \end{aligned}\right\} \tag{4.0.1}$$

But then we would be able to derive the false result that the anomaly for the gauge-invariant axial current is zero. So there has to be an inconsistency ('t Hooft & Veltman (1972a)). More complicated problems in a similar vein arise when treating supersymmetric theories (Jones & LeVeille (1982)).

In this chapter we will start by stating the axioms for d-dimensional integration given by Wilson (1973). These are sufficient to prove uniqueness. Our calculation of a one-loop graph in Section 3.5 was in fact a realization of the uniqueness proof for one particular integral. Then we will construct an explicit definition of d-dimensional integration. The vector space on which we work is in fact infinite dimensional.

Unfortunately, the definition gives a divergent result in most cases, so we will next have to find a powerful enough extension (Section 4.2). We then prove some standard properties (Section 4.3). One particular result involves finding a definition of the metric tensor on an infinite-dimensional space such that its trace is d rather than infinity.

Then we will be in a position to derive some useful formulae (Sections 4.4 and 4.5) for use in Feynman graph calculations. Finally we will show how to define Dirac matrices; this is obviously important if we are to be able to calculate consistently graphs containing the Adler–Bell–Jackiw anomaly.

The utility of a precise definition such as we give is that if inconsistencies arise at some stage, then one can always go back to first principles to discover the error.

4.1 Definition and axioms

Let d be a complex number. We wish to define an operation that we may regard as integration over a d-dimensional space:

$$\int \mathrm{d}^d\mathbf{p}\, f(\mathbf{p}). \tag{4.1.1}$$

Here $f(\mathbf{p})$ is any given function of a vector $\mathbf{p}$, which is in the d-dimensional space. We will suppose that the space is Euclidean. (Minkowski space is regarded as a one-dimensional time together with a $(d-1)$-dimensional Euclidean space.) Following Wilson (1973) we will give an explicit definition in which the space is actually infinite dimensional; it is the integration operation that gives the dimensionality. Making d a positive integer n will effectively insert a δ-function in the integration that will force all vectors involved in defining the function $f(\mathbf{p})$ to lie in some n-dimensional subspace.

What properties must we impose on a functional of f in order to regard it as d-dimensional integration? The following properties or axioms (due to Wilson (1973)) are natural and are necessary in applications to Feynman graphs:

(1) Linearity: For any complex numbers a and b

$$\int \mathrm{d}^d\mathbf{p}[af(\mathbf{p}) + bg(\mathbf{p})] = a\int \mathrm{d}^d\mathbf{p}\, f(\mathbf{p}) + b\int \mathrm{d}^d\mathbf{p}\, g(\mathbf{p}). \tag{4.1.2}$$

(2) Scaling: For any number s

$$\int \mathrm{d}^d\mathbf{p}\, f(s\mathbf{p}) = s^{-d}\int \mathrm{d}^d\mathbf{p}\, f(\mathbf{p}). \tag{4.1.3}$$

(3) Translation invariance: For any vector $\mathbf{q}$

$$\int \mathrm{d}^d\mathbf{p}\, f(\mathbf{p}+\mathbf{q}) = \int \mathrm{d}^d\mathbf{p}\, f(\mathbf{p}). \tag{4.1.4}$$

We will also require rotational covariance of our results.

Linearity is true of any integration, while translation and rotation invariance are basic properties of a Euclidean space, and the scaling property embodies the d-dimensionality.

Not only are the above three axioms necessary, but they also ensure that integration is unique, aside from an overall normalization (Wilson (1973)). In fact, they determine the usual integration measure in an integer-dimensional space (again up to normalization). The proof is simple:

Use linearity to expand $f(\mathbf{p})$ in terms of a set of basis functions. Choose a basis such as the functions

$$f_{s,\mathbf{q}}(\mathbf{p}) = \exp[-s^2(\mathbf{p}+\mathbf{q})^2]. \tag{4.1.5}$$

Then the integral of a basis function can be written in terms of the integral of one single function:

$$\int \mathrm{d}^d\mathbf{p} f_{s,\mathbf{q}}(\mathbf{p}) = s^{-d}\int \mathrm{d}^d\mathbf{p}\exp(-\mathbf{p}^2). \tag{4.1.6}$$

The integral of this one function sets the normalization. It is natural to require that the value be the usual one in integer dimensions and that we can write

$$\int \mathrm{d}^{d_1}\mathbf{p}\,\mathrm{d}^{d_2}\mathbf{q}\exp(-\mathbf{p}^2-\mathbf{q}^2) = \int \mathrm{d}^{d_1+d_2}\mathbf{k}\exp(-\mathbf{k}^2). \tag{4.1.7}$$

Thus the normalization is given by

$$\int \mathrm{d}^d\mathbf{p}\exp(-\mathbf{p}^2) = \pi^{d/2}. \tag{4.1.8}$$

An abstract uniqueness theorem is not sufficient for us. We also need an explicit formula so that a d-dimensional integral can be written as a sequence of ordinary integrals. This will be important in allowing us to prove standard properties of the integration. In addition it ensures that there exists a self-consistent definition. It is *a priori* possible that no consistent definition exists; the uniqueness theorem only applies if the integration operation exists.

A function $f(\mathbf{p})$ that we integrate could in principle be any function of the components of its vector argument. However, we do not, *a priori*, know the meaning of the components of, say, a vector in 3.99 dimensions. We will soon see that there are in fact infinitely many components. In practice, we will work with rotationally covariant functions. So we will assume that f is a tensor function of a finite set of vectors: $\mathbf{p}, \mathbf{q}_1, \ldots, \mathbf{q}_N$ say. For example, a scalar function is a function only of scalar products

$$f = f(p^2, \mathbf{p}\cdot\mathbf{q}_1, {q_1}^2, \ldots). \tag{4.1.9}$$

Thus f is in fact an ordinary function of scalar numbers, rather than some more complicated kind of function. Of course the values of the scalar products lie in restricted ranges. Thus:

$$\mathbf{q}_a^2 \geq 0,$$
$$|\mathbf{q}_a \cdot \mathbf{q}_b| \leq \mathbf{q}_a^2 \mathbf{q}_b^2. \tag{4.1.10}$$

A tensor function is obtained by writing explicit tensors in terms of the vectors $\mathbf{p}, \mathbf{q}_1, \ldots, \mathbf{q}_N$ and of the metric tensor δ^{ij}, with scalar coefficients. For example, we might have

$$f^{ij}(\mathbf{p}, \mathbf{q}) = q^i p^j f_a(\mathbf{p}^2, \mathbf{p}\cdot\mathbf{q}, \mathbf{q}^2) + \delta^{ij} f_b(\mathbf{p}^2, \mathbf{p}\cdot\mathbf{q}, \mathbf{q}^2). \tag{4.1.11}$$

Such functions are the most general that we need to consider. (We will see later how to handle the antisymmetric tensor $\varepsilon_{\kappa\lambda\mu\nu}$ and the Dirac γ-matrices.)

To give a realization of the objects $\mathbf{p}, \mathbf{q}_1, \ldots,$ we assume that they are vectors in an ordinary vector space. The space must be infinite dimensional, as we will show in a moment. So we define the vectors each to be an infinite sequence of components, $\mathbf{p} = (p^1, p^2, \ldots)$, just as we can define a three-dimensional vector $\vec{\mathbf{V}}$ as a sequence of three components (V^1, V^2, V^3). The metric is given by:

$$\mathbf{p}\cdot\mathbf{q} = p^1 q^1 + p^2 q^2 + \cdots.$$

The reason for the infinite dimensionality is that an integral with, say, $d = 3.99$ can be used not only as a regulator for a physical theory in a space-time of dimension $d_0 = 4$, but also as a regulator for a model theory in any higher dimension, e.g., $d_0 = 5$ or 6 or .... The vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots$ in (4.1.9) can be thought of as momenta of external particles, and our vector space must be large enough to accommodate d_0 linearly independent momenta. Since d_0 is arbitrary, we are forced to infinite dimension.

To define the d-dimensional integral of a scalar function, we find a finite-dimensional subspace containing all the $\mathbf{q}_j$'s. Then we write $\mathbf{p}$ as a component $\mathbf{p}_{\|}$ in this space and an orthogonal component $\mathbf{p}_{\mathrm{T}}$:

$$\begin{aligned} \mathbf{p} &= \mathbf{p}_{\|} + \mathbf{p}_{\mathrm{T}} \\ &= \sum_{j=1}^{J} p^j \mathbf{e}_j + \mathbf{p}_{\mathrm{T}}. \end{aligned} \tag{4.1.12}$$

The 'parallel space', in which lie the $\mathbf{q}_j$'s, is spanned by an orthonormal basis $\mathbf{e}_j$ (with $j = 1, \ldots, J$). We define the integral over $\mathbf{p}$ to be the ordinary J-dimensional integral over $\mathbf{p}_{\|}$ performed after integration in $d - J$ dimensions over $\mathbf{p}_{\mathrm{T}}$:

$$\int \mathrm{d}^d\mathbf{p}\, f(\mathbf{p}) = \int \mathrm{d}p^1 \cdots \mathrm{d}p^J \int \mathrm{d}^{d-J}\mathbf{p}_{\mathrm{T}}\, f(\mathbf{p}). \tag{4.1.13}$$

Since $f(\mathbf{p})$ does not depend on the direction of $\mathbf{p}_T$ we now can define

$$\int d^{d-J}\mathbf{p}_T f(\mathbf{p}) = K_{d-J}\int_0^\infty dp_T p_T^{d-J-1} f(\mathbf{p}). \qquad (4.1.14)$$

Here K_ν (with $\nu = d - J$) is effectively the area of the surface of a hypersphere in ν dimensions. The value of K_ν is obtained by considering the special case where f is chosen to be a Gaussian – see (4.1.8) – with the result

$$K_\nu = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)}. \qquad (4.1.15)$$

Hence we have a definition of d-dimensional integration in terms of ordinary integration:

$$\int d^d\mathbf{p} f(\mathbf{p}) = \frac{2\pi^{(d-J)/2}}{\Gamma((d-J)/2)}\int d^J\mathbf{p}_{\|}\int_0^\infty dp_T p_T^{d-J-1} f(\mathbf{p}). \qquad (4.1.16)$$

We must check that the result is independent of the choice of the subspace of the $\mathbf{p}_{\|}$. We must extend the definition to handle the divergences at $\mathbf{p}_T = 0$ when d is small, which we will do in Section 4.2. Then in Section 4.3 we will prove important properties of our definition. But first there are a couple of details to clear up.

The J-dimensional subspace of $\mathbf{p}_{\|}$'s is chosen subject only to the requirement that it include all $\mathbf{q}_j$'s. So it is possible to extend the space to include extra dimensions. To show this has no effect on the value of the integral we must prove

$$K_\nu\int_0^\infty dp\, p^{\nu-1} g(p^2) = \int_{-\infty}^\infty dk\, K_{\nu-1}\int_0^\infty dp_T p_T^{\nu-2} g(p_T^2 + k^2) \qquad (4.1.17)$$

for any function, g, which depends on a scalar argument. This equation is true since the right-hand side is

$$\frac{2\pi^{(\nu-1)/2}}{\Gamma((\nu-1)/2)}\int_0^\infty dp\, p^{\nu-1} g(p^2)\int_0^1 dx\, x^{(\nu-3)/2}(1-x)^{-1/2}, \qquad (4.1.18)$$

where $p_T^2 = xp^2$ and $k^2 = (1-x)p^2$.

To show that different choices of the 'parallel' subspace have no effect on the value of the integral, we merely extend both spaces to a common larger space. The sole problem is that there may be a divergence in (4.1.18) at $x = 0$; this we will cover by Section 4.2.

Up till now we have supposed $f(\mathbf{p})$ is a scalar function. If it is a tensor $f^{ij\cdots}(\mathbf{p})$, we work component-by-component. To define, say, the component

$$\int d^d\mathbf{p} f^{12}(\mathbf{p}),$$

we take the parallel space to include the 1- and 2-directions and any vectors $\mathbf{q}_j$ on which f^{ij} depends. Then we proceed as before.

For example, suppose $f^{ij}(\mathbf{p}) = p^i p^j g(\mathbf{p}^2)$, where g is a scalar function. Then

$$\int f^{12}(\mathbf{p}) \mathrm{d}^d \mathbf{p} = \int \mathrm{d}p^1 \mathrm{d}p^2 \int \mathrm{d}^{d-2}\mathbf{p}_\mathrm{T} p^1 p^2 g[(p^1)^2 + (p^2)^2 + \mathbf{p}_\mathrm{T}^2] = 0,$$

while

$$\begin{aligned}\int \mathrm{d}^d \mathbf{p} f^{11}(\mathbf{p}^2) &= \int_{-\infty}^{\infty} \mathrm{d}p_\parallel \int \mathrm{d}^{d-1}\mathbf{p}_\mathrm{T} p_\parallel^2 g(p_\parallel^2 + \mathbf{p}_\mathrm{T}^2) \\ &= \frac{2\pi^{d/2}}{d\Gamma(d/2)} \int_0^\infty \mathrm{d}p p^{d+1} g(p^2) \\ &= \frac{1}{d} \int \mathrm{d}^d \mathbf{p} p^2 g(p^2).\end{aligned}$$

Generalizing this result, we see that

$$\int \mathrm{d}^d \mathbf{p} p^i p^j g(\mathbf{p}^2) = \frac{\delta^{ij}}{d} \int \mathrm{d}^d \mathbf{p} p^2 g(\mathbf{p}^2).$$

More general cases are treated in Section 4.3.

4.2 Continuation to small d

The convergence of the definition (4.1.16) is d-dependent at $p_\mathrm{T} = 0$ and $p_\mathrm{T} = \infty$. It improves at $p_\mathrm{T} = \infty$ when d gets smaller, but it improves at $p_\mathrm{T} = 0$ when d gets bigger. Even for a function that decreases exponentially at large p, and that is analytic for finite p, the defining integral has a divergence if the transverse space has a dimension $d - J \leq 0$; this is forced to happen if d is negative or zero. So our first task in this section is to find an explicit formula for the continuation of (4.1.16) to arbitrarily negative d. We will see that the p_T-integral has poles whenever $(d - J)/2$ is zero or a negative integer, but that these are cancelled by the zeros in $1/\Gamma((d - J)/2)$.

We will then be able to adopt the resulting formula as a definition of the d-dimensional integral of a function for which (4.1.16) converges for no value of d. An examp'e of such a function is

$$f(\mathbf{p}) = \frac{1}{(\mathbf{q}_1 + \mathbf{p})^2 + (\mathbf{q}_2 + \mathbf{p})^2 + m^2}.$$

The parallel space must be at least two-dimensional to accommodate $\mathbf{q}_1$ and $\mathbf{q}_2$, so we may set $J = 2$. Then the transverse integral converges at $p_\mathrm{T} = 0$ only if $d > J = 2$, while the complete integral converges at $p = \infty$ only if $d < 2$.

We will have a definition that defines $\int d^d\mathbf{p} f(\mathbf{p})$ for all small enough d. For larger values of d we define the integral by analytic continuation. In general there will be ultra-violet poles at certain values of d – just as in the Feynman graph we computed in Sections 3.5 and 3.6.

To explicitly define the continuation to small d, it is sufficient to consider a function $f(\mathbf{p}^2)$. Let us assume that $f \to 0$ rapidly enough as $p \to \infty$ that

$$\int d^d\mathbf{p} f(\mathbf{p}^2) \equiv \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dp\, p^{d-1} f(p^2) \tag{4.2.1}$$

converges at $p \to \infty$ for some positive value of d. We also assume that $f(\mathbf{p}^2)$ is analytic at $p = 0$. Then (4.2.1) converges and is analytic in d for some range $0 < \operatorname{Re} d < d_{\max}$. We define the integral for all other values of d by analytic continuation in d. Explicit formulae for the continuation to smaller d's are constructed by adding and subtracting the leading behavior at $p \to 0$. For example, the following formula gives the integral in the range $-2 < \operatorname{Re} d < d_{\max}$:

$$\int d^d\mathbf{p} f(\mathbf{p}^2) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \left\{ \int_C^\infty dp\, p^{d-1} f(p^2) + \int_0^C dp\, p^{d-1} [f(p^2) - f(0)] + f(0) C^d/d \right\}. \tag{4.2.2}$$

This is independent of the arbitrary constant C.

When $-2 < \operatorname{Re} d < 0$ we may let $C \to \infty$ to obtain

$$\int d^d\mathbf{p} f(\mathbf{p}^2) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dp\, p^{d-1} [f(p^2) - f(0)], \tag{4.2.3}$$

while at $d = 0$ the zero in $1/\Gamma(d/2)$ is cancelled by the pole term to give

$$\int d^0\mathbf{p} f(\mathbf{p}^2) = f(0). \tag{4.2.4}$$

We extend this procedure to continue to $-2l - 2 < \operatorname{Re} d < -2l$ for any positive integer l:

$$\int d^d\mathbf{p} f(\mathbf{p}^2) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dp\, p^{d-1} \{ f(p^2) - f(0) - p^2 f'(0) - \cdots \ldots - (p^2)^l f^{(l)}(0)/l! \},$$

$$\int d^{-2l}\mathbf{p} f(\mathbf{p}^2) = (-\pi)^{-l} f^{(l)}(0). \tag{4.2.5}$$

This equation gives us the integral when $-2l - 2 < \operatorname{Re} d < -2l$ on the assumption that the original formula (4.2.1) converges when d is just greater than zero. Suppose now that (4.2.1) diverges at $p = \infty$ for all positive values of d, but that f is power behaved as $p \to \infty$. Then it is sensible to adopt (4.2.5) as the definition of the integral. This particular definition is very important since we will use dimensional continuation to regulate Feynman graphs that are ultra-violet divergent at $d = 4$. The definition (4.1.16) applied to a Feynman graph frequently has a negative number $d - 4$ of transverse dimensions in order to ensure ultra-violet convergence of the complete integral. Then we may apply the definition (4.2.5) to the p_T-integral with d replaced by $d - J$.

Another obstacle to continuation in d is sometimes that $f(p^2)$ is not analytic at $p^2 = 0$ but has a power-law singularity there. We may generalize the derivation of (4.2.5) to write down a formula for the continuation of the integral.

An example of the use of (4.2.5) as a definition is given by choosing

$$f(p^2) = (p^2 + A)/(p^2 + B),$$

where A and B are numbers. The definition (4.2.1) diverges for all d, but with $l = 1$, (4.2.5) gives us a definition valid for $-2 < \mathrm{Re}\, d < 0$:

$$\int \mathrm{d}^d\mathbf{p} f(\mathbf{p}) = \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty \mathrm{d}p^2\, p^{d-2} \left[\frac{p^2 + A}{p^2 + B} - \frac{A}{B} \right].$$

The integral can be explicitly computed to give:

$$\int \mathrm{d}^d\mathbf{p} (\mathbf{p}^2 + A)/(\mathbf{p}^2 + B) = (\pi B)^{d/2}(A/B - 1)\Gamma(1 - d/2),$$

which can be continued to all d.

Suppose f has a power-law singularity, as for example

$$f = \frac{1}{(\mathbf{p} + \mathbf{q})^2(\mathbf{p}^2 + m^2)}.$$

The definition (4.1.16) of the integral of this function converges if $2 < d < 4$. To continue it to lower d we must subtract the power behavior at $\mathbf{p} = -\mathbf{q}$, just as we did for singularities at $p_T = 0$, or at $p = 0$ in (4.2.2). Then we can define the integral of say

$$\frac{\mathbf{p}^2}{(\mathbf{p} + \mathbf{q})^2(\mathbf{p}^2 + m^2)}.$$

One result of all these definitions is that the integral of a power of p is zero:

$$\int \mathrm{d}^d\mathbf{p}(\mathbf{p}^2)^\alpha = 0 \tag{4.2.6}$$

for any value of α (integer or not). It should not be thought that there is any choice in (4.2.6). It follows from (*a*) the explicit continuation of (4.1.16) to small d, and (*b*) application of the continued formula as a definition of $\int \mathrm{d}^d\mathbf{p}(\mathbf{p}^2)^\alpha$.

Consistency of the formalism also requires (4.2.6). For suppose that $f(\mathbf{p}) = (\mathbf{p}^2)^\alpha/(\mathbf{p}^2 + m^2)$. Then when $-2\alpha - 2 < d \le -2\alpha$ we have

$$\int \mathrm{d}^d\mathbf{p} f(\mathbf{p}^2) = \int \mathrm{d}^d\mathbf{p}[f(\mathbf{p}^2) - (\mathbf{p}^2)^\alpha/m^2].$$

If linearity is to be true then we have (4.2.6).

We could subtract out the singularity differently, by a function that is not just a power of p. But then, for example, the simplification obtained in (4.2.2) by taking $C \to \infty$ would no longer occur.

Observe that if in the first of our definitions (4.1.16), we take f to be a positive-definite function, then the integral is positive. But when the integral is continued away from the region where this definition converges, then the subtraction terms mean that the integrand is no longer positive definite, so that the integral need not be positive.

At the end of Section 4.1, we proved that the value of a d-dimensional integral does not depend on how we split the integral into an ordinary integral over some integer-dimensional 'parallel space' and a spherically symmetric integral over the remaining dimensions. We let J be the dimension of the parallel space. Then the proof consists of examining what happens when J is increased by one. Ultimately we had to prove (4.1.17), which is a property of ordinary integrals. We assumed $d > J$, so that there were no subtraction terms. To generalize the result to the case that $d - J$ is not positive, we must prove that

$$\begin{aligned} K_{d-J} \int_0^\infty \mathrm{d}p\, p^{d-J-1}\left[f(p^2) - \sum_{n=0}^{[J/2-d/2]} f^{(n)}(0)p^{2n}/n!\right] \\ = K_{d-J-1} \int_{-\infty}^\infty \mathrm{d}k \int_0^\infty \mathrm{d}p_\mathrm{T} p_\mathrm{T}^{d-J-2}\Bigg[f(k^2 + p_\mathrm{T}^2) \\ - \sum_{n=0}^{[(J+1-d)/2]} f^{(n)}(k^2)p_\mathrm{T}^{2n}/n!\Bigg]. \end{aligned} \tag{4.2.7}$$

Here the symbol $[a]$ denotes the largest integer smaller than a. To prove the equation we change variables on the right-hand side to x and p, where

$p_\mathrm{T}^2 = xp^2$ and $k^2 = (1-x)p^2$. For the right-hand side we get

$$K_{d-J-1}\int_0^1 \mathrm{d}x\, x^{(d-J-3)/2}(1-x)^{-1/2}\int_0^\infty \mathrm{d}p\, p^{d-J-1} \times$$
$$\times\left[f(p^2) - \sum_{n=0}^{[(J+1-d)/2]} f^{(n)}((1-x)p^2)x^n p^{2n}/n!\right]$$
$$= \tfrac{1}{2}K_{d-J-1}\int_0^1 \mathrm{d}x\, x^{(d-J-3)/2}(1-x)^{-1/2}\int_0^\infty \mathrm{d}p^2 (p^2)^{d/2-J/2-1} \times$$
$$\times\left\{\left[f(p^2) - \sum_{n=0}^{[J/2-d/2]} f^{(n)}(0)p^{2n}/n!\right]\right.$$
$$\left. - \sum_{n=0}^{[(J+1-d)/2]}\left[f^{(n)}((1-x)p^2)\frac{x^n p^{2n}}{n!} - \sum_{j=n}^{[(J-d)/2]} f^{(j)}(0)\frac{(1-x)^{j-n}x^n p^{2j}}{n!(n-j)!}\right]\right\}. \tag{4.2.8}$$

Here we have added and subtracted

$$\sum_{n=0}^{[(J-d)/2]} f^{(n)}(0)p^{2n}/n!,$$

so that the integral over p^2 of p^{d-J-2} times each square bracket term is convergent. After scaling p^2 by $(1-x)$, we get:

$$\tfrac{1}{2}K_{d-J-1}\int_0^1 \mathrm{d}x\, x^{(d-J-3)/2}(1-x)^{-1/2}\int_0^\infty \mathrm{d}p^2(p^2)^{d/2-J/2-1} \times$$
$$\times\left\{\left[f(p^2) - \sum_{n=0}^{[J/2-d/2]} f^{(n)}(0)p^{2n}/n!\right]\right.$$
$$- \sum_{n=0}^{[(J+1-d)/2]} \frac{x^n(1-x)^{J/2-d/2-n}p^{2n}}{n!}$$
$$\left.\left[f^{(n)}(p^2) - \sum_{j=n}^{[(J-d)/2]} f^{(j)}(0)\frac{p^{2j}}{(n-j)!}\right]\right\}.$$

Integration by parts in the p-integral gives

$$K_{d-J-1}\int_0^1 \mathrm{d}x\, x^{(d-J-3)/2}(1-x)^{-1/2} \times$$
$$\times\left[1 - \sum_{n=0}^{[(J+1-d)/2]} x^n(1-x)^{J/2-d/2-n}\frac{(1+J/2-d/2)!}{n!(1+J/2-d/2-n)!}\right]\times$$
$$\times\int_0^\infty \mathrm{d}p\, p^{d-J-1}\left[f(p^2) - \sum_{n=0}^{[J/2-d/2]} f^{(n)}(0)p^{2n}/n!\right].$$

An integration by parts on the x-integral is used to show that it equals $\Gamma(1/2)\Gamma[(d-J-1)/2]/\Gamma[(d-J)/2]$, from which the required result follows.

4.3 Properties

Property 1. *Axioms*: The definitions (4.1.16) and (4.2.5) satisfy Wilson's axioms (4.1.2), etc., for d-dimensional integration.

Proof. We reduced d-dimensional integration to ordinary integration so linearity follows from linearity of ordinary integration. We must choose the $\mathbf{p}_{\parallel}$ space to be large enough that it is the same for both functions f and g in (4.1.2). Our explicit continuation (4.2.5) to arbitrary negative d ensures that reducing the dimension of the transverse space is no problem.

Scaling and rotation covariance are explicit properties of all our definitions.

Translation invariance is valid for ordinary integration, so it follows from definition (4.1.16) provided the $\mathbf{p}_{\parallel}$ space is big enough to include the vector $\mathbf{q}$ used in the axiom (4.1.4).

Property 2.

$$\int \mathrm{d}^d\mathbf{p}\frac{(\mathbf{p}^2)^\alpha}{(\mathbf{p}^2+M^2)^\beta} = \pi^{d/2}M^{d+2\alpha-2\beta}\frac{\Gamma(\alpha+d/2)\Gamma(\beta-\alpha-d/2)}{\Gamma(d/2)\Gamma(\beta)}. \tag{4.3.1}$$

Proof. Immediate from (4.2.5). Note that this implies that the integral of a power of p^2 is zero, since $\Gamma(\beta)\sim 1/\beta$ as $\beta\to 0$.

Property 2*a*.

$$\int \mathrm{d}^d\mathbf{p}(\mathbf{p}^2)^\alpha = 0. \tag{4.3.1a}$$

Proof. Already done.

Property 3.

$$\frac{\partial}{\partial\mathbf{q}}\int \mathrm{d}^d\mathbf{p}f(\mathbf{p},\mathbf{q},\ldots) = \int \mathrm{d}^d\mathbf{p}\frac{\partial}{\partial\mathbf{q}}f(\mathbf{p},\mathbf{q},\cdots). \tag{4.3.2}$$

Proof. Contract with a vector $\delta\mathbf{q}$ which projects out the derivative with respect to the component of $\mathbf{q}$ in the $\delta\mathbf{q}$ direction. Then make the parallel space (of $\mathbf{p}_{\parallel}$'s) big enough to include $\delta\mathbf{q}$ and use (4.3.2) on ordinary integrals. This is true for all $\delta\mathbf{q}$.

Property 4.

$$\delta^{ij}\delta_{ij} = d. \tag{4.3.3}$$

Proof and definition of δ_{ij}. Now δ^{ij} is defined to be the component form of a contravariant tensor with $\delta^{ij} = 1$ if $i = j$ and zero otherwise. The obvious definition of the covariant tensor δ_{ij} is as the inverse matrix, i.e., the same thing. This gives $\delta^{ij}\delta_{ij} = \infty$. However in an infinite-dimensional space, there is space for a different definition.

A contravariant tensor may be defined by specifying its components. But a covariant tensor ω is fundamentally a linear function acting on covariant tensors: $\omega(\mathbf{T})$. We can write $\omega(\mathbf{T}) = \omega_{ij}T^{ij}$ only if the sum converges.

We need the covariant δ (which we symbolize by δ_{ij}) to be rotation invariant, and to give $\delta(\mathbf{T}) = T^{ii}$ whenever the sum exists. We would also like contraction with δ_{ij} to commute with integration. For example

$$\left.\begin{aligned} &\delta_{ij}\int \mathrm{d}^d\mathbf{p}\,p^i p^j f(\mathbf{p}^2) = \delta_{ij}\int \mathrm{d}^d\mathbf{p}\,\delta^{ij}\mathbf{p}^2 f(\mathbf{p}^2)/d \\ \text{and}\qquad &\delta_{ij}\int \mathrm{d}^d\mathbf{p}\,p^i p^j f(\mathbf{p}^2) = \int \mathrm{d}^d\mathbf{p}\,\mathbf{p}^2 f(\mathbf{p}^2). \end{aligned}\right\} \tag{4.3.4}$$

Since we have an infinite sum, we cannot immediately apply linearity to prove this equation.

Let us define

$$\delta(\mathbf{T}) = \frac{d\Gamma(d/2)}{\pi^{d/2}}\int \mathrm{d}^d\mathbf{p}\sum_{i,j} T^{ij}p_i p_j \delta(\mathbf{p}^2 - 1). \tag{4.3.5}$$

Whenever $\sum T^{ii}$ converges, this definition gives

$$\delta(\mathbf{T}) = \sum T^{ii}$$

But if the sum diverges, then it is possible to get a finite value for $\delta(\mathbf{T})$. In particular,

$$\delta(\delta^{ij}) = \frac{d\Gamma(d/2)}{\pi^{d/2}}\int \mathrm{d}^d\mathbf{p}\,\mathbf{p}^2\delta(\mathbf{p}^2 - 1) = d,$$

as required. The definition is rotationally invariant. Commutation of contraction with δ_{ij} and integration will now be a consequence of commutativity of two integrals – which we will prove later.

Property 5. Integration by parts:

$$\int \mathrm{d}^d\mathbf{p}\,\partial f(\mathbf{p})/\partial p^i = 0. \tag{4.3.6}$$

Proof. Work component-by-component. Contract with an arbitrary vector k:

$$\int \mathrm{d}^d\mathbf{p}k^i \frac{\partial}{\partial p^i} f.$$

Then, to define this integral, we must put $\mathbf{k}$ in the parallel space, and we can use the proof of (4.3.6) for ordinary integration in the space parallel to $\mathbf{k}$.

Property 6. To define integration over two (or more) variables:

$$\int \mathrm{d}^d\mathbf{p}\mathrm{d}^d\mathbf{k} f(\mathbf{p},\mathbf{k};\mathbf{q}_1,\ldots\mathbf{q}_N)$$

we must choose to calculate one integral then the other, according to the rules already stated.

For this definition to be sensible we need the result to be independent of the order of integration:

$$\int \mathrm{d}^d\mathbf{p} \int \mathrm{d}^d\mathbf{k} f = \int \mathrm{d}^d\mathbf{k} \int \mathrm{d}^d\mathbf{p} f. \tag{4.3.7}$$

We could also allow the dimensions of the $\mathbf{p}$- and $\mathbf{k}$-integrals to be different. Then exchange of order of integration $\int \mathrm{d}^d\mathbf{p} \int \mathrm{d}^{d'}\mathbf{k} \to \int \mathrm{d}^{d'}\mathbf{k} \int \mathrm{d}^d\mathbf{p}$. is allowed only if $d = d'$, or if f is independent of $\mathbf{p}\cdot\mathbf{k}$.

Proof. It is sufficient to consider the case that there are no $\mathbf{q}_i$'s, so that $f = f(\mathbf{p}^2, \mathbf{p}\cdot\mathbf{k}, \mathbf{k}^2)$. (If there are $\mathbf{q}_i$'s, then we take out a finite-dimensional integral for both $\mathbf{k}$ and $\mathbf{p}$ which spans all $\mathbf{q}_i$'s and then we apply the theorem to the remaining dimensions.)

The left-hand side of (4.3.7) is

$$\frac{4\pi^{d-1/2}}{\Gamma(d/2)\Gamma((d-1)/2)} \int_0^\infty \mathrm{d}p \int_{-\infty}^\infty \mathrm{d}k_1 \int_0^\infty \mathrm{d}k_\mathrm{T} p^{d-1} k_\mathrm{T}^{d-2} f(p^2, k_1 p, k_1^2 + k_\mathrm{T}^2), \tag{4.3.7L}$$

while the right-hand side is

$$\frac{4\pi^{d-1/2}}{\Gamma(d/2)\Gamma((d-1)/2)} \int_0^\infty \mathrm{d}k \int_{-\infty}^\infty \mathrm{d}p_1 \int_0^\infty \mathrm{d}p_\mathrm{T} p_\mathrm{T}^{d-2} k^{d-1} f(p_1^2 + p_\mathrm{T}^2, p_1 k, k^2). \tag{4.3.7R}$$

Here k_1 is the component of $\mathbf{k}$ parallel to $\mathbf{p}$, while p_1 is the component of $\mathbf{p}$ parallel to $\mathbf{k}$. Change variables to, say, $\mathbf{p}^2$, $\mathbf{k}^2$, and $z = \mathbf{p}\cdot\mathbf{k}/(pk) = p_1/\sqrt{(p_1^2 + p_\mathrm{T}^2)} = k_1/\sqrt{(k_1^2 + k_\mathrm{T}^2)}$, with the result that both (4.3.7L) and (4.3.7R) are equal to

$$\frac{4\pi^{d-1/2}}{\Gamma(d/2)\Gamma((d-1)/2)} \int_0^\infty \mathrm{d}p\, p^{d-1} \int_0^\infty \mathrm{d}k\, k^{d-1} \int_{-1}^1 \mathrm{d}z (1-z^2)^{(d-3)/2} f(p^2, pkz, k^2). \tag{4.3.7C}$$

The theorem is thus proved in the case that (4.3.7L) and (4.3.7R) are both convergent. Note that it is not a trivial consequence of the corresponding result for integer-dimensional integration.

If the dimensions of the integrations are not the same, then let the **k** integral have dimension d'. The left-hand side gives

$$\frac{4\pi^{(d+d'-1)/2}}{\Gamma(d/2)\Gamma((d'-1)/2)}\int_0^\infty \mathrm{d}p\, p^{d-1}\int_0^\infty \mathrm{d}k\, k^{d'-1}\int_{-1}^1 \mathrm{d}z(1-z^2)^{(d'-3)/2}f,$$

which in general is not the same as the corresponding expression for the right-hand side. But if f is independent of z, then the z-integral can be computed explicitly. The result is

$$\int \mathrm{d}^{d'}\mathbf{k}\,\mathrm{d}^d\mathbf{p}\, f(\mathbf{p}^2,\mathbf{k}^2) = \frac{4\pi^{(d+d')/2}}{\Gamma(d/2)\Gamma(d'/2)}\int_0^\infty \mathrm{d}p\, p^{d-1}\int_0^\infty \mathrm{d}k\, k^{d'-1} f(p^2,k^2). \tag{4.3.8}$$

A problem is that if d is not positive, we must make subtractions as in (4.2.5). These are clearly asymmetric between the two orders (4.3.7L) and (4.3.7R) of performing the original integral (with now $d' = d$); in practice, d will be the number of dimensions transverse to the external vectors $\mathbf{q}_1, \ldots, \mathbf{q}_N$. In applications to Feynman graphs d will therefore be negative in order to regulate UV divergences. So we must use (4.2.5) to define the integrals. Then (4.3.7) does not give (4.3.7L), (4.3.7R) and (4.3.7C).

We solve this problem by defining an auxiliary integral with a convergence factor, say

$$I(a,d) = \int \mathrm{d}^d\mathbf{p}\int \mathrm{d}^{d'}\mathbf{k}\, f(\mathbf{p},\mathbf{k})\exp[-a(\mathbf{p}^2+\mathbf{k}^2)]. \tag{4.3.9}$$

Assume f is power-behaved at infinity. Then for all d, (4.3.7L or R) is UV convergent. Moreover, if $d > 1$ then both (4.3.7L) and (4.3.7R) are IR convergent without subtractions. The function $I(a,d)$ is analytic in a and d. Continue down to small enough d that (4.3.7) is UV convergent. Then $I(a,d)$ is given both by (4.3.7L) and (4.3.7R) with f replaced by $f\exp[-a(\mathbf{p}^2+\mathbf{k}^2)]$, and with subtractions made. Now set $a = 0$ to prove the theorem (4.3.7).

Property 7.

$$\int \mathrm{d}^d\mathbf{k}\int \mathrm{d}^{d'}\mathbf{p}\, f(\mathbf{p}^2+\mathbf{k}^2) = \int \mathrm{d}^{d+d'}\mathbf{q}\, f(\mathbf{q}^2). \tag{4.3.10}$$

Proof. Since f is independent of $\mathbf{p}\cdot\mathbf{k}$, the previous theorem shows that the left-hand side is independent of the order of integration, even if the dimensions of the **p**- and **k**-integrals are different. Then use (4.3.9) and change variables to $q = (\mathbf{p}^2 + \mathbf{k}^2)^{1/2}$ and $x = \mathbf{p}^2/\mathbf{q}^2$.

Property 8.

$$\int d^d\mathbf{p}\, p^{i_1} \cdots p^{i_t} g(\mathbf{p}^2) = \begin{cases} 0, & \text{if } t \text{ is odd}, \\ T^{i_1 \cdots i_t} A_t[g], & \text{if } t \text{ is even}, \end{cases} \tag{4.3.11}$$

with

$$T^{i_1 \cdots i_t} = [\delta^{i_1 i_2}\delta^{i_3 i_4} \cdots \delta^{i_{t-1} i_t} + \cdots \\ + \text{all permutations of the } i\text{'s}]/t!, \tag{4.3.12}$$

and

$$\begin{aligned} A_t(g) &= \frac{\Gamma(d/2)\Gamma(t/2 + 1/2)}{\Gamma(1/2)\Gamma(d/2 + t/2)} \int d^d\mathbf{p}(\mathbf{p}^2)^{t/2} g(\mathbf{p}^2) \\ &= \frac{2\pi^{d/2}\Gamma(t/2 + 1/2)}{\Gamma(1/2)\Gamma(d/2 + t/2)} \int_0^\infty dp\, p^{d+t-1} g(p^2). \end{aligned} \tag{4.3.13}$$

Proof. If t is odd, antisymmetry of the integrand under $\mathbf{p} \to -\mathbf{p}$ makes the integral over the 'parallel' space zero.

Antisymmetry under reversal of one component of **p**, symmetry under permutations of the i's, and rotation invariance give the general form (4.3.11) and (4.3.12). Computation of one component (say, $i_1 = i_2 = \cdots = i_t = 1$) then gives (4.3.13).

Examples.

$$\int d^d\mathbf{p}\, p^i p^j g(\mathbf{p}^2) = (1/d)\delta^{ij} \int d^d\mathbf{p}\, \mathbf{p}^2 g(\mathbf{p}^2), \tag{4.3.14}$$

$$\int d^d\mathbf{p}\, p^i p^j p^k p^l g(\mathbf{p}^2) = \frac{(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})}{d(d+2)} \int d^d\mathbf{p} |\mathbf{p}|^4 g(\mathbf{p}^2). \tag{4.3.15}$$

Property 9. Consider an integral

$$I(\mathbf{p}_1, \ldots, \mathbf{p}_J) = \int d^d\mathbf{k}\, f(\mathbf{k}, \mathbf{p}_1, \ldots, \mathbf{p}_J) \tag{4.3.16}$$

which is UV convergent by power-counting at $d = 4$; that is $f = O(1/k^{4+a})$ as **k** goes to infinity in any direction, for some positive number a. Then the integral is analytic in d and in the parameters $\mathbf{p}_i$, when d is close to four, if the integrand is analytic. If the $\mathbf{p}_i$'s lie in the first four dimensions, then the integral at $d = 4$ has the same value as the ordinary four-dimensional integral of f.

Example Suppose f has the form

$$f = \bar{f}(\bar{\mathbf{k}}, \bar{\mathbf{p}}_1, \ldots, \bar{\mathbf{p}}_J)\exp(-A\hat{\mathbf{k}}^2 + 2A\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}), \tag{4.3.17}$$

where for any vector $\mathbf{v}$ we let $\bar{\mathbf{v}}$ be its projection onto the first four dimensions and let $\hat{\mathbf{v}}$ be its projection onto the remaining dimensions. Then we let $\bar{I}$ be the ordinary four-dimensional integral of $\bar{f}$. By use of our definitions of d-dimensional integration, we have

$$\begin{aligned} I &= \int \mathrm{d}^d\mathbf{k}\, f \\ &= \bar{I}\int \mathrm{d}^{d-4}\hat{\mathbf{k}}\exp(-A\hat{\mathbf{k}}^2 + 2A\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}) \\ &= \bar{I}(\pi/A)^{d/2-2}\exp(A\hat{\mathbf{p}}^2). \end{aligned} \tag{4.3.18}$$

This is manifestly analytic in d and $\hat{\mathbf{p}}$. If we set $d = 4$, the integral becomes

$$I = \bar{I}\exp(A\hat{\mathbf{p}}^2).$$

Notice that there is no restriction on $\hat{\mathbf{p}}$, even though $\hat{\mathbf{p}} = 0$ in four dimensions. However, if we let $d \to 4$ and $\hat{\mathbf{p}} \to 0$, the limit is smooth.

Proof of Property 9. The proof is easily made by examining the definition of the d-dimensional integral in terms of ordinary integrals. As usual we divide the space into a finite-dimensional parallel space big enough to contain $\mathbf{p}_1, \ldots, \mathbf{p}_J$, and into a transverse space containing the remaining dimensions. It is convenient to choose the parallel space to have an odd number $2N + 1$ of dimensions. Then:

$$\begin{aligned} I = \int_{-\infty}^{\infty}\mathrm{d}k_1\cdots\int_{-\infty}^{\infty}\mathrm{d}k_{2N+1}\frac{\pi^{(d-1)/2-N}}{\Gamma((d-1)/2-N)}\int_0^{\infty}\mathrm{d}k_{\mathrm{T}}^2(k_{\mathrm{T}}^2)^{(d-3)/2-N}\times \\ \times\left[F(k_1, k_2, \ldots, k_{2N+1}, k_{\mathrm{T}}^2) - \sum_{n=0}^{N-2}F^{(n)}(k_1, \ldots, k_{2N+1}, 0)k_{\mathrm{T}}^{2n}/n!\right]. \end{aligned} \tag{4.3.19}$$

Here we have used F to denote f considered as a function of the first $2N + 1$ components of $\mathbf{k}$ and of k_{T}^2. Since f is an analytic function of $\mathbf{k}$, it can be expanded in powers of k_{T}^2.

As required by the definition, we have subtracted off a power series in k_{T}^2, to give convergence at $k_{\mathrm{T}} = 0$. We use $F^{(n)}$ to denote the nth derivative of F with respect to k_{T}^2. The integrand of the k_{T}^2 integral behaves as k_{T}^{d-5}, so we have convergence at $k_{\mathrm{T}} = 0$ if $d > 3$. Since f is analytic, there are no other singularities at finite k, and the only other possible source of a divergence is from large k. The subtractions do not introduce a divergence provided that $d < 5$. Moreover, we have assumed that $f = O(1/k^{4+a})$ as $k \to \infty$, so that there are no other large k divergences when d is close to four. Hence (4.3.19) converges and is analytic in a neighborhood of $d = 4$.

Now the integral will depend on the $\mathbf{p}_i$'s only through the Lorentz scalars $\mathbf{p}_a \cdot \mathbf{p}_b$ (with $1 \le a \le b \le J$). To determine this dependence, it is sufficient to keep the $\mathbf{p}_i$'s within some fixed J-dimensional subspace. Since (4.3.19) is a perfectly finite integral, it is an analytic function of the $\mathbf{p}_i$'s.

To determine the value of the integral when $d = 4$ and when the $\mathbf{p}_i$'s are in the first four dimensions, we use the freedom to vary the dimension of the 'parallel' space in the definition (4.3.19). Let us now make it four dimensional. We will obtain an integral of the form:

$$I = \int_{-\infty}^{\infty} \mathrm{d}k_1 \ldots \mathrm{d}k_4 \frac{2\pi^{d/2-2}}{\Gamma(d/2-2)} \int_0^{\infty} \mathrm{d}\hat{k}\hat{k}^{d-5} \hat{f}(k_1, k_2, k_3, k_4, \hat{k}^2) \quad (4.3.20)$$

if $d > 4$. When we let $d \to 4$, the integral over $\hat{k}$ is singular at $\hat{k} = 0$; the resulting divergence cancels the zero of the inverse Γ-function to give

$$I(d=4) = \int \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \hat{f}(k_1, k_2, k_3, k_4, 0), \quad (4.3.21)$$

as required.

We may alternatively continue from $d < 4$ using

$$I = \int \mathrm{d}k_1 \ldots \mathrm{d}k_4 \frac{2\pi^{d/2-2}}{\Gamma(d/2-2)} \int_0^{\infty} \mathrm{d}\hat{k}\hat{k}^{d-5} [\hat{f}(k_1, \ldots, k_4, \hat{k}^2) - \hat{f}(k_1, \ldots, k_4, 0)]. \quad (4.3.22)$$

The singularity at $\hat{k} = 0$ is cancelled, but as $d \to 4$ we get a divergence at $\hat{k} = \infty$ which gives the same result (4.3.21).

Comment In this proof we used the freedom to alter the dimension of the parallel space. To show that the integral is well-behaved at $d = 4$, it was convenient to choose the parallel space to have an odd dimension. But to compute the actual value at $d = 4$, it was convenient to choose the parallel space to have an even dimension, specifically, four. It is instructive to see the equivalence in a simple non-trivial case. (The general case was summarized at the end of Section 4.2.)

Suppose $3 < d < 4$. Then define

$$I_1 = \frac{\pi^{d/2-2}}{\Gamma(d/2-2)} \int_0^{\infty} \mathrm{d}\hat{k}^2 (\hat{k}^2)^{d/2-3} f(k_1, k_2, k_3, k_4, \hat{k}^2), \quad (4.3.23)$$

$$I_2 = \int_{-\infty}^{\infty} \mathrm{d}k_5 \frac{\pi^{(d-5)/2}}{\Gamma(\frac{1}{2}(d-5))} \int_0^{\infty} \mathrm{d}k_T^2 (k_T^2)^{(d-7)/2} [f(k_1, \ldots, k_4, k_5^2 + k_T^2) - f(k_1, \ldots, k_4, k_5^2)]. \quad (4.3.24)$$

Our definition of the d-dimensional integral of f tells us that

$$I=\int_{-\infty}^{\infty}\mathrm{d}k_1\ldots\mathrm{d}k_4 I_1=\int\mathrm{d}k_1\ldots\mathrm{d}k_4 I_2, \tag{4.3.25}$$

so we must prove that $I_1=I_2$.

To do this we change variables in I_2, by setting $k_\mathrm{T}^2=x\hat{k}^2$ and $k_5^2=(1-x)\hat{k}^2$ to obtain:

$$\begin{aligned}I_2&=\frac{\pi^{(d-5)/2}}{\Gamma(d/2-5/2)}\int_0^1\mathrm{d}x\int_0^\infty\mathrm{d}\hat{k}^2(\hat{k}^2)^{d/2-3}(1-x)^{-1/2}x^{d/2-7/2}\times\\&\qquad\times[f(k_1,\ldots,k_4,\hat{k}^2)-f(k_1,\ldots,k_4,\hat{k}^2(1-x))]\\&=\frac{\pi^{(d-5)/2}}{\Gamma(d/2-5/2)}\int_0^1\mathrm{d}x\int_0^\infty\mathrm{d}\hat{k}^2\hat{k}^{d-6}(1-x)^{-1/2}x^{d/2-7/2}\times\\&\quad\times\{[f(k_1,\ldots,k_4,\hat{k}^2)-f(k_1,\ldots,k_4,0)]\\&\qquad-[f(k_1,\ldots,k_4,\hat{k}^2(1-x))-f(k_1,\ldots,k_4,0)]\}.\end{aligned} \tag{4.3.26}$$

In the last line we subtracted and added $f(k_1,\ldots,k_4,0)$, so that we can integrate seperately each term in square brackets. In particular, we have

$$\begin{aligned}&\int\mathrm{d}\hat{k}^2\hat{k}^{d-6}[f(k_1,\ldots,k_4,\hat{k}^2(1-x))-f(k_1,\ldots,k_4,0)]\\&\qquad=(1-x)^{2-d/2}\int\mathrm{d}\hat{k}^2\hat{k}^{d-6}[f(k_1,\ldots,k_4,\hat{k}^2)-f(k_1,\ldots,k_4,0)].\end{aligned}$$

Comparison with the definition (4.3.23) of I_1 shows that

$$\begin{aligned}\frac{I_2}{I_1}=\frac{\pi^{-1/2}\Gamma(d/2-2)}{\Gamma(d/2-5/2)}\int_0^1\mathrm{d}x[(1-x)^{-1/2}x^{d/2-7/2}\\-x^{d/2-7/2}(1-x)^{3/2-d/2}].\end{aligned} \tag{4.3.27}$$

The integral is in fact the analytic continuation from $d>5$ of a beta-function, so that it equals $\Gamma(d/2-5/2)\Gamma(1/2)/\Gamma(d/2-2)$. The required result $I_1=I_2$ follows.

Property 10. Multiple integrals are correct at $d=4$.

Consider the integral

$$I(\mathbf{p}_1,\ldots,\mathbf{p}_N)=\int\mathrm{d}^d\mathbf{k}_1\ldots\mathrm{d}^d\mathbf{k}_L f(\mathbf{k}_1,\ldots,\mathbf{k}_L,\mathbf{p}_1,\ldots,\mathbf{p}_N). \tag{4.3.28}$$

This might represent a Feynman graph with N external lines and L loops. Then the $\mathbf{p}_i$'s and $\mathbf{k}_i$'s represent momentum vectors. Suppose that at $d=4$ the integral is completely convergent – in particular that there are no ultraviolet ($k\to\infty$) divergences or subdivergences. If we restrict the $\mathbf{p}_i$'s to the first four dimensions and set $d=4$, then I is the ordinary four-dimensional integral of f.

Proof. For each vector $\mathbf{v}$, we define the projections $\bar{\mathbf{v}}$ and $\hat{\mathbf{v}}$, onto the physical and unphysical dimensions, as before. The result to be proved is that if $\mathbf{p}_i = \bar{\mathbf{p}}_i$ for each of the $\mathbf{p}_i$'s then I defined as the limit as $d \to 4$ of the dimensionally regulated integral is identical to the ordinary integral. We can split each integral over a $\mathbf{k}_i$ into an ordinary four-dimensional integral over $\bar{\mathbf{k}}_i$ and a $(d-4)$-dimensional integral over $\hat{\mathbf{k}}_i$. The result to be proved is then that

$$\lim \int \mathrm{d}^{d-4}\hat{\mathbf{k}}_1 \ldots \mathrm{d}^{d-4}\hat{\mathbf{k}}_L f(\mathbf{k}_1, \ldots, \mathbf{k}_L, \bar{\mathbf{p}}_1, \ldots, \bar{\mathbf{p}}_N) = f(\bar{\mathbf{k}}_1, \ldots, \bar{\mathbf{k}}_L, \bar{\mathbf{p}}_1, \ldots, \bar{\mathbf{p}}_N) \tag{4.3.29}$$

as $d \to 4$.

This formula is proved by doing all but the integral over $\hat{\mathbf{k}}_1$. Let the result be $I_{(1)}$:

$$I_{(1)}(\hat{\mathbf{k}}_1) = \int \mathrm{d}^{d-4}\hat{\mathbf{k}}_2 \ldots \mathrm{d}^{d-4}\hat{\mathbf{k}}_L f; \tag{4.3.30}$$

its only dependence on $\hat{\mathbf{k}}_1$ is via its length. We then have that the left-hand side of (4.3.29) is

$$\int \mathrm{d}^{d-4}\hat{\mathbf{k}}_1 I_{(1)} = I_{(1)}(0) = \int \mathrm{d}^{d-4}\hat{\mathbf{k}}_2 \ldots \mathrm{d}^{d-4}\hat{\mathbf{k}}_L f(\bar{\mathbf{k}}_1, \mathbf{k}_2, \ldots, \mathbf{k}_L, \bar{\mathbf{p}}_1, \ldots, \bar{\mathbf{p}}_N),$$

by use of the Property 9. Notice that the dependence on $\mathbf{k}_1$ is on its first four dimensions. We can then repeat this process to show that

$$I_{(1)}(0) = \int \mathrm{d}^{d-4}\hat{\mathbf{k}}_3 \ldots \mathrm{d}^{d-4}\hat{\mathbf{k}}_L f(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \mathbf{k}_3, \ldots).$$

Another $L-2$ repetitions give (4.3.29), from which the desired property follows.

4.4 Formulae for Minkowski space

In this section we derive a collection of results that are useful for Feynman graph calculations.

4.4.1 *Schwinger parameters*

To convert an arbitrary graph in d dimensions to a parametric integral, we first rewrite each propagator using

$$\frac{1}{(m^2 - k^2)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{d}x\, x^{\alpha-1} \exp[-x(m^2 - k^2)]. \tag{4.4.1}$$

Then we perform the momentum integrals. Since all Feynman graphs are of the form of a polynomial in momenta times a product of simple scalar propagators, we only have to calculate d-dimensional integrals of the form:

$$I_n^{\mu_1\ldots\mu_n}(A,B) = \int d^d k\, k^{\mu_1}\ldots k^{\mu_n} \exp[-(-Ak^2 - 2B\cdot k)]. \quad (4.4.2)$$

Here A depends only on the parameters introduced by (4.4.1), while B^μ depends on these parameters and also linearly on the other momenta (both external and loop momenta).

By linearity we can find I_n by differentiating I_0:

$$I_n^{\mu_1\ldots\mu_n} = \prod_{j=1}^{n}\left(\frac{1}{2}\frac{\partial}{\partial B_{\mu_j}}\right)\int d^d k \exp(Ak^2 + 2B\cdot k). \quad (4.4.3)$$

(This uses linearity of d-dimensional integration.) We find I_0 by using the translation $k^\mu \to k^\mu - B^\mu/A$, the scaling $k \to k\, A^{-1/2}$, and Wick rotation:

$$\begin{aligned} I_0(A,B) &= \int d^d k \exp(Ak^2 + 2B\cdot k) \\ &= i(\pi/A)^{d/2} \exp(-B^2/A). \end{aligned} \quad (4.4.4)$$

Thus

$$\begin{aligned} I_1^\mu &= \int d^d k\, k^\mu \exp(Ak^2 + 2B\cdot k) \\ &= i(\pi/A)^{d/2} \exp(-B^2/A)(-B^\mu/A), \end{aligned} \quad (4.4.5)$$

$$\begin{aligned} I_2^{\mu\nu} &= \int d^d k\, k^\mu k^\nu \exp(Ak^2 + 2B\cdot k) \\ &= i(\pi/A)^{d/2} \exp(-B^2/A)(B^\mu B^\nu/A^2 - \tfrac{1}{2}g^{\mu\nu}/A), \end{aligned} \quad (4.4.6)$$

$$\begin{aligned} I_3^{\lambda\mu\nu} &= \int d^d k\, k^\lambda k^\mu k^\nu \exp(Ak^2 + 2B\cdot k) \\ &= i(\pi/A)^{d/2} e^{-B^2/A}\left[\frac{-B^\lambda B^\mu B^\nu}{A^3} + \frac{(B^\lambda g^{\mu\nu} + B^\mu g^{\lambda\nu} + B^\nu g^{\lambda\mu})}{2A^2}\right], \end{aligned} \quad (4.4.7)$$

$$\begin{aligned} I_4^{\kappa\lambda\mu\nu} &= \int d^d k\, k^\kappa k^\lambda k^\mu k^\nu \exp(Ak^2 + 2B\cdot k) \\ &= i(\pi/A)^{d/2}\, e^{-B^2/A}\left[\frac{B^\kappa B^\lambda B^\mu B^\nu}{A^4} - \frac{(B^\kappa B^\lambda g^{\mu\nu} + \text{five similar})}{2A^3}\right. \\ &\qquad \left. + \frac{(g^{\kappa\lambda}g^{\mu\nu} + \text{two similar})}{4A^2}\right]. \end{aligned} \quad (4.4.8)$$

Each of the loop-momentum integrals is performed in this way. At each stage the momenta only appear quadratically and linearly in the exponent.

4.4.2 *Feynman parameters*

It is also common to use

$$1/(AB) = \int_0^1 \mathrm{d}x/[Ax + B(1-x)]^2 \tag{4.4.9}$$

and its generalizations:

$$\frac{1}{A^\alpha B^\beta \cdots E^\varepsilon} = \frac{\Gamma(\alpha+\beta+\cdots\varepsilon)}{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\varepsilon)} \times$$
$$\times \int_0^1 \mathrm{d}x\,\mathrm{d}y\cdots\mathrm{d}z\,\delta(1-x-y-\cdots z) \times$$
$$\times \frac{x^{\alpha-1}y^{\beta-1}\cdots z^{\varepsilon-1}}{(Ax+By+\cdots Ez)^{\alpha+\beta+\cdots+\varepsilon}}. \tag{4.4.10}$$

Here A, $B,\ldots,E$ represent the denominators of the propagators of a Feynman graph. The resulting momentum integrals have the form

$$J^{\mu_1\cdots\mu_n}_{n,\alpha} = \int \mathrm{d}^d k \frac{k^{\mu_1}\cdots k^{\mu_n}}{[-k^2 - 2p\cdot k + C]^\alpha}. \tag{4.4.11}$$

Application of (4.4.1) and the results (4.4.4)–(4.4.8), etc., gives

$$J_0 \equiv \int \mathrm{d}^d k/(-k^2 - 2p\cdot k + C)^\alpha$$
$$= \mathrm{i}\pi^{d/2}(C+p^2)^{d/2-\alpha}\Gamma(\alpha - d/2)/\Gamma(\alpha), \tag{4.4.12}$$

$$J_1^\mu \equiv \int \mathrm{d}^d k\, k^\mu/(-k^2 - 2p\cdot k + C)^\alpha$$
$$= \mathrm{i}\pi^{d/2}(C+p^2)^{d/2-\alpha}(-p^\mu)\Gamma(\alpha - d/2)/\Gamma(\alpha), \tag{4.4.13}$$

$$J_2^{\mu\nu} \equiv \int \mathrm{d}^d k\, k^\mu k^\nu/(-k^2 - 2p\cdot k + C)^\alpha$$
$$= \mathrm{i}\pi^{d/2}(C+p^2)^{d/2-\alpha} \times$$
$$\times [\Gamma(\alpha - d/2)p^\mu p^\nu - \Gamma(\alpha - 1 - d/2)g^{\mu\nu}(C+p^2)/2]/\Gamma(\alpha). \tag{4.4.14}$$

4.5 Dirac matrices

The Dirac matrices satisfy the following properties:

(1) Anticommutation relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}1. \tag{4.5.1}$$

(2) Hermiticity:

$$\gamma^{\mu\dagger} = \gamma_\mu = \begin{cases} \gamma^\mu & \text{if } \mu = 0, \\ -\gamma^\mu & \text{if } \mu \geq 1. \end{cases} \tag{4.5.2}$$

When we use dimensional regularization, the Lorentz indices range over an infinite set of values, so we need infinite-dimensional matrices to represent the algebra (4.5.1). We will also need a trace operation:

$$\operatorname{tr} 1 = f(d),$$

so that the representation behaves as if its dimension were $f(d)$. We must require $f(d_0)$ to be the usual value at the physical space-time dimension, $d = d_0$. Usually this means $f(4) = 4$.

The trace is a linear operation on the matrices which we will define later. In an even integer dimension $d = 2\omega$, the standard representation of the γ^μ's has dimension 2^ω. However, it is not necessary to choose $f(d) = 2^{d/2}$. The variation $f(d) - f(d_0)$ is only relevant for a divergent graph, so, by Chapter 7, any change in $f(d)$ amounts to a renormalization-group transformation. It is usually convenient to set $f(d) = f(d_0)$ for all d.

To set up a formalism for dimensionally regularized γ-matrices, we must treat the following issues:

(1) We must exhibit a representation of the anticommutation relations; this will ensure consistency.
(2) The formulae for the trace of an arbitrary product of γ^μ's must be derived.
(3) While a knowledge of the γ^μ's alone is sufficient for QCD and QED, we must show how to define a γ_5 so that we can treat chiral symmetries. This will also give us a definition of the antisymmetric tensor $\varepsilon_{\kappa\lambda\mu\nu}$.

The following construction gives a representation:

Let ω be a positive integer, and suppose inductively that we have defined a 2^ω dimensional representation $\gamma^\mu_{(\omega)}$ of the algebra (4.5.1) for $0 \leq \mu \leq 2\omega - 1$. We define the infinite dimensional γ^μ for $0 \leq \mu \leq 2\omega - 1$ by having a sequence of $\gamma^\mu_{(\omega)}$'s down the diagonal, and zeros everywhere else:

$$\gamma^\mu = \begin{pmatrix} \gamma^\mu_{(\omega)} & 0 & \\ 0 & \gamma^\mu_{(\omega)} & \\ & & \ddots \end{pmatrix}. \tag{4.5.3}$$

We will construct the next higher representation $\gamma^\mu_{(\omega+1)}$ of dimension $2^{\omega+1}$, with $0 \leq \mu \leq 2\omega + 1$. In order that (4.5.3) apply independently of ω, we must choose

$$\gamma^{\mu}_{(\omega+1)} = \begin{pmatrix} \gamma^{\mu}_{(\omega)} & 0 \\ 0 & \gamma^{\mu}_{(\omega)} \end{pmatrix} \quad \text{if } 0 \le \mu \le 2\omega - 1.$$

This satisfies the anticommutation relations (4.5.1) and the hermiticity relation (4.5.2), provided that $0 \le \mu$, $\nu \le 2\omega - 1$. Our task then is to find $\gamma^{\mu}_{(\omega+1)}$ for $\mu = 2\omega$ and $2\omega + 1$.

Notice that the induction starts with $\omega = 1$. We can define

$$\gamma^{0}_{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^{1}_{(1)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.5.4}$$

Given the 2^{ω}-dimensional representation $\gamma^{\mu}_{(\omega)}$ we define another matrix

$$\hat{\gamma}_{(\omega)} = \mathrm{i}^{\omega-1}\gamma^{0}_{(\omega)} \cdots \gamma^{2\omega-1}_{(\omega)}. \tag{4.5.5}$$

Observe that

$$\hat{\gamma}^{\dagger}_{(\omega)} = \hat{\gamma}_{(\omega)}, \quad \hat{\gamma}^{2}_{(\omega)} = 1, \quad \{\hat{\gamma}_{(\omega)}, \hat{\gamma}^{\mu}_{(\omega)}\} = 0. \tag{4.5.6}$$

Also, when at $\omega = 2$, we have $\hat{\gamma} = \gamma_5$, in the usual notation for Dirac matrices at $d = 4$. We define

$$\gamma^{2\omega}_{(\omega+1)} = \begin{pmatrix} 0 & \hat{\gamma}_{(\omega)} \\ -\hat{\gamma}_{(\omega)} & 0 \end{pmatrix},$$

$$\gamma^{2\omega+1}_{(\omega)} = \begin{pmatrix} 0 & \mathrm{i}\hat{\gamma}_{(\omega)} \\ \mathrm{i}\hat{\gamma}_{(\omega)} & 0 \end{pmatrix}. \tag{4.5.7}$$

It is easy to check that (4.5.1) and (4.5.2) are satisfied for $0 \le \mu$, $\nu \le 2\omega + 1$.

We now have an explicit representation of the Dirac matrices for any ω, and for the infinite-dimensional case, because of (4.5.3).

Standard manipulations involving the anticommutation relations are valid independently of d. Two useful results are:

$$\gamma^{\mu}\gamma_{\mu} = \tfrac{1}{2}\{\gamma^{\mu}, \gamma_{\mu}\}1 = g^{\mu}_{\mu}1 = d1, \tag{4.5.8}$$

$$\begin{aligned} \gamma^{\mu}\gamma_{\nu}\gamma_{\mu} &= 2g_{\mu\nu}\gamma^{\mu} - \gamma^{\mu}\gamma_{\mu}\gamma_{\nu} \\ &= (2-d)\gamma_{\nu}. \end{aligned} \tag{4.5.9}$$

We also need traces of γ-matrices, in graphs with fermion loops. The trace of a matrix is linear:

$$\operatorname{tr}(aA + bB) = a\operatorname{tr}(A) + b\operatorname{tr}(B), \tag{4.5.10}$$

and is cyclic:

$$\operatorname{tr}(AB) = \operatorname{tr}(BA). \tag{4.5.11}$$

Here A and B are any product of γ-matrices, and a and b are any numbers.

These properties, together with the value of tr 1, define the trace of any linear combination of products of γ-matrices.

For example,

$$\begin{aligned}\operatorname{tr}(\gamma^\mu\gamma^\nu) &= \operatorname{tr}(\gamma^\nu\gamma^\mu) \quad \text{(cyclicity)}\\ &= \operatorname{tr}(-\gamma^\mu\gamma^\nu + 2g^{\mu\nu}1) \quad \text{(anticommutation)}\\ &= -\operatorname{tr}(\gamma^\mu\gamma^\nu) + 2g^{\mu\nu}\operatorname{tr}1 \quad \text{(linearity)},\end{aligned}$$

so we have the usual result

$$\operatorname{tr}(\gamma^\mu\gamma^\nu) = g^{\mu\nu}\operatorname{tr}1. \tag{4.5.12}$$

Similarly

$$\operatorname{tr}(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu) = (g^{\kappa\lambda}g^{\mu\nu} - g^{\kappa\mu}g^{\lambda\nu} + g^{\kappa\nu}g^{\lambda\mu})\operatorname{tr}1. \tag{4.5.13}$$

The trace of the product of an odd number of γ-matrices is zero. For example

$$\begin{aligned} d\operatorname{tr}\gamma^\lambda &= \operatorname{tr}(\gamma^\kappa\gamma_\kappa\gamma^\lambda)\\ &= -\operatorname{tr}(\gamma^\kappa\gamma^\lambda\gamma_\kappa) + 2\operatorname{tr}\gamma^\lambda\\ &= -\operatorname{tr}(\gamma_\kappa\gamma^\kappa\gamma^\lambda) + 2\operatorname{tr}\gamma^\lambda,\end{aligned}$$

so $\operatorname{tr}\gamma^\lambda = 0$.

It should be possible to make a more constructive definition of the trace, along the lines of (4.3.5). It is necessary to check consistency. We can find a formula for the trace of any number of γ-matrices – generalizing (4.5.13). It is true for any finite-dimensional representation, $\gamma^\mu_{(\omega)}$, so it agrees with the algebraic properties. Linearity defines the trace of more general products. We must also check that contracting with $g_{\mu\nu}$ commutes with the trace. This can be checked directly.

A possible explicit definition of the trace of a matrix with components M_{ij} is

$$\operatorname{tr}M_{ij} = (\operatorname{tr}1)\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N} M_{jj}. \tag{4.5.14}$$

This definition exploits the fact that each matrix γ^μ is an infinite set of copies of a finite-dimensional $\gamma^\mu_{(\omega)}$ strung along the diagonal. Since the γ^μ's are independent of d, the only possible d-dependence is in the choice of the value of tr 1.

4.6 γ_5 and $\varepsilon_{\kappa\lambda\mu\nu}$

In four dimensions, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\varepsilon_{\kappa\lambda\mu\nu}$ is a totally antisymmetric Lorentz-invariant tensor with $\varepsilon_{0123} = 1$. We need γ_5, for example, to define the axial current $\bar{\psi}\gamma^\mu\gamma_5\psi$. The ε-tensor comes in because $\gamma_5 = i\varepsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu/4!$, and we have the trace formula:

$$\operatorname{tr}\gamma^5\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = i\varepsilon^{\kappa\lambda\mu\nu}\operatorname{tr}1 = -i\varepsilon_{\kappa\lambda\mu\nu}\operatorname{tr}1.$$

The appropriate definition changes when we go to two dimensions: Instead of γ_5 we have $\hat{\gamma}_{(1)} = \gamma^0\gamma^1$, and instead of $\varepsilon_{\kappa\lambda\mu\nu}$ we have $\varepsilon_{\mu\nu}$, for which $\varepsilon_{01} = 1 = -\varepsilon_{10}$, $\varepsilon_{00} = \varepsilon_{11} = 0$.

To continue dimensionally, we might expect γ_5 to satisfy

$$\{\gamma_5, \gamma^\mu\} = 0,$$

just as in four dimensions. But then, as we will see in Chapter 13, the only consistent result for γ_5 is that it has zero trace when multiplied by any string of γ^μ's. Thus we do not have a regularization involving the usual γ_5.

A consistent definition is obtained by writing

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\varepsilon_{\kappa\lambda\mu\nu}/4!, \tag{4.6.1}$$

$$\varepsilon_{\kappa\lambda\mu\nu} = \begin{cases} 1 \text{ if } (\kappa\lambda\mu\nu) \text{ is an even permutation of } (0123), \\ -1 \text{ if } (\kappa\lambda\mu\nu) \text{ is an odd permutation of } (0123), \\ 0 \text{ otherwise.} \end{cases} \tag{4.6.2}$$

This definition is not Lorentz invariant on the full space, but only on the first four dimensions. We have

$$\begin{aligned} \{\gamma_5, \gamma^\mu\} &= 0, \quad \text{if } \mu = 0, 1, 2, 3, \\ [\gamma_5, \gamma^\mu] &= 0, \quad \text{otherwise}, \\ (\gamma_5)^2 &= 1, \quad \gamma_5^\dagger = \gamma_5. \end{aligned} \tag{4.6.3}$$

The lack of full Lorentz invariance is a nuisance, but it does give the correct axial anomaly ('t Hooft & Veltman (1972a)).