sponding expressions involving only x-derivatives without changing the equivalence class of $\mathbf{v}$. For instance, u_t is replaced by uu_x, u_{xt} by $uu_{xx} + u_x^2$, u_{tt} by $u^2u_{xx} + 2uu_x^2$ and so on. Thus every symmetry is uniquely equivalent to one with characteristic $Q = Q(x, t, u, u_x, u_{xx}, \ldots)$. The infinitesimal condition (5.4) for invariance is then

$$D_t Q = uD_x Q + u_x Q, \tag{5.10}$$

which must be satisfied for all solutions. To calculate second order symmetries, we require $Q = Q(x, t, u, u_x, u_{xx})$, so (5.10) becomes, upon substituting for u_t according to the equation and simplifying,

$$\frac{\partial Q}{\partial t} - u\frac{\partial Q}{\partial x} + u_x^2\frac{\partial Q}{\partial u_x} + 3u_x u_{xx}\frac{\partial Q}{\partial u_{xx}} = u_x Q.$$

By the method of characteristics, cf. (2.12), the most general solution of this linear, first order partial differential equation is

$$Q = u_x R\left(x + tu, u, t + \frac{1}{u_x}, \frac{u_{xx}}{u_x^3}\right),$$

where R is an arbitrary function of its arguments. Which of these generalized symmetries correspond to geometric symmetries of the type discussed in Chapter 2? For this to be the case, the characteristic Q must be of the form $Q = \phi - u_x\xi - uu_x\tau$, where ϕ, ξ and τ depend only on x, t and u, and where $\mathbf{v} = \xi\partial_x + \tau\partial_t + \phi\partial_u$ is the corresponding infinitesimal generator. Thus

$$Q = u_x\psi(x + tu, u) + (tu_x + 1)\phi(x + tu, u),$$

for some ψ, and where $-\xi - u\tau = \psi + t\phi$. Thus there is quite a lot of freedom in the forms of ξ and τ; however, if $\xi + u\tau = 0 = \phi$, then the evolutionary form of $\mathbf{v}$ is trivial, $Q = 0$, so every geometric symmetry is equivalent to one in which $\tau = 0$, i.e.

$$\mathbf{v} = -(\psi + t\phi)\partial_x + \phi\partial_u.$$

If we restrict our attention to projectable symmetries, then it can be shown that this subgroup is generated by the following eight vector fields:

$$\begin{array}{ll}
\partial_x, & t\partial_x - \partial_u, \\
\partial_t, & x\partial_t + u^2\partial_u, \\
x\partial_x + t\partial_t, & xt\partial_x + t^2\partial_t - (x + tu)\partial_u, \\
x\partial_x + u\partial_u, & x^2\partial_x + xt\partial_t + (x + tu)u\partial_u.
\end{array}$$

The preceding example might give the reader an overly optimistic assessment of the computational complexity of the problem of computing generalized symmetries. In practice, given a system of differential equations, the computation of all generalized symmetries of a given order is inherently feasible, but only after a considerable investment of time and computational dexterity on the part of the investigator. The following example, which is still relatively easy, should give a better idea of what is required.

Example 5.8. Here we compute all third order generalized symmetries of Burgers' equation, which we take in potential form

$$u_t = u_{xx} + u_x^2. \tag{5.11}$$

We take our infinitesimal generator in evolutionary form $\mathbf{v} = Q\partial_u$, where we assume Q depends on $x, t, u, u_x, u_{xx}, u_{xxx}$. The symmetry condition (5.4) is

$$D_t Q = D_x^2 Q + 2u_x D_x Q. \tag{5.12}$$

Since this is only required to hold on solutions, we can substitute for any t derivatives of u therein using (5.11) and its prolongations. Upon analyzing (5.12) in detail, we can read off the coefficients of the various derivatives of u in descending order. The coefficients of the fifth order derivative u_{xxxxx} cancel, so we proceed to terms involving u_{xxxx}. From the only term involving u_{xxxx}^2, we see that Q is affine in u_{xxx},

$$Q = \alpha(t)u_{xxx} + Q'(x, t, u, u_x, u_{xx}),$$

where α depends only on t due to the other terms involving u_{xxxx}.

Proceeding to the terms involving the third order derivative u_{xxx}, we find

$$6\alpha u_{xx}u_{xxx} + \alpha_t u_{xxx} = Q'_{u_{xx}u_{xx}}u_{xxx}^2 + 2Q'_{u_x u_{xx}}u_{xx}u_{xxx} + 2Q'_{uu_{xx}}u_x u_{xxx} + 2Q'_{xu_{xx}}u_{xxx}.$$

Thus Q' is affine in u_{xx}, with

$$Q' = 3\alpha u_x u_{xx} + (\tfrac{1}{2}\alpha_t x + \beta)u_{xx} + Q''(x, t, u, u_x),$$

where $\beta = \beta(t)$ is a function of t alone. The coefficient of u_{xx}^2 in (5.12) now reads

$$6\alpha u_x + \alpha_t x + 2\beta = Q''_{u_x u_x},$$

hence

$$Q = \alpha(u_{xxx} + 3u_x u_{xx} + u_x^3) + (\tfrac{1}{2}\alpha_t x + \beta)(u_{xx} + u_x^2) + A(x, t, u)u_x + B(x, t, u).$$

The only other terms involving u_{xx} are

$$(3\alpha_t u_x + \tfrac{1}{2}\alpha_{tt}x + \beta_t)u_{xx} = (2A_u u_x + 3\alpha_t u_x + 2A_x)u_{xx}.$$

Thus A does not depend on u, and

$$A = \tfrac{1}{8}\alpha_{tt}x^2 + \tfrac{1}{2}\beta_t x + \gamma,$$

where $\gamma = \gamma(t)$ is yet another function of t. The coefficient of u_x^2 now implies

$$B(x, t, u) = \rho(x, t)e^{-u} + \sigma(x, t),$$

with ρ and σ to be determined. The coefficient of u_x reads

$$\tfrac{1}{8}\alpha_{ttt}x^2 + \tfrac{1}{2}\beta_{tt}x + \gamma_t = 2\sigma_x + \tfrac{1}{4}\alpha_{tt},$$

so

$$\sigma(x,t) = \tfrac{1}{48}\alpha_{ttt}x^3 + \tfrac{1}{8}\beta_{tt}x^2 + (\tfrac{1}{2}\gamma_t - \tfrac{1}{8}\alpha_{tt})x + \delta,$$

where $\delta = \delta(t)$. The remaining terms in (5.12), which do not involve derivatives of u, are just

$$\rho_t e^{-u} + \sigma_t = \rho_{xx}e^{-u} + \sigma_{xx}.$$

Thus $\rho(x, t)$ is any solution to the heat equation $\rho_t = \rho_{xx}$, while using the above form of σ, we conclude

$$\alpha_{tttt} = 0, \qquad \beta_{ttt} = 0, \qquad \gamma_{tt} = \tfrac{1}{2}\alpha_{ttt}, \qquad \delta_t = \tfrac{1}{4}\beta_{tt}.$$

Thus α and β are, respectively, cubic and quadratic polynomials in t,

$$\alpha(t) = c_9t^3 + c_8t^2 + c_7t + c_6, \qquad \beta(t) = c_5t^2 + c_4t + c_3,$$

where $c_3, \dots, c_9$ are arbitrary constants, whence

$$\gamma(t) = \tfrac{3}{2}c_9t^2 + c_2t + c_1, \qquad \delta(t) = \tfrac{1}{2}c_5t + c_0,$$

for further constants c_0, c_1, c_2.

Assembling all the information we have obtained, we conclude that every third order generalized symmetry of the potential Burgers' equation has as its characteristic Q a linear, constant-coefficient combination of the following ten "basic" characteristics

$$\begin{aligned}
Q_0 &= 1,\\
Q_1 &= u_x,\\
Q_2 &= tu_x + \tfrac{1}{2}x,\\
Q_3 &= u_{xx} + u_x^2,\\
Q_4 &= t(u_{xx} + u_x^2) + \tfrac{1}{2}xu_x,\\
Q_5 &= t^2(u_{xx} + u_x^2) + txu_x + (\tfrac{1}{2}t + \tfrac{1}{4}x^2),\\
Q_6 &= u_{xxx} + 3u_xu_{xx} + u_x^3,\\
Q_7 &= t(u_{xxx} + 3u_xu_{xx} + u_x^3) + \tfrac{1}{2}x(u_{xx} + u_x^2),\\
Q_8 &= t^2(u_{xxx} + 3u_xu_{xx} + u_x^3) + tx(u_{xx} + u_x^2) + (\tfrac{1}{2}t + \tfrac{1}{4}x^2)u_x,\\
Q_9 &= t^3(u_{xxx} + 3u_xu_{xx} + u_x^3) + \tfrac{3}{2}t^2x(u_{xx} + u_x^2) + (\tfrac{3}{2}t^2 + \tfrac{3}{4}tx^2)u_x\\
&\quad + (\tfrac{3}{4}tx + \tfrac{1}{8}x^3),
\end{aligned} \tag{5.13}$$

plus the infinite family of characteristics

$$Q_\rho = \rho(x,t)e^{-u},$$

where ρ is an arbitrary solution to the heat equation. Of these characteristics, the first six, $Q_0, \dots, Q_5$, and the characteristics Q_ρ correspond to the geometric symmetries computed in Example 2.42. For example, Q_4 is equivalent to

$$\tilde{Q}_4 = tu_t + \tfrac{1}{2}xu_x,$$

which is the characteristic corresponding to the vector field

$$-\tfrac{1}{2}\mathbf{v}_4 = -\tfrac{1}{2}x\partial_x - t\partial_t$$

generating the scaling group of symmetries.

One could continue in this fashion to compute higher and higher order generalized symmetries, but the computations grow rapidly more and more involved. The reader might try fourth order characteristics $Q = Q(x, t, u, \ldots, u_{xxxx})$ to gain a feeling for this phenomenon. In Section 5.2 we will discover a more systematic means of finding these symmetries.

Group Transformations

What is the group of transformations corresponding to a generalized vector field? If $\mathbf{v}$ is a genuine generalized vector field, its one-parameter group $\exp(\varepsilon\mathbf{v})$ can no longer act geometrically on the underlying domain $M \subset X \times U$ since the coefficients of $\mathbf{v}$ depend on derivatives of u, which are also being transformed. Nor can we define a prolonged group action on any finite jet space $M^{(n)}$ since the coefficients of $\mathrm{pr}^{(n)}\,\mathbf{v}$ will depend on still higher order derivatives of u than appear in $M^{(n)}$. The easiest way to resolve this dilemma is to define an action of the group $\exp(\varepsilon\mathbf{v})$ on a space of smooth functions as follows:† First replace $\mathbf{v}$ by its evolutionary representative $\mathbf{v}_Q$ as above and consider the system of evolution equations

$$\frac{\partial u}{\partial \varepsilon} = Q(x, u^{(m)}), \tag{5.14}$$

where Q is the characteristic of $\mathbf{v}$. The solution (provided it exists) to the Cauchy problem $u(x, 0) = f(x)$ will determine the group action:

$$[\exp(\varepsilon\mathbf{v}_Q)f](x) \equiv u(x, \varepsilon).$$

Here we are forced to assume that the solution to this Cauchy problem is uniquely determined provided the initial data $f(x)$ is chosen in some appropriate space of functions, at least for ε sufficiently small. The resulting flow $\exp(\varepsilon\mathbf{v}_Q)$ will then be on the given function space. Of course, the verification of this hypothesis leads to some very difficult problems on existence and uniqueness of solutions to systems of evolution equations which lie far beyond the scope of this book. Our results are, barring a resolution of these problems, of a somewhat formal nature, but nevertheless will have direct

† See also Exercise 5.8 for an alternative method.

practical applications. Note that our uniqueness assumption implies that $\exp(\varepsilon \mathbf{v}_Q)$ determines a local one-parameter group of transformations on the function space.

Example 5.9. Let $p = 2$, $q = 1$ with coordinates (x, y, u) and consider the translation group G generated by $\mathbf{v} = \partial_x$. The induced action of G on functions $u = f(x, y)$, as defined in Section 2.2, is

$$[\exp(\varepsilon \mathbf{v})f](x, y) = f(x - \varepsilon, y).$$

The evolutionary form of $\mathbf{v}$ is the generalized vector field $\mathbf{v}_0 = -u_x \partial_u$. The associated one-parameter group is determined by solving the Cauchy problem

$$\partial u/\partial \varepsilon = -u_x, \qquad u(x, y, 0) = f(x, y).$$

The solution is

$$[\exp(\varepsilon \mathbf{v}_0)f](x, y) = u(x, y, \varepsilon) = f(x - \varepsilon, y).$$

Thus $\mathbf{v}$ and $\mathbf{v}_0$ generate the same action, and in this sense are equivalent vector fields.

Theorem 5.10. *The evolutionary vector field* $\mathbf{v} = \mathbf{v}_Q$ *is a symmetry of the system of differential equations* Δ *if and only if the corresponding group* $\exp(\varepsilon \mathbf{v})$ *transforms solutions of the system to other solutions.*

Remark. This theorem is of course subject to various technical assumptions, namely

(1) Δ is a totally nondegenerate system as in our blanket hypothesis.
(2) The system of evolution equations appropriate to $\mathbf{v}$ is uniquely solvable in some space of functions which includes all the (local) solutions to Δ.
(3) A certain system of linear equations (5.15) appearing in the proof has unique solutions.

Proof. Let $u_\varepsilon = \exp(\varepsilon \mathbf{v})f$. (N.B.: the ε subscript is *not* a derivative.) If u_ε is a parametrized family of solutions, then

$$0 = \frac{\partial}{\partial \varepsilon} \Delta_\nu(x, u_\varepsilon^{(n)}) = \sum_{\alpha, J} D_J Q_\alpha(x, u_\varepsilon^{(n)}) \frac{\partial \Delta_\nu}{\partial u_J^\alpha}(x, u_\varepsilon^{(n)}) = \operatorname{pr} \mathbf{v}_Q[\Delta_\nu(x, u_\varepsilon^{(n)})].$$

Setting $\varepsilon = 0$ verifies (5.4). Conversely, suppose (5.5) holds. Assume that for ε sufficiently small, the only solution $v = (v^1, \dots, v^l)$ of the linear system of evolution equations

$$\frac{\partial v^\nu}{\partial \varepsilon} = \sum_\mu \mathscr{D}_{\nu\mu} v^\mu = \sum_{\mu, J} P_{\nu\mu}^J(x, u_\varepsilon^{(m)}(x)) v_J^\mu, \qquad \nu = 1, \dots, l, \tag{5.15}$$

with zero initial values $v(x, 0) \equiv 0$ is the zero solution $v(x, \varepsilon) \equiv 0$. Then (5.5) and the above computation imply that if $u = f(x)$ is a solution to Δ then

$v^\nu(x, \varepsilon) = \Delta_\nu(x, u_\varepsilon^{(n)})$ satisfies this initial value problem, and hence $\Delta_\nu(x, u_\varepsilon^{(n)}) = 0$ for all ε, proving that u_ε is a solution. $\square$

If $P[u]$ is any differential function, and $u(x, \varepsilon)$ a smooth solution to (5.14), then it is not difficult to see that

$$\frac{d}{d\varepsilon} P[u] = \sum_{\alpha, J} D_J Q_\alpha[u] \frac{\partial P}{\partial u_J^\alpha} = \operatorname{pr} \mathbf{v}_Q(P).$$

In other words, $\operatorname{pr} \mathbf{v}_Q(P)$ determines the infinitesimal change in P under the one-parameter group generated by $\mathbf{v}_Q$:

$$P[\exp(\varepsilon \mathbf{v}_Q) f] = P[f] + \varepsilon \operatorname{pr} \mathbf{v}_Q(P)[f] + \mathrm{O}(\varepsilon^2). \tag{5.16}$$

As in (1.18), we can continue to expand in powers of ε, leading to the Lie series

$$P[\exp(\varepsilon \mathbf{v}_Q) f] = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} (\operatorname{pr} \mathbf{v}_Q)^n P[f]. \tag{5.17}$$

(Here $(\operatorname{pr} \mathbf{v}_Q)^2(P) = \operatorname{pr} \mathbf{v}_Q[\operatorname{pr} \mathbf{v}_Q(P)]$, etc.) In particular, if $P[u] = u$, then (5.17) provides a (formal) Lie series solution to the evolutionary system (5.14). (We will not try to analyze the actual convergence of (5.17); see the following example.)

Example 5.11. Let $p = q = 1$ and consider the generalized vector field $\mathbf{v} = u_{xx} \partial_u$. The corresponding one-parameter group will be obtained by solving the Cauchy problem

$$\frac{\partial u}{\partial \varepsilon} = u_{xx}, \qquad u(x, 0) = f(x), \tag{5.18}$$

the solution being $u(x, \varepsilon) = \exp(\varepsilon \mathbf{v}) f(x)$. Thus exponentiating the generalized vector field $\mathbf{v} = u_{xx} \partial_u$ is equivalent to solving the heat equation!

Several difficulties will be immediately apparent to any reader familiar with this problem. First, for $\varepsilon < 0$ we are dealing with the "backwards heat equation", which is a classic ill-posed problem and may not even have solutions. Thus we should only expect to have a "semi-group" of transformations generated by $\mathbf{v}$. Secondly, as an example due to Tikhonov makes clear, unless we impose some growth conditions the solution will *not* in general be unique. Furthermore, if $P[u] = u$ in (5.17), we obtain the (formal) series solution

$$u(x, \varepsilon) = f(x) + \varepsilon \frac{\partial^2 f}{\partial x^2} + \frac{\varepsilon^2}{2!} \frac{\partial^4 f}{\partial x^4} + \cdots$$

to (5.18). However, as shown by Kovalevskaya, even if f is analytic, this Lie series for u may not converge. In fact, it will converge only if f is an entire analytic function satisfying the growth condition $|f(x)| \leqslant C \exp(Kx^2)$ for positive constants C, K. (These are the same growth conditions needed to ensure uniqueness of solutions.) This example gives a good indication of some of the difficulties associated with rigorously implementing our exponentiation of generalized vector fields.

Symmetries and Prolongations

The connection between generalized symmetries of systems of differential equations and their prolongations is based on the following important characterization of evolutionary vector fields. It says that except for the trivial translation fields $\partial/\partial x^i$, evolutionary vector fields are uniquely determined by the fact that they commute with the operations of total differentiation.

Lemma 5.12. *If* $\mathbf{v}_Q$ *is an evolutionary vector field, then*

$$\operatorname{pr} \mathbf{v}_Q[D_i P] = D_i[\operatorname{pr} \mathbf{v}_Q(P)], \qquad i = 1, \ldots, p, \tag{5.19}$$

for all $P \in \mathcal{A}$. *Conversely, given a vector field*

$$\mathbf{v}^* = \sum_{k=1}^{p} \xi^k[u] \frac{\partial}{\partial x^k} + \sum_{\alpha=1}^{q} \sum_J \phi_\alpha^J[u] \frac{\partial}{\partial u_J^\alpha}$$

for some $\xi^i, \phi_\alpha^J \in \mathcal{A}$, *we have* $[\mathbf{v}^*, D_i] = 0$ *for* $i = 1, \ldots, p$, *if and only if*

$$\mathbf{v}^* = \operatorname{pr} \mathbf{v}_Q + \sum_{k=1}^{p} c_k \frac{\partial}{\partial x^k}$$

for some $Q \in \mathcal{A}^q$, $c_1, \ldots, c_p \in \mathbb{R}$.

Proof. Note first the commutation relation

$$\frac{\partial}{\partial u_J^\alpha}(D_i P) = D_i\left(\frac{\partial P}{\partial u_J^\alpha}\right) + \frac{\partial P}{\partial u_{J\backslash i}^\alpha}, \tag{5.20}$$

where $J\backslash i$ is obtained by deleting one i from the multi-index J. (If i does not occur in J, this term is zero by convention.) This implies

$$\begin{aligned}\operatorname{pr} \mathbf{v}_Q[D_i P] &= \sum_{\alpha,J} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha}[D_i P] \\ &= \sum_{\alpha,J} D_J Q_\alpha \cdot D_i\left(\frac{\partial P}{\partial u_J^\alpha}\right) + \sum_{\alpha,J} D_J Q_\alpha \frac{\partial P}{\partial u_{J\backslash i}^\alpha}.\end{aligned}$$

Relabelling $J\backslash i$ as J in the second summation (so J becomes J, i), this is easily seen to equal

$$D_i[\operatorname{pr} \mathbf{v}_Q(P)] = \sum_{\alpha,J} D_J Q_\alpha \cdot D_i\left(\frac{\partial P}{\partial u_J^\alpha}\right) + \sum_{\alpha,J} D_i D_J Q_\alpha \cdot \frac{\partial P}{\partial u_J^\alpha}.$$

To prove the converse, we have by (5.20)

$$D_i \cdot \mathbf{v}^* - \mathbf{v}^* \cdot D_i = \sum_{k=1}^{p} D_i \xi^k \frac{\partial}{\partial x^k} + \sum_{J,\alpha} (D_i \phi_\alpha^J - \phi_\alpha^{J,i}) \frac{\partial}{\partial u_J^\alpha}.$$

This vanishes if and only if $D_i \xi^k = 0$ for all i, k, and $\phi_\alpha^{J,i} = D_i \phi_\alpha^J$ for all i, J, α. Thus each ξ^k is necessarily a constant and, by induction, $\phi_\alpha^J = D_J Q_\alpha$, where $Q_\alpha = \phi_\alpha^0$ is the coefficient of ∂_{u^α}.

Theorem 5.13. *If* $\mathbf{v}_Q$ *is a symmetry of the system* Δ, *then it is also a symmetry of any prolongation* $\Delta^{(k)}$.

PROOF. All the equations in $\Delta^{(k)}$ are of the form $D_J \Delta_\nu = 0$. By the lemma,

$$\operatorname{pr} \mathbf{v}_Q(D_J \Delta_\nu) = D_J(\operatorname{pr} \mathbf{v}_Q(\Delta_\nu)) = 0$$

whenever u is a solution since $\operatorname{pr} \mathbf{v}_Q(\Delta_\nu)$ vanishes on solutions by assumption. □

The Lie Bracket

As with ordinary vector fields, there is a Lie bracket between generalized vector fields, which, owing to the appearance of derivatives of u in their coefficient functions, must arise from the form of their prolongations. As with the usual Lie bracket, the easiest definition is as a commutator, but it can also be related to the corresponding one-parameter groups (see Exercise 5.7).

Definition 5.14. Let $\mathbf{v}$ and $\mathbf{w}$ be generalized vector fields. Their *Lie bracket* $[\mathbf{v}, \mathbf{w}]$ is the unique generalized vector field satisfying

$$\operatorname{pr}[\mathbf{v}, \mathbf{w}](P) = \operatorname{pr} \mathbf{v}[\operatorname{pr} \mathbf{w}(P)] - \operatorname{pr} \mathbf{w}[\operatorname{pr} \mathbf{v}(P)] \tag{5.21}$$

for all differential functions $P \in \mathscr{A}$.

There is a slight complication here in that it is not obvious that the right-hand side of (5.21) really is the prolongation of a generalized vector field. However, this follows from the explicit formulae for the Lie bracket.

Proposition 5.15. (a) *Let* $\mathbf{v}_Q$ *and* $\mathbf{v}_R$ *be evolutionary vector fields. Then their Lie bracket* $[\mathbf{v}_Q, \mathbf{v}_R] = \mathbf{v}_S$ *is also an evolutionary vector field with characteristic*

$$S = \operatorname{pr} \mathbf{v}_Q(R) - \operatorname{pr} \mathbf{v}_R(Q). \tag{5.22}$$

In (5.22), $\operatorname{pr} \mathbf{v}_Q$ *acts component-wise on* $R \in \mathscr{A}^q$, *with entries* $\operatorname{pr} \mathbf{v}_Q(R_k)$, *and conversely.*

(b) *More generally, if*

$$\mathbf{v} = \sum_i \xi^i[u] \frac{\partial}{\partial x^i} + \sum_\alpha \phi_\alpha[u] \frac{\partial}{\partial u^\alpha}, \qquad \mathbf{w} = \sum_i \eta^i[u] \frac{\partial}{\partial x^i} + \sum_\alpha \psi_\alpha[u] \frac{\partial}{\partial u^\alpha},$$

then

$$[\mathbf{v}, \mathbf{w}] = \sum_{i=1}^p \{\operatorname{pr} \mathbf{v}(\eta^i) - \operatorname{pr} \mathbf{w}(\xi^i)\} \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \{\operatorname{pr} \mathbf{v}(\psi_\alpha) - \operatorname{pr} \mathbf{w}(\phi_\alpha)\} \frac{\partial}{\partial u^\alpha}. \tag{5.23}$$

Moreover, if $\mathbf{v}$ *has characteristic* Q *and* $\mathbf{w}$ *has characteristic* R, *then* $[\mathbf{v}, \mathbf{w}]$ *has characteristic* S *as given by* (5.22).

PROOF. In the basic formula (5.21) with $\mathbf{v} = \mathbf{v}_Q$, $\mathbf{w} = \mathbf{v}_R$, the coefficient of $\partial/\partial u^\alpha$ in $\mathrm{pr}\,[\mathbf{v}, \mathbf{w}]$ is clearly given by the α-th component S_α of (5.22). Thus to prove part (a), it suffices to show that $[\mathrm{pr}\,\mathbf{v}_Q, \mathrm{pr}\,\mathbf{v}_R]$ is an evolutionary vector field, which will necessarily imply that it agrees with $\mathrm{pr}\,\mathbf{v}_S$. This immediately follows from Lemma 5.12: since both $\mathrm{pr}\,\mathbf{v}_Q$ and $\mathrm{pr}\,\mathbf{v}_R$ commute with all total derivatives, the same is true of their commutator, which contains no terms involving any $\partial/\partial x^i$. This proves part (a). Part (b) follows from the prolongation formula (5.8), and is left to the reader. □

For example, if $\mathbf{v} = uu_x\partial_u$, $\mathbf{w} = u_{xx}\partial_u$, then

$$[\mathbf{v}, \mathbf{w}] = (\mathrm{pr}\,\mathbf{v}(u_{xx}) - \mathrm{pr}\,\mathbf{w}(uu_x))\partial_u = 2u_x u_{xx}\partial_u.$$

Proposition 5.16. *The Lie bracket between generalized vector fields has the usual properties of*

(a) Bilinearity:

$$[c\mathbf{u} + c'\mathbf{v}, \mathbf{w}] = c[\mathbf{u}, \mathbf{w}] + c'[\mathbf{v}, \mathbf{w}], \qquad c, c' \in \mathbb{R},$$

(b) Skew-Symmetry:

$$[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}],$$

(c) Jacobi Identity:

$$[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] = 0,$$

for any generalized vector fields $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Indeed, these properties clearly hold when we replace each vector field by its prolongation, and this suffices to prove their validity. The commutator definition (5.21) of the Lie bracket immediately implies:

Proposition 5.17. *The set of generalized symmetries of a nondegenerate system of differential equations forms a Lie algebra.*

Example 5.18. In certain cases, this result can be used to construct new generalized symmetries from known ones. For example, consider the list of symmetries (5.13) of the potential Burgers' equation. Using $\mathbf{v}_i$ to denote the symmetry with characteristic Q_i, we conclude that $[\mathbf{v}_i, \mathbf{v}_j]$ is a symmetry with characteristic $\mathrm{pr}\,\mathbf{v}_i(Q_j) - \mathrm{pr}\,\mathbf{v}_j(Q_i)$ for any i, j. For example,

$$\mathrm{pr}\,\mathbf{v}_6(Q_7) - \mathrm{pr}\,\mathbf{v}_7(Q_6) = -\tfrac{3}{2}(u_{xxxx} + 4u_x u_{xxx} + 3u_{xx}^2 + 6u_x^2 u_{xx} + u_x^4)$$

gives the characteristic Q_{10} of a *new*, fourth order symmetry $\mathbf{v}_{10} \equiv -\tfrac{3}{2}[\mathbf{v}_6, \mathbf{v}_7]$ of Burgers' equation. This process can be repeated indefinitely, so $[\mathbf{v}_7, \mathbf{v}_{10}]$ will be a fifth order symmetry and so on. Thus Burgers' equation has an infinite collection of generalized symmetries depending on progressively higher and higher order derivatives of u. In Fuchssteiner's terminology, $\mathbf{v}_7$ is known as a "master symmetry" for the potential Burger's equation. (See Section 5.2.)

Evolution Equations

Consider a system of evolution equations

$$\frac{\partial u}{\partial t} = P[u] \tag{5.24}$$

in which $P[u] = P(x, u^{(n)}) \in \mathscr{A}^q$ depends on $x \in \mathbb{R}^p$, $u \in \mathbb{R}^q$ and x-derivatives of u only. Substituting according to (5.24) and its derivatives, we see that any evolutionary symmetry must be equivalent to one whose characteristic $Q[u] = Q(x, t, u^{(m)})$ depends only on x, t, u and the x-derivatives of u. On the other hand, (5.24) itself can be thought of as the equations for the flow $\exp(t\mathbf{v}_P)$ of the evolutionary vector field with characteristic P. The symmetry criterion (5.4), which in this case is

$$D_t Q_\nu = \mathrm{pr}\, \mathbf{v}_Q(P_\nu), \qquad \nu = 1, \ldots, q, \tag{5.25}$$

is readily seen to be equivalent to the following Lie bracket condition on the two generalized vector fields, generalizing the correspondence between symmetries of systems of first order ordinary differential equations and the Lie bracket of the corresponding vector fields.

Proposition 5.19. *An evolutionary vector field $\mathbf{v}_Q$ is a symmetry of the system of evolution equations $u_t = P[u]$ if and only if*

$$\frac{\partial \mathbf{v}_Q}{\partial t} + [\mathbf{v}_P, \mathbf{v}_Q] = 0 \tag{5.26}$$

holds identically in $(x, t, u^{(m)})$. (Here $\partial \mathbf{v}_Q/\partial t$ denotes the evolutionary vector field with characteristic $\partial Q/\partial t$.)

Proof. Note that according to the prolongation of the system of evolution equations, the derivative $u^\alpha_{J,t} = \partial u^\alpha_J/\partial t$ evolves according to $u^\alpha_{J,t} = D_J P_\alpha[u]$. Using this and the formula for the total derivative, it is easy to see that on solutions

$$D_t Q_\nu = \frac{\partial Q_\nu}{\partial t} + \sum_{\alpha, J} u^\alpha_{J,t} \frac{\partial Q_\nu}{\partial u^\alpha_J} = \frac{\partial Q_\nu}{\partial t} + \mathrm{pr}\, \mathbf{v}_P(Q_\nu),$$

since Q_ν only depends on x-derivatives of u. Thus (5.25) is equivalent to the equation

$$\frac{\partial Q}{\partial t} + \mathrm{pr}\, \mathbf{v}_P(Q) = \mathrm{pr}\, \mathbf{v}_Q(P),$$

which, as there are no more t-derivatives of u present, must hold identically in x, t and u. The equivalence with (5.26) follows easily from the formula (5.22) for the Lie bracket. □