38 Metastable vacua in quantum field theory (QFT)

In this chapter, we generalize to quantum field theory (QFT) [373, 374] the methods to evaluate barrier penetration effects in the semi-classical limit, as described in Chapter 37. In quantum mechanics (QM), we have shown that barrier penetration is associated with classical motion in imaginary time. Therefore, we consider QFT here in its Euclidean formulation.

In the representation of QFT in terms of field integrals, in the semi-classical limit, barrier penetration is related to finite action solutions (instantons) of the classical field equations [384]. We first characterize such solutions. We then explain how to evaluate the instanton contributions at leading order, the main new problem arising from ultraviolet (UV) divergences [375].

We have argued that the lifetime of metastable states is related to the imaginary part of the 'ground state' energy. However, for later purpose, it is useful to calculate the imaginary part not only of the vacuum amplitude, but also of correlation functions. In the case of the vacuum amplitude, we find that the instanton contribution is proportional to the space–time volume. Therefore, dividing by the volume, we obtain the probability per unit time and unit volume of a metastable pseudo-vacuum to decay.

We first discuss a scalar QFT with a ϕ^4 interaction, generalization of the quartic anharmonic oscillator considered in Sections 37.1–37.4, in two and three dimensions, dimensions in which the theory is super-renormalizable. We then consider more general scalar QFTs, of a form analogous to the quantum models discussed in Section 37.5 [380].

We calculate instanton contributions, at leading order, explicitly in the ϕ^4 theory in dimension 4, the dimension in which the theory is renormalizable. Several new problems arise. With help of Sobolev's inequalities, we prove in Section A38.2 that the massive field equation has no instanton solution in dimension 4, and that the relevant instanton is a solution of the massless field equation. Therefore, we first study the massless ϕ^4 QFT, and comment at the end about the massive theory. The price to pay for such a simplification is the appearance of some subtle infrared (IR) problems. In the leading order calculation, in addition to the mass renormalization already met in the super-renormalizable case, the one-loop coupling constant renormalization has to be taken into account. This feature, together with the scale invariance of the classical theory leads to the appearance of an effective coupling constant at the scale of the instanton and, therefore, the calculation of the contribution of the instanton depends on global renormalization group (RG) properties of the theory.

Finally, Section 38.8 is devoted to a brief discussion of a speculative cosmological application of these results.

In the appendix, we discuss virial theorems, Sobolev inequalities relevant to the properties of the classical solutions of the ϕ^4 field theory, RG properties and conformal invariance relevant to the instanton calculations in the $\phi^4_{d=4}$ field theory.

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38.1 The ϕ^4 QFT for negative coupling

We consider the d-dimensional QFT for a scalar field ϕ corresponding, in the classical approximation, to the Euclidean action

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\big(\nabla\phi(x)\big)^2 + \tfrac{1}{2}m^2\phi^2(x) + \frac{1}{4!}g m^{4-d}\phi^4(x)\right], \qquad (38.1)$$

m being the mass, and g the dimensionless coupling constant (the power of m that appears in front of the interaction term ϕ^4 takes care of the dimension).

The complete n-point correlation function has the field-integral representation

$$Z^{(n)}(x_1,\ldots,x_n) = \int [\mathrm{d}\phi(x)]\,\phi(x_1)\phi(x_2)\cdots\phi(x_n)\exp[-\mathcal{S}(\phi)]. \qquad (38.2)$$

We normalize the field integral with respect to the vacuum amplitude (partition function) at $g=0$ to avoid introducing a non-trivial g dependence through the normalization. Following the method described in Section 37.3, we assume that we start from positive values of g, and proceed by analytic continuation to define the field integral for g negative. The imaginary part of correlation functions is given by the difference between the continuations above and below the negative g-axis. For g small, only non-constant saddle points contribute to the imaginary part. Therefore, we look for instanton configurations, corresponding to non-constant finite action solutions of the Euclidean field equations, and evaluate their contributions.

38.1.1 Instantons: Classical solutions and classical action

The instanton solutions. The field equation corresponding to the action (38.1) is

$$\left(-\nabla^2 + m^2\right)\phi_c(x) + \tfrac{1}{6}g m^{4-d}\phi_c^3(x) = 0\,. \qquad (38.3)$$

We set (g is negative)

$$\phi_c(x) = (-6/g)^{1/2} m^{d/2-1}\varphi(mx)\,. \qquad (38.4)$$

In terms of φ, the classical action (38.1) reads

$$\mathcal{S}(\varphi) = -\frac{6}{g}\int \mathrm{d}^d x \left[\tfrac{1}{2}\big(\nabla\varphi(x)\big)^2 + \tfrac{1}{2}\varphi^2(x) - \tfrac{1}{4}\varphi^4(x)\right]. \qquad (38.5)$$

The function φ satisfies the parameter-free-field equation

$$\left(-\nabla^2 + 1\right)\varphi(x) - \varphi^3(x) = 0\,. \qquad (38.6)$$

It can be shown (for details, see Section A38.2) that the solution with the smallest action is spherically symmetric [385]. Therefore, we choose an arbitrary origin x_0, and set

$$r = |x - x_0|\,, \quad \varphi(x) = f(r). \qquad (38.7)$$

The function f satisfies the non-linear differential equation,

$$\left[-\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^2 - \frac{d-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + 1\right] f(r) - f^3(r) = 0\,. \qquad (38.8)$$

Interpreting r as a (real) time, we note that the equation describes the motion of a particle in a potential $U(f)$, with

$$U(f) = -\tfrac{1}{2}f^2 + \tfrac{1}{4}f^4, \qquad (38.9)$$

submitted, in addition, to a viscous damping force, due to the first derivative.

The finite-action solutions must satisfy the boundary condition

$$f(r) \to 0\,, \quad \text{for} \quad r \to \infty\,. \tag{38.10}$$

Equation (38.8) shows that, if $f(r)$ goes to 0 for $r \to +\infty$, it goes exponentially. The equation has solutions even in r, which are thus determined by the value of f at the origin. For a generic value of $f(0)$, the corresponding solution, for $r \to \infty$, tends towards a minimum of the potential $f = \pm 1$. The condition (38.10) is only satisfied for a discrete set of initial values of $f(0)$. Moreover, it can be shown that the minimal action solution corresponds to the function for which $|f(0)|$ is minimal in the set, and which only vanishes at infinity. The values [387],

$$\text{for } d = 2, \quad f(0) = 2.20620086465074607(1), \qquad A = 35.10268957367896(1), \tag{38.11}$$

$$\text{for } d = 3, \quad f(0) = 4.3373876799769943(1), \qquad A = 113.38350781527714(1) \tag{38.12}$$

correspond to the suitable numerical solution of the field equation (38.8). Moreover, due to translation symmetry, equation (38.3) has a family of degenerate saddle points $\phi_c(x)$ depending on d parameters $x_{0\mu}$ (equation (38.7)).

The instanton action. Since g is dimensionless, the corresponding classical action has the general form

$$\mathcal{S}(\phi_c) \equiv \mathcal{S}(\varphi) = S/g\,, \quad \text{with } S = -A\,, \tag{38.13}$$

and the scaling arguments of Section A38.1 [386] lead to the relations

$$A = \frac{6}{d}\int [\nabla\varphi(x)]^2\,\mathrm{d}^d x = \frac{3}{2}\int \varphi^4(x)\mathrm{d}^d x = \frac{6}{4-d}\int \varphi^2(x)\mathrm{d}^d x\,, \tag{38.14}$$

which show that A is positive. We note that these relations can only be satisfied for $d < 4$, and that the dimension 4 is singular (see Section A38.2).

38.1.2 The Gaussian integration for $d < 4$

To perform the Gaussian integration in the neighbourhood of the saddle point, we have to examine the spectrum of the differential operator, second functional derivative of $\mathcal{S}$, given by (in quantum bra–ket matrix-element notation)

$$\langle x\,|\mathbf{M}|\,x'\rangle \equiv \left.\frac{\delta^2\mathcal{S}}{\delta\phi(x)\delta\phi(x')}\right|_{\phi=\phi_c} = \left[\left(-\nabla_x^2 + m^2\right) + \tfrac{1}{2}gm^{4-d}\phi_c^2(x)\right]\delta^{(d)}(x-x'), \tag{38.15}$$

$$= \left[\left(-\nabla_x^2 + m^2\right) - 3m^2\varphi^2(mx)\right]\delta^{(d)}(x-x'). \tag{38.16}$$

Differentiating the equation of motion (38.3) with respect to x_μ, we infer that, as expected, the d partial derivatives $\partial_\mu\phi_c(x)$ ($\partial_\mu \equiv \partial/\partial_\mu$) are eigenvectors of $\mathbf{M}$ with vanishing eigenvalue:

$$\left(-\nabla^2 + m^2\right)\partial_\mu\phi_c(x) + \tfrac{1}{2}gm^{4-d}\phi_c^2(x)\partial_\mu\phi_c(x) = 0 \quad \Longleftrightarrow \quad \mathbf{M}\,\partial_\mu\phi_c = 0\,. \tag{38.17}$$

As in QM (Section 37.4.1), to sum over all saddle points, we have to factorize the field integration into an integration over the d collective coordinates $x_{0\mu}$ (which has to be done exactly), and an integration over the remaining field modes [388]. For this purpose, we can, for example, apply the identity (37.25) to each variable $x_{0,\mu}$.

The Jacobian. As the result (37.30) shows, this factorization leads to the determinant of $\mathbf{M}$ in the subspace orthogonal to the zero eigenvalue sector, and to a Jacobian J, which, at leading order, is

$$J=\prod_{\mu=1}^{d}\|\partial_\mu\phi_c(x)\|=\left[-\frac{6}{dg}\int \mathrm{d}^d x\big(\nabla\varphi(x)\big)^2\right]^{d/2}=\left(\frac{-A}{g}\right)^{d/2}, \tag{38.18}$$

where the invariance under rotation (38.7), and the first relation (38.14) have been used. Moreover, a factor $(2\pi)^{-1/2}$ from the Gaussian integration, is generated for each variable. Note one important feature of this expression: each translation symmetry broken by the instanton solution (each component of x_0) has generated a factor $(-g)^{-1/2}$.

Wave function arguments, of the kind used for the Schrödinger equation, show that $\partial_\mu\phi_c$ is not one ground state of $\mathbf{M}$. One state has a negative eigenvalue and, therefore, the final result is real as expected. In Section A38.2, we give a proof of this property using Sobolev inequalities.

Correlation functions. In expression (38.2), one can replace, at leading order, the field $\phi(x)$ by $\phi_c(x)$ in the product $\prod_{i=1}^{n}\phi(x_i)$. Collecting all factors, one finds,

$$\operatorname{Im} Z^{(n)}(x_1,\ldots,x_n)=\frac{1}{2i}\left(\frac{A}{2\pi}\right)^{d/2}\Omega\frac{\mathrm{e}^{A/g}}{(-g)^{(d+n)/2}}F_n(x_1,\ldots,x_n), \tag{38.19}$$

with

$$F_n(x_1,\ldots,x_n)=m^{d+n(d-2)/2}6^{n/2}\int \mathrm{d}^d x_0\prod_{i=1}^{n}f\big(m(x_i-x_0)\big), \tag{38.20}$$

and

$$\langle x\,|\mathbf{M}_0|\,x'\rangle=\left(-\nabla_x^2+m^2\right)\delta^{(d)}(x-x'), \tag{38.21a}$$

$$\Omega^{-2}=\det{}'\,\mathbf{M}\mathbf{M}_0^{-1}\big|_{m=1}=\lim_{\epsilon\to 0}\epsilon^{-d}\det\left[(\mathbf{M}+\epsilon)\,\mathbf{M}_0^{-1}\right]\big|_{m=1}. \tag{38.21b}$$

While, for the vacuum amplitude, the integration over x_0 generates a factor proportional to the volume, for non-trivial correlation functions the integration restores translation invariance.

Discussion. A few comments concerning expression (38.19) are in order here. We have obtained a result for the complete correlation functions, improperly normalized, for convenience, with respect to the free QFT. However, because $\phi_c(x)$ is proportional to $1/\sqrt{-g}$, the imaginary part of the n-point function increases with n for g small. This shows that, at leading order, the correlation functions, normalized with respect to the partition function corresponding to the complete action (38.1), have the same behaviour as those renormalized with respect to the free QFT.

Moreover, for the same reason, when we consider a complete n-point function, the imaginary part coming from disconnected parts is subleading by at least one power of g. If we denote by $W^{(n)}(x_1,\ldots,x_n)$ the connected n-point function, we thus find, at leading order,

$$\operatorname{Im} W^{(n)}\sim\operatorname{Im} Z^{(n)},$$

a result that is consistent with the observation that the explicit expression (38.19) is indeed connected.

Vertex functions are derived from connected correlation functions by first subtracting the reducible contributions, which involve functions with a smaller number of arguments, and which are, therefore, negligible at leading order, and then by amputating the remaining part. Again, for the same reason, only the real part of the propagator matters; therefore, to amputate expression (38.19), we must simply multiply it by the product of the inverse free propagators corresponding to each external line. Introducing $\tilde{f}$, the Fourier transform of f, and denoting the n-point vertex function in the Fourier representation by $\tilde{\Gamma}^{(n)}$, one obtains

$$\frac{\operatorname{Im}\tilde{\Gamma}^{(n)}(p_1,\ldots,p_n)}{m^{d-n(d/2+1)}} \sim -\frac{1}{2i}\left(\frac{A}{2\pi}\right)^{d/2}\Omega\frac{\mathrm{e}^{A/g}}{(-g)^{(d+n)/2}}\prod_{i=1}^{n}\sqrt{6}\tilde{f}\left(\frac{p_i}{m}\right)\left(p_i^2+m^2\right). \quad (38.22)$$

The structure, at leading order, of the imaginary part of the n-point vertex function is particularly simple. In particular, it only depends on the square of the momenta p_i, and not of their scalar products.

Up to this point, the discussions of the ϕ^4 QFT and of the anharmonic oscillator have been remarkably similar. Now comes one significant difference: the determinant of the operator $\mathbf{M}$ is actually UV divergent, and one has to deal with this new problem.

38.1.3 UV divergences and renormalization for $d<4$

A regularization is required to define the ϕ^4 theory in dimensions $d<4$, and then, a mass counter-term must be added to the classical action. After the introduction of higher derivatives and a momentum cut-off Λ (Section 8.4.2), the regularized action $\mathcal{S}_\Lambda$ takes the form:

$$\mathcal{S}_\Lambda(\phi)=\int \mathrm{d}^d x\left[\tfrac{1}{2}\phi(x)\left(-\nabla^2+\nabla^4/\Lambda^2+r\right)\phi(x)+\tfrac{1}{4!}g\phi^4(x)\right], \quad (38.23)$$

where $r=m^2+\delta m^2(\Lambda)$, $\delta m^2(\Lambda)$ being the mass counter-term (Section 9.2.1).

At large cut-off Λ, the additional term

$$\frac{1}{\Lambda^2}\int \phi(x)\nabla^4\phi(x)\mathrm{d}^d x$$

modifies the equation of motion but, when $\Lambda\to\infty$, the modification vanishes like $1/\Lambda^2$. On the other hand, the counter-term increases with the cut-off, but is proportional to at least one power of g. Hence, because we take the small g limit before taking the large cut-off limit, the counter-term does not contribute to the classical equation of motion.

If we then calculate the contribution of the counter-term to the classical action, we find that the one-loop counter-term, which is proportional to g, gives a contribution of order 1 in g, because $\phi_c(x)$ is proportional to $1/\sqrt{-g}$. Therefore, it generates an additional multiplicative factor.

The operator $\mathbf{M}$ (equation (38.15)) in the regularized theory is given by

$$\langle x\,|\mathbf{M}|\,x'\rangle=\left[\left(-\nabla^2+\nabla^4/\Lambda^2+m^2\right)+\tfrac{1}{2}gm^{4-d}\phi_c^2(x)\right]\delta^{(d)}(x-x'). \quad (38.24)$$

We expand its determinant in powers of $\phi_c^2(x)$, using the identity $\operatorname{tr}\ln=\ln\det$,

$$\begin{aligned}\ln\det\mathbf{M}&=\ln\det\left(-\nabla^2+\nabla^4/\Lambda^2+m^2\right)\\&\quad-\sum_{k=1}^{\infty}\frac{1}{k}\operatorname{tr}\left[\tfrac{1}{2}gm^{4-d}\phi_c^2(x)\left(\nabla^2-\nabla^4/\Lambda^2-m^2\right)^{-1}\right]^k. \quad (38.25)\end{aligned}$$

The first term is cancelled by the free determinant $\det \mathbf{M}_0$. All terms for $k \geq 2$, are UV finite in two and three dimensions. Finally, the $k = 1$ term is

$$\tfrac{1}{2} \operatorname{tr} g m^{4-d} \phi_c^2(x) \left(-\nabla^2 + \nabla^4/\Lambda^2 + m\right)^{-1} = \tfrac{1}{2} g m^{4-d} D \int \mathrm{d}^d x \, \phi_c^2(x) \,, \tag{38.26}$$

in which D is the regularized free propagator at coinciding arguments,

$$D = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d p}{p^2 + p^4/\Lambda^2 + m^2}. \tag{38.27}$$

The determinant of the operator $\mathbf{M}$ thus contains a factor that diverges at large cut-off as

$$\exp\left[-\tfrac{1}{4} g m^{4-d} D \int \mathrm{d}^d x \, \phi_c^2(x)\right]. \tag{38.28}$$

This factor exactly cancels the infinite factor coming from the mass counter-term, in such a way that the final expression for the imaginary part is finite. The fact that really we have to calculate $\det' \mathbf{M}$ does not change the argument, because UV divergences are insensitive to the omission of a finite number of eigenvalues of $\mathbf{M}$, as the second form (38.21b) explicitly shows [375].

The decay of the false vacuum. The special case $n = 0$ corresponds to the imaginary part of the vacuum amplitude. We have expanded perturbation theory around the minimum $\phi = 0$ of the potential. The perturbative ground state corresponds to a wave functional concentrated around small fields. However, because we have expanded around a relative minimum of the potential, this state is actually metastable. We have calculated its decay rate due to barrier penetration. We note that the integral over $x_{0\mu}$ in equation (38.20) yields a space–time volume factor. To obtain a finite decay amplitude, we have to divide the result by this volume factor. We thus obtain the probability per unit time and *unit volume* for the metastable state ('false vacuum') of the theory to decay. Some implications of such a result are discussed in a slightly more general context in Section 38.8.

38.2 General potentials: Instanton contributions

We now extend the analysis of Section 37.5 to analogous scalar field theories using the techniques developed in Section 38.1 [380]. We consider a Euclidean action of the form

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\left(\nabla\phi(x)\right)^2 + g^{-1} V\left(\phi(x)\sqrt{g}\right)\right], \tag{38.29}$$

in which the polynomial potential $V(\phi)$ has one stable and one metastable minimum, and is of the type discussed in Section 37.5. Assuming that, at some initial time, the quantum state corresponds to a field concentrated around the metastable minimum of the potential, the 'false' vacuum, we want to evaluate, in the semi-classical limit, the probability for the false vacuum to decay into the true vacuum of the theory. The calculation, at leading order, again involves the determination of an instanton solution, a factorization of the integral over collective coordinates, and a remaining Gaussian integration around the instanton.

38.2.1 Calculation of the instanton contribution

We define the field in such a way that the metastable minimum corresponds to $\phi = 0$. The discussion of the existence of an instanton solution is similar to the one given in Section 38.1. A theorem establishes, under mild assumptions, that spherically symmetric solutions give the minimal action [385]. Therefore, we look for such a solution and set

$$r = |x - x_0|\,, \quad f(r) = \sqrt{g}\phi_c(x)\,.$$

The classical equation of motion reduces to

$$\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{d-1}{r}\frac{\mathrm{d}f}{\mathrm{d}r} = V'\big(f(r)\big). \tag{38.30}$$

This is the equation governing the motion of a particle in a potential $-V(f)$ and submitted to a viscous damping force. We denote by f_+ the absolute minimum of the potential. The solution depends on its value at the origin $f(0)$. If we choose $f(0)$ too close to f_+, $f(r) - f(0)$ remains small until r becomes very large. When r is large, the damping force is small, the energy is almost conserved, and the particle overshoots. If $f(0)$ is too close to 0, the particle loses too much energy and, therefore, undershoots the asymptotic value $f(r)$ then corresponding to the maximum of $V(f)$. Thus, somewhere in between, we expect to find values $f(0)$, which correspond to solutions that vanish at infinity and, therefore, have finite actions.

The virial theorem, derived in Section A38.1, implies that the corresponding action is positive:

$$\mathcal{S}(\phi_c) = S/g\,, \tag{38.31}$$

with

$$S = \frac{1}{d}\int \big(\nabla f(x)\big)^2 \mathrm{d}^d x > 0\,. \tag{38.32}$$

Moreover, we also derive in Section A38.1 that the operator $\mathbf{M}$ with kernel

$$\langle x\,|\mathbf{M}|\,x'\rangle = \delta^2\mathcal{S}/\delta\phi(x)\delta\phi(x')\big|_{\phi=\phi_c}\,,$$

has one and only one negative eigenvalue.

Again, we factorize the field integration measure in an integration over x_0, and an integration over the other field modes. This generates a Jacobian J which, as we have shown in Section 38.1, at leading order is given by

$$J = \prod_{\mu=1}^{d} \|\partial_\mu\phi_c\| = \left[\frac{1}{d}\int \mathrm{d}^d x\,\big(\nabla\phi_c(x)\big)^2\right]^{d/2} = \left(\frac{S}{g}\right)^{d/2}, \tag{38.33}$$

where equations (38.31, 38.32) are used.

The remaining details of the calculation can be borrowed from the ϕ^4 example, and lead to an explicit expression for the imaginary part of the n-point correlation function.

Renormalization: A few remarks. If the QFT is super-renormalizable or renormalizable, the renormalized theory can be generated in the following way: after regularizing the theory, we proceed by induction, adding the counter-terms that render the theory finite order by order in a loop expansion, that is, here an expansion in powers of g. The renormalized action $\mathcal{S}_\mathrm{r}(\phi)$ takes the form

$$\mathcal{S}_\mathrm{r}(\phi) = \mathcal{S}_0(\phi\sqrt{g})/g + \mathcal{S}_1(\phi\sqrt{g}) + \cdots + g^{L-1}\mathcal{S}_L(\phi\sqrt{g}) + \cdots\,.$$

For the instanton calculation, at leading order, only the one-loop counter-terms are needed. To evaluate them, we expand the generating functional of vertex (1PI) functions (or effective potential) $\Gamma(\phi)$ up to one-loop order using the regularized action and calculate its divergent part.

In Chapter 7, we have derived the one-loop expression (equation (7.93))

$$\Gamma_{1\,\text{loop}}(\phi) = \mathcal{S}(\phi) + \tfrac{1}{2}\operatorname{tr}\ln\frac{\delta^2\mathcal{S}}{\delta\phi(x)\delta\phi(x')}. \tag{38.34}$$

To render vertex functions finite, we have to subtract to the regularized action the divergent part of the one-loop contribution of $\Gamma(\phi)$:

$$\mathcal{S}_1(\phi\sqrt{g}) = -\tfrac{1}{2}\left(\operatorname{tr}\ln\frac{\delta^2\mathcal{S}}{\delta\phi\delta\phi}\right)_{\text{div.}}.$$

When evaluated for $\phi = \phi_c$, this contribution exactly cancels the divergence in the determinant coming from the Gaussian integration around the saddle point. This argument can be generalized to arbitrary orders.

38.3 The ϕ^4 QFT in dimension 4

In dimension 4, the ϕ^4 QFT is just renormalizable, and, as we show in Section A38.2, only the massless field equations have instanton solutions. This leads to a set of new problems which we now examine. We first consider the massless theory, which is simpler, although it has some subtle IR problems. In particular, the barrier penetration is rather peculiar, since it is not generated by the potential but only by the kinetic term of the action.

We explain the leading order calculation of instanton contribution for the one-component ϕ^4 theory, but the extension to the $O(N)$-symmetric model is simple, and the explicit expressions can be found in the literature [375].

The classical Euclidean action of the massless theory ϕ^4 theory can be written as

$$\mathcal{S}(\phi) = \int \mathrm{d}^4x \left[\tfrac{1}{2}\big(\nabla\phi(x)\big)^2 + \tfrac{1}{4}g\phi^4(x)\right], \tag{38.35}$$

and the corresponding field equation reads

$$-\nabla^2\phi(x) + g\phi^3(x) = 0\,. \tag{38.36}$$

Note the different normalization of the coupling constant. To return to the convention used elsewhere, one has to substitute $g \mapsto g/6$.

We know that the solution of minimal action is spherically symmetric, and thus we set

$$\phi(x) = \frac{1}{\sqrt{-g}}f(r), \quad \text{with } r = |x - x_0|\,. \tag{38.37}$$

We then obtain the differential equation

$$-\left[\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^2 + \frac{3}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right]f(r) = f^3(r). \tag{38.38}$$

The classical action is scale invariant (actually, the QFT is conformal invariant, see Section A38.4). If $\phi(x)$ is an instanton solution to the field equation, then the rescaled functions $\psi(x) = \phi(x/\lambda)/\lambda$ (λ is a scale parameter) are also solutions.

This property suggests the following parametrization,

$$f(r) = \mathrm{e}^{-t}\, h(t), \quad \text{with } r = \mathrm{e}^{t}\,, \tag{38.39}$$

which transforms equation (38.38) into

$$\ddot{h}(t) = h(t) - h^3(t). \tag{38.40}$$

We recognize the equation of motion of the anharmonic oscillator in Chapter 37, whose solution is (equation (37.16)),

$$h_c(t) = \pm\frac{\sqrt{2}}{\cosh(t - t_0)}. \tag{38.41}$$

The solution $\phi_c(x)$ of equation (38.36) is then

$$f(r) = \pm\frac{2\sqrt{2}\lambda}{1 + \lambda^2 r^2}, \tag{38.42a}$$

$$\Rightarrow \phi_c(x) = \pm\frac{1}{\sqrt{-g}}\frac{2\sqrt{2}\lambda}{1 + \lambda^2\left(x - x_0\right)^2}, \tag{38.42b}$$

where we have defined $\lambda = \mathrm{e}^{-t_0}$. The corresponding classical action $\mathcal{S}(\phi_c)$ is

$$\mathcal{S}(\phi_c) = S/g\,, \quad \text{with } S = -8\pi^2/3\,. \tag{38.43}$$

With the usual normalization of g, one finds $S = -16\pi^2$.

Because the classical theory is scale invariant, the instanton solution now depends on a scale parameter λ, in addition to the four translation parameters $x_{0\mu}$. Therefore, we have to introduce five collective coordinates to calculate the instanton contribution.

38.4 Instanton contributions at leading order

General strategy. The second functional derivative of the action at the saddle point is

$$\langle x\,|\mathbf{M}|\,x'\rangle = \left.\frac{\delta^2\mathcal{S}}{\delta\phi(x)\delta\phi(x')}\right|_{\phi=\phi_c} = \left[-\nabla^2 - \frac{24\lambda^2}{\left(1+\lambda^2 x^2\right)^2}\right]\delta^{(4)}(x - x'). \tag{38.44}$$

To determine the eigenvalues of the operator $\mathbf{M}$, one has to solve a four-dimensional Schrödinger equation with a spherically symmetric potential. We note, at this stage, two serious problems. The operator $\mathbf{M}$ has formally, as expected, five eigenvectors, $\nabla\phi_c(x)$ and $(\mathrm{d}/\mathrm{d}\lambda)\phi_c(x)$ with eigenvalue 0, but the last of these eigenvectors is not normalizable with the natural measure of the problem

$$\int\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_c(x)\right)^2 \mathrm{d}^4x = \infty\,. \tag{38.45}$$

This is an IR problem which arises, because the theory is massless.

Moreover, the mass counter-term, which has to be added to the action, and has the form

$$\tfrac{1}{2}\delta m_0^2\int \mathrm{d}^4x\,\phi_c^2(x) = \infty\,, \tag{38.46}$$

is also IR divergent, and this IR divergence is expected to cancel an IR divergence coming from $\det\mathbf{M}$.

Thus, in general, we need some kind of IR regularization. In the particular case of dimensional regularization, this problem is postponed to the two-loop order.

These problems will be solved in several steps. First, we realize that we do not need the eigenvalues of $\mathbf{M}$, but only the determinant $\det' \mathbf{M}\mathbf{M}_0^{-1}$ (equations (38.21)). We can multiply $\mathbf{M}$ and $\mathbf{M}_0$ by the same operator. A specific choice which makes full use of the scale invariance of the classical theory, then transforms $\mathbf{M}$ into an operator whose eigenvalues can be calculated analytically. Because the calculations are somewhat tedious, we only indicate here the main steps, without giving all details.

The transformation. We extend the transformation (38.39) to arbitrary fields, setting

$$\phi(x) = \mathrm{e}^{-t}\, h(t,\hat{n}), \text{ with } t = \ln|x|, \quad \hat{n}_n = x^\mu/|x| \,. \tag{38.47}$$

The classical action $\mathcal{S}(\phi) = \tilde{\mathcal{S}}(h)$ then becomes

$$\tilde{\mathcal{S}}(h) = \int \mathrm{d}t\, \mathrm{d}\Omega \left\{ \tfrac{1}{2} \left[\dot{h}(t,\hat{n}) - h(t,\hat{n}) \right]^2 + h(t,\hat{n}) \mathbf{L}^2 h(t,\hat{n}) + \tfrac{1}{4} g h^4(t,\hat{n}) \right\}. \tag{38.48}$$

The symbol $\int \mathrm{d}\Omega$ means integration over the angular variables $\hat{n}$, and $\mathbf{L}^2$ is the square of the angular momentum operator with eigenvalues $l(l+2)$ and degeneracy $(l+1)^2$. The expression (38.48) can be rewritten as

$$\tilde{\mathcal{S}}(h) = \int \mathrm{d}t\, \mathrm{d}\Omega \left\{ \tfrac{1}{2} \left[\left(\dot{h}(t,\hat{n})\right)^2 + h(t,\hat{n}) \left(\mathbf{L}^2 + 1\right) h(t,\hat{n}) \right] + \tfrac{1}{4} g h^4(t,\hat{n}) \right\}, \tag{38.49}$$

because the integral of the cross term $\dot{h}h$ vanishes due to the boundary conditions.

With the parametrization

$$\lambda = \mathrm{e}^{-t_0}, \qquad \mathbf{x}_0 = \mathrm{e}^{t_0}\, \mathbf{v}\,,$$

the classical solution (38.42*b*) transforms into

$$h_c(t) = \pm \frac{2\,(-2/g)^{1/2}}{\mathrm{e}^{(t-t_0)} - 2\mathbf{v}\cdot\mathbf{n} + \mathrm{e}^{-(t-t_0)}(\mathbf{v}^2+1)}. \tag{38.50}$$

We note that, in these new variables, translations take a complicated form, unlike dilatation which simply corresponds to a translation of the variable t.

The second derivative of the classical action at the saddle point now takes the form (for $t_0 = x_{0\mu} = 0$)

$$\mathbf{M} = \frac{\delta^2 \mathcal{S}}{\delta h_c \delta h_c} = -\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 + \mathbf{L}^2 + 1 - \frac{6}{\cosh^2 t}. \tag{38.51}$$

The natural measure associated to this Hamiltonian problem is

$$\int \mathrm{d}t\, \mathrm{d}\Omega\,,$$

which, in the original variables, is

$$\int \frac{\mathrm{d}^4 x}{\mathbf{x}^2}.$$

This measure is not translation invariant, and thus the Jacobian resulting from the introduction of collective coordinates, and the determinant depend individually on $x_{0\mu}$. However, the product of the corresponding contributions to the final result should not, thus we perform the calculation for $x_{0\mu} = 0$.

38.4.1 The Jacobian

With the new measure, $\mathrm{d}\phi_c/\mathrm{d}\lambda$ is normalizable. Indeed,

$$J_1 = \left[\int \frac{\mathrm{d}^4 x}{\mathbf{x}^2}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_c(x)\right)^2\right]^{1/2}, \tag{38.52}$$

$$= \left[\frac{16\pi^2}{(-g)}\int_0^\infty r\mathrm{d}r \frac{\left(1-\lambda^2 r^2\right)^2}{(1+\lambda^2 r^2)^4}\right]^{1/2}. \tag{38.53}$$

This leads to a first factor:

$$J_1 = \frac{1}{\lambda}\sqrt{\frac{8}{3}}\frac{\pi}{\sqrt{-g}}. \tag{38.54}$$

The second Jacobian J_2 is generated by the collective coordinates $x_{0\mu}$:

$$J_2 = \left[\frac{1}{4}\int \frac{\mathrm{d}^4 x}{\mathbf{x}^2}\sum_{\mu=1}^{4}\left(\partial_\mu \phi_c(x)\right)^2\right]^2, \tag{38.55}$$

$$= \frac{1}{g^2}\left[16\pi^2\int_0^\infty \frac{r^3 \mathrm{d}r\lambda^6}{(1+\lambda^2 r^2)^4}\right]^2 = \frac{\lambda^4}{g^2}\times\frac{16}{9}\pi^4, \tag{38.56}$$

where rotation invariance has been used. The complete Jacobian J is thus

$$J = J_1 J_2 = \frac{\lambda^3}{(-g)^{5/2}}\pi^5 \times \frac{32\sqrt{2}}{9\sqrt{3}}. \tag{38.57}$$

38.4.2 The determinant

In the angular momentum basis, for each value l of the angular momentum, the component of the operator (38.51) reads

$$M_l = -\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 + (1+l)^2 - \frac{6}{\cosh^2 t}. \tag{38.58}$$

Using equation (37.31), it is possible to calculate the determinant of M_l. One finds

$$\det\left(M_l+\varepsilon\right)\left(M_{0l}+\varepsilon\right)^{-1} = \frac{\sqrt{\varepsilon+(l+1)^2}-1}{\sqrt{\varepsilon+(l+1)^2}+2}\frac{\sqrt{\varepsilon+(l+1)^2}-2}{\sqrt{\varepsilon+(l+1)^2}+1]}, \tag{38.59}$$

in which M_{0l} is the operator of the corresponding free theory. As we know, this determinant is UV divergent and has to be renormalized. However, we first calculate formally the unrenormalized determinant:

$$l \geq 2: \qquad \det M_l M_{0l}^{-1} = \frac{l(l-1)}{(l+2)(l+3)}, \tag{38.60}$$

$$l = 1: \qquad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\det\left(M_1+\varepsilon\right)\left(M_{01}+\varepsilon\right)^{-1} = \frac{1}{48}, \tag{38.61}$$

$$l = 0: \qquad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\det\left(M_{l=0}+\varepsilon\right)\left(M_{0l=0}+\varepsilon\right)^{-1} = -\frac{1}{12}. \tag{38.62}$$

As expected the determinant is negative, and we obtain the formal expression

$$\det{}' \mathbf{M}\mathbf{M}_0^{-1} = -\frac{1}{12} \times \left(\frac{1}{48}\right)^4 \times \prod_{l=2}^{\infty} \left[\frac{l(l-1)}{(l+2)(l+3)}\right]^{(l+1)^2} . \tag{38.63}$$

Renormalization. In these variables, the UV divergences appear as divergences of the infinite product on l. As an intermediate step, we use a maximum value L of l as a cut-off. From the general analysis, we know the UV divergent part of $\ln\det \mathbf{M}$ is completely contained in the two first terms of the expansion in powers of ϕ_c^2. Therefore, we proceed in the following way: the determinant of the operator

$$\mathbf{M}(s) = -\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 - \frac{s(s+1)}{\cosh^2 t}\,, \tag{38.64}$$

is exactly known:

$$\det\left[\mathbf{M}(s)+z\right]\left[\mathbf{M}_0+z\right]^{-1} = \frac{\Gamma(1+\sqrt{z})\Gamma(\sqrt{z})}{\Gamma(1+s+\sqrt{z})\Gamma(\sqrt{z}-s)}. \tag{38.65}$$

Setting:

$$s(s+1) = 6\gamma\,, \tag{38.66}$$

one expands $\ln\det \mathbf{M}(s)$ in powers of γ. One infers from this expansion, the expansion up to second order of $\ln\det \mathbf{M}$ in powers of the potential $-6/\cosh^2 t$ in the representation (38.59). One then subtracts these two terms from $\ln\det \mathbf{M}$ as obtained from the representation (38.63).

One verifies that, indeed, the large L limit of the subtracted quantity,

$$\begin{aligned}\left\{\det{}' \mathbf{M}\mathbf{M}_0^{-1}\right\}_{\text{ren.}}^{-1/2} &= \lim_{L\to+\infty} i2\sqrt{3}\times(48)^2 \prod_{l=2}^{L}\left[\frac{(l+2)(l+3)}{(l-1)}\right]^{(l+1)^2/2} \prod_{l=0}^{L} \mathrm{e}^{-3(l+1)} \\ &\quad \times \prod_{l=0}^{L} \mathrm{e}^{-18(l+1)^2}\left[\sum_{k=l+1}^{\infty}\frac{1}{k^2} - \frac{1}{l+1} - \frac{1}{2(l+1)^2}\right],\end{aligned} \tag{38.67}$$

is finite. We set:

$$\left\{\det{}' \mathbf{M}\mathbf{M}_0^{-1}\right\}_{\text{ren.}}^{-1/2} = iC_1\,. \tag{38.68}$$

Taking into account the Jacobians, the factor $(2\pi)^{-1/2}$ for each collective mode, the factor $(2i)^{-1}$, and a factor 2 for the two saddle points, one obtains a first factor C_2 of the form

$$C_2 = \frac{\lambda^3}{(-g)^{5/2}} \times \pi^5 \times \frac{32\sqrt{2}}{9\sqrt{3}} \times \frac{C_1}{(2\pi)^{5/2}}, \tag{38.69}$$

which we write as

$$C_2 = C_3\lambda^3/(-g)^{5/2}\,. \tag{38.70}$$

We then have to add to the classical action the two terms we have subtracted above from $\ln\det \mathbf{M}$. However, we can now write them in the normal space representation, regularized as we have regularized the perturbative correlation functions, and take into account the one-loop counter-terms. The first term in the expansion in powers of ϕ_c^2 is exactly cancelled by the mass counter-term, as we have already discussed. The second term in the expansion, which is the one-loop contribution to the four-point function, is logarithmically divergent. In Section 38.5, we calculate explicitly the finite difference between this term and the coupling constant counter-term that cancels the divergence.

38.5 Coupling constant renormalization

The terms we want to calculate involve the renormalized four-point function. We first choose a renormalization scheme for the field based on minimal subtraction (MS) after dimensional regularization. The renormalization constants have been calculated in Section 10.5. Note the different normalization of the coupling constant. The contribution $\delta\mathcal{S}_2$, which we have to add to the action, coming from the subtraction of $\ln\det\mathbf{M}$ and the one-loop coupling renormalization constant, is

$$\delta\mathcal{S}_2 = \frac{9}{4}\frac{N_d}{\varepsilon}g^2\int\phi_c^4(x)\mathrm{d}^4x - \frac{9}{4}g^2\,\mathrm{tr}\left[\phi_c^2\left(-\nabla^2\right)^{-1}\phi_c^2\left(-\nabla^2\right)^{-1}\right], \tag{38.71}$$

in which N_d is the usual loop factor,

$$N_d = 2(4\pi)^{-d/2}/\Gamma(d/2)\,, \tag{38.72}$$

and $d = 4-\varepsilon$. The expression can be rewritten as

$$\delta\mathcal{S}_2 = -\frac{9}{4}g^2\int\mathrm{d}^4x\,\mathrm{d}^4y\,\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{ip(x-y)}\,\phi_c^2(x)\phi_c^2(y)$$
$$\times\lim_{d\to4}\left(\int\frac{\mathrm{d}^dq}{(2\pi)^d}\frac{\mu^\varepsilon}{q^2(p-q)^2} - \frac{N_d}{\varepsilon}\right), \tag{38.73}$$

in which μ is the renormalization scale.

The integral over $\mathbf{q}$ has been performed in Section 10.1 (see equation (10.6)),

$$\int\frac{\mathrm{d}^dq}{(2\pi)^d}\frac{1}{q^2(p-q)^2} - \frac{N_d}{\varepsilon} = \frac{1}{8\pi^2}\left(\frac{1}{2}-\ln p\right) + O(\varepsilon)\,. \tag{38.74}$$

We also introduce the Fourier transform of the function $f^2(r)$ (for $\lambda = 1$) ($f(r)$ being defined by equation (38.42a)),

$$v(p) = \frac{1}{(2\pi)^4}\int\mathrm{d}^4x\frac{8\,\mathrm{e}^{ipx}}{(1+x^2)^2}. \tag{38.75}$$

The solution $\phi_c(x)$ depends on the scale λ. Rescaling the variables x, y, and p, we can then write the complete expression more explicitly as

$$\delta\mathcal{S}_2 = -\frac{9\pi^2}{2}\int\mathrm{d}^4p\,v^2(p)\left[\tfrac{1}{2} - \ln\left(\lambda p/\mu\right)\right]. \tag{38.76}$$

From the definition of $v(p)$, we derive after a short calculation,

$$\int\mathrm{d}^4p\,v^2(p) = \frac{2}{(3\pi^2)}, \tag{38.77}$$

$$\int\mathrm{d}^4p\,\ln p\,v^2(p) = \frac{2}{3\pi^2}\left(\ln 2 + \gamma + \frac{1}{6}\right), \tag{38.78}$$

in which γ is Euler's constant: $\gamma = -\psi(1) = 0.577215\ldots$. We then obtain

$$\delta\mathcal{S}_2 = 3\ln\lambda/\mu - \ln C_4\,, \tag{38.79}$$

with

$$\ln C_4 = 1 - 3\ln 2 - 3\gamma\,. \tag{38.80}$$

We note that the right-hand side of equation (38.79) now depends on the scale parameter λ. The interpretation of this result is the following: the coupling constant renormalization breaks the scale invariance of the classical theory and, therefore, the scale parameter λ remains in the expression. Moreover, the term proportional to $\ln\lambda$ together with the contribution from the classical action can be rewritten as

$$\frac{8\pi^2}{3g} - 3\ln\lambda/\mu = \frac{8\pi^2}{3g(\lambda)} + O\left(g\right), \tag{38.81}$$

in which $g(\lambda)$ is the effective coupling at the scale λ, solution of the RG equation

$$\frac{\mathrm{d}g(\lambda)}{\mathrm{d}\ln\lambda} = \beta\left[g(\lambda)\right], \tag{38.82}$$

with

$$\beta(g) = \frac{9}{8\pi^2}g^2 + O\left(g^3\right). \tag{38.83}$$

This property is expected. The renormalization of the perturbative expansion renders the instanton contribution, before integration over dilatation, finite. Therefore, this contribution should satisfy a RG equation, and the coupling constant g can be present only in the combination $g(\lambda)$, since λ fixes the scale in the calculation (for details, see Section A38.3).

38.6 The imaginary part of the n-point function

The complete contribution to the imaginary part of the n-point function then takes the form

$$\begin{aligned}&\operatorname{Im} Z^{(n)}(x_1,\ldots,x_n)\\ &\underset{g\to 0_-}{\sim} C_5\int \mathrm{d}^4x_0\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda}\lambda^4\prod_{i=1}^{n}\frac{2\sqrt{2}\lambda}{1+\lambda^2\left(x_i-x_0\right)^2}\frac{\mathrm{e}^{8\pi^2/3g(\lambda)}}{(-g)^{(n+5)/2}},\end{aligned} \tag{38.84}$$

where we have set:

$$C_5 = C_3C_4\,.$$

To calculate the Fourier transform of the expression (38.84), we introduce

$$u(p) = 2\sqrt{2}\int \mathrm{e}^{ipx}\,\frac{\mathrm{d}^4x}{1+x^2}. \tag{38.85}$$

Then, after factorizing the δ-function of momentum conservation, one obtains

$$\operatorname{Im}\tilde{Z}^{(n)}(p_1,\ldots,p_n) \sim \frac{C_5}{(-g)^{(n+5)/2}}\int_0^\infty \mathrm{d}\lambda\,\lambda^{3-3n}\,\mathrm{e}^{8\pi^2/3g(\lambda)}\prod_{i=1}^{n}u(p_i/\lambda). \tag{38.86}$$

The corresponding expression for vertex functions is

$$\operatorname{Im}\tilde{\Gamma}^{(n)}(p_1,\ldots,p_n) \sim \frac{C_5}{(-g)^{(n+5)/2}}\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda}\lambda^{4-n}\,\mathrm{e}^{8\pi^2/3g(\lambda)}\prod_{i=1}^{n}(p_i^2/\lambda^2)\,u\,(p_i/\lambda). \tag{38.87}$$

One verifies that $p^2u(p)$ goes to a constant for $|p|$ small.

In contrast to the super-renormalizable case, because the theory is only renormalizable, the final result is not totally explicit, since it involves a final integration over dilatations whose convergence is not obvious. Let us now discuss this point.

The small instanton contribution. Small instantons correspond to λ large. For λ large, the integral behaves like

$$\int^{\infty} \mathrm{d}\lambda\, \lambda^{3-n}\, \mathrm{e}^{8\pi^2/3g(\lambda)} \tag{38.88}$$

and, therefore, we have to examine the behaviour of $g(\lambda)$ for λ large. From equation (38.83), we see that the theory is UV asymptotically free, because for g is negative, that is, $g(\lambda)$ goes to zero for λ large. Thus, perturbation theory is applicable, and we can use the approximation (38.81). The argument remains true even if we take g slightly complex. Thus, the integral has the form

$$\int^{\infty} \mathrm{d}\lambda\, \lambda^{-n}. \tag{38.89}$$

We note that the power behaviour in λ depends explicitly on the coefficient of the g^2 term of the $\beta(g)$-function. Without the contribution coming from $g(\lambda)$, the integral (38.89) would have a UV divergence similar to the one found in the corresponding perturbative expansion. Due to the additional power of λ coming from $g(\lambda)$, only the vacuum amplitude is divergent.

The convergence of the dilatation integral is thus better than expected: indeed, the renormalization constants are now themselves given by divergent series and are complex for g negative. Their imaginary part contributes directly to the imaginary part of $\tilde{\Gamma}^{(n)}(p_1,\ldots,p_n)$ for $n \le 4$.

In the ϕ^6 QFT in dimension 3, for example, these contributions cancel the divergences coming from the integral over λ. By contrast, here the integrals over λ are finite at this order. In particular, this implies that, in the MS scheme, the imaginary parts of the renormalization constants vanish at leading order. In another renormalization scheme (*e.g.* fixed-momentum subtraction), these imaginary parts are finite at leading order.

The large instanton contribution. We now examine the convergence of the λ integral for λ small. The behaviour of $g(\lambda)$ is unknown. On the other hand, it is easy to verify that the factors $u(p_i/\lambda)$ decrease exponentially for λ small. Thus, if the behaviour of $g(\lambda)$ does not cancel this decrease, the integrals converge, and it is justified to replace $g(\lambda)$ by the expansion (38.81). For the vacuum amplitude, this argument does not apply, and so the result is unknown.

This analysis shows that, although this calculation seems to be a simple formal extension of the calculation for lower dimensions, coupling constant renormalization introduces a set of new problems, which are not all completely under control. The fact that the theory is massless only makes matters worse. Consideration of the massive theory improves the situation in this respect, but the instanton calculation becomes more involved.

38.7 The massive theory

In Section A38.2, we show that the massive field equation has no instanton solutions, and that the minimum of the action is obtained from the massless theory. To study the massive theory, we thus start from the instanton solution of the massless theory with its scale parameter λ. However, a problem appears: as explained in Section 38.4, the integral of ϕ_c^2 is IR divergent.

Thus, it is necessary to modify the field configuration at large distances, by connecting it smoothly to the solution of the massive free equation with mass m. The idea is to define a configuration which, up to a distance R, $\lambda R \gg 1$, $mR \ll 1$, is $\lambda\phi_c(\lambda x)$, and for $|x| > R$ is proportional to the free massive solution. An analogous problem is met in Chapter 42 in the case of multi-instanton configurations. Although the theory is no longer scale invariant, λ has to be kept as a collective coordinate. The mass term then acts as an IR cut-off, and restrict the domain of integration in λ to values large compared to m. The classical action has the form

$$\mathcal{S}_m(\phi_c) = -\frac{1}{g}\left(\frac{8\pi^2}{3} + 8\pi^2\frac{m^2}{\lambda^2}\ln\frac{\lambda}{m}\right), \quad \text{for} \quad \lambda \gg m\,, \tag{38.90}$$

where the $\ln m$ term is directly related to the initial IR divergence of the ϕ^2 integral.

The remaining part of the calculation closely follows the calculation for the massless case, and the reader is referred to the literature for details [389].

In the massless theory, the instanton contribution to the vacuum energy could not be evaluated without some knowledge of the non-perturbative IR behaviour of the RG β-function. In the massive theory the problem is absent, because the λ integral is cut at a scale $m/\sqrt{-g}$. For correlation functions, the integral is cut by the largest between momenta and $m/\sqrt{-g}$. This implies that the limits $m \to 0$ and $g \to 0$ do not commute.

38.8 Cosmology: The decay of the false vacuum

In previous sections, we have determined the probability for a 'false vacuum' of a QFT to decay through barrier penetration. It had been speculated [390] that such a phenomenon could be linked to the dynamics of the early Universe. When the Universe started to cool down, some symmetries started to be spontaneously broken. However, some region might have been trapped in the wrong phase. The false vacuum must eventually decay in the true vacuum, but if the process is slow enough, it might have occurred at a much later time when the Universe was already cool. This kind of physics speculation can be studied by an instanton approach [391].

According to the previous discussion, if the Universe is in the wrong vacuum, there is some probability at each point in space for some bubble of true vacuum to be created, and if the bubble is large enough, it becomes favourable for it to expand, eventually absorbing the whole space. To discuss what happens once a bubble has been created, it is useful to consider first the analogous problem for a quantum particle.

Quantum particle. In the example of a quantum particle, a semi-classical description of the decay process is the following: a particle is sitting in the well of the potential corresponding to the metastable minimum. At a given time, it makes a quantum jump and reappears outside of the barrier at the point where the potential has the same value as in the bottom of the well, with zero velocity (by energy conservation). Then its further trajectory can be entirely described by classical mechanics.

QFT. We apply the same ideas to the field theoretical model we discuss in this section. At time 0, the system makes a quantum jump. According to the previous discussion, the value of the field at time 0 is then (with the choice $x_{0\mu} = 0$)

$$\phi(t = 0, \mathbf{x}) = \phi_c(x_d = 0, \mathbf{x}), \qquad (\mathbf{x} = x_1, \dots, x_{d-1})\,, \tag{38.91}$$

and its time derivative vanishes,

$$\left.\frac{\partial}{\partial t}\phi(t, \mathbf{x})\right|_{t=0} = 0\,. \tag{38.92}$$

At a later time, $\phi(t,\mathbf{x})$ then obeys the *real-time* field equation,

$$\left[\nabla_x^2 - (\partial/\partial t)^2\right]\phi(t,\mathbf{x}) = \frac{1}{\sqrt{g}}V'\big(\sqrt{g}\phi(t,\mathbf{x})\big). \tag{38.93}$$

The first equation (38.91) tells us that the same function describes the form of the instanton in Euclidean space, and its shape in ordinary $(d-1)$ space when it materializes. We now consider the continuation in real time of the solution of the Euclidean field equation $\phi_c[(\mathbf{x}^2 - t^2)^{1/2}]$ (since $\phi_c(r)$ is an even function, the sign in front of the square root is irrelevant). It satisfies the conditions (38.91, 38.92), and clearly obeys the field equation (38.93). Therefore, it is the solution of our problem for positive times.

Since the size of the bubble is given by microphysics, the interior of the bubble corresponds to small values of r on a macroscopic scale,

$$0 \le \mathbf{x}^2 - t^2 = r^2 \ll 1\,.$$

Therefore, after a short time, the bubble starts expanding at almost the speed of light.

A38 Instantons: Additional remarks

Here we prove a few simple relations and inequalities concerning instanton solutions, which we have used in several places in the chapter, discuss the RG properties of instanton contributions, and the conformal invariance of the $\phi^4_{d=4}$ massless QFT.

A38.1 Virial theorem

We consider the general action, with polynomial potential,

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\bigl(\nabla\phi(x)\bigr)^2 + \mathcal{V}\bigl(\phi(x)\bigr)\right], \tag{A38.1}$$

and assume that the field equation has a finite action solution $\phi_c(x)$. If the action $\mathcal{S}(\phi_c)$ is finite, so is the action $\mathcal{S}(\phi_c,\lambda)$ for $\phi(x) = \phi_c(\lambda x)$, where λ is an arbitrary constant. If we now change variables in the action setting $\lambda x = x'$, we obtain [386]

$$\mathcal{S}(\phi_c,\lambda) = \lambda^{2-d}\int \tfrac{1}{2}\bigl(\nabla\phi_c(x)\bigr)^2 \mathrm{d}^d x + \lambda^{-d}\int \mathcal{V}\bigl(\phi_c(x)\bigr)\mathrm{d}^d x\,. \tag{A38.2}$$

Since $\phi_c(x)$ satisfies the field equation, the variation of the action vanishes for $\lambda = 1$:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{S}(\phi_c,\lambda)\Big|_{\lambda=1} = 0 \;\Rightarrow\; (d-2)\int \tfrac{1}{2}\bigl(\nabla\phi_c(x)\bigr)^2\mathrm{d}^d x + d\int \mathcal{V}\bigl(\phi_c(x)\bigr)\mathrm{d}^d x = 0\,. \tag{A38.3}$$

The classical action $\mathcal{S}(\phi_c)$ can thus be expressed in terms of the kinetic term only,

$$\mathcal{S}(\phi_c) = \frac{1}{d}\int \bigl(\nabla\phi_c(x)\bigr)^2 \mathrm{d}^d x > 0\,, \tag{A38.4}$$

a form that shows that $\mathcal{S}(\phi_c)$ is always positive.

It is also interesting to calculate the second derivative of $\mathcal{S}(\phi_c,\lambda)$,

$$\frac{\mathrm{d}^2}{(\mathrm{d}\lambda)^2}\mathcal{S}(\phi_c,\lambda)\Big|_{\lambda=1} = (2-d)\int \bigl(\nabla\phi_c(x)\bigr)^2\mathrm{d}^d x\,. \tag{A38.5}$$

For $d \ge 2$, the solution is not a local minimum of the action, and the operator defined by

$$\langle x\,|\mathbf{M}|\,x'\rangle = \frac{\delta^2\mathcal{S}}{\delta\phi(x)\delta\phi(x')}\Big|_{\phi=\phi_c},$$

has at least one negative eigenvalue.

Moreover, a general theorem [385] states that $\phi_c(x)$ corresponds to an absolute minimum of $\mathcal{S}(\phi)$ at fixed integral of the potential $\int \mathrm{d}^d x\,\mathcal{V}(\phi_c(x))$. If $\mathbf{M}$ has two negative eigenvalues, one can then find a linear combination of the corresponding two eigenvectors which, added to $\phi_c(x)$, leaves at first-order the integral of the potential unchanged, and decreases $\mathcal{S}(\phi)$. This contradicts the theorem. Thus $\mathbf{M}$ has at most one negative eigenvalue. Since the equation ($A38.5$) shows that $\mathbf{M}$ has at least one negative eigenvalue, it has one and only one.

Special potentials: Other relation. We consider potentials of the special form ($N > 2$),

$$\mathcal{V}(\phi) = \tfrac{1}{2} m^2 \phi^2 + g\phi^N . \tag{A38.6}$$

If the action $\mathcal{S}(\phi_c)$ is finite, so is $\mathcal{S}(\phi_c/\lambda)$. Again, if ϕ_c is a solution, the derivative of $\mathcal{S}(\phi_c/\lambda)$ for $\lambda = 1$ vanishes. This yields

$$\left.\frac{\mathrm{d}\mathcal{S}(\phi_c/\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=1} = -2 \int \frac{1}{2} \left[(\nabla\phi_c(x))^2 + m^2\phi_c^2(x) \right] \mathrm{d}^d x - Ng \int \phi_c^N(x) \mathrm{d}^d x = 0 . \tag{A38.7}$$

Combining the relations (A38.3) and (A38.7), for $2d - N(d-2) \neq 0$, one infers

$$\mathcal{S}(\phi_c) = \frac{N-2}{2d - N(d-2)} m^2 \int \phi_c^2(x) \mathrm{d}^d x . \tag{A38.8}$$

Since equation (A38.4) implies $\mathcal{S}(\phi_c) > 0$, equation (A38.8) can only be satisfied for $2 < N < 2d/(d-2)$. Therefore, the existence of instanton solutions implies that the QFT must be super-renormalizable.

In the marginal renormalizable case $N = 2d/(d-2)$ $((N = 3, d = 6), (N = 4, d = 4), (N = 6, d = 3))$, one derives

$$m^2 \int \phi_c^2(x) \mathrm{d}^d x = 0 ,$$

which is only satisfied for $m = 0$. The massive field equations have also no instanton solution, only the massless equations do. We examine this problem more thoroughly in Section A38.2, in the example $N = 4$.

A38.2 Sobolev inequalities

We consider the functional

$$R(\varphi) = \frac{\left\{ \int \mathrm{d}^d x \left[\left(\nabla\varphi(x)\right)^2 + \varphi^2(x) \right] \right\}^2}{\int \varphi^4(x) \mathrm{d}^d x} . \tag{A38.9}$$

For dimensions $d \le 4$, Sobolev inequalities imply [392]

$$R(\varphi) \ge R > 0 . \tag{A38.10}$$

In addition, for $d < 4$, there exists a spherically symmetric, zero-free function $\varphi_c(x)$, such that $R(\varphi_c) = R$, and which is a solution of the variational equation

$$\left.\frac{\delta R}{\delta\varphi(x)}\right|_{\varphi=\varphi_c} = 0 . \tag{A38.11}$$

Dimension smaller than 4. The equation (A38.11) has the explicit form

$$\left(-\nabla^2 + 1\right) \varphi(x) - \varphi^3(x) K = 0 , \tag{A38.12}$$

in which we have defined

$$K = \int \mathrm{d}^d x \left[\left(\nabla\varphi_c(x)\right)^2 + \varphi_c^2(x) \right] \Big/ \int \varphi_c^4(x) \mathrm{d}^d x . \tag{A38.13}$$

This equation is, up to a rescaling of $\varphi_c(x)$, the equation of motion (38.6). Both equations become identical if we choose the scale of $\varphi_c(x)$, which is otherwise arbitrary, such that

$$K = 1 \ \Rightarrow f(x) = \varphi_c(x) \,. \tag{A38.14}$$

For each instanton solution, we have derived the identities (38.14). Combining them with $K = 1$, we obtain

$$A = \frac{3}{2}\int \mathrm{d}^d x\, f^4(x) = \frac{3}{2}R(\varphi_c)\,. \tag{A38.15}$$

The smallest action solution thus corresponds to the minimum of $R(\varphi)$:

$$A = 3R/2\,, \tag{A38.16}$$

and the solution $f(x)$ we are looking for is given by

$$f(x) = \varphi_c(x), \quad \text{for} \quad K = 1\,. \tag{A38.17}$$

The introduction of the functional $R(\varphi)$ has the following advantage: the action (38.1) is obviously not bounded from below. But, if the fields $\phi(x)$ are restricted to be solutions of the equation of motion with finite action, then the action can be related to the functional $R(\varphi)$, which is bounded from below for all fields.

We then derive the property that the operator $\mathbf{M}$ has one and only one negative eigenvalue from the form of $R(\varphi)$, and the assumption that φ_c corresponds to an absolute minimum of R. The operator defined by the kernel

$$\left.\frac{\delta^2 R}{\delta\varphi(x)\delta\varphi(x')}\right|_{\varphi=\varphi_c}, \tag{A38.18}$$

is positive. Calculating the second functional derivative of R explicitly, one obtains

$$\left.\frac{\delta^2 R}{\delta\varphi(x)\delta\varphi(x')}\right|_{\varphi=\varphi_c} = 4\left\{\left[-\nabla^2 + 1 - 3\varphi_c^2(x)\right]\delta^{(d)}(x-x') + \frac{2\varphi_c^3(x)\varphi_c^3(x')}{\int \varphi_c^4(y)\mathrm{d}^d y}\right\}. \tag{A38.19}$$

We have again set $K = 1$. We will now express $\mathbf{M}$ in terms of $f(x)$, or $\varphi_c(x)$, for $m = 1$,

$$\langle x\,|\mathbf{M}|\,x'\rangle = \left[-\nabla^2 + 1 - 3\varphi_c^2(x)\right]\delta^{(d)}(x-x'). \tag{A38.20}$$

Therefore, we have derived the relation

$$\langle x\,|\mathbf{M}|\,x'\rangle = \frac{1}{4}\left.\frac{\delta^2 R}{\delta\varphi(x)\delta\varphi(x')}\right|_{\varphi=\varphi_c} - 2\left(\frac{\varphi_c^3(x')\varphi_c^3(x)}{\int \varphi_c^4(y)\mathrm{d}^d y}\right). \tag{A38.21}$$

(i) Since $R(\varphi)$ is invariant in the change $\varphi_c(x)$ in $\lambda\varphi_c(x)$, φ_c is an eigenvector of $\delta^2 R/(\delta\varphi_c)^2$ with eigenvalue 0, thus

$$\int \varphi_c(x')\varphi_c(x)\,\langle x\,|\mathbf{M}|\,x'\rangle\,\mathrm{d}^d x\,\mathrm{d}^d x' = -2\int \varphi_c^4(x) < 0\,. \tag{A38.22}$$

The operator $\mathbf{M}$ has at least one negative eigenvalue.

(ii) Since $\mathbf{M}$ is the sum of a positive operator and a projector of rank 1, it can have at most one negative eigenvalue.

Indeed, if it had two negative eigenvalues, one could find a linear combination of the corresponding two eigenvectors which would decrease $\mathbf{M}$ at an average of the projector fixed. This would imply that the kernel $\delta^2 R/\delta\varphi(x)\delta\varphi(x')|_{\varphi=\varphi_c}$ does not define a positive operator.

We conclude that $\mathbf{M}$ has *one and only one* negative eigenvalue.

Dimension 4. Let us calculate $R[\varphi(\lambda x)]$ for $d \le 4$. Then, changing λx in x' in the various integrals, we obtain

$$R\left[\varphi(\lambda x)\right] = \frac{\left\{\int \mathrm{d}^d x \left[\lambda^{2-d}\left(\nabla\varphi(x)\right)^2 + \lambda^{-d}\varphi^2(x)\right]\right\}^2}{\lambda^{-d}\int \varphi^4(x)\mathrm{d}^d x}. \qquad (A38.23)$$

We can now write

$$R = \min_{\{\varphi(x)\}}, \min_{\lambda} R\left[\varphi(\lambda x)\right]. \qquad (A38.24)$$

The minimum in λ of expression (A38.23) is obtained for

$$\lambda = \left[\frac{d}{(4-d)}\frac{\int \varphi^2(x)\mathrm{d}^d x}{\int\left(\nabla\varphi(x)\right)^2\mathrm{d}^d x}\right]^{1/2}, \qquad (A38.25)$$

and equation (A38.24) becomes

$$R = \min_{\{\varphi(x)\}} \frac{16}{d^{d/2}(4-d)^{(4-d)/2}} \frac{\left(\int\left(\nabla\varphi(x)\right)^2\mathrm{d}^d x\right)^{d/2}\left(\int\varphi^2(x)\mathrm{d}^d x\right)^{2-d/2}}{\int\varphi^4(x)\mathrm{d}^d x}. \qquad (A38.26)$$

For $d = 4$, we note that the solution is $\lambda = \infty$, and expression (A38.26) is just the equivalent of expression (A38.9) for the massless ϕ^4 QFT. Since, for $d = 4$, the massless ϕ^4 is scale invariant, the contribution of the mass term can be arbitrarily decreased by a rescaling of the variable x.

We can draw two interesting conclusions from this analysis: the minimal value of $R(\varphi)$ is the same in four dimensions for the massive and the massless theory. The same applies to the ϕ^4 action.

The minimum of $R(\varphi)$ is obtained from a solution of the massless field equation. The massive field equation has no solution.

These remarks explain a number of peculiarities of the ϕ^4 QFT in four dimensions, which we have discussed in Section 38.3.

A38.3 Instantons and RG equations

We now briefly describe a few RG properties of the instanton contributions in the $\phi^4_{d=4}$ QFT (see Section 38.3).

The instanton contribution to the n-point vertex function can be written as

$$\mathrm{Im}\,\tilde{\Gamma}^{(n)}(p_i;\mu,g) = \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda} F^{(n)}(p_i;\mu,g,\lambda), \qquad (A38.27)$$

in which μ represents the subtraction scale, and λ the dilatation parameter. The counter-terms that renormalize the perturbative expansion, also render $F^{(n)}$ finite for reasons we have explained. Therefore, $F^{(n)}$ satisfies the RG equation,

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\eta(g)\right] F^{(n)}(p_i;\mu,g,\lambda) = 0. \qquad (A38.28)$$

Integrated by the method of characteristics, the equation yields

$$F^{(n)}(p_i;\mu;g;\lambda) = Z^{-n/2}(\tau)F^{(n)}(p_i;\mu\tau,g(\tau),\lambda), \tag{A38.29}$$

with the definitions:

$$\begin{aligned}\ln\tau &= \int_g^{g(\tau)} \frac{\mathrm{d}g'}{\beta(g')},\\ \ln Z &= \int_g^{g(\tau)} \frac{\eta(g')}{\beta(g')}.\end{aligned} \tag{A38.30}$$

The coupling constant $g(\tau)$ is the effective coupling constant at the scale τ.

Dimensional analysis implies the scaling relation

$$F^{(n)}(p_i;\mu;g;\lambda) = \lambda^{4-n}F^{(n)}(p_i/\lambda,\mu/\lambda,g,1). \tag{A38.31}$$

Applied to the right-hand side of equation ($A38.29$), this identity yields

$$F^{(n)}(p_i;\mu;g;\lambda) = Z^{-n/2}(\tau)\lambda^{4-n}F^{(n)}\left(\frac{p_i}{\lambda},\frac{\mu\tau}{\lambda},g(\tau)\right). \tag{A38.32}$$

Finally, the choice

$$\tau = \lambda/\mu$$

leads to the relation

$$F^{(n)}(p_i;\mu;g;\lambda) = [Z(\lambda/\mu)]^{-n/2}\,\lambda^{4-n}F^{(n)}\left[p_i/\lambda;g(\lambda/\mu)\right]. \tag{A38.33}$$

A38.4 Conformal invariance

In Section 38.3, the scale invariance of the classical $\phi^4_{d=4}$ field theory has made it possible to obtain an analytic instanton solution. Moreover, by introducing the special coordinates (t,n_μ), we have been able to use the results obtained for the anharmonic oscillator in Chapter 37 and calculate explicitly the instanton contribution at leading order. Actually, the scale invariant classical $\phi^4_{d=4}$ QFT is also conformal invariant (see Section A13.3). This property, which also holds for other scale invariant field theories like gauge theories, can be used more directly to calculate the instanton contribution [384]. The conformal group is isomorphic to $SO(5,1)$. It is expected that the minimal action solution will be invariant under $SO(5)$, the maximal compact subgroup of $SO(5,1)$. A stereographic mapping of $\mathbb{R}^4$ onto the sphere S_4 can be used to simplify the $SO(5)$ transformations. One sets:

$$\xi^\mu = \frac{2x^\mu}{1+\mathbf{x}^2},\quad \xi^5 = \frac{1-\mathbf{x}^2}{1+\mathbf{x}^2} \quad\Rightarrow\quad \sum_{a=1}^{5}\xi^a\xi^a = 1\,. \tag{A38.34}$$

Correspondingly, one can introduce a field that has simple transformation properties under $SO(5)$. In the $\phi^4_{d=4}$ QFT, the conformal transformation properties of the field ϕ lead to set:

$$\phi(x) = \frac{1}{1+\mathbf{x}^2}\psi(x). \tag{A38.35}$$

We then express the classical action (38.35) in terms of these new variables by performing the transformations in two steps: first we keep the variables x^μ, but now considered as coordinates on S_4, and only perform the substitution ($A38.35$).

The metric $g_{\mu\nu}$ on S_4 in the coordinates x^μ is

$$g_{\mu\nu} = 4\frac{\delta_{\mu\nu}}{\left(1+\mathbf{x}^2\right)^2}. \tag{A38.36}$$

The invariant measure on the sphere involves the square root of the determinant of the metric $\mathbf{g}$ (see Section 28.3.1):

$$\sqrt{\det \mathbf{g}} = \frac{16}{\left(1+\mathbf{x}^2\right)^4}. \tag{A38.37}$$

Finally, after an integration by parts, the kinetic term can be rewritten as

$$\int \mathrm{d}^4 x\ \left(\nabla\phi(x)\right)^2 = \int \mathrm{d}^4 x \left[\frac{\left(\nabla\psi(x)\right)^2}{\left(1+\mathbf{x}^2\right)^2} + \frac{8\psi^2(x)}{\left(1+\mathbf{x}^2\right)^4}\right]. \tag{A38.38}$$

The classical action then reads

$$\mathcal{S}(\psi) = \int \mathrm{d}^4 x \sqrt{\det \mathbf{g}} \left(\tfrac{1}{8}\sum_{\mu,\nu} g^{\mu\nu}(x)\partial_\mu\psi(x)\partial_\nu\psi(x) + \tfrac{1}{4}\psi^2(x) + \tfrac{1}{64}g\psi^4(x)\right). \tag{A38.39}$$

In this covariant form (see Section 28.3), the change of coordinates ($A38.34$) is straightforward and hardly necessary. One solution of minimal action is a constant:

$$\psi^2(x) = -1/8g\,. \tag{A38.40}$$

The classical action is then proportional to the surface of S_4 which is $8\pi^2/3$. The operator $\mathbf{M}$, second functional derivative of the action, is given by (see Section 3.5)

$$\mathbf{M} = \tfrac{1}{4}\mathbf{L}^2 - 1\,, \tag{A38.41}$$

in which $\mathbf{L}$ is the angular momentum in five space dimensions. The eigenvalues of $\mathbf{L}^2$ are $l(l+3)$ with the degeneracy

$$\delta_l = \frac{1}{6}\frac{(2l+3)\Gamma(l+3)}{\Gamma(l+1)}. \tag{A38.42}$$

The form of $\mathbf{M}$ shows that it has 0 as eigenvalue, corresponding to $l=1$, with degeneracy 5, in agreement with the considerations of Section 38.4. We leave it up to the reader, as an exercise, to verify other results.