Fully developed isotropic turbulence: Symmetries and exact identities

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We consider the regime of fully developed isotropic and homogeneous turbulence of the Navier-Stokes equation with a stochastic forcing. We present two *gauge* symmetries of the corresponding Navier-Stokes field theory and derive the associated general Ward identities. Furthermore, by introducing a local source bilinear in the velocity field, we show that these symmetries entail an infinite set of *exact* and *local* relations between correlation functions. They include in particular the Kármán-Howarth relation and another exact relation for a pressure-velocity correlation function recently derived in G. Falkovich, I. Fouxon, and Y. Oz [J. Fluid Mech. **644**, 465 (2010)] that we further generalize.

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I. INTRODUCTION

The application of field-theoretic methods to Navier-Stokes (NS) turbulence has a long history [1,2]. The most systematic way to implement them consists in deriving the generating functional $\mathcal{Z}$ of velocity correlation and response functions under the form of a functional integral, using the standard Martin-Siggia-Rose-Janssen-de Dominicis procedure (recalled below). The resulting action $\mathcal{S}$ involves not only the original fields—velocity $\vec{v}(t,\vec{x})$ and pressure $p(t,\vec{x})$—but also the associated response fields $\bar{\vec{v}}(t,\vec{x})$ and $\bar{p}(t,\vec{x})$. The symmetries of the model and of the field theory play a crucial role, especially regarding the renormalization properties of the model, since symmetries can yield important nonrenormalization theorems. From this point of view, Kolmogorov exact result for the three-velocity correlation function should follow from a symmetry of the NS field theory, but this is not the way it is usually presented.

In this work, we perform a detailed analysis of the symmetries of the NS field theory. We review the well-known Galilean invariance and the gauged (in time) version of this symmetry. Furthermore, we exhibit a gauged (in time) shift symmetry of the response velocity field that was not, to the best of our knowledge, identified yet and which constitutes the first main contribution of this paper. Both these symmetries yield very useful Ward identities that we derive. We show in Ref. [3] that, in the framework of a nonperturbative renormalization group analysis, they eventually lead to a first-principles understanding of multiscaling.

Furthermore, by adding in the generating functional $\mathcal{Z}$, a source term bilinear in the velocity field $\exp(\int_{t,\vec{x}} v_\alpha L_{\alpha\beta} v_\beta)$, we show that the response velocity shift symmetry can be fully gauged, both in time and space. The functional Ward identity ensuing from this gauge symmetry yields infinitely many exact relations among correlation and response functions. This constitutes the second main contribution of this paper. We show in particular that the Kármán-Howarth identity (which roots the four-fifths Kolmogorov exact result for the third order structure function), emerges as a consequence of this gauge symmetry. This general Ward identity also allows us to recover the exact relation, involving a pressure-velocity correlation function, derived in Ref. [4], and to generalize it.

Let us briefly expound on the context of these symmetry studies. The Galilean invariance is a fundamental property of the theory of turbulence, and its implications have been widely studied and discussed, in particular in field-theoretic descriptions. General Ward identities for time-independent Galilean invariance were derived early, e.g., in Refs. [5,6]. Time-dependent forms, also referred to as time-gauged, or extended, Galilean invariance were also introduced, e.g., in Refs. [5,7–9], mainly to derive exact results on the dimensions of composite operators in the operator product expansion. Functional Ward identities associated with both the gauged and nongauged Galilean symmetry were also obtained in Refs. [10,11] via the formalism of gauge fixing and Slavnov-Taylor identities related to the ensuing Becchi-Rouet-Stora (BRS) symmetry.

It is worth noting that a time-gauged Galilean symmetry was also unveiled in Ref. [12] in a different but closely related context: in the field theory associated with the Kardar-Parisi-Zhang equation [13], describing the stochastic growth and roughening of interfaces. This stochastic field theory shares several common features with the NS field theory (for a detailed discussion of the corresponding Ward identities, see Refs. [14,15]). Besides the time-gauged Galilean invariance, it is endowed with a time-gauged shift symmetry, highlighted in Ref. [14], reminiscent of the one we bring out here for the NS field theory, although it applies in the KPZ context to the original field instead of the response field. Furthermore, as already mentioned, this symmetry admits, for the NS field theory, a fully gauged form provided the bilinear source term $\vec{v}\cdot L\cdot\vec{v}$ is added, which yields more stringent Ward identities.

II. NAVIER-STOKES EQUATION WITH STOCHASTIC FORCING

To describe fully developed isotropic and homogeneous turbulence, one usually considers the forced Navier-Stokes equation:

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha = -\frac{1}{\rho}\partial_\alpha p + \nu\nabla^2 v_\alpha + f_\alpha, \tag{1}$$

where the velocity field $\vec{v}$, the pressure field p, and the forcing $\vec{f}$ depend on the space-time coordinates $(t,\vec{x})$ and with ν the kinematic viscosity and ρ the density of the fluid. The presence of a forcing $\vec{f}$ is essential to balance the dissipative nature of the (unforced) NS equation and maintain a turbulent steady state. We consider in the following incompressible flows, satisfying

$$\partial_\alpha v_\alpha = 0. \tag{2}$$

Within the inertial range, correlation functions are expected to be universal and thus insensitive to the precise mechanism of forcing at the integral scale L. One can hence conveniently average over various stochastic forcings $\vec{f}$ with a Gaussian probability distribution, chosen of zero mean and variance,

$$\langle f_\alpha(t,\vec{x}) f_\beta(t',\vec{x}\,')\rangle = 2\delta(t-t')N_{\alpha\beta}(|\vec{x}-\vec{x}\,'|), \tag{3}$$

where the forcing $N_{\alpha\beta}$ is exerted at the integral scale L.

III. NAVIER-STOKES FIELD THEORY

The NS equation in the presence of a stochastic forcing $\vec{f}$ stands as a Langevin equation. One can resort to the standard Martin-Siggia-Rose-Janssen-de Dominicis procedure [16–18] to derive the associated field theory. It may be achieved in two different but equivalent ways. Either the pressure field may be eliminated since the incompressibility constraint (2) uniquely fixes the pressure in terms of $\vec{f}$ and $\vec{v}$ as the solution of the Poisson equation

$$-\frac{1}{\rho}\partial_\alpha\partial_\alpha p = \partial_\alpha v_\beta \partial_\beta v_\alpha - \partial_\alpha f_\alpha \tag{4}$$

for given boundary conditions or, alternatively, the pressure field may be kept and one introduces Martin-Siggia-Rose response fields $\bar{v}_\alpha$ and $\bar{p}$ to enforce both the equation of motion (1) and the incompressibility constraint (2), which amounts in effect to encoding Eq. (4). We follow this latter route, because the pressure sector turns out to be very simple to handle since it is not renormalized (see Sec. IV A). Moreover, this procedure ensures the analyticity of the effective action Γ, whereas eliminating the pressure requires one to deal with nonanalytic (in wave vectors) transverse projectors. As derived in Appendix A, once the response fields are introduced, the stochastic forcing can be integrated out and one obtains the generating functional

$$\mathcal{Z}[\vec{J},\bar{\vec{J}},K,\bar{K}] = \int \mathcal{D}\vec{v}\,\mathcal{D}p\,\mathcal{D}\bar{\vec{v}}\,\mathcal{D}\bar{p}\; e^{-\mathcal{S}[\vec{v},\bar{\vec{v}},p,\bar{p}]+\mathcal{J}[\vec{v},\bar{\vec{v}},p,\bar{p}]}, \tag{5}$$

where the term $\mathcal{J}$ contains the sources $\vec{J}$, K, $\bar{\vec{J}}$, and $\bar{K}$ for the velocity, pressure, and response fields

$$\mathcal{J}[\vec{v},\bar{\vec{v}},p,\bar{p}] = \int_{\mathbf{x}}\{\vec{J}\cdot\vec{v} + \bar{\vec{J}}\cdot\bar{\vec{v}} + Kp + \bar{K}\bar{p}\} \tag{6}$$

with the notation $\mathbf{x} \equiv (t,\vec{x})$ and $\int_{\mathbf{x}} \equiv \int dt d^d\vec{x}$ and where the NS action $\mathcal{S}$ is given by

$$\begin{aligned}\mathcal{S}[\vec{v},\bar{\vec{v}},p,\bar{p}] = \int_{\mathbf{x}}\Big\{&\bar{p}(\mathbf{x})\,\partial_\alpha v_\alpha(\mathbf{x}) + \bar{v}_\alpha(\mathbf{x})\Big[\partial_t v_\alpha(\mathbf{x})\\ &+ v_\beta(\mathbf{x})\partial_\beta v_\alpha(\mathbf{x}) + \frac{1}{\rho}\partial_\alpha p(\mathbf{x}) - \nu\nabla^2 v_\alpha(\mathbf{x})\Big]\Big\}\\ &- \int_{t,\vec{x},\vec{x}'} \bar{v}_\alpha(t,\vec{x})N_{\alpha\beta}(|\vec{x}-\vec{x}'|)\bar{v}_\beta(t,\vec{x}').\end{aligned} \tag{7}$$

We introduce for later purpose the notation φ_i, $i=1,\ldots,4$, which stands for the fields $\vec{v}$, $\bar{\vec{v}}$, p, and $\bar{p}$, respectively, and j_i, $i=1,\ldots,4$, which stands for the sources $\vec{J}$, $\bar{\vec{J}}$, K, and $\bar{K}$, respectively.

Field expectation values in the presence of the external sources j_i are obtained as functional derivatives of $\mathcal{W} = \log\mathcal{Z}$ as

$$u_\alpha(\mathbf{x}) = \langle v_\alpha(\mathbf{x})\rangle = \frac{\delta\mathcal{W}}{\delta J_\alpha(\mathbf{x})}, \quad \bar{u}_\alpha(\mathbf{x}) = \langle \bar{v}_\alpha(\mathbf{x})\rangle = \frac{\delta\mathcal{W}}{\delta\bar{J}_\alpha(\mathbf{x})}$$

and similarly for the pressure fields, for which, for simplicity, the same notation can be kept for the fields and their average values,

$$p(\mathbf{x}) \equiv \langle p(\mathbf{x})\rangle = \frac{\delta\mathcal{W}}{\delta K(\mathbf{x})}, \quad \bar{p}(\mathbf{x}) \equiv \langle\bar{p}(\mathbf{x})\rangle = \frac{\delta\mathcal{W}}{\delta\bar{K}(\mathbf{x})}.$$

The effective action $\Gamma[\vec{u},\bar{\vec{u}},p,\bar{p}]$ is defined as the Legendre transform of $\mathcal{W}$:

$$\Gamma[\vec{u},\bar{\vec{u}},p,\bar{p}] + \mathcal{W}[\vec{J},\bar{\vec{J}},K,\bar{K}] = \mathcal{J}[\langle\varphi_i\rangle]. \tag{8}$$

IV. SYMMETRIES AND RELATED WARD IDENTITIES

In this section, we analyze three gauge symmetries of the NS action (7), the time-gauged Galilean and response velocity field shift symmetries, and also gauged shifts of the pressure fields. More precisely, for each of these symmetries, the different terms of the NS action $\mathcal{S}$ are either invariant or have a linear variation in the fields, and this entails important nonrenormalization theorems and general Ward identities. The mechanism is standard: One performs as a change of variables in the functional integral $\mathcal{Z}$, Eq. (5), the (infinitesimal) transformation of the fields corresponding to the symmetry under study. Since it must leave $\mathcal{Z}$ unaltered, one obtains the equality $\langle\delta\mathcal{S}\rangle = \langle\delta\mathcal{J}\rangle$. Then, because the variation of the action is linear in the fields, $\langle\delta\mathcal{S}[\varphi_i]\rangle = \delta\mathcal{S}[\langle\varphi_i\rangle]$ and

$$\begin{aligned}\langle\delta\mathcal{J}\rangle = \Big\langle\sum_i j_i\delta\varphi_i\Big\rangle = \sum_i\frac{\delta\Gamma}{\delta\langle\varphi_i\rangle}\langle\delta\varphi_i\rangle = \sum_i\frac{\delta\Gamma}{\delta\langle\varphi_i\rangle}\delta\langle\varphi_i\rangle\\ \equiv \delta\Gamma\end{aligned} \tag{9}$$

using the definition (8) of the Legendre transform. The last two equalities hold because the considered field transformations are all affine in the fields. The resulting identity hence imposes that the variation of the effective action Γ remains identical to the variation of the bare one $\mathcal{S}$ (in terms of the average fields). Thus, apart from the linearly varying terms, Γ must be invariant under the considered symmetry, and these (nonsymmetric) terms cannot be renormalized. These identities are now derived in detail for the three gauge symmetries.

A. Nonrenormalization of the pressure sector

Let us begin with the simplest identities, the field equations for the pressure sector. We consider the infinitesimal field transformation $p(t,\vec{x}) \to p(t,\vec{x}) + \epsilon(t,\vec{x})$ which describes a gauged shift of the pressure field[1] (and could be used to gauge it away). This change of variables leaves the functional integral (5) unchanged, and thus one deduces

$$0 = \int_{\mathbf{x}} \left\langle -\frac{\bar{v}_\alpha(\mathbf{x})\partial_\alpha \epsilon(\mathbf{x})}{\rho} + K(\mathbf{x})\epsilon(\mathbf{x}) \right\rangle. \tag{10}$$

Since this equality holds for all infinitesimal $\epsilon(t,\vec{x})$, this yields

$$K(\mathbf{x}) = \frac{\delta \Gamma}{\delta p(\mathbf{x})} = -\frac{\partial_\alpha \bar{u}_\alpha(\mathbf{x})}{\rho} = \frac{\delta S}{\delta p(\mathbf{x})}\bigg|_{\varphi_i = \langle \varphi_i \rangle}, \tag{11}$$

which means that the dependence in $p(t,\vec{x})$ of both the effective action Γ and the bare one $\mathcal{S}$ are identical. A similar identity can be derived for the response pressure field $\bar{p}$, considering the infinitesimal gauged field transformation $\bar{p}(t,\vec{x}) \to \bar{p}(t,\vec{x}) + \bar{\epsilon}(t,\vec{x})$, which reads

$$\bar{K}(\mathbf{x}) = \frac{\delta \Gamma}{\delta \bar{p}(\mathbf{x})} = \partial_\alpha u_\alpha(\mathbf{x}) = \frac{\delta S}{\delta \bar{p}(\mathbf{x})}\bigg|_{\varphi_i = \langle \varphi_i \rangle}. \tag{12}$$

One concludes that the whole pressure sector is not renormalized.

B. Time-gauged Galilean symmetry

We review in this section the time-gauged (or time-dependent) Galilean symmetry and rederive the corresponding functional Ward identity. More precisely, the variation of the NS action (7) under a time-gauged Galilean transformation is linear in the fields, and this entails a Ward identity, which is stronger than the usual nongauged one. Let us hence consider the infinitesimal time-gauged Galilean transformation $\mathcal{G}[\vec{\epsilon}(t)]$, defined as

$$\begin{aligned}
\delta v_\alpha(\mathbf{x}) &= -\dot{\epsilon}_\alpha(t) + \epsilon_\beta(t)\partial_\beta v_\alpha(\mathbf{x}),\\
\delta \bar{v}_\alpha(\mathbf{x}) &= \epsilon_\beta(t)\partial_\beta \bar{v}_\alpha(\mathbf{x}),\\
\delta p(\mathbf{x}) &= \epsilon_\beta(t)\partial_\beta p(\mathbf{x}),\\
\delta \bar{p}(\mathbf{x}) &= \epsilon_\beta(t)\partial_\beta \bar{p}(\mathbf{x}),
\end{aligned} \tag{13}$$

where $\dot{\epsilon}_\alpha = \partial_t \epsilon_\alpha$. The NS action is invariant under $\mathcal{G}(\vec{\epsilon})$ with an arbitrary constant vector $\vec{\epsilon}$, which corresponds to a translation in space, and also under $\mathcal{G}(\vec{\epsilon}\, t)$, which corresponds to the usual (nongauged) Galilean transformation.

To analyze the variation of the NS action under a generic transformation $\mathcal{G}[\vec{\epsilon}(t)]$, let us define, following the standard geometric interpretation [9], a Galilean scalar density, as a quantity $\psi(\mathbf{x})$ which transforms under (13) as $\delta\psi(\mathbf{x}) = \epsilon_\beta(t)\partial_\beta \psi(\mathbf{x})$. This definition implies that the integral over $\vec{x}$ of a scalar density is invariant under $\mathcal{G}$ since $\int_{\vec{x}} \delta\psi = 0$. The sum and the product of two scalar densities are scalar densities. The gradient of a scalar density is also a scalar density, whereas its time derivative is not. However, the Lagrangian time derivative, defined by

$$D_t \psi(\mathbf{x}) \equiv \partial_t \psi(\mathbf{x}) + v_\beta(\mathbf{x})\partial_\beta \psi(\mathbf{x}), \tag{14}$$

is a covariant time derivative since $D_t\psi$ remains a scalar density if ψ is.

The three fields $\bar{\vec{v}}$, p, and $\bar{p}$ are scalar densities by definition of the transformation (13), as is the gradient of the velocity field $\vec{v}$. In contrast, neither $\vec{v}$ nor $\partial_t \vec{v}$ are scalar densities. One concludes that all the terms in the NS action (7) are invariant under the time-gauged transformation $\mathcal{G}[\vec{\epsilon}(t)]$, apart from the term proportional to the Lagrangian time derivative of the velocity $D_t v_\alpha(\mathbf{x})$. Nevertheless, although this is not strictly a scalar density as its transform includes an additional contribution proportional to $\ddot{\epsilon}_\alpha(t)$, the variation of the corresponding term in the action is linear in the fields. In summary, the overall variation of the action $\mathcal{S}$ is

$$\delta \mathcal{S} = \delta \int_{\mathbf{x}} \bar{v}_\alpha(\mathbf{x}) D_t v_\alpha(\mathbf{x}) = -\int_{\mathbf{x}} \epsilon_\alpha(t)\partial_t^2 \bar{v}_\alpha(\mathbf{x}). \tag{15}$$

Hence, performing the transformation $\mathcal{G}[\vec{\epsilon}(t)]$ in the functional integral (5), one obtains

$$\begin{aligned}
\langle \delta \mathcal{S} \rangle = \langle \delta \mathcal{J} \rangle &= \int_{\mathbf{x}} \left\{ -\dot{\epsilon}_\alpha(t) J_\alpha(\mathbf{x}) + \epsilon_\beta(t) \sum_i j_i \partial_\beta \langle \varphi_i \rangle \right\}\\
&= \delta\Gamma,
\end{aligned} \tag{16}$$

which, since the equality is valid for all infinitesimal $\vec{\epsilon}(t)$, yields the following Ward identity:

$$\begin{aligned}
\int_{\vec{x}} \bigg\{ &[\delta_{\alpha\beta}\partial_t + \partial_\alpha u_\beta(\mathbf{x})] \frac{\delta\Gamma}{\delta u_\beta(\mathbf{x})} + \partial_\alpha \bar{u}_\beta(\mathbf{x}) \frac{\delta\Gamma}{\delta \bar{u}_\beta(\mathbf{x})}\\
&+ \partial_\alpha p(\mathbf{x}) \frac{\delta\Gamma}{\delta p(\mathbf{x})} + \partial_\alpha \bar{p}(\mathbf{x}) \frac{\delta\Gamma}{\delta \bar{p}(\mathbf{x})} \bigg\} = -\int_{\vec{x}} \partial_t^2 \bar{u}_\alpha(\mathbf{x}).
\end{aligned} \tag{17}$$

Thus, both variations of the effective action and of the bare one are identical. This entails that, apart from the term $\int_{\mathbf{x}} \bar{u}_\alpha(\mathbf{x}) D_t u_\alpha(\mathbf{x})$ which is not renormalized and remains equal to its bare expression, Γ is invariant under time-gauged Galilean transformations.

C. Time-gauged response fields shift symmetry

Let us show that another class of transformations yields nonrenormalization theorems because again the corresponding variation of the NS action $\mathcal{S}$ is linear in the fields. They consist in time-gauged shifts of the response fields, represented by the infinitesimal transformation[2]

$$\begin{aligned}
\delta \bar{v}_\alpha(\mathbf{x}) &= \bar{\epsilon}_\alpha(t),\\
\delta \bar{p}(\mathbf{x}) &= v_\beta(\mathbf{x})\bar{\epsilon}_\beta(t).
\end{aligned} \tag{18}$$

This transformation leaves in particular the combination $\bar{v}_\alpha v_\beta \partial_\beta v_\alpha + \bar{p}\partial_\alpha v_\alpha$ invariant. The overall variation of the action stems from the term $\int_{\mathbf{x}} \bar{v}_\alpha \partial_t v_\alpha$ and from the forcing

[1] The NS action is strictly invariant under a time-dependent shift of the pressure $p(t,\vec{x}) \to p(t,\vec{x}) + \epsilon(t)$, as recalled, e.g., in Ref. [19].

[2] The NS action is strictly invariant under a constant shift of the velocity response field, as noted, e.g., in Ref. [6].

term and is linear in the fields:

$$\delta\mathcal{S} = \int_{\mathbf{x}} \bar{\epsilon}_\beta(t)\partial_t v_\beta(\mathbf{x}) + 2\int_{t,\vec{x},\vec{x}'} \bar{\epsilon}_\alpha(t) N_{\alpha_\beta}(\vec{x}-\vec{x}')\bar{v}_\beta(t,\vec{x}'). \tag{19}$$

Hence, performing the change of variables (18) in the functional integral (5), one concludes that the variations of both the bare and effective actions coincide and deduces the following Ward identity:

$$\int_{\vec{x}} \left\{ \frac{\delta\Gamma}{\delta\bar{u}_\alpha(\mathbf{x})} + u_\alpha(\mathbf{x})\frac{\delta\Gamma}{\delta\bar{p}(\mathbf{x})} \right\} = \int_{\vec{x}} \partial_t u_\alpha(\mathbf{x}) + 2\int_{\vec{x},\vec{x}'} N_{\alpha_\beta}(\vec{x}-\vec{x}')\bar{u}_\beta(t,\vec{x}'). \tag{20}$$

Again, this implies that, apart from the term $\int_{\mathbf{x}} \bar{u}_\alpha \partial_t u_\alpha$ and the forcing term that are not renormalized, the effective action Γ is invariant under time-gauged response fields shift transformations.

D. General structure of the effective action Γ

Let us summarize the previous analysis of the symmetries of the NS field theory. The effective action Γ may be written as

$$\begin{aligned}\Gamma[\vec{u},\bar{\vec{u}},p,\bar{p}] = &\int_{\mathbf{x}} \left\{ \bar{u}_\alpha \left(\partial_t u_\alpha + u_\beta\partial_\beta u_\alpha + \frac{\partial_\alpha p}{\rho} \right) + \bar{p}\partial_\alpha u_\alpha \right\} \\ &- \int_{t,\vec{x},\vec{x}'} \{\bar{u}_\alpha(t,\vec{x}) N_{\alpha\beta}(\vec{x}-\vec{x}')\bar{u}_\beta(t,\vec{x}')\} \\ &+ \tilde{\Gamma}[\vec{u},\bar{\vec{u}}],\end{aligned} \tag{21}$$

where the explicit terms are not renormalized and thus keep their bare forms and the functional $\tilde{\Gamma}$ only depends on the velocity fields and is invariant under time-gauged Galilean and response velocity shift transformations.

V. EXACT RELATIONS IN THE PRESENCE OF A LOCAL BILINEAR SOURCE

In this section we consider the same model in the presence of a local source $L_{\alpha\beta}$ for the quadratic operator $v_\alpha(\vec{x},t)v_\beta(\vec{x},t)$. The advantage of adding such a source is that the response fields shift symmetry (18) can be completely gauged (both in *time and space*). The ensuing functional Ward identity yields in particular the well-known exact Kármán-Howarth relation [20], from which can be derived the four-fifths Kolmogorov law for the $S^{(3)}$ structure function [21]. This Ward identity also entails the exact relation for a pressure-velocity correlation function derived in Ref. [4] and further allows one to extend it. More generally, it constitutes a full functional relation from which can be deduced an infinite set of exact local relations which, to the best of our knowledge, were not obtained before. We first derive this Ward identity for the effective action Γ. However, since in the literature the known relations are expressed in terms of the connected correlation functions, we also formulate this functional relation for the corresponding generating functional.

A. Local Ward identity for the effective action

In order to deduce Ward identities, we consider the generalized generating functional in the presence of a local source $L_{\alpha\beta}(\mathbf{x})$ for the composite operator $v_\alpha(\mathbf{x})v_\beta(\mathbf{x})$,

$$\mathcal{Z}[\vec{J},\bar{\vec{J}},K,\bar{K},L] = \int \mathcal{D}\vec{v}\,\mathcal{D}p\,\mathcal{D}\bar{\vec{v}}\,\mathcal{D}\bar{p}\; e^{-\mathcal{S}[\vec{v},\bar{\vec{v}},p,\bar{p}]+\mathcal{J}_L[\vec{v},\bar{\vec{v}},p,\bar{p}]} \tag{22}$$

with the new source term

$$\mathcal{J}_L[\vec{v},\bar{\vec{v}},p,\bar{p}] = \int_{\mathbf{x}} \{\vec{J}\cdot\vec{v} + \bar{\vec{J}}\cdot\bar{\vec{v}} + Kp + \bar{K}\bar{p} + \vec{v}\cdot L\cdot\vec{v}\} \tag{23}$$

using matrix notation $\vec{v}\cdot L\cdot\vec{v} \equiv v_\alpha(\mathbf{x})L_{\alpha\beta}(\mathbf{x})v_\beta(\mathbf{x})$. We now consider a shift in the response fields gauged both in space and time:

$$\begin{aligned}\delta\bar{v}_\alpha(\mathbf{x}) &= \bar{\epsilon}_\alpha(t,\vec{x}),\\ \delta\bar{p}(\mathbf{x}) &= v_\beta(\mathbf{x})\bar{\epsilon}_\beta(t,\vec{x}).\end{aligned} \tag{24}$$

None of the terms of the NS action (7) are invariant under this transformation, but their variations are still linear in the fields. Thus, using the same mechanism, one can derive Ward identities, which are now completely local, i.e., no longer integrated over space, since the relation $\langle\delta\mathcal{S}\rangle = \langle\delta\mathcal{J}_L\rangle$ obtained when performing the change of variables (24) in the generating functional (22) is now valid for all $\bar{\vec{\epsilon}}(t,\vec{x})$. One obtains

$$\left\langle -\partial_t v_\alpha - \frac{1}{\rho}\partial_\alpha p + \nu\nabla^2 v_\alpha - \partial_\beta(v_\alpha v_\beta) + \bar{J}_\alpha + \bar{K}v_\alpha + \int_{\vec{x}'} (2N_{\alpha\beta}(\vec{x}-\vec{x}')\bar{v}_\beta(t,\vec{x}')) \right\rangle = 0. \tag{25}$$

The key role of the new source term is that the average $\langle\partial_\beta(v_\alpha v_\beta)\rangle$ can now be expressed as a derivative with respect to $L_{\alpha\beta}$. The precise form of the generalized Legendre transform in the presence of this source is the same as without it:

$$\Gamma[\vec{u},\bar{\vec{u}},p,\bar{p},L] + \mathcal{W}[\vec{J},\bar{\vec{J}},K,\bar{K},L] = \int_{\mathbf{x}} (\vec{J}\cdot\vec{u} + \bar{\vec{J}}\cdot\bar{\vec{u}} + Kp + \bar{K}\bar{p}), \tag{26}$$

that is (as usual with composite operators), we *do not perform* the Legendre transform with respect to the corresponding source. It follows that $\langle v_\alpha v_\beta\rangle = \frac{\delta\mathcal{W}}{\delta L_{\alpha\beta}} = -\frac{\delta\Gamma}{\delta L_{\alpha\beta}}$. Moreover, the Ward identities ensuing from the gauged shifts of the pressure fields are unchanged in the presence of the source L. Hence, using the explicit form of $\frac{\delta\Gamma}{\delta\bar{p}}$, Eq. (12), the identity (25) reads

$$\frac{\delta\Gamma}{\delta\bar{u}_\alpha} = \partial_t u_\alpha + \frac{1}{\rho}\partial_\alpha p - \nu\nabla^2 u_\alpha - \partial_\beta\left(\frac{\delta\Gamma}{\delta L_{\alpha\beta}}\right) - u_\alpha\partial_\beta u_\beta - 2\int_{\vec{x}'} N_{\alpha\beta}(\vec{x}-\vec{x}')\bar{u}_\beta(t,\vec{x}'). \tag{27}$$

This functional identity has several important properties. As already emphasized, this identity is *local* in space and time. Moreover, it implies that vertex functions involving the response field $\bar{\vec{u}}$ (that is, generalized response functions) can be expressed in terms of vertex functions involving the composite operator $v_\alpha v_\beta$ in a simple way. More precisely, it means that $\Gamma[\vec{u},\bar{\vec{u}},p,\bar{p},L]$ does not depend on L and $\bar{\vec{u}}$ independently. Furthermore, let us point out that the parameter ν explicitly enters this identity, whereas it was not present in the previous one (20) (integrated on space). Let us finally mention that the Ward identity for Γ stemming from the Galilean symmetry in the presence of the source L keeps the same form as without it, except for the source L which is added to the fields transforming as Galilean scalar densities (see below for $\mathcal{W}$).

B. Local Ward identity for connected correlation functions

Let us now write the same Ward identity for the generating functional of connected correlation functions $\mathcal{W}$. Equation (25) can alternatively be expressed as the following Ward identity:

$$-\partial_t\frac{\delta\mathcal{W}}{\delta J_\alpha}-\frac{1}{\rho}\partial_\alpha\frac{\delta\mathcal{W}}{\delta K}+\nu\nabla^2\frac{\delta\mathcal{W}}{\delta J_\alpha}+\bar{J}_\alpha+\bar{K}\frac{\delta\mathcal{W}}{\delta J_\alpha}-\partial_\beta\frac{\delta\mathcal{W}}{\delta L_{\alpha\beta}}+\int_{\vec{x}'}\left\{2\frac{\delta\mathcal{W}}{\delta\bar{J}_\beta(t,\vec{x}')}N_{\alpha\beta}(\vec{x}-\vec{x}')\right\}=0. \tag{28}$$

For completeness, we derive the time-gauged Galilean Ward identity for $\mathcal{W}$ in the presence of L. As previously, performing the change of variables (13) in the generating functional (22) yields the Ward identity:

$$\int_{\vec{x}}\left\{\partial_t^2\frac{\delta\mathcal{W}}{\delta\bar{J}_\alpha}+\partial_t J_\alpha+J_\beta\partial_\alpha\frac{\delta\mathcal{W}}{\delta J_\beta}+K\partial_\alpha\frac{\delta\mathcal{W}}{\delta p}+\bar{K}\partial_\alpha\frac{\delta\mathcal{W}}{\delta\bar{p}}+2\partial_t\left(L_{\alpha\beta}\frac{\delta\mathcal{W}}{\delta J_\beta}\right)+L_{\beta\gamma}\partial_\alpha\left(\frac{\delta\mathcal{W}}{\delta L_{\beta\gamma}}\right)\right\}=0. \tag{29}$$

This identity is very similar to the previous Ward identity (17) (but formulated here in terms of $\mathcal{W}$) except that L has to be also considered as a scalar density with respect to Galilean transformations.

C. Kármán-Howarth relation

We now show that the Ward identity (28) entails the Kármán-Howarth relation. For this, we differentiate this relation with respect to $J_\beta(t_y,\vec{y})$ and evaluate the resulting identity at zero external sources. This yields

$$0=(\nu\Delta_x-\partial_{t_x})\langle v_\alpha(\mathbf{x})v_\beta(\mathbf{y})\rangle-\frac{1}{\rho}\partial_\alpha^x\langle p(\mathbf{x})v_\beta(\mathbf{y})\rangle-\partial_\gamma^x\langle v_\alpha(\mathbf{x})v_\gamma(\mathbf{x})v_\beta(\mathbf{y})\rangle+2\int_{\vec{x}'}\langle\bar{v}_\gamma(\mathbf{x})v_\beta(\mathbf{y})\rangle N_{\alpha\gamma}(\vec{x}-\vec{x}'). \tag{30}$$

As shown in Appendix A, the term involving $N_{\alpha\gamma}$ can be simply expressed in term of the force as $\langle f_\gamma(\mathbf{x})v_\beta(\mathbf{y})\rangle$. We now set $\alpha=\beta$ and sum over α. By homogeneity, $\partial_\alpha^x=-\partial_\alpha^y$ when acting on averages, and thus the term proportional to the pressure vanishes due to the incompressibility constraint.[3]

[3]Note that the velocity field v_α is unconstrained in the functional integral (22). However, since the response pressure sector is not renormalized, the incompressibility constraint is in practice always satisfied on average at zero external sources.

Symmetrizing in $\mathbf{x}$ and $\mathbf{y}$ and considering equal times $t_x=t_y\equiv t$ eventually yields

$$-\partial_t\langle v_\alpha(\mathbf{x})v_\alpha(\mathbf{y})\rangle+\nu(\Delta_x+\Delta_y)\langle v_\alpha(\mathbf{x})v_\alpha(\mathbf{y})\rangle-\partial_\gamma^x\langle v_\alpha(\mathbf{x})v_\gamma(\mathbf{x})v_\alpha(\mathbf{y})\rangle-\partial_\gamma^y\langle v_\alpha(\mathbf{y})v_\gamma(\mathbf{y})v_\alpha(\mathbf{x})\rangle+\langle f_\alpha(\mathbf{x})v_\alpha(\mathbf{y})\rangle+\langle f_\alpha(\mathbf{y})v_\alpha(\mathbf{x})\rangle=0, \tag{31}$$

which identifies with the Kármán-Howarth relation [20]. The four-fifths Kolmogorov law [22] stating that the third-order structure function exactly obeys the Kolmogorov scaling follows from this relation [21].

D. Relation for the pressure-velocity correlation function of Ref. [4]

We now show that the Ward identity (28) also yields another exact relation, recently derived in Ref. [4]. For this, let us differentiate twice Eq. (28) with respect to $L_{\mu\nu}(t_y,\vec{y})$ and $J_\beta(t_z,\vec{z})$ and evaluate at zero external sources to obtain

$$0=(\nu\Delta^x-\partial_{t_x})\langle v_\alpha(\mathbf{x})v_\mu(\mathbf{y})v_\nu(\mathbf{y})v_\beta(\mathbf{z})\rangle-\frac{1}{\rho}\langle p(\mathbf{x})v_\mu(\mathbf{y})v_\nu(\mathbf{y})v_\beta(\mathbf{z})\rangle+\langle f_\alpha(\mathbf{x})v_\mu(\mathbf{y})v_\nu(\mathbf{y})v_\beta(\mathbf{z})\rangle-\partial_\gamma^x\langle v_\alpha(\mathbf{x})v_\gamma(\mathbf{x})v_\mu(\mathbf{y})v_\nu(\mathbf{y})v_\beta(\mathbf{z})\rangle, \tag{32}$$

where the term proportional to $N_{\alpha\beta}$ was expressed in terms of the forcing $\vec{f}$ following Appendix A. We now set $\mu=\nu$ and $\alpha=\beta$, sum over μ and α, and eventually choose coinciding space points $\vec{x}=\vec{z}$ and coinciding times $t_x=t_y=t_z$, which gives

$$\langle v_\alpha(\mathbf{x})v^2(\mathbf{y})[\nu\Delta^x-\partial_{t_x}]v_\alpha(\mathbf{x})\rangle-\frac{1}{\rho}\langle v^2(\mathbf{y})v_\alpha(\mathbf{x})\partial_\alpha^x p(\mathbf{x})\rangle+\langle f_\alpha(\mathbf{x})v_\alpha(\mathbf{x})v^2(\mathbf{y})\rangle-\langle v_\alpha(\mathbf{x})v_\gamma(\mathbf{x})v^2(\mathbf{y})\partial_\gamma^x v_\alpha(\mathbf{x})\rangle=0. \tag{33}$$

Notice that the correlation functions in this relation are connected since they all originate in differentiating the $\mathcal{W}$ functional. However, one can easily show that the nonconnected parts either vanish or factorize into $\langle\vec{v}^2\rangle$ times the left-hand side of the Kármán-Howarth relation, which hence also vanishes. Thus, Eq. (33) holds for both the connected and nonconnected correlation functions. In the stationary regime, the term involving the time derivative is proportional to $\partial_{t_x}\langle v^2(\mathbf{x})v^2(\mathbf{y})\rangle$ and thus vanishes, and one obtains

$$\nu\langle v_\alpha(\mathbf{x})\Delta^x v_\alpha(\mathbf{x})v^2(\mathbf{y})\rangle-\frac{1}{\rho}\partial_\alpha^x\langle v^2(\mathbf{y})v_\alpha(\mathbf{x})p(\mathbf{x})\rangle+\langle f_\alpha(\mathbf{x})v_\alpha(\mathbf{x})v^2(\mathbf{y})\rangle-\frac{1}{2}\partial_\alpha^x\langle v_\alpha(\mathbf{x})v^2(\mathbf{x})v^2(\mathbf{y})\rangle=0 \tag{34}$$

using the incompressibility constraint.[3] Equation (34) coincides with the exact relation for the pressure-velocity correlation function derived in Ref. [4]. According to the authors of Ref. [4], this relation can be simplified in the inertial range using arguments based on the decoupling of the large- and small-scale fields as:

$$\langle \vec{v}(\vec{r})p(\vec{r})v^2(0)\rangle \propto \vec{r}. \tag{35}$$

Other exact relations among (connected or nonconnected) correlation functions involving one pressure field and three velocity fields can be simply generated from Eq. (32). For instance, setting $\mu=\alpha$ and $\nu=\beta$, summing over μ and ν, and eventually choosing coinciding space points $\vec{x}=\vec{z}$ and coinciding times $t_x=t_y=t_z$ yields

$$0=\left\langle v_\alpha(\mathbf{y})\vec{v}(\mathbf{y}).\vec{v}(\mathbf{x})\left[(\nu\Delta^x-\partial_{t_x})v_\alpha(\mathbf{x})-\frac{1}{\rho}\partial_\alpha^x p(\mathbf{x})\right]\right\rangle + \langle f(\mathbf{x}).v(\mathbf{y})\vec{v}(\mathbf{y}).\vec{v}(\mathbf{x})\rangle-\langle v_\alpha(\mathbf{y})\vec{v}(\mathbf{y}).\vec{v}(\mathbf{x})v_\gamma(\mathbf{x})\partial_\gamma^x v_\alpha(\mathbf{x})\rangle. \tag{36}$$

Using the same arguments put forward in Ref. [4], this relation also can be simplified in the stationary regime and in the inertial range, as detailed in Appendix B. We find:

$$\frac{1}{\rho}\left\langle\partial_\alpha^x p(\mathbf{x})v_\alpha(0)\vec{v}(0).\vec{v}(\mathbf{x})\right\rangle = -\frac{d+2}{2d}\langle\epsilon\vec{v}^2\rangle+\langle\vec{f}.\vec{v}\,\vec{v}^2\rangle-\frac{d}{64}F(x^2)-\frac{1}{32}F'(x^2)x^2, \tag{37}$$

where

$$\langle v_\alpha(\mathbf{x})[\vec{v}(0)\cdot\vec{v}(\mathbf{x})]^2\rangle=\left[\frac{1}{32}F(x^2)+\frac{1}{d}\langle\epsilon\vec{v}^2\rangle\right]x_\alpha. \tag{38}$$

Thus, one concludes that if F does not diverge when $x\to 0$, then Eq. (37) implies that at leading order in $|\vec{x}|$, $\langle\partial_\alpha^x p(\mathbf{x})v_\alpha(0)\vec{v}(0)\cdot\vec{v}(\mathbf{x})\rangle$ is a constant, whereas if F diverges, then it follows that

$$\frac{1}{\rho}\left\langle\partial_\alpha^x p(\mathbf{x})v_\alpha(0)\vec{v}(0)\cdot\vec{v}(\mathbf{x})\right\rangle\simeq-\frac{1}{2}\partial_\alpha^x\langle v_\alpha(\mathbf{x})(\vec{v}(0)\cdot\vec{v}(\mathbf{x}))^2\rangle. \tag{39}$$

VI. CONCLUSION

In this paper, we analyze the symmetries of the NS field theory. We revisit in particular the time-gauged form of the Galilean symmetry and unveil a time-gauged response velocity shift symmetry, which, to the best of our knowledge, was not yet identified. We derive the related general (functional) Ward identities. Furthermore, we show, by introducing a local source term bilinear in the velocity, that the related Ward identities yield an infinite set of *exact* and *local* relations between correlation functions. They include in particular the Kármán-Howarth relation and a similar identity for the pressure-velocity correlation function recently derived in Ref. [4]. Furthermore, we generalize this latter identity.

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APPENDIX A: FIELD THEORY FOR THE NS EQUATION

In this appendix, we recall the derivation of the field theory associated with the NS equation in the presence of a stochastic stirring force $\vec{f}$, following the standard Janssen-de Dominicis procedure [17,18]. Let us consider the mean value of an observable $\mathcal{O}$, which is a functional of the velocity field, with respect to the distribution of the stochastic force. It is given by

$$\langle\mathcal{O}[\vec{v}]\rangle\propto\int\mathcal{D}\vec{f}\,\mathcal{O}[\vec{v}_f]\mathcal{P}[\vec{f}\,], \tag{A1}$$

where $\vec{v}_f$ denotes the incompressible solution of the NS equation for the specific realization $\vec{f}$ of the force (for fixed initial conditions, which are neglected as they are assumed to play no role in the stationary regime). The normalizations—proportionality constants—need not be specified either. The probability distribution $\mathcal{P}[\vec{f}\,]$ of the force is Gaussian and assumed to be given by

$$\mathcal{P}[\vec{f}\,]\propto\exp\left[-\frac{1}{4}\int_{t,\vec{x},\vec{x}'}f_\alpha(t,\vec{x})(N^{-1})_{\alpha\beta}(|\vec{x}-\vec{x}'|)f_\beta(t,\vec{x}')\right]. \tag{A2}$$

The mean value (A1) can be expressed as a functional integral over the velocity and pressure fields as

$$\langle\mathcal{O}[\vec{v}]\rangle\propto\int\mathcal{D}\vec{f}\,\mathcal{D}\vec{v}\,\mathcal{O}[\vec{v}]\mathcal{P}[\vec{f}\,]\delta(\vec{v}-\vec{v}_f)$$
$$\propto\int\mathcal{D}\vec{f}\,\mathcal{D}\vec{v}\,\mathcal{D}p\,\mathcal{O}[\vec{v}]\mathcal{P}[\vec{f}\,]\,\delta(\partial_\alpha v_\alpha)\times\delta(\mathcal{E}_\alpha) \tag{A3}$$

with

$$\mathcal{E}_\alpha\equiv\partial_t v_\alpha+v_\beta\partial_\beta v_\alpha+\frac{1}{\rho}\partial_\alpha p-\nu\nabla^2 v_\alpha-f_\alpha, \tag{A4}$$

where the second line of (A3) holds in Itō's discretization for which the transformation has a constant Jacobian independent of the fields. One introduces the Lagrange multipliers $\tilde{\vec{v}}$ and $\tilde{p}$ to enforce the equation of motion and the incompressibility constraint in the following way:

$$\langle\mathcal{O}[\vec{v}]\rangle\propto\int\mathcal{D}\vec{f}\,\mathcal{D}\vec{v}\,\mathcal{D}p\,\mathcal{D}\tilde{\vec{v}}\,\mathcal{D}\tilde{p}\,\mathcal{O}[\vec{v}]\mathcal{P}[\vec{f}\,]\times\exp\left\{-i\int_{\mathbf{x}}\tilde{v}_\alpha\,\mathcal{E}_\alpha-i\int_{\mathbf{x}}\tilde{p}\,\partial_\alpha v_\alpha\right\}. \tag{A5}$$

The functional integral on the force $\vec{f}$ then can be carried out and one obtains

$$\langle\mathcal{O}[\vec{v}]\rangle\propto\int\mathcal{D}\vec{v}\,\mathcal{D}p\,\mathcal{D}\tilde{\vec{v}}\,\mathcal{D}\tilde{p}\,\mathcal{O}[\vec{v}]\,e^{-\mathcal{S}[\vec{v},\tilde{\vec{v}},p,\tilde{p}]} \tag{A6}$$

with the action

$$\mathcal{S}[\vec{v},\tilde{\vec{v}},p,\tilde{p}] = i\int_{\mathbf{x}}\left\{\tilde{p}\,\partial_\alpha v_\alpha + \tilde{v}_\alpha\left[\partial_t v_\alpha + v_\beta\partial_\beta v_\alpha + \frac{1}{\rho}\partial_\alpha p - \nu\nabla^2 v_\alpha\right]\right\} + \int_{t,\vec{x},\vec{x}'}\tilde{v}_\alpha(t,\vec{x})N_{\alpha\beta}(|\vec{x}-\vec{x}'|)\tilde{v}_\beta(t,\vec{x}'). \tag{A7}$$

We finally redefine the fields as $\bar{p}\equiv i\tilde{p}$ and $\bar{\vec{u}}\equiv i\tilde{\vec{u}}$ and introduce sources $\vec{J}$, K, $\bar{\vec{J}}$, and $\bar{K}$ for the velocity, the pressure, and the response fields, respectively. One can hence write the Janssen-de Dominicis generating functional as

$$\mathcal{Z}[\vec{J},\bar{\vec{J}},K,\bar{K}] = \int\mathcal{D}\vec{v}\,\mathcal{D}p\,\mathcal{D}\bar{\vec{v}}\,\mathcal{D}\bar{p}\;e^{-\mathcal{S}[\vec{v},\bar{\vec{v}},p,\bar{p}]+\mathcal{J}[\vec{v},\bar{\vec{v}},p,\bar{p}]}, \tag{A8}$$

where $\mathcal{S}$ and $\mathcal{J}$ are given by Eqs. (7) and (6), respectively.

Let us now calculate the mean value of a quantity which depends linearly on the forcing:

$$\langle f_\alpha(t,\vec{x})\mathcal{O}[\vec{v}]\rangle \propto \int\mathcal{D}\vec{f}\,\mathcal{P}[\vec{f}]\,f_\alpha(t,\vec{x})\mathcal{O}[\vec{v}_f], \tag{A9}$$

with the same normalization factor as in Eq. (A1). The Janssen-de Dominicis procedure can be applied almost in the same way, yielding

$$\langle f_\alpha(t,\vec{x})\mathcal{O}[\vec{v}]\rangle \propto \int\mathcal{D}\vec{f}\;\mathcal{D}\vec{v}\,\mathcal{D}p\,\mathcal{D}\tilde{\vec{v}}\,\mathcal{D}\tilde{p}\;\,f_\alpha(t,\vec{x})\mathcal{O}[\vec{v}] \times\mathcal{P}[\vec{f}\,]\exp\left\{-i\int_{\mathbf{x}}\tilde{v}_\alpha\,\mathcal{E}_\alpha - i\int_{\mathbf{x}}\tilde{p}\,\partial_\alpha v_\alpha\right\}. \tag{A10}$$

One can now perform the Gaussian integral, taking into account that the quantity to be averaged depends on f_α, to obtain

$$\langle f_\alpha(t,\vec{x})\mathcal{O}[\vec{v}]\rangle = 2\int_{\vec{x}'}N_{\alpha\beta}(|\vec{x}-\vec{x}'|)\langle\bar{v}_\beta(t,\vec{x}')\mathcal{O}[\vec{v}]\rangle. \tag{A11}$$

The averages of quantities linear in $\vec{f}$ are thus related to response functions.

APPENDIX B: SIMPLIFICATION OF EQ. (36)

We consider Eq. (36), which reads:

$$0 = \left\langle v_\alpha(\mathbf{y})\vec{v}(\mathbf{y})\cdot\vec{v}(\mathbf{x})\left[(\nu\Delta^x - \partial_{t_x})v_\alpha(\mathbf{x}) - \frac{1}{\rho}\partial^x_\alpha p(\mathbf{x})\right]\right\rangle + \langle\vec{f}(\mathbf{x})\cdot\vec{v}(\mathbf{y})\vec{v}(\mathbf{y})\cdot\vec{v}(\mathbf{x})\rangle - \langle v_\alpha(\mathbf{y})\vec{v}(\mathbf{y})\cdot\vec{v}(\mathbf{x})v_\gamma(\mathbf{x})\partial^x_\gamma v_\alpha(\mathbf{x})\rangle. \tag{B1}$$

The term involving the time derivative is proportional to the time derivative of $\langle(\vec{v}(\mathbf{x}).\vec{v}(\mathbf{y}))^2\rangle$ and thus vanishes in the stationary regime. The force and the velocity are large-scale fields, and thus $\langle\vec{f}(\mathbf{x})\cdot\vec{v}(\mathbf{y})\vec{v}(\mathbf{y})\cdot\vec{v}(\mathbf{x})\rangle \simeq \langle\vec{f}\cdot\vec{v}\,\vec{v}^2\rangle$. Using that the local dissipation operator $\epsilon = -\nu v_\alpha(\mathbf{x})\partial_x^2 v_\alpha(\mathbf{x})$ is a small-scale field enables one to write $\langle v_\alpha(\mathbf{y})\vec{v}(\mathbf{y})\cdot\vec{v}(\mathbf{x})\nu\Delta^x v_\alpha(\mathbf{x})\rangle = -\langle\epsilon\vec{v}^2\rangle/d$. Finally, by introducing, as in Ref. [4], $u_\alpha = v_\alpha(\mathbf{x}) - v_\alpha(0)$ and $V_\alpha = v_\alpha(\mathbf{x}) + v_\alpha(0)$ that are, respectively, small- and large-scale fields, one obtains

$$\langle v_\alpha(\mathbf{y})(\vec{v}(\mathbf{x})\cdot\vec{v}(\mathbf{y}))^2\rangle = \tfrac{1}{32}\langle u_\alpha\vec{V}^4\rangle + \tfrac{1}{32}\langle u_\alpha\vec{u}^4\rangle - \tfrac{1}{16}\langle u_\alpha\vec{u}^2\vec{V}^2\rangle. \tag{B2}$$

The term $\langle u_\alpha\vec{u}^4\rangle$ is negligible in the inertial range. The term $\langle u_\alpha\vec{V}^4\rangle$ was not present in the other relation Eq. (34) and hence not discussed in Ref. [4]. This function vanishes when $|\vec{x}|\to 0$ and thus $\langle u_\alpha\vec{V}^4\rangle = F(x^2)x_\alpha$ with F a function that cannot diverge faster than $|\vec{x}|^{-1}$. Gathering the previous terms leads to Eq. (39).

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