all $g \in \tilde{G}_x$. A smooth function $F: U \to N$, where U is some open subset of M, is called *locally G-invariant* if for each $x \in U$ there is a neighbourhood $\tilde{G}_x \subset G_x$ of e in G such that $F(g \cdot x) = F(x)$ for all $g \in \tilde{G}_x$. F is called *globally G-invariant* (even though it is only defined on an open subset of M) if $F(g \cdot x) = F(x)$ for all $x \in U$, $g \in G$ such that $g \cdot x \in U$ also.

Example 2.13. Let G be the group of horizontal translations

$$(x, y) \mapsto (x + \varepsilon, y)$$

in $\mathbb{R}^2$. Then the line segment

$$\{(x, y): y = 0, \ -1 < x < 1\}$$

is locally G-invariant, but not G-invariant.

Similarly, the function

$$\zeta(x, y) = \begin{cases} 0, & y \leqslant 0, \quad \text{or} \quad y > 0 \quad \text{and} \quad x > 0, \\ e^{-1/y}, & y > 0 \quad \text{and} \quad x < 0, \end{cases}$$

is smooth and locally G-invariant on $U = \mathbb{R}^2 \setminus \{(0, y): y \geqslant 0\}$, since

$$\zeta(x + \varepsilon, y) = \zeta(x, y)$$

for $|\varepsilon| < |x|$; ζ is clearly not globally G-invariant.

Proposition 2.14. *Let $N \subset M$ be a submanifold of M. Then N is locally G-invariant if and only if for each $x \in N$, $\mathfrak{g}|_x \subset TN|_x$. In other words, N is locally G-invariant if and only if the infinitesimal generators $\mathbf{v}$ of G are everywhere tangent to N.*

The proof is left to the reader; see Exercise 2.1.

Invariants and Functional Dependence

Often we are interested in determining precisely "how many" invariants a given group of transformations has. To make this problem precise, we first note that if $\zeta^1(x), \ldots, \zeta^k(x)$ are invariants (either local or global) of a group of transformations, and $F(z^1, \ldots, z^k)$ is any smooth function, then $\zeta(x) = F(\zeta^1(x), \ldots, \zeta^k(x))$ will also be an invariant (of the same sort). Such an invariant adds no new knowledge to the given problem, and is termed "functionally dependent" on the preceding invariants $\zeta^1, \ldots, \zeta^k$. In practice, we need only classify functionally independent invariants of a group action, the other invariants all being obtained by relations of the above form.

Definition 2.15. Let $\zeta^1(x), \ldots, \zeta^k(x)$ be smooth, real-valued functions defined on a manifold M. Then

(a) $\zeta^1, \ldots, \zeta^k$ are called *functionally dependent* if for each $x \in M$ there is a neighbourhood U of x and a smooth real-valued function $F(z^1, \ldots, z^k)$, not identically zero on any open subset of $\mathbb{R}^k$, such that

$$F(\zeta^1(x), \ldots, \zeta^k(x)) = 0 \tag{2.9}$$

for all $x \in U$.

(b) $\zeta^1, \ldots, \zeta^k$ are called *functionally independent* if they are not functionally dependent when restricted to any open subset $U \subset M$; in other words, if $F(z^1, \ldots, z^k)$ is such that (2.9) hold for all x in some open $U \subset M$, then $F(z^1, \ldots, z^k) \equiv 0$ for all z in some open subset of $\mathbb{R}^k$ (which is contained in the image of U).

For example, the functions x/y and $xy/(x^2 + y^2)$ are functionally dependent on $\{(x, y)\colon y \neq 0\}$ since

$$\frac{xy}{x^2 + y^2} = \frac{x/y}{1 + (x/y)^2} = f\left(\frac{x}{y}\right)$$

there. On the other hand, x/y and $x + y$ are functionally independent where defined, since if $F(x + y, x/y) \equiv 0$ for (x, y) in any open subset of $\mathbb{R}^2$, then, by the inverse function theorem, the image set contains an open subset of $\mathbb{R}^2$ on which $F = 0$.

Note that functional dependence and functional independence do not exhaust the range of possibilities except in the case of analytic functions, where the vanishing of (2.9) in some open set implies its vanishing everywhere. For example, the smooth functions

$$\eta(x, y) = x, \qquad \zeta(x, y) = \begin{cases} x, & y \leqslant 0, \\ x + e^{-1/y}, & y > 0, \end{cases}$$

are dependent on the lower half plane $\{y < 0\}$, independent on the upper half plane $\{y > 0\}$, but *neither* on the entire (x, y)-plane. Finally, we note that $\zeta^1, \ldots, \zeta^k$ may be locally functionally dependent, but there may be no nonzero function $F(z^1, \ldots, z^k)$ such that (2.9) holds for all x in M. For instance, the image $\{(\zeta^1(x), \ldots, \zeta^k(x))\colon x \in M\}$ may be dense in some open subset of $\mathbb{R}^k$, so (2.9) would only hold with $F \equiv 0$ there.

The classical necessary and sufficient condition that $\zeta^1(x), \ldots, \zeta^k(x)$ be functionally dependent is that their $k \times m$ Jacobian matrix $(\partial\zeta^i/\partial x^j)$ be of rank $\leqslant k - 1$ everywhere. (See the notes at the end of this chapter regarding the proof of this result.)

Theorem 2.16. *Let $\zeta = (\zeta^1, \ldots, \zeta^k)$ be a smooth function from M to $\mathbb{R}^k$. Then $\zeta^1(x), \ldots, \zeta^k(x)$ are functionally dependent if and only if $d\zeta|_x$ has rank strictly less than k for all $x \in M$.*

The basic theorem regarding number of independent invariants of a group of transformations is the following.

Theorem 2.17. *Let G act semi-regularly on the m-dimensional manifold M with s-dimensional orbits. If $x_0 \in M$, then there exist precisely $m - s$ functionally independent local invariants $\zeta^1(x), \ldots, \zeta^{m-s}(x)$ defined in a neighbourhood of x_0. Moreover, any other local invariant of the group action defined near x_0 is of the form*

$$\zeta(x) = F(\zeta^1(x), \ldots, \zeta^{m-s}(x)) \tag{2.10}$$

for some smooth function F. If the action of G is regular, then the invariants can be taken to be globally invariant in a neighbourhood of x_0.

PROOF. Using Frobenius' Theorem 1.43, we can find flat local coordinates $y = \psi(x)$ near x_0 for the system of vector fields $\mathfrak{g}$ spanned by the infinitesimal generators of G, such that the orbits of G are the slices $\{y^1 = c_1, \ldots, y^{m-s} = c_{m-s}\}$. Then the new coordinates $y^1 = \zeta^1(x), \ldots, y^{m-s} = \zeta^{m-s}(x)$ themselves are local invariants for G, being constant on each slice. Moreover, any other invariant of G must also be constant on these slices, and hence a function of $y^1, \ldots, y^{m-s}$ only. Finally, if G acts regularly, we can choose our flat coordinate chart such that each orbit intersects it in at most one slice. In this case, $y^1, \ldots, y^{m-s}$ actually form global invariants. □

In classical terminology, the invariants constructed in this theorem are called a *complete set of functionally independent invariants.* We have shown that once we have found such a complete set, any other invariant of G can be expressed as a function of these invariants. There is an analogous result for invariant subvarieties.

Proposition 2.18. *Let G act semi-regularly on M and let $\zeta^1(x), \ldots, \zeta^{m-s}(x)$ be a complete set of functionally independent invariants defined on an open subset $W \subset M$. If a subvariety $\mathscr{S}_F = \{x: F(x) = 0\}$ is G-invariant, then for each solution $x_0 \in \mathscr{S}_F$ there is a neighbourhood $\tilde{W} \subset W$ of x_0, and an "equivalent" G-invariant function $\tilde{F}(x) = \tilde{F}(\zeta^1(x), \ldots, \zeta^{m-s}(x))$ whose solution set coincides with that of F in $\tilde{W}$:*

$$\mathscr{S}_F \cap \tilde{W} = \mathscr{S}_{\tilde{F}} \cap \tilde{W} = \{x \in \tilde{W}: \tilde{F}(\zeta^1(x), \ldots, \zeta^{m-s}(x)) = 0\}.$$

PROOF. Note first that we can complete the set of invariants $y^1 = \zeta^1(x), \ldots, y^{m-s} = \zeta^{m-s}(x)$ to be flat local coordinates $y = (y^1, \ldots, y^m)$ for G near x_0. In fact, the remaining coordinates $\hat{y} = (y^{m-s+1}, \ldots, y^m)$ can be chosen from among the given coordinates $(x^1, \ldots, x^m)$ so $\hat{y} = \hat{x} = (x^{i_1}, \ldots, x^{i_s})$. For example, if $\partial(\zeta^1, \ldots, \zeta^{m-s})/\partial(x^1, \ldots, x^{m-s}) \neq 0$ at x_0, then we can set $\hat{x} = (x^{m-s+1}, \ldots, x^m)$. Thus the change of coordinates is of the form $y = \psi(x) = (\zeta(x), \hat{x})$, in which $\zeta(x)$ denotes the invariants and $\hat{x}$ are called *parametric variables.* We write $F(x) = F^*(y) = F^*(\zeta(x), \hat{x})$ in terms of these coordinates, so $F^* = F \circ \psi^{-1}$. Set

$$\tilde{F}(\zeta(x)) = F^*(\zeta(x), \hat{x}_0),$$

where $\hat{x}_0$ is the value of the parametric variables $\hat{x}$ at x_0. Since $\mathscr{S}_F$ is G-invariant, and the orbits of G in these coordinates are the common level sets (or slices) $\{\zeta(x) = c\}$ of the invariants, we find $F^*(\zeta(x), \hat{x}) = 0$ if and only if $F^*(\zeta(x), \hat{x}_0) = 0$ since both points lie in the same slice. $\square$

Note that unless F itself is G-invariant, the corresponding $\tilde{F}$ will not be the same function; only their solution sets coincide. For instance, in the case presented in Example 2.9, $F(x, y) = x^4 + x^2y^2 + y^2 - 1$ has the same solution set as the SO(2)-invariant function $\tilde{F}(x, y) = x^2 + y^2 - 1$ even though they clearly disagree elsewhere.

Methods for Constructing Invariants

It remains to show how one finds the invariants of a given group action. First suppose G is a one-parameter group of transformations acting on M, with infinitesimal generator

$$\mathbf{v} = \xi^1(x)\frac{\partial}{\partial x^1} + \cdots + \xi^m(x)\frac{\partial}{\partial x^m}$$

expressed in some given local coordinates. A local invariant $\zeta(x)$ of G is a solution of the linear, homogeneous first order partial differential equation

$$\mathbf{v}(\zeta) = \xi^1(x)\frac{\partial \zeta}{\partial x^1} + \cdots + \xi^m(x)\frac{\partial \zeta}{\partial x^m} = 0. \tag{2.11}$$

Theorem 2.17 says that if $\mathbf{v}|_x \neq 0$, then there exist $m - 1$ functionally independent invariants, hence $m - 1$ functionally independent solutions of the partial differential equation (2.11) in a neighbourhood of x_0.

The classical theory of such equations shows that the general solution of (2.11) can be found by integrating the corresponding *characteristic system* of ordinary differential equations, which is

$$\frac{dx^1}{\xi^1(x)} = \frac{dx^2}{\xi^2(x)} = \cdots = \frac{dx^m}{\xi^m(x)}. \tag{2.12}$$

The general solution of (2.12) can be written in the form

$$\zeta^1(x^1, \ldots, x^m) = c_1, \ldots, \zeta^{m-1}(x^1, \ldots, x^m) = c_{m-1},$$

in which $c_1, \ldots, c_{m-1}$ are the constants of integration, and the $\zeta^i(x)$ are functions independent of the c_j's. It is then easily seen that the functions $\zeta^1, \ldots, \zeta^{m-1}$ are the required functionally independent solutions to (2.11). Any other invariant, i.e. any other solution of (2.11), will necessarily be a function of $\zeta^1, \ldots, \zeta^{m-1}$. We illustrate this technique with a couple of examples.

Example 2.19. (a) Consider the rotation group SO(2), which has infinitesimal generator $\mathbf{v} = -y\partial_x + x\partial_y$. The corresponding characteristic system is

$$\frac{dx}{-y} = \frac{dy}{x}.$$

This first order ordinary differential equation is easily solved; the solutions are $x^2 + y^2 = c$ for c an arbitrary constant. Thus, $\zeta(x, y) = x^2 + y^2$, or any function thereof, is the single independent invariant of the rotation group.

(b) Consider the vector field

$$\mathbf{v} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + (1 + z^2)\frac{\partial}{\partial z}$$

defined on $\mathbb{R}^3$. Note that $\mathbf{v}$ never vanishes, so we can find two independent invariants of the one-parameter group generated by $\mathbf{v}$, in a neighbourhood of any point in $\mathbb{R}^3$. The characteristic system in this case is

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{1 + z^2}.$$

The first of these two equations was solved in part (a), so one of the invariants is the radius $r = \sqrt{x^2 + y^2}$. To find the other invariant, note that r is a constant for all solutions of the characteristic system, so we can replace x by $\sqrt{r^2 - y^2}$ before integrating. This leads to the equation

$$\frac{dy}{\sqrt{r^2 - y^2}} = \frac{dz}{1 + z^2},$$

which has solution

$$\arcsin\frac{y}{r} = \arctan z + k$$

for k an arbitrary constant. Thus

$$\arctan z - \arcsin\frac{y}{r} = \arctan z - \arctan\frac{y}{x}$$

is a second independent invariant for $\mathbf{v}$. A slightly simpler expression comes by taking the tangent of this invariant, which is $(xz - y)/(yz + x)$, so

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \zeta = \frac{xz - y}{yz + x}$$

provide a complete set of functionally independent invariants (provided $yz \neq -x$). As usual, any function of r and ζ is also an invariant, so, for instance

$$\tilde{\zeta} = \frac{r}{\sqrt{1 + \zeta^2}} = \frac{x + yz}{\sqrt{1 + z^2}}$$

is also an invariant, which in conjunction with r forms yet another pair of independent invariants. (This iterative technique of using knowledge of some invariants to simplify the computation of the remaining invariants is extremely useful for solving characteristic systems in general.)

The computation of independent invariants for r-parameter groups of transformations when $r > 1$ can get very complicated. If $\mathbf{v}_k = \sum \xi_k^i(x)\partial/\partial x^i$, $k = 1, \ldots, r$, form a basis for the infinitesimal generators, then the invariants are found by solving the system of homogeneous, linear, first order partial differential equations

$$\mathbf{v}_k(\zeta) = \sum_{i=1}^{m} \xi_k^i(x)\frac{\partial \zeta}{\partial x^i} = 0, \qquad k = 1, \ldots, r.$$

In other words, each invariant ζ must be a *joint invariant* of all the vector fields $\mathbf{v}_1, \ldots, \mathbf{v}_k$. One way to proceed is to first compute the invariants of one of the vector fields, say $\mathbf{v}_1$. Since any joint invariant ζ must in particular be an invariant of $\mathbf{v}_1$, we can write ζ as some function of the computed invariants of $\mathbf{v}_1$. Thus, we should re-express the remaining vector fields $\mathbf{v}_2$, $\ldots, \mathbf{v}_r$ using the invariants of $\mathbf{v}_1$ as coordinates, and then find joint invariants of these "new" $r - 1$ vector fields. The procedure then works inductively, leading eventually to the joint invariants of all the vector fields expressed in terms of the joint invariants of the first $r - 1$ of them. The process will become clearer in an example.

Example 2.20. Consider the vector fields

$$\mathbf{v} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \qquad \mathbf{w} = 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (z^2 + 1 - x^2 - y^2)\frac{\partial}{\partial z}$$

on $\mathbb{R}^3$. These were considered in Example 1.42, where it was shown that they generate a two-parameter abelian group of transformations on $\mathbb{R}^3$, which is regular on $M = \mathbb{R}^3 \backslash (\{x = y = 0\} \cup \{x^2 + y^2 = 1, z = 0\})$. An invariant $\zeta(x, y, z)$ is a solution to the pair of equations $\mathbf{v}(\zeta) = 0 = \mathbf{w}(\zeta)$. First note that independent invariants of $\mathbf{v}$ are just $r = \sqrt{x^2 + y^2}$ and z. We now re-express $\mathbf{w}$ in terms of r and z,

$$\mathbf{w} = 2rz\frac{\partial}{\partial r} + (z^2 + 1 - r^2)\frac{\partial}{\partial z}.$$

Since ζ must be a function of the invariants r, z of $\mathbf{v}$, it must be a solution to the differential equation

$$\mathbf{w}(\zeta) = 2rz\frac{\partial \zeta}{\partial r} + (z^2 + 1 - r^2)\frac{\partial \zeta}{\partial z} = 0.$$

The characteristic system here is

$$\frac{dr}{2rz} = \frac{dz}{z^2 + 1 - r^2}.$$

Solving this ordinary differential equation, we find that

$$\zeta = \frac{z^2 + r^2 + 1}{r} = \frac{x^2 + y^2 + z^2 + 1}{\sqrt{x^2 + y^2}}$$

is the single independent invariant of this group. (This result was given in Example 1.42, but without the details of the intervening calculation.)

2.2. Groups and Differential Equations

Suppose we are considering a system $\mathscr{S}$ of differential equations involving p independent variables $x = (x^1, \ldots, x^p)$, and q dependent variables $u = (u^1, \ldots, u^q)$. The solutions of the system will be of the form $u = f(x)$, or, in components, $u^\alpha = f^\alpha(x^1, \ldots, x^p)$, $\alpha = 1, \ldots, q$.† Let $X = \mathbb{R}^p$, with coordinates $x = (x^1, \ldots, x^p)$, be the space representing the independent variables, and let $U = \mathbb{R}^q$, with coordinates $u = (u^1, \ldots, u^q)$, represent the dependent variables. A symmetry group of the system $\mathscr{S}$ will be a local group of transformations, G, acting on some open subset $M \subset X \times U$ in such a way that "G transforms solutions of $\mathscr{S}$ to other solutions of $\mathscr{S}$". Note that we are allowing arbitrary nonlinear transformations of both the independent and dependent variables in our definition of symmetry.

To proceed rigorously, we must explain exactly how a given transformation g in the Lie group G transforms a function $u = f(x)$. We begin by identifying the function $u = f(x)$ with its graph

$$\Gamma_f = \{(x, f(x)) : x \in \Omega\} \subset X \times U,$$

where $\Omega \subset X$ is the domain of definition of f. Note that Γ_f is a certain p-dimensional submanifold of $X \times U$. If $\Gamma_f \subset M_g$, the domain of definition of the group transformation g, then the transform of Γ_f by g is just

$$g \cdot \Gamma_f = \{(\tilde{x}, \tilde{u}) = g \cdot (x, u) : (x, u) \in \Gamma_f\}.$$

The set $g \cdot \Gamma_f$ is not necessarily the graph of another single-valued function $\tilde{u} = \tilde{f}(\tilde{x})$. However, since G acts smoothly and the identity element of G leaves Γ_f unchanged, by suitably shrinking the domain of definition Ω of f we ensure that for elements g near the identity, the transform $g \cdot \Gamma_f = \Gamma_{\tilde{f}}$ is the graph of some single-valued smooth function $\tilde{u} = \tilde{f}(\tilde{x})$. We write $\tilde{f} = g \cdot f$ and call the function $\tilde{f}$ the *transform* of f by g.

Example 2.21. Let $p = 1$, $q = 1$, so $X = \mathbb{R}$, with a single independent variable x, and $U = \mathbb{R}$ with a single dependent variable u. (We are thus in the situation of a single ordinary differential equation involving a single function $u = f(x)$.)

† We will consistently employ Latin subscripts or superscripts to refer to the independent variables and Greek subscripts or superscripts to refer to the dependent variables.

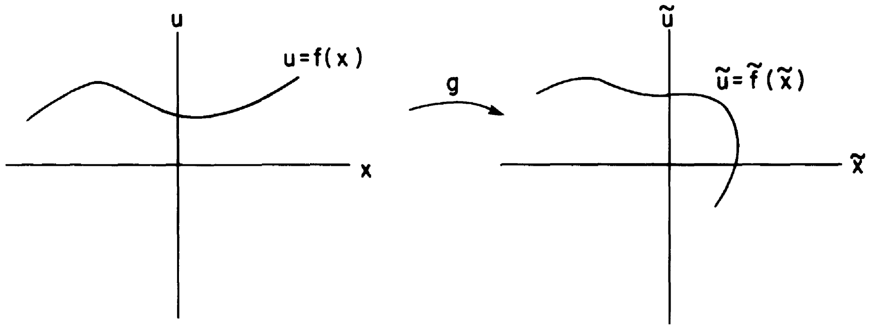

Figure 6. Action of a group transformation on a function.

Let $G = \mathrm{SO}(2)$ be the rotation group acting on $X \times U \simeq \mathbb{R}^2$. The transformations in G are given by

$$(\tilde{x}, \tilde{u}) = \theta \cdot (x, u) = (x \cos \theta - u \sin \theta, x \sin \theta + u \cos \theta). \tag{2.13}$$

Suppose $u = f(x)$ is a function, whose graph is a subset $\Gamma_f \subset X \times U$. The group SO(2) acts on f by rotating its graph. Clearly, if the angle θ is sufficiently large, the rotated graph $\theta \cdot \Gamma_f$ will no longer be the graph of a single-valued function. However, if $f(x)$ is defined on a finite interval $a \leqslant x \leqslant b$, and $|\theta|$ is not too large, then $\theta \cdot \Gamma_f$ will be the graph of a well-defined function $\tilde{u} = \tilde{f}(\tilde{x})$, with $\Gamma_{\tilde{f}} = \theta \cdot \Gamma_f$.

As a specific example, consider the linear function

$$u = f(x) = ax + b.$$

The graph of f is a straight line, so its rotation through angle θ will be another straight line, which, as long as it is not vertical, will be the graph of another linear function $\theta \cdot f = \tilde{f}$, the transform of f by the rotation through angle θ. To find the precise formula for $\theta \cdot f$, note that by (2.13) a point $(x, u) = (x, ax + b)$ on the graph of f is rotated to the point

$$(\tilde{x}, \tilde{u}) = (x \cos \theta - (ax + b) \sin \theta, x \sin \theta + (ax + b) \cos \theta).$$

In order to find $\tilde{u} = \tilde{f}(\tilde{x})$, we must eliminate x from this pair of equations; this is possible provided $\cot \theta \neq a$ (in particular, for θ sufficiently near 0), so that the graph is not vertical. We find

$$x = \frac{\tilde{x} + b \sin \theta}{\cos \theta - a \sin \theta},$$

hence $\theta \cdot f = \tilde{f}$ is given by

$$\tilde{u} = \tilde{f}(\tilde{x}) = \frac{\sin \theta + a \cos \theta}{\cos \theta - a \sin \theta} \tilde{x} + \frac{b}{\cos \theta - a \sin \theta},$$

which, as we noticed earlier, is again a linear function.

In general, the procedure for finding the transformed function $\tilde{f} = g \cdot f$ is much the same as in this elementary example. Suppose the transformation g is given in coordinates by

$$(\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\Xi_g(x, u), \Phi_g(x, u)),$$

for smooth functions Ξ_g, Φ_g. Then the graph $\Gamma_{\tilde{f}} = g \cdot \Gamma_f$ of $g \cdot f$ is given parametrically by the equations

$$\begin{aligned} \tilde{x} &= \Xi_g(x, f(x)) = \Xi_g \circ (\mathbb{1} \times f)(x), \\ \tilde{u} &= \Phi_g(x, f(x)) = \Phi_g \circ (\mathbb{1} \times f)(x), \end{aligned} \qquad x \in \Omega.$$

Here $\mathbb{1}$ denotes the identity function of X, so $\mathbb{1}(x) = x$, and $\times$ is the Cartesian product of functions. To find $\tilde{f} = g \cdot f$ explicitly, we must eliminate x from these two systems of equations. Since for $g = e$, $\Xi_e \circ (\mathbb{1} \times f) = \mathbb{1}$, we know that, provided g is sufficiently near the identity, the Jacobian matrix of $\Xi_g \circ (\mathbb{1} \times f)$ is nonsingular and hence by the inverse function theorem we can locally solve for x:

$$x = [\Xi_g \circ (\mathbb{1} \times f)]^{-1}(\tilde{x}).$$

Substitution into the second system yields the required equation for the transform $g \cdot f$:

$$g \cdot f = [\Phi_g \circ (\mathbb{1} \times f)] \circ [\Xi_g \circ (\mathbb{1} \times f)]^{-1}, \tag{2.14}$$

which holds whenever the second factor is invertible. This general formula is slightly complicated, but this was to be expected from our experience with just linear functions and the rotation group.

Example 2.22. Consider the special case in which the group G transforms just the independent variables x. Thus the transformations in G take the special form

$$(\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\Xi_g(x), u),$$

in which Ξ_g is, in fact, a diffeomorphism of X with $\Xi_g^{-1} = \Xi_{g^{-1}}$ where defined. If $\Gamma_f = \{(x, f(x))\}$ is the graph of a smooth function, then its transform $g \cdot \Gamma_f = \{g \cdot (x, f(x))\}$ is *always* the graph of a smooth function. Indeed

$$(\tilde{x}, \tilde{u}) = g \cdot (x, f(x)) = (\Xi_g(x), f(x)).$$

Thus we can easily eliminate x by inverting Ξ_g, with

$$\tilde{u} = \tilde{f}(\tilde{x}) = f(\Xi_g^{-1}(\tilde{x})) = f(\Xi_{g^{-1}}(\tilde{x})).$$

For example, if G is a group of translations

$$(x, u) \mapsto (x + \varepsilon a, u), \qquad \varepsilon \in \mathbb{R},$$

for $a \in X$ fixed, then the transform of the function $u = f(x)$ is the translate

$$\tilde{u} = \tilde{f}(\tilde{x}) = f(\tilde{x} - \varepsilon a)$$

of f.

The same sort of result holds in the more general case of a *projectable* or *fiber-preserving* group of transformations, in which the action on the independent variables does not depend on the dependent variables:

$$g \cdot (x, u) = (\Xi_g(x), \Phi_g(x, u)).$$

For example, the one-parameter group

$$g_\varepsilon\colon (x, t, u) \mapsto (x + 2\varepsilon t, t, e^{-\varepsilon x - \varepsilon^2 t} u), \qquad \varepsilon \in \mathbb{R},$$

arises as a symmetry group of the heat equation. (See Example 2.41.) If $u = f(x, t)$ is any functon, then its transform by g_ε is

$$\tilde{u} = e^{-\varepsilon x - \varepsilon^2 t} \cdot u = e^{-\varepsilon x - \varepsilon^2 t} \cdot f(x, t),$$

which must now be written in terms of $(\tilde{x}, \tilde{t}) = g_\varepsilon \cdot (x, t) = (x + 2\varepsilon t, t)$. Therefore

$$\begin{aligned} \tilde{u} &= e^{-\varepsilon(\tilde{x} - 2\varepsilon\tilde{t}) - \varepsilon^2 \tilde{t}} \cdot f(\tilde{x} - 2\varepsilon\tilde{t}, \tilde{t}) \\ &= e^{-\varepsilon\tilde{x} + \varepsilon^2 \tilde{t}} \cdot f(\tilde{x} - 2\varepsilon\tilde{t}, \tilde{t}) \end{aligned}$$

is the transformed function in this particular case. (Note the disparity with the expressions for the group transformations themselves. The reader is advised to do several examples to gain familiarity with how this works in practice.)

We can now give a rigorous definition of the concept of a symmetry group of a system of differential equations.

Definition 2.23. Let $\mathscr{S}$ be a system of differential equations. A *symmetry group* of the system $\mathscr{S}$ is a local group of transformations G acting on an open subset M of the space of independent and dependent variables for the system with the property that whenever $u = f(x)$ is a solution of $\mathscr{S}$, and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system. (By *solution* we mean any smooth solution $u = f(x)$ defined on any subdomain $\Omega \subset X$.)

For example, in the case of the ordinary differential equation $u_{xx} = 0$, the rotation group SO(2) considered in Example 2.21 is obviously a symmetry group, since the solutions are all linear functions and SO(2) takes any linear function to another linear function. Another easy example is given by the heat equation $u_t = u_{xx}$. Here the group of translations

$$(x, t, u) \mapsto (x + \varepsilon a, t + \varepsilon b, u), \qquad \varepsilon \in \mathbb{R},$$

is a symmetry group since $u = f(x - \varepsilon a, t - \varepsilon b)$ is a solution to the heat equation whenever $u = f(x, t)$ is. The reader might enjoy checking that the group presented at the end of Example 2.22 is also a symmetry group of the