of G. *In* (4.15), $\operatorname{Div} \xi$ *denotes the total divergence of the p-tuple* $\xi = (\xi^1, \ldots, \xi^p)$, cf. (4.4).

PROOF. For each $g \in G$, the group transformation

$$(\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\Xi_g(x, u), \Phi_g(x, u))$$

can be regarded as a change of variables, so by the same reasoning as led to (4.6) we can rewrite the symmetry condition (4.13) in the form

$$\int_\Omega L(\tilde{x}, \operatorname{pr}^{(n)}(g \cdot f)(\tilde{x})) \det \mathrm{J}_g(x, \operatorname{pr}^{(1)} f(x))\, dx = \int_\Omega L(x, \operatorname{pr}^{(n)} f(x))\, dx,$$

where the Jacobian matrix has entries

$$\mathrm{J}_g^{ij}(x, u^{(1)}) = D_i \Xi_g^j(x, u^{(1)}).$$

Since this is required to hold for all subdomains Ω and all functions $u = f(x)$, the integrands must agree pointwise:

$$L(\operatorname{pr}^{(n)} g \cdot (x, u^{(n)})) \det \mathrm{J}_g(x, u^{(1)}) = L(x, u^{(n)}) \tag{4.16}$$

for all $(x, u^{(n)}) \in M^{(n)}$. To obtain the infinitesimal version of (4.16) we set $g = g_\varepsilon = \exp(\varepsilon \mathbf{v})$ and differentiate with respect to ε. We need the formula

$$\frac{d}{d\varepsilon}[\det \mathrm{J}_{g_\varepsilon}(x, u^{(1)})] = \operatorname{Div} \xi(\operatorname{pr}^{(1)} g_\varepsilon \cdot (x, u^{(1)})) \det \mathrm{J}_{g_\varepsilon}(x, u^{(1)}) \tag{4.17}$$

expressing the fact that the divergence of a vector field measures the rate of change of volume under the induced flow. Indeed, if we replace u by a function $f(x)$, then (4.17) reduces to the identity of Exercise 1.36 for the reduced vector field $\tilde{\mathbf{v}} = \sum_{i=1}^p \xi^i(x, f(x))\, \partial/\partial x^i$.

Using (4.17) and (2.21), the derivative of (4.16) with respect to ε when $g = g_\varepsilon = \exp(\varepsilon \mathbf{v})$ is

$$(\operatorname{pr}^{(n)} \mathbf{v}(L) + L \operatorname{Div} \xi) \det \mathrm{J}_{g_\varepsilon} = 0, \tag{4.18}$$

the expression in parentheses being evaluated at $(\tilde{x}, \tilde{u}_\varepsilon^{(n)}) = \operatorname{pr}^{(n)} g_\varepsilon \cdot (x, u^{(n)})$. In particular, at $\varepsilon = 0$, g_ε is the identity map and we have proved the necessity of (4.15) for G to be a variational symmetry group. Conversely, if (4.15) holds everywhere, then (4.18) holds for ε sufficiently small. The left-hand side of (4.18), though, is just the derivative of the left-hand side of (4.16) (for $g = g_\varepsilon$) with respect to ε; thus, integrating from 0 to ε we prove (4.16) for g sufficiently near the identity. The usual connectivity arguments complete the proof of (4.16) for all $g \in G$, and hence the theorem. □

Example 4.13. For an easy illustration of Theorem 4.12, we re-derive the result of Example 4.11. The infinitesimal generator of the horizontal translation group is ∂_x, with prolongation $\operatorname{pr}^{(1)} \partial_x = \partial_x$. Also $\xi(x, u) = 1$, so $D_x \xi = 0$. Thus for $\mathscr{L} = \int_a^b L(u, u_x)\, dx$, we find

$$\operatorname{pr}^{(1)} \partial_x(L) + L D_x \xi = 0,$$

trivially, so (4.15) is verified. Another easy example is the arc-length integral $\mathscr{L}_0[u] = \int_a^b \sqrt{1+u_x^2}\,dx$, with $\mathbf{v} = -u\partial_x + x\partial_u$ the generator of the rotation group. We have

$$\mathrm{pr}^{(1)}\,\mathbf{v} = -u\partial_x + x\partial_u + (1+u_x^2)\partial_{u_x},$$

and $\xi = -u$, so

$$\mathrm{pr}^{(1)}\,\mathbf{v}(L) + LD_x\xi = (1+u_x^2)\frac{\partial}{\partial u_x}\sqrt{1+u_x^2} - \sqrt{1+u_x^2}\cdot u_x \equiv 0.$$

Thus Theorem 4.12 implies the geometrically obvious fact that arc-length is unchanged by a rigid rotation.

Symmetries of the Euler–Lagrange Equations

In both of the preceding examples, as the reader may readily check, invariance of the given variational integral under a group of symmetries implies that the associated Euler–Lagrange equations are also invariant under the group. This result holds in general.

Theorem 4.14. *If G is a variational symmetry group of the functional $\mathscr{L}[u] = \int_{\Omega_0} L(x, u^{(n)})\,dx$, then G is a symmetry group of the Euler–Lagrange equations $\mathsf{E}(L) = 0$.*

Intuitively, what is happening is that if $g \in G$ and $u = f(x)$ is an extremal of $\mathscr{L}[u]$, then clearly $\tilde{u} = g \cdot f(\tilde{x})$ (provided it is defined) is an extremal of the transformed variational problem $\tilde{\mathscr{L}}[\tilde{u}]$ coming from $(\tilde{x}, \tilde{u}) = g\cdot(x, u)$. But if G is a variational symmetry group, $\tilde{\mathscr{L}}[\tilde{u}] = \mathscr{L}[\tilde{u}]$, hence $g \cdot f$ is also an extremal of $\mathscr{L}$. The problem is that there are also non-extremal solutions of the Euler–Lagrange equations. One approach would be to use the change of variables formula from Theorem 4.8. Rather than belabour the point here, we refer the reader to Theorem 5.53 for a direct, computational proof.

It is *not* true that every symmetry group of the Euler–Lagrange equations is also a variational symmetry group of the original variational problem! The most common counterexamples are given by groups of scaling transformations.

Example 4.15. *The Wave Equation.* We return to the wave equation $u_{tt} = u_{xx} + u_{yy}$, whose symmetry group was found in Example 2.43. The wave equation is the Euler–Lagrange equation for the variational problem

$$\mathscr{L}[u] = \iiint [\tfrac{1}{2}u_t^2 - \tfrac{1}{2}u_x^2 - \tfrac{1}{2}u_y^2]\,dx\,dy\,dt.$$

Let us find out which of the symmetries listed in (2.65) are variational symmetries of $\mathscr{L}$. We use $L = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2$ in the infinitesimal criterion

(4.15). The translations are easily found to be variational symmetries since their prolongations have no effect on the derivatives of u. Next consider a rotation group generator, say $\mathbf{r}_{xy} = -y\partial_x + x\partial_y$. The term $\operatorname{Div} \xi$ in (4.15) vanishes. Moreover, $\operatorname{pr}^{(1)} \mathbf{r}_{xy} = \mathbf{r}_{xy} - u_y\partial_{u_x} + u_x\partial_{u_y}$, hence (4.15) reads

$$\operatorname{pr}^{(1)} \mathbf{r}_{xy}(L) = u_y u_x - u_x u_y = 0,$$

so $\mathscr{L}$ is rotationally invariant under the group generated by $\mathbf{r}_{xy}$. A similar computation shows that $\mathbf{r}_{xt}$ and $\mathbf{r}_{yt}$ also generate variational symmetry groups. Turning to the dilatational subgroup, we have

$$\operatorname{pr}^{(1)} \mathbf{d} = x\partial_x + y\partial_y + t\partial_t - u_x\partial_{u_x} - u_y\partial_{u_y} - u_t\partial_{u_t},$$

and, in this case,

$$\operatorname{Div} \xi = D_x(x) + D_y(y) + D_t(t) = 3.$$

Therefore

$$\operatorname{pr}^{(1)} \mathbf{d}(L) + L \operatorname{Div} \xi = L,$$

so $\mathbf{d}$ does *not* generate a variational symmetry group of $\mathscr{L}$. However, if we modify the dilatational generator to be

$$\mathbf{m} \equiv \mathbf{d} - \tfrac{1}{2}u\partial_u = x\partial_x + y\partial_y + t\partial_t - \tfrac{1}{2}u\partial_u,$$

then

$$\operatorname{pr}^{(1)} \mathbf{m}(L) + L \operatorname{Div} \xi = -3L + 3L = 0,$$

so $\mathbf{m}$ does generate a variational symmetry group. Finally, consider an inversional group, say that generated by

$$\mathbf{i}_x = (x^2 - y^2 + t^2)\partial_x + 2xy\partial_y + 2xt\partial_t - xu\partial_u.$$

We have

$$\begin{aligned}\operatorname{pr}^{(1)} \mathbf{i}_x = \mathbf{i}_x &- (u + 3xu_x + 2yu_y + 2tu_t)\partial_{u_x} \\ &+ (2yu_x - 3xu_y)\partial_{u_y} - (2tu_x + 3xu_t)\partial_{u_t},\end{aligned}$$

and

$$\operatorname{Div} \xi = D_x(x^2 - y^2 + t^2) + D_y(2xy) + D_t(2xt) = 6x.$$

Therefore

$$\operatorname{pr}^{(1)} \mathbf{i}_x(L) + L \operatorname{Div} \xi = uu_x - 3x(u_t^2 - u_t^2 - u_y^2) + 6xL = uu_x,$$

and hence $\mathbf{i}_x$ is not a variational symmetry according to Definition 4.10. Neither are the other two one-parameter inversional subgroups of the symmetry group (2.65) nor is any linear combination thereof. Finally, if $\alpha(x, y, t)$ is a solution of the wave equation, with symmetry generator $\mathbf{v}_\alpha = \alpha\partial_u$, we find

$$\operatorname{pr}^{(1)} \mathbf{v}_\alpha(L) = \alpha_t u_t - \alpha_x u_x - \alpha_y u_y,$$

so we get a variational symmetry if and only if α is a constant. Thus the variational symmetry group of L is generated by the translations, the "rotations", the scaling group generated by $\mathbf{m}$, and the group generated by ∂_u. Theorem 4.12 assures us that there are no other variational symmetries. (Of course, this could be checked directly by solving (4.15) for the coefficients of the infinitesimal generator $\mathbf{v}$.)

Proposition 4.16. *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are variational symmetries of* $\mathscr{L}[u]$, *then so is their Lie bracket* $[\mathbf{v}, \mathbf{w}]$.

The proof is left to the reader. (See Exercise 4.1.)

Reduction of Order

As we've seen in Section 2.5, knowledge of a one-parameter symmetry group of a single ordinary differential equation allows us to reduce the order of the equation by one. In this section, we will see that knowledge of a one-parameter group of *variational* symmetries for the Euler–Lagrange equation of some variational problem allows us to reduce the order of the equation by *two*! In effect, the variational structure of the differential equation and the symmetry group doubles the power of Lie's integration theory.

The easiest way to see how this happens is to use the invariance of the Euler–Lagrange equations under changes of variable as presented in Theorem 4.8, which allows us to change both independent and dependent variables without affecting the variational nature of the problem. Thus, let $x, u \in \mathbb{R}$, and let $\mathscr{L}[u]$ be an n-th order variational problem with $2n$-th order Euler–Lagrange equations. Suppose $\mathbf{v} = \xi(x, u)\partial_x + \phi(x, u)\partial_u$ is the infinitesimal generator of a one-parameter group of variational symmetries of $\mathscr{L}$. Note that by the definition of variational symmetry, $\mathbf{v}$ will remain a variational symmetry under a change of both independent and dependent variables. As in Section 2.5, we now introduce *particular* new variables $y = \eta(x, u)$, $w = \zeta(x, u)$ so that $\mathbf{v}$ takes the elementary form $\tilde{\mathbf{v}} = \partial/\partial w$ in these new coordinates. Let $\tilde{\mathscr{L}}[w] = \int \tilde{L}(y, w^{(n)})\, dy$ be the corresponding variational problem in the (y, w) variables. According to the above remarks, $\tilde{\mathbf{v}}$ remains a variational symmetry of $\tilde{\mathscr{L}}$, so by the infinitesimal criterion (4.15) we have

$$\mathrm{pr}^{(n)}\, \tilde{\mathbf{v}}(\tilde{L}) = \partial \tilde{L}/\partial w = 0,$$

hence $\tilde{L} = \tilde{L}(y, w_y, w_{yy}, \ldots)$ is independent of w. The Euler–Lagrange equation for $\tilde{\mathscr{L}}$ thus takes the form

$$0 = \mathrm{E}_w(\tilde{L}) = \sum_{j=1}^{n} (-D_y)^j \frac{\partial \tilde{L}}{\partial w_j} = -D_y \left\{ \sum_{j=0}^{n-1} (-D_y)^j \frac{\partial \tilde{L}}{\partial w_{j+1}} \right\}, \tag{4.19}$$

where $w_j = d^j w/dy^j$. Thus the expression in the brackets is constant, independent of y, and hence a *first integral* of the Euler–Lagrange equations. (This constitutes our first real encounter with Noether's theorem.)

Note further that if we introduce the new dependent variable $v = w_y$, so $v_j = d^j v/dy^j = w_{j+1}$, the expression in brackets can be written as the variational derivative of $\tilde{\mathscr{L}}[v] = \int \hat{L}(y, v^{(n-1)})\, dy$, where

$$\hat{L}(y, v, \ldots, v_{n-1}) = \tilde{L}(y, w_y, \ldots, w_n).$$

Every solution $w = f(y)$ of the original $2n$-th order Euler–Lagrange equation corresponds to a solution $v = h(y)$ of the $(2n-2)$-nd order equation

$$\mathsf{E}_v(\hat{L})(y, v^{(2n-2)}) = \lambda \tag{4.20}$$

for some constant λ (depending on the initial conditions), where w is recovered by quadrature:

$$w = \int h(y)\, dy + c.$$

Note that we can write (4.20) as a pure Euler–Lagrange equation for

$$\hat{\mathscr{L}}_\lambda[v] = \int [\hat{L}(y, v^{(n-1)}) - \lambda v]\, dy.$$

(Alternatively, λ can be thought of as a Lagrange multiplier, so we are minimizing $\hat{\mathscr{L}}[v]$ subject to the constraint $\int v\, dy = 0$, say; see Courant and Hilbert, [1; p. 218].)

Theorem 4.17. *Let $p = q = 1$. Let $\mathscr{L}[u]$ be an n-th order variational problem with Euler–Lagrange equation of order $2n$. Suppose G is a one-parameter group of variational symmetries of $\mathscr{L}$. Then there exists a one-parameter family of variational problems $\hat{\mathscr{L}}_\lambda[v]$ of order $n-1$, with Euler–Lagrange equations of order $2n-2$, such that every solution of the Euler–Lagrange equation for $\mathscr{L}[u]$ can be found by quadrature from the solutions to the Euler–Lagrange equation for $\hat{\mathscr{L}}_\lambda[v]$, $\lambda \in \mathbb{R}$.*

Example 4.18. In the case of first order variational problems, as in (4.14), knowledge of a one-parameter group of variational symmetries allows us to integrate the second order Euler–Lagrange equation

$$\mathsf{E}(L) = \frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} = 0 \tag{4.21}$$

completely by quadratures. (For a general one-parameter symmetry group of a second order ordinary differential equation, we can only expect to reduce to a first order equation.) Thus if L is independent of u, (4.21) reduces to $D_x(\partial L/\partial u_x) = 0$, hence

$$\frac{\partial L}{\partial u_x}(x, u_x) = \lambda$$

for some constant λ. We can solve this implicit relation for $u_x = F(x, \lambda)$, so the general solution is

$$u = \int F(x, \lambda)\, dx + c.$$

If $L(u, u_x)$ is independent of x, we can reduce to the previous case by using the hodograph change of variables of Example 4.9: $y = u$, $w = x$. A somewhat more direct approach, however, is to note that if we multiply the Euler–Lagrange equation by u_x, we can find a first integral

$$0 = u_x \mathsf{E}(L) = u_x \frac{\partial L}{\partial u} - u_x^2 \frac{\partial^2 L}{\partial u\, \partial u_x} - u_x u_{xx} \frac{\partial^2 L}{\partial u_x^2} = D_x\left(L - u_x \frac{\partial L}{\partial u_x}\right).$$

Thus

$$L(u, u_x) - u_x \frac{\partial L}{\partial u_x}(u, u_x) = \lambda$$

defines $u_x = F(u, \lambda)$ implicitly as a function of u and λ, which we can integrate to recover the solution of the Euler–Lagrange equation:

$$\int \frac{du}{F(u, \lambda)} = x + c.$$

The method can be extended to multi-parameter groups, but, unless they are abelian, we cannot in general expect to reduce the order by two at each stage. (See Exercise 4.11 and the later development of Hamiltonian systems in Chapter 6.) Here we content ourselves with an illustrative example.

Example 4.19. *Kepler's Problem.* We show how the above procedure can be used to immediately integrate the two-dimensional version of Kepler's problem of a mass under a central gravitational force field. The functional is

$$\mathscr{L}[x, y] = \int \left[\tfrac{1}{2}(x_t^2 + y_t^2) - U(r)\right] dt,$$

in which $(x(t), y(t))$ are the coordinates of the mass, $r^2 = x^2 + y^2$, and U is the potential function; for the three-dimensional gravitational attraction of a mass moving in the (x, y)-plane, $U(r) = -\gamma/r$. The Euler–Lagrange equations are

$$x_{tt} = -\frac{x}{r} U'(r), \qquad y_{tt} = -\frac{y}{r} U'(r).$$

Clearly, $\mathscr{L}$ is invariant under the abelian two-parameter group of time translations and spatial rotations with infinitesimal generators ∂_t and $x\partial_y - y\partial_x$ respectively. Introducing the polar coordinates (r, θ, t), we see that these vector fields become ∂_t and ∂_θ. By analogy with the case of a one-parameter group, this says that we should regard r as the new independent variable and θ and t as the new dependent variables, which should effect a reduction of the second order system to a system solvable by quadratures. Note first that

$$x_t = \frac{1}{t_r}(\cos\theta - r\sin\theta \cdot \theta_r), \qquad y_t = \frac{1}{t_r}(\sin\theta + r\cos\theta \cdot \theta_r),$$

hence in polar coordinates

$$\mathscr{L} = \int \left[\frac{1}{2t_r}(1 + r^2\theta_r^2) - t_r U(r) \right] dr,$$

which is, as expected, independent of both t and θ. The Euler–Lagrange equations can thus be immediately integrated once, leading to

$$\frac{1}{2t_r^2}(1 + r^2\theta_r^2) + U(r) = \lambda, \qquad \frac{r^2\theta_r}{t_r} = \mu,$$

where λ, μ are constants. Note that if we revert back to t as the independent variable, the first equation gives the well-known conservation of energy, while the second equation is just Kepler's second law, $r^2\theta_t = \mu$, that the mass sweeps out equal areas in equal times. Retaining r as the independent variable, however, we can eliminate t_r from these two equations,

$$(2\lambda\mu^{-2}r^4 - 2\mu^{-2}r^4 U(r) - r^2)\left(\frac{d\theta}{dr}\right)^2 = 1,$$

hence

$$\theta = \int \frac{dr}{r(2\lambda\mu^{-2}r^2 - 2\mu^{-2}r^2 U(r) - 1)^{1/2}} + \theta_0.$$

In particular, if $U(r) = -\gamma r^{-1}$, we can integrate this explicitly,

$$\theta - \theta_0 = \arcsin\left[\frac{p}{\varepsilon}\left(\frac{1}{p} - \frac{1}{r}\right)\right],$$

where

$$\varepsilon^2 = 1 + \frac{2\mu^2\lambda}{\gamma^2}, \qquad p = \frac{\mu^2}{\gamma}.$$

Thus the orbits are conic sections

$$r = \frac{p}{1 - \varepsilon\sin(\theta - \theta_0)}$$

of eccentricity ε. Similarly, we can determine t by a single quadrature:

$$t = \int \frac{r^2\theta_r}{\mu}\, dr + t_0 = \int \frac{r\, dr}{(2\lambda r^2 - 2r^2 U(r) - \mu^2)^{1/2}} + t_0,$$

which, in the gravitational case, yields

$$t = \frac{s}{2\lambda} - \frac{\gamma}{(2\lambda)^{3/2}} \log(\sqrt{2\lambda}s + 2\lambda r + \gamma), \qquad s = \sqrt{2\lambda r^2 + 2\gamma r - \mu^2}.$$

We have thus completely solved Kepler's problem by quadratures.

4.3. Conservation Laws

Consider a system of differential equations $\Delta(x, u^{(n)}) = 0$. A *conservation law* is a divergence expression

$$\operatorname{Div} P = 0 \tag{4.22}$$

which vanishes for all solutions $u = f(x)$ of the given system. Here $P = (P_1(x, u^{(n)}), \ldots, P_p(x, u^{(n)}))$ is a p-tuple of smooth functions of x, u and the derivatives of u, and $\operatorname{Div} P = D_1 P_1 + \cdots + D_p P_p$ is its total divergence.

For example, in the case of Laplace's equation, some conservation laws are readily apparent. First of all, the equation itself is a conservation law since

$$\Delta u = \operatorname{Div}(\nabla u) = 0$$

for all solutions u. Multiplying Laplace's equation by $u_i = \partial u/\partial x^i$ yields p further conservation laws.

$$0 = u_i \Delta u = \sum_{j=1}^{p} D_j\left(u_i u_j - \tfrac{1}{2}\delta_i^j \sum_{k=1}^{p} u_k^2\right).$$

Later we will see how to establish yet more conservation laws.

In the case of a system of ordinary differential equations involving a single independent variable $x \in \mathbb{R}$, a conservation law takes the form $D_x P = 0$ for all solutions $u = f(x)$ of the system. This requires that $P(x, u^{(n)})$ be *constant* for all solutions of the system. Thus a conservation law for a system of ordinary differential equations is equivalent to the classical notion of a *first integral* or *constant of the motion* of the system. As we will see, (4.22) is the appropriate generalization of this concept to partial differential equations, and includes familiar concepts of conservation of mass, energy, momentum, etc. arising in physical applications.

In a dynamical problem, one of the independent variables is distinguished as the time t, the remaining variables $x = (x^1, \ldots, x^p)$ being spatial variables. In this case a conservation law takes the form

$$D_t T + \operatorname{Div} X = 0,$$

in which Div is the spatial divergence of X with respect to $x^1, \ldots, x^p$. The *conserved density*, T, and the associated *flux*, $X = (X_1, \ldots, X_p)$, are functions of x, t, u and the derivatives of u with respect to both x and t. In this situation, it is easy to see that for certain types of solutions, the conserved density, when integrated, provides us with a constant of the motion of the system. More specifically, suppose $\Omega \subset \mathbb{R}^p$ is a spatial domain, and $u = f(x, t)$ a solution defined for all $x \in \Omega$, $a \leqslant t \leqslant b$. Consider the functional

$$\mathscr{T}_\Omega[f](t) = \int_\Omega T(x, t, \operatorname{pr}^{(n)} f(x, t))\, dx, \tag{4.23}$$

which, for fixed f, Ω, depends on t alone. The basic conservative property of T states that $\mathscr{T}_\Omega[f]$ depends only on the initial values of f at $t = a$ and the boundary values of f on $\partial\Omega$.

Proposition 4.20. *Suppose T, X are the conserved density and flux for a conservation law of a given system of differential equations. Then for any bounded domain $\Omega \subset \mathbb{R}^p$ with smooth boundary $\partial\Omega$, and any solution $u = f(x, t)$ defined for $x \in \Omega$, $a \leqslant t \leqslant b$, the functional* (4.23) *satisfies*

$$\mathscr{T}_\Omega[f](t) - \mathscr{T}_\Omega[f](a) = -\int_a^t \int_{\partial\Omega} X(x, \tau, \mathrm{pr}^{(n)} f(x, \tau)) \cdot dS\, d\tau. \quad (4.24)$$

Conversely, if (4.24) *holds for all such domains and solutions $u = f(x, t)$, then T, X define a conservation law.*

Proof. By the divergence theorem

$$\frac{d}{dt}\mathscr{T}_\Omega[f](t) = \int_\Omega D_t T(x, t, \mathrm{pr}^{(n+1)} f)\, dx = -\int_{\partial\Omega} X(x, t, \mathrm{pr}^{(n)} f) \cdot dS.$$

Then (4.24) follows upon integration. The converse follows by differentiating (4.24) with respect to t, yielding

$$\int_\Omega \{D_t T(x, t, \mathrm{pr}^{(n+1)} f) + \mathrm{Div}\, X(x, t, \mathrm{pr}^{(n+1)} f)\}\, dx = 0.$$

Since this holds for arbitrary subdomains, the integrand itself must vanish, proving the converse. □

Corollary 4.21. *If $\Omega \subset \mathbb{R}^p$ is bounded, and $u = f(x, t)$ is a solution such that $X(x, t, \mathrm{pr}^{(n)} f(x, t)) \to 0$ as $x \to \partial\Omega$, then $\mathscr{T}_\Omega[f]$ is a constant, independent of t.*

Usually, $X(x, t, 0) \equiv 0$, so one requires that the solution $f(x, t)$ vanish sufficiently rapidly as $x \to \partial\Omega$ (so that there is no flux over $\partial\Omega$), or, if Ω has unbounded components, as $|x| \to \infty$.

Example 4.22. Perhaps the most graphic physical illustration of the relationship between conserved densities and fluxes comes from the equations of compressible, inviscid fluid motion. Let $x \in \mathbb{R}^3$ represent the spatial coordinates, and $u = u(x, t) \in \mathbb{R}^3$ the velocity of a fluid particle at position x and time t. Further let $\rho(x, t)$ be the density, and $p(x, t)$ the pressure; in the particular case of isentropic (constant entropy) flow, pressure $p = P(\rho)$ will depend on density alone. The equation of continuity takes the form

$$\rho_t + \mathrm{Div}(\rho u) = 0,$$

where $\mathrm{Div}(\rho u) = \sum_j \partial(\rho u^j)/\partial x^j$ is the spatial divergence, while momentum balance yields the three equations

$$\frac{\partial u^i}{\partial t} + \sum_{j=1}^3 u^j \frac{\partial u^i}{\partial x^j} = -\frac{1}{\rho}\frac{\partial p}{\partial x^i}, \qquad i = 1, 2, 3.$$

The equation of continuity is already in the form of a conservation law, with density $T = \rho$ and flux $X = \rho u$. This leads to the integral equation for the conservation of mass

$$\frac{d}{dt}\int_\Omega \rho\, dx = -\int_{\partial\Omega} \rho u \cdot n\, dS.$$

Here $\int_\Omega \rho\, dx$ is clearly the mass of fluid within the domain Ω, while $\rho u \cdot n$, with n the unit normal to $\partial\Omega$, is the instantaneous mass flux of fluid out of a point on the boundary $\partial\Omega$. Thus we see that the net change in mass inside Ω equals the flux of fluid into Ω. In particular, if the normal component of velocity $u \cdot n$ on $\partial\Omega$ vanishes, there is no net change in mass within the domain Ω, and we have a law of conservation of mass:

$$\int_\Omega \rho\, dx = \text{constant}.$$

The momentum balance equations, coupled with the continuity equation, yield three further conservation laws

$$D_t(\rho u^i) + \sum_{j=1}^{3} D_j(\rho u^i u^j + p\delta_i^j) = 0, \qquad i = 1, 2, 3.$$

In integrated form, these are the laws of conservation of linear momentum

$$\frac{d}{dt}\int_\Omega \rho u^i\, dx = -\int_{\partial\Omega} (\rho u^i(u \cdot n) + pn_i)\, dS, \qquad i = 1, 2, 3,$$

(n_i being the i-th component of the normal n). The first term in the boundary integral denotes the transport of momentum ρu^i due to the flow across the surface $\partial\Omega$, while the second term is the net change in momentum due to the pressure across $\partial\Omega$. In this way $X_j = \rho u^i u^j + p\delta_i^j$ does represent the components of momentum flux. Finally, if the flow is isentropic, we can introduce the internal energy $W(\rho) = \int \rho^{-2} P(\rho)\, d\rho$ per unit mass, measuring the work done by the fluid against the pressure. The law of conservation of energy takes the form

$$D_t[\tfrac{1}{2}\rho|u|^2 + \rho W(\rho)] + \operatorname{Div}[(\tfrac{1}{2}\rho|u|^2 + P(\rho) + \rho W(\rho))u] = 0;$$

or, in integrated form,

$$\frac{d}{dt}\int_\Omega [\tfrac{1}{2}\rho|u|^2 + \rho W(\rho)]\, dx = -\int_{\partial\Omega} (\tfrac{1}{2}\rho|u|^2 + P(\rho) + \rho W(\rho))u \cdot n\, dS.$$

Here $\int_\Omega \frac{1}{2}\rho|u|^2\, dx$ is the kinetic energy, while $\int_\Omega \rho W(\rho)\, dx$ the internal (potential) energy of the fluid. The surface integral represents the transport of kinetic and potential energy across $\partial\Omega$ together with the rate of working due to the pressure across the boundary. In particular, if $u \cdot n = 0$ on $\partial\Omega$, both mass and energy are conserved, while momentum is conserved if the pressure $p = 0$ on $\partial\Omega$ also.