40 Large order behaviour of perturbation theory

In quantum field theory (QFT), the main *analytic tool* to calculate physical quantities is the perturbative expansion. Quite early, Dyson argued, with intuitive arguments [400], that perturbative expansions correspond to divergent series.

This problem started to be mathematically investigated, using the Schrödinger equation, in some models of quantum mechanics (QM) with polynomial potentials [401]. Dyson's intuition was confirmed and, moreover, the connection between barrier penetration effects in the semi-classical limit and large behaviour in perturbation theory in QM was realized. Later, it was proposed to study the problem within a path integral formulation [402]. However, a systematic study of the problem was triggered by Lipatov [384], using the field-integral representation of the $\phi^4_{d=4}$ field theory [403].

In Chapter 37, we have studied the analytic structure of the ground-state energy $E(g)$ of the quartic anharmonic oscillator. We have argued that $E(g)$ is analytic in a cut-plane. For g small, $E(g)$ can be calculated as a perturbative series in g:

$$E(g) = \sum_k E_k g^k. \tag{40.1}$$

On the cut, for g small and negative, its imaginary part $\operatorname{Im} E(g)$ can be calculated by instanton methods. In this chapter, we show how the behaviour of $\operatorname{Im} E(g)$ for $g \to 0_-$ is related to the behaviour of the coefficients E_k for k large [404, 405]. The method is then generalized to the class of potentials for which we have calculated instanton contributions. The same method can readily be applied to boson field theories, using the results of Chapter 38, while the extension to field theories involving fermions, like quantum electrodynamics (QED), requires additional considerations.

A general conclusion is that, in QFT, all perturbative series, expanded in terms of a loop-expansion parameter, are divergent series.

40.1 QM

We first examine two situations where we have found instantons. We then argue that, for other analytic potentials, complex solutions to the Euclidean equation of motion are also relevant [380].

40.1.1 Real instantons

The quartic anharmonic oscillator. We first consider the ground-state energy $E(g)$ of the Hamiltonian of the quartic anharmonic oscillator (37.1),

$$H = -\tfrac{1}{2}\left(\mathrm{d}/\mathrm{d}q\right)^2 + \tfrac{1}{2}q^2 + \tfrac{1}{4}gq^4.$$

Since $E(g)$ is analytic in the cut-plane and behaves like $g^{1/3}$ for $|g|$ large, it has the Cauchy representation

$$E(g) = \frac{1}{2} + \frac{g}{\pi}\int_{-\infty}^{0} \frac{\operatorname{Im} E(g')\,\mathrm{d}g'}{g'(g'-g)}\,. \tag{40.2}$$

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Expanding the integrand in powers of g, one obtains the integral representation for the perturbative coefficients

$$E_k = \frac{1}{\pi}\int_{-\infty}^{0} \frac{\operatorname{Im} E(g)\,\mathrm{d}g}{g^{k+1}}\,, \quad \text{for } k > 0\,. \tag{40.3}$$

When k, the order in the expansion, becomes large, due to the factor g^{-k} the dispersion integral (40.3) is dominated by the small negative g values.

In Section 37.4 (equation (37.37)), we have evaluated $\operatorname{Im} E(g)$ for g small and negative. We use here the result to estimate the large k behaviour of E_k:

$$E_k \underset{k\to\infty}{\sim} \frac{1}{\pi}\int^{0-} \left(\frac{8}{\pi}\right)^{1/2} \frac{1}{\sqrt{-g}}\frac{\mathrm{e}^{4/3g}}{g^{k+1}}\left[1+O(g)\right]\mathrm{d}g\,. \tag{40.4}$$

The explicit integration yields

$$E_k = (-1)^{k+1}\sqrt{6/\pi^3}\,(3/4)^k\,\Gamma(k+1/2)\left[1+O\left(1/k\right)\right]. \tag{40.5}$$

This result confirms that the perturbative series is divergent for all values of $g > 0$, and *determines the nature of the divergence*. Successive corrections to the semi-classical result yield a series in powers of g which, integrated, generate a systematic expansion in powers of $1/k$.

General holomorphic potentials. The same argument is applicable to the situation described in Section 37.5. We can calculate the energy of the metastable state in power series of the coupling constant g by making a systematic expansion around the relative minimum of the potential. On the other hand, we can, as previously mentioned, derive from the knowledge of the imaginary part of the energy level for small coupling, an estimate of the behaviour of the perturbative coefficients at large order. We consider the action

$$\mathcal{S}(q) = \int \mathrm{d}t\,\left[\tfrac{1}{2}\dot{q}^2(t) + g^{-1}V\big(q(t)\sqrt{g}\big)\right], \tag{40.6}$$

where g is the loop expansion parameter. The analogue of the dispersion integral (40.3) is

$$E_k \sim \frac{1}{\pi}\int_0^{\infty}\frac{\operatorname{Im} E(g)}{g^{k+1}}\mathrm{d}g\,.$$

The behaviour of $\operatorname{Im} E(g)$ for g small is given by the expression (37.95). Integrating near $g = 0$, one obtains (x_+ is a zero of $V(x)$),

$$E_k \sim -\frac{x_+}{2\pi^{3/2}}\exp\left[\int_0^{x_+}\left(\frac{1}{\sqrt{2V(x)}} - \frac{1}{x}\right)\mathrm{d}x\right] A^{-(k+1/2)}\Gamma(k+1/2), \tag{40.7}$$

where A is the instanton action

$$A = 2\int_0^{x_+}\sqrt{2V(x)}\,\mathrm{d}x\,. \tag{40.8}$$

We now see general features emerging: at large orders, the perturbative coefficients E_k behave like

$$E_k \underset{k\to\infty}{\sim} C k^{b-1} k!\,A^{-k}. \tag{40.9}$$

The universal factor $k!$, which is characteristic of a semi-classical or loop expansion, implies that the perturbation series is a divergent series. The factor A^{-k}, where A is the action of the classical solution, is common to all quantities calculated with the same Hamiltonian. The power k^b depends, in particular, on the number of continuous symmetries broken by the classical solution, but it is also on the specific quantity that is expanded. This can be verified by explicitly calculating the imaginary parts of the energy of the excited levels and using equation (40.3). The parameter b is in general a half integer. Finally, the constant multiplicative factor C depends in a more complicated way on all the specific features of the expanded quantity.

Discussion. In both examples, we have been able to derive the large-order behaviour of perturbation series from the decay rate, due to barrier penetration, of a metastable minimum of the potential. For the potentials considered in Section 37.5, the action A is positive and, therefore, all terms in the perturbative expansion have the same sign. The same property holds for the quartic anharmonic oscillator in the metastable case, that is, when g is negative. However, for g positive, in which case perturbation series has been expanded around an absolute minimum of the potential, we observe that the perturbative coefficients oscillate in sign. Also, we note that for $g > 0$, the instanton solution becomes purely imaginary. This suggests how one can derive the large behaviour in the generic stable case.

40.1.2 Complex instantons

So far, we have characterized the large-order behaviour of perturbation theory in two cases: in the generic case, in which we expand around a relative minimum of the potential, and in one special case in which we have expanded around an absolute minimum of the potential, but which by analytic continuation in the coupling constant can be transformed into a relative minimum. We now consider actions of the form (40.6), in which the potential $V(q)$ is still an entire function of q, and satisfies the condition

$$V(q) = \tfrac{1}{2}q^2 + O\left(q^3\right),$$

and we assume that perturbation theory is expanded around $q = 0$, the absolute minimum of the potential. Then, no real instanton solutions can be found. Following the example of the anharmonic oscillator, we thus assume that we can introduce parameters in the potential which make an analytic continuation to a metastable situation possible. We obtain the large-order behaviour from the expression (40.7). We then use the inverse analytic continuation to return to the initial situation. It is plausible that the analytic continuation of the expression (40.7) still gives the large behaviour of the initial expansion.

We can now formulate the rules of the large-order behaviour calculation directly in the initial theory. Complex instanton solutions, with, in general, complex (or exceptionally negative) action [380] are associated to the complex zeros (at finite or infinite distance) of the potential $V(q)$. These instantons are candidates to contribute to the large-order behaviour. In the expression (40.7), we see that the action(s) with the smallest modulus (when the action is complex, there will be at least two complex conjugate actions) gives the leading contribution to the large-order behaviour. Note that the difference we have found between the anharmonic oscillator and the metastable case is generic. In the stable case, the classical action is not real positive, and the perturbative coefficients at large order involve an order-dependent phase factor.

Such a property plays an essential role for the summability of divergent series (see Section 41.1.2).

40.1.3 Degenerate classical minima

The preceding discussion does not apply directly to the case of potentials with non-continuously connected degenerate minima (see Chapter 39), for example, the potentials $x^2(1-x)^2$, or $\cos(x)$. Let us indeed consider such a potential as the limit of a potential which has two minima with very close values of the potential. From the explicit form of the large-order behaviour (40.7), we note that the classical action (40.8) has a limit, which can be identified as being *twice the action of the instanton that connects the two minima of the potential*. This property generalizes to the field-theory examples. However, the calculation of the determinant generated by expanding to quadratic order around the saddle point leads to a new problem.

This is illustrated by the QM example: the integral in expression (40.7) diverges when x_+ is an extremum of the potential. This property can be understood in the following way. When the values at the two minima approach each other, the time spent by the instanton path close to the second minimum of the potential diverges. Therefore, fluctuations which tend to change this time leave the action almost stationary. Correspondingly, one eigenvalue of the operator $\delta^2\mathcal{S}/\delta q(t)\delta q(t')|_{q=q_c}$ goes to 0, and this explains the divergence of expression (40.7) in this case. It becomes necessary to let the separation time fluctuate and, therefore, to introduce an additional time collective coordinate. In this way, one can one derive the correct answer [406–408]. Let us also note that here, like in the case of relative minima, the instanton action is positive. Summing the perturbative expansion becomes a more difficult problem, which we discuss in Chapter 42.

40.2 Scalar field theories: The example of the ϕ^4 field theory

In Chapter 38, we have shown how to evaluate the contribution of instantons to the decay rate of metastable states. These results can be applied to large-order behaviour estimates. In a general scalar boson field theory, if instanton solutions can be found, the same arguments applied to n-point correlation functions lead to [384, 380]

$$\left\{Z^{(n)}(x_1,\ldots,x_n)\right\}_k \underset{k\to\infty}{\sim} \sum_{\substack{\text{dominant}\\ \text{saddle points}}} C_n(x_1,\ldots,x_n) k^{b-1} S^{-k} k!\,, \tag{40.10}$$

in which

(i) S is the instanton action which, in general, is complex;

(ii) $b=\frac{1}{2}(n+\delta)$, δ being the number of symmetries broken by the classical solution;

(iii) $C_n(x_1,\ldots,x_n)$, which does not depend on k, contains the whole dependence in the arguments of the correlation function.

In the case of the ϕ^4 field theory, the discontinuity across the cut of the n-point function reads (equation (38.19), *where the instanton action is* $S=-A$)

$$\operatorname{disc.} Z^{(n)}(x_1,\ldots,x_n) \underset{g\to 0_-}{\sim} \left(\frac{-S}{2\pi}\right)^{d/2} \Omega \frac{\mathrm{e}^{-S/g}}{(-g)^{(d+n)/2}} F_n(x_1,\ldots,x_n), \tag{40.11}$$

with

$$\Omega = \left(\det M' M_0^{-1}\right)^{-1/2}_{\text{ren.}}\,,$$

and

$$F_n(x_1,\ldots,x_n) = m^{d+n(d-2)/2} 6^{n/2} \int \mathrm{d}^d x_0 \prod_{i=1}^{n} f\big(m(x_i - x_0)\big). \tag{40.12}$$

Table 40.1

The coefficients β_k of the coupling constant RG-function $\beta(g)$ divided by the large-order estimate, in the case of the $O(N)$-symmetric $(\boldsymbol{\phi}^2)^2_{d=3}$ field theory.

k	2	3	4	5	6	7
$N=0$	3.53	1.55	1.185	1.022	0.967	0.951
$N=1$	3.98	1.75	1.32	1.120	1.050	1.023
$N=2$	4.82	2.09	1.53	1.29	1.20	1.15
$N=3$	6.14	2.58	1.86	1.55	1.41	1.35

Using previous arguments, we can immediately translate this result into the large-order behaviour estimate for correlation functions,

$$\left\{Z^{(n)}(x_1,\ldots,x_n)\right\}_k = \frac{1}{2i\pi}\int_{-\infty}^{0}\frac{\mathrm{d}g}{g^{k+1}}\ \mathrm{disc.}\, Z^{(n)}(x_1,\ldots,x_n),$$

and, therefore,

$$\left\{Z^{(n)}(x_1,\ldots,x_n)\right\}_k \underset{k\to\infty}{\sim} \frac{1}{2i\pi}\frac{\Omega}{(2\pi)^{d/2}}F_n(x_1,\ldots,x_n)(-1)^k\frac{\Gamma\big(k+(d+n)/2\big)}{(-S)^{n/2+k}}. \quad (40.13)$$

Example: the renormalization group β-function in the $(\boldsymbol{\phi}^2)^2$ field theory in dimension 3. The large-order behaviour has been determined numerically by solving the field equations to determine the instanton action S, and then by evaluating the determinant [409]. The predictions of the asymptotic formulae can be compared with the available terms of the series (see Section 41.3.1). The agreement is rather good and strongly suggests that the large-order behaviour estimates are indeed correct (see Table 40.1).

40.3 The $(\boldsymbol{\phi}^2)^2$ field theory in dimension 4 and $4-\varepsilon$

As a by-product of the calculation of the instanton contribution in Sections 38.3–38.6, one can evaluate the semi-classical contribution to the large-order behaviour in the $(\boldsymbol{\phi}^2)^2$ field theory in four dimensions. However, because the theory is exactly renormalizable, at order k, as a consequence of their large momenta properties, some diagrams grow themselves like $k!$, generating additional contributions to the large-order behaviour (*e.g.*, see Section 18.5). Moreover, infrared (IR) singularities in the massless theory also yield contributions of order $k!$, but with different signs.

40.3.1 Semi-classical contribution

The instanton contribution to the large-order behaviour for vertex functions is given by

$$\left\{\tilde{\Gamma}^{(n)}(p_1,\ldots,p_n)\right\}_k = \frac{1}{\pi}\int_{-\infty}^{0}\frac{\mathrm{Im}\,\tilde{\Gamma}^{(n)}(p_1,\ldots,p_n)}{g^{k+1}}\mathrm{d}g\,. \quad (40.14)$$

This yields a result of the form

$$\left\{\tilde{\Gamma}^{(n)}(p_1,\ldots,p_n)\right\}_k \underset{k\to\infty}{\sim} \tilde{C}_n(p_1,\ldots,p_n)\int^{0-}\frac{\mathrm{e}^{8\pi^2/3g}}{(-g)^{n+5/2}}\frac{\mathrm{d}g}{g^{k+1}}\,. \quad (40.15)$$

After integration, one obtains

$$\left\{\tilde{\Gamma}^{(n)}(p_1,\dots,p_n)\right\}_k \sim \tilde{C}_n(p_1,\dots,p_n)(-1)^k\left(\frac{3}{8\pi^2}\right)^{n+3+k}\Gamma(k+n/2+5/2). \quad (40.16)$$

From this expression, it is simple to derive the semi-classical contribution to the large order behaviour of various renormalization-group (RG) functions in, for example, the fixed momentum subtraction scheme. A comparison between large order behaviour and explicit calculations can be found in Table 40.2, in the case of the RG β-function.

Finally, note that, in the massive theory, the calculation is slightly modified because the integral over the collective dilatation coordinate is cut at a scale of order $m\sqrt{k}$ (see Section 38.7).

Table 40.2

The coefficients β_k of the RG β-function divided by the semi-classical asymptotic estimate, in the case of the $O(N)$-symmetric $((\boldsymbol{\phi})^2)^2_{d=4}$ field theory.

k	2	3	4	5
$N=1$	0.10	0.66	1.08	1.57
$N=2$	0.06	0.49	0.87	1.32
$N=3$	0.04	0.33	0.66	1.09

40.3.2 Ultraviolet (UV) and infrared (IR) (renormalons) contributions

So far, an implicit assumption in the large-order behaviour calculation has been that the singularities of correlation functions come entirely, in the neighbourhood of the origin, from barrier penetration effects. If this assumption is certainly correct in QM, if there is convincing evidence that it is valid for super-renormalizable theories, it is much more questionable for renormalizable theories, in the absence of a finite UV cut-off, or for massless renormalizable theories. We first explain the large momentum problem, and then the IR problem of massless theories [410].

UV singularities: Renormalons [411]. If the semi-classical analysis is valid for the cut-off regularized field theory, it becomes somewhat formal for the renormalized theory in the infinite cut-off limit. We have already seen that even in the straightforward calculation, non-trivial questions arise about the global RG properties of the theory. A direct investigation of the perturbative expansion raises new questions, and suggests that UV singularities yield additional contributions to the large-order behaviour.

Let us consider the $O(N)$-symmetric $(\boldsymbol{\phi}^2)^2$ field theory, where the field $\boldsymbol{\phi}$ is an N-component vector, in dimension 4. The action has the form

$$\mathcal{S}(\boldsymbol{\phi}) = \int \mathrm{d}^4x \left[\frac{1}{2}\big(\nabla\boldsymbol{\phi}(x)\big)^2 + \frac{m^2}{2}\boldsymbol{\phi}^2(x) + \frac{g}{4}\big(\boldsymbol{\phi}^2(x)\big)^2\right]. \quad (40.17)$$

In Section 18.5, we have shown that, at order $1/N$ in the large N expansion, the renormalized two-point function is given by a divergent integral, because the integrand has a pole, corresponding to the Landau ghost.

We briefly recall the argument. The $1/N$ contribution to the two-point function in the massive renormalized theory is

$$F_2(p) = \frac{2g}{(2\pi)^4} \int \frac{\mathrm{d}^4 q}{[(p+q)^2 + m^2]\,[1 + NgB_\mathrm{r}(q)]} - \text{subtractions}\,, \tag{40.18}$$

where the renormalized 'bubble' diagram is given by

$$B_\mathrm{r}(p) = \frac{1}{(2\pi)^4} \int \frac{\mathrm{d}^4 q}{[(p+q)^2 + m^2]\,(q^2 + m^2)} - \text{subtraction}\,. \tag{40.19}$$

For $|p| \to \infty$, $B_\mathrm{r}(p)$ behaves like

$$B_\mathrm{r}(p) \sim \frac{1}{8\pi^2} \ln(m/|p|)\,. \tag{40.20}$$

Therefore, the sum of the bubble diagrams which appears in expression (40.18) has a singularity for g small (which justifies the large momentum approximation) and positive, at a momentum

$$|p| \sim m\,\mathrm{e}^{8\pi^2/\mathbf{N}g}, \quad \text{for } g \to 0_+\,. \tag{40.21}$$

Since the theory is IR-free, and not UV asymptotically free, this singularity occurs for positive values of the coupling constant. Once this sum of bubbles is inserted into expression (40.18), it produces a cut for g small and positive. More precisely, after subtraction, and for q large, the integrand of F_2 at large momenta behaves like

$$\int_{|q|\gg 1} \frac{\mathrm{d}q}{q^3} \left[1 + \frac{Ng}{8\pi^2} \ln(m/q)\right]^{-1} + \cdots\ . \tag{40.22}$$

The change of variables $t = \ln(q/m)$ transforms the expression (40.22) into

$$\int^{\infty} \mathrm{d}t\ \mathrm{e}^{-2t} \frac{1}{1 - Ngt/(8\pi^2)}. \tag{40.23}$$

This yields an imaginary contribution to the correlation functions for g small and positive of the form $\exp(-16\pi^2/Ng)$. Alternatively, by expanding expression (40.18) in powers of g, we obtain the contribution of individual diagrams containing bubble insertions. These diagrams behave like $(N/16\pi^2)^k k!$ at large order k. Therefore, in contrast to super-renormalizable theories in which an individual diagram behaves like an exponential of k, and the $k!$ comes from the number of diagrams, here, some individual diagrams give a $k!$ contribution, without the sign oscillations characteristic of the semi-classical result.

Further investigations show that, if a non-perturbative contribution exists, it should satisfy the homogeneous RG equations. For simplicity, we consider the example of a dimensionless ratio of correlation functions $R(p/m, g)$ without anomalous dimensions,

$$\left(m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g}\right) R(p/m, g) = 0\,. \tag{40.24}$$

The RG equation implies that the function $R(p/m, g)$ is actually a function of only one variable $s(g)p/m$, in which $s(g)$ then satisfies

$$\beta(g)s'(g) = s(g), \tag{40.25}$$

which after integration yields

$$s(g) \sim \exp\left[\int^{g} \frac{\mathrm{d}g'}{\beta(g')}\right]. \tag{40.26}$$

For g small, $s(g)$ behaves like

$$\beta(g) = \beta_2 g^2 + O\left(g^3\right), \quad \text{with} \quad \beta_2 = \frac{N+8}{8\pi^2}, \tag{40.27}$$

$$s(g) \underset{g\to 0}{\propto} g^{-\beta_3/\beta_2^2}\,\mathrm{e}^{-1/\beta_2 g}\,. \tag{40.28}$$

Since the correlation function depends only on the mass squared, only $s^2(g)$ enters the calculation, and the contribution to the large-order behaviour has the form

$$\int_0 \frac{s(g)}{g^{k+1}}\mathrm{d}g \propto (\beta_2/2)^k\Gamma(k+1+2\beta_3/\beta_2^2), \tag{40.29}$$

a result which coincides, in the large N limit, with the contribution that we obtained from the set of bubble diagrams. This potential contribution has to be compared with the semi-classical result (40.16).

In fact, this problem is related to the question of the existence of the renormalized $(\boldsymbol{\phi}^2)^2$ field theory in four dimensions. If the theory does not exist, then probably the sum of perturbation theory is complex for g positive, and these singular terms, sometimes called *renormalon* effects, are the small coupling evidence of this situation. More generally, the existence of renormalons shows that the perturbation series is not Borel summable and does not define unique correlation functions (see Section 41.1).

Massless renormalizable theories: IR renormalons [412]. Again, we illustrate the problem with the $(\boldsymbol{\phi}^2)^2$ field theory in the large N limit. We now work in a massless theory with fixed cut-off Λ. We evaluate the contribution of the small momentum region to the mass renormalization constant. The bubble diagram (40.19) behaves like

$$I(p) \sim \frac{1}{8\pi^2}\ln(\Lambda/p).$$

The sum of bubbles yields a contribution to the mass renormalization proportional to

$$\int^{\Lambda} \frac{\mathrm{d}^4 q}{q^2\bigl(1+NgI(q)\bigr)} = \int \frac{\mathrm{d}^4 q}{q^2\bigl(1+\frac{N}{8\pi^2}g\ln(\Lambda/q)\bigr)}\,.$$

Expanded in powers of g, the integral yields a contribution of order $(-1)^k(N/16\pi^2)^k k!$ for large order k. This contribution has the sign oscillations of the semi-classical term. More generally, for finite N one finds $(-\beta_2/2)^k k!$. IR singularities yield an additional Borel summable contribution to the large-order behaviour.

For massless but asymptotically free theories, the role of the IR and UV regions are interchanged. UV renormalons are expected to yield additional singularities to the Borel transform on the real negative axis, while IR contributions destroy Borel summability. When these theories have real instantons like QCD or the $CP(N-1)$ models (see Section 39.5, 39.6), the Borel transform has also semi-classical singularities on the real positive axis.

Table 40.3

Sum of the successive terms of the ε-expansion of γ and η for $\varepsilon = 1$ and $N = 1$.

k	0	1	2	3	4	5
γ	1.000	1.1667	1.2438	1.1948	1.3384	0.8918
η	0.0...	0.0...	0.0185	0.0372	0.0289	0.0545

40.3.3 Wilson–Fisher's ε-expansion

In the theory of critical phenomena, many universal physical quantities have been calculated as power series of $\varepsilon = 4 - d$, where d is the space dimension (Section 15.4). The physical dimensions correspond to $\varepsilon = 1, 2$, that is, three and two dimensions. As Table 40.3 illustrates, the ε-expansion generates divergent series. Divergent series can be used for small values of the argument. However, only a finite number of terms of the series can then be taken into account. The last added term gives an indication of the size of the irreducible error. Therefore, for the critical exponents γ and η we conclude from the series displayed in Table 40.3,

$$\gamma = 1.244 \pm 0.050\,, \quad \eta = 0.037 \pm 0.008\,,$$

where the errors are only indicative of the uncertainty about the value.

The precision of the estimates can only be improved if the ε expansion is Borel summable (see Section 41.1), and thus free of renormalon singularties.

In the minimal subtraction (MS) scheme, the RG functions have a simple form (see Section 10.4): only the RG β-function depends on ε with the explicit dependence,

$$\beta(g, \varepsilon) = -\varepsilon g + \beta(g, 0), \text{ with } \beta(g, 0) = \frac{N+8}{48\pi^2} g^2 + O(g^3).$$

The large-order behaviour of the ε-expansion can only be guessed because, as discussed previously, it vanishes at leading order. A calculation of the next order would be necessary, and this has not yet been done. Since, at leading order, the fixed point constant $g^*(\varepsilon)$ solution of $\beta(g, \varepsilon) = 0$ is

$$g^*(\varepsilon) = 48\pi^2 \varepsilon/(N+8) + O(\varepsilon^2)\,,$$

except if for some unknown reason the accident of leading order persists, the ε-expansion is likely to involve a factor $(-3/(N+8))^k k!$ multiplied by an unknown power of k.

Finally, we note that, at leading order in the $1/N$ expansion for the Wilson–Fisher ε-expansion, and thus also for suitably defined RG functions like by the MS scheme, the renormalon singularities cancel. We conjecture on this basis, and on the basis of the numerical evidence presented in Chapter 41, that the ε-expansion of universal quantities is free of renormalon singularities, and can be Borel summable.

40.4 Field theories with fermions

In the case of boson field theories, we have related the large-order behaviour of perturbation theory to the decay of the false vacuum for, in general, non-physical values of the coupling constant. Therefore, we expect some modifications if we consider a system of self-interacting fermions, or of fermions interacting with bosons that themselves have no self-interaction. (The first case can be reduced to the second one by introducing an auxiliary boson field but additional difficulties arise.) Indeed, the Pauli principle renders the decay of a false vacuum more difficult, because several fermions cannot occupy the same state to create a classical field, and this effect is especially strong in low dimensions. Note that if the bosons have self-interactions, these interactions drive the decay of the vacuum, and fermions no longer play a role, at least at leading order.

Seen from the point of view of integrals, the difference between fermions and bosons is also immediately apparent. We have shown that the simple integral counting the number of Feynman diagrams, which is also the ϕ^4 field theory in $d = 0$ dimensions, already has the characteristic $k!$ behaviour at large orders.

By contrast, let us consider an example of a zero-dimensional fermion theory, the integral over a finite number of fermion degrees of freedom,

$$I(\lambda) = \int \prod_{i=1}^{N} \left(\mathrm{d}\bar{\xi}_i \mathrm{d}\xi_i\right) \exp\left[\sum_{i,j} \bar{\xi}_i D_{ij} \xi_j + \lambda \sum_{i,j,k,l} C_{ijkl} \bar{\xi}_i \bar{\xi}_j \xi_k \xi_l\right]. \tag{40.30}$$

The quantities ξ_i and $\bar{\xi}_i$ are anticommuting (Grassmann) variables and D_{ij} and C_{ijkl} are sets of real or complex numbers. Because we assume a finite number of anticommuting variables, the expansion of the exponential yields a polynomial, and thus, $I(\lambda)$ is a polynomial in λ, by contrast with the boson case.

40.4.1 Example of a Yukawa-like QFT

We now consider the vacuum amplitude or partition function of the Yukawa-like theory with Dirac fermions $\bar{\psi}(x)$, $\psi(x)$, and a scalar boson $\phi(x)$:

$$\mathcal{Z} = \int [\mathrm{d}\phi(x)] \left[\mathrm{d}\bar{\psi}(x)\right] [\mathrm{d}\psi(x)] \exp\left[-\mathcal{S}(\phi, \bar{\psi}, \psi)\right], \tag{40.31}$$

in which the action is

$$\begin{aligned}\mathcal{S}(\phi, \bar{\psi}, \psi) = \int \mathrm{d}^d x \Big[& -\bar{\psi}(x)\big(\not{\partial} + M + \sqrt{g}\phi(x)\big)\psi(x) \\ & + \tfrac{1}{2}\big(\nabla\phi(x)\big)^2 + \tfrac{1}{2}m^2\phi^2(x)\Big].\end{aligned} \tag{40.32}$$

The parameter g is a loop expansion parameter. Since a fermion field has no classical limit, the expression (40.31) is not directly suited to the study of the vacuum decay. In fact, we expect the fermion fields to generate an effective interaction for the boson field $\phi(x)$, and this effective interaction will lead to the decay of the vacuum. This suggests that one should integrate over the ψ and $\bar{\psi}$ variables and study the instantons of the effective theory for $\phi(x)$. In addition, the zero-dimensional example has shown that the fermion integration gives some hints about the analytic structure of the theory. The integration over ψ and $\bar{\psi}$ yields ($\ln\det = \operatorname{tr}\ln$)

$$\begin{aligned}\mathcal{Z} = \int [\mathrm{d}\phi(x)] \exp\Big\{ & -\tfrac{1}{2} \int \mathrm{d}^d x \left[\big(\nabla\phi(x)\big)^2 + m^2\phi^2(x)\right] \\ & + \operatorname{tr}\ln\left[\not{\partial} + M + \sqrt{g}\phi(x)\right]\Big\}.\end{aligned} \tag{40.33}$$

We now face a new difficulty, arising from the integration: the generated effective action is not local in $\phi(x)$, and leads to non-local field equations. However, because we are concerned only with the determination of the large behaviour, we can simplify the effective action. The determinant generated by the fermion integration is, at least for the class of relevant $\phi(x)$ fields, an entire function of the coupling constant $\sqrt{g}$. Therefore, essential singularities can only be generated by the infinite range of the ϕ-integration. It is thus sufficient to evaluate the the determinant, generated by the field integration, for large fields $\phi(x)$ [413]. This situation has to be contrasted with what would have happened if $\psi(x)$ and $\bar{\psi}(x)$ would have been commuting variables. The integration then would have generated the inverse of the determinant function which has singularities for all zeros in g of the determinant. These singularities would have yielded essential singularities in the coupling constant after integration. Finally, we note that this difference, determinant versus inverse determinant, is responsible for the minus sign for each fermion loop in perturbation theory which makes cancellations possible.

40.4.2 Evaluation of the fermion determinant for large fields

As a preparatory exercise, we first solve a similar problem in which, however, the additional complication due to the spin structure is absent.

The Fredholm determinant of a Schrödinger operator for large potentials. We consider a Schrödinger operator, in d space dimensions, for smooth potentials of the form $\lambda V(x)$, with $V(x) \geq 0$, in the limit $\lambda \to +\infty$. We want to evaluate the logarithm of its normalized Fredholm determinant,

$$\Sigma(\lambda) = \ln\det\left\{\left[-\nabla^2 + \lambda V(x)\right]\left[-\nabla^2 + \mu^2\right]^{-1}\right\}, \tag{40.34}$$

where μ is a mass parameter. On intuitive grounds, we expect the determinant to converge towards a local functional. To derive this property, we differentiate the determinant with respect to λ. Using the identity $\mathrm{d}(\ln\det\Omega) = \mathrm{tr}\,\mathrm{d}\Omega\,\Omega^{-1}$, we find

$$\Sigma'(\lambda) = \int \mathrm{d}^d x\, V(x)\,\langle x|\left[-\nabla^2 + \lambda V\right]^{-1}|x\rangle\,,$$

in the quantum bra–ket notation. When λ becomes large, the operator $\left[-\nabla^2 + \lambda V\right]^{-1}$ converges towards a local operator. Thus, we can replace V operator by its expectation value $V(x)$ in the state $|x\rangle$. In the Fourier representation,

$$\Sigma'(\lambda) \sim \frac{1}{(2\pi)^d}\int \mathrm{d}^d x\, V(x)\frac{\mathrm{d}^d p}{p^2 + \lambda V(x)} = \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\lambda^{d/2-1}\int \mathrm{d}^d x\, V^{d/2}(x), \tag{40.35}$$

a result formally valid for $d < 2$. The integration over λ yields

$$\Sigma(\lambda) \sim -\frac{\Gamma(-d/2)}{(4\pi)^{d/2}}\int \mathrm{d}^d x\,\big(\lambda V(x)\big)^{d/2} + \text{ constant}\,. \tag{40.36}$$

For $d \geq 2$, we know that the expression (40.34), which has the form of a one-loop diagram in a scalar field theory, has to be renormalized. For $d = 2$, a mass counter-term has to be added. One then obtains the evaluation

$$\begin{aligned}\Sigma(\lambda) &\sim \lim_{d\to 2}\left\{-\frac{1}{(4\pi)^{d/2}}\int \mathrm{d}^d x\left[\Gamma(-d/2)\big(\lambda V(x)\big)^{d/2} + \Gamma(1-d/2)\lambda V(x)\right]\right\},\\ &\sim -\frac{1}{4\pi}\int \mathrm{d}^2 x\,\lambda V(x)\ln\big(\lambda V(x)\big).\end{aligned} \tag{40.37}$$

In the same limit, for $d = 3$, one obtains

$$\Sigma(\lambda) \sim -\frac{1}{12\pi}\int \mathrm{d}^3 x\,\big(\lambda V(x)\big)^{3/2}. \tag{40.38}$$

For $d = 4$, a counter-term quadratic in $V(x)$ is required. Then, one finds

$$\Sigma(\lambda) \sim \frac{1}{32\pi^2}\int \mathrm{d}^4 x\,\big(\lambda V(x)\big)^2\ln\big(\lambda V(x)\big). \tag{40.39}$$

The fermion determinant. A similar method can be used to evaluate the contribution of the fermion determinant

$$\Sigma(\lambda) \equiv \ln\det\left(\not{\partial} + \lambda V\right)\left(\not{\partial} + M\right)^{-1} \tag{40.40}$$

where we have set $\lambda V(x) = M + \sqrt{g}\phi(x)$, and $V(x)$ is assumed to be smooth.

Again, we differentiate with respect to λ, and obtain

$$\Sigma'(\lambda) = \mathrm{tr}_\gamma \int \mathrm{d}^d x\, V(x)\, \langle x| \left[\not\partial + \lambda V\right]^{-1} |x\rangle \,.$$

In the large λ limit, the operator becomes local, and, again, we replace the operator V by its expectation value in the state $|x\rangle$. In the Fourier representation, we obtain

$$\begin{aligned}\Sigma'(\lambda) &= \frac{1}{(2\pi)^d}\,\mathrm{tr}_\gamma \int \mathrm{d}^d x\, V(x) \int \mathrm{d}^d p\, \left[i\not p + \lambda V(x)\right]^{-1} \\ &= \frac{1}{(2\pi)^d}\,\mathrm{tr}_\gamma \int \mathrm{d}^d x\, V(x) \int \frac{\mathrm{d}^d p\big(-i\not p + \lambda V(x)\big)}{p^2 + \lambda^2 V^2(x)}.\end{aligned}$$

Then, the trace over γ matrices can be taken, $\mathrm{tr}\,\not p = 0$, and we set $\mathrm{tr}\,\mathbf{1} = N$. The remaining part of the calculation is analogous to the case of the Schrödinger equation. Integrating, one finds

$$\Sigma'(\lambda) = \frac{N}{(4\pi)^{d/2}}\Gamma(1-d/2)\lambda^{d-1} \int \mathrm{d}^d x\, |V(x)|^d.$$

Finally, integrating over λ and returning to the initial parametrization, one obtains the large field behaviour,

$$\Sigma \sim -\frac{N}{2}\frac{\Gamma(-d/2)}{(4\pi)^{d/2}} \int \mathrm{d}^d x\, |M + \sqrt{g}\phi(x)|^d \sim -\frac{N}{2}\frac{\Gamma(-d/2)}{(4\pi)^{d/2}} g^{d/2} \int \mathrm{d}^d x\, |\phi(x)|^d. \quad (40.41)$$

40.4.3 The large-order behaviour

We now determine the essential singularity of the field theory at $g = 0$ from the properties of the effective local action

$$\mathcal{S}_{\text{eff.}}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\big(\nabla\phi(x)\big)^2 + \tfrac{1}{2}m^2\phi^2(x) + \frac{N}{2}\frac{\Gamma(-d/2)}{(4\pi)^{d/2}} g^{d/2}\phi^d(x)\right]. \quad (40.42)$$

It should be understood that for d even, the necessary counter-terms are provided to render the action finite. We have to look for instanton solutions of the corresponding field equations. Since this particular model is not interesting in itself, we will not solve the field equations explicitly, but only assume the existence of a solution. We then rescale the field ϕ,

$$\phi(x) \mapsto \phi(x) g^{-d/2(d-2)} \quad (40.43)$$

to factorize the dependence on g in front of the classical action. The classical action, calculated for a solution, thus takes the form

$$\mathcal{S}(\phi_c) = (A/g)^{d/d-2}, \quad (40.44)$$

where A does not depend on g. Introducing this form into the Cauchy representation, we find

$$\mathcal{Z}_k \underset{k\to\infty}{\sim} \int_0 \frac{\mathrm{e}^{-(A/g)^{d/d-2}}}{g^{k-1}}\mathrm{d}g\,. \quad (40.45)$$

The integration yields the large-order estimate

$$\mathcal{Z}_k \sim A^{-k}\Gamma\left[k(d-2)/d\right]. \tag{40.46}$$

We observe that, as expected, this theory is less divergent than a self-interacting boson field theory. The boson result is recovered (in a cut-off field theory) for d large, because the Pauli principle becomes decreasingly effective when the dimension increases. For $d = 2$, the expression (40.46) becomes

$$\mathcal{Z}_k \sim A^{-k}(\ln k)^k, \tag{40.47}$$

in agreement with a rigorous bound [414] that states $|\mathcal{Z}_k| < (k!\ )^{\varepsilon}$ for all $\varepsilon > 0$.

Remark. To compare the contributions of boson and fermion interactions, we have implicitly assumed a loop expansion. Then, at each order, the boson contributions always dominate the large-order behaviour. If we group the diagrams differently, this may no longer be the case. Let us again consider the theory defined by the action (40.32) in four dimensions. In four dimensions, this theory cannot be renormalized without the addition of a $\lambda\phi^4$ counter-term. Thus, renormalization requires introducing a boson self-interaction. But it is consistent with renormalization to consider λ as being of order g^2. Then, both interaction terms $\bar{\psi}\psi\phi$ and ϕ^4 give contributions of the same order to the large-order behaviour.

40.4.4 The example of QED

Potentially quite interesting applications of the preceding analysis are gauge theories. While non-Abelian theories involves the additional problem of degenerate classical minima [415] (see also Section 40.1.3 and Chapter 42), QED [416, 417] has been more extensively studied.

The action has formally the same structure as in the Yukawa theory, but one additional complication then arises. The fermion integration yields the determinant (see Section 21.3),

$$D(e) = \det(\not{D} + m), \quad \text{with } \not{D} = \sum_{\mu} \gamma_{\mu}\mathrm{D}_{\mu}\,, \text{ and } \mathrm{D}_{\mu} = \partial_{\mu} + ieA_{\mu}(x). \tag{40.48}$$

To estimate $D(e)$ for large charge e, we can use the equation (we assume a dimension d even, see for example, Section A21.3)

$$D^2(e) = \det\left(m^2 - \sum_{\mu} \mathrm{D}_{\mu}^2 - \tfrac{1}{2}e\sum_{\mu,\nu}\sigma_{\mu\nu}F_{\mu\nu}(x)\right). \tag{40.49}$$

In the large e limit, the last term, which is of order e, is negligible with respect to D_{μ}^2 which is of order e^2 (N_d is the loop factor):

$$\ln D(e) \sim \tfrac{1}{2}N_d \operatorname{tr}\ln\left(m^2 - \sum_{\mu}\mathrm{D}_{\mu}^2\right). \tag{40.50}$$

However, the determination of the large-coupling constant behaviour is more subtle than before. A direct calculation of the determinant has not been performed. One difficulty is related to the property that, due to gauge invariance, the gauge degree of freedom of the gauge field cannot be considered as slowly varying.

In particular, the constant field approximation is not meaningful. Moreover, since perturbation theory implies gauge fixing, it is sufficient to calculate the determinant in a fixed gauge.

Therefore, it has been conjectured, on the basis of studying the determinant for special gauge fields, that the behaviour of the determinant is given for large e by

$$\ln D(e) \sim C(d) \int \mathrm{d}^d x \big| e [A_\mathrm{T}]_\mu(x) \big|^d, \quad \text{with } C^{-1}(d) = d(4\pi)^{(d-1)/2} \Gamma\big((d+1)/2\big),$$

where $[A_\mathrm{T}]_\mu$ is the transverse part of A_μ:

$$[A_\mathrm{T}]_\mu(x) = A_\mu(x) - \nabla^{-2} \partial_\mu \sum_\nu \partial_\nu A_\nu(x).$$

This result is gauge invariant, as it should, but not local, except in the gauge $\sum_\mu \partial_\mu A_\mu = 0$. It agrees for $d = 2$ with the exact result (30.59) obtained from the Abelian anomaly ($C(2) = 1/2\pi$). For $d = 4$, the case of physical interest, $C(4) = 1/12\pi^2$. The effective classical field theory then is scale invariant. Arguments related to conformal invariance can be used to construct some ansatz for the instanton solutions. Two kind of solutions have been explored in Refs. [416] and [417]. Taking the minimal action solution, one obtains an evaluation of the form

$$\mathcal{Z}_k \sim (-1)^k A^{-k} \Gamma(k/2)\,, \quad A = 4.886\,, \tag{40.51}$$

the expansion parameter being $\alpha = e^2/4\pi$. It is worth mentioning that this evaluation is probably not very useful as a practical method to predict new orders in QED, for several reasons. First, the theory is not asymptotically free and thus has a potential renormalon problem, which can be understood by inserting in a Feynman diagram the one-loop corrected photon propagator. Second, the cancellation coming from the sign of fermion loops does not seem to be very effective at low orders. Therefore, an alternative calculation, which leads to a large-order behaviour at a fixed number of fermion loops, seems to be more useful. Predictions of this kind made for diagrams with one fermion loop, seem to agree better with numerical estimates [416].

A40 large-order behaviour: Additional remarks

large-order behaviour for simple integrals. We consider the integral of Chapter 37,

$$I(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[-\left(\tfrac{1}{2}x^2 + \tfrac{1}{4}gx^4\right)\right] \mathrm{d}x \,. \tag{A40.1}$$

We can expand $I(g)$ in a power series,

$$I(g) = \sum_0^{\infty} I_k g^k . \tag{A40.2}$$

The coefficients I_k enumerate the number of vacuum Feynman diagrams with the proper weights in a ϕ^4 field theory. In Section 37.2, we have evaluated the imaginary part of $I(g)$ for g negative by the steepest descent method. The contributions of the non-trivial saddle points yield

$$\operatorname{Im} I(g) \underset{g\to 0_-}{\sim} 2^{-1/2}\, \mathrm{e}^{1/4g} \,. \tag{A40.3}$$

Therefore, for k large, I_k behaves as

$$I_k \underset{k\to\infty}{\sim} \frac{1}{\pi\sqrt{2}} (-4)^k (k-1)! \,. \tag{A40.4}$$

The result suggests the following interpretation of the large-order behaviour formulae obtained in Chapter 40: in the case of the anharmonic oscillator, and the ϕ^4 field theory, the number of Feynman diagrams is of the order of $4^k k!$, for k large, and a typical diagram behaves at large orders as $(4S)^{-k}$, where k is the order but also, up to an additive constant, the number of loops, and S the instanton action.

QM: Other perturbative expansions. Although we have only discussed large-order behaviour estimates for loop expansions, it is possible to generalize the analysis for perturbative expansions in other parameters. For example, we consider the action

$$\mathcal{S}(q) = \int \left[\tfrac{1}{2}\dot{q}(t)^2 + \tfrac{1}{2}q^2(t) + \lambda V\big((q(t)\big)\right] \mathrm{d}t \,, \text{ with } V(q) = \sum_2^{2N} V_n q^n ,$$

and we want to evaluate the large-order behaviour of the expansion in powers of λ.

The divergence of the perturbative expansion is a consequence of the infinite range of the q integration. Therefore, it is dominated by the large q behaviour of V, and thus related to the instantons associated to the action in which $V(q)$ is replaced by its term of highest degree. Then, after a rescaling of $q(t)$, we find that the classical solution $q_c(t)$ takes the form

$$q_c(t) = \lambda^{-1/(2N-2)} f(t) \,. \tag{A40.5}$$

The term of degree n in $V(q)$ gives a contribution to the classical action proportional to $\lambda^{1-n/(2N-2)}$. For $\lambda \to 0$, one verifies that the term of highest degree gives indeed the largest contribution to the action. The saddle point in λ in the dispersion relation for large-order k is of the order

$$\lambda \sim k^{-(N-1)} .$$

Thus, the term of degree n in the potential generates a factor of the form

$$\exp\left[c_n k^{2/(n+2-2N)}\right] ,$$

which is relevant, at leading order, only for $n \geq 2N-2$.