heat equation, meaning that

$$u = e^{-\varepsilon x + \varepsilon^2 t} f(x - 2\varepsilon t, t)$$

is a solution of the heat equation whenever $u = f(x, t)$ is.

One of the obvious advantages of knowing a symmetry group of a system of differential equations is that we can construct new solutions of the system from known ones. Namely, if we know $u = f(x)$ is a solution, then according to the definition, $\tilde{u} = g \cdot f(\tilde{x})$ is also a solution for any group element $g$, so we have the possibility of constructing whole families of solutions just by transforming a known solution by all possible group elements. For example, in the case of the above symmetry group of the heat equaton, starting with the trivial constant solution $u = c$, we deduce the existence of a two-parameter family of exponential solutions

$$u(x, t) = ce^{-\varepsilon x + \varepsilon^2 t}.$$

We could further subject these to the translation group, but in this case no new solutions are obtained. The reader might try seeing what happens to other known solutions of the heat equation, e.g. the fundamental solution, under the above group actions.

The primary goal of this chapter is to establish a workable criterion that can be readily checked to determine whether a given group of transformations is or is not a symmetry group of a given system of differential equations. This criterion will be infinitesimal, in direct analogy with the criterion of Theorem 2.8 for systems of algebraic equations. In fact, once we have developed the appropriate geometrical setting for studying systems of differential equations, we will be able to directly invoke Theorem 2.8 to establish an infinitesimal criterion of invariance. Once we have this criterion in hand, not only will we be able to simply check whether a given group is a symmetry group of our system of differential equations, we will actually be able to compute the most general symmetry group of the system through a series of fairly routine calculations.

## 2.3. Prolongation

Before implementing our program of finding symmetries of differential equations by employing analogues of the infinitesimal methods for algebraic equations discussed in Section 2.1, we need to replace the somewhat nebulous notion of a "system of differential equations" by a concrete geometric object determined by the vanishing of certain functions. To do this we need to "prolong" the basic space $X \times U$ representing the independent and dependent variables under consideration to a space which also represents the various partial derivatives occurring in the system. This construction is a greatly simplified version of the theory of jet bundles occurring in the differential-geometric theory of partial differential equations. So as to avoid the introduction of too much extraneous machinery, we work exclusively in Euclidean space here. (See Section 3.5 for a generalization.)

Given a smooth real-valued function $f(x) = f(x^1, \ldots, x^p)$ of $p$ independent variables, there are

$$p_k \equiv \binom{p+k-1}{k}$$

different $k$-th order partial derivatives of $f$. We employ the multi-index notation

$$\partial_J f(x) = \frac{\partial^k f(x)}{\partial x^{j_1} \partial x^{j_2} \cdots \partial x^{j_k}}$$

for these derivatives. In this notation, $J = (j_1, \ldots, j_k)$ is an *unordered* $k$-tuple of integers, with entries $1 \leqslant j_\kappa \leqslant p$ indicating which derivatives are being taken. The *order* of such a multi-index, which we denote by $\# J \equiv k$, indicates how may derivatives are being taken. More generally, if $f\colon X \to U$ is a smooth function from $X \simeq \mathbb{R}^p$ to $U \simeq \mathbb{R}^q$, so $u = f(x) = (f^1(x), \ldots, f^q(x))$, there are $q \cdot p_k$ numbers $u_J^\alpha = \partial_J f^\alpha(x)$ needed to represent all the different $k$-th order derivatives of the components of $f$ at a point $x$. We let $U_k \equiv \mathbb{R}^{q \cdot p_k}$ be the Euclidean space of this dimension, endowed with coordinates $u_J^\alpha$ corresponding to $\alpha = 1, \ldots, q$, and all multi-indices $J = (j_1, \ldots, j_k)$ of order $k$, designed so as to represent the above derivatives. Furthermore, set $U^{(n)} = U \times U_1 \times \cdots \times U_n$ to be the Cartesian product space, whose coordinates represent all the derivatives of functions $u = f(x)$ of all orders from 0 to $n$. Note that $U^{(n)}$ is a Euclidean space of dimension

$$q + qp_1 + \cdots + qp_n = q\binom{p+n}{n} \equiv qp^{(n)}.$$

A typical point in $U^{(n)}$ will be denoted by $u^{(n)}$, so $u^{(n)}$ has $q \cdot p^{(n)}$ different components $u_J^\alpha$ where $\alpha = 1, \ldots, q$, and $J$ runs over all unordered multi-indices $J = (j_1, \ldots, j_k)$ with $1 \leqslant j_k \leqslant p$ and $0 \leqslant k \leqslant n$. (By convention, for $k = 0$ there is just one such multi-index, denoted by 0, and $u_0^\alpha$ just refers to the component $u^\alpha$ of $u$ itself.)

**Example 2.24.** Consider the case $p = 2$, $q = 1$. Then $X \simeq \mathbb{R}^2$ has coordinates $(x^1, x^2) = (x, y)$, and $U \simeq \mathbb{R}$ has the single coordinate $u$. The space $U_1$ is isomorphic to $\mathbb{R}^2$ with coordinates $(u_x, u_y)$ since these represent all the first order partial derivatives of $u$ with respect to $x$ and $y$. Similarly, $U_2 \simeq \mathbb{R}^3$ has coordinates $(u_{xx}, u_{xy}, u_{yy})$ representing the second order partial derivatives of $u$, and, in general, $U_k \simeq \mathbb{R}^{k+1}$, since there are $k + 1$ distinct $k$-th order partial derivatives of $u$, namely $\partial^k u/\partial x^i \partial y^{k-i}$, $i = 0, \ldots, k$. Finally, the space $U^{(2)} = U \times U_1 \times U_2 \simeq \mathbb{R}^6$, with coordinates $u^{(2)} = (u; u_x, u_y; u_{xx}, u_{xy}, u_{yy})$, represents all derivatives of $u$ with respect to $x$ and $y$ of order at most 2.

Given a smooth function $u = f(x)$, so $f: X \to U$, there is an induced function $u^{(n)} = \mathrm{pr}^{(n)} f(x)$, called the $n$-th *prolongation* of $f$, which is defined by the equations

$$u_J^\alpha = \partial_J f^\alpha(x).$$

Thus $\mathrm{pr}^{(n)} f$ is a function from $X$ to the space $U^{(n)}$, and for each $x$ in $X$, $\mathrm{pr}^{(n)} f(x)$ is a vector whose $q \cdot p^{(n)}$ entries represent the values of $f$ and all its derivatives up to order $n$ at the point $x$. For example, in the case $p = 2$, $q = 1$ discussed above, given $u = f(x, y)$, the second prolongation $u^{(2)} = \mathrm{pr}^{(2)} f(x, y)$ is given by

$$(u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}) = \left(f; \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}; \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x\, \partial y}, \frac{\partial^2 f}{\partial y^2}\right), \tag{2.15}$$

all evaluated at $(x, y)$. (Another way of looking at the $n$-th prolongation $\mathrm{pr}^{(n)} f(x)$ is that it represents the Taylor polynomial of degree $n$ for $f$ at the point $x$, since the derivatives of order $\leqslant n$ determine the Taylor polynomial and vice versa.)

The total space $X \times U^{(n)}$, whose coordinates represent the independent variables, the dependent variables *and* the derivatives of the dependent variables up to order $n$ is called the $n$-th order *jet space* of the underlying space $X \times U$. (The $n$-th prolongation $\mathrm{pr}^{(n)} f(x)$ is also known as the *n-jet* of $f$, but we will stick to the more suggestive term "prolongation".) Often we are not interested in differential equations defined over all of $X \times U$, but only on some open subset $M \subset X \times U$. In this case, we define the $n$-jet space

$$M^{(n)} \equiv M \times U_1 \times \cdots \times U_n$$

of $M$. If $u = f(x)$ is a function whose graph lies in $M$, the $n$-th prolongation $\mathrm{pr}^{(n)} f(x)$ is a function whose graph lies in the $n$-jet space $M^{(n)}$.

## Systems of Differential Equations

A system $\mathscr{S}$ of $n$-th order differential equations in $p$ independent and $q$ dependent variables is given as a system of equations

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, l,$$

involving $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ and the derivatives of $u$ with respect to $x$ up to order $n$. The functions $\Delta(x, u^{(n)}) = (\Delta_1(x, u^{(n)}), \ldots, \Delta_l(x, u^{(n)}))$ will be assumed to be smooth in their arguments, so $\Delta$ can be viewed as a smooth map from the jet space $X \times U^{(n)}$ to some $l$-dimensional Euclidean space,

$$\Delta: X \times U^{(n)} \to \mathbb{R}^l.$$

The differential equations themselves tell where the given map $\Delta$ vanishes on $X \times U^{(n)}$, and thus determine a subvariety

$$\mathscr{S}_\Delta = \{(x, u^{(n)}): \Delta(x, u^{(n)}) = 0\} \subset X \times U^{(n)}$$

of the total jet space. We can identify the system of differential equations with its corresponding subvariety, thereby realizing the "abstract" relations among the various derivatives of $u$ determined by the system as some concrete, geometrical subset $\mathscr{S}_\Delta$ of the jet space $X \times U^{(n)}$. We will use the same symbol "$\Delta$" as shorthand for both the system of differential equations $\Delta(x, u^{(n)}) = 0$ and the map $\Delta: X \times U^{(n)} \to \mathbb{R}^l$ which determines it. This should not be the cause of any confusion.

From this point of view, a smooth *solution* of the given system of differential equations is a smooth function $u = f(x)$ such that

$$\Delta_\nu(x, \mathrm{pr}^{(n)} f(x)) = 0, \qquad \nu = 1, \ldots, l,$$

whenever $x$ lies in the domain of $f$. This is just a restatement of the fact that the derivatives $\partial_J f^\alpha(x)$ of $f$ must satisfy the algebraic constraints imposed by the system of differential equations. This condition is equivalent to the statement that the graph of the prolongation $\mathrm{pr}^{(n)} f(x)$ must lie entirely within the subvariety $\mathscr{S}_\Delta$ determined by the system:

$$\Gamma_f^{(n)} \equiv \{(x, \mathrm{pr}^{(n)} f(x))\} \subset \mathscr{S}_\Delta = \{\Delta(x, u^{(n)}) = 0\}.$$

We can thus take an $n$-th order system of differential equations to *be* a subvariety $\mathscr{S}_\Delta$ in the $n$-jet space $X \times U^{(n)}$ and a solution to *be* a function $u = f(x)$ such that the graph of the $n$-th prolongation $\mathrm{pr}^{(n)} f$ is contained in the subvariety $\mathscr{S}_\Delta$. So far we have not done anything but reformulate the basic problem of finding solutions of systems of differential equations in a more geometrical form, ideally suited to our investigation into symmetry groups thereof. It is perhaps worthwhile pausing at this point to consider a simple example.

**Example 2.25.** Consider the case of Laplace's equation in the plane

$$u_{xx} + u_{yy} = 0. \tag{2.16}$$

Here $p = 2$ since there are two independent variables $x$ and $y$, and $q = 1$ since there is one dependent variable $u$. Also $n = 2$ since the equation is second order, so we are in the situation described in Example 2.24. In terms of the coordinates $(x, y; u; u_x, u_y; u_{xx}, u_{xy}, u_{yy})$ of $X \times U^{(2)}$, (2.16) defines a linear subvariety (a "hyperplane") there, and this is the set $\mathscr{S}_\Delta$ for Laplace's equation. A solution $u = f(x, y)$ must satisfy

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

for all $(x, y)$. This is clearly the same as requiring that the graph of the second prolongation $\mathrm{pr}^{(2)} f$ lie in $\mathscr{S}_\Delta$. For instance, if

$$f(x, y) = x^3 - 3xy^2,$$

then (using (2.15))

$$\mathrm{pr}^{(2)} f(x, y) = (x^3 - 3xy^2; 3x^2 - 3y^2, -6xy; 6x, -6y, -6x),$$

which lies in $\mathscr{S}_\Delta$ since the fourth and sixth entries add up to 0: $6x + (-6x) = 0$.

## Prolongation of Group Actions

Now suppose $G$ is a local group of transformations acting on an open subset $M \subset X \times U$ of the space of independent and dependent variables. There is an induced local action of $G$ on the $n$-jet space $M^{(n)}$, called the $n$-th *prolongation* of $G$ (or, more correctly, the $n$-th prolongation of the action of $G$ on $M$) and denoted $\mathrm{pr}^{(n)} G$. This prolongation is defined so that it transforms the derivatives of functions $u = f(x)$ into the corresponding derivatives of the transformed function $\tilde{u} = \tilde{f}(\tilde{x})$. More rigorously, suppose $(x_0, u_0^{(n)})$ is a given point in $M^{(n)}$. Choose any smooth function $u = f(x)$ defined in a neighbourhood of $x_0$, whose graph lies in $M$, and has the given derivatives at $x_0$:

$$u_0^{(n)} = \mathrm{pr}^{(n)} f(x_0), \quad \text{i.e.} \quad u_{J0}^\alpha = \partial_J f^\alpha(x_0).$$

For example, $f$ might be the $n$-th order Taylor polynomial at $x_0$ corresponding to the given values $u_0^{(n)}$:

$$f^\alpha(x) = \sum_J \frac{u_{J0}^\alpha}{\tilde{J}!}(x - x_0)^J, \qquad \alpha = 1, \ldots, q. \tag{2.17}$$

(Here the sum is over all multi-indices $J = (j_1, \ldots, j_k)$ with $0 \leqslant k \leqslant n$; also

$$(x - x_0)^J = (x^{j_1} - x_0^{j_1})(x^{j_2} - x_0^{j_2}) \cdots (x^{j_k} - x_0^{j_k}).$$

Further, given $J$, set $\tilde{J} = (\tilde{j}_1, \ldots, \tilde{j}_p)$, where $\tilde{j}_i$ equals the number of $j_\kappa$'s which equal $i$. For instance, if $J = (1, 1, 1, 2, 4, 4)$, $p = 4$, $k = 6$, then $\tilde{J} = (3, 1, 0, 2)$. With this notation $\tilde{J}! \equiv \tilde{j}_1! \tilde{j}_2! \cdots \tilde{j}_p!$.)

If $g$ is an element of $G$ sufficiently near the identity, the transformed function $g \cdot f$ as given by (2.14) is defined in a neighbourhood of the corresponding point $(\tilde{x}_0, \tilde{u}_0) = g \cdot (x_0, u_0)$, with $u_0 = f(x_0)$ being the zeroth order components of $u_0^{(n)}$. We then determine the action of the prolonged group transformation $\mathrm{pr}^{(n)} g$ on the point $(x_0, u_0^{(n)})$ by evaluating the derivatives of the transformed function $g \cdot f$ at $\tilde{x}_0$; explicitly

$$\mathrm{pr}^{(n)} g \cdot (x_0, u_0^{(n)}) = (\tilde{x}_0, \tilde{u}_0^{(n)}),$$

where

$$\tilde{u}_0^{(n)} \equiv \mathrm{pr}^{(n)}(g \cdot f)(\tilde{x}_0). \tag{2.18}$$

It is a relatively straight-forward exercise to check, using the chain rule, that this definition of $\mathrm{pr}^{(n)} g \cdot (x_0, u_0^{(n)})$ depends only on the derivatives of $f$ at $x_0$ up to order $n$, i.e. on $(x_0, u_0^{(n)})$ itself, and hence is independent of the choice of representative function $f$ for $(x_0, u_0^{(n)})$. Thus the prolonged group action is well defined. Again, put more succinctly, to define the action of $\mathrm{pr}^{(n)} g$ on a point in $M^{(n)}$, choose a function whose derivatives agree with the given values; transform the function according to (2.14), and re-evaluate the derivatives.

**Example 2.26.** Let $p = q = 1$, so $X \times U \simeq \mathbb{R}^2$, and consider the action of the rotation group SO(2) as discussed in Example 2.21. We calculate here the first prolongation $\mathrm{pr}^{(1)}\,\mathrm{SO}(2)$. Note first that $X \times U^{(1)} \simeq \mathbb{R}^3$, with coordinates $(x, u, u_x)$. Given a function $u = f(x)$, the first prolongation is

$$\mathrm{pr}^{(1)} f(x) = (f(x), f'(x)).$$

Now given a point $(x^0, u^0, u_x^0) \in X \times U^{(1)}$, and a rotation in SO(2) characterized by the angle $\theta$, we wish to find the corresponding transformed point

$$\mathrm{pr}^{(1)}\,\theta\cdot(x^0, u^0, u_x^0) = (\tilde{x}^0, \tilde{u}^0, \tilde{u}_x^0)$$

(provided it exists). Choose the linear Taylor polynomial

$$f(x) = u^0 + u_x^0(x - x^0) = u_x^0\cdot x + (u^0 - u_x^0 x^0)$$

as a representative function, noting that

$$f(x^0) = u^0, \qquad f'(x^0) = u_x^0,$$

as required. According to the calculations of Example 2.21, the transform of $f$ by a rotation through angle $\theta$ is the linear function

$$\tilde{f}(\tilde{x}) = \theta\cdot f(\tilde{x}) = \frac{\sin\theta + u_x^0\cos\theta}{\cos\theta - u_x^0\sin\theta}\tilde{x} + \frac{u^0 - u_x^0 x^0}{\cos\theta - u_x^0\sin\theta},$$

which is well defined provided $u_x^0 \neq \cot\theta$. Then

$$\tilde{x}^0 = x^0\cos\theta - u^0\sin\theta,$$

hence

$$\tilde{u}^0 = \tilde{f}(\tilde{x}^0) = x^0\sin\theta + u^0\cos\theta,$$

as we already knew. As for the first order derivative, we find

$$\tilde{u}_x^0 = \tilde{f}'(\tilde{x}^0) = \frac{\sin\theta + u_x^0\cos\theta}{\cos\theta - u_x^0\sin\theta}.$$

Therefore, dropping the 0-superscripts, we find that the prolonged action $\mathrm{pr}^{(1)}\,\mathrm{SO}(2)$ on $X \times U^{(1)}$ is given by

$$\mathrm{pr}^{(1)}\,\theta\cdot(x, u, u_x) = \left(x\cos\theta - u\sin\theta,\ x\sin\theta + u\cos\theta,\ \frac{\sin\theta + u_x\cos\theta}{\cos\theta - u_x\sin\theta}\right), \tag{2.19}$$

which is defined for $|\theta| < |\operatorname{arccot} u_x|$. Note that even though SO(2) is a linear, globally defined group of transformations, its first prolongation is both nonlinear and only locally defined. From this relatively simple example the reader can appreciate the complexity of the operation of prolonging a group of transformations!

The reader will note that in the above example, the first prolongation $\mathrm{pr}^{(1)}\,G$ acts on the original variables $(x, u)$ exactly the same way that $G$ itself does; only the action on the derivative $u_x$ provides new information. This remark holds in general. Namely, given the $n$-th prolongation $\mathrm{pr}^{(n)} G$ acting on the variables $(x, u^{(n)})$, if we restrict our attention to just the derivatives up to order $k \leqslant n$, so just look at the variables $(x, u^{(k)})$, then the action of $\mathrm{pr}^{(n)} G$ there agrees with the earlier prolongation $\mathrm{pr}^{(k)} G$. In particular, for $k = 0$, $\mathrm{pr}^{(0)} G$ agrees with $G$ itself, acting on $M^{(0)} = M$. This result can be stated more precisely by defining a natural projection $\pi_k^n \colon M^{(n)} \to M^{(k)}$, where $\pi_k^n(x, u^{(n)}) = (x, u^{(k)})$, $u^{(k)}$ just consisting of the components $u_J^\alpha$, $\# J \leqslant k$ of $u^{(n)}$ itself. For example, if $p = 2$, $q = 1$,

$$\pi_1^2(x, y; u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}) = (x, y; u; u_x, u_y),$$

while

$$\pi_0^2(x, y; u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}) = (x, y; u).$$

We then have

$$\pi_k^n \circ \mathrm{pr}^{(n)} g = \mathrm{pr}^{(k)} g, \qquad n \geqslant k, \tag{2.20}$$

for any group element $g \in G$. Another way of looking at this remark is that if we already know the $k$-th order prolonged group action $\mathrm{pr}^{(k)} G$, then to compute the $n$-th order prolongation $\mathrm{pr}^{(n)} G$ we need only find how the derivatives $u_J^\alpha$ of orders $k < \# J \leqslant n$ transform, since the action on $k$-th and lower order derivatives is already determined.

## Invariance of Differential Equations

Suppose we are given an $n$-th order system of differential equations, or, equivalently, a subvariety $\mathscr{S}_\Delta$ of the jet space $M^{(n)} \subset X \times U^{(n)}$. A symmetry group of this system was defined to be a local group of transformations $G$ acting on $M \subset X \times U$ which transforms solutions of the system to other solutions. We will establish the connection between this symmetry condition and the geometric condition that the corresponding subvariety $\mathscr{S}_\Delta$ be invariant under the prolonged group action $\mathrm{pr}^{(n)} G$. This observation will effectively reduce the problem of determining symmetry groups of differential equations to the more tractable problem of determining when some subvariety (in this case $\mathscr{S}_\Delta$) is invariant under some local group of transformations (in this case the prolonged group $\mathrm{pr}^{(n)} G$). In this way all the tools developed in Section 2.1 for symmetries of algebraic equations are at our disposal for the study of symmetries of differential equations. This alone should demonstrate the effectiveness of our geometric reformulation of the notion of differential equation which has been developed in this section.

**Theorem 2.27.** *Let $M$ be an open subset of $X \times U$ and suppose $\Delta(x, u^{(n)}) = 0$ is an $n$-th order system of differential equations defined over $M$, with corresponding subvariety $\mathscr{S}_\Delta \subset M^{(n)}$. Suppose $G$ is a local group of transformations acting on $M$ whose prolongation leaves $\mathscr{S}_\Delta$ invariant, meaning that when-*

*ever* $(x, u^{(n)}) \in \mathscr{S}_\Delta$, *we have* $\mathrm{pr}^{(n)} g \cdot (x, u^{(n)}) \in \mathscr{S}_\Delta$ *for all* $g \in G$ *such that this is defined. Then* $G$ *is a symmetry group of the system of differential equations in the sense of Definition* 2.23.

PROOF. The proof just consists of untangling the various definitions. Suppose $u = f(x)$ is a local solution to $\Delta(x, u^{(n)}) = 0$. This means that the graph

$$\Gamma_f^{(n)} = \{(x, \mathrm{pr}^{(n)} f(x))\}$$

of the prolongation $\mathrm{pr}^{(n)} f$ lies entirely within $\mathscr{S}_\Delta$. If $g \in G$ is such that the transformed function $g \cdot f$ is well defined, the graph of its prolongation, namely $\Gamma_{g \cdot f}^{(n)}$, is the same as the transform of the graph of $\mathrm{pr}^{(n)} f$ by the prolonged group transformation $\mathrm{pr}^{(n)} g$:

$$\Gamma_{g \cdot f}^{(n)} = \mathrm{pr}^{(n)} g(\Gamma_f^{(n)}).$$

(This is just a restatement of the basic formula (2.18) defining the prolonged group action.) Now, since $\mathscr{S}_\Delta$ is invariant under $\mathrm{pr}^{(n)} g$, the graph of $\mathrm{pr}^{(n)}(g \cdot f)$ again lies entirely in $\mathscr{S}_\Delta$. But this is just another way of saying that the transformed function $g \cdot f$ is a solution to the system $\Delta$. □

Later (Theorem 2.71) we will present a converse to this result, subject to some additional hypotheses on the system itself.

## Prolongation of Vector Fields

As with the group transformations themselves, we can also define the prolongation of the corresponding infinitesimal generators. Indeed, these will just be the infinitesimal generators of the prolonged group action.

**Definition 2.28.** Let $M \subset X \times U$ be open and suppose $\mathbf{v}$ is a vector field on $M$, with corresponding (local) one-parameter group $\exp(\varepsilon \mathbf{v})$. The $n$-th *prolongation* of $\mathbf{v}$, denoted $\mathrm{pr}^{(n)} \mathbf{v}$, will be a vector field on the $n$-jet space $M^{(n)}$, and is defined to be the infinitesimal generator of the corresponding prolonged one-parameter group $\mathrm{pr}^{(n)}[\exp(\varepsilon \mathbf{v})]$. In other words,

$$\mathrm{pr}^{(n)} \mathbf{v}|_{(x, u^{(n)})} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathrm{pr}^{(n)} [\exp(\varepsilon \mathbf{v})](x, u^{(n)}) \tag{2.21}$$

for any $(x, u^{(n)}) \in M^{(n)}$.

Note that since the coordinates $(x, u^{(n)})$ on $M^{(n)}$ consists of the independent variables $(x^1, \ldots, x^p)$ and all derivatives $u_J^\alpha$ of the dependent variables up to order $n$, a vector field on $M^{(n)}$ will in general take the form

$$\mathbf{v}^* = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_J \phi_\alpha^J \frac{\partial}{\partial u_J^\alpha},$$

the latter sum ranging over all multi-indices $J$ of orders $0 \leqslant \# J \leqslant n$; the coefficient functions $\xi^i$, $\phi_\alpha^J$ could depend on all the variables $(x, u^{(n)})$. In the case $\mathbf{v}^*$ is the prolongation $\mathrm{pr}^{(n)}\,\mathbf{v}$ of a vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u)\frac{\partial}{\partial u^\alpha},$$

the coefficients $\xi^i$, $\phi_\alpha^J$ of $\mathbf{v}^* = \mathrm{pr}^{(n)}\,\mathbf{v}$ will be determined by the coefficients $\xi^i$, $\phi_\alpha$ of $\mathbf{v}$ itself. According to (2.20), the prolonged group action $\mathrm{pr}^{(n)}[\exp(\varepsilon\mathbf{v})]$, when restricted to just the zeroth order variables $x$, $u$ of $M^{(0)} = M$, agrees with the ordinary group action $\exp(\varepsilon\mathbf{v})$ on $M$. Therefore the coefficients $\xi^i$ and $\phi_\alpha^0 = \phi_\alpha$ of $\mathbf{v}^* = \mathrm{pr}^{(n)}\,\mathbf{v}$ must agree with the corresponding coefficients $\xi^i$, $\phi_\alpha$ of $\mathbf{v}$ itself. Thus

$$\mathrm{pr}^{(n)}\,\mathbf{v} = \sum_{i=1}^{p} \xi^i\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J\frac{\partial}{\partial u_J^\alpha}, \tag{2.22}$$

where $\xi^i$, $\phi_\alpha = \phi_\alpha^0$ come directly from $\mathbf{v}$. Moreover, if $\# J = k$, the coefficient $\phi_\alpha^J$ of $\partial/\partial u_J^\alpha$ will only depend on $k$-th and lower order derivatives of $u$, $\phi_\alpha^J = \phi_\alpha^J(x, u^{(k)})$, since, again by (2.20), the corresponding group transformations of $k$-th order derivatives only involve $k$-th and lower order derivatives. This can be stated formally, using the projection maps in (2.20), as

$$d\pi_k^n(\mathrm{pr}^{(n)}\,\mathbf{v}) = \mathrm{pr}^{(k)}\,\mathbf{v}, \qquad n \geqslant k, \tag{2.23}$$

where $\mathrm{pr}^{(0)}\,\mathbf{v} = \mathbf{v}$ for $k = 0$. This indicates the possibility of recursively constructing the various prolongations of a given vector field. Our principal remaining task, then, is to find a general formula for the coefficients $\phi_\alpha^J$ of the prolongation of a vector field. Before tackling this question, however, we look at a simple example and then draw some general conclusions on the computation of symmetry groups of differential equations.

**Example 2.29.** Consider the rotation group $\mathrm{SO}(2)$ acting on $X \times U \simeq \mathbb{R}^2$ as discussed in Examples 2.21 and 2.26. The corresponding infinitesimal generator is

$$\mathbf{v} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u},$$

with

$$\exp(\varepsilon\mathbf{v})(x, u) = (x\cos\varepsilon - u\sin\varepsilon,\ x\sin\varepsilon + u\cos\varepsilon)$$

being the rotation through angle $\varepsilon$. The first prolongation takes the form

$$\begin{aligned}&\mathrm{pr}^{(1)}[\exp(\varepsilon\mathbf{v})](x, u, u_x)\\ &\quad= \left(x\cos\varepsilon - u\sin\varepsilon,\ x\sin\varepsilon + u\cos\varepsilon,\ \frac{\sin\varepsilon + u_x\cos\varepsilon}{\cos\varepsilon - u_x\sin\varepsilon}\right).\end{aligned}$$

According to (2.21), the first prolongation of $\mathbf{v}$ is obtained by differentiating these expressions with respect to $\varepsilon$ and setting $\varepsilon = 0$. An easy computation shows that

$$\mathrm{pr}^{(1)}\,\mathbf{v} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1 + u_x^2)\frac{\partial}{\partial u_x}. \tag{2.24}$$

Note that in accordance with (2.23) the first two terms in $\mathrm{pr}^{(1)}\,\mathbf{v}$ agree with those in $\mathbf{v}$ itself.

## Infinitesimal Invariance

Combining Theorems 2.27 and 2.8, we immediately deduce the important infinitesimal condition for a group $G$ to be a symmetry group of a given system of differential equations. Of course, to apply the latter theorem, we need a corresponding maximal rank condition for the system of differential equations.

**Definition 2.30.** Let

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, l,$$

be a system of differential equations. The system is said to be of *maximal rank* if the $l \times (p + qp^{(n)})$ Jacobian matrix

$$\mathrm{J}_\Delta(x, u^{(n)}) = \left(\frac{\partial \Delta_\nu}{\partial x^i}, \frac{\partial \Delta_\nu}{\partial u_J^\alpha}\right)$$

of $\Delta$ with respect to all the variables $(x, u^{(n)})$ is of rank $l$ whenever $\Delta(x, u^{(n)}) = 0$.

Thus, for instance, Laplace's equation

$$\Delta = u_{xx} + u_{yy} = 0$$

is of maximal rank, since the Jacobian matrix with respect to all the variables $(x, y; u; u_x, u_y; u_{xx}, u_{xy}, u_{yy})$ in $X \times U^{(2)}$ (cf. Example 2.25) is

$$\mathrm{J}_\Delta = (0, 0; 0; 0, 0; 1, 0, 1),$$

which is clearly of rank 1 everywhere. However, the rather silly equivalent equation

$$\tilde{\Delta} = (u_{xx} + u_{yy})^2 = 0$$

is *not* of maximal rank, since

$$\mathrm{J}_{\tilde{\Delta}} = (0, 0; 0; 0, 0; 2(u_{xx} + u_{yy}), 0, 2(u_{xx} + u_{yy}))$$

vanishes whenever $(u_{xx} + u_{yy})^2 = 0$.

The maximal rank condition is not much of a restriction, since according to Lemma 1.12 if the subvariety $\mathcal{S}_\Delta = \{\Delta(x, u^{(n)}) = 0\}$ is a regular submanifold of $M^{(n)}$, then there is an (algebraically) equivalent system of