

Non-relativistic conformal symmetries in fluid mechanics

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Abstract

The symmetries of a free incompressible fluid span the Galilei group, augmented with independent dilations of space and time. When the fluid is compressible, the symmetry is enlarged to the expanded Schrödinger group, which also involves, in addition, Schrödinger expansions. While incompressible fluid dynamics can be derived as an appropriate non-relativistic limit of a conformally-invariant relativistic theory, the recently discussed Conformal Galilei group, obtained by contraction from the relativistic conformal group, is *not* a symmetry. This is explained by the subtleties of the non-relativistic limit.

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I. INTRODUCTION

Non-relativistic conformal symmetries, much studied in recent times [1, 2, 3, 4, 5, 6], are two-fold.

The usual one is Schrödinger symmetry [8, 9], highlighted by dilations and expansion,

$$D : \quad t^* = \lambda^2 t, \quad \mathbf{r}^* = \lambda \mathbf{r}, \quad (1)$$

$$K : \quad t^* = \Omega(t) t, \quad \mathbf{r}^* = \Omega(t) \mathbf{r}, \quad \Omega(t) = \frac{1}{1 - \kappa t}, \quad (2)$$

where $\lambda > 0$, $\kappa \in \mathbb{R}$ [8]. These transformations close, with time translations $t^* = t + \epsilon$, into an $O(2, 1)$ group. Note that (i) time is dilated twice w.r.t. space, i.e. the dynamical exponent is $z = 2$, while (ii) space and time expansions share the common factor $\Omega(t)$. (iii) Schrödinger symmetry typically arises for massive systems, and involves the one-parameter central extension of the Galilei group.

The second type, called “Alt” [10] or Conformal Galilean (CG) Symmetry [2, 3, 4, 5, 6, 7, 11], is more subtle. It also has an $O(2, 1)$ subgroup, generated by time translations, augmented with

$$\tilde{D} : \quad t^* = \lambda t, \quad \mathbf{r}^* = \lambda \mathbf{r}, \quad (3)$$

$$\tilde{K} : \quad t^* = \Omega(t) t, \quad \mathbf{r}^* = \Omega^2(t) \mathbf{r}, \quad (4)$$

with the same $\Omega(t)$ as for Schrödinger.

The characteristic features of this second type of non-relativistic conformal symmetry are (i) space and time are dilated equally ($z = 1$) as in a relativistic theory, (ii) under new expansions time and space have different factors, Ω and Ω^2 , respectively; (iii) the CG group also contains accelerations,

$$\mathbf{A} : \quad t^* = t, \quad \mathbf{r}^* = \mathbf{r} - \frac{1}{2} \mathbf{a} t^2, \quad (5)$$

where $\mathbf{a} \in \mathbb{R}^D$. Moreover, (iv) this extension only allows a vanishing mass.

Owing to masslessness, it is more difficult to find physical systems which exhibit this kind of symmetry [23]. In [3, 12, 13] it was suggested that *incompressible fluid motion*,

$$\nabla \cdot \mathbf{v} = 0, \quad \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P, \quad (6)$$

where \mathbf{v} is the velocity field, P the pressure, and the constant density is taken to be $\rho = 1$, could be an example.

There is, however, a curious disagreement among the published statements :

Firstly, considering the non-relativistic limit of the relativistic Navier-Stokes equations, Bhattacharyya Minwalla and Wadia (BMW) [12] derive the infinitesimal symmetry,

$$\begin{aligned} Dv^j &= (-2t\partial_t - \mathbf{r} \cdot \nabla - 1)v^j, \\ A_i v^j &= -t\delta_{ij} + \frac{1}{2}t^2\partial_i v^j, \\ B_i v^j &= \delta_{ij} - t\partial_i v^j, \\ H v^j &= -\partial_t v^j, \\ P_i v^j &= -\partial_i v^j, \\ M_{ik} v^j &= \delta_{ij}v^k - \delta_{kj}v^i - (x^k\partial_i - x^i\partial_k)v^j. \end{aligned} \tag{7}$$

Their algebra contains (i) Schrödinger dilations, D in (1), (ii) accelerations, A_i , but (iii) contains no expansions.

Fouxon and Oz [13] find instead that the system is scale-invariant with respect to dilations with any dynamical exponent z . CG-expansions, \tilde{K} in (4), are broken, but would be restored by a suitable modification of the system, namely when

$$\nabla \cdot \mathbf{v}(t, \mathbf{r}) = -3a(t), \tag{8}$$

$$\partial_t \mathbf{v}(t, \mathbf{r}) + (\mathbf{v} \cdot \nabla) \mathbf{v}(t, \mathbf{r}) = -\nabla P - a(t)\mathbf{v} \tag{9}$$

for some function $a(t)$ of time alone.

Yet other people claim [3] that the system (6) is CG-symmetric.

At last, all these statements are in sharp contrast with what happens for a compressible fluid, which has a mass-centrally extended Schrödinger symmetry [6, 14].

The aim of this Note is to clarify and complete these results.

First we define : *a symmetry is a transformation which carries a solution of the equations of motion into a solution of these same equations.* In detail, let us assume that ψ , the physical field, belongs to some linear space, say H , and the equation of motion is $E(\psi) = 0$. Then consider a space-time transformation $(\mathbf{r}, t) \rightarrow (\mathbf{r}^*, t^*)$, implemented as

$$\psi^*(\mathbf{r}, t) = f(t^*, \mathbf{r}^*)\psi(\mathbf{r}^*, t^*) + g(t^*, \mathbf{r}^*), \tag{10}$$

[where f is a linear operator acting on ψ and $g \in H$ a shift], is a symmetry if ψ^* satisfies $E(\psi^*) = 0$ whenever $E(\psi) = 0$. For example, f is a numerical factor when ψ is a scalar, or a matrix if ψ is a vector, etc. [24].

II. SYMMETRIES OF THE INCOMPRESSIBLE FLUID EQUATIONS

The equations (6), which describe an incompressible fluid with no viscosity, are plainly translation and rotation symmetric. A Galilean boost, $\mathbf{B} : t^* = t$, $\mathbf{r}^* = \mathbf{r} + \mathbf{b}t$, implemented as $\mathbf{v}^*(t, \mathbf{r}) = \mathbf{v}(t^*, \mathbf{r}^*) - \mathbf{b}$, also leaves the equations of motion invariant. Consistently with Ref. [13], the “free” system, $P = 0$, is also invariant under dilations with *any* dynamical exponent z ,

$$D^{(z)} : \quad t^* = \lambda^z t, \quad \mathbf{r}^* = \lambda \mathbf{r}, \quad (11)$$

Attempting to implement a z -dilation as $\mathbf{v}^* = \lambda^a \mathbf{v}$ for a suitable exponent a , we find,

$$\partial_t \mathbf{v}^* + (\mathbf{v}^* \cdot \nabla) \mathbf{v}^* = \lambda^{a+z} \{ \partial_t \mathbf{v} \} + \lambda^{2a+1} \{ (\mathbf{v} \cdot \nabla) \mathbf{v} \}.$$

The two terms scale in the same way if $a = z - 1$, i.e., when

$$\mathbf{v}^*(t, \mathbf{r}) = \lambda^{z-1} \mathbf{v}(t^*, \mathbf{r}^*). \quad (12)$$

Then the l.h.s. of the Euler equation in (6) is multiplied by λ^{2z-1} ,

$$\partial_t \mathbf{v}^* + (\mathbf{v}^* \cdot \nabla) \mathbf{v}^* = \lambda^{2z-1} \left(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right). \quad (13)$$

Therefore, the system is invariant if the pressure changes as $P^* = \lambda^{2(z-1)} P$.

The incompressibility condition is also preserved, $\nabla \cdot \mathbf{v}^* = \lambda^z \nabla \cdot \mathbf{v} = 0$.

Consider now general expansions of the form

$$K^{(\alpha)} : \quad t^* = \Omega(t) t, \quad \mathbf{r}^* = \Omega^\alpha(t) \mathbf{r}, \quad (14)$$

where α is some integer and try $\mathbf{v}^*(t, \mathbf{r}) = \Omega^\delta \mathbf{v}(t^*, \mathbf{r}^*) + \beta \Omega^\tau \mathbf{r}^*$, where δ, β, τ are to be determined. Then

$$\begin{aligned} \partial_t \mathbf{v}^* + (\mathbf{v}^* \cdot \nabla) \mathbf{v}^* &= \Omega^{\delta+2} \left(\partial_{t^*} \mathbf{v} + \Omega^{\alpha+\delta-2} (\mathbf{v} \cdot \nabla^*) \mathbf{v} \right) + \\ &\Omega^{\delta+1} (\delta \kappa + \beta \Omega^{\alpha+\tau-1}) \mathbf{v} + \beta \Omega^{\tau+1} (\tau \kappa + \alpha \kappa + \beta \Omega^{\tau+\alpha-1}) \mathbf{r}^* + \Omega^{\delta+1} (\alpha \kappa + \beta \Omega^{\alpha+\tau-1}) (\mathbf{r}^* \cdot \nabla^*) \mathbf{v}, \end{aligned}$$

so that we must have $\alpha = \delta = 1$, $\tau = 0$, $\beta = -\kappa$. Thus, *the only expansion which preserves the free incompressible Euler equations is that of Schrödinger*, $K \equiv K^{(1)}$ in (2), implemented as

$$\mathbf{v}^*(t, \mathbf{r}) = \Omega(t) \mathbf{v}(t^*, \mathbf{r}^*) - \kappa \mathbf{r}^*. \quad (15)$$

This implementation multiplies the free incompressible Euler equations by the common factor Ω^3 .

For the incompressibility condition $\nabla \cdot \mathbf{v} = 0$ we find, however, that under a Schrödinger expansion,

$$\nabla \cdot \mathbf{v}^* = \Omega \nabla^* \cdot (\Omega \mathbf{v} - \kappa \mathbf{r}^*) = \Omega^2 \nabla^* \cdot \mathbf{v} - D\kappa \Omega, \quad (16)$$

so that *the invariance w.r.t. Schrödinger expansions is broken*. The symmetry of incompressible hydrodynamics is, therefore, the *Galilei group, augmented with arbitrary dilations of space and time*.

CG expansions, $K^{(2)} \equiv \tilde{K}$ in (4), must, consequently, be broken. This is confirmed by calculating,

$$\mathbf{v}^*(t, \mathbf{r}) = \mathbf{v}(t^*, \mathbf{r}^*) - 2\kappa\Omega \mathbf{r} = \mathbf{v}(t^*, \mathbf{r}^*) - \frac{2\kappa \mathbf{r}^*}{\Omega}. \quad (17)$$

The modified incompressibility condition, (8), is correct with $D = 3$ and $a(t) = 2\kappa\Omega(t)$,

$$\nabla \cdot \mathbf{v} = -6\kappa\Omega. \quad (18)$$

However,

$$\begin{aligned} \partial_t \mathbf{v}^* + (\mathbf{v}^* \cdot \nabla) \mathbf{v}^* &= \Omega^2 \left(\partial_{t^*} \mathbf{v} + (\mathbf{v} \cdot \nabla^*) \mathbf{v} \right) - 2\kappa\Omega \mathbf{v}^* - 2\kappa^2 \mathbf{r}^* \\ &= \Omega^2 \left(\partial_{t^*} \mathbf{v} + (\mathbf{v} \cdot \nabla^*) \mathbf{v} \right) - 2\kappa\Omega \mathbf{v} + 2\kappa^2 \Omega^2 \mathbf{r}. \end{aligned} \quad (19)$$

The last-but-first term here would fit into the framework (9) but the last term breaks the invariance [25].

For accelerations, (5), we get,

$$\mathbf{v}^*(t, \mathbf{r}) = \mathbf{v}(t^*, \mathbf{r}^*) + \mathbf{a} t, \quad (20)$$

$$\nabla \cdot \mathbf{v}^* = 0, \quad \partial_t \mathbf{v}^* + (\mathbf{v}^* \cdot \nabla) \mathbf{v}^* = \left\{ \partial_{t^*} \mathbf{v} + (\mathbf{v} \cdot \nabla^*) \mathbf{v} \right\} + \mathbf{a}. \quad (21)$$

Accelerations can, therefore, be accommodated if the extra term is absorbed into the pressure, $P^* = P - \mathbf{a} \cdot \mathbf{r}^*$, as suggested in Ref. [13]. But do we get a symmetry? We argue that

no, since (6) and (21) *do not describe the same system* : while (6) describes fluid motion in empty space, (21) describe it in a *constant force field*. This is analogous to that the free fall of a projectile on Earth is different from free motion. In our opinion, the correct interpretation is that the transformation maps “conformally” one system into the other [17] — just like the inertial force can compensate terrestrial gravitation in a freely falling lift.

Both the BMW algebra (7) and the CGA, (3)-(4)-(5) are legitimate algebras. They are both subalgebras of the conformal Milne algebra, $\mathbf{cmil}(D)$, eqn #(4.62) of Ref. [6],

$$\left(\kappa t^2 + z\lambda t + \epsilon\right)\partial_t + \left(\boldsymbol{\omega} \times \mathbf{r} + 2\kappa t \mathbf{r} + \lambda \mathbf{r} - \frac{1}{2}t^2 \mathbf{a} + \boldsymbol{\beta}t + \boldsymbol{\gamma}\right) \cdot \boldsymbol{\nabla} \quad (22)$$

[where we used the obvious notation for rotations in $D = 3$].

The BMW algebra (7) is obtained for $z = 2$ and with no expansions, $\kappa = 0$, and CGA is obtained for $z = 1$ and also includes CG expansions and accelerations, (4) and (5), respectively. However, as explained above, *none* of these algebras is a symmetry of the incompressible equations (6).

Let us now complete (6) by adding the dissipation term $\nu \boldsymbol{\nabla}^2 \mathbf{v}$ where ν is the shear viscosity, i.e., consider the incompressible Navier-Stokes equations

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0, \quad \partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v} = -\boldsymbol{\nabla} P + \nu \boldsymbol{\nabla}^2 \mathbf{v}. \quad (23)$$

Ignoring space and time translations, we only study boosts and conformal transformations.

Boosts, implemented as before, leave the viscosity term invariant, $\nu \boldsymbol{\nabla}^2 \mathbf{v}^* = \nu \boldsymbol{\nabla}^{*2} \mathbf{v}$. Then for a z -dilation, implemented as $\mathbf{v}^*(t, \mathbf{r}) = \lambda^a \mathbf{v}(t^*, \mathbf{r}^*)$, we have

$$\partial_t \mathbf{v}^* + (\mathbf{v}^* \cdot \boldsymbol{\nabla}) \mathbf{v}^* - \nu \boldsymbol{\nabla}^2 \mathbf{v}^* = \lambda^{z+a} \left\{ \partial_{t^*} \mathbf{v} \right\} + \lambda^{2a+1} \left\{ (\mathbf{v} \cdot \boldsymbol{\nabla}^*) \mathbf{v} \right\} - \lambda^{a+2} \left\{ \nu \boldsymbol{\nabla}^{*2} \mathbf{v} \right\}.$$

From here we infer $a = 1$ and $z = 2$. Thus, for the incompressible NS flow, the dissipation term reduces dilation symmetry to Schrödinger dilations only.

The added dissipation term is also consistent with Schrödinger expansions (2), which still scale the incompressible Euler equations by Ω^3 ,

$$\partial_t \mathbf{v}^* + (\mathbf{v}^* \cdot \boldsymbol{\nabla}) \mathbf{v}^* - \nu \boldsymbol{\nabla}^2 \mathbf{v}^* = \Omega^3 \left(\partial_{t^*} \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla}^*) \mathbf{v} - \nu \boldsymbol{\nabla}^{*2} \mathbf{v} \right) = 0. \quad (24)$$

Due to the non-invariance of the incompressibility condition, $\boldsymbol{\nabla} \cdot \mathbf{v}^* \neq 0$, the full Schrödinger symmetry is, nevertheless, broken to (Galilei) \times (Schrödinger dilations), cf. Table I.

	free equations	with dissipation
dilatation	dilatation with arbitrary z	Schr dilatation ($z = 2$)
expansion	no expansion	no expansion
max symmetry	Galilei + arbitrary dilatation	Galilei + Schr dilatation

TABLE I: *Symmetries of an incompressible fluid*

At last, accelerations, (5), also leave invariant the incompressible NS term $(\nabla^*)^2$, and carry the empty-space NS equations into a constant-field background equations as with no dissipation.

In conclusion,

- we disagree with the claim, made in Refs. [3, 12, 13], that accelerations would be symmetries. The transformation (8)-(9) is, from our point of view, *not* a symmetry ; it is rather a transformation from one system to another one. Moreover,

1. The BMW algebra (7) is *incomplete* in that it only contains $z = 2$ dilations and misses the others;
2. Ref. [13] does have all dilations, but we found that CG expansions, \tilde{K} in (4); don't satisfy neither our symmetry definition, nor the one, in Eqns (8) and (9), proposed by these authors [except when the unjustified transformation rule in Footnote [24] is assumed].
3. CGA is, thereforee, *not a symmetry*, due to the failure of CGA expansions, \tilde{K} . Note that the authors of [3] also miss dilatations with $z \neq 1$. It is also worth mentioning that the CGA and BMW algebras are different, contradicting what is said in [3].

Let us insist that if our disagreement about accelerations can be considered interpretational, that about *expansions* is fundamental. CGA expansions, \tilde{K} in (4), are *never* symmetries of fluid mechanics, in any sense.

These questions are further discussed in Section IV from a different point of view.

III. SYMMETRIES OF COMPRESSIBLE FLUIDS

For the sake of comparison, we now shortly discuss compressible fluids [6, 14] from our present point of view. Compressible fluid motion with no external forces is described by the

Navier-Stokes equations,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (25)$$

$$\rho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}) = -\nabla P + \nu \nabla^2 \mathbf{v} + (\zeta + \frac{1}{3}\nu) \nabla(\nabla \cdot \mathbf{v}), \quad (26)$$

where ρ is the density, ν and ζ the shear and the bulk viscosity, respectively [15, 16]. When the fluid is incompressible, $\nabla \cdot \mathbf{v} = 0$, the last term disappears, and we recover the equations (23).

Let us first assume that there is no dissipation, $\nu = \zeta = 0$, and also that the motion is isentropic, $\nabla P = \rho \nabla V'(\rho)$ for some function $V(\rho)$ of the density called the enthalpy [15].

Galilean invariance can be shown as before, completing the previous implementation (15) by $\rho^*(t, \mathbf{r}) = \rho(t^*, \mathbf{r}^*)$.

The “free” system, $P = 0$, is, again, scale invariant under dilations with any dynamical exponent z , $D^{(z)}$ in (11), implemented as in (12), completed with $\rho^* = \lambda^b \rho$. This follows from

$$\rho^*(\partial_t \mathbf{v}^* + (\mathbf{v}^* \cdot \nabla) \mathbf{v}^*) - \nu \nabla^2 \mathbf{v}^* = \lambda^b \rho (\lambda^{z+a} \partial_{t^*} \mathbf{v} + \lambda^{2a+1} (\mathbf{v} \cdot \nabla^*) \mathbf{v}),$$

which still requires $\mathbf{v}^*(t, x) = \lambda^{z-1} v(t^*, x^*)$, while b is left undetermined. Given b , the pressure has to scale as $P^* = \lambda^{2z-2+b} P$. In the polytropic case, $P \propto (\gamma - 1) \rho^\gamma$, for example, this requires

$$\gamma = 1 + \frac{2(z-1)}{b}. \quad (27)$$

Conversely, giving γ fixes b .

Next, the same proof as above shows that the free compressible Euler equations, (26) with $P = 0$, only allow Schrödinger expansions, $K^{(1)} \equiv K$ in (2). This is because the key requirement of equal scaling of the two terms is unchanged. Completing the implementation (15) by $\rho^* = \Omega^\sigma \rho$, a tedious calculation shows, however, that choosing $\sigma = D$, i.e.,

$$\rho^*(t, \mathbf{r}) = \Omega^D \rho(t^*, \mathbf{r}^*), \quad (28)$$

cancels the unwanted term $-D\kappa\Omega$ in the incompressibility equation (16) [promoted to the continuity equation (25)].

The Euler equation (26) with $\nu = \zeta = 0$ scales in turn, for $P = 0$, by the factor Ω^{3+D} . To preserve the invariance under expansions, the pressure has to scale as $P^* = \Omega^{2+D} P$. In

the polytropic case, this fixes the exponent as $\gamma = 1 + 2/D$, which is (27) with $z = 2$ and $b = D$, and is consistent with previous results [6, 14].

In conclusion, for a free compressible fluid, the symmetry is the full *expanded Schrödinger algebra*, \widetilde{sch} , generated by [6] #(4.14),

$$X = (\kappa t^2 + \mu t + \varepsilon) \frac{\partial}{\partial t} + (\boldsymbol{\omega} \times \mathbf{r} + \kappa t \mathbf{r} + \lambda \mathbf{r} + \boldsymbol{\beta} t + \boldsymbol{\gamma}) \cdot \nabla, \quad (29)$$

where $\boldsymbol{\omega} \in \text{so}(D)$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^D$, and $\kappa, \mu, \lambda, \varepsilon \in \mathbb{R}$ are respectively infinitesimal rotations, boosts, spatial translations, inversions, time dilations, space dilations, and time translations.

Do dilations and expansions combine into a closed algebra? Commuting Schrödinger expansions with z -dilations, we have,

$$[D^{(z)}, K] = zK, \quad [D^{(z)}, H] = -zH, \quad [K, H] = D^{(2)} \equiv D, \quad (30)$$

so that an $o(2, 1)$ is only obtained when $z = 2$, when $D^{(2)} = D \equiv D^{(sch)}$. For this value of z (29) reduces to the Schrödinger algebra sch i.e., the $\mu = 2\lambda$ subalgebra of the expanded Schrödinger algebra \widetilde{sch} . Choosing the potential to be consistent with $z = 2$, yields the Schrödinger symmetry in the polytropic case, cf. [6, 14].

For $z \neq 2$ i.e. $\mu \neq 2\lambda$, we only get a closed subalgebra when expansion are eliminated. Then (29) reduces to the Galilei algebra, augmented with z -dilations [6].

Furthermore, implementing accelerations as in (20) completed with $\rho(t, \mathbf{r}) = \rho(t^*, \mathbf{r}^*)$, changes the l.h.s. of the free Euler equation into

$$\rho^* (\partial_t \mathbf{v}^* + (\mathbf{v}^* \cdot \nabla) \mathbf{v}^*) = \rho (\partial_{t^*} \mathbf{v} + (\mathbf{v} \cdot \nabla^*) \mathbf{v}) + \mathbf{a} \rho, \quad (31)$$

which are the compressible Euler equations in a constant external field. Eqn. (31) trivially generalizes (21). The ρ -derivative terms in continuity equation cancel and (25) is acceleration-invariant.

Let us now restore the (manifestly Galilei invariant) viscosity term,

$$\nu \nabla^2 \mathbf{v} + \left(\zeta + \frac{1}{3}\nu\right) \nabla (\nabla \cdot \mathbf{v}). \quad (32)$$

Both terms in (32) behaves nicely : instead of reducing the dilations to $z = 2$ as in the incompressible case, they allow any dynamical exponent, fixing the scaling of ρ to $b = 2 - z$. The pressure would have to scale as $P^* = \lambda^z P$. Expansions are, however, broken, leaving us with a (Galilei) \times (arbitrary dilation) symmetry [14]. (In the incompressible case, freezing ρ to a constant value would require $b = 0$, yielding, once again, $z = 2$). The situation is summarized in Table II.

	free equations	with dissipation
dilatation	z arbitrary	z arbitrary
expansion	Schr expansion ($\alpha = 1$)	no expansion
max symmetry	expanded Schrödinger \widetilde{Sch}	Galilei + arbitrary dilatation

TABLE II: *Symmetries of a compressible fluid*

IV. HOW IS RELATIVISTIC CONFORMAL SYMMETRY LOST ?

To get further insight, let us review the derivation of the non-relativistic system [13, 19]. The starting point is to write the equations of relativistic conformal hydrodynamics as conservation of the energy-momentum tensor [13, 19],

$$\partial_\nu T^{\mu\nu} = 0, \quad T^\mu_\mu = 0. \quad (33)$$

With $T_{\mu\nu} = aT^4(\eta_{\mu\nu} + 4u_\mu u_\nu)$, where T is the temperature and u_μ is the four-velocity of the fluid, this becomes

$$u^\alpha \partial_\alpha \xi = -\frac{1}{3} \partial_\nu u^\nu, \quad u^\alpha \partial_\alpha u^\mu = -\partial^\mu \xi + \frac{1}{3} u^\mu \partial_\nu u^\nu, \quad (34)$$

where $\xi = \ln T$.

To derive the non-relativistic limit, we express our equations in terms of the 3-velocity,

$$(1 - (v/c)^2) \left[\delta_{ik} - \frac{2}{3c^2} \frac{v_i v_k}{[1 - \frac{1}{3}(v/c)^2]} \right] \partial_k \xi + \frac{1}{c^2} \left(\partial_t v_i + v_j \partial_j v_i - \frac{1}{3} \left(\frac{1}{[1 - \frac{1}{3}(v/c)^2]} \right) v_i \partial_k v_k \right) = 0, \quad (35)$$

$$\partial_t \xi + \frac{2}{3} \left(\frac{1}{[1 - \frac{1}{3}(v/c)^2]} \right) v_i \partial_i \xi + \frac{1}{3} \left(\frac{1}{[1 - \frac{1}{3}(v/c)^2]} \right) \partial_i v_i = 0. \quad (36)$$

Keeping the leading terms only as $c \rightarrow \infty$ would yield

$$\partial_i \xi = 0, \quad (37)$$

$$\partial_t \xi + \frac{1}{3} \nabla \cdot \mathbf{v} = 0, \quad (38)$$

The first equation here requires that the temperature be homogenous over the whole space, and only depend on time. The second equation is a generalization of the incompressibility

equations $\nabla \cdot \mathbf{v} = 0$ [to which it reduces when the temperature is constant]. No Euler equation is obtained at this order, though. This “simple non-relativistic limit” is *unsatisfactory* therefore, since it does *not* yield the correct equations of non-relativistic hydrodynamics.

Interestingly, if the temperature homogeneous, (37), the second line in (35) yields

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \frac{1}{3} \mathbf{v} (\nabla \cdot \mathbf{v}) = 0, \quad (39)$$

which *is* an Euler-type equation (26) with $\rho = 1$, no viscosity ($\nu = 0$) no pressure ($P = 0$) but with an extra term, $-\frac{1}{3} \mathbf{v} (\nabla \cdot \mathbf{v}) = \partial_t \xi \mathbf{v}$, completed with (37) and (38). For constant temperature $T = T_0$, we would get in particular the free incompressible Euler equations (6). Let us also stress that (39) comes not from the leading, only from the c^{-2} term.

Fouxon and Oz propose, instead, a *different* kind of NR limit reminiscent of the “Jackiw-Nair limit” encountered before in non-commutative mechanics [20]. Their clue is *not* to keep the term $\xi = \ln T$ finite, but put rather

$$P = c^2 \xi = c^2 \ln T \quad (40)$$

and require that P , identified with the pressure, remains finite as $c \rightarrow \infty$. Doing so allows them to recover the *incompressible Euler equations with pressure*,

$$\nabla \cdot \mathbf{v} = 0, \quad \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla P = 0. \quad (41)$$

So far so good. But what about symmetries ?

Firstly, it is an easy matter to prove [13] that the relativistic conformal group $O(4, 2)$ is a symmetry of the relativistic system (33) [alias (34)]. This is true, in particular, for [relativistic] *special conformal conformal transformations*,

$$\Phi^\mu(x, b) = \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2}, \quad (42)$$

[where x and b are four-vectors], implemented on the fields as

$$u_\mu(x, b) = (1 + 2b \cdot x + b^2 x^2) (\partial_\mu \Phi^\alpha) u_\alpha(\Phi), \quad T(x, b) = \frac{T(\Phi)}{1 + 2b \cdot x + b^2 x^2}. \quad (43)$$

It is also true that the contraction $c \rightarrow \infty$ of the relativistic conformal group is the conformal Galilei Group [2, 21]. A special conformal transformation becomes, in particular, a CG expansion, \tilde{K} in (4) with $\kappa = b^0/c$, and an acceleration, \mathbf{A} in (5) with $a^i = b^i/c$.

- For the expansions we get, for example,

$$\mathbf{v}^*(t, \mathbf{r}) = \mathbf{v}(t^*, \mathbf{r}^*) + 2 \frac{\mathbf{a}t - \kappa \mathbf{r}}{1 - \kappa t}, \quad T^*(t, \mathbf{r}) = \frac{T(t^*, \mathbf{r}^*)}{(1 - \kappa t)^2}. \quad (44)$$

Note that the implementation on the velocity is consistent with (17).

Then a straightforward calculation shows that

$$\partial_t \xi^* + \frac{1}{3} \nabla \cdot \mathbf{v}^* = \Omega^2 \left(\partial_{t^*} \xi + \frac{1}{3} \nabla^* \cdot \mathbf{v} \right), \quad \nabla \xi^* = \Omega^2 (\nabla^* \xi),$$

so that the leading-in- c [but physically uninteresting] equations (38)-(37) are *invariant* under a CG expansion \tilde{K} in (4).

The next (c^{-2}) order term yields, however, the free Euler-type equations (39), whose “Euler part” is, as seen before, *invariant* under *Schrödinger* but not under CG expansions. The new term, $\mathbf{v}(\nabla \cdot \mathbf{v})$, breaks, however both expansions. The full system (37)-(38)-(39), obtained by c^{-2} truncation, has therefore *no* expansion symmetry.

But what about the incompressible Euler system (41) derived by the “Jackiw-Nair type” NR limit (40) ? Using (44) the pressure transforms as

$$P^*(t, \mathbf{r}) = P(t^*, \mathbf{r}^*) - 2c^2 \ln(1 - \kappa t). \quad (45)$$

The extra term here (which drops out, however, from ∇P) diverges as $c^2 \rightarrow \infty$, so finiteness of P already rules out expansions. Furthermore,

$$\nabla \cdot \mathbf{v}^* = \Omega^2 \nabla^* \cdot \mathbf{v} - \frac{6\kappa}{1 - \kappa t}, \quad (46)$$

$$\begin{aligned} \partial_t \mathbf{v}^* + (\mathbf{v}^* \cdot \nabla) \mathbf{v}^* - \frac{1}{3} \mathbf{v}^* (\nabla \cdot \mathbf{v}^*) + \nabla P^* = \\ \Omega^2 \left(\partial_{t^*} \mathbf{v} + (\mathbf{v} \cdot \nabla^*) \mathbf{v} - \frac{1}{3} \mathbf{v} (\nabla^* \cdot \mathbf{v}) + \nabla^* P \right) - \frac{2\kappa^2}{(1 - \kappa t)^2} \mathbf{r} + \frac{1}{3} \frac{2\kappa}{(1 - \kappa t)^3} \mathbf{r} (\nabla^* \cdot \mathbf{v}) \end{aligned} \quad (47)$$

If P must remain finite, then $\nabla^* \cdot \mathbf{v} = 0$ and the last term drops out. Then (46) and (47) reduce precisely to (18) and (19), leading to the same conclusion as before : *expansions are broken and can not be restored as suggested in [13]*.

Thus, while the “simple limit” (37)-(38) would have the CG-expansion symmetry obtained by contraction, the tricky “JN-type” limit breaks it.

Now we can explain also the other peculiarities.

- As said already, the relativistic system (34) is symmetric also under the space part of special conformal transformations, (42)-(43); their contraction is the acceleration (5), which

acts on NR space-time and fields as in (20). A similar calculation as above [whose details are omitted] yields, furthermore, that accelerations

1. are symmetries for the leading-in- c system (37)-(38);
 2. leave invariant the incompressibility condition but shift the Euler equation, as in (21) ;
- A peculiar feature of incompressible hydrodynamics is that not only CG (or Schrödinger), but *all dilations* act as symmetries. Remarkably, this can also be explained from considering the R \rightarrow NR transition. It is enough to discuss time dilations alone,

$$D^\infty : t^* = \lambda t, \quad \mathbf{r}^* = \mathbf{r}, \quad (48)$$

since all values of the dynamical exponent can be obtained by combining D^∞ with \tilde{D} . Firstly, for the non-relativistic system, we check that time dilation, (48),

1. implemented as

$$v^* = \lambda^p v, \quad \xi^* = \lambda^{p-1} \xi \quad (49)$$

is a symmetry for the leading-in- c system (37)-(38) for any p ;

2. leave the full incompressible system (41) invariant, when

$$\mathbf{v}^*(t, \mathbf{r}) = \lambda \mathbf{v}(t^*, \mathbf{r}^*), \quad P^*(t, \mathbf{r}) = \lambda^2 P(t^*, \mathbf{r}^*) \quad (50)$$

3. is *not* a symmetry for the relativistic system. For the implementation (50), for example,

$$\begin{aligned} \partial_t \xi^* + \frac{2c^2}{3c^2 - \mathbf{v}^{*2}} (\mathbf{v}^* \cdot \nabla) \xi^* + \frac{c^2}{3c^2 - \mathbf{v}^{*2}} \nabla \cdot \mathbf{v}^* = \\ \lambda^3 \left\{ \partial_{t^*} \xi + \frac{2c^2}{3c^2 - \lambda^2 \mathbf{v}^2} \mathbf{v} \cdot \nabla^* \xi \right\} + \lambda \frac{c^2}{3c^2 - \lambda^2 \mathbf{v}^2} \nabla^* \cdot \mathbf{v} \end{aligned}$$

and

$$\begin{aligned} \partial_t v_i^* + (\mathbf{v}^* \cdot \nabla) v_i^* - \frac{c^2 - \mathbf{v}^{*2}}{3c^2 - \mathbf{v}^{*2}} v_i^* (\nabla \cdot \mathbf{v}^*) + (c^2 - \mathbf{v}^{*2}) \left[\delta_{ik} - \frac{2v_i^* v_k^*}{3c^2 - \mathbf{v}^{*2}} \right] \partial_k \xi^* = \\ \lambda^2 \left\{ \partial_{t^*} v_i + (\mathbf{v} \cdot \nabla^*) v_i - \frac{c^2 - \lambda^2 \mathbf{v}^2}{3c^2 - \lambda^2 \mathbf{v}^2} v_i (\nabla^* \cdot \mathbf{v}) + (c^2 - \lambda^2 \mathbf{v}^2) \left[\delta_{ik} - \lambda^2 \frac{2v_i v_k}{3c^2 - \lambda^2 \mathbf{v}^2} \right] \partial_k \xi \right\} \end{aligned}$$

which is obviously not a symmetry if $\lambda \neq 1$. If $c \rightarrow \infty$ so that $P = c^2 \xi$ remains finite, however, then all *symmetry breaking terms drop out*, and both equations scale homogeneously. In other words, *the relativistic “no-symmetry” (48) becomes a non-relativistic symmetry.*

V. CONCLUSION

In this paper, we have carried out a systematic study of the conformal symmetries of non-relativistic fluids. The conclusion is that the system admits various Schrödinger-type, but no CGA-type symmetries, completing and partly contradicting recently publicized statements. In the compressible case, the new freedom of scaling the density as in (28) allows us to overcome the “rigidity” of the density and to restore the symmetry w.r.t. *Schrödinger expansions*, K in (2), which had been broken by the incompressibility condition.

Interesting insight can be gained when the non-relativistic systems are derived from relativistic conformally invariant hydrodynamics. Then the leading-in- c order system does carry the CG symmetry, obtained by contraction from the relativistic conformal group. This system has, however, limited physical interest; and incompressible hydrodynamics is derived by another, more subtle limit [13, 19], which *owing to mixing different c-powers*, does *not* carry the CG symmetry. The general dilation symmetry of non-relativistic hydrodynamics is also explained from this point of view.

Our definition of a symmetry was based on the *equations of motion* alone. For a Lagrangian system however, a symmetry can also be defined as a *transformation which changes the Lagrangian by a mere surface term*. This definition is clearly stronger as it implies the first one, but not vice versa : if for example, the Lagrangian is multiplied by a constant factor then the equations of motion are preserved. Note that it is only the second type of symmetries which implies, through Noether’s theorem, conserved quantities. We call it, therefore, a *Noetherian symmetry*.

A compressible fluid with no dissipation can be derived from a Lagrangian [6, 14, 15]. The approaches based on the Lagrangian and on the field equations, respectively, lead to identical conclusions in the free case, but to different ones in the presence of a polytropic potential. In the Lagrangian approach, z is not more arbitrary, but fixed by the polytropic exponent.

In the compressible case, the Hamiltonian structure induced by the Lagrangian [14] could be used to provide another argument against the CGA. One has indeed [14]

$$\{(boost)_i, (\text{momentum})_j\} = (\text{mass}) \delta_{ij}, \quad (\text{mass}) = \int d^3\mathbf{r} \rho > 0, \quad (51)$$

showing that compressible fluids realize the *one-parameter (mass) central extension* of the

Galilei group. But CGA is only consistent with (mass)= 0 [2] and can not be, therefore, a symmetry.

Putting $\rho = \rho_0$ in the compressible Lagrangian makes the system singular, making the Lagrangian approach problematic; Hamiltonian structure should be determined using a reduction [18].

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- [1] The number of papers published on the subject is rapidly growing. It is, therefore, impossible to give credit to all important contributions. See, e.g. K. Balasubramanian, and J. McGreevy, “Gravity duals for non-relativistic CFTs,” *Phys. Rev. Lett.* **101** (2008) 061601;
- D. T. Son, “Toward and AdS/cold atom correspondence: a geometric realization of the Schrödinger symmetry,” *Phys. Rev.* **D78** (2008) 046003;
- C. Leiva, M. S. Plyushchay, “Conformal symmetry of relativistic and nonrelativistic systems and AdS/CFT correspondence,” *Ann. Phys. (N.Y.)* **307** (2003) 372;
- W. D. Goldberger “AdS/CFT duality for non-relativistic field theory.” *JHEP* **0903** 069 (2009);
- J.L.B. Barbon, C. A. Fuertes, “On the spectrum of nonrelativistic AdS/CFT,” *JHEP* **0809** 030 (2008);
- J. Maldacena, D. Martelli, Y. Tachikawa, “Comments on string theory backgrounds with non-relativistic conformal symmetry,” *JHEP* **0810** 072 (2008);
- A. V. Galajinsky, “Remark on quantum mechanics with conformal Galilean symmetry,” *Phys. Rev.* **D78** (2008) 087701;
- C. Duval, M. Hassaïne, and P. A. Horváthy, “The geometry of Schrödinger symmetry in non-relativistic CFT,” *Ann. Phys. (N.Y.)* **324** (2009) 1158.

- [2] J. Lukierski, P. C. Stichel, and W. J. Zakrzewski, “Exotic Galilean conformal symmetry and its dynamical realizations,” Phys. Lett. A **357**, 1 (2006). [arXiv:hep-th/0511259].
J. Lukierski, P. C. Stichel, and W. J. Zakrzewski, “Acceleration-Extended Galilean Symmetries with Central Charges and their Dynamical Realizations,” Phys. Lett. B **650** (2007) 203; [arXiv:hep-th/0702179];
J. Lukierski, P. C. Stichel, and W. J. Zakrzewski, “Acceleration-Enlarged Symmetries in Non-relativistic Space-Time with a Cosmological Constant,” arXiv:0710.3093 [hep-th].
P. Stichel, “Conformal Galilean-type algebras, massless particles and gravitation,” arXiv:0908.1303 [hep-th].
- [3] A. Bagchi, and R. Gopakumar, “Galilean Conformal Algebras and AdS/CFT,” JHEP **0907** (2009) 037 [arXiv:0902.1385 [hep-th]].
- [4] M. Alishahiha, A. Davody, and A. Vahedi, “On AdS/CFT of Galilean Conformal Field Theories,” arXiv:0903.3953 [hep-th].
- [5] D. Martelli, and Y. Tachikawa, “Comments on Galilean conformal field theories and their geometric realization,” arXiv:0903.5184 [hep-th].
- [6] C. Duval and P. A. Horvathy, “Non-relativistic conformal symmetries and Newton-Cartan structures,” J. Phys. A **42** 465206 (2009), arXiv:0904.0531
- [7] A. Hosseiny and S. Rouhani, “Affine Extension of Galilean Conformal Algebra in 2+1 Dimensions,” arXiv:0909.1203 [hep-th].
- [8] R. Jackiw, “Introducing scaling symmetry,” Phys. Today, **25** (1972) 23.
U. Niederer, “The maximal kinematical symmetry group of the free Schrödinger equation,” Helv. Phys. Acta **45** (1972) 802 .
C. R. Hagen, “Scale and conformal transformations in Galilean-covariant field theory,” Phys. Rev. D**5** (1972) 377 .
G. Burdet, M. Perrin, and P. Sorba, “About the Non-Relativistic Structure of the Conformal Algebra,” Commun. Math. Phys. **34** (1973) 85 .
C. Duval, “Quelques procédures géométriques en dynamique des particules,” Thèse d’Etat (1982), unpublished.
- [9] C. Duval, G. Burdet, H.P. Künzle, and M. Perrin, “Bargmann Structures And Newton-Cartan Theory,” Phys. Rev. D**31** (1985) 1841.
C. Duval, “Nonrelativistic Conformal Symmetries and Bargmann Structures,” in *Conformal*

Groups and Related Symmetries. Physical Results and Mathematical Background, Clausthal 1985, (A.O. Barut & H.D. Doebner Eds), p. 162. Lecture Notes in Physics **261**, Springer-Verlag (1986).

C. Duval, G. W. Gibbons, and P. A. Horváthy, “Celestial Mechanics, Conformal Structures and Gravitational Waves,” Phys. Rev. **D43** 3907 (1991).

- [10] M. Henkel, “Schrödinger invariance and strongly anisotropic critical systems,” J. Stat. Phys. **75** (1994) 1023; “Local Scale Invariance and Strongly Anisotropic Equilibrium Critical Systems,” Phys Rev. Lett. **78** (1997) 1940; “Phenomenology of local scale invariance: from conformal invariance to dynamical scaling.” Nucl. Phys. B **641** (2002) 405;
M. Henkel and J. Unterberger, “Schroedinger invariance and space-time symmetries,” Nucl. Phys. **B 660** (2003) 407. [arXiv:hep-th/0302187].
- [11] J. Negro, M. A. del Olmo, An A. Rodríguez-Marco, “Nonrelativistic conformal groups,” J. Math. Phys. **38** (1997) 3786, and *ibid.* 3810.
- [12] S. Bhattacharyya, S. Minwalla and S. R. Wadia, “The Incompressible Non-Relativistic Navier-Stokes Equation from Gravity,” JHEP **0908** (2009) 059 [arXiv:0810.1545 [hep-th]].
- [13] I. Fouxon, and Y. Oz, “CFT Hydrodynamics: symmetries, exact solutions and gravity,” JHEP **0903** 120 (2009)
- [14] M. Hassaïne and P. Horvathy, “Field-dependent symmetries of a non-relativistic fluid model.” Ann. Phys. (N. Y.) **282** 218 (2000) [math-ph/9904022]; “Symmetries of fluid dynamics with polytropic exponent.” Phys. Lett. **A 279** (2001) 215 [hep-th/0009092].
L. O’Raifeartaigh and V. V. Sreedhar, “The maximal kinematical invariance group of fluid dynamics and explosion-implosion duality,” Annals Phys. **293** (2001) 215.
- [15] R. Jackiw, *Lectures on fluid dynamics*. CRM series in Mathematical Physics. Springer (2002);
R. Jackiw, V. P. Nair, S. Y. Pi and A. P. Polychronakos, “Perfect fluid theory and its extensions,” J. Phys. A **37** (2004) R327 [arXiv:hep-ph/0407101].
- [16] L. Landau, E. Lifchitz, *Mécanique des fluides*. Physique Théorique. tome 6. 2nd Edition.
Mir: Moscou (1989).
- [17] C. Duval and P. A. Horvathy and L. Palla, “Conformal properties of Chern-Simons vortices in external fields.” Phys. Rev. **D50** 6658 (1994) [hep-ph/9405229]. For more details see, e.g.,
P. A. Horvathy and P. Zhang, “Vortices in (abelian) Chern-Simons gauge theory,” Physics Reports **481** 83 (2009). Sects. 4.3 and 6.2. arXiv:0811.2094 [hep-th].

- [18] J. Marsden and A. Weinstein, *Physica* **7D**, 305 (1983);
P. J. Morrison, “Hamiltonian description of an ideal fluid”, *Rev. Mod. Phys.* **70**, 467 (1998).
- [19] I. Fouxon and Y. Oz, “Conformal Field Theory as Microscopic Dynamics of Incompressible Euler and Navier-Stokes Equations,” *Phys. Rev. Lett.* **101** (2008) 261602 [arXiv:0809.4512 [hep-th]].
- [20] R. Jackiw and V. P. Nair, “Anyon spin and the exotic central extension of the planar Galilei group,” *Phys. Lett.* **B 480**, 237 (2000) [hep-th/0003130]; C. Duval and P. A. Horváthy, *Phys. Lett.* **B 547**, 306 (2002) [hep-th/0209166].
- [21] A. O. Barut, “Conformal group → Schrödinger group → dynamical group – the maximal kinematical group of the massive Schrödinger particle”, *Helv. Phys. Acta* **46** (1973) 496.
P. Havas, and J. Plebański, “Conformal extensions of the Galilei group and their relation to the Schrödinger group,” *J. Math. Phys.* **19**, 482 (1978).
- [22] R. Cherniha and M. Henkel “The exotic Conformal Galilei Algebra and nonlinear partial differential equations,” (to be published).
- [23] Recent results [22] indicate that, within a large class, *no* physical system can carry the CG symmetry.
- [24] If ψ is a complex field [a wave function], the usual implementation is $\psi^*(\mathbf{r}, t) = f(\mathbf{r}^*, t^*)\psi(\mathbf{r}^*, t^*)$, where f is a complex function. Decomposing into module and phase, $\sqrt{\rho}e^{i\theta}$ and $f = Fe^{iG}$, changes this into

$$\rho^*(\mathbf{r}, t) = F^2\rho(\mathbf{r}^*, t^*), \quad \theta^*(\mathbf{r}, t) = \theta(\mathbf{r}^*, t^*) + G(\mathbf{r}^*, t^*).$$

Similarly, for a vector field (like the velocity field, \mathbf{v}), rotations are implemented through f , which is a rotation matrix, etc.

- [25] Fouxon and Oz [13] argue that the CG-expansion symmetry (4) *could be restored* by the modification (9). This would require the pressure to transform as

$$P^*(t, \mathbf{r}) = P(t^*, \mathbf{r}^*) + \frac{\kappa^2}{\Omega^2}\mathbf{r}^{*2},$$

which is quite an arbitrary rule which is, furthermore, *inconsistent* with the one, (45) following from the relativistic framework.