M/G, which otherwise just looks like a copy of the real line:

$$M/G = \{y > 0\} \cup \{y < 0\} \cup \{0_+\} \cup \{0_-\}.$$

A basic neighbourhood of the "origin" 0_+ is just $U_+ = \{y\colon y = 0_+ \text{ or } 0 < |y| < \delta_+\}$ for some constant $\delta_+ > 0$, a typical neighbourhood of 0_- being $U_- = \{y\colon y = 0_- \text{ or } 0 < |y| < \delta_-\}$, $\delta_- > 0$. Clearly $U_+ \cap U_- \neq \emptyset$ no matter what δ_+ and δ_- are, so the points 0_+ and 0_- do not satisfy the Hausdorff separation property. (A more physically relevant example of a non-Hausdorff quotient space is provided by the scaling groups G^α of Example 3.19 when $\alpha < 0$; see Exercise 3.14.)

Proposition 3.21. *Let G act regularly on the manifold M with s-dimensional orbits.*

(a) *A smooth function $F\colon M \to \mathbb{R}^l$ is G-invariant if and only if there is a smooth function $\tilde{F} = F/G\colon M/G \to \mathbb{R}^l$ such that $\tilde{F}(x) = F[\pi(x)]$ for all $x \in M$.*
(b) *A smooth n-dimensional submanifold $N \subset M$ is G-invariant if and only if there is a smooth $(n - s)$-dimensional submanifold $\tilde{N} = N/G \subset M/G$ such that $\tilde{N} = \pi[N]$ and hence $N = \pi^{-1}[\tilde{N}]$.*
(c) *A subvariety $\mathscr{S}_F = \{x\colon F(x) = 0\}$ defined by a smooth function $F\colon M \to \mathbb{R}^l$ is G-invariant if and only if there is a smooth subvariety $\mathscr{S}_{\tilde{F}} = \{y\colon \tilde{F}(y) = 0\}$ defined by $\tilde{F}\colon M/G \to \mathbb{R}^l$ such that $\mathscr{S}_{\tilde{F}} = \pi[\mathscr{S}_F]$. (In this case, it is not necessarily true that $\tilde{F} = F \circ \pi$ unless F itself happens to be G-invariant.)*

This proposition is just a global restatement of Theorem 2.17 and Proposition 2.18, and we leave the details of the construction to the reader.

Dimensional Analysis

In the case of scaling groups, the preceding constructions provide an easy proof of the so-called Pi theorem, which forms the foundation of the method of dimensional analysis. In any physical problem, there are certain fundamental physical quantities, such as length, time, mass, etc., which can all be scaled independently of each other. Let $z^1, \ldots, z^r$ denote these quantities, so the group under consideration transforms according to

$$(z^1, \ldots, z^r) \mapsto (\lambda_1 z^1, \ldots, \lambda_r z^r),$$

where the scaling factors $\lambda = (\lambda_1, \ldots, \lambda_r)$ are arbitrary positive real numbers. Thus the underlying group is just the Cartesian products of r copies of the multiplicative group $\mathbb{R}^+$ of positive real numbers. There also exist certain derived physical quantities, such as velocity, force, fluid density and so on, which also scale under a rescaling of the fundamental physical units. Calling these quantities $x = (x^1, \ldots, x^m)$ and, assuming dimensional homogeneity, the action of our scaling group on the derived quantities takes the form

$$\lambda \cdot (x^1, \ldots, x^m) = (\lambda_1^{\alpha_{11}} \lambda_2^{\alpha_{21}} \cdots \lambda_r^{\alpha_{r1}} x^1, \ldots, \lambda_1^{\alpha_{1m}} \lambda_2^{\alpha_{2m}} \cdots \lambda_r^{\alpha_{rm}} x^m), \qquad (3.28)$$

in which the exponents α_{ij}, $i = 1, \ldots, r$, $j = 1, \ldots, m$, are prescribed by the physical dependence of the quantities x^j on the fundamental units y^i. For example, if y^1 denotes length, y^2 time and y^3 mass, then changes in velocity v, being the ratio of length over time, and fluid density ρ, the ratio of mass to volume (or length cubed), are given by

$$\lambda \cdot v = \lambda_1 \lambda_2^{-1} v, \qquad \lambda \cdot \rho = \lambda_1^{-3} \lambda_3 \rho.$$

If a derived quantity remains unchanged under the given scalings, then it is called *dimensionless*. The first part of the Pi theorem tells how many independent dimensionless quantities exist, this being determined by the number of independent invariants of the underlying group action.

In general, certain functional relations of the form $F(x^1, \ldots, x^m) = 0$ among the derived quantities are posited. For instance, for waves in deep water, the velocity v might be determined as a function of wave length l and the gravitational acceleration g. (In this simple model, we ignore surface tension and other effects.) Such a relation is called *unit-free* if it is unchanged under a rescaling of the fundamental quantities. Such unit-free relations are often of great physical significance. The second part of the Pi theorem states that any such unit-free relation can be re-expressed solely in terms of the dimensionless combinations of the derived physical quantities. For instance, in our example, if $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ represent the scalings in length, time and mass respectively, then

$$\lambda \cdot (v, l, g) = (\lambda_1 \lambda_2^{-1} v, \lambda_1 l, \lambda_1 \lambda_2^{-2} g).$$

Obviously, the only dimensionless quantity here is the *Froude number* v^2/lg, or powers thereof. Thus any unit-free relation determining wave speed as a function of wave length and gravitational acceleration must take the form

$$v = c\sqrt{lg},$$

in which only the constant c remains to be determined. We now state the Pi theorem in general.

Theorem 3.22. *Let $z^1, \ldots, z^r$ be fundamental physical quantities which scale independently according to $z^i \mapsto \lambda_i z^i$. Let $x^1, \ldots, x^m$ be the derived quantities scaling according to* (3.28) *for some $r \times m$ matrix of constants $A = (\alpha_{ij})$. Let s be the rank of A. Then there exist $m - s$ independent dimensionless "power products"*

$$\pi^k = (x^1)^{\beta_{1k}} (x^2)^{\beta_{2k}} \cdots (x^m)^{\beta_{mk}}, \qquad k = 1, \ldots, m - s, \tag{3.29}$$

with the property that any other dimensionless quantity can be written as a function of $\pi^1, \ldots, \pi^{m-s}$. If $F(x^1, \ldots, x^m) = 0$ is any unit-free relation among the given derived quantities, then there is an equivalent relation $\tilde{F}(x^1, \ldots, x^m) = 0$ which can be expressed solely in terms of the above dimensionless power products:

$$\tilde{F} = \tilde{F}(\pi^1, \ldots, \pi^{m-s}) = 0.$$

Proof. Consider the positive octant of $\mathbb{R}^m$, $M = \{x = (x^1, \ldots, x^m): x^i > 0, i = 1, \ldots, m\}$. If $G = \mathbb{R}^+ \times \cdots \times \mathbb{R}^+$ is the r-fold Cartesian product of the multiplicative group $\mathbb{R}^+$, then (3.28) determines a global action of G on M. The infinitesimal generators of this action are found by differentiating (3.28) with respect to λ_i and setting $\lambda_1 = \cdots = \lambda_r = 1$. We find

$$\mathbf{v}_i = \alpha_{i1} x^1 \frac{\partial}{\partial x^1} + \alpha_{i2} x^2 \frac{\partial}{\partial x^2} + \cdots + \alpha_{im} x^m \frac{\partial}{\partial x^m}$$

to be the generator corresponding to the i-th copy of $\mathbb{R}^+$ in G. The dimension of the span of $\mathbf{v}_1, \ldots, \mathbf{v}_r$ at $x \in M$ is clearly the same as the rank of the matrix $A = (\alpha_{ij})$, namely s, hence G has s-dimensional orbits. Global invariants for G on the entire octant M are given by power products of the form (3.29) provided $\mathbf{v}_i(\pi^k) = 0$ for $i = 1, \ldots, r$. This holds if and only if the exponents β_{jk} in (3.29) satisfy the linear system

$$\sum_{j=1}^{m} \alpha_{ij} \beta_{jk} = 0, \qquad i = 1, \ldots, r. \tag{3.30}$$

There are $m - s$ linearly independent solutions to this system leading to $m - s$ functionally independent power products. Moreover, the power products uniquely determine the orbits of G on M. Indeed, if $\pi^k(x) = \pi^k(\tilde{x})$ for all k, set $x^j = e^{t_j} \tilde{x}^j$ for $j = 1, \ldots, m$. The exponents t_j satisfy the linear system $\sum_j t_j \beta_{jk} = 0$ for $k = 1, \ldots, r$. Since we have constructed a basis for the null space of A, this is true if and only if there exist real numbers $s_1, \ldots, s_r$ such that $t_j = \sum_i s_i \alpha_{ij}$. But then $x = \lambda \cdot \tilde{x}$ where $\lambda_i = e^{s_i}$, and hence x and $\tilde{x}$ lie in the same orbit of G. Since each orbit is thus a common level set of the global invariants $\pi^1, \ldots, \pi^{m-s}$, the action of G is automatically regular and $\pi^1, \ldots, \pi^{m-s}$ provide global coordinates on the quotient manifold G/M, which can be identified with the positive octant of $\mathbb{R}^{m-s}$. The second part of the theorem now follows immediately from part (c) of Proposition 3.21. □

Example 3.23. Assume that the resistance D of an object immersed in a fluid is determined by a unit-free function of fluid density ρ, fluid velocity v, object diameter d, and fluid viscosity μ. Letting $\lambda_1, \lambda_2, \lambda_3$ be the scaling parameters of length, time and mass respectively, we obtain the consequent scaling of the relevant derived quantities

$$\lambda \cdot (\rho, v, d, \mu, D) = (\lambda_1^{-3} \lambda_3 \rho, \lambda_1 \lambda_2^{-1} v, \lambda_1 d, \lambda_1^{-1} \lambda_2^{-1} \lambda_3 \mu, \lambda_1 \lambda_2^{-2} \lambda_3 D).$$

(For instance, D is written in units of length $\times$ mass/(time)^2, etc.) The matrix A in this case takes the form

$$A = \begin{pmatrix} -3 & 1 & 1 & -1 & 1 \\ 0 & -1 & 0 & -1 & -2 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix},$$

which is of rank 3. There are thus $5 - 3 = 2$ independent dimensionless power products. To determine them, according to (3.30) we need to analyze the null space of A, which is spanned by the column vectors $(1, 1, 1, -1, 0)^T$, $(-1, -2, -2, 0, 1)^T$. These correspond to the independent power products

$$\pi_1 = R = \frac{\rho v d}{\mu}, \qquad \pi_2 = K = \frac{D}{\rho v^2 d^2},$$

the first of which is the *Reynolds number* for the flow. According to the Pi theorem, any unit-free relation between our five quantities must be of the form $F(R, K) = 0$, or, upon solving for K, the resistance is given by $D = \rho v^2 d^2 h(R)$, where h is a function whose form remains to be determined.

3.5. Group-Invariant Prolongations and Reduction

In the basic program to rigorously implement the general procedure for constructing group-invariant solutions to differential equations, there is one primary hurdle that must be overcome. In general, if the system of differential equations Δ is defined over an open subset $M \subset X \times U$ of the space of independent and dependent variables upon which the symmetry group G acts regularly, then the reduced system of differential equations Δ/G for the G-invariant solutions will naturally live on the quotient manifold M/G. The difficulty is that although M/G has the structure of a smooth manifold, it will not in general be an open subset of any Euclidean space and so our earlier construction of jet spaces and prolonged group actions is no longer applicable. In practice, however, we work in local coordinate charts, and so we can make the more modest assumption that there are $p + q - s$ functionally-independent invariants on M, $\eta^1(x, u), \ldots, \eta^{p+q-s}(x, u)$, determining global cooordinates on M/G, which can therefore be viewed as an open subset of the Euclidean space $\mathbb{R}^{p+q-s}$. Here s denotes the dimension of the orbits of G. At this point, a second difficulty arises in that there is in general no natural way of distinguishing which of the invariants η^j will be the new independent variables and which will be the new dependent variables. If G acts projectably, then, as we have seen, there are precisely $p - s$ invariants which depend only on x, which can be designated as the new independent variables, the remaining q invariants becoming the new dependent variables. In the general case, however, there is no way of determining new independent and dependent variables in a consistent manner, and one is forced to make an arbitrary choice among the given invariants, assigning $p - s$ of them the role of independent variables $y = (y^1, \ldots, y^{p-s})$, and the remaining q the role of dependent variables $v = (v^1, \ldots, v^q)$. In this way, we have forced M/G to be a subset of the Euclidean space $Y \times V \simeq \mathbb{R}^{p-s} \times \mathbb{R}^q$, and hence can determine an explicit form for the reduced system Δ/G by regarding v as a function of y. Already, the roles of independent and dependent variables are starting to blur. Different assignations of invariants will lead to seemingly different expressions for the reduced system, but—and this must be emphasized—these will all be equivalent under an interchange of the roles of independent and dependent variables reminiscent of the hodograph transformation of fluid dynamics.

So far, this would be all right for our purposes, were it not for yet another complication. Once we have selected the new independent variables y and dependent variables v for our reduced system Δ/G, there is no guarantee that a given function $v = h(y)$ will correspond to a smooth, single-valued function $u = f(x)$; vice versa, there exists the possibility of G-invariant functions $u = f(x)$ not corresponding to smooth functions of the form $v = h(y)$ relative to the given choice of independent and dependent variables. The problem in both instances is that a function on one of the spaces may give rise to a "function" with infinite derivatives or multiple values on the other space, and these are excluded by our perhaps artificial division of the coordinates into independent and dependent variables. This point is perhaps made more clear through the use of an illustrative example.

Example 3.24. Suppose $p = 2$, $q = 1$ so $M = X \times U$ has coordinates (x, t, u). Consider the one-parameter group of translations G: $(x, t, u) \mapsto (x, t + \varepsilon, u)$. Suppose instead of choosing the natural invariants x and u to coordinatize $M/G = \mathbb{R}^2$ we were to choose the invariants $y = x + u$ and $v = u$, with y the new independent and v the new dependent variable. Any function $v = h(y)$ on $M/G = Y \times V$ will determine a two-dimensional G-invariant submanifold of M, given by the equation $u = h(x + u)$, but unless $h'(y) \neq 1$, this equation will not determine u explicitly as a function of x (and t). For example, the function $v = y$ in M/G corresponds to the vertical plane $\{x = 0\}$, which is certainly not the graph of a function $u = f(x, t)$. On the other hand, the G-invariant function $u = -x$ reduces to the vertical line $y = 0$, which is not the graph of a function of the form $v = h(y)$. Although this example is somewhat artificial, the phenomena may be unavoidable in more complicated situations.

The principal reason for all these technical complications is our attachment to the distinction between independent and dependent variables, a stance which becomes increasingly untenable in light of the above reasoning. If we abandon this prejudice, the general construction of the reduced system for group-invariant solutions becomes very natural. Once the basic coordinate-free construction has been made, the technicalities involved in introducing particular independent and dependent variables, both on M and M/G, can be handled with a minimum of difficulty. We therefore commence this section with a coordinate-free reformulation of our basic jet space construction, this time valid for arbitrary manifolds, not just open subsets of Euclidean space.

Extended Jet Bundles

Let us begin by looking a bit closer at our earlier constructions of the jet space $M^{(n)}$ for $M \subset X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$. Each point $(x_0, u_0^{(n)}) \in M^{(n)}$ is determined by the derivatives of a smooth function $u = f(x)$ whose graph passes through the base point $z_0 = (x_0, u_0) \in M$, with $u_0^{(n)} = \mathrm{pr}^{(n)} f(x_0)$. Two such functions are said to be n-th *order equivalent* at z_0 if they determine the same point in $M^{(n)}|_{z_0} \equiv \{(x, u^{(n)}): (x, u) = z_0\}$; in other words, f and $\tilde{f}$ are n-th order equivalent at (x_0, u_0) if their derivatives up to order n agree:

$$\partial_J f^\alpha(x_0) = \partial_J \tilde{f}^\alpha(x_0), \qquad \alpha = 1, \ldots, q, \quad 0 \leqslant \#J \leqslant n.$$

From this point of view, the jet space $M^{(n)}|_{z_0}$ can be regarded as the set of n-th order equivalence classes on the space of all smooth functions $u = f(x)$ whose graphs pass through $z_0 = (x_0, u_0)$.

Thus the important object is not the function f but rather its graph $\Gamma_f = \{(x, f(x))\}$, which is a p-dimensional submanifold of M. However, not every p-dimensional submanifold of M is the graph of a smooth function, so not every such submanifold passing through the point $z_0 \in M$ will determine a point in $M^{(n)}|_{z_0}$. The goal here is to "extend" the jet space $M^{(n)}|_{z_0}$ to include those submanifolds with "vertical tangents". The implicit function theorem tells us which submanifolds are the graphs of smooth functions.

Proposition 3.25. *A p-dimensional submanifold $\Gamma \subset M \subset X \times U$ is the graph of a smooth function $u = f(x)$ if and only if Γ satisfies the properties of being*

(a) Transverse. *For each $z_0 = (x_0, u_0) \in \Gamma$, Γ intersects the vertical space $U_{z_0} = \{(x_0, u): u \in U\}$ transversally, meaning $T\Gamma|_{z_0} \cap TU_{z_0}|_{z_0} = \{0\}$.*
(b) Single-Valued. *Γ intersects each vertical space U_{z_0}, $z_0 \in M$, in at most one point.*

Of course, if we change coordinates on M, the requisite vertical planes will change, so a submanifold which is the graph of a function in one coordinate system may not be in another. For instance, the parabola $u = x^2$ is the graph of a function when $x \in \mathbb{R}$ is the independent and $u \in \mathbb{R}$ the dependent variable, but if we let $y = u$ be the new independent and $v = x$ the new dependent variable, then the parabola fails the transversality condition at the origin and the single-valuedness condition for all $y > 0$. However, if we have a sufficiently small p-dimensional submanifold Γ, we can always arrange local coordinates so that Γ is the graph of a function.

Once we allow arbitrary changes in the independent and dependent variables, it is senseless to exclude certain p-dimensional submanifolds merely because they happen to violate the transversality or single-valuedness conditions in some given set of coordinates. From this standpoint, the role of functions $u = f(x)$ is now played by *arbitrary* p-dimensional submanifolds $\Gamma \subset M$. At this point, we see that we have freed ourselves entirely from our dependence on Euclidean coordinates (x, u) and the definitions to follow will make sense for arbitrary $(p + q)$-dimensional manifolds M.

Definition 3.26. Let Γ and $\tilde{\Gamma}$ be regular p-dimensional submanifolds of the smooth manifold M. We say that Γ and $\tilde{\Gamma}$ have n-th *order contact* at a common point $z_0 \in \Gamma \cap \tilde{\Gamma}$ if and only if there exists a local coordinate chart W containing $z_0 = (x_0, u_0)$ with coordinates $(x, u) = (x^1, \ldots, x^p, u^1, \ldots, u^q)$, such that $\Gamma \cap W$ and $\tilde{\Gamma} \cap W$ coincide with the graphs of smooth functions $u = f(x)$ and $u = \tilde{f}(x)$ which are n-th order equivalent at z_0: $\mathrm{pr}^{(n)} f(x_0) = \mathrm{pr}^{(n)} \tilde{f}(x_0)$.

It is not difficult to see that the property of having n-th order contact at z_0 is independent of the choice of local coordinates at z_0, provided only that Γ and $\tilde{\Gamma}$ are both transverse to the vertical space U_{z_0} and hence are locally the graphs of smooth functions. Clearly n-th order contact determines an equivalence relation on the set of p-dimensional submanifolds passing through a point.

Definition 3.27. Let M be a smooth manifold and p a fixed integer with $0 < p < \dim M$. The *extended jet space* $M_*^{(n)}|_z$, is defined as the set of equivalence classes of the set of all p-dimensional submanifolds passing through z under the equivalence relation of n-th order contact. The *extended jet bundle* is the union of all these spaces: $M_*^{(n)} = \bigcup_{z \in M} M_*^{(n)}|_z$.

If $\Gamma \subset M$ is any p-dimensional submanifold, and $z \in \Gamma$, the n-th *prolongation* $\mathrm{pr}^{(n)} \Gamma|_z \in M_*^{(n)}|_z$ is the equivalence class determined by Γ. If Γ and $\tilde{\Gamma}$ have n-th order contact at z_0, they certainly have k-th order contact for any $k < n$, so there is a natural projection $\pi_k^n\colon M_*^{(n)} \to M_*^{(k)}$, $\pi_k^n(\mathrm{pr}^{(n)} \Gamma) = \mathrm{pr}^{(k)} \Gamma$. In particular, $M_*^{(0)} \simeq M$. The next result makes precise in what sense the extended jet space is the "completion" of the ordinary jet space (in the same way that projective space is the "completion" of Euclidean space).

Theorem 3.28. *If M is a smooth $(p + q)$-dimensional manifold, then the extended jet bundle $M_*^{(n)}$ determined by p-dimensional submanifolds is a smooth $p + q\binom{p+n}{n}$-dimensional manifold. If $\Gamma \subset M$ is any regular p-dimensional submanifold, its prolongaton $\mathrm{pr}^{(n)} \Gamma$ is a regular p-dimensional submanifold of $M_*^{(n)}$. If $\tilde{M} \subset M$ is a local coordinate chart, which determines a local choice of independent and dependent variables (x, u), then the subspace*

$$\tilde{M}^{(n)}|_z \equiv \{\mathrm{pr}^{(n)} \Gamma|_z \colon z \in \Gamma,\ T\Gamma|_z \cap TU_z|_z = \{0\}\}$$

determined by the transverse submanifolds Γ passing through z, is an open dense subset of the extended jet space $M_*^{(n)}|_z$. Moreover, the union of all such subspaces, $\tilde{M}^{(n)}$, is isomorphic to the ordinary Euclidean jet space: $\tilde{M}^{(n)} \simeq \tilde{M} \times U_1 \times \cdots \times U_n = \{(x, u^{(n)})\colon (x, u) \in \tilde{M}\}$. If $\Gamma \subset \tilde{M}$ coincides with the graph of a smooth function $u = f(x)$, then under the above identification its prolongation $\mathrm{pr}^{(n)} \Gamma \subset \tilde{M}^{(n)} \subset \tilde{M}_*^{(n)}$ coincides with the graph of the prolongation of f: $\mathrm{pr}^{(n)} \Gamma = \{(x, \mathrm{pr}^{(n)} f(x))\}$.

Thus, except for the singular subvariety $\mathscr{V}^{(n)}|_z \equiv \tilde{M}_*^{(n)}|_z \setminus \tilde{M}^{(n)}|_z$, consisting of the prolongations of nontransverse submanifolds, the exended jet space looks just like the ordinary jet space discussed in Chapter 2. The proof of this theorem is not difficult; an illustrative example should indicate how one would fill in the details in general.

Example 3.29. Let $M \subset \mathbb{R}^2$ be open and let $p = 1$, so we are considering one-dimensional submanifolds (curves) in M. Two curves determine the same point in $M_*^{(n)}|_{z_0}$ if and only if in some local coordinates near $z_0 = (x_0, u_0)$ they are given as the graphs of functions $u = f(x)$, $u = \tilde{f}(x)$ with the same derivatives up to order n at x_0:

$$u_0 = f(x_0) = \tilde{f}(x_0), f'(x_0) = \tilde{f}'(x_0), \ldots, f^{(n)}(x_0) = \tilde{f}^{(n)}(x_0).$$

In the case $n = 1$, then, the curves Γ and $\tilde{\Gamma}$ have first order contact at z_0 if and only if they are tangent at z_0. Thus $M_*^{(1)}|_{z_0}$ is given by the set of all tangent lines to curves passing through z_0. Since every such line is determined by the angle θ it makes with the horizontal, varying from $\theta = 0$ to $\theta = \pi$, we can identify $M_*^{(1)}|_{z_0}$ with the circle S^1, where the "angular" coordinate satisfies $0 \leqslant \theta < \pi$. Topologically, then, $M^{(1)} \simeq M \times S^1$. Choosing coordinates (x, u) on M, the Euclidean jet space $M^{(1)}|_{z_0}$ is the subset of $M_*^{(1)}|_{z_0}$ given by those curves whose tangent line is not vertical, i.e. $\theta \neq \pi/2$. We can identify this subset with the usual jet space $\{(x, u, u_x)\}$ by setting $u_x = \tan \theta$.

Turning to the case $n = 2$, we see that two curves Γ and $\tilde{\Gamma}$ have second order contact at z_0 if and only if they osculate at z_0, i.e. have the same tangent *and* curvature there. Thus $M_*^{(2)}|_{z_0}$ can be identified with the set of all circles of positive radius passing through z_0, including the degenerate straight lines. I claim that this space is topologically equivalent to a Möbius band! The natural projection $\pi_1^2\colon M_*^{(2)}|_{z_0} \to M_*^{(1)}|_{z_0} \simeq S^1$ associates the common tangent line to any pair of osculating curves, so the inverse image of a point $\theta \in S^1$ (i.e. a tangent line through z_0) is isomorphic to $\mathbb{R}$, the additional coordinate being the signed curvature of the curve with the given tangent direction. Thus, locally $M_*^{(2)}|_{z_0}$ looks like the Cartesian product of a piece of S^1 with $\mathbb{R}$. However, if we fix the curvature, but let θ increase from 0 to π, in essence we

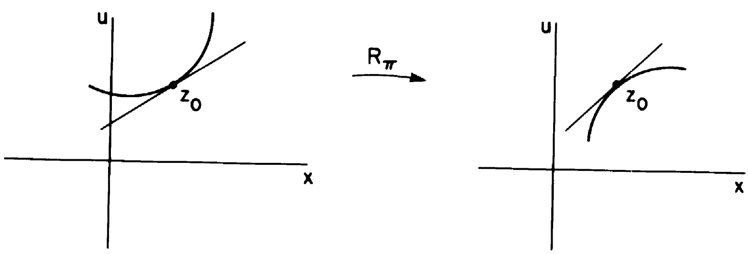

Figure 9. Extended jet space M_*^2 for $M \subset \mathbb{R}^2$.

are rotating the given curve through an angle of π. The result is a curve with the same tangent, but whose curvature has changed sign! Thus as the circle S^1 is traversed, the copy of $\mathbb{R}$ over each point twists once, and we have a Möbius band. In local coordinates (x, u), the open dense subset $M^{(2)}|_{z_0} \subset M_*^{(2)}|_{z_0}$ is obtained by cutting the band along the line $\theta = \pi/2$, the result being isomorphic to the two-dimensional plane $U_1 \times U_2 = \{(u_x, u_{xx})\}$.

Further results on the structure of the extended jet space are given in Exercise 3.16 and Olver, [2].

Differential Equations

Definition 3.30. Let M be a smooth manifold with extended jet bundle $M_*^{(n)}$ determined by p-dimensional submanifolds. A *system of differential equations* over M is determined by a closed subvariety $\mathscr{S}_\Delta^* \subset M_*^{(n)}$. A *solution* to the system is a p-dimensional submanifold Γ whose prolongation lies entirely within the subvariety: $\mathrm{pr}^{(n)}\,\Gamma \subset \mathscr{S}_\Delta^*$.

If we choose a local coordinate chart $\tilde{M} \subset M$, and concentrate on the subset $\tilde{M}^{(n)} \subset \tilde{M}_*^{(n)}$, then we reduce to our previous concept of a system of differential equations: $\mathscr{S}_\Delta \equiv \mathscr{S}_\Delta^* \cap \tilde{M}^{(n)} = \{(x, u^{(n)}): \Delta(x, u^{(n)}) = 0\}$ for some set of smooth functions $\Delta\colon \tilde{M}^{(n)} \to \mathbb{R}^l$. A transverse, single-valued submanifold Γ, which thus is the graph of a smooth function $u = f(x)$, is a solution in the above sense if and only if the corresponding function f is a solution in the traditional sense: $\Delta(x, \mathrm{pr}^{(n)} f(x)) = 0$. In addition we are allowing the possibility of both multiply-valued solutions and solutions with vertical tangents (infinite derivatives) provided they are in some sense "limits" of classical solutions. Any "traditional" system of differential equations defined over an open subset M of Euclidean space $X \times U$ can always be made into such an "extended" system by taking the closure of its subvariety in $M_*^{(n)}$: $\mathscr{S}_\Delta^* = \overline{\mathscr{S}_\Delta}$.

Example 3.31. Consider the nonlinear wave equation $u_t + uu_x = 0$. Here the underlying space is $M \simeq \mathbb{R}^2 \times \mathbb{R}$, with coordinates (x, t, u), and the equation determines a subvariety in the first jet space $M^{(1)} \simeq \mathbb{R}^5$ with coordinates $(x, t; u; u_x, u_t)$. The extended jet space $M_*^{(1)}|_z$ is, as in the previous example, equivalent to the set of all planes passing through $z = (x, t, u)$. Each plane is uniquely determined by its normal direction $\mathbf{n} = (\lambda, \mu, \nu)$, and two nonzero normal vectors $\mathbf{n}, \tilde{\mathbf{n}}$ determine the same plane if and only if they are scalar multiples, $\tilde{\lambda} = \kappa\lambda$, $\tilde{\mu} = \kappa\mu$, $\tilde{\nu} = \kappa\nu$. The entries $[\lambda, \mu, \nu]$ of the normal vector thus provide "homogeneous" coordinates on $M_*^{(1)}|_z$ (which is isomorphic to $\mathbb{RP}^2$).

A function $u = f(x, t)$ determines a two-dimensional submanifold Γ_f with normal $\mathbf{n} = (-f_x, -f_t, 1)$, so $\lambda = -f_x$, $\mu = -f_t$, $\nu = 1$ form one set of homogeneous coordinates for $\mathrm{pr}^{(1)}\,\Gamma_f$. A more general submanifold $\Gamma =$

$\{F(x, t, u) = 0\}$ has normal $\mathbf{n} = \nabla F$, hence $\mathrm{pr}^{(1)} \Gamma = \{(x, t; u; [F_x, F_t, F_u])\}$ in our coordinates. In particular, Γ and Γ_f have the same tangent plane if and only if their homogeneous coordinates are equivalent: $F_x = -\kappa f_x$, $F_t = -\kappa f_t$, $F_u = \kappa$, from which we deduce the familiar formulae $u_x = -F_x/F_u$, $u_t = -F_t/F_u$. Consequently, $M^{(1)}|_z$ is the open subset of $M_*^{(1)}|_z$ where the third homogeneous coordinate ν does not vanish, and in this case $u_x = -\lambda/\nu$, $u_t = -\mu/\nu$.

If we substitute these expressions into the equation, we find an explicit formula for the extended subvariety

$$\mathscr{S}_\Delta^* = \{(x, t; u; [\lambda, \mu, \nu]) : \lambda + u\mu = 0\}.$$

A solution is then a two-dimensional submanifold $\Gamma = \{F(x, t, u) = 0\}$ with $\mathrm{pr}^{(1)} \Gamma \subset \mathscr{S}_\Delta^*$, meaning $\partial_t F + u \partial_x F = 0$ ($\nabla F \neq 0$). This equation can now be solved directly by the characteristic methods of Section 2.1, leading to $F = F(x - tu, u)$ for the general solution. Alternatively, we can use a "hodograph" coordinate change and choose new independent variables t and u and new dependent variable x. Note that $x_t = -\mu/\lambda$, $x_u = -\nu/\lambda$, so in the $(t, u; x; x_t, x_u)$ coordinates on $M_*^{(1)}$, the equation becomes $x_t = u$, with elementary solution $x = tu + h(u)$, h an arbitrary function of u. Note that although this choice of coordinates leads to globally defined solutions, in the original coordinates $(x, t; u)$ solutions can become multiply-valued, leading to the familiar phenomena of wave breaking. (In our present interpretation, these multiply-valued functions *remain* solutions, while for physical applications, one would replace them by shock solutions.)

Group Actions

If $g: M \to M$ is any diffeomorphism and $\Gamma \subset M$ is a p-dimensional submanifold, then $g \cdot \Gamma = \{g \cdot x : x \in \Gamma\}$ is also a p-dimensional submanifold. Moreover, g preserves the equivalence relation of n-th order contact, so there is an induced diffeomorphism $\mathrm{pr}^{(n)} g$ of the extended jet space $M_*^{(n)}$:

$$\mathrm{pr}^{(n)} g(\mathrm{pr}^{(n)} \Gamma|_z) \equiv \mathrm{pr}^{(n)} (g \cdot \Gamma)|_{g \cdot z}, \qquad z \in \Gamma. \tag{3.31}$$

Thus for any local group of transformations G acting on M, there is an induced action $\mathrm{pr}^{(n)} G$, the n-th *prolongation* of G, on $M_*^{(n)}$. In any local coordinate chart $\tilde{M} \subset M$, this action agrees with our earlier notion of prolongation on the corresponding Euclidean jet space $\tilde{M}^{(n)} \subset \tilde{M}_*^{(n)}$. Note especially that since *any* p-dimensional submanifold, transverse or not, is now being regarded as the graph of a "function", we no longer have to worry about domains of definition of the prolonged group action; if g is defined on M_g, then $\mathrm{pr}^{(n)} g$ is defined on all of $M_{g*}^{(n)}$. In particular, if G is a global group of transformations, its prolongation to $M_*^{(n)}$ is *still* a global group of transformations. (Compare this with Example 2.26.)