Resurgent Trans-series in Hopf-Algebraic Dyson-Schwinger Equations

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M. Borinsky & GD, 2005.04265; M. Borinsky, GD, M. Meynig, 2104.00593

O. Costin & GD, 1904.11593, 2003.07451, 2009.01962, 2108.01145, ...

[DOE Division of High Energy Physics]

Motivation

- Kreimer-Connes:

[perturbative] QFT renormalisation $\longleftrightarrow$ Hopf algebra stucture

$\Rightarrow$ enables perturbative computations to very high order

- Écalle: resurgent asymptotics

[perturbative] series $\longrightarrow$ [perturbative + non-perturbative] trans-series

$\Rightarrow$ non-perturbative physics encoded in perturbative physics

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comment: the connection between *perturbative* and *non-perturbative* physics can be probed most efficiently at high orders of perturbation theory

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- in quantum mechanics and quantum field theory perturbation theory is very successful, but it is typically divergent
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• e.g. Stark effect (1919 Nobel Prize): electron energy levels are modified by an applied weak electric field $\mathcal{E}$

$$\text{perturbation theory :} \quad E_{\text{pert}}(\mathcal{E}) = E_0 + E_1\mathcal{E}^2 + E_2\mathcal{E}^4 + E_3\mathcal{E}^6 + \dots$$

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- this formal perturbative series is asymptotic: $E_n \sim (2n)!$
- but there is also a "non-perturbative" effect: ionization

$$E(\mathcal{E}) = E_{\rm pert}(\mathcal{E}) + i\,\exp\left[-\frac{\rm const}{\mathcal{E}}\right] E^{\rm ion.}_{\rm pert}(\mathcal{E}) + \dots$$

- resurgence $\Rightarrow$ these two aspects are intimately related

Resurgent Trans-Series

- Écalle: *resurgent functions* closed under all operations:

(Borel transform) + (analytic continuation) + (Laplace transform)

- building blocks: *trans-monomial elements*: x, $e^{-\frac{1}{x}}$, $\ln(x)$
- in physics applications: semi-classical trans-series:

$$f(x) \sim \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, x^p}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{1}{x}\right]\right)^k}_{\text{non-perturbative}} \underbrace{(\ln[x])^l}_{\text{logarithm powers}}$$

- new: analytic continuation encoded in trans-series
- new: trans-series coefficients $c_{k,l,p}$ are highly correlated
- theorems in ODEs, PDEs, difference eqs.; evidence in QM, matrix models; being explored in QFT, string theory, ...

"Resurgence"

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

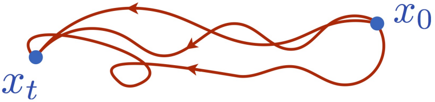
J. Écalle

grand conjecture: resurgent trans-series solve all "natural problems"

$$\text{QM} : \int \mathcal{D}x(t) \exp\left[\frac{i}{\hbar} S[x(t)]\right]$$

$$\text{QFT} : \int \mathcal{D}\phi(x) \exp\left[\frac{i}{g} S[\phi(x)]\right]$$

$$\langle x_t | e^{-i\hat{H}t/\hbar} | x_0 \rangle =$$



- Feynman path integral = generating function of Feynman diagrams
- rich combinatorial, algebraic & graph-theoretic structure
- Feynman path integral = generating function of non-perturbative saddles
- new perspective: Feynman path integral = a trans-series

• renormalization makes resurgence in quantum field theory more interesting

• recent progress for semiclassical QFT and lattice QFT

• here: invoke Hopf algebra structure of the perturbative renormalization of QFT

Resurgence in QFT ?

review: GD & M. Ünsal (1603.04924)

• renormalization makes resurgence in quantum field theory more interesting

• recent progress for semiclassical QFT and lattice QFT

• here: invoke Hopf algebra structure of the perturbative renormalization of QFT

Q1: do the Dyson-Schwinger equations contain **all** information (perturbative & non-perturbative) about a QFT?

Q2: [how] can one decode non-perturbative information?

Q3: is there a natural Hopf algebraic formulation of Écalle's "bridge equations" which relate the perturbative and non-perturbative features ?

Nonlinear ODEs from Dyson-Schwinger Equations

- Broadhurst/Kreimer 1999/2000; Kreimer/Yeats 2006:

for certain QFTs the renormalization group equations can be reduced to coupled nonlinear ODEs for the anomalous dimension in terms of the renormalized coupling

- resurgence is deeply understood for (nonlinear) ODEs (Écalle, Costin, Kruskal, Ramis, Takei, Sauzin, Fauvet, ...)

n^{th} order nonlinear ODE: $\mathbf{y}' = \mathbf{f}(z, \mathbf{y}) \quad , \quad z \in \mathbb{C} \quad , \quad \mathbf{y} \in \mathbb{C}^n$

• normal form:

$$\mathbf{y}' = \mathbf{f}_0(z) - \hat{\Lambda}\mathbf{y} - \frac{1}{z}\hat{B}\mathbf{y} + \mathbf{g}(z, \mathbf{y})$$

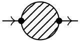
$\Rightarrow$ *generic* resurgent trans-series

$$\mathbf{y} = \mathbf{y}_0 + \sum_{\mathbf{k}} \sigma_1^{k_1} \dots \sigma_n^{k_n} \, e^{-(\mathbf{k}\cdot\lambda)z} \, z^{-\mathbf{k}\cdot\beta} \sum_l a_{\mathbf{k};l} \, z^{-l}$$

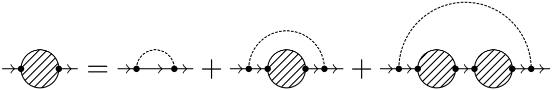
• discontinuities of singularities of Borel transform $\mathcal{B}\mathbf{y}$ at $\mathbb{N}\,\lambda_j$ are all related by a network of algebraic “bridge equations”

• in principle, even the original ODE can be reconstructed from a formal solution

- renormalised fermion self-energy

$$\Sigma(q) :=$$  $$= \not{q}\, \Sigma(q^2)$$

- Dyson-Schwinger equation

 $+ \cdots -$ subtractions

- anomalous dimension $\gamma(\alpha)$ ($\alpha \equiv$ <u>renormalised coupling</u>):

$$\gamma(\alpha) = \frac{d}{d \ln q^2} \ln \left(1 - \Sigma(q^2)\right)\Bigg|_{q^2=\mu^2}$$

- renormalisation group $\Rightarrow$ non-linear ODE (1st order)

$$2\gamma = -\alpha - \gamma^2 + 2\alpha\, \gamma \frac{d}{d\alpha}\gamma$$

Perturbative Solution (rescale: $C(x) := \frac{1}{2}\gamma(-4x)$)

$$\left[C(x)\left(2\,x\,\frac{d}{dx}-1\right)-1\right]C(x)=-x$$

- perturbative solution: $C(x)=\sum_{n=1}^{\infty} C_n x^n$ (OEIS: A000699)

$$C_n = [1, 1, 4, 27, 248, 2830, 38232, 593859, 10401712, 202601898, \dots]$$

- combinatorics: generating function for "connected chord diagrams"

- large order asymptotics

$$C_n \sim e^{-1}\frac{2^{n+\frac{1}{2}}\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{2\pi}}\left(1-\frac{\frac{5}{2}}{2\left(n-\frac{1}{2}\right)}-\frac{\frac{43}{8}}{2^2\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)}-\dots\right)$$

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- missing boundary condition parameter ?

Écalle: formal series $\to$ trans-series : $\quad C(x)=\sum_{k=0}^{\infty}\sigma^k\, C^{(k)}(x)$

- expand $C(x) = C^{(0)}(x) + \sigma\, C^{(1)}(x) + \sigma^2\, C^{(2)}(x) + \dots$
- $C^{(0)}(x) =$ previous formal perturbative series solution
- linear (in)homogeneous equations for $C^{(k)}(x)$, $k \geq 1$ (2)

$$
\begin{aligned}
C^{(1)}(x) \quad &= \quad \frac{1}{\sqrt{2\pi}} \frac{\sqrt{x}}{C^{(0)}(x)} \exp\left[-\frac{\left(C^{(0)}(x)+1\right)^2}{2x}\right] \\
&\sim \quad \frac{e^{-1}}{\sqrt{2\pi}} \frac{e^{-1/(2x)}}{\sqrt{x}} \left[1 - \frac{5}{2}\,x - \frac{43}{8}\,x^2 - \frac{579}{16}\,x^3 - \dots\right]
\end{aligned}
$$

- note: $C^{(1)}(x) =$ (instanton factor) $\times$ (fluctuation series)
- resurgence: $C^{(1)}(x)$ expressed in terms of $C^{(0)}(x)$

Trans-series Solution

M. Borinsky & GD, 2005.04265

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- note: $C^{(1)}(x) =$ (instanton factor) $\times$ (fluctuation series)
- resurgence: $C^{(1)}(x)$ expressed in terms of $C^{(0)}(x)$
- characteristic signature of resurgent structure:

$$
C_n^{(0)} \sim e^{-1} \frac{2^{n+\frac{1}{2}}\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{2\pi}} \left(1 - \frac{\frac{5}{2}}{2\left(n-\frac{1}{2}\right)} - \frac{\frac{43}{8}}{2^2\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)} - \ldots\right)
$$

Resurgent structure

- large order asymptotics of $C_n^{(1)}$ coefficients

$$C_n^{(1)} \sim -2e^{-2} \frac{2^{n+\frac{3}{2}}\,\Gamma\left(n+\frac{3}{2}\right)}{2\pi} \left(1 - \frac{5}{2\left(n+\frac{1}{2}\right)} - \frac{\frac{11}{2}}{2^2\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)} - \dots \right)$$

- next nonperturbative solution ($\xi(x) \equiv \frac{1}{\sqrt{x}}\, e^{-1/(2x)}$):

$$C^{(2)}(x) \sim \xi(x)^2 \frac{e^{-2}}{2\pi} \left[\frac{1}{x} - 5 - \frac{11}{2} x - \frac{97}{2} x^2 - \dots \right]$$

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- continues to all orders $\Rightarrow$ all-orders summation

$$C(x)=\left[\exp\left(\sigma\,\xi(x)f(x,y)\,\frac{\partial}{\partial y}\right)\cdot y\right]_{y=C^{(0)}(x)}$$

generating function : $\quad f(x,y)\equiv\frac{1}{\sqrt{2\pi}}\,\frac{x}{y}\,\exp\left[-\frac{1}{2x}\,y(y+2)\right]$

- note: no logarithms, just powers of x and $e^{-1/(2x)}$

Resurgence in the 4 dimensional massless Yukawa Model

• trans-series: the (asymptotic) perturbative solution to the nonlinear ODE for the anomalous dimension can be extended to a trans-series which resums all non-perturbative orders

$$C(x) = C^{(0)}(x) + \sigma\, C^{(1)}(x) + \sigma^2\, C^{(2)}(x) + \dots$$

• non-perturbative terms $C^{(k)}(x)$ $(k \geq 1) \longleftrightarrow$ singularities of the Borel transform of the perturbative series

• resurgence: all non-perturbative terms are encoded in the original formal perturbative series $C^{(0)}(x)$

- physically more interesting quantum field theory

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{g}{3!}\,\phi^3 \qquad , \qquad a := \frac{g^2}{(4\pi)^3}$$

- asymptotically free; $d = 6$ critical dimension
- conformal field theory
- known non-perturbative physics: Lipatov instanton
- renormalon-like bubble-chain diagrammatic structures
- multi-component fields → percolation, Lee-Yang edge singularity, critical exponents, ...
- β function & anomalous dims computed to 4 loops (Gracey 2015); now 5 loops (Gracey et al, 2021)

Hopf Algebra Analysis of 6 dim. Scalar ϕ^3 Theory

- Broadhurst/Kreimer: <u>3rd order</u> ODE for anomalous dimension, with <u>quartic nonlinearity</u>

$$a = 8a^3\gamma(a)\left(\gamma(a)^2\gamma'''(a) + \gamma'(a)^3 + 4\gamma(a)\gamma'(a)\gamma''(a)\right)$$
$$+4a^2\gamma(a)\left(2(\gamma(a)-3)\gamma(a)\gamma''(a) + (\gamma(a)-6)\gamma'(a)^2\right)$$
$$+2a\gamma(a)\left(2\gamma(a)^2 + 6\gamma(a) + 11\right)\gamma'(a) - \gamma(a)(\gamma(a)+1)(\gamma(a)+2)(\gamma(a)+3)$$

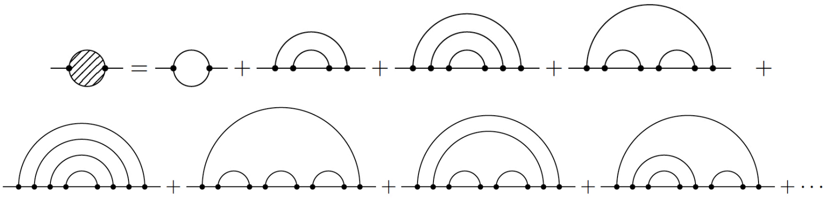
- formal perturbative solution:

$$\gamma_{\rm Hopf}(a) := \sum_{n=1}^{\infty} (-1)^n \frac{A_n}{6^{2n-1}} a^n$$
$$\sim -\frac{a}{6} + 11\frac{a^2}{6^3} - 376\frac{a^3}{6^5} + 20241\frac{a^4}{6^7} - 1427156\frac{a^5}{6^9} + \dots$$

$$A_n = \{1, 11, 376, 20241, 1427156, 121639250, 12007003824, \dots\}$$

- no known combinatorial interpretation of A_n: OEIS A051862

Perturbative Diagrammatics of the Hopf Approximation



$$\gamma_{\text{rainbow}}(a) \quad = \quad -\frac{a}{6} + 11\frac{a^2}{6^3} - 206\frac{a^3}{6^5} + 4711\frac{a^4}{6^7} - 119762\frac{a^5}{6^9} + \ldots$$

$$\gamma_{\text{chain}}(a) \quad \sim \quad -\frac{a}{6} + 11\,\frac{a^2}{6^3} - 170\,\frac{a^3}{6^5} + 3450\,\frac{a^4}{6^7} - 87864\,\frac{a^5}{6^9} + \ldots$$

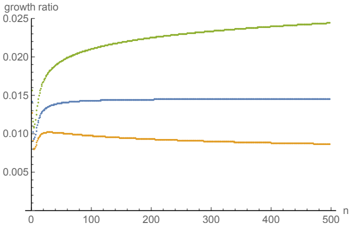
$$\gamma_{\text{Hopf}}(a) \quad \sim \quad -\frac{a}{6} + 11\frac{a^2}{6^3} - 376\frac{a^3}{6^5} + 20241\frac{a^4}{6^7} - 1427156\frac{a^5}{6^9} + \ldots$$

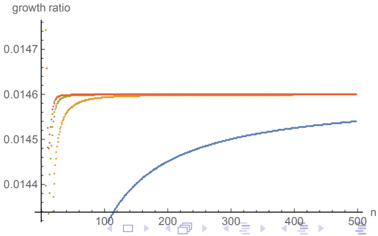
Large Order Behavior

Borinsky, GD, Meynig, 2104.00593

- Broadhurst/Kreimer: $A_n \sim 12^n \, \Gamma\left(n+2\right)$
- factorially divergent
- with more data $\to$ subleading resurgent corrections

$$A_n \sim S_1 \, 12^n \, \Gamma\left(n + \frac{23}{12}\right) \left(1 - \frac{\frac{97}{48}}{\left(n+\frac{11}{12}\right)} - \frac{\frac{53917}{13824}}{\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)} - \frac{\frac{3026443}{221184}}{\left(n-\frac{13}{12}\right)\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)} - \dots \right) + \dots \quad , \quad n \to \infty$$





Borel Analysis

- Dyson-Schwinger equation factorizes ($G(x) := \gamma(-3x)$):

$$\left[G(x)\left(2\,x\,\frac{d}{dx}-1\right)-1\right]\left[G(x)\left(2\,x\,\frac{d}{dx}-1\right)-2\right]\left[G(x)\left(2\,x\,\frac{d}{dx}-1\right)-3\right]G(x)=3x$$

- Borel transform

$$G^{\text{pert}}(x) \sim 6\sum_{n=1}^{\infty}\frac{A_n}{12^n}\,x^n \quad\longrightarrow\quad \mathcal{B}^{\text{pert}}(t) := 6\sum_{n=1}^{\infty}\frac{A_n}{12^n}\frac{t^n}{n!}$$

- Borel radius of convergence $= 1$
- formal perturbative series by formal Laplace transform

$$G^{\text{pert}}(x) = \frac{1}{x}\int_0^{\infty} dt\, e^{-t/x}\mathcal{B}^{\text{pert}}(t)$$

- singularities of $\mathcal{B}^{\text{pert}}(t)$ encode non-perturbative physics $\sim e^{-\frac{1}{x}}$

Trans-series Analysis

- Dyson-Schwinger eqn is 3rd order $\Rightarrow$ 3 b.c. parameters?
- perturbative series $G^{\text{pert}}(x)$ has no b.c. parameters

$$G(x) \sim G^{\text{pert}}(x) + \sigma\, G^{\text{non-pert}}(x)$$

- $G^{\text{non-pert}}(x)$ is *beyond all orders* in perturbation theory:

$$G^{\text{non-pert}}(x) \sim x^{\beta}\, e^{-\lambda/x}\, (1 + O(x))$$

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- linearize equation for $G^{\rm non-pert}(x) \longrightarrow$ 3 solutions

$$\vec{\lambda} = (1, 2, 3) \qquad , \qquad \vec{\beta} = \left(-\frac{23}{12}, +\frac{1}{6}, -\frac{11}{4}\right)$$

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- <u>linearize</u> equation for $G^{\rm non-pert}(x) \longrightarrow 3$ solutions

$$\vec{\lambda} = (1, 2, 3) \qquad , \qquad \vec{\beta} = \left(-\frac{23}{12}, +\frac{1}{6}, -\frac{11}{4}\right)$$

- recall generic trans-series solution:

$$G(x) = G^{\rm pert}(x) + \sum_{\vec{k}\geq 0, |\vec{k}|>0} \sigma_1^{k_1}\sigma_2^{k_2}\sigma_3^{k_3}\, x^{\vec{k}\cdot\vec{\beta}}\, e^{-\vec{k}\cdot\vec{\lambda}/x}\, \mathcal{F}_{\vec{k}}(x)$$

- trans-series parameters, σ_j: 3 “missing” b.c. parameters

$\Rightarrow$ three <u>resonant</u> Borel singularities at $t = 1, 2, 3$

Resurgence Relations in Large-Order Behavior

- fluctuations about the 3 "seed" Borel singularities:

$$x^{\vec{k}\cdot\vec{\beta}}\, e^{-\vec{k}\cdot\vec{\lambda}/x}\, \mathcal{F}_{\vec{k}}(x) \sim x^{\vec{k}\cdot\vec{\beta}}\, e^{-\vec{k}\cdot\vec{\lambda}/x} \sum_{n=0}^{\infty} a_n^{\vec{k}} x^n$$

$$\begin{aligned}
\{a_n^{(1,0,0)}\} &= \left\{-1, \frac{97}{48}, \frac{53917}{13824}, \frac{3026443}{221184}, \frac{32035763261}{382205952}, \dots\right\} \\
\{a_n^{(0,1,0)}\} &= \left\{-1, \frac{151}{24}, -\frac{63727}{3456}, \frac{7112963}{82944}, -\frac{7975908763}{23887872}, \dots\right\} \\
\{a_n^{(0,0,1)}\} &= \left\{-1, \frac{227}{48}, \frac{1399}{4608}, \frac{814211}{73728}, \frac{3444654437}{42467328}, \dots\right\}
\end{aligned}$$

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\end{aligned}$$

- we have seen the $a_n^{(1,0,0)}$ before !

$$A_n^{\text{pert}} \sim S_1\, 12^n\, \Gamma\left(n + \frac{23}{12}\right)\left(1 - \frac{\frac{97}{48}}{\left(n + \frac{11}{12}\right)} - \frac{\frac{53917}{13824}}{\left(n - \frac{1}{12}\right)\left(n + \frac{11}{12}\right)} - \dots\right)$$

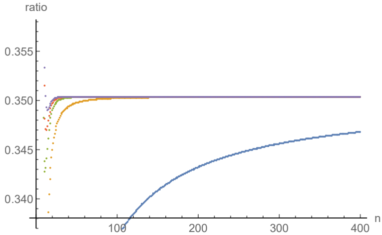
- large-order/low-order resurgence relations

Large-Order Behavior About First Borel Singularity

- large order behavior of coefficients of $\mathcal{F}_{(1,0,0)}(x)$:

$$a_n^{(1,0,0)} \sim 24\, S_1 \Gamma\left(n+\frac{23}{12}\right)\left[1-\frac{\frac{49}{12}}{\left(n+\frac{11}{12}\right)}-\frac{\frac{13235}{3456}}{\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)}\right.$$

$$\left.-\frac{\frac{43049}{3456}}{\left(n-\frac{13}{12}\right)\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)}-\frac{\frac{2496477497}{23887872}}{\left(n-\frac{25}{12}\right)\left(n-\frac{13}{12}\right)\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)}-\dots\right]$$



- what is the physics of these subleading coefficients ?

Resonant Resurgence: Logarithmic Trans-series Terms

- general trans-series: *trans-monomials* x, $e^{-1/x}$, $\log(x)$
- 4 dim. Yukawa trans-series involved only x and $e^{-1/x}$
- 6 dim. ϕ^3 theory has resonant Borel singularities $\rightarrow$ logs
- generic trans-series solution:

$$G(x) = G^{\rm pert}(x) + \sum_{\vec{k}\geq 0, |\vec{k}|>0} \sigma_1^{k_1}\sigma_2^{k_2}\sigma_3^{k_3}\, x^{\vec{k}\cdot\vec{\beta}}\, e^{-\vec{k}\cdot\vec{\lambda}/x}\, \mathcal{F}_{\vec{k}}(x)$$

- $\vec{\lambda} = (1, 2, 3) \Rightarrow$ different $\vec{k}$ can lead to same exponent
- re-organize as an exponentially graded trans-series

$$G(x) \sim G_{(0)}(x) + e^{-\frac{1}{x}} G_{(1)}(x) + e^{-\frac{2}{x}} G_{(2)}(x) + e^{-\frac{3}{x}} G_{(3)}(x) + e^{-\frac{4}{x}} G_{(4)}(x) + \dots$$

Logarithmic Trans-series Terms

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- $G_{(0)}(x) = G_{\text{pert}}(x)$, the formal perturbative series
- equation for $G_{(1)}(x)$ is *linear* and *homogeneous*

$$G_{(1)}(x) = \sigma_1 \, x^{-23/12} \, \mathcal{F}_{(1,0,0)}(x)$$

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- the equation for $G_{(k)}(x)$ is *linear* and <u>*inhomogeneous*</u> for $k \geq 2$
- we need both homogeneous and particular solution

Logarithmic Trans-series Terms

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$$G_{(2)} \sim \left(\frac{1}{x^{23/12}}\right)^2 \left[-2\sigma_1^2 \frac{1}{x} \mathcal{F}_{(2,0,0)}(x) + x^4 \left(\sigma_1^2 \frac{21265}{2304} \log(x) + \sigma_2\right) \mathcal{F}_{(0,1,0)}(x)\right]$$

- $\mathcal{F}_{(0,1,0)}(x)$ is the second “seed” fluctuation from before
- note appearance of $\log(x)$ term multiplying $\mathcal{F}_{(0,1,0)}(x)$
- $\mathcal{F}_{(2,0,0)}(x)$ is a new fluctuation series

$$\mathcal{F}_{(2,0,0)}(x) \sim 1 - \frac{49}{12} x - \frac{13235}{3456} x^2 - \frac{43049}{3456} x^3 - \frac{2496477497}{23887872} x^4 - 0.x^5 - \frac{3315185066507813}{247669456896} x^6 - \dots$$

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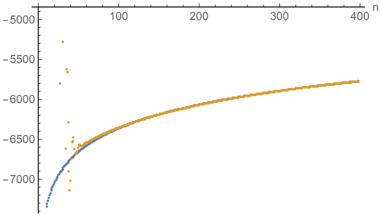
recall:
$$a_n^{(1,0,0)} \sim 24\, S_1 \Gamma\left(n + \frac{23}{12}\right) \left[1 - \frac{\frac{49}{12}}{\left(n + \frac{11}{12}\right)} - \frac{\frac{13235}{3456}}{\left(n - \frac{1}{12}\right)\left(n + \frac{11}{12}\right)} - \frac{\frac{43049}{3456}}{\left(n - \frac{13}{12}\right)\left(n - \frac{1}{12}\right)\left(n + \frac{11}{12}\right)} - \frac{\frac{2496477497}{23887872}}{\left(n - \frac{25}{12}\right)\left(n - \frac{13}{12}\right)\left(n - \frac{1}{12}\right)\left(n + \frac{11}{12}\right)} - \dots\right]$$

- resurgently related to the large order growth of $a_n^{(1,0,0)}$
- higher exponential orders $\longrightarrow$ higher powers of $\log(x)$

Logarithmic Large-Order Growth

- corresponding $\log(n)$ in the large-order growth of $a_n^{(1,0,0)}$

$$a_n^{(1,0,0)} \sim 24 S_1 \Gamma\left(n+\frac{23}{12}\right)\left[1-\frac{\frac{49}{12}}{\left(n+\frac{11}{12}\right)}-\frac{\frac{13235}{3456}}{\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)}\right.$$
$$-\frac{\frac{43049}{3456}}{\left(n-\frac{13}{12}\right)\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)}-\frac{\frac{2496477497}{23887872}}{\left(n-\frac{25}{12}\right)\left(n-\frac{13}{12}\right)\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)}$$
$$\left.+\frac{d-2\,\frac{21265}{2304}\log(n)}{\left(n-\frac{37}{12}\right)\left(n-\frac{25}{12}\right)\left(n-\frac{13}{12}\right)\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)}+\dots\right]$$



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“Multi-Instanton” Trans-series from Dyson-Schwinger Equation

$$G(x) \sim G_{(0)}(x) + e^{-\frac{1}{x}} G_{(1)}(x) + e^{-\frac{2}{x}} G_{(2)}(x) + e^{-\frac{3}{x}} G_{(3)}(x) + e^{-\frac{4}{x}} G_{(4)}(x) + \ldots$$

- trans-series for anomalous dimension of 6 dim ϕ^3 QFT

$$G(x) \sim \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} c_{n,k,l}\, x^{n+1-k} \left(\frac{e^{-1/x}}{x^{23/12}}\right)^k (\log(x))^l$$

- the ϕ_6^3 trans-series has the form of a semi-classical trans-series constructed via an infinite sum of non-perturbative instantons, including their fluctuations and interactions

- $\log(x)$ terms arise from interactions between instantons and anti-instantons (“bions”): starts at “two-instanton” order

- but the ϕ_6^3 trans-series was derived without any mention of instantons or a semi-classical limit; it came directly from the perturbative Hopf algebraic formalism

Summary: Resurgence in the 6 dimensional Scalar ϕ^3 Theory

- 3rd order ODE with 4th order non-linearity $\Rightarrow$ much richer non-perturbative structure than the 4 dim Yukawa model

- "resonant resurgence": both large-order behavior and trans-series structure reveal logarithmic effects characteristic of interactions between instantons and anti-instantons

- large order/low order resurgence relations

- non-perturbative information encoded in the formal perturbative series

Conclusions

$$\boxed{\text{perturbative Hopf algebra renormalisation}}$$

$$\text{resurgent} \Downarrow \text{analysis}$$

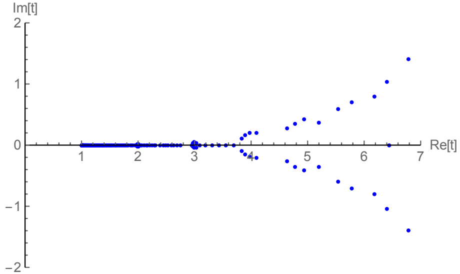
$$\boxed{\text{non-perturbative completion}}$$

$$G(x) \sim G_{(0)}(x) + e^{-\frac{1}{x}} G_{(1)}(x) + e^{-\frac{2}{x}} G_{(2)}(x) + e^{-\frac{3}{x}} G_{(3)}(x) + e^{-\frac{4}{x}} G_{(4)}(x) + \dots$$

- multi-component fields ? (Gracey, 2015; Giombi et al, ...)
- relation with instantons and renormalons ?
- more general Dyson-Schwinger equations ? (Bellon/Rossi 2020)
- 2d σ models, Chern-Simons, SUSY, QED, QCD, ... ?
- big question: does there exist a "natural" Hopf algebraic non-perturbative trans-series structure ?

Borel Singularities

- need analytic continuation of finite order $\mathcal{B}^{\text{pert}}(t)$
- Padé = ratio of polynomials $\Rightarrow$ only pole singularities
- Padé poles of Borel transform



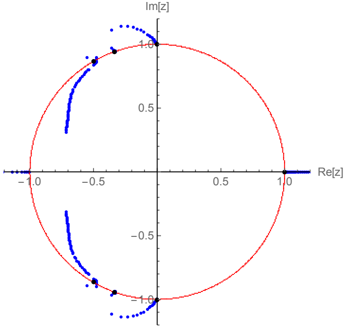
- Padé: branch cut = an accumulation of poles
- this obscures the possible existence of physical higher Borel singularities, associated with higher-order instantons

Resurgence & Padé-Conformal-Borel [Costin & GD: 2003.07451, 2009.01962]

- "hidden" Borel singularities resolved by a conformal map

$$z = \frac{1-\sqrt{1-t}}{1+\sqrt{1-t}} \qquad \longleftrightarrow \qquad t = \frac{4z}{(1+z)^2}$$

- maps the cut Borel plane into the unit disk



- z branch points lie on the unit circle: probe separately
- reveals the first 4 "instantons"