The new equation involving w and y is therefore

$$(w - y)\frac{dw}{dy} + w^2 = yw.$$

This has two families of solutions; either $w = y$ or $dw/dy = -w$, the latter integrating to $w = ce^{-y}$ for some constant c. Reverting back to the original variables, we obtain two homogeneous first order equations, as guaranteed by the form of (2.94):

$$\frac{du}{dx} = \frac{u}{x}, \quad \text{or} \quad \frac{du}{dx} = ce^{-u/x}.$$

The first has solutions $u = kx$; the second has implicit solutions

$$\int \frac{dy}{ce^{-y} - y} = \log x + k, \tag{2.96}$$

where $y = u/x$. Here (2.96) is the "general" solution to (2.95), the linear functions being a one-parameter family of singular solutions.

The integration of (2.95) by our earlier method is quite a bit more tricky. As in Example 2.46, we set $y = u/x$, $\tilde{w} = \log x$, so that in terms of y and w the infinitesimal generator is $\partial/\partial\tilde{w}$. We further have

$$u_x = \frac{1 + y\tilde{w}_y}{\tilde{w}_y}, \qquad xu_{xx} = \frac{\tilde{w}_y^2 - \tilde{w}_{yy}}{\tilde{w}_y^3},$$

so the equation takes the form of a first order Riccati equation

$$\frac{dz}{dy} = (y + 1)z^2 + z \tag{2.97}$$

for $z = d\tilde{w}/dy$. The solution proceeds either by using general methods for integrating Riccati equations, or, more expediently, by noticing that it admits a one-parameter symmetry group with generator $\mathbf{w} = (z + yz^2)\partial_z$. Therefore, by Theorem 2.48, $R = (z + yz^2)^{-1}$ is an integrating factor for (2.97). We find

$$T(y, z) = y + \log(y + z^{-1}) = \tilde{c}$$

to be the integral, hence the solutions of (2.97) are given by $z = (ce^{-y} - y)^{-1}$. Recalling the definition of $z = \tilde{w}_y$, we see that we can integrate this latter expression to recover the general solution (2.96) to (2.95). The singular solutions $u = kx$ do not appear in this case since they do not correspond to functions of the form $\tilde{w} = h(y)$. They can be found by choosing alternative coordinates, e.g. $\hat{w} = \log u$ instead of $\tilde{w}$.

One interesting point is that the symmetry group of the Riccati equation (2.97) generated by $\mathbf{w}$ does *not* appear to have a counterpart for the original equation. (In fact, it can be shown that the scaling group is the *only* symmetry group of (2.95); cf. Exercise 2.25a.) Thus, reducing the order of an ordinary differential equation may result in an equation with new symmetries, whereby the order can be yet further reduced!

Multi-parameter Symmetry Groups

If an ordinary differential equation $\Delta(x, u^{(n)}) = 0$ is invariant under an r-parameter group, then intuition tells us that we should be able to reduce the order of the equation by r. In one sense, this somewhat naïve presumption is correct, but the problem may be that we cannot reconstruct the solution of the original n-th order equation from that of the reduced $(n - r)$-th order equation by quadratures alone. More specifically, suppose G is an r-parameter group of transformations acting on $M \subset X \times U$. *Assume*, for simplicity, that the r-th prolongation $\mathrm{pr}^{(r)} G$ acting on $M^{(r)}$ has r-dimensional orbits. (More degenerate cases can be treated analogously, although technical complications may arise.) Since $M^{(r)}$ is $(r + 2)$-dimensional, this means that locally there exist exactly two independent r-th order differential invariants of G, say

$$y = \eta(x, u^{(r)}), \qquad w = \zeta(x, u^{(r)}). \tag{2.98}$$

Note that every further prolongation $\mathrm{pr}^{(n)} G$ also has r-dimensional orbits. (This is because they project down to the orbits of $\mathrm{pr}^{(r)} G$ in $M^{(r)}$, so are at least r-dimensional, but G itself is r-dimensional, so the orbits can never have more than r dimensions; see Exercise 3.17.) Therefore, $\mathrm{pr}^{(n)} G$ has $n - r + 2$ independent differential invariants, which by Proposition 2.53 we can take to be

$$y, w, dw/dy, \ldots, d^{n-r}w/dy^{n-r}.$$

If $\Delta(x, u^{(n)}) = 0$ is invariant under the entire symmetry group G, then by Proposition 2.18 there is an equivalent equation

$$\tilde{\Delta}(y, w, dw/dy, \ldots, d^{n-r}w/dy^{n-r}) = 0 \tag{2.99}$$

involving only the invariants of $\mathrm{pr}^{(n)} G$. In this sense, we have reduced the n-th order system for u as a function of x to an $(n - r)$-th order system for w as a function of y.

The principal problem at this juncture is that it is unclear how we determine the solution $u = f(x)$ of the original system from the general solution $w = h(y)$ of the reduced system (2.99). Using the expressions (2.98) for the invariants y, w, we find that we must solve an auxiliary r-th order equation

$$\zeta(x, u^{(r)}) = h[\eta(x, u^{(r)})] \tag{2.100}$$

to determine u. This auxiliary equation, being expressed in terms of differential invariants, retains G as an r-parameter symmetry group. However, in contrast to the one-parameter situation, there is no assurance that we will be able to integrate (2.100) completely by quadratures, thereby explicitly determining the solution of our original equation. The difficulty in this regard is apparent in the following example.

Example 2.59. Recalling Example 1.58(c), consider the action of SL(2) as the projective group

$$(x, u) \mapsto ((\alpha x + \beta)/(\gamma x + \delta), u)$$

on the line. The infinitesimal generators are

$$\mathbf{v}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{v}_2 = x\frac{\partial}{\partial x}, \qquad \mathbf{v}_3 = x^2\frac{\partial}{\partial x},$$

from which we see that u and its Schwarzian derivative,

$$y = u, \qquad w = 2u_x^{-3}u_{xxx} - 3u_x^{-4}u_{xx}^2$$

form a complete set of functionally independent invariants for the prolongation $\mathrm{pr}^{(3)}\,\mathrm{SL}(2)$.

We conclude that any differential equation $\Delta(x, u^{(n)}) = 0$ which is invariant under the full projective group is equivalent to an $(n-3)$-rd order equation

$$\Delta\left(y, w, \frac{dw}{dy}, \ldots, \frac{d^{n-3}w}{dy^{n-3}}\right) = 0, \tag{2.101}$$

involving only the invariants of $\mathrm{pr}^{(n)}\,\mathrm{SL}(2)$. For instance, since

$$\frac{dw}{dy} = \frac{dw/dx}{u_x} = \frac{2u_x^2u_{xxxx} - 12u_xu_{xx}u_{xxx} + 12u_{xx}^3}{u_x^6},$$

any fourth order equation admitting SL(2) as a symmetry group is equivalent to one of the form

$$2u_x^2u_{xxxx} - 12u_xu_{xx}u_{xxx} + 12u_{xx}^3 = u_x^6H(u, u_x^{-4}(2u_xu_{xxx} - 3u_{xx}^2)).$$

The reduced system (2.101) in this case is the first order equation $dw/dy = H(y, w)$.

However, once we have solved the reduced equations (2.101) for $w = h(y)$, we are left with the task of determining the corresponding solutions $u = f(x)$ by solving the auxiliary equation

$$2u_xu_{xxx} - 3u_{xx}^2 = u_x^4h(u) \tag{2.102}$$

obtained by substituting for y and w. This equation remains invariant under SL(2), so we can use this knowledge to try to integrate it. In particular, it is invariant under the translation subgroup generated by ∂_x, which has invariants $y = u$, $z = u_x$, in terms of which (2.102) reduces to

$$2z\frac{d^2z}{dy^2} - \left(\frac{dz}{dy}\right)^2 = z^2h(y).$$

This latter equation is invariant under the scale group $(y, z) \mapsto (y, \lambda z)$ (reflecting the symmetry of (2.102) under the scale group $(x, u) \mapsto (\lambda^{-1}x, u)$) and hence can be reduced to a first order Riccati equation

$$2\frac{dv}{dy} + v^2 = h(y) \tag{2.103}$$

for $v = (\log z)_y = z_y/z$. However, at this point we are stuck. We have already used the translational and scaling symmetries to reduce (2.102) to a first order equation, but there is no remnant of the inversional symmetries generated by $\mathbf{v}_3$ which can be used to integrate the standard Riccati equation (2.103). Thus the best that can be said is that the solution of n-th order differential equation invariant under the projective group can be found from the general solution of a reduced $(n-3)$-rd order equation by using two quadratures *and* the solution of an auxiliary first order Riccati equation.

This whole example is illustrative of an important point. If we reduce the order of an ordinary differential equation using only a subgroup of the full symmetry group, we may very well lose any additional symmetry properties present in the full group. Only special types of subgroups, namely the normal subgroups presented in Exercise 1.24, will enable us to retain the full symmetry properties under reduction. Before discussing this case, it helps to return to symmetries of algebraic equations once again.

Let G be an r-parameter group acting on $M \subset \mathbb{R}^m$ and let $H \subset G$ be a subgroup. Suppose that $\eta(x) = (\eta^1(x), \ldots, \eta^{m-s}(x))$ form a complete set of functionally independent invariants of H. If H happens to be a *normal* subgroup, meaning that $ghg^{-1} \in H$ whenever $g \in G$, $h \in H$, then there is an induced action of G on the subset $\tilde{M} \subset \mathbb{R}^{m-s}$ determined by these invariants $y = (y^1, \ldots, y^{m-s}) = \eta(x)$:

$$\tilde{g} \cdot y = \tilde{g} \cdot \eta(x) = \eta(g \cdot x), \qquad g \in G, \quad x \in M. \tag{2.104}$$

Note that for any $h \in H$

$$\tilde{g} \cdot \eta(hx) = \eta(g \cdot hx) = \eta(\hat{h} \cdot gx) = \eta(gx) = \tilde{g} \cdot \eta(x),$$

where $\hat{h} = ghg^{-1} \in H$; from this it is easy to see that this action on $\tilde{M}$ is well defined. (In fact, H acts trivially on $\tilde{M}$, so (2.104) actually defines an action of the quotient group G/H; see Exercise 3.11.)

According to Proposition 2.18, any H-invariant subset of M can be written as the zero set $\mathscr{S}_F = \{F(x) = 0\}$ of some H-invariant function $F(x) = \tilde{F}(\eta(x))$. It is not hard to see that, assuming H is a normal subgroup, $\mathscr{S}_F \subset M$ is invariant under the full group G if and only if the reduced subvariety $\mathscr{S}_{\tilde{F}} = \{y : \tilde{F}(y) = 0\} \subset \tilde{M}$ is invariant under the induced action of G on $\tilde{M}$.

For the infinitesimal version, let us introduce s further variables $\hat{x} = (\hat{x}^1, \ldots, \hat{x}^s)$ completing $y = \eta(x)$ to a set of local coordinates $(y, \hat{x})$ on M. Using the infinitesimal criterion of normality from Exercise 1.24(b), we see that each infinitesimal generator of G must be of the form

$$\mathbf{v}_k = \sum_{i=1}^{m-s} \eta_k^i(y) \frac{\partial}{\partial y^i} + \sum_{j=1}^{s} \hat{\xi}_k^j(y, \hat{x}) \frac{\partial}{\partial \hat{x}^j}, \qquad k = 1, \ldots, r, \tag{2.105}$$

in these coordinates, where each η^i is independent of the parametric variables $\hat{x}$. Thus $\mathbf{v}_k$ reduces to a vector field

$$\tilde{\mathbf{v}}_k = \sum_{i=1}^{m-s} \eta_k^i(y) \frac{\partial}{\partial y^i}, \qquad k = 1, \ldots, r,$$

generating the reduced action of G on $\tilde{M}$. These we can use to check the invariance of the reduced subvariety $\mathscr{S}_{\tilde{F}}$, and hence that of $\mathscr{S}_F$.

Similar results hold for differential equations. Assume, as above, that the r-parameter group G acts on $M \subset X \times U \simeq \mathbb{R}^2$ and suppose $H \subset G$ is an s-parameter subgroup whose prolongation $\mathrm{pr}^{(s)} H$ has s-dimensional orbits in $M^{(s)}$. (As before, degenerate cases can also be treated if required.) Let $y = \eta(x, u^{(s)})$, $w = \zeta(x, u^{(s)})$ be a complete set of functionally independent differential invariants for H on $M^{(s)}$, with corresponding invariants $w^{(m)} = \zeta^{(m)}(x, u^{(s+m)})$ on $M^{(s+m)}$, $m \geq 0$. Any n-th order ordinary differential equation admitting H as a symmetry group can be written in the form

$$\Delta(x, u^{(n)}) = \tilde{\Delta}(\eta(x, u^{(s)}), \zeta^{(n-s)}(x, u^{(n)})) = \tilde{\Delta}(y, w^{(n-s)}) = 0,$$

using only the invariants $y, w, \ldots, d^{n-s}w/dy^{n-s}$ of $\mathrm{pr}^{(n)} H$. Moreover, since H is a normal subgroup of G, there is an induced action of G on $\tilde{M} \subset Y \times W \simeq \mathbb{R}^2$, with

$$\begin{aligned} \tilde{g} \cdot (y, w) &= \tilde{g} \cdot (\eta(x, u^{(s)}), \zeta(x, u^{(s)})) \\ &= (\eta(\mathrm{pr}^{(s)} g \cdot (x, u^{(s)})), \zeta(\mathrm{pr}^{(s)} g \cdot (x, u^{(s)}))), \qquad g \in G, \end{aligned} \tag{2.106}$$

cf. (2.104). Similarly, the action of G on $M^{(n)}$ reduces to an action of G on the space $\tilde{M}^{(n-s)}$ determining the derivatives of w with respect to y. It is not too difficult to see that *this* reduced action coincides with the prolongation of the action of G on $\tilde{M}$ defined by (2.106); in other words

$$\mathrm{pr}^{(n-s)} \tilde{g} \cdot (\eta(x, u^{(s)}), \zeta^{(n-s)}(x, u^{(n)})) = (\eta(\mathrm{pr}^{(s)} g \cdot (x, u^{(s)})), \zeta^{(n-s)}(\mathrm{pr}^{(n)} g \cdot (x, u^{(n)}))).$$

(To check this, look at what happens to a representative smooth H-invariant function $u = f(x)$.)

Translating our earlier results for algebraic equations, we deduce the following result on normal subgroups of symmetry groups of ordinary differential equations.

Theorem 2.60. *Let $H \subset G$ be an s-parameter normal subgroup of a Lie group of transformations acting on $M \subset X \times U \simeq \mathbb{R}^2$ such that $\mathrm{pr}^{(s)} H$ has s-dimensional orbits in $M^{(s)}$. Let $\Delta(x, u^{(n)}) = 0$ be an n-th order ordinary differential equation admitting H as a symmetry group, with corresponding reduced equation $\tilde{\Delta}(y, w^{(n-s)}) = 0$ for the invariants $y = \eta(x, u^{(s)})$, $w = \zeta(x, u^{(s)})$ of H. There is an induced action of the quotient group G/H on $\tilde{M} \subset Y \times W$ and Δ admits all of G as a symmetry group if and only if the H-reduced equation $\tilde{\Delta}$ admits the quotient group G/H as a symmetry group.*

An especially important example is the case of a two-parameter symmetry group. Here, owing to the special structure of two-dimensional Lie groups, we can use the preceding theorem to carry out the reduction in order by two using only quadratures.

Theorem 2.61. *Let $\Delta(x, u^{(n)}) = 0$ be an n-th order ordinary differential equation invariant under a two-parameter symmetry group G. Then there is an $(n - 2)$-nd order equation $\hat{\Delta}(z, v^{(n-2)}) = 0$ with the property that the general solution to Δ can be found by a pair of quadratures from the general solution to $\hat{\Delta}$.*

Proof. According to Exercise 1.21, we can find a basis $\{\mathbf{v}, \mathbf{w}\}$ for any two-dimensional Lie algebra $\mathfrak{g}$ with the property

$$[\mathbf{v}, \mathbf{w}] = k\mathbf{v} \tag{2.107}$$

for some constant k. (In fact, k can be taken to be 0 if $\mathfrak{g}$ is abelian and 1 in all other cases.) The one-parameter subgroup H generated by $\mathbf{v}$ is then a normal subgroup of G, with one-parameter quotient group G/H. To effect the reduction of Δ, we begin by determining first order differential invariants $y = \eta(x, u)$, $w = \zeta(x, u, u_x)$ for H using our earlier methods. By Proposition 2.56, our n-th order equation is equivalent to an $(n - 1)$st order equation $\Delta(y, w^{(n-1)}) = 0$; moreover, once we know the solution $w = h(y)$ of this latter equation, we can reconstruct the solution to Δ by solving the corresponding first order equation (2.94) using a single quadrature. Since H is normal, the reduced equation $\tilde{\Delta}$ is invariant under the action of G/H on the variables (y, w), and hence we can employ our earlier methods for one-parameter symmetry groups to reduce the order yet again by one. On an infinitesimal level, suppose $(x, y, w) = (x, \eta(x, u), \zeta(x, u, u_x))$ form local coordinates on some subset of $M^{(1)}$. (If x happens to be one of the invariants, we can replace it by u or some combination $\gamma(x, u)$.) As in (2.105), normality, as expressed by (2.107), implies that the vector field $\mathbf{w}$ has first prolongation

$$\mathrm{pr}^{(1)} \mathbf{w} = \alpha(x, y, w)\partial_x + \beta(y, w)\partial_y + \psi(y, w)\partial_w$$

in terms of these coordinates, and hence reduces to a vector field

$$\tilde{\mathbf{w}} = \beta(y, w)\partial_y + \psi(y, w)\partial_w$$

on the space $\tilde{M}$, generating the quotient group action of G/H. Theorem 2.60 assures us that $\tilde{\mathbf{w}}$ remains a symmetry of the preliminary reduced system $\tilde{\Delta}$, and hence we can reduce $\tilde{\Delta}$ in order by one using either the method of differential invariants or that of changing coordinates to straighten out $\tilde{\mathbf{w}}$, leading to a differential equation $\hat{\Delta}(z, v^{(n-2)}) = 0$ of order $n - 2$. This completes the reduction procedure, and the proof of the theorem. $\square$

Example 2.62. Consider a second order differential equation of the form

$$x^2 u_{xx} = H(xu_x - u), \tag{2.108}$$

where H is a given function. This equation admits the two-parameter symmetry group

$$(x, u) \mapsto (\lambda x, u + \varepsilon x), \qquad \varepsilon \in \mathbb{R}, \quad \lambda \in \mathbb{R}^+,$$

with infinitesimal generators $\mathbf{v} = x\partial_u$ and $\mathbf{w} = x\partial_x$. Note that $[\mathbf{v}, \mathbf{w}] = -\mathbf{v}$, so the generators are in the correct order to take advantage of Theorem 2.61. According to the basic procedure, we need to first determine the invariants of $\mathbf{v}$, which are x and $w = xu_x - u$, in terms of which (2.108) reduces to the first order equation

$$x\frac{dw}{dx} = H(w).$$

This latter equation is separable, with implicit solution

$$\int \frac{dw}{H(w)} = \log x + c,$$

reflecting the fact that it remains invariant under the reduced group $(x, w) \mapsto (\lambda x, w)$ determined by $\tilde{\mathbf{w}} = x\partial_x$, as guaranteed by the method. From this solution, rewritten in explicit form $w = h(x)$, we reconstruct the general solution to (2.108) by solving the linear equation.

$$xu_x - u = h(x).$$

The integrating factor is $1/x^2$, as can be determined directly from the form of the underlying symmetry group generated by $\mathbf{v}$, and hence we find

$$u = x\left(\int x^{-2}h(x)\,dx + k\right)$$

is the general solution to (2.108).

It is instructive to see what would have happened if, unheeded by the general procedure, we had tried to integrate (2.108) by considering the two one-parameter groups in the reverse order. In this case, the invariants for $\mathbf{w}$ are $y = u$, $z = xu_x$, whence $z_y = xu_x^{-1}u_{xx} + 1$, and the equation reduces to

$$z\left(\frac{dz}{dy} - 1\right) = H(z - y). \tag{2.109}$$

However, at this stage there is no symmetry property of (2.109) which reflects the symmetry of (2.108) under the group generated by $\mathbf{v}$. This shows that it *is* important to do our reduction procedure in the right order, otherwise we may not end up with the solution. Reversing the procedure, we are left with the intriguing possibility of being able to integrate an $(n-1)$-st order equation by first changing it into an n-th order equation with several symmetries, whose order can then be reduced substantially. For instance, we can solve (2.109) by first substituting $y = u$, $z = xu_x$, which changes it to (2.108), and then integrating the latter equation. This point will be investigated in greater detail in the exercises at the end of the chapter.

Solvable Groups

Turning to yet higher-dimensional symmetry groups, we find, as evidenced by the example of the projective group, that for $r \geqslant 3$, invariance of an n-th order equation under an r-parameter group will not in general imply that we can find the general solution by quadratures from the solution of the corresponding reduced $(n - r)$-th order equation. The problem is that there is not in general a sufficient supply of normal subgroups to ensure the continued applicability of Theorem 2.60 and the reduction procedure for one-parameter groups at each stage. This motivates the following definition of those groups which can be used to fully reduce or "solve" an equation to the extent promised by their dimensionality.

Definition 2.63. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then G is *solvable* if there exists a chain of Lie subgroups

$$\{e\} = G^{(0)} \subset G^{(1)} \subset G^{(2)} \subset \cdots \subset G^{(r-1)} \subset G^{(r)} = G$$

such that for each $k = 1, \ldots, r$, $G^{(k)}$ is a k-dimensional subgroup of G and $G^{(k-1)}$ is a normal subgroup of $G^{(k)}$. Equivalently, there exists a chain of subalgebras

$$\{0\} = \mathfrak{g}^{(0)} \subset \mathfrak{g}^{(1)} \subset \mathfrak{g}^{(2)} \subset \cdots \subset \mathfrak{g}^{(r-1)} \subset \mathfrak{g}^{(r)} = \mathfrak{g}, \tag{2.110}$$

such that for each k, $\dim \mathfrak{g}^{(k)} = k$ and $\mathfrak{g}^{(k-1)}$ is a normal subalgebra of $\mathfrak{g}^{(k)}$:

$$[\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k)}] \subset \mathfrak{g}^{(k-1)}.$$

The requirement for solvability is equivalent to the existence of a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ of $\mathfrak{g}$ such that

$$[\mathbf{v}_i, \mathbf{v}_j] = \sum_{k=1}^{j-1} c_{ij}^k \mathbf{v}_k \quad \text{whenever} \quad i < j.$$

Note that any abelian Lie algebra, i.e. one for which the Lie bracket is always zero, is trivially solvable. Any two-dimensional Lie algebra is solvable, since using the basis (2.107), we can set $\mathfrak{g}^{(1)}$ to be the one-dimensional subalgebra generated by $\mathbf{v}$ to produce the chain $\{0\} = \mathfrak{g}^{(0)} \subset \mathfrak{g}^{(1)} \subset \mathfrak{g}^{(2)} = \mathfrak{g}$. The simplest example of a nonsolvable Lie algebra is the three-dimensional algebra $\mathfrak{sl}(2)$.

Theorem 2.64. *Let $\Delta(x, u^{(n)}) = 0$ be an n-th order ordinary differential equation. If Δ admits a solvable r-parameter group of symmetries G such that for $1 \leqslant k \leqslant r$ the orbits of $\mathrm{pr}^{(k)} G^{(k)}$ have dimension k, then the general solution of Δ can be found by quadratures from the general solution of an $(n - r)$-th order differential equation $\tilde{\Delta}(y, w^{(n-r)}) = 0$. In particular, if Δ admits an n-parameter solvable group of symmetries, then (subject to the above technical restrictions) the general solution to Δ can be found by quadratures alone.*

PROOF. The proof proceeds by induction along the chain of subalgebras (2.110) guaranteed by the solvability of G. At the k-th stage, we have used the invariance of Δ under the k-dimensional subalgebra $\mathfrak{g}^{(k)}$ to reduce it to an $(n-k)$-th order equation

$$\tilde{\Delta}^{(k)}(y, w^{(n-k)}) = 0,$$

in which $y, w, dw/dy, \ldots, d^{n-k}w/dy^{n-k}$ form a complete set of functionally independent differential invariants for the n-th prolongation $\mathrm{pr}^{(n)}\, G^{(k)}$; in particular, $y = \eta(x, u^{(k)})$, $w = \zeta(x, u^{(k)})$ form a complete set of invariants of the k-th prolongation of $G^{(k)}$. We also can reconstruct the general solution $u = f(x)$ from the general solution $w = h(y)$ of $\tilde{\Delta}^{(k)}$ by a series of quadratures.

To pass to the $(k+1)$-st stage, consider a generator $\mathbf{v}_{k+1}$ of $\mathfrak{g}^{(k+1)}$ which does not lie in $\mathfrak{g}^{(k)}$. Since $\mathfrak{g}^{(k)}$ is a normal subalgebra of $\mathfrak{g}^{(k+1)}$, (2.105) says that $\mathrm{pr}^{(k)}\, \mathbf{v}_{k+1}$ takes the form

$$\mathrm{pr}^{(k)}\, \mathbf{v}_{k+1} = \mathrm{pr}^{(k-2)}\, \mathbf{v}_{k+1} + \alpha(y, w)\frac{\partial}{\partial y} + \psi(y, w)\frac{\partial}{\partial w}$$

$$\equiv \mathrm{pr}^{(k-2)}\, \mathbf{v}_{k+1} + \tilde{\mathbf{v}}_{k+1},$$

in which $\mathrm{pr}^{(k-2)}\, \mathbf{v}_{k+1}$ depends on the noninvariant coordinates $x, u, \ldots, u_{k-2}$ needed to complete y, w to a coordinate system on $M^{(k)}$.

Theorem 2.60 says that the original equaton Δ is invariant under all of $\mathfrak{g}^{(k+1)}$ if and only if the reduced equation $\tilde{\Delta}^{(k)}$ is invariant under the reduced vector field $\tilde{\mathbf{v}}_{k+1}$, which allows us to implement our reduction procedure for $\tilde{\Delta}^{(k)}$ using the vector field $\tilde{\mathbf{v}}_{k+1}$. Namely, we set

$$\hat{y} = \hat{\eta}(y, w), \qquad \hat{w} = \hat{\zeta}(y, w, w_y)$$

to be independent invariants of the first prolongation $\mathrm{pr}^{(1)}\, \tilde{\mathbf{v}}_{k+1}$. Then $\hat{y}, \hat{w}, d\hat{w}/d\hat{y}, \ldots, d^{n-k-1}\hat{w}/d\hat{y}^{n-k-1}$ form a complete set of invariants for the $(n-k)$-th prolongation $\mathrm{pr}^{(n-k)}\, \tilde{\mathbf{v}}_{k+1}$. Since $\tilde{\Delta}^{(k)}$ determines an invariant subvariety of this group, there is an equivalent equation

$$\hat{\Delta}^{(k+1)}(\hat{y}, \hat{w}^{(n-k-1)}) = 0$$

depending only on the invariants of $\mathrm{pr}^{(n-k)}\, \tilde{\mathbf{v}}_{k+1}$. Moreover, to reconstruct the solutions to $\tilde{\Delta}^{(k)}$ from those, $\hat{w} = \hat{h}(\hat{y})$, to $\hat{\Delta}^{(k+1)}$, we need only solve the first order equation

$$\hat{\zeta}(y, w, w_y) = \hat{h}[\hat{\eta}(y, w)].$$

This is invariant under the one-parameter group generated by $\tilde{\mathbf{v}}_{k+1}$, and hence can be integrated by quadrature. This completes the induction step, and thus proves the theorem. □

Example 2.65. Consider the third order equation

$$u_x^5 u_{xxx} = 3u_x^4 u_{xx}^2 + u_{xx}^3. \qquad (2.111)$$

There is a three-parameter group of symmetries, generated by the vector fields

$$\mathbf{v}_1 = \partial_u, \qquad \mathbf{v}_2 = \partial_x, \qquad \mathbf{v}_3 = u\partial_x,$$

which is solvable since

$$[\mathbf{v}_1, \mathbf{v}_2] = 0, \qquad [\mathbf{v}_1, \mathbf{v}_3] = \mathbf{v}_2, \qquad [\mathbf{v}_2, \mathbf{v}_3] = 0.$$

Thus (2.111) can be solved by quadratures. We proceed to implement the reduction procedure given in the proof of Theorem 2.64. First, for $\mathfrak{g}^{(1)}$ generated by $\mathbf{v}_1$, we have invariants x, $v = u_x$, in terms of which (2.111) reduces to

$$v^5 v_{xx} = 3v^4 v_x^2 + v_x^3. \tag{2.112}$$

The second vector field $\mathbf{v}_2$ maintains its form $\tilde{\mathbf{v}}_2 = \partial_x$ when written using the invariants of $\mathfrak{g}^{(1)}$, so to reduce (2.112) for $\mathfrak{g}^{(2)} = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ we need the invariants

$$y = v, \qquad w = v_x, \qquad w_y = v_{xx}/v_x,$$

of $\operatorname{pr}^{(2)} \tilde{\mathbf{v}}_2$. In terms of these, (2.112) reduces to the first order equation

$$y^5 w_y = 3y^4 w + w^2. \tag{2.113}$$

This last Riccati equaton should retain one further symmetry corresponding to the vector field $\mathbf{v}_3$. Indeed, in terms of x, $y = u_x$, $w = u_{xx}$,

$$\operatorname{pr}^{(2)} \mathbf{v}_3 = u\partial_x - y^2\partial_y - 3yw\partial_w,$$

and the reduced vector field

$$\tilde{\mathbf{v}}_3 = -y^2\partial_y - 3yw\partial_w$$

is a symmetry of (2.113). We can thus integrate (2.113) by setting $t = -1/y$, $z = w/y^3$ (in terms of which $\tilde{\mathbf{v}}_3 = -\partial_t$), so (2.113) becomes

$$\frac{dz}{dt} = z^2.$$

Thus $z = 1/(c - t)$, or, in terms of the invariants of $\tilde{\mathbf{v}}_2$,

$$w = \frac{y^4}{cy + 1}.$$

To find v, we need to solve the autonomous equation

$$\frac{dv}{dx} = \frac{v^4}{cv + 1},$$

(autonomy being guaranteed by invariance under $\mathbf{v}_2$). We find the implicit solution

$$6(x - \tilde{c})v^3 + 3cv + 2 = 6(x - \tilde{c})u_x^3 + 3cu_x + 2 = 0.$$