hence

$$\mathscr{J}(\mathscr{D}, \mathscr{E}; P, Q, R) = 0. \tag{7.31}$$

From this it is easy to check that

$$\mathscr{J}(a\mathscr{D} + b\mathscr{E}, a\mathscr{D} + b\mathscr{E}; P, Q, R) = 0,$$

for any $a, b \in \mathbb{R}$. □

**Corollary 7.21.** *Let $\mathscr{D}$ and $\mathscr{E}$ be Hamiltonian differential operators. Then $\mathscr{D}$, $\mathscr{E}$ form a Hamiltonian pair if and only if*

$$\mathrm{pr}\, \mathbf{v}_{\mathscr{D}\theta}(\Theta_{\mathscr{E}}) + \mathrm{pr}\, \mathbf{v}_{\mathscr{E}\theta}(\Theta_{\mathscr{D}}) = 0, \tag{7.32}$$

*where*

$$\Theta_{\mathscr{D}} = \tfrac{1}{2} \int \{\theta \wedge \mathscr{D}\theta\}\, dx, \qquad \Theta_{\mathscr{E}} = \tfrac{1}{2} \int \{\theta \wedge \mathscr{E}\theta\}\, dx$$

*are the functional bi-vectors representing the respective Poisson brackets.*

Indeed, we have

$$\mathscr{J}(\mathscr{D}, \mathscr{D}; P, Q, R) = \langle \mathrm{pr}\, \mathbf{v}_{\mathscr{D}\theta}(\Theta_{\mathscr{D}}); P, Q, R \rangle,$$

so evaluating (7.32) at $P, Q, R$ reproduces (7.31). □

**Example 7.22.** Consider the Hamiltonian operators $\mathscr{D}$, $\mathscr{E}$ connected with the Korteweg–de Vries equation. We have

$$\mathrm{pr}\, \mathbf{v}_{\mathscr{E}\theta}(\Theta_{\mathscr{D}}) = \mathrm{pr}\, \mathbf{v}_{\mathscr{E}\theta} \int \tfrac{1}{2}\{\theta \wedge \theta_x\}\, dx = 0$$

trivially, while,

$$\mathrm{pr}\, \mathbf{v}_{\mathscr{D}\theta}(\Theta_{\mathscr{E}}) = \mathrm{pr}\, \mathbf{v}_{\mathscr{D}\theta} \int \{\tfrac{1}{2}\theta \wedge \theta_{xxx} + \tfrac{1}{3}u\theta \wedge \theta_x\}\, dx = \int \{\tfrac{1}{3}\theta_x \wedge \theta \wedge \theta_x\}\, dx$$
$$= 0$$

by the properties of the wedge product. Thus $\mathscr{D}$ and $\mathscr{E}$ form a Hamiltonian pair.

Incidentally, when we discuss Hamiltonian pairs, we are always excluding the trivial case when one operator is a constant multiple of the other. In the case of systems (as opposed to scalar equations) we must impose an additional constraint on one of the operators, $\mathscr{D}$, owing to the appearance of its inverse in the form of the recursion operator.

**Definition 7.23.** A differential operator $\mathscr{D}\colon \mathscr{A}^r \to \mathscr{A}^s$ is *degenerate* if there is a nonzero differential operator $\tilde{\mathscr{D}}\colon \mathscr{A}^s \to \mathscr{A}$ such that $\tilde{\mathscr{D}} \cdot \mathscr{D} \equiv 0$.

For example, the matrix operator

$$\mathcal{D} = \begin{pmatrix} D_x^3 & -D_x^2 \\ D_x^2 & -D_x \end{pmatrix}$$

is degenerate (and Hamiltonian), since if $\tilde{\mathcal{D}} = (1, -D_x)$, then $\tilde{\mathcal{D}} \cdot \mathcal{D} \equiv 0$. It is not difficult to see that degeneracy is strictly a matrix phenomenon; any nonzero scalar operator $\mathcal{D}: \mathcal{A} \to \mathcal{A}$ is automatically nondegenerate. (A useful criterion for nondegeneracy is given in Exercise 7.14.)

We are now in a position to state the main theorem on bi-Hamiltonian systems.

**Theorem 7.24.** *Let*

$$u_t = K_1[u] = \mathcal{D}\delta\mathcal{H}_1 = \mathcal{E}\delta\mathcal{H}_0$$

*be a bi-Hamiltonian system of evolution equations. Assume that the operator $\mathcal{D}$ of the Hamiltonian pair is nondegenerate. Let $\mathcal{R} = \mathcal{E} \cdot \mathcal{D}^{-1}$ be the corresponding recursion operator, and let $K_0 = \mathcal{D}\delta\mathcal{H}_0$. Assume that for each $n = 1, 2, \ldots$ we can recursively define*

$$K_n = \mathcal{R}K_{n-1}, \qquad n \geqslant 1,$$

*meaning that for each $n$, $K_{n-1}$ lies in the image of $\mathcal{D}$. Then there exists a sequence of functionals $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots$ such that*

(i) *for each $n \geqslant 1$, the evolution equation*

$$u_t = K_n[u] = \mathcal{D}\delta\mathcal{H}_n = \mathcal{E}\delta\mathcal{H}_{n-1} \tag{7.33}$$

*is a bi-Hamiltonian system;*

(ii) *the corresponding evolutionary vector fields $\mathbf{v}_n = \mathbf{v}_{K_n}$ all mutually commute:*

$$[\mathbf{v}_n, \mathbf{v}_m] = 0, \qquad n, m \geqslant 0;$$

(iii) *the Hamiltonian functionals $\mathcal{H}_n$ are all in involution with respect to either Poisson bracket:*

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{D}} = 0 = \{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{E}}, \qquad n, m \geqslant 0, \tag{7.34}$$

*and hence provide an infinite collection of conservation laws for each of the bi-Hamiltonian systems* (7.33).

Before proving the theorem, some remarks are in order. Although as it stands, the result is quite powerful, there is one annoying defect; namely the fact that we must assume at each stage that we can apply the recursion operator to $K_{n-1}$ to produce $K_n$, i.e. prove that $K_{n-1}$ lies in the image of $\mathcal{D}$. In most examples known to date, this always seems to be the case, but it would be nice to have a general proof of this. (The argument given in Theorem 5.31 seems to be fairly specific to the Korteweg–de Vries equation.) At the moment, though, except in some special instances, this seems to be the best that we can do. A second problem is that of determining whether all the Hamiltonian functionals $\mathscr{H}_n$ are independent; in practice this is usually easy to see from the leading terms of the corresponding evolution equations.

The proof itself rests on the following technical lemma:

**Lemma 7.25.** *Suppose $\mathscr{D}$, $\mathscr{E}$ form a Hamiltonian pair with $\mathscr{D}$ nondegenerate. Let $P, Q, R, \in \mathscr{A}^q$ satisfy*

$$\mathscr{E}P = \mathscr{D}Q, \qquad \mathscr{E}Q = \mathscr{D}R. \tag{7.35}$$

*If $P = \delta\mathscr{P}$, $Q = \delta\mathscr{Q}$ are variational derivatives of functionals $\mathscr{P}, \mathscr{Q} \in \mathscr{F}$, then so is $R = \delta\mathscr{R}$ for some $\mathscr{R} \in \mathscr{F}$.*

Before proving the lemma, let us see how it is used to prove the theorem. For each $n \geq 0$, we let $K_n = \mathscr{D}Q_n$ where, by assumption, $Q_n \in \mathscr{A}^q$ is a well-defined $q$-tuple of differential functions. By the lemma, if $Q_{n-1} = \delta\mathscr{H}_{n-1}$, $Q_n = \delta\mathscr{H}_n$ are variational derivatives, so is $Q_{n+1} = \delta\mathscr{H}_{n+1}$ for some $\mathscr{H}_{n+1} \in \mathscr{F}$. Since we already know $Q_0 = \delta\mathscr{H}_0$, $Q_1 = \delta\mathscr{H}_1$ are of this form, the existence of the functionals $\mathscr{H}_n$, $n \geq 0$ follows by an easy induction. This proves part (i).

Part (ii) follows from part (iii) using (7.22), so we concentrate on (iii). According to (7.4)

$$\{\mathscr{H}_n, \mathscr{H}_m\}_{\mathscr{D}} = \operatorname{pr} \mathbf{v}_m(\mathscr{H}_n),$$

and

$$\{\mathscr{H}_n, \mathscr{H}_m\}_{\mathscr{E}} = \operatorname{pr} \mathbf{v}_{m+1}(\mathscr{H}_n),$$

so

$$\{\mathscr{H}_n, \mathscr{H}_m\}_{\mathscr{D}} = \{\mathscr{H}_n, \mathscr{H}_{m-1}\}_{\mathscr{E}}.$$

We now use the skew-symmetry of the Poisson brackets to work our way down. If $n < m$, then

$$\{\mathscr{H}_n, \mathscr{H}_m\}_{\mathscr{D}} = \{\mathscr{H}_n, \mathscr{H}_{m-1}\}_{\mathscr{E}} = \{\mathscr{H}_{n+1}, \mathscr{H}_{m-1}\}_{\mathscr{D}} = \cdots = \{\mathscr{H}_k, \mathscr{H}_k\} = 0,$$

where $k$ is the integer part of $(m - n)/2$ and the final bracket is the $\mathscr{D}$-Poisson bracket if $m - n$ is even, or the $\mathscr{E}$-Poisson bracket if $m - n$ is odd. This proves (7.34) and completes the proof of the theorem. □

Proof of Lemma 7.25. Referring back to the derivation of (7.11) from the Jacobi identity (7.3) we recall that the large number of cancellations resulted from the fact that, at that stage, the $q$-tuples $P, Q, R$ were assumed to be variational derivatives of functionals and, consequently, their Fréchet derivatives were all self-adjoint operators. If we were to drop this initial assumption and carry through the computation for arbitrary $q$-tuples $P, Q, R \in \mathscr{A}^q$, we would find an identity of the form

$$\mathscr{K}(\mathscr{D}, \mathscr{D}; P, Q, R) = \mathscr{L}(\mathscr{D}, \mathscr{D}; P, Q, R) + \mathscr{J}(\mathscr{D}, \mathscr{D}; P, Q, R),$$

where

$$\mathscr{K}(\mathscr{D}, \mathscr{E}; P, Q, R) =$$

$$\tfrac{1}{2}\left\{\operatorname{pr}\mathbf{v}_{\mathscr{D}R}\int P\cdot\mathscr{E}Q\,dx + \operatorname{pr}\mathbf{v}_{\mathscr{D}P}\int Q\cdot\mathscr{E}R\,dx + \operatorname{pr}\mathbf{v}_{\mathscr{D}Q}\int R\cdot\mathscr{E}P\,dx\right.$$

$$\left. + \operatorname{pr}\mathbf{v}_{\mathscr{E}R}\int P\cdot\mathscr{D}Q\,dx + \operatorname{pr}\mathbf{v}_{\mathscr{E}P}\int Q\cdot\mathscr{D}R\,dx + \operatorname{pr}\mathbf{v}_{\mathscr{E}Q}\int R\cdot\mathscr{D}P\,dx\right\}, \qquad (7.36)$$

$\mathscr{J}(\mathscr{D}, \mathscr{D}; P, Q, R)$ is as in (7.30) and $\mathscr{L}$ is the quadratic version of the bilinear expression

$$\mathscr{L}(\mathscr{D}, \mathscr{E}; P, Q, R) =$$

$$-\tfrac{1}{2}\int\{\mathscr{D}P\cdot(\mathrm{D}_Q - \mathrm{D}_Q^*)\mathscr{E}R + \mathscr{D}Q\cdot(\mathrm{D}_R - \mathrm{D}_R^*)\mathscr{E}P + \mathscr{D}R\cdot(\mathrm{D}_P - \mathrm{D}_P^*)\mathscr{E}Q$$

$$+ \mathscr{E}P\cdot(\mathrm{D}_Q - \mathrm{D}_Q^*)\mathscr{D}R + \mathscr{E}Q\cdot(\mathrm{D}_R - \mathrm{D}_R^*)\mathscr{D}P + \mathscr{E}R\cdot(\mathrm{D}_P - \mathrm{D}_P^*)\mathscr{D}Q\}\,dx.$$

The above identity has a bilinear counterpart

$$\mathscr{K}(\mathscr{D}, \mathscr{E}; P, Q, R) = \mathscr{L}(\mathscr{D}, \mathscr{E}; P, Q, R) + \mathscr{J}(\mathscr{D}, \mathscr{E}; P, Q, R), \qquad (7.37)$$

which holds for arbitrary $P, Q, R \in \mathscr{A}$ and arbitrary skew-adjoint operators $\mathscr{D}, \mathscr{E}$. In particular, if $\mathscr{D}, \mathscr{E}$ form a Hamiltonian pair, then the $\mathscr{J}$ term in (7.37) vanishes.

Now replace $P$ by $S = \delta\mathscr{S}$, $R$ by $T = \delta\mathscr{T}$ in (7.37) and assume that $P$, $Q$, $R$ are related by (7.35). Since $Q$, $S$ and $T$ are all variational derivatives of functionals, Theorem 5.92, (7.37) and (7.31) imply that

$$0 = \mathscr{L}(\mathscr{D}, \mathscr{E}; Q, S, T) = \mathscr{K}(\mathscr{D}, \mathscr{E}; Q, S, T). \qquad (7.38)$$

Moreover using (7.35), (7.36) and the skew-adjointness of $\mathscr{D}$ and $\mathscr{E}$ we find (upon rearranging terms)

$$\mathscr{K}(\mathscr{D}, \mathscr{E}; Q, S, T) = \tfrac{1}{2}\{\mathscr{K}(\mathscr{E}, \mathscr{E}; P, S, T) + \mathscr{K}(\mathscr{D}, \mathscr{D}; R, S, T)\}.$$

Now $\mathscr{E}$ is Hamiltonian and $P$, $S$, $T$ are all variational derivatives of functionals, so $\mathscr{K}(\mathscr{E}, \mathscr{E}; P, S, T)$ is just the Jacobi identity for $\mathscr{E}$ and hence vanishes. We still don't know that $R$ is the variational derivative of some functional, so we cannot make the same claim for $\mathscr{K}(\mathscr{D}, \mathscr{D}; R, S, T)$. However, by (7.37), (7.38) and Theorem 5.92 (for $S$ and $T$) we find

$$0 = \mathscr{K}(\mathscr{D}, \mathscr{D}; R, S, T) = \mathscr{L}(\mathscr{D}, \mathscr{D}; R, S, T)$$

$$= \int \mathscr{D}T\cdot(\mathrm{D}_R - \mathrm{D}_R^*)\mathscr{D}S\,dx = -\int T\cdot\mathscr{D}(\mathrm{D}_R - \mathrm{D}_R^*)\mathscr{D}S\,dx.$$

Note that the operator $\mathscr{D}(\mathrm{D}_R - \mathrm{D}_R^*)\mathscr{D}$ is skew-adjoint. Since this identity holds for arbitrary variational derivatives $S = \delta\mathscr{S}$, $T = \delta\mathscr{T}$, Proposition 5.88

and the substitution principle of Exercise 5.42 imply that

$$\mathscr{D}\cdot(\mathsf{D}_R - \mathsf{D}_R^*)\cdot\mathscr{D} = 0.$$

Finally, the nondegeneracy hypothesis on $\mathscr{D}$ allows us to conclude that

$$\mathscr{D}\cdot(\mathsf{D}_R - \mathsf{D}_R^*) = 0.$$

Taking adjoints, we have

$$-(\mathsf{D}_R^* - \mathsf{D}_R)\cdot\mathscr{D} = 0.$$

One further application of the nondegeneracy of $\mathscr{D}$ allows us to conclude that $R$ satisfies the Helmholtz conditions $\mathsf{D}_R^* = \mathsf{D}_R$, and hence, by Theorem 5.92, is the variational derivative of some functional. □

## Recursion Operators

We have seen that given a bi-Hamiltonian system, the operator $\mathscr{R} = \mathscr{E}\cdot\mathscr{D}^{-1}$, when applied successively to the initial equation $K_0 = \mathscr{D}\delta\mathscr{H}_0$, produces an infinite sequence of generalized symmetries of the original system (subject to the technical assumptions contained in Theorem 7.24). It is still not clear that $\mathscr{R}$ is a true recursion operator for the system, in the sense that whenever $\mathbf{v}_Q$ is a generalized symmetry, so is $\mathbf{v}_{\mathscr{R}Q}$. So far we only know it for symmetries with $Q = K_n$ for some $n$. In order to establish this more general result, we need a formula for the infinitesimal change of the Hamiltonian operator itself under a Hamiltonian flow.

**Lemma 7.26.** *Let $u_t = K = \mathscr{D}\delta\mathscr{H}$ be a Hamiltonian system of evolution equations with corresponding vector field $\mathbf{v}_K = \hat{\mathbf{v}}_{\mathscr{H}}$. Then*

$$\operatorname{pr}\hat{\mathbf{v}}_{\mathscr{H}}(\mathscr{D}) = \mathsf{D}_K\cdot\mathscr{D} + \mathscr{D}\cdot\mathsf{D}_K^*,$$

Proof. Let $L = \delta\mathscr{H}$, so $K = \mathscr{D}L$. Let $P = \delta\mathscr{P}$, $Q = \delta\mathscr{Q}$ be arbitrary variational derivatives. By the Jacobi identity for $\mathscr{D}$, in the form (7.11), and (7.7), (5.33),

$$\begin{aligned}\int [P\cdot\operatorname{pr}\hat{\mathbf{v}}_{\mathscr{H}}(\mathscr{D})Q]\,dx &= \int [P\cdot\operatorname{pr}\hat{\mathbf{v}}_{\mathscr{Q}}(\mathscr{D})L - Q\cdot\operatorname{pr}\hat{\mathbf{v}}_{\mathscr{P}}(\mathscr{D})L]\,dx \\ &= \int \{P\cdot[\operatorname{pr}\hat{\mathbf{v}}_{\mathscr{Q}}(K) - \mathscr{D}\operatorname{pr}\hat{\mathbf{v}}_{\mathscr{Q}}(L)] \\ &\qquad - Q\cdot[\operatorname{pr}\hat{\mathbf{v}}_{\mathscr{P}}(K) - \mathscr{D}\operatorname{pr}\hat{\mathbf{v}}_{\mathscr{P}}(L)]\}\,dx\end{aligned}$$

$$= \int \{P \cdot [\mathsf{D}_K(\mathscr{D}Q) - \mathscr{D}\mathsf{D}_L(\mathscr{D}Q)]$$

$$- Q \cdot [\mathsf{D}_K(\mathscr{D}P) - \mathscr{D}\mathsf{D}_L(\mathscr{D}P)]\} \, dx$$

$$= \int [P \cdot \mathsf{D}_K(\mathscr{D}Q) - Q \cdot \mathsf{D}_K(\mathscr{D}P)] \, dx$$

$$= \int [P \cdot (\mathsf{D}_K\mathscr{D} + \mathscr{D}\mathsf{D}_K^*)Q] \, dx.$$

The next to last equality follows from the fact that $\mathscr{D}\mathsf{D}_L\mathscr{D}$ is self-adjoint since $\mathscr{D}$ is skew-adjoint, while $\mathsf{D}_L$ is self-adjoint since $L = \delta\mathscr{H}$ is a variational derivative. The result now follows by the substitution principle. □

**Theorem 7.27.** *Let $u_t = K = \mathscr{D}\delta\mathscr{H}_1 = \mathscr{E}\delta\mathscr{H}_0$ be a bi-Hamiltonian system of evolution equations. Then the operator $\mathscr{R} = \mathscr{E} \cdot \mathscr{D}^{-1}$ is a recursion operator for the system.*

Proof. We must verify the infinitesimal criterion (5.43). Using the previous lemma, we have

$$\mathscr{R}_t = \operatorname{pr} \mathbf{v}_K(\mathscr{R}) = \operatorname{pr} \mathbf{v}_K(\mathscr{E}) \cdot \mathscr{D}^{-1} - \mathscr{E} \cdot \mathscr{D}^{-1} \cdot \operatorname{pr} \mathbf{v}_K(\mathscr{D}) \cdot \mathscr{D}^{-1}$$

$$= (\mathsf{D}_K\mathscr{E} + \mathscr{E}\mathsf{D}_K^*) \cdot \mathscr{D}^{-1} - \mathscr{E} \cdot \mathscr{D}^{-1}(\mathsf{D}_K\mathscr{D} + \mathscr{D}\mathsf{D}_K^*)\mathscr{D}^{-1}$$

$$= \mathsf{D}_K\mathscr{R} - \mathscr{R}\mathsf{D}_K.$$

This verifies (5.42). □

Notice that in this theorem, we did not require that $(\mathscr{D}, \mathscr{E})$ form a Hamiltonian pair—only that each individual operator be Hamiltonian. Thus the recursion operator remains valid in more general situations. However, without the assumption that $(\mathscr{D}, \mathscr{E})$ form a Hamiltonian pair, it is unclear whether the symmetries $\mathbf{v}_n$, $n = 0, 1, 2, \ldots$, determined by recursion, are Hamiltonian or not.

**Example 7.28.** The *Boussinesq equation*, which we take in the form

$$u_{tt} = \tfrac{1}{3}u_{xxxx} + \tfrac{4}{3}(u^2)_{xx},$$

arises in a model for uni-directional propagation of long waves in shallow water (despite the fact that it admits waves travelling in both directions!). It can be converted into an equivalent evolutionary system

$$u_t = v_x, \qquad v_t = \tfrac{1}{3}u_{xxx} + \tfrac{8}{3}uu_x, \tag{7.39}$$

which turns out to be bi-Hamiltonian. The first Hamiltonian formulation is easy to discern. We take

$$\mathscr{D} = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix}$$

as the Hamiltonian operator (which trivially satisfies the Jacobi identity since it has constant coefficients) and Hamiltonian functional

$$\mathscr{H}_1[u, v] = \int \left(-\tfrac{1}{6}u_x^2 + \tfrac{4}{9}u^3 + \tfrac{1}{2}v^2\right) dx.$$

The second Hamiltonian structure is not so obvious. The Hamiltonian functional is

$$\mathscr{H}_0[u, v] = \int \tfrac{1}{2}v \, dx,$$

the Hamiltonian operator being

$$\mathscr{E} = \begin{pmatrix} D_x^3 + 2uD_x + u_x & 3vD_x + 2v_x \\ 3vD_x + v_x & \tfrac{1}{3}D_x^5 + \tfrac{5}{3}(uD_x^3 + D_x^3 \cdot u) - (u_{xx}D_x + D_x \cdot u_{xx}) + \tfrac{16}{3}uD_x \cdot u \end{pmatrix}.$$

Even the proof that $\mathscr{E}$ is Hamiltonian is a rather laborious computation. The associated bi-vector is

$$\Theta_{\mathscr{E}} = \tfrac{1}{2}\int \{\theta \wedge \theta_{xxx} + 2u\theta \wedge \theta_x + 2v\theta \wedge \zeta_x - 4v\theta_x \wedge \zeta$$
$$+ \tfrac{1}{3}\zeta \wedge \zeta_{xxxxx} + \tfrac{4}{3}u\zeta \wedge \zeta_{xxx} - 2u\zeta_x \wedge \zeta_{xx} + \tfrac{16}{3}u^2\zeta \wedge \zeta_x\} \, dx,$$

where $\boldsymbol{\theta} = (\theta, \zeta)$, and $\theta$ and $\zeta$ are the basic uni-vectors corresponding to $u$ and $v$ respectively. Evaluating (7.18) (for $\mathscr{E}$), we use the fact that

$$\operatorname{pr} \mathbf{v}_{\mathscr{E}\boldsymbol{\theta}}(u) = \theta_{xxx} + 2u\theta_x + u_x\theta + 3v\zeta_x + 2v_x\zeta,$$
$$\operatorname{pr} \mathbf{v}_{\mathscr{E}\boldsymbol{\theta}}(v) = 3v\theta_x + v_x\theta + \tfrac{1}{3}\zeta_{xxxxx} + \tfrac{10}{3}u\zeta_{xxx} + 5u_x\zeta_{xx} + (3u_{xx} + \tfrac{16}{3}u^2)\zeta_x$$
$$+ (\tfrac{2}{3}u_{xxx} + \tfrac{16}{3}uu_x)\zeta,$$

and a lot of integration by parts—the reader may wish to try his or her skill at this! The proof that $\mathscr{D}$, $\mathscr{E}$ form a Hamiltonian pair is easier; since $\mathscr{D}$ has constant coefficients, $\operatorname{pr} \mathbf{v}_{\mathscr{E}\boldsymbol{\theta}}(\Theta_{\mathscr{D}}) = 0$, so we only need to verify

$$\operatorname{pr} \mathbf{v}_{\mathscr{D}\boldsymbol{\theta}}(\Theta_{\mathscr{E}}) = 0, \quad \text{where} \quad \operatorname{pr} \mathbf{v}_{\mathscr{D}\boldsymbol{\theta}}(u) = \zeta_x, \qquad \operatorname{pr} \mathbf{v}_{\mathscr{D}\boldsymbol{\theta}}(v) = \theta_x.$$

There is thus a whole hierarchy of conservation laws and commuting flows for the Boussinesq equation. The recursion operator is

$$\mathscr{R} = \mathscr{E} \cdot \mathscr{D}^{-1} =$$
$$\begin{pmatrix} 3v + 2v_xD_x^{-1} & D_x^2 + 2u + u_xD_x^{-1} \\ \tfrac{1}{3}D_x^4 + \tfrac{10}{3}uD_x^2 + 5u_xD_x + 3u_{xx} + \tfrac{16}{3}u^2 + (\tfrac{2}{3}u_{xxx} + \tfrac{16}{3}uu_x)D_x^{-1} & 3v + v_xD_x^{-1} \end{pmatrix}$$

and we can apply $\mathcal{R}$ successively to the right-hand side of (7.39) to obtain the symmetries. The first stage in this recursion is the flow

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathcal{E}\delta\mathcal{H}_1 = \mathcal{D}\delta\mathcal{H}_2$$

$$= \begin{pmatrix} \tfrac{1}{3}u_{xxxxx} + \tfrac{10}{3}uu_{xxx} + \tfrac{25}{3}u_x u_{xx} + \tfrac{20}{3}u^2 u_x + 5vv_x \\ \tfrac{1}{3}v_{xxxxx} + \tfrac{10}{3}uv_{xxx} + 5u_x v_{xx} + \tfrac{10}{3}u_{xx}v_x + \tfrac{5}{3}u_{xxx}v + \tfrac{20}{3}u^2 v_x + \tfrac{40}{3}uu_x v \end{pmatrix},$$

with consequent conservation law

$$\mathcal{H}_2[u, v] = \int (\tfrac{1}{3}u_{xx}v_{xx} + \tfrac{10}{3}uu_{xx}v + \tfrac{5}{2}u_x^2 v + \tfrac{20}{9}u^3 v + \tfrac{5}{6}v^3)\, dx.$$

Alternatively, one can start out with the translational Hamiltonian symmetry

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \mathcal{E}\delta\hat{\mathcal{H}}_0 = \mathcal{D}\delta\hat{\mathcal{H}}_1,$$

where

$$\hat{\mathcal{H}}_0[u, v] = \int u\, dx, \qquad \hat{\mathcal{H}}_1[u, v] = \int uv\, dx,$$

are both conserved. By Theorem 7.27, applying $\mathcal{R}$ to this symmetry leads to a *second* hierarchy of commuting flows and consequent conservation laws, the first of which is

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathcal{E}\delta\hat{\mathcal{H}}_1 = \mathcal{D}\delta\hat{\mathcal{H}}_2 = \begin{pmatrix} v_{xxx} + 4uv_x + 4u_x v \\ \tfrac{1}{3}u_{xxxxx} + 4uu_{xxx} + 8u_x u_{xx} + \tfrac{32}{3}u^2 u_x + 4vv_x \end{pmatrix},$$

where

$$\hat{\mathcal{H}}_2[u, v] = \int (\tfrac{1}{6}u_{xx}^2 - 2uu_x^2 + \tfrac{8}{9}u^4 + 2uv^2 - \tfrac{1}{2}v_x^2)\, dx$$

is yet another conservation law. (At each stage, one needs to know that the operator $\mathcal{D}$ can be inverted, but this is proved in a similar fashion as the Korteweg–de Vries case.)

## Notes

Although Hamiltonian systems of ordinary differential equations have been of paramount importance in the theory of both classical and quantum mechanics, the extension of these ideas and techniques to infinite-dimensional systems governed by systems of evolution equations has been very slow in maturing. The main reason for this delay has been the insistence on using canonical coordinates for finite-dimensional systems, which always exist by virtue of Darboux' theorem; these coordinates, however, do not appear to exist for the evolutionary systems of interest, and hence familiarity with the more general concept of a Poisson structure is a prerequisite here. (The infinite-dimensional version of Darboux' theorem proved by Weinstein, [1], does not seem to apply in this context.)

The correct formulation of a Hamiltonian structure for evolution equations was based on two significant developments. Arnol'd, [1], [2], showed that the Euler equations for ideal fluid flow could be viewed as a Hamiltonian system on the infinite-dimensional group of volume-preserving diffeomorphisms of the underlying space using the Lie–Poisson bracket (as generalized to the corresponding infinite-dimensional Lie algebra). Arnol'd wrote his Hamiltonian structure in Lagrangian (moving) coordinates; the Eulerian version was first discovered by Kuznetsov and Mikhailov, [1]. The version presented here, including the derivation of the conservation laws, is based on Olver, [5] and Ibragimov, [1; §25.3]. (The Poisson bracket presented here, while formally correct, fails to incorporate boundary effects, and needs to be slightly modified when discussing solutions over bounded domains; see Lewis, Marsden, Montgometry and Ratiu, [1], for a discussion of this point.) Subsequently, Marsden and Weinstein, [2], showed that the Lagrangian and Eulerian Poisson brackets were indeed the same. This method of Arnol'd has been applied with great success to determine the Hamiltonian structures of many of the systems of differential equations arising in fluid mechanics, plasma physics, etc. These have been used for proving new nonlinear stability results for these complicated systems; see Holm, Marsden, Ratiu and Weinstein, [1], and the references therein.

The second important development in the general theory was the discovery by Gardner, [1], that the Korteweg–de Vries equation could be written as a completely integrable Hamiltonian system. This idea was further developed by Zakharov and Fadeev, [1], Gel'fand and Dikii, [1], [2], [3], and Lax, [3]. Adler, [1], showed that the (first) Hamiltonian structure of the Korteweg–de Vries equation could be viewed as a formal Lie–Poisson structure on the infinite-dimensional Lie algebra of pseudo-differential operators on the real line, and extended these results to more general soliton equations having Lax representations, including the Boussinesq equation of Example 7.28. See Lax, [1], Dickey, [1], and Newell, [1].

Early versions of the theory of Hamiltonian systems of evolution equations were restricted by their insistence on introducing canonical coordinates; see Broer, [1], for a representative of this approach. The general concept of a Hamiltonian system of evolution equations first surfaces in the work of Magri, [1], Vinogradov, [2], Kupershmidt, [1], and Manin, [1]. Further developments, including the simplified techniques for verifying the Jacobi identity, appear in Gel'fand and Dorfman, [1], Olver, [4] and Kosmann-Schwarzbach, [3]. The computational methods based on functional multi-vectors presented here are a slightly modified version of the methods introduced in the second of these papers. The operator $\mathrm{pr}\,\mathbf{v}_{\mathscr{D}\theta}$ in (7.18) is the same as the Schouten bracket with the bi-vector $\Theta$ determining the Poisson bracket, an approach favoured by Magri, [1], and Gel'fand and Dorfman, [1]. (See Exercise 6.20 for the finite-dimensional version of this bracket and Olver, [10], for a general infinite-dimensional form.) See Dubrovin and Novikov, [1], [2], [3], for a remarkable connection between first order Hamiltonian operators and Riemannian geometry. See Astashov and Vinogradov, [1], Cooke, [1], Dorfman, [1], Doyle, [1], and Olver, [12], for classification results on low order Hamiltonian operators.

The basic theorem on bi-Hamiltonian systems is due to Magri, [1], [2], who was also the first to publish the second Hamiltonian structure for the Korteweg–de Vries and other equations. Magri's method was developed by Gel'fand and Dorfman, [1], [2], and Fuchssteiner and Fokas, [1]. The second Hamiltonian structures of other soliton equations were found by Adler, [1], and Gel'fand and Dikii, [4], with further developments by Kupershmidt and Wilson, [1]. The concept of a bi-Hamiltonian system has also recently surfaced in work on finite-dimensional Hamiltonian systems, in which families of conservation laws are constructed using a related recursive procedure; see Hojman and Harleston, [1], Crampin, [1], Olver, [15], and the references therein. See Olver and Nutku, [1], and Olver, [15], for examples of bi-Hamiltonian systems which do not satisfy the compatability conditions. Strangely, these examples are "even more integrable" than the compatible ones.

The proof of the basic Theorem 7.24 on bi-Hamiltonian systems is based on that of Gel'fand and Dorfman, [1]. The annoying technical hypothesis on the invertibility of the operator $\mathscr{D}$ at each stage is not particularly satisfying. However, it is possible to drop this hypothesis if $\mathscr{D}$ happens to be a constant-coefficient operator; the proof relies on the exactness of an infinite-dimensional generalization of a complex due to Lichnerowicz, [1], based on the Schouten bracket; see Olver, [13]. The theory of bi-Hamiltonian systems played a key role in motivating the important subjects of $R$-matrices and quantum groups; see Semenov-Tian-Shanski, [1], Drinfel'd, [1], and Kosmann-Schwarzbach, [4].

## Exercises

*7.1. Let $p = q = 1$. Find all Hamiltonian operators of the form

$$D_x^3 + PD_x + Q,$$

where $P$ and $Q$ are differential functions. (Try $P$, $Q$ just depending on $u$ and $u_x$ first.) (Gel'fand and Dorfman, [1].)

7.2. Let $p = 1$, $q = 3$, with dependent variables $u, v, w$. Let

$$\mathscr{D} = \begin{bmatrix} 0 & 0 & D_x \\ 0 & D_x & 0 \\ D_x & 0 & D_x^3 + 2uD_x + u_x \end{bmatrix}.$$

Prove that $\mathscr{D}$ is Hamiltonian. (Adler, [1].)