equation has real solutions

$$v = 3c\,\mathrm{sech}^2[\tfrac{1}{2}\sqrt{c}y + \delta],$$

provided the wave speed c is positive. These produce the celebrated "one soliton" solutions

$$u(x, t) = 3c\,\mathrm{sech}^2[\tfrac{1}{2}\sqrt{c}(x - ct) + \delta]$$

to the Korteweg–de Vries equation. (If $c = 0$, we also obtain the singular stationary solution $u = -12(x + \delta)^{-2}$.) More generally, if we only require u to be bounded, we obtain the periodic "cnoidal wave" solutions

$$u(x, t) = a\,\mathrm{cn}^2[\lambda(x - ct) + \delta] + m,$$

where cn is the Jacobi elliptic function of modulus $k = \sqrt{(r_3 - r_2)/(r_3 - r_1)}$, $a = r_3 - r_2$, $\lambda = \sqrt{(r_3 - r_1)/6}$, $m = r_2$ and $r_1 < r_2 < r_3$ are the roots of the cubic polynomial on the right-hand side of (3.12).

(b) *Galilean-Invariant Solutions.* Next look at the one-parameter group of Galilean boosts generated by $t\partial_x + \partial_u$. Here, for $t > 0$, $y = t$ and $v = tu - x$ are independent invariants, from which we calculate

$$u = y^{-1}(x + v), \qquad u_x = y^{-1}, \qquad u_{xxx} = 0, \qquad u_t = y^{-2}(yv_y - v - x),$$

where x is the parametric variable. The reduced equation is simply $dv/dy = 0$, so the general Galilean-invariant solution is $u = (x + \delta)/t$ for δ an arbitrary constant.

A more interesting class of solutions with Galilean-like invariance can be found by adding a time translational component to this group. The generator $t\partial_x + a\partial_t + \partial_u$, $a \neq 0$, has global invariants

$$y = x - \tfrac{1}{2}bt^2, \qquad v = u - bt,$$

where $b = 1/a$. We have

$$u = v + bt, \qquad u_x = v_y, \qquad u_{xxx} = v_{yyy}, \qquad u_t = -btv_y + b,$$

so the reduced equation is

$$v_{yyy} + vv_y + b = 0.$$

This integrates once, leading to a second-order equation

$$v_{yy} + \tfrac{1}{2}v^2 + by + c = 0$$

known as the *first Painlevé transcendent*. Its solutions $v = h(y)$ are meromorphic in the entire complex plane, but are essentially new functions not expressible in terms of standard special functions. The corresponding solutions of the Korteweg–de Vries equation take the form

$$u(x, t) = h(x - \tfrac{1}{2}bt^2) + bt.$$

(c) *Scale-Invariant Solutions.* Finally consider the group of scaling symmetries

$$(x, t, u) \mapsto (\lambda x, \lambda^3 t, \lambda^{-2}u).$$

Invariants on the half space $\{t > 0\}$ are

$$y = t^{-1/3}x, \qquad v = t^{2/3}u.$$

We find

$$u_x = t^{-1}v_y, \qquad u_{xxx} = t^{-5/3}v_{yyy}, \qquad u_t = -\tfrac{1}{3}t^{-5/3}(yv_y + 2v),$$

so that the reduced equation is

$$v_{yyy} + vv_y - \tfrac{1}{3}yv_y - \tfrac{2}{3}v = 0.$$

It is by no means obvious how to solve this third order ordinary differential equation directly. However, motivated by a transformation discovered by Miura, [1], for the Korteweg–de Vries equation itself (see Exercise 5.11), let us set

$$v = \frac{dw}{dy} - \tfrac{1}{6}w^2.$$

The equation for w is

$$\begin{aligned} 0 &= w_{yyyy} - \tfrac{1}{3}ww_{yyy} - \tfrac{1}{3}ww_y^2 - \tfrac{1}{6}w^2w_{yy} + \tfrac{1}{18}w^3w_y - \tfrac{1}{3}yw_{yy} + \tfrac{1}{9}yww_y - \tfrac{2}{3}w_y + \tfrac{1}{9}w^2 \\ &= (D_y - \tfrac{1}{3}w)(w_{yyy} - \tfrac{1}{6}w^2w_y - \tfrac{1}{3}yw_y - \tfrac{1}{3}w). \end{aligned}$$

Therefore every solution to the "modified" third-order equation

$$w_{yyy} - \tfrac{1}{6}w^2w_y - \tfrac{1}{3}yw_y - \tfrac{1}{3}w = 0$$

gives rise to a scale-invariant solution of the Korteweg–de Vries equation by the above transformation. The latter equation can be integrated once:

$$w_{yy} = \tfrac{1}{18}w^3 + \tfrac{1}{3}yw + k$$

for some constant k. This equation is the *second Painlevé transcendent*, which shares similar properties to the first. See Ince, [1], for an extensive discussion of these equations.

Example 3.5. For the Euler equations of three-dimensional incompressible ideal fluid flow,

$$\begin{aligned} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla p, \\ \nabla\cdot\mathbf{u} &= 0, \end{aligned} \tag{3.13}$$

there are four independent variables: $\mathbf{x} = (x, y, z)$ and t, so we can discuss solutions which are invariant under one-, two- and three-parameter subgroups of the full symmetry group, which was determined in Example 2.45. Here we look at a couple of such subgroups, leading to solutions of physical or mathematical interest. In all cases, the group will contain a one-parameter subgroup of either uniform, time-independent translations in a fixed direction, which we may as well take to be the z-axis, or rotations around a fixed axis, again taken as the z-axis.

(a) *Translationally-Invariant Solutions.* For solutions invariant under the translation group generated by ∂_z, the three-dimensional Euler equations (3.13) reduce to their two-dimensional counterparts, which have the same form but with $\mathbf{u} = (u, v)$, p depending only on $\mathbf{x} = (x, y)$, t, together with an equation

$$w_t + uw_x + vw_y = 0 \tag{3.14}$$

for the vertical component of the velocity, which can be integrated by solving the characteristic equation $dt = dx/u = dy/v$. Of course, the two-dimensional Euler equations are still far too difficult to solve explicitly, so we look for solutions invariant under a second one-parameter group.

For the time-dependent translational group G_β generated by $\mathbf{v}_\beta = \beta\partial_y + \beta_t\partial_v - \beta_{tt}y\partial_p$, $\beta(t) \neq 0$, invariants are given by

$$t, \qquad x, \qquad \tilde{u} = u, \qquad \tilde{v} = v - (\beta_t/\beta)y, \qquad \tilde{p} = p + \tfrac{1}{2}(\beta_{tt}/\beta)y^2,$$

with t, x being independent variables. The reduced system

$$\tilde{u}_t + \tilde{u}\tilde{u}_x = -\tilde{p}_x, \qquad \tilde{v}_t + \tilde{u}\tilde{v}_x + (\beta_t/\beta)\tilde{v} = 0, \qquad \tilde{u}_x + (\beta_t/\beta) = 0, \tag{3.15}$$

is readily solved:

$$\tilde{u} = [-\beta_t x + \sigma_t]/\beta, \qquad \tilde{v} = h(\beta x - \sigma)/\beta,$$
$$\tilde{p} = [(\tfrac{1}{2}\beta\beta_{tt} - \beta_t^2)x^2 + (2\beta_t\sigma_t - \beta\sigma_{tt})x + \tau]/\beta^2,$$

where $\sigma(t)$, $\tau(t)$ are arbitrary functions of t, and h is an arbitrary function of the single invariant $\beta(t)x - \sigma(t)$ for the second equation in (3.15). Thus we obtain the G_β-invariant solution

$$u = [-\beta_t x + \sigma_t]/\beta, \qquad v = [\beta_t y + h(\beta x - \sigma)]/\beta,$$
$$p = [(\tfrac{1}{2}\beta\beta_{tt} - \beta_t^2)x^2 - \tfrac{1}{2}\beta\beta_{tt}y^2 + (2\beta_t\sigma_t - \beta\sigma_{tt})x + \tau]/\beta^2,$$

of the two-dimensional Euler equations. In particular, if $\beta(t) \equiv 1$, we can further solve (3.14) explicitly, with

$$u = \sigma_t, \qquad v = h(x - \sigma(t)), \qquad w = H(x - \sigma(t), y - th(x - \sigma(t))),$$
$$p = -\sigma_{tt}x + \tau(t),$$

being the three-dimensional solution invariant under the group of translations in the y- and z-directions. (Here $\sigma(t)$, $\tau(t)$, $h(\xi)$, $H(\xi, \eta)$ are arbitrary smooth functions.)

Although this determines all solutions of the two-dimensional Euler equations which are invariant under the group generated by $\mathbf{v}_\beta$, it is instructive to see what happens if we try to determine the more specialized solutions which are invariant under the two-parameter group generated by $\mathbf{v}_\beta$ and $\mathbf{v}_\alpha = \alpha\partial_x + \alpha_t\partial_u - \alpha_{tt}x\partial_p$. Invariants are

$$\tilde{u} = u - (\alpha_t/\alpha)x, \qquad \tilde{v} = v - (\beta_t/\beta)y, \qquad \tilde{p} = p + (\alpha_{tt}/2\alpha)x^2 + (\beta_{tt}/2\beta)y^2,$$

which are functions of the sole remaining independent variable t. The reduced system of ordinary differential equation is

$$\tilde{u}_t + (\alpha_t/\alpha)\tilde{u} = 0 = \tilde{v}_t + (\beta_t/\beta)\tilde{v}, \tag{3.16a}$$

plus the divergence-free condition

$$(\alpha_t/\alpha) + (\beta_t/\beta) = 0. \tag{3.16b}$$

An important point here is that *unless* $\alpha(t) = k/\beta(t)$ for some constant k, (3.16b) is inconsistent and there are *no* solutions to the reduced equations. In other words, there is no guarantee in general that the reduced system of differential equations for some symmetry group be consistent, and hence no guarantee that any such solutions exist.

(b) *Rotationally-Invariant Solutions.* For the group of rotations about the z-axis, generated by $-y\partial_x + x\partial_y - v\partial_u + u\partial_v$, invariants are provided by $r = \sqrt{x^2 + y^2}$, z, t, p and the cylindrical components of the velocity $\hat{u} = u\cos\theta + v\sin\theta$, $\hat{v} = -u\sin\theta + v\cos\theta$, $\hat{w} = w$. The reduced equations are

$$\begin{aligned} \hat{u}_t + \hat{u}\hat{u}_r + \hat{w}\hat{u}_z - r^{-1}\hat{v}^2 &= -p_r, \\ \hat{v}_t + \hat{u}\hat{v}_r + \hat{w}\hat{v}_z + r^{-1}\hat{u}\hat{v} &= 0, \\ \hat{w}_t + \hat{u}\hat{w}_r + \hat{w}\hat{w}_z \qquad &= -p_z, \\ (r\hat{u})_r + (r\hat{w})_z &= 0, \end{aligned} \tag{3.17}$$

cf. Berker, [1]. If we further assume translational invariance under ∂_z, so $\hat{u}$, $\hat{v}$, $\hat{w}$, p are independent of z, then we can solve (3.17) explicitly:

$$\hat{u} = \sigma_t/r, \qquad \hat{v} = r^{-1}h[\tfrac{1}{2}r^2 - \sigma(t)], \qquad \hat{w} = \tilde{h}[\tfrac{1}{2}r^2 - \sigma(t)],$$

$$p = -\sigma_{tt}\log r - \tfrac{1}{2}r^{-2}\sigma_t^2 + \int_0^r s^{-3}h[\tfrac{1}{2}s^2 - \sigma(t)]^2\,ds + \tau(t),$$

where $\sigma(t)$, $\tau(t)$ and $h(\xi)$, $\tilde{h}(\xi)$ are arbitrary functions. These are the most general solutions depending only on t and the cylindrical radius r.

Are there solutions which are completely rotationally-invariant, i.e. have the full SO(3) invariance group? Although SO(3) acts projectably on $\mathbb{R}^3 \times \mathbb{R}^3$ via $(\mathbf{x}, \mathbf{u}) \mapsto (R\mathbf{x}, R\mathbf{u})$, $R \in \mathrm{SO}(3)$, and regularly with three-dimensional orbits on an open subset of $\mathbb{R}^3 \times \mathbb{R}^3$, the projected group action $\mathbf{x} \mapsto R\mathbf{x}$ on $\mathbb{R}^3$ has only two-dimensional orbits. In this case the transversality conditions (3.33) are violated and we are unable to construct a reduced system $\Delta/\mathrm{SO}(3)$. Another way to see this is to look at the invariants for SO(3), which are t, $|\mathbf{x}|$, $\mathbf{x}\cdot\mathbf{u}$, $|\mathbf{u}|$, p, and note that there are one too many independent and one too few dependent variables to carry through the reduction procedure. Thus no SO(3)-invariant solutions can be constructed by this technique.

As a final example, we look directly at the Euler equations in cylindrical coordinates (3.17). This system has a number of symmetry groups, most of which come from symmetry groups for the full Euler equations (3.13). There is, however, one additional symmetry generator, $\mathbf{v}^* = r^{-2}(\hat{v}^{-1}\partial_{\hat{v}} - \partial_p)$, which does not come from a symmetry of (3.13)! Thus, reducing a system Δ by a known symmetry group G may lead to a system Δ/G with *additional* symmetry properties not shared by the original system. Let us look for solutions invariant under the one-parameter group generated by $\partial_t - \mathbf{v}^*$, which has invariants r, z, $\hat{u}$, $\hat{w}$, $\omega = \tfrac{1}{2}r^2\hat{v}^2 + t$, $q = p - r^{-2}t$. These satisfy the reduced system

$$\begin{aligned} \hat{u}\hat{u}_r + \hat{w}\hat{u}_z - 2r^{-3}\omega &= -q_r, & \hat{u}\omega_r + \hat{w}\omega_z &= 1, \\ \hat{u}\hat{w}_r + \hat{w}\hat{w}_z &= -q_z, & (r\hat{u})_r + (r\hat{w})_z &= 0. \end{aligned} \tag{3.18}$$

This system is still too complicated to solve in general; however, following Kapitanskii, [1], we look for solutions with the ansatz

$$\hat{u} = \hat{u}(r), \qquad \omega = \omega(r), \qquad \hat{w} = \xi(r)z + \eta(r).$$

The first and last equations in (3.18) imply

$$\xi = -\hat{u}_r - r^{-1}\hat{u}, \qquad q_r = -\hat{u}\hat{u}_r + 2r^{-3}\omega.$$

Differentiating the third equation with respect to r, we find the compatibility condition

$$0 = -q_{rz} = (\hat{u}\xi_r + \xi^2)_r z + (\hat{u}\eta_r + \xi\eta)_r,$$

hence $\hat{u}\xi_r + \xi^2 = k$, $\hat{u}\eta_r + \xi\eta = l$ are constant. Using the above formula for ξ, we find that $\hat{u}$ must satisfy the ordinary differential equation

$$\hat{u}\hat{u}_{rr} - \hat{u}_r^2 - r^{-1}\hat{u}\hat{u}_r - 2r^{-2}\hat{u}^2 + k = 0.$$

This equation admits a two-parameter group of symmetries, generated by $\mathbf{w} = r\partial_r + u\partial_u$, $\tilde{\mathbf{w}} = r^{-1}\partial_r - r^{-2}\hat{u}\partial_{\hat{u}}$, and hence can be integrated using the methods of Section 2.5. For $k < 0$ we have

$$\hat{u}(r) = ar^{-1}\cosh(br^2 + \delta),$$

a, b, δ arbitrary constants, hence

$$\omega(r) = -\frac{\arctan\exp(br^2 + \delta)}{2ab^2}, \qquad \xi(r) = -2ab\sinh(br^2 + \delta),$$

$$q(r, z) = -\tfrac{1}{2}kz^2 - lz - \tfrac{1}{2}[\hat{u}(r)]^2 + \int_{r_0}^{r} 2s^{-3}\,\omega(s)\,ds,$$

and, from the last equation of (3.18),

$$\eta(r) = -2b\sinh(br^2 + \delta),$$

and $l = -4ab^2$. Thus the general such solution is

$$\hat{u} = \hat{u}(r), \quad \hat{v} = r^{-1}\sqrt{2\omega(r) - 2t}, \quad \hat{w} = \xi(r)z + \eta(r), \quad p = tr^{-2} + q(r, z),$$

with $\hat{u}$, ω, ξ, η, q being as above. Kapitanskii notes that since $\hat{v}$ is given by a square root, these solutions can be arranged to provide solutions of the Euler equations on cylindrical domains which blow up in finite time ($|\nabla\mathbf{u}| \to \infty$) even though the normal component of u on the boundary is smooth for all t. The reason, of course, is that the singularity $\omega(r) = t$ can be arranged to cross the boundary without affecting the normal component of $\mathbf{u}$. (Similar sorts of behaviour can be arranged for the simpler translationally-invariant solutions.) This observation, therefore, does not answer the outstanding problem of whether smooth solutions to the three-dimensional Euler equations can develop singularities after a finite time.

3.3. Classification of Group-Invariant Solutions

In general, to each s-parameter subgroup H of the full symmetry group G of a system of differential equations in $p > s$ independent variables, there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is not usually feasible to list all possible group-invariant solutions to the system. We need an effective, systematic means of classifying these solutions, leading to an "optimal system" of group-invariant solutions from which every other such solution can be derived. Since elements $g \in G$ not in the subgroup H will transform an H-invariant solution to some other group-invariant solution, only those solutions not so related need be listed in our optimal system. The basic result is the following:

Proposition 3.6. *Let G be the symmetry group of a system of differential equations Δ and let $H \subset G$ be an s-parameter subgroup. If $u = f(x)$ is an H-invariant solution to Δ and $g \in G$ is any other group element, then the transformed function $u = \tilde{f}(x) = g \cdot f(x)$ is a $\tilde{H}$-invariant solution, where $\tilde{H} = gHg^{-1}$ is the conjugate subgroup to H under g.*

The proof follows directly from Exercise 2.3 using the graph Γ_f as the invariant subset. As a consequence of this result, the problem of classifying group-invariant solutions reduces to the problem of classifying subgroups of the full symmetry group G under conjugation, We thus need to study the conjugacy map $h \mapsto ghg^{-1}$ on a Lie group in detail, after which we will return to our original classification problem.

The Adjoint Representation

Let G be a Lie group. For each $g \in G$, group conjugation $K_g(h) \equiv ghg^{-1}$, $h \in G$, determines a diffeomorphism on G. Moreover, $K_g \circ K_{g'} = K_{gg'}$, $K_e = \mathbb{1}_G$, so K_g determines a global group action of G on itself, with each conjugacy map K_g being a group homomorphism: $K_g(hh') = K_g(h)K_g(h')$, etc. The differential dK_g: $TG|_h \to TG|_{K_g(h)}$ is readily seen to preserve the right-invariance of vector fields, and hence determines a linear map on the Lie algebra of G, called the *adjoint representation*:

$$\operatorname{Ad} g(\mathbf{v}) \equiv dK_g(\mathbf{v}), \qquad \mathbf{v} \in \mathfrak{g}. \tag{3.19}$$

Note that the adjoint representation gives a global linear action of G on $\mathfrak{g}$:

$$\operatorname{Ad}(g \cdot g') = \operatorname{Ad} g \circ \operatorname{Ad} g', \qquad \operatorname{Ad} e = \mathbb{1}.$$

If $\mathbf{v} \in \mathfrak{g}$ generates the one-parameter subgroup $H = \{\exp(\varepsilon\mathbf{v})\colon \varepsilon \in \mathbb{R}\}$, then by (1.22) $\operatorname{Ad} g(\mathbf{v})$ is easily seen to generate the conjugate one-parameter subgroup $K_g(H) = gHg^{-1}$. This remark readily generalizes to higher dimensional subgroups using the fact that they are completely determined by their one-parameter subgroups.

Proposition 3.7. *Let H and $\tilde{H}$ be connected, s-dimensional Lie subgroups of the Lie group G with corresponding Lie subalgebras $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ of the Lie algebra $\mathfrak{g}$ of G. Then $\tilde{H} = gHg^{-1}$ are conjugate subgroups if and only if $\tilde{\mathfrak{h}} = \operatorname{Ad} g(\mathfrak{h})$ are conjugate subalgebras.*

The adjoint representation of a Lie group on its Lie algebra is often most easily reconstructed from its infinitesimal generators. If $\mathbf{v}$ generates the one-parameter subgroup $\{\exp(\varepsilon\mathbf{v})\}$, then we let $\operatorname{ad}\mathbf{v}$ be the vector field on $\mathfrak{g}$ generating the corresponding one-parameter group of adjoint transformations:

$$\operatorname{ad}\mathbf{v}|_{\mathbf{w}} \equiv \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \operatorname{Ad}(\exp(\varepsilon\mathbf{v}))\mathbf{w}, \qquad \mathbf{w}\in\mathfrak{g}. \tag{3.20}$$

A fundamental fact is that the infinitesimal adjoint action agrees (up to sign) with the Lie bracket on $\mathfrak{g}$:

Proposition 3.8. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For each $\mathbf{v}\in\mathfrak{g}$, the adjoint vector $\operatorname{ad}\mathbf{v}$ at $\mathbf{w}\in\mathfrak{g}$ is*

$$\operatorname{ad}\mathbf{v}|_{\mathbf{w}} = [\mathbf{w},\mathbf{v}] = -[\mathbf{v},\mathbf{w}], \tag{3.21}$$

where we are using the identification of $T\mathfrak{g}|_{\mathbf{w}}$ with $\mathfrak{g}$ itself since $\mathfrak{g}$ is a vector space.

PROOF. We identify $\mathfrak{g} \simeq TG|_e$. Using (3.20), the definition (3.19) of the adjoint representation, and the right-invariance of $\mathbf{w}$, we find

$$\begin{aligned}\operatorname{ad}\mathbf{v}|_{\mathbf{w}} &= \lim_{\varepsilon\to 0}\frac{dK_{\exp(\varepsilon\mathbf{v})}[\mathbf{w}|_e] - \mathbf{w}|_e}{\varepsilon}\\ &= \lim_{\varepsilon\to 0}\frac{d\exp(\varepsilon\mathbf{v})[\mathbf{w}|_{\exp(-\varepsilon\mathbf{v})}] - \mathbf{w}|_e}{\varepsilon}.\end{aligned}$$

If we replace ε by $-\varepsilon$, this last expression is the same as the definition (1.57) of the Lie derivative of $\mathbf{w}$ with respect to $\mathbf{v}$, so (3.21) follows from Proposition 1.64. □

Note. In most references, the adjoint map $\operatorname{ad}\mathbf{v}|_{\mathbf{w}}$ has the other sign $+[\mathbf{v},\mathbf{w}]$. The reason is our choice in Chapter 1 of right-invariant vector fields to define the Lie algebra, rather than the more traditional left-invariant vector fields. (The reasons for this choice were discussed in Exercise 1.33.) In this book, we will consistently use (3.21) for the infinitesimal adjoint action.

In the case $G \subset \mathrm{GL}(n)$ is a matrix Lie group with Lie algebra $\mathfrak{g}\subset\mathfrak{gl}(n)$, the above formulae are particularly easy to verify. Since $K_A(B) = ABA^{-1}$, where $A, B\in G$ are $n\times n$ matrices, the adjoint map is also given by conjugation

$$\operatorname{Ad} A(X) = AXA^{-1}, \qquad A\in G, \quad X\in\mathfrak{g}.$$

Letting $A = e^{\varepsilon Y}$, $Y \in \mathfrak{g}$, and differentiating with respect to ε, we find

$$\operatorname{ad} Y|_X = YX - XY = [X, Y],$$

agreeing with the commutator bracket on $\mathfrak{gl}(n)$.

Example 3.9. Let $G = \mathrm{SO}(3)$ be the group of rotations in $\mathbb{R}^3$. The Lie algebra $\mathfrak{so}(3)$ is spanned by the matrices

$$A^x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A^y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad A^z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

generating the one-parameter subgroups

$$R^x_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad R^y_\theta = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$R^z_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

of counterclockwise rotations about the coordinate axes. The adjoint action of, say, R^x_θ on the generator A^y can be found by differentiating the product $R^x_\theta R^y_\varepsilon R^x_{-\theta}$ with respect to ε and setting $\varepsilon = 0$. We find

$$\operatorname{Ad} R^x_\theta(A^y) = \begin{bmatrix} 0 & -\sin\theta & \cos\theta \\ \sin\theta & 0 & 0 \\ -\cos\theta & 0 & 0 \end{bmatrix} = \cos\theta \cdot A^y + \sin\theta \cdot A^z,$$

and similarly,

$$\operatorname{Ad} R^x_\theta(A^x) = A^x, \qquad \operatorname{Ad} R^x_\theta(A^z) = -\sin\theta \cdot A^y + \cos\theta \cdot A^z.$$

Thus the adjoint action of the subgroup R^x_θ of rotations around the x-axis in physical space is the same as the group of rotations around the A^x-axis in the Lie algebra space $\mathfrak{so}(3)$. Similar remarks apply to the other subgroups, so if $R \in \mathrm{SO}(3)$ is any rotation matrix relative to the given (x, y, z)-coordinates in $\mathbb{R}^3$, its adjoint map $\operatorname{Ad} R$ acting on $\mathfrak{so}(3) \simeq \mathbb{R}^3$ has the same matrix representation R relative to the induced basis $\{A^x, A^y, A^z\}$. (The fact that the adjoint representation of $\mathrm{SO}(3)$ agrees with its natural physical representation is accidental and will *not* hold for other matrix Lie groups.) Finally, the infinitesimal generators of the adjoint action are found by differentiation; for example,

$$\operatorname{ad} A^x|_{A^y} = \left.\frac{d}{d\theta}\right|_{\theta=0} \operatorname{Ad}(R^x_\theta)A^y = A^z,$$

which agrees with the commutator

$$[A^y, A^x] = A^x A^y - A^y A^x = A^z.$$

Conversely, if we know the infinitesimal adjoint action ad $\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ on itself, we can reconstruct the adjoint representation Ad G of the underlying Lie group, either by integrating the system of linear ordinary differential equations

$$\frac{d\mathbf{w}}{d\varepsilon} = \operatorname{ad}\mathbf{v}|_{\mathbf{w}}, \qquad \mathbf{w}(0) = \mathbf{w}_0, \tag{3.22}$$

with solution

$$\mathbf{w}(\varepsilon) = \operatorname{Ad}(\exp(\varepsilon\mathbf{v}))\mathbf{w}_0,$$

or, perhaps more simply, by summing the Lie series (cf. (1.19))

$$\begin{aligned}\operatorname{Ad}(\exp(\varepsilon\mathbf{v}))\mathbf{w}_0 &= \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(\operatorname{ad}\mathbf{v})^n(\mathbf{w}_0) \\ &= \mathbf{w}_0 - \varepsilon[\mathbf{v}, \mathbf{w}_0] + \frac{\varepsilon^2}{2}[\mathbf{v}, [\mathbf{v}, \mathbf{w}_0]] - \cdots. \end{aligned} \tag{3.23}$$

(The convergence of (3.23) follows since (3.22) is a linear system of ordinary differential equations, for which (3.23) is the corresponding matrix exponential.)

Example 3.10. The Lie algebra spanned by $\mathbf{v}_1 = \partial_x$, $\mathbf{v}_2 = \partial_t$, $\mathbf{v}_3 = t\partial_x + \partial_u$, $\mathbf{v}_4 = x\partial_x + 3t\partial_t - 2u\partial_u$ generates the symmetry group of the Korteweg–de Vries equation. To compute the adjoint representation, we use the Lie series (3.23) in conjunction with the commutator table in Example 2.44. For instance

$$\begin{aligned}\operatorname{Ad}(\exp(\varepsilon\mathbf{v}_2))\mathbf{v}_4 &= \mathbf{v}_4 - \varepsilon[\mathbf{v}_2, \mathbf{v}_4] + \tfrac{1}{2}\varepsilon^2[\mathbf{v}_2, [\mathbf{v}_2, \mathbf{v}_4]] - \cdots \\ &= \mathbf{v}_4 - 3\varepsilon\mathbf{v}_2.\end{aligned}$$

In this manner, we construct the table

Ad	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4 - \varepsilon\mathbf{v}_1$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 - \varepsilon\mathbf{v}_1$	$\mathbf{v}_4 - 3\varepsilon\mathbf{v}_2$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2 + \varepsilon\mathbf{v}_1$	$\mathbf{v}_3$	$\mathbf{v}_4 + 2\varepsilon\mathbf{v}_3$
$\mathbf{v}_4$	$e^{\varepsilon}\mathbf{v}_1$	$e^{3\varepsilon}\mathbf{v}_2$	$e^{-2\varepsilon}\mathbf{v}_3$	$\mathbf{v}_4$

(3.24)

with the (i, j)-th entry indicating $\operatorname{Ad}(\exp(\varepsilon\mathbf{v}_i))\mathbf{v}_j$.

Classification of Subgroups and Subalgebras

Definition 3.11. Let G be a Lie group. An *optimal system* of s-parameter subgroups is a list of conjugacy inequivalent s-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of s-parameter subalgebras forms an *optimal system* if every s-parameter subalgebra of $\mathfrak{g}$ is equivalent to a unique member of the list under some element of the adjoint representation: $\tilde{\mathfrak{h}} = \operatorname{Ad} g(\mathfrak{h})$, $g \in G$.

Proposition 3.7 says that the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras, and so we concentrate on the latter. Unfortunately, this problem can still be quite complicated, and, for once, infinitesimal techniques do not seem to be overly useful.

For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by a nonzero vector in $\mathfrak{g}$. Although some sophisticated techniques are available for Lie algebras with additional structure, in essence this problem is attacked by the naïve approach of taking a general element $\mathbf{v}$ in $\mathfrak{g}$ and subjecting it to various adjoint transformations so as to "simplify" it as much as possible. We treat a couple of illustrative examples.

Example 3.12. Consider the symmetry algebra $\mathfrak{g}$ of the Korteweg–de Vries equation, whose adjoint representation was determined in Example 3.10. Given a nonzero vector

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4,$$

our task is to simplify as many of the coefficients a_i as possible through judicious applications of adjoint maps to $\mathbf{v}$.

Suppose first that $a_4 \neq 0$. Scaling $\mathbf{v}$ if necessary, we can assume that $a_4 = 1$. Referring to table (3.24), if we act on such a $\mathbf{v}$ by $\operatorname{Ad}(\exp(-\tfrac{1}{2}a_3\mathbf{v}_3))$, we can make the coefficient of $\mathbf{v}_3$ vanish:

$$\mathbf{v}' = \operatorname{Ad}(\exp(-\tfrac{1}{2}a_3\mathbf{v}_3))\mathbf{v} = a_1'\mathbf{v}_1 + a_2'\mathbf{v}_2 + \mathbf{v}_4$$

for certain scalars a_1', a_2' depending on a_1, a_2, a_3. Next we act on $\mathbf{v}'$ by $\operatorname{Ad}(\exp(\tfrac{1}{3}a_2'\mathbf{v}_2))$ to cancel the coefficient of $\mathbf{v}_2$, leading to $\mathbf{v}'' = a_1'\mathbf{v}_1 + \mathbf{v}_4$, and finally by $\operatorname{Ad}(\exp(a_1'\mathbf{v}_1))$ to cancel the remaining coefficient, so that $\mathbf{v}$ is equivalent to $\mathbf{v}_4$ under the adjoint representation. In other words, every one-dimensional subalgebra generated by a $\mathbf{v}$ with $a_4 \neq 0$ is equivalent to the subalgebra spanned by $\mathbf{v}_4$.

The remaining one-dimensional subalgebras are spanned by vectors of the above form with $a_4 = 0$. If $a_3 \neq 0$, we scale to make $a_3 = 1$, and then act on $\mathbf{v}$ by $\operatorname{Ad}(\exp(a_1\mathbf{v}_2))$, so that $\mathbf{v}$ is equivalent to $\mathbf{v}' = a_2'\mathbf{v}_2 + \mathbf{v}_3$ for some a_2'. We