RENORMALIZING APPROACH TO THE THEORY OF DEVELOPED TURBULENCE:
INFRARED ASYMPTOTIC OF SCALING FUNCTIONS

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The problem of infrared discrepancies of the velocity correlation functions in stochastic hydrodynamics is examined within the framework of the quantum field renormalization group method.

## 1. INTRODUCTION

An important problem of the theory of developed incompressible fluid (gas) turbulence is giving a foundation to the phenomenological Kolmogorov–Obukhov theory [1, 2] within the framework of an accurate micromodel. Ordinarily, the stochastic Navier–Stokes equation [2] is considered such

$$\nabla_t \varphi_i = \nu_0 \Delta \varphi_i - \partial_i p + F_i, \ \nabla_t \equiv \partial_t + \varphi\partial, \tag{1}$$

where $\varphi$ is the transverse velocity field; p and F are the pressure and a transverse external random force per unit mass (they all depend on $x \equiv t, \vec{x}$); and $\nu_0$ is the coefficient of viscosity. A Gaussian distribution with $\langle F\rangle = 0$ and the correlator

$$\langle F_s(x)F_\ell(x')\rangle = (2\pi)^{-4}\int d\omega d\vec{k}\, P_{s\ell}(\vec{k})d(k)e^{i\vec{k}(\vec{x}-\vec{x}')-i\omega(t-t')}, \tag{2}$$

is assumed for F, where $P_{s\ell}(\vec{k}) = \delta_{s\ell} - k_s k_\ell/k^2$ is the transverse projector, $k \equiv |\vec{k}|$, and

$$d(k) = g_0\nu_0^3 k(k^2+m^2)^{-\varepsilon}, \tag{3}$$

can be selected as d(k) [3], where m is the reciprocal inverse scale of turbulence; $g_0$ assures arbitrariness (for given m) of the pumping power (per unit mass)

$$E = (2\pi)^{-3}\int d\vec{k}\, d(k). \tag{4}$$

We limit ourselves to the domain $0 < \varepsilon \le 2$ by considering the real value $\varepsilon = 2$ [then $d(k) \sim \delta(\vec{k})$ for small m], and an ultraviolet (UV) cutoff is understood in (4) and $\Lambda \equiv E^{1/4}\nu_0^{-3/4}$ is the inverse dissipative length.

According to the Kolmogorov–Obukhov theory, the correlation functions of the models (1) and (2) in the inertial interval $m \ll j \ll \Lambda$, $\nu_0 m^2 \ll \omega \ll \nu_0\Lambda^2$ posses the following two properties: 1) they depend on the total pumping power E but not on the details of its "apparatus," in particular the quantity m. The property 1) is formulated in the classical monographs [1, 2] without any constraints (we call this the modification IA), it is considered in some modern papers (see [4]) that there is no m in the static (simultaneous) correlators, but it can be present in the dynamic (modification IB); 2) they are independent of $\nu_0$, which denotes the presence of scaling for fixed E, $\nu_0$ with the dimensionalities

$$\Delta_\varphi = -1/3, \ \Delta_F = 1/3, \ \Delta_\omega = -\Delta_t = 2/3, \ \Delta_m = 1. \tag{5}$$

The method of quantum-field renormalization group (RG) was applied to the problem in [3] (see [5-7] also). In this approach, the theory of turbulence was examined analogously to the theory of critical behavior, an exact $\varepsilon$-expansion was obtained for the "critical indices," and they agree with (5) for real $\varepsilon = 2$.

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Hypothesis 1 was not discussed in these papers, since it is a question therein of the asymptotic of scaling functions as $m \to 0$ for finite $\varepsilon$, while all the calculations in [3, 5-7] were performed within the framework of $\varepsilon$-expansion and with $m = 0$.

In this paper we examine the problem with $m \neq 0$ and outside the framework of the $\varepsilon$-expansion. A certain infrared perturbation theory is developed which, in combination with the RG results for composite operators, permits proof of the hypothesis IB for $0 < \varepsilon < 2$ (IA is known to be false).

For elucidation of the method of summing the leading infrared (IR) singularities in Sec. 4, we will follow the research of L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev (sent to Zh. Éksp. Teor. Fiz.), where formula (33) was obtained in first-order perturbation theory for a "hard" propagator.

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## 2. RENORMALIZATION, THE RG EQUATION, AND THE SCALING FUNCTIONS

The stochastic problem (1), (2) is equivalent to the quantum theory of two transverse fields $\Phi = \varphi, \varphi'$ with the operator [3] (see [5] also)

$$S(\Phi) = \varphi' D_F \varphi'/2 + \varphi'[-\nabla_t \varphi + \nu_0 \Delta \varphi], \tag{6}$$

$D_F$ is the correlator of (2) with function (3), integration is with respect to x and summation over the vector symbols is understood. The theory (6) is logarithmic (nondimensionality $g_0$) for $\varepsilon = 0$, the UV divergences have the form of poles in $\varepsilon$ in the correlators, which must be eliminated by going over to the renormalization variables g, $\nu$ (the renormalization of $\Phi$, m is not required) by using one renormalizing constant $Z_\nu(g, \varepsilon)$:

$$\nu_0 = \nu Z_\nu, \quad g_0 = g M^{2\varepsilon} Z_\nu^{-3} \tag{7}$$

M is the normalizing mass. The renormalized correlators are finite as $\varepsilon \to 0$ in every order in g. For fixed $g_0$, $\nu_0$, and m they are independent of M, as is expressed by the RG equation. We write it in the example of a dual correlator $D = \langle\varphi\varphi\rangle$ in $\omega$, p representation (in the vector symbols D is a multiple to the transverse projector):

$$[\mathcal{D}_M + \beta(g)\partial_g - \gamma_\nu(g)\mathcal{D}_\nu]\, D(\omega, p, g, \nu, m, M) = 0. \tag{8}$$

Here and henceforth, $\mathcal{D}_x \equiv x\partial_x$ for any variable x, and $\widetilde{\mathcal{D}}_M$ denotes the operator $\mathcal{D}_M$ for fixed $g_0$, $\nu_0$, m and the RG functions

$$\beta(g) = \widetilde{\mathcal{D}}_M g, \quad \gamma_\nu(g) = \widetilde{\mathcal{D}}_M \ln Z_\nu \tag{9}$$

are connected by virtue of (7) by the relationship

$$\beta(g) = g[-2\varepsilon + 3\gamma_\nu(g)]. \tag{10}$$

There is an infrared stable fixed point $\beta(g_*) = 0$, $g_* > 0$ in the theory, and by virtue of (10) the value $\gamma_\nu(g_*) = 2\varepsilon/3$ is determined exactly (without corrections $\varepsilon^2$, etc.).

From dimensional analysis

$$D = \nu p^{-3} R(s, g, z, u); \; s = p/M, \; z = \omega/\nu p^2, \; u = m/p. \tag{11}$$

Substitution of (11) into (8) yields

$$[\mathcal{D}_s - \beta(g)\partial_g - \gamma_\nu(g)\mathcal{D}_z] \ln R + \gamma_\nu(g) = 0. \tag{12}$$

Solving (12), we have

$$R(s, g, z, u) = R(1, \bar{g}, \bar{z}, u)\Psi^{-1}, \; \bar{z} = z\Psi, \; \Psi = (s^{2\varepsilon}\bar{g}/g)^{1/3}, \tag{13}$$

where $\bar{g} = \bar{g}(s, g)$, $\bar{z} = \bar{z}(s, g, z)$, and $\bar{u} \equiv u$ are first integrals of the homogeneous equation (12) with normalization g, z, u for s = 1. In a similar manner, for the static (simultaneous) correlator

$$D_{st} = (2\pi)^{-1}\int d\omega\, D = \nu^2 p^{-1} R(s, g, u) \tag{14}$$

it is possible to obtain

$$R(s,g,u)=R(1,\bar{g},u)\psi^2. \tag{15}$$

Because $\bar{g} \to g_*$ as $s \to 0$, the formulas (13) and (15) predict the asymptotic D, $D_{st}$ for $s \to 0$ and an arbitrary fixed u:

$$\begin{gathered} D=a_0^{1/3}\rho^{-3-2\varepsilon/3}f(\bar{z},u), \quad \bar{z}=\omega a_0^{-1/3}\rho^{-2+2\varepsilon/3}, \\ D_{st}=a_0^{2/3}\rho^{-1-4\varepsilon/3}f(u), \end{gathered} \tag{16}$$

where $a_0 \equiv gM^{2\varepsilon}\nu^3/g_* = g_0\nu_0{}^3/g_*$ [the second equality is a corollary of (7)], and f are the scaling fucntions

$$f(\bar{z},u)=R(1,g_*,\bar{z},u), \; f(u)=R(1,g_*,u). \tag{17}$$

For $0 < \varepsilon < 2$, the evaluation of (4) yields $a_0 \sim E\Lambda^{2\varepsilon-4}$ (to the accuracy of corrections not essential for $m \ll \Lambda$) so that (16) and (17) describe scaling for fixed $\nu_0$, E with dimensionalities in agreement with (5) for real $\varepsilon = 2$ [3, 5].

The additional condition $u \ll 1$ corresponds to the inertial interval, the hypothesis IA denotes finiteness of function (17) for u = 0 and finite $\varepsilon$, and IB the finiteness of just f(u). The scaling fucntions are not fixed at all by the RG equation; they can be evaluated in the form of $\varepsilon$ expansions, as $f(u) = \Sigma\varepsilon^n f_n(u)$, say. It is known from diagram analysis that the coefficients $f_n(u)$ have weak singularities of the $u\ln u$ type as $u \to 0$, i.e., are finite for u = 0 (the second-order $\varepsilon$-expansion $f(\bar{z}, 0)$ is calculated in [7]). However, this does not prove hypothesis I for finite $\varepsilon$ since, for arbitrarily small $\varepsilon > 0$ there are diagrams divergent as $m \to 0$. As in the critical behavior theory [8], the Wilson operator expansion [9] can be utilized; for $x \equiv (x_1 + x_2)/2 = \text{const}$, $r = |\vec{x}_1 - \vec{x}_2| \to 0$, $\tau \equiv t_1 - t_2 \to 0$

$$\Phi(x_1)\Phi(x_2) \cong \sum_i c_i(\tau, r, m)F_i(x), \tag{18}$$

where $c_i$ are analytic numerical functions as $m \to 0$, and $F_i(x)$ are all possible local composite renormalized operators. The correlator $\langle\Phi\Phi\rangle$ is obtained by averaging (18), in whose right side the means $\langle F_i\rangle$ occur. We will discuss renormalization of the composite operators in more detail in Sec. 3; now it is just important to us that a critical dimensionality $\Delta_i$ can be associated with each $F_i$ and $\langle F_i\rangle \sim m^{\Delta_i}$. After substitution of (17), the contributions $u^{\Delta_i}$ to the scaling functions correspond to the operator $F_i$ and the operators with $\Delta_i < 0$ are "dangerous" (from the viewpoint of hypothesis I). Because of the Galilean invariance of the model (6), certain dimensionalities are evaluated exactly, in particular

$$\Delta[\varphi^n] = n\Delta_\varphi = n(1-2\varepsilon/3) \tag{19}$$

(see Sec. 3 for more detail). It is seen that all $\varphi^n$ become dangerous for $\varepsilon > 3/2$. We consider that there are no other dangerous operators for $\varepsilon \le 2$ (this is proved in [6] for operators with $\Delta_i(\varepsilon = 0) \le 4$, in the general case it is confirmed indirectly by the results in Sec. 4); then for $0 < \varepsilon < 3/2$ the functions (17) are finite for u = 0 and hypothesis I is valid. For $\varepsilon > 3/2$ all the $\varphi^n$ are dangerous and their contributions must be summed. Such a summation is executed in Sec. 4. It turns out that the contributions $\varphi^n$ drop out in the static f(u) and, therefore, hypothesis IB is satisfied (IA is known to be false) for $0 < \varepsilon < 2$.

## 3. CRITICAL DIMENSIONALITIES OF COMPOSITE OPERATORS

In this section we discuss briefly the RG equation for composite operators [9] and we obtain (19) for the dimensionalities of the operators $\varphi^n$.

Let $F_i(x)$, i = 1, ..., N be a full set of local composite operators having an identical canonical dimensionality for $\varepsilon = 0$. Addition of the component $a^{(0)}F \equiv \Sigma_i \int dx a_i^{(0)}(x)F_i(x)$ to operation (6) generates new divergences (all divergences are UV in this section), and multiplicative renormalization $F_i = Z_{ij}F_j^{ren}$ assures the finiteness of the correlators with one $F^{ren}$ and any nonzero number of fields $\Phi$. The renormalized sources $a \equiv a_i(x)$ defined by the equality

$$a^{(0)}F = aF^{ren}. \tag{20}$$

are considered independent finite parameters. Let us take the average of (20) with the weight exp S; let us go over to the renormalized parameters on the right by using (7), by applying M (see Sec. 2), we obtain the RG equation $\tilde{\mathcal{D}}_M\langle aF^{ren}\rangle = 0$. Writing $\tilde{\mathcal{D}}_M$ in terms of the renormalized paramgers g, $\nu$, $a$, M we obtain the following system of equations (because of the independence of ):

$$(L\delta_{ij}+\gamma_{ij})\langle F_j^{\text{РЕН}}\rangle = 0, \quad L \equiv \mathcal{D}_M + \beta\partial_g - \gamma_\nu \mathcal{D}_\nu \tag{21}$$

with matrices of anomalous dimensionality $\gamma = Z^{-1}\tilde{\mathcal{D}}_M Z$. Equations (21) are separated for the means of the operators $F_i' = U_{ij}F_j^{ren}$ if $\gamma' = U\gamma U^{-1}$ is diagonal

$$(L+\gamma_i')\langle F_i'\rangle = 0, \quad i = 1,\dots,N \tag{22}$$

($\gamma_i'$ is the diagonal element of $\gamma'$; there is no sum over i). From dimensional analysis $\langle F_i'\rangle = m^{d_i}\nu^{d_i^\omega}R_i(g, m/M)$, where $d_i$ and $d_i^\omega$ are the frequency and summary canonical dimensionalities of $F_i'$ [5], substituting into (22) and operating by analogy with Sec. 2, we obtain for the asymptotic $m/M \to 0$

$$\langle F_i'\rangle = c(g,\nu,M,\varepsilon)\, m^{\Delta_i}, \tag{23}$$

i.e., $F_i'$ has a definite critical dimensionality

$$\Delta_i \equiv d_i - \gamma_\nu(g_*)d_i^\omega + \gamma_i'(g_*). \tag{24}$$

Certain dimensionalities can be determined without evaluating diagrams by using Ward identities expressing the Galilean invariance of the models [6]. They assure UV finiteness of the operator for the system $\{F_i\}$ (summation over i without integration over x)

$$a_i^{(0)}(x)\,\partial F_i(x)/\partial\varphi(x)+\dots \tag{25}$$

(the three dots denote contributions occurring from operators containing $\partial_t$; see [6]; they do not affect the subsequent discussion and are omitted because of awkwardness).

Let $\{F_i\}$ be a specific system of operators including $F_1 = \varphi^n(x)$ (a tensor of rank n) with a certain n. We first probe that $F_1$ is not introduced during renormalization to other operators, i.e.,

$$Z_{11} = 1;\ Z_{i1} = 0,\ i \neq 1. \tag{26}$$

We assume that this is true for systems $\{\bar{F}_i\}$ with $\bar{F}_1 = \varphi^{n-1}$, i.e., $\bar{Z}_{i1} = \delta_{i1}$ [this is evident for $\bar{F} = \varphi(x)$]. Substitution of the specific system $\{F_i\}$ in (25) yields the UV finiteness of the expression

$$a_1^{(0)}\varphi^{n-1} + \sum_{i\neq 1} c_i\bar{F}_i$$

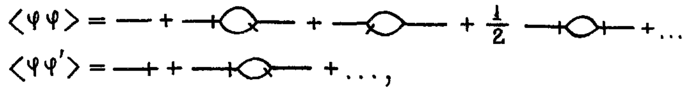
with a certain $c_i$. The coefficients for the independent renormalized operators $\bar{F}_i^{ren}$ are finite; by virtue of the induction assumption, the coefficient of $\bar{F}_i^{ren}$ equals the coefficient of $\bar{F}_1 = \varphi^{n-1}$, which proves finiteness of the latter.

Thus, $a_1^{(0)} = a_iZ_{i1}^{-1}$ is finite. By virtue of the independence of the $a_i$, all the $Z_{i1}^{-1}$, which is denoted by $Z_{i1}^{-1} = \delta_{i1}$ in the scheme of minimal subtractions [9]. The same property is true then for Z [the desired (26)], U, $U^{-1}$, and for $\gamma$ and $\gamma'$ we have $\gamma_{11} = \gamma_1 = 0$ in place of the first equality of (26). Taking account of the canonical dimensionalities $\varphi^n$: $d_1 = nd$, $d_1^\omega = nd^\omega_\varphi$ ($d_\varphi = d^\omega_\varphi = 1$ [5]) and $\gamma^\nu(g_*) = 2\varepsilon/3$, we find the critical dimensionality $F_1'$ from (24) ("associated with $F_1$" in the terminology of [5]): $\Delta_1 = n\Delta_\varphi$, as was declared in (19).

Strictly speaking, $\varphi^n$ itself does not denote a definite critical dimensionality since it contains an "impurity" of the operators $F_i'$ with $i > 1$ that we consider safe for $\varepsilon < 2$ (see Sec. 2) and do not take into account. Neglecting the operators $F \neq \varphi^n$ in (18), by virtue of (26) and the same property of U, we do not lose the contribution $\sim m^{\Delta_1}$, i.e., such an approximation is self-consistent.

## 4. INFRARED PERTURBATION THEORY

Let us consider the principal singularities of perturbation theory diagrams in $g_0$ in the model (6) as $m \to 0$ in an example of the dual correlator $\langle\varphi\varphi\rangle$ and the response function $\langle\varphi\varphi'\rangle$. In a single-loop approximation

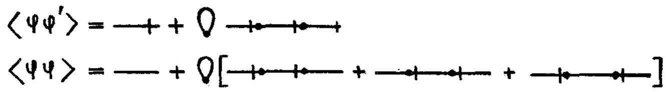


(27)

the line without the canellation denotes the free propagator $\langle\varphi\varphi\rangle_0$, the cancelled line $\langle\varphi\varphi'\rangle_0$ (the cancelled terminus is $\varphi'$), they are proportional to the transverse propagator with the scalar coefficients $\Delta_{\varphi\varphi'} = \Delta^*_{\varphi'\varphi} = (-i\omega + \nu_0\kappa^2)^{-1}$, $\Delta_{\varphi\varphi} = \Delta_{\varphi\varphi'}\, d(\kappa)\, \Delta_{\varphi'\varphi}$. Let $\tilde{p}$ be a four-dimensional external impulse in the diagrams ($\tilde{p}_0 = \omega$, $\tilde{p}_i = p_i$, $i = 1, 2, 3$); $\tilde{k}$ is an integration impulse flowing along the lines $\varphi\varphi$ in the loops; then $\tilde{p} - \tilde{k}$ flows over the second line. We consider $\tilde{p}$ to be in the inertial interval $\tilde{p} \gg m$ (this writing denotes $p \gg m$, $\tilde{p}_0 = \omega \gg \nu_0 m^2$). Let us introduce a fixed boundary impulse $\tilde{p} \gg \mu \gg m$ separating the domain of soft ($\tilde{k} \le \mu$) and hard ($\tilde{k} \ge \mu$) impulses. The integral over $\tilde{k}$ in the loops is divided into two domains: 1) both lines of loops are hard; 2) one is soft and the other hard (two soft are forbidden because of $\tilde{p} \gg \mu$). The singularity in m is generated by the soft line $\varphi\varphi$. Neglecting the dependence of the verties and the hard line on the soft $\tilde{k}$, its contribution is extracted in the form of the multiplier

$$(2\pi)^{-4}\int_{\tilde{k}<\mu} d\tilde{k}\,\langle\varphi\varphi\rangle_0(\tilde{k}) \cong \langle\varphi^2(x)\rangle_0 \sim g_0 m^{2-2\varepsilon}, \tag{28}$$

which diverges as $m \to 0$ for $\varepsilon > 1$ ($\cong$ denotes equality to the accuracy of terms finite as $m \to 0$ for any $\varepsilon$; the subscript "0" is the zeroth approximation in $g_0$). In such an approximation we obtain from (27)



(29)

where the shortened line denotes (28).

The procedure for extraction of singularities in m being considered corresponds to an operator expansion of the type (18): to first order of $g_0$ there appears $\langle\varphi^2\rangle_0$; in the next, $\langle\varphi^4\rangle_0$ appears, and the first correction to $\langle\varphi^2\rangle_0$, in the long run all the completely dressed soft blocks appear:

$$(2\pi)^{-8n}\int_{\tilde{k}_i<\mu} d\tilde{k}_1\ldots d\tilde{k}_{2n}\langle\varphi(\tilde{k}_1)\ldots\varphi(\tilde{k}_{2n})\rangle \cong \langle\varphi^{2n}(x)\rangle \tag{30}$$

(the means of the odd powers equal 0). Taking account of the dependence of the hard lines and the vertices thereon on the soft impulses means taking account of operators in (18) with products that we consider safe for $\varepsilon < 2$. Summing perturbation theory series for $\langle\varphi^{2n}\rangle$ by using the RG yields (see Sec. 3) $\langle\varphi^{2n}\rangle \sim m^{2n(1-2\varepsilon/3)}$, where for $\varepsilon > 3/2$ all $\langle\varphi^{2n}\rangle$ are dangerous and their contributions must be summed.

Let us present a simple method for their summation. We represent the field $\Phi = \varphi, \varphi'$ in (6) in the form of the sum $\Phi = \Phi_s + \Phi$ of soft and hard components; then $\int D\Phi\ldots = \int D\Phi_s \times \int D\Phi_h\ldots$. The mean hard fields in which we are interested are $D_h = \int D\Phi\Phi_h\Phi_h \exp S(\Phi)$ (in model (6) the normalizing factor is $\int D\Phi \exp S(\Phi) = 1$ because of the absence of vacuum diagrams [5]). We rewrite the determination of $D_h$ in the form

$$D_h = \int D\Phi_s D_h(\Phi_s)\int D\Phi_h \exp S(\Phi), \tag{31}$$

where

$$D_h(\Phi_s) = \frac{\int D\Phi_h \Phi_h\Phi_h \exp S(\Phi)}{\int D\Phi_h \exp S(\Phi)}. \tag{32}$$

In meaning (32) is a hard propagator in a fixed external soft field, and (31) is the average in the statistics of $\Phi_s$, determined by the full $S(\Phi)$. Upon substitution of $\Phi = \Phi_s + \Phi_h$ into

(6) the interaction $\varphi'(\varphi\partial)\varphi$ breeds several components, we keep $\varphi'_h(\varphi_h\partial)\varphi_h + \varphi'_h(\varphi_s\partial)\varphi_h$, in our approximation, then the quantities (32) are exact propagators of model (6) with the replacement $\partial_t \to \partial_t + \varphi_s\partial$ and the IR cutoff $\mu$ of all the integrations in the loops. Neglecting the weak inhomogeneity $\varphi_s$ (this is equivalent to neglecting the dependence of the hard lines and vertices on the soft impulses in the diagram language), we replace the correlators $\langle\varphi_s(x_1)\ \dots\ \varphi_s(x_n)\rangle$ in (31) by $\langle\varphi_s(x)\ \dots\ \varphi_s(x)\rangle$ that agree with (30). This operation corresponds to replacing $\varphi_s(x)$ in (32) by the random variable $V \equiv \varphi_s(0)$, $D_h(\Phi_s)$ are transformed into exact propagators of model (6) with the IR cutoff $\mu$ and the replacement $\omega \to \omega - vp$ in the $\omega$, p representation (going over to it became possible by virtue of the homogeneity of v). Functional averaging of (31) in $\Phi_s$ is replaced by the averaging $\langle\langle\ \dots\ \rangle\rangle$ in v: $\langle\langle v^n\rangle\rangle = \langle s^n(x)\rangle$; consequently there follows from (31) and (32):

$$\langle\Phi\Phi\rangle = \langle\langle D(\omega - vp, p; \mu)\rangle\rangle, \tag{33}$$

where $D(\omega, p; \mu)$ are propagators $\langle\Phi\Phi\rangle$ of model (6) with the IR cutoff $\mu$ that is consequently finite for $m = 0$.

There are no visible reasons for the disappearance of the dependence on m as $m \to 0$ in the dynamic objects (33), i.e., hypothesis IA (see Sec. 1) is not satisfied even in the UV pumping domain $2 > \varepsilon > 3/2$. The passage over to static objects corresponds to integration of (33) with respect to $\omega$ [see (14)], the dependence on the shear vp vanishes; $D_{st}(p; \mu)$ is finite as $m \to 0$ and independent of m; this proves hypothesis IB for $2 > \varepsilon > 0$ [the corrections to (33) and $D_{st}(p, \mu)$ have the form $m^\Delta$ with $\Delta > 0$ because of composite operators with derivatives; it is impossible to believe (33) in the domain $\varepsilon > 2$ since the operators $\varphi\Delta\varphi$ and others with $\Delta < 0$ are known to be manifest].

## 5. COMPUTATION OF IR PERTURBATION THEORY DIAGRAMS

From the physical viewpoint, approximation (33) describes interaction of turbulent vortices of the scale $k \geq \mu$ and their transfer as entire vortices having the external turbulence scale $k \simeq m$. As all approximations of such kind (see the papers cited in [4]), (33) contains an explicit dependence on the separating impulse $\mu$. Certainly the dependence on $\mu$ vanishes when all the contributions to the operator expansion (18) are taken into account, but it is desirable to have a method of calculation for which $\mu$ would not appear in the intermediate stages. We propose such a method in an example of scalar theory $g\varphi^3$ [the extension to model (6) is obvious but awkward because of the vector symbols] in a space of dimensionality $d = 6 - 2\varepsilon$; the parameter $\varepsilon = 3 - d/2$ is analogous to $\varepsilon$ in the exponent of (3). Let $D = \langle\varphi\varphi\rangle$ in the lowest-order perturbation theory $D^{-1} = p^2 + m^2 - gI/2(2\pi)^\alpha$,

$$I = \int d\vec{k}\,\Delta(k)\Delta(p-k), \quad \Delta(k) \equiv (k^2 + m^2)^{-1}. \tag{34}$$

Let us introduce the separating impulse $m \ll \mu \ll p$, integration in (34) is divided into a domain in which both impulses k, p − k are hard (we denote the corresponding contribution by $I_h$) and two domains with one soft and one hard impulse (two contributions $I_s$). We expand the hard propagator in the soft impulse in $I_s$

$$I_s = \sum_{n=0}^{\infty} a_n b_n, \quad b_n = \int_{k \leq \mu} d\vec{k}\,\Delta(k) k^{2n}. \tag{35}$$

Representation (35) corresponds to an operator expansion of type (18): "hard" multipliers $a_n = a_n(p, m)$ are analytic in $m^2$, the "soft" $b_n$ are independent of p, $b_0$ yields a contribution to $\langle\varphi^2\rangle$, $b_1$ to $\{\varphi\Delta\varphi\}$, etc. Evaluation of $b_0$ yields

$$b_0 = \pi^{3-\varepsilon}\Gamma(\varepsilon - 2)m^{4-2\varepsilon} + \frac{\pi^{3-\varepsilon}}{\Gamma(3-\varepsilon)}\mu^{4-2\varepsilon}\sum_{n=0}^{\infty}\frac{(-1)^n(m/\mu)^{2n}}{(2-n-\varepsilon)}. \tag{36}$$

Similarly to $b_0$, the remaining $b_n$ contain a component independent of $\mu$ that diverges for large $\varepsilon$ (namely $\varepsilon > n + 2$) and analytic contributions in $m^2$ that vanish as $\mu \to \infty$ for $\varepsilon > n + 2$. Let us agree to discard these analytic contributions, i.e., let us define $b_0 = \pi^{3-\varepsilon}\Gamma(\varepsilon - 2)m^{4-2\varepsilon}$, etc. This can be realized technically by setting $\mu = \infty$ in $b_n$ and calculating them by prescribed dimensional regularization (DR). Indeed, the DR response for integrals of type $b_n$ is indeed defined as the analytic continuation in $\varepsilon$ from a domain in which the limit $\mu \to \infty$ exists [10]. In order not to alter the response for I, we agree to understand $I_h$ as $I_h = I - 2I_s$, where $I_s$ is calculated by DR rules for $\mu = \infty$. In such an understanding $I_h$ is independent of $\mu$ and finite as $m \to 0$ for any $\varepsilon$, its value can be obtained for $m = 0$ by calculating I itself directly for $m = 0$ by the DR rules

$$I_{pp}(m=0) = \int d\vec{k}\, k^{-2}(p-k)^{-2} = \frac{\pi^{3-\varepsilon}\Gamma^2(2-\varepsilon)\Gamma(\varepsilon-1)}{\Gamma(4-2\varepsilon)}\, p^{2-2\varepsilon}. \tag{37}$$

The extension to multiloop diagrams is made directly: the diagram is represented by the sum of all modifications with hard and soft lines, the hard lines are expanded in a series in soft impulses, "soft" blocks of the type $b_n$ are evaluated for $\mu = \infty$ by the DR rules, integration in the hard blocks is understood as the "total minus the soft"; hence the hard blocks, including the original graphs with all the hard lines, are calculated (neglecting contributions analytic in $m^2$) directly for $m = \mu = 0$ by DR rules for massless diagrams of the type (37).

The absence of the $\mu$ cutoff and utilization of standard dimensional regularization formulas [10] significantly simplify the practical computations.

The extension to the model (6) is not difficult. In particular, $\langle\langle v^n \rangle\rangle = \langle \varphi_s{}^n \rangle$ in approximation (33) is replaced by $\langle \varphi^n \rangle$, while $D(\omega, p; \mu)$ is evaluated (neglecting analytic contributions in $m^2$) directly for $m = \mu = 0$ by the DR rules and is described adequately by an $\varepsilon$-expression. Expanding (33) afterward in $\langle \varphi^n \rangle$, we find an $\varepsilon$-expansion of the coefficients $c_n$ $(m = 0)$ in (18) exactly in a $\omega$, p-representation.

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