1.33. In most treatments of Lie groups, the Lie algebra is defined as the space of *left*-invariant vector fields on the Lie group rather than the right-invariant vector fields employed here. In this problem we compare these two approaches.

(a) Define what is meant by a left-invariant vector field on a Lie group G. Prove that the space of all left-invariant vector fields on G forms a Lie algebra, denoted $\mathfrak{g}_L$, which we can identify with $TG|_e$.

(b) Using subscripts L and R to denote the two Lie algebras, prove that $[\mathbf{v}, \mathbf{w}]_L = -[\mathbf{v}, \mathbf{w}]_R$ where $\mathbf{v}, \mathbf{w}$ are identified with their values at e.

(c) Let G act on a manifold M as in Definition 1.25. If $\mathbf{v} \in \mathfrak{g}_L$ or $\mathfrak{g}_R$, we can define the corresponding infinitesimal generator $\psi(\mathbf{v})$ on M. Show that while (1.47) holds for right-invariant vector fields it is *false* for left-invariant vector fields. What happens to the Lie bracket formula (1.47) in this case? On the other hand, prove that if $\mathbf{v}$ is a left-invariant vector field and $\Psi_g(x) = \Psi(g, x)$, then

$$d\Psi_g(\psi(\mathbf{v})|_x) = \psi(\mathbf{v})|_{g\cdot x},$$

i.e. the infinitesimal generators of the action of G on M behave naturally with respect to the group transformations. Show that this is *false* for right-invariant vector fields.

(d) How does this all change if we let G act on M on the *right*, i.e. set $x \cdot g = \Psi(x, g)$ with $x \cdot (g \cdot h) = (x \cdot g) \cdot h$?
(Marsden, Ratiu and Weinstein, [1].)

1.34. Let $\boldsymbol{\alpha} = (\alpha, \beta, \gamma)$ be a vector field on $\mathbb{R}^3$ with $\nabla \cdot \boldsymbol{\alpha} = 0$. Use the homotopy operator (1.69) to construct a vector field $\boldsymbol{\lambda}$ with $\nabla \times \boldsymbol{\lambda} = \boldsymbol{\alpha}$. Similarly, if $\nabla \times \boldsymbol{\lambda} = \mathbf{0}$, find a function f with $\boldsymbol{\lambda} = \nabla f$.

1.35. (a) Let $\mathbf{v}$ and $\mathbf{w}$ be vector fields, ω a one-form. Prove that

$$\mathbf{v}\langle \omega; \mathbf{w}\rangle = \langle \mathbf{v}(\omega); \mathbf{w}\rangle + \langle \omega; [\mathbf{v}, \mathbf{w}]\rangle.$$

(b) More generally, if ω is a k-form prove that

$$\mathbf{v}\langle \omega; \mathbf{w}_1, \ldots, \mathbf{w}_k\rangle = \langle \mathbf{v}(\omega); \mathbf{w}_1, \ldots, \mathbf{w}_k\rangle + \sum_{i=1}^{k} \langle \omega; \mathbf{w}_1, \ldots, [\mathbf{v}, \mathbf{w}_i], \ldots, \mathbf{w}_k\rangle.$$

(c) Deduce that

$$\mathbf{v}(\mathbf{w} \,\lrcorner\, \omega) = \mathbf{w} \,\lrcorner\, \mathbf{v}(\omega) + [\mathbf{v}, \mathbf{w}] \,\lrcorner\, \omega.$$

1.36. Let $\omega = dx^1 \wedge \cdots \wedge dx^m$ be the volume m-form on $\mathbb{R}^m$ and let $\mathbf{v} = \sum \xi^i(x)\partial/\partial x^i$ be a vector field.

(a) Prove that the Lie derivative of the volume form is $\mathbf{v}(\omega) = \operatorname{div} \xi \cdot \omega$, where $\operatorname{div} \xi = \sum \partial \xi^i/\partial x^i$ is the ordinary divergence.

(b) Prove that the flow $\phi_\varepsilon = \exp(\varepsilon \mathbf{v})$ generated by $\mathbf{v}$ preserves volume, meaning $\operatorname{Vol}(\phi_\varepsilon[S]) = \operatorname{Vol}(S)$ for any $S \subset \mathbb{R}^m$ such that $\phi_\varepsilon(x)$ is defined for all $x \in S$, if and only if $\operatorname{div} \xi = 0$ everywhere.

1.37. Let $\partial_i = \partial/\partial x^i$, and dx^i, $i = 1, \ldots, m$, be the standard bases for $T\mathbb{R}^m$ and $T^*\mathbb{R}^m$ respectively. Let ω be an r-form on $\mathbb{R}^m$. Prove the following formulae:

$$\partial_k \,\lrcorner\, (dx^l \wedge \omega) = -dx^l \wedge (\partial_k \,\lrcorner\, \omega), \quad \text{whenever} \quad k \neq l,$$

$$\partial_k \,\lrcorner\, (dx^k \wedge \omega) = \omega - dx^k \wedge (\partial_k \,\lrcorner\, \omega),$$

$$\sum_{k=1}^{m} \partial_k \,\lrcorner\, (dx^k \wedge \omega) = (m - r)\omega.$$

CHAPTER 2

Symmetry Groups of Differential Equations

The symmetry group of a system of differential equations is the largest local group of transformations acting on the independent and dependent variables of the system with the property that it transform solutions of the system to other solutions. The main goal of this chapter is to determine a useful, systematic, computational method that will explicitly determine the symmetry group of any given system of differential equations. We restrict our attention to connected local Lie groups of symmetries, leaving aside problems involving discrete symmetries such as reflections, in order to take full advantage of the infinitesimal techniques developed in the preceding chapter. Before pressing on to the case of differential equations, it is vital that we deal adequately with the simpler situation presented by symmetry groups of systems of algebraic equations, and this is done in the first section. Section 2.2 investigates the precise definition of a symmetry group of a system of differential equations, which requires knowledge of how the group elements actually transform the solutions. The corresponding infinitesimal method rests on the important concept of "prolonging" a group action to the spaces of derivatives of the dependent variables represented in the system. The key "prolongation formula" for an infinitesimal generator of a group of transformations, given in Theorem 2.36, then provides the basis for the systematic determination of symmetry groups of differential equations. Applications to physically important partial differential equations, including the heat equation, Burgers' equation, the Korteweg-de Vries equation and Euler's equations for ideal fluid flow are presented in Section 2.4.

In the case of ordinary differential equations, Lie showed how knowledge of a one-parameter symmetry group allows us to reduce the order of the equation by one. In particular, a first order equation with a known one-parameter symmetry group can be integrated by a single quadrature. The situation is more delicate in the case of higher dimensional symmetry groups; it is not in general possible to reduce the order of an equation invariant under an r-parameter symmetry group by r using only quadratures. We will discuss in detail how the theory proceeds for multi-parameter symmetry groups of higher order equations and systems of ordinary differential equations.

The last section of this chapter deals with some more technical mathematical issues, and may safely be omitted by an application-oriented reader at first. The basic converse to the theorem on existence of symmetry groups says when one can conclude that every (continuous) symmetry group has been obtained by the above methods. Besides the algebraic maximal rank condition, an additional existence result known as "local solvability" is required. In the case of analytic systems, these questions are related to the problem of existence of noncharacteristic directions for the system, relative to which the Cauchy–Kovalevskaya existence theorem is applicable. Such systems are designated as "normal systems", but there do exist "abnormal systems", of which several examples are presented. The correct understanding of these matters will be crucial to the formulation and proof of Noether's theorems relating symmetry groups and conservation laws to be presented in Chapter 5.

2.1. Symmetries of Algebraic Equations

Before considering symmetry groups of differential equations, it is essential that we deal properly with the conceptually simpler case of symmetry groups of systems of algebraic equations. By a "system of algebraic equations" we mean a system of equations

$$F_\nu(x) = 0, \qquad \nu = 1, \ldots, l,$$

in which $F_1(x), \ldots, F_l(x)$ are smooth real-valued functions defined for x in some manifold M. (Note that the adjective "algebraic" is only used to distinguish this case from the case of systems of differential equations; it does *not* mean that the F_ν must be polynomials—just any differentiable functions.) A *solution* is a point $x \in M$ such that $F_\nu(x) = 0$ for $\nu = 1, \ldots, l$. A *symmetry group* of the system will be a local group of transformations G acting on M with the property that G transforms solutions of the system to other solutions. In other words, if x is a solution, g a group element and $g \cdot x$ is defined, then we require that $g \cdot x$ also be a solution. In this section we will be primarily concerned with finding easily verifiable conditions that a given group of transformations be a symmetry group of such a system.

Invariant Subsets

More generally, we can look at symmetry groups of arbitrary subsets of the given manifold.

Definition 2.1. Let G be a local group of transformations acting on a manifold M. A subset $\mathcal{S} \subset M$ is called *G-invariant*, and G is called a *symmetry group* of $\mathcal{S}$, if whenever $x \in \mathcal{S}$, and $g \in G$ is such that $g \cdot x$ is defined, then $g \cdot x \in \mathcal{S}$.

Example 2.2. Let $M = \mathbb{R}^2$.

(a) If G_c is the one-parameter group of translations

$$(x, y) \mapsto (x + c\varepsilon, y + \varepsilon), \qquad \varepsilon \in \mathbb{R},$$

where c is some fixed constant, then the lines $x = cy + d$ are easily seen to be G_c-invariant, being precisely the orbits of G_c. It can also be readily seen that any invariant subset of $\mathbb{R}^2$ is just the union of some collection of such lines. For example, the strip $\{(x, y): k_1 < x - cy < k_2\}$ is G_c-invariant.

(b) As a second elementary example, let G^α be the one-parameter group of scale transformations

$$(x, y) \mapsto (\lambda x, \lambda^\alpha y), \qquad \lambda > 0,$$

where α is a constant. The origin $(0, 0)$ is a G^α-invariant subset, as are the positive and negative x- and y-axes, e.g. $\{(x, 0): x > 0\}$. Also, the axes themselves, being unions of invariant subsets, are also invariant. Thus the subvariety $\{(x, y): xy = 0\}$ consisting of both coordinate axes is invariant. Other invariant sets are of the form $y = k|x|^\alpha$ for $x > 0$ or $x < 0$, and unions of these orbits of G^α.

In most of our applications, the set $\mathcal{S}$ will be the set of solutions or *subvariety* determined by the common zeros of a collection of smooth functions $F = (F_1, \ldots, F_l)$,

$$\mathcal{S} = \mathcal{S}_F = \{x: F_\nu(x) = 0, \nu = 1, \ldots, l\}.$$

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are G-invariant sets, so are $\mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{S}_1 \cap \mathcal{S}_2$.

Invariant Functions

Besides looking at the symmetries of the solution set of a system of algebraic equations, we can look at the symmetries of the function $F(x)$ which defines them.

Definition 2.3. Let G be a local group of transformations acting on a manifold M. A function $F: M \to N$, where N is another manifold, is called a *G-invariant function* if for all $x \in M$ and all $g \in G$ such that $g \cdot x$ is defined,

$$F(g \cdot x) = F(x).$$

A real-valued G-invariant function $\zeta: M \to \mathbb{R}$ is simply called an *invariant* of G. Note that $F: M \to \mathbb{R}^l$ is G-invariant if and only if each component F_ν of $F = (F_1, \ldots, F_l)$ is an invariant of G.

Example 2.4. (a) Let G_c be the group of translations in the plane presented in Example 2.2(a). Then the function

$$\zeta(x, y) = x - cy$$

is an invariant of G_c since

$$\zeta(x + c\varepsilon, y + \varepsilon) = \zeta(x, y)$$

for all ε. In fact, it is not difficult to see that *every* invariant of this translation group is of the form $\tilde{\zeta}(x, y) = f(x - cy)$, where f is a smooth function of the single variable $x - cy$.

(b) For the scaling group

$$G^1\colon (x, y) \mapsto (\lambda x, \lambda y), \qquad \lambda > 0,$$

the function

$$\zeta(x, y) = x/y$$

is an invariant defined on the upper and lower half planes $\{y \neq 0\}$. Other invariants include the angular coordinate $\theta = \tan^{-1}(y/x)$ which is smooth on $\mathbb{R}^2 \setminus \{(x, y)\colon x \leqslant 0\}$, say, but not globally single-valued, and the function

$$\tilde{\zeta}(x, y) = xy/(x^2 + y^2),$$

which is smooth everywhere except at the origin. There is, in this case, no smooth, nonconstant, globally defined invariant of G^1. Similar remarks apply to the more general scaling groups G^α of Example 2.2(b) when $\alpha > 0$.

If $F\colon M \to \mathbb{R}^l$ is a G-invariant function, then clearly every level set of F is a G-invariant subset of M. However, it is *not* true that if the set of zeros of a smooth function, $\{x\colon F(x) = 0\}$, is an invariant subset of M, then the function itself is invariant. For instance, as we saw in the previous example $\{(x, y)\colon xy = 0\}$ is an invariant subset of the scaling group G^1. However, $F(x, y) = xy$ is not an invariant function for this group since

$$F(\lambda x, \lambda y) = \lambda^2 xy \neq F(x, y)$$

for $\lambda \neq 1$. However, if *every* level set of F is invariant, then F is an invariant function.

Proposition 2.5. *If G acts on M, and $F\colon M \to \mathbb{R}^l$ is a smooth function, then F is a G-invariant function if and only if every level set $\{F(x) = c\}$, $c \in \mathbb{R}^l$, is a G-invariant subset of M.*

The proof of this result is left to the reader. Thus in Example 2.4(a), the lines $x = cy + d$ are just the level sets of the G_c-invariant function $\zeta(x, y) = x - cy$ and hence are automatically G_c-invariant subsets. Alternatively, the G_c-invariance of ζ follows from the fact that each level set is G_c-invariant, indeed an orbit of G_c.

Another way of looking at the preceding observations is that the "symmetry group" of the solution set $\mathcal{S}_F = \{F(x) = 0\}$ of some system of algebraic equations is, in general, larger than the "symmetry group" of the function F determining it. Here "symmetry group" means, somewhat imprecisely, the largest group of transformations leaving the subvariety or function invariant. For algebraic equations, such a group will not usually be finite dimensional, but the idea underlying these remarks should be clear. The importance of widening our concept of symmetry to those of the solution set, rather than the defining functions, will become evident when we treat symmetry groups of differential equations, and will lead to a much wider variety of symmetry groups.

Infinitesimal Invariance

The great power of Lie group theory lies in the crucial observation that one can replace the complicated, nonlinear conditions for the invariance of a subset or function under the group transformations themselves by an equivalent linear condition of infinitesimal invariance under the corresponding infinitesimal generators of the group action. This infinitesimal criterion will be readily verifiable in practice, and will thereby provide the key to the explicit determination of the symmetry groups of systems of differential equations. Its importance cannot be overemphasized. We begin with the simpler case of an invariant function. Here the infinitesimal criterion for invariance follows directly from the basic formula describing how functions change under the flow generated by a vector field.

Proposition 2.6. *Let G be a connected group of transformations acting on the manifold M. A smooth real-valued function $\zeta\colon M \to \mathbb{R}$ is an invariant function for G if and only if*

$$\mathbf{v}(\zeta) = 0 \quad \textit{for all} \quad x \in M, \tag{2.1}$$

and every infinitesimal generator $\mathbf{v}$ of G.

Proof. According to (1.17), if $x \in M$,

$$\frac{d}{d\varepsilon}\zeta(\exp(\varepsilon\mathbf{v})x) = \mathbf{v}(\zeta)[\exp(\varepsilon\mathbf{v})x]$$

whenever $\exp(\varepsilon\mathbf{v})x$ is defined. Setting $\varepsilon = 0$ proves the necessity of (2.1). Conversely, if (2.1) holds everywhere, then

$$\frac{d}{d\varepsilon}\zeta(\exp(\varepsilon\mathbf{v})x) = 0$$

where defined, hence $\zeta(\exp(\varepsilon\mathbf{v})x)$ is a constant for the connected, local one-parameter subgroup $\exp(\varepsilon\mathbf{v})$ of $G_x = \{g \in G\colon g\cdot x \text{ is defined}\}$. But by (1.40), every element of G_x can be written as a finite product of exponentials of infinitesimal generators $\mathbf{v}_i$ of G, hence $\zeta(g \cdot x) = \zeta(x)$ for all $g \in G_x$. □

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathfrak{g}$, the Lie algebra of infinitesimal generators of G, then Proposition 2.6 says that $\zeta(x)$ is an invariant if and only if $\mathbf{v}_k(\zeta) = 0$ for $k = 1, \ldots, r$. In local coordinates,

$$\mathbf{v}_k = \sum_{i=1}^{m} \xi_k^i(x) \frac{\partial}{\partial x^i},$$

so ζ must be a solution to the homogeneous system of linear, first order partial differential equatons

$$\mathbf{v}_k(\zeta) = \sum_{i=1}^{m} \xi_k^i(x) \frac{\partial \zeta}{\partial x^i} = 0, \qquad k = 1, \ldots, r. \tag{2.2}$$

Example 2.7. For the translation group G_c of Example 2.4(a) the infinitesimal generator is $\mathbf{v} = c\partial_x + \partial_y$. Then

$$\mathbf{v}(x - cy) = (c\partial_x + \partial_y)(x - cy) = c - c = 0,$$

so the infinitesimal criterion is satisfied. A similar computation verifies the infinitesimal criterion (2.1) for the invariants of the scale group G^α, whose infinitesimal generator is $x\partial_x + \alpha y \partial_y$.

For the case of the solution set of a system of algebraic equations $F(x) = 0$, the infinitesimal criterion of invariance requires additional conditions to be placed on the defining functions F, namely the maximal rank condition of Definition 1.7. (If F happens to be a G-invariant function, then by Proposition 2.5 this maximal rank condition can be dropped, but in general it is essential.)

Theorem 2.8. *Let G be a connected local Lie group of transformations acting on the m-dimensional manifold M. Let $F\colon M \to \mathbb{R}^l$, $l \leqslant m$, define a system of algebraic equations*

$$F_\nu(x) = 0, \qquad \nu = 1, \ldots, l,$$

and assume that the system is of maximal rank, *meaning that the Jacobian matrix $(\partial F_\nu / \partial x^k)$ is of rank l at every solution x of the system. Then G is a symmetry group of the system if and only if*

$$\mathbf{v}[F_\nu(x)] = 0, \qquad \nu = 1, \ldots, l, \quad \textit{whenever} \quad F(x) = 0, \tag{2.3}$$

for every infinitesimal generator $\mathbf{v}$ of G. (Note especially that (2.3) is required to hold only for solutions x of the system.)

Proof. The necessity of (2.3) follows from differentiating the identity

$$F(\exp(\varepsilon \mathbf{v})x) = 0,$$

in which x is a solution, and $\mathbf{v}$ is an infinitesimal generator of G, with respect to ε and setting $\varepsilon = 0$.

To prove sufficiency, let x_0 be a solution to the system. Using the maximal rank condition, we can choose local coordinates $y = (y^1, \ldots, y^m)$ such that $x_0 = 0$ and F has the simple form $F(y) = (y^1, \ldots, y^l)$, cf. Theorem 1.8. Let

$$\mathbf{v} = \xi^1(y)\frac{\partial}{\partial y^1} + \cdots + \xi^m(y)\frac{\partial}{\partial y^m}$$

be any infinitesimal generator of G, expressed in the new coordinates. Condition (2.3) means that

$$\mathbf{v}(y^\nu) = \xi^\nu(y) = 0, \qquad \nu = 1, \ldots, l, \tag{2.4}$$

whenever $y^1 = y^2 = \cdots = y^l = 0$. Now the flow $\phi(\varepsilon) = \exp(\varepsilon\mathbf{v})\cdot x_0$ of $\mathbf{v}$ through $x_0 = 0$ satisfies the system of ordinary differential equations

$$\frac{d\phi^i}{d\varepsilon} = \xi^i(\phi(\varepsilon)), \qquad \phi^i(0) = 0, \quad i = 1, \ldots, m.$$

By (2.4) and the uniqueness of solutions to this initial-value problem, we conclude that $\phi^\nu(\varepsilon) = 0$ for $\nu = 1, \ldots, l$, and ε sufficiently small. We have thus shown that if x_0 is a solution to $F(x) = 0$, $\mathbf{v}$ is an infinitesimal generator of G, and ε is sufficiently small, then $\exp(\varepsilon\mathbf{v})x_0$ is again a solution to the system. Since the solution set $\mathscr{S}_F = \{x\colon F(x) = 0\}$ is closed, the group property (1.13) and continuity of $\exp(\varepsilon\mathbf{v})$ allows us to draw the same conclusion for all $g = \exp(\varepsilon\mathbf{v})$ in the connected one-parameter subgroup of G_{x_0} generated by $\mathbf{v}$. Another application of (1.40), similar to that in the proof of Proposition 2.6, completes the proof of the theorem in general. □

Example 2.9. Let $G = \mathrm{SO}(2)$ be the rotation group in the plane, with infinitesimal generator $\mathbf{v} = -y\partial_x + x\partial_y$. The unit circle $S^1 = \{x^2 + y^2 = 1\}$ is an invariant subset for SO(2) as it is the solution set of the invariant function $\zeta(x, y) = x^2 + y^2 - 1$; indeed

$$\mathbf{v}(\zeta) = -2xy + 2xy = 0$$

everywhere, so (2.3) is verified on the unit circle itself. The maximal rank condition does hold for ζ since its gradient $\nabla\zeta = (2x, 2y)$ does not vanish on S^1, but, as remarked before the theorem, since ζ is already an invariant function, we don't really need to check this.

As a less trivial example, consider the function

$$F(x, y) = x^4 + x^2y^2 + y^2 - 1.$$

We have

$$\mathbf{v}(F) = -4x^3y - 2xy^3 + 2x^3y + 2xy = -2xy(x^2 + 1)^{-1}F(x, y),$$

hence $\mathbf{v}(F) = 0$ whenever $F = 0$. Moreover,

$$\nabla F = (4x^3 + 2xy^2, 2x^2y + 2y)$$

vanishes only when $x = y = 0$, which is not a solution to $F(x, y) = 0$, hence the maximal rank condition is verified. We conclude that the solution set $\{(x,y): x^4 + x^2y^2 + y^2 = 1\}$ is a rotationally-invariant subset of $\mathbb{R}^2$. Indeed, we can factor F as

$$x^4 + x^2y^2 + y^2 - 1 = (x^2 + 1)(x^2 + y^2 - 1),$$

hence the solution set is just the unit circle. Note that $F(x, y)$ is *not* an SO(2)-invariant function in this case; in fact, the other level sets of F are *not* rotationally invariant.

Finally, to appreciate the importance of the maximal rank condition, consider the function

$$H(x, y) = y^2 - 2y + 1.$$

The solution set $\{H(x, y) = 0\}$ is just the horizontal line $\{y = 1\}$ which is certainly not rotationally invariant. However,

$$\mathbf{v}(H) = 2xy - 2x = 2x(y - 1) = 0$$

whenever $H(x, y) = 0$, so the infinitesimal condition (2.3) does hold in this case. The problem is that $\nabla H = (0, 2y - 2)$ vanishes everywhere on the solution set, so that the maximal rank condition fails to hold.

The maximal rank condition needed to apply our infinitesimal symmetry criterion will play a key role in the development of the theory, both for algebraic and differential equations. We will subsequently need several elementary consequences of this condition, which we state here for ease of reference. The proofs are outlined in Exercise 2.5.

Proposition 2.10. *Let $F: M \to \mathbb{R}^l$ be of maximal rank on the subvariety $\mathcal{S}_F = \{x: F(x) = 0\}$. Then a real-valued function $f: M \to \mathbb{R}$ vanishes on $\mathcal{S}_F$ if and only if there exist smooth functions $Q_1(x), \ldots, Q_l(x)$ such that*

$$f(x) = Q_1(x)F_1(x) + \cdots + Q_l(x)F_l(x), \tag{2.5}$$

for all $x \in M$.

Again, the maximal rank condition is essential. For example, suppose $F(x, y) = y^2 - 2y + 1$. Then the function $f(x, y) = y - 1$ vanishes for all solutions of $F(x, y) = 0$, namely $\mathcal{S}_F = \{y = 1\}$, but there is no smooth function $Q(x, y)$ such that $f(x, y) = Q(x, y)F(x, y)$.

Proposition 2.10 says that we can replace the infinitesimal criterion (2.3) for invariance by the equivalent condition

$$\mathbf{v}(F_\nu) = \sum_{\mu=1}^{l} Q_{\nu\mu}(x)F_\mu(x), \qquad \nu = 1, \ldots, l, \qquad x \in M, \tag{2.6}$$

for functions $Q_{\nu\mu}: M \to \mathbb{R}$, $\mu, \nu = 1, \ldots, l$, to be determined. This was indeed how we proved invariance in the second case in Example 2.9, with $Q(x, y) =$

$-2xy/(x^2 + 1)$. *Both* (2.3) and (2.6) are useful conditions for checking invariance, and will both be employed in various examples.

The functions $Q_\nu(x)$ in (2.5) are not in general uniquely determined. For example, let.

$$F_1(x, y, z) = x, \qquad F_2(x, y, z) = y,$$

so the solution set $\mathscr{S} = \{F_1 = F_2 = 0\}$ is the z-axis in $\mathbb{R}^3$. The function

$$f(x, y, z) = xz + y^2$$

vanishes on $\mathscr{S}$, and indeed can be written both as

$$f = zF_1 + yF_2, \quad \text{or} \quad f = (z - y)F_1 + (x + y)F_2.$$

In general, if

$$f(x) = \sum_\nu Q_\nu(x)F_\nu(x) = \sum_\nu \tilde{Q}_\nu(x)F_\nu(x),$$

then the differences $R_\nu(x) = Q_\nu(x) - \tilde{Q}_\nu(x)$ satisfy the homogeneous system

$$\sum_{\nu=1}^{l} R_\nu(x)F_\nu(x) = 0 \tag{2.7}$$

for all $x \in M$. The folowing provides a useful necessary condition for such functions.

Proposition 2.11. *Let $F\colon M \to \mathbb{R}^l$ be of maximal rank on $\mathscr{S}_F = \{F(x) = 0\}$. Suppose $R_1(x), \ldots, R_l(x)$ are real-valued functions satisfying* (2.7) *for all $x \in M$. Then $R_\nu(x) = 0$ for all $x \in \mathscr{S}_F$. Equivalently, there exist functions $S_\nu^\mu(x)$,* for *$\nu, \mu = 1, \ldots, l$, such that*

$$R_\nu(x) = \sum_{\mu=1}^{l} S_\nu^\mu(x)F_\mu(x), \qquad x \in M. \tag{2.8}$$

Moreover, the S_ν^μ can be chosen to be skew-symmetric in their indices:

$$S_\nu^\mu(x) = -S_\mu^\nu(x),$$

in which case (2.8) *is necessary and sufficient for* (2.7) *to hold everywhere.*

Local Invariance

It is also useful to introduce the concept of a locally-invariant function or subset for a group of transformations. In this case we only require invariance for group transformations sufficiently near the identity.

Definition 2.12. Let G be a local group of transformations, acting on the manifold M. A subset $\mathscr{S} \subset M$ is called *locally G-invariant* if for every $x \in \mathscr{S}$ there is a neighbourhood $\tilde{G}_x \subset G_x$ of the identity in G such that $g \cdot x \in \mathscr{S}$ for