Next consider the variational differential of a functional one-form, which we take in canonical form $\omega = \int \{P \cdot du\} \, dx$. We find

$$\delta\omega = \int \left\{ \sum_\alpha \sum_{\beta, J} \frac{\partial P_\alpha}{\partial u_J^\beta} \, du_J^\beta \wedge du^\alpha \right\} dx = \int \{\mathsf{D}_P(du) \wedge du\} \, dx,$$

where D_P is the Fréchet derivative of P, cf. (5.32). As in (5.120), we can integrate by parts a second time, leading to the canonical form

$$\delta\omega = \tfrac{1}{2} \int \{du \wedge (\mathsf{D}_P^* - \mathsf{D}_P) \, du\} \, dx.$$

In particular, ω is closed if and only if D_P is a self-adjoint differential operator. Exactness of the variational complex, coupled with the explicit form for the homotopy operator (5.122), thus gives the complete solution to the problem of characterizing the image of the Euler–Lagrange operator.

Theorem 5.92. *Let $P[u] \in \mathscr{A}^p$ be defined over a vertically star-shaped region $M \subset X \times U$. Then P is the Euler–Lagrange expression for some variational problem $\mathscr{L} = \int L \, dx$, i.e. $P = \mathsf{E}(L)$, if and only if the Fréchet derivative D_P is self-adjoint: $\mathsf{D}_P^* = \mathsf{D}_P$. In this case, a Lagrangian for P can be explicitly constructed using the homotopy formula*

$$L[u] = \int_0^1 u \cdot P[\lambda u] \, d\lambda. \tag{5.123}$$

Example 5.93. Let $p = q = 1$. The functional

$$\mathscr{L}[u] = \int (\tfrac{1}{2} u_{xx}^2 - u u_x^2) \, dx$$

has Euler–Lagrange expression

$$\mathsf{E}(L)[u] = P[u] = u_{xxxx} + 2u u_{xx} + u_x^2.$$

The Fréchet derivative of P is the ordinary differential operator

$$\mathsf{D}_P = D_x^4 + 2u D_x^2 + 2u_x D_x + 2u_{xx},$$

which is easily seen to be self-adjoint. If, on the other hand, we were just given P, we could reconstruct a variational problem using (5.123),

$$\mathscr{L}[u] = \int \left\{ \int_0^1 u(\lambda u_{xxxx} + 2\lambda^2 u u_{xx} + \lambda^2 u_x^2) \, d\lambda \right\} dx$$

$$= \int \{\tfrac{1}{2} u u_{xxxx} + \tfrac{2}{3} u^2 u_{xx} + \tfrac{1}{3} u u_x^2\} \, dx.$$

The Lagrangian, while not the same as the original one, is still equivalent, since

$$\tfrac{1}{2} u u_{xxxx} + \tfrac{2}{3} u^2 u_{xx} + \tfrac{1}{3} u u_x^2 = \tfrac{1}{2} u_{xx}^2 - u u_x^2 + D_x(\tfrac{1}{2} u u_{xxx} - \tfrac{1}{2} u_x u_{xx} + \tfrac{2}{3} u^2 u_x).$$

We have thus solved Helmholtz' version of the inverse problem of the calculus of variations: characterizing those q-tuples $P[u] \in \mathcal{A}^q$ which are Euler–Lagrange expressions. (The conditions requiring the self-adjointness of the operator D_P are often referred to as the *Helmholtz conditions*.) Although the solution is very neat, from the wider perspective of determining which systems of differential equations $\Delta = 0$ arise from variational principles, it is somewhat unsatisfactory. If one happens to write the equations in the "wrong" order, say $\Delta_1 = E_2(L)$, $\Delta_2 = E_1(L)$, etc., then the Helmholtz conditions for Δ will not hold, and the variational structure of the system will remain undiscovered. Even more difficult to detect will be when Δ is *equivalent* to a set of Euler–Lagrange equations, so $\Delta = A \cdot E(L)$ for some invertible $q \times q$ matrix of differential functions A, or, even more generally, $\Delta = \mathcal{D}E(L)$ for some differential operator $\mathcal{D}$. The solution to the general equivalence problem is unknown, even in the case when A is constant matrix! (Some special cases have been considered—see the notes at the end of the chapter.)

Higher Euler Operators

Although the D-complex was perhaps simpler to write down, the construction of a suitable homotopy operator is considerably more complicated. The usual de Rham formula no longer works, and we are forced to introduce the so-called "higher Euler operators". These arise most naturally through a detailed analysis of the fundamental integration by parts formula (4.39) used in the proof of Noether's theorem.

Definition 5.94. For each $1 \leqslant \alpha \leqslant q$ and each multi-index J, the *higher Euler operators* E_α^J are defined so that the formula

$$\text{pr } \mathbf{v}_Q(P) = \sum_{\alpha=1}^{q} \sum_J D_J(Q_\alpha \cdot E_\alpha^J(P)) \qquad (5.124)$$

holds for every evolutionary vector field $\mathbf{v}_Q$ and every differential function $P \in \mathcal{A}$.

   The fact that (5.124) serves to uniquely determine these operators can perhaps best be appreciated through an example.

Example 5.95. Let $p = q = 1$, so there are Euler operators $E^{(0)}$, $E^{(1)}$, $E^{(2)}$, etc., satisfying

$$\text{pr } \mathbf{v}_Q(P) = Q\,E^{(0)}(P) + D_x(Q\,E^{(1)}(P)) + D_x^2(Q\,E^{(2)}(P)) + \cdots \qquad (5.125)$$

for general $P = P(x, u, u_x, \ldots)$. Suppose $P = P(x, u, u_x, u_{xx})$ depends only on second order derivatives, so

$$\text{pr } \mathbf{v}_Q(P) = Q\frac{\partial P}{\partial u} + D_x Q\frac{\partial P}{\partial u_x} + D_x^2 Q\frac{\partial P}{\partial u_{xx}}.$$

To rewrite this in the form (5.125), we must integrate the second and third terms by parts:

$$D_x Q \cdot \frac{\partial P}{\partial u_x} = -Q \cdot D_x \frac{\partial P}{\partial u_x} + D_x \left(Q \frac{\partial P}{\partial u_x} \right),$$

$$D_x^2 Q \cdot \frac{\partial P}{\partial u_{xx}} = Q \cdot D_x^2 \frac{\partial P}{\partial u_{xx}} - 2D_x \left(Q \cdot D_x \frac{\partial P}{\partial u_{xx}} \right) + D_x^2 \left(Q \frac{\partial P}{\partial u_{xx}} \right).$$

Comparing with (5.125), we see that for such P,

$$\mathsf{E}^{(0)}(P) = \frac{\partial P}{\partial u} - D_x \frac{\partial P}{\partial u_x} + D_x^2 \frac{\partial P}{\partial u_{xx}},$$

$$\mathsf{E}^{(1)}(P) = \frac{\partial P}{\partial u_x} - 2D_x \frac{\partial P}{\partial u_{xx}}, \qquad \mathsf{E}^{(2)}(P) = \frac{\partial P}{\partial u_{xx}}.$$

If we carry through the same procedure for general P, we find that (5.125) holds provided we set

$$\mathsf{E}^{(k)}(P) = \sum_{l=k}^{\infty} \binom{l}{k} (-D_x)^{l-k} \frac{\partial P}{\partial u_l},$$

so that

$$\mathsf{E}^{(0)}(P) = \frac{\partial P}{\partial u} - D_x \frac{\partial P}{\partial u_x} + D_x^2 \frac{\partial P}{\partial u_{xx}} - D_x^3 \frac{\partial P}{\partial u_{xxx}} + \cdots$$

agrees with the usual Euler operator, while

$$\mathsf{E}^{(1)}(P) = \frac{\partial P}{\partial u_x} - 2D_x \frac{\partial P}{\partial u_{xx}} + 3D_x^2 \frac{\partial P}{\partial u_{xxx}} - 4D_x^3 \frac{\partial P}{\partial u_{xxxx}} + \cdots,$$

$$\mathsf{E}^{(2)}(P) = \frac{\partial P}{\partial u_{xx}} - 3D_x \frac{\partial P}{\partial u_{xxx}} + 6D_x^2 \frac{\partial P}{\partial u_{xxxx}} - 10D_x^3 \frac{\partial P}{\partial u_{xxxxx}} + \cdots,$$

and so on.

To state the general formula for the higher Euler operators, we need some further multi-index notation. Let I, J be unordered multi-indices of the type introduced in Chapter 2. We say $J \subset I$ if all the indices in J appear in I. We write $I \backslash J$ for the remaining indices in I. For example, if $p = 4$, $J = (1, 1, 2, 4)$ is contained in $I = (1, 1, 1, 2, 4, 4)$ and $J \backslash I = (1, 4)$. Given $I = (i_1, \ldots, i_n)$, let $\tilde{I} = (\tilde{i}_1, \ldots, \tilde{i}_p)$ be the "transposed" ordered multi-index, where $\tilde{i}_j$ equals the number of occurrences of the integer j in I; for the above example, $\tilde{I} = (3, 1, 0, 2)$ since there are three 1's, one 2, no 3's and two 4's in I. Set $I! = \tilde{I}! = \tilde{i}_1! \tilde{i}_2! \cdots \tilde{i}_p!$, and define the multinomial coefficient $\binom{I}{J} = I!/(J!(I \backslash J)!)$ when $J \subset I$; 0 otherwise. In the above example, $I! = 3! \cdot 1! \cdot 0! \cdot 2! = 12$, $\binom{I}{J} = 12/(2 \cdot 1) = 6$.

Proposition 5.96. *Let* $1 \leqslant \alpha \leqslant q$, $\#J \geqslant 0$. *Then*

$$E_\alpha^J(P) = \sum_{I \supset J} \binom{I}{J} (-D)_{I \setminus J} \frac{\partial P}{\partial u_I^\alpha} \tag{5.126}$$

for all $P \in \mathcal{A}$.

PROOF. First note that

$$R \cdot D_I Q = \sum_{J \subset I} \binom{I}{J} D_J (Q \cdot (-D)_{I \setminus J} R) \tag{5.127}$$

for any I, a formula which is easy to prove by induction starting with the Leibniz rule $R D_i Q = D_i(QR) - Q D_i R$. Evaluating the left-hand side of (5.124), and using (5.127), we find

$$\text{pr } \mathbf{v}_Q(P) = \sum_{\alpha, I} D_I Q_\alpha \cdot \frac{\partial P}{\partial u_I^\alpha} = \sum_{\alpha, I} \sum_{J \subset I} D_J \left(Q_\alpha \cdot \binom{I}{J} (-D)_{I \setminus J} \frac{\partial P}{\partial u_I^\alpha} \right).$$

Interchanging the order of summation proves (5.126) and hence the uniqueness of the Euler operators. In particular, for $J = 0$, $E_\alpha^0 = E_\alpha$ agrees with the usual Euler operator. $\square$

Example 5.97. Let $p = 2$, $q = 1$ and let x, y denote the independent variables. Then, for instance,

$$E^{(x)}(P) = \frac{\partial P}{\partial u_x} - 2D_x \frac{\partial P}{\partial u_{xx}} - D_y \frac{\partial P}{\partial u_{xy}}$$

$$+ 3D_x^2 \frac{\partial P}{\partial u_{xxx}} + 2D_x D_y \frac{\partial P}{\partial u_{xxy}} + D_y^2 \frac{\partial P}{\partial u_{xyy}} - \cdots,$$

$$E^{(xy)}(P) = \frac{\partial P}{\partial u_{xy}} - 2D_x \frac{\partial P}{\partial u_{xxy}} - 2D_y \frac{\partial P}{\partial u_{xyy}}$$

$$+ 3D_x^2 \frac{\partial P}{\partial u_{xxxy}} + 4D_x D_y \frac{\partial P}{\partial u_{xxyy}} + 3D_y^2 \frac{\partial P}{\partial u_{xyyy}} - \cdots,$$

and so on.

Actually, for theoretical purposes, the precise formula for the E_α^J is not important; what is important is that they are uniquely determined by the integration by parts formula (5.124). As a first application, we find the explicit expression for the divergence in the key formula (4.39) used in Noether's theorem.

Proposition 5.98. *Let* $Q \in \mathcal{A}^q$, $L \in \mathcal{A}$. *Then*

$$\text{pr } \mathbf{v}_Q(L) = Q \cdot E(L) + \text{Div } A, \tag{5.128}$$

where

$$A_k = \sum_{\alpha=1}^{q} \sum_{\#I \geqslant 0} \frac{\tilde{\imath}_k + 1}{\#I + 1} D_I[Q_\alpha \mathsf{E}_\alpha^{I,k}(L)], \qquad k = 1, \ldots, p. \qquad (5.129)$$

PROOF. We compute

$$\text{Div } A = \sum_{\alpha=1}^{q} \sum_{\#I \geqslant 0} \sum_{k=1}^{p} \frac{\tilde{\imath}_k + 1}{\#I + 1} D_{I,k}[Q_\alpha \mathsf{E}_\alpha^{I,k}(L)].$$

Now change the summation variable to be $J = (I, k)$, so $\tilde{\imath}_k + 1 = \tilde{\jmath}_k$ and $\#I + 1 = \#J = \sum \tilde{\jmath}_k$. Thus the coefficient of $D_J[Q_\alpha \mathsf{E}_\alpha^J(L)]$ is unity. Comparing this with (5.124), we see that only the terms $Q_\alpha \mathsf{E}_\alpha(L)$ corresponding to $\#J = 0$ are missing. Thus (5.128) follows immediately.   □

The higher Euler operators are also intimately connected with the total derivatives.

Proposition 5.99. *Let* $1 \leqslant \alpha \leqslant q$, $1 \leqslant i \leqslant p$, $\#J \geqslant 0$. *Then*

$$\mathsf{E}_\alpha^J(D_i P) = \begin{cases} \mathsf{E}_\alpha^{J \setminus i}(P) & \text{if } i \subset J, \\ 0 & \text{if } i \not\subset J, \end{cases} \qquad (5.130)$$

for any $P \in \mathscr{A}$.

PROOF. Although this can be proved directly from (5.126), it is simpler to use the uniqueness properties of (5.124). We have

$$\text{pr } \mathbf{v}_Q(D_i P) = \sum_{\alpha, J} D_J[Q_\alpha \mathsf{E}_\alpha^J(D_i P)].$$

On the other hand, by (5.19), this equals

$$D_i \text{ pr } \mathbf{v}_Q(P) = \sum_{\alpha, K} D_i D_K[Q_\alpha \mathsf{E}_\alpha^K(P)].$$

Replacing K by $J = (K, i)$ and comparing the two expressions immediately gives (5.130) by uniqueness.   □

Corollary 5.100. *A differential function* P *is an "n-th order divergence", i.e. there exist* $Q_I \in \mathscr{A}$, $\#I = n$, *such that* $P = \sum D_I Q_I$, *if and only if* $\mathsf{E}_\alpha^J(P) = 0$ *for all* $\alpha = 1, \ldots, q$, $0 \leqslant \#J \leqslant n - 1$.

The Total Homotopy Operator

As in our proof of the Poincaré lemma in Section 1.5, the construction of the homotopy operator for the D-complex rests on a formula for the Lie derivative of a total differential form with respect to an evolutionary vector field. To establish this result, we begin by noting that any operator, such as a total

derivative, higher Euler operator or prolonged vector field, which acts on the space $\mathscr{A}$ of differential functions, can be made to act coefficient-wise on the total differential forms. For example, if $\omega = \sum P_I \, dx^I$, $P_I \in \mathscr{A}$, then

$$\text{pr } \mathbf{v}_Q(\omega) = \sum \text{pr } \mathbf{v}_Q(P_I) \, dx^I. \tag{5.131}$$

In particular, the total differential can be written as

$$D\omega = \sum_{i=1}^{p} D_i(dx^i \wedge \omega) = \sum_{i=1}^{p} dx^i \wedge D_i\omega, \tag{5.132}$$

the D_i's acting only on the coefficients of ω.

The first goal in our construction is to establish a formula mimicking (1.65), but for the total differential. Thus we need to find "interior product" operators $I_Q \colon \bigwedge_k \to \bigwedge_{k-1}$, $k = 1, \ldots, p$, $Q \in \mathscr{A}^q$, such that

$$\text{pr } \mathbf{v}_Q(\omega) = DI_Q(\omega) + I_Q(D\omega) \tag{5.133}$$

for any $\omega \in \bigwedge_r$, $0 < r < p$. It turns out that this *total interior product* can be written most succinctly in terms of the higher Euler operators:

$$I_Q(\omega) = \sum_{\alpha=1}^{q} \sum_{\#I \geqslant 0} \sum_{k=1}^{p} \frac{\tilde{i}_k + 1}{p - r + \#I + 1} D_I \left\{ Q_\alpha \mathsf{E}_\alpha^{I,k}\left(\frac{\partial}{\partial x^k} \lrcorner \, \omega \right) \right\}, \qquad \omega \in \bigwedge_r. \tag{5.134}$$

Before proving that this does satisfy (5.133), we look at a couple of special cases.

Example 5.101. If $\omega = L \, dx^1 \wedge \cdots \wedge dx^p$, then $I_Q(\omega) \in \bigwedge_{p-1}$ and hence has the form

$$I_Q(\omega) = \sum_{k=1}^{p} (-1)^{k-1} A_k \, dx^{\hat{k}}.$$

Since $(-1)^{k-1} \, dx^{\hat{k}} = \partial_{x^k} \lrcorner (dx^1 \wedge \cdots \wedge dx^p)$, (5.134) implies that

$$A_k = \sum_{\alpha, I} \frac{\tilde{i}_k + 1}{\#I + 1} D_I[Q_\alpha \mathsf{E}_\alpha^{I,k}(L)],$$

which recovers the divergence terms (5.129) in (5.128), which we can rewrite in "homotopy form"

$$\text{pr } \mathbf{v}_Q(\omega) = D(I_Q(\omega)) + Q \cdot \mathsf{E}(\omega). \tag{5.135}$$

Example 5.102. Let $r = p - 1$, so ω is of the form

$$\omega = \sum_{k=1}^{p} (-1)^{k-1} P_k \, dx^{\hat{k}}.$$

The $(p-2)$-form $I_Q(\omega)$ has the form

$$I_Q(\omega) = \sum_{j < k} (-1)^{j+k-1} R_{jk} \, dx^{\widehat{jk}},$$

where

$$R_{jk} = \sum_{\alpha=1}^{q} \sum_{\#I \geq 0} D_I \left\{ Q_\alpha \left(\frac{\tilde{i}_j + 1}{\#I + 2} E_\alpha^{I,j}(P_k) - \frac{\tilde{i}_k + 1}{\#I + 2} E_\alpha^{I,k}(P_j) \right) \right\}.$$

The Lie derivative formula (5.133) takes the form

$$\text{pr } \mathbf{v}_Q(P_k) = \sum_{j=1}^{p} D_j R_{jk} + A_k, \tag{5.136}$$

where A_k is given by (5.129) when $L = \text{Div } P$, which, using (5.130), is

$$A_k = \sum_{\alpha, I} \sum_{l \subset I} \frac{\tilde{i}_k + 1}{\#I + 1} D_I[Q_\alpha E_\alpha^{I,k \setminus l}(P_l)]. \tag{5.137}$$

(We leave to the reader the direct verification of (5.136).)

The proof of (5.133) is perhaps the most complex calculation of this book. (However, the present proof of the exactness of the D-complex is much easier than previous computational proofs!) We begin by analyzing the right-hand side using (5.132):

$$I_Q(D\omega) = \sum_{l=1}^{p} I_Q[D_l(dx^l \wedge \omega)]$$

$$= \sum_{\alpha, I} \sum_{k,l=1}^{p} \frac{\tilde{i}_k + 1}{p - r + \#I} D_I \left\{ Q_\alpha E_\alpha^{I,k} \left[\frac{\partial}{\partial x^k} \lrcorner D_l(dx^l \wedge \omega) \right] \right\}, \tag{5.138}$$

since $D\omega$ is an $(r + 1)$-form. The principal constituent in (5.138) is the interior summation

$$\sum_{k,l=1}^{p} (\tilde{i}_k + 1) E_\alpha^{I,k} \left[\frac{\partial}{\partial x^k} \lrcorner D_l(dx^l \wedge \omega) \right]$$

$$= \sum_{k,l=1}^{p} (\tilde{i}_k + 1) E_\alpha^{I,k} \left[D_l \left(\frac{\partial}{\partial x^k} \lrcorner (dx^l \wedge \omega) \right) \right], \tag{5.139}$$

which we break into two pieces according to whether $k = l$ or $k \neq l$. If $k \neq l$, then by (5.130), $E_\alpha^{I,k} \cdot D_l = E_\alpha^{I \setminus l, k}$, where, by convention, this operator is 0 if l does not appear in I. Also, according to Exercise 1.37,

$$\frac{\partial}{\partial x^k} \lrcorner (dx^l \wedge \omega) = -dx^l \wedge \left(\frac{\partial}{\partial x^k} \lrcorner \omega \right), \qquad k \neq l.$$

We conclude that

$$E_\alpha^{I,k} \left[\frac{\partial}{\partial x^k} \lrcorner D_l(dx^l \wedge \omega) \right] = -E_\alpha^{I \setminus l, k} \left[dx^l \wedge \left(\frac{\partial}{\partial x^k} \lrcorner \omega \right) \right] \qquad \text{whenever} \quad k \neq l.$$

$$\tag{5.140}$$

The case $k = l$ is a bit more delicate. First note that $E_\alpha^{I,k} \cdot D_k = E_\alpha^I$, so the relevant sum is

$$\sum_{k=1}^p (\tilde{\imath}_k + 1) E_\alpha^{I,k} \left[\frac{\partial}{\partial x^k} \, \lrcorner \, D_k (dx^k \wedge \omega) \right]$$

$$= \sum_{k=1}^p E_\alpha^I \left[\frac{\partial}{\partial x^k} \, \lrcorner \, (dx^k \wedge \omega) \right] + \sum_{k=1}^p \tilde{\imath}_k E_\alpha^I \left[\frac{\partial}{\partial x^k} \, \lrcorner \, (dx^k \wedge \omega) \right]. \quad (5.141)$$

On the right-hand side of (5.141), we use the two further identities of Exercise 1.37:

$$\sum_{k=1}^p \frac{\partial}{\partial x^k} \, \lrcorner \, (dx^k \wedge \omega) = (p - r)\omega$$

in the first summation, and

$$\frac{\partial}{\partial x^k} \, \lrcorner \, (dx^k \wedge \omega) = \omega - dx^k \wedge \left(\frac{\partial}{\partial x^k} \, \lrcorner \, \omega \right)$$

in the second. This yields

$$\sum_{k=1}^p (\tilde{\imath}_k + 1) E_\alpha^{I,k} \left[\frac{\partial}{\partial x^k} \, \lrcorner \, D_k (dx^k \wedge \omega) \right]$$

$$= (p - r + \#I) E_\alpha^I(\omega) - \sum_{k=1}^p \tilde{\imath}_k E_\alpha^I \left[dx^k \wedge \left(\frac{\partial}{\partial x^k} \, \lrcorner \, \omega \right) \right] \quad (5.142)$$

since $\sum_{k=1}^p \tilde{\imath}_k = \#I$. Combining (5.139), (5.140) and (5.142), we conclude that

$$\sum_{k,\bar{l}=1}^p (\tilde{\imath}_k + 1) E_\alpha^{I,k} \left[\frac{\partial}{\partial x^k} \, \lrcorner \, D_l (dx^l \wedge \omega) \right]$$

$$= (p - r + \#I) E_\alpha^I(\omega) - \sum_{k,\bar{l}=1}^p (\tilde{\imath}_k + 1 - \delta_l^k) E_\alpha^{I\setminus l,k} \left[dx^l \wedge \left(\frac{\partial}{\partial x^k} \, \lrcorner \, \omega \right) \right]. \quad (5.143)$$

This is our key identity in the proof of (5.133).

We now have

$$I_Q(D\omega) = \sum_{\alpha,I} D_I(Q_\alpha E_\alpha^I(\omega))$$

$$- \sum_{\alpha,I} \sum_{k,l} \frac{\tilde{\imath}_k + 1 - \delta_l^k}{p - r + \#I} D_l \left\{ Q_\alpha E_\alpha^{I\setminus l,k} \left[dx^l \wedge \left(\frac{\partial}{\partial x^k} \, \lrcorner \, \omega \right) \right] \right\}.$$

By (5.124), the first summation on the right-hand side is just pr $v_Q(\omega)$, so to complete the proof of (5.133), we need only identify the second summation with

$$- DI_Q(\omega) = - \sum_{l=1}^p D_l \left\{ dx^l \wedge \sum_{\alpha,J} \sum_k \frac{\tilde{\jmath}_k + 1}{p - r + \#J + 1} D_J \left[Q_\alpha E_\alpha^{J,k} \left(\frac{\partial}{\partial x^k} \, \lrcorner \, \omega \right) \right] \right\}$$

$$= - \sum_{\alpha,J} \sum_{k,l} \frac{\tilde{\jmath}_k + 1}{p - r + \#J + 1} D_{J,l} \left[Q_\alpha E_\alpha^{J,k} \left(dx^l \wedge \left(\frac{\partial}{\partial x^k} \, \lrcorner \, \omega \right) \right) \right].$$

But these two summations agree upon changing the multi-index summation variable from J to $I = (J, l)$, noting that $\tilde{\imath}_k = \tilde{\jmath}_k + \delta_l^k$, $\#I = \#J + 1$. This completes the proof of (5.133).

We now specialize (5.133) to the case of the scaling vector field pr $\mathbf{v}_u$ introduced earlier in the proof of the exactness of the variational complex. Note that if $P[u] = P(x, u^{(n)})$ is any smooth differential function defined on a vertically star-shaped domain, then

$$\frac{d}{d\lambda} P[\lambda u] = \sum_{\alpha, J} u_J^\alpha \frac{\partial P}{\partial u_J^\alpha}[\lambda u] = \frac{1}{\lambda} \text{pr } \mathbf{v}_u(P)[\lambda u],$$

where the notation means that we first apply pr $\mathbf{v}_u$ to P and then evaluate at λu. Integrating, we find

$$P[u] - P[0] = \int_0^1 \text{pr } \mathbf{v}_u(P)[\lambda u] \frac{d\lambda}{\lambda},$$

where $P[0] = P(x, 0)$ is a function of x alone. Since pr $\mathbf{v}_u$ acts coefficient-wise on a total differential form $\omega(x, u^{(n)})$, we have the analogous formula

$$\omega[u] - \omega[0] = \int_0^1 \text{pr } \mathbf{v}_u(\omega)[\lambda u] \frac{d\lambda}{\lambda}, \tag{5.144}$$

where $\omega[0] = \omega(x, 0)$ is an ordinary differential form on the base space X. If we now use (5.133) in the case $Q = u$, whereby

$$\mathsf{I}_u(\omega) = \sum_{\alpha=1}^q \sum_I \sum_{k=1}^p \frac{\tilde{\imath}_k + 1}{p - r + \#I + 1} D_I \left\{ u^\alpha \mathsf{E}_\alpha^{I,k} \left(\frac{\partial}{\partial x^k} \lrcorner \omega \right) \right\}, \tag{5.145}$$

we obtain the homotopy formula

$$\omega[u] - \omega[0] = \mathsf{DH}(\omega) + \mathsf{H}(\mathsf{D}\omega), \tag{5.146}$$

where the *total homotopy operator* is

$$\mathsf{H}(\omega) = \int_0^1 \mathsf{I}_u(\omega)[\lambda u] \frac{d\lambda}{\lambda}, \tag{5.147}$$

meaning that we first evaluate $\mathsf{I}_u(\omega)$ and then replace u by λu wherever it occurs. Except for the extra term $\omega[0]$ this would suffice to prove the exactness of the D-complex. However, $\omega[0]$ is an ordinary differential form on $\Omega = M \cap \{u = 0\}$, so provided Ω is also star-shaped we can use the ordinary Poincaré homotopy operator (1.69), with

$$\omega[0] - \omega_0 = dh(\omega[0]) + h(d\omega[0]), \tag{5.148}$$

where $\omega_0 = 0$ if $r > 0$, while $\omega_0 = f(0)$ if $\omega[0] = f(x)$ is a function, $r = 0$. For such forms, the total derivatives D_i and the partial derivatives $\partial/\partial x^i$ are the same, so we can replace the differential d by the total differential D. Combining (5.146) and (5.148), we obtain

$$\omega - \omega_0 = \mathsf{DH}^*(\omega) + \mathsf{H}^*(\mathsf{D}\omega), \tag{5.149}$$

for $\omega \in \bigwedge_r$, $0 \leqslant r < p$, where

$$H^*(\omega) = H(\omega) + h(\omega[0])$$

is the combined homotopy operator. Now that we have established (5.149), the proof of Theorem 5.80 is straightforward.

The homotopy formula (5.146) also extends to total p-forms $\omega = L[u]\, dx^1 \wedge \cdots \wedge dx^p$ if we utilize the modified formula (5.135) for the Lie derivative. Translating the differential form language, we see that if $L[u] = L(x, u^{(n)})$ is defined over a totally star-shaped domain $M \subset X \times U$, then

$$L[u] = \text{Div } B^*[u] + \int_0^1 u \cdot E(L)[\lambda u]\, d\lambda, \tag{5.150}$$

where B^* is the sum of the p-tuples $B[u] \in \mathscr{A}^p$ and $b(x)$ with entries

$$B_k(u) = \int_0^1 \sum_{\alpha=1}^q \sum_I \frac{\tilde{\imath}_k + 1}{\#I + 1} D_I(u^\alpha E_\alpha^{I,k}(L)[\lambda u])\, d\lambda,$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 1, \ldots, p. \tag{5.151}$$

$$b_k(x) = \int_0^1 x^k L(\lambda x, 0)\, d\lambda,$$

In particular, if $E(L) = 0$, then $L = \text{Div } B^*$ with B^* as above. Thus we have an explicit formula expressing any null Lagrangian as a divergence.

Example 5.103. Let $p = 2$, $q = 1$. Consider the null Lagrangian $L = u_x u_{yy}$. According to (5.150), $L = D_x A + D_y B$, where

$$A = \int_0^1 \left\{ u E^{(x)}(L) + D_x(u E^{(xx)}(L)) + \tfrac{1}{2} D_y(u E^{(xy)}(L)) + \cdots \right\} d\lambda,$$

$$B = \int_0^1 \left\{ u E^{(y)}(L) + \tfrac{1}{2} D_x(u E^{(xy)}(L)) + D_y(u E^{(yy)}(L)) + \cdots \right\} d\lambda,$$

where the differential functions $E^{(x)}(L)$, $E^{(y)}(L)$, etc. are to be evaluated at λu. In the case $L = u_x u_{yy}$, we have

$$E^{(x)}(L) = u_{yy}, \qquad E^{(y)}(L) = -2u_{xy}, \qquad E^{(yy)}(L) = u_x,$$

and all the other terms in A and B vanish. (See Example 5.97.) Thus

$$A = \int_0^1 u(\lambda u_{yy})\, d\lambda = \tfrac{1}{2} u u_{yy},$$

$$B = \int_0^1 u(-2\lambda u_{xy}) + D_y[u(\lambda u_x)]\, d\lambda = -\tfrac{1}{2} u u_{xy} + \tfrac{1}{2} u_x u_y,$$

satisfy the above divergence identity. Even from this relatively simple example, it is easy to see how the homotopy formula (5.151) can rapidly become unmanageable. In practice, it is often easier to determine the divergence form directly by inspection, using (5.151) only as a last resort.