and the Poisson bracket is

$$\{F, H\} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial y} + \frac{\partial F}{\partial r}\frac{\partial H}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial y}\frac{\partial H}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial H}{\partial r} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q}.$$

Further, the Hamiltonian system splits into

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = 0, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p} + \frac{\partial H}{\partial y} = y,$$

and

$$\frac{dy}{dt} = -\frac{\partial H}{\partial q} - \frac{\partial H}{\partial r} = -V'(r), \qquad \frac{dr}{dt} = \frac{\partial H}{\partial y} = 2y - p. \tag{6.30}$$

The solution to the first pair,

$$p = a, \qquad q = \int y(t)\, dt + b,$$

(a, b constant) can be determined from the solutions to the second pair (6.30). These form a reduced Hamiltonian system relative to the reduced Poisson bracket $\{\tilde{F}, \tilde{H}\} = \tilde{F}_r \tilde{H}_y - \tilde{F}_y \tilde{H}_r$ for functions of y and r, with the energy (6.29) obtained by fixing $p = a$. Presently, we will see how the two-dimensional system (6.30) can be explicitly integrated, thereby solving the original two-particle system (6.28).

As with the general reduction method for ordinary differential equations, if the vector field $\hat{\mathbf{v}}_P$ associated with the first integral P is too complicated, it may not be possible to explicitly find the change of variables that straightens it out, and so the reduction method cannot be completed. (Of course, the fact that P is a first integral certainly allows a reduction in order by one in all cases.) For example, if the Hamiltonian $H(x)$ is time-independent, it provides a first integral, but straightening out its corresponding vector field $\hat{\mathbf{v}}_H$ is the same problem as solving the Hamiltonian system itself! In this special case, however, the fact that $\hat{\mathbf{v}}_H$ is equivalent to the time translational symmetry generator ∂_t allows us to reduce the order by two provided we are willing to go to a time-dependent Hamiltonian framework.

Proposition 6.37. *Let $\dot{x} = J\nabla H$ be a Hamiltonian system in which $H(x)$ does not depend on t. Then there is a reduced, time-dependent Hamiltonian system in two fewer variables, from whose solutions those of the original system can be found by quadrature.*

Proof. The reduction in order by two *per se* is easy. First, since H is constant, we can restrict to a level set $H(x) = c$, reducing the order by one. Furthermore, the resulting system remains autonomous and so can be reduced in order once more using the method in Example 2.67. The problem is that unless we choose our coordinates more astutely, the system resulting from this reduction will not be of Hamiltonian form in any obvious way.

The easiest way to proceed is to first introduce the coordinates (p, q, y) used in the proof of Darboux' Theorem 6.22, relative to which the original system takes the form

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dy^i}{dt} = \sum_{j=1}^{m-2} \tilde{J}^{ij}(y)\frac{\partial H}{\partial y^j}, \qquad i = 1, \ldots, m-2.$$

Assume that $\partial H/\partial p \neq 0$, so that we can solve the equation $w = H(p, q, y)$ locally for $p = K(w, q, y)$. (If $\partial H/\partial p = 0$ everywhere, q is a first integral and we can use the previous reduction procedure!) We take t, w and y to be the new dependent variables and q the new independent variable, in terms of which the system takes the form

$$\frac{dt}{dq} = \frac{1}{\partial H/\partial p} = \frac{\partial K}{\partial w}, \qquad \frac{dw}{dq} = 0, \tag{6.31a}$$

$$\frac{dy^i}{dq} = \sum_{j=1}^{m-2} \tilde{J}^{ij}(y)\frac{\partial H/\partial y^j}{\partial H/\partial p} = \sum_{j=1}^{m-2} \tilde{J}^{ij}(y)\frac{\partial K}{\partial y^j}. \tag{6.31b}$$

The system (6.31b) is Hamiltonian using the reduced Poisson bracket corresponding to the structure matrix $\tilde{J}(y)$ and Hamiltonian function $K(w, q, y)$. For each fixed value of w, once we have solved (6.31b) we can determine the remaining variable $t(q)$ from (6.31a) by a single quadrature. This completes the procedure. □

Example 6.38. In the case of an autonomous Hamiltonian system

$$q_t = \partial H/\partial p, \qquad p_t = -\partial H/\partial q,$$

in the plane, we can use this method to explicitly integrate it. We first solve $w = H(p, q)$ for one of the coordinates, say p, in terms of q and w, which is constant. The first equation, then, leaves an autonomous equation for q, which we can solve by quadrature. For example, in the case of a simple pendulum $H(p, q) = \frac{1}{2}p^2 + (1 - \cos q)$, so on the level curve $H = \omega + 1$, $p = \sqrt{2(\omega + \cos q)}$. The remaining equation

$$dq/dt = p = \sqrt{2(\omega + \cos q)}$$

can be solved in terms of Jacobi elliptic functions

$$q(t) = 2 \sin^{-1}\{\mathrm{sn}(k^{-1}(t + \delta), k)\},$$

where sn has modulus $k = \sqrt{2/(\omega + 1)}$.

Similarly, in the case of the two-particle system on the line from Example 6.36, setting $H(y, r) = \omega + \frac{1}{4}p^2$, we find

$$y = \tfrac{1}{2}p \pm \sqrt{\omega - V(r)}.$$

Thus we recover the solution just by integrating

$$\frac{dr}{dt} = 2y - p = \pm 2\sqrt{\omega - V(r)}.$$

Example 6.39. Consider the equations of rigid body motion (6.18), which were realized as a Hamiltonian system on $\mathfrak{so}(3)^*$. The distingiushed function $C(u) = |u|^2$ naturally reduces the order by one by restriction to a level set or co-adjoint orbit. Provided the moments of inertia I_1, I_2, I_3 are not all equal, the Hamiltonian itself provides a second independent first integral. We conclude that the integral curves of this Hamiltonian vector field are determined by the intersection of a sphere $\{C(u) = |u|^2 = c\}$ and an ellipsoid $\{H(u) = \omega\}$ forming the common level set of these two first integrals. The explicit solutions can be determined by eliminating two of the variables, say u^2 and u^3, from the pair of equations $C(u) = c$, $H(u) = \omega$. Proposition 6.37 then guarantees that the one remaining equation for $u^1 = y$ is autonomous, and hence can be integrated. It turns out to be of the form

$$\frac{dy}{dt} = \sqrt{\alpha(\beta^2 - y^2)(\gamma^2 - y^2)},$$

and hence the solutions can be written in terms of elliptic functions. See Whittaker, [1; Chap. 6] for the explicit formula and a geometric interpretation.

Reduction Using Multi-parameter Groups

As we already saw in the case of Euler–Lagrange equations (cf. Exercise 4.11), it is not true that a Hamiltonian system which admits an r-parameter symmetry group can be fully reduced in order by $2r$, even if the group is solvable. In the Hamiltonian case, however, we can actually determine the degree of reduction which can be effected. Interestingly, this question is closely tied with the structure of the co-adjoint action of the symmetry group on its Lie algebra. We begin by considering an example.

Example 6.40. Let $M = \mathbb{R}^4$ with canonical coordinates $(p, \tilde{p}, q, \tilde{q})$ and consider a Hamiltonian function of the form $H(\tilde{p}, pe^{\tilde{q}}, t)$. Hamilton's equations are

$$\frac{dp}{dt} = 0, \qquad \frac{dq}{dt} = e^{\tilde{q}} H_r, \qquad \frac{d\tilde{p}}{dt} = -pe^{\tilde{q}} H_r, \qquad \frac{d\tilde{q}}{dt} = H_{\tilde{p}}, \tag{6.32}$$

where $r = pe^{\tilde{q}}$ is the second argument of H. They admit a *two*-parameter solvable symmetry group, generated by

$$\mathbf{v} = \partial_q, \qquad \mathbf{w} = -p\partial_p + q\partial_q + \partial_{\tilde{q}},$$

which correspond to the two first integrals $P = p$, $Q = pq + \tilde{p}$. Nevertheless, we can in general only reduce the order of (6.32) by two! There are, in fact, four distinct ways in which this reduction can be effected, and we examine them in turn.

(1) The simplest approach is to use the two first integrals directly and restrict to a common level set. Let $s = pq + \tilde{p}$, so s and p are constant. If we treat $p, \tilde{p}, r, s$ as the new variables (which is valid provided $p \neq 0$) then the reduced system is

$$\frac{d\tilde{p}}{dt} = -rH_r, \qquad \frac{dr}{dt} = rH_{\tilde{p}}, \tag{6.33}$$

which is Hamiltonian using the reduced Poisson bracket $\{F, H\} = r(F_r H_{\tilde{p}} - F_{\tilde{p}} H_r)$. However, there is no residual symmetry property of (6.33) reflecting the invariance of (6.32) under $\mathbf{v}$ and $\mathbf{w}$, so barring any special structure of the Hamiltonian function $H(\tilde{p}, r, t)$ (e.g. time-independence) we cannot reduce any further.

(2) Alternatively, we can employ the reduction procedure of Theorem 6.35 using the Hamiltonian symmetry $\mathbf{v}$. Here the coordinates are already in the appropriate form, with $y = (\tilde{p}, \tilde{q})$. Fixing p, we see that once we have solved the third and fourth equation in (6.32) for $\tilde{p}$ and $\tilde{q}$, we can determine q by quadrature. The reduced system for $\tilde{p}$ and $\tilde{q}$ is canonically Hamiltonian, but there is no symmetry or first integral of it which comes from the second Hamiltonian symmetry group of the full system. Again the order can only be reduced by two.

(3) Reduction using the symmetry group generated by $\mathbf{w}$ leads to a similar conclusion. The relevant flat coordinates are $s = pq + \tilde{p}$, $\tilde{q}$, $r = pe^{\tilde{q}}$ and $z = qe^{-\tilde{q}}$ in terms of which $\mathbf{w} = \partial_{\tilde{q}}$, $H = H(s - rz, r, t) = \tilde{H}(r, s, z, t)$. The system is now

$$\frac{ds}{dt} = 0, \qquad \frac{d\tilde{q}}{dt} = \tilde{H}_s, \qquad \frac{dr}{dt} = -\tilde{H}_z, \qquad \frac{dz}{dt} = \tilde{H}_r. \tag{6.34}$$

Fixing s, the third and fourth equation form a Hamiltonian system, the solutions of which determine $\tilde{q}$ by quadrature. Again, no symmetry or integral reflecting the original invariance under $\mathbf{v}$ remains.

(4) The final possibility is to ignore the Hamiltonian structure of (6.32) entirely and reduce using the symmetry procedure of Section 2.5. Noting that $[\mathbf{v}, \mathbf{w}] = \mathbf{v}$, we first reduce using $\mathbf{v}$, which is trivial. Namely the first, third and fourth equations of (6.32), once solved, will determine $q(t)$ by quadrature. This third order system remains invariant under the reduced vector field $\tilde{\mathbf{w}} = -p\partial_p + \partial_{\tilde{q}}$. We set $r = pe^{\tilde{q}}$ and use $r, \tilde{p}, \tilde{q}$ as variables. The result is identical with (6.33), using which we can determine $\tilde{q}$ (and hence q) by quadrature. As in part (2), no further reduction is possible in general!

Finally, note that for certain special initial conditions, e.g. $p = 0$, we can actually compute the solution by quadrature alone. Thus the degree of reduction possible would appear to depend both on the structure of the symmetry group and the precise initial conditions desired for the solution.

Hamiltonian Transformation Groups

Throughout the following discussion, the underlying symmetry group will be assumed to be Hamiltonian in the following strict sense.

Definition 6.41. Let M be a Poisson manifold. Let G be a Lie group with structure constants c_{ij}^k, $i, j, k = 1, \ldots, r$, relative to some basis of its Lie algebra $\mathfrak{g}$. The functions $P_1, \ldots, P_r \colon M \to \mathbb{R}$ generate a *Hamiltonian action* of G on M provided their Poisson brackets satisfy the relations

$$\{P_i, P_j\} = -\sum_{k=1}^{r} c_{ij}^k P_k, \qquad i, j = 1, \ldots, r.$$

Note that by (6.8), the corresponding Hamiltonian vector fields $\hat{\mathbf{v}}_i = \hat{\mathbf{v}}_{P_i}$ satisfy the same commutation relations (up to sign)

$$[\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j] = \sum_{k=1}^{r} c_{ij}^k \mathbf{v}_k,$$

and therefore generate a local action of G on M by Theorem 1.57. Given a Hamiltonian system on M, we will say that G is a *Hamiltonian symmetry group* if each of its generating functions P_i is a first integral, $\{P_i, H\} = 0$, $i = 1, \ldots, r$, which implies that each $\hat{\mathbf{v}}_i$ generates a one-parameter symmetry group.

As we saw in Section 2.5 and Exercise 3.12, any first order system of differential equations on a manifold M which admits a regular symmetry group G reduces to a first order system on the quotient manifold M/G. (Of course, if G is not solvable, we will not be able to reconstruct the solutions to the original system from those of the reduced system by quadrature, but we ignore this point at the moment.) In the case M is a Poisson manifold, and G a Hamiltonian group of transformations, the quotient manifold naturally inherits a Poisson structure, relative to which the reduced system is Hamiltonian. Moreover, the degree of degeneracy of the Poisson bracket on M/G will determine how much further we can reduce the system using any distinguished functions on the quotient space.

Theorem 6.42. *Let G be a Hamiltonian group of transformations acting regularly on the Poisson manifold M. Then the quotient manifold M/G inherits a Poisson structure so that whenever $\tilde{F}, \tilde{H} \colon M/G \to \mathbb{R}$ correspond to the G-invariant functions F, $H \colon M \to \mathbb{R}$, their Poisson bracket $\{\tilde{F}, \tilde{H}\}_{M/G}$ corresponds to the G-invariant function $\{F, H\}_M$. Moreover, if G is a Hamiltonian symmetry group for a Hamiltonian system on M, then there is a reduced Hamiltonian system on M/G whose solutions are just the projections of the solutions of the system on M.*

PROOF. First note that the fact that the Poisson bracket $\{F, H\}$ of two G-invariant functions remains G-invariant is a simple consequence of the

Jacobi identity and the connectivity of G; we find, for $i = 1, \ldots, r$,

$$\hat{\mathbf{v}}_i(\{F, H\}) = \{\{F, H\}, P_i\} = \{\{F, P_i\}, H\} + \{F, \{H, P_i\}\} = 0$$

since F and H are invariant, verifying the infinitesimal invariance condition (2.1). Thus the Poisson bracket is well defined on M/G; the verification that it satisfy the properties of Definition 6.1 is trivial.

Now if $H: M \to \mathbb{R}$ has G as a Hamiltonian symmetry group, then H is automatically a G-invariant function: $\hat{\mathbf{v}}_i(H) = \{H, P_i\} = 0$ since each P_i is, by assumption, a first integral. Let $\tilde{H}: M/G \to \mathbb{R}$ be the corresponding function on the quotient manifold. To prove that the corresponding Hamiltonian vector fields are related, $d\pi(\hat{\mathbf{v}}_H) = \hat{\mathbf{v}}_{\tilde{H}}$, where $\pi: M \to M/G$ is the natural projection, it suffices to note that by (1.24)

$$d\pi(\hat{\mathbf{v}}_H)(\tilde{F}) \circ \pi = \hat{\mathbf{v}}_H[\tilde{F} \circ \pi] = \{\tilde{F} \circ \pi, H\}_M$$

for any $\tilde{F}: M/G \to \mathbb{R}$. But this equals

$$\{\tilde{F}, \tilde{H}\}_{M/G} \circ \pi = \hat{\mathbf{v}}_{\tilde{H}}(\tilde{F}) \circ \pi$$

by the definition of the Poisson bracket on M/G, and hence proves the correspondence. □

Example 6.43. Consider the Euclidean space $\mathbb{R}^6$ with canonical coordinates $(p, q) = (p^1, p^2, p^3, q^1, q^2, q^3)$. The functions

$$P_1 = q^2 p^3 - q^3 p^2, \qquad P_2 = q^3 p^1 - q^1 p^3, \qquad P_3 = q^1 p^2 - q^2 p^1,$$

satisfy the bracket relations

$$\{P_1, P_2\} = P_3, \qquad \{P_2, P_3\} = P_1, \qquad \{P_3, P_1\} = P_2,$$

and hence generate a Hamiltonian action of the rotation group SO(3) on $\mathbb{R}^6$, which is, in fact, given by $(p, q) \mapsto (Rp, Rq)$, $R \in \mathrm{SO}(3)$. This action is regular on the open subset $M = \{(p, q): p, q \text{ are linearly independent}\}$, with three-dimensional orbits and global invariants

$$\xi(p, q) = \tfrac{1}{2}|p|^2, \qquad \eta(p, q) = p \cdot q, \qquad \zeta(p, q) = \tfrac{1}{2}|q|^2.$$

We can thus identify the quotient manifold with the subset $M/G \simeq \{(x, y, z): x > 0, z > 0, y^2 < 4xz\}$ of $\mathbb{R}^3$, where $x = \xi$, $y = \eta$, $z = \zeta$ are the new coordinates.

How do we compute the reduced Poisson bracket on M/G? According to (6.10), we need only compute the basic Poisson brackets between the corresponding invariants ξ, η, ζ using the Poisson bracket on M, and re-expressing them in terms of the invariant themselves. For instance, since

$$\{\xi, \eta\} = \sum_{i=1}^{3} \left(\frac{\partial \xi}{\partial q^i} \frac{\partial \eta}{\partial p^i} - \frac{\partial \xi}{\partial p^i} \frac{\partial \eta}{\partial q^i} \right) = -\sum_{i=1}^{3} (p^i)^2 = -2\xi,$$

we have $\{x, y\}_{M/G} = -2x$. Similarly the bracket relations $\{\xi, \zeta\} = -\eta$, $\{\eta, \zeta\} = -2\zeta$ on M lead to the structure functions $\{x, z\}_{M/G} = -y$,

$\{y, z\}_{M/G} = -2z$ on M/G. The structure matrix on M/G is thus

$$J/G = \begin{bmatrix} 0 & -2x & -y \\ 2x & 0 & -2z \\ y & 2z & 0 \end{bmatrix},$$

with Poisson bracket

$$\{\tilde{F}, \tilde{H}\} = -2x(\tilde{F}_x\tilde{H}_y - \tilde{F}_y\tilde{H}_x) - y(\tilde{F}_x\tilde{H}_z - \tilde{F}_z\tilde{H}_x) - 2z(\tilde{F}_y\tilde{H}_z - \tilde{F}_z\tilde{H}_y).$$

Any Hamiltonian system on M admitting the angular momenta P_i as first integrals will reduce to a Hamiltonian system on M/G. For example, the general Kepler problem of a mass moving in a central force field with potential $V(r)$ is such a candidate. Here the Hamiltonian function is the energy $H(p, q) = \frac{1}{2}|p|^2 + V(|q|)$. The reduced system on M/G is obtained by rewriting H in terms of the invariants and then using the given Poisson bracket to reconstruct the Hamiltonian vector field. We find the reduced Hamiltonian $\tilde{H}(x, y, z) = x + \tilde{V}(z)$, where $\tilde{V}(z) = V(\sqrt{2z})$, and reduced system

$$x_t = -y\tilde{V}'(z), \qquad y_t = 2x - 2z\tilde{V}'(z), \qquad z_t = y. \tag{6.35}$$

(The reader may enjoy deriving this directly from Hamilton's equations on M.)

Now M/G is three-dimensional, so there is at least one distinguished function. This is easily seen to be $C(x, y, z) = 4xz - y^2$, which is an invariant of any Hamiltonian system on M/G. (In the original variables, $C = |p \times q|^2$.) The hyperboloids $4xz - y^2 = k^2$, being the level sets of C, are the leaves of the symplectic foliation, and hence we can restrict (6.35) to any such leaf. Using (x, z) as coordinates, we find the fully reduced system

$$x_t = -\sqrt{4xz - k^2}\,\tilde{V}'(z), \qquad z_t = \sqrt{4xz - k^2}, \tag{6.36}$$

which is Hamiltonian relative to the induced Poisson bracket $\{\tilde{F}, \tilde{H}\} = -\sqrt{4xz - k^2}(\tilde{F}_x\tilde{H}_z - \tilde{F}_z\tilde{H}_x)$ on the hyperboloid. This final two-dimensional system can be solved by method of Proposition 6.37, so we can solve the reduced system (6.35) by quadrature. However, at this stage we cannot use this solution to integrate the original central force problem because SO(3) is not a solvable group. But, as we will soon see, this difficulty can be circumvented by an alternative approach to the reduction rocedure.

The Momentum Map

The above approach to the reduction problem, while geometrically appealing, leaves something to be desired from a computational standpoint. The problem is that we are concentrating initially on the more complicated aspect of a Hamiltonian symmetry group, namely the group transformations, and ignoring the first integrals, which are also present, until after the symmetry reduction has been effected, at which point they manifest their presence as distinguished functions. A more logical approach would be to use the first integrals at the outset, restricting the system to a common level set thereof, and then completing the reduction by using any residual symmetry properties of the resulting system. This turns out to be equivalent to the above procedure, but now we stand a better chance of being able to reconstruct the solution to the original system by quadratures alone.

The first step here is to organize the first integrals furnished by a Hamiltonian group of symmetries in a more natural framework. It is here that the dual to the Lie algebra of the symmetry group and, subsequently, the co-adjoint action makes its appearance.

Definition 6.44. Let G be a Hamiltonian group of transformations acting on the Poisson manifold M, generated by the real-valued functions $P_1, \ldots, P_r$. The *momentum map* for G is the smooth map $P: M \to \mathfrak{g}^*$ given by

$$P(x) = \sum_{i=1}^{r} P_i(x)\omega_i,$$

in which $\{\omega_1, \ldots, \omega_r\}$ are the dual basis to $\mathfrak{g}^*$ for the basis $\{\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_r\}$ of $\mathfrak{g}$ relative to which the structure constants c_{ij}^k were computed.

The key property of the momentum map, which explains why we allowed it to take values in $\mathfrak{g}^*$, is its invariance (or, more correctly, "equivariance") with respect to the co-adjoint representation of G on $\mathfrak{g}^*$.

Proposition 6.45. *Let $P: M \to \mathfrak{g}^*$ be the momentum map determined by a Hamiltonian group action of G on the Poisson manifold M. Then*

$$P(g \cdot x) = \mathrm{Ad}^* g(P(x)) \tag{6.37}$$

for all $x \in M$, $g \in G$.

PROOF. As usual, it suffices to prove the infinitesimal form of this identity, which is

$$dP(\hat{\mathbf{v}}_j|_x) = \mathrm{ad}^*\, \hat{\mathbf{v}}_j|_{P(x)}, \qquad x \in M, \tag{6.38}$$

for any generator $\hat{\mathbf{v}}_j \in \mathfrak{g}$, $j = 1, \ldots, r$, of G. If we identify $T\mathfrak{g}^*|_{P(x)}$ with $\mathfrak{g}^*$ itself, then

$$dP(\hat{\mathbf{v}}_j|_x) = \sum_{i=1}^{r} \hat{\mathbf{v}}_j(P_i)\omega_i = \sum_{i=1}^{r} \{P_i, P_j\}(x)\omega_i = -\sum_{i,k=1}^{r} c_{ij}^k P_k(x)\omega_i,$$

cf. (1.24), (6.4) and Definition 6.41. By (6.24) this expression is the same as the right-hand side of (6.38).

To prove (6.37), we note that if $g = \exp(\varepsilon\hat{\mathbf{v}}_j)$ and we differentiate with respect to ε, then we recover (6.38) at $\tilde{x} = \exp(\varepsilon\hat{\mathbf{v}}_j)x$. Since this holds at all $\tilde{x}$, the usual connectivity arguments prove that (6.37) holds in general. □

Example 6.46. Consider the Hamiltonian action of SO(3) on $\mathbb{R}^6$ presented in Example 6.43. The momentum map is

$$P(p, q) = (q^2 p^3 - q^3 p^2)\omega_1 + (q^3 p^1 - q^1 p^3)\omega_2 + (q^1 p^2 - q^2 p^1)\omega_3,$$

where $\{\omega_1, \omega_2, \omega_3\}$ are the basis of $\mathfrak{so}(3)^*$ of Example 6.9. Note that if we identify $\mathfrak{so}(3)^*$ with $\mathbb{R}^3$, $P(p, q) = q \times p$ is the same as the cross product of vectors in $\mathbb{R}^3$. In this case, SO(3) acts on $\mathfrak{so}(3)^*$ by rotations, and the equivariance of the momentum map is just a restatement of the rotational invariance of the cross product: $R(q \times p) = (Rq) \times (Rp)$ for $R \in \mathrm{SO}(3)$.

Now, as remarked earlier, any Hamiltonian system with G as a Hamiltonian symmetry group naturally restricts to a system of ordinary differential equations on the common level set $\{P_i(x) = c_i\}$ of the given first integrals. Note that these common level sets are just the level sets of the momentum map, denoted $\mathscr{S}_\alpha = \{x: P(x) = \alpha\}$ where $\alpha = \sum c_i \omega_i \in \mathfrak{g}^*$. Moreover, the reduced system will automatically remain invariant under the *residual symmetry group*

$$G_\alpha \equiv \{g \in G: g \cdot \mathscr{S}_\alpha \subset \mathscr{S}_\alpha\}$$

of group elements leaving the chosen level set invariant. There is an easy characterization of this residual group.

Proposition 6.47. *Let $P: M \to \mathfrak{g}^*$ be the momentum map associated with a Hamiltonian group action. Then the residual symmetry group of a level set $\mathscr{S}_\alpha = \{P(x) = \alpha\}$ is the isotropy subgroup of the element $\alpha \in \mathfrak{g}^*$:*

$$G_\alpha = \{g \in G: \mathrm{Ad}^* g(\alpha) = \alpha\}.$$

Moreover, if $g \in G$ has the property that it takes one point $x \in \mathscr{S}_\alpha$ to a point $g \cdot x \in \mathscr{S}_\alpha$, then $g \in G_\alpha$, and has this property for all $x \in \mathscr{S}_\alpha$.

PROOF. By definition, $g \in G_\alpha$ if and only if $P(g \cdot x) = \alpha$ whenever $P(x) = \alpha$. But, by the equivariance of P,

$$\alpha = P(g \cdot x) = \mathrm{Ad}^* g(P(x)) = \mathrm{Ad}^* g(\alpha),$$

so g is in the isotropy subgroup of α. The second statement easily follows from this identity. □

Note that the residual Lie algebra corresponding to G_α is the *isotropy subalgebra* $\mathfrak{g}_\alpha \equiv \{\mathbf{v} \in \mathfrak{g}: \mathrm{ad}^*\, \mathbf{v}|_\alpha = 0\}$, which is readily computable. In particular, the dimension of G_α can be computed as the dimension of its Lie algebra $\mathfrak{g}_\alpha$. For instance, if G is an abelian Lie group, its co-adjoint representation is trivial, $\mathrm{Ad}^* g(\alpha) = \alpha$ for all $g \in G$, $\alpha \in \mathfrak{g}^*$, hence $G_\alpha = G$ for every α. Therefore any Hamiltonian system admitting an abelian Hamiltonian symmetry group remains invariant under the full group, even on restriction to a common level set $\mathscr{S}_\alpha$. This will imply that we can always reduce such a system in order by $2r$, twice the dimension of the group. As a second example, consider the two-parameter solvable group of Example 6.40. Here the momentum map is

$$P(p, q, \tilde{p}, \tilde{q}) = p\omega_1 + (pq + \tilde{p})\omega_2,$$

where $\{\omega_1, \omega_2\}$ are a basis of $\mathfrak{g}^*$ dual to the basis $\{\mathbf{v}, \mathbf{w}\}$ of $\mathfrak{g}$. The co-adjoint representation of $g = \exp(\varepsilon_1 \mathbf{v} + \varepsilon_2 \mathbf{w})$ is found to be

$$\mathrm{Ad}^* g(c_1\omega_1 + c_2\omega_2) = e^{-\varepsilon_2} c_1 \omega_1 + (\varepsilon_1 \varepsilon_2^{-1}(e^{-\varepsilon_2} - 1)c_1 + c_2)\omega_2$$

(with appropriate limiting values if $\varepsilon_2 = 0$). Thus the isotropy subgroup of $\alpha = c_1\omega_1 + c_2\omega_2$ is just $\{e\}$ unless $c_1 = 0$, in which case it is all of G. Thus we expect that the restriction of a Hamiltonian system with symmetry group G to a level set $\mathscr{S}_\alpha = \{p = c_1, pq + \tilde{p} = c_2\}$ will retain no residual symmetry group unless $c_1 = 0$, in which case the entire group G will remain. This is precisely what we observed in Example 6.40.

Once we have restricted the Hamiltonian system to the level set $\mathscr{S}_\alpha$, the idea is then to utilize the methods of Section 2.5 to reduce further using the residual symmetry group G_α. Under certain regularity assumptions on the group action, the quotient manifold $\mathscr{S}_\alpha/G_\alpha$, on which the fully reduced system will live, has a natural identification as a Poisson submanifold of M/G. Thus the fully reduced system inherits a Hamiltonian structure itself. In particular, if the residual group G_α is solvable (rather than G itself being solvable) we can reconstruct the solutions to the original system on $\mathscr{S}_\alpha$ by quadrature from those of the fully reduced system on $\mathscr{S}_\alpha/G_\alpha$. The general result follows:

Theorem 6.48. *Let M be a Poisson manifold and G a regular Hamiltonian group of transformations. Let $\alpha \in \mathfrak{g}^*$. Assume that the momentum map $P\colon M \to \mathfrak{g}^*$ is of maximal rank everywhere on the level set $\mathscr{S}_\alpha = P^{-1}\{\alpha\}$, and that the residual symmetry group G_α acts regularly on the submanifold $\mathscr{S}_\alpha$. Then there is a natural immersion ϕ making $\mathscr{S}_\alpha/G_\alpha$ into a Poisson submanifold of M/G in such a way that the diagram*

$$\begin{tikzcd} & M \arrow[dr, "\pi"] & \\ \mathscr{S}_\alpha \arrow[ur, "i"] \arrow[dr, "\pi_\alpha"'] & & M/G \\ & \mathscr{S}_\alpha/G_\alpha \arrow[ur, "\phi"'] & \end{tikzcd} \tag{6.39}$$

commutes. (Here π and π_α are the natural projections and i the immersion realizing $\mathscr{S}_\alpha$ as a submanifold of M.) Moreover, any Hamiltonian system on M which admits G as a Hamiltonian symmetry group naturally restricts to systems on the other spaces in (6.39), which are Hamiltonian on M/G and $\mathscr{S}_\alpha/G_\alpha$, and which are related by the appropriate maps. In particular, we obtain a Hamiltonian system on $\mathscr{S}_\alpha/G_\alpha$ by first restricting to $\mathscr{S}_\alpha$ and then projecting using π_α.

Proof. Assume G is a global group of transformation, although the proof is easily modified to incorporate the local case. According to the diagram, if $z = \pi_\alpha(x) \in \mathscr{S}_\alpha/G_\alpha$, then we should define $\phi(z) = \pi(x) \in M/G$. Note that