gives a seventh order generalized symmetry. As the following result shows, we can continue this recursive procedure indefinitely, leading to higher and higher order generalized symmetries.

Theorem 5.31. *Let $Q_0 = u_x$. For each $k \geqslant 0$, the differential polynomial $Q_k = \mathscr{R}^k Q_0$ is a total x-derivative, $Q_k = D_x R_k$, and hence we can recursively define $Q_{k+1} = \mathscr{R} Q_k$. Each Q_k is the characteristic of a symmetry of the Korteweg–de Vries equation.*

In fact, by Theorem 5.36, the vector fields $\mathbf{v}_k = \mathbf{v}_{Q_k}$ determine an infinite collection of mutually commuting flows

$$u_t = Q_k[u] = u_{2k+1} + \cdots,$$

called the "higher order Korteweg–de Vries equations". All of the above vector fields are thus symmetries of any one of these remarkable evolution equations.

Proof of Theorem 5.31. We proceed by induction on k, so assume that $Q_k = D_x R_k$ for some $R_k \in \mathscr{A}$. From the form of the recursion operator,

$$Q_{k+1} = D_x^2 Q_k + \tfrac{2}{3} u Q_k + \tfrac{1}{3} u_x D_x^{-1} Q_k = D_x[D_x Q_k + \tfrac{1}{3} u D_x^{-1} Q_k + \tfrac{1}{3} D_x^{-1}(u Q_k)].$$

If we can prove that $uQ_k = D_x S_k$ for some differential polynomial $S_k \in \mathscr{A}$, we will have proved that $Q_{k+1} = D_x R_{k+1}$, where R_{k+1} is the above expression in brackets, which will complete the induction step.

To prove this fact, note first that the formal adjoint of the recursion operator $\mathscr{R}$ is[†]

$$\mathscr{R}^* = D_x^2 + \tfrac{2}{3} u - \tfrac{1}{3} D_x^{-1} \cdot u_x = D_x^{-1} \mathscr{R} D_x.$$

We use this to integrate the expression uQ_k by parts, cf. (5.77), so

$$uQ_k = u\mathscr{R}^k[u_x] = u_x \cdot (\mathscr{R}^*)^k[u] + D_x A_k$$

for some differential function $A_k \in \mathscr{A}$. On the other hand, using a further integration by parts

$$u_x(\mathscr{R}^*)^k[u] = u_x \cdot D_x^{-1} \mathscr{R}^k[u_x] = u_x \cdot D_x^{-1} Q_k = -uQ_k + D_x B_k$$

for some $B_k \in \mathscr{A}$. Substituting into the previous identity, we conclude

$$uQ_k = D_x S_k, \quad \text{where} \quad S_k = \tfrac{1}{2}(A_k + B_k),$$

proving the claim. □

There were two other geometrical symmetry groups of the Korteweg–de Vries equation. The characteristic of the Galilean group $t\partial_x - \partial_u$ is $1 + tu_x$, and

$$3\mathscr{R}(1 + tu_x) = 2u + xu_x + 3t(u_{xxx} + uu_x),$$

[†] See the beginning of the following section.

which is equivalent to the scaling symmetry group. This latter characteristic is not a total derivative, so we cannot re-apply the recursion operator to get a meaningful generalized symmetry. However, the resulting "nonlocal symmetry" is important, as the following subsection demonstrates.

Example 5.32. Another example where the inverse of the total derivative is required is the physical form of Burgers' equation

$$v_t = v_{xx} + vv_x. \tag{5.46}$$

Note that if u satisfies the potential form (5.11), then $v = 2u_x$ satisfies (5.46); u is a "potential" for the physical velocity v. Thus, given a recursion operator $\mathscr{R}$, for the potential Burgers' equation, the transformed operator $\hat{\mathscr{R}} = D_x \mathscr{R} D_x^{-1}$ should be a recursion operator for the physical equation (5.46). (Why?) We conclude that

$$\hat{\mathscr{R}}_1 = D_x + \tfrac{1}{2}v + \tfrac{1}{2}v_x D_x^{-1}, \qquad \hat{\mathscr{R}}_2 = t\hat{\mathscr{R}}_1 + \tfrac{1}{2}x + \tfrac{1}{2}D_x^{-1}, \tag{5.47}$$

are recursion operators for Burgers' equation (5.46), a fact that can be verified directly. The resulting hierarchies of generalized symmetries are directly related to those of the potential Burgers' equation via the "potential" transformation $v = 2u_x$.

Master Symmetries

As in Example 5.18, time-dependent generalized symmetries can often used as an effective alternative to the recursion operator method for generating infinite hierarchies of symmetries of evolution equations. In this guise, they are called "master symmetries." A *master symmetry*, then, is a generalized (or even nonlocal) vector field $\mathbf{w}$ with the property that whenever $\mathbf{v}_Q$ is a generalized symmetry of the evolution equation, so is the Lie bracket $[\mathbf{w}, \mathbf{v}_Q]$. Note that any symmetry of the system satisfies this property, so to be really interesting, the master symmetry should produce *new* symmetries, mapping, say, the n-th member of the hierarchy of symmetries to the $(n+1)$-st one, as the Burgers' example does.

In the theory of master symmetries, the following extension of the concept of a recursion operator plays a key role. (See (5.40) for notation.)

Definition 5.33. An operator $\mathscr{R}\colon \mathscr{A}^q \to \mathscr{A}^q$ is said to be *hereditary* if it satisfies

$$\mathbf{v}_{\mathscr{R}P}[\![\mathscr{R}]\!] = \mathscr{R}\cdot \mathbf{v}_P[\![\mathscr{R}]\!], \tag{5.48}$$

for all differential functions $P[u] \in \mathscr{A}^q$.

Almost all known recursion operators satisfy the hereditary property, including the recursion operators for the potential and ordinary Burgers' equations, and the Korteweg–de Vries equation. To see this in the case of

$\mathscr{R} = D_x + u_x$, we compute the (1, 1)-Lie derivative

$$\mathbf{v}_P[\![\mathscr{R}]\!] = D_x P + [\mathscr{R}, \mathsf{D}_P].$$

Moreover, by (5.35), $\mathsf{D}_{D_x P} = D_x \mathsf{D}_P$, whereas by (5.34), $\mathsf{D}_{u_x P} = u_x \mathsf{D}_P + P D_x$. Hence (5.48) reduces to

$$\begin{aligned} &D_x(D_x + u_x)P + [D_x + u_x, (D_x + u_x)\mathsf{D}_P + PD_x] \\ &\quad = (D_x + u_x)\{D_x P + [D_x + u_x, \mathsf{D}_P]\}, \end{aligned}$$

which can be straightforwardly verified. A similar, but more complicated computation, which is left to the reader, proves that the recursion operators for Burgers' equation and the Korteweg–de Vries equation also satisfy the hereditary property.

Proposition 5.34. *Suppose $\mathscr{R}$ is an hereditary operator. Let Q_0 be a differential function such that $\mathbf{v}_{Q_0}[\![\mathscr{R}]\!] = 0$, so that $\mathscr{R}$ is a recursion operator for the evolution equation $u_t = Q_0$. Then $\mathscr{R}$ is also a recursion operator for each of the evolution equations in the hierarchy $u_t = Q_k = \mathscr{R}^k Q_0$, $k = 0, 1, 2, \ldots$.*

Proof. According to the preceding remarks, to prove $\mathscr{R}$ is a recursion operator for the evolution equation $u_t = Q_k$, we need only prove the vanishing of the Lie derivative

$$\mathbf{v}_{Q_k}[\![\mathscr{R}]\!] = \mathbf{v}_{\mathscr{R}^k Q_0}[\![\mathscr{R}]\!] = \mathscr{R}^k \cdot \mathbf{v}_{Q_0}[\![\mathscr{R}]\!] = 0,$$

the second equality following directly from the hereditary condition (5.48). □

The most common application of Proposition 5.34 is when $\mathscr{R}$ is a hereditary operator which does not depend explicitly on x, and the "seed" function for the hierarchy is $Q_0 = u_x$, the characteristic of the translational group $x \mapsto x - \varepsilon$, with associated evolution equation $u_t = u_x$. It is easy to see that *any* operator $\mathscr{R}$ which does not explicitly depend on x satisfies $\mathbf{v}_{Q_0}[\![\mathscr{R}]\!] = 0$, and so satisfies the basic hypothesis of Proposition 5.34. Indeed, if $\mathscr{R}$ does not depend on x, then, on solutions to $u_t = u_x$, we have $\mathscr{R}_t = \mathscr{R}_x$, the latter being obtained by applying D_x coefficient-wise to $\mathscr{R}$. Moreover, $\mathsf{D}_{u_x} = D_x$, so, in this case the Lie derivative (5.48) reduces to

$$\mathbf{v}_{u_x}[\![\mathscr{R}]\!] = \mathscr{R}_x + [\mathscr{R}, D_x] = 0.$$

Therefore, if $\mathscr{R}$ is an x-independent hereditary operator, Proposition 5.34 proves that $\mathscr{R}$ defines a recursion operator for the hierarchy $Q_k = \mathscr{R}^k(u_x)$. Both the Burgers' and Korteweg–de Vries equations fit into this framework.

Master symmetries arise from applying the recursion operator associated with the hierarchy to an appropriate scaling symmetry. Here "scaling" requires that the recursion operator itself scales under the associated one-parameter group.

Definition 5.35. A generalized vector field $\tilde{\mathbf{v}}_0$ is called a *scaling symmetry* for the operator $\mathscr{R}$ if $\tilde{\mathbf{v}}_0[\![\mathscr{R}]\!] = \lambda\mathscr{R}$ for some constant λ.

Examples are the time-independent parts of the scaling symmetries for the potential Burgers' and Korteweg–de Vries equations—see below. The basic result on master symmetries is the following.

Theorem 5.36. *Suppose $\mathscr{R}$ is an hereditary operator. Let $\mathbf{v}_0 = \mathbf{v}_{Q_0}$ be an evolutionary vector field such that $\mathbf{v}_0[\![\mathscr{R}]\!] = 0$, and let $\tilde{\mathbf{v}} = \mathbf{v}_{\tilde{Q}_0}$ be a scaling symmetry for both the operator $\mathscr{R}$ and the vector field $\mathbf{v}_0$, meaning that $[\tilde{\mathbf{v}}_0, \mathbf{v}_0] = \mu\mathbf{v}_0$ for some constant μ. Then $\mathscr{R}\tilde{\mathbf{v}}_0 = \mathbf{v}_{\mathscr{R}\tilde{Q}_0}$ is a master symmetry of the hierarchy of evolution equations $u_t = Q_k = \mathscr{R}^k(Q_0)$. In fact, if we set $\mathbf{v}_k = \mathbf{v}_{Q_k}$ and $\tilde{\mathbf{v}}_k = \mathbf{v}_{\tilde{Q}_k}$, where $\tilde{Q}_k = \mathscr{R}^k(\tilde{Q}_0)$, then*

$$[\mathbf{v}_j, \mathbf{v}_k] = 0, \qquad [\tilde{\mathbf{v}}_j, \mathbf{v}_k] = (\mu + k\lambda)\mathbf{v}_{j+k}, \qquad [\tilde{\mathbf{v}}_j, \tilde{\mathbf{v}}_k] = \lambda(k-j)\tilde{\mathbf{v}}_{j+k}.$$

Proof. According to Proposition 5.34, $\mathscr{R}$ is a recursion operator for the hierarchy, hence $\mathbf{v}_j[\![\mathscr{R}]\!] = 0$, $j \geqslant 0$. The first identity is proved using the Leibniz rule (5.41) and the fact that $\mathscr{R}$ is hereditary:

$$[\mathbf{v}_j, \mathbf{v}_k] = [\mathbf{v}_j, \mathscr{R}^k\mathbf{v}_0] = \mathscr{R}^k[\mathbf{v}_j, \mathbf{v}_0] = \mathscr{R}^{j+k}[\mathbf{v}_0, \mathbf{v}_0] = 0.$$

Similarly, to prove the second identity,

$$\begin{aligned}[\mathbf{v}_j, \tilde{\mathbf{v}}_k] = [\mathbf{v}_j, \mathscr{R}^k\tilde{\mathbf{v}}_0] &= -\mathscr{R}^k[\tilde{\mathbf{v}}_0, \mathscr{R}^j\mathbf{v}_0] = -\mathscr{R}^k\{\tilde{\mathbf{v}}_0[\![\mathscr{R}^j]\!]\mathbf{v}_0 + \mathscr{R}^j[\tilde{\mathbf{v}}_0, \mathbf{v}_0]\} \\ &= (j\lambda + \mu)\mathscr{R}^{j+k}\mathbf{v}_0.\end{aligned}$$

The third identity is left to the reader. □

In the case of the potential Burgers' equation, the scaling symmetry is provided by the time-independent part, $\tilde{Q}_0 = xu_x$, of the geometric scaling symmetry (which was called Q_4 in (5.13)). The basic master symmetry, then, has characteristic $\tilde{Q}_1 = \mathscr{R}(xu_x) = x(u_{xx} + u_x^2) + u_x$, which is (except for the extra term u_x, which doesn't play any role since it commutes with everything) the time-independent part of the symmetry with characteristic $2Q_7$ in (5.13). Theorem 5.36 reconfirms the bracket relations noted earlier.

In the Korteweg–de Vries case, the scaling symmetry has characteristic $\tilde{Q}_0 = xu_x + 2u$, which, again, is the time-independent part of the standard geometric scaling symmetry. The basic master symmetry, then, has characteristic

$$\begin{aligned}\tilde{Q}_1 = \mathscr{R}\tilde{Q}_0 &= (D_x^2 + \tfrac{2}{3}u + \tfrac{1}{3}u_xD_x^{-1})(xu_x + 2u) \\ &= x(u_{xxx} + uu_x) + 4u_{xx} + \tfrac{4}{3}u^2 + \tfrac{1}{3}u_xD_x^{-1}(u),\end{aligned}$$

where we wrote $xu_x + 2u = D_x(xu) + u$ to obtain the final expression. The occurrence of the "integral" $D_x^{-1}(u)$ of u in the resulting formula means that $\tilde{Q}_1$ is not an ordinary differential function. Thus, the master symmetry of the

Korteweg–de Vries equation is not an ordinary generalized symmetry, but rather a "nonlocal vector field," and one must be a bit careful in computations which involve it. It is remarkable that its commutator with any of the higher order Korteweg–de Vries equations is a local evolution equation, a fact that follows from Theorem 5.31 and Theorem 5.36. Despite its importance, we will not pursue the theory of such nonlocal symmetries any further here.

Pseudo-differential Operators

In our discussion of recursion operators, we have already encountered the formal inverse D_x^{-1} of the total derivative operator D_x. By allowing the inverse of D_x, and higher order powers thereof, into the picture, we are naturally led to enlarge our space of differential operators. The net result is a formal, but very useful version of the calculus of pseudo-differential operators in one independent variable.

We begin by recalling the elementary properties of the ring of differential operators in one independent variable x and (just for simplicity) one dependent variable u, most of which have already been used. A *differential operator* is a finite sum

$$\mathscr{D} = \sum_{i=0}^{n} P_i[u] D_x^i, \tag{5.49}$$

where the coefficients $P_i[u]$ are differential functions. The differential operator $\mathscr{D}$ has *order* n provided its leading coefficient is not identically zero: $P_n \not\equiv 0$. A differential operator of order 0 is given by a single differential function, $\mathscr{D} = P_0[u]$ and is referred to as a *multiplication operator*; it is important in what follows to distinguish between differential functions and the multiplication operators they determine.

The (noncommutative) multiplication of differential operators is completely described by the obvious formula

$$D_x^i \cdot D_x^j = D_x^{i+j}, \tag{5.50}$$

valid for $i, j \geqslant 0$, and the elementary Leibniz rule, which begins with the derivational property of D_x:

$$D_x \cdot Q = Q D_x + Q', \qquad Q' = D_x Q. \tag{5.51}$$

(In (5.51), the differential function Q is to be viewed as a multiplication operator.) By induction, we find the general Leibniz rule

$$D_x^n \cdot Q = \sum_{k=0}^{n} \binom{n}{k} Q^{(k)} D_x^{n-k}, \tag{5.52}$$

where $Q^{(k)} = D_x^k Q$. The two rules (5.50) and (5.52) allow us, by linearity, to define the product of any two differential operators. Therefore, in algebraic language, the space of all differential operators forms a noncommutative ring, the identity operator being the multiplication operator determined by the constant function 1.

A differential operator (5.49), then, is a polynomial in the total derivative D_x with differential functions for coefficients. A pseudo-differential operator will be the analogous "Laurent series" (as in complex analysis) obtained by admitting negative powers of D_x.

Definition 5.37. A (formal) *pseudo-differential operator* is a formal infinite series

$$\mathscr{D} = \sum_{i=-\infty}^{n} P_i[u] D_x^i, \tag{5.53}$$

whose coefficients P_i are differential functions. We say that $\mathscr{D}$ has order n provided its leading coefficient is not identically zero: $P_n \not\equiv 0$.

By convention, the zero pseudo-differential operator is said to have order $-\infty$. (The reason for this will appear shortly.) The recursion operator $\mathscr{R} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1}$ for the Korteweg–de Vries equation discussed above provides an example of a pseudo-differential operator of order 2.

An important remark: Given a differential function $P[u]$, while it is clear how to define $\mathscr{D}(P)$ when $\mathscr{D}$ is a differential operator, there is no obvious analogue when $\mathscr{D}$ is a general pseudo-differential operator. We will not even try to make sense of any action of pseudo-differential operators on differential functions here.

Just as we multiply ordinary differential operators, we can define a compatible multiplication process for pseudo-differential operators. We treat D_x^{-1} as the inverse of D_x, so that $D_x^{-1} \cdot D_x = 1 = D_x \cdot D_x^{-1}$; more generally, we continue to allow the product rule (5.50), for arbitrary integers i, j. Now consider the Leibniz rule (5.51). If we multiply (5.51) on both the left and the right by the operator D_x^{-1}, and rearrange terms, we deduce the identity

$$D_x^{-1} \cdot Q = Q D_x^{-1} - D_x^{-1} \cdot Q' D_x^{-1}. \tag{5.54}$$

If we apply identity (5.54) to the second term on the right-hand side (with $Q' = D_x Q$ replacing Q), we deduce

$$D_x^{-1} \cdot Q = Q D_x^{-1} - Q' D_x^{-2} + D_x^{-1} \cdot Q'' D_x^{-2}.$$

Applying (5.54) to the third term on the right-hand side of this latter equation, and continuing, we deduce the general rule

$$D_x^{-1} \cdot Q = Q D_x^{-1} - Q' D_x^{-2} + Q'' D_x^{-2} - \cdots = \sum_{i=0}^{\infty} (-1)^i Q^{(i)} D_x^{-i-1}, \tag{5.55}$$

which allows us to commute the multiplication operator determined by Q with the operator D_x^{-1} and explains why we allow infinite series in our original definition of pseudo-differential operators. An obvious induction allows us to prove a general *Leibniz rule* for commuting multiplication and differentiation operators,

$$D_x^n \cdot Q = \sum_{i=0}^{\infty} \frac{n(n-1)\cdots(n-k+1)}{k!} Q^{(k)} D_x^{n-k}, \tag{5.56}$$

which is valid for *any* integral value of n. (If $n \geqslant 0$, all but finitely many of the terms in (5.56) vanish, and hence this formula does reduce to the differential operator version (5.52)). Extending formulas (5.50) and (5.56) by linearity allows us to express the product of any two pseudo-differential operators as a pseudo-differential operator. For example,

$$\begin{aligned}(D_x + u^2 D_x^{-1})(D_x^2 + u D_x^{-1}) = D_x^3 + u^2 D_x + u + u_x D_x^{-1} + u^3 D_x^{-2} - u^2 u_x D_x^{-3} \\ + u^2 u_{xx} D_x^{-4} + \cdots .\end{aligned}$$

See Exercise 5.19 for an alternative version of the product formula for pseudo-differential operators.

The product of pseudo-differential operators is associative and linear, but not commutative. Indeed, the commutator of two pseudo-differential operators is defined as $[\mathscr{D}, \mathscr{E}] = \mathscr{D} \cdot \mathscr{E} - \mathscr{E} \cdot \mathscr{D}$, and, like the commutator of (scalar) differential operators, obeys the order relation

$$\operatorname{order}[\mathscr{D}, \mathscr{E}] \leqslant \operatorname{order} \mathscr{D} + \operatorname{order} \mathscr{E} - 1. \tag{5.57}$$

The embedding of the ring of differential operators into the larger ring of pseudo-differential operators has many remarkable consequences, due to the additional structure of the latter domain.

Theorem 5.38. *Every nonzero pseudo-differential operator has an inverse.*

Proof. Suppose $\mathscr{D}$ is a pseudo-differential operator of order n as in (5.53), with nonzero leading coefficient P_n. The inverse $\mathscr{E} = \mathscr{D}^{-1}$ will have order $-n$, and take the form

$$\mathscr{E} = \frac{1}{P_n} D_x^{-n} + Q_1 D_x^{-n-1} + Q_2 D_x^{-n-2} + \cdots .$$

Substituting these two expressions into the equation $\mathscr{D} \cdot \mathscr{E} = 1$ leads to a system of equations for the coefficients Q_k of $\mathscr{E}$:

$$P_n Q_1 + n P_n D_x\left(\frac{1}{P_n}\right) + \frac{P_{n-1}}{P_n} = 0,$$

$$\begin{aligned}&P_n Q_2 + n P_n D_x Q_1 + P_{n-1} Q_1 \\ &\quad + \binom{n}{2} P_n D_x^2\left(\frac{1}{P_n}\right) + (n-1) P_{n-1} D_x\left(\frac{1}{P_n}\right) + \frac{P_{n-2}}{P_n} = 0,\end{aligned}$$

and so on. The first equation can be solved for Q_1; plugging this formula into the second allows us to solve for Q_2 in terms of the P_j, etc., etc.

The resulting pseudo-differential operator $\mathscr{E}$ is, by construction, a right inverse for $\mathscr{D}$. To prove that it is also a left inverse, we multiply the equation $\mathscr{D}\cdot\mathscr{E} = 1$ on the right by $\mathscr{D}$, producing $\mathscr{D}\cdot\mathscr{E}\cdot\mathscr{D} = \mathscr{D}$; hence

$$\mathscr{D}\cdot(\mathscr{E}\cdot\mathscr{D} - 1) = 0.$$

Since $\mathscr{D} \neq 0$, the latter equation proves that $\mathscr{E}\cdot\mathscr{D} = 1$, and hence $\mathscr{E}$ is also a left inverse for $\mathscr{D}$. Here we make use of the easy fact that the space of pseudo-differential operators has no zero divisors, so $\mathscr{D}\cdot\mathscr{F} = 0$ if and only if $\mathscr{D} = 0$ or $\mathscr{F} = 0$, a result which becomes obvious by looking at the leading order term of the product of any two nonzero pseudo-differential operators. □

For example, the inverse of the first order differential operator $\mathscr{D} = D_x + u$ has the form

$$\mathscr{D}^{-1} = D_x^{-1} - uD_x^{-2} + (u_x + u^2)D_x^{-3} - (u_{xx} + 3uu_x + u^3)D_x^{-4} + \cdots.$$

It can be proved that the coefficient of D_n^{-n} in $(D_x + u)^{-1}$ is $(-1)^{n-1}(D_x + u)^{n-2}u$ for $n \geqslant 2$, cf. Exercise 5.20.

Theorem 5.38 proves that the space of pseudo-differential operators forms a skew (noncommutative) field. Yet there is even more structure available: Not only can we take inverses, but also roots.

Theorem 5.39. *Every pseudo-differential operator of order $n > 0$ has an n-th root.*

Proof. The proof is similar to that of Theorem 5.38. Suppose $\mathscr{D}$ is a pseudo-differential operator of the form (5.53), with nonzero leading coefficient P_n. The n-th root $\mathscr{E} = \sqrt[n]{\mathscr{D}}$ will be a first order pseudo-differential operator of the form

$$\mathscr{E} = \sqrt[n]{P_n}D_x + Q_0 + Q_{-1}D_x^{-1} + Q_{-2}D_x^{-2} + \cdots.$$

Substituting into the equation $\mathscr{E}^n = \mathscr{D}$ leads to a system of equations for the coefficients Q_k of $\mathscr{E}$, which, in analogy with the proof of Theorem 5.38, can be recursively solved for the Q_k. □

For example, the square root of the second order operator $\mathscr{D} = D_x^2 + u$ has the form

$$\sqrt{\mathscr{D}} = D_x + \tfrac{1}{2}uD_x^{-1} - \tfrac{1}{4}u_xD_x^{-2} + \tfrac{1}{8}(u_{xx} - u^2)D_x^{-3} + \cdots.$$

Theorem 5.39 allows us to define the fractional powers $\mathscr{D}^{i/n} = (\sqrt[n]{\mathscr{D}})^i$, for all integers i, of any n-th order pseudo-differential operator. We note that, just as with the ordinary powers, the fractional powers of a pseudo-differential oprator commute: $[\mathscr{D}^{i/n}, \mathscr{D}^{j/n}] = 0$. The fractional powers play a key role in Gel'fand and Dikii's approach, [1], [3], to soliton equations such as the Korteweg–de Vries equation.

Formal Symmetries

Consider an n-th order evolution equation

$$u_t = K[u] = K(x, u^{(n)}), \tag{5.58}$$

which does not explicitly depend on t. According to (5.27), a t-independent evolutionary vector field $\mathbf{v}_Q$ determines a symmetry of this equation if and only if the flows commute; this condition has the infinitesimal formulation

$$\operatorname{pr} \mathbf{v}_K(Q) - \operatorname{pr} \mathbf{v}_Q(K) = 0. \tag{5.59}$$

The key observation is that the symmetry condition (5.59) can, in turn, be converted into an equation involving differential operators by taking its Fréchet derivative. (We are effectively "relinearizing" the symmetry condition.)

Lemma 5.40. *Let $u_t = K[u]$ be an evolution equation, and let $Q[u]$ be a differential function. The Fréchet derivative of $Q_t = \operatorname{pr} \mathbf{v}_K(Q) = \mathsf{D}_Q(K)$ has the form*

$$\mathsf{D}_{Q_t} = \mathsf{D}_{\operatorname{pr} \mathbf{v}_K(Q)} = \operatorname{pr} \mathbf{v}_K(\mathsf{D}_Q) + \mathsf{D}_Q \cdot \mathsf{D}_K. \tag{5.60}$$

The proof follows directly from Definition 5.24 of the Fréchet derivative and is left to the reader. Note that, on solutions to $u_t = K$, the first term on the right-hand side of (5.60), which is the (ordinary) Lie derivative of the differential operator D_Q with respect to $\mathbf{v}_K$, can also be written as the time derivative $(\mathsf{D}_Q)_t$ of D_Q. □

Taking the Fréchet derivative of our symmetry condition (5.59) and using (5.60), we deduce the *operator symmetry condition*

$$\operatorname{pr} \mathbf{v}_K(\mathsf{D}_Q) - \operatorname{pr} \mathbf{v}_Q(\mathsf{D}_K) - [\mathsf{D}_K, \mathsf{D}_Q] = 0. \tag{5.61}$$

Condition (5.61) is almost identical to the original symmetry condition (5.59) since the Fréchet derivative of a differential function is zero if and only if the function depends on x only, so that if (5.61) holds, (5.59) is satisfied up to a function of x.

We now concentrate on equations having generalized symmetries of high order, meaning that the order m of the symmetry is much greater that the order n of the evolution equation: $m \gg n$. Note that, according to (5.57), the differential operators in the operator symmetry condition (5.61) have respective orders m, n and $m + n - 1$. Therefore, if $m \gg n$, the dominant (highest order) terms in (5.61) are the first and third, which, on solutions to $u_t = K$, are

$$(\mathsf{D}_Q)_t - [\mathsf{D}_K, \mathsf{D}_Q] = \operatorname{pr} \mathbf{v}_K(\mathsf{D}_Q) + [\mathsf{D}_Q, \mathsf{D}_K].$$

According to Definition 5.28, the latter operator is nothing but the (1, 1)-Lie derivative, cf. (5.40) (but with K playing the role of Q), of the differential operator D_Q with respect to the evolutionary vector field $\mathbf{v}_K$ determined by our evolution equation (5.58). According to (5.61), these terms must vanish modulo a differential operator of order n.

We can generalize this condition by replacing the Fréchet derivative operator D_Q by an arbitrary (pseudo-) differential operator.

Definition 5.41. Let $u_t = K[u]$ be an n-th order evolution equation. A pseudo-differential operator $\mathscr{D}$ of order m is called a *formal symmetry of rank k* if the (1, 1)-Lie derivative $\mathbf{v}_K[\![\mathscr{D}]\!]$ has order at most $n + m - k$; explicitly,

$$\operatorname{order}(\mathscr{D}_t + [\mathscr{D}, \mathsf{D}_K]) \leqslant n + m - k. \tag{5.62}$$

As an immediate consequence of (5.61), we see that genuine symmetries provide formal symmetries whose rank is the same as the order of the symmetry.

Proposition 5.42. *If $Q(x, u^{(m)})$ is the characteristic of an m-th order generalized symmetry of an evolution equation, then its Fréchet derivative D_Q is a formal symmetry of the equation of order m and of rank m.*

Our goal is to determine explicit conditions on our evolution equation (5.58) such that it will possess a formal symmetry operator of a prescribed rank. As we shall see, the higher the rank of the formal symmetry, the more restrictive the conditions imposed on the evolution equation. Eventually, the existence of a formal symmetry of a high enough rank will impose so many conditions on the evolution equation that it will possess a formal symmetry of infinite rank! Indeed, since the condition (5.44) that an operator $\mathscr{R}$ be a recursion operator for the evolution equation $u_t = K$ is that the (1, 1)-Lie derivative of $\mathscr{R}$ vanish, our convention that the zero pseudo-differential operator has order $-\infty$ implies that a recursion operator is the same as a formal symmetry of rank ∞. As we have seen, any evolution equation possessing a recursion operator has an infinite hierarchy of generalized symmetries, and, is, in an appropriate sense, an integrable evolution equation. We therefore propose the following symmetry-based definition of integrability.

Definition 5.43. An evolution equation is called *integrable* if it possesses a nonconstant formal symmetry of rank ∞.

(Note that a constant multiplication operator $\mathscr{D} = c$ is trivially a formal symmetry of order ∞.) For example, it is known that a second order evolution equation is integrable if and only if it has a formal symmetry of rank 5, and a third order evolution equation in which u_{xxx} occurs linearly is integrable if and only if it has a formal symmetry of rank 8. However, there is, as yet, no proof that a general evolution equation having a formal symmetry of sufficiently high rank is integrable in the sense of Definition 5.43, nor are there any realistic estimates of what "sufficiently high" might mean.

Lemma 5.44. *If $\mathscr{D}_1$ and $\mathscr{D}_2$ are formal symmetries of ranks k_1 and k_2, respectively, then their sum $\mathscr{D}_1 + \mathscr{D}_2$ and their product $\mathscr{D}_1 \cdot \mathscr{D}_2$ are formal symmetries of rank (at least) $k = \min\{k_1, k_2\}$.*