

Exact Renormalization Group Equations. An Introductory Review.

C. Bagnuls*and C. Bervillier†
C. E. Saclay, F91191 Gif-sur-Yvette Cedex, France

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Abstract

We critically review the use of the exact renormalization group equations (ERGE) in the framework of the scalar theory. We lay emphasis on the existence of different versions of the ERGE and on an approximation method to solve it: the derivative expansion. The leading order of this expansion appears as an excellent textbook example to underline the nonperturbative features of the Wilson renormalization group theory. We limit ourselves to the consideration of the scalar field (this is why it is an introductory review) but the reader will find (at the end of the review) a set of references to existing studies on more complex systems.

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*Service de Physique de l'Etat Condensé

†Service de Physique Théorique e-mail: bervil@spht.saclay.cea.fr

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1 Introduction

“The formal discussion of consequences of the renormalization group works best if one has a differential form of the renormalization group transformation. Also, a differential form is useful for the investigation of properties of the ε expansion to all orders (...) A longer range possibility is that one will be able to develop approximate forms of the transformation which can be integrated numerically; if so one might be able to solve problems which cannot be solved any other way.” [1]

By “exact renormalization group equation (ERGE)”, we mean the continuous (i.e. not discrete) realization of the Wilson renormalization group (RG) transformation of the action in which no approximation is made and also no expansion is involved with respect to some small parameter of the action¹. Its formulation — under a differential form — is known since the early seventies [3, 1, 4]. However, due to its complexity (an integro-differential equation), its study calls for the use of approximation (and/or truncation) methods. For a long time it was natural to use a perturbative approach (based on the existence of a small parameter like the famous ε -expansion for example). But, the standard perturbative field theory (e.g. see [5]) turned out to be more efficient and, in addition the defenders of the nonperturbative approach have turned towards the discrete formulation of the RG due to the problem of the “stiff” differential equation (see a discussion following a talk given by Wegner [6]). This is why it is only since the middle of the eighties that substantial studies have been carried out via:

- the truncation procedures in the scaling field method [7, 8] (extended studies of [9])
- the explicit consideration of the local potential approximation [10] and of the derivative expansion [11]
- an appealing use, for field theoreticians, of the ERGE [12]

In the nineties there has been a rapid growth of studies in all directions, accounting for scalar (or vector) fields, spinor, gauge fields, finite temperature, supersymmetry, gravity, etc...

In this paper we report on progress in the handling of the ERGE. Due to the abundance of the literature on the subject and because this is an introductory report, we have considered in detail the various versions of the ERGE only in the scalar (or vector) case². This critical review must also be seen as an incitement to look at the original papers of which we give a list as complete as possible.

Let us mention that the ERGE is almost ignored in most of the textbooks on the renormalization group except notably in [16] (see also [17]).

2 Exact Renormalization Group Equations

2.1 Introduction

There are four representations of the ERGE: the functional differential equation, the functional integral, the infinite set of partial differential equations for the couplings $u_n(\mathbf{p}_1, \dots, \mathbf{p}_n; t)$ (eq. 11.19 of [1]) which are popularly known because they introduce the famous “beta” functions $\beta_n(\{u_n\})$:

$$\frac{du_n}{dt} = \beta_n(\{u_n\})$$

¹On the meaning of the word “exact”, see a discussion following the talk by Halperin in [2].

²However, because of the review by Wetterich and collaborators in this volume, we have not reported on their work as it deserves. The reader is invited to refer to their review [13] and also to [14, 15] from which he could realize the rich variety of calculations that may be done within the nonperturbative framework of the ERGE.

and the infinite hierarchy of the ordinary differential equations for the scaling fields $\mu_i(t)$ [8] (the RG-scale parameter t is defined in section 2.2 and the scaling fields $\mu_i(t)$ in section 2.2.4).

In this review we shall only consider the functional differential representation of the ERGE.

There is not a unique form of the ERGE, each form of the equation is characterized by the way the momentum cutoff Λ is introduced. (In perturbative RG, this kind of dependence is known as the regularization-scheme dependence or “scheme dependence” in short [18].) The important point is that the various forms of the ERGE embody a unique physical content in the sense that they all preserve the same physics at large distances and, via the recourse to a process of limit, yield the same physics at small distances (continuum limits). The object of this part is to present the main equations used in the literature and connections between their formally (but not necessarily practically) equivalent forms. We do not derive them in detail here since one may find the derivations in several articles or reviews (that we indicate below).

Although the (Wilson) renormalization group theory owes much to the statistical physics as recently stressed by M. E. Fisher [19] we adopt here the notations and the language of field theory.

Before considering explicitly the various forms of the ERGE (in sections 2.4-2.6), we find it essential to fix the notations and to remind some fundamental aspects of the RG.

2.2 Notations, reminders and useful definitions

We denote by Λ the momentum cutoff and the RG-“time” t is defined by $\frac{\Lambda}{\Lambda_0} = e^{-t}$ in which Λ_0 stands for some initial value of Λ .

We consider a scalar field $\phi(\mathbf{x})$ with \mathbf{x} the coordinate vector in an Euclidean space of dimension d . The Fourier transformation of $\phi(\mathbf{x})$ is defined as:

$$\phi(\mathbf{x}) = \int_p \phi_p e^{i\mathbf{p}\cdot\mathbf{x}}$$

in which

$$\int_p \equiv \int \frac{d^d p}{(2\pi)^d} \quad (1)$$

and ϕ_p stands for the function $\phi(\mathbf{p})$ where \mathbf{p} is the momentum vector (wave vector).

The norms of \mathbf{x} and \mathbf{p} are noted respectively x and p ($x \equiv \sqrt{\mathbf{x}\cdot\mathbf{x}}$). However, when no confusion may arise we shall denote the vectors \mathbf{x} and \mathbf{p} by simply x and p as in (1) for example. Sometimes the letters k and q (or \mathbf{k} and \mathbf{q}) will refer also to momentum variables.

It is useful to define K_d as the surface of the d -dimensional unit sphere divided by $(2\pi)^d$, i.e.:

$$K_d = \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma(\frac{d}{2})} \quad (2)$$

We shall also consider the case where the field has N components $\phi = (\phi_1, \dots, \phi_N)$ that we shall also denote generically by ϕ_α .

The action $S[\phi]$ (the Hamiltonian divided by $k_B T$ for statistical physics) is a general semi-local functional of ϕ . Semi-local means that, when it is expanded³ in powers of ϕ , $S[\phi]$ involves only powers of $\phi(\mathbf{x})$ and of its derivatives with respect to x^μ (that we denote $\partial^\mu \phi$ or even $\partial \phi$ instead of $\partial \phi / \partial x^\mu$). This characteristics is better expressed in the momentum-space (or wave-vector-space). So we write:

³We do not need to assume this expansion as the unique form of $S[\phi]$ in general.

$$S[\phi] = \sum_n^\infty \int_{p_1 \dots p_n} u_n(\mathbf{p}_1, \dots, \mathbf{p}_n) \phi_{p_1} \dots \phi_{p_n} \hat{\delta}(\mathbf{p}_1 + \dots + \mathbf{p}_n) \quad (3)$$

in which $\hat{\delta}(\mathbf{p}) \equiv (2\pi)^d \delta^d(\mathbf{p})$ is the d -dimensional delta-function:

$$\delta^d(\mathbf{x}) = \int_p e^{i\mathbf{p}\cdot\mathbf{x}}$$

Notice that the $O(1)$ symmetry $\phi \rightarrow -\phi$, also called the Z_2 symmetry, is not assumed neither here nor in the following sections except when it is explicitly mentioned.

The u_n 's are invariant under permutations of their arguments.

For the functional derivative with respect to ϕ , we have the relation:

$$\frac{\delta}{\delta \phi_p} = \int d^d x e^{i\mathbf{p}\cdot\mathbf{x}} \frac{\delta}{\delta \phi(\mathbf{x})}$$

so that in performing the functional derivative with respect to ϕ_p we get rid of the π -factors involved in the definition (1), e.g.:

$$\begin{aligned} \frac{\delta}{\delta \phi_p} & \left[\int_{p_1 \dots p_n} u_n(\mathbf{p}_1, \dots, \mathbf{p}_n) \phi_{p_1} \dots \phi_{p_n} \hat{\delta}(\mathbf{p}_1 + \dots + \mathbf{p}_n) \right] = \\ & n \int_{p_1 \dots p_{n-1}} u_n(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{p}) \phi_{p_1} \dots \phi_{p_{n-1}} \hat{\delta}(\mathbf{p}_1 + \dots + \mathbf{p}_{n-1} + \mathbf{p}) \end{aligned}$$

in order to lighten the notations we sometimes will write $\frac{\delta}{\delta \phi}$ instead of $\frac{\delta}{\delta \phi(\mathbf{x})}$ when no confusion may arise.

Let us also introduce:

- The generating functional $Z[J]$ of Green's functions:

$$Z[J] = \mathcal{Z}^{-1} \int \mathcal{D}\phi \exp \{-S[\phi] + J \cdot \phi\} \quad (4)$$

in which

$$J \cdot \phi \equiv \int d^d x J(\mathbf{x}) \phi(\mathbf{x})$$

$J(\mathbf{x})$ is an external source, and \mathcal{Z} is a normalization such that $Z[0] = 1$. Indeed \mathcal{Z} is the partition function:

$$\mathcal{Z} = \int \mathcal{D}\phi \exp \{-S[\phi]\} \quad (5)$$

- The generating functional $W[J]$, of connected Green functions, is related to $Z[J]$ as follows:

$$W[J] = \ln(Z[J]) \quad (6)$$

Notice that if one defines \mathcal{W}

$$e^{\mathcal{W}} = \mathcal{Z}$$

then \mathcal{W} is minus the free energy.

- The Legendre transformation which defines the generating functional $\Gamma[\Phi]$ of the one-particle-irreducible (1PI) Green functions (or simply vertex functions):

$$\begin{aligned}\Gamma[\Phi] + W[J] - J \cdot \Phi &= 0 \\ \frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} \Big|_J &= J(x)\end{aligned}\tag{7}$$

in which we have introduced the notation Φ to make a distinction between this and the (dummy) field variable ϕ in (4). In the following we shall not necessarily make this distinction.

2.2.1 Dimensions

First let us define our conventions relative to the usual (i.e. engineering) dimensions.

In the following, we refer to a system of units in which the dimension of a length scale L is -1 :

$$[L] = -1$$

and a momentum scale like Λ has the dimension 1:

$$[\Lambda] = 1$$

The usual classical dimension of the field (in momentum unit):

$$d_\phi^c = [\phi] = \frac{1}{2}(d - 2)\tag{8}$$

is obtained by imposing that the coefficient of the kinetic term $\int d^d x (\partial\phi(x))^2$ in S is dimensionless [it is usually set to $\frac{1}{2}$].

The dimension of the field is not always given by (8). Indeed one knows that the field may have an anomalous dimension:

$$d_\phi^a = \frac{1}{2}(d - 2 + \eta)\tag{9}$$

with η a non-zero constant defined with respect to a non-trivial fixed point.

In RG theory, the dimension of the field depends on the fixed point in the vicinity of which the (field) theory is considered. Hence we introduce an adjustable dimension of the field, d_ϕ , which controls the scaling transformation of the field:

$$\phi(s\mathbf{x}) = s^{-d_\phi} \phi(\mathbf{x})\tag{10}$$

For the Fourier transformation, we have:

$$\phi(s\mathbf{p}) = s^{d_\phi - d} \phi(\mathbf{p})\tag{11}$$

Since the dimension of any dimensioned (in the classical meaning for ϕ) quantity is expressed in term of a momentum scale, we use Λ to reduce all dimensioned quantities into dimensionless quantities. In the following we deal with dimensionless quantities and, in particular, the notation p will refer to a dimensionless momentum variable. However sometimes, for the sake of clarity, we will need to reintroduce the explicit Λ -dependence, e.g. via the ratio p/Λ .

It is also useful to notice that, with a dimensionless p , the following derivatives are equivalent (the derivative is taken at constant dimensioned momentum):

$$\frac{\partial}{\partial t} = -\Lambda \frac{\partial}{\partial \Lambda} = p \frac{\partial}{\partial p}$$

2.2.2 Transformations of the field variable

In order to discuss invariances in RG theory, it is useful to consider a general transformation of the field which leaves invariant the partition function. Following [20, 21], we replace ϕ_p by ϕ'_p such that

$$\phi'_p = \phi_p + \sigma \Psi_p[\phi] \quad (12)$$

where σ is infinitesimally small and Ψ_p a function which may depend on all Fourier components of ϕ . Then one has

$$S[\phi'] = S[\phi] + \sigma \int_p \Psi_p[\phi] \frac{\delta S[\phi]}{\delta \phi_p}$$

Moreover we have:

$$\int \mathcal{D}\phi' = \int \mathcal{D}\phi \frac{\partial \{\phi'\}}{\partial \{\phi\}} = \int \mathcal{D}\phi \left(1 + \sigma \int_p \frac{\delta \Psi_p[\phi]}{\delta \phi_p} \right)$$

The transformation must leave the partition function \mathcal{Z} [eq. (5)] invariant. Therefore one obtains

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\phi' \exp \{-S[\phi']\} \\ &= \int \mathcal{D}\phi \exp \{-S[\phi] - \sigma \mathcal{G}_{\text{tra}} \{\Psi\} S[\phi]\} \end{aligned}$$

with

$$\mathcal{G}_{\text{tra}} \{\Psi\} S[\phi] = \int_p \left(\Psi_p \frac{\delta S}{\delta \phi_p} - \frac{\delta \Psi_p}{\delta \phi_p} \right) \quad (13)$$

which indicates how the action transforms under the infinitesimal change (12):

$$\frac{dS}{d\sigma} = \mathcal{G}_{\text{tra}} \{\Psi\} S$$

In the case of N components, the expression (13) generalizes obviously:

$$\mathcal{G}_{\text{tra}} \{\Psi\} S[\phi] = \sum_{\alpha=1}^N \int_p \left(\Psi_p^\alpha \frac{\delta S}{\delta \phi_p^\alpha} - \frac{\delta \Psi_p^\alpha}{\delta \phi_p^\alpha} \right) \quad (14)$$

2.2.3 Rescaling

We consider an infinitesimal change of (momentum) scale:

$$p \rightarrow p' = s p = (1 + \sigma) p \quad (15)$$

with σ infinitesimally small. Introducing the rescaling operator of [20, 21], the consequence on $S[\phi]$ is written as

$$S \rightarrow S' = S + \sigma \mathcal{G}_{\text{dil}} S \quad (16)$$

and thus:

$$\frac{dS}{d\sigma} = \mathcal{G}_{\text{dil}} S \quad (17)$$

Considering S as given by (3), then $\Delta S = \sigma \mathcal{G}_{\text{dil}} S$ may be expressed by gathering the changes induced by (15) on the various factors in the sum, namely:

1 the differential volume $\prod_{i=1}^n d^d p_i = (1 + \sigma)^{-nd} \prod_{i=1}^n d^d p'_i$ induces a change ΔS_1 which may be written as

$$\Delta S_1 = -\sigma \left(d \int_p \phi_p \frac{\delta}{\delta \phi_p} \right) S$$

2 the couplings $u_n(\mathbf{p}_1, \dots, \mathbf{p}_n) = u_n(s^{-1}\mathbf{p}'_1, \dots, s^{-1}\mathbf{p}'_n)$ induce a change ΔS_2 which may be written as

$$\Delta S_2 = -\sigma \left(\int_p \phi_p \mathbf{p} \cdot \partial'_p \frac{\delta}{\delta \phi_p} \right) S$$

where the prime on the derivative symbol (∂'_p) indicates that the momentum derivative does not act on the delta-functions.

3 the delta-functions $\hat{\delta}(\mathbf{p}_1 + \dots + \mathbf{p}_n) = \hat{\delta}(s^{-1}\mathbf{p}'_1 + \dots + s^{-1}\mathbf{p}'_n)$ induce a change ΔS_3 which may be written as

$$\Delta S_3 = \sigma (dS)$$

4 the field itself $\phi(\mathbf{p}) = \phi(s^{-1}\mathbf{p}')$ induces a change ΔS_4 which, according to (11), is dictated by

$$\phi(s^{-1}\mathbf{p}') = (1 + \sigma)^{d-d_\phi} \phi(\mathbf{p})$$

Hence, to the first order in σ , we have

$$\Delta S_4 = \sigma \left[(d - d_\phi) \int_p \phi_p \frac{\delta}{\delta \phi_p} \right] S$$

Summing the four contributions, $\Delta S = \sum_{i=1}^4 \Delta S_i$ we obtain:

$$\Delta S = \sigma \left(dS - \int_p \phi_p \mathbf{p} \cdot \partial'_p \frac{\delta}{\delta \phi_p} - d_\phi \int_p \phi_p \frac{\delta}{\delta \phi_p} \right) S$$

This expression may be further simplified by allowing the momentum derivative ∂_p to act also on the delta-functions (this eliminates the prime and absorbs the term dS). We thus may write

$$\Delta S = -\sigma \left(\int_p \phi_p \mathbf{p} \cdot \partial_p \frac{\delta}{\delta \phi_p} + d_\phi \int_p \phi_p \frac{\delta}{\delta \phi_p} \right) S$$

hence:

$$\mathcal{G}_{\text{dil}} S = - \left(\int_p \phi_p \mathbf{p} \cdot \partial_p \frac{\delta}{\delta \phi_p} + d_\phi \int_p \phi_p \frac{\delta}{\delta \phi_p} \right) S \quad (18)$$

As in the case of \mathcal{G}_{tra} , the generalization to N components is obvious [see eq. (14)].

The writing of the action of \mathcal{G}_{dil} [eq. (18)] may take on two other forms in the literature:

1. due to the possible integration by parts of the first term:

$$\mathcal{G}_{\text{dil}} S = \left(\int_p (\mathbf{p} \cdot \partial_p \phi_p) \frac{\delta}{\delta \phi_p} + (d - d_\phi) \int_p \phi_p \frac{\delta}{\delta \phi_p} \right) S \quad (19)$$

2. if one explicitly performs the derivative with respect to \mathbf{p} acting on the δ -functions:

$$\mathcal{G}_{\text{dil}} S = dS - \left(\int_p \phi_p \mathbf{p} \cdot \partial'_p \frac{\delta}{\delta \phi_p} + d_\phi \int_p \phi_p \frac{\delta}{\delta \phi_p} \right) S \quad (20)$$

in which now ∂'_p does not act on the δ -functions.

An other expression of the operator \mathcal{G}_{dil} may be found in the literature [22], it is:

$$\mathcal{G}_{\text{dil}} = d - \Delta_\partial - d_\phi \Delta_\phi \quad (21)$$

where $\Delta_\phi = \phi \cdot \frac{\delta}{\delta \phi}$ is the ‘phi-ness’ counting operator: it counts the number of occurrences of the field ϕ in a given vertex and Δ_∂ may be expressed as

$$\Delta_\partial = d + \int_p \phi_p \mathbf{p} \cdot \partial_p \frac{\delta}{\delta \phi_p} \quad (22)$$

i.e. the momentum scale counting operator $+d$. Operating on a given vertex it counts the total number of derivatives acting on the fields ϕ [22].

Notice that we have not introduced the anomalous dimension η of the field. This is because it naturally arises at the level of searching for a fixed point of the ERGE. As we indicate in the following section, the introduction of η is related to an invariance.

2.2.4 Linearized RG theory

Following Wegner [21], we write the ERGE under the following formal form

$$\frac{dS}{dt} = \mathcal{G} S$$

Near a fixed point S^* (such that $\mathcal{G} S^* = 0$) we have:

$$\frac{d(S^* + \Delta S)}{dt} = \mathcal{L} \Delta S + \mathcal{Q} \Delta S$$

in which the RG operator has been separated into a linear \mathcal{L} and a quadratic \mathcal{Q} parts.

The eigenvalue equation:

$$\mathcal{L} \mathcal{O}_i^* = \lambda_i \mathcal{O}_i^*$$

defines scaling exponents λ_i and a set (assumed to be complete) of eigenoperators \mathcal{O}_i^* . Hence we have for any $S(t)$:

$$S(t) = S^* + \sum_i \mu_i(t) \mathcal{O}_i^*$$

In which μ_i are the “scaling fields” [23] which in the linear approximation satisfy:

$$\frac{d\mu_i(t)}{dt} = \lambda_i \mu_i(t)$$

Which yields:

$$S(t) = S^* + \sum_i \mu_i(0) t^{\lambda_i} \mathcal{O}_i^* \quad (23)$$

Scaling operators There are three kinds of operators associated to well defined eigenvalues and called “scaling operators” in [23, 21] which are classified as follows⁴:

- $\lambda_i > 0$, the associated scaling field μ_i (or the operator \mathcal{O}_i^*) is *relevant* because it brings the action away from the fixed point.
- $\lambda_i < 0$, the associated scaling field μ_i is *irrelevant* because it decays to zero when $t \rightarrow \infty$ and $S(t)$ finally approaches S^* .
- $\lambda_i = 0$, the associated scaling field μ_i is *marginal* and $S^* + \mu_i \mathcal{O}_i^*$ is a fixed point for any μ_i . This latter property may be destroyed beyond the linear order⁵.

In critical phenomena, the relevant scaling fields alone are responsible for the scaling form of the physical quantities: e.g., in its scaling form, the free energy depends only on the scaling fields [21]. The irrelevant scaling fields induce corrections to the scaling form [21].

To be specific, the positive eigenvalue of a critical fixed point (once unstable), say λ_1 , is the inverse of the correlation length critical exponent ν and the less negative eigenvalue is equal to (minus) the subcritical exponent ω .

In modern field theory, only the relevant (or marginally relevant) scaling fields are of interest in the continuum limit [1]: they correspond to the renormalized couplings (or masses) of field theory the continuum limit of which being defined “at” the considered fixed point (see section 2.10).

Redundant operators and reparametrization invariance In addition to scaling operators, there are redundant operators [20, 21]. They come out due to invariances of the RG [20, 21] (see also [24]). Thus they can be expressed in the form $\mathcal{G}_{\text{tra}} \{\Phi\} S^*$ and the associated exponents λ_i (which, in general, have nonuniversal values) are spurious since the free energy does not depend on the corresponding redundant fields μ_i (by construction of the transformation generator \mathcal{G}_{tra} which leaves the partition function invariant, see section 2.2.2).

Although unphysical, the redundant fields cannot be neglected. For example, a well known redundant operator is⁶ $\frac{\delta S^*}{\delta \phi_0}$ which may be written under the form $\mathcal{G}_{\text{tra}} \{\Phi\} S^*$ with $\Phi_q = \hat{\delta}(q)$. and, most often, has the eigenvalue $\lambda = \frac{1}{2}(d - 2 + \eta)$. Since $\lambda > 0$ for $d = 3$, this operator is relevant with respect to the Wilson-Fisher [26] (i.e. Ising-like for $d = 3$) fixed point although it is not physical. Indeed, as pointed out by Hubbard and Schofield in [27], the fixed point becomes unstable in presence of a $\int \phi^3(x)$ term which, however, may be eliminated by the substitution $\phi_0 \rightarrow \phi_0 + \mu$, which is controlled by the operator $\frac{\delta S^*}{\delta \phi_0}$ since:

$$S^*(\phi_0 + \mu) = S^*(\phi_0) + \mu \frac{\delta S^*}{\delta \phi_0} + O(\mu^2)$$

This redundant operator is not really annoying because it is sufficient to consider actions that are even functional of ϕ (Z_2 -symmetric) to get rid of $\frac{\delta S^*}{\delta \phi_0}$.

Less obvious and more interesting for field theory is the following redundant operator:

$$\mathcal{O}_1 = \int_q \left[\frac{\delta^2 S}{\delta \phi_q \delta \phi_{-q}} - \frac{\delta S}{\delta \phi_q} \frac{\delta S}{\delta \phi_{-q}} + \phi_q \frac{\delta S}{\delta \phi_q} \right] \quad (24)$$

⁴In perturbative field theory, where the implicit fixed point is Gaussian, the scaling fields are called super-renormalizable, nonrenormalizable and strictly renormalizable respectively.

⁵This is the case of the renormalized ϕ^4 -coupling constant for $d = 4$ with respect to the Gaussian fixed point: it is marginal in the linear approximation and irrelevant beyond. It is marginally irrelevant.

⁶A clear explanation of this may be found in [25] p.101–102.

which has been studied in detail by Riedel *et al* [8]. When the RG transforms the field variable linearly (as in the present review), \mathcal{O}_1 has, once and for all, the eigenvalue $\lambda_1 = 0$; it is absolutely marginal [8].

\mathcal{O}_1 is redundant because it may be written under the form $\mathcal{G}_{\text{tra}} \{\Phi\} S$ with:

$$\Phi_q = \phi_q - \frac{\delta S}{\delta \phi_{-q}}$$

The redundant character of \mathcal{O}_1 is related to the invariance of the RG transformation under a change of the overall normalization of ϕ [28, 29]. This invariance is also called the “reparametrization invariance” [22].

The most general realization of this symmetry [29] is not linear, this explains why \mathcal{O}_1 is so complicated (otherwise, in the case of a linear realization of the invariance, \mathcal{O}_1 would reduce to simply $\int_q \phi_q \frac{\delta S}{\delta \phi_q}$).

As consequences of the reparametrization invariance [28, 29]:

- a line of equivalent fixed points exists which is parametrized by the normalization of the field,
- a field-rescaling parameter that enters in the ERGE must be properly adjusted in order for the RG transformation to have a fixed point (the exponent η takes on a specific value).

We illustrate these two aspects with the Gaussian fixed point in section 2.7 after having written down the ERGE.

- Due to the complexity of \mathcal{O}_1 , truncations of the ERGE (in particular the derivative expansion, see section 4.1) may easily violate the reparametrization invariance [29], in which case the line of equivalent fixed points becomes a line of inequivalent fixed points yielding different nonuniversal values for the exponent η . However the search for a vestige of the invariance may be used to determine the best approximation for η [29, 11, 30]. In the case where the invariance is manifestly linearly realized and momentum independent (as when a regularization with a sharp cutoff is utilized for example, see below), then the derivative expansion may preserve the invariance and as a consequence, η is uniquely defined (see section 4.1).

To understand why η must take on a specific value, it is helpful to think of a linear eigenvalue problem. “*The latter may have apparently a solution for each arbitrary eigenvalue, but the fact that we can choose the normalization of the eigenvector at will over-determines the system, making that only a discrete set of eigenvalues are allowed.*” [30] (see section 4.1)

2.3 Principles of derivation of the ERGE

The Wilson RG procedure is carried out in two steps [1] (see also [31] for example):

1. an integration of the fluctuations $\phi(\mathbf{p})$ over the range $e^{-t} < |p| \leq 1$ which leaves the partition function (5) invariant,
2. a change of the length scale by a factor e^{-t} in all linear dimensions to restore the original scale Λ of the system, i. e. $\mathbf{p} \rightarrow \mathbf{p}' = e^t \mathbf{p}$

For infinitesimal value of t , step 2 corresponds to a change in the effective action [see eqs (15, 16 and 17 with $\sigma = t$)]

$$S \rightarrow S' = S + t\mathcal{G}_{\text{dil}}S \quad (25)$$

inducing a contribution to

$$\dot{S} \equiv \frac{\partial S}{\partial t} \quad (26)$$

which is equal to $\mathcal{G}_{\text{dil}}S$.

The step of reducing the number of degrees of freedom (step 1) is the main part of the RG theory. It is sometimes called “coarse grain decimation” by reference to a discrete realization of the RG transformation or Kadanoff’s transformation [32], it is also called sometimes “blocking” or “coarsening”. It carries its own arbitrariness due to the vague notion of “block”, i.e. in the present review the unprecised way of separating the high from the low momentum frequencies. Step 1 may be roughly introduced as follows.

We assume that the partition function may be symbolically written as

$$\mathcal{Z} = \prod_{p \leq 1} \int \mathcal{D}\phi_p \exp \{-S[\phi]\} \quad (27)$$

then after performing the integrations of step 1, we have:

$$\mathcal{Z} = \prod_{p \leq e^{-t}} \int \mathcal{D}\phi_p \exp \{-S'[\phi]\}$$

with

$$\exp \{-S''[\phi; t]\} = \prod_{e^{-t} < p \leq 1} \int \mathcal{D}\phi_p \exp \{-S[\phi]\} \quad (28)$$

$S'[\phi; t]$ is named the Wilson effective action. By considering an infinitesimal value of t , one obtains an evolution equation for S under a differential form, i.e. an explicit expression for \dot{S} .

As indicated by Wegner [20, 10], the infinitesimal “blocking” transformation of S (step 1) may sometimes⁷ be expressed as a transformation of the field of the form introduced in section 2.2.2. Hence the general expression of the ERGE may formally be written as follows [20, 10]:

$$\dot{S} = \mathcal{G}_{\text{dil}}S + \mathcal{G}_{\text{tra}}\{\Psi\}S$$

in which Ψ has different expressions depending on the way one introduces the cutoff Λ . For example, in the case of the Wilson ERGE [see eq. (30) below], Ψ has the following form [21]:

$$\Psi_p = (c + 2p^2) \left(\phi_p - \frac{\delta S}{\delta \phi_{-p}} \right)$$

As it is introduced just above in (27-28), the cutoff is said sharp or hard⁸. It is known that a sharp boundary in momentum space introduces non-local interactions in position space [1] which one would like to avoid. Nevertheless, a differential ERGE has been derived [4] which has been used several times with success under an approximate form. Indeed, in the leading approximation of the derivative expansion (local potential approximation), most of the differences between a sharp and smooth cutoff disappear, and moreover, as stressed by Morris [33], the difficulties induced by the sharp cutoff may be circumvented by considering the Legendre transformation (7) (see section 2.6).

⁷When the cutoff is smooth.

⁸It corresponds to a well defined boundary between low and high momentum frequencies, to be opposed to a smooth cutoff which corresponds to a blurred boundary.

2.4 Wegner-Houghton's sharp cutoff version of the ERGE

The equation has been derived in [4], one may also find an interesting detailed presentation in [34], it reads:

$$\dot{S} = \lim_{t \rightarrow 0} \frac{1}{2t} \left[\int'_p \ln \left(\frac{\delta^2 S}{\delta \phi_p \delta \phi_{-p}} \right) - \int'_p \frac{\delta S}{\delta \phi_p} \frac{\delta S}{\delta \phi_{-p}} \left(\frac{\delta^2 S}{\delta \phi_p \delta \phi_{-p}} \right)^{-1} \right] + \mathcal{G}_{\text{dil}} S + \text{const} \quad (29)$$

in which we use (26) and the prime on the integral symbol indicates that the momenta are restricted to the shell $e^{-t} < |p| \leq 1$, and $\mathcal{G}_{\text{dil}} S$ corresponds to any of the eqs (18–20) with $d_\phi(t)$ set to a constant in [4].

The explicit terms correspond to the step 1 (decimation or coarsening) of section 2.3, while $\mathcal{G}_{\text{dil}} S$ refers to step 2 (rescaling) as indicated in section 2.2.3. The additive constant may be neglected in field theory [due to the normalization of (4)].

2.5 Smooth cutoff versions of the ERGE

2.5.1 Wilson's incomplete integration

The first expression of the exact renormalization group equation under a differential form has been presented as far back as 1970 [3] before publication in the famous Wilson and Kogut review [1] (see chapt. 11). The step 1 (decimation) of this version (referred to below as the Wilson ERGE) consists in an “incomplete” integration in which large momenta are more completely integrated than small momenta (see chapt. 11 of [1] and also [16] p. 70 for the details).

The Wilson RG equation in our notations reads (with the change $\mathcal{H} \rightarrow -S$ compared to [1]):

$$\dot{S} = \mathcal{G}_{\text{dil}} S + \int_p (c + 2p^2) \left(\frac{\delta^2 S}{\delta \phi_p \delta \phi_{-p}} - \frac{\delta S}{\delta \phi_p} \frac{\delta S}{\delta \phi_{-p}} + \phi_p \frac{\delta S}{\delta \phi_p} \right) \quad (30)$$

Some short comments relative to (30):

The term $\mathcal{G}_{\text{dil}} S$ (which comes out of the rescaling step 2) is given by one of the eqs. (18–21) but, in [1], the choice $d_\phi = d/2$ has been made. The somewhat mysterious function c (denoted $\frac{d\rho}{dt}$ in [1]) must be adjusted in such a way as to obtain a useful fixed point [1, 28]. This adjustment is related to the reparametrization invariance (see section 2.7 for an example). Notice that c is precisely introduced in (30) in front of the operator \mathcal{O}_1 of eq. (24) which controls the change of normalization of the field (see section 2.2.4). Indeed, in the vicinity of the fixed point, we have [11]:

$$c = 1 - \frac{\eta}{2} \quad (31)$$

and most often c is considered as a constant (as in [35, 11] for example). Notice that the unusual (for field theory) choice⁹ $d_\phi = d/2$ in [1], leads to the same anomalous dimension (9) at the fixed point.

2.5.2 Polchinski's equation

With a view to study field theory, Polchinski [12] has derived his own smooth cutoff version of the ERGE (see also section 2.10.2). A general ultraviolet (UV) cutoff function $K(p^2/\Lambda^2)$ is introduced (we momentarily restore the dimensions) with the property that it vanishes rapidly when $p > \Lambda$. (Several kinds of explicit functions K may be chosen, the sharp cutoff would be introduced with the Heaviside step function $K(x) = \Theta(1 - x)$.) The Euclidean action reads:

⁹See ref. [20, 16] for a further explanation of this choice.

$$S[\phi] \equiv \frac{1}{2} \int_p \phi_p \phi_{-p} p^2 K^{-1}(p^2/\Lambda^2) + S_{\text{int}}[\phi] \quad (32)$$

Compared to [12], the “mass” term has been incorporated into $S_{\text{int}}[\phi]$, this does not corrupt in any way the eventual analysis of the massive theory because the RG naturally generates quadratic terms in ϕ and the massive or massless character of the (field) theory is not defined at the level of eq. (32) but a posteriori in the process of defining the continuum limit (modern conception of the renormalization of field theory, see sections 2.10.1 and 3.4.1).

Polchinski’s ERGE is obtained from the requirement that the coarsening step (step 1 of section 2.3) leaves the generating functional $Z[J]$ [eq. (4)] invariant. The difficulty of dealing with an external source is circumvented by imposing that $J(p) = 0$ for $p > \Lambda$. As in [1], the derivation relies upon the writing of an (ad hoc) expression under the form of a complete derivative with respect to the field in such a way as to impose $dZ[J]/d\Lambda = 0$. The original form of Polchinski’s equation accounts only for the step 1 and reads (for more details on this derivation, see for example [36, 37]):

$$\Lambda \frac{dS_{\text{int}}}{d\Lambda} = \frac{1}{2} \int_p p^{-2} \Lambda \frac{dK}{d\Lambda} \left(\frac{\delta S_{\text{int}}}{\delta \phi_p} \frac{\delta S_{\text{int}}}{\delta \phi_{-p}} - \frac{\delta^2 S_{\text{int}}}{\delta \phi_p \delta \phi_{-p}} \right) \quad (33)$$

then, considering the rescaling (step 2) and the complete action, the Polchinski ERGE is (see for example [38]):

$$\dot{S} = \mathcal{G}_{\text{dil}} S - \int_p K'(p^2) \left(\frac{\delta^2 S}{\delta \phi_p \delta \phi_{-p}} - \frac{\delta S}{\delta \phi_p} \frac{\delta S}{\delta \phi_{-p}} + \frac{2p^2}{K(p^2)} \phi_p \frac{\delta S}{\delta \phi_p} \right) \quad (34)$$

in which all quantities are dimensionless and $K'(p^2)$ stands for $dK(p^2)/dp^2$.

Let us mention that one easily arrives at eq. (33) using the observation that the two following functionals:

$$Z[J] = \int \mathcal{D}\phi \exp \left\{ -\frac{1}{2} \phi \cdot \Delta^{-1} \cdot \phi - S[\phi] + J \cdot \phi \right\} \quad (35)$$

and

$$Z'[J] = \int \mathcal{D}\phi \exp \left\{ -\frac{1}{2} \phi_1 \cdot \Delta_1^{-1} \cdot \phi_1 - \frac{1}{2} \phi_2 \cdot \Delta_2^{-1} \cdot \phi_2 - S[\phi_1 + \phi_2] + J \cdot (\phi_1 + \phi_2) \right\} \quad (36)$$

are equivalent (up to a multiplicative factor) provided that $\Delta = \Delta_1 + \Delta_2$ and $\phi = \phi_1 + \phi_2$ (see appendix¹⁰ 10 of [5] and also [39]).

2.5.3 Redundant free ERGE

In [39] it is proposed to make the effective dimension of the field d_ϕ [defined in (10)], which enters Polchinski’s ERGE (34) via the rescaling part $\mathcal{G}_{\text{dil}} S$ [see eqs (18–21)], depend on the momentum \mathbf{p} in such a way as to keep unchanged, along the RG flows, the initial quadratic part $S_0[\phi] \equiv \frac{1}{2} \int_p \phi_p \phi_{-p} p^2 K^{-1}(p^2)$. This additional condition imposed on the ERGE would completely eliminate the ambiguities in the definition (the invariances) of the RG transformation leaving no room for any redundant operators (see also chapt 5 of [16]).

If we understand correctly the procedure, it is similar to (but perhaps more general than) that proposed in [8] to eliminate the redundant operator (24). One fixes the arbitrariness associated to the invariance (reparametrization invariance of section 2.2.4) in order that any RG flow remains orthogonal to the redundant direction(s).

¹⁰Or appendix 24 of the first edition in 1989.

The authors of [39] do not specify what happens in the case where the unavoidable truncation (or approximation) used to study the ERGE breaks some invariances¹¹. In fact, as already mentioned at the end of section 2.2.4, the freedom associated with the redundant directions allows us to search for the region of minimal break of invariance in the space of interactions [29, 11, 30]. If this freedom is suppressed one may not obtain, for universal quantities such as the critical exponents, the optimal values compatible with the approximation (or truncation) used.

2.6 ERGE for the Legendre Effective Action

The first derivation¹² of an ERGE for the Legendre effective action $\Gamma[\Phi]$ [defined in eq. (7)] has been carried out with a sharp cutoff by Nicoll and Chang [40] (see also [41]). Their aim was to simplify the obtention of the ε -expansion from the ERGE. More recent obtentions of this equation with a smooth cutoff are due to Bonini, D'Attanasio and Marchesini [42], Wetterich [43], Morris [33] and Ellwanger [44].

A striking fact arises with the Legendre transformation: the running cutoff Λ (or intermediate-scale momentum cutoff, i.e. associated to the “time” t) acts as an IR cutoff and physical Green functions are obtained in the limit $\Lambda \rightarrow 0$ [45, 46, 47, 43, 33, 42, 44]. The reason behind this result is simple to understand. The generating functional $\Gamma[\Phi]$ is obtained by integrating out *all* modes (from $\Lambda_0 = \infty$ to 0). If an intermediate cutoff $\Lambda \ll \Lambda_0$ is introduced and integration is performed only in the range $[\Lambda, \Lambda_0]$, then for the integrated modes (thus for the effective $\Gamma_\Lambda[\Phi]$), Λ is an IR cutoff while for the unintegrated modes (for the effective $S_\Lambda[\phi]$), Λ is an UV cutoff. There is an apparent second consequence: contrary to the ERGE for $S_\Lambda[\phi]$, the ERGE for $\Gamma_\Lambda[\Phi]$ will depend on both an IR (Λ) and an UV (Λ_0) cutoffs¹³. However it is possible to send Λ_0 to infinity, see section 2.6.2 for a short discussion of this point.

2.6.1 Sharp cutoff version

In [40] an ERGE for the Legendre transformation $\Gamma[\Phi]$ [eq. 7] is derived (See also [41]). It reads:

$$\dot{\Gamma} = \mathcal{G}_{\text{dil}}\Gamma + \frac{1}{2} \int \frac{d\Omega}{(2\pi)^d} \ln \left\{ \Gamma_{q,-q} - \int_p \int_{p'} \Gamma_{qp} (\Gamma^{-1})_{pp'} \Gamma_{p',-q} \right\} \quad (37)$$

in which:

$$\Gamma_{kk'} = \frac{\delta^2 \Gamma}{\delta \Phi_k \delta \Phi_{k'}}$$

the momentum q lies on the shell $q = |\mathbf{q}| = 1$ while the integrations on p and p' are performed inside the shell $[\Lambda, \Lambda_0/\Lambda]$ where Λ_0 is some initial cutoff ($\Lambda_0 > \Lambda$) and Ω is the surface of the d -dimensional unit sphere [$\Omega = (2\pi)^d K_d$].

One sees that Λ is like an IR cutoff and that (37) depends on the initial cutoff Λ_0 .

2.6.2 Smooth cutoff version

We adopt notations which are close to the writing of (32) and we consider the Wilson effective action with an “additive” [43] IR cutoff Λ such that:

¹¹In particular it is known that the derivative expansion developed within ERGE with smooth cutoffs breaks the reparametrization invariance.

¹²The first mention to the Legendre transformation in the expression of the ERGE has been formulated by E. Brézin in a discussion following a talk by Halperin [2].

¹³One could think a priori that as a simple differential equation, the ERGE would be instantaneous (would depend only on Λ and not on some initial scale Λ_0), but it is an integro-differential equation.

$$S_\Lambda[\phi] \equiv \frac{1}{2} \int_p \phi_p \phi_{-p} C^{-1}(p, \Lambda) + S_{\Lambda_0}[\phi] \quad (38)$$

in which $C(p, \Lambda)$ is an additive infrared cutoff function which is small for $p < \Lambda$ (tending to zero as $p \rightarrow 0$) and $p^2 C(p, \Lambda)$ should be large for $p > \Lambda$ [22]. Due to the additive character of the cutoff function, $S_{\Lambda_0}[\phi]$ is the entire action (involving the kinetic term contrary to eq. (32) and to [33] where the cutoff function was chosen multiplicative). In this section, because C is naturally dimensioned¹⁴ [contrary to K in (32)], all the dimensions are implicitly restored in order to keep the same writing as in the original papers. The ultra-violet regularization is provided by Λ_0 and needs not to be introduced explicitly (see [33] and below). The Legendre transformation is defined as:

$$\Gamma[\Phi] + \frac{1}{2} \int_p \Phi_p \Phi_{-p} C^{-1}(p, \Lambda) = -W[J] + J \cdot \Phi$$

in which $W[J]$ and Φ are defined as usual [see eq (7)] from (38).

Then the ERGE reads:

$$\dot{\Gamma} = \mathcal{G}_{\text{dil}} \Gamma + \frac{1}{2} \int_p \frac{1}{C} \Lambda \frac{\partial C}{\partial \Lambda} (1 + C \Gamma_{p,-p})^{-1} \quad (39)$$

When the field is no longer a pure scalar but carries some supplementary internal degrees of freedom and becomes a vector, a spinor or a gauge field etc..., a more compact expression using the trace of operators is often used:

$$\dot{\Gamma} = \mathcal{G}_{\text{dil}} \Gamma + \frac{1}{2} \text{tr} \left[\frac{1}{C} \Lambda \frac{\partial C}{\partial \Lambda} \cdot \left(1 + C \cdot \frac{\delta^2 \Gamma}{\delta \Phi \delta \Phi} \right)^{-1} \right] \quad (40)$$

which, for example, allows us to include the generalization to N components in a unique writing ($\frac{\delta^2 \Gamma}{\delta \Phi \delta \Phi}$ has then two supplementary indices α and β corresponding to the derivatives with respect to the fields Φ_α and Φ_β , the trace is relative to both the momenta and the indices).

The equations (39,40) may be obtained, as in [33], using the trick of eqs.(35 and 36), but see also [43, 42].

For practical computations it is actually often quite convenient (for example, see section 4.2) to write the flow equation (39) as follows [14]:

$$\dot{\Gamma} = \mathcal{G}_{\text{dil}} \Gamma + \frac{1}{2} \text{tr} \tilde{\partial}_t \ln \left(C^{-1} + \frac{\delta^2 \Gamma}{\delta \Phi \delta \Phi} \right) \quad (41)$$

with $\tilde{\partial}_t \equiv -\Lambda \frac{\partial}{\partial \Lambda}$ acting only on C and not on Γ , i.e. $\tilde{\partial}_t = (\partial C^{-1} / \partial t) (\partial / \partial C^{-1})$.

Wetterich's expression of the ERGE [43] is identical¹⁵ to eqs. (39, 40, 41). Its originality is in the choice of the cutoff function C^{-1} : to make the momentum integration in (39) converge, a cutoff function is introduced such that only a small momentum interval $p^2 \approx \Lambda^2$ effectively contributes [43] (see also the review in [14]). This feature, which avoids an explicit UV regularization, allows calculations in models where the UV regularization is a delicate matter (e.g. non-Abelian gauge theories). In fact, as noticed in [33], the ERGE only requires momenta $p \approx \Lambda$ and should not depend on $\Lambda_0 \gg \Lambda$ at all. Indeed, once a finite ERGE is obtained, the flow equation for $\Gamma_\Lambda[\Phi]$ is finite and provides us with an “ERGE”-regularization scheme which is specified by the flow equation, the choice of the infrared cutoff function C and the “initial condition” Γ_Λ [14]. Most often, there is no need for any UV regularization and the limit $\Lambda_0 \rightarrow \infty$ may be taken

¹⁴It is not harmless that C is dimensionful because the anomalous dimension η may be a part of its dimension (see [22] and section 3.1.3)

¹⁵The correspondence between Wetterich's notations and ours is as follows: $\Lambda \rightarrow k$; $C \rightarrow 1/R_k$; $t \rightarrow -t$.

safely. In this case, the cutoff function chosen by Wetterich [43] has the following form (up to some factor):

$$C(p, \Lambda) = \frac{1 - f(\frac{p^2}{\Lambda^2})}{p^2 f(\frac{p^2}{\Lambda^2})} \quad (42)$$

$$f(x) = e^{-2ax^b} \quad (43)$$

in which the two parameters a and b may be adjusted to vary the smoothness of the cutoff function.

Although it is almost an anticipation on the expansions (local potential approximation and derivative expansion) considered in the parts to come, we find it worthwhile indicating here the exact equation satisfied by the effective potential which is often used by Wetterich and co-workers (see their review [13] in this volume). Thus following Wetterich [43], we write the effective (Legendre) action $\Gamma[\Phi]$ for $O(N)$ -symmetric systems as follows¹⁶:

$$\begin{aligned} \Gamma[\Phi] &= \int d^d x \left[U(\rho) + \frac{1}{2} \partial^\mu \Phi_\alpha \mathcal{Z}(\rho, -\square) \partial_\mu \Phi_\alpha + \frac{1}{4} \partial^\mu \rho \mathcal{Y}(\rho, -\square) \partial_\mu \rho \right] \\ \rho &= \frac{1}{2} \Phi^2 \end{aligned}$$

where the symbol \square stands for $\partial_\mu \partial^\mu$ and acts only on the right (summation over repeated indices is assumed). Then the exact evolution equation for the effective potential $U(\rho)$ [43] follows straightforwardly from (41):

$$\dot{U} = \frac{1}{2} \int_p \frac{1}{C^2} \Lambda \frac{\partial C}{\partial \Lambda} \left[\frac{N-1}{M_0} + \frac{1}{M_1} \right] + dU - d_\phi U' \quad (44)$$

$$M_0 = \mathcal{Z}(\rho, p^2) p^2 + C^{-1} + U' \quad (45)$$

$$M_1 = [\mathcal{Z}(\rho, p^2) + \rho \mathcal{Y}(\rho, p^2)] p^2 + C^{-1} + U' + 2\rho U'' \quad (46)$$

in which \dot{U} stands for $\partial U(\rho, t)/\partial t$ and U' and U'' refer to the first and second (respectively) derivatives with respect to ρ . The two last terms in (44) come from $\mathcal{G}_{\text{dil}}\Gamma$ which was not explicitly considered in [43].

The interest of dealing with an additive cutoff function is that one may easily look for the classes of C that allow a *linear* realization of the reparametrization invariance [22]. It is found that [22] C must be chosen as:

$$C(p, \Lambda) = \Lambda^{-2} \left(\frac{p^2}{\Lambda^2} \right)^k$$

with k an integer such that $k > d/2 - 1$ to have UV convergence. With this choice, the derivative expansion preserves the reparametrization invariance and η is uniquely defined [22] (see part 4.1.2). On the contrary, because the cutoff function corresponding to eqs. (42, 43) has an exponential form, the derivative expansion does not provide us with a uniquely defined value of η (see section 4.3).

Sharp cutoff limit It is possible to obtain the sharp cutoff limit from eqs.(39, 40) provided one is cautious in dealing with the value at the origin of the Heaviside function $\theta(0)$ (which is *not* equal to $\frac{1}{2}$) [33, 48]. One obtains the sharp cutoff limit of the flow equation [48]:

$$\dot{\Gamma} = \mathcal{G}_{\text{dil}}\Gamma + \frac{1}{2} \int_p \frac{\delta(p-1)}{\gamma(p)} \left[\hat{\Gamma} \cdot \left(1 + G \cdot \hat{\Gamma} \right)^{-1} \right] (\mathbf{p}, -\mathbf{p}) \quad (47)$$

¹⁶It is customary to introduce the variable ρ for $O(N)$ systems because this allows better convergences in some cases (see section 3.3) but in the particular case $N = 1$ the symmetry assumption is not required.

in which the field independent full inverse propagator $\gamma(p)$ has been separated from the two-point function :

$$\frac{\delta^2 \Gamma[\Phi]}{\delta \Phi \delta \Phi} (\mathbf{p}, \mathbf{p}') = \gamma(p) \hat{\delta}(\mathbf{p} + \mathbf{p}') + \hat{\Gamma}[\Phi](\mathbf{p}, \mathbf{p}')$$

so that $\hat{\Gamma}[0] = 0$, and $G(p) = \theta(p - 1)/\gamma(p)$.

2.7 Equivalent fixed points and reparametrization invariance.

To illustrate the line of equivalent fixed points which arises when the reparametrization invariance is satisfied (see the end of section 2.2.4), we consider here the pure Gaussian case of the Wilson ERGE (see also the appendix of [1] and [30]). No truncation is needed to study the Gaussian case, the analysis below is thus exact.

The action is assumed to have the following pure quadratic form:

$$S_G[\phi] = \frac{1}{2} \int_p \phi_p \phi_{-p} R(p^2)$$

The effect of \mathcal{G}_{dil} in (30) yields (the prime denotes the derivative with respect to p^2):

$$\mathcal{G}_{\text{dil}} S_G = \left(\frac{d}{2} - d_\phi \right) \int_p \phi_p \phi_{-p} R - \int_p \phi_p \phi_{-p} p^2 R'$$

while the remaining part (coarsening) gives (up to neglected constant terms):

$$\mathcal{G}_{\text{tra}} S_G = \int_p \phi_p \phi_{-p} [(c + 2p^2)(R - R^2) - p^2 R']$$

Adding the two contributions to \dot{S}_G , choosing $d_\phi = \frac{d}{2}$ as in [1] and imposing that the fixed point is reached ($\dot{S}_G = 0$), one obtains:

$$\begin{aligned} c &= 1 \\ R^*(p^2) &= \frac{zp^2}{e^{-2p^2} + zp^2} \end{aligned}$$

This is a line of (Gaussian) fixed points parametrized by z . To reach this line, the parameter c must be adjusted to 1 [i.e., following eq. (31), $\eta = 0$], the fixed points on the line are equivalent.

The same analysis may be done with the same kind of conclusions with the Polchinski ERGE (see [30]). For the Wegner-Houghton version (29), the situation is very different in nature [30]. The same kind of considerations yields:

$$R^*(p^2) = p^{2-\eta}$$

in which η is undetermined. “This phenomenon is of quite different nature as the similar one described above. There the whole line shares the same critical properties, here it does not; there we have well-behaved actions throughout the fixed line, here nearly all of them are terribly non-local (in the sense that we cannot expand the action integrand in a power series of p^2). What happens is that we have one physical FP (the $\eta = 0$ case) and a line of spurious ones.” [30]

2.8 Approximations and truncations

How to deal with integro-differential equations is not known in general. In the case of the RG equations, one often had recourse to perturbative expansions such as the usual perturbation in powers of the coupling but also the famous ε -expansion (where $\varepsilon = 4 - d$), the $1/N$ -expansion or expansions in the dimensionality $2 - d$, let us mention also an expansion exploiting the smallness of the critical exponent¹⁷ η [50].

When no small parameter can be identified or when one does not want to consider a perturbative approach, one must truncate the number of degrees of freedom involved in order to reduce the infinite system of coupled differential equations to a finite system. The way truncations are introduced is of utmost importance as one may learn from the development of the scaling field method (see [8]). The ERGE is a useful starting point to develop approximate approach. If from the beginning it has been exclusively seen as a useful tool for the investigation of the ε -expansion [51], two complementary approaches to nonperturbative truncations have been proposed:

- an expansion of the effective action in powers of the derivatives of the field [52, 11]:

$$S[\phi] = \int d^d x \left\{ V(\phi) + \frac{1}{2} Z(\phi) (\partial_\mu \phi)^2 + O(\partial^4) \right\} \quad (48)$$

which is explicitly considered in this review (see sections 3 and 4.1)

- an expansion in powers of the field for the functional $\Gamma[\Phi]$ [53] (see also [54, 55]):

$$\Gamma[\Phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \left(\prod_{k=0}^n d^d x_k \Phi(x_k) \right) \Gamma^{(n)}(x_1, \dots, x_n) \quad (49)$$

The flow equations for the 1PI n -point functions $\Gamma^{(n)}$ are obtained by functional differentiation of the ERGE. The distinction between S in (48) and Γ in (49) is not essential, we can introduce the two approximations for both S and Γ .

Because the derivative expansion corresponds to small values of p , it is naturally (quantitatively) adapted to the study of the large distance or low energy physics like critical phenomena or phase transition. We will see, however, that at a qualitative level it is suitable to a general discussion of many aspects of field theory (like the continuum limit, see part 3). Obviously, when bound state formation or nonperturbative momentum dependences are studied, the expansion (49) seems better adapted (see, for examples, [56, 57]).

Only a few terms of such series will be calculable in practice, since the number of invariants increases rapidly for higher orders (see section 4.1).

2.9 Scaling field representation of the ERGE

To date the most expanded approximate method for the solution of the ERGE has been developed by Golner and Riedel in [9] (see also [7] and especially [8]) from the scaling-field representation. The idea is to introduce the expansion (23) into the ERGE which is thus transformed into an infinite hierarchy of nonlinear ordinary differential equations for scaling fields. For evident reasons¹⁸, the fixed point chosen for S^* is the Gaussian fixed point. Approximate solutions may be obtained by using truncations and iterations. The approximations appear to be effective also in calculations of properties, like phase diagrams, scaling functions, crossover phenomena and so on (see [8]).

¹⁷The expansion requires also a truncation in powers of the field and some re-expansion of $\varepsilon/4 = 1 - d/4$ in powers of $\eta^{1/2}$ [49].

¹⁸The eigenoperators \mathcal{O}_i^* and eigenvalues λ_i of the Gaussian fixed point can be determined exactly.

Because they are unjustly not often mentioned in the literature, we find it fair to extract from [7] the following estimates for $N = 1$ (see the paper for estimates corresponding to other values of N):

$$\begin{aligned}\nu &= 0.626 \pm 0.009 \\ \eta &= 0.040 \pm 0.007 \\ \omega &= 0.855 \pm 0.07 \\ \omega_2 &= 1.67 \pm 0.11 \\ \omega_5 &= 2.4 \pm 0.4\end{aligned}$$

in which ω_2 is the second correction-to-scaling exponent and the subscript “5” in ω_5 refers to a ϕ^5 interaction present in the action and which would be responsible for correction-to-scaling terms specific to the critical behavior of fluids, as opposed to the Ising model which satisfies the symmetry Z_2 , see [58] and also section 3.2.3.

2.10 Renormalizability, continuum limits and the Wilson theory

2.10.1 Wilson’s continuum limit

In the section 12.2 of [1], a nonperturbative realization of the renormalization of field theory is schematically presented. The illustration is done with the example of a fixed point with one relevant direction (i.e. a fixed point which controls the criticality of magnetic systems in zero magnetic field). The resulting renormalized field theory is purely massive (involving only one parameter: a mass).

We find satisfactory¹⁹ the presentation of this continuum limit by Morris in [59] (relative to the discussion of his fig. 3, see fig. 3 of the present paper for an illustration with actual RG trajectories), and we reproduce it here just as it is.

“In the infinite dimensional space of bare actions, there is the so-called critical manifold, which consists of all bare actions yielding a given massless continuum limit. Any point on this manifold – i.e. any such bare action – flows under a given RG towards its fixed point; local to the fixed point, the critical manifold is spanned by the infinite set of irrelevant operators. The other directions emanating out of the critical manifold at the fixed point, are spanned by relevant and marginally relevant perturbations (with RG eigenvalues $\lambda_i > 0$ and $\lambda_i = 0$, respectively). [In the example of [1] and in fig. 3, there is only one relevant perturbation.] Choosing an appropriate parametrization of the bare action, we move a little bit away from the critical manifold. The trajectory of the RG will to begin with, move towards the fixed point, but then shoot away along one of the relevant directions towards the so-called high temperature fixed point which represents an infinitely massive quantum field theory.”

To obtain the continuum limit, and thus finite masses, one must now tune the bare action back towards the critical manifold and at the same time, reexpress physical quantities in renormalised terms appropriate for the diverging correlation length. In the limit that the bare action touches the critical manifold, the RG trajectory splits into two: a part that goes right into the fixed point, and a second part that emanates out from the fixed point along the relevant directions. This path is known as a Renormalised Trajectory [1] (RT). The effective actions on this path are ‘perfect actions’ [60].”

The continuum limit so defined “at” a critical fixed point has been used by Wilson to show that the ϕ^6 -field-theory in three dimensions has a nontrivial continuum limit involving

¹⁹Except the expression “critical manifold, which consists of all bare actions yielding a given massless continuum limit”, see section 3.4.1.

no coupling constant renormalization [61] (i.e. the continuum limit involves only a mass as renormalized parameter, but see also [62]). The fixed point utilized in the circumstances is the Wilson-Fisher (critical) fixed point. In fact, exactly the same limit would have been obtained starting with a ϕ^4 - or a ϕ^8 -bare-theory, since it is the symmetry of the bare action which is important and not the specific form chosen for the initial (bare) interaction (ϕ^4 , ϕ^6 , or ϕ^8 are all elements of the same Z_2 -symmetric scalar theory, see section 3.4.1 for more details).

It is noteworthy that one (mainly) exclusively presents the Wilson continuum limit as it is illustrated in [1], i.e. relatively to a critical point which possesses only one relevant direction. Obviously one may choose any fixed point with several relevant directions (as suggested in the Morris presentation reproduced above). The relevant parameters provide the renormalized couplings of the continuum limit. For example the Gaussian fixed point in three dimensions for the scalar theory (a tricritical fixed point with two relevant directions) yields a continuum limit which involves two renormalized parameters (see an interesting discussion with Brézin following a talk given by Wilson [63]). Indeed, that continuum limit is nothing but the so-called ϕ^4 -field theory used successfully in the investigation, by perturbative means, of the critical properties of statistical systems below four dimensions and which is better known as the “*Field theoretical approach to critical phenomena*” [64] (see [5] for a review). The scalar field theory below four dimensions defined “at” the Gaussian fixed point involves a mass and a (renormalized) ϕ^4 -coupling “constant”.

“At” the Gaussian fixed one may also define a massless renormalized theory. To reach this massless theory, one must inhibit the direction of instability²⁰ of the Gaussian fixed point toward the massive sector. As a consequence, the useful space of bare interactions is limited to the critical manifold alluded to above by Morris (there is one parameter to be adjusted in the bare action). The discussion is as previously but with one dimension less for the space of bare interactions: the original whole space is replaced by the critical submanifold and this latter by the tricritical submanifold. Notice that the massless continuum limit so defined really involves a scale dependent parameter: the remaining relevant direction of the Gaussian fixed point which corresponds to the ϕ^4 -renormalized coupling (see section 3.4.1 for more details). It differs however from the massless theories sketchily defined by Morris as *fixed point theories* [48, 65, 66, 67, 59, 68]. Most certainly no mass can be defined right at a fixed point [67, 59] (there is scale invariance there and a mass would set a scale) but at the same time the theory would also have no useful parameter at all (no scale dependent parameter) since, by definition, right at a fixed point nothing changes, nothing happens, there is nothing to describe.

An important aspect of the Wilson continuum limit is the resulting self-similarity emphasized rightly in several occasions by Morris and collaborators [65, 67, 59, 68, 69] (see section 3.4.1). This notion expresses the fact that in a properly defined continuum limit, the effects of the infinite number of degrees of freedom involved in a field theory are completely represented by a (very) small number of flowing (scale dependent) parameters (the relevant parameters of a fixed point): the system is self-similar in the sense that it is exactly (completely) described by the same finite set of parameters seen at different scales (see section 3.4.1). This is exactly what one usually means by renormalizability in perturbative field theory. However the question is nonperturbative in essence. For example, the ϕ^4 -field theory in four dimensions is perturbatively renormalizable, but it is not self-similar at any scale and especially in the short distance regime due to the UV “renormalon” problems [70, 71] (for a review see [72]) which prevent the perturbatively renormalized coupling constant to carry exactly all the effects of the other (an infinite number) degrees of freedom: it is not a relevant parameter for a fixed point (see section 2.10.2, 3.4.1 and [73, 59]).

²⁰The relevant direction that points towards the most stable high-temperature fixed point.

2.10.2 Polchinski's effective field theories

There has been a renewed interest of field theoreticians in the ERGE since Polchinski's paper [12] in 1984. From the properties of the RG flows generated by an ERGE (see section 2.5.2) and by using only “*very simple bounds on momentum integrals*”, Polchinski presented an easy proof of the perturbative renormalizability of the ϕ^4 field theory in four dimensions (see also [74, 42]). This paper had a considerable success. One may understand the reasons of the resulting incipient interest of field theoreticians in the ERGE, let us cite for example:

“*Proofs of renormalizability to all orders in perturbation theory were notoriously long and complicated (involving notions of graph topologies, skeleton expansions, overlapping divergences, the forest theorem, Weinberg's theorem, etc.), ...*” [37].

The enthusiasm of some was so great that one has sometimes referred to Polchinski's presentation, really based on the Wilson RG theory, as the Wilson-Polchinski theory. However the strategy relative to the construction of the continuum limit (modern expression for “renormalizability”) is rather opposite to the ideas of Wilson because *they are perturbative in essence*. Indeed, while the reference to a fixed point is *essential* in the Wilson construction of the continuum limit, Polchinski does not need any explicit reference to a fixed point, but in fact refers *implicitly* to the Gaussian fixed point²¹. A classification of parameters as relevant, irrelevant and marginal is given using a purely classical dimensional analysis (referring to the Gaussian fixed point, see [75] for example). In the Polchinski view, the marginal parameters are then considered as being relevant although in some cases (as the scalar case, see footnote 5) they may actually be (marginally) irrelevant.

The arguments may then lead to confusions. The notion of relevant parameter (the natural candidate for the renormalized parameter), which, in the Wilson theory represents an unstable direction of a fixed point (one goes away from the fixed point which thus in the occasion displays an ultraviolet stability or attractivity) has been replaced in the Polchinski point of view (for the ϕ^4 field) by the least irrelevant parameter which controls the final approach to a fixed point (one goes toward the fixed point which thus presents an infrared stability or attractivity). Thus the renormalized coupling resulting from the “proof” controls only the infrared (large distances or low energy) regime of the scalar theory. The field theory so constructed is actually an “effective” field theory [75, 76] (valid in the infrared regime) and not a field theory well defined in the short distance regime (e.g. even after the “proof” the ϕ^4 field theory in four dimensions remains trivial due to the lack of ultraviolet stable fixed point).

Of course, if by chance the Gaussian fixed point is ultraviolet stable (asymptotically free field theories), then the marginal coupling is truly a relevant parameter for the Gaussian fixed point and the perturbatively constructed field theory exists beyond perturbation (in the Wilson sense of section 2.10.1). In that case, one may use the Polchinski approach to prove the existence of a continuum limit (see some references in [75]).

It is fair to specify that, in several occasions in [12], Polchinski has emphasized the perturbative character of his proof which is *only* equivalent to (but simpler than) the usual perturbative proof. In order to be clear, let us precisely indicate the weak point of Polchinski's arguments which is clearly expressed in the discussion of the fig. 2 of [12], p. 274 one may read:

“*We can proceed in this way, thus defining the bare coupling λ_4^0 as a function of λ_4^R , Λ_R , and Λ_0 . Now take $\Lambda_0 \rightarrow \infty$ holding Λ_R and λ_4^R fixed.*”

The objection is that, in Wilson's theory (i.e. nonperturbatively) it is impossible to make sense to the second sentence without an explicit reference to an (eventually nontrivial) ultraviolet stable fixed point. In perturbation theory, however, no explicit reference to a fixed point is needed since, order by order, terms proportional to, say $(\Lambda_R/\Lambda_0)^2$, give exactly zero in the limit “ $\Lambda_0 \rightarrow \infty$ holding Λ_R and λ_4^R fixed”. However this limit introduces singularities in the

²¹As in perturbation theory.

perturbative expansion: the famous ultraviolet renormalons which make it ambiguous to resum the perturbative series for the scalar field in four dimensions (for a review on the renormalons see [72]). See section 3.4.1 for more details.

Actually, using the *nonperturbative* framework of the ERGE to present a proof of the *perturbative* renormalizability might be seen as a misunderstanding of the Wilson theory.

3 Local potential approximation: A textbook example

3.1 Introduction

The local potential approximation (LPA) of the ERGE (the momentum-independent limit of the ERGE) allows to consider all powers of ϕ . The approximation still involves an infinite number of degrees of freedom which are treated on the same footing within a nonlinear partial differential equation for a simple function $V(\phi)$ [V is the (local) potential, ϕ is assumed to be a constant field and thus, except the kinetic term, the derivatives $\partial\phi$ of $\phi(x)$ are all neglected in the ERGE].

LPA of the ERGE is the continuous version of the Wilson approximate recursion formula [77, 26] (see also [1], p. 117) which is a discrete (and approximate) realization of the RG (the momentum scale of reference is reduced by a factor two). As shown by Felder [78], LPA is also similar to a continuous version of the hierarchical model [79].

This approximation has been first considered in [80] (see also [35]) from the sharp cutoff version of the ERGE of Wegner and Houghton [4], it has been rederived by Tokar [81] by using approximate functional integrations and rediscovered by Hasenfratz and Hasenfratz [10].

LPA amounts to assuming that the action $S[\phi]$ reduces to the following form:

$$S[\phi] = \int d^d x \frac{z}{2} (\partial_\mu \phi)^2 + V(\phi) \quad (50)$$

in which z is a pure number (a constant usually set equal to 1) and , to set the ideas, $V(\phi)$ is a simple function of ϕ_0 , it has the form (symbolically)

$$V(\phi) = \sum_n u_n (\phi_0)^n \delta(0) \quad (51)$$

The infinity carried by the delta function would be absent in a treatment at finite volume, it reflects the difficulties of selecting one mode out of a continuum set, such ill-defined factors may be removed within a rescaling of ϕ [34]. In the derivation of the approximate equation, instead of using (51) we find it convenient to deal with

$$V(\phi) = \sum_n u_n \int_{p_1 \dots p_n} \phi_{p_1} \dots \phi_{p_n} \hat{\delta}(\mathbf{p}_1 + \dots + \mathbf{p}_n) \quad (52)$$

in which the u_n 's do not depend on the momenta [see eq. (3)] and to project onto the zero modes ϕ_0 of ϕ at the end of the calculation.

Eq. (50) is identical to eq. (48) in which $Z(\phi)$ is set equal to the constant z . LPA may also be considered as the zeroth order of a systematic expansion in powers of the (spatial) derivative of the field (derivative expansion) [11] (see part 4).

In the following, φ stands for ϕ_0 and primes denote derivatives with respect to φ (at fixed t):

$$\begin{aligned} \varphi &\equiv \phi_0 \\ V'(\varphi, t) &= \left. \frac{\partial V}{\partial \varphi} \right|_t \end{aligned} \quad (53)$$

Frequently, as in the present review, $V'(\varphi, t)$ is replaced by $f(\varphi, t)$.

Let us consider the expressions of LPA for the various ERGE's introduced in part 2.

3.1.1 Sharp cutoff version

The derivation of the local potential approximation for eq. (29) [80] (sharp cutoff version of the LPA of the ERGE) is well known, we only give the result (for more details see [10, 34]). It reads:

$$\dot{V} = \frac{K_d}{2} \ln [z + V''] + dV - d_\phi \varphi V' \quad (54)$$

in which K_d is given by (2). The non logarithmic terms come from the contribution $\mathcal{G}_{\text{dil}}S$ in (29). As usual in field theory, we neglect the field independent contributions (“const” in (29)]) to the effective potential V .

The t -dependence is entirely carried by the coefficient $u_n(t)$ in (52) while z is considered as being independent of t . This condition is required for consistency of the approximation (it prevents the ERGE from generating contribution to the kinetic term: there is no wave function renormalization). Writing down explicitly this condition (namely $\dot{z} = 0$) provides us with the relation:

$$d_\phi = \frac{d-2}{2}$$

in other word $\eta = 0$, i.e. the anomalous part of the dimension of the field is zero. This is a characteristic feature of the LPA.

The dependence on z in (54) may be removed (up to an additive constant) by the simplest (or naive) change of normalization of the field $\varphi \rightarrow \varphi\sqrt{z}$. [The exact version (29) is invariant under the same change.] In order to avoid the useless additive constant terms generated in (54), it is frequent to write down the evolution equation for the derivative $f = V'$, it comes [10]:

$$\dot{f} = \frac{K_d}{2} \frac{f''}{z + f'} + \left(1 + \frac{d}{2}\right) f + \left(1 - \frac{d}{2}\right) \varphi f' \quad (55)$$

Notice that one could eliminate the factor K_d by the change $f(\varphi, t) \rightarrow \lambda f(\varphi/\lambda, t)$ with $K_d \cdot \lambda^2 = 1$.

It is interesting also to write down the ERGE in the same approximation when the number of components N of the field is variable. With a view to eventually consider large values of N , it is convenient to redefine the action and the field as follows:

$$S \rightarrow N S \left[\frac{\phi}{\sqrt{N}} \right]$$

then in the case of $O(N)$ symmetric potential, the LPA of (29) yields:

$$\dot{V} = \frac{K_d}{2N} \left\{ (N-1) \ln \left[z + \frac{V'}{\varphi} \right] + \ln [z + V''] \right\} + dV - d_\phi \varphi V' \quad (56)$$

or [10]

$$\dot{f} = \frac{K_d}{2N} \left\{ (N-1) \frac{\varphi f' - f}{z\varphi^2 + \varphi f} + \frac{f''}{z + f'} \right\} + \left(1 + \frac{d}{2}\right) f + \left(1 - \frac{d}{2}\right) \varphi f' \quad (57)$$

It may be also useful to express that, in the $O(N)$ symmetric case, V is a function of φ^2 . By setting $s = \varphi^2$ and $u = 2dV/ds$, one obtains [34]:

$$\dot{u} = \frac{K_d}{N} \left[\frac{3u' + 2su''}{1 + u + 2su'} + (N-1) \frac{u'}{1+u} \right] + 2u + (2-d)su' \quad (58)$$

3.1.2 The Wilson (or Polchinski) version

Due to the originality in introducing the arbitrary scaling parameter c , it is worthwhile writing down explicitly the LPA of eq. (30). This equation has first been derived in [35].

From the same lines as previously, it comes:

$$\dot{V} = c \left[V'' - (V')^2 + \varphi V' \right] + dV - d_\phi \varphi V'$$

in which c is determined by the condition implying no wave function renormalization ($\dot{z} = 0$) which reads:

$$d - 2 - 2d_\phi + 2c = 0$$

For the Wilson choice $d_\phi = d/2$ [1], it comes $c = 1$ and from (31), $\eta = 0$ (as it must). Consequently the LPA of (30) is [35]:

$$\dot{V} = V'' - (V')^2 + \left(1 - \frac{d}{2}\right) \varphi V' + dV \quad (59)$$

which, for the derivative $f = V'$ yields [38]:

$$\dot{f} = f'' - 2ff' + \left(1 + \frac{d}{2}\right) f + \left(1 - \frac{d}{2}\right) \varphi f'$$

Notice that, contrary to the sharp cutoff version, this equation (as the exact version) is not invariant under the simplest rescaling of the field $\varphi \rightarrow \varphi\sqrt{z}$.

In the $O(N)$ -case, it comes:

$$\dot{V} = \frac{1}{N} V'' - (V')^2 + \frac{N-1}{N} \frac{V'}{\varphi} + \left(1 - \frac{d}{2}\right) \varphi V' + dV$$

Polchinski's version In the LPA, eq. (34) yields exactly the same partial differential equation as previously [38] [eq. (59)]. For general N , it has been studied by Comellas and Travesset [34] under the following form:

$$\dot{u} = \frac{2s}{N} u'' + \left[1 + \frac{2}{N} + (2-d)s - 2su\right] u' + (2-u)u$$

in which $s = \frac{1}{2}\varphi^2$ and $u = 2dV/ds$ [the definition of the variables is different from (58)].

3.1.3 The Legendre transformed version

Sharp cutoff version LPA for the Legendre transformed ERGE has been first written down by Nicoll, Chang and Stanley [82] (see also [45]) with a sharp cutoff.

From eq. (37), it is easy to verify that one obtains the same equation as in the Wegner-Houghton case (also for general N). This is because at this level of approximation the effective potential coincides with its Legendre transformation (the Helmholtz potential coincides with the free energy). Also, the sharp cutoff limit leading to eq. (47) yields the correct LPA (54) [48].

Smooth cutoff version With a smooth cutoff, the LPA of the Legendre version of the ERGE [eq. (39)] keeps the integro-differential form except for particular choices of the cutoff function and of the dimension d [e.g. see eq. (65) below]. This is clearly an inconvenience.

From (44) but without introducing the substitution $\varphi \rightarrow \rho = \frac{1}{2}\varphi^2$, we may easily write down the LPA for general N , it reads:

$$\dot{V} = -\frac{1}{N} \int_p \frac{1}{\tilde{C}^2} \left(\left[p^2 \tilde{C}' + \tilde{C} \right] \right) \left(\frac{N-1}{M'_0} + \frac{1}{M'_1} \right) + dV - d_\phi \varphi V' \quad (60)$$

$$M'_0 = p^2 + \tilde{C}^{-1} + V'/\varphi \quad (61)$$

$$M'_1 = p^2 + \tilde{C}^{-1} + V'' \quad (62)$$

in which \tilde{C} is the dimensionless version of the cutoff function C of section 2.6.2 with $C = \Lambda^{-2+\eta} \tilde{C}(p^2)$ and $\tilde{C}' = d\tilde{C}(p^2)/dp^2$ (p is there also dimensionless). Notice that, following [22], we have introduced the anomalous dimension η by anticipation of the anomalous scaling behavior satisfied by the field in the close vicinity of a non trivial fixed point. In the approximation presently considered, η vanishes and does not appear in (60) but it would have an effect at higher orders of the derivative expansion (see section 4.1).

With the particular choice of cutoff function given by (42, 43), eq. (60) may be written as follows:

$$\dot{V} = \frac{K_d}{4N} \left[(N-1) L_0^d(V'/\varphi) + L_0^d(V'') \right] + dV - d_\phi \varphi V' \quad (63)$$

in which:

$$L_0^d(w) = 2(2a)^{\frac{2-d}{2b}} \int_0^\infty dy y^{\frac{d-2}{2b}} \frac{e^{-y}}{(1-e^{-y})} \frac{1}{\left[1 + \left(\frac{2a}{y} \right)^{\frac{1}{b}} e^{-y} w \right]}$$

In the sharp cutoff limit $b \rightarrow \infty$ one has:

$$L_0^d(w) = 2 \ln(1+w) + \text{const}$$

in which “const” is infinite, neglecting this usual infinity, one sees that (63) gives back the expression (54) for $z=1$. In order to avoid the infinite “const”, it is preferable to consider the flow equation for the derivative $f = V'$, in which case the function $L_1^d(w) = -\frac{\partial}{\partial w} L_0^d(w)$ appears in the equation:

$$\dot{f} = \frac{K_d}{4N} \left[(N-1) (f'/\varphi - f/\varphi^2) L_1^d(f/\varphi) + f'' L_1^d(f') \right]$$

See [83] for more details on the function $L_n^d(w)$.

An interesting expression of the flow equation for the Legendre transformed action Γ is obtained from the smooth cutoff version of Morris [22] [see eqs. (39, 40)] with a pure power law (dimensionless) cutoff function of the form:

$$\tilde{C}(p^2) = p^{2k} \quad (64)$$

For $d=3$, $k=1$ and $N=1$, the LPA reads [22]:

$$\dot{V} = -\frac{1}{\sqrt{2+V''}} + 3V - \frac{1}{2}\varphi V' \quad (65)$$

and for general N [66]:

$$\dot{V} = -\frac{1}{\sqrt{2+V''}} - \frac{N-1}{\sqrt{2+V'/\varphi}} + 3V - \frac{1}{2}\varphi V' \quad (66)$$

The choice of the power law cutoff function (64) is dictated by the will to *linearly* realize the reparametrization invariance [22]. With the sharp cutoff, the power law cutoff is the only known cutoff that satisfies the conditions required [22, 30] to preserve this invariance.

3.2 The quest for fixed points

Fixed points are essential in the RG theory. In field theory, they determine the nature of the continuum limits; in statistical physics they control the large distance physics of a critical system.

A fixed point is a solution of the equation

$$\dot{V}^* = 0 \quad (67)$$

From the forms of the equations involved (see the preceding section), it is easy to see that $V = 0$ (or $V = \text{const}$) is always a solution of the fixed point equation. This is the Gaussian fixed point. There are two other trivial fixed points which are only accounted for with the Wilson (or Polchinski) version. Following Zumbach [84, 85, 86], let us write the eq. (59) for the quantity $\mu(\varphi, t) = \exp(-V(\varphi, t))$:

$$\dot{\mu} = \mu'' + \left(1 - \frac{d}{2}\right) \varphi \mu' + d \mu \ln \mu \quad (68)$$

from which the following trivial fixed point solutions are evident (the notations differ from those used in [84, 85, 86]):

- $\mu_G^* = 1$, the Gaussian fixed point mentioned above
- $\mu_{HT}^* = \exp\left(-\frac{1}{2}\varphi^2 + \frac{1}{d}\right)$, the high-temperature (or infinitely massive) fixed point.
- $\mu_{LT}^* = 0$, the low-temperature fixed point.

We are more interested in nontrivial fixed points. But notice that in general, there are two generic ways fixed points can appear as N or d is varied [58]:

- (a) splitting off from existing fixed points (bifurcation)
- (b) appearing in pairs in any region.

In the case (a), the signature is the approach to marginality of some operator representing a perturbation on an existing fixed point. The classic example is the Wilson-Fisher fixed point [26] which bifurcates from the Gaussian as d goes below four. The study of LPA (in the scalar case) yields no other kind of fixed point, this is why we consider first the vicinity of the Gaussian fixed point.

3.2.1 The Gaussian fixed point

A study of the properties of the Gaussian fixed point may easily be realized by linearization of the flow equations in the vicinity of the origin.

In this linearization, all the LPA equations mentioned in section 3.1 reduce to a unique equation. Considering the derivative $f(\varphi)$ of the potential $V(\varphi)$ and a small deviation $g(\varphi)$ to a fixed point solution $f^*(\varphi)$:

$$f(\varphi) = f^*(\varphi) + g(\varphi)$$

and choosing $f^*(\varphi) \equiv 0$ (the Gaussian fixed point) the equations linearized in g yields²² the unique equation [10, 30]:

$$\dot{g} = g'' + \left(1 - \frac{d}{2}\right) \varphi g' + \left(1 + \frac{d}{2}\right) g \quad (69)$$

²²Up to some change of normalization for eq. (54) and eq. (65) what is authorized in the cases of the sharp cutoff and of the power law cutoff due to (evident, see section 3.1) reparametrization invariance.

If one sets [10]:

$$g(\varphi, t) = e^{\lambda t} \alpha h(\beta \varphi)$$

with

$$\alpha = \frac{4}{d-2}, \quad \beta = \left(\frac{d-2}{4} \right)^{\frac{1}{2}}$$

then (69) reads:

$$h'' - 2\varphi h' + 2 \frac{2+d-2\lambda}{d-2} h = 0 \quad (70)$$

Polynomial form of the potential If we request the effective potential to be bounded by polynomials²³ then eq. (70) identifies [10] with the differential equation of Hermite's polynomials of degree $n = 2k - 1$ for the set of discrete values of λ satisfying:

$$\frac{2+d-2\lambda_k}{d-2} = 2k - 1 \quad k = 1, 2, 3, \dots \quad (71)$$

since $f(\varphi, t)$ is an odd function of φ .

The same kind of considerations may be done for general N , in which case the Hermite polynomials are replaced by the Laguerre polynomials [80, 49]. Since the discussion is similar for all N , we limit ourselves here to a discussion of the simple case $N = 1$.

Eigenvalues: From (71), the eigenvalues are defined by:

$$\lambda_k = d - k(d-2) \quad k = 1, 2, 3, \dots \quad (72)$$

then it follows that

- for $d = 4$: $\lambda_k = 4 - 2k \quad k = 1, 2, 3, \dots$, there are two non-negative eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = 0$
- for $d = 3$: $\lambda_k = 3 - k \quad k = 1, 2, 3, \dots$, there are three non-negative eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 1$ et $\lambda_3 = 0$

Eigenfunctions: If we denote by $\chi_k(\varphi)$ the eigenfunctions associated to the eigenvalue λ_k , it comes:

- $\chi_1^+ = \varphi$, $\chi_2^+ = \varphi^3 - \frac{3}{2}\varphi$, $\chi_3^+ = \varphi^5 - 5\varphi^3 + \frac{15}{4}\varphi$, \dots , whatever the spatial dimensionality d . The superscript “+” is just a reminder of the fact that the eigenfunctions are defined up to a global factor and thus the functions $\chi_k^-(\varphi) = -\chi_k^+(\varphi)$ are also eigenfunctions with the same eigenvalue λ_k . This seemingly harmless remark gains in importance after the following considerations.

To decide whether the marginal operator (associated with the eigenvalue equal to zero, i.e. λ_2 in four dimensions, or λ_3 in three dimensions) is relevant or irrelevant, one must go beyond the linear approximation. The analysis is presented in [10] for $d = 4$. If one considers a RG flow along χ_2^+ such that $g_2(\varphi, t) = c(t)\chi_2^+(\varphi)$, then one obtains, for small c : $c(t) = c(0)[1 - Ac(0)t]$ with $A > 0$. Hence the marginal parameter decreases as t grows. As is well known, in four

²³There are other possibilities, see below “Nonpolynomial...”.

dimensions the marginal parameter is irrelevant. However, if one considers the direction opposite to χ_2^+ (i.e. χ_2^-) then the evolution corresponds to changing $c \rightarrow -c$. This gives, for small values of c : $c(t) = c(0)[1 + A|c(0)|t]$ and the parameter becomes relevant. The parameter c is the renormalized ϕ^4 coupling constant u_R and it is known that in four dimensions the Gaussian fixed point is IR stable for $u_R > 0$ but IR unstable for $u_R < 0$ (if the corresponding action was positive for all φ , one could say that the ϕ_4^4 -field theory with a negative coupling is asymptotically free, see section 3.4.1).

For d slightly smaller than four, λ_2 is positive and the Gaussian fixed point becomes IR unstable in the direction χ_2^+ (and remains IR unstable along χ_2^-). The instability in the direction χ_2^+ is responsible for the appearance of the famous Wilson-Fisher fixed point which remains the only known nontrivial fixed point until d becomes smaller than 3 in which case it appears a second non trivial fixed point which bifurcates from the Gaussian fixed point. Any dimension d_k corresponding to $\lambda_k = 0$, is a border dimension below which a new fixed point appears [78] by splitting off from the Gaussian fixed point. Eq. (72) gives²⁴:

$$d_k = \frac{2k}{k-1}, \quad k = 2, 3, \dots, \infty$$

Nonpolynomial form of the potential As pointed out by Halpern and Huang [88] (see also [89]), there exist nonpolynomial eigenfunctions for the Gaussian fixed point. In four dimensions these nonpolynomial eigenpotentials have the asymptotic form $\exp(c\varphi^2)$ for large φ and provides the Gaussian fixed point with new relevant directions (with positive eigenvalues). From trivial, the scalar field theory in four dimensions would become physically non trivial due to asymptotic freedom and some effort have been made with a view to understand the physical implications of that finding [90].

Unfortunately, as stressed by Morris [69] (see also [65]), the finding of Halpern and Huang implies a continuum of eigenvalues and this is opposite to the usual formulation of the RG theory as it is applied to field theory where the eigenvalues take on quantized values. Indeed, usually there are a finite number (preferably small) of relevant (renormalized) parameters, and it is precisely that property which is essential in the renormalization of field theory: if the number of relevant parameters is finite the theory is said renormalizable otherwise it is not. Let us emphasize that, there is no mathematical error in the work of Halpern and Huang (see the reply of Halpern and Huang [91] which maintain their position except for the “line of fixed points”²⁵), the key point is that the theory of “renormalization” for nonpolynomial potentials does not exist. We come back to this discussion in section 3.4.1 where we illustrate, among other notions, the notion of self-similarity which is rightly so dear to Morris [69, 65].

3.2.2 Non trivial fixed points

As one may see from the equations presented in section 3.1, the fixed point equation (67) is a second order non linear differential equation. Hence a solution would be parametrized by two arbitrary constants. One of these two constants may easily be determined if $V^*(\varphi)$ is expected to be an even function of φ [$O(1)$ symmetry] then $V''(0) = 0$ may be imposed²⁶. It remains one free parameter: a one-parameter family of (nontrivial) fixed points are solutions to the differential equation. But there is not an infinity of physically acceptable fixed points.

As first²⁷ indicated by Hasenfratz and Hasenfratz (private communication of H. Leutwyler)

²⁴This is a result already known from the ϵ -expansion framework [87, 80].

²⁵About the infinity of nontrivial fixed points, see also [92].

²⁶Or if it is an odd function of φ then $V'''(0) = 0$ may be chosen as condition.

²⁷A discussion on the singular fixed point solutions in the case $N = \infty$ similar to that mentioned here for LPA, may be found in [4].

[10], studied in detail by Felder²⁸ [78] then by Filippov and Breus [93, 94, 49] and by Morris [95, 22], all but a finite number of the solutions in the family are singular at some φ_c . By requiring the physical fixed point to be defined for all φ then the acceptable fixed points (if they exist) may be all found by adjusting one parameter in $V(\varphi)$ (see fig. 1). For even fixed points, this parameter is generally chosen to be $V''(0)$ ($= \sigma^*$ in the following). For $N = 1$ the situation is as follows:

- $d \geq 4$, no fixed point is found except for $\sigma^* = 0$ (Gaussian fixed point).
- $3 \leq d < 4$, one fixed point (the Wilson-Fisher fixed point [26]) is found for a nonzero value of σ^* which depends on the equation considered. For $d = 3$ one has:

$$\begin{aligned}\sigma^* &= -0.461533 \dots [10, 96, 65] (-0.4615413727 \dots \text{in [97]}) \text{ with eq. (54),} \\ \sigma^* &= -0.228601293102 \dots \text{in [38] (or at } V^*(0) = 0.0762099 \dots [94, 49]) \text{ with eq. (59),} \\ \sigma^* &= -0.5346 \dots [22] \text{ with eq. (65).}\end{aligned}$$

- As indicated previously, a new nontrivial fixed point emanates from the origin (the Gaussian fixed point) below each dimensional threshold $d_k = 2k/(k-1)$, $k = 2, 3, \dots, \infty$ [78].

We show in fig. 1, in the case $d = 3$, how the physical fixed point is progressively discovered by adjusting $\sigma = V''(0)$ to σ^* after several tries (shooting method). The knowledge of the behavior of the solution for large φ (obtained from the flow equation studied) greatly facilitates the numerical determination of σ^* and of the fixed point solution $V^*(\varphi)$ (for example see [22, 95]). For an indication on the numerical methods one can use, references [85, 66, 30] are interesting.

3.2.3 Critical exponents in three dimensions

Once the fixed point has been located, the first idea that generally occurs to someone is to calculate the critical exponents. There is only one exponent to calculate (e.g. ν) since $\eta = 0$. The other exponents are deduced from ν by the scaling relations (e.g. $\gamma = 2\nu$)²⁹. The best way to calculate the exponents is to linearize the flow equation in the vicinity of the fixed point and to look at the eigenvalue problem. One obtains as in the case of the Gaussian fixed point a linear second order differential equation. For example with the Wilson (or Polchinski) version (59), setting $V(\varphi, t) = V^* + e^{\lambda t} v(\varphi)$, one obtains the eigenvalue equation:

$$v'' + \left[\left(1 - \frac{d}{2} \right) \varphi - 2V'^* \right] v' + (d - \lambda) v = 0 \quad (73)$$

As Morris explains in the case of eq. (65) [22], “(… again one expects solutions to (73) labelled by two parameters, however by linearity one can choose $v(0) = 1$ (arbitrary normalization of the eigenvectors) and by symmetry $v'(0) = 0$ (or by asymmetry and linearity: $v(0) = 0$ and $v'(0) = 1$). Thus the solutions are unique, given λ . Now for large φ , $v(\varphi)$ is generically a superposition³⁰ of $v_1 \sim \varphi^{2(d-\lambda)/(d+2)}$ and of $v_2 \sim \exp(\frac{d+2}{4}\varphi^2)$. Requiring zero coefficient for the latter restricts the allowed values of λ to a discret set”.

The reason for which the exponential must be eliminated is the same as previously mentioned in section 24 to discard nonpolynomial forms of the potential.

For $d = 3$, the Wilson-Fisher fixed point possesses just one positive eigenvalue λ_1 corresponding to the correlation length exponent ($\nu = 1/\lambda_1$) and infinitely many negative eigenvalues. In

²⁸Who demonstrates that, for $d = 3$, there is only one nontrivial fixed point.

²⁹The right relation is $\gamma = \nu(2 - \eta)$.

³⁰To obtain this behavior, use the large φ behavior of the fixed point potential $V^*(\varphi) \simeq \frac{1}{2}\varphi^2$ coming from eq.(59), see [30].

the symmetric case, the less negative λ_2 corresponds to the first correction-to-scaling exponent $\omega = -\lambda_2$ while λ_3 provides us with the second $\omega_2 = -\lambda_3$ and so on. In the asymmetric case, which is generally not considered (see however [38]), one may also associate the first negative eigenvalue λ_1^{as} to the first non-symmetric correction-to-scaling exponent $\omega_5 = -\lambda_1^{\text{as}}$ (the subscript “5” refers to the ϕ^5 interaction term in the action responsible for this kind of correction, see [58]).

Another way of numerically determining the (leading) eigenvalues is the shooting method. One chooses an initial (simple) action and tries to approach the fixed point (one parameter of the initial action must be finely adjusted). When the flow approaches very close to the fixed point, its rate of approach is controlled by ω (i.e. $-\lambda_2$). The adjustment cannot be perfect and the flow ends up going away from the fixed point along the relevant direction with a rate controlled by $\lambda_1 = 1/\nu$.

The first determination of ν and ω has been made by Parola and Reatto [98] from eq (54) [in the course of constructing a unified theory of fluids, for a review see [99]]. They found³¹ (for $d = 3$ and $N = 1$):

$$\nu = 0.689, \quad \omega = 0.581$$

estimates which are compatible with the well known results of Hasenfratz and Hasenfratz [10] obtained from eq. (55) using the shooting method:

$$\nu = 0.687(1), \quad \omega = 0.595(1)$$

The error is only indicative of the numerical inaccuracy of solving the differential equation [10]. Since then, the above results have been obtained several times. The first estimate of ω_2 has been given in [96] from the same equation and using again the shooting method:

$$\omega_2 \simeq 2.8$$

No error was given due to the difficulty of approaching the fixed point along the second irrelevant direction (two parameters of the initial action must be adjusted [96, 73]). More accurate estimations of this exponent may be found in [34, 101].

We also mention an estimate of ω_5 from the same eq. (54) [102]:

$$\omega_5 \simeq 1.69$$

In themselves the estimates of critical exponents in the local potential approximation do not present a great interest except as first order estimates in a systematic expansion see section 4.1. It is however interesting to notice that the LPA estimates are not unique but depend on the equation studied. Hence with the Legendre version (65) it comes [22]:

$$\nu = 0.6604, \quad \omega = 0.6285$$

which is closer to the “best” values. And the closest to the “best” are obtained from the Wilson (or Polchinski) version [38, 30, 34]:

$$\nu = 0.6496, \quad \omega = 0.6557$$

LPA estimates of exponents at various values of N and d have been published (see for example [34]). It is also worth mentioning that LPA gives the exact exponents up to $O(\varepsilon)$ [81, 103].

³¹In order to appreciate the quality of the estimates the reader may refer to the so-called best values given, for example, in [100].

3.2.4 Other dimensions $2 < d < 3$

If for $2 < d < 3$ multicritical fixed points appear at the dimensional thresholds $d_k = 2k/(k-1)$, $k = 2, 3, \dots, \infty$, the Wilson-Fisher (critical) fixed point (once unstable) still exists and the same analysis as above for $d = 3$ could have been done to estimate the critical exponents. However this kind of calculations have not been performed in the LPA despite some considerations relative to $2 < d < 3$ [93, 94, 49].

The special case of $d = 2$ does not yield the infinite set of nonperturbative and multicritical fixed points expected following the conformal field theories but only periodic solutions corresponding to critical sine-Gordon models [104]. This is due to having discarded the nonlocal contributions which are not small for $d = 2$ ($\eta = 1/4$ is not small) [93, 94, 49].

3.3 Truncations

One may try to find solutions to the ERGE within LPA by expanding the potential in powers of the constant field variable φ . Although it is obviously not a convenient way of studying a non linear partial differential equation, this programme is interesting because the study of the ERGE necessarily requires some sort of truncation (or approximation). It is an opportunity to study the simplest truncation scheme in the simple configuration of the LPA in as much as there are complex systems (e.g. gauge theories) for which the truncation in the powers of the field seems inevitable [105].

Margaritis, Ódor and Patkós [106] for arbitrary N and Haagensen et al [107] for $2 < d < 4$ have tried this kind of truncation on eq. (56). The idea is as follows. One expands $V(\varphi, t)$ in powers of φ :

$$V(\varphi, t) = \sum_{m=1}^{\infty} c_m(t) \varphi^m$$

and one reports within the flow equation to obtain, e.g. for the fixed point equation, an infinite system of equations for the coefficients c_i . That system may be truncated at order M (i.e. $c_i \equiv 0$ for $i > M$) to get a finite easily solvable set of equations (from which solutions may be obtained analytically [107]). By considering larger and larger values of M , one may expect to observe some convergence.

As one could think, the method does not generate very good results: an apparent new (but spurious) fixed point is found [106]. Moreover, the estimation of critical exponents (associated to the Wilson-Fisher fixed point which, nevertheless, is identified) shows a poor oscillatory convergence [106]. Indeed Morris [95] has shown that this poor convergence is due to the proximity of a singularity in the complex plane of φ .

Surprisingly, the truncation procedure considered just above works very well in the case of $O(N)$ -symmetric systems. It appears that if one considers the variable $\rho = \frac{1}{2} \sum_{\alpha} \varphi_{\alpha}^2$ and expands $V(\rho, t)$ about the location ρ_0 of the minimum of the potential [83]

$$\left. \frac{dV(\rho, t)}{d\rho} \right|_{\rho=\rho_0} = 0$$

then one obtains an impressive apparent convergence [108] toward the correct LPA value of the exponents (the method works also within the derivative expansion [83, 108]). The convergence of the method has been studied in [103] [with eq. (54)] and further in [55] [with the Legendre effective action (65)] where it is shown that the truncation scheme associated to the expansion around the minimum of the potential (called co-moving scheme in [103, 55]) actually does not converge but finally, at a certain large order, leads also to an oscillatory behavior.

We have seen that the estimates of the critical exponents in the LPA depend on the ERGE chosen (see section 3.2.3). It is thought [105] however that for a given ERGE, the estimates

should not depend on the form of the cutoff function (the dependence would occur only at next-to-leading order in the derivative expansion [38]). Nevertheless, the truncation in powers of the field may violate this “scheme independence” and affect the convergence of the truncation. It is the issue studied in [105] with three smooth cutoff functions: hyperbolic tangent, exponential and power-law. An improvement of the convergence is proposed by adjustment of the smoothness of the cutoff function.

3.4 A textbook example

Despite its relative defect in precise quantitative predictions, the local potential approximation of the ERGE is more than simply the zeroth order of a systematic expansion in powers of derivatives (see section 4.1). In the first place it is a pedagogical example of the way infinitely many degrees of freedom are accounted for in RG theory. Almost all the characteristics of the RG theory are involved in the LPA. The only lacking features are related to phenomena highly correlated to the non local parts neglected in the approximation. For example in two dimensions, where $\eta = \frac{1}{4}$ is not particularly small, LPA is unable to display the expected fixed point structure [93, 94, 49, 104]. But, when η is small (especially for $d = 4$ and $d = 3$), one expects the approximation to be qualitatively correct on all aspects of the RG theory. One may thus trust the results presented in section 3.2 on the search for nontrivial fixed points in four and three dimensions.

It is a matter of fact that much of studies on RG theory are limited to the vicinity of a fixed point. This is easily understood due to the universality of many quantities (exponents, amplitude ratios, scaled equation of state, etc...) that occurs there. However this limitation greatly curtails the possibilities that RG theory offers. The fixed points and their local properties (relevant and irrelevant directions) are not the only interesting aspects of the RG theory. Let us simply quote, as an example, the crossover phenomenon which reflects the competition between two fixed points. But what is worse than a simple limitation in the use of the theory, is the resulting misinterpretation of the theory. This is particularly true with respect to the definition of the continuum limit of field theory and its relation to the study of critical phenomena. Let us specify a bit this point (more details may be found in [96, 73])

It is often expressed that the continuum limit of field theory is defined “at” a fixed point and that it is sufficient to look at its relevant directions to get the renormalized parameters, i.e. a simple linear study of the RG theory in the vicinity of the fixed point would be sufficient to define the continuum limit (see for example in [65, 67, 59]). This is not wrong but incomplete and, actually void of practical meaning. Indeed it is not enough emphasized (or understood) in the literature that, for example, although defined “at” the Gaussian fixed point, the field theoretic approach to critical phenomena [64, 5] is finally applied “at” the Wilson-Fisher fixed point which, in three dimensions, lies far away from the Gaussian fixed point. Then if the renormalized coupling of the ϕ_3^4 -field theory was only defined by the linear properties of the RG theory in the vicinity of the Gaussian fixed point, one would certainly not be able to discover the nontrivial Wilson-Fisher fixed point by perturbative means.

Actually, the relevant directions of a fixed point provide us with exclusively the number and the nature of the renormalized parameters involved in the continuum limit. But the most important step of the continuum limit is the determination of the actual scale dependence (say, the beta functions) of those renormalized parameters. It is at this step that the recourse to RG theory actually makes sense: the attractive RG flow “*that emanates out from the fixed point along the relevant direction* [1]” results from the effect of infinitely many degrees of freedom and the problem of determining this flow is nonperturbative in essence. In the Wilson space \mathcal{S} of the couplings $\{u_n\}$, the flow in the continuum runs along a submanifold (of dimension one if the fixed point has only one relevant direction) which is entirely plunged in \mathcal{S} . The writing of

the corresponding action (the “perfect action” [60]) would require the specification of infinitely many conditions on the action. Because it is an hopeless task to find the “perfect action”³², one gives up all idea of a determination of the initial absolute scale dependence³³ and one has recourse to the process of fine tuning some parameter of the (bare) action in the vicinity of the (Gaussian) fixed point (or in the vicinity of a critical or tricritical etc.. surface in \mathcal{S}). This allows us to approach the ideal (or “perfect”) flow one is looking for and which runs along a trajectory tangent to a relevant direction. This flow is determined under a differential form (beta function) since the initial condition is not specified (the “perfect action” is not known) but it results from the effect of all degrees of freedom involved in the theory (by virtue of the relevant direction).

The simplicity of the ERGE in the LPA allows us to visualize the evolution of RG trajectories in the space \mathcal{S} of the couplings $\{u_n\}$ and thus to illustrate and qualitatively discuss many aspects directly related to the nonperturbative character of the RG theory. In particular the approach to the continuum limit and the various domains of attraction of a fixed point.[96, 73, 101]

3.4.1 Renormalization group trajectories

Following [96, 73], one considers an initial simple potential [rather its derivative with respect to φ , see (53)], say³⁴:

$$f(\varphi, 0) = u_2(0)\varphi + u_4(0)\varphi^3 + u_6(0)\varphi^5 \quad (74)$$

corresponding to a point of coordinates $(u_2(0), u_4(0), u_6(0), 0, 0, \dots)$ in \mathcal{S} , and after having numerically determined the associated solution $f(\varphi, t)$ of Eq. (55) at a varying “time” t , one concretely represents the RG trajectories (entirely plunged in \mathcal{S}) by numerically evaluating the derivatives of f at the origin ($\varphi = 0$) corresponding to: $u_2(t)$, $u_4(t)$, $u_6(t)$ etc...

We then are able to visualize the actual RG trajectories by means of projections onto the planes $\{u_2, u_4\}$ or $\{u_4, u_6\}$ (for example) of the space \mathcal{S} .

For the sake of shortness we limit ourselves to a rapid presentation of figures. The reader is invited to read the original papers.

To approach the Wilson-Fisher fixed point in three dimensions (or $3 \leq d < 4$) starting with (74), it is necessary to adjust the initial value of one coupling (e.g. $u_2(0)$) to a critical value $u_2^c [u_4(0), u_6(0)]$ (see fig. 2). This is because the Wilson-Fisher fixed point has one relevant direction (which must be thwarted). In such a case the fixed point controls the large distance properties of a critical system and the potential corresponding to (74) with $u_2(0) = u_2^c [u_4(0), u_6(0)]$ represents some physical system at criticality. The initial $f(\varphi, 0)$ lies in the critical submanifold \mathcal{S}_c which is locally orthogonal to the relevant eigendirection of the fixed point. As already mentioned in section 2.10.1, the renormalized trajectory (RT) T_0 that emerges from the Wilson-Fisher fixed point tangentially to the relevant direction allows to define a massive continuum limit³⁵ (see fig. 3).

In three dimensions, the Gaussian fixed point has two relevant directions. There a field theory involving two (renormalized) parameters (a mass-like and a ϕ^4 -like coupling) may be constructed. A purely massless field theory may also be constructed by choosing the relevant direction lying in the critical surface (corresponding to the eigenfunction χ_2^+ of section 3.2.1). One obtains a one-parameter theory which interpolates between the Gaussian and the Wilson-Fisher fixed points (the renormalized submanifold T_1 of fig. 4). As already mentioned in section 2.10.1, this scale dependent massless theory contradicts the Morris view of massless theories as

³²However a truncated expression of the perfect action could be useful in actual studies of field theory defined on a lattice.

³³Called the functional form of the scale dependence in [73].

³⁴The normalization of the couplings u_n are here modified ($u_n \rightarrow n u_n$) compared to (52).

³⁵The nontrivial continuum limit proposed by Wilson in [61] for the so-called ϕ_3^6 -field-theory which involves no coupling constant renormalization but a mass (and a wave function renormalization [62] which cannot be evidenced in the present approximation).

fixed point theories (thus scale invariant) [48, 65, 66, 67, 59, 68]. In fact a sensible massless (renormalized) scalar theory involves only a ϕ^4 -like coupling “constant”, say g , as parameter. But g is not at all constant it is scale dependent and this is usually expressed via the beta function

$$\beta(g) = \mu \frac{dg}{d\mu}$$

in which μ is some momentum scale of reference and the function $\beta(g)$ is defined relatively to the flow running along the attractive submanifold T_1 (the slowest flow [109, 73] in the critical submanifold \mathcal{S}_c). [73]

The fact that we get a unique scale dependent parameter (renormalizability) illustrates well the notion of “*self-similarity*” which means that the cooperative effect of the infinite number of degrees of freedom may be reproduced by means of a finite set of (effective or renormalized) scale dependent parameters (the system “looks like” the same when probed at different scales). In the case of the massless scalar theory, the number of renormalized parameters is equal to one: a coupling “constant”³⁶.

Fig. 4 shows other RG trajectories (different to those attracted to T_1), see the caption and [96, 73] for more details.

Less known are the RG trajectories drawn on fig. 5 in the sector $u_4 < 0$ [111, 101]. They correspond to:

1. The attractive submanifold T''_1 which emerges from the Gaussian fixed point and is tangent to the eigenfunction χ_2^- of section 3.2.1 (symmetric to T_1 which emerges from the Gaussian fixed point tangentially to χ_2^+). Along this submanifold the flows run toward larger and larger negative values of u_4 . It is interesting to know that this submanifold still exists in four dimensions [101], it still corresponds to χ_2^- and the associated eigenvalue is zero but the nonlinear analysis shows that the associated operator (ϕ^4 -like) is marginally relevant. Hence the ϕ_4^4 field theory with $u_4 < 0$ would be asymptotically free if the action had not the wrong sign for large values of φ (the negative sign of the ϕ_4^4 term which is the only dominant term of the “perfect action” in the vicinity of the Gaussian fixed point). It is worthwhile mentioning that this relevant direction of the Gaussian fixed point in four dimensions is different from those discovered by Halpern and Huang [88] precisely because of the self-similarity displayed in the present case and which is a consequence of the discretization of the eigenvalues presented in section 3.2.1 in the case of polynomial interactions [69, 65].

The attractive submanifold T''_1 is endless in the infrared direction (no nontrivial fixed point lies in the sector $u_4 < 0$, see section 3.2.2). It is customary to say in that case that T''_1 is associated to a first order transition. This is because without a fixed point the correlation length ξ remains finite. However, although finite, the existence and the length of T''_1 suggest that ξ may be very large. Following Zumbach in a study of the Stiefel nonlinear sigma model³⁷ within LPA [84, 112], one may refer in the circumstances to “*almost second order phase transition*” or equivalently to “*weakly first order phase transition*” (see [113, 101] for supplementary details).

2. The “tricritical” attractive submanifold which approaches the Gaussian fixed point asymptotically tangentially to χ_3^+ (the submanifold tangent to χ_3^- also exists but is not drawn)

³⁶It is, perhaps, useful to mention here that the perturbative expression of the β -function (up to one loop order in [109] and up to two loops in [110]) has been re-obtained from the ERGE once expanded with respect to the number of loops (perturbative study).

³⁷The Stiefel nonlinear sigma model is a generalization of the Heisenberg model with the field a real $N \times P$ matrix and the action is $O(N) \times O(P)$ invariant [84, 112, 86].

necessitates the adjustment of two parameters in the initial action (because in three dimensions the Gaussian fixed point is twice unstable). See the figure caption for more comments and [101].

The same configuration displayed by fig. 5 has been obtained also by Tetradis and Litim [114] while studying analytical solutions of an ERGE in the LPA for the $O(N)$ -symmetric scalar theory in the large N limit. But they were not able to determine “*the region in parameter space which results in first order transitions*” [114]. Fig. 5 shows this region for the scalar theory in three dimensions.

Fig. 6 shows RG trajectories in the critical surface for $d = 4$ and $u_4 > 0$. The verification of this configuration was the main aim of the paper by Hasenfratz and Hasenfratz [10]. As emphasized by Polchinski [12], in the infrared regime all the trajectories approach a submanifold of dimension one (on which runs the slowest RG flow [109, 73]) before plunging into the Gaussian fixed point. This pseudo renormalized trajectory allows to make sense to the notion of effective field theory (a theory which only makes sense below some finite momentum scale). Because there is no fixed point other than Gaussian, it is not possible to adjust the initial action in such a way that the deviations between the actual RG trajectories and the ideal one dimensional submanifold be reduced to zero at arbitrary large momentum scales. Those irreducible deviations are responsible for the presence of the so-called UV renormalon singularities [70, 71] (for a review see [72]) in the perturbative construction of the ϕ_4^4 field theory. Indeed as explained in [73], the perturbative approach selects the ideal flow (the slowest) and sets a priori equal to zero all possible deviation (this is possible order by order in perturbation) without caring about the genuine physical momentum scale dependence (that requires the explicit reference to a relevant parameter relative to another fixed point to be well accounted for). Especially, it is argued in [73] that the UV renormalon singularities would be absent in the β -function calculated within the minimal subtraction scheme of perturbation theory simply because in that case the scale of reference μ is completely artificial (has no relation with a genuine momentum scale except the dimension). See also [59], for a discussion of the UV renormalon singularities with the help of an ERGE.

A study of the attraction of RG flows to infrared stable submanifolds are also presented in [115].

3.4.2 Absence of infrared divergences

It is known that the perturbation expansion of the massless scalar-field theory with $d < 4$ involves infrared divergences. However, it has been shown [116] that theory “develops by itself” an infinite number of non-perturbative terms that are adapted to make it well defined beyond perturbation. One may show that these purely nonperturbative terms are related to the critical parameter u_2^ξ mentioned above [117]. In the nonperturbative framework of the ERGE there is never infrared divergences and it is thus particularly well adapted to treat problems which are known to develop infrared singularities in the perturbative approach (e.g. Goldstone modes in the broken symmetry phase of $O(N)$ scalar theory and in general super renormalizable massless theories).

3.4.3 Other illustrations

Limit $N = \infty$, large N Exact results are accessible with LPA. The limit $N = \infty$ corresponds to the model of Berlin and Kac [118, 31] which can be exactly solved. Wegner and Houghton [4] have shown that, in this limit their equation (29) is identical to the limit $N = \infty$ of the LPA. A first order non-linear differential equation is then obtained and studied in [4] (see also [34]). Comparison with the exact model is completely satisfactory. More recently Breus and Filippov

[94] and then Comellas and Travesset [34] have studied the large N limit of the LPA for the Wilson (or Polchinski) ERGE (and for the Wegner-Houghton in [34]) and find also agreement with the exact results. However D'Attanasio and Morris [119] have pointed out that the Large N limit of the LPA for the Polchinski (or Wilson) ERGE is not exact although it may give correct results. See also [114] in which analytical solutions in the Large N limit of the LPA for the Legendre ERGE are obtained and discussed..

The RG flow is gradient flow LPA allows to easily illustrate the property that the RG flow is gradient flow. This property is important because if the RG flow is gradient flow then only fixed points are allowed (limit cycles or more complicated behavior are excluded) and the eigenvalues of the linearized RG in the vicinity of a fixed point are real [120, 121]. The conditions for gradient flow are that the beta functions $\beta_i(\{u\})$ [The infinite set of differential renormalization group equations: $\dot{u}_i = \beta_i(\{u\})$] may be written in terms of a non-singular metric $g_{ij}(\{u\})$ and a scalar function $c(\{u\})$ [120]:

$$\beta_i(\{u\}) = - \sum_j g_{ij}(\{u\}) \frac{\partial c(\{u\})}{\partial u_j}$$

If $g_{ij}(\{u\})$ is a positive-definite metric, the function $c(\{u\})$ is monotonically decreasing along the RG flows:

$$\dot{c} = \sum_i \beta_i \frac{\partial c}{\partial u_i} = - \sum_{i,j} g_{ij} \frac{\partial c}{\partial u_i} \frac{\partial c}{\partial u_j} \leq 0$$

Following Zumbach [85, 86] one may easily verify that the local potential approximation of the Wilson (or Polchinski) ERGE, written in terms of $\mu(\varphi, t) = \exp(-V(\varphi, t))$ [eq. (68)] may be expressed as a gradient flow:

$$g(\varphi) \dot{\mu} = - \frac{\delta \mathcal{F}[\mu]}{\delta \mu}$$

where

$$\begin{aligned} g(\varphi) &= \exp \left[-\frac{1}{4} (d-2) \varphi^2 \right] \\ \mathcal{F}[\mu] &= \int d\varphi g(\varphi) \left\{ \frac{1}{2} \mu'' + \frac{d}{4} \mu^2 (1 - 2 \ln \mu) \right\} \end{aligned}$$

It has then been shown [122] (see also [67]) that a c-function may be defined as (A is a normalization factor)

$$c = \frac{1}{A} \ln \left(\frac{4\mathcal{F}}{d} \right)$$

which satisfies in any d the two first properties of Zamolodchikov's c-function [123] and has a counting property which generalizes the third property.

Let us mention also that a c-function has been obtained in the framework of the truncation in powers of the field of section 3.3 by Haagensen et al [107, 97] and that Myers and Periwal [124] have proposed a new form of the ERGE which is similar to a gradient flow.

Triviality bounds It has been argued [125] that an upper bound on the Higgs mass may be estimated from the only trivial character of the scalar field theory in four dimensions. The idea may be roughly illustrated by the following relation:

$$\frac{\Lambda_{max}}{m} = \int_g^\infty \frac{dx}{\beta(x)} \tag{75}$$

in which g is the (usual renormalized) ϕ^4 -coupling of the massive theory, m is the mass parameter and Λ_{max} the maximum value that the momentum-scale of reference of the scalar theory can take on (it is associated with an infinite value of the coupling g since no nontrivial fixed point exists —this is the consequence of triviality). Hence there is a finite relation between m and Λ_{max} .

In order to determine the (triviality) upper bound for the Higgs mass m_H (which then replaces m), one usually refers to the ratio $R = \frac{m_H}{m_W}$ in which m_W is the mass of the vector boson W of the standard model of the electro-weak interaction [126]. The ratio R expresses as a function of both g (the scalar coupling) and G (the gauge coupling):

$$R = \frac{m_H}{m_W} = f(g, G) \quad (76)$$

It appears that, at fixed G , R is an increasing function of g [127]. For example, at tree level it comes:

$$R^2 = 8 \frac{g}{G^2} \quad (77)$$

Knowing G and m_W from experiments (usually one considers $G^2 \sim 0.4$ and $m_W \sim 80$ GeV, see [128] for example), the calculation of $f(g, G)$ would thus allow us to estimate the triviality bound on m_H from the following inequality:

$$R \leq f(\infty, G) \quad (78)$$

However because g becomes infinite at Λ_{max} , the question of determining a bound on, say, the Higgs mass is highly a nonperturbative issue. Hasenfratz and Nager [127] using the LPA of the Wegner-Houghton ERGE have shown how one can proceed to estimate that bound nonperturbatively (see also [129]).

Principle of naturalness. Some authors have invoked a “concept of naturalness” to argue that fundamental scalar fields may not exist. Initiated by Wilson [130], this concept would require [131] “*the observable properties of a theory to be stable against minute variations of the fundamental parameters*”.

A different concept of naturalness, brought up with a view to eliminate non asymptotically free field theories, would be that [132] “*the effective interactions (· · ·) at a low energy scale μ_1 should follow from the properties (· · ·) at a much higher energy scale μ_2 without the requirement that various different parameters at the energy scale μ_2 match with an accuracy of the order of $\frac{\mu_1}{\mu_2}$. That would be unnatural. On the other hand, if at the energy scale μ_2 some parameters would be very small, say $\alpha(\mu_2) = O(\frac{\mu_1}{\mu_2})$, then this may still be natural · · ·*

Anyway, the two expressions have the same consequence for the scalar field theory which appears non natural. Let us illustrate this point with the help of the LPA.

With a view to make the scalar field theory in four dimensions (i.e. ϕ_4^4) non trivial, a non Gaussian fixed point is required. Assuming that a nontrivial fixed point exists in four dimensions, the procedure of construction of the resulting continuum limit “at” this fixed point would be similar to that described in fig. 3 for the purely massive field theory in three dimensions. We have seen that in order to approach the RT T_0 , at least one parameter ($u_2(0)$) of the (bare) initial action as to be finely tuned³⁸ to a nonzero value (u_2^c). Moreover an infinitely small deviation from the actual u_2^c results in a drastic change in the scale dependence of the effective renormalized parameter. This adjustment is unnatural: how can we justify the origin of numbers

³⁸It is clear that the large number of digits in the determination of u_2^c (e.g. $-0.299586913 \dots$ as indicated in the caption of fig. 2 or similarly in the determination of $\sigma^* = -0.228601293102 \dots$ in [38]) is not indicative of the accuracy in the determination of a physical parameter in the LPA but is required to get as close as possible to the critical surface (consistently within LPA).

like u_2^c ? This unnatural adjustment will be required each time one defines a continuum limit “at” a non trivial fixed point.

On the contrary, an asymptotically free field theory would appear natural because the adjustment of the initial action is made with respect to the Gaussian fixed point (the nonuniversal parameters like $u_2(0)$ are adjusted to zero!).

Dynamical generation of masses. “*… one essential element of this systematic theory (a satisfying synthesis of the theories of weak, electromagnetic and strong interactions) has remained obscure: we must take the mass of the leptons and quarks as input parameters, without any real idea of where they come from. … the search for a truly natural theory of the quark masses must continue.*” [133]

In order to give masses to the intermediate vector bosons while preserving symmetry, one has imagined the occurrence of a spontaneous symmetry breaking mechanism. In the standard model of electro-weak interaction the symmetry-breaking mechanism is associated with the introduction of a scalar field (ϕ_4^4) in the model and the vector bosons acquire masses via the Higgs mechanism [134]. The conceptual difficulty with this model is that one has introduced a peculiar kind of interaction (the scalar field is self-interacting) which is not asymptotically free. The Higgs mechanism finally appears to be convenient in the range of energy scales over which the standard model seems to work but conceptually unnatural and not generalizable to higher energies (ϕ_4^4 does not make sense above some energy). The other possibility is that the symmetry-breaking mechanism occurs *dynamically*, that is to say without need for introducing scalars but simply because the gauge fields are interacting fields [135, 131].

In the process of generating masses *dynamically*, the main interesting feature of a non-abelian-gauge-invariant field theory is not actually asymptotic freedom but its infrared diseases associated with the presence of IR “renormalons” in perturbative series (or equivalently the existence of a ghost in the infrared regime [70], for a review see [72]). It is very likely that those IR “renormalons” convey the lack of any infrared stable fixed point in the critical (i.e. massless) surface. This means that a scale dependent coupling constant G of a purely *massless* (gauge invariant) theory is not defined below some momentum scale Λ_{min} . It is thus expected that the appearance of massive particles below Λ_{min} can proceed from the existence of a symmetric (massless) theory at momentum-scales larger than Λ_{min} [70, 131, 135]. Let us illustrate this point with the scalar theory and LPA.

The usual scalar field theory in three dimensions is well defined but does not present a great interest with respect to a mass generation because:

- either the theory is massive and the mass is a given parameter.
- or the theory is purely massless but defined as such at any (momentum) scale in the range $]0, \infty[$ (interpolation between two fixed points).

On the contrary, as shown in fig. 5, the massless theory becomes asymptotically free in the sector $u_4 < 0$ (the fact that the action has the wrong sign is not important for our illustrative purposes). But more importantly, there is no infrared stable fixed point to allow the scale dependence to be defined at any scale along this RT. Consequently, as close to the Gaussian fixed point as any trajectory would be initialized, the resulting trajectory will, after a finite “time”, end up going away from the critical surface, i.e. within the massive sector. Finally masses would have been generated from the momentum scale dependence of a purely massless theory.

This mechanism illustrated here for masses may also occur for any symmetry breaking parameter.

4 Further developments

4.1 Next-to-leading order in the derivative expansion

The LPA considered in the preceding sections is the zeroth order of a derivative expansion first proposed as a systematic expansion by Golner [11]. To be fair, the first use of the derivative expansion (or rather the gradient expansion [31]) was by Myerson [52] in conjunction with an expansion in powers of the field. A line of fixed points with $\eta \cong 0.045$ was obtained.

The genuine derivative expansion is a functional power series expansion of the Wilson effective action in powers of momenta so that all powers of the field are included at each level of the approximation. The idea is to expand the action $S[\phi; t]$ in powers of momenta [11]:

$$S[\phi; t] = S^{(0)}[\phi; t] + S^{(2)}[\phi; t] + \sum_{i=1}^3 S_i^{(4)}[\phi; t] + \dots$$

where

$$\begin{aligned} S_i^{(2k)}[\phi; t] &= \sum_n a_{in}^{(2k)}(t) H_{in}^{(2k)}[\phi], \\ H_{in}^{(2k)}[\phi] &= \int_{q_1} \dots \int_{q_n} h_i^{(2k)}(\mathbf{q}_1, \dots, \mathbf{q}_n) \hat{\delta}(\mathbf{q}_1 + \dots + \mathbf{q}_n) \phi_{q_1} \dots \phi_{q_n}, \\ H_0^{(0)}[\phi] &= \delta(0) \end{aligned}$$

and the $h_i^{(2k)}(\mathbf{q}_1, \dots, \mathbf{q}_n)$ are homogenous monomials in $\{\mathbf{q}_j\}$ of degree $2k$, with the index i present when needed to keep track of degeneracies. Because of the momentum conserving δ function we have, for spatially isotropic systems, only one linearly independent functional of degree 2: $h^{(2)} = \mathbf{q}_1 \cdot \mathbf{q}_2$, and three of degree 4: $h_1^{(4)} = (\mathbf{q}_1 \cdot \mathbf{q}_2)^2$, $h_2^{(4)} = (\mathbf{q}_1 \cdot \mathbf{q}_2)(\mathbf{q}_1 \cdot \mathbf{q}_3)$, $h_3^{(4)} = (\mathbf{q}_1 \cdot \mathbf{q}_2)(\mathbf{q}_3 \cdot \mathbf{q}_4)$, since all powers of q_j^2 can be re-expressed in terms of powers of $\mathbf{q}_i \cdot \mathbf{q}_j$, $i \neq j$. This is better seen in the position space where the expansion up to third order may be written as follows:

$$\begin{aligned} S[\phi] = \int d^d x \left\{ V(\phi, t) + \frac{1}{2} Z(\phi, t) (\partial_\mu \phi)^2 + H_1(\phi, t) (\partial_\mu \phi)^4 \right. \\ \left. + H_2(\phi, t) (\square \phi)^2 + H_3(\phi, t) (\partial_\mu \phi)^2 (\square \phi) + \dots \right\} \end{aligned}$$

on which expression the integrations by parts allow to easily identify the linearly dependent functionals (as previously the symbol \square stands for $\partial_\mu \partial^\mu$).

It remains to substitute this expansion into the ERGE chosen among those described in section 2. To our knowledge the derivative expansion has only been explicitly written down up to the first order. This produces two coupled nonlinear partial differential equations for V and Z .

For the sake of clarity we limit ourselves to a detailed discussion of the equations for the Polchinski version of the ERGE [of section 2.5.2, eq. (34)], the other forms of the ERGE³⁹ are considered in section 4.3.

The Polchinski version of the ERGE at first order of the derivative expansion yields the following coupled equations [30] (see also [38]):

³⁹The discussion of the Wilson formulation given by eq. (30) is very similar to that of Polchinski and will not be considered explicitly here (see [11] for details).

$$\begin{aligned}\dot{f} &= 2K'(0)ff' - \left(\int K'\right)f'' - \left(\int p^2 K'\right)Z' + \frac{d+2-\eta}{2}f - \frac{d-2+\eta}{2}\varphi f', \\ \dot{Z} &= 2K'(0)fZ' + 4K'(0)f'Z + 2K''(0)f'^2 - \left(\int K'\right)Z'' - 4K^{-1}(0)K'(0)f' \\ &\quad - \eta Z - \frac{d-2+\eta}{2}\varphi Z',\end{aligned}$$

with $\varphi \equiv \phi_0$ and $f(\varphi) \equiv V'(\varphi)$. As previously defined in section 2.5.2, K' stands for $dK(p^2)/dp^2$ and $\int K' \equiv \int_p K'(p^2)$ etc....

It is convenient to perform the following rescalings

$$\varphi \rightarrow \sqrt{-\int K'} \varphi, \quad f \rightarrow \frac{\sqrt{-\int K'}}{-K'(0)} f, \quad Z \rightarrow K^{-1}(0)Z;$$

so that,

$$\dot{f} = -2ff' + f'' + AZ' + \frac{d+2-\eta}{2}f - \frac{d-2+\eta}{2}\varphi f', \quad (79)$$

$$\dot{Z} = -2fZ' - 4f'Z + 2Bf'^2 + Z'' + 4f' - \eta Z - \frac{d-2+\eta}{2}\varphi Z', \quad (80)$$

where

$$A \equiv \frac{(-K'(0))(-\int p^2 K')}{(-\int K')}, \quad B \equiv \frac{K''(0)}{(-K'(0))^2}.$$

Compared to [30], we have set $K(0) = 1$. These conventions coincide also with those of [38].

Eqs. (79, 80) show that all cutoff (scheme) dependence at order p^2 is reduced to a two-parameter family (A, B) while at zeroth order [eq. (79) with $Z' = 0$] there is no explicit dependence. In general the scheme (cutoff) dependence can be absorbed into $2k$ parameters at k -th order in the derivative expansion [38]. The set of eqs. (79, 80) has been considered first by Ball et al in [38] with a view to study the scheme dependence of the estimates of critical exponents and reexamined by Comellas [30] who emphasizes (following a remark by Morris [68]) the importance of the breaking of the reparametrization invariance [48, 68] in estimating the critical exponents [11] (see also [29, 52] and section 2.2.4). Let us report on this important aspect of the eqs. (79, 80).

4.1.1 Fixed points, η and the breaking of the reparametrization invariance

The distribution of the fixed points of eqs. (79, 80) solution of $\dot{f}^* = \dot{Z}^* = 0$ is identical to that of the leading order (LPA discussed in section 3.2) except for $d = 2$ (see sections 3.2.4 and 4.1.3). Let us simply present the case of the Wilson-Fisher fixed point for $d = 3$.

Following [30] and in accordance with the discussion of section 3.2.2, to get the non trivial fixed point we impose the following boundary conditions:

$$f^*(0) = 0, \quad (81)$$

$$Z^{*\prime}(0) = 0, \quad (82)$$

$$f^*(\varphi) \sim \frac{2-\eta}{2}\varphi + C\varphi^{\frac{d-2+\eta}{d+2-\eta}} + \dots, \quad \text{as } \varphi \rightarrow \infty \quad (83)$$

$$Z^*(\varphi) \sim D + \dots, \quad \text{as } \varphi \rightarrow \infty \quad (84)$$

where C and D are arbitrary constants. The first two conditions (81, 82) come from imposing Z_2 -symmetry, while the last two come directly from the fixed point eqs. (79, 80), once we require the solutions to exist for the whole range $0 \leq \varphi < \infty$.

Hence we have three free parameters (C , D , η) which are reduced to one after imposing eqs. (81, 82). The remaining arbitrary parameter, e.g. $z = Z(0)$, generates a line of (Wilson-Fisher) fixed points (one fixed point for each normalization z). In principle these fixed points are equivalent as a consequence of the reparameterization invariance (see section 2.2.4) and there is a corresponding unique value of η (for any fixed point of the line). Consequently, if the reparameterization invariance was preserved one could get rid of the arbitrary parameter z by setting it equal to 1. Unfortunately, due to the derivative expansion, this is not the case: eqs. (79, 80) violate the reparameterization invariance and the estimates of η (and of ν) *depend* on z .

In order to get the best estimates for η , one can adjust z in such a way as to get an almost realized reparameterization invariance [52, 11, 30, 29]. The analysis is not simple [30] due to the additional effects of the two cutoff parameters A and B . Finally estimates of the critical and subcritical exponents are proposed ($d = 3$ and $N = 1$) [30]:

$$\begin{aligned}\eta &= 0.042 \\ \nu &= 0.622 \\ \omega &= 0.754\end{aligned}$$

It is interesting to compare these estimates with those obtained by Golner in [11] from the Wilson version of the ERGE:

$$\eta = 0.024 \pm 0.007 \tag{85}$$

$$\nu = 0.617 \pm 0.008 \tag{86}$$

The equations are essentially similar in both cases and the difference in the estimations of η surely originates from the way the cutoff is introduced and used in the Polchinski case.

Although those two sets of values are close to the best values (see footnote 31), the procedure which involves z as adjustable parameter is less attractive than if η was uniquely defined at each order of the derivative expansion. It is thus interesting to look for the conditions of preservation of the reparameterization invariance.

4.1.2 Reparametrization invariance linearly realized and preserved

With a view to control the preservation of the reparameterization invariance, one may impose it evidently, i.e. linearly, via a particular choice of cutoff function and try to keep this realization through the derivative expansion [30]. This is what has been done in [22] for the Legendre version of the ERGE (see below). For the smooth cutoff version of the ERGE, the only acceptable cutoff function is power-law like [68, 30] (otherwise the cutoff should be sharp [68, 48]). Unfortunately, for the Polchinski version, the symmetry is broken at finite order in the derivative expansion and the regulators do not regulate, at least not in a finite order in the derivative expansion [33, 30, 68, 38, 48, 104]. Now considering the Legendre version of the ERGE of section 2.6 is sufficient to overcome this difficulty [33, 68, 48, 104, 38].

The smooth cutoff Legendre version and the derivative expansion By choosing a power-law cutoff function $\tilde{C}(q^2) = q^{2k}$ in eq. (39), one is sure that the derivative expansion will preserve the reparameterization invariance [33, 22] and that the exponent η will be unambiguously defined.

Let us expand the Legendre (effective) action $\Gamma[\Phi]$ as follows:

$$\Gamma[\Phi] = \int d^d x \left\{ U(\varphi, t) + \frac{1}{2} Z(\varphi, t) (\partial_\mu \Phi)^2 \right\}$$

in which φ is independent on x .

For $d = 3$ and $k = 1$, the first order of the derivative expansion yields (after a long but straightforward computation) the following two coupled equations for U and Z [22]:

$$\begin{aligned}\dot{U} &= -\frac{1-\eta/4}{\sqrt{Z}\sqrt{U''+2\sqrt{Z}}} + 3U - \frac{1}{2}(1+\eta)\varphi U' \\ \dot{Z} &= -\frac{1}{2}(1+\eta)\varphi Z' - \eta Z + \left(1 - \frac{\eta}{4}\right) \left\{ \frac{1}{48} \frac{24ZZ'' - 19(Z')^2}{Z^{3/2}(U'' + 2\sqrt{Z})^{3/2}} \right. \\ &\quad \left. - \frac{1}{48} \frac{58U'''Z'\sqrt{Z} + 57(Z')^2 + (Z''')^2Z}{Z(U'' + 2\sqrt{Z})^{5/2}} + \frac{5}{12} \frac{(U''')^2Z + 2U'''Z'\sqrt{Z} + (Z')^2}{\sqrt{Z}(U'' + 2\sqrt{Z})^{7/2}} \right\} \end{aligned} \quad (87)$$

As expected, the search for a non trivial fixed point solution for these equations (a solution which is nonsingular up to $\varphi \rightarrow \infty$) produces a unique solution with an unambiguously defined η [22]:

$$\eta = 0.05393 \quad (88)$$

The linearization about this fixed point yields the eigenvalues:

$$\nu = 0.6181 \quad (89)$$

$$\omega = 0.8975 \quad (90)$$

and also a zero eigenvalue $\lambda = 0$ [22] which corresponds to the redundant operator \mathcal{O}_1 [eq. (24)] responsible for the moving along the line of equivalent fixed points. This is, of course, an expected confirmation of the preservation of the reparametrization invariance.

A generalization of the above equations (87) to the $O(N)$ symmetric scalar field theory has been done by Morris and Turner in [66]. There, estimates of η , ν and ω are provided for various values of N and it is shown that the derivative expansion reproduces exactly known results at special values $N = \infty, -2, -4, \dots$ and an interesting discussion on the numerical methods used is presented in their appendix.

The sharp cutoff Legendre version and the derivative expansion The sharp cutoff is the other kind of regularization which allows a linear realization of the reparametrization invariance [33, 68, 38, 48]. As in the previous case of the power-law form of the cutoff function, the derivative expansion performed with the ERGE satisfied by the Wilson effective action $S[\phi]$ with a sharp cutoff induces singularities which can be avoided by considering the Legendre transformed $\Gamma[\Phi]$ [33, 48]. But the Taylor expansion in the momenta must be replaced by an expansion in terms of homogeneous functions of momenta of integer degree [48] (momentum-scale expansion). A systematic series of approximations — the $O(p^M)$ approximations — results [48].

Although not absolutely necessary, an additional expansion and truncation in powers of φ (avoiding the truncation of the potential) have been performed in [48] due to the complexity of the equations⁴⁰. There, as in the previous case of the smooth cutoff, the zero eigenvalue corresponding to the redundant operator \mathcal{O}_1 is found. The estimates for the exponents, however, are worse than those obtained with smooth cutoff in [22] [see eqs.(88-90)] presumably due to the truncation of the field dependence [48]:

$$\eta = 0.0660$$

$$\nu = 0.612$$

$$\omega = 0.91$$

⁴⁰Especially $\sim \varphi^8$ terms and higher have been discarded in non-zero momentum pieces.

The set of equations (87) together with the sharp cutoff version of the momentum expansion [$O(p^1)$] have also been studied in [65] where, in particular, universal quantities other than the exponents (universal coupling ratios) have been estimated.

4.1.3 Studies in two dimensions

In two dimensions it is expected that an infinite set of non-perturbative multicritical fixed points exists corresponding to the unitary minimal series of $(p, p+1)$ conformal field theories with $p = 3, 4, \dots, \infty$ [136]. As mentioned in section 3.2.4, this infinite set cannot be obtained at the level of LPA with which only periodic solutions could be obtained [104]. Using the Legendre ERGE at first order of the derivative expansion with a power law cutoff [the equations are obtained similarly to (87) but for $d = 2$], Morris in [104] (see also [68]) has found the first ten fixed points (and only these) and computed the corresponding critical exponents (and other quantities). The comparison with the exact results of the conformal field theory is satisfactory (in consideration of the low — the lowest — order of approximation). A similar study has been done using the Polchinski ERGE (at first order of the derivative expansion) by Kubyshin et al [137] using the same iteration technique as in [38].

4.2 A field theorist's self-consistent approach

There is an efficient short cut for obtaining the ERGE satisfied by the (Legendre) effective action. It is based on the observation that this (exact) equation [see (40 or 41)] may be obtained from the one loop (unregularized, thus formal) expression of the effective action, which reads (up to a field independent term within the logarithm):

$$\Gamma[\Phi] = S[\Phi] + \frac{1}{2} \text{tr} \ln \left(\frac{\delta^2 S}{\delta \phi \delta \phi} \Big|_{\phi=\Phi} \right) + \text{higher loop-order}, \quad (91)$$

by using the following practical rules:

1. add the infrared cutoff function $C(p, \Lambda)$ of (38) to the action S , eq. (91) then becomes:

$$\Gamma[\Phi] = \frac{1}{2} \int_p \Phi_p \Phi_{-p} C^{-1}(p, \Lambda) + S[\Phi] + \frac{1}{2} \text{tr} \ln \left(C^{-1} + \frac{\delta^2 S}{\delta \phi \delta \phi} \right) \Big|_{\phi=\Phi} + \dots$$

2. redefine $\tilde{\Gamma}[\Phi] = \Gamma[\Phi] - \frac{1}{2} \int_p \Phi_p \Phi_{-p} C^{-1}(p, \Lambda)$, then:

$$\tilde{\Gamma}[\Phi] = S[\Phi] + \frac{1}{2} \text{tr} \ln \left(C^{-1} + \frac{\delta^2 S}{\delta \phi \delta \phi} \right) \Big|_{\phi=\Phi} + \dots$$

3. perform the derivative with respect to Λ , (only the cutoff function is concerned) and forget about the higher loop contributions:

$$\partial_t \tilde{\Gamma} = \frac{1}{2} \text{tr} \left[\frac{1}{C} \Lambda \frac{\partial C}{\partial \Lambda} \cdot \left(1 + C \cdot \frac{\delta^2 S[\Phi]}{\delta \Phi \delta \Phi} \right)^{-1} \right]$$

4. replace⁴¹ the action S , in the right hand side of the latter equation, by the effective action $\tilde{\Gamma}$ to get eqs. (40, 41), the dilatation part $\mathcal{G}_{\text{dil}} \tilde{\Gamma}$ being obtained from usual (engineering) dimensional considerations⁴².

⁴¹This step is often referred to as the “renormalization group improvement” of the one loop effective action.

⁴²But do not forget to introduce the anomalous dimension of the field in order to get an eventual nontrivial fixed point.

It is noteworthy that the above rules have been heuristically first used [138, 139] to obtain the local potential approximation of the ERGE for the (Legendre) effective action. However the main interest of the above considerations is that they allow introducing the (infra-red) cutoff function independently of S , via the so-called “*proper time*” (or “*heat kernel*” or “*operator*”) regularization [140]. This kind of regularization is introduced at the level of eq. (91) via the general identity:

$$\text{tr} \ln \left(\frac{A}{B} \right) = - \int_0^\infty \frac{ds}{s} \text{tr} (e^{-sA} - e^{-sB})$$

Forgetting again about the field-independent part (and, momentarily, about the ultra-violet regularization needed for $s \rightarrow 0$) one introduces an infrared cutoff function $F_\Lambda(s)$ within the proper time integral representation of the logarithm of $A = \frac{\delta^2 S}{\delta \phi \delta \phi} \Big|_{\phi=\Phi}$:

$$\frac{1}{2} \text{tr} \ln A \longrightarrow -\frac{1}{2} \int_0^\infty \frac{ds}{s} F_\Lambda(s) \text{tr} e^{-sA}$$

The function $F_\Lambda(s)$ must tend to zero sufficiently rapidly for large values of s in order to suppress the small momentum modes and should be equal to 1 for $\Lambda = 0$.

Then following the rules 3-4 above applied on Γ (i.e., not on $\tilde{\Gamma}$), one obtains a new kind of ERGE⁴³ for the effective action [141]:

$$\partial_t \Gamma = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \Lambda \frac{\partial F_\Lambda(s)}{\partial \Lambda} \exp \left[-s \frac{\delta^2 \Gamma}{\delta \Phi \delta \Phi} \right] \quad (92)$$

There are apparently two advantages of using this kind of ERGE:

- the regularization preserves the symmetry of the action [142]
- the derivative expansion is slightly easier to perform than in the conventional approach and one may preserve the reparametrization invariance [141].

In [141], Bonanno and Zappalà have considered the next-to-leading order of the derivative expansion of (92) (while in [143] only a pseudo derivative expansion, in which the wave-function renormalization function $Z(\phi, t)$ is field-independent⁴⁴, was used). They have chosen $F_\Lambda(s)$ in such a way that the integro-differential character of the ERGE disappears and they have tested the preservation of the reparametrization invariance. Moreover a scheme dependence parameter, related to the cutoff width, is at hand in this framework which, presumably, will allow someone to look at the best possible convergence of the derivative expansion when higher orders will be considered.

Another kind of regularization related to this “self-consistent” approach should be mentioned here. It consists in introducing the cutoff function in eq. (91) in-between the momentum integration [expressing the trace] and the logarithm. This procedure has been considered in [144] at the level of the local potential approximation. However the ERGE keeps its integro-differential character and the study of [144] has then been conducted within the constraining polynomial expansion method of section 3.3.

⁴³Notice that, because one performs a derivative with respect to Λ , the essential contribution to $\partial_t \Gamma$ comes from the integration over a small range of values of s (corresponding to the rapid decreasing of $F_\Lambda(s)$), hence an ultraviolet regularization is not needed provided that the resulting RG equation be finite.

⁴⁴See below.

4.3 Other studies up to first order of the derivative expansion

In this section we mention studies of the derivative expansion which, although interesting, do not consider explicitly the reparametrization invariance.

Filippov and Radievskii [145] have obtained a set of two coupled equations that look like eqs. (79, 80) but their numerical studies were based on an approximation which consists in neglecting the term corresponding to AZ' in (79). They, nevertheless, present interesting estimates of critical exponents for several values of d in the range $[2, 3.5]$.

As already mentioned, Ball et al [38] have studied the “scheme” dependence with the help of the Polchinski ERGE at first order in the derivative expansion (without considering explicitly the breaking of the reparametrization invariance). They have used an interesting simple iteration procedure to determine the fixed point.

Bonanno et al [146] have presented a sharp version of the coupled differential equation for V and Z which however yields a negative value of η in three dimensions. The authors claim that this failure is not to be searched in an intrinsic weakness of the sharp cutoff. A previous attempt had been done with the sharp cutoff version and a two loop perturbative anomalous dimension was obtained by means of a polynomial truncation in the field dependence [147]. Re-obtention of two loop results from the derivative expansion of the ERGE satisfied by the effective (Legendre) action (with smooth cutoff) are also described in [138, 83, 110].

Pseudo derivative expansion Tetradis and Wetterich [83] have initiated an original strategy to obtain systematic accurate estimates on, say, critical quantities already from the lowest order of the derivative expansion. The idea is based on the smallness of η and may be roughly described as follows. At lowest order of the derivative expansion (LPA), one assumes that $Z(0, t)$ already depends on t . One then determines an approximate t -dependence (assuming η is small) from the momentum dependence of the exact propagator $\Gamma^{(2)}$. Hence η is not equal to zero even at the lowest order of the derivative expansion and this yields an “improved” LPA. The next order would amount to consider an explicit φ -dependent $Z(\varphi, t)$ and the following order higher derivatives of the field in the action. This is not a genuine derivative expansion and it does not account for the reparametrization invariance. Nevertheless the approach seems efficient considering the estimates obtained at the leading order of that pseudo derivative expansion. Let us first quote, for $d = 3$ and $N = 1$, the results found with the supplementary help of a truncation in powers of the field associated to an expansion around the minimum of the potential [83]: $\nu = 0.638$, $\eta = 0.045$, $\gamma = 1.247$, $\beta = 0.333$ and without truncation in the field dependence [148]: $\nu = 0.643$, $\eta = 0.044$, $\gamma = 1.258$, $\beta = 0.336$, $\delta = 4.75$. In this latter work, the scaled equation of state has been calculated using this pseudo derivative expansion. For more details on this approach see the review by Berges et al [13] in this volume.

A study for $d = 2$ has also been achieved [149] following the spirit of [83] (i.e. with a truncation) with a view to discuss the Kosterlitz-Thouless phase transition [150]. The aim of the authors was to show the power of the ERGE compared to the perturbative approach (due to IR singularities). It is amazing to notice the excellent estimation of η ($\simeq 0.24$ instead of $\frac{1}{4}$) obtained in this work knowing that η was assumed to be small and that the truncation was crude. As indicated by the authors, this result may be accidental. An extension of this work may be found in [151].

4.4 Convergence of the derivative expansion?

Comparing the estimates of the critical exponents obtained at first order of the derivative expansion to that of, e.g., the ε -expansion, one can easily see that the derivative expansion is potentially much more effective than the perturbative (field theoretical) approach. But, to date,

it is not known whether it converges or not. Morris and Tighe [152] have considered this question at one and two loop orders for different cases of regularization (cutoff) functions and for either the Wilson (— Polchinski) or Legendre effective action. It is found that the Legendre flow equation converges at one and two loops: slowly with sharp cutoff (as a momentum-scale expansion), and rapidly in the case of a smooth exponential cutoff (but, in this latter case, the reparametrization invariance is not satisfied, see above). The Wilson (— Polchinski) version and the Legendre flow equation with power law cutoff function do not converge.

It is possible that the derivative expansion gives rise to asymptotic series which would be Borel summable. This is deduced from the knowledge of an exact solution for the effective potential for QED₂₊₁ in a particular inhomogeneous external magnetic field, from which it has been shown that the derivative expansion (known at any order) is a divergent but Borel summable asymptotic series [153].

The annoying perspective that the derivative expansion does not converge has prevailed on Golner to look for a method of successive approximations for the ERGE that is not based on power series expansion [154].

4.5 Other models, other ERGE's, other studies...

Up to this point, we have presented in some details various aspects (derivation, invariances, approximations, truncations, calculations)⁴⁵ of the ERGE for scalar systems. Since these issues are also encountered for more complex systems (but with, potentially, a more interesting physical content), in this section we limit ourselves to mentioning the existence of studies based on the ERGE relative to models different from the pure scalar theory⁴⁶. Most often these models involve more structure due to supplementary internal degrees of freedom. Formally, the master equations keep essentially the same general forms as described in section 2 [owing to the trace symbol as used in (40)]. The equations involved in the studies actually show their differences when approximations are effectively considered. The studies are characterized by the action S considered, the cutoff function chosen and the approximation applied on the ERGE. In consideration of the large number of publications and the variety of models studied, we choose to classify them according to the increasing degree of complexity of the model with respect to the field: scalar (or vector), spinor and gauge field.

Other ERGE's involving pure scalars (or vectors) We have already mentioned the Stiefel model studied in the LPA in [84, 112] let us quote also the Boson systems [157], the nucleation and spinodal decomposition [158], the interface unbinding transitions arising in wetting phenomena [159], the roughening transition [160], transitions in magnets (non-collinear spin ordering, frustrated) studied in [161], disordered systems [162], the quantum tunnelling effect [163], the well developed turbulence [164] and even the one-quantum-particle system [165]. More developed are the numerous studies of scalar theories at finite temperature [166]. Reviews on the finite temperature framework may be found in [14, 167] though they cover more than scalars.

Other ERGE's It exists some studies involving pure spinors [168] and also some mixing scalars and spinors [169], reviews may be found in [170].

In addition, a rich literature on ERGE deals with systems in presence of gauge fields: pure gauge fields [57, 171], gauge fields with scalars [172] and with spins [46, 173], supersymmetric gauge fields [174] and gravity [175]. Reviews on this theme are listed in [176].

⁴⁵Let us quote, in addition, the issue of scheme dependence, already mentioned in the text, several aspects of which are considered in [155].

⁴⁶One may also refer to a recent review by Aoki [156].

Despite the great number of studies done up to now on the ERGE, its systematic use in nonperturbative calculations and in describing nonuniversalities is still in its infancy. A better mastery of invariances within the truncation procedure, the extension of series (this has required some time in perturbation theory), the consideration of more complex and realistic models with a view to obtain estimates of useful physical quantities will necessitate much more investigations in the future.

Figure captions

1. Three solutions of the sharp cutoff fixed point equation for $f(\phi_0) = V'(\phi_0)$ and $d = 3$ [eq. (55) with $\dot{f} = 0$]. All (here two) but one ($\sigma^* = -0.4615337\cdots$) of the solutions are singular at some (not fixed) φ_c . The parameter $\sigma = V''(0)$ is adjusted to σ^* by requiring the physical fixed point to be defined for all ϕ_0 (in the text φ stands for ϕ_0).
2. Determination by the shooting method of the initial critical value $u_2^c = -0.299586913\cdots$ corresponding to the initial values $u_4(0) = 3$ and $u_n(0) = 0$ for $n > 4$ [from eq. (55) with $d = 3$]. Open circles indicate the initial points chosen in the canonical surface (representing simple actions) of \mathcal{S} . The illustration is made via projections onto the plane $[u_2, u_4]$. The determination of u_2^c is made by iterations (shooting method) according to increasing labels. Arrows indicate the infrared direction (decreasing of the momentum-scale of reference). The RG trajectories follow two opposite directions according to whether $u_2(0) > u_2^c$ (labels 1, 3, 5) or $u_2(0) < u_2^c$ (labels 2, 4, 6). The Wilson-Fisher (once infrared unstable) fixed point (full circle) is only reached when $u_2(0) = u_2^c$ (dashed curve). The corresponding RG trajectory lies in the critical surface \mathcal{S}_c of codimension 1.
3. Illustration of the simplest nonperturbative continuum limit in three dimensions [from eq. (55) with $d = 3$]. Approach to the purely massive “renormalized trajectory” T_0 (dot-dashed curve) by RG trajectories initialized at $u_4(0) = 3$ and $u_n(0) = 0$ for $n > 4$ and $(u_2(0) - u_2^c) \rightarrow 0^+$ (open circles). The trajectories drawn correspond to $\log(u_2(0) - u_2^c) = -1, -2, -3, -4, -5, -6$. When $u_2(0) = u_2^c$ the trajectories do not leave the critical surface and approach the Wilson-Fisher fixed point (full circle), as in figure 2. But, “*moving a little bit away from the critical manifold, the trajectory of the RG will to begin with, move towards the fixed point, but then shoot away along [...] the relevant direction towards the so-called high temperature fixed point...*” (see text, section 2.10.1 and [59]).
4. Projection onto the plane (u_4, u_6) of some remarkable RG trajectories for $u_4(0) > 0$ [from eq. (55) with $d = 3$]. Full lines represent trajectories on the critical surface \mathcal{S}_c . The arrows indicate the directions of the RG flows on the trajectories. The submanifold T_1 of one dimension to which are attracted the trajectories with small values of $u_4(0)$ and which links the Gaussian fixed point to the Wilson-Fisher fixed point corresponds to the renormalized trajectory on which is defined the continuum limit of the massless field theory in three dimensions. For larger values of $u_4(0)$ the RG trajectories approach the Wilson-Fisher fixed point from the opposite side, they correspond to the Ising model. The dotted line T_2 plunging into the Wilson-Fisher fixed point does not lie on \mathcal{S}_c but represents a RG trajectory approaching the Wilson-Fisher fixed point along the second less irrelevant direction (lying in a space of codimension 2). The corresponding critical behavior is characterized by the absence of the first kind of correction to scaling (that corresponding to the exponent ω), it would be representative of some Ising models with spin $s = 1/2$. The two trajectories that leave the Wilson-Fisher fixed point (dashed lines) correspond to the unique relevant eigendirection (with two ways, due to the arbitrary normalization, associated with the two phases of the critical point, they also correspond to two massive RT’s). The open circles represent initial simple actions.
5. Projection onto the plane $\{u_2, u_4\}$ of some remarkable RG trajectories for $u_4(0) < 0$ [from eq. (55) with $d = 3$]. Black circles represent the Gaussian and Wilson-Fisher fixed points. The arrows indicate the directions of the RG flows on the trajectories. The ideal trajectory T_1 (dot line) which interpolates between the two fixed points represents the RT corresponding to the so-called ϕ_3^4 renormalized field theory in three dimensions (usual RT for $u_4 > 0$). White circles represent the projections onto the plane of initial critical

actions. For $u_4(0) > 0$, the effective actions (e.g. initialized at B') run toward the Wilson-Fisher fixed point asymptotically along the usual RT. Instead, for $u_4(0) < 0$ and according to the initial values of the parameters of higher order (u_6, u_8 , etc.), the RG trajectories either (A) meet an endless RT emerging from the Gaussian fixed point T''_1 (dashed curve) and lying entirely in the sector $u_4 < 0$ or (B) meet the usual RT to reach the Wilson-Fisher fixed point. The frontier which separates these two very different cases (A and B) corresponds to initial actions lying on the tri-critical subspace (white square C) that are sources of RG trajectories flowing toward the Gaussian fixed point asymptotically along the tricritical (pseudo) RT. Notice that the coincidence of the initial point B with the RG trajectory starting at point A is not real (it is accidentally due to the projection onto a plane of the trajectories lying in a space of infinite dimension). See text for a discussion and [111, 101, 114].

6. RG trajectories on the critical surface \mathcal{S}_c obtained from integration of eq. (55) with $d = 4$ (projection onto the plane (u_4, u_6)). Open circles indicate the initial points chosen on the canonical surface of \mathcal{S}_c (of codimension 1). The two lines which come from the upper side of the figure are RG trajectories initialized at $u_4(0) = 20$ and $u_4(0) = 40$ respectively. The arrows indicate the infrared direction. The trajectories are attracted to a submanifold of dimension one before plunging into the Gaussian fixed point. This pseudo renormalized trajectory (it has no well defined beginning) allows to make sense to the notion of effective (here massless) field theory. Strictly speaking, the continuum limit does not exist due to the lack of another (nontrivial fixed point) which would allow the scale dependence (of the renormalized parameter along the RT) to be defined in the whole range of scale $]0, \infty[$. See text for a discussion (from [73]).

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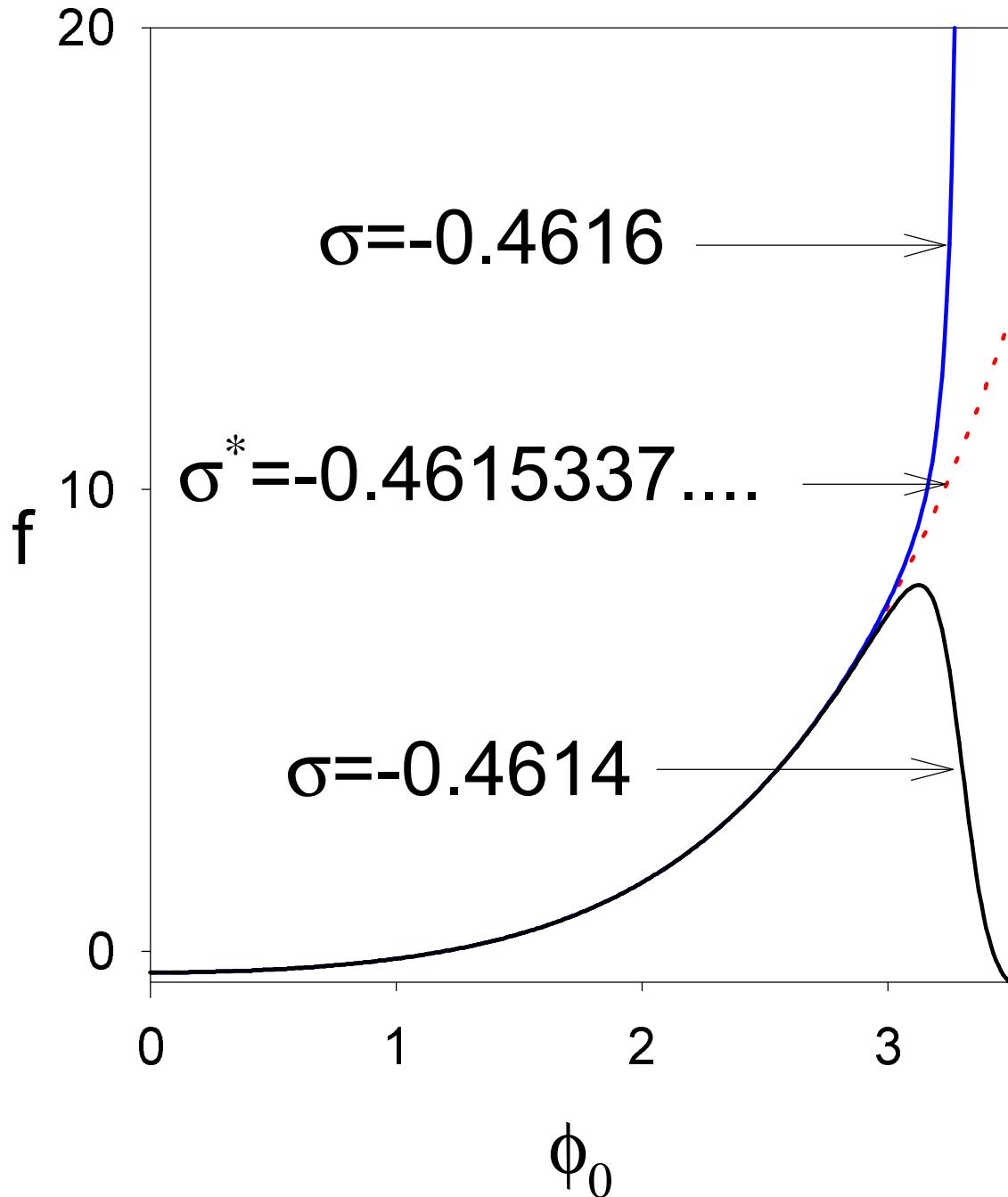


Figure 1

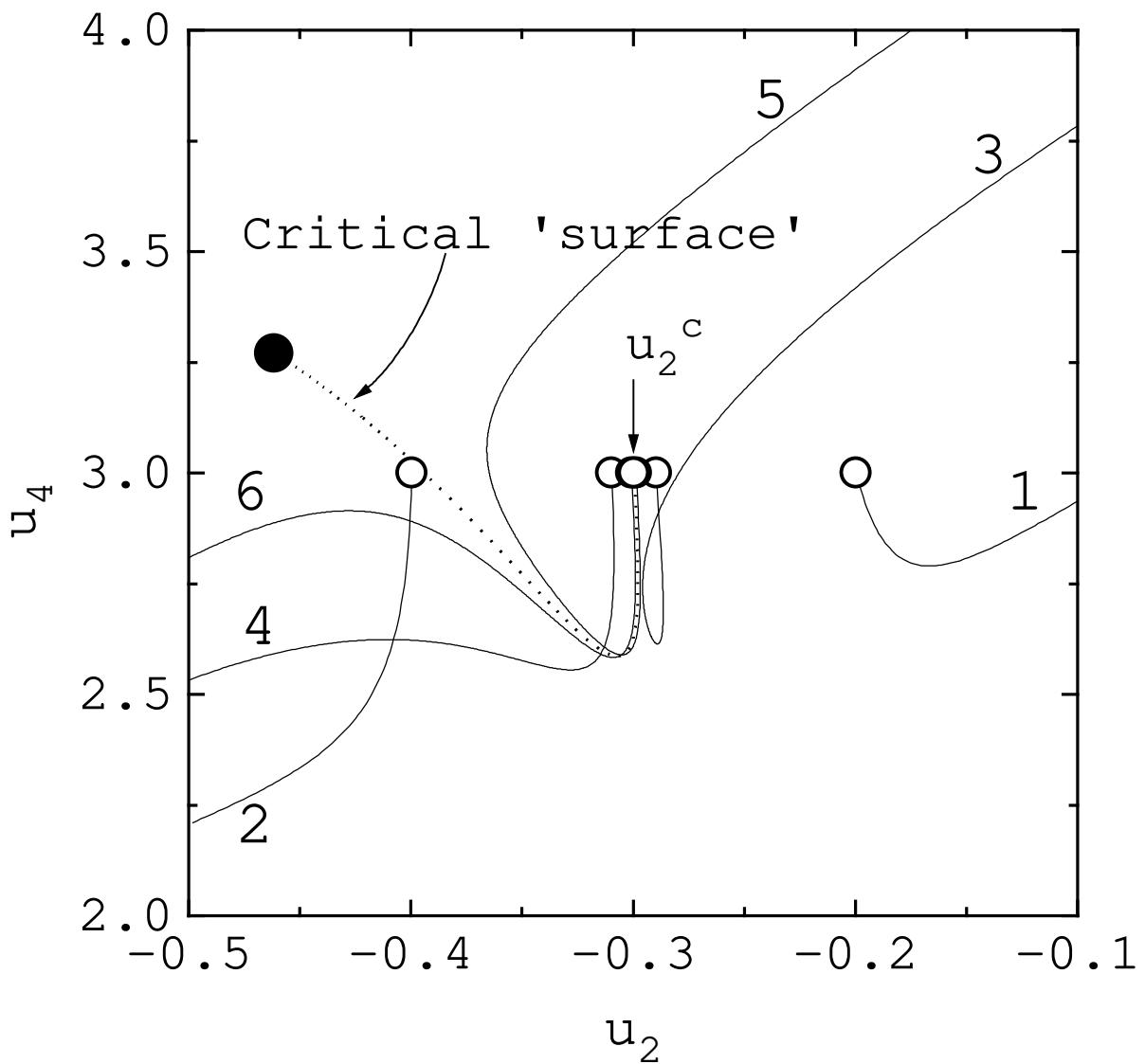


Figure 2

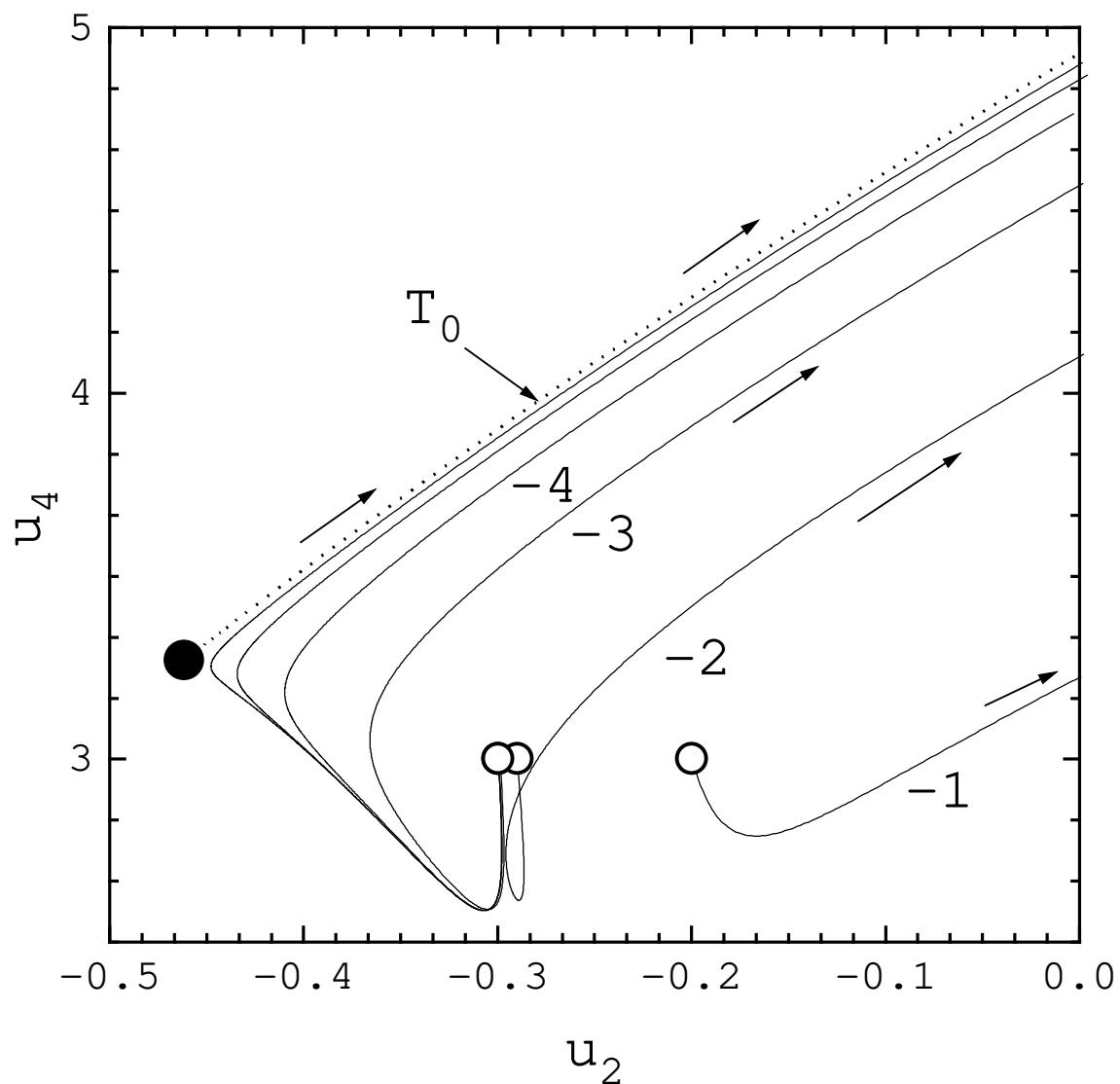


Figure 3

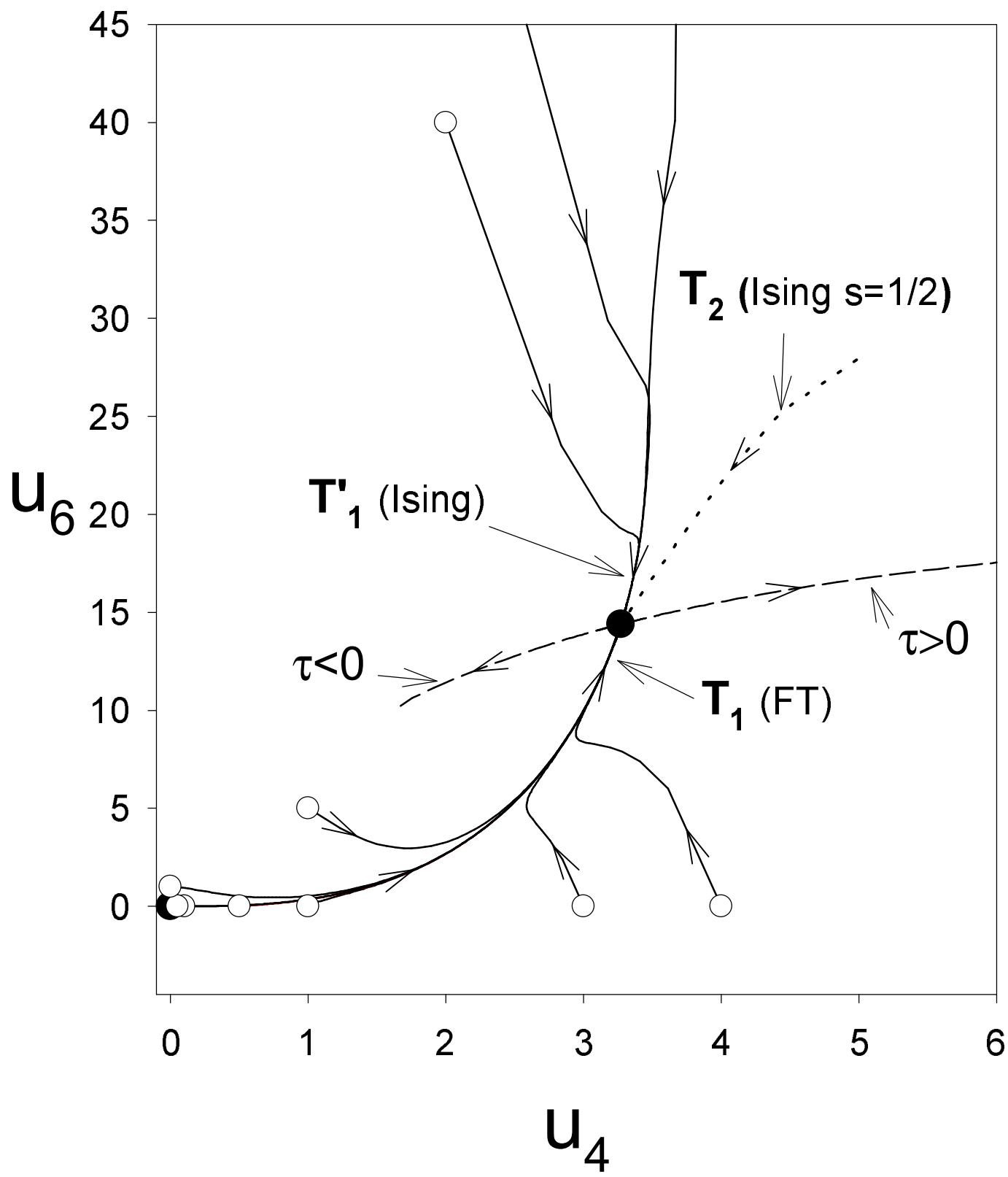


Figure 4

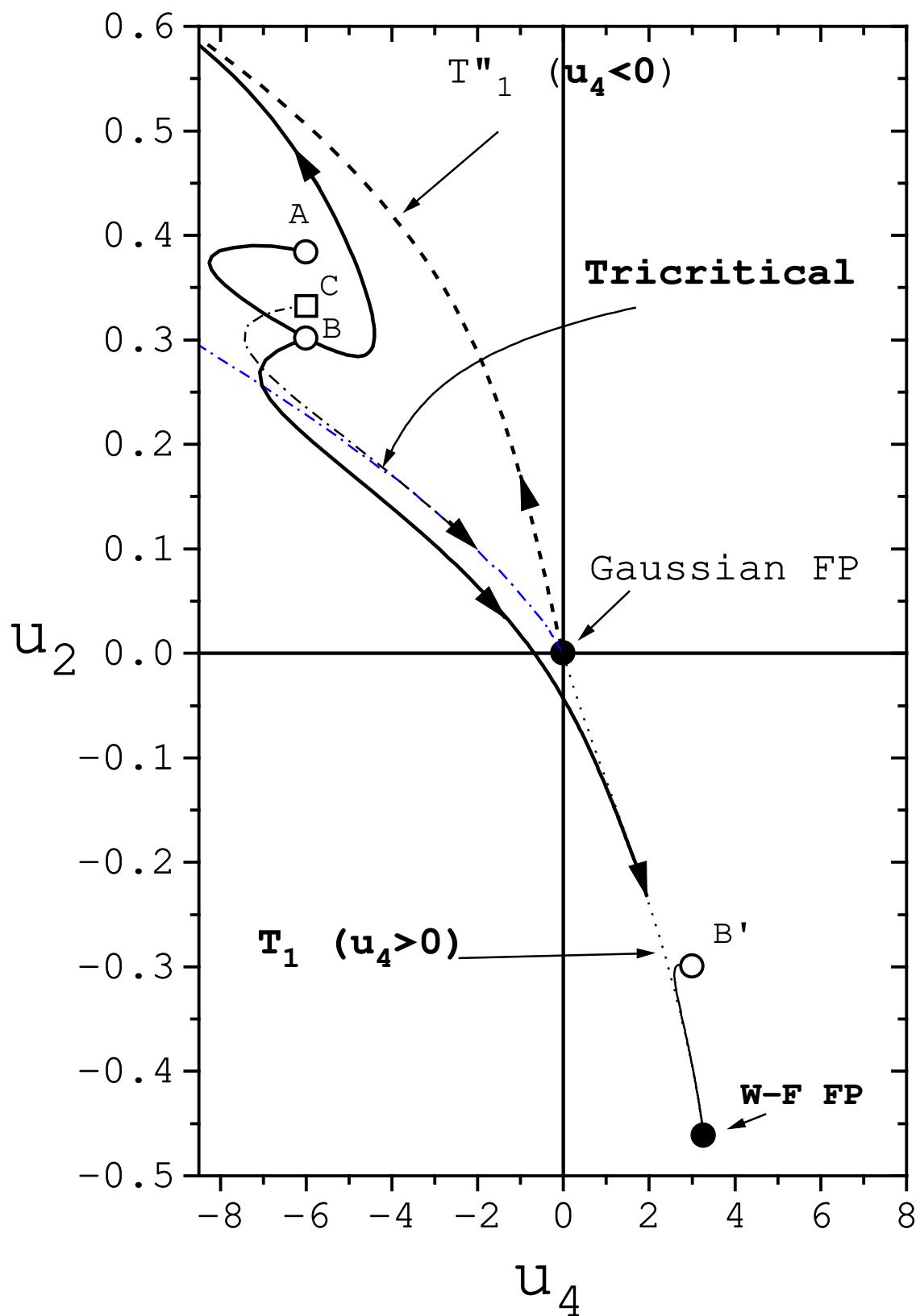


Figure 5

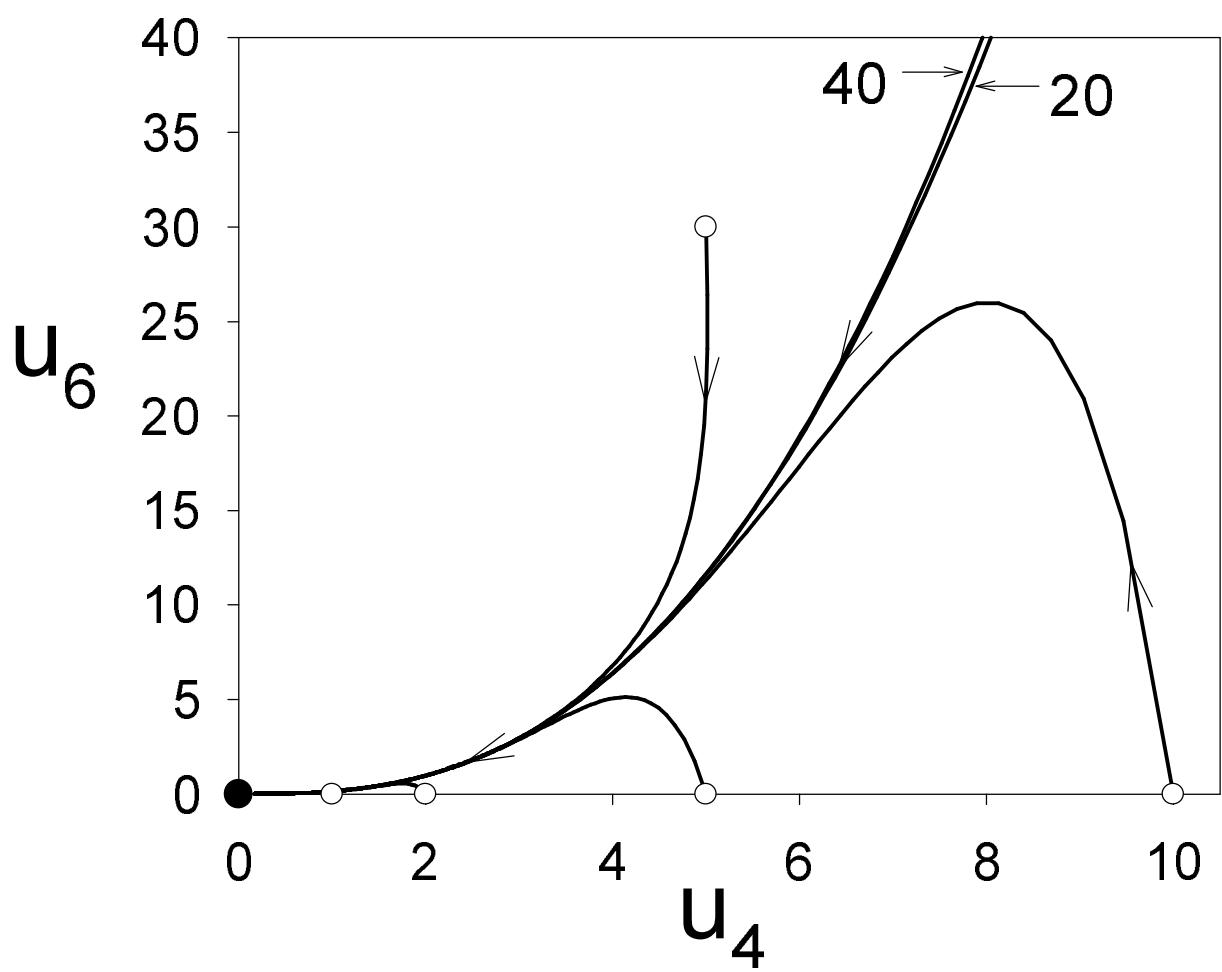


Figure 6

