can further act on $\mathbf{v}'$ by the group generated by $\mathbf{v}_4$; this has the net effect of scaling the coefficients of $\mathbf{v}_2$ and $\mathbf{v}_3$:

$$\mathbf{v}'' = \mathrm{Ad}(\exp(\varepsilon\mathbf{v}_4))\mathbf{v}' = a_2' e^{3\varepsilon}\mathbf{v}_2 + e^{-2\varepsilon}\mathbf{v}_3.$$

This is a scalar multiple of $\mathbf{v}''' = a_2' e^{5\varepsilon}\mathbf{v}_2 + \mathbf{v}_3$, so, depending on the sign of a_2', we can make the coefficient of $\mathbf{v}_2$ either $+1$, -1 or 0. Thus any one-dimensional subalgebra spanned by $\mathbf{v}$ with $a_4 = 0, a_3 \neq 0$ is equivalent to one spanned by either $\mathbf{v}_3 + \mathbf{v}_2$, $\mathbf{v}_3 - \mathbf{v}_2$ or $\mathbf{v}_3$. The remaining cases, $a_3 = a_4 = 0$, are similarly seen to be equivalent either to $\mathbf{v}_2$ ($a_2 \neq 0$) or to $\mathbf{v}_1$ ($a_2 = a_3 = a_4 = 0$). The reader can check that no further simplifications are possible.

Recapitulating, we have found an optimal system of one-dimensional subalgebras to be those spanned by

$$\begin{aligned}
&\text{(a)} & \mathbf{v}_4 &= x\partial_x + 3t\partial_t - 2u\partial_u,\\
&(\text{b}_1) & \mathbf{v}_3 + \mathbf{v}_2 &= t\partial_x + \partial_t + \partial_u,\\
&(\text{b}_2) & \mathbf{v}_3 - \mathbf{v}_2 &= t\partial_x - \partial_t + \partial_u,\\
&(\text{b}_3) & \mathbf{v}_3 &= t\partial_x + \partial_u,\\
&\text{(c)} & \mathbf{v}_2 &= \partial_t,\\
&\text{(d)} & \mathbf{v}_1 &= \partial_x,
\end{aligned} \tag{3.25}$$

This list can be reduced slightly if we admit the discrete symmetry $(x, t, u) \mapsto (-x, -t, u)$, not in the connected component of the identity of the full symmetry group, which maps $\mathbf{v}_3 - \mathbf{v}_2$ to $\mathbf{v}_3 + \mathbf{v}_2$, thereby reducing the number of inequivalent subalgebras to five.

Example 3.13. Consider the six-dimensional symmetry algebra $\mathfrak{g}$ of the heat equation (2.55), which is generated by the vector fields

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = u\partial_u, \qquad \mathbf{v}_4 = x\partial_x + 2t\partial_t,$$
$$\mathbf{v}_5 = 2t\partial_x - xu\partial_u, \qquad \mathbf{v}_6 = 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u.$$

(For the moment we are ignoring the trivial infinite-dimensional subalgebras coming from the linearity of the heat equation.) From the commutator table for this algebra, we obtain the following table:

Ad	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_4$	$e^{\varepsilon}\mathbf{v}_1$	$e^{2\varepsilon}\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_5$	$\mathbf{v}_1 - \varepsilon\mathbf{v}_3$	$\mathbf{v}_2 + 2\varepsilon\mathbf{v}_1 - \varepsilon^2\mathbf{v}_3$	$\mathbf{v}_3$
$\mathbf{v}_6$	$\mathbf{v}_1 + 2\varepsilon\mathbf{v}_5$	$\mathbf{v}_2 - 2\varepsilon\mathbf{v}_3 + 4\varepsilon\mathbf{v}_4 + 4\varepsilon^2\mathbf{v}_6$	$\mathbf{v}_3$

Ad	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_1$	$\mathbf{v}_4 - \varepsilon\mathbf{v}_1$	$\mathbf{v}_5 + \varepsilon\mathbf{v}_3$	$\mathbf{v}_6 - 2\varepsilon\mathbf{v}_5 - \varepsilon^2\mathbf{v}_3$
$\mathbf{v}_2$	$\mathbf{v}_4 - 2\varepsilon\mathbf{v}_2$	$\mathbf{v}_5 - 2\varepsilon\mathbf{v}_1$	$\mathbf{v}_6 - 4\varepsilon\mathbf{v}_4 + 2\varepsilon\mathbf{v}_3 + 4\varepsilon^2\mathbf{v}_2$
$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_4$	$\mathbf{v}_4$	$e^{-\varepsilon}\mathbf{v}_5$	$e^{-2\varepsilon}\mathbf{v}_6$
$\mathbf{v}_5$	$\mathbf{v}_4 + \varepsilon\mathbf{v}_5$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_6$	$\mathbf{v}_4 + 2\varepsilon\mathbf{v}_6$	$\mathbf{v}_5$	$\mathbf{v}_6$

where the (i, j)-th entry gives $\mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i))\mathbf{v}_j$.

Let $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_6\mathbf{v}_6$ be an element of $\mathfrak{g}$, which we shall try to simplify using suitable adjoint maps. A key observation here is that the function $\eta(\mathbf{v}) = (a_4)^2 - 4a_2a_6$ is an invariant of the full adjoint action: $\eta(\mathrm{Ad}\, g(\mathbf{v})) = \eta(\mathbf{v})$, $\mathbf{v} \in \mathfrak{g}$, $g \in G$. The detection of such an invariant is important since it places restrictions on how far we can expect to simplify $\mathbf{v}$. For example, if $\eta(\mathbf{v}) \neq 0$, then we cannot simultaneously make a_2, a_4 and a_6 all zero through adjoint maps; if $\eta(\mathbf{v}) < 0$ we cannot make either a_2 or a_6 zero!

To begin the classification process, we concentrate on the a_2, a_4, a_6 coefficients of $\mathbf{v}$. If $\mathbf{v}$ is as above, then

$$\tilde{\mathbf{v}} = \sum_{i=1}^{6} \tilde{a}_i\mathbf{v}_i = \mathrm{Ad}(\exp(\alpha\mathbf{v}_6)) \circ \mathrm{Ad}(\exp(\beta\mathbf{v}_2))\mathbf{v}$$

has coefficients

$$\begin{aligned} \tilde{a}_2 &= a_2 - 2\beta a_4 + 4\beta^2 a_6, \\ \tilde{a}_4 &= 4\alpha a_2 + (1 - 8\alpha\beta)a_4 - 4\beta(1 - 4\alpha\beta)a_6, \\ \tilde{a}_6 &= 4\alpha^2 a_2 + 2\alpha(1 - 4\alpha\beta)a_4 + (1 - 4\alpha\beta)^2 a_6. \end{aligned} \tag{3.26}$$

There are now three cases, depending on the sign of the invariant η:

Case 1. If $\eta(\mathbf{v}) > 0$, then we choose β to be either real root of the quadratic equation $4a_6\beta^2 - 2a_4\beta + a_2 = 0$, and $\alpha = a_6/(8\beta a_6 - 2a_4)$ (which is always well defined). Then $\tilde{a}_2 = \tilde{a}_6 = 0$, while $\tilde{a}_4 = \sqrt{\eta(\mathbf{v})} \neq 0$, so $\mathbf{v}$ is equivalent to a multiple of $\tilde{\mathbf{v}} = \mathbf{v}_4 + \tilde{a}_1\mathbf{v}_1 + \tilde{a}_3\mathbf{v}_3 + \tilde{a}_5\mathbf{v}_5$. Acting further by adjoint maps generated respectively by $\mathbf{v}_5$ and $\mathbf{v}_1$ we can arrange that the coefficients of $\mathbf{v}_5$ and $\mathbf{v}_1$ in $\tilde{\mathbf{v}}$ vanish. Therefore, every element with $\eta(\mathbf{v}) > 0$ is equivalent to a multiple of $\mathbf{v}_4 + a\mathbf{v}_3$ for some $a \in \mathbb{R}$. No further simplifications are possible.

Case 2. If $\eta(\mathbf{v}) < 0$, set $\beta = 0$, $\alpha = -a_4/4a_2$ to make $\tilde{a}_4 = 0$. Acting on $\mathbf{v}$ by the group generated by $\mathbf{v}_4$, we can make the coefficients of $\mathbf{v}_2$ and $\mathbf{v}_6$ agree, so $\mathbf{v}$ is equivalent to a scalar multiple of $\tilde{\mathbf{v}} = (\mathbf{v}_2 + \mathbf{v}_6) + \tilde{a}_1\mathbf{v}_1 + \tilde{a}_3\mathbf{v}_3 + \tilde{a}_5\mathbf{v}_5$. Further use of the groups generated by $\mathbf{v}_1$ and $\mathbf{v}_5$ show that $\tilde{\mathbf{v}}$ is equivalent to a scalar multiple of $\mathbf{v}_2 + \mathbf{v}_6 + a\mathbf{v}_3$ for some $a \in \mathbb{R}$.

Case 3. If $\eta(\mathbf{v}) = 0$, there are two subcases. If not all of the coefficients a_2, a_4, a_6 vanish, then we can choose α and β in (3.26) so that $\tilde{a}_2 \neq 0$, but $\tilde{a}_4 = \tilde{a}_6 = 0$, so $\mathbf{v}$ is equivalent to a multiple of $\tilde{\mathbf{v}} = \mathbf{v}_2 + \tilde{a}_1\mathbf{v}_1 + \tilde{a}_3\mathbf{v}_3 + \tilde{a}_5\mathbf{v}_5$. Suppose $\tilde{a}_5 \neq 0$. Then we can make the coefficients of $\mathbf{v}_1$ and $\mathbf{v}_3$ zero using the groups generated by $\mathbf{v}_1$ and $\mathbf{v}_2$, while the group generated by $\mathbf{v}_4$ independently scales the coefficients of $\mathbf{v}_2$ and $\mathbf{v}_5$. Thus such a $\mathbf{v}$ is equivalent to a multiple of either $\mathbf{v}_2 + \mathbf{v}_5$ or $\mathbf{v}_2 - \mathbf{v}_5$. If, on the other hand, $\tilde{a}_5 = 0$, then the group generated by $\mathbf{v}_5$ can be used to reduce $\mathbf{v}$ to a vector of the form $\mathbf{v}_2 + a\mathbf{v}_3$, $a \in \mathbb{R}$.

The last remaining case occurs when $a_2 = a_4 = a_6 = 0$, for which our earlier simplifications were unnecessary. If $a_1 \neq 0$, then using groups generated by $\mathbf{v}_5$ and $\mathbf{v}_6$ we can arrange $\mathbf{v}$ to become a multiple of $\mathbf{v}_1$. If $a_1 = 0$, but $a_5 \neq 0$, we can first act by any map $\operatorname{Ad}(\exp(\varepsilon\mathbf{v}_2))$ to get a nonzero coefficient in front of $\mathbf{v}_1$, reducing to the previous case. The only remaining vectors are the multiples of $\mathbf{v}_3$, on which the adjoint representation acts trivially.

In summary, an optimal system of one-dimensional subalgebras of the heat algebra is provided by those generated by

$$
\begin{array}{llll}
\text{(a)} & \mathbf{v}_4 + a\mathbf{v}_3, & \eta > 0, & a \in \mathbb{R}, \\
\text{(b)} & \mathbf{v}_2 + \mathbf{v}_6 + a\mathbf{v}_3, & \eta < 0, & a \in \mathbb{R}, \\
\text{(c1)} & \mathbf{v}_2 - \mathbf{v}_5, & \eta = 0, & \\
\text{(c2)} & \mathbf{v}_2 + \mathbf{v}_5, & \eta = 0, & \\
\text{(d)} & \mathbf{v}_2 + a\mathbf{v}_3, & \eta = 0, & a \in \mathbb{R}, \\
\text{(e)} & \mathbf{v}_1, & \eta = 0, & \\
\text{(f)} & \mathbf{v}_3, & \eta = 0. &
\end{array}
\tag{3.27}
$$

Again, the discrete symmetry $(x, t, u) \mapsto (-x, t, u)$ will map $\mathbf{v}_2 - \mathbf{v}_5$ to $\mathbf{v}_2 + \mathbf{v}_5$, and the list is reduced by one.

Inclusion of the additional infinite-dimensional symmetry algebra $\{\mathbf{v}_\alpha = \alpha(x, t)\partial_u\}$, α a solution to the heat equation, does not essentially alter this classification. If $\mathbf{v} + \mathbf{v}_\alpha$ is in this larger algebra, with $\mathbf{v} \neq 0$ in the above six-dimensional heat algebra, then we can always find $\mathbf{v}_\beta = \beta(x, t)\partial_u$ such that $\operatorname{Ad}(\exp(\mathbf{v}_\beta))(\mathbf{v} + \mathbf{v}_\alpha) = \mathbf{v}$. For instance, if $\mathbf{v} = \mathbf{v}_1 = \partial_x$, then

$$\beta(x, t) = -\int_0^x \alpha(y, t)\, dy - \int_0^t \alpha_x(0, s)\, ds$$

will do. (The reader should check that β is a solution to the heat equation.) The only remaining vectors not equivalent to ones in the six-dimensional algebra are thus those of the form $\mathbf{v}_\alpha$ only. We will not attempt to classify these vectors as they do *not* lead to group-invariant solutions to the heat equation.

Once we have classified one-dimensional subalgebras of a Lie algebra, we can go on to find optimal systems of higher dimensional subalgebras. Lack of space precludes us from pursuing this interesting problem any further here, so we refer the reader to Ovsiannikov, [3; § 14.8], for some of the techniques available.

Classification of Group-Invariant Solutions

Definition 3.14. An *optimal system* of s-parameter group-invariant solutions to a system of differential equations is a collection of solutions $u = f(x)$ with the following properties:

(i) Each solution in the list is invariant under some s-parameter symmetry group of the system of differential equations.
(ii) If $u = \tilde{f}(x)$ is any other solution invariant under an s-parameter symmetry group, then there is a further symmetry g of the system which maps $\tilde{f}$ to a solution $f = g \cdot \tilde{f}$ on the list.

Proposition 3.15. *Let G be the full symmetry group of a system of partial differential equations Δ. Let $\{H_\alpha\}$ be an optimal system of s-parameter subgroups of G. Then the collecton of all H_α-invariant solutions, for H_α in the optimal system, forms an optimal system of s-parameter group-invariant solutions to Δ.*

The proof is immediate from Proposition 3.6. Moreover, our earlier classification of subalgebras is now directly applicable to the classification of group-invariant solutions.

Example 3.16. For the Korteweg–de Vries equation, we've already done all the work to provide a complete list of invariant solutions in our earlier treatment of Example 3.4. Indeed, according to our optimal system of one-dimensional subalgebras (3.25) of the full symmetry algebra, we need only find group-invariant solutions for the one-parameter subgroups generated by: (a) $\mathbf{v}_4$—scaling; (b) $\mathbf{v}_3 + \mathbf{v}_2$—modified Galilean boosts; (c) $\mathbf{v}_3$—Galilean boosts; (d) $\mathbf{v}_2$—time translations; and (e) $\mathbf{v}_1$—space translations. All of these except the last were determined in Example 3.4, to which we refer to reader. The space translationally-invariant solutions are all constant, and hence trivially appear among the other solutions. Any other group-invariant solution of the Korteweg–de Vries equation can thus be found by transforming one of the solutions of Example 3.4 by an appropriate group element.

For example, the travelling wave solutions, which correspond to the symmetry group generated by $\mathbf{v}_2 + c\mathbf{v}_1 = \partial_t + c\partial_x$ can be recovered from the stationary solutions $u = f(x)$, invariant under the group generated by $\mathbf{v}_2 = \partial_t$. Referring to table (3.24), we see that

$$\mathrm{Ad}(\exp(c\mathbf{v}_3))\mathbf{v}_2 = \mathbf{v}_2 + c\mathbf{v}_1,$$

where $\mathbf{v}_3 = t\partial_x + \partial_u$ generates the one-parameter Galilean symmetry group for the Korteweg–de Vries equation. According to Proposition 3.6, if $u = f(x)$ is any stationary solution, then $\tilde{f} = \exp(c\mathbf{v}_3)f$ will be a travelling wave solution with velocity c. Using the formulas in Example 2.64, we see that

$$\tilde{f}(x, t) = f(x - ct) + c,$$

where $f(x)$ is any elliptic functon satisfying

$$f''' + ff' = 0.$$

In particular, if

$$u = f_0(x) = 3c\ \mathrm{sech}^2(\tfrac{1}{2}\sqrt{c}x + \delta) - c,$$

which, as the reader can check, is a stationary solution to the Korteweg–de Vries equation for any $c > 0$, we recover the one-soliton solution with velocity c.

Example 3.17. Finally, we look at the classification of the group-invariant solutions to the heat equation $u_t = u_{xx}$. The construction of the group-invariant solutions for each of the one-dimensional subgroups in the optimal system (3.27) proceeds in the same fashion as in Example 3.3, and we merely list the results.

(a) $$\mathbf{v}_4 + a\mathbf{v}_3 = x\partial_x + 2t\partial_t + 2au\partial_u.$$

Invariants are $y = x/\sqrt{t}$, $v = t^{-a}u$; the reduced equation is $v_{yy} + \frac{1}{2}yv_y - av = 0$, and the invariant solutions are our earlier parabolic cylinder solutions

$$u(x, t) = t^a e^{-x^2/8t}\left\{k\mathrm{U}\left(2a + \tfrac{1}{2}, \frac{x}{\sqrt{2t}}\right) + \tilde{k}\mathrm{V}\left(2a + \tfrac{1}{2}, \frac{x}{\sqrt{2t}}\right)\right\}.$$

(b) $$\mathbf{v}_2 + \mathbf{v}_6 + a\mathbf{v}_3 = 4tx\partial_x + (4t^2 + 1)\partial_t - (x^2 + 2t - a)u\partial_u.$$

Invariants are

$$y = (4t^2 + 1)^{-1/2}x, \qquad v = (4t^2 + 1)^{1/4}u \cdot \exp\left\{(4t^2 + 1)^{-1}tx^2 + \frac{a}{2}\arctan(2t)\right\}.$$

The reduced equation is

$$v_{yy} + (a + y^2)v = 0.$$

Invariant solutions are expressed in terms of parabolic cylinder functions with imaginary arguments (Abramowitz and Stegun, [1; § 19.17])

$$u(x, t) = (4t^2 + 1)^{-1/4}\left\{k\mathrm{W}\left(-\frac{a}{2}, \frac{x}{\sqrt{8t^2 + 2}}\right)\right.$$
$$\left. + \tilde{k}\mathrm{W}\left(-\frac{a}{2}, \frac{-x}{\sqrt{8t^2 + 2}}\right)\right\}\exp\left\{\frac{-tx^2}{4t^2 + 1} - \frac{a}{2}\arctan(2t)\right\}.$$

(c) $$\mathbf{v}_2 - \mathbf{v}_5 = \partial_t - 2t\partial_x + xu\partial_u.$$

Invariants are:

$$y = x + t^2, \qquad v = u\exp(-xt - \tfrac{2}{3}t^3).$$

The reduced equation is Airy's equation

$$v_{yy} = yv.$$

Solutions are written in terms of Airy functions

$$u(x, t) = \{k \operatorname{Ai}(x + t^2) + \tilde{k} \operatorname{Bi}(x + t^2)\} \exp(xt + \tfrac{2}{3}t^3).$$

The corresponding invariant solutions for $\mathbf{v}_2 + \mathbf{v}_5$ are obtained by replacing x by $-x$.

(d) $$\mathbf{v}_2 + a\mathbf{v}_3 = \partial_t + au\partial_u.$$

Invariants are x, $v = e^{-at}u$, and the reduced equation $v_{xx} = av$ leads to the solutions

$$u(x, t) = \begin{cases} k e^{at} \cosh(\sqrt{a}x + \delta), & a > 0, \\ kx + \tilde{k}, & a = 0, \\ k e^{at} \cos(\sqrt{-a}x + \delta), & a < 0. \end{cases}$$

For the two remaining subalgebras, that generated by $\mathbf{v}_1$ has only constants for its invariant solutions, which already appear in (d), and that generated by $\mathbf{v}_3$ has no invariant solutions. Thus the above solutions constitute an optimal system of group-invariant solutions to the heat equation whereby any other group-invariant solution can be found by transforming one of these solutions by a suitable group element—see page 120.

In our previous encounter with this problem, Example 3.3, we determined group invariant solutions for a couple of subgroups not appearing in the optimal system (3.27). By the general theory, these solutions can be derived from the above solutions by suitable group transformations. For example, since

$$\operatorname{Ad}[\exp(-\tfrac{1}{2}c\mathbf{v}_5)](\mathbf{v}_2 + c\mathbf{v}_1) = \mathbf{v}_2 + \tfrac{1}{4}c^2\mathbf{v}_3,$$

we could have found the travelling wave solutions by transforming the solutions invariant under $\mathbf{v}_2 + a\mathbf{v}_3$, $a = c^2/4$, by the Galilean boost $\exp(\tfrac{1}{2}c\mathbf{v}_5)$, and indeed $u = k e^{\sqrt{a}x + at} + \tilde{k} e^{-\sqrt{a}x + at}$ gets changed into the travelling wave solution $u = k + \tilde{k} e^{-c(x - ct)}$ when $a = c^2/4$.

3.4. Quotient Manifolds

In order to provide a rigorous formulation of the basic method for finding group-invariant solutions of systems of differential equations outlined in Section 3.1, we need to gain a better understanding of the geometry underlying these constructions. The concept of the quotient manifold of a smooth manifold under a regular group of transformations will provide the natural setting for all group-invariant objects. Ultimately, we will see how the reduced system of differential equations for the group-invariant solutions naturally lives on the quotient manifold. We begin by discussing this quotient manifold in general.

Let G be a local group of transformations acting on a smooth manifold M. There is an induced equivalence relation among the points of M, with x being equivalent to y if they lie in the same orbit of G. Let M/G denote the set of equivalence classes, or, equivalently, the set of orbits of G. The projection $\pi: M \to M/G$ associates to each x in M its equivalence class $\pi(x) \in M/G$, which can be identified with the orbit of G passing through x. In particular, $\pi(g \cdot x) = \pi(x)$ for any $g \in G$ such that $g \cdot x$ is defined. Conversely, given a point $w \in M/G$, $\pi^{-1}\{w\}$ will be the orbit determined by w, realized as a subset of M. The quotient space M/G has a natural topology obtained by requiring that the projection $\pi[U]$ of an open subset $U \subset M$ is open in M/G.

In general, the quotient space M/G will be an extremely complicated topological space with no readily comprehensible structure. However, if we further require G to act *regularly* on M, then we can endow M/G with the structure of a smooth manifold. If M is an m-dimensional manifold and G has s-dimensional orbits, then the quotient manifold M/G will be of dimension $m - s$.† Thus the quotient manifold construction has the effect of reducing the dimension by s, the dimension of the orbits of G.

Once we have constructed the quotient manifold, the general philosophy is that any object on M which is invariant under the action of G will have a natural counterpart on the lower-dimensional quotient manifold M/G whose properties completely characterize the original object on M. As a first example, consider a G-invariant function $F: M \to \mathbb{R}^l$. Since $F(g \cdot x) = F(x)$ whenever $g \cdot x$ is defined, F is constant along the orbits of G. Therefore there is a well-defined function $\tilde{F} = F/G: M/G \to \mathbb{R}^l$ such that $\tilde{F}(\pi(x)) = F(x)$ whenever $x \in M$. Conversely, if $\tilde{F}: M/G \to \mathbb{R}^l$ then the function $F: M \to \mathbb{R}^l$ given by $F(x) = \tilde{F}(\pi(x))$, $x \in M$ is clearly a G-invariant function on M. There is thus a one-to-one correspondence between G-invariant functions on M and arbitrary functions on M/G. Note further that in any local coordinate chart, the functions defined on M/G depend on s fewer variables than their counterparts on M merely because we have reduced the dimension of the underlying manifold by s. Thus projection to the quotient manifold has the net effect of reducing the number of degrees of freedom by s, the dimension of the orbits of the group action.

Theorem 3.18. *Let M be a smooth m-dimensional manifold. Suppose G is a local group of transformations which acts regularly on M with s-dimensional orbits. Then there exists a smooth $(m - s)$-dimensional manifold, called the* quotient manifold *of M by G and denoted M/G, together with a projection $\pi: M \to M/G$, which satisfy the following properties.*

(a) *The projection π is a smooth map between manifolds.*
(b) *The points x and y lie in the same orbit of G in M if and only if $\pi(x) = \pi(y)$.*

† It may, however, not be a Hausdorff manifold; see the subsequent discussion.

(c) *If* $\mathfrak{g}$ *denotes the Lie algebra of infinitesimal generators of the action of* G, *then the linear map*

$$d\pi \colon TM|_x \to T(M/G)|_{\pi(x)}$$

is onto, with kernel $\mathfrak{g}|_x = \{\mathbf{v}|_x \colon \mathbf{v} \in \mathfrak{g}\}$.

PROOF. As above, M/G is simply the set of all orbits of G on M. Coordinate charts on M/G are constructed using the flat local coordinate charts on M provided by Frobenius' Theorem 1.43 using the regularity of G. The local coordinates $y_\alpha = (y_\alpha^1, \ldots, y_\alpha^m)$ on such a chart U_α are such that each orbit intersects U_α in at most one slice $\mathcal{O} \cap U_\alpha = \{y_\alpha^1 = c_\alpha^1, \ldots, y_\alpha^{m-s} = c_\alpha^{m-s}\}$, the constants $c_\alpha^1, \ldots, c_\alpha^{m-s}$ uniquely determining the orbit $\mathcal{O}$. The corresponding coordinate chart V_α on M/G is defined as the set of all orbits with nonempty intersection with U_α, so

$$V_\alpha = \{w \in M/G \colon \pi^{-1}\{w\} \cap U_\alpha \neq \varnothing\}.$$

Local coordinates on V_α are determined by the slice coordinates $y_\alpha^1, \ldots, y_\alpha^{m-s}$; in other words the coordinate map $\tilde{\chi}_\alpha \colon V_\alpha \to \mathbb{R}^{m-s}$ is defined so that $\tilde{\chi}_\alpha(w) = (c_\alpha^1, \ldots, c_\alpha^{m-s})$ when $\pi^{-1}\{w\}$ intersects U_α in the slice determined by $y_\alpha^1 = c_\alpha^1, \ldots, y_\alpha^{m-s} = c_\alpha^{m-s}$. Clearly the projection $\pi \colon M \to M/G$ is smooth in these coordinates since $\pi(y_\alpha^1, \ldots, y_\alpha^m) = (y_\alpha^1, \ldots, y_\alpha^{m-s})$ for $y_\alpha \in U_\alpha$. Furthermore,

$$d\pi\left[\frac{\partial}{\partial y_\alpha^i}\right] = \begin{cases} \partial/\partial y_\alpha^i, & i = 1, \ldots, m-s, \\ 0, & i = m-s+1, \ldots, m, \end{cases}$$

so $d\pi \colon TM|_{y_\alpha} \to T(M/G)|_{\pi(y_\alpha)}$ is onto, with kernel spanned by $\partial/\partial y_\alpha^{m-s+1}, \ldots, \partial/\partial y_\alpha^m$, which is the same as the span of the infinitesimal generators $\mathfrak{g}$ of G at y_α.

The only remaining point is to prove that the overlap functions $\tilde{\chi}_\beta \circ \tilde{\chi}_\alpha^{-1}$ are smooth on the intersection $V_\alpha \cap V_\beta$ of two local coordinate charts on M/G. This is more or less clear if the corresponding flat coordinate charts U_α and U_β are sufficiently small, and intersect on M, but this latter possibility need not occur. However, a fairly straightforward argument based on the connectivity of the orbits of G can be applied here, and the result holds; the details are given in Palais, [1]. □

In order to see a little more clearly what the local coordinates on the quotient manifold mean, consider some general local coordinates $x = (x^1, \ldots, x^m)$ on M. Theorem 2.17 shows that by possibly shrinking the coordinate chart, we can find a complete set of functionally independent invariants $\eta^1(x), \ldots, \eta^{m-s}(x)$, with each orbit intersecting the chart in at most one connected component, which is a level set $\{\eta^1(x) = c_1, \ldots, \eta^{m-s}(x) = c_{m-s}\}$. The constants $c_1, \ldots, c_{m-s}$ uniquely determine the orbit, then, and hence can be chosen as new local coordinates on the quotient manifold M/G, which agree with a set of flat coordinates used in the proof of the theorem. Thus *local coordinates on the quotient manifold* M/G *are provided by a*

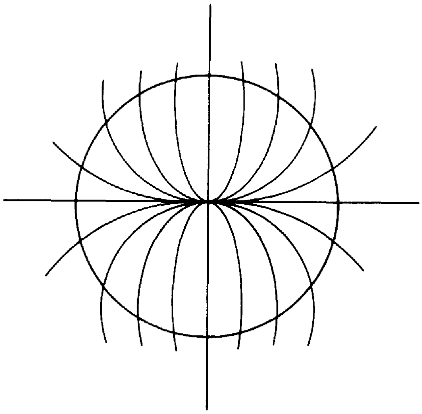

Figure 7. Quotient manifold for $\mathbb{R}^2/G^2$.

complete set of functionally independent invariants for the group action:

$$y^1 = \eta^1(x), \ldots, y^{m-s} = \eta^{m-s}(x).$$

Example 3.19. Consider the group of scale transformations

$$G^2\colon (x, y) \mapsto (\lambda x, \lambda^2 y), \qquad \lambda > 0.$$

The action is regular on $M = \mathbb{R}^2\backslash\{0\}$, the orbits being semi-parabolas $y = kx^2$ for $x > 0$ or $x < 0$ and the positive and negative y-axis. Since each orbit is uniquely determined by its point of intersection with the unit circle $S^1 = \{x^2 + y^2 = 1\}$, we can identify M/G^2 with S^1. A local coordinate on M/G^2 is provided by the group invariant y/x^2 for $x > 0$ or $x < 0$, or by x^2/y if $y > 0$ or $y < 0$, giving four overlapping coordinate charts on M/G. (A better choice of coordinate on M/G is perhaps given by the multiply-valued "angular" invariant $\theta = \arctan(y/x^2)$.) Clearly, no global coordinate chart valid for all nonzero (x, y) can be found in this case.

The same construction works for any of the two-dimensional scaling groups $G^\alpha\colon (x, y) \mapsto (\lambda x, \lambda^\alpha y)$ provided $\alpha > 0$, the "angular" invariant $\theta = \arctan(y/x^\alpha)$ providing the identification $M/G^\alpha \simeq S^1$. (The case $\alpha < 0$ is discussed in Exercise 3.14.)

As remarked earlier, one technical difficulty that can arise is that the quotient manifold M/G may not satisfy the Hausdorff separation property, and so we are naturally led to consider a more general notion of manifold than is usual. In other words, although M/G will always satisfy the requirements (a) and (b) in Definition 1.1, there may exist distinct points y and $\tilde{y}$ in M/G which cannot be "separated" by open neighbourhoods, i.e. if U is *any* neighbourhood of y and $\tilde{U}$ *any* neighbourhood of $\tilde{y}$, then $U \cap \tilde{U} \neq \varnothing$. One can develop the

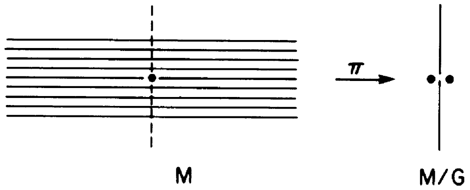

Figure 8. Non-Hausdorff quotient manifold.

entire theory of manifolds omitting the Hausdorff axiom, in which case, as shown by Palais, [1], the quotient manifold construction keeps one in the same "category". Alternatively, the approach most often taken in practice is to remove non-Hausdorff "singularities" in M/G by restricting attention to a suitably small open submanifold $\tilde{M}$ of the original manifold M with $\tilde{M}/G \subset M/G$ an open, Hausdorff submanifold. For instance, $\tilde{M}$ might be a coordinate chart on which we construct a complete set of functionally independent invariants, in which case $\tilde{M}/G$ will have global coordinates provided by these invariants, and so can be realized as an open subset of the Euclidean space $\mathbb{R}^{m-s}$.

Example 3.20. Consider the vector field $\mathbf{v} = (x^2 + y^2)\partial_x$ on $\mathbb{R}^2$. The one-parameter group G generated by $\mathbf{v}$ takes the form $(\tilde{x}, \tilde{y}) = \exp(\varepsilon\mathbf{v})(x, y)$, where $\tilde{y} = y$ and

$$\tilde{x} = \begin{cases} y\tan(\varepsilon y + \arctan(x/y)), & y \neq 0, \\ x/(1 - \varepsilon x), & y = 0. \end{cases}$$

The orbits of G consist of

(a) The origin $(0, 0)$,
(b) The horizontal lines $\{y = c\}$ with $c \neq 0$,
(c) The positive x-axis $\{y = 0, x > 0\}$,
(d) The negative x-axis $\{y = 0, x < 0\}$.

Thus G acts regularly on $M = \mathbb{R}^2\backslash\{0\}$. The quotient manifold M/G is one-dimensional and looks like a copy of the real line but with two "infinitely close" origins! Indeed, the single invariant of G is the coordinate y, each orbit not on the x-axis being uniquely determined by its vertical displacement. Thus the point in M/G determined by the horizontal lines (b) are coordinatized by $\pi(x, y) = y$, $x \neq 0$, so we can identify this part of M/G with the positive and negative real axes, corresponding to the images of the upper and lower half-planes respectively under the projection $\pi: M \to M/G$. However, there must be *two* points in M/G corresponding to the positive and negative x-axes in $\mathbb{R}^2$; we denote these by 0_+ and 0_- respectively, which can be viewed as two distinct, but infinitely close, origins of the quotient manifold