4.8. The Emden–Fowler equation is

$$\frac{d^2u}{dx^2} + \frac{2}{x}\frac{du}{dx} + u^5 = 0.$$

(a) Determine a variational problem such that the Emden–Fowler equation is the Euler–Lagrange equation thereof. (*Hint*: Multiply by x^2.)
(b) Find a simple variational scaling symmetry and use this to integrate the Emden–Fowler equation.
(Dresner, [1; p. 14], Logan, [1; p. 52], Rosenau, [1].)

4.9. Prove that the damped harmonic oscillator $m\ddot{x} + a\dot{x} + kx = 0$, $m \neq 0$, can be made into the form of an Euler–Lagrange equation by multiplying by $\exp(at/m)$. Prove that the vector field $\mathbf{v} = \partial_t - (ax/2m)\partial_x$ generates a one-parameter group of variational symmetries. Use this to integrate the equation by quadrature. How does this method compare in effort with the usual method of solving linear ordinary differential equations? (Logan, [1; p. 57]; see also Exercise 5.48.)

4.10. Consider an n-th order ordinary differential equation, on $M \subset X \times U \simeq \mathbb{R}^2$,

$$\frac{d^n u}{dx^n} = H(x, u^{(n-1)}).$$

Prove that the first integrals of this equation are the same as the invariants of the one-parameter group generated by

$$\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \cdots + u_{n-1}\frac{\partial}{\partial u_{n-2}} + H(x, u^{(n-1)})\frac{\partial}{\partial u_{n-1}}$$

acting on the jet space $M^{(n-1)}$. Find the solution to $u_{xx} = u$ using this remark. (Cohen [1; pp. 86, 99].)

4.11. Consider a variational problem of the form $\mathscr{L} = \int L(x, u_x^{-1}u_{xx})\,dx$, for $x, u \in \mathbb{R}$.
(a) Prove that the two-parameter group $(x, u) \mapsto (x, au + b)$, $a \neq 0$, is a variational symmetry group.
(b) What is the Euler–Lagrange equation for $\mathscr{L}$?
(c) Show how the Euler–Lagrange equation can be integrated twice using the translational invariance, but that the resulting second order equation is not in general scale-invariant.
(d) Do the same for the scaling symmetry.
(e) Integrate the Euler–Lagrange equation twice by using the two first integrals given by Noether's theorem, but show again that one cannot in general reduce the order any further.
(f) What happens if one uses the methods of Section 2.5 on the equation?

This shows that, whereas a one-parameter variational symmetry group will in general allow one to reduce a system of Euler–Lagrange equations by two, a two-parameter variational symmetry group does *not* in general allow one to reduce the order by four! (This problem will be taken up in a Hamiltonian framework in Chapter 6.)

4.12. Show that if $\mathscr{L}$ is a variational problem depending on a single independent and single dependent variable, and $\mathscr{L}$ is invariant under a two-parameter *abelian* group of symmetries, then one can reduce the order of the corresponding Euler–Lagrange equation by four.

4.13. (a) Suppose $\Delta = \mathsf{E}(L) = 0$ forms the Euler–Lagrange equations of some variational problem, and G is a regular group of variational symmetries (or even divergence symmetries) acting on M. Is the reduced system $\Delta/G = 0$ for the G-invariant solutions of Δ necessarily the Euler–Lagrange equations for some variational problem on the quotient manifold M/G? See Anderson and Fels, Symmetry reduction of variational bicomplexes and the principle of symmetric criticality, *Amer. J. Math.* **119 (1997) 609–670, for details.

(b) Find variational principles for the equations for the group-invariant solutions to the Korteweg–de Vries equation (2.66) found in Example 3.4, using the substitution $u = v_x$ to first put the Korteweg–de Vries equation itself into variational form.

4.14. The heat equation $u_t = u_{xx}$ cannot be put into variational form (except through some artificial tricks—see Exercises 5.36 and 5.37). Prove, however, that the equation for the scale-invariant solutions is equivalent to an Euler–Lagrange equation. (*Hint*: Look for an appropriate function to multiply it by.) Generalize to higher dimensions. (Thus reduction by a symmetry group will usually maintain a variational structure if there is one to begin with, but may also introduce a variational structure where none existed before!)

4.15. Suppose $p = 1$, $q = 2$ and we have a functional

$$\mathscr{L}[u, \tilde{u}] = \int L(x, u, \tilde{u}, u_x, \tilde{u}_x, \ldots)\, dx.$$

Consider the "hodograph" change of variables $y = \tilde{u}$, $v = u$, $\tilde{v} = x$, and let

$$\tilde{\mathscr{L}}[v, \tilde{v}] = \int \tilde{L}(y, v, \tilde{v}, v_y, \tilde{v}_y, \ldots)\, dy$$

be the transformed functional. Prove that the corresponding Euler–Lagrange equations are related by the formula

$$\mathsf{E}_u(L) = \tilde{u}_x \mathsf{E}_v(\tilde{L}), \qquad \mathsf{E}_{\tilde{u}}(L) = -u_x \mathsf{E}_v(\tilde{L}) - \mathsf{E}_{\tilde{v}}(\tilde{L}).$$

4.16. Use Noether's theorem to give an alternative proof of the Reduction Theorem 4.17 that does not directly rely on a change of variables. Apply your result to Exercises 4.8 and 4.9.

CHAPTER 5

Generalized Symmetries

The symmetry groups of differential equations or variational problems considered so far in this book have all been local transformation groups acting "geometrically" on the space of independent and dependent variables. E. Noether was the first to recognize that one could significantly extend the application of symmetry group methods by including derivatives of the relevant dependent variables in the transformations (or, more correctly, their infinitesimal generators). More recently, these "generalized symmetries"† have proved to be of importance in the study of nonlinear wave equations, where it appears that the possession of an infinite number of such symmetries is a characterizing property of "solvable" equations, such as the Korteweg–de Vries equation, which have "soliton" solutions and can be linearized either directly or via inverse scattering.

The first section of this chapter presents the basic theory of generalized vector fields and the associated group transformations, which are now found by solving the Cauchy problem for some associated system of evolution equations. The determination of the generalized symmetries of a system of differential equations is essentially the same as before, although the intervening calculations usually are far more complicated. A second approach to this problem is through the use of a recursion operator, which will generate infinite families of symmetries at once. These are presented in the second

† Some authors have mistakenly attributed the introduction of these symmetries to the work of Lie and Bäcklund, and have given the misleading misnomer of "Lie-Bäcklund transformations". (In particlar, they are *not* the same as true Bäcklund transformations, which do *not* have group properties.) We have chosen the term "generalized symmetry" rather than "Noether transformation" since the latter already has acquired several other meanings in the context of variational problems. A fuller discussion of the curious history of these symmetries appears in the notes at the end of the chapter.

section. For linear systems, recursion operators and symmetries are essentially the same objects, while for nonlinear equations, only very special "solvable" equations appear to have recursion operators.

Many of our earlier applications of geometrical symmetries remain valid for generalized symmetries. In particular, Noether's theorem now provides a complete one-to-one correspondence between one-parameter groups of generalized variational symmetries of some functional and the conservation laws of its associated Euler–Lagrange equations. Thus, one can hope to completely classify conservation laws by constructive symmetry group methods. In particular, the recursion operator interpretation of symmetry groups of linear systems leads at once to infinite families of conservation laws depending on higher order derivatives in very general situations. Recent results have further crystallized the roles of trivial symmetries and conservation laws in the Noether correspondence for totally nondegenerate systems, with the consequence that each nontrivial variational symmetry group gives rise to a nontrivial conservation law, and conversely. Under-determined systems fall under the ambit of Noether's second theorem, which relates infinite-dimensional groups of variational symmetries to dependencies among the Euler–Lagrange equations themselves. All these will be discussed in detail in the third section of this chapter.

Underlying much of our algebraic manipulations involving symmetries, conservation laws, differential operators and the like, a subject best described as the "formal variational calculus", is a certain complex, called the variational complex, doing for the variational calculus what the de Rham complex does for ordinary vector calculus on manifolds. There are three fundamental results which motivate the consideration of this complex: the first is the characterization of the kernel of the Euler operator as the space of total divergences; the second is the characterization, in Theorem 4.24, of the space of null divergences (trivial conservation laws of the second kind) as "total curls"; the third is Helmholtz's version of the inverse problem of the calculus of variations which states when a given set of differential equations forms the Euler–Lagrange equations for some variational problem. All of these results are manifestations of the exactness of the full variational complex at different stages. Although each result could be proved as it stands, the variational complex, whose fundamental role in the geometric theory of the calculus of variations is becoming more and more apparent, provides the unifying theme behind them, and the complete proof of exactness of it is not much more difficult to obtain. Thus we have devoted the last section of this chapter to a self-contained exposition of this complex, together with a much simplified proof of exactness thereof.

5.1. Generalized Symmetries of Differential Equations

Consider a vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u)\frac{\partial}{\partial u^\alpha}$$

defined on some open subset M of the space of independent and dependent variables $X \times U$. Provided the coefficient functions ξ^i, ϕ_α depend only on x and u, $\mathbf{v}$ will generate a (local) one-parameter group of transformations $\exp(\varepsilon\mathbf{v})$ acting pointwise on the underlying space M of the type discussed in detail in the previous chapters. A significant generalization of the notion of symmetry group is obtained by relaxing this geometrical assumption, and allowing the coefficient functions ξ^i, ϕ_α to also depend on derivatives of u. In this chapter, we will explore the many consequences of such an extension of the notion of symmetry.

Differential Functions

Before proceeding with the development of the theory of generalized vector fields, it is useful to introduce some notation. Throughout this chapter $M \subset X \times U$ will denote a fixed connected open subset of the space of independent and dependent variables. The prolongations $M^{(n)} \subset X \times U^{(n)}$ are then open subsets of the corresponding jet spaces, with $(x, u^{(n)}) \in M^{(n)}$ if and only if $(x, u) \in M$. We let $\mathscr{A}$ denote the space of smooth functions $P(x, u^{(n)})$ depending on x, u and derivatives of u up to some finite, but unspecified order n, defined for $(x, u^{(n)}) \in M^{(n)}$. The functions in $\mathscr{A}$ are called *differential functions* (in analogy with the differential polynomials of differential algebra). Each differential function is thus a smooth function $P: M^{(n)} \to \mathbb{R}$ for some (finite) n. If $m \geqslant n$, then $P(x, u^{(n)})$ can also be viewed as a function on $M^{(m)}$ since the coordinates $(x, u^{(n)})$ form part of the coordinates $(x, u^{(m)})$ on $M^{(m)}$. If we do not care as to precisely how many derivatives of u that P depends on, we will write $P[u] = P(x, u^{(n)})$ for P, where the square brackets will serve to remind us that P depends on x, u *and* derivatives of u. We further define $\mathscr{A}^l$ to be the vector space of l-tuples of differential functions, $P[u] = (P_1[u], \ldots, P_l[u])$, where each $P_j \in \mathscr{A}$.

Note that $\mathscr{A}$ is an algebra, meaning that we can add differential functions and multiply them together. There are also a number of fundamental differential operators on $\mathscr{A}$ which we have already encountered. Both the partial derivatives $\partial/\partial x^i$ and $\partial/\partial u_J^\alpha$ take a differential function to another differential function, but in general do not preserve the order of derivatives on which they depend. For instance, $P = u_{xxx} + xuu_x$ depends on third order derivatives, but $\partial P/\partial u = xu_x$ only depends on first order derivatives. Similarly, the total derivatives $D_j: \mathscr{A} \to \mathscr{A}$ are linear maps, with $D_j P[u]$ depending on $(n+1)$-st order derivatives when $P[u] = P(x, u^{(n)})$ depends on n-th order derivatives. Two other important operators are the total divergence Div: $\mathscr{A}^p \to \mathscr{A}$ and the Euler operator $\mathsf{E}: \mathscr{A} \to \mathscr{A}^q$ defined in the preceding chapter.

Generalized Vector Fields

Definition 5.1. A *generalized vector field* will be a (formal) expression of the form

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i[u] \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha[u] \frac{\partial}{\partial u^\alpha} \tag{5.1}$$

in which ξ^i and ϕ_α are smooth differential functions.

Thus, for example,

$$\mathbf{v} = x u_x \frac{\partial}{\partial x} + u_{xx} \frac{\partial}{\partial u}$$

is a generalized vector field in the case $p = q = 1$. For the moment, we will avoid any discussion of the precise meaning of such an object, but work with such generalized vector fields as if they were ordinary vector fields. Thus, in accordance with the prolongation formula of Theorem 2.36, we can define the *prolonged* generalized vector field

$$\mathrm{pr}^{(n)}\, \mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q} \sum_{\#J \leqslant n} \phi_\alpha^J[u] \frac{\partial}{\partial u_J^\alpha},$$

whose coefficients are determined by the formula

$$\phi_\alpha^J = D_J\left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha\right) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha, \tag{5.2}$$

with the same notation as before. Thus, in our previous example,

$$\mathrm{pr}^{(1)}\, \mathbf{v} = x u_x \frac{\partial}{\partial x} + u_{xx} \frac{\partial}{\partial u} + [u_{xxx} - (x u_{xx} + u_x) u_x] \frac{\partial}{\partial u_x},$$

the coefficient of $\partial/\partial u_x$ being computed as

$$D_x(u_{xx} - x u_x^2) + x u_x u_{xx} = D_x(u_{xx}) - D_x(x u_x) u_x.$$

Since all the prolongations of $\mathbf{v}$ have the same general expression for their coefficient functions ϕ_α^J, it is helpful to pass to the "infinite" prolongation, and take care of *all* the derivatives at once. Specifically, given a generalized vector field $\mathbf{v}$, its *infinite prolongation* (or *prolongation* for short) is the formally infinite sum

$$\mathrm{pr}\, \mathbf{v} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J} \phi_\alpha^J \frac{\partial}{\partial u_J^\alpha}, \tag{5.3}$$

where each ϕ_α^J is given by (5.2), and the sum in (5.3) now extends over *all* multi-indices $J = (j_1, \ldots, j_k)$ for $k \geqslant 0$, $1 \leqslant j_\kappa \leqslant p$. Note that if $P[u] = P(x, u^{(n)})$ is any differential function, $\operatorname{pr} \mathbf{v}(P) = \operatorname{pr}^{(n)}\mathbf{v}(P)$ is again a differential function. In particular, since P depends on only finitely many derivatives of u, only finitely many terms in the sum (5.3) are ever required to compute $\operatorname{pr} \mathbf{v}(P)$. Thus questions about the "convergence" of (5.3) never arise.

Whatever the geometrical significance of a generalized vector field (a subject we will explore in depth later in this section) the formal condition that it be an "infinitesimal symmetry" of a system of differential equations is clear.

Definition 5.2. A generalized vector field $\mathbf{v}$ is a *generalized infinitesimal symmetry* of a system of differential equations

$$\Delta_\nu[u] = \Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, l,$$

if and only if

$$\operatorname{pr} \mathbf{v}[\Delta_\nu] = 0, \qquad \nu = 1, \ldots, l, \tag{5.4}$$

for every smooth solution $u = f(x)$.

This is the direct analogue of the infinitesimal symmetry criterion in Theorems 2.31 and 2.72. According to the latter result, we need to make some nondegeneracy assumptions on the system Δ. Note that by the preceding discussion if the coefficients of $\mathbf{v}$ depend on m-th order derivatives $u^{(m)}$, then the left-hand sides of (5.4) will in general depend on $(m + n)$-th order derivatives. Thus if we are going to require (5.4) to vanish for all solutions of the system, we must impose nondegeneracy conditions not only on the system Δ itself but also on all its prolongations $\Delta^{(k)}$, $k = 0, 1, \ldots$. To avoid always restating this hypothesis, we will assume it throughout this chapter.

Blanket Hypothesis. *Unless stated otherwise, all systems of differential equations are assumed to be totally nondegenerate in the sense of Definition* 2.83; *namely they, and all their prolongations, are of maximal rank and locally solvable.*

In particular, if Δ is a normal, analytic system, as discussed in Section 2.6, then Δ satisfies this hypothesis. In this case (5.4) holds for all solutions if and only if there exist differential operators $\mathscr{D}_{\nu\mu} = \sum P_{\nu\mu}^J D_J$, $P_{\nu\mu}^J \in \mathscr{A}$, such that

$$\operatorname{pr} \mathbf{v}(\Delta_\nu) = \sum_{\mu=1}^{l} \mathscr{D}_{\nu\mu}\Delta_\mu \tag{5.5}$$

for all functions $u = f(x)$. (See Exercise 2.33.) Both (5.4) and (5.5) are useful versions of the basic infinitesimal criterion for a generalized symmetry group.

Example 5.3. Consider the heat equation

$$\Delta[u] = u_t - u_{xx} = 0.$$

The generalized vector field $\mathbf{v} = u_x \partial_u$ has prolongation

$$\operatorname{pr} \mathbf{v} = u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + u_{xxx} \frac{\partial}{\partial u_{xx}} + \cdots .$$

Thus

$$\operatorname{pr} \mathbf{v}(\Delta) = u_{xt} - u_{xxx} = D_x(u_t - u_{xx}) = D_x \Delta,$$

and hence according to (5.5) $\mathbf{v}$ is a generalized symmetry of the heat equation. More generally, any generalized vector field of the form $\mathbf{v} = \mathscr{D}[u]\partial_u$, where $\mathscr{D}$ is any linear, constant-coefficient differential operator, is easily seen to be a generalized symmetry of the heat equation.

Evolutionary Vector Fields

Among all the generalized vector fields, those in which the coefficients $\xi^i[u]$ of the $\partial/\partial x^i$ are zero play a distinguished role.

Definition 5.4. Let $Q[u] = (Q_1[u], \ldots, Q_q[u]) \in \mathscr{A}^q$ be a q-tuple of differential functions. The generalized vector field

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha[u] \frac{\partial}{\partial u^\alpha}$$

is called an *evolutionary vector field*, and Q is called its *characteristic*.

Note that according to (5.2), the prolongation of an evolutionary vector field takes a particularly simple form:

$$\operatorname{pr} \mathbf{v}_Q = \sum_{\alpha, J} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha}. \tag{5.6}$$

Any generalized vector field $\mathbf{v}$ as in (5.1) has an associated *evolutionary representative* $\mathbf{v}_Q$ in which the characteristic Q has entries

$$Q_\alpha = \phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha, \qquad \alpha = 1, \ldots, q, \tag{5.7}$$

where $u_i^\alpha = \partial u^\alpha / \partial x^i$. (See (2.48).) These two generalized vector fields determine essentially the same symmetry.

Proposition 5.5. *A generalized vector field $\mathbf{v}$ is a symmetry of a system of differential equations if and only if its evolutionary representative $\mathbf{v}_Q$ is.*

PROOF. According to the alternative form (2.50) of the prolongation formula,

$$\operatorname{pr} \mathbf{v}[\Delta_\nu] = \operatorname{pr} \mathbf{v}_Q[\Delta_\nu] + \sum_{i=1}^{p} \xi^i D_i \Delta_\nu. \tag{5.8}$$

The second set of terms vanishes on all solutions to Δ, so the proposition follows easily from Definition 5.2. □

For example, the symmetry $u_x \partial_u$ of the heat equation is just the evolutionary representative of the translational symmetry generator $-\partial_x$. Similarly, the Galilean generator $-2t\partial_x + xu\partial_u$ has evolutionary representative $(2tu_x + xu)\partial_u$, which, as the reader can check, is also a symmetry of the heat equation.

We will distinguish between the symmetries discussed in Chapter 2 and the true generalized symmetries here by referring to the former as *geometric symmetries* since they act geometrically on the underlying space $X \times U$. (Another suggestive name in use is *point transformations*.) According to the previous example, every geometric symmetry has an evolutionary representative with characteristic depending on at most first order derivatives. However, not every first order evolutionary symmetry comes from a geometrical group of transformations; the characteristic must be of the specific form (5.7), with ξ^i and ϕ_α depending only on x and u.

Equivalence and Trivial Symmetries

Note that if $\mathbf{v}_Q$ is an evolutionary vector field and the q-tuple Q vanishes on solutions of the system Δ then by (5.6) all the coefficients of the prolongation $\operatorname{pr} \mathbf{v}_Q$ also vanish on all solutions. Therefore $\mathbf{v}_Q$ is automatically a generalized symmetry of the system Δ. Such symmetries are called *trivial*, and we are primarily interested in nontrivial symmetries of the system. A generalized symmetry is *trivial* if its evolutionary form is. Two generalized symmetries $\mathbf{v}$ and $\tilde{\mathbf{v}}$ are called *equivalent* if their difference $\mathbf{v} - \tilde{\mathbf{v}}$ is a trivial symmetry of the system. This induces an equivalence relation on the space of generalized symmetries of the given system; moreover, we will classify symmetries up to equivalence so by a *symmetry* of the system we really mean a whole equivalence class of generalized symmetries, each differing from the other by a trivial symmetry. For example, in the case of the heat equation, the time translation symmetry ∂_t, its evolutionary form $-u_t\partial_u$ and the generalized symmetry $-u_{xx}\partial_u$ are all equivalent, and for all practical purposes determine the self-same symmetry group.

Example 5.6. Let's look at the case of a system of first order ordinary differential equations

$$\frac{du^\alpha}{dt} = P_\alpha(t, u), \qquad \alpha = 1, \ldots, q. \tag{5.9}$$

Suppose we are interested in finding generalized symmetries

$$\mathbf{v} = \tau(t, u, u_t, \ldots)\frac{\partial}{\partial t} + \sum_{\alpha=1}^{q} \phi_\alpha(t, u, u_t, \ldots)\frac{\partial}{\partial u^\alpha}.$$

We simplify the computation by replacing $\mathbf{v}$ by its evolutionary representative

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha(t, u, u_t, \ldots)\frac{\partial}{\partial u^\alpha}, \quad \text{where} \quad Q_\alpha = \phi_\alpha - \tau u_t^\alpha.$$

Moreover, for solutions $u = f(t)$, the system (5.9) provides expressions for the derivatives du^α/dt solely in terms of u and t. Differentiating (5.9) will similarly lead to expressions for all higher order derivatives $d^k u^\alpha/dt^k$ in terms of just u and t. Under the above notion of equivalence, we are allowed to substitute these expressions into Q, leading to an equivalent vector field of the simple form

$$\mathbf{w} = \sum_{\alpha=1}^{q} \tilde{Q}_\alpha(t, u)\frac{\partial}{\partial u^\alpha}.$$

In other words, for systems of first order ordinary differential equations, any generalized symmetry is always equivalent to a geometric symmetry in which only the dependent variables are transformed.

Computation of Generalized Symmetries

In principle, the computation of generalized symmetries of a given system of differential equations proceeds in the same way as the earlier computations of geometric symmetries, but with the following added features: First we should put the symmetry in evolutionary form $\mathbf{v}_Q$—this has the effect of reducing the number of unknown functions from $p + q$ to just q, while simultaneously simplifying the computation of the prolongation $\operatorname{pr} \mathbf{v}_Q$. One must then *a priori* fix the order of derivatives on which the characteristic $Q(x, u^{(m)})$ may depend. The basic trade-off in this regard is that the more derivatives of u that Q depends on, the more possible generalized symmetries there are to be found, but, on the other hand, the more tedious and time-consuming it will be to solve the ensuing symmetry equations. Of course, such an approach cannot hope to find all generalized symmetries (unless one can treat evolutionary vector fields depending on all orders of derivatives $u^{(m)}$ simultaneously) but taking m not too large will often yield important information on the general form of the symmetries. Finally one must deal with the occurrence of trivial symmetries; the easiest way to handle these is to eliminate any superfluous derivatives in Q by substitution using the prolongations of the system, as was done in the preceding example.

Example 5.7. Consider the elementary nonlinear wave equation

$$u_t = uu_x.$$

Suppose $\mathbf{v}_Q = Q[u]\partial_u$ is a generalized symmetry in evolutionary form. Note that we can replace any t-derivatives of u occurring in Q by their corre-