12

Renormalization of gauge theories

It is important to show that renormalization of a gauge theory can be accomplished without violating its gauge invariance. Gauge invariance is physically important; among other things it is used (via the Ward identities) to show that the unphysical states decouple ('t Hooft (1971a)).

In Chapter 9 we considered the case that the basic Lagrangian of a theory is invariant under a global symmetry, as opposed to a gauge symmetry, such as we will be investigating in this chapter. We showed that the counterterm Lagrangian is also invariant under the symmetry. Suppose now that the basic Lagrangian is invariant under a gauge symmetry. One might suppose that the counterterms are also invariant under the symmetry, just as for a global symmetry. This is not true, however, since the introduction of gauge fixing (as explained in Sections 2.12 and 2.13) destroys manifest gauge invariance of the Lagrangian. One might instead point out that the theory with gauge fixing is BRS invariant and deduce that the counterterms are BRS invariant. This deduction is false. To see this, we recall that an ordinary internal symmetry relates Green's functions with certain external fields to other Green's functions differing only by change of symmetry labels. However, BRS symmetry relates a field to a composite field (2.13.1). This wrecks the proof of BRS invariance of counterterms except in an abelian theory, where the Faddeev–Popov ghost is a free field.

Before treating the non-abelian case, let us examine an abelian theory, QED. The Lagrangian is

$$\begin{aligned}\mathscr{L} = & -\tfrac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(\mathrm{i}\not{\partial} + e_{\mathrm{R}}\not{A} - M)\psi \\ & -\tfrac{1}{4}(Z_3 - 1)F_{\mu\nu}^2 + (Z_2 - 1)\bar{\psi}(\mathrm{i}\not{\partial} + e_{\mathrm{R}}\not{A} - M)\psi \\ & - Z_2\bar{\psi}(M_0 - M)\psi - (1/2\xi)\partial\cdot A^2. \qquad (12.0.1)\end{aligned}$$

Here e_{R} and M are the renormalized coupling and mass, while A_μ, ψ and $\bar{\psi}$ are the renormalized fields. We have chosen to include only counterterms invariant under the gauge transformation

$$\left.\begin{aligned}\psi &\to \mathrm{e}^{\mathrm{i}e_{\mathrm{R}}\omega}\psi, \\ \bar{\psi} &\to \mathrm{e}^{-\mathrm{i}e_{\mathrm{R}}\omega}\bar{\psi}, \\ A_\mu &\to A_\mu + \partial_\mu\omega.\end{aligned}\right\} \qquad (12.0.2)$$

https://doi.org/10.1017/CBO9780511622656.012 Published online by Cambridge University Press

A special case of our later results will show that coefficients of the possible gauge non-invariant counterterms actually vanish. (We will write $e_R = \mu^{2-d/2}e$ if dimensional regularization is used, with e being the dimensionless renormalized charge.) In terms of unrenormalized quantities $A^0_\mu \equiv Z_3^{1/2}A_\mu$, $\psi_0 \equiv Z_2^{1/2}\psi$, $e_0 \equiv Z_3^{-1/2}e_R$, and $\xi_0 \equiv \xi Z_3$, we have

$$\mathscr{L} = -\tfrac{1}{4}(\partial_\mu A_{0\nu} - \partial_\nu A_{0\mu})^2 + \bar{\psi}_0(\mathrm{i}\not\partial + e_0\not A_0 - M_0)\psi_0 - (1/2\xi_0)\partial\cdot A_0^2. \tag{12.0.3}$$

The counterterm for the photon–electron vertex is, from (12.0.1),

$$(Z_2 - 1)e_R\bar{\psi}\not A\psi, \tag{12.0.4}$$

which has a coefficient proportional to the wave-function renormalization. It is easy to verify that to one-loop order this result is correct. Our later results prove it to all orders.

If the non-abelian theory of (2.11.7) and (2.11.8) were renormalized by gauge-invariant counterterms, then the same relation of the $\bar{\psi}\not A\psi$ counterterm to the quark field strength renormalization Z_2 would hold. It is easily verified by explicit calculation that this is false. As we pointed out above, the reason that the counterterms need not be BRS invariant is that the BRS invariance is not a symmetry that relates elementary fields to elementary fields. Even so, the counterterms are such that the Lagrangian is BRS invariant after renormalization, but under a renormalized BRS symmetry, as we will show in this chapter.

One result will be that whereas the gauge transformation of a fermion field in QED is given by $\delta\psi = \mathrm{i}e_R\omega\psi$, with e_R being the renormalized charge, the BRS transformation of a fermion field in QCD is $\delta\psi = \mathrm{i}g_R c^a t^a X\psi\delta\lambda$, where X is a divergent renormalization factor. The composite operator $\delta\psi$ is finite.

There are a number of strategies for proving renormalizability. Before explaining them, let us remark that the aim is to show that a finite theory exists which has gauge-invariance properties. The gauge invariance is exhibited by Ward identities. It is possible to choose counterterms in such a way that gauge invariance does not hold. For example, we could add a term $(A^\mu A_\mu)^2$ to the QED Lagrangian. Since the coefficient of this term is dimensionless, we obtain a finite theory by adding appropriate extra infinite counterterms. But this theory is not gauge invariant. This implies, for example, that negative metric states do not decouple from physical processes, and the theory is unphysical. The proofs state that it is possible to choose counterterms so that gauge invariance holds.

Several approaches can be distinguished:

1. *Invariant regulator.* Use an ultra-violet regulator that does not break gauge symmetry, for example dimensional regularization. Then Ward identities are true when the Lagrangian is given by (2.11.7), (2.12.5), and (2.12.9). Allow all parameters and fields to get renormalized. The theorems to be proved are that this is sufficient to make the theory finite when the regulator is removed. This is the traditional approach. The advantages center around the manifest preservation of gauge invariance. The disadvantages are that chiral symmetries cannot be regulated gauge invariantly; this symptomizes the fact that not all chiral symmetries can be preserved after quantization – see Chapter 13.

2. *Gauge invariant regulator $+MS$.* Let us again use dimensional regularization (or another gauge-invariant cut-off). But now let us choose to renormalize each separate graph by the forest formula or by the recursive method (as given in Chapter 5). We do not explicitly constrain the counterterms to satisfy gauge invariance; so in general we have violated gauge invariance. But if we use minimal subtraction then the counterterms are gauge invariant. The reason is simple: since we will prove that we can renormalize the theory gauge invariantly, the lowest-order counterterm that is not gauge invariant must be finite, after summing over all graphs of this order. But in minimal subtraction the only finite counterterm is zero. This method is of great use when renormalizing the complicated non-local operators that appear in generalized operator product expansions (Collins & Soper (1981)). We can renormalize the graphs without explicitly investigating the Ward identities. The disadvantage is that the method is closely tied to a specific renormalization prescription.

3. *Non-invariant regulator plus non-invariant counterterms.* One can use any regulator and adjust overall counterterms, if possible, to satisfy all the Ward identities (Piguet & Rouet (1981), Symanzik (1970a), 't Hooft (1971a), and Piguet & Sibold (1982a, b, c)).

There are two different forms of the Ward identities, either of which may be used. There are the Ward identities derived in Section 2.13 for Green's functions, and there are the ones for the 1PI graphs, as derived by Lee (1976). Our approach will use the BRS identities for Green's functions together with a combination of approaches 1 and 2. It is based on the treatment of Brandt (1976). Most other treatments have used the identities for the 1PI graphs.

We will restrict our attention to the simplest theories, like QCD. More general cases – with chiral or supersymmetries – are not treated here. See Chapter 13 for references.

12.1 Statement of results

For simplicity we will mainly treat one case: a theory of a gauge field coupled to a Dirac field, with the gauge-fixing term being the usual one, $-\sum_a (\partial\cdot A^a)^2/(2\xi)$. We assume that the gauge group is simple (in the mathematical sense); physically, this implies that there is only one independent gauge coupling. With a $U(1)$ gauge group and one or more Dirac fields, this theory is quantum electrodynamics. If the gauge group is $SU(3)$ and the matter fields are in the triplet representation, then we have quantum chromodynamics.

The result to be proved is that the Green's functions are made finite by renormalizing the values of all the parameters in the basic Lagrangian (2.11.7). These parameters are the gauge coupling, the fermion masses M, the field strength renormalizations, and the gauge-fixing parameter.

The resulting Lagrangian expressed in terms of renormalized fields is

$$\mathscr{L} = -\tfrac{1}{4}Z_3 G^{a2}_{\mu\nu} + \sum_i Z^{(i)}_2 \bar{\psi}_i(\mathrm{i}\not{D} - M^{(i)}_0)\psi_i \\ -(1/2\xi)\partial\cdot A^{a2} + \tilde{Z}\partial_\mu \bar{c}^a D^\mu C_a. \tag{12.1.1}$$

We have allowed the fermion fields to be in several irreducible representations of the gauge group labelled by i. There are separate field-strength renormalizations $Z^{(i)}_2$ and bare masses $M^{(i)}_0$ for each representation. The covariant derivative is

$$D_\mu\psi = (\partial_\mu + \mathrm{i}g_0 A^a_{0\mu}t^a)\psi \\ = (\partial_\mu + \mathrm{i}g_\mathrm{R} X\tilde{Z}^{-1}t^a A^a_\mu)\psi, \tag{12.1.2}$$

where g_0 and g_R are the bare and renormalized couplings. (With dimensional regularization we write $g_\mathrm{R} = \mu^{2-d/2}g$, with g dimensionless.) Following Lee (1976) we write the bare coupling as

$$g_0 = \frac{X}{\tilde{Z}Z_3^{1/2}}g_\mathrm{R}, \tag{12.1.3}$$

so that the coupling of the gauge field to the ghost is Xg_R:

$$\tilde{Z}\partial_\mu\bar{c}^a D^\mu c_a = \tilde{Z}\partial_\mu\bar{c}^a\partial^\mu c_a + g_\mathrm{R}Xc_{abc}(\partial_\mu\bar{c}^a)c^b A^\mu_c. \tag{12.1.4}$$

The field strength tensor is

$$G^a_{\mu\nu} = Z_3^{-1/2}G^a_{(0)\mu\nu} \\ \equiv Z_3^{-1/2}(\partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu} - g_0 c_{abc}A^b_{0\mu}A^c_{0\nu}) \\ = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_\mathrm{R}X\tilde{Z}^{-1}c_{abc}A^b_\mu A^c_\nu. \tag{12.1.5}$$

Observe that in accordance with the results to be proved, the coefficient of the gauge-fixing term $(\partial\cdot A)^2$ is finite, when the renormalized field is used.

This gives a gauge-fixing term $-(\partial\cdot A_0)^2/(2\xi_0)$, when expressed in terms of the bare field, with

$$\xi_0 = Z_3\xi. \tag{12.1.6}$$

The main theorem to be proved is:

Theorem 1. The renormalizations

$$M_0^{(i)},\ X,\ \tilde{Z},\ Z_2^{(i)},\ \text{and } Z_3 \tag{12.1.7}$$

can be chosen so that Green's functions of A, ψ, $\bar{\psi}$, c, and $\bar{c}$ are finite.

To prove this result we will use the Ward identities for BRS invariance. These involve a number of composite fields, which we also need to prove finite. The counterterms for these composite operators are related to the basic counterterms (12.1.7). We will prove:

Theorem 2.

$$\delta_{\mathrm{BRS}}(\text{renormalized field})/\delta\lambda_{\mathrm{R}}$$

is finite. That is, its Green's functions with any number of renormalized fields are finite. We define the renormalized BRS transformation $\delta_{\mathrm{BRS}}/\delta\lambda_{\mathrm{R}}$ as follows:

(1) Let the BRS transformation δ_{BRS} be defined by (2.13.1), (2.13.2) with g and ξ replaced by g_0 and ξ_0 and with the fields replaced by unrenormalized fields (i.e., $A \to A_0$, etc). Then the Lagrangian (12.1.1) is BRS invariant.

(2) Define $\delta\lambda_{\mathrm{R}} = \delta\lambda Z_3^{-1/2}\tilde{Z}^{-1/2}$. Then

$$\left.\begin{aligned}
\delta_{\mathrm{R}}\psi &= \delta_{\mathrm{BRS}}\psi/\delta\lambda_{\mathrm{R}} = -\mathrm{i}g_{\mathrm{R}}Xt^a\psi c_a,\\
\delta_{\mathrm{R}}A_\mu^a &= \delta_{\mathrm{BRS}}A_\mu^a/\delta\lambda_{\mathrm{R}} = \partial_\mu c_a\tilde{Z} + g_{\mathrm{R}}c_{abc}c^bA_\mu^cX,\\
\delta_{\mathrm{R}}c^a &= \delta_{\mathrm{BRS}}c^a/\delta\lambda_{\mathrm{R}} = -\tfrac{1}{2}g_{\mathrm{R}}Xc_{abc}c^bc^c = -\tfrac{1}{2}g_{\mathrm{R}}Xc\wedge c,\\
\delta_{\mathrm{R}}\bar{c}^a &= \delta_{\mathrm{BRS}}\bar{c}^a/\delta\lambda_{\mathrm{R}} = \frac{1}{\xi}\partial\cdot A^a.
\end{aligned}\right\} \tag{12.1.8}$$

Renormalized Ward identities follow from the unrenormalized ones by multiplication by $\delta\lambda/\delta\lambda_{\mathrm{R}}$. The operators appearing in them are finite, because of Theorem 2.

Certain auxiliary operators are useful for reasons which only become apparent in proving Theorems 1 and 2.

$$\mathcal{O}_a = \tilde{Z}\,\Box\, c^a + \partial^\mu(c^bA_\mu^c)g_{\mathrm{R}}Xc_{abc}, \tag{12.1.9}$$

$$\mathcal{B}_{a\mu} = (\tilde{Z}/X - 1)\partial_\mu\bar{c}^a/g_{\mathrm{R}} + c_{abc}\bar{c}_bA_\mu^c, \tag{12.1.10}$$

$$\bar{c}\wedge\bar{c} = c_{abc}\bar{c}_b\bar{c}_c. \tag{12.1.11}$$

The operator $\mathcal{O}_a$ is zero by the ghost equations of motion. We will prove:

Theorem 3. Green's functions of $\mathcal{O}_a$, $\mathcal{B}_{a\mu}$, and $\bar{c} \wedge \bar{c}$ with any number of basic fields are finite.

Theorem 4. Green's functions of $\mathcal{B}_{a\mu}$ with $\delta_{\mathrm{R}}\phi$ and any number (greater than zero) of basic fields are finite, where ϕ is any basic field.

Theorem 5. Green's functions of $\bar{c} \wedge \bar{c}$ with one or two $\delta_{\mathrm{R}}\phi$'s and any number (bigger than zero) of basic fields are finite.

The last few results have no intuitive appeal. They will be needed as part of an inductive proof of the important Theorem 1. We will also find it convenient to use CPT invariance of the theory (after dimensional regularization). Now reversal of one time and one space component is equivalent to reversal of components 0, 1, 2, and 3 with a spatial rotation. So to obtain the TP part of CPT, we need only consider reversal of the 0 and 1 components only. Therefore, we define

$$\theta^\mu_\nu = \begin{cases} -1, & \text{if } \mu = \nu = 0 \text{ or } 1, \\ +1, & \text{if } \mu = \nu \geq 2, \\ 0, & \text{otherwise.} \end{cases} \tag{12.1.12}$$

Let the fields transform under CPT as

$$\left.\begin{aligned} \psi(x) &\to \gamma^1 \bar{\psi}^{\mathrm{T}}(\theta x), \\ A^a(x) &\to A^a(\theta x), \\ c^a(x) &\to c^a(\theta x), \\ \bar{c}^a(x) &\to \bar{c}^a(\theta x). \end{aligned}\right\} \tag{12.1.13}$$

Then the theory is CPT invariant. (We use γ-matrices in which $\gamma^0 = \gamma^{0\mathrm{T}} = \gamma^{0*}$, and $\gamma^{\mu\dagger} = \gamma_\mu$.) Notice that the ghost field, c_a, transforms to itself, rather than to the antighost field, $\bar{c}^a$, even though these fields might be regarded as complex conjugate fields.

12.2 Proof of renormalizability

12.2.1 *Preliminaries*

The Ward identities of a gauge theory provide relations between different Green's functions. However, the identities mostly relate Green's functions of elementary fields to Green's functions containing the composite fields listed in (12.1.8)–(12.1.11). However, to prove renormalizability, we actually need relations between Green's functions of elementary fields only. Consequently proofs tend to be rather indirect and long.

The following references contain a representative selection of the proofs in the literature: 't Hooft (1971a, b), Taylor (1971), Slavnov (1972), 't Hooft & Veltman (1972b), Lee & Zinn-Justin (1972), Becchi, Rouet & Stora (1975), Lee (1976), Itzykson & Zuber (1980), and Piguet & Rouet (1981). A great simplification was introduced by the discovery by Becchi *et al.* of their symmetry. However, the proofs still are mostly rather inexplicit. The proof to be given in this section gives all the steps needed to go from the basic Ward identities to the relations between the counterterms. The method follows that given by Brandt (1976) and Cvitanovic (1977). A point at which many proofs became rather inexplicit turns out in this method to be the point at which the operator $\mathscr{B}_{a\mu}$ (defined in (12.1.10)) makes its appearance. It is an unobvious operator to use, but its use is essential to completing the proof that the gauge coupling to matter fields is the same as the self-coupling of the gauge field.

The proof is by induction on the number, N, of loops. We assume that all graphs with less than N loops have been successfully renormalized by counterterms of the form implied by the Lagrangian (12.1.1). We also require that Green's functions of the composite operators considered in Theorems 2 to 5 are also finite up to $N-1$ loops if the indicated counterterms are used. The induction starts with tree graphs, which need no counterterms.

Our strategy is as follows:

(1) At each order below N loops we have values of the five independent renormalizations $\tilde{Z}$, Z_2, Z_3, M_0, and X. For each 1PI Green's function with an overall divergence a value of the overall counterterm is hence computed at each order less than N. Partition this counterterm into a set of counterterms to cancel the overall divergences of the individual graphs for the Green's function.

(2) Compute N-loop contributions to $\tilde{Z}$, Z_2, Z_3, M_0, and X by imposing renormalization conditions on certain Green's functions. The Ward identities will be true, but it is not immediate that the many other 1PI graphs are finite. This is done at step (3).

(3) Using these Ward identities show that Theorems 1 to 5 hold for N-loop graphs.

Step (1) is technical but important. Its use is that, in order to say that the only divergence of an N-loop graph is the overall divergence, we must have subtracted off its subdivergences. However, for individual graphs the constraints imposed by gauge invariance do not hold. Consider, for example, the two-loop graph, Fig. 12.2.1. Its self-energy subgraph needs a counterterm of the form $A_1 g_{\mu\nu} k^2 - B_1 k_\mu k_\nu$.

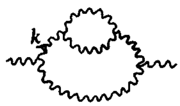


Fig. 12.2.1.

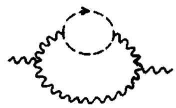


Fig. 12.2.2.

Similarly the subgraph of Fig. 12.2.2 has a counterterm $A_2 g_{\mu\nu} k^2 - B_2 k_\mu k_\nu$. As we will see, the Ward identities imply that the total self-energy counterterm is obtained from the term $-\frac{1}{4}(Z_3 - 1)(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$ in $\mathscr{L}$. It follows that $A_1 + A_2 = B_1 + B_2$. However, the relation is false for the separate graphs, i.e., we have $A_1 \neq B_1$, $A_2 \neq B_2$.

At steps (2) and (3), where we discuss the N-loop counterterms, we then know that the only divergences are overall ones. Moreover, we know this without having to check on intricate series of cancellations between different graphs. But for the purposes of finding the constraints imposed by the Ward identities on the N-loop counterterms, it is convenient to consider a single overall counterterm for the sum of all N-loop graphs for a given 1PI Green's function. Having obtained these constraints, we decompose the N-loop counterterms into individual portions for each graph; this enables us to continue the induction at the next order.

If we use minimal subtraction, the counterterms can be obtained graph-by-graph without worrying about the constraints imposed by Ward identities. These constraints will be satisfied automatically. For example, in a general renormalization prescription the counterterms to the subgraphs in Figs. 12.2.1 and 12.2.2 have the form

$$A_i = g^2[a_i/(d-4) + a_i'],$$
$$B_i = g^2[b_i/(d-4) + b_i'],$$

where a_i' and b_i' are finite and depend on the renormalization prescription. The Ward identities tell us that we can renormalize the divergences by a transverse counterterm

$$(a_1 + a_2)/(d-4) + a_1' + a_2' = (b_1 + b_2)/(d-4) + b_1' + b_2'.$$

Evidently $a_1 + a_2 = b_1 + b_2$, and $a_1' + a_2' = b_1' + b_2'$. The first equation must always be satisfied, the second must be imposed by choice of renormalization prescription. Minimal subtraction with $a_1' = a_2' = b_1' = b_2' = 0$ always satisfies these equations.

12.2.2 *Choice of counterterms*

We now assume that step (1) has been carried out. The next step is to pick a set of 1PI Green's functions to fix $\tilde{Z}$, Z_2, Z_3, M_0, and X. This is somewhat arbitrary, but our choice will determine the form of the rest of the proof. We choose the following set:

(1) The fermion self-energy has divergences proportional to $\not{p}$ and to 1. These are cancelled by counterterms in Z_2 and M_0.
(2) $\tilde{Z}$ is chosen to cancel the p^2 divergence in the ghost self-energy.
(3) X is chosen to make the ghost–gluon coupling finite as far as the c_{abc} part is concerned.
(4) Z_3 is chosen to cancel the part of the divergence of the gluon's self-energy that is proportional to $-g_{\mu\nu}k^2 + k_\mu k_\nu$.

Next we will examine the Green's functions used in Theorems 1 to 5 to check for possible divergences at N-loop order. Since all divergences at lower order are cancelled, the possible remaining divergences are overall divergences of N-loop 1PI Green's functions. These are Green's functions with either elementary external lines or with insertions of the various composite operators we use. The dimension of a 1PI Green's function must be zero or greater in order that it have a non-negative degree of divergence and thus be potentially divergent. The contributions to such a Green's function are (a) N-loop basic graphs, (b) graphs with counterterms to cancel subdivergences, (c) an overall N-loop counterterm derived from our knowledge of $\tilde{Z}$, Z_2, Z_3, M_0, and X. We must prove that the sum of these contributions is finite.

12.2.3 *Graphs with external derivatives*

There is a derivative on a ghost line where it exits from an interaction. Thus the 1PI graphs of Fig. 12.2.3 have negative degree of divergence even

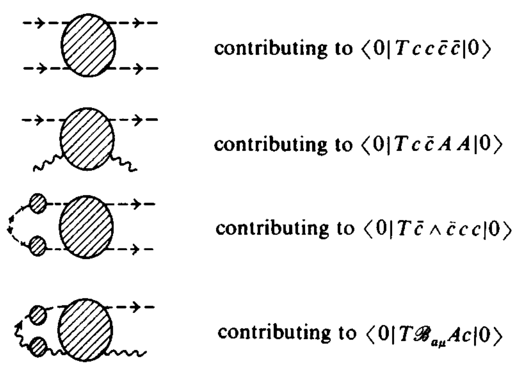


Fig. 12.2.3. Graphs with negative degree of divergence and non-negative dimension.

though their dimension is zero. Subdivergences are cancelled and no overall counterterms are present, so the corresponding 1PI Green's functions are finite.

The same argument shows that the ghost self-energy needs no mass counterterm, but only a field-strength renormalization $\tilde{Z}$.

12.2.4 *Graphs finite by equations of motion*

Consider Green's functions of $\mathcal{O}_a$ with basic fields. These are finite. For example,

$$\begin{aligned}\langle 0|T\mathcal{O}_a(x)\bar{c}^b(y)|0\rangle &= \tilde{Z}\,\Box_x\langle 0|Tc_a(x)\bar{c}^b(y)|0\rangle \\ &\quad + g_R X c_{adc}\frac{\partial}{\partial x^\mu}\langle 0|Tc_d(x)A^{c\mu}(x)\bar{c}^b(y)|0\rangle \\ &= -\mathrm{i}\delta^{ab}\delta^{(d)}(x-y).\end{aligned} \tag{12.2.1}$$

The only graphs for the left-hand side with an N-loop 1PI subgraph are of the form Fig. 12.2.4(*a*) and (*b*). The graph (*a*) has a ghost self-energy made finite by its wave-function counterterm. Graph (*b*) needs a counterterm in $\mathcal{O}_a$ proportional to $\Box c_a$. Such a counterterm is graph (*c*), which has the N-loop contribution to the $\tilde{Z}\Box c_a$ term in $\mathcal{O}_a$. Finiteness of (12.2.1) shows that this is the correct counterterm. Similarly, the other potentially divergent Green's function of $\mathcal{O}_a$, viz.,

$$\langle 0|T\mathcal{O}_a\bar{c}^b A^c|0\rangle$$

is finite.

Fig. 12.2.4. Overall-divergent graphs for Green's functions of $\mathcal{O}_a$

12.2.5 *Gluon self-energy*

We have the Ward identity

$$\begin{aligned}0 &= \delta\langle 0|T\partial\cdot A^a(x)\bar{c}^b(y)|0\rangle/\delta\lambda_R \\ &= (1/\xi)\langle 0|T\partial\cdot A^a(x)\partial\cdot A^b(y)|0\rangle - \langle 0|T\mathcal{O}^a(x)\bar{c}^b(y)|0\rangle \\ &= \frac{1}{\xi}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\langle 0|TA^{a\mu}(x)A^{b\nu}(y)|0\rangle + \mathrm{i}\delta^{(d)}(x-y).\end{aligned} \tag{12.2.2}$$

The second term involves $\mathcal{O}_a$ because of our definition of $\delta_R A^a_\mu$. In the last line we used (12.2.1) on $\mathcal{O}_a$, and remembered that Green's functions of $\partial\cdot A$ have the derivatives outside the time-ordering, by definition.

Fig. 12.2.5. Gluon self-energy.

The only possible divergences in $\langle 0|TA_\mu^a(x)A_\nu^b(y)|0\rangle$ are in the N-loop self-energy graphs (Fig. 12.2.5), and they consist of terms proportional to $g_{\mu\nu}M^2$ or to $k_\mu k_\nu$. Remember that we used Z_3 to cancel any divergence proportional to $g_{\mu\nu}k^2 - k_\mu k_\nu$. Both the remaining divergences are absent, by (12.2.2). Thus no renormalization of the gauge parameter is needed, and no gluon mass is needed. The form of (12.2.2) is just as in QED. It implies that the gluon self-energy is purely transverse, so that the longitudinal part of the propagator is unchanged by higher-order correction. Hence, we can write:

$$\int d^dx e^{iq\cdot x}\langle 0|TA_\mu^a(x)A_\nu^b(0)|0\rangle = \frac{i\delta_{ab}}{q^2+i\varepsilon}\left[\frac{-g_{\mu\nu}+q_\mu q_\nu/q^2}{1+\Pi(q^2)} - \frac{\xi q_\mu q_\nu}{q^2}\right]. \tag{12.2.3}$$

12.2.6 *BRS transformation of* A_μ^a.

Consider the Ward identity

$$\begin{aligned} 0 &= \delta\langle 0|TA_\mu^a(x)\bar{c}^b(y)|0\rangle/\delta\lambda_R \\ &= (1/\xi)\langle 0|TA_\mu^a(x)\partial\cdot A^b(y)|0\rangle - \langle 0|T\delta_R A^a(x)\bar{c}^b(y)|0\rangle. \end{aligned} \tag{12.2.4}$$

The first term we have just proved to be finite. The only potentially divergent graphs for the second term are shown in Fig. 12.2.6. Equation (12.2.4) shows their sum to be finite.

The $\bar{c}(y)A(z)$ Green's function (Fig. 12.2.7) of $\delta_R A$ is also possibly divergent (logarithmically). But we have:

$$\begin{aligned} &\frac{\partial}{\partial x^\mu}\langle 0|T\delta_R A^{a\mu}(x)\bar{c}^b(y)A_\nu^c(z)|0\rangle \\ &\quad = \langle 0|T\mathcal{O}^a(x)\bar{c}^b(y)A_\nu^c(z)|0\rangle \\ &\quad = -i\delta^{(d)}(x-y)\delta^{ab}\langle 0|A_\nu^c(z)|0\rangle \\ &\quad = 0. \end{aligned} \tag{12.2.5}$$

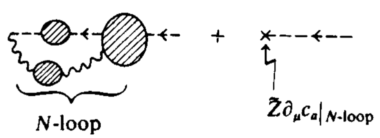


Fig. 12.2.6. Divergent graphs for right-hand side of (12.2.4).

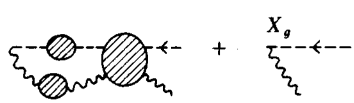


Fig. 12.2.7. Graphs for $\langle 0|T\delta_R A^{a\mu}(x)\bar{c}^b(y)A_\nu^c(z)|0\rangle$.

Applying $\partial/\partial x^\mu$ is equivalent to multiplying by k_μ in momentum space. Since the divergence is at most logarithmic, (12.2.5) shows that there is no divergence at all.

12.2.7 *Gluon self-interaction*

Now

$$\begin{aligned} 0 &= \delta\langle 0|TA^a_\mu(x)A^b_\nu(y)\bar{c}^c(z)|0\rangle/\delta\lambda_R \\ &= \langle 0|T\delta_R A^a_\mu(x)A^b_\nu(y)\bar{c}^c(z)|0\rangle \\ &\quad - \langle 0|TA^a_\mu(x)\delta_R A^b_\nu(y)\bar{c}^c(z)|0\rangle \\ &\quad - \langle 0|TA^a_\mu(x)A^b_\nu(y)\delta_R\bar{c}^c(z)|0\rangle \\ &= \text{finite} + \frac{1}{\xi}\frac{\partial}{\partial z^\lambda}\langle 0|TA^a_\mu A^b_\nu A^{c\lambda}|0\rangle, \end{aligned} \tag{12.2.6}$$

where we used the previous result. The three-gluon vertex is linearly divergent, and we have a counterterm equal to $(Z_3X\tilde{Z}^{-1} - 1)$ times the lowest-order vertex. There is no possible left-over divergence that satisfies (12.2.6).

Similarly

$$\frac{\partial}{\partial z^\nu}\langle 0|TA_\kappa(w)A_\lambda(x)A_\mu(y)A^\nu(z)|0\rangle = \text{finite}. \tag{12.2.7}$$

Here the only potentially divergent N-loop 1PI subgraphs are as in Fig. 12.2.8. We have just seen that the divergences in Fig. 12.2.8(*b*) are cancelled by the counterterm for the triple gluon coupling. Since the divergence in graph (*a*) is logarithmic, (12.2.7) proves that it is exactly cancelled by the counterterm in the four-gluon interaction.

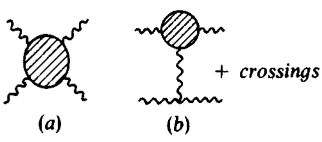


Fig. 12.2.8. Potentially divergent graphs for four-point Green's function of gluon.

12.2.8 $\delta_R c$

The only Green's function of $\delta_R c$ that could be divergent is

$$\langle 0|T\delta_R c\,\bar{c}\,\bar{c}|0\rangle. \tag{12.2.8}$$

But

$$\begin{aligned}0 &= \delta\langle 0|Tc^a(x)\bar{c}^b(y)\bar{c}^c(z)|0\rangle \\ &= \langle 0|T\delta_R c^a(x)\bar{c}^b(y)\bar{c}^c(z)|0\rangle \\ &\quad -(1/\xi)\langle 0|Tc^a(x)\partial\cdot A^b(y)\bar{c}^c(z)|0\rangle \\ &\quad +(1/\xi)\langle 0|Tc^a(x)\bar{c}^b(y)\partial\cdot A^c(z)|0\rangle,\end{aligned} \tag{12.2.9}$$

so finiteness of (12.2.8) follows from finiteness of the ghost-gluon vertex, which is a renormalization condition.

12.2.9 *Quark–gluon interaction, $\delta\psi$, $\delta\bar{\psi}$; introduction of $\mathscr{B}_{a\mu}$*

Consider the Ward identity

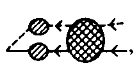
$$\begin{aligned}0 &= \delta\langle 0|T\psi(x)\bar{\psi}(y)\bar{c}(z)|0\rangle/\delta\lambda_R \\ &= \langle 0|T\delta_R\psi\bar{\psi}\bar{c}|0\rangle - \langle 0|T\psi\delta_R\bar{\psi}\bar{c}|0\rangle + \langle 0|T\psi\bar{\psi}(\partial\cdot A/\xi)|0\rangle.\end{aligned}$$

Finiteness of the last term follows if we can prove $\delta_R\psi = -ig_R X t^a\psi c_a$ finite. (Note that $\delta_R\bar{\psi}$ is related to $\delta_R\psi$ by the CPT transformation of (12.1.13).) Now X was defined by requiring the ghost-gluon vertex to be finite. An explicit proof, which we now give, brings in all the remaining operators listed in (12.1.9)–(12.1.11), and in particular $\mathscr{B}_{a\mu}$.

In Fig. 12.2.9 we list all Green's functions still to be proved finite. Observe that $\delta_R\bar{c} = \partial\cdot A/\xi$, so that it is finite if the gauge field is. Also Green's functions of $\mathscr{B}_{a\mu}$ or $\bar{c}\wedge\bar{c}$ with $\delta_R\phi$ and any number greater than zero of basic fields are finite, if the Green's functions of Figs. 12.2.7 to 12.2.9 are finite. (Here $\delta_R\phi$ is the BRS variation of any elementary field ϕ.) This proves Theorems 4 and 5, so it remains to prove finiteness of the Green's functions illustrated in Fig. 12.2.9.

The idea behind the proof is to examine the right-most vertex on the ghost line in Fig 12.2.9(a). This comes from the following term in the interaction Lagrangian:

$$\int c_{abc}g_R(\partial^\mu\bar{c}^a)c^bA^c_\mu\,d^4x. \tag{12.2.10}$$

(*a*) , logarithmically divergent, contributing to $\langle 0|T\delta_R\psi\bar{\psi}\bar{c}|0\rangle$

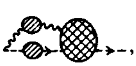
(*b*) , logarithmically divergent, contributing to $\langle 0|T\mathscr{B}_{a\mu}c|0\rangle$

Fig. 12.2.9. Green's functions not yet proved finite.

The derivative is on the line entering the vertex graph. By integrating by parts, we see that the vertex equals

$$-g_R c_{abc}\int d^4x[\bar{c}^a(\partial_\mu c^b)A^{c\mu} + \bar{c}^a c^b \partial\cdot A^c]. \qquad (12.2.11)$$

In the first term the derivative is outside the loop-momentum integrals, so the degree of divergence is reduced by one. Factoring out the field $\partial_\mu c^b$ on the external line gives the basic vertex $c_{abc}\bar{c}^b A^c_\mu$ in the operator $\mathscr{B}_{a\mu}$. The second term in (12.2.11) contains $\partial\cdot A$ which we shall relate to something else by use of Ward identities.

We first prove finiteness of $\mathscr{B}_{a\mu}$, by formalizing the argument leading to (12.2.11). This is done in an unobvious way:

$$\begin{aligned}
\Box_z\langle 0|T\delta_R c^a(x)\bar{c}^b(y)\bar{c}^c(z)|0\rangle
&= (1/X)\{\tilde{Z}\,\Box_z\langle 0|T\delta_R c^a\bar{c}^b\bar{c}^c|0\rangle \\
&\qquad + g_R X c_{cde}\langle 0|T\delta_R c^a\bar{c}^b(A^e_\mu\partial^\mu\bar{c}^d)|0\rangle\} \\
&\quad + g_R\langle 0|T\delta_R c^a\bar{c}^b(c_{cde}\bar{c}^d\partial\cdot A^e)|0\rangle \\
&\quad - g_R\frac{\partial}{\partial z^\mu}\langle 0|T\delta_R c^a\bar{c}^b\mathscr{B}^\mu_c(z)|0\rangle \\
&= i g_R c_{acd}\langle 0|Tc^d(x)\bar{c}^b(y)|0\rangle\delta^{(d)}(x-z) \\
&\quad + \tfrac{1}{2}g_R\langle 0|T\delta_R c^a\bar{c}^b\delta_R(c_{cde}\bar{c}^d\bar{c}^e)|0\rangle \\
&\quad - g_R\frac{\partial}{\partial z^\mu}\langle 0|T\delta_R c^a\bar{c}^b\mathscr{B}^\mu_c(z)|0\rangle \\
&= \text{finite} - \tfrac{1}{2}g_R\langle 0|T\delta_R c^a\partial\cdot A^b(\bar{c}\wedge\bar{c})^c|0\rangle \\
&\quad - g_R\frac{\partial}{\partial z^\mu}\langle 0|T\delta_R c^a\bar{c}^b\mathscr{B}^\mu_c(z)|0\rangle. \qquad (12.2.12)
\end{aligned}$$

The next-to-last line follows by the antighost equation of motion. The last line uses a Ward identity plus the nilpotence property

$$\delta_R[\delta_R c^a] = 0. \qquad (12.2.13)$$

Now the second term on the last line is finite (by Theorem 5), and the left-hand side of (12.2.12) is finite. So the last term on the right is finite. The only possible uncancelled divergence is of the form of Fig. 12.2.10, from which finiteness of $\mathscr{B}_{a\mu}$ follows.

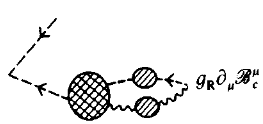


Fig. 12.2.10. Only possibly divergent graph for (12.2.12).

Finiteness of $\delta_R \psi$ follow by the same manipulations applied to $\langle 0|T\delta_R \psi \bar{\psi} \bar{c}|0\rangle$:

$$\begin{aligned}\Box_z \langle 0|T\delta_R \psi(x)\bar{\psi}(y)\bar{c}^a(z)|0\rangle \\ &= g_R \langle 0|T t^a \psi(x)\bar{\psi}(y)|0\rangle \delta^{(d)}(x-z) \\ &\quad -\tfrac{1}{2} g_R \langle 0|T\delta_R \psi \delta_R \bar{\psi}(\bar{c}\wedge \bar{c})|0\rangle \qquad (12.2.14) \\ &\quad - g_R \frac{\partial}{\partial z^\mu}\langle 0|T\delta_R \psi(x)\bar{\psi}(y)\mathscr{B}_a^\mu(z)|0\rangle.\end{aligned}$$

12.3 More general theories

In the last section we proved renormalizable the simplest gauge theories: that is, those with a gauge group with a single component and with fermion matter fields. More general cases can easily be treated by the same methods. The general result is that renormalizations are needed for each independent coupling in the basic Lagrangian and for the field strength for each irreducible field multiplet. Let us examine some specific generalizations.

12.3.1 *Bigger gauge group*

The gauge group can in general be a product of several components: $G = \prod_{i=1}^{n} G_i \otimes U(1)^\nu$. Here each G_i is a simple compact non-abelian group (like $SU(N)$), and there are ν abelian $U(1)$'s. For each G_i and for each $U(1)$ factor there is an independent coupling g_i. When we perform gauge fixing there will be a multiplet of ghost fields for each component of the gauge group. The proof of renormalizability will need no change. It will relate the renormalizations of the couplings within each multiplet. Thus for each of the $n + \nu$ components of the gauge group there are renormalization factors X_i, Z_{3i}, and $\tilde{Z}_i$, for the coupling, the gauge field and the ghost field. In addition there are the usual renormalizations for the matter fields.

There are some special features of the abelian case which we will treat in Section 12.9.

12.3.2 *Scalar matter*

The part of the Lagrangian for a scalar field coupled to gauge fields is

$$D^\mu \phi^\dagger D_\mu \phi - V(\phi^\dagger, \phi). \qquad (12.3.1)$$

Here V is a function of ϕ and $\phi^\dagger$ that is invariant under the gauge group, and D_μ is the usual covariant derivative.

For example, consider an $SU(2)$ gauge theory in which ϕ is a doublet under the gauge group. Then the most general renormalizable form of V is

$$V = m^2\phi^\dagger\phi + \tfrac{1}{4}\lambda(\phi^\dagger\phi)^2, \tag{12.3.2}$$

while

$$\begin{aligned} D_\mu\phi^\dagger D\phi &= |\partial_\mu\phi|^2 - ig A^{a\mu}\phi^\dagger t^a \overleftrightarrow{\partial}_\mu\phi + g^2 A^a_\mu A^{b\mu}\phi^\dagger t^a t^b\phi \\ &= |\partial_\mu\phi|^2 - ig A^a_\mu\phi^\dagger t^a \overleftrightarrow{\partial}^\mu\phi + \tfrac{1}{4}g^2 A^{a2}\phi^\dagger\phi. \end{aligned} \tag{12.3.3}$$

Wave-function and mass renormalization are used to make the propagator finite, and the $\phi^\dagger\partial\phi A$ coupling is proved finite as for a fermion. The self-coupling of the ϕ-field is made finite by a renormalization of the $\lambda\phi^\dagger\phi^2$ term, which is the only four-point coupling invariant under global $SU(2)$ transformations. One further Green's function, viz., $\langle 0|TAA\phi^\dagger\phi|0\rangle$ has a potential logarithmic divergence. It is proved finite by the Ward identity:

$$\begin{aligned} 0 &= \delta\langle 0|T\bar{c}A\phi^\dagger\phi|0\rangle \\ &= (1/\xi)\langle 0|T\partial\cdot AA\phi^+\phi|0\rangle + \langle 0|T\bar{c}\delta A\phi^\dagger\phi|0\rangle \\ &\quad + \langle 0|T\bar{c}A\delta\phi^\dagger\phi|0\rangle + \langle 0|T\bar{c}A\phi^\dagger\delta\phi|0\rangle. \end{aligned} \tag{12.3.4}$$

12.3.3 *Spontaneous symmetry breaking*

Consider the abelian Higgs model as an easy example. Its basic Lagrangian is

$$\mathscr{L}_{\text{basic}} = -\tfrac{1}{4}F^2_{\mu\nu} + (\partial_\mu - ieA_\mu)\phi^\dagger(\partial^\mu + ieA^\mu)\phi - \tfrac{1}{4}\lambda^2(\phi^\dagger\phi - f^2/2\lambda^2)^2. \tag{12.3.5}$$

The symmetry is spontaneously broken with $\langle 0|\phi|0\rangle = f/(\lambda\sqrt{2})$, in lowest order. So we write $\phi = [f/\lambda + (\phi_1 + i\phi_2)]/\sqrt{2}$:

$$\begin{aligned} \mathscr{L}_{\text{basic}} = &-\tfrac{1}{4}F^2_{\mu\nu} + (e^2f^2/2\lambda^2)A^2 + (ef/\lambda)A_\mu\partial^\mu\phi_2 + \tfrac{1}{2}\partial\phi_1^2 + \tfrac{1}{2}\partial\phi_2^2 - f^2\phi_1^2/4 \\ &- \lambda^2(\phi_1^2 + \phi_2^2)^2/16 - \lambda f\phi_1(\phi_1^2 + \phi_2^2)/4 \\ &+ eA_\mu\phi_1\partial^\mu\phi_2 + e^2A^2(\phi_1^2 + \phi_2^2 + 2f\phi_1/\lambda)/2. \end{aligned} \tag{12.3.6}$$

If we quantize with the simple gauge fixing term

$$\mathscr{L}_{\text{gf}} = -\frac{1}{2\xi}(\partial\cdot A)^2, \tag{12.3.7}$$

then renormalization is covered by the discussion of Section 9.2, where we treated spontaneous symmetry breaking in a non-gauge theory. We first renormalize in the unbroken theory (i.e., with $f^2 \to -f^2$). We need wave-function renormalizations Z_2 and Z_3, coupling renormalization $\lambda^2 \to \lambda_R^2 Y$, and mass renormalization (which is effectively $f^2 \to f^2 Z_m$). Since the Faddeev–Popov ghost is a free field, the gauge-coupling renormalization is $X = 1$. After spontaneous symmetry breaking the same renormalizations make the Green's functions finite. Unfortunately the $A_\mu \partial^\mu \phi_2$ term in the free Lagrangian makes the Feynman rules rather messy, so it is convenient to use another gauge condition. We will discuss this in Section 12.5.

12.4 Gauge dependence of counterterms

To quantize and renormalize a gauge theory, we choose to fix the gauge in a particular way. It is important to show that physical quantities are independent of this choice. On the other hand, Green's functions of the elementary fields can certainly be gauge dependent. Since it is the Green's functions with which we work when renormalizing the theory, we must understand their gauge dependence and its effect on the renormalizations.

One important class of physical quantities which we will treat is the set of Green's functions of gauge-invariant operators. Also important is the S-matrix for physical states. In a spontaneously broken theory there are particle states that couple to the elementary fields of the theory. Some such states are physical, and we must prove that their S-matrix is gauge independent. But other states, like the Faddeev–Popov ghost, are unphysical. We will not attempt a complete treatment here.

When a gauge symmetry is unbroken there may even be no states coupled to the elementary fields. Indeed, it is commonly expected that in QCD colored states are confined. Certainly, within perturbation theory there are severe infra-red problems in obtaining the S-matrix for quarks and gluons. However, S-matrix elements of hadrons can be obtained from gauge-invariant Green's functions. Consider, for example:

$$\langle 0 | T j^a_{5\kappa}(w) j^b_{5\lambda}(x) j^c_{5\mu}(y) j^d_{5\nu}(z) | 0 \rangle, \tag{12.4.1}$$

where $j^a_{5\mu}$ is the axial isospin current. Application of the LSZ formalism will give the S-matrix for $\pi\pi \rightarrow \pi\pi$ scattering. Similarly the gauge-invariant field $\psi_i \psi_j \psi_k \varepsilon_{ijk}$ is an interpolating field for baryons. Here ψ_i is a quark field and i its color label.

There are two sorts of gauge-dependence that we will consider. The first is where we change the gauge-fixing function F_a to a different function of the fields. A specific application of our general results for this case will be given in Section 12.5 for the R_ξ-gauge. The second type of gauge-dependence is a variation of the parameter ξ. Although variation of ξ by a factor λ is equivalent to variation of F_a by a factor $\lambda^{-1/2}$, there are a number of special simplifications that will lead us to treat this second case first.

The use of BRS invariance will make our computation of gauge dependence very simple.

12.4.1 *Change of* ξ

The Green's functions and the renormalization factors in general depend on ξ. We can use the action principle to compute the dependence on ξ of a Green's function $\langle 0|TX|0\rangle$. Here X denotes any product of local fields with no explicit dependence on ξ. Then

$$\begin{aligned}\frac{\partial}{\partial\xi}\langle 0|TX|0\rangle &= \mathrm{i}\int \mathrm{d}^4y\langle 0|T{:}\frac{\partial\mathscr{L}}{\partial\xi}(y){:}X|0\rangle \\ &= \mathrm{i}\sum_Y \frac{\partial Y}{\partial\xi}\int \mathrm{d}^4y\langle 0|T{:}\frac{\partial\mathscr{L}}{\partial Y}(y){:}X|0\rangle \\ &\quad + \frac{\mathrm{i}}{2\xi^2}\int \mathrm{d}^4y\langle 0|T{:}F_a^2(y){:}X|0\rangle. \end{aligned} \tag{12.4.2}$$

In the first term we use Y to denote any of the independent renormalization factors. As usual, we use the symbol: : to indicate subtraction of the vacuum expectation value. Next we use the form of the BRS transformations to write

$$(1/\xi)F_a^2 = \delta(\bar{c}_aF_a) - \bar{c}_a\delta F_a, \tag{12.4.3}$$

and substitute for F_a^2 in (12.4.2). We can use the Ward identity

$$\delta\langle 0|T\bar{c}FX|0\rangle = 0$$

and the equation of motion

$$\langle 0|T\bar{c}_a\delta F_a(y)X|0\rangle = -\mathrm{i}\sum_a\langle 0|T\bar{c}_a\delta X/\delta\bar{c}_a(y)|0\rangle$$

to give

$$\begin{aligned}\frac{\partial}{\partial\xi}\langle 0|TX|0\rangle &= \mathrm{i}\sum_Y \frac{\partial Y}{\partial\xi}\int \mathrm{d}^4y\langle 0|Y{:}\frac{\partial\mathscr{L}}{\partial Y}{:}X|0\rangle \\ &\quad -\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4y\langle 0|T{:}\bar{c}_aF_a{:}\delta X|0\rangle + (N_c/2\xi)\langle 0|TX|0\rangle. \end{aligned} \tag{12.4.4}$$

Here N_c is the number of factors of the ghost field in X.

Suppose first that we choose to use the same counterterms for all values of ξ as at some particular value, say $\xi = 0$. Then Green's functions of elementary fields will not be finite except at this special value. But if we take a Green's function of gauge-invariant operators then (12.4.4) indicates that the Green's function is gauge independent, for such operators contain no ghosts.

Next consider a Green's function of elementary fields:

$$\langle 0|TX|0\rangle = \langle 0|T\prod_{i=1}^{N}\phi_i(x_i^\mu)|0\rangle. \tag{12.4.5}$$

The renormalizations in $\mathscr{L}$ are adjusted to keep Green's functions finite for any value of ξ. Then in (12.4.4) we find the ξ-dependence of the counterterms by requiring the

$$\sum_Y \frac{\partial Y}{\partial \xi} : \frac{\partial \mathscr{L}}{\partial Y} :$$

term to be the sum of the counterterms needed to cancel the divergences of $-:\bar{c}_a F_a:/(2\xi)$.

We consider the ultra-violet divergences of

$$\langle 0|T{:}\bar{c}_a F_a(y){:}\delta X|0\rangle. \tag{12.4.6}$$

Divergences in (12.4.6) occur either when y coincides with some set of interaction vertices or when it coincides with one of the x_i's, which are the positions of fields in X. We can renormalize Green's functions of $:\bar{c}_a F_a:$ with elementary fields by adding to it an operator $:D(y):$. Since we choose X to be a product of elementary fields, the only remaining divergences in (12.4.6) are when y coincides with the position x of a BRS varied field $\delta\phi(x)$ in δX. To renormalize this divergence we need counterterms to (12.4.6) of the form

$$\langle 0|T{:}D(y){:}\delta X|0\rangle + \sum_{i=1}^{N}\delta^{(4)}(y-x_i)\langle 0|TX|0\rangle|_{\phi_i(x_i)\to -\mathrm{i}E_{\phi_i}(x_i)} \tag{12.4.7}$$

The counterterm when y coincides with $\delta\phi(x)$ has been written as $-\mathrm{i}E_\phi(x)\delta(y-x)$, there are possibly derivatives of the δ-function, and the normalization factor $-\mathrm{i}$ will be convenient later.

By use of Ward identities and equations of motion, we can write (12.4.7) as

$$-\langle 0|T{:}\delta D(y){:}X|0\rangle - \sum_\phi \langle 0|T{:}E_\phi(y)\delta S/\delta\phi(y){:}\,X|0\rangle. \tag{12.4.8}$$

This is the most general form of counterterm needed to keep the Green's function finite as ξ varies. It is even the correct form for the change in counterterms caused by a change in the form of F_a, as we will see in Section 12.4.4. To proceed any further we must do power-counting to determine which counterterms actually occur.

We now restrict attention to the gauge condition $F_a = \partial\cdot A_a$. The only divergent elementary Green's functions of $:\bar{c}_a F_a:$ are illustrated by their

Fig. 12.4.1. Lowest-order graph for $\langle 0|Tc:\bar{c}_a F_a:|0\rangle$.

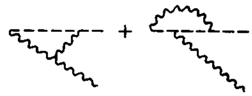


Fig. 12.4.2. Lowest-order graph for $\langle 0|Tc\,A:\bar{c}_a E_a:|0\rangle$

lowest-order cases in Figs. 12.4.1 and 12.4.2. The divergence of the first graph has a factor of the derivative at the ghost interaction. The counterterm for Fig. 12.4.2 is proportional to $(\partial_\mu \bar{c}_a)A_a^\mu = \text{derivative} - \bar{c}_a F_a$. Since we integrate over all y, the only contribution to the counterterm operator D is proportional to $\bar{c}_a F_a$.

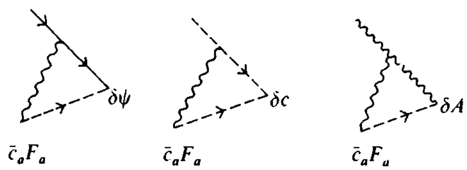


Fig. 12.4.3. Lowest-order graphs for $:\bar{c}_a F_a:$ with BRS variation of a field.

The divergence with a BRS-varied field $\delta\phi$ is logarithmic and proportional to ϕ, as in the graphs of Fig. 12.4.3. So we have $E_\phi \propto \phi$. This corresponds to a variation with ξ of the wave-function renormalizations with g_0 and M_0 fixed. For example,

$$\frac{\partial Z_2}{\partial \xi}\frac{\partial \mathscr{L}}{\partial Z_2} = \frac{1}{2}\frac{\partial \ln Z_2}{\partial \xi}(\psi \delta S/\delta\psi + \bar{\psi}\delta S/\delta\bar{\psi}) \tag{12.4.9}$$

A complication arises since we know ξ is not renormalized:

$$\begin{aligned}\frac{\partial Z_3}{\partial \xi}\frac{\partial \mathscr{L}}{\partial Z_3} &= \frac{1}{2}\frac{\partial \ln Z_3}{\partial \xi} A_\mu^a \frac{\delta}{\delta A_\mu^a}(S + F_a^2/2\xi) \\ &= \frac{1}{2}\frac{\partial \ln Z_3}{\partial \xi}(A\cdot g\delta S/\delta A + F_a^2/2\xi).\end{aligned} \tag{12.4.10}$$

Putting all this work together, we find

$$\begin{aligned}\frac{\partial}{\partial \xi}\langle 0|TX|0\rangle = &-\frac{\mathrm{i}}{2\xi}\left(1 + \frac{\partial \ln Z_3}{\partial \ln \xi}\right)\int \mathrm{d}^4 y \langle 0|T:\bar{c}_a F_a(y):\delta X|0\rangle \\ &- \langle 0|TX|0\rangle\Bigg\{N_\psi \frac{\partial \ln Z_2}{\partial \xi} + \frac{N_A}{2}\frac{\partial \ln Z_3}{\partial \xi} \\ &+ N_c\left[\frac{\partial \ln(\tilde{Z}Z_3^{1/2})}{\partial \xi} + \frac{\xi}{2}\right]\Bigg\}.\end{aligned} \tag{12.4.11}$$

This is just the counterterm structure we need. The operator $\bar{c}_a F_a$ is multiplicatively renormalized by a factor $1 + \xi \partial \ln Z_3/\partial \xi$. (In an abelian theory no renormalization is needed since $\bar{c}_a$ is a free field; so Z_3 is independent of ξ.) The other terms (where N_ϕ is the number of external ϕ fields of X) come from renormalizations of 1PI graphs including both $\bar{c}_a F_a$ and a $\delta\phi$.

Note that $\partial \ln Z_3/\partial \xi$ occurs in two places. This comes from assuming that the gauge-fixing term is $-F_a^2/(2\xi) = -\partial\cdot A^2/(2\xi)$ with no extra renormalization factor. We proved this from the Ward identity:

$$\begin{aligned} 0 &= \delta\langle 0|T\bar{c}_a(x)F_b(y)|0\rangle \\ &= (1/\xi)\langle 0|TF_a(x)F_b(y)|0\rangle + \langle 0|T\bar{c}_a\delta F_b|0\rangle \\ &= (1/\xi)\langle 0|TF_a(x)F_b(y)|0\rangle - \mathrm{i}\delta(x-y)\delta_{ab}, \end{aligned} \tag{12.4.12}$$

where the last line follows from the ghost equation of motion. Since $F_a = \partial\cdot A_a$, its Green's functions are finite, and so ξ is finite.

We can use (12.4.11) to prove gauge invariance of the S-matrix by picking out the residue when we go to the particle pole for each external line.

If we use the correct, ξ-dependent counterterms in $\mathscr{L}$, then a gauge-invariant operator like

$$(G^a_{\mu\nu})^2 = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0 Z_3^{1/2} c_{abc} A^b_\mu A^c_\nu)^2 \tag{12.4.13}$$

is actually ξ-dependent; from (12.4.11) we see that is proportional to Z_3^{-1}. But the bare operator

$$(G^a_{(0)\mu\nu})^2 = (\partial_\mu A^a_{(0)\nu} - \partial_\nu A^a_{(0)\mu} + g_0 c_{abc} A^b_{(0)\mu} A^c_{(0)\nu})^2$$

is gauge independent. Both operators have UV divergences, so that they must be renormalized.

12.4.2 *Change of F_a*

Let us assume that F_a depends on a parameter κ. This allows us to interpolate between two different gauge-fixing conditions. We follow the same method as for the ξ-dependence. The change in the Lagrangian is

$$\begin{aligned} \frac{\partial\mathscr{L}}{\partial\kappa} &= \sum_Y \frac{\partial Y}{\partial\kappa}\frac{\partial\mathscr{L}}{\partial Y} - \frac{1}{\xi}F_a\frac{\partial F_a}{\partial\kappa} - \bar{c}_a\delta\frac{\partial F_a}{\partial\kappa} \\ &= \sum_Y \frac{\partial Y}{\partial\kappa}\frac{\partial\mathscr{L}}{\partial Y} - \delta\left(\bar{c}_a\frac{\partial F_a}{\partial\kappa}\right). \end{aligned} \tag{12.4.14}$$

So

$$\begin{aligned} \frac{\partial}{\partial\kappa}\langle 0|TX|0\rangle &= \mathrm{i}\sum_Y \frac{\partial Y}{\partial\kappa}\int \mathrm{d}^4y\langle 0|T{:}\frac{\partial\mathscr{L}(y)}{\partial\kappa}{:}X|0\rangle \\ &\quad + \mathrm{i}\int \mathrm{d}^4y\langle 0|T{:}\bar{c}_a\frac{\partial F_a}{\partial\kappa}{:}\delta X|0\rangle. \end{aligned} \tag{12.4.15}$$

The counterterms for the second term have exactly the same form as we met in the ξ-dependence of a Green's function, i.e., the form (12.4.8). Until the form of F_a is specified we can make no further statement about the necessary counterterms. We will examine an example of this in Section 12.5.

In the most general case we can take a Green's function of gauge-invariant operators and omit any κ-dependence of the counterterms in $\mathscr{L}$. Then the Green's functions are gauge-independent: $\partial\langle 0|TX|0\rangle/\partial\kappa = 0$.

12.5 $\mathbf{R}_\xi$-gauge

To eliminate the mixing between the gauge field and the scalar field in spontaneously broken gauge theories one can use the R_ξ-gauge devised by 't Hooft (1971b), Fujikawa, Lee & Sanda (1972), and Yao (1973). Let us use the Higgs model of (12.3.5) as an example. The R_ξ-gauge is defined by using the gauge-fixing Lagrangian

$$\mathscr{L}_{\mathrm{gf}} = -(1/2\xi)(\partial\cdot A - \kappa\xi m\phi_2)^2. \tag{12.5.1}$$

We use the parameter κ to interpolate between the R_ξ-gauge, where $\kappa = 1$, and the gauges we used in Section 12.2, where $\kappa = 0$. Also, $m = ef/\lambda$.

An interacting Faddeev–Popov ghost is needed, even in an abelian theory (which is the only case we will explicitly treat):

$$\mathscr{L}_{\mathrm{gc}} = \partial_\mu\bar{c}\partial^\mu c - \kappa\xi m^2\bar{c}c - e\xi m\kappa\phi_1\bar{c}c. \tag{12.5.2}$$

The BRS variations of the fields are:

$$\left.\begin{aligned}
\delta A_\mu &= \partial_\mu c,\\
\delta\phi_2 &= -(m + e\phi_1)c,\\
\delta\phi_1 &= e\phi_2 c,\\
\delta\bar{c} &= (\partial\cdot A - \xi\kappa m\phi_2)/\xi = F/\xi,\\
\delta c &= 0.
\end{aligned}\right\} \tag{12.5.3}$$

Observe first of all that the extra couplings relative to the gauge $\kappa = 0$ are all super-renormalizable, so that the renormalizations of the dimensionless couplings can remain unchanged. These are X (which equals unity in an abelian theory), Y, Z_2, and Z_3. To determine what further renormalizations are needed, we use the method of Section 12.4.2, with $\partial F/\partial\kappa = -\xi m\phi_2$.

Thus we need the counterterms to

$$-\mathrm{i}\xi m\int \mathrm{d}^4y\langle 0|T{:}\bar{c}\phi_2(y){:}\delta X|0\rangle. \tag{12.5.4}$$

The only divergent graphs contain $\bar{c}\phi_2$, the BRS variation $\delta\phi_1$, and no other external lines. They have the form of Fig. 12.5.1, which is logarithmi-

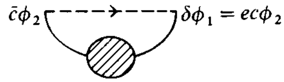


Fig. 12.5.1. Divergent graphs for (12.5.4).

cally divergent. Notice that interactions on the ghost line would make the graph convergent. The counterterm to (12.5.4) therefore has the form

$$C\int \mathrm{d}^4 y\langle 0|T\delta X/\delta\phi_1(y)|0\rangle$$
$$= \mathrm{i}C\int \mathrm{d}^4 y\langle 0|T{:}\delta S/\delta\phi_1(y){:}X|0\rangle. \qquad (12.5.5)$$

This counterterm is generated by a shift of ϕ_1 in the Lagrangian:

$$\mathscr{L}(\phi_1 + m/e, \phi_2, A, c, \bar{c}) \to \mathscr{L}(\phi_1 + Bm/e, \phi_2, A, c, \bar{c}) \qquad (12.5.6)$$

with

$$\frac{m}{e}\frac{\partial B}{\partial \kappa} = C. \qquad (12.5.7)$$

As far as renormalization is concerned, the effect of using the new gauge condition is to generate an ultra-violet divergence in the vacuum expectation value of the scalar field. Although this might appear strange, it is permitted to happen since the field is not gauge invariant.

Gauge-invariant Green's functions are unchanged, of course, as is the S-matrix. The ghost field, ϕ_2, and the longitudinal part of A_μ all couple to unphysical states. In this abelian theory, $F_{\mu\nu}$ is a gauge-invariant field which couples to the transverse part of the gluon, while

$$\begin{aligned} Z_m Z_2 \phi^\dagger\phi &= Z_m Z_2[(\phi_1 + Bm/e)^2 + \phi_2^2] \\ &= Z_m Z_2 (Bm/e)^2 + \phi_1[2(Bm/e)Z_m Z_2] + \cdots \end{aligned}$$

is gauge invariant and renormalized, and couples to the ϕ_1-particle.

The Lagrangian can be written out in terms of the fields, to exhibit the interactions. It is rather fearsome-looking. We set $\kappa = 1$ and find

$$\begin{aligned} \mathscr{L} = &-\tfrac{1}{4}Z_3 F_{\mu\nu}^2 - (1/2\xi)\partial\cdot A^2 + \tfrac{1}{2}m^2 B^2 Z_2 A_\mu^2 + \partial\bar{c}\partial c \\ &- m^2 B\xi\bar{c}c + A_\mu\partial^\mu\phi_2 m(Z_2 B - 1) \\ &+ \tfrac{1}{2}Z_2(\partial\phi_1^2 + \partial\phi_2^2) - \tfrac{1}{8}\phi_1^2 m^2\lambda^2 e^{-2}(3B^2 Y - Z_m) \\ &- \tfrac{1}{2}\phi_2^2 m^2[\xi + \tfrac{1}{4}\lambda^2 e^{-2}(B^2 Y - Z_m)] \\ &- \tfrac{1}{16}\lambda^2 Y(\phi_1^2 + \phi_2^2)^2 - \tfrac{1}{4}\lambda^2 m e^{-1} BY\phi_1(\phi_1^2 + \phi_2^2) \\ &- \tfrac{1}{4}\lambda^2 m^3 e^{-3} B(B^2 Y - Z_m)\phi_1 \\ &+ eZ_2 A_\mu \phi_1 \overleftrightarrow{\partial^\mu}\phi_2 + \tfrac{1}{2}e^2 Z_2 A_\mu^2(\phi_1^2 + \phi_2^2) \\ &+ emBZ_2\phi_1 A_\mu^2 - \xi m\phi_1\bar{c}c. \end{aligned} \qquad (12.5.8)$$

The same methods can be applied to a non-abelian theory with extra complications in the structure of the vertices – see Fujikawa, Lee & Sanda (1972).

12.6 Renormalization of gauge-invariant operators

In applications such as to deep-inelastic scattering we need the operator product expansion of gauge-invariant operators. It is natural to assume that only gauge-invariant operators appear in the expansion. Moreover, in using the operator product expansion, we need the anomalous dimensions of the operators. To compute these, we need the expression for the renormalized operators in terms of the bare operators. Again, it is natural to assume that only gauge-invariant operators are needed.

Both assumptions, taken literally, are false (Dixon & Taylor (1974), and Kluberg-Stern & Zuber (1975)). We will first treat the renormalization problem, where certain gauge-variant operators mix with gauge-invariant operators. As we will see, the gauge-variant operators vanish in physical matrix elements (Joglekar & Lee (1976), Joglekar (1977a and b)). We will simplify many parts of the proof by working in coordinate space. Then we will apply the same methods to find the operators that appear in the operator product expansion.

In the case of an ordinary global symmetry (like Lorentz invariance or isospin), an operator that is invariant under the symmetry mixes only with invariant operators. In the case of a gauge theory, we break the invariance of the action by gauge-fixing, leaving only a BRS invariance. But a gauge-invariant operator even mixes with operators that are not BRS invariant. How can this be?

To answer this question, we consider an operator $\mathcal{O}$ that is invariant under some given transformation. Let C be the sum of its counterterms, and assume the action is invariant. Then the Green's functions are invariant under the transformation, so

$$\begin{aligned} 0 &= \delta\langle 0|TX(\mathcal{O}+C)|0\rangle \\ &= \langle 0|T\delta X(\mathcal{O}+C)|0\rangle + \langle 0|TX\,\delta(\mathcal{O}+C)|0\rangle \\ &= \langle 0|T\delta X(\mathcal{O}+C)|0\rangle + \langle 0|TX\,\delta C|0\rangle. \end{aligned} \tag{12.6.1}$$

Now C is defined so that Green's functions of $\mathcal{O}+C$ with elementary fields are finite. For an ordinary symmetry, the variation of an elementary field is again an elementary field, so the first term on the right of (12.6.1) is finite. Hence the other term, an arbitrary Green's function of δC, is finite. If we use minimal subtraction, the counterterms are powers of $1/\varepsilon$, so δC is finite only if it is zero. Thus counterterms to an invariant operator are invariant.

But when δ represents a BRS transformation the variation of an elementary field is composite. So finiteness of $\langle 0|T\delta X(\mathcal{O}+C)|0\rangle$ does not follow from finiteness of the elementary Green's functions of $O+C$. Hence the counterterms to a gauge-invariant operator need not be BRS invariant, let alone gauge invariant.

This argument also shows us how to handle the problem. Let X be a product $\phi_1(x_1)\ldots\phi_N(x_N)$ of local fields. Then the only divergences of $\langle 0|T\delta X(\mathcal{O}(y)+C(y))|0\rangle$ that lack a counterterm are when y coincides with the position of a BRS-varied operator $\delta\phi_i(x_i)$ in δX. Hence, provided y is not equal to any x_i, $\langle 0|T\delta X(\mathcal{O}(y)+C(y))|0\rangle$ is finite. From (12.6.1) we then see that $\langle 0|TX\delta C|0\rangle$ is then finite. But if we use minimal subtraction this means that it is zero. Hence an arbitrary Green's function $\langle 0|TX\delta C|0\rangle$ of δC with elementary fields is zero. Thus, as an operator, $\delta_{\mathrm{BRS}}C=0$: the counterterms to a gauge-invariant operator are BRS invariant.

One can try verifying this theorem by explicit calculations (e.g. Kluberg-Stern & Zuber (1975)). These apparently contradict the theorem. However, the non-invariant counterterms must correspond to operators that vanish by the equations of motion. As usual, the treatment of derivatives of fields in covariant perturbation theory implicitly generates commutator terms. After Fourier transformation into momentum space the non-invariant operators are not manifestly zero.

What we are actually interested in are gauge-invariant operators rather than merely BRS-invariant operators, for it is only the gauge-invariant operators that have physical significance independently of the method of gauge-fixing. So we must show how it is that the gauge-variant operators that mix with $\mathcal{O}$ do not enter into physical quantities.

Let $\mathcal{O}_1,\ldots,\mathcal{O}_n$ be the set of gauge-invariant operators that mix with $\mathcal{O}$. These are the operators which have the same transformations under global symmetries (e.g. Lorentz, isospin) as $\mathcal{O}$, and whose dimension is at most that of $\mathcal{O}$. Choose this set so that it is linearly independent (after use of the equations of motion). There are three other classes of operators that mix with $\mathcal{O}$:

Class A These operators, A_i, are the BRS variation of some operator: $A_i=\delta\hat{A}_i$

Class B These operators, B_i, vanish by the equations of motion.

Class C Any other operators that mix with $\mathcal{O}$, and that are not linear combinations of A_i's, B_i's, and $\mathcal{O}_i$'s.

The nilpotence of BRS transformations ensures that $\delta A_i=0$ up to terms vanishing by the equations of motion.

The most obvious classes of BRS-invariant operators are the gauge-invariant operators $\mathcal{O}_i$ and the BRS transformation of operators $\delta\hat{A}_i$. The first result to prove is that there are no others; i.e., there are no operators of class C. The second result is that the renormalization matrix expressing the renormalized operators $[\mathcal{O}]$, $[A]$, and $[B]$ in terms of unrenormalized operators is triangular:

$$\begin{pmatrix}[\mathcal{O}]\\ [\mathbf{A}]\\ [\mathbf{B}]\end{pmatrix} = \begin{pmatrix} Z_{\mathcal{O}\mathcal{O}} & Z_{\mathcal{O}A} & Z_{\mathcal{O}B}\\ 0 & Z_{AA} & Z_{AB}\\ 0 & 0 & Z_{BB}\end{pmatrix}\begin{pmatrix}\mathcal{O}\\ \mathbf{A}\\ \mathbf{B}\end{pmatrix}. \tag{12.6.2}$$

It is easy to prove this second result. The only operators that can be counterterms to a class B operator must themselves vanish by the equations of motion, i.e., they are of class B. A Green's function of a class A operator $\delta\hat{A}_i$ can be written as

$$\langle 0|T\delta\hat{A}_i(y)X|0\rangle = -\langle 0|T\hat{A}_i\delta X|0\rangle. \tag{12.6.3}$$

Thus if the counterterm to $\hat{A}_i$ is $C(\hat{A}_i)$ then

$$\langle 0|T[\hat{A}_i + C(\hat{A}_i)]\delta X|0\rangle$$

is finite if y equals none of the x_i's. Hence the counterterm to $\delta\hat{A}_i$ is $\delta C(\hat{A}_i)$, modulo terms vanishing by the equations of motion. That is, the counterterms to $A_i = \delta\hat{A}_i$ are class A and class B only.

The proof that there are no class C operators is somewhat complicated. It is essentially a mathematical exercise in homology theory. We refer the reader to Joglekar (1977a, b) and Joglekar & Lee (1976) for proofs.

The importance of these results is as follows: both class A and class B operators vanish in physical matrix elements. For class B this is because of the equations of motion. For class A we obtain the matrix element by the LSZ reduction formula from a Green's function $\langle 0|TA_iX|0\rangle$. But there is a Ward identity

$$\langle 0|T\delta\hat{A}_iX|0\rangle = -\langle 0|T\hat{A}_i\delta X|0\rangle. \tag{12.6.4}$$

Whenever X is gauge invariant we get zero. But we may also use elementary field operators in X. A term in δX with a varied field operator $\delta\phi(x)$ has no physical particle pole for this line. This can be seen by examining the possible Feynman graphs, and observing that $\delta\phi$ contains a ghost field.

If the renormalization matrix in (12.6.2) were not triangular then renormalized operators of classes A and B would be non-zero on-shell, even though the unrenormalized operators are zero. The triangularity ensures that for physical matrix elements we have

$$[\mathcal{O}] = Z_{\mathcal{O}\mathcal{O}}\mathcal{O}. \tag{12.6.5}$$

Thus we can disregard the non-invariant operators.

12.6.1 *Caveat*

In practical calculations with Feynman graphs one must beware of taking (12.6.5) too glibly. Consider the calculation of $[G^{a2}_{\mu\nu}]$ in QCD. The only gauge-invariant operator to mix with it is $\bar{\psi}\psi$, so we can write (12.6.5) as

$$\begin{pmatrix} [G^{a2}_{\mu\nu}] \\ [\bar{\psi}\psi] \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{pmatrix} \begin{pmatrix} G^{a2}_{(0)\mu\nu} \\ \bar{\psi}\psi \end{pmatrix}. \tag{12.6.6}$$

The Z_{21}-coefficient is zero because $\bar{\psi}\psi$ has lower dimension than G^2.

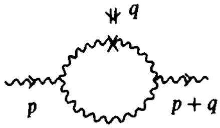


Fig. 12.6.1. One-loop graph for $\langle 0|TAA[G^2_{\mu\nu}]|0\rangle$.

We compute the one-loop term in Z_{11} from the graph of Fig. 12.6.1. There are various tensor structures for the counterterms. The $G^{a2}_{\mu\nu}$ counterterm is proportional to

$$g_{\mu\nu} p\cdot(p+q) - (p+q)_\mu p_\nu.$$

Tensors that vanish on-shell are, for example,

$$g_{\mu\nu}p^2 - p_\mu p_\nu, \quad g_{\mu\nu}(p+q)^2 - (p+q)_\mu(p+q)_\nu.$$

To simplify the calculations it is tempting to set $q=0$. But then these three structures are equal and it is not possible to separate the three coefficients. If one sets $p^2 = 0 = (p+q)^2$ so that p and q are on-shell, and then multiplies the graph by polarization vectors $\varepsilon^\mu \varepsilon'^\nu$ satisfying $\varepsilon\cdot p = 0 = \varepsilon'\cdot(p+q)$, then one has taken a physical matrix element and only the $G^{a2}_{\mu\nu}$ counterterm survives.

The calculations of Kluberg-Stern & Zuber (1975) are at zero momentum, and they have to go to some effort to overcome the above problems.

12.7 Renormalization-group equation

Renormalization-group equations are derived in gauge theories just as in any other theory. One feature that is easy to overlook when making calculations is the variation of the gauge-fixing parameter under a renormalization group transformation. We consider the theory (12.1.1) and let G_{N_c,N_f,N_g} be the renormalized Green's function with N_g external gluons, N_f fermions, N_f antifermions, N_c ghosts, and N_c antighosts. The corresponding bare Green's function is

$$G^{(0)}_{N_c,N_f,N_g} = \tilde{Z}^{N_c} Z_2^{N_f} Z_3^{N_g/2} G_{N_c,N_f,N_g},$$

and it is RG invariant:

$$
\begin{aligned}
0 &= \mu \frac{\mathrm{d}}{\mathrm{d}\mu} G^{(0)} \\
&= \left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} - \gamma_M M \frac{\partial}{\partial M} - \delta \xi \frac{\partial}{\partial \xi} \right) G^{(0)}.
\end{aligned} \tag{12.7.1}
$$

Here β, γ_M and δ are obtained by requiring g_0, M_0, and $\xi_0 = Z_3 \xi$ to be RG invariant.

In the minimal subtraction scheme we have

$$
\left.
\begin{aligned}
g_0 &= \mu^{2-d/2} g \left[1 + \sum_{n=1}^{\infty} a_n(g)/(4-d)^n \right], \\
Z_M &= 1 + \sum_{n=1}^{\infty} b_n(g)/(4-d)^n, \\
Z_i &= 1 + \sum_{n=1}^{\infty} c_{i,n}(g, \xi)/(4-d)^n.
\end{aligned}
\right\} \tag{12.7.2}
$$

Here Z_i stands for Z_2, Z_3, or $\tilde{Z}$. The renormalization-group coefficients are

$$
\left.
\begin{aligned}
\beta &= (d/2-2) g_0 / (\partial g_0 / \partial g) \\
&= (d/2-2) g + \tfrac{1}{2} g \partial a_1 / \partial g, \\
\gamma_M &= \beta \partial \ln Z_M / \partial g = -\tfrac{1}{2} g \partial b_1 / \partial g, \\
\gamma_i &= \beta \partial \ln Z_i / \partial g = -\tfrac{1}{2} g \partial c_{i,1} / \partial g, \\
\delta &= \gamma_3 = -\tfrac{1}{2} g \partial c_{3,1} / \partial g.
\end{aligned}
\right\} \tag{12.7.3}
$$

Similar formulae hold in any other subtraction scheme.

Observe that, by the results of Section 12.4, g_0 and Z_M are independent of ξ so that β and γ_M are independent of ξ. But Z_2, Z_3 and $\tilde{Z}$ depend on ξ, so that γ_2, γ_3, $\tilde{\gamma}$ and δ also depend on ξ. As a special case, in an abelian theory, $g_0 = \mu^{2-d/2} g Z_3^{-1/2}$, so that Z_3, γ_3 and δ are independent of ξ.

The renormalization-group equations for renormalized Green's functions are then

$$
\begin{aligned}
0 &= \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu} + N_c \tilde{\gamma} + N_f \gamma_2 + \tfrac{1}{2} N_g \gamma_3 \right) G \\
&= \left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} - \gamma_M M \frac{\partial}{\partial M} - \delta \xi \frac{\partial}{\partial \xi} + N_c \tilde{\gamma} + N_f \gamma_2 + \tfrac{1}{2} N_g \gamma_3 \right) G.
\end{aligned} \tag{12.7.4}
$$

Since the anomalous dimensions are ξ-dependent, the solutions of the RG equation are a little complicated. The most convenient gauge to use is the Landau gauge $\xi = 0$, for then ξ does not vary when a renormalization group transformation is made. Tarasov Vladimirov & Zharkov (1980) have computed β to three-loop order in this theory.

12.8 Operator-product expansion

Consider the Green's function

$$M(y; x_1, \ldots, x_N) = \langle 0 | Tj(y)j(0)X | 0 \rangle. \tag{12.8.1}$$

where j is a renormalized gauge-invariant operator, and X is a product of local fields at points $x_1, \ldots, x_N$. We have the Wilson expansion

$$M(y; x_1, \ldots, x_N) \sim \sum_i C_i(y) \langle 0 | T[\mathcal{O}_i(0)] X | 0 \rangle. \tag{12.8.2}$$

The sum is over all renormalized operators of the appropriate dimension. This expansion is proved in a gauge theory the same way as in any other theory.

In the application to deep-inelastic scattering in Chapter 13, we will take X to be a product of two gauge-invariant operators and use the LSZ reduction formula to obtain a matrix element:

$$\begin{aligned} W &\equiv \langle P | Tj(y)j(0) | P \rangle \\ &= \sum_i C_i \langle P | [\mathcal{O}_i(0)] | P \rangle. \end{aligned} \tag{12.8.3}$$

We would like to show that only gauge-invariant operators need be included in (12.8.3). Now the proof of the operator-product expansion (in Chapter 10) treats the right-hand side of the expansion in the same way as renormalization counterterms. So the method we applied to the renormalization of gauge-invariant operators also applies to the Wilson expansion. The operators $\mathcal{O}_i$ are either gauge invariant, are the BRS variation of something, or vanish by the equations of motion. Then if we keep $y \neq 0$ in (12.8.3) (as is the case in applications) we only need gauge-invariant operators.

The renormalized operator j has, according to Section 12.6, the form

$$j = j_{\rm GI} + \delta_{\rm R} \hat{A} + B, \tag{12.8.4}$$

where $j_{\rm GI}$ is gauge invariant and B vanishes by the equation of motion. Hence if we take a matrix element of $j(y)j(0)$, like (12.8.3), (or if we take a gauge-invariant Green's function of $j(y)j(0)$), then we can drop the A and B terms, so that

$$\langle P | Tj(y)j(0) | P \rangle = \langle P | Tj_{\rm GI}(y) j_{\rm GI}(0) | P \rangle. \tag{12.8.5}$$

We now follow our proof in Section 12.6, starting with the Ward identity

$$\begin{aligned} 0 &= \delta_{\rm BRS} \langle 0 | Tj(y)j(0)X | 0 \rangle \\ &= \langle 0 | Tj(y)j(0)\delta X | 0 \rangle. \end{aligned} \tag{12.8.6}$$

Here we assumed that the positions y_μ, $x_{i\mu}$, and 0 are all distinct, so that

$$\delta j = \delta B = 0$$

by the equations of motion. But if $y \to 0$, then none of the $x_{i\mu}$'s are at the origin, so that:

$$\langle 0|Tj(y)j(0)\delta X|0\rangle \sim \sum_i C_i(y)\langle 0|T[\mathcal{O}_i(0)]\delta X|0\rangle$$
$$= -\sum_i C_i(y)\langle 0|T[\delta\mathcal{O}_i(0)]X|0\rangle. \qquad (12.8.7)$$

But the left-hand side is zero, by (12.8.6), while the right-hand side is its leading behavior as $y \to 0$ and is therefore also zero. If an arbitrary Green's function of an operator is zero, then the operator itself is zero, i.e.,

$$\sum_i C_i\,[\delta\mathcal{O}_i] = 0. \qquad (12.8.8)$$

Hence the operators needed in the expansion are those that are BRS invariant. As in Section 12.6, this means they are either gauge invariant or of classes A and B. In physical matrix elements like (12.8.3) we can therefore restrict the operators to be gauge invariant.

The practical use of this result is to compute the Wilson coefficients by taking the state $|P\rangle$ to be a state of one on-shell quark or gluon. The infrared divergences are regulated by going to $d > 4$. The Wilson coefficients and the gauge-invariant renormalization counterterms of the $\mathcal{O}_i$'s can be unambiguously obtained provided care is taken to separate the IR from the UV divergences (both of which appear as poles at $d = 4$).

12.9 Abelian theories: with and without photon mass

Consider QED with a possible mass term for the photon. The Lagrangian can be expressed in terms of unrenormalized or renormalized fields:

$$\mathcal{L}_{\text{inv}} = -\tfrac{1}{4}(F_{\mu\nu}^{(0)})^2 + \tfrac{1}{2}m_0^2(A_\mu^{(0)})^2 + \bar{\psi}_0(\mathrm{i}\not{D} - M_0)\psi_0$$
$$= -\tfrac{1}{4}Z_3F_{\mu\nu}^2 + \tfrac{1}{2}m_0^2Z_3A_\mu^2 + Z_2\bar{\psi}(\mathrm{i}\not{D} - M_0)\psi. \qquad (12.9.1)$$

Here we have a single vector field A_μ, and we let

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad (12.9.2)$$

$$D_\mu\psi = (\partial_\mu - \mathrm{i}e_0A_\mu^{(0)})\psi$$
$$= (\partial_\mu - \mathrm{i}e_0Z_3^{1/2}A_\mu)\psi. \qquad (12.9.3)$$

We will call A_μ the photon. Without a mass term for the photon, the Lagrangian (12.9.1) is invariant under the gauge transformation

$$\left.\begin{aligned} A_\mu &\to A_\mu + \partial_\mu\omega, \\ \psi &\to \psi\exp(\mathrm{i}e_0Z_3^{1/2}\omega), \\ \bar{\psi} &\to \bar{\psi}\exp(-\mathrm{i}e_0Z_3^{1/2}\omega). \end{aligned}\right\} \qquad (12.9.4)$$

With a mass term the Lagrangian is not invariant, but the theory can be consistently treated, as we will see.

In contrast, a non-abelian theory with a gluon mass term is not consistent. For example, there are unphysical states – the Faddeev–Popov ghosts – which violate the spin-statistics theorem, and the negative metric gluon states. In the massless theory these cancel in sums over intermediate states, but in a massive non-abelian theory they do not cancel. These problems do not occur in the abelian theory with a massive photon.

12.9.1 *BRS treatment of massive photon*

Even with a mass term for the photon, the Lagrangian is BRS invariant, if we use the standard gauge fixing. We write

$$\mathscr{L} = \mathscr{L}_{\text{inv}} - (1/2\xi)\partial\cdot A^2 + \partial_\mu \bar{c}\partial^\mu c - \xi m_0^2 Z_3 \bar{c}c. \tag{12.9.5}$$

and define BRS variations

$$\begin{gathered}\delta A_\mu = \partial_\mu c, \quad \delta\psi = \mathrm{i}e_0 Z_3^{1/2} c\psi, \quad \delta\bar{\psi} = -\mathrm{i}e_0 Z_3^{1/2}\bar{\psi}c, \\ \delta c = 0, \quad \delta\bar{c} = -\partial\cdot A/\xi.\end{gathered} \tag{12.9.6}$$

Then $\mathscr{L}$ is BRS invariant, aside from an irrelevant total divergence. Since the Faddeev–Popov ghost is a free field, it can be omitted without affecting any physics (except gravity).

We may use the general methods of Section 12.2, with the result that the theory is renormalizable. Since the ghost is a free field, the renormalization factors X and $\tilde{Z}$ are both unity. Hence

$$e_0 = e_{\mathrm{R}} Z_3^{-1/2} \tag{12.9.7}$$

and the renormalization of e is gauge independent. (If we use minimal subtraction we write $e_{\mathrm{R}} = \mu^{2-d/2}e$.) We now find that $e_0 A_\mu^{(0)} = e_{\mathrm{R}} A_\mu$, and that the covariant derivative and gauge transformations simplify:

$$D_\mu\psi = (\partial_\mu - \mathrm{i}e_{\mathrm{R}} A_\mu)\psi, \tag{12.9.8}$$

$$\psi \to \mathrm{e}^{\mathrm{i}e_{\mathrm{R}}\omega}\psi. \tag{12.9.9}$$

Using this method we also find that $m_0^2 = Z_3^{-1} m^2$, so that the ghost field has finite mass. Then the Lagrangian is

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{4}Z_3 F_{\mu\nu}^2 + \tfrac{1}{2}m^2 A_\mu^2 - (1/2\xi)\partial\cdot A^2 \\ &+ Z_2\bar{\psi}(\mathrm{i}\partial\!\!\!/ + eA\!\!\!/ - M_0)\psi,\end{aligned} \tag{12.9.10}$$

with the ghost ignored, since it is a free field. Note that there is no counterterm for the photon mass.

Similar methods apply in theories where the gauge group is a product of a non-abelian group and one or more $U(1)$ factors. The Weinberg–Salam theory of weak and electromagnetic interactions is a simple case – with gauge group $SU(2)\otimes(1)$.

12.9.2 *Elementary treatment of abelian theory with photon mass*

The BRS methods are much more sophisticated than necessary for the abelian theory, so we will now treat the theory by elementary methods. First we write the free photon propagator for the perturbation theory of the Lagrangian (12.9.10):

$$\begin{aligned} D_{\mu\nu} &= \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon}(-g_{\mu\nu} + k_\mu k_\nu/m^2) - \frac{\mathrm{i}}{m^2}\frac{k_\mu k_\nu}{k^2 - \xi m^2 + \mathrm{i}\varepsilon} \\ &= \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + \frac{(1-\xi)k_\mu k_\nu}{k^2 - \xi m^2 + \mathrm{i}\varepsilon}\right]. \end{aligned} \tag{12.9.11}$$

If $m = 0$, then we cannot remove the gauge-fixing term, for then the propagator does not exist. (This is the same as taking the limit $\xi \to \infty$.) But if m is non-zero, then the propagator exists when the gauge-fixing term is removed. However it behaves like $\mathrm{i}k_\mu k_\nu/k^2 m^2$, rather than $1/k^2$. So the theory with $m = 0$ and $\xi = \infty$ has worse divergences than usual and is not manifestly renormalizable. Even so, physical quantities are independent of ξ and the theory is renormalizable if ξ is finite, as we will see. Hence enough cancellations are present as $\xi \to \infty$ that the theory remains renormalizable if we only compute physical quantities.

Our treatment makes extensive use of the equations of motion:

$$\begin{aligned} \mathscr{L}_{A_\mu} &\equiv \frac{\Delta S}{\Delta A_\mu} \\ &\equiv Z_3 m_0^2 A^\mu + Z_3(\Box A^\mu - \partial^\mu\partial\cdot A) + \partial^\mu\partial\cdot A/\xi + e_0 Z_3^{1/2}\bar\psi\gamma^\mu\psi Z_2 \\ &= 0, \\ \mathscr{L}_{\bar\psi} &\equiv \frac{\Delta S}{\Delta\bar\psi} \equiv Z_2(\mathrm{i}\not{D} - M_0)\psi = 0, \\ \mathscr{L}_\psi &\equiv \frac{\Delta S}{\Delta\psi} \equiv Z_2\bar\psi(-\mathrm{i}\overleftarrow{\not{D}} - M_0) = 0. \end{aligned} \tag{12.9.12}$$

We have not assumed the relation (12.9.7). Taking the divergence of the gauge-field equation of motion gives

$$0 = (\Box/\xi + m^2)\partial\cdot A. \tag{12.9.13}$$

Here, we have used the invariance of the theory under the global transformations, which are (12.9.4) with constant ω. The Noether current for this invariance is the electromagnetic current:

$$j^\mu = -e_0 Z_3^{1/2} Z_2 \bar{\psi}\gamma^\mu\psi. \tag{12.9.14}$$

This is conserved, by the electron's equation of motion.

The result (12.9.13) that $\partial \cdot A$ is a free field is important for three reasons:

(1) It would otherwise be difficult to interpret the longitudinal part of A_μ, which has unphysical properties.
(2) The ξ-dependent part of the propagator is confined to the $k_\mu k_\nu$ term, which only contributes to $\partial \cdot A$. Then the ξ-dependence decouples from physics if $\partial \cdot A$ is free.
(3) Similarly the bad ultra-violet behavior when $\xi \to \infty$ is decoupled from physics.

To make these statements precise we will examine the Green's functions. We will prove directly:

(1) Ward identities,
(2) $m_0^2 = Z_3^{-1} m^2$, so that no mass counterterm is needed for the photon,
(3) no counterterms are needed for the $\partial \cdot A^2$ term,
(4) $e_0 = e_R Z_3^{-1/2}$, with e_R finite,
(5) Z_2 but not Z_3 depends on ξ,
(6) the S-matrix is independent of ξ.

We will also compute the exact ξ-dependence of Z_2 when minimal subtraction is used, and we will compute the exact ξ-dependence of the residue of the electron's propagator pole.

12.9.3 *Ward identities*

Green's functions of $\mathscr{L}_A$, $\mathscr{L}_\psi$ and $\mathscr{L}_{\bar{\psi}}$ are non-zero, since the derivatives in these operators are implicitly taken outside the time-ordering. To obtain the Ward identity corresponding to (12.9.13), we use identities like

$$\langle 0| T\mathscr{L}_{A_\mu}(x) X |0\rangle = \mathrm{i}\langle 0| T\frac{\Delta X}{\Delta A_\mu(x)}|0\rangle. \tag{12.9.15}$$

Now

$$(\Box/\xi + m^2)\partial \cdot A = \partial_\mu \mathscr{L}_{A_\mu} - \mathrm{i}e\bar{\psi}\mathscr{L}_{\bar{\psi}} + \mathrm{i}e\mathscr{L}_\psi \psi. \tag{12.9.16}$$

We choose $e_0 = e_R Z_3^{-1/2}$, $m_0^2 = Z_3^{-1} m^2$. This gives a Lagrangian of the form (12.9.10). There are no counterterms for A^2 and $\partial \cdot A^2$, and the counterterm for $\bar{\psi}\not{A}\psi$ is proportional to the counterterm $Z_2 - 1$ for $\bar{\psi}\not{\partial}\psi$. We will prove later that this is correct. But for the moment we will choose to have our Lagrangian in this form. Our gambit is that if extra counterterms should be needed, then they will not be available and the Green's functions will diverge.

From (12.9.16) it follows that

$$
\begin{aligned}
(\Box_x/\xi + m^2)&\langle 0|T\partial\cdot A(x)X|0\rangle \\
&= \mathrm{i}\frac{\partial}{\partial x^\mu}\langle 0|T\Delta X/\Delta A_\mu(x)|0\rangle \\
&\quad + e_\mathrm{R}\langle 0|T\bar{\psi}(x)\Delta X/\Delta\bar{\psi}(x)|0\rangle - e_\mathrm{R}\langle 0|T\psi(x)\Delta X/\Delta\psi(x)|0\rangle \\
&= \mathrm{i}\frac{\delta_{\text{gauge}}}{\delta\omega(x)}\langle 0|TX|0\rangle,
\end{aligned}
\tag{12.9.17}
$$

where the gauge variation is computed according to (12.9.4).

A convenient way to write these results is as follows:

(1) Let ϕ be a free scalar boson field of mass $\xi^{1/2}m$.

(2) Let $\hat{\delta}$(field) = gauge variation of 'field' with parameter ϕ, i.e.,

$$
\begin{aligned}
\hat{\delta}A_\mu &= \partial_\mu\phi, \\
\hat{\delta}\psi &= \mathrm{i}e_\mathrm{R}\phi\psi, \\
\hat{\delta}\bar{\psi} &= -\mathrm{i}e_\mathrm{R}\phi\bar{\psi}.
\end{aligned}
$$

(3) Then (12.9.17) is

$$
(1/\xi)\langle 0|T\partial\cdot AX|0\rangle = \langle 0|T\phi(x)\hat{\delta}X|0\rangle. \tag{12.9.18}
$$

The field ϕ is similar to the Faddeev–Popov ghost except for being a boson (this will be important later). In a non-abelian theory the nilpotence of the BRS transformation is crucial to proving Ward identities and is proved using the anticommutation of c with itself. A field introduced in the same way as ϕ has to be the ghost field and must be a fermion. In (12.9.18) we can choose ϕ to be a boson.

12.9.4 *Counterterms proportional to A^2 and $\partial\cdot A^2$*

Let us apply (12.9.17) to the case $X = A_\nu(y)$:

$$
(\Box_x/\xi + m^2)\frac{\partial}{\partial x^\mu}\langle 0|TA^\mu(x)A_\nu(y)|0\rangle = \mathrm{i}\frac{\partial}{\partial x^\nu}\delta^{(d)}(x-y). \tag{12.9.19}
$$

This equation implies that no counterterms proportional to A^2 or to $\partial\cdot A^2$ are needed. For suppose otherwise. Then consider the lowest order in which there is a divergence in the photon propagator. The divergence comes from the insertion of a divergent self-energy graph in the free photon propagator. The divergence has the form of Fig. 12.9.1, where the cross denotes the counterterm to this divergence. Power-counting indicates that

https://doi.org/10.1017/CBO9780511622656.013 Published online by Cambridge University Press

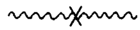


Fig. 12.9.1. Counterterm to photon's self-energy.

the only divergences are proportional to $k_\mu k_\nu$ and $g_{\mu\nu}$. A divergence $g_{\mu\nu}k^2 - k_\mu k_\nu$ is cancelled by the wave-function renormalization Z_3 for the photon. Insertion of the extra counterterms in the propagator gives a divergent contribution to the left-hand side of (12.9.19). But the left-hand side is finite. So there are in fact no divergences. Hence $m_0^2 = Z_3^{-1}m^2$. It also follows that the photon self-energy is transverse.

Applying (12.9.17) to $X = A_\mu(y)A_\nu(z)$ and to $A_\kappa(w)A_\lambda(y)A_\nu(z)$ shows that no counterterms cubic or quartic in A are needed.

12.9.5 *Relation between e_0 and Z_3*

Apply (12.9.17) to $X = \psi(y)\bar{\psi}(z)$:

$$(\Box_x/\xi + m^2)\frac{\partial}{\partial x^\mu}\langle 0|TA^\mu(x)\psi\bar{\psi}|0\rangle$$
$$= e\langle 0|T\psi(y)\bar{\psi}(z)|0\rangle[\delta(x-y)-\delta(x-z)]. \qquad (12.9.20)$$

We assumed $e_0 = Z_3^{-1/2}e_{\mathrm{R}}$. If this is not the correct counterterm, then let the ψ–$\bar{\psi}$–A 1PI vertex first diverge at N-loops. Then the left-hand side of (12.9.20) diverges at order e_{R}^{2N+1}, while there is a counterterm $Z_2 - 1$ available to make the right-hand side finite. Hence the left-hand side does not diverge.

12.9.6 *Gauge dependence*

The gauge variation of a Green's function $\langle 0|TX|0\rangle$ is

$$\frac{\partial}{\partial\xi}\langle 0|TX|0\rangle = \frac{\mathrm{i}}{2\xi^2}\int \mathrm{d}^4x\langle 0|T[\partial\cdot A^2(x) - \langle 0|\partial\cdot A^2|0\rangle]X|0\rangle$$
$$+\mathrm{i}\frac{\partial \ln Z_2}{\partial\xi}\int \mathrm{d}^4x\langle 0|T\bar{\psi}\mathscr{L}_{\bar{\psi}}X|0\rangle$$
$$= \frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x\langle 0|T\partial\cdot A(x)\phi(x)\hat{\delta}X|0\rangle$$
$$- N_\psi\langle 0|TX|0\rangle\partial\ln Z_2/\partial\xi$$
$$= \frac{\mathrm{i}}{2}\int \mathrm{d}^4x\langle 0|T[\phi^2(x) - \langle 0|\phi^2|0\rangle]\hat{\delta}^2X|0\rangle$$
$$- N_\psi\langle 0|TX|0\rangle\partial\ln Z_2/\partial\xi. \qquad (12.9.21)$$

Here N_ψ is the number of ψ fields in X (which equals the number of $\bar{\psi}$ fields).

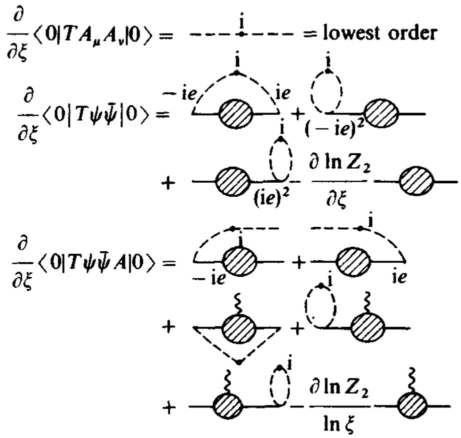


Fig. 12.9.2. Examples of (12.9.21).

The disappearance of the factor $\frac{1}{2}$ in the last equality is due to the identical nature of the two ϕ fields. Also, $\phi^2 - \langle 0|\phi^2|0\rangle$ appears, rather than ϕ^2, because the pure vacuum graphs never occur on the left-hand side. (A better derivation uses two different fields ϕ_1 and ϕ_2 for each application of the Ward identities.) Notice that we cannot derive this equation if ϕ is a fermion, for then $\phi^2 = 0$. So, in a non-abelian theory, the derivation must be replaced by the more complicated one which leads to the weaker (12.4.4). The graphical structure of (12.9.21) for a few simple cases is shown in Fig. 12.9.2.

From (12.9.21) it is immediate that all Green's functions of gauge-invariant fields are ξ-independent.

The only graphs with UV divergences are those with the simple loop (Fig. 12.9.3) attached to a $\bar\psi$ or a ψ vertex together with those with the $\partial \ln Z_2/\partial\xi$ factor. We therefore find that

$$-\frac{\partial \ln Z_2}{\partial \xi} + \mathrm{i}e_{\mathrm{R}}^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - \xi m^2)^2} \tag{12.9.22}$$

is UV finite. Hence

$$\frac{\partial \ln Z_2}{\partial \xi} = -\frac{e^2}{16\pi^2}\Gamma(2 - d/2)\left(\frac{\xi m^2}{4\pi\mu^2}\right)^{d/2-2} + \text{finite}.$$

Fig. 12.9.3. All UV divergent graphs contributing to (12.9.21) contain this loop.

If we use minimal subtraction then

$$Z_2 = Z_2(\xi = 0)\exp\left[\frac{e^2\xi}{8\pi^2(d-4)}\right]. \qquad (12.9.23)$$

To obtain the ξ-dependence of the S-matrix, we recall the LSZ formula. It tells us to consider the corresponding Green's function, and pick out the poles in its external momenta. Since we use transverse polarization ($\varepsilon\cdot k = 0$) for photons this picks out graphs with the loop Fig. 12.9.3 and the $\partial \ln Z_2/\partial\xi$ terms. Then

$$S = \prod_{\substack{\text{ext}\\ \text{Fermions}}} \left(\frac{\not p - M_{\text{ph}}}{\mathrm{i}z_2^{1/2}}\right) \prod_{\substack{\text{ext}\\ \text{photons}}} \left(\frac{p^2 - m_{\text{ph}}^2}{\mathrm{i}z_3^{1/2}}\right)\langle 0|T\tilde{X}|0\rangle. \qquad (12.9.24)$$

From Fig. 12.9.2 for $\langle 0|T\psi\bar{\psi}|0\rangle$ and for $\langle 0|TA_\mu A_\nu|0\rangle$, we see that the physical masses and the residue of the photon pole are ξ-independent, while the gauge dependence of z_2 exactly cancels the gauge dependence of the particle pole coefficient of $\langle 0|T\tilde{X}|0\rangle$.

We can compute explicitly the ξ-dependence of the residue, z_2, of the pole of the fermion propagator, if we use minimal subtraction:

$$\begin{aligned}\frac{\partial \ln z_2}{\partial \xi} &= \lim_{d\to 4}(12.9.22)\\ &= \frac{e^2}{16\pi^2}\left[\gamma + \ln\left(\frac{\xi m^2}{4\pi\mu^2}\right)\right],\end{aligned}$$

i.e.,

$$z_2 = z_2(\xi = 0)\exp\left\{\frac{e^2\xi}{16\pi^2}\left[\gamma - 1 + \ln\left(\frac{\xi m^2}{4\pi\mu^2}\right)\right]\right\}. \qquad (12.9.25)$$

Since $\xi_0 = \xi Z_3$, $e_0^2 = Z_3^{-1}e^2$, and $m_0^2 = Z_3^{-1}m^2$, the combinations $e^2\xi$ and ξm^2 in (12.9.25) are RG invariant.

12.9.7 *Renormalization-group equation*

Using our knowledge of the renormalization of e_0, ξ_0, and m_0^2 we find the RG equation for a Green's function of N_f ψ's, N_f $\bar{\psi}$'s, and N_A A's to be

$$0 = (\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial e} - \gamma_M M\frac{\partial}{\partial M} + \gamma_3 m^2\frac{\partial}{\partial m^2} - \gamma_3\xi\frac{\partial}{\partial\xi} + N_f\gamma_2 + \tfrac{1}{2}N_A\gamma_3)G, \qquad (12.9.26)$$

with

$$\beta = \tfrac{1}{2}e\gamma_3 + (d/2 - 2)e.$$

If we use minimal subtraction then the only ξ-dependent coefficient is γ_2, and from (12.9.23) we find that

$$\gamma_2 = \gamma_2(\xi = 0) + \xi e^2/8\pi^2. \tag{12.9.27}$$

The results (12.9.23), (12.9.25), and (12.9.27) were derived by Lautrup (1976) and by Collins (1975a).

12.10 Unitary gauge for massive photon

The unitary gauge is the limit $\xi \to \infty$. Since the S-matrix and gauge-invariant operators are ξ-independent, this limit exists for them. But for gauge-variant operators there are severe ultra-violet divergences. Thus the limits $\xi \to \infty$, $d \to 4$ do not commute.

We may take $\xi \to \infty$ first, in the regulated theory. The resulting UV divergences at $d = 4$ may be cancelled by extra counterterms beyond those that we have already considered. Since the S-matrix is gauge-independent, all these counterterms must vanish by the equations of motion.