Trivial Conservation Laws

There are two distinct ways in which a conservation law could trivially hold. In the *first kind* of triviality, the p-tuple P itself in (4.22) vanishes for all solutions of the given system. This type of triviality is usually easy to eliminate by solving the system and its prolongations $\Delta^{(k)}$ for certain of the variables u_J^α in terms of the remaining variables, and substituting for these distinguished variables wherever they occur. For example, in the case of an evolution equation $u_t = P(x, u^{(n)})$, we can always solve for any time derivative of u, e.g. u_{tt}, u_{xt}, etc., solely in terms of x, u and spatial derivatives of u. As a net result, any dynamical conservation law is equivalent, up to the addition of a trivial conservation law of the first kind, to a conservation law in which the conserved density T depends only on x, t, u and spatial derivatives of u. For evolution equations, this is the usual form of a conservation law.

Example 4.23. Consider the system of first order evolution equations

$$u_t = v_x, \qquad v_t = u_x,$$

which is equivalent to the one-dimensional wave equation $u_{tt} = u_{xx}$. The expression

$$D_t(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2) - D_x(u_t u_x) = u_t(u_{tt} - u_{xx}) = 0$$

is clearly a conservation law. According to the above remarks, we can replace the conserved density and flux by ones depending on spatial derivatives, resulting in the equivalent conservation law

$$D_t(\tfrac{1}{2}u_x^2 + \tfrac{1}{2}v_x^2) - D_x(u_x v_x) = 0.$$

These differ by the trivial conservation law

$$D_t(\tfrac{1}{2}u_t^2 - \tfrac{1}{2}v_x^2) + D_x(v_x u_x - u_t u_x) = 0,$$

whose density and flux both vanish on solutions of the system.

A second possible type of triviality occurs when the divergence identity

$$\operatorname{Div} P = 0$$

holds for *all* functions $u = f(x)$, regardless of whether they solve the given system of differential equations. For example, in the case $p = 2$ the identity

$$D_x(u_y) - D_y(u_x) \equiv 0$$

clearly holds for any smooth function $u = f(x, y)$, and hence provides a trivial conservation law of the *second kind* for any partial differential equation involving $u = f(x, y)$. A less obvious example is the identity

$$D_x(u_y v_z - u_z v_y) + D_y(u_z v_x - u_x v_z) + D_z(u_x v_y - u_y v_x) \equiv 0$$

involving Jacobian determinants. Any such p-tuple $P(x, u^{(n)})$, whose divergence vanishes identically, is called a *null divergence*. The conservation law offered by any null divergence does not depend on the particular structure of any given system of differential equations, and we are thus justified in labelling these laws as trivial.

As with the Poincaré lemma, which characterizes the kernel of the ordinary divergence operator (cf. Example 1.62), there is a similar characterization of all null divergences/trivial conservation laws of the second kind.

Theorem 4.24. *Suppose $P = (P_1, \ldots, P_p)$ is a p-tuple of smooth functions depending on $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ and derivatives of u, defined on all of the jet space $X \times U^{(n)}$. Then P is a null divergence*: $\operatorname{Div} P \equiv 0$ *if and only if there exist smooth functions Q_{jk}, $j, k = 1, \ldots, p$, depending on x, u and derivatives of u, such that*

$$Q_{jk} = -Q_{kj}, \qquad j, k = 1, \ldots, p, \tag{4.25}$$

and

$$P_j = \sum_{k=1}^{p} D_k Q_{jk}, \qquad j = 1, \ldots, p, \tag{4.26}$$

for all $(x, u^{(n)})$.

In particular, if $p = 3$, Theorem 4.24 says that

$$\operatorname{Div} P = D_1 P_1 + D_2 P_2 + D_3 P_3 \equiv 0$$

if and only if P is a "total curl": $P = \operatorname{Curl} Q$, i.e.

$$P_1 = D_2 Q_3 - D_3 Q_2, \qquad P_2 = D_3 Q_1 - D_1 Q_3, \qquad P_3 = D_1 Q_2 - D_2 Q_1.$$

(Here we identify $Q_{12} = -Q_{21}$ with Q_3, etc.) For our previous example,

$$P = (u_y v_z - u_z v_y,\, u_z v_x - u_x v_z,\, u_x v_y - u_y v_x),$$

and the corresponding Q is (uv_x, uv_y, uv_z).

Although for any fixed function $u = f(x)$, Theorem 4.24 reduces to the Poincaré lemma, the fact that the resulting Q_{jk} can be taken to depend just on x, u and derivatives of u for *all* such functions is a considerably more delicate matter. The proof turns out to be rather complicated, and will be deferred until Section 5.4, when we have considerably more algebraic machinery at our disposal.

In general, a *trivial conservation law* will be, by definition, a linear combination of trivial laws of the above two kinds. In other words $\operatorname{Div} P = 0$ is a trivial conservation law of the system if and only if there exist functions Q_{jk} satisfying (4.25) such that (4.26) holds for all solutions of Δ. Two conservation laws P and $\tilde{P}$ are *equivalent* if they differ by a trivial conservation law, so $\tilde{P} = P + R$ where R is trivial. We will only be interested in classifying conservation laws up to equivalence, so by "conservation law" in general we really mean "equivalence class of conservation laws".

Characteristics of Conservation Laws

Consider a conservation law of a totally nondegenerate system of differential equations $\Delta(x, u^{(n)}) = 0$. According to Exercise 2.35, Div P vanishes on all solutions of the system if and only if there exist functions $Q_\nu^J(x, u^{(m)})$ such that

$$\operatorname{Div} P = \sum_{\nu, J} Q_\nu^J D_J \Delta_\nu. \tag{4.27}$$

for all (x, u). Now, each of the terms in (4.27) can be integrated by parts; for example, if $1 \leqslant j \leqslant p$,

$$Q_\nu^j D_j \Delta_\nu = D_j(Q_\nu^j \Delta_\nu) - D_j(Q_\nu^j)\Delta_\nu.$$

In this way, we obtain an equivalent identity

$$\operatorname{Div} P = \operatorname{Div} R + \sum_{\nu=1}^{l} Q_\nu \Delta_\nu \equiv \operatorname{Div} R + Q \cdot \Delta,$$

in which the l-tuple $Q = (Q_1, \ldots, Q_l)$ has entries

$$Q_\nu = \sum_J (-D)_J Q_\nu^J, \tag{4.28}$$

and $R = (R_1, \ldots, R_p)$ (whose precise expression is not required here) depends linearly on the components Δ_ν of the given system of differential equations and their total derivatives. Thus R is a trivial conservation law (of the first kind), and if we replace P by $P - R$, we have an equivalent conservation law of the special form

$$\operatorname{Div} P = Q \cdot \Delta. \tag{4.29}$$

We call (4.29) the *characteristic form* of the conservation law (4.27), and the l-tuple $Q = (Q_1, \ldots, Q_l)$ the *characteristic* of the given conservation law.

In general, unless $l = 1$ the characteristic of a given conservation law is not uniquely determined, this stemming from the fact that the Q_ν in (4.29) are not uniquely determined. Note that if Q and $\tilde{Q}$ both satisfy (4.29) for the same P, then $Q \cdot \Delta = \tilde{Q} \cdot \Delta$. Since Δ is nondegenerate, Proposition 2.11 implies that $Q - \tilde{Q}$ vanishes on all solutions. This motivates the definition of a *trivial characteristic* Q as one which vanishes for all solutions of the system. Two characteristics Q and $\tilde{Q}$ are *equivalent* if they differ by a trivial characteristic, so $Q = \tilde{Q}$ for all solutions $u = f(x)$ to Δ. In general, characteristics are only determined up to equivalence.

Example 4.25. In order to find the characteristic for the conservation law of the wave equation in Example 4.23, we need to rewrite the left-hand side in the form (4.27), which is

$$D_t(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2) - D_x(u_x u_t) = u_t D_t(u_t - v_x) + u_t D_x(v_t - u_x).$$

Therefore, according to (4.28), the characteristic is

$$Q = (-D_t(u_t), -D_x(u_t)) = (-u_{tt}, -u_{xt}),$$

and there is an equivalent conservation law in characteristic form, which is found by integrating by parts:

$$D_t(\tfrac{1}{2}u_x^2 - \tfrac{1}{2}u_t^2 + u_t v_x) + D_x(-u_t v_t) = -u_{tt}(u_t - v_x) - u_{xt}(v_t - u_x).$$

It is important to note that replacing the t-derivatives by x-derivatives in this conservation law will, as in Example 4.23, lead to an equivalent conservation law, but that this will *not* in general remain in characteristic form. In the present example, the conserved density is equivalent to $\frac{1}{2}u_x^2 + \frac{1}{2}v_x^2$, the flux to $-u_x v_x$, but the resulting conservation law

$$D_t(\tfrac{1}{2}u_x^2 + \tfrac{1}{2}v_x^2) + D_x(-u_x v_x) = u_x(u_{xt} - v_{xx}) + v_x(v_{xt} - u_{xx})$$

is definitely not in characteristic form. In general, *replacing a conservation law by an equivalent one does not maintain the characteristic form.*

Furthermore, this last conservation law has as its characteristic

$$\tilde{Q} = (-D_x(u_x), -D_x(v_x)) = (-u_{xx}, -v_{xx}),$$

which is *not* the same as our previous characteristic. However, the difference

$$Q - \tilde{Q} = (-(u_{tt} - u_{xx}), -(v_{tt} - v_{xx}))$$

is a trivial characteristic, since it vanishes for all solutions of the system. Thus two equivalent conservation laws can have equivalent, but not identical, characteristics. We finally note that the characteristic forms of the two conservation laws are different, that of the latter law being

$$D_t(\tfrac{1}{2}u_x^2 + \tfrac{1}{2}v_x^2) + D_x(u_x v_x - u_x u_t - v_x v_t) = -u_{xx}(u_t - v_x) - v_{xx}(v_t - u_x).$$

Thus, a *single* (equivalence class of) *conservation laws may have more than one characteristic form.* Finally, we remark that one can add any null divergence to any of the above conservation laws without affecting its validity, or the form(s) of the characteristic.

The preceding example should give the reader a good idea of the algebraic complexity of the general relationship between characteristics and conservation laws. Nevertheless, if we restrict our attention to normal, nondegenerate systems (in particular, normal analytic systems), there is a one-to-one correspondence between equivalence classes of conservation laws and equivalence classes of characteristics, so that each conservation law is uniquely determined by its characteristic and vice versa, provided one keeps the equivalence relations in mind. This result forms the cornerstone for much of the general theory and classification of conservation laws, including Noether's theorem. (Counterexamples in the case of abnormal systems will be discussed in Section 5.3.)

Theorem 4.26. *Let $\Delta(x, u^{(n)}) = 0$ be a normal, totally nondegenerate system of differential equations. Let the p-tuples P and $\tilde{P}$ determine conservation laws with respective characteristics Q and $\tilde{Q}$. Then P and $\tilde{P}$ are equivalent conservation laws if and only if Q and $\tilde{Q}$ are equivalent characteristics.*

Clearly the theorem reduces to proving that a conservation law in characteristic form (4.29) is trivial if and only if its characteristic Q is trivial. Several complications arise because there are two types of triviality for conservation laws which must be treated. The proof itself is quite complicated, and the reader may at first be well advised to skip ahead to Section 4.4 at this point.

As a warm-up exercise for the general proof, we begin with the simple case of a single n-th order ordinary differential equation

$$\Delta(x, u^{(n)}) = \Delta(x, u, u_1, \ldots, u_n) = 0,$$

in which $u_k = d^k u/dx^k$ are the derivatives of the single dependent variable u. A conservation law in characteristic form is

$$D_x P = Q \cdot \Delta,$$

in which $Q(x, u^{(m)})$ is a single function of x, u and the derivatives of u. Note that in this case, the only trivial conservation laws of the second kind are the constants, so the proof will be considerably simplified. First suppose P is a trivial conservation law. Since Δ is nondegenerate,

$$P = \sum_{k=0}^{l} A_k \cdot D_x^k \Delta + c$$

for certain functions A_k, and $c \in \mathbb{R}$. By Leibniz' rule,

$$\begin{aligned} D_x P &= \sum_{k=0}^{l} [D_x A_k \cdot D_x^k \Delta + A_k \cdot D_x^{k+1} \Delta] \\ &= (D_x A_0) \cdot \Delta + \sum_{k=1}^{l} (A_{k-1} + D_x A_k) \cdot D_x^k \Delta + A_l \cdot D_x^{l+1} \Delta. \end{aligned}$$

Equating this to $Q \cdot \Delta$, we find that

$$(D_x A_0 - Q) \cdot \Delta + \sum_{k=1}^{l} (A_{k-1} + D_x A_k) \cdot D_x^k \Delta + A_l \cdot D_x^{l+1} \Delta = 0$$

for all x, u. Now, the prolongations $\Delta^{(l+1)}$ of Δ are assumed to be of maximal rank. According to Proposition 2.11, then, a linear combination of the functions $\Delta, D_x\Delta, \ldots, D_x^{l+1}\Delta$ determining $\Delta^{(l+1)}$ will vanish identically if and only if the coefficients vanish for all solutions of $\Delta^{(l+1)}$. Thus we find

$$A_l = 0, \qquad A_{k-1} + D_x A_k = 0, \qquad k = 1, 2, \ldots, l,$$

and

$$D_x A_0 - Q = 0,$$

whenever $u = f(x)$ is a solution to Δ. An easy induction shows that $A_k = 0$ on solutions to Δ for $k = l, l-1, \ldots, 1, 0$, and hence the final equation requires Q to vanish for all solutions too. This means Q is a trivial characteristic, and hence we've proved that a trivial conservation law necessarily has a trivial characteristic.

In order to prove the converse, we need to solve our equation Δ for the highest order derivative,

$$u_n = \Gamma(x, u, \ldots, u_{n-1}), \tag{4.30}$$

which can be done in a neighbourhood of any point $(x_0, u_0^{(n)})$ at which Δ is normal, which, in the present circumstance means $\partial\Delta(x_0, u_0^{(n)})/\partial u_n \neq 0$. Before continuing, its is important to note that replacing Δ by the algebraically equivalent equation $u_n = \Gamma$ does not affect the structure of the space of conservation laws:

Lemma 4.27. *Suppose Δ and $\tilde{\Delta}$ are two totally nondegenerate systems of partial differential equations which are algebraically equivalent in the sense that their corresponding subvarieties $\mathscr{S}_\Delta$ and $\mathscr{S}_{\tilde{\Delta}}$ in the jet space $M^{(n)}$ coincide*:

$$\mathscr{S}_\Delta = \{(x, u^{(n)})\colon \Delta(x, u^{(n)}) = 0\} = \mathscr{S}_{\tilde{\Delta}} = \{(x, u^{(n)})\colon \tilde{\Delta}(x, u^{(n)}) = 0\}.$$

A p-tuple P is then a conservation law for Δ if and only if it is a conservation law for $\tilde{\Delta}$. It is trivial as a conservation law for Δ if and only if it is trivial as a conservation law for $\tilde{\Delta}$. If $\operatorname{Div} P = Q \cdot \Delta$ *is in characteristic form for Δ, it is also in characteristic form for $\tilde{\Delta}$*: $\operatorname{Div} P = \tilde{Q} \cdot \tilde{\Delta}$. *Finally, Q is a trivial characteristic for Δ if and only if $\tilde{Q}$ is a trivial characteristic for $\tilde{\Delta}$.*

Proof. The statement that a function $R(x, u^{(n)})$ vanishes for all solutions of Δ is, by local solvability, equivalent to saying that $R(x, u^{(n)}) = 0$ whenever $(x, u^{(n)}) \in \mathscr{S}_\Delta$. Clearly this requirement is independent of the particular functions Δ or $\tilde{\Delta}$ used to characterize the subvariety $\mathscr{S}_\Delta$ (or its prolongations). This trivial observation is sufficient to prove all the statements in the lemma save the last one. The requirement that Q be a trivial characteristic means that the expression $Q \cdot \Delta = R$ vanish to *second order* on some appropriate prolongation $\Delta^{(k)}$ of Δ. (This means that both R and all its partial derivatives $\partial R/\partial x^i$, $\partial R/\partial u_J^\alpha$ vanish on the prolonged subvariety $\mathscr{S}_{\Delta^{(k)}}$.) Again, this geometric condition is clearly independent of the particular functions Δ or $\tilde{\Delta}$ used to characterize $\mathscr{S}_\Delta$, and hence also $\mathscr{S}_{\Delta^{(k)}} = \mathscr{S}_{\tilde{\Delta}^{(k)}}$. □

Returning to our proof of Theorem 4.26 in the ordinary differential equation case, we are trying to show that if Q is a trivial characteristic, then $D_x P = Q \cdot \Delta$ is necessarily a trivial conservation law. By Lemma 4.27, we can assume that Δ has the form (4.30). Moreover, differentiating (4.30) and substituting we can find expressions for higher order derivatives u_{n+k}, $k \geqslant 0$, in terms of $x, u, \ldots, u_{n-1}$. These can be substituted into the conservation law P, leading to an equivalent conservation law $P^*(x, u, \ldots, u_{n-1})$ depending on only $(n-1)$-st and lower order derivatives of u.

Now in the general case, as Example 4.25 made clear, replacing a conservation law by an equivalent one does not necessarily preserve the characteristic form itself, and so we have no reason to expect $D_x P^* = 0$ to be in characteristic form. However, since P^* only depends on $(n-1)$-st and lower order derivatives, the only way n-th and higher order derivatives appear in

$$D_x P^* = \frac{\partial P^*}{\partial x} + u_1 \frac{\partial P^*}{\partial u} + \cdots + u_n \frac{\partial P^*}{\partial u_{n-1}}$$

is in the final term. Thus, by local solvability, $D_x P^* = 0$ on solutions if and only if

$$D_x P^* = Q^*(u_n - \Gamma),$$

where $Q^* = \partial P/\partial u_{n-1}$ is the characteristic which, by the first half of the theorem, is equivalent to the original characteristic Q, and is hence also trivial. Moreover, Q^* only depends on $(n-1)$-st and lower order derivatives of u, so the only way that it can be trivial is if it vanishes identically, $Q^* = \partial P/\partial u_{n-1} \equiv 0$. This implies $D_x P^* \equiv 0$, and hence P^* is a trivial conservation law of the second kind. (In the present case this means P^* is a constant!) Thus P is also trivial, and the theorem is proved in this special case.

The proof of Theorem 4.26 in the general case proceeds along similar lines, although the details, especially in the second part of the proof, become much more complicated. First suppose that P is a trivial conservation law, so that by the nondegeneracy of Δ, there exist functions $A_{i\nu}^J(x, u^{(m)})$ such that

$$P_i = \sum_{\nu, J} A_{i\nu}^J D_J \Delta_\nu + R_i, \qquad i = 1, \ldots, p, \tag{4.31}$$

where $R = (R_1, \ldots, R_p)$ is a null divergence. In this case

$$\operatorname{Div} P = \sum_{i,\nu,J} \{D_i A_{i\nu}^J \cdot D_J \Delta_\nu + A_{i\nu}^J D_i D_J \Delta_\nu\}.$$

Assuming P is in characteristic form, we equate this latter expression to $Q \cdot \Delta$, thereby obtaining a linear combination of the derivatives $D_K \Delta_\nu$ which vanishes identically in x and u. Again, by the maximal rank condition on the prolongations of Δ, Proposition 2.11 requires that the coefficient of each derivative $D_K \Delta_\nu$ must vanish whenever $u = f(x)$ is a solution to the system. An easy induction along the same lines as in the ordinary differential equation case shows that each coefficient $A_{i\nu}^J$ must vanish whenever u is a solution, and, finally, each $Q_\nu = 0$ whenever $u = f(x)$ is a solution to Δ. Thus a trivial conservation law necessarily has a trivial characteristic and the first half of the theorem is proved.

To prove the converse, we first need to use a change of independent variables $(y, t) = \psi(x)$ which makes the system Δ equivalent to one in Kovalevskaya form

$$u_{nt}^\nu \equiv \frac{\partial^n u^\nu}{\partial t^n} = \Gamma_\nu(y, t, \widetilde{u^{(n)}}), \qquad \nu = 1, \ldots, q, \tag{4.32}$$

the Γ_ν depending on all derivatives u_J^α up to order n except the u_{nt}^α. (See Theorem 2.79. The extension to the more general Kovalevskaya form (2.123) is not difficult, but the notation is more complicated, so this will be left to the reader.) Using Lemma 4.27, it suffices to prove that if $Q = (Q_1, \ldots, Q_q)$

is a trivial characteristic for a system in Kovalevskaya form, then the corresponding conservation law is trivial.

Lemma 4.28. *If Δ is in Kovalevskaya form* (4.32), *and P is a conservation law, then there exists an equivalent conservation law $\bar{P}$ such that*

$$\operatorname{Div} \bar{P} = D_t \bar{T} + \operatorname{Div}_y \bar{Y} = \sum_{\nu=1}^{q} \bar{Q}_\nu(y, t, \widetilde{u^{(m)}})\{u^\nu_{nt} - \Gamma_\nu\}, \tag{4.33}$$

where $\operatorname{Div}_y = D_{y^1} Y_1 + \cdots + D_{y^{p-1}} Y_{p-1}$ denotes the "spatial part" of the total divergence. The new characteristic $\bar{Q} = (\bar{Q}_1, \ldots, \bar{Q}_q)$ depends only on y, t, and derivatives $u^\alpha_{jt,J}$ of orders $j + \# J \leqslant m$ such that $j < n$, which we denote by $\widetilde{u^{(m)}}$; in other words, there are no t-derivatives of u of order $\geqslant n$ occurring in $\bar{Q}$.

Proof. First, as in Section 2.6, each prolongation of a system in Kovalevskaya form (4.32) will provide formulae for the higher-order t-derivatives $u^\alpha_{kt,K}$, $k \geqslant n$, in terms of the derivatives $u^\beta_{jt,J}$, $j < n$, of order strictly less than n in t. If the original conservation law P depends on n-th or higher order t-derivatives of u, we can substitute these formulae and thereby replace P by an equivalent conservation law $\hat{P}$ which is independent of the derivatives $u^\alpha_{kt,K}$, $k \geqslant n$. Hence, the divergence of $\hat{P}$ has the form

$$\begin{aligned} \operatorname{Div} \hat{P} = D_t \hat{T} + \operatorname{Div}_y \hat{Y} &= \sum_{\nu,K} \frac{\partial \hat{T}}{\partial u^\nu_{(n-1)t,K}} u^\nu_{nt,K} + R \\ &= \sum_{\nu,K} \frac{\partial \hat{T}}{\partial u^\nu_{(n-1)t,K}} D_K\{u^\nu_{nt} - \Gamma_\nu\} + \hat{R}, \end{aligned} \tag{4.34}$$

where neither R nor $\hat{R}$ depend on the derivatives $u^\alpha_{nt,K}$. However, to be a conservation law, $\operatorname{Div} \hat{P}$ must vanish on the system (4.32), and this implies that $\hat{R}$ vanishes on (4.32), which is not possible unless $\hat{R} \equiv 0$ vanishes identically. Therefore, (4.34) reduces to an identity

$$\operatorname{Div} \hat{P} = \sum_{\nu,K} Z^K_\nu D_K\{u^\nu_{nt} - \Gamma_\nu\}, \tag{4.35}$$

where the coefficients $Z^K_\nu = \partial \hat{T} / \partial u^\nu_{(n-1)t,K}$ do not depend on the derivatives $u^\alpha_{nt,K}$. Integrating (4.35) by parts, we recover the equivalent conservation law (4.33), with characteristic

$$\bar{Q}_\nu = \sum_K (-D)_K Z^K_\nu. \tag{4.36}$$

Since the multi-indices K in (4.36) only refer to y-derivatives, $\bar{Q}_\nu$ is independent of the n-th order t-derivatives $u^\alpha_{nt,J}$. This proves Lemma 4.28. $\square$

To complete the proof of Theorem 4.26, suppose Q is a trivial characteristic for the conservation law

$$\operatorname{Div} P = Q \cdot \Delta.$$

Replace P by the equivalent conservation law $\bar{P}$ as given by Lemma 4.28. According to the direct half of Theorem 4.26 (which has already been proven!), since P and $\bar{P}$ are equivalent conservation laws, the associated characteristics Q and $\bar{Q}$ are equivalent; hence, $Q - \bar{Q} = 0$ on Δ. But Q already vanishes on Δ, and, according to Lemma 4.28, $\bar{Q}$ does not depend on the derivatives $u^\alpha_{kt,K}$ for $k \geqslant n$. Therefore, the only way that $\bar{Q}$ can vanish on Δ is if it vanishes identically. This means that the conservation law $\bar{P}$ is a trivial conservation law of the second kind (a null divergence); hence P, being equivalent to $\bar{P}$, is also trivial. This completes the proof that, for a system in Kovalevskaya form (and hence any normal system), trivial characteristics necessarily come from trivial conservation laws. □

4.4. Noether's Theorem

The general principle relating symmetry groups and conservation laws was first determined by E. Noether, [1], who stated it in almost complete generality. The version presented in this section is the one most familiar to physicists and engineers, requiring only knowledge of ordinary symmetry group theory as developed in Chapter 2, but is far from the most comprehensive version of Noether's theorem available. We will return to this topic in Section 5.3, where the complete, general form of Noether's theorem, which subsumes the present version, will be proved. Nevertheless, the result here is still of great practical use, and we will illustrate its effectiveness with a number of examples of physical importance.

Theorem 4.29. *Suppose G is a (local) one-parameter group of symmetries of the variational problem $\mathscr{L}[u] = \int L(x, u^{(n)})\, dx$. Let*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u)\frac{\partial}{\partial u^\alpha} \tag{4.37}$$

be the infinitesimal generator of G, and

$$Q_\alpha(x, u) = \phi_\alpha - \sum_{i=1}^{p} \xi^i u^\alpha_i, \qquad u^\alpha_i = \partial u^\alpha/\partial x^i,$$

the corresponding characteristic of $\mathbf{v}$, *as in* (2.48). *Then $Q = (Q_1, \ldots, Q_q)$ is also the characteristic of a conservation law for the Euler–Lagrange equations $\mathsf{E}(L) = 0$; in other words, there is a p-tuple $P(x, u^{(m)}) = (P_1, \ldots, P_p)$ such that*

$$\operatorname{Div} P = Q \cdot \mathsf{E}(L) = \sum_{\nu=1}^{q} Q_\nu \mathsf{E}_\nu(L) \tag{4.38}$$

is a conservation law in characteristic form for the Euler–Lagrange equations $\mathsf{E}(L) = 0$.

PROOF. We substitute the prolongation formula (2.50) into the infinitesimal invariance criterion (4.15), to find

$$\begin{aligned} 0 &= \mathrm{pr}^{(n)}\, \mathbf{v}(L) + L\, \mathrm{Div}\, \xi \\ &= \mathrm{pr}^{(n)}\, \mathbf{v}_Q(L) + \sum_{i=1}^{p} \xi^i D_i L + L \sum_{i=1}^{p} D_i \xi^i \\ &= \mathrm{pr}^{(n)}\, \mathbf{v}_Q(L) + \mathrm{Div}(L\xi), \end{aligned}$$

where $L\xi$ is the p-tuple with components $(L\xi^1, \ldots, L\xi^p)$. The first term in this equation can be integrated by parts:

$$\begin{aligned} \mathrm{pr}^{(n)}\, \mathbf{v}_Q(L) &= \sum_{\alpha, J} D_J Q_\alpha \frac{\partial L}{\partial u_J^\alpha} \\ &= \sum_{\alpha, J} Q_\alpha \cdot (-D)_J \frac{\partial L}{\partial u_J^\alpha} + \mathrm{Div}\, A \\ &= \sum_{\alpha=1}^{q} Q_\alpha \mathrm{E}_\alpha(L) + \mathrm{Div}\, A, \end{aligned}$$

where $A = (A_1, \ldots, A_p)$ is some p-tuple of functions depending on Q, L and their derivatives whose precise form is not required here. We have proved that

$$\mathrm{pr}^{(n)}\, \mathbf{v}_Q(L) = Q \cdot \mathrm{E}(L) + \mathrm{Div}\, A \tag{4.39}$$

for some A. Therefore,

$$0 = Q \cdot \mathrm{E}(L) + \mathrm{Div}(A + L\xi), \tag{4.40}$$

and (4.38) holds with $P = -(A + L\xi)$. This completes the proof of Noether's theorem. □

From this standpoint, the essence of Noether's theorem is reduced to the integration by parts formula (4.39). To find the explicit expression for the resulting conservation law $P = -(A + L\xi)$, we thus need to find the general formula for A in terms of L and the characteristic Q of the symmetry. The general formula appears in Proposition 5.98; here we look at the case of first order variational problems in detail. (An alternative approach is to construct P directly from the basic formula (4.38) once the characteristic Q is known. This somewhat *ad hoc* tecnique is often useful in practice, when the general formula is rather cumbersome to apply directly.) If $L(x, u^{(1)})$ depends only on first order derivatives, then

$$\mathrm{pr}^{(1)}\, \mathbf{v}_Q(L) = \sum_{\alpha=1}^{q} \left\{ Q_\alpha \frac{\partial L}{\partial u^\alpha} + \sum_{i=1}^{p} D_i Q_\alpha \frac{\partial L}{\partial u_i^\alpha} \right\}.$$

Only the second batch of summands need to be integrated by parts, so we find (4.39) holds with $A_i = \sum_\alpha Q_\alpha\, \partial L/\partial u_i^\alpha$. Thus we have the following version of Noether's theorem for first order variational problems.