16 Critical domain: Universality, ε-expansion

In Chapter 15, we have derived the scaling behaviour of correlation functions at criticality, that is, $T = T_c$. We now study the *critical domain*, which is defined by the property that the correlation length is large with respect to the microscopic scale, but finite.

By a simple generalization of the formalism of Chapter 9 to dimensions $d < 4$, we first demonstrate strong scaling above T_c: in the critical domain above T_c, all correlation functions, after rescaling, can be expressed in terms of universal correlation functions, in which the scale of distance is provided by the correlation length.

However, because the correlation length is singular at T_c, in this formalism, the critical temperature cannot be crossed, making a consistent description of the whole critical domain impossible. Therefore, we then expand correlation functions, with some care, in formal power series of the deviation $(T - T_c)$ from the critical temperature. The coefficients are critical correlation functions involving $\phi^2(x)$ insertions at zero momentum (as one can infer from the analysis of Section 9.10), whose scaling behaviour has been derived in Section 15.6. Summing the expansion, one obtains renormalization group (RG) equations valid for $T \neq T_c$. In addition, to crossing the critical temperature, while avoiding critical singularities, it is necessary to break the symmetry of the Hamiltonian explicitly. We thus add a linear coupling to a small external magnetic field to the interactions. We then derive RG equations in a magnetic field, or at fixed magnetization (the field expectation value $\langle\phi(x)\rangle$). In this way, we are able to continuously connect correlation functions above and below T_c, and derive scaling properties in the whole critical domain.

In the first part of the chapter, we restrict the discussion to Ising-like systems. In Section 16.6, we generalize the results to an N-component order parameter $\boldsymbol{\phi}(x)$. In Section 16.8, we show how to expand the universal two-point function when T approaches T_c, using the short distance expansion (SDE), discussed in Section 11.3.

In Section 16.9, we report a few terms of the ε expansion ($\varepsilon = 4 - d$ is the deviation from dimension 4) of various universal quantities. Calculations at fixed dimension and summation of perturbative expansions are described in Sections 41.3–41.5.

Finally, in Section 16.10, we briefly describe the application of the SDE to the determination of critical exponents.

Background material and additional references can be found, for example, in Ref. [50].

The temperature. The temperature is coupled to the total Hamiltonian or configuration energy. Therefore, a variation of the temperature generates a variation of all terms contributing to the effective Hamiltonian. However, as we have shown in Section 15.2, the most relevant contribution (the most infrared (IR) singular) corresponds to the $\phi^2(x)$ operator. Therefore, we take the difference $\tau = r - r_c$ between the coefficient of ϕ^2 in the Hamiltonian (15.41) and its critical value r_c as a linear measure of the deviation from the critical temperature, and parametrize the effective Hamiltonian (15.41) as

$$\mathcal{H}(\phi) = \int \left[\tfrac{1}{2}\big(\nabla_\Lambda \phi(x)\big)^2 + \tfrac{1}{2}(r_c + \tau)\phi^2(x) + \tfrac{1}{4!}g\Lambda^\varepsilon \phi^4(x)\right] \mathrm{d}^d x \,, \tag{16.1}$$

with $|\tau| \ll \Lambda^2$. Dimensional analysis, which results from the Gaussian renormalization, then yields for vertex (one-particle irreducible or 1PI) functions the relation

$$\tilde{\Gamma}^{(n)}(p_i; \tau, g, \Lambda) = \Lambda^{d-n(d-2)/2}\tilde{\Gamma}^{(n)}(p_i\Lambda^{-1}; \tau\Lambda^{-2}, g, 1). \tag{16.2}$$

Quantum Field Theory and Critical Phenomena 5E. Fifth edition. Jean Zinn-Justin, Oxford University Press (2021). DOI: 10.1093/oso/9780198834625.003.0016

16.1 Strong scaling above T_c: The renormalized theory

We first describe a few properties of the critical behaviour above T_c. For $T > T_c$, because the theory is massive, it is possible to introduce a special renormalized quantum field theory (QFT) with renormalization conditions imposed at zero momentum. Generalizing to any dimension $d \le 4$ the formalism of Sections 9.2 and 9.3, we define renormalized vertex functions (see equations (9.30)) by the renormalization conditions at zero momentum,

$$\tilde{\Gamma}_{\mathrm{r}}^{(2)}(p; m_{\mathrm{r}}, g_{\mathrm{r}}) = m_{\mathrm{r}}^2 + p^2 + O(p^4), \tag{16.3}$$

$$\tilde{\Gamma}_{\mathrm{r}}^{(4)}(p_i = 0; m_{\mathrm{r}}, g_{\mathrm{r}}) = m_{\mathrm{r}}^{\varepsilon} g_{\mathrm{r}} \,. \tag{16.4}$$

The condition (16.4) is written in such a way that g_{r} is dimensionless. The renormalized theory is derived from the initial microscopic theory by taking the large cut-off limit at g_{r} and m_{r} fixed:

$$\tilde{\Gamma}_{\mathrm{r}}^{(n)}(p_i; m_{\mathrm{r}}, g_{\mathrm{r}}) = \lim_{\Lambda\to\infty} Z^{n/2}(m_{\mathrm{r}}/\Lambda, g_{\mathrm{r}})\tilde{\Gamma}^{(n)}(p_i; \tau, g, \Lambda). \tag{16.5}$$

A simple dimensional analysis then yields for the renormalized vertex functions, the relation

$$\tilde{\Gamma}_{\mathrm{r}}^{(n)}(p_i; m_{\mathrm{r}}, g_{\mathrm{r}}) = m_{\mathrm{r}}^{d-(n/2)(d-2)}\tilde{\Gamma}_{\mathrm{r}}^{(n)}(p_i\,/m_{\mathrm{r}}; 1, g_{\mathrm{r}}\,). \tag{16.6}$$

The correlation length ξ, which characterizes the decay of the connected two-point function, is proportional to m_{r}^{-1}: it fixes the scale in the theory with renormalization conditions at zero momentum.

Note that equation (16.5) holds for any dimension $d \le 4$ and, therefore, some consequences of the equation are valid beyond the ε-expansion. This is one reason why this formalism is especially useful.

In Section 9.5, we have introduced the Callan–Symanzik equations, and in the following sections proved that are satisfied by the renormalized vertex functions in four dimensions. Below four dimensions, the proof is much simpler, because only the mass is divergent. With the renormalization conditions (16.3, 16.4), one thus derives

$$\begin{aligned}\left[m_{\mathrm{r}}\frac{\partial}{\partial m_{\mathrm{r}}} + \beta(g_{\mathrm{r}})\frac{\partial}{\partial g_{\mathrm{r}}} - \frac{n}{2}\eta(g_{\mathrm{r}})\right]\tilde{\Gamma}_{\mathrm{r}}^{(n)}(p_i; m_{\mathrm{r}}, g_{\mathrm{r}})\\ = m_{\mathrm{r}}^2\,[2-\eta(g_{\mathrm{r}})]\,\tilde{\Gamma}_{\mathrm{r}}^{(1,n)}(0; p_i; m_{\mathrm{r}}, g_{\mathrm{r}}),\end{aligned} \tag{16.7}$$

in which the renormalized ϕ^2 insertion is specified by the condition

$$\tilde{\Gamma}_{\mathrm{r}}^{(1,2)}(0; 0, 0; m_{\mathrm{r}}, g_{\mathrm{r}}) = 1\,. \tag{16.8}$$

As usual, we have given to different RG functions the same name $\{\beta, \eta\}$ because they play the same role in the equations. We examine some consequences of equation (16.7) in Section 16.1.3.

16.1.1 Microscopic and renormalized coupling constants

We first discuss the relation between microscopic and renormalized coupling constants. Several remarks are in order here:

(i) Since, in dimension $d < 4$, the ϕ^4 QFT is super-renormalizable, all correlation functions have a large cut-off limit after a simple mass renormalization, that is, after one has taken the correlation length (or the physical mass) as a parameter. The coupling constant and the field renormalizations are finite and, therefore are not required, in general.

If one renormalizes only the mass, one obtains a finite theory which is a function of the bare coupling constant $g\Lambda^\varepsilon$, the coefficient of $\phi^4(x)$ in the Hamiltonian. Therefore, perturbation theory is really an expansion in powers of $g(\Lambda/m_\mathrm{r})^\varepsilon$. It only makes sense if this ratio is kept fixed, which implies that the coupling constant $g\Lambda^\varepsilon$ (which characterizes the deviation from the Gaussian fixed point) goes to zero with the inverse correlation length as m_r^ε. This is indeed what is implicitly assumed in the conventional QFT renormalization theory. By contrast, in a generic situation, as envisaged in the theory of the critical phenomena, the coupling constant, which is related to microscopic parameters of the theory, is fixed. This means, as we already stressed in Chapter 15, that after introduction by rescaling of the cut-off Λ, $g \mapsto g\Lambda^\varepsilon$, g remains finite (see action (15.41)) when $\Lambda \to \infty$. Therefore, in the critical domain, that is, in the large cut-off limit, the expansion parameter becomes large for $d < 4$, and the mass renormalized perturbation theory becomes useless. This is the reason why it is necessary to introduce a field renormalization and a new expansion parameter g_r which, as we will verify, remains finite in this limit.

(ii) We note that by this method (introduction of the renormalized theory), we have taken the large cut-off limit in the following order: first $m \ll \Lambda g^{1/\varepsilon} \ll \Lambda$, and then $\Lambda g^{1/\varepsilon} \to \infty$, instead of $m \ll \Lambda \propto \Lambda g^{1/\varepsilon}$. For an application to critical phenomena, we have to assume that the result is the same at leading order.

The renormalized coupling constant. The RG β-function in equation (16.7) is given by equation (9.37),

$$\beta(g_\mathrm{r}) = m_\mathrm{r} \left.\frac{\mathrm{d}}{\mathrm{d}m_\mathrm{r}}\right|_{\Lambda,g} g_\mathrm{r}(g, m_\mathrm{r}/\Lambda). \tag{16.9}$$

We now introduce the variable $\lambda = m_\mathrm{r}/\Lambda$. Considering g_r as a function of λ at g *fixed*, we can rewrite equation (16.9) as

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda} g_\mathrm{r}(\lambda) = \beta(g_\mathrm{r}(\lambda)). \tag{16.10}$$

This equation, which is similar to equation (9.98), is a flow equation for $g_\mathrm{r}(\lambda)$. When the correlation length increases, the ratio m_r/Λ decreases, and thus λ goes to zero. The renormalized coupling constant is driven towards an IR stable zero of $\beta(g_\mathrm{r})$ if such a zero exists. Assuming the existence of such an IR fixed point,

$$\beta(g_\mathrm{r}^*) = 0\,, \quad \text{with} \quad \omega = \beta'(g_\mathrm{r}^*) > 0\,, \tag{16.11}$$

one can integrate equation (16.10) in the neighbourhood of g_r^*, and estimate $g_\mathrm{r} - g_\mathrm{r}^*$:

$$|g_\mathrm{r} - g_\mathrm{r}^*| \propto (m_\mathrm{r}/\Lambda)^\omega. \tag{16.12}$$

Therefore, because the bare coupling constant, which is associated with microscopic physics, is fixed in the critical domain, the renormalized, or effective coupling constant at correlation scale, is close to the IR fixed point value.

16.1.2 Strong scaling

We first evaluate the behaviour of the renormalization constant Z defined by equation (9.38),

$$m_\mathrm{r}\left.\frac{\mathrm{d}}{\mathrm{d}m_\mathrm{r}}\right|_{g,\Lambda} \ln Z(g_\mathrm{r}, m_\mathrm{r}/\Lambda) = \eta(g_\mathrm{r}). \tag{16.13}$$

Using the parameter $\lambda = m_\mathrm{r}/\Lambda$ and the function $g_\mathrm{r}(\lambda)$ of equation (16.10), we rewrite equation (16.13) as

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln Z(\lambda) = \eta(g_\mathrm{r}(\lambda)). \tag{16.14}$$

In the case of an IR fixed point, for $\lambda = m_\mathrm{r}/\Lambda$ small, at leading order one can then replace g_r by g_r^*. One infers

$$Z \propto (m_\mathrm{r}/\Lambda)^\eta, \tag{16.15}$$

where η is the value (assumed finite) of the function $\eta(g)$ at the fixed point:

$$\eta \equiv \eta(g_\mathrm{r}^*). \tag{16.16}$$

We also assume that the renormalized vertex functions $\tilde{\Gamma}_\mathrm{r}^{(n)}(p_i; m_\mathrm{r}, g_\mathrm{r}^*)$ are finite.

The parameter Λ provides a (momentum) scale for all quantities. Once the form of the critical behaviour has been determined, all dimensional parameters (here momenta, deviation from T_c, and correlation length) can be rescaled to eliminate Λ (see equation (16.2)).

Assuming the existence of an IR fixed point, combining equations (16.5), (16.6), and (16.15), one then find, in the critical domain, the scaling relation [80, 132, 51]

$$\tilde{\Gamma}^{(n)}(p_i, \tau, g, \Lambda = 1) \underset{m_\mathrm{r} \ll 1,\ |p_i| \ll 1}{\sim} m_\mathrm{r}^{d-(n/2)(d-2+\eta)} \tilde{\Gamma}_\mathrm{r}^{(n)}(p_i / m_\mathrm{r}; 1, g_\mathrm{r}^*). \tag{16.17}$$

Equation (16.17) provides a proof of *strong scaling* and *universality* in the whole critical domain above T_c. The initial effective local QFT may depend on an infinite number of parameters (see Chapter 8). The ϕ^4 QFT, viewed as an effective microscopic theory where all irrelevant contributions are omitted, depends only on the way the cut-off is introduced and explicitly on p_i, g, and τ. The right-hand side of equation (16.17) only involves the renormalized correlation functions at $g_\mathrm{r} = g_\mathrm{r}^*$, that is, functions of ratios p_i/m_r. Moreover, the result (16.17) holds for any fixed dimension $d \le 4$. Resorting to the ε-expansion has been avoided. It has only been necessary to assume the existence of an IR fixed point. Of course, within the ε-expansion, using the results of Chapter 10 and the remark of Section 9.10.1, we immediately obtain $\beta(g_\mathrm{r})$ at leading order and thus g_r^* and ω,

$$g_\mathrm{r}^* = \frac{16\pi^2}{3}\varepsilon + O(\varepsilon^2), \quad \omega = \varepsilon + O(\varepsilon^2). \tag{16.18}$$

However, as first suggested by Parisi [146], it is also possible to numerically analyse the perturbative expansion in powers of g_r at fixed dimension 3 or 2 (see Chapter 41). Such an analysis convincingly demonstrates the existence of an IR fixed point, and makes a precise determination of critical exponents possible, as the numerical results reported in Section 41.5 illustrate.

Finally, equations (9.39–9.42) can be used to characterize the divergence of m_r^{-1}, and thus the correlation length, as a function of the temperature or bare mass. This behaviour is derived in Section 16.3 by another, simpler, method. It confirms that the relation between m_r and τ is singular. Therefore, within the formalism of this section, based on the introduction of a zero-momentum renormalized theory, crossing the critical temperature is impossible. Indeed, all correlation functions are parametrized in terms of the correlation length, which is singular at T_c. To avoid this problem, in the next section we introduce a different formalism which is a natural extension of the formalism of Chapter 15.

A few remarks.

(i) We have shown that the renormalized coupling constant g_r has a finite limit $g_\mathrm{r} = g_\mathrm{r}^*$, although the expansion parameter $g\Lambda^\varepsilon$ becomes infinite. In addition, precisely at g_r^*, the field renormalization diverges as equation (16.15) shows. The conclusion is that the QFT, at the IR fixed point, behaves even below four dimensions like a renormalizable QFT, and a complete renormalization is indeed required.

(ii) A second somewhat related remark is that when $g\Lambda^\varepsilon$ varies from 0 to infinity, g_r varies from 0 to g_r^*. This property suggests that no renormalized theory exists for $g_\mathrm{r} > g_\mathrm{r}^*$. Since $g_\mathrm{r}^* = O(\varepsilon)$, this argument again confirms that it is unlikely that a non-trivial ϕ^4 QFT consistent on all scales exists in four dimensions. On the other hand, to construct the renormalized theory, one takes the large cut-off limit at $g\Lambda^\varepsilon$ as fixed. This procedure is only legitimate if the bare RG has only one IR fixed point. Otherwise, other non-trivial theories might be obtained by sending the cut-off to infinity at g fixed. This point is further discussed in Section 24.1.2.

16.1.3 Large momentum behaviour

We return to equation (16.7) in order to investigate the behaviour of correlation functions at large momenta, in the critical domain $m_\mathrm{r} \ll |p_i| \ll \Lambda$. At leading order, we can replace, in equation (16.7), g_r by its IR fixed point value g_r^*:

$$\left(m_\mathrm{r}\frac{\partial}{\partial m_\mathrm{r}} - \frac{n}{2}\eta\right)\tilde{\Gamma}_\mathrm{r}^{(n)}(p_i; m_\mathrm{r}, g_\mathrm{r}^*) = m_\mathrm{r}^2(2-\eta)\tilde{\Gamma}_\mathrm{r}^{(1,n)}(0; p_i; m_\mathrm{r}, g_\mathrm{r}^*). \tag{16.19}$$

It follows from the scaling relation (16.17) that the large momentum behaviour is directly related to the approach to the critical theory $m_\mathrm{r} = 0$. Therefore, in order to extract some information from equation (16.19), it is necessary to return to the framework of the ε-expansion. An extension of Weinberg's theorem indicates that, order by order in g_r and ε, the right-hand side of equation (16.19) is then negligible at large (non-exceptional) momenta, or small masses, as one might naively guess from the factor m_r^2 that has been factorized. By contrast, below four dimensions at a fixed dimension, there always exists an order at which the right-hand side ceases to be negligible. This is interpreted as being a consequence of expanding perturbation theory around the wrong fixed point, the IR unstable Gaussian fixed point. The correct asymptotic behaviour of correlation functions is different, because it is governed by the non-trivial point $g_\mathrm{r} = g_\mathrm{r}^*$. Therefore, asymptotically the renormalized vertex functions satisfy

$$\left(m_\mathrm{r}\frac{\partial}{\partial m_\mathrm{r}} - \frac{n}{2}\eta\right)\tilde{\Gamma}_\mathrm{r}^{(n)}(p_i, m_\mathrm{r}) \underset{|p_i|\gg m_\mathrm{r}}{\approx} 0\,. \tag{16.20}$$

Combined with the relation (16.6), this leads to [80, 132, 51]:

$$\tilde{\Gamma}_\mathrm{r}^{(n)}(\lambda p_i; m_\mathrm{r}) \underset{|p_i|\gg m_\mathrm{r}}{\sim} \lambda^{d-(n/2)(d-2+\eta)}\tilde{\Gamma}_\mathrm{r}^{(n)}(p_i; m_\mathrm{r}). \tag{16.21}$$

Using then the relation (16.5) between microscopic and renormalized correlation functions, one recovers the scaling behaviour of equation (15.60) derived in Section 15.5.

In this approach, the large momentum behaviour of the right-hand side of equation (16.19) can also be analysed and, therefore, corrections to the leading behaviour can be calculated, using the SDE (see Section 16.8).

Remark. If g_r is fixed with $g_\mathrm{r} < g_\mathrm{r}^*$, then the large momentum behaviour of correlation functions is given by the UV fixed point $g_\mathrm{r} = 0$, that is, by perturbation theory. However, as we have already discussed, this corresponds to a situation where the bare coupling constant g goes to zero as $(m_\mathrm{r}/\Lambda)^\varepsilon$, which is a non-generic situation.

16.2 Critical domain: Homogeneous RG equations

To characterize the behaviour of correlation functions in the whole critical domain, we now use a different strategy. First, we consider the field integral representation of n-point correlation functions corresponding to the Hamiltonian (16.1). If we formally expand it in powers of τ, we obtain an expansion in terms of *critical* correlation functions with $\frac{1}{2}\int \mathrm{d}^d x\, \phi^2(x)$ (the most IR singular part of the energy operator) insertions (see also Section 9.10). As a consequence, to be able to define these correlation functions by their perturbative expansion, we now have to return to the framework developed in Sections 15.4 and 15.6 and to the ε-expansion.

However, even so, because the insertion of $\int \mathrm{d}^d x\, \phi^2(x)$ is the insertion of the Fourier transform of $\phi^2(x)$ at zero momentum, the corresponding correlation functions are still IR divergent. Therefore, in the Hamiltonian (16.1), we first replace the constant τ by a field $\tau(x)$. We can then expand the vertex functions $\Gamma^{(n)}$, as functions of space variables, in powers of the field $\tau(x)$:

$$\Gamma^{(n)}(x_i;\tau,g,\Lambda)=\sum_{l=0}^{\infty}\frac{1}{l!}\int \mathrm{d}^d y_1\cdots \mathrm{d}^d y_l\,\tau(y_1)\cdots\tau(y_l)\Gamma^{(l,n)}(y_j;x_i;\tau=0,g,\Lambda). \quad (16.22)$$

As already noted in Section 9.10, by acting with the functional differential operator $\int \mathrm{d}^d y\, \tau(y)\delta/\delta\tau(y)$ on equation (16.22), one generates in the right-hand side a factor l in front of $\Gamma^{(l,n)}$. One then verifies that equation (15.67) implies (see equation (9.81)):

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}-\frac{n}{2}\eta(g)-\eta_2(g)\int \mathrm{d}^d y\,\tau(y)\frac{\delta}{\delta\tau(y)}\right]\Gamma^{(n)}(x_i;\tau,g,\Lambda)=0\,. \quad (16.23)$$

To calculate $\Gamma^{(n)}$, it is possible to perform a partial summation of perturbation theory in order to introduce the propagator $[-\nabla^2+\tau(x)]^{-1}$, for example, by using the loop expansion. In this new perturbative expansion, the constant τ limit can be taken and leads to a massive theory: no IR divergence is generated anymore. Then, equation (16.23), in the Fourier representation, becomes [72]

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}-\frac{n}{2}\eta(g)-\eta_2(g)\tau\frac{\partial}{\partial\tau}\right]\tilde{\Gamma}^{(n)}(p_i;\tau,g,\Lambda)=0\,. \quad (16.24)$$

Equation (16.24) is the formal analogue of equation (9.82), and differs from equation (16.7), which also applies to the non-critical (*i.e.* massive) theory, by the property that it is *homogeneous*.

16.3 Scaling properties above T_c

Within this new formalism, we first discuss again the behaviour of correlation functions in the critical domain above T_c, which we have already examined in Section 16.1. We integrate equation (16.24) using the method of characteristics. In addition to the functions $g(\lambda)$ and $Z(\lambda)$ of equations (15.51), we now define a function $\tau(\lambda)$ such that [51],

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}(p_i;\tau(\lambda),g(\lambda),\lambda\Lambda)\right]=0\,, \quad (16.25)$$

and which is determined by imposing consistency with equation (16.24).

The consistency equations are:

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda) = \beta(g(\lambda)), \qquad g(1) = g\,, \tag{16.26}$$

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \tau(\lambda) = -\eta_2(g(\lambda)), \qquad \tau(1) = \tau\,, \tag{16.27}$$

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln Z(\lambda) = \eta\big(g(\lambda)\big), \qquad Z(1) = 1\,. \tag{16.28}$$

Dimensional analysis yields

$$\tilde{\Gamma}^{(n)}\big(p_i; \tau(\lambda), g(\lambda), \lambda\Lambda\big) = (\lambda\Lambda)^{d-n(d-2)/2} \tilde{\Gamma}^{(n)}\big(p_i/\lambda\Lambda; \tau(\lambda)/\lambda^2\Lambda^2, g(\lambda), 1\big). \tag{16.29}$$

The critical domain is defined, in particular, by $|\tau| \ll \Lambda^2$ and this is the source of the IR singular behaviour observed in perturbation theory. If the equation for λ,

$$\tau(\lambda) = \lambda^2 \Lambda^2, \tag{16.30}$$

has a solution, then the theory at scale λ is no longer critical. Combining equations (16.25–16.30), one finds

$$\tilde{\Gamma}^{(n)}(p_i; \tau, g, \Lambda) = Z^{-n/2}(\lambda) m^{(d-n(d-2)/2)} \tilde{\Gamma}^{(n)}(p_i/m; 1, g(\lambda), 1), \tag{16.31}$$

with the notation

$$m = \lambda\Lambda\,. \tag{16.32}$$

The solution of equation (16.27) can be written as

$$\tau(\lambda) = \tau \exp\left[-\int_1^\lambda \frac{\mathrm{d}\sigma}{\sigma} \eta_2\big(g(\sigma)\big)\right]. \tag{16.33}$$

Substituting this relation into equation (16.30) and introducing the function (15.70),

$$\nu(g) = [\eta_2(g) + 2]^{-1}\,,$$

one then obtains

$$\ln(\tau/\Lambda^2) = \int_1^\lambda \frac{\mathrm{d}\sigma}{\sigma} \frac{1}{\nu(g(\sigma))}. \tag{16.34}$$

One looks for a solution λ in the limit $\tau/\Lambda^2 \ll 1$. Since $\nu(g)$ is a positive function, at least for g small enough, as one can verify in the explicit expression (15.74), from equation (16.34), one infers that the value of the parameter λ is small and, thus, that $g(\lambda)$ is close to the IR fixed point g^*. In this limit, $\nu(g(\sigma))$ can be replaced, at leading order, by the exponent $\nu = \nu(g^*)$ and equation (16.34) implies

$$\ln(\tau/\Lambda^2) \sim \frac{1}{\nu} \ln \lambda\,. \tag{16.35}$$

Equation (16.28) then yields

$$Z(\lambda) \propto \lambda^\eta. \tag{16.36}$$

Finally, taking the large Λ, or the small λ limit, and using equations (16.35) and (16.36) in equation (16.31), one finds that vertex functions have the scaling form

$$\tilde{\Gamma}^{(n)}(p_i;\tau,g,\Lambda=1) \underset{\substack{\tau\ll 1\\ |p_i|\ll 1}}{\sim} m^{(d-n(d-2+\eta)/2)} G_+^{(n)}(p_i/m), \tag{16.37}$$

where $G_+^{(n)}$ are universal functions, and

$$m(\Lambda=1)=\xi^{-1}\propto\tau^{\nu}. \tag{16.38}$$

Equations (16.37) and (16.17) express the same scaling property. However, one additional result has been obtained. Equation (16.37) shows that the quantity m is proportional to the inverse correlation length (the physical mass in particle physics). Equation (16.38) then shows that the divergence of the correlation length $\xi=m^{-1}$ at T_c is characterized by the exponent ν (a result that could also have been derived with the formalism of Section 16.1).

From equation (16.37), one derives the scaling form of connected correlation functions:

$$\tilde{W}^{(n)}(p_i;\tau,g,\Lambda=1) \underset{\substack{\tau\ll 1\\ |p_i|\ll 1}}{\sim} m^{(d-n(d+2-\eta)/2)} H_+^{(n)}(p_i/m), \tag{16.39}$$

$$W^{(n)}(x_i;\tau,g,\Lambda=1) \underset{\substack{\tau\ll 1\\ |x_i|\gg 1}}{\sim} m^{(d-2+\eta)n/2} I_+^{(n)}(mx_i). \tag{16.40}$$

For $\tau\neq 0$, the correlation functions are finite at zero momentum and behave like

$$\tilde{W}^{(n)}(0;\tau,g,\Lambda)\propto\tau^{\nu(d-n(d+2-\eta)/2)}. \tag{16.41}$$

In particular, $n=2$ corresponds to the magnetic susceptibility χ. Therefore,

$$\chi=\tilde{W}^{(2)}(p=0;\tau,g,\Lambda)\propto\tau^{-\nu(2-\eta)}. \tag{16.42}$$

The exponent which characterizes the divergence of χ is usually denoted by γ. Equation (16.41) establishes the relation between exponents

$$\gamma=\nu(2-\eta). \tag{16.43}$$

Finally, for the critical theory to exist when τ goes to zero, different powers of τ have to cancel in equation (16.37). From this observation, one recovers equation (15.60) in the form

$$\tilde{\Gamma}^{(n)}(\lambda p_i;\tau,g,\Lambda=1) \underset{\tau^{\nu}\ll\lambda|p_i|\ll 1}{\propto} \lambda^{d-n(d-2+\eta)/2}. \tag{16.44}$$

16.4 Correlation functions with ϕ^2 insertions

A differentiation of the field integral with respect to $\tau(x)$, before taking the uniform τ limit, generates correlation functions with $\left[\frac{1}{2}\phi^2(x)\right]$ insertions (insertions of the Hamiltonian or configuration energy density, in the statistical formulation). By differentiating equation (16.23) with respect to $\tau(y_1), \ldots, \tau(y_l)$, before taking the same limit, one derives RG equations for the corresponding vertex functions. One verifies that the resulting equation, except for $l = 2$, $n = 0$, is

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\eta(g) - \eta_2(g)\left(l + \tau\frac{\partial}{\partial\tau}\right)\right]\tilde{\Gamma}^{(l,n)}(q_i; p_j; \tau, g, \Lambda) = 0\,. \tag{16.45}$$

The equation can be solved exactly in the same way as equation (16.24). We set $\Lambda = 1$. Then for $\tau \ll 1$, $|q_i| \ll 1$, $|p_j| \ll 1$, one finds the scaling form ($m \propto \tau^\nu$),

$$\tilde{\Gamma}^{(l,n)}(q_i; p_j; \tau, g, \Lambda = 1) \sim m^{[d-l/\nu-n(d-2+\eta)/2]}G_+^{(l,n)}(q_i/m; p_j/m). \tag{16.46}$$

The discussion then exactly follows the lines of Section 16.3, and we do not repeat it here. Rather, we focus on the case $n = 0$, $l = 2$, which corresponds to the energy density two-point correlation function. Starting from equation (15.67), and using the method described previously, one obtains

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \eta_2(g)\left(2 + \tau\frac{\partial}{\partial\tau}\right)\right]\tilde{\Gamma}^{(2,0)}(q; \tau, g, \Lambda) = \Lambda^{-\varepsilon}B(g). \tag{16.47}$$

The function $\Lambda^{-\varepsilon}C_2(g)$ of equation (15.75) is still a solution of the inhomogeneous equation. Equation (16.46) is then replaced by

$$\tilde{\Gamma}^{(2,0)}(q; \tau, g, \Lambda = 1) \underset{\substack{\tau\ll 1\\ |q|\ll 1}}{\sim} m^{(d-2/\nu)}G_+^{(2,0)}(q\,/m) + C_2(g). \tag{16.48}$$

At zero momentum, $\tilde{\Gamma}^{(2,0)}(q = 0)$ is also the second derivative of the free energy (the connected vacuum amplitude) with respect to the temperature τ, that is, the *specific heat.* Its behaviour for $T \to T_{c+}$, is thus

$$\tilde{\Gamma}^{(2,0)}(0; \tau, g, \Lambda = 1) \sim A^+\tau^{-(2-\nu d)} + C_2(g). \tag{16.49}$$

The specific heat exponent is called α. Therefore, we have established the scaling relation

$$\alpha = 2 - \nu d\,. \tag{16.50}$$

Integrating $\tilde{\Gamma}^{(2,0)}$ twice with respect to τ, one obtains the thermodynamic potential density,

$$\mathcal{G}(\tau, g, \Lambda) = \lim_{\Omega\to\infty}\Omega^{-1}\Gamma(\tau, g, \Lambda),$$

where Ω is the volume, for a vanishing magnetization M (and thus also the free energy in zero magnetic field). It has the expansion,

$$\begin{aligned}&\mathcal{G}(M = 0, \tau, g, \Lambda)\\ &\quad= \Lambda^d\left[C_0(g) + C_1(g)\,\frac{\tau}{\Lambda^2} + C_2(g)\frac{\tau^2}{2\Lambda^4} + \frac{A^+}{(2-\alpha)(1-\alpha)}\left(\frac{\tau}{\Lambda^2}\right)^{2-\alpha}\right],\end{aligned} \tag{16.51}$$

valid for $|\tau| \ll \Lambda^2$. The three first terms correspond to the beginning of the small τ expansion of the regular part of the free energy and depend explicitly on g through three functions C_0, C_1, and C_2, while the fourth term characterizes the leading behaviour of the singular part of the free energy and is universal (it still depends on the normalization of the temperature but this normalization cancels in appropriate ratios).

16.5 Scaling properties in a magnetic field and below T_c

In Sections 16.3 and 16.4, we have determined the behaviour of correlation functions in the critical domain, above T_c. We now examine the behaviour in the critical domain in the ordered phase ($M \neq 0$).

In order to pass continuously from the disordered ($T > T_c$) to the ordered phase ($T < T_c$), it is necessary to add an interaction that explicitly breaks the symmetry to the Hamiltonian. We thus consider correlation functions in the presence of an external magnetic field. Actually, it is more convenient to consider correlation functions at fixed magnetization. Therefore, we first discuss the relation between field and magnetization, called the *equation of state*.

As we have already noted in the mean-field analysis, in the ordered phase some qualitative differences appear between systems which have a discrete and a continuous symmetry. We consider first the case of the ϕ^4 (Ising-like) theory with a discrete $\mathbb{Z}_2$ symmetry. In Section 16.6, we discuss the N-vector model with $O(N)$ symmetry and $(\boldsymbol{\phi}^2)^2$ interaction, an example of continuous symmetry.

16.5.1 The equation of state

The equation of state relates the magnetization M, expectation value of $\phi(x)$ in a constant magnetic field H, to the temperature. The thermodynamic potential density, as a function of M, is by definition

$$\mathcal{G}(M, \tau, g, \Lambda) = \sum_{n=0}^{\infty} \frac{M^n}{n!} \tilde{\Gamma}^{(n)}(p_i = 0; \tau, g, \Lambda). \tag{16.52}$$

The magnetic field H is given by

$$H = \frac{\partial \mathcal{G}}{\partial M} = \sum_{n=1}^{\infty} \frac{M^n}{n!} \tilde{\Gamma}^{(n+1)}(p_i = 0; \tau, g, \Lambda). \tag{16.53}$$

Noting that $n \equiv M(\partial/\partial M)$, one immediately derives, from the RG equation (16.24), the RG equation [51]

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \frac{1}{2}\eta(g) \left(1 + M \frac{\partial}{\partial M}\right) - \eta_2(g) \tau \frac{\partial}{\partial \tau}\right] H(M, \tau, g, \Lambda) = 0\,. \tag{16.54}$$

To integrate equation (16.54) by the method of characteristics, one introduces, in addition to $g(\lambda)$, $t(\lambda)$ and $Z(\lambda)$ of equations (16.26–16.28), a new function $M(\lambda)$ that satisfies

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln M(\lambda) = -\tfrac{1}{2}\eta\left[g(\lambda)\right], \quad \text{with } M(1) = M\,. \tag{16.55}$$

A comparison between equations (16.55) and (16.28) shows that $M(\lambda)$ is given by

$$M(\lambda) = M Z^{-1/2}(\lambda).$$

The solution of equation (16.54) can then be written as

$$H(M, \tau, g, \Lambda) = Z^{-1/2}(\lambda) H\left[M(\lambda), \tau(\lambda), g(\lambda), \lambda\Lambda\right]. \tag{16.56}$$

Dimensional analysis shows that

$$H(M,\tau,g,\Lambda)=\Lambda^{3-\varepsilon/2}H(M/\Lambda^{1-\varepsilon/2},\tau/\Lambda^2,g,1). \tag{16.57}$$

Applying the relation to the right-hand side of equation (16.56), one obtains

$$H(M,\tau,g,\Lambda)=(\lambda\Lambda)^{3-\varepsilon/2}Z^{-1/2}(\lambda)H[M(\lambda)/(\lambda\Lambda)^{1-\varepsilon/2},\tau(\lambda)/\lambda^2\Lambda^2,g(\lambda),1]. \tag{16.58}$$

Again one can use the arbitrariness of λ to move outside the critical domain, in order to remove the critical singularities in the right-hand side of equation (16.58). Here, a natural choice is

$$M(\lambda)=(\lambda\Lambda)^{1-\varepsilon/2}, \tag{16.59}$$

which implies, using the solution of equation (16.55), that

$$\ln(M/\Lambda^{1-\varepsilon/2})=\frac{1}{2}\int_1^\lambda\frac{\mathrm{d}\sigma}{\sigma}\left[d-2+\eta(g(\sigma))\right]. \tag{16.60}$$

In the critical domain, the magnetization is small:

$$M\ll\Lambda^{1-\varepsilon/2}.$$

For $d\geq 2$ and g small, the expression $d-2+\eta(g)$ is positive because $\eta(g)$ is positive. This again implies that λ is small and thus $g(\lambda)$ is close to g^*. In this limit, equation (16.60) implies

$$M\Lambda^{\varepsilon/2-1}\propto\lambda^{(d-2+\eta)/2}. \tag{16.61}$$

Form equation (16.27), one infers

$$\tau(\lambda)/\lambda^2\propto\tau\lambda^{-1/\nu}, \tag{16.62}$$

and we have already seen that

$$Z(\lambda)\propto\lambda^\eta. \tag{16.63}$$

Finally, replacing $\tau(\lambda)$ and $Z(\lambda)$ by their asymptotic forms (16.62) and (16.63), and using equation (16.61) to eliminate λ, one concludes that, in the critical domain, the equation of state takes the general *scaling form*, proposed by Widom [122],

$$H(M,\tau,g,1)\sim M^\delta f(\tau M^{-1/\beta}), \tag{16.64}$$

where $f(x)$ is, up to normalizations, a *universal function* and

$$\beta=\frac{\nu}{2}(d-2+\eta), \tag{16.65}$$

$$\delta=\frac{d+2-\eta}{d-2+\eta}. \tag{16.66}$$

Equations (16.65) and (16.66) relate the traditional critical exponents that characterize the vanishing of the spontaneous magnetization, and the singular relation between magnetic field and magnetization at T_c, respectively, to the exponents η and ν introduced previously.

Valid for $d<4$, these latter two relations seem to be inconsistent with the values of the mean-field exponents for $d>4$. To understand this point, it is necessary to remember that for $d>4$, g^* vanishes, and all terms in H, except the term linear in M, come from corrections to expression (16.64) (see Section 17.1).

16.5.2 Properties of the universal function $f(x)$

(i) Griffith's analyticity [148]: equation (16.53) shows that H has a regular expansion in odd powers of M for $\tau > 0$. This implies that when the variable x becomes large and positive, $f(x)$ has the expansion

$$f(x) = \sum_{p=0}^{\infty} a_p x^{\gamma - 2p\beta}. \tag{16.67}$$

(ii) For $M \neq 0$, when τ vanishes the theory remains massive. In the loop expansion, the corresponding propagator is massive. It follows that one can expand $\Gamma(M, \tau)$ and, therefore, $H(M, \tau)$ in powers of τ without meeting IR divergences. Therefore, $f(x)$ is infinitely differentiable at $x = 0$.

(iii) The appearance of a spontaneous magnetization below T_c, implies that the function $f(x)$ has a zero for $x = x_0$ with $x_0 < 0$,

$$f(x_0) = 0\,, \quad x_0 < 0\,. \tag{16.68}$$

Then, equation (16.64) leads, for $\tau < 0$, to the relation

$$M = |x_0|^{-\beta}\,(-\tau)^{\beta}, \quad \text{for } H = 0\,. \tag{16.69}$$

Equation (16.69) exhibits the singular behaviour of the spontaneous magnetization when the temperature approaches the critical temperature from below.

16.5.3 Correlation functions for non-vanishing magnetization

We now examine the behaviour of correlation functions in a field. All expressions are again be written for Ising-like systems. The generalization to the N-vector model with $O(N)$ symmetry is briefly discussed in Section 16.6.

The vertex functions, at fixed magnetization M, are obtained by expanding the generating functional $\Gamma(M)$ of vertex functions around $M(x) = M$,

$$\Gamma^{(n)}(x_1, \ldots, x_n; \tau, M, g, \Lambda) = \frac{\delta^n}{\delta M(x_1) \cdots \delta M(x_n)} \Gamma(M, \tau, g, \Lambda)\bigg|_{M(x)=M}. \tag{16.70}$$

The expansion in powers of M of the right-hand side of the equation, after Fourier transformation, then yields

$$\tilde{\Gamma}^{(n)}(p_1, \ldots, p_n; \tau, M, g, \Lambda) = \sum_{s=0}^{\infty} \frac{M^s}{s!} \tilde{\Gamma}^{(n+s)}(p_1, \ldots, p_n, 0, \ldots, 0; \tau, 0, g, \Lambda). \tag{16.71}$$

From the RG equations satisfied by the correlation functions in zero magnetization (equations (16.24)), one derives [51]

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \frac{1}{2}\eta(g)\left(n + M\frac{\partial}{\partial M}\right) - \eta_2(g)\tau\frac{\partial}{\partial \tau}\right] \tilde{\Gamma}^{(n)}(p_i; \tau, M, g, \Lambda) = 0\,. \tag{16.72}$$

The equation can be solved by exactly the same method as equation (16.54). One obtains the universal scaling form of vertex functions,

$$\tilde{\Gamma}^{(n)}(p_i; \tau, M, g, \Lambda = 1) \sim m^{[d - n(d-2+\eta)/2]} G^{(n)}(p_i/m, \tau m^{-1/\nu}), \tag{16.73}$$

for $|p_i| \ll 1$, $|t| \ll 1$, $M \ll 1$ and with here the definition

$$m = M^{\nu/\beta}. \tag{16.74}$$

The right-hand side of equation (16.73) now depends on two different length or mass scales: $m = M^{\nu/\beta}$ and τ^{ν}. The scaling forms of correlation functions differ by the powers of m as in equations (16.39) and (16.40).

16.5.4 Correlation functions in zero field below T_c: Spontaneous symmetry breaking

We have argued previously that, for $M \neq 0$, correlation functions are regular functions of τ near $\tau = 0$. Therefore, it is possible to cross the critical temperature and to then take the zero external magnetic field limit. In the limit, M becomes the *spontaneous magnetization* which is given, as a function of τ, by equation (16.69). After elimination of M in favour of τ in equation (16.73), one finds, below T_c ($\tau < 0$) in the critical domain, a behaviour of the form

$$\tilde{\Gamma}^{(n)}(p_i; \tau, M(\tau, H=0), g, 1) \sim m^{d-n(d-2+\eta)/2} G_-^{(n)}(p_i/m), \tag{16.75}$$

with

$$m = |x_0|^{-\nu} (-\tau)^{\nu}. \tag{16.76}$$

Therefore, vertex and connected correlation functions have exactly the same scaling behaviour above and below T_c [51]. In particular, since traditionally one defines below T_c,

$$m^{-1} = \xi \propto (-\tau)^{-\nu'}, \quad \left[\tilde{\Gamma}^{(2)}(0)\right]^{-1} = \chi \sim (-\tau)^{-\gamma'}, \tag{16.77}$$

this establishes the relations $\nu' = \nu$ and $\gamma' = \gamma$. However, the universal functions $G_+^{(n)}$ and $G_-^{(n)}$ are different.

The extension of these considerations to the functions with ϕ^2 insertions, $\tilde{\Gamma}^{(l,n)}$ is straightforward. In particular, the same method yields the singular behaviour of the specific heat below T_c:

$$\tilde{\Gamma}^{(2,0)}(q=0, M(H=0,\tau)) - \Lambda^{-\varepsilon} C_2(g) \underset{\text{for } \tau \to 0_-}{\sim} A^-(-\tau)^{-\alpha}, \tag{16.78}$$

which, similarly, proves that the exponents above and below T_c are the same: $\alpha' = \alpha$.

Note that the constant term $\Lambda^{-\varepsilon} C_2(g)$ which depends explicitly on g is the same above and below T_c, in contrast with the coefficient of the singular part.

The derivation of the equality of exponents above and below T_c, relies on the existence of a path which avoids the critical point, along which the correlation functions are regular, and the RG equations everywhere satisfied.

Remark. The universal functions characterizing the behaviour of correlation functions in the critical domain still depend on the normalization of the physical parameters τ, H, M, distances or momenta. Physical quantities that are independent of these normalizations are *universal pure numbers*. In addition to critical exponents, examples are provided by the ratios of the amplitudes of the singularities above and below T_c, like A^+/A^- for the specific heat.

16.6 The N-vector model

We now generalize the results to models in which the order parameter is an N-component vector, and which have symmetries such that the Landau–Ginzburg–Wilson Hamiltonian has still the form of a ϕ^4-like QFT. We first consider a simple but important example: the $O(N)$ symmetric model and then discuss briefly the general situation.

16.6.1 The $O(N)$ symmetric N-vector model: IR fixed point

The $O(N)$ symmetric effective Hamiltonian has the form

$$\mathcal{H}(\boldsymbol{\phi}) = \int \left[\tfrac{1}{2}\big(\nabla\boldsymbol{\phi}(x)\big)^2 + \tfrac{1}{2}(r_c+\tau)\boldsymbol{\phi}^2(x) + \tfrac{1}{4!}g\Lambda^{\varepsilon}\big(\boldsymbol{\phi}^2(x)\big)^2\right]\mathrm{d}^d x\,, \tag{16.79}$$

in which $\boldsymbol{\phi}$ is an N-component vector.

Above T_c and in zero field, the RG equations have exactly the same form as in the Ising-like case $N=1$. The RG β-function has the expansion

$$\beta(g) = -\varepsilon g + \frac{N+8}{48\pi^2}g^2 + O(g^3, g^2\varepsilon). \tag{16.80}$$

At leading order in ε, $\beta(g)$ has a zero g^* which is an IR fixed point:

$$g^* = \frac{48\pi^2}{N+8}\varepsilon + O(\varepsilon^2), \tag{16.81}$$

$$\omega \equiv \beta'(g^*) = \varepsilon + O(\varepsilon^2) \tag{16.82}$$

and, therefore, all the scaling relations derived for $T \geq T_c$ in zero field in Sections 15.5, 15.6 can also be proved for the N-vector model with $O(N)$ symmetry. We give the expressions of the other RG functions at leading order in Section 16.7, by specializing expressions obtained for the general N-vector model.

16.6.2 Correlation functions in a field or below T_c

The addition of a magnetic field term in an $O(N)$ symmetric Hamiltonian has several consequences. First, the magnetization and the magnetic field are now vectors. The scaling forms derived previously apply to the modulus of these vectors. Second, the continuous $O(N)$ symmetry of the Hamiltonian is broken linearly in the dynamical variables by the addition of a magnetic field (see Chapter 13). Since the field and the magnetization distinguish one direction in vector space, there now exist 2^n n-point functions, each spin being either along the magnetization or transverse to it.

As we have shown in Chapter 13, these different correlation functions are related by a set of identities, called Ward–Takahashi (WT) identities, which have been discussed there in a general framework. In the example of the $O(N)$ symmetry, we recall here the simplest one, involving the two-point function at zero momentum, also verified in the mean-field approximation in Section 14.5.3. In terms of $\tilde{\Gamma}_{\mathrm{T}}$, the inverse two-point function transverse to $\mathbf{M}$ at zero momentum or the transverse susceptibility, it reads

$$\tilde{\Gamma}_{\mathrm{T}}(p=0) = \chi_{\mathrm{T}}^{-1} = \frac{H}{M}\,. \tag{16.83}$$

We recognize, in a different notation, equation (13.43). As we have already discussed in Section 13.4, it follows from this equation that if H goes to 0 below T_c, H/M, and, therefore, $\tilde{\Gamma}_{\mathrm{T}}$ at zero momentum vanish. This last result implies the existence of $(N-1)$ Goldstone modes corresponding to the spontaneous breaking of the $O(N)$ symmetry with a residual $O(N-1)$ symmetry.

Finally, note that the inverse longitudinal two-point function $\tilde{\Gamma}_{\mathrm{L}}(p)$ is singular at zero momentum in zero field below T_c, as one can infer from its Feynman graph expansion (Fig. 16.1). This IR singularity is not generated by critical fluctuations but by the Goldstone modes. It is characteristic of continuous symmetries. It implies that the longitudinal two-point correlation function does not decrease exponentially at large distance.

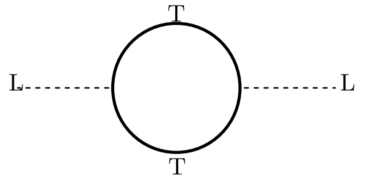

Fig. 16.1 One-loop Goldstone mode contribution to $\tilde{\Gamma}_{\mathrm{L}}$

We discuss this problem more thoroughly in Chapter 19. In particular, the correlation length (16.76) becomes a crossover scale between critical behaviour and Goldstone mode dominated behaviour at larger distances.

Emergent symmetries. Previous results also apply to models in which the Hamiltonian has a symmetry smaller than the group $O(N)$, but is still such that the effective Hamiltonian has the form (16.79), because the quadratic and quartic group invariants are uniquely determined. Then, the $O(N)$ symmetry is dynamically generated in the critical domain. Only a close examination of the leading corrections to the critical behaviour reveals the difference. We have already encountered such a phenomenon: the hypercubic symmetry of the lattice has led to a $O(d)$ continuum space symmetry in the critical domain.

16.7 The general N-vector model

One can find interesting physical systems in which the effective Hamiltonian is not $O(N)$ invariant. A first category consists in systems in which there are several correlation lengths. In such situations generically, when the temperature varies, only one correlation length becomes infinite at a time. Then, the components of the dynamic variables which are non-critical do not contribute to the IR singularities. They can be integrated out in much the same way as the auxiliary fields in Pauli–Villars's regularization scheme of Section 8.4.2. The effect is to renormalize the effective local Hamiltonian for the critical components. This remark is related to the decoupling theorem of particle physics [47]. Therefore, one can restrict the discussion to theories with only one correlation length.

Models with one correlation length. These models consist in systems in which the Hamiltonian is invariant under a symmetry group, subgroup of $O(N)$, which admits a unique quadratic invariant $\boldsymbol{\phi}^2$ but several quartic invariants.

A general Hamiltonian in this case has the form

$$\mathcal{H}(\boldsymbol{\phi}) = \int \mathrm{d}^d x \bigg\{ \tfrac{1}{2} \left[(\nabla \boldsymbol{\phi}(x))^2 + (r_c + \tau) \boldsymbol{\phi}^2(x) \right] + \frac{\Lambda^\varepsilon}{4!} \sum_{i,j,k,l=1}^{N} g_{ijkl} \phi_i(x) \phi_j(x) \phi_k(x) \phi_l(x) \bigg\}. \tag{16.84}$$

The symmetry implies that the two-point vertex function $\tilde{\Gamma}^{(2)}_{ij}$ is necessarily proportional to the unit matrix:

$$\tilde{\Gamma}^{(2)}_{ij}(p, \tau, g) = \delta_{ij} \tilde{\Gamma}^{(2)}(p, \tau, g), \tag{16.85}$$

and that the symmetric tensor g_{ijkl} has special properties, which in perturbation theory, take the form of successive algebraic conditions (see Section 10.6).

16.7.1 RG equations

In Section 10.6, we have discussed the renormalization of a general ϕ^4 QFT, and derived the corresponding RG equations. We can apply the formalism here [149, 51].

We first sketch the derivation of the RG equations for a multi-component critical theory. Since the field amplitude renormalization constant is independent of the components, the relation between bare and renormalized vertex functions takes the form

$$\tilde{\Gamma}^{(n)}_{\mathrm{r};i_1,i_2,\ldots,i_n}(p_k, m_\mathrm{r}, g_\mathrm{r}, \mu) = Z^{n/2}\tilde{\Gamma}^{(n)}_{i_1,i_2,\ldots,i_n}(p_k, \tau, g, \Lambda), \tag{16.86}$$

in which g stands for g_{ijkl} and g_r for $g_{\mathrm{r};ijkl}$.

Differentiating the equation with respect to Λ at g_r, m_r and μ fixed, one obtains the RG equation [149],

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \sum_{i',j',k',l'} \beta_{i'j'k'l'}\frac{\partial}{\partial g_{i'j'k'l'}} - \frac{n}{2}\eta(g) - \eta_2(g)\tau\frac{\partial}{\partial\tau}\right)\tilde{\Gamma}^{(n)}_{i_1,i_2,\ldots,i_n} = 0\,, \tag{16.87}$$

with the definition

$$\sum_{i',j',k',l'} \beta_{i'j'k'l'}\frac{\partial g_{\mathrm{r};ijkl}}{\partial g_{i'j'k'l'}} = -\varepsilon g_{\mathrm{r};ijkl}\,. \tag{16.88}$$

These equations can be integrated by the same method as before. One introduces scale-dependent coupling constants $g_{ijkl}(\lambda)$ obeying the flow equation

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}g_{ijkl}(\lambda) = \beta_{ijkl}\big(g(\lambda)\big). \tag{16.89}$$

The large distance properties of such theories are then governed by fixed points, solution of the equations

$$\beta_{ijkl}(g^*) = 0\,,\ \forall\, i,j,i,k,l\,. \tag{16.90}$$

The local stability properties of fixed points are governed by the eigenvalues of the matrix

$$M_{ijkl,i'j'k'l'} = \frac{\partial\beta_{ijkl}(g^*)}{\partial g_{i'j'k'l'}}. \tag{16.91}$$

If the real parts of all eigenvalues are positive, the fixed point is locally stable. The global properties depend on the complete solutions of equation (16.89), which determine the basin of attraction in coupling space of each IR stable fixed point. We do not discuss this problem further here and refer to the literature where a number of specific models have been considered (see Refs. [51, 149–152]).

The $\phi_i(x)\phi_j(x)$ insertions. It is also useful to consider correlation functions with $\frac{1}{2}\phi_i(x)\phi_j(x)$ insertions. Their renormalization involves a multiplication of each insertion by the matrix $\zeta^{(2)}_{ij,kl}$. This leads to the RG equation

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \sum_{i',j',k',l'} \beta_{i'j'k'l'}\frac{\partial}{\partial g_{i'j'k'l'}} - \frac{n}{2}\eta(g) - \eta_2(g)\tau\frac{\partial}{\partial\tau}\right]\tilde{\Gamma}^{(l,n)}_{j_1k_1\ldots j_lk_l,i_1\ldots i_n}$$
$$- \sum_{m=1}^{l}\eta^{(2)}_{j_mk_m,b_mc_m}\tilde{\Gamma}^{(l,n)}_{j_1k_1\ldots b_mc_m\ldots j_lk_l,i_1\ldots i_n} = 0\,, \tag{16.92}$$

with the definition

$$\eta^{(2)}_{ij,kl} = -\sum_{i',j',k',l'}\beta_{i'j'k'l'}\left(\frac{\partial\boldsymbol{\zeta}^{(2)}}{\partial g_{i'j'k'l'}}\left[\boldsymbol{\zeta}^{(2)}\right]^{-1}\right)_{ij,kl}. \tag{16.93}$$

Since the insertions of $\boldsymbol{\phi}^2$, which are generated by a differentiation with respect to τ, are multiplicatively renormalized, the matrix $\eta^{(2)}_{ij,kl}(g)$ has δ_{kl} as eigenvector, and

$$\sum_k \eta^{(2)}_{ij,kk}(g) = \eta_2(g)\delta_{ij}\,. \tag{16.94}$$

RG functions. At one-loop order, the RG functions are given by (equations (10.86) and (10.87))

$$\beta_{ijkl} = -\varepsilon g_{ijkl} + \frac{1}{16\pi^2}\sum_{m,n}(g_{ijmn}g_{mnkl} + 2 \text{ terms}) + O(g^3), \tag{16.95}$$

$$\eta(g)\delta_{ij} = \frac{1}{24}\frac{1}{(8\pi^2)^2}\sum_{k,l,m} g_{iklm}g_{jklm} + O(g^3), \tag{16.96}$$

$$\eta^{(2)}_{ij,kl} = -\frac{1}{16\pi^2}g_{ijkl} + O(g^2). \tag{16.97}$$

The condition (16.94) then implies (equations (10.68) and (10.69))

$$\sum_k g_{ijkk} = \gamma_1\delta_{ij}\,, \quad \sum_{k,l,m} g_{iklm}g_{jklm} = \gamma_2\delta_{ij}\,, \tag{16.98}$$

and summing over i, j, one can rewrite equation (16.96) as

$$\eta(g) = \frac{1}{(8\pi^2)^2}\frac{\gamma_2}{24} + O(g^3, g^2\varepsilon). \tag{16.99}$$

16.7.2 Stability of the $O(N)$ symmetric fixed point

Among the possible fixed points, one always finds, in addition to the trivial Gaussian fixed point, the $O(N)$ symmetric fixed point. We can study its local stability at leading order in ε. We first specialize the expressions (16.95–16.97) to the case of the $(\boldsymbol{\phi}^2)^2$ QFT with $O(N)$ symmetry. We then have to substitute

$$g_{ijkl} = \frac{g}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{16.100}$$

After a short calculation, the expression (16.80) of the β-function is recovered and, in addition,

$$\eta(\tilde{g}) = \frac{(N+2)}{72}\tilde{g}^2 + O(\tilde{g}^3), \tag{16.101}$$

where the notation (10.64) has been used:

$$\tilde{g} = N_d g\,, \quad N_d = 2(4\pi)^{-d/2}/\tilde{\Gamma}(d/2).$$

Introducing the identity matrix $\mathbf{I}$ and the projector $\mathbf{P}$,

$$I_{ij,kl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{16.102}$$

$$P_{ij,kl} = \frac{1}{N}\delta_{ij}\delta_{kl}\,, \tag{16.103}$$

one can express the matrix $\eta^{(2)}_{ij,kl}$ as

$$\eta^{(2)} = -(N\mathbf{P} + 2\mathbf{I})\frac{\tilde{g}}{6} + O(\tilde{g}^2). \tag{16.104}$$

The trace of the matrix $\eta^{(2)}$ yields $\eta_2(g)$. The second eigenvalue of the matrix $\eta_2'(g)$, given by its traceless part, corresponds to a symmetry breaking mass term, and as we have discussed at the beginning of this section, describes the crossover from a situation with one correlation length to a situation in which some components of the order parameter decouple. It is traditional to introduce a new function $\varphi(g)$ and to parametrize it as

$$\eta_2'(g) = \frac{\varphi(g)}{\nu(g)} - 2\,. \tag{16.105}$$

The fixed point value $\varphi = \varphi(g^*)$ is called the *crossover exponent* [153]. Finally, the stability conditions are given by the eigenvalues of the matrix (16.91). Setting

$$g_{ijkl} = g^*_{ijkl} + s_{ijkl}\,, \tag{16.106}$$

at leading order one finds

$$(Ms)_{ijkl} = -\varepsilon s_{ijkl} + \frac{\varepsilon}{N+8}(\delta_{ij}s_{mmkl} + 5 \text{ terms} + 12 s_{ijkl}). \tag{16.107}$$

Taking s_{ijkl} proportional to g^*_{ijkl}, one recovers the exponent ω. More generally, the eigenvectors can be classified according to their trace properties. We expand

$$s_{ijkl} = u g^*_{ijkl} + (v_{ij}\delta_{kl} + 5 \text{ terms}) + w_{ijkl}\,, \tag{16.108}$$

in which the tensors v_{ij} and w_{ijkl} are traceless:

$$\sum_i v_{ii} = 0\,, \quad \sum_k w_{ijkk} = 0\,. \tag{16.109}$$

The three eigenvalues corresponding to u, w, and v are, respectively,

$$\omega = \varepsilon + O(\varepsilon^2),\ \omega_{\text{anis.}} = \varepsilon\frac{4-N}{N+8} + O(\varepsilon^2),\ \ \omega' = \frac{8\varepsilon}{N+8} + O(\varepsilon^2). \tag{16.110}$$

The perturbation proportional to v_{ij} does not satisfy the trace condition (16.98). Therefore, it lifts the degeneracy between the correlation lengths of the different components of the order parameter. It induces a crossover to a situation in which some components decouple. However, one easily verifies that the corresponding eigenvalue ω' leads to effects subleading for ε small with respect to the eigenvalue $\eta_2'(g^*)$. Within the class of interactions satisfying equation (16.98), the relevant eigenvalue is $\omega_{\text{anis.}}$. We find the very interesting result that the $O(N)$ symmetric fixed point is stable against any perturbation for N smaller than some value N_c [149, 51]. This is an example of *emergent symmetry*: correlation functions in the critical domain have a larger symmetry than microscopic correlation functions. The calculation of $\omega_{\text{anis.}}$ at order ε, in models with cubic anisotropy [150], yields

$$N_c = 4 - 2\varepsilon + O(\varepsilon^2). \tag{16.111}$$

16.7.3 Gradient flow

Denoting g_α a set of variables (here the set of coupling constants) parametrizing a manifold, the flow equation

$$\beta_\alpha(g) = \sum_\beta T_{\alpha\beta}(g) \frac{\partial U}{\partial g_\beta}, \tag{16.112}$$

in which the function U is the potential, defines a gradient flow for g_α if $T_{\alpha\beta}$ is a symmetric and positive matrix. The form of the equation is invariant under a change of parametrization (more precisely a diffeomorphism) of the manifold.

It can be verified that the β-function, at two-loop order as given by (16.95), can indeed be written as [154]

$$\beta_{ijkl} = \frac{\partial U(g)}{\partial g_{ijkl}}\,,$$

with

$$\begin{aligned} U(g) = -\tfrac{1}{2}\varepsilon \sum_{i,j,k,l} g_{ijkl} g_{ijkl} + \frac{1}{(4\pi)^2} \sum_{i,j,k,l,m,n} g_{ijkl} g_{klmn} g_{mnij} \\ + \frac{1}{(4\pi)^4} \sum_{i,j,k,l,m,n,p,q} \left(\tfrac{3}{2} g_{ijkl} g_{ijmn} g_{pqkm} g_{pqln} + \tfrac{1}{12} g_{ijkl} g_{ijkm} g_{npql} g_{npqm}\right). \end{aligned} \tag{16.113}$$

At three-loop order, the RG flow equation for g_{ijkl} takes the more general form (16.112), where the matrix T is positive, at least for all g_{ijkl} small.

The structure of gradient flow implies that the RG flow follows curves of monotonous decrease of the potential U, the fixed points are extrema of the function U, and the stable fixed point corresponds to the lowest value of the potential. Moreover, the matrix of derivatives of the β-function at a fixed point is symmetric, and thus has real eigenvalues.

At leading order in ε, a fixed point g^* is solution of the equation

$$\begin{aligned} \varepsilon g^*_{ijkl} &= \frac{1}{16\pi^2} \sum_{m,n} (g^*_{ijmn} g^*_{mnkl} + \text{ 2 terms}) \\ \Rightarrow\ \varepsilon \sum_{i,j,k,l} g^*_{ijkl} g^*_{ijkl} &= \frac{3}{(4\pi)^2} \sum_{i,j,k,l,m,n} g^*_{ijkl} g^*_{klmn} g^*_{mnij} \end{aligned}$$

and, therefore, the value of the potential can then be written as

$$U(g^*) = -\tfrac{1}{6}\varepsilon \sum_{i,j,k,l} g^*_{ijkl} g^*_{ijkl} + O(\varepsilon^4).$$

Comparing with the expression (16.99) for the RG function $\eta(g)$, we find the relation between the potential at the fixed point and the corresponding exponent,

$$\eta = \frac{1}{6N} \frac{1}{(4\pi)^4} \sum_{i,j,k,l} g^*_{ijkl} g^*_{ijkl} + O(\varepsilon^3) = -\frac{1}{N\varepsilon} \frac{1}{(4\pi)^4} U(g^*) + O(\varepsilon^3).$$

Therefore, at leading order in ε, the stable fixed point, which corresponds to the lowest value of the potential, also corresponds to the largest value of the exponent η. Numerical estimates suggest that this property remains true beyond the ε-expansion [152].

16.8 Asymptotic expansion of the two-point function

In the critical domain, when points are separated by distances much smaller than the correlation length ξ, the correlation functions tend towards the correlation functions of the critical theory ($T = T_c$), for example,

$$\tilde{\Gamma}^{(2)}(p) \underset{\xi^{-1} \ll p \ll \Lambda}{\propto} p^{2-\eta}. \tag{16.114}$$

The right-hand side is actually the first term of an asymptotic expansion in the variable $p\xi$ for $p\xi$ large. The leading term has been obtained by using the property that, at large non exceptional momenta, the derivative $\partial\tilde{\Gamma}^{(n)}(p_1, \ldots, p_n)/\partial\tau$ is asymptotically negligible with respect to $\tilde{\Gamma}^{(n)}(p_1, \ldots, p_n)$. However, since

$$\frac{\partial}{\partial\tau}\tilde{\Gamma}^{(n)}(p_1, \ldots, p_n) = \tilde{\Gamma}^{(1,n)}(0; p_1, \ldots, p_n),$$

the derivative $\partial\tilde{\Gamma}^{(n)}/\partial\tau$ cannot be evaluated with the same method, because the momenta are exceptional. As we have explained in Section 11.3, it is necessary to use the SDE of operator products.

16.8.1 Asymptotic expansion from SDE

We focus on the two-point function. We have to evaluate $\tilde{\Gamma}^{(1,2)}(0; p, -p)$. However, we cannot apply directly the SDE to $\tilde{\Gamma}^{(1,2)}$ because this would involve $\tilde{\Gamma}^{(2,0)}$, which requires additional renormalizations.

Therefore, we differentiate once more with respect to τ,

$$\frac{\partial^2}{(\partial\tau)^2}\tilde{\Gamma}^{(2)}(p) = \tilde{\Gamma}^{(2,2)}(0, 0; p, -p). \tag{16.115}$$

To $\tilde{\Gamma}^{(2,2)}$ we can now apply the SDE,

$$\tilde{\Gamma}^{(2,2)}(0, 0; p, -p) \sim B(p)\tilde{\Gamma}^{(3,0)}(0, 0, 0), \quad \text{for } \xi^{-1} \ll p \ll \Lambda\,. \tag{16.116}$$

As shown in Section 11.4, $B(p)$ satisfies an RG equation that can be obtained by applying the differential operator

$$D \equiv \Lambda\frac{\partial}{\partial\Lambda} + \beta(g) - \left(\frac{1}{\nu(g)} - 2\right)\tau\frac{\partial}{\partial\tau}, \tag{16.117}$$

on both sides of equation (16.116) and using the RG equations (16.45).

One finds

$$\left[D + \nu^{-1}(g) - 2 - \eta(g)\right] B(p) \sim 0\,. \tag{16.118}$$

For $\tau = 0$ and $g = g^*$, the equation becomes

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \frac{1}{\nu} - 2 - \eta\right) B(p) \sim 0\,. \tag{16.119}$$

Equation (16.116) shows also that $B(p)$ has canonical dimension ε. It follows that

$$B(p) \propto \Lambda^{\varepsilon}(p/\Lambda)^{2-\eta-(1-\alpha)/\nu}. \tag{16.120}$$

A differentiation of equation (16.49) with respect to τ yields

$$\tilde{\Gamma}^{(3,0)}(0,0,0) \sim \Lambda^{-\varepsilon}(\tau/\Lambda^2)^{-1-\alpha}. \tag{16.121}$$

Finally, integrating equation (16.115) twice with respect to τ, and using the set of equations (16.116), (16.120) and (16.121), one obtains an expansion of the form [155]

$$\tilde{\Gamma}^{(2)}(p) \underset{\xi^{-1} \ll p \ll \Lambda}{=} p^{2-\eta}(a + b\tau\, p^{-1/\nu} + c\tau^{1-\alpha} p^{-(1-\alpha)/\nu} + \cdots), \tag{16.122}$$

a result partially anticipated on physical grounds [156]. Successive corrections to expression (16.122) can then be obtained by using systematically the SDE beyond leading order.

Note that the effect of the differentiation with respect to τ has been simply to generate the regular terms in the temperature that have an order in τ comparable to the singular terms for $\varepsilon \to 0$.

16.8.2 *Next to leading terms in a field or below T_c*

It is also possible to obtain expressions in a field or below T_c by expanding correlation functions in powers of the magnetization and applying the SDE to each term. The results now differ between Ising-like systems and the N-vector model. For Ising-like systems, one finds [155],

$$\tilde{\Gamma}^{(2)}(p,\tau,M) = p^{2-\eta}\left[a + b\tau/p^{1/\nu} + G_1(\tau/M^{1/\beta})(p/M^{\nu/\beta})^{-(1-\alpha)/\nu} + \cdots\right], \tag{16.123}$$

in which the function G_1 can be related to the free energy and thus the equation of state by

$$G_1(x) = \int_1^\infty \mathrm{d}s\, s^{\delta-1/\beta}\left[f'(0) - f'(x/s^{1/\beta})\right] + \frac{f'(0)}{\delta - (1/\beta) + 1}. \tag{16.124}$$

In the $O(N)$ symmetric case, the SDE involves the second operator of dimension 2,

$$\mathcal{O}_{ij}\left[\boldsymbol{\phi}(x)\right] = \phi_i(x)\phi_j(x) - \frac{\delta_{ij}}{N}\boldsymbol{\phi}^2(x). \tag{16.125}$$

This operator is multiplicatively renormalizable. Correlation functions with the insertion of $\mathcal{O}_{ij}$ satisfy the RG equations,

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\eta(g) - \left(\frac{\varphi(g)}{\nu(g)} - 2\right)\right]\tilde{\Gamma}^{(n)}_{\mathcal{O}_{ij}}(p_i; g, \Lambda) = 0\,, \tag{16.126}$$

in which the RG function $\varphi(g)$ has been defined in equation (16.105). As a consequence, a new term is present at the same order in the asymptotic expansion of the two-point function, which becomes [155],

$$\begin{aligned}\tilde{\Gamma}^{(2)}(p,\tau,M) = p^{2-\eta}\Big\{&\left[a + b\tau/p^{1/\nu} + G_1(\tau/M^{1/\beta})(p/M^{\nu/\beta})^{-(1-\alpha)/\nu}\right]\delta_{ij}\\ &+G_2(\tau/M^{1/\beta})(p/M^{\nu/\beta})^{-d+\varphi/\nu}\left(\frac{\delta_{ij}}{N} - \frac{M_iM_j}{M^2}\right)\Big\} + \cdots\,,\end{aligned} \tag{16.127}$$

in which φ is the crossover exponent, and G_2 a new universal function which may be calculated in an ε-expansion. At order ε, it is given by

$$G_2(x) = 1 + \frac{\varepsilon}{2(N+8)}\left[(x+3)\ln(x+3) - (x+1)\ln(x+1)\right] + O(\varepsilon^2). \tag{16.128}$$

16.9 Some universal quantities as ε expansions

Many models with second order phase transitions have been investigated, the effective QFTs identified and then various universal quantities calculated by field theoretical methods. We can report here only a limited number of significant results. Therefore, we consider only the important example of the $O(N)$ symmetric N-vector model described by the Hamiltonian (16.79),

$$\mathcal{H}(\boldsymbol{\phi}) = \int \left\{ \tfrac{1}{2} \left[\nabla \boldsymbol{\phi}(x)\right]^2 + \tfrac{1}{2}\left(r_c + \tau\right) \boldsymbol{\phi}^2(x) + \tfrac{1}{4!} g \Lambda^{\varepsilon} \left(\boldsymbol{\phi}^2(x)\right)^2 \right\} \mathrm{d}^d x,$$

in which $\boldsymbol{\phi}$ is an N-component field.

We describe results for critical exponents, the equation of state, and a few amplitude ratios. We discuss more thoroughly critical exponents, because they make a detailed and precise comparison between QFT, other theoretical methods and experiments possible.

In QFT, universal quantities have been calculated by two methods: the ε expansion, initiated by Wilson and Fisher [75], which we have systematically discussed in previous chapters, and perturbation theory at fixed dimension (see Sections 16.1 and 41.3.1). In both cases, the expansion is divergent for all values of the expansion parameter. The rate of divergence can be obtained from instanton calculus, as we explain in Chapter 40. There exist methods to deal with divergent series (see Sections 41.1–41.2). In the case of the ϕ^4 QFT, methods based on the Borel transformation has been extensively used (Chapter 40).

Initially, many quantities have been calculated up to order ε^2 by elementary techniques [75, 157]. The order ε^3 has required more involved QFT techniques [158]. Later, a series of ingenious tricks, and the systematic use of computer algebra, within the framework of dimensional regularization and the minimal subtraction scheme (see Chapter 10), have led to a determination of critical exponents in the N-vector model, up to five [159], then six-loop order for all N [160], and even seven loops for $N = 1$ [161].

16.9.1 RG functions. Critical exponents

Although the RG functions of the $(\boldsymbol{\phi}^2)^2$ theory and, therefore, the critical exponents are known up to six-loop order for the $O(N)$ model and up to seven loops for $N = 1$, we give here, for illustration, only three successive terms in the expansion, referring to the literature for higher order results. In terms of the variable

$$\tilde{g} = N_d\, g\,, \quad N_d = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\,, \tag{16.129}$$

the RG functions $\beta(\tilde{g})$ and $\eta_2(\tilde{g})$ at three-loop order, $\eta(\tilde{g})$ at four-loop order are

$$\begin{aligned} \beta(\tilde{g}) = {} & -\varepsilon\tilde{g} + \frac{(N+8)}{6}\tilde{g}^2 - \frac{(3N+14)}{12}\tilde{g}^3 \\ & + \frac{\left[33N^2 + 922N + 2960 + 96(5N+22)\zeta(3)\right]}{12^3}\tilde{g}^4 + O\left(\tilde{g}^5\right), \end{aligned} \tag{16.130}$$

$$\eta(\tilde{g}) = \frac{(N+2)}{72}\tilde{g}^2 \left[1 - \frac{(N+8)}{24}\tilde{g} + \frac{5(-N^2+18N+100)}{576}\tilde{g}^2\right] + O\left(\tilde{g}^5\right), \tag{16.131}$$

$$\eta_2(\tilde{g}) = -\frac{(N+2)}{6}\tilde{g}\left[1 - \frac{5}{12}\tilde{g} + \frac{(5N+37)}{48}\tilde{g}^2\right] + O\left(\tilde{g}^4\right), \tag{16.132}$$

in which $\zeta(s)$ is Riemann's ζ-function, $\zeta(3) = 1.20205690315\ldots$.

The zero $\tilde{g}^*(\varepsilon)$ of the β-function then is

$$\tilde{g}^*(\varepsilon) = \frac{6\varepsilon}{(N+8)}\left[1 + \frac{3(3N+14)}{(N+8)^2}\varepsilon + \left(\frac{18(3N+14)^2}{(N+8)^4} - \frac{33N^2 + 922N + 2960 + 96(5N+22)\zeta(3)}{8(N+8)^3}\right)\varepsilon^2\right] + O\left(\varepsilon^4\right). \tag{16.133}$$

The values of the critical exponents η, γ and the correction exponent ω,

$$\eta = \eta(\tilde{g}^*), \quad \gamma = \frac{2-\eta}{2+\eta_2(\tilde{g}^*)}, \quad \omega = \beta'(\tilde{g}^*),$$

follow:

$$\eta = \frac{\varepsilon^2(N+2)}{2(N+8)^2}\left\{1 + \frac{(-N^2+56N+272)}{4(N+8)^2}\varepsilon + \frac{1}{16(N+8)^4}\left[-5N^4 - 230N^3 + 1124N^2 + 17920N + 46144 - 384(5N+22)(N+8)\zeta(3)\right]\varepsilon^2\right\} + O\left(\varepsilon^5\right), \tag{16.134}$$

$$\gamma = 1 + \frac{(N+2)}{2(N+8)}\varepsilon + \frac{(N+2)}{4(N+8)^3}\left(N^2+22N+52\right)\varepsilon^2 + \frac{(N+2)}{8(N+8)^5} \times \left[N^4 + 44N^3 + 664N^2 + 2496N + 3104 - 48(5N+22)(N+8)\zeta(3)\right]\varepsilon^3 + O\left(\varepsilon^4\right), \tag{16.135}$$

$$\omega = \varepsilon - \frac{3(3N+14)}{(N+8)^2}\varepsilon^2 + \frac{\left[33N^2 + 922N + 2960 + 96(5N+22)\zeta(3)\right]}{4(N+8)^3}\varepsilon^3 - 18\frac{(3N+14)^2}{(N+8)^4}\varepsilon^3 + O\left(\varepsilon^4\right). \tag{16.136}$$

All other exponents can be obtained from the scaling relations derived in Sections 16.3–16.5. Note that the β function at order g^4 involves $\zeta(3)$. At higher orders $\zeta(5)$ and $\zeta(7)$ successively appear. In Table 40.3, we give the values of the critical exponents γ and η obtained by simply adding the successive terms of the ε expansion for $\varepsilon = 1$ and $N = 1$. We immediately observe a striking phenomenon: the sums first seem to settle near some reasonable value, and then begin to diverge with increasing oscillations.

We argue in Chapter 40 that the ε expansion is divergent for all values of ε.

Remark. The definition of the β function by minimal subtraction has an intrinsic meaning, unlike other definitions, since then $\beta(g)$ can be expressed in terms of $\omega(\varepsilon)$. Setting

$$\beta(g) = g\bigl(-\varepsilon + s(g)\bigr),$$

then

$$g(s) = \frac{s}{\beta_2}\exp\left[\int_0^s \mathrm{d}s'\left(\frac{1}{\omega(s')} - \frac{1}{s'}\right)\right].$$

16.9.2 The scaling equation of state

The scaling equation of state provides an interesting example of a universal function. Its ε-expansion has been obtained up to order ε^2 for arbitrary N [147], and order ε^3 for $N = 1$ [162]. We set

$$H = M^{\delta} f(x = \tau/M^{1/\beta}), \tag{16.137}$$

in which the normalizations of x and the function $f(x)$ are such that

$$f(0) = 1\,, \quad f(-1) = 0\,. \tag{16.138}$$

It is also convenient to set

$$y = x + 1\,, \quad z = x + 3\,, \quad \rho = z/4y\,. \tag{16.139}$$

The expansion up to order ε^2 is then

$$f(x) = 1 + x + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + O\left(\varepsilon^3\right), \tag{16.140}$$

with

$$f_1(x) = \frac{1}{2(N+8)}\left[(N-1)y\ln y + 3z\ln z - 9y\ln 3 + 6x\ln 2\right], \tag{16.141a}$$

$$\begin{aligned} f_2(x) = {} & \left[\frac{1}{2(N+8)}\right]^2 \{[N - 1 + 6\ln 2 - 9\ln 3 + (N-1)\ln y]\,[3z\ln z + (N-1)y\ln y \\ & +6x\ln 2 - 9y\ln 3] + \tfrac{1}{2}(10-N)y\left(\ln^2 z - \ln^2 3\right) + 36\left(\ln^2 z - y\ln^2 3 + x\ln^2 2\right) \\ & - 54\ln 2\,(\ln z + x\ln 2 - y\ln 3) + 3\ln\frac{27}{4}(N-1)y\ln y + \frac{-4N^2 + 17N + 212}{N+8} \\ & \times [z\ln z + 2x\ln 2 - 3y\ln 3] + (N-1)y\ln y\ln z - \tfrac{1}{2}N(N-1)y\ln^2 y \\ & + \frac{N-1}{N+8}(19N + 92)y\ln y - 2(N-1)\,[(x+6)J_1(x) - 6yJ_1(0)] \\ & -6(N-1)\,[J_2(x) - yJ_2(0)] + 4(N-1)\,[J_3(x) - yJ_3(0)]\}\,, \end{aligned} \tag{16.141b}$$

where

$$J_i(x) = I_i(\rho), \tag{16.142}$$

and

$$I_1(\rho) = \int_0^{\infty} \frac{\mathrm{d}u\ln u}{u(1-u)}\left[(1 - u/\rho)^{1/2}\,\theta(\rho - u) - 1\right], \tag{16.143}$$

$$I_2(\rho) = \rho\frac{\mathrm{d}}{\mathrm{d}\rho}I_1(\rho), \tag{16.144}$$

$$I_3(\rho) = I_1(\rho) + 2I_2(\rho). \tag{16.145}$$

The expressions (16.141) are not uniform, and valid only for x of order 1. For x large, that is, for small magnetization M, the magnetic field has a regular expansion in odd powers of M, that is, in the variable $x^{-\beta}$ (Section 16.5). Therefore, it is convenient to introduce Josephson's parametrization [163], which leads to a representation uniform in both limits.

16.9.3 Parametric representation of the equation of state

We set

$$x = x_0 \left(1-\theta^2\right)\theta^{-1/\beta}, \quad \theta > 0\,, \tag{16.146}$$

where x_0 is an arbitrary positive constant. More directly, we can parametrize M and t in terms of two variables R and θ, setting

$$\begin{aligned} M &= R^{\beta}\theta\,, \\ \tau &= x_0 R\left(1-\theta^2\right), \\ H &= R^{\beta\delta} h(\theta)\,. \end{aligned} \tag{16.147}$$

Then, the function

$$h(\theta) = \theta^{\delta} f\left(x(\theta)\right) \tag{16.148}$$

is an odd function of θ regular near $\theta = 1$, which is x small, and near $\theta = 0$ which is x large. For the special choice

$$x_0 = 3\,(3/2)^{1/2\beta-1}\,, \tag{16.149}$$

the equation of state at order ε takes the rather simple form

$$h(\theta) = \theta\left(3-2\theta^2\right)\left[1+\frac{\varepsilon(N-1)}{2(N+8)}\ln\left(3-2\theta^2\right)\right] + O\left(\varepsilon^2\right). \tag{16.150}$$

For $N = 1$, the equation is especially simple and corresponds to the so-called linear parametric model in which $h(\theta)$ is a cubic odd function of θ. One verifies that it is still possible to adjust x_0 at order ε^2 to preserve this form. However, at order ε^3, which is also known for $N = 1$, the introduction of a term proportional to θ^5 becomes necessary. One finds

$$h(\theta) = h_0\theta\left(b^2-\theta^2\right)\left(1+c\theta^2\right) + O\left(\varepsilon^4\right), \tag{16.151}$$

in which h_0 is the field normalization constant, and b, c are given by

$$b^2 = \frac{3}{2}\left(1-\frac{\varepsilon^2}{12}\right), \qquad c = -\frac{\varepsilon^3}{18}\left(\zeta(3)+\frac{I-1}{4}\right), \tag{16.152}$$

with

$$I = \int_0^1 \mathrm{d}x \frac{\ln\left[x(1-x)\right]}{1-x(1-x)} = \tfrac{4}{9}\pi^2 - \tfrac{2}{3}\psi'(1/3) = -2.3439072386\ldots\,. \tag{16.153}$$

The constant x_0 is given by

$$x_0 = b^{1/\beta}/\left(b^2-1\right). \tag{16.154}$$

Remark. In the case $N > 1$, the function $h(\theta)$ has still a singularity on the coexistence curve, due to the presence of Goldstone modes in the ordered phase. The nature of this singularity can be obtained from the study of the non-linear σ-model, in Chapter 19. We show that the behaviour of correlation functions below T_c in a theory with a spontaneously broken continuous symmetry is governed by the zero temperature IR fixed point. Therefore, the coexistence curve singularities can be obtained from a low temperature expansion (for details, see Sections 19.11 and 19.12).

In all cases, as already stated, the essential property of the parametric representation is that it automatically satisfies the different requirements about the regularity properties of the equation of state and leads to uniform approximations.

The comparison with the numerical results for the Ising model ($N = 1$) and the Heisenberg model $N = 3$ in three dimensions shows that the successive ε and ε^2 corrections improve the mean-field approximation.

From the parametric representation of the equation of state, it is also possible to derive a representation for the singular part of the free energy density. Setting

$$F(M,\tau) \equiv \Omega^{-1}\Gamma_{\text{sing.}}(M,\tau) = R^{2-\alpha} g(\theta), \tag{16.155}$$

(Ω is the volume) one finds for $g(\theta)$ the differential equation

$$h(\theta)\left(1-\theta^2+2\beta\theta^2\right) = 2(2-\alpha)\theta g(\theta) + \left(1-\theta^2\right) g'(\theta). \tag{16.156}$$

The integration constant is obtained by requiring the regularity of $g(\theta)$ at $\theta = 1$. Note that if one expands

$$h(\theta)\left(1-\theta^2+2\beta\theta^2\right) = X_0 + X_1(1-\theta^2) + X_2(1-\theta^2)^2 + O\left((1-\theta^2)^3\right),$$

then for $\alpha \to 0$

$$g(\theta) \sim -\frac{X_2}{2\alpha}(1-\theta^2)^2. \tag{16.157}$$

In the same way, the inverse magnetic susceptibility is given by

$$\chi^{-1} = R^{\gamma} g_2(\theta), \tag{16.158}$$

with

$$g_2(\theta)\left(1-\theta^2+2\beta\theta^2\right) = 2\beta\delta\theta h(\theta) + \left(1-\theta^2\right) h'(\theta). \tag{16.159}$$

In particular, these expressions can be used to calculate various universal ratios of amplitudes.

16.9.4 Amplitude ratios

Some simple universal numbers, apart from critical exponents, have been calculated: ratios of amplitudes of singularities near T_c [164–167]. We first consider two examples which can be derived directly from the equation of state.

The specific heat. The singular part of the specific heat C_H, that is, the $\phi^2(x)$ two-point correlation function at zero momentum, behaves near T_c as

$$C_H = A^{\pm}\left|\tau\right|^{-\alpha}, \quad \tau = T - T_c \to \pm 0\,. \tag{16.160}$$

The ratio A^+/A^- is universal. It is directly related to the function $g(\theta)$ defined by equation (16.156):

$$\frac{A^+}{A^-} = \left(b^2-1\right)^{2-\alpha}\frac{g(0)}{g(b)}. \tag{16.161}$$

At order ε^2, one finds

$$\begin{aligned}\frac{A^+}{A^-} = 2^{\alpha-2}N\Big\{&1+\varepsilon+\big[3N^2+26N+100+(4-N)(N-1)\zeta(2)\\ &-6(5N+22)\zeta(3)-9(4-N)\lambda\big]\frac{\varepsilon^2}{2(N+8)^2}\Big\} + O\left(\varepsilon^3\right),\end{aligned} \tag{16.162}$$

with

$$\zeta(2) = \pi^2/6 = 1.64493406684\ldots\,,$$

while λ is defined in terms of the integral I given in equation (16.153):

$$\lambda = -I/2 = \tfrac{1}{3}\psi'(1/3) - \tfrac{2}{9}\pi^2 = 1.17195361934\ldots\,.$$

The evaluation (16.157) shows that, for α small, this is a poor representation since $A^+/A^- = 1 + O(\alpha)$. A better representation then is (we give only the two first terms),

$$\frac{A^+}{A^-} = 2^\alpha (1 - K\alpha/\varepsilon), \quad \text{with} \quad K = \frac{1}{2}(N+8) + \frac{N^2 + 4N + 28}{2(N+8)}\varepsilon + O(\varepsilon^2).$$

The magnetic susceptibility. The magnetic susceptibility in zero field can also be calculated from the function $g_2(\theta)$ defined by equation (16.159). As we know, below T_c, the susceptibility diverges for systems with Goldstone modes. We restrict ourselves, therefore, to $N = 1$. Defining

$$\chi = C^\pm |\tau|^{-\gamma}, \quad \text{for } \tau \to \pm 0, \tag{16.163}$$

one obtains

$$\frac{C^+}{C^-} = \frac{2\left(1 + cb^2\right)\left(b^2 - 1\right)^{1-\gamma}}{\left[1 - b^2\left(1 - 2\beta\right)\right]} \tag{16.164a}$$

$$= \frac{2^{\gamma+1}}{6\beta - 1}\left[1 + \left(\frac{2\lambda + 1}{4} - \zeta(3)\right)\frac{\varepsilon^3}{12}\right] + O\left(\varepsilon^4\right). \tag{16.164b}$$

The ratio C^+/C^- can be expressed, at order ε^2, entirely in terms of critical exponents. This form follows naturally from the parametric representation of the equation of state. The ε^3 relative correction is of the order of only 3%.

The correlation length. We define here the correlation length in terms of the ratio of the two first moments of the two-point correlation function:

$$\tilde{\Gamma}^{(2)}(p) = \tilde{\Gamma}^{(2)}(0)\left(1 + \xi_1^2 p^2\right) + O\left(p^4\right). \tag{16.165}$$

The function ξ_1^2 has the scaling form

$$\xi_1^2(M, \tau) = M^{-2\nu/\beta} f_\xi(\tau/M^{1/\beta}). \tag{16.166}$$

Otherwise, it shares all the properties of the equation of state. It can be written in parametric form as

$$\xi_1^2(M, \tau) = R^{-2\nu} g_\xi(\theta). \tag{16.167}$$

At order ε, for $N = 1$, for example, one finds

$$g_\xi(\theta) = g_\xi(0)\left(1 - \frac{5}{18}\varepsilon\theta^2\right) + O\left(\varepsilon^2\right). \tag{16.168}$$

Setting in zero field,

$$\xi_1 = f_1^\pm |\tau|^{-\nu}, \quad \text{for } \tau \to \pm 0, \tag{16.169}$$

quantity that exists only for $N = 1$, one can use the determination of g_ξ to calculate the ratio

$$f_1^+/f_1^- = 2^\nu \left[1 + \frac{5}{24}\varepsilon + \frac{1}{432}\left(\frac{295}{24} + 2I\right)\varepsilon^2\right] + O\left(\varepsilon^3\right), \tag{16.170}$$

in which the constant I is given by equation (16.153).

An additional universal constant. To the relation between exponents,

$$2 - \alpha = d\nu$$

is associated a universal combination, which involves only amplitudes of singularities when T_c is approached from above,

$$R_\xi^+ = f_1^+ \left(\alpha A^+\right)^{1/d}. \tag{16.171}$$

Indeed, from the definitions (16.155) and (16.169), one infers

$$\left(R_\xi^+\right)^d = (1-\alpha)(2-\alpha)\tau^{\alpha-2}F(0,\tau)\tau^{\nu d}(\xi_1)^d = (1-\alpha)(2-\alpha)F(0,\tau)(\xi_1)^d,$$

where the last product is normalization independent. The ε-expansion of R_ξ^+ is

$$\left(R_\xi^+\right)^d = \sigma_d \frac{N}{2}\nu(1-\alpha)\left[1+\eta\left(\frac{-11}{2}+\frac{14}{3}\lambda\right)\right] + O\left(\varepsilon^3\right), \tag{16.172}$$

with

$$\sigma_d = \Gamma(1+\varepsilon/2)\Gamma(1-\varepsilon/2)N_d$$

(the loop factor N_d has been defined in equation (16.129)).

Other universal ratios. Clearly, it is possible to define an infinite number of other universal ratios. We give here a few other examples, which have been considered in the literature. Let us first define some additional amplitudes. On the critical isotherm, the correlation length behaves as

$$\xi_1 = f_1^c / H^{2/(d+2-\eta)}, \tag{16.173}$$

the magnetic susceptibility as

$$\chi = C^c / H^{1-1/\delta}; \tag{16.174}$$

the spontaneous magnetization vanishes as

$$M = B\left(-\tau\right)^\beta, \tag{16.175}$$

and the spin–spin correlation function in momentum space at T_c behaves as

$$\chi(p) = \left[\tilde{\Gamma}^{(2)}(p)\right]^{-1} = D p^{\eta-2}. \tag{16.176}$$

One can then define the following universal ratio,

$$R_c = \alpha A^+ C^+ / B^2, \tag{16.177}$$

which corresponds to the relation between exponents

$$\alpha + 2\beta + \gamma = 2\,.$$

Indeed, using this relation, one verifies that R_c is proportional to $F(0,\tau)M^{-2}\chi$, which is normalization independent. The ε-expansion of R_c is,

$$R_c = \frac{N}{N+8}2^{-2\beta-1}\varepsilon\left[1+\left(1-\frac{30}{(N+8)^2}\right)\varepsilon\right] + O\left(\varepsilon^3\right). \tag{16.178}$$

One can also construct the three following combinations [165]:

$$Q_1 = C^c \delta / \left(B^{\delta-1} C^+\right)^{1/\delta} , \tag{16.179}$$

$$Q_2 = \left(f_1^c / f_1^+\right)^{2-\eta} C^+ / C^c , \tag{16.180}$$

$$Q_3 = D \left(f_1^+\right)^{2-\eta} / C^+ , \tag{16.181}$$

which correspond to the relations $\gamma = \beta(\delta - 1)$, the explicit expression of δ and $\gamma = \nu(2-\eta)$. Moreover, Q_1 and Q_3 are normalization independent, because $H\chi/M$ and $p\xi$, respectively, are. For Q_2, this property follows immediately from the definition. Thus, all three quantities are *universal*.

The quantity Q_1 is related to R_χ defined in Ref. [166],

$$R_\chi = Q_1^{-\delta} . \tag{16.182}$$

Their ε-expansions can be written as

$$R_\chi = 3^{(\delta-3)/2} 2^{\gamma+(1-\delta)/2} \left[1 + \left(\frac{2\lambda+1}{4} - \zeta(3)\right) \frac{\varepsilon^3}{18}\right] + O\left(\varepsilon^4\right) , \tag{16.183}$$

$$Q_2 = 1 + \frac{\varepsilon}{18} + \left(\frac{23}{9} + \frac{4}{3}\lambda\right) \frac{\varepsilon^2}{54} + O\left(\varepsilon^3\right) , \tag{16.184}$$

$$Q_3 = 1 - \left(\frac{8}{3}\lambda + 5\right) \frac{\varepsilon^2}{216} + O\left(\varepsilon^3\right) . \tag{16.185}$$

Numerical results are given in Table 41.6, and compared with various high temperature (HT) series and experimental determinations.

It is worth mentioning that universal ratios of amplitudes of corrections to the leading critical behaviour have also been calculated. We expand any physical quantity, for $\tau = T - T_c$ small, as

$$f(\tau) = A_f \left|\tau\right|^{-\lambda_f} \left(1 + a_f \left|\tau\right|^{\theta} + \cdots\right) , \tag{16.186}$$

where the correction exponent θ (sometimes also called Δ_1) is given by

$$\theta = \omega\nu . \tag{16.187}$$

The ratio of correction amplitudes a_{f_1}/a_{f_2} corresponding to two different quantities f_1 and f_2 is also universal. A few such ratios have been calculated. One example is the ratio of corrections involving the correlation length and the susceptibility above T_c:

$$\frac{a_\chi^+}{a_\xi^+} = 2 \left\{1 - \frac{\varepsilon}{N+8} - \left[\frac{2\lambda}{3(N+8)} - \frac{N^2 - 15N - 124}{2(N+8)^3}\right] \varepsilon^2\right\} + O\left(\varepsilon^3\right) . \tag{16.188}$$

16.10 Conformal bootstrap

At an IR fixed point, a QFT is scale invariant, but is also expected to be conformal invariant (Section A13.3). In dimension 2, the conformal group is infinite dimensional, and this has made a systematic construction of many conformal invariant theories possible. However, in higher dimensions, the conformal group reduces to $SO(d+1,1)$, and only determines the form of the two- and three-point functions.

In recent years, a new method has been developed to determine critical exponents in dimensions $d > 2$. It combines conformal invariance and short distance expansion (Section 11.3) [91]. Describing the whole method precisely goes beyond this work, but a few simple elements can be given, in the example of the $\mathbb{Z}_2$ symmetric universality class, which contains the Ising model.

At the IR fixed point, all scaling operators can be classified according to their scaling dimension, spin and $\mathbb{Z}_2$ parity. Moreover, the scaling operators can be further divided into primaries, and descendants obtained by differentiation of the primaries.

SDE. The SDE of the product of two primary operators involves only primary operators $O_i(x)$ and descendants and, therefore, can be expressed only in terms of primaries in the form

$$O_i(x)O_j(y) = \sum_k f_{ijk} C_{ijk}(y, \partial_y) O_k(y),$$

where the functions C_{ijk} are determined by conformal invariance, spins and dimensions of the primary operators, and the f_{ijk} are additional constants, which play the role of structure constants of the operator algebra.

Moreover, in the case of the Ising-model universality class, the $\mathbb{Z}_2$ parity further restricts the set of operators in the SDE.

Associativity of the SDE. One considers the expectation value of the product of four primaries,

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle ,$$

and applies the SDE first on O_1O_2 and O_3O_4, and then on O_1O_4 and O_3O_2. In both cases, the four-point function is reduced to a combination of products of two-point functions. Assuming the associativity of the SDE, one can identify the two expansions. Presumably, the resulting equations determine a discrete set of parameters corresponding to $\mathbb{Z}_2$ symmetric critical theories.

Truncating the expansion, ordered according to increasing scaling dimensions, one then verifies that the resulting equations yield inequalities involving all the parameters of the conformal theory, in particular, scaling dimensions. The most recent results [435] constrain exponents with a remarkable precision, and yield values consistent with other theoretical estimates (Sections 41.5 and 41.6).