based on the construction of invariants for the given group action. This is illustrated in Section 3.2 by a number of interesting examples, including the heat equation, the Korteweg–de Vries equation and the Euler equations of ideal fluid flow. Further examples are indicated in the exercises at the end of the chapter as well as the cited references. The third section deals with the problem of classifying group-invariant solutions. Since there are almost always an infinite number of different symmetry groups one might employ to find invariant solutions, a means of determining which groups give fundamentally different types of invariant solutions is essential for gaining a complete understanding of the solutions which might be available. Since any transformation in the full symmetry group will take a solution to another solution, we need only find invariant solutions which are not related by a transformation in the full symmetry group. This classification problem can be solved by looking at the adjoint representation of the symmetry group on its Lie algebra, and includes an analogous classification of the different subgroups of the full symmetry group.

The remaining two sections of this chapter are devoted to a rigorous presentation of the theoretical basis of the group-invariant solution method and can safely be omitted if one is only interested in applying these techniques. A rigorous, global geometrical setting for these results is provided by the quotient manifold of a manifold under some regular group of transformations. Each point on the quotient manifold will correspond to an orbit of the group, so the quotient manifold has, essentially, s fewer dimensions where s is the dimension of the orbits of the group. Group-invariant objects on the original manifold will have natural counterparts on the quotient manifold which serve to completely characterize them. In particular, a system of partial differential equations which is invariant under the given transformation group will have a corresponding reduced system of differential equations on the quotient manifold, the number of independent variables having thereby been reduced by r. Solutions of the reduced system will correspond to group-invariant solutions of the original system. The one complicating detail in this method is that even when the original manifold is an open subset of some Euclidean space, the quotient manifold is not in any natural way an open subset of a "reduced" Euclidean space, so our earlier construction of jet spaces and symmetry groups is not immediately applicable. At this point there are two routes available. The more concrete avenue of attack would be to restrict to suitably smaller open subsets of Euclidean space, thereby forcing the quotient manifold to also be the subset of some Euclidean space through a choice of new independent and dependent variables on it. However, in this approach, constructions become very unpleasantly coordinate-dependent and lose much of their innate simplicity. The more abstract approach, and the one adopted here, is to generalize our construction of jet spaces, prolongations and differential equations to arbitrary smooth manifolds. This is done by "completing" the ordinary jet spaces so as to include "functions" determined by arbitrary p-dimensional submanifolds, which may be multiply-valued or have infinite derivatives. Although this method requires a fair amount of abstraction and mathematical sophistication just to state the definitions, the principal results on group-invariant solutions retain their strong geometrical flavour and, as far as the proofs are concerned, become practically trivial. The more technically complicated local coordinate picture is then straightforwardly derived from this abstract reformulation of the reduction procedure.

3.1. Construction of Group-Invariant Solutions

Consider a system of partial differential equations Δ defined over an open subset $M \subset X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$ of the space of independent and dependent variables. Let G be a local group of transformations acting on M. Roughly, a solution $u = f(x)$ of the system is said to be G-invariant if it remains unchanged by all the group transformations in G, meaning that for each $g \in G$, the functions f and (provided it is defined) $g \cdot f$ agree on their common domains of definition. For example, the fundamental solution $u = \log(x^2 + y^2)$ for the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ is invariant under the one-parameter rotation group $\mathrm{SO}(2)$: $(x, y, u) \mapsto (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta, u)$, acting on the independent variables x, y. More rigorously, we can define a *G-invariant solution* of a system of partial differential equations as a solution $u = f(x)$ whose graph $\Gamma_f \equiv \{(x, f(x))\} \subset M$ is a locally G-invariant subset of M; see Definition 2.12.

If G is a symmetry group of a system of partial differential equations Δ, then, under some additional regularity assumptions on the action of G, we can find all the G-invariant solutions to Δ by solving a reduced system of differential equations, denoted by Δ/G, which will involve fewer independent variables than the original system Δ. To see how this reduction is effected, we begin by making the simplifying assumption that G act *projectably* on M. This means that the transformations in G all take the form $(\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\Xi_g(x), \Phi_g(x, u))$, i.e. the changes in the independent variables x do not depend on the dependent variables u. (More general nonprojectable group actions will be treated subsequently, but the basic technique is the same.) There is then a projected group action $\tilde{x} = g \cdot x = \Xi_g(x)$ on an open subset $\Omega \subset X$. We make the regularity assumption that both the action of G on M and the projected action of G on Ω are *regular* in the sense of Definition 1.26, and that the orbits of both of these actions have the same dimension s, where s is strictly less than p, the number of independent variables in the system. (The case $s = p$ is fairly trivial, while if $s > p$, no G-invariant functions exist. Usually s will be the same as the dimension of G itself, but this need not be the case.) Under these assumptions, Theorem 2.17 implies that locally there exist $p - s$ functionally independent invariants $y^1 = \eta^1(x), \ldots, y^{p-s} = \eta^{p-s}(x)$ of the projected group action on $\Omega \subset X$. Each of these functions is also an invariant of the full group action on M, and, furthermore, we can find q additional invariants of the action of G on M, of the form $v^1 = \zeta^1(x, u), \ldots, v^q = \zeta^q(x, u)$, which, together with the η's provide a complete set of $p + q - s$ functionally independent invariants for G on M. We write this complete collection of invariants concisely as

$$y = \eta(x), \qquad v = \zeta(x, u). \tag{3.1}$$

In the construction of the reduced system of differential equations for the G-invariant solutions to Δ, the y's will play the role of the new independent variables, and the v's the role of the new dependent variables. Note in particular that there are s fewer independent variables $y^1, \ldots, y^{p-s}$ which will appear in this reduced system, where s is the dimension of the orbits of G.

There is now a one-to-one correspondence between G-invariant functions $u = f(x)$ on M and arbitrary functions $v = h(y)$ involving the new variables. To explain this correspondence, we begin by invoking the implicit function theorem to solve the system $y = \eta(x)$ for $p - s$ of the independent variables, say $\tilde{x} = (x^{i_1}, \ldots, x^{i_{p-s}})$, in terms of the new variables $y^1, \ldots, y^{p-s}$ and the remaining s old independent variables, denoted as $\hat{x} = (x^{j_1}, \ldots, x^{j_s})$. Thus we have the solution

$$\tilde{x} = \gamma(\hat{x}, y) \tag{3.2}$$

for some well-defined function γ. The first $p - s$ of the old independent variables $\tilde{x}$ are known as *principal variables*, and the remaining s of these variables $\hat{x}$ are the *parametric variables*, as they will, in fact, enter parametrically into all the subsequent formulae. The precise manner in which one splits the variables x into principal and parametric variables is restricted only by the requirement that the $(p - s) \times (p - s)$ submatrix $(\partial\eta^j/\partial\tilde{x}^i)$ of the full Jacobian matrix $\partial\eta/\partial x$ is invertible, so that the implicit function theorem is applicable; otherwise, the choice is entirely arbitrary. We need to make a further *transversality* assumption on the action of G on M, cf. (3.35), that allows us to solve the other system of invariants $v = \zeta(x, u)$ for all of the dependent variables $u^1, \ldots, u^q$ in terms of $x^1, \ldots, x^p$, and $v^1, \ldots, v^q$, and hence in terms of the new variables y, v and the parametric variables $\hat{x}$:

$$u = \tilde{\delta}(x, v) = \tilde{\delta}(\hat{x}, \gamma(\hat{x}, y), v) \equiv \delta(\hat{x}, y, v) \tag{3.3}$$

near any point $(x_0, u_0) \in M$.

If $v = h(y)$ is any smooth function, then (3.3) coupled with (3.1) produces a corresponding G-invariant function on M, of the form

$$u = f(x) = \delta(\hat{x}, \eta(x), h(\eta(x))). \tag{3.4}$$

Conversely, if $u = f(x)$ is any G-invariant function on M, then it is not too difficult to see that there necessarily exists a function $v = h(y)$ such that f and the corresponding function (3.4) locally agree. Thus, we have seen how G-invariance of functions serves to decrease the number of variables upon which they depend.

We are now interested in finding all the G-invariant solutions to some system of partial differential equations

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, l.$$

In other words, we want to know when a function of the form (3.4) corresponding to a function $v = h(y)$ is a solution to Δ. This will impose certain constraints on the function h; these are found by computing the formulae for the derivatives of a function of the form (3.4) with respect to x in terms of the derivatives of $v = h(y)$ with respect y, and then substituting these into the system of differential equations Δ. Thus we need to know how the derivatives of functions $v = h(y)$ are related to the derivatives of the corresponding G-invariant function $u = f(x)$. However, this is an easy application of the chain rule. Differentiating (3.4) with respect to x leads to a system of equations of the form

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[\delta(\hat{x}, y, v)] = \frac{\partial \delta}{\partial \hat{x}} + \frac{\partial \delta}{\partial y}\frac{\partial \eta}{\partial x} + \frac{\partial \delta}{\partial v}\frac{\partial v}{\partial y}\frac{\partial \eta}{\partial x},$$

since $y = \eta(x)$. Here, $\partial u/\partial x$, etc., denote Jacobian matrices of first order derivatives of the indicated variables. Moreover, using (3.2), we can rewrite $\partial\eta/\partial x$ in terms of y and the parametric variables $\hat{x}$. Thus we obtain an equation of the form

$$\partial u/\partial x = \delta_1(\hat{x}, y, v, \partial v/\partial y)$$

expressing the first order derivatives of any G-invariant function u with respect to x in terms of y, v, the first order derivatives of v with respect to y, *plus* the parametric variables $\hat{x}$. Continuing to differentiate using the chain rule, and substituting according to (3.2) whenever necessary, we are led to general formulae

$$u^{(n)} = \delta^{(n)}(\hat{x}, y, v^{(n)}),$$

for all the derivatives of such a u with respect to x up to order n in terms of y, v, the derivatives of v with respect to y up to order n, and the ubiquitous parametric variables $\hat{x}$. At this point, it is worth considering a specific example.

Example 3.1. Consider the one-parameter scaling group

$$(x, t, u) \mapsto (\lambda x, \lambda t, u), \qquad \lambda \in \mathbb{R}^+,$$

acting on $X \times U \simeq \mathbb{R}^3$. On the upper half space $M \equiv \{t > 0\}$, the action is regular, with global independent invariants $y = x/t$ and $v = u$. If we treat v as a function of y, we can compute formulae for the derivatives of u with respect to x and t in terms of y, v and the derivatives of v with respect to y, along with a single parametric variable, which we designate to be t, so that x will be the corresponding principal variable. We find, using the chain rule, that if

$u = v = v(y) = v(x/t)$, then

$$u_x = t^{-1} v_y, \qquad u_t = -t^{-2} x v_y = -t^{-1} y v_y.$$

Further differentiations yield the second order formulae

$$u_{xx} = t^{-2} v_{yy}, \qquad u_{xt} = -t^{-2}(y v_{yy} + v_y), \qquad u_{tt} = t^{-2}(y^2 v_{yy} + 2 y v_y), \tag{3.5}$$

and so on.

Once the relevant formulae relating derivatives of u with respect to x to those of v with respect to y have been determined, the reduced system of differential equations for the G-invariant solutions to the system Δ is found by substituting these expressions into the system wherever they occur. In general, this leads to a system of equations of the form

$$\tilde{\Delta}_\nu(\hat{x}, y, v^{(n)}) = 0, \qquad \nu = 1, \ldots, l,$$

still involving the parametric variables $\hat{x}$. If G is actually a symmetry group for Δ, this resulting system will in fact always be *equivalent* to a system of equations, denoted

$$(\Delta/G)_\nu(y, v^{(n)}) = 0, \qquad \nu = 1, \ldots, l,$$

which are independent of the parametric variables, and thus constitute a genuine system of differential equations for v as a function of y. This is the reduced system Δ/G for the G-invariant solutions to the system Δ. Every solution $v = h(y)$ of Δ/G will correspond, via (3.4), to a G-invariant solution to Δ, and, moreover, every G-invariant solution can be constructed in this manner.

Example 3.2. The one-dimensional wave equation $u_{tt} - u_{xx} = 0$ is invariant under the scaling group presented in Example 3.1. To construct the corresponding scale-invariant solutions, we need only substitute the derivative formulae (3.5) into the wave equation, and solve the resulting ordinary differential equation. Upon substituting, we find the equation

$$t^{-2}(y^2 v_{yy} + 2 y v_y - v_{yy}) = 0.$$

As promised by the general theory, this equation is equivalent to an equation

$$(y^2 - 1) v_{yy} + 2 y v_y = 0$$

in which the parametric variable t no longer appears. This latter ordinary differential equation is the reduced equation for the scale-invariant solutions to the wave equation. It is easily integrated, with general solution

$$v = c \log|(y - 1)/(y + 1)| + c',$$

where c, c' are arbitrary constants. Replacing the variables y, v in the solution by their expressions in terms of x, t, u, we deduce the general scale-invariant solution to the wave equation (for the particular scaling symmetry group under consideration) to be

$$u = c \log|(x - t)/(x + t)| + c'.$$

For the reader's convenience, we summarize the basic computational procedures for finding group-invariant solutions of a given system of partial differential equations from the beginning. We list the steps in order, starting with the computation of the symmetry group.

(I) Find all the infinitesimal generators $\mathbf{v}$ of symmetry groups of the system using the basic prolongation methods from Chapter 2, specifically the infinitesimal criterion (2.25).

(II) Decide on the "degree of symmetry" s of the invariant solutions. Here $1 \leqslant s \leqslant p$ will correspond to the dimension of the orbits of some subgroup of the full symmetry group. The reduced systems of differential equations for the invariant solutions will depend on $p - s$ independent variables. Thus to reduce the system of partial differential equations to a system of ordinary differential equations, we need to choose $s = p - 1$. In general, the smaller s is the more invariant solutions there will be, but the harder the reduced system Δ/G will be to solve explicitly.

(III) Find all s-dimensional subgroups G of the full symmetry group found in part I. This is equivalent (Theorem 1.51) to finding all s-dimensional subalgebras of the full Lie algebra of infinitestimal symmetries $\mathbf{v}$. To each such subgroup or subalgebra there will correspond a set of group-invariant solutions reflecting the symmetries inherent in G itself. The problem of classifying subalgebras of a given Lie algebra will be explored in detail in Section 3.3. (In principle, an s-dimensional subgroup G may have orbits of dimension smaller than s, and, as we remarked earlier, it is the dimension of the orbits which matters. In practice, however, this mode of degeneracy rarely occurs, so we can content ourselves with fixing the dimension of the subgroup.)

(IV) Fixing the symmetry group G, we construct a complete set of functionally independent invariants, as in Section 2.1, which we divide into two classes

$$\begin{aligned} y^1 &= \eta^1(x, u), \ldots, y^{p-s} = \eta^{p-s}(x, u), \\ v^1 &= \zeta^1(x, u), \ldots, v^q = \zeta^q(x, u), \end{aligned} \tag{3.6}$$

corresponding to the new independent and dependent variables respectively. If G acts projectably, the choice of independent and dependent variables is prescribed by requiring the η^i's to be independent of u; in the more general case, there is quite a bit of freedom in this choice, and different choices will lead to seemingly different reduced systems, all of which are related by some form of "hodograph" transformation.

(V) Provided G acts transversally, (cf. Proposition 3.37) we can solve (3.6) for $p - s$ of the x's, which we denote by $\tilde{x}$, and all of the u's in terms of y, v and the remaining s parametric variables $\hat{x}$.

$$\tilde{x} = \gamma(\hat{x}, y, v), \qquad u = \delta(\hat{x}, y, v). \tag{3.7}$$

Furthermore, considering v as a function of y we can use (3.6), (3.7) and the chain rule to differentiate and thereby find expressions for the x-derivatives of any G-invariant u in terms of y, v, y-derivatives of v and the parametric variables $\hat{x}$:

$$u^{(n)} = \delta^{(n)}(\hat{x}, y, v^{(n)}). \tag{3.8}$$

(VI) Substitute the expressions (3.7), (3.8) into the system $\Delta(x, u^{(n)}) = 0$. The resulting system of equations will always be equivalent to a system of differential equations for $v = h(y)$ independent of the parametric variables $\hat{x}$:

$$\Delta/G(y, v^{(n)}) = 0. \tag{3.9}$$

At this stage we have constructed the reduced system of differential equations for the G-invariant solutions.

(VII) Solve the reduced system (3.9). For each solution $v = h(y)$ of Δ/G there corresponds a G-invariant solution $u = f(x)$ of the original system, which is given implicitly by the relation

$$\zeta(x, u) = h[\eta(x, u)]. \tag{3.10}$$

Repeating steps IV through VII for each symmetry group G determined in step III will yield a complete set of group-invariant solutions for our systems.

3.2. Examples of Group-Invariant Solutions

Before attempting to prove that the basic procedure for constructing group-invariant solutions outlined in the preceding section works, we will illustrate the method with some systematic examples, constructing group-invariant solutions of the Korteweg–de Vries, heat and Euler equations. These will lead naturally into the problem of how to classify group-invariant solutions in such a way as to find "all" such solutions with a minimum of computational difficulty. Before addressing this question, however, we begin with our examples.

Example 3.3. *The Heat Equation.* The symmetry group of the heat equation

$$u_t = u_{xx}$$

was computed in Example 2.41; it consisted of a six-parameter group of symmetries particular to the heat equation itself plus an infinite-dimensional subgroup stemming from the linearity of the equation. For each one-parameter subgroup of the full symmetry group there will be a corresponding class of group-invariant solutions which will be determined from a reduced ordinary differential equation, whose form will in general depend on the particular subgroup under investigation.

(a) *Travelling Wave Solutions.* In general, travelling wave solutions to a partial differential equation arise as special group-invariant solutions in which the group under consideration is a translation group on the space of independent variables. In the present example, consider the translation group

$$(x, t, u) \mapsto (x + c\varepsilon, t + \varepsilon, u), \qquad \varepsilon \in \mathbb{R},$$

generated by $\partial_t + c\partial_x$, in which c is a fixed constant, which will determine the speed of the waves. Global invariants of this group are

$$y = x - ct, \qquad v = u, \tag{3.11}$$

so that a group-invariant solution $v = h(y)$ takes the familiar form $u = h(x - ct)$ determining a wave of unchanging profile moving at the constant velocity c. Solving for the derivatives of u with respect to x and t in terms of those of v with respect to y we find

$$u_t = -cv_y, \qquad u_x = v_y, \qquad u_{xx} = v_{yy},$$

and so on. Substituting these expressions into the heat equation, we find the reduced ordinary differential equation for the travelling wave solutions to be

$$-cv_y = v_{yy}.$$

The general solution of this linear, constant coefficient equation is

$$v(y) = k\, e^{-cy} + l$$

for k, l arbitrary constants. Substituting back according to (3.11), we find the most general travelling wave solution to the heat equation to be an exponential of the form

$$u(x, t) = k\, e^{-c(x-ct)} + l.$$

(b) *Scale-Invariant (Similarity) Solutions.* There are two one-parameter groups of scaling symmetries of the heat equation, and we consider a linear combination

$$x\partial_x + 2t\partial_t + 2au\partial_u, \qquad a \in \mathbb{R},$$

of their infinitesimal generators, which corresponds to the scaling group

$$(x, t, u) \mapsto (\lambda x, \lambda^2 t, \lambda^{2a} u), \qquad \lambda \in \mathbb{R}^+.$$

On the half space $\{(x, t, u): t > 0\}$, global invariants of this one-parameter group are provided by the functions

$$y = x/\sqrt{t}, \qquad v = t^{-a}u.$$

Solving for the derivatives of u in terms of those of v, we find

$$u = t^a v,$$

$$u_x = t^{a-1/2}v_y, \qquad u_{xx} = t^{a-1}v_{yy},$$

$$u_t = -\tfrac{1}{2}xt^{a-3/2}v_y + at^{a-1}v = t^{a-1}(-\tfrac{1}{2}yv_y + av).$$

Here we are treating t as the parametric variable, and we have succeeded in expressing the relevant derivatives of u with respect to x and t in terms of y, v, the derivatives of v with respect to y, and the parametric variable t as in (3.8).

Substituting these expressions into the heat equation, we find

$$t^{a-1}v_{yy} = t^{a-1}(-\tfrac{1}{2}yv_y + av).$$

As guaranteed by the general theory, this equation is equivalent to one in which the parametric variable t does not occur, namely

$$v_{yy} + \tfrac{1}{2}yv_y - av = 0,$$

which forms the reduced equation for the scale-invariant solutions. The solutions of this linear ordinary differential equation can be written in terms of parabolic cylinder functions. Indeed, if we set

$$w = v\exp(\tfrac{1}{8}y^2),$$

then w satisfies a scaled form of Weber's differential equation,

$$w_{yy} = [(a + \tfrac{1}{4}) + \tfrac{1}{16}y^2]w.$$

The general solution of this equation is

$$w(y) = k\mathrm{U}\left(2a + \tfrac{1}{2}, \frac{y}{\sqrt{2}}\right) + \tilde{k}\mathrm{V}\left(2a + \tfrac{1}{2}, \frac{y}{\sqrt{2}}\right),$$

where $\mathrm{U}(b, z)$, $\mathrm{V}(b, z)$ are parabolic cylinder functions, cf. Abramowitz and Stegun, [1; § 19.1]. Thus the general scale-invariant solution to the heat equation takes the form

$$u(x, t) = t^a\, e^{-x^2/8t}\left\{k\mathrm{U}\left(2a + \tfrac{1}{2}, \frac{x}{\sqrt{2t}}\right) + \tilde{k}\mathrm{V}\left(2a + \tfrac{1}{2}, \frac{x}{\sqrt{2t}}\right)\right\}.$$

Particular values of a lead to special scale-invariant solutions which are expressible in terms of elementary functions. For instance, if $a = 0$, we obtain the probability solution

$$u(x, t) = k^* \operatorname{erf}(x/\sqrt{2t}) + \tilde{k}^*,$$

where erf is the error function. Since $\mathrm{U}(-n - \tfrac{1}{2}, z) = e^{-z^2/4}\,\mathrm{He}_n(z)$ where He_n is the n-th Hermite polynomial, if $a = -(n + 1)/2$ we obtain the solutions

$$u(x, t) = t^{-(n+1)/2}\, e^{-x^2/4t}\,\mathrm{He}_n(x/\sqrt{2t}),$$

which include the source solution ($n = 0$). Similarly, the relation $\mathrm{V}(n + \tfrac{1}{2}, z) = \sqrt{2/\pi}e^{z^2/4}\,\mathrm{He}_n^*(z)$, where $\mathrm{He}_n^*(z) = (-i)^n\,\mathrm{He}_n(iz)$, leads to the heat polynomials (see Widder, [1])

$$x, \quad x^2 + 2t, \quad x^3 + 6xt, \quad \text{etc.},$$

as special scale-invariant solutions.

(c) *Galilean-Invariant Solutions.* The one-parameter group of Galilean boosts, generated by $\mathbf{v}_5 = 2t\partial_x - xu\partial_u$ has global invariants $y = t$, $v = u\exp(x^2/4t)$ on the upper half space $\{t > 0\}$. We find

$$u_t = \left(v_y + \frac{x^2}{4t^2}v\right)e^{-x^2/4t}, \qquad u_{xx} = \left(\frac{x^2}{4t^2} - \frac{1}{2t}\right)v\, e^{-x^2/4t}.$$

Therefore, for the heat equation the reduced equation for Galilean-invariant solutions is a *first order* ordinary differential equation $2yv_y + v = 0$, despite the fact that the heat equation was a second order partial differential equation. The solution is $v(y) = k/\sqrt{y}$. Hence the most general Galilean-invariant solution is a scalar multiple of the source solution,

$$u(x, t) = \frac{k}{\sqrt{t}}e^{-x^2/4t}.$$

which we earlier found as a scale-invariant solution. Thus a given solution may be invariant under more than one subgroup of the full symmetry group.

We can clearly extend this list of group-invariant solutions by considering further one-parameter subgroups obtained from more general linear combinations of the infinitesimal generators of the full symmetry group. At the moment, however, without some means of classifying these solutions, it is somewhat pointless to continue. Once we have determined the correct classification procedure, we will return to this question and find (in a sense) the most general group-invariant solutions to the heat equation. See Example 3.17.

Example 3.4. The symmetry group of the Korteweg–de Vries equation

$$u_t + u_{xxx} + uu_x = 0$$

was computed in Example 2.44. Let us look at particular group-invariant solutions.

(a) *Travelling Wave Solutions.* Here the group is the same translational group already looked at in the previous example. In terms of the invariants $y = x - ct$, $v = u$, the reduced equation is

$$v_{yyy} + vv_y - cv_y = 0.$$

This can be immediately integrated once,

$$v_{yy} + \tfrac{1}{2}v^2 - cv = k,$$

and a second integration is performed after multiplying by v_y:

$$\tfrac{1}{2}v_y^2 = -\tfrac{1}{6}v^3 + \tfrac{1}{2}cv^2 + kv + l, \tag{3.12}$$

where k and l are arbitrary constants. The general solution can be written in terms of elliptic functions, $u = \wp(x - ct + \delta)$, δ being an arbitrary phase shift. If $u \to 0$ sufficiently rapidly as $|x| \to \infty$, then $k = l = 0$ in (3.12). This