for some well-determined function δ_2. The n-th order case is very similar, replacing (3.41) we have a formula of the form

$$\mathrm{D}_x^n \zeta = \frac{\partial v}{\partial y} \mathrm{D}_x^n \eta + \cdots, \tag{3.42}$$

where the omitted terms depend on lower order total derivatives of η, as well as y-derivatives of v up to order n. Furthermore

$$\mathrm{D}_x^n \zeta = \frac{\partial \zeta}{\partial u} \frac{\partial^n u}{\partial x^n} + \cdots,$$

the omitted terms depending on $(n-1)$-st and lower order derivatives of u. Thus (3.42) is of the form

$$\left(\frac{\partial \zeta}{\partial u} - \frac{\partial v}{\partial y}\frac{\partial \eta}{\partial u}\right)\frac{\partial^n u}{\partial x^n} = \tilde{\delta}_n\left(x, u, \ldots, \frac{\partial^{n-1} u}{\partial x^{n-1}}, \frac{\partial v}{\partial y}, \ldots, \frac{\partial^n v}{\partial y^n}\right).$$

As before, we can invert the matrix on the left, and substitute for all the derivatives of u which appear on the right their formerly computed expressions, leading to an explicit formula

$$u^{(n)} = \delta^{(n)}(\hat{x}, y, v^{(n)}) \tag{3.43}$$

for the n-th order derivatives of u with respect to x in terms of y, v and the derivatives of v with respect to y up to order n, plus the ubiquitous parametric variables $\hat{x}$.

These formulae serve to parametrize the invariant space, so that if $\pi(z) = w$, then

$$\tilde{I}^{(n)}|_z = \{(x, u^{(n)}) : u^{(n)} = \delta^{(n)}(\hat{x}, y, v^{(n)}) \text{ for some } (y, v^{(n)}) \in \widetilde{(M/G)}^{(n)}|_w\},$$

so that for each fixed $(y, v^{(n)})$ satisfying (3.38), the corresponding orbit of $\mathrm{pr}^{(n)} G$ in $\tilde{I}^{(n)}$ is parametrized by $\hat{x}$. Moreover, the projection $\pi^{(n)}(x, u^{(n)})$ of such a point is simply the point $(y, v^{(n)}) \in (M/G)^{(n)}$ obtained by omitting the variables $\hat{x}$.

Finally, let $\Delta(x, u^{(n)}) = 0$ be a system of partial differential equations on M which determines a subvariety $\mathscr{S}_\Delta \subset M^{(n)}$. The G-invariant solutions of this system (provided they exist) will have prolongations in the intersection $\mathscr{S}_\Delta \cap I^{(n)}$. If we further require each such G-invariant solution $u = f(x)$ to correspond to a smooth function $v = h(y)$ on the quotient manifold, we must further restrict to the subspace $\tilde{I}^{(n)}$ and look at $\mathscr{S}_\Delta \cap \tilde{I}^{(n)}$. Since $\tilde{I}^{(n)}$ is parametrized by $\hat{x}$, y, $v^{(n)}$, as given by (3.43), we can find this intersection by re-expressing Δ in terms of $\hat{x}$, y, $v^{(n)}$, so $\tilde{\Delta}(\hat{x}, y, v^{(n)}) = \Delta(x, u^{(n)})$ whenever (3.43) holds, whereby

$$\mathscr{S}_\Delta \cap \tilde{I}^{(n)} = \{(\hat{x}, y, v^{(n)}) \in \tilde{I}^{(n)} : \tilde{\Delta}(\hat{x}, y, v^{(n)}) = 0\}.$$

Furthermore, if G is a symmetry group of Δ, $\mathscr{S}_\Delta \cap \tilde{I}^{(n)}$ is locally invariant under the prolonged group action $\mathrm{pr}^{(n)} G$, for which $(y, v^{(n)})$ form a complete set of independent invariants. By Proposition 2.18, there is an equivalent set of equations, which we call Δ/G, which are independent of the parametric variables $\hat{x}$, so

$$\mathscr{S}_\Delta \cap \tilde{I}^{(n)} = \{(\hat{x}, y, v^{(n)}) \in \tilde{I}^{(n)}: \Delta/G(y, v^{(n)}) = 0\}.$$

From the above form of the projection $\pi^{(n)}$, we immediately conclude that the part of the reduced system in $(\widetilde{M/G})^{(n)}$, namely

$$\mathscr{S}_{\Delta/G} = \pi^{(n)}(\mathscr{S}_\Delta \cap \tilde{I}^{(n)}) = \{(y, v^{(n)}): \Delta/G(y, v^{(n)}) = 0\}, \tag{3.44}$$

is given by the equations Δ/G. Thus we have completely justified our procedure of Section 3.1. Theorem 3.36, when completed with Proposition 3.40, shows that we have proved the following rigorous version of the construction of group-invariant solutions to systems of partial differential equations.

Theorem 3.41. *Let G be a local group of transformations acting regularly and transversally on $M \subset X \times U$ with globally defined independent invariants, whereby $M/G \subset Y \times V$. Let Δ be a system of partial differential equations defined over M for which G is a symmetry group. Then there is a reduced system of differential equations Δ/G over M/G, determined by* (3.44), *with the property that any G-invariant function $u = f(x)$ on M corresponding to a well-defined function $v = h(y)$ on M/G will be a solution to Δ if and only if its representative h is a solution to Δ/G.*

(Suitable changes of coordinates on M/G will lead to all the G-invariant solutions to Δ, even those which might originally be nontransverse for the original choice of coordinates.) This completes our development of the theory and justification of the group reduction procedure.

Notes

Despite numerous claims that the concept of a group-invariant solution of a system of partial differential equations did not originate in its full generality until the 1950's, Lie, in one of his last papers, [6], actually did introduce the present general method for finding such solutions. Lie was concerned with solutions to systems of partial differential equations invariant under groups of contact transformations, but his results include the local versions of the present reduction theorems. In Section 65 of the above-mentioned paper he proves that the solutions to a partial differential equation in two independent variables, which are invariant under a one-parameter group, can all be found by solving a related ordinary differential equation. The generalization to systems of partial differential equations invariant under multi-parameter groups, i.e. our Theorem 3.41, is stated and proved in Section 76 of the same paper, but, as far as I am aware, has never before been referred to in any of the literature on this subject!

Lie died before he could make any application of his discovery. Much later, A. J. A. Morgan, [1], and Michal, [1], restated the special case of Lie's result for one-parameter symmetry groups. Subsequently, Ovsiannikov, [1], [2], reproved the general case, again unaware of the earlier work of Lie. Prior to these rediscoveries, a number of special instances of group-invariant solutions, especially similarity solutions, appeared sporadically in the literature, but without any indication that they were special cases of a much more general theory. The first such construction of which I am aware is in a paper of Boltzmann, [1]. After the turn of the century, similarity solutions appear extensively in the work of Prandtl and Blasius, and, later, Falkner and Skan, on boundary layers in fluid mechanics; see Birkhoff, [2; Chap. 5], for these and other references, as well as a discussion of the history of the Pi Theorem 3.22 from dimensional analysis. Sedov, [1], gave great emphasis to the applicability of scaling groups of symmetries and the consequential similarity solutions in the theory of dimensional analysis of complicated systems. (A good modern introduction to the use of similarity methods in engineering applications is the book of Seshadri and Na, [1].) It remained for Birkhoff, [2], to champion the use of more general symmetry groups for constructing explicit solutions to partial differential equations, and thereby directly inspire the rediscovery of Lie's method.

Since Ovsiannikov began his extensive investigations, the reduction method for constructing group-invariant solutions to partial differential equations has become the focus of much research activity, first in the Soviet Union, and, subsequently, in Europe and the United States. There is by now a large body of Soviet papers on the symmetry properties and explicit solutions for the equations of fluid mechanics, including the recent work of Kapitanskii, [1], [2], mentioned in the text; see Ovsiannikov, [3; p. 391] for a complete bibliography. (Alternative techniques for constructing explicit solutions in fluid mechanics can be found in Berker, [1].) The appearance of extra symmetries after performing a group reduction noticed by Kapitanskii has also been looked at by Rosen, [2].

Group-invariant solutions have been used to great effect in the description of the asymptotic behaviour of much more general solutions to systems of partial differential equations. The book of Barenblatt, [1], gives a good introduction to the applications to hyperbolic equations. In the same vein, Ablowitz and Kodama, [1], have given a rigorous analysis of the asymptotic behaviour of solutions to the Korteweg–de Vries equation, proving that any solution decaying to 0 at $\pm\infty$ ultimately breaks up into a finite number of distinct solitons (travelling waves) plus a dispersive tail decaying like the second Painlevé transcendent solution described here. (Incidentally, the complete classification of the group-invariant solutions of the Korteweg–de Vries equation appeared first in Kostin, [1].) Related ideas appear in the St.-Venant problem in elasticity—see Ericksen, [1].

The general connection between completely integrable (soliton) equations such as the Korteweg–de Vries equation and ordinary differential equations of Painlevé type using the mechanism of group-invariance was first conjectured by Ablowitz, Ramani and Segur, [1]. Proofs of certain special cases of the general conjecture, which gives a quite useful test for "integrability", were given by Ablowitz, Ramani and Segur, [2], and McLeod and Olver, [1]. Recently this method has been significantly extended by Weiss, Tabor and Carnevale, [1].

The rigorous foundation of the general method for constructing group-invariant solutions based on Palais' monograph, [1], using quotient manifolds first appeared in Olver, [2]. The present treatment is a much simplified version of this theory. (See also Vinogradov, [5].) For more details of the theory of extended jet bundles, see Golubitsky and Guillemin, [1; p. 172ff.], and Olver, [2].

The adjoint representation of a Lie group on its Lie algebra was known to Lie. Its use in classifying group-invariant solutions appears in Ovsiannikov, [2; § 86], and [3; § 20]. The latter reference contains more details on how to perform the classification of subgroups of a Lie group under the adjoint action. The method has received extensive development by Patera, Winternitz and Zassenhaus: see [1] and the references therein for many examples of optimal systems of subgroups for the important Lie groups of mathematical physics. The classification of the symmetry algebra of the heat equation is originally due to Weisner, [1], in his investigation of the connections between Lie groups and special functions. See also Kalnins and Miller, [1], where this classification is applied to the problem of separation of variables.

A generalization of the concept of a group-invariant solution known as a partially-invariant solution was introduced by Ovsiannikov, [2; § 17], [3; Chap. 6]. In essence, a partially-invariant solution is one whose graph, while not fully invariant under the group transformations, gets mapped into a submanifold of dimension strictly less than $p + r$ thereby. Here p is the number of independent variables and r the dimension of the orbits of G. (Note that a general function's graph would get mapped into a $(p + r)$-dimensional manifold under all the group transformations.) In certain cases these, too, can be found explicitly by solving a reduced system of differential equations in fewer independent variables, but the intervening calculations are quite a bit more complicated than in the fully invariant case. The interested reader can refer to the above-mentioned works of Ovsiannikov for a full development of this theory, and the thesis of Ondich, [1], for more recent developments.

A second possible generalization was proposed by Bluman and Cole, [1], and Ames, [1; Vol. 2, § 2.10], and called the "nonclassical method" for group-invariant solutions. Here one requires not that the entire subvariety $\mathscr{S}_\Delta^*$ be $\mathrm{pr}^{(n)}$ G-invariant, but only that its intersection with the invariant space, $\mathscr{S}_\Delta^* \cap I_*^{(n)}$, be $\mathrm{pr}^{(n)}$ G-invariant. Although this method does lead to reduced equations, it is a little *too* general in that, once we admit the prolongations of the equations into the picture, *every* group of transformations on M satisfies this requirement and, conversely, *every* solution of the system can be obtained in this manner. See Olver and Rosenau, [1], [2]. A direct method for finding explicit solutions to partial differential equations introduced by Clarkson and Kruskal, [1], has proved very effective. Levi and Winternitz, [1], noted its connection with the nonclassical method, while Clarkson and Nucci, [1], showed that it is not quite as general as the nonclassical approach. Galaktionov, [1], gives a further promising generalization called "nonlinear separation." See Olver, [16], for a review of the available methods.

EXERCISES

3.1. Consider the axially symmetric wave equation $u_{tt} - u_{xx} - (1/x)u_x = 0$.
   (a) What is the symmetry group?
   (b) Find and classify the group-invariant solutions.
   (c) What is the fundamental solution to this equation?

3.2. The BBM equation $u_t + u_x + uu_x - u_{xxt} = 0$ arises as a model equation for the uni-directional propagation of long waves in shallow water.
   (a) What is the symmetry group of this equation?
   (b) Find group-invariant solutions corresponding to the various one-parameter subgroups found in part (a). (McLeod and Olver, [1].)

3.3. Determine the scale-invariant solutions to Boltzmann's problem $u_t = (uu_x)_x$, the solutions of which represent diffusion of some material in a medium, the rate of diffusion of which is proportional to the concentration of the material. What other types of group-invariant solutions exist? (Dresner, [1; §4.1].)

*3.4. Discuss the group-invariant solutions to the two-dimensional wave equation. Can you classify them?

3.5. Discuss the scale-invariant solutions to the two-dimensional Euler equations of ideal fluid flow. (The reduced equations are *not*, as far as I know, soluble in closed form!)

3.6. Assume that the fluid resistance of an object is determined by the density of the fluid ρ, the velocity of the object v, the object diameter d, and the compressibility $\rho^{-2}\,d\rho/dp$ of the fluid. Let $c^2 = dp/d\rho$ denote the sound speed. Prove that $D = \rho v^2 d^2 f(M)$, where $M = v/c$ is the Mach number of the fluid. ($M < 1$ corrresponds to subsonic motion, $M > 1$ to supersonic motion.) (Birkhoff, [2; p. 92].)

3.7. In 1947, G. I. Taylor determined the energy released from the first atomic explosion in New Mexico by applying a similarity analysis to the photographs of it. In an expanding spherical shock wave, the radius R will depend on time t, energy E released, and the ambient air density ρ_0 and pressure p_0. Assuming dimensional homogeneity, prove that

$$R = \left(\frac{t^2 E}{\rho_0}\right)^{1/5} h\left[p_0\left(\frac{t^6 E^2}{\rho_0^3}\right)^{1/5}\right]$$

for some function $h(\zeta)$. (For t small, the argument ζ of h is small, so we can approximate $R \simeq h_0 t^{2/5} E^{1/5} \rho_0^{-1/5}$, $h_0 = h(0)$; this was the relation used by Taylor.) (G. I. Taylor, [1], [2].)

*3.8. Find the orbits of the adjoint representation of the Euclidean groups E(2) and E(3). (See Exercise 1.29.)

*3.9. Prove that every subalgebra of the Korteweg–de Vries symmetry algebra (2.68) is uniquely equivalent to one subalgebra in the optimal system consisting of 0, the one-dimensional subalgebras (3.25), the subalgebras spanned by

$$\{\mathbf{v}_1, \mathbf{v}_4\}, \{\mathbf{v}_2, \mathbf{v}_4\}, \{\mathbf{v}_3, \mathbf{v}_4\}, \{\mathbf{v}_1, \mathbf{v}_3\}, \{\mathbf{v}_1, \mathbf{v}_2 + \mathbf{v}_3\}, \{\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_3\},$$

$$\{\mathbf{v}_1, \mathbf{v}_2\}, \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}, \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}, \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\},$$

and the full symmetry algebra itself.

3.10. Consider the differential equation $\Delta[u] = u_{xy} = 0$, on $M = X \times U \simeq \mathbb{R}^3$.
(a) Prove that the one-parameter group G of translations in the x-direction, $(x, y, u) \mapsto (x + \varepsilon, y, u)$ is a symmetry group.
(b) Show that the reduced equation Δ/G on M/G is vacuous, so any function on M/G determines a G-invariant solution to Δ.
(c) More generally, prove that if G is a regular symmetry group of a system of differential equations Δ, and its invariant space $I^{(n)}$ is a subset of the corresponding subvariety $\mathscr{S}_\Delta \subset M^{(n)}$, then *every* function on M/G gives rise to a G-invariant solution to Δ. How might the condition $I^{(n)} \subset \mathscr{S}_\Delta \subset M^{(n)}$ be checked in practice? (See also Exercise 3.18.)

3.11. (a) Suppose G is a symmetry group of the system Δ and $H \subset G$ is a normal subgroup acting regularly on $M \subset X \times U$. Prove that the reduced system Δ/H is invariant under the quotient group G/H acting on M/H.
(b) Suppose $p = 2$, $q = 1$ and $\Delta(x, u^{(n)}) = 0$ is a single n-th order partial differential equation. Prove that if Δ is invariant under an $(n + 1)$-parameter solvable Lie group, then all group-invariant solutions corresponding to a particular one-parameter subgroup can be found by quadrature.

3.12. Suppose $\mathbf{v}$ is a vector field on the smooth manifold M and $\dot{x} = \xi(x)$ the system of ordinary differential equations describing the flow of $\mathbf{v}$. Suppose G acts regularly on M, and is a symmetry group of this system. Prove that there is an induced vector field $\tilde{\mathbf{v}} = d\pi(\mathbf{v})$ on the quotient manifold M/G whose flow corresponds to that of $\mathbf{v}$ on M. Discuss how this result applies to Theorem 2.66.

3.13. Let G be a Lie group and $H \subset G$ a closed subgroup, which acts on G itself by right translation: $g \mapsto g \cdot h$, $h \in H$. Prove that the quotient space G/H is a smooth manifold. What is G/H in the case $G = \mathrm{SO}(3)$ and $H \simeq \mathrm{SO}(2)$, a one-parameter subgroup of rotations about a fixed axis?

3.14. (a) Consider the scaling group G: $(x, y) \mapsto (\lambda x, \lambda^{-1} y)$ acting regularly on $M = \mathbb{R}^2 \backslash \{0\}$. Prove that the quotient manifold is not Hausdorff and discuss its structure.
(b) Do the same problem for the scaling group of symmetries of the Korteweg–de Vries equation. (Olver, [2].)

3.15. Let $p = 2$, $q = 1$ and consider the one-parameter group G: $(x, y, u) \mapsto (x + \varepsilon, y + \varepsilon u, u)$. Prove that G acts transversally everywhere, but there are no nonconstant globally-defined G-invariant functions.

3.16. The p-Grassmann bundle of an m-dimensional manifold M, $m \geqslant p$, is defined so that over each point $x \in M$, $\mathrm{Grass}(p, M)|_x = \mathrm{Grass}(p, TM|_x)$ is the Grassmann manifold of p-planes in the tangent space $TM|_x$. (See Exercise 1.2.) Prove that this is the same as the first extended jet bundle: $\mathrm{Grass}(p, M) \simeq M_*^{(1)}$. (Olver, [2].)

3.17. Let G be a group of transformations acting on $M \subset X \times U$ with prolongation $\mathrm{pr}^{(n)}\, G$ acting on $M^{(n)}$.
(a) Prove that the dimension of the orbits of $\mathrm{pr}^{(n)}\, G$ is greater than or equal to the dimension of the orbits of G. Give an example where the strict inequality holds.
(b) Prove that if G is an r-parameter group, and G has r-dimensional orbits, then the same is true of $\mathrm{pr}^{(n)}\, G$.
(c) Prove that if G has r-dimensional orbits, then $I^{(n)} = (I^{(1)})^{(n-1)}$, where $(I^{(1)})^{(n-1)}$ denotes the $(n-1)$-st prolongation of the invariant space $I^{(1)} \subset M^{(1)}$ as determined by Definition 2.81. Interpret Corollary 2.54 in light of this result.

*3.18. *Explicit Characterization of the Invariant Space*
(a) Let $p = q = 1$, and let G be a regular one-parameter group of transformations acting on $M \subset X \times U$ with a single global invariant $\zeta(x, u)$. Prove that the invariant space $I^{(n)} \subset M^{(n)}$ is defined by the equations

$$I^{(n)} = \{(x, u^{(n)}): D_x^k \zeta = 0,\ k = 1, 2, \ldots, n\}.$$

(b) Let $q = 1$ but p be arbitrary. Let G be a regular one-parameter group of transformations acting on M with global invariants $\eta^1(x, u), \ldots, \eta^p(x, u)$. Prove that

$$I^{(1)} = \{(x, u^{(1)}): \mathsf{D}(\eta^1, \ldots, \eta^p)/\mathsf{D}(x^1, \ldots, x^p) = 0\},$$

where the defining equation stands for the $p \times p$ "total Jacobian determinant"

$$\frac{\mathsf{D}(\eta^1, \ldots, \eta^p)}{\mathsf{D}(x^1, \ldots, x^p)} = \det(D_i \eta^j).$$

What if G is an r-parameter group?
(c) Generalize part (b) to give an explicit characterization of the invariant space $I^{(n)}$ in general.
(d) How is this result related to Theorem 3.38?

3.19. Show that the vector field $\mathbf{v} = 2t\partial_x + \partial_t + 8t\partial_u$ is not a symmetry of the equation $u_{tt} = uu_{xx}$, but nevertheless one can use the method of this chapter to find solutions which are invariant under the one-parameter group generated by $\mathbf{v}$. Show that none of these arise among the standard group-invariant solutions. Explain. (See the following exercise.) (Olver and Rosenau, [2].)

*3.20. *The Nonclassical Method for Group-Invariant Solutions.* In Bluman and Cole, [1], the following method is proposed as a generalization of the reduction method for finding group-invariant solutions.
(a) Let Δ be an n-th order system of partial differential equations over M with corresponding subvariety $\mathscr{S}_\Delta^* \subset M_*^{(n)}$. Let G be a regular group of transformations acting on M with invariant space $I_*^{(n)} \subset M_*^{(n)}$. Prove that if the intersection $\mathscr{S}_\Delta^* \cap I_*^{(n)}$ is invariant under $\mathrm{pr}^{(n)}\, G$, then there is a reduced sys-

tem of differential equations Δ/G on the quotient manifold M/G such that all the solutions to Δ/G give rise to G-invariant solutions to Δ and conversely. (Note especially that $\mathscr{S}_\Delta^*$ itself does *not* have to be invariant under $\mathrm{pr}^{(n)}\, G$, so these groups are more general than symmetry groups as defined in Chapter 2.)

(b) Interpret Exercise 3.19 in light of this result.

(c) Let Δ be *any* system of differential equations and G *any* (regular) group of transformations. Prove that a suitable prolongation of $\mathscr{S}_\Delta^* \cap I_*^{(n)}$ (as per Definition 2.81) is always $\mathrm{pr}^{(n)}\, G$-invariant. Thus one can use *any* group to effect the reduction of part (a). (*Hint*: Use the prolongation formula (2.50) and the characterization of the invariant space in Theorem 3.38.)

(d) Conversely, show that if $u = f(x)$ is any solution to Δ, then there exists a group G leading to f by the reduction method of part (a).

(Olver and Rosenau, [1], [2].)

3.21. Let $\mathbf{v}$ be a vector field on $M \subset X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$, and let $Q = (Q_1, \ldots, Q_q)$ be the characteristic of $\mathbf{v}$ in the (x, u)-coordinates, cf. (2.48). Let $y = Y(x, u)$, $v = \Psi(x, u)$ be a change of coordinates on M, and let $\tilde{Q} = (\tilde{Q}_1, \ldots, \tilde{Q}_q)$ be the characteristic of $\mathbf{v}$ in the new (y, v)-coordinates on M. Prove that $\tilde{Q}$ is related to Q by the change of variables formula

$$\tilde{Q}_\beta = \sum_{\alpha=1}^{q} Q_\alpha \left(\frac{\partial \Psi^\beta}{\partial u^\alpha} - \sum_{j=1}^{p} \frac{\partial Y^j}{\partial u^\alpha} \frac{\partial v^\beta}{\partial y^j} \right), \qquad \beta = 1, \ldots, q.$$

*3.22. (a) Find an optimal system of all higher-dimensional subalgebras of the six-dimensional heat algebra (2.55).

(b) Show that, in contrast to the one-dimensional case in Example 3.13, the subalgebras in part (a) do *not* form an optimal system for the full infinite-dimensional heat symmetry algebra.

(Svinolupov and Sokolov, [1].)

CHAPTER 4

Symmetry Groups and Conservation Laws

In the study of systems of differential equations, the concept of a conservation law, which is a mathematical formulation of the familiar physical laws of conservation of energy, conservation of momentum and so on, plays an important role in the analysis of basic properties of the solutions. In 1918, Emmy Noether proved the remarkable result that for systems arising from a variational principle, every conservation law of the system comes from a corresponding symmetry property.† For example, invariance of a variational principle under a group of time translations implies the conservation of energy for the solutions of the associated Euler–Lagrange equations, and invariance under a group of spatial translations implies conservation of momentum. This basic principle constitutes the first fundamental result in the study of classical or quantum-mechanical systems with prescribed groups of symmetries. Noether's method is the principal systematic procedure for constructing conservation laws for complicated systems of partial differential equations.

For the applicability of Noether's theorem, one needs some form of variational structure in the system under consideration. The first section of this chapter gives a rudimentary introduction to the relevant aspects of the calculus of variations, of which the construction of the Euler–Lagrange equations characterizing the minimizers of a variational problem is the most important. Beyond this, not many of the results from the calculus of variations will be required, so the student interested in further studying this important field of mathematics would be well advised to consult any of the standard reference books on the subject. Not every symmetry group of a system of Euler–

† There is now an English translation of Noether's paper, [1], available. The reader is *strongly* urged to read this essential work.

Lagrange equations will give rise to a conservation law; one needs the group to satisfy an additional "variational" property of leaving the variational integral in a certain sense invariant. Section 4.2 develops the theory of variational symmetries, illustrated by a number of examples. In the case of a system of ordinary differential equations in variational form, the variational character of a symmetry group *doubles* the effectiveness of the reduction procedure presented in Section 2.5, so that a system of Euler–Lagrange equations which admits a one-parameter group of variational symmetries can be reduced in order by two.

The third section of this chapter is devoted to the systematic development of the theory of conservation laws of systems of differential equations. An important complication here is the existence of trivial conservation laws, which apply to any system of differential equations and in essence provide no new information on the behaviour of solutions to the particular system being considered. Pending some proofs to be given at the end of Chapter 5, we are able to completely characterize such trivial laws. Each nontrivial conservation law is, for normal systems of differential equations, uniquely characterized by a certain function, called its characteristic. Once we have the connection between conservation laws and their characteristics well in hand, the proof of the so-called "classical form" of Noether's theorem is immediate. In the second half of Section 4.4 we apply the constructions embodied in Noether's theorem to determine conservation laws for a number of systems of physical importance. Lack of space, however, precludes us from applying these conservation laws to the direct study of properties of solutions of the systems, which include global existence results, decay estimates, scattering theory, crack and dislocation problems, stability of solutions and so on; these can be found in the references discussed at the end of the chapter.

4.1. The Calculus of Variations

As usual, to keep things as simple as possible, we will work in Euclidean space, with $X = \mathbb{R}^p$, with coordinates $x = (x^1, \ldots, x^p)$ representing the independent variables, and $U = \mathbb{R}^q$, with coordinates $u = (u^1, \ldots, u^q)$ the dependent variables in our problem. (Extension of these local results to variational problems over smooth manifolds is not difficult; however, global results require the introduction of topological tools—see Anderson, [1] or Vinogradov, [1].). Let $\Omega \subset X$ be an open, connected subset with smooth boundary $\partial\Omega$. A *variational problem* consists of finding the extrema (maxima or minima) of a *functional*

$$\mathscr{L}[u] = \int_\Omega L(x, u^{(n)})\, dx$$

in some class of functions $u = f(x)$ defined over Ω. The integrand $L(x, u^{(n)})$, called the *Lagrangian* of the variational problem $\mathscr{L}$, is a smooth function of