Corollary 4.30. *Suppose $\mathscr{L}[u] = \int L(x, u^{(1)})\,dx$ is a first order variational problem, and* $\mathbf{v}$ *as in* (4.37) *is a variational symmetry. Then*

$$P_i = \sum_{\alpha=1}^{q} \phi_\alpha \frac{\partial L}{\partial u_i^\alpha} + \xi^i L - \sum_{\alpha=1}^{q} \sum_{j=1}^{p} \xi^j u_j^\alpha \frac{\partial L}{\partial u_i^\alpha}, \qquad i = 1, \dots, p, \tag{4.41}$$

form the components of a conservation law $\operatorname{Div} P = 0$ *for the Euler–Lagrange equations* $\mathsf{E}(L) = 0$.

Example 4.31. Consider a system of n particles moving in $\mathbb{R}^3$ subject to a potential force field. The kinetic energy of this system takes the form

$$K(\dot{\mathbf{x}}) = \tfrac{1}{2} \sum_{\alpha=1}^{n} m_\alpha |\dot{\mathbf{x}}^\alpha|^2,$$

where m_α is the mass and $\mathbf{x}^\alpha = (x^\alpha, y^\alpha, z^\alpha)$ the position of the α-th particle. The potential energy $U(t, x)$ will depend on the specific problem; for instance,

$$U(t, \mathbf{x}) = \sum \gamma_{\alpha\beta} |\mathbf{x}^\alpha - \mathbf{x}^\beta|^{-1}$$

might depend only on the pairwise gravitational interaction between masses, or (if $n = 1$) we may have the central gravitational force of Kepler's problem. Newton's equations of motion

$$m_\alpha \mathbf{x}_{tt}^\alpha = -\nabla_\alpha U \equiv -(U_{x^\alpha}, U_{y^\alpha}, U_{z^\alpha}), \qquad \alpha = 1, \dots, n,$$

are in variational form, being the Euler–Lagrange equations for the action integral $\int_{-\infty}^{\infty} (K - U)\,dt$.

A vector field

$$\mathbf{v} = \tau(t, \mathbf{x}) \frac{\partial}{\partial t} + \sum_\alpha \xi^\alpha(t, \mathbf{x}) \cdot \frac{\partial}{\partial \mathbf{x}^\alpha} \equiv \tau \frac{\partial}{\partial t} + \sum_\alpha \left(\xi^\alpha \frac{\partial}{\partial x^\alpha} + \eta^\alpha \frac{\partial}{\partial y^\alpha} + \zeta^\alpha \frac{\partial}{\partial z^\alpha} \right)$$

will generate a variational symmetry group if and only if

$$\operatorname{pr}^{(1)} \mathbf{v}(K - U) + (K - U) D_t \tau = 0 \tag{4.42}$$

for all $(t, \mathbf{x})$. Noether's theorem immediately provides a corresponding conservation law or first integral

$$T = \sum_{\alpha=1}^{n} m_\alpha \xi^\alpha \cdot \dot{\mathbf{x}}^\alpha - \tau E = \text{constant}, \tag{4.43}$$

where $E = K + U$ is the total energy of the system. In this example, we investigate what form the potential must take so that certain groups of direct physical interest be variational symmetries, and deduce the form of the corresponding conservation law.

First, the group of time translations has generator $\mathbf{v} = \partial_t$. Since $\operatorname{pr}^{(1)} \mathbf{v} = \mathbf{v}$, (4.42) holds if and only if $\partial U/\partial t = 0$, i.e. U does not depend explicitly on t. The resulting conservation law is just the energy E. Invariance of a physical system under time translations generally implies conservation of energy. Next consider the group of simultaneous translations of all the particles in a fixed direction $\mathbf{a} \in \mathbb{R}^3$. The group $\mathbf{x}^\alpha \mapsto \mathbf{x}^\alpha + \varepsilon\mathbf{a}$ has generator $\mathbf{v} = \sum_\alpha \mathbf{a} \cdot \partial/\partial\mathbf{x}^\alpha$. Again $\mathrm{pr}^{(1)}\,\mathbf{v} = \mathbf{v}$, so (4.42) holds if and only if $\mathbf{v}(U) = 0$ meaning that U is translationally invariant in the given direction. The corresponding first integral is the linear momentum

$$\sum_\alpha m_\alpha \mathbf{a} \cdot \dot{\mathbf{x}}^\alpha = \text{constant}.$$

Again, in most physical systems translational invariance implies conservation of linear momentum. As a last example, consider the group of simultaneous rotations of all the masses about some fixed axis which, for simplicity, we take as the z-axis. The generator of this group is

$$\mathbf{v} = \sum_\alpha \left(x^\alpha \frac{\partial}{\partial y^\alpha} - y^\alpha \frac{\partial}{\partial x^\alpha} \right), \qquad \mathrm{pr}^{(1)}\,\mathbf{v} = \mathbf{v} + \sum_\alpha \left(\dot{x}^\alpha \frac{\partial}{\partial \dot{y}^\alpha} - \dot{y}^\alpha \frac{\partial}{\partial \dot{x}^\alpha} \right).$$

Note that $\mathrm{pr}^{(1)}\,\mathbf{v}(K) = 0$, hence rotations form a variational symmetry group if and only if U is rotationally invariant: $\mathbf{v}(U) = 0$. The conservation law is that of angular momentum

$$\sum_\alpha m_\alpha (x^\alpha \dot{y}^\alpha - y^\alpha \dot{x}^\alpha) = \text{constant}.$$

Again, in general, rotational invariance implies conservation of angular momentum. For example, the n-body problem admits all seven symmetries and thus has conservation of energy, linear and angular momentum, while Kepler's problem only retains energy and angular momentum; the translational invariance no longer holds since one mass has been fixed at the origin.

Example 4.32. *Elastostatics.* In elasticity, conservation laws take on an added importance because they provide nontrivial path-independent integrals, thereby allowing one to investigate singularities such as cracks by integrating appropriate quantities far away from them. Let $x \in \Omega \subset \mathbb{R}^p$ represent the material coordinates of an elastic body in some reference configuration, and $u \in \mathbb{R}^q$ the spatial coordinates representing the deformation, so $u(x)$ is the deformed position of the initial point x. Thus, in physical applications, $p = q = 2$ or 3 for planar or three-dimensional elasticity. In the hyperelastic theory, assuming the absence of body forces, the equilibrium deformations are determined as minima of the energy functional

$$\mathscr{W}[u] = \int_\Omega W(x, u^{(1)})\, dx$$

subject to appropriate boundary conditions on $\partial\Omega$. In most cases W, the *stored energy function*, will depend on material coordinates, deformation, and deformation gradient $\nabla u = (\partial u^\alpha/\partial x^i)$, the last measuring the strain due to the deformation.† The precise form of the stored energy function will depend on

† There do, however, exist theories of "higher grade" materials, allowing dependence of W on higher order derivatives.

the constitutive assumptions governing the type of elastic material of which the body is composed. Nevertheless, certain universal physical constraints will impose certain general restrictions on the form of W. Each of these constraints will appear in the guise of a variational symmetry group of $\mathscr{W}$, and then Noether's theorem will immediately lead to corresponding conservation laws, valid for general elastic materials.

First of all, since W is independent of any external forces, it presumably does not depend on the frame of reference of the observer. This means that W must be invariant under the Euclidean group

$$\mathrm{E}(q)\colon \quad u \mapsto Ru + a, \qquad a \in \mathbb{R}^q, \quad R \in \mathrm{SO}(q),$$

in the spatial variables. Translational invariance implies $W = W(x, \nabla u)$ is independent of u; the corresponding conservation laws are just

$$\sum_{i=1}^{p} D_i(\partial W/\partial u_i^\alpha) = 0, \qquad \alpha = 1, \ldots, q,$$

which are nothing but the Euler–Lagrange equations themselves, expressed in divergence form. The rotational invariance of W:

$$W(x, R\nabla u) = W(x, \nabla u), \qquad R \in \mathrm{SO}(q),$$

leads to conservation laws

$$\sum_{i=1}^{p} D_i\left\{u^\alpha \frac{\partial W}{\partial u_i^\beta} - u^\beta \frac{\partial W}{\partial u_i^\alpha}\right\} = 0, \qquad \alpha, \beta = 1, \ldots, q,$$

whose characteristics are those of the infinitesimal rotations $u^\alpha\, \partial_{u^\beta} - u^\beta\, \partial_{u^\alpha}$.

Further conservation laws can result if we impose additional restrictions on the type of elastic material. For instance, if the body is homogeneous, $W = W(\nabla u)$ does not depend on x. Invariance under the translation group $x \mapsto x + a$, $a \in \mathbb{R}^p$, leads to p further conservation laws.

$$\sum_{i=1}^{p} D_i\left(\sum_{\alpha=1}^{q} u_j^\alpha \frac{\partial W}{\partial u_i^\alpha} - \delta_i^j W\right) = 0,$$

the components of which form Eshelby's celebrated *energy-momentum tensor*. When integrated around the tip of a crack it determines the associated energy-release rate. For a homogeneous, isotropic material, the symmetry group $x \mapsto Qx$, $Q \in \mathrm{SO}(p)$, which requires $W(\nabla u \cdot Q) = W(\nabla u)$, leads to $\frac{1}{2}p(p-1)$ further laws

$$\sum_{i=1}^{p} D_i\left[\sum_{\alpha=1}^{q} (x^j u_k^\alpha - x^k u_j^\alpha)\frac{\partial W}{\partial u_i^\alpha} + (\delta_i^j x^k - \delta_i^k x^j) W\right] = 0$$

corresponding to the infinitesimal generators $x^k\, \partial/\partial x^j - x^j\, \partial/\partial x^k$. Further interesting conservation laws can be found by imposing still more restrictions on the nature of the stored energy function W. Restricting to a homogeneous material, if $W(\nabla u)$ is an algebraically homogeneous function of degree n, so

$$W(\lambda \nabla u) = \lambda^n W(\nabla u), \qquad \lambda > 0,$$

for all ∇u, then the scaling group

$$(x, u) \mapsto (\lambda x, \lambda^{(n-p)/n} u), \qquad \lambda > 0,$$

is a variational symmetry group since

$$\int_{\tilde{\Omega}} W(\nabla \tilde{u})\, d\tilde{x} = \int_{\Omega} W(\lambda^{-p/n} \nabla u) \lambda^p \, dx = \int_{\Omega} W(\nabla u)\, dx.$$

(If we just scale x or u individually, we have a symmetry of the Euler–Lagrange equations, but *not* in general a variational symmetry.) The infinitesimal generator of this group is

$$\sum_{i=1}^{p} x^i \frac{\partial}{\partial x^i} + \frac{n-p}{n} \sum_{\alpha=1}^{q} u^\alpha \frac{\partial}{\partial u^\alpha},$$

so the conservation law is

$$\sum_{i=1}^{p} D_i \left\{ \frac{n-p}{n} \sum_{\alpha=1}^{q} u^\alpha \frac{\partial W}{\partial u_i^\alpha} + x^i W - \sum_{j=1}^{p} \sum_{\alpha=1}^{q} x^j u_j^\alpha \frac{\partial W}{\partial u_i^\alpha} \right\} = 0.$$

Now in practice, the algebraic homogeneity assumption on W is rather special. For a general function W, then, a slightly modified form of the above conservation law yields the divergence identity

$$\sum_{i=1}^{p} D_i \left\{ \sum_{\alpha=1}^{q} u^\alpha \frac{\partial W}{\partial u_i^\alpha} + x^i W - \sum_{j=1}^{p} \sum_{\alpha=1}^{q} x^j u_j^\alpha \frac{\partial W}{\partial u_i^\alpha} \right\} = pW,$$

which was used by Knops and Stuart, [1], to prove uniqueness of equilibrium solutions corresponding to homogeneous deformations. (See Exercise 5.35 for a general theorem of this type.) If W is algebraically homogeneous of degree p, then there is a full conformal group of variational symmetries. The infinitesimal generators of the inversional transformations take the form

$$\sum_{j=1}^{p} (x^i x^j - \tfrac{1}{2} \delta_j^i |x|^2) \frac{\partial}{\partial x^j}$$

with corresponding conservation laws

$$\sum_{i=1}^{p} D_i C_i^j \equiv \sum_{i=1}^{p} D_i \left\{ \sum_{k=1}^{p} (x^j x^k - \tfrac{1}{2} \delta_j^k |x|^2) \left(\sum_{\alpha=1}^{q} u_k^\alpha \frac{\partial W}{\partial u_i^\alpha} - \delta_i^k W \right) \right\} = 0.$$

Again, if W is not homogeneous, these turn into divergence identities:

$$\sum_{i=1}^{p} D_i C_i^j = x^j \left[pW - \sum_{\alpha, k} u_k^\alpha \frac{\partial W}{\partial u_k^\alpha} \right].$$

This method of using symmetries of special variational problems to construct useful divergence identities for more general functionals is quite promising. See Pucci and Serrin, [1], and van der Vorst, [1], for further developments and applications.

Divergence Symmetries

A cursory inspection of the proof of Noether's theorem reveals that the hypothesis that the vector field $\mathbf{v}$ generate a group of variational symmetries is overly restrictive for us to deduce the existence of a conservation law. This inspires the following relaxation of the definition of a variational symmetry group.

Definition 4.33. Let $\mathscr{L}[u] = \int L\,dx$ be a functional. A vector field $\mathbf{v}$ on $M \subset X \times U$ is an *infinitesimal divergence symmetry* of $\mathscr{L}$ if there exists a p-tuple $B(x, u^{(m)}) = (B_1, \ldots, B_p)$ of functions of x, u and derivatives of u such that

$$\operatorname{pr}^{(n)} \mathbf{v}(L) + L \operatorname{Div} \xi = \operatorname{Div} B \tag{4.44}$$

for all x, u in M.

Compare Theorem 4.12 for the motivation and notation for the "infinitesimal criterion" (4.44). In particular, if $B = 0$ we recover our previous notion of variational symmetry. Each infinitesimal divergence symmetry of a variational problem generates a one-parameter group $g_\varepsilon = \exp(\varepsilon \mathbf{v})$ of transformations on M, but the precise symmetry properties of such groups of *divergence symmetries* is less transparent than for the ordinary groups of variational symmetries. However, we do have the following generalization of Theorem 4.14.

Theorem 4.34. *If* $\mathbf{v}$ *is an infinitesimal divergence symmetry of a variational problem, then* $\mathbf{v}$ *generates a symmetry group of the associated Euler–Lagrange equations.*

The proof of this result is deferred until Section 5.3, when a generalization will be developed. In practice, then, to determine divergence symmetries of a given variational problem, one first computes the general symmetry group of the corresponding Euler–Lagrange equations. It is then a fairly straightforward matter to check which linear combination of these symmetries satisfies the additional criterion (4.44) so as to actually be a divergence symmetry. (See also Proposition 5.55.)

The statement of Noether's Theorem 4.29 remains the same if we replace variational symmetry by divergence symmetry in the hypothesis: the characteristic Q of the infinitesimal divergence symmetry remains the characteristic of a conservation law of the Euler–Lagrange equations. The only thing that changes in the proof is the incorporation of the extra term $\operatorname{Div} B$ stemming from (4.44) in the formulae so that, for instance, (4.40) is replaced by

$$Q \cdot E(L) + \operatorname{Div}(A + L\xi) = \operatorname{Div} B.$$

Thus the conclusion (4.38) holds, with $P = B - A - L\xi$ in this case.

Example 4.35. Let us look at the invariance of the Lagrangian $K - U$ for a system of n masses under Galilean boosts:

$$(t, \mathbf{x}^\alpha) \mapsto (t, \mathbf{x}^\alpha + \varepsilon t \mathbf{a}),$$

where $\mathbf{a} \in \mathbb{R}^3$. The infinitesimal generator of this action has prolongation

$$\mathrm{pr}^{(1)}\,\mathbf{v} = \sum_{\alpha=1}^{n} \left(t\mathbf{a} \cdot \frac{\partial}{\partial \mathbf{x}^\alpha} + \mathbf{a} \cdot \frac{\partial}{\partial \dot{\mathbf{x}}^\alpha} \right);$$

thus

$$\mathrm{pr}^{(1)}\,\mathbf{v}(L) = \sum_{\alpha=1}^{n} m_\alpha \mathbf{a} \cdot \dot{\mathbf{x}}^\alpha - t \sum_{\alpha=1}^{n} \mathbf{a} \cdot \nabla_\alpha U.$$

This never vanishes identically (unless $\mathbf{a} = 0$), so the Galilean boost is never an ordinary variational symmetry. However, the first term in $\mathrm{pr}^{(1)}\,\mathbf{v}(L)$ is a divergence, namely $D_t(\sum m_\alpha \mathbf{a} \cdot \mathbf{x}^\alpha)$, so $\mathbf{v}$ generates a group of divergence symmetries provided U is translationally invariant in the direction of $\mathbf{a}$. The associated first integral is

$$\sum_\alpha m_\alpha \mathbf{a} \cdot \mathbf{x}^\alpha - t \sum_\alpha m_\alpha \mathbf{a} \cdot \dot{\mathbf{x}}^\alpha.$$

The first summation when divided by the total mass $\sum m_\alpha$ determines the position of the centre of mass of the system in the direction $\mathbf{a}$, while the second is just the linear momentum in the same direction. We thus find that if U is translationally invariant in a given direction, not only is the linear momentum in that direction a constant, but the centre of mass in that direction is a linear function of t:

$$\text{Centre of Mass} = t(\text{Momentum})/(\text{Mass}) + c.$$

In particular, if U is invariant under the complete translation group in $\mathbb{R}^3$, the centre of mass of any such system moves linearly in a fixed direction.

Example 4.36. Return to the wave equation in two spatial dimensions considered in Examples 2.43 and 4.15. It has already been shown that, of the full group of symmetries of the wave equation, the translations, rotations and (modified) dilatations are symmetries of the associated variational problem. It is now seen that the inversions, while not variational symmetries in the strict sense, are divergence symmetries. In the case of $\mathbf{i}_x$, we have

$$\mathrm{pr}^{(1)}\,\mathbf{i}_x(L) + L \,\mathrm{Div}\, \xi = uu_x = D_x(\tfrac{1}{2}u^2).$$

There are thus ten conservation laws for the wave equation arising from geometrical symmetry groups—three from translations, three from rotations, one dilatational and, finally, three inversional conservation laws. In the following table, we just list the ten conserved densities, leaving the reader to determine the associated fluxes.

Symmetry	Characteristic	Conserved Density
Translations	u_x	$P_x = u_x u_t$
	u_y	$P_y = u_y u_t$
	u_t	$E = \frac{1}{2}(u_x^2 + u_y^2 + u_t^2)$
Rotations	$xu_y - yu_x$	$A = xP_y - yP_x$
	$xu_t + tu_x$	$M_x = xE + tP_x$
	$yu_t + tu_y$	$M_y = yE + tP_y$
Dilatations	$xu_x + yu_y + tu_t + \frac{1}{2}u$	$D = xP_x + yP_y + \frac{1}{2}uu_t + tE$
Inversions	$(x^2 - y^2 + t^2)u_x + 2xyu_y + 2xtu_t + xu$	$I_x = xD - yA + \frac{1}{2}xuu_t + tM_x$
	$2xyu_x + (y^2 - x^2 + t^2)u_y + 2ytu_t + yu$	$I_y = yD - xA + \frac{1}{2}yuu_t + tM_y$
	$2xtu_x + 2ytu_y + (x^2 + y^2 + t^2)u_t + tu$	$I_t = (x^2 + y^2)E - \frac{1}{2}u^2 + 2tD - t^2E$

Consequently, if $u(x, y, t)$ is any global solution to the wave equation decaying sufficiently rapidly as $x^2 + y^2 \to \infty$, then the spatial integrals of each of the above densities is a constant, independent of t. Thus we obtain conservation of energy

$$\mathscr{E} = \iint E \, dx \, dy = \text{constant}$$

and similar statements about linear momenta $\mathscr{P}_x$ and $\mathscr{P}_y$ (the integrals of P_x and P_y) and angular momentum $\mathscr{A}$. The hyperbolic rotations yield the linear dependence of the associated energy moments on t; for instance,

$$-\iint xE \, dx \, dy = \mathscr{P}_x t + \mathscr{C}$$

for some constant $\mathscr{C}$, where $\mathscr{P}_x$ is the constant linear momentum. The dilatational group leads to the useful identity

$$-\frac{d}{dt}\iint \tfrac{1}{2}u^2 \, dx \, dy = \iint (xP_x + yP_y) \, dx \, dy + \mathscr{E}t + \hat{\mathscr{C}},$$

for $\hat{\mathscr{C}}$ constant. The three inversional conservation laws, e.g.

$$\iint [(x^2 + y^2)E - \tfrac{1}{2}u^2) \, dx \, dy = \mathscr{E}t^2 + 2\hat{\mathscr{C}}t + \mathscr{C}^*,$$

while less physically motivated, are of key importance in the development of scattering theory for both linear and nonlinear wave equations.

Finally, there are the symmetry generators $\mathbf{v}_\alpha = \alpha(x, y. t)\partial_u$ stemming from the linearity of the equation. These satisfy

$$\mathrm{pr}^{(1)} \mathbf{v}_\alpha(L) = \alpha_t u_t - \alpha_x u_x - \alpha_y u_y = D_t(\alpha_t u) - D_x(\alpha_x u) - D_y(\alpha_y u),$$

since α is a solution to the wave equation. Thus, except in the special case of constant α, these are not variational symmetries in the sense of Definition

4.10; they do generate divergence symmetries. The corresponding conservation laws are the reciprocity relations

$$\begin{aligned} D_t(\alpha u_t - \alpha_t u) - D_x(\alpha u_x - \alpha_x u) - D_y(\alpha u_y - \alpha_y u) \\ = \alpha(u_{tt} - u_{xx} - u_{yy}) - u(\alpha_{tt} - \alpha_{xx} - \alpha_{yy}) = 0, \end{aligned}$$

vanishing whenever α and u both solve the wave equation. In integrated form this law is just Green's formula, as applied to the wave operator. (See Section 5.3 for a general discussion of reciprocity relations.)

Notes

The calculus of variations has its origins in the work of Euler and the Bernoullis in the eighteenth century, the operator bearing Euler's name first appearing in 1744. However, it was not until the work of Weierstrass and Hilbert in the latter half of the nineteenth century that some semblance of rigor appeared in the subject. The book by Gel'fand and Fomin, [1], gives a reasonable introduction to the calculus of variations, of which we are only using the most elementary ideas here. Conservation laws are of even older origin, although the idea of conservation of energy was not conceptualized until the work of Helmholtz in the 1840's. (See Elkana, [1], for an interesting study of the historical development of this idea.) See Whitham, [2; § 6.1], for a more detailed development of the conservation laws of fluid mechanics outlined in Example 4.22.

In this book I have not attempted to present any of the numerous applications of conservation laws to the study of differential equations, but have concentrated just on their systematic derivation using the symmetry group method of Noether. Lax, [2], uses conservation laws (called "entropy-flux pairs" in this context) to prove global existence theorems and determine realistic conditions for shock wave solutions to hyperbolic systems. This is further developed in DiPerna, [1], [2], where extra conservation laws are applied to the decay of shock waves and further existence theorems. Conservation laws have been applied to problems of stability by Benjamin, [1], and Holm, Marsden, Ratiu and Weinstein, [1]. Morawetz, [1] and Strauss, [1], use them in scattering theory. In elasticity, conservation laws (or, rather, their path-independent integral form—see Exercise 4.2) are of key importance in the study of cracks and dislocations; see the papers in Bilby, Miller and Willis, [1]. Knops and Stuart, [1], have used them to prove uniqueness theorems for elastic equilibria. The above is only a small sampling of all the applications which have appeared.

Trivial conservation laws were known for a long time by people in general relativity. Those of the second kind go under the name of "strong conservation laws" since they hold regardless of the underlying field equations; see the review papers of J. G. Fletcher, [1], and Goldberg, [1]. The characteristic form of a conservation law appears in Steudel, [1], but the connection between trivial characteristics and trivial conservation laws of Theorem 4.26 is due to Alonso, [1]. See Vinogradov, [5], and Olver, [11], for related results.

The concept of a variational symmetry, including the basic infinitesimal criterion (4.15), is due to Lie, [7], from his early theory of integral invariants. The first people to notice a connection between symmetries and conservation laws were Jacobi, [1], and later, Schütz, [1]. Engel, [1], developed the correspondence between the conservation of linear and angular momenta and linear motion of the centre of mass with invariance under translational, rotational and Galilean symmetries in the context of classical mechanics. Klein and Hilbert's investigations into Einstein's theory of general relativity inspired Noether to her remarkable paper, [1], in which both the concept of a variational symmetry group and the connection with conservation laws were set down in complete generality. The version of Noether's theorem appearing in this chapter is only a special case of her more general theorem, to be discussed in Section 5.3. The extension of Noether's methods to include divergence symmetries is due to Bessel-Hagen, [1].

Thus by 1922 all the machinery for a detailed, systematic investigation into the symmetry properties and consequent conservation laws of the important equations of mathematical physics was available. Strangely enough, this did not occur until quite recently. One possible explanation is that the constructive infinitesimal methods of Lie for computing symmetry groups were never quite reconciled with the theorem of Noether. In any event, the next significant reference to Noether's paper is in a review article by the physicist Hill, [1], in which the special case of Noether's theorem discussed in this chapter was presented, with implications that this was all Noether had actually proved on the subject. Unfortunately, the next twenty years saw a succession of innumerable papers either re-deriving the basic Noether Theorem 4.29 or purporting to generalize it, while in reality only reproving Noether's original result or special cases thereof. The mathematical physics literature to this day abounds with such papers, and it would be senseless to list them here. (I know of close to 50 such references, but I am certain many more exist!) Some references can be found in the book of Logan, [1], (which again only treats the special form of Noether's theorem for classical symmetry groups) and also other references mentioned below.

The lack of investigation into and appreciation of Noether's theorem has had some interesting consequences. Eshelby's energy-momentum tensor, which has much importance in the study of cracks and dislocations in elastic media, was originally found using *ad hoc* techniques, Eshelby, [1]. It was not related to symmetry properties of the media, as in Example 4.32, until the work of Günther, [1], and Knowles and Sternberg, [1]. An extension to the equations of linear elastodynamics was made by D. C. Fletcher, [1]. Subsequently, Olver, [8], [9], [14], found further undetected symmetries of the equations of linear elasticity, with consequent new conservation laws. Similarly, the important identities of Morawetz, [1], used in scattering theory for the wave equation were initially derived from scratch. Subsequently Strauss, [1], showed how these were related to the conformal invariance of the equation. (The further conservation laws to be found in Chapter 5 have yet to be applied here.) A similar development holds for the work of Baker and Tavel, [1], on conservation laws in optics, and no doubt further examples can be found.

The use of variational symmetry groups to reduce the order of ordinary differential equations which are the Euler–Lagrange equations of some variational problem presented in Theorem 4.17 is not as well known as its Hamiltonian counterpart, Theorem 6.35. A version of Theorem 4.17 for Lagrangians depending on only first order derivatives of the dependent variables is given in Whittaker, [1; p. 55], but I was unable to locate a reference to the full statement of this theorem in the literature.

Exercises

4.1. Let $\mathscr{L}$ be a functional. Prove that if $\mathbf{v}$ and $\mathbf{w}$ generate one-parameter variational symmetry groups of $\mathscr{L}$, then so does their Lie bracket $[\mathbf{v}, \mathbf{w}]$.

4.2. Suppose $p = 2$ and $D_x P + D_y Q = 0$ is a conservation law for a system of differential equations. Prove that if $u(x, y)$ is any solution to the system, the line integral

$$\int_C Q(x, y, u^{(m)})\, dx - P(x, y, u^{(m)})\, dy$$

does not depend on the path C. Generalize to $p > 2$.

4.3. If the case of a mechanical system, such as that in Example 4.31, time-translational invariance implies conservation of energy, space-translational invariance implies conservation of linear momentum (in the given direction) while, as in Example 4.35, Galilean invariance implies linear motion of the centre of mass. Prove that if a system admits laws of conservation of energy and the linear motion of the centre of mass, then it automatically admits the law of conservation of linear momentum as well. (Schütz, [1].)

4.4. The BBM equation $u_t + u_x + uu_x - u_{xxt} = 0$ can be put into variational form by letting $u = v_x$. Find three conservation laws of this equation using Noether's theorem. (Olver, [3].)

4.5. The equation $u_{tt} = u_{xxxx}$ describes the vibrations of a rod. Compute symmetries and conservation laws of this equation using Noether's theorem.

*4.6. Prove that Maxwell's equations, in both the physical form of Exercise 2.16(a) and the potential form of Exercise 2.16(b) are Euler–Lagrange equations. Find the variational principle in each case. Which of the symmetries of Exercise 2.16 lead to conservation laws and what are these laws? (Pohjanpelto, [1], [2].)

*4.7. Find a variational principle for Navier's equations (2.127) of linear elasticity. Discuss symmetries and the associated conservation laws, including triviality, in this instance. Do the same for the abnormal system (2.118). (Olver, [9].)