International Journal of Modern Physics A, Vol. 9, No. 8 (1994) 1261–1286

A GEOMETRICAL INTERPRETATION OF RENORMALIZATION GROUP FLOW*

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Received 17 May 1993

The renormalization group (RG) equation in D-dimensional Euclidean space, $\mathbf{R}^D$, is analyzed from a geometrical point of view. A general form of the RG equation is derived which is applicable to composite operators as well as tensor operators (on $\mathbf{R}^D$) which may depend on the Euclidean metric. It is argued that physical N-point amplitudes should be interpreted as rank N covariant tensors on the space of couplings, $\mathcal{G}$, and that the RG equation can be viewed as an equation for Lie transport on $\mathcal{G}$ with respect to the vector field generated by the β functions of the theory. In one sense it is nothing more than the definition of a Lie derivative. The source of the anomalous dimensions can be interpreted as being due to the change of the basis vectors on $\mathcal{G}$ under Lie transport. The RG equation acts as a bridge between Euclidean space and coupling constant space in that the effect on amplitudes of a diffeomorphism of $\mathbf{R}^D$ (that of dilations) is completely equivalent to a diffeomorphism of $\mathcal{G}$ generated by the β functions of the theory. A form of the RG equation for operators is also given. These ideas are developed in detail for the example of massive $\lambda\varphi^4$ theory in four dimensions.

1. Introduction

Geometry and physics seem inextricably linked in our attempts to understand the fundamental forces of nature, not only in Einstein's geometrical interpretation of the gravitational force but also in the geometrical approach to the gauge interactions of particle physics which has introduced concepts such as fiber bundles and connections into elementary particle theory, together with their associated structures of instantons and monopoles. Many aspects of modern physics have a geometrical interpretation and can be described in a co-ordinate-free manner. In conjunction with this emphasis on geometry is the importance of symmetry, which has played an ever-increasing role in Twentieth Century physics. Symmetries in particle physics can be classified into two types — internal and space symmetries. If the dynamics of a physical system are determined by an action which has a symmetry then one

PACS Numbers: 03.70.+k and 11.10.Gh

*Work carried out with an Alexander von Humboldt research stipend.

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can immediately deduce the existence of quantities that remain conserved under all possible circumstances. This concept, however, is laced with subtleties as it sometimes happens that a symmetry which is present in the classical action is violated by quantum processes and an anomaly ensues.

The renormalization group (RG) equation was first introduced into particle physics with the observation that quantum effects can violate an important space–time symmetry of classical electromagnetism, that of conformal invariance. This is not a symmetry of the Minkowski metric, it only preserves angles not lengths, but it is a symmetry of the classical action for electromagnetism, in the energy regime $E \gg 0.5$ MeV, where the electron mass can be ignored. It was observed in Refs. 1 and 2 that the electromagnetic charge picks up a scale dependence through quantum effects and that the derivative of the electromagnetic coupling with respect to the scale should be an analytic function of the coupling itself, the β function. More important than the couplings, however, are the physical amplitudes in a quantum field theory. Amplitudes depend not only on the couplings but also on a number of points in space (Minkowski or Euclidean) which must be labeled by some co-ordinates, x_i (usually Cartesians in flat space). It was pointed out by Callan and Symanzik[3] that the variation in the couplings under a change in the scale at which they are defined can always be compensated for by a change in the separation of the points, or alternatively a rescaling of the co-ordinates x_i, so that the vacuum amplitudes remain invariant. This results in an inhomogeneous partial differential equation for the amplitudes — the RG equation. This concept was further extended to a homogeneous equation by 't Hooft and Weinberg.[4] Meanwhile, the RG equation was raised to central importance in statistical physics through the work of Wilson.[5]

The idea that I wish to present in this paper is the following. A rescaling of points in Euclidean space, $\mathbf{R}^D$, is a diffeomorphism and is generated by the vector $\vec{\mathbf{D}} = x^\mu \partial_\mu$. If the space of couplings is also considered to be a finite-dimensional, differentiable manifold, $\mathcal{G}$, then the amplitudes can be interpreted as covariant tensors on $\mathcal{G}$ and one can interpret the rescaling of the couplings as a diffeomorphism of $\mathcal{G}$, and it is generated by a vector which is given by the β functions of the theory. The RG equation then expresses the fact that the change in amplitudes under the diffeomorphism of $\mathbf{R}^D$ generated by $\vec{\mathbf{D}}$, keeping the couplings fixed, is exactly the same as the change effected by a diffeomorphism of $\mathcal{G}$, generated by the vector $\vec{\beta}$, keeping the spatial points x_i fixed. This will be proven in Sec. 3. Alternatively, if the RG equation is written as a differential equation involving $\kappa \frac{\partial}{\partial \kappa}|_g$ rather than $x^\mu \frac{\partial}{\partial x^\mu}|_g$, using standard naive scaling arguments, it becomes nothing more than the *definition* of a Lie derivative on $\mathcal{G}$ with respect to the vector field $\vec{\beta}$. The terms in the RG equation involving anomalous dimensions are interpreted as coming from the change in the basis for covectors, dg^a, as we move along the RG trajectory.

Of course quantum field theory is famous for being plagued by "infinities" which, at least for renormalizable theories, can be "tamed" by a regularization procedure. This requires the introduction of "bare" couplings, $g_0^a(g, \varepsilon)$, which are analytic functions of the renormalized couplings, g^a, and a regularization parameter or parameters, ε, e.g. for a cutoff, Λ, $\varepsilon = \kappa/\Lambda$, where κ is a renormalization point and for dimensional regularization $\varepsilon = 4 - D$, where D is the dimension of space or space–time. It will further be argued in Sec. 3 that g_0^a and g^a can be thought of as different co-ordinate systems on $\mathcal{G}$. The matrix $\frac{\partial g_0^a}{\partial g^b}$ tells us how to transform tensors (amplitudes) between co-ordinate systems. g_0^a enter on a different footing from g^a, however, in that they depend on the regularization parameter whereas g^a do not. The "infinities" of quantum field theory are then viewed as being due to the fact that the co-ordinate transformation between g_0^a and g^a is singular when the regularization parameter is removed. This is not a disaster; singularities in co-ordinate transformations are common in differential geometry. For example the event horizon of a black hole in the Schwarzchild metric was for a long time viewed as being "singular" because, in polar co-ordinates, the metric appears to be degenerate there. Then a co-ordinate system was discovered, Kruskal co-ordinates, in which the metric is perfectly regular at the event horizon. The transformation between polar co-ordinates and Kruskal co-ordinates in the Schwarzchild metric is singular at the event horizon but the metric, and physics, is perfectly regular there. The apparent singularity in polars is just due to a bad choice of co-ordinates. Of course polar co-ordinates are fine for describing space outside of the event horizon. The singularity at the origin, however, is a real physical singularity of space–time, indicating that some new physics must enter here. One must be very careful to distinguish between genuine singularities and singularities that are merely due to the choice of co-ordinates.

The basic idea presented here, that the RG equation should be viewed in terms of a co-ordinate transformation on the space of couplings, was inspired by O'Connor and Stephens.[6]

The main results will be described in Sec. 3, where it is shown that the RG equation for vacuum amplitudes can be interpreted as a Lie derivative on the space of couplings and an expression for the RG equation for operators is also presented. As a preliminary, Sec. 2 gives a discussion of the role of the stress tensor in the RG equation and a version of the RG equation is derived by demanding invariance of functional integrals under a specific change in integration variable, associated with dilations of Euclidean space. Section 4 illustrates the ideas with a detailed analysis of massive $\lambda\varphi^4$ with a constant source, where the space $\mathcal{G}$ is three-dimensional. Finally Sec. 5 gives a summary and outlook.

2. Conformal Transformations and the Renormalization Group Equation

This section is based on the introductory part of Ref. 7. Consider a field theory in flat D-dimensional Euclidean space. In principle there are an infinite number of operators that can be constructed out of the fields, each of which introduces a coupling, but the criterion of renormalizability requires that only a finite number of these couplings be independent.[5] This means that, within the *a priori* infinite-dimensional space of coupling constants, the theory can be formulated on a finite n-dimensional subspace which will be denoted by $\mathcal{G}$. It will be assumed that $\mathcal{G}$ is, at least locally, a differentiable manifold and we will denote the renormalized couplings (co-ordinates on $\mathcal{G}$) by g^a, $a = 1, \dots, n$. Bare quantities will be represented by a subscript o. Thus the bare couplings are denoted by g_0^a. It will be assumed that a regularization procedure is imposed which renders bare quantities very large but still finite. This is because bare quantities appear in some of the following formulae and we do not want to be manipulating infinite quantities. Ultimately, of course, all bare quantities disappear from physical amplitudes. The couplings g^a will be taken to be real and massless. If the theory contains any couplings which are massive these can always be made massless by multiplying by appropriate powers of the renormalization mass scale, κ. Questions about the global structure of $\mathcal{G}$ will not be addressed here.

The physics is given by the renormalized N-point vacuum amplitudes (or correlation functions in statistical physics)

$$A^{(N)}(x_1, \dots, x_N) = \langle \hat{A}_1(x_1) \cdots \hat{A}_N(x_N) \rangle \,, \tag{1}$$

where x_i, $i = 1, \dots, N$, are positions in $\mathbf{R}^D$ with $x_i \neq x_j$, $\forall i \neq j$ and $\hat{A}_i$ are physical operators. In many of the following expressions we shall freely interchange between operator and functional integral formalism. The former will be represented as $\hat{A}_i$ whereas the corresponding quantity appearing in functional integrals, which can be thought of as classical functions of the fields, will be denoted by A_i without a circumflex. The amplitudes (1) can be expressed in terms of the couplings g^a, but their form depends on the choice of couplings and the renormalization scheme chosen and does not necessarily bear any simple relation to the g^a. For example in scalar $\lambda\varphi^4$ theory in four space dimensions the relation between the coupling, $\lambda(\kappa)$, at the renormalization point, κ, and the four-point amplitude, in momentum space at the symmetric point $p_\mu p_\nu = \frac{\kappa^2}{3}(4\delta_{\mu\nu} - 1)$, depends on the renormalization scheme. Using momentum subtraction they are the same (because they are defined to be, for $p^2 = \kappa^2$ but not otherwise), but using dimensional regularization and minimal subtraction they are not the same. Even at $p^2 = \kappa^2$ the four-point amplitude $\langle \hat{\varphi}(p_1)\hat{\varphi}(p_2)\hat{\varphi}(p_3)\hat{\varphi}(p_4)\rangle$ is a complicated, though analytic, function of $\lambda(\kappa)$ which is only approximately equal to λ for small λ. From this it is clear that the specific choice of co-ordinates on $\mathcal{G}$ is not physically important (though, in practice, a clever choice makes calculations much simpler).

The theory can be probed by distorting it slightly to see how it responds. Let $\vec{X}$ be an arbitrary vector field on $\mathbf{R}^D$ (generating a diffeomorphism of space). For general $\vec{X}$ the amplitudes will change under this diffeomorphism, and the response is described by the insertion of the stress operator, $\hat{T}_{\mu\nu}(x)$, into the amplitudes (in Lorentzian signature this would, of course, be called the energy–momentum tensor). Let us see how this comes about. Express the amplitudes in the form of functional integrals over the bare fields of the theory, which will be denoted by $\varphi_0(x)$ but need not be scalars:

$$A^{(N)}(x_1,\dots,x_N)=\int\rho[\varphi_0]A_1(x_1)\cdots A_N(x_N)e^{-S_0[g_0,\varphi_0,\partial\varphi_0,\gamma]}/Z_0[g_0,\gamma]\,. \tag{2}$$

Here $\gamma_{\mu\nu}$ is the Euclidean metric on $\mathbf{R}^D$ and $A_i(x_i)$ depends on the renormalized fields $\varphi(x_i)$ and possibly also on $\gamma_{\mu\nu}$, for example if $A_i(x_i)$ involves contractions of $\partial_\mu\varphi$ or if φ is a vector field and $A_i(x_i)$ are composite operators. $\rho[\varphi_0]$ is the measure. The notation $\rho[\varphi_0]$ is used purely for convenience, to simplify subsequent formulae, and is not intended to imply that the measure is anything other than the usual one more conventionally written as $[d\varphi_0]$ or $\mathcal{D}\varphi_0$. $Z_0[g_0,\gamma]$ is the partition function,

$$Z_0[g,\varepsilon,\gamma]=Z_0[g_0,\gamma]=\int\rho[\varphi_0]e^{-S_0[g_0,\varphi_0,\partial\varphi_0,\gamma]}\,, \tag{3}$$

and the action is the Hamiltonian density integrated over $\mathbf{R}^D$,

$$S_0[g,\varepsilon,\varphi,\partial\varphi,\gamma]=S_0[g_0,\varphi_0,\partial\varphi_0,\gamma]=\int d^D\tilde{y}H_0(g_0,\varphi_0(y),\,\partial\varphi_0(y),\,\gamma(y))\,, \tag{4}$$

where $d^D\tilde{y}=\sqrt{\gamma}d^Dy$ is the measure in D-dimensional Euclidean space. The subscript 0 here is to emphasize that it is the *bare* action that appears exponentiated in functional integrals. When doing perturbation theory this is split into a renormalized action plus counterterms and is then considered to be a function of renormalized fields and couplings but this split has no physical significance, just as the renormalized couplings have no intrinsic physical meaning. The action can then be thought as a function of the renormalized couplings, as indicated in Eq. (4), and the regularization parameter(s) — the latter dependence being purely through the counterterms. The bare subscript on the partition function (3) could be removed by using a measure defined in terms of the renormalized fields rather than the bare fields,

$$Z[g,\varepsilon,\gamma]=\int\rho[\varphi]e^{-S_0[g,\varepsilon,\varphi,\partial\varphi,\gamma]}\,, \tag{5}$$

thus introducing a factor which cancels in all amplitudes.

Now consider the response of the amplitudes to a deformation generated by a vector field, $\vec{X}$, on $\mathbf{R}^D$:

$$\mathcal{L}_{\vec{X}}A^{(N)}(x_1,\dots,x_N)=\sum_{i=1}^{N}\int\rho[\varphi_0]A_1(x_1)\cdots\{\mathcal{L}_{\vec{X}}A_i(x_i)\}\cdots A_N(x_N)e^{-S_0}/Z_0\,, \tag{6}$$

where $\mathcal{L}_{\vec{X}}$ represents Lie differentiation with respect to $\vec{X}$. The partition function itself is unchanged by this deformation. This is because $\mathcal{L}_{\vec{X}}S_0$ vanishes since the action is integrated over $\mathbf{R}^D$ and the fields are assumed to vanish at infinity.

We shall use two identities to re-express the right hand side of Eq. (6). Firstly

$$\mathcal{L}_{\vec{X}} A_i = \frac{\delta A_i}{\delta\varphi}\mathcal{L}_{\vec{X}}\varphi + \frac{\delta A_i}{\delta(\partial_\mu\varphi)}\mathcal{L}_{\vec{X}}(\partial_\mu\varphi) + \frac{\delta A_i}{\delta\gamma^{\mu\nu}}(\mathcal{L}_{\vec{X}}\gamma)^{\mu\nu}\,, \tag{7}$$

which is just the Leibniz rule for differentiation allowing for the possibility that $A_i(x_i)$ might depend on $\partial_\mu\varphi(x_i)$ as well as $\varphi(x_i)$ and may have explicit metric dependence. The Lie derivative of the inverse metric is given by the usual formula,

$$(\mathcal{L}_{\vec{X}}\gamma)^{\mu\nu} = X^\tau\partial_\tau\gamma^{\mu\nu} - \gamma^{\mu\tau}\partial_\tau X^\nu - \gamma^{\tau\nu}\partial_\tau X^\mu\,. \tag{8}$$

In Cartesian co-ordinates, with $\gamma_{\mu\nu} = \delta_{\mu\nu}$, this reduces to the familiar deformation

$$(\mathcal{L}_{\vec{X}}\gamma)^{\mu\nu} = -\partial^\mu X^\nu - \partial^\nu X^\mu\,. \tag{9}$$

The second identity follows from the fact that functional integrals should be invariant under reparameterizations of the integration variables, φ_0, so that

$$\begin{aligned}\int \rho[\varphi_0]\int d^D\tilde{y}A_1(x_1)\cdots A_N(x_N)\bigg\{&\frac{\delta S_0}{\delta\varphi_0(y)}\delta\varphi_0(y) + \frac{\delta S_0}{\delta(\partial_\mu\varphi_0(y))}\delta(\partial_\mu\varphi_0(y))\\ &-\rho^{-1}\frac{\delta\rho}{\delta\varphi_0(y)}\delta\varphi_0(y)\bigg\}e^{-S_0} = \sum_{i=1}^N\int\rho[\varphi_0]\int d^D\tilde{y}A_1(x_1)\cdots\\ &\times\left\{\frac{\delta A_i(x_i)}{\delta\varphi_0(y)}\delta\varphi_0(y) + \frac{\delta A_i(x_i)}{\delta(\partial_\mu\varphi_0(y))}\delta(\partial_\mu\varphi_0(y))\right\}\cdots A_N(x_N)e^{-S_0}\,,\end{aligned} \tag{10}$$

where we have allowed for the possibility that the measure might not be invariant. Clearly the bare field φ_0 in the curly brackets on the right hand side of this equation can be replaced by the renormalized field φ since the renormalization coefficients just cancel. Applying this identity to the variation $\delta\varphi_0 = \varepsilon\mathcal{L}_{\vec{X}}\varphi_0$, with ε independent of position, and using the Leibniz rule in the form of Eq. (7) to replace all functional variations with respect to fields with Lie derivatives and metric variations gives

$$\begin{aligned}&\int\rho[\varphi_0]\int d^D\tilde{y}A_1(x_1)\cdots A_N(x_N)\left\{\mathcal{L}_{\vec{X}}H_0 - \left(\frac{\delta H_0}{\delta\gamma^{\mu\nu}} - \frac{1}{2}M_{\mu\nu}\right)(\mathcal{L}_{\vec{X}}\gamma)^{\mu\nu}\right\}\bigg|_y e^{-S_0}\\ &\quad= \sum_{i=1}^N\int\rho[\varphi_0]A_1(x_1)\cdots\left\{\mathcal{L}_{\vec{X}}A_i - \frac{\delta A_i}{\delta\gamma^{\mu\nu}}(\mathcal{L}_{\vec{X}}\gamma)^{\mu\nu}\right\}\bigg|_{x_i}\cdots A_N(x_N)e^{-S_0}\,,\end{aligned} \tag{11}$$

where we have defined

$$\frac{1}{2}M_{\mu\nu}(\mathcal{L}_{\vec{X}}\gamma)^{\mu\nu} = -\rho^{-1}\frac{\delta\rho}{\delta\varphi_0}\mathcal{L}_{\vec{X}}\varphi_0(y)\,.$$

Since $\delta\varphi_0 = \varepsilon \mathcal{L}_{\vec{X}}\varphi_0$ is linear in φ_0 and $\frac{1}{2}(M_{\mu\nu}\mathcal{L}_{\vec{X}}\gamma)^{\mu\nu}$ is just the Jacobian of this transformation, one can expect that $M_{\mu\nu}$ is independent of φ_0, i.e. $\hat{M}_{\mu\nu}$ is a multiple of the identity.

One may worry here about what happens when φ is a spinor field because it is difficult to find a consistent definition of the Lie derivative of a spinor field for an arbitrary diffeomorphism, $\vec{X}$. Howver in this section we will only really be interested in the case where $\vec{X}$ is a conformal Killing vector for the Euclidean metric, $\gamma_{\mu\nu}$, and the Lie derivative of a spinor with respect to a conformal Killing vector can be well defined.[a]

The classical stress tensor (density) is defined as

$$T_{\mu\nu} = 2\frac{\delta H_0}{\delta\gamma^{\mu\nu}} \tag{12}$$

and this is just the second term on the left hand side of Eq. (11). The reason why there is no bare subscript on $T_{\mu\nu}$ will be explained shortly. The first term in (11) can be dropped because it is a total derivative. To see this note that since H_0 is a D-form (a density) we have

$$\mathcal{L}_{\vec{X}}H_0 = i_{\vec{X}}dH_0 + di_{\vec{X}}H_0 = di_{\vec{X}}H_0 \tag{13}$$

and hence is a total derivative. Now dividing (11) by $Z_0[g,\varepsilon,\gamma]$ in order to get genuine amplitudes gives

$$\begin{aligned}\mathcal{L}_{\vec{X}}A^{(N)}(x_1,\dots,x_N) = -\frac{1}{2}\int d^D\tilde{y}(\mathcal{L}_{\vec{X}}\gamma)^{\mu\nu}(y)&\\ \times\langle(\hat{T}_{\mu\nu}(y) - \hat{M}_{\mu\nu}(y))\hat{A}_1(x_1)\cdots\hat{A}_N(x_N)\rangle&\\ + \sum_{i=1}^{N}\left\langle\hat{A}_1(x_1)\cdots\left[\frac{\delta\hat{A}_i}{\delta\gamma^{\mu\nu}}(\mathcal{L}_{\vec{X}}\gamma)^{\mu\nu}\right]\Bigg|_{x_i}\cdots\hat{A}_N(x_N)\right\rangle&.\end{aligned} \tag{14}$$

When $N = 0$ this gives

$$\int d^D\tilde{y}(\mathcal{L}_{\vec{X}}\gamma)^{\mu\nu}(y)(\langle\hat{T}_{\mu\nu}(y)\rangle - M_{\mu\nu}(y)) = 0\,, \tag{15}$$

i.e. up to total derivatives we have that $M_{\mu\nu}$ is just the expectation value of $\langle\hat{T}_{\mu\nu}\rangle$. An integration by parts can be performed in this equation and using Eq. (9) with an arbitrary diffeomorphism $\vec{X}$ we deduce that, in Cartesians,

$$\langle\partial_\mu\hat{T}'^{\mu}{}_{\nu}(y)\rangle = 0\,, \tag{16}$$

[a] I am grateful to Norbert Dragon for an illuminating discussion on this point.

where we have defined $\hat{T}'_{\mu\nu} = \hat{T}_{\mu\nu} - \hat{M}_{\mu\nu}$. Then (14) can be written

$$\mathcal{L}_{\vec{X}} A^{(N)}(x_1, \ldots, x_N) = -\frac{1}{2} \int d^D \tilde{y} (\mathcal{L}_{\vec{X}} \gamma)^{\mu\nu}(y) \langle \hat{T}'_{\mu\nu}(y) \hat{A}_1(x_1) \cdots \hat{A}_N(x_n) \rangle$$
$$+ \sum_{i=1}^{N} \left\langle \hat{A}_1(x_1) \cdots \left[\frac{\delta \hat{A}_i}{\delta \gamma^{\mu\nu}} (\mathcal{L}_{\vec{X}} \gamma)^{\mu\nu} \right] \Bigg|_{x_i} \cdots \hat{A}_N(x_N) \right\rangle . \quad (17)$$

Note that if the $\hat{A}_i$ are composite operators, e.g. $\hat{A}_i = \partial_\mu \hat{\varphi} \partial | \nu \hat{\varphi} \gamma^{\mu\nu}$, the terms $\frac{\delta \hat{A}_i}{\delta \gamma^{\mu\nu}} |_{x_i}$ can give a nonzero contribution to Eq. (17).

Equation (17) can be interpreted as a quantum-mechanical statement of translational invariance and is often called the Ward identity associated with this symmetry. It is one half of the RG equation and will be used extensively later on. For the moment a short digression will be made on the properties of the quantum stress operator, $\hat{T}'_{\mu\nu}$.

Classically translation invariance of the Euclidean metric ensures, via Noether's theorem, the existence of D conserved currents, when the equations of motion are used. In Cartesian co-ordinates,

$$\partial_\mu T^{C\mu}{}_\nu = 0 , \qquad \nu = 1, \ldots , D . \quad (18)$$

Quantum-mechanically, we must take the existence of a stress operator obeying Eq. (17) as an *assumption*.[8] The stress operator, $\hat{T}'_{\mu\nu}$, appearing in (17) must be the regularized stress operator. It satisfies the quantum version of (18),

$$\partial_\mu \hat{T}'^{\mu}{}_\nu = 0 \quad (19)$$

(modulo the equations of motion). It is defined (in flat space) so that $\langle \hat{T}'_{\mu\nu} \rangle = 0$.

In quantum field theory conserved currents do not get renormalized. This means that for any global symmetry of the action, which does not acquire an anomaly upon quantization, the bare conserved current $\hat{J}^\mu_0$ is equal to the renormalized current $\hat{J}^\mu$. In other words the renormalization constant Z_J, which is defined by

$$\hat{J}^\mu_0 = Z_J \hat{J}^\mu , \quad (20)$$

is unity (rotational invariance demands that Z_J be the same for every component of the vector J^μ). To see this in the case of translational invariance consider Eq. (17), which is the mathematical expression of this symmetry. Assuming translational invariance to hold at both the bare and the renormalized level, this equation must be true for both bare and renormalized operators. Consider Eq. (17) for bare operators, $\hat{A}_{0i}$, which are independent of the metric (the inclusion of metric dependence does not affect the argument)

$$\sum_{i=1}^{n}\langle\hat{A}_{01}(x_1)\cdots(\mathcal{L}_{\vec{X}}\hat{A}_{0i}(x_i))\cdots\hat{A}_{0N}(x_N)\rangle$$

$$= -\frac{1}{2}\int d^D\tilde{y}(\mathcal{L}_{\vec{X}}\gamma)^{\mu\nu}(y)\langle\hat{T}'_{0\mu\nu}(y)\hat{A}_{01}(x_1)\cdots\hat{A}_{0N}(x_n)\rangle \tag{21}$$

$$\Leftrightarrow \left(\prod_{i=1}^{N} Z_{A_i}\right)\sum_{i=1}^{n}\langle\hat{A}_1(x_1)\cdots(\mathcal{L}_{\vec{X}}A_i(x_i))\cdots\hat{A}_N(x_N)\rangle$$

$$= -\frac{1}{2}\left(\prod_{i=1}^{N} Z_{A_i}\right)Z_T\int d^D\tilde{y}(\mathcal{L}_{\vec{X}}\gamma)^{\mu\nu}(y)\langle\hat{T}'_{\mu\nu}(y)\hat{A}_1(x_1)\cdots\hat{A}_N(x_n)\rangle\,, \tag{22}$$

where Z_T is the renormalization constant for the stress tensor, $\hat{T}_{0\mu\nu} = Z_T\hat{T}_{\mu\nu}$, and Z_{A_i} are the renormalization constants for the operators $\hat{A}_i$, $\hat{A}_{0i} = Z_{A_i}\hat{A}_i$ (for simplicity it is assumed that there is no operator mixing between the $\hat{A}_i$'s). Putting (17) for the renormalized operators into Eq. (22) gives the nonrenormalization theorem $Z_T = 1$. Thus we have, as a consequence of the assumption that translational invariance holds at both the bare and renormalized level,

$$\hat{T}_{0\mu\nu} = \hat{T}_{\mu\nu} \tag{23}$$

(modulo the equations of motion). This is why there is no bare subscript on the left hand side of Eq. (12).

Equation (23) is rather subtle in practice for (at least) two reasons. Firstly, for example in scalar $\lambda\varphi^4$ theory in four dimensions, the "naive" stress tensor is not a finite operator and one must construct an "improved" stress tensor.[9] Equation (23) can be demonstrated explicitly for scalar fields; see Sec. 4. For gauge theories the question of the existence of a finite stress operator is addressed in Refs. 10 and 11, where it is argued that no improvement terms, other than those already necessary for scalar fields, are required. This is connected with the classical conformal invariance of gauge field theories. A second subtlety associated with (23) is that it assumes translational invariance at the bare level,

$$\partial_\mu\hat{T}_0^{\mu}{}_{\nu} = 0\,, \tag{24}$$

i.e. bare quantities are translationally invariant. This means that whatever regularization procedure is adopted it must respect translational invariance.[9] In particular if we use (23) then we cannot use a cutoff! Fortunately there exist regularization procedures that preserve translational invariance, e.g. Pauli–Villars or dimensional regularization. Nevertheless this is a rather unsatisfactory aspect of the analysis presented here. The philosophy of this paper is to try to develop a description of the RG in a co-ordinate independent manner, i.e. in a way that is independent of the regularization scheme chosen; yet here we find ourselves immediately restricting our choice of schemes. Of course this is only an intermediate step and in the end all physical amplitudes are independent of bare quantities. However, the results of Sec. 3 do not rely on the stress tensor as such, so this is not an important restriction.

Now let us return to the Ward identity (17). The real power of this equation lies in choosing particular diffeomorphisms, $\vec{X}$, which are related to the symmetries of the Euclidean metric γ. In particular, we can obtain very powerful results by taking $\vec{X}$ to be the generators of the conformal group, SO(1, $D+1$), in D Euclidean dimensions. Thus, for isometries (translations or rotations) of γ, the right hand side of (17) vanishes, and this equation tells us that all physical amplitudes are translationally and rotationally invariant.

The response of the system to changes in scale is given by taking $\vec{X}$ to be the vector which generates dilations. In Cartesians this is

$$\vec{X} = \vec{D} = x^\mu \partial_\mu \qquad \text{with} \qquad (\mathcal{L}_{\vec{D}}\gamma)^{\mu\nu} = -2\gamma^{\mu\nu} \,. \tag{25}$$

In this case Eq. (17) reduces to

$$\begin{aligned} \mathcal{L}_{\vec{D}} A^{(N)}(x_1, \ldots, x_N) = \int d^D \tilde{y} \langle \hat{T}'(y) \hat{A}_1(x_1) \cdots \hat{A}_N(x_n) \rangle \\ - \sum_{i=1}^{N} \langle \hat{A}_1(x_1) \cdots \{\Delta_\gamma \hat{A}_i(x_i)\} \cdots \hat{A}_N(x_n) \rangle \,, \end{aligned} \tag{26}$$

where $\hat{T}' = \hat{T}'^\mu{}_\mu$ is the trace of the stress operator and $\Delta_\gamma \hat{A}_i = 2\gamma^{\mu\nu}(\frac{\delta \hat{A}_i}{\delta \gamma^{\mu\nu}})$.

Now we shall eliminate $\hat{T}'$ to obtain the RG equation. Following Ref. 7 we consider the operators

$$\hat{\Phi}_a(x) = \frac{\partial \hat{H}_0(x)}{\partial g^a} \,, \tag{27}$$

where $\hat{H}_0(x)$ is the bare Hamiltonian density, to be a basis for all relevant or marginal operators of the theory, i.e. any relevant or marginal operator, which is a scalar (or more precisely a density) in $\mathbf{R}^D$, can be written as a linear combination of $\hat{\Phi}_a(x)$. There is one other operator which must be included in order to have a complete basis and this is an operator proportional to the equations of motion of the theory since this linear combination is obtained not by varying a coupling in the action but by varying the fields. Let

$$\begin{aligned} E_0(x) &= \varphi_0(x) \left\{ \frac{\delta S_0}{\delta \varphi_0(x)} - \frac{\partial}{\partial x^\mu} \left(\frac{\delta S_0}{\delta(\partial_\mu \varphi_0(x))} \right) \right\} \\ &= \varphi(x) \left\{ \frac{\delta S_0}{\delta \varphi(x)} - \frac{\partial}{\partial x^\mu} \left(\frac{\delta S_0}{\delta(\partial_\mu \varphi(x))} \right) \right\} ; \end{aligned} \tag{28}$$

then the operator $\hat{E}_0(x)$ has canonical mass dimension D. When inserted into amplitudes $A^{(N)}(x_1,\ldots,x_N)$ it gives

$$\langle \hat{E}_0'(x)\hat{A}_1(x_1)\cdots\hat{A}_N(x_N)\rangle = \sum_{i=1}^{N}\langle \hat{A}_1(x_1)\cdots\{\Delta_{\varphi(x)}\hat{A}_i(x_i)\}\cdots\hat{A}_N(x_N)\rangle\,, \tag{29}$$

where we have defined

$$\Delta_{\varphi(x)}A_i(x_i) = \varphi(x)\left\{\frac{\delta A_i(x_i)}{\delta(\varphi(x))} - \partial_\mu\left(\frac{\delta A_i(x_i)}{\delta(\partial_\mu\varphi(x))}\right)\right\}. \tag{30}$$

Here $\hat{E}_0'(x) = \hat{E}_0(x) - \langle\hat{E}_0(x)\rangle$, where $\langle\hat{E}_0(x)\rangle$ can be obtained by using (10) with $N=0$:

$$\langle\hat{E}_0(x)\rangle = \left\langle \rho^{-1}\frac{\delta\rho}{\delta\varphi_0(x)}\varphi_0(x)\right\rangle. \tag{31}$$

For general N Eq. (29) is just Eq. (10) with $\delta\varphi_0(y) = \varepsilon\delta^{(N)}(x-y)\varphi_0(x)$, ε a constant, and integrated by parts using Cartesian co-ordinates.

The right hand side of (29) vanishes unless $x = x_i$ for some i. In particular if $\hat{A}_i(x_i) = \hat{\varphi}(x_i)$ then

$$\langle\hat{E}_0'(x)\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_N)\rangle = \sum_{i=1}^{N}\delta^{(N)}(x-x_i)\langle\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_N)\rangle\,. \tag{32}$$

In the ensuing analysis Eq. (29) will only ever appear integrated over x, so it will be convenient to define

$$\Delta_\varphi A_i(x_i) = \int d^D\tilde{x}\,\Delta_{\varphi(x)}A_i(x_i) = \int d^D\tilde{x}\,\varphi(x)\left\{\frac{\delta A_i(x_i)}{\delta(\varphi(x))} - \partial_\mu\left(\frac{\delta A_i(x_i)}{\delta(\partial_\mu\varphi(x))}\right)\right\}. \tag{33}$$

Note that applying the same arguments to (29) as were used on the Ward identity (17) to derive the nonrenormalization theorem (23) shows that $\hat{E}_0(x) = \hat{E}(x)$, i.e. the "equation of motion operator" does not get renormalized.[12] This is analogous to a Ward identity in that the equations of motion themselves constitute a linear relation between operators which must hold at both the bare and the renormalized level and thus prevent $\hat{E}(x)$ from getting renormalized. However this linear relation does not represent a symmetry of the action and it would therefore be an abuse of language to call it a Ward identity [unless one wants to promote reparameterization invariance of functional integrals, Eq. (10), to a symmetry of the theory].

To incorporate tensor (or spinor) operators would require a larger basis than $\hat{\Phi}_a$ but this will not be necessary here. A consequence of the definition of $\hat{\Phi}_a(x)$ is

$$\partial_a\hat{\Phi}_b = \partial_b\hat{\Phi}_a\,. \tag{34}$$

The set of operators $\hat{\Phi}_a$ can be thought of as forming an operator-valued one-form, $\hat{\Phi} = \hat{\Phi}_a dg^a = d\hat{H}_0$, in the cotangent space, $T^*(\mathcal{G})$, of the space of couplings. Using (34) gives $d\hat{\Phi} = 0$, where d represents the exterior derivative on $T^*(\mathcal{G})$, i.e. the one-form $\hat{\Phi}$ is closed because it is, at least locally, exact. This is a direct consequence of the definition (27) and can be thought of as representing the fact that $\hat{\Phi}_a$ are an operational analog of a co-ordinate basis for real-valued one-forms, e.g. for dx^μ, $d^2x^\mu = 0$. Of course dg^a constitute a co-ordinate basis for ordinary real-valued one-forms on $T^*(\mathcal{G})$ as usual.

Now consider what happens to the amplitude (1) under a variation of the renormalized couplings. In general the renormalized operators $\hat{A}_i$, as well as the action itself, will depend on the couplings, so

$$\partial_b\langle \hat{A}_1(x_1)\cdots\hat{A}_N(x_N)\rangle = \sum_{i=1}^{N}\langle \hat{A}_1(x_1)\cdots\{\partial_b\hat{A}_i(x_i)\}\cdots\hat{A}_N(x_N)\rangle - \int d^D\tilde{y}\langle\hat{\Phi}'_b(y)\hat{A}_1(x_1)\cdots\hat{A}_N(x_N)\rangle\,, \tag{35}$$

where $\hat{\Phi}'_b = \hat{\Phi}_b - \langle\hat{\Phi}_b\rangle$. The term involving $\langle\hat{\Phi}_b\rangle$ appears here because the variation of the couplings in the normalization factor $Z_0[g_0,\varepsilon,\gamma]^{-1}$, which is necessary for the definition of physical amplitudes, introduces factors of $\langle\frac{\partial\hat{H}_0(y)}{\partial g^a}\rangle$ into the amplitudes on the right hand side.

Now the trace of the stress operator, $\hat{T}$, can be expanded as a linear combination $\hat{T}(z) = \beta^b(g)\hat{\Phi}_b(z) - e(g)\hat{E}(z)$ for some real functions, $\beta^b(g)$ and $e(g)$, of g^a. It will be convenient to define $\hat{\Theta}(z) = \beta^b(g)\hat{\Phi}_b(z)$ so that $\hat{T}(z) = \hat{\Theta}(z) - e(g)\hat{E}(z)$. Contracting (35) with the functions β^b yields B:

$$\beta^b\partial_b\langle \hat{A}_1(x_1)\cdots\hat{A}_N(x_N)\rangle = \sum_{i=1}^{N}\langle \hat{A}_1(x_1)\cdots\{(\beta^b\partial_b - e\Delta_\varphi)\hat{A}_i(x_i)\}\cdots\hat{A}_N(x_N)\rangle - \int d^D\tilde{y}\langle\hat{T}'(y)\hat{A}_1(x_1)\cdots\hat{A}_N(x_n)\rangle\,, \tag{36}$$

where Eq. (29) has been used. We now have two equations both involving the expectation value of $\hat{T}'$ with the $\hat{A}_i$, (26) and (36). Combining these equations to eliminate $\hat{T}'$ results in the following version of the RG equation:

$$\mathcal{L}_{\vec{D}}\langle \hat{A}_1(x_1)\cdots\hat{A}_N(x_N)\rangle + \sum_{i=1}^{N}\langle \hat{A}_1(x_1)\cdots\{\Delta_\gamma\hat{A}_i(x_i)\}\cdots\hat{A}_N(x_N)\rangle = \langle(\beta^b\partial_b - e\Delta_\varphi)\{\hat{A}_1(x_1)\cdots\hat{A}_N(x_N)\}\rangle - \beta^b\partial_b\langle\hat{A}_1(x_1)\cdots\hat{A}_N(x_N)\rangle\,. \tag{37}$$

This can be related to the more familiar form of the RG equation in the following manner. Consider a scalar field and let $\hat{A}_i(x_i) = \hat{\varphi}^{p_i}(x_i)$, with p_i a positive integer, be monomials of the fundamental fields, so that $\Delta_\varphi \hat{A}_i(x_i) = p_i \hat{A}_i(x_i)$ and $\Delta_\gamma \hat{A}_i(x_i) = 0$. Now, following Ref. 7, identify $e(g)$ with the canonical mass dimension, $d_{\hat{\varphi}}$, of the operator $\hat{\varphi}$ and $\beta^b \partial_b \hat{A}_i(x_i)$ with minus the anomalous dimension of $\hat{A}_i$ so that

$$\beta^b \partial_b \hat{A}_i(x_i) = -\gamma_{\hat{A}_i} \hat{A}_i(x_i) \tag{38}$$

($\gamma_{\hat{A}}$ may be a matrix if there is operator mixing). Finally note that, for scalar operators using Cartesian co-ordinates, $\mathcal{L}_{\vec{D}} \hat{A}_i(x_i) = x_i^\mu \frac{\partial}{\partial x_i^\mu} \hat{A}_i(x_i)$. Then the RG equation emerges in the standard homogeneous form

$$\begin{aligned}\sum_{i=1}^{N} \left(x_i^\mu \frac{\partial}{\partial x_i^\mu} + d_{\hat{A}_i} + \gamma_{\hat{A}_i} \right) &\langle \hat{A}_1(x_1) \cdots \hat{A}_N(x_N) \rangle \\ &+ \beta^b \partial_b \langle \hat{A}_1(x_1) \cdots \hat{A}_N(x_N) \rangle = 0\,,\end{aligned} \tag{39}$$

where $d_{\hat{A}_i} = p_i d_{\hat{\varphi}}$ is the canonical dimension of $\hat{A}_i = \hat{\varphi}^{p_i}$. β^a are thus seen to be nothing other than the usual β functions of the theory, for the couplings g^a:

$$\beta^a = \kappa \frac{dg^a}{d\kappa}\,. \tag{40}$$

Since the couplings, g^a, are defined to be massless, these β functions include the canonical dimensions for the couplings. For example, if the theory involves a mass, m^2, then the corresponding massless coupling is $\tilde{m}^2 = m^2 \kappa^{-2}$ and $\beta^{\tilde{m}^2} = (-2 + \delta)\tilde{m}^2$, where δ is the usual anomalous mass dimension, $\kappa \frac{dm^2}{d\kappa} = \delta m^2$. Note that Eq. (39) is the version of the RG equation for connected amplitudes (or Green's functions $G^{(N)}$), not for proper vertices (one particle irreducible functions, $\Gamma^{(N)}$) which would require the opposite sign for $d_{\hat{A}_i} + \gamma_{\hat{A}_i}$.

Returning to the general form of the RG equation (37), identifying $e(g)$ with the canonical dimension of the field, $d_{\hat{\varphi}}$, finally gives

$$\begin{aligned}&\mathcal{L}_{\vec{D}} \langle \hat{A}_1(x_1) \cdots \hat{A}_N(x_N) \rangle + \sum_{i=1}^{N} \langle \hat{A}_1(x_1) \cdots \{\Delta_\gamma \hat{A}_i(x_i)\} \cdots \hat{A}_N(x_N) \rangle \\ &\quad = \langle (\beta^b \partial_b - d_{\hat{\varphi}} \Delta_\varphi) \{\hat{A}_1(x_1) \cdots \hat{A}_N(x_N)\} \rangle - \beta^b \partial_b \langle \hat{A}_1(x_1) \cdots \hat{A}_N(x_N) \rangle\,.\end{aligned} \tag{41}$$

with $\Delta_\gamma A_i$ defined in Eq. (26) and $\Delta_\varphi A_i$ in (33). This is the main result of this section. It is a slight generalization of the usual RG equation for amplitudes. It includes the possibility of $\hat{A}_i$ being Euclidean tensors or even spinors, since the Lie derivative with respect to the dilation generator, $\mathcal{L}_{\vec{D}}$, automatically tells us how to handle tensor (and spinor) indices, as well as allowing for the possibility that $\hat{A}_i$ might be composite operators involving the Euclidean metric $\gamma_{\mu\nu}$. Clearly this equation is applicable to theories with more than one field, φ; one merely sums $d_{\hat{\varphi}} \Delta_\varphi$ over the fields.

3. The Renormalization Group Equation as a Lie Derivative

In this section, it will be demonstrated that the RG equation (41) can be interpreted as an equation for Lie derivatives of amplitudes. The Lie derivative of an amplitude with respect to the dilation generator, $\vec{D}$, on Euclidean space is exactly the same as the Lie derivative with respect to the vector $\vec{\beta} = \beta^a \partial_a$ on the space of couplings. We shall first give a simple proof and then relate the result to the analysis of the previous section.

Since $\hat{\Phi}_a$ are a basis for rotationally invariant operators of dimension D it suffices to consider amplitudes of these operators. All other amplitudes for Euclidean scalars (more precisely densities) which are relevant or marginal operators can be obtained from linear combinations of these basic operators. Consider, therefore, amplitudes of the form

$$\Phi^{(N)}_{a_1\cdots a_N}(x_1, \dots , x_N) = \langle \hat{\Phi}_{a_1}(x_1) \cdots \hat{\Phi}_{a_N}(x_N) \rangle \,. \tag{42}$$

The usual derivation of the RG equation relies on the simple fact that all physical amplitudes should be independent of the renormalization point κ. However $\Phi^{(N)}_{a_1\cdots a_N}(x_1, \dots , x_N)$ cannot be independent of κ in general. If we chose a different parametrization of the renormalized couplings (a different co-ordinate system on $\mathcal{G}$), which will be denoted by primes $g^{a'}(g)$, then $\Phi^{(N)}_{a_1\cdots a_N}(x_1, \dots , x_N)$ transforms as a rank N covariant tensor,

$$\Phi^{(N)}_{a_1\cdots a_N}(x_1, \dots , x_N) = \left(\frac{\partial g^{a'_1}}{\partial g^{a_1}}\right) \cdots \left(\frac{\partial g^{a'_N}}{\partial g^{a_N}}\right) \hat{\Phi}^{(N)}_{a'_1\cdots a'_N}(x_1, \dots , x_N) \,, \tag{43}$$

and these cannot both be independent of κ, since $\frac{\partial g^{a'_i}}{\partial g^{a_i}}$ is not, in general. The object that ought to be independent of κ is the reparametrization-invariant amplitude

$$\langle \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_N) \rangle = \langle \hat{\Phi}_{a_1}(x_1) \cdots \hat{\Phi}_{a_N}(x_N) \rangle dg^{a_1} \cdots dg^{a_N} \,. \tag{44}$$

The correct statement that amplitudes are independent of the renormalization point is

$$\kappa \frac{d}{d\kappa} \langle \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_N) \rangle = 0 \,. \tag{45}$$

Allowing for the fact that $\langle \hat{\Phi}_{a_1}(x_1) \cdots \hat{\Phi}_{a_N}(x_N) \rangle$ are also functions of the g^a gives

$$\begin{aligned} -\kappa \frac{\partial}{\partial \kappa}\bigg|_g \langle \hat{\Phi}_{a_1}(x_1) \cdots \hat{\Phi}_{a_N}(x_N) \rangle &= \beta^b \partial_b \langle \hat{\Phi}_{a_1}(x_1) \cdots \hat{\Phi}_{a_N}(x_N) \rangle \\ &\quad + \sum_{i=1}^{N} (\partial_{a_i} \beta^b) \langle \hat{\Phi}_{a_1}(x_1) \cdots \hat{\Phi}(x_i)_b \cdots \hat{\Phi}_{a_N}(x_N) \rangle \,, \end{aligned} \tag{46}$$

where we have used

$$\kappa \frac{d}{d\kappa}(dg^a) = d\left(\kappa \frac{dg^a}{d\kappa}\right) = d\beta^a = \frac{\partial \beta^a}{\partial g^b} dg^b \,. \tag{47}$$

In co-ordinate free notation this is

$$-\kappa\frac{\partial}{\partial\kappa}\bigg|_g \langle\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_N)\rangle = \mathcal{L}_{\vec{\beta}}\langle\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_N)\rangle\,, \tag{48}$$

where $\mathcal{L}_{\vec{\beta}}$ represents Lie differentiation on $\mathcal{G}$ with respect to the vector field $\vec{\beta}$. Thus the RG equation is nothing other than the *definition* of a Lie derivative provided that it is appreciated that the amplitudes are tensors on $T^*(\mathcal{G})$, and not scalars [the minus sign in (48) is standard in the definition of a Lie derivative; see e.g. Ref. 13]. This analysis makes it clear that the anomalous dimensions have the geometrical interpretation of arising from the change in dg^a [which are a basis for real valued one-forms on $T^*(\mathcal{G})$] as we move along the vector field $\vec{\beta}$.

Equation (48) can be expressed in terms of the dilation vector on Euclidean space, $\vec{D}$, using the usual scaling arguments. To see this first note that $\hat{\Phi}_a$ are densities therefore in Cartesian co-ordinates

$$\mathcal{L}_{\vec{D}}\hat{\Phi}_{a_i}(x_i) = \frac{\partial}{\partial x_i^\mu}(x_i^\mu\hat{\Phi}_{a_i}(x_i)) = \left(x_i^\mu\frac{\partial}{\partial x_i^\mu} + D\right)\hat{\Phi}_{a_i}(x_i)\,. \tag{49}$$

Now by definition, Eq. (27), $\hat{\Phi}_a(x)$ has mass dimension D thus the usual naive scaling arguments give

$$\begin{aligned}\kappa\frac{\partial}{\partial\kappa}\bigg|_g \langle\hat{\Phi}_{a_1}(x_1)\cdots\hat{\Phi}_{a_N}(x_N)\rangle &= \sum_{i=1}^{N}\left(x_i^\mu\frac{\partial}{\partial x_i^\mu} + D\right)\langle\hat{\Phi}(x_1)_{a_1}\cdots\hat{\Phi}_{a_N}(x_N)\rangle \\ &= \mathcal{L}_{\vec{D}}\langle\hat{\Phi}_{a_i}(x_i)\cdots\hat{\Phi}_{a_N}(x_N)\rangle\,;\end{aligned} \tag{50}$$

hence

$$\mathcal{L}_{\vec{D}}\langle\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_N)\rangle = -\mathcal{L}_{\vec{\beta}}\langle\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_N)\rangle\,. \tag{51}$$

This is the main result of this paper. The RG equation is seen as a bridge connecting a particular diffeomorphism (dilations generated by the conformal Killing vector, $\vec{D}$) of Euclidean space, $\mathbf{R}^D$, with a diffeomorphism (generated by the associated vector $\vec{\beta}$) of the space of couplings $\mathcal{G}$. [I apologize for the minus sign in Eq. (51). This is due to the usual field theory definition of β^a as being derivatives of the couplings with respect to the renormalization mass κ. Had they been defined as derivatives with respect to a length then this equation would have had a plus sign on the right hand side.]

This can be related to the analysis of the previous section and Eq. (41) in the following manner. Let $\hat{A}_i(x_i) = \hat{\Phi}_{a_i}(x_i)$ in (41). Now consider the meaning of $\Delta_\gamma\hat{\Phi}_a$. Using the definition of $\hat{\Phi}_a$ in Eq. (27) and assuming that metric variations commute with variations of the couplings leads to

$$\Delta_\gamma\hat{\Phi}_a = 2\gamma^{\mu\nu}\frac{\delta}{\delta\gamma^{\mu\nu}}\left(\frac{\partial\hat{H}_0}{\partial g^a}\right) = \frac{\partial}{\partial g^a}\left(2\gamma^{\mu\nu}\frac{\delta\hat{H}_0}{\delta\gamma^{\mu\nu}}\right) = \partial_a\hat{T}\,. \tag{52}$$

Note that (34) implies that

$$\partial_a \hat{T} = (\partial_a \beta^b)\hat{\Phi}_b + \beta^b \partial_b \hat{\Phi}_a - d_{\hat{\varphi}} \partial_a \hat{E} = \partial_a \hat{\Theta} - d_{\hat{\varphi}} \partial_a \hat{E} \,, \tag{53}$$

where the previous identification of $e(g)$ with $d_{\hat{\varphi}}$ has been made. Now assuming that field variations commute with coupling variations gives

$$\Delta_\varphi \hat{\Phi}_a = \Delta_\varphi \partial_a \hat{H}_0 = \partial_a \Delta_\varphi \hat{H}_0 = \partial_a \hat{E} \,, \tag{54}$$

where (29) and (33) have been used. Putting all this together (41) becomes

$$\begin{aligned} \pounds_{\tilde{D}} \langle \hat{\Phi}_{a_1}(x_1) \cdots \hat{\Phi}_{a_N}(x_N) \rangle = & \sum_{i=1}^{N} \langle \hat{\Phi}_{a_1}(x_1) \cdots \{\beta^b \partial_b \hat{\Phi}_{a_i}(x_i)\} \cdots \hat{\Phi}_{a_N}(x_N) \rangle \\ & - \beta^b \partial_b \langle \hat{\Phi}_{a_1}(x_1) \cdots \hat{\Phi}_{a_N}(x_N) \rangle \\ & - \sum_{i=1}^{N} \langle \hat{\Phi}_{a_1}(x_1) \cdots \{\partial_{a_i} \hat{\Theta}(x_i)\} \cdots \hat{\Phi}_{a_N}(x_N) \rangle \\ = & -\beta^b \partial_b \langle \hat{\Phi}_{a_1}(x_1) \cdots \hat{\Phi}_{a_N}(x_N) \rangle \\ & - \sum_{i=1}^{N} (\partial_{a_i} \beta^b) \langle \hat{\Phi}_{a_1}(x_1) \cdots \hat{\Phi}_b(x_i) \cdots \hat{\Phi}_{a_N}(x_N) \rangle \,, \end{aligned} \tag{55}$$

which is just Eq. (51). From this analysis it is clear that the source of the anomalous dimensions in the RG equation is the term involving $\partial_{a_i} \beta^b$ in this equation, which comes from Lie transport of the basis dg^a (this term actually contains both the canonical dimensions and the anomalous dimensions because the couplings are defined to be dimensionless).

Another way of expressing this idea is to define the matrix of renormalization coefficients, $Z_a{}^b$, by

$$\hat{\Phi}_{0a} = Z_a{}^b(g) \hat{\Phi}_b \,, \tag{56}$$

where $\hat{\Phi}_{0a}$ constitute the bare basis

$$\hat{\Phi}_{0a} = \frac{\partial \hat{H}_0}{\partial g_0^a} \,. \tag{57}$$

The matrix $Z_a{}^b$ depends on the renormalization scheme, of course. For example in dimensional regularization it is a function not only of the renormalized couplings, g^a, but also of the dimension $D = 4 - \varepsilon$ and can be expanded as a series of poles in ε. Clearly the definition of the renormalized basis (27) also implies that

$$\left[Z_a{}^b(g) = \frac{\partial g^b}{\partial g_0^a} \qquad \Leftrightarrow \qquad dg_0^a Z_a{}^b(g) = dg^b \right] , \tag{58}$$

so we can write

$$\hat{\Phi}(x) = \hat{\Phi}_{0a}(x)dg_0^a = \hat{\Phi}_a(x)dg^a\,. \tag{59}$$

We now demand that the bare operators be independent of the renormalization point. There is a slight subtlety here, though, in that the bare couplings are defined to be massless. This means that they change, under changes in κ, by their *canonical* dimensions, for example for a mass the massless bare coupling is $\tilde{m}_0^2 = m_0^2\kappa^{-2}$ and $\kappa\frac{dm_0^2}{d\kappa} = 0$ requires $\kappa\frac{d\tilde{m}_0^2}{d\kappa} = -2\tilde{m}_0^2$. We shall denote the canonical dimension of the coupling g^a by d_a; thus $\kappa\frac{dg_0^a}{d\kappa} = -d_a g_0^a$, where there is no sum over a. As in the previous argument for amplitudes, it is the operator-valued one-form $\hat{\Phi}_0 = \hat{\Phi}_{0a}dg_0^a$ which is independent of κ. Thus

$$\kappa\frac{d\hat{\Phi}_{0a}}{d\kappa} - d_a\hat{\Phi}_{0a} = 0 \qquad \Leftrightarrow \qquad \kappa\frac{d\hat{\Phi}_a}{d\kappa} + \Gamma_a{}^b\hat{\Phi}_b = 0 \tag{60}$$

or

$$\left(\kappa\frac{\partial}{d\kappa}\bigg|_g + \beta^b\partial_b\right)\hat{\Phi}_a + \Gamma_a{}^b\hat{\Phi}_b = 0\,, \tag{61}$$

where the matrix, $\Gamma_a{}^b$, is defined by

$$\Gamma_a{}^b = \frac{\partial\beta^b}{\partial g^a}\,. \tag{62}$$

This should be distinguished from the matrix of anomalous dimensions,

$$\gamma^a{}_b = (Z^{-1})^a{}_c\left(\frac{dZ^c{}_b}{d\kappa}\right). \tag{63}$$

Equation (60) is a RG equation for the operators $\hat{\Phi}_a$. One must be careful in evaluating amplitudes involving this equation, however, since neither $\kappa\frac{\partial}{d\kappa}|_g$ nor $\beta^b\partial_b$ can be pulled outside of expectation values separately, though the combination can be since S_0 is independent of κ.

It is crucial to this interpretation that massless couplings are used. In particular this means that

$$\vec{\beta} = \beta^a\frac{\partial}{\partial g^a} = \beta_0^a\frac{\partial}{\partial g_0^a}\,, \tag{64}$$

where $\beta_0^a = \kappa\frac{dg_0^a}{d\kappa} = -d_a$ are just the canonical dimensions. Had massive couplings been used, then β_0^a would be zero since the bare massive couplings are defined to be independent of κ, but β^a are nonzero in general. One can hardly transform from a nonzero vector to one which vanishes using a co-ordinate transformation! (This has nothing to do with the "singularities" of the renormalization program; the regulator is still in place and so the co-ordinate transformation is still nonsingular.) The important quantity is the total dimension, canonical plus anomalous, and to split it up spoils general co-ordinate invariance. Stated differently, the redefinition

$m^2 \to \tilde{m}^2 = m^2\kappa^{-2}$ is *not* a co-ordinate transformation because κ is not a co-ordinate.

We end this section with some comments on the matrix of dimensions (canonical plus anomalous), $\Gamma^b{}_a = \frac{\partial \beta^b}{\partial g^a}$, which appears in (55) and (60). This matrix plays a very important role near a critical point in statistical mechanics — its eigenvalues determine the relevant and irrelevant directions in the space of interactions. A linear combination of operators corresponding to a positive eigenvalue (negative mass dimension of a coupling) tends to blow up for large κ and decrease for small κ; such an operator is termed irrelevant and corresponds to a nonrenormalizable interaction in field theory. A negative or zero eigenvalue (relevant or marginal operator) corresponds to a renormalizable interaction in field theory. Thus for a renormalizable field theory near a fixed point, the matrix of dimensions had better have only nonpositive eigenvalues, but this restriction is not necessary away from the fixed point. This is related to the triviality problem for $\lambda\varphi^4$ in four dimensions. In $D = 4 - \varepsilon$ dimensions the canonical dimension of the coupling, λ, is ε and in perturbation theory β starts off at order λ^2, $\beta^\lambda = -\varepsilon\lambda + \frac{3}{16\pi^2}\lambda^2 + \cdots$ and $\partial_\lambda \beta^\lambda = -\varepsilon + \frac{3}{8\pi^2}\lambda + \cdots$. Provided $\varepsilon > 0$ this is negative for small λ but when $\varepsilon = 0$ it is positive. This does not of course mean that the theory is nonrenormalizable in four dimensions because at the fixed point $\lambda = \lambda_* = 0$, $(\partial_\lambda\beta^\lambda)_* = 0$, i.e. λ is a marginal coupling at the trivial fixed point in four dimensions.

In a theory in which Γ has complex eigenvalues one would have operators with complex anomalous dimensions which would presumably correspond to a non-unitary theory. It is shown in Ref. 14 that this circumstance is automatically avoided if the β functions are derivable from a potential function.

4. An Example — $\lambda\varphi^4$ Theory

The ideas presented in the previous sections will now be exemplified in a concrete model — that of massive $\lambda\varphi^4$ in $D = 4 - \varepsilon$ Euclidean dimensions with a constant source. The renormalization procedure adopted will be that of dimensional regularization and minimal substraction. This section is a generalization of the concepts developed in Ref. 12.

The action is

$$S_0 = \int d^{4-\varepsilon}\tilde{x}\left(\frac{1}{2}(\partial\varphi_0(x))^2 + \frac{1}{2}m_0^2\varphi_0^2(x) + j_0\varphi_0(x) + \frac{1}{4!}\lambda_0\varphi_0^4(x)\right). \tag{65}$$

The space $\mathcal{G}$ is thus three-dimensional and is parametrized by three renormalized couplings,

$$g^1 = \lambda\,, \quad g^2 = \tilde{j} = j\kappa^{-3+\varepsilon/2}\,, \qquad g^3 = \tilde{m}^2 = m^2\kappa^{-2}\,, \tag{66}$$

where, as usual in dimensional regularization, λ is dimensionless. The massless bare couplings are

$$g_0^1 = \tilde{\lambda}_0 = \lambda_0\kappa^{-\varepsilon}\,, \quad g_0^2 = \tilde{j}_0 = j_0\kappa^{-3+\varepsilon/2}\,, \quad g_0^3 = \tilde{m}_0^2 = m_0^2\kappa^{-2}\,. \tag{67}$$

The basis operators, $\hat{\Phi}_{0a}(x)$ of Eq. (27), can thus be presented as a three-component vector (in Cartesian co-ordinates so that $\sqrt{\gamma}=1$),

$$\hat{\Phi}_0(x) = \begin{pmatrix} \hat{\Phi}_{\tilde{\lambda}_0} \\ \hat{\Phi}_{\tilde{j}_0} \\ \hat{\Phi}_{\tilde{m}_0^2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4!}\kappa^{\varepsilon}\varphi_0^4(x) \\ \kappa^{3-\varepsilon/2}\varphi_0(x) \\ \frac{1}{2}\kappa^{2-\varepsilon/2}\varphi_0^2(x) \end{pmatrix} . \tag{68}$$

$\hat{\Phi}_0$, as defined here, differs, by factors of the coupling constants, from the Q_0 of Ref. 12. This is because (68) is defined so as to have a natural interpretation as a covector in $T^*(\mathcal{G})$, which transforms covariantly under general co-ordinate transformations on $\mathcal{G}$, whereas multiplying by couplings spoils this interpretation.

The operator $\partial^2\varphi_0^2(x)$, which is also of dimension $4-\varepsilon$ and could in principle mix with $\hat{\Phi}(x)$, has been omitted. More will be said concerning this operator later but for the moment we merely note that it is a total derivative and does not contribute to the action. The only other operator with the same dimensions as $\hat{\Phi}_0(x)$ is $\varphi_0\partial^2\varphi_0(x)$ but rather than use this directly the equations of motion can be used as the remaining linearly independent operator,

$$E_0(x) = \varphi_0\left(-\partial^2\varphi_0 + m_0^2\varphi_0 + j_0 + \frac{1}{6}\lambda_0\varphi_0^3\right)\Big|_x . \tag{69}$$

This is convenient because, as explained in Sec. 2, the equations of motion do not get renormalized; $\hat{E}_0 = \hat{E}$.

The renormalization constants, $Z_a{}^b$, relate bare to renormalized operators

$$\hat{\Phi}_0(x) = Z\hat{\Phi}(x) = Z\begin{pmatrix} \hat{\Phi}_{\lambda} \\ \hat{\Phi}_{\tilde{j}} \\ \hat{\Phi}_{\tilde{m}^2} \end{pmatrix} . \tag{70}$$

The matrix Z can be determined as follows. Relate the bare couplings g_0^a to their renormalized counterparts in the usual way,

$$\lambda_0 = z_\lambda^{-1}(\lambda,\varepsilon)\kappa^\varepsilon\lambda\,, \quad j_0 = z_1^{-1}(\lambda,\varepsilon)j\,, \quad m_0^2 = z_2^{-1}(\lambda,\varepsilon)m^2\,. \tag{71}$$

Then demanding that λ_0, j_0 and m_0^2 be independent of κ gives

$$\begin{aligned} \kappa\frac{d\lambda}{d\kappa} &= -\varepsilon\lambda + \beta \qquad &&\text{with} \qquad \frac{\beta}{\lambda} = z_\lambda^{-1}\left(\kappa\frac{dz_\lambda}{d\kappa}\right), \\ \kappa\frac{d\tilde{j}}{d\kappa} &= \left(-3+\frac{\varepsilon}{2}+\gamma\right)\tilde{j} \qquad &&\text{with} \qquad \gamma = z_1^{-1}\left(\kappa\frac{dz_1}{d\kappa}\right), \\ \kappa\frac{d\tilde{m}^2}{d\kappa} &= (-2+\delta)\tilde{m}^2 \qquad &&\text{with} \qquad \delta = z_2^{-1}\left(\kappa\frac{dz_2}{d\kappa}\right), \end{aligned} \tag{72}$$

where β, γ and δ are functions of λ only, independent of ε. $\beta(\lambda)$ is the usual four-dimensional β function which starts with λ^2 in perturbation theory. The vector $\vec{\beta}$ of Eq. (40) is thus

$$\vec{\beta} = \begin{pmatrix} -\varepsilon\lambda + \beta \\ \left(-3 + \frac{\varepsilon}{2} + \gamma\right)\tilde{j} \\ (-2+\delta)\tilde{m}^2 \end{pmatrix} . \tag{73}$$

Equations (71) and (72) now give

$$\begin{aligned} d\tilde{\lambda}_0 &= -\frac{\varepsilon z_\lambda^{-1}\lambda}{(-\varepsilon\lambda+\beta)} d\lambda\,, \\ d\tilde{j}_0 &= z_1^{-1} d\tilde{j} - \frac{\gamma z_1^{-1}\tilde{j}}{(-\varepsilon\lambda+\beta)} d\lambda\,, \\ d\tilde{m}_0^2 &= z_2^{-1} d\tilde{m}^2 - \frac{\delta z_2^{-1}\tilde{m}^2}{(-\varepsilon\lambda+\beta)} d\lambda\,. \end{aligned} \tag{74}$$

Using (58),

$$\left[Z_a{}^b(g) = \frac{\partial g^b}{\partial g_0^a} \quad \Leftrightarrow \quad dg_0^a Z_a{}^b(g) = dg^b \right],$$

immediately leads to

$$(Z^{-1})_a{}^b = \frac{\partial g_0^b}{\partial g^a} = \begin{pmatrix} -\frac{\varepsilon z^{-1}\lambda}{(-\varepsilon\lambda+\beta)} & -\frac{\gamma z_1^{-1}\tilde{j}}{(-\varepsilon\lambda+\beta)} & -\frac{\delta z_2^{-1}\tilde{m}^2}{(-\varepsilon\lambda+\beta)} \\ 0 & z_1^{-1} & 0 \\ 0 & 0 & z_2^{-1} \end{pmatrix} \tag{75}$$

or

$$Z = \begin{pmatrix} -\frac{(-\varepsilon\lambda+\beta)z_\lambda}{\varepsilon\lambda} & -\frac{\gamma\tilde{j}z_\lambda}{\varepsilon\lambda} & -\frac{\delta z_\lambda \tilde{m}^2}{\varepsilon\lambda} \\ 0 & z_1 & 0 \\ 0 & 0 & z_2 \end{pmatrix} . \tag{76}$$

The matrix of dimensions, Γ, follows easily from (73):

$$\Gamma_a{}^b = \frac{\partial \beta^b}{\partial g^a} = \begin{pmatrix} -\varepsilon + \dot{\beta} & \dot{\gamma}\tilde{j} & \dot{\delta}\tilde{m}^2 \\ 0 & -3 + \frac{\varepsilon}{2} + \gamma & 0 \\ 0 & 0 & -2+\delta \end{pmatrix} , \tag{77}$$

where a dot denotes differentiation with respect to λ. This should be compared with the matrix of anomalous dimensions

$$\gamma^a{}_b = (Z^{-1})^a{}_c\kappa\left(\frac{dZ^c{}_b}{d\kappa}\right)$$
$$= \begin{pmatrix} \dot\beta & \left[\dot\gamma + \frac{3(-1+\varepsilon/2)\gamma}{(-\varepsilon\lambda+\beta)}\right]\tilde{j} & \left[\dot\delta + \frac{2(-1+\varepsilon/2)\delta}{(-\varepsilon\gamma+\beta)}\right]\tilde{m}^2 \\ 0 & \gamma & 0 \\ 0 & 0 & \delta \end{pmatrix}. \tag{78}$$

It is worth noting that $\hat\Phi_\lambda$ is *not* simply $\frac{1}{4!}\hat\varphi^4$. It can be shown, using the techniques of Ref. 12, that it mixes with the equation of motion operator, $\hat E$. In fact

$$\hat\Phi_\lambda = \frac{\kappa^\varepsilon}{4!}\hat\varphi^4 - \frac{\gamma}{(-\varepsilon\lambda+\beta)}\hat E\,,$$
$$\hat\Phi_{\tilde{j}} = \kappa^{3-\varepsilon/2}\hat\varphi\,, \tag{79}$$
$$\hat\Phi_{m^2} = \frac{1}{2}\kappa^{2-\varepsilon/2}\hat\varphi^2\,,$$

where $\hat\varphi^4$, $\hat\varphi^2$ and $\hat\varphi$ are the renormalized operators. We will not prove these relations here, since they will not be needed, but merely refer to Brown's paper.[12]

Let us now turn to the consideration of the stress tensor. Variation of the metric in (65) produces the classical stress tensor

$$T^C_{\mu\nu} = \partial_\mu\varphi_0\partial_\nu\varphi_0 - \delta_{\mu\nu}\left\{\frac{1}{2}(\partial\varphi_0)^2 + \frac{1}{2}m_0^2\varphi_0^2 + j_0\varphi_0 + \frac{1}{4!}\lambda_0\varphi_0^4\right\}, \tag{80}$$

where $\gamma_{\mu\nu}$ has been set to $\delta_{\mu\nu}$ after the variation. As is well known, this gives rise to an infinite operator,[9] but it can be made finite by adding the "improvement" term

$$T_{\mu\nu} = \partial_\mu\varphi_0\partial_\nu\varphi_0 - \delta_{\mu\nu}\left\{\frac{1}{2}(\partial\varphi_0)^2 + \frac{1}{2}m_0^2\varphi_0^2 + j_0\varphi_0 + \frac{1}{4!}\lambda_0\varphi_0^4\right\}$$
$$- \frac{1}{4}\left(\frac{2-\varepsilon}{3-\varepsilon}\right)(\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2)\varphi_0^2\,. \tag{81}$$

The improvement term can be obtained by introducing the Riemann curvature, $\mathcal{R}$, adding a coupling to the curvature, $\frac{1}{2}\mathcal{R}\varphi_0^2$, with coefficient $\frac{1}{4}(\frac{2-\varepsilon}{3-\varepsilon})$ to (65), and then varying the metric before setting $\mathcal{R} = 0$ to give Euclidean space. Of course this new coupling should also run, but it is omitted here for simplicity. It can be treated quite consistently and is done so in Ref. 15, but it is sufficient for our purposes to know that it can be done.

Now $T_{\mu\nu}$ is conserved (modulo the equations of motion) and finite, i.e. $T_{0\mu\nu} = T_{\mu\nu}$. Note that an arbitrary finite multiple of $(\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2)z_2^{-1}\varphi_0^2$ can be added to (81) without affecting either of these properties, but we choose not to do this because (81) as it stands is manifestly independent of κ and any further finite additions would spoil this feature (see below).

From (81) the trace of the stress tensor is, using the equations of motion, (69), to eliminate $\varphi_0\partial^2\varphi_0$,

$$T = \left(-3+\frac{\varepsilon}{2}\right)j_0\varphi_0 - 2\left(\frac{1}{2}m_0^2\varphi_0^2\right) - \varepsilon\left(\frac{1}{4!}\lambda_0\varphi_0^4\right) + \left(-1+\frac{\varepsilon}{2}\right)E_0\,. \tag{82}$$

The coefficients in this equation are just given by minus the classical dimensions of the couplings and minus the canonical dimension of φ for E_0. It is a simple matter to use the matrix $Z^a{}_b$ to convert this to an expression involving only renormalized quantities,

$$\begin{aligned}\hat{T} &= \left(-3+\frac{\varepsilon}{2}+\gamma\right)\tilde{j}\hat{\Phi}_{\tilde{j}} + (-2+\delta)\frac{\tilde{m}^2}{2}\hat{\Phi}_{\tilde{m}^2} + (-\varepsilon\lambda+\beta)\hat{\Phi}_\lambda + \left(-1+\frac{\varepsilon}{2}\right)E\\ &= \left(-3+\frac{\varepsilon}{2}+\gamma\right)j\hat{\varphi} + (-2+\delta)\frac{m^2}{2}\hat{\varphi}^2 + (-\varepsilon\lambda+\beta)\frac{1}{4!}\kappa^\varepsilon\hat{\varphi}^4 + \left(-1+\frac{\varepsilon}{2}-\gamma\right)\hat{E}\,,\end{aligned} \tag{83}$$

which is the same as the expression in Ref. 12. Thus

$$\hat{T}_0 = \hat{T} = \beta^a\hat{\Phi}_a - \left(1-\frac{\varepsilon}{2}\right)\hat{E} = \hat{\Theta} - \left(1-\frac{\varepsilon}{2}\right)\hat{E}\,, \tag{84}$$

which is the expansion of $\hat{T}$ used in Sec. 2, with the coefficient of $\hat{E}$ being the canonical dimension of the field, as was shown in that section. Since the equations of motion do not get renormalized it immediately follows that $\hat{\Theta}$ is also not renormalized, $\hat{\Theta}_0 = \hat{\Theta}$.

It is straightforward to show, using (74), that

$$\beta^a\partial_a g_0^b = -d_a g_0^a \qquad (\text{no sum over } a)\,. \tag{85}$$

This is an explicit example of Eq. (64) of the previous section, the statement of the fact that

$$\vec{\beta} = \beta^a\frac{\partial}{\partial g^a} = \beta_0^a\frac{\partial}{\partial g_0^a}\,, \quad \text{where} \quad \beta_0^a = \beta^b\left(\frac{\partial g_0^a}{\partial g^b}\right) = -d_a\,, \tag{86}$$

i.e. the change from bare to renormalized couplings is just a co-ordinate transformation in this language (it is a singular co-ordinate transformation when $\varepsilon = 0$, but this need not worry us!).

Note that $\hat{T}_0$ and $\hat{T}$ are the *same* operator. The bare stress operator is equal to the renormalized stress operator, once the improvement term (which corresponds to the purely classical conformal coupling to the curvature) is added. This is an explicit example of the nonrenormalization theorem discussed in Sec. 2.

Since $T_{0\mu\nu}$ should be independent of the renormalization point, κ, we can immediately deduce from (23) that

$$\kappa\frac{d\hat{T}_{\mu\nu}}{d\kappa}=0\,, \tag{87}$$

and, in particular,

$$\left[\kappa\frac{d\hat{T}}{d\kappa}=0 \qquad \Leftrightarrow \qquad \kappa\frac{d\hat{\Theta}}{d\kappa}=0\right]. \tag{88}$$

It is worthwhile making a comment about the operator $\partial^2\hat{\varphi}^2$ at this point. In Ref. 12 this operator was included in the basis set and a renormalization coefficient introduced for it. This coefficient has the effect of modifying the factor $\frac{1}{4}(\frac{2-\varepsilon}{3-\varepsilon})$ in the stress tensor, (81), but does not affect the form of the trace, (83). However it does affect the κ dependence and results in a stress operator for which $\kappa\frac{d\hat{T}_{\mu\nu}}{d\kappa}\neq 0$ but instead depends (in a finite way) on the new renormalization constant. This is the usual ambiguity in the definition of the stress operator, but here we turn this ambiguity into a virtue and use the freedom it gives to *choose* a stress operator satisfying (87). This requires omitting the operator $\partial^2\hat{\varphi}^2$ from the basis altogether, in flat space, and considering $\partial^2\hat{\varphi}^2$, to be not fundamental, but rather derivable from the more fundamental operator $\hat{\varphi}^2$. This is somewhat similar to the idea of "primary" and "descendent" fields in two-dimensional conformal field theory.[8] In this language $\hat{\varphi}^2$ would be considered to be a primary operator and $\partial^2\hat{\varphi}^2$ to be a descendant, though of course there is as yet no general theory of the classification of such objects in anything other than two dimensions.

In a curved space the story becomes much more complicated and more terms, involving invariants of the Riemann tensor, must be introduced for a consistent description of the theory.[15] One would not expect (87) to hold in a curved space since the curvature introduces another length scale into the theory and the right hand side would involve tensor operators which incorporate the Riemann tensor.

It is not difficult to verify that

$$\beta^a\partial_a\hat{\Phi}_{\tilde{j}}=-\gamma\hat{\Phi}_{\tilde{j}}\,,\quad \beta^a\partial_a\hat{\Phi}_{\tilde{m}^2}=-\delta\hat{\Phi}_{\tilde{m}^2}\,, \tag{89}$$

as claimed in Sec. 2. The anomalous dimension of $\hat{\Phi}_\lambda$ is, however, more complicated:

$$\beta^a\partial_a\hat{\Phi}_\lambda=-\left\{\dot{\beta}\hat{\Phi}_\lambda+\left(\dot{\gamma}-\frac{3(1-\varepsilon/2)}{(-\varepsilon\lambda+\beta)}\gamma\right)\tilde{j}\hat{\Phi}_{\tilde{j}}-\left(\dot{\delta}+\frac{2(1-\varepsilon/2)}{(-\varepsilon\lambda+\beta)}\delta\right)\tilde{m}^2\hat{\Phi}_{\tilde{m}^2}\right\}. \tag{90}$$

We note here also the interesting relation

$$\beta^a\partial_a\hat{T}=-(4-\varepsilon)\hat{\Theta}=-(4-\varepsilon)\beta^a\hat{\Phi}_a=-(4-\varepsilon)(\hat{T}+d_{\hat{\varphi}}\hat{E})\,. \tag{91}$$

This equation is perhaps most easily obtained by expressing $\hat{T}$ in terms of bare quantities and using Eq. (91).

The coefficient $4-\varepsilon$ is the canonical dimension of $\hat{T}$. The operator $\hat{T}$ is, of course, an operator-valued scalar on $\mathcal{G}$, not a tensor, so there is no contribution from dg^a to interpret as anomalous. Equation (91) can be thought of as giving the *total* dimension of $\hat{T}$, canonical plus anomalous. This is yet another statement of the nonrenormalization of the stress tensor — its anomalous dimension, and therefore the anomalous dimension of the trace, vanishes modulo the equations of motion.

Since $\hat{\Phi}_a$ are a linearly independent basis of operators one can decompose (91) and deduce that

$$\partial_a \hat{T} = -(4-\varepsilon)\hat{\Phi}_a \,, \tag{92}$$

a formula that can also be checked explicitly for each $\hat{\Phi}_a$ independently. This formula shows that the basis $\hat{\Phi}_a$ can be determined purely from the knowledge of the trace of the stress operator, $\hat{T}$, *without* the Hamiltonian, H_0. Thus $\hat{T}$ contains all the information of the theory since all physical amplitudes can be calculated from a knowledge of this operator.

5. Conclusions

It has been argued that the RG equation for physical amplitudes can be given a geometrical interpretation in the sense that it may be viewd as an equation for Lie transport. In terms of Lie transport on the space of couplings it reduces to no more than the definition of a Lie derivative, but its real significance lies in the way that it ties a particular diffeomorphism of Euclidean space, that of dilations, to the diffeomorphisms of the space of couplings generated by the vector field $\vec{\beta}$ through Eq. (51):

$$\mathcal{L}_{\vec{D}}\langle\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_N)\rangle = -\mathcal{L}_{\vec{\beta}}\langle\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_N)\rangle \,.$$

It is crucial to this interpretation that N-point amplitudes be viewed as rank N covariant tensors on the space of couplings, and Eq. (51) states that the two diffeomorphisms are completely equivalent. From this point of view the anomalous dimension terms in the RG equation are seen as coming from the change in the basis dg^a under Lie transport along the trajectories of $\vec{\beta}$. Since Lie derivatives are the natural geometric way in which to describe symmetries, this picture gives a clearer insight into the structure of the RG equation and the manner in which conformal symmetry, and perhaps also other space symmetries, may be broken. For instance one might postulate the existence of a β function, $\mathcal{L}_{\beta\vec{X}}$ associated with a more general diffeomorphism, $\vec{X}$, of space and consider the equation

$$\mathcal{L}_{\vec{X}}\langle\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_N)\rangle = -\mathcal{L}_{\beta\vec{X}}\langle\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_N)\rangle \,. \tag{93}$$

This would presumably necessitate the introduction of position-dependent couplings, a concept which has already proved to be of some use in understanding the RG equation.[16]

An alternative version of the RG equation, for operators rather than amplitudes, is given by Eq. (61),

$$\left(\kappa \frac{\partial}{d\kappa}\bigg|_g + \beta^b \partial_b\right)\hat{\Phi}_a + \Gamma_a{}^b \hat{\Phi}_b = 0\,,$$

but it must be stressed that neither $\kappa\frac{\partial}{d\kappa}|_g$ nor $\beta^b\partial_b$ can be pulled out of expectation values separately since the action, S_0, is not invariant under either separately.

The emphasis here is on a covariant description of the RG equation and the notion that one is free to choose any set of renormalized co-ordinates that one wishes in a description of physical amplitudes.

A generalized form of the RG equation, for arbitrary operators rather than just the basis $\hat{\Phi}_a$, is given by Eq. (41) which allows for the possibility of composite operators, which may depend on the Euclidean metric, as well as operators which are Euclidean tensors rather than scalars.

The example of massive $\lambda\varphi^4$ with a constant source has been analyzed in detail where the role of the stress tensor is clearly displayed. Equation (92) seems very important in this respect. It states that complete knowledge of the basis operators $\hat{\Phi}_a$, and therefore of the entire theory, is in principle obtainable just from the trace of the stress tensor without any knowledge of the underlying Hamiltonian. This seems to be a general property, not specific to this theory, and it would be interesting to examine this in other examples. In particular the subtleties associated with gauge theories have not been addressed here. The definition of a renormalized stress tensor for a gauge theory is fraught with difficulties but it is to be hoped that, since gauge theories are one of the jewels in the crown of the geometric picture of physics and the analysis presented here is fundamentally geometric in approach, gauge theories should fit very naturally into this picture, but this is not yet clear.

Acknowledgment

It is a pleasure to thank Denjoe O'Connor for the many discussions on the geometric nature of the RG.

References

1. E. C. G. Stuckelberg and A. Petermann, *Helv. Phys. Acta.* **26**, 499 (1953).
2. M. Gell-Mann and F. Low, *Phys. Rev.* **98**, 1300 (1954).
3. C. G. Callan, *Phys. Rev.* **D2**, 1541 (1970); K. Symanzik, *Commun. Math. Phys.* **18**, 227 (1970).
4. S. Weinberg, *Phys. Rev.* **D8**, 3497 (1973); G. 't Hooft, *Nucl. Phys.* **B61**, 455 (1973).
5. K. G. Wilson and J. Kogut, *Phys. Rep.* **C12**, 75 (1974).
6. D. O.'Connor and C. R. Stephens, "Geometry the renormalization group and gravity," in *Directions in General Relativity*, Proc. 1993 Int. Symposium, Maryland, Vol. 1, eds. B. L. Hu, M. P. Ryan Jr. and C. V. Vishveshwara, (C.U.P., 1993).
7. A. B. Zamolodchikov, *Rev. Math. Phys.* **1**, 197 (1990).
8. A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Nucl. Phys.* **B241**, 333 (1984).

9. C. G. Callan, S. Coleman and R. Jackiw, *Ann. Phys.* **59**, 42 (1970).
10. D. Z. Freedman, I. J. Muzinich and E. J. Weinberg, *Ann. Phys.* **87**, 95 (1974).
11. S. D. Joglekar, *Ann. Phys.* **100**, 395 (1976).
12. L. S. Brown, *Ann. Phys.* **126**, 135 (1980).
13. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure Of Space–Time* (C.U.P., 1973).
14. D. J. Wallace and R. K. P. Zia, *Phys. Lett.* **A48**, 325 (1974); *Ann. Phys.* **92**, 142 (1975).
15. L. S. Brown and J. C. Collins, *Ann. Phys.* **130**, 215 (1980).
16. I. T. Drummond and G. M. Shore, *Phys. Rev.* **D19**, 1134 (1979); H. Osborn, *Phys. Lett.* **B222**, 97 (1989); I. Jack and H. Osborn, *Nucl. Phys.* **343**, 647 (1990); G. M. Shore, *Nucl. Phys.* **B362**, 85 (1991).