$\pi_\alpha(x) = \pi_\alpha(\hat{x})$ if and only if $x = g \cdot \hat{x}$ for some $g \in G_\alpha$, but this means $\pi(x) = \pi(\hat{x})$ and hence ϕ is well defined. Similarly, ϕ is one-to-one since if $x, \hat{x} \in \mathscr{S}_\alpha$ and $\pi(x) = \pi(\hat{x})$, then $x = g \cdot \hat{x}$ for some $g \in G$; according to Proposition 6.47, $g \in G_\alpha$, and hence $\pi_\alpha(x) = \pi_\alpha(\hat{x})$. Finally, ϕ is an immersion, meaning $d\phi$ has maximal rank everywhere, since $d\phi \circ d\pi_\alpha = d\pi \circ di$, and by Proposition 6.47,

$$\ker d\pi_\alpha = \mathfrak{g}_\alpha = \mathfrak{g} \cap T\mathscr{S}_\alpha = \ker(d\pi \circ di).$$

Let $\tilde{H}: M/G \to \mathbb{R}$ correspond to the G-invariant function $H: M \to \mathbb{R}$, so by Theorem 6.42 the corresponding Hamiltonian systems are related: $\hat{\mathbf{v}}_{\tilde{H}} = d\pi(\hat{\mathbf{v}}_H)$. We also know that $\hat{\mathbf{v}}_H$ is everywhere tangent to the level set $\mathscr{S}_\alpha$, and hence there is a reduced vector field $\tilde{\mathbf{v}}$ on $\mathscr{S}_\alpha$ with $\hat{\mathbf{v}}_H = di(\tilde{\mathbf{v}})$ there. Moreover, as $\hat{\mathbf{v}}_H$ has G as a symmetry group, $\tilde{\mathbf{v}}$ retains G_α as a residual symmetry group and there is thus a well-defined vector field $\mathbf{v}^* = d\pi_\alpha(\tilde{\mathbf{v}})$ on the quotient manifold $\mathscr{S}_\alpha/G_\alpha$. Furthermore, this vector field agrees with the restriction of $\hat{\mathbf{v}}_{\tilde{H}}$ to the submanifold $\phi(\mathscr{S}_\alpha/G_\alpha)$ since

$$d\phi(\mathbf{v}^*) = d\phi \circ d\pi_\alpha(\tilde{\mathbf{v}}) = d\pi \circ di(\tilde{\mathbf{v}}) = d\pi(\hat{\mathbf{v}}_H) = \hat{\mathbf{v}}_{\tilde{H}}$$

there.

This last argument proves that *every* Hamiltonian vector field on M/G is everywhere tangent to $\phi(\mathscr{S}_\alpha/G_\alpha)$. Proposition 6.19 then implies that ϕ makes $\mathscr{S}_\alpha/G_\alpha$ into a Poisson submanifold of M/G and, moreover, the restriction of a Hamiltonian vector field $\hat{\mathbf{v}}_{\tilde{H}}$ on M/G to $\mathscr{S}_\alpha/G_\alpha$ (i.e. $\mathbf{v}^*$) is Hamiltonian with respect to the induced Poisson structure. This completes the proof of the theorem and hence the reduction procedure. □

If M is symplectic, then it is not true that M/G is necessarily symplectic. However, it is possible to show that the submanifolds $\mathscr{S}_\alpha/G_\alpha$ form the leaves of the symplectic foliation of M/G! (See Exercise 6.14.)

Example 6.49. Consider the abelian Hamiltonian symmetry group G acting on $\mathbb{R}^6$, with canonical coordinates $(p, q) = (p^1, p^2, p^3, q^1, q^2, q^3)$, generated by the functions $P = p^3$, $Q = q^1 p^2 - q^2 p^1$. The corresponding Hamiltonian vector fields

$$\mathbf{v}_1 = \frac{\partial}{\partial q^3} \quad \text{and} \quad \mathbf{v}_2 = p^1 \frac{\partial}{\partial p^2} - p^2 \frac{\partial}{\partial p^1} + q^1 \frac{\partial}{\partial q^2} - q^2 \frac{\partial}{\partial q^1}$$

generate a two-parameter abelian group of transformations. Any Hamiltonian function of the form $H(\rho, \sigma, \gamma, \zeta, t)$, where $\rho = \sqrt{(q^1)^2 + (q^2)^2}$, $\sigma = \sqrt{(p^1)^2 + (p^2)^2}$, $\gamma = q^1 p^2 - q^2 p^1$, $\zeta = p^3$, has G as a symmetry group; in particular, $H = \frac{1}{2}|p|^2 + V(\rho)$, a cylindrically symmetrical energy potential, is such a function.

The method of Proposition 6.48 will allow us to reduce the order of such a Hamiltonian system by four. (And, if H does not depend on t, we can integrate the entire system by quadratures.) First we restrict to the level set

$\mathscr{S} = \{P = \zeta, Q = \gamma\}$ for ζ, γ constant. If we use cylindrical coordinates

$$q = (\rho \cos \theta, \rho \sin \theta, z), \qquad p = (\sigma \cos \psi, \sigma \sin \psi, \zeta),$$

for q and p, then

$$\gamma = \rho\sigma \sin(\psi - \theta) = \rho\sigma \sin \phi,$$

where $\phi = \psi - \theta$. In terms of the variables ρ, θ, ϕ, z,† the Hamiltonian system, when restricted to $\mathscr{S}$, takes the form

$$\rho_t = \cos \phi \cdot H_\sigma, \qquad \phi_t = \sin \phi(\sigma^{-1} H_\rho - \rho^{-1} H_\sigma) \tag{6.40a}$$

$$\theta_t = \rho^{-1} \sin \phi H_\sigma + H_\gamma, \qquad z_t = H_\zeta, \tag{6.40b}$$

the subscripts on H denoting partial derivatives. These variables are also designed so that on $\mathscr{S}$, $\mathbf{v}_1 = \partial_z$, $\mathbf{v}_2 = \partial_\theta$. Theorem 6.48 guarantees that (6.40) is invariant under the reduced symmetry group of $\mathscr{S}$, which, owing to the abelian character of G, is all of G itself. This is reflected in the fact that neither z nor θ appears explicitly on the right-hand sides of (6.40). Thus, once we have determined $\rho(t)$ and $\phi(t)$ to solve the first two equations, $\theta(t)$ and $z(t)$ are determined by quadrature.

Moreover, Theorem 6.48 says that (6.40a) forms a Hamiltonian system in its own right. Fixing γ and ζ, let

$$\hat{H}(\rho, \phi, t) = H(\rho, \gamma/(\rho \sin \phi), \gamma, \zeta, t)$$

be the reduced Hamiltonian. Note that

$$\{\rho, \phi\} = -\gamma\rho^{-1}\sigma^{-2} = -\gamma^{-1}\rho \sin^2 \phi.$$

An easy computation using the chain rule shows that (6.40a) is the same as

$$\rho_t = -\gamma^{-1}\rho \sin^2\phi \cdot \hat{H}_\phi, \qquad \phi_t = \gamma^{-1}\rho \sin^2\phi \cdot \hat{H}_\rho, \tag{6.41}$$

which is indeed Hamiltonian. In particular, if H (and hence $\hat{H}$) is independent of t we can, in principle, integrate (6.41) by quadrature and hence solve the original system. (In practice, however, even for simple functions H, the intervening algebraic manipulations may prove to be overly complex.)

In general, if a Hamiltonian system is invariant under an r-parameter *abelian* Hamiltonian symmetry group, one can reduce the order by $2r$. This is because the residual symmetry group is always the entire abelian group itself owing to the triviality of the co-adjoint action. A $2n$-th order Hamiltonian system with an n-parameter abelian Hamiltonian symmetry group, or, equivalently possessing n first integrals $P_1(x), \ldots, P_n(x)$ which are *in involution*:

$$\{P_i, P_j\} = 0 \quad \text{for all } i, j,$$

† These, of course, are not universally valid local coordinates; if $\rho = 0$ we must use slightly different variables.

is called a *completely integrable* Hamiltonian system since, in principle, its solutions can be determined by quadrature alone. Actually, much more can be said about such completely integrable systems and the topic forms a significant chapter in the classical theory of Hamiltonian mechanics.

Example 6.50. Consider the group of simultaneous rotations $(p, q) \mapsto (Rp, Rq)$, $R \in \mathrm{SO}(3)$ acting on $\mathbb{R}^6$. In Example 6.43 this was shown to be a Hamiltonian group action generated by the components of the angular momentum vector $\boldsymbol{\omega} = q \times p$. Any Hamiltonian function of the form $H(|p|, |q|, p \cdot q)$ will be rotationally-invariant and thereby generate a Hamiltonian system with SO(3) as a Hamiltonian symmetry group. On the subset $M = \{(p, q)\colon q \times p \neq 0\}$, SO(3) acts regularly with three-dimensional orbits. According to Theorem 6.48, we will be able to reduce any such Hamiltonian system in degree by a total of four; three from the reduction to a common level set $\mathscr{S}_{\boldsymbol{\omega}} = \{q \times p = \boldsymbol{\omega}\}$ and one further degree from the residual symmetry group $G_{\boldsymbol{\omega}} \simeq \mathrm{SO}(2)$ of rotations around the $\boldsymbol{\omega}$-axis.

Before charging ahead with the reduction, it will help to make a small observation. By the equivariance of the momentum map $P\colon \mathbb{R}^6 \to \mathfrak{so}(3)^* \simeq \mathbb{R}^3$, $P(p, q) = q \times p = \boldsymbol{\omega}$, we see that $R \in \mathrm{SO}(3)$ maps the level set $\mathscr{S}_{\boldsymbol{\omega}}$ to the level set $R \cdot \mathscr{S}_{\boldsymbol{\omega}} = \mathscr{S}_{R\boldsymbol{\omega}}$. Thus we can choose R so as to make $\boldsymbol{\omega} = (0, 0, \omega)$, $\omega > 0$, point in the direction of the positive z-axis. All other solutions, except those of zero angular momentum, which must be treated separately, can be found by suitably rotating these solutions. If $\boldsymbol{\omega}$ is of this form, both p and q must lie in the xy-plane. We use polar coordinates (ρ, θ) for q and (σ, ψ) for p (as in the previous example). Choosing three of these as local coordinates on $\mathscr{S}_{\boldsymbol{\omega}}$ (and ignoring singular points) we obtain the reduced system

$$\rho_t = \cos\phi \cdot H_\sigma + \rho H_\tau, \quad \sigma_t = -\cos\phi \cdot H_\rho - \sigma H_\tau, \quad \theta_t = \rho^{-1} \sin\phi \cdot H_\sigma. \tag{6.42}$$

Here ϕ denotes the angle from q to p, so $\omega = \rho\sigma \sin\phi$, and $\tau = p \cdot q = \rho\sigma\cos\phi$; subscripts on H denote partial derivatives. The residual symmetry group of rotations around the z-axis is generated by

$$\mathbf{v} = -q^2 \frac{\partial}{\partial q^1} + q^1 \frac{\partial}{\partial q^2} - p^2 \frac{\partial}{\partial p^1} + p^1 \frac{\partial}{\partial p^2} = \frac{\partial}{\partial \theta},$$

and is reflected in the fact that θ does not appear on the right-hand side of the equations in (6.42). We can thus determine $\theta(t)$ by a single quadrature from the solutions to the fully reduced system

$$\rho_t = \cos\phi \cdot \hat{H}_\sigma, \qquad \sigma_t = -\cos\phi \cdot \hat{H}_\rho, \tag{6.43}$$

which are Hamilton's equations for

$$\hat{H}(\rho, \sigma) = H(\rho, \sigma, \rho\sigma\cos\phi), \quad \text{where} \quad \omega^2 = \rho\sigma\sin\phi.$$

We leave it to the reader to check that the appropriate Poisson bracket is

$$\{\hat{F}, \hat{H}\} = \cos\phi \left(\frac{\partial \hat{F}}{\partial \rho} \frac{\partial \hat{H}}{\partial \sigma} - \frac{\partial \hat{F}}{\partial \sigma} \frac{\partial \hat{H}}{\partial \rho} \right).$$

In particular, if $\hat{H}$ is independent of t, we can integrate (6.43) by quadrature, leading to a full solution to the original system. The reader might check that the present procedure is, more or less, equivalent to our integration of the Kepler problem in Example 4.19.

NOTES

Hamiltonian mechanics and the closely allied concept of a Poisson bracket have their origins in the original investigations of Poisson, Hamilton, Ostrogradskii and Liouville in the nineteenth century; see Whittaker, [1; p. 264] for details on the historical development of the classical theory, which relied exclusively on the canonical coordinates (p, q) of a symplectic structure on $\mathbb{R}^{2n}$. Besides the classical work of Whittaker, [1], good general references for the theory of Hamiltonian mechanics in the symplectic framework include the books of Abraham and Marsden, [1], Arnol'd, [3], Goldstein, [1], and Arnol'd and Novikov [1].

The more general notion of a Poisson structure first appears in Lie's theory of "function groups" (which predates his theory of Lie groups!) and the integration of systems of first order linear partial differential equations; see Lie, [4; Vol. 2, Chap. 8], Forsyth, [1; Vol. 5, § 137], and Carathéodory, [1; Chap. 9] for this theory. Lie already proved the general Darboux Theorem 6.22 for a Poisson structure of constant rank, and called the distinguished functions "ausgezeichnete functionen", which Forsyth translates as "indicial functions". In this book, I have chosen to use Carathéodory's translation of Lie's term. Recently, Weinstein, [3], proposed the less historically motivated term "Casimir function" for these objects, which has become the more popular terminology of late. Lie's theory was, by and large, forgotten by both the mathematics and physics communities. Poisson structures were re-introduced, more or less independently, by Dirac, [1], Jost, [1], Sudarshan and Mukunda, [1], and, in its present form, Lichnerowicz, [1], [2], Marsden and Weinstein, [2], and Weinstein, [3]. They are of considerable importance in both mathematical physics and differential geometry.

Lie was also well aware of the Poisson bracket associated with the dual of a Lie algebra and its connections with the co-adjoint representation. The explicit formula for this Lie–Poisson bracket can be found in Lie, [4; Vol. 2, p. 294]. This bracket too was forgotten until the 1960's, when it was rediscovered by Berezin, [1], and used by Kirillov, [1], Kostant, [1], and Souriau, [1], in connection with representation theory and geometric quantization. This bracket then bore the name of one or more of the above authors until Weinstein, [2], pointed out its much earlier appearance in Lie's work; the name "Lie–Poisson bracket" first appears in Marsden and Weinstein, [2]. The connection between rigid body motion and the Lie–Poisson bracket on SO(3) is due to Arnol'd, [2]. There is also, of course, a Lie–Poisson bracket corresponding to the left-invariant vector fields on a Lie group; perhaps surprisingly, it differs from the right-invariant version merely by a sign. In fact, to be geometrically accurate, the rigid body bracket of

Example 6.9 is really the left-invariant Lie–Poisson bracket—see Marsden, [2]. See Whittaker, [1; §69], Goldstein, [1; Chap. 4], for classical developments of these equations and Holmes and Marsden, [1], for a more detailed exposition along the lines of this chapter. See Weinstein, [4], for applications to stability theory. See Marsden, [2], for a survey of recent developments.

The reduction of Hamiltonian systems with symmetry has a long history, and most of the techniques, including Jacobi's "elimination of the node" appear in their classical form in Whittaker, [1]. The modern approach to this theory has its origins in the paper of Smale, [1], where the present version of the momentum map was introduced. Further developments due to Souriau, [1], and Meyer, [1], led to the fully developed Marsden and Weinstein, [1], approach to the reduction procedure. The treatment in this chapter is a slightly simplified and slightly less general version of the Marsden–Weinstein theory. Completely integrable Hamiltonian systems, which we've only touched on, have been a subject of immense importance throughout the history of classical (and quantum) mechanics. Most of these examples, such as rigid body motion in $\mathbb{R}^3$ and the Kepler problem, have been known for a long time, but the Toda lattice of Exercise 6.11 is of more recent origin. Manakov, [1], has shown the complete integrability of rigid body motion in $\mathbb{R}^n$. Generalizing the notion of complete integrability to include systems whose integrals are not in involution, as in Exercise 6.12, has been popularized in recent years by Mishchenko and Fomenko, [1], and Kozlov, [1].

Exercises

6.1. Suppose $P(x, t)$ is a first integral of a time-independent Hamiltonian system. Prove that the derivatives $\partial P/\partial t$, $\partial^2 P/\partial t^2$, etc. are also all first integrals. (Whittaker, [1; p. 336].)

6.2. Suppose $\dot{x} = J\nabla H(x)$ is a time-independent Hamiltonian system. Suppose $\hat{\mathbf{v}}_P$ is a Hamiltonian symmetry of the system corresponding to a time-independent function $P(x)$. Prove that for any solution $x(t)$ of the system, $P(x(t)) = at + b$ is a linear function of t. How does this compare with Theorem 6.33? Prove that if the Hamiltonian system has a fixed point x_0, then $a = 0$ and P is actually a first integral as it stands.

6.3. Suppose $\hat{\mathbf{v}}_P$ is a Hamiltonian symmetry of the Hamiltonian system $\dot{x} = J\nabla H$. Let $f(s)$ be any real-valued function of the real variable s. Prove that $f(P(x))\hat{\mathbf{v}}_P$ is again a Hamiltonian symmetry and find the corresponding first integral.

6.4. Let M be a Poisson manifold of constant rank. Prove that a function $C\colon M \to \mathbb{R}$ is a distinguished function if and only if C is constant on the leaves of the symplectic foliation of M. Does this generalize to the case of nonconstant rank? (Weinstein, [3].)

6.5. Discuss the Lie–Poisson bracket and co-adjoint orbits for the Lie algebra $\mathfrak{sl}(2)$.

6.6. Determine the Lie–Poisson bracket for the Euclidean groups E(2) and E(3). What does it look like when restricted to a co-adjoint orbit?

6.7. Suppose $\{F, H\}$ is a Poisson bracket on $\mathbb{R}^m$ whose structure functions $J^{ij}(x)$ depend linearly on $x \in \mathbb{R}^m$. Prove that this bracket determines a Lie–Poisson structure on $\mathbb{R}^m$.

6.8. Solve the Hamiltonian system corresponding to the Hamiltonian function $H(p, \tilde{p}, q, \tilde{q}) = \frac{1}{2}\tilde{p}^2 + \frac{1}{2}\tilde{q}^{-2}(p^2 - 1)$ on $\mathbb{R}^4$ with canonical Poisson bracket. (Whittaker, [1; p. 314].)

6.9. For a conservative mechanical system, discuss the process of choosing coordinates that fix the centre of mass and the angular momentum of the system in light of our general group-reduction procedure.

*6.10. The motion of n identical point vortices in the plane is governed by the canonical Hamiltonian system in $M = \mathbb{R}^{2n}$ corresponding to the Hamiltonian function

$$H(p, q) = \sum_{i \neq j} \gamma_i \gamma_j \log[(p^i - p^j)^2 + (q^i - q^j)^2]$$

in which (p^i, q^i) is the planar coordinates of the i-th vortex and γ_i its strength. Prove that the Euclidean group E(2) of simultaneous translations and rotations of the vortices forms a symmetry group of this system. Show that each infinitesimal generator of this group is a Hamiltonian vector field, and determine the corresponding conserved quantity. Show that, however, the entire group E(2) is *not* a Hamiltonian symmetry group in the strict sense of Definition 6.41. For what values of n is the vortex problem completely integrable? (Kozlov, [1; p. 15].)

6.11. The three-particle periodic *Toda lattice* is governed by the Hamiltonian system with Hamiltonian function

$$H(p, q) = \tfrac{1}{2}|p|^2 + y^1 + y^2 + y^3, \qquad p, q \in \mathbb{R}^3,$$

where

$$y^1 = e^{q^1 - q^2}, \qquad y^2 = e^{q^2 - q^3}, \qquad y^3 = e^{q^3 - q^1},$$

and we use the canonical Poisson structure on $\mathbb{R}^6$. Prove that the functions

$$P(p, q) = p^1 + p^2 + p^3, \qquad Q(p, q) = p^1 p^2 p^3 - p^1 y^2 - p^2 y^3 - p^3 y^1,$$

are first integrals, and hence the Toda lattice is a completely integrable Hamiltonian system. Is it possible to explicitly integrate it? (Toda, [1; § 2.10]).

6.12. Suppose a Hamiltonian system on a $2n$-dimensional symplectic manifold is invariant under an n-parameter solvable Hamiltonian transformation group. Prove that the solutions whose initial conditions cause the n integrals to all *vanish* can (in principle) be found by quadrature, generalizing Example 6.40. (Mishchenko and Fomenko, [1], Kozlov, [1].)

6.13. Let $M = \mathbb{R}^{2n}$ with the canonical Poisson structure. Discuss the reduction Theorem 6.48 for the symmetry group $\mathfrak{so}(2)$ whose action is generated by the energy of a harmonic oscillator, $H(p, q) = \frac{1}{2}(|p|^2 + |q|^2)$. (Arnol'd, [3; p. 377].)

*6.14. Let G and M satisfy the hypotheses of Theorem 6.48. Prove that if M is symplectic, the quotient manifold $\mathscr{S}_\alpha / G_\alpha$ is also symplectic. (Marsden and Weinstein, [1].)

6.15. Corresponding to a Hamiltonian system

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}, \qquad \frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}, \qquad H = H(p, q, t),$$

in canonical form is the Hamilton–Jacobi partial differential equation

$$\frac{\partial u}{\partial t} + H\left(\frac{\partial u}{\partial x}, x, t\right) = 0.$$

Prove that the vector field $\mathbf{v} = A(t, x, \partial u/\partial x)\partial_u$ is a generalized symmetry of the Hamilton–Jacobi equation if and only if $A(t, q, p)$ is a first integral of Hamilton's equations. (Fokas, [1].)

6.16. Suppose $\mathscr{L}[u] = \int L(t, u^{(n)})\,dt$ is a functional involving $t \in \mathbb{R}$, $u \in \mathbb{R}$. Define a change of variables

$$q^1 = u, \qquad q^2 = u_t, \ldots, \qquad q^n = u_{n-1} = d^{n-1}u/dt^{n-1},$$

$$p^1 = \frac{\partial L}{\partial u_t} - D_t\frac{\partial L}{\partial u_{tt}} + \cdots + (-D_t)^{n-1}\frac{\partial L}{\partial u_n},$$

$$p^2 = \frac{\partial L}{\partial u_{tt}} - D_t\frac{\partial L}{\partial u_{ttt}} + \cdots + (-D_t)^{n-2}\frac{\partial L}{\partial u_n},$$

$$\vdots$$

$$p^n = \frac{\partial L}{\partial u_n}.$$

Finally, let

$$H(p, q) = -L + p^1q^2 + p^2q^3 + \cdots + p^{n-1}q^n + p^n u_n,$$

where $u_n = d^n u/dt^n$ is determined implicitly from the equation for p^n. Prove that $u(t)$ satisfies the Euler–Lagrange equation for $\mathscr{L}$ if and only if $(p(t), q(t))$ satisfy Hamilton's equations for H relative to the canonical Poisson bracket. (Whittaker, [1; p. 266].)

*6.17. *Integral Invariants.* Let M be a Poisson manifold and $\hat{\mathbf{v}}_H$ a Hamiltonian vector field on M. If $S \subset M$ is any subset, let $S(t) = \{\exp(t\hat{\mathbf{v}}_H)x : x \in S\}$, assuming the Hamiltonian flow $\exp(t\hat{\mathbf{v}}_H)$ at time t is defined over all of S. A differential k-form ω on M is called an (absolute) *integral invariant* of the Hamiltonian system determined by $\hat{\mathbf{v}}_H$ if $\int_{S(t)}\omega = \int_S \omega$ for all k-dimensional compact submanifolds $S \subset M$ (with boundary).

(a) Prove that ω is an integral invariant if and only if $\hat{\mathbf{v}}_H(\omega) = 0$ on all of M.

(b) Prove that if $F(x)$ is any first integral of the Hamiltonian system, the one-form dF is an integral invariant. Does the converse hold?

(c) Suppose M is symplectic and $K(x) = J(x)^{-1}$ is as in Proposition 6.15 in terms of local coordinates x. Prove that the differential two-form

$$\Omega = \tfrac{1}{2}\sum_{i,j=1}^{m} K_{ij}(x)\,dx^i \wedge dx^j$$

is independent of the choice of local coordinates, and is an integral invariant of *any* Hamiltonian system with the given Poisson structure.

(d) Conversely, prove that a two-form Ω on M determines a symplectic Poisson structure if and only if it is closed: $d\Omega = 0$, and of maximal rank.
(e) Prove that if ω and ζ are absolute integral invariants for $\hat{\mathbf{v}}_H$, so is $\omega \wedge \zeta$.
(f) Prove *Liouville's theorem* that any Hamiltonian system in $\mathbb{R}^{2n}$ with canonical Poisson bracket (6.1) is volume preserving:

$$\mathrm{Vol}(S(t)) = \mathrm{Vol}(S), \qquad S \subset \mathbb{R}^{2n}.$$

(See Exercise 1.36.) (Cartan, [1], Arnol'd, [3].)

*6.18. Let $\hat{\mathbf{v}}_H$ be a Hamiltonian vector field on the Poisson manifold M.
(a) Prove that if the k-form ω is an integral invariant and $\mathbf{w}$ generates a symmetry group, then the $(k-1)$-form $\mathbf{w} \lrcorner\, \omega$ is an integral invariant. Also show that the Lie derivative $\mathbf{w}(\omega)$ is an integral invariant.
(b) If M is symplectic, prove that any Hamiltonian system with two *non*-Hamiltonian symmetries has a first integral. What about Hamiltonian symmetries?
(c) Discuss the Hamiltonian vector field corresponding to this first integral. (Rosencrans, [1].)

*6.19. Let N be a smooth manifold, and $M = T^*N$ its cotangent bundle. Then there is a natural symplectic structure on $M = T^*N$ which can be described in any of the following equivalent ways.
(a) Let $q = (q^1, \ldots, q^n)$ be local coordinates on N. Then $T^*N|_q$ is spanned by the basic one-forms $dq^1, \ldots, dq^n$, so that $\omega \in T^*N|_q$ can be written as $\omega = \sum p^i\, dq^i$; hence (q, p) determine the local coordinates on T^*N. Define

$$\{F, H\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i}\frac{\partial H}{\partial q^i}.$$

Prove that $\{\ ,\ \}$ is a Poisson bracket, which is well-defined on all of T^*N.
(b) Let $\pi\colon T^*N \to N$ be the projection. The canonical one-form θ on $M = T^*N$ is defined so that for any tangent vector $\mathbf{v} \in TM|_\omega$ at $\omega \in M = T^*N$,

$$\langle \theta; \mathbf{v} \rangle = \langle \omega; d\pi(\mathbf{v}) \rangle.$$

Prove that in the local coordinates of part (a), $\theta = \sum p^i\, dq^i$. Therefore, its exterior derivative $\Omega = d\theta$, as in Exercise 6.17(c), defines the Poisson bracket on T^*M.
(c) Let $\mathbf{v}$ be a vector field on N with flow $\exp(\varepsilon\mathbf{v})\colon N \to N$. Prove that the pull-back $\exp(\varepsilon\mathbf{v})^*$ defines a flow on $M = T^*N$. What is its infinitesimal generator? If $H\colon T^*N \to \mathbb{R}$ is any function, prove that there is a unique vector field $\hat{\mathbf{v}}_H$ on $M = T^*N$ such that

$$\hat{\mathbf{v}}_H(\langle \omega; \mathbf{v} \rangle|_x) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} H[\exp(\varepsilon\mathbf{v})^*(x, \omega)]$$

for all $(x, \omega) \in T^*N$, $\mathbf{v}$ a vector field on N, and that this vector field is the Hamiltonian vector field associated with H relative to the above symplectic structure.

*6.20. *Multi-vectors.* The dual objects to differential forms on a manifold are called "multi-vectors" and are defined as alternating, k-linear, real-valued maps on the cotangent space $T^*M|_x$, varying smoothly from point to point.

(a) Prove that a "uni-vector" (i.e. $k = 1$) is the same as a vector field.
(b) Prove that in a local coordinate chart, any k-vector is of the form

$$\theta = \sum_I h_I(x)\partial_{i_1} \wedge \cdots \wedge \partial_{i_k},$$

where the sum is over all strictly increasing multi-indices I, $\partial_i = \partial/\partial x^i$ form a basis for the tangent space $TM|_x$, and h_I depends smoothly on x.
(c) Let $J(x)$ be the structure matrix for a Poisson bracket on M. Prove that

$$\Theta = \tfrac{1}{2} \sum_{i,j=1}^{m} J^{ij}(x)\partial_i \wedge \partial_j$$

determines a bi-vector, defined so that

$$\langle \Theta; dH, dF \rangle = \{H, F\} \qquad (*)$$

for any pair of real-valued functions H, F.
(d) Prove that if θ is a k-vector and ζ an l-vector, then there is a uniquely defined $(k + l - 1)$-vector $[\theta, \zeta]$, called the *Schouten bracket* of θ and ζ, determined by the properties that $[\cdot,\cdot]$ is bilinear, skew-symmetric: $[\theta, \zeta] = (-1)^{kl}[\zeta, \theta]$, satisfies the Leibniz rule

$$[\theta, \zeta \wedge \eta] = [\theta, \zeta] \wedge \eta + (-1)^{kl+l}\zeta \wedge [\theta, \eta],$$

and agrees with the ordinary Lie bracket in the case θ and ζ are vector fields (uni-vectors). What is the analogue of the Jacobi identity for the Schouten bracket?
(e) Prove that if Θ is a bi-vector then the bracket between functions H and F defined by $(*)$ is necessarily bilinear and alternating; it satisfies the Jacobi identity if and only if the tri-vector $[\Theta, \Theta]$ obtained by bracketing Θ with itself vanishes identically:

$$[\Theta, \Theta] \equiv 0. \qquad (**)$$

Thus the definition of a Poisson structure on a manifold M is equivalent to choosing a bi-vector satisfying $(**)$. (Lichnerowicz, [1], [2].)

6.21. Are the orbits of the adjoint representation of a Lie group on its Lie algebra always even dimensional? Discuss.

*6.22. Suppose M is a Poisson manifold and G a local group of transformations each of which is a Poisson map.
(a) Are the infinitesimal generators of G necessarily Hamiltonian vector fields?
(b) Is G a Hamiltonian transformation group?
(c) Discuss the reduction procedure of Theorem 6.48 for such symmetry groups.

CHAPTER 7

Hamiltonian Methods for Evolution Equations

The equilibrium solutions of the equations of nondissipative continuum mechanics are usually found by minimizing an appropriate variational integral. Consequently, smooth solutions will satisfy the Euler–Lagrange equations for the relevant functional and one can employ the group-theoretic methods in the Lagrangian framework discussed in Chapters 4 and 5. However, when presented with the full dynamical problem, one encounters systems of evolution equations for which the Lagrangian viewpoint, even if applicable, no longer is appropriate or natural to the problem. In this case, the Hamiltonian formulation of systems of evolution equations assumes the natural variational role for the system.

Historically, though, the recognition of the correct general form for an infinite-dimensional generalization to evolution equations of the classical concept of a finite-dimensional Hamiltonian system has only recently been acknowledged. Part of the problem was the excessive reliance on canonical coordinates, guaranteed by Darboux' theorem in finite dimensions, but no longer valid for evolutionary systems. The Poisson bracket approach adopted here, however, does readily generalize in this context. The principal innovations needed to convert a Hamiltonian system of ordinary differential equations (6.14) to a Hamiltonian system of evolution equations are:

(i) replacing the Hamiltonian function $H(x)$ by a Hamiltonian functional $\mathscr{H}[u]$,
(ii) replacing the vector gradient operation ∇H by the variational derivative $\delta\mathscr{H}$ of the Hamiltonian functional, and
(iii) replacing the skew-symmetric matrix $J(x)$ by a skew-adjoint differential operator $\mathscr{D}$, which may depend on u.

The resulting Hamiltonian system will take the form