Geometry of renormalization group flows in theory space

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Renormalization group (RG) flows in theory space (the space of couplings) are generated by a vector field—the β function. Using a specific metric ansatz in theory space and certain methods employed largely in the context of general relativity, we examine the nature of the expansion, shear and rotation of geodesic RG flows. The expansion turns out to be a negative quantity inversely related to the norm of the β function. This implies the focusing of the flows towards the fixed points of a given field theory. The evolution equation for the expansion along geodesic RG flows is written down and analyzed. We illustrate the results for a scalar field theory with a $j\phi$ coupling and pointers to other areas are briefly mentioned.

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In the theory of quantized fields, the notion of the renormalization group (RG) plays a central role in understanding certain nonperturbative issues. It is a well-known fact that an ambiguity arises in the choice of the infinite part of a regularized Feynman amplitude. This necessitates the choice of a renormalization scheme, which, naturally, implies the existence of a scale or a renormalization point. The requirement that physical amplitudes, say the one-particle irreducible amplitudes, are independent of a choice of scale leads to the concept of the renormalization group. The RG transformations are finite renormalizations and the RG equation essentially follows from the scale invariance of the physical Green functions.

The fact that the RG equation can be written as a *geometric statement* was first noted by Lassig in [1] and later by Dolan [2,3]. The idea there was to visualize the β function as a vector field in the space of couplings (this is the space we refer to as "theory space"). Theory space, constructed with the couplings as the coordinates can be thought of as a manifold in its own right. Each point on such a manifold defines a set of values for the various couplings which appear in a given Lagrangian. A curve in theory space, therefore, represents the flow of couplings. Finite renormalization generates such a flow and thereby gives rise to RG trajectories that have the β function as the tangent vector at each point. The RG equation, in its theory space incarnation, can be expressed as a Lie derivative of the n-point function with respect to this vector field. The change in physical amplitudes due to a diffeomorphism of R^D (scale transformations of the spatial variables) at fixed couplings, generated by the dilatation generator **D** is equivalent to the change in amplitudes due to a diffeomorphism of theory space, generated by the β function at fixed spatial positions. This fact lies at the heart of the geometric form of the RG equation. As is well known, the definition of the Lie derivative does not require the concept of a metric or a connection. However, we do need such structures in order to define and analyze a geometry. The natural question which one therefore has to address is—how do we define a metric in theory space?

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There have been proposals for such a metric. One of them is related to the spatial integral of a two-point function [4], which, in general, could as well be a two-point function of composite operators. We also need to impose the crucial fact that spatial points are well separated. This enables us to avoid the possible divergences of amplitudes which always come up in quantum field theory. The above proposal is akin to that due to Zamolodchikov [5], which he and later authors utilized in order to prove the c theorem for two-dimensional field theories.

Given a definition of a metric we can now investigate the *geometry* of theory space through the ensuing connection and curvature. This has been worked out for several field-theoretic models such as scalar field theories with a $j\phi$ term [6], $\lambda\phi^4$ theory, the one-dimensional Ising model [7], the O(N) model (for large N) [8] and $N=2$ supersymmetric Yang-Mills theory [9]. The location of zeros of the β function that correspond to fixed points for the flows (and also imply conformally invariant field theories) coincide, rather surprisingly, with points of diverging Ricci curvature in theory space.

The fact that the β function is a vector field in theory space can improve our understanding of the nature of RG flows in a novel way if we employ the techniques and results primarily used in proving the singularity theorems in general relativity (GR). This is what we aim to do in this article. We shall also point out that there are some worthwhile directions which may be pursued in greater depth in the future.

Firstly, let us recall the analysis from Riemannian geometry and GR which we will use extensively. This is based on the fact that the covariant derivative of the velocity field in a Riemannian or pseudo–Riemannian spacetime is a second rank tensorial object. Therefore $v_{\mu;\nu}$ can be split into it's symmetric traceless, trace and antisymmetric parts. These three parts constitute the shear, expansion and rotation of the geodesic flow. Along the families of flow lines, one can therefore write down the differential equations for each of these quantities. It turns out that these equations are coupled and quite difficult to solve completely. However, for simplistic scenarios, there do exist solutions which have been analyzed in some amount of detail.

In Riemannian geometry and GR one does not quite solve these equations. For general, possibly non-geodesic flows, one has the Raychaudhuri identity given by

$$\nabla_\alpha(v^\mu \nabla_\mu v^\beta) = (\nabla_\alpha v^\mu)(\nabla_\mu v^\beta) + v^\mu \nabla_\alpha \nabla_\mu v^\beta. \qquad (1)$$

In the case of affinely parametrized geodesic curves, the left-hand side is zero which, thereby, gives us the Raychaudhuri equation. Utilizing the split of $v_{\mu;\nu}$ into shear, rotation and expansion we obtain equations for each of these quantities. The isotropic increase of the cross-sectional area containing a family of geodesics is termed as expansion. Shear corresponds to an anisotropic deformation of this area (such as circle $\rightarrow$ ellipse) and rotation involves a twist in the area (imagine twisting a thick rope made out of individual threads). These notions, though familiar to the relativist, are illustrated in Figs. 1(a)–1(c).

As an example, we have the equation for the expansion

$$\frac{d\theta}{d\lambda} + \frac{1}{n-1}\theta^2 + 2\sigma^2 - 2\omega^2 = -R_{\mu\nu}\xi^\mu \xi^\nu \qquad (2)$$

which, along with the other equations for the shear and rotation, comprise the full set of nonlinear, coupled ordinary differential equations (ODEs). n here is the dimension of space (for Euclidean signature) or spacetime (for Lorentzian signature). The equation for the expansion is analyzed in a couple of ways. Here we state one of them (for details see [10,11]). It is easy to see that a redefinition of θ in terms of the quantity $\theta = (n-1)(F'/F)$ converts the equation into a second order linear ODE (note that the first order equation was a nonlinear first order ODE, known in mathematics as a Riccati equation). Thereafter, by imposing $\omega_{\mu\nu} = 0$ (which is an allowed solution of the evolution equation for rotation) we find that the existence of zeros in a solution crucially depends on the positivity (or nullity) of the quantity $R_{\mu\nu}\xi^\mu\xi^\nu$. Translating $R_{\mu\nu}$ into $T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}$ via the Einstein field equation one obtains a condition on the matter stress energy known as the "strong energy condition" (this condition is the statement that $T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} \geqslant 0$). This (and some weaker versions) is, therefore, a requirement on the existence of zeros in F and therefore an infinity in θ. Since θ is related to the rate of change of the cross-sectional area of the family of geodesics (the cross section being normal to the tangent vector field) one finds that an infinity in θ (more precisely a negative infinity) at a finite value of the affine parameter implies a vanishing of this area and therefore an intersection of the set of geodesics. Such an intersection is termed focusing of geodesics and is a precursor to the existence of spacetime singularities in the sense of geodesic incompleteness, which, in some cases, does coincide with the other definition of singularities, i.e., divergent curvatures.

Having stated the notion of focusing in the context of spacetime geodesics we now return to quantum field theory. The RG flows are integral curves of the β function in theory space, with the added fact that they obey the RG equation. The curves may be geodesic or non-geodesic. If they are geodesic then too they may be affinely or non-affinely parametrized. The overall analysis of the previous paragraphs can therefore be translated to the case of RG flows which may or may not be geodesic.

It has been shown by Dolan [9] that the RG equation for a class of theories (exceptions and more general discussion with an example are available in [13]) can indeed be written as a condition on the β function. More precisely, β^a (we use Latin indices for theory space) is a conformal Killing vector. As mentioned before, this was previously assumed by Lassig

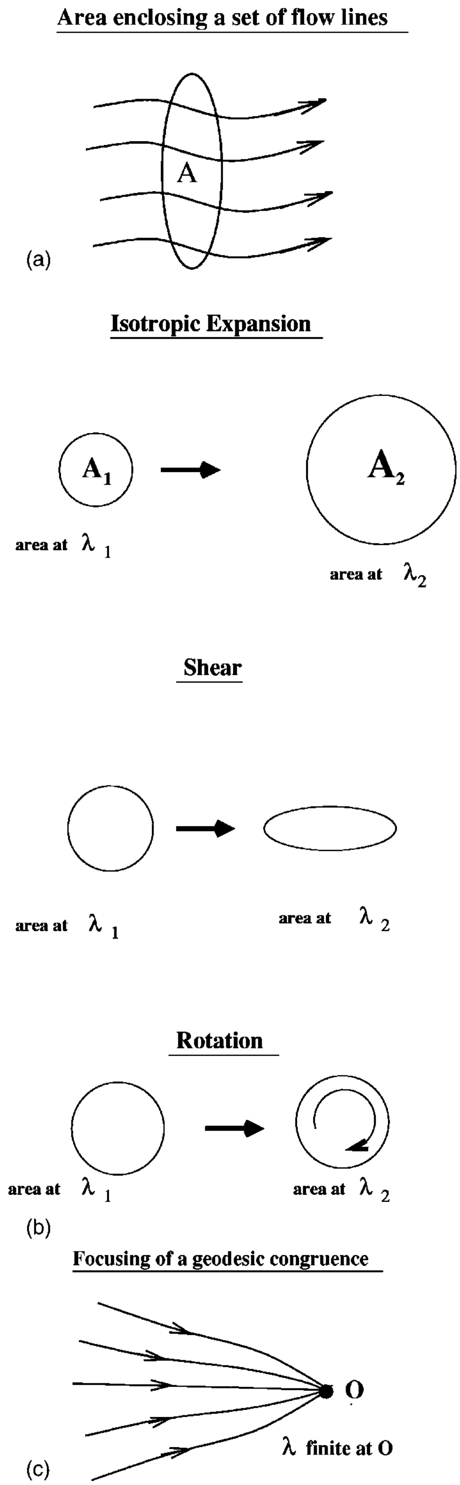

FIG. 1. (a) A family or congruence of flow lines generated by a vector field. The cross-sectional area A encloses these lines. (b) Isotropic expansion, shear and rotation of a family of flow lines explained in terms of the changes in the cross-sectional area. (c) Focusing of geodesics within a finite value of λ.

[1] in order to demonstrate that the RG equation can be written (in theory space) in terms of a Lie derivative of the Green function with respect to β^a.

This condition is

$$\nabla_a \beta_b + \nabla_b \beta_a = -D g_{ab} \tag{3}$$

where $g_{ab} = \int \langle \Phi_a(y) \Phi_b(0) \rangle d^D y$ is the O'Connor-Stephens metric and D is the dimension of Euclidean space in which the field theory is defined. Φ_a are, in general, composite operators and are defined after taking care of a subtraction of $\langle \Phi_a \rangle$ (for details on this see [6]). The metric in theory space is essentially the Fisher-Rao metric [based on the Fisher information matrix—(for details see [7]) used in statistics to compare probability distributions].

The above equation follows in a straightforward way, from the RG equation for the two-point function:

$$\kappa \partial_\kappa \langle \Phi_a(x) \Phi_b(y) \rangle + \beta^c \partial_c \langle \Phi_a(x) \Phi_b(y) \rangle + \partial_a \beta^c \langle \Phi_c(x) \Phi_b(y) \rangle + \partial_b \beta^c \langle \Phi_a(x) \Phi_c(y) \rangle = 0. \tag{4}$$

Using the facts that couplings are scaled to be dimensionless, the $\Phi_a(x)$ have canonical mass dimension D and the scaling argument one can arrive at the equation

$$\begin{aligned} &\left(x^\mu \frac{\partial}{\partial x^\mu} + y^\mu \frac{\partial}{\partial y^\mu} \right) \langle \Phi_a(x) \Phi_b(y) \rangle + 2D \langle \Phi_a(x) \Phi_b(y) \rangle \\ &\quad = -\beta^c \partial_c \langle \Phi_a(x) \Phi_b(y) \rangle - \partial_a \beta^c \langle \Phi_c(x) \Phi_b(y) \rangle \\ &\qquad - \partial_b \beta^c \langle \Phi_a(x) \Phi_c(y) \rangle. \end{aligned} \tag{5}$$

Integrating over all y and using translational invariance we get

$$\beta^c \partial_c g_{ab} + (\partial_a \beta^c) g_{cb} + (\partial_b \beta^c) g_{ca} = -D g_{ab} \tag{6}$$

which will finally yield (after some simple manipulations) the condition given earlier.

In this condition, if we use the following decomposition:

$$\nabla_a \hat{\beta}_b = \sigma_{ab} + \omega_{ab} + \frac{1}{n-1} h_{ab} \theta \tag{7}$$

where $\hat{\beta}^a = \beta^a / \sqrt{\beta^a \beta_a}$, $h_{ab} = g_{ab} - \hat{\beta}_a \hat{\beta}_b$ and the quantities σ_{ab} (shear), ω_{ab} (rotation) and θ (expansion) are functions of λ. Taking the trace of the above expression with respect to h^{ab} we arrive at the rather striking result that

$$\theta = -\frac{D(n-1)}{2\beta}. \tag{8}$$

This is a general result. It states that RG flows must necessarily converge (θ negative). They will converge to a point where the norm of the β function is zero, which means (since the norm here is a positive quantity and we are working in a Euclidean space) that the flows *focus where the field theory has a fixed point* (i.e., all the β functions—each component of the vector field β^a tends to zero).

The shear is given by the symmetric, traceless part of the gradient of the normalized β function. It turns out to be

$$\sigma_{ab} = \frac{1}{2} (\nabla_a \hat{\beta}_b + \nabla_b \hat{\beta}_a) + h_{ab} \frac{D}{2\beta}. \tag{9}$$

In a similar way, the antisymmetric part leads to the rotation, given through the expression

$$\omega_{ab} = \frac{1}{2} (\nabla_a \hat{\beta}_b - \nabla_b \hat{\beta}_a). \tag{10}$$

It is also possible that along certain directions in theory space the β functions are zero but they remain non-zero along others. For example, in a three-dimensional theory space the geodesics that lie entirely on a two dimensional section may converge to a point where the β functions along the coordinate directions which define this section, vanish. If one has parallel sections where geodesics focus, then the expansion does not tend to negative infinity—it is negative nevertheless but we get a curve along which the convergence happens. One can relate this to shear and rotation which are nonzero if even one of the β functions is finite.

Knowing the β function and the metric in the coupling space , one may therefore calculate these quantities. We shall do this in detail later when we discuss a specific example.

The condition that β^a be a conformal Killing vector translates into the following under contraction with β^b. We have

$$\hat{\beta}^b \nabla_b \beta^a = -\nabla^a \beta - D \hat{\beta}^a. \tag{11}$$

This equation is a consequence derived from the RG equation.

The evolution equations for the quantities θ, σ_{ab} and ω_{ab} are the usual Raychaudhuri equations. However, the major difference from ordinary geodesic flows in geometry and RG flows in theory space is the fact that RG flows obey an extra condition—notably that the β-function vector field is conformally Killing. This determines θ universally for all RG flows. Let us now write down the Raychaudhuri equation for the expansion in terms of the β function. This turns out to be (for $\sigma_{ab} = 0$, $\omega_{ab} = 0$ which, are, consistent solutions of the equations for σ_{ab} and ω_{ab})

$$\frac{D(n-1)}{2} \left[\frac{d\beta}{d\lambda} + \frac{D}{2} \right] = -R_{ab} \beta^a \beta^b. \tag{12}$$

Note that this is a relation between the curvature properties of theory space and the β function. For geodesic flows both the conformal Killing vector condition and the above equation has to hold good simultaneously.

One can comment qualitatively on the relation between curvature properties of theory space and the existence of zeros of the β function. A divergence in R_{ab} must be killed by a zero in β^a because the left-hand side of the Raychaudhuri equation is finite. Thus it is possible that the existence of a curvature singularity in theory space indicates a fixed point of the underlying field theory.

In a coupling space which is Einstein, i.e., $R_{ab} = \Lambda g_{ab}$, the above equation takes the form

$$\frac{d\beta}{d\lambda}+\frac{D}{2}=-C\beta^2 \tag{13}$$

where $C=2\Lambda/D(n-1)$. The equation can easily be solved to obtain the norm β of the β-function vector field in terms of the affine parameter. We have two cases to deal with $\Lambda>0$ and $\Lambda<0$. The expressions for β are

$$\Lambda>0; \quad \beta=\sqrt{\frac{D}{2C}}\tan\sqrt{\frac{D}{2C}}(C_1-C\lambda) \tag{14}$$

$$\Lambda<0; \quad \beta=-\sqrt{\frac{D}{2|C|}}\tanh\sqrt{\frac{D}{2|C|}}(C_1+|C|\lambda) \tag{15}$$

where $C=2\Lambda/D(n-1)$ and C_1 is an arbitrary constant.

The $\Lambda>0$ expression is defined over the interval $0\leqslant(C_1-C\lambda)\leqslant\pi/2$, or between π and $3\pi/2$ and so on. This is to ensure non-negativity of the norm of the β function vector field. Similarly, for the $\Lambda<0$ expression the interval is $-\infty\leqslant(|C|\lambda+C_1)\leqslant 0$. The focal points of the RG flows are those where $\beta=0$ and these are seen to occur at finite values of the parameter λ.

For a theory space of zero Ricci curvature, it is easy to see that β is linearly related to the parameter λ.

It is worth noting that these expressions for β are conclusions which are theory independent—it is possible that there could be many theories for which the theory space is Einstein. The existence of zeros at finite values of Λ also follow largely from the expressions for β—these correspond to the focal points of these RG flows.

The equations for σ_{ab} and ω_{ab} are the same as the usual Raychaudhuri equations except for the fact that the expression for θ needs to be substituted. As in the usual case σ_{ab}, ω_{ab} both equal to zero are valid solutions of these equations.

We shall now illustrate the above results by working out the details for the simplest of field theories—a theory involving a single scalar with a $j\phi$ coupling (where j is a constant). The action is

$$S(\phi;j,m^2)=\int d^Dx\left(\frac{1}{2}\phi(-\nabla^2+m^2)\phi+j\phi\right). \tag{16}$$

From the action one can find out the corresponding $w(j,m^2)$ $[W=\int d^Dxw$ and $Z=\exp(-W)]$ after integrating over ϕ. Subsequently, the line element (for a Lagrangian linear in couplings the metric $g_{ab}=-\partial_a\partial_b w$ [8]) is given as

$$ds^2=(-\partial_a\partial_b w)dx^a dx^b=(m^2)d\xi^2+\frac{1}{2}\frac{1}{(4\pi)^{D/2}}m^{2(D/2)-2}\times\Gamma\left(2-\frac{D}{2}\right)(dm^2)^2. \tag{17}$$

The mass-squared m^2 and the quantity $\xi=\langle\xi\rangle=-j/m^2$ are the two "couplings" (in a generalized sense the mass is also attributed the status of a coupling). The theory space here is two-dimensional and we shall assume the Euclidean coordinate space in which the theory is defined as a flat Euclidean space of dimension D.

The metric given above can be transformed into the form

$$ds^2=dr^2+r^{4/D}d\chi^2 \tag{18}$$

where

$$\chi=2\sqrt{\pi}\left[\frac{D}{4}\sqrt{\frac{2}{\Gamma\left(2-\frac{D}{2}\right)}}\right]^{2/D}\xi \tag{19}$$

$$r=\frac{4}{D}\sqrt{\frac{\Gamma\left(2-\frac{D}{2}\right)}{2}\left(\frac{m^2}{4\pi}\right)^{D/4}}. \tag{20}$$

The Ricci scalar is given as

$$R=-\frac{2(2-D)}{D^2r^2}. \tag{21}$$

Notice that for $D=2$ the Ricci scalar is zero and the space is also flat (a coordinate transformation will reduce the polar coordinate form for $D=2$ to $ds^2=dx^2+dy^2$). The geometry for $D\neq 2$ is, however, nontrivial. For $D>2$ one can easily show that the line elements can be written in the form $ds^2=dr^2+a^2(r)d\chi^2$ (analogous to a Euclidean, two-dimensional cosmological line element).

The β-functions can be obtained by using the scaling properties of the couplings with respect to a parameter, the values of which define the renormalization point. These turn out to be

$$\beta^r=-\frac{D}{2}r, \quad \beta^\chi=-\left(\frac{D}{2}-1\right)\chi. \tag{22}$$

The norm of the β function in the coupling space given by $g_{ab}\beta^a\beta^b$ provides us with the "normalized" β functions which we use while defining the expansion. It has been shown that for $D=2$ the flows are geodesic. The trajectories correspond to radial curves which focus at the origin. The expansion of the geodesic congruences (for $D=2$) also indicate such a behavior:

$$\theta=-\frac{1}{\beta}=-\frac{1}{r} \tag{23}$$

which implies that as $r\to 0$, $\theta\to-\infty$, which is the focal point of the congruence.

Note also that these β functions satisfy the conformal Killing vector condition. By solving them we can obtain the integral curves $r(\lambda)$ and $\chi(\lambda)$. The result should tally with the Raychaudhuri equation for the expansion θ written down above. One can check easily that it does. It is also easy to see that the shear and rotation for these flows in $D=2$ are both identically equal to zero.

Let us now briefly frame the notions mentioned above for the non-linear sigma model which describes strings. Recall that one of the basic statements of string theory has been to view the metric appearing in the sigma model as a *coupling*. The dynamic geometry of GR is a *derived* concept in string theory. It is easy to see that if the metric is a coupling, then the space of metrics is the corresponding theory space. Therefore, the Raychaudhuri equation can be used to analyze the nature of the RG flows in this ''superspace.'' For simplicity, let us qualitatively look at bosonic strings. Here the metric $g_{\mu\nu}$, the anti-symmetric tensor potential $(B_{\mu\nu})$ and the dilaton ϕ are the background fields to which the string couples. The full theory space $(\mathbf{g},\mathbf{B},\phi)$ is therefore a space of functions. The β functionals for each of these ''couplings'' are known from the RG analysis of the nonlinear σ model. As a simple example, one may think of a minisuperspace model with only a single function $a(t)$ specifying the metric and a dilaton $\phi(t)$ as a second coupling. The space of theories is therefore the space of $a(t),\phi(t)$—a function space. It is therefore necessary to evaluate the metric in this space which is obtainable from the two-point functions of the nonlinear sigma model. Once we know the metric, we can draw conclusions on geodesic flows, curvature properties of theory space and the Raychaudhuri identity. An analysis somewhat in the same vein was carried out in [12] where attractive and repulsive behavior of the RG trajectories in theory space (with a minisuperspace ansatz) was discussed.

In models of particle physics the evolution of couplings, the nature and extrapolation of RG flows and their intersection do play an important role in understanding the unification of interactions. It is certainly worth investigating whether the methods and results presented here can be of any use there.

Let us conclude by summarizing the main results:

It has been shown that the expansion for a RG flow is a negative quantity depending directly on coordinate and theory space dimensions and inversely on the norm of the β function in theory space. The shear and rotation for these flows may or may not be zero. Both these facts result from the geometric version of the RG equation.

The Raychaudhuri equation for the expansion has been rewritten in terms of the β function and it serves as an additional constraint on the β function. RG flows are generated by conformally Killing vector fields and their evolution (for geodesic flows) is indeed governed by the Raychaudhuri equations.

The salient feature of the analysis presented here is that methods and conclusions obtained in coordinate space in Riemannian geometry and GR can be used (with a few appropriate modifications) to analyze the geometry and physics in the space of couplings of a given quantum field theory. It is not yet clear whether this approach helps in some way in understanding quantum field theory or whether it is just an exercise that will remain an exclusively aesthetic endeavor. However, we should note that the results on the nature of RG flows in theory space do not pertain to any *specific* field theory but are applicable to the class of field theories for which the β function satisfies the conformal Killing condition. For theories where the conformal Killing vector character of the β function does not hold (such as the one-dimensional Ising model as shown in [13]) one will have to write down the full set of Raychaudhuri equations (without assuming specific forms of σ_{ab} or θ) for geodesic RG flows. Then one may proceed to further the analysis. A crucial assumption in all this is the metric ansatz which, however, has been used extensively in this context and follows from statistics. Additionally, the fact that the theory space in the QFT of strings coincides with the actual space of metrics is a problem worth pursuing. So is the application in realistic models of particle physics. Further detailed calculations along the each of these directions, as well as more nontrivial examples illustrating the ideas presented here, will be communicated elsewhere.

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