# Acronyms

| | |
|---|---|
| BBGKY | Bogoliubov-Born-Green-Kirkwood-Yvon |
| CAM | Coherent anomaly method |
| CE | Chapman-Enskog |
| EOM | Equation of motion |
| F-P | Fokker-Planck |
| IS | Israel-Stewart |
| KBM | Krylov, Bogoliubov, and Mitropolski |
| KdV | Kortweg-de Vries |
| l.h.s. | left-hand side |
| LHC | Large Hadron Collider |
| L-P | Lindstedt-Poincaré |
| MVOC | Method of variation of constants |
| QCD | Quantum chromodynamics |
| QFT | Quantum field theory |
| QGP | QUARK-gluon plasma |
| r.h.s. | right-hand side |
| RBE | Relativistic Boltzmann equation |
| RG | Renormalization-group |
| RG/E | Renormalization-group/envelope |
| RHIC | Relativistic Heavy Ion Collider |
| RTA | Relaxation-time approximation |
| TDGL | Time-dependent Ginzburg-Landau equation |

# Chapter 1
# Introduction: Reduction of Dynamics, Notion of Effective Theories, and Renormalization Groups

## 1.1 Reduction of Dynamics of a Simple Equation and the Notion of Effective Theory

Let us start with solving the following simple equation of a damped oscillator in classical mechanics:

$$\ddot{x} + 2\epsilon\dot{x} + x = 0, \tag{1.1}$$

where the positive parameter $\epsilon$ describes the strength of the friction, and is assumed to be small, say, less than 1. Actually, we know the exact solution to Eq. (1.1) as

$$x(t) = A(t)\sin\phi(t), \quad (A(t) := \bar{A}\mathrm{e}^{-\epsilon t},\ \phi(t) := \omega t + \bar{\theta}), \tag{1.2}$$

where $\omega := \sqrt{1-\epsilon^2}$ with $\bar{A}$ and $\bar{\theta}$ being constant. One sees that the angular velocity $\omega$ becomes small and the amplitude $A(t)$ shows a slow damping along with time owing to the friction $\epsilon$. These are well known facts.

Now, let us dare to solve Eq. (1.1) by applying a simple perturbative expansion:

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots, \tag{1.3}$$

substitution of which to (1.1) and the subsequent comparison of the coefficients of $\epsilon^n$ $(n = 0,\ 1, \dots)$ leads to series of equations for $x_n(t)$, with $\ddot{x}_0 + x_0 = 0$. If we take

$$x_0(t) = A\sin(t+\theta), \quad (A,\ \theta;\ \text{constant}) \tag{1.4}$$

as the zeroth order solution, the first-order equation reads

$$Lx_1 = -A\cos(t+\theta), \quad \left(L := \frac{d^2}{dt^2} + 1\right), \tag{1.5}$$

which is formally the equation of motion (Newton equation) for a forced oscillation with the external force with the same frequency as that of the intrinsic one. Then




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there arises a resonance phenomenon, which leads to an ever increasing amplitude of the oscillation because no friction is present;

$$x_1 = -At\sin(t+\theta), \tag{1.6}$$

which is proportional to $t$ and called a **secular term**. As will be worked out in Sect. 2.2, the second-order solution $x_2(t)$ contains similar terms, and the perturbative solution up to the second order is given by

$$x(t) = A\sin(t+\theta) - \epsilon At\sin(t+\theta) + \epsilon^2\frac{A}{2}\{t^2\sin(t+\theta) - t\cos(t+\theta)\}, \tag{1.7}$$

which shows that the amplitude of the oscillator increases with a polynomial of time owing to the secular terms. This behavior is quite different from that of the exact solution (1.2) of the damped oscillator, and hence a disaster.

We remark that the appearance of the secular terms are attributed to the fact that the zero modes of the linear operator $L$ constitutes the inhomogeneous term in the higher-order equations.

Nevertheless, one might have recognized that the perturbative solution (1.7) is actually nothing but the first few terms of the expansion of the exact solution in terms of powers of $\epsilon$: Indeed one can make the following manipulation,

$$\begin{aligned} x &\simeq A(1-\epsilon\cdot t+\epsilon^2/2\cdot t^2)\sin((1-\epsilon^2/2)t+\theta) \\ &\simeq A\exp(-\epsilon t)\sin(\sqrt{1-\epsilon^2}t+\theta), \end{aligned} \tag{1.8}$$

where one may have recognized that the secular terms are 'renormalized' into the slowly-varying amplitude and the shifted angular velocity.

Here, we see the typical problems in the (naïve) perturbative expansion of the solution of differential (or dynamical) equations; If the homogeneous equation is expressed as $Lx = 0$ and the linear operator $L$ has zero modes, a naïve perturbation method of the solution gives rise to secular terms, which may give a valid description in a local domain but would lead to an inadequate or even disastrous result in a global domain.

Then the problem can be how to circumvent the appearance of secular terms and/or resum the seemingly divergent perturbation series of the solution. Furthermore, the secular terms appearing in the perturbation theory inherently contain small parameters, and are expected to be 'renormalized' into slow modes such as, say, some amplitudes or phases. Therefore it is most desirable to be able to extract not only the slow variables but also the dynamical equations that describe the slow motions explicitly. Indeed, the amplitude $A(t)$ and the phase $\phi(t)$ in the damped oscillator discussed above satisfy the following simple equations,

$$\frac{dA}{dt} = -\epsilon A, \quad \frac{d\phi}{dt} = 1 - \frac{1}{2}\epsilon^2. \tag{1.9}$$

This means that the intrinsic dynamics in the respective (time) scales is revealed, and also the reduction of the degrees of freedom is achieved. In other words, one may say that the respective **effective theories** in **different energy/time scales** are extracted.

One of the central aims of this monograph is to make an introductory account of the so-called renormalization-group method [1, 2] in a geometrical way [3–6], and thereby present elementary methods to achieve these tasks, *i.e.*, reduction of dynamics and construction of effective theories in a not only systematic but also elementary manner.

## 1.2 Notion of Effective Theories and Renormalization Group in Physical Sciences

Extracting low-energy slow dynamics with long wave lengths is of fundamental significance in physical sciences since the birth of physics; it was actually one of the essential ingredients of the method developed by Galileo Galiley [7], who invented the method of experimentation in which conjectures are tested by actively modifying Nature, effectively used mathematics, and utilized the notion of *idealization* as is seen in the discovery of the law of inertia that holds when resistance by the environment can be neglected. The last point, which is often overseen, is of essential importance for the success of the method of Galilei, and mostly based on a separation of (energy) scales in modern languages. The recognition of separation of scales constitutes the basis of the notion of (infrared) effective theories and the renormalization-group method in various fields of physical sciences.

The concept of the renormalization group (RG) was introduced by Stuckelberg and Petermann [8] as well as Gell-Mann and Low [9] in relation to an ambiguity in the renormalization procedure of the perturbation series in quantum field theory (QFT). The significance of the RG equation was greatly emphasized by Bogoliubov and Shirkov [10, 11]; see also [12]. However, the essential nature of the RG is non-perturbative [13–16]. Subsequently, as is well known, the machinery of the RG has been applied to various problems in QFT and statistical physics with a great success [17–19].

The essence of the RG in QFT and statistical physics may be stated as follows [13, 14, 18, 19]: Let $\Gamma(\phi, \boldsymbol{g}(\Lambda), \Lambda)$ be the effective action (or thermodynamic potential) obtained by integration of the field variable with the energy scale down to $\Lambda$ from infinity or a very large cutoff $\Lambda_0$. Here $\boldsymbol{g}(\Lambda)$ is a collection of the coupling constants including the wave-function renormalization constant defined at the energy scale at $\Lambda$. Then the RG equation may be expressed as a simple fact that the effective action as a functional of the field variable $\phi$ should be the same, irrespective to how much the integration of the field variable is achieved, *i.e.*, $\Gamma(\phi, \boldsymbol{g}(\Lambda), \Lambda) = \Gamma(\phi, \boldsymbol{g}(\Lambda'), \Lambda')$. If we take the limit $\Lambda' \to \Lambda$, we have

$$d\Gamma(\phi, \boldsymbol{g}(\Lambda), \Lambda)/d\Lambda = 0, \tag{1.10}$$

which is the Wilson RG equation [13], or the flow equation in the Wegner's terminology [14]; notice that Eq. (1.10) is rewritten as

$$\frac{\partial \Gamma}{\partial \boldsymbol{g}} \cdot \frac{d\boldsymbol{g}}{d\Lambda} = -\frac{\partial \Gamma}{\partial \Lambda}. \tag{1.11}$$

If the number of the coupling constants is finite, the theory is called renormalizable. In this case, the functional space of the theory does not change in the flow given by the variation of $\Lambda$; one may say that the flow has an invariant manifold.

Owing to the very non-perturbative nature, the RG has at least two merits: (A) Construction of infrared effective actions, (B) Resummation of the perturbation series.

(A) Finding effective degrees of freedom and extract the reduced dynamics of the effective variables in fact have constituted and still constitute the core of various fields of theoretical physics. A notable aspect of the RG is that the RG equation gives a systematic tool for obtaining the infrared effective theories with fewer degrees of freedom than in the original Lagrangian relevant in the high-energy region [13]. This is a kind of reduction of the dynamics.

(B) Applying the RG equation of Gell-Mann-Low type [9–11] to perturbative calculations up to first lowest orders, a resummation in the infinite order of diagrams of some kind can be achieved. That is, the RG method gives a powerful resummation method [12].

An appearance of diverging series is a common phenomenon in every mathematical science not restricted in QFT, and some convenient resummation methods are needed and developed [20–22]. Deducing a slow and long-wave length motion is one of the basic problems in almost all the fields of physics. The problems may be collectively called the reduction problem of dynamics. The RG method might be a unified method for the reduction of dynamics.

## 1.3 The Renormalization Group Method in Global and Asymptotic Analysis

It was an Illinois group [1] and Bricmont and Kupiainen [23] who showed that the RG equations can be used for a global and asymptotic analysis of ordinary and partial differential equations, and thereby give a reduction theory of dynamical systems of some types.

Whereas the theory of Bricmont and Kupiainen [23] is based on a scaling transformation (block transformation) applied to nonlinear diffusion equations in a rigorous manner, a unique feature of the perturbative approach proposed by the Illinois group [1] is to allow secular terms to appear. Then introducing an intermediate time $\tau$ like a renormalization point in QFT, they rewrite the perturbative solution by 'renormalizing' the integral constants reminiscent of the procedure in QFT. Next declaring that the renormalized solution should not depend on $\tau$, they write down a Gell-Mann-Low like equation with respect to $\tau$, and finally mentioning that one is entitled to equate $\tau$ to the physical time $t$, they succeed in obtaining a global solution of differential equations.

Subsequently, one of the present authors [3–5] formulated the RG method in terms of the classical theory of envelopes [24], where a geometrical interpretation was given on the RG method[1] in terms of the classical theory of envelopes in elementary differential geometry. He also formulated a short-cut prescription for the renormalization-group (RG) method without introducing an intermediate time $\tau$, which procedure was adopted in [1] but might have been somewhat mysterious to those who were not familiar with the renormalization group in physics. A detailed account of the geometrical formulation of the RG method is presented in Chap. 4. It will be also emphasized in Chaps. 4 and 5 that the RG method as formulated in [5, 6] and presented in this monograph may be viewed as a natural extension of the asymptotic method by Krylov, Bogoliubov, and Mitropolski for non-linear oscillators [25]. The conventional resummation methods and asymptotic analysis applied to differential equations are reviewed in Chap. 3. The formulation of the RG method in terms of envelope *surfaces* is given in [4], and an asymptotic analysis of partial differential equations such as Barlenblatt equation [26], Swift-Hohenberg equations [27], a damped Kuramoto-Sivashinsky equation [28–31].

It was also elucidated in [5, 6] that the RG method by the Illinois group can be nicely reformulated so that it provides us with a powerful systematic reduction theory of dynamics. In particular, it gives an elementary realization of the geometrical scenario of the reduction of dynamics proposed by Yoshiki Kuramoto [30, 32]: Some time ago, Kuramoto revealed the universal structure of all the existing perturbative methods for reduction of evolution equations [30, 32]; when a reduction of evolution equation is possible, the unperturbed equation admits neutrally stable solutions, and succeeded in describing the reduction of dynamics in a geometrical manner, without recourse to any particular mathematical theory as given in [33, 34].

In Chap. 5, the RG method is reformulated in a non-perturbative way, and then a comprehensive formulation [6] of the reduction theory of dynamics based on the perturbative RG method is given in terms of the notion of attractive/invariant manifold [33, 34]. Then, in this chapter, a fully systematic reduction theory is developed for generic system that contains zero modes in the homogeneous linear operator.[2] A notable point is that the formulation is developed for the case in which the linear operator is not semi-simple and has a Jordan cell structure as well as the semi-simple case [6]; examples of the non semi-simple case include a soliton-soliton interaction described by the KdV equation [6] and the extraction of the final speed of the Benney equation [36], which is treated in Chap. 5 for the first time in the RG method.

[1] We shall call the method by the Illinois group simply the renormalization-group method or RG method in short.

[2] We refer to the work by Gorban and Karlin [35], in which they present a unique reduction theory of dynamics with an emphasis on the notion of invariant manifold and show an extensive applications of it to physical and chemical kinetics.

The RG method has been applied to quite a wide class of problems by many authors; see review articles by some of the original authors [2] for some references on the subsequent works in some stage. To mention some; Graham [37] derived a rotationally invariant amplitude equation appearing in the problem of pattern formation. Sasa [38] derived a diffusion type phase equation, and Maruo, Nozaki, and Yoshimori [39] derived Kuramoto-Sivashinsky equation [28, 29]. The discrete RG method based on the notion of the discrete envelopes was developed by Kunihiro and Matsukidaira [40], where a global and asymptotic analysis of discrete systems was discussed and thereby an optimized discretization scheme of differential equations was proposed. The method was applied to analyze asymptotic behavior of the non-linear equations appearing in cosmology [41, 42]. Boyanovsky and de Vega also and their collaborators [43] apply the RG method to discuss anomalous transport and relaxation phenomena in the early universe and quark-gluon plasma (QGP). The RG method was also shown to be a powerful tool to resum divergent perturbation series appearing in problems of quantum mechanics [44–46]. Possible relation between renormalizability and integrability of Hamilton systems was discussed by Yamaguchi and Nambu [47].

As some effort for more rigorously formulate the Illinois RG method, we can refer to [48–52], although there should be more works of importance, in particular by others.

## 1.4 Derivation of Stochastic Equations and Fluid Dynamic Limit of Boltzmann Equation

Statistical physics, in particular, non-equilibrium statistical physics, is a collection of theories of how to reduce the dynamics of many-body systems to ones with fewer variables, since the work of Boltzmann [53]. The time-*irreversible* Boltzmann equation [55] , which is written solely in terms of the single-particle distribution function for dilute gas systems [54], can be derived from the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy [55] , which is equivalent to Liouville equation hence time-*reversible*; Bogoliubov [54] showed that the dilute-gas dynamics as described by the hierarchy of the many-body distribution functions has an *attractive/invariant manifold* [34] spanned by the one-particle distribution function . In fact, a sketch for deriving Euler equation from the Boltzmann equation as the RG equation was given in [56]. In [57], the RG method was applied to derive the Boltzmann equation from the BBGKY hierarchy where the essential importance of the setting of the 'initial' condition was elucidated.

The Boltzmann equation in turn can be further reduced to the fluid dynamic equation (Navier-Stokes equation) by a perturbation theory like Chapman-Enskog method or Bogoliubov's method [25, 54]. In [57], an attempt of the derivation of the Euler equation from the classical non-relativistic Boltzmann equation with the use of the RG method was presented, which is, unfortunately, based on an inadequate inner product in the functional space composed of distribution functions. Subsequently a derivation of the dissipative fluid dynamics from the Boltzmann equation with the corrected inner product was reported in [58]. A comprehensive derivation of the Navier-Stokes equation based on [58] is presented in Chap. 8 on the basis of the geometrical formulation of the RG method [3, 5, 6].

Recently, there is a growing interest in the relativistic fluid dynamics [59, 60]. One might get surprised to find that there is no established relativistic dissipative fluid dynamic equations, as is detailed in Chap.10: For instance, there is an ambiguity of the choice of the rest frame of the fluid [61, 62] owing to the energy-mass equivalence inherent in the relativistic theory. Another problem is that the relativistic counter part of the Navier-Stokes equation, which is called the first-order (dissipative) equation, suffers from the loss of the causality because the equation becomes parabolic as the diffusion equation is. Furthermore, it is found that the thermal equilibrium state can be unstable if some relativistic dissipative fluid dynamic equations are applied [63].

A promising way of deriving the fluid dynamics that is free from these problems may be to start with the relativistic Boltzmann equation, which does not have such drawbacks, and derive the fluid dynamic limit of it by adopting some reduction theory of dynamics [64–68]. It should be naturally recognized, however, that the crucial point in such a project is how powerful and reliable is the reduction theory to be adopted. Popular methods among them are the multiple-scale method and/or use of a truncated functional space ; see [67, 68], for instance. In fact, the problem is rather involved because the derivation of a causal relativistic dissipative fluid dynamic equation must incorporate some excited modes as well as the usual zero modes originating from the conservation laws of energy-momenta and conserved charges. The excited modes reflect relatively short time dynamics, which is called the mesoscopic dynamics. One must say, however, there was no reliable reduction theory that properly identifies the appropriate excited modes as well as the zero modes.

In Chap. 9, a general theory for constructing mesoscopic dynamics is presented as an extension of the RG method utilizing the notion of envelopes, which is called the doublet scheme [69]. Then the doublet scheme in the RG method is applied to derive the relativistic second-order dissipative fluid dynamics in Chap. 12, which is extended to the reactive-multiple-component case in Chap. 15. An accurate and efficient numerical scheme is presented in Chap. 14, where numerical calculations are worked out for single component system.

The construction of mesoscopic dynamics in the non-relativistic case will be made in Chap. 16, where numerical results of the transport coefficients and the relaxation times are also presented using the derived microscopic expressions of them, and thereby some critical comparison is also made with the results obtained in the relaxation-time approximation that is commonly used in the current literature.

Langevin equation [70–76] is a kind of kinetic equations, and can be reduced to the time-irreversible Fokker-Planck equation [73–78], as is the fluid dynamics can be derived as an asymptotic dynamics of the Boltzmann equation. In Chap. 7, the RG method is applied to derive the Fokker-Planck equation from a generic multiplicative Langevin equation.

All the contents of the monograph are virtually based on the authors original works except for some review parts that include Chap. 2 in which a description is given on how secular terms appear ubiquitously in the perturbation theory, and Chap.3 in which an account of various conventional methods for the asymptotic analysis is made with a focus on their universal aspect that they all utilize the solvability condition of a linear inhomogeneous equation to make up some techniques with which the appearance of secular terms is circumvented. The other review part is Chap. 10, where a comprehensive account of the derivation of the relativistic dissipative fluid dynamics based on the Chapman-Enskog and Israel-Stewart methods is given with some comments.

The presentation of the monograph, at least in the first part, is intentionally made as pedagogical as possible so that not only researchers who are not familiar with the RG theory in physics but also undergraduate students with minimal mathematical backgrounds such as linear differential equations and linear algebra may appreciate and understand the method. Moreover we believe that simple but classical examples that are worked out will help the reader to understand what the RG method does in a geometrical way.

Conversely, the monograph is never intended to be a systematic review not only on the RG method and but also the relativistic dissipative fluid dynamics either with respect to its foundations or applications. In fact the literature on these subjects is so large that it is virtually impossible and beyond the authors' ability to make a systematic review on the current status on these subjects. Therefore we apologize those whose important articles are not cited in this monograph in advance.

# Part I
# Geometrical Formulation of Renormalization-Group Method and Its Extention for Global and Asymptotic Analysis with Examples