differential equations $\tilde{\Delta}(x, u^{(n)}) = 0$ such that

$$\mathcal{S}_\Delta = \mathcal{S}_{\tilde{\Delta}} = \{\tilde{\Delta}(x, u^{(n)}) = 0\},$$

and $\tilde{\Delta}$ is of maximal rank.

Theorem 2.31. *Suppose*

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, l,$$

is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M, and

$$\mathrm{pr}^{(n)}\, \mathbf{v}[\Delta_\nu(x, u^{(n)})] = 0, \quad \nu = 1, \ldots, l, \quad \textit{whenever} \quad \Delta(x, u^{(n)}) = 0, \tag{2.25}$$

for every infinitesimal generator $\mathbf{v}$ of G, then G is a symmetry group of the system.

The proof, as remarked above, is immediate from Theorems 2.8 and 2.27. Again, as for Theorem 2.27, it will be shown in Section 2.6 that, provided the system Δ satisfy certain additional "local solvability" conditions, (2.25) is in fact both necessary and sufficient for G to be a symmetry group. In this case, *all* (connected) symmetry groups can be systematically determined through an analysis of the infinitesimal criterion (2.25), as will be seen in numerous examples in Section 2.4. As a consequence of the maximal rank condition and Proposition 2.10, we see that we can replace (2.25) by the equivalent condition that there exist functions $Q_{\nu\mu}(x, u^{(n)})$ such that

$$\mathrm{pr}^{(n)}\, \mathbf{v}[\Delta_\nu(x, u^{(n)})] = \sum_{\mu=1}^{l} Q_{\nu\mu}(x, u^{(n)})\Delta_\mu(x, u^{(n)}) \tag{2.26}$$

holds identically in $(x, u^{(n)}) \in M^{(n)}$. Both (2.25) and (2.26) are useful when analyzing the infinitesimal criterion of invariance.

Example 2.32. Let $G = \mathrm{SO}(2)$ acting on $X \times U = \mathbb{R}^2$ as in Examples 2.29, 2.26 and 2.21. Consider the first order ordinary differential equation

$$\Delta(x, u, u_x) = (u - x)u_x + u + x = 0. \tag{2.27}$$

Note that the Jacobian matrix referred to in Definition 2.30 is

$$\mathrm{J}_\Delta = \left(\frac{\partial \Delta}{\partial x}, \frac{\partial \Delta}{\partial u}, \frac{\partial \Delta}{\partial u_x}\right) = (1 - u_x, 1 + u_x, u - x),$$

which is of rank 1 everywhere. Applying the infinitesimal generator of $\mathrm{pr}^{(1)}\,\mathrm{SO}(2)$, as calculated in (2.24), to (2.27), we find

$$\begin{aligned}
\mathrm{pr}^{(1)}\, \mathbf{v}(\Delta) &= -u\frac{\partial \Delta}{\partial x} + x\frac{\partial \Delta}{\partial u} + (1 + u_x^2)\frac{\partial \Delta}{\partial u_x} \\
&= -u(1 - u_x) + x(1 + u_x) + (1 + u_x^2)(u - x) \\
&= u_x[(u - x)u_x + u + x] \\
&= u_x\Delta.
\end{aligned}$$

Therefore $\mathrm{pr}^{(1)}\mathbf{v}(\Delta) = 0$ whenever $\Delta = 0$, and the infinitesimal criterion of invariance (2.25) is verified. We conclude that the rotation group SO(2) transforms solutions of (2.27) to other solutions. More geometrically, if $u = f(x)$ is a solution, and we rotate the graph of f by any angle θ, the resulting function is again a solution. Indeed, changing to polar coordinates

$$x = r\cos\theta, \qquad u = r\sin\theta,$$

(2.27) becomes

$$\frac{dr}{d\theta} = r.$$

The solutions are thus (pieces of) logarithmic spirals $r = ce^{\theta}$ for c constant. Obviously, rotating any one of these spirals produces another spiral of the same form, so SO(2) is indeed a symmetry group. (The choice of polar coordinates, and the fact that the equation could be readily solved in these coordinates, is, as we shall see, no accident.)

The Prolongation Formula

In light of Theorem 2.31, which connects symmetry groups of a system of differential equations with the infinitesimal criterion of invariance of the system under the prolonged infinitesimal generators of the group, the principal task remaining for us is to find an explicit formula for the prolongation of a vector field. Despite the daunting complexity of the prolonged group action, as determined by (2.18), the prolonged vector fields have a relatively simple, easily computable expression.

Before tackling the general case, it is helpful to illustrate the basic method in a couple of simpler situations. We first investigate the prolongation of a one-parameter group of transformations which acts solely on the independent variables in our system of differential equations. In other words, consider the vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x)\frac{\partial}{\partial x^i}$$

on the space $M \subset X \times U$. The group transformations $g_\varepsilon = \exp(\varepsilon\mathbf{v})$ are therefore of the form

$$(\tilde{x}, \tilde{u}) = g_\varepsilon \cdot (x, u) = (\Xi_\varepsilon(x), u)$$

discussed in Example 2.22, in which the components $\tilde{x}^i = \Xi^i_\varepsilon(x)$ satisfy

$$\left.\frac{d\Xi^i_\varepsilon(x)}{d\varepsilon}\right|_{\varepsilon=0} = \xi^i(x), \tag{2.28}$$

cf. (1.48). For simplicity, we consider the case of a single dependent variable $u \in \mathbb{R}$, although the discussion readily generalizes to several dependent variables.

The first jet space $M^{(1)}$ has coordinates $(x, u^{(1)}) = (x^i, u, u_j)$, where $u_j \equiv \partial u/\partial x^j$. The prolonged group action is found as follows: if $(x, u^{(1)})$ is any point in $M^{(1)}$, and $u = f(x)$ is any function with $u_j = \partial f/\partial x^j$, $j = 1, \ldots, p$, then $\mathrm{pr}^{(1)}\, g_\varepsilon \cdot (x, u^{(1)}) = (\tilde{x}, \tilde{u}^{(1)})$, where $\tilde{x} = \Xi_\varepsilon(x)$, $\tilde{u} = u$, and $\tilde{u}_j$ are the derivatives of the transformed function $\tilde{f}_\varepsilon = g_\varepsilon \cdot f$, which, according to (2.14), is given by

$$\tilde{u} = \tilde{f}_\varepsilon(\tilde{x}) = f[\Xi_\varepsilon^{-1}(\tilde{x})] = f[\Xi_{-\varepsilon}(\tilde{x})].$$

(Here we have used the fact that $g_\varepsilon^{-1} = g_{-\varepsilon}$ wherever defined.) Thus

$$\tilde{u}_j = \frac{\partial \tilde{f}_\varepsilon}{\partial \tilde{x}^j}(\tilde{x}) = \sum_{k=1}^{p} \frac{\partial f}{\partial x^k}(\Xi_{-\varepsilon}(\tilde{x})) \cdot \frac{\partial \Xi_{-\varepsilon}^k}{\partial \tilde{x}^j}(\tilde{x}).$$

But $\Xi_{-\varepsilon}(\tilde{x}) = x$, hence

$$\tilde{u}_j = \sum_{k=1}^{p} \frac{\partial \Xi_{-\varepsilon}^k}{\partial \tilde{x}^j}(\Xi_\varepsilon(x)) u_k \tag{2.29}$$

gives the explicit formula for the prolonged group action on the first order derivatives.

To find the infinitesimal generator of $\mathrm{pr}^{(1)}\, g_\varepsilon$, we must differentiate the formulas for the prolonged transformations with respect to ε and set $\varepsilon = 0$. Thus

$$\mathrm{pr}^{(1)}\, \mathbf{v} = \sum_{i=1}^{p} \xi^i(x) \frac{\partial}{\partial x^i} + \sum_{j=1}^{p} \phi^j(x, u^{(1)}) \frac{\partial}{\partial u_j}, \tag{2.30}$$

where $\xi^i(x)$ is as before (since $\mathrm{pr}^{(1)}\, g_\varepsilon$ transforms x and u just as g_ε does) and, by (2.21),

$$\phi^j(x, u^{(1)}) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \sum_{k=1}^{p} \frac{\partial \Xi_{-\varepsilon}^k}{\partial \tilde{x}^j}(\Xi_\varepsilon(x)) u_k.$$

Since all the functions are smooth, we can interchange the order of differentiation, and so obtain two types of terms multiplying u_k: first, those of the form

$$\frac{\partial}{\partial \tilde{x}^j}\left[\frac{d\Xi_{-\varepsilon}^k}{d\varepsilon}\right](\Xi_\varepsilon(x))\bigg|_{\varepsilon=0} = \frac{\partial}{\partial x^j}\left[\frac{d\Xi_{-\varepsilon}^k}{d\varepsilon}\bigg|_{\varepsilon=0}\right](x) = -\frac{\partial \xi^k}{\partial x^j}(x),$$

where we have used (2.28) and the fact that at $\varepsilon = 0$, $\Xi_0(x) = x$ is the identity; second, those involving two x-derivatives of $\Xi_{-\varepsilon}$:

$$\sum_l \frac{\partial^2 \Xi_{-\varepsilon}^k}{\partial \tilde{x}^j \partial \tilde{x}^l}(\Xi_{-\varepsilon}(x)) \frac{d\Xi_\varepsilon^l}{d\varepsilon}(x)\bigg|_{\varepsilon=0} = 0,$$

which vanish since $\Xi_0(x)$ is the identity, hence at $\varepsilon = 0$ all second order x-derivatives of Ξ_ε vanish. Therefore,

$$\phi^j(x, u, u_x) = -\sum_{k=1}^{p} \frac{\partial \xi^k}{\partial x^j} \cdot u_k \tag{2.31}$$

provides the basic prolongation formula for $\mathrm{pr}^{(1)}\, \mathbf{v}$ in (2.30).

Example 2.33. Let $p = 2$, $q = 1$, and consider the vector field

$$\mathbf{v} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

on $X \simeq \mathbb{R}^2$ with coordinates (x, y). According to (2.31), the first prolongation of $\mathbf{v}$ is the vector field

$$\mathrm{pr}^{(1)}\,\mathbf{v} = \mathbf{v} + \phi^x\frac{\partial}{\partial u_x} + \phi^y\frac{\partial}{\partial u_y}, \tag{2.32}$$

where

$$\phi^x = -\frac{\partial\xi}{\partial x}u_x - \frac{\partial\eta}{\partial x}u_y, \qquad \phi^y = -\frac{\partial\xi}{\partial y}u_x - \frac{\partial\eta}{\partial y}u_y.$$

For example, in the case of the rotation group

$$(x, y, u) \mapsto (x\cos\varepsilon - y\sin\varepsilon, x\sin\varepsilon + y\cos\varepsilon, u)$$

on X, the infinitesimal generator is

$$\mathbf{v} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Here $\xi = -y$, $\eta = x$, and hence $\mathbf{v}$ has first prolongation

$$\mathrm{pr}^{(1)}\,\mathbf{v} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - u_y\frac{\partial}{\partial u_x} + u_x\frac{\partial}{\partial u_y}.$$

(The first prolongation of the rotation group,

$$(x, y, u, u_x, u_y) \mapsto (x\cos\varepsilon - y\sin\varepsilon, x\sin\varepsilon + y\cos\varepsilon, u,$$
$$u_x\cos\varepsilon - u_y\sin\varepsilon, u_x\sin\varepsilon + u_y\cos\varepsilon),$$

can be reconstructed either by integrating $\mathrm{pr}^{(1)}\,\mathbf{v}$, as in (1.7), or directly from formula (2.29).)

It is useful to consider one other special case before proceeding to the general prolongation formula. Again we stick to one dependent variable u, but now look at groups that transform only the dependent variable:

$$(\tilde{x}, \tilde{u}) = g_\varepsilon\cdot(x, u) = (x, \Phi_\varepsilon(x, u)).$$

This has infinitesimal generator $\mathbf{v} = \phi(x, u)\partial_u$, where

$$\phi(x, u) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\Phi_\varepsilon(x, u).$$

If $u = f(x)$ is a function, the transformed function $\tilde{f}_\varepsilon = g_\varepsilon\cdot f$ is, according to (2.14), just

$$\tilde{u} = \tilde{f}_\varepsilon(x) = \Phi_\varepsilon(x, f(x)). \tag{2.33}$$

To find the prolonged group action, we differentiate:

$$\tilde{u}_j = \frac{\partial \tilde{f}_\varepsilon}{\partial x^j}(x) = \frac{\partial}{\partial x^j}\{\Phi_\varepsilon(x, f(x))\} = \frac{\partial \Phi_\varepsilon}{\partial x^j}(x, f(x)) + \frac{\partial f}{\partial x^j}(x)\frac{\partial \Phi_\varepsilon}{\partial u}(x, f(x)),$$

hence $\mathrm{pr}^{(1)}\, g_\varepsilon \cdot (x, u^{(1)}) = (x, \tilde{u}^{(1)})$, where

$$\tilde{u}_j = \frac{\partial \Phi_\varepsilon}{\partial x^j} + u_j \frac{\partial \Phi_\varepsilon}{\partial u}. \tag{2.34}$$

The infinitesimal generator

$$\mathrm{pr}^{(1)}\, \mathbf{v} = \mathbf{v} + \sum_{j=1}^{p} \phi^j(x, u^{(1)}) \frac{\partial}{\partial u_j}$$

of $\mathrm{pr}^{(1)}\, g_\varepsilon$ is obtained from (2.34) by differentiating with respect to ε and setting $\varepsilon = 0$, just as in our previous computations. Thus,

$$\phi^j(x, u^{(1)}) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \tilde{u}_j = \frac{\partial \phi}{\partial x^j} + u_j \frac{\partial \phi}{\partial u}. \tag{2.35}$$

This gives the prolongation formula in this special case. For example, if $p = 2$, with independent variables x, y,

$$\mathrm{pr}^{(1)}\left[xu^2 \frac{\partial}{\partial u}\right] = xu^2 \frac{\partial}{\partial u} + (u^2 + 2xuu_x)\frac{\partial}{\partial u_x} + 2xuu_y \frac{\partial}{\partial u_y}.$$

Higher order prolongations of either (2.31) or (2.35) are found by further differentiating the relevant group prolongation formula. To give a general version of this, and in preparation for the general form of the prolongation theorem, we need to introduce the concept of a total derivative.

Total Derivatives

The preceding formulae (2.35) for the prolongation of a vector field of the form $\phi(x, u)\partial_u$ can be "simplified" by making the following observation. If $u = f(x)$ is any function, then $\phi^j(x, u^{(1)})$, when evaluated on f and its first order derivatives, is just the derivative of $\phi(x, f(x))$ with respect to x:

$$\phi^j(x, \mathrm{pr}^{(1)} f(x)) = \frac{\partial}{\partial x^j}[\phi(x, f(x))].$$

(Indeed, this is essentially how the ϕ^j were found.) In other words, $\phi^j(x, u^{(1)})$ is obtained from $\phi(x, u)$ by differentiating it with respect to x^j, while treating u as a function of x. The resulting derivative is called the *total derivative* of ϕ with respect to x^j, and denoted

$$\phi^j(x, u^{(1)}) = D_j \phi(x, u) = \frac{\partial \phi}{\partial x^j} + u_j \frac{\partial \phi}{\partial u}.$$

(The term "total" derivative is to distinguish $D_j\phi$ from the "partial" derivative $\partial\phi/\partial x^j$.) The definition of total derivative extends naturally to functions depending on $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ and derivatives u_J^α of u.

Definition 2.34. Let $P(x, u^{(n)})$ be a smooth function of x, u and derivatives of u up to order n, defined on an open subset $M^{(n)} \subset X \times U^{(n)}$. The *total derivative* of P with respect to x^i is the unique smooth function $D_i P(x, u^{(n+1)})$ defined on $M^{(n+1)}$ and depending on derivatives of u up to order $n + 1$, with the property that if $u = f(x)$ is any smooth function

$$D_i P(x, \mathrm{pr}^{(n+1)} f(x)) = \frac{\partial}{\partial x^i}\{P(x, \mathrm{pr}^{(n)} f(x))\}.$$

In other words, $D_i P$ is obtained from P by differentiating P with respect to x^i while treating all the u^α's and their derivatives as functions of x.

Proposition 2.35. *Given $P(x, u^{(n)})$, the i-th total derivative of P has the general form*

$$D_i P = \frac{\partial P}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_J u_{J,i}^\alpha \frac{\partial P}{\partial u_J^\alpha}, \tag{2.36}$$

where, for $J = (j_1, \ldots, j_k)$,

$$u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x^i} = \frac{\partial^{k+1} u^\alpha}{\partial x^i \partial x^{j_1} \cdots \partial x^{j_k}}. \tag{2.37}$$

In (2.36) *the sum is over all J's of order $0 \leqslant \#J \leqslant n$, where n is the highest order derivative appearing in P.*

The proof is a straightforward applicaton of the chain rule. For example, in the case $X = \mathbb{R}^2$, with coordinates (x, y), and $U = \mathbb{R}$, there are two total derivatives D_x, D_y, with

$$D_x P = \frac{\partial P}{\partial x} + u_x \frac{\partial P}{\partial u} + u_{xx} \frac{\partial P}{\partial u_x} + u_{xy} \frac{\partial P}{\partial u_y} + u_{xxx} \frac{\partial P}{\partial u_{xx}} + \cdots,$$

$$D_y P = \frac{\partial P}{\partial y} + u_y \frac{\partial P}{\partial u} + u_{xy} \frac{\partial P}{\partial u_x} + u_{yy} \frac{\partial P}{\partial u_y} + u_{xxy} \frac{\partial P}{\partial u_{xx}} + \cdots.$$

Thus, if $P = xuu_{xy}$, then

$$D_x P = uu_{xy} + xu_x u_{xy} + xuu_{xxy}, \qquad D_y P = xu_y u_{xy} + xuu_{xyy}.$$

Higher order total derivatives are defined in analogy with our notation for higher order partial derivatives. Explicitly, if $J = (j_1, \ldots, j_k)$ is a k-th order multi-index, with $1 \leqslant j_\kappa \leqslant p$ for each κ, then the J-th total derivative is denoted

$$D_J = D_{j_1} D_{j_2} \cdots D_{j_k}.$$

(The explicit expressions for $D_J P$ in terms of the partial derivatives of P with respect to u_J^α rapidly become unmanageable.) Note that, as with partial derivatives, the order of differentiation for total derivatives of smooth functions is immaterial. Thus, for the above example,

$$D_x D_y P = D_y D_x P = u_y u_{xy} + u u_{xyy} + x(u_{xy}^2 + u_x u_{xyy} + u_y u_{xxy} + u u_{xxyy}).$$

The General Prolongation Formula

Theorem 2.36. *Let*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

be a vector field defined on an open subset $M \subset X \times U$. The n-th prolongation of $\mathbf{v}$ *is the vector field*

$$\mathrm{pr}^{(n)} \mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q} \sum_{J} \phi_\alpha^J(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha} \tag{2.38}$$

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$, the second summation being over all (unordered) multi-indices $J = (j_1, \ldots, j_k)$, with $1 \leqslant j_\kappa \leqslant p$, $1 \leqslant k \leqslant n$. The coefficient functions ϕ_α^J of $\mathrm{pr}^{(n)} \mathbf{v}$ *are given by the following formula*:

$$\phi_\alpha^J(x, u^{(n)}) = D_J\left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha\right) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha, \tag{2.39}$$

where $u_i^\alpha = \partial u^\alpha / \partial x^i$, and $u_{J,i}^\alpha = \partial u_J^\alpha / \partial x^i$, cf. (2.37).

Proof. We first prove the formula for first order derivatives, so $n = 1$ to begin with. Let $g_\varepsilon = \exp(\varepsilon \mathbf{v})$ be the corresponding one-parameter group, with transformations having the formula

$$(\tilde{x}, \tilde{u}) = g_\varepsilon \cdot (x, u) = (\Xi_\varepsilon(x, u), \Phi_\varepsilon(x, u)),$$

wherever defined. Note that

$$\begin{aligned} \xi^i(x, u) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Xi_\varepsilon^i(x, u), \qquad i = 1, \ldots, p, \\ \phi_\alpha(x, u) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Phi_\varepsilon^\alpha(x, u), \qquad \alpha = 1, \ldots, q, \end{aligned} \tag{2.40}$$

where Ξ_ε^i, Φ_ε^α are the components of Ξ_ε, Φ_ε. Given $(x, u^{(1)}) \in M^{(1)}$, let $u = f(x)$ be any representative function, so that $u^{(1)} = \mathrm{pr}^{(1)} f(x)$, or, explicitly,

$$u^\alpha = f^\alpha(x), \qquad u_i^\alpha = \partial f^\alpha(x) / \partial x^i.$$

According to (2.14), for ε sufficiently small, the transform of f by the group element g_ε is well defined (at least if the domain of definition of f is a suitably small neighbourhood of x), and is given by

$$\tilde{u} = \tilde{f}_\varepsilon(\tilde{x}) = (g_\varepsilon \cdot f)(\tilde{x}) = [\Phi_\varepsilon \circ (\mathbb{1} \times f)] \circ [\Xi_\varepsilon \circ (\mathbb{1} \times f)]^{-1}(\tilde{x}).$$

Using the chain rule, the Jacobian matrix $\mathrm{J}\tilde{f}_\varepsilon(x) = (\partial \tilde{f}_\varepsilon^\alpha / \partial \tilde{x}^i)$ is then

$$\mathrm{J}\tilde{f}_\varepsilon(\tilde{x}) = \mathrm{J}[\Phi_\varepsilon \circ (\mathbb{1} \times f)](x) \cdot \{\mathrm{J}[\Xi_\varepsilon \circ (\mathbb{1} \times f)](x)\}^{-1} \tag{2.41}$$

(provided the inverse is defined), since

$$x = [\Xi_\varepsilon \circ (\mathbb{1} \times f)]^{-1}(\tilde{x}).$$

Writing out the matrix entries of $\mathrm{J}\tilde{f}_\varepsilon(\tilde{x})$ thus provides explicit formulae for the first prolongation $\mathrm{pr}^{(1)}\, g_\varepsilon$.

To find the infinitesimal generator $\mathrm{pr}^{(1)}\, \mathbf{v}$, we must differentiate (2.41) with respect to ε and set $\varepsilon = 0$. Recall first that if $M(\varepsilon)$ is any invertible square matrix of functions of ε, then

$$\frac{d}{d\varepsilon}[M(\varepsilon)^{-1}] = -M(\varepsilon)^{-1} \frac{dM(\varepsilon)}{d\varepsilon} M(\varepsilon)^{-1}.$$

Also note that since $\varepsilon = 0$ corresponds to the identity transformation,

$$\Xi_0(x, f(x)) = x, \qquad \Phi_0(x, f(x)) = f(x), \tag{2.42}$$

so if I denotes the $p \times p$ identity matrix,

$$\mathrm{J}[\Xi_0 \circ (\mathbb{1} \times f)](x) = \mathrm{I}, \qquad \mathrm{J}(\Phi_0 \circ (\mathbb{1} \times f)](x) = \mathrm{J}f(x).$$

Now, differentiating (2.41) and setting $\varepsilon = 0$, we find, using Leibniz' rule,

$$\begin{aligned} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathrm{J}\tilde{f}_\varepsilon(\tilde{x}) &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathrm{J}[\Phi_\varepsilon \circ (\mathbb{1} \times f)](x) - \mathrm{J}f(x) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathrm{J}[\Xi_\varepsilon \circ (\mathbb{1} \times f)](x) \\ &= \mathrm{J}[\phi \circ (\mathbb{1} \times f)](x) - \mathrm{J}f(x) \cdot \mathrm{J}[\xi \circ (\mathbb{1} \times f)](x). \end{aligned}$$

In the second equality, $\xi = (\xi^1, \ldots, \xi^p)^T$, and $\phi = (\phi_1, \ldots, \phi_q)^T$ are column vectors and we have used (2.40). The matrix entries of this last formula will give the coefficient functions ϕ_α^k of $\partial/\partial u_k^\alpha$ in $\mathrm{pr}^{(1)}\, \mathbf{v}$. Namely, the (α, k)-th entry is

$$\phi_\alpha^k(x, \mathrm{pr}^{(1)} f(x)) = \frac{\partial}{\partial x^k}[\phi_\alpha(x, f(x))] - \sum_{i=1}^p \frac{\partial f^\alpha}{\partial x^i} \cdot \frac{\partial}{\partial x^k}[\xi^i(x, f(x))].$$

Thus, by the definition of total derivative,

$$\begin{aligned} \phi_\alpha^k(x, u^{(1)}) &= D_k[\phi_\alpha(x, u)] - \sum_{i=1}^p D_k[\xi^i(x, u)] u_i^\alpha \\ &= D_k\left[\phi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha\right] + \sum_{i=1}^p \xi^i u_{ki}^\alpha, \end{aligned} \tag{2.43}$$

where $u_{ki}^\alpha = \partial^2 u^\alpha / \partial x^k \partial x^i$. This proves (2.39) in the case $n = 1$.

To prove the theorem in general, we proceed by induction. The key remark is that the $(n+1)$-st jet space $M^{(n+1)}$ can be viewed as a subspace of the first jet space $(M^{(n)})^{(1)}$ of the n-th jet space. This is because each $(n+1)$-st order derivative u_J^α can be viewed as a first order derivative of an n-th order derivative. (This can be done in general in several ways.) It is instructive to look at an illustrative example. For $p = 2$, $q = 1$, the first jet space $M^{(1)}$ has coordinates $(x, y; u; u_x, u_y)$. If we view (u_x, u_y) as *new* dependent variables, say $u_x = v$, $u_y = w$, then $M^{(1)}$ looks just like an open subset of $X \times \tilde{U}$, where X is still two-dimensional, but now $\tilde{U}$ has three dependent variables u, v and w. Thus the first jet space of $M^{(1)}$, i.e. $(M^{(1)})^{(1)}$, will be an open subset of $X \times \tilde{U}^{(1)}$, with coordinates $(x, y; u; v, w; u_x, u_y, v_x, v_y, w_x, w_y)$. Now remembering that $v = u_x$ and $w = u_y$, we see that $M^{(2)} \subset (M^{(1)})^{(1)}$ is the subspace defined by the relations

$$v = u_x, \qquad w = u_y, \qquad v_y = w_x,$$

in $X \times \tilde{U}^{(1)}$, determined by the superfluous variables u_x, u_y in $(M^{(1)})^{(1)}$ and the equality of mixed second order partial derivatives of u.

With this point of view, the inductive procedure for determining $\mathrm{pr}^{(n)}\,\mathbf{v}$ from $\mathrm{pr}^{(n-1)}\,\mathbf{v}$ is as follows; we regard $\mathrm{pr}^{(n-1)}\,\mathbf{v}$ as a vector field on $M^{(n-1)}$ (of a certain special type) and so by our first order prolongation formula can prolong it to $(M^{(n-1)})^{(1)}$. We then restrict the resulting vector field to the subspace $M^{(n)}$, and this will determine the n-th prolongation $\mathrm{pr}^{(n)}\,\mathbf{v}$. (Of course, we must check that the restriction is possible, but this will follow from the explicit formula.) Now the new "n-th order" coordinates in $(M^{(n-1)})^{(1)}$ are given by $u_{J,k}^\alpha = \partial u_J^\alpha/\partial x^k$, where $J = (j_1, \ldots, j_{n-1})$, $1 \leqslant k \leqslant p$, and $1 \leqslant \alpha \leqslant q$. According to (2.43), the coefficient of $\partial/\partial u_{J,k}^\alpha$ in the first prolongation of $\mathrm{pr}^{(n-1)}\,\mathbf{v}$ is therefore

$$\phi_\alpha^{J,k} = D_k \phi_\alpha^J - \sum_{i=1}^{p} D_k \xi^i \cdot u_{J,i}^\alpha. \tag{2.44}$$

(As we will see, (2.44) provides a useful recursion relation for the coefficient functions of $\mathrm{pr}^{(n)}\,\mathbf{v}$.) It now suffices to check that the formula (2.39) solves the recursion relation (2.44) in closed form. By induction, we find

$$\begin{aligned}
\phi_\alpha^{J,k} &= D_k\left\{D_J\left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha\right) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha\right\} - \sum_{i=1}^{p} D_k\xi^i \cdot u_{J,i}^\alpha \\
&= D_k D_J\left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha\right) + \sum_{i=1}^{p} (D_k\xi^i \cdot u_{J,i}^\alpha + \xi^i u_{J,ik}^\alpha) - \sum_{i=1}^{p} D_k\xi^i \cdot u_{J,i}^\alpha \\
&= D_k D_J\left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha\right) + \sum_{i=1}^{p} \xi^i u_{J,ik}^\alpha,
\end{aligned}$$

where $u_{J,ik}^\alpha = \partial^2 u_J^\alpha/\partial x^i \partial x^k$. Thus $\phi_\alpha^{J,k}$ is of the form (2.39), and the induction step is completed. □

Example 2.37. Let's repeat the case of the rotation group SO(2) acting on $X \times U \simeq \mathbb{R} \times \mathbb{R}$ with infinitesimal generator

$$\mathbf{v} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u};$$

see Examples 2.26 and 2.29. In this case $\phi = x$, $\xi = -u$, so the first prolongation

$$\mathrm{pr}^{(1)}\,\mathbf{v} = \mathbf{v} + \phi^x \frac{\partial}{\partial u_x}$$

is given by

$$\phi^x = D_x(\phi - \xi u_x) + \xi u_{xx} = D_x(x + uu_x) - uu_{xx} = 1 + u_x^2.$$

Thus we recover the result of (2.24). The coefficient function ϕ^{xx} of $\partial/\partial u_{xx}$ in $\mathrm{pr}^{(2)}\,\mathbf{v}$ is found using either (2.39)

$$\phi^{xx} = D_x^2(\phi - \xi u_x) + \xi u_{xxx} = D_x^2(x + uu_x) - uu_{xxx} = 3u_x u_{xx},$$

or the recursion formula (2.44)

$$\phi^{xx} = D_x\phi^x - u_{xx}D_x\xi = D_x(1 + u_x^2) + u_x u_{xx} = 3u_x u_{xx}.$$

Thus the infinitesimal generator of the second prolongation $\mathrm{pr}^{(2)}$ SO(2) acting on $X \times U^{(2)}$ is

$$\mathrm{pr}^{(2)}\,\mathbf{v} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1 + u_x^2)\frac{\partial}{\partial u_x} + 3u_x u_{xx}\frac{\partial}{\partial u_{xx}}.$$

(The derivation of this formula directly from the action $\mathrm{pr}^{(2)}$ SO(2) is, needless to say, considerably more complicated.)

Using the infinitesimal criterion of invariance of Theorem 2.31, we immediately deduce that the ordinary differential equation $u_{xx} = 0$ has SO(2) as a symmetry group, since

$$\mathrm{pr}^{(2)}\,\mathbf{v}(u_{xx}) = 3u_x u_{xx} = 0$$

whenever $u_{xx} = 0$. This is just a restatement of the geometric fact that rotations take straight lines to straight lines. For another geometric illustration, consider the function

$$\kappa(x, u^{(2)}) = u_{xx}(1 + u_x^2)^{-3/2}.$$

An easy computation shows that

$$\mathrm{pr}^{(2)}\,\mathbf{v}(\kappa) = 0$$

for all u_x, u_{xx}, hence by Proposition 2.6, κ is an invariant of $\mathrm{pr}^{(2)}$ SO(2):

$$\kappa(\mathrm{pr}^{(2)}\,\theta\cdot(x, u^{(2)})) = \kappa(x, u^{(2)})$$