Thus, if $N \subset M$ is a Poisson submanifold, any Hamiltonian vector field $\hat{v}_H$ on M is everywhere tangent to N and thereby naturally restricts to a Hamiltonian vector field $\hat{v}_{\tilde{H}}$ on N, where $\tilde{H} = H|N$ is the restriction of H to N and we are using the induced Poisson structure on N to compute $\hat{v}_{\tilde{H}}$. If we are only interested in solutions to the Hamiltonian system corresponding to H on M with initial conditions x_0 in N, we can restrict to the Hamiltonian system corresponding to $\tilde{H}$ on N without loss of information, thereby reducing the order of the system. In particular, as far as finding particular solutions of the Hamiltonian system goes, we may as well restrict to the *minimal* Poisson submanifolds of M in which the initial data lies. According to the next theorem, these are always symplectic submanifolds, so every Hamiltonian system can be reduced to one in which the Poisson bracket is symplectic.

Theorem 6.21. *Let M be a Poisson manifold. The system of Hamiltonian vector fields $\mathscr{H}$ on M is integrable, so through each point $x \in M$ there passes an integral submanifold N of $\mathscr{H}$ satisfying $TN|_y = \mathscr{H}|_y$ for each $y \in N$. Each integral submanifold is a symplectic submanifold of M, and, collectively, these submanifolds determine the* symplectic foliation *of the Poisson manifold M. Moreover, if $H: M \to \mathbb{R}$ is any Hamiltonian function, and $x(t) = \exp(t\hat{v}_H)x_0$ any solution to the corresponding Hamiltonian system, with initial data $x_0 \in N$, then $x(t) \in N$ remains in a single integral submanifold N for all t.*

PROOF. This is a direct consequence of the variable rank Frobenius' Theorem 1.41. The involutiveness of $\mathscr{H}$ follows from the fact that the Lie bracket of two Hamiltonian vector fields is again a Hamiltonian vector field, (6.8). The rank-invariance of $\mathscr{H}$ is given in Corollary 6.17. □

Thus each Poisson manifold naturally splits into a collection of even-dimensional symplectic submanifolds—the *leaves* of the symplectic foliation. The dimension of any such leaf N equals the rank of the Poisson structure at any point $y \in N$, so if M has nonconstant rank, the symplectic leaves will be of varying dimensions. For example, in the case of $\mathfrak{so}(3)^*$ the leaves are just the spheres S^2_ρ centred at the origin, together with the singular point $u = 0$. Any Hamiltonian system on M naturally restricts to any symplectic leaf. If we are only interested in the dynamics of particular solutions, then, we could effectively restrict our attention to the single symplectic submanifold in which our solution lies. For instance, the solutions of the equations (6.18) of rigid body motion naturally live on the spheres $|u| = \rho$.

Darboux' Theorem

If we restrict attention to the places where the Poisson structure is of constant rank (in particular, on the open submanifold where its rank achieves

its maximum) the geometric picture underlying the symplectic foliation simplifies considerably. In fact, as with the constant rank version of Frobenius' Theorem 1.43, we can introduce flat local coordinates which make the foliation of a particularly simple, canonical form. This is the content of Darboux' theorem.

Theorem 6.22. *Let M be an m-dimensional Poisson manifold of constant rank $2n \leqslant m$ everywhere. At each $x_0 \in M$ there exist canonical local coordinates $(p, q, z) = (p^1, \dots, p^n, q^1, \dots, q^n, z^1, \dots, z^l)$, $2n + l = m$, in terms of which the Poisson bracket takes the form*

$$\{F, H\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q^i} \right).$$

The leaves of the symplectic foliation intersect the coordinate chart in the slices $\{z^1 = c_1, \dots, z^l = c_l\}$ determined by the distinguished coordinates z.

PROOF. If the rank of the Poisson structure is 0 everywhere, there is nothing to prove. Indeed, the Poisson bracket is trivial: $\{F, H\} \equiv 0$ for all F, H, and any set of local coordinates $z = (z^1, \dots, z^l)$, $l = m$, satisfies the conditions of the theorem. Otherwise, we proceed by induction on the "half-rank" n.

Since the rank at x_0 is nonzero, we can find real-valued functions F and P on M whose Poisson bracket does not vanish at x_0:

$$\{F, P\}(x_0) = \hat{\mathbf{v}}_P(F)(x_0) \neq 0.$$

In particular, $\hat{\mathbf{v}}_P|_{x_0} \neq 0$, so we can use Proposition 1.29 to straighten out $\hat{\mathbf{v}}_P$ in a neighbourhood U of x_0 and thereby find a function $Q(x)$ satisfying

$$\hat{\mathbf{v}}_P(Q) = \{Q, P\} = 1$$

for all $x \in U$. (In the notation of Proposition 1.29, Q would be the coordinate y^1.) Since $\{Q, P\}$ is constant, (6.8) and (6.13) imply that

$$[\hat{\mathbf{v}}_P, \hat{\mathbf{v}}_Q] = \hat{\mathbf{v}}_{\{Q,P\}} = 0$$

for all $x \in U$. On the other hand, $\hat{\mathbf{v}}_Q(Q) = \{Q, Q\} = 0$, so $\hat{\mathbf{v}}_P$ and $\hat{\mathbf{v}}_Q$ form a commuting, linearly independent pair of vector fields defined on U. If we set $p = P(x)$, $q = Q(x)$, then Frobenius' Theorem 1.43 allows us to complete p, q to form a system of local coordinates $(p, q, y^3, \dots, y^m)$ on a possibly smaller neighbourhood $\tilde{U} \subset U$ of x_0 with $\hat{\mathbf{v}}_p = \partial_q$, $\hat{\mathbf{v}}_q = -\partial_p$ there. The bracket relations $\{p, q\} = 1$, $\{p, y^i\} = 0 = \{q, y^i\}$, $i = 3, \dots, m$, imply that the structure matrix takes the form

$$J(p, q, y) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \tilde{J}(p, q, y) \end{pmatrix},$$

where $\tilde{J}$ has entries $\tilde{J}^{ij} = \{y^i, y^j\}$, $i, j = 3, \dots, m$. Finally, we prove that $\tilde{J}$ is actually independent of p and q, and hence forms the structure matrix of a

Poisson bracket in the y variables of rank two less than that of J, from which the induction step is clear. To prove the claim, we just use the Jacobi identity and the above brackets relations; for instance

$$\frac{\partial \tilde{J}^{ij}}{\partial q} = \{\tilde{J}^{ij}, p\} = \{\{y^i, y^j\}, p\} = 0,$$

and similarly for the p derivative. □

Example 6.23. Let us compute the canonical coordinates for the Lie–Poisson bracket on $\mathfrak{so}(3)^*$. According to the proof of Darboux' theorem, we need only find functions $P(u)$, $Q(u)$ whose Poisson bracket is identically 1. Here the function $z = u^3$ generates the rotational vector field $\hat{\mathbf{v}}_3 = u^2\partial_1 - u^1\partial_2$, which can be straightened out using the polar angle $\theta = \arctan(u^2/u^1)$ provided $(u^1, u^2) \neq (0, 0)$. We find $\{\theta, z\} = \hat{\mathbf{v}}_3(\theta) = -1$, hence θ and z will provide canonical coordinates on the symplectic spheres $S^2_\rho = \{|u| = \rho\}$. Indeed, an easy calculation shows that if we re-express $F(u)$ and $H(u)$ in terms of θ, z and ρ, then the Lie–Poisson bracket is simply $\{F, H\} = F_z H_\theta - F_\theta H_z$. In other words, while the symplectic leaves in $\mathfrak{so}(3)^*$ are spheres, canonical coordinates are provided by cylindrical coordinates z, θ!

The Co-adjoint Representation

In the case of a Lie–Poisson bracket on the dual to a Lie algebra $\mathfrak{g}^*$, the induced symplectic foliation has a particularly nice interpretation in terms of the dual to the adjoint representation of the underlying Lie group G on the Lie algebra $\mathfrak{g}$. (See Section 3.3.)

Definition 6.24. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The *co-adjoint action* of a group element $g \in G$ is the linear map $\mathrm{Ad}^* g\colon \mathfrak{g}^* \to \mathfrak{g}^*$ on the dual space satisfying

$$\langle \mathrm{Ad}^* g(\omega); \mathbf{w} \rangle = \langle \omega; \mathrm{Ad}\, g^{-1}(\mathbf{w}) \rangle \tag{6.23}$$

for all $\omega \in \mathfrak{g}^*$, $\mathbf{w} \in \mathfrak{g}$. Here $\langle\ ;\ \rangle$ is the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, and $\mathrm{Ad}\, g$ the adjoint action of g on $\mathfrak{g}$.

If we identify the tangent space $T\mathfrak{g}^*|_\omega$, $\omega \in \mathfrak{g}^*$, with $\mathfrak{g}^*$ itself, and similarly for $\mathfrak{g}$, then the infinitesimal generators of the co-adjoint action are determined by differentiating (6.23):

$$\langle \mathrm{ad}^* \mathbf{v}|_\omega; \mathbf{w} \rangle = -\langle \omega; \mathrm{ad}\, \mathbf{v}|_{\mathbf{w}} \rangle = \langle \omega; [\mathbf{v}, \mathbf{w}] \rangle, \tag{6.24}$$

for $\mathbf{v}, \mathbf{w} \in \mathfrak{g}$, $\omega \in \mathfrak{g}^*$ (cf. (3.21)).

The fundamental result that connects the co-adjoint action with the Lie–Poisson bracket is the following:

Theorem 6.25. *Let G be a connected Lie group with co-adjoint representation Ad^*G on $\mathfrak{g}^*$. Then the orbits of Ad^*G are precisely the leaves of the symplectic foliation induced by the Lie–Poisson bracket on $\mathfrak{g}^*$. Moreover, for each $g \in G$, the co-adjoint map Ad^*g is a Poisson mapping on $\mathfrak{g}^*$ preserving the leaves of the foliation.*

PROOF. Let $\mathbf{v} \in \mathfrak{g}$ and consider the linear function $H(\omega) = H_{\mathbf{v}}(\omega) = \langle \omega; \mathbf{v} \rangle$ on $\mathfrak{g}^*$. Note that for $\omega \in \mathfrak{g}^*$, the gradient $\nabla H(\omega)$, considered as an element of $T^*\mathfrak{g}^*|_\omega \simeq \mathfrak{g}$, is just $\mathbf{v}$ itself. Using the intrinsic definition (6.17) of the Lie–Poisson bracket, we find

$$\begin{aligned} \hat{\mathbf{v}}_H(F)(\omega) &= \{F, H\}(\omega) = \langle \omega; [\nabla F(\omega), \nabla H(\omega)] \rangle \\ &= \langle \omega; [\nabla F(\omega), \mathbf{v}] \rangle = \langle \omega; \operatorname{ad} \mathbf{v}(\nabla F(\omega)) \rangle \\ &= -\langle \operatorname{ad}^* \mathbf{v}(\omega); \nabla F(\omega) \rangle \end{aligned}$$

for any $F\colon \mathfrak{g}^* \to \mathbb{R}$. On the other hand,

$$\hat{\mathbf{v}}_H(F)(\omega) = \langle \hat{\mathbf{v}}_H|_\omega; \nabla F(\omega) \rangle$$

is uniquely determined by its action on all such functions. We conclude that the Hamiltonian vector field determined by the linear function $H = H_{\mathbf{v}}$ coincides, up to sign, with the infinitesimal generator of the co-adjoint action determined by $\mathbf{v} \in \mathfrak{g}$: $\hat{\mathbf{v}}_H = -\operatorname{ad}^* \mathbf{v}$. Thus the corresponding one-parameter groups satisfy

$$\exp(t\hat{\mathbf{v}}_H) = \mathrm{Ad}^*[\exp(-t\mathbf{v})].$$

Proposition 6.16 and the usual connectivity arguments show that $\mathrm{Ad}^* g$ is a Poisson mapping for each $g \in G$.

Moreover, the subspace $\mathscr{H}|_\omega$, $\omega \in \mathfrak{g}^*$, is spanned by the Hamiltonian vector fields $\hat{\mathbf{v}}_H$ corresponding to all such linear functions $H = H_{\mathbf{v}}$, $\mathbf{v} \in \mathfrak{g}$, hence $\mathscr{H}|_\omega = \operatorname{ad}^* \mathfrak{g}|_\omega$ coincides with the space spanned by the corresponding infinitesimal generators $\operatorname{ad}^* \mathbf{v}|_\omega$. Since $\operatorname{ad}^* \mathfrak{g}|_\omega$ is precisely the tangent space to the co-adjoint orbit of G through ω, which is connected, we immediately conclude that this co-adjoint orbit is the corresponding integral submanifold of $\mathscr{H}$. □

Corollary 6.26. *The orbits of the co-adjoint representation of G are even-dimensional submanifolds of $\mathfrak{g}^*$.*

Example 6.27. In the case of the rotation group SO(3), the co-adjoint orbits are the spheres $S^2_\rho \subset \mathfrak{so}(3)^*$ determined in Example 6.20. Indeed, according to Example 3.9, the adjoint representation of a rotation matrix $R \in \mathrm{SO}(3)$ on the Lie algebra $\mathfrak{so}(3) \simeq \mathbb{R}^3$ coincides with the rotation R itself relative to the standard basis: $\mathrm{Ad}\, R(\mathbf{v}) = R\mathbf{v}$, $\mathbf{v} \in \mathfrak{so}(3)$. Thus the co-adjoint action Ad^*R of R on $\mathfrak{so}(3)^*$ has matrix representation $\mathrm{Ad}^*R = (R^{-1})^T = R$ relative to the corresponding dual basis on $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$, and the co-adjoint representation of SO(3) coincides with its usual action on $\mathbb{R}^3$ under the above identifications. In particular, the co-adjoint orbits are precisely the spheres S^2_ρ, $\rho \geqslant 0$.

6.3. Symmetries, First Integrals and Reduction of Order

For a system of ordinary differential equations in Lagrangian form, i.e. the Euler–Lagrange equations associated to some variational problem, Noether's theorem provides a connection between one-parameter variational symmetry groups of the system and conservation laws or first integrals. Moreover, the knowledge of such a first integral allows us to reduce the order of the system by two in the case of a one-parameter symmetry group, indicating that we only need find half as many symmetries as the order of the system in order to integrate it entirely by quadrature. All of these statements carry over to the Hamiltonian framework, and, in fact, arise in a far more natural geometric setting than our original Lagrangian results. In this section we discuss the general theory of symmetry and reduction for finite-dimensional Hamiltonian systems.

First Integrals

Consider a system of ordinary differential equations in Hamiltonian form

$$\frac{dx}{dt} = J(x)\nabla H(x, t), \tag{6.25}$$

where $H(x, t)$ is the Hamiltonian function and $J(x)$ the structure matrix determining the Poisson bracket. In this case, first integrals are readily characterized using the Poisson bracket.

Proposition 6.28. *A function $P(x, t)$ is a first integral for the Hamiltonian system* (6.25) *if and only if*

$$\frac{\partial P}{\partial t} + \{P, H\} = 0 \tag{6.26}$$

for all x, t. In particular, a time-independent function $P(x)$ is a first integral if and only if $\{P, H\} = 0$ everywhere.

PROOF. Let $\mathbf{v}_H$ be the Hamiltonian vector field determining (6.25). Then, by (1.17), if $x(t)$ is any solution to Hamilton's equations,

$$\frac{d}{dt} P(x(t), t) = \frac{\partial P}{\partial t}(x(t), t) + \mathbf{v}_H(P)(x(t), t).$$

Thus $dP/dt = 0$ along solutions if and only if (6.26) holds everywhere.

□

Some first integrals are immediately apparent from the form of (6.26).

Corollary 6.29. *If $x_t = J\nabla H$ is any Hamiltonian system with time-independent Hamiltonian function $H(x)$, then $H(x)$ itself is automatically a first integral.*

Corollary 6.30. *If $x_t = J\nabla H$ is a Hamiltonian system, then any distinguished function $C(x)$ for the Poisson bracket determined by J is automatically a first integral.*

The first integrals supplied by the distinguished functions arise from degeneracies in the Poisson bracket itself and are *not* governed by any intrinsic symmetry properties of the particular Hamiltonian system under investigation. If the Poisson bracket is symplectic, only the constants are distinguished functions and Corollary 6.30 provides no new information. For a Poisson structure of constant rank, the common level sets of the distinguished functions are the leaves of the symplectic foliation, so Corollary 6.30 is just a restatement of Theorem 6.21 that any solution is contained in a single symplectic leaf.

Hamiltonian Symmetry Groups

For systems of Euler–Lagrange equations, first integrals arise from variational symmetry groups; for Hamiltonian systems this role is played by the one-parameter *Hamiltonian symmetry groups* whose infinitesimal generators (in evolutionary form) are Hamiltonian vector fields. First, it is easy to show that any first integral leads to such a symmetry group.

Proposition 6.31. *Let $P(x, t)$ be a first integral of a Hamiltonian system. Then the Hamiltonian vector field $\hat{\mathbf{v}}_P$ determined by P generates a one-parameter symmetry group of the system.*

PROOF. Note first that since the structure matrix $J(x)$ does not depend on t, the Hamiltonian vector field associated with $\partial P/\partial t$ is just the t-derivative $\partial \hat{\mathbf{v}}_P/\partial t$ of that associated with P. Thus the Hamiltonian vector field associated with the combination $\partial P/\partial t + \{P, H\}$ occurring in (6.26) is, using (6.8),

$$\partial \hat{\mathbf{v}}_P/\partial t + [\hat{\mathbf{v}}_H, \hat{\mathbf{v}}_P].$$

If P is a first integral, this last vector field vanishes, which is just the condition (5.26) that $\hat{\mathbf{v}}_P$ generate a symmetry group. □

In particular, if $H(x)$ is time-independent, the associated symmetry group is generated by $\hat{\mathbf{v}}_H$, which is equivalent to the generator ∂_t of the symmetry group of time translations reflecting the autonomy of the Hamiltonian system. For a distinguished function $C(x)$, the corresponding symmetry is trivial: $\hat{\mathbf{v}}_C \equiv 0$.

Example 6.32. Consider the equations of a harmonic oscillator $p_t = -q$, $q_t = p$, which form a Hamiltonian system on $M = \mathbb{R}^2$ relative to the canonical Poisson bracket. The Hamiltonian function $H(q, p) = \frac{1}{2}(p^2 + q^2)$ is thus a first integral, reflecting the fact that the solutions move on the circles $p^2 + q^2 = \text{constant}$.

Not every Hamiltonian symmetry group corresponds directly to a first integral. For example, on $\tilde{M} = M \backslash \{(p, 0): p \leqslant 0\}$, the vector field $\mathbf{w} = -(p^2 + q^2)^{-1}(p\partial_p + q\partial_q)$ generates a symmetry group. Moreover, $\mathbf{w} = \hat{\mathbf{v}}_{\tilde{P}}$ is Hamiltonian for $\tilde{P}(p, q) = \arctan(q/p)$. But $\tilde{P}$ is *not* a first integral; in fact $\tilde{P}(p(t), q(t)) = t + \theta_0$, a linear function of t, whenever $(p(t), q(t))$ solves the system.

The problem here, and more generally, is that there is *not* a one-to-one correspondence between Hamiltonian vector fields and their corresponding Hamiltonian functions. For example, the function $P(p, q, t) = \arctan(q/p) - t$, which *is* a first integral for the oscillator, has the same Hamiltonian vector field $\hat{\mathbf{v}}_P = \mathbf{w} = \hat{\mathbf{v}}_{\tilde{P}}$ as $\tilde{P}$. More generally, we can add any *time-dependent distinguished function* $C(x, t)$ (meaning that for each fixed t, C is a distinguished function) to a given function P without changing the form of its Hamiltonian vector field. Once we recognize the possibility of modifying the function determining a Hamiltonian symmetry group, we can readily prove a converse to the preceding proposition. This forms the Hamiltonian version of Noether's theorem.

Theorem 6.33. *A vector* $\mathbf{w}$ *generates a Hamiltonian symmetry group of a Hamiltonian system of ordinary differential equations if and only if there exists a first integral* $P(x, t)$ *so that* $\mathbf{w} = \hat{\mathbf{v}}_P$ *is the corresponding Hamiltonian vector field. A second function* $\tilde{P}(x, t)$ *determines the same Hamiltonian symmetry if and only if* $\tilde{P} = P + C$ *for some time-dependent distinguished function* $C(x, t)$.

PROOF. The second statement follows immediately from Definition 6.3 of a distinguished function applied to the difference $P - \tilde{P}$. To prove the first part, let $\mathbf{w} = \hat{\mathbf{v}}_{\tilde{P}}$ for some function $\tilde{P}(x, t)$. The symmetry condition (5.26) implies that the Hamiltonian vector field associated with the function $\partial\tilde{P}/\partial t + \{\tilde{P}, H\}$ vanishes everywhere, and hence this combination must be a time-dependent distinguished function $\tilde{C}(x, t)$:

$$\frac{d\tilde{P}}{dt} = \frac{\partial \tilde{P}}{\partial t} + \{\tilde{P}, H\} = \tilde{C}.$$

Set $C(x, t) = \int_0^t \tilde{C}(x, \tau)\, d\tau$, so that C is also distinguished. Moreover, for solutions $x(t)$ of the Hamiltonian system,

$$\frac{dC}{dt} = \frac{\partial C}{\partial t} + \{C, H\} = \tilde{C}.$$

It is now easy to see that the modified function $P = \tilde{P} - C$ has the same Hamiltonian vector field, $\hat{\mathbf{v}}_P = \mathbf{w}$, and provides a first integral: $dP/dt = 0$ on solutions. □

In particular, if the Poisson bracket is symplectic, the only time-dependent distinguished functions are functions $C(t)$ which depend only on t. In this case the theorem states that the Hamiltonian vector field $\hat{\mathbf{v}}_{\tilde{P}}$ generates a symmetry group if and only if there is a function $C(t)$ such that $P(x, t) = \tilde{P}(x, t) - C(t)$ is a first integral. Note that even though both $H(x)$ and $\tilde{P}(x)$ might be t-independent, the first integral $P(x, t) = \tilde{P}(x) - C(t)$ may be required to depend on t! (Indeed, this was precisely the case in Example 6.32.) See Exercise 6.2 for further information on this case.

Example 6.34. The equations of motion of n point masses subject to pairwise potential interactions discussed in Example 4.31 can be put into a canonical Hamiltonian form. We use the positions $\mathbf{q}_i = (x^i, y^i, z^i)$ and momenta $\mathbf{p}_i = (\xi^i, \eta^i, \zeta^i) = m_i\dot{\mathbf{q}}_i$, $i = 1, \ldots, n$, as canonical coordinates. The Hamiltonian function is the total energy

$$H(p, q) = K(p) + U(q) = \sum_{i=1}^{n} \frac{|\mathbf{p}_i|^2}{2m_i} + \sum_{i<j} m_i m_j V(|\mathbf{q}_i - \mathbf{q}_j|),$$

where the potential $V(r)$ depends only on the distance between the two masses. The equations of motion are thus

$$\frac{d\mathbf{p}_i}{dt} = -\frac{\partial U}{\partial \mathbf{q}_i}, \qquad \frac{d\mathbf{q}_i}{dt} = \frac{\mathbf{p}_i}{m_i}, \qquad i = 1, \ldots, n.$$

Several geometrical symmetry groups are immediately apparent. Simultaneous translation of all the masses in a given direction $\mathbf{a} = (a^1, a^2, a^3)$ is generated by the Hamiltonian vector field

$$\hat{\mathbf{v}}_P = \sum_{i=1}^{n} \left(a^1 \frac{\partial}{\partial x^i} + a^2 \frac{\partial}{\partial y^i} + a^3 \frac{\partial}{\partial z^i} \right),$$

and corresponds to the first integral $P = \sum_i \mathbf{a} \cdot \mathbf{p}_i$ representing the linear momentum in the given direction. Similarly, the group SO(3) of simultaneous rotations of the masses about the origin leads to the integrals of angular momentum. For example, $Q = \sum_i (x^i \eta^i - y^i \xi^i)$, the angular momentum about the z-axis, generates the symmetry group

$$\hat{\mathbf{v}}_Q = \sum_{i=1}^{n} \left(x^i \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial x^i} + \xi^i \frac{\partial}{\partial \eta^i} - \eta^i \frac{\partial}{\partial \xi^i} \right)$$

of simultaneous rotations about the z-axis. Besides the six momentum integrals, the constancy of the Hamiltonian function itself implies conservation of energy. Three further first integrals are provided by the uniform motion of the centre of mass, and these lead to three further Hamiltonian symmetry

groups. For example, in the x-direction we have

$$R = \sum_{i=1}^{n} m_i x^i - t \sum_{i=1}^{n} \xi^i = \text{constant},$$

and hence

$$\hat{\mathbf{v}}_R = -\sum_{i=1}^{n} \left(t \frac{\partial}{\partial x^i} + m_i \frac{\partial}{\partial \xi^i} \right)$$

generates a one-parameter symmetry group of Galilean boosts.

Reduction of Order in Hamiltonian Systems

The use of symmetry groups to effect a reduction in order of a Hamiltonian system of ordinary differential equations parallels the methods for Euler–Lagrange equations of Section 4.3, but with the added advantage of an immediate geometrical interpretation. We first remark that if the underlying Poisson bracket is degenerate, we can always restrict attention to a single symplectic leaf. Thus each nonconstant distinguished function will reduce the order of the system by one. Other kinds of first integrals, which generate nontrivial Hamiltonian symmetry groups, can then be used to reduce the order by *two*. For simplicity, we restrict our attention to time-independent first integrals.

Theorem 6.35. *Suppose $\hat{\mathbf{v}}_P \neq 0$ generates a Hamiltonian symmetry group of the Hamiltonian system $\dot{x} = J\nabla H$ corresponding to the time-independent first integral $P(x)$. Then there is a reduced Hamiltonian system involving two fewer variables with the property that every solution of the original system can be determined using a single quadrature from those of the reduced system.*

PROOF. The construction is the same as the initial step in the proof of Darboux' Theorem 6.22. We introduce new variables $p = P(x)$, $q = Q(x)$, $y = (y^1, \dots, y^{m-2}) = Y(x)$ which straighten out the symmetry, so $\hat{\mathbf{v}}_P = \partial_q$ in the (p, q, y)-coordinates. In terms of these coordinates, the structure matrix has the form

$$J(p, q, y) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & a \\ 0 & -a^T & \tilde{J} \end{bmatrix},$$

where $a(p, q, y)$ is a row vector of length $m - 2$ and $\tilde{J}(p, y)$ is an $(m - 2) \times (m - 2)$ skew-symmetric matrix, which is independent of q, and for each fixed value of p is the structure matrix for a Poisson bracket in the y variables. (If $y = (y^1, \dots, y^{m-2})$ are chosen as flat coordinates as in the proof of Darboux' theorem, then $a = 0$ and $\tilde{J}(y)$ is independent of p also, as we saw earlier. However, to effect the reduction procedure this is not necessary, and, indeed,

may be impractical to achieve.) The proofs of the above statements on the form of the structure matrix follow as in the "flat" case.

The reduced system will be Hamiltonian with respect to the reduced structure matrix $\tilde{J}(p, y)$ for any fixed value of the first integral $p = P(x)$. Note that in terms of the (p, q, y) coordinates

$$0 = \{p, H\} = -\hat{\mathbf{v}}_P(H) = -\partial H/\partial q,$$

hence $H = H(p, y)$ also only depends on p and y. Therefore Hamilton's equations take the form

$$\frac{dp}{dt} = 0, \tag{6.27a}$$

$$\frac{dq}{dt} = -\frac{\partial H}{\partial p} + \sum_{j=1}^{m-2} a^j(p, y)\frac{\partial H}{\partial y^j}, \tag{6.27b}$$

$$\frac{dy^i}{dt} = \sum_{j=1}^{m-2} \tilde{J}^{ij}(p, y)\frac{\partial H}{\partial y^j}, \qquad i = 1, \dots, m-2. \tag{6.27c}$$

The first equation says that p is a constant (as it should be). Fixing a value of p, we see that the $(m-2)$ equations (6.27c) form a Hamiltonian system relative to the reduced structure matrix $\tilde{J}(p, y)$ and the Hamiltonian function $H(p, y)$; this is the reduced system referred to in the statement of the theorem. Finally, (6.27b), which governs the time evolution of the remaining coordinate q, can be integrated by a single quadrature once we know the solution to the reduced system (6.27c), since the right-hand side does not depend on q. □

Example 6.36. Let $M = \mathbb{R}^4$ with canonical Poisson bracket and consider a Hamiltonian function of the form

$$H(p_1, p_2, q_1, q_2) = \tfrac{1}{2}(p_1^2 + p_2^2) + V(q_1 - q_2).$$

The corresponding Hamiltonian system

$$\frac{dq_1}{dt} = p_1, \quad \frac{dq_2}{dt} = p_2, \quad \frac{dp_1}{dt} = -V'(q_1 - q_2), \quad \frac{dp_2}{dt} = V'(q_1 - q_2), \tag{6.28}$$

determines the motion of two particles of unit mass on a line whose interaction comes from a potential $V(r)$ depending on their relative displacements. This system admits an obvious translational invariance $\mathbf{v} = \partial_{q_1} + \partial_{q_2}$; the corresponding first integral is the linear momentum $p_1 + p_2$. According to the theorem, we can reduce the order of the system by two if we introduce new coordinates

$$p = p_1 + p_2, \qquad q = q_1, \qquad y = p_1, \qquad r = q_1 - q_2,$$

which straighten out $\mathbf{v} = \partial_q$. In these variables, the Hamiltonian function is

$$H(p, y, r) = y^2 - py + \tfrac{1}{2}p^2 + V(r), \tag{6.29}$$