then we have the homotopy-like formula

$$\exp(\varepsilon\mathbf{v})^*[\omega|_{\exp(\varepsilon\mathbf{v})x}] - \omega|_x = dh_{\mathbf{v}}^{\varepsilon}(\omega)|_x + h_{\mathbf{v}}^{\varepsilon}(d\omega)|_x, \tag{1.67}$$

which is valid for any manifold M, any differential form ω, any vector field $\mathbf{v}$ and all $\varepsilon \in \mathbb{R}$ such that $\exp(\varepsilon\mathbf{v})x$ is defined.

We are now in a position to prove the Poincaré lemma (Theorem 1.61) by constructing a homotopy operator over the star-shaped domain $M \subset \mathbb{R}^m$. Note that the scaling vector field $\mathbf{v}_0 = \sum x^i \partial/\partial x^i$ has flow $\exp(\varepsilon\mathbf{v})x = e^{\varepsilon}x$, which, for $x \in M$, remains in M for all $\varepsilon \leqslant 0$. If $\omega = \sum \alpha_I(x)\, dx^I$ is a k-form defined on all of M, then for $\varepsilon \leqslant 0$,

$$\exp(\varepsilon\mathbf{v}_0)^*[\omega|_{\exp(\varepsilon\mathbf{v}_0)x}] = \sum_I \alpha_I(e^{\varepsilon}x)e^{k\varepsilon}\, dx^I,$$

since $\exp(\varepsilon\mathbf{v}_0)^*(dx^i) = d(e^{\varepsilon}x^i) = e^{\varepsilon}\, dx^i$. We can write this formula in a simpler manner if we denote ω by $\omega[x]$, whereby

$$\exp(\varepsilon\mathbf{v}_0)^*\omega[x] = \omega[e^{\varepsilon}x] = \sum \alpha_I(e^{\varepsilon}x)d(e^{\varepsilon}x^{i_1}) \wedge \cdots \wedge d(e^{\varepsilon}x^{i_k}).$$

(In other words, we substitute $e^{\varepsilon}x^i$ for each x^i wherever it occurs in ω, including the differentials dx^i.) In this special case, (1.67) with $\mathbf{v} = \mathbf{v}_0$ reads

$$\omega[e^{\varepsilon}x] - \omega[x] = dh_0^{\varepsilon}(\omega) + h_0^{\varepsilon}(d\omega), \tag{1.68}$$

where, for $\varepsilon \leqslant 0$,

$$h_0^{\varepsilon}(\omega) \equiv \int_0^{\varepsilon} (\mathbf{v}_0 \,\lrcorner\, \omega)[e^{\tilde{\varepsilon}}x]\, d\tilde{\varepsilon} = -\int_{\exp\varepsilon}^{1} (\mathbf{v}_0 \,\lrcorner\, \omega)[\lambda x]\frac{d\lambda}{\lambda},$$

(using the change of variables $\lambda = e^{\tilde{\varepsilon}}$). Now let $\varepsilon \to -\infty$. If ω is a k-form, and $k > 0$, then $\omega[e^{\varepsilon}x] \to 0$ as $\varepsilon \to -\infty$. Thus (1.68) reduces to the homotopy formula (1.66) with homotopy operator

$$h(\omega) = \int_0^1 (\mathbf{v}_0 \,\lrcorner\, \omega)[\lambda x]\frac{d\lambda}{\lambda}. \tag{1.69}$$

(Note that in this formula, we first compute the interior product $\mathbf{v}_0 \,\lrcorner\, \omega$ and then evaluate at λx.) If, however, $k = 0$, so ω is a smooth function $f(x)$, (1.68) reduces to the alternative formula

$$f(x) - f(0) = dh(f) + h(df) = h(df)$$

in the limit as $\varepsilon \to -\infty$, leading to the initial injection $\mathbb{R} \to \bigwedge_0$ in the de Rham complex. We have thus completed the proof of the Poincaré lemma.

Example 1.67. Consider a planar star-shaped domain $M \subset \mathbb{R}^2$. If

$$\omega = \alpha(x, y)\, dx + \beta(x, y)\, dy$$

is any one-form, then

$$\mathbf{v}_0 \,\lrcorner\, \omega = (x\partial_x + y\partial_y) \,\lrcorner\, \omega = x\alpha(x, y) + y\beta(x, y).$$

Therefore, the function $h(\omega)$ obtained by applying our homotopy operator (1.69) to ω is

$$h(\omega) = \int_0^1 \{\lambda x \alpha(\lambda x, \lambda y) + \lambda y \beta(\lambda x, \lambda y)\} \frac{d\lambda}{\lambda}$$
$$= \int_0^1 \{x \alpha(\lambda x, \lambda y) + y \beta(\lambda x, \lambda y)\}\, d\lambda.$$

Similarly, applying h to a two-form leads to the one-form

$$h[\gamma(x, y)\, dx \wedge dy] = \int_0^1 \{\lambda^2 x \gamma(\lambda x, \lambda y)\, dy - \lambda^2 y \gamma(\lambda x, \lambda y)\, dx\} \frac{d\lambda}{\lambda}$$
$$= -\left\{\int_0^1 \lambda y \gamma(\lambda x, \lambda y)\, d\lambda\right\} dx + \left\{\int_0^1 \lambda x \gamma(\lambda x, \lambda y)\, d\lambda\right\} dy,$$

the differentials dx and dy not being affected by the λ-integration. In particular, for the above one-form,

$$d\omega = (\beta_x - \alpha_y)\, dx \wedge dy,$$

so the homotopy formula (1.66) reduces to two formulae, for α and β, the first of which is

$$\alpha(x, y) = \frac{\partial}{\partial x} \int_0^1 \{x \alpha(\lambda x, \lambda y) + y \beta(\lambda x, \lambda y)\}\, d\lambda$$
$$- \int_0^1 \lambda y [\beta_x(\lambda x, \lambda y) - \alpha_y(\lambda x, \lambda y)]\, d\lambda.$$

The reader may enjoy directly verifying this latter statement. In particular, if $d\omega = 0$, then $\omega = df$ where $f = h(\omega)$ is as above. (A similar kind of result holds for two-forms.)

Integration and Stokes' Theorem

Although it is not a subject central to the theme of this book, it would be unfair to omit a brief discussion of integration and Stokes' theorem from our introduction to differential forms. Indeed, differential forms arise as "the objects one integrates on manifolds". To define integration, we need to first *orient* the m-dimensional manifold M with a nonvanishing m-form ω defined over all of M. A second nonvanishing m-form $\tilde{\omega}$ defines the same orientation if it is a positive scalar multiple of ω at each point. There are precisely two orientations on such a manifold M. (Not every manifold is orientable, for instance, a Möbius band is not.) In particular, we can orient $\mathbb{R}^m$ (and any open subset thereof) by choosing the volume form $dx^1 \wedge \cdots \wedge dx^m$. A map

$F: \tilde{M} \to M$ between two oriented m-dimensional manifolds is *orientation-preserving* if the pull-back of the orientation form on M determines the same orientation on $\tilde{M}$ as the given one. If M is oriented, then we can cover M by orientation-preserving coordinate charts $\chi_\alpha: U_\alpha \to V_\alpha$ whose overlap functions $\chi_\beta \circ \chi_\alpha^{-1}$ are orientation-preserving diffeomorphisms on $\mathbb{R}^m$.

If M is an oriented m-dimensional manifold, we can define the integral $\int_M \omega$ of any m-form ω on M. In essence, we chop up M into component *oriented* coordinate charts and add up the individual integrals

$$\int_{U_\alpha} \omega = \int_{V_\alpha} (\chi_\alpha^{-1})^* \omega = \int_{V_\alpha} f(x)\, dx^1 \wedge \cdots \wedge dx^m,$$

the latter integral being an ordinary multiple integral over $V_\alpha \subset \mathbb{R}^m$. The change of variables formula for multiple integrals assures us that this definition is coordinate-free. More generally, $\int_M \omega = \int_{\tilde{M}} F^*\omega$ whenever $F: \tilde{M} \to M$ is orientation-preserving.

Stokes' theorem relates integrals of m-forms over a compact m-dimensional manifold M to integrals of $(m-1)$-forms over the boundary ∂M. The simplest manifold with boundary is the upper half space $\mathbb{H}^m \equiv \{(x^1, \ldots, x^m): x^m \geq 0\}$ of $\mathbb{R}^m$, with $\partial \mathbb{H}^m = \{(x^1, \ldots, x^{m-1}, 0)\} \simeq \mathbb{R}^{m-1}$. Any other manifold with boundary is defined using coordinate charts $\chi_\alpha: U_\alpha \to V_\alpha$ where $V_\alpha \subset \mathbb{H}^m$ is open, meaning $V_\alpha = \mathbb{H}^m \cap \tilde{V}_\alpha$ where $\tilde{V}_\alpha \subset \mathbb{R}^m$ is open. The boundary of the chart is $\partial U_\alpha = \chi_\alpha^{-1}[\partial V_\alpha]$, $\partial V_\alpha = V_\alpha \cap \partial \mathbb{H}^m$, and ∂M is the union of all such boundaries of coordinate charts. Thus ∂M is a smooth $(m-1)$-dimensional manifold, without boundary.

The boundary of $\mathbb{H}^m$ is given an "induced" orientation $(-1)^m\, dx^1 \wedge \cdots \wedge dx^{m-1}$ from the volume form $dx^1 \wedge \cdots \wedge dx^m$ determining the orientation of $\mathbb{H}^m$ itself. If M is an oriented manifold with boundary, then ∂M inherits an induced orientation so that any oriented coordinate chart $\chi_\alpha: U_\alpha \to V_\alpha$ on M restricts to an oriented coordinate chart $\partial\chi_\alpha: \partial U_\alpha \to \partial V_\alpha$ on ∂M. With these definitions, we can state the general form of Stokes' theorem.

Theorem 1.68. *Let M be a compact, oriented, m-dimensional manifold with boundary ∂M. Let ω be a smooth $(m-1)$-form defined on M. Then $\int_{\partial M} \omega = \int_M d\omega$.*

Using the identification of the differential d with the usual vector differential operations in $\mathbb{R}^3$, the reader can check that Theorem 1.68 reduces to the usual forms of Stokes' theorem and the Divergence theorem of vector calculus. More generally, there is an intimate connection between the de Rham complex, Stokes' theorem and the underlying topology of M, but this would lead us too far afield to discuss any further, and so we conclude our brief introduction to this subject.

Notes

In this chapter we have only been able to give the briefest of introductions to the vast and important subjects of Lie groups and differentiable manifolds. There are a number of excellent books which can be profitably studied by the reader interested in delving further into these areas, including those by Warner, [1], Boothby, [1], Thirring, [1; Chap. 2] and Miller, [2]. Pontryagin, [1], is useful as a reference for the local Lie group approach to the subject and includes many otherwise hard-to-find proofs of important theorems. Many other works could be mentioned as well.

Historically, the two subjects of differentiable manifolds and Lie groups have been closely intertwined throughout their development, each inspiring further work in the other. Lie himself, though, drew his original motivation from the spectacular success of Galois' group theory applied to the solution of polynomial equations and sought to erect a similar theory for the solution of differential equations using *his* theory of continuous groups. Although Lie fell sort of this goal (the more refined Picard–Vessiot theory being the correct "Galois theory of differential equations"—see Pommaret, [2]), his seminal influence in all aspects of the subject continues to this day. Remarkably, Lie arrived at the fundamental concept of a (local) Lie group through his research into the analysis of systems of partial differential equations (predating Frobenius), leading to the concept of a "function group," which we will encounter under the name "Poisson structure" in Chap. 6, and then, finally, to Lie groups. The interested reader is well advised to look up Hawkins' fascinating and illuminating historical essays, [1], [2], [3], on the early history of Lie groups.

In Lie's time, all Lie groups were local groups and arose concretely as groups of transformations on some Euclidean space. The global, abstract approach was quite slow in maturing, and the first modern definition of a manifold with coordinate charts appears in Cartan, [2]. (Cartan himself played a fundamental role in the history of Lie groups; his definition of manifold was inspired by Weyl's book, [1], on Riemann surfaces as well as closely related ideas of Schreier, [1].) The passage from the local Lie group to the present-day definition using manifold theory was also accomplished by Cartan, [2]. Cartan also introduced the concept of the simply-connected covering group of a Lie group and noted, in [3], that the simply-connected covering group of $SL(2, \mathbb{R})$ is not a subgroup of any matrix group $GL(n)$. (Interestingly, there is a realization of this global group as an open subset of $\mathbb{R}^3$ due to Bargmann, [1].) A more accessible example of a Lie group which cannot be realized as a group of matrices can be found in Birkhoff, [1].

Lie's fundamental tool in his theory was the infinitesimal form of a Lie group, now called the Lie algebra. In its local version, the correspondence between a Lie group and the right- (or left-)invariant vector fields forming its Lie algebra is known as the first fundamental theorem of Lie. The reconstruction of a local Lie group from its Lie algebra is known as Lie's second fundamental theorem; a proof not relying on Ado's theorem can be found in Pontryagin, [1; Theorem 89]. The construction of a global Lie group from its Lie algebra, though, is due to Cartan, [3]; see also Pontryagin, [1; Theorem 96]. The proof based on the contemporaneous theorem of Ado [1] (see also Jacobson, [1; Chap. 6]), outlined here is more recent. Lie's third fundamental theorem states that the structure constants determine the Lie algebra, and hence the Lie group. The complete proof of the general correspondence between subgroups of a Lie group and subalgebras of its Lie algebra can be found in Warner, [1; Theorems 3.19 and 3.28]. Theorem 1.19 on closed subgroups of Lie groups is due to Cartan, [2]; see Warner, [1; Theorem 3.42] for the proof. Theorem 1.57 on the reconstruction of a transformation group from its infinitesimal generators dates back to Lie; see Pontryagin, [1; Theorem 98] for a proof. The definition used here of a regular group of transformations is based on the monograph of Palais, [1], and is further developed in Chapter 3.

While vector fields have their origins in the study of mathematical physics, the modern geometrical formulation owes much to the work of Poincaré, [1], whose influence, like Lie's, pervades the entire subject. The notation for a vector field employed here and throughout modern differential geometry, however, comes from Lie's notation for the infinitesimal generators of a group of transformations. Flows of vector fields arise, naturally enough, in fluid mechanics; see Wilczynski, [1], for an early connection between their physical and group-theoretic interpretations.

Frobenius' Theorem 1.43 originally appears as a theorem on the nature of the solutions to certain systems of homogeneous, first order, linear partial differential equations; see Frobenius, [1], and the discussion of invariants in Section 2.1. Its translation into a theorem a differential geometry first appears in Chevalley's influential book, [1], on Lie groups. (In this book, most of the modern definitions and theorems in this subject are assembled together for the first time.) A proof of the semi-regular version of Frobenius' theorem can be found in Narasimhan, [1], and Warner, [1; Theorem 1.64], and, using a modern method due to Weinstein, in Abraham and Marsden, [1; p. 93]. The extension of this result to systems of vector fields of varying rank is due to Hermann, [1], [2]. This result has subsequently been generalized much further—see Sussmann, [1]—but much work, especially on the structure of the singular sets, remains to be done. In these and other references, the terms "distribution" or "differential system" have been applied to what we have simply called a system of vector fields. (The former is especially confusing as it appears in a completely unrelated context in functional analysis.) Furthermore, our term "integrable" is more commonly referred to as "completely integrable", but this latter term has very different connotations in the study of Hamiltonian systems, which motivates our choice of the former.

Differential forms have their origins in the work of Grassmann and the attempts to find a multi-dimensional generalization of Stokes' theorem. In the hands of Poincaré and Cartan they became a powerful tool for the study of differential geometry, topology and differential equations. Already in Poincaré, [1], we find the basic concepts of wedge product (p. 25), interior product (p. 33), the differential and the multi-dimensional form of Stokes' theorem (p. 10) as well as the lemma bearing his name. See also Cartan, [1], for further developments and applications to differential equations. The concept of the Lie derivative, though, while in essence due to Poincaré (see also Cartan, [1; p. 82]), was first formally defined by Schouten and his coworkers; see Schouten and Struik, [1; p. 142]. This last reference also contains the basic homotopy formula proof of the Poincaré lemma for the first time. Finally, the connections with topology stemming from de Rham's theorem can be found in Warner, [1; Chap. 5], and Bott and Tu, [1].

EXERCISES

1.1. *Real projective m-space* is defined to be the set of all lines through the origin in $\mathbb{R}^{m+1}$. Specifically, we define an equivalence relation on $\mathbb{R}^{m+1}\setminus\{0\}$ by setting $x \sim y$ if and only if $x = \lambda y$ for some nonzero scalar λ. Then $\mathbb{RP}^m$ is the set of equivalence classes in $\mathbb{R}^{m+1}\setminus\{0\}$.
(a) Prove that $\mathbb{RP}^m$ is a manifold of dimension m by exhibiting coordinate charts.
(b) Prove that $\mathbb{RP}^1 \simeq S^1$ are the same smooth manifolds, and exhibit a diffeomorphism.
(c) Let S^m be the unit sphere in $\mathbb{R}^{m+1}$. Prove that the map $F: S^m \to \mathbb{RP}^m$ which associates to $x \in S^m$ its equivalence class in $\mathbb{RP}^m$ is a smooth covering map. What is the inverse image $F^{-1}\{z\}$ of a point $z \in \mathbb{RP}^m$?

*1.2. *Grassmann Manifolds*: Let $0 < m < n$.
(a) Prove that the space $\mathrm{GL}(m, n)$ of all $m \times n$ matrices of maximal rank is an analytic manifold of dimension $m \cdot n$.
(b) Let $\mathrm{Grass}(m, n)$ denote the set of all m-dimensional subspaces of $\mathbb{R}^n$. Show that $\mathrm{Grass}(m, n)$ can be given the structure of an analytic manifold of dimension $m(n - m)$. (*Hint*: To any basis of such an m-dimensional subspace, associate the matrix in $\mathrm{GL}(m, n)$ whose rows are the basis. Show that the basis can be chosen so that this matrix has the same m columns as the identity matrix; the remaining entries will give local coordinates for $\mathrm{Grass}(m, n)$.)
(c) Let $F: \mathrm{GL}(m, n) \to \mathrm{Grass}(m, n)$ be the map assigns to a matrix A the subspace of $\mathbb{R}^n$ spanned by its rows. Prove that F is an analytic map between manifolds.
(d) Prove that $\mathrm{Grass}(m, n)$ and $\mathrm{Grass}(n - m, n)$ are diffeomorphic manifolds. In particular, $\mathrm{Grass}(1, n) \simeq \mathrm{Grass}(n - 1, n) \simeq \mathbb{RP}^{n-1}$.

1.3. Let $\phi(t) = ((\sqrt{2} + \cos 2t)\cos 3t,\ (\sqrt{2} + \cos 2t)\sin 3t, \sin 2t)$ for $0 \leqslant t \leqslant 2\pi$. Prove that the image of ϕ is a regular closed curve in $\mathbb{R}^3$—a "trefoil knot".

1.4. Let

$$\phi(u, v) = \left(2\cos u + v \sin\frac{u}{2}\cos u, 2\sin u + v\sin\frac{u}{2}\sin u, v\cos\frac{u}{2}\right)$$

for $0 \leqslant u < 2\pi$, $-1 < v < 1$. Prove that ϕ is a regular immersion, whose image is a Möbius band in $\mathbb{R}^3$.

1.5. Prove that if $N \subset M$ is a compact submanifold, then N is a regular submanifold. (Boothby, [1; page 79].)

1.6. Prove that the m-dimensional sphere S^m is simply connected if $m \geqslant 2$. What about real projective space $\mathbb{RP}^m$? (See Exercise 1.1.)

1.7. Let $M = \mathbb{R}^2 \backslash \{0\}$. Prove that $(x, y) \mapsto (e^x \cos y, e^x \sin y)$ defines a covering map from $\mathbb{R}^2$ onto M, hence $\mathbb{R}^2$ is the simply-connected cover of $\mathbb{R}^2 \backslash \{0\}$.

1.8. Let $M = \mathbb{R}^2 \backslash \{0\}$. Prove that $\Psi(\varepsilon, (r, \theta)) = (re^{-\varepsilon} + (1 - e^{-\varepsilon}), \theta + \varepsilon)$, $\varepsilon \in \mathbb{R}$, written in polar coordinates, determines a one-parameter group of transformations. What is its infinitesimal generator? Prove that every orbit is a regular submanifold of M, but the group action is *not* regular.

1.9. Consider the system of vector fields

$$\mathbf{v}_1 = x\partial_y - y\partial_x + z\partial_w - w\partial_z, \qquad \mathbf{v}_2 = z\partial_x - x\partial_z + w\partial_y - y\partial_w,$$

on the unit sphere $S^3 \subset \mathbb{R}^4$.

(a) Prove that $\{\mathbf{v}_1, \mathbf{v}_2\}$ form an integrable system. What are the integral submanifolds in S^3?

(b) Let $\pi: S^3 \backslash \{(0, 0, 0, 1)\} \to \mathbb{R}^3$ be stereographic projection (as in Example 1.3). What are the vector fields $d\pi(\mathbf{v}_1)$ and $d\pi(\mathbf{v}_2)$ on $\mathbb{R}^3$? What are their integral submanifolds?

1.10. Is it possible to construct a system of three vector fields $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ on $\mathbb{R}^3$ such that $[\mathbf{u}, \mathbf{v}] = 0 = [\mathbf{u}, \mathbf{w}]$, but $[\mathbf{v}, \mathbf{w}] \neq 0$? Is it possible to construct an integrable system with the above commutation relations? If so, what would the integral submanifolds of such a system look like?

1.11. Prove that the vector field

$$\mathbf{v} = (-y - 2z(x^2 + y^2))\partial_x + x\partial_y + x(x^2 + y^2 - z^2 - 1)\partial_z$$

does not form a regular system on $\mathbb{R}^3$. Prove that any integral curve of $\mathbf{v}$ lies in one of the tori looked at in Example 1.42. Prove that the flow generated by $\mathbf{v}$, when restricted to one of the above tori, is isomorphic to either the rational or irrational flow on the torus, depending on the size of the torus.

1.12. Suppose $\mathbf{v}$ is a smooth linear map on the space of smooth functions defined near a point $x \in M$ which satisfies (1.20–1.21). Prove that $\mathbf{v}$ is a tangent vector to M at x. (Warner, [1; p. 12].)

*1.13. Let $M = \mathbb{R}^2$ and consider the system of vector fields $\mathscr{H}$ spanned by $\mathbf{v}_0 = \partial_x$ and all vector fields of the form $f(x)\partial_y$ where $f: \mathbb{R} \to \mathbb{R}$ is any smooth function such that all derivatives $f^{(n)}(0) = 0$ vanish for all $n = 0, 1, 2, \ldots$.

(a) Prove that $\mathscr{H}$ is involutive.

(b) Prove that $\mathscr{H}$ has *no* integral submanifold passing through any point $(0, y)$ on the y-axis.

(c) How do you reconcile this with Frobenius' Theorem 1.40 or 1.41? (Nagano, [1].)

*1.14. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be a finite, involutive system of vector fields on a manifold M. Prove that the system is always rank-invariant. (Thus Theorem 1.40 is a special case of Theorem 1.41). (Hermann, [1].)

1.15. Prove that the set of all nonsingular upper triangular matrices forms a Lie group $\mathrm{T}(n)$. What is its Lie algebra?

1.16. Consider the $2n \times 2n$ matrix

$$\mathrm{J} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix},$$

where each I is an $n \times n$ identity matrix. The symplectic group $\mathrm{Sp}(n)$ is defined to be the set of all $2n \times 2n$ matrices A such that $A^T \mathrm{J} A = \mathrm{J}$. Prove that $\mathrm{Sp}(n)$ is a Lie group and compute its dimension. What is its Lie algebra?

1.17. Prove that if $H \subset G$ is a connected one-parameter subgroup of a Lie group G then H is isomorphic to either $\mathrm{SO}(2)$ or $\mathbb{R}$.

1.18. Prove that if G and H are Lie groups, then their Cartesian product $G \times H$ is also a Lie group.

1.19. Let G and H be Lie groups and suppose G acts (globally) on H as a group of transformations, via $h \mapsto g \cdot h$, $g \in G$, $h \in H$, with $g \cdot (h_1 \cdot h_2) = (g \cdot h_1) \cdot (g \cdot h_2)$. Define the *semi-direct product* of G and H, denoted $G \ltimes H$, to be the Lie group whose manifold structure is just the Cartesian product $G \times H$, but whose group multiplication is given by

$$(g, h) \cdot (\tilde{g}, \tilde{h}) = (g \cdot \tilde{g}, h \cdot (g \cdot \tilde{h})).$$

(a) Prove that $G \ltimes H$ is a Lie group.
(b) How is the Lie algebra of $G \ltimes H$ related to those of G and H?
(c) Prove that the Euclidean group $\mathrm{E}(m)$, consisting of all the translations and rotations of $\mathbb{R}^m$, is a semi-direct product of the rotation group $\mathrm{SO}(m)$ with the vector group $\mathbb{R}^m$, $\mathrm{SO}(m)$ acting on $\mathbb{R}^m$ as a group of rotations. (See also Exercise 1.29.)

1.20. Let $V = \{(x, y) : |x| < 1\} \subset \mathbb{R}^2$ and define the map $m: V \times V \to \mathbb{R}^2$ by

$$m(x, y; z, w) = (xz + x + z, xw + w + y(z + 1)^{-1}), \qquad (x, y), (z, w) \in V.$$

Prove that m determines a multiplication map making V into a local Lie group by constructing an inverse map $i: V_0 \to V$ on a suitable subdomain V_0. What is the Lie algebra of V?

1.21. Prove that every two-dimensional Lie algebra is either (a) abelian (all brackets are 0) or (b) isomorphic to the Lie algebra with basis $\{\mathbf{v}, \mathbf{w}\}$ satisfying $[\mathbf{v}, \mathbf{w}] = \mathbf{w}$. Find a 2×2 matrix representation of the Lie algebra in part (b). Find the corresponding simply-connected Lie group. Construct a local group isomorphism from the local Lie group of Exercise 1.20 to this global Lie group. (Jacobson, [1; p. 11].)

1.22. Prove that $\mathbb{R}^3$ forms a Lie algebra with Lie bracket determined by the vector cross product: $[\mathbf{v}, \mathbf{w}] = \mathbf{v} \times \mathbf{w}$, $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. What are the structure constants for this Lie algebra with respect to the standard basis of $\mathbb{R}^3$? Prove that this Lie algebra is isomorphic to $\mathfrak{so}(3)$, the Lie algebra of the three-dimensional rotation group. Show that the isomorphism can be constructed so that a given vector $\mathbf{v} \in \mathbb{R}^3$ corresponds to the infinitesimal generator of the one-parameter group of right-handed rotations about the axis in the direction of $\mathbf{v}$.

*1.23. Prove that every complex Lie group contains a two-dimensional subgroup. Is the same true for real Lie groups? (Cohen, [1; p. 50].)

*1.24. A subgroup H of a group G is called *normal* if for every $h \in H$ and every $g \in G$, $g^{-1}hg \in H$. Let G/H denote the set of equivalence classes of G, where g and $\hat{g}$ are equivalent if and only if $g = \hat{g}h$ for some $h \in H$.

(a) Prove that if $H \subset G$ is normal, then G/H can be given the structure of a group in a natural way.

(b) Prove that a Lie subgroup H of a Lie group G is normal if and only if its Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ has the property that $[\mathbf{v}, \mathbf{w}] \in \mathfrak{h}$ whenever $\mathbf{v} \in \mathfrak{g}$ and $\mathbf{w} \in \mathfrak{h}$.

(c) Prove that if $H \subset G$ is a normal Lie subgroup, the *quotient group* G/H is a Lie group with Lie algebra $\mathfrak{g}/\mathfrak{h}$. Explain.

(d) Find all normal subgroups of the two-dimensional Lie groups of Exercise 1.21.

(e) Does SO(3) have any normal subgroups?

*1.25. Let G be a Lie group. The *commutator subgroup* H is defined to be the subgroup generated by the elements $ghg^{-1}h^{-1}$ for $g, h \in G$.

(a) Prove that H is a Lie subgroup of G, and that the Lie algebra of H is the *derived subalgebra* of $\mathfrak{g}$, given by $\mathfrak{h} = \{[\mathbf{v}, \mathbf{w}]: \mathbf{v}, \mathbf{w} \in \mathfrak{g}\}$.

(b) Prove that the commutator subgroup of SO(3) is SO(3) itself. What about SO(m)?

(c) What are the commutator subgroups of the two-dimensional Lie groups discussed in Exercise 1.21?

(d) Is every element in H of the form $ghg^{-1}h^{-1}$?

1.26. Prove Proposition 1.24. (*Hint*: Show that $\bigcup U^k$ is both open and closed in G.) (Warner, [1; p. 93].)

1.27. Let G be a Lie group with Lie algebra $\mathfrak{g}$.

(a) Prove that the exponential map $\exp\colon \mathfrak{g} \to G$ is a local diffeomorphism from a neighbourhood of $0 \in \mathfrak{g}$ to a neighbourhood of the identity in G.

(b) Prove the "normal coordinate" formula (1.40). (Warner, [1; pp. 103, 109].)

*1.28. Let SL(2) denote the Lie group of 2×2 real matrices of determinant $+1$, and $\mathfrak{sl}(2)$ its Lie algebra.

(a) Let $A \in \mathfrak{sl}(2)$. Prove that

$$\exp(A) = \begin{cases} (\cosh \delta)\mathrm{I} + (\delta^{-1} \sinh \delta)A, & \delta = \sqrt{-\det A}, \quad \det A < 0, \\ (\cos \delta)\mathrm{I} + (\delta^{-1} \sin \delta)A, & \delta = \sqrt{\det A}, \quad \det A > 0. \end{cases}$$

What about the case $\det A = 0$?

(b) Consider the matrix $M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathrm{SL}(2)$, where $\lambda \neq 0$. Prove that M lies on exactly one one-parameter subgroup of SL(2) if $\lambda > 0$, on infinitely many one-parameter subgroups if $\lambda = -1$, and on *no* one-parameter subgroup if $-1 \neq \lambda < 0$. (This shows that the exponential map $\exp\colon \mathfrak{g} \to G$ is, in general, neither one-to-one nor onto!) (Helgason, [1; p. 126].)

*1.29. A diffeomorphism $\psi: \mathbb{R}^m \to \mathbb{R}^m$ is called an *isometry* if it preserves distance, i.e. $|d\psi(\mathbf{v})| = |\mathbf{v}|$ for all $\mathbf{v} \in T\mathbb{R}^m|_x$, $x \in \mathbb{R}^m$, where $|\cdot|$ is the usual Euclidean metric $\sum (dx^i)^2$, i.e.

$$|\mathbf{v}|^2 = \sum (\xi^i)^2, \qquad \mathbf{v} = \sum \xi^i \partial/\partial x^i.$$

(a) Prove that a vector field $\mathbf{v} = \sum \xi^i(x)\partial/\partial x^i$ on $\mathbb{R}^m$ generates a one-parameter group of isometries if and only if its coefficient functions satisfy the system of partial differential equations

$$\frac{\partial \xi^i}{\partial x^j} + \frac{\partial \xi^j}{\partial x^i} = 0, \qquad i \neq j, \qquad \frac{\partial \xi^i}{\partial x^i} = 0, \quad i = 1, \ldots, m.$$

(b) Prove that the (connected) group of isometries of $\mathbb{R}^m$, called the Euclidean group $\mathrm{E}(m)$, is generated by translations and rotations, and hence is an $m(m + 1)/2$-dimensional Lie group.

(c) What if $|\cdot|$ is replaced by some non-Euclidean metric? For example, consider the Lorentz metric $(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2$ on $\mathbb{R}^4$. (Eisenhart, [1; Chap. 6].)

*1.30. A diffeomorphism $\psi: \mathbb{R}^m \to \mathbb{R}^m$ is called a *conformal transformation* if $|d\psi(\mathbf{v})| = \lambda(x)|\mathbf{v}|$ for all $\mathbf{v} \in T\mathbb{R}^m|_x$, $x \in \mathbb{R}^m$, where λ is some scalar-valued function of x, and $|\mathbf{v}|$ is as in Exercise 1.29.

(a) Prove that a vector field $\mathbf{v} = \sum \xi^i(x)\partial/\partial x^i$ on $\mathbb{R}^m$ generates a one-parameter group of conformal transformations if and only if it satisfies

$$\frac{\partial \xi^i}{\partial x^j} + \frac{\partial \xi^j}{\partial x^i} = 0, \quad i \neq j, \qquad \frac{\partial \xi^i}{\partial x^i} = \mu(x), \quad i = 1, \ldots, m, \tag{$*$}$$

for some undetermined function $\mu(x)$.

(b) Prove that if $m \geqslant 3$ then the conformal group of $\mathbb{R}^m$ is an $(m + 1) \times (m + 2)/2$-dimensional Lie group. Find its infinitesimal generators. (*Hint*: Prove that $(*)$ implies that *all* third order derivatives of the coefficient functions are identically zero.) What about $m = 2$?

(c) Prove that the inversion $I(x) = x/|x|^2$, $0 \neq x \in \mathbb{R}^m$ is a conformal transformation on $\mathbb{R}^m \backslash \{0\}$.

(d) Prove that for $m \geqslant 3$, the group of conformal transformations is generated by the groups of translations and rotations, the scaling group $x \mapsto \lambda x$, $\lambda > 0$, and the inversion of part (c).

(e) Discuss the case of the conformal group for the Lorentz metric in $\mathbb{R}^4$. (See Exercise 1.29(c).) (Eisenhart, [1; Chap. 6].)

*1.31. Let $\pi: S^m \backslash \{(0, \ldots, 0, 1)\} \to \mathbb{R}^m$ be stereographic projection from the unit sphere in $\mathbb{R}^{m+1}$. Prove that if $A \in \mathrm{SO}(m + 1)$ is any rotation of S^m, then A induces a conformal transformation $\pi \circ A \circ \pi^{-1}$ of $\mathbb{R}^m$. Which of the conformal transformations constructed in Exercise 1.30 correspond to rotations of S^m?

1.32. Let G be a local group of transformations acting on a smooth manifold M. For each $x \in M$, the *isotropy group* is defined to be $G^x = \{g \in G: g \cdot x = x\}$. Prove that G^x is a (local) subgroup of G with Lie algebra $\mathfrak{g}^x = \{\mathbf{v} \in \mathfrak{g}: \mathbf{v}|_x = 0\}$. Find the isotropy subgroups and subalgebras of the rotation group $\mathrm{SO}(3)$ acting on $\mathbb{R}^3$. Suppose $y = g \cdot x$. How is the isotropy subgroup G^y related to G^x?