so the equation takes the form

$$\frac{dw}{dy} = \frac{1}{F(y) - y}.$$

This has the solution

$$w = \int \frac{dy}{F(y) - y} + c,$$

which in turn defines u implicitly as a function of x once we set $w = \log x$, $y = u/x$. For example, if the equation is

$$\frac{du}{dx} = \frac{u^2 + 2xu}{x^2} = \left(\frac{u}{x}\right)^2 + 2\frac{u}{x},$$

so $F(y) = y^2 + 2y$, then, in the coordinates $y = u/x$, $w = \log x$, we have

$$\frac{dw}{dy} = \frac{1}{y^2 + y}.$$

The solution is

$$w = -\log(1 + y^{-1}) + c,$$

or, in terms of the original variables,

$$\log x = -\log\left(1 + \frac{x}{u}\right) + c.$$

This can be solved explicitly for u as a function of x:

$$u = \frac{x^2}{\tilde{c} - x},$$

where $\tilde{c} = e^c$.

Although the answer is of course the same, the above procedure is not quite the usual one learned in a first course in ordinary differential equations. Here the roles of w and y are reversed, with w being the new independent variable. For many first order equations, it is often expedient to adopt this latter strategy. In the present case, we can drop the logarithm and treat x and $y = u/x$ as the new variables. Then

$$\frac{du}{dx} = \frac{d}{dx}(xy) = x\frac{dy}{dx} + y,$$

and we obtain the solution in the form

$$\int \frac{dy}{F(y) - y} = \int \frac{dx}{x} = \log x + c.$$

The equivalence of the methods is clear. Finally, note that in general the origin $u = x = 0$ is a singular point, corresponding to the point where $\mathbf{v}$ vanishes.

Example 2.47. Let G be the rotation group SO(2), whose infinitesimal generator

$$\mathrm{pr}^{(1)}\,\mathbf{v} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1 + u_x^2)\frac{\partial}{\partial u_x}$$

was computed in Example 2.29. It is a straightforward computation to check that any equation of the form

$$\frac{du}{dx} = \frac{u + xH(r)}{x - uH(r)}, \tag{2.82}$$

where $H(r) = H(\sqrt{x^2 + u^2})$ is any function of the radius, admits SO(2) as a symmetry group. Polar coordinates r, θ, with $x = r\cos\theta$, $u = r\sin\theta$, are the new coordinates satisfying (2.78) since $\mathbf{v} = \partial/\partial\theta$ in these coordinates. Furthermore,

$$\frac{du}{dx} = \frac{du/dr}{dx/dr} = \frac{\sin\theta + r\theta_r\cos\theta}{\cos\theta - r\theta_r\sin\theta}.$$

Substituting into (2.82) and solving for $d\theta/dr$, we find

$$\frac{d\theta}{dr} = \frac{1}{r}H(r),$$

hence

$$\theta = \int \frac{H(r)}{r}\,dr + c$$

is the general solution. For example, if $H(r) = 1$, we have the equation of Example 2.32.

An alternative method for solving first order equations invariant under a one-parameter group is based on the construction of an integrating factor. We rewrite (2.74) as a total differential equation

$$P(x, u)\,dx + Q(x, u)\,du = 0, \tag{2.83}$$

so $F = -P/Q$. The equation is *exact* provided $\partial P/\partial u = \partial Q/\partial x$, and in this case we can find the solution in implicit form $T(x, u) = c$ by requiring

$$\frac{\partial T}{\partial x} = P, \qquad \frac{\partial T}{\partial u} = Q.$$

(This assumes that the domain M is simply-connected.) If (2.83) is not exact, we must search for an integrating factor $R(x, u)$ such that when we multiply by R we do obtain an exact equation.

Theorem 2.48. *Suppose the equation $P\,dx + Q\,du = 0$ has a one-parameter symmetry group with infinitesimal generator $\mathbf{v} = \xi\partial_x + \phi\partial_u$. Then the function*

$$R(x, u) = \frac{1}{\xi(x, u)P(x, u) + \phi(x, u)Q(x, u)} \tag{2.84}$$

is an integrating factor.

Proof. Using the infinitesimal criterion of invariance (2.76), we find that **v** is a symmetry of (2.83) if and only if

$$\left(\xi\frac{\partial P}{\partial x}+\phi\frac{\partial P}{\partial u}\right)Q-\left(\xi\frac{\partial Q}{\partial x}+\phi\frac{\partial Q}{\partial u}\right)P+\frac{\partial\phi}{\partial x}Q^2-\left(\frac{\partial\phi}{\partial u}-\frac{\partial\xi}{\partial x}\right)PQ-\frac{\partial\xi}{\partial u}P^2=0. \tag{2.85}$$

The condition that R be an integrating factor is

$$\frac{\partial}{\partial u}(RP)=\frac{\partial}{\partial x}(RQ).$$

Substituting the formula for R, this becomes

$$R^2\left\{\phi\left(Q\frac{\partial P}{\partial u}-P\frac{\partial Q}{\partial u}\right)-\frac{\partial\xi}{\partial u}P^2-\frac{\partial\phi}{\partial u}PQ\right\}$$
$$=R^2\left\{\xi\left(P\frac{\partial Q}{\partial x}-Q\frac{\partial P}{\partial x}\right)-\frac{\partial\xi}{\partial x}PQ-\frac{\partial\phi}{\partial x}Q^2\right\}.$$

Comparison with the symmetry condition (2.85) proves the theorem. □

For example, in the case of the rotation group, the equation takes the general form

$$(u+xH(r))\,dx+(uH(r)-x)\,du=0.$$

The integrating factor is then

$$\frac{1}{-u(u+xH)+x(uH-x)}=\frac{-1}{x^2+u^2}.$$

For example, let $H(r)=1$, so we have

$$(u+x)\,dx+(u-x)\,du=0.$$

Multiplying by $(x^2+u^2)^{-1}$ we get an exact equation

$$0=\frac{u+x}{x^2+u^2}\,dx+\frac{u-x}{x^2+u^2}\,du=d\left[\tfrac{1}{2}\log(x^2+u^2)-\arctan\frac{u}{x}\right],$$

hence we re-derive the logarithmic spiral solutions $r=ce^\theta$ found in Example 2.32.

Note that if

$$\xi P+\phi Q\equiv 0$$

for all (x,u), then the integrating factor does not exist. This happens precisely in the case (2.81) when the computation of the symmetry group invariants is the same problem as solving the ordinary differential equation itself. In this case, both the invariant method and the integrating factor method fail to provide solutions.

In practice, the integrating factor method is perhaps easier to implement in that we do not need to find the solutions η, ζ to the auxiliary pair of partial differential equations (2.78). However, if one must consider a large number of equations all with the same symmetry group, this slight advantage is nullified by the relative difficulty of finding potentials T for each of the requisite exact differentials.

Higher Order Equations

Symmetry groups can also be used to aid in the solution of higher order ordinary differential equations. The integration method based on the invariants of the group extends straightforwardly. Let

$$\Delta(x, u^{(n)}) = \Delta(x, u, u_x, \ldots, u_n) = 0, \tag{2.86}$$

where $u_n \equiv d^n u/dx^n$, be a single n-th order differential equation involving the single dependent variable u. The basic result in this case is that if we know a one-parameter symmetry group of this equation, then we can reduce the order of the equation by one.

To see this, we first choose coordinates $y = \eta(x, u)$, $w = \zeta(x, u)$ as in (2.78) such that the group transforms into a group of translations with infinitesimal generator $\mathbf{v} = \partial/\partial w$. Employing the chain rule, we can express the derivatives of u with respect to x in terms of y, w and the derivatives of w with respect to y,

$$\frac{d^k u}{dx^k} = \delta_k\left(y, w, \frac{dw}{dy}, \ldots, \frac{d^k w}{dy^k}\right),$$

for certain functions δ_k. Substituting these expressions into our equation, we find an equivalent n-th order equation

$$\tilde{\Delta}(y, w^{(n)}) = \tilde{\Delta}(y, w, w_y, \ldots, w_n) = 0 \tag{2.87}$$

in terms of the new coordinates y and w. Moreover, since the original system (2.86) has the invariance group G, so does the transformed system. In terms of the (y, w)-coordinates, the infinitesimal generator has trivial prolongation

$$\mathrm{pr}^{(n)}\,\mathbf{v} = \mathbf{v} = \partial/\partial w.$$

The infinitesimal criteron of invariance implies

$$\mathrm{pr}^{(n)}\,\mathbf{v}(\tilde{\Delta}) = \frac{\partial\tilde{\Delta}}{\partial w} = 0 \quad \text{whenever} \quad \tilde{\Delta}(y, w^{(n)}) = 0.$$

This means, as in Proposition 2.18, that there is an equivalent equation

$$\hat{\Delta}\left(y, \frac{dw}{dy}, \ldots, \frac{d^n w}{dy^n}\right) = 0$$

which is independent of w, i.e. $\tilde{\Delta}(y, w^{(n)}) = 0$ if and only if $\hat{\Delta}(y, w^{(n)}) = 0$. Now we have accomplished our goal; setting $z = w_y$ we have an $(n-1)$-st order equation for z,

$$\hat{\Delta}(y, z, \ldots, d^{n-1}z/dy^{n-1}) = \hat{\Delta}(y, z^{(n-1)}) = 0, \tag{2.88}$$

whose solutions provide the general solution to our original equation. Namely, if $z = h(y)$ is a solution of (2.88), then $w = \int h(y)\, dy + c$ is a solution of (2.87), and hence, by replacing w and y by their expressions in terms of x and u, implicitly defines a solution of the original equation.

Example 2.49. As an elementary example, consider the case of a second order equation in which x does not occur explicitly,

$$\Delta(u, u_x, u_{xx}) = 0.$$

This equation is clearly invariant under the group of translations in the x-direction, with infinitesimal generator $\partial/\partial x$. In order to change this into the vector field $\partial/\partial w$, corresponding to translations of the dependent variable, it suffices to reverse the roles of dependent and independent variable, so we set $y = u$, $w = x$. Then

$$\frac{du}{dx} = \frac{1}{w_y}, \qquad \frac{d^2u}{dx^2} = -\frac{w_{yy}}{w_y^3},$$

so our equation becomes

$$\Delta\left(y, \frac{1}{w_y}, -\frac{w_{yy}}{w_y^3}\right) = 0,$$

which is a first order equation for $z = w_y$:

$$\hat{\Delta}(y, z, z_y) \equiv \Delta(y, z^{-1}, -z^{-3}z_y) = 0.$$

For example, to solve

$$u_{xx} - 2uu_x = 0,$$

we have the corresponding first order equation

$$-z^{-3}z_y - 2yz^{-1} = 0$$

for $z = dw/dy = (du/dx)^{-1}$. This can easily be solved by separation, with solution

$$z = (y^2 + c)^{-1}.$$

Thus, if $c = c'^2 > 0$, we find

$$w = \int z\, dy = \frac{1}{c'} \arctan\frac{y}{c'} + \tilde{c},$$

or, in terms of x and u,

$$u = c' \tan(c'x + d), \qquad d = -\tilde{c}c'.$$

(For $c < 0$, we have a hyperbolic tangent, for $c = 0$, we get the limiting solution $u = -(x + d)^{-1}$.)

Example 2.50. Consider a homogeneous second order linear equation

$$u_{xx} + p(x)u_x + q(x)u = 0. \tag{2.89}$$

This is clearly invariant under the group of scale transformations

$$(x, u) \mapsto (x, \lambda u),$$

with infinitesimal generator $\mathbf{v} = u\partial_u$. Coordinates (y, w) which straighten out $\mathbf{v}$ are given by $y = x$, $w = \log u$ (provided $u \neq 0$), with $\mathbf{v} = \partial_w$ in these coordinates. We have

$$u = e^w, \qquad u_x = w_x e^w, \qquad u_{xx} = (w_{xx} + w_x^2)e^w,$$

so the equation becomes

$$w_{xx} + w_x^2 + p(x)w_x + q(x) = 0,$$

which is independent of w. We have thus reconstructed the well-known transformation between a linear second order equation and a first order Riccati equation; namely $z = w_x = u_x/u$ changes (2.89) into the Riccati equation

$$z_x = -z^2 - p(x)z - q(x).$$

Differential Invariants

Besides trying to determine the most general symmetry group of a given differential equation, we can turn the whole procedure around and ask the complementary question: What is the most general type of differential equation which admits a given group as a group of symmetries? An answer to this question will not only provide us with a catalogue of large classes of ordinary differential equations which can be integrated by a common method, but also familiarity with the various types of equations which arise from known groups will aid in the recognition of symmetry groups for other equations.

According to Section 2.2, an n-th order ordinary differential equation $\Delta(x, u^{(n)}) = 0$ admits a group G as a symmetry group if and only if the corresponding subvariety $\mathscr{S}_\Delta \subset M^{(n)}$ is invariant under the n-th prolongation $\mathrm{pr}^{(n)} G$. Furthermore, according to Proposition 2.18, there is an equivalent equation $\tilde{\Delta} = 0$ describing the subvariety $\mathscr{S}_\Delta$, where $\tilde{\Delta}$ depends only on the invariants of the group action, which in this case is $\mathrm{pr}^{(n)} G$. The invariants of a prolonged group action play an important role in this procedure, and are known as "differential invariants".

Definition 2.51. Let G be a local group of transformations acting on $M \subset X \times U$. An n-th order *differential invariant* of G is a smooth function $\eta: M^{(n)} \to \mathbb{R}$, depending on x, u and derivatives of u, such that η is an invariant of the prolonged group action $\mathrm{pr}^{(n)}\, G$:

$$\eta(\mathrm{pr}^{(n)}\, g \cdot (x, u^{(n)})) = \eta(x, u^{(n)}), \qquad (x, u^{(n)}) \in M^{(n)},$$

for all $g \in G$ such that $\mathrm{pr}^{(n)}\, g \cdot (x, u^{(n)})$ is defined.

Although the definition makes sense when there are several independent and several dependent variables, we will primarily be interested in the ordinary differential equation case $p = q = 1$.

Example 2.52. Suppose $G = \mathrm{SO}(2)$ is the rotation group acting on $X \times U \simeq \mathbb{R}^2$ with generator $\mathbf{v} = -u\partial_x + x\partial_u$. The first order differential invariants are the ordinary invariants of the first prolongation $\mathrm{pr}^{(1)}\,\mathrm{SO}(2)$, which has infinitesimal generator

$$\mathrm{pr}^{(1)}\,\mathbf{v} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1 + u_x^2)\frac{\partial}{\partial u_x}.$$

If we relabel the variables (x, u, u_x) by (x, y, z), then we are precisely in the situation covered by Example 2.19(b). Translating the result obtained there into the present context, we find that the functions

$$y = \sqrt{x^2 + u^2} \quad \text{and} \quad w = \frac{xu_x - u}{x + uu_x} \tag{2.90}$$

provide a complete set of first order differential invariants for SO(2). For second order invariants, we would also include the curvature invariant κ found in Example 2.37. Any other second order differential invariant must be a function of these three independent invariants.

For higher order differential invariants there is an easy short cut which allows us to construct all differential invariants from knowledge of the lowest order ones.

Proposition 2.53. *Let G be a group of transformations acting on $M \subset X \times U \simeq \mathbb{R}^2$. Suppose $y = \eta(x, u^{(n)})$ and $w = \zeta(x, u^{(n)})$ are n-th order differential invariants of G. Then the derivative*

$$\frac{dw}{dy} = \frac{dw/dx}{dy/dx} \equiv \frac{D_x\zeta}{D_x\eta} \tag{2.91}$$

is an $(n + 1)$-st order differential invariant for G.

PROOF. The proof requires the following formula. Let $\zeta(x, u^{(n)})$ be any smooth function and $\mathbf{v} = \xi\partial_x + \phi\partial_u$ any vector field. Then

$$\mathrm{pr}^{(n+1)}\,\mathbf{v}(D_x\zeta) = D_x[\mathrm{pr}^{(n)}\,\mathbf{v}(\zeta)] - D_x\xi \cdot D_x\zeta. \tag{2.92}$$

Using the alternative formulation (2.50) of the prolongation formula, we see that

$$\mathrm{pr}^{(n+1)}\,\mathbf{v}(D_x\zeta) = \mathrm{pr}^{(n+1)}\,\mathbf{v}_Q(D_x\zeta) + \xi D_x^2\zeta,$$

while

$$D_x[\mathrm{pr}^{(n)}\,\mathbf{v}(\zeta)] = D_x[\mathrm{pr}^{(n)}\,\mathbf{v}_Q(\zeta)] + D_x(\xi D_x\zeta).$$

Therefore (2.92) reduces to the simpler formula

$$\mathrm{pr}^{(n+1)}\,\mathbf{v}_Q(D_x\zeta) = D_x[\mathrm{pr}^{(n)}\,\mathbf{v}_Q(\zeta)].$$

This latter formula is a special case of a general commutation rule for vector fields and total derivatives—which will be proved in Lemma 5.12. (It is, however, not difficult for the reader to prove directly here.)

Proceeding to the proof of (2.91), let $\mathbf{v}$ be any infinitesimal generator of G. Using (2.92), and the fact that $\mathrm{pr}^{(n+1)}\,\mathbf{v}$ is a derivation,

$$\begin{aligned}\mathrm{pr}^{(n+1)}\,\mathbf{v}\left[\frac{dw}{dy}\right] &= \frac{1}{(D_x\eta)^2}\{\mathrm{pr}^{(n+1)}\,\mathbf{v}(D_x\zeta)\cdot D_x\eta - D_x\zeta\cdot\mathrm{pr}^{(n+1)}\,\mathbf{v}(D_x\eta)\}\\ &= \frac{1}{(D_x\eta)^2}\{D_x[\mathrm{pr}^{(n)}\,\mathbf{v}(\zeta)]\cdot D_x\eta - D_x\xi\cdot D_x\zeta\cdot D_x\eta\\ &\qquad - D_x\zeta\cdot D_x[\mathrm{pr}^{(n)}\,\mathbf{v}(\eta)] + D_x\zeta\cdot D_x\xi\cdot D_x\eta\}\\ &= 0\end{aligned}$$

since $\mathrm{pr}^{(n)}\,\mathbf{v}(\zeta) = 0 = \mathrm{pr}^{(n)}\,\mathbf{v}(\eta)$ by assumption. Thus dw/dy is infinitesimally invariant under the action of $\mathrm{pr}^{(n+1)}\,G$, and hence by Proposition 2.6 is an invariant. □

Corollary 2.54. *Suppose G is a one-parameter group of transformations acting on $M \subset X \times U \simeq \mathbb{R}^2$. Let $y = \eta(x, u)$ and $w = \zeta(x, u, u_x)$ be a complete set of functionally independent invariants of the first prolongation $\mathrm{pr}^{(1)}\,G$. Then the derivatives*

$$y,\ w,\ dw/dy, \ldots, d^{n-1}w/dy^{n-1}$$

provide a complete set of functionally independent invariants for the n-th prolongation $\mathrm{pr}^{(n)}\,G$ for $n \geqslant 1$.

To check the independence, it suffices to note that the k-th derivative d^kw/dy^k depends explicitly on $u_{k+1} = d^{k+1}u/dx^{k+1}$, and hence is independent of the previous invariants $y, w, \ldots, d^{k-1}w/dy^{k-1}$, which are only functions of $x, u, \ldots, u_k$.

Example 2.55. Return to the second order invariants of the rotation group SO(2) discussed in the previous Example 2.52. It follows from Corollary 2.54 that y, w and the derivative

$$\frac{dw}{dy} = \frac{dw/dx}{dy/dx} = \frac{\sqrt{x^2+u^2}}{(x+uu_x)^3}[(x^2+u^2)u_{xx} - (1+u_x^2)(xu_x - u)]$$

form a complete set of functionally independent invariants for the second prolongation $\mathrm{pr}^{(2)}\,\mathrm{SO}(2)$. Note that this means *any* other second order differential invariant of the rotation group can be written in terms of y, w and dw/dy; for instance, the curvature invariant found previously has expression

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}} = \frac{w_y}{(1+w^2)^{3/2}} + \frac{w}{y(1+w^2)^{1/2}},$$

as the reader can check.

Once we know the differential invariants for a group of transformations acting on $M \subset X \times U$, we can determine the structure of all differential equations which admit the given group as a symmetry group. In the case G is a one-parameter group, we thus know all equations which can be integrated using G.

Proposition 2.56. *Let G be a local group of transformations acting on $M \subset X \times U$. Assume $\mathrm{pr}^{(n)} G$ acts semi-regularly on an open subset of $M^{(n)}$, and let $\eta^1(x, u^{(n)}), \ldots, \eta^k(x, u^{(n)})$ be a complete set of functionally independent n-th order differential invariants. An n-th order differential equation $\Delta(x, u^{(n)}) = 0$ admits G as a symmetry group if and only if there is an equivalent equation*

$$\tilde{\Delta}(\eta^1(x, u^{(n)}), \ldots, \eta^k(x, u^{(n)})) = 0$$

involving only the differential invariants of G. In particular, if G is a one-parameter group of transformations, any n-th order differential equation having G as a symmetry group is equivalent to an $(n-1)$-st order equation

$$\tilde{\Delta}(y, w, dw/dy, \ldots, d^{n-1}w/dy^{n-1}) = 0 \tag{2.93}$$

involving the invariants $y = \eta(x, u)$, $w = \zeta(x, u, u_x)$ of $\mathrm{pr}^{(1)} G$ and their derivatives.

The proof is immediate from Proposition 2.18 and Corollary 2.54. □

Example 2.57. For example, we can completely classify all first and second order differential equations admitting the rotation group SO(2) as a symmetry group. Any first order equation invariant under SO(2) is equivalent to an equation involving only the invariants (2.90). Solving for w, we find every such equation takes the form

$$\frac{xu_x - u}{x + uu_x} = H(\sqrt{x^2 + u^2})$$

for some function H. But this is precisely the form (2.82) discussed in Example 2.47 once we solve for u_x. Thus (2.82) is the most general first order ordinary differential equation invariant under the rotation group SO(2).

Similarly, any second order equation invariant under SO(2) is equivalent to one involving y, w and the curvature $\kappa = u_{xx}(1 + u_x^2)^{-3/2}$, i.e.

$$u_{xx} = (1 + u_x^2)^{3/2} H\left(\sqrt{x^2 + u^2}, \frac{xu_x - u}{x + uu_x}\right),$$

where $H(y, w)$ is any function of the first order invariants. This can be integrated once, as in Example 2.47, by setting $r = \sqrt{x^2 + u^2}$, $\theta = \arctan(u/x)$. We find

$$w = r\theta_r, \qquad \kappa = \frac{r\theta_{rr} + r^2\theta_r^3 + 2\theta_r}{(1 + r^2\theta_r^2)^{3/2}},$$

the latter being the expression for the curvature of a curve $\theta = \theta(r)$ expressed in polar coordinates. Thus the equation becomes a first order equation

$$r\frac{dz}{dr} = (1 + r^2z^2)^{3/2}H(r, rz) - (r^2z^3 + 2z)$$

involving only $z = d\theta/dr$, from which we can determine $\theta(r) = \int z(r)\, dr + c$.

The preceding proposition also indicates an alternative method for reducing the order of a differential equation invariant under a one-parameter group by using the differential invariants of the group. Namely, the differential equation $\Delta(x, u^{(n)}) = 0$ must be equivalent to an equation (2.93) involving only the invariants $y, w, \ldots, d^{n-1}w/dy^{n-1}$ of the n-th prolongation of G. But (2.93) is automatically an $(n - 1)$-st order equation for w as a function of y, so that merely by re-expressing the original equation in terms of the given list of differential invariants, we have automatically reduced its order by one. *Moreover*, once we know the solution $w = h(y)$ of the reduced equation (2.93), the solution of the original equation is found by integrating the auxiliary first order equation

$$\zeta(x, u, u_x) = h[\eta(x, u)] \tag{2.94}$$

obtained by substituting for y and w their expressions in terms of the original variables x and u. Since (2.94) depends only on the invariants y and w of $\mathrm{pr}^{(1)} G$, it clearly has G as a one-parameter symmetry group and hence can be integrated by the methods for first order equations discussed previously. We have thus, by a completely different method, re-established the basic fact that an ordinary differential equation invariant under a one-parameter group can be reduced in order by one.

Example 2.58. Consider the second order equation

$$x^2u_{xx} + xu_x^2 = uu_x. \tag{2.95}$$

This is invariant under the scaling group $(x, u) \mapsto (\lambda x, \lambda u)$.

Let us first try to integrate (2.95) using the method of differential invariants. We find that the invariants of the second prolonged group action are

$$y = \frac{u}{x}, \qquad w = u_x, \qquad \frac{dw}{dy} = \frac{x^2u_{xx}}{xu_x - u}.$$