*5.34. Prove that if the Euler–Lagrange equations $\mathsf{E}(L) = 0$ admit a trivial conservation law corresponding to a nontrivial variational symmetry, then they are necessarily under-determined and admit an infinite family of such laws. (Compare Exercise 5.29.)

5.35. Let $\mathscr{L}[u] = \int L[u]\, dx$ be a functional, and let $\mathbf{v}$ generate a one-parameter (generalized) group which does not leave $\mathscr{L}$ invariant but rather multiplies it by a scalar factor. (For example, the scaling symmetries for the wave equation are of this type.) Prove that there is a divergence identity of the form $\operatorname{Div} P[u] = L[u]$ which holds whenever u is a solution of the Euler–Lagrange equations $\mathsf{E}(L) = 0$. (Steudel, [2].)

*5.36. One trick used to construct variational principles for arbitrary systems of differential equations is the following. If $\Delta_\nu[u] = 0$, $\nu = 1, \ldots, l$ is the system, we let $\mathscr{L}[u] = \int \frac{1}{2}|\Delta[u]|^2\, dx = \int \frac{1}{2}\sum_\nu (\Delta_\nu[u])^2\, dx$ (called a "weak Lagrangian structure" by Anderson and Ibragimov, [1; § 14.3]).

(a) Prove that the Euler–Lagrange equations for $\mathscr{L}$ take the form $\delta\mathscr{L} = \mathsf{D}_\Delta^*(\Delta) = 0$. Thus solutions of $\Delta = 0$ are solutions of the Euler–Lagrange equation for $\mathscr{L}$, but the converse is not true in general. What is $\mathscr{L}$ and $\delta\mathscr{L}$ in the case of the heat equation $u_t = u_{xx}$?

(b) Prove that if $\mathbf{v}_Q$ is any generalized symmetry of Δ, then one can construct a conservation law for Δ with characteristic $\mathsf{D}_\Delta(Q)$ using the techniques of Noether's theorem, but the conservation law is always trivial! Thus *this* method for finding variational principles in practice leads only to trivial conservation laws.

*5.37. A second method for finding variational principles for arbitrary systems of differential equations $\Delta = (\Delta_1, \ldots, \Delta_l) = 0$ is to introduce auxiliary variables $v = (v^1, \ldots, v^l)$ and consider the problem $\mathscr{L}[u, v] = \int v \cdot \Delta[u]\, dx$.

(a) Prove that the Euler–Lagrange equations for $\mathscr{L}$ are $\Delta = 0$ and $\mathsf{D}_\Delta^*(v) = 0$.

(b) Find variational symmetries and conservation laws for the heat equation $u_t = u_{xx}$ by this method. How does one intepret these results physically?

(Bateman, [2], Atherton and Homsy, [1].)

5.38. Define a pseudo-variational symmetry $\mathbf{v}$ to be a generalized vector field that satisfies (5.84) only on solutions u of the Euler–Lagrange equations. Prove that to every pseudo-variational symmetry of a normal variational problem there corresponds a conservation law, but that there is also always a true variational symmetry giving rise to the same law. How is the true variational symmetry related to the pseudo-variational symmetry?

5.39. Let $\mathscr{D}\colon \mathscr{A}^r \to \mathscr{A}^s$ be a matrix differential operator. Prove that $\mathscr{D}Q = 0$ for all $Q \in \mathscr{A}^r$ if and only if $\mathscr{D} = 0$ is the zero operator.

5.40. Let $\mathscr{D}$ be a scalar differential operator.

(a) Prove that $\mathscr{D}$ is self-adjoint if and only if $\mathscr{D}$ can be written in the form

$$\mathscr{D} = \sum_{\# J \text{ even}} (D_J \cdot P_J + P_J \cdot D_J)$$

for certain differential functions P_J.

(b) Prove that $\mathscr{D}$ is skew-adjoint if and only if

$$\mathscr{D} = \sum_{\# J \text{ odd}} (D_J \cdot P_J + P_J \cdot D_J)$$

for certain P_J.

(c) In the case $p = 1$, prove that $\mathscr{D}$ is self-adjoint if and only if

$$\mathscr{D} = \sum_j D_x^j Q_j D_x^j$$

for certain differential functions Q_j. Does this result generalize to the case $p \geqslant 2$?

5.41. Let $p = q = 1$, and let $P(x, u^{(2n+1)})$ be a differential function. Prove that the Fréchet derivative D_P is skew-adjoint, $\mathsf{D}_P^* = -\mathsf{D}_P$, if and only if P is linear in u, $u_x, \ldots, u_{2n+1}$. Is this true if $p > 1$?

*5.42. *The Substitution Principle.* For subsequent applications in Chapter 7, we will require a slight generalization of the technical vanishing results such as those in Corollary 5.68 and Lemma 5.86. The basic problem is that one has an expression depending on one or more q-tuples of differential functions $Q^1, \ldots, Q^k \in \mathscr{A}^q$ and, possibly, their *total* derivatives, with the property that it vanish whenever each $Q^i = \delta \mathscr{Q}_i$ is a variational derivative of some functionals $\mathscr{Q}_1, \ldots, \mathscr{Q}_k \in \mathscr{F}$. The conclusion to be drawn is that the same expression will vanish no matter what the Q^i are, variational derivatives or otherwise.

More specifically, the reader should prove the following.

(a) Let $P \in \mathscr{A}^q$ be a fixed q-tuple of differential functions. Then $\int P \cdot Q \, dx = 0$ whenever $Q = \delta \mathscr{Q} \in \mathscr{A}^q$ is a variational derivative if and only if $P = 0$, and hence $\int P \cdot Q \, dx = 0$ for all $Q \in \mathscr{A}^q$.

(b) Let $\mathscr{D}\colon \mathscr{A}^q \to \mathscr{A}^r$ be an $r \times q$ matrix of differential operators. Then $\mathscr{D}Q = 0$ whenever $Q = \delta \mathscr{Q} \in \mathscr{A}^q$ is a variational derivative if and only if $\mathscr{D} = 0$, and hence $\mathscr{D}Q = 0$ for all $Q \in \mathscr{A}^q$.

(c) Suppose $\mathscr{D}\colon \mathscr{A}^q \to \mathscr{A}^q$ is a differential operator. Then $\int Q \cdot \mathscr{D}R \, dx = 0$ for all variational derivatives $Q = \delta \mathscr{Q}$, $R = \delta \mathscr{R} \in \mathscr{A}^q$ if and only if $\mathscr{D} = 0$, and hence $\int Q \cdot \mathscr{D}R \, dx = 0$ for all $Q, R \in \mathscr{A}^q$.

*5.43. Let $p = q = 1$. Prove that if $L = L(u^{(n)})$ does not explicitly depend on x, then $\int u_x \mathsf{E}(L) \, dx = 0$. This shows that one must be careful with the above "substitution principle", since the following "slight" generalization of part (a) is *not* true: If $P \in \mathscr{A}^q$, and $\int P \cdot Q \, dx = 0$ for all $Q(u^{(n)}) = \delta \mathscr{Q} \in \mathscr{A}^q$ which do not depend explicitly on x, then $P = 0$. The above ($P = u_x$) is a definite counterexample. Is this the only such counterexample?

5.44. Let $P(u^{(m)})$ be a q-tuple of homogeneous differential functions of degree $\alpha \neq -1$: $P(\lambda u^{(m)}) = \lambda^\alpha P(u^{(m)})$. Prove that $P = \mathsf{E}(L)$ for some Lagrangian L if and only if $\mathsf{E}(u \cdot P) = (\alpha + 1)P$. Is this true if $\alpha = -1$?
(Olver and Shakiban, [1], Shakiban, [1].)

*5.45. Prove the Helmholtz Theorem 5.45 directly, without using variational forms: If $P \in \mathscr{A}^q$ has self-adjoint Fréchet derivative, then $P = \mathsf{E}(L)$ where L is given by (5.123). Conversely, if $P = \mathsf{E}(L)$ for some L, then D_P is self-adjoint.

5.46. (a) Show that a single evolution equation $u_t = P[u]$, $u \in \mathbb{R}$, is never the Euler–Lagrange equation for a variational problem. Is the same true for systems of evolution equations?

(b) One common trick to put a single evolution equation into variational form is to replace u by a potential function v with $u = v_x$, yielding $v_{xt} = P[v_x]$. Show that the Korteweg–de Vries equation becomes the Euler–Lagrange equation of some functional in this way. Which of the geometrical and generalized symmetries of the Korteweg–de Vries equation yield conservation laws via Noether's theorem?

(c) Find necessary and sufficient conditions on P that the trick of part (b) yields an Euler–Lagrange equation.

(d) A second trick to convert such an evolution equation is to just differentiate with respect to x: $u_{xt} = D_x P[u]$. Prove that this yields an Euler–Lagrange equation if and only if $D_x P$ depends only on x, u_x, u_{xx}, ... (not u), and the equation is equivalent to an evolution equation $w_t = Q[w]$, $w = u_x$, for which the trick in part (b) is applicable.

*5.47. (a) Prove that if $Q(x, u^{(n)})$ is any system of differential functions which satisfies the Helmholtz conditions $\mathsf{D}_Q = \mathsf{D}_Q^*$, and $P(\lambda, x, u^{(m)})$ any one-parameter family of q-tuples of differential functions such that

$$P(0, x, u^{(m)}) = f(x), \qquad P(1, x, u^{(m)}) = u,$$

for some fixed $f(x)$, then

$$L = \int_0^1 \frac{\partial P}{\partial \lambda} \cdot Q(x, P^{(n)})\, d\lambda$$

is a Lagrangian for Q: $\mathsf{E}(L) = Q$.

(b) This method is useful for finding variational principles for systems of differential equations not defined on the entire jet space $M^{(n)}$. For example, let $p = q = 1$, $Q = u_x^{-2} u_{xx} + u_{yy}$, and use $P(\lambda) = (1 - \lambda)x + \lambda u$ to find a variational principle for Q. Why does the classical construction (5.123) break down in this case? (Horndeski, [1].)

*5.48. Given a differential equation $\Delta[u] = 0$, the *multiplier problem* of the calculus of variations is to determine a nonvanishing differential function $Q[u]$ such that $0 = Q \cdot \Delta = \mathsf{E}(L)$ is the Euler–Lagrange equation for some variational problem.

(a) Prove that if $\Delta[u] = u_{xx} - H(x, u, u_x)$ is a second order ordinary differential equation, then $Q(x, u, u_x)$ is such a multiplier if and only if Q satisfies the partial differential equation

$$\frac{\partial Q}{\partial x} + u_x \frac{\partial Q}{\partial u} + \frac{\partial}{\partial u_x}(QH) = 0.$$

Conclude that any second order ordinary differential equation is always locally equivalent to an Euler–Lagrange equation of some first order variational problem. (See Exercises 4.8 and 4.9.)

(b) Find all multipliers and corresponding variational problems for the equation $u_{xx} - u_x = 0$.

(c) Discuss the case of a higher order differential equation. (Hirsch, [1]; see Douglas, [1], and Anderson and Thompson, [1], for generalizations to systems of ordinary differential equations, and Anderson and Duchamp, [2], for second order partial differential equations.)

**5.49. Here we generalize the formulae in Theorem 4.8 for the action of the Euler operator under a change of variables, where now the new variables can depend on derivatives of the old variables. Let $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ be the original variables and $\mathscr{L}[u] = \int L(x, u^{(n)})\, dx$ be a variational problem with Euler–Lagrange expression $\mathsf{E}(L)$. Suppose $y = (y^1, \ldots, y^p)$ and $w = (w^1, \ldots, w^q)$ are new variables, with $y = P(x, u^{(m)})$, $w = Q(x, u^{(m)})$, for certain differential functions $P \in \mathscr{A}^p$, $Q \in \mathscr{A}^q$. Let $\tilde{\mathscr{L}}[w] = \int \tilde{L}(y, w^{(l)})\, dy$ be the corresponding variational problem. Prove that

$$\mathsf{E}_{u^\alpha}(L) = \sum_{\beta=1}^{q} \frac{\mathsf{D}(P_1, \ldots, P_p, Q_\beta)}{\mathsf{D}(x^1, \ldots, x^p, u^\alpha)} \mathsf{E}_{w^\beta}(\tilde{L}).$$

Here the coefficient of E_{w^β} is a differential operator, given by a determinantal formula

$$\frac{\mathsf{D}(P_1, \ldots, P_p, Q_\beta)}{\mathsf{D}(x^1, \ldots, x^p, u^\alpha)} = \det \begin{bmatrix} D_1 P_1 & \cdots & D_p P_1 & \mathsf{D}^*_{P_1,\alpha} \\ \vdots & & \vdots & \vdots \\ D_1 P_p & \cdots & D_p P_p & \mathsf{D}^*_{P_p,\alpha} \\ D_1 Q_\beta & \cdots & D_p Q_\beta & \mathsf{D}^*_{Q_\beta,\alpha} \end{bmatrix},$$

in which

$$\mathsf{D}_{P,\alpha} = \sum_J \frac{\partial P}{\partial u_J^\alpha} D_J$$

is the Fréchet derivative of P with respect to u^α, $\mathsf{D}^*_{P,\alpha}$ its adjoint, and in the expansion of the determinant, the differential operators are written *first* in any product. For example,

$$\frac{\mathsf{D}(P, Q)}{\mathsf{D}(x, u)}(R) = \det\begin{pmatrix} D_x P & \mathsf{D}^*_P \\ D_x Q & \mathsf{D}^*_Q \end{pmatrix} R = \mathsf{D}^*_Q(D_x P \cdot R) - \mathsf{D}^*_P(D_x Q \cdot R).$$

Discuss how (4.7) is a special case. Try some specific examples, e.g. $y = x$, $w = u_x$, to see how this works in practice.

5.50. An n-th *order divergence* is a differential function $P[u]$ such that

$$P = \sum_{\#I=n} D_I Q_I$$

for certain differential functions Q_I. For example,

$$u_{xx}u_{yy} - u_{xy}^2 = D_x^2(-\tfrac{1}{2}u_y^2) + D_x D_y(u_x u_y) + D_y^2(-\tfrac{1}{2}u_x^2)$$

is a second order divergence.

(a) Prove that P is an n-th order divergence if and only if $\mathsf{E}_\alpha^I(P) = 0$ for all $\alpha = 1, \ldots, q$, $0 \leqslant \#I \leqslant n-1$. (See Corollary 5.100.)

(b) Show that

$$u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx}$$

is a second order divergence, and

$$u_{xxx}v_{yyy} - 3u_{xxy}v_{xyy} + 3u_{xyy}v_{xxy} - u_{yyy}v_{xxx}$$

is a third order divergence. Can you generalize this result? (Olver, [6].)

*5.51. Suppose $\Delta[u] = 0$ is a homogeneous system of differential equations in the sense that $\Delta[\lambda u] = \lambda^\alpha \Delta[u]$ for all $\lambda \in \mathbb{R}$, where α is the degree of homogeneity. Prove that if $\operatorname{Div} P = 0$ is a conservation law with *trivial* characteristic for such a system, then P itself is a trivial conservation law. (*Hint*: First reduce to the case of a homogeneous conservation law $P[\lambda u] = \lambda^\beta P[u]$. Use the homotopy formula (5.151) to reconstruct P from the characteristic form $\operatorname{Div} P = Q \cdot \Delta$, where $Q = 0$ whenever $\Delta = 0$, Q homogeneous.) (Olver, [11].)

5.52. (a) If $\mathscr{L}[u] = \int L[u]\,dx$ is a functional, and $\mathbf{v}_Q$ an evolutionary vector field, prove that the prolonged action

$$\operatorname{pr} \mathbf{v}_Q(\mathscr{L}) \equiv \int \operatorname{pr} \mathbf{v}_Q(L)\,dx$$

gives a well-defined map on the space $\mathscr{F}$ of functionals.

(b) Prove that the action is effective, i.e., $\operatorname{pr} \mathbf{v}_Q(\mathscr{L}) = 0$ for all $\mathscr{L} \in \mathscr{F}$ if and only if $Q = 0$. Similarly prove that $\operatorname{pr} \mathbf{v}_Q(\mathscr{L}) = 0$ for all $Q \in \mathscr{A}^q$ if and only if $\mathscr{L} = 0$ in $\mathscr{F}$.

(c) Generalize this to define the Lie derivative of a functional form with respect to an evolutionary vector field. Prove a homotopy formula generalizing (1.67) or (5.122) to functional forms.

5.53. Let $\hat{\omega}$ be a vertical k-form and $\mathbf{v}_Q$ an evolutionary vector field with flow $\exp(\varepsilon \mathbf{v}_Q)$ determined by (5.14). Define a suitable action $\exp(\varepsilon \mathbf{v}_Q)_* \hat{\omega}$ of this flow on $\hat{\omega}$ and prove the Lie derivative formula

$$\operatorname{pr} \mathbf{v}_Q(\hat{\omega}) = \frac{d}{d\varepsilon}[\exp(\varepsilon \mathbf{v}_Q)_* \hat{\omega}].$$

(As always, assume existence and uniqueness for the relevant initial value problem.) Can the same be done if we use the definition of Exercise 5.8 for the flow generated by $\mathbf{v}_Q$?

CHAPTER 6

Finite-Dimensional Hamiltonian Systems

The guiding concept of a Hamiltonian system of differential equations forms the basis of much of the more advanced work in classical mechanics, including motion of rigid bodies, celestial mechanics, quantization theory and so on. More recently, Hamiltonian methods have become increasingly important in the study of the equations of continuum mechanics, including fluids, plasmas and elastic media. In this book, we are concerned with just one aspect of this vast subject, namely the interplay between symmetry groups, conservation laws and reduction in order for systems in Hamiltonian form. The Hamiltonian version of Noether's theorem has a particularly attractive geometrical flavour, which remains somewhat masked in our previous Lagrangian framework.

No previous knowledge of Hamiltonian mechanics will be assumed, so our first order of business will be to make precise the concept of a Hamiltonian system of differential equations. In this chapter, we concentrate on the more familiar, and conceptually easier case of systems of ordinary differential equations. Once we have mastered these, the generalizations to systems of evolution equations to be taken up in Chapter 7 will be quite natural. There are, at the outset, several different approaches to Hamiltonian mechanics, and that adopted here is slightly novel. It is important to realize the necessity of a coordinate-free treatment of "Hamiltonian structures" which does not assume the introduction of special canonical coordinates (the p's and q's of the elementary classical mechanics texts). Admittedly, one always has the temptation to simplify matters as much as possible, and, for finite-dimensional systems of constant rank, Darboux' theorem says that we could, if desired, always introduce such coordinates, with the attendant simplification in the formulae, but this may not always be the most natural or straightforward approach to the problem. Besides, in the infinite-dimensional version of this theory to be discussed later, such a result is no longer available; hence, if we are to receive a proper grounding in the finite-dimensional theory to make the ascension to infinite dimensions and evolution equations, we must cast aside the crutch of canonical coordinates and approach the Hamiltonian structure from a more intrinsic standpoint.

Even so, there are still several coordinate-free approaches to Hamiltonian mechanics. The one that requires the least preparatory work in differential geometric foundations concentrates on the Poisson bracket as the fundamental object of study. This has the advantage of avoiding differential forms almost entirely, and proceeding directly to the heart of the subject. In addition, the Poisson bracket approach admits Hamiltonian structures of varying rank (in a sense to be defined shortly), which have proved important in recent work on collective motion and stability. It includes as an important special case the Lie–Poisson bracket on the dual to a Lie algebra which plays a key role in representation theory and geometric quantization, as well as providing the theoretical basis for the general theory of reduction of Hamiltonian systems with symmetry.

6.1. Poisson Brackets

Given a smooth manifold M, a Poisson bracket on M assigns to each pair of smooth, real-valued functions $F, H\colon M \to \mathbb{R}$ another smooth, real-valued function, which we denote by $\{F, H\}$. There are certain basic properties that such a bracket operation must satisfy in order to qualify as a Poisson bracket. We state these properties initially in the simple, coordinate-free manner. Subsequently, local coordinate versions will be found, which, especially if M is an open subset of some Euclidean space, could equally well be taken as the defining properties for a Poisson bracket.

Definition 6.1. A *Poisson bracket* on a smooth manifold M is an operation that assigns a smooth real-valued function $\{F, H\}$ on M to each pair F, H of smooth, real-valued functions, with the basic properties:

(a) *Bilinearity*:

$$\{cF + c'P, H\} = c\{F, H\} + c'\{P, H\}, \quad \{F, cH + c'P\} = c\{F, H\} + c'\{F, P\},$$

for constants $c, c' \in \mathbb{R}$,

(b) *Skew-Symmetry*:

$$\{F, H\} = -\{H, F\},$$

(c) *Jacobi Identity*:

$$\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{\{H, P\}, F\} = 0,$$

(d) *Leibniz' Rule*:

$$\{F, H \cdot P\} = \{F, H\} \cdot P + H \cdot \{F, P\}.$$

(Here $\cdot$ denotes the ordinary multiplication of real-valued functions.) In all these equations, F, H and P are arbitrary smooth real-valued functions on M.

A manifold M with a Poisson bracket is called a *Poisson manifold*, the bracket defining a *Poisson structure* on M. The notion of a Poisson manifold is slightly more general than that of a symplectic manifold, or manifold with Hamiltonian structure; in particular, the underlying manifold M need not be even-dimensional. This is borne out by the standard examples from classical mechanics.

Example 6.2. Let M be the even-dimensional Euclidean space $\mathbb{R}^{2n}$ with coordinates $(p, q) = (p^1, \ldots, p^n, q^1, \ldots, q^n)$. (In physical problems, the p's represent momenta and the q's positions of the mechanical objects.) If $F(p, q)$ and $H(p, q)$ are smooth functions, we define their Poisson bracket to be the function

$$\{F, H\} = \sum_{i=1}^{n} \left\{ \frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q^i} \right\}. \tag{6.1}$$

This bracket is clearly bilinear and skew-symmetric; the verifications of the Jacobi identity and the Leibniz rule are straightforward exercises in vector calculus which we leave to the reader. We note the particular bracket identities

$$\{p^i, p^j\} = 0, \qquad \{q^i, q^j\} = 0, \qquad \{q^i, p^j\} = \delta^i_j, \tag{6.2}$$

in which i and j run from 1 to n, and δ^i_j is the Kronecker symbol, which is 1 if $i = j$ and 0 otherwise. (In (6.2) we are viewing the coordinates themselves as functions on M.)

More generally, we can determine a Poisson bracket on any Euclidean space $M = \mathbb{R}^m$. Just let $(p, q, z) = (p^1, \ldots, p^n, q^1, \ldots, q^n, z^1, \ldots, z^l)$ be the coordinates, so $2n + l = m$, and define the Poisson bracket between two functions $F(p, q, z)$, $H(p, q, z)$ by the same formula (6.1). In particular, if the function $F(z)$ depends on the z's only, then $\{F, H\} = 0$ for all functions H. Such functions, in particular the z^k's themselves, are known as *distinguished functions* or *Casimir functions* and are characterized by the property that their Poisson bracket with any other function is always zero. We must supplement the basic coordinate bracket relations (6.2) by the additional relations

$$\{p^i, z^k\} = \{q^i, z^k\} = \{z^j, z^k\} = 0 \tag{6.3}$$

for all $i = 1, \ldots, n$, and $j, k = 1, \ldots, l$. Although this example appears to be somewhat special, Darboux' Theorem 6.22 will show that locally, except at singular points, every Poisson bracket looks like this. We therefore call (6.1) the *canonical Poisson bracket*.

Definition 6.3. Let M be a Poisson manifold. A smooth, real-valued function $C\colon M \to \mathbb{R}$ is called a *distinguished function* if the Poisson bracket of C with any other real-valued function vanishes identically, i.e. $\{C, H\} = 0$ for all $H\colon M \to \mathbb{R}$.

In the case of the canonical Poisson bracket (6.1) on $\mathbb{R}^{2n}$, the only distinguished functions are the constants, which always satisfy the requirements of the definition. At the other extreme, if the Poisson bracket is completely trivial, i.e. $\{F, H\} = 0$ for every F, H, then every function is distinguished.

Hamiltonian Vector Fields

Let M be a Poisson manifold, so the Poisson bracket satisfies the basic requirements of Definition 6.1. Concentrating for the moment on just the bilinearity and Leibniz rule, note that given a smooth function H on M, the map $F \mapsto \{F, H\}$ defines a derivation on the space of smooth functions F on M, and hence by (1.20), (1.21) determines a vector field on M. This observation leads to a fundamental definition.

Definition 6.4. Let M be a Poisson manifold and $H\colon M \to \mathbb{R}$ a smooth function. The *Hamiltonian vector field* associated with H is the unique smooth vector field $\hat{\mathbf{v}}_H$ on M satisfying

$$\hat{\mathbf{v}}_H(F) = \{F, H\} = -\{H, F\} \tag{6.4}$$

for every smooth function $F\colon M \to \mathbb{R}$. The equations governing the flow of $\hat{\mathbf{v}}_H$ are referred to as *Hamilton's equations* for the "Hamiltonian" function H.

Example 6.5. In the case of the canonical Poisson bracket (6.1) on $\mathbb{R}^m$, $m = 2n + l$, the Hamiltonian vector field corresponding to $H(p, q, z)$ is clearly

$$\hat{\mathbf{v}}_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} \right). \tag{6.5}$$

The corresponding flow is obtained by integrating the system of ordinary differential equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}, \qquad \frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}, \qquad i = 1, \ldots, n, \tag{6.6}$$

$$\frac{dz^j}{dt} = 0, \qquad j = 1, \ldots, l, \tag{6.7}$$

which are Hamilton's equations in this case. In the nondegenerate case $m = 2n$, we have just (6.6), which is the canonical form of Hamilton's equations in classical mechanics. More generally, (6.7) just adds in the constancy of the distinguished coordinates z^j under the flow. In particular, if H depends only on the distinguished coordinates z, its Hamiltonian flow is completely trivial. This remark holds in general: A function C on a Poisson manifold is distinguished if and only if its Hamiltonian vector field $\hat{\mathbf{v}}_C = 0$ vanishes everywhere.

There is a fundamental connection between the Poisson bracket of two functions and the Lie bracket of their associated Hamiltonian vector fields, which forms the basis of much of the theory of Hamiltonian systems.

Proposition 6.6. *Let M be a Poisson manifold. Let $F, H: M \to \mathbb{R}$ be smooth functions with corresponding Hamiltonian vector fields $\hat{\mathbf{v}}_F$, $\hat{\mathbf{v}}_H$. The Hamiltonian vector field associated with the Poisson bracket of F and H is, up to sign, the Lie bracket of the two Hamiltonian vector fields*:

$$\hat{\mathbf{v}}_{\{F,H\}} = -[\hat{\mathbf{v}}_F, \hat{\mathbf{v}}_H] = [\hat{\mathbf{v}}_H, \hat{\mathbf{v}}_F]. \tag{6.8}$$

PROOF. Let $P: M \to \mathbb{R}$ be any other smooth function. Using the commutator definition of the Lie bracket, we find

$$\begin{aligned}
[\hat{\mathbf{v}}_H, \hat{\mathbf{v}}_F]P &= \hat{\mathbf{v}}_H \cdot \hat{\mathbf{v}}_F(P) - \hat{\mathbf{v}}_F \cdot \hat{\mathbf{v}}_H(P) \\
&= \hat{\mathbf{v}}_H\{P, F\} - \hat{\mathbf{v}}_F\{P, H\} \\
&= \{\{P, F\}, H\} - \{\{P, H\}, F\} \\
&= \{P, \{F, H\}\} \\
&= \hat{\mathbf{v}}_{\{F,H\}}(P),
\end{aligned}$$

where we have made use of the Jacobi identity, the skew-symmetry of the Poisson bracket, and the definition (6.4) of a Hamiltonian vector field. Since P is arbitrary, this suffices to prove (6.8). □

Example 6.7. Let $M = \mathbb{R}^2$ with coordinates (p, q) and canonical Poisson bracket $\{F, H\} = F_q H_p - F_p H_q$. For a function $H(p, q)$, the corresponding Hamiltonian vector field is $\hat{\mathbf{v}}_H = H_p \partial_q - H_q \partial_p$. Thus for $H = \frac{1}{2}(p^2 + q^2)$ we have $\hat{\mathbf{v}}_H = p\partial_q - q\partial_p$, whereas for $F = pq$, $\hat{\mathbf{v}}_F = q\partial_q - p\partial_p$. The Poisson bracket of F and H is $\{F, H\} = p^2 - q^2$, which has Hamiltonian vector field $\hat{\mathbf{v}}_{\{F,H\}} = 2p\partial_q + 2q\partial_p$. This agrees with the commutator $[\hat{\mathbf{v}}_H, \hat{\mathbf{v}}_F]$, as the reader can verify.

The Structure Functions

To determine the general local coordinate picture for a Poisson manifold, we first look at the Hamiltonian vector fields. Let $x = (x^1, \ldots, x^m)$ be local coordinates on M and $H(x)$ a real-valued function. The associated Hamiltonian vector field will be of the general form $\hat{\mathbf{v}}_H = \sum_{i=1}^m \xi^i(x)\partial/\partial x^i$, where the coeffi-