

Neglected for many years following Noether's prescient work, generalized symmetries have recently been found to be of importance in the study of nonlinear partial differential equations which, like the Korteweg–de Vries equation, can be viewed as "completely integrable systems". The existence of infinitely many generalized symmetries, usually found via the recursion operator methods of Section 5.2, appears to be intimately connected with the possibility of linearizing the system, either directly through some change of variables, or, more subtly, through some form of inverse scattering method. Thus, the generalized symmetry approach, which is amenable to direct calculation as with ordinary symmetries, provides a systematic means of recognizing these remarkable equations and thereby constructing an infinite collection of conservation laws for them. (The construction of the related scattering problem requires different techniques such as the prolongation methods of Wahlquist and Estabrook, [1].) A systematic method for determining evolution equations having recursion operators, and hence classifying "integrable" systems, is provided by the theory of formal symmetries.

A number of the applications of symmetry group methods to partial differential equations are most naturally done using some form of Hamiltonian structure. The finite-dimensional formulation of Hamiltonian systems of ordinary differential equations is well known, but in preparation for the more recent theory of Hamiltonian systems of evolution equations, we are required to take a slightly novel approach to the whole subject of Hamiltonian mechanics. Here we will de-emphasize the use of canonical coordinates (the p 's and q 's of classical mechanics) and concentrate instead on the Poisson bracket as the cornerstone of the subject. The result is the more general concept of a Poisson structure, which is easily extended to include the infinite-dimensional theory of Hamiltonian systems of evolution equations. An important special case of a Poisson structure is the Lie–Poisson structure on the dual to a Lie algebra, originally discovered by Lie, and more recently used to great effect in geometric quantization, representation theory, and fluid and plasma mechanics. In this general approach to Hamiltonian mechanics, conservation laws can arise not only from symmetry properties of the system, but also from degeneracies of the Poisson bracket itself. In the finite-dimensional set-up, each one-parameter Hamiltonian symmetry group allows us to reduce the order of a system by two. In its modern formulation, the degree of reduction available for multi-parameter symmetry groups is given by the general theory of Marsden and Weinstein, which is based on the concept of a momentum map to the dual of the symmetry Lie algebra. In more recent work, there has been a fair amount of interest in systems of differential equations which possess not just one, but *two* distinct (but compatible) Hamiltonian structures. For such a "bi-Hamiltonian system", there is a direct recursive means of constructing an infinite hierarchy of mutually commuting flows (symmetries) and consequent conservation laws, indicating the system's complete integrability. Most of the soliton equations, as well as

some of the finite-dimensional completely integrable Hamiltonian systems, are in fact bi-Hamiltonian systems.

Underlying much of the theory of generalized symmetries, conservation laws, and Hamiltonian structures for evolution equations is a subject known as the “formal calculus of variations”, which constitutes a calculus specifically devised for answering a wide range of questions dealing with complicated algebraic identities among objects such as the Euler operator from the calculus of variations, generalized symmetries, total derivatives and more general differential operators, and several generalizations of the concept of a differential form. The principal result in the formal variational calculus is the local exactness of a certain complex—called the “variational complex”—which is in a sense the proper generalization to the variational or jet space context of the de Rham complex from algebraic topology. In recent years, this variational complex has been seen to play an increasingly important role in the development of the algebraic and geometric theory of the calculus of variations. Included as special cases of the variational complex are:

- (1) a solution to the “inverse problem of the calculus of variations”, which is to characterize those systems of differential equations which are the Euler–Lagrange equations for some variational problem;
- (2) the characterization of “null Lagrangians”, meaning those variational integrals whose Euler–Lagrange expressions vanish identically, as total divergences; and
- (3) the characterization of trivial conservation laws, also known as “null divergences”, as “total curls”.

Each of these results is vital to the development of our applications of Lie groups to the study of conservation laws and Hamiltonian structures for evolution equations. Since it is not much more difficult to provide the proof of exactness of the full variational complex, Section 5.4 is devoted to a complete development of this proof and application to the three special cases of interest.

Although the book covers a wide range of different applications of Lie groups to differential equations, a number of important topics have necessarily been omitted. Most notable among these omissions is the connection between Lie groups and separation of variables. There are two reasons for this: first, there is an excellent, comprehensive text—Miller, [3]—already available; second, except for special classes of partial differential equations, such as Hamilton–Jacobi and Helmholtz equations, the precise connections between symmetries and separation of variables is not well understood at present. This is especially true in the case of systems of linear equations, or for fully nonlinear separation of variables; in neither case is there even a good definition of what separation of variables really entails, let alone how one uses symmetry properties of the system to detect coordinate systems in which separation of variables is possible. I have also not attempted to cover any of the vast area of representation theory, and the consequent applications to

special function theory; see Miller, [1] or Vilenkin, [1]. Bifurcation theory is another fertile ground for group-theoretic applications; I refer the reader to the lecture notes of Sattinger, [1], and the references therein. Applications of symmetry groups to numerical analysis are given extensive treatment in Shokin, [1], and Dorodnitsyn, [1]. Applications to control theory can be found in van der Schaft, [1], and Ramakrishnan and Schaettler, [1]. See Maeda, [1], and Levi and Winternitz, [2], for applications to difference and differential-difference equations. Extensions of the present methods to boundary value problems for partial differential equations can be found in the books of Bluman and Cole, [1], and Seshadri and Na, [1], and to free boundary problems in Benjamin and Olver, [1]. Although I have given an extensive treatment to generalized symmetries in Chapter 5, the related concept of contact transformations introduced by Lie has not been covered, as it seems much less relevant to the equations arising in applications, and, for the most part, is subsumed by the more general theory presented here; see Anderson and Ibragimov, [1], Bluman and Kumei, [2], and the references therein for these types of transformation groups. Finally, we should mention the use of Lie group methods for differential equations arising in geometry, including, for example, motions in Riemannian manifolds, cf. Ibragimov, [1], or symmetric spaces and invariant differential operators associated with them, cf. Helgason, [1], [2].

Notes to the Reader

The guiding principle in the organization of this book has been so as to enable the reader whose principal goal is to apply Lie group methods to concrete problems to learn the basic computational tools and techniques as quickly as possible and with a minimum of theoretical diversions. At the same time, the computational applications have been placed on a solid theoretical foundation, so that the more mathematically inclined reader can readily delve further into the subject. Each chapter following the first has been arranged so that the applications and examples appear as quickly as feasible, with the more theoretical proofs and explanations coming towards the end. Even should the reader have more theoretical goals in mind, though, I would still strongly recommend that they learn the computational techniques and examples first before proceeding to the general theory. It has been said that it is far easier to abstract a general mathematical theory from a single well-chosen example than it is to apply an existing abstract theory to a specific example, and this, I believe, is certainly the case here.

For the reader whose main interest is in applications, I would recommend the following strategy for reading the book. The principal question is how much of the introductory theory of manifolds, vector fields, Lie groups and Lie algebras (which has, for convenience, been collected together in Chapter 1 and Section 2.1), really needs to be covered before one can proceed to the applications to differential equations starting in Section 2.2. The answer is, in fact, surprisingly little. Manifolds can for the most part be thought of locally, as open subsets of a Euclidean space \mathbb{R}^m in which one has the freedom to change coordinates as one desires. Geometrical symmetry groups will just be collections of transformations on such a subset which satisfy certain elementary group axioms allowing one to compose successive symmetries, take inverses, etc. The key concept in the subject is the infinitesimal generator of a

symmetry group. This is a vector field (of the type already familiar in vector calculus or fluid mechanics) on the underlying manifold or subset of \mathbb{R}^n whose associated flow coincides with the one-parameter group it generates. One can regard the entire group of symmetries as being generated in this manner by composition of the basic flows of its infinitesimal generators. Thus a familiarity with the basic notation for and correspondence between a vector field and its flow is the primary concept required from Chapter 1. The other key result is the infinitesimal criterion for a system of algebraic equations to be invariant under such a group of transformations, which is embodied in Theorem 2.8. With these two tools, one can plunge ahead into the material on differential equations starting in Section 2.2, referring back to further results on Lie groups or manifolds as the need arises.

The generalization of the infinitesimal invariance criterion to systems of differential equations rests on the important technique of “prolonging” the group transformations to include not only the independent variables and dependent variables appearing in the system, but also the derivatives of the dependent variables. This is most easily accomplished in a geometrical manner through the introduction of spaces whose coordinates represent these derivatives: the “jet spaces” of Section 2.3. The key formula for computing symmetry groups of differential equations is the prolongation formula for the infinitesimal generators in Theorem 2.36. Armed with this formula (or, at least the special cases appearing in the following example) and the corresponding infinitesimal invariance criterion, one is ready to compute the symmetry groups of well-nigh any system of ordinary or partial differential equations which may arise. Several illustrative examples of the basic computational techniques required are presented in Section 2.4; readers are also advised to try their hands at some additional examples, either those in the exercises at the end of Chapter 2, or some system of differential equations of their own devising.

At this juncture, a number of options as to what to pursue next present themselves. For the devotee of ordinary differential equations, Section 2.5 provides a detailed summary of the basic method of Lie for integrating these equations using symmetry groups. See also Sections 4.2 and 6.3 for the case of ordinary differential equations with some form of variational structure, either in Lagrangian or Hamiltonian form. Those interested in determining explicit group-invariant solutions to partial differential equations can move directly on to Chapter 3. There the basic method for computing these solutions through reduction is outlined in Section 3.1 and illustrated by a number of examples in Section 3.2. The third section of this chapter addresses the problem of classification of these solutions, and does require some of the more sophisticated results on Lie algebras from Section 1.4. The final two sections of Chapter 3 are devoted to the underlying theory for the reduction method, and are not required for applications, although a discussion of the important Pi theorem from dimensional analysis does appear in Section 3.4.

The reader whose principal interest is in the derivation of conservation laws using Noether's theorem can move directly from Section 2.4 to Chapter 4, which is devoted to the "classical" form of this result. A brief review of the most basic concepts required from the calculus of variations is presented in Section 4.1. The introduction of symmetry groups and the basic infinitesimal invariance criterion for a variational integral is the subject of Section 4.2, along with the reduction procedures available for ordinary differential equations which are the Euler–Lagrange equations for some variational problem. The third section introduces the general notion of a conservation law. Here the treatment is more novel; the guiding concept is the correspondence between conservation laws and their "characteristics", although the technically complicated proof of Theorem 4.26 can be safely omitted on a first reading. Once one learns to deal with conservation laws in characteristic form, the statement and implementation of Noether's theorem is relatively straightforward.

Beginning with Chapter 5, a slightly higher degree of mathematical sophistication is required, although one can still approach much of the material on generalized symmetries and conservation laws from a purely computational point of view with only a minimum of the Lie group machinery. The most difficult algebraic manipulations have been relegated to Section 5.4, where the variational complex is developed in its full glory for the true aficionado. Incidentally, Section 5.4, along with Chapter 7 on Hamiltonian structures for evolution equations are the only parts of the book where the material on differential forms in Section 1.5 is used to any great extent. Despite their seeming complexity, the proofs in Section 5.4 are a substantial improvement over the current versions available in the literature.

Chapter 6 on finite-dimensional Hamiltonian systems is by-and-large independent of much of the earlier material in the book. Up through the reduction method for one-parameter symmetry groups, not very much of the material on Lie groups is required. However, the Marsden–Weinstein reduction theory for multi-parameter symmetry groups does require some of the more sophisticated results on Lie algebras from Sections 1.4 and 3.3. Chapter 7 depends very much on an understanding of the Poisson bracket approach to Hamiltonian mechanics adopted in Chapter 6, and, to a certain extent, the formal variational calculus methods of Section 5.4. Nevertheless, acquiring a basic agility in the relevant computational applications is not that difficult.

The exercises which appear at the end of each chapter vary considerably in their range of difficulty. A few are fairly routine calculations based on the material in the text, but a substantial number provide significant extensions of the basic material. The more difficult exercises are indicated by an asterisk; one or two, signaled by a double asterisk, might be better classed as miniature research projects. A number of the results presented in the exercises are new; otherwise, I have tried to give the most pertinent references where appropriate. Here references have been selected more on the basis of direct relevance for the problem as stated rather than on the basis of historical precedence.

At the end of each chapter is a brief set of notes, mostly discussing the historical details and references for the results discussed there. While I cannot hope to claim full historical accuracy, these notes do represent a fair amount of research into the historical roots of the subject. I have tried to determine the origins and subsequent history of many of the important developments in the area. The resulting notes undoubtedly reflect a large number of personal biases, but, I hope, may provide the groundwork for a more serious look into the fascinating and, at times, bizarre history of this subject, a topic which is well worth the attention of a true historian of mathematics. Although I have, for the most part, listed what I determined to be significant papers in the historical development of the subject, owing to the great duplication of efforts over the decades, I have obviously been unable to provide an exhaustive listing of all the relevant references. I sincerely apologize to those authors whose work does play a significant role in the development, but was inadvertently missed from these notes.

CHAPTER 1

Introduction to Lie Groups

Roughly speaking, a Lie group is a “group” which is also a “manifold”. Of course, to make sense of this definition, we must explain these two basic concepts and how they can be related. Groups arise as an algebraic abstraction of the notion of symmetry; an important example is the group of rotations in the plane or three-dimensional space. Manifolds, which form the fundamental objects in the field of differential geometry, generalize the familiar concepts of curves and surfaces in three-dimensional space. In general, a manifold is a space that locally looks like Euclidean space, but whose global character might be quite different. The conjunction of these two seemingly disparate mathematical ideas combines, and significantly extends, both the algebraic methods of group theory and the multi-variable calculus used in analytic geometry. This resulting theory, particularly the powerful infinitesimal techniques, can then be applied to a wide range of physical and mathematical problems.

The goal of this chapter is to provide the reader with a relatively quick and painless introduction to the theory of manifolds and Lie groups in a form that will be conducive to applications to differential equations. No prior knowledge of either group theory or differential geometry is required, but a good background in basic analysis (i.e. “advanced calculus”), including the inverse and implicit function theorems, will be assumed. Of necessity, proofs of most of the “hard” theorems in Lie group theory will be omitted; references can be found in the notes at the end of the chapter.

Throughout this chapter, I have tried to strike a balance between the local coordinate picture, in which a manifold just looks like an open subset of some Euclidean space, and the more modern global formulation of the theory. Each has its own particular uses and advantages, and it would be a mistake to emphasize one or the other unduly. The applications-oriented

reader might question the inclusion of the global framework here, since admittedly most of the applications of the theory presented in this book take place on open subsets of Euclidean space. Suffice it to say that the geometrical insight and understanding offered by the general notion of a manifold amply repays the relatively slight effort required to gain familiarity with the definition. However, if the reader still remains unconvinced, they can replace the word “manifold” wherever it occurs by “open subset of Euclidean space” without losing too much of the flavour or range of applicability of the theory. With this approach, they should concentrate on the sections on local Lie groups (which were, indeed, the way Lie himself thought of Lie groups) and use these as the principal objects of study.

The first section gives a basic outline of the general concept of a manifold, the second doing the same for Lie groups, both local and global. In practice Lie groups arise as groups of symmetries of some object, or, more precisely, as local groups of transformations acting on some manifold; the second section gives a brief look at these. The most important concept in the entire theory is that of a vector field, which acts as the “infinitesimal generator” of some one-parameter Lie group of transformations. This concept is fundamental for both the development of the theory of Lie groups and the applications to differential equations. It has the crucial effect of replacing complicated nonlinear conditions for the symmetry of some object under a group of transformations by easily verifiable linear conditions reflecting its infinitesimal symmetry under the corresponding vector fields. This technique will be explored in depth for systems of algebraic and differential equations in Chapter 2. The notion of vector field then leads to the concept of a Lie algebra, which can be thought of as the infinitesimal generator of the Lie group itself, the theory of which is developed in Section 1.4. The final section of this chapter gives a brief introduction to differential forms and integration on manifolds.

1.1. Manifolds

Throughout most of this book, we will be primarily interested in objects, such as differential equations, symmetry groups and so on, which are defined on open subsets of Euclidean space \mathbb{R}^m . The underlying geometrical features of these objects will be independent of any particular coordinate system on the open subset which might be used to define them, and it becomes of great importance to free ourselves from the dependence on particular local coordinates, so that our objects will be essentially “coordinate-free”. More specifically, if $U \subset \mathbb{R}^m$ is open and $\psi: U \rightarrow V$, where $V \subset \mathbb{R}^n$ is open, is any diffeomorphism, meaning that ψ is an infinitely differentiable map with infinitely differentiable inverse, then objects defined on U will have equivalent counterparts on V . Although the precise formulae for the object on U and its

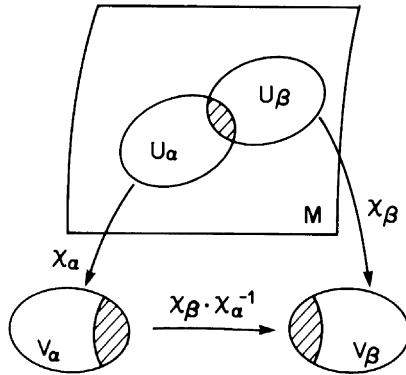


Figure 1. Coordinate charts on a manifold.

counterpart on V will, in general, change, the essential underlying properties will remain the same. Once we have freed ourselves from this dependence on coordinates, it is a small step to the general definition of a smooth manifold. From this point of view, manifolds provide the natural setting for studying objects that do not depend on coordinates.

Definition 1.1. An m -dimensional manifold is a set M , together with a countable collection of subsets $U_\alpha \subset M$, called *coordinate charts*, and one-to-one functions $\chi_\alpha: U_\alpha \rightarrow V_\alpha$ onto connected open subsets $V_\alpha \subset \mathbb{R}^m$, called *local coordinate maps*, which satisfy the following properties:

(a) The coordinate charts cover M :

$$\bigcup_{\alpha} U_\alpha = M.$$

(b) On the overlap of any pair of coordinate charts $U_\alpha \cap U_\beta$ the composite map

$$\chi_\beta \circ \chi_\alpha^{-1}: \chi_\alpha(U_\alpha \cap U_\beta) \rightarrow \chi_\beta(U_\alpha \cap U_\beta)$$

is a smooth (infinitely differentiable) function.

(c) If $x \in U_\alpha$, $\tilde{x} \in U_\beta$ are distinct points of M , then there exist open subsets $W \subset V_\alpha$, $\tilde{W} \subset V_\beta$, with $\chi_\alpha(x) \in W$, $\chi_\beta(\tilde{x}) \in \tilde{W}$, satisfying

$$\chi_\alpha^{-1}(W) \cap \chi_\beta^{-1}(\tilde{W}) = \emptyset.$$

The coordinate charts $\chi_\alpha: U_\alpha \rightarrow V_\alpha$ endow the manifold M with the structure of a topological space. Namely, we require that for each open subset $W \subset V_\alpha \subset \mathbb{R}^m$, $\chi_\alpha^{-1}(W)$ be an open subset of M . These sets form a *basis* for the topology on M , so that $U \subset M$ is open if and only if for each $x \in U$ there is a neighbourhood of x of the above form contained in U ; so $x \in \chi_\alpha^{-1}(W) \subset U$ where $\chi_\alpha: U_\alpha \rightarrow V_\alpha$ is a coordinate chart containing x , and W is an open subset