in the past 15 years. See Holm, [1], for an extensive list of early references up to 1976.

The basic method for computing symmetry groups, using the prolongation formula for their infinitesimal generators, dates back to Lie. Indeed, the recursive form (2.44) of the prolongation formula appears in Lie, [2; § 11], [3; § 1]; see also Eisenhart, [2; equation (28.12)]. The explicit formula (2.39), though, first appears in Olver, [2]. The alternative form (2.50) using the characteristic of such a vector field will be discussed in more detail in Chapter 5; Seshadri and Na, [1; § 3.2(e)] use it as an alternative method for computing ordinary symmetries. In this book, I have chosen to adopt an extremely simplified version of the theory of jets, due in its modern form to Ehresmann, [1], [2], which serves to clearly delineate the geometric foundations of the prolongation theory. A readable account of the more abstract, differential-geometric approach to jets can be found in Golubitsky and Guillemin, [1]; see also Section 3.5. Needless to say, all the results stated here have many alternative restatements and reformulations, using more and more technical and abstract mathematical machinery, a pointless exercise enjoyed by a number of researchers. The net result, of course, is always the same no matter how one tries to dress it up; the unfortunate reader of these versions comes away thoroughly confused, learning nothing of the ease and efficacy of applying this theory to concrete problems. I hope that this book has, for the most part, avoided such pitfalls, and that the use of local coordinates and illustrative examples will genuinely educate the reader interested in applications.

By now, the literature on examples of explicit computations of symmetry groups of specific systems of differential equations has grown too voluminous to attempt to list here. The reader can find references in the book of Ovsiannikov, [3], as well as an extensive, but by no means complete, bibliography in Steinberg, [2]. The calculation of the symmetry group of the Euler equations is due to Buchnev, [1]. The derivation of the Hopf–Cole transformation (actually originally due to Forsyth, [1; Vol. 6, p. 102]—see also Whitham, [2; Chap. 4]) using group-theoretic methods can be found in Kumei and Bluman, [1], along with generalizations. The actual computations for finding the symmetry group of a given system of differential equations are quite mechanical, and are thus amenable to implementation on a computer using a symbolic manipulation program. Several versions have been developed, including Rosenau and Schwarzmeier, [1], Steinberg, [1], Rosencrans, [2], Schwarz, [1], and Champagne, Hereman and Winternitz, [1]. The latter reference contains a complete survey (as of 1991) of available packages, including a discussion of their strengths and weaknesses. (A more up-to-date survey by W. Hereman will appear in *Euromath Bull.* **2** in summer, 1993.)

There are several alternative approaches to the theory of symmetry groups of differential equations worth mentioning. For linear equations, Kalnins, Miller, Boyer and others (see Miller, [2], [3]) have emphasized the use of differential operators rather than vector fields to determine symmetries. The relation between their method and Lie's is made clear in Section 5.2. Ames, [1], proposed a method based on the group transformations themselves, circumventing the introduction of the infinitesimal generators, but this seems to have limited applicability. Poloszny and Rubel, [1], and Rubel, [1], use a method based on the theory of "motions" of a differential equation. Seshadri and Na, [1], make the point that one can considerably simplify the computation of symmetries if one imposes, *a priori*, restrictions on the form of the group, e.g., that it be projectable or a scaling group. An approach based on differential forms was proposed by Harrison and Estabrook, [1], and developed by Edelen, [1], who also describes symbolic manipulation programs based on this method; see also Gragert, Kersten and Martini, [1]. The resuts are the same as the present approach, but their method suffers the drawback of having to first re-express a system of partial differential equations as the integrability conditions for a set of differential forms before one can proceed to the computation of symmetries. Nevertheless, the method can be useful, especially for constructing Bäcklund transformations. An extension of the present infinitesimal method to free boundary problems can be found in Benjamin and Olver, [1].

The more technical matters raised in the final secton of this chapter have only recently been seen to be of importance for symmetry group methods. The connection between existence of solutions and the theory of characteristics dates back to the work of Kovalevskaya, [1]. (The present development of this theory most closely parallels the presentation in Petrovskii, [1].) Bourlet, [1], was the first to demonstrate the existence of systems which could not be solved using the Cauchy–Kovalevskaya theorem in any direction, but did not pursue the matter. Subsequently a number of researchers in the last century, including Delassus, [1], and Riquier, [1], developed quite elaborate existence theorems for systems of partial differential equations generalizing the Cauchy–Kovalevskaya theorem. However, it was not until Finzi, [1], proved the important Lemma 2.85 (see also Hadamard, [1; §25a]) that the true connections between solvability and integrability conditions became evident. The consequent definitions of over- and under-determined systems proposed here are new; see also Olver, [11]. Normality is a more classical concept; see also Vinogradov, [4], for a more technical version of this definition. Although there are definite connections between our definitions and the Spencer, Goldschmidt, *et al.* theory of over-determined systems of partial differential equations, the present terminology is *more* precise. Comparing with the definitions in Pommaret, [1, §V.6.6], (which are for linear systems only) we find Pommaret's underdetermined systems to always have fewer equations than unknowns, whereas his over-determined systems include *both* the under- and over-determined systems of Definition 2.86. These issues are also closely related to questions on the "degree of determinancy" of a system of partial differential equations, which arise in relativity, and were discussed, but never fully resolved, by Cartan and Einstein, [1]. The question of local solvability of systems of partial differential equations is closely connected with the general Riquier existence theory, see Ritt, [1; Chap. 8] for a discussion. Nirenberg, [1; p. 15] proves the local solvability of fairly general types of elliptic systems. Nonsolvability due to integrability conditions was recognized in the last century; the Lewy type of nonsolvable C^∞ systems is much more recent. See Lewy, [1], and Nirenberg, [1; p. 8] for examples. Applications of these results to symmetry group theory have appeared previously in Olver, [2], [7], [11], and Vinogradov, [5].

Exercises

2.1. Let G be a local group of transformations acting on the manifold M.
  (a) Prove that a subset $\mathcal{S} \subset M$ is G-invariant if and only if $\mathcal{S} = \bigcup \mathcal{O}$ is a union of orbits of G.
  (b) Prove Proposition 2.14.
  (c) Prove that a function $F\colon M \to \mathbb{R}^l$ is G-invariant if and only if F is constant on the orbits of G.
  (d) Prove that the only invariants of the irrational flow on the torus are the constant functions.

2.2. Let G be the one-parameter group of transformations of $\mathbb{R}^3$ generated by the vector field $\mathbf{v}$ of Exercise 1.11. Prove that G has only one independent global invariant.

2.3. Let G act on the manifold M, and let $H \subset G$ be a subgroup. Prove that if $\mathcal{S} \subset M$ is a (locally) H-invariant subset, and $g \in G$ is defined on all of $\mathcal{S}$, then $g \cdot \mathcal{S} = \{g \cdot x \colon x \in \mathcal{S}\}$ is (locally) invariant under the conjugate subgroup $gHg^{-1} = \{ghg^{-1}\colon h \in H\}$.

2.4. A system of submanifolds of M is called G-invariant if the group elements g map one submanifold to another submanifold in the system. For example, the set of parallel lines $\{y = kx + b\}$, k fixed, is invariant under *any* translation group of $\mathbb{R}^2$. Prove that the level sets of a function $F\colon M \to \mathbb{R}^l$ are invariant under the transformation group G if and only if $\mathbf{v}(F) = H(F)$ for every infinitesimal generator $\mathbf{v}$ of G, where H, depending on $\mathbf{v}$, is some function defined on the range of F. (Eisenhart, [2; p. 82].)

2.5. (a) Prove the local version of Proposition 2.10.
  (b) Prove the global version using a partition of unity—see Kahn [1; Theorem 1.4].
  (c) Prove Proposition 2.11. (*Hint*: Use Theorem 1.8.)
  (d) Prove that if $R_i(x)$, $i = 1, \ldots, p$, are smooth, then

$$\sum_{i=1}^{p} R_i(x)(x^i - c_i) = \sum_{i=1}^{p} a_i x^i + b$$

is an affine function of x if and only if $R_i(x) = a_i + S_i(x)$, where

$$\sum_{i=1}^{p} S_i(x)(x^i - c_i) \equiv 0,$$

or, equivalently,

$$S_i(x) = \sum_{j=1}^{p} S_{ij}(x)(x^j - c_j)$$

where $S_{ij} = -S_{ji}$.

2.6. Let $X = \mathbb{R}$, $U = \mathbb{R}$ and consider the one-parameter group

$$g_\varepsilon: (x, u) \mapsto (x \cos(r\varepsilon) - u \sin(r\varepsilon), x \sin(r\varepsilon) + u \cos(r\varepsilon)),$$

where $r^2 = x^2 + u^2$. Let $u = f(x)$ be a function defined for all $x \in \mathbb{R}$. Prove that for *any* $\varepsilon \neq 0$, the transformed function $\tilde{u} = g_\varepsilon \cdot f(x)$ is *not* a globally defined function for $\tilde{x} \in \mathbb{R}$. How does this affect our construction of the prolonged group action?

2.7. Find the determining equations for the symmetry group of the nonlinear wave equation $u_t = uu_x$ and determine some particular symmetry groups. Do your results change if the coefficient of u_x is replaced by $f(u)$ for some function f? (See also Example 5.7.)

2.8. The Fokker-Planck equation is

$$u_t = u_{xx} + (xu)_x = u_{xx} + xu_x + u.$$

Find the symmetry group and interpret geometrically. Use the group transformations to determine some particular solutions to this equation. (Bluman and Cole, [2; § 2.10].)

2.9. Find the symmetry group of the telegraph equation $u_{tt} = u_{xx} + u$. Compare this group with that of the equivalent first order system $u_t + u_x = v$, $v_t - v_x = u$.

2.10. Groups of higher order equations and their equivalent first order systems are not always comparable. For instance, compute the symmetry group of the two-dimensional wave equation $u_{tt} = u_{xx}$, and compare this with the symmetry group of the equivalent system $u_t = v$, $u_x = w$, $v_t = w_x$, $v_x = w_t$. What about the two-dimensional Laplace equation? (Olver, [2], Ibragimov, [1; § 17.1].)

*2.11. Prove that the symmetry group (2.65) of the two-dimensional wave equation (omitting the trivial linear symmetries $u \mapsto \lambda u + \alpha(x, t)$) is locally isomorphic to the group SO(3, 2) of linear isometries $z \mapsto Rz$ of $\mathbb{R}^5$ with metric $(dz^1)^2 + (dz^2)^2 + (dz^3)^2 - (dz^4)^2 - (dz^5)^2$. (See Exercise 1.29.) (Miller, [3; p. 223].)

2.12. Find the symmetry group of the m-dimensional heat equation $u_t = \Delta u$, $x \in \mathbb{R}^m$. How does it compare with the one-dimensional case? (Goff, [1].)

2.13. Discuss the symmetry group of the Helmholtz equation $\Delta u + \lambda u = 0$ for λ a fixed constant, $x \in \mathbb{R}^3$. (Miller, [3; § 3.1].)

2.14. Discuss the symmetry group of the biharmonic equation $\Delta^2 u = \Delta(\Delta u) = 0$, $x \in \mathbb{R}^m$. How is it related to the symmetry group for Laplace's equation?

*2.15. Prove that the symmetry group for the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{u} \in \mathbb{R}^2$ or $\mathbb{R}^3$, ν is the viscosity, is the same as that of the corresponding system of Euler equations ($\nu = 0$). (Buchnev, [1], Lloyd, [1].)

*2.16. (a) Maxwell's equations for the electric field $E \in \mathbb{R}^3$ and the magnetic field $B \in \mathbb{R}^3$ take the vector form

$$E_t = \nabla \times B, \qquad B_t = -\nabla \times E, \qquad \nabla \cdot E = 0, \qquad \nabla \cdot B = 0.$$

Discuss the symmetries of this system.

(b) An equivalent formulation is obtained by introducing the vector potential A with $B = \nabla \times A$, and noting that $\nabla \times (A_t + E) = 0$, hence there exists a scalar potential ϕ satisfying $A_t + E = \nabla\phi$. The resulting system is

$$\frac{\partial^2 A}{\partial t^2} + \nabla \times (\nabla \times A) = \nabla \frac{\partial \phi}{\partial t}, \qquad \nabla \cdot \frac{\partial A}{\partial t} = \Delta\phi.$$

How does the symmetry group of this latter system compare with the previous form of Maxwell's equations? (Ovsiannikov, [3; p. 394], Fushchich and Nikitin, [1], [2], and Pohjanpelto, [1].)

*2.17. Perform a symmetry analysis of Navier's equations (2.127) of linear isotropic elasticity. Discuss the difference between the two- and three-dimensional cases. Do your results depend on the values of the Lamé moduli λ and μ? (Olver, [9].)

2.18. *Group Classification.* Often a system of differential equations arising from a physical problem will involve some arbitrary functions whose precise forms depend on the specific physical system under consideration. For example, the general equation of nonlinear heat conduction takes the form

$$u_t = D_x(K(u)u_x), \tag{$*$}$$

where $K(u)$ depends on the particular type of conductor being modelled. There are often good physical motivations for studying those equations in which the form of the arbitrary functions provides a larger symmetry group than would otherwise be applicable. The problem of determining such functions is known as the *group classification problem.* Perform a group classification on the nonlinear heat conduction equation by proving:

(a) If K is arbitrary (i.e. not any of the following special forms), then $(*)$ has a three-parameter symmetry group.

(b) If $K(u) = (au + b)^m$ for $m \neq -\frac{4}{3}$, $a \neq 0$, the symmetry group is four-dimensional.

(c) For $K(u) = ce^{\alpha u}$, there is a four-parameter group.

(d) For $K(u) = (au + b)^{-4/3}$, $a \neq 0$, there is a five-parameter group.

(e) For $K(u)$ constant the group is infinite-dimensional.

(Ovsiannikov, [3; pp. 68–73], Lisle, [1]; see also Lie, [2].)

2.19. Consider a first order homogeneous linear partial differential equation

$$\sum_{i=1}^{p} \xi^i(x) \frac{\partial u}{\partial x^i} = 0, \tag{$*$}$$

and let $\mathbf{v} = \sum \xi^i(x)\partial_i$ be the corresponding vector field.

(a) Show that $\mathbf{w} = \sum \eta^i(x)\partial_i$ generates a one-parameter symmetry group if and only if $[\mathbf{v}, \mathbf{w}] = \gamma\mathbf{v}$ for some scalar-valued function $\gamma(x)$.

(b) Suppose $p = 2$. Show that if $\mathbf{w}$ generates a nontrivial symmetry group, meaning $\mathbf{w} \neq \lambda\mathbf{v}$ for some function $\lambda(x)$, then we can find the general solution to $(*)$ by quadrature (provided we know the invariants of $\mathbf{w}$).

(c) What about $p \geqslant 3$?

(Lie, [5; p. 434].)

2.20. Prove that the system $u_x = 0$, $u_y + xu_z = 0$ is not locally solvable. Prove that the group generated by $\mathbf{v} = x\partial_z$ is a symmetry group, but $\mathbf{v}$ does not satisfy the infinitesimal criterion (2.120).

*2.21. Suppose the differential equation $P(x, u^{(n)}) = 0$, $x \in \mathbb{R}^p$, $u \in \mathbb{R}$, admits an infinite-dimensional symmetry group with generators $\rho(x)\partial_u$ where ρ is an arbitrary solution to a linear differential equation $\Delta[\rho] = 0$. Prove that P is equivalent to a nonhomogeneous version of the same equation: $\Delta[u] = f(x)$. (Kumei and Bluman, [1].)

*2.22. (a) Prove that a differential equation $P(x, u^{(n)}) = 0$ is equivalent to a *linear* differential equation $\Delta[\tilde{u}] = f(\tilde{x})$ under a change of variables $x = \Xi(\tilde{x}, \tilde{u})$, $u = \Phi(\tilde{x}, \tilde{u})$ if and only if it admits an infinite-dimensional abelian symmetry group with generators of the form

$$\mathbf{v} = \rho(\Xi(\tilde{x}, \tilde{u}), \Phi(\tilde{x}, \tilde{u}))\left\{\frac{\partial \Xi}{\partial \tilde{u}}\frac{\partial}{\partial \tilde{x}} + \frac{\partial \Phi}{\partial \tilde{u}}\frac{\partial}{\partial \tilde{u}}\right\},$$

where $\rho(x, u)$ is an arbitrary solution to a linear differential equation. (*Hint*: Change variables and use the previous exercise.)

(b) Discuss our derivation of the Hopf–Cole transformation in Example 2.42 in light of this result.

(c) Apply this technique to linearize the Thomas equation

$$u_{xt} + \alpha u_t + \beta u_x + \gamma u_x u_t = 0,$$

which arises in the study of chemical exchange processes.

(d) Apply this technique to the potential form $u_t = u_x^{-2}u_{xx}$ of the nonlinear diffusion equation $v_t = D_x(v^{-2}v_x)$ of importance in porous media flow and solid state physics.

(Kumei and Bluman, [1], Whitham [2; p. 95], Rosen, [1], Fokas and Yortsos, [1], Bluman and Kumei, [1].)

*2.23. Two evolution equations, $u_t = P(x, u^{(n)})$ and $v_s = Q(y, v^{(m)})$, are said to be *related* if there exists a change of variables

$$t = T(s, y), \qquad x = \Xi(s, y), \qquad u = \Phi(s, y, v),$$

changing one into an equation equivalent to the other.

(a) Prove that if $u_t = P$ is related to $u_s = Q$, then

$$\mathbf{v} = \frac{\partial T}{\partial s}\frac{\partial}{\partial t} + \frac{\partial \Xi}{\partial s}\frac{\partial}{\partial x} + \frac{\partial \Phi}{\partial s}\frac{\partial}{\partial u}$$

is a symmetry of $u_t = P$.

(b) Prove that if

$$\mathbf{v} = \tau(t)\frac{\partial}{\partial t} + \xi(t, x)\frac{\partial}{\partial x} + \phi(t, x, u)\frac{\partial}{\partial u} \qquad (*)$$

is a symmetry of $u_t = P$, then there is a related evolution equation $v_s = Q$ with $\mathbf{v} = \partial_s$ in the new coordinates. (In fact, for a large class of evolution equations, (*) is the most general symmetry, so we have a one-to-one correspondence between related evolution equations and symmetries.)

(c) Find a transformation relating the Korteweg–de Vries equation $u_t = u_{xxx} + uu_x$ and the equation $v_s = v_{yyy} + vv_y + 1$. (Kalnins and Miller, [2].)

2.24. Let $\alpha \in \mathbb{R}$. Find the most general first order ordinary differential equation invariant under the scaling group $(x, u) \mapsto (\lambda x, \lambda^\alpha u)$, $\lambda > 0$. How are these equations solved by quadrature?

2.25. (a) Prove that the scaling group $(x, u) \mapsto (\lambda x, \lambda u)$ is the only continuous symmetry group of equation (2.95). (Hint: First change coordinates to straighten out the scaling generator.)
(b) Prove that the second order equation $u_{xx} = xu + \tan(u_x)$ has *no* continuous symmetry groups! (Cohen, [1; p. 206].)

2.26. (a) Prove that the symmetry group of the equation $u_{xx} = 0$ is eight-dimensional, generated by

$$\begin{array}{llll} \partial_x, & x\partial_x, & u\partial_x, & xu\partial_x + u^2\partial_u, \\ \partial_u, & x\partial_u, & u\partial_u, & x^2\partial_x + xu\partial_u. \end{array}$$

Prove that the corresponding group is the projective group in the plane, namely,

$$(x, u) \mapsto \left(\frac{ax + bu + c}{\alpha x + \beta u + \gamma}, \frac{dx + eu + f}{\alpha x + \beta u + \gamma}\right),$$

where $\det \begin{vmatrix} a & b & c \\ d & e & f \\ \alpha & \beta & \gamma \end{vmatrix} \neq 0$. Interpret these transformations geometrically.
(b) For $n \geqslant 3$, prove that the symmetry group of $d^n u/dx^n = 0$ is $(n + 4)$-dimensional.
(Lie, [3], Markus, [1].)

**2.27. (a) Prove that a second order ordinary differential equation admits a symmetry group of dimension at most 8. Moreover, if the equation has an eight-parameter symmetry group, then it can be transformed into the elementary equation $u_{xx} = 0$. Show, in particular, that a homogeneous linear second order ordinary differential equation has an eight-parameter symmetry group, and find the explicit transformation that reduces it to $u_{xx} = 0$. (Note that you will have to know a basis for the solution space to the equation, so this method is of no use for solving the equation!)
(b) Prove that, for $n \geqslant 3$, an n-th order ordinary differential equation has at most an $(n + 4)$-parameter symmetry group. (Lie, [5].)
Interestingly, although the corresponding maximal dimension for symmetry groups of second order systems of ordinary differential equations is known, a significant open problem is to determine the maximal dimension of the symmetry group of a higher order system of ordinary differential equations. Bounds are known, cf. González–Gascon and González–López, [1], but there are no known systems that achieve the bounds.

2.28. Prove that SL(2) is not a solvable Lie group. How about SO(3)?

*2.29. Consider the ordinary differential equation Δ: $u_x^2 - 4u = 0$.
(a) Prove that Δ is of maximal rank everywhere.
(b) Prove that all prolongations $\Delta^{(m)}$ are of maximal rank provided $u \neq 0$. However, $u_{xxx}^2 = 0$ is a combination of the equations in $\Delta^{(5)}$, but $u_{xxx} = 0$ is *not*. Discuss.
(Ritt, [1; p. 79].)

2.30. Suppose we know the general solution to the first order ordinary differential equation $u_x = F_0(x, u)$. This knowledge implies that we can, using the methods of Section 2.5, integrate any first order ordinary differential equation invariant under the one-parameter group generated by the vector field $\mathbf{v}_0 = \partial_x + F_0(x, u)\partial_u$. Determine the most general first order equation $u_x = F_1(x, u)$ admitting $\mathbf{v}_0$ as a symmetry. This method can then be iterated to give a "tower" of first order ordinary differential equations $u_x = F_n(x, u)$, $n \geqslant 0$, the n-th equation providing a symmetry, and hence an integration method, for the $(n + 1)$-st equation. Investigate this method for the translation, scaling and rotation groups. Describe the tower that starts with a first order linear ordinary differential equation. (Beyer, [1].)

*2.31. *Equations Invariant under "Nonlocal Symmetries"*. Lest the reader think that all methods for integrating ordinary differential equations reduce to the invariance of the equation under some symmetry group of the type presented here, we offer the following cautionary problems.

(a) An *exponential vector field* is a formal expression of the form

$$\mathbf{v}^* = e^{\int P(x,u)\,dx}\left(\xi(x, u)\frac{\partial}{\partial x} + \phi(x, u)\frac{\partial}{\partial u}\right),$$

where $\int P(x, u)\,dx$ is, formally, the integral of the function $P(x, u)$, once we choose a function $u = f(x)$. Thus

$$D_x\left[\int P(x, u)\,dx\right] = P(x, u), \qquad D_x^2\left[\int P(x, u)\,dx\right] = D_x P,$$

and so on. Substituting $\mathbf{v}^*$ into the prolongation formula (2.50) (with ξ replaced by $e^{\int P\,dx}\xi$, ϕ by $e^{\int P\,dx}\phi$), prove that

$$\mathrm{pr}^{(n)}\,\mathbf{v}^* = e^{\int P(x,u)\,dx}\cdot \mathbf{v}^{(n)},$$

where $\mathbf{v}^{(n)}$ is an ordinary vector field on $M^{(n)}$. For example, if $\mathbf{v}^* = e^{\int u\,dx}\,\partial_u$, then

$$\begin{aligned}\mathrm{pr}^{(n)}\,\mathbf{v}^* &= \sum_{k=0}^{n} D_x^k[e^{\int u\,dx}]\frac{\partial}{\partial u_k}\\ &= e^{\int u\,dx}[\partial_u + u\partial_{u_x} + (u_x + u^2)\partial_{u_{xx}} + \cdots].\end{aligned}$$

(b) Prove that one can choose differential invariants for an exponential vector field of the form

$$y = \eta(x, u), \qquad w = \zeta(x, u, u_x), \qquad w_n = d^n w/dy^n, \qquad n = 1, 2, \ldots,$$

just as for an ordinary vector field. What are the third order differential invariants of $e^{\int u\,dx}\partial_u$?

(c) Prove that an ordinary differential equation $\Delta(x, u^{(n)}) = 0$ which is invariant under an exponential vector field: $\mathrm{pr}^{(n)}\,\mathbf{v}^*(\Delta) = 0$ whenever $\Delta = 0$, can be reduced in order by one. Use this to reduce the equation

$$u_{xx} - uu_x = H(x, u_x - \tfrac{1}{2}u^2)$$

to a first order ordinary differential equation.

(d) Conversely, prove that if Δ can be reduced in order by one by setting $v = \gamma(x, u, u_x)$, then Δ is invariant under the exponential vector field

$$\mathbf{v}^* = \exp\left[-\int \frac{\partial\gamma/\partial u}{\partial\gamma/\partial u_x}\,dx\right]\frac{\partial}{\partial u}.$$

(e) How might these symmetries arise in practice? Consider the "wrong" reduction procedure used in Example 2.62 to obtain (2.109). We would like to say that the other symmetry $\mathbf{v}$ of (2.108) remains a symmetry of (2.109). However, $\mathrm{pr}^{(1)}\,\mathbf{v} = x\partial_u + \partial_{u_x} = x(\partial_y + \partial_z)$ is not a well-defined vector field in the (y, z)-coordinates. Prove that $\mathrm{pr}^{(1)}\,\mathbf{v}$ is an *exponential* vector field in these coordinates. (*Hint*: Show $x = \exp(\int z^{-1}\,dy)$.) Moreover, it remains a symmetry of (2.109). Use this informaton to complete the integration of (2.108).

*2.32. (a) Prove that the second order equation

$$u_{xx} = D_x[(x + x^2)e^u]$$

has a trivial symmetry group.

(b) Show that, nevertheless, the equation can be explicitly solved by quadrature.

(c) Does the equation have any exponential symmetries? (See Exercise 2.31.) (Ibragimov, [2].)

2.33. Let $\mathsf{M}(\omega)$ be a $q \times q$ matrix of homogeneous n-th order polynomials in ω. Prove that $\det \mathsf{M}(\omega) = 0$ for all ω if and only if there is a vector $\sigma(\omega)$ of homogeneous polynomials such that $\mathsf{M}(\omega)\sigma(\omega) = 0$ for all ω. What is the minimal degree of the polynomials required for σ? Generalize to the case when the polynomials in M are homogeneous, but not all of the same degree. (Finzi, [1].)

2.34. Is a system of evolution equations always normal?

2.35. Let $\Delta_\nu(x, u^{(n)})$, $\nu = 1, \ldots, l$, be a totally nondegenerate system of differential equations. Prove that a function $Q(x, u^{(m)}) = 0$ vanishes for all solutions $u = f(x)$ to Δ if and only if there exist differential operators $\mathscr{D}_\nu = \sum_J P_\nu^J(x, u^{(m)})D_J$, $\nu = 1, \ldots, l$, such that $Q = \sum_\nu \mathscr{D}_\nu \Delta_\nu$ for all functions $u = f(x)$. (*Hint*: Use Proposition 2.10.)

CHAPTER 3

Group-Invariant Solutions

When one is confronted with a complicated system of partial differential equations arising from some physically important problem, the discovery of any explicit solution whatsoever is of great interest. Explicit solutions can be used as models for physical experiments, as benchmarks for testing numerical methods, etc., and often reflect the asymptotic or dominant behaviour of more general types of solutions. The methods used to find group-invariant solutions, generalizing the well-known techniques for finding similarity solutions, provide a systematic computational method for determining large classes of special solutions. These group-invariant solutions are characterized by their invariance under some symmetry group of the system of partial differential equations; the more symmetrical the solution, the easier it is to construct. The fundamental theorem on group-invariant solutions roughly states that the solutions which are invariant under a given r-parameter symmetry group of the system can all be found by solving a system of differential equations involving r fewer independent variables than the original system. In particular, if the number of parameters is one less than the number of independent variables in the physical system, $r = p - 1$, then all the corresponding group-invariant solutions can be found by solving a system of *ordinary differential equations*. In this way, one reduces an intractable set of partial differential equations to a simpler set of ordinary differential equations which one might stand a chance of solving explicitly. In practical applications, these group-invariant solutions can, in most instances, be effectively found and, often, are the only explicit solutions which are known.

This chapter is organized so that the applications-oriented reader can immediately learn the practical implementation of the method of constructing group-invariant solutions without having to delve into the theoretical foundations needed to justify the method. Section 3.1 outlines the method,