42 Multi-instantons in quantum mechanics (QM)

In general, a linear combination of instanton solutions is not a solution of the imaginary-time equations of motion, because the equations are not linear. Moreover, in QM, all solutions of the classical equations can depend only on one time collective coordinate (in this respect, in field theory, the situation is different, see Sections 39.5, [399]). However, a linear combination of largely separated instantons (a *multi-instanton* configuration) renders the action almost stationary, because each instanton solution differs from a constant solution by only exponentially small corrections at large distances (in field theory this is only true if the theory is massive). In the context of QM, we examine the possible contributions of such multi-instanton configurations. However, the generalization to quantum field theory (QFT) is not trivial, and is, to a large extent, still to be worked out, even if some progress has been reported [452].

In several simple situations, multi-instantons are expected to play a role. In the case in which instantons are found, when calculating the contribution to $\operatorname{tr} \mathrm{e}^{-\beta H}$ at finite β, we have always kept only the solution that describes the classical trajectory once. We have argued that the other solutions, in which the trajectory is described n-times, have in the large β limit an action n-times larger, and, therefore, give subleading contributions to the path integral. In the large β limit, these configurations have the properties one expects from n-instantons. However, there is a subtlety: naively, one expects these configurations to give contributions of order β, because a classical solution depends only on one time parameter. On the other hand, the ground state energy E_0 has an expansion of the form

$$E_0 = E_0^{(0)} + E_0^{(1)} + \cdots \,,$$

in which $E_0^{(0)}$ and $E_0^{(1)}$ are the perturbative and one-instanton contributions, respectively, and the dots represent possible multi-instanton contributions. The ground state energy has been derived from a semi-classical calculation, at β large, of the partition function, which then has the form

$$\operatorname{tr} \mathrm{e}^{-\beta H} \sim \mathrm{e}^{-\beta E_0} \sim \mathrm{e}^{-\beta E_0^{(0)}} \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \left(E_0^{(1)}\right)^n . \tag{42.1}$$

Thus, the existence of a one-instanton contribution to the energy implies the existence of n-instanton contributions to the partition function proportional to β^n instead of β.

Another example is provided by large-order behaviour estimates of perturbation theory for potentials with degenerate minima (Section 40.1.3). When one starts from a situation in which the minima are almost degenerate, one obtains, in the degenerate limit, a contribution that has the interpretation of the contribution of two, infinitely separated, instantons, multiplied by an infinite multiplicative coefficient. The divergence has the following interpretation: in the degenerate limit, the fluctuations that correspond to changing the distance between largely separated instanton and anti-instanton, induce a vanishingly small variation of the action. It follows that, to correctly calculate the limit, one has to introduce a second collective coordinate which describes these fluctuations, although there is no corresponding symmetry of the action.

Quantum Field Theory and Critical Phenomena 5E. Fifth edition. Jean Zinn-Justin, Oxford University Press (2021). © Jean Zinn-Justin. DOI: 10.1093/oso/9780198834625.003.0042

It can then also be understood where, in the first example, the factor β^n comes from. Although a given classical trajectory can only generate a factor β, these new configurations depend on n independent collective coordinates over which one has to integrate.

To summarize, multi-instanton contributions do exist. However, multi-instantons are not solutions of the classical equation of motion. They correspond to configurations of largely separated instantons connected in a way that has to be examined. They become solutions of the equation of motion only asymptotically, in the limit of infinite separation, and depend on n times more collective coordinates than the one-instanton configuration.

In Sections 42.1 and 42.2, we first return to two examples that we have already discussed in Chapter 39: the quartic *double-well* potential, and the *periodic cosine* potential. We then discuss general potentials with degenerate minima. We also calculate the large order behaviour in the case of the $O(\nu)$-symmetric anharmonic oscillator.

The determination, at leading order, of the many-instanton contributions has led to conjecture the exact form of the semi-classical expansion for potentials with degenerate minima, generalizing the exact Bohr–Sommerfeld quantization condition. Although a rigorous analysis of the problem relies on Wentzel–Kramers–Brillouin (WKB) exact methods and resurgence theory, the original multi-instanton viewpoint remains quite intuitive, and could still be useful in QFT, which cannot be described in terms of differential equations.

The appendix contains some technical remarks about the calculation of multi-instanton contributions, a simple example of a non Borel-summable expansion, and a discussion of the generalized Bohr–Sommerfeld quantization condition within the framework of Schrödinger's equation and WKB approximation.

42.1 The quartic double-well potential

We first consider the Hamiltonian of the double-well potential (see Section 39.1 for details),

$$H = -\tfrac{1}{2}\left(\mathrm{d}/\mathrm{d}q\right)^2 + V(q\sqrt{g})/g, \quad \text{with} \quad V(q) = \tfrac{1}{2}q^2(1-q)^2, \quad g > 0\,. \tag{42.2}$$

We derive properties of the spectrum by considering both the partition function $\operatorname{tr} \mathrm{e}^{-\beta H}$ and a twisted partition function $\mathrm{P} \operatorname{tr} \mathrm{e}^{-\beta H}$, where P is the reflection operator corresponding to the symmetry $q \mapsto g^{-1/2} - q$ of the Hamiltonian.

The partition function is given by the path integral (equation (39.8))

$$\mathcal{Z}(\beta) = \int [\mathrm{d}q(t)] \exp\left[-\mathcal{S}(q)\right], \text{ with } \mathcal{S}(q) = \int_{-\beta/2}^{\beta/2} \left[\tfrac{1}{2}\dot{q}^2(t) + V\big(q(t)\sqrt{g}\big)/g\right] \mathrm{d}t\,, \tag{42.3}$$

and the paths satisfy periodic boundary conditions: $q(-\beta/2) = q(\beta/2)$. The path integral representation of the twisted partition function $\operatorname{tr} \mathrm{P}\, \mathrm{e}^{-\beta H}$ differs by the boundary conditions, which are $q(-\beta/2) + q(\beta/2) = g^{-1/2}$.

In the infinite β limit, the instanton solutions are (equation (39.11)),

$$q_c(t) = f\big(\mp(t-t_0)\big)/\sqrt{g}\,, \quad \text{with} \quad f(t) = 1/(1+\mathrm{e}^{-t}) = 1 - f(-t)\,, \tag{42.4}$$

where the constant t_0 characterizes the instanton position, and the corresponding classical action is (equation (39.12))

$$\mathcal{S}(q_c) = \frac{1}{g}\int \mathrm{d}t \left[\tfrac{1}{2}\dot{f}^2(t) + V\big(f(t)\big)\right] = \frac{1}{6g}\,. \tag{42.5}$$

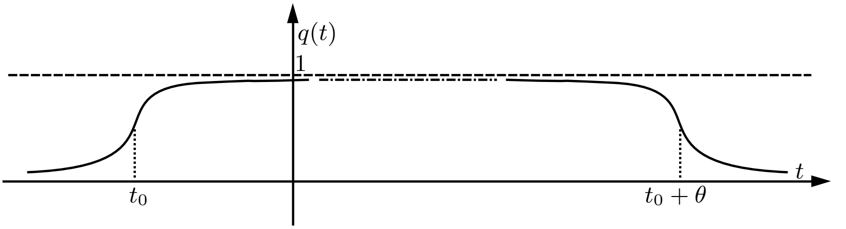

Fig. 42.1 A two-instanton configuration

42.1.1 The two-instanton configuration

We define a two-instanton (really an instanton–anti-instanton) configuration, as a sufficiently differentiable path, which depends on an additional time parameter, characterizing the separation between instantons. For large separations, it must decompose into the superposition of two instantons (Fig. 42.1), and minimize the variation of the action.

For this purpose, we could introduce a constraint in the path integral fixing the separation between instantons and solve the equation of motion with a Lagrange multiplier for the constraint (see Sections 37.4.1 and A42.2.1). Instead, we use a method which, at least at leading order is simpler, and shows that the result is universal [406, 407, 408].

We consider a path $q_c(t)$, sum of instantons separated by a distance $\theta > 0$, up to an additive constant adjusted in such a way as to satisfy the boundary conditions. It is convenient to introduce the notation

$$\underline{u}(t) = f(t+\theta/2), \quad v(t) = 1 - \underline{u}(-t) = \underline{u}(t-t_0)\,, \tag{42.6}$$

where $f(t)$ is the function (42.4). We then consider the path $q_c(t)$, with

$$q_c(t)\sqrt{g} = \underline{u}(t) + \underline{u}(-t) - 1 = \underline{u}(t) - v(t) \tag{42.7}$$

(Again time translation $t \mapsto t + t_0$ generates a set of degenerate configurations). The path has the following properties: it is continuous and differentiable and, when θ is large, it differs, near each instanton, from the instanton solution only by exponentially small terms of order $\mathrm{e}^{-\theta}$. Although the calculation of the corresponding action is simple, we describe it stepwise to show the generality of the ansatz. The action of the path (42.7) is

$$\mathcal{S}(q_c) = \int \mathrm{d}t \left[\tfrac{1}{2}\dot{q}_c^2(t) + V\big(q_c(t)\sqrt{g}\big)/g\right] = \frac{2}{6g} + \frac{1}{g}\Sigma(\underline{u}, v), \text{ with}$$
$$\Sigma(\underline{u}, v) = \int \mathrm{d}t \left[-\dot{\underline{u}}(t)\dot{v}(t) + V\big(\underline{u}(t) - v(t)\big) - V\big(\underline{u}(t)\big) - V\big(v(t)\big)\right], \tag{42.8}$$

where equation (42.5) for $\underline{u}$ and v has been used. Since $q_c(t)$ is even, the integration can be restricted to the region $t < 0$, where $v(t)$ is small. After an integration by parts of the term $\dot{\underline{u}}(t)\dot{v}(t)$, we find

$$\Sigma(\underline{u}, v) = -2v(0)\dot{\underline{u}}(0) + 2\int_{-\infty}^{0} \mathrm{d}t \left[v(t)\ddot{\underline{u}}(t) + V\big(\underline{u}(t) - v(t)\big) - V\big(\underline{u}(t)\big) - V\big(v(t)\big)\right].$$

We now expand in powers of v, and take into account the equation of motion for $\underline{u}$. The leading terms is of order $\mathrm{e}^{-\theta}$, and we thus stop at order v^2. We obtain

$$\Sigma(\underline{u}, v) = -2v(0)\dot{\underline{u}}(0) + 2\int_{-\infty}^{0} \mathrm{d}t \left[\tfrac{1}{2}v^2(t)V''\big(\underline{u}(t)\big) - V\big(v(t)\big)\right]. \tag{42.9}$$

The function v decreases exponentially away from the origin. Therefore, the main contributions to the integral come from the neighbourhood of $t = 0$, where $V''(\underline{u}) \sim 1$. Moreover, $V(v)$ can be replaced at leading order by $\frac{1}{2}v^2$, and the two terms cancel.

The only term left is the integrated contribution,

$$v(0)\underline{\dot{u}}(0) \sim \mathrm{e}^{-\theta},$$

and thus

$$\mathcal{S}(q_c) = g^{-1}\left[\tfrac{1}{3} - 2\,\mathrm{e}^{-\theta} + O\left(\mathrm{e}^{-2\theta}\right)\right]. \tag{42.10}$$

It will become clearer later why we need the classical action only up to order $\mathrm{e}^{-\theta}$ (a contribution that we call later *instanton interaction*). It is also useful to keep β large, but finite, in the calculation. Symmetry between θ and $\beta - \theta$ then implies

$$\mathcal{S}(q_c) = g^{-1}\left[\tfrac{1}{3} - 2\,\mathrm{e}^{-\theta} - 2\,\mathrm{e}^{-(\beta-\theta)}\right]. \tag{42.11}$$

As a verification, we calculate the extremum of $\mathcal{S}(q)$ at β fixed and obtain

$$\theta_c = \beta/2, \;\Rightarrow\; \mathcal{S}(q_c) = g^{-1}\left[\tfrac{1}{3} - 4\,\mathrm{e}^{-\beta/2} + O\left(\mathrm{e}^{-\beta}\right)\right].$$

In Chapter 39, for the same Hamiltonian, we have found for β large (equation (39.14))

$$\mathcal{S}(q_c) = g^{-1}\left[\tfrac{1}{6} - 2\,\mathrm{e}^{-\beta} + O\left(\mathrm{e}^{-2\beta}\right)\right]. \tag{42.12}$$

Both results are consistent. Indeed, to compare them, we have to replace β by $\beta/2$ in equation (42.12) and multiply the action by a factor 2, since the action corresponds to a trajectory described twice in the total time β.

The variation of the action. We now show that if we infinitesimally (for θ large) modify the configuration to further decrease the variation of the action, the change $r(t)$ of the path will be of order $\mathrm{e}^{-\theta}$, and the variation of the action of order $\mathrm{e}^{-2\theta}$, at least. Setting

$$q(t) = q_c(t) + r(t)\,, \tag{42.13}$$

and expanding the action up to second order in $r(t)$, we find,

$$\begin{aligned}\mathcal{S}(q_c + r) = \mathcal{S}(q_c) &+ \int\left[\dot{q}_c(t)\dot{r}(t) + \frac{1}{\sqrt{g}}V'\big(q_c(t)\sqrt{g}\big)r(t)\right]\mathrm{d}t \\ &+ \frac{1}{2}\int \mathrm{d}t\,\left[\dot{r}^2(t) + V''(q_c\sqrt{g})r^2(t)\right] + O\left([r(t)]^3\right).\end{aligned} \tag{42.14}$$

In the term linear in $r(t)$, we integrate by parts $\dot{r}(t)$ in order to use the property that $q_c(t)$ approximately satisfies the equation of motion. In the term proportional to $r^2(t)$, we replace V'' by 1, since we expect $r(t)$ to be large only far from the instantons. We then verify that the term linear in r is of order $\mathrm{e}^{-\theta}$, while the quadratic term is of order 1. A shift of r, to eliminate the linear term in r, would then generate a contribution of order $\mathrm{e}^{-2\theta}$, which is negligible at the order we calculate.

42.1.2 The n-instanton configuration and action

An n-instanton configuration corresponds to a succession of n instantons, separated by times θ_i, such that

$$\textstyle\sum_{i=1}^{n}\theta_i = \beta\,. \tag{42.15}$$

At leading order, we only need to take into account 'interactions' between nearest neighbours. This is an essential simplifying feature of one-dimensional QM. The classical action $\mathcal{S}_c(\theta_i)$ can then be directly inferred from expression (42.11):

$$\mathcal{S}_c(\theta_i) = \frac{1}{g}\left[\frac{n}{6} - 2\sum_{i=1}^{n}\mathrm{e}^{-\theta_i} + O\left(\mathrm{e}^{-(\theta_i+\theta_j)}\right)\right]. \tag{42.16}$$

Other interactions are negligible, because they are of higher order in $\mathrm{e}^{-\theta}$.

Note that, for n even, the n-instanton configuration contributes to $\operatorname{tr} \mathrm{e}^{-\beta H}$, while for n odd it contributes to $\operatorname{tr} P \mathrm{e}^{-\beta H}$ (P is the reflection operator). Then, calculating

$$\mathcal{Z}_\epsilon = \tfrac{1}{2} \operatorname{tr} \left[(1 + \epsilon P) \, \mathrm{e}^{-\beta H} \right], \tag{42.17}$$

we obtain for $\epsilon = +1$ and $\epsilon = -1$ contributions to the even and odd eigenstate energies, respectively.

Remark. Since one keeps in the action all terms of order $\mathrm{e}^{-\beta}$, one expects to find the contributions not only to the two lowest eigenvalues, but also to all eigenvalues that remain finite when g goes to zero (see the remark after equation (37.95) in Section 37.7).

42.1.3 The n-instanton contribution

We have calculated the n-instanton action. We now evaluate, at leading order, the contribution to the path integral of the neighbourhood of the classical path. Although the path is not a solution of the equation of motion, we have defined it in such a way that the linear terms can be neglected in the Gaussian integration. The second derivative of the action at the classical path,

$$M(t', t) = \left[-\mathrm{d}_t^2 + V''\bigl(\sqrt{g} q_c(t)\bigr) \right] \delta(t - t') \tag{42.18}$$

has the form of a quantum Hamiltonian, with a potential that consists of n largely separated wells, almost identical to the well arising in the one-instanton problem. Therefore, at leading order, the corresponding spectrum is the spectrum arising in the one-instanton problem n-times degenerate. Corrections are exponentially small in the separation (for details see Section A42.1). Moreover, by introducing n collective time variables, we suppress n times the eigenvalue 0, and generate the Jacobian of the one-instanton case to the power n. Therefore, the n-instanton contribution to $\mathcal{Z}_\epsilon$ (equation (42.17)) reduces to

$$\mathcal{Z}_\epsilon^{(n)} = \mathrm{e}^{-\beta/2} \frac{\beta}{n} \left(\frac{\mathrm{e}^{-1/6g}}{\sqrt{\pi g}} \right)^n \int_{\theta_i \ge 0} \delta\left(\sum \theta_i - \beta \right) \prod_i \mathrm{d}\theta_i \exp\left(\frac{2}{g} \sum_{i=1}^{n} \mathrm{e}^{-\theta_i} \right). \tag{42.19}$$

All factors have already been explained, except the factor β which comes from the integration over a global time translation, and the factor $1/n$, which arises because the configuration is invariant under a cyclic permutation of the θ_i. The factor $\mathrm{e}^{-\beta/2}$ is the usual normalization factor.

We define the quantity κ, the 'fugacity' of the instanton gas,

$$\kappa = \frac{\epsilon}{\sqrt{\pi g}} \mathrm{e}^{-1/6g}, \tag{42.20}$$

which is half the one-instanton contribution at leading order. It is then convenient to introduce the sum $\Sigma(\beta, g)$ of the leading order n-instanton contributions:

$$\Sigma(\beta, g) = \mathrm{e}^{-\beta/2} + \sum_{n=1}^{\infty} \mathcal{Z}_\epsilon^{(n)}(\beta, g)\,. \tag{42.21}$$

If one neglects the instanton interaction (the dilute gas approximation), one can integrate over the θ_i's, and calculate the sum. One finds

$$\Sigma(\beta, g) = \mathrm{e}^{-\beta/2} \left(1 + \beta \sum_{n=1}^{\infty} \frac{\kappa^n}{n} \frac{\beta^{n-1}}{(n-1)!} \right) = \mathrm{e}^{-\beta(1/2 - \kappa)}\,. \tag{42.22}$$

We recognize the perturbative and one-instanton contributions, at leading order, to $E_\epsilon(g)$, the ground state and the first excited state energies:

$$E_\epsilon(g) = \frac{1}{2} + O(g) - \frac{\epsilon}{\sqrt{\pi g}}\, \mathrm{e}^{-1/6g}\big(1 + O(g)\big). \tag{42.23}$$

Instanton interactions. To go beyond the one-instanton approximation, we have to take into account the interaction between instantons. We then face a problem: examining expression (42.19), we discover that the interaction between instantons is *attractive*.

Therefore, for g small, the dominant contributions to the integral come from paths in which the instantons are close. For such paths, the concept of instanton is no longer meaningful, since such paths cannot be distinguished from the fluctuations around the constant, or the one-instanton solution.

Actually, this phenomenon is consistent with the large-order behaviour analysis, which has indicated that the perturbative expansion in the case of potentials with degenerate minima is not Borel summable. An ambiguity is expected at the two-instanton order. If the perturbative expansion is ambiguous at the two-instanton order, it seems impossible to calculate a correction of the same order or smaller. To proceed any further, first we must define the sum of perturbation theory. Remarkably enough, for $g < 0$, the perturbative expansion of the double-well potential is, up to a normalization and the change of g in $-g$, identical to the perturbative expansion of the $O(2)$ anharmonic oscillator, which is Borel summable [447, 453]. Therefore, we define the sum of the perturbation series as the analytic continuation of this Borel sum from g negative to $g = |g| \pm i0$. This corresponds in the Borel transformation to integrate above or below the real positive axis. We then note that, for g *negative*, the interaction between instantons becomes *repulsive*, and the expression (42.19) is defined. Therefore, we calculate, for g small and negative, both the sum of the perturbation series and the instanton contributions, and perform an analytic continuation to g positive of all quantities consistently. In the same way, the perturbative expansion around each multi-instanton configuration is also not Borel summable, and must be summed by the same procedure.

42.1.4 The calculation

We start from the n-instanton contribution (42.19),

$$\mathcal{Z}_\epsilon^{(n)} \sim \frac{\beta}{n}\, \mathrm{e}^{-\beta/2}\, \kappa^n \int_{\theta_i \ge 0} \delta\left(\sum \theta_i - \beta\right) \prod_{i=1}^{n} \mathrm{d}\theta_i \exp\left(\frac{2}{g} \sum_{i=1}^{n} \mathrm{e}^{-\theta_i}\right). \tag{42.24}$$

To factorize the integral over the variables θ_i, we replace the δ-function by the integral representation,

$$\delta\left(\sum_{i=1}^{n} \theta_i - \beta\right) = \frac{1}{2i\pi} \int_{-i\infty-\eta}^{i\infty-\eta} \mathrm{d}s\, \exp\left[-s\left(\beta - \sum_{i=1}^{n} \theta_i\right)\right], \text{ with } \eta > 0\,. \tag{42.25}$$

In terms of the function

$$I(s) = \int_0^{+\infty} \mathrm{e}^{s\theta - \mu\, \mathrm{e}^{-\theta}}\, \mathrm{d}\theta\,, \tag{42.26}$$

the integral (42.24) can be rewritten as

$$\mathcal{Z}_\epsilon^{(n)} \sim \frac{\beta\, \mathrm{e}^{-\beta/2}}{2i\pi} \frac{\kappa^n}{n} \int_{-i\infty-\eta}^{i\infty-\eta} \mathrm{d}s\ \mathrm{e}^{-\beta s}\, [I(s)]^n\,, \quad \text{with } \mu = -2/g\,. \tag{42.27}$$

By giving to s a small negative real part, we have ensured the convergence of the integral (42.26). To evaluate the integral (42.26), we set

$$\mu\,\mathrm{e}^{-\theta} = t \tag{42.28}$$

and the integral becomes

$$I(s) = \int_0^{\mu} \frac{\mathrm{d}t}{t}\left(\frac{\mu}{t}\right)^s \mathrm{e}^{-t} = \int_0^{+\infty} \frac{\mathrm{d}t}{t}\left(\frac{\mu}{t}\right)^s \mathrm{e}^{-t} + O\left(\mathrm{e}^{-\mu}/\mu\right), \tag{42.29}$$

for μ positive and large, that is, $g \to 0_-$. Up to an exponentially small correction, we thus obtain

$$I(s) \sim \mu^s \Gamma(-s)\,. \tag{42.30}$$

The generating function $\Sigma(\beta, g)$ (equation (42.21)) of the leading order multi-instanton contributions then becomes

$$\begin{aligned}\Sigma(\beta, g) &= -\frac{\beta\,\mathrm{e}^{-\beta/2}}{2i\pi}\int_{-i\infty-\eta}^{i\infty-\eta} \mathrm{d}s\ \mathrm{e}^{-\beta s}\sum_n \frac{\kappa^n}{n}\mu^{ns}\left[\Gamma(-s)\right]^n \\ &= -\frac{\beta\,\mathrm{e}^{-\beta/2}}{2i\pi}\int_{-i\infty-\eta}^{i\infty-\eta} \mathrm{d}s\ \mathrm{e}^{-\beta s}\ln\left[1-\kappa\mu^s\Gamma(-s)\right].\end{aligned} \tag{42.31}$$

We set

$$E = s + 1/2\,, \quad \phi(E) = 1 - \kappa\mu^{E-1/2}\Gamma(1/2 - E). \tag{42.32}$$

We then integrate $\beta\,\mathrm{e}^{-\beta s}$ by parts, and obtain

$$\Sigma(\beta, g) = -\frac{1}{2i\pi}\int_{-i\infty}^{+i\infty} \mathrm{d}E\,\mathrm{e}^{-\beta E}\,\frac{\phi'(E)}{\phi(E)}. \tag{42.33}$$

The asymptotic behaviour of the Γ-function (given by the Stirling formula) ensures the convergence of the integral and, moreover, the contour can be deformed to enclose the poles of the integrand in the half-plane $\mathrm{Re}(E) > 0$. Integrating, we obtain the sum of residues

$$\Sigma(\beta, g) = \sum_{N\geq 0} \mathrm{e}^{-\beta E_N}\,, \tag{42.34}$$

where the energies E_N are solutions of the spectral equation [408],

$$\phi(E) = 1 - \kappa\mu^{E-1/2}\Gamma(1/2 - E) = 0\,. \tag{42.35}$$

Since κ is small, a zero E of this equation is close to a pole of $\Gamma(1/2 - E)$:

$$E_N = N + \tfrac{1}{2} + O(\kappa)\,, \quad N \geq 0\,. \tag{42.36}$$

We then expand the solutions of equation (42.35) as a power series of κ:

$$E_N(g) = \sum E_N^{(n)}(g)\kappa^n. \tag{42.37}$$

We obtain, at once, the many-instanton contributions to all energy levels $E_N(g)$ of the double-well potential, at leading order. It is convenient to write equation (42.35) as

$$\frac{\mathrm{e}^{-1/6g}}{\sqrt{2\pi}}\left(-\frac{2}{g}\right)^{E}\Gamma(\tfrac{1}{2}-E)=-\epsilon i \;\Leftrightarrow\; \frac{\cos\pi E}{\pi}=\epsilon i\frac{\mathrm{e}^{-1/6g}}{\sqrt{2\pi}}\left(-\frac{2}{g}\right)^{E}\frac{1}{\Gamma(\frac{1}{2}+E)}. \quad (42.38)$$

For example, the one-instanton contribution is

$$E_N^{(1)}(g)=-\frac{\epsilon}{N!}\left(\frac{2}{g}\right)^{N+1/2}\frac{\mathrm{e}^{-1/6g}}{\sqrt{2\pi}}\big(1+O(g)\big). \quad (42.39)$$

The two-instanton contribution is then,

$$E_N^{(2)}(g)=\frac{1}{(N!)^2}\left(\frac{2}{g}\right)^{2N+1}\frac{\mathrm{e}^{-1/3g}}{2\pi}\left[\ln(-2/g)-\psi(N+1)+O\left(g\ln g\right)\right], \quad (42.40)$$

where ψ is the logarithmic derivative of the Γ-function.

The appearance of a factor $\ln g$ can be simply understood by noting that the interaction terms are only relevant for $g^{-1}\,\mathrm{e}^{-\theta}$ of order 1, that is, θ of order $-\ln g$.

More generally, it can be verified that the n-instanton contribution has, at leading order, the form

$$E_N^{(n)}(g)=-\left(\frac{2}{g}\right)^{n(N+1/2)}\left(\frac{\mathrm{e}^{-1/6g}}{\sqrt{2\pi}}\right)^{n}\left[P_n^N\left(\ln(-g/2)\right)+O\left(g\left(\ln g\right)^{n-1}\right)\right], \quad (42.41)$$

in which $P_n^N(\sigma)$ is a polynomial of degree $(n-1)$. For example, for $N=0$, one finds

$$P_2(\sigma)=\sigma+\gamma\,,\quad P_3(\sigma)=\frac{3}{2}(\sigma+\gamma)^2+\frac{\pi^2}{12}\,, \quad (42.42)$$

in which γ is Euler's constant $\gamma=-\psi(1)=0.577215....$

Analytic continuation. When we perform our analytic continuation from g negative to g positive, two things happen: the Borel sums become complex, with an imaginary part exponentially smaller by about a factor $\mathrm{e}^{-1/3g}$ than the real part. Simultaneously, the function $\ln(-2/g)$ also becomes complex and yields an imaginary part $\pm i\pi$. Since the sum of all the contributions is real, the imaginary parts should cancel (a sign of a resurgent structure). This argument leads to an evaluation of the imaginary part of the Borel sum of the perturbation series, or of the expansion around one instanton, for example.

From the imaginary part of P_2, we infer

$$\operatorname{Im}E^{(0)}(g)\sim\frac{1}{\pi g}\,\mathrm{e}^{-1/3g}\operatorname{Im}\left[P_2\big(\ln(-g/2)\big)\right], \quad (42.43)$$

andv therefore,

$$\operatorname{Im}E^{(0)}(g)\sim-\frac{1}{g}\,\mathrm{e}^{-1/3g}\,. \quad (42.44)$$

Using a dispersion relation to calculate the coefficients $E_k^{(0)}$ of the expansion of

$$E^{(0)}(g)=\sum_k E_k^{(0)}g^k, \quad (42.45)$$

we obtain the large-order behaviour of the perturbative expansion,

$$E_k^{(0)}\underset{k\to\infty}{\sim}\frac{1}{\pi}\int_0^\infty \operatorname{Im}\left[E^{(0)}(g)\right]\frac{\mathrm{d}g}{g^{k+1}},\;\Rightarrow\; E_k^{(0)}\sim-\frac{1}{\pi}3^{k+1}k!\,. \quad (42.46)$$

From the imaginary part of P_3, we derive the large-order behaviour of the expansion of

$$E^{(1)}(g) = -\frac{1}{\sqrt{\pi g}}\,\mathrm{e}^{-1/6g}\left(1+\sum^{\infty} E_k^{(1)} g^k\right). \tag{42.47}$$

We express that the imaginary parts of $E^{(1)}(g)$ and $E^{(3)}(g)$ cancel at leading order:

$$\operatorname{Im} E^{(1)}(g) \sim -\left(\frac{\mathrm{e}^{-1/6g}}{\sqrt{\pi g}}\right)^3 \operatorname{Im}\left[P_3\big(\ln(-g/2)\big)\right]. \tag{42.48}$$

We derive the coefficients $E_k^{(1)}$ from the dispersion integral,

$$E_k^{(1)} = \frac{1}{\pi}\int_0^\infty \left\{\operatorname{Im}\left[E^{(1)}(g)\right]\sqrt{\pi g}\,\mathrm{e}^{1/6g}\right\}\frac{\mathrm{d}g}{g^{k+1}}. \tag{42.49}$$

Then, using equations (42.42) and (42.48), we find

$$E_k^{(1)} \sim -\frac{1}{\pi}\int_0^\infty 3\left[\ln(2/g)+\gamma\right]\mathrm{e}^{-1/3g}\,\frac{\mathrm{d}g}{g^{k+2}}. \tag{42.50}$$

Finally, at leading order for k large, we can replace g by its saddle point value $1/3k$ in $\ln g$, and obtain

$$E_k^{(1)} = -\frac{3^{k+2}}{\pi}k!\left[\ln 6k+\gamma+O\left(\frac{\ln k}{k}\right)\right]. \tag{42.51}$$

Both results (42.46) and (42.51) have been shown to agree with the numerical behaviour of the corresponding series by calculating about 100 terms of the series.

The behaviour of the real part of P_2 for g has also been verified numerically [395].

42.2 The periodic cosine potential

The structure of the low-lying levels of the periodic cosine potential are discussed in Section 39.2, and we borrow the results here.

To avoid the proliferation of big integer factors, it is convenient to use a specific normalization of the coupling constant. We write the Hamiltonian as

$$H = -\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 + \frac{1}{16g}\left(1-\cos 4q\sqrt{g}\right). \tag{42.52}$$

For g small, the spectrum has a band structure. The unitary operator T, which translates wave functions by one period $\pi/2\sqrt{g}$ of the potential,

$$T\psi(q) \equiv \psi(q+\pi/2\sqrt{g}),$$

commutes with the Hamiltonian, and can thus be diagonalized simultaneously. An eigenstate in a band, that we denote by $|N,\varphi,g\rangle$, is characterized by the eigenvalue $\mathrm{e}^{i\varphi}$ of T, and the corresponding eigenfunction $\psi_N(\varphi,g,q)$ of H:

$$T|N,\varphi,g\rangle = \mathrm{e}^{i\varphi}\,|N,\varphi,g\rangle,\quad H|N,\varphi,g\rangle = \mathcal{E}_N(\varphi,g)|N,\varphi,g\rangle. \tag{42.53}$$

For g small, the eigenvalue $\mathcal{E}_N(\varphi,g)$ is a periodic function of the angle φ, and $\mathcal{E}_N(\varphi,g) = N+1/2+O(g)$.

The partition function in the φ-sector. We define the partition function in the sector corresponding to an angle φ as the sum

$$\mathcal{Z}(\beta,\varphi,g)=\sum_{N=0}\mathrm{e}^{-\beta\mathcal{E}_N(\varphi,g)}\,. \tag{42.54}$$

In particular, for β large,

$$\mathcal{Z}(\beta,\varphi,g)\underset{\beta\to\infty}{\sim}\mathrm{e}^{-\beta\mathcal{E}_0(\varphi,g)}\,. \tag{42.55}$$

We also define the quantity (equation (A39.8))

$$\begin{aligned}\mathcal{Z}_l(\beta,g)&\equiv\operatorname{tr}\left(T^l\,\mathrm{e}^{-\beta H}\right)\\&=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\sum_N\mathrm{e}^{-\beta\mathcal{E}_N(\varphi,g)}\,\mathrm{e}^{il\varphi}=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\,\mathcal{Z}(\beta,\varphi,g)\,\mathrm{e}^{il\varphi}\,.\end{aligned} \tag{42.56}$$

The inverse of last relation is,

$$\mathcal{Z}(\beta,\varphi,g)=\sum_{l=-\infty}^{+\infty}\mathrm{e}^{-il\varphi}\,\mathcal{Z}_l(\beta,g). \tag{42.57}$$

The path integral representation of $\mathcal{Z}_l(\beta,g)$ is

$$\mathcal{Z}_l(\beta,g)=\int_{q(\beta/2)=q(-\beta/2)+l\pi/2\sqrt{g}}[\mathrm{d}q(t)]\exp\left[-\mathcal{S}(q)\right], \tag{42.58}$$

with

$$\mathcal{S}(q)=\int_{-\beta/2}^{\beta/2}\mathrm{d}t\left[\frac{1}{2}\dot{q}^2(t)+\frac{1}{16g}\left(1-\cos 4q(t)\sqrt{g}\right)\right]. \tag{42.59}$$

The factor $\mathrm{e}^{-il\varphi}$ can be incorporated in the path integral, since

$$-il\varphi=-\frac{2\sqrt{g}}{\pi}\varphi\left[q(\beta/2)-q(-\beta/2)\right]=-\frac{2i\sqrt{g}}{\pi}\int_{-\beta/2}^{+\beta/2}\mathrm{d}t\,\dot{q}(t)\,.$$

This corresponds to adding to $\mathcal{S}(q)$ the integral of a local density,

$$\mathcal{S}(q)\mapsto\mathcal{S}(q)+\frac{2i\sqrt{g}}{\pi}\int_{-\beta/2}^{+\beta/2}\mathrm{d}t\,\dot{q}(t)\,. \tag{42.60}$$

The sum over l then is obtained by summing over all periodic trajectories. In the infinite β limit, the configurations for which $q(\beta/2)-q(-\beta/2)$ is not a multiple of the period are suppressed, since their classical action is necessarily infinite. The sum (42.57) can thus be expressed without restriction on the path $q(t)$ as

$$\mathrm{e}^{-\beta\mathcal{E}_0(\varphi,g)}\underset{\beta\to\infty}{\sim}\int[\mathrm{d}q(t)]\exp\left[-\mathcal{S}(q)-i\frac{2\sqrt{g}}{\pi}\varphi\int_{-\beta/2}^{+\beta/2}\mathrm{d}t\,\dot{q}(t)\right]. \tag{42.61}$$

These expressions have natural generalizations in the case of the θ-vacuum of the $CP(N-1)$ model, and non-Abelian gauge theories (see Sections 39.5 and 39.6).

Remark. In an expansion of an eigenvalue $\mathcal{E}_N(\varphi,g)$ in a Fourier series,

$$\mathcal{E}_N(\varphi,g)=\sum_{l=-\infty}^{+\infty}E_N(l,g)\,\mathrm{e}^{il\varphi},\quad E_N(l,g)=E_N(-l,g)\,, \tag{42.62}$$

for g small, $E_{N,l}(g)$ is dominated by l-instanton contributions. In particular, for the ground state energy $\mathcal{E}_0(\varphi,g)$ in the φ sector, , the $l=1$ term behaves like

$$E_0(l=1,g)\sim\frac{1}{\sqrt{\pi g}}\,\mathrm{e}^{-1/2g}\,. \tag{42.63}$$

Multi-instanton configurations. This example, and the double-well potential example differ in one important aspect. In the double-well case, each configuration is a succession of instantons and anti-instantons. By contrast, here, at each step, the instanton can go to the next minimum of the potential, or the preceding one. Therefore, we assign a sign $\epsilon = +1$ to an instanton and a sign $\epsilon = -1$ to an anti-instanton. A simple calculation, similar to the calculation of Section 42.1 (for details, see Section A42.2), yields the interaction between two consecutive instantons of types ϵ_1 and ϵ_2 separated by a distance θ_{12}:

$$\frac{2\epsilon_1\epsilon_2}{g}\,\mathrm{e}^{-\theta_{12}}\,. \tag{42.64}$$

The interaction between instantons of the same kind is repulsive, while it is attractive for different kinds. We denote the one-instanton contribution at leading order by

$$\kappa = \frac{1}{\sqrt{\pi g}}\,\mathrm{e}^{-1/2g}\,. \tag{42.65}$$

With this notation, the n-instanton contribution reads

$$\mathcal{Z}^{(n)}(\beta,\varphi,g) = \beta\,\mathrm{e}^{-\beta/2}\,\frac{\kappa^n}{n}\int_{\theta_1\geq 0}\delta\Bigl(\sum_{i=1}^{n}\theta_i - \beta\Bigr)J_n(\theta_i), \tag{42.66}$$

with

$$J_n(\theta_i) = \sum_{\epsilon_i=\pm 1}\exp\Bigl(\sum_{i=1}^{n} -\frac{2}{g}\epsilon_i\epsilon_{i+1}\,\mathrm{e}^{-\theta_i} - i\epsilon_i\varphi\Bigr). \tag{42.67}$$

The additional term $-i\epsilon_i\varphi$ originates from the expression (42.57). We have identified ϵ_{n+1} and ϵ_1.

In contrast with the example of the double-well potential, the interaction between instantons contains both attractive and repulsive terms. Thus, we have to begin with g complex to perform the analytic continuation of both the Borel sums and the instanton contributions.

Following the same steps as in the case of the double-well potential, we obtain

$$\begin{aligned}\mathcal{Z}^{(n)}(\beta,\varphi,g) = {}& \frac{\beta}{2i\pi}\,\mathrm{e}^{-\beta/2}\,\frac{\kappa^n}{n}\oint \mathrm{d}s\;\mathrm{e}^{-\beta s}\,\Gamma^n(-s)\\ &\times\sum_{\{\epsilon_i=\pm 1\}}\exp\Bigl[\sum_{i=1}^{n} -i\epsilon_i\varphi - s\,\ln(\epsilon_i\epsilon_{i+1}g/2)\Bigr].\end{aligned} \tag{42.68}$$

We introduce the notation

$$\sigma = \ln(g/2)\,, \tag{42.69}$$

and choose to make the analytic continuation from above so that

$$\ln\left(\tfrac{1}{2}g\epsilon_i\epsilon_{i+1}\right) = \sigma - \tfrac{1}{2}i\pi(1-\epsilon_i\epsilon_{i+1}). \tag{42.70}$$

The expression (42.68) can then be written as

$$\begin{aligned}\mathcal{Z}^{(n)}(\beta,\varphi,g) \sim {}& \frac{\beta}{2i\pi}\,\mathrm{e}^{-\beta/2}\,\frac{\kappa^n}{n}\oint \mathrm{d}s\;\mathrm{e}^{-\beta s}\left([\Gamma(-s)\,\mathrm{e}^{-\sigma s}\right)^n\\ &\times\sum_{\{\epsilon_i=\pm 1\}}\exp\Bigl[\sum_{i=1}^{n} -i\epsilon_i\varphi + \tfrac{1}{2}i\pi s(1-\epsilon_i\epsilon_{i+1})\Bigr].\end{aligned} \tag{42.71}$$

The summation over the set $\{\epsilon_i\}$ corresponds to the calculation of the partition function of a one-dimensional Ising model, whose transfer matrix is

$$\mathbf{M} = \begin{bmatrix} \mathrm{e}^{-i\varphi} & \mathrm{e}^{i\pi s} \\ \mathrm{e}^{i\pi s} & \mathrm{e}^{i\varphi} \end{bmatrix}. \tag{42.72}$$

The sum then is $\operatorname{tr}\mathbf{M}^n$. The eigenvalues $m_\pm$ of $\mathbf{M}$ are

$$m_\pm = \cos\varphi \pm \left(\mathrm{e}^{2i\pi s} - \sin^2\varphi\right)^{1/2}. \tag{42.73}$$

The expression (42.71) can then be written as

$$\mathcal{Z}^{(n)}(\beta,\varphi,g) \sim \frac{\beta\,\mathrm{e}^{-\beta/2}}{2i\pi}\frac{\kappa^n}{n}\oint \mathrm{d}s\;\mathrm{e}^{-\beta s}\left[\Gamma(-s)\,\mathrm{e}^{-\sigma s}\right]^n\left(m_+^n + m_-^n\right). \tag{42.74}$$

The sum $\Sigma(\beta,g)$ of all leading order multi-instanton contributions can be calculated and yields,

$$\begin{aligned}\Sigma(\beta,g) = \mathrm{e}^{-\beta/2} - \frac{\beta\,\mathrm{e}^{-\beta/2}}{2i\pi}\oint \mathrm{d}s\;\mathrm{e}^{-\beta s}\ln\left\{\left[1-\kappa\Gamma(-s)\,\mathrm{e}^{-\sigma s}\,m_+(s)\right]\right.\\ \left.\times\left[1-\kappa\Gamma(-s)\,\mathrm{e}^{-\sigma s}\,m_-(s)\right]\right\}.\end{aligned} \tag{42.75}$$

The argument of the logarithm can also be written as

$$\begin{aligned}&\left[1-\kappa\Gamma(-s)\,\mathrm{e}^{-\sigma s}\,m_+(s)\right]\left[1-\kappa\Gamma(-s)\,\mathrm{e}^{-\sigma s}\,m_-(s)\right]\\ &\quad = 1-\kappa\Gamma(-s)\,\mathrm{e}^{-\sigma s}\left[2\cos\varphi + \kappa\frac{2i\pi\,\mathrm{e}^{(i\pi-\sigma)s}}{\Gamma(1+s)}\right].\end{aligned} \tag{42.76}$$

An integration by parts of $\beta\,\mathrm{e}^{-\beta s}$ in the integral (42.75) yields the final result:

$$\Sigma(\beta,g) = \sum_{N=0}^{\infty}\mathrm{e}^{-\beta E_N(g)}, \tag{42.77}$$

in which $E_N(g) = \frac{1}{2} + s_N(\kappa,\sigma)$ is a solution expandable in powers of κ of the equation [408],

$$\left(\frac{2}{g}\right)^{-E}\frac{\mathrm{e}^{1/2g}}{\Gamma(\frac{1}{2}-E)} + \left(-\frac{2}{g}\right)^{E}\frac{\mathrm{e}^{-1/2g}}{\Gamma(\frac{1}{2}+E)} = \frac{2\cos\varphi}{\sqrt{2\pi}}. \tag{42.78}$$

Note the symmetry in the change $g,E \mapsto -g,-E$. However, this symmetry is slightly misleading, because the equation is actually quadratic in $\Gamma(\frac{1}{2}-E)$ and only one root, corresponding to m_+, is relevant for $g > 0$.

42.3 General potentials with degenerate minima

We now consider a general analytic potential having two degenerate minima located at the origin and another point x_0:

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^2 + O\left(x^3\right), \\ V(x) &= \tfrac{1}{2}\omega^2(x-x_0)^2 + O\left((x-x_0)^3\right). \end{aligned} \tag{42.79}$$

For definiteness, we assume $\omega > 1$.

In such a situation, the classical equation of motion have instanton solutions connecting the two minima of the potential. However, there is no ground state degeneracy beyond the classical level. Therefore, the one-instanton solution does not contribute anymore to the path integral. Only periodic classical paths are relevant; the leading contribution now comes from the two-instanton configuration.

To determine the potential between instantons and the normalization of the path integral, it is convenient to first calculate the contribution at β finite of a trajectory described n times, and then take the large β limit of the expression. Using the expressions derived in Section 37.5, we infer

$$\left\{\operatorname{tr} \mathrm{e}^{-\beta H}\right\}_{(n)} = i(-1)^n \frac{\beta}{n\sqrt{\pi g}} \sqrt{\frac{\omega C}{n(1+\omega)}}\, \mathrm{e}^{-\omega\beta/[2n(1+\omega)]}\, \mathrm{e}^{-nA(\beta)/g}, \tag{42.80}$$

with the definitions

$$C = x_0^2 \omega^{2/(1+\omega)} \exp\left\{\frac{2\omega}{1+\omega}\left[\int_0^{x_0} \mathrm{d}x \left(\frac{1}{\sqrt{2V(x)}} - \frac{1}{x} - \frac{1}{\omega(x_0-x)}\right)\right]\right\}, \tag{42.81}$$

and

$$A(\beta) = 2\int_0^{x_0} \sqrt{2V(x)}\mathrm{d}x - 2C\frac{(1+\omega)}{\omega}\, \mathrm{e}^{-(\beta/n)\omega/(1+\omega)} + \cdots. \tag{42.82}$$

Note that n does not have the same meaning here as in Section 42.1. Since ω is different from 1, the one-instanton configuration does not contribute and n instead counts the number of instanton and anti-instanton pairs in the terminology of Section 42.1. Therefore, n here actually corresponds to $2n$ in the $\omega = 1$ limit.

42.3.1 The n-instanton action

We denote by θ_i the successive amounts of time the classical trajectory spends near x_0, and φ_i the amounts it spends near the origin. The n-instanton action takes the form

$$A(\theta_i, \varphi_j) = na - 2\sum_{i=1}^{n} \left(C_1\, \mathrm{e}^{-\omega\theta_i} + C_2\, \mathrm{e}^{-\varphi_i}\right) \tag{42.83}$$

with $\sum_{i=1}^{n}(\theta_i + \varphi_i) = \beta$ and

$$a = 2\int_0^{x_0} \sqrt{2V(x)}\mathrm{d}x\,. \tag{42.84}$$

By comparing the value of the action at the saddle point

$$\theta_i = \frac{\beta}{n(1+\omega)}, \qquad \varphi_i = \frac{\omega\beta}{n(1+\omega)} \tag{42.85}$$

with the expression (42.82), we see that we can choose

$$C_1 = C/\omega\,, \qquad C_2 = C \tag{42.86}$$

by adjusting the definitions of θ and φ.

42.3.2 The n-instanton contribution

The n-instanton contribution then has the form

$$\left\{\operatorname{tr} e^{-\beta H}\right\}_{(n)} = \beta\, e^{-\beta/2} \frac{e^{-na/g}}{(\pi g)^n} N_n \times \int_{\theta_i,\varphi_i \geq 0} \delta\left(\sum_i (\theta_i + \varphi_i) - \beta\right) \exp\left[\sum_{i=1}^{n} \frac{1}{2}(1-\omega)\theta_i - \frac{1}{g} A(\theta,\varphi)\right]. \quad (42.87)$$

The additional term $\sum_i \frac{1}{2}(1-\omega)\theta_i$ in the integrand comes from the determinant generated by the Gaussian integration around the classical path. The normalization can be obtained by evaluating the integral over the variables θ_i and φ_i using the steepest descent method, and comparing the result with expression (42.80). The result is

$$N_n = \frac{(C\sqrt{\omega})^n}{n}. \quad (42.88)$$

The factor $1/n$ comes from the symmetry of the action under cyclic permutations of the θ_i and φ_i.

We set

$$\kappa = \frac{e^{-a/g}}{\pi g} C\sqrt{\omega}\,, \quad \mu = -\frac{2C}{g}\,. \quad (42.89)$$

As in Section 42.1, we introduce an integral representation for the δ-function:

$$\delta\left(\sum_{i=1}^{n} (\theta_i + \varphi_i) - \beta\right) = \frac{1}{2i\pi} \int_{-i\infty-\eta}^{+i\infty+\eta} ds\, \exp\left[-s\beta + s\sum_{i=1}^{n} (\theta_i + \varphi_i)\right], \quad \text{for } n \geq 1\,. \quad (42.90)$$

The expression (42.87) can be rewritten as

$$\left\{\operatorname{tr} e^{-\beta H}\right\}_{(n)} = \beta\, e^{-\beta/2} \frac{\kappa^n}{n} \frac{1}{2i\pi} \int_{-i\infty-\eta}^{+i\infty-\eta} ds\, e^{-s\beta} \left[I(s)J(s)\right]^n, \quad (42.91)$$

where we have defined

$$I(s) = \int_0^{+\infty} e^{\theta s - \mu\, e^{-\theta}}\, d\theta\,, \quad (42.92)$$

$$J(s) = \int_0^{+\infty} \exp\left\{\left[\tfrac{1}{2}(1-\omega) + s\right]\theta - \frac{\mu}{\omega}\, e^{-\omega\theta}\right\} d\theta\,. \quad (42.93)$$

The integrals can be calculated explicitly in the small g limit:

$$I(s) = \Gamma(-s)\mu^s, \quad (42.94)$$

$$J(s) = \frac{1}{\sqrt{\mu\omega}} \Gamma\left(\tfrac{1}{2} - (s + \tfrac{1}{2})/\omega\right) \left(\frac{\mu}{\omega}\right)^{(s+1/2)/\omega}. \quad (42.95)$$

We denote by $\Sigma(\beta, g)$, the generating functional of the many-instanton contributions.

Summing over n and integrating by parts, we obtain $\Sigma(\beta, g)$ as a sum of residues:

$$\Sigma(\beta, g) = \sum_\alpha e^{-\beta E_\alpha}, \quad (42.96)$$

in which the values $E_\alpha = \frac{1}{2} + s_\alpha$ are the solutions expandable for g small of the equation

$$\left(\frac{\mu}{\omega}\right)^{E/\omega} \Gamma(\tfrac{1}{2} - E/\omega)\mu^E \Gamma(\tfrac{1}{2} - E) \frac{e^{-a/g}}{2\pi} = -1\,. \quad (42.97)$$

We now note that we find two series of energy levels corresponding to the poles of the two Γ-functions:

$$E_N = N + \tfrac{1}{2} + O(\kappa)\,, \tag{42.98}$$
$$E_N = (N + \tfrac{1}{2})\omega + O(\kappa)\,. \tag{42.99}$$

The same expression contains the instanton contributions to the two different sets of eigenvalues.

One can verify that the many-instanton contributions are singular for $\omega = 1$. But if one directly sets $\omega = 1$ in equation (42.97), one obtains

$$\mu^{2E}\Gamma^2(\tfrac{1}{2} - E)\frac{\mathrm{e}^{-a/g}}{2\pi} = -1\,,$$

equation that can be rewritten as

$$\mu^{E}\Gamma(\tfrac{1}{2} - E)\frac{\mathrm{e}^{-a/(2g)}}{\sqrt{2\pi}} = \pm i\,. \tag{42.100}$$

This is exactly the set of two equations obtained in Section 42.1.

42.3.3 Large-order estimates of perturbation theory

The expression (42.97) can be used to determine the large-order behaviour of perturbation theory by calculating the imaginary part of the leading instanton contribution, and using a dispersion integral as we have done in Section 40.1.1. For the energy $E_N(g) = N + \frac{1}{2} + O(g)$, one finds

$$\operatorname{Im} E_N(g) \sim K_N g^{-(N+1/2)(1+1/\omega)}\,\mathrm{e}^{-a/g}, \tag{42.101}$$

with

$$\begin{aligned} K_N = {} & \frac{(-1)^{N+1}}{2\pi N!}\omega^{-(N+1/2)/\omega}\,(2C)^{(N+1/2)(1+1/\omega)}\sin[\pi(N+\tfrac{1}{2})(1+1/\omega)] \\ & \times \Gamma[\tfrac{1}{2} - (N+\tfrac{1}{2})/\omega]. \end{aligned} \tag{42.102}$$

From the imaginary part of $E_N(g)$, one infers at large-order k:

$$E_{Nk} \sim K_N \frac{\Gamma\big(k + (N+1/2)(1+1/\omega)\big)}{a^{k+(N+1/2)(1+1/\omega)}}\big(1 + O\left(k^{-1}\right)\big). \tag{42.103}$$

Note that, in contrast with the instanton contribution to the real part, the expression is uniform in the limit $\omega = 1$ and yields the result (42.46).

42.4 The $O(\nu)$-symmetric anharmonic oscillator

As a last example, we consider the analytic continuation to negative coupling of the energy levels of the quartic anharmonic oscillator with $O(\nu)$ symmetry, corresponding to the Hamiltonian

$$H = -\tfrac{1}{2}\nabla^2 + \tfrac{1}{2}\mathbf{q}^2 + g\left(\mathbf{q}^2\right)^2, \quad \mathbf{q} \in \mathbb{R}^\nu . \tag{42.104}$$

We explicitly discuss the $\nu = 2$ example, but the extension to other values of ν is then simple.

The $O(2)$-symmetric anharmonic oscillator. The instanton solution can be written as

$$\mathbf{q}(t) = \mathbf{u} f(t)\,, \tag{42.105}$$

in which $\mathbf{u}$ is a fixed unit vector. The one-instanton contribution to the ground state energy is

$$E^{(1)}(g) = \frac{4i}{g}\,\mathrm{e}^{1/3g}\left(1 + O\left(g\right)\right), \quad \text{for } g \to 0_- \,. \tag{42.106}$$

From a calculation of the instanton interaction, one infers the n-instanton action:

$$A(\theta_i) = -\frac{1}{3}n - 4\sum_i \mathrm{e}^{-\theta_i}\cos\varphi_i\,, \tag{42.107}$$

in which θ_i is the distance between two successive instantons and φ_i the angle between them:

$$\cos\varphi_i = \mathbf{u}_i \cdot \mathbf{u}_{i+1}\,. \tag{42.108}$$

It is useful to consider the quantity (Sections 3.4 and 14.3),

$$\mathrm{tr}\left[R(\alpha)\,\mathrm{e}^{-\beta H}\right] = \int [\mathrm{d}\mathbf{q}(t)]\exp\left(-\mathcal{S}(q)\right), \quad \text{with} \quad \hat{\mathbf{q}}(-\beta/2)\cdot\hat{\mathbf{q}}(\beta/2) = \cos\alpha\,. \tag{42.109}$$

The matrix $R(\alpha)$ is a rotation matrix, which rotates vectors by an angle α. It leads to the boundary condition that $\mathbf{q}(t)$ at initial and final times should differ by an angle α.

The right-hand side of equation (42.109) can be rewritten as

$$\mathrm{tr}\left[R(\alpha)\,\mathrm{e}^{-\beta H}\right] = \sum_{l,N} \mathrm{e}^{-il\alpha - \beta E_{l,N}}\,. \tag{42.110}$$

In this expression, l is the angular momentum. The boundary condition imposed on the path integral (42.109) implies a constraint on the many-instanton configuration:

$$\sum_{i=1}^{n} \varphi_i = \alpha\,, \tag{42.111}$$

constraint which can be implemented through the identity

$$\delta\left(\sum_{i=1}^{n}\varphi_i - \alpha\right) = \frac{1}{2\pi}\sum_{l=-\infty}^{+\infty}\exp\left[il\left(\sum_{i=1}^{n}\varphi_i\right) - i\alpha l\right]. \tag{42.112}$$

The n-instanton contribution to expression (42.110) then takes the form

$$\left\{\operatorname{tr}\left[R(\alpha)\,\mathrm{e}^{-\beta H}\right]\right\}_{(n)} \sim \frac{\kappa^n}{2i\pi n}\beta\,\mathrm{e}^{-\beta}\int \mathrm{d}s\ \mathrm{e}^{-s\beta}\sum_{l=-\infty}^{+\infty}\mathrm{e}^{-il\alpha}\left[I_l(s)\right]^n, \tag{42.113}$$

where we have set

$$\kappa = \frac{4i}{g}\,\mathrm{e}^{1/3g}, \quad \mu = -\frac{4}{g}, \tag{42.114}$$

$$I_l(s) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\int_{-\infty}^{+\infty}\mathrm{d}\theta\ \exp\left(s\theta + il\varphi - \mu\,\mathrm{e}^{-\theta}\cos\varphi\right). \tag{42.115}$$

We denote by $\Sigma_l(\beta,g)$ the generating function of n-instanton contributions at fixed angular momentum l:

$$\Sigma_l(\beta,g) = \sum_n \frac{\kappa^n}{2i\pi n}\beta\,\mathrm{e}^{-\beta}\int \mathrm{d}s\ \mathrm{e}^{-s\beta}\left[I_l(s)\right]^n. \tag{42.116}$$

To evaluate $I_l(s)$, we first integrate over θ:

$$I_l(s) = \mu^s\Gamma(-s)\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\,\mathrm{e}^{il\varphi}\left(\cos\varphi\right)^s. \tag{42.117}$$

Finally, performing the last integral and using various relations among Γ functions, we obtain

$$I_l(s) = \mu^s\,\mathrm{e}^{\frac{1}{2}i\pi(s+1)}\,\frac{\Gamma(\frac{1}{2}(l-s))2^{-s-1}}{\Gamma(1+\frac{1}{2}(l+s))}. \tag{42.118}$$

It is easy to verify that

$$I_l(s) = I_{-l}(s)\,. \tag{42.119}$$

From equation (42.116), we now derive the result

$$\operatorname{tr}\mathrm{e}^{-\beta H_l} = \sum_{N=0}^{\infty}\mathrm{e}^{-\beta E_{N,l}}, \tag{42.120}$$

with $E_{N,l} = s_{N,l} + 1$ being the solution of the equation

$$\mathrm{e}^{1/(3g)}\left(-\frac{2}{g}\right)^E \mathrm{e}^{i\pi(E+l)/2}\,\frac{\Gamma\left(\frac{1}{2}(l+1-E)\right)}{\Gamma\left(\frac{1}{2}(l+1+E)\right)} = 1\,, \tag{42.121}$$

which satisfies

$$E_{N,l} = l + 2N + 1 + O(g)\,, \quad N \ge 0\,. \tag{42.122}$$

Note that checks about these expressions are provided by the surprising perturbative relation between the $O(2)$ anharmonic oscillator with negative coupling and the quartic double-well potential [408, 453, 454].

The $O(\nu)$-symmetric Hamiltonian. One can extend this result to the general $O(\nu)$ case since, at fixed angular momentum l, the Hamiltonian depends only on the combination $l+\nu/2$. Hence, making in equation (42.121) the corresponding substitution, one obtains

$$i\,\mathrm{e}^{1/(3g)}\left(-\frac{2}{g}\right)^E \mathrm{e}^{i\pi(E+l+\nu/2)/2}\,\frac{\Gamma\left(\frac{1}{2}(l+\nu/2-E)\right)}{\Gamma\left(\frac{1}{2}(l+\nu/2+E)\right)} = 1\,. \tag{42.123}$$

At leading order in κ, one recovers the imaginary part of the energy levels for g small and negative:

$$\operatorname{Im} E_{N,l} \underset{g\to 0_-}{=} -\frac{1}{N!}\frac{1}{\Gamma(\frac{1}{2}\nu+l+N)}\left(\frac{2}{g}\right)^{(\nu/2)+l+2N} \mathrm{e}^{1/3g}\bigl(1+O(g)\bigr). \tag{42.124}$$

Using the Cauchy formula, one can derive from this expression large-order estimates of perturbation theory. At next order in κ, one obtains the two-instanton contribution which is related by the same dispersion relation to the large-order behaviour of the perturbative expansion around one instanton.

42.5 Generalized Bohr–Sommerfeld quantization formula

So far, we have considered instanton contributions only at leading order. However, the form of the result is extremely suggestive and has led to a conjecture [408] about the general form of the semi-classical expansion for potentials with degenerate minima, proved to a large extent since [395, 455], using exact WKB methods [456] and resurgence theory [457].

To be specific, we explain the conjecture for the double-well potential, although it can be easily generalized to the other problems discussed in this chapter (see Section A42.4).

We introduce the function

$$D(E,g)=E+\sum_{k=1}^{\infty} g^k D_{k+1}(E), \tag{42.125}$$

in terms of which the perturbation expansion for an energy level $E_N^{(0)}$ can be obtained by inverting

$$N+\tfrac{1}{2}=D(E^{(0)},g). \tag{42.126}$$

This is the form of the usual exact Bohr–Sommerfeld quantization condition.

One verifies that, as a power series, $D(E,g)=-D(-E,-g)$.

In the case of the double-well potential, to take into account instanton contributions, we generalize the Bohr–Sommerfeld quantization condition as

$$\frac{1}{\sqrt{2\pi}}\Gamma(\tfrac{1}{2}-D)\,(-2/g)^{D(E,g)}\,\mathrm{e}^{-A(g,E)/2}=\pm i\,, \tag{42.127}$$

with

$$A(g,E)=\frac{1}{3g}+\sum_{k=1}^{\infty} g^k A_{k+1}(E), \tag{42.128}$$

where again, in the sense of power series, $A(g,E)=-A(-g,-E)$. The functions $A(E,g)$ and $D(E,g)$ can be calculated by expanding the corresponding WKB series, which are expansions at gE fixed, for E small (for details see Section A42.4). The coefficients $D_k(E)$ and $A_k(E)$ are polynomials of degree k in E.

If we solve equation (42.127) in the one-instanton approximation, and substitute into equation (42.125), we find ($\epsilon=\pm 1$)

$$E=E^{(0)}(g)-\epsilon\left(\frac{2}{g}\right)^N \frac{1}{N!}\frac{\mathrm{e}^{-A(g,E^{(0)})/2}}{\sqrt{\pi g}}\frac{\partial D}{\partial E}\left(E^{(0)}\right)^{-1}. \tag{42.129}$$

Thus the knowledge of the two functions D and A is equivalent to the knowledge and the perturbative and one-instanton expansions for all levels and to all orders.

If we now systematically expand equation (42.127), we find for the energy level $E_N(g) = N + 1/2 + O(g)$ the following expansion,

$$E_N(g) = \sum_0^{\infty} E_l^{(0)} g^l + \sum_{n=1}^{\infty} \frac{1}{g^{Nn}} \left(\frac{1}{\sqrt{\pi g}}\, \mathrm{e}^{-1/6g} \right)^n \sum_{k=0}^{n-1} (\ln(-2/g))^k \sum_{l=0}^{\infty} e_{nkl} g^l . \quad (42.130)$$

All the series in powers of g appearing in this expansion are determined by the perturbative expansion of A and D. This structure has found an explanation in the framework of the theory of *resurgent* functions.

The function $A(E, g)$ has initially been determined at this order by a combination of analytic and numerical calculations.

However, later, it has been proved for the double well and cosine potentials [455], using differential equation techniques, the remarkable relation

$$\frac{\partial E}{\partial D} = -6Dg - 3g^2 \frac{\partial A}{\partial g} ,$$

which reduces the determination of both functions to the much simpler determination of the perturbative spectral function $D(E, g)$.

Moreover, we conjecture that all these series have to be summed for g negative first, and the value of each instanton contribution for g positive is then obtained by analytic continuation. The property that the infinite number of perturbation series around all instantons are related may, at least in QM, simplify the problem of the summation of the many-instanton contributions.

In Section A42.4, we give the generalized Bohr–Sommerfeld quantization formulae for other potentials with degenerate minima.

A42 Additional remarks

A42.1 Multi-instantons: The determinant

We can express the operator M defined by equation (42.18) as

$$M = -\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 + 1 + \sum_{i=1}^{n} v(t - t_i), \tag{A42.1}$$

in which $v(t)$ is a potential localized around $t = 0$, and t_i are the positions of the instantons:

$$v(t) = O\left(\mathrm{e}^{-|t|}\right), \quad \text{for } |t| \to \infty\,. \tag{A42.2}$$

We want to evaluate

$$\det(M\,M_0^{-1}) = \det\left\{1 + \left[-(\mathrm{d}/\mathrm{d}t)^2 + 1\right]^{-1} \sum_{i=1}^{n} v(t - t_i)\right\}. \tag{A42.3}$$

Using the identity $\ln\det = \operatorname{tr}\ln$, we expand the right-hand side in powers of $v(t)$ and find

$$\begin{aligned}\ln\det MM_0^{-1} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int \Bigg[&\Delta(u_1 - u_2) \sum_{i_1=1}^{n} v(u_2 - t_{i_1})\Delta(u_2 - u_3)\cdots\Delta(u_k - u_1) \\ &\times \sum_{i_k=1}^{n} v(u_1 - t_{i_k})\Bigg] \prod_{j=1}^{k} \mathrm{d}u_j\,,\end{aligned} \tag{A42.4}$$

with the definition

$$\Delta(t) = \left\langle 0\middle|\left[-(\mathrm{d}/\mathrm{d}t)^2 + 1\right]^{-1}\middle|t\right\rangle \sim \tfrac{1}{2}\,\mathrm{e}^{-|t|}, \quad \text{for } 1 \ll t \ll \beta\,. \tag{A42.5}$$

It is clear from the behaviour of $v(t)$ and $\Delta(t)$ that, when the instantons are largely separated, only the terms in which one retains the same instanton contribution from each potential survive. Therefore,

$$\begin{aligned}\ln\det MM_0^{-1} = n\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int \Delta(u_1 - u_2)v(u_2)\cdots\Delta(u_k - u_1)v(u_1) \prod_{j=1}^{k} \mathrm{d}u_j\,, \\ \text{for } |t_i - t_j| \gg 1\,.\end{aligned} \tag{A42.6}$$

We recognize n times the logarithm of the one-instanton determinant.

A42.2 The instanton interaction

Like in Section 42.3, we assume that the potential has two degenerate minima located at the points $x = 0$ and $x = x_0$, with

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^2 + O\left(x^3\right) \\ V(x) &= \tfrac{1}{2}\omega^2 \left(x - x_0\right)^2 + O\left((x - x_0)^3\right). \end{aligned} \tag{A42.7}$$

The one-instanton solution $q_c(t)$, which goes from 0 to $q_0 = x_0/\sqrt{g}$, can be written as

$$q_c(t) = f(t)/\sqrt{g}\ . \tag{A42.8}$$

We choose the function $f(t)$ in such a way that it satisfies

$$\begin{aligned} x_0 - f(t) &\sim \sqrt{C}\,\mathrm{e}^{-\omega t}/\omega\,, \quad \text{for } t \to +\infty\,, \\ f(t) &\sim \sqrt{C}\,\mathrm{e}^{t}, \quad \text{for } t \to -\infty\,. \end{aligned} \tag{A42.9}$$

By solving the equation of motion, it is easy to calculate the constant

$$C = x_0^2 \omega^{2/(1+\omega)} \exp\left\{\frac{2\omega}{1+\omega}\left[\int_0^{x_0} \mathrm{d}x \left(\frac{1}{\sqrt{2V(x)}} - \frac{1}{x} - \frac{1}{\omega(x_0 - x)}\right)\right]\right\}. \tag{A42.10}$$

We recognize the constant (42.81).

We now construct instanton–anti-instanton pair configurations $q(t)$, which correspond to trajectories starting from, and returning to, $q = q_0$ or $q = 0$. Since we want also to deal with the case of two successive instantons, we assume, but only in this case, that $V(x)$ is an even function and has, therefore, a third minimum at $x = -x_0$.

Following the discussion of Section 42.2, we take as a two-instanton configuration:

$$q_1(t) = \frac{1}{\sqrt{g}}\big(f_+(t) + \epsilon f_-(t)\big), \quad \epsilon = \pm 1\,, \tag{A42.11}$$

with

$$f_+(t) = f(t - \theta/2), \quad f_-(t) = f(-t - \theta/2), \tag{A42.12}$$

where θ parametrizes the instanton separation. The case $\epsilon = 1$ corresponds to an instanton–anti-instanton pair starting from $q = q_0$ at time $-\infty$, approaching $q = 0$ at intermediate times, and returning to q_0. The case $\epsilon = -1$ corresponds to a sequence of two instantons going from $-q_0$ to q_0. Moreover, for the classical trajectory that, instead, goes from the origin to q_0 and back, we choose

$$q_2(t) = \left[f(t + \theta/2) + f(\theta/2 - t) - x_0\right]/\sqrt{g}\ . \tag{A42.13}$$

To calculate the classical action corresponding to $q_1(t)$, we separate the action into two parts, corresponding at leading order to the two instanton contributions:

$$\mathcal{S}(q_1) = \mathcal{S}_+\ (q_1) + \mathcal{S}_-(q_1), \tag{A42.14}$$

with

$$\begin{aligned} \mathcal{S}_+(q_1) &= \int_0^{+\infty} \left[\tfrac{1}{2}\dot{q}_1^2(t) + V\big(\sqrt{g}q_1(t)\big)/g\right] \mathrm{d}t, \\ \mathcal{S}_-(q_1) &= \int_{-\infty}^{0} \left[\tfrac{1}{2}\dot{q}_1^2(t) + V\big(\sqrt{g}q_1(t)\big)/g\right] \mathrm{d}t\,. \end{aligned} \tag{A42.15}$$

The value $t = 0$ of the separation point is somewhat arbitrary and can be replaced by any value which remains finite when θ goes to infinity. We then use the properties that, for θ large, $f_+(t)$ is small for $t < 0$, and $f_-(t)$ is small for $t > 0$, to expand both terms. For example, for $\mathcal{S}_+$, we find

$$\begin{aligned}\mathcal{S}_+(q_1) = \frac{1}{g}\int_0^{+\infty} \mathrm{d}t \Big\{ \left[\tfrac{1}{2}\dot f_+^2(t) + V\big(f_+(t)\big)\right] + \epsilon \left[\dot f_-(t)\dot f_+(t) + V'\big(f_+(t)\big) f_-(t)\right] \\ + \tfrac{1}{2}\left[\dot f_-^2(t) + V''\big(f_+(t)\big) f_-^2(t)\right]\Big\}. \end{aligned} \qquad (A42.16)$$

Since $f_-(t)$ decreases exponentially, only values of t small compared to $\theta/2$ contribute to the last term of equation (A42.16), which is proportional to V''. For such values of t, we note that

$$\tfrac{1}{2}V''\big(f_+(t)\big) f_-^2(t) \sim V\big(f_-(t)\big). \qquad (A42.17)$$

For the terms linear in $f_-(t)$, we integrate by parts the kinetic term, and use the equation of motion

$$\ddot f(t) = V'\left[f(t)\right]. \qquad (A42.18)$$

Only the integrated term survives and yields

$$\int_0^{+\infty} \mathrm{d}t \left[\dot f_-(t)\dot f_+(t) + V'\big(f_+(t)\big) f_-(t)\right] = -\dot f(-\theta/2) f(-\theta/2). \qquad (A42.19)$$

The contribution $\mathcal{S}_-$ can be evaluated by exactly the same method. We note that the sum of the two contributions reconstructs twice the classical action a. We then find

$$\mathcal{S}(q_1) = \frac{1}{g}\left[2a - 2\epsilon f(-\theta/2)\dot f(-\theta/2) + \cdots\right], \qquad (A42.20)$$

with

$$a = \int_0^{x_0} \sqrt{2V(x)}\,\mathrm{d}x\,. \qquad (A42.21)$$

Finally, replacing, for θ large, f by its asymptotic form (A42.9), we obtain the classical action

$$\mathcal{S}(q_1) = g^{-1}\left[2a - 2\epsilon\, C\,\mathrm{e}^{-\theta} + O\left(\mathrm{e}^{-2\theta}\right)\right], \qquad (A42.22)$$

and, therefore, the instanton interaction.

The calculation of the classical action corresponding to $q_2(t)$ follows the same steps, and one finds

$$\mathcal{S}(q_2) = \frac{1}{g}\left\{2a - 2\left[f(\theta/2) - x_0\right]\dot f(\theta/2) + \cdots\right\}, \qquad (A42.23)$$

expression which, for θ large, is equivalent to

$$\mathcal{S}(q_2) = \frac{1}{g}\left[2a - 2(C/\omega)\,\mathrm{e}^{-\omega\theta}\right]. \qquad (A42.24)$$

Finally, in the case of a finite time interval β with periodic boundary conditions, we can combine both results to find the action of a periodic trajectory passing close to $q = 0$ and $q = q_0$:

$$\mathcal{S}(q) = g^{-1}\left[2a - 2C\left(\mathrm{e}^{-\beta+\theta} + \mathrm{e}^{-\omega\theta}/\omega\right)\right], \qquad (A42.25)$$

in agreement with equations (42.83, 42.86).

A42.2.1 Multi-instantons from constraints

Although multi-instanton configurations do not correspond to solutions of the equation of motion, it is nevertheless possible to modify the classical action by introducing constraints, and integrating over all possible constraints, generalizing the method of Section 37.4.1. The main problem with such a method is to find a set of constraints that are both theoretically acceptable and convenient for practical calculations.

One can, for instance, fix the positions of the instantons by introducing in the path integral (in the example of the double-well potential):

$$1 = \int \prod_{i=1}^{n} \left[\int \mathrm{d}t\, \dot{q}^2_{\epsilon_i}(t-t_i)\right] \delta\left[\int \mathrm{d}t\, \dot{q}_{\epsilon_i}(t-t_i)\big(q(t)-q_{\epsilon_i}(t-t_i)\big)\right] \mathrm{d}t_i\,, \qquad (A42.26)$$

where t_i are the instanton positions, and ϵ_i a successions of $\pm$ indicating instantons and anti-instantons. One then uses an integral representation of the δ-functions, so that the path integral becomes

$$\begin{gathered}\left(\frac{\|\dot{q}_+\|^2}{2i\pi}\right)^n \int \prod_{i=1}^{n} \mathrm{d}t_i \mathrm{d}\kappa_i \int [\mathrm{d}q(t)] \prod_{i=1}^{n} \exp\left[-\mathcal{S}(q,\kappa_i)\right], \quad \text{with} \\ \mathcal{S}(q,\kappa_i) = \mathcal{S}(q) + \sum_{i=1}^{n} \kappa_i \int \mathrm{d}t\, \dot{q}_{\epsilon_i}(t-t_i)\big(q(t)-q_{\epsilon_i}(t-t_i)\big).\end{gathered} \qquad (A42.27)$$

The arguments of Section 37.4.1 can then be generalized to recover the results of Section 42.1.

A42.3 A simple example of non-Borel summability

We illustrate here the problem of non-Borel summability with the example of a simple integral, which shares some of the features of the problem in QM which we have studied in the chapter. We consider the function

$$I(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}q\, \exp\left[-\frac{1}{g} V(q\sqrt{g})\right], \qquad (A42.28)$$

where $V(x)$ is an entire function with an absolute minimum at $x=0$ with $V(0)=0$. For g small, $I(g)$ can be calculated by steepest descent, expanding V around $q=0$. This yields an expansion of the form,

$$I(g) = \sum_{k\ge 0} I_k g^k. \qquad (A42.29)$$

One can express the integral ($A42.28$) as a generalized Borel or Laplace transform:

$$I(g) = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}q \int \mathrm{d}t\, \delta\left[V(q\sqrt{g})-t\right] \mathrm{e}^{-t/g}\,. \qquad (A42.30)$$

Interverting the q and t integration, we can integrate over q and find,

$$I(g) = \frac{1}{\sqrt{2\pi g}} \int_0^{\infty} \mathrm{d}t\ \mathrm{e}^{-t/g} \sum_i \frac{1}{|V'[x_i(t)]|}, \qquad (A42.31)$$

in which $\{x_i(t)\}$ are the solutions of the equation

$$V[x_i(t)] = t\,. \qquad (A42.32)$$

When the function $V(x)$ is monotonous both for x positive and negative, equation (A42.32) has two solutions for all values of t, and equation (A42.31) is directly the Borel representation of the function $I(g)$, which has a Borel summable power series expansion.

By contrast, we now assume that $V(x)$ has a second local minimum which gives a negligible contribution to $I(g)$ for g small. A simple example is

$$V(x) = \tfrac{1}{2}x^2 - \tfrac{1}{3a}x^3(1+a) + \tfrac{1}{4a}x^4, \quad \tfrac{1}{2} < a < 1\,, \tag{A42.33}$$

which has a minimum at $x = 1$. Between its two minima, the potential $V(x)$ has a maximum, located at $x = a$, whose contribution dominates the large-order behaviour of the expansion in powers of g:

$$I_k \underset{k\to\infty}{\propto} \Gamma(k)\,[V(a)]^{-k}\,, \qquad V(a) > 0\,, \tag{A42.34}$$

(in the example (A42.33) $V(a) = a^2(1-a/2)/6$), and the series is not Borel summable.

The *naive* Borel transform of $I(g)$ is obtained by retaining in equation (A42.31) only the roots of equation (A42.32) that exist for t small. The singularities of the Borel transform then correspond to the zeros of $V''(x)$.

For the potential (A42.33), the expression (A42.31) has the form (θ is the step function)

$$\begin{aligned} I(g) = \frac{1}{\sqrt{2\pi g}} \int_0^{+\infty} \mathrm{d}t\, \mathrm{e}^{-t/g} \Bigg[&\frac{1}{\left|V'\big(x_1(t)\big)\right|} + \frac{\theta\big(V(a)-t\big)}{\left|V'\big(x_2(t)\big)\right|} + \frac{\theta\big(V(a)-t\big)\theta\big(t-V(1)\big)}{\left|V'\big(x_3(t)\big)\right|} \\ &+ \frac{\theta\big(t-V(1)\big)}{\left|V'\big(x_4(t)\big)\right|} \Bigg], \end{aligned} \tag{A42.35}$$

with the definitions (see Fig. 42.2) $x_1(t) \le 0 \le x_2(t) \le a \le x_3(t) \le 1 \le x_4(t)$.

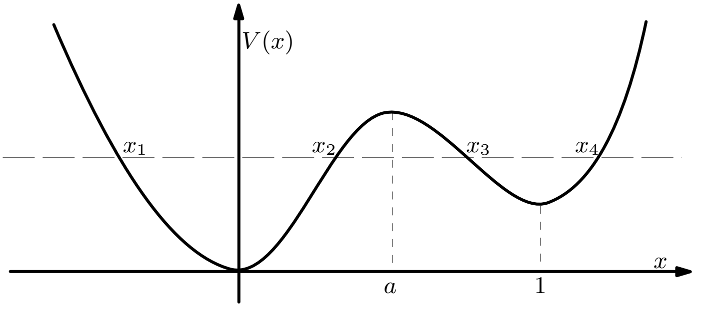

Fig. 42.2 The four roots of equation (A42.32)

The idea of the analytic continuation is to integrate each contribution up to $t = +\infty$ following a contour which passes below or above the cut along the positive real axis. This means that we consider $x_2(t)$ to be solution of the equation

$$V[x_2(t)] = t \pm i\epsilon\,. \tag{A42.36}$$

The sign is arbitrary. Let us, for instance, choose the positive sign. We then have to subtract this additional contribution. We proceed in the same way for $x_3(t)$ for $t > V(a)$.

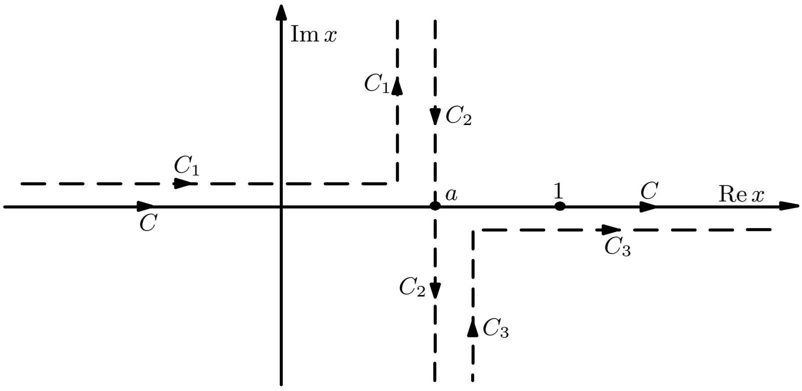

Fig. 42.3 The different contours in the x plane

Since $x_2(t)$ and $x_3(t)$ meet at $t = V(a)$, the analytic continuation corresponds to take for $x_3(t)$ the other solution of

$$V[x_3(t)] = t \mp i\epsilon \,. \tag{A42.37}$$

Therefore, we have to subtract from the total expression the contributions of two roots of the equation. But it is easy to verify that this is just the contribution of the saddle point located at $x = a$, which corresponds to a maximum of the potential.

Therefore, we have succeeded in writing expression (*A*42.35) as the sum of three saddle point contributions (see Fig. 42.3). There is some arbitrariness in the decomposition, which here corresponds to the choice $\epsilon = \pm 1$.

In the complex x plane, we have replaced the initial contour C on the real positive axis, by a sum of three contours C_1, C_2, and C_3, corresponding to the three saddle points located at $0, a, 1$.

A42.4 Multi-instantons and WKB approximation

We now give a more general form of the conjecture presented in Section 42.5, and indicate how a few properties of the functions $A(E, g)$ and $D(E, g)$ can be obtained from the 'exact' WKB expansion [456].

A42.4.1 The conjecture

The double-well potential. For the double-well potential, we have conjectured the spectral equation (42.127),

$$\frac{1}{\sqrt{2\pi}}\Gamma\left(\tfrac{1}{2} - D\right)(-2/g)^{D(E,g)}\,\mathrm{e}^{-A(g,E)/2} = \epsilon i \,, \tag{A42.38}$$

where the functions $D(E, g)$ and $A(E, g)$ have the expansions (42.125) and (42.128), respectively. The conjecture can be generalized to potentials with two asymmetric wells:

$$\begin{aligned}&\Gamma\left(\tfrac{1}{2} - D_1(E,g)\right)\left(-\frac{2C_1}{g}\right)^{D_1(E,g)}\Gamma\left(\tfrac{1}{2} - D_2(E,g)\right)\left(-\frac{2C_2}{g}\right)^{D_2(E,g)}\frac{\mathrm{e}^{-A(g,E)}}{2\pi}\\ &\quad = -1 \,,\end{aligned} \tag{A42.39}$$

where $D_1(E, g)$ and $D_2(E, g)$ are determined by the perturbative expansions around each of the two minima of the potential, and C_1, C_2 are numerical constants.

The cosine potential. For the cosine potential $\frac{1}{16}(1-\cos 4q)$, the conjecture reads

$$\left(\frac{2}{g}\right)^{-D(E,g)} \frac{\mathrm{e}^{A(E,g)/2}}{\Gamma(\frac{1}{2}-D(E,g))} + \left(\frac{-2}{g}\right)^{D(E,g)} \frac{\mathrm{e}^{-A(g,E)/2}}{\Gamma(\frac{1}{2}+D(E,g))} = \frac{2\cos\varphi}{\sqrt{2\pi}}. \qquad (A42.40)$$

The $O(\nu)$-symmetric anharmonic oscillator. For the $O(\nu)$ anharmonic oscillator, the conjecture

$$i\,\mathrm{e}^{-A(E,g)}\left(-\frac{2}{g}\right)^{D} \mathrm{e}^{i\pi(D+l+\nu/2)/2}\,\frac{\Gamma\left(\frac{1}{2}(l+\nu/2-D)\right)}{\Gamma\left(\frac{1}{2}(l+\nu/2+D)\right)} = 1\,, \qquad (A42.41)$$

corresponds to the expansion of the energy levels for $g<0$ and, in particular, yields the instanton contributions to the large-order behaviour.

A42.4.2 The WKB expansion

We consider the Schrödinger equation,

$$-\tfrac{1}{2}\psi''(q) + \left(V(q\sqrt{g})/g\right)\psi(q) = E\psi(q). \qquad (A42.42)$$

We first assume that $q=0$ is the absolute minimum of $V(q)$ and, moreover, $V(q)\sim\frac{1}{2}q^2$, for q small. In terms of $x=\sqrt{g}q$, the equation can be rewritten as

$$-g^2\psi''(x) + 2V(x)\psi(x) = 2gE\psi(x). \qquad (A42.43)$$

The WKB expansion is an expansion for $g\to 0$ at Eg fixed, in contrast with the perturbative expansion where E is fixed. It can be constructed by going over to the corresponding Riccati equation, setting

$$S(x) = -g\psi'(x)/\psi(x), \qquad (A42.44)$$

where S satisfies

$$gS'(x) - S^2(x) + S_0^2(x) = 0\,, \quad S_0^2(x) = 2V(x) - 2gE\,. \qquad (A42.45)$$

One then expands systematically in powers of g, at Eg fixed, starting from $S(x)=S_0(x)$. It is convenient to decompose $S(x)$ into an odd and even part, setting,

$$S(x,g,E) = S_+(x,g,E) + S_-(x,g,E)\,, \quad S_\pm(x,-g,-E) = \pm S_\pm(x,g,E)\,. \qquad (A42.46)$$

It follows that

$$gS_-'(x) - S_+^2(x) - S_-^2(x) + S_0^2(x) = 0\,, \qquad (A42.47)$$

$$gS_+'(x) - 2S_+(x)S_-(x) = 0\,. \qquad (A42.48)$$

It is then possible to express the wave function in terms of S_+ only:

$$\psi(x) = \left(S_+(x)\right)^{-1/2}\exp\left[-\frac{1}{g}\int_{x_0}^{x}\mathrm{d}x'\,S_+(x')\right]. \qquad (A42.49)$$

The spectrum can then be determined by the condition

$$\frac{1}{2i\pi}\oint_C \mathrm{d}z\,\frac{\psi'(z)}{\psi(z)} = N\,, \qquad (A42.50)$$

where N is the number of nodes of the eigenfunction, and C a contour which encloses them.

In the semi-classical limit, C encloses the cut of $S_0(x)$, which joins the two turning points solutions of $S_0(x) = 0$ (x_1, x_2 in Fig. 42.2). In terms of S_+, equation (A42.50) becomes

$$-\frac{1}{2i\pi g}\oint_C \mathrm{d}z\, S_+(z) = N + \tfrac{1}{2}\,. \qquad (A42.51)$$

If we replace S_+ by its WKB expansion, and expand each term in a power series of Eg, we obtain the function $D(E,g)$ (the perturbative expansion):

$$-\frac{1}{2i\pi g}\oint_C \mathrm{d}z\, S_+(z) = D(E,g) = -D(-E,-g)\,. \qquad (A42.52)$$

Potentials with degenerate minima. In the case of potentials with degenerate minima, two functions D_1 and D_2 (in the notation of equation (A42.39)) appear, corresponding to the expansions around each minimum. An additional contour integral arises corresponding to barrier penetration effects. The expansion for Eg small of its WKB expansion yields the function $A(g,E)$:

$$\begin{aligned}\frac{1}{g}\oint_{C'} \mathrm{d}z\, S_+(z) = A(E,g) + \ln(2\pi) - \sum_{i=1}^{2} \ln\Gamma\bigl(\tfrac{1}{2} - D_i(E,g)\bigr)\\ + D_i(E,g)\ln(-2g/C_i)\,,\end{aligned} \qquad (A42.53)$$

where C' encloses $[x_2, x_3]$ in Fig. 42.2. In the WKB expansion, the functions $\Gamma(\frac{1}{2} - D_i)$ have to be replaced by their asymptotic expansion for D_i large. Still a calculation of $A(E,g)$ at a finite order in g requires the WKB expansion and the asymptotic expansion of the Γ function only at a finite order.

$O(\nu)$-symmetric potentials. These expressions can be generalized to the case of $O(\nu)$-symmetric potentials. The perturbative expansion can be obtained by inverting a relation of the form

$$\mu + 2N + 1 = D(E,g,\mu), \quad \mu = l + \nu/2 - 1\,, \qquad (A42.54)$$

where the function $D(E,g,\mu)$ is given by a contour integral surrounding all zeros of the wave function on the real axis (including the negative real axis) of the even part (in the sense of equation (A42.46)) of $-g(\psi_l'/\psi_l + (\nu-1)/2|q|)$. The following properties can then be verified

$$D(E,g,\mu) = -D(-E,-g,\mu), \quad D(E,g,\mu) = D(E,g,-\mu), \qquad (A42.55)$$

and the coefficient of order g^k in the expansion of D is a polynomial of degree $[(k+1)/2]$ in μ. In the WKB expansion, the functions D and A again correspond to different contour integrals around turning points.