5

The perturbative renormalization group

In this chapter we shall explore the consequences of a system possessing two renormalization group fixed points, and a parameter ϵ (for example, related to the number of dimensions of space) such that, as its value is varied, the fixed points actually collide. When the two fixed points are sufficiently close it is then possible to deduce universal properties at one fixed point in terms of those at the other. Such an analysis is the basis of the ϵ-expansion and many other similar techniques. It also allows us to describe the properties of fixed points with exactly marginal scaling variables.

To see how this works, let us consider a simple example with a single coupling u. For simplicity, we consider an infinitesimal renormalization group (see p.47). Without loss of generality, we may assume that the fixed points are at $u = 0$ and $u = u^*$. Since the beta function is supposed to be analytic in u, we may assume that, for sufficiently small u, its form is determined by its zeroes, and the renormalization group equation therefore has the form

$$du/d\ell = -ku(u - u^*), \tag{5.1}$$

where k is some constant of order unity. Notice that the renormalization group eigenvalues at $u = 0$ and $u = u^*$ are ku^* and $-ku^*$ respectively, and that they are both $O(\epsilon)$ if $u^* = O(\epsilon)$. If $ku^* > 0$, the fixed point at $u = 0$ will be unstable, and, for $u > 0$, the flow will be towards the non-trivial fixed point at $u = u^*$. If we now have some other coupling u_1, say, which is a scaling variable at the $u = 0$ fixed point, in this simple example its renormalization group equation will have the form for small u and u_1

$$du_1/d\ell = y_1 u_1 + b' u u_1 + \cdots, \tag{5.2}$$

where b' is some constant. Therefore the renormalization group eigenvalue of u_1 at the non-trivial fixed point is, to this order, simply $y_1 + bu'^*$, that is, it is changed by an amount $O(\epsilon)$.

Figure 5.1. The operator product expansion.

Although very simple, this idea is the basis of the ϵ-expansion approach which gives a systematic approximation scheme for the computation of critical exponents. As long as sufficient information is known about the 'trivial' fixed point, all the properties of the non-trivial fixed point may be obtained as an expansion in ϵ. In this book, however, we shall concentrate on only the first order corrections, since they may be obtained rather directly without recourse to much formal development. However, their calculation does depend on a fundamental property of fixed point correlation functions which has not yet been described.

5.1 The operator product expansion

We discussed in Section 3.8 how the correlation functions at a fixed point could be described in terms of scaling operators $\phi_i(r)$, whose two-point correlations have a pure power behaviour. Consider now a correlation function

$$\langle \phi_i(r_1)\phi_j(r_2)\,\Phi\rangle, \tag{5.3}$$

where Φ denotes some arbitrary product of other operators $\prod_l \phi_l(r_l)$. We are interested in how this correlation function behaves in the limit when $|r_1 - r_2|$ is much smaller than the separations $|r_1 - r_l|$ with $l > 2$, as illustrated in Figure 5.1. The idea of the operator product expansion is that, in this limit, (5.3) may be replaced by a sum of the form

$$\sum_k C_{ijk}(r_1 - r_2)\langle \phi_k((r_1 + r_2)/2)\,\Phi\rangle, \tag{5.4}$$

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where the sum is over all scaling operators ϕ_k. To see why this should be true, consider replacing ϕ_i and ϕ_j by suitable linear combinations of 'bare' operators S_a, as discussed in Section 3.8. Then the product $\phi_i\phi_j$ may also be expressed in terms of this set, since, viewed from very far away, this is still a local quantity. Each of these may then be re-expressed as a linear combination of the ϕ_k. This argument, although appealing, is at best heuristic. It is therefore comforting to know that the operator product expansion has been proved to be valid to all orders in perturbation theory in simple models. For exactly solvable models, like the two-dimensional Ising model, its validity has also been demonstrated.

The important point about this result is that the functions $C_{ijk}(r_1 - r_2)$ in (5.4) are *independent* of Φ. For that reason, the operator product expansion may be written

$$\phi_i(r_1)\phi_j(r_2) = \sum_k C_{ijk}(r_1 - r_2)\,\phi_k((r_1 + r_2)/2). \tag{5.5}$$

However, it must be remembered that the above equality is only valid in a 'weak' sense, that is, when both sides of the equation are inserted into correlation functions $\langle \ldots \Phi \rangle$ with all the other operators in Φ much further away. It should also be remarked that the choice of the argument of ϕ_k as the midpoint of r_1 and r_2 is motivated only by considerations of symmetry, and is in fact arbitrary. It is, for example, quite permissible to take the argument of ϕ_k to be r_1. The difference between these two cases involves the Taylor expansion of ϕ_k about r_1. Therefore, to be correct, in the sum over k on the right hand side we should include not just the scaling operators ϕ_k but all possible derivatives of them. These derivative operators would not enter, coupled to scaling fields, in the hamiltonian, since they would give zero on summing over the whole system, but they do enter into the operator product expansion. Fortunately, this subtlety will not trouble us in deriving the renormalization group equations, at least to the order required. Because ϕ_i, ϕ_j and ϕ_k are scaling operators, the form of the function $C_{ijk}(r_1 - r_2)$ is in fact completely determined† by the

† In fact, (5.6) is valid only when all the operators transform as scalars under rotations. The appropriate generalisation to non-scalars is straightforward but very cumbersome, at least for dimension $d > 2$. Fortunately, the most

homogeneity relation (3.59).

$$C_{ijk}(r_1 - r_2) = \frac{c_{ijk}}{|r_1 - r_2|^{x_i + x_j - x_k}}. \tag{5.6}$$

The quantities c_{ijk} are pure numbers called the *operator product expansion coefficients.* Their numerical value depends on the way the operators are normalised: if we fix this by requiring that $\langle \phi_i(r_1)\phi_i(r_2)\rangle = |r_1 - r_2|^{-2x_i}$, for example, the c_{ijk} are then *universal.* As we shall see in the next section, it is these quantities which determine the first order corrections to the renormalization group equations.

5.2 The perturbative renormalization group

Following the discussion in the introduction to this chapter, let us therefore consider a fixed point hamiltonian $\mathcal{H}^*$ which is perturbed by a number of scaling fields, so that the partition function is

$$Z = \mathrm{Tr} e^{-\mathcal{H}^* - \sum_i g_i \sum_r a^{x_i} \phi_i(r)}. \tag{5.7}$$

The factors of a^{x_i} are needed because the coupling constants g_i, being linear combinations of the K_a on the lattice, are dimensionless, and the scaling operators have dimension $(\text{length})^{-x_i}$ if we wish to normalise them as above. Let us expand this in powers of the couplings g_i:

$$\begin{aligned} Z = Z^* \Bigg[1 &- \sum_i g_i \int \langle \phi_i(r)\rangle \frac{d^d r}{a^{d-x_i}} \\ &+ \tfrac{1}{2} \sum_{ij} g_i g_j \int \langle \phi_i(r_1)\phi_j(r_2)\rangle \frac{d^d r_1 d^d r_2}{a^{2d - x_i - x_j}} \\ &- \tfrac{1}{3!} \sum_{ijk} g_i g_j g_k \int \langle \phi_i(r_1)\phi_j(r_2)\phi_k(r_3)\rangle \frac{d^d r_1 d^d r_2 d^d r_3}{a^{3d - x_i - x_j - x_k}} + \cdots \Bigg], \end{aligned} \tag{5.8}$$

where all correlation functions are to be evaluated with respect to the fixed point hamiltonian $\mathcal{H}^*$. A number of comments should be made at this point. First, we have adopted a continuum notation in which $\sum_r$ is replaced by $\int (d^d r / a^d)$, where a is the microscopic short distance cut-off (for example, the lattice spacing). This is

relevant operators are scalars, so we may ignore this complication.

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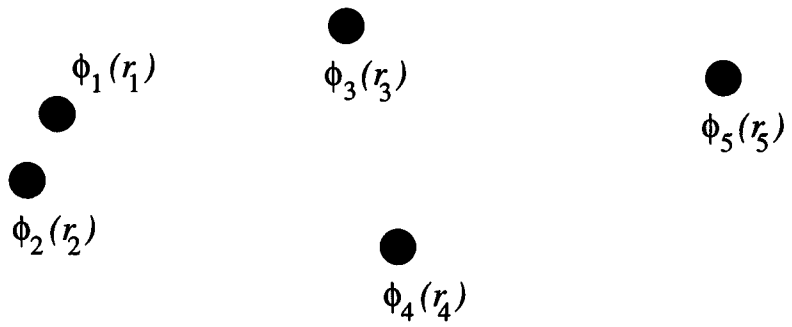

Figure 5.2. Gas picture representing a typical term in (5.7).

justified since the details of the lattice are supposed to be irrelevant. Second, (5.8) purports to represent the power series expansion about a critical point, where it known to be singular. Such an expansion should not exist, and this is made manifest by the fact that the integrals are in fact divergent in the infinite volume limit. The solution to this problem is to first place the system in a finite box, of linear size L. All integrals are then finite, and in fact Z is now non-singular because thermodynamic singularities emerge only in the thermodynamic limit. The drawback to this is that the correlation functions are sensitive to this infrared cut-off, and, for example, the 2-point functions will in general not have a simple power-law form. However, it will turn out that in order to implement the renormalization group, all we need are the short-distance singularities, given by the operator product expansion, which is independent of any boundary conditions.

The expansion in (5.8) may be thought of as a kind of low-density expansion of a gas in the grand canonical ensemble (see Figure 5.2). The gas contains particles of different species, labelled by i, each with its own fugacity g_i. The partition function is given by a sum of one-body terms, two-body terms, and so on. The analogue of the Boltzmann weight of a given configuration is the appropriate correlation function. Note that, although it is always possible to redefine our operators by $\phi_i \to \phi_i - \langle \phi_i \rangle$, so that they have zero expectation value in the infinite volume limit, this will not necessarily be true in a finite box. Similarly, although as discussed in Section 11.2, the 2-point functions vanish when $i \neq j$

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in the infinite volume, this is also not necessarily true at finite L. Therefore, we shall keep all the appropriate terms in (5.8).

We now implement the renormalization group by changing the microscopic cut-off $a \to ba$, with $b = 1 + \delta\ell$, and asking how the couplings g_i should be changed in order to preserve the partition function Z. The length a appears in three ways in (5.8):

- Explicitly, through the divisors a^{d-x_i};
- Implicitly, since the integrals have potential short-distance divergences as the points approach each other. On the lattice, this would be regulated by the lattice itself. In the continuum approach, it is more practical to introduce a rotationally invariant cut-off, and to insist that all integrals should be restricted to $|r_i - r_j| > a$. In the interacting gas picture, this corresponds to introducing a hard core repulsion of radius a between the particles. This cut-off, although crude, is quite sufficient for the first order calculation of the renormalization group functions.
- Through the dependence on the system size L, in the dimensionless ratio L/a.

If we choose an infinitesimal rescaling of a, with $\delta\ell \ll 1$, the effects of these different kinds of dependence may be considered separately, as they will contribute additively to the beta functions. The first dependence is simple: the rescaling is compensated in every term in (5.8) by a rescaling

$$g_i \to (1 + \delta\ell)^{d-x_i} g_i \sim g_i + (d - x_i) g_i \delta\ell. \tag{5.9}$$

The effect of the change in the cut-off may be evaluated by use of the operator product expansion (5.5). To see this explicitly, consider the simplest case, of the second order term in (5.8). After changing $a \to a(1 + \delta l)$, we may break up the integral as

$$\int_{|r_1 - r_2| > a(1+\delta\ell)} = \int_{|r_1 - r_2| > a} - \int_{a(1+\delta\ell) > |r_1 - r_2| > a}. \tag{5.10}$$

The first term simply gives back the original contribution to Z. The second term may be expressed using the operator product expansion, as

$$\tfrac{1}{2} \sum_{ij} \sum_{k} c_{ijk} a^{x_k - x_i - x_j} \int_{a(1+\delta\ell) > |r_1 - r_2| > a} \langle \phi_k((r_1 + r_2)/2) \rangle \frac{d^d r_1 d^d r_2}{a^{2d - x_i - x_j}}. \tag{5.11}$$

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The integral gives a factor $S_d a^d \delta\ell$, where S_d is the area of a hypersphere of unit radius in d dimensions.† This term may then be compensated by making the change

$$g_k \to g_k - \tfrac{1}{2} S_d \sum_{ij} c_{ijk} g_i g_j \, \delta\ell. \tag{5.12}$$

In fact, because of the structure of the operator product expansion (the fact that it is independent of all the other operators Φ in (5.3)), a similar change works to all orders in the expansion (5.8). It is simply a matter of keeping track of the combinatorics to verify that this is the case. Finally, there is the dependence on a which arises through the finite size L. However, it is important *not* to change L, otherwise we should have done nothing at all! As a result, $N^{-1} \equiv a/L$ flows according to the finite-size scaling arguments of Section 4.4.

Putting together the two contributions to the renormalization of the coupling constants,

$$dg_k/d\ell = (d - x_k) g_k - \tfrac{1}{2} S_d \sum_{ij} c_{ijk} g_i g_j + \cdots. \tag{5.13}$$

The renormalization group eigenvalues at the trivial fixed point are $y_k = d - x_k$, in accord with the fundamental scaling relation (3.55). The somewhat unpleasant factor of $\frac{1}{2} S_d$ may be removed by a simple uniform rescaling $g_k \to (2/S_d) g_k$ of the couplings, and this we shall assume to have already been done from now on. The final form of the perturbative renormalization group equations is therefore

$$\boxed{dg_k/d\ell = y_k g_k - \sum_{ij} c_{ijk} g_i g_j + \cdots} \tag{5.14}$$

Note that we have not assumed that the scaling operators are normalised in any particular manner in deriving this (merely that this normalisation is independent of a). In practical applications, it is useful to have this freedom. The higher order terms on the right hand side arise from situations in which three or more integration points approach within a distance a of each other. Such contributions are evidently very difficult to evaluate using the hard core

† Explicitly $S_d = 2\pi^{d/2}/\Gamma(d/2)$, although its value will never be needed in the sequel.

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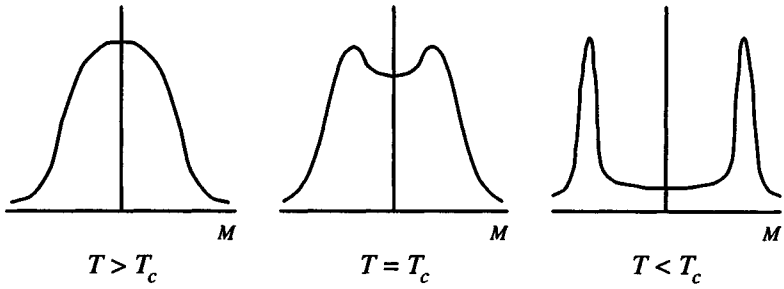

Figure 5.3. Schematic histograms of the block magnetisation in high dimensions.

cut-off used above. At this point, more sophisticated approaches, such as the use of renormalized quantum field theory, show their advantage. However, a considerable amount of formal apparatus needs to be erected before these methods yield useful results, and this goes beyond the remit of this volume.

5.3 The Ising model near four dimensions

We shall now apply the above methods to the well-known and important example of the critical Ising model near four dimensions. In higher dimensions, block spin transformations of the type described in Section 3.1 become less and less reliable. One reason for this (apart from the proliferation of possible couplings) is seen if one studies typical histograms of the total magnetisation $\sum_i s_i$ of a block immersed in a large system, shown schematically in Figure 5.3. Close to the critical point, the distribution gets less and less peaked around two values as the dimension increases. It therefore becomes less reasonable to model this situation with block spins which take only the values ± 1. Instead, it is more appropriate to consider *continuous spins* $S(r)$, which are real variables which can, in principle, take any value. However, we expect their distribution still to be peaked around the typical values ± 1. One way to model this is to write down a continuous spin hamiltonian,

of the form

$$\mathcal{H} = -\tfrac{1}{2}\sum_{r,r'} J(r-r')S(r)S(r') - H\sum_r S(r) + \lambda\sum_r (S(r)^2-1)^2, \tag{5.15}$$

where λ is a number of order unity. The partition function is now

$$Z = \int \prod_r dS(r)\, e^{-\mathcal{H}}. \tag{5.16}$$

This modification of the original problem may seem quite barbaric and arbitrary, but, in the spirit of universality, it is quite justified as long as we remain within the domain of attraction of the same critical fixed point. In the end, the justification for this assumption has to be in a comparison of the final answers with those obtained by other methods. In addition, we must be content with calculating only universal quantities, and not, for example, the critical temperature. Since the microscopic parameters for real systems are, in any case, not completely known, in practice this is no great limitation.

An advantage of using a continuous spin variable is that the space continuum limit is more transparent. Since we are interested in only the long wavelength modes of $S(r)$, we may further approximate

$$\begin{aligned}\sum_{r,r'} J(r-r')S(r)S(r') \approx \\ \sum_{r,r'} J(r-r')S(r)\left[S(r) + (r-r')\cdot\nabla S + \tfrac{1}{2}(r-r')^2\nabla^2 S + \cdots\right] \\ = J\sum_r \left(S(r)^2 - R^2a^2(\nabla S(r))^2\right) + \cdots,\end{aligned} \tag{5.17}$$

after symmetrisation and integration by parts. As in Chapter 2, $J = \sum_r J(r)$ and $R^2J = \sum_r r^2 J(r)$. The last step is to approximate the sum on r by an integral. The resultant hamiltonian has the form

$$\mathcal{H} = \int \left[\tfrac{1}{2}Ja^2R^2(\nabla S)^2 - (2\lambda + J)S^2 + \lambda S^4 - HS\right]\frac{d^dr}{a^d}. \tag{5.18}$$

Finally, since S is a continuous variable, we may rescale it so that the coefficient of the $(\nabla S)^2$ term is $\frac{1}{2}$: $S(r)^2 \to (a^{d-2}/JR^2)S(r)^2$.

The hamiltonian may then be written as

$$\mathcal{H} = \int [\tfrac{1}{2}(\nabla S)^2 + ta^{-2}S^2 + ua^{d-4}S^4 + ha^{-d/2-1}S]d^d r, \quad (5.19)$$

where† $t = -(2\lambda + J)/JR^2$, $u = \lambda/J^2R^4$, and $h = -H/J^{1/2}R$. The form of the hamiltonian in (5.19) is sometimes called the Landau–Ginzburg–Wilson model.

There are three dimensionless couplings here: t, h and u. We may trade J, R and H for them as coordinates in the space of all couplings. The infinite set of other couplings may now be thought of as multiplying all the other terms involving higher powers of S and higher numbers of derivatives, neglected in writing (5.19). As will be seen, these are all irrelevant near four dimensions. We may now apply the renormalization group to lowest order by letting $a \to ba$, and asking how we should change the parameters so as to keep (5.19) invariant. This is simply

$$t' = b^2 t \quad (5.20)$$

$$h' = b^{d/2+1} h \quad (5.21)$$

$$u' = b^{4-d} u. \quad (5.22)$$

5.4 The Gaussian fixed point

There is an obvious fixed point of these equations at $t = h = u = 0$. This is called the *Gaussian* fixed point, because the fixed point hamiltonian is then $\mathcal{H}^* = \frac{1}{2}\int (\nabla S)^2 d^d r$, and the partition function is simply a Gaussian integral. There is an obvious difference in the properties of this fixed point depending on whether d is greater than, or less than, four.

For $d > 4$, u is irrelevant, and the fixed point has two relevant variables, one in the thermal sector and one magnetic, as expected of a fixed point which is to describe the critical Ising model. The eigenvalues are simply $y_t = 2$ and $y_h = d/2 + 1$, and, following the arguments in Section 3.5 we may then deduce the values of the various exponents. These are shown in Table 5.1, together with predictions of mean field theory, which, as we argued in Section 2.4, should give the correct exponents for $d > 4$. Note that

† The reader who is puzzled why the value of t is apparently always negative is referred to the discussion on p.95.

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Table 5.1. *Exponents at the Gaussian fixed point and in mean field theory.*

Exponent	Gaussian	MFT
α	$2 - d/2$	0
β	$(d-2)/4$	$\frac{1}{2}$
γ	1	1
δ	$(d+2)/(d-2)$	3
ν	$\frac{1}{2}$	$\frac{1}{2}$
η	0	0

agreement is perfect when $d = 4$, and that the exponents (γ, ν, η) which may be related to the correlation function all agree.‡ But something is wrong in the free energy. For $d > 4$, the predictions of mean field theory are all more singular than those of the Gaussian fixed point. In addition, the hyperscaling relation (3.52) is seen to fail.

This failure of conventional renormalization group scaling may be understood in several ways. In terms of the homogeneous scaling law (3.40) for the singular part of the free energy, it must be that u is *dangerously* irrelevant, that is, it may not simply be set equal to zero (see p.49). This is most spectacularly seen in the spontaneous magnetisation: if we simply minimise the hamiltonian in (5.19) we find that $\langle S \rangle \propto (-t/u)^{1/2}$. This is consistent with the mean field result $\beta = \frac{1}{2}$, but it certainly makes no sense to set $u = 0$ in this result.

Alternatively, we can consider the free energy as calculated by the iterated inhomogeneous transformation law (3.60). The sum consists of two parts, one coming from the vicinity of the Gaussian fixed point, which will give rise to contributions scaling with the exponents in the second column of Table 5.1, and a piece from the renormalization group trajectory far from the fixed point, where mean field theory should apply. Since the mean field values are more singular than the Gaussian ones, this contribution actually dominates. This argument does, however, suggest that the Gaus-

‡ Although the susceptibility is given by a suitable derivative of the free energy, it is also proportional to $\sum_r \langle s(r)s(0) \rangle$.

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sian values should be observed as subleading corrections to the mean field terms – a feature which is not present in unadorned mean field theory.

When $d < 4$, u is relevant at the Gaussian fixed point, and this is no longer the appropriate one to describe the critical point of the Ising model. However, if the value of u happens to be small in a particular system, we may use the cross-over theory of Section 4.2. The cross-over exponent is $\phi = (4-d)/2$, and the cross-over region is given by $|t| \sim u^{2/(4-d)}$. If we express this in terms of ξ and R we then recover the Ginzburg criterion of Section 2.4.

5.5 The Wilson–Fisher fixed point

When the dimension d is slightly smaller than four, the scaling field u is only slightly relevant at the Gaussian fixed point, and therefore we might hope to find another nearby to which it flows, using the perturbative methods described in Section 5.2. In order to do this it is necessary to compute the operator product expansion coefficients at the Gaussian fixed point. We first have to learn how to compute correlation functions in the Gaussian model, with hamiltonian $\mathcal{H} = \frac{1}{2}\int(\nabla S)^2 d^d r$. Fortunately this is very simple. Since the reduced hamiltonian must be dimensionless, S must have dimensions $(\text{length})^{-d/2+1}$. Therefore the 2-point correlation function $\langle S(r_1)S(r_2)\rangle$ must be proportional to $|r_1 - r_2|^{-d+2}$. Since we may always redefine the normalisation of S so that the coefficient of this power law is unity, we therefore take

$$\langle S(r_1)S(r_2)\rangle = \frac{1}{|r_1 - r_2|^{d-2}}. \tag{5.23}$$

Higher correlation functions $\langle S(r_1)\ldots S(r_{2n})\rangle$ in the Gaussian model are given by the following rule (see Appendix): sum over all possible ways of dividing the set of points $(r_1, \ldots, r_{2n})$ into n pairs, each pair corresponding to a two-point correlation function, or *Wick contraction*. For example

$$\langle S(r_1)S(r_2)S(r_3)S(r_4)\rangle = \langle S(r_1)S(r_2)\rangle\langle S(r_3)S(r_4)\rangle + \langle S(r_1)S(r_3)\rangle\langle S(r_2)S(r_4)\rangle + \langle S(r_1)S(r_4)\rangle\langle S(r_2)S(r_3)\rangle. \tag{5.24}$$

If we decide to represent the two-point function $\langle S(r_1)S(r_2)\rangle$ by a line connecting r_1 and r_2, the terms on the right hand side of (5.24) are represented by the diagrams in Figure 5.4. The above

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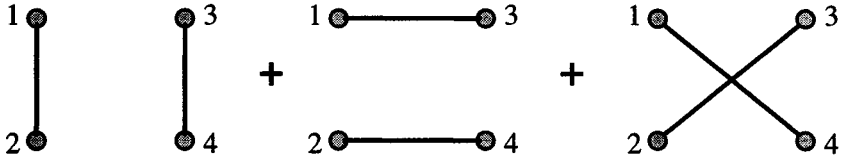

Figure 5.4. Diagrams for the 4-point correlation function in the Gaussian model.

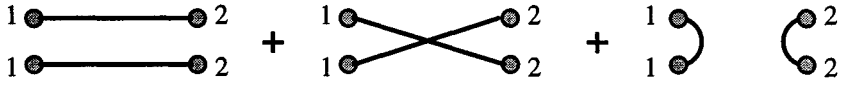

Figure 5.5. Diagrams for the $\langle S^2 S^2 \rangle$ correlation function.

rule is equivalent to *Wick's theorem* in quantum field theory. It still holds when some of the points coincide. To enumerate all the possible pairings in this case, it is convenient to imagine the points as slightly separated, and then take the limit of coincidence at the end. For example, to compute $\langle S^2(r_1)S^2(r_2)\rangle$, we split the points as shown in Figure 5.5. There are three distinct ways of contracting these points in pairs. The first two each give rise to a contribution $\langle S(r_1)S(r_2)\rangle^2$. The third is different, however, and gives $\langle S^2\rangle^2$, independent of $|r_1 - r_2|$. Such terms do not contribute to the renormalization group equations, and it is convenient to remove them by defining *normal ordered* operators by

$$:S^2: \equiv S^2 - \langle S^2 \rangle \tag{5.25}$$

$$:S^4: \equiv S^4 - 3\langle S^2 \rangle S^2, \tag{5.26}$$

and so on. This merely corresponds to taking a slightly different basis of scaling operators at the Gaussian fixed point. In particular, the coefficient of $:S^2:$ in the hamiltonian changes according to $t \to t' = t + 3ua^{d-2}\langle S^2 \rangle$, which now changes sign as the temperature is varied. With the normal ordered operators, Wick's theorem is now modified by the rule: ignore all diagrams in which both ends of a line are connected to the same point. This considerably reduces the number of diagrams, without losing any of the physics.

Wick's theorem also applies to the operator product expansion. For example, consider the operator product $:S(r_1)^2::S(r_2)^2:$. This means that we should consider all possible correlation functions

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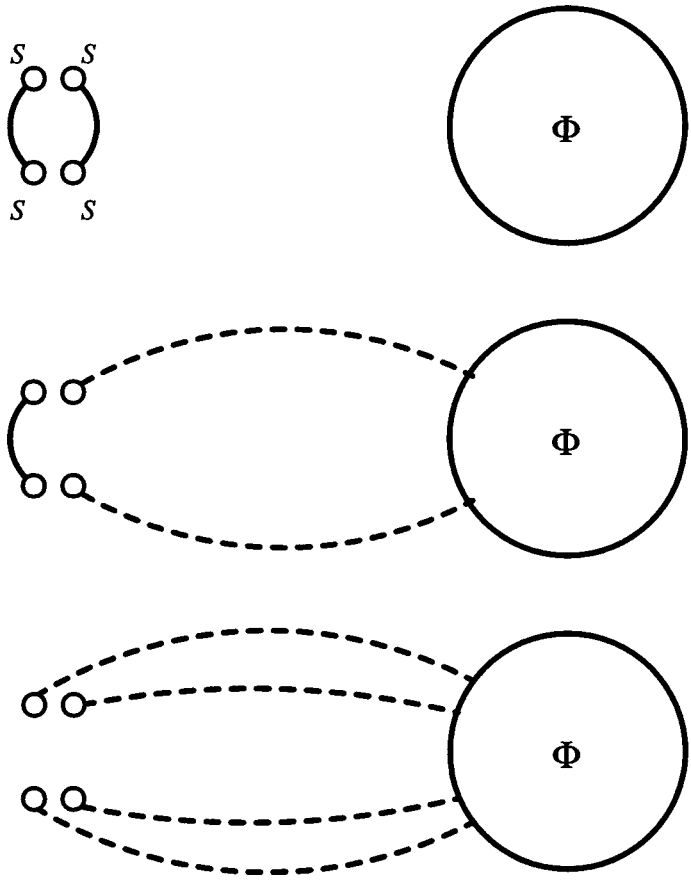

Figure 5.6. Operator product expansion of S^2 and S^2.

of the form $\langle : S(r_1)^2 :: S(r_2)^2 : \Phi \rangle$. Applying Wick's theorem to this gives a number of terms, depending on how many of the $S(r_1)$ are connected to the $S(r_2)$. These possibilities are illustrated in Figure 5.6. If both the operators at r_1 are connected to operators at r_2, there is no further possibility of connection to the operators in Φ. Thus, from the point of view of Φ, this pairing will behave as the insertion of the unit operator at r_1. Similarly, if just one of the $S(r_1)$ is connected onto one of the $S(r_2)$, this leaves a product $S(r_1)S(r_2)$ each of whose factors must be connected onto an operator in Φ. From a distance, this operator will then look like

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$:S(r_1)^2:$. Finally, if none of the operators at r_1 or r_2 are connected to each other, they must all be connected to operators in Φ, and, from a distance, they will look like $:S(r_1)^4:$. Once again, using normal ordering allows us to neglect all contributions where operators at the same point are connected. Proceeding in this manner, we find that the required operator product expansion has the form

$$:S(r_1)^2::S(r_2)^2:=\frac{2}{r_{12}^{2d-4}}+\frac{4}{r_{12}^{d-2}}:S(r_1)^2:+:S(r_1)^4:+\cdots, \tag{5.27}$$

where $r_{12}=|r_1-r_2|$, and the neglected terms include derivative operators. The only important quantities in (5.27) are the operator product expansion coefficients $(2,4,1,\ldots)$, and we see that they are very simply determined by counting arguments. It is similarly straightforward to evaluate the other operator product expansions. Denoting $:S^n:$ by ϕ_n, and focussing only on the coefficients themselves, the relevant operator product expansions may be written schematically as

$$\phi_1\cdot\phi_1=1+\phi_2 \tag{5.28}$$

$$\phi_1\cdot\phi_2=2\phi_1+\phi_3 \tag{5.29}$$

$$\phi_1\cdot\phi_4=4\phi_3+\cdots \tag{5.30}$$

$$\phi_2\cdot\phi_2=2+4\phi_2+\phi_4 \tag{5.31}$$

$$\phi_2\cdot\phi_4=12\phi_2+8\phi_4+\cdots \tag{5.32}$$

$$\phi_4\cdot\phi_4=24+96\phi_2+72\phi_4+\cdots. \tag{5.33}$$

The neglected terms in the above correspond to values of $n>4$, and to derivative operators, both of which will turn out to be irrelevant. Notice that the operator $\phi_3=:S^3:$ has been generated. For reasons to be explained below on p.101, this may be ignored. In fact, very few of the operator product expansion coefficients shown in (5.28–5.33) actually enter into the subsequent calculation. To illustrate further the computation of one which does, consider the coefficient of ϕ_4 in the operator product $\phi_4\cdot\phi_4$. One of the contractions is shown in Figure 5.7. Two of the factors $S(r_1)$ must be connected onto two of the $S(r_2)$, leaving four S's over. The two $S(r_1)$ may be chosen in $4\cdot3$ ways, and similarly for the two $S(r_2)$. But this method of reasoning will double count, since there are 2! equivalent permutations of the lines which connect r_1 and r_2. Thus the correct coefficient is $(4\cdot3)^2/2=72$.

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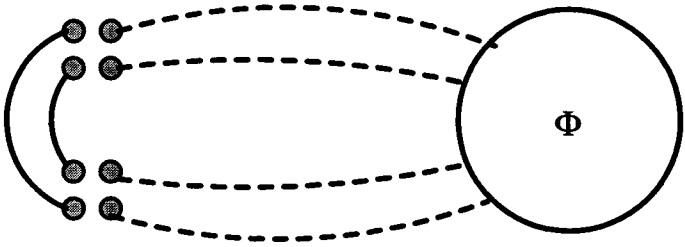

Figure 5.7. Contribution to S^4 in the operator product expansion of S^4 with S^4.

We are now in a position to write down the renormalization group equations, using (5.14):

$$dh/d\ell = (d/2+1)h - 2\cdot 2ht + \cdots \tag{5.34}$$

$$dt/d\ell = 2t - h^2 - 4t^2 - 2\cdot 12tu - 96u^2 + \cdots \tag{5.35}$$

$$du/d\ell = \epsilon u - t^2 - 2\cdot 8tu - 72u^2 + \cdots, \tag{5.36}$$

where $\epsilon = 4-d$. The method for obtaining these is straightforward: for example, to find the terms contributing to $dt/d\ell$ we simply look at where ϕ_2 occurs on the right hand side of the operator product expansions in (5.28–5.33). Note that contributions where $i \neq j$ in (5.14) are counted twice, since $c_{ijk} = c_{jik}$, and therefore appear with factors of two.

If we now suppose that ϵ is small, we see that, apart from the Gaussian fixed point, there is another in which $h = 0$, $u = u^* = \epsilon/72 + O(\epsilon^2)$, and $t = O(\epsilon^2)$. This is called the *Wilson–Fisher fixed point.* The thermal eigenvalue at this new fixed point is given by linearising (5.35):

$$dt/d\ell = 2t - 24u^*t + \cdots, \tag{5.37}$$

so that $y_t = 2 - (24/72)\epsilon + O(\epsilon^2)$. Finally, this gives for the correlation length exponent $\nu = 1/y_t$

$$\boxed{\nu = \tfrac{1}{2} + \tfrac{1}{12}\epsilon + O(\epsilon^2)} \tag{5.38}$$

We see that the $O(\epsilon)$ corrections to the exponents are given by simple ratios of operator product expansion coefficents at the Gaussian fixed point. In fact, as stated earlier, very few of the coefficients calculated in (5.28–5.33)) enter into the final result. With hindsight, we could therefore have saved even more effort. Because ϕ_1 does not appear in the operator product expansion of ϕ_1

Table 5.2. *Estimates for critical exponents of the two-dimensional Ising model from the resummed ϵ-expansion at order ϵ^5, compared with exact results.*

	ϵ-expansion	Exact
ν	0.99 ± 0.04	1
η	0.26 ± 0.05	0.25

and ϕ_4, the $O(\epsilon)$ correction to the magnetic exponent vanishes, so $\eta = 0$ to this order. In fact, it does receive a contribution at $O(\epsilon^2)$.

The usefulness of the ϵ-expansion

The computation of the critical exponents beyond the first order requires knowledge of the higher order terms in the renormalization group equations (5.14). These do not follow simply from the operator product expansion, and require more sophisticated calculations of Feynman diagrams which are beyond the scope of this book. However, in any case, one might question the numerical accuracy of an expansion scheme in powers of a parameter ϵ which ultimately has to be set equal to one (or two) to obtain results of direct physical significance. In fact, in many cases, the ϵ-expansion does remarkably well if carried out to sufficiently high orders. It is believed, however, that it is only an asymptotic expansion. That is, for any non-zero value of ϵ, no matter how small, the series will eventually give inaccurate results if truncated at too high an order. There are, nevertheless, a number of efficient resummation methods available which yield strongly convergent estimates, even for $d = 2$. The best values for the Ising model ($n = 1$) universality class in two and three dimensions are shown in Tables 5.2 and 5.3, and compared with the results of other accurate methods: a direct field-theoretic renormalization group calculation, high-temperature expansions, and exact results.†

Even in other cases where the ϵ-expansion fares poorly from a

† There appears to be, in fact, a small but persistent discrepancy between the high-temperature expansion estimates in three dimensions and those of the RG, which has not been satisfactorily explained.

Table 5.3. *Estimates for critical exponents of the three-dimensional Ising model from the resummed ϵ-expansion at order ϵ^5, compared with resummed direct RG and high-temperature expansion methods. The last estimate for η follows from those for γ and ν and the scaling relation $\gamma = \nu(2-\eta)$.*

	ϵ-expansion	Direct RG	High-T expansion
ν	0.6305 ± 0.0025	0.6300 ± 0.0015	0.633 ± 0.0013
η	0.037 ± 0.003	0.031 ± 0.004	0.042 ± 0.005

numerical point of view, it nevertheless provides an unambiguous classification of possible fixed points in a given system, the number and nature of the various relevant scaling variables at each fixed point, and the existence of scaling laws between the various critical exponents. All of these are robust predictions of the method which hold independently of its ability to make quantitative statements. This should be contrasted with real space methods where such results may depend strongly on the precise transformation chosen.

Irrelevant operators

We now discuss an important point which has been ignored so far. The appearance of the S^4 term in the continuous spin hamiltonian was crucial to understanding the role of four dimensions, yet it seemed to emerge from the somewhat arbitrary choice of $(S^2-1)^2$ as a weight function in (5.15). What if we had chosen some other? In general, such a weight function should be an even function of S, and we may therefore consider expanding it as a power series $\sum_n g_{2n} S^{2n}$. If we make the same rescaling of S as before, in the continuum version the coupling g_{2n} will come along with a factor $a^{n(d-2)-d}$. Thus, at the Gaussian fixed point where all the g_{2n} vanish, this coupling has a renormalization group eigenvalue $2n-(n-1)d$. If d is near 4, then all couplings with $2n > 4$ are irrelevant at the Gaussian fixed point, so that they give only correction to scaling terms. Since the non-trivial fixed point is, for small ϵ, close to the Gaussian fixed point, we expect the eigenvalues of these operators to be modified only to $O(\epsilon)$ at worst, and so they will

remain irrelevant at the new fixed point.

One might object that g_6 becomes marginal exactly in three dimensions, where we hope eventually to apply the theory, and therefore it may not be neglected. However, this criticism is not well-founded, because, as we showed above, the relevant fixed point for $d < 4$ is not the Gaussian, but rather the Wilson–Fisher fixed point, and we should really ask whether g_6 is relevant there. In order to study this question in greater depth, let us calculate the renormalization group eigenvalue of g_6 to first order in ϵ. According to (5.14) this involves evaluating the coefficient of ϕ_6 appearing in the product of ϕ_4 and ϕ_6. By the same counting arguments as above, this is $4!6!/4!2!2! = 180$, so the renormalization group equation is

$$dg_6/d\ell = (6 - 2d)g_6 - 360ug_6 + \cdots, \tag{5.39}$$

where none of the neglected terms affect the $O(\epsilon)$ calculation. Expanding in ϵ we then see that the eigenvalue at the non-trivial fixed point is

$$-2 + 2\epsilon - 5\epsilon + O(\epsilon^2), \tag{5.40}$$

where the third term comes from the $O(u^*)$ correction. We see that this quite overwhelms the second term, and that u_6 is actually *less* relevant in $4 - \epsilon$ dimensions than in $d = 4$! Of course, such a result might not survive to higher orders in ϵ, but all indications are that it does.

One consequence of this calculation is that the corrections to scaling terms coming from the higher powers of S are not expected to be very important. The dominant correction to scaling comes from the irrelevant variable $u^* - u$, which has eigenvalue $y = -\epsilon + O(\epsilon^2)$. This will give rise to corrections to scaling of the order $|t|^{|y|/y_t} \approx |t|^{1/2}$ in three dimensions.

Redundant operators

Let us now return to the question of the S^3 term, which according to (5.30), is generated by the renormalization group when $h \neq 0$, even if it is not present in the original hamiltonian (5.19). Power counting would suggest that such an operator is relevant near $d = 4$; how, then, may it be ignored? If we explicitly include such

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a term in the hamiltonian, it then has the form

$$\mathcal{H} = \int [\tfrac{1}{2}(\nabla S)^2 + tS^2 + u_3 S^3 + uS^4 + hS] d^d r, \tag{5.41}$$

where the factors of a have been suppressed for clarity. It is now clear that the S^3 term may be removed by a suitable shift $S \to S + \text{const.}$, at the cost of redefining t and h. Since S is nothing but an integration variable over which the trace is performed in the partition function, we are clearly at liberty to make such a shift without altering the physics in any way. In fact, if we make an infinitesimal uniform shift $S \to S + \delta S$, the first order change in the hamiltonian density is

$$(2tS + 3u_3 S^2 + 4uS^3 + h)\delta S, \tag{5.42}$$

which corresponds to a redefinition of the parameters

$$h \to h + 2t\delta S \tag{5.43}$$

$$t \to t + 3u_3 \delta S \tag{5.44}$$

$$u_3 \to u_3 + 4u\delta S. \tag{5.45}$$

Thus, within the 4-dimensional subspace of renormalization group flows parametrised by (h, t, u_3, u), there is a family of one-dimensional curves generated by (5.43–5.45) along which the actual physics does not change. The tangent vectors to these curves are called redundant scaling variables, and the operators to which they couple in the hamiltonian (in this case $2tS + 3u_3 S^2 + 4uS^3$) are called *redundant operators.*

Thus, although S^3 is apparently relevant, it may be eliminated by a suitable change in h and t. Since this shift is itself of $O(u)$, however, it does not affect the calculation of the critical exponents to first order in ϵ. The redundant operator described above is not the only one. In fact, any suitably local redefinition of the variable S will correspond to such a direction in parameter space. Within the even subspace, such a redefinition has the form

$$S \to S + a_1 S^3 + a_2 S^5 + \cdots + b_1 \nabla^2 S + b_2 S^2 \nabla^2 S + \cdots \tag{5.46}$$

and it may be seen to introduce only operators which are irrelevant near $d = 4$. Because of this redundancy, the physical space of irrelevant operators is smaller than it might appear, but, it is still very large! In the Ising universality class, only the S^3 operator is superficially relevant but also redundant.

5.6 Logarithmic corrections in $d = 4$

When $d = 4$, the scaling variable u is marginal. While $d = 4$ is not physically relevant, it does provide a simple example in which the consequences of the existence of such a marginal variable may be illustrated. For $d=4$, the renormalization group equations reduce to

$$du/d\ell = -72u^2 + \cdots \tag{5.47}$$

$$dt/d\ell = 2t - 24ut + \cdots, \tag{5.48}$$

where all the inessential terms have been dropped. We see that u is in fact marginally *irrelevant*, so that the large distance behaviour is still controlled by the Gaussian fixed point. However, as u will flow to zero only very slowly, we might expect modifications of the behaviour inferred from the Gaussian fixed point. This is indeed the case.

The equation for u may be integrated straightforwardly to give

$$u(\ell) = \frac{u(0)}{1 + 72u(0)\ell}. \tag{5.49}$$

The transformation law (3.26) for the singular part of the free energy, suitably generalised to the case of an infinitesimal renormalization group transformation, implies that

$$f_s(t, u) = e^{-d\ell} f_s\left(t(\ell), u(\ell)\right). \tag{5.50}$$

As usual, we choose $\ell = \ell_0$, so that $t(\ell_0) = t_0 = O(1)$. We may find ℓ_0 by observing that

$$\begin{aligned} \ln(t_0/t) = \int_t^{t_0} \frac{dt'}{t'} &= \int_0^{\ell_0} \left[2 - \frac{24u(0)}{1 + 72u(0)\ell}\right] d\ell \\ &= 2\ell_0 - \tfrac{1}{3}\ln\left[1 + 72u(0)\ell_0\right]. \end{aligned} \tag{5.51}$$

This transcendental equation cannot be solved explicitly, but, for large t_0/t, it may be iterated to give the asymptotic expansion

$$\ell_0 = \tfrac{1}{2}\ln(t_0/t) + \tfrac{1}{6}\ln\left[1 + 36u(0)\ln(t_0/t)\right] + \cdots. \tag{5.52}$$

From (5.50)

$$f_s(t, u(0)) = e^{-4\ell_0} f_s(t_0, u(\ell_0)). \tag{5.53}$$

In estimating the right hand side we should be mindful that u may be a dangerous irrelevant variable, in which case it is not permissible simply to set $u(\ell_0) = 0$. Since the right hand side is to be evaluated outside the critical region, we may use the mean

field results of Chapter 2 to estimate that $f_s(t_0, u(\ell_0)) \propto t_0^2/u(\ell_0)$. Using the value of $u(\ell_0)$ from (5.49) then yields

$$f_s(t, u(0)) \propto t^2 \left[1 + 36u(0) \ln(t_0/t)\right]^{1/3}, \tag{5.54}$$

When $u(0) = 0$, we recover the results of the Gaussian fixed point, but for $u(0) \neq 0$ there are multiplicative *logarithmic corrections* to scaling. In particular, for the specific heat we find

$$C \propto |\ln(t/t_0)|^{1/3}, \tag{5.55}$$

as $t \to 0$. This type of behaviour is a typical consequence of the existence of a marginally irrelevant variable. Note that the power of the logarithm in (5.55) was simply determined by the ratio of operator product expansion coefficients.

Although the above example was in four dimensions, there are physical examples of universality classes which have $d_c = 3$ as their upper critical dimension, and which are therefore expected to exhibit such logarithmic corrections. Examples are the tricritical Ising model (see Ex. 2.6), the theta-point in linear polymers (see Section 9.5), and uniaxial dipolar ferromagnets. However, such logarithmic corrections are very difficult to see in practice, either experimentally or in numerical simulations. The asymptotic behaviour like that in (5.55) is apparent only when $36u(0)\ln(t_0/t) \gg 1$, and, since $u(0)$ is not universal, it is difficult to predict when this will set in. Even trying to fit the data with the more accurate form (5.54) can be misleading, because the higher terms in the asymptotic expansion (5.52) lead to contributions which are down only by powers of $\ln(t_0/t)$, rather than powers of t as in the normal case.

5.7 The $O(n)$ model near four dimensions

So far we have considered only the universality class of the critical Ising model. All the above calculations may be generalised quite easily to the $O(n)$ model or n-vector model introduced in Section 1.2. Instead of Ising spins taking the values ± 1, these models have n-component spins $\mathbf{s}(r)$, normalised so that $\mathbf{s}(r)^2 = 1$. The continuous spin version of this is a simple generalisation of that for the Ising model ($n = 1$):

$$\mathcal{H} = \int [\tfrac{1}{2}(\nabla \mathbf{S})^2 + ta^{-2}\mathbf{S}^2 + ua^{d-4}(\mathbf{S}^2)^2 + ha^{-d/2-1}S_1] d^d r. \tag{5.56}$$

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The ϵ-expansion for this model is very similar to that carried out in the Ising case. The only difference lies in the values of the various operator product expansion coefficients.

In the Gaussian model, the various components of $\mathbf{S}$ are decoupled, so the 2-point correlation function has the form

$$\langle S_i(r_1)S_j(r_2)\rangle = \frac{\delta_{ij}}{r_{12}^{d-2}}. \tag{5.57}$$

As in the Ising case, there are only two important operator product expansion coefficients which enter the calculation to first order in ϵ. The first is the coefficient of S_iS_i in the product of S_iS_i and $(S_jS_j)(S_kS_k)$ (where the summation convention is assumed). There are two types of contribution, one where both S_i are connected either to both S_j or to both S_k, and another where one S_i is connected to an S_j and one to an S_k. The first type of contribution involves $\delta_{ij}\delta_{ij} = n$, and it is not difficult to see that the coefficient is 4. The second is independent of n. Its evaluation is rather more complicated, but in fact we have to do no more work, because we already know from the previous section that the result is 12 when $n = 1$. For arbitrary n, then, this operator product expansion coefficient must be $4(n+2)$. A similar method works for the coefficient of $(\mathbf{S}^2)^2$ in its operator product with itself. In this case, the term proportional to n is easily seen to be $8n$, and so, comparing with the case $n = 1$, we find the general result $8(n+8)$.

The important terms in the renormalization group equations for the $O(n)$ model are therefore

$$du/d\ell = \epsilon u - 8(n+8)u^2 \cdots \tag{5.58}$$

$$dt/d\ell = 2t - 8(n+2)ut + \cdots. \tag{5.59}$$

The fixed point is now at $u^* = \epsilon/8(n+8)$, and the thermal eigenvalue at the non-trivial fixed point is

$$y_t = \nu^{-1} = 2 - \frac{n+2}{n+8}\epsilon + O(\epsilon^2). \tag{5.60}$$

The fact that $u^* \to 0$ as $n \to \infty$ suggests that the above result may be exact in this limit. In fact, this turns out to be the case, as may be verified by an independent calculation as follows. For large n, $\mathbf{S}^2 = \sum_a S_a^2$ is the sum of a large number of random variables and, by the central limit theorem, it should have a normal distribution. In particular, we may replace the interaction term $(\mathbf{S}^2)^2$ in

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(5.56) by its cumulant $3\langle \mathbf{S}^2\rangle \mathbf{S}^2$. This reduces the hamiltonian to a Gaussian form, so that $\langle \mathbf{S}^2\rangle$ may be calculated self-consistently using the rules given in the Appendix:

$$\langle \mathbf{S}^2\rangle = n \int_{\mathrm{BZ}} \frac{1}{k^2 + t_0 + 3u\langle \mathbf{S}^2\rangle} \frac{d^d k}{(2\pi)^d}. \tag{5.61}$$

A subscript has been added to the reduced temperature to stress that it is defined relative to the *mean field* critical temperature. This equation in principle determines $\langle \mathbf{S}^2\rangle$ for each value of t_0. Once this is solved, the correlation length is then given by $\xi^{-2} = t + 3u\langle \mathbf{S}^2\rangle$. Therefore (5.61) may be rewritten as

$$\begin{aligned} \xi^{-2} &= t_0 + 3nu \int_{\mathrm{BZ}} \frac{1}{k^2 + \xi^{-2}} \frac{d^d k}{(2\pi)^d} \qquad (5.62) \\ &= t_0 + 3nu \int_{\mathrm{BZ}} \frac{1}{k^2} \frac{d^d k}{(2\pi)^d} - 3nu \int_{\mathrm{BZ}} \frac{\xi^{-2}}{k^2(k^2 + \xi^{-2})} \frac{d^d k}{(2\pi)^d}. \end{aligned}$$

Setting $\xi^{-2} = 0$ determines the actual critical temperature. Note that the fluctuations depress it below the mean field value, as expected. When $d > 4$, the last integral is finite when $\xi^{-2} = 0$, so that

$$\xi^{-2} = t_0 - t_{0c} + O(\xi^{-2}). \tag{5.63}$$

As a result $\xi^{-2} \propto t = t_0 - t_{0c}$, corresponding to the expected mean field value $\nu = \frac{1}{2}$. However, when $d < 4$ this is no longer permissible. Instead, we may argue that the ultraviolet cut off on large k is no longer relevant, and it may be removed. In that case, simple power counting shows that, close to the critical point,

$$\xi^{-2} = t - \text{const.}\, \xi^{2-d} + \cdots, \tag{5.64}$$

at least when $2 < d < 4$. In that case, the left hand side is asymptotically negligible, and

$$\nu = 1/(d-2), \tag{5.65}$$

in agreement with (5.60). Note that $\nu \to \infty$ as $d \to 2+$. This is related to the fact that $d = 2$ is the lower critical dimensionality for this model (in fact, for all $n > 2$), as we shall see in Section 6.1. Although it has no experimental relevance, the $n \to \infty$ limit provides a non-trivial testing ground for theoretical approaches, and forms the starting point for the $1/n$ expansion, which gives an approximation scheme complementary to the ϵ-expansion.

5.8 Cubic symmetry breaking

As a final introductory example of the power of the perturbative renormalization group, we consider the case of cubic symmetry breaking fields acting on the $O(n)$ model above. This is a famous example of the power of the renormalization group analysis in a situation where it predicts rather different behaviour from that suggested by mean field theory. Cubic fields are important for the discussion of, for example, XY or Heisenberg magnets where the magnetic ions are arranged in a crystal of cubic symmetry. In that case, crystalline fields will break the symmetry of the hamiltonian down from the full $O(n)$ rotational group acting on the spin vector $\mathbf{S}$, to its cubic subgroup. This allows further terms to appear in the continuous spin hamiltonian, consistent with the reduced symmetry. In principle, these could be very complicated. However, near four dimensions, we need consider only terms up to and including fourth order in S, since higher powers would be irrelevant. In addition to the terms already present in the $O(n)$ model, the only other allowed term has the form $v\sum_i S_i^4$, so that the relevant hamiltonian is (suppressing powers of a)

$$\mathcal{H} = \int \left[\tfrac{1}{2}(\nabla S)^2 + t\sum_i S_i^2 + u\sum_{ij} S_i^2 S_j^2 + v\sum_i S_i^4\right] d^d r. \quad (5.66)$$

To write the first order renormalization group equations for u and v, we need the appropriate operator product expansion coefficients. The operator product of $\sum_{ij} S_i^2 S_j^2$ with itself has already been computed in the previous section: it is $8(n+8)$. The term $\sum_i S_i^4$ behaves, at the Gaussian fixed point, like a sum of n decoupled Ising-like S^4 terms. Therefore its operator product expansion with itself can only give this term back, with the same coefficient of 72 we found in the Ising case. The only new operator product expansion which need be evaluated, therefore, is that of $\sum_{ij} S_i^2 S_j^2$ with $\sum_k S_k^4$. There are two types of contribution. The first is where either both S_i or both S_j are connected to an S_k: there are $2\cdot 2\cdot 4\cdot 3/2 = 24$ ways of doing this, and the residual operator has the form $\sum_{ijk}\delta_{ik}\delta_{ik}S_j^2 S_k^2$, so this contributes to the renormalization of u. In the second type, one S_i and one S_j each connect to an S_k: there are $2\cdot 2\cdot 4\cdot 3 = 48$ ways of doing this. The remaining factors have the form $\sum_{ijk}\delta_{ik}\delta_{jk}S_i S_j S_k^2$, so this contributes to the renormalization of v. The renormalization group

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equations for the couplings u and v are therefore

$$du/d\ell = \epsilon u - 8(n+8)u^2 - 48uv + \cdots \tag{5.67}$$

$$dv/d\ell = \epsilon v - 96uv - 72v^2 + \cdots, \tag{5.68}$$

where the neglected terms are higher order in u, v and ϵ. Apart from the Gaussian fixed point at $u = v = 0$, these equations have three non-trivial fixed points of $O(\epsilon)$:

- $(u, v) = (\epsilon/8(n+8), 0)$. This is the usual fixed point of the $O(n)$ model, in the absence of cubic fields;
- $(u, v) = (0, \epsilon/72)$ corresponds to n decoupled Ising models;
- $(u, v) = (\epsilon/24n, (n-4)\epsilon/72n)$. This is new, and is called the *cubic fixed point.*

In order to understand which fixed point controls the critical behaviour, we may perform a linear analysis of their relative stabilities. At the decoupled Ising fixed point, the eigenvalue corresponding to u is $\epsilon - 48v^* = \epsilon/3$, so this fixed point is always unstable. On the other hand, at the $O(n)$ fixed point, the eigenvalue corresponding to v is $\epsilon - 96u^* = (n-4)\epsilon/(n+8)$, which is unstable only when $n > 4$. The scaling variables at the cubic fixed point are non-trivial combinations of $u - u^*$ and $v - v^*$. The eigenvalues are therefore those of the matrix of derivatives

$$\begin{pmatrix} \epsilon - 16(n+8)u^* - 48v^* & -48u^* \\ -96v^* & \epsilon - 144v^* - 96u^* \end{pmatrix}, \tag{5.69}$$

which gives $(-\epsilon, (4-n)\epsilon/3n)$.

For $n < n_c = 4 - O(\epsilon)$, therefore, cubic symmetry-breaking fields are irrelevant, and lead only to additional corrections to scaling. For $n > n_c$, however, the result depends on the sign of v. For $v > 0$, the critical renormalization group trajectory flows to the cubic fixed point. This is the case illustrated in Figure 5.8. In order to compute the thermal eigenvalue at this new fixed point, we need to consider the renormalization group equation for t. However, both the required operator product expansion coefficients have been calculated already, in the Ising (5.37) and $O(n)$ (5.59) limits. It follows immediately that the renormalization group equation is

$$dt/d\ell = 2t - 8(n+2)ut - 24vt + \cdots, \tag{5.70}$$

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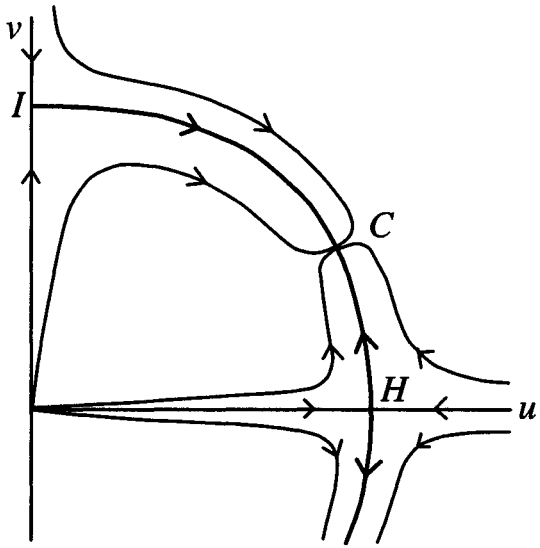

Figure 5.8. RG flows for cubic symmetry-breaking when $n > n_c$.

which gives the thermal eigenvalue as $y_t^{\text{cubic}} = 2 - 2(n-1)\epsilon/3n + O(\epsilon^2)$.

On the other hand, when $v < 0$ initially, the renormalization group trajectories ultimately flow into the region $u + v < 0$, where the renormalized hamiltonian no longer has a stable minimum. In this case, it may be argued that the transition is first order. Such a transition, which would be second order according to mean field theory, is called a *fluctuation-driven* first order transition.

Exercises

5.1 Consider the continuous spin tricritical Ising model, in which the dominant interaction is proportional to $:S^6:$. Show that the upper critical dimension is $d_c = 3$, and calculate the renormalization group eigenvalues to first order in $\epsilon = 3 - d$, using the same method as discussed in Section 5.5. [Note that there are two thermal eigenvalues to be calculated.] How does the specific heat behave in three dimensions?

5.2 Generalise the above calculation to the case of the $O(n)$ model, and verify that the $n \to \infty$ limit of your result agrees

with that found by an independent method which replaces the $(\mathbf{S}^2)^3$ term by something proportional to $\langle \mathbf{S}^2 \rangle^2 \mathbf{S}^2$.

5.3 At a fluid critical point (as opposed to that of an Ising ferromagnet), there is no reason to eliminate, on the basis of symmetry, an S^5 term in the corresponding continuous spin model. Calculate the renormalization group eigenvalue corresponding to such a term to first order in ϵ, along the lines of the calculation on p.101, and show that this suggests that such a term does not give rise to important corrections to scaling in three dimensions.

5.4 What are the logarithmic corrections to the dependence of the correlation length on the reduced temperature t for the $O(n)$ model in $d = 4$? Check this by comparison with an explicit calculation in the limit $n \to \infty$.

5.5 The operator which breaks the $O(3)$ symmetry of the Heisenberg model down to uniaxial symmetry is proportional to $S_z^2 - \frac{1}{2}(S_x^2 + S_y^2)$. Calculate its renormalization group eigenvalue, and hence the corresponding cross-over exponent, to first order in ϵ.

5.6 Consider the general case of the cross-over between two nearby fixed points, described to lowest order by the renormalization group equations $du/d\ell = \epsilon u - bu^2$ and, for the external magnetic field, $dh/d\ell = h(y + b'u)$. By solving these equations explicitly and using the transformation law for the correlation function derived on p.51, derive a form for the correlation function which explicitly exhibits the cross-over from a power-law characteristic of one fixed point at short distances, to the other at large distances.