8 Ultraviolet divergences: Effective field theory (EFT)

In Chapter 7, we have explained how, in quantum field theory (QFT), physical quantities can be calculated as power series in the various interactions. We consider now specifically *local* relativistic QFTs: the action that appears in the field integral is the space-time integral of a classical Lagrangian density, function of fields, and their derivatives, taken at the same point. We show that, as a consequence of locality (an essential property), infinities appear in *perturbative calculations*, due to short distance singularities, or after Fourier transformation, to integrals diverging at large momenta: one speaks of ultraviolet (UV) divergences. These divergences are peculiar to local QFT: in contrast to classical mechanics or non-relativistic quantum mechanics with a finite number of particles, a straightforward construction of a relativistic quantum theory of point-like objects with contact interactions is impossible.

Therefore, a local QFT, in a straightforward formulation, is an *incomplete theory*. It must, eventually (perhaps at the Planck's scale?), be embedded in some non-local theory, which renders the full theory finite, but where the non-local effects affect only short distance properties (an operation sometimes called UV completion). The impossibility to define a QFT without an explicit reference to an external short scale is an indication of a *non-decoupling* between short and long-distance physics.

From the more technical viewpoint of field integration, the Gaussian measure corresponding to a free field theory does not constrain the fields to be even continuous and, therefore, expectation values of products of fields taken at the same point as they appear in the perturbative expansion are not defined.

In Section 8.2, we investigate the form of divergences in a general local QFT, to all orders in perturbation theory. The analysis is based on *power counting* arguments. It leads to a classification of interactions as being super-renormalizable, renormalizable, and non-renormalizable. The relevance of this classification will appear in Chapter 9, where we introduce renormalization and renormalization group (RG).

To characterize the nature of divergences, one modifies the QFT at large momentum (the cut-off scale), or equivalently at short distance, in such a way that a finite perturbative expansion (based on Feynman diagrams) can be defined. The modifications must be such that when the momentum cut-off is sent to infinity, the original action is formally recovered. This procedure is called *regularization* (the dimensional regularization is of different nature, and has no direct physics interpretation, see Chapter 10). The regularization that renders the perturbative expansion finite is a substitute for the necessary short-distance structure, which, either is too complicated (as in statistical physics), or is unknown, as in particle physics. It makes it possible to isolate well-defined divergent parts of diagrams and deal with them with renormalization, as we explain in Chapter 9.

The physical interpretation of the divergences of QFT leads to the concept of *EFT* (see also Section 9.11): QFT is an EFT, local approximation, valid at large distance or low energy, of a more fundamental non-local finite theory.

Note that *one also calls EFT a QFT that is a low energy approximation of an embedding QFT with heavier masses*, which act as cut-offs [47].

Remark. We discuss deeply quantum and relativistic physics (or statistical physics) , and thus set everywhere $\hbar = c = 1$.

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8.1 Gaussian expectation values and divergences: The scalar field

The n-point correlation function for a scalar field ϕ is given by a field integral of the form,

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \mathcal{Z}^{-1}\int[\mathrm{d}\phi]\,\phi(x_1)\cdots\phi(x_n)\,\mathrm{e}^{-\mathcal{S}(\phi)}, \quad \text{with } \mathcal{Z}=\int[\mathrm{d}\phi]\,\mathrm{e}^{-\mathcal{S}(\phi)}, \tag{8.1}$$

where $\mathcal{Z}$ is the partition function, or vacuum amplitude.

One could have expected that, as in non-relativistic quantum mechanics, $\mathcal{S}(\phi)$ can be chosen as a *local* action, space integral of Euclidean *classical Lagrangian density*, function of $\phi(x)$ and its derivatives. Examples are translation invariant QFTs, in d space dimensions (including Euclidean time), with an $O(d)$ invariant action, of the form,

$$\mathcal{S}(\phi)=\int \mathrm{d}^d x\,\left[\tfrac{1}{2}\big(\nabla\phi(x)\big)^2+\tfrac{1}{2}m^2\phi^2(x)+V_{\mathrm{I}}\big(\phi(x)\big)\right], \tag{8.2}$$

($\nabla\equiv(\partial/\partial x_1,\ldots,\partial/\partial x_d)$) where m is the ϕ mass at leading order, and $V_{\mathrm{I}}(\phi)$ a polynomial.

Locality and unavoidable UV divergences. As we have explained in Section 7.2, the field integral can be calculated, *perturbatively*, by keeping a quadratic term, here a free field action of the form

$$\mathcal{S}_0(\phi)=\tfrac{1}{2}\int \mathrm{d}^d x\,\left[\big(\nabla\phi(x)\big)^2+m^2\phi^2(x)\right], \tag{8.3}$$

in the exponential and expanding the partition function in a power series of $V_{\mathrm{I}}(\phi)$, reducing the calculation to an infinite sum of Gaussian integrals (see Section 7.2.2).

The Gaussian expectation value of a monomial of degree n contributing to $V_{\mathrm{I}}(\phi)$,

$$\langle\phi^n(x)\rangle_0 \propto \int[\mathrm{d}\phi]\phi^n(x)\,\mathrm{e}^{-\mathcal{S}_0(\phi)},$$

can be evaluated, using Wick's theorem (7.18), in terms of the Gaussian two-point correlation function corresponding to the action (8.3),

$$\Delta(x)=\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^d p\,\mathrm{e}^{ipx}}{p^2+m^2}\,.$$

It thus involves contributions of the form

$$\left\langle\phi^2(x)\right\rangle_0=\Delta(x=0)=\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^d p}{p^2+m^2}, \tag{8.4}$$

which are infinite for $d\geq 2$. One call such divergences *UV divergences*, because they follow from the large momentum behaviour of perturbative contributions.

This divergence is related to the property that the typical fields that contribute to the Gaussian field integral are not continuous. Indeed, in d dimensions,

$$\tfrac{1}{2}\left\langle\big(\phi(x)-\phi(y)\big)^2\right\rangle_0=\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^d p\,\left(1-\mathrm{e}^{ip(x-y)}\right)}{p^2+m^2}\underset{|x-y|\to 0}{\propto}|x-y|^{2-d},$$

and thus, the fields are continuous only for $d<2$ (quantum mechanics). Expectation values of product of fields at coinciding points (local monomials) are always divergent.

Note that, in the case of relativistic derivative couplings, divergences of the form,

$$\left\langle\big(\nabla\phi(x)\big)^2\right\rangle_0=\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^d p\,p^2}{p^2+m^2},$$

may appear. In order to be defined, these quantities require the fields contributing to the field integral to be differentiable.

8.2 Divergences of Feynman diagrams: Power counting

We have shown that a straightforward perturbation expansion is plagued by UV divergences, revealing that *any local, interacting QFT is an incomplete theory*, because *short- and large- distance physics do not decouple.* However, before discussing the methods to deal with this essential problem (regularization, renormalization, and RG), we first characterize the general structure and nature of the UV divergences of all Feynman diagrams generated by polynomial local interactions [25].

8.2.1 UV dimension of fields and interaction vertices

Propagator: Large momentum behaviour. Quite generally, we assume that the propagator (after continuation to Euclidean time) of the field ϕ, in the Fourier representation,

$$\left\langle \tilde{\phi}(p)\tilde{\phi}(-p)\right\rangle_0 = \tilde{\Delta}(p),$$

behaves for large momenta as,

$$\left|\tilde{\Delta}(\lambda p)\right| \underset{\lambda\to+\infty}{\sim} C\lambda^{-\sigma}, \quad \text{with } \sigma\,,\ C>0\,. \tag{8.5}$$

For the action (8.2) $\sigma = 2$, but, for later purpose, we need to generalize this behaviour.

Then, from the viewpoint of the functional Gaussian measure, the space of fields contributing to the field integral can be inferred from the short-distance behaviour of the Gaussian two-point function. One finds

$$\tfrac{1}{2}\left\langle \left(\phi(x)-\phi(y)\right)^2\right\rangle_0 = \frac{1}{(2\pi)^d}\int \mathrm{d}^d p\left(1-\mathrm{e}^{ip(x-y)}\right)\tilde{\Delta}(p) \underset{|x-y|\to 0}{\propto} |x-y|^{\sigma-d},$$

for $\sigma > d$. Typical fields contributing to the field integral are continuous only for $\sigma > d$ and are k times continuously differentiable if

$$\sigma > d + 2k\,. \tag{8.6}$$

In particular, as we have shown, the expectation value of any derivative-free interaction $V_{\mathrm{I}}(\phi)$ is finite only if the fields contributing to the field integral are continuous.

UV field dimension. We now define the dimensions $[x]$ of space and, correspondingly, $[\nabla]$ of derivative or momentum, and the canonical, or UV dimension $[\phi]$ of the field ϕ, appropriated to the large momentum behaviour (equation (8.5)) by

$$[x]=-1\,,\quad [\nabla]=1\,,\quad [\phi]=\tfrac{1}{2}(d-\sigma). \tag{8.7}$$

Then, the leading order contribution from $V_{\mathrm{I}}(\phi)$ is finite only if $[\phi]$ is negative.

Interaction vertex UV dimensions. Quite generally, we assume that the perturbation V_{I} is a linear combination of local monomials of degree n in the field ϕ of the form,

$$\mathcal{V}^{\alpha}_{k,n}(\phi) = \int \mathrm{d}^d x\, V^{\alpha}_{k,n}\left(\phi, x\right), \tag{8.8}$$

where $V^{\alpha}_{k,n}$ involves k differentiations (the index α reflects the property that, at k, n fixed, one can find sometimes several monomials), for example,

$$\phi^3(x)(n=3,k=0),\ \phi^4(x)\ (n=4,k=0),\ \phi^6(x)\ (n=6,k=0),$$
$$\phi^2(x)\nabla^2\phi^2(x)\ (n=4,k=2)\ldots\,.$$

We call a monomial of the form $\mathcal{V}^{\alpha}_{k,n}(\phi)$ a vertex, because, for $n \geq 3$, it is represented by a vertex in Feynman diagrams.

It is natural then to define the dimension $[\mathcal{V}]$ of a vertex $\mathcal{V}(\phi)$ by

$$[\mathcal{V}] = -d + k + n[\phi]. \tag{8.9}$$

In terms of the Fourier components $\tilde{\phi}(p)$ of the fields $\phi(x)$, and taking into account translation invariance, we write the vertex $\mathcal{V}(\phi)$ (in symbolic notation) as

$$\mathcal{V}(\phi) \propto \int \prod_{i=1}^{n} \mathrm{d}^d p_i \, \delta^{(d)}(p_1 + p_2 + \cdots + p_n) \, p_{i_1} p_{i_2} \cdots p_{i_k} \tilde{\phi}(p_1) \cdots \tilde{\phi}(p_n),$$

where $[\tilde{\phi}] = [\phi] - d$. For simplicity, we have written all expressions for only one field ϕ, but the same divergence analysis applies to several scalar fields that have propagators with the same large momentum behaviour.

8.2.2 Vertex functions: Power counting, superficial degree of divergence,

We consider only 1PI diagrams (see Section 7.10) contributing to vertex functions. In the Fourier representation, each vertex contains a δ-function of momentum conservation (translation invariance). The number of independent integration momenta in a Feynman diagram, taking into account momentum conservation at vertices, thus equals the number of loops. This follows directly from one of the definitions of the number of loops L in a diagram given in Section 7.9. Finally, a vertex multiplies the numerator of a Feynman diagram by the product of k momenta.

Therefore, if all integration momenta in a diagram γ are scaled by a factor λ, for $\lambda \to \infty$, the diagram is scaled by a factor $\lambda^{\delta(\gamma)}$ with

$$\delta(\gamma) = dL - I\sigma + \sum_a v_a k_a \,, \tag{8.10}$$

in which v_a is the number of vertices of type a with k_a derivatives, and I the number of internal lines corresponding to propagators Δ joining the different vertices.

The calculation of $\delta(\gamma)$ is called *power counting*, and $\delta(\gamma)$ is called the *superficial degree of divergence* of the diagram γ. For $\delta(\gamma) \geq 0$, the Feynman diagram diverges. If $\delta(\gamma)$ is negative, the diagram is superficially convergent but, beyond one-loop, it may still have divergences coming from subdiagrams.

Example. In the example of the ϕ^3 field theory in $d = 6$ dimensions, since $\sigma = 2$ and $k = 0$, expression (8.10) yields $\delta(\gamma) = 6L - 2I$.

At one-loop $L = 1$, $\delta(\gamma) = 6 - 2I$ and, for $I = 1, 2, 3$, the diagrams are divergent in agreement with equation (8.32). For $I > 3$, all one-loop diagrams are convergent.

Fig. 8.1 exhibits a diagram in the same theory that is superficially convergent ($L = 2$, $I = 7$), but which, inside the dotted box, contains a divergent subdiagram.

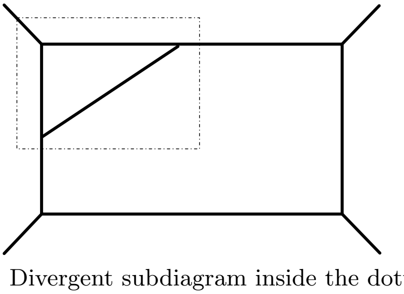

Fig. 8.1 Divergent subdiagram inside the dotted box

Other expressions. By using several topological relations on graphs, it is possible to express $\delta(\gamma)$ in different forms.

Combining equation (8.10) with the relation (7.96) written in the form,

$$L = I - \sum_a v_a + 1\,, \tag{8.11}$$

one can eliminate L and finds

$$\delta(\gamma) = d + 2I[\phi] + \sum_a (k_a - d)v_a\,. \tag{8.12}$$

We now consider a diagram γ contributing to a vertex function with E (for external line) fields ϕ. We denote by n^a the number of fields ϕ belonging to a vertex a, inside a diagram γ. They satisfy the relation

$$E + 2I = \sum_a n^a v_a\,. \tag{8.13}$$

The relation has a simple interpretation: each internal line connects two vertices while an external line is attached only to one vertex. Fig. 8.2 gives an example.

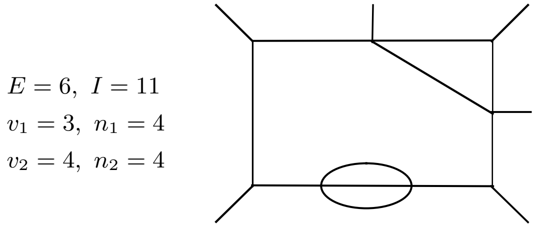

Fig. 8.2 Diagram illustrating equation (8.13)

Combining equation (8.13) with the relation (8.12), one can eliminate I in $\delta(\gamma)$, and obtain

$$\delta(\gamma) = d - E[\phi] + \sum_a v_a[\mathcal{V}_a], \tag{8.14}$$

where $[\mathcal{V}_a]$ is the dimension of the vertex a (equation (8.9)):

$$[\mathcal{V}_a] = -d + k_a + n_a[\phi]$$

and $[\phi]$ the dimension of ϕ (equation (8.7)). Equation (8.14) leads directly to a classification of scalar field theories, according to the degree of divergence of Feynman diagrams.

8.3 Classification of interactions in scalar quamtum field theories

We classify vertices from the point of view of power counting. We use the word *renormalizable*, a concept that will be justified in Chapter 9. For a scalar field theory, $\sigma = 2$ and, thus, $[\phi] = \frac{1}{2}(d-2)$. Then, the dimension of a vertex is (equation (8.9)),

$$[\mathcal{V}] = -d + k + \tfrac{1}{2}n(d-2), \tag{8.15}$$

where k is even in scalar QFTs.

8.3.1 Classification of vertices

(i) *Super-renormalizable vertices (or interactions).* If the dimension (8.9) of a vertex is negative, that is,

$$-d + k + n(d-2)/2 < 0\,, \tag{8.16}$$

the vertex is called *super-renormalizable* or, in the RG terminology, *relevant* (see Section 15.2). Adding such a vertex to a Feynman diagram decreases the superficial degree of divergence (8.14). The condition (8.16) then implies:

for $d = 2$, $k = 0$, and the interaction ϕ^n is super-renormalizable for all n;

for $d = 3$, the condition becomes $k + n/2 - 3 < 0$, and then $k = 0$ and $n < 6$;

for $d = 4$, the condition is $k + n - 4 < 0$, and then again $k = 0$ and $n < 4$;

for $d = 5$, $k = 0$ and $3n/2 < 5$, and only ϕ^3 is super-renormalizable. For $d \geq 6$, no interaction is super-renormalizable.

(ii) *Renormalizable vertices.* This is the RG marginal case where the dimension of the vertex vanishes. Adding such a vertex to a Feynman diagram does not change the superficial degree of divergence. The condition (8.16) then implies:

for $d = 2$, $k = 2$ and all interactions with two derivatives are renormalizable;

for $d = 3$, only the ϕ^6 interaction is renormalizable;

for $d = 4$, the ϕ^4 interaction is renormalizable;

for $d > 4$, only the ϕ^3 interaction is renormalizable in $d = 6$ dimensions.

(ii) *Non-renormalizable (or RG irrelevant) vertices.* The remaining vertices, when added to a diagram, increase the superficial degree of divergence.

8.3.2 Classification of field theories

We now classify field theories based on the form (8.14) of the superficial degree of divergence. The classification follows directly from the classification of vertices.

Non-renormalizable theories. If at least one vertex $\mathcal{V}$ has a positive dimension, $[\mathcal{V}] > 0$, then the degree of divergence of diagrams contributing to any vertex function can be rendered arbitrarily large by increasing the number v of vertices of this type. A field theory with such a vertex is not renormalizable, as will be discussed in Chapter 9.

Super-renormalizable theories. When all vertices have strictly negative dimensions, only a finite number of Feynman diagrams are superficially divergent. The corresponding field theory is called super-renormalizable.

Example. In the ϕ^4 field theory, in dimension $d = 3$,

$$\delta(\gamma) = 3 - \tfrac{1}{2}E - v\,.$$

The superficially divergent diagrams are listed in Fig. 8.3.

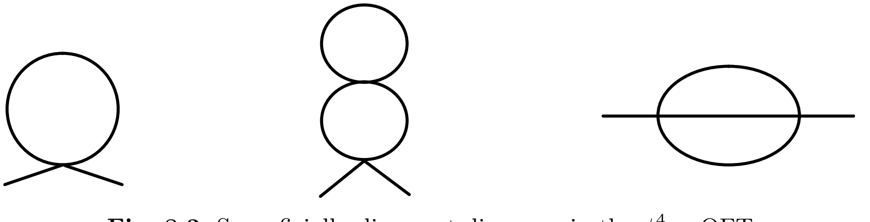

Fig. 8.3 Superficially divergent diagrams in the $\phi^4_{d=3}$ QFT

Renormalizable theories. Renormalizable QFTs are characterized by the property that at least one vertex has dimension 0, and no vertex has a positive dimension. Then, an infinite number of diagrams has a positive superficial degree of divergence, but the maximal degree of divergence at E fixed is bounded and is independent of the number of insertions of the vertices of dimension 0. In addition, if all dimensions of fields $[\phi]$ are strictly positive, only a finite number of vertex functions are superficially divergent.

If at least one field has dimension 0 (this happens with a scalar field in dimension 2), the situation is more complicated: the degree of divergence is bounded, but an infinite number of vertex functions are superficially divergent. Generically, this leads to field theories depending on an infinite number of parameters, except when some symmetry relates them (see Chapters 19 and 29).

Finally, no stable field theory with polynomial interaction exists above dimension 4. For particle physics, dimension 4 is also the physical dimension (at least for macroscopic dimensions). It is not known whether this is a mere coincidence or has a deeper meaning.

8.4 Momentum regularization

The perturbative expansion in all local, relativistic QFTs is plagued by UV divergences. These divergences signal a non-decoupling of short and large distance scales, a situation totally unusual in physics. Therefore, a QFT is an incomplete theory, which has eventually to be embedded within a finite non-local microscopic theory. However, in particle physics, such a theory is unknown. In macroscopic phase transitions, the microscopic structure is often known, but may be too complicated, and contain irrelevant features.

The momentum *regularization method* consists in *artificially* modifying the short distance, or large Euclidean momentum structure of the QFT to render the perturbative expansion finite. The modification is largely arbitrary and, in particle physics, not physical, since it violates unitarity. Momentum regularization works in the continuum, compared to lattice methods (see Section 8.7), and at fixed dimension, unlike dimensional regularization (which, moreover, has no physics interpretation, see Chapter 10). From the viewpoint of field integration, the regularized action selects a space of more regular fields (at least continuous to render theories with derivative-free interactions finite). Such modified theories make physical sense only if their large distance (or low energy-momentum) properties are, at least to a great extent, insensitive to the detailed form of the modification (a property also called *universality*). Renormalization theory and RG are the relevant tools to investigate this problem.

8.4.1 Effective field theory: Regularization

In order to limit the number of parameters, we consider only QFTs symmetric in the reflection $\phi \to -\phi$ (but we exhibit a different example in Section 8.5).

To give a meaning to perturbation theory, we modify the action, which we write now, quite generally, as

$$\mathcal{S}(\phi) = \mathcal{S}_{\mathrm{G}}(\phi) + \mathcal{V}_{\mathrm{I}}(\phi), \tag{8.17}$$

with

$$\mathcal{S}_{\mathrm{G}}(\phi) = \tfrac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, \phi(x) K(x-y)\phi(y), \quad \mathcal{V}_{\mathrm{I}}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2} v_0 \phi^2(x) + V_{\mathrm{I}}(\phi, x)\right], \tag{8.18}$$

where v_0 is a parameter, and K can be a positive local differential operator, polynomial in ∇^2 or even, if required, a non-local kernel (see Section A8.3), which cuts the momentum integration in Feynman diagrams, and $V_{\mathrm{I}}(\phi, x)$ a polynomial (for simplicity) in ϕ and its derivatives at point x, with degree $n \geq 4$.

For generic values of the parameter v_0, the physical mass (or the inverse correlation length equation defined by ($A7.18$)) is of the order of the momentum cut-off and, thus, non-physical. Therefore, the parameter v_0 has to be tuned in such a way that the physical mass becomes much smaller than the cut-off. In macroscopic phase transitions, this amounts to adjusting the temperature close to the critical temperature (see Chapter 14). This problem is specific to scalar QFTs.

The kernel K is an artificial substitute for the real short-distance structure that renders the theory finite. We assume that the kernel K has the Fourier representation,

$$K(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d p \; \mathrm{e}^{-ipx} \, \tilde{K}(p), \quad \tilde{K}(p) = m^2 + p^2 + O(p^4). \tag{8.19}$$

Moreover, we assume that $\tilde{K}(p)$ is an entire function of p^2, positive for $p^2 > 0$, which has zeros only on the p^2 negative semi-axis $p^2 \le 0$, and which increases fast enough for $|p| \to \infty$. Indeed, singularities in momentum variables generate, after Fourier transformation, contributions to the large-distance behaviour of the propagator and regularization should modify the QFT only at short distance. Examples are described in Sections A8.2 (based on Schwinger's proper-time representation [53], see Section A8.1) and A8.3.

We define the perturbative expansion by keeping $\mathcal{S}_{\mathrm{G}}$ in the exponential and expanding the remainder, reducing the calculation of the field integral (8.1) to an infinite sum of Gaussian expectation values. As a consequence of Wick's theorem, all perturbative contributions can be expressed in terms of the Gaussian two-point function.

Power counting. The Gaussian two-point function, or propagator, in the Fourier representation, is now

$$\tilde{\Delta}(p) = 1/\tilde{K}(p). \tag{8.20}$$

If we assume that $\tilde{K}(p) \propto |p|^\sigma$, we can use the power counting arguments of Section 8.2 and investigate under which condition the perturbative expansion is finite.

In the specific example of a QFT with derivative-free interactions, finiteness of $\langle \phi^2(0) \rangle$ (see also the integral (8.4)) then yields the necessary condition $\sigma > d$ and, therefore, $[\phi] < 0$. The condition is also sufficient since equation (8.12) implies,

$$\delta(\gamma) = d - d \sum_a v_a + 2I[\phi] < 0 \,,$$

because $[\phi] < 0$ and $v_a \ge 1$.

In the case of interactions with derivatives, using equation (8.10) and the topological relation (8.11), it is convenient to rewrite the superficial (or global) degree of divergence of a Feynman diagram γ (equation (8.14)) as

$$\delta(\gamma) = (d - \sigma)L + \sum_a v_a \, (k_a - \sigma) + \sigma \,,$$

where L is the number of loops and v_a the number of vertices of type a and k_a the number of derivatives at the vertex (Section 8.2).

To render all diagrams finite, it is thus necessary to choose $\sigma > d + \sup_a k_a$, since both L and v_a can increase indefinitely. Moreover, since $L \ge 1$, and one at least of the v_a is positive, one can then always satisfy $\delta(\gamma) < 0$ for all diagrams. One recovers the regularity condition (8.6) for the fields contributing to the field integral.

UV divergences from quantization. Momentum cut-offs deal with divergences caused by the infinite number of degrees of freedom of fields, a property that leads also to an RG. With some care they preserve, beyond space–time symmetries, all linear symmetries of the initial action. However, they do not remove, in general, divergences related to quantization problems and order of quantum operators in local products (*e.g.* see Chapter 19).

8.4.2 Terms quadratic in the fields with higher derivatives

We now exhibit a special class of regularizations that render perturbation theory finite when, in the action, all non-renormalizable interactions are omitted (see Section 8.3.1). It is characterized by the polynomial form of $\tilde{K}(p)$ in the expression (8.19).

We consider again the action (8.17),

$$\mathcal{S}(\phi) = \mathcal{S}_{\mathrm{G}}(\phi) + \mathcal{V}_{\mathrm{I}}(\phi). \tag{8.21}$$

To improve the convergence of Feynman diagrams at large momentum, we choose

$$\mathcal{S}_{\mathrm{G}}(\phi) = \tfrac{1}{2}\int \mathrm{d}^d x\, \phi(x)(-\nabla^2 + m^2)\prod_{i=1}^{s}(1 - \nabla^2/M_i^2)\phi(x), \tag{8.22}$$

with $M_i^2 \gg m^2$. In the limit where all M_i's become infinite, the unregularized propagator is recovered. This is the spirit of Pauli–Villars's regularization scheme [48].

The choice corresponds to the kernel (equation (8.55)),

$$\tilde{K}(p) = (p^2 + m^2)\prod_{i=1}^{s}(1 + p^2/M_i^2). \tag{8.23}$$

The degree s must be chosen large enough to render all Feynman diagrams convergent. In the example of derivative-free interactions, we have shown that this implies $2s+2 > d$.

The corresponding propagator (8.20) cannot be derived from a Hermitian Hamiltonian and, thus, is non-physical for particle physics. Indeed, the Hermiticity of the Hamiltonian leads to Källen–Lehmann's representation representation (6.60) for the two-point function. If the propagator is, as above, a rational fraction, it must be a sum of poles with positive residues and thus, cannot decrease faster than $1/p^2$.

However, the modification (8.23), unlike more general modifications (Section A8.3), can be implemented also in Minkowski space (*i.e*, in real time), because the regularized propagators decrease in all complex p^2 directions.

8.4.3 Regulator fields

Still in the framework of renormalizable theories, momentum regularization has another implementation, which, in the simplest cases equivalent (but not for gauge theories), based on the introduction of regulator fields.

To regularize the action (8.21), one introduces additional dynamical scalar fields ϕ_i, $i = 1, \ldots, s$, and considers the modified action

$$\begin{aligned}\mathcal{S}(\phi, \phi_i) = \tfrac{1}{2}\int \mathrm{d}^d x \Big[\phi(x)\left(-\nabla^2 + m^2\right)\phi(x) + \sum_{i=1}^{s}\frac{1}{z_i}\phi_i(x)\left(-\nabla^2 + M_i^2\right)\phi_i(x)\Big] \\ + \mathcal{V}\big(\phi + \textstyle\sum_i \phi_i\big).\end{aligned} \tag{8.24}$$

With the action (8.24), any internal ϕ propagator is replaced by the sum of the ϕ propagator and all the ϕ_i propagators $z_i/(p^2 + M_i^2)$. For an appropriate choice of the constants z_k, after integration over the regulator fields, the form (8.23) is recovered, as a simple calculation shows.

In the field integral

$$\int [\mathrm{d}\phi] \prod_{i=1}^{s} [\mathrm{d}\phi_i] \exp\left[-\mathcal{S}(\phi, \phi_i)\right], \tag{8.25}$$

we first change variables, setting $\phi(x) = \phi'(x) - \sum_{i=1}^{s} \phi_i(x)$. The action then becomes

$$\mathcal{S}(\phi', \phi_i) = \tfrac{1}{2} \int \mathrm{d}^d x \Big[\left(\phi'(x) - \textstyle\sum_i \phi_i(x)\right) \left(-\nabla^2 + m^2\right) \left(\phi'(x) - \textstyle\sum_i \phi_i(x)\right)$$
$$+ \sum_{i=1}^{s} \frac{1}{z_i} \phi_i(x) \left(-\nabla^2 + M_i^2\right) \phi_i(x) \Big] + \mathcal{V}(\phi').$$

The Gaussian integration over the fields ϕ_i can be performed explicitly. A straightforward calculation leads, as expected, to

$$\mathcal{S}(\phi) = \tfrac{1}{2} \int \mathrm{d}^d x \, \phi(x) K(-\nabla^2) \phi(x) + \mathcal{V}(\phi),$$

with

$$[K]^{-1} = \left(-\nabla^2 + m^2\right)^{-1} + \sum_i z_i \left(-\nabla^2 + M_i^2\right)^{-1}.$$

It is then possible to choose the coefficients z_i in such a way that

$$\left[\frac{1}{-\nabla^2 + m^2} + \sum_i \frac{z_i}{-\nabla^2 + M_i^2}\right]^{-1} = \left(-\nabla^2 + m^2\right) \prod_i \frac{\left(-\nabla^2 + M_i^2\right)}{\left(-m^2 + M_i^2\right)}. \tag{8.26}$$

This corresponds to the choice

$$\tilde{K}(p) = (p^2 + m^2) \prod_i \frac{\left(p^2 + M_i^2\right)}{\left(-m^2 + M_i^2\right)}.$$

Note that the first condition, $1 + \sum_i z_i = 0$ shows that some z_i must be negative, and thus correspond to non-physical fields, as Källen–Lehmann's representation implies (see Section 6.6). One then integrates over imaginary values of the corresponding fields.

8.5 Example: The $\phi^3_{d=6}$ field theory at one-loop order

In most of the chapter, we consider only actions even in the field. However, a simple illustration of the analysis of the nature of divergences of Feynman diagrams is provided by the renormalizable ϕ^3 scalar QFT in dimension 6. The ϕ^3 QFT is non-physical because the potential is not bounded from below. However, it has a well-defined perturbative expansion where this non-perturbative pathology is not visible. Moreover, it provides a simplified model of vacuum metastability (see Chapter 38). For g imaginary, it makes sense beyond perturbation theory in lower dimensions and describes the universal properties of the Yang–Lee edge singularity of the Ising model [49] (see Section 8.5.3).

In terms of a scalar field ϕ, the regularized action has the form

$$\mathcal{S}(\phi) = \int \mathrm{d}^6 x \left[\tfrac{1}{2}\phi(x) K(-\nabla^2)\phi(x) + \tfrac{1}{2} m^2 \phi^2(x) + \tfrac{1}{3!} g\, \phi^3(x)\right.$$
$$\left. + v_1 \phi(x) + \tfrac{1}{2} v_2 \phi^2(x)\right], \tag{8.27}$$

where m, v_1, v_2, and g, which is dimensionless, are constants.

At leading order $v_1 = v_2 = 0$, and $\phi = 0$ corresponds to the local minimum of the potential. Then, m is the physical mass in the tree approximation.

However, at higher orders, the ϕ^3 interaction shifts the mass and field expectation value to large non-physical values and the parameters v_1 and v_2 have to be tuned to maintain, for example, $\langle\phi\rangle = 0$ and the physical mass much smaller than the cut-off.

Tree approximation. In the tree approximation (Section 7.9.1), the generating functional of vertex (or 1PI) functions $\Gamma(\varphi)$ reduces to $\mathcal{S}(\varphi)$. At this order, in the infinite cut-off limit, the vertex or inverse two-point function is

$$\Gamma^{(2)}_{\text{tree}}(x,y) = \left(-\nabla^2 + m^2\right)\delta^{(6)}(x-y),$$

and, after Fourier transformation,

$$\widetilde{\Gamma}^{(2)}_{\text{tree}}(p) = p^2 + m^2. \tag{8.28}$$

More generally, the Fourier components of the n-point vertex functions are

$$\widetilde{\Gamma}^{(3)}_{\text{tree}}(p_1, p_2, -p_1 - p_2) = g\,, \quad \widetilde{\Gamma}^{(n)}_{\text{tree}}(p_1, \ldots, p_n) = 0\,, \quad \text{for } n > 3\,. \tag{8.29}$$

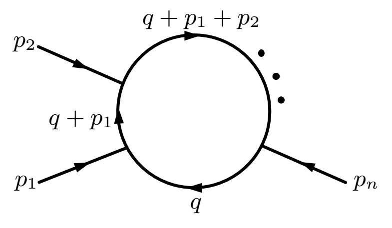

Fig. 8.4 The one-loop contribution to the n-point vertex function

8.5.1 Perturbation theory at one-loop order

The one-loop contribution $\Gamma_1(\varphi)$ to the functional $\Gamma(\varphi)$ has been derived in Section 7.9.2 (see also Section 7.7). From equation (7.93), and before regularization, one infers,

$$\Gamma_1(\varphi) = \tfrac{1}{2}\operatorname{tr}\ln\left[1 + g\left(-\nabla^2 + m^2\right)^{-1}\varphi\right]. \tag{8.30}$$

The expansion of $\Gamma_1(\varphi)$ in powers of φ generates the one-loop contributions to the vertex functions $\Gamma^{(n)}$. After Fourier transformation, one finds

$$\begin{aligned}\widetilde{\Gamma}^{(n)}_{1\,\text{loop}}(p_1, \ldots, p_n) = -\frac{(n-1)!}{2}(-g)^n \int \frac{\mathrm{d}^6 q}{(2\pi)^d}\frac{1}{q^2+m^2}\frac{1}{(q+p_1)^2+m^2}\cdots \\ \times \frac{1}{(q+p_1+\cdots+p_{n-1})^2+m^2},\end{aligned} \tag{8.31}$$

an expression represented by the Feynman diagram in Fig. 8.4.

For large-momentum q, the integrand in expression (8.31) behaves like $1/q^{2n}$, and the integral thus diverges for $2n \le 6$, that is, for the one, two and three-point functions. To determine the divergent contributions explicitly, we expand the integrand in a Taylor series in the external momenta. It is easy to verify, using dimensional analysis, that the coefficients of the terms of global degree k in the momenta are given by integrals which diverge only for $6 \ge k + 2n$.

Therefore, the divergent part of a one-loop contribution to the n-point function is a polynomial of degree $(6-2n)$.

The essential observation is that, since the divergences are polynomials in the external momenta, the divergent contribution $\Gamma_1^{\text{div.}}(\varphi)$ to the functional $\Gamma(\varphi)$ is *local*, that is, it takes the form of the space integral of a function of the field and its derivatives, like the action itself (see Section A8.2 for a direct calculation).

To isolate more precisely a divergent part, we use one of the regularization methods discussed in Section 8.4.2. Cutting the momentum integral according to a regularization (8.23) with $M_i = \Lambda$, $s = 3$, one finds

$$\begin{aligned}
\widetilde{\Gamma}^{(1)}_{1\,\text{loop}} &= \frac{g}{2^7\pi^3}\left[\tfrac{1}{4}\Lambda^4 - \tfrac{1}{4}m^2\Lambda^2 + m^4\ln(\Lambda/m) + O(1)\right], \\
\widetilde{\Gamma}^{(2)}_{1\,\text{loop}} &= -\frac{g^2}{2^7\pi^3}\left[\tfrac{1}{10}\Lambda^2 - \left(2m^2 + p^2/3\right)\ln(\Lambda/m) + O(1)\right], \\
\widetilde{\Gamma}^{(3)}_{1\,\text{loop}} &= \frac{g^3}{2^6\pi^3}\ln(\Lambda/m) + O(1).
\end{aligned} \tag{8.32}$$

The three divergent one-loop diagrams are displayed in Fig. 8.5. One verifies that the dimension $d = 6$ is special in the following sense: the vertex functions that diverge are all those that are already non-vanishing in the tree approximation (a term linear in ϕ can be added to the action (8.27) by translating ϕ by a constant). Moreover, the divergent terms and the tree approximation have the same momentum dependence.

By contrast, for dimensions $d \geq 8$, for example, the four-point function, which vanishes in the tree approximation, is also divergent.

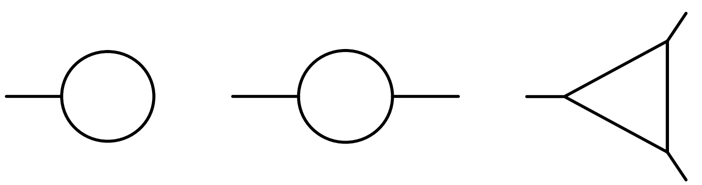

Fig. 8.5 Vertex functions: Divergent one-loop diagrams in the $\phi^3_{d=6}$ QFT

8.5.2 Analysis of the divergences at one-loop order

For the dimension $d = 6$, the divergent parts of the one-loop vertex functions have the structure of the initial action: $\Gamma_1^{\text{div.}}(\varphi)$, the divergent part at one-loop of $\Gamma(\varphi)$ has the structure

$$\begin{aligned}
\Gamma_1^{\text{div.}}(\varphi) = \int \mathrm{d}^6x \Big[&\tfrac{1}{2}g^2 a_0(\Lambda)\big(\nabla\varphi(x)\big)^2 + g a_1(\Lambda)\varphi(x) + \tfrac{1}{2}g^2 a_2(\Lambda)\varphi^2(x) \\
&+ \tfrac{1}{3!}g^3 a_3(\Lambda)\varphi^3(x)\Big].
\end{aligned} \tag{8.33}$$

The functions $a_i(\Lambda)$ follow from equations (8.32) and are, therefore, defined only up to additive finite parts.

For later purpose (the minimal subtraction scheme), it is convenient to give a canonical definition of the divergent part of a Feynman diagram as the sum of the divergent terms in the asymptotic expansion in a dimensionless parameter. Choosing here Λ/m, one finds

$$\begin{aligned}
2^7\pi^3 a_0(\Lambda) &= \tfrac{1}{3}\ln(\Lambda/m), \\
2^7\pi^3 a_1(\Lambda) &= \tfrac{1}{4}\Lambda^4 - \tfrac{1}{4}m^2\Lambda^2 + m^4\ln(\Lambda/m), \\
2^7\pi^3 a_2(\Lambda) &= -\tfrac{1}{10}\Lambda^2 + 2m^2\ln(\Lambda/m), \\
2^7\pi^3 a_3(\Lambda) &= 2\ln(\Lambda/m).
\end{aligned} \tag{8.34}$$

We first note that the field expectation value is now divergent and given by

$$\frac{\delta\Gamma}{\delta\varphi(x)} = m^2\varphi + ga_1 + O(g^3) = 0\,.$$

This leads to shift the field to cancel this expectation value, by setting

$$\varphi(x) = \varphi_{\mathrm{r}}(x) - ga_1(\Lambda)/m^2,$$

or, alternatively, to cancel the linear term by choosing $v_1 = -ga_1(\Lambda)$.

Then, in order for the field to have finite correlations, it is necessary to renormalize it to cancel the divergent coefficient a_0. One sets

$$\varphi(x) = \sqrt{Z}\varphi_{\mathrm{r}}(x), \quad \text{with} \quad Z = 1 - g^2 a_0(\Lambda).$$

The next problem is the physical mass: if $v_2 = 0$, the mass is of order Λ and, thus, non-physical. This provides an example of the fine-tuning problem. It is necessary to cancel this divergence by adjusting the coefficient of φ^2, for example, by setting

$$v_2 = -g^2 a_2(\Lambda).$$

After the affine transformation,

$$\varphi(x) = Z^{1/2}\varphi_{\mathrm{r}}(x) - ga_1(\Lambda),$$

and the tuning the mass parameter, only the logarithmic divergence of the contribution to the interaction remains. We define,

$$g_{\mathrm{r}} = g + g^3 a_3(\Lambda), \tag{8.35}$$

and then, the new vertex functional $\Gamma(\varphi_r)$, at one-loop order and for Λ large, is

$$\Gamma(\varphi_{\mathrm{r}}) = \int \mathrm{d}^6x \left[\tfrac{1}{2}\varphi_r(x)(-\nabla^2)\phi(x) + \tfrac{1}{2}m^2\varphi_{\mathrm{r}}^2(x) + \tfrac{1}{3!}g_{\mathrm{r}}\,\varphi_r^3(x)\right] + \Gamma_1(\varphi_{\mathrm{r}}) - \Gamma_1^{\mathrm{div.}}(\varphi_{\mathrm{r}})$$
$$+ O(\text{two loops}),$$

an expression that now has a limit, *at g_{r} fixed*, when the cut-off becomes infinite.

Fixing g_{r} when the cut-off varies (this is the viewpoint adopted in the renormalization theory) amounts to an *additional fine tuning*. By contrast, if the bare coupling is fixed, g_{r} becomes a logarithmically running coupling constant, the running being a consequence of the non-decoupling of scales. This running is described by RG equations (see Section 9.11).

Finally, note that a change in the definition of the divergent part changes $\Gamma_1^{\mathrm{div.}}(\varphi_{\mathrm{r}})$ by a finite local polynomial, and the conclusions are unchanged.

8.5.3 Universal properties of Yang–Lee's edge singularity

In classical statistical physics, the Yang–Lee edge singularity is a singularity of the partition function of the Ising model for a small imaginary magnetic field when the temperature approaches the critical temperature from above [49]. Its universal properties are described by an $i\phi^3$ field theory corresponding to the unregularized action

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\big(\nabla\phi(x)\big)^2 + \tfrac{1}{2}r\phi^2(x) + \tfrac{i}{3!}g\,\phi^3(x)\right], \tag{8.36}$$

where g and r are real constants. The action has a reflection symmetry $\mathcal{S}^*(\phi) = \mathcal{S}(-\phi)$. It leads to a well-defined field integral. The $d = 1$ example (quantum mechanics) has been thoroughly investigated. The perturbative expansion is related by analytic continuation to the one discussed in Section 8.5.

8.6 Operator insertions: Generating functionals, power counting

So far, we have analysed the divergences of field correlation functions. However, various physical problems involve correlation functions of non-linear local polynomials of the field, called hereafter composite fields or for historical reasons composite operators (this terminology comes from the operator formulation of QFT). Typical examples include

$$\mathcal{O}(\phi; x) \equiv \phi^2(x),\ \phi^4(x),\ [\nabla\phi(x)]^2 \ldots\ .$$

One insertion of an operator $\mathcal{O}(\phi)$ yields the correlation functions

$$\langle \mathcal{O}(\phi; y)\phi(x_1)\cdots\phi(x_n)\rangle\,.$$

Such correlation functions can, in principle, be obtained from the field correlation functions by letting various points coincide. However, since with the initial functional measure fields are not continuous (or if $\mathcal{O}$ involves derivatives, sufficiently differentiable), this limit is singular and leads, in momentum space, to additional integrations that generate new divergences. Therefore, it is necessary to analyse the vertex functions with operator insertions, from the point of view of power counting, separately.

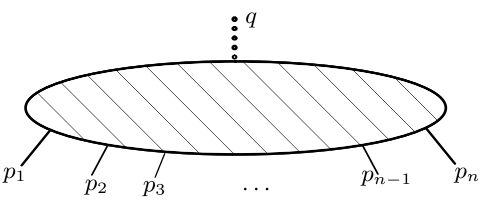

Fig. 8.6 The vertex function $\langle \mathcal{O}(q)\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_n)\rangle$ (Fourier representation)

Generating functional. We consider Euclidean action $\mathcal{S}(\phi)$, to which we add a source (which is a space-dependent coupling constant) for the local operator $\mathcal{O}(\phi; x)$:

$$\mathcal{S}_g(\phi) = \mathcal{S}(\phi) + \int \mathrm{d}^d x\, g(x)\mathcal{O}(\phi; x). \tag{8.37}$$

With the action $\mathcal{S}_g(\phi)$,

$$\mathcal{Z}(J, g) = \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}_g(\phi) + \int J(x)\phi(x)\mathrm{d}^d x\right] \tag{8.38}$$

becomes the generating functional for correlation functions with $\mathcal{O}$ insertions. The correlation functions with one operator $\mathcal{O}(\phi; x)$ insertion can be derived from the generating functional $\delta\mathcal{Z}/\delta g(x)$, taken at $g = 0$, by functional differentiation:

$$\langle \mathcal{O}(\phi; y)\phi(x_1)\cdots\phi(x_n)\rangle = -\left[\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\frac{\delta}{\delta g(y)}\mathcal{Z}(J, g)\right]\Bigg\|_{J=g=0}. \tag{8.39}$$

More generally, successive differentiations with respect to $g(x)$ yield generating functionals of correlation functions with multiple operator insertions.

After Legendre transformation of $\mathcal{W}(J,g) = \ln \mathcal{Z}(J,g)$ with respect to $J(x)$, one obtains the vertex functional $\Gamma(\varphi, g)$. The generating functional of vertex functions with one $\mathcal{O}(\phi; y)$ insertion, $\Gamma^{(n)}_{\mathcal{O}}$, is then

$$\left.\frac{\delta\Gamma(\varphi,g)}{\delta g(y)}\right|_{g=0} = \sum_{n=1}^{\infty}\frac{1}{n!}\int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_n\, \varphi(x_1)\cdots\varphi(x_n)\Gamma^{(n)}_{\mathcal{O}}(y; x_1,\ldots,x_n).$$

The corresponding Feynman diagrams have the structure displayed in Fig. 8.6. After Fourier transformation, they are just ordinary diagrams with one additional vertex $\mathcal{O}(\phi)$, except that an additional momentum enters the diagram at the vertex so that total momentum there is no longer conserved.

Power counting. When all integration momenta are scaled by a factor λ, in the limit $\lambda \to \infty$, all external momenta become negligible. Therefore, asymptotically, momentum is conserved even at the vertex corresponding to the operator insertion: the power counting of vertex functions with one operator insertion is the same as with one vertex insertion. We assign to an operator $\mathcal{O}(\phi)$ the dimension

$$[\mathcal{O}] = k + \sum n[\phi], \tag{8.40}$$

in which k is the number of derivatives in the operator, and n the number of fields. This definition differs from the definition of the dimension of the corresponding vertex (equation (8.9)) by d.

The expression (8.14) of the superficial degree of divergence δ_γ of an 1PI diagram γ is then modified in the case of the insertion of the product of operators $\mathcal{O}_1(x_1)\cdots\mathcal{O}_r(x_r)$, and becomes

$$\delta_\gamma(\mathcal{O}_1\cdots\mathcal{O}_r) = d - E[\phi] + \sum_a v_a[\mathcal{V}_a] + [\mathcal{O}_1] + \cdots + [\mathcal{O}_r] - rd\,. \tag{8.41}$$

For example, for $d = 4$, one insertion of $\phi^m(x)$ in the ϕ^4 QFT with $\sigma = 2$ yields

$$[\phi^m] = m \ \Rightarrow\ \delta_\gamma = 4 - E + m - 4\,. \tag{8.42}$$

In the same theory, the n-point vertex function with $l\,\phi^2$ insertions,

$$\Gamma^{(n,l)}(x_1,\ldots,x_n; y_1,\ldots,y_l) = \left\langle \phi(x_1)\cdots\phi(x_n)\phi^2(y_1)\cdots\phi^2(y_l)\right\rangle_{1\mathrm{PI}},$$

has the degree of divergence

$$\delta = 4 - n - 2l\,. \tag{8.43}$$

The new divergent correlation functions are displayed in Fig. 8.7.

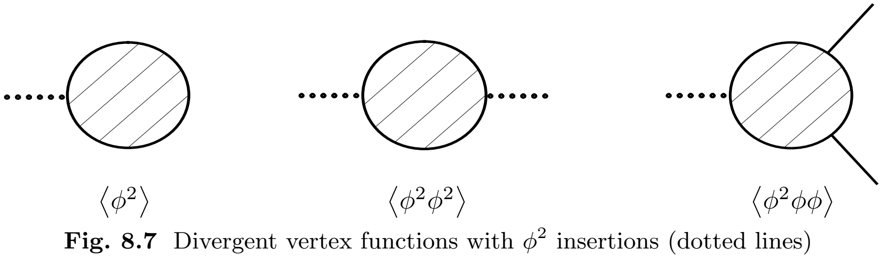

Fig. 8.7 Divergent vertex functions with ϕ^2 insertions (dotted lines)

Change of field parametrization. We show in Section 6.5.4 that the S-matrix of particle physics is formally invariant under local field transformations of the form

$$\phi(x) \mapsto \phi(x) + u_1\phi^3(x) + u_2\phi^5(x)\cdots \tag{8.44}$$

Similarly, in the theory of critical phenomena, a change of this nature in the parametrization of the order parameter does not affect universal properties. One may then wonder about the meaning of correlation functions of a field in a specific parametrization. However, the coefficients u_n have a dimension $u_n \propto \Lambda^{-2n}$, and are thus very small. Therefore, the effect of a change of parametrization, like the addition of non-renormalizable interactions, simply induces a change in the parametrization of the renormalizable action (see Section 11.1).

8.7 Lattice regularization. Classical statistical physics

We shall meet examples where Pauli–Villars's regularization does not work: field theories where quantization problems occur due products of non-commuting operators. This includes actions that have a definite geometric character like models on homogeneous spaces (for example, the non-linear σ-model) or gauge theories. As we have shown in Chapter 3, these problems already appear in path integrals of simple quantum mechanics. In particular, the forms of the propagator and of the interaction terms may not be independent. In this case, when the propagator is regularized, new more singular interactions have to be added to the action to preserve the symmetry and, as a few examples will illustrate, a class of one-loop diagrams cannot be regularized. Other regularization methods are needed. In many cases, lattice regularization of the Euclidean field theory (imaginary time) can be used [51, 52]. The advantages are the following:

(i) Lattice regularization indeed corresponds to a specific choice of quantization.

(ii) It is the only established regularization that always has a meaning beyond perturbation theory. Moreover, the regularized theory has the form of a statistical lattice model, interesting in its own right, because it can be studied by various standard methods of statistical physics.

(iii) It preserves most global or local symmetries, with the exception of the space $O(d)$ symmetry which is replaced by a hypercubic symmetry, but this is not a major issue, as we argue in Chapters 14.4 and 15.

One obvious drawback is that it leads to very cumbersome perturbative calculations.

Scalar field theories. As a regularization, we may define the field integral (6.44) as the formal limit of an integral in which both time and space are discretized. We introduce a d-dimensional hypercubic lattice, with lattice spacing a, and use as dynamical variables the values of the field $\phi(x)$ at all lattice sites. To regularize the scalar field theory with field and action (8.2), one replaces the continuum by a discrete space lattice, which we choose here, for simplicity, to be hypercubic.

Derivatives $\partial_\mu\phi$ are replaced by finite differences, for example,

$$\partial_\mu\phi(x) \mapsto \nabla^{\text{lat}}_\mu\phi = \frac{1}{a}\left[\phi(x+an_\mu) - \phi(x)\right],$$

where x is a lattice site and n_μ the unit vector in the μ direction.

The regularized lattice action (8.2) then takes the form

$$\mathcal{S}_{\text{reg.}}(\phi) = a^d \sum_{x\in(a\mathbb{Z})^d} \left[\frac{1}{2}\sum_{\mu=1}^{d} \left[\nabla^{\text{lat}}_\mu\phi(x)\right]^2 + V\big(\phi(x)\big)\right]. \tag{8.45}$$

Correlation functions of lattice variables have also, as a formal limit, the continuum ϕ-field correlation functions.

The regularized theory can be considered as a statistical lattice model and the discretized action as the corresponding configuration energy. For $d \geq 2$, such a model does not generate, in general, a continuum limit. However, if the model has a continuous phase transition, and if one parameter, which plays the role of the temperature, is chosen asymptotically close to the transition value (which corresponds to the critical temperature), then a continuum limit can be defined.

The study of the continuum limit leads to establish relations between renormalization theory, as explained in the present chapter and Chapter 9, and the statistical theory of continuous phase transitions, which is discussed in Chapters 14–18, devoted to critical phenomena.

Fourier representation. The Fourier transform $\tilde{\phi}(p)$ of the field is a periodic function of cyclic momentum variables p_μ:

$$\tilde{\phi}(p) = \left(\frac{a}{2\pi}\right)^d \sum_{x\in(a\mathbb{Z})^d} \phi(x)\,\mathrm{e}^{-ip\cdot x} \Leftrightarrow \phi(x) = \int \mathrm{d}^d p\,\tilde{\phi}(p)\,\mathrm{e}^{ip\cdot x}, \tag{8.46}$$

which can, therefore, be restricted to a Brillouin zone, for example,

$$-\pi/a \leq p_\mu < \pi/a\,, \quad \text{for } \mu = 1, \ldots, d\,.$$

The corresponding propagator $\tilde{\Delta}_\mathrm{B}(p)$ is then given by

$$\tilde{\Delta}_\mathrm{B}^{-1}(p) = m^2 + \frac{2}{a^2}\sum_{\mu=1}^{d}\bigl(1-\cos(ap_\mu)\bigr). \tag{8.47}$$

In the momentum representation, Feynman diagrams become periodic functions of the momentum components, with period $2\pi/a$. In the small lattice spacing limit, the continuum propagator is recovered since

$$\tilde{\Delta}_\mathrm{B}^{-1}(p) = m^2 + p^2 - \frac{1}{12}\sum_{\mu} a^2 p_\mu^4 + O\left(a^4 p_\mu^6\right). \tag{8.48}$$

We note that hypercubic symmetry implies $O(d)$ symmetry at order p^2 and, thus, at large distance.

8.8 Effective QFT. The fine-tuning problem

Since in a QFT defined by a field integral involving a local classical action, divergences are unavoidable, we are forced to completely reconsider the very notion of local QFT.

First, we have to assume that a QFT is an incomplete theory, necessarily embedded in another finite non-local theory, but whose non-locality is confined to very short distances (the assumption of short range interactions for macroscopic phase transitions).

The problem of small masses, or large correlation lengths. For reasons that must be understood, the initial non-local theory generates particles with very low mass compared to the inverse microscopic scale. For example, the microscopic theory could imply gauge invariance and some form of chiral symmetry, which would explain low mass vector particles and fermions.

However, the problem of scalar particles is more complex. Goldstone bosons, which result from spontaneous symmetry breaking, are massless, but no such fundamental scalar particle has been observed. Supersymmetry (see Chapter 27) has been proposed, because it relates bosons to fermions but, so far, no sign of supersymmetry has been found in experiments (by 2020). Therefore, a problem of fine tuning remains, which involves adjusting precisely one parameter of the QFT to generate a small scalar mass. For macroscopic phase transitions, the divergence of the correlation length (equivalent to an inverse mass) is obtained by tuning the temperature at the critical temperature.

At the scale of the physical masses (the *physical scale*), the non-locality is only visible through the presence of UV divergences, which indicate a *non-decoupling of scales*. Therefore, at the physical scale, physics can be derived from *regularized local EFT*, also called Landau–Ginzburg–Wilson theory in the context of critical phenomena (see Section 15.1). We discuss the example of scalar field theories, but the concept extends to QFTs with fermions and gauge fields, relevant for particle physics.

The EFT is an approximation to a real microscopic theory, suitable to describe only large-distance or low-energy–momentum physics. It has the form of a QFT but where, in equation (8.19), $\tilde{K}(p)$ increases faster that any power for $|p| \to \infty$, and in expression (8.17), $\mathcal{V}_{\mathrm{I}}(\phi)$ is a *linear combination of all local monomials* of the form (8.8), only restricted by symmetries (for simplicity, we consider only QFTs symmetrical in $\phi \mapsto -\phi$).

8.8.1 Effective action and perturbative assumption

We thus assume that the effective action has the general form (we include the contributions of the regularization terms),

$$\mathcal{S}(\phi) = \sum_{n\geq 2, k\geq 0, \alpha} \frac{1}{n!} a^{k-d} g^{\alpha}_{k,n} \mathcal{V}^{\alpha}_{k,n}(\phi), \tag{8.49}$$

where a is the microscopic scale (the lattice spacing for lattice models), and the powers of a are such that ϕ, $\mathcal{S}(\phi)$ and all coefficients $g^{\alpha}_{k,n}$ are dimensionless. The field ϕ is normalized such that $g_{2,2} = 1$. Generically, the coefficients are of order 1, and this, to some extent, characterizes the microscopic scale.

The perturbative assumption. The concept of EFT is based on the assumption of the relevance of some form of perturbation theory, an assumption for which there is empirical evidence. In particular, this means that the deviations from a free massless theory (the RG Gaussian fixed point, see Section 15.2), with the action

$$\mathcal{S}^*(\phi) = \tfrac{1}{2} a^{2-d} \int \mathrm{d}^d x \left(\nabla_x \phi(x)\right)^2, \tag{8.50}$$

are small (massless because masses are small with respect to the inverse microscopic length $1/a$), in some qualitative sense, which soon will become clearer. Since $\mathcal{S}^*(\phi)$ is the leading term, we change the field normalization, setting

$$\phi(x) = a^{(d-2)/2} \phi'(x),$$

giving the field a momentum dimension $(d-2)/2$. Then, the Gaussian action becomes

$$\mathcal{S}^*(\phi) = \tfrac{1}{2} \int \mathrm{d}^d x \left(\nabla_x \phi(x)\right)^2, \tag{8.51}$$

and the coefficient of $\mathcal{V}^{\alpha}_{k,n}(\phi)$ is changed to $a^{k-d+n(d-2)/2} g^{\alpha}_{kn}/n!$.

In the theory of continuous phase transitions, this general scheme can be justified, to a large extent. In particle physics, it is an educated guess justified by its consequences.

8.8.2 Gaussian renormalization, dimensional analysis

To describe only large-distance physics, it is more convenient to take the physical scale as a reference, instead of the microscopic scale. We thus rescale distances $x \mapsto x'$, with

$$x = \lambda x', \quad \phi(x) = \lambda^{(2-d)/2}\phi'(x), \quad \text{with } \lambda \gg 1\,, \tag{8.52}$$

where the ϕ renormalization is such that $\mathcal{S}^*$ remains invariant. Therefore, this transformation can be called a *Gaussian renormalization*. It has the form of an RG transformation, which has the Gaussian action $\mathcal{S}^*(\phi)$ as a fixed point (see Section 15.2).

Gaussian scaling of interactions and cut-off. The coefficient of $\mathcal{V}^\alpha_{k,n}(\phi)$ becomes

$$\lambda^{d-k-n(d-2)/2} a^{k-d+n(d-2)/2} g^\alpha_{k,n}/n!\,. \tag{8.53}$$

In the physical scale, $\Lambda = \lambda/a$ is a large momentum, characteristic of the microscopic scale. Instead of studying physics at large distances, or small momenta, one now studies physics for Λ large.

The scale parameter Λ is also the *cut-off* scale (in the QFT terminology), the scale at which the momentum integrals, in the Fourier representation of the perturbative expansion, must be cut, because the local expansion breaks down below the scale $1/\Lambda$ at which non-localities, which render the initial theory finite, become important.

The coefficient of the monomial $\mathcal{V}^\alpha_{k,n}$ (equation (8.8)) is transformed into

$$\Lambda^{d-n(d-2)/2-k} g^\alpha_{k,n}/n!\,. \tag{8.54}$$

8.8.3 The quadratic action and the fine-tuning problem

We collect all terms of the action quadratic in ϕ. The sum can be rewritten as

$$\mathcal{S}_2(\phi) = \tfrac{1}{2}\int \mathrm{d}^d x\, \phi(x)\, \mathbf{K}\, \phi(x), \tag{8.55}$$

where $\mathbf{K}$ is an operator, which has a derivative expansion and, therefore, in the momentum basis is diagonal. Its Fourier representation has a Fourier expansion of the form ($g_{0,2} > 0$ is assumed),

$$\mathbf{K} \mapsto \ \tilde{K}(p) = g_{0,2}\Lambda^2 + p^2 + g_{4,2}p^4/\Lambda^2 - g_{6,2}p^6/\Lambda^4 + \cdots\,. \tag{8.56}$$

For Λ large, $\tilde{K}(p)$ is dominated by the first term, $g_{0,2}\Lambda^2$. At leading order, in the absence of interactions, the physical mass m is of order $\sqrt{g_{0,2}}\Lambda$ and, for $g_{0,2} = O(1)$, non-physical. This is a first example of the *fine-tuning problem.* Indeed, a mass much smaller than the cut-off requires,

$$m = \sqrt{g_{0,2}}\Lambda \ll \Lambda\,, \text{ or } g_{0,2} = m^2/\Lambda^2 \ll 1\,.$$

If this condition is satisfied, for $|p| = O(m)$ (the physical domain), all terms of degree p^2 and higher in the expansion vanish for $\Lambda \to \infty$.

In the presence of interactions, the fine-tuning issue remains, but $g_{0,2}$ must be tuned to be close to a different value (in renormalization theory, this is related to mass renormalization, see Section 9.2). In the same way, in macroscopic phase transitions, the correlation length is large with respect to the microscopic scale only for temperatures close to the critical temperature (see Chapter 14). This implies a tuning of the temperature.

8.9 The emergence of renormalizable field theories

The cut-off behaviour (8.54) of the monomial $\mathcal{V}^{\alpha}_{k,n}$ is directly opposite to its momentum or mass (infrared) dimension,

$$[\mathcal{V}^{\alpha}_{k,n}] = -d + n(d-2)/2 + k\,. \tag{8.57}$$

The study of the relative strength of the different terms in the action, at the physical scale, is thus reduced again to dimensional analysis or *power counting*. The analysis can also be formulated in terms of the *stability with respect to local perturbations of the free massless theory* [51], or Gaussian fixed point, in the RG terminology. Terms with a positive power of Λ, like $\Lambda^2 \int \mathrm{d}^d x\, \phi^2(x)$ (negative dimension), which contribute to $\mathcal{S}_2$ (equation (8.55) and have already been discussed, are the most important ones; they are called *relevant*. Dimensionless terms are called *marginal*, and terms that vanish for large Λ (positive dimension) are called *irrelevant*. Since the analysis depends on space dimension, we deal with different dimensions separately.

Dimension $d = 4$. Only the $\int \mathrm{d}^d x\, \phi^2(x)$ term has a negative dimension and this leads to the issue of fine tuning. Then, the ϕ^4 interaction is dimensionless (the $\int \mathrm{d}^d x (\nabla\phi)^2$ term is dimensionless by construction) and, thus, survives at a large distance. In the terminology of renormalization theory, it is called *renormalizable* (see Chapter 9), and in the RG terminology *marginal*. All other contributions have positive dimensions and thus vanish for $\Lambda \to \infty$. Therefore, one expects to be able to describe, at leading order, the large-distance, or equivalently small-mass and small-momentum physics with the *renormalizable* action, which we parametrize now as

$$\mathcal{S}(\phi) = \int \mathrm{d}^4 x \left[\tfrac{1}{2}\big(\nabla\phi(x)\big)^2 + \tfrac{1}{2} r\phi^2(x) + \frac{1}{4!} g\phi^4(x)\right], \tag{8.58}$$

where r and $g \geq 0$ are two constants. However, as we have already discussed in Section 8.1, one has to retain at least the few first terms of the expansion of $\tilde{K}(p)$ in powers of p^2 to render perturbation theory finite and thus to satisfy the condition (8.6), $\sigma > 4$.

In this parametrization, the physical mass vanishes for a special value r_c of the parameter r and the condition that the mass should be much smaller than Λ is equivalent to the condition $|r - r_c| \ll \Lambda^2$, which involves a fine tuning of r.

The analysis shows that the *condition of renormalizability*, discovered empirically in particle physics, *is not a law of nature*, but is simply the consequence of a few general assumptions that we have described. Moreover, these assumptions are known to be satisfied for a large class of macroscopic critical phenomena (see Chapters 14 and 15).

Dimension $d = 3$. For $d = 3$, the field has dimension $\frac{1}{2}$. The leading interaction is ϕ^4, which has dimension -1 and thus is proportional to Λ. The interaction ϕ^6 is dimensionless. All other contributions have positive dimensions and vanish for $\Lambda \to \infty$.

If only the ϕ^2 coefficient is tuned, in order to generate a small mass, generically, the interaction diverges with the cut-off Λ. Retaining the leading interaction, one finds the action

$$\mathcal{S}(\phi) = \int \mathrm{d}^3 x \left[\tfrac{1}{2}\big(\nabla\phi(x)\big)^2 + \tfrac{1}{2} r\phi^2(x) + \frac{1}{4!} g\Lambda\phi^4(x)\right],$$

where g is dimensionless (and a cut-off is implied). This strong coupling situation is typical of critical phenomena when only the temperature can be adjusted.

By contrast, by tuning the ϕ^4 coupling such that $g = O(m/\Lambda)$, where m is the physical mass, and, in addition, cancelling the ϕ^6 interaction, from the viewpoint of renormalization theory, one obtains a *super-renormalizable* QFT. Then, after tuning of r and at $g\Lambda$ fixed, the resulting theory is finite in the infinite cut-off ($\Lambda \to \infty$) limit.

Otherwise, by tuning only the coefficient of ϕ^4 (in addition to the coefficient of ϕ^2) to a critical value, one obtains a theory in which the interaction is dominated by the marginal ϕ^6 term and the field theory is renormalizable. This field theory describes a *tricritical behaviour* in critical phenomena.

Dimension $d = 2$. This dimension is peculiar because the dimension of the field vanishes. All derivative-free terms are proportional to Λ^2 and terms with two derivatives are dimensionless, corresponding to renormalizable interactions. This case requires a specific discussion.

8.9.1 Non-renormalizable interactions: The example of four dimensions

Non-renormalizable interactions, like the ϕ^6 interaction, which cannot be dealt with by renormalization theory, appear in the EFT framework as quite innocuous, because they are multiplied by negative powers of Λ. They lead to very weak interactions. However, when added as a perturbation in the ϕ^4 QFT, they generate increasing divergences (this is the reason why they have been excluded in the construction of renormalizable QFTs). These divergences are partially cancelled by the powers of Λ (8.54). Moreover, a study of the *renormalization of local monomials of the fields*, also called composite operators [51], shows that the contributions that do not vanish for $\Lambda \to \infty$, or infinite cut-off, simply renormalize (like counter-terms, Section 9.2.1) the parameters of the renormalizable part of the action. For example, the operators of dimension 6 generate divergent contributions of dimension 2 and 4 that renormalize the coefficients of terms already present in the action, and a contribution decreasing as $1/\Lambda^2$, up to powers of logarithms, with the cut-off (see Section 11.1.4).

Field parametrization. It is quite possible that, in the microscopic theory, the field is just a parametrization of some manifold (see also Chapters 19, 21–29). Therefore, it could be replaced by any other field ϕ' related to ϕ by

$$\phi'(x) = \phi(x) + c_1\phi^2(x) + c_2\phi^3(x) + \cdots\,, \tag{8.59}$$

where the c_n are constants of order 1 (note that this *strictly local* change of variables is defined only with lattice regularization). However, after the rescaling (8.52), at the physical scale c_n transforms into $c_n\Lambda^{-n(d-2)/2}$ and leads to changes only in the parametrization of the renormalizable and non-renormalizable parts of the action. Therefore, the renormalized perturbation theory is not affected.

Beyond perturbation theory. The analysis of the hierarchy of interactions is based here on power counting, or Gaussian renormalization, equivalent to mean-field theory (see Chapter 14). Its validity, beyond leading order perturbation theory, relies on the assumption that a global RG analysis, which takes into account the scale non-decoupling, would not qualitatively modify the hierarchy of interactions. Such a global study requires solving, to some extent, *functional RG equations* [61–63]. However, there is empirical evidence that, for four-dimensional QFT as relevant for particle physics, as well as for a large class of two- and three-dimensional QFTs, relevant to the theory of macroscopic phase transitions, the general hierarchy of interactions is not modified, the irrelevant or non-renormalizable interactions remain irrelevant, and a much simpler perturbative RG is sufficient. In four dimensions, the Gaussian scaling is then only modified by logarithms. Still, the triviality issue emerges (see Sections 9.11 and 9.12).

A8 Technical details

A8.1 Schwinger's proper-time representation

We first establish a formal representation of the one-loop contribution to the generating functional of vertex functions $\Gamma(\varphi)$. In Section 7.9, we have shown that $\Gamma_{1\,\text{loop}}(\varphi)$ is given by

$$\Gamma_{1\,\text{loop}}(\varphi) = \frac{1}{2}\operatorname{tr}\left[\ln\frac{\delta^2\mathcal{S}}{\delta\varphi(x_1)\delta\varphi(x_2)} - \ln\left.\frac{\delta^2\mathcal{S}}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\varphi=0}\right]. \tag{A8.1}$$

For example, if $\mathcal{S}(\phi)$ is

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\big(\nabla\phi(x)\big)^2 + \tfrac{1}{2}m^2\phi^2(x) + U\big(\phi(x)\big)\right], \tag{A8.2}$$

where $U(\phi)$ is a polynomial, the second functional derivative $\delta^2\mathcal{S}/\delta\varphi(x_1)\delta\varphi(x_2)$ takes the form of a quantum Hamiltonian:

$$M(x_1,x_2) \equiv \frac{\delta^2\mathcal{S}}{\delta\varphi(x_1)\delta\varphi(x_2)} = \left[-\nabla^2 + m^2 + U''\big(\varphi(x_1)\big)\right]\delta^{(d)}(x_1-x_2). \tag{A8.3}$$

We also define

$$M_0(x_1,x_2) \equiv \left.\frac{\delta^2\mathcal{S}}{\delta\varphi\delta\varphi}\right|_{\varphi=0} = \left(-\nabla^2 + m^2\right)\delta^{(d)}(x_1-x_2). \tag{A8.4}$$

The general identity,

$$\operatorname{tr}\left(\ln M - \ln M_0\right) = -\int_0^\infty \frac{\mathrm{d}t}{t}\operatorname{tr}\left(\mathrm{e}^{-tM} - \mathrm{e}^{-tM_0}\right), \tag{A8.5}$$

then leads to a compact representation of the one-loop functional $\Gamma_{1\,\text{loop}}(\varphi)$ as an integral over Schwinger's proper time:

$$\Gamma_{1\,\text{loop}}(\varphi) = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\operatorname{tr}\left(\mathrm{e}^{-tM} - \mathrm{e}^{-tM_0}\right). \tag{A8.6}$$

The methods that are used in quantum mechanics to calculate the statistical operator $\mathrm{e}^{-\beta H}$, can again be used here.

A8.2 Regularization and one-loop divergences

In the representation (A8.6), large momentum divergences appear as divergences at $t = 0$ and, therefore, the determination of one-loop divergences is reduced to the small t expansion of the diagonal matrix elements $\langle x|\,\mathrm{e}^{-tM}\,|x\rangle$ (we use the bra–ket notation of quantum mechanics) for a Schrödinger-like operator M. This is a problem we have already faced in Section 2.2, and which can be solved, for example, by using Schrödinger's equation.

The expression (A8.6) can be regularized by the various methods explained in the chapter. For instance, we can multiply it by a cutting factor $\rho(t\Lambda^2)$ and we then recover the regularization defined by equation (A8.23).

Schwinger's proper-time regularization. Schwinger's proper-time regularization consists in simply cutting the t integral at a small value ε. Setting, for convenience,

$$M = H + m^2\,, \quad M_0 = H_0 + m^2,$$

we obtain the regularized expression

$$\Gamma^{\text{reg.}}_{1-\text{loop}}(\varphi) = -\frac{1}{2}\int_\varepsilon^\infty \frac{\mathrm{d}t}{t}\,\mathrm{e}^{-m^2 t}\,\mathrm{tr}\left(\mathrm{e}^{-tH} - \mathrm{e}^{-tH_0}\right). \qquad (A8.7)$$

For illustration purpose, we expand for t small the integrand in the case of the action ($A8.2$). Setting

$$U''\big(\phi(x)\big) = V(x),$$

we first expand the solution of the Schrödinger equation,

$$\left[-\nabla_x^2 + V(x)\right]\langle x|\,\mathrm{e}^{-tH}\,|x'\rangle = -\frac{\partial}{\partial t}\langle x|\,\mathrm{e}^{-tH}\,|x'\rangle\,. \qquad (A8.8)$$

We set

$$\langle x|\,\mathrm{e}^{-tH}\,|x'\rangle = \mathrm{e}^{-\sigma(x,x';t)}\,. \qquad (A8.9)$$

The Schrödinger equation then leads to

$$\nabla^2\sigma(t,x) - \big(\nabla\sigma(t,x)\big)^2 + V(x) = \frac{\partial\sigma(t,x)}{\partial t}\,. \qquad (A8.10)$$

The function σ has, for $t \to 0$, an expansion of the form

$$\sigma(t,x) = \frac{1}{4t}(x-x')^2 + \frac{d}{2}\ln 4\pi t + A(x,x')t + B(x,x')t^2 + C(x,x')t^3 + O\left(t^4\right). \quad (A8.11)$$

We obtain for the coefficients A, B, and C the equations,

$$\begin{aligned} A + (\mathbf{x}-\mathbf{x}')\cdot\nabla A &= V(x), \\ 2B + (\mathbf{x}-\mathbf{x}')\cdot\nabla B &= \nabla^2 A\,, \\ 3C + (\mathbf{x}-\mathbf{x}')\cdot\nabla C &= \nabla^2 B - (\nabla A)^2\,. \end{aligned} \qquad (A8.12)$$

The solutions for A and B are

$$\begin{aligned} A(x,x') &= \int_0^1 \mathrm{d}s\, V\big(x' + s(x-x')\big), \\ B(x,x') &= \int_0^1 \mathrm{d}s\, s(1-s)\nabla^2 V\big(x' + s(x-x')\big). \end{aligned} \qquad (A8.13)$$

It follows that

$$A(x,x) = V(x), \quad B(x,x) = \tfrac{1}{6}\nabla^2 V(x), \quad C(x,x) = -\tfrac{1}{2}\big(\nabla V(x)\big)^2 + \tfrac{1}{20}\nabla^4 V(x).$$

The divergent part of the regularized expression ($A8.7$) comes from the contribution I_ε of the lower bound, near which we can use the small t expansion. Total derivatives disappear in the trace. After an integration by parts, one obtains

$$I_\varepsilon = -\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\int_\varepsilon \frac{\mathrm{d}t\,\mathrm{e}^{-m^2 t}}{t^{1+d/2}}\int \mathrm{d}^d x\left\{-V(x)t + \tfrac{1}{2}V^2(x)t^2 - \tfrac{1}{6}\left[V^3(x) + \tfrac{1}{2}\left(\nabla V(x)\right)^2\right]t^3\right\} + O\left(t^4\right). \qquad (A8.14)$$

Finally, keeping only the contribution of the lower bound of the t integration, one finds

$$I_\varepsilon = \frac{1}{2}\frac{1}{(4\pi)^{d/2}}\Bigg\{-\frac{\varepsilon^{1-d/2}}{1-d/2}\int \mathrm{d}^d x\, V(x) + \frac{1}{2}\frac{\varepsilon^{2-d/2}}{2-d/2}\int \mathrm{d}^d x\left(V^2(x)+2m^2V(x)\right)$$
$$-\frac{1}{6}\frac{\varepsilon^{3-d/2}}{3-d/2}\int \mathrm{d}^d x\left[V^3(x)+3m^2V^2(x)+3m^4V(x)+\tfrac{1}{2}\left(\nabla V(x)\right)^2\right]\Bigg\}+\cdots \quad (A8.15)$$

When d is an even integer, $\varepsilon^0/0$ has to be replaced by $\ln(1/\varepsilon)$. This expression gives all divergences for $d \le 6$.

For example, we can apply this result to the interaction $U(\phi) = g\phi^3/3!$ in six dimensions. To compare with other regularizations, we set $\varepsilon = 1/\Lambda^2$. Then,

$$\Gamma^{\mathrm{div.}}_{1\,\mathrm{loop}}(\varphi) = \frac{1}{2^7\pi^3}\int \mathrm{d}^6 x\left\{\frac{\Lambda^4}{2}g\varphi(x) - \frac{\Lambda^2}{2}\left[g^2\varphi^2(x)+gm^2\varphi(x)\right]\right.$$
$$\left.+\frac{1}{3}\ln\frac{\Lambda}{m}\left[g^3\varphi^3(x)+3g^2m^2\varphi^2(x)+3gm^4\varphi(x)+\frac{g^2}{2}\left(\nabla\varphi(x)\right)^2\right]\right\}. \quad (A8.16)$$

This leads to the results of equations (8.33) and (8.34).

For the interaction $U(\phi) = g\phi^4/4!$ in four dimensions, the result is

$$\Gamma^{\mathrm{div.}}_{1\,\mathrm{loop}} = \frac{1}{32\pi^2}\left\{\frac{\Lambda^2}{2}g\int \mathrm{d}^4 x\,\varphi^2(x) - \ln\frac{\Lambda}{m}\int \mathrm{d}^4 x\left[\frac{g^2}{4}\varphi^4(x)+gm^2\varphi^2(x)\right]\right\}. \quad (A8.17)$$

An identical expression will be recovered in Section 9.3, equation (9.29). Both in equations (A8.16) and (A8.17), we have defined the divergent part of $\Gamma(\varphi)$ as the sum of the divergent terms in the asymptotic expansion for Λ/m large.

An application of equation (A8.15) to $d = 2$ and a general interaction $U(\phi)$ yields

$$\Gamma^{\mathrm{div.}}_{1\,\mathrm{loop}} = \frac{1}{4\pi}\ln\frac{\Lambda}{m}\int \mathrm{d}^2 x\, U''\big(\varphi(x)\big). \quad (A8.18)$$

Although, in these examples, the results can easily be recovered from the Feynman graph expansion, in more complicated cases, in which symmetries play an essential role, this method can be quite useful to evaluate divergences of one-loop diagrams.

ζ-function regularization. A variant of the preceding regularization method is to replace expression (A8.6) by

$$\Gamma^{\mathrm{reg.}}_{1\,\mathrm{loop}}(\varphi) = -\frac{1}{2\Gamma(1+\mu)}\int_0^\infty \mathrm{d}t\, t^{\mu-1}\,\mathrm{tr}\left(\mathrm{e}^{-tM}-\mathrm{e}^{-tM_0}\right), \quad (A8.19)$$

and to take, after analytic continuation in μ, the limit $\mu = 0$, in the spirit of the dimensional regularization.

We again consider the example of the ϕ^4 field theory in four dimensions and calculate $\Gamma_{1\,\mathrm{loop}}$ per unit volume for a constant field φ (V is the volume and $V \to \infty$):

$$\frac{1}{V}\Gamma^{\mathrm{reg}}_{1\,\mathrm{loop}}(\varphi) = -\frac{1}{2\Gamma(1+\mu)}\int_0^\infty \mathrm{d}t\, t^{\mu-1}\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\left[\mathrm{e}^{-t\left(p^2+m^2+g\varphi^2/2\right)} - (\varphi=0)\right].$$

The integration over the momentum p yields

$$\frac{1}{V}\Gamma^{\mathrm{reg}}_{1\,\mathrm{loop}}(\varphi) = -\frac{1}{32\pi^2\Gamma(1+\mu)}\int_0^\infty \mathrm{d}t\, t^{\mu-3}\left[\mathrm{e}^{-t\left(m^2+g\varphi^2/2\right)} - (\varphi=0)\right]. \qquad (A8.20)$$

The integration over t can then also be performed:

$$\frac{1}{V}\Gamma^{\mathrm{reg.}}_{1\,\mathrm{loop}}(\varphi) = -\frac{1}{32\pi^2\mu(\mu-1)(\mu-2)}\left[\left(m^2+g\varphi^2/2\right)^{(2-\mu)} - (\varphi=0)\right]. \qquad (A8.21)$$

Finally, expanding the expression for μ small and keeping only the divergent and finite parts, one obtains (to be compared with expression (A8.17))

$$\frac{1}{V}\Gamma^{\mathrm{reg}}_{1\,\mathrm{loop}}(\varphi) = \frac{1}{64\pi^2}\left(m^2+\frac{g}{2}\varphi^2\right)^2\left[-\frac{1}{\mu}+\ln\left(m^2+\frac{g}{2}\varphi^2\right)-\frac{3}{2}\right] - (\varphi=0). \qquad (A8.22)$$

The coefficient of the divergent part differs by a factor 2 from the one obtained in dimensional regularization: at leading order $\mu \sim \varepsilon/2$ (see, *e.g.*, equations (10.32)).

A8.3 More general momentum regularizations

Regularizations based on Schwinger's proper-time representation (see equations (A8.7, A8.19)) suggest the more general form of a regularized propagator,

$$\Delta_{\mathrm{B}}(p) = \int_0^\infty \mathrm{d}t\,\rho(t\Lambda^2)\,\mathrm{e}^{-t(p^2+m^2)}, \qquad (A8.23)$$

in which the function $\rho(t)$ is positive and satisfies the condition

$$|1-\rho(t)| < C\,\mathrm{e}^{-\sigma t}\ (\sigma>0) \text{ for } t\to+\infty\,.$$

By choosing a function $\rho(t)$ that decreases fast enough for $t \to 0$, the behaviour of the propagator can be arbitrarily improved. If $\rho(t) = O(t^n)$, the behaviour (8.23) is recovered. Another example is

$$\rho(t) = \theta(t-1), \qquad (A8.24)$$

$\theta(t)$ being the step function, which leads to exponential decrease:

$$\Delta_{\mathrm{B}}(p) = \frac{\mathrm{e}^{-(p^2+m^2)/\Lambda^2}}{p^2+m^2}. \qquad (A8.25)$$

As the example (A8.25) shows, it is possible to find propagators without non-physical singularities in this more general class, but they do not follow from a Hamiltonian formalism because continuation to real time becomes impossible. On the other hand, arbitrary local interaction monomials can be regularized.