

17 Critical phenomena: Corrections to scaling behaviour

In Chapters 15 and 16, while deriving the scaling behaviour of correlation functions, we have always kept only the leading term in the critical region. We examine now the different corrections to the leading behaviour [168–170, 80].

For instance, when we have solved the renormalization group (RG) equations, we have so far replaced the effective coupling constant $g(\lambda)$ at scale λ by g^* , neglecting the small difference $g(\lambda) - g^*$, which vanishes only if $g = g^*$. Moreover, to establish RG equations, we have neglected corrections subleading by powers of Λ , and effects of other couplings of higher canonical dimensions. Subleading terms related to the motion of $g(\lambda)$, which give the leading corrections for ε small, can easily be derived from the solutions of the RG equations studied previously, and are discussed first. The situations below and at four dimensions (the upper-critical dimension) have to be examined separately. The second type of corrections involves additional considerations, and is examined in the second part of the chapter.

The last section is devoted to one physical application, provided by systems with strong dipolar forces, which have 3 as upper-critical dimension.

17.1 Corrections to scaling: Generic dimensions

Dimensions $d < 4$. In dimensions $d < 4$, to characterize the corrections to scaling due to an initial value of the ϕ^4 coupling g different from the fixed point value g^* , it is convenient to solve the RG equations (16.24) by a slightly different method, which is based on introducing a set of coupling constant-dependent renormalizations:

$$\begin{cases} \ln \tilde{Z}(g) = - \int_{g^*}^g \frac{dg'}{\beta(g')} [\eta(g') - \eta], \\ \tilde{M}(g) = M Z^{-1/2}(g), \\ \tilde{\tau}(g) = \tau \exp \left[\int_{g^*}^g \frac{dg'}{\beta(g')} \left(\frac{1}{\nu(g')} - \frac{1}{\nu} \right) \right], \end{cases} \quad (17.1)$$

and a new coupling constant \tilde{g} , which characterizes the deviation of g from g^* :

$$\tilde{g} = (g - g^*) \exp \left[\int_{g^*}^g dg' \left(\frac{\omega}{\beta(g')} - \frac{1}{(g' - g^*)} \right) \right]. \quad (17.2)$$

($\omega = \beta'(g^*)$, see equation (16.11).) Then, in the vertex functions in the Fourier representation, we substitute

$$\begin{aligned} \tilde{\Gamma}^{(n)}(p_i; \tau, M, g, \Lambda) &= \tilde{Z}^{-n/2}(g) \tilde{\Gamma}^{(n)}(p_i; \tilde{\tau}(g), \tilde{M}(g), g^*, \Lambda) \\ &\times C^{(n)}(p_i; \tilde{\tau}(g), M(g), \tilde{g}, \Lambda). \end{aligned} \quad (17.3)$$

The function $C^{(n)}$ satisfies the boundary condition,

$$C^{(n)}(p_i; \tau, M; 0, \Lambda) = 1. \quad (17.4)$$

The finite renormalizations (17.1) eliminate the trivial deviations from the fixed point theory that correspond simply to a finite renormalizations of the different scaling variables. Equation (16.72) then implies

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \omega \tilde{g} \frac{\partial}{\partial \tilde{g}} - \frac{\eta}{2} M \frac{\partial}{\partial M} - \left(\frac{1}{\nu} - 2 \right) \tau \frac{\partial}{\partial \tau} \right] C^{(n)}(p_i, \tau, M, \tilde{g}, \Lambda) = 0. \quad (17.5)$$

Solving this equation by expanding $C^{(n)}$ in powers of \tilde{g} , one obtains

$$C^{(n)}(p_i, \tau, M, \tilde{g}, 1) = 1 + \sum_{s=1}^{\infty} \tilde{g}^s \tau^{s\omega\nu} D_s^{(n)}(p_i \tau^{-\nu}, \tau M^{-1/\beta}), \quad (17.6)$$

in which Λ has been set equal to 1.

Therefore, the exponent ω , which characterizes the approach to the fixed point, also characterizes the leading corrections to the critical behaviour.

For models with one coupling constant, $\beta(g)$ has the form

$$\beta(g) = -\varepsilon g + ag^2 + O(g^3, g^2\varepsilon). \quad (17.7)$$

In such a case, one always finds

$$\omega = \varepsilon + O(\varepsilon^2), \quad \omega\nu = \varepsilon/2 + O(\varepsilon^2). \quad (17.8)$$

Note that, to render all terms in the expansion (17.6) dimensionless, in the sense of scaling dimensions, we can assign to \tilde{g} the dimension $-\omega$.

Dimension $d = 4$. In exactly four dimensions, the situation is more subtle, because the $\phi^4(x)$ operator is marginal, and the approach to the fixed point is only logarithmic. This question is examined in next section.

Scaling for $d > 4$. So far, we have considered corrections to scaling for $d < 4$. In four dimensions or above, the fixed point corresponds to $g^* = 0$, that is, to the Gaussian fixed point and, therefore, the leading contributions to all correlation functions, except the two-point function, come from corrections to scaling, since these functions vanish at the fixed point. It is simple to verify that this special feature of the Gaussian fixed point explains the apparent contradiction between some RG predictions like relation between exponents involving explicitly the dimension d (called hyperscaling relations) and mean-field exponents: it is necessary to take into account the dimension of the ϕ^4 coupling constant g which, according to the preceding discussion, is $-\omega$:

$$\omega = d - 4, \quad \text{for } d > 4. \quad (17.9)$$

Let us consider, for example, the mean-field equation of state, valid for all dimensions $d > 4$:

$$H = \tau M + \frac{1}{6} g M^3.$$

The magnetization has dimension $(d-2)/2$ and the magnetic field H dimension $(d+2)/2$ in agreement with the general expressions for the exponents given by equations (16.65) and (16.66). These values are consistent with the product τM , since τ has dimension 2. Then, the dimension of gM^3 is $4-d+3(d-2)/2=(d+2)/2$, which is indeed the dimension of H .

17.2 Logarithmic corrections at the upper-critical dimension

The upper-critical dimension is the dimension at which deviations from mean-field theory appear (and the effective quantum field theory is just renormalizable). For the ϕ^4 field theory, this dimension is 4. In this dimension, there generally exists a marginal operator, here $\int \phi^4(x) d^d x$ and, therefore, as we have indicated in the general discussion of Section 16.1, logarithmic corrections to the mean-field behaviour are expected. We have already examined some consequences in the framework of particle physics in Sections 9.11 and 9.12, in particular, the issue of *triviality*. By contrast, the dimension 4 is not of physical relevance for statistical problems. However, its study is of special pedagogical value, because exact predictions can be derived from RG arguments. Because the infrared (IR) fixed point corresponds to $g^* = 0$, no assumption about the fixed point theory is required. Finally, we note some physical systems have $d = 3$ as upper-critical dimension, for example, tricritical systems (Section 14.8), or ferroelectrics with dipolar uniaxial long range forces (see Section 17.5).

We will study here only the equation of state and the specific heat, the generalization to correlation functions being straightforward. For $\varepsilon = 0$, the relation (16.58) becomes,

$$H(M, \tau, g, \Lambda) = Z^{-1/2}(\lambda)(\lambda\Lambda)^3 H(M(\lambda)/\lambda\Lambda, \tau(\lambda)/\lambda^2\Lambda^2, g(\lambda), 1). \quad (17.10)$$

The various functions are given by,

$$\ln \lambda = \int_g^{g(\lambda)} \frac{dg'}{\beta(g')}, \quad (17.11)$$

$$\ln Z(\lambda) = \int_g^{g(\lambda)} dg' \frac{\eta(g')}{\beta(g')}, \quad (17.12)$$

$$\ln(\tau(\lambda)/\tau) = - \int_g^{g(\lambda)} dg' \frac{\eta_2(g')}{\beta(g')}, \quad (17.13)$$

and $M(\lambda)/M = Z^{-1/2}(\lambda)$.

In the $(\phi^2)^2$ field theory, ϕ being a N -component vector field, that is, in the $O(N)$ symmetric N -vector model, for g small, the expansions of the RG functions are (equations (10.89), (10.90), and (10.91), but with a different normalization of the coupling g),

$$\begin{aligned} \beta(g) &= \frac{(N+8)}{6} \frac{g^2}{8\pi^2} - \frac{(3N+14)}{12} \frac{g^3}{(8\pi^2)^2} O(g^4), \\ \eta(g) &= \frac{(N+2)}{72} \left(\frac{g}{8\pi^2} \right)^2 + O(g^3), \\ \eta_2(g) &= - \frac{(N+2)}{6} \frac{g}{8\pi^2} + O(g^2). \end{aligned} \quad (17.14)$$

We now introduce a physical momentum scale $\mu \ll \Lambda$, like the correlation length when it is finite, and choose then $\lambda = \mu/\Lambda$. The solution of equation (17.11) in this regime,

$$\ln \lambda = - \frac{48\pi^2}{(N+8)g(\lambda)} + \frac{3(3N+14)}{(N+8)^2} \ln(g(\lambda)) + O(1),$$

yields the effective coupling

$$g(\lambda) \equiv g_r \underset{\lambda \rightarrow 0}{=} \frac{48\pi^2}{N+8} \frac{1}{|\ln \lambda|} - \frac{3(3N+14)}{(N+8)^2} \frac{\ln |\ln \lambda|}{(\ln \lambda)^2} + O\left(\frac{1}{(\ln \lambda)^2}\right). \quad (17.15)$$

The factor $Z(\lambda)$ yields inessential finite renormalizations of H and M given by

$$\ln Z(\lambda) = \ln \zeta(g) + \frac{2\pi^2}{3} \frac{N+2}{N+8} g_r + O(g_r^2),$$

with

$$\ln \zeta(g) = \int_g^0 dg' \frac{\eta(g')}{\beta(g')}.$$

Therefore,

$$M(\lambda) = M \zeta^{-1/2}(g) [1 + O(g_r)].$$

Also,

$$\tau(\lambda) = T(g)(g(\lambda))^{(N+2)/(N+8)} \tau [1 + O(g_r)], \quad (17.16)$$

with

$$\ln T(g) = \int_g^0 dg' \left[\frac{N+2}{(N+8)g'} - \frac{\eta_2(g')}{\beta(g')} \right].$$

Since g_r is small, $H(\lambda)$ in the right-hand side of equation (17.10), as well all other physical quantities, can be calculated from perturbation theory. Note here the power of the RG method: we started from a theory with a coupling constant of order 1 and perturbative coefficients increasing like powers of $\ln(\tau/\Lambda^2)$ or equivalent. Direct perturbation theory is obviously useless. In contrast, in the effective theory at scale λ , the coupling constant $g(\lambda)$ is small, and the perturbative coefficients are of order 1.

The field H has the perturbative expansion,

$$H = \tau M + \frac{g}{6} M^3 + \dots .$$

We renormalize H , M , and τ by the factors $\zeta^{1/2}(g), T(g)$. Then, from the relations (17.10) and (17.15), one derives

$$H(M, \tau, g, \Lambda = 1) = \tau M g_r^{(N+2)/(N+8)} + \frac{1}{6} g_r M^3 + \dots . \quad (17.17)$$

At T_c (*i.e.*, $\tau = 0$), the physical length scale can be chosen such that $M(\lambda)/\lambda = 1$ (with $\Lambda = 1$), and thus $\lambda \propto M/\Lambda$. It follows that

$$H \propto M^3 / |\ln M| . \quad (17.18)$$

For $|\tau| \neq 0$, it is more convenient to use the length scale defined by

$$\tau(\lambda) = \lambda^2 \Rightarrow g(\lambda) \propto 1 / \ln |\tau|. \quad (17.19)$$

Then, the spontaneous magnetization is given by ($\tau < 0$)

$$M \propto (|\tau|)^{1/2} |\ln |\tau||^{3/(N+8)}, \quad (17.20)$$

and the susceptibility in zero field by

$$\chi^{-1} \propto |\tau| |\ln |\tau||^{-[(N+2)/(N+8)]}. \quad (17.21)$$

Finally, the specific heat satisfies the RG equation,

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \eta_2(g) \left(2 + \tau \frac{\partial}{\partial \tau} \right) \right] \tilde{\Gamma}^{(2,0)} = B(g), \quad (17.22)$$

and $B(g)$ has the expansion

$$B(g) = \frac{N}{16\pi^2} + O(g). \quad (17.23)$$

It is simple to verify that a function $C_2(g)$, solution of equation (17.22), and thus of

$$\beta(g)C'_2(g) - 2\eta_2(g)C_2(g) = B(g),$$

is necessarily singular at the origin. For example, one can take a solution of the form

$$C_2(g) = \frac{3N}{(N-4)} \frac{1}{g} + O(1), \quad \text{for } N \neq 4. \quad (17.24)$$

For $N = 4$, an additional logarithmic singularity is present. The combination $\tilde{\Gamma}^{(2,0)} - C_2(g)$, solution of the homogeneous RG equation, is, for g small, dominated by the pole of $C_2(g)$:

$$\tilde{\Gamma}^{(2,0)}(0; \tau, g, \Lambda) - C_2(g) \sim -\frac{3N}{(N-4)} \frac{1}{g(\lambda)} \zeta^{-2}(\lambda), \quad (17.25)$$

with

$$\zeta(\lambda) = \exp \left[\int_g^{g(\lambda)} \frac{dg'}{\beta(g')} \eta_2(g') \right] \propto [g(\lambda)]^{-(N+2)/(N+8)}. \quad (17.26)$$

Collecting all factors, one obtains the behaviour of the specific heat:

$$\tilde{\Gamma}^{(2,0)}(0; \tau, g, \Lambda) - C_2(g) \propto |\ln \tau|^{(4-N)/(N+8)} \left[1 + O\left(\frac{\ln |\ln \tau|}{\ln \tau}\right) \right]. \quad (17.27)$$

Particle physics. For the viewpoint of particle physics, the renormalization by a logarithmic factor in equation (17.16) is irrelevant compared to the Gaussian renormalization, and does not alleviate the fine-tuning problem. However, the equation analogous to equation (17.13) in the *renormalized theory*, describing a finite renormalization associated with a change of physical scale, is useful.

Remark. It is apparent from these expressions that a parametrization in terms of the variables τ or M leads to rather complicated expressions. A more efficient way of writing all these results is to introduce a parametric representation in terms of the effective coupling constant $g(\lambda)$ and to calculate λ in terms of $g(\lambda)$ from equation (17.11). We parametrize $\beta(g)$ as

$$\beta(g) = \beta_2 g^2 + \beta_3 g^3 + O(g^4), \quad (17.28)$$

and set

$$s = g(\lambda). \quad (17.29)$$

Then, λ is given by

$$\lambda = s^{-\beta_3/\beta_2^2} e^{-1/\beta_2 s} \tilde{\lambda}(s). \quad (17.30)$$

In equation (17.30), the function $\tilde{\lambda}(s)$ is a regular function of s for s small. The renormalization factor $Z(\lambda)$ is a regular function of s . Finally, equation (17.13) yields

$$\frac{\tau(\lambda)}{\lambda^2 \Lambda^2} = \frac{\tau}{\Lambda^2} s^{[2\nu_1 + \beta_3/\beta_2]/\beta_2} e^{2/\beta_2 s} [\tilde{\tau}(s)]^{-1}, \quad (17.31)$$

in which $\tilde{\tau}(s)$ is a regular function of s , and $\nu(g)$ has been parametrized as

$$\nu(g) = (2 + \eta_2(g))^{-1} = \tfrac{1}{2} + \nu_1 g + O(g^2). \quad (17.32)$$

Then, we determine λ from the condition (17.19), and equation (17.31) parametrizes τ as a function of s . At leading order, all critical behaviours are described by a singular factor of the form occurring in equations (17.30) or (17.31), multiplied by regular series in $s = g(\lambda)$.

17.3 Irrelevant operators and the question of universality

In generic dimensions, we now examine the contributions coming from irrelevant operators (see also Section 11.1) [168–170]. We again stress that these operators have been found to be irrelevant at the Gaussian fixed point, near four dimensions. We still rely on the assumption that dimensions vary continuously when the IR fixed point moves away from the Gaussian fixed point. Finally, the analysis is local, we consider only the neighbourhood of the fixed point.

We first recall power counting arguments for a general theory with an action $\mathcal{S}(\phi)$ (for details, see Chapters 8 and 11):

$$\mathcal{S}(\phi) = \int d^d x \left[\tfrac{1}{2} (\nabla \phi(x))^2_\Lambda + \tfrac{1}{2} r \phi^2(x) + \sum_\alpha u_\alpha \mathcal{O}_\alpha(\phi, x) \right], \quad (17.33)$$

in which $\mathcal{O}_\alpha(\phi, x)$ is a local monomial of degree n_α in ϕ with k_α derivatives. The dimension $[u_\alpha]$ of the coupling constant u_α is then (consequence of equation (8.15))

$$[u_\alpha] = d - k_\alpha - \tfrac{1}{2} n_\alpha (d - 2). \quad (17.34)$$

We treat all interactions in action (17.33) in perturbation theory. Order by order in the loop expansion, we evaluate the divergent part of the corresponding Feynman diagrams, and add counter-terms to the action to render the theory finite. Since the action (17.33) contains all possible monomials in the field, any counter-term is a linear combination of the operators $\mathcal{O}_\alpha(\phi)$.

In order for a product of constants u_β to appear in a counter-term proportional to an operator \mathcal{O}_α , it is necessary and generically sufficient that the condition

$$\Delta = d - [\mathcal{O}_\alpha] - \sum_l [u_{\beta_l}] \geq 0, \quad (17.35)$$

is satisfied. Then, the coefficient $\delta u_\alpha(\Lambda)$ of the counter-term proportional to \mathcal{O}_α diverges like a positive power of the cut-off Λ (or a power of logarithm if $\Delta = 0$),

$$\delta u_\alpha(\lambda) \sim \Lambda^\Delta. \quad (17.36)$$

We now return to the question of irrelevant operators. We restrict ourselves to dimensions $d = 4 - \varepsilon$, $\varepsilon > 0$ and small, since as we have seen, this is the only situation in which a reliable analysis is possible. The first operators we have, for example, in mind are ϕ^6 , $\phi^2(\nabla\phi)^2\dots$, which are operators of dimension 6 in four dimensions.

To introduce the cut-off Λ in the effective field theory, we rescale the lengths and the field ϕ (equations (15.40) and (15.41)). Therefore, each coupling constant u_α is the product of a dimensionless quantity g_α by a power of the cut-off:

$$u_\alpha = g_\alpha \Lambda^{-\delta_\alpha}, \quad (17.37)$$

where δ_α , as given by equation (17.34), has the form

$$\delta_\alpha = -d + k_\alpha + \frac{1}{2}n_\alpha(d-2). \quad (17.38)$$

If δ_α is positive, the corresponding operator \mathcal{O}_α leads to a non-renormalizable theory, and we have already stated that it is irrelevant. In the tree approximation, the statement follows from equation (17.37): the operator gives contributions vanishing with a power of the cut-off. However, in higher orders, the statement is less trivial, since divergences at large cut-off coming from the momentum integration may cancel the powers coming from the coupling constants. To understand what happens, it is necessary to analyse the counter-terms generated by these operators using equations (17.35) and (17.36).

The total power Δ' of the cut-off, which multiplies the operator \mathcal{O}_α in a counter-term, is the sum of the power Δ generated by the divergence of perturbation theory (equation (17.35)), and the powers already present in the coefficients u_β (equation (17.37)):

$$\Delta' = \Delta - \sum_l \delta_{\beta_l} = \Delta + \sum_l [u_{\beta_l}], \quad (17.39)$$

and, therefore, using the definition (17.35),

$$\Delta' = d - [\mathcal{O}_\alpha] = [u_\alpha]. \quad (17.40)$$

The conclusion is simple: due to the divergences of perturbation theory, irrelevant operators indeed give non-vanishing contributions, but these contributions can be cancelled by changing the amplitudes of the relevant or marginal terms in the Hamiltonian, because $\Delta' \geq 0$ is equivalent to $[u_\alpha] \geq 0$.

Example. The leading new corrections come from operators \mathcal{O}_i^6 which have dimension 6 in four dimensions. The corresponding interactions have the form

$$\Lambda^{2\varepsilon-2} \int d^d x \phi^6(x), \quad \Lambda^{\varepsilon-2} \int d^d x (\phi(x) \nabla \phi(x))^2, \quad \Lambda^{-2} \int d^d x (\nabla^2 \phi(x))^2.$$

In terms of the renormalized operators, they have the expansion

$$\int d^d x \mathcal{O}_i^6(x) = \int d^d x \left\{ \sum_j Z_{ij} [\mathcal{O}_j^6(x)]_r + \sum_j A_{ij} [\mathcal{O}_j^4(x)]_r + B [\phi^2(x)]_r \right\}.$$

We have denoted by $\mathcal{O}_j^4(x)$ the two operators of dimension 4 in four dimensions: ϕ^4 and $(\nabla\phi)^2$. In the framework of the ε -expansion, the coefficients Z_{ij} diverge like powers of $\ln \Lambda$, A_{ij} like Λ^2 and B like Λ^4 , up to powers of $\ln \Lambda$.

Taking into account the powers of Λ in front of the interaction terms, we see that only the contributions proportional to operators of dimensions 4 and 2 are divergent. If we cancel these contributions by subtracting to the operators of dimension 6 a suitable combination of bare operators of dimensions 4 and 2 (bare and renormalized operators are linearly related), we obtain the true new corrections which decrease like $\Lambda^{-2+O(\varepsilon)}$.

Discussion. The analysis also clarifies the interpretation of the constants r and g , which parametrize the ϕ^4 Hamiltonian. These are not the parameters that are generated directly by the microscopic theory but, instead, effective parameters taking into account the effect of neglected irrelevant operators. However, the analysis of previous chapters is, at leading order, not modified. Indeed, the change in ϕ^2 corresponds only to a modification of the critical temperature which is a non-universal quantity. Moreover, below four dimensions, we have shown that many physical quantities (universal quantities) do not depend on g either, since g can be replaced by its fixed point value g^* . Finally, a change in the cut-off procedure corresponds generally to a change in the coefficients of the irrelevant part of the propagator ($\phi\Delta^2\phi\dots$). The effect of such a change is obtained from the previous analysis also. We can now clarify the concept of universality: below four dimensions all dimensionless quantities in which g can be replaced by g^* , the IR fixed point value, and which do not depend on the normalizations of the field ϕ , the deviation from the critical temperature t , and of the magnetic field are universal. Obvious examples are ratios of amplitude of singularities below and above T_c , ratios of amplitudes of leading corrections to scaling, the rescaled equation of state (relation between H and M), the renormalized correlation functions as defined in Chapters 15 and 16, and so on.

Emergent symmetries. A simple consequence of the analysis is the following: if one adds an irrelevant operator to a renormalizable Hamiltonian, which breaks its symmetry, then the symmetry of the critical theory is broken if and only if the irrelevant operator generates by renormalization relevant or marginal operators breaking the symmetry. An application is the following: on the lattice, operators of the form $\sum_\mu \int \phi(x) \partial_\mu^4 \phi(x) d^d x$ which break the $O(d)$ symmetry of the effective $\phi^4(x)$ action are present. However, these operators have a hypercubic symmetry and, since the only relevant operators they can generate, like $\int (\nabla \phi)^2 d^d x$, due to the hypercubic symmetry, are $O(d)$ symmetric, the critical theory has an *emergent $O(d)$ symmetry*.

By contrast, the addition of a naively irrelevant term like $\int \phi^5(x) d^d x$ to a Hamiltonian, which is symmetric in the reflection $\phi \mapsto -\phi$, generates relevant terms linear in ϕ , which are equivalent to the addition of a magnetic field, and breaks the reflection symmetry.

17.4 Corrections coming from irrelevant operators. Improved action

For simplicity, we consider the effect, in the critical theory, of an operator \mathcal{O}_α at first order only in the corresponding coupling constant u_α . The following discussion applies in the framework of the ε -expansion, and relies on the results of Chapter 11 concerning the renormalization of composite operators.

17.4.1 Corrections to scaling

In Section 17.3, we have shown that an operator \mathcal{O}_α gives contributions equivalent to all operators of lower canonical dimensions. For example, $\phi^8(x)$ first generates effects equivalent to $\phi^2(x)$, $\phi^4(x)$ and $(\nabla\phi(x))^2$ and all operators of dimension 6, and then genuine new corrections. To isolate these corrections, it is necessary to subtract from the operator a linear combination of all operators which have a lower dimensions at $d = 4$, that is, to perform an additive renormalization.

However, we can omit all operators that are total derivatives, since only the space integrals appear in the action. We define a subtracted operator $\bar{\mathcal{O}}_\alpha(\phi)$ by

$$\bar{\mathcal{O}}_\alpha(\phi) = \mathcal{O}_\alpha(\phi) - \sum_{\beta \text{ such that } [\mathcal{O}_\beta] < [\mathcal{O}_\alpha]} C_{\alpha\beta}(\Lambda, g) \mathcal{O}_\beta(\phi). \quad (17.41)$$

Let us again illustrate this point with the example of the operators of dimension 6, like $\phi^6(x)$. One then subtracts a linear combination of operators of dimensions 2 and 4:

$$\phi^6(x) = \phi^6(x) - C_1 \phi^2(x) - C_2 [\nabla \phi(x)]^2 - C_3 \phi^4(x).$$

The coefficients C_1 , C_2 , and C_3 can be determined by a set of renormalization conditions at zero momentum:

$$\begin{aligned} \tilde{\Gamma}_{\phi^6}^{(2)}(p) &= O(p^4 \times \text{powers of } \ln p), \text{ for } p \rightarrow 0, \\ \tilde{\Gamma}_{\phi^6}^{(4)}(p_i = 0) &= 0. \end{aligned} \quad (17.42)$$

The first condition implies, in particular, that the critical temperature is not changed. These conditions are not affected by IR divergences, because the correlation functions with an operator insertion have positive dimensions.

After such additive renormalizations, the bare operators are related to the completely renormalized operators \mathcal{O}_α^r by

$$\mu^{-\delta_\alpha} \int d^d x \mathcal{O}_\alpha^r(x) = \sum_\beta Z_{\alpha\beta}(g, \Lambda/\mu) \Lambda^{-\delta_\beta} \int d^d x \bar{\mathcal{O}}_\beta(x), \quad (17.43)$$

in which μ is the renormalization scale. Additional renormalization conditions at scale μ for the insertion of renormalized operators determine the matrix $Z_{\alpha\beta}$.

The relation between correlation functions $\tilde{\Gamma}_{\bar{\mathcal{O}}_\alpha}^{(n)}$ with one $\Lambda^{-\delta_\alpha} \int d^d x \bar{\mathcal{O}}_\alpha$ insertion, and the renormalized functions with $\mu^{-\delta_\alpha} \int d^d x \mathcal{O}_\alpha(x)$ insertion, then is

$$\sum_\beta Z_{\alpha\beta} Z^{n/2} \tilde{\Gamma}_{\bar{\mathcal{O}}_\beta}^{(n)}(p_i; g, \Lambda) = \left[\tilde{\Gamma}_{\mathcal{O}_\alpha}^{(n)}(p_i, g_r, \mu) \right]_r. \quad (17.44)$$

This leads to the RG equations,

$$\sum_\beta \left\{ \left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2} \eta(g) \right] \delta_{\alpha\beta} - \eta_{\alpha\beta}(g) \right\} \tilde{\Gamma}_{\bar{\mathcal{O}}_\beta}^{(n)} = 0, \quad (17.45)$$

with

$$\eta_{\alpha\beta}(g) = - \sum_\gamma \left(\Lambda \frac{\partial}{\partial \Lambda} Z_{\alpha\gamma} \right) (Z^{-1})_{\gamma\beta}. \quad (17.46)$$

The effects of the insertions of the operators $\bar{\mathcal{O}}_\alpha$ are then governed by the eigenvalues η_α of the matrix $\eta_{\alpha\beta}(g^*)$ [171]. From the relation (17.43), we infer that the renormalization matrix $Z_{\alpha\beta}$ has the form

$$Z_{\alpha\beta} = \delta_{\alpha\beta} (\Lambda/\mu)^{\delta_\alpha} (1 + O(g)). \quad (17.47)$$

Therefore, the eigenvalues η_α are at leading order given by

$$\eta_\alpha = \delta_\alpha + O(\varepsilon).$$

The genuine new contributions of the irrelevant operators of canonical dimension $d + \delta_\alpha$ in four dimensions, are suppressed by powers $\Lambda^{-\delta_\alpha+O(\varepsilon)}$ of the cut-off Λ . For operators of dimension 6, $\delta_6 = 2$. In an infinitesimal neighbourhood of dimension 4, these operators remain irrelevant. Our general analysis, which is based upon the idea that the critical behaviour of ferromagnetic systems can be described by an effective ϕ^4 field theory remains valid beyond the ε -expansion, as long as this property remains true.

Important irrelevant effects. In some cases, the irrelevant effects may be especially important. An example is provided by systems where the initial theory has only a discrete symmetry, while the symmetry of the critical theory is continuous. In the low temperature phase, the critical theory has Goldstone-mode singularities. These singularities are suppressed by corrections due to irrelevant operators.

17.4.2 Fixed point in Hamiltonian space and improved action

We have noted that by adding to the $\phi^4(x)$ field theory irrelevant interactions, we could modify correlation functions by terms in the ε -expansion of order $1/\Lambda^2$ (up to logarithms). In the right-hand side of the RG equations (15.46), we have neglected terms of the same order. Symanzik [90] has given a perturbative proof that, by adding to the Hamiltonian the proper linear combination of irrelevant operators, it is possible to cancel exactly these corrections. The coefficients of the linear combination are functions of the ϕ^4 coupling constant g . For example, the complete set of operators of dimension 6 can be used to cancel exactly the corrections of order $1/\Lambda^2$ in the right-hand side of the RG equations (15.46), the operators of dimension 8 to cancel the order $1/\Lambda^4$ (up to logarithms), and so on. An infinite iteration of this procedure leads to a theory that depends on only one ϕ^4 coupling constant, and which satisfies the RG equations exactly. It is actually a ‘renormalized’ theory constructed without using the renormalization procedure, but by considering an infinite sequence of Hamiltonians.

From the general RG point of view, the Hamiltonians which lead to correlation functions satisfying RG equations exactly belong to a one parameter line in Hamiltonian space which goes from the Gaussian fixed point to the non-trivial IR fixed point.

Conversely, by directly constructing an RG for a cut-off field theory, it is possible to prove the existence of the renormalized field theory [59].

As an application, Symanzik [90] has advocated the use of such improved actions (adding, for example, all terms of dimension 6) for numerical lattice simulations. However, it should be mentioned, that the applications of this ingenious idea (in particular, in numerical simulations for particle physics) require very precise numerical data and large lattices. Indeed, because the improved action involves more extended interactions on the lattice (like second nearest neighbours), the effective size of the lattice is reduced, increasing finite size effects. But statistics then becomes a serious problem, and more complicated interactions slow down numerical calculations.

17.5 Application: Uniaxial systems with strong dipolar forces

In Section 17.2, we have stressed that, for critical systems at the upper-critical dimension, the RG leads to exact predictions. Unfortunately, in the case of the N -vector model, the upper-critical dimension is 4 and, therefore, the predictions are not useful for macroscopic phase transitions. The main application is numerical physics, for example, the Higgs sector ($N = 4$) of the Standard Model of particle physics has been investigated numerically.

However, the critical dimensions change with long-range forces [172, 173]. Therefore, here we describe another system, for which precise measurements have been made, and which has dimension 3 as the upper-critical dimension: a uniaxial ferromagnet or ferroelectric system with strong dipolar forces [174, 169, 175].

Dipolar forces. We consider a spin system in d dimensions in which the d -component spins S^μ interact both through short range and dipolar forces:

$$-\beta\mathcal{H}(\mathbf{S}) = \sum_{\mathbf{x}, \mathbf{x}'} \left[\sum_{\mu, \nu} V_{\mu\nu}(\mathbf{x} - \mathbf{x}') S_\mathbf{x}^\mu S_{\mathbf{x}'}^\nu + \gamma (\mathbf{S}_\mathbf{x} \cdot \nabla_\mathbf{x})(\mathbf{S}_{\mathbf{x}'} \cdot \nabla_{\mathbf{x}'}) \frac{1}{|\mathbf{x} - \mathbf{x}'|^{d-2}} \right], \quad (17.48)$$

$V_{\mu\nu}(\mathbf{x})$ being the short range potential, and γ a parameter. We assume that the long range dipolar forces are strong enough to play a role in the part of the critical domain accessible to experiments. In addition, we assume that the lattice is strongly anisotropic in such a way that only one component of the spin \mathbf{S} is critical, and the effective Hamiltonian can be simplified into

$$-\beta\mathcal{H}(S) = \sum_{\mathbf{x}, \mathbf{x}'} S_\mathbf{x} S_{\mathbf{x}'} \left(V(\mathbf{x} - \mathbf{x}') + \gamma \partial_z \partial_{z'} \frac{1}{|\mathbf{x} - \mathbf{x}'|^{d-2}} \right), \quad (17.49)$$

in which S now denotes the component of the spin vector \mathbf{S} along the $z \equiv x_d$ axis.

After Fourier transformation, the Hamiltonian can be written as

$$-\beta\mathcal{H}(S) = \int d^d q \tilde{S}(\mathbf{q}) \tilde{S}(-\mathbf{q}) \left(\tilde{V}(\mathbf{q}) + \tilde{\gamma} \frac{q_z^2}{q^2} \right), \quad (17.50)$$

where $\tilde{\gamma}$ is proportional to γ and $\tilde{V}(\mathbf{q})$ is a regular function of \mathbf{q} which, due to hypercubic symmetry, has the expansion

$$\tilde{V}(\mathbf{q}) = \tilde{V}(0) + \frac{1}{6} \nabla^2 \tilde{V}(0) \mathbf{q}^2 + O(q^4). \quad (17.51)$$

In the critical domain, in which $|\mathbf{q}|$ is small, the two terms coming from the short range potential and the dipolar forces are of the same order of magnitude:

$$q^2 \sim q_z^2 / q^2. \quad (17.52)$$

This implies that q_z , the z component of the vector \mathbf{q} , is much smaller than the other components \mathbf{q}_\perp :

$$|q_z| \sim (\mathbf{q}_\perp)^2. \quad (17.53)$$

Therefore, we can simplify further the interaction potential. Finally, in the case of an even spin distribution, we can reproduce the configuration energy by an effective Hamiltonian $\mathcal{H}(\phi)$ of the form (in terms of ‘bare’ parameters τ_0, A_0, u_0), in the Fourier representation,

$$\begin{aligned}\mathcal{H}(\phi) = & \frac{1}{2} \int d^d q \phi(-\mathbf{q}) (\mathbf{q}_\perp^2 + A_0^2 q_z^2 / \mathbf{q}_\perp^2 + r_c + \tau_0) \phi(\mathbf{q}) \\ & + \frac{u_0}{4!} \int d^d q_1 \cdots d^d q_4 \delta^{(d)}(\sum \mathbf{q}_i) \phi(\mathbf{q}_1) \cdots \phi(\mathbf{q}_4),\end{aligned}\quad (17.54)$$

where r_c is the critical value and τ_0 parametrizes the deviation from the critical temperature.

The upper-critical dimension. Usual power counting is modified, because space is no longer isotropic. In units of the transverse components of \mathbf{q} , the dimension of q_z is 2 (equation (17.53)):

$$[q_z] = 2 \Rightarrow [z] = -2.$$

The volume element in configuration space $d\mathbf{x}_\perp dz$ has thus canonical dimension $-(d+1)$. This implies that power counting analysis is the same as in the conventional ϕ^4 theory in $(d+1)$ dimensions. In particular, the upper-critical dimension is given by

$$d+1=4 \Rightarrow d=3.$$

The RG methods thus lead to exact analytic predictions in the physical dimension $d=3$.

RG equations. From expression (17.54), we infer the propagator (A and τ are the renormalized parameters),

$$\tilde{\Delta}(\mathbf{q}) = \frac{\mathbf{q}_\perp^2}{(\mathbf{q}_\perp^2)^2 + A^2 q_z^2 + \tau \mathbf{q}_\perp^2}. \quad (17.55)$$

Diagrams calculated with this propagator are regular for \mathbf{q} small, therefore,

$$\int d^d q \phi(-\mathbf{q}) \frac{q_z^2}{\mathbf{q}_\perp^2} \phi(\mathbf{q}),$$

never appears as a counter-term.

The renormalized Hamiltonian thus has the form ($Z, Z_2, Z_g, \delta m^2$ are renormalization constants)

$$\begin{aligned}\mathcal{H}_r = & \frac{1}{2} \int d^d q \phi(-\mathbf{q}) (Z \mathbf{q}_\perp^2 + A^2 q_z^2 / \mathbf{q}_\perp^2 + \delta m^2 + Z_2 \tau) \phi(\mathbf{q}) \\ & + \frac{\mu^\varepsilon}{4!} A g Z_g(g) \int d^d q_1 \cdots d^d q_4 \delta^{(d)}(\sum \mathbf{q}_i) \phi(\mathbf{q}_1) \cdots \phi(\mathbf{q}_4),\end{aligned}\quad (17.56)$$

in which μ is the renormalization scale and ε now is defined as

$$d = 3 - \varepsilon. \quad (17.57)$$

We introduce also a bare dimensionless coupling constant

$$u_0 = \Lambda^\varepsilon A_0 g_0. \quad (17.58)$$

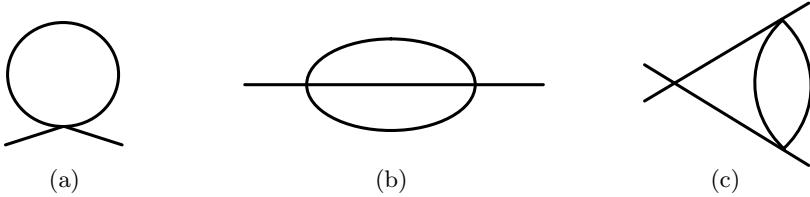


Fig. 17.1 Diagrams needed at two-loop order

The relation between bare and renormalized correlation functions reads

$$\tilde{\Gamma}_r^{(n)}(p_i; \tau, g, A, \mu) = Z^{n/2} \tilde{\Gamma}^{(n)}(p_i; \tau_0, g_0, A_0, \Lambda). \quad (17.59)$$

In addition, comparing expressions (17.54) and (17.56), one finds

$$A = Z^{1/2} A_0. \quad (17.60)$$

RG equations follow:

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g_0) \frac{\partial}{\partial g_0} + \frac{1}{2} \eta(g_0) \left(A_0 \frac{\partial}{\partial A_0} - n \right) - \eta_2(g_0) \tau_0 \frac{\partial}{\partial \tau_0} \right] \tilde{\Gamma}^{(n)}(p_i; g_0, A_0, \tau_0, \Lambda) = 0. \quad (17.61)$$

Two-loop calculation of RG functions. For reasons explained in Section 15.4, for practical calculations, we use the renormalized theory and minimal subtraction (see Chapter 10). In what follows, to simplify the notation we omit the index r on renormalized functions. The renormalized RG equations are formally identical to equations (17.61):

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \frac{1}{2} \eta(g) \left(A \frac{\partial}{\partial A} - n \right) - \eta_2(g) \tau \frac{\partial}{\partial \tau} \right] \tilde{\Gamma}^{(n)}(p_i; g, A, \tau, \Lambda) = 0, \quad (17.62)$$

but the RG functions are different at two-loop order.

The calculations are here somewhat similar to the dynamic calculations as described in Section A36, because if we identify the z direction with time, the propagators have the same denominators. The combinatorial factors of Feynman diagrams are those of the ϕ^4 field theory. Only the expressions of the diagrams differ. We need only their values at vanishing external z components of momenta, and this simplifies the integration over momentum variables in the z direction (called ω hereafter). In Fig. 17.1, we display the three diagrams needed at two-loop order. In the critical (or massless) theory, in terms of the Fourier transform of the propagator,

$$\tilde{\Delta}(\omega, k) = \frac{1}{k^2 + \omega^2/k^2},$$

the expressions are

$$\begin{aligned} (a) &\equiv \frac{1}{(2\pi)^d} \int d\omega d^{d-1}k \tilde{\Delta}(\omega, k) \tilde{\Delta}(\omega, p+k) \\ &= \frac{1}{(2\pi)^{d-1}} \frac{1}{2} \int \frac{d^{d-1}k}{k^2 + (p+k)^2} = \frac{1}{(16\pi)^{(d-1)/2}} \Gamma(\varepsilon/2) p^{-\varepsilon}. \\ (b) &\equiv \frac{1}{(2\pi)^{2d}} \int d\omega_1 d\omega_2 d^{d-1}k_1 d^{d-1}k_2 \tilde{\Delta}(\omega_1, k_1) \tilde{\Delta}(\omega_2, k_2) \tilde{\Delta}(\omega_1 + \omega_2, k_1 + k_2 + p) \\ &= \frac{1}{(2\pi)^{d-2}} \frac{1}{4} \int \frac{d^{d-1}k_1 d^{d-1}k_2}{k_1^2 + k_2^2 + (k_1 + k_2 + p)^2} = \frac{1}{(16\pi)^{d-1}} \frac{3}{8} \left(\frac{16}{27} \right)^{(d-1)/2} \Gamma(2-d) p^{2d-4}. \end{aligned}$$

After integration over the corresponding ω variables, the diagram (c) takes the form

$$(c) = \frac{1}{(2\pi)^{2(d-1)}} \frac{1}{8} [C_1(p_1, p_2) + C_1(p_2, p_1) + C_2(p_1, p_2)], \quad \text{with}$$

$$C_1(p, q) = \int \frac{d^{d-1}k_1 d^{d-1}k_2}{[k_1^2 + (k_1 + p + q)^2][k_1^2 + k_2^2 + (k_1 + k_2 + p)^2]},$$

$$C_2(p, q) = \int \frac{d^{d-1}k_1 d^{d-1}k_2}{[(k_1 + p + q)^2 + k_2^2 + (k_1 + k_2 + p)^2]} \frac{1}{[k_1^2 + k_2^2 + (k_1 + k_2 + p)^2]}.$$

The final result is

$$C_1(\varepsilon) \simeq \frac{1}{(16\pi)^{d-1}} \left(\frac{2}{\sqrt{3}} \right)^\varepsilon \frac{\Gamma(1+\varepsilon)}{\varepsilon^2} (|p_1 + p_2|)^{-2\varepsilon} + O(1),$$

$$C_2(\varepsilon) = \frac{2}{3} \frac{1}{(16\pi)^2 \varepsilon} + O(1).$$

The renormalization constants Z_g , Z and Z_2 are then determined up to order g^2 :

$$Z_g = 1 + \frac{3N_d}{\varepsilon} g + \left(\frac{9}{\varepsilon^2} - \frac{3}{\varepsilon} \ln \frac{4}{3} - \frac{2}{\varepsilon} \right) N_d^2 g^2 + O(g^3), \quad (17.63)$$

$$Z = 1 - \frac{2}{27} N_d^2 \frac{g^2}{\varepsilon} + O(g^3), \quad (17.64)$$

$$Z_2^{-1} = 1 - N_d \frac{g}{\varepsilon} + \left(-\frac{1}{\varepsilon^2} + \frac{1}{3\varepsilon} + \frac{1}{2\varepsilon} \ln \frac{4}{3} \right) N_d^2 g^2 + O(g^3), \quad (17.65)$$

with

$$N_d = (16\pi)^{\varepsilon/2-1} \Gamma(1 + \varepsilon/2). \quad (17.66)$$

The RG functions follow:

$$\beta(g) = -\varepsilon \left[\frac{d}{dg} \ln(gZ_g Z^{-3/2}) \right]^{-1}, \quad (17.67)$$

$$\eta(g) = \beta(g) \frac{d}{dg} \ln Z(g), \quad (17.68)$$

$$\eta_2(g) = \frac{1}{\nu(g)} - 2 = \frac{d}{dg} \ln(Z_2 Z^{-1}). \quad (17.69)$$

Therefore, setting $\tilde{g} = N_d g$, one finds

$$N_d \beta(\tilde{g}) = -\varepsilon \tilde{g} + 3\tilde{g}^2 - \left(-6 \ln \frac{4}{3} + \frac{34}{9} \right) \tilde{g}^3 + O(\tilde{g}^4), \quad (17.70)$$

$$\eta(\tilde{g}) = \frac{4}{27} \tilde{g}^2 + O(\tilde{g}^3), \quad (17.71)$$

$$\eta_2(\tilde{g}) = -\tilde{g} + \left(\frac{14}{27} + \ln \frac{4}{3} \right) \tilde{g}^2 + O(\tilde{g}^3). \quad (17.72)$$

Scaling behaviour below three dimensions. Dimensional analysis in the critical theory yields

$$\begin{aligned} \tilde{\Gamma}^{(n)}(\lambda \mathbf{p}_\perp, \rho p_z; \tau, g, A, \mu) &= \lambda^{n+(n-2)(1-d)/2} \rho^{(2-n)/2} \\ &\times \tilde{\Gamma}^{(n)}(\mathbf{p}_\perp, p_z; \tau/\lambda^2, g, A\rho/\lambda^2, \mu/\lambda). \end{aligned} \quad (17.73)$$

In $d = 3 - \varepsilon$ dimensions, the model has an IR fixed point $g^*(\varepsilon)$. At the fixed point, one finds

$$\tilde{\Gamma}^{(n)}(\mathbf{p}_\perp, p_z, \tau, A = \mu = 1) = \tau^{\gamma - (n-2)d_\phi} \tilde{\Gamma}^{(n)}(\mathbf{p}_\perp \tau^\nu, p_z / \tau^{\nu(2-\eta/2)}), \quad (17.74)$$

with

$$\gamma = \nu(2 - \eta), \quad d_\phi = \frac{1}{2}(d - 1 + \eta).$$

At two-loop order, the fixed point coupling constant and the critical exponents are

$$\tilde{g}^*(\varepsilon) = \frac{\varepsilon}{3} + \left(\frac{2}{9} \ln \frac{4}{3} + \frac{34}{243} \right) \varepsilon^2 + O(\varepsilon^3), \quad (17.75)$$

$$\eta = \frac{4}{243} \varepsilon^2 + O(\varepsilon^3), \quad (17.76)$$

$$\nu^{-1} = 2 - \frac{\varepsilon}{3} - \left(\frac{1}{9} \ln \frac{4}{3} + \frac{20}{243} \right) \varepsilon^2 + O(\varepsilon^3), \quad (17.77)$$

$$\omega = \varepsilon - \left(\frac{2}{3} \ln \frac{4}{3} + \frac{34}{81} \right) \varepsilon^2 + O(\varepsilon^3). \quad (17.78)$$

Logarithmic corrections to mean-field behaviour in three dimensions. In three dimensions, the RG equations can be solved by the method indicated in Section 17.2.

For the effective coupling constant at scale λ , one finds

$$\frac{1}{g(\lambda)} \underset{\lambda \rightarrow 0}{\sim} \frac{3}{16\pi} \ln \frac{1}{\lambda} \left[1 - 2 \frac{(17 + 27 \ln 4/3)}{81} \frac{\ln |\ln \lambda|}{|\ln \lambda|} + O\left(\frac{1}{\ln \lambda}\right) \right]. \quad (17.79)$$

A short calculation then yields, for example, the susceptibility in zero field,

$$\chi^{-1} \sim C_{\pm} |\ln \tau|^{-1/3} \left[1 + \frac{1}{243} (108 \ln(4/3) + 41) \frac{\ln |\ln \tau|}{|\ln \tau|} + O\left(\frac{1}{|\ln \tau|}\right) \right], \quad (17.80)$$

or the specific heat,

$$C = A_{\pm} |\ln \tau|^{1/3} \left[1 - \frac{1}{243} (108 \ln(4/3) + 41) \frac{\ln |\ln \tau|}{|\ln \tau|} + O\left(\frac{1}{|\ln \tau|}\right) \right], \quad (17.81)$$

with the universal ratio

$$\frac{A_+}{A_-} = \frac{1}{4}. \quad (17.82)$$

The specific heat has been measured in a high precision experiment on the dipolar Ising ferromagnet LiTbF₄ by Ahlers *et al* [176]. Fitting the specific heat by

$$\begin{aligned} C_+ &= \frac{A_+}{b^z} \{ [1 + b \ln(a/|\tau|)]^z - 1 \} + B, \\ C_- &= \frac{A_-}{b^{z'}} \left\{ [1 + b \ln(a/|\tau|)]^{z'} - \frac{1}{4} \right\} + B, \end{aligned}$$

they find

$$\begin{aligned} \frac{A_+}{A_-} &= 0.244 \pm 0.009, \\ z = z' &= 0.336 \pm 0.024, \end{aligned}$$

results which agree nicely with the theoretical predictions.