6

Composite operators

In Chapter 2 we met a number of equations involving products of field operators at the same point. Examples are given by the equations of motion (2.1.10) and the Ward identities (2.7.6). These products we will call composite operators. When computed directly they have ultra-violet divergences: the product $\phi(x)\phi(y)$ makes unambiguous sense if x is not equal to y, but if x equals y then we have $\phi(x)^2$, which diverges. Since the equations of motion and the Ward identities express fundamental properties of the theory, it is useful to construct finite, renormalized composite operators with which to express these same properties.

It could be argued that there is no need to have renormalized equations of motion. One could say that one only actually needs the equations of motion in the regulated theory, where they are finite. A situation of practical importance where we actually do need renormalized composite operators is the operator-product expansion, to be discussed in Chapter 10. This is used in a phenomenological situation such as deep-inelastic scattering (Chapter 14) where we wish to compute the behavior of a Green's function when some of its external momenta get large. Equivalently, we need to know how a product of operators, like $\phi(x)\phi(y)$, behaves as $x \to y$.

This information is contained in the operator-product expansion of Wilson (1969) which has the form

$$\phi(x)\phi(y) \sim C_1(x-y)1 + C_{\phi^2}(x-y)[\phi(y)^2] + \cdots . \tag{6.0.1}$$

Here the symbol $[A(x)]$ denotes the renormalized operator corresponding to an unrenormalized composite operator $A(x)$. The coefficients $C(x-y)$ are c-numbers, and each has a subscript which labels the operator that it multiplies.

Therefore in this chapter we show how to renormalize Green's functions of composite operators, e.g.,

$$\begin{aligned}
&\langle 0|T\phi(x)\phi(y)\phi^2(z)|0\rangle, \\
&\langle 0|T\phi(w)\phi(x)\phi(y)^2\phi(z)^2|0\rangle, \\
&\langle 0|T\phi(x)^2\phi(y)^2|0\rangle.
\end{aligned} \tag{6.0.2}$$

We will first motivate the use of composite operators by seeing how the operator-product expansion arises in a low-order graph. Then we will examine the divergences that appear in low-order graphs for composite operators. We will see that we must expect multiplicative renormalization:

$$[\phi^2] = Z_{\phi^2}\phi^2, \tag{6.0.3}$$

where $[\phi^2]$ is finite as the UV cut-off is removed, while Z_{ϕ^2} is a divergent renormalization factor. The unrenormalized operator $\phi(x)^2$ is divergent when the cut-off is removed.

These examples will provide motivation to define renormalized composite operators by application to Feynman graphs of the same R-operation that we defined in Chapter 5. After discussion of a number of technical issues, we will derive some basic properties of the renormalized operators, including the equations of motion and the Ward identities.

6.1 Operator-product expansion

We will postpone a complete treatment of the operator-product expansion to Chapter 10. Here we merely wish to motivate our definition of composite operators with an example of their use.

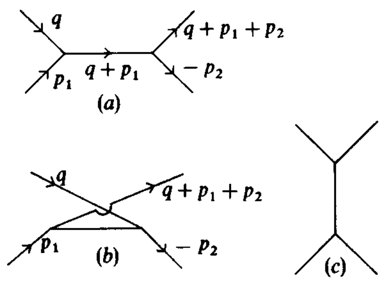


Fig. 6.1.1. Take $q \to \infty$ in these graphs to obtain the lowest-order example of the operator-product expansion.

Consider the graphs of Fig. 6.1.1 for the four-point function in ϕ^3 theory. We let q^μ go to infinity with p_1 and p_2 fixed, and with the ratios of the components of q fixed. Then $|q^2| \to \infty$. We expand the graphs in powers of q^2 to find:

$$\text{Fig. 6.1.1}(a) + (b) \sim \left[\frac{\mathrm{i}}{(p_1^2 - m^2)}\frac{\mathrm{i}}{(p_2^2 - m^2)}\right]\frac{2\mathrm{i}g^2}{(q^2)^3}, \tag{6.1.1a}$$

$$\text{Fig. 6.1.1}(c) \sim \left[\frac{-g^2}{(p_1^2-m^2)(p_2^2-m^2)((p_1+p_2)^2-m^2)}\right]\frac{\mathrm{i}g}{(q^2)^2}\times$$
$$\times\left[1-\frac{2q\cdot(p_1+p_2)}{q^2}+\frac{(2m^2-(p_1+p_2)^2)}{q^2}+\frac{4q\cdot(p_1+p_2)^2}{(q^2)^2}\right]. \tag{6.1.1b}$$

In each term the dependence of p_1 and p_2 has factorized. We now show that this is a case of an operator-product expansion like (6.0.1) (after Fourier transformation into momentum space).

In (6.1.1a) the factor in square brackets is in fact the value of the lowest-order graph for the following Green's function:

$$\langle 0|T\tilde{\phi}(-p_1)\tilde{\phi}(-p_2)\phi^2(0)/2|0\rangle$$
$$=\int \mathrm{d}^4x\int \mathrm{d}^4y \exp(-\mathrm{i}p_1\cdot x-\mathrm{i}p_2\cdot y)\langle 0|T\phi(x)\phi(y)\phi^2(0)/2|0\rangle. \tag{6.1.2}$$

We have not Fourier transformed the $\phi^2(0)$ operator, but have set it at the origin. If we had made the Fourier transform, then we would merely pick up a momentum-conservation δ-function, which we do not have in (6.1.1a).

To understand the appearance of the operator $\phi^2(0)/2$ in (6.1.2), we may find a functional-integral formula for the Green's functions that appear in this equation. Since $\phi^2(0)$ means the product of two fields at the same space-time point, such a formula follows from our work in Section 2.2. It is

$$\langle 0|T\phi(x)\phi(y)\phi^2(0)/2|0\rangle = \mathcal{N}\int[\mathrm{d}A]A(x)A(y)\tfrac{1}{2}A^2(0)\mathrm{e}^{\mathrm{i}S}. \tag{6.1.3}$$

The Feynman rules for this Green's function can then be derived. They are the usual ones for the Green's function $\langle 0|T\phi(x)\phi(y)|0\rangle$ with the addition that each graph contains exactly one special vertex for the $\phi^2(0)/2$ operator. The lowest-order graph is shown in Fig. 6.1.2, where the special $\phi^2/2$ vertex is indicated by a cross. The value of the vertex is unity, for the explicit factor 1/2 in $\phi^2/2$ gets cancelled. This happens in exactly the same way as the 1/4! in the ϕ^4 interaction or the 1/3! in the ϕ^3 interaction gets cancelled to leave a value $-\mathrm{i}g$ for an interaction vertex.

The operator $\phi^2(0)/2$ is our first example of a composite operator (or composite field). By this term we mean, in general, a product of elementary

Fig. 6.1.2. Lowest-order graph for two-point function of ϕ^2.

fields (or their derivatives) at the same point. It is the properties of such operators that we will investigate in this chapter.

We can write

$$\text{Fig. 6.1.1}(a) + (b) \sim \frac{2ig^2}{(q^2)^3} \langle 0 | T\tilde{\phi}(p_1)\tilde{\phi}(p_2)\phi(0)^2/2 | 0 \rangle, \qquad (6.1.4a)$$

This is illustrated in diagrams by Fig. 6.1.3.

In similar fashion, we derive an operator formula for (6.1.1*b*):

$$\text{Fig. 6.1.1}(c) \sim \frac{ig^2}{(q^2)^2} \left[1 - \frac{2iq^\mu}{q^2} \frac{\partial}{\partial x^\mu} + \frac{(2m^2 + \Box)}{q^2} - \frac{4q^\mu q^\nu \partial_\mu \partial_\nu}{q^4} \right] \times$$
$$\times \langle 0 | T\tilde{\phi}(p_1)\tilde{\phi}(p_2)\phi(x) | 0 \rangle |_{x=0}. \qquad (6.1.4b)$$

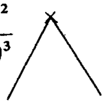
$(a) + (b) \sim \dfrac{2ig^2}{(q^2)^3}$ 

Fig. 6.1.3. Generation of terms in the operator-product expansion from the graphs of Fig. 6.1.1(*a*) and (*b*).

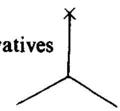
$(c) \sim \dfrac{ig^2}{(q^2)^2} \sum$ coefficients $\times$ derivatives 

Fig. 6.1.4. Generation of terms in the operator-product expansion from the graph of Fig. 6.1.1(*c*)

Here the form of the first square-bracket factor means that we need an elementary field $\phi(x)$, rather than a composite field. The p_1's and p_2's in the numerators in the second square-bracket factor have turned themselves into derivatives with respect to x; we set $x = 0$ at the end. Equation (6.1.4*b*) is illustrated in Fig. 6.1.4.

The form of (6.1.4) suggests the following formula:

$$\langle 0 | T\tilde{\phi}(q)\phi(0)\tilde{\phi}(p_1)\tilde{\phi}(p_2) | 0 \rangle$$
$$\sim \sum_i C_i(q) \langle 0 | T\mathcal{O}_i(0)\tilde{\phi}(p_1)\tilde{\phi}(p_2) | 0 \rangle. \qquad (6.1.5)$$

The sum is over a set of local operators $\mathcal{O}_i$. Each of these is either the elementary field ϕ, one of its derivatives, or a composite operator such as ϕ^2. The coefficients $C_i(q)$ are called the Wilson coefficients. In Chapter 10 we will generalize this result to all orders. We will have an expansion for any Green's function with large momentum on some of its external lines.

Now higher-order corrections to the Green's functions of the composite operators such as the $\phi^2(0)$ that appears in (6.1.4*a*) have ultra-violet divergences beyond those appearing in Green's functions of elementary fields. We will see this in the next section, Section 6.2. To obtain an operator product expansion, like (6.1.5), with finite coefficients, we will need to renormalize the composite operators. This particular problem will occupy most of this chapter.

6.2 Renormalization of composite operators: examples

6.2.1 *Renormalization of* ϕ^2

The Feynman rules for Green's functions of unrenormalized composite operators can be derived from the functional integral, in the presence of an ultra-violet cut-off. In coordinate space they are the usual rules, modified only by having several external fields at the same point. For example, we consider

$$\langle 0|T\phi(x)\phi(y)\phi^2(z)/2|0\rangle \tag{6.2.1}$$

in ϕ^3 theory in six-dimensional space-time. The connected graphs up to order g^2 are shown in Fig. 6.2.1. As before, the vertex for $\phi^2/2$ is denoted by a cross.

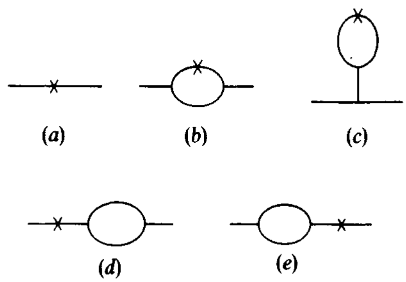


Fig. 6.2.1. Renormalization of the operator ϕ^2.

To work in momentum space, we Fourier transform, as usual, and define

$$\begin{aligned} G &= \langle 0|T\tilde{\phi}(p)\tilde{\phi}(q)\phi(z)^2/2|0\rangle \\ &= \int d^d x d^d y \exp[i(p_1\cdot x + p_2\cdot y]\langle 0|T\phi(x)\phi(y)\phi(z)^2/2|0\rangle \end{aligned} \tag{6.2.2}$$

The lowest-order graph, Fig. 6.2.1(*a*), is equal to

$$G_a = \frac{i}{(p_1^2 - m^2 + i\varepsilon)}\frac{i}{(p_2^2 - m^2 + i\varepsilon)}. \tag{6.2.3}$$

Observe that the factor 1/2 in the operator $\phi^2/2$ is cancelled, just like the 1/3! that comes with the interaction vertices.

Let us now turn to the one-loop graphs of Fig. 6.2.1. They are all divergent: Fig. 6.2.1(*b*) is logarithmically divergent, while the remaining graphs, Fig. 6.2.1(*c*) to (*e*), are quadratically divergent (all at $d=6$). The divergences in the last two graphs, Figs. 6.2.1(*d*) and (*e*), involve self-energy corrections only, so these divergences are cancelled by the usual wave-

(a) (b)

Fig. 6.2.2. Counterterm graphs for Fig. 6.2.1(d) and (e).

function and mass counterterms, Fig. 6.2.2. In these and other graphs in this chapter, we indicate counterterms by a heavy dot and an insertion of a composite operator by a cross.

The remaining two graphs have no counterterm from the interaction, and they are both divergent. For Fig. 6.2.1(b) we get

$$
\begin{aligned}
G_b &= \frac{\mathrm{i}}{(p_1^2 - m^2 + \mathrm{i}\varepsilon)}\frac{\mathrm{i}}{(p_2^2 - m^2 + \mathrm{i}\varepsilon)} \times \\
&\times \left\{ \frac{\mathrm{i}g^2\mu^{6-d}}{(2\pi)^d} \int \mathrm{d}^d k \frac{1}{(k^2 - m^2 + \mathrm{i}\varepsilon)[(k-p_1)^2 - m^2 + \mathrm{i}\varepsilon][(k+p_2)^2 - m^2 + \mathrm{i}\varepsilon]} \right\} \\
&= \frac{\mathrm{i}}{(p_1^2 - m^2)}\frac{\mathrm{i}}{(p_2^2 - m^2)}\frac{g^2}{64\pi^3}\Gamma(3 - d/2) \times \int_0^1 \mathrm{d}x \times \\
&\times \int_0^{1-x} \mathrm{d}y \left[\frac{m^2 - p_1^2 y(1-x-y) - p_2^2 x(1-x-y) - (p_1+p_2)^2 xy}{4\pi\mu^2} \right]^{d/2-3}
\end{aligned}
\tag{6.2.4}
$$

and for Fig. 6.2.1 (c) we get

$$
\begin{aligned}
G_c &= \frac{-g\mu^{3-d/2}}{(p_1^2 - m^2)(p_2^2 - m^2)[(p_1+p_2)^2 - m^2]} \times \\
&\quad \times \frac{\mathrm{i}g\mu^{3-d/2}}{2(2\pi)^d} \int \mathrm{d}^d k \frac{1}{(k^2 - m^2)[(p_1 + p_2 + k)^2 - m^2]} \\
&= \frac{-g\mu^{3-d/2}}{(p_1^2 - m^2)(p_2^2 - m^2)[(p_1+p_2)^2 - m^2]} \times \\
&\quad \times \frac{-g\mu^{d/2-3}}{128\pi^3}\Gamma(2 - d/2) \int_0^1 \mathrm{d}x \frac{[m^2 - (p_1+p_2)^2 x(1-x)]^{d/2-2}}{(4\pi\mu^2)^{d/2-3}}.
\end{aligned}
\tag{6.2.5}
$$

Note that there is a symmetry factor 1/2 in this last equation. The fact that the sum of (6.2.4) and (6.2.5) diverges means that the operator $\phi^2(0)$ is not finite.

For use in the operator-product expansion we do not need precisely the operator ϕ^2. Rather, we need some local operator similar to ϕ^2 that is finite. This indicates that we should define a renormalized operator by subtraction of the divergences. Let us agree to use minimal subtraction. Then the counterterm graphs are obtained by replacing each divergent loop by

Fig. 6.2.3. Counterterm graphs for Fig. 6.2.1(*b*) and (*c*).

minus its pole part, as illustrated in Fig. 6.2.3. Thus the counterterm graph for Fig. 6.2.1(*b*) is

$$\frac{\mathrm{i}^2}{(p_1^2-m^2)(p_2^2-m^2)}\frac{g^2}{64\pi^3(d-6)}, \tag{6.2.6}$$

and the counterterm graph for Fig. 6.2.1(*c*) is

$$\frac{-g\mu^{3-d/2}}{(p_1^2-m^2)(p_2^2-m^2)[(p_1+p_2)^2-m^2]}\left\{\frac{g\mu^{d/2-3}}{64\pi^3(d-6)}\left[m^2-\tfrac{1}{6}(p_1+p_2)^2\right]\right\}. \tag{6.2.7}$$

The positioning of the factors of μ is such that the counterterm in curly brackets has exactly the same dimension as the loop to which it is a counterterm.

We thus find the renormalized values at $d=6$:

$$\begin{aligned} R(G_b) &= \frac{\mathrm{i}^2}{(p_1^2-m^2)(p_2^2-m^2)}\frac{g^2}{64\pi^3}\left\{-\frac{\gamma}{2}-\int_0^1 \mathrm{d}x\,\times\right. \\ &\quad \left.\times\int_0^{1-x}\mathrm{d}y\ln\left[\frac{m^2-(p_1^2y+p_2^2x)(1-x-y)-(p_1+p_2)^2xy}{4\pi\mu^2}\right]\right\}, \end{aligned} \tag{6.2.8}$$

$$\begin{aligned} R(G_c) &= \frac{-g}{(p_1^2-m^2)(p_2^2-m^2)[(p_1+p_2)^2-m^2]}\left(\frac{-g}{128\pi^3}\right)\times \\ &\quad \times\left\{(\gamma-1)\left[m^2-\tfrac{1}{6}(p_1+p_2)^2\right]+\int_0^1\mathrm{d}x\left[m^2-(p_1+p_2)^2x(1-x)\right]\times\right. \\ &\quad \left.\times\ln\left[\frac{m^2-(p_1+p_2)^2x(1-x)}{4\pi\mu^2}\right]\right\}. \end{aligned} \tag{6.2.9}$$

To interpret these renormalizations we observe that the counterterms are vertices for $\phi^2(0)$ in (6.2.6) and for $(m^2+\Box/6)\phi$ in (6.2.7). Thus

$$\begin{aligned} &G_a+R(G_b)+R(G_c)+R(G_d)+R(G_e) \\ &=\left[1+\frac{g^2}{64\pi^3(d-6)}\right]\langle 0|T\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tfrac{1}{2}\phi^2(0)|0\rangle \\ &\quad+\frac{g\mu^{d/2-3}}{64\pi^3(d-6)}\langle 0|T\tilde{\phi}(p_1)\tilde{\phi}(p_2)(m^2+\tfrac{1}{6}\Box)\phi(0)|0\rangle+O(g^4). \end{aligned} \tag{6.2.10}$$

So what we are computing is a Green's function of the operator

$$\tfrac{1}{2}[\phi^2] \equiv \left[1 + \frac{g^2}{64\pi^3(d-6)}\right]\tfrac{1}{2}\phi^2 + \frac{g\mu^{d/2-3}}{64\pi^3(d-6)}(m^2 + \tfrac{1}{6}\Box)\phi + \text{higher order.} \tag{6.2.11}$$

We use the square brackets on the left-hand side to denote a renormalized operator. In subsequent sections we will see that this result generalizes to all orders: a renormalized operator $[\phi^2]$ can be defined to all orders by a formula of the form:

$$\tfrac{1}{2}[\phi^2] = Z_a \tfrac{1}{2}\phi^2 + \mu^{d/2-3} Z_b m^2 \phi + \mu^{d/2-3} Z_c \Box\phi, \tag{6.2.12}$$

where the Z's depend only on g and d.

Such renormalized operators were defined by Zimmermann (1973a). He called them normal products and used the notation $N[\phi^2]$. His definition differed from ours only in that he used BPHZ renormalization instead of minimal subtraction.

Observe that in order to obtain a degree of divergence of at least zero, the operators that appear as counterterms in (6.2.11) or (6.2.12) have dimension less than or equal to that of the original operator ϕ^2. This is a general phenomenon. Moreover, the only operators of such dimension are those actually appearing in (6.2.12). We may write the renormalization as a matrix equation in the following form:

$$\begin{pmatrix} \tfrac{1}{2}[\phi^2] \\ \phi \\ \Box\phi \end{pmatrix} = \begin{pmatrix} Z_a & \mu^{d/2-3} Z_b m^2 & \mu^{d/2-3} Z_c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tfrac{1}{2}\phi^2 \\ \phi \\ \Box\phi \end{pmatrix}. \tag{6.2.13}$$

Here we have used the fact that ϕ and $\Box\phi$ are finite. The operators ϕ and $\Box\phi$ are said to mix with ϕ^2 under renormalization, because the off-diagonal elements Z_b and Z_c are non-zero. Moreover, no further operators are needed in the renormalizations, so ϕ^2, ϕ and $\Box\phi$ are said to form a closed set under renormalization.

6.2.2 *Renormalization of* $\phi^2(x)\phi^2(y)$

Sometimes we need Green's functions involving two or more composite operators. A simple example is

$$\begin{aligned} &\langle 0| T\tfrac{1}{2}[\tilde{\phi}^2](p)\tfrac{1}{2}[\phi^2](0)|0\rangle \\ &= \int \mathrm{d}^d x\, e^{-ip\cdot x} \langle 0| T\tfrac{1}{2}[\phi^2](x)\tfrac{1}{2}[\phi^2](0)|0\rangle, \end{aligned} \tag{6.2.14}$$

for which the lowest-order graph is Fig. 6.2.4. The renormalizations of

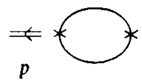


Fig. 6.2.4. Lowest-order graph for $\langle 0|T[\phi^2]/2[\phi^2]/2|0\rangle$.

(6.2.11) do not appear until the next order, so the graph has the value

$$-\frac{1}{2(2\pi)^d}\int d^d k \frac{1}{(k^2-m^2)[(p+k)^2-m^2]}. \tag{6.2.15}$$

This is ultra-violet divergent, even in free-field theory. Even though the operators $\phi^2(x)$ and $\phi^2(0)$ on the right of (6.2.14) are well-defined, we have to integrate through the point $x=0$. At $x=0$ there is a singularity, which is not integrable if $d \geq 4$. We may nevertheless define a finite Green's function by adding a local counterterm:

$$\begin{aligned}\langle 0|T\tfrac{1}{2}[\phi^2](x)\tfrac{1}{2}[\phi^2](0)|0\rangle_R = {} & \langle 0|T\tfrac{1}{2}[\phi^2](x)\tfrac{1}{2}[\phi^2](0)|0\rangle \\ & - C(x)\langle 0|1|0\rangle. \end{aligned}\tag{6.2.16}$$

with

$$C(x) = (m^2 + \Box/6)\delta^{(d)}(x)\frac{(-i\mu^{d-6})}{64\pi^3(d-6)} + O(g^2). \tag{6.2.17}$$

Once more we have used minimal subtraction at $d=6$.

6.3 Definitions

We define renormalized Green's functions of composite operators by applying the R-operation to the Feynman graphs, just as we did for Green's functions of elementary fields in Sections 5.3 and 5.5. We will need to show that the counterterms generate multiplicative renormalizations of the operators (e.g., (6.2.12)). This is similar to what we did in Section 5.6, where we showed that the counterterms in elementary Green's functions are generated by counterterms in the Lagrangian. Our motivation for starting with the graph-by-graph renormalization is again to allow a simple treatment of the problems of subdivergences. We do not need a new proof that the counterterms for operator insertions are local; our original proof suffices.

As before, we have a choice of many renormalization prescriptions. The ones that are most useful for subsequent developments are the BPHZ scheme – see Zimmermann (1973a) – and minimal subtraction – see Breitenlohner & Maison (1977a, b, c) and Collins (1975b). In any case we have a subtraction operator $T(G)$ which is applied to a graph G (after removal of subdivergences) in order to extract the divergent part of G. In the

BPHZ scheme, $T(G)$ gives the first $\delta(G)$ terms in the Taylor expansion about zero external momentum, where $\delta(G)$ is, as usual, its degree of divergence.

In the case of minimal subtraction we must state how the unit of mass μ is treated. As can be seen from the examples in Section 6.2, we must always arrange to compute the pole part of a quantity whose dimension does not vary with d. So suppose we need $T_{\text{MS}}(G)$ for a 1PI graph G of dimension $\mu^{B(d-d_0)}$. Then we define

$$T = \mu^{B(d-d_0)}\{\text{pole part of } G\mu^{B(d_0-d)}\}, \tag{6.3.1}$$

where, as usual, d_0 is the physical space-time dimension.

The definition of a renormalized Green's function by the R-operation is rather abstract, and we will now show that it amounts to adding counterterm operators. In the one-loop examples of Section 6.2 this was rather obvious. In the general case, we start from the formula for renormalization of an arbitrary Green's function (see (5.3.7)–(5.3.9)):

$$R(G) = G + \sum_{\gamma} C_\gamma(G). \tag{6.3.2}$$

Here, as usual, the sum is over all subgraphs γ that consist of a set of disjoint 1PI subgraphs $\gamma_1, \ldots, \gamma_n$. Each γ_i is replaced by its overall counterterm vertex, generated as in Section 5.3.

Let us distinguish the various γ_i's that occur, according to the number of composite operator insertions that they contain. Consider, for example, Fig. 6.3.1, which illustrates the renormalization of $\langle 0|T\phi\ \phi\ \phi^2/2|0\rangle$ in the ϕ^3 theory in six dimensions. There is a one-loop subgraph γ_a for the three-point function. This is renormalized by its counterterm C_{γ_a} in the Lagrangian. There is also a two-loop subgraph γ_b which contains the composite operator vertex. (This has a subdivergence, which must be subtracted.) The counterterm C_{γ_b} can be considered as

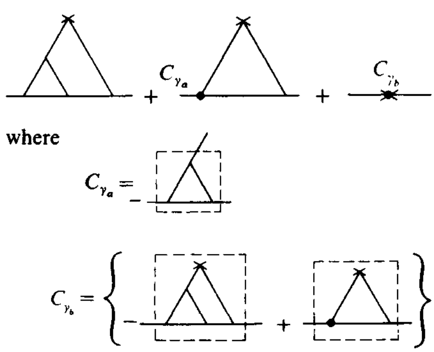


Fig. 6.3.1. Renormalization of two-loop graph with insertion of composite operator.

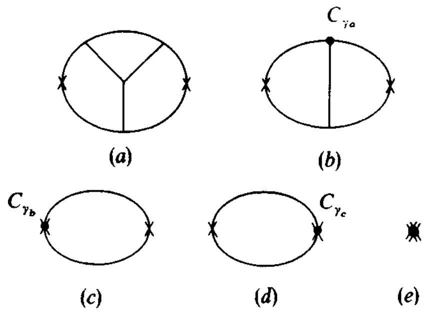


Fig. 6.3.2. Renormalization of three-loop graph for $\langle 0|T[\phi^2]/2[\phi^2]/2|0\rangle$.

generated by an $O(g^4)$ term in the renormalization factor Z_a.

Next consider Fig. 6.3.2, a graph of order g^4 for the Green's function $\langle 0|T[\phi^2]/2[\phi^2]/2|0\rangle$. In addition to an overall logarithmic divergence, it has three divergent subgraphs. One subgraph γ_a is renormalized by a counterterm in the interaction Lagrangian. The other two subgraphs γ_b and γ_c each look like γ_b of Fig. 6.3.1, and are renormalized by the same counterterm. These two counterterms are generated from

$$\langle 0|T\tfrac{1}{2}[\phi^2](x)\tfrac{1}{2}[\phi^2](y)|0\rangle = Z_a^2\langle 0|T\tfrac{1}{2}\phi^2\tfrac{1}{2}\phi^2|0\rangle + \text{other terms from } Z_b \text{ and } Z_c, \quad (6.3.3)$$

by expanding each Z_a to $O(g^4)$ and picking out the terms for γ_b. Finally, the overall counterterm is obtained. It gives a term of $O(g^4)$ in the $C(x)$ of (6.2.16).

These arguments generalize easily to arbitrary graphs. It suffices to consider a renormalized Green's function of one composite operator

$$\langle 0|T\prod_{i=1}^{N}\phi(x_i)[A(y)]|0\rangle. \quad (6.3.4)$$

We define the renormalization by the recursive formula (6.3.2), and we wish to prove that this equals

$$\sum_B Z_{AB}\langle 0|T\prod_{i=1}^{N}\phi(x_i)B(y)|0\rangle. \quad (6.3.5)$$

Here Z_{AB} is a renormalization factor whose value is given by writing, in analogy to (5.6.5),

$$[A] = \sum_B Z_{AB}B = \sum_G \frac{1}{N_G!}D(G). \quad (6.3.6)$$

Here G is any 1PI basic graph that includes a vertex for A. It has N_G external lines in addition to the vertex for A, and $D(G)$ is the operator corresponding to the overall counterterm $C(G)$. The factor $1/N_G!$ is just like the $1/N(G)$ in (5.6.5), to organize the symmetry factors.

Consider a graph for (6.3.4). We investigate one of its counterterms $C_\gamma(G)$. The subgraph γ consists of 1PI graphs $\gamma_1, \ldots, \gamma_n$. A γ_j that does not contain the vertex for A is replaced by $C(\gamma_j)$, which corresponds to one of the counterterms in $\mathscr{L}$. A γ_j which does contain the vertex for A must be one of the G's that are summed over in (6.3.6), so the counterterm C_{γ_j} is generated by one of the counterterm operators on the right of (6.3.6).

Now sum over all graphs G for our Green's function and expand each $R(G)$ by (6.3.2). The sum over 1PI subgraphs γ_j that correspond to counterterms in $\mathscr{L}$ can be done independently of the sum over the subgraphs giving the counterterms for the operator vertex. The result is then the desired result (6.3.5).

6.4 Operator mixing

We have seen that a renormalized composite operator $[A]$ is expressed in terms of unrenormalized operators by

$$[A] = \sum_B Z_{AB} B. \tag{6.4.1}$$

In the case of $[\phi^2]$ we saw that the operators that were needed as counterterms had the same or lower dimension. Let us now demonstrate this for the general case.

The proof is essentially dimensional analysis. Let G be a 1PI graph containing a vertex for A and having the same number N_B of external lines as a particular operator B. Now B is a product of N_B fields with a certain number D_B of derivatives. A counterterm can only be generated if the degree of divergence $\delta(G)$ is at least D_B. Now in a renormalizable theory all couplings have non-negative dimension, so

$$\begin{aligned}\delta(G) &= \dim(G) - \dim(\text{couplings}) \\ &\le \dim(G).\end{aligned} \tag{6.4.2}$$

On the other hand, since $Z_{AB}B$ is a possible counterterm we have

$$\dim(G) = D_B + \dim(Z_{AB}). \tag{6.4.3}$$

But we only need B as a counterterm if $D_B \le \delta(G)$, so

$$\dim(Z_{AB}) \ge 0$$

for every counterterm. This means that the maximum dimension of a counterterm operator is the dimension of A, as we wished to prove.

In Section 5.8.1 we examined the dependence of counterterms in the

Lagrangian on mass parameters. Provided we used minimal subtraction this dependence was polynomial, with the mass behaving as a dimensional coupling in determining allowed counterterms. The same argument applies here. The result is that Z_{AB} is a polynomial in masses (and super-renormalizable couplings) times the inevitable power of the unit of mass μ. The coefficients of the polynomial are dimensionless functions of the dimensionless couplings and of d. A typical example of this is given by our calculation in Section 6.2 of the renormalization of $[\phi^2]$ – see (6.2.11) and (6.2.12).

6.5 Tensors and minimal subtraction

Suppose we use minimal subtraction to define $[(\partial\phi)^2]$ and $[\partial_\mu\phi\partial_\nu\phi]$. It is tempting to suppose that

$$[(\partial\phi)^2] = g^{\mu\nu}[\partial_\mu\phi\partial_\nu\phi]. \tag{6.5.1}$$

This supposition is in fact false, as we will now demonstrate. This means that the taking of a trace does not commute with taking a finite part, in general. We will explain the significance of this fact.

The lowest-order graph (in ϕ^3 theory) for either operator is Fig. 6.5.1. The 1PI part for $\partial\phi^2/2$ before renormalization is

$$\begin{aligned}G(p) &= \frac{ig\mu^{3-d/2}}{2(2\pi)^d}\int d^dk \frac{k\cdot(k+p)}{(k^2-m^2)[(p+k)^2-m^2]} \\ &= \frac{g\mu^{d/2-3}}{128\pi^3}\int_0^1 dx\left[\frac{m^2-p^2x(1-x)}{4\pi\mu^2}\right]^{d/2-3} \times \\ &\quad \times \{\Gamma(2-d/2)p^2x(1-x)[m^2-p^2x(1-x)] \\ &\quad + (d/2)\Gamma(1-d/2)[m^2-p^2x(1-x)]^2\}\end{aligned} \tag{6.5.2}$$

For $\partial_\mu\phi\partial_\nu\phi/2$ we have

$$\begin{aligned}G_{\mu\nu}(p) &= \frac{ig\mu^{3-d/2}}{2(2\pi)^d}\int d^dk \frac{k_\mu(k+p)_\nu}{(k^2-m^2)[(p+k)^2-m^2]} \\ &= \frac{g\mu^{d/2-3}}{128\pi^3}\int_0^1 dx\left[\frac{m^2-p^2x(1-x)}{4\pi\mu^2}\right]^{d/2-3} \times \\ &\quad \times \{\Gamma(2-d/2)p_\mu p_\nu x(1-x)[m^2-p^2x(1-x)] \\ &\quad + \tfrac{1}{2}g_{\mu\nu}\Gamma(1-d/2)[m^2-p^2x(1-x)]^2\}.\end{aligned} \tag{6.5.3}$$

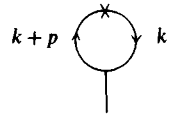


Fig. 6.5.1. One-loop graph for the operators in (6.5.1).

Manifestly

$$g^{\mu\nu}G_{\mu\nu} = G \tag{6.5.4}$$

for the unrenormalized Green's functions. This has to be true since $\partial\phi^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$.

We can renormalize by minimal subtraction.

$$R[G(p)] = G(p) - \frac{g\mu^{d/2-3}}{128\pi^3(d-6)}\left[-3m^4 + \tfrac{4}{3}m^2p^2 - \tfrac{1}{6}p^4\right], \tag{6.5.5a}$$

$$R[G_{\mu\nu}(p)] = G_{\mu\nu}(p) - \frac{g\mu^{d/2-3}}{128\pi^3(d-6)}\left[p_\mu p_\nu(\tfrac{1}{3}m^2 - \tfrac{1}{15}p^2) - g_{\mu\nu}(\tfrac{1}{2}m^4 - \tfrac{1}{6}m^2p^2 + \tfrac{1}{60}p^4)\right]. \tag{6.5.5b}$$

Thus

$$R[G(p)] - g^{\mu\nu}R[G_{\mu\nu}(p)] = -\frac{g\mu^{d/2-3}}{128\pi^3}\left[\tfrac{1}{2}m^4 - \tfrac{1}{6}m^2p^2 + \tfrac{1}{60}p^4\right]. \tag{6.5.6}$$

The reason why contraction with $g^{\mu\nu}$ does not commute with the subtraction of the pole is simply that taking the trace introduces d-dependence. Thus:

$$g^{\mu\nu} \text{ pole part of } \frac{g_{\mu\nu}}{d-6} = \frac{d}{d-6},$$

$$\text{pole part of } \frac{g^{\mu\nu}g_{\mu\nu}}{d-6} = \text{pole } \frac{d}{d-6} = \frac{6}{d-6}.$$

We must evidently be careful to specify whether a trace is inside or outside of the renormalization. The need to do this is characteristic of dimensional regularization. Which place to put the trace depends on the problem under consideration.

The problem arises whenever we have to consider a tensor of rank at least 2. (It could also arise in connection with taking a trace of Dirac γ-matrices except that we choose the trace of the unit Dirac matrix to be independent of d.) We have discovered that our renormalized operators do not have all the properties that the bare operators do. The lack of commutativity of the trace and the finite-part operation is related to a physical effect, that there is an anomaly in the Ward identity for scale transformations – see Callan (1970), Symanzik (1970b) and Brown (1980).

If we were to use, say, zero-momentum subtractions (BPH or BPHZ), then the trace and the finite-part operation would commute – as can be checked from our example. So it might appear that zero-momentum subtraction provides a better all-purpose definition of renormalized operators than does minimal subtraction. However, some of the properties we will prove when using minimal subtraction now disappear or become more complicated.

For example, the equations of motion which we will prove in Section 6.6 are only true if the mass terms are oversubtracted. This turns out to prevent some Ward identities from being true when the simplest renormalized operators are used, whereas they are true in their simplest form when using minimal subtraction. We will prove this in Section 6.6 also. The moral is that one cannot completely eliminate the problems.

It is possible (Collins (1975b)) to construct a definition of, say, $[\partial_\mu\phi\partial_\nu\phi/2]$ which uses minimal subtraction and for which $g^{\mu\nu}[\partial_\mu\phi\partial_\nu\phi/2] = [(\partial\phi)^2]$. This is done by writing tensors in terms of Lorentz-irreducible components. Thus we write a second-rank tensor $M_{\mu\nu}$ as the sum of an antisymmetric term, a symmetric traceless term, and an invariant term:

$$M_{\mu\nu} = \tfrac{1}{2}(M_{\mu\nu} - M_{\nu\mu}) + \tfrac{1}{2}(M_{\mu\nu} + M_{\nu\mu} - (2/d)g_{\mu\nu}M^\kappa_\kappa) + (1/d)g_{\mu\nu}M^\kappa_\kappa.$$

The subtraction procedure is applied to each term separately. This definition loses other properties of the renormalized products. For example, conservation of energy and momentum is a consequence of the fact that the bare energy-momentum tensor $\theta_{\mu\nu}$ has zero divergence: $\partial^\mu\theta_{\mu\nu} = 0$. If we define a renormalized energy-momentum tensor $[\theta_{\mu\nu}]$ by our original definition of minimal subtraction, then this is the same as the bare $\theta_{\mu\nu}$ up to allowed redefinitions and it is conserved. But if we construct $[\theta_{\mu\nu}]$ by the procedure just suggested, then it is not conserved.

6.6 Properties

One of our motivations for working out the theory of renormalization of composite operators was that in Chapter 2 we had proved equations of motion and Ward identities. These results involved unrenormalized composite operators. So now that we have defined renormalized composite operators, we must prove the equations of motion and Ward identities expressed in terms of these renormalized operators. This is particularly important for the Ward identities, for these express the symmetry properties of the theory.

In this section we will derive a number of useful properties of the renormalized operators. Some properties will be purely technical, while others will be the actual equations of motion and Ward identities. Our proof will be given for the case that the operators are renormalized by minimal subtraction. A typical proof of some equation starts by observing that the corresponding equation is true for the unrenormalized operators. Renormalization is almost the same procedure applied to both sides of the unrenormalized equation, so the main problem is to find the places where the renormalization procedure is not identical for the two sides.

It is always possible to make the theorems false by changes in the renormalization prescription. The point of using minimal subtraction is that it is a universal prescription that preserves almost all of the desirable properties. (The reason is that it amounts, roughly, to defining each counterterm by the requirement 'remove exactly the singularity'.) These properties are relations between different operators.

The other standard renormalization prescription that preserves most of these relations is the BPH (or BPHZ) method of zero-momentum subtraction. In fact, the proofs were first given using the BPHZ prescription (Zimmermann (1973a, b), Lowenstein (1971) and Lam (1972)). However the use of minimal subtraction is better for gauge theories because of their infrared singularities. The proofs were given in this case by Collins (1975b) and Breitenlohner & Maison (1977a, b, c). All these works are rather technical. However, the basic ideas are simple.

Property 1. *Linearity*:

$$a[A] + b[B] = [aA + bB], \tag{6.6.1a}$$

where a and b are pure numbers, while A and B are composite operators. This equation is to be interpreted as an equation for Green's functions of the operator. That is, if X is any product of renormalized operators (elementary or composite), then

$$a\langle 0|T[A]X|0\rangle + b\langle 0|T[B]X|0\rangle = \langle 0|T[aA + bB]X|0\rangle \tag{6.6.1b}$$

Proof. This property is almost obvious. If A and B have different numbers of external legs (e.g. ϕ^2 and ϕ^4), then there is no simple way of defining the right-hand side of (6.6.1) except as being the left-hand side. But if A and B have the same fields, like $(\partial\phi)^2$ and ϕ^2, then the Feynman graphs for $\langle 0|TAX|0\rangle$, $\langle 0|TBX|0\rangle$ and $\langle 0|T(aA + bB)X|0\rangle$ are the same; the differences are only in the placement of powers of momentum. The equation corresponding to (6.6.1b) is true for the basic graphs (i.e., without counterterms). To obtain the renormalized Green's functions we apply the forest formula to each graph. The terms in the forest formula are the same, since the graphs for the three Green's functions are the same. Then (6.6.1) follows from linearity of the subtraction operators T_γ.

Comments. (1) It is necessary to be pedantic about this proof because: (a) it is a prototype for less trivial cases, and (b) it fails for the case of zero-momentum subtractions. The reason for the failure is that the T_γ operation is then not linear. For example, $[(\partial\phi)^2]$ and $[(\partial\phi)^2 + m^2\phi^2]$ need two extra subtractions compared with $[\phi^2]$, because the degree of divergence is two higher. So we can only have

$$[(\partial\phi)^2]_{\mathrm{BPH}} + m^2[\phi^2]_{\mathrm{BPH}} = [(\partial\phi)^2 + m^2\phi^2]_{\mathrm{BPH}} \tag{6.6.2}$$

if the $[\phi^2]$ operator is oversubtracted.

(2) The coefficients a and b in (6.6.1) must be independent of d, for otherwise taking a pole part is non-linear. Furthermore, we cannot use (6.6.1) to show, for example, $g^{\mu\nu}[\partial_\mu\phi\partial_\nu\phi] = [(\partial\phi)^2]$. As we saw, this equation is in fact false. The proof fails because it can only be applied to the case that we sum over a finite number of operators. It does not automatically apply to infinite summations. However, when we defined dimensional regularization in Chapter 4, we saw that our vectors and tensors have to have infinitely many components.

Property 2. *Differentiation is distributive.* Let A be the composite operator

$$A = \prod_{j=1}^{n} \phi_j(x)$$

where each ϕ_j is an elementary field or one of its derivatives. Then

$$\frac{\partial}{\partial x^\mu}[A] = \left[\frac{\partial A}{\partial x^\mu}\right] = \sum_{i=1}^{n}\left[\frac{\partial\phi_i}{\partial x^\mu}\prod_{j\neq i}\phi_i(x)\right]. \tag{6.6.3a}$$

Again this equation is to be interpreted for Green's functions:

$$\frac{\partial}{\partial x^\mu}\langle 0|T[A(x)]X|0\rangle = \langle 0|T[\partial A/\partial x^\mu]X|0\rangle. \tag{6.6.3b}$$

Proof. Let p^μ be the momentum leaving the Green's function (6.6.3b) at the vertex for A. Then the derivative $\partial/\partial x^\mu$ gives a factor $-ip^\mu$. The point of (6.6.3) is to state that we get the same results whether or not we take p^μ inside the finite-part operation. To prove the equation, it is enough to observe that this statement is true for the basic subtraction operator T_γ.

Comments. (1) We can contract μ with an index in A. Thus we have

$$g^{\mu\nu}\partial_\mu[\phi\partial_\nu\phi] = g^{\mu\nu}[\partial_\mu(\phi\partial_\nu\phi)] = [(\partial\phi)^2] + [\phi\Box\phi]. \tag{6.6.4}$$

Since the overall derivative merely gives a factor $-ip_\mu$ there is no possibility of introducing extra d-dependence by contracting with $g^{\mu\nu}$. This is in contrast to the case considered in Section 6.5.

(2) Note that derivatives are always implicitly taken outside of the time ordering. Thus:

$$\langle 0|T(\phi\partial_\mu\phi)(x)\tfrac{1}{2}\phi^2(y)|0\rangle = \lim_{z\to x}\frac{\partial}{\partial x^\mu}\langle 0|T\phi(z)\phi(x)\tfrac{1}{2}\phi^2(y)|0\rangle. \tag{6.6.5}$$

This gives the simplest Feynman rules in momentum space, with each derivative of a field giving a factor of momentum on the corresponding line. The lowest-order graph for the Fourier transform of (6.6.5) is

$$\int \mathrm{d}^d x \mathrm{d}^d y e^{ip\cdot x + iq\cdot y} \langle 0 | T(\phi \partial_\mu \phi)(x) \tfrac{1}{2}\phi^2(y) | 0 \rangle$$

$$= (2\pi)^d \delta^{(d)}(p+q) \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{ik_\mu}{(k^2 - m^2)[(k-p)^2 - m^2]}. \tag{6.6.6}$$

Property 3. *Simple equation of motion.* Decompose the action into a basic action and a counterterm action:

$$\mathscr{S} = \mathscr{S}_\mathrm{b} + \mathscr{S}_\mathrm{ct}, \tag{6.6.7}$$

just like the decomposition (5.1.1) of the Lagrangian, except that $\mathscr{S}_\mathrm{b}$ includes both the free and interaction terms: $\mathscr{S}_\mathrm{b} = \int \mathrm{d}^d x (\mathscr{L}_0 + \mathscr{L}_\mathrm{b})$. Then define functional derivatives with respect to renormalized fields

$$\mathscr{S}_\phi(x) \equiv \frac{\delta \mathscr{S}}{\delta \phi(x)} = \frac{\partial \mathscr{L}}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathscr{L}}{\partial \partial_\mu \phi(x)}, \tag{6.6.8}$$

$$\mathscr{S}_\phi^{(\mathrm{b})}(x) = \frac{\delta \mathscr{S}_\mathrm{b}}{\delta \phi(x)} = \mathscr{S}_\phi \text{ with counterterms omitted.} \tag{6.6.9}$$

We already know the unrenormalized equation of motion (2.5.5)

$$\langle 0 | T \mathscr{S}_\phi(x) X | 0 \rangle = i \frac{\delta}{\delta \phi(x)} \langle 0 | T X | 0 \rangle. \tag{6.6.10}$$

Now we wish to prove the renormalized equation

$$\langle 0 | T[\mathscr{S}_\phi^\mathrm{b}(x)] X | 0 \rangle = i \frac{\delta}{\delta \phi(x)} \langle 0 | T X | 0 \rangle, \tag{6.6.11}$$

from which follows the operator equation

$$[\mathscr{S}_\phi^\mathrm{b}] = 0. \tag{6.6.12}$$

Comments. (1) The functional derivatives are to be treated in a purely formal sense.

(2) Even though $[\mathscr{S}_\phi(x)]$ is zero as an operator, its Green's functions (6.6.11) are non-zero because we define them by taking derivatives outside the time-ordering. Bringing them inside gives equal-time commutators (Section 2.5).

(3) Signs for fermions are easiest to determine by examining the derivation in Chapter 2.

Example. In the ϕ^3 theory

$$\mathscr{L} = Z(\partial\phi)^2/2 - m_0^2 Z\phi^2/2 - g_0 Z^{3/2}\phi^3/6,$$

we have

$$\mathscr{S}_\phi = -Z\Box\phi - m_0^2 Z\phi - \tfrac{1}{2}g_0 Z^{3/2}\phi^2, \tag{6.6.13}$$

$$[\mathscr{S}_\phi^{\mathrm{b}}] = -\Box\phi - m^2\phi - \tfrac{1}{2}\mu^{3-d/2}g[\phi^2]. \tag{6.6.14}$$

Then cases of (6.6.11) are

$$\begin{aligned}\langle 0|T[\mathscr{S}_\phi^{\mathrm{b}}(x)]\phi(y)|0\rangle &= (-\Box_x - m^2)\langle 0|T\phi(x)\phi(y)|0\rangle \\ &\quad - \tfrac{1}{2}\mu^{3-d/2}g\langle 0|T[\phi^2(x)]\phi(y)|0\rangle \\ &= \mathrm{i}\delta^{(d)}(x-y),\end{aligned} \tag{6.6.15}$$

$$\begin{aligned}&\langle 0|T[\mathscr{S}_\phi^{\mathrm{b}}(x)][\phi^3(y)]\phi(z)\phi(w)|0\rangle_{\mathrm{R}} \\ &\quad = 3\mathrm{i}\delta^{(d)}(x-y)\langle 0|T[\phi^2(y)]\phi(z)\phi(w)|0\rangle \\ &\qquad + \mathrm{i}\delta^{(d)}(x-z)\langle 0|T[\phi^3(y)]\phi(w)|0\rangle \\ &\qquad + \mathrm{i}\delta^{(d)}(x-w)\langle 0|T[\phi^3(y)]\phi(z)|0\rangle.\end{aligned} \tag{6.6.16}$$

Proof. A general proof of (6.6.11) is rather complicated because of the arbitrary number of fields. To show the main points it is sufficient to prove one case, (6.6.15) in ϕ^3 theory. The problem in proving the renormalized equation from the unrenormalized equation is that the Feynman graphs are different for the different terms.

We write

$$\begin{aligned}\mathscr{S}_\phi^{\mathrm{b}} &= \mathscr{S}_{0,\phi} + \mathscr{S}_{\mathrm{int},\phi}^{\mathrm{b}} \\ &= -(\Box + m^2)\phi - \tfrac{1}{2}g\mu^{3-d/2}\phi^2.\end{aligned} \tag{6.6.17}$$

Examples of low-order graphs are given in Figs. 6.6.1 and 6.6.2. The counterterms are those arising from the action.

In momentum space we evidently have

$$\langle 0|T\tilde{\mathscr{S}}_{0,\phi}(p)\tilde{\phi}(q)|0\rangle = (p^2 - m^2)\langle 0|T\tilde{\phi}(p)\tilde{\phi}(q)|0\rangle. \tag{6.6.18}$$

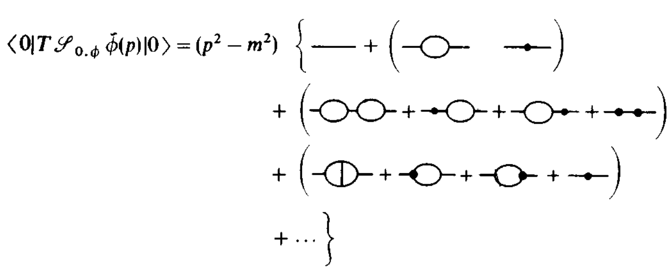


Fig. 6.6.1. Low-order graphs for (6.6.15) with free part of action.

$\langle 0|T[\mathscr{S}^{\mathrm{b}}_{\mathrm{int},\phi}]\tilde{\phi}(p)|0\rangle =$

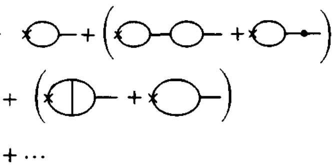


Fig. 6.6.2. Low-order graphs for (6.6.15) with interaction part of action.

The $p^2 - m^2$ multiplying the free propagator attached to $\tilde{\phi}(p)$ cancels the denominator of the propagator. Thus in the graph where the other end of the propagator is $\tilde{\phi}(q)$ we obtain the right-hand side of (6.6.15).

In all the remaining graphs the other end of the propagator is an interaction vertex, either a basic interaction or a counterterm. In fact we obtain

$$\langle 0|\, T\tilde{\mathscr{S}}_{\mathrm{int},\phi}(p)\tilde{\phi}(q)\,|0\rangle \tag{6.6.19}$$

where

$$\begin{aligned}\mathscr{S}_{\mathrm{int},\phi}(x) &= \mathscr{S}_{\phi} - \mathscr{S}_{0,\phi} \\ &= -(Z-1)\Box\phi - (m_0^2 Z - m^2)\phi - \tfrac{1}{2}g_0 Z^{3/2}\phi^2.\end{aligned} \tag{6.6.20}$$

This is exactly what we must obtain in order that the unrenormalized equation of motion (6.6.10) is true. Notice that because (6.6.18) is finite, so is (6.6.19). We must now prove that the counterterms in (6.6.20) are precisely those that are needed to give the operator $[S^{\mathrm{b}}_{\mathrm{int},\phi}] = -\frac{1}{6}g\mu^{3-d/2}[\phi^2]$ renormalized according to our standard prescription for composite operators.

Now, the renormalization prescription is precisely to add to the basic term $S^{\mathrm{b}}_{\mathrm{int},\phi} = -\frac{1}{6}g\mu^{3-d/2}\phi^2$ a series of counterterm operators whose coefficients are pure poles at $d = 6$, so as to make its Green's functions finite. But this is precisely (6.6.20). The relation between the counterterms can be seen from a comparison of Figs. 6.6.1 and 6.6.3.

$\langle 0|T\mathscr{S}^{\mathrm{b}}_{\mathrm{int}\phi}\tilde{\phi}(p)|0\rangle =$

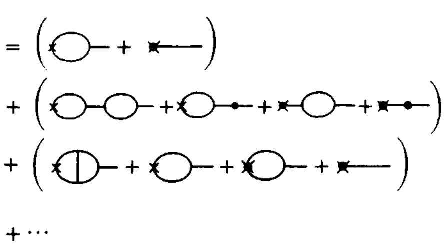


Fig. 6.6.3. Renormalization of Fig. 6.6.2.

Property 4. *Equation of motion times operator.* With the same notation as before we have

$$\langle 0|T[A\mathscr{S}_\phi(x)]X|0\rangle_R = i\langle 0|TA(x)\frac{\delta}{\delta\phi(x)}X|0\rangle_R, \qquad (6.6.21)$$

where A is any product of operators at the same point. Hence

$$[A\mathscr{S}_\phi]=0. \qquad (6.6.22)$$

Comments and examples. (1) This property is crucial to proving Ward identities.

(2) All operators appearing on the right of (6.6.21) are to be renormalized.

(3) In ϕ^3 theory, cases of (6.6.21) are

$$\begin{aligned}&\langle 0|T[-\phi\Box\phi-m^2\phi^2-\tfrac{1}{2}g\phi^3](x)\phi(y)\phi(z)|0\rangle\\&\quad=i\delta(x-y)\langle 0|T\phi(y)\phi(z)|0\rangle+(y\leftrightarrow z),\end{aligned} \qquad (6.6.23)$$

$$\begin{aligned}&\langle 0|T[-\phi^2\Box\phi-m^2\phi^3-\tfrac{1}{2}g\phi^4](x)[\phi^2](y)\phi(z)|0\rangle_R\\&\quad=2i\delta(x-y)\langle 0|T[\phi^3](y)\phi(z)|0\rangle\\&\qquad+i\delta(x-z)\langle 0|T[\phi^2](y)[\phi^2](z)|0\rangle_R.\end{aligned} \qquad (6.6.24)$$

Proof. The unrenormalized version of (6.6.21) follows almost directly from the previous property in its unrenormalized version. This in turn follows from the functional-integral solution of the theory, as shown in Section 2.5. We have

$$\begin{aligned}&\langle 0|TA(x)\mathscr{S}_\phi(x)X|0\rangle\\&\quad=i\langle 0|TA(x)\frac{\delta X}{\delta\phi(x)}|0\rangle+i\langle 0|T\frac{\delta A(x)}{\delta\phi(x)}X|0\rangle.\end{aligned} \qquad (6.6.25)$$

Then we use the fact that in dimensional regularization

$$\delta^{(d)}(0)=\int d^dp\,1=0, \qquad (6.6.26)$$

according to the results in Chapter 4. This enables us to eliminate the $\delta A(x)/\delta\phi(x)$ term.

The renormalized equation of motion (6.6.21) can be proved by generalizing the method for the previous property. It is enough to consider the example (6.6.23). Low-order graphs for the left-hand side of (6.6.23) are shown in Fig. 6.6.4. The $(-\Box-m^2)$ factor in $\mathscr{S}_{0,\phi}$ cancels an attached propagator. If the other end attaches to an external field (viz., $\phi(y)$ or $\phi(z)$), then we have a contribution to the right-hand side. If it attaches to an interaction then the negative of a contribution with $\mathscr{S}_{int}$ is obtained, such as Fig. 6.6.5. Since these manipulations do not change the one-particle-

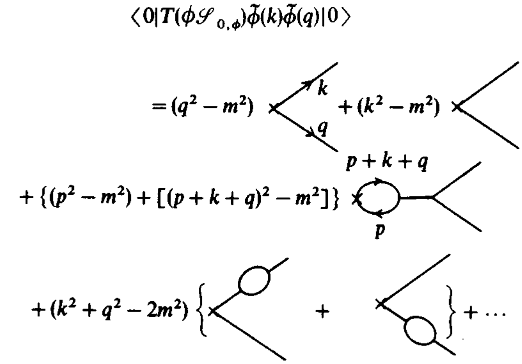


Fig. 6.6.4. Low-order graphs for the left-hand side of (6.6.23), using free action.

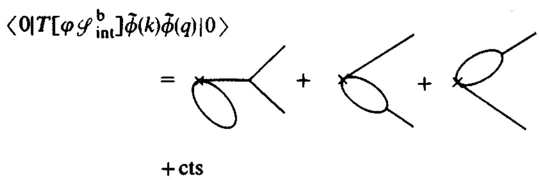


Fig. 6.6.5. Low-order graphs for the left-hand side of (6.6.23), using interaction part of action.

(ir)reducibility structure, renormalization can be performed without changing the result.

Since the $\mathscr{S}_{0,\phi}$ terms need renormalization two subtleties arise:

(1) Since $\square\phi = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, the ambiguity about the placement of $g^{\mu\nu}$ is relevant. To preserve the derivation it must come before the renormalization is performed.
(2) In the BPHZ scheme $m^2\phi^2$ must be oversubtracted otherwise we cannot use linearity to combine $-\phi\square\phi$ and $-m^2\phi^2$.

The case of (6.6.24) involves two further subtleties illustrated by Fig. 6.6.6. In the first graph (a) the $(-\square - m^2)$ multiplies the line coming back to a $\phi(x)$ factor. This term gives zero after use of (6.6.26). The second graph (b) has two 1PI loops separated by a line. In the basic graph, the $q^2 - m^2$ factor cancels the propagator to give graph (c), which has a different reducibility structure. The first two counterterm graphs give the obvious counterterm graphs in Fig. 6.6.6(d). These correspond to the first two counterterms in Fig. 6.6.6(b). But the last counterterm in (b) has two vertices while the corresponding graph (e) has one vertex. Their operator structure is different: graph (e) is another counterterm to renormalize the $x \to y$ singularity of $\phi^4(x)\,\phi^2(y)$. Even so the

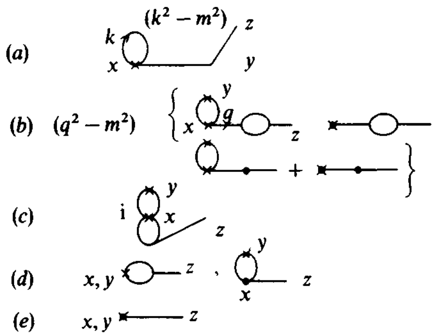


Fig. 6.6.6. Some graphs for (6.6.24).

two counterterm graphs must be equal. They are both a single free propagator times a pole part coefficient times a polynomial in momentum times the same power of the unit of mass. They both make the complete Green's function finite.

The general proof is rather tedious and can be found in Collins (1975b). This proof was given for the minimal subtraction scheme, but also works for the BPH(Z) scheme. In the original BPHZ proof, by Lam (1972), of (6.6.21), there is no treatment of this complication, that the counterterms for the two sides are not in manifest correspondence – i.e., that the forests are different.

Property 5. *Ward identities*: We will use the notation of Section 2.6 for transformations under potential symmetry operations. Let

$$\phi_j \to \phi_j + \delta\phi_j$$

be an infinitesimal transformation of the fields under which the basic Lagrangian transforms as

$$\mathscr{L}_{\text{basic}} \to \mathscr{L}_{\text{basic}} + \Delta_b + \partial_\mu Y^\mu_b.$$

We are restricting our attention to the transformations generated by one particular generator of a group. Thus, as compared to Section 2.6, we now drop the index 'α', which labelled the generators. The subscript 'b' on Δ_b and Y_b indicates that we are considering transformations on the basic Lagrangian (i.e., without counterterms). In the equations below, we will add in the counterterms by use of our standard renormalization scheme. Note also that in setting up the renormalized Green's functions, in Section 2.8, we defined a free Lagrangian $\mathscr{L}_0$ and a basic interaction Lagrangian $\mathscr{L}_b$. We now work with their sum: $\mathscr{L}_{\text{basic}} = \mathscr{L}_0 + \mathscr{L}_b$.

We proved earlier the unrenormalized Ward identity (2.7.6). The renormalized Ward identity is

$$\frac{\partial}{\partial x^\mu}\langle 0|T[j_b^\mu(x)]X|0\rangle_R$$
$$= \langle 0|T[\Delta_b(x)]X|0\rangle_R - i\langle 0|T\delta_b\phi(x)\frac{\delta X}{\delta\phi(x)}|0\rangle_R. \tag{6.6.27}$$

We will have to prove it. From it follows

$$\partial_\mu[j_b^\mu] = [\Delta_b] \tag{6.6.28}$$

and by integration over all x:

$$0 = \int d^4x\langle 0|T[\Delta^b(x)]X|0\rangle_R - i\delta\langle 0|TX|0\rangle_R. \tag{6.6.29}$$

Of course $\Delta = 0$ for a symmetry. The current is

$$[j_b^\mu] = \sum_j\left[\frac{\partial\mathscr{L}_{\text{basic}}}{\partial\partial_\mu\phi_j}\delta_b\phi_j\right] - [Y_b^\mu]. \tag{6.6.30}$$

Proof. In defining Y_b^μ and Δ_b, we have used the basic Lagrangian, i.e., the one with the counterterms omitted. This is because we use the operation symbolized by square brackets to generate the counterterms. The proof of (6.6.27)–(6.6.30) follows the usual proof of Noether's theorem, but using the previously proved properties to write it directly in terms of renormalized operators.

First we use linearity and distributivity to obtain

$$\begin{aligned}-[\Delta_b] + [\partial_\mu j_b^\mu] &= \sum_j \partial_\mu\left[\left(\frac{\partial\mathscr{L}_{\text{basic}}}{\partial\partial_\mu\phi_j}\right)\delta\phi_j - Y_b^\mu\right] + [\partial_\mu Y_b^\mu - \delta\mathscr{L}_{\text{basic}}]\\ &= \sum_j\left[\partial_\mu\frac{\partial\mathscr{L}_{\text{basic}}}{\partial\partial_\mu\phi_j}\delta\phi_j\right] - \sum_j\left[\frac{\partial\mathscr{L}_{\text{basic}}}{\partial\phi_j}\delta\phi_j\right]\\ &= -\sum_j[\delta\phi_j\mathscr{S}_{\phi_j}^b].\end{aligned} \tag{6.6.31}$$

From this the Ward identity (6.6.27) follows by the equation of motion (6.6.22).

We exchanged the order of renormalization and tracing over μ to write $\partial_\mu[j_b^\mu] = [\partial_\mu j_b^\mu]$. This is permitted – see our remarks below (6.6.4).

Comments (1) The theorem appears to give an unrestricted proof of the renormalized Ward identities. This appearance is false, since there are symmetries that can and often do have anomalous breaking – see Chapter 13. Such symmetries are dilatation and conformal symmetries and chiral and supersymmetries. The potential for such anomalies can be seen by computing Δ^b in a regulated theory; it contains a non-zero coefficient which vanishes as $d \to 4$. Such is the case for conformal transformations and for chiral symmetries (where the transformations involve γ_5 or $\varepsilon_{\kappa\lambda\mu\nu}$ explicitly). Minimal subtraction is then not easily applicable and the properties we used in the proof are false.

(2) Corresponding problems appear with any other regulator and with any other renormalization scheme (Piguet & Rouet (1981)). Minimal subtraction confines the problems to cases with anomalies.

Property 6. *Non-renormalization of current*: Consider an exact internal symmetry (such as the symmetry that gives electric charge conservation). Compute the corresponding unrenormalized current j^μ from the complete Lagrangian. Now j^μ contains counterterms derived from the counterterm Lagrangian. We will now prove that these make j^μ finite and that

$$j^\mu = [j^\mu_b]. \tag{6.6.32}$$

Comments This theorem does not apply to space-time symmetries – see Callan, Coleman & Jackiw (1970), Freedman, Muzinich & Weinberg (1974), Collins (1976), Brown & Collins (1980) and Joglekar (1976) for the case of the energy-momentum tensor. It also cannot be extended to the case of a non-conserved current unless the breaking term has dimension below that of $\mathscr{L}$ (Symanzik (1970a)). Furthermore, the proof does not apply directly if the transformation $\delta\phi_j$ is non-linear in ϕ_j. It also needs generalization for gauge theories.

Proof. Both j^μ and $[j^\mu_b]$ consist of the basic current j^μ_b plus some minimal subtraction counterterms. The difference

$$\varepsilon^\mu = j^\mu - [j^\mu_b] \tag{6.6.33}$$

is a series of pure pole terms, and we wish to prove it vanishes. Each term has dimension 3 or less (at $d = 4$), since the currents have dimension 3.

Now both j^μ_b and $[j^\mu_b]$ satisfy the same Ward identity, so

$$\partial_\mu \langle 0 | T\varepsilon^\mu(x) X | 0 \rangle = 0. \tag{6.6.34}$$

Thus $\partial_\mu \varepsilon^\mu = 0$, without use of equations of motion; any need to use the equations of motion would give a non-zero right-hand side to (6.6.34). In the absence of gauge fields, it is impossible to construct such a term. The theorem is thus proved.

In the presence of gauge fields, such terms do exist. For example, in quantum electrodynamics we have a counterterm to the electromagnetic current proportional to $\partial_\nu F^{\mu\nu}$, where $F_{\mu\nu}$ is the field-strength tensor $\partial_\mu A_\nu - \partial_\nu A_\mu$. With non-abelian gauge fields one might have $\varepsilon_\mu \propto \varepsilon_{\kappa\lambda\mu\nu}\partial^\kappa(A_a^\lambda A_b^\nu)$, but the presence of the $\varepsilon_{\kappa\lambda\mu\nu}$ indicates a chiral symmetry, which in any case needs special treatment. Moreover, in a non-abelian theory, we must also take account of the constraints imposed by gauge invariance, which is a subject we will not treat until Chapter 12.

The energy-momentum tensor also has possible counterterms, like $(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\phi^2$ – see the references quoted above.

6.7 Differentiation with respect to parameters in $\mathscr{L}$

Consider Green's function derived from the bare classical Lagrangian

$$\mathscr{L} = Z(\partial A)^2/2 - m_B^2 A^2/2 - g_B A^4/4!$$

by using the functional integral:

$$G_N \equiv \langle 0|T\phi(x_1)\dots\phi(x_N)|0\rangle = \frac{\displaystyle\int[\mathrm{d}A]A(x_1)\dots A(x_N)\mathrm{e}^{\mathrm{i}\int\mathscr{L}}}{\displaystyle\int[\mathrm{d}A]\mathrm{e}^{\mathrm{i}\int\mathscr{L}}}. \tag{6.7.1}$$

Differentiating with respect to g_B gives

$$\begin{aligned}\frac{\partial G_N}{\partial g_B} &= \frac{\displaystyle\int[\mathrm{d}A]A(x_1)\dots A(x_N)\left(\frac{-\mathrm{i}}{4!}\int\mathrm{d}^4y A^4(y)\right)\mathrm{e}^{\mathrm{i}\int\mathscr{L}}}{\displaystyle\int[\mathrm{d}A]\mathrm{e}^{\mathrm{i}\int\mathscr{L}}} \\ &\quad - \frac{\left\{\displaystyle\int[\mathrm{d}A]A(x_1)\dots A(x_N)\mathrm{e}^{\mathrm{i}\int\mathscr{L}}\right\}\left\{\displaystyle\int[\mathrm{d}A]\left(\frac{-\mathrm{i}}{4!}\right)\int\mathrm{d}^4yA^4(y)\mathrm{e}^{\mathrm{i}\int\mathscr{L}}\right\}}{\left\{\displaystyle\int[\mathrm{d}A]\mathrm{e}^{\mathrm{i}\int\mathscr{L}}\right\}^2} \\ &= \int\mathrm{d}^4y\left(\frac{-\mathrm{i}}{4!}\right)\langle 0|T\phi(x_1)\dots\phi(x_N)(\phi^4(y) - \langle 0|\phi^4(y)|0\rangle)|0\rangle \\ &= \mathrm{i}\int\mathrm{d}^4y\langle 0|T\phi(x_1)\dots\phi(x_N)\left(\frac{\partial\mathscr{L}(y)}{\partial g_B} - \langle 0|\frac{\partial\mathscr{L}(y)}{\partial g_B}|0\rangle\right)|0\rangle.\end{aligned} \tag{6.7.2}$$

Similar formulae hold for differentiation with respect to the other parameters Z or m_B^2.

The renormalized equivalents of those equations are also useful. One use will be to show, within perturbation theory, that terms quadratic in the fields can be shifted between free and interaction Lagrangians without affecting the Green's functions.

First consider differentiation with respect to the renormalized coupling, g. We differentiate the renormalized Green's function

$$G_N(x_1,\ldots,x_N)=\sum_{\Gamma}\left[\Gamma+\sum_{\gamma}C_{\gamma}(\Gamma)\right]. \tag{6.7.3}$$

Applied to each basic graph Γ in this formula, the differentiation just gives

$$\frac{-\mathrm{i}}{4!}\int \mathrm{d}^d y\langle 0|T\phi(y)^4\phi(x_1)\ldots\phi(x_N)|0\rangle|_{\text{no counterterms}}. \tag{6.7.4}$$

Renormalization of (6.7.4) produces a set of counterterms isomorphic to those in (6.7.3). So

$$\frac{\partial}{\partial g}G_N=\frac{-\mathrm{i}}{4!}\int \mathrm{d}^d y\langle 0|T\{[\phi(y)^4]-\langle 0|[\phi(y)^4]|0\rangle\}\phi(x_1)\ldots\phi(x_N)|0\rangle. \tag{6.7.5}$$

The subtraction of the vacuum expectation value of $[\phi^4]$ comes about because no vacuum bubbles are used in G_N. Thus each ϕ^4 vertex in $\partial G_N/\partial g$ is connected to some external line.

Suppose we let the basic Lagrangian be

$$\mathscr{L}_{\text{basic}}=z(\partial\phi)^2/2-m^2\phi^2/2-g\phi^4/4! \tag{6.7.6}$$

and let the free and interaction Lagrangians be

$$\mathscr{L}_0=(\partial\phi)^2/2-m_1^2\phi^2/2 \tag{6.7.7a}$$

$$\mathscr{L}_{\mathrm{I}}=(z-1)(\partial\phi)^2/2-(m^2-m_1^2)\phi^2/2-g\phi^4/4!, \tag{6.7.7b}$$

with $m_1^2+m_2^2=m^2$. Notice that we have allowed the $(\partial\phi)^2$ term to have an arbitrary coefficient. We choose to put some of the terms quadratic in ϕ into the interaction, so that we can derive an equation for $\partial G_N/\partial z$ or $\partial G_N/\partial m^2$ like (6.7.5). Then we will show we can move the quadratic terms to the $\mathscr{L}_0$ without changing the Green's functions.

Differentiation with respect to z or m^2 gives

$$\frac{\partial}{\partial z}G_N=\frac{\mathrm{i}}{2}\int \mathrm{d}^d y\langle 0|T\{[(\partial\phi)^2]-\langle 0|[(\partial\phi)^2]|0\rangle\}\phi(x_1)\ldots\phi(x_N)|0\rangle, \tag{6.7.8}$$

$$\frac{\partial}{\partial m^2}G_N=\frac{-\mathrm{i}}{2}\int \mathrm{d}^d y\langle 0|T\{[\phi(y)^2]-\langle 0|[\phi(y)^2]|0\rangle\}\phi(x_1)\ldots\phi(x_N)|0\rangle. \tag{6.7.9}$$

However, we may want to put all of $(\partial\phi)^2 z/2 - m^2\phi^2/2$ into the free Lagrangian. In this case the free propagator is

$$\mathrm{i}/(zp^2 - m^2).$$

We wish to prove that (6.7.8) and (6.7.9) remain valid.

In this case $\partial/\partial m^2$ applied to an unrenormalized graph Γ gives a sum over terms in which each propagator is differentiated

$$\frac{\partial}{\partial m^2}\frac{\mathrm{i}}{zp^2 - m^2} = -\mathrm{i}\left(\frac{\mathrm{i}}{zp^2 - m^2}\right)^2. \tag{6.7.10}$$

This gives us the same result for the unrenormalized graphs as the right-hand side of (6.7.9). Next, we differentiate a counterterm graph $C_\gamma(\Gamma)$. Either a propagator is differentiated, so that the $-\mathrm{i}$ in (6.7.9) gives the basic vertex for $[\phi^2(y)]$, or a counterterm $C(\gamma_1)$ is differentiated. In this second case there is also a counterterm graph $C_\gamma(\partial\Gamma/\partial m^2)$ with a term $\partial\gamma_1/\partial m^2$

Now

$$C(\partial\gamma_1/\partial m^2) = \partial C(\gamma_1)/\partial m^2 \tag{6.7.11}$$

in the minimal subtraction scheme. (The reason is that both are defined to be pure poles times μ to a power – the same for both graphs.) We thus obtain all the counterterm graph for the right-hand side of (6.7.9).

Similarly (6.7.8) is true if $z(\partial\phi)^2$ is all in the free Lagrangian $\mathscr{L}_0$.

We thus see that, for any renormalized parameter λ in the Lagrangian $\mathscr{L}$, we have

$$\frac{\partial}{\partial\lambda}G_N = \mathrm{i}\int \mathrm{d}^d y \langle 0|T\left\{\left[\frac{\partial\mathscr{L}_{\mathrm{basic}}}{\partial\lambda}(y)\right] - \left\langle 0\left|\left[\frac{\partial\mathscr{L}^{\mathrm{basic}}}{\partial\lambda}(y)\right]\right|0\right\rangle\right\}\phi(x_1)\dots\phi(x_N)|0\rangle \tag{6.7.12}$$

From this result we can see that the Lagrangian (6.7.6) is equivalent to the one with unit kinetic term

$$\mathscr{L}'_{\mathrm{b}} = (\partial\phi')^2/2 - m'^2\phi'^2/2 - g'\phi'^4/4! \tag{6.7.13}$$

by a scaling of the field, with

$$\phi = z^{-1/2}\phi', \tag{6.7.14a}$$

$$m^2 = zm'^2, \tag{6.7.14b}$$

$$g = z^2 g'. \tag{6.7.14c}$$

The proof is to write (6.7.6) as

$$\mathscr{L}_{\mathrm{basic}} = z(\partial\phi)^2/2 - m'^2 z\phi^2/2 - g'z^2\phi^4/4!. \tag{6.7.15}$$

Then differentiation of a renormalized Green's function G_N of ϕ with respect to z gives

$$\begin{aligned} z\mathrm{d}G_N/\mathrm{d}z &= z\partial G_N/\partial z|_{\text{fixed}\, m,g} \\ &\quad + m^2\partial G_N/\partial m^2|_{\text{fixed}\, z,g} + 2g\partial G_N/\partial g|_{\text{fixed}\, z,m} \\ &= \mathrm{i}\int \mathrm{d}^4 y\langle 0|T\{[z(\partial\phi)^2/2 - m^2\phi^2/2 - 2g\phi^4/4!] \\ &\qquad - \text{vacuum expectation value}\}\ \phi(x_1)\ldots\phi(x_N)|0\rangle \\ &= \frac{\mathrm{i}}{2}\int \mathrm{d}^4 y(\langle 0|T\{[\phi\mathscr{S}^{\mathrm{b}}_{\phi}] - \text{vacuum expectation value}\}\times \\ &\qquad \times \phi(x_1)\ldots\phi(x_N)|0\rangle). \end{aligned} \tag{6.7.16}$$

We now use the equation of motion (6.6.21) with $A = \phi$ to give

$$z\mathrm{d}G_N/\mathrm{d}z = -NG_N/2. \tag{6.7.17}$$

From this it follows that

$$G_N|_{z,g,m^2} = z^{-N/2}G_N|_{z\to 1, g\to g', m^2\to m'^2}$$

i.e.,

$$\langle 0|T\phi(x_1)\ldots\phi(x_N)|0\rangle = z^{-N/2}\langle 0|T\phi'(x_1)\ldots\phi'(x_N)|0\rangle, \tag{6.7.18}$$

exactly as we would expect. The proof is non-trivial only because we are shifting terms between the free and interaction Lagrangians. Thus we must ensure that counterterms do not go astray.

6.8 Relation of renormalizations of ϕ^2 and m^2

Observe that at order g^2 the renormalization factor Z_a for ϕ^2 in (6.2.13) and (6.2.11) is the inverse of the renormalization factor m_B^2/m^2. This relation is true to all orders, as we will now prove. (We are now back in ϕ^3 theory at $d = 6$.)

We use the renormalized formula

$$\begin{aligned} m^2\frac{\partial}{\partial m^2}G_N &= \mathrm{i}\int \mathrm{d}^4 y\langle 0|T\left\{m^2\frac{\partial\mathscr{L}}{\partial m^2} - \text{vacuum expectation value}\right\}\times \\ &\qquad \times\phi(x_1)\ldots\phi(x_N)|0\rangle \\ &= \frac{-\mathrm{i}}{2}m^2\int \mathrm{d}^4 y\langle 0|T\{[\phi^2] - \langle 0|[\phi^2]|0\rangle\}\phi(x_1)\ldots\phi(x_N)|0\rangle. \end{aligned} \tag{6.8.1}$$

But we also have

$$\mathscr{L} = (\partial\phi_0)^2/2 - m^2 Z_m\phi_0^2/2 - g_0\phi_0^3/6, \tag{6.8.2}$$

so that

$$m^2 \partial \mathscr{L}/\partial m^2 = -m^2 Z_m \phi_0^2/2. \tag{6.8.3}$$

Hence

$$\frac{\partial}{\partial m^2} G_N = -\frac{i}{2} m^2 Z_m \int d^d y \langle 0 | T\{\phi_0^2 - \langle 0|\phi_0^2|0\rangle\} \phi(x_1) \ldots \phi(x_N) |0\rangle. \tag{6.8.4}$$

Therefore

$$\int d^d y m^2 [\phi^2] = \int d^d y m_0^2 \phi_0^2, \tag{6.8.5}$$

from which the desired result follows.

Generalizations of this method can be found in Brown (1980) and Brown & Collins (1980).