Chapter 2
Naïve Perturbation Method for Solving Ordinary Differential Equations and Notion of Secular Terms

2.1 Introduction

If a differential equation has a small parameter ϵ, one may be inclined to try to represent the solution by a power series of ϵ by applying the naïve perturbation theory to solve it. However, surprisingly enough, such a simple-minded method often leads to a disastrous result showing an unreasonable divergent behavior after a long time, or in a global domain of the independent variables. One of the typical causes of this undesired behavior is due to secular terms.

In this chapter, we analyze a few simple differential equations that have a small parameter ϵ, and introduce the notion of a secular term, and then demonstrate that secular terms are to appear in general in the naïve perturbation series of solutions of ordinary differential equations.

The presentation of this chapter is quite pedagogical and intended to be an elementary introduction to standard methods for solving linear inhomogeneous ordinary differential equations, say, in the undergraduate level. Indeed we present a detailed account of the Lagrange's method of variation of constants in the appendix of this chapter.

2.2 A Simple Example: Damped Oscillator

Let us first consider the following simple equation

$$m\frac{d^2x}{dt^2} = -kx - \kappa\frac{dx}{dt}. \tag{2.1}$$

This is a Newton equation for a particle with mass m in a harmonic potential

$$U(x) = \frac{1}{2}kx^2 \tag{2.2}$$

giving the mechanical force $-kx$ in accordance with the Hooke's law, and the second term in the right-hand side (r.h.s.) represents the air resistance (or friction) supposed

T. Kunihiro et al., *Geometrical Formulation of Renormalization-Group Method as an Asymptotic Analysis*, Fundamental Theories of Physics 206,
https://doi.org/10.1007/978-981-16-8189-9_2

to be proportional to the velocity

$$v = \frac{dx}{dt} =: \dot{x} \tag{2.3}$$

with a coefficient κ. Defining the angular velocity by

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{2.4}$$

and making a replacement

$$\omega_0 t \ \to \ t, \tag{2.5}$$

Equation (2.1) is converted to a simple form as

$$\ddot{x} \, + \, 2\epsilon\dot{x} \, + \, x = 0, \quad \left(\epsilon := \frac{\kappa}{2m\omega_0} \, > \, 0\right) \tag{2.6}$$

with

$$\ddot{x} := \frac{d^2x}{dt^2}. \tag{2.7}$$

Since Eq. (2.6) is a second-order linear differential equation, there exist two independent solutions [21, 22], which may be obtained by inserting

$$x = \mathrm{e}^{\lambda t} \tag{2.8}$$

into (2.6); λ is found to be

$$\lambda = -\epsilon \pm i\sqrt{1-\epsilon^2} =: \lambda_\pm. \tag{2.9}$$

The general solution $x(t)$ of (2.6) is given by a linear combination of the independent solutions,

$$\mathrm{e}^{\lambda_\pm t} = \mathrm{e}^{-\epsilon t}\mathrm{e}^{\pm i\omega t}, \quad (\omega := \sqrt{1-\epsilon^2}), \tag{2.10}$$

as

$$x(t) = a\mathrm{e}^{-\epsilon t}\mathrm{e}^{i\omega t} + a^*\mathrm{e}^{-\epsilon t}\mathrm{e}^{-i\omega t}, \tag{2.11}$$

where the reality of $x(t)$ has been taken into account.

If we parametrize the coefficient as

$$a = -\frac{i}{2}\bar{A}e^{i\theta} \tag{2.12}$$

we have

$$x(t) = A(t)\sin\phi(t), \quad A(t) := \bar{A}e^{-\epsilon t}, \;\; \phi(t) := \omega t + \bar{\theta}, \tag{2.13}$$

where $\bar{A}$ and $\bar{\theta}$ are both constant real numbers.

If it were not for the resistance, *i.e.*, $\epsilon = 0$, then the amplitude $A(t)$ is time-independent and the angular velocity $\omega = 1$ with the period

$$T = \frac{2\pi}{\omega} = 2\pi. \tag{2.14}$$

In the following, we also use the word "frequency" in place of angular velocity if any misunderstanding will not be expected.

Owing to the resistance proportional to ϵ, the amplitude $A(t)$ decreases exponentially in time and the ω becomes smaller with a longer period.

Although we know the exact solution to (2.6), we shall dare to apply a naïve perturbative expansion [22] assuming that ϵ is small:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \tag{2.15}$$

Inserting (2.15) into (2.6), we have

$$\ddot{x}_0 + x_0 + \epsilon(\ddot{x}_1 + x_1) + \epsilon^2(\ddot{x}_2 + x_2) + \cdots = -2\epsilon(\dot{x}_0 + \epsilon\dot{x}_1 + \cdots). \tag{2.16}$$

Equating the coefficients of ϵ^n $(n = 0, 1, \ldots)$, we have a series of equations as follows,

$$\ddot{x}_0 + x_0 = 0, \quad \ddot{x}_1 + x_1 = -2\dot{x}_0, \quad \ddot{x}_2 + x_2 = -2\dot{x}_1, \tag{2.17}$$

and so on.

If we define the linear operator

$$L = \frac{d^2}{dt^2} + 1, \tag{2.18}$$

all the perturbative equations are expressed as

$$Lx_n = -2\dot{x}_{n-1}, \quad (n = 0, 1, \ldots) \tag{2.19}$$

with $x_{-1} = 0$.

The zeroth order solution may be given by

$$x_0 = \bar{A}\sin(t + \theta) \tag{2.20}$$

with $\bar{A}$ and θ being integral constants. Then the first-order equation has a form of an inhomogeneous equation as

$$\ddot{x}_1 + x_1 = -2\bar{A}\cos(t+\theta) \equiv F(t). \tag{2.21}$$

The inhomogeneous equation (2.21) may be solved using the Lagrange's method of variation of constants [21], an account of which is given below in Sect. 2.5. Using the two independent solutions

$$x^{(1)}(t) = \cos(t+\theta) \quad \text{and} \quad x^{(2)}(t) = \sin(t+\theta) \tag{2.22}$$

to the unperturbed equation,[1] we set

$$x_1 = C_1(t)x^{(1)}(t) + C_2(t)x^{(2)}(t) = C_1(t)\cos(t+\theta) + C_2(t)\sin(t+\theta) \tag{2.23}$$

with a constraint

$$\dot{C}_1 x^{(1)}(t) + \dot{C}_2 x^{(2)}(t) = 0, \tag{2.24}$$

where

$$\dot{C}_i := \frac{dC_i}{dt}, \qquad (i = 1,\ 2). \tag{2.25}$$

Inserting (2.23) into (2.21), we have

$$\dot{C}_1\dot{x}^{(1)}(t) + \dot{C}_2\dot{x}^{(2)}(t) = F(t), \tag{2.26}$$

where (2.24) has been utilized.

The set of Eqs. (2.24) and (2.26) for $\dot{C}_1$ and $\dot{C}_2$ yields

$$\dot{C}_1 = -F(t)x^{(2)}(t)/W(t), \quad \dot{C}_2 = F(t)x^{(1)}(t)/W(t), \tag{2.27}$$

where $W(t)$ is the Wronskian

$$W(t) = \begin{vmatrix} x^{(1)}(t) & x^{(2)}(t) \\ \dot{x}^{(1)}(t) & \dot{x}^{(2)}(t) \end{vmatrix} = \begin{vmatrix} \cos(t+\theta) & \sin(t+\theta) \\ -\sin(t+\theta) & \cos(t+\theta) \end{vmatrix} = 1. \tag{2.28}$$

Thus we have

$$\dot{C}_1 = \bar{A}\sin(2t+2\theta), \quad \dot{C}_2 = -\bar{A}(1+\cos(2t+2\theta)). \tag{2.29}$$

[1] Although the choice $x^{(1)}(t) = \cos t$ and $x^{(2)}(t) = \sin t$ also will do, θ has been introduced for later convenience.

A simple integration leads to

$$C_1 = -\frac{1}{2}\bar{A}\,\cos(2t+2\theta)+\alpha, \quad C_2 = -\bar{A}t - \frac{1}{2}\bar{A}\,\sin(2t+2\theta)+\beta \quad (2.30)$$

with α and β being constants. Thus we have for $x_1(t)$

$$x_1 = -\bar{A}t\sin(t+\theta)+\left(\alpha-\frac{\bar{A}}{2}\right)\cos(t+\theta)+\beta\sin(t+\theta), \quad (2.31)$$

where the last two terms satisfy the unperturbed equation and may be discarded because we are interested in deriving a general solution to (2.6); the prefactors of the two terms are interpreted to be renormalized into $\bar{A}$ and θ in $x_0(t)$. Thus we arrive at

$$x_1 = -\bar{A}t\,\sin(t+\theta), \quad (2.32)$$

which is proportional to time and called a **secular term** [21, 22].

This result can be understood in physical terms: Equation (2.21) is of the same form as that for a forced oscillator with no resistance but with an external force $f_{\mathrm{ex}}(t) = -2\bar{A}\cos(t+\theta)$ having the same frequency as the intrinsic one. Then it is expected that a resonance phenomenon occurs with an ever growing amplitude because of no resistance, which reflects in the appearance of the very secular term.

We can proceed to the second-order equation, which now reads

$$\ddot{x}_2 + x_2 = F_1(t) + F_2(t) \quad (2.33)$$

with

$$F_1(t) := 2\bar{A}\sin(t+\theta), \quad F_2(t) := 2\bar{A}t\,\cos(t+\theta). \quad (2.34)$$

Since (2.33) is a linear equation with $F_{1,2}(t)$ being the inhomogeneous terms, the solution is given as a sum $x_2(t) = x_2^{(1)} + x_2^{(2)}$ of the solutions $x_2^{(i)}$ to the same equations but with $F(t)$ being replaced by $F_i(t)$, respectively:

$$\ddot{x}_2^{(i)} + x_2^{(i)} = F_i(t), \quad (i = 1,\ 2). \quad (2.35)$$

Applying the method of variation of constants as before, we have

$$\begin{aligned} x_2^{(1)}(t) &= -\bar{A}t\,\cos(t+\theta) + \frac{\bar{A}}{2}\sin(t+\theta), \\ x_2^{(2)}(t) &= \frac{\bar{A}}{2}t^2\,\sin(t+\theta) + \frac{\bar{A}}{2}t\,\cos(t+\theta) - \frac{\bar{A}}{4}\sin(t+\theta), \end{aligned} \quad (2.36)$$

where the respective last term may be discarded because it is a solution to the homogeneous equation as was done before and implicitly for other similar terms even in the present case.

Thus summing all the terms obtained so far, we arrive at the perturbative solution to (2.6) in the second order as

$$\begin{aligned} x(t) &\simeq x_0(t) \;+\; \epsilon x_1(t) \;+\; \epsilon^2 x_2(t) \\ &= \bar{A}\sin(t+\theta) - \epsilon \bar{A} t \sin(t+\theta) \\ &\qquad\qquad + \epsilon^2 \frac{\bar{A}}{2}\{t^2 \sin(t+\theta) - t\cos(t+\theta)\} =: \bar{x}(t). \end{aligned} \tag{2.37}$$

Notice the appearance of the secular terms given by polynomials of t that cause the amplitude to ever increase with time contrary to the exact solution (2.13) in which the amplitude decreases in time exponentially. Furthermore we see that the powers of t in the secular terms in the perturbative (would-be) corrections becomes worse in higher orders.

The reason of the appearance of the secular terms can be traced back to the fact that the **zero modes** of the linear operator $L = d^2/dt^2 + 1$ in the unperturbed terms appear as the inhomogeneous term in the perturbative equations.

Nevertheless, it is noteworthy that the secular terms in (2.37) can be absorbed into the amplitude and phase with a weak time dependence of the unperturbed solution up to ϵ^3 as

$$\begin{aligned} \bar{x}(t) &\simeq \bar{A}\left(1 - \epsilon t + \frac{\epsilon^2}{2} t^2\right) \sin\left(\left(1 - \frac{\epsilon^2}{2}\right) t + \theta\right) \\ &\simeq \bar{A} \mathrm{e}^{-\epsilon t} \sin(\sqrt{1-\epsilon^2}\, t + \theta). \end{aligned} \tag{2.38}$$

Thus one sees that the perturbative solution (2.37) actually represents the first few terms of the expanded formula of the exact solution (2.13) with respect to ϵ. Furthermore the amplitude $A(t)$ of the damped oscillator just discussed satisfies the following equation

$$\dot{A} = -\epsilon A, \tag{2.39}$$

which shows that the time variation of A is proportional to ϵ and hence small.

What we have seen are typical phenomena and problems in naïve perturbation method for solving differential equations. When the inhomogeneous terms in the naïve perturbative equations contain zero modes of the linear operator L of the homogeneous equation, the appearance of secular terms are inevitable, which actually may be resummed into nonsingular expressions. Furthermore, it seems that the secular terms may be renormalized into the integral constants in the unperturbed solutions. Then it would be desirable to extract such slow motions and write down explicitly the equations to describe the slow motions.

2.3 Motion of a Particle in an Anharmonic Potential: Duffing Equation

In the previous section, we have dealt with a linear differential equation. In this section and the following one, we shall apply the perturbative expansion to nonlinear equations [21, 22].

Let us consider a particle with a mass m in a potential

$$U(x) = \frac{m\omega^2}{2}x^2 + \frac{k'}{4}x^4. \tag{2.40}$$

The equation of motion (EOM) reads

$$m\ddot{x} = -\frac{dU}{dx} = -m\omega^2 x - k'x^3, \tag{2.41}$$

which is called Duffing equation . It is found that the mechanical energy

$$E = \frac{m}{2}\dot{x}^2 + \frac{x^2}{2} + \frac{\epsilon}{4}x^4 \tag{2.42}$$

is conserved, *i.e.*, time-independent. In fact,

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + \frac{dU}{dx}\dot{x} = \dot{x}\left[m\ddot{x} + \frac{dU}{dx}\right] = 0 \tag{2.43}$$

on account of (2.41). We can give a simple qualitative argument on the behavior of $x(t)$. Because the kinetic energy is semi-positive definite, we have an inequality $\frac{1}{2}m\dot{x}^2 = E - U(x) \geq 0$, leading to

$$\frac{m\omega^2}{2}x^2 + \frac{k'}{4}x^4 \leq E. \tag{2.44}$$

which implies that $|x(t)|$ is bounded for $E > 0$;

$$|x(t)|^2 \leq (m\omega^2/k')\left[\sqrt{1 + Ek'^2/(m\omega)^2} - 1\right]. \tag{2.45}$$

Now making a replacement $\omega t \to t$ and defining

$$\epsilon := \frac{k'}{m\omega^2}, \tag{2.46}$$

Equation (2.41) is cast into the following form,

$$\ddot{x} + x = -\epsilon x^3. \tag{2.47}$$

In the present analysis, we assume that the strength of the anharmonic potential is small, *i.e.*,

$$|\epsilon| < 1. \tag{2.48}$$

2.3.1 Exact Solution of Duffing Equation

Using the constancy of E, the time dependence of $x(t)$ can be exactly obtained in terms of elliptic functions[22]. Firstly, we note that Eq. (2.42) implies that

$$\left(\frac{dx}{dt}\right)^2 = 2E - x^2 - \frac{1}{2}\epsilon x^4, \tag{2.49}$$

which is reduced to a first-order equation as

$$\frac{dx}{dt} = \pm\sqrt{2E - x^2 - \epsilon x^4/2}. \tag{2.50}$$

Let x varies from x_0 to x as time does from t_0 to t. Then Eq. (2.50) is readily integrated out as

$$\int_{t_0}^{t} dt' = \pm \int_{x_0}^{x} \frac{dx'}{\sqrt{2E - x'^2 - \epsilon x'^4/2}}, \tag{2.51}$$

or

$$t - t_0 = \pm \int_{x_0}^{x} \frac{dx'}{\sqrt{2E - x'^2 - \epsilon x'^4/2}}. \tag{2.52}$$

To convert the integrand into a convenient form, we first determine the amplitude A, *i.e.*, the maximum value of x:

$$\frac{\epsilon}{2}A^4 + A^2 - 2E = 0, \tag{2.53}$$

which is solved for A^2 as

$$\begin{aligned} A^2 &= \frac{1}{\epsilon}\left(-1 + \sqrt{1 + 4\epsilon E}\right) \simeq \frac{1}{\epsilon}\left[-1 + \left(1 + 2\epsilon E - \frac{1}{8}16\epsilon^2 E^2 + \cdots\right)\right] \\ &\simeq 2E(1 - \epsilon E), \end{aligned} \tag{2.54}$$

where the expansion formula $\sqrt{1+x} = 1 + x/2 - x^2/8 + \cdots$ has been used in the last equality. Using Eq. (2.53) for A^2, the function in the integrand is rewritten as

$$2E - x^2 - \frac{\epsilon}{2}x^4 = 2E - x^2 - \frac{\epsilon}{2}x^4 - \left(2E - A^2 - \frac{\epsilon}{2}A^4\right)$$
$$= \frac{\epsilon}{2}\left(A^2 - x^2)(x^2 + A^2 + \frac{2}{\epsilon}\right). \tag{2.55}$$

Then changing the integration variable by

$$x' = A\cos\phi', \tag{2.56}$$

Equation (2.52) is converted to a simple form as

$$t - t_0 = \mp\frac{1}{\sqrt{1+\epsilon A^2}}\int_{\phi_0}^{\phi}\frac{d\phi'}{\sqrt{1-k^2\sin^2\phi'}}, \qquad (x_0 = \sin\phi_0), \tag{2.57}$$

where

$$k^2 := \frac{\epsilon A^2/2}{1+\epsilon A^2}. \tag{2.58}$$

The integral in (2.57) is expressed in terms of the incomplete elliptic integral of the first kind

$$F(\phi, k) = \int_0^{\phi}\frac{d\phi'}{\sqrt{1-k^2\sin^2\phi'}} \tag{2.59}$$

with k being the modulus. Then from (2.57), the period T is given by the time needed for the phase to change from 0 to 2π;

$$T = \frac{1}{\sqrt{1+\epsilon A^2}}\int_0^{2\pi}\frac{d\phi'}{\sqrt{1-k^2\sin^2\phi'}} = \frac{1}{\sqrt{1+\epsilon A^2}}4\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$
$$= \frac{1}{\sqrt{1+\epsilon A^2}}4F(\pi/2, k) = \frac{4}{\sqrt{1+\epsilon A^2}}K(k), \tag{2.60}$$

where we have introduced the complete elliptic integral of the first kind

$$K(k) := F(\pi/2, k). \tag{2.61}$$

When ϵ is small, *i.e.*,

$$|\epsilon|A^2 < 1, \tag{2.62}$$

we have an approximate formula for $K(k)$ using the expansion $(1-x)^{-1/2} = 1 + x/2 + 3x^2/8 + \cdots$ as

$$K(k) \simeq \int_0^{\pi/2} d\phi' \left[1 + \frac{k^2}{2}\sin^2\phi' + \frac{3k^4}{8}\sin^4\phi' + \cdots\right]$$
$$= \frac{\pi}{2}[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \cdots], \tag{2.63}$$

where we have used the Wallis's formula with n being a positive integer

$$\int_0^{\pi/2} d\phi \sin^{2n}\phi = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}. \tag{2.64}$$

Thus an approximate formula for $K(k)$ is given as

$$K(k) \simeq \frac{\pi}{2}\left[1 + \frac{\epsilon}{8}A^2\right]. \tag{2.65}$$

Hence

$$T \simeq 2\pi\left(1 - \frac{\epsilon}{2}A^2\right)\left(1 + \frac{\epsilon}{8}A^2\right) \simeq 2\pi\left[1 - \frac{3\epsilon}{8}A^2\right], \tag{2.66}$$

accordingly the frequency is given by

$$\Omega \equiv \frac{2\pi}{T} \simeq \frac{2\pi}{2\pi(1 - \frac{3\epsilon}{8}A^2)} \simeq 1 + \frac{3\epsilon}{8}A^2. \tag{2.67}$$

We see that the anharmonic force makes the frequency larger and the period shorter. This is physically plausible results because for a large amplitude, the nonlinear term in the restoring force $-k'x^3$ would become significant, and a part of which effects may be absorbed into a *renormalization of the spring constant* $m\omega^2$ and hence the frequency.

2.3.2 Naïve Perturbation Theory Applied to Duffing Equation

Although the exact solution is known as has been shown above, we here dare to apply a naïve perturbative expansion

$$x = x_0 + \epsilon x_1 + \epsilon x_2 + \cdots, \tag{2.68}$$

assuming that ϵ is small. Inserting this expansion into (2.47), we have

$$L(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) = -\epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^3, \tag{2.69}$$

with $L = \frac{d^2}{dt^2} + 1$ defined in (2.18). Then equating the coefficients of ϵ^n $(n = 0, 1, \ldots)$, we have a series of equations as follows,

$$Lx_0 = 0, \quad Lx_1 = -x_0^3, \quad Lx_2 = -3x_0^2 x_1, \tag{2.70}$$

and so on. The zero-th order solution may be given by

$$x_0 = \bar{A}\cos(t + \theta) \tag{2.71}$$

with $\bar{A}$ and θ being integral constants.

The first-order equation has a form of an inhomogeneous equation which reads

$$\ddot{x}_1 + x_1 = -\bar{A}^3 \cos^3(t + \theta) = -\frac{\bar{A}^3}{4}\left(3\cos\phi(t) + \cos 3\phi(t)\right), \tag{2.72}$$

with

$$\phi(t) = t + \theta. \tag{2.73}$$

Since Eq. (2.72) is a linear equation, the particular solution to it is given as a linear combination $x_1 = x_1^{(1)}(t) + x_2^{(2)}(t)$ of those to the inhomogeneous equations with homogeneous terms, $\cos\phi(t)$ and $\cos 3\phi(t)$, respectively:

$$\ddot{x}_1^{(1)} + x_1^{(1)} = -\frac{3\bar{A}^3}{4}\cos\phi(t), \tag{2.74}$$

$$\ddot{x}_1^{(2)} + x_1^{(2)} = -\frac{\bar{A}^3}{4}\cos 3\phi(t). \tag{2.75}$$

These equations are of the same form as that of a forced oscillator without resistance with external forces proportional to $\cos\phi(t)$ and $\cos 3\phi(t)$, the former of which causes a resonance phenomenon described by a secular term as before. Although the particular solution can be obtained by the method of variation of constants , here we make a short cut by assuming that

$$x_1^{(1)} = at\cos\phi(t) + bt\sin\phi(t). \tag{2.76}$$

Inserting this ansatz to (2.74), we have

$$\ddot{x}_1^{(1)} + x_1^{(1)} = -2a\sin\phi(t) + 2b\cos\phi(t) = -\frac{3\bar{A}^3}{4}\cos\phi(t), \tag{2.77}$$

which gives

$$a = 0 \quad \text{and} \quad b = -3\bar{A}^3/8 \tag{2.78}$$

and hence

$$x_1^{(1)} = -\frac{3\bar{A}^3}{8} t \sin\phi(t). \tag{2.79}$$

On the other hand, a simple application of the method of variation of constants (2.75) gives

$$x_1^{(2)} = \frac{\bar{A}^3}{32} \cos 3\phi(t). \tag{2.80}$$

Thus we have for the particular solution $x_1(t)$

$$x_1(t) = -\frac{3\bar{A}^3}{8} t \sin\phi(t) + \frac{\bar{A}^3}{32} \cos 3\phi(t). \tag{2.81}$$

The approximate solution up to this order now reads

$$x(t) \simeq \bar{A}\cos\phi(t) - \frac{3\bar{A}^3}{8}\epsilon\left[t \sin\phi(t) - \frac{1}{12}\cos 3\phi(t)\right], \tag{2.82}$$

which contains a secular term causing the amplitude to grow infinitely when $t \to \infty$. This is a physically absurd result and a quite opposite behavior to the exact solution that is bounded because of the restoring (confining) force.

Nevertheless we can make the following argument which suggests a possible origin of the appearance of the secular term. Since

$$(3\bar{A}^2/8)\,\epsilon t \simeq \sin[(3\epsilon t \bar{A}^2/8] \tag{2.83}$$

for sufficiently small ϵt, the secular term and the unperturbed one in (2.82) can be combined into

$$\begin{aligned} \bar{A}\cos\phi(t) - \frac{3\bar{A}^3}{8}\epsilon t \sin\phi(t) &\simeq \bar{A}\cos\phi(t) - \sin(3\bar{A}^3\epsilon t/8) \sin\phi(t) \\ &\simeq \bar{A}\cos[\phi(t) + 3\epsilon t\bar{A}^2/8] \\ &= \bar{A}\cos[(1 + 3\epsilon\bar{A}^2/8)t + \theta]. \end{aligned} \tag{2.84}$$

Thus we have

$$x(t) \simeq \bar{A}\cos[\Omega t + \theta] + \epsilon\frac{\bar{A}^3}{32}\cos 3\phi(t), \tag{2.85}$$

with

$$\Omega := 1 + \frac{3}{8}\epsilon\bar{A}^2, \tag{2.86}$$

which coincides with the shifted frequency (2.67) extracted from the exact solution for small ϵ.

It is also to be noted that even if a part of the effects of the nonlinear term could be absorbed through the *renormalization of the spring constant*, it is not the whole effects of it; the nonlinear term also gives rise to an additional term with *higher harmonics*.

2.4 van der Pol Equation

The final example is the van der Pol equation, which describes a self-sustained oscillation [22];

$$\ddot{x} + x = \epsilon\,(1 - x^2)\,\dot{x}, \tag{2.87}$$

where ϵ is supposed to be small so as to the perturbation analysis is valid. It is known that the van der Pol equation admits a limit cycle; see [79] for a pedagogical explanation on what a limit cycle is in non-linear oscillators.

Let us apply the perturbative expansion to (2.87) as before,

$$x(t) = x_0(t) + \epsilon\, x_1(t) + \epsilon^2\, x_2(t) + \cdots . \tag{2.88}$$

Inserting this expansion to (2.87), we have the following series of equations with $L = \frac{d^2}{dt^2} + 1$,

$$Lx_0 = 0, \tag{2.89}$$

$$Lx_1 = (1 - x_0^2)\,\dot{x}_0, \tag{2.90}$$

$$Lx_2 = (1 - x_0^2)\,\dot{x}_1 - 2x_0 x_1 \dot{x}_0, \tag{2.91}$$

and so on.

We take for the zeroth solution

$$x_0(t) = A\cos(t + \theta). \tag{2.92}$$

Then Eq. (2.90) becomes

$$Lx_1 = -A\left(1 - \frac{A^2}{4}\right)\sin\phi(t) + \frac{A^3}{4}\sin 3\phi(t), \quad (\phi(t) = t + \theta). \tag{2.93}$$

Again the first term of r.h.s. is a zero mode of L, and hence the particular solution to this equation will contain a secular term. As was done in the previous section, the application of the method of variation of constants leads to the solution as

$$x_1(t) = t\frac{A}{2}\left(1 - \frac{A^2}{4}\right)\cos\phi(t) \;-\; \frac{A^3}{32}\sin 3\phi(t). \tag{2.94}$$

Thus up to the second order, we have

$$x(t) \simeq A\cos\phi(t) \;+\; \epsilon\left[\frac{tA}{2}\left(1 - \frac{A^2}{4}\right)\cos\phi(t) - \frac{A^3}{32}\sin 3\phi(t)\right], \tag{2.95}$$

which contains a secular term proportional to t. Again the secular term can be absorbed into the unperturbed solution, and then (2.95) is rewritten as

$$x(t) \simeq \mathcal{A}(t)\cos\phi(t) - \epsilon\frac{A^3}{32}\sin 3\phi(t). \tag{2.96}$$

with

$$\mathcal{A}(t) := A\left[1 + \frac{\epsilon t}{2}\left(1 - \frac{A^2}{4}\right)\right], \tag{2.97}$$

which happens to satisfy the following equation up to ϵ^2

$$\frac{d\mathcal{A}}{dt} = \epsilon\frac{A}{2}\left(1 - \frac{A^2}{4}\right) \simeq \epsilon\frac{\mathcal{A}}{2}\left(1 - \frac{\mathcal{A}^2}{4}\right). \tag{2.98}$$

If one takes the last equality literally, it is readily verified that if $\mathcal{A}(0) \neq 0$, $\lim_{t\to\infty}\mathcal{A}(t) = 2$, which means that the van der Pol equation admits a limit cycle with a radius 2; this result is in accordance with the behavior of the solution obtained in numerical calculations and in resummation methods to be introduced later.

It is also noteworthy that the effects of the nonlinear term also create higher-harmonics terms that could not be absorbed or renormalized away to the zeroth order solution as might have been possible for the secular terms.

2.5 Concluding Remarks

We have seen typical phenomena and problems in the naïve perturbation method for solving differential expansions using a few examples. When the inhomogeneous terms in the naïve perturbative equations contain **zero modes** of the linear operator L of the homogeneous equation, the appearance of secular terms is inevitable, which actually do not appear in the exact solutions.

A rather generic phenomenon in such a resummation of the perturbation series is that the *resummed terms* may be *renormalized into* the integral constants in the unperturbed solution, like the amplitudes and/or phases, which should contain a small parameter and thus would describe slow motions (or long-wave length phenomena), *i.e.*, a kind of collective motion. Then it would be desirable to explicitly extract such slow variables and the dynamical equations to describe their slow motions.

In fact, many methods have been developed to avoid the appearance of secular terms and in the same time achieve a *resummation* of the seemingly divergent series with secular terms, which include Poincare-Lighthil-Luo, Krylov-Bogoliubov-Mitropolsky method, the reductive perturbation theory and so on [21, 22]. All of these existing methods are based on some techniques with which the appearance of secular terms or singular terms are circumvented.

The renormalization-group (RG) method that we are going to introduce is one of such methods: A unique feature of the method, however, lies in the fact that it allows the appearance of secular terms in contrast to the conventional resummation methods. Another merit of the RG method is that it provides us with an elementary but constructive method to extract the reduced dynamical equations explicitly for the slow motions embedded in the original dynamical equation.

Appendix: Method of Variation of Constants

Let us consider the following n-th order equation with the coefficients $a_i(t)$ ($i = 1, 2, \ldots, n$) and the inhomogeneous term $b(t)$:

$$x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_{n-1}(t)\dot{x} + a_n(t)x = b(t), \tag{2.99}$$

where $x^{(k)} := d^k x/dt^k$.

The standard procedure of the method of variation of constants (MVOC) goes as follows [21]. Let $x_1(t), x_2(t), \ldots, x_n(t)$ be a set of independent solutions of the homogeneous equation

$$Lx_i := x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_{n-1}(t)\dot{x} + a_n(t)x = 0 \tag{2.100}$$

with $i = 1, 2, \ldots, n$. The general solution to (2.100) is given by

$$x(t) = \sum_{i=1}^{n} C_i x_i(t), \tag{2.101}$$

with C_i ($i = 1, 2, \ldots, n$) being arbitrary constants. In the MVOC, we start with the following ansatz to a solution to (2.100),

$$x(t) = \sum_{i=1}^{n} C_i(t) x_i(t), \tag{2.102}$$

where the coefficients $C_i(t)$'s are now all functions of t in contrast to (2.101) where they are constants, hence the name of MVOC.