

Scale and conformal invariance in field theory: a physical counterexample

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In this note, we illustrate how the two-dimensional theory of elasticity provides a physical example of field theory displaying scale but not conformal invariance.

I. INTRODUCTION

In the quantum field theory literature, scale invariance is often assumed to imply conformal invariance, provided the theory is local. Furthermore, both invariances are usually considered equivalent to the tracelessness of the stress-energy tensor. These widely held convictions, sustained by the difficulty of finding counterexamples, are actually incorrect.

Coleman and Jackiw [1] clarified this issue in the case of four space-time dimensions, showing that conformal invariance is not in general guaranteed by the presence of scale invariance. A systematic analysis of the problem for arbitrary dimensionality D was then performed by Polchinski [2], who achieved the same conclusion for any $D \neq 2$. In the particular case $D = 2$, however, Polchinski proved that scale invariance implies conformal invariance under broad conditions. In the following, we will focus on this interesting dimensionality, providing a physical example in which the implication does not hold.

Let us now summarize the observations presented in Ref. [2]. Given a symmetric and conserved stress-energy tensor $T_{\mu\nu}(x)$, the property of scale invariance can be equivalently formulated in terms of its trace as

$$T_\mu{}^\mu(x) = -\partial_\mu K^\mu(x), \quad (1)$$

where $K^\mu(x)$ is some local operator. Conformal invariance further requires the existence of another local operator $L(x)$ such that

$$K_\mu(x) = -\partial_\mu L(x) \quad \Rightarrow \quad T_\mu{}^\mu(x) = \partial_\mu \partial^\mu L(x). \quad (2)$$

The above property is then equivalent to the tracelessness of the stress-energy tensor, because one can define the ‘improved’ tensor

$$\Theta_{\mu\nu}(x) = T_{\mu\nu}(x) + \partial_\mu \partial_\nu L(x) - g_{\mu\nu} \partial_\rho \partial^\rho L(x), \quad (3)$$

which is both conserved and traceless. As properly emphasized in Ref. [2], most of the physically relevant theories display both scale and conformal invariance because they do not have any non-trivial candidate for K_μ . We will see in the following how this is the crucial ingredient in our counterexample.

Besides these general remarks, Polchinski also refined an argument by Zamolodchikov [3], demonstrating that scale invariance implies conformal invariance in $D = 2$. The proof consists of defining another kind of ‘improved’

stress-energy tensor $\Theta'_{\mu\nu}(x)$, whose trace is shown to have a vanishing two-point function:

$$\langle \Theta'_\mu{}^\mu(x) \Theta'_\sigma{}^\sigma(0) \rangle = 0. \quad (4)$$

The sufficient condition for constructing $\Theta'_{\mu\nu}(x)$ is a discrete spectrum of scaling dimensions, and, together with the assumption of reflection positivity, (4) implies the vanishing of the trace $\Theta'_\mu{}^\mu$ itself. Actually, under the above hypotheses the two ‘improved’ tensors $\Theta_{\mu\nu}(x)$ and $\Theta'_{\mu\nu}(x)$ coincide.

II. THE MODEL

Let us now introduce a physical example in which scale invariance does not imply conformal invariance. This is the theory of elasticity [4] in two dimensions, defined by the Euclidean action

$$\mathcal{S} = \int d^2x \mathcal{L} = \frac{1}{2} \int d^2x \{2g u_{\mu\nu} u^{\mu\nu} + k (u_\sigma{}^\sigma)^2\}, \quad (5)$$

where $u_{\mu\nu} = \frac{1}{2} (\partial_\mu u_\nu + \partial_\nu u_\mu)$ is the so-called strain tensor, built with the ‘displacement fields’ u_μ . Greek indices run over 1, 2 and we use the summation convention. The coefficients g and $k + g$ represent, respectively, the shear modulus and the bulk modulus of the described material.

The action (5) is invariant under translations, rotations and dilatations, provided the fields u_μ transform under rotations $x'^\mu = \Lambda_\nu^\mu x^\nu$ as vectors

$$u'_\mu(x') = \Lambda_\mu{}^\nu u_\nu(x), \quad (6)$$

while no change is required for fields under dilatations. The canonical stress-energy tensor

$$T_{\mu\nu}^c = \frac{\partial \mathcal{L}}{\partial (\partial^\mu u_\sigma)} \partial_\nu u_\sigma - g_{\mu\nu} \mathcal{L} \quad (7)$$

associated to (5) is traceless but not symmetric. However, a symmetric and conserved tensor $T_{\mu\nu}$ can be conventionally constructed via the Belinfante prescription:

$$T_{\mu\nu} = T_{\mu\nu}^c + \partial^\rho B_{\rho\mu\nu}, \quad (8)$$

where

$$B_{\rho\mu\nu} = \frac{i}{2} \left\{ \frac{\partial \mathcal{L}}{\partial(\partial^\rho u_\sigma)} S_{\nu\mu} u_\sigma + \frac{\partial \mathcal{L}}{\partial(\partial^\mu u_\sigma)} S_{\rho\nu} u_\sigma + \frac{\partial \mathcal{L}}{\partial(\partial^\nu u_\sigma)} S_{\rho\mu} u_\sigma \right\} = -B_{\mu\rho\nu}. \quad (9)$$

$S_{\mu\nu}$ is an antisymmetric tensor, taking values in the representations of the Lorentz group, which expresses the variation of the field multiplet $\phi = \{u_\mu\}$ under infinitesimal rotations $x'^\mu \simeq x^\mu + \omega^\mu_\nu x^\nu$:

$$\phi'(x') \simeq \left(I - \frac{i}{2} \omega_{\rho\nu} S^{\rho\nu} \right) \phi(x).$$

In our case the fields transform according to the vector representation (6), and the only non-vanishing Euclidean components of $S_{\mu\nu}$ act as

$$\begin{aligned} S_{12} u_1 &= -S_{21} u_1 = i u_2 \\ S_{12} u_2 &= -S_{21} u_2 = -i u_1 \end{aligned}$$

It follows from (8) that the trace of the stress-energy tensor can be cast in the form (1)

$$T_\mu^\mu = -\partial^\mu K_\mu \quad \text{with} \quad K_\mu = -B_{\mu\rho}^\rho, \quad (10)$$

in agreement with the scale invariance of the theory. In order to investigate whether the additional property (2), equivalent to conformal invariance, is also attained, it is now convenient to explicitly write K_μ in Euclidean coordinates. We have

$$\begin{aligned} K_1 &= \partial_1 \left[-\frac{k}{2} u_1^2 + \frac{g}{2} u_2^2 \right] - (k+2g) u_1 \partial_2 u_2 + g u_2 \partial_2 u_1 \\ K_2 &= \partial_2 \left[\frac{g}{2} u_1^2 - \frac{k}{2} u_2^2 \right] - (k+2g) u_2 \partial_1 u_1 + g u_1 \partial_1 u_2. \end{aligned} \quad (11)$$

It appears from (11) that K_μ cannot be entirely reduced to a gradient[5], therefore the necessary condition (2) for conformal invariance does not hold and the stress-energy tensor cannot be ‘improved’ to be traceless.

The lack of conformal invariance in (5) becomes manifest if we write the action in complex coordinates $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$. We have

$$\mathcal{S} = \frac{1}{2} \int d^2 z \left\{ (k+g) [\partial \bar{u} + \bar{\partial} u]^2 + 4g (\partial u)(\bar{\partial} \bar{u}) \right\}, \quad (12)$$

where $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$ and $u = u_1 - iu_2$, $\bar{u} = u_1 + iu_2$. The transformation (6) under rotations translates into the requirement that the fields u and \bar{u} have spins $s_u = 1$ and $s_{\bar{u}} = -1$, while both their scaling dimensions Δ_u , $\Delta_{\bar{u}}$ have to vanish in order to ensure scale invariance. These properties are obtained by assigning the conformal weights

$$h_u = \bar{h}_{\bar{u}} = \frac{1}{2}, \quad \bar{h}_u = h_{\bar{u}} = -\frac{1}{2}, \quad (13)$$

which are defined through

$$\Delta = h + \bar{h}, \quad s = h - \bar{h}.$$

It is then easy to see that (12) is not invariant under a conformal transformation $z \rightarrow w = f(z)$, $\bar{z} \rightarrow \bar{w} = \bar{f}(\bar{z})$, where the fields transform as $\phi \rightarrow (f')^{-h} (\bar{f}')^{-\bar{h}} \phi$.

Conformal invariance is only recovered in the unphysical case of zero bulk modulus $k+g=0$, when (12) describes a conformal field theory with central charge $c=2$. This is the familiar situation in which the two fields u and \bar{u} are not required to transform under rotations, and the corresponding symmetry is then enlarged from $O(2)$ to $O(2) \times O(2)$. It is worth stressing that all conformal weights (13) now vanish, therefore this particular case cannot be described by simply evaluating the previous results in the limit $k+g \rightarrow 0$, but must be separately treated. In fact, the canonical stress-energy tensor (7) is now already symmetric, therefore no Belinfante construction has to be implemented, and conformal invariance immediately follows from the tracelessness of $T_{\mu\nu}^c$.

As a final remark, we will now show that the above observations for generic values of $k+g$ are not in contradiction with the statement, proven in [2], that the stress-energy tensor can be nevertheless ‘improved’ to a certain degree, in order to obtain a vanishing two-point function of its trace, as in (4). However, this is not associated with the vanishing of the trace itself, because the theory under examination turns out to be not reflection positive, as we shall illustrate below.

Let us first notice that the trace of $T_{\mu\nu}$ can be expressed at quantum level as

$$T_\mu^\mu = (k+g) [: \partial \bar{u} \partial \bar{u} : + : \bar{\partial} u \bar{\partial} u : + 2 : \partial \bar{u} \bar{\partial} u :] + -g [: \partial u \bar{\partial} \bar{u} : - : u \partial \bar{\partial} \bar{u} : - : \bar{u} \partial \bar{\partial} u :], \quad (14)$$

where the symbol ‘:’ indicates normal ordering. By using Wick theorem and the explicit expressions

$$\begin{aligned} \langle u(z) u(w) \rangle &= \frac{k+g}{4\pi g (k+2g)} \frac{\bar{z} - \bar{w}}{z - w}, \\ \langle \bar{u}(z) \bar{u}(w) \rangle &= \frac{k+g}{4\pi g (k+2g)} \frac{z - w}{\bar{z} - \bar{w}}, \\ \langle u(z) \bar{u}(w) \rangle &= \frac{k+g - (k+3g) \log(z-w)(\bar{z}-\bar{w})}{4\pi g (k+2g)}, \end{aligned} \quad (15)$$

it is then straightforward to check that the two-point function of (14) does not vanish, being

$$\langle T_\mu^\mu(z) T_\sigma^\sigma(0) \rangle = \frac{-32g^2(k+g)(k+3g)}{[4\pi g(k+2g)]^2} \frac{1}{z^2 \bar{z}^2}. \quad (16)$$

However, we can guess the kind of improvement to be performed on $T_{\mu\nu}$, by observing that the operator K_μ defined in (10) can be partially reduced to a gradient, as

$$\begin{aligned} K_z &= \partial(g u \bar{u}) - \frac{k+g}{2} u \bar{\partial} u - \frac{k+3g}{2} u \partial \bar{u}, \\ K_{\bar{z}} &= \bar{\partial}(g u \bar{u}) - \frac{k+g}{2} \bar{u} \partial \bar{u} - \frac{k+3g}{2} \bar{u} \bar{\partial} u. \end{aligned} \quad (17)$$

It is then natural to define $\Theta'_{\mu\nu}$ as in (3), with $L = -g u \bar{u}$, and it can be easily checked that the two-point function of its trace indeed vanishes

$$\langle \Theta'_\mu{}^\mu(z) \Theta'_\sigma{}^\sigma(0) \rangle = 0, \quad (18)$$

although the trace itself does not

$$\begin{aligned} \Theta'_\mu{}^\mu &= (k+g) [: \partial \bar{u} \partial \bar{u} : + : \bar{\partial} u \bar{\partial} u :] + \\ &\quad + 2(k+3g) : \partial \bar{u} \bar{\partial} u : . \end{aligned} \quad (19)$$

suggesting therefore the failure of reflection positivity. In fact, in a reflection positive theory eq. (18) should imply the vanishing of any two-point function involving $\Theta'_\mu{}^\mu$, but several counterexamples occur here, for instance

$$\langle \Theta'_\mu{}^\mu(z) : \partial u \partial u : (0) \rangle = -\frac{k+g}{2\pi^2 g(k+2g)} \frac{1}{z^4}.$$

The lack of reflection positivity can be equivalently seen as non-unitarity in Minkowski coordinates. In fact, the Hamiltonian associated to (5) is not positive definite, being expressed as

$$\begin{aligned} H &= \frac{1}{2} \int dx \left\{ \frac{1}{k+2g} \pi_t^2 + g (\partial_x u_t)^2 + \right. \\ &\quad \left. - \frac{1}{g} [\pi_x - (k+g) \partial_x u_t]^2 - (k+2g) (\partial_x u_x)^2 \right\}, \end{aligned} \quad (20)$$

where the conjugate momenta to u_t, u_x are given by

$$\begin{aligned} \pi_t &= (k+2g) \partial_t u_t, \\ \pi_x &= g \partial_x u_x + (k+g) \partial_x u_t. \end{aligned}$$

The negative signs in (20) are produced by the $(1, -1)$ signature of the Minkowski target space $\{u_t, u_x\}$, which follows from the transformation property (6).

III. COMMENTS

In concluding this note, it is worth emphasizing that the lack of conformal invariance in the discussed example entirely originates from the transformation property (6) of the fields under rotations. This is what makes the canonical stress-energy tensor $T_{\mu\nu}^c$ not symmetric and provides the non-trivial expression (10) for the trace of the symmetrized tensor $T_{\mu\nu}$. The symmetrization procedure always respects scale invariance, since the Belinfante prescription only contributes with the divergence of a local function K_μ to the trace $T_\mu{}^\mu$, as shown in (10). However, K_μ has in general a non-trivial vectorial structure, and therefore $T_{\mu\nu}$ cannot be made traceless, consistently with the fact that the theory is not conformally invariant.

Finally, it is interesting also to notice that the non-unitarity of the theory, which reconciles our observations with Polchinski's proof, is itself a consequence of the vectorial nature of the fields u_μ . In fact, covariance requires both worldsheet and target space to have the same Lorentzian signature, making the Hamiltonian (20) not positive definite. A similar phenomenon takes place also in electrodynamics and string theory, where the Lorentzian signature of target space produces negative-norm states in the Hilbert space. However, in those cases the presence of a gauge invariance allows one to recover unitarity in the subspace of physical states.

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[5] As clearly explained in [2], no further freedom in the definition of the stress-energy tensor can modify this conclusion.