

## CONNECTION ON THE THEORY SPACE\*

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### 1. Introduction

The idea of the theory space is essential for qualitative understanding of quantum field theory<sup>1</sup>; only through the notions of relevance, irrelevance, and universality, we understand the meaning of the continuum limit of a field theory. But our use of the theory space has so far been mostly qualitative. It is the purpose of this short article to treat the theory space seriously and introduce a quantitative structure.<sup>2-4</sup>

### 2. Covariant Quantities in the Theory Space

We consider renormalized field theories in  $D$  dimensional euclidean space parametrized by  $g^i (i = 1, \dots, N)$ . The parameters  $g^i$  give local coordinates of the theory space. We take the origin  $g^i = 0 (\forall i)$  to be the UV fixed point. The renormalization group (RG) describes the change of the theory under scale change; under the transformation  $R_l$  the renormalization scale is fixed, but the coordinate distance  $r$  transforms into  $re^{-l}$ . The RG equations of the parameters are given by

$$\frac{d}{dl} g^i = \beta^i(g) \equiv x_i g^i + O(g^2), \quad (2.1)$$

where the scale dimension  $x_i$  of the parameter  $g^i$  is positive. Note that the beta function  $\beta^i(g)$  is a vector field on the theory space.

Each point in the theory space is a renormalized field theory which has an infinite number of linearly independent composite fields. Let  $\{\Phi_a\}_g$  be a complete basis of composite fields at  $g$ . The space of composite fields make a linear space, and therefore the composite fields make an infinite dimensional vector bundle over the theory space. The choice of the local basis is by no means unique, and two bases  $\{\Phi_a\}_g$  and  $\{\Phi'_a\}_g$  are related by an invertible infinite dimensional matrix  $N(g)$  as

$$\Phi'_a = (N(g))_a^b \Phi_b. \quad (2.2)$$

Therefore, the correlation function of  $n$  composite fields

$$\langle \Phi_{a_1}(r_1) \dots \Phi_{a_n}(r_n) \rangle_g \quad (2.3)$$

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forms a rank- $n$  tensor field on the theory space.

We can write the RG equations of the composite fields as

$$\frac{d}{dl} \Phi_a = (\Gamma(g))_a^b \Phi_b, \quad (2.4)$$

where the matrix

$$(\Gamma(g))_a^b = y_a \delta_a^b + O(g) \quad (2.5)$$

gives the scale dimensions  $y_a$  and anomalous dimensions of the renormalized fields. The precise meaning of the above RG equations is given by the following RG equations for correlation functions:

$$\begin{aligned} \frac{d}{dl} \langle \Phi_{a_1}(r_1) \dots \Phi_{a_n}(r_n) \rangle_g &\equiv \left( -\sum_{k=1}^n r_{k,\mu} \frac{\partial}{\partial r_{k,\mu}} + \beta^i \frac{\partial}{\partial g^i} \right) \langle \Phi_{a_1}(r_1) \dots \Phi_{a_n}(r_n) \rangle_g \\ &= \sum_{k=1}^n \Gamma_{a_k}^b \langle \Phi_{a_1}(r_1) \dots \Phi_b(r_k) \dots \Phi_{a_n}(r_n) \rangle_g. \end{aligned} \quad (2.6)$$

Among the infinite number of composite fields,  $N$  composite scalar fields stand out; they are the fields  $\mathcal{O}_i$  conjugate to the parameters  $g^i$ . The notion of conjugacy should be familiar from thermodynamics, where the external magnetic field  $H$  has magnetization  $M$  as its conjugate field. The expectation value of  $\mathcal{O}_i$  gives the derivative of the free energy density  $F(g)$  with respect to  $g^i$ :

$$\langle \mathcal{O}_i \rangle_g = \frac{\partial}{\partial g^i} F(g). \quad (2.7)$$

Since the free energy density satisfies the canonical RG equation

$$\frac{d}{dl} F(g) = DF(g) \quad (2.8)$$

( $D$  is the dimensionality of space), we find, from Eqs. (2.1) and (2.8), that

$$\frac{d}{dl} \mathcal{O}_i = D\mathcal{O}_i - \frac{\partial \beta^j}{\partial g^i} \mathcal{O}_j. \quad (2.9)$$

We will define the conjugate fields more precisely in sect. 3.

### 3. Variational Formula

We now consider the  $g$ -dependence of the correlation functions, which are tensor fields on the theory space. In analogy to thermodynamics and statistical mechanics, the derivative of a correlation function with respect to  $g^i$  should be given as a volume integral of the conjugate field  $\mathcal{O}_i$ :

$$\begin{aligned} &- \frac{\partial}{\partial g^i} \langle \Phi_{a_1}(r_1) \dots \Phi_{a_n}(r_n) \rangle_g \\ &= \lim_{\epsilon \rightarrow 0} \left[ \int_{|r-r_k| \geq \epsilon} d^D r \langle (\mathcal{O}_i(r) - \langle \mathcal{O}_i \rangle_g) \Phi_{a_1}(r_1) \dots \Phi_{a_n}(r_n) \rangle_g \right. \\ &\quad \left. + \sum_{k=1}^n \left( c_{i,a_k}^b(g) - \int_{1 \geq r \geq \epsilon} \frac{d^D r}{\text{vol}(S^{D-1})} C_{i,a_k}^b(r; g) \right) \langle \Phi_{a_1}(r_1) \dots \Phi_b(r_k) \dots \Phi_{a_n}(r_n) \rangle_g \right], \end{aligned} \quad (3.1)$$

where  $\text{vol}(S^{D-1})$  is the volume of a unit  $D-1$  sphere. Two remarks are in order. First, due to the short distance singularity

$$\mathcal{O}_i(r)\Phi_a(0) = \frac{1}{\text{vol}(S^{D-1})} C_{i,a}{}^b(r; g)\Phi_b + \mathcal{O}\left(\frac{1}{r^D}\right), \quad (3.2)$$

the integral needs subtractions. In the above operator product expansion (OPE), we keep only composite fields  $\Phi_b$  whose scale dimension is at most  $y_a$ , since we need to subtract only the unintegrable part. Second, we need the finite counterterms  $c_{i,a}{}^b(g)$  not only to compensate the arbitrariness of the subtraction scheme, but also to satisfy covariance in the theory space. Under a change of basis, Eq. (2.2), the subtracted integral on the right-hand side of Eq. (3.1) is covariant, but the left-hand side is non-covariant due to the naked derivative. Hence, covariance demands the finite counterterms  $c_{i,a}{}^b(g)$  which transform as a connection

$$c_i(g) \rightarrow c'_i(g) = N(g)(\partial_i + c_i(g))N^{-1}(g) \quad (3.3)$$

under the change of basis Eq. (2.2). This is how the connection  $c_i$  arises naturally in the theory space. We call Eq. (3.1) as a variational formula.

#### 4. Consistency Conditions

It is difficult to *derive* the variational formula from first principles. Instead, we *demand* the conjugate fields  $\mathcal{O}_i$  to satisfy Eq. (2.7) and the variational formula Eq. (3.1). It is an assumption that such fields exist. Nonetheless, we can check the consistency of Eq. (3.1).

First, we demand that Eq. (3.1) be consistent with the RG equations (2.1), (2.4), and (2.9). Then, we obtain

$$C_i(r=1; g) = \frac{\partial}{\partial g^i} \Psi(g) + [c_i(g), \Psi(g)] + \beta^j(g)\Omega_{ji}(g), \quad (4.1)$$

where

$$\Psi(g) \equiv \Gamma(g) + \beta^i(g)c_i(g) \quad (4.2)$$

is a covariant tensor field, and the curvature  $\Omega_{ij}$  is defined by

$$\Omega_{ij} \equiv \partial_i c_j - \partial_j c_i + [c_i, c_j]. \quad (4.3)$$

Eq. (4.1) gives a geometric expression of the singular part of the OPE coefficients.

Second, we demand that Eq. (3.1) satisfy the Maxwell relation:

$$\partial_i \partial_j = \partial_j \partial_i. \quad (4.4)$$

This assures the consistency of using Eq. (3.1) recursively to evaluate higher order derivatives. Eq. (4.4) gives the curvature in terms of a double integral over a finite domain:

$$\Omega_{ij} \langle \Phi \rangle_g = \int_{r \leq 1} d^D r \text{ F.P.} \int_{r' \leq 1} d^D r' \langle \mathcal{O}_j(r) \left( \mathcal{O}_i(r') - \frac{C_i(r')}{\text{vol}(S^{D-1})} \right) \Phi(0) - (i \leftrightarrow j) \rangle_g, \quad (4.5)$$

where F.P. denotes taking the integrable part with respect to  $r$ .

## 5. Conclusions

By studying the variational formula, Eq. (3.1), we have found that the geometrical quantities on the theory space, i.e., the vector field  $\beta^i(g)$  (beta functions), the rank-two tensor field  $\Psi_a^b(g)$  (scale and anomalous dimensions), and the connection  $c_i$  (finite counterterms), determine the short-distance singularities of the theory, as is given by Eqs. (4.1) and (4.5). These are analogues of the well-known relation between the renormalization constants and the beta functions and anomalous dimensions in the minimal subtraction scheme.<sup>5</sup> It is important to note that the connection  $c_i$  has more than its geometric significance: the connection gives the finite counterterms in the variational formula (3.1).

One promising place where we can apply the variational formula and the connection  $c_i$  of sect. 3 is in closed string field theory. To understand the background independence of closed string field theory, we must find the relation between two different formulations based upon two conformal field theories. This requires comparing string states (or vertex operators) in one conformal field theory to those in the other conformal field theory. Hence, we need a connection to parallel transport states from one theory to the other. We have applied the variational formula of sect. 3 to the space of conformal field theories<sup>6,7</sup>; in particular we have studied the properties of the connection  $c_i$  and its two modifications. It is not yet clear which precise connection is the right choice for closed string field theory, but a connection, together with the variational formula (3.1), is expected to play an important role.<sup>8</sup>

## References

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