11 Renormalization of local polynomials. Short-distance expansion (SDE)

In this chapter, we discuss two related topics: the renormalization of local polynomials (or local operators) of the field [82], and the SDE of the product of local operators, in space dimension 4 for simplicity. Both problems are related, since one can consider the insertion of a product of operators $A(x)B(y)$ as a regularization by point splitting of the operator $A[\frac{1}{2}(x+y)]B[\frac{1}{2}(x+y)]$. Therefore, in the limit $x \to y$, we expect the product to become dominated by a linear combination of the local operators which appear in the renormalization of the product AB, with singular coefficients, functions of $(x-y)$, replacing the usual cut-off dependent renormalization constants.

We first discuss the renormalization of local polynomials from the viewpoint of power counting. We use the relations between bare and renormalized operators to establish Callan–Symanzik (CS) equations for the insertion of operators of dimension 4 in the $\phi^4_{d=4}$ quantum field theory (QFT) [83]. We show that, in a QFT, there exist linear relations between operators, due to the equations of motion and relations derived in Section 7.5.

Then, starting with Section 11.3, we establish the existence of a SDE [84–87] for the product of two basic fields, and discuss the SDE at leading order in the ϕ^4 QFT.

We pass from short-distance behaviour to large-momentum behaviour, and derive a CS equation for the coefficient of the expansion at leading order [88]. Finally, we briefly discuss the generalization of this analysis to the SDE beyond leading order, to the SDE of arbitrary operators and to the light cone expansion (LCE) [89], which appears in the study of the large-momentum behaviour of real-time correlation functions (in contrast to Euclidean or imaginary time). In the whole chapter, we work in space-time dimension 4.

11.1 Renormalization of operator insertions

In Section 9.6, we showed how to renormalize insertions of the monomial $\phi^2(x)$. We could have considered other local monomials like $\phi^4(x)$, $(\nabla\phi(x))^2$... They all generate new divergences which have to be cancelled by additional renormalizations.

In Section 8.6, we explained how to calculate the superficial degree of divergence of the insertion of a local monomial (or operator) $\mathcal{O}(\phi,x)$ by adding a source $g_{\mathcal{O}}(x)$ for the operator in the action:

$$\mathcal{S}(\phi) \mapsto \mathcal{S}(\phi) + \int \mathrm{d}^4x\, \mathcal{O}(\phi,x) g_{\mathcal{O}}(x).$$

With this choice of sign, differentiation of $\mathcal{W}(J, g_{\mathcal{O}})$ with respect to $g_{\mathcal{O}}(x)$ generates insertions of $-\mathcal{O}(\phi,x)$ in connected correlation functions. However, since $\delta\Gamma/\delta g_{\mathcal{O}}(x) = -\delta\mathcal{W}/\delta g_{\mathcal{O}}(x)$, $\delta\Gamma/\delta g_{\mathcal{O}}(x)$ corresponds to the insertion of $\mathcal{O}(\phi,x)$ in vertex functions.

As a convention, we assign a canonical dimension $[g_{\mathcal{O}}]$ to the source $g_{\mathcal{O}}(x)$, opposite to the dimension of the vertex associated to $\mathcal{O}(\phi)$, and thus related to the dimension $[\mathcal{O}(\phi)]$ of the operator $\mathcal{O}(\phi,x)$ by

$$[g_{\mathcal{O}}] = 4 - [\mathcal{O}(\phi)].$$

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Then, the implications of power counting and renormalization theory can be summarized as: the sum of counter-terms needed to render $\Gamma(\phi, g_{\mathcal{O}})$ finite is the most general local functional of $\phi(x)$ and $g_{\mathcal{O}}(x)$, allowed by power counting.

More precisely, it is *the most general linear combination of all vertices in $\phi(x)$ and $g_{\mathcal{O}}(x)$ of non-positive dimensions.*

In particular, this implies that an operator of a given dimension inserted once, in general, mixes with all operators of equal or lower dimension under renormalization. It is thus natural to study the renormalization of operators of increasing dimension.

We first verify this assertion in the case of the insertion of $\phi^2(x)$ in a ϕ^4 QFT in four dimensions, which we have already discussed in Chapter 9.

Regularization. For simplicity, in this chapter we assume an implicit momentum cut-off regularization, and denote the cut-off by Λ.

11.1.1 The $\phi^2(x)$ insertion

We use the conventions of Section 9.2 for the bare and renormalized actions:

$$\mathcal{S}(\phi) = \int \mathrm{d}^4x \left[\frac{1}{2}\left(\nabla\phi(x)\right)^2 + \frac{1}{2}r\phi^2(x) + \frac{1}{4!}g\phi^4(x)\right], \tag{11.1}$$

$$\mathcal{S}_\mathrm{r}(\phi_\mathrm{r}) = \int \mathrm{d}^4x \left[\frac{1}{2}Z\left(\nabla\phi_\mathrm{r}(x)\right)^2 + \frac{1}{2}\left(m_\mathrm{r}^2 + Z\delta r\right)\phi_\mathrm{r}^2(x) + \frac{1}{4!}g_\mathrm{r}Z_g\phi_\mathrm{r}^4(x)\right]. \tag{11.2}$$

In the ϕ^4 field theory in four dimensions, the operator $\phi^2(x)$ has dimension $[\phi^2] = 2$.

Then, we denote by $t(x)$ the source for $\phi^2(x)$. Its dimension is $[t] = 2$.

Then, in addition to the vertices involving only the field ϕ, the following vertices arise as counter-terms:

$$\int t(x)\phi^2(x)\mathrm{d}^4x\,, \quad \text{which has dimension } 0,$$

$$\int t^2(x)\mathrm{d}^4x\,, \quad \text{which also has dimension } 0,$$

$$\int t(x)\mathrm{d}^4x\,, \quad \text{which has dimension } -2\,.$$

The renormalized action then has the form (a and b are constants)

$$\mathcal{S}_\mathrm{r}(\phi_\mathrm{r}, t) = \mathcal{S}_\mathrm{r}(\phi_\mathrm{r}) + \tfrac{1}{2}Z_2\int \mathrm{d}^4x\, t(x)\phi_\mathrm{r}^2(x) + \int \mathrm{d}^4x\left[\tfrac{1}{2}at^2(x) + bt(x)\right]. \tag{11.3}$$

The last two terms only contribute to the vacuum amplitude. Expression (11.3) implies a set of relations between bare and renormalized generating functionals. First, for complete correlation functions,

$$\mathcal{Z}_\mathrm{r}(J,t) = \mathcal{Z}(J/\sqrt{Z}, tZ_2/Z)\exp\left[-\int \mathrm{d}^4x\big(\tfrac{1}{2}at(x)^2 + bt(x)\big)\right]. \tag{11.4}$$

For the connected functions, this implies

$$\mathcal{W}_\mathrm{r}(J,t) = \mathcal{W}(J/\sqrt{Z}, tZ_2/Z) - \int \mathrm{d}^4x\big(\tfrac{1}{2}at(x)^2 + bt(x)\big). \tag{11.5}$$

After Legendre transformation with respect to J, one obtains

$$\Gamma_\mathrm{r}(\varphi,t) = \Gamma(\varphi\sqrt{Z}, tZ_2/Z) + \int \mathrm{d}^4x\big(\tfrac{1}{2}at(x)^2 + bt(x)\big). \tag{11.6}$$

Expanding in powers of t and φ, one recovers the relations between bare and renormalized vertex functions described in Sections 9.7–9.10.

11.1.2 Operators of dimensions 3 and 4

The ϕ^3 insertion. To discuss the insertion of the $\phi^3(x)$ operator which has dimension 3, we introduce a source $t(x)$ that has dimension 1. The renormalized action $\mathcal{S}_\mathrm{r}(\phi, t)$ then has the form ($a_1, \ldots, a_6$ are constants)

$$\mathcal{S}_\mathrm{r}(\phi, t) = \mathcal{S}_\mathrm{r}(\phi) + \int \mathrm{d}^4 x \left[t(x) \left(\tfrac{1}{3!} Z_3 \phi^3(x) + a_1 \phi(x) \right) + a_2 \left(\nabla t(x) \right)^2 \right.$$
$$\left. + t^2(x) \left(a_3 \phi^2(x) + a_4 \right) + a_5 t^3(x) \phi(x) + a_6 t^4(x) \right] . \tag{11.7}$$

In particular, the expression shows that the operator $\phi^3(x)$ mixes with $\phi(x)$ under renormalization, that the double insertion of ϕ^3 generates a counter-term proportional to $\phi^2 \ldots$. Therefore, to be able to write the equivalent of relations (11.4–11.6), we have to introduce explicitly a source for $\phi^2(x)$. We leave it to the reader to determine the renormalized action with sources for ϕ^3 and ϕ^2, and the relations between renormalized and bare vertex functions. We postpone the discussion of the CS equations of correlation functions with ϕ^3 insertion until Section 11.2, in order to be able to incorporate the information provided by the field quantum equation of motion.

Operators of dimension 4 [83]. Quite generally, if an operator has a dimension strictly smaller than the space dimension d, its source has a strictly positive dimension, and the renormalized action is a polynomial in the source. If a source is coupled to operators of dimension d, corresponding to vertices of dimension 0 (here $\phi^4(x)$, $(\nabla\phi(x))^2$, an infinite series in the source is generated by the renormalization procedure, together with all operators of lower or equal dimensions.

For example, in $\phi^4_{d=4}$ QFT, if one inserts an operator of dimension 4 once, one has to consider the mixing of all linearly independent operators of dimensions 4 and 2 (parity in ϕ excludes odd dimensions). For example, the four operators

$$\begin{aligned} \mathcal{O}_1(\phi) &= \tfrac{1}{2} m_\mathrm{r}^2 \phi^2(x), \qquad & \mathcal{O}_2(\phi) &= -\tfrac{1}{2} \nabla^2 \left(\phi^2(x) \right), \\ \mathcal{O}_3(\phi) &= \tfrac{1}{2} \left[\nabla \phi(x) \right]^2 , \qquad & \mathcal{O}_4(\phi) &= \tfrac{1}{4!} \phi^4(x), \end{aligned} \tag{11.8}$$

form a basis of linearly independent operators, which mix under renormalization. There exists another operator $\phi(x)\nabla^2\phi(x)$ of dimension 4, but it is a linear combination of $\mathcal{O}_2$ and $\mathcal{O}_3$:

$$\tfrac{1}{2} \nabla^2 \left(\phi^2(x) \right) = \phi(x) \nabla^2 \phi(x) + \left[\nabla \phi(x) \right]^2 .$$

The operator $\mathcal{O}_1(\phi)$ and, therefore, also the operator $\mathcal{O}_2(\phi)$, are multiplicatively renormalizable.

We can thus express the relation between bare and renormalized correlation functions $\Gamma^{(n)}_{\mathcal{O}_i}$ with $\mathcal{O}_i$ insertion as

$$\left\{ \Gamma^{(n)}_{\mathcal{O}_i} \right\}_\mathrm{r} = Z^{n/2} \sum_j Z_{ij} \Gamma^{(n)}_{\mathcal{O}_j} . \tag{11.9}$$

The renormalization matrix Z_{ij} has the form

$$\begin{pmatrix} (Z_2/Z) \mathbf{1}_2 & 0 \\ \mathbf{B} & \mathbf{A} \end{pmatrix} ,$$

in which $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices. We have used for the renormalization of ϕ^2 the notation of Section 9.2.

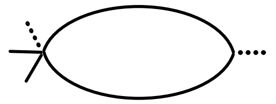

Fig. 11.1 Divergent contribution to $\phi^2(x)\phi^4(y)$ insertion (dotted lines)

CS equations. From equation (11.9), we can derive CS equation for $\{\Gamma^{(n)}_{\mathcal{O}_i}\}_{\mathrm{r}}$. However, here some care is required. The CS operation involves ϕ^2 insertions and the product, for example, $\{\phi^4 x|]_{\mathrm{r}}\{\phi^2(y)\}_{\mathrm{r}}$ inserted in a correlation function is not finite: since the source for ϕ^4 has dimension 0, and the source for ϕ^2 dimension 2, the product of ϕ^2 by both sources has dimension 4. Fig. 11.1 displays the first divergent diagram.

This implies

$$\tfrac{1}{4!}\left\{\left\{\phi^4(x)\right\}_{\mathrm{r}}\left\{\phi^2(y)\right\})_{\mathrm{r}}\right\}_{\mathrm{r}} = \tfrac{1}{4!}\left\{\phi^4(x)\right\}_{\mathrm{r}}\left\{\phi^2(y)\right\}_{\mathrm{r}} + C_4\delta^{(4)}(x-y)\left\{\phi^2(x)\right\}_{\mathrm{r}}, \quad (11.10)$$

in which C_4 is a new renormalization constant. Identity (11.10) is only true as an insertion in an n-point correlation function for $n \neq 0$.

After Fourier transformation, and for an insertion of ϕ^2 at zero momentum, equation (11.10) becomes

$$\left\{\left\{\tilde{\mathcal{O}}_4(p)\right\}_{\mathrm{r}}\left\{\tilde{\mathcal{O}}_1(0)\right\}_{\mathrm{r}}\right\}_{\mathrm{r}} = \left\{\tilde{\mathcal{O}}_4(p)\right\}_{\mathrm{r}}\left\{\tilde{\mathcal{O}}_1(0)\right\}_{\mathrm{r}} + C_4\left\{\tilde{\mathcal{O}}_1(p)\right\}_{\mathrm{r}}. \quad (11.11)$$

A similar equation holds for the operator $\mathcal{O}_3$.

We now apply the CS operator, $m_{\mathrm{r}}\partial/\partial m_{\mathrm{r}}$, at g and Λ fixed, on equation (11.9).

A new set of RG functions is generated involving the matrix

$$\tilde{\eta}_{ij}(g_{\mathrm{r}}, \Lambda/m_{\mathrm{r}}) = \sum_k \left(m_{\mathrm{r}}\frac{\partial}{\partial m_{\mathrm{r}}}Z_{ik}\right)\left[Z^{-1}\right]_{kj}. \quad (11.12)$$

As a consequence of relation (11.11), two elements, $\tilde{\eta}_{31}$ and $\tilde{\eta}_{41}$, of the matrix $\tilde{\eta}_{ij}$ are not finite when the cut-off becomes infinite. Their divergent part cancels the divergences coming from the insertion of ϕ^2 as represented by equation (11.11). Defining then,

$$\eta_{i1}(g_{\mathrm{r}}) = \tilde{\eta}_{i1} - m_{\mathrm{r}}^2\sigma(g_{\mathrm{r}})C_i\,, \quad (11.13)$$

we now obtain two finite RG functions.

For the other matrix elements, we just set

$$\eta_{ij} = \tilde{\eta}_{ij}\,. \quad (11.14)$$

The CS equations then read

$$\left\{\left[m_{\mathrm{r}}\frac{\partial}{\partial m_{\mathrm{r}}} + \beta(g_{\mathrm{r}})\frac{\partial}{\partial g_{\mathrm{r}}} - \frac{n}{2}\eta(g_{\mathrm{r}})\right]\delta_{ij} - \eta_{ij}(g_{\mathrm{r}})\right\}\left\{\tilde{\Gamma}^{(n)}_{\mathcal{O}_j}\right\}_{\mathrm{r}}(p) = m_{\mathrm{r}}^2\,\sigma(g_{\mathrm{r}})\left\{\tilde{\Gamma}^{(1,n)}_{\mathcal{O}_i}\right\}_{\mathrm{r}}(0;p). \quad (11.15)$$

The matrix η_{ij} has the form

$$\eta_{ij} = \begin{bmatrix}\mathbf{c} & 0\\ \mathbf{b} & \mathbf{a}\end{bmatrix},$$

in which $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are 2×2 matrices, and $\mathbf{c}$ is diagonal:

$$\mathbf{c} = \begin{bmatrix}2+\eta_2 & 0\\ 0 & \eta_2\end{bmatrix}.$$

This completes the discussion of one insertion of the operators of dimension 4 in ϕ_4^4 field theory. It reveals the general features of the insertion of any other operator of higher dimension.

Double insertion of operators of dimension 4. Let us now briefly discuss the double $\phi^4(x)$ or $(\nabla\phi(x))^2$ insertion. It is similar to the $\phi^4\phi^2$ insertion. The relation between product of renormalized operators and renormalized product is

$$\left\{\left\{\phi^4(x)\right\}_{\mathrm{r}}\left\{\phi^4(y)\right\}_{\mathrm{r}}\right\}_{\mathrm{r}} = \left\{\phi^4(x)\right\}_{\mathrm{r}}\left\{\phi^4(y)\right\}_{\mathrm{r}} + \delta^{(4)}(x-y)\sum_{i=1}^{4} D_{4i}\left\{\mathcal{O}_i\big(\phi(x)\big)\right\}_{\mathrm{r}}$$
$$+\nabla\delta^{(4)}(x-y)E_4\nabla\mathcal{O}_1\left\{\phi(x)\right\}_{\mathrm{r}} + \nabla^2\delta^{(4)}(x-y)F_4\left\{\mathcal{O}_1\big(\phi(x)\big)\right\}_{\mathrm{r}}, \tag{11.16}$$

in which D_{4i}, E_4 and F_4 are new renormalization constants. A similar equation is valid for $[\nabla\phi(x)]^2$. Again, equation (11.16) is valid only as an insertion.

11.1.3 Operator insertion: General case

Power counting arguments, based on the dimension of operators and sources, imply quite generally that if $\mathcal{O}(\phi,x)$ is an operator of canonical dimension $[\mathcal{O}(\phi)] = D$, then it renormalizes as

$$\left\{\mathcal{O}(\phi,x)\right\}_{\mathrm{r}} = \sum_{\alpha:\ D_\alpha=[\mathcal{O}_\alpha]\le D} Z_\alpha \mathcal{O}_\alpha(\phi,x), \tag{11.17}$$

where Z_α are renormalization constants.

If we now consider the product of two operators $\mathcal{O}(\phi)$ and $\mathcal{O}'(\phi)$ of dimensions D and D' at different points x and y, then,

$$\left\{\left\{\mathcal{O}(\phi,x)\right\}_{\mathrm{r}}\left\{\mathcal{O}'(\phi,y)\right\}_{\mathrm{r}}\right\}_{\mathrm{r}} = \left\{\mathcal{O}(\phi,x)\right\}_{\mathrm{r}}\left\{\mathcal{O}'(\phi,y)\right\}_{\mathrm{r}}$$
$$+\sum_{\alpha\,:\,[\mathcal{O}_\alpha]+[P_\alpha]\le D+D'-d} C_\alpha\left\{\mathcal{O}_\alpha(\phi,x)\right\}_{\mathrm{r}} P_\alpha(\nabla)\delta^{(4)}(x-y), \tag{11.18}$$

in which $P_\alpha(\nabla)$ is a polynomial in ∇. For example, in $\phi^4_{d=4}$ field theory, the product $\{\phi^6(x)\}_{\mathrm{r}}\{\phi^8(y)\}_{\mathrm{r}}$ involves all operators of dimension lower than or equal to 10.

11.1.4 Operator insertion and effective field theory

In Section 8.9, we have argued that to describe large scale physics, the effective action could be reduced to its renormalizable part. The argument was based on an analysis at leading order in perturbation theory; for dimensional reasons, the non-renormalizable interactions are strongly suppressed at large distance. However, beyond leading order, the insertion of these interactions generate strong divergences, which seem to cancel the suppression factor. These divergences were used in the traditional presentation of renormalization theory as an argument to exclude them.

However, we are now in a position to examine more precisely the effect of such insertions. We can invert the relations (11.17) in the form

$$\mathcal{O}(\phi,x) = \sum_{\alpha:\ D_\alpha\le D} Z'_\alpha\left\{\mathcal{O}_\alpha(\phi,x)\right\}_{\mathrm{r}}, \tag{11.19}$$

where $D_\alpha = [\mathcal{O}_\alpha]$ and Z'_α is another combination of renormalization constants. From power counting, we know that the behaviour of Z'_α for a large cut-off Λ is proportional to Λ^{D-D_α}, up to powers of $\ln\Lambda$.

The operator $\mathcal{O}(\phi, x)$ is multiplied by a factor Λ^{4-D}. Therefore, the contribution of the operator $\{\mathcal{O}_\alpha(\phi, x)\}_\mathrm{r}$ is multiplied by a factor Λ^{4-D_α} up to logarithms. One infers that:

(i) the operators corresponding to non-negative powers of Λ, $D_\alpha \le 4$, are those that are already present in the renormalizable action;

(ii) all other operators are suppressed by powers of Λ.

The conclusion is that the higher order contributions of non-renormalizable interactions renormalize the coefficients of the renormalizable action, that is, the field normalization and the ϕ^2 and ϕ^4 coefficients, and add subleading contributions, only modified by powers of logarithms compared the leading order contributions.

For example, in the $\phi^4_{d=4}$ QFT, the leading contributions, for Λ large, correspond to the operators of dimension 6, like ϕ^6, and are suppressed by a factor $1/\Lambda^2$ up to powers of $\ln\Lambda$ to any finite order in perturbation theory. For example,

$$\frac{1}{\Lambda^2}\phi^6(x) = C_1(\Lambda)\{\phi^2(x)\}_\mathrm{r} + C_2(\Lambda)\{\phi^4(x)\}_\mathrm{r} + C_3(\Lambda)\left\{\left(\nabla\phi(x)^2\right\}_\mathrm{r} + \sum_\alpha D_\alpha\left\{\mathcal{O}_{6,\alpha}(\phi, x)\right\}_\mathrm{r},$$

where C_1 grows like Λ^2 up to powers of logarithms, C_2, C_3 grow like powers of logarithms, the $\{\mathcal{O}_{6,\alpha}(\phi, x)\}_\mathrm{r}$ are all operators of dimension 6, and the coefficients D_α decrease like $1/\Lambda^2$ up to logarithms, to any finite order in perturbation theory.

A more precise analysis, useful for numerical simulations of QFTs, also shows that the coefficients of the non-renormalizable interactions can be adjusted in order to cancel the leading corrections for Λ large to the renormalized functions, facilitating in this way the approach to the continuum limit, and leading to the concept of *improved action* [90].

11.2 Quantum field equations

We have discussed the renormalization of local polynomials. However, not all renormalizations are independent in a given field theory, because quantum field equations and other identities discussed in Section 7.5 imply relations between operators.

11.2.1 Insertion of the ϕ^3 operator

We return to the example of the $\phi^3(x)$ operator in the framework of the ϕ^4 field theory.

We consider the action

$$\begin{aligned}\mathcal{S}(\phi, t, u)\\ &= \int \mathrm{d}^4x\left[\tfrac{1}{2}\left(\nabla_\Lambda\phi(x)\right)^2 + \tfrac{1}{2}r\phi^2(x) + \tfrac{1}{4!}g\phi^4(x) + \tfrac{1}{2}t(x)\phi^2(x) + \tfrac{1}{3!}u(x)\phi^3(x)\right]. \quad (11.20)\end{aligned}$$

(∇_Λ is a momentum regularized form of ∇.) A differentiation of the partition function, corresponding to the action, with respect to the external fields $u(x)$ and $t(x)$ generates $-\frac{1}{3!}\phi^3(x)$ and $-\frac{1}{2}\phi^2(x)$ insertions, respectively.

The simplest relations are derived from the identity (see Section 7.5),

$$\int [\mathrm{d}\phi]\left[\frac{\delta\mathcal{S}(\phi, t, u)}{\delta\phi(x)} - J(x)\right]\exp\left(-\mathcal{S}(\phi, t, u) + \int \mathrm{d}^4y\, J(y)\phi(y)\right) = 0\,.$$

Since we have introduced sources for ϕ^2 and ϕ^3, we can use them to express the terms ϕ^2 and ϕ^3 in $\delta\mathcal{S}/\delta\phi$ as functional derivatives with respect to $t(x)$ and $u(x)$. The partition function then satisfies

$$\left[-\nabla^2_\Lambda + r + t(x)\right]\frac{\delta\mathcal{Z}}{\delta J(x)} - g\frac{\delta\mathcal{Z}}{\delta u(x)} - u(x)\frac{\delta\mathcal{Z}}{\delta t(x)} = J(x)\mathcal{Z}(J, t, u). \quad (11.21)$$

The functional $\mathcal{W} = \ln \mathcal{Z}$ then satisfies the equation

$$\left[-\nabla_\Lambda^2 + r + t(x)\right] \frac{\delta \mathcal{W}}{\delta J(x)} - g \frac{\delta \mathcal{W}}{\delta u(x)} - u(x) \frac{\delta \mathcal{W}}{\delta t(x)} = J(x). \tag{11.22}$$

The Legendre transformation is straightforward. The property (7.65) implies

$$\frac{\delta \mathcal{W}}{\delta u(x)} = -\frac{\delta \Gamma}{\delta u(x)}, \quad \frac{\delta \mathcal{W}}{\delta t(x)} = -\frac{\delta \Gamma}{\delta t(x)}\,,$$

and one finds

$$\left[-\nabla_\Lambda^2 + r + t(x)\right] \varphi(x) + g \frac{\delta \Gamma}{\delta u(x)} + u(x) \frac{\delta \Gamma}{\delta t(x)} - \frac{\delta \Gamma}{\delta \varphi(x)} = 0\,. \tag{11.23}$$

Setting $t(x) = u(x) = 0$, we derive a first consequence of the relation. Since

$$\left.\frac{\delta \Gamma}{\delta u(x)}\right|_{u=t=0} = \Gamma_{\phi^3}(\varphi, x), \tag{11.24}$$

where $\Gamma_{\phi^3}(\varphi, x)$ is the generating functional of ϕ-field vertex functions with one $\frac{1}{3!}\phi^3(x)$ insertion, then

$$g\, \Gamma_{\phi^3}(\varphi, x) = \frac{\delta \Gamma}{\delta \varphi(x)} - \left(-\nabla_\Lambda^2 + r\right) \varphi(x). \tag{11.25}$$

The relation shows that, up to explicit subtractions affecting only the $\langle \phi^3 \phi \rangle$ vertex function, the insertion of ϕ^3 is equivalent to the insertion of ϕ itself.

The diagrammatic interpretation of equation (11.25) is simple: the insertion of ϕ^3 is indistinguishable from the addition of a ϕ^4 vertex with one of the lines attached to the vertex being an external line. However, diagrams without a ϕ^4 vertex cannot be generated, and this explains the subtractions (see Fig. 11.2).

Fig. 11.2 ϕ^3 insertion

We now introduce the generating functional of renormalized vertex functions

$$\Gamma_{\mathrm{r}}(\varphi) = \Gamma(\varphi\sqrt{Z})$$

and, thus,

$$\frac{\delta \Gamma_{\mathrm{r}}(\varphi)}{\delta \varphi(x)} = \sqrt{Z}\, \frac{\delta \Gamma(\varphi\sqrt{Z})}{\delta \varphi(x)}.$$

We insert this relation into equation (11.25),

$$\left(-\nabla_\Lambda^2 + r\right) Z \varphi(x) + g\sqrt{Z}\, \Gamma_{\phi^3}(\varphi\sqrt{Z}, x) = \frac{\delta \Gamma_{\mathrm{r}}(\varphi)}{\delta \varphi(x)}. \tag{11.26}$$

The right-hand side is finite in the infinite cut-off limit. We now also introduce in the equation the renormalization constants of the $\phi^4_{d=4}$ QFT, as defined by equations (11.1) and (11.2), and obtain

$$\left[-Z\nabla^2 + m_{\mathrm{r}}^2 + Z\delta r\right] Z\varphi(x) + g_{\mathrm{r}}(Z_g/Z^{3/2})\Gamma_{\phi^3}(\varphi\sqrt{Z}, x) = \frac{\delta\Gamma_{\mathrm{r}}(\varphi)}{\delta\varphi(x)}. \tag{11.27}$$

This relation shows that all ϕ-field vertex functions with one insertion of the operator $Z_g\phi_{\mathrm{r}}^3(x)$ are finite, except the $\langle\phi^3\phi\rangle$ vertex function which needs two additional subtractions. We determine the corresponding renormalization constants by imposing, in the Fourier representation,

$$\left\{\tilde{\Gamma}^{(1)}_{\phi^3}\right\}_{\mathrm{r}}(p,-p)\Big|_{p=0} = 0\,,$$
$$\frac{\partial}{\partial p^2}\left\{\tilde{\Gamma}^{(1)}_{\phi^3}\right\}_{\mathrm{r}}(p,-p)\Big|_{p=0} = 0\,.$$

The coefficient of degree n in φ in equation (11.27) then yields explicitly (q is the argument of ϕ^3),

$$\tilde{\Gamma}^{(n+1)}_{\mathrm{r}}(q;p_1,\ldots,p_n) = g_{\mathrm{r}}\left\{\tilde{\Gamma}^{(n)}_{\phi^3}\right\}_{\mathrm{r}}(q;p_1,\ldots,p_n) + \delta_{n1}\left(p^2+m_{\mathrm{r}}^2\right). \tag{11.28}$$

With this definition $\{\tilde{\Gamma}^{(3)}_{\phi^3}\}_{\mathrm{r}}$ satisfies the renormalization condition

$$\left\{\tilde{\Gamma}^{(3)}_{\phi^3}\right\}_{\mathrm{r}}(0;0,0,0) = 1\,. \tag{11.29}$$

Equation (11.23) also contains information about multiple insertions of ϕ^3. For example, after some algebraic manipulations, for two insertions, one finds

$$\begin{aligned} &g^2\Gamma_{\phi^3\phi^3}(\varphi,x_1,x_2) + g\delta^{(4)}(x_1-x_2)\Gamma_{\phi^2}(\varphi,x_1) + \left(-\nabla^2_\Lambda + r\right)\delta^{(4)}(x_1-x_2) \\ &\quad = \frac{\delta^2\Gamma}{\delta\varphi(x_1)\delta\varphi(x_2)}\,. \end{aligned} \tag{11.30}$$

The equation relates two insertions of ϕ^3 to two insertions of ϕ, again with subtraction terms, which now involve the insertion of ϕ^2.

11.2.2 Other relations: Renormalization of operators of dimension 4

We have shown in Section 7.5 that more general equations are obtained by performing infinitesimal changes of variables. We can use them to establish relations between operators. For example, in the change of field variables,

$$\phi(x) \mapsto \phi'(x),\ \text{with}\ \phi'(x) = \phi(x) + \varepsilon(x)\phi(x),$$

the variation of the action (11.1) in the presence of a source is

$$\delta\left[\mathcal{S}(\phi) - \int \mathrm{d}^4y\, J(y)\phi(y)\right] = \int \mathrm{d}^4x\,\varepsilon(x)\left[\phi(x)\left(-\nabla^2_\Lambda + r\right)\phi(x) + \frac{g}{3!}\phi^4(x) - J(x)\phi(x)\right].$$

We define the operator $O(x) \equiv \phi(x)\left(-\nabla^2_\Lambda + r\right)\phi(x)$. From the invariance of the field integral, it follows that

$$\mathcal{W}_O(J,x) + \frac{g}{3!}\mathcal{W}_{\phi^4}(J,x) = J(x)\frac{\delta\mathcal{W}}{\delta J(x)}\,. \tag{11.31}$$

The discussion of the large cut-off limit of the equation, or the corresponding one obtained after Legendre transformation,

$$\Gamma_O(\phi, x) + \frac{g}{3!}\Gamma_{\phi^4}(\phi, x) = \varphi(x)\frac{\delta\Gamma}{\delta\varphi(x)}, \tag{11.32}$$

is more subtle than in the case of equation (11.25): the operator $O(x)$, which in Pauli–Villars regularization is

$$O(x) \equiv \phi(x)\left(-\nabla^2 + r + \alpha_1\nabla^4/\Lambda^2 - \alpha_2\nabla^6/\Lambda^3 + \cdots\right)\phi(x),$$

contains operators of canonical dimensions larger than 4 divided by powers of the cut-off. We have discussed in Section 11.1.4 the insertion of non-renormalizable (irrelevant) operators in four dimensions in the large cut-off limit (we return to the problem in dimension $d = 4 - \varepsilon$ in Section 17.3). The conclusion is that, in the large cut-off limit, the operator $\phi(-\nabla_\Lambda^2 + r)\phi$ is equivalent, at leading order, to a linear combination of all operators of dimensions 4 and 2.

Equation (11.32) implies, after renormalization, the identity satisfied by the operators $\{\mathcal{O}_i(\phi)\}_\mathrm{r}$ as defined in equations (11.8) and (11.9),

$$\sum_{i=1}^{4} C_i(g_\mathrm{r})\left\{\tilde{\Gamma}^{(n)}_{\mathcal{O}_i}\right\}_\mathrm{r}(q; p_1, \ldots, p_n) = \sum_{m=1}^{n}\tilde{\Gamma}^{(n)}_r(p_1, \ldots, p_m + q, \ldots, p_n). \tag{11.33}$$

To explicitly define the insertions of the operators $\mathcal{O}_i$, in the spirit of Chapter 9 we impose the renormalization conditions,

$$\begin{aligned}
\left\{\tilde{\Gamma}^{(2)}_{\mathcal{O}_1}\right\}_\mathrm{r}(0; 0, 0) &= m_\mathrm{r}^2\,, \\
\left\{\tilde{\Gamma}^{(2)}_{\mathcal{O}_3}\right\}_\mathrm{r}(q; p_1, p_2) &= -p_1\cdot p_2 + O\left(p^4\right), \\
\left\{\tilde{\Gamma}^{(4)}_{\mathcal{O}_3}\right\}_\mathrm{r}(0; 0, 0, 0, 0) &= 0\,, \\
\left\{\tilde{\Gamma}^{(2)}_{\mathcal{O}_4}\right\}_\mathrm{r}(q; p_1, p_2) &= O\left(p^4\right), \\
\left\{\tilde{\Gamma}^{(4)}_{\mathcal{O}_4}\right\}_\mathrm{r}(0; 0, 0, 0, 0) &= 1\,.
\end{aligned} \tag{11.34}$$

We can then calculate the coefficients $C_i(g_\mathrm{r})$ only from $\{\Gamma^{(n)}_{\mathcal{O}_1}\}_\mathrm{r}$ and its derivatives at zero momentum.

11.3 Short-distance expansion of operator products

Several chapters are devoted to the discussion of the large or small momentum behaviour of correlation functions. Our essential tools are the CS or RG equations. However, these equations lead to a direct characterization of the behaviour of

$$\tilde{\Gamma}^{(n)}(\rho p_1 + r_1, \rho p_2 + r_2, \ldots, \rho p_n + r_n),$$

for $\rho \to \infty$, only for *non-exceptional momenta*, that is, provided all, not empty strictly subsets of momenta p_i have a non-vanishing sum. If we denote by I the indices of momenta in such a subset, the condition can be written as

$$\sum_{i\in I\neq\emptyset} p_i \neq 0\,, \quad \forall I \not\equiv \{1, 2, \ldots, n\}\,.$$

For exceptional momenta, a new tool is required: the SDE of product of operators [84, 85, 87]. In this section, we mainly discuss the SDE of the product of two fields at leading order, but we shall indicate how the method can be generalized to more complicated examples. It is entirely based on the theory of renormalization of local monomials we have just described.

Definition. We consider the vertex function

$$\Gamma^{(n+2)}(x+v/2, x-v/2, y_1, \dots, y_n) = \langle \phi(x+v/2)\phi(x-v/2)\phi(y_1)\cdots\phi(y_n)\rangle_{1\mathrm{PI}}\,, \tag{11.35}$$

in which all arguments are fixed, except the vector v. We want to investigate the $|v| \to 0$ limit. In a QFT that is sufficiently regularized (see Section A8.3), one can expand the product of fields $\phi(x+v/2)\phi(x-v/2)$ in powers of v,

$$\begin{aligned}\phi\left(x+v/2\right)\phi\left(x-v/2\right) = \phi^2(x) + \frac{1}{4}\sum_{\mu_1,\mu_2} v_{\mu_1}v_{\mu_2}\left[\phi(x)\partial_{\mu_1}\partial_{\mu_2}\phi(x) - \partial_{\mu_1}\phi(x)\partial_{\mu_2}\phi(x)\right] \\ + \sum_{n=2}^{\infty}\frac{1}{2^{2n}}\frac{1}{2n!}\sum_{\mu_1,\dots,\mu_{2n}} v_{\mu_1}\cdots v_{\mu_{2n}}\mathcal{O}_{\mu_1\dots\mu_{2n}}\left[\phi(x)\right],\end{aligned} \tag{11.36}$$

in which $\mathcal{O}_{\mu_1\dots\mu_{2n}}(\phi)$ is a local operator quadratic in ϕ, with $2n$ derivatives.

If we now insert expansion (11.36) into a correlation function, even after field renormalization, all terms of the expansion in general diverge when the cut-off is removed. As we have extensively discussed in Section 11.1, the various local monomials appearing in expansion (11.36) have to be renormalized. Therefore, we have to expand each bare operator on a basis of renormalized operators. In terms of renormalized operators $\{\mathcal{O}^\alpha(\phi)\}_\mathrm{r}$, the expansion (11.36) takes the form

$$\phi_\mathrm{r}(x+v/2)\phi_\mathrm{r}(x-v/2) = \sum_\alpha C_\alpha(v,\Lambda)\left\{\mathcal{O}^\alpha(x)\right\}_\mathrm{r}, \tag{11.37}$$

in which, at cut-off Λ fixed, the coefficients $C_\alpha(v,\Lambda)$ are regular, even functions of v. An operator $\{\mathcal{O}^\alpha(x)\}_\mathrm{r}$ receives contributions from bare operators of equal or higher dimensions. We conclude that, for $|v|$ small, the coefficient functions $C_\alpha(v,\Lambda)$ behave like

$$C_\alpha(\lambda v,\Lambda) \sim \lambda^{[\mathcal{O}_\alpha]-2[\phi]}, \quad \text{for } \lambda \to 0\,. \tag{11.38}$$

When the cut-off becomes infinite, the coefficients of the expansion of C_α in powers of v, being renormalization constants, diverge, and if C_α has a limit, the limiting function is singular at $v = 0$. The coefficients are functions of the cut-off, which, for $v = 0$, diverge in a way predicted by power counting. Since v is small but non-vanishing, the coefficients grow with the cut-off until Λ is of order $1/|v|$. In this range, all contributions to a given coefficient are of the same order, up to powers of logarithms, because powers of v compensate the powers of Λ. Therefore, at least in perturbation theory, the ordering of operators consequence of equation (11.38) survives, the operators of lowest dimensions dominate the expansion (11.37) for $|v|$ small, and the behaviour of the limiting coefficients $C_\alpha(v)$ is given by equation (11.38), up to powers of $\ln v$.

The expansion (11.37) converges towards the SDE of the product of two operators ϕ.

11.3.1 SDE at leading order

Further insight into the structure of the SDE is gained by realizing that the product $\phi_\mathrm{r}(x+v/2)\phi_\mathrm{r}(x-v/2)$ can be considered as a regularization by point splitting of the local monomials $\{\phi^2(x)\}_\mathrm{r}$. Let us investigate this point more precisely in the framework of the $\phi^4_{d=4}$ QFT. We then expect

$$\phi_\mathrm{r}(x+v/2)\phi_\mathrm{r}(x-v/2) \underset{|\mathbf{v}|\to 0}{\sim} C_1(v)\left\{\phi^2(x)\right\}_\mathrm{r} . \tag{11.39}$$

The singularities of $C_1(v)$ for $|\mathbf{v}|$ small should be directly related to the divergences of the renormalization constant Z_2 that renders $\{\phi^2(x)\}_\mathrm{r}$ finite.

In what follows, it becomes convenient to treat the product $\frac{1}{2}\phi_\mathrm{r}(x+v/2)\phi_\mathrm{r}(x-v/2)$ as a local monomial, depending on the point x and a parameter v, in particular, from the point of view of connectivity and one-particle irreducibility. To make this explicit, we set

$$\Xi(x,v) = \tfrac{1}{2}\phi_\mathrm{r}(x+v/2)\phi_\mathrm{r}(x-v/2). \tag{11.40}$$

However, the relation between connected correlation functions is then affected, for example,

$$\begin{aligned}\left\langle \Xi(x,v)\phi_\mathrm{r}(y_1)\cdots\phi_\mathrm{r}(y_n)\right\rangle_\mathrm{c} &= \tfrac{1}{2}\left\langle \phi_\mathrm{r}(x+v/2)\phi_\mathrm{r}(x-v/2)\phi_\mathrm{r}(y_1)\cdots\phi_\mathrm{r}(y_n)\right\rangle_\mathrm{c}\\ &\quad+\tfrac{1}{2}\sum_{I\cup J=(1,\ldots,n)}\left\langle \phi_\mathrm{r}(x+v/2)\phi_\mathrm{r}(y_{i_1})\cdots\phi_\mathrm{r}(y_{i_p})\right\rangle_\mathrm{c}\\ &\quad\times\left\langle \phi_\mathrm{r}(x-v/2)\phi_\mathrm{r}(y_{j_1})\cdots\phi_\mathrm{r}(y_{j_{n-p}})\right\rangle_\mathrm{c} ,\end{aligned} \tag{11.41}$$

in which I and J are all non-empty partitions of $(1,\ldots,n)$.

Rather than writing explicitly the corresponding relations between vertex functions, we give the relation in terms of generating functionals. Denoting the generating functional of correlation functions with one insertion of $\Xi(x,v)$ by

$$\mathcal{Z}_\Xi(x,v;J) = \frac{1}{2}\frac{\delta^2\mathcal{Z}(J)}{\delta J(x+v/2)\delta J(x-v/2)}, \tag{11.42}$$

we find for, connected correlation functions with obvious notation,

$$\begin{aligned}\mathcal{W}_\Xi(x,v;J) &= \mathcal{Z}_\Xi(x,v;J)\mathcal{Z}^{-1}(J)\\ &= \frac{1}{2}\frac{\delta^2\mathcal{W}(J)}{\delta J(x+v/2)\delta J(x-v/2)} + \frac{1}{2}\frac{\delta\mathcal{W}}{\delta J(x+v/2)}\frac{\delta\mathcal{W}}{\delta J(x-v/2)} ,\end{aligned} \tag{11.43}$$

and, finally, for the generating functional of vertex functions,

$$\Gamma_\Xi(x,v;\varphi) = -\frac{1}{2}\varphi(x+v/2)\varphi(x-v/2) - \frac{1}{2}\left[\frac{\delta^2\Gamma}{\delta\varphi(x+v/2)\delta\varphi(x-v/2)}\right]^{-1} . \tag{11.44}$$

where inverse means inverse in the sense of kernels.

Note that the equation is analogous to equation (7.107), which arises in the proof the one-line irreducibility of $\Gamma(\varphi)$.

To now ensure the limit (11.39), we determine $C_1(v)$ by imposing that the insertion of the operator $-\frac{1}{2}C_1^{-1}(v)\Xi(x,v)$ in the two-point function satisfies, for any v, the renormalization condition (9.33). Defining

$$C_1^{-1}(v)\int \mathrm{d}^4x\,\mathrm{d}^4y_1\,\mathrm{d}^4y_2\ \mathrm{e}^{ipx+iq_1y_1+iq_2y_2}\ \left.\frac{\delta^2\{\Gamma_\Xi(x,v;\varphi)\}_\mathrm{r}}{\delta\varphi(y_1)\delta\varphi(y_2)}\right|_{\varphi=0}$$
$$= -(2\pi)^4\delta^{(4)}(p+q_1+q_2)\tilde\Gamma_\mathrm{r}^{(1,2)}(v;p;q_1,q_2), \tag{11.45}$$

we impose

$$\tilde\Gamma_\mathrm{r}^{(1,2)}(v;0;0,0)=1\,. \tag{11.46}$$

Setting $q_1=q_2=0$ in equation (11.45), we derive

$$C_1(v)=-\left.\frac{\delta^2\{\Gamma_\Xi(0,v;\varphi)\}_\mathrm{r}}{\delta\tilde\varphi(0)\delta\tilde\varphi(0)}\right|_{\varphi=0}, \tag{11.47}$$

in which $\tilde\varphi(q)$ is the Fourier transform of $\varphi(x)$.

By differentiating equation (11.44) twice with respect to φ, we relate the right-hand side of equation (11.47) to the φ-field vertex functions, and find

$$C_1(v)=1-\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{-ikv}\left[\widetilde{W}_\mathrm{r}^{(2)}(k)\right]^2\tilde\Gamma_\mathrm{r}^{(4)}(k,-k,0,0). \tag{11.48}$$

The equation can also be rewritten as

$$C_1(v)=1+\tfrac{1}{2}m_\mathrm{r}^4\langle\phi_\mathrm{r}(v/2)\phi_\mathrm{r}(-v/2)\tilde\phi_\mathrm{r}(0)\tilde\phi_\mathrm{r}(0)\rangle_\mathrm{c}\,. \tag{11.49}$$

We have introduced a mixed connected correlation function, $\tilde\phi(p)$ being the Fourier transform of the field $\phi(x)$ and used the renormalization conditions (9.30).

The coefficient $C_1(v)$ is defined in such a way that the renormalized operator $\phi_\mathrm{r}(x+v/2)\phi_\mathrm{r}(x-v/2)C_1^{-1}(v)$ then converges towards the operator $\{\phi_\mathrm{r}^2(x)\}_\mathrm{r}$, correctly normalized.

For example, the implication for the four-point function is

$$C_1(v)\tilde\Gamma_\mathrm{r}^{(1,2)}(p;q_1,q_2)\underset{|\mathbf{v}|\to0}{\sim}1-\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{-ikv}\,\widetilde{W}_\mathrm{r}^{(2)}(p/2-k)\widetilde{W}_\mathrm{r}^{(2)}(p/2+k)$$
$$\times\,\tilde\Gamma_\mathrm{r}^{(4)}(p/2+k,p/2-k,q_1,q_2). \tag{11.50}$$

The neglected contributions are of order v^2, up to powers of $\ln v$, at any finite order in the perturbative expansion.

Renormalization theory tells us that equation (11.39) is valid as long as the replacement of the operator product by $\phi^2(x)$ does not generate new renormalizations. Therefore, we can use equation (11.39) in all $\phi(x)$ and $\phi^2(x)$ correlation functions except for:

(i) the two-point function $\langle\phi(x+v/2)\phi(x-v/2)\rangle$, which leads to $\langle\phi^2(x)\rangle$ and

(ii) the four-point function $\langle\phi(x+v/2)\phi(x-v/2)\phi^2(y)\rangle$, which leads to the ϕ^2 two-point function $\langle\phi^2(x)\phi^2(y)\rangle$.

Both require an additional additive renormalization. The strategy in such cases is to first apply the CS differential operator on the correlation function to generate additional $\int\phi^2(x)\mathrm{d}x$ insertions until relation (11.39) can be used. As a consequence, the SDE is modified by contributions which are solutions of the homogeneous CS equations.

11.3.2 One-loop calculation of the leading coefficient of the SDE

Equation (11.48) can be used to calculate the coefficient function $C_1(v)$ in perturbation theory. At one-loop order, it is sufficient to replace correlation functions by their tree level values, and the bare parameters by renormalized parameters:

$$C_1(v) = 1 - \frac{g}{2} \int \frac{\mathrm{d}^4 k}{16\pi^2} \frac{\mathrm{e}^{-ikv}}{\left(k^2 + m_\mathrm{r}^2\right)^2} + O\left(g^2\right). \tag{11.51}$$

It is clear from this expression that, at one-loop order, $C_1(v)$ has the form

$$C_1(v) = A \ln\left(m_\mathrm{r}|v|\right) + B + O\left(v^2\right). \tag{11.52}$$

A frequently useful idea to extract an asymptotic expansion of this form is to calculate the Mellin transform $\mu(\alpha)$ of the function:

$$\mu(\alpha) = \int_0 \mathrm{d}v\, v^{\alpha-1} C_1(v), \tag{11.53}$$

in which v is the length $|\mathbf{v}|$ of the vector $\mathbf{v}$. For $\alpha \to 0$, the expansion (11.52) then implies

$$\mu(\alpha) = -\frac{A}{\alpha^2} + \left(A \ln m_\mathrm{r} + B\right) \frac{1}{\alpha} + O\left(1\right). \tag{11.54}$$

Applying this technique, one has to evaluate

$$f(\alpha) = \int_0 \mathrm{d}v\, v^{\alpha-1} \int \frac{\mathrm{d}^4 k}{16\pi^2} \frac{\mathrm{e}^{-ikv}}{\left(k^2 + m_\mathrm{r}^2\right)^2}. \tag{11.55}$$

As usual, we rewrite the integral as

$$f(\alpha) = \int_0 \mathrm{d}v\, v^{\alpha-1} \int_0^\infty t\, \mathrm{d}t \frac{\mathrm{d}^4 k}{16\pi^2}\, \mathrm{e}^{-t\left(k^2+m_\mathrm{r}^2\right)-ikv}\,. \tag{11.56}$$

Finally, integrating over k, v, and t, in this order, we obtain

$$f(\alpha) = \frac{1}{8}\left(\frac{2}{m_\mathrm{r}}\right)^\alpha \frac{\Gamma^2(1+\alpha/2)}{\alpha^2}. \tag{11.57}$$

An expansion for $\alpha \sim 0$ yields the coefficients A and B:

$$C_1(v) = 1 - \frac{g}{16}\left[-\ln\left(\frac{|v| m_\mathrm{r}}{2}\right) + \psi(1)\right] + O\left(g^2\right), \tag{11.58}$$

in which $\psi(z)$ is the logarithmic derivative of the function $\Gamma(z)$.

11.4 Large-momentum expansion of the SDE coefficients: CS equations

To the behaviour at short distance of the product of fields $\phi(x+v/2)\phi(x-v/2)$ is associated, after Fourier transformation, the behaviour at large relative momentum k of the product $\tilde{\phi}(p/2-k)\tilde{\phi}(p/2+k)$. However, some information is lost in the transformation. The large-momentum behaviour is only sensitive to the singular part of the short-distance behaviour. For instance, after Fourier transformation, the constant terms in the asymptotic expansion of $C_1(v)$ yield terms proportional to $\delta^{(4)}(k)$, which do not contribute to the large-momentum behaviour. As a consequence, the algebraic structure is, for the same reason, somewhat simplified.

We consider the example of equation (11.50). We introduce the Fourier transform

$$\tilde{C}_1(k) = \int \mathrm{e}^{ikv}\, C_1(v)\mathrm{d}^4 v\,. \tag{11.59}$$

After Fourier transformation, for k large, equation (11.50) yields

$$\tilde{\Gamma}^{(4)}_{\mathrm{r}}(p/2+k, p/2-k, q_1, q_2) \sim -2\tilde{C}_1(k)\left[\tilde{\Gamma}^{(2)}_{\mathrm{r}}(k)\right]^2 \tilde{\Gamma}^{(1,2)}_{\mathrm{r}}(p; q_1, q_2), \tag{11.60}$$

in which the neglected terms are of order $1/k^2$ up to powers of $\ln k$ at any finite order in perturbation theory.

More generally, due to momentum conservation, the disconnected contributions in equation (11.43) coming from $\delta\mathcal{W}/\delta J(x+v/2)\delta\mathcal{W}/\delta J(x-v/2)$ do not contribute to the large-momentum behaviour.

In addition, the expansion of the right-hand side of equation (11.44) in powers of φ yields two types of contributions: one contribution which becomes one-line irreducible after amputation of the lines corresponding to the fields $\phi(x+v/2)$ and $\phi(x-v/2)$, and the other contributions, which remain reducible. Fig. 11.3 gives the example of the six-point function.

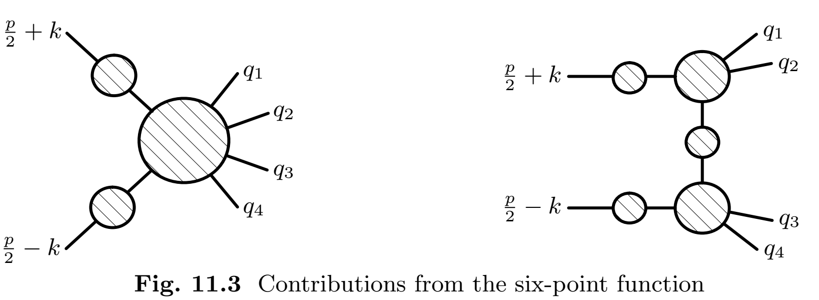

Fig. 11.3 Contributions from the six-point function

Due to momentum conservation in the reducible terms, the propagator that connects the vertex functions carries a momentum of order k, for k large. The corresponding contributions are thus suppressed by a factor $1/k^2$ (up to powers of $\ln k$), and can be neglected at leading order.

Therefore, at leading order, one finds, for $n>0$,

$$\tilde{\Gamma}^{(n+2)}_{\mathrm{r}}\left(\frac{p}{2}+k, \frac{p}{2}-k, q_1, \ldots, q_n\right) \underset{|k|\to\infty}{\sim} -2\tilde{C}_1(k)\left[\tilde{\Gamma}^{(2)}_{\mathrm{r}}(k)\right]^2 \tilde{\Gamma}^{(1,n)}_{\mathrm{r}}(p; q_1, \ldots, q_n). \tag{11.61}$$

As explained in Section 11.3, the equation generalizes to ϕ^2 insertions, for $n > 0$ or $l > 1$,

$$\begin{aligned}&\tilde{\Gamma}_{\mathrm{r}}^{(l,n+2)}\,(p_1,\ldots,p_l;p/2+k,p/2-k,q_1,\ldots,q_n)\\&\quad\sim -2\tilde{C}_1(k)\left[\tilde{\Gamma}_{\mathrm{r}}^{(2)}(k)\right]^2\tilde{\Gamma}_{\mathrm{r}}^{(l+1,n)}(p_1,\ldots,p_l,p;q_1,\ldots,q_n).\end{aligned} \tag{11.62}$$

Callan–Symanzik equation for the first coefficient of the SDE [88]. From now on, all quantities are assumed to be renormalized and *we omit the subscript* r.

By comparing equations (11.48) and (11.61), we note that in the momentum representation the relevant function is $\tilde{\Gamma}^{(4)}(k,-k,0,0)$, which we call $B(k)$ in what follows:

$$B(k)\equiv\tilde{\Gamma}^{(4)}(k,-k,0,0)\underset{|\mathbf{k}|\to\infty}{\sim}-2C_1(k)\left[\tilde{\Gamma}^{(2)}(k)\right]^2. \tag{11.63}$$

Equation (11.61) becomes

$$\tilde{\Gamma}^{(n+2)}(p/2+k,p/2-k,q_1,\ldots,q_n)\sim B(k)\tilde{\Gamma}^{(1,n)}(p;q_1,\ldots,q_n). \tag{11.64}$$

Similarly,

$$\tilde{\Gamma}^{(1,n+2)}(0;p/2+k,p/2-k,q_1,\ldots,q_n)\sim B(k)\tilde{\Gamma}^{(2,n)}(0,p;q_1,\ldots,q_n). \tag{11.65}$$

We introduce for the CS differential operator the notation

$$\mathrm{D}\equiv m_{\mathrm{r}}\frac{\partial}{\partial m_{\mathrm{r}}}+\beta(g)\frac{\partial}{\partial g}. \tag{11.66}$$

We write the CS equations for $\tilde{\Gamma}^{(n+2)}$ as

$$\left[\mathrm{D}-\tfrac{1}{2}(n+2)\eta(g)\right]\tilde{\Gamma}^{(n+2)}(\ldots)=m_r^2\sigma(g)\tilde{\Gamma}^{(1,n+2)}(0;\ldots). \tag{11.67}$$

We now use equations (11.64) and (11.65) in the large $|k|$ limit:

$$\left[\mathrm{D}-\tfrac{1}{2}(n+2)\eta(g)\right]\Big(B(k)\tilde{\Gamma}^{(1,n)}(p;q_1,\ldots,q_n)\Big)=m_{\mathrm{r}}^2\sigma(g)B(k)\tilde{\Gamma}^{(2,n)}(0,p;q_1,\ldots,q_n).$$

Using the CS equation for $\tilde{\Gamma}^{(1,n)}$,

$$\mathrm{D}\tilde{\Gamma}^{(1,n)}=\left[\tfrac{1}{2}n\eta(g)+\eta_2(g)\right]\tilde{\Gamma}^{(1,n)}+m_{\mathrm{r}}^2\sigma(g)\tilde{\Gamma}^{(2,n)}(0,\ldots), \tag{11.68}$$

we finally obtain an equation for $B(k)$:

$$\left[\mathrm{D}-\eta(g)+\eta_2(g)\right]B(k)\sim 0\,. \tag{11.69}$$

We can also write an equation for $\tilde{C}_1(k)$, using relation (11.63):

$$\left[\mathrm{D}+\eta(g)+\eta_2(g)\right]\tilde{C}_1(k/m_{\mathrm{r}},g)\sim 0\,. \tag{11.70}$$

We can compare this equation with equations (9.38) and (9.39), which imply

$$\left[\mathrm{D}-\eta(g)-\eta_2(g)\right]Z_2(\Lambda/m_{\mathrm{r}},g)=0\,. \tag{11.71}$$

These equations confirm that $\tilde{C}_1(k/m_{\mathrm{r}},g)$ and $Z_2^{-1}(\Lambda/m_{\mathrm{r}},g)$ play similar roles.

11.5 SDE beyond leading order. General operator product

In this section also, all quantities except the operators are assumed to be renormalized.

The $\phi\phi$ product. The product $\phi(x+v/2)\phi(x-v/2)$ is not only a regularization of $\phi^2(x)$ by point splitting, but also of operators of higher dimensions which can be obtained by differentiation.

At next order, which means taking in expansion (11.37) all operators of dimensions 2 and 4 into account, the SDE of $\phi(x+v/2)\phi(x-v/2)$ is an expansion of a regularized bare operator of dimension 4 on a basis of renormalized operators of dimensions 2 and 4, as discussed in previous sections. Indeed, differentiating the expansion (11.37) twice with respect to v, one obtains

$$\begin{aligned}\frac{1}{4}\Big[\phi\left(x+\frac{v}{2}\right)\partial_\mu\partial_\nu\phi\left(x-\frac{v}{2}\right)+\phi\left(x-\frac{v}{2}\right)\partial_\mu\partial_\nu\phi\left(x+\frac{v}{2}\right)-\partial_\mu\phi\left(x+\frac{v}{2}\right)\partial_\nu\phi\left(x-\frac{v}{2}\right)\\ -\partial_\mu\phi\left(x-\frac{v}{2}\right)\partial_\nu\phi\left(x+\frac{v}{2}\right)\Big]=\sum_\alpha\partial_\mu\partial_\nu C_\alpha(v)\left\{\mathcal{O}^\alpha\big(\phi(x)\big)\right\}_{\mathrm{r}}.\end{aligned}\tag{11.72}$$

The product in the right-hand side can be considered as a form regularized by point splitting of $\frac{1}{2}\left(\phi(x)\partial_\mu\partial_\nu\phi(x)-\partial_\mu\phi(x)\partial_\nu\phi(x)\right)$, which is a linear combination of operators of dimension 4, and spins 0 and 2. In Section 11.1, we have discussed the renormalization of operators of dimension 4 and spin 0. The operators of spin 2 introduce two new linearly independent operators, which can be chosen to be the traceless part of $\partial_\mu\phi(x)\partial_\nu\phi(x)$ and $\partial_\mu\partial_\nu\left(\phi^2(x)\right)$. Rotation invariance in space implies that operators of different spins do not mix under renormalization. Therefore, in addition to the relations (11.9), we have

$$\begin{aligned}\mathcal{O}_5\big(\phi(x)\big)&\equiv\partial_\mu\phi(x)\partial_\nu\phi(x)-\tfrac{1}{4}\delta_{\mu\nu}\left(\nabla\phi(x)\right)^2\\ &=Z_5^{-1}\left\{\big(\mathcal{O}_5\big(\phi(x)\big)\right\}_{\mathrm{r}}-B_5\left(\partial_\mu\partial_\nu-\tfrac{1}{4}\delta_{\mu\nu}\nabla^2\right)\left\{\phi^2(x)\right\}_{\mathrm{r}}.\end{aligned}\tag{11.73}$$

We can impose the renormalization conditions

$$\begin{aligned}\left\{\tilde\Gamma^{(2)}_{\mathcal{O}_5}\right\}_{\mathrm{r}}(q;p_1,p_2)&=-\left(p_1^\mu p_2^\nu+p_1^\nu p_2^\mu-\tfrac{1}{2}\delta_{\mu\nu}p_1p_2\right)+O\left(p^4\right),\\ \left\{\tilde\Gamma^{(4)}_{\mathcal{O}_5}\right\}_{\mathrm{r}}(0;0,0,0,0)&=0\,.\end{aligned}\tag{11.74}$$

We can now use exactly the same strategy as for the leading term in the SDE. We insert the expansion (11.72), truncated by omitting all operators of dimension larger than 4, which correspond to vanishing coefficients, in the two- and four-point correlation functions. We go over to the vertex functions in the way explained in Section 11.3, that is, considering the product of fields in (11.72) as a local monomial. Finally, we use the renormalization conditions (11.34) and (11.74) to determine all the coefficients $\partial_\mu\partial_\nu C_\alpha(v)$ in the truncated SDE.

Note that we have lost no information by differentiating twice, since only $C_1(v)$, which has been already determined has a constant part when $|v|$ goes to zero. In this limit, the coefficients $\partial_\mu\partial_\nu C_\alpha(v)$ have singularities in v that are similar to the divergences in Λ/m of the renormalization constants which appear in the expansion of operators $\mathcal{O}_5$ on a basis of dimension 4 renormalized operators. To determine the asymptotic behaviour of the coefficients $C_\alpha(v)$, we then have to establish RG equations for them, by introducing the SDE in the CS equations for vertex functions. We have to worry about the SDE expansion in the presence of a $\phi^2(x)$ insertion. We have to use the analogue of equation (11.10), the difference being that the new renormalization constant, which appears in front of the contact term, is now a function of v.

We conclude that the coefficients of the SDE satisfy RG equations which are formally identical to the relations between the renormalization constants and the finite RG functions which arise in the CS equations for the operators of dimension 4 and spins 0 and 2, in complete analogy with the correspondence between equations (11.70) and (11.71).

SDE of products of arbitrary local operators. For general local operators A and B, we expect

$$A(x+v/2)B(x-v/2) = \sum_{\alpha} C^{\alpha}_{AB}(v)\mathcal{O}^{\alpha}_{\mathrm{r}}(x), \tag{11.75}$$

in which at any finite order in perturbation theory

$$C^{\alpha}_{AB}(\lambda v) \sim \lambda^{[\mathcal{O}^{\alpha}]-[A]-[B]} \quad \text{for} \quad \lambda \to 0\,, \tag{11.76}$$

up to powers of $\ln\lambda$ and the $\{\mathcal{O}^{\alpha}(x)\}_{\mathrm{r}}$ form a complete basis of local operators.

Let us consider the example of $A \equiv B \equiv \left\{\phi^2(x)\right\}_{\mathrm{r}}$. The product $\phi^2(x+v/2)\phi^2(x-v/2)$ is a regularized form of $\phi^4(x)$; however, $\phi^4(x)$ by renormalization mixes with all operators of dimension 4 and $\phi^2(x)$. Power counting implies that, among these operators, $\phi^2(x)$ has the most divergent coefficient. Therefore, at leading order,

$$\left\{\phi^2(x+v/2)\right\}_{\mathrm{r}}\left\{\phi^2(x-v/2)\right\}_{\mathrm{r}} \underset{|\mathbf{v}|\to 0}{\sim} C^{1}_{\phi^2\phi^2}(v)\left\{\phi^2(x)\right\}_{\mathrm{r}}\,, \tag{11.77}$$

in which the coefficient can be determined by using the renormalization condition for $\{\phi^2\}_{\mathrm{r}}$:

$$\tilde{\Gamma}^{(1,2)}(q;p_1,p_2) = 1\,, \quad \text{for} \quad q = p_1 = p_2 = 0\,,$$

and expressing that equation (11.77) should be exact when inserted into the two-point function at the subtraction point where all momenta vanish.

It is then simple to derive RG equations for this new coefficient by inserting the relation into the CS equations for the vertex functions $\tilde{\Gamma}^{(l,n)}$.

We do not go into further detail, since the discussion becomes rather technical. The most important idea to keep in mind is the complete parallelism between the SDE of operator products and the renormalization equations of the corresponding local monomials.

11.6 Light-cone expansion of operator products

After analytic continuation to real time, the length of the vector squared, x^2, may vanish, while the vector x_μ remains finite. In such a situation the relevant expansion for a product of fields is no longer the SDE but, instead, the light-cone expansion (LCE) [89].

It is necessary to classify all operators not only according to their canonical dimensions, but also to their spin s, which characterizes their transformation properties under space rotations. The LCE takes the form

$$\phi(x+v/2)\phi(x-v/2) = \sum_{\alpha,s} C^{s}_{\alpha}\left(v^2\right) P^{s}_{\mu_1\ldots\mu_s}(v)\mathcal{O}^{s,\alpha}_{\mu_1\ldots\mu_s}(x). \tag{11.78}$$

The polynomial $P^{s}_{\mu_1\ldots\mu_s}(v)$ is a homogeneous, traceless for $s>0$, polynomial of the vector v_μ, and the operators $\mathcal{O}^{s,\alpha}_{\mu_1\ldots\mu_s}$ form a complete basis of local operators.

For example,

$$P^{2}_{\mu_1,\mu_2}(v) = v_{\mu_1}v_{\mu_2} - \delta_{\mu_1\mu_2}v^2/d\,.$$

When v^2 goes to zero with v_μ finite, the polynomials P^s have a finite limit and, therefore, the coefficients $C^s_\alpha(v^2)$ contain the whole non-trivial dependence on v^2. The analysis, already performed for the SDE, can be extended, and shows that in perturbation theory $C^s_\alpha(v^2)$ behaves as

$$C^s_\alpha(\lambda^2 v^2) \sim \lambda^{[\mathcal{O}^{s,\alpha}]-2[\phi]-s}, \quad \text{up to powers of } \ln\lambda\,, \text{ for } \lambda \to 0\,. \tag{11.79}$$

Therefore, the important parameter is no longer the dimension of the operator $\mathcal{O}^{s,\alpha}$, but the quantity

$$\tau = [\mathcal{O}^{s,\alpha}] - s\,, \tag{11.80}$$

called the *twist*.

The operators of lowest twist dominate the LCE of product of operators.

In the expansion (11.78), the lowest twist is 2, which is the twist of $\phi^2(x)$. Each new factor $\phi(x)$ increases the twist by one unit, while additional derivatives either increase the twist or leave it unchanged.

Therefore, the most general operator of twist 2 has the form

$$\phi(x)\,(\partial_{\mu_1}\cdots\partial_{\mu_2}\cdots\partial_{\mu_n} - \text{ traces})\,\phi(x),$$

or is a combination of derivatives of twist 2 operators.

Operators of twist 2 and spin s, since they are the operators of lowest dimension for a given spin, renormalize among themselves. Using previous considerations about the SDE, it is simple to write RG equations for the corresponding coefficients $C^s_\alpha(v^2)$.