x, u and various derivatives of u. The precise specification of the class of functions over which $\mathscr{L}$ is to be extremized will depend both on boundary conditions which might be pertinent to the physical problem, as well as differentiability conditions required of the extremals $u = f(x)$.

As a simple example, the problem of finding a curve of minimum length joining two points (a, b) and (c, d) in the plane can be cast into variational form as follows. Assume that the minimizing curve is given as the graph of a function $u = f(x)$. The length of such a curve is

$$\mathscr{L}[u] = \int_a^c \sqrt{1 + u_x^2}\, dx.$$

The variational problem consists of minimizing $\mathscr{L}$ over the space of differentiable functions $u = f(x)$, say, such that $b = f(a)$ and $d = f(c)$.

The precise degree of smoothness required of the extrema of a given variational problem, the space of functions being extremized over, and the appropriate norm(s) are quite delicate matters in general and quickly lead into advanced topics in nonlinear functional analysis. The complex issues involved are not directly relevant to our immediate area of inquiry, however, and we therefore adopt the admittedly oversimplifying assumption of only considering smooth (C^∞) extremals of a variational problem. Extending our results on symmetry groups and conservation laws to more general types of functions must then be done on a case by case basis.

The Variational Derivative

In finite dimensions, the extrema of a smooth real-valued function $f(x)$, $x \in \mathbb{R}^m$, are determined by looking at the points where the gradient $\nabla f(x)$ vanishes. The gradient itself is found by seeing how f changes under small changes in x:

$$\langle \nabla f(x), y \rangle = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(x + \varepsilon y),$$

where $\langle x, y \rangle$ is the usual inner product on $\mathbb{R}^m$. For functionals $\mathscr{L}[u]$, the role of the gradient is played by the "variational derivative" of $\mathscr{L}$. To construct this object, we look at how $\mathscr{L}$ changes under small "variations" in u. The inner product $\langle \cdot\,, \cdot \rangle$ on $\mathbb{R}^m$ is replaced by the L^2 inner product

$$\langle f, g \rangle = \int_\Omega f(x) \cdot g(x)\, dx = \int_\Omega \sum_{\alpha=1}^{q} f^\alpha(x) g^\alpha(x)\, dx$$

between vector-valued functions $f, g\colon \mathbb{R}^p \to \mathbb{R}^q$. This motivates the following definition:

Definition 4.1. Let $\mathscr{L}[u]$ be a variational problem. The *variational derivative* of $\mathscr{L}$ is the unique q-tuple

$$\delta\mathscr{L}[u] = (\delta_1\mathscr{L}[u], \ldots, \delta_q\mathscr{L}[u]),$$

with the property that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathscr{L}[f + \varepsilon\eta] = \int_\Omega \delta\mathscr{L}[f(x)]\cdot\eta(x)\,dx \tag{4.1}$$

whenever $u = f(x)$ is a smooth function defined on Ω, and $\eta(x) = (\eta^1(x), \ldots, \eta^q(x))$ is a smooth function with compact support in Ω, so that $f + \varepsilon\eta$ still satisfies any boundary conditions that might be imposed on the space of functions over which we are extremizing $\mathscr{L}$. The component $\delta_\alpha\mathscr{L} = \delta\mathscr{L}/\delta u^\alpha$ is the *variational derivative of* $\mathscr{L}$ *with respect to* u^α.

Proposition 4.2. *If* $u = f(x)$ *is an extremal of* $\mathscr{L}[u]$, *then*

$$\delta\mathscr{L}[f(x)] = 0, \qquad x \in \Omega. \tag{4.2}$$

Proof. Since f is an extremal, for any η of compact support in Ω, $f + \varepsilon\eta$ lies in the same function space, so, as a function of ε, $\mathscr{L}[f + \varepsilon\eta]$ must have an extremum at $\varepsilon = 0$. Therefore, by elementary calculus, (4.1) must vanish for all η of compact support in Ω, hence (4.2) must hold everywhere. (The same argument proves the uniqueness of $\delta\mathscr{L}$.) □

The general formula for the variational derivative is not difficult to find. First of all, interchanging the order of differentiation and integration (which is justified under our assumption of smoothness)

$$\begin{aligned}\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathscr{L}[f + \varepsilon\eta] &= \int_\Omega \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L(x, \mathrm{pr}^{(n)}(f + \varepsilon\eta)(x))\,dx\\ &= \int_\Omega \left\{\sum_{\alpha,J} \frac{\partial L}{\partial u_J^\alpha}(x, \mathrm{pr}^{(n)}f(x))\cdot\partial_J\eta^\alpha(x)\right\}dx,\end{aligned}$$

where the u_J^α are as usual the partial derivatives of u^α, and $\partial_J\eta^\alpha$ the corresponding derivatives of η^α. Since η has compact support, we can use the divergence theorem to integrate the latter expression by parts, with the boundary terms on $\partial\Omega$ vanishing. Each partial derivative $\partial/\partial x^j$, when applied to the derivatives $\partial L/\partial u_J^\alpha$ of the Lagrangian, becomes the total derivative D_j since L depends on x through the function $u = f(x)$ also—see Definition 2.34. Therefore

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathscr{L}[f + \varepsilon\eta] = \int_\Omega \left\{\sum_{\alpha=1}^q \left[\sum_J (-D)_J \frac{\partial L}{\partial u_J^\alpha}(x, \mathrm{pr}^{(n)} f(x))\right]\eta^\alpha(x)\right\}dx,$$

where, for $J = (j_1, \ldots, j_k)$,

$$(-D)_J = (-1)^k D_J = (-D_{j_1})(-D_{j_2})\cdots(-D_{j_k}).$$

The operator appearing in the preceding formula is of key importance in the calculus of variations.

Definition 4.3. For $1 \leqslant \alpha \leqslant q$, the α-th *Euler operator* is given by

$$\mathsf{E}_\alpha = \sum_J (-D)_J \frac{\partial}{\partial u_J^\alpha}, \tag{4.3}$$

the sum extending over all multi-indices $J = (j_1, \ldots, j_k)$ with $1 \leqslant j_\kappa \leqslant p$, $k \geqslant 0$. Note that to apply E_α to any given function $L(x, u^{(n)})$ of u and its derivatives, only finitely many terms in the summation are required, since L depends on only finitely many derivatives u_J^α.

Thus, according to our calculation, the variational derivative of $\mathscr{L}[u] = \int_\Omega L(x, u^{(n)})\, dx$ is found by applying the Euler operator to the Lagrangian: $\delta\mathscr{L}[u] = \mathsf{E}(L)$, where $\mathsf{E}(L) = (\mathsf{E}_1(L), \ldots, \mathsf{E}_q(L))$. Proposition 4.2 provides the classical necessary conditions for smooth extremals of a variational problem.

Theorem 4.4. *If $u = f(x)$ is a smooth extremal of the variational problem $\mathscr{L}[u] = \int_\Omega L(x, u^{(n)})\, dx$, then it must be a solution of the* Euler–Lagrange equations

$$\mathsf{E}_\nu(L) = 0, \qquad \nu = 1, \ldots, q.$$

Of course, not every solution to the Euler–Lagrange equations is an extremal. The other solutions furnish other types of critical points for the functional.

Example 4.5. Let us look at the special case $p = q = 1$, so we are considering a single function $u = f(x)$ of a single independent variable. The Euler operator here takes the form

$$\mathsf{E} = \sum_{j=0}^{\infty} (-D_x)^j \frac{\partial}{\partial u_j} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - \cdots,$$

where D_x is the total derivative with respect to x, and $u_j = d^j u/dx^j$. The Euler–Lagrange equation for an n-th order variational problem

$$\mathscr{L}[u] = \int_a^b L(x, u^{(n)})\, dx$$

takes the form

$$0 = \mathsf{E}(L) = \frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} + D_x^2 \frac{\partial L}{\partial u_{xx}} - \cdots + (-1)^n D_x^n \frac{\partial L}{\partial u_n}.$$

It is a $2n$-th order ordinary differential equation provided L satisfies the nondegeneracy condition $\partial^2 L/\partial u_n^2 \neq 0$. In particular, for a first order variational problem, $L = L(x, u, u_x)$, we recover the familiar second order Euler–

Lagrange equation

$$0 = \frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} = \frac{\partial L}{\partial u} - \frac{\partial^2 L}{\partial x \, \partial u_x} - u_x \frac{\partial^2 L}{\partial u \, \partial u_x} - u_{xx} \frac{\partial^2 L}{\partial u_x^2}.$$

Thus, for our curve length minimizing problem, the Euler–Lagrange equation is

$$-D_x\left(\frac{u_x}{\sqrt{1 + u_x^2}}\right) = -\frac{u_{xx}}{(1 + u_x^2)^{3/2}} = 0.$$

The solutions are all straight lines $u = mx + k$, and these are the only smooth candidates for the minimization problem.

Example 4.6. Perhaps the most famous variational problem comes from Dirichlet's principle for Laplace's equation $\Delta u = 0$. Here we set

$$\mathscr{L}[u] = \int_\Omega \tfrac{1}{2}|\nabla u|^2 \, dx = \int_\Omega \tfrac{1}{2} \sum_{i=1}^{p} u_i^2 \, dx,$$

where $u_i = \partial u/\partial x^i$ and $\Omega \subset X \simeq \mathbb{R}^p$. The Euler–Lagrange equation is

$$0 = \mathsf{E}(L) = \sum_{i=1}^{p} (-D_i) \frac{\partial L}{\partial u_i} = -\sum_{i=1}^{p} D_i(u_i) = -\Delta u,$$

which agrees with Laplace's equation up to sign. Further examples will appear later in this chapter.

Null Lagrangians and Divergences

Occasionally, for a given variational problem the Euler–Lagrange equations vanish identically and so every function is a possible extremal of the problem. For example, if

$$\mathscr{L}[u] = \int_a^b u u_x \, dx,$$

then

$$\delta \mathscr{L} = u_x - D_x(u) \equiv 0$$

for all u. In this case the variational problem is trivial, since by the fundamental theorem of calculus

$$\mathscr{L}[u] = \int_a^b D_x(\tfrac{1}{2}u^2) \, dx = \tfrac{1}{2}u^2 \Big|_{x=a}^{b},$$

so any function $u = f(x)$ satisfying the relevant boundary conditions will give the same value for $\mathscr{L}$.

The situation readily generalizes to the case of several independent variables. If $x = (x^1, \dots, x^p)$ and $P(x, u^{(n)}) = (P_1(x, u^{(n)}), \dots, P_p(x, u^{(n)}))$ is a p-tuple of smooth functions of x, u and the derivatives of u, we define the *total divergence* of P to be the function

$$\operatorname{Div} P = D_1 P_1 + D_2 P_2 + \cdots + D_p P_p, \tag{4.4}$$

where each D_j is the total derivative with respect to x^j. For instance, if $p = 2$, and $P = (uu_y, uu_x)$, then

$$\operatorname{Div} P = D_x(uu_y) + D_y(uu_x) = 2uu_{xy} + 2u_x u_y.$$

If a Lagrangian $L(x, u^{(n)})$ can be written as a divergence, so $L = \operatorname{Div} P$ for some p-tuple P, then, by the divergence theorem,

$$\mathscr{L}[u] = \int_\Omega L\, dx = \int_{\partial\Omega} P \cdot dS$$

for any function $u = f(x)$ and any bounded domain Ω with smooth boundary $\partial\Omega$. Thus $\mathscr{L}[f]$ depends only on the boundary behaviour of $u = f(x)$, and will be unaffected by the variations η used in the definition of the variational derivative. Therefore the Euler–Lagrange equations for such a functional are identically 0. Remarkably, these are the only such examples of "null Lagrangians".

Theorem 4.7. *A function $L(x, u^{(n)})$ of x, u and the derivatives of u, defined everywhere on $X \times U^{(n)}$, is a* null Lagrangian, *meaning that the Euler-Lagrange equations $\mathsf{E}(L) \equiv 0$ vanish identically for all x, u, if and only if it is a total divergence: $L = \operatorname{Div} P$, for some p-tuple of functions $P = (P_1, \dots, P_p)$ of x, u and the derivatives of u.*

PROOF. The proof that $\mathsf{E}(\operatorname{Div} P) \equiv 0$ follows from the above remarks, or by direct computation; see Section 5.4. To prove the converse, suppose $L(x, u^{(n)})$ is a null Lagrangian, and consider the derivative

$$\frac{d}{d\varepsilon} L(x, \varepsilon u^{(n)}) = \sum_{\alpha, J} u_J^\alpha \frac{\partial L}{\partial u_J^\alpha}(x, \varepsilon u^{(n)}).$$

Each term in this sum can be integrated by parts; for example

$$u_i^\alpha \frac{\partial L}{\partial u_i^\alpha} = D_i(u^\alpha) \frac{\partial L}{\partial u_i^\alpha} = -u^\alpha D_i \frac{\partial L}{\partial u_i^\alpha} + D_i\left(u^\alpha \frac{\partial L}{\partial u_i^\alpha}\right),$$

leading to an expression of the form

$$\begin{aligned} \frac{d}{d\varepsilon} L(x, \varepsilon u^{(n)}) &= \sum_{\alpha=1}^q u^\alpha \sum_J (-D)_J \frac{\partial L}{\partial u_J^\alpha}(x, \varepsilon u^{(n)}) + \operatorname{Div} \hat{P}(\varepsilon; x, u^{(2n)}) \\ &= u \cdot \mathsf{E}(L)(x, \varepsilon u^{(2n)}) + \operatorname{Div} \hat{P}(\varepsilon; x, u^{(2n)}) \end{aligned}$$

for some p-tuple $\hat{P}$ of functions of x, u and derivatives of u whose precise form is not of importance. (However, see (5.150), (5.151).) Since $\mathsf{E}(L) \equiv 0$, we can integrate with respect to ε,

$$L(x, u^{(n)}) - L(x, 0) = \operatorname{Div} \tilde{P}, \quad \text{where} \quad \tilde{P}(x, u^{(2n)}) = \int_0^1 \hat{P}(\varepsilon; x, u^{(2n)})\, d\varepsilon.$$

Finally, since L is defined on all of $\mathbb{R}^p$, we can always find a p-tuple $p(x)$ of ordinary functions of x such that $\operatorname{div} p(x) = L(x, 0)$, so the theorem holds with $P = \tilde{P} + p$. □

A corollary of Theorem 4.7 is the basic fact that two Lagrangians L and $\tilde{L}$ have the same Euler–Lagrange expression, $\mathsf{E}(L) = \mathsf{E}(\tilde{L})$, if and only if they differ by a divergence,

$$L = \tilde{L} + \operatorname{Div} P.$$

Invariance of the Euler Operator

Since the Euler–Lagrange equations determine the extremals of a variational problem, their solution set should remain unchanged by a change of variables. This suggests that the Euler operator itself should be, more or less, invariant under a change of variables. Here we derive the basic formula expressing this fact.

Note first that if

$$\tilde{x} = \Xi(x, u), \qquad \tilde{u} = \Phi(x, u), \tag{4.5}$$

is any change of variables, there is an induced change of variables

$$\tilde{u}^{(n)} = \Phi^{(n)}(x, u^{(n)})$$

for the derivatives, given by prolongation. Thus, given a function $u = f(x)$, (4.5) implicitly defines the transformed function $\tilde{u} = \tilde{f}(\tilde{x})$ (provided the conditions required by the implicit function theorem are satisfied). Each functional

$$\mathscr{L}[f] = \int_\Omega L(x, \operatorname{pr}^{(n)} f(x))\, dx$$

will be transformed into a new form

$$\tilde{\mathscr{L}}[\tilde{f}] = \int_{\tilde{\Omega}} \tilde{L}(\tilde{x}, \operatorname{pr}^{(n)} \tilde{f}(\tilde{x}))\, d\tilde{x}.$$

In this latter integral, the transformed domain

$$\tilde{\Omega} = \{\tilde{x} = \Xi(x, f(x)) \colon x \in \Omega\}$$

will depend not only on the original domain Ω, but also on the precise function $u = f(x)$ on which $\mathscr{L}$ is being evaluated. The formula for the new Lagrangian follows readily from the change of variables formula for multiple integrals:

$$L(x, \mathrm{pr}^{(n)} f(x)) = \tilde{L}(\tilde{x}, \mathrm{pr}^{(n)} \tilde{f}(\tilde{x})) \det \mathsf{J}(x, \mathrm{pr}^{(1)} f(x)), \tag{4.6}$$

whenever $(\tilde{x}, \tilde{u})$ are given by (4.5), where J is the Jacobian matrix with entries

$$\mathsf{J}^{ij}(x, \mathrm{pr}^{(1)} f(x)) = \frac{\partial}{\partial x^j}[\Xi^i(x, f(x))] = D_j \Xi^i(x, \mathrm{pr}^{(1)} f(x)), \qquad i, j = 1, \ldots, p,$$

corresponding to the function f. Here we are assuming, for simplicity, that the change of variables is orientation-preserving, so $\det \mathsf{J}(x) > 0$.

Theorem 4.8. *Let $L(x, u^{(n)})$ and $\tilde{L}(\tilde{x}, \tilde{u}^{(n)})$ be two Lagrangians related by the change of variables formula* (4.5), (4.6). *Then*

$$\mathsf{E}_{u^\alpha}(L) = \sum_{\beta=1}^{q} \mathsf{F}_{\alpha\beta}(x, u^{(1)}) E_{\tilde{u}^\beta}(\tilde{L}), \qquad \alpha = 1, \ldots, q, \tag{4.7}$$

whenever $(\tilde{x}, \tilde{u}^{(n)})$ and $(x, u^{(n)})$ are so related, where $\mathsf{F}_{\alpha\beta}$ is the determinant of the following $(p + 1) \times (p + 1)$ matrix

$$\mathsf{F}_{\alpha\beta} = \det \begin{bmatrix} D_1\Xi^1 & \cdots & D_p\Xi^1 & \partial\Xi^1/\partial u^\alpha \\ \vdots & & \vdots & \vdots \\ D_1\Xi^p & \cdots & D_p\Xi^p & \partial\Xi^p/\partial u^\alpha \\ D_1\Phi^\beta & \cdots & D_p\Phi^\beta & \partial\Phi^\beta/\partial u^\alpha \end{bmatrix}. \tag{4.8}$$

PROOF. Let $u = f(x)$ be a given function defined over a domain Ω and let $\tilde{u} = \tilde{f}(\tilde{x})$, $\tilde{x} \in \tilde{\Omega}$, be the corresponding function in the transformed variables, which is usually well defined as long as Ω is sufficiently small. For ε sufficiently small, the perturbations $u_\varepsilon = f(x, \varepsilon) = f(x) + \varepsilon\eta(x)$, η of compact support in Ω, have corresponding expressions $\tilde{u} = \tilde{f}(\tilde{x}, \varepsilon)$ determined implicitly

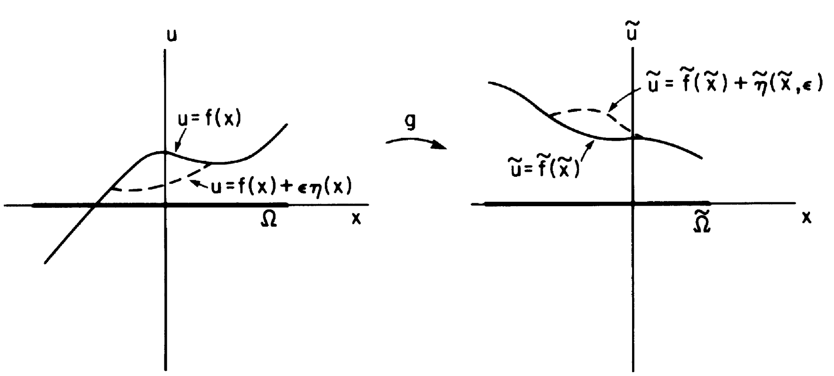

Figure 10. Change of coordinates for the variation of a functional.

by the relations

$$\tilde{x} = \Xi(x, f(x) + \varepsilon\eta(x)), \qquad \tilde{u} = \Phi(x, f(x) + \varepsilon\eta(x)). \tag{4.9}$$

An important point is that since η has compact support in Ω, each $\tilde{f}(\tilde{x}, \varepsilon) = \tilde{f}(\tilde{x}) + \tilde{\eta}(\tilde{x}, \varepsilon)$ is defined over a common domain

$$\tilde{\Omega} = \{\tilde{x} = \Xi(x, f(x)) : x \in \Omega\}$$

independent of ε, and $\tilde{\eta}$ has compact support in $\tilde{\Omega}$. The variational derivative of $\mathscr{L}$ was determined by differentiating $\mathscr{L}[f + \varepsilon\eta]$ with respect to ε at $\varepsilon = 0$; similarly, by a slight generalization of the argument leading to the formula (4.3) for the Euler operator, we find

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \tilde{\mathscr{L}}[\tilde{f}] = \int_{\tilde{\Omega}} \mathsf{E}_{\tilde{u}}(\tilde{L}) \cdot \left.\frac{\partial \tilde{f}}{\partial \varepsilon}\right|_{\varepsilon=0} d\tilde{x}, \tag{4.10}$$

where $\mathsf{E}_{\tilde{u}}(\tilde{L})$ is evaluated at $\tilde{u} = \tilde{f}$. We now need to evaluate $\partial\tilde{f}/\partial\varepsilon$.

Keeping in mind that when variations of $\tilde{\mathscr{L}}$ are computed, the base variables $\tilde{x}$ are not allowed to depend on ε, we find from (4.9) that

$$0 = \sum_{j=1}^{p} D_j \Xi^i \frac{\partial x^j}{\partial \varepsilon} + \sum_{\alpha=1}^{q} \frac{\partial \Xi^i}{\partial u^\alpha} \eta^\alpha,$$

hence, by Cramer's rule,

$$\left.\frac{\partial x^j}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{-1}{\det \mathsf{J}} \sum_{i=1}^{p} K_{ij} \sum_{\alpha=1}^{q} \frac{\partial \Xi^i}{\partial u^\alpha} \eta^\alpha,$$

where K_{ij} is the (i, j)-th cofactor of the Jacobian matrix $\mathsf{J}(x)$. Therefore,

$$\begin{aligned}
\left.\frac{\partial \tilde{f}^\beta}{\partial \varepsilon}\right|_{\varepsilon=0} &= \sum_{\alpha=1}^{q} \frac{\partial \Phi^\beta}{\partial u^\alpha} \eta^\alpha + \sum_{j=1}^{p} D_j \Phi^\beta \left.\frac{\partial x^j}{\partial \varepsilon}\right|_{\varepsilon=0} \\
&= \frac{1}{\det \mathsf{J}} \sum_{\alpha=1}^{q} \left\{ \frac{\partial \Phi^\beta}{\partial u^\alpha} \det \mathsf{J} - \sum_{i,j=1}^{p} D_j \Phi^\beta \cdot K_{ij} \frac{\partial \Xi^i}{\partial u^\alpha} \right\} \eta^\alpha.
\end{aligned}$$

The reader can recognize the expression in brackets as the column expansion of the determinant (4.8) along the last column, where the intermediary summation $\sum_j D_j \Phi^\beta \cdot K_{ij}$ is the row expansion of the $(i, p + 1)$-st minor along its last row. Thus

$$\left.\frac{\partial \tilde{f}^\beta}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{1}{\det \mathsf{J}} \sum_{\alpha=1}^{q} \mathsf{F}_{\alpha\beta} \eta^\alpha.$$

Substituting into (4.10) and changing variables, we find

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \tilde{\mathscr{L}}[\tilde{f}] = \int_{\Omega} \left\{ \sum_{\alpha,\beta=1}^{q} \mathsf{F}_{\alpha\beta} \mathsf{E}_{\tilde{u}^\beta}(\tilde{L}) \cdot \eta^\alpha \right\} dx.$$

On the other hand, this must equal

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathscr{L}[f + \varepsilon\eta] = \int_{\Omega} \left\{ \sum_{\alpha=1}^{q} \mathsf{E}_{u^\alpha}(L) \eta^\alpha \right\} dx,$$

from which (4.7) follows since η is arbitrary. $\square$

Example 4.9. In the case $p = q = 1$,

$$L(x, u^{(n)}) = \tilde{L}(\tilde{x}, \tilde{u}^{(n)}) D_x \Xi(x, u^{(1)}),$$

and (4.7) simplifies to

$$\mathsf{E}_u(L) = \frac{\partial(\Xi, \Phi)}{\partial(x, u)} \mathsf{E}_{\tilde{u}}(\tilde{L}), \tag{4.11}$$

where we need only the Jacobian determinant

$$\frac{\partial(\Xi, \Phi)}{\partial(x, u)} = \det\begin{pmatrix} \partial\Xi/\partial x & \partial\Xi/\partial u \\ \partial\Phi/\partial x & \partial\Phi/\partial u \end{pmatrix}$$

of partial derivatives of Ξ and Φ. Indeed, the determinant appearing in (4.8),

$$\mathsf{F}_{11}(x) = \det\begin{pmatrix} D_x\Xi & \partial\Xi/\partial u \\ D_x\Phi & \partial\Phi/\partial u \end{pmatrix} = \det\begin{pmatrix} \partial\Xi/\partial x + u_x\partial\Xi/\partial u & \partial\Xi/\partial u \\ \partial\Phi/\partial x + u_x\partial\Phi/\partial u & \partial\Phi/\partial u \end{pmatrix}$$

equals the above determinant by an elementary column operation. For example, if

$$\mathscr{L}[u] = \int_a^b \tfrac{1}{2}u_x^2 \, dx,$$

and we use the hodograph transformation $\tilde{x} = u$, $\tilde{u} = x$, then

$$\tilde{\mathscr{L}}[\tilde{u}] = \int_{\tilde{a}}^{\tilde{b}} \tfrac{1}{2}(\tilde{u}_{\tilde{x}})^{-2}\tilde{u}_{\tilde{x}} \, d\tilde{x} = \int_a^b (2\tilde{u}_{\tilde{x}})^{-1} \, d\tilde{x},$$

and

$$\mathsf{E}_u(L) = -u_{xx} = -\tilde{u}_{\tilde{x}}^{-3}\tilde{u}_{\tilde{x}\tilde{x}} = -\mathsf{E}_{\tilde{u}}(\tilde{L}),$$

verifying (4.11). However, in general we cannot replace the total derivatives in (4.8) by partial derivatives. (See Exercise 4.15.)

4.2. Variational Symmetries

In order to apply group methods in the calculus of variations, we need to make precise the notion of a symmetry group of a functional

$$\mathscr{L}[u] = \int_{\Omega_0} L(x, u^{(n)}) \, dx. \tag{4.12}$$

The groups considered here will be local groups of transformations G acting on an open subset $M \subset \Omega_0 \times U \subset X \times U$. As discussed in detail in Chapter 2, if $u = f(x)$ is a smooth function defined over a suitably small subdomain $\Omega \subset \Omega_0$, such that the graph of f lies in M, each transformation g in G sufficiently close to the identity will transform f to another smooth function

$\tilde{u} = \tilde{f}(\tilde{x}) = g \cdot f(\tilde{x})$ defined over $\tilde{\Omega} \subset \Omega_0$. (Note that unless G is projectable, $\tilde{\Omega}$ will, in general, not only depend on g, but also on f itself.) A symmetry group G will be one that, roughly speaking, leaves the variational integral $\mathscr{L}$ unchanged for all such f.

Definition 4.10. A local group of transformations G acting on $M \subset \Omega_0 \times U$ is a *variational symmetry group* of the functional (4.12) if whenever Ω is a subdomain with closure $\bar{\Omega} \subset \Omega_0$, $u = f(x)$ is a smooth function defined over Ω whose graph lies in M, and $g \in G$ is such that $\tilde{u} = \tilde{f}(\tilde{x}) = g \cdot f(\tilde{x})$ is a single-valued function defined over $\tilde{\bar{\Omega}} \subset \Omega_0$, then

$$\int_{\tilde{\Omega}} L(\tilde{x}, \mathrm{pr}^{(n)} \tilde{f}(\tilde{x}))\, d\tilde{x} = \int_{\Omega} L(x, \mathrm{pr}^{(n)} f(x))\, dx. \tag{4.13}$$

Example 4.11. Consider the case when $X = \mathbb{R}$ and we have a first order variational problem

$$\mathscr{L}[u] = \int_a^b L(x, u, u_x)\, dx. \tag{4.14}$$

If L does not depend on x, then the translation group $(x, u) \mapsto (x + \varepsilon, u)$ is a variational symmetry group of $\mathscr{L}$. Indeed, since $\tilde{x} = x + \varepsilon$, $\tilde{u} = u$, if $u = f(x)$ is any function defined over a smaller subinterval $[c, d] \subset (a, b)$, then $\tilde{u} = \tilde{f}(\tilde{x}) = f(\tilde{x} - \varepsilon)$ is defined over $[\tilde{c}, \tilde{d}] = [c + \varepsilon, d + \varepsilon]$, which, for ε sufficiently small, is still a subinterval of (a, b). To verify (4.13), we have

$$\int_{\tilde{c}}^{\tilde{d}} L(\tilde{f}(\tilde{x}), \tilde{f}'(\tilde{x}))\, d\tilde{x} = \int_{\tilde{c}}^{\tilde{d}} L(f(\tilde{x} - \varepsilon), f'(\tilde{x} - \varepsilon))\, d\tilde{x} = \int_c^d L(f(x), f'(x))\, dx,$$

using a change of variables.

Infinitesimal Criterion of Invariance

In accordance with our usual *modus operandi*, we now find the analogous infinitesimal criterion for the invariance of a variational problem under a group of transformations. Again this condition will be necessary and sufficient for a connected group of transformations to be a symmetry group of the variational problem.

Theorem 4.12. *A connected group of transformations G acting on $M \subset \Omega_0 \times U$ is a variational symmetry group of the functional* (4.12) *if and only if*

$$\mathrm{pr}^{(n)}\, \mathbf{v}(L) + L\, \mathrm{Div}\, \xi = 0 \tag{4.15}$$

for all $(x, u^{(n)}) \in M^{(n)}$ and every infinitesimal generator

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u)\frac{\partial}{\partial u^\alpha}$$