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Resurgence and semiclassical expansion in two-dimensional large- N sigma models

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ABSTRACT: The resurgence structure of the 2d $O(N)$ sigma model at large N is studied with a focus on an IR momentum cutoff scale a that regularizes IR singularities in the semiclassical expansion. Transseries expressions for condensates and correlators are derived as series of the dynamical scale Λ (nonperturbative exponential) and coupling λ_μ renormalized at the momentum scale μ . While there is no ambiguity when $a > \Lambda$, we find for $a < \Lambda$ that the nonperturbative sectors have new imaginary ambiguities besides the well-known renormalon ambiguity in the perturbative sector. These ambiguities arise as a result of an analytic continuation of transseries coefficients to small values of the IR cutoff a below the dynamical scale Λ . We find that the imaginary ambiguities are cancelled each other when we take all of them into account. By comparing the semiclassical expansion with the transseries for the exact large- N result, we find that some ambiguities vanish in the $a \rightarrow 0$ limit and hence the resurgence structure changes when going from the semiclassical expansion to the exact result with no IR cutoff. An application of our approach to the $\mathbb{C}P^{N-1}$ sigma model is also discussed. We find in the compactified model with the \mathbb{Z}_N twisted boundary condition that the resurgence structure changes discontinuously as the compactification radius is varied.

KEYWORDS: Field Theories in Lower Dimensions, Resummation, Sigma Models

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1 Introduction

Quantum field theory (QFT) is arguably the pinnacle of human knowledge of the microscopic world. It is the language used in the standard model of particle physics and condensed matter physics. Despite its success in many areas of physics, there is no rigorous definition of QFT in continuum yet. There has recently been, however, a considerable effort to use the resurgence theory [1–11] to study QFT and to give a continuum definition of it. There have been particularly a lot of progress in quantum mechanics: the resurgent structure in double-well and periodic potentials [12–19], the valley method [20–24], exact quantization conditions and constructive resurgence [25–35], bion cancellation mechanism and the other cancellation mechanics [36–54]. The resurgent structure of two-dimensional quantum field theories has been also investigated: sigma models [55–70], principle chiral models [84–86], and 2D Yang-Mills theory [87]. This paper is a continuation of such an effort to study QFT in the framework of resurgence theory.

In most of continuum quantum field theories, the exact solution is not known, and one of the most reliable approaches may be the semiclassical method, which uses a transseries with powers of nonperturbative exponential e^{-A/λ_μ} ($A : \text{const.}$) and divergent power series in coupling λ_μ , where the subscript μ indicates that the coupling constant is renormalized at the momentum scale μ . Schematically, the expectation value of an observable \mathcal{O} may be written as

$$\langle \mathcal{O} \rangle = \sum_{l=0}^{\infty} e^{-lA/\lambda_\mu} C_l(\lambda_\mu), \quad C_l(\lambda_\mu) = \sum_{n=0}^{\infty} c_{(l,n)} \lambda_\mu^n. \quad (1.1)$$

The $l = 0$ sector is the usual perturbative expansion around the trivial vacuum, while the $l > 0$ sector is a small-coupling expansion around the l -th nonperturbative background configuration. The latter contains an essential singularity e^{-1/λ_μ} in the complex λ_μ plane, and therefore it is nonperturbative in nature. Generically $C_l(\lambda_\mu)$ are divergent asymptotic series: the sum over all n is neither convergent nor Borel summable. Typically the expansion coefficients are factorially divergent $c_{(l,n)} \sim n!$, and this gives rise to singularities in the Borel plane. These Borel singularities are sometimes located on the real positive axis in the Borel plane, which leads to an imaginary ambiguity of the resultant Borel resummation. They are expected to cancel with each other when all the ambiguities (including those from discontinuous jumps of Stokes constants) are taken into account. As a consequence, it connects the perturbative and nonperturbative contributions in the physical quantities via the imaginary ambiguities. This is one of the final goals of the application of the resurgence theory: resurgence theory attempts to find out a nontrivial relation between the perturbative and nonperturbative sectors that enables us to extract the nonperturbative information just from the perturbative series through the Borel resummation and its imaginary ambiguity.

The Borel singularities of the factorially divergent perturbative series can be classified into two classes. The first one comes from the proliferation of the number of Feynman diagrams. Ambiguities associated with this type of Borel singularities are found to be cancelled by nonperturbative contributions of saddle point configurations such as a bion (a fractional instanton-anti-instanton composite) [55, 56]. The second one is the “renormalon” type [88, 89], whose imaginary ambiguities can not naively be cancelled by those of saddle point configurations. In general, the renormalon type ambiguity is related to contributions of a selected set of the Feynman diagrams with loop momentum integrations written in terms of the renormalized coupling constant. They also contribute a factorially divergent series, whose Borel resummation gives an imaginary ambiguity. In particular, ambiguities of renormalon type remain even at large N , while those associated with the proliferation of diagrams vanish in the large- N limit. There is a conjecture that the renormalon can be identified as a certain semiclassical object (e.g. bion) in the compactified spacetime in the Euclidean path integral formulation [55, 56], on which both the affirmative and negative arguments have been developed [66–70]. The way how the renormalon ambiguity is cancelled out by nonperturbative contributions is still an open question.

In this paper, we would like to shed light on these issues by revisiting the $O(N)$ non-linear sigma model in two dimensions at large N . The resurgence structure and renormalons in the $O(N)$ sigma model have been studied by using method such as the large- N

analysis and integrability [71–83]. The model is an asymptotically free theory with a mass gap, and many important properties can be exactly solved at large N . In this paper, we mainly focus on the condensate $\langle \delta D^2 \rangle$ of fluctuations of an auxiliary field $D(x)$, as the perturbative expansion is known to have a renormalon. Intended to simulate the massless perturbation theory, we compute the transseries coefficients $c_{(l,n)}$ using the transseries expansion of the momentum integrand, which can be exactly determined in the large- N limit, in powers of the nonperturbative exponential $e^{-2\pi/\lambda_p} = \Lambda/p$ and the coupling constant $\lambda_p = 2\pi/\log(p/\Lambda)$ renormalized at momentum scale p . In this approach, we encounter IR divergences like in the massless perturbation theory, since the model is perturbatively massless even though there is a nonperturbative mass gap. We obtain transseries and examine both renormalons and IR divergences in the model with our semiclassical ansatz.

In the rest of this section, we outline the paper and highlight our main results. In section 2, we briefly review how renormalon type ambiguities appear in a generic asymptotically free theory. In section 3, we review the $O(N)$ sigma model at large N and write down the exact large- N expression of the condensate $\langle \delta D^2 \rangle$. The asymptotic series of the exact result contains a divergent perturbative expansion, which is not Borel summable. The singularity in the Borel plane gives an renormalon type ambiguity of order Λ^4 . Since the exact expression is real and unambiguous, such a renormalon ambiguity must be cancelled. The question we would like to address is how the cancellation of the renormalon occurs in the context of the semiclassical expansion. In section 4.1, we argue that the expected form for the semiclassical (s.c.) expansion of the condensate can be written as

$$\langle \delta D^2 \rangle \underset{\text{s.c.}}{\sim} \sum_{l=0}^{\infty} e^{-4\pi l/\lambda_\mu} \sum_{n=0}^{\infty} \lambda_\mu^{n+1} \int \frac{d^d p}{(2\pi)^d} h_{(l,n)}(p), \quad (1.2)$$

in d dimensions. It is beyond the scope of this paper to derive the semiclassical expansion by explicitly computing the perturbation series around the vacuum and nontrivial backgrounds. Instead, we simply deduce the semiclassical form (1.2) from the exact solution at large N . In our semiclassical ansatz, the momentum integral becomes IR divergent for higher orders in l since the model is perturbatively massless. We therefore need to introduce an IR momentum cutoff a ($|p| > a$). In each sector labeled by l (of order $e^{-4\pi l/\lambda_\mu} = (\Lambda/\mu)^{2l}$), the sum over n can contain a factorially divergent series, but their Borel resummations have no imaginary ambiguities at any l when the IR cutoff is larger than the dynamical scale $a > \Lambda$. In section 4.2, we show that when the IR cutoff is small, $0 < a < \Lambda$, the imaginary ambiguities arise at order Λ^0 , Λ^4 , and Λ^8 . This shows that the presence of imaginary ambiguities depends on the IR cutoff a . In section 4.3, we investigate the origin of the imaginary ambiguities by explicitly computing the semiclassical expansion up to order Λ^8 . We first identify that the ambiguity at Λ^0 is the well-known renormalon in perturbation theory. We then show that this imaginary ambiguity is cancelled by the combined imaginary ambiguities that come from order Λ^4 and Λ^8 in the semiclassical ansatz, and not only from order Λ^4 as previously known for the transseries of the exact large- N result with $a = 0$. This is one of our main results in this paper. In section 5, we examine the result of the semiclassical ansatz by comparing it with the exact result with $a = 0$. We find that the ambiguities at order Λ^8 can be understood as the result

of an analytic continuation in λ_a below the dynamical mass $a < \Lambda$, where $\lambda_a < 0$. We also find how to obtain $a \rightarrow 0$ limit. In section 6, we generalize the discussion to the correlation function $\langle \delta D(x) \delta D(0) \rangle$. In section 7, we discuss the generalization to the $\mathbb{C}P^{N-1}$ sigma model including the case of the \mathbb{Z}_N twisted periodic boundary condition. We show that the resurgence structure changes discontinuously when each Kaluza-Klein mass (Matsubara frequency) $2\pi n/L$ ($n \in \mathbb{Z}$) becomes smaller than Λ as we vary the compactification period L . Section 8 gives conclusion and discussion.

2 Renormalons in asymptotically free theories

In this section, let us briefly review how renormalons appear in two-point functions and condensates in asymptotic free theories. Suppose that there is a renormalized coupling constant λ_μ depending on the renormalization scale μ as determined by the renormalization group equation

$$\mu \frac{\partial}{\partial \mu} \frac{2\pi}{\lambda_\mu} = \beta(\lambda_\mu), \quad (2.1)$$

where $\beta(\lambda_\mu)$ is the beta function, whose expansion takes the form

$$\beta(\lambda_\mu) = \beta_0 + \beta_1 \lambda_\mu + \beta_2 \lambda_\mu^2 + \dots . \quad (2.2)$$

We assume that the first coefficient is positive $\beta_0 > 0$ so that the coupling constant λ_μ becomes small for $\mu \rightarrow \infty$, i.e. the model is asymptotically free.¹ We define the dynamically generated scale Λ as the scale at which the renormalized coupling constant diverges

$$\lambda_\mu(\mu/\Lambda) \xrightarrow{\mu \rightarrow +\Lambda} \infty. \quad (2.3)$$

Note that λ_μ is a function of μ/Λ . In the following, we assume that the model has no parameter with a mass scale other than Λ .

Suppose that we are interested in the two-point function of a local operator \mathcal{O}

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = \int \frac{d^d p}{(2\pi)^d} e^{ip \cdot x} \Delta(p), \quad (2.4)$$

where $\Delta(p)$ is the (Euclidean) momentum space propagator, that is, the correlation function of the Fourier transform $\tilde{\mathcal{O}}(p)$ of $\mathcal{O}(x)$

$$\langle \tilde{\mathcal{O}}(p) \tilde{\mathcal{O}}(p') \rangle = \delta(p + p') \Delta(p). \quad (2.5)$$

We assume that $\Delta(p)$ has no singularity as a function of $p_i \in \mathbb{R}^d$ and the integral (2.4) is well-defined. In particular, the regularity of the propagator at the origin implies that there is a mass gap in this model. The semiclassical expansion with respect to the renormalized coupling constant λ_μ would give the following transseries expression for the propagator

$$\Delta(p) = p^{[\Delta]} \sum_{l=0}^{\infty} \exp\left(-\frac{2\pi\sigma_l}{\lambda_\mu}\right) h_l\left(\frac{p}{\mu}, \lambda_\mu\right), \quad h_l\left(\frac{p}{\mu}, \lambda_\mu\right) = \sum_{n=0}^{\infty} a_{ln}\left(\frac{p}{\mu}\right) \lambda_\mu^n, \quad (2.6)$$

¹In this paper, the symbol λ_μ stands for the renormalized 't Hooft coupling, for which $\beta_0 = 1$ in the $O(N)$ and $\mathbb{C}P^{N-1}$ sigma models.

where $[\Delta]$ is the mass dimension of the propagator $\Delta(p)$ related to that of the operator \mathcal{O} as $[\Delta] = 2[\mathcal{O}] - d$, σ_l ($0 < \sigma_1 < \sigma_2 < \dots$) are the nonperturbative exponents, $a_{ln}(p/\mu)$ are the n -th expansion coefficients in the l -th nonperturbative sector. In the following, we call expressions like (2.6) “semiclassical ansatz” since transseries of this type would be the expected form obtained in the semiclassical expansion: the l -th sector is the contribution from the l -th saddle point configuration characterized by the action $S_l \propto \sigma_l$ and h_l stands for the power series obtained through the perturbative expansion around the saddle point configuration. For simplicity, we assume that the semi-classical expansion for the propagator (2.6) is convergent for $p > \mu > \Lambda$. This assumption is true in the large- N case discussed below. Using the renormalization group, we can change the renormalization scale from μ to p . Then the transseries for the propagator becomes

$$\Delta(p) = p^{[\Delta]} \sum_{l=0}^{\infty} \exp\left(-\frac{2\pi\sigma_l}{\lambda_p}\right) \tilde{h}_l(\lambda_p), \quad \tilde{h}_l(\lambda_p) = \sum_{n=0}^{\infty} b_{ln} \lambda_p^n. \quad (2.7)$$

In this expression, the expansion coefficients b_{ln} are constants without p -dependence since there is no other mass scale. Note that each term in this transseries is singular at $p = \Lambda$ due to the singularity of the renormalized coupling λ_p . The singularity structure becomes more manifest by expanding the renormalized coupling constant λ_p in powers of the one-loop coupling constant λ'_p as²

$$\lambda_p(p) = \lambda'_p - \frac{\beta_1}{\beta_0} \lambda'^2_p \log \frac{4\pi}{\lambda'_p} + \dots \quad \text{with} \quad \lambda'_p = \frac{2\pi}{\beta_0 \log p/\Lambda}. \quad (2.8)$$

Then, the transseries (2.7) would be rewritten as

$$\Delta(p) = p^{[\Delta]} \sum_{l=0}^{\infty} \left(\frac{\Lambda}{p}\right)^{\beta_0\sigma_l} f_l(\lambda'_p), \quad f_l(\lambda'_p) = \lambda'^{\alpha} \sum_{n=0}^{\infty} c_{ln} \lambda'^n, \quad (2.9)$$

where $\alpha = 2\pi\sigma_l\beta_1/\beta_0$ and c_{ln} are functions of $\log 4\pi/\lambda'_p$. Because of the asymptotic freedom, this transseries expression can also be viewed as the large- p expansion of the propagator. If the function $f_l(\lambda'_p)$ is divergent in the limit $\lambda'_p \rightarrow \infty$, the l -th term of the transseries (2.9) has a singularity at $p = \Lambda$ originating from that of the renormalized coupling λ_p . Due to this singularity at $p = \Lambda$, each term in the transseries for the two point function

$$\langle \mathcal{O}(x)\mathcal{O}(0) \rangle_a = \sum_{l=0}^{\infty} \int_{|p|>a} \frac{d^d p}{(2\pi)^d} e^{ip \cdot x} p^{[\Delta]} \left(\frac{\Lambda}{p}\right)^{\beta_0\sigma_l} f_l(\lambda'_p), \quad (2.10)$$

has an ambiguity depending on the regularization if the singularity at $p = \Lambda$ is contained in the integration domain $|p| > a$. Here we have introduced an IR cutoff scale a to regularize the singularity at $p = 0$ in the integration for each term in the transseries (2.9). Such an IR cutoff is always necessary in the semi-classical computation in a perturbatively massless

²In the large- N sigma model discussed below, λ'_p is identified with λ_p since the correction is subleading in the large- N limit.

model even though there is a dynamically generated mass gap.³ On the other hand, the existence of the mass gap guarantee that there is a well-defined limit $a \rightarrow 0$ for the full two point function. Since the propagator itself has no singularity, all the ambiguities cancel out and the $a \rightarrow 0$ limit is regular.

We can associate the ambiguity from the singularity at $p = \Lambda$ with a singularity in the Borel plane. For simplicity, let us focus on the case $x \rightarrow 0$, where the two point function reduces to the condensate

$$\langle \mathcal{O}(x)\mathcal{O}(0) \rangle_a \rightarrow \langle \mathcal{O}(0)^2 \rangle_{\tilde{a},a} = \int_{a < |p| < \tilde{a}} \frac{d^d p}{(2\pi)^d} \Delta(p), \quad (2.11)$$

where we have introduced another cutoff scale \tilde{a} to regularize the UV divergence. Assume that the transseries for the propagator (2.9) has the following Borel resummed form

$$\Delta(p) = 2\pi p^2 \sum_{l=0}^{\infty} \left(\frac{\Lambda}{p}\right)^{\beta_0 \sigma_l} \int_0^{\infty} dt \left(\frac{\Lambda}{p}\right)^t P_l(t). \quad (2.12)$$

By a change of variable and some manipulation (see appendix A), we can rewrite the condensate as

$$\langle \mathcal{O}(0)^2 \rangle_{\tilde{a},a} = C \mu^{2[\mathcal{O}]} \sum_{l=0}^{\infty} \left(\frac{\Lambda}{\mu}\right)^{\beta_0 \sigma_l} \int_0^{\infty} dt \left(\frac{\Lambda}{\mu}\right)^t B_l(t) \quad \text{with} \quad C = \frac{d \log \frac{\mu}{\Lambda}}{(4\pi)^{\frac{d}{2}} \Gamma(d/2 + 1)}. \quad (2.13)$$

The function $B_l(t)$ are given by⁴

$$B_l(t) = \frac{1}{e_l} \left[\left(\frac{\mu}{a}\right)^{e_l} f_l \left(\frac{e_l \lambda'_\mu}{t + t_a} \right) - \left(\frac{\mu}{\tilde{a}}\right)^{e_l} f_l \left(\frac{e_l \lambda'_\mu}{t + t_{\tilde{a}}} \right) \right], \quad (2.14)$$

where

$$e_l = \beta_0 \sigma_l - 2[\mathcal{O}], \quad t_p = e_l \frac{\log p/\Lambda}{\log \mu/\Lambda}. \quad (2.15)$$

If $f_l(\lambda'_p)$ is divergent in the limit $\lambda'_p \rightarrow \infty$ ($p \rightarrow \Lambda$), the Borel transform B_l has a singularity at $t = -t_a$. This singularity is on the integration contour and gives rise to an ambiguity if t_a is negative, i.e. the IR cutoff scale a is smaller than the dynamically generated scale Λ . We can see that this singularity and the corresponding ambiguity do not vanish even in the large- N limit. For example, the singularity and the corresponding ambiguity is independent of N in the $O(N)$ sigma model since $\beta_0 = 1$, $\sigma_l = 2l$. The factorial divergence of the perturbation series can also be seen from the fact that the Taylor expansion of $B_l(t')$ around $t = 0$ has a finite radius of convergence due to the singularity. In this way, the singularity of the renormalized coupling constant at $p = \Lambda$ results in renormalon type ambiguities. In the next section, we will explicitly examine these renormalon ambiguities in the $O(N)$ sigma model in the large- N limit.

³Instead of the IR momentum cutoff, we may introduce other deformations such as mass deformations, chemical potentials or background fields like the Ω -background.

⁴This ‘‘Borel transform’’ $B_l(t)$ has a λ'_μ -dependence. The standard coupling independent Borel transform will be denoted as $\mathcal{B}_l(t)$ in section 4.3. We will use the same symbol t for the variables of $B_l(t)$ and $\mathcal{B}_l(t)$ although they are not exactly identical.

3 $O(N)$ sigma model at large N

In this section, we give a brief review of the $O(N)$ sigma model in two dimensions at large N in order to establish our notations and to write down the exact expression for the correlation functions and the condensate. More comprehensive reviews on this subject can be found in refs. [74, 89, 90].

In the two-dimensional $O(N)$ sigma model, the target space is the unit sphere in Euclidean N -dimensional space. The action is given by

$$S = \frac{1}{2g^2} \int d^2x \left[(\partial_i \phi^a)^2 + D \left\{ (\phi^a)^2 - 1 \right\} \right], \quad (3.1)$$

where ϕ^a with $a = 1 \dots N$ are real scalar fields and the field D is a Lagrange multiplier field that imposes the constraint, $(\phi^a)^2 = 1$. The parameter g is a bare coupling constant that needs to be renormalized. The theory is asymptotically free, has a mass gap, and is therefore a good toy model for the Yang-Mills theory in four dimensions.

The expectation value of the Lagrange multiplier field, $\langle D \rangle$, serves as the mass for the ϕ fields. At large N , the mass gap $\sqrt{\langle D \rangle}$ can be computed exactly by looking for the saddle point of the effective potential for D . Assuming that D is a constant and integrating ϕ^a , we obtain the effective potential for D as

$$V_{\text{eff}}(D) = \frac{N}{2} \left[\int \frac{d^2p}{(2\pi)^2} \log \left(p^2 + D \right) - \frac{D}{\lambda} \right], \quad (3.2)$$

where $\lambda = g^2 N$ is the 't Hooft coupling that is kept finite in the large- N limit. After subtracting the UV divergence and renormalizing the coupling, the effective potential becomes

$$V_{\text{eff}}(D) = -\frac{N}{8\pi} D \left(\log \frac{D}{\Lambda^2} - 1 \right), \quad (3.3)$$

where the renormalization group (RG)-invariant dynamical scale Λ is defined by the renormalized 't Hooft coupling λ_μ at the renormalization scale μ as

$$\Lambda = \mu \exp \left(-\frac{2\pi}{\lambda_\mu} \right), \quad (3.4)$$

in the MS-bar scheme. The effective potential gives the unique minimum at

$$\langle D \rangle = \Lambda^2. \quad (3.5)$$

Let us consider two-point correlation functions of the fluctuation field $\delta D(x)$ of the Lagrange multiplier field $D(x)$ around the expectation value $\langle D \rangle = \Lambda^2$. Since the correlation function is nontrivial only at the next-to-leading order of $1/N$ expansion, we choose a normalization

$$D(x) = \Lambda^2 + \frac{\delta D(x)}{\sqrt{N}}. \quad (3.6)$$

At the leading order in the large- N limit, the two-point correlation function $\Delta(p)$ of the fluctuation field $\delta D(x)$ in the momentum space (propagator) is given as

$$\Delta(p) \equiv \left[\frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \frac{1}{(q^2 + \Lambda^2) ((q+p)^2 + \Lambda^2)} \right]^{-1} = \frac{8\pi \sqrt{p^2 (p^2 + 4\Lambda^2)}}{s_p}, \quad (3.7)$$

where s_p is the function of p defined as

$$s_p = 4 \log \left(\sqrt{\frac{p^2}{4\Lambda^2} + 1} + \sqrt{\frac{p^2}{4\Lambda^2}} \right) \quad \left(= 4 \operatorname{arcsinh} \frac{p}{2\Lambda} \right). \quad (3.8)$$

The correlation function in the position space can be obtained by the Fourier transformation

$$\langle \delta D(x) \delta D(0) \rangle = \int \frac{d^2 p}{(2\pi)^2} e^{ip \cdot x} \Delta(p). \quad (3.9)$$

This is a well-defined UV (and IR) convergent integral. However, it becomes UV divergent in the limit $x = 0$

$$\langle \delta D^2 \rangle \equiv \lim_{x \rightarrow 0} \langle \delta D(x) \delta D(0) \rangle \rightarrow \infty. \quad (3.10)$$

This quantity appears as one of the operator basis O_n of the operator product expansion

$$D(x) D(0) = \sum_n F_n(x) O_n, \quad (3.11)$$

where $F_n(x)$ are the coefficient functions. For that reason, we are interested in the limit $x \rightarrow 0$ and call the quantity as a condensate, in analogy to the gluon condensate in QCD. To regularize the UV divergence, we introduce the UV cutoff \tilde{a} to limit the momentum integration $|p| < \tilde{a}$

$$\langle \delta D^2 \rangle_{\tilde{a}} \equiv \int_{|p| < \tilde{a}} \frac{d^2 p}{(2\pi)^2} \Delta(p). \quad (3.12)$$

Changing the variable from $|p|$ to $s = s_p$, we obtain

$$\langle \delta D^2 \rangle_{\tilde{a}} = 2\Lambda^4 \int_0^{s_{\tilde{a}}} ds \frac{\cosh s - 1}{s} = 2\Lambda^4 \operatorname{Chin}(s_{\tilde{a}}), \quad (3.13)$$

where $\operatorname{Chin}(s_{\tilde{a}})$ is an entire function of $s_{\tilde{a}}$ related to the hyperbolic cosine integral Chi and Euler's constant γ_E as

$$\operatorname{Chin}(s_{\tilde{a}}) = \operatorname{Chi}(s_{\tilde{a}}) - \log(s_{\tilde{a}}) - \gamma_E. \quad (3.14)$$

This is the regular and well-defined exact result in the large- N limit [74]. In the next section, instead of directly evaluating the integral (3.12), we use the large p/Λ expansion of the integrand (3.12) to simulate the semiclassical expansion, which has an IR divergence and a renormalon type ambiguity.

4 Semiclassical expansion

In section 4.1, we first expand the propagator (3.7) into a transseries of $\Lambda^2/p^2 = \exp(-4\pi/\lambda_p)$ and λ_p in order to imitate massless perturbation theory around the vacuum and nontrivial backgrounds. We then discuss IR divergences and imaginary ambiguities in the expansion in section 4.2, and finally compute the semiclassical expansion up to order Λ^8 in section 4.3.

4.1 Expansion of the propagator in powers of Λ^2/p^2

Here we consider the $x \rightarrow 0$ limit of the correlation function, i.e. the condensate, of the fluctuation of the Lagrange multiplier field $\delta D(x)$ in eq. (3.12).

In most of interesting theories like QCD, the gap equation to generate the mass gap is not known explicitly, contrary to the two-dimensional large- N $O(N)$ model. In such a situation, we can use only the weak coupling perturbation theory with massless fields. We are interested in studying properties of perturbation theory and associated resurgence structure when only perturbative series with massless fields are available. In order to mimic such a situation, we use the large p^2/Λ^2 expansion of the propagator $\Delta(p)$ to obtain a transseries in powers of $\Lambda^2/p^2 = \exp(-4\pi/\lambda_p)$ and λ_p . In this way, we can study quantities such as the condensate as if we perform massless field perturbation theory on various backgrounds corresponding to possible nonperturbative saddle points. Hence we wish to expand the propagator $\Delta(p)$ in eq. (3.7) in powers of Λ^2/p^2 . The asymptotic behavior for $\Lambda^2 \ll p^2$ of the denominator s_p of the propagator is given by

$$s_p = 4 \log \left(\sqrt{\frac{p^2}{4\Lambda^2} + 1} + \sqrt{\frac{p^2}{4\Lambda^2}} \right) = \frac{8\pi}{\lambda_p} + u_p, \quad (4.1)$$

where the leading term is the inverse coupling λ_p renormalized at the momentum scale p ,⁵

$$\lambda_p \equiv \frac{2\pi}{\log(p/\Lambda)}, \quad (4.2)$$

and the remaining term u_p can be expanded in a power of Λ^2/p^2

$$u_p = 4 \log \left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\Lambda^2}{p^2}} \right) = \frac{4\Lambda^2}{p^2} - \frac{6\Lambda^4}{p^4} + \mathcal{O}(\Lambda^6). \quad (4.3)$$

Thus, we obtain a power series expansion for large momenta as a power series in $u_p \lambda_p / 8\pi$

$$\Delta(p) = p^2 \lambda_p \sqrt{1 + \frac{4\Lambda^2}{p^2}} \sum_{n=0}^{\infty} \left(-\frac{u_p \lambda_p}{8\pi} \right)^n, \quad (4.4)$$

which is convergent if $u_p \lambda_p / (8\pi) < 1$. We can expand $\sqrt{1 + 4\Lambda^2/p^2}$ and u_p in powers of Λ/p to obtain

$$\Delta(p) = p^2 \sum_{l=0}^{\infty} \left(\frac{\Lambda}{p} \right)^{2l} f_l(\lambda_p), \quad (4.5)$$

⁵In the large- N limit, we do not distinguish the full renormalized coupling λ_p and the one-loop coupling λ'_p used in section 2 since the higher order coefficients of the beta function in (2.2) are of order $1/N$.

where $f_l(\lambda_p)$ is a polynomial of degree $l+1$. A convenient way to derive the explicit forms of $f_l(\lambda_p)$ is to use the Borel resummed form of $\Delta(p)$

$$\Delta(p) = 2\pi p^2 \sum_{l=0}^{\infty} \left(\frac{\Lambda}{p}\right)^{2l} \int_0^\infty dt \left(\frac{\Lambda}{p}\right)^t P_l(t), \quad (4.6)$$

where $P_l(t)$ is a polynomial of t

$$P_l(t) \equiv \frac{(-1)^l}{l!} \left[(t+l+1)^{(l)} - 4l(t+l)^{(l-1)} \right] \quad \text{with} \quad (a)^{(l)} = \frac{\Gamma(a+l)}{\Gamma(a)}. \quad (4.7)$$

From this expression, we find that $f_l(\lambda_p)$ can be obtained as

$$f_l(\lambda_p) = P_l(\Lambda \partial_\Lambda) \lambda_p. \quad (4.8)$$

To obtain the condensate, we need to perform the momentum integral (3.12). We now use the large momentum expansion (4.5) to all momentum regions, including $p < \Lambda$ region. This is intended to imitate the calculation with massless fields, even though the large momentum expansion is valid only for $|p| \gg \Lambda$. Then we need to introduce an IR regularization, which is achieved by a momentum cutoff at a ($|p| > a$). The condensate is now given as

$$\langle \delta D^2 \rangle_{\tilde{a},a} \underset{\text{s.c.}}{=} \sum_{l=0}^{\infty} \Lambda^{2l} C_{2l}, \quad (4.9)$$

with

$$C_{2l} = \int_{a < |p| < \tilde{a}} \frac{d^2 p}{(2\pi)^2} p^{2-2l} f_l(\lambda_p), \quad (4.10)$$

with $f_l(\lambda_p)$ in eq. (4.5). In this work, we call the transseries expression (4.9) the semiclassical ansatz (s.c.), since this would be the transseries obtained through the semiclassical expansion of the path integral. We have introduced a UV cutoff at \tilde{a} and IR cutoff at a in momentum integration in order to eliminate the UV and IR divergences. However, it is not clear if the semiclassical ansatz for $\langle \delta D^2 \rangle_{\tilde{a},a}$ gives the exact expression in the limit $a \rightarrow 0$, since the series in powers of Λ may not be convergent for $\Lambda > a$ and the ordering of summation and integration is exchanged. We will come back to this point in section 5.

Using the relation

$$\frac{\lambda_p}{4\pi} = \left[\frac{4\pi}{\lambda_{\tilde{a}}} + \log \left(\frac{p^2}{\tilde{a}^2} \right) \right]^{-1} = \sum_{n=0}^{\infty} \left(\frac{\lambda_{\tilde{a}}}{4\pi} \right)^{n+1} \left[-\log \left(\frac{p^2}{\tilde{a}^2} \right) \right]^n, \quad (4.11)$$

we can expand the integrand in (4.10) in powers of the coupling $\lambda_{\tilde{a}}$ at scale \tilde{a} to

$$C_{2l} = \sum_{n=0}^{\infty} \lambda_{\tilde{a}}^{n+1} c_{(2l,n)}, \quad (4.12)$$

whose explicit computations for $l = 0, \dots, 4$ are given in the appendix B.

The power series in eq. (4.12) can contain factorially divergent parts, which have a precise meaning by the Borel resummation. If such divergent series are Borel non-summable, the associated imaginary ambiguities should be of the renormalon type, since only renormalon type ambiguities are expected to remain in the large- N limit. The $l = 0$ terms $c_{(0,n)}$ correspond to the usual perturbative expansion on the trivial vacuum. The physical interpretation of $c_{(2l,n)}$ for higher $l > 0$ is that it is a contribution of the fluctuation at order $\lambda_{\tilde{a}}^{n+1}$ around a possible semi-classical configuration $(\Lambda/\tilde{a})^{2l} \sim e^{-4\pi l/\lambda_{\tilde{a}}}$, although we have no understanding of such a semiclassical configuration explicitly.

4.2 Infrared divergence and imaginary ambiguities

It is evident that there are three issues with the semiclassical expansion obtained above due to the IR behavior. The first one is that the integral C_{2l} is IR divergent when $l \geq 2$ due to the factor p^{2-2l} in the integrand, which requires an IR cutoff a . We need to take the limit $a \rightarrow 0$ at the end of the calculation. The second issue then arises when the IR cutoff is small $a \ll \Lambda$, because the semiclassical ansatz above involves a power series in Λ^2/a^2 and requires a care to take the limit $a \rightarrow 0$. We will come back to this point in section 5. The third issue is that there is a possible singularity at $p = \Lambda$ due to the terms involving the renormalized coupling constant $\lambda_p = 4\pi/\log(p^2/\Lambda^2)$ in eq. (4.10). In fact, the renormalon ambiguity in the usual perturbation theory is due to this type of singularity in the integrand of C_0 . Below we identify these singularities in the integrand of all C_{2l} . Changing variables from p to $\tilde{t} = \log(\tilde{a}^2/p^2) = 4\pi/\lambda_{\tilde{a}} - 4\pi/\lambda_p$, we can rewrite it as

$$C_{2l} = \frac{1}{4\pi} \int_0^{\log(\tilde{a}^2/a^2)} d\tilde{t} \left(\tilde{a}^2 e^{-\tilde{t}} \right)^{2-l} f_l \left(\frac{4\pi}{4\pi/\lambda_{\tilde{a}} - \tilde{t}} \right). \quad (4.13)$$

This form resembles the Borel resummation of a divergent perturbative series. For $l < 2$ we can take $a \rightarrow 0$ at this point, and C_{2l} becomes a Borel resummation. For $l \geq 2$, we cannot take $a \rightarrow 0$ due to the IR divergence. Since $f_l(\lambda_p)$ is a polynomial of order $l + 1$ and hence the integrand has a pole at $\tilde{t} = 4\pi/\lambda_{\tilde{a}}$. If $a < \Lambda$, this pole is on the integration contour of (4.13) since $0 < 4\pi/\lambda_{\tilde{a}} < \log(\tilde{a}^2/a^2)$.

In order to circumvent the poles, we use an analytic continuation of the coupling $\lambda_{\tilde{a}}$ to the complex plane. After the integration over t , we then analytically continue back to the real axis in two different directions:

$$\lambda_{\tilde{a}} \rightarrow \lambda_{\tilde{a}} \pm i\epsilon, \quad (4.14)$$

with $\epsilon > 0$, or equivalently $\Lambda \rightarrow \Lambda(1 \pm i\epsilon')$ with $\epsilon' = 2\pi\epsilon/\lambda_{\tilde{a}}^2 > 0$. We then take ϵ to zero in the end. This can be understood as a deformation of the integration contour in eq. (4.13) in the upper or lower t -plane.

The deformation of the integration contour can give rise to an ambiguity, since the imaginary part of C_{2l} depends on whether we take $\lambda_{\tilde{a}} + i\epsilon$ or $\lambda_{\tilde{a}} - i\epsilon$. The imaginary ambiguities, however, should cancel once we sum over all l , regardless of the prescription. We can find the imaginary ambiguities by computing the residue. Using eq. (4.13) and computing up to order Λ^8 , we find that our semiclassical ansatz (4.9) as a whole is indeed

free of imaginary ambiguity:

$$\text{Im} \left\langle \delta D^2 \right\rangle_{\tilde{a},a} \underset{\text{s.c.}}{=} \pm \pi \left[\left(\tilde{a}^2 e^{-\frac{4\pi}{\lambda_{\tilde{a}}}} \right)^2 \Lambda^0 - 2\Lambda^4 + \left(\tilde{a}^2 e^{-\frac{4\pi}{\lambda_{\tilde{a}}}} \right)^{-2} \Lambda^8 \right] \theta(\Lambda - a) = 0. \quad (4.15)$$

We show that only the three terms, C_0 , C_4 , and C_8 , have non-zero residues at $t = 4\pi/\lambda_{\tilde{a}}$ that give rise to the imaginary ambiguities in section 4.3. We also show that the first term at order Λ^0 in the bracket corresponds to the renormalon ambiguity due to the Borel resummation of the divergent perturbative series on the trivial vacuum. Thus, the ansatz (4.9) gives a surprising result that the renormalon ambiguity on the trivial vacuum (order Λ^0) is cancelled not solely by the ambiguity from the term at order Λ^4 as one would naively expect, but the combination of the terms at order Λ^4 and Λ^8 .

4.3 Perturbative expansion around vacuum and nontrivial background

In this section, we compute the coefficients C_{2l} of the expansion (4.9) and investigate the origin of each ambiguity in eq. (4.15). We first take a large IR cutoff, $\Lambda \ll a < \tilde{a}$, where the expansion in powers of Λ^2/p^2 ($a < p < \tilde{a}$) of the integrand is convergent and well-defined. This allows us to obtain unambiguous C_{2l} without any imaginary parts. We then take a small cutoff $a < \Lambda$. As explained in the previous section, we use an analytic continuation of $\lambda_{\tilde{a}}$ (or Λ) as (4.14) to avoid a possible singularity at $p = \Lambda$. Depending on the sign of $\pm i\epsilon$, we show that C_{2l} picks up an imaginary part in accordance with eq. (4.15).

The integral for C_{2l} gives

$$C_{2l} = \int_a^{\tilde{a}} \frac{dp}{2\pi} p^{3-2l} f_l(\lambda_p) = \mathcal{C}_{2l}(p)|_{a}^{\tilde{a}} = \mathcal{C}_{2l}(\tilde{a}) - \mathcal{C}_{2l}(a), \quad (4.16)$$

where we have defined $\mathcal{C}_{2l}(p)$ as an indefinite integral of the p -integration. We call $\mathcal{C}_{2l}(\tilde{a})$ and $\mathcal{C}_{2l}(a)$ as the UV and IR contributions, respectively, although only the difference is unambiguously defined.

We now compute C_{2l} for $l = 0, \dots, 4$. In the semiclassical expansion, one would first need to compute the coefficients c_{2l} of perturbative expansion, and then (Borel) resum it to obtain $C_{2l} = \sum_{n=0}^{\infty} \lambda_{\tilde{a}}^{n+1} c_{(2l,n)}$. We demonstrate this for the case of $l = 0$ here, and the rest in appendix B. Alternatively we can directly compute C_{2l} from eq. (4.10).

The leading contribution, the term at order Λ^0 , is given as

$$c_{(0,n)} = \int_{a < |p| < \tilde{a}} \frac{d^2 p}{(2\pi)^2} p^2 \left(\frac{1}{4\pi} \log \frac{\tilde{a}^2}{p^2} \right)^n = \frac{\tilde{a}^4}{(8\pi)^{n+1}} \left[\Gamma(n+1) - \Gamma \left(n+1, 2 \log \frac{\tilde{a}^2}{a^2} \right) \right], \quad (4.17)$$

where $\Gamma(n+1, \alpha)$ is the incomplete Gamma function

$$\Gamma(n+1, \alpha) = \int_{\alpha}^{\infty} dt e^{-t} t^n. \quad (4.18)$$

If we turn off the IR cutoff $a \rightarrow 0$, the second term vanishes, and we arrive at the known perturbative result. If we keep an arbitrary IR cutoff a , then we have $C_0 = \mathcal{C}_0(\tilde{a}) - \mathcal{C}_0(a)$ with

$$\mathcal{C}_0(p) = \tilde{a}^4 \sum_{n=0}^{\infty} \left(\frac{\lambda_{\tilde{a}}}{8\pi} \right)^{n+1} \Gamma \left(n+1, 2 \log \frac{\tilde{a}^2}{p^2} \right). \quad (4.19)$$

This is a divergent asymptotic series since $\Gamma(n+1, \alpha) \sim n!$ for large n . Applying the Borel resummation, we obtain

$$\mathcal{C}_0(p) = -p^4 \int_0^\infty dt \frac{e^{-t}}{t - \frac{8\pi}{\lambda_p}} = p^4 e^{-8\pi/\lambda_p} \left[\gamma_E + \log\left(-\frac{8\pi}{\lambda_p}\right) - \text{Ein}\left(-\frac{8\pi}{\lambda_p}\right) \right], \quad (4.20)$$

where $\text{Ein}(z)$ denotes the entire function defined as⁶

$$\text{Ein}(z) = \int_0^z dt \frac{1 - e^{-t}}{t}. \quad (4.22)$$

Due to the branch cut of $\log(-8\pi/\lambda_p) = \log(-2 \log p^2/\Lambda^2)$, the function $\mathcal{C}_0(p)$ is ambiguous for $p > \Lambda$

$$\text{Im } \mathcal{C}_0(p) = \pm \pi p^4 \exp\left(-\frac{8\pi}{\lambda_p}\right) \theta(p - \Lambda) = \pm \pi \Lambda^4 \theta(p - \Lambda). \quad (4.23)$$

The total imaginary ambiguity at the leading order can be then expressed as

$$\text{Im } C_0 = \text{Im } \mathcal{C}_0(\tilde{a}) - \text{Im } \mathcal{C}_0(a) = \pm \{\pi - \pi \theta(a - \Lambda)\} \Lambda^4 = \pm \pi \Lambda^4 \theta(\Lambda - a), \quad (4.24)$$

where we have assumed that the UV scale \tilde{a} is always larger than Λ . While there is a usual renormalon ambiguity when $a < \Lambda$, the imaginary ambiguity is absent when $a > \Lambda$. In this paper we focus on the case when the infrared cutoff is small, $a < \Lambda$, because this is when the well-known renormalon ambiguity arises in perturbation theory. We show below how this ambiguity is cancelled in our semi-classical expansion.

The Λ^2 and Λ^6 can be readily computed without any imaginary ambiguities. For notational simplicity, we use v_p defined as

$$v_p \equiv \frac{4\pi}{\lambda_p} = \log \frac{p^2}{\Lambda^2}, \quad (4.25)$$

instead of λ_p . At order Λ^2 , we have

$$\mathcal{C}_2(p) = \int dp \frac{4p(-1 + v_p)}{v_p^2} = \frac{2p^2}{v_p}, \quad (4.26)$$

while at Λ^6 , we obtain

$$\mathcal{C}_6(p) = \int dp \frac{-48 - 24v_p + 20v_p^2 + 24v_p^3}{3p^3 v_p^4} = \frac{8 + 2v_p - 12v_p^2}{3p^2 v_p^3}. \quad (4.27)$$

We thus find that the IR contribution $\mathcal{C}_2(a)$ goes to zero, while $\mathcal{C}_6(a)$ diverges as a goes to zero. Note that each of the integrands for $\mathcal{C}_2(p)$ and $\mathcal{C}_6(p)$ has a pole at $p = \Lambda$ but the residue is zero and hence it does not give any ambiguities.

⁶The standard exponential integral $\text{Ei}(z)$ is related to the entire function $\text{Ein}(z)$ as

$$\text{Ei}(z) = \gamma_E + \log z - \text{Ein}(-z), \quad (4.21)$$

At order Λ^4 , we have

$$\mathcal{C}_4(p) = \int dp \frac{8 - 2v_p - 4v_p^2}{pv_p^3} = -2 \log(v_p) - \frac{2 - v_p}{v_p^2}. \quad (4.28)$$

This term is also IR divergent $\mathcal{C}_4(a) \rightarrow \infty$ ($a \rightarrow 0$). Moreover the logarithm gives rise to the imaginary ambiguity when $v_p < 0$:

$$\text{Im } C_4 = \text{Im } \mathcal{C}_4(\tilde{a}) - \text{Im } \mathcal{C}_4(a) = \mp 2\pi\theta(\Lambda - a). \quad (4.29)$$

Compared to the renormalon ambiguity (4.24), this ambiguity at order Λ^4 has the opposite sign but its magnitude is twice as large, so the renormalon ambiguity is not cancelled if we stop the calculation at this order.

At order Λ^8 , we have

$$\begin{aligned} \mathcal{C}_8(p) &= \int dp \frac{96 + 120v_p + 22v_p^2 - 59v_p^3 - 60v_p^4}{3p^5v_p^5} \\ &= \frac{1}{\Lambda^4} \left[-\text{Ein}\left(\frac{8\pi}{\lambda_p}\right) + \log\left(\frac{8\pi}{\lambda_p}\right) + \gamma_E \right] - \frac{24 + 24v_p - 13v_p^2 - 33v_p^3}{6p^4v_p^4}. \end{aligned} \quad (4.30)$$

At this order, the logarithm remains as in the case of $C_0(p)$. Therefore it has the imaginary ambiguity when $v_a < 0$ or $a < \Lambda$:

$$\text{Im } C_8 = \pm\theta(\Lambda - a)\frac{\pi}{\Lambda^4}. \quad (4.31)$$

Using eq. (B.5) in appendix, we can write the perturbative expansion as

$$\mathcal{C}_8(p) \supset \frac{1}{\tilde{a}^4} \sum_{n=0}^{\infty} \left(-\frac{\lambda_{\tilde{a}}}{8\pi}\right)^{n+1} \Gamma\left(n+1, -2 \log \frac{\tilde{a}^2}{p^2}\right) = -\frac{1}{p^4} \int_0^\infty dt \frac{e^{-t}}{t + \frac{8\pi}{\lambda_p}}. \quad (4.32)$$

The integrand has a pole at $t = -8\pi/\lambda_p$ and the residue gives the imaginary ambiguity of eq. (4.30). One should note that the t -plane pole for $\mathcal{C}_{2l}(p)$ is at $t = -8\pi/\lambda_p$, in contrast to $t = 8\pi/\lambda_p$ for $\mathcal{C}_0(p)$ in (4.20).

By using eq. (4.8), we can show that $\mathcal{C}_{2l}(p)$ for general l is given by

$$\mathcal{C}_{2l}(p) = p^{4-2l} \int_0^\infty dt \left(\frac{\Lambda}{p}\right)^t \frac{P_l(t)}{4 - 2l - t} = -P_l(\Lambda \partial_\Lambda) \left[\Lambda^{-2l+4} \Gamma\left(0, (l-2) \log \frac{p^2}{\Lambda^2}\right) \right]. \quad (4.33)$$

From this expression, we can check that there is no ambiguity for $l \geq 5$.

We now combine all the results up to order Λ^8 obtained above and write the semiclassical expansion of the condensate for any values of $\tilde{a} > \Lambda$ and $a \neq \Lambda$

$$\begin{aligned} \langle \delta D^2 \rangle_{\tilde{a},a} &\underset{\text{s.c.}}{=} \sum_{l=0}^{\infty} \Lambda^{2l} \left[\{C_{2l}(\tilde{a})\} - \{C_{2l}(a)\} \right] & (4.34) \\ &= \Lambda^0 \left[\tilde{a}^4 \left\{ e^{-8\pi/\lambda_{\tilde{a}}} \text{Ei} \left(\frac{8\pi}{\lambda_{\tilde{a}}} \right) \right\} - a^4 \left\{ e^{-8\pi/\lambda_a} \text{Ei} \left(\frac{8\pi}{\lambda_a} \right) \right\} \pm i\pi\Lambda^4 \theta(\Lambda - a) \right] \\ &\quad + \Lambda^2 \left[\tilde{a}^2 \left\{ \frac{\lambda_{\tilde{a}}}{2\pi} \right\} - a^2 \left\{ \frac{\lambda_a}{2\pi} \right\} \right] \\ &\quad + \Lambda^4 \left[\tilde{a}^0 \left\{ \frac{\lambda_{\tilde{a}}}{4\pi} - \frac{\lambda_{\tilde{a}}^2}{8\pi^2} - 2 \log \left(\frac{4\pi}{\lambda_{\tilde{a}}} \right) \right\} - a^0 \left\{ \frac{\lambda_a}{4\pi} - \frac{\lambda_a^2}{8\pi^2} - 2 \log \left| \frac{4\pi}{\lambda_a} \right| \right\} \mp 2\pi i \theta(\Lambda - a) \right] \\ &\quad + \Lambda^6 \left[\frac{1}{\tilde{a}^2} \left\{ -\frac{\lambda_{\tilde{a}}}{\pi} + \frac{\lambda_{\tilde{a}}^2}{24\pi^2} + \frac{\lambda_{\tilde{a}}^3}{24\pi^3} \right\} - \frac{1}{a^2} \left\{ -\frac{\lambda_a}{\pi} + \frac{\lambda_a^2}{24\pi^2} + \frac{\lambda_a^3}{24\pi^3} \right\} \right] \\ &\quad + \Lambda^8 \left[\frac{1}{\tilde{a}^4} \left\{ e^{8\pi/\lambda_{\tilde{a}}} \text{Ei} \left(-\frac{8\pi}{\lambda_{\tilde{a}}} \right) + \frac{11\lambda_{\tilde{a}}}{8\pi} + \frac{13\lambda_{\tilde{a}}^2}{96\pi^2} - \frac{\lambda_{\tilde{a}}^3}{16\pi^3} - \frac{\lambda_{\tilde{a}}^4}{64\pi^4} \right\} \right. \\ &\quad \left. - \frac{1}{a^4} \left\{ e^{8\pi/\lambda_a} \text{Ei} \left(-\frac{8\pi}{\lambda_a} \right) + \frac{11\lambda_a}{8\pi} + \frac{13\lambda_a^2}{96\pi^2} - \frac{\lambda_a^3}{16\pi^3} - \frac{\lambda_a^4}{64\pi^4} \right\} \pm \frac{i\pi}{\Lambda^4} \theta(\Lambda - a) \right] \\ &\quad + \mathcal{O}(\Lambda^{10}), \end{aligned}$$

where the exponential integral $\text{Ei}(z)$ is defined as

$$\text{Ei}(z) = \gamma_E + \log z - \text{Ein}(-z) = \gamma_E + \log z - \int_0^{-z} dt \frac{1 - e^{-t}}{t}. \quad (4.35)$$

The imaginary ambiguity at each order depends on the value of the IR cutoff a as denoted by the Heaviside step function $\theta(\Lambda - a)$. For a large IR cutoff $a > \Lambda$, there is no ambiguity at any order. Once we take the small cutoff $a < \Lambda$, imaginary ambiguities appear at order Λ^0 , Λ^4 , and Λ^8 . We have identified the imaginary ambiguity at order Λ^0 as the renormalon ambiguity in perturbation theory on the trivial vacuum. The imaginary ambiguities at order Λ^4 and Λ^8 also arise when $a < \Lambda$, and the combination of the two cancels the renormalon ambiguity, leaving the semiclassical expansion free of imaginary ambiguities as a whole. This result agrees with eq. (4.15).

Using the general form C_{2l} in eq. (4.33), the all-order transseries can be written as

$$\langle \delta D^2 \rangle_{\tilde{a},a} \underset{\text{s.c.}}{=} \mu^4 \sum_{l=0}^{\infty} \left(\frac{\Lambda}{\mu} \right)^{2l} \int_0^{\infty} dt \left(\frac{\Lambda}{\mu} \right)^t \mathcal{B}_l(t). \quad (4.36)$$

The Borel transform $\mathcal{B}_l(t)$ is given by

$$\mathcal{B}_l(t) = \frac{1}{2} \mu^{-2\eta_l(t)} \left[\tilde{a}^{2\eta_l(t)} - a^{2\eta_l(t)} \right] \frac{P_l(t)}{\eta_l(t)}, \quad \text{with} \quad \eta_l(t) = 2 - l - \frac{t}{2}, \quad (4.37)$$

where $P_l(t)$ is the polynomial given in eq. (4.7). Since the Borel transform $\mathcal{B}_l(t)$ has no pole on the positive real axis, there is not ambiguity in this expression. However, the integral converges only when $a > \Lambda$. For $a < \Lambda$ ($\lambda_a < 0$), the Borel resummation for the

a -dependent term must be performed along the negative real axis⁷, or equivalently, the Borel resummation must be rewritten as

$$\langle \delta D^2 \rangle_{\tilde{a},a} \underset{\text{s.c.}}{=} \mu^4 \sum_{l=0}^{\infty} \left(\frac{\Lambda}{\mu}\right)^{2l} \int_{-\infty}^{\infty} dt \left(\frac{\Lambda}{\mu}\right)^t \tilde{\mathcal{B}}_l(t), \quad (4.39)$$

with

$$\tilde{\mathcal{B}}_l(t) = \frac{1}{2} \mu^{-2\eta_l(t)} \left[\tilde{a}^{2\eta_l(t)} \theta(t) + a^{2\eta_l(t)} \theta(-t) \right] \frac{P_l(t)}{\eta_l(t)}, \quad (4.40)$$

where $\theta(t)$ is the step function. In this case, $\tilde{\mathcal{B}}_l(t)$ with $l = 0, 2, 4$ have singularities at $t = 4, 0, -4$ and give rise to the imaginary ambiguities at order Λ^0 , Λ^4 and Λ^8 , respectively. It is worth noting that the singularity on the negative real axis on the Borel plane is relevant when $a < \Lambda$. This is related to the fact that the condensate contains terms with the negative coupling constant $\lambda_a = 2\pi/\log(a/\Lambda)$ and the non-perturbative factors $(\Lambda/a)^{2l}$ that become more dominant for higher l . This is a typical situation in which renormalons give rise to imaginary ambiguities.

Although the condensate in the semiclassical expansion is real for any value of the IR cutoff a , the expansion is convergent only if the IR cutoff is large, $a \gg \Lambda$. When the cutoff is small, $a \ll \Lambda$, the terms in the series for the IR contribution, $C_l(a)$, becomes divergent for $l \geq 2$. This tells us that we need to consider taking $a \gg \Lambda$ in order to sum over l . We can take the limit $a \rightarrow 0$ only after summing over l . We now discuss this procedure in the next section.

5 Transseries from the exact result

We reanalyze the exact result for the condensate $\langle \delta D^2 \rangle$ in eq. (3.13), in order to understand the newly found imaginary ambiguities at higher powers of Λ and to take the $a \rightarrow 0$ limit properly in our result in eq. (4.34). To compare the exact result with the semiclassical expansion eq. (4.34), we now work out the transseries representation of the exact result in eq. (3.13). From dimensional reasons, the condensate is a function of a single variable Λ/\tilde{a} apart from the factor Λ^4

$$\langle \delta D^2 \rangle_{\tilde{a}} = \Lambda^4 F(s_{\tilde{a}}) = 2\Lambda^4 \int_0^{s_{\tilde{a}}} ds \frac{\cosh s - 1}{s}, \quad (5.1)$$

where the upper end $s_{\tilde{a}}$ of the integral is a function of Λ^2/\tilde{a}^2 as defined in eq. (4.4)

$$s_{\tilde{a}} = \frac{8\pi}{\lambda_{\tilde{a}}} + u_{\tilde{a}}, \quad u_{\tilde{a}} = 4 \log \left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\Lambda^2}{\tilde{a}^2}} \right). \quad (5.2)$$

⁷The integration path of the Borel resummation must be chosen depending on the sign (or, more precisely, argument) of the variable as

$$\sum_{n=0}^{\infty} a_n \lambda^n = \begin{cases} + \int_0^{+\infty} dt e^{-t/\lambda} B(t) & \text{for } \lambda > 0 \\ - \int_{-\infty}^0 dt e^{-t/\lambda} B(t) & \text{for } \lambda < 0 \end{cases} \quad \text{with} \quad B(t) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n)} t^{n-1}. \quad (4.38)$$

As given in eq. (C.1) in appendix C, the variable $u_{\tilde{a}}$ can be expanded in powers of Λ^2/\tilde{a}^2 with the finite radius of convergence. On the other hand, the function $F(s_{\tilde{a}})$ can be expanded in powers of $u_{\tilde{a}}$ with a finite radius of convergence. Therefore, we find that contributions from the integration region $8\pi/\lambda_{\tilde{a}} < s < s_{\tilde{a}}$ in the integral representation in eq. (5.1) of the exact solution gives a power series in Λ^2/\tilde{a}^2 . Moreover, it is easy to see that each l -th order terms $\Lambda^{2l}/\tilde{a}^{2l}$ contains only up to l powers of $\lambda_{\tilde{a}}$. The remaining term, however, gives a divergent power series in $\lambda_{\tilde{a}}$ and needs to be Borel resummed. In fact, the contribution from $0 < s < 8\pi/\lambda_{\tilde{a}}$ can be rewritten into the Borel resummation of the factorially divergent series

$$F\left(\frac{8\pi}{\lambda_{\tilde{a}}}\right) = -\frac{\tilde{a}^4}{\Lambda^4} \int_0^\infty dt \frac{e^{-t}}{t - \frac{8\pi}{\lambda_{\tilde{a}}} \pm i0} + \left[2\log\left(\frac{\lambda_{\tilde{a}}}{8\pi}\right) - 2\gamma_E \mp i\pi \right] - \frac{\Lambda^4}{\tilde{a}^4} \int_0^\infty dt \frac{e^{-t}}{t + \frac{8\pi}{\lambda_{\tilde{a}}}}. \quad (5.3)$$

We note that the first term is the result of Borel resummation of Borel-nonsummable divergent power series and has an imaginary ambiguity, which is cancelled by the imaginary ambiguity in the second term [74]. The third term is the result of Borel resummation of Borel-summable divergent series without imaginary ambiguity. Combining contributions from the integration region $8\pi/\lambda_{\tilde{a}} < s < s_{\tilde{a}}$, we obtain up to terms of order Λ^8 as

$$\begin{aligned} \langle \delta D^2 \rangle_{\tilde{a}} &= \Lambda^0 \tilde{a}^4 \left\{ - \int_0^\infty dt \frac{e^{-t}}{t - \frac{8\pi}{\lambda_{\tilde{a}}} \pm i0} \right\} + \Lambda^2 \tilde{a}^2 \left\{ \frac{\lambda_{\tilde{a}}}{2\pi} \right\} \\ &\quad + \Lambda^4 \left\{ \frac{\lambda_{\tilde{a}}}{4\pi} - \frac{\lambda_{\tilde{a}}^2}{8\pi^2} + 2\log\left(\frac{\lambda_{\tilde{a}}}{8\pi}\right) - 2\gamma_E \mp i\pi \right\} + \frac{\Lambda^6}{\tilde{a}^2} \left\{ -\frac{\lambda_{\tilde{a}}}{\pi} + \frac{\lambda_{\tilde{a}}^2}{24\pi^2} + \frac{\lambda_{\tilde{a}}^3}{24\pi^3} \right\} \\ &\quad + \frac{\Lambda^8}{\tilde{a}^4} \left\{ - \int_0^\infty dt \frac{e^{-t}}{t + \frac{8\pi}{\lambda_{\tilde{a}}}} + \frac{11\lambda_{\tilde{a}}}{8\pi} + \frac{13\lambda_{\tilde{a}}^2}{96\pi^2} - \frac{\lambda_{\tilde{a}}^3}{16\pi^3} - \frac{\lambda_{\tilde{a}}^4}{64\pi^4} \right\} + \mathcal{O}\left(\frac{\Lambda^{10}}{\tilde{a}^6}\right). \end{aligned} \quad (5.4)$$

This is the Borel resummed transseries for the exact result without an IR cutoff, which is valid for $\tilde{a} \gg \Lambda$. We can see that the ambiguity structure of this transseries without the IR cutoff is different from that with the IR cutoff (4.34).

Although the condensate itself has no IR divergence, we can introduce the IR cutoff a for the momentum integration in order to compare the result of the semiclassical ansatz with the exact result

$$\langle \delta D^2 \rangle_{\tilde{a},a} = \Lambda^4 \{ F(s_{\tilde{a}}) - F(s_a) \}. \quad (5.5)$$

The contribution $F(s_a)$ is defined by the integral representation in eq. (5.1), with the upper end of integration given by s_a instead of $s_{\tilde{a}}$. We find that it is expandable in power series of a/Λ as given in eq. (C.3) in appendix C. In particular, $F(s_a) \rightarrow 0$ in the limit of $a \rightarrow 0$:

$$F(s_a) = \frac{a}{\Lambda} + \mathcal{O}\left(\frac{a^2}{\Lambda^2}\right). \quad (5.6)$$

On the other hand, the function $F(s_a)$ has an interesting analytic structure. It has a Borel resummed transseries form for $a \gg \Lambda$ whose functional form is precisely identical to that in eq. (5.4). In this region, the Borel non-summable divergent series in $\lambda_{\tilde{a}} > 0$ gives imaginary ambiguities which cancel those from the contribution $F(s_{\tilde{a}})$.

To understand the result in eq. (4.34) of the semiclassical ansatz, let us first consider the Borel resummed transseries valid for $a \gg \Lambda$. It consists of a series in powers of Λ^2/a^2 , whose l -th power coefficient is a (divergent) power series of λ_a , in exactly the same form as that in eq. (5.4) with a replacing \tilde{a} . If we take the coefficient of each term of $(\Lambda/a)^{2l}$ and analytically continue each coefficient to the region $a < \Lambda$, we find the following formal expression similar to a transseries

$$\begin{aligned} \Lambda^4 F(s_a)_{\text{formal}} = & \Lambda^0 a^4 \left\{ - \int_0^\infty dt \frac{e^{-t}}{t - \frac{8\pi}{\lambda_a}} \right\} + \Lambda^2 a^2 \left\{ \frac{\lambda_a}{2\pi} \right\} \\ & + \Lambda^4 \left\{ \frac{\lambda_a}{4\pi} - \frac{\lambda_a^2}{8\pi^2} + 2 \log \left(\frac{-\lambda_a}{8\pi} \right) - 2\gamma_E \pm i\pi \right\} + \frac{\Lambda^6}{a^2} \left\{ - \frac{\lambda_a}{\pi} + \frac{\lambda_a^2}{24\pi^2} + \frac{\lambda_a^3}{24\pi^3} \right\} \\ & + \frac{\Lambda^8}{a^4} \left\{ - \int_0^\infty dt \frac{e^{-t}}{t + \frac{8\pi}{\lambda_a} \mp i0} + \frac{11\lambda_a}{8\pi} + \frac{13\lambda_a^2}{96\pi^2} - \frac{\lambda_a^3}{16\pi^3} - \frac{\lambda_a^4}{64\pi^4} \right\} + \mathcal{O}\left(\frac{\Lambda^{10}}{a^6}\right). \end{aligned} \quad (5.7)$$

Since $\lambda_a = 4\pi/\log(a^2/\Lambda^2) < 0$ for $a < \Lambda$, the Λ^0 term becomes Borel summable, whereas the Λ^8 term becomes Borel nonsummable, resulting in an imaginary ambiguity. We also need an analytic continuation for the Λ^4 term. Thus this formal transseries exhibits imaginary ambiguities in the Λ^4 and Λ^8 terms. We now observe that the result of the semiclassical ansatz in eq. (4.34) is precisely recovered as the difference of $F(s_{\tilde{a}})$ in eq. (5.4) and this formal transseries $F(s_a)$ in eq. (5.7). In the semiclassical ansatz, we note that only the difference between the UV and IR contributions is determined.

Now we can understand the imaginary ambiguities found for $a < \Lambda$ in eq. (4.34) using the semiclassical ansatz in eq. (4.10). In the semiclassical ansatz, we first expand the momentum integrand in powers of Λ^2/p^2 which is valid only for $p^2 \gg \Lambda^2$. We then evaluate the momentum integral of each powers of Λ^2/p^2 using an IR cutoff $|p| > a$. As a result, the IR contribution $\mathcal{C}_{2l}(a)$ for the Λ^{2l} term involves powers of $(\Lambda/a)^{2l}$. However, we are using the expansion in powers of Λ^2/p^2 outside of its validity, when we take the IR cutoff a smaller than the dynamical mass Λ . This is the reason why we obtain the imaginary ambiguity corresponding to the Borel non-summable series in $\lambda(a)$ at order Λ^8/a^4 in eq. (5.7) of the formal transseries $F(s_a)_{\text{formal}}$.

In order to properly take the limit of $a \rightarrow 0$ of the result in eq. (4.34) of the semiclassical ansatz, we need to first continue a from the region $a < \Lambda$ to the region $a \gg \Lambda$, where the transseries would be convergent. Then the formal transseries becomes a well-defined transseries and gives back an analytic function defined in eq. (3.13):

$$F(s_a) = 2 \int_0^{s_a} ds \frac{\cosh s - 1}{s} = 2 \text{Chin}(s_a). \quad (5.8)$$

After obtaining the analytic function, we can safely continue it to the region $a < \Lambda$ and find that

$$\lim_{a \rightarrow 0} F(s_a) = 0. \quad (5.9)$$

Thus the final result of the $a \rightarrow 0$ limit is that we can neglect the contribution $F(s_a)_{\text{formal}}$ altogether, including those imaginary ambiguities contained in $F(s_a)_{\text{formal}}$. We also note

that the imaginary ambiguities in the IR contribution $F(s_a)$ _{formal} changes from the Λ^8 term to the Λ^0 term in the process of the analytic continuation to the region $a > \Lambda$. It is interesting to note that the function $F(s_a)$ is an example of functions of the renormalized coupling λ_a that can be continued analytically beyond the Landau singularity at $a = \Lambda$ to the negative values $\lambda_a < 0$ exhibiting an entirely different behavior [68] compared to the region $\lambda_a > 0$: power expandable in a/Λ in the region $a < \Lambda$, and Borel resummations of divergent power series in λ_a as coefficients of power series in Λ/a in the region $a \gg \Lambda$.

6 Two point function

So far we have seen the resurgence structure of the condensate $\langle \delta D^2 \rangle$. A similar but more complicated structure can be seen in the transseries expansion of the two point function

$$\langle \delta D(x) \delta D(0) \rangle = \int \frac{d^2 p}{(2\pi)^2} e^{i\mathbf{p}\cdot\mathbf{x}} \Delta(p). \quad (6.1)$$

In the following we assume that $1/x$ is larger than Λ ($\Lambda x < 1$) for simplicity. A convenient way to obtain the transseries expansion of two point function is to use its relation to the condensate with a UV cutoff \tilde{a}

$$\langle \delta D(x) \delta D(0) \rangle = \int_0^\infty d\tilde{a} x J_1(\tilde{a}x) \langle \delta D^2 \rangle_{\tilde{a}}, \quad (6.2)$$

which can be shown by using the property of the Bessel functions $J_l(px)$

$$\int \frac{d^2 p}{(2\pi)^2} e^{i\mathbf{p}\cdot\mathbf{x}} f(p) = \int_0^\infty \frac{dp}{2\pi} J_0(px) f(p) = \int_0^\infty d\tilde{a} \left[x J_1(\tilde{a}x) \int_0^{\tilde{a}} \frac{dp}{2\pi} f(p) \right], \quad (6.3)$$

for any function $f(p)$. As in the previous case, it is necessary to introduce an IR cutoff a to obtain each term in the transseries. As we have seen above, the transseries for the condensate with UV cutoff \tilde{a} and IR cutoff a can be written as

$$\langle \delta D^2 \rangle_{\tilde{a},a} = \frac{1}{2} \sum_{l=0}^{\infty} \Lambda^{2l} \int_0^\infty dt \Lambda^t \left[\tilde{a}^{2\eta_l(t)} - a^{2\eta_l(t)} \right] \frac{P_l(t)}{\eta_l(t)}, \quad \text{for } \Lambda < a < \tilde{a}, \quad (6.4)$$

where $P_l(t)$ and $\eta_l(t)$ are given by

$$P_l(t) = \frac{(-1)^l}{l!} \left[(t+l+1)^{(l)} - 4l(t+l)^{(l-1)} \right], \quad \eta_l(t) = 2 - l - \frac{t}{2}, \quad (6.5)$$

where $(a)^{(l)} = \Gamma(a+l)/\Gamma(a) = a(a+1)\cdots(a+l-1)$ denotes the Pochhammer symbol. Note that $P_l(t)$ are polynomials of t and have no singularity. From this expression and the relation (6.2), we obtain the transseries for the two point function with IR cutoff $a > \Lambda$ as

$$\begin{aligned} \langle \delta D(x) \delta D(0) \rangle_a &= \int_a^\infty d\tilde{a} x J_1(\tilde{a}x) \langle \delta D^2 \rangle_{\tilde{a},a} \\ &= \frac{\Lambda^4}{2} \sum_{l=0}^{\infty} \int_0^\infty dt \left(\frac{\Lambda^2 x^2}{4} \right)^{-\eta_l(t)} \left[\frac{\Gamma(\eta_l(t))}{\Gamma(1-\eta_l(t))} - \mathcal{F}_l(ax, t) \right] P_l(t), \end{aligned} \quad (6.6)$$

$n \setminus l$	0	1	2	3	4	5	6	7	...
0	-1	0	2	0	-1	0	0	0	...
1	1	-2	-1	4	-1	-2	1	0	...
:									

Table 1. Coefficients $A_{l,n}$.

where

$$\begin{aligned} \mathcal{F}_l(ax, t) &= \frac{1}{\eta_l(t)} {}_1F_2\left(\eta_l(t); 1, 1 + \eta_l(t), -\frac{a^2 x^2}{4}\right) \left(\frac{a^2 x^2}{4}\right)^{\eta_l(t)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \frac{1}{\eta_l(t) + n} \left(\frac{a^2 x^2}{4}\right)^{\eta_l(t)+n}. \end{aligned} \quad (6.7)$$

We can show that each integrand in eq. (6.6) has no pole on the positive real axis on the complex t -plane and hence the Borel resummation gives a finite result with no ambiguity. Therefore, it would be possible to obtain a closed form for the two point function with $a = 0$ by an analytic continuation.

Next, let us consider what becomes of each term in the transseries when the IR cutoff a becomes smaller than Λ . To obtain the correct series in each non-perturbative sector for $a < \Lambda$, the Borel resummation for the a -dependent term must be performed along the negative real axis of the t -plane. In other words, the integral must be modified as

$$\langle \delta D(x) \delta D(0) \rangle_a = \frac{\Lambda^4}{2} \sum_{l=0}^{\infty} \int_{-\infty}^{\infty} dt \left(\frac{\Lambda^2 x^2}{4}\right)^{-\eta_l(t)} \left[\frac{\Gamma(\eta_l(t))}{\Gamma(1 - \eta_l(t))} \theta(t) + \mathcal{F}_l(ax, t) \theta(-t) \right] P_l(t),$$

where $\theta(t)$ is the step function. Since each integrand has singularities at points such that $\eta_l(t) = -n$ ($n = 0, 1, 2, \dots$), i.e. $t = 2(2 - l + n)$, we need to regularize the integral. Although we can regularize the integral by shifting the integration contour as $\text{Im } t = \pm\epsilon$, the result depends on the sign of t . Each singularity gives rise to the ambiguity

$$\text{Im} \langle \delta D(x) \delta D(0) \rangle_a = \pm \pi \Lambda^4 \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} A_{l,n} \left(\frac{\Lambda^2 x^2}{4}\right)^n, \quad (6.8)$$

where the coefficients $A_{l,n}$ are given by

$$A_{l,n} = (-1)^{l+n} \frac{1}{(n!)^2} \left[\binom{2n+4}{l} - 4 \binom{2n+2}{l-1} \right], \quad (6.9)$$

where $\binom{p}{q} = \frac{\Gamma(p+1)}{\Gamma(q+1)\Gamma(p-q)}$ denotes the binomial coefficient. Summing over $n = 0, 1, 2, \dots$, we find that each term in the transseries has an ambiguity that is a non-trivial function of Λx

$$\text{Im} \langle \delta D(x) \delta D(0) \rangle_a|_{l=0} = \pm \pi \Lambda^4 J_0(\Lambda x), \quad \text{Im} \langle \delta D(x) \delta D(0) \rangle_a|_{l=1} = \pm \pi \Lambda^5 x J_1(\Lambda x), \quad \dots,$$

where $J_l(\Lambda x)$ are the Bessel functions. For higher l , the ambiguities can be determined as follows. Let G_l and H_l be functions of Λx defined as

$$G_l = \Lambda^4 \sum_{n=0}^{\infty} \frac{(-1)^{l+n}}{(n!)^2} \binom{2n+4}{l} \left(\frac{\Lambda^2 x^2}{4}\right)^n, \quad H_l = \Lambda^4 \sum_{n=0}^{\infty} \frac{(-1)^{l+n}}{(n!)^2} \binom{2n+2}{l-1} \left(\frac{\Lambda^2 x^2}{4}\right)^n. \quad (6.10)$$

We can show that these function satisfy the recursion relations

$$G_{l+1} = -\frac{1}{l+1} [\Lambda \partial_\Lambda - l] G_l, \quad H_{l+1} = -\frac{1}{l} [\Lambda \partial_\Lambda - (l+1)] H_l. \quad (6.11)$$

Starting with the initial terms $H_0 = 0$ and $G_0 = -H_1 = \Lambda^4 J_0(\Lambda x)$, we can determine G_l , H_l and the ambiguity of the two point funciton

$$\text{Im} \langle \delta D(x) \delta D(0) \rangle_a = \pm \pi \sum_{l=0}^{\infty} (G_l - 4H_l) = \pm \pi \sum_{l=0}^{\infty} \Lambda^{2l} P_l(\Lambda \partial_\Lambda) [\Lambda^{4-2l} J_0(\Lambda x)]. \quad (6.12)$$

On the other hand, summing over $l = 0, 1, 2, \dots$ and using the binomial theorem $\sum_q \binom{p}{q} z^q = (1+z)^p$, we can show that the total ambiguity cancel (see table 1)

$$\text{Im} \langle \delta D(x) \delta D(0) \rangle_a = 0. \quad (6.13)$$

As in the case of the condensate, the singularities on the negative real axis of the Borel plane ($t < 0$) are relevant to the cancellation of the imaginary ambiguities for $a < \Lambda$.

7 $\mathbb{C}P^{N-1}$ sigma model

It is straightforward to apply our computations above to the $\mathbb{C}P^{N-1}$ sigma model

$$\mathcal{L} = \frac{1}{g^2} \left[\sum_{a=1}^N |\mathcal{D}_i \phi^a|^2 + D (|\phi^a|^2 - 1) \right], \quad (7.1)$$

where D is a Lagrange multiplier, $\mathcal{D}_i \phi^a = (\partial_i + iA_i) \phi^a$ is the covariant derivative and A_i is an auxiliary U(1) gauge field. Here we compute the cancellation of the imaginary ambiguities following section 4.2.

Integrating out the complex scalar fields ϕ^a with the ansatz $A_\mu = 0$ and $D = \text{const.}$, we obtain the same effective potential as (3.3), whose minimum is given by

$$\langle D \rangle = \Lambda^2 = \mu^2 e^{-\frac{4\pi}{\lambda_\mu}}, \quad (7.2)$$

where $\lambda_\mu = g_\mu^2 N$ is the 't Hooft coupling renormalized at μ . Like the $O(N)$ sigma model, the theory is asymptotically free, and the mass gap at large N is identical to that in the $O(N)$ model in eq. (3.4).

In addition to the condensate of the auxiliary field $\langle \delta D^2 \rangle$, we can consider the condensate of field strength, which takes the form

$$\langle F_{\mu\nu}^2 \rangle = -\frac{8\pi}{N} \int \frac{d^2 p}{(2\pi)^2} p^2 \sqrt{\frac{p^2}{p^2 + 4\Lambda^2}} \frac{1}{s_p} + \mathcal{O}(N^{-2}), \quad (7.3)$$

with $s_p = 4 \operatorname{arcsinh}(p/2\Lambda)$. We can explicitly perform the integral to obtain the exact expression in the large- N limit

$$\left\langle F_{\mu\nu}^2 \right\rangle_{\tilde{a}} = \frac{2}{N} \Lambda^4 \left[4 \operatorname{Chin}\left(\frac{s_{\tilde{a}}}{2}\right) - \operatorname{Chin}(s_{\tilde{a}}) \right], \quad (7.4)$$

where \tilde{a} is the UV cutoff, and $\operatorname{Chin}(x)$ is the entire function defined by the integral in (3.13).

On the other hand, the transseries expression with an IR cutoff $a > \Lambda$ takes the form

$$\left\langle F_{\mu\nu}^2 \right\rangle_{\tilde{a},a} = -\frac{1}{2N} \sum_{l=0}^{\infty} \Lambda^{2l} \int_0^\infty dt \Lambda^t \left[\tilde{a}^{2\eta_l(t)} - a^{2\eta_l(t)} \right] \frac{\tilde{P}_l(t)}{\eta_l(t)}, \quad (7.5)$$

where $\tilde{P}_l(t)$ and $\eta_l(t)$ are given by

$$\tilde{P}_l(t) = \frac{(-1)^l}{\Gamma(l+1)} \frac{\Gamma(t+2l+1)}{\Gamma(t+l+1)}, \quad \eta_l(t) = 2-l-\frac{t}{2}. \quad (7.6)$$

For $a > \Lambda$, there is no singularity on the positive real axis on the Borel plane and hence the exact expression (7.4) can be obtained by an analytic continuation to $a \rightarrow 0$.

If we consider the continuation of the transseries to the region where $a < \Lambda$, the Borel resummation should be modified as

$$\left\langle F_{\mu\nu}^2 \right\rangle_{\tilde{a},a} = -\frac{1}{2N} \sum_{l=0}^{\infty} \Lambda^{2l} \int_{-\infty}^\infty dt \Lambda^t \left[\tilde{a}^{2\eta_l(t)} \theta(t) - a^{2\eta_l(t)} \theta(-t) \right] \frac{P_l(t)}{\eta_l(t)} \quad (7.7)$$

In this case, the terms with $l = 0, 1, 3, 4$ have imaginary ambiguities associated with the poles at

$$t = 4 - 2l \quad (l = 0, 1, 3, 4). \quad (7.8)$$

The term with $l = 2$ also has an ambiguity since it contains $\log \lambda_{\tilde{a}}$ and $\log \lambda_a$. Although each term has an imaginary ambiguities, $\left\langle F_{\mu\nu}^2 \right\rangle$ has no imaginary part due to the cancellation

$$\operatorname{Im} \left\langle F_{\mu\nu}^2 \right\rangle_{\tilde{a},a} = \frac{\pm \pi}{N} \left[\left(\tilde{a} e^{-\frac{2\pi}{\lambda_{\tilde{a}}}} \right)^4 - 4 \left(\tilde{a} e^{-\frac{2\pi}{\lambda_{\tilde{a}}}} \right)^2 \Lambda^2 + 6 \Lambda^4 - 4 \left(\tilde{a} e^{-\frac{2\pi}{\lambda_{\tilde{a}}}} \right)^{-2} \Lambda^6 + \left(\tilde{a} e^{-\frac{2\pi}{\lambda_{\tilde{a}}}} \right)^{-4} \Lambda^8 \right] = 0.$$

We next look at the compactified model on $\mathbb{R} \times S^1$ with the \mathbb{Z}_N symmetric twisted boundary conditions. We first take the circumference of the compactified dimension L small $L\Lambda \ll 1$ but fixed in the large N limit $NL\Lambda \gg 1$. This conventional large- N limit is different from the Abelian large N limit $NL\Lambda \ll 1$ where the monopole-instantons can be computed [91]. We impose the twisted boundary conditions on the field as

$$\phi^a(x_1 + nL, x_2) = e^{inLm^a} \phi^a(x_1, x_2), \quad \text{with } n \in \mathbb{R} \text{ and } m^a = 2\pi a/(NL), \quad (7.9)$$

where the coordinates of \mathbb{R} and S^1 are denoted by x_1 and x_2 , respectively. We set the periodic boundary conditions for the auxiliary field D and the gauge field. The effective action for the auxiliary field D is given as

$$V_{\text{eff}}(D) = \frac{1}{2L} \sum_{a=1}^N \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{dp_2}{2\pi} \log \left[(k_n^a)^2 + p_2^2 + D \right] - \frac{D}{2g^2}, \quad (7.10)$$

where the Matsubara frequency is given as $k_n^a = 2\pi n/L + m^a$. At large N , we obtain the same effective action as \mathbb{R}^2 :

$$\begin{aligned} V_{\text{eff}}(D) &= \frac{N}{2} \frac{1}{NL} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{dp_2}{2\pi} \log \left[\left(\frac{2\pi n}{NL} \right)^2 + p_2^2 + D \right] - \frac{D}{2g^2} \\ &\rightarrow \frac{N}{2} \int_{\mathbb{R}^2} \frac{d^2 p}{(2\pi)^2} \log(p^2 + D) - \frac{D}{2g^2}. \end{aligned} \quad (7.11)$$

This is a consequence of the volume independence at large N [65]. Therefore the gap equation is unchanged, and we obtain the same mass gap as before in eq. (3.4).

To compute the condensate, we need to write the momentum integral in (7.3) as

$$\langle F_{\mu\nu}^2 \rangle = -\frac{8\pi}{NL} \sum_{n \in \mathbb{Z}} \int_a^{\tilde{a}} \frac{dp_2}{2\pi} p^2 \sqrt{\frac{p^2}{p^2 + 4\Lambda^2}} \frac{1}{s_p} + \mathcal{O}(N^{-2}), \quad (7.12)$$

with $p = \sqrt{(2\pi n/L)^2 + p_2^2}$. We still need the momentum cutoff a due to the IR divergence in the semiclassical expansion. To compute the imaginary ambiguities for small L , we only need to look at the zero Matsubara mode, because the nonzero Matsubara mode acts as a large momentum cutoff for and eliminates the pole in the momentum integral. Following section 4.2, we can compute the imaginary ambiguities as

$$\begin{aligned} \text{Im} \langle \delta D^2 \rangle_{\tilde{a},a}^{\mathbb{R} \times S^1} &= \frac{\pm\pi}{L} \left[2 \left(\tilde{a} e^{-\frac{2\pi}{\lambda_{\tilde{a}}}} \right)^3 + 2 \left(\tilde{a} e^{-\frac{2\pi}{\lambda_{\tilde{a}}}} \right) \Lambda^2 - 2 \left(\tilde{a} e^{-\frac{2\pi}{\lambda_{\tilde{a}}}} \right)^{-1} \Lambda^4 - 2 \left(\tilde{a} e^{-\frac{2\pi}{\lambda_{\tilde{a}}}} \right)^{-3} \Lambda^6 \right] \theta(\Lambda - a), \end{aligned} \quad (7.13)$$

for the condensate in the $O(N)$ sigma model while

$$\begin{aligned} \text{Im} \langle F_{\mu\nu}^2 \rangle_{\tilde{a},a}^{\mathbb{R} \times S^1} &= \frac{\pm\pi}{NL} \left[2 \left(\tilde{a} e^{-\frac{2\pi}{\lambda_{\tilde{a}}}} \right)^3 - 6 \left(\tilde{a} e^{-\frac{2\pi}{\lambda_{\tilde{a}}}} \right) \Lambda^2 + 6 \left(\tilde{a} e^{-\frac{2\pi}{\lambda_{\tilde{a}}}} \right)^{-1} \Lambda^4 - 2 \left(\tilde{a} e^{-\frac{2\pi}{\lambda_{\tilde{a}}}} \right)^{-3} \Lambda^6 \right] \theta(\Lambda - a), \end{aligned} \quad (7.14)$$

for the condensate in $\mathbb{C}P^{N-1}$ model. They both vanish but have a different structure than the case of \mathbb{R}^2 . The first term in eq. (7.14) is computed in ref. [67] and they agree.

In the calculation above, we have assumed that the only zero mode is relevant to the imaginary ambiguity. However, we have to take into account the contributions of the higher Matsubara modes to see how the results in the compact and non-compact cases are related. For that purpose, it is convenient to consider the imaginary ambiguity of the correlation functions. For the two point function of the fluctuations of the auxiliary field, it is convenient to use the Poisson resummation formula

$$\sum_{n \in \mathbb{Z}} f(2\pi n/L) = \sum_{\nu \in \mathbb{Z}} \frac{1}{L} \int \frac{dp}{2\pi} e^{ip\nu L} f(p). \quad (7.15)$$

The two point function of the auxiliary field δD in the compactified case is given by

$$\langle \delta D(x) \delta D(0) \rangle_a = 8\pi \sum_{\nu \in \mathbb{Z}} \int_a^\infty \frac{d^2 p}{(2\pi)^2} e^{ip(x+\nu L)} \frac{\sqrt{p^2(p^2 + 4\Lambda^2)}}{s_p} + \mathcal{O}(N^{-1}), \quad (7.16)$$

where we have fixed the position of the first operator $\delta D(x)$ at $(x_1, x_2) = (x, 0)$ for simplicity. From the ambiguity of the two point function on \mathbb{R}^2 in (6.8), we obtain the ambiguity of the $\mathcal{O}(\Lambda^l)$ term for $a < \Lambda$ as

$$\text{Im}\langle\delta D(x)\delta D(0)\rangle_a|_l = \pm\pi\sum_{\nu\in\mathbb{Z}}\left[G_l(x+\nu L)-4H_l(x+\nu L)\right], \quad (7.17)$$

where the functions G_l and H_l are defined in (6.10). By using the Poisson resummation formula, the summation over the integer ν can be rewritten back into the Kaluza Klein momentum number n . For example, $G_0 = -H_1 = \Lambda^4 J_0(\Lambda x)$ can be rewritten as

$$\pi\sum_{\nu\in\mathbb{Z}}G_0(x+\nu L) = -\pi\sum_{\nu\in\mathbb{Z}}H_1(x+\nu L) = \frac{\Lambda^3}{R}\sum_{n\in\mathbb{Z}}\frac{e^{-i\frac{n}{R}x}}{\sqrt{1-\frac{n^2}{R^2\Lambda^2}}}\theta\left(\Lambda^2-\frac{n^2}{R^2}\right), \quad (7.18)$$

where $R = L/2\pi$ is the compactification radius. The higher order terms can also be determined by using the recursion relation (6.11) as

$$\text{Im}\langle\delta D(x)\delta D(0)\rangle_a|_l = \pm\pi\sum_{n\in\mathbb{Z}}\Lambda^{2l}P_l(\Lambda\partial_\Lambda)\left[\frac{\Lambda^{3-2l}}{R}\frac{e^{-i\frac{n}{R}x}}{\sqrt{1-\frac{n^2}{R^2\Lambda^2}}}\theta\left(\Lambda^2-\frac{n^2}{R^2}\right)\right], \quad (7.19)$$

where $P_l(t)$ is the polynomial given in eq. (6.5). The step function $\theta(\Lambda^2 - n^2/R^2)$ in the imaginary ambiguity (7.19) implies that Stokes phenomena occur every time one of Kaluza Klein masses (Matsubara frequencies) n/R becomes smaller than the scale Λ . In particular, the ambiguity of the perturbative part ($l=0$) changes from $\mathcal{O}(\Lambda^3/R)$ to $\mathcal{O}(\Lambda^4)$ due to the infinitely many Stokes phenomena which occur as the compactification radius R is varied from zero to infinity

$$\text{Im}\langle\delta D(x)\delta D(0)\rangle_a|_{l=0} = \pm\begin{cases} \Lambda^3/R & \text{for } R < \Lambda^{-1} \\ \Lambda^4 + \dots & \text{for } R \rightarrow \infty \end{cases}. \quad (7.20)$$

This explains the discrepancy between the imaginary ambiguities in the models on \mathbb{R}^2 and $\mathbb{R} \times S^1$ with the \mathbb{Z}_n twisted boundary condition [67].

8 Conclusions

We have studied the resurgence structure of the condensate and two-point function in the $O(N)$ sigma model at large N using the semi-classical expansion. We have deduced the semi-classical ansatz in eq. (4.9) from the exact solution at large N by using an expansion in powers of Λ^2/p^2 and a small-coupling λ_p expansion before performing the momentum integral. The expansion suffers from the renormalon and IR divergences, both of which are typical in the semiclassical expansion in QFT. In order to circumvent the IR problem at higher order, we have introduced the IR cutoff a in the momentum integral.

We have shown that the leading term of our semi-classical expansion agrees with the well-known perturbative result. It does not have IR divergences but gives rise to the renormalon ambiguity. We stress that we did not construct the semi-classical expansion

using nontrivial saddle-point configurations in the theories: as far as we know, there is no known saddles in the theory, and thus no one has computed beyond the leading perturbative result. Instead what our analysis may suggest is that if we could compute the semi-classical expansion in a systematic way, then we would have to compute up to order Λ^8 , rather than Λ^4 as previously thought, to see the cancellation of the renormalon ambiguity.

We have also examined the result in eq. (4.34) of the semiclassical ansatz comparing it to the exact solution. We find that the result of the semiclassical ansatz can be understood in terms of the transseries expansion of the exact result at large N . To understand the behavior as a function of the IR cutoff a , we have first taken the transseries expansion of the condensate in powers of Λ^2/a^2 and λ_a . Their coefficients $c_{(2l,n)}$ at order Λ^{2l}/a^{2l-4} and λ_a^n can be analytically continued in λ_a to the region $a < \Lambda$ where $\lambda_a < 0$, which reproduces the result of the semiclassical ansatz. In this way, the imaginary ambiguity at order Λ^8/a^4 can be understood as coming from the Borel resummation of the IR contribution. We have also found that the limit of $a \rightarrow 0$ of the result of the semiclassical ansatz can be taken if we first make an analytic continuation to $a \gg \Lambda$ and then sum over l (the Borel resummation) and over n of the transseries to obtain an analytic function, which leads to the correct $a \rightarrow 0$ limit. With this procedure we have been able to recover the exact result at large N .

We have computed the cancellation of the renormalon in the semiclassical expansion in other models, such as the $O(N)$ and $\mathbb{C}P^{N-1}$ models on $\mathbb{R} \times S^1$ at large N . They turn out to be more complicated as multiple nonperturbative sectors give rise to imaginary ambiguities. In particular, we found that there exist infinitely many Stokes phenomena which occur as the compactification radius R is varied from zero to infinity. It is important to further investigate these models in the future.

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A Generic structure of Borel transform for condensate

In this appendix, we derive the Borel transforms for the condensate (2.11)

$$\langle \mathcal{O}(0)^2 \rangle|_{a,\tilde{a}} = \int_{|a| < |p| < |\tilde{a}|} \frac{d^d p}{(2\pi)^d} \Delta(p) = \sum_{l=0}^{\infty} \int_{|a| < |p| < |\tilde{a}|} \frac{d^d p}{(2\pi)^d} p^{[\Delta]} \left(\frac{\Lambda}{p}\right)^{\beta_0 \sigma_l} f_l(\lambda'_p),$$

where $\lambda'_p = 2\pi/(\beta_0 \log \frac{p}{\Lambda})$. By the change of variable,

$$p = \Lambda \left(\frac{\mu}{\Lambda}\right)^v, \quad \left(v = \frac{\lambda'_\mu}{\lambda'_p} = \frac{\log p/\Lambda}{\log \mu/\Lambda}\right), \quad (\text{A.1})$$

the condensate (A.1) can be rewritten as

$$\langle \mathcal{O}(0)^2 \rangle|_{\tilde{a},a} = C \Lambda^{2[\mathcal{O}]} \sum_{l=0}^{\infty} \int_{v_a}^{v_{\tilde{a}}} dv \left(\frac{\Lambda}{\mu}\right)^{e_l v} f_l(\lambda'_\mu/v) \quad \text{with} \quad C = \frac{d \log \frac{\mu}{\Lambda}}{(4\pi)^{\frac{d}{2}} \Gamma(d/2 + 1)}, \quad (\text{A.2})$$

where

$$e_l = \beta_0 \sigma_l - 2[\mathcal{O}], \quad v_p = \frac{\log p/\Lambda}{\log \mu/\Lambda}. \quad (\text{A.3})$$

To rewrite (A.2) into a Borel resummed form, let us decompose the integral as

$$\int_{v_a}^{v_{\tilde{a}}} dv \left(\frac{\Lambda}{\mu}\right)^{e_l v} f_l(\lambda'_\mu/v) = \int_{v_a}^{\infty} dv \left(\frac{\Lambda}{\mu}\right)^{e_l v} f_l(\lambda'_\mu/v) - \int_{v_{\tilde{a}}}^{\infty} dv \left(\frac{\Lambda}{\mu}\right)^{e_l v} f_l(\lambda'_\mu/v). \quad (\text{A.4})$$

Then, by change of variables $v = t/e_l + v_a$ and $v = t/e_l + v_{\tilde{a}}$, we obtain

$$\Lambda^{2[\mathcal{O}]} \int_{v_a}^{v_{\tilde{a}}} dv \left(\frac{\Lambda}{\mu}\right)^{e_l v} f_l(\lambda'_\mu/v) = \mu^{2[\mathcal{O}]} \left(\frac{\Lambda}{\mu}\right)^{\beta_0 \sigma_l} \int_0^{\infty} dt \left(\frac{\Lambda}{\mu}\right)^t B_l(t), \quad (\text{A.5})$$

with

$$B_l(t) = \frac{1}{e_l} \left[\left(\frac{\mu}{a}\right)^{e_l} f_l\left(\frac{e_l \lambda'_\mu}{t + e_l v_a}\right) - \left(\frac{\mu}{\tilde{a}}\right)^{e_l} f_l\left(\frac{e_l \lambda'_\mu}{t + e_l v_{\tilde{a}}}\right) \right]. \quad (\text{A.6})$$

B Perturbative expansions

In this appendix, we explicitly calculate the expansion coefficients $c_{(2l,n)}$. Defining $c_{(2l,n)} = c_{(2l,n)}(\tilde{a}) - c_{(2l,n)}(a)$, we have

$$\begin{aligned} c_{(0,n)}(p) &= \int dp \frac{2p^3 t_p^n}{(4\pi)^{n+1}} \\ &= \frac{\tilde{a}^4}{(8\pi)^{n+1}} \Gamma(n+1, 2t_p) \end{aligned} \quad (\text{B.1})$$

$$\begin{aligned} c_{(2,n)}(p) &= \int dp \frac{4p^2 (t_p^n - nt_p^{n-1})}{(4\pi)^{n+1}} \\ &= \frac{2p^2 t_p^n}{(4\pi)^{n+1}} \end{aligned} \quad (\text{B.2})$$

$$\begin{aligned} c_{(4,n)}(p) &= \int dp \frac{2(-2t_p^n - nt_p^{n-1} + 2(n)_2 t_p^{n-2})}{(4\pi)^{n+1} p} \\ &= \frac{2t_p^{n+1}}{(4\pi)^{n+1} (n+1)} + \frac{t_p^n - 2nt_p^{n-1}}{(4\pi)^{n+1}} \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} c_{(6,n)}(p) &= \int dp \frac{4(6t_p^n + 5nt_p^{n-1} - 3(n)_2 t_p^{n-2} - 2(n)_3 t_p^{n-3})}{3(4\pi)^{n+1} p^3} \\ &= \frac{2(-6t_p^n + nt_p^{n-1} + 2(n)_2 t_p^{n-2})}{3(4\pi)^{n+1} p^2} \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned} c_{(8,n)}(p) &= \int dp \frac{-60t_p^n - 59nt_p^{n-1} + 11(n)_2 t_p^{n-2} + 20(n)_3 t_p^{n-3} + 4(n)_4 t_p^{n-4}}{3(4\pi)^{n+1} p^5} \\ &= \frac{(-1)^{n+1}}{\tilde{a}^4 (8\pi)^{n+1}} \Gamma(n+1, -2t_p) + \frac{33t_p^n + 13nt_p^{n-1} - 12(n)_2 t_p^{n-2} - 4(n)_3 t_p^{n-3}}{6(4\pi)^{n+1} p^4}, \end{aligned} \quad (\text{B.5})$$

where $t_p = \log(\tilde{a}^2/p^2)$ and $(n)_k = n(n-1)(n-2)\dots(n-k+1)$ is the falling factorial. To compute the integrals, we have used the property of the incomplete Gamma function

$$\Gamma(n+1, z) = n\Gamma(n, z) + z^n e^{-z}. \quad (\text{B.6})$$

To sum $C_{2l} = \sum_{n=0}^{\infty} \lambda^{n+1} c_{(2l,n)}$, we set $x = \lambda t_p / 4\pi$ and use

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\log(1-x) \quad \text{and} \quad \frac{d^k}{dx^k} x^n = (n)_k x^{n-k}. \quad (\text{B.7})$$

We find that they are equivalent to the ones computed in section 4.3, as they should, up to a constant for C_4 .

C Power series expansion in Λ^2

In this appendix, we show the transseries expansion of the condensate. We first expand the variable $u_{\tilde{a}}$ in powers of Λ^2/\tilde{a}^2 as

$$u_{\tilde{a}} = -4 \sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{l=1}^{\infty} \frac{(2l-3)!!}{l!2} \left(\frac{-2\Lambda^2}{\tilde{a}^2} \right)^l \right)^k. \quad (\text{C.1})$$

Summation over l is convergent when $2\Lambda/\tilde{a} < 1$, and the sum over k is also convergent when $\sqrt{1 + (4\Lambda^2/\tilde{a}^2)} < 2$.

We can expand the function $F(s_{\tilde{a}})$ around $F(8\pi/\lambda(\tilde{a}))$ in powers of $u_{\tilde{a}} = s_{\tilde{a}} - 8\pi/\lambda(\tilde{a})$ as

$$\begin{aligned} F(s_{\tilde{a}}) - F\left(\frac{8\pi}{\lambda_{\tilde{a}}}\right) &= \int_{\frac{8\pi}{\lambda_{\tilde{a}}}}^{\frac{8\pi}{\lambda_{\tilde{a}}} + u_{\tilde{a}}} ds \frac{e^s - 2 + e^{-s}}{s} \\ &= \sum_{m=1}^{\infty} \frac{(u_{\tilde{a}})^m}{m} \left[\frac{\tilde{a}^4}{\Lambda^4} \sum_{n=0}^{m-1} \frac{(-1)^n}{(m-n-1)!} \left(\frac{\lambda_{\tilde{a}}}{8\pi}\right)^n + 2 \left(-\frac{\lambda_{\tilde{a}}}{8\pi}\right)^m - \frac{\Lambda^4}{\tilde{a}^4} \sum_{n=0}^{m-1} \frac{(-1)^m}{(m-n-1)!} \left(\frac{\lambda_{\tilde{a}}}{8\pi}\right)^n \right]. \end{aligned} \quad (\text{C.2})$$

The radius of convergence is given by $|u_{\tilde{a}}| < 8\pi/\lambda_{\tilde{a}}$. Therefore we can obtain the transseries expansion of the condensate using the power series expansion (C.2) together with the Borel resummed transseries expansion of $F(8\pi/\lambda_{\tilde{a}})$ in (5.3).

To find the power series expansion in a/Λ of the function $F(s_a)$ for the IR contribution of the condensate, we first expand $F(s_a)$ in powers of s_a and then s_a in powers of a/Λ as

$$\begin{aligned} F(s_a) &= \int_0^{s_a} ds \left[\frac{e^s - 1}{s} + \frac{e^{-s} - 1}{s} \right] = \sum_{k=1}^{\infty} \frac{1}{(2k)!k} s_a^{2k} \\ &= \sum_{k=1}^{\infty} \frac{1}{(2k)!k} \left\{ \sum_{m=1}^{\infty} \frac{4(-1)^{m-1}}{m} \left(\frac{a}{2\Lambda} + \sum_{l=1}^{\infty} \frac{(-1)^{l-1}(2l-3)!!}{l!2^l} \left(\frac{a}{2\Lambda} \right)^{2l} \right)^m \right\}^k, \end{aligned} \quad (\text{C.3})$$

where the sum over m is convergent if $a < 2\Lambda$, the sum over l is convergent if $\sqrt{1+a^2/(2\Lambda)^2} + a/(2\Lambda) < 2$.

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