must have the same number of equations as unknowns (dependent variables) to stand any chance of being transformed into a system of Kovalevskaya form, and we restrict our attention here to such systems.

It turns out that it suffices to consider changes of variable of the simple form

$$t = \psi(x), \qquad y = (y^1, \ldots, y^{p-1}) = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^p), \quad (2.124)$$

in which ψ is a smooth, real-valued function with nonzero gradient, $\nabla\psi(x_0) \neq 0$ at the point x_0 under investigation, and i is chosen so that $\partial\psi(x_0)/\partial x^i \neq 0$ so the change of variables (2.124) is locally invertible. Note that the initial hyperplane $\{t = t_0\}$ in the (y, t) coordinate comes from the level set $S = \{x\colon \psi(x) = t_0\}$ in the original coordinates, so the Cauchy problem in the x-coordinates consists of prescribing initial data on the hypersurface S. Under the change of variables (2.124), there is a corresponding system

$$\tilde{\Delta}_\nu(y, t, u^{(n)}) = 0, \qquad \nu = 1, \ldots, q, \quad (2.125)$$

involving y, t and derivatives of u with respect to y and t up to order n obtained by re-expressing the x-derivatives of u in terms of the y and t derivatives. We can apply the Cauchy-Kovalevskaya theorem to the transformed system (2.125) *provided* we can solve it for the n-th order t-derivatives u^α_{nt} in terms of y, t and the remaining derivatives $\widetilde{u^{(n)}}$. By the implicit function theorem, this is possible in a neighbourhood of a point $(y_0, t_0, u_0^{(n)})$ provided the $q \times q$ matrix M with entries

$$\mathsf{M}_{\alpha\nu} = \partial\tilde{\Delta}_\nu(y_0, t_0, u_0^{(n)})/\partial u^\alpha_{nt}, \qquad \alpha, \nu = 1, \ldots, q,$$

is nonsingular: $\det \mathsf{M} \neq 0$.

Let us see what this matrix M looks like. If u^α_J is any n-th order x-derivative of u, then by the chain rule,

$$u^\alpha_J = \frac{\partial^n u^\alpha}{\partial x^{j_1} \cdots \partial x^{j_n}} = \frac{\partial\psi}{\partial x^{j_1}} \cdot \frac{\partial\psi}{\partial x^{j_2}} \cdot \ldots \cdot \frac{\partial\psi}{\partial x^{j_n}} \cdot \frac{\partial^n u^\alpha}{\partial t^n} + \cdots \equiv (\nabla\psi)_J u^\alpha_{nt} + \cdots,$$

where the omitted terms involve various n-th and lower order derivatives of u^α with respect to y and t, except the key derivative u^α_{nt}. Therefore, if we form the $q \times q$ matrix $\mathsf{M}(\omega) = \mathsf{M}_\Delta(\omega; x_0, u_0^{(n)})$ whose entries are the homogeneous polynomials

$$\mathsf{M}_{\alpha\nu}(\omega) = \sum_{\#J=n} \frac{\partial\Delta_\nu}{\partial u^\alpha_J}(x_0, u_0^{(n)}) \cdot \omega_J, \qquad \alpha, \nu = 1, \ldots, q, \quad (2.126)$$

of degree n depending on $\omega = (\omega_1, \ldots, \omega_p)$, with $\omega_J \equiv \omega_{j_1}\omega_{j_2} \cdots \omega_{j_n}$, then the above matrix is obtained by evaluating $\mathsf{M}(\omega)$ at $\omega = \nabla\psi(x_0)$.

Definition 2.75. Let Δ be an n-th order system of differential equations having the same number of equations as unknowns. Given a point $(x_0, u_0^{(n)}) \in \mathscr{S}_\Delta$, form the $q \times q$ matrix of polynomials (2.126). A nonzero p-tuple ω is said to define a *noncharacteristic direction* (respectively *characteristic direction*) to Δ at $(x_0, u_0^{(n)})$ if $\mathsf{M}(\omega)$ is nonsingular (respectively singular). A hypersurface $S = \{\psi(x) = c\}$, $\nabla\psi \neq 0$, is called *noncharacteristic* at $(x_0, u_0^{(n)})$ if $\omega = \nabla\psi(x_0)$ determines a noncharacteristic direction there.

In particular, if the highest order derivatives in the system $\Delta(x, u^{(n)}) = 0$ occur linearly with coefficients only depending on x, then the matrix $\mathsf{M}(\omega)$ determining the characteristic directions depends only on x_0, so we can omit reference to the particular solution $u_0^{(n)}$ and refer unambiguously to a characteristic or noncharacteristic direction at x_0 itself. This is the case occurring most frequently in physical systems.

Our earlier considerations show that we can apply the Cauchy-Kovalevskaya theorem to the Cauchy problem provided the initial data lies on a noncharacteristic hypersurface.

Theorem 2.76. *If $\Delta(x, u^{(n)}) = 0$ is an analytic system of differential equations and S is a noncharacteristic, analytic hypersurface for Δ at $(x_0, u_0^{(n)})$, then there exists a local analytic solution to the Cauchy problem*

$$\Delta(x, u^{(n)}) = 0,$$

$$\frac{\partial^k u}{\partial n^k} = h_k(x), \qquad x \in S, \quad k = 0, \ldots, n-1,$$

in a neighbourhood of x_0. Here the h_k are analytic functions on S, and $\partial/\partial n$ denotes the normal derivative for S.

Example 2.77. (a) In the case of the one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = H(x, t, u, u_x, u_t),$$

a direction $\omega = (\tau, \xi)$ is characteristic if and only if

$$\tau^2 - c^2\xi^2 = 0.$$

For c constant, we recover the familiar characteristic curves

$$\psi(x, t) = x \pm ct = k.$$

Any curve not tangent to these lines can be used for valid Cauchy data.

(b) The equations of linear isotropic elasticity are known as *Navier's equations.* In two dimensions they take the form

$$\begin{aligned} (2\mu + \lambda)u_{xx} + \mu u_{yy} + (\mu + \lambda)v_{xy} &= 0, \\ (\mu + \lambda)u_{xy} + \mu v_{xx} + (2\mu + \lambda)v_{yy} &= 0, \end{aligned} \tag{2.127}$$

where λ and μ are constants known as the *Lamé moduli.* The 2×2 matrix $\mathsf{M}(\xi, \eta) = \mathsf{M}(\omega)$ determining the characteristics has the form

$$\mathsf{M}(\xi, \eta) = \begin{pmatrix} (2\mu + \lambda)\xi^2 + \mu\eta^2 & (\mu + \lambda)\xi\eta \\ (\mu + \lambda)\xi\eta & \mu\xi^2 + (2\mu + \lambda)\eta^2 \end{pmatrix}.$$

Then $\omega = (\xi, \eta)$ is characteristic if and only if

$$\det \mathsf{M}(\xi, \eta) = (2\mu + \lambda)\mu(\xi^2 + \eta^2)^2 = 0.$$

Thus unless $\mu = 0$ or $2\mu + \lambda = 0$, in which case *every* direction is characteristic, there are no real characteristic directions to Navier's equations. Note that the case $\mu = 0, \lambda = 1$ yields the leading order terms in the not locally-solvable system (2.118), hence this latter system has every direction characteristic.

Normal Systems

Corollary 2.74 will provide an immediate solution to the local solvability problem for an analytic system provided we can find at least one noncharacteristic direction to the system at the point $(x_0, u_0^{(n)})$ of interest. As Example 2.77(b) makes clear, not every system of partial differential equations satisfies this basic requirement, so we need to distinguish those systems which do.

Definition 2.78. A system of q differential equations $\Delta(x, u^{(n)}) = 0$ in q dependent variables $u = (u^1, \ldots, u^q)$ is *normal* at the point $(x_0, u_0^{(n)}) \in \mathscr{S}_\Delta$ if there exists at least one noncharacteristic direction ω for Δ there. The system is *normal* if it is normal at each point of $\mathscr{S}_\Delta$.

Theorem 2.79. *A system of differential equations is normal at $(x_0, u_0^{(n)})$ if and only if there is a change of variables $(y, t) = \chi(x)$ transforming it into a system in Kovalevskaya form near $(y_0, t_0) = \chi(x_0)$.*

Corollary 2.80. *If a system of differential equations is both analytic and normal at $(x_0, u_0^{(n)})$ then it is locally solvable at $(x_0, u_0^{(n)})$.*

We just change variables and invoke Corollary 2.74 for the resulting Kovalevskaya system. Later we will see that Corollary 2.80 admits a converse!

Prolongation of Differential Equations

Definition 2.81. Let

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, l,$$

be an n-th order system of differential equations defined by the vanishing of a smooth function $\Delta\colon M^{(n)} \to \mathbb{R}^l$. The k-th *prolongation* of this system is the $(n + k)$-th order system of differential equations

$$\Delta^{(k)}(x, u^{(n+k)}) = 0$$

obtained by differentiating the equations in Δ in all possible ways up to order k. In other words, $\Delta^{(k)}$ consists of the $\binom{p+k-1}{k}\cdot l$ equations

$$D_J\Delta_\nu(x, u^{(n+k)}) = 0,$$

where $\nu = 1, \ldots, l$, and J runs over all multi-indices of orders $0 \leqslant \#J \leqslant k$.

For example, the first prolongation of the heat equation

$$u_t = u_{xx}$$

is the third order system

$$u_t = u_{xx}, \qquad u_{xt} = u_{xxx}, \qquad u_{tt} = u_{xxt}.$$

The second prolongation appends the additional fourth order equations

$$u_{xxt} = u_{xxxx}, \qquad u_{xtt} = u_{xxxt}, \qquad u_{ttt} = u_{xxtt},$$

and so on.

Proposition 2.82. *If $u = f(x)$ is a smooth solution of a system $\Delta(x, u^{(n)}) = 0$, then it is also a solution to every prolongation of the system $\Delta^{(k)}(x, u^{(n+k)}) = 0$, $k = 0, 1, 2, \ldots$.*

Definition 2.83. A system of differential equatons is called *totally nondegenerate* if it and all its prolongations are both of maximal rank and locally solvable.

As we will see in a moment, any analytic system in Kovalevskaya form, and hence any normal analytic system, is always totally nondegenerate. Surprisingly, in the case of analytic systems with the same number of equations as unknowns, these are the only totally nondegenerate systems; if an analytic system is not normal, some prolongation of it is either not of maximal rank or not locally solvable. The C^∞ case is more complicated, owing to the appearance of the Lewy phenomenon of nonexistence.

Theorem 2.84. *An analytic system of differential equations*

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, q,$$

involving the same number of equations as dependent variables $u^1, \ldots, u^q$, is totally nondegenerate if and only if it is normal.

Proof. The Cauchy-Kovalevskaya theorem immediately proves that any normal system is totally nondegenerate. Indeed, by choosing a noncharacteristic direction, we can assume that the system is in Kovalevskaya form

(2.121). The k-th prolongation of such a system takes the form

$$u^\alpha_{(n+l)t,J} = D_t^l D_J\{\Gamma_\alpha(y, t, \widetilde{u^{(n)}})\}, \tag{2.128}$$

where $J = (j_1, \ldots, j_i)$ runs over all multi-indices with $1 \leqslant j_\kappa \leqslant p-1$, $l + i \leqslant k$. Also D_J denotes the corresponding i-th order total derivative with respect to $y = (y^1, \ldots, y^{p-1})$, and $u^\alpha_{(n+l)t,J} = D_J[u^\alpha_{(n+l)t}]$. The right-hand side of (2.128) depends on derivatives $u^\beta_{mt,K}$ where $m < n + l$. We can therefore inductively solve for the derivatives $u^\beta_{mt,K}$, $m \geqslant n$, in terms of y, t and derivatives $u^\gamma_{jt,L}$ with $j < n$. Thus (2.128) is equivalent to a system of the form

$$u^\alpha_{(n+l)t,J} = \Gamma^{J,l}_\alpha(y, t, \widetilde{u^{(n+k)}}), \tag{2.129}$$

in which $l + \#J \leqslant k$ and $\widetilde{u^{(n+k)}}$ denotes all derivatives of u up to order $n + k$ except those involving n or more t-derivatives. The maximal rank condition for $\Delta^{(k)}$ follows easily since the submatrix of the full Jacobian matrix for (2.129), cf. Definition 2.30, corresponding to all the partial derivatives

$$\frac{\partial}{\partial u^\beta_{mt,K}}[u^\alpha_{(n+l)t,J} - \Gamma^{J,l}_\alpha], \qquad m \geqslant n,$$

is the identity matrix.

The local solvability of (2.129) follows from the Cauchy-Kovalevskaya theorem. We can specify the derivatives $\widetilde{u_0^{(n+k)}}$ at a point y_0, t_0 arbitrarily; the values of the remaining derivatives in $u_0^{(n+k)}$ will then be determined by the prolonged system itself. Let $h^\alpha_m(y)$, $m = 0, \ldots, n-1$, $\alpha = 1, \ldots, q$, be analytic functions taking the prescribed values

$$\partial_J h^\alpha_m(y_0) = u^\alpha_{mt,J0}, \qquad \#J \leqslant n + k - m,$$

at (y_0, t_0). Let $u = f(y, t)$ be the analytic solution to the resulting Cauchy problem given by the Cauchy–Kovalevskaya theorem. Then

$$\partial_J \partial_t^m f(y_0, t_0) = u^\alpha_{mt,J0}$$

for $\#J + m \leqslant n + k$: for $m < n$ this follows from the definition of h^α_m, while for $m \geqslant n$ this follows since both $\mathrm{pr}^{(n+k)} f(y_0, t_0)$ and $(y_0, t_0, u_0^{(n+k)})$ satisfy the k-th prolongation of Δ at this point. Thus $u = f(y, t)$ gives the solution to the local solvability problem for $\Delta^{(k)}$ at $(y_0, t_0, u_0^{(n+k)})$.

The proof of the converse in Theorem 2.84 rests on a beautiful result due to Finzi.

Lemma 2.85. *Suppose*

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, q,$$

is an n-th order system of differential equations. Then Δ has no noncharacteristic directions at $(x_0, u_0^{(n)})$ if and only if there exist homogeneous k-th order differential operators

$$\mathscr{D}_\nu = \sum_{\#J=k} P^J_\nu D_J, \qquad \nu = 1, \ldots, q,$$

not all zero at $(x_0, u_0^{(n)})$, such that at $(x_0, u_0^{(n)})$ the combination

$$\sum_{\nu=1}^{q} \mathscr{D}_\nu \Delta_\nu \equiv Q(x_0, u_0^{(n+k-1)}) \tag{2.130}$$

depends only on derivatives of u of order at most $n + k - 1$.

Moreover, if there are no noncharacteristic directions for Δ for all $(x, u^{(n)})$ in some relatively open subset $\mathscr{S}_\Delta \cap V$, V open in $M^{(n)}$, then the differential operators $\mathscr{D}_\nu$ depend smoothly on $(x, u^{(n)})$. In this case, (2.130) *holds for all $(x, u^{(n+k)}) \in M^{(n+k)}$ which project to $(x, u^{(n)}) = \pi_n^{n+k}(x, u^{(n+k)}) \in \mathscr{S}_\Delta \cap V$.*

The point of the lemma is the following. Ordinarily, if Δ is an n-th order system of differential equations and $\mathscr{D}_1, \ldots, \mathscr{D}_q$ are k-th order differential operators, one would expect the linear combination $\sum \mathscr{D}_\nu \Delta_\nu$ to depend on $(n + k)$-th order derivatives of the u's. However, in the case Δ has only characteristic directions, one can find certain nontrivial k-th order differential operators $\mathscr{D}_\nu$ such that the combination $\sum \mathscr{D}_\nu \Delta_\nu$ depends on *only* $(n + k - 1)$-st and lower order derivatives, and hence the condition $\sum \mathscr{D}_\nu \Delta_\nu = 0$, which must hold for all solutions, provides an additional integrability condition on $(n + k - 1)$-st order derivatives of the u's which is not directly deduced from the $(k - 1)$-st order prolongaton $\Delta^{(k-1)}$. Conversely, if a system has some nontrivial integrability conditions, Finzi's lemma implies that there cannot be any noncharacteristic directions for the system. We can now appreciate why the system (2.118) failed to have noncharacteristic directions: it is for the same reason that it is not locally solvable! The further ramifications of this result will be explored after we discuss the proof.

PROOF OF LEMMA 2.85. According to Definition 2.75, the system Δ has only characteristic directions at a point if and only if the associated $q \times q$ matrix $\mathsf{M}(\omega)$ of n-th degree polynomials in $\omega = (\omega_1, \ldots, \omega_p)$ is singular for all values of ω:

$$\det \mathsf{M}(\omega) \equiv 0, \qquad \omega \in \mathbb{R}^p.$$

A relatively easy result from linear algebra (see Exercise 2.32) says that this is true if and only if there exists a row vector $\sigma(\omega) = (\sigma^1(\omega), \ldots, \sigma^q(\omega)) \not\equiv 0$ of homogeneous polynomials in ω such that

$$\sigma(\omega) \cdot \mathsf{M}(\omega) \equiv 0 \tag{2.131}$$

for all ω. In our case, suppose

$$\sigma^\nu(\omega) = \sum_{\#J=k} P_\nu^J \omega_J.$$

Then the coefficients P_ν^J of the σ_ν will serve as the coefficients of the operators $\mathscr{D}_\nu$ in (2.130). Indeed, it can easily be seen that if $\#J = k$,

$$D_J[\Delta_\nu(x, u^{(n)})] = \sum_{\alpha=1}^{q} \sum_{\#K=n} \frac{\partial \Delta_\nu}{\partial u_K^\alpha} u_{J,K}^\alpha + \cdots,$$

where $u^\alpha_{J,K}$ denotes the $(n+k)$-th order derivative $D_J(u^\alpha_K)$, and the omitted terms all depend on derivatives of orders at most $n+k-1$. Thus

$$\sum_{\nu=1}^{q} \mathscr{D}_\nu \Delta_\nu = \sum_{\nu=1}^{q} \sum_{\alpha=1}^{q} \sum_{\#J=k} \sum_{\#K=n} P^J_\nu \frac{\partial \Delta_\nu}{\partial u^\alpha_K} u^\alpha_{J,K} + Q(x, u^{(n+k-1)}) \qquad (2.132)$$

for some well-defined Q depending on at most $(n+k-1)$-st order derivatives of u. On the other hand, the α-th entry of the product (2.131) of σ and M is, by (2.126),

$$\sum_{\nu=1}^{q} \sum_{\#J=k} \sum_{\#K=n} P^\nu_J \frac{\partial \Delta_\nu}{\partial u^\alpha_K} \omega_J \omega_K \equiv 0$$

at $(x_0, u_0^{(n)})$. Since $u^\alpha_{J,K}$ is also completely symmetric in the indices in J, K, we conclude that at $(x_0, u_0^{(n)})$, the leading summation in (2.132) vanishes, and hence (2.130) holds. The smooth dependence of the differential operators $\mathscr{D}_\nu$ on $(x, u^{(n)})$ if there are no non-characteristic directions in any open subset of $\mathscr{S}_\Delta$ follows because if $\mathsf{M}(\omega) = \mathsf{M}(\omega; x, u^{(n)})$ depends smoothly on the parameters $(x, u^{(n)})$, the polynomials $\sigma(\omega) = \sigma(\omega; x, u^{(n)})$ can also be chosen to depend smoothly on the same parameters.

To prove the converse, it suffices to note that (2.130) can never occur for a system in Kovalevskaya form. Indeed any combination $\sum \mathscr{D}_\nu \Delta_\nu$ with k-th order operators $\mathscr{D}_\nu$ not all zero will always depend on $(n+k)$-th order derivatives, namely the k-th order derivatives of the u^α_{nt}. Thus if $\Delta(x, u^{(n)}) = 0$ has a noncharacteristic direction at $(x_0, u_0^{(n)})$, we can choose coordinates so that the system is in Kovalevskaya form, and hence (2.130) does not hold. □

Suppose a system of differential equations Δ satisfies the hypotheses of Lemma 2.85, so it is not normal at the point $(x_0, u_0^{(n)})$. There are then integrability conditions of the form (2.130) in which some linear combination of equations in the k-th prolongation $\Delta^{(k)}$ depends on at most $(n+k-1)$-st order derivatives. At this stage, two distinct possibilities arise.

(a) The integrability condition $\sum \mathscr{D}_\nu \Delta_\nu = 0$ vanishes at $(x_0, u_0^{(n)})$ by virtue of the algebraic relations among the $(n+k-1)$-st and lower order derivatives already established by $\Delta^{(k-1)}$, or

(b) The integrability condition $\sum \mathscr{D}_\nu \Delta_\nu = 0$ is genuine, not being an algebraic consequence of $\Delta^{(k-1)}$, and introduces a further relation among $(n+k-1)$-st and lower order derivatives.

We formalize this dichotomy into a definition of under-determined and over-determined systems, respectively.

Definition 2.86. Let Δ be an n-th order system of differential equations. Let $(x_0, u_0^{(n)})$ be initial values satisfying the system.

(a) Δ is *over-determined* at $(x_0, u_0^{(n)})$ if for some $k \geqslant 0$ there exist homogeneous k-th order differential operators $\mathscr{D}_1, \ldots, \mathscr{D}_q$, not all zero, such that the linear combination $\sum \mathscr{D}_\nu \Delta_\nu = Q$ of equations in $\Delta^{(k)}$, at the point $(x_0, u_0^{(n)})$, depends only on derivatives of u of order at most $n + k - 1$, *and* the linear combination Q does not vanish as an algebraic consequence of $\Delta^{(k-1)}$.

(b) Δ is *under-determined* at $(x_0, u_0^{(n)})$ if (i) there exists at least one set of homogeneous k-th order operators $\mathscr{D}_1, \ldots, \mathscr{D}_q$, not all zero, with $\sum \mathscr{D}_\nu \Delta_\nu = Q$ depending on at most $(n + k - 1)$-st order derivatives at the point x_0, and (ii) whenever $\mathscr{D}_1, \ldots, \mathscr{D}_q$ satisfy the conditions in part (i), the resulting Q vanishes as an algebraic consequence of the previous prolongation $\Delta^{(k-1)}$.

More succinctly, an over-determined system is one in which there are nontrivial integrability conditions. In this case, some prolongation $\Delta^{(k-1)}$ is *not* locally solvable since we can find a point $(x_0, u_0^{(n+k-1)}) \in \mathscr{S}_{\Delta^{(k-1)}}$ which does not satisfy the new integrability condition introduced by $\Delta^{(k)}$. On the other hand, an under-determined system is one in which the equations in some prolongation $\Delta^{(k)}$ are algebraically dependent, so the maximal rank condition cannot hold. In either case, the system is not totally nondegenerate. The third type of system, the normal systems, are then in a very definite sense precisely determined, and hence (in the analytic case) are the only totally nondegenerate systems; all others are either under- or over-determined. This completes the proof of Theorem 2.84. □

Example 2.87. (a) Consider the second order system (2.118). As it stands the system is over-determined since

$$D_y(u_{xx} + v_{xy} + v_x) - D_x(u_{xy} + v_{yy} - u_x) = v_{xy} + u_{xx},$$

which depends on second order derivatives, but does not vanish as an algebraic consequence of (2.118). On the other hand, if we omit the lower order terms, the system

$$u_{xx} + v_{xy} = 0, \qquad u_{xy} + v_{yy} = 0,$$

which corresponds to Navier's equations (2.127) when $\mu = 0$, $\lambda = 1$, is under-determined, since the combination

$$D_y(u_{xx} + v_{xy}) - D_x(u_{xy} + v_{yy}) \equiv 0$$

vanishes identically. In this latter case the general solution

$$u(x, y) = \phi_y(x, y) + cx, \qquad v(x, y) = -\phi_x(x, y),$$

depends on an arbitrary function $\phi(x, y)$. As a matter of fact, this holds for every under-determined system Δ—there is at least one arbitrary function depending on *all* the independent variables in the form of the general solution. In this case the Cauchy problem does *not* uniquely determine the solution, whereas in the over-determined case there does not, in general, exist a solution to the Cauchy problem. Thus for analytic systems, the normal systems are again precisely determined, here from the viewpoint of the Cauchy problem; over- or under-determined systems are characterized by their lack of existence or uniqueness respectively.

Notes

The system of partial differential equations for the invariants of a local group of transformations pre-dates Lie's work, having arisen in the problem of Pfaff. Its integration was studied by Jacobi, Mayer, Darboux, Lie and, finally, Frobenius, [1], who proved the general result on the existence of functionally independent solutions. See Forsyth, [1; Vol. 1], or Carathéodory, [1], for a discussion of the classical approaches to this problem. The connection with the corresponding characteristic system of ordinary differential equations is also classical; Kamke, [1; vol. 2, § D4] gives a treatment closest in spirit to that given here, along with other methods of integration—see also Ince, [1; § 2.7].

The concepts of functional independence and dependence are classical, but, surprisingly, most standard proofs of the basic Theorem 2.16 are remarkably deficient, usually assuming that the rank of the differential $d\zeta$ is constant. A modern proof of this result, not requiring extra hypotheses, appears in Narasimhan, [1; Theorem 1.4.14]. An alternative proof can be based on a theorem of A. B. Brown, [1] (see also Milnor, [1; p. 11]) that states that the set $\{\zeta(x): x \in M, \text{rank } d\zeta|_x < k\}$ of *critical values* of a smooth map $\zeta: M \to \mathbb{R}^k$ contains no open subset of $\mathbb{R}^k$, together with a theorem of Whitney (see Kahn, [1; Theorem 1.5]) that states that any closed subset $K \subset \mathbb{R}^k$ can be given as the set of zeros, $K = \{z: F(z) = 0\}$, of some smooth function $F: \mathbb{R}^k \to \mathbb{R}$. To prove Theorem 2.16, then, assuming rank $d\zeta < k$ everywhere, we set $K = \zeta[\bar{U}]$, where $U \subset M$ is any open set with compact closure, and choose F as in Whitney's theorem. Brown's theorem says that F does not vanish on any open subset of $\mathbb{R}^k$, and hence satisfies the requirements of Definition 2.15 for functional dependence. (This direct proof does not, to my knowledge, appear in the literature!)

In the case of analytic systems, the maximal rank condition for Theorem 2.8 can be relaxed to hold only "almost everywhere" on the subvariety $\mathscr{S}_F$, allowing the possibility of singularities. This result, which is not hard to prove, does not, however, seem to generalize to the C^∞ case. A similar generalization for Theorems 2.31 and 2.72 can thus also be proved for analytic systems of differential equations, allowing some singularities in the subvariety $\mathscr{S}_\Delta$.

Lie originally formulated his theory of continuous groups expressly for the study of differential equations, but was well aware of the applicability of his powerful infinitesimal method to the study of invariants and algebraic equations. See Lie, [4], for the algebraic and geometric side of his work. Historical accounts of Lie's work and influence appear in Hawkins, [1], [2], [3], and Wussing, [1; § III.3]. Most of Lie's work on ordinary differential equations appears in his collected papers; the book [5] does not really do justice to the full extent of his discoveries. The key to Lie's approach to integrating higher order ordinary differential equations was his complete classification (up to change of variable) of all transformation groups on the complex plane $\mathbb{C}^2$. Using this he was able to exhaustively list all possible reductions in order for a single ordinary differential equation; see Lie, [3], for these results, along with many explicit examples of interest. Lie's results in [3] include the results of Section 2.5 on multi-parameter symmetry groups of higher order ordinary differential equations; all of the other treatments in the earlier literature, including Cohen, [1], Ince, [1; Chap. 3], Markus, [1], and Ovsiannikov, [3; § 8], only do the case of one- and two-parameter groups. See Krause and Michel, [1], for more details on the kinds of symmetry groups admitted by ordinary differential equations. Theorem 2.68 on solvable symmetry groups of first order systems of ordinary differential equatons, though, is due to Bianchi, [1; § 167]; see also Eisenhart, [2; § 36]. This result clearly includes the corresponding Theorem 2.64 on higher order equations, but I was unable to find an explicit statement of the latter result in the literature; see Bluman and Kumei, [2], for further developments.

Most of Lie's work on symmetry groups of partial differential equations was concerned with linear systems of first order equations, which, by the method of characteristics, are essentially equivalent to systems of ordinary differential equations. However, in [2] and [6], Lie did look into symmetries of higher order partial differential equations. In [2; Part 1], Lie computes the symmetry groups of a number of second order partial differential equations in two independent variables, including the heat equation whose symmetry group appears at the end of § 13. This group was recomputed by Appell, [1], and, in the higher dimensional case, Goff, [1]. Lie's work on higher order partial differential equations, however, was not developed at all by other researchers, one possible reason being that, in contrast to the case of ordinary differential equations, knowledge of the symmetry group of a system of partial differential equations did *not* aid one in determining the general solution to the system. (One intriguing possibility, though, is the "group splitting method" of Vessiot, [1]; see Ovsiannikov, [3, § 26], for a modern presentation.) The only other early work on symmetries of partial differential equations of which I am aware is the work of Bateman, [1], Cunningham, [1], and Carmichael, [1], on the symmetries of the wave equation and Maxwell's equations. Apart from this, and despite the availability of Noether's theorem after 1918, work on the theory and applications of symmetry groups of partial differential equations came to a complete standstill; it was not until the appearance of Birkhoff's book, [2], on hydrodynamics, that group methods in the study of the important partial differential equations of mathematical physics began to revive. Under the leadership of Ovsiannikov, [1], [2], in the late 1950's and 1960's, the Soviet school made great progress in the study of symmetry groups of many of these systems. Interest in the methods in the West grew through the works of Bluman and Cole, [1], [2], and the books of Ames, [1], resulting in a great surge of research activity in these areas