

of  $V_\alpha$ . In terms of this topology, the third requirement in the definition of a manifold is just a restatement of the Hausdorff separation axiom: If  $x \neq \tilde{x}$  are points in  $M$ , then there exist open sets  $U$  containing  $x$  and  $\tilde{U}$  containing  $\tilde{x}$  such that  $U \cap \tilde{U} = \emptyset$ . In Chapter 3, we will have occasion to drop this property and consider non-Hausdorff manifolds. Many of the results of the other chapters remain true in this more general context, but as this introduces some technical complications we will work exclusively with Hausdorff manifolds except in the relevant sections of Chapter 3.

The degree of differentiability of the overlap functions  $\chi_\beta \circ \chi_\alpha^{-1}$  determines the degree of smoothness of the manifold  $M$ . We will be primarily interested in *smooth manifolds*, in which the overlap functions are smooth, meaning  $C^\infty$ , diffeomorphisms on open subsets of  $\mathbb{R}^m$ . If we require the overlap functions  $\chi_\beta \circ \chi_\alpha^{-1}$  to be real analytic functions, then  $M$  is called an *analytic manifold*. Most classical examples of manifolds are in fact analytic. Alternatively, we can weaken the differentiability requirements and consider  $C^k$ -manifolds, in which the overlap functions are only required to have continuous derivatives up to order  $k$ . Many of our results hold under these weaker differentiability requirements, but to avoid keeping track of precisely how many continuous derivatives are needed at each stage, we simply stick to the case of smooth or, occasionally, analytic manifolds. The weakening of our differentiability hypotheses is left to the interested reader. We begin by illustrating the general definition of a manifold with a few elementary examples.

**Example 1.2.** The simplest  $m$ -dimensional manifold is just Euclidean space  $\mathbb{R}^m$  itself. There is a single coordinate chart  $U = \mathbb{R}^m$ , with local coordinate map given by the identity:  $\chi = 1: \mathbb{R}^m \rightarrow \mathbb{R}^m$ . More generally, any open subset  $U \subset \mathbb{R}^m$  is an  $m$ -dimensional manifold with a single coordinate chart given by  $U$  itself, with local coordinate map the identity again. Conversely, if  $M$  is any manifold with a single global coordinate chart  $\chi: M \rightarrow V \subset \mathbb{R}^m$ , we can identify  $M$  with its image  $V$ , an open subset of  $\mathbb{R}^m$ .

**Example 1.3.** The unit sphere

$$S^2 = \{(x, y, z): x^2 + y^2 + z^2 = 1\}$$

is a good example of a nontrivial two-dimensional manifold realized as a surface in  $\mathbb{R}^3$ . Let

$$U_1 = S^2 \setminus \{(0, 0, 1)\}, \quad U_2 = S^2 \setminus \{(0, 0, -1)\}$$

be the subsets obtained by deleting the north and south poles respectively. Let

$$\chi_\alpha: U_\alpha \rightarrow \mathbb{R}^2 \simeq \{(x, y, 0)\}, \quad \alpha = 1, 2,$$

be stereographic projections from the respective poles, so

$$\chi_1(x, y, z) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right), \quad \chi_2(x, y, z) = \left( \frac{x}{1+z}, \frac{y}{1+z} \right).$$

It can be easily checked that on the overlap  $U_1 \cap U_2$ ,

$$\chi_1 \circ \chi_2^{-1}: \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}^2 \setminus \{0\}$$

is a smooth diffeomorphism, given by the inversion

$$\chi_1 \circ \chi_2^{-1}(x, y) = \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).$$

The Hausdorff separation property follows easily from that of  $\mathbb{R}^3$ , so  $S^2$  is a smooth, indeed analytic, two-dimensional manifold. The unit sphere is a particular case of the general concept of a surface in  $\mathbb{R}^3$ , which historically provided the principal motivating example for the development of the general theory of manifolds.

**Example 1.4.** An easier example is the unit circle

$$S^1 = \{(x, y): x^2 + y^2 = 1\},$$

which is similarly seen to be a one-dimensional manifold with two coordinate charts. Alternatively, we can identify a point on  $S^1$  with its angular coordinate  $\theta$ , where  $(x, y) = (\cos \theta, \sin \theta)$ , with two angles being identified if they differ by an integral multiple of  $2\pi$ .

The Cartesian product

$$T^2 = S^1 \times S^1$$

of  $S^1$  with itself is a two-dimensional manifold called a *torus*, and can be thought of as the surface of an inner tube. (See Example 1.6.) The points on  $T^2$  are given by pairs  $(\theta, \rho)$  of angular coordinates, with two pairs being identified if they differ by integral multiples of  $2\pi$ . In other words,  $(\theta, \rho)$  and  $(\tilde{\theta}, \tilde{\rho})$  describe the same point on  $T^2$  if and only if

$$\theta - \tilde{\theta} = 2k\pi \quad \text{and} \quad \rho - \tilde{\rho} = 2l\pi,$$

for integers  $k, l$ . Thus  $T^2$  can be covered by three coordinates charts

$$U_1 = \{(\theta, \rho): 0 < \theta < 2\pi, 0 < \rho < 2\pi\},$$

$$U_2 = \{(\theta, \rho): \pi < \theta < 3\pi, \pi < \rho < 3\pi\},$$

$$U_3 = \{(\theta, \rho): \pi/2 < \theta < 5\pi/2, \pi/2 < \rho < 5\pi/2\}.$$

The first overlap function is

$$\chi_1 \circ \chi_2^{-1}(\theta, \rho) = \begin{cases} (\theta, \rho), & \pi < \theta < 2\pi, & \pi < \rho < 2\pi, \\ (\theta - 2\pi, \rho), & 2\pi < \theta < 3\pi, & \pi < \rho < 2\pi, \\ (\theta, \rho - 2\pi), & \pi < \theta < 2\pi, & 2\pi < \rho < 3\pi, \\ (\theta - 2\pi, \rho - 2\pi), & 2\pi < \theta < 3\pi, & 2\pi < \rho < 3\pi \end{cases}$$

on the intersection  $U_1 \cap U_2$ , which is the set of all  $(\theta, \rho)$  with neither  $\theta$  nor  $\rho$  being an integral multiple of  $\pi$ . More generally, an  $m$ -dimensional torus is given by the  $m$ -fold Cartesian product  $T^m = S^1 \times \cdots \times S^1$  of  $S^1$  with itself.

In general, if  $M$  and  $N$  are smooth manifolds of dimension  $m$  and  $n$  respectively, then their Cartesian product  $M \times N$  is easily seen to be a smooth  $(m + n)$ -dimensional manifold. If  $\chi_\alpha: U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^m$ , and  $\tilde{\chi}_\beta: \tilde{U}_\beta \rightarrow \tilde{V}_\beta \subset \mathbb{R}^n$  are coordinate charts on  $M$  and  $N$  respectively then their Cartesian products

$$\chi_\alpha \times \tilde{\chi}_\beta: U_\alpha \times \tilde{U}_\beta \rightarrow V_\alpha \times \tilde{V}_\beta \subset \mathbb{R}^m \times \mathbb{R}^n \simeq \mathbb{R}^{m+n}$$

provide coordinate charts on  $M \times N$ . The verification of the requirements of Definition 1.1 for  $M \times N$  are left to the reader.

## Change of Coordinates

Besides the basic coordinate charts  $\chi_\alpha: U_\alpha \rightarrow V_\alpha$  given in the definition of  $M$ , one can always adjoin many additional coordinate charts  $\chi: U \rightarrow V \subset \mathbb{R}^m$ , subject to the requirement that they be *compatible* with the given charts. This means that for each  $\alpha$ ,  $\chi \circ \chi_\alpha^{-1}$  is smooth on the intersection  $\chi_\alpha(U \cap U_\alpha)$ . Thus, restriction of a given set of local coordinates  $\chi_\alpha$  to a smaller chart  $\tilde{U}_\alpha \subset U_\alpha$  will also be a valid coordinate chart. An additional possibility is to compose a given local coordinate map  $\chi_\alpha: U_\alpha \rightarrow V_\alpha$  with any diffeomorphism  $\psi: V_\alpha \rightarrow \tilde{V}_\alpha$  of  $\mathbb{R}^m$ . Such a diffeomorphism is referred to as a *change of coordinates*. Since both  $\chi_\alpha$  and  $\psi \circ \chi_\alpha$  are equally valid local coordinates on the chart  $U_\alpha$ , any property of  $M$ , or object defined on  $M$ , must be independent of any particular choice of local coordinates. (Of course, the explicit formulae for the given object may change when going from one coordinate chart to another, but the intrinsic characterization of the object remains coordinate-free.) If we choose to define an object on a manifold using its formula in a given coordinate chart, we must then check that the definition is actually independent of the particular coordinates used. This will require an investigation into how the object behaves under changes of coordinates. Often, as computations are most easily done in local coordinates, the choice of a special coordinate chart in which the object of interest takes a particularly simple form will enable us to considerably simplify many of these computations. The use of this basic technique will become clearer as we continue.

Often one expands the collection of coordinate charts to include all those compatible with the defining charts. The resulting collection, called a *maximal collection* of charts or *atlas* on  $M$ , still satisfies the basic properties (a), (b), (c) of Definition 1.1 (but, of course, is no longer countable!). The easy details of proving that two charts, compatible with the defining charts, are mutually compatible, are left to the reader.

Usually, in talking about local coordinates on a manifold, we will dispense with explicit reference to the map  $\chi_\alpha$  defining the local coordinate chart, and speak as if the local coordinate expressions were identical with the corresponding points on the manifold itself. Thus, we will say “let  $x = (x^1, \dots, x^m)$  be local coordinates on  $M$ ”, which, more precisely, means that there is a local coordinate chart  $\chi_\alpha: U_\alpha \rightarrow V_\alpha$ , with  $U_\alpha \subset M$  open,  $V_\alpha \subset \mathbb{R}^m$  open, and such

that each  $p$  in  $U_\alpha$  has local coordinates  $x = \chi_\alpha(p)$ . Since  $\chi_\alpha$  is one-to-one, we can clearly identify  $p$  with its local coordinate expression  $x$ . By the compatibility condition, we know that  $y = (y^1, \dots, y^m)$  are also local coordinates if and only if on the overlap of the two coordinate charts there is a diffeomorphism  $y = \psi(x)$  defined on an open subset of  $\mathbb{R}^m$  relating the two coordinates. For example, in the case of the circle  $S^1$ , the angle  $-\pi < \theta = \arctan(y/x) < \pi$  is a local coordinate on  $S^1 \setminus \{(-1, 0)\}$ . The ratio  $\rho = y/x$  is a local coordinate on  $S^1 \cap \{x > 0\}$ . On the overlap, the change of coordinates is given by  $\rho = \tan \theta$ , which is a diffeomorphism from the interval  $(-\pi/2, \pi/2)$  to  $\mathbb{R}$ .

## Maps Between Manifolds

If  $M$  and  $N$  are smooth manifolds, a map  $F: M \rightarrow N$  is said to be *smooth* if its local coordinate expression is a smooth map in every coordinate chart. In other words, for every coordinate chart  $\chi_\alpha: U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^m$  on  $M$  and every chart  $\tilde{\chi}_\beta: \tilde{U}_\beta \rightarrow \tilde{V}_\beta \subset \mathbb{R}^n$  on  $N$ , the composite map

$$\tilde{\chi}_\beta \circ F \circ \chi_\alpha^{-1}: \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is a smooth map wherever it is defined (i.e. on the subset  $\chi_\alpha[U_\alpha \cap F^{-1}(\tilde{U}_\beta)]$ ). In other words, a smooth map is of the form  $y = F(x)$ , where  $F$  is a smooth function on the open subsets giving local coordinates  $x$  on  $M$  and  $y$  on  $N$ .

**Example 1.5.** An easy example is provided by the map  $f: \mathbb{R} \rightarrow S^1$ ,  $f(t) = (\cos t, \sin t)$ . In terms of the angular coordinate  $\theta$  on  $S^1$ ,  $f$  is a linear function:  $\theta = t \bmod 2\pi$ , and so is clearly smooth.

**Example 1.6.** For a less trivial example we show how the torus  $T^2$  can be mapped smoothly into  $\mathbb{R}^3$ . Define  $F: T^2 \rightarrow \mathbb{R}^3$  by

$$F(\theta, \rho) = ((\sqrt{2} + \cos \rho) \cos \theta, (\sqrt{2} + \cos \rho) \sin \theta, \sin \rho).$$

Then  $F$  is clearly smooth in  $\theta$  and  $\rho$ , and one-to-one. The image of  $F$  is the toroidal surface in  $\mathbb{R}^3$  given by the single equation

$$x^2 + y^2 + z^2 + 1 = 2\sqrt{2(x^2 + y^2)}.$$

Thus  $T^2$  can be realized as a surface in  $\mathbb{R}^3$ . The local coordinates  $(\theta, \rho)$  on  $T^2$  serve as a parameterization of the image in  $\mathbb{R}^3$ .

## The Maximal Rank Condition

**Definition 1.7.** Let  $F: M \rightarrow N$  be a smooth mapping from an  $m$ -dimensional manifold  $M$  to an  $n$ -dimensional manifold  $N$ . The *rank* of  $F$  at a point  $x \in M$  is the rank of the  $n \times m$  Jacobian matrix  $(\partial F^i / \partial x^j)$  at  $x$ , where  $y = F(x)$  is

expressed in any convenient local coordinates near  $x$ . The mapping  $F$  is of *maximal rank* on a subset  $S \subset M$  if for each  $x \in S$  the rank of  $F$  is as large as possible (i.e. the minimum of  $m$  and  $n$ ).

The reader can easily check that the definition of the rank of  $F$  at  $x$  does not depend on the particular local coordinates chosen on  $M$  or on  $N$ . For example, the rank of  $F(x, y) = xy$  on  $\mathbb{R}^2$  is 1 at all points except the origin  $(0, 0)$  since its Jacobian matrix  $(F_x, F_y) = (y, x)$  is nonzero except at  $x = y = 0$ . (Here and elsewhere, subscripts denote derivatives, so  $F_x = \partial F / \partial x$ , etc.)

**Theorem 1.8.** *Let  $F: M \rightarrow N$  be of maximal rank at  $x_0 \in M$ . Then there are local coordinates  $x = (x^1, \dots, x^m)$  near  $x_0$ , and  $y = (y^1, \dots, y^n)$  near  $y_0 = F(x_0)$  such that in these coordinates  $F$  has the simple form*

$$y = (x^1, \dots, x^m, 0, \dots, 0), \quad \text{if } n > m,$$

or

$$y = (x^1, \dots, x^n), \quad \text{if } n \leq m.$$

This theorem is an easy consequence of the implicit function theorem—see Boothby, [1; Theorem II.7.1] for the proof. It is the first illustration of our contention that one can significantly simplify objects (in this case functions) on manifolds through a judicious choice of local coordinates.

## Submanifolds

The previous examples of surfaces in  $\mathbb{R}^3$ —the sphere and the torus—are special cases of the general notion of a submanifold. Naïvely, given a smooth manifold  $M$ , a submanifold  $N \subset M$  should be a subset which is also a smooth manifold in its own right. However, this preliminary definition can be interpreted in several fundamentally different ways, so we need to be more careful. There are also several methods of describing submanifolds, either implicitly by the vanishing of some smooth functions, as was the case with the sphere, or parametrically by some local parametrization, as we did initially with the torus. Both methods are very useful; we begin though with the latter, which leads to a more general notion of submanifolds.

**Definition 1.9.** Let  $M$  be a smooth manifold. A *submanifold* of  $M$  is a subset  $N \subset M$ , together with a smooth, one-to-one map  $\phi: \tilde{N} \rightarrow N \subset M$  satisfying the maximal rank condition everywhere, where the *parameter space*  $\tilde{N}$  is some other manifold and  $N = \phi(\tilde{N})$  is the image of  $\phi$ . In particular, the dimension of  $N$  is the same as that of  $\tilde{N}$ , and does not exceed the dimension of  $M$ .

The map  $\phi$  is often called an *immersion*, and serves to define a parametrization of the submanifold  $N$ . Often such a submanifold is referred to as an *immersed submanifold*, to emphasize the difference between this definition and other notions of submanifold. In this book, the term “submanifold” without qualifications always refers to “immersed submanifold” as in the above definition. The maximal rank condition is needed to ensure that  $N$  does not have singularities. For instance, the function  $\phi(t) = (t^2, t^3)$  is a smooth map from  $\mathbb{R}$  to  $\mathbb{R}^2$ , but the image of  $\phi$  is the curve  $y^2 = x^3$ , which has a cusp at  $(0, 0)$ . The Jacobian matrix is  $\dot{\phi}(t) = (2t, 3t^2)$ , which is not of maximal rank at  $t = 0$ , indicating the appearance of a singularity in the image of  $\phi$ .

The following series of examples will indicate some of the possibilities for submanifolds which are allowed by this definition. As the reader will see, although the maximal rank condition does have the effect of eliminating singularities like cusps, general submanifolds can still exhibit rather bizarre properties.

**Example 1.10.** In all of these examples of submanifolds, the parameter space  $\tilde{N} = \mathbb{R}$  is the real line, with  $\phi: \mathbb{R} \rightarrow M$  parameterizing a one-dimensional submanifold  $N = \phi(\mathbb{R})$  of some manifold  $M$ .

(a) Let  $M = \mathbb{R}^3$ . Then

$$\phi(t) = (\cos t, \sin t, t)$$

defines a circular helix spiralling up the  $z$ -axis. Here  $\phi$  is clearly one-to-one, and  $\dot{\phi} = (-\sin t, \cos t, 1)$  never vanishes, so the maximal rank condition holds.

(b) Let  $M = \mathbb{R}^2$ , and

$$\phi(t) = ((1 + e^{-t}) \cos t, (1 + e^{-t}) \sin t).$$

Then as  $t \rightarrow \infty$ ,  $N$  spirals in to the unit circle  $x^2 + y^2 = 1$ . Similarly,  $\tilde{\phi}(t) = (e^{-t} \cos t, e^{-t} \sin t)$  defines a logarithmic spiral at the origin.

(c) Let  $M = \mathbb{R}^2$  again, and consider the map

$$\hat{\phi}(t) = (\sin t, 2 \sin(2t)).$$

Then  $\hat{\phi}$  parametrizes a figure eight, which is a curve with self-intersections; namely  $\hat{\phi}(t) = (0, 0)$  whenever  $t$  is an integral multiple of  $\pi$ . By slightly modifying this example, for instance

$$\phi(t) = (\sin(2 \arctan t), 2 \sin(4 \arctan t)),$$

we can arrange that the parametrization is one-to-one, with the curve passing through the origin just once. The maximal rank condition holds everywhere. The image of  $\phi$  is again the figure eight, so we have a parametrization of a submanifold with “apparent” self-intersections, even though the immersion  $\phi$  is one-to-one. Note that the same figure eight can be parametrized in a

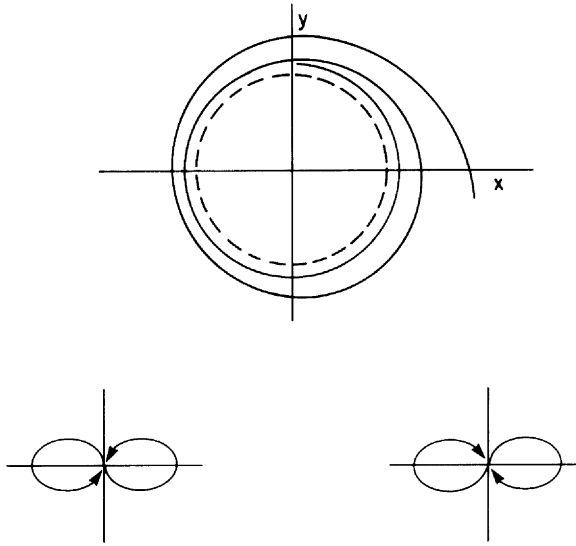


Figure 2. Examples of submanifolds.

different, inequivalent way:

$$\tilde{\phi}(t) = (-\sin(2 \arctan t), 2 \sin(4 \arctan t)).$$

The image of  $\tilde{\phi}$  is the same, but the composition  $\phi \circ \tilde{\phi}^{-1}: \mathbb{R} \rightarrow \mathbb{R}$  is *not* a continuous map!

This example shows that in general we must specify not only the subset  $N \subset M$ , but also the immersion  $\phi: \tilde{N} \rightarrow M$  in order to define a submanifold unambiguously.

(d) Let  $M = T^2$  be the two-dimensional torus with angular coordinates  $(\theta, \rho)$ . Let  $\phi: \mathbb{R} \rightarrow T^2$  be the curve  $\phi(t) = (t, \omega t)$ , where  $\omega$  is some fixed real number and the coordinates are taken modulo integral multiples of  $2\pi$  as before. Note that  $\dot{\phi} = (1, \omega)$ , so the maximal rank condition is satisfied. If  $\omega = p/q$  is a rational number, then  $\phi$  is not one-to-one; indeed  $\phi(t + 2\pi q) = \phi(t)$ , so the image of  $\phi$  is a closed curve on  $T^2$ . It can be realized as a one-dimensional manifold by using  $\tilde{N} = S^1$  as the parametrizing manifold, with  $\tilde{\phi}(\theta) = (q\theta, \omega q\theta)$ ,  $\theta \in S^1$  (provided  $p$  and  $q$  have no common factors). If  $\omega$  is irrational,  $\phi$  itself is one-to-one and the image curve  $N = \phi(\mathbb{R})$  can, without too much difficulty, be shown to be a dense submanifold of  $T^2$  whose closure is the entire torus. (See Boothby, [1; p. 86] for the details.) An analogous example can be constructed in  $\mathbb{R}^3$  by following  $\phi$  with the map  $F: T^2 \rightarrow \mathbb{R}^3$  given in Example 1.6. Thus for  $\omega$  irrational,

$$\phi(t) = ((\sqrt{2} + \cos \omega t) \cos t, (\sqrt{2} + \cos \omega t) \sin t, \sin \omega t)$$

parametrizes a one-dimensional submanifold of  $\mathbb{R}^3$  whose closure is the entire two-dimensional toroidal surface  $x^2 + y^2 + z^2 + 1 = 2\sqrt{2(x^2 + y^2)}$ .

## Regular Submanifolds

The latter two examples in 1.10 are perhaps more pathological than what one might wish to consider as submanifolds. Although, as we will see, there are good reasons for retaining Definition 1.9 as the general definition of a submanifold, it is also helpful to distinguish a class of examples, the regular or embedded submanifolds, which corresponds perhaps more accurately to one's intuitive notion of submanifold.

**Definition 1.11.** A *regular submanifold*  $N$  of a manifold  $M$  is a submanifold parametrized by  $\phi: \tilde{N} \rightarrow M$  with the property that for each  $x$  in  $N$  there exist arbitrarily small open neighbourhoods  $U$  of  $x$  in  $M$  such that  $\phi^{-1}[U \cap N]$  is a connected open subset of  $\tilde{N}$ .

In Example 1.10, (a) and (b) are both regular submanifolds, whereas (c) and (d) (for irrational  $\omega$ ) are not. In case (c), any neighbourhood  $U$  of  $(0, 0)$  will contain both the piece of the curve passing through  $(0, 0)$  together with the two “ends” of the curve coming back to the origin. In other words,  $\phi^{-1}[U]$  consists of at least three disjoint open intervals  $(-\infty, a)$ ,  $(b, c)$ ,  $(d, +\infty)$  with  $a < b < 0 < c < d$ . Similarly in case (d), if  $\omega$  is irrational and  $U$  is any open subset of  $T^2$ , then  $\phi^{-1}[U]$  consists of an infinite collection of disjoint open intervals.

As a consequence of the Implicit Function Theorem 1.8 we obtain a local-coordinate characterization of regularity.

**Lemma 1.12.** An  $n$ -dimensional submanifold  $N \subset M$  is regular if and only if for each  $x_0 \in N$  there exist local coordinates  $x = (x^1, \dots, x^m)$  defined on a neighbourhood  $U$  of  $x_0$  such that

$$N \cap U = \{x: x^{n+1} = \dots = x^m = 0\}.$$

Such a coordinate chart is called a *flat coordinate chart* on  $M$ . Note that, in view of this lemma, for regular submanifolds  $N \subset M$  we can dispense with the parametrizing manifold  $\tilde{N}$  and just treat  $N$  as a manifold in its own right. Namely the flat local coordinates  $x = (x^1, \dots, x^m)$  on  $U \subset M$  induce local coordinates, namely  $\tilde{x} = (x^1, \dots, x^n)$ , on  $U \cap N$ . The parametrization thereby is replaced by the natural inclusion  $N \subset M$ .

## Implicit Submanifolds

Instead of defining a surface  $S$  in  $\mathbb{R}^3$  parametrically, an alternative method is to define it *implicitly* by the vanishing of a smooth function:

$$S = \{F(x, y, z) = 0\}.$$

If we assume that the gradient  $\nabla F = (F_x, F_y, F_z)$  never vanishes on  $S$ , then by the implicit function theorem, at each point  $(x_0, y_0, z_0)$  in  $S$  we can solve for one of the variables  $x$ ,  $y$  or  $z$  in terms of the other two. Thus if  $F_z(x_0, y_0, z_0) \neq 0$ , there is a neighbourhood  $U_\alpha$  of  $(x_0, y_0, z_0)$  such that in  $U_\alpha$ ,  $S$  is given as the graph  $z = f(x, y)$  of some smooth function  $f$  defined on an open subset  $\tilde{V}_\alpha \subset \mathbb{R}^2$ . This permits us to define a local coordinate chart on  $S$  by projecting along the  $z$ -axis; in other words, set  $\tilde{U}_\alpha = S \cap U_\alpha$ , with  $\chi_\alpha: \tilde{U}_\alpha \rightarrow \tilde{V}_\alpha$ ,  $\chi_\alpha(x, y, z) = (x, y)$ . Similar constructions apply if  $F_y$  or  $F_x$  is non-zero. On the overlap  $\tilde{U}_\alpha \cap \tilde{U}_\beta$ , if  $\tilde{U}_\alpha$  is given by  $z = f(x, y)$ , so  $\chi_\alpha(x, y, z) = (x, y)$ , and  $\tilde{U}_\beta$  by  $y = h(x, z)$ , say, so  $\chi_\beta(x, y, z) = (x, z)$ , then

$$\chi_\beta \circ \chi_\alpha^{-1}(x, y) = \chi_\beta(x, y, f(x, y)) = (x, f(x, y)),$$

which is clearly smooth with smooth inverse  $\chi_\alpha \circ \chi_\beta^{-1}(x, z) = (x, h(x, z))$ . Thus  $S$  is a two-dimensional submanifold of  $\mathbb{R}^3$ . This motivates the general concept of an *implicitly defined submanifold*.

**Theorem 1.13.** *Let  $M$  be a smooth  $m$ -dimensional manifold, and  $F: M \rightarrow \mathbb{R}^n$ ,  $n \leq m$ , be a smooth map. If  $F$  is of maximal rank on the subset  $N = \{x: F(x) = 0\}$ , then  $N$  is a regular,  $(m - n)$ -dimensional submanifold of  $M$ .*

The proof of this theorem follows easily from the implicit function theorem using arguments similar to the above case of surfaces in  $\mathbb{R}^3$ . Indeed, Theorem 1.8 says that we can choose local coordinates  $x = (x^1, \dots, x^m)$  on  $M$  near each  $x_0 \in N$  such that  $F(x) = (x^1, \dots, x^n)$ . Thus, in terms of these coordinates,  $N = \{x^1 = \dots = x^n = 0\}$ , and so the  $x$ 's provide the flat local coordinates for  $N$  near  $x_0$ . Moreover, the latter  $m - n$  components  $(x^{n+1}, \dots, x^m)$  then provide local coordinates on  $N$  itself. In particular, this proves that  $N$  is a regular submanifold. Note especially that we do not require that the rank of  $F$  be maximal everywhere on  $M$ —this condition is only needed on the subset  $N$  where  $F$  vanishes. If, however,  $F$  is of maximal rank everywhere, then *every* level set of  $F$ ,  $\{x: F(x) = c\}$ , is a regular  $(m - n)$ -dimensional submanifold of  $M$ .

For example,  $F(x, y, z) = x^2 + y^2 + z^2 - 2\sqrt{2(x^2 + y^2)}$  is of maximal rank everywhere on  $\mathbb{R}^3$  except on the  $z$ -axis (where it is not even smooth) and the circle  $\{x^2 + y^2 = 2, z = 0\}$ . The level sets  $\{(x, y, z): F(x, y, z) = c\}$  are tori for  $-2 < c < \sqrt{2} - 2$ , and like spheres with indented dimples on the  $z$ -axis for  $c \geq \sqrt{2} - 2$ . For  $c = -2$ , the level set is the circle  $\{x^2 + y^2 = 2, z = 0\}$ , on which the gradient of  $F$  vanishes. This example shows the importance of both the differentiability and the maximal rank conditions for the validity of the theorem.

## Curves and Connectedness

A *curve*  $C$  on a smooth manifold  $M$  is parametrized by a smooth map  $\phi: I \rightarrow M$  where  $I$  is a subinterval of  $\mathbb{R}$ . In local coordinates,  $C$  is defined by  $m$  functions  $x = \phi(t) = (\phi^1(t), \dots, \phi^m(t))$ . Note that we are *not* requiring that

$\phi$  be one-to-one—so a curve can have self-intersections, or be of maximal rank—so a curve can have singularities like cusps. In consequence, curves are more general than one-dimensional submanifolds. A particularly degenerate curve occurs when  $\phi(t) \equiv x_0$  for all  $t$ , for some fixed  $x_0$ , so  $C$  consists of just one point. A *closed curve* is one whose endpoints coincide:  $\phi(a) = \phi(b)$ , with  $I = [a, b]$ , a closed interval.

A topological space is *connected* if it cannot be written as the disjoint union of two open sets. Since any manifold looks locally like Euclidean space, it is not difficult to prove that any connected manifold is *pathwise connected*, meaning that there is a smooth curve joining any pair of points. For our purposes, it will be very useful to impose, from the outset, the requirement that all manifolds under consideration are connected.

**Blanket Hypothesis.** *Unless explicitly stated otherwise, all manifolds (submanifolds, etc.) are assumed to be connected.*

This will avoid constantly restating the connectedness condition in the statement of our results.

A manifold  $M$  is *simply-connected* if every closed curve  $C \subset M$  can be continuously deformed to a point. This is equivalent to the existence of a continuous map

$$H: [0, 1] \times [0, 1] \rightarrow M$$

such that  $H(t, 0) = x_0$  for all  $0 \leq t \leq 1$ , while  $H(t, 1)$ ,  $0 \leq t \leq 1$  parametrizes  $C$ . For example,  $\mathbb{R}^m$  is simply-connected, while  $\mathbb{R}^2 \setminus \{0\}$  is not, as there is no way to continuously contract the unit circle to a point without passing through the origin. (On the other hand,  $\mathbb{R}^m \setminus \{0\}$  is simply connected for  $m \geq 3$ .) If  $M$  is any manifold, there exists a simply-connected *covering manifold*  $\tilde{M} \rightarrow M$ , where the covering map  $\pi$  is onto and a local diffeomorphism. For example, the simply-connected cover of the unit circle  $S^1$  is the real line  $\mathbb{R}$  with covering map  $\pi(t) = (\cos t, \sin t)$ ,  $t \in \mathbb{R}$ .

## 1.2. Lie Groups

At first sight, a Lie group appears to be a somewhat unnatural marriage between on the one hand the algebraic concept of a group, and on the other hand the differential-geometric notion of a manifold. However, as we shall soon see, this combination of algebra and calculus leads to powerful techniques for the study of symmetry which are not available for, say, finite groups.<sup>†</sup> We begin by recalling the definition of an abstract group.

<sup>†</sup> Witness, for instance, the recent complete classification of finite simple groups (Gorenstein, [1]); the corresponding problem for Lie groups was solved before the turn of the century.