for $\hat{\omega}, \hat{\omega}' \in \hat{\bigwedge}^k$, the space of vertical k-forms over M, $\hat{\eta} \in \hat{\bigwedge}^l$ and c, c' constants. Moreover, the proof of the Poincaré lemma immediately extends to this situation to prove exactness of the "vertical complex" over suitable subdomains $M \subset X \times U$.

Theorem 5.82. *Let $M \subset X \times U$ be vertically star-shaped. Then the vertical complex*

$$\hat{\bigwedge}^0 \xrightarrow{\hat{d}} \hat{\bigwedge}^1 \xrightarrow{\hat{d}} \hat{\bigwedge}^2 \xrightarrow{\hat{d}} \cdots$$

is exact. In other words, for $k > 0$ a vertical k-form $\hat{\omega}$ is closed: $\hat{d}\hat{\omega} = 0$, if and only if it is exact: $\hat{\omega} = \hat{d}\hat{\eta}$ for some $(k-1)$-form $\hat{\eta}$. For $k = 0$, a 0-form or differential function is $\hat{d}$-closed if and only if it is a function of x only.

Note that although any given vertical form depends on only finitely many variables, the entire vertical complex never terminates since we can keep bringing in higher and higher order derivatives of u to construct nonzero vertical k-forms for any $k \geqslant 0$.

The proof of Theorem 5.82 uses the same homotopy operator as was used in the ordinary Poincaré lemma, but adapted to the infinity of variables u_J^α. The basic scaling vector field is $\operatorname{pr} \mathbf{v}_u = \sum u_J^\alpha \partial/\partial u_J^\alpha$, the infinite prolongation of the evolutionary vector field $\mathbf{v}_u = \sum u^\alpha \partial/\partial u^\alpha$. There is a well-defined interior product between such vector fields and vertical forms, with $\{\partial/\partial u_J^\alpha\}$ and $\{du_J^\alpha\}$ being the dual bases of the relevant tangent and cotangent spaces. (Note that since vertical forms are required to be finite sums (5.108), we can allow infinite sums in our vector fields, since in computing $\operatorname{pr} \mathbf{v}_u \,\lrcorner\, \hat{\omega}$, say, only finitely many terms in the full prolongation of $\mathbf{v}_u$ are needed.) The formula for the homotopy operator corresponding to (1.69) is then

$$\hat{h}(\hat{\omega}) = \int_0^1 \{\operatorname{pr} \mathbf{v}_u \,\lrcorner\, \hat{\omega}[\lambda u]\} \frac{d\lambda}{\lambda}, \tag{5.110}$$

and we find for $\hat{\omega} \in \hat{\bigwedge}^k$, $k > 0$,

$$\hat{\omega} = \hat{d}\hat{h}(\hat{\omega}) + \hat{h}(\hat{d}\hat{\omega}).$$

In (5.110), $\hat{\omega}[u]$ indicates the dependence of $\hat{\omega}$ on u and all its derivatives, so to find $\hat{\omega}[\lambda u]$ we replace each u_J^α appearing in $\hat{\omega}$ (either explicitly or as a differential) by λu_J^α. Taking the interior product and then integrating out the λ's completes the determination of $\hat{h}(\hat{\omega})$. (In particular, there is no singularity in the integrand at $\lambda = 0$.)

Example 5.83. Let $p = q = 1$. If $\hat{\omega} = xu_x\, du \wedge du_x$, then $\hat{d}\hat{\omega} = x\, du_x \wedge du \wedge du_x = 0$, so $\hat{\omega}$ is closed. To find a one-form $\hat{\eta}$ such that $\hat{\omega} = \hat{d}\hat{\eta}$ we need only evaluate

$$\begin{aligned} \hat{\eta} = \hat{h}(\hat{\omega}) &= \int_0^1 \{\operatorname{pr} \mathbf{v}_u \,\lrcorner\, [x(\lambda u_x)\, d(\lambda u) \wedge d(\lambda u_x)]\} \frac{d\lambda}{\lambda} \\ &= \int_0^1 \lambda^2 \{xuu_x\, du_x - xu_x^2\, du\}\, d\lambda \\ &= \tfrac{1}{3}(xuu_x\, du_x - xu_x^2\, du), \end{aligned}$$

which is correct.

Each vertical k-form determines an alternating k-linear map from the space of *vertical vector fields* $\mathbf{v}^* = \sum Q_\alpha^J \partial/\partial u_J^\alpha$ to the space $\mathscr{A}$ of differential functions; in particular, it determines an alternating multi-linear map on the space T_0 of evolutionary vector fields. The precise formula is written using determinants, as in (1.49), so if $\hat{\omega}$ is given by (5.108), then

$$\langle \hat{\omega}; \operatorname{pr} \mathbf{v}_1, \ldots, \operatorname{pr} \mathbf{v}_k \rangle = \sum_{\boldsymbol{\alpha}, \mathbf{J}} P_{\mathbf{J}}^{\boldsymbol{\alpha}} \det(D_{J_i} Q_{\alpha_i}^j), \tag{5.111}$$

where $Q^j \in \mathscr{A}^q$ is the characteristic of $\mathbf{v}_j$ and the determinant is of the $k \times k$ matrix with the indicated (i, j) entry. For instance,

$$\begin{aligned}\langle x u_{xx}\, du \wedge du_x; \operatorname{pr} \mathbf{v}_Q, \operatorname{pr} \mathbf{v}_R \rangle &= x u_{xx} \det\begin{pmatrix} Q & R \\ D_x Q & D_x R \end{pmatrix} \\ &= x u_{xx}(Q D_x R - R D_x Q).\end{aligned}$$

Total Derivatives of Vertical Forms

For each $i = 1, \ldots, p$, the total derivative D_i can be thought of as a kind of vector field on the infinite jet space. As such, we can allow it to act on vertical forms as a "Lie derivative", which is determined by the following rules:

(a) *Linearity*:

$$D_i(c\hat{\omega} + c'\hat{\omega}') = c D_i \hat{\omega} + c' D_i \hat{\omega}', \qquad c, c' \in \mathbb{R}, \tag{5.112a}$$

(b) *Derivation*:

$$D_i(\hat{\omega} \wedge \hat{\eta}) = (D_i \hat{\omega}) \wedge \hat{\eta} + \hat{\omega} \wedge (D_i \hat{\eta}), \tag{5.112b}$$

(c) *Commutation with the Vertical Differential*:

$$D_i(\hat{d}\hat{\omega}) = \hat{d}(D_i \hat{\omega}), \tag{5.112c}$$

together with its well-established action on differential functions. (See (1.59), (1.60), (1.61).) In particular, D_i acts on the basic forms by $D_i\, du_J^\alpha = d(D_i u_J^\alpha) = du_{J,i}^\alpha$. The action is easy to reconstruct from these properties. For example,

$$\begin{aligned} D_x(x u_{xx}\, du \wedge du_x) &= D_x(x u_{xx})\, du \wedge du_x + x u_{xx} D_x(du) \wedge du_x \\ &\quad + x u_{xx}\, du \wedge D_x(du_x) \\ &= (x u_{xxx} + u_{xx})\, du \wedge du_x + x u_{xx}\, du \wedge du_{xx}, \end{aligned} \tag{5.113}$$

the middle term vanishing since $D_x(du) = du_x$. The proof that (5.112) determine a well-defined action of D_i is not difficult; in essence, it is a direct consequence of the same uniqueness property of the ordinary Lie derivative. One key property is that the total derivative is compatible with the evaluation of vertical forms on *evolutionary* vector fields:

$$D_i \langle \hat{\omega}; \operatorname{pr} \mathbf{v}_1, \ldots, \operatorname{pr} \mathbf{v}_k \rangle = \langle D_i \hat{\omega}; \operatorname{pr} \mathbf{v}_1, \ldots, \operatorname{pr} \mathbf{v}_k \rangle, \tag{5.114}$$

whenever $1 \leqslant i \leqslant p$, $\hat{\omega} \in \hat{\bigwedge}^k$, $\mathbf{v}_i = \mathbf{v}_{Q^i}$, $Q^i \in \mathscr{A}^q$. Thus, for example, the D_x derivative of $xu_{xx}(QD_xR - RD_xQ)$ agrees with the evaluation of the two-form (5.113) on $\mathrm{pr}\,\mathbf{v}_Q$ and $\mathrm{pr}\,\mathbf{v}_R$. The proof of (5.114) rests on the Lie derivative formulae in Exercise 1.35 together with the fact (5.19) that total derivatives commute with evolutionary vector fields.

Functionals and Functional Forms

Actually, what we are really interested in are the "functional versions" of our vertical forms, which are related to them just as functionals are related to differential functions. Although the basic notion of a functional appeared in its traditional guise in Chapter 4, subsequent developments necessitate a more algebraic approach to these fundamental objects of the calculus of variations. Each differential function $L \in \mathscr{A}$ determines a functional $\mathscr{L}[u] = \int_\Omega L[u]\,dx$ defined over any region $\Omega \subset X$ in its domain of definition. Provided we ignore boundary contributions (say by considering only functions $u = f(x)$ vanishing sufficiently rapidly near the boundary) a second function $\tilde{L} \in \mathscr{A}$ will determine the same functional, i.e. $\int_\Omega L[u]\,dx = \int_\Omega \tilde{L}[u]\,dx$ for all such u, if and only if it differs from L by a total divergence:

$$\tilde{L} = L + \mathrm{Div}\,P, \quad \text{for some} \quad P \in \mathscr{A}^p. \tag{5.115}$$

This is the essential content of Theorem 4.7 in the case that $P[u] = 0$ on $\partial\Omega$. Condition (5.115) does not any longer depend on the domain Ω, and determines an equivalence relation on the space of differential functions. Specifically, L and $\tilde{L}$ are *equivalent*, and determine the same functional, provided (5.115) holds. Each functional is thereby uniquely determined by an equivalence class of differential functions and conversely. It is reasonable, therefore, to *define* the space of *functionals*, denoted $\mathscr{F}$, as the set of equivalence classes of the space $\mathscr{A}$ of differential functions under the equivalence relation (5.115). Put another way, $\mathscr{F} = \mathscr{A}/\mathrm{Div}(\mathscr{A}^p)$ is the quotient vector space of $\mathscr{A}$ under the subspace of total divergences, i.e. the "cokernel" of the total divergence map $\mathrm{Div}\colon \mathscr{A}^p \to \mathscr{A}$. The natural projection from $\mathscr{A}$ to $\mathscr{F}$ which associates to each differential function L its equivalence class or functional will be denoted suggestively by an integral sign, so $\int L\,dx \in \mathscr{F}$ is the functional, or equivalence class, corresponding to $L \in \mathscr{A}$. In particular, $\int L\,dx = 0$ if and only if $L = \mathrm{Div}\,P$ for some P. This allows us the freedom of "integrating functionals by parts":

$$\int (P \cdot D_i Q)\,dx = -\int (Q \cdot D_i P)\,dx, \qquad P, Q \in \mathscr{A}.$$

(From our earlier standpoint, the image of the total divergence can be identified with the image of $\mathrm{D}\colon \bigwedge_{p-1} \to \bigwedge_p$, where $L[u]$ corresponds to the p-form $L[u]\,dx = L[u]\,dx^1 \wedge \cdots \wedge dx^p$. We can identify $\mathscr{F}$, the space of functionals, with the cokernel $\mathscr{F} \simeq \bigwedge_p / \mathrm{D} \bigwedge_{p-1}$, the projection of $\hat{\omega} = L\,dx$ being the functional $\int \hat{\omega} = \int L\,dx$. Indeed, if we were pursuing a truly coordinate-free presentation, we should be working with $\bigwedge_p$, the space of total p-forms, rather than $\mathscr{A}$, the space of differential functions. Note also that we could thus complete the D-complex by appending the trivially exact piece $\bigwedge_{p-1} \xrightarrow{\mathrm{D}} \bigwedge_p \to \mathscr{F} \to 0$, but this is not as interesting as the full variational complex.)

One important point is that whereas the space $\mathscr{A}$ of differential functions is an algebra, the same is no longer true of the space $\mathscr{F}$ of functionals, since we cannot multiply functionals together in any natural way. For example, the differential functions u_x and u_{xxx} both determine trivial functionals: $\int u_x \, dx = 0 = \int u_{xxx} \, dx$, but their product $u_x u_{xxx}$ is *not* a divergence, and hence $\mathscr{L} = \int u_x u_{xxx} \, dx \neq 0$ is not a trivial functional. Indeed, $\delta\mathscr{L} = -2u_{xxxx} \neq 0$, hence by Theorem 4.7, $\mathscr{L} \neq 0$. Of course we can still take constant coefficient linear combinations of functionals, so $\mathscr{F}$ is a vector space.

Similarly, we define an equivalence relation on the space $\hat{\bigwedge}^k$ of vertical k-forms, with $\hat{\omega}$ equivalent to $\hat{\omega}'$ if they differ by a total divergence

$$\hat{\omega} = \hat{\omega}' + \operatorname{Div} \hat{\eta} = \hat{\omega}' + \sum_{i=1}^{p} D_i \hat{\eta}_i, \qquad \hat{\eta}_i \in \hat{\bigwedge}^k,$$

where D_i acts on $\hat{\eta}_i$ according to (5.112). The space of equivalence classes is the space of *functional k-forms*, denoted

$$\bigwedge_*^k = \hat{\bigwedge}^k / \operatorname{Div}(\hat{\bigwedge}^k)^p.$$

The natural projection from $\hat{\bigwedge}^k$ to $\bigwedge_*^k$ is again denoted by an integral sign, so $\int \hat{\omega} \, dx$ stands for the equivalence class containing $\hat{\omega} \in \hat{\bigwedge}^k$. In particular, $\int \operatorname{Div} \hat{\eta} \, dx = 0$ for any p-tuple of vertical k-forms $\hat{\eta}$. Coupled with the derivational property of the total derivative (5.112b), this gives the integration by parts formula

$$\int \hat{\omega} \wedge D_i \hat{\eta} \, dx = -\int (D_i \hat{\omega}) \wedge \hat{\eta} \, dx, \qquad \hat{\omega} \in \hat{\bigwedge}^k, \quad \hat{\eta} \in \hat{\bigwedge}^l. \tag{5.116}$$

Example 5.84. Let $p = q = 1$ and consider the functional two-form

$$\omega = \int \{u_x \, du \wedge du_{xx}\} \, dx.$$

We can integrate this by parts using the fact that $du_{xx} = D_x(du_x)$, so by (5.116)

$$\begin{aligned} \omega &= -\int \{D_x(u_x \, du) \wedge du_x\} \, dx = -\int \{(u_{xx} \, du + u_x \, du_x) \wedge du_x\} \, dx \\ &= -\int \{u_{xx} \, du \wedge du_x\} \, dx. \end{aligned}$$

It doesn't help, though, to try a second integration by parts, since we get

$$\omega = +\int \{D_x(u_{xx} \, du) \wedge du\} \, dx = +\int \{u_{xx} \, du_x \wedge du\} \, dx,$$

which is exactly the same form as before.

Just as we are not allowed to multiply functionals, there is *no* well-defined wedge product between functional forms, since if

$$\hat{\omega} = \tilde{\omega} + \operatorname{Div} \eta \quad \text{and} \quad \hat{\theta} = \tilde{\theta} + \operatorname{Div} \zeta,$$

are equivalent forms, there is no guarantee that $\hat{\omega} \wedge \hat{\theta}$ and $\tilde{\omega} \wedge \tilde{\theta}$ are equivalent. In the above example, $du_{xx} = D_x(du_x)$ is trivial, but the functional two-form ω is not trivial. (See Proposition 5.88.)

Each functional form is an alternating multi-linear map from the space of evolutionary vector fields to the space of functionals, defined so that

$$\langle \omega; \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle = \int \langle \hat{\omega}; \operatorname{pr} \mathbf{v}_1, \ldots, \operatorname{pr} \mathbf{v}_k \rangle \, dx, \qquad \mathbf{v}_i \in T_0, \tag{5.117}$$

whenever $\omega = \int \hat{\omega} \, dx$, $\hat{\omega} \in \hat{\bigwedge}^k$. This is well defined by virtue of (5.114). For example, if $\omega = \int \{u_x \, du \wedge du_{xx}\} \, dx$ as above, then

$$\langle \omega; \mathbf{v}_Q, \mathbf{v}_R \rangle = \int u_x (Q D_x^2 R - R D_x^2 Q) \, dx.$$

What is slightly less obvious is that this action uniquely determines ω:

Lemma 5.85. *If ω and ω' are functional k-forms, then $\omega = \omega'$ if and only if $\langle \omega; \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle = \langle \omega'; \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle$ for every set of evolutionary vector fields $\mathbf{v}_1, \ldots, \mathbf{v}_k$.*

The proof rests on a more basic result:

Lemma 5.86. *Suppose $u \in \mathbb{R}^q$ and $v \in \mathbb{R}^r$ are both dependent variables depending on $x \in \mathbb{R}^p$. Suppose $\mathscr{L}[u, v] = \int L(x, u^{(n)}, v^{(n)}) \, dx$ is a functional with the property that $\mathscr{L}[u, Q[u]] = 0$ for all differential r-tuples $Q \in \mathscr{A}^r$ depending on x, u and derivatives of u. Then $\mathscr{L}[u, v] = 0$ as a functional in u and v.*

Proof. An equivalent way of stating this result is to say that if for every $Q \in \mathscr{A}^r$

$$L[u, Q[u]] = \operatorname{Div} P_Q[u]$$

for some $P_Q \in \mathscr{A}^p$ depending on x, u and derivatives of u, then

$$L[u, v] = \operatorname{Div} P^*[u, v]$$

for some p-tuple P^* depending on x, u, v and derivatives of u and v. In particular,

$$L[u, Q[u]] = \operatorname{Div} P^*[u, Q[u]],$$

where P^* depends on Q and its total derivatives alone. (The same is not necessarily true of P_Q, especially if it was constructed using the method of proof of Theorem 4.7.)

To prove this result, let $Q, R \in \mathscr{A}^r$. Then by the methods used to determine the variational derivative

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathscr{L}[u, Q + \varepsilon R] = \int \mathsf{E}_v(L)[u, Q] \cdot R \, dx,$$

where $\mathsf{E}_v(L)$ denotes the variational derivative of L with respect to v. By Corollary 5.68, $\mathsf{E}_v(L)[u, Q[u]] \equiv 0$ for all Q, hence $\mathsf{E}_v(L)[u, v] = 0$ for all u, v. Similarly, differentiating $\mathscr{L}[u + \varepsilon P[u], Q[u + \varepsilon P[u]]]$ with respect to ε at $\varepsilon = 0$ and using the vanishing of $\mathsf{E}_v(L)$, we find $\mathsf{E}_u(L) \equiv 0$. Theorem 4.7 immediately implies $L[u, v] = \operatorname{Div} P^*[u, v]$ for some P^*, proving the lemma. □

To prove Lemma 5.85, we need only show that

$$\mathscr{L}[u; Q^1, \dots, Q^k] \equiv \langle \omega; \mathbf{v}_1, \dots, \mathbf{v}_k \rangle = 0$$

for all $Q^\nu \in \mathscr{A}^q$, $\nu = 1, \dots, k$, if and only if $\hat{\omega} = \operatorname{Div} \hat{\eta}$ for some p-tuple of vertical forms $\hat{\eta}$. Lemma 5.86 implies that

$$\langle \hat{\omega}; \operatorname{pr} \mathbf{v}_1, \dots, \operatorname{pr} \mathbf{v}_k \rangle = \operatorname{Div} P^*[u; Q^1, \dots, Q^k],$$

where P^* depends on $Q^1, \dots, Q^k$ and their total derivatives only. As it stands, the components P_j^* of P^* can certainly be chosen to be linear in each Q^ν, but may not be alternating functions thereof. However, if we replace P^* by its "skew-symmetrization",

$$\hat{P}^*[u; Q^1, \dots, Q^k] = \frac{1}{k!} \sum_\pi (-1)^\pi P^*[u; Q^{\pi 1}, \dots, Q^{\pi k}],$$

the sum being over all permutations π of $\{1, \dots, k\}$, we maintain the condition

$$\langle \hat{\omega}; \operatorname{pr} \mathbf{v}_1, \dots, \operatorname{pr} \mathbf{v}_k \rangle = \operatorname{Div} \hat{P}^*[u; Q^1, \dots, Q^k].$$

Moreover, each component of $\hat{P}^*$ is an alternating, multi-linear function of the Q^ν's and their total derivatives, and hence can be identified with a vertical k-form

$$\hat{P}_j^*[u; Q^1, \dots, Q^k] = \langle \hat{\eta}_j; \operatorname{pr} \mathbf{v}_1, \dots, \operatorname{pr} \mathbf{v}_k \rangle.$$

Since this holds for all such $Q^1, \dots, Q^k$, we conclude that $\hat{\omega} = \operatorname{Div} \hat{\eta}$, and the lemma is proved. □

Let us look in more detail at the cases of functional one- and two-forms. Any one-form

$$\omega = \int \left\{ \sum_{\alpha, J} P_\alpha^J[u] \, du_J^\alpha \right\} dx$$

is determined by a finite collection of differential functions P_α^J, but the P_α^J are not uniquely determined by ω. Indeed, since $du_J^\alpha = D_J du^\alpha$, we can integrate each summand by parts, leading to the simpler expression

$$\omega = \int \left\{ \sum_{\alpha=1}^{q} P_\alpha[u]\, du^\alpha \right\} dx \equiv \int \{P \cdot du\}\, dx, \quad \text{where} \quad P_\alpha = \sum_J (-D)_J P_\alpha^J, \tag{5.118}$$

called the *canonical* form of ω. It is not hard to see that each functional one-form *does* have a uniquely determined canonical form.

Proposition 5.87. *Let $\omega = \int \{P \cdot du\}\, dx$ and $\tilde{\omega} = \int \{\tilde{P} \cdot du\}\, dx$ be functional one-forms in canonical form, so $P, \tilde{P} \in \mathscr{A}^q$. Then $\omega = \tilde{\omega}$ if and only if $P = \tilde{P}$.*

PROOF. It suffices to show that a functional one-form $\omega = 0$ if and only if the p-tuple P appearing in its canonical form vanishes identically. Evaluating (5.118) on an arbitrary vector field, we have

$$\langle \omega; \mathbf{v}_Q \rangle = \int (P \cdot Q)\, dx.$$

According to Lemma 5.85, $\omega = 0$ if and only if this vanishes for all such $\mathbf{v}_Q$, but by Corollary 5.68 this occurs if and only if $P = 0$, proving the result. □

Next consider the case of functional two-forms, the most general one of which is

$$\omega = \int \left\{ \sum_{\substack{\alpha,\beta \\ J,K}} P_{\alpha\beta}^{JK}[u]\, du_J^\alpha \wedge du_K^\beta \right\} dx,$$

the sum as usual being finite. To simplify the vertical two-form in the integrand, we rewrite $du_J^\alpha = D_J\, du^\alpha$ and integrate by parts. This leads to an expression of the form

$$\omega = \int \left\{ \sum_{\alpha,\beta,I} P_{\alpha\beta}^{I}[u]\, du^\alpha \wedge du_I^\beta \right\} dx,$$

where the $P_{\alpha\beta}^I$ are determined from the $P_{\alpha\beta}^{JK}$ and their derivatives. Define the differential operators

$$\tilde{\mathscr{D}}_{\alpha\beta} = \sum_I P_{\alpha\beta}^I[u] D_I,$$

whereby the above expression can be written as

$$\omega = \int \left\{ \sum_{\alpha,\beta=1}^{q} du^\alpha \wedge \tilde{\mathscr{D}}_{\alpha\beta}\, du^\beta \right\} dx, \tag{5.119}$$

or, using a more compact matrix notation,

$$\omega = \int \{du \wedge \tilde{\mathscr{D}}\, du\}\, dx.$$

As it stands, though, the matrix differential operator $\tilde{\mathscr{D}} = (\tilde{\mathscr{D}}_{\alpha\beta})$ is *not* uniquely determined by ω. Indeed, (5.119) can be integrated by parts, leading to an equivalent expression

$$\omega = \int \left\{ \sum_{\alpha,\beta} \tilde{\mathscr{D}}^*_{\alpha\beta}(du^\alpha) \wedge du^\beta \right\} dx = -\int \left\{ \sum_{\alpha,\beta} du^\beta \wedge \tilde{\mathscr{D}}^*_{\alpha\beta}(du^\alpha)\, dx \right\}$$

involving the adjoint $\tilde{\mathscr{D}}^* = (\tilde{\mathscr{D}}^*_{\beta\alpha})$ of $\tilde{\mathscr{D}}$. If we set $\mathscr{D} = \tilde{\mathscr{D}} - \tilde{\mathscr{D}}^*$, so $\mathscr{D}\colon \mathscr{A}^q \to \mathscr{A}^q$ is a skew-adjoint differential operator: $\mathscr{D}^* = -\mathscr{D}$, then ω has the *canonical form*

$$\omega = \tfrac{1}{2} \int \{du \wedge \mathscr{D}\, du\}\, dx, \qquad \mathscr{D}^* = -\mathscr{D}. \tag{5.120}$$

Its value on a pair of evolutionary vector fields is then

$$\langle \omega; \mathbf{v}_Q, \mathbf{v}_R \rangle = \tfrac{1}{2} \int \{Q \cdot \mathscr{D}R - R \cdot \mathscr{D}Q\}\, dx = \int \{Q \cdot \mathscr{D}R\}\, dx$$

since $\mathscr{D}$ is skew-adjoint. *This* canonical form is uniquely determined by ω.

Proposition 5.88. *Let* $\omega = \frac{1}{2}\int \{du \wedge \mathscr{D}(du)\}\, dx$, $\tilde{\omega} = \frac{1}{2}\int \{du \wedge \tilde{\mathscr{D}}(du)\}\, dx$ *be functional two-forms in canonical form, so* $\mathscr{D}$ *and* $\tilde{\mathscr{D}}$ *are skew-adjoint* $q \times q$ *matrix differential operators. Then* $\omega = \tilde{\omega}$ *if and only if* $\mathscr{D} = \tilde{\mathscr{D}}$.

Proof. By Lemma 5.85, it suffices to prove that if $\mathscr{D}\colon \mathscr{A}^q \to \mathscr{A}^q$ is skew-adjoint, then $\int (Q \cdot \mathscr{D}R)\, dx = 0$ for all q-tuples $Q, R \in \mathscr{A}^q$ if and only if $\mathscr{D} = 0$. Corollary 5.68 implies that $\mathscr{D}R = 0$ for all R, which implies that $\mathscr{D} = 0$. (See Exercise 5.39.) □

The Variational Differential

Definition 5.89. Let $\omega = \int \hat{\omega}\, dx$ be a functional k-form corresponding to the vertical k-form $\hat{\omega}$. The *variational differential* of ω is the functional $(k+1)$-form corresponding to the vertical differential of ω:

$$\delta\omega = \int (\hat{d}\hat{\omega})\, dx. \tag{5.121}$$

The commutativity relation (5.112c) assures us that this operator is well defined on the spaces of functional forms. The basic properties follow at once from those of the vertical differential, so we immediately have an exact *variational complex.*

Theorem 5.90. *Let* $M \subset X \times U$ *be vertically star-shaped. The variational differential determines an exact complex*

$$0 \to \Lambda^0_* \xrightarrow{\delta} \Lambda^1_* \xrightarrow{\delta} \Lambda^2_* \xrightarrow{\delta} \Lambda^3_* \xrightarrow{\delta} \cdots$$

on the spaces of functional forms on M. In other words, a functional form is closed: $\delta\omega = 0$, *if and only if it is exact:* $\omega = \delta\eta$.

PROOF. The homotopy formula (5.110) immediately projects to a homotopy formula for the variational differential: if ω is any functional k-form, $k > 0$, then

$$\omega = \delta h(\omega) + h(\delta\omega),$$

where, for $\omega = \int \hat{\omega}\, dx$,

$$h(\omega) = \int \hat{h}(\hat{\omega})\, dx = \int \left\{ \int_0^1 (\mathrm{pr}\, \mathbf{v}_u \lrcorner\, \hat{\omega}[\lambda u]) \frac{d\lambda}{\lambda} \right\} dx. \tag{5.122}$$

This also extends to the case when $k = 0$, i.e. ω is a functional, since $\hat{\omega}$ only differs from $\hat{d}\hat{h}(\hat{\omega}) + \hat{h}(\hat{d}\hat{\omega})$ by a function of x alone, and any such function determines a trivial functional. This suffices to prove Theorem 5.90 in all cases. □

Example 5.91. Consider the functional two-form

$$\omega = \int \{u_{xxx}\, du \wedge du_x\}\, dx.$$

(Note that ω is not quite in canonical form, which would be

$$\tfrac{1}{2} \int \{du \wedge (2u_{xxx}\, du_x + u_{xxxx}\, du)\}\, dx$$

corresponding to the skew-adjoint operator $2u_{xxx}D_x + u_{xxxx}$.) The variational derivative is the functional three-form

$$\delta\omega = \int \{du_{xxx} \wedge du \wedge du_x\}\, dx.$$

This form is trivial: integrating by parts, we see

$$\begin{aligned} \delta\omega &= -\int \{du_{xx} \wedge D_x(du \wedge du_x)\}\, dx \\ &= -\int \{du_{xx} \wedge du_x \wedge du_x + du_{xx} \wedge du \wedge du_{xx}\}\, dx = 0. \end{aligned}$$

Equivalently, $du_{xxx} \wedge du \wedge du_x = D_x(du_{xx} \wedge du \wedge du_x)$ is a total x-derivative. (Another way to see this is to note that the evaluation of the corresponding vertical three-form on a triple of evolutionary vector fields is an x-derivative:

$$\begin{aligned} \langle du \wedge du_x \wedge du_{xxx};\, \mathrm{pr}\, \mathbf{v}_P, \mathrm{pr}\, \mathbf{v}_Q, \mathrm{pr}\, \mathbf{v}_R \rangle &= \det \begin{bmatrix} P & Q & R \\ D_x P & D_x Q & D_x R \\ D_x^3 P & D_x^3 Q & D_x^3 R \end{bmatrix} \\ &= D_x \left\{ \det \begin{bmatrix} P & Q & R \\ D_x P & D_x Q & D_x R \\ D_x^2 P & D_x^2 Q & D_x^2 R \end{bmatrix} \right\}.) \end{aligned}$$

To compute a one-form η whose variational differential is ω, we use the homotopy formula (5.122),

$$\eta = h(\omega) = \int \left\{ \int_0^1 \lambda^2 (u u_{xxx}\, du_x - u_x u_{xxx}\, du)\, d\lambda \right\} dx$$
$$= \int \{\tfrac{1}{3} u u_{xxx}\, du_x - \tfrac{1}{3} u_x u_{xxx}\, du\}\, dx.$$

This has the canonical form

$$\eta = \int \{(-\tfrac{1}{3} u u_{xxxx} - \tfrac{2}{3} u_x u_{xxx})\, du\}\, dx,$$

and, indeed

$$\delta\eta = \int \{-\tfrac{1}{3} u\, du_{xxxx} \wedge du - \tfrac{2}{3} u_x\, du_{xxx} \wedge du - \tfrac{2}{3} u_{xxx}\, du_x \wedge du\}\, dx$$

can be shown to be equal to ω through a couple of integrations by parts.

The exactness of the variational complex at the $\bigwedge_*^1$-stage is of especial importance since it provides the afore-mentioned solution to the inverse problem of the calculus of variations. To see this, we need to first relate the variational differential to the variational derivative. If $\mathscr{L} = \int L\, dx$ is a functional, which we regard as an element of $\bigwedge_*^0$, then its variational differential is the functional one-form

$$\delta\mathscr{L} = \int \{\hat{d}L\}\, dx = \int \left\{ \sum_{\alpha=1}^{q} \sum_J \frac{\partial L}{\partial u_J^\alpha}\, du_J^\alpha \right\} dx.$$

As in (5.118), we can integrate this latter form by parts, leading to the canonical form

$$\delta\mathscr{L} = \int \left\{ \sum_{\alpha=1}^{q} \left(\sum_J (-D)_J \frac{\partial L}{\partial u_J^\alpha} \right) du^\alpha \right\} dx = \int \{\mathsf{E}(L) \cdot du\}\, dx,$$

cf. (4.3). Proposition 5.87 implies that $\delta\mathscr{L}$ can be uniquely identified with the Euler–Lagrange expression $\mathsf{E}(L)$, and this provides the connection between the variational differential and our previous notation for the variational derivative. (Indeed, if we interpret the differentials du^α as infinitesimal variations in the u^α, with corresponding variations $du_J^\alpha = D_J\, du^\alpha$ in the derivatives, then the above computation is the same as the traditional determination of the Euler–Lagrange equations from the definition of the variational derivative.) Exactness of the variational complex at the $\bigwedge_*^0$-stage, then, is equivalent to Theorem 4.7 that a functional is trivial if and only if its variational derivative vanishes identically.

We can thus "glue" the D-complex to the complex determined by the variational differential to obtain the full *variational complex*

$$0 \to \mathbb{R} \to \textstyle\bigwedge_0 \xrightarrow{\mathsf{D}} \bigwedge_1 \xrightarrow{\mathsf{D}} \cdots \xrightarrow{\mathsf{D}} \bigwedge_{p-1} \xrightarrow{\mathsf{D}} \bigwedge_p \xrightarrow{\mathsf{E}} \bigwedge_*^1 \xrightarrow{\delta} \bigwedge_*^2 \xrightarrow{\delta} \cdots,$$

which is exact over totally star-shaped domains $M \subset X \times U$.