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Introduction

When beginning students first encounter ordinary differential equations they are, more often than not, presented with a bewildering variety of special techniques designed to solve certain particular, seemingly unrelated types of equations, such as separable, homogeneous or exact equations. Indeed, this was the state of the art around the middle of the nineteenth century, when Sophus Lie made the profound and far-reaching discovery that these special methods were, in fact, all special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of symmetries. This observation at once unified and significantly extended the available integration techniques, and inspired Lie to devote the remainder of his mathematical career to the development and application of his monumental theory of continuous groups. These groups, now universally known as Lie groups, have had a profound impact on all areas of mathematics, both pure and applied, as well as physics, engineering and other mathematically-based sciences. The applications of Lie's continuous symmetry groups include such diverse fields as algebraic topology, differential geometry, invariant theory, bifurcation theory, special functions, numerical analysis, control theory, classical mechanics, quantum mechanics, relativity, continuum mechanics and so on. It is impossible to overestimate the importance of Lie's contribution to modern science and mathematics.

Nevertheless, anyone who is already familiar with one of these modern manifestations of Lie group theory is perhaps surprised to learn that its original inspirational source was the field of differential equations. One possible cause for the general lack of familiarity with this important aspect of Lie group theory is the fact that, as with many applied fields, the Lie groups that do arise as symmetry groups of genuine physical systems of differential equations are often not particularly elegant groups from a purely mathematical viewpoint, being neither semi-simple, nor solvable, nor any of the other special classes of Lie groups so popular in mathematics. Moreover, these groups often act nonlinearly on the underlying space (taking us outside the domain of representation theory) and can even be only locally defined, with the transformations making sense only for group elements sufficiently near the identity. The relevant group actions, then, are much closer in spirit to Lie's original formulation of the subject in terms of local Lie groups acting on open subsets of Euclidean space, and runs directly counter to the modern tendencies towards abstraction and globalization which have enveloped much of present-day Lie group theory. Historically, the applications of Lie groups to differential equations pioneered by Lie and Noether faded into obscurity just as the global, abstract reformulation of differential geometry and Lie group theory championed by E. Cartan gained its ascendency in the mathematical community. The entire subject lay dormant for nearly half a century until G. Birkhoff called attention to the unexploited applications of Lie groups to the differential equations of fluid mechanics. Subsequently, Ovsiannikov and his school began a systematic program of successfully applying these methods to a wide range of physically important problems. The last two decades have witnessed a veritable explosion of research activity in this field, both in the applications to concrete physical systems, as well as extensions of the scope and depth of the theory itself. Nevertheless, many questions remain unresolved, and the full range of applicability of Lie group methods to differential equations is yet to be determined.

Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. In the classical framework of Lie, these groups consist of geometric transformations on the space of independent and dependent variables for the system, and act on solutions by transforming their graphs. Typical examples are groups of translations and rotations, as well as groups of scaling symmetries, but these certainly do not exhaust the range of possibilities. The great advantage of looking at continuous symmetry groups, as opposed to discrete symmetries such as reflections, is that they can all be found using explicit computational methods. This is not to say that discrete groups are not important in the study of differential equations (see, for example, Hejhal, [1], and the references therein), but rather that one must employ quite different methods to find or utilize them. Lie's fundamental discovery was that the complicated nonlinear conditions of invariance of the system under the group transformations could, in the case of a continuous group, be replaced by equivalent, but far simpler, *linear* conditions reflecting a form of "infinitesimal" invariance of the system under the generators of the group. In almost every physically important system of differential equations, these infinitesimal symmetry conditions—the so-called defining equations of the symmetry group of the system—can be explicitly solved in closed form and thus the most general continuous symmetry group of the system can be explicitly determined. The entire procedure consists of rather mechanical computations, and, indeed, several symbolic manipulation computer programs have been developed for this task.

Once one has determined the symmetry group of a system of differential equations, a number of applications become available. To start with, one can directly use the defining property of such a group and construct new solutions to the system from known ones. The symmetry group thus provides a means of classifying different symmetry classes of solutions, where two solutions are deemed to be equivalent if one can be transformed into the other by some group element. Alternatively, one can use the symmetry groups to effect a classification of families of differential equations depending on arbitrary parameters or functions; often there are good physical or mathematical reasons for preferring those equations with as high a degree of symmetry as possible. Another approach is to determine which types of differential equations admit a prescribed group of symmetries; this problem is also answered by infinitesimal methods using the theory of differential invariants.

In the case of ordinary differential equations, invariance under a one-parameter symmetry group implies that we can reduce the order of the equation by one, recovering the solutions to the original equation from those of the reduced equation by a single quadrature. For a single first order equation, this method provides an explicit formula for the general solution. Multi-parameter symmetry groups engender further reductions in order, but, unless the group itself satisfies an additional "solvability" requirement, we may not be able to recover the solutions to the original equation from those of the reduced equation by quadratures alone. If the system of ordinary differential equations is derived from a variational principle, either as the Euler–Lagrange equations of some functional, or as a Hamiltonian system, then the power of the symmetry group reduction method is effectively doubled. A one-parameter group of "variational" symmetries allows one to reduce the order of the system by two; the case of multi-parameter symmetry groups is more delicate.

Unfortunately, for systems of partial differential equations, the symmetry group is usually of no help in determining the general solution (although in special cases it may indicate when the system can be transformed into a more easily soluble system such as a linear system). However, one can use general symmetry groups to explicitly determine special types of solutions which are themselves invariant under some subgroup of the full symmetry group of the system. These "group-invariant" solutions are found by solving a reduced system of differential equations involving fewer independent variables than the original system (which presumably makes it easier to solve). For example, the solutions to a partial differential equation in two independent variables which are invariant under a given one-parameter symmetry group are all found by solving a system of ordinary differential equations. Included among these general group-invariant solutions are the classical similarity solutions coming from groups of scaling symmetries, and travelling wave solutions reflecting some form of translational invariance in the system, as well as many other explicit solutions of direct physical or mathematical importance. For many nonlinear systems, these are the *only* explicit, exact solutions which are available, and, as such, play an important role in both the mathematical analysis and physical applications of the system.

In 1918, E. Noether proved two remarkable theorems relating symmetry groups of a variational integral to properties of its associated Euler–Lagrange equations. In the first of these theorems, Noether shows how each one-parameter variational symmetry group gives rise to a conservation law of the Euler–Lagrange equations. Thus, for example, conservation of energy comes from the invariance of the problem under a group of time translations, while conservation of linear and angular momenta reflect translational and rotational invariance of the system. Chapter 4 is devoted to the so-called classical form of Noether's theorem, in which only the geometrical types of symmetry groups are used. Noether herself proved a far more general result and gave a one-to-one correspondence between symmetry groups and conservation laws. The general result necessitates the introduction of "generalized symmetries" which are groups whose infinitesimal generators depend not only on the independent and dependent variables of the system, but also the derivatives of the dependent variables. The corresponding group transformations will no longer act geometrically on the space of independent and dependent variables, transforming a function's graph point-wise, but are nonlocal transformations found by integrating an evolutionary system of partial differential equations. Each one-parameter group of symmetries of a variational problem, either geometrical or generalized, will give rise to a conservation law, and, conversely, every conservation law arises in this manner. The simplest example of a conserved quantity coming from a true generalized symmetry is the Runge–Lenz vector for the Kepler problem, but additional recent applications, including soliton equations and elasticity, has sparked a renewed interest in the general version of Noether's theorem. In Section 5.3 we prove a strengthened form of Noether's theorem, stating that for "normal" systems there is in fact a one-to-one correspondence between *nontrivial* variational symmetry groups and *nontrivial* conservation laws. The condition of normality is satisfied by most physically important systems of differential equations; abnormal systems are essentially those with nontrivial integrability conditions. An important class of abnormal systems, which do arise in general relativity, are those whose variational integral admits an infinite-dimensional symmetry group depending on an arbitrary function. Noether's second theorem shows that there is then a nontrivial relation among the ensuing Euler–Lagrange equations, and, consequently, nontrivial symmetries giving rise to only trivial conservation laws. Once found, conservation laws have many important applications, both physical and mathematical, including existence results, shock waves, scattering theory, stability, relativity, fluid mechanics, elasticity and so on. See the notes on Chapter 4 for a more extensive list, including references.