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The Renormalisation Group Equation As An Equation For Lie Transport Of Amplitudes *

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ABSTRACT

It is shown that the renormalisation group (RG) equation can be viewed as an equation for Lie transport of physical amplitudes along the integral curves generated by the β -functions of a quantum field theory. The anomalous dimensions arise from Lie transport of basis vectors on the space of couplings. The RG equation can be interpreted as relating a particular diffeomorphism of flat space-(time), that of dilations, to a diffeomorphism on the space of couplings generated by the vector field associated with the β -functions.

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§1 Introduction

Recently there has been much interest in the possibility of putting differential geometrical structures such as a metric and connection on the space of interactions of a quantum field theory. In particular metrics on the space of interactions have been considered in [1] [2] and [3] and the question of connections has been addressed in [4]. A more primitive concept than covariant differentiation on a manifold is that of Lie differentiation and the purpose of this paper is to point out that away from a fixed point, when the β -functions are non-zero, the renormalisation group (RG) equation can be interpreted as an equation for Lie transport along the vector field on the space of couplings defined by the β -functions of the theory. That the RG equation may be understood without the necessity of introducing a connection on the space of couplings should not come as a surprise, since the usual derivations of this equation do not require introducing any extra structures, such as a connection, onto the space of couplings.

The idea presented in this paper is the following. A rescaling in Euclidean space, \mathbf{R}^D , is a diffeomorphism and is generated by the vector $\vec{\mathbf{D}} = x^\mu \partial_\mu$. If the space of couplings is also considered to be a finite dimensional, differentiable manifold, \mathcal{G} , then physical amplitudes can be interpreted as co-variant tensors on \mathcal{G} and one can view the rescaling of the couplings as a diffeomorphism of \mathcal{G} which is generated by the vector given by the β -functions of the theory. The renormalisation group equation then expresses the fact that the change in amplitudes under the diffeomorphism of \mathbf{R}^D generated by $\vec{\mathbf{D}}$, keeping the couplings fixed, is exactly the same as the change effected by a diffeomorphism of \mathcal{G} , generated by the vector $\vec{\beta}$, keeping the spatial points x_i fixed. Alternatively, if the RG equation is written as a differential equation involving $\kappa \frac{\partial}{\partial \kappa}|_g$ rather than $x^\mu \frac{\partial}{\partial x^\mu}|_g$, using standard naive scaling arguments, it becomes nothing more than the *definition* of a Lie derivative on \mathcal{G} with respect to the vector field $\vec{\beta}$. The terms in the RG equation involving anomalous dimensions are interpreted as coming from the change in the basis for co-vectors, dg^a , as we move along the RG trajectory.

Of course quantum field theory is famous for being plagued by “infinities” which, at least for renormalised theories, can be “tamed” by a regularisation procedure. This requires the introduction of “bare” couplings, $g_o^a(g, \epsilon)$, which are analytic functions of the renormalised couplings, g^a , and a regularisation parameter or parameters, ϵ . e.g. for a cut-off, Λ , $\epsilon = \kappa/\Lambda$ where κ is a renormalisation point and for dimensional regularisation $\epsilon = 4 - D$ where D is the dimension of space or space-time. The point of view adopted here is that g_o^a and g^a can be thought of as different co-ordinate systems on \mathcal{G} . The matrix $\frac{\partial g_o^a}{\partial g^b}$ tells us how to transform tensors (amplitudes) between co-ordinate systems. g_o^a enter on a different footing from g^a however in that they depend on the regularisation parameter whereas g^a do not. The “infinities” of quantum field theory are then viewed as being due to the fact that the co-ordinate transformation between g_o^a and g^a is singular when the regularisation parameter is removed, but this is not an insurmountable problem as long as theory is renormalisable. One must be very careful to distinguish between genuine singularities and singularities that are merely due to the choice of co-ordinates.

The basic idea presented here, that the RG equation should be viewed in terms of a co-ordinate transformation on the space of couplings, was inspired by O’Connor and

Stephens [3].

§2 The Renormalisation Group Equation As A Lie Derivative

It will now be demonstrated that the RG equation can be interpreted as an equation for Lie derivatives of amplitudes. The Lie derivative of an amplitude with respect to the dilation generator, \vec{D} , on Euclidean space is exactly the same as the Lie derivative with respect to the vector $\vec{\beta} = \beta^a \partial_a$ on the space of couplings.

Consider a field theory in flat D -dimensional Euclidean space. In principle there are an infinite number of operators that can be constructed out of the fields, each of which introduces a coupling, but the criterion of renormalisability requires that only a finite number of these couplings is independent, [5]. This means that, within the a priori infinite dimensional space of coupling constants, the theory can be formulated on a finite n -dimensional subspace which will be denoted by \mathcal{G} . It will be assumed that \mathcal{G} is, at least locally, a differentiable manifold and we will denote the renormalised couplings (coordinates on \mathcal{G}) by $g^a; a = 1, \dots, n$. Bare quantities will be represented by a subscript o . Thus the bare couplings are denoted by g_o^a . It will be assumed that a regularisation procedure is imposed which renders bare quantities very large but still finite. This is because bare quantities appear in some of the following formulae and we do not want to be manipulating infinite quantities. Ultimately, of course, all bare quantities disappear from physical amplitudes. The couplings g^a will be taken to be real and massless. If the theory contains any couplings which are massive these can always be made massless by multiplying by appropriate powers of the renormalisation mass scale, κ . Questions about the global structure of \mathcal{G} will not be addressed here.

Following ref. [2] we consider the operators

$$\hat{\Phi}_a(x) = \frac{\partial \hat{H}_o(x)}{\partial g^a}, \quad (1)$$

where $\hat{H}_o(x)$ is the bare Hamiltonian density, to be a basis for all relevant or marginal operators of the theory, i.e. any relevant or marginal operator, which is a scalar (or more precisely a density) in \mathbf{R}^D , can be written as a linear combination of $\hat{\Phi}_a(x)$.

Since $\hat{\Phi}_a$ are a basis for all relevant (and marginal) rotationally invariant operators it suffices to consider amplitudes of these operators. All other amplitudes for Euclidean scalars (more precisely densities) which are relevant or marginal operators can be obtained from linear combinations of these basic operators. Consider, therefore, amplitudes of the form

$$\hat{\Phi}_{a_1 \dots a_N}^{(N)}(x_1, \dots, x_N) = \langle \hat{\Phi}_{a_1}(x_1) \cdots \hat{\Phi}_{a_N}(x_N) \rangle. \quad (2)$$

Of course $\hat{\Phi}_a$ are, in general, composite operators and renormalisation of amplitudes involving multi-insertions of composite operators is a subtle problem involving the removal of new infinities which are not present in amplitudes involving only the elementary fields. A clear treatment of this problem is given in [6]. However, these problems can be avoided

here since we shall always assume that $x_i \neq x_j \forall i \neq j$ in amplitudes, so these extra infinities never arise.

The usual derivation of the renormalisation group equation relies on the simple fact that all physical amplitudes should be independent of the renormalisation point κ . However $\hat{\Phi}_{a_1 \dots a_N}^{(N)}(x_1, \dots x_N)$ cannot be independent of κ in general. If we chose a different parameterisation of the renormalised couplings (a different co-ordinate system on \mathcal{G}), which will be denoted by primes $g^{a'}(g)$, then $\hat{\Phi}_{a'_1 \dots a'_N}^{(N)}(x_1, \dots x_N)$ transforms as a rank N co-variant tensor,

$$\hat{\Phi}_{a_1 \dots a_N}^{(N)}(x_1, \dots x_N) = \left(\frac{\partial g^{a'_1}}{\partial g^{a_1}} \right) \dots \left(\frac{\partial g^{a'_N}}{\partial g^{a_N}} \right) \hat{\Phi}_{a'_1 \dots a'_N}^{(N)}(x_1, \dots x_N), \quad (3)$$

and these cannot both be independent of κ , since $\frac{\partial g^{a'_i}}{\partial g^{a_i}}$ is not, in general. The object that ought to be independent of κ is the reparameterisation invariant amplitude

$$\langle \hat{\Phi}(x_1) \dots \hat{\Phi}(x_N) \rangle = \langle \hat{\Phi}_{a_1}(x_1) \dots \hat{\Phi}_{a_N}(x_N) \rangle dg^{a_1} \dots dg^{a_N}. \quad (4)$$

The correct statement that amplitudes are independent of the renormalisation point is,

$$\kappa \frac{d}{d\kappa} \langle \hat{\Phi}(x_1) \dots \hat{\Phi}(x_N) \rangle = 0. \quad (5)$$

Allowing for the fact that $\langle \hat{\Phi}_{a_1}(x_1) \dots \hat{\Phi}_{a_N}(x_N) \rangle$ are also functions of the g^a gives

$$\begin{aligned} -\kappa \frac{\partial}{\partial \kappa} \Big|_g \langle \hat{\Phi}_{a_1}(x_1) \dots \hat{\Phi}_{a_N}(x_N) \rangle &= \\ \beta^b \partial_b \langle \hat{\Phi}_{a_1}(x_1) \dots \hat{\Phi}_{a_N}(x_N) \rangle + \sum_{i=1}^N (\partial_{a_i} \beta^b) \langle \hat{\Phi}_{a_1}(x_1) \dots \hat{\Phi}_{a_i}(x_i)_b \dots \hat{\Phi}_{a_N}(x_N) \rangle, \end{aligned} \quad (6)$$

where we have used

$$\kappa \frac{d}{d\kappa} (dg^a) = d \left(\kappa \frac{dg^a}{d\kappa} \right) = d\beta^a = \frac{\partial \beta^a}{\partial g^b} dg^b. \quad (7)$$

In co-ordinate free notation this is

$$-\kappa \frac{\partial}{\partial \kappa} \Big|_g \langle \hat{\Phi}(x_1) \dots \hat{\Phi}(x_N) \rangle = \mathcal{L}_{\vec{\beta}} \langle \hat{\Phi}(x_1) \dots \hat{\Phi}(x_N) \rangle, \quad (8)$$

where $\mathcal{L}_{\vec{\beta}}$ is the Lie derivative on \mathcal{G} with respect to the vector field, $\vec{\beta}$. Thus the renormalisation group equation is nothing other than the *definition* of a Lie derivative provided that it is appreciated that the amplitudes are tensors on $T^*(\mathcal{G})$, and not scalars (the minus sign in (8) is standard in the definition of a Lie derivative, see e.g. [7]). This analysis makes it clear that the anomalous dimensions have the geometrical interpretation of arising from the change in dg^a (which are a basis for real valued one-forms on $T^*(\mathcal{G})$) as we move along the vector field $\vec{\beta}$.

Equation (8) can be expressed in terms of the dilation vector on Euclidean space, $\vec{\mathbf{D}}$, using the usual scaling arguments. To see this first note that $\hat{\Phi}_a$ are densities (D -forms) in \mathbf{R}^D therefore, in Cartesian co-ordinates, the Lie derivative of $\hat{\Phi}_a(x)$ with respect to the vector $\vec{\mathbf{D}}$ is

$$\mathcal{L}_{\vec{\mathbf{D}}} \hat{\Phi}_{a_i}(x_i) = (di_{\vec{\mathbf{D}}} + i_{\vec{\mathbf{D}}} d) \hat{\Phi}_a(x) = di_{\vec{\mathbf{D}}} \hat{\Phi}_a(x) = \frac{\partial}{\partial x_i^\mu} \left(x_i^\mu \hat{\Phi}_{a_i}(x_i) \right) = \left(x_i^\mu \frac{\partial}{\partial x_i^\mu} + D \right) \hat{\Phi}_{a_i}(x_i), \quad (9)$$

where d and $i_{\vec{\mathbf{D}}}$ here represent exterior derivative and contraction on \mathbf{R}^D . Now by definition, equation (1), $\hat{\Phi}_a(x)$ has mass dimension D thus the usual naive scaling arguments give

$$\begin{aligned} \kappa \frac{\partial}{\partial \kappa} \Big|_g < \hat{\Phi}_{a_1}(x_1) \cdots \hat{\Phi}_{a_N}(x_N) > &= \sum_{i=1}^N \left(x_i^\mu \frac{\partial}{\partial x_i^\mu} + D \right) < \hat{\Phi}(x_1)_{a_1} \cdots \hat{\Phi}_{a_N}(x_N) > \\ &= \mathcal{L}_{\vec{\mathbf{D}}} < \hat{\Phi}_{a_1}(x_1) \cdots \hat{\Phi}_{a_N}(x_N) >, \end{aligned} \quad (10)$$

hence

$$\mathcal{L}_{\vec{\mathbf{D}}} < \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_N) > = -\mathcal{L}_{\vec{\beta}} < \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_N) >. \quad (11)$$

This is the geometrical form of the RG equation, in co-ordinate free notation. It is seen to relate a particular diffeomorphism (dilations generated by the conformal Killing vector, $\vec{\mathbf{D}}$) of Euclidean space, \mathbf{R}^D , to a diffeomorphism (generated by the associated vector $\vec{\beta}$) of the space of couplings \mathcal{G} .

Another way of expressing this idea is to define the matrix of renormalisation coefficients, Z_a^b , by

$$\hat{\Phi}_{oa} = Z_a^b(g) \hat{\Phi}_b, \quad (12)$$

where $\hat{\Phi}_{oa}$ constitute the bare basis

$$\hat{\Phi}_{oa} = \frac{\partial \hat{H}_o}{\partial g_o^a}. \quad (13)$$

The matrix Z_a^b depends on the renormalisation scheme, of course. For example in dimensional regularisation it is a function not only of the renormalised couplings, g^a , but also of the dimension $D = 4 - \epsilon$ and can be expanded as a series of poles in ϵ . Clearly the definition of the renormalised basis (1) also implies that

$$Z_a^b(g) = \frac{\partial g^b}{\partial g_o^a} \quad \Leftrightarrow \quad dg_o^a Z_a^b(g) = dg^b \quad (14)$$

so we can write

$$\hat{\Phi}(x) = \hat{\Phi}_{oa}(x) dg_o^a = \hat{\Phi}_a(x) dg^a. \quad (15)$$

We now demand that the bare operators be independent of the renormalisation point. There is a slight subtlety here, though, in that the bare couplings are defined to be massless.

This means that they change, under changes in κ , by their *canonical* dimensions. e.g. for a mass the massless bare coupling is $\tilde{m}_o^2 = m_o^2 \kappa^{-2}$ and $\kappa \frac{dm_o^2}{d\kappa} = 0$ requires $\kappa \frac{d\tilde{m}_o^2}{d\kappa} = -2\tilde{m}_o^2$. We shall denote the canonical dimension of the coupling g^a by d_a , thus $\kappa \frac{dg_o^a}{d\kappa} = -d_a g_o^a$ where there is no sum over a . As in the previous argument for amplitudes the operator-valued one form $\hat{\Phi}_o = \hat{\Phi}_{oa} dg_o^a$ is independent of κ . Thus

$$\kappa \frac{d\hat{\Phi}_{oa}}{d\kappa} - d_a \hat{\Phi}_{oa} = 0, \quad \Leftrightarrow \quad \kappa \frac{d\hat{\Phi}_a}{d\kappa} + \Gamma_a^b \hat{\Phi}_b = 0 \quad (16)$$

or

$$\left(\kappa \frac{\partial}{d\kappa} \Big|_g + \beta^b \partial_b \right) \hat{\Phi}_a + \Gamma_a^b \hat{\Phi}_b = 0, \quad (17)$$

where the matrix, Γ_a^b , is defined by

$$\Gamma_a^b = \frac{\partial \beta^b}{\partial g^a}. \quad (18)$$

This includes the matrix of anomalous dimensions,

$$\gamma^a_b = (Z^{-1})^a_c \kappa \left(\frac{dZ^c_b}{d\kappa} \right), \quad (19)$$

but they are not equal because the couplings are dimensionless. Equation (17) is a renormalisation group equation for the operators $\hat{\Phi}_a$. One must be careful in evaluating amplitudes involving this equation however since neither $\kappa \frac{\partial}{d\kappa} \Big|_g$ nor $\beta^b \partial_b$ can be pulled outside of expectation values separately, though the combination can be since the action $S_o = \int d^D x H_o(x)$, which appears in the functional integral representation of amplitudes, is independent of κ .

It is crucial to this interpretation that massless couplings are used. In particular this means that

$$\vec{\beta} = \beta^a \frac{\partial}{\partial g^a} = \beta_o^a \frac{\partial}{\partial g_o^a}, \quad (20)$$

where $\beta_o^a = \kappa \frac{dg_o^a}{d\kappa} = -d_a$ are just the canonical dimensions. Had massive couplings been used, then β_o^a would be zero since the bare massive couplings are defined to be independent of κ , but β^a are non-zero in general. One cannot transform from a non-zero vector to one which vanishes using a co-ordinate transformation. (This has nothing to do with the “singularities” of the renormalisation program, the regulator is still in place so the co-ordinate transformation is still non-singular.) The important quantity is the total dimension, canonical plus anomalous, and to split it up spoils general co-ordinate invariance. Stated differently, the redefinition $m^2 \rightarrow \tilde{m}^2 = m^2 \kappa^{-2}$ is *not* a co-ordinate transformation because κ is not a co-ordinate.

§3 Conclusions

The emphasis here is on a co-variant description of the RG equation and the notion that one is free to choose any set of renormalised co-ordinates that one wishes in a description of physical amplitudes. It has been argued that the renormalisation group equation can be given a geometrical interpretation in the sense that it may be viewed as an equation for Lie transport. In terms of Lie transport on the space of couplings it reduces to no more than the definition of a Lie derivative, but its real significance lies in the way that it ties a particular diffeomorphism of Euclidean space, that of dilations, to the diffeomorphism of the space of couplings generated by the vector field $\vec{\beta}$ through equation (11)

$$\mathcal{L}_{\vec{D}} \langle \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_N) \rangle = -\mathcal{L}_{\vec{\beta}} \langle \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_N) \rangle, \quad (11)$$

where $\mathcal{L}_{\vec{D}}$ is the Lie derivative on Euclidean space and $\mathcal{L}_{\vec{\beta}}$ the Lie derivative on the space of couplings. It is crucial to this interpretation that N -point amplitudes be viewed as rank N co-variant tensors on the space of couplings, and equation (11) states that the two diffeomorphisms are completely equivalent. From this point of view the anomalous dimension terms in the RG equation are seen as coming from the change in the basis dg^a under Lie transport along the trajectories of $\vec{\beta}$. Since Lie derivatives are the natural geometric way in which to describe symmetries, this picture gives a clearer insight into the structure of the RG equation and the manner in which conformal symmetry, and perhaps also other space symmetries, may be broken. For instance one might postulate the existence of a β -function, \mathcal{L}_{β_X} associated with a more general diffeomorphism, \vec{X} , of space and consider the equation

$$\mathcal{L}_{\vec{X}} \langle \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_N) \rangle = -\mathcal{L}_{\beta_X} \langle \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_N) \rangle. \quad (21)$$

This would presumably necessitate the introduction of position dependent couplings, a concept which has already proved to be of some use in understanding the RG equation [8].

An alternative version of the RG equation, for operators rather than amplitudes, is given by equation (17),

$$\left(\kappa \frac{\partial}{d\kappa} \Big|_g + \beta^b \partial_b \right) \hat{\Phi}_a + \Gamma_a{}^b \hat{\Phi}_b = 0, \quad (17)$$

but it must be stressed that neither $\kappa \frac{\partial}{d\kappa}|_g$ nor $\beta^b \partial_b$ can be pulled out of expectation values separately since S_o is not invariant under either separately.

It is a pleasure to thank Denjoe O'Connor for many discussions on the geometric nature of the RG.

After completion of this work the author's attention was drawn to reference [9] where the RG equation is also discussed in terms of Lie derivatives by introducing a connection on the space of couplings. I am grateful to Chris Stephens for informing me of this reference.

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