Lemma 5.45. *If a pseudo-differential operator $\mathscr{D}$ is a formal symmetry of rank k, its inverse $\mathscr{D}^{-1}$ is a formal symmetry of rank k. If $\mathscr{D}$ has order $m > 0$, then any (fractional) power $\mathscr{D}^{i/m}$ is also a formal symmetry of the same rank k.*

Thus, we can replace a (pseudo-) differential operator $\mathscr{D}$ of positive order $m > 0$ by its m-th root $\sqrt[m]{\mathscr{D}}$, which is a pseudo-differential operator of order 1, without losing the formal symmetry property. In fact, all the formal symmetry operators of a given rank (if any exist) can be completely characterized by a first order formal symmetry of that rank.

Theorem 5.46. *If an evolution equation of order $n \geqslant 2$ possesses a nonconstant formal symmetry of rank k, then it has a first order formal symmetry of rank k. Conversely, if*

$$\mathscr{D} = Q_1 D_x + Q_0 + Q_{-1} D_x^{-1} + \cdots \tag{5.63}$$

is a first order pseudo-differential operator which is a formal symmetry of rank k, then every formal symmetry of rank k has the form

$$\tilde{\mathscr{D}} = c_m \mathscr{D}^m + c_{m-1} \mathscr{D}^{m-1} + \cdots + c_{m-k+2} \mathscr{D}^{m-k+2} + \mathscr{E}, \tag{5.64}$$

where $c_m, \ldots, c_{m-k+2}$ are arbitrary constants, and $\mathscr{E}$ is any pseudo-differential operator of order at most $m - k + 1$.

Proof. The proof relies on a lemma characterizing the leading terms of a formal symmetry of an evolution equation. Note first that, by (5.57), *any* pseudo-differential operator is a formal symmetry of rank 1, so we only get interesting information starting at rank 2.

Lemma 5.47. *Let $\mathscr{D} = P_m[u] D_x^m + \cdots$ be an m-th order pseudo-differential operator which is a formal symmetry of rank $k \geqslant 2$ of the n-th order evolution equation $u_t = K[u]$, where $n \geqslant 2$. Then its leading coefficient is $P_m = c(\partial K/\partial u_n)^{m/n}$ for some constant $c \neq 0$.*

Proof. Define $K_n = \partial K/\partial u_n$. Since $\mathsf{D}_K = K_n D_x^n + \cdots$, the Leibniz rule (5.56) shows that the commutator in the formal symmetry condition (5.62) has leading term

$$[\mathsf{D}_K, \mathscr{D}] = (n K_n D_x P_m - m P_m D_x K_n) D_x^{m+n-1} + \cdots .$$

Since $n \geqslant 2$, the other term $\mathscr{D}_t = \mathrm{pr}\, \mathbf{v}_K(P_m) D_x^m + \cdots$ is of lower order. Therefore, $\mathscr{D}$ will be a formal symmetry of rank 2 (or more) if and only if $n K_n D_x P_m = m P_m D_x K_n$ or, equivalently, $D_x(K_n^{-m/n} P_m) = 0$. This suffices to prove the lemma. □

To prove the first part of Theorem 5.46, if the formal symmetry has positive order $m > 0$, then, according to Lemma 5.45, its m-th root provides a first order formal symmetry of the same rank. If its order is negative, $m < 0$, then

$\mathscr{D}^{-1/m}$ is the required first order formal symmetry. Finally, if $\mathscr{D}$ has order 0, then according to Lemma 5.47, $\mathscr{D} = c + \hat{\mathscr{D}}$ for some constant c and where $\hat{\mathscr{D}}$ is a pseudo-differential operator of order -1. Since c is trivially a formal symmetry of rank ∞, $\hat{\mathscr{D}}$ is a formal symmetry of rank k, and hence $\hat{\mathscr{D}}^{-1}$ provides the required first order formal symmetry.

To prove the second part, Lemma 5.47 implies that the first order formal symmetry (5.63) has leading term $\mathscr{D} = c\sqrt[n]{K_n}D_x + \cdots$ for some nonzero constant c. Similarly, any m-th order formal symmetry of order $k \geqslant 2$ has leading term $\tilde{\mathscr{D}} = \tilde{c}_m K_n^{m/n} D_x^m + \cdots$. Let $c_m = \tilde{c}_m/c^m$. Then $\hat{\mathscr{D}} = \tilde{\mathscr{D}} - c_m\mathscr{D}^m$ is a pseudo-differential operator of order $\hat{m} < m$; moreover, by the linearity of the Lie derivative, $\hat{\mathscr{D}}$ satisfies order$(\hat{\mathscr{D}}_t + [\hat{\mathscr{D}}, \mathsf{D}_K]) \leqslant n + m - k$, and hence $\hat{\mathscr{D}}$ is a formal symmetry of rank $k - (m - \hat{m})$. The proof now proceeds by an obvious induction on m, the order of the formal symmetry. □

As an illustration of the basic techniques, we discuss the problem of classifying integrable second order evolution equations of the particular form

$$u_t = u_{xx} + P(x, u, u_x). \tag{5.65}$$

(The classification of general second order evolution equations is handled by similar methods, although the calculations are more complicated. See the review paper of Shabat, Mikhailov and Sokolov, [1], for a more comprehensive treatment.) The Fréchet derivative of the right-hand side of (5.65) is

$$\mathsf{D}_K = D_x^2 + P_{u_x}D_x + P_u. \tag{5.66}$$

Consider a first order pseudo-differential operator which defines a formal symmetry of this equation. According to Lemma 5.47, we can, without loss of generality, assume that its leading coefficient is unity, and so the operator has the form

$$\mathscr{D} = D_x + Q_0 + Q_1 D_x^{-1} + Q_2 D_x^{-2} + \cdots, \tag{5.67}$$

where $Q_0, Q_1, \ldots$ are differential functions to be determined so as to satisfy the formal symmetry conditions. In order that $\mathscr{D}$ be a formal symmetry of rank k, the pseudo-differential operator

$$\mathscr{E} = \mathsf{v}_K[\![\mathscr{D}]\!] = \mathscr{D}_t + [\mathscr{D}, \mathsf{D}_K]$$

appearing in (5.62) must have order at most $3 - k$; the fact that $\mathscr{D}$ has leading coefficient 1 implies that it is already a formal symmetry of rank 2, so $\mathscr{E}$ has order at most 1. Requiring that the coefficients of the successive powers of D_x in $\mathscr{E}$ vanish will impose a series of increasingly stringent conditions on the operator, and, eventually, on the equation (5.65) itself, that are necessary for the existence of a formal symmetry of progressively higher and higher orders.

First, the coefficient of D_x in $\mathscr{E}$ is

$$D_x P_{u_x} - 2D_x Q_0.$$

To have a formal symmetry of rank 3, we require

$$Q_0 = \tfrac{1}{2}P_{u_x}. \tag{5.68}$$

(We can ignore the additive constant since, as remarked above, adding a constant to $\mathscr{D}$ does not affect its formal symmetry property.) Next, the coefficient of $D_x^0 = 1$ in $\mathscr{E}$ is

$$D_t Q_0 + D_x P_u - 2D_x Q_1 - P_{u_x} D_x Q_0.$$

In order to have a formal symmetry of rank 4, this quantity must vanish, which, in view of our normalization (5.68) of Q_0, means

$$\begin{aligned} 2D_x Q_1 &= D_t Q_0 + D_x P_u - P_{u_x} D_x Q_0 \\ &= \tfrac{1}{2} P_{u_x u_x} D_x(u_{xx} + P) + \tfrac{1}{2} P_{uu_x}(u_{xx} + P) + D_x P_u - \tfrac{1}{2} P_{u_x} D_x P_{u_x} \qquad (5.69) \\ &= D_x\left[\tfrac{1}{2} P_{u_x u_x}(u_{xx} + P) + P_u - \tfrac{1}{4} P_{u_x}^2\right] + \tfrac{1}{2}(P_{uu_x} - D_x P_{u_x u_x})(u_{xx} + P). \end{aligned}$$

In order that (5.69) be soluble for the differential function Q_1, the right-hand side must lie in the image of the total derivative D_x:

$$\begin{aligned} &(P_{uu_x} - D_x P_{u_x u_x})(u_{xx} + P) \\ &\quad = -P_{u_x u_x u_x} u_{xx}^2 + (P_{uu_x} - P_{xu_x u_x} - u_x P_{uu_x u_x} - P P_{u_x u_x u_x}) u_{xx} \\ &\qquad + (P_{uu_x} - P_{xu_x u_x} - u_x P_{uu_x u_x}) P \in \operatorname{im} D_x. \end{aligned} \tag{5.70}$$

Applying Theorem 4.7 (or, by inspection), we first see that this expression will certainly not lie in the image of D_x unless the coefficient of u_{xx}^2 vanishes. Therefore, P must be a quadratic polynomial in u_x,

$$P(x, u, u_x) = \alpha(x, u) u_x^2 + \beta(x, u) u_x + \gamma(x, u). \tag{5.71}$$

Thus, we immediately deduce a strong restriction on the type of evolution equations (5.65) which admit formal symmetries. Any equation (5.65) which admits a formal symmetry of rank 4 is necessarily of the form

$$u_t = u_{xx} + \alpha(x, u) u_x^2 + \beta(x, u) u_x + \gamma(x, u). \tag{5.72}$$

In particular, according to Proposition 5.42, only these types of equations can possibly admit generalized symmetries of order 4 or more (including admitting recursion operators). Plugging (5.71) into (5.70), and incorporating the terms involving u_{xx} into a total derivative, leads to a rather messy quadratic polynomial in u_x (the cubic terms all cancel) which must lie in the image of D_x. We could continue to analyze the general case directly, but the analysis is fairly complex. However, a simple observation will dramatically simplify our calculations.

Suppose we change variables in our evolution equation, replacing u by $v = \varphi(x, u)$, where φ is a smooth function, with $\varphi_u \neq 0$. (Note that the formal symmetry property, being given by the (1, 1)-Lie derivative, is unaffected by changes of variables.) Then

$$v_t = \varphi_u u_t, \quad v_x = \varphi_u u_x + \varphi_x, \quad v_{xx} = \varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{xu} u_x + \varphi_{xx}. \tag{5.73}$$

Therefore, given an equation of the form (5.72), the equation for v will take the same form

$$v_t = v_{xx} + \hat{\alpha}(x, v)v_x^2 + \hat{\beta}(x, v)v_x + \hat{\gamma}(x, v),$$

whose coefficients are related to those of the equation (5.72) for u according to

$$\alpha = \frac{\varphi_{uu}}{\varphi_u} + \hat{\alpha}\varphi_u,$$

$$\beta = 2\frac{\varphi_{xu}}{\varphi_u} + 2\hat{\alpha}\varphi_x + \hat{\beta},$$

$$\gamma = \frac{\varphi_{xu} + \hat{\alpha}\varphi_x^2 + \hat{\beta}\varphi_x + \hat{\gamma}}{\varphi_u}.$$

In particular, if we choose φ so that $\varphi_{uu} = \alpha\varphi_u$, we can eliminate the v_x^2 term in (5.73). Therefore, if, in our classification of evolution equations admitting formal symmetries, we are allowed to change variables, we can, without loss of generality (and using u instead of v), assume that the coefficient α in P is zero, so we need only consider evolution equations of the quasi-linear form:

$$u_t = u_{xx} + \beta(x, u)u_x + \gamma(x, u). \tag{5.74}$$

For such equations, the formal symmetry condition (5.70) dramatically simplifies to

$$\beta_u(u_{xx} + \beta u_x + \gamma) = D_x(\beta_u u_x) - \beta_{uu}u_x^2 + (\beta\beta_u - \beta_{xu})u_x + \beta_u\gamma \in \operatorname{im} D_x.$$

The coefficient of u_x^2 must vanish, hence $\beta(x, u) = a(x)u + b(x)$, and we require

$$\begin{aligned} &a^2uu_x + (ab - a_x)u_x + a\gamma(x, u) \\ &\qquad = D_x[\tfrac{1}{2}a^2u^2 + (ab - a_x)u] - aa_xu^2 - (a_xb + ab_x - a_{xx})u + a\gamma \in \operatorname{im} D_x. \end{aligned}$$

Therefore,

$$\gamma = a_xu^2 + \frac{a_xb + ab_x - a_{xx}}{a}u + c(x).$$

We can yet further simplify this equation by incorporating the linear change of variables $u \mapsto a(x)u + b(x) - 2(a'(x)/a(x))$. The resulting equation is of the form

$$u_t = u_{xx} + uu_x + h(x). \tag{5.75}$$

The higher rank conditions for formal symmetries of (5.75) are all automatically satisfied; indeed (5.75) is a simple modification of the usual Burgers' equation (5.46) and therefore possesses a recursion operator, cf. Exercise 5.14. Thus, we have proved that every evolution equation of the form (5.65) which admits a formal symmetry of rank 4 (or, as in Proposition 5.42, a generalized symmetry of order 4 or more) is necessarily integrable, admits a recursion operator and generalized symmetries of arbitrarily high order and is, in fact, equivalent to the Burgers'-type equation (5.75). Applying the inverse transformation $u \mapsto \psi(x, u)$ to (5.75) will produce the most general integrable equation of the form (5.65).

A similar analysis will give a complete classification of all integrable second order evolution equations $u_t = Q(x, u, u_x, u_{xx})$. A key simplification comes from allowing a sufficiently wide variety of changes of variables to simplify the analysis as much as possible. The appropriate class consists not solely of the changes of dependent variable $v = \varphi(x, u)$, but, rather, all first order contact transformations. (See Bluman and Kumei, [2], and Ibragimov, [1].) The final result of this analysis, cf. Mikhailov, Shabat and Sokolov, [1], is the following.

Theorem 5.48. *Every second order evolution equation which admits a formal symmetry of rank* 5 or *more is integrable and is equivalent, under a contact transformation, to one of the following*:

$$u_t = u_{xx} + q(x)u,$$
$$u_t = u_{xx} + uu_x + h(x),$$
$$u_t = (u^{-2}u_x + \alpha xu + \beta u)_x,$$
$$u_t = (u^{-2}u_x)_x + 1.$$

5.3. Generalized Symmetries and Conservation Laws

The correspondence between ordinary variational symmetries and conservation laws of systems of Euler–Lagrange equations readily generalizes, a fact recognized even by Noether herself. In fact, once we admit generalized symmetries into the picture, Noether's theorem provides a *one-to-one* correspondence between variational symmetries and conservation laws. In this section we develop this result in the form due to Bessel-Hagen. (See Exercise 5.33 for Noether's original version.) The basic computational results depend on the concept of the adjoint of a differential operator.

Adjoints of Differential Operators

If

$$\mathscr{D} = \sum_J P_J[u]D_J, \qquad P_J \in \mathscr{A},$$

is a differential operator, its (formal) *adjoint* is the differential operator $\mathscr{D}^*$ which satisfies

$$\int_\Omega P \cdot \mathscr{D}Q \, dx = \int_\Omega Q \cdot \mathscr{D}^* P \, dx \tag{5.76}$$

for every pair of differential functions $P, Q \in \mathscr{A}$ which vanish when $u \equiv 0$, every domain $\Omega \subset \mathbb{R}^p$ and every function $u = f(x)$ of compact support in Ω. An easy integration by parts shows that

$$\mathscr{D}^* = \sum_J (-D)_J \cdot P_J,$$

meaning that for any $Q \in \mathscr{A}$,

$$\mathscr{D}^* Q = \sum_J (-D)_J [P_J Q].$$

For example, if

$$\mathscr{D} = D_x^2 + u D_x,$$

then its adjoint is

$$\mathscr{D}^* = (-D_x)^2 + (-D_x)\cdot u = D_x^2 - u D_x - u_x.$$

Similarly, a matrix differential operator $\mathscr{D}: \mathscr{A}^k \to \mathscr{A}^l$ with entries $\mathscr{D}_{\mu\nu}$ has adjoint $\mathscr{D}^*: \mathscr{A}^l \to \mathscr{A}^k$ with entries $\mathscr{D}^*_{\mu\nu} = (\mathscr{D}_{\nu\mu})^*$, the adjoint of the transposed entries of $\mathscr{D}$. Note that $(\mathscr{D}\mathscr{E})^* = \mathscr{E}^*\mathscr{D}^*$ for any operators $\mathscr{D}$, $\mathscr{E}$. An operator $\mathscr{D}$ is *self-adjoint* if $\mathscr{D}^* = \mathscr{D}$; it is *skew-adjoint* if $\mathscr{D}^* = -\mathscr{D}$. For example, $D_x^2 + u$ is self-adjoint, while $D_x^3 + 2uD_x + u_x$ is skew-adjoint. Note that (5.76) is equivalent to the integration by parts formula

$$P \cdot \mathscr{D} Q = Q \cdot \mathscr{D}^* P + \operatorname{Div} A, \tag{5.77}$$

where $A \in \mathscr{A}^p$ is a bilinear expression involving P, Q and their derivatives, with coefficients depending on x, u and derivatives of u. Equivalently

$$\mathsf{E}(P \cdot \mathscr{D} Q) = \mathsf{E}(Q \cdot \mathscr{D}^* P), \tag{5.78}$$

where E is the Euler operator, cf. Theorem 4.7.

Note that if $P \in \mathscr{A}^l$, its Fréchet derivative has adjoint $\mathsf{D}_P^*: \mathscr{A}^l \to \mathscr{A}^q$, which, using (5.32), has entries

$$(\mathsf{D}_P^*)_{\nu\mu} = \sum_J (-D)_J \cdot \frac{\partial P_\mu}{\partial u_J^\nu}, \qquad \mu = 1, \ldots, l, \quad \nu = 1, \ldots, q. \tag{5.79}$$

For example, if $P = u_{xx} + u_x^2$,

$$\mathsf{D}_P = D_x^2 + 2u_x D_x, \qquad \mathsf{D}_P^* = D_x^2 - 2D_x \cdot u_x = D_x^2 - 2u_x D_x - 2u_{xx}.$$

Although (5.79) bears some similarity to the Euler operator, it is in fact a differential operator, not a differential function, and is thus quite different. However, if $P \in \mathscr{A}$,

$$\mathsf{E}(P) = \left(\sum_J (-D)_J \frac{\partial P}{\partial u_J^\nu}\right) = \mathsf{D}_P^*(1),$$

1 denoting the constant differential function. We note finally the important formula for the variational derivative of the product of two functions

$$\mathsf{E}(P \cdot Q) = \mathsf{D}_P^*(Q) + \mathsf{D}_Q^*(P), \qquad P, Q \in \mathscr{A}^l, \tag{5.80}$$

which follows from the Leibniz rule:

$$\mathsf{E}_\nu(P\cdot Q)=\sum_{\mu=1}^{l}\left\{\sum_J(-D)_J\left[\frac{\partial P_\mu}{\partial u_J^\nu}\cdot Q_\mu\right]+\sum_J(-D)_J\left[\frac{\partial Q_\mu}{\partial u_J^\nu}\cdot P_\mu\right]\right\}.$$

Characteristics of Conservation Laws

Before restricting our attention to Euler–Lagrange equations, we look at conservation laws in general again. Recall that every conservation law of a system of differential equations Δ is equivalent to one in characteristic form

$$\operatorname{Div} P=Q\cdot\Delta=\sum_{\nu=1}^{l}Q_\nu\Delta_\nu. \tag{5.81}$$

Using the notion of a Fréchet derivative, we readily obtain necessary and sufficient conditions for a given l-tuple Q to be the characteristic of a conservation law.

Proposition 5.49. *Let $\Delta=0$ be a system of differential equations. An l-tuple $Q\in\mathscr{A}^l$ is the characteristic of a conservation law if and only if*

$$\mathsf{D}_\Delta^*(Q)+\mathsf{D}_Q^*(\Delta)=0 \tag{5.82}$$

for all (x,u).

PROOF. According to Theorem 4.7, $Q\cdot\Delta$ is a total divergence (5.81) if and only if $\mathsf{E}(Q\cdot\Delta)=0$. Thus (5.82) follows at once from the product rule (5.80). □

In particular, a necessary condition for Q to be the characteristic of a conservation law for Δ is

$$\mathsf{D}_\Delta^*(Q)=0\qquad\text{for all solutions to }\Delta, \tag{5.83}$$

since $\mathsf{D}_Q^*(\Delta)=0$ automatically on solutions. This simplified form of (5.82) can often be used effectively to eliminate many possible l-tuples Q from consideration as characteristics of conservation laws, and thus readily lead to a complete classification of conservation laws for the system.

Example 5.50. Consider Burgers' equation in physical form

$$u_t=u_{xx}+uu_x.$$

If $\tilde{Q}[u]\in\mathscr{A}$ is the characteristic of a conservation law, then we can always replace t-derivatives of u by x-derivatives using the equation, so there is an equivalent characteristic of the form $Q(x,t,u,u_x,\ldots,u_n)$, $u_n=\partial^n u/\partial x^n$. Let us see what (5.83) says about the form of Q. For Burgers' equation,

$$\mathsf{D}_\Delta=D_t-D_x^2-uD_x-u_x,\quad\text{so}\quad \mathsf{D}_\Delta^*=-D_t-D_x^2+uD_x.$$

The leading order terms in (5.83) are

$$\mathsf{D}_\Delta^*(Q) = \frac{\partial Q}{\partial u_n}(-u_{n,t} - u_{n+2}) + \cdots = -2\frac{\partial Q}{\partial u_n}u_{n+2} + \cdots,$$

on solutions, the omitted terms depending on $(n+1)$-st and lower order x-derivatives of u. Thus (5.83) implies that $\partial Q/\partial u_n = 0$, so Q actually only depends on $(n-1)$-st and lower order derivatives of u. Proceeding by induction, we conclude that $Q = q(x, t)$ cannot depend on u or its derivatives in any nontrivial way. Moreover,

$$\mathsf{D}_\Delta^*(q) = q_t - q_{xx} + uq_x = 0$$

if and only if q is a constant. Thus the only nontrivial conservation law for Burgers' equation has a constant for its characteristic; the corresponding law is the equation itself:

$$D_t(u) + D_x(-u_x - \tfrac{1}{2}u^2) = 0.$$

Variational Symmetries

As with the geometrical form of Noether's theorem discussed in Chapter 4, the general form of Noether's theorem will only provide a correspondence between conservation laws and *variational symmetries.* These are defined in analogy with the divergence symmetries of Definition 4.33.

Definition 5.51. A generalized vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha \frac{\partial}{\partial u^\alpha}$$

is a *variational symmetry* of the functional $\mathscr{L}[u] = \int L(x, u^{(n)})\,dx$ if and only if there exists a p-tuple $B[u] \in \mathscr{A}^p$ of differential functions such that

$$\mathrm{pr}\,\mathbf{v}(L) + L\,\mathrm{Div}\,\xi = \mathrm{Div}\,B \tag{5.84}$$

for all x, u. (Here $\xi = (\xi^1, \ldots, \xi^p)$ is as in (4.15).)

We first show that we can effectively restrict our attention to variational symmetries which are in evolutionary form.

Proposition 5.52. *A generalized vector field* $\mathbf{v}$ *is a variational symmetry of* $\mathscr{L}[u]$ *if and only if its evolutionary representative* $\mathbf{v}_Q$ *is.* (*Note*: *This statement is false if we omit the divergence term* Div B *in our definition* (5.84).)

PROOF. Using the basic prolongation formula (5.8),

$$\begin{aligned}\mathrm{pr}\,\mathbf{v}(L) + L\,\mathrm{Div}\,\xi &= \mathrm{pr}\,\mathbf{v}_Q(L) + \sum_{i=1}^{p} \xi^i D_i L + L \sum_{i=1}^{p} D_i \xi^i \\ &= \mathrm{pr}\,\mathbf{v}_Q(L) + \sum_{i=1}^{p} D_i(\xi^i L).\end{aligned}$$

Therefore, (5.84) holds if and only if

$$\operatorname{pr} \mathbf{v}_Q(L) = \operatorname{Div} \tilde{B}, \tag{5.85}$$

where $\tilde{B}_i = B_i - L\xi^i$. □

As with ordinary symmetries, every generalized variational symmetry of a variational problem is necessarily a symmetry of the corresponding Euler–Lagrange equations. (The converse of this statement remains *not* true in general.)

Theorem 5.53. *If the generalized vector field* $\mathbf{v}$ *is a variational symmetry of* $\mathscr{L}[u] = \int L(x, u^{(n)})\, dx$, *then* $\mathbf{v}$ *is a generalized symmetry of the Euler–Lagrange equations* $\mathsf{E}(L) = 0$.

The proof is based on the following important commutation formula.

Lemma 5.54. *Suppose* $L \in \mathscr{A}$, $Q \in \mathscr{A}^q$. *Then*

$$\mathsf{E}[\operatorname{pr} \mathbf{v}_Q(L)] = \operatorname{pr} \mathbf{v}_Q[\mathsf{E}(L)] + \mathsf{D}_Q^* \mathsf{E}(L). \tag{5.86}$$

Proof. According to the integration by parts formula (4.39) and the identity (5.80),

$$\mathsf{E}[\operatorname{pr} \mathbf{v}_Q(L)] = \mathsf{E}[Q \cdot \mathsf{E}(L)] = \mathsf{D}_{\mathsf{E}(L)}^*[Q] + \mathsf{D}_Q^*[\mathsf{E}(L)].$$

We now need the important result that $\Delta = \mathsf{E}(L)$ is an Euler–Lagrange expression if and only if its Fréchet derivative is a self-adjoint differential operator: $\mathsf{D}_\Delta^* = \mathsf{D}_\Delta$. This fundamental theorem, which is the variational analogue of the equality of mixed partial derivatives, and constitutes the solution to the inverse problem of the calculus of variations, will be proved in Section 5.4. (See Theorem 5.92.) Assuming this result, (5.86) follows easily from (5.33) since

$$\mathsf{D}_{\mathsf{E}(L)}^*[Q] = \mathsf{D}_{\mathsf{E}(L)}[Q] = \operatorname{pr} \mathbf{v}_Q[\mathsf{E}(L)]. \qquad \square$$

Proof of Theorem 5.53. By Propositions 5.5 and 5.52 we can replace $\mathbf{v}$ by its evolutionary form $\mathbf{v}_Q$ without affecting the validity of the theorem. If $\mathbf{v}_Q$ is a variational symmetry, (5.85) implies that the left-hand side of (5.86) vanishes. But D_Q^* is a linear differential operator, hence the symmetry condition (5.5) for $\Delta = \mathsf{E}(L)$ holds, completing the proof. □

Thus to find all the variational symmetries of a system of Euler–Lagrange equations, it suffices to use the methods of Sections 5.1 or 5.2 to construct symmetries of the Euler–Lagrange equations and then check which of them satisfy the additional variational requirement (5.84). Actually, we don't need to re-apply $\operatorname{pr} \mathbf{v}$ to the Lagrangian, or even know precisely what the Lagrangian is, since we can use the following intrinsic characterization of a variational symmetry.

Proposition 5.55. *Let $\Delta = 0$ be a system of differential equations whose Fréchet derivative is self-adjoint: $\mathsf{D}_\Delta^* = \mathsf{D}_\Delta$, so Δ is the Euler–Lagrange equations for some variational problem.*[†] *An evolutionary vector field $\mathbf{v}_Q$ is a variational symmetry thereof if and only if*

$$\operatorname{pr} \mathbf{v}_Q(\Delta) + \mathsf{D}_Q^*(\Delta) = 0 \tag{5.87}$$

for all x, u.

The proof is immediate from the preceding calculations and the solution to the inverse problem in Theorem 5.92. □

Group Transformations

Assuming that the variational symmetry is in evolutionary form, we can deduce that the corresponding group transformations leave the functional itself invariant in the following sense.

Proposition 5.56. *Given the relevant existence and uniqueness results on the Cauchy problem for the associated system of evolution equations, a generalized vector field $\mathbf{v}_Q$ is a variational symmetry of the functional $\mathscr{L}_{\Omega_0}[u] = \int_{\Omega_0} L(x, u^{(n)})\, dx$ if and only if for every subdomain $\Omega \subset \Omega_0$ and every function $u = f(x)$ in the appropriate function space*

$$\mathscr{L}_\Omega[\exp(\varepsilon \mathbf{v}_Q) f] = \mathscr{L}_\Omega[f] + \mathscr{B}_{\partial\Omega}[\varepsilon, f], \tag{5.88}$$

where $\mathscr{B}_{\partial\Omega}$ depends only on the values of $\exp(\varepsilon \mathbf{v}_Q) f$ *and its derivatives on the boundary $\partial\Omega$.*

Another way of interpreting this result is that a generalized vector field $\mathbf{v}_Q$ is a variational symmetry of a functional $\mathscr{L}$ if and only if $\mathscr{L}$ determines a conservation law for the system of evolution equations $u_t = Q$ prescribing the flow of $\mathbf{v}_Q$.

PROOF. Differentiating (5.88) with respect to ε, we find

$$\int_\Omega \operatorname{pr} \mathbf{v}_Q(L)\, dx = \int_{\partial\Omega} B \cdot dS = \int_\Omega (\operatorname{Div} B)\, dx$$

for some $B \in \mathscr{A}^p$ depending on u and its derivatives; both sides of this latter identity are to be evaluated at $u = \exp(\varepsilon \mathbf{v}_Q) f$. Since this holds for an arbitrary subdomain Ω, we conclude the equality of the integrands,

$$\operatorname{pr} \mathbf{v}_Q(L) = \operatorname{Div} B,$$

verifying the infinitesimal criterion (5.85). The converse follows upon integration with respect to ε. □

[†] This assumes the restriction on the domain M of Theorem 5.92.