$$\frac{\partial u}{\partial t} = \mathscr{D} \cdot \delta \mathscr{H}[u].$$

There are certain restrictions, based on the Jacobi identity for the associated Poisson bracket, that a differential operator $\mathscr{D}$ must satisfy in order to qualify as a true Hamiltonian operator. These are described in Section 7.1; in their original form they are hopelessly complicated to work with but, using the theory of "functional multi-vectors", an efficient, simple computational algorithm for determining when an operator $\mathscr{D}$ is Hamiltonian is devised. The second section explores the standard applications of symmetry groups and conservation laws to Hamiltonian systems of evolution equations. The main tool is the Hamiltonian form of Noether's theorem. Applications to the Korteweg–de Vries equation and the Euler equations of ideal fluid flow are presented.

The final section deals with the recent theory of bi-Hamiltonian systems. Occasionally, as in the case of the Korteweg–de Vries equation, one runs across a system of evolution equations which can be written in Hamiltonian form in two distinct ways. In this case, subject to a mild compatibility condition, the system will necessarily have an infinite hierarchy of mutually commuting conservation laws and consequent Hamiltonian flows, generated by a recursion operator based on the two Poisson brackets, and hence can be viewed as a "completely integrable" Hamiltonian system. Such systems have many other remarkable properties, including soliton solutions, linearization by inverse scattering and so on. A new proof of the basic theorem on bi-Hamiltonian systems is given here, along with some applications.

7.1. Poisson Brackets

Recall first the basic set-up of the formal variational calculus presented in Section 5.4. Let $M \subset X \times U$ be an open subset of the space of independent and dependent variables $x = (x^1, \ldots, x^p)$ and $u = (u^1, \ldots, u^q)$. The algebra of differential functions $P(x, u^{(n)}) = P[u]$ over M is denoted by $\mathscr{A}$, and its quotient space under the image of the total divergence is the space $\mathscr{F}$ of functionals $\mathscr{P} = \int P\, dx$.

The main goal of this section is to make precise what we mean by a system of evolution equations

$$u_t = K[u] = K(x, u^{(n)}), \qquad K \in \mathscr{A}^q,$$

being a Hamiltonian system. Here K depends only on spatial variables x and spatial derivatives of u; t is singled out to play a special role. To do this, we need to pursue analogies to the various components of (6.14) in the present context. Firstly, the role of the Hamiltonian function in (6.14) should be played by a Hamiltonian *functional* $\mathscr{H} = \int H\, dx \in \mathscr{F}$. Therefore, we must replace the gradient operation by the "functional gradient" or variational derivative $\delta\mathscr{H} \in \mathscr{A}^q$ of $\mathscr{H}$. The remaining ingredient is the analogue of the skew-symmetric matrix $J(x)$ which serves to define the Poisson bracket. Here we need a linear operator $\mathscr{D}: \mathscr{A}^q \to \mathscr{A}^q$ on the space of q-tuples of differential functions, which, in most instances, will be a linear $q \times q$ matrix differential operator, and may depend on x, u and derivatives of u. To qualify as a Hamiltonian operator, $\mathscr{D}$ must enjoy further properties, which are found by looking at the corresponding Poisson bracket.

In finite dimensions, the Poisson brackets of two functions is a function which depends bilinearly on the respective gradients, the coefficients being determined by the Hamiltonian matrix $J(x)$, cf. (6.12). Thus, for evolution equations, the Poisson bracket of two functionals must be a *functional* depending bilinearly on the respective variational derivatives. Clearly, for a candidate Hamiltonian operator $\mathscr{D}$, the correct expression for the corresponding Poison bracket has the form

$$\{\mathscr{P}, \mathscr{Q}\} = \int \delta\mathscr{P} \cdot \mathscr{D}\delta\mathscr{Q}\, dx, \tag{7.1}$$

whenever $\mathscr{P}, \mathscr{Q} \in \mathscr{F}$ are functionals. Of course, the Hamiltonian operator $\mathscr{D}$ must satisfy certain further restrictions in order that (7.1) be a true Poisson bracket.

Definition 7.1. A linear operator $\mathscr{D}: \mathscr{A}^q \to \mathscr{A}^q$ is called *Hamiltonian* if its Poisson bracket (7.1) satisfies the conditions of *skew-symmetry*

$$\{\mathscr{P}, \mathscr{Q}\} = -\{\mathscr{Q}, \mathscr{P}\}, \tag{7.2}$$

and the *Jacobi identity*

$$\{\{\mathscr{P}, \mathscr{Q}\}, \mathscr{R}\} + \{\{\mathscr{R}, \mathscr{P}\}, \mathscr{Q}\} + \{\{\mathscr{Q}, \mathscr{R}\}, \mathscr{P}\} = 0, \tag{7.3}$$

for all functionals $\mathscr{P}, \mathscr{Q}, \mathscr{R} \in \mathscr{F}$.

If we compare Definition 7.1 with the finite-dimensional version of Definition 6.1, we see that two of the earlier conditions have been dropped. Of these, bilinearity is apparent from the form (7.1) of the bracket. The Leibniz rule has no counterpart in this situation, since, as we saw in Section 5.4, there is no well-defined multiplication between functionals. However, the principal use of Leibniz' rule was to deduce the existence of a Hamiltonian vector field from a real-valued function H, satisfying (6.4). This *does* carry over to the functional case:

Proposition 7.2. *Let $\mathscr{D}$ be a Hamiltonian operator with Poisson bracket* (7.1). *To each functional $\mathscr{H} = \int H\, dx \in \mathscr{F}$, there is an evolutionary vector field $\hat{\mathbf{v}}_{\mathscr{H}}$, called the* Hamiltonian vector field *associated with $\mathscr{H}$, which satisfies*

$$\operatorname{pr} \hat{\mathbf{v}}_{\mathscr{H}}(\mathscr{P}) = \{\mathscr{P}, \mathscr{H}\} \tag{7.4}$$

for all functionals $\mathscr{P} \in \mathscr{F}$. Indeed, $\hat{\mathbf{v}}_{\mathscr{H}}$ has characteristic $\mathscr{D}\delta\mathscr{H} = \mathscr{D}\mathsf{E}(H)$.

PROOF. Let $\mathscr{P} = \int P\,dx$, $P \in \mathscr{A}$. Then, using the integration by parts formula (5.128), we find

$$\{\mathscr{P}, \mathscr{H}\} = \int \mathsf{E}(P) \cdot \mathscr{D}\mathsf{E}(H)\,dx = \int \mathrm{pr}\,\mathbf{v}_{\mathscr{D}\mathsf{E}(H)}(P)\,dx = \mathrm{pr}\,\mathbf{v}_{\mathscr{D}\mathsf{E}(H)}\left(\int P\,dx\right).$$

(See Exercise 5.52). Thus (7.4) holds provided $\hat{\mathbf{v}}_{\mathscr{H}} = \mathbf{v}_{\mathscr{D}\mathsf{E}(H)}$. □

The Hamiltonian flow corresponding to a functional $\mathscr{H}[u]$ is obtained by exponentiating the corresponding Hamiltonian vector field $\hat{\mathbf{v}}_{\mathscr{H}}$. According to (5.14), then, a Hamiltonian system of evolution equations takes the form

$$\frac{\partial u}{\partial t} = \mathscr{D} \cdot \delta\mathscr{H}, \tag{7.5}$$

where δ is the variational derivative, and $\mathscr{D}$ the Hamiltonian operator. Note the complete analogy with the finite-dimensional Hamiltonian system (6.14). Before proceeding to examples of Hamiltonian systems, we need to have some reasonably straightforward means of determining when a given operator $\mathscr{D}$ is Hamiltonian. To begin with, the requirement imposed by the skew-symmetry of the Poisson bracket is immediate, being the infinite-dimensional verison of the skew-symmetry of the matrix J in (6.14).

Proposition 7.3. *Let $\mathscr{D}$ be a $q \times q$ matrix differential operator with bracket* (7.1) *on the space of functionals. Then the bracket is skew-symmetric, i.e.* (7.2) *holds, if and only if $\mathscr{D}$ is skew-adjoint: $\mathscr{D}^* = -\mathscr{D}$.*

PROOF. If $\mathscr{P} = \int P\,dx$, $\mathscr{Q} = \int Q\,dx$, then (7.2) can be written as

$$\int \mathsf{E}(P) \cdot \mathscr{D}\mathsf{E}(Q)\,dx = -\int \mathsf{E}(Q) \cdot \mathscr{D}\mathsf{E}(P)\,dx. \tag{7.6}$$

Using the definition (5.76) of the adjoint $\mathscr{D}^*$, we find (7.6) is equivalent to

$$\int \mathsf{E}(P) \cdot (\mathscr{D} + \mathscr{D}^*)\mathsf{E}(Q)\,dx = 0. \tag{7.7}$$

If (7.7) holds for all $P, Q \in \mathscr{A}$, then, as in the proof of Proposition 5.88, using the "substitution principle" which was enunciated in Exercise 5.42, we must have $\mathscr{D} + \mathscr{D}^* = 0$. □

The Jacobi Identity

At first sight, the direct verification of the Jacobi identity (7.3), even for the simplest skew-adjoint operators, appears a hopelessly complicated computational task. However, a considerable simplification is effected by utilizing some of our basic results from the formal variational calculus, bringing this problem within the realm of feasibility. An even further simplification can be made by introducing a version of the functional forms of Section 5.4 (although here they are, in a sense, "dual" objects), after which the verification of the Jacobi identity becomes a more or less routine computation.

Let $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{R}$ be functionals, with variational derivatives $\delta\mathscr{P} = P$, $\delta\mathscr{Q} = Q$, $\delta\mathscr{R} = R$ in $\mathscr{A}^q$. (Note the change of notation: P is no longer the integrand for $\mathscr{P}$!) With this notation, the first term in the Jacobi identity (7.3) becomes

$$\{\{\mathscr{P}, \mathscr{Q}\}, \mathscr{R}\} = \operatorname{pr} \hat{\mathbf{v}}_{\mathscr{R}}\left(\int P \cdot \mathscr{D}Q \, dx\right) = \int \operatorname{pr} \mathbf{v}_{\mathscr{D}R}(P \cdot \mathscr{D}Q) \, dx.$$

Using Leibniz' rule, and the Lie derivative formula (5.38), this is

$$\{\{\mathscr{P}, \mathscr{Q}\}, \mathscr{R}\} = \int \{\operatorname{pr} \mathbf{v}_{\mathscr{D}R}(P) \cdot \mathscr{D}Q + P \cdot \operatorname{pr} \mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q + P \cdot \mathscr{D}[\operatorname{pr} \mathbf{v}_{\mathscr{D}R}(Q)]\} \, dx. \tag{7.8}$$

From the formula (5.33) connecting the Lie derivative and Fréchet derivative, the first term in this expression is

$$\int \operatorname{pr} \mathbf{v}_{\mathscr{D}R}(P) \cdot \mathscr{D}Q \, dx = \int \mathsf{D}_P(\mathscr{D}R) \cdot \mathscr{D}Q \, dx.$$

Similarly, if we use the fact that $\mathscr{D}$ is skew-adjoint, the third term has an analogous form

$$\int P \cdot \mathscr{D}[\operatorname{pr} \mathbf{v}_{\mathscr{D}R}(Q)] \, dx = -\int \mathscr{D}P \cdot \mathsf{D}_Q(\mathscr{D}R) \, dx. \tag{7.9}$$

The second and third components in the Jacobi identity contribute similar expressions; for example, $\{\{\mathscr{Q}, \mathscr{R}\}, \mathscr{P}\}$ contains the terms

$$\int \mathsf{D}_Q(\mathscr{D}P) \cdot \mathscr{D}R \, dx \quad \text{and} \quad -\int \mathscr{D}Q \cdot \mathsf{D}_R(\mathscr{D}P) \, dx. \tag{7.10}$$

But according to Theorem 5.92, the Fréchet derivative of $Q = \delta\mathscr{Q}$ is a self-adjoint differential operator, so the first integral in (7.10) equals

$$\int \mathscr{D}P \cdot \mathsf{D}_Q(\mathscr{D}R) \, dx,$$

and cancels the integral (7.9) when substituted into the Jacobi identity. Thus, once we have expanded the Jacobi identity in this way, six of the terms cancel and we are left with the equivalent form

$$\int [P \cdot \operatorname{pr} \mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q + R \cdot \operatorname{pr} \mathbf{v}_{\mathscr{D}Q}(\mathscr{D})P + Q \cdot \operatorname{pr} \mathbf{v}_{\mathscr{D}P}(\mathscr{D})R] \, dx = 0, \tag{7.11}$$

which must vanish for all P, Q, R which are variational derivatives of functionals.

At this stage, a further simplification arises. Note that the integrand in (7.11) depends only on P, Q and R and their *total* derivatives. According to our general "substitution principle" announced in Exercise 5.42, this expression vanishes for all variational derivatives $P = \delta\mathscr{P}$, $Q = \delta\mathscr{Q}$, $R = \delta\mathscr{R}$ if and only if it vanishes for *arbitrary* q-tuples $P, Q, R \in \mathscr{A}^q$. We have thus proved

Proposition 7.4. *Let $\mathscr{D}$ be a skew-adjoint $q \times q$ matrix differential operator. Then the bracket* (7.1) *satisfies the Jacobi identity if and only if* (7.11) *vanishes for all q-tuples $P, Q, R \in \mathscr{A}^q$.*

Corollary 7.5. *If $\mathscr{D}$ is a skew-adjoint $q \times q$ matrix differential operator whose coefficients do not depend on u or its derivatives, then $\mathscr{D}$ is automatically a Hamiltonian operator.*

In fact, in this case $\operatorname{pr} \mathbf{v}_Q(\mathscr{D}) = 0$ for any evolutionary vector field $\mathbf{v}_Q$, so (7.11) is trivially satisfied. □

Example 7.6. The Korteweg–de Vries equation

$$u_t = u_{xxx} + uu_x$$

can in fact be written in Hamiltonian form in two distinct ways. Firstly, we see

$$u_t = D_x(u_{xx} + \tfrac{1}{2}u^2) = \mathscr{D}\delta\mathscr{H}_1,$$

where $\mathscr{D} = D_x$ and

$$\mathscr{H}_1[u] = \int [-\tfrac{1}{2}u_x^2 + \tfrac{1}{6}u^3]\,dx$$

is one of the classical conservation laws. Note that $\mathscr{D}$ is certainly skew-adjoint, and hence by Corollary 7.5 is automatically Hamiltonian. The Poisson bracket is

$$\{\mathscr{P}, \mathscr{Q}\} = \int \delta\mathscr{P} \cdot D_x(\delta\mathscr{Q})\,dx. \tag{7.12}$$

(To gain a true appreciation of the efficacy of our formal variational methods, the reader might try verifying the Jacobi identity for (7.12) directly!).

The second Hamiltonian form is a bit less obvious. We find

$$u_t = (D_x^3 + \tfrac{2}{3}uD_x + \tfrac{1}{3}u_x)u = \mathscr{E}\delta\mathscr{H}_0,$$

where

$$\mathscr{H}_0[u] = \int \tfrac{1}{2}u^2\,dx$$

is another of the conserved quantities, and

$$\mathscr{E} = D_x^3 + \tfrac{2}{3}uD_x + \tfrac{1}{3}u_x.$$

It is easy to prove that $\mathscr{E}$ is skew-adjoint; to prove the Jacobi identity we look at (7.11). The first term there is

$$\int P \operatorname{pr} \mathbf{v}_{\mathscr{E}(R)}(\mathscr{E})Q \, dx = \int P[\tfrac{2}{3}(\mathscr{E}R)Q_x + \tfrac{1}{3}(\mathscr{E}R)_x Q] \, dx$$

$$= \int [\tfrac{2}{3}PR_{xxx}Q_x + \tfrac{1}{3}PR_{xxxx}Q + \tfrac{4}{9}uPR_xQ_x + \tfrac{2}{9}u_xPRQ_x$$

$$+ \tfrac{2}{9}uPR_{xx}Q + \tfrac{1}{3}u_xPR_xQ + \tfrac{1}{9}u_xPRQ] \, dx,$$

where we are using x-subscripts as abbreviations for *total* derivatives: $P_x = D_xP$, $P_{xx} = D_x^2P$, etc. We must add in the corresponding expressions stemming from the other two terms in (7.11) and then prove that the resulting integrand is a null Lagrangian, i.e. a total D_x-derivative, no matter what P, Q and R are. This is true, and we leave the proof to the reader, since later we will find a much simpler proof of this fact. We conclude that

$$\{\mathscr{P}, \mathscr{Q}\} = \int [\delta\mathscr{P} \cdot (D_x^3 + \tfrac{2}{3}uD_x + \tfrac{1}{3}u_x)\delta\mathscr{Q}] \, dx \tag{7.13}$$

does define a Poisson bracket on the space of functionals.

Although the above computation is quite a bit more tractable than the direct verification of the Jacobi identity, it still requires quite a lot of computational stamina even in such a relatively simple example. An even more radical simplification can be effected if we employ a theory of multilinear maps similar to that developed in Section 5.4. (An applications-oriented reader may wish to skip ahead to Section 7.2 at this stage.)

Functional Multi-vectors

In finite dimensions, multivectors are the dual objects to differential forms. In Exercise 6.20 it was shown how to develop the theory of finite-dimensional Poisson structures based on the theory of multivectors. Here we introduce the analogous objects for infinite-dimensional Hamiltonian systems of evolution equations. Since we are working with open subsets of Euclidean space $M \subset X \times U$, the theory of functional multi-vectors is identical with that of functional forms developed in Section 5.4. The only reason that we employ different terminology and notation is that, from a more global standpoint, the transformation rules for these objects under changes of variables are *not* the same; functional forms transform like Euler–Lagrange expressions whereas functional multi-vectors are more like evolutionary vector fields. Except for this distinction (which will not actually occur in this book) these objects are the same.

Recall that each functional k-form determines an alternating k-linear map from the space T_0 of evolutionary vector fields to the space $\mathscr{F}$ of functionals.

Similarly, a *functional k-vector* will be determined by an alternating k-linear map from the "dual" space $\bigwedge_*^1$ of *functional* one-forms to $\mathscr{F}$. Since each evolutionary vector field is uniquely determined by its characteristic, we can identify T_0 with $\mathscr{A}^q$, the space of q-tuples of differential functions on M. Similarly, according to Proposition 5.87, each functional one-form is uniquely determined by its canonical form, and hence we can also identify $\bigwedge_*^1$ with $\mathscr{A}^q$. Under these two identifications (which depend on the precise Euclidean coordinates on M), we obtain the identification of functional multi-vectors and forms.

Each functional form arises from a vertical form, so correspondingly each functional multi-vector arises from a vertical multi-vector. To preserve the notational distinction between the two, we use the notation θ_J^α for the "uni-vector" corresponding to the one-form du_J^α; thus

$$\langle \theta_J^\alpha; P \rangle = D_J P_\alpha \quad \text{whenever} \quad P = (P_1, \ldots, P_q) \in \mathscr{A}^q.$$

(From now on we replace $\bigwedge_*^1$ by $\mathscr{A}^q$.) Note that we could identify θ_J^α with the derivation $\partial/\partial u_J^\alpha$, which could be adopted as an alternative notation for multi-vectors, but one that is heavier and, later, slightly confusing. A general functional k-vector is thus a finite sum

$$\Theta = \int \left\{ \sum_{\boldsymbol{\alpha}, \mathbf{J}} R_{\mathbf{J}}^{\boldsymbol{\alpha}}[u] \theta_{J_1}^{\alpha_1} \wedge \cdots \wedge \theta_{J_k}^{\alpha_k} \right\} dx,$$

with coefficients $R_{\mathbf{J}}^{\boldsymbol{\alpha}} \in \mathscr{A}$; it defines the k-linear map

$$\langle \Theta; P^1, \ldots, P^k \rangle = \int \left[\sum_{\boldsymbol{\alpha}, \mathbf{J}} R_{\mathbf{J}}^{\boldsymbol{\alpha}} \det(D_{J_i} P_{\alpha_i}^j) \right] dx, \qquad P^j \in \mathscr{A}^q,$$

cf. (5.111). The total derivatives act as Lie derivatives on the vertical k-vectors, with $D_i(\theta_J^\alpha) = \theta_{J,i}^\alpha$, cf. (5.112). The space $\bigwedge_k^*$ of functional k-vectors is then the quotient space of the space of *vertical k-vectors* (i.e. finite sums of wedge products of the θ_J^α with coefficients in $\mathscr{A}$) under the image of total divergence. By Lemma 5.85, every functional k-vector is uniquely determined by its values on the space of q-tuples of differential functions. In this way, all of the theorems and examples of functional forms discussed in Section 5.4 carry over to functional multi-vectors once we replace du_J^α by its counterpart θ_J^α and the vector fields $\mathrm{pr}\, \mathbf{v}_Q$ by their characteristics $Q \in \mathscr{A}^q$.†

For example, any functional uni-vector

$$\gamma = \int \left\{ \sum_{\alpha, J} R_J^\alpha \theta_J^\alpha \right\} dx$$

† One tricky point in this theory is that while the spaces $\bigwedge_*^k$ and $\bigwedge_k^*$ of functional forms and multi-vectors are defined in a dual manner, they are *not* naturally dual vector spaces for any $k > 1$! This is a reflection of our inability to define a multiplication on the space $\mathscr{F}$ of functionals.

can be put into *canonical form*

$$\gamma = \int \{R \cdot \theta\}\, dx = \int \left\{ \sum_{\alpha=1}^{q} R_\alpha \theta^\alpha \right\} dx, \qquad R_\alpha = \sum_J (-D)_J R_J^\alpha,$$

by an integration by parts. (Thus we can identify $\bigwedge_1^*$ with $\mathcal{T}_0$, the space of evolutionary vector fields so γ corresponds to $\mathbf{v}_R$!) Similarly, any functional bi-vector has the *canonical form*

$$\Theta = \frac{1}{2} \int \{\theta \wedge \mathscr{D}\theta\}\, dx = \frac{1}{2} \int \left\{ \sum_{\alpha,\beta=1}^{q} \theta^\alpha \wedge \mathscr{D}_{\alpha\beta}\theta^\beta \right\} dx, \tag{7.14}$$

where $\mathscr{D} = (\mathscr{D}_{\alpha\beta})$ is a skew-adjoint $q \times q$ matrix differential operator; see Proposition 5.88. Such a bi-vector defines the bilinear map

$$\langle \Theta; P, Q \rangle = \frac{1}{2} \int (P \cdot \mathscr{D}Q - Q \cdot \mathscr{D}P)\, dx = \int (P \cdot \mathscr{D}Q)\, dx, \qquad P, Q \in \mathscr{A}^q,$$

where we used the skew-adjoint nature of $\mathscr{D}$. In particular, if P and Q are variational derivatives (or differentials if we revert back to $\bigwedge_*^1$),

$$\langle \Theta; \delta\mathscr{P}, \delta\mathscr{Q} \rangle = \int (\delta\mathscr{P} \cdot \mathscr{D}\delta\mathscr{Q})\, dx$$

reproduces the bracket $\{\mathscr{P}, \mathscr{Q}\}$ determined by the skew-adjoint operator $\mathscr{D}$. For example, the Poisson bracket for the second Hamiltonian operator $\mathscr{E}$ of the Korteweg–de Vries example is represented by the bi-vector

$$\Theta = \frac{1}{2} \int \{\theta \wedge \mathscr{E}(\theta)\}\, dx = \frac{1}{2} \int \{\theta \wedge \theta_{xxx} + \tfrac{2}{3} u\theta \wedge \theta_x\}\, dx, \tag{7.15}$$

the term involving $\theta \wedge \theta$ trivially vanishing.

The Jacobi identity provides a natural example of a functional tri-vector. Note that in its original form, the left-hand side of (7.3) is clearly an alternating, tri-linear function of the variational derivatives $\delta\mathscr{P}$, $\delta\mathscr{Q}$, $\delta\mathscr{R}$. Therefore (7.11), although it may not appear to be, is an alternating tri-linear function of the q-tuples P, Q, R and hence determines a functional tri-vector, which we denote by

$$\Psi = \frac{1}{2} \int \{\theta \wedge \mathrm{pr}\, \mathbf{v}_{\mathscr{D}\theta}(\mathscr{D}) \wedge \theta\}\, dx, \tag{7.16}$$

so that $\langle \Psi; P, Q, R \rangle$ is the left-hand side of (7.11). (See also Exercise 7.12.) It remains to explain the notation in (7.16).

The "vector field" $\mathbf{v}_{\mathscr{D}\theta}$ is a formal evolutionary vector field whose characteristic is the q-tuple

$$(\mathscr{D}\theta)_\alpha = \sum_{\beta=1}^{q} \mathscr{D}_{\alpha\beta}\theta^\beta$$

of vertical uni-vectors; thus formally, using (5.6) as a model,

$$\operatorname{pr} \mathbf{v}_{\mathscr{D}\theta} = \sum_{\alpha, J} D_J \left(\sum_\beta \mathscr{D}_{\alpha\beta} \theta^\beta \right) \frac{\partial}{\partial u_J^\alpha}.$$

In particular, if $R \in \mathscr{A}$ is any differential function,

$$\operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}(R) = \sum_{\alpha, J} \frac{\partial R}{\partial u_J^\alpha} D_J \left(\sum_\beta \mathscr{D}_{\alpha\beta} \theta^\beta \right)$$

is a vertical uni-vector. For example, in the case of the second Hamiltonian operator for the Korteweg–de Vries equation, we have

$$\operatorname{pr} \mathbf{v}_{\mathscr{E}\theta}(u) = \mathscr{E}\theta = \theta_{xxx} + \tfrac{2}{3}u\theta_x + \tfrac{1}{3}u_x\theta,$$

$$\operatorname{pr} \mathbf{v}_{\mathscr{E}\theta}(u_x) = D_x(\mathscr{E}\theta) = \theta_{xxxx} + \tfrac{2}{3}u\theta_{xx} + u_x\theta_x + \tfrac{1}{3}u_{xx}\theta,$$

and so on.

Secondly, we can let $\operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}$ act on a differential operator, say $\mathscr{D}$, as a Lie derivative; the resulting object will be a differential operator whose coefficients are functional uni-vectors in that they involve θ_J^α's. For example,

$$\begin{aligned} \operatorname{pr} \mathbf{v}_{\mathscr{E}\theta}(\mathscr{E}) &= \operatorname{pr} \mathbf{v}_{\mathscr{E}\theta}(D_x^3 + \tfrac{2}{3}uD_x + \tfrac{1}{3}u_x) \\ &= \tfrac{2}{3} \operatorname{pr} \mathbf{v}_{\mathscr{E}\theta}(u)D_x + \tfrac{1}{3} \operatorname{pr} \mathbf{v}_{\mathscr{E}\theta}(u_x) \\ &= \tfrac{2}{3}(\theta_{xxx} + \tfrac{2}{3}u\theta_x + \tfrac{1}{3}u_x\theta)D_x + \tfrac{1}{3}(\theta_{xxxx} + \tfrac{2}{3}u\theta_{xx} + u_x\theta_x + \tfrac{1}{3}u_{xx}\theta). \end{aligned}$$

Finally, we apply $\operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}(\mathscr{D})$ to θ itself by a combination of differentiating and wedging in the obvious manner. For instance, the tri-vector for the Jacobi identity corresponding to $\mathscr{E}$ is

$$\begin{aligned} &\tfrac{1}{2} \int \{\theta \wedge \operatorname{pr} \mathbf{v}_{\mathscr{E}\theta}(\mathscr{E}) \wedge \theta\}\, dx \\ &\quad = \int \{\tfrac{1}{3}\theta \wedge \theta_{xxx} \wedge \theta_x + \tfrac{2}{9}u\theta \wedge \theta_x \wedge \theta_x + \tfrac{1}{9}u_x\theta \wedge \theta \wedge \theta_x + \tfrac{1}{6}\theta \wedge \theta_{xxxx} \wedge \theta \\ &\qquad + \tfrac{1}{9}u\theta \wedge \theta_{xx} \wedge \theta + \tfrac{1}{6}u_x\theta \wedge \theta_x \wedge \theta + \tfrac{1}{18}u_{xx}\theta \wedge \theta \wedge \theta\}\, dx \\ &\quad = -\tfrac{1}{3} \int \{\theta \wedge \theta_x \wedge \theta_{xxx}\}\, dx \end{aligned}$$

by the basic properties of the wedge product. This final tri-vector is also trivial, as can be seen by a simple integration by parts:

$$\begin{aligned} \int \{\theta \wedge \theta_x \wedge \theta_{xxx}\}\, dx &= -\int \{D_x(\theta \wedge \theta_x) \wedge \theta_{xx}\}\, dx \\ &= -\int \{\theta_x \wedge \theta_x \wedge \theta_{xx} + \theta \wedge \theta_{xx} \wedge \theta_{xx}\}\, dx = 0. \end{aligned}$$

Note that according to this notation, if we evaluate Ψ, as given in (7.16), on $P, Q, R \in \mathscr{A}^q$ we obtain six terms, the first two of which are

$$\frac{1}{2}\int [P \operatorname{pr} \mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q - Q \operatorname{pr} \mathbf{v}_{\mathscr{D}R}(\mathscr{D})P]\, dx.$$

By Exercise 7.12, since $\mathscr{D}$ is a skew-adjoint differential operator, so is $\operatorname{pr} \mathbf{v}_Q(\mathscr{D})$ for any evolutionary vector field $\mathbf{v}_Q$. Thus these two particular terms are equal and combine to give the first term in the Jacobi identity (7.11). Thus, as claimed above, $\langle \Psi; P, Q, R\rangle$ is the same as the left-hand side of (7.11). Moreover, using Lemma 5.85 (or, rather, its counterpart for functional multi-vectors), we see that vanishing of (7.11) is equivalent to the triviality of the tri-vector Ψ.

Proposition 7.7. *Let $\mathscr{D}$ be a skew-adjoint $q \times q$ matrix differential operator. Then $\mathscr{D}$ is Hamiltonian if and only if the functional tri-vector* (7.16) *vanishes:* $\Psi = 0$.

There is one final simplification available. Let us extend the definition of the prolonged "vector field" $\operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}$ to the space of vertical uni-vectors by setting

$$\operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}(\theta_J^\alpha) = 0$$

for all α, J and extending it to act as a derivation on vertical multi-vectors. Thus we can write the integrand of (7.16) as

$$-\operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}(\theta \wedge \mathscr{D}\theta) = \theta \wedge \operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}(\mathscr{D}) \wedge \theta, \tag{7.17}$$

the minus sign coming from the fact that we have interchanged a wedge product of θ's. Moreover, $\operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}$, being evolutionary, commutes with total differentiation:

$$\operatorname{pr} \mathbf{v}_{\mathscr{D}\theta} \cdot D_k = D_k \cdot \operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}, \qquad k = 1, \ldots, p,$$

cf. (5.19), even when it acts on vertical multi-vectors. Therefore, if $\Phi = \int \tilde{\Phi}\, dx$ is any functional k-vector, we can unambiguously define the $(k+1)$-vector

$$\operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}(\Phi) = \int \operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}(\tilde{\Phi})\, dx.$$

(See Exercise 5.52.) In particular, the bi-vector Θ determining the Poisson bracket, (7.14), can be acted on this way, and by (7.17) we find

$$\operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}(\Theta) = \frac{1}{2}\int \{\operatorname{pr} \mathbf{v}_{\mathscr{D}\theta}(\theta \wedge \mathscr{D}\theta)\}\, dx = -\Psi,$$

agrees, up to sign, with the tri-vector corresponding to the Jacobi identity. We have thus proved