Regularization, renormalization, and renormalization groups

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Chapter 1

Regularization, renormalization, and renormalization groups: relationships and epistemological aspects

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This chapter confronts renormalization used in quantum field theory and that used in critical phenomena studies in statistical mechanics or dynamical systems theory. Regularization that cures spurious divergences is distinguished from renormalization transformations allowing to compute actual physical divergences. The former generates a group, and is also encountered in singular perturbation analyses in nonlinear physics. The latter generates a semi-group, and is implemented as a flow in a space of models; its analysis, focusing on fixed points and their neighborhood, allows to determine asymptotic scaling behavior, to delineate universality classes and to assess model structural stability (or instability, i.e. crossovers). The renormalization group can be seen as a symmetry group and a general covariant formulation is proposed. Aspects presented here show that renormalization theory has emulated a shift of focus from the investigation of outcomes of a given model to the analysis of models themselves, by relating models of the same system at different scales or grouping models of different systems exhibiting the same large-scale behavior. So doing, not only (subjective and partial) models are distinguished from underlying physical systems, but also intrinsic physical features can be derived from model comparison and classification.

1.1. Introduction

A first motivation of the present chapter is to investigate the relations between renormalization techniques encountered in very different domains of theoretical physics. The use of one and the same term of "renormalization" seems confusing when comparing its contents in quantum field theory (QFT) with renormalization methods exploited for studying critical phe-

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nomena in statistical mechanics or the onset of chaos in dynamical systems theory. They respectively involve cutoffs or local averages, without or with rescaling, a single transformation or a transformation iterated to infinity with relevant large-scale information lying in the limiting behavior of the ensuing flow in a space of models[a]. The approach is perturbative in QFT whereas it is non perturbative in statistical mechanics and dynamical systems theory. The associated 'renormalization group' (RG) has a different structure, being respectively a group (in general non commutative) or a semi-group (in general commutative). Finally, the divergences asking for renormalization differ in nature, being in short either spurious ultra-violet (UV) divergences of the theory (to be cured) or actual infra-red (IR) divergences observed experimentally (to be computed); they come respectively from unphysical features of the theory at very small scales or from the presence of fluctuations and correlations at all scales that build up an anomalous macroscopic behavior. I here propose to distinguish:
— renormalization understood in the sense of *regularization*, used mainly in quantum field theory to cure UV divergences,
— renormalization implemented as a *renormalization flow*, used to investigate critical phenomena and universal scaling properties.

But there are also conceptual and historical links between the different renormalization approaches, e.g. they all rely on the computation of effective ('renormalized') parameters and exhibit a group structure related to some scale invariance property. The relation between QFT and critical phenomena and their respective renormalizations has been already discussed by K.G. Wilson,[51] after he had successfully applied renormalization to understand Kondo effect (effect of a magnetic impurity in a non magnetic metal). I shall discuss in this paper several additional points:
– the distinction between regularization and renormalization flow (§ 1.2);
– QFT-inspired renormalization to manage with singular perturbations and singular dependence in initial or boundary conditions (§ 1.3);
– principles of real-space RG generating a flow in a space of models (§ 1.4);
– RG flow seen as a 'generalized dynamical system' where the rescaling factor (or number of iterations) plays the role of time; its analysis around renormalization fixed points allows to compute the critical exponents. Remarkably, its global analysis achieves a partition of the space of models into universality classes, identifies the possible origins of crossovers, and provides a systematic way for investigating model structural stability (§ 1.5);

[a]Note that one speaks of a 'theory' in QFT and of a 'model' in statistical mechanics.

– mathematical structures of RG (e.g. Lie group and symmetry group), that I propose to extend into a general covariant formulation (§ 1.6);
– change of scope associated with RG, shifting the investigations from the phase space of a given model to a space of models (§ 1.7).

1.2. Renormalization as a way of regularization

1.2.1. *Historical background*

The first meaning of renormalization, encountered in hydrodynamics more than a century ago, refers to a method in which some indirect, external or spurious effects are dealt with through a redefinition of the parameters of the model. This idea originates in the renormalization of the mass of a moving body into a moving fluid.[9,17] The adjective *"renormalized"* here describes a coefficient or a quantity computed by adding to its actual value (*'bare'* value) in the initial model $\mathcal{M}$ of the system[b] $\mathcal{S}$ some contributions of the same dimension, coming from additional phenomena taking place in $\mathcal{S}$ or from interactions between $\mathcal{S}$ and its environment.

A typical instance where this basic renormalization is fruitful is the case where some parameters of the initial theoretical model have no experimental reality: their values are affected by the inadequacies of the model (e.g. they diverge) due to an unavoidable lack of information on small scale mechanisms, and they should be replaced by effective ones in order to reproduce correctly the observed reality: the measured values correspond to these effective parameters. An acknowledged example is the renormalization of the mass of the classical electron. In classical electrodynamics, the electromagnetic mass m_∞ is defined by $m_\infty c^2 = U$ where U is the total electromagnetic energy of the electron; m_∞ diverges in the limit where the electron is described as a point-like charge[c]. Renormalization here appears as a formal

[b] We here meet a leitmotiv of the present chapter, that renormalization methods underlined and exploited in a deep way: the careful distinction between the system $\mathcal{S}$ and its models, observer-dependent insofar as constructing a model $\mathcal{M}$ requires to choose a theoretical framework, hence to introduce minimum scales under which another theory has to be used.[1]

[c] An intuitive understanding of the problem can be grasped by considering the model where the charge Q is described as a homogeneous distribution in a sphere of radius a, hence involving a uniform charge density $\rho_a(\vec{r}) = \rho_a$ if $r < a$ else 0, such that $Q = 4\pi\rho_a a^3/3$. Letting the scale $a \to 0$ corresponds to a mathematical idealization extending artificially the model beyond its natural range of validity, from which the divergence of ρ_a as $a \to 0$ originates. The mathematical theory of 'generalized functions' has been precisely devised by L. Schwartz and others to handle such singular situations: the charge density is properly written as a distribution $\rho_{a\to 0}(\vec{r}) = Q\delta(\vec{r})$. What makes

operation aiming at reestablishing the operational validity of the point-like model. It consists in compensating the divergence of m_∞ by an infinite mass m' in order to obtain an effective finite mass $m_{eff} = m_\infty - m'$, giving the experimental value of the mass. m_{eff} can rather be termed a *regularized* quantity in which the spurious divergence due to the model inadequacy at infinitely small scales has been discarded.

1.2.2. *Ultra-violet divergences and cutoffs*

In QFT, renormalization mainly refers to a procedure intending to eliminate the so-called "*UV divergences*" arising in perturbative calculations with respect to interaction terms (the zero order being the free-field theory). The adjective "ultra-violet" refers to high-energy and large-momentum domain in the conjugate space, corresponding to structures and mechanisms with short characteristic lengths and times. As in the above examples, § 1.2.1, the origin of UV divergences is the inadequacy of the explicit microscopic description that encompasses an unlimited number of interactions of arbitrarily small scale and high energy (e.g. arbitrarily fast creation and annihilation of electron-positon pairs[d]). The occurrence of ultra-violet divergences thus appears as a feature of the *theory*. Regularization intends to compensate the unphysical infinite contributions in perturbative expansions by adequately modifying the initial model, so as to obtain finite contributions and, all things considered, a regular model.

A standard way to implement regularization is to introduce an *UV cutoff* $\Lambda = (\Lambda_q, \Lambda_\omega)$ and to ignore large-momentum and large-frequency components with $q \equiv ||\vec{q}|| > \Lambda_q$ and $|\omega| > \Lambda_\omega$ (those producing unphysical divergences). Mechanisms involving these components, i.e. high energy processes, are no longer described explicitly but taken into account by introducing additional low-momentum and low-frequency contributions. By construction, the effective model thus obtained has only components with $q \leq \Lambda_q$ and $|\omega| \leq \Lambda_\omega$ (hence no longer UV divergences) and it induces the same observable consequences as the initial model; this is the very prescription defining what the additional terms should be, the obvious difficulty being to explicitly determine the terms that fulfill this requirement.

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sense physically is not the infinite value of $\rho_{a\to 0}$ at the point-charge location $\vec{r} = 0$ but the integrated distribution over an observation sphere of radius r_0 since $\int_{r<r_0} \rho_{a\to 0}(\vec{r}) d\vec{r} = Q$ (independently of r_0, that plays the role of a cutoff, as in § 1.2.2).

[d] A similar difficulty would arise in chemistry, associated with the lack of knowledge about elementary reactions, involving transient species of arbitrarily short lifetimes.

UV cutoffs are here used to circumvent ill-defined small-scale mechanisms and stick to a well-controlled effective theory; we shall see in § 1.3.2 that they are also useful to reduce the number of microscopic degrees of freedom (replaced by a smaller number of effective degrees of freedom) and are centrally involved in renormalization-group studies in statistical physics.

1.2.3. *Renormalizability*

In quantum electrodynamics, a theory is said to be *renormalizable* if its regularization, that is, adding effective terms to counterbalance apparent divergences due to the inadequacy of the theory at high energies, amounts to a modification of the coupling constants (charges) and masses into "renormalized" ones.[6] When regularization is implemented with an UV cutoff, renormalizability means that the overall influence of the modes left aside by the cutoff can be reproduced by changing the parameters which control the equations for the remaining modes. Renormalizability appears as a kind of internal consistency of the theory, as its space-time scales can be bounded from below without increasing the dimension of the parameter space. It is still discussed whether renormalizability has to be prescribed as a criterion of validity when constructing a quantum field theory.

1.2.4. *Emergence of the notion of renormalization group*

The so-called *renormalization group* has been introduced in 1953 by Stueckelberg and Petermann[49] to relate one to another the different regularized models $\mathcal{M}_\Lambda$ obtained with different UV cutoffs Λ. One may indeed introduce a transformation $\mathcal{R}_{\Lambda_2,\Lambda_1}$ relating the regularized models $\mathcal{M}_{\Lambda_1}$ and $\mathcal{M}_{\Lambda_2}$ according to $\mathcal{M}_{\Lambda_2} = \mathcal{R}_{\Lambda_2,\Lambda_1}[\mathcal{M}_{\Lambda_1}]$. In other words, the various possible regularizations $[\mathcal{R}_{\Lambda,\infty}]_{\Lambda>0}$, each associated with a cutoff parameter Λ, are formally related through $\mathcal{R}_{\Lambda_2,\infty} = \mathcal{R}_{\Lambda_2,\Lambda_1} \circ \mathcal{R}_{\Lambda_1,\infty}$. Obvious transitivity (the models are regularized versions of the same theory) implies that the set of operators $\mathcal{R}$ has a group-theoretic structure $\mathcal{R}_{\Lambda_3,\Lambda_2} \circ \mathcal{R}_{\Lambda_2,\Lambda_1} = \mathcal{R}_{\Lambda_3,\Lambda_1}$ which explains the term "renormalization group". In case when the transformation $\mathcal{R}_{\Lambda_2,\Lambda_1}$ depends only on the ratio[e] Λ_1/Λ_2, namely $\mathcal{R}_{\Lambda_2,\Lambda_1} = \widetilde{\mathcal{R}}_{\Lambda_1/\Lambda_2}$, the constructive relation $\widetilde{\mathcal{R}}_k[\mathcal{M}_\Lambda] = \mathcal{M}_{\Lambda/k}$ implies the simpler group-theoretic relation $\widetilde{\mathcal{R}}_{k_1} \circ \widetilde{\mathcal{R}}_{k_2} = \widetilde{\mathcal{R}}_{k_1 k_2}$.

Soon after, a similar method was exploited by Gell-Mann and Low[28] to

[e]For simplicity, we here use a shortcut in the notation: being a bound on 4-vectors, Λ hence k have 4 components, and the ratio Λ_1/Λ_2 should be considered component-wise.

study the behavior of quantum electrodynamics at small space-time scales. They first pointed out the relevance of using different effective parameters according to the scale κ of the description, which led them to introduce apparent charges $(e_\kappa)_\kappa$ for the electron. Again, the breakthrough was to relate one to another the different values $(e_\kappa)_\kappa$, more precisely their square $E_Q = (e_{\kappa=\sqrt{Q}})^2$, by a *renormalization transformation*:

$$R_{k,Q}[E_Q] = E_{Q/k} \tag{1.1}$$

A non-commutative group-like structure shows up in the relation

$$R_{k_2,Q/k_1} \circ R_{k_1,Q} = R_{k_1 k_2,Q} \tag{1.2}$$

Commutativity fails as $R_{k_2,Q/k_1} \circ R_{k_1,Q} = R_{k_1,Q/k_2} \circ R_{k_2,Q} \neq R_{k_1,Q} \circ R_{k_2,Q/k_1}$ (in fact, the right-hand-side is not defined, and the structure is not a group but a groupoid in which the internal composition law is not total). Such a renormalization group is said to be *non-autonomous* because the associated renormalization equation, obtained by differentiating the relation $R_{k,Q}[E_Q] = E_{Q/k}$ with respect to k and taking $k = 1$, appears to be a non-autonomous dynamical system:

$$Q\ \left(\frac{dE_Q}{dQ}\right)(Q) = -\ \left(\frac{dR_{k,Q}}{dk}\right)_{(k=1)}\ [E_Q] \equiv \psi\left(E_Q, \frac{m^2}{Q}\right) \tag{1.3}$$

where m is the mass of the electron, or equivalently:

$$\kappa^2\ \frac{de_\kappa^2}{d\kappa^2}\ =\ \psi\left(e_\kappa^2, \frac{m^2}{\kappa^2}\right) \tag{1.4}$$

(note that the right-hand-side depends explicitly on κ). The difficult step is to compute the function ψ from the quantum electrodynamic description of the electron. ψ contains the information required to determine the correspondence $\kappa \mapsto e_\kappa$. The limiting case e_∞ corresponds to the bare charge that appears in the initial theory; it does not correspond to a physical (i.e. measurable) quantity because the electron cannot be removed from the electromagnetic field and its charge measured in a bare situation. By contrast, any renormalized charge e_κ is a physical quantity corresponding to the charge measured at a κ-dependent energy level.

1.2.5. *Renormalization equations*

Scale consistency of the regularization expresses in the renormalization group-structure (1.2) or equivalently in its differential form (1.3-1.4) called

a *renormalization equation*. It ensures basically that the arbitrary cut-off choice has no impact on the investigated physical features and in fact contains all the information to determine them. A change of notation $R_{k,Q}[E_Q] \equiv \bar{g}(k, Q, E_Q)$ translates the group relation (1.2) into the equations derived by Bogoliubov and Shirkov:[3]

$$\bar{g}(x, y, g) = \bar{g}[x/k, y/k, \bar{g}(k, y, g)] \tag{1.5}$$

expressing the functional self-similarity of the regularization. Derivation with respect to k in $k = 1$ yields the so-called Callan-Symanzik equations:[46]

$$\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \beta(y, g)\frac{\partial}{\partial g}\right]\bar{g}(x, y, g) = 0 \tag{1.6}$$

where

$$\beta(y, g) = \frac{\partial \bar{g}}{\partial x}(x = 1, y, g) \tag{1.7}$$

These first advances, now fifty years old, have been followed by a huge amount of developments, giving rise to new issues and new viewpoints on renormalization, for which I refer, among others, to papers by Rivasseau,[44] Ebrahimi-Fard and Kreimer[20] or Connes,[14] and a book of proceedings edited by Duplantier and Rivasseau.[18]

1.3. Renormalization to compute singular limits

1.3.1. *Renormalization to handle non-commuting limits*

A common feature of all renormalization techniques is to deal with *singular limits*. In QFT, an ill-defined small-scale limit $\lim_{a\to 0} m_\infty(a) = \infty$ is regularized, currently using a cutoff Λ, so that $m_\Lambda(a)$ remains finite (and independent of the scale a) as $a \to 0$; one would like here to compute $\lim_{\Lambda\to 0} m_\Lambda$. In statistical physics, the issue is to compute the temperature dependence of a thermodynamic limit[f] (size $N \to \infty$ at constant density) and its critical behavior $\lim_{T\to T_c}\lim_{N\to\infty}$ (differing from $\lim_{N\to\infty}\lim_{T\to T_c}$). In dynamical systems theory, similar non commuting limits arise in computing the parameter μ dependence of the asymptotic behavior (infinite time) and its bifurcating behavior $\lim_{\mu\to\mu_c}\lim_{t\to\infty}$.

The idea of renormalization is to evidence an invariant behavior when the limits are performed *jointly*, the parameter being gradually renormalized as $1/\Lambda$, N or t goes to infinity. The unique way to get a non trivial

[f]Indeed, Lee-Yang theorem assessed that phase transitions are well-defined only in the thermodynamic limit (as free energy singularities) and have no rigorous meaning in a finite system.[36,54]

limiting behavior expresses through renormalization equation and the point is that it contains all the desired information on the investigated behavior.

1.3.2. *Renormalization approach to singular perturbations*

Singular perturbations refer to situations depending on a small dimensionless parameter ϵ in such a way that the behavior for $\epsilon = 0$ differs qualitatively from the limiting behavior $\epsilon \to 0$. Acknowledged examples are anharmonic oscillators (Duffing and Van der Pol oscillators) where the damping goes to 0 with ϵ, hydrodynamics in the inviscid limit (viscosity $\nu \sim \epsilon \to 0$), differential equations where ϵ is a factor in front of the highest-order derivative, or secular divergences (originally in celestial mechanics) where a resonance between the perturbation and the unperturbed system builds up a slow unbounded drift (we here face again non commuting limits $\lim_{\epsilon\to 0}\lim_{t\to\infty} \neq \lim_{t\to\infty}\lim_{\epsilon\to 0}$). As in critical phenomena, divergences arise at large scales and correspond to a real feature; but perturbation series asking for regularization are quite similar to QFT divergences issues.

Let us denote θ a control parameter and investigate the θ-dependence of the asymptotic behavior of the solution $f(\epsilon, \theta, t)$ of an ϵ-dependent dynamical system. The singularity reflects in the lack of uniform convergence of the straightforward perturbation series:

$$f(\epsilon, \theta, t) = \sum_{n=0}^{\infty} \epsilon^n f_n(\theta, t) \tag{1.8}$$

for instance $\lim_{t\to\infty} f_n(\theta, t) = \infty$ at fixed n and θ. It means that the perturbation, although of small amplitude and controlled over any finite time, is amplified and propagated up to have macroscopic non perturbative consequences, that makes the ϵ-expansion inconsistent at long times: the relative order of the successive terms is not preserved along the time. It is thus impossible to investigate the asymptotic behavior $t \to \infty$ using the plain expansion (1.8). The principle of the renormalization approach is to integrate diverging contributions in a reparametrization:

$$f(\epsilon, \theta, t) = g[\epsilon, \Theta(\epsilon, \theta, t), t] \tag{1.9}$$

where the renormalized parameter $\Theta(\epsilon, \theta, t)$ is such that the expansion of $g(\epsilon, \Theta, t)$ in powers of ϵ is uniformly convergent with respect to Θ and t. This prescription requires a solubility condition, expressing the multi-scale consistency of the renormalized perturbation expansion, to be satisfied: the fast components should actually be fast, meaning that they have no slowly

evolving consequences, and slow modes should involve slow variables only. In this way, the asymptotic behavior can be faithfully captured at finite orders of the ϵ-expansion of $g(\epsilon, \Theta, t)$.[10] The solubility condition appears as a renormalization equation, thus bridging this approach with QFT renormalization (§ 1.2.5). Other common features with QFT regularization are that a single step $\theta \to \Theta$ will be in general sufficient, no rescaling is involved, and the procedure is not intended to be iterated to infinity.

1.3.3. *Singular dependence to initial conditions*

A formal identity can be drawn between the QFT renormalization equations, § 1.2.5, and the regularization of the solution $f(t; t_0, f_0)$ of a differential equation that depends in a singular way on the initial condition (f_0, t_0) with $f(t_0; t_0, f_0) = f_0$. Namely, f_0 tends to ∞ as one tries to let $t_0 \to -\infty$: some cutoff or regularization is thus required, as in QFT.

Let us note $z = e^t$, $z_0 = e^{t_0}$ and $f(t; t_0, f_0) = F(z; z_0, f_0)$. The group structure of the solution (a temporal flow) implies that

$$F(z; z_0, f_0) = F[z; z_1, F(z_1; z_0, f_0)] \tag{1.10}$$

When $F(z; z_0, f_0) = g(z/z_0, f_0)$, it comes $g(x, f_0) = g[x/k, g(k, f_0)]$ to be compared to the homogeneous (y-independent) version of (1.6). The same functional self-similarity is observed, expressing here the consistency relation between the initial conditions at different times required to describe one and the same solution. It will be exploited to replace the singular initial condition in $z_0 = 0$ by a finite one at some $z_1 > 0$. This intermediary z_1 plays the role of a cutoff and its specific value is of no importance provided the corresponding initial condition is chosen according to (1.10), namely $f_1 = F(z_1; z_0, f_0)$. This procedure for regularizing a singular dependence to initial condition has been successfully applied to anomalous diffusion equation[29] and several other partial differential equations.[11,30] Let us underline that renormalization is here applied only once, and to the solution itself; it thus strongly differs from the RG procedure applied to the evolution rules that will be discussed in the next section, § 1.4.2.

1.4. Renormalization of critical phenomena

1.4.1. *From QFT to critical phenomena*

In the context of critical phenomena, RG approaches have been developed to compute large-scale behaviors, singular insofar as scaling exponents com-

puted by mean-field methods dramatically differ from the measured ones; one also speaks of *anomalous* exponents. The discrepancy clearly originates in the very notion of *criticality*: the system exhibits long-range correlations; accordingly, the slightest fluctuations are able to have a macroscopic impact and never self-average out.[29,37] Denoting ξ the correlation length and a a typical molecular length scale, all the fluctuations in the range $[a, \xi]$ matter (with $\xi \to \infty$ at the critical point whereas ξ is of the same order than a far away from the critical point). Experimental signatures are the divergences at the critical temperature T_c of susceptibilities, transport coefficients and other response functions, and singularities at $T = T_c$ of thermodynamic functions, e.g. free energy or order parameter; in the critical region where $\xi(t) \gg a$, all these quantities satisfy scaling laws with respect to $T - T_c$ with anomalous exponents.[38]

The conceptual unity of RG across different domains of theoretical physics has already been underlined, by bridging RG principles and techniques between QFT and statistical physics[25,52,55] but also between statistical physics and dynamical systems (analogy between spatial and temporal settings)[29,37] and even probability theory, understanding criticality as a statistical pathology departing from the central limit theorem range of validity.[10,34] Focusing here on the passage from QFT to statistical physics, several common features can be underlined, mainly UV cutoffs and the computation of effective parameters. But there are also deep differences, already mentioned in the introduction, § 1.1, that we shall now detail; they are mainly related to the systematic exploitation of the renormalization (semi)group-structure insofar as it generates a flow in the space of models.

1.4.2. *Principles of renormalization-group analysis*

The issue in critical phenomena studies is to account for the macroscopic impact of fluctuations at all scales. Renormalization proceeds by dividing the correlation range into subranges that can be recursively managed with by means of *coarse-graining*[g], that is, by integrating out the smallest-scale degrees of freedom and interactions, and capturing their overall influence

[g]In one case, namely fully developed turbulence driven by an energy input at a macroscopic scale, the relevant renormalization scheme is to integrate out the influence of large scale onto smaller ones, traveling the Richardson cascade downwards, down to dissipation scale; indeed, criticality here appears when the dissipation scale tends to 0, since then small-scale singularities develop in finite time, whereas the macroscopic scales are prescribed in the setting, e.g. the size of an obstacle or the scale at which energy is injected. This unique situation is known as *inverse renormalization-group*.[27]

through effective terms at higher scales. When the model is expressed in the conjugate space, a coarse-graining corresponds to a UV cutoff integrating out modes with large wave vectors, as in QFT. However, it is usually assumed[h] in (classical) statistical physics that the model exhibits a minimal length scale $a = 2\pi/\Lambda_q$, e.g. the cell size in spin lattice models or more generally some molecular length scale. Contrary to regularizing cutoffs in QFT, cutoffs involved in statistical physics are *finite ones*, in the sense that they integrate out modes in a finite window $\Lambda_q/k < q \leq \Lambda_q$ for some scale factor $k > 1$. In real space, coarse-graining is implemented through a local average over cells of linear size ka. Considering the coarse-grained cells as the elementary units in the renormalized description reduces the number N of elements by a factor k^d where d is the space dimension.

Another departure from QFT renormalization is to supplement the coarse-graining (equivalently the cutoff) with *rescalings* intending to make the renormalized configurations the most similar possible to the original ones. The lattice cell size a (equivalently the upper bound Λ_q) is preserved by means of a rescaling of all lengths by the factor k. Other rescalings, in the first place of the local state variable $\vec{s}$ (e.g. the spin value in case of spin lattice models) might be needed to preserve physical invariants.

The central step of renormalization is to describe how the model should change when we change the description level in order to correspond to the *same real system*. It expresses as a transformation $\phi \to \mathcal{R}_k\phi$ of the model rule ϕ, that is a "microscopic" state function (defined on the microscopic phase space of the model) which thoroughly determines the microscopic behavior: think of a reduced Hamiltonian $\phi = \mathcal{H}/k_BT$ where T is the temperature and k_B the Boltzmann constant; other examples of model rules are the action in a quantum field theory, the evolution map for a discrete dynamical system, the vector-field for a continuous dynamical system, or the transition probabilities for a diffusion process. In summary (details and case studies can be found in literature[8,10,22,23,29,37,52]) the renormalization procedure acts *jointly* on several quantities:

$N \to Nk^{-d}$	which defines the rescaling factor k,
$\vec{r} \to \vec{r}/k$	in the real space,
$\vec{q} \to k\vec{q}$	in the conjugate space,
$\vec{s} \to \vec{s}/k^w$	(suitable rescaling to preserve physical invariants),
$\phi \to \mathcal{R}_k\phi$	(renormalization operator) with $\mathcal{R}_{k_1} \circ \mathcal{R}_{k_2} = \mathcal{R}_{k_1 k_2}$.

[h]We mean that investigations start directly with a lattice or finite-resolution model, without working out its derivation from more fundamental (e.g. quantum) physics.

It makes sense to compare the initial model with the renormalized one since they have the same (apparent) resolution. A single renormalization is already fruitful since it achieves dimensional reduction, as well as regularization insofar as the correlation length ξ is reduced due to space rescaling: $\xi(\mathcal{R}_k\phi) = \xi(\phi)/k$ hence the model is less critical. But renormalization becomes really fruitful when iterated. A renormalized model of size N (e.g. with N elements) corresponds to an original model of size $N_0 = k^d N$: the larger the original system, the more times its model at a given scale a could be renormalized. At fixed N, $\lim_{k\to\infty}(\mathcal{R}_k\phi)(N)$ describes the thermodynamic limit of the model ruled by ϕ. This idea of *iterated* coarse-grainings has been proposed by Kadanoff,[35] and it amounts to consider the degrees of freedom not all at once but hierarchically and recursively. In case of critical phenomena, renormalization should be iterated indefinitely to take into account fluctuations and correlations at all scales.

As can be guessed from the equivalence between coarse-grainings and cutoffs, renormalization can be implemented either in real space or in conjugate space. Real-space RG[7] is intuitively meaningful hence easier to devise; it is preferable for systems where real-space geometry plays a central role, e.g. for percolation and polymer conformation studies.[16] RG implemented in the conjugate space is technically more efficient; in particular, the space dimension d appears as a numerical parameter, that allows to implement a perturbation approach with respect to $\epsilon = 4 - d$ (critical exponents take their mean-field values for[i] $d > d_c = 4$), the so-called ϵ-expansion.[53]

Renormalization can also be exploited to investigate critical dynamics, replacing coarse-graining in real space by a local time averaging over k time steps, while the flow analysis follows the same steps. RG methods have for instance been very fruitful to investigate universal scaling at the onset of chaos in the period-doubling scenario.[13,15,21,37] Spatio-temporal extension as well has been developed to investigate dynamic critical phenomena.[32] RG can also be seen as an iterated multiscale approach allowing to derive in a rigorous, systematic and constructive way effective low-dimensional equations describing large-scale behavior of the system under consideration; an example is the derivation of Navier-Stokes hydrodynamic equations from Boltzmann kinetic equation.[10,31]

[i]For $d = d_c = 4$, the critical exponents still take their mean-field values but additional logarithmic terms slightly modify the scaling laws.

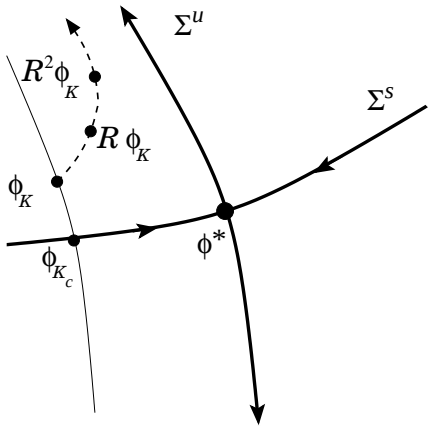

Fig. 1.1. Renormalization flow around an hyperbolic fixed-point ϕ^* in the space of model rules $\{\phi\}$. The critical manifold coincides with the stable manifold Σ^s. The critical scaling of a one-parameter family $(\phi_K)_K$ is obtained by determining the critical point K_c such that $\phi_{K_c} \in \Sigma^s$ and identifying the renormalization action on the family with its action along the unstable manifold Σ^u. A universality class is thus associated with each hyperbolic fixed point. Relevant directions are the unstable ones (along Σ^u) while contributions to ϕ along Σ^s have no large-scale consequences.

1.4.3. *RG computation of critical exponents*

Let us now unravel the 'magic of the renormalization-group'[26] leading from general principles to numerical values for the critical exponents. Having devised the renormalization operator $\mathcal{R}_k$, the next step is the determination of its fixed points ϕ^* satisfying the fixed-point equation $\mathcal{R}_k\phi^* = \phi^*$ and corresponding to exact self-similarity. Since $\xi[\mathcal{R}_k(\phi)] = \xi(\phi)/k$, the correlation length $\xi^* = \xi(\phi^*)$ satisfies $\xi^* = \xi^*/k$ hence either $\xi^* = 0$ (trivial fixed point corresponding to an uncorrelated system) or $\xi^* = \infty$, associated with criticality. Critical fixed points are those having moreover both stable and unstable directions ('hyperbolic' fixed points). The operators $\mathcal{R}_k$ are precisely designed to extract the universal large-scale behavior of a critical system. Each renormalization eliminates the specific small-scale details in order to highlight the self-similar properties common to all the models of the same universality class: all models converging to the same fixed point ϕ^* upon the action of $\mathcal{R}_k$ exhibit the same large-scale features: those of ϕ^*. The stable manifold Σ^s of ϕ^* is thus the location of the critical systems:

$$\forall\ \phi \in \Sigma^s, \qquad \lim_{n\to\infty} \mathcal{R}_{k_0}^n \phi = \lim_{k\to\infty} \mathcal{R}_k \phi = \phi^* \tag{1.11}$$

To grasp the principle of RG computation of critical exponents, we shall consider a model rule ϕ_K depending on a control parameter K, and assume (at the cost of some approximations to be checked afterwards) that renormalization action can be transferred onto a transformation $K \to r_k(K)$, i.e. $\mathcal{R}_k\phi_K \approx \phi_{r_k(K)}$. The critical point is determined by $r_k(K_c) \approx K_c$, with $\phi_{K_c} \in \Sigma^s$ (see Fig. 1.1). Plugging the investigated scaling law $\xi(\phi_K) \sim |K - K_c|^{-\nu}$ for the correlation length into the equation $\xi(\mathcal{R}_k\phi_K) = \xi(\phi_K)/k$ expressing the renormalization action on ξ yields $|r_k(K) - K_c|^{-\nu} \sim |K - K_c|^{-\nu}/k$. Then, the first order approximation $r_k(K) - K_c \approx r_k(K) - r_k(K_c) \approx r'_k(K_c)[K - K_c]$ yields $k = |r'_k(K_c)|^\nu$ from which follows the value of the exponent:

$$\nu = \frac{\ln k}{\ln |r'_k(K_c)|} \tag{1.12}$$

The next and last step is to show that $|r'_k(K_c)|$ coincides with the largest eigenvalue of $D\mathcal{R}_k(\phi^*)$. Expanding $\mathcal{R}_k\phi_K \approx \phi_{r_k(K)}$ around $K = K_c$ yields:

$$D\mathcal{R}_k(\phi_{K_c}) \,.\, \frac{\partial \phi_K}{\partial K}(K_c) \,.\, (K - K_c) \approx \frac{\partial \phi_K}{\partial K}(K_c) \,.\, r'_k(K_c) \,.\, (K - K_c) \tag{1.13}$$

Provided ϕ_{K_c} is enough close to ϕ^*, this equation reduces to:

$$D\mathcal{R}_k(\phi^*) \,.\, \frac{\partial \phi_K}{\partial K}(K_c) \approx \frac{\partial \phi_K}{\partial K}(K_c) \,.\, r'_k(K_c) \tag{1.14}$$

Applying the projection P^u onto the unstable direction of $D\mathcal{R}_k(\phi^*)$ to this equation evidences the 'transversality condition' $P^u\,[\frac{\partial \phi_K}{\partial K}(K_c)] \neq 0$ required for the validity of the analysis and finally yields the equality $r'_k(K_c) = k^{\gamma_1}$ where k^{γ_1} is the most unstable eigenvalue of $D\mathcal{R}_k(\phi^*)$.

In case when the renormalization operator involves $m > 1$ rescaling factors $(k_1, \ldots, k_m)$, there is in general a unique one-parameter family $\bar{k} = (k^{\alpha_1}, k^{\alpha_2}, \ldots, k^{\alpha_m})$ leading to a non trivial limit as $k \to \infty$, and this very fact brings a lot of information on the scaling exponents describing the asymptotic behavior. Let us consider the example of diffusion processes. Aiming at determining the leading behavior at large scales of a random walk in $\mathbf{R}^d$, renormalization procedure is defined by the action on the transition probability $p(\vec{r}, t)$ (probability density of being in $\vec{r}$ at time t when starting from the origin at time $t = 0$) of a family $[\mathcal{R}_{k,K}]_{k\geq 1, K\geq 1}$ of operators

$$[\mathcal{R}_{k,K}(p)](\vec{r}, t) \equiv k^d \; p(k\vec{r}, Kt) \tag{1.15}$$

The mean-square displacement $\mathcal{D}(p, t) \equiv \langle(\vec{r}(t) - \vec{r}(0))^2\rangle$ is transformed according to $k^{2n}\; \mathcal{D}[\mathcal{R}^n_{k,K}p, t] = \mathcal{D}(p, K^n t)$ after n iterations. In case of normal diffusion, only the iteration of $\mathcal{R}_{k,K=k^2}$ leads to a non trivial limit,[10,37]

yielding the asymptotic diffusion law $\mathcal{D}(p,t) \sim Dt$ (for $t \to \infty$) of the process. Starting with different assumptions on the statistics of the elementary steps (e.g. Lévy distributed), the non trivial limit is obtained with $K = k^\alpha$, yielding $\mathcal{D}(p,t) \sim |t|^{2/\alpha}$ for $t \to \infty$.

1.5. Structural stability and universality classes

1.5.1. *Renormalization as a flow in a space of models*

The above technical steps in fact lead to a huge methodological shift: instead of studying the outcome of a given model $\mathcal{M}$, for example computing the value of state functions and their variations with respect to variables and parameters changes, renormalization analysis focuses on the transformation of the model following a change in the description scale and on the *renormalization flow* it generates. In other words, renormalization defines a dynamical system in the space of model rules $\{\phi\}$: either a discrete one if only an isolated map $\mathcal{R}$ is defined and iterated, the number of iterations being the analog of (discrete) time; either a continuous one if $\mathcal{R}_k$ depends smoothly on a continuous rescaling factor k. In this latter case, denoting $\kappa = \ln k$ and $\widetilde{\mathcal{R}}_\kappa = \mathcal{R}_k$, the renormalization group structure is that of a one-parameter flow $\phi(\kappa;\kappa_0;\phi_0) = \widetilde{\mathcal{R}}_{\kappa-\kappa_0}\phi_0$. In its differential form, the dynamical system writes

$$\frac{d\phi}{d\kappa}(\kappa) = \mathcal{L}[\phi(\kappa)] \quad \text{with} \quad \mathcal{L} = \frac{d\widetilde{\mathcal{R}}_\kappa}{d\kappa}(\kappa = 0) \tag{1.16}$$

As we shall see, a trajectory represents the relation between models of the same system at different scales, whereas comparing different trajectories allows to investigate model structural stability and universality classes.

1.5.2. *Relevant and irrelevant contributions*

Due to the renormalization group structure, the eigenvectors $(h_j)_{j\geq 1}$ of $D\mathcal{R}_k(\phi^*)$ do not depend on k and the eigenvalues have the form k^γ, where $\gamma > 0$ for the unstable directions and $\gamma < 0$ for the stable ones[j]. A parametrization of the model rules is available in the neighborhood of ϕ^*:

$$\phi = \phi^* + \sum_{j=1}^{\infty} \varphi_j\, h_j + \mathcal{O}(\varphi^2) \quad \Rightarrow \quad \mathcal{R}_k\phi = \phi^* + \sum_{j=1}^{\infty} k^{\gamma_j}\, \varphi_j\, h_j + \mathcal{O}(\varphi^2) \tag{1.17}$$

[j] Complex eigenvalues would generate oscillatory corrections to scaling.

where the labeling $j \to \varphi_j$ is chosen according to the ordering $\gamma_j \geq \gamma_{j+1}$. Hence, renormalization reduces in the linear approximation to a scale transformation r_k acting component-wise according to $r_k^{(j)}(\varphi_j) = k^{\gamma_j}\,\varphi_j$. The scaling behavior of any state function A follows. Indeed, denoting k^α the rescaling factor for the values of A (e.g. $\alpha = 1$ if A is a length or $\alpha = 2w$ if A is the correlation function of the local state variable $\vec{s}$), its transformation upon renormalization writes exactly $A(N, \mathcal{R}_k\phi) = k^{-\alpha}A(Nk^d, \phi)$ and in the linear approximation

$$A[N, (\varphi_j)_{j\geq 1}] = k^\alpha\; A[Nk^{-d}, (k^{\gamma_j}\varphi_j)_{j\geq 1}] \tag{1.18}$$

The signs of the exponents $(\gamma_j)_{j\geq 1}$ are crucial. The direction associated to γ_j is said to be: *relevant* if $\gamma_j > 0$, *marginal* if $\gamma_j = 0$, *irrelevant* if $\gamma_j < 0$. Let us explain this terminology in case when $\gamma_1 > \gamma_2 > \gamma_3 = 0 > \gamma_4$. The coefficient φ_1, associated to the maximal exponent $\gamma_1 > 0$, appears to be the control parameter of the phenomenon; in particular, critical behavior is observed for $\varphi_1 \to 0$. Taking $k = \varphi_1^{-1/\gamma_1}$ (if $\varphi_1 > 0$) leads to:

$$A(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \varphi_1^{-\alpha/\gamma_1}\; A(1, \varphi_2\,\varphi_1^{-\gamma_2/\gamma_1}, \varphi_3, \varphi_4\,\varphi_1^{-\gamma_4/\gamma_1}) \tag{1.19}$$

The value of φ_4 does not play any role on the limiting behavior $\varphi_1 \to 0$, hence φ_4 is an irrelevant parameter. Typically, $A(1, z, \varphi_3, 0)$ behaves as a power law $|z|^{\mu^\pm}$ for z tending to $\pm\infty$, hence the leading behavior when $\varphi_1 \to 0^+$ writes $A(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \sim \varphi_1^{-\eta}$ where:

- if $\varphi_2 = 0$, $A(\varphi_1, 0, \varphi_3, \varphi_4) \sim \varphi_1^{\frac{-\alpha}{\gamma_1}}\; \Theta^0(\varphi_3)$ hence $\eta = \frac{\alpha}{\gamma_1}$;
- if $\varphi_2 > 0$, $A(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \sim \varphi_1^{\frac{-(\alpha+\gamma_2)\mu^+}{\gamma_1}}\; \varphi_2^{\mu^+}\; \Theta^+(\varphi_3)$ hence $\eta = \frac{\alpha+\gamma_2\mu^+}{\gamma_1}$;
- if $\varphi_2 < 0$, $A(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \sim \varphi_1^{\frac{-(\alpha+\gamma_2)\mu^-}{\gamma_1}}\; |\varphi_2|^{\mu^-}\,\Theta^-(\varphi_3)$ hence $\eta = \frac{\alpha+\gamma_2\mu^-}{\gamma_1}$,

where Θ^0, Θ^+ and Θ^- are regular functions of their arguments. Similar scaling laws are deduced for the various derivatives of A. Hence the relevant directions control the values of critical exponents. A *crossover* between different scaling laws is observed when φ_2 varies from negative values to positive ones, passing through $\varphi_2 = 0$ (other instances of crossover are investigated in the next subsection). The marginal directions do not affect the exponent values but intervene in the prefactors. The irrelevant directions have no influence at all on the leading scaling behavior; they only provide corrections to the asymptotic scaling behavior. Additional corrections come from the nonlinear terms in the renormalization action.

Let us illustrate the above discussion with the ferromagnetic transition: we have $\varphi_1 = T_c - T$ and $\varphi_2 = h$ (magnetic field). Taking for A the magnetization m, we have in zero field $m(T, h = 0) \sim (T_c - T)^\beta$ with $\beta = -\alpha/\gamma_1$ (here $\alpha < 0$ hence $\beta > 0$). In non-vanishing magnetic field, the behavior of the magnetization with respect to $T_c - T$ is regular: this corresponds to $\mu^\pm = -\alpha/\gamma_2$, so that $m(T = T_c, h) \sim |h|^\delta$ with $\delta = -\alpha/\gamma_2$ (hence $\delta > 0$). Taking now for A the correlation length, with $\alpha = 1$, we get $\xi(T, h = 0) \sim |T - T_c|^{-\nu}$ with $\nu = 1/\gamma_1$: we recover the expression (1.12), namely $\nu = \ln k / \ln k^{\gamma_1}$.

1.5.3. *Crossovers*

A *crossover* refers to the failure to reach the scaling regime prescribed by the action of $\mathcal{R}$ along the most unstable direction of the critical fixed point ϕ^*. Different origins can be distinguished, that will put forward related notions: crossover exponent, crossover scale and crossover parameter.

(i) The underlying system is of finite size L that prevents it to reach the asymptotic universal regime. We then observe *finite-size scaling*: a scaling law $A(K) \sim |K - K_c|^{-\alpha}$ becomes $A(L, K) \sim L^\beta F[L|K - K_c|^\nu]$ where ν is the exponent of the correlation length $\xi(K) \sim |K - K_c|^{-\nu}$. Indeed, $\xi(K)$ is the unique other characteristic length of the system, hence a scaling argument implies that F depends on the unique variable $L/\xi(K)$. The behavior of the scaling function should be $F(z) \sim z^{-\alpha/\nu}$ as $z \to 0$ so as to recover the infinite-size scaling as $L \to \infty$, from which follows $\beta = \alpha/\nu$. In renormalization terms, the finite size of the system is directly related to the number of renormalizations that can be performed starting from the initial model, i.e. to the length of the trajectory in the space of model rules.

(ii) $D\mathcal{R}(\phi^*)$ has more than one unstable direction hence more than one relevant control parameter, say K and h (think of a ferromagnetic system where K is related to the temperature and h is the magnetic field); we then observe *bi-critical scaling*.[41] RG arguments that related critical exponents to the unstable eigenvalue of $D\mathcal{R}_k(\phi^*)$ extend to the situation where there is two unstable eigenvalues, as exposed above in § 1.5.2. Bi-critical scaling and finite-size scaling in fact express through very analogous scaling relations, the role of L being played by $1/h$ where $h = 0$ recovers the standard scaling. In the case of a ferromagnet,[29,38] the joint scaling of the magnetization with respect to the magnetic field h and the temperature difference $T - T_c$ write $m(h, T) \sim |T - T_c|^\beta M(h|T - T_c|^\Delta)$ with a finite value $M(0)$ and

$M(z \to \infty) \sim z^{-\beta/\Delta}$; in a similar way, the singular part of the free energy writes $f_{sing}(h,T) \sim |T-T_c|^{2-\alpha} F(h|T-T_c|^{\Delta})$ with $F(0)$ finite and $F(z \to \infty) \sim z^{-(2-\alpha)/\Delta}$. For $|h|.|T-T_c|^{\Delta} \ll 1$, the leading order describes the behavior with respect to T at $h=0$, e.g. $f_{sing}(h=0,T) \sim |T-T_c|^{2-\alpha}$. For $|h|.|T-T_c|^{\Delta} \gg 1$, what dominates is the behavior with respect to h at $T=T_c$, namely $m(h,T_c) \sim |h|^{-\beta/\Delta}$. The exponent Δ is called the *crossover exponent* with $\Delta = -\beta/\delta$ so as to recover the scaling law $m(h,T_c) \sim |h|^{\delta}$.

(iii) $\mathcal{R}$ has several critical fixed points, say ϕ_1^* and ϕ_3^*. In the generic case, ϕ_2^* does not belong to $\Sigma^s(\phi_1^*)$ (see Fig. 1.1) hence the fixed point that influences the trajectory upon the action of $\mathcal{R}$ depends on the initial condition ϕ but also dramatically on the number n of iterations of $\mathcal{R}$. Each fixed point describes a different form of collective behavior observed when $\mathcal{R}^n\phi$ comes close to the corresponding fixed point. The number n is directly related to the observation scale, i.e. the system or sample size L according to $k^n a \sim L$ if a is the resolution at which the initial model and its rule ϕ describe the real system. Two scaling behaviors will thus be observed according to the system size, that evidence a *crossover scale* L^* at which occurs a fixed-point dominance exchange ($L^* \sim k^{n^*}a$ such that $\mathcal{R}^{n^*}\phi$ becomes more sensitive to ϕ_2^* than to ϕ_1^*). Detailed understanding of such a crossover requires a global analysis of the renormalization flow. An example is the behavior of the sum (divided by N) of N independent random variables distributed according to a truncated Lévy distribution: at moderately large values of N, the dominant behavior is described by the anomalous central limit theorem for Lévy distributions while at very large values of N, the standard central limit theorem applies since the random variables have a finite variance.[48] We see that such a crossover might be avoided in a system of finite size $L < L^*$ where the ultimate influence of the second fixed point ϕ_2^* will never been felt.

(iv) $\mathcal{R}$ has several critical fixed points that have disjoint basins of attraction; in this case, the crossover is sharper than in (iii) and the initial location ϕ in the space of models fully determines to which critical fixed point the trajectory $\mathcal{R}^n\phi$ converges upon the renormalization action, i.e. which one determines its universality class and critical exponents. Such a crossover is observed if the rule depends on a parameter ϵ (the *crossover parameter*) determining to which basin of attraction it belongs: the system belongs to different universality classes for different values of ϵ. An example has been

proposed by Fisher,[22] considering a spin lattice with a pair Hamiltonian

$$\mathcal{H}(\vec{s},\vec{s}') = K(s_1 s_1' + s_2 s_2' + s_3 s_3') + \epsilon K(s_2 s_2' + s_3 s_3') \tag{1.20}$$

where $\vec{s} = (s_1, s_2, s_3)$. For $\epsilon = 0$, critical exponents are those of Heisenberg spins with 3 components; for $\epsilon = -1$, those of Ising spins with 1 component; for $\epsilon = \infty$, those of XY-spins with 2 components.

(v) Disorder, noise or random terms qualitatively affect the asymptotic scaling behavior:[33] the degrees of freedom associated with randomness might generate additional unstable directions or even a novel fixed-point. A well-known example is the random field Ising model where the presence of quenched randomness induces a new critical behavior.[40,45] Renormalization methods are well-suited to investigate the structural stability of the renormalization flow itself with respect to disorder or noise; a relevant framework to appreciate the relative influence of the different unstable directions is random dynamical systems and large deviations theory.[2,4,13]

1.5.4. *Mapping spaces of models: universality classes*

Strictly speaking, the basin of attraction of the fixed point ϕ^* upon renormalization action is embedded in the stable manifold $\Sigma^s(\phi^*)$ (also called the critical manifold). Its universality class is composed of one-parameter families $(\phi_K)_K$ crossing transversally the stable manifold in $K = K_c$ and exhibiting the same critical behavior for $K \to K_c$; indeed, as exposed in § 1.4.3, this behavior can be mapped onto the action of $\mathcal{R}$ along the unstable manifold $\Sigma^u(\phi^*)$. Accordingly, all real systems whose critical behavior involves a transverse one-parameter family of models will exhibit the same critical exponents related to the eigenvalues of $D\mathcal{R}(\phi^*)$. By expressing the relationship between models at different scales, the renormalized flow picture allows to describe quantitatively the actual physical divergences arising at critical points in infinite size (i.e., in the thermodynamic limit or in the asymptotic regime) and to prove their universality. The full non linear analysis, if tractable, would yield a partition of the space of models into universality classes, each associated with an ideally critical and self-similar model (a renormalization fixed point) and grouping both models at different scales of the same system (along a RG trajectory) and models at a given scale of different systems sharing the same critical behavior.

1.5.5. *Renormalization and structural stability*

By shifting the study from the phase space to a space of models, renormalization offers a way to investigate structural stability issues: two models differing in a relevant parameter will see their difference increases upon renormalization while differences in irrelevant terms are damped out, corresponding to structural stability. Renormalization can be exploited to check the robustness of a model, as regards the asymptotic scaling behavior, with respect to enlarging both the parameter space (adding new terms to the model rule ϕ) or the phase space of the model (e.g. passing from a discrete to a continuous space of states[47]). Renormalization is also efficient to appreciate the influence of disorder on phase transitions,[33,40,45] of noise on the onset of chaos,[2,13] or of a stochastic forcing term on the large-scale behavior of the solutions of partial differential equations, e.g. Landau-Ginzburg equation,[5] hydrodynamic equations (Navier-Stokes equations[24]) or equations describing the growth of an interface (Kardar-Parisi-Zhang equation[50]); the stochastic term can represent an external noise or the influence of degrees of freedom not explicitly accounted for in the model. Renormalization approaches, allowing to discriminate essential noise terms for irrelevant ones, thus offer a constructive and qualitative way to investigate the structural stability of deterministic models and the robustness of their prediction with respect to external or internal noise.

1.6. Mathematical structures of renormalization

A common feature of renormalization methods presented in § 1.3 and § 1.4 is the semi-group structure of the renormalization flow. It expresses jointly at several levels (variables, state functions, fields, model rules) and leads to technical simplifications in case of a continuous structure (Lie group). Investigating the renormalization group structure moreover evidences its relations with symmetry groups and hints at possible generalizations.

1.6.1. *Renormalization group structure and action*

Let us denote $\bar{z} = (z_1, \ldots, z_m)$ a set of *independent* extensive variables of the model, typically chosen among the numbers of particles, linear sizes, durations, maximal amplitudes, time and space variable; $\bar{k} = (k_1, \ldots, k_m)$ the associated rescaling factors (with $k_i \geq 1$); and $\mathcal{R}_{\bar{k}}$ the renormalization operator acting on the model rules ϕ. Internal consistency requires that

renormalization procedures associated with the same rescalings coincide:

$$\mathcal{R}_{\bar{k}_1} \circ \mathcal{R}_{\bar{k}_2} = \mathcal{R}_{\bar{k}_1 \times \bar{k}_2} \qquad (\times \text{ being performed pairwise}) \quad (1.21)$$

The set $\{\mathcal{R}_{\bar{k}}, k_1 \geq 1, \ldots, k_m \geq 1\}$ of renormalization operators is thus endowed with a multiplicative semi-group structure[k] homomorphic to $([1,\infty]^m, \times)$; it would be additive, homomorphic to $([0,\infty]^m, +)$ by replacing k_i with $\kappa_i = \ln k_i$. This homomorphism ensures that the renormalization group is *commutative*. Rescalings involved in $\mathcal{R}_{\bar{k}}$ should be performed so as to satisfy the group relation when iterated: for instance, rescaling a function $x \mapsto f(x)$ writes $(k \cdot f)(x) = kf(x/k)$ so that $(k_1 k_2) \cdot f = k_1 \cdot (k_2 \cdot f)$. Internal consistency also prescribes the rescaling of related variables so as to preserve physical invariants; for instance, linear size, distances and space positions are rescaled by the same factor, say k_1, whereas the number of particles is rescaled by the factor k_1^d in dimension d and the local state variable $\vec{s}$ by a factor k_1^w (see § 1.4.2). More generally, we shall denote $\Lambda_{\bar{k}}^{(x)}$ the rescaling factor acting on the quantity x, e.g. $\Lambda_{\bar{k}}^{(\vec{r})}\vec{r} = \vec{r}/k_1$ and $\Lambda_{\bar{k}}^{(N)}N = N/k_1^d$, and by definition of the variables $\bar{z}$ of reference, $\Lambda_{\bar{k}}^{(z_i)}z_i = z_i/k_i$. For any variable x, the set $[\Lambda_{\bar{k}}^{(x)}]_{\bar{k}}$ has the same multiplicative semi-group structure; in particular, in case of a single independent rescaling factor k, it simply writes $\Lambda_k^{(x)} = k^{-\alpha(x)}$ where the exponent $\alpha(x)$ depends on the physical nature of x, e.g. $\Lambda_k^{(G)} = k^{-2w}$ for the correlation function $G(\vec{r}) = \langle \vec{s}(0).\vec{s}(r) \rangle$. Physical quantities are transformed upon renormalization according to:

$$A(\mathcal{R}_{\bar{k}}\phi, \bar{z}) = \Lambda_{\bar{k}}^{(A)}\ A(\phi, \bar{k} \times \bar{z}) \quad \text{or} \quad A(\mathcal{R}_{\bar{k}}\phi, \Lambda_{\bar{k}}^{(\bar{z})}\bar{z}) = \Lambda_{\bar{k}}^{(A)}\ A(\phi, \bar{z}) \quad (1.22)$$

and more generally, $A(\mathcal{R}_{\bar{k}}\phi, \Lambda_{\bar{k}}^{(\bar{z})}\bar{z}, \Lambda_{\bar{k}}^{(\vec{r})}\vec{r}, \Lambda_{\bar{k}}^{(x)}x) = \Lambda_{\bar{k}}^{(A)}\ A(\phi, \bar{z}, \vec{r}, x)$ for a field $A(\vec{r}, x)$. These relations define a transformation $\mathbf{R}_{\bar{k}}$ according to:

$$\mathbf{R}_{\bar{k}}[A(\phi)] = A(\mathcal{R}_{\bar{k}}\phi) \quad (1.23)$$

so that $(\mathbf{R}_{\bar{k}}A)(\Lambda_{\bar{k}}^{(\bar{z})}\bar{z}, \Lambda_{\bar{k}}^{(\vec{r})}\vec{r}, \Lambda_{\bar{k}}^{(x)}x) = \Lambda_{\bar{k}}^{(A)}\ A(\bar{z}, \vec{r}, x)$. Consistency ensures an homomorphic semi-group structure $\mathbf{R}_{\bar{k}_1} \circ \mathbf{R}_{\bar{k}_2} = \mathbf{R}_{\bar{k}_1 \times \bar{k}_2}$. Invariance upon renormalization simply writes $\mathcal{R}_{\bar{k}}\phi* = \phi*$, hence

$$\mathbf{R}_{\bar{k}}A^* = A^*$$

where $A^* = A(\phi^*)$, and corresponds to self-similarity. In this framework, renormalizability means that it is possible to transfer the renormalization

[k]Regularization, § 1.2, has a group structure; by contrast, the set of renormalization transformations has generically a *semi-group* structure, since coarse-graining occurs with a loss of information and only rescaling factors $k_i \geq 1$ are relevant (or $k_i \leq 1$ for conjugate variables, e.g. $\vec{q} \to k\vec{q}$ (wave vector) whereas $\vec{r} \to \vec{r}/k$ (position) with $k \geq 1$).

$\phi \to \mathcal{R}_{\bar{k}}\phi$ of the model rule ϕ, hence of all the state variables A, onto a transformation $K \to r_{\bar{k}}(K)$ of its parameters. It comes:

$$\mathcal{R}_{\bar{k}}[\phi(K)] = \phi[r_{\bar{k}}(K)] \quad \text{and} \quad \mathbf{R}_{\bar{k}}[A(K)] = A[r_{\bar{k}}(K)] \tag{1.24}$$

Again, consistency ensures a semi-group structure $r_{\bar{k}_1} \circ r_{\bar{k}_2} = r_{\bar{k}_1 \times \bar{k}_2}$.

1.6.2. *Renormalization group as a Lie group*

Focusing on the semi-group structure is of special interest when it is a continuous semi-group endowed with a differentiable structure, namely a *Lie semi-group*. In this case, the whole analysis can be performed in the Lie algebra generated by the infinitesimal renormalization operators. Let us first consider the case of a one-dimensional Lie semi-group $(\mathcal{R}_k)_{k\geq 1}$. This semi-group and its action can be thoroughly described in terms of the properties of the *infinitesimal generator*

$$\mathcal{L} \equiv \frac{d\mathcal{R}_k}{dk}(k=1) \tag{1.25}$$

acting locally in scales. Fixed-point equations $[\forall k \geq 1, \mathcal{R}_k\phi^* = \phi^*]$ equivalently express in a compact way: $\mathcal{L}\phi^* = 0$. Linear analysis of the renormalization action around ϕ^* can be performed on $\mathcal{L}$, according to:

$$[\forall k \geq 1, \quad D\mathcal{R}_k(\phi^*)(\varphi) = k^\gamma\, \varphi\,] \quad \Longleftrightarrow \quad D\mathcal{L}(\phi^*)(\varphi) = \gamma\, \varphi \tag{1.26}$$

All the operators $[D\mathcal{R}_k(\phi^*)]_{k\geq 1}$ commute, hence have the same eigenvectors, that are also eigenvectors of $D\mathcal{L}(\phi^*)$. Due to the group structure, any eigenvalue of $D\mathcal{R}_k(\phi^*)$ is necessarily of the form k^γ, where the exponent γ is independent of k and can be determined from the spectrum of $D\mathcal{L}(\phi^*)$.

The formulas extend straightforwardly to the case when the renormalization transformation involves $m > 1$ rescaling factors $\bar{k} = (k_1, \ldots, k_m)$. The associated Lie algebra is m-dimensional and it is generated by m infinitesimal renormalization operators $\mathcal{L}_j = (\partial\mathcal{R}_{\bar{k}}/\partial k_j)(\bar{k}_0)$ where $\bar{k}_0 = (1 \ldots 1)$. It is to note that $\mathcal{R}_{\bar{k}} = \prod_{j=1}^m \mathcal{Q}_{k_j}$ where $\mathcal{Q}_{k_j} = \mathcal{R}_{1,\ldots,k_j,\ldots 1}$. These operators $(\mathcal{Q}_{k_j})_{j=1\ldots m}$ commute so that $\mathcal{L}_j = (d\mathcal{Q}_{k_j}/dk_j)(k_j = 1)$. The m operators $(\mathcal{L}_j)_{j=1\ldots m}$ commute hence have the same eigenvectors, which coincide with the eigenvectors of any $\mathcal{R}_{\bar{k}}$ being thus $\bar{k}$-independent.

Note that although the different generators $\mathcal{L}_j$ commute, the limits $k_j \to \infty$ (or at the level of the model, the limits $z_j \to \infty$) do not commute. Typically, there is a unique way of performing joint rescalings, namely a unique direction $\sum_{j=1}^m \gamma_j \mathcal{L}_j = \bar{\gamma} \cdot \bar{\mathcal{L}}$ in the multidimensional Lie algebra,

such that the iterates under renormalization action converge to a non trivial limit. The remarkable point is that this direction $\bar{\gamma} \cdot \bar{\mathcal{L}}$ gives access to the critical exponents characterizing the asymptotic behavior. Let us consider again the RG study of diffusion processes, § 1.4.3. The infinitesimal generators associated to (1.15) write:[10,37]

$$\begin{array}{ll} B_0 = (d\mathcal{R}_{1,K}/dK)(K=1) & [B_0 p](t,\vec{r}) = t(\partial_t p)(t,\vec{r}) \\ B_i = (\partial \mathcal{R}_{k,1}/\partial k_i)(k_i=1) & [B_i p](t,\vec{r}) = [(1+x_i.\nabla_i)p](t,\vec{r}) \end{array} \tag{1.27}$$

showing explicitly the commutative character of the Lie group. But the limits $t \to \infty$ and $x \to \infty$ do not commute. The renormalization iterates converge towards a non trivial limit only for a special relation $k_{x_i} = k_t^{\gamma_i}$ between the rescaling factors, corresponding to a single direction $A_t + \vec{\gamma}.A_{\vec{r}}$ in the Lie algebra, where $\gamma_i \equiv \gamma$ is the anomalous exponent $\langle \vec{r}(t)^2 \rangle \sim t^{2\gamma}$.

1.6.3. *Renormalization groups and symmetry groups*

Renormalization action and symmetries are described in the same formalism of group theory. RG is besides nothing but a special symmetry group expressing the system scale invariance. Since the renormalization operator $\mathcal{R}$ acts upon the representations of the system (that is, its models) while preserving the system itself (that is, its physical reality), it should also preserve its symmetries. In order that the original model and its renormalized versions share the same symmetries properties, the renormalization operator has to commute with all symmetry transformations that leave the system (and its model) invariant. Accordingly, symmetries appear as additional constraints, but also as a guideline in defining $\mathcal{R}$ and the relevant subspace of models in which to consider its action.

Let us note that a marginal direction of the RG is in most cases associated to a continuous symmetry $(S_a)_{a>0}$ of the system. If $(S_a)_{a>0}$ is a one-parameter[1] symmetry group (with $S_a \circ S_b = S_{ab}$) and ϕ^* a fixed point of $\mathcal{R}$, then $[\forall a > 0,\ \mathcal{R}(S_a\phi^*) = S_a\phi^*]$. Either ϕ^* is symmetric, i.e. invariant under the action of the symmetry group $[\forall a > 0, S_a\phi^* = \phi^*]$. Either its orbit under the action of the symmetry group is non trivial and defines a one-parameter family of renormalization fixed points $(S_a\phi^*)_{a>0}$, and $\partial_a[S_a\phi^*](a=1)$ is a *marginal direction* of $D\mathcal{R}(\phi^*)$ associated with the eigenvalue 1 (it corresponds to a displacement along the curve $(S_a\phi^*)_{a>0}$ of

[1]This case fully extends to continuous symmetry groups of higher dimension; the assertions that do not involve the derivative $\partial_a S_a$ remain valid for a discrete symmetry group.

fixed points in the space of models). Moreover, in this latter case, the eigenvalues of $D\mathcal{R}(S_a\phi^*)$ are obtained through the action of $DS_a(\phi^*)$ from those of $D\mathcal{R}(\phi^*)$. In other words, we get a set of fixed points and a corresponding set of eigenspaces fibered over the symmetry group.

An important correlate is the symmetry properties of universality classes. Models belonging to the same universality class are related by renormalization-group action to the same renormalization fixed-point ϕ^*. If ϕ^* is invariant under all symmetries that are preserved by renormalization, so is the universality class, meaning that either the models of the class are symmetric or their images under the action of the symmetry group belong to the same universality class. If ϕ^* is not symmetric, the image of the class under the action of a symmetry transformation S is the universality class associated with the image $S\phi^*$.

1.6.4. *Covariant formulation of renormalization*

Renormalization group can be seen as a *symmetry group*: we are going to show that RG action on state functions formally parallels the basic covariance properties of tensors upon the action of the group of rotations. The transformation upon a rotation Θ in real space of a vector field $A(.)$ writes

$$\mathbf{\Theta}A(\vec{r}) \equiv \Theta[A(\Theta^{-1}\vec{r})] \text{ or equivalently } \mathbf{\Theta}A(\Theta\vec{r}) \equiv \Theta[A(\vec{r})] \tag{1.28}$$

More generally, the transformation of a tensor field $A(.)$ of order m writes

$$(\mathbf{\Theta}A)_{i_1,\ldots,i_m}(\vec{r}) \equiv \sum_{j_1,\ldots,j_m} \Theta_{i_1 j_1}\ldots\Theta_{i_m j_m} A_{j_1,\ldots,j_m}(\Theta^{-1}\vec{r}) \tag{1.29}$$

We have seen in § 1.6.1 a similar consistency requirement between the transformation upon renormalization of variables (e.g. space position $\vec{r}$ and size N) state functions or fields, and model rule ϕ, leading to define for each quantity A a transformation $\mathbf{R}_{\bar{k}}$ such that

$$\mathbf{R}_{\bar{k}}[A(\phi)] = A(\mathcal{R}_{\bar{k}}\phi) \tag{1.30}$$

or more explicitly

$$\mathbf{R}_{\bar{k}}A(\Lambda_{\bar{k}}^{(\bar{z})}\bar{z}, \Lambda_{\bar{k}}^{(\vec{r})}\vec{r}, \Lambda_{\bar{k}}^{(x)}x) = \Lambda_{\bar{k}}^{(A)}\ A(\bar{z}, \vec{r}, x) \tag{1.31}$$

The nonlinear dependence of $\mathbf{R}_{\bar{k}}A$ with respect to A is quite weak: it is involved in the value of the rescaling factor $\Lambda_{\bar{k}}^{(A)}$, that is identical for all quantities of the same physical nature, i.e. of the same dimension.

Isotropy of the tensor field, meaning its invariance upon rotation simply expresses $\mathbf{\Theta}A = A$ for any rotation θ. In a similar way, scale invariance

writes $\mathbf{R}_{\bar{k}}A = A$ for any set of rescaling factors $\bar{k}$. Note that if the model rule is a renormalization fixed point ϕ^*, i.e. $\mathcal{R}_{\bar{k}}\phi^* = \phi^*$, then all state functions are invariant upon renormalization; but a state function could be scale invariant, i.e. $\mathbf{R}_{\bar{k}}A = A$, in a model that is not fully invariant upon renormalization. In the same way, the trivial vector field everywhere equal to 0 is isotropic whatever the system is, whereas the system has to be itself isotropic in order that all vector fields are invariant upon any rotation. For example, if there is a single independent rescaling factor k and $\Lambda_{\bar{k}}^{(A)} = k^{-\alpha}$, scale invariance writes: $k^\alpha\ A(\vec{r}) = A(k\vec{r})$; if $A(r)$ is the mass contained in a sample of radius r of the scale invariant system, α is its fractal dimension. Renormalization thus expresses a covariance property, with scale invariance replacing Galilean invariance and fractal geometry replacing Euclidean geometry.

Roughly, a representation describes a set of observable consequences of the group action. The relation $\mathbf{R}_{\bar{k}}[A(\phi)] = A(\mathcal{R}_{\bar{k}}\phi)$ defines a representation $\mathcal{R}_{\bar{k}} \mapsto \mathbf{R}_{\bar{k}}$ of the renormalization group in the space of state functions. The consistency of renormalization, acting at the same time on the variables (time, space), the state (configuration, field) and the functional that rules the system equation or evolution (action, Hamiltonian, vector field, transition probabilities), expresses through *commutative diagrams*:

$$\begin{array}{ccc} \phi & \longrightarrow & \mathcal{R}_{\bar{k}}\phi \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathbf{R}_{\bar{k}}A \end{array} \qquad \begin{array}{ccc} \phi & \longrightarrow & \mathcal{R}_{\bar{k}}\phi \\ \uparrow & & \uparrow \\ K & \longrightarrow & r_{\bar{k}}(K) \end{array} \tag{1.32}$$

Accordingly, RG could be naturally embedded in the framework of category theory. The simplest explicit example is the temporal RG for the period-doubling scenario.[15,21,37] The variable is here the time t, and $Rt = t/2$ (namely $k = 2$). The state is here the position x in the phase space $[-1, 1]$ and $\mathbf{R}x = \lambda x$ where the rescaling factor λ involves the evolution law f according to $\lambda = f(t = 1, x = 1)$. Finally, the renormalization writes $(\mathcal{R}f)(t, x) = \lambda f(2t, x/\lambda) = \mathbf{R}[f(R^{-1}t, \mathbf{R}^{-1}x)]$. These multi-level renormalization transformations can be summarized under the form of a

commutative diagram as follows:

$$\begin{array}{ccc} (t,x) & \xrightarrow{f} & f(t,x) \\ (R,\mathbf{R})\downarrow & & \downarrow\ \mathbf{R} \\ (Rt,\mathbf{R}x) & \xrightarrow{\mathcal{R}f} & \mathbf{R}[f(t,x)] = \mathcal{R}f(Rt,\mathbf{R}x) \end{array}$$

The general structure of RG can finally be formulated as follows. It acts on a fiber bundle $(\mathcal{F}_g)_{g\in\mathcal{G}}$ over an arbitrary set $\mathcal{G}$, possibly ordered or even partly ordered. The renormalization operator $\mathcal{R}_{g_1 g_0}$ is a transformation from $\mathcal{F}_{g_0}$ to $\mathcal{F}_{g_1}$ that satisfies the semi-group relation $\mathcal{R}_{g_2 g_1} \circ \mathcal{R}_{g_1 g_0} = \mathcal{R}_{g_2 g_0}$ for any triplet g_0, g_1 and g_2 in $\mathcal{G}$ and possibly a constraint $g_2 \geq g_1 \geq g_0$ (the chain rule being ill-defined if there is no ordering or an inconsistent ordering between g_0, g_1 and g_2), and $\mathcal{R}_{gg}(\phi) = \phi$ for any $g \in \mathcal{G}$ and $\phi \in \mathcal{F}_g$. Moreover, the chain rule consistency imposes a constraint on the domains and images: $\mathrm{Im}[\mathcal{R}_{g_1 g_0}] \subset \mathrm{Dom}[\mathcal{R}_{g_2 g_1}] \subset \mathcal{F}_{g_1}$. These operators $\mathcal{R}$ define a parallel transport on the fiber bundle. The semi-group structure of this extended RG is homomorphic to the semi-group structure embedded in $\Gamma = \mathcal{G} \times \mathcal{G}$, with $(g_2, g_1) \times (g_1, g_0) = (g_2, g_0)$ and the ordering consistency. The RG is actually a group if and only if the transformations $\mathcal{R}_{g_1 g_0}$ are invertible, with inverse $\mathcal{R}_{g_0 g_1}$; it is then homomorphic to the full group Γ.

1.7. Conclusion: a deep epistemological shift

The usual approach in theoretical physics is to consider a given model and try to extract as much information as possible by studying the solution (evolution, equilibrium state) ruled by this model. But such an approach is unavoidably flawed by the idealizations and approximations involved in devising the model, that is, a limited representation of the real system involving arbitrary choice of description scales and keeping some degrees of freedom while ignoring other details. Renormalization approach is totally different. In focusing on the dependence on the observation scale and expressing how our models and theories change when we change the scale of our description, renormalization gives access to a more objective knowledge about the system. An illustration of this idea is a fractal curve: its length $l(a)$ is a quantity depending on the observation scale a; by contrast, the relation $l(ka) = k^{1-D_f} l(a)$ between the lengths $l(a)$ and $l(ka)$ measured

at different scales a and ka involves an intrinsic quantity D_f, namely the fractal dimension of the curve.

Replacing the study of a given model by the study of the renormalization flow is naturally suited to investigate structural stability of the models and robustness of their predictions. This allows to split this space into universality classes grouping models at different scales of a given system, and also, at a given scale, models of different systems exhibiting the same large-scale behavior. In this way, RG is able to transform qualitative information (the belonging to the same universality class) into quantitative information (the values of the critical exponents and expression of scaling functions); conversely, it enlarges the scope of numerical computation of the critical exponents by assessing their universality and the structural stability of the model used in the computation. RG has thus a deep impact on the epistemological level, changing our way to consider and devise models, hence changing our very theoretical approach of real systems.

RG theory today offers several perspectives[42] and novel directions to be further explored. It seems promising to investigate the physical interpretation of RG flow features other than hyperbolic fixed points. Asymptotic behavior of renormalization trajectories more complicated than the convergence towards a fixed point, e.g. a limit cycle or a chaotic behavior, are indeed possible.[39,53] Renormalization group analysis could be generalized using recent extensions of Lie group theory, e.g. local group structures that would allow to extend RG into an assembly of local versions[m] in the spirit of gauge theories extending global symmetries into space-dependent local versions. One could envision RG based on other symmetries than scale invariance, e.g. to handle any pair of non commuting limits (without necessarily one being associated to time or size); renormalization would then relate models of the same system along another parameter axis than space or time scales. Another direction could be to develop RG extensions on the basis of general covariance properties within the framework of category theory. Let us finally mention recent and promising insights into the mathematical structures of renormalization in QFT: the Feynman graphs

[m]This idea is reminiscent of the notion of 'scale covariance' proposed by Pocheau[43] in the context of fully developed turbulence, amounting to consider local reference frames, and to express more refined scale invariance properties as covariance properties with respect to changes of the local reference frame; let us also mention the somehow related 'generalized scale invariance' introduced by Dubrulle,[19] expressed with respect to a generalized scale variable defined as a combination of the length scale and other relevant quantities.

describing the successive contributions to the perturbation expansion form an Hopf algebra, with expected relations to non commutative geometry,[9] and the RG appears as a one-parameter subgroup of a larger group with a rich mathematical structure related to Galois theory.[14,20]

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