the general definition that there *is* only one one-dimensional Lie algebra, namely $\mathfrak{g} = \mathbb{R}$, with necessarily trivial Lie bracket.

Example 1.47. Here we compute the Lie algebra of the general linear group $\mathrm{GL}(n)$. Note that since $\mathrm{GL}(n)$ is n^2-dimensional, we can identify the Lie algebra $\mathfrak{gl}(n) \simeq \mathbb{R}^{n^2}$ with the space of all $n \times n$ matrices. Indeed, coordinates on $\mathrm{GL}(n)$ are provided by the matrix entries x_{ij}, $i, j = 1, \ldots, n$, so the tangent space to $\mathrm{GL}(n)$ at the identity is the set of all vector fields

$$\mathbf{v}_A|_{\mathrm{I}} = \sum_{i,j} a_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_{\mathrm{I}},$$

where $A = (a_{ij})$ is an arbitrary $n \times n$ matrix. Now given $Y = (y_{ij}) \in \mathrm{GL}(n)$, the matrix $R_Y(X) = XY$ has entries

$$\sum_{k=1}^{n} x_{ik} y_{kj}.$$

Therefore, according to (1.35), we find

$$\begin{aligned} \mathbf{v}_A|_Y &= dR_Y(\mathbf{v}_A|_{\mathrm{I}}) \\ &= \sum_{l,m} \sum_{i,j} a_{ij} \frac{\partial}{\partial x_{ij}} \left(\sum_k x_{lk} y_{km} \right) \frac{\partial}{\partial x_{lm}} \\ &= \sum_{i,j,m} a_{ij} y_{jm} \frac{\partial}{\partial x_{im}}, \end{aligned}$$

or, in terms of $X \in \mathrm{GL}(n)$,

$$\mathbf{v}_A|_X = \sum_{i,j} \left(\sum_k a_{ik} x_{kj} \right) \frac{\partial}{\partial x_{ij}}. \tag{1.37}$$

To compute the Lie bracket:

$$\begin{aligned} [\mathbf{v}_A, \mathbf{v}_B] &= \sum_{\substack{i,j,k \\ l,m,p}} \left\{ a_{lp} x_{pm} \frac{\partial}{\partial x_{lm}} (b_{ik} x_{kj}) - b_{lp} x_{pm} \frac{\partial}{\partial x_{lm}} (a_{ik} x_{kj}) \right\} \frac{\partial}{\partial x_{ij}} \\ &= \sum_{i,j,k} \left[\sum_l (b_{il} a_{lk} - a_{il} b_{lk}) \right] x_{kj} \frac{\partial}{\partial x_{ij}} \\ &= \mathbf{v}_{[A,B]}, \end{aligned}$$

where $[A, B] \equiv BA - AB$ is the *matrix commutator*. Therefore, *the Lie algebra* $\mathfrak{gl}(n)$ *of the general linear group* $\mathrm{GL}(n)$ *is the space of all* $n \times n$ *matrices with the Lie bracket being the matrix commutator.*

One-Parameter Subgroups

Suppose $\mathfrak{g}$ is the Lie algebra of a Lie group G. The next result shows that there is a one-to-one correspondence between one-dimensional subspaces of $\mathfrak{g}$ and (connected) one-parameter subgroups of G.

Proposition 1.48. *Let $\mathbf{v} \neq 0$ be a right-invariant vector field on a Lie group G. Then the flow generated by $\mathbf{v}$ through the identity, namely*

$$g_\varepsilon = \exp(\varepsilon \mathbf{v})e \equiv \exp(\varepsilon \mathbf{v}) \tag{1.38}$$

is defined for all $\varepsilon \in \mathbb{R}$ and forms a one-parameter subgroup of G, with

$$g_{\varepsilon+\delta} = g_\varepsilon \cdot g_\delta, \qquad g_0 = e, \qquad g_\varepsilon^{-1} = g_{-\varepsilon}, \tag{1.39}$$

isomorphic to either $\mathbb{R}$ itself or the circle group SO(2). *Conversely, any connected one-dimensional subgroup of G is generated by such a right-invariant vector field in the above manner.*

Proof. For ε, δ sufficiently small, (1.39) follows from the right-invariance of $\mathbf{v}$ and (1.26):

$$\begin{aligned} g_\delta \cdot g_\varepsilon &= R_{g_\varepsilon}(g_\delta) = R_{g_\varepsilon}[\exp(\delta \mathbf{v})e] \\ &= \exp[\delta \cdot dR_{g_\varepsilon}(\mathbf{v})] R_{g_\varepsilon}(e) \\ &= \exp(\delta \mathbf{v}) g_\varepsilon \\ &= \exp(\delta \mathbf{v}) \exp(\varepsilon \mathbf{v}) e \\ &= \exp[(\delta + \varepsilon)\mathbf{v}]e = g_{\delta+\varepsilon}. \end{aligned}$$

Thus g_ε is at least a local one-parameter subgroup. In particular, $g_0 = e$, and $g_{-\varepsilon} = g_\varepsilon^{-1}$ for ε small. Furthermore, g_ε is defined at least for $-\frac{1}{2}\varepsilon_0 \leqslant \varepsilon \leqslant \frac{1}{2}\varepsilon_0$, for some $\varepsilon_0 > 0$, so we can inductively define

$$g_{m\varepsilon_0+\varepsilon} = g_{m\varepsilon_0} \cdot g_\varepsilon, \qquad -\tfrac{1}{2}\varepsilon_0 \leqslant \varepsilon \leqslant \tfrac{1}{2}\varepsilon_0,$$

for m an integer. The above calculation shows that g_ε is a smooth curve in G satisfying (1.39) for all ε, δ, proving that the flow is globally defined and forms a subgroup. If $g_\varepsilon = g_\delta$ for some $\varepsilon \neq \delta$, then it is not hard to show that $g_{\varepsilon_0} = e$ for some least positive $\varepsilon_0 > 0$, and that g_ε is periodic with period ε_0, i.e. $g_{\varepsilon+\varepsilon_0} = g_\varepsilon$. In this case $\{g_\varepsilon\}$ is isomorphic to SO(2) (take $\theta = 2\pi\varepsilon/\varepsilon_0$). Otherwise $g_\varepsilon \neq g_\delta$ for all $\varepsilon \neq \delta$, and $\{g_\varepsilon\}$ is isomorphic to $\mathbb{R}$.

Conversely, if $H \subset G$ is a one-dimensional subgroup, we let $\mathbf{v}|_e$ be any nonzero tangent vector to H at the identity. Using the isomorphism (1.36) we extend $\mathbf{v}$ to a right-invariant vector field on all of G. Since H is a subgroup it follows that $\mathbf{v}|_h$ is tangent to H at any $h \in H$, and therefore H is the integral curve of $\mathbf{v}$ passing through e. This proves the converse. □

Example 1.49. Suppose $G = \mathrm{GL}(n)$ with Lie algebra $\mathrm{gl}(n)$, the space of all $n \times n$ matrices with commutator as the Lie bracket. If $A \in \mathrm{gl}(n)$, then the corresponding right-invariant vector field $\mathbf{v}_A$ on $\mathrm{GL}(n)$ has the expression (1.37). The one-parameter subgroup $\exp(\varepsilon \mathbf{v}_A)e$ is found by integrating the system of n^2 ordinary differential equations

$$\frac{dx_{ij}}{d\varepsilon} = \sum_{k=1}^{n} a_{ik} x_{kj}, \qquad x_{ij}(0) = \delta_j^i, \qquad i, j = 1, \ldots, n,$$

involving the matrix entries of A. The solution is just the matrix exponential $X(\varepsilon) = e^{\varepsilon A}$, which is the one-parameter subgroup of $\mathrm{GL}(n)$ generated by a matrix A in $\mathfrak{gl}(n)$.

Example 1.50. Consider the torus T^2 with group multiplication

$$(\theta, \rho)\cdot(\theta', \rho') = (\theta + \theta', \rho + \rho') \bmod 2\pi.$$

Clearly the Lie algebra of T^2 is spanned by the right-invariant fields $\partial/\partial\theta$, $\partial/\partial\rho$ with trivial Lie bracket: $[\partial_\theta, \partial_\rho] = 0$. Let

$$\mathbf{v}_\omega = \partial_\theta + \omega\partial_\rho$$

for some $\omega \in \mathbb{R}$. Then the corresponding one-parameter subgroup is

$$\exp(\varepsilon\mathbf{v}_\omega)(0, 0) = (\varepsilon, \varepsilon\omega) \bmod 2\pi, \qquad \varepsilon \in \mathbb{R},$$

which is precisely the subgroup H_ω discussed on pages 17–18. In particular, if ω is rational, H_ω is a closed, one-parameter subgroup isomorphic to $\mathrm{SO}(2)$, while if ω is irrational, H_ω is a dense subgroup isomorphic to $\mathbb{R}$. This shows that it is rather difficult in general to tell the precise character of a one-parameter subgroup just from knowledge of its infinitesimal generator.

Subalgebras

In general a *subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a vector subspace which is closed under the Lie bracket, so $[\mathbf{v}, \mathbf{w}] \in \mathfrak{h}$ whenever $\mathbf{v}, \mathbf{w} \in \mathfrak{h}$. If H is a Lie subgroup of a Lie group G, any right-invariant vector field $\mathbf{v}$ on H can be extended to a right-invariant vector field on G. (Just set $\mathbf{v}|_g = dR_g(\mathbf{v}|_e)$, $g \in G$.) In this way the Lie algebra $\mathfrak{h}$ of H is realized as a subalgebra of the Lie algebra $\mathfrak{g}$ of G. The correspondence between one-parameter subgroups of a Lie group G and one-dimensional subspaces $\mathfrak{h}$ (subalgebras) of its Lie algebra $\mathfrak{g}$ generalizes to provide a complete one-to-one correspondence between Lie subgroups of G and subalgebras of $\mathfrak{g}$.

Theorem 1.51. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $H \subset G$ is a Lie subgroup, its Lie algebra is a subalgebra of $\mathfrak{g}$. Conversely, if $\mathfrak{h}$ is any s-dimensional subalgebra of $\mathfrak{g}$, there is a unique connected s-parameter Lie subgroup H of G with Lie algebra $\mathfrak{h}$.*

The main idea in the proof of this theorem can be outlined as follows. Let $\mathbf{v}_1, \ldots, \mathbf{v}_s$ be a basis of $\mathfrak{h}$, which defines a system of vector fields on G. Since $\mathfrak{h}$ is a subalgebra, each Lie bracket $[\mathbf{v}_i, \mathbf{v}_j]$ is again an element of $\mathfrak{h}$, and hence in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$. Thus $\mathfrak{h}$ defines an involutive system of vector fields on G. Furthermore, it is easily seen that at each point $g \in G$, $\{\mathbf{v}_1|_g, \ldots, \mathbf{v}_s|_g\}$ are linearly independent tangent vectors, so the system is semi-regular. By Frobenius' Theorem 1.40, there is a maximal s-dimensional submanifold of this system passing through e, and this submanifold is the Lie subgroup H

corresponding to $\mathfrak{h}$. It is not too hard to check that H is indeed a subgroup, and we know it is a submanifold. The main technical complication in the proof comes from showing the group operations of multiplication and inversion induced from those of G are smooth in the manifold structure of H. The interested reader can look at Warner, [1; Theorem 3.19] for the complete proof.

Example 1.52. The preceding theorem greatly simplifies the computation of Lie algebras of Lie groups which can be realized as Lie subgroups of the general linear group $\mathrm{GL}(n)$. Namely, if $H \subset \mathrm{GL}(n)$ is a subgroup, then its Lie algebra $\mathfrak{h}$ will be a subalgebra of the Lie algebra $\mathfrak{gl}(n)$ of all $n \times n$ matrices, with Lie bracket being the matrix commutator. Moreover, we can find $\mathfrak{h} \simeq TH|_e$ just by looking at all one-parameter subgroups of $\mathrm{GL}(n)$ which are contained in H:

$$\mathfrak{h} = \{A \in \mathfrak{gl}(n) \colon e^{\varepsilon A} \in H \text{ for } \varepsilon \in \mathbb{R}\}.$$

For example, to find the Lie algebra of the orthogonal group $\mathrm{O}(n)$, we need to find all $n \times n$ matrices A such that

$$(e^{\varepsilon A})(e^{\varepsilon A})^T = \mathrm{I}.$$

Differentiating with respect to ε and setting $\varepsilon = 0$ we find

$$A + A^T = 0.$$

Therefore $\mathfrak{so}(n) = \{A \colon A \text{ is skew-symmetric}\}$ is the Lie algebra of both $\mathrm{O}(n)$ and $\mathrm{SO}(n)$. Lie algebras of other matrix Lie groups are found similarly.

We have seen that there is a general one-to-one correspondence between subalgebras of the Lie algebra of a given Lie group and connected Lie subgroups of the same group. In particular, every subalgebra of $\mathfrak{gl}(n)$ gives rise to a matrix Lie group, i.e. a Lie subgroup of $\mathrm{GL}(n)$. More generally, if $\mathfrak{g}$ is any finite-dimensional (abstract) Lie algebra, the question arises as to whether there is a corresponding Lie group G with the given space $\mathfrak{g}$ as its Lie algebra. The answer to this question is affirmative, and, in fact, reduces to the matrix case by the following important theorem of Ado.

Theorem 1.53. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{gl}(n)$ for some n.*

As a direct consequence of Ado's theorem and the latter half of Theorem 1.22 we deduce the fundamental correspondence between Lie groups and Lie algebras.

Theorem 1.54. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then there exists a unique connected, simply-connected Lie group G^* having $\mathfrak{g}$ as its Lie algebra. Moreover, if G is any other connected Lie group with Lie algebra $\mathfrak{g}$, then $\pi \colon G^* \to G$ is the simply-connected covering group of G.*

Indeed, we need only realize $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}(n)$ for some n, and take $\tilde{G}$ to be the corresponding Lie subgroup of $\mathrm{GL}(n)$. Then G^* will be the simply-connected covering group of $\tilde{G}$, guaranteed by Theorem 1.22.

It is important to emphasize that it is *not* true that every Lie group G is isomorphic to a subgroup of $\mathrm{GL}(n)$ for some n. In particular, the simply-connected covering group of $\mathrm{SL}(2, \mathbb{R})$ is *not* realizable as a matrix Lie group!

The Exponential Map

The *exponential map* $\exp: \mathfrak{g} \to G$ is obtained by setting $\varepsilon = 1$ in the one-parameter subgroup generated by $\mathbf{v}$:

$$\exp(\mathbf{v}) \equiv \exp(\mathbf{v})e.$$

One readily proves that the differential

$$d\exp: T\mathfrak{g}|_0 \simeq \mathfrak{g} \to TG|_e \simeq \mathfrak{g}$$

of exp at 0 is the identity map. (See Exercise 1.27.) Thus, by the inverse function theorem, exp determines a local diffeomorphism from $\mathfrak{g}$ onto a neighbourhood of the identity element in G. Consequently, every group element g sufficiently close to the identity can be written as an exponential: $g = \exp(\mathbf{v})$ for some $\mathbf{v} \in \mathfrak{g}$. In general, $\exp: \mathfrak{g} \to G$ is globally neither one-to-one nor onto. (See Exercise 1.28.) However, using Proposition 1.24, we can always write any group element g as a finite product of exponentials

$$g = \exp(\mathbf{v}_1)\exp(\mathbf{v}_2)\cdots\exp(\mathbf{v}_k)$$

for some $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathfrak{g}$. The net effect of this observation is that the proof of the invariance of some object under the entire Lie group reduces to a proof of its invariance just under one-parameter subgroups of G, which in turn will be implied by a form of "infinitesimal invariance" under the corresponding infinitesimal generators in $\mathfrak{g}$. With a little more work, we can actually reduce to just proving "invariance" under a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ of $\mathfrak{g}$, with any group element being expressible in the form

$$g = \exp(\varepsilon^1 \mathbf{v}_{i_1})\exp(\varepsilon^2 \mathbf{v}_{i_2})\cdots\exp(\varepsilon^k \mathbf{v}_{i_k}) \tag{1.40}$$

for suitable $\varepsilon^j \in \mathbb{R}$, $1 \leqslant i_j \leqslant r$, $j = 1, \ldots, k$. (See Exercise 1.27.)

Lie Algebras of Local Lie Groups

Turning to the local version we consider a local Lie group $V \subset \mathbb{R}^r$ with multiplication $m(x, y)$. The corresponding right multiplication map $R_y: V \to \mathbb{R}^r$ is $R_y(x) = m(x, y)$. A vector field $\mathbf{v}$ on V is right-invariant if and only if

$$dR_y(\mathbf{v}|_x) = \mathbf{v}|_{R_y(x)} = \mathbf{v}|_{m(x,y)}$$

whenever x, y and $m(x, y)$ are in V. As in the case of global Lie groups, it is easy to check that any right-invariant vector field is determined uniquely by its value at the origin (identity element), $\mathbf{v}|_x = dR_x(\mathbf{v}|_0)$, and hence the Lie algebra $\mathfrak{g}$ for the local Lie group V, determined as the space of right-invariant vector fields on V, is an r-dimensional vector space. In fact, we can determine $\mathfrak{g}$ directly from the formula for the group multiplication.

Proposition 1.55. *Let $V \subset \mathbb{R}^r$ be a local Lie group with multiplication $m(x, y)$, $x, y \in V$. Then the Lie algebra $\mathfrak{g}$ of right-invariant vector fields on V is spanned by the vector fields*

$$\mathbf{v}_k = \sum_{i=1}^{r} \xi_k^i(x)\frac{\partial}{\partial x^i}, \qquad k = 1, \ldots, r,$$

where

$$\xi_k^i(x) = \frac{\partial m^i}{\partial x^k}(0, x). \tag{1.41}$$

Here the m^i's are the components of m, and the $\partial/\partial x^k$ denote derivatives with respect to the first set of r variables in $m(x, y)$, after which the values $x = 0$, $y = x$ are to be substituted.

PROOF. Since $R_y(x) = m(x, y)$, we have

$$\mathbf{v}_k|_y = dR_y\left(\sum_{i=1}^{r} \xi_k^i(0)\frac{\partial}{\partial x^i}\right) = \sum_{i,j} \xi_k^i(0)\frac{\partial m^j}{\partial x^i}(0, y)\frac{\partial}{\partial x^j}.$$

Thus it suffices to prove that

$$\xi_k^i(0) = \delta_k^i,$$

i.e.

$$\frac{\partial m^i}{\partial x^k}(0, 0) = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

But this follows trivially from the fact that $m(x, 0) = x$ is the identity map. $\square$

Example 1.56. Consider the local Lie group of Example 1.21. The Lie algebra $\mathfrak{g}$ is one-dimensional, spanned by the vector field $\xi(x)\partial_x$, where, by (1.41),

$$\xi(x) = \frac{\partial m}{\partial x}(0, x) = (x - 1)^2.$$

Thus $\mathbf{v} = (x - 1)^2\partial_x$ is the unique independent right-invariant vector field on V. Note that the local group homomorphism $\chi\colon \mathbb{R} \to V$ of Example 1.23 maps the invariant vector field ∂_t on $\mathbb{R}$ to $-\mathbf{v} = d\chi(\partial_t)$.

Structure Constants

Suppose $\mathfrak{g}$ is any finite-dimensional Lie algebra, so by Theorem 1.54 $\mathfrak{g}$ is the Lie algebra of some Lie group G. If we introduce a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ of $\mathfrak{g}$, then the Lie bracket of any two basis vectors must again lie in $\mathfrak{g}$. Thus there are certain constants c_{ij}^k, $i, j, k = 1, \ldots, r$, called the *structure constants* of $\mathfrak{g}$ such that

$$[\mathbf{v}_i, \mathbf{v}_j] = \sum_{k=1}^{r} c_{ij}^k \mathbf{v}_k, \qquad i, j = 1, \ldots, r. \tag{1.42}$$

Note that since the $\mathbf{v}_i$'s form a basis, if we know the structure constants, then we can recover the Lie algebra $\mathfrak{g}$ just by using (1.42) and the bilinearity of the Lie bracket. The conditions of skew-symmetry and the Jacobi identity place further constraints on the structure constants:

(i) *Skew-symmetry*

$$c_{ij}^k = -c_{ji}^k, \tag{1.43}$$

(ii) *Jacobi identity*

$$\sum_{k=1}^{r} (c_{ij}^k c_{kl}^m + c_{li}^k c_{kj}^m + c_{jl}^k c_{ki}^m) = 0. \tag{1.44}$$

Conversely, it is not difficult to show that any set of constants c_{ij}^k which satisfy (1.43), (1.44) are the structure constants for some Lie algebra $\mathfrak{g}$.

Of course, if we choose a new basis of $\mathfrak{g}$, then in general the structure constants will change. If $\hat{\mathbf{v}}_i = \sum_j a_{ij} \mathbf{v}_j$, then

$$\hat{c}_{ij}^k = \sum_{l,m,n} a_{il} a_{jm} b_{nk} c_{lm}^n, \tag{1.45}$$

where (b_{ij}) is the inverse matrix to (a_{ij}). Thus two sets of structure constants determine the same Lie algebra if and only if they are related by (1.45). Consequently, from Theorem 1.54 we see that there is a one-to-one correspondence between equivalence classes of structure constants c_{ij}^k satisfying (1.43), (1.44) and connected, simply-connected Lie groups G whose Lie algebras have the given structure constants relative to some basis. Thus, in principle, the entire theory of Lie groups reduces to a study of the algebraic equations (1.43), (1.44); however, this is perhaps an excessively simplistic point of view!

Commutator Tables

The most convenient way to display the structure of a given Lie algebra is to write it in tabular form. If $\mathfrak{g}$ is an r-dimensional Lie algebra, and $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathfrak{g}$, then the *commutator table* for $\mathfrak{g}$ will be the $r \times r$ table whose (i, j)-th entry expresses the Lie bracket $[\mathbf{v}_i, \mathbf{v}_j]$. Note that the table is always skew-symmetric since $[\mathbf{v}_i, \mathbf{v}_j] = -[\mathbf{v}_j, \mathbf{v}_i]$; in particular, the diagonal entries are all zero. The structure constants can be easily read off the commutator table; namely c_{ij}^k is the coefficient of $\mathbf{v}_k$ in the (i, j)-th entry of the table.

For example, if $\mathfrak{g} = \mathfrak{sl}(2)$, the Lie algebra of the special linear group $SL(2)$, which consists of all 2×2 matrices with trace 0, and we use the basis

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then we obtain the commutator table

	A_1	A_2	A_3
A_1	0	A_1	$-2A_2$
A_2	$-A_1$	0	A_3
A_3	$2A_2$	$-A_3$	0

Thus, for example,

$$[A_1, A_3] = A_3 A_1 - A_1 A_3 = -2A_2,$$

and so on. The structure constants are

$$c_{12}^1 = c_{23}^3 = 1 = -c_{21}^1 = -c_{32}^3, \qquad c_{13}^2 = -2 = -c_{31}^2,$$

with all other c_{jk}^i's being zero.

Infinitesimal Group Actions

Suppose G is a local group of transformations acting on a manifold M via $g \cdot x = \Psi(g, x)$ for $(g, x) \in \mathcal{U} \subset G \times M$. There is then a corresponding "infinitesimal action" of the Lie algebra $\mathfrak{g}$ of G on M. Namely, if $\mathbf{v} \in \mathfrak{g}$ we define $\psi(\mathbf{v})$ to be the vector field on M whose flow coincides with the action of the one-parameter subgroup $\exp(\varepsilon\mathbf{v})$ of G on M. This means that for $x \in M$,

$$\psi(\mathbf{v})|_x = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Psi(\exp(\varepsilon\mathbf{v}), x) = d\Psi_x(\mathbf{v}|_e), \tag{1.46}$$

where $\Psi_x(g) \equiv \Psi(g, x)$. Note further that since

$$\Psi_x \circ R_g(h) = \Psi(h \cdot g, x) = \Psi(h, g \cdot x) = \Psi_{g \cdot x}(h)$$

wherever defined, we have

$$d\Psi_x(\mathbf{v}|_g) = d\Psi_{g \cdot x}(\mathbf{v}|_e) = \psi(\mathbf{v})|_{g \cdot x}$$

for any $g \in G_x$. It follows from property (1.32) of the Lie bracket that ψ is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on M:

$$[\psi(\mathbf{v}), \psi(\mathbf{w})] = \psi([\mathbf{v}, \mathbf{w}]), \qquad \mathbf{v}, \mathbf{w} \in \mathfrak{g}. \tag{1.47}$$

Therefore the set of all vector fields $\psi(\mathbf{v})$ corresponding to $\mathbf{v} \in \mathfrak{g}$ forms a Lie algebra of vector fields on M. Conversely, given a finite-dimensional Lie algebra of vector fields on M, there is always a local group of transformations whose infinitesimal action is generated by the given Lie algebra.

Theorem 1.57. *Let $\mathbf{w}_1, \dots, \mathbf{w}_r$ be vector fields on a manifold M satisfying*

$$[\mathbf{w}_i, \mathbf{w}_j] = \sum_{k=1}^{r} c_{ij}^k \mathbf{w}_k, \qquad i, j = 1, \dots, r,$$

for certain constants c_{ij}^k. *Then there is a Lie group G whose Lie algebra has the given c_{ij}^k as structure constants relative to some basis $\mathbf{v}_1, \dots, \mathbf{v}_r$, and a local group action of G on M such that $\psi(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, \dots, r$, where ψ is defined by* (1.46).

Usually we will omit explicit reference to the map ψ and identify the Lie algebra $\mathfrak{g}$ with its image $\psi(\mathfrak{g})$, which forms a Lie algebra of vector fields on M. In this language, we recover $\mathfrak{g}$ from the group transformations by the basic formula

$$\mathbf{v}|_x = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \exp(\varepsilon \mathbf{v})x, \qquad \mathbf{v} \in \mathfrak{g}. \tag{1.48}$$

A vector field $\mathbf{v}$ in $\mathfrak{g}$ is called an *infinitesimal generator* of the group action G. Theorem 1.57 says that if we know infinitesimal generators $\mathbf{w}_1, \dots, \mathbf{w}_r$, which form a basis for a Lie algebra, then we can always exponentiate to find a local group of transformations whose Lie algebra coincides with the given one.

Example 1.58. Lie proved that up to diffeomorphism there are precisely three finite-dimensional Lie algebras of vector fields on the real line $M = \mathbb{R}$. These are

(a) The algebra spanned by ∂_x: This generates an action of $\mathbb{R}$ on M as a one-parameter group of translations: $x \mapsto x + \varepsilon$.

(b) The two-dimensional Lie algebra spanned by ∂_x and $x\partial_x$, the second vector field generating the group of dilatations $x \mapsto \lambda x$: Note that

$$[\partial_x, x\partial_x] = \partial_x,$$

so this Lie algebra is isomorphic to the 2×2 matrix Lie algebra spanned by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This generates the Lie group of all upper triangular matrices of the form

$$A = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}, \qquad \alpha > 0.$$

The corresponding action on $\mathbb{R}$ is the group $x \mapsto \alpha x + \beta$ of affine transformations; we leave it to the reader to check that this indeed defines a Lie group action, whose infinitesimal generators agree with the given ones.

(c) The three-dimensional algebra spanned by

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = x\partial_x, \qquad \mathbf{v}_3 = x^2\partial_x,$$

the third vector field generating the local group of "inversions"

$$x \mapsto \frac{x}{1 - \varepsilon x}, \qquad |\varepsilon| < \frac{1}{x}.$$

The commutator table for this Lie algebra is as follows:

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	0	$\mathbf{v}_1$	$2\mathbf{v}_2$
$\mathbf{v}_2$	$-\mathbf{v}_1$	0	$\mathbf{v}_3$
$\mathbf{v}_3$	$-2\mathbf{v}_2$	$-\mathbf{v}_3$	0

If we replace $\mathbf{v}_3$ by $-\mathbf{v}_3 = -x^2\partial_x$, then we find the same commutator table as $\mathfrak{sl}(2)$ with basis

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

There is thus a local action of the special linear group SL(2) on the real line with ∂_x, $x\partial_x$ and $-x^2\partial_x$ serving as the infinitesimal generators. It is not difficult to see that this group action is just the *projective group*

$$x \mapsto \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2),$$

being the real analogue of the complex group of linear fractional transformations.

1.5. Differential Forms

Originally developed as a tool for the multi-dimensional generalization of Stokes' theorem, differential forms play a fundamental role in the topological aspects of differential geometry. Although in this book I have tended to de-emphasize the use of differential forms, there are several occasions, most notably the variational complex of Section 5.4, in which the language of differential forms is especially effective. This section provides a rapid introduction to the theory of differential forms for the reader who is interested in pursuing these more theoretical aspects of the subject. We begin with the basic definition.

Definition 1.59. Let M be a smooth manifold and $TM|_x$ its tangent space at x. The space $\bigwedge_k T^*M|_x$ of *differential k-forms at x* is the set of all k-linear