In particular, if $Q[u] = Q(x, u^{(m)})$ does not depend explicitly on t, then (5.26) reduces to the condition that the two vector fields $\mathbf{v}_P$ and $\mathbf{v}_Q$ commute:

$$[\mathbf{v}_P, \mathbf{v}_Q] = 0. \tag{5.27}$$

It is not difficult to show that, under certain existence and uniqueness hypotheses, this condition is equivalent to the condition that the corresponding one-parameter symmetry groups commute:

$$\exp(\varepsilon \mathbf{v}_P) \exp(\tilde{\varepsilon} \mathbf{v}_Q) f = \exp(\tilde{\varepsilon} \mathbf{v}_Q) \exp(\varepsilon \mathbf{v}_P) f \tag{5.28}$$

where defined. (See Exercise 5.7.) Consequently, we have the reciprocity relation that, provided $P, Q \in \mathscr{A}^q$ only depend on x, u and x-derivatives of u, the vector field $\mathbf{v}_Q$ is a generalized symmetry of the system $u_t = P$ if and only if $\mathbf{v}_P$ is a generalized symmetry of $u_t = Q$. In particular, for P as above, the vector field $\mathbf{v}_P$ itself is always a symmetry of $u_t = P[u]$. Indeed, $\mathbf{v}_P$ is equivalent to the evolutionary form of the time translation symmetry group generated by ∂_t, stemming from the "autonomy" of the evolution equation. The reader may try rechecking the symmetry condition (5.26) for the symmetries of Burgers' equation found in Example 5.8.

5.2. Recursion Operators, Master Symmetries and Formal Symmetries

The method of Section 5.1 provides a systematic means of determining all the generalized symmetries of a given order of a system of differential equations, but suffers the drawback that the order of derivatives on which the coefficients of the symmetry depend must be specified in advance. Thus the method cannot simultaneously generate *all* generalized symmetries of the system. In this section we explore a second method for generating symmetries based on the notion of a recursion operator. While this method cannot provide an exhaustive classification of all possible symmetries without further analysis, it does provide a mechanism for generating infinite hierarchies of generalized symmetries, depending on higher order derivatives of u, in one step. Unfortunately, while the verification that a given operator does determine a recursion operator is fairly straightforward, in contrast to the previous method, this technique is not fully constructive. The deduction of the form of the recursion operator (if it exists) requires a certain amount of inspired guesswork, often based on the form of lower order symmetries determined by the earlier method.

Definition 5.20. Let Δ be a system of differential equations. A *recursion operator* for Δ is a linear operator $\mathscr{R}: \mathscr{A}^q \to \mathscr{A}^q$ in the space of q-tuples of differential functions with the property that whenever $\mathbf{v}_Q$ is an evolutionary symmetry of Δ, so is $\mathbf{v}_{\tilde{Q}}$ with $\tilde{Q} = \mathscr{R}Q$.

Thus if we are fortunate enough to know a recursion operator $\mathscr{R}$ for a system of differential equations, we can generate an infinite family of symmetries from any one symmetry $\mathbf{v}_{Q_0}$ merely by applying $\mathscr{R}$ successively to the characteristic Q_0; in other words, each $Q_j = \mathscr{R}^j Q_0$, $j = 0, 1, 2, \ldots$ is the characteristic of a generalized symmetry. Often, but not always, $\mathscr{R}$ will be a $q \times q$ matrix of differential operators.

Example 5.21. As an easy example, we show that the differential operator $\mathscr{R}_1 = D_x$ is a recursion operator for the heat equation $u_t = u_{xx}$. Now $\mathbf{v}_Q$ is a generalized symmetry of the heat equation if and only if $D_t Q = D_x^2 Q$ on all solutions. Then $\tilde{Q} = D_x Q$ is also the characteristic of a symmetry since

$$D_t \tilde{Q} - D_x^2 \tilde{Q} = (D_t - D_x^2) D_x Q = D_x(D_t Q - D_x^2 Q) = 0$$

on solutions. Thus, starting with the basic scaling symmetry $u\partial_u$ we can generate a whole hierarchy of generalized symmetries by recursively applying $\mathscr{R}_1$; we find $u_x = \mathscr{R}_1(u)$, $u_{xx} = \mathscr{R}_1^2(u)$, etc. are all characteristics of generalized symmetries of the heat equation. Put another way, the "flow" generated by the heat equation commutes with the "flow" determined by the evolution equations $u_t = \partial^k u / \partial x^k$, $k \geqslant 0$, cf. (5.28).

By the same arguments, the t-derivative D_t is also a recursion operator, but it is trivially related to D_x since $D_t Q = D_x^2 Q$ whenever Q is the characteristic of a symmetry. (In general, $\mathscr{R}^m$ is trivially a recursion operator whenever $\mathscr{R}$ is.) There is, however, a second recursion operator not related to D_x, namely $\mathscr{R}_2 = tD_x + \frac{1}{2}x$. To see this, we find

$$(D_t - D_x^2)(tD_x + \tfrac{1}{2}x)Q = (tD_x + \tfrac{1}{2}x)(D_t Q - D_x^2 Q),$$

so $\tilde{Q} = \mathscr{R}_2 Q$ gives a symmetry whenever Q does. We thus obtain a double infinity of generalized symmetries of the heat equation, by applying $\mathscr{R}_1$ or $\mathscr{R}_2$ successively to $Q_0 = u$. These have characteristics

$$\begin{aligned} &Q_0 = u, \qquad Q_1 = \mathscr{R}_1[u] = u_x, \qquad Q_2 = \mathscr{R}_2[u] = tu_x + \tfrac{1}{2}xu, \\ &Q_3 = \mathscr{R}_1^2[u] = u_{xx}, \qquad Q_4 = \mathscr{R}_2\mathscr{R}_1[u] = tu_{xx} + \tfrac{1}{2}xu_x, \\ &Q_5 = \mathscr{R}_2^2[u] = t^2 u_{xx} + txu_x + (\tfrac{1}{2}t + \tfrac{1}{4}x^2)u, \\ &Q_6 = \mathscr{R}_1^3[u] = u_{xxx}, \qquad Q_7 = \mathscr{R}_2\mathscr{R}_1^2[u] = tu_{xxx} + \tfrac{1}{2}xu_{xx}, \quad \text{etc.} \end{aligned} \tag{5.29}$$

(Note that since $\mathscr{R}_1\mathscr{R}_2 = \mathscr{R}_2\mathscr{R}_1 + \frac{1}{2}$, if we are only interested in independent characteristics, it doesn't matter in which order $\mathscr{R}_1$ and $\mathscr{R}_2$ are applied.) The first six of these, $Q_0, \ldots, Q_5$, are (up to sign) the characteristics of the geometric symmetries of the heat equation computed in Example 2.41; the rest of these are genuine generalized symmetries.

The above results for the heat equation actually generalize to arbitrary linear systems of differential equations as follows.

Proposition 5.22. *Let $\Delta[u] = 0$ be a linear system of differential equations, with Δ denoting a linear differential operator. A second linear differential operator $\mathscr{R}\colon \mathscr{A}^q \to \mathscr{A}^q$ not depending on u or its derivatives is a recursion operator for Δ if and only if $Q = \mathscr{R}[u]$ is the characteristic of a "linear" generalized symmetry to the system.*

In other words, for a linear system every generalized symmetry whose characteristic depends linearly on u and its derivatives determines a recursion operator and conversely. For linear systems, then, the whole theory of (linear) symmetries could be developed using the recursion operators as the fundamental objects. This is the approach favoured by Miller, [3], and his co-workers. Proposition 5.22 provides the link between their approach and the more geometrical Lie theory developed in this book. (The latter has the advantage of simultaneously treating nonlinear systems, which are not covered by the operator method.)

The proof of Proposition 5.22 is easy. If $\mathscr{R}$ is a recursion operator, then $Q = \mathscr{R}[u]$ trivially gives a symmetry since $Q_0 = u$ is the characteristic for the trivial scaling symmetry group $(x, u) \mapsto (x, \lambda u)$ stemming from the linearity of the system. Conversely, if $\mathbf{v}_Q$ is a symmetry, by (5.4) and the linearity of Δ,

$$\operatorname{pr} \mathbf{v}_Q(\Delta[u]) = \Delta[Q] = \Delta\mathscr{R}[u]$$

on all solutions. Total nondegeneracy of Δ implies the existence of a differential operator $\tilde{\mathscr{R}}$ satisfying $\Delta\mathscr{R}[u] = \tilde{\mathscr{R}}\Delta[u]$ for all u, cf. (5.5). It is easily seen that since Δ and $\mathscr{R}$ are independent of u, we can choose $\tilde{\mathscr{R}}$ to also be independent of u, and $\Delta\mathscr{R} = \tilde{\mathscr{R}}\Delta$ identically. Thus if $\tilde{Q} = \mathscr{R}Q$, where Q is the characteristic of a symmetry, so $\Delta[Q] = 0$, then $\Delta[\tilde{Q}] = \tilde{\mathscr{R}}\Delta[Q] = 0$ on solutions, and $\tilde{Q}$ provides another symmetry. □

Example 5.23. For the two-dimensional wave equation, the ten-parameter conformal symmetry group was derived in Example 2.43; the corresponding characteristics are given in Example 4.36. According to Proposition 5.22 there are ten recursion operators, namely

$$\begin{aligned}
&D_x, \quad D_y, \quad D_t, && \text{(translations)}\\
&\mathscr{R}_{xy} = xD_y - yD_x, \quad \mathscr{R}_{xt} = tD_x + xD_t, \quad \mathscr{R}_{yt} = tD_y + yD_t, && \text{(rotations)}\\
&\mathscr{D} = xD_x + yD_y + tD_t, && \text{(dilatation)}\\
&\mathscr{I}_x = (x^2 - y^2 + t^2)D_x + 2xyD_y + 2xtD_t + x, && \\
&\mathscr{I}_y = 2xyD_x + (y^2 - x^2 + t^2)D_y + 2ytD_t + y, && \text{(inversions)}\\
&\mathscr{I}_t = 2xtD_x + 2ytD_t + (x^2 + y^2 + t^2)D_t + t. &&
\end{aligned} \tag{5.30}$$

Applying successive products of these operators to $Q_0 = u$ leads to vast numbers of generalized symmetries of the wave equation; for example

$$\mathscr{R}_{xy}\mathscr{R}_{xt}[u] = xtu_{xy} - ytu_{xx} + x^2u_{yt} - xyu_{xt} - yu_t,$$

and so on. There are a number of dependencies among the resulting symmetries stemming from relations among the operators, e.g.

$$\mathscr{R}_{xy}D_t - \mathscr{R}_{xt}D_y + \mathscr{R}_{yt}D_x = 0.$$

In his thesis, Delong, [1], proves that there are $(2k+1)(2k+2)(2k+3)/6$ independent k-th order symmetries generated by these recursion operators; for instance, there are 35 independent second order symmetries like the above example. Recently, Shapovalov and Shirokov, [1], proved that every generalized symmetry of the wave equation can be obtained in this way.

Fréchet Derivatives

For nonlinear systems, there is an analogous criterion for a differential operator to be a recursion operator, but to state it we need to introduce the notion of the (formal) Fréchet derivative of a differential function.

Definition 5.24. Let $P[u] = P(x, u^{(n)}) \in \mathscr{A}^r$ be an r-tuple of differential functions. The *Fréchet derivative* of P is the differential operator $\mathsf{D}_P\colon \mathscr{A}^q \to \mathscr{A}^r$ defined so that

$$\mathsf{D}_P(Q) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} P[u + \varepsilon Q[u]] \tag{5.31}$$

for any $Q \in \mathscr{A}^q$.

In other words, to evaluate $\mathsf{D}_P(Q)$ we replace u (and its derivatives) in P by $u + \varepsilon Q$ and differentiate the resulting expression with respect to ε. For example, if $P[u] = u_x u_{xx}$, then

$$\mathsf{D}_P(Q) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (u_x + \varepsilon D_x Q)(u_{xx} + \varepsilon D_x^2 Q) = u_x D_x^2 Q + u_{xx} D_x Q,$$

so $\mathsf{D}_P = u_x D_x^2 + u_{xx} D_x$. This calculation easily generalizes and shows that the Fréchet derivative of a general r-tuple $P = (P_1, \ldots, P_r)$ is the $q \times r$ matrix differential operator with entries

$$(\mathsf{D}_P)_{\mu\nu} = \sum_J (\partial P_\mu / \partial u_J^\nu) D_J, \qquad \mu = 1, \ldots, r, \quad \nu = 1, \ldots, q, \tag{5.32}$$

the sum being over all multi-indices J. In particular, if $P = \Delta[u]$ is a linear differential polynomial, then $\mathsf{D}_P = \Delta$ is the same as the differential operator determining it. There is an intimate connection between the Fréchet derivative and evolutionary vector fields.

Proposition 5.25. *If $P \in \mathscr{A}^r$ and $Q \in \mathscr{A}^q$, then*

$$\mathsf{D}_P(Q) = \operatorname{pr} \mathbf{v}_Q(P). \tag{5.33}$$

This follows directly from the formulae (5.6), (5.32). Alternatively, one can remark that both sides determine the infinitesimal variation in P under the action of the one-parameter group generated by $\mathbf{v}_Q$, cf. (5.16), and hence must agree. □

The Fréchet derivative is a derivation, meaning that it satisfies a Leibniz rule.

Proposition 5.26. *The Fréchet derivative of the product of two differential functions is*

$$\mathsf{D}_{PQ} = P\,\mathsf{D}_Q + Q\,\mathsf{D}_P. \tag{5.34}$$

The proof of this and the following proposition follows directly from the definition (5.31) of the Fréchet derivative.

Proposition 5.27. *Let $P \in \mathscr{A}^s$. Let $\mathscr{D}$ be a constant coefficient $r \times s$ matrix differential operator, so that $\mathscr{D}P \in \mathscr{A}^r$. Then the Fréchet derivative of $\mathscr{D}P$ is the product of two differential operators*

$$\mathsf{D}_{\mathscr{D}P} = \mathscr{D} \cdot \mathsf{D}_P. \tag{5.35}$$

For example, the Fréchet derivative of the total derivative $D_x P$ of a differential function P is $\mathsf{D}_{D_x P} = D_x \cdot \mathsf{D}_P$. Thus, the Fréchet derivative of $P = uu_{xx} + u_x^2 = D_x(uu_x) = D_x^2(\frac{1}{2}u^2)$ is $\mathsf{D}_P = uD_x^2 + 2u_x D_x + u_{xx} = D_x \cdot (uD_x + u_x) = D_x^2 \cdot u$.

Lie Derivatives of Differential Operators

Recall that the Lie derivative of an ordinary geometric object with respect to a vector field, as introduced in Section 1.5, represents the infinitesimal change in the object under the flow induced by the vector field. Analogous concepts exist for generalized vector fields; here we are particularly interested in the Lie derivative of a differential operator with respect to an evolutionary vector field. If $\mathbf{v}_Q$ is an evolutionary vector field, and $\mathscr{D} = \sum P_K[u] D_K$ a differential operator whose coefficients may depend on u, then we define the *Lie derivative* of $\mathscr{D}$ to be the infinitesimal change in $\mathscr{D}$ under the one-parameter group $\exp(t\mathbf{v}_Q)$. This is computed by evaluating the "time" derivative of $\mathscr{D}$:

$$\mathscr{D}_t = \sum_K D_t(P_K) D_K, \tag{5.36}$$

on solutions to the associated evolution equation $u_t = Q$, leading to the formula

$$\operatorname{pr} \mathbf{v}_Q(\mathscr{D}) = \sum_K \operatorname{pr} \mathbf{v}_Q(P_K) D_K. \tag{5.37}$$

For example, if $\mathscr{D} = D_x^3 + 2uD_x + u_x$, and $\mathbf{v}_Q = u_{xx}\partial_u$, then

$$\text{pr}\,\mathbf{v}_Q(\mathscr{D}) = 2\,\text{pr}\,\mathbf{v}_Q(u)D_x + \text{pr}\,\mathbf{v}_Q(u_x) = 2u_{xx}D_x + u_{xxx}.$$

The one-parameter group $\exp(t\mathbf{v}_Q)$ in this case is found by solving the heat equation $u_t = u_{xx}$ and $\text{pr}\,\mathbf{v}_Q(\mathscr{D})$ represents the infinitesimal change in $\mathscr{D}$ when $u(x, t)$ is a solution to the heat equation. Alternatively, we can identify $\text{pr}\,\mathbf{v}_Q(\mathscr{D})$ with the time derivative $\mathscr{D}_t = 2u_tD_x + u_{xt}$ of $\mathscr{D}$ (note that D_x does not depend on t) evaluated on solutions to the heat equation.

This definition of the Lie derivative extends to matrix differential operators, where now $\text{pr}\,\mathbf{v}_Q$ acts on the individual entries. A straightforward computation proves that the Lie derivative is a derivation satisfying the following Leibniz rule:

$$\text{pr}\,\mathbf{v}_Q(\mathscr{D}P) = \text{pr}\,\mathbf{v}_Q(\mathscr{D})\cdot P + \mathscr{D}[\text{pr}\,\mathbf{v}_Q(P)]. \tag{5.38}$$

For $q \times q$ matrix differential operators, there is a second type of Lie derivative which arises from a different "tensorial" interpretation of the operator. Here, instead of letting the differential operator act on q-tuples of differential functions, we regard it as a linear transformation on the space of evolutionary vector fields. Specifically, if $\mathscr{R} = (\mathscr{R}^\alpha_\beta)$ is a $q \times q$ matrix differential operator, and $\mathbf{v}_Q$ an evolutionary vector field with characteristic $Q = (Q_1, \dots, Q_q)^T$ (viewed as a column vector), we define $\mathscr{R}(\mathbf{v}_Q)$ to be the evolutionary vector field with characteristic $\mathscr{R}Q$, so[†]

$$\mathscr{R}(\mathbf{v}_Q) = \mathbf{v}_{\mathscr{R}Q}. \tag{5.39}$$

Note that, although an evolutionary vector field is prescribed by a q-tuple of differential functions, its transformation properties under a change of variables are different (just as the transformation rules for ordinary vector fields are different from those of points).

As before, the Lie derivative of the operator $\mathscr{R}$ with respect to an evolutionary vector field is defined to be its infinitesimal change under the one-parameter group generated by the vector field; however, since the operator acts on vector fields, the formula in this case is different. Indeed, in analogy with the formula for the Lie derivative of an ordinary vector field with respect to another vector field given in Proposition 1.64, the Lie derivative of an evolutionary vector field $\mathbf{v}_P$ with respect to another vector field $\mathbf{v}_Q$ will be provided by the Lie bracket $[\mathbf{v}_Q, \mathbf{v}_P]$ of Definition 5.14. (This can, of course, be justified directly.) To compute the infinitesimal change of the operator $\mathscr{R}$ with respect to a given evolutionary vector field $\mathbf{v}_Q$, we determine the Lie derivative of the vector field $\mathscr{R}\mathbf{v}_P = \mathbf{v}_{\mathscr{R}P}$, where P is an arbitrary q-tuple of differential functions, with respect to $\mathbf{v}_Q$. This is given by the Lie bracket

[†] In the language of tensor analysis, we are interpreting the differential operator $\mathscr{R}$ as a (1, 1)-tensor, since it maps vector fields to vector fields.

$[\mathbf{v}_Q, \mathbf{v}_{\mathscr{R}P}]$, which, according to Proposition 5.15, has characteristic

$$\begin{aligned}\mathrm{pr}\,\mathbf{v}_Q(\mathscr{R}P) - \mathrm{pr}\,\mathbf{v}_{\mathscr{R}P}(Q) &= \mathrm{pr}\,\mathbf{v}_Q(\mathscr{R})P + \mathscr{R}\,\mathrm{pr}\,\mathbf{v}_Q(P) - \mathrm{pr}\,\mathbf{v}_{\mathscr{R}P}(Q)\\ &= \{\mathrm{pr}\,\mathbf{v}_Q(\mathscr{R}) + \mathscr{R}\mathrm{D}_Q - \mathrm{D}_Q\mathscr{R}\}P\\ &\quad + \mathscr{R}\{\mathrm{pr}\,\mathbf{v}_Q(P) - \mathrm{pr}\,\mathbf{v}_P(Q)\},\end{aligned}$$

where we used (5.33) and (5.38). In the final formula, the second set of terms just gives the infinitesimal change in P; hence the first set of terms give the required formula for this alternative type of Lie derivative of the operator $\mathscr{R}$ with respect to $\mathbf{v}_Q$, for which we introduce the following notation and terminology.

Definition 5.28. Let $\mathscr{R}$ be a $q \times q$ matrix differential operator. The (1, 1)-*Lie derivative* of $\mathscr{R}$ with respect to a generalized vector field $\mathbf{v}_Q$ is the differential operator

$$\mathbf{v}_Q[\![\mathscr{R}]\!] = \mathscr{R}_t + [\mathscr{R}, \mathrm{D}_Q] = \mathrm{pr}\,\mathbf{v}_Q(\mathscr{R}) + [\mathscr{R}, \mathrm{D}_Q]. \tag{5.40}$$

(In (5.40), the time derivative $\mathscr{R}_t$ is evaluated on solutions to the associated evolution equation $u_t = Q$.)

The preceding computation proves the Leibniz rule

$$[\mathbf{v}_Q, \mathbf{v}_{\mathscr{R}P}] = [\mathbf{v}_Q, \mathscr{R}\mathbf{v}_P] = \mathbf{v}_Q[\![\mathscr{R}]\!]\mathbf{v}_P + \mathscr{R}[\mathbf{v}_Q, \mathbf{v}_P], \tag{5.41}$$

for the (1, 1)-Lie derivative.

Finally, we remark that, owing to their invariant definitions, the two kinds of Lie derivatives are unaffected by changes of variables. For instance, if we replace the dependent variable u by $v = \varphi(x, u)$, then the evolutionary vector field with characteristic $Q[u]$ in the u-coordinates has characteristic $\tilde{Q}[v] = \varphi_u \cdot Q[u]$ in the v-coordinates, where φ_u denotes the $q \times q$ Jacobian matrix with entries $\partial\varphi^\alpha/\partial u^\beta$; see Exercise 3.21. Consequently, a $q \times q$ matrix differential operator $\mathscr{R}$ mapping vector fields to vector fields will change to $\tilde{\mathscr{R}} = \varphi_u \cdot \mathscr{R} \cdot \varphi_u^{-1}$ in the v-coordinates. The (1, 1)-Lie derivative is invariant under such transformations, as can also be proved directly, cf. Exercise 5.16.

Criteria for Recursion Operators

Formula (5.33) readily leads to a general characterization of recursion operators.

Theorem 5.29. *Suppose $\Delta[u] = 0$ is a system of differential equations. If $\mathscr{R}: \mathscr{A}^q \to \mathscr{A}^q$ is a linear operator such that*

$$\mathrm{D}_\Delta \cdot \mathscr{R} = \tilde{\mathscr{R}} \cdot \mathrm{D}_\Delta \tag{5.42}$$

for all solutions u to Δ, where $\tilde{\mathscr{R}}: \mathscr{A}^q \to \mathscr{A}^q$ is a linear differential operator, then $\mathscr{R}$ is a recursion operator for the system.

PROOF. According to (5.33), an evolutionary vector field $\mathbf{v}_Q$ is a symmetry of Δ if and only if

$$\mathsf{D}_\Delta(Q) = \mathrm{pr}\,\mathbf{v}_Q(\Delta) = 0$$

for all solutions to Δ. If $\mathscr{R}$ satisfies (5.42), and $\tilde{Q} = \mathscr{R}Q$, then

$$\mathsf{D}_\Delta \tilde{Q} = \mathsf{D}_\Delta(\mathscr{R}Q) = \tilde{\mathscr{R}}(\mathsf{D}_\Delta Q) = 0$$

for all solutions. Thus $\tilde{Q}$ is also a symmetry, and the theorem follows. □

For example, suppose $\Delta[u] = u_t - K[u]$ is an evolution equation. Then $\mathsf{D}_\Delta = D_t - \mathsf{D}_K$. If $\mathscr{R}$ is a recursion operator, then it is not hard to see that the operator $\tilde{\mathscr{R}}$ in (5.42) must be the same as $\mathscr{R}$. Therefore, the condition (5.42) in this case reduces to the commutator condition

$$\mathscr{R}_t = [\mathsf{D}_K, \mathscr{R}] \tag{5.43}$$

for a recursion operator of an evolution equation. Note that condition (5.43) is the same as requiring that the (1, 1)-Lie derivative (5.40) of $\mathscr{R}$ with respect to the evolutionary vector field $\mathbf{v}_K$ vanishes:

$$\mathbf{v}_K[\![\mathscr{R}]\!] = 0. \tag{5.44}$$

Therefore, a recursion operator is nothing but an operator (or, rather, a (1, 1)-tensor) which is invariant under the flow of the evolution equation $u_t = K$!

Remark. Condition (5.43) has the form of a *Lax pair*, of fundamental importance to the inverse scattering approach to soliton theories; see Lax, [1], Dickey, [1], and Newell, [1]. However, in most cases, the recursion operator is not the usual spectral operator appearing in the Lax pair, but, rather, the "squared eigenfunction operator".

Example 5.30. Return to the potential Burgers' equation $u_t = K = u_{xx} + u_x^2$ for which we computed generalized symmetries in Examples 5.8 and 5.18. The structure of the resulting characteristics strongly suggests that, like the heat equation, Burgers' equation has two recursion operators. Inspection of Q_0, Q_1, Q_3, Q_6 and Q_{10} leads us to conjecture that $\mathscr{R}_1 = D_x + u_x$ is a recursion operator since $Q_1 = \mathscr{R}_1 Q_0$, $Q_3 = \mathscr{R}_1 Q_1$, etc. To prove this, we note that the Fréchet derivative for right-hand side of Burgers' equation (5.11) is

$$\mathsf{D}_K = D_x^2 + 2u_x D_x.$$

We must verify (5.43). The time derivative of the first recursion operator $\mathscr{R}_1$ on solutions to Burgers' equation is the multiplication operator

$$(\mathscr{R}_1)_t = (D_x + u_x)_t = u_{xt} = u_{xxx} + 2u_x u_{xx}.$$

On the other hand, the commutator is computed using Leibniz' rule for differential operators:

$$\begin{aligned}[\mathsf{D}_K, \mathscr{R}_1] &= (D_x^2 + 2u_x D_x)(D_x + u_x) - (D_x + u_x)(D_x^2 + 2u_x D_x)\\ &= (D_x^3 + 3u_x D_x^2 + 2(u_{xx} + u_x^2)D_x + u_{xxx} + 2u_x u_{xx})\\ &\quad - (D_x^3 + 3u_x D_x^2 + 2(u_{xx} + u_x^2)D_x)\\ &= u_{xxx} + 2u_x u_{xx}.\end{aligned}$$

Comparing these two verifies (5.43) and proves that $\mathscr{R}_1$ is a recursion operator for the potential Burgers' equation.

There is thus an infinite hierarchy of symmetries, with characteristics $\mathscr{R}_1^k Q_0$, $k = 0, 1, 2, \ldots$. For example, the next characteristic after Q_{10} in the sequence is

$$\begin{aligned}Q_{15} = \mathscr{R}_1 Q_{10} = u_{xxxxx} &+ 5u_x u_{xxxx} + 10u_{xx}u_{xxx} + 10u_x^2 u_{xxx}\\ &+ 15u_x u_{xx}^2 + 10u_x^3 u_{xx} + u_x^5.\end{aligned}$$

To obtain the characteristics depending on x and t, we require a second recursion operator, which, by inspection, we guess to be

$$\mathscr{R}_2 = t\mathscr{R}_1 + \tfrac{1}{2}x = tD_x + tu_x + \tfrac{1}{2}x.$$

Using the fact that $\mathscr{R}_1$ satisfies (5.43), we find

$$(\mathscr{R}_2)_t = t(\mathscr{R}_1)_t + \mathscr{R}_1 = t[\mathsf{D}_K, \mathscr{R}_1] + \mathscr{R}_1,$$

whereas

$$\begin{aligned}[\mathsf{D}_K, \mathscr{R}_2] &= t[\mathsf{D}_K, \mathscr{R}_1] + [D_x^2 + 2u_x D_x, \tfrac{1}{2}x]\\ &= t[\mathsf{D}_K, \mathscr{R}_1] + (D_x + u_x) = t[\mathsf{D}_K, \mathscr{R}_1] + \mathscr{R}_1,\end{aligned}$$

proving that $\mathscr{R}_2$ is also a recursion operator. There is thus a doubly infinite hierarchy of generalized symmetries of Burgers' equation, with characteristics $\mathscr{R}_2^l \mathscr{R}_1^k Q_0$, $k, l \geqslant 0$. For instance, $Q_2 = \mathscr{R}_2 Q_0$, $Q_4 = \mathscr{R}_2 \mathscr{R}_1 Q_0$, and so on.

The Korteweg–de Vries Equation

In the case of nonlinear equations, we often have to expand the class of possible recursion operators to include formal "pseudo-differential" operators. As an example, we prove that the operator

$$\mathscr{R} = D_x^2 + \tfrac{2}{3}u + \tfrac{1}{3}u_x D_x^{-1}$$

is a recursion operator for the Korteweg–de Vries equation, in the form

$$u_t = u_{xxx} + uu_x. \tag{5.45}$$

(This is the same as (2.66) under the change of variable $x \mapsto -x$.) The integral operator D_x^{-1} will *only* be defined on those differential functions which are total derivatives, so if $Q = D_x R$, then we set $R = D_x^{-1} Q$.† (Actually, this only defines $D_x^{-1} Q$ up to an additive constant, which we can normalize by requiring $R(0, 0) = 0$.) If $\mathbf{v}_Q$ is a generalized symmetry, then $\mathscr{R}Q$ will only be defined if $Q = D_x R$ for some $R \in \mathscr{A}$. Thus, we could run into difficulties in trying to obtain a full hierarchy of symmetries $\mathscr{R}^k Q$, $k = 0, 1, 2, \ldots$.

Before addressing these difficulties, we first show that formally $\mathscr{R}$ is a recursion operator. The relevant Fréchet derivative is

$$\mathsf{D}_K = D_x^3 + u D_x + u_x,$$

and we will prove that, on solutions of the Korteweg–de Vries equation, (5.43) holds. First,

$$\mathscr{R}_t = \tfrac{2}{3} u_t + \tfrac{1}{3} u_{xt} D_x^{-1} = \tfrac{2}{3}(u_{xxx} + u u_x) + \tfrac{1}{3}(u_{xxxx} + u u_{xx} + u_x^2) D_x^{-1}.$$

Next,

$$\mathsf{D}_K \mathscr{R} = D_x^5 + \tfrac{5}{3} u D_x^3 + \tfrac{10}{3} u_x D_x^2 + (3 u_{xx} + \tfrac{2}{3} u^2) D_x + \tfrac{5}{3}(u_{xxx} + u u_x)$$
$$+ \tfrac{1}{3}(u_{xxxx} + u u_{xx} + u_x^2) D_x^{-1}.$$

On the other hand, since $D_x \cdot u = u D_x + u_x$, we have $D_x^{-1} \cdot (u D_x + u_x) = u$. Therefore,

$$\mathscr{R} \mathsf{D}_K = D_x^5 + \tfrac{5}{3} u D_x^3 + \tfrac{10}{3} u_x D_x^2 + (3 u_{xx} + \tfrac{2}{3} u^2) D_x + (u_{xxx} + u u_x).$$

Therefore, the required commutator is

$$[\mathsf{D}_K, \mathscr{R}] = \tfrac{2}{3}(u_{xxx} + u u_x) + \tfrac{1}{3}(u_{xxxx} + u u_{xx} + u_x^2) D_x^{-1},$$

which, on comparison with the formula for $\mathscr{R}_t$, verifies (5.43) and proves that $\mathscr{R}$ is a recursion operator for the Korteweg–de Vries equation.

If we start applying $\mathscr{R}$ successively to the translational symmetry $-\partial_x$, with characteristic $Q_0 = u_x$, we first obtain

$$Q_1 = \mathscr{R} Q_0 = u_{xxx} + u u_x,$$

which is equivalent to the characteristic u of the translational symmetry $-\partial_t$. Noting that $Q_1 = D_x(u_{xx} + \frac{1}{2} u^2)$, we find

$$Q_2 = \mathscr{R} Q_1 = u_{xxxxx} + \tfrac{5}{3} u u_{xxx} + \tfrac{10}{3} u_x u_{xx} + \tfrac{5}{6} u^2 u_x$$

to be the characteristic of a genuine generalized symmetry, as the reader can check. Similarly

$$Q_3 = \mathscr{R} Q_2 = u_{xxxxxxx} + \tfrac{7}{3} u u_{xxxxx} + 7 u_x u_{xxxx} + \tfrac{35}{3} u_{xx} u_{xxx} + \tfrac{35}{18} u^2 u_{xxx}$$
$$+ \tfrac{70}{9} u u_x u_{xx} + \tfrac{35}{18} u_x^3 + \tfrac{35}{54} u^3 u_x$$

† More generally, one might try defining $D_x^{-1} P = \int_0^x P(x, u^{(n)})\, dx$, but this takes us outside the class of differential functions.