cient functions $\xi^i(x)$, which depend on H, are to be determined. Let $F(x)$ be a second smooth function. Using (6.4), we find

$$\{F, H\} = \hat{\mathbf{v}}_H(F) = \sum_{i=1}^{m} \xi^i(x) \frac{\partial F}{\partial x^i}.$$

But, again by (6.4),

$$\xi^i(x) = \hat{\mathbf{v}}_H(x^i) = \{x^i, H\},$$

so this formula becomes

$$\{F, H\} = \sum_{i=1}^{m} \{x^i, H\} \frac{\partial F}{\partial x^i}. \tag{6.9}$$

On the other hand, using the skew-symmetry of the Poisson bracket, we can turn this whole procedure around and compute the latter set of Poisson brackets in terms of the particular Hamiltonian vector fields $\hat{\mathbf{v}}_i = \hat{\mathbf{v}}_{x^i}$ associated with the local coordinate functions x^i; namely

$$\{x^i, H\} = -\{H, x^i\} = -\hat{\mathbf{v}}_i(H) = -\sum_{j=1}^{m} \{x^j, x^i\} \frac{\partial H}{\partial x^j},$$

the last equality following from a second application of (6.9), with H replacing F and x^i replacing H. Thus we obtain the basic formula

$$\{F, H\} = \sum_{i=1}^{m} \sum_{j=1}^{m} \{x^i, x^j\} \frac{\partial F}{\partial x^i} \frac{\partial H}{\partial x^j} \tag{6.10}$$

for the Poisson bracket. In other words, to compute the Poisson bracket of any pair of functions in some given set of local coordinates, it suffices to know the Poisson brackets between the coordinate functions themselves. These basic brackets,

$$J^{ij}(x) = \{x^i, x^j\}, \qquad i, j = 1, \ldots, m, \tag{6.11}$$

are called the *structure functions* of the Poisson manifold M relative to the given local coordinates, and serve to uniquely determine the Poisson structure itself. For convenience, we assemble the structure functions into a skew-symmetric $m \times m$ matrix $J(x)$, called the *structure matrix* of M. Using ∇H to denote the (column) gradient vector for H, the local coordinate form (6.10) for the Poisson bracket can be written as

$$\{F, H\} = \nabla F \cdot J \nabla H. \tag{6.12}$$

For example, in the case of the canonical bracket (6.1) on $\mathbb{R}^m$, $m = 2n + l$, the structure matrix has the simple form

$$J = \begin{pmatrix} 0 & -\mathrm{I} & 0 \\ \mathrm{I} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

relative to the (p, q, z)-coordinates, where I is the $n \times n$ identity matrix.

The Hamiltonian vector field associated with $H(x)$ has the form

$$\hat{\mathbf{v}}_H = \sum_{i=1}^{m} \left(\sum_{j=1}^{m} J^{ij}(x) \frac{\partial H}{\partial x^j} \frac{\partial}{\partial x^i} \right), \tag{6.13}$$

or, in matrix notation, $\hat{\mathbf{v}}_H = (J \nabla H) \cdot \partial_x$, ∂_x being the "vector" with entries $\partial/\partial x^i$. Therefore, in the given coordinate chart, Hamilton's equations take the form†

$$\frac{dx}{dt} = J(x) \nabla H(x). \tag{6.14}$$

Alternatively, using (6.9), we could write this in the "bracket form"

$$\frac{dx}{dt} = \{x, H\},$$

the i-th component of the right-hand side being $\{x^i, H\}$. A system of first order ordinary differential equations is said to be a *Hamiltonian system* if there is a Hamiltonian function $H(x)$ and a matrix of functions $J(x)$ determining a Poisson bracket (6.12) whereby the system takes the form (6.14). Of course, we need to know which matrices $J(x)$ are the structure matrices for Poisson brackets.

Proposition 6.8. *Let $J(x) = (J^{ij}(x))$ be an $m \times m$ matrix of functions of $x = (x^1, \ldots, x^m)$ defined over an open subset $M \subset \mathbb{R}^m$. Then $J(x)$ is the structure matrix for a Poisson bracket $\{F, H\} = \nabla F \cdot J \nabla H$ over M if and only if it has the properties of:*

(a) Skew-Symmetry:

$$J^{ij}(x) = -J^{ji}(x), \qquad i, j = 1, \ldots, m,$$

(b) Jacobi Identity:

$$\sum_{l=1}^{m} \{J^{il} \partial_l J^{jk} + J^{kl} \partial_l J^{ij} + J^{jl} \partial_l J^{ki}\} = 0, \qquad i, j, k = 1, \ldots, m, \tag{6.15}$$

for all $x \in M$. (Here, as usual, $\partial_l = \partial/\partial x^l$.)

Proof. In its basic form (6.12) the Poisson bracket is automatically bilinear and satisfies Leibniz' rule. The skew-symmetry of the structure matrix is clearly equivalent to the skew-symmetry of the bracket. Thus we need only verify the equivalence of (6.15) with the Jacobi identity. Note that by (6.10), (6.11)

$$\{\{x^i, x^j\}, x^k\} = \sum_{l=1}^{m} J^{lk}(x) \partial_l J^{ij}(x),$$

† More generally, we can allow $H(x, t)$ to depend on t as well, which leads to a time-dependent Hamiltonian vector field; see Section 6.3.

so (6.15) is equivalent to the Jacobi identity for the coordinate functions x^i, x^j and x^k. More generally, for $F, H, P\colon M \to \mathbb{R}$,

$$\begin{aligned}\{\{F,H\},P\} &= \sum_{k,l=1}^{m} J^{lk}\frac{\partial}{\partial x^l}\left\{\sum_{i,j=1}^{m} J^{ij}\frac{\partial F}{\partial x^i}\frac{\partial H}{\partial x^j}\right\}\frac{\partial P}{\partial x^k}\\ &= \sum_{i,j,k,l}\left\{J^{lk}\frac{\partial J^{ij}}{\partial x^l}\frac{\partial F}{\partial x^i}\frac{\partial H}{\partial x^j}\frac{\partial P}{\partial x^k}\right.\\ &\quad\left. + J^{lk}J^{ij}\left(\frac{\partial^2 F}{\partial x^l\partial x^i}\frac{\partial H}{\partial x^j}\frac{\partial P}{\partial x^k} + \frac{\partial F}{\partial x^i}\frac{\partial^2 H}{\partial x^l\partial x^j}\frac{\partial P}{\partial x^k}\right)\right\}.\end{aligned}$$

Summing cyclically on F, H, P, we find that the first set of terms vanishes by virtue of (6.15), while the remaining terms cancel due to the skew-symmetry of the structure matrix. □

Note that we could just as well take the requirements of Proposition 6.8 on the structure matrix as the definition of a Poisson bracket (6.12) in a local coordinate chart. The conditions (6.15) guaranteeing the Jacobi identity form a large system of *nonlinear* partial differential equations which the structure functions must satisfy. In particular, any constant skew-symmetric matrix J trivially satisfies (6.15) and thus determines a Poisson bracket.

The Lie–Poisson Structure

One of the most important examples of a Poisson structure is that associated with an r-dimensional Lie algebra $\mathfrak{g}$. Let c_{ij}^k, $i, j, k = 1, \dots, r$, be the structure constants of $\mathfrak{g}$ relative to a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$. Let V be another r-dimensional vector space, with coordinates $x = (x^1, \dots, x^r)$ determined by a basis $\{\omega_1, \dots, \omega_r\}$. Define the *Lie-Poisson bracket* between two smooth functions $F, H\colon V \to \mathbb{R}$ to be

$$\{F,H\} = \sum_{i,j,k=1}^{r} c_{ij}^k x^k \frac{\partial F}{\partial x^i}\frac{\partial H}{\partial x^j}. \tag{6.16}$$

This clearly takes the form (6.10) with linear structure functions $J^{ij}(x) = \sum_{k=1}^r c_{ij}^k x^k$. The verification of the properties of Proposition 6.8 for the structure matrix follows easily from the basic properties (1.43), (1.44) of the structure constants; in particular, (6.15) reduces to the Jacobi identity (1.44), as the reader can easily verify.

There is a more intrinsic characterization of the Lie–Poisson bracket. First, recall that if V is any vector space and $F\colon V \to \mathbb{R}$ a smooth, real-valued function, then the gradient $\nabla F(x)$ at any point $x \in V$ is naturally an element of the dual vector space V^* consisting of all (continuous) linear functions on V. Indeed, by definition,

$$\langle \nabla F(x); y\rangle = \lim_{\varepsilon\to 0}\frac{F(x+\varepsilon y) - F(x)}{\varepsilon}$$

for any $y \in V$, where $\langle\ ;\ \rangle$ is the natural pairing between V and its dual V^*. Keeping this in mind, we identify the vector space V used in our initial construction of the Lie–Poisson bracket with the dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$, $\{\omega_1, \dots, \omega_r\}$ being the dual basis to $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$. If $F\colon \mathfrak{g}^* \to \mathbb{R}$ is any smooth function, then its gradient $\nabla F(x)$ is an element of $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$ (since $\mathfrak{g}$ is finite-dimensional). Then the Lie–Poisson bracket has the coordinate-free form

$$\{F, H\}(x) = \langle x; [\nabla F(x), \nabla H(x)]\rangle, \qquad x \in \mathfrak{g}^*, \tag{6.17}$$

where $[\ ,\ ]$ is the ordinary Lie bracket on the Lie algebra $\mathfrak{g}$ itself; the proof is left to the reader. If $H\colon \mathfrak{g} \to \mathbb{R}$ is any function, the associated system of Hamilton's equations takes the form

$$\frac{dx^i}{dt} = \sum_{j,k=1}^{r} c_{ij}^k x^k \frac{\partial H}{\partial x^j}, \qquad i = 1, \dots, r,$$

in which the coordinates x^k themselves appear explicitly.

Example 6.9. Consider the three-dimensional Lie algebra $\mathfrak{so}(3)$ of the rotation group $SO(3)$. Using the basis $\mathbf{v}_1 = y\partial_z - z\partial_y$, $\mathbf{v}_2 = z\partial_x - x\partial_z$, $\mathbf{v}_3 = x\partial_y - y\partial_x$ of infinitesimal rotations around the x-, y- and z-axes of $\mathbb{R}^3$ (or their matrix counterparts), we have the commutation relations $[\mathbf{v}_1, \mathbf{v}_2] = -\mathbf{v}_3$, $[\mathbf{v}_3, \mathbf{v}_1] = -\mathbf{v}_2$, $[\mathbf{v}_2, \mathbf{v}_3] = -\mathbf{v}_1$. Let ω_1, ω_2, ω_3 be a dual basis for $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$ and $u = u^1\omega_1 + u^2\omega_2 + u^3\omega_3$ a typical point therein. If $F\colon \mathfrak{so}(3)^* \to \mathbb{R}$, then its gradient is the vector

$$\nabla F = \frac{\partial F}{\partial u^1}\mathbf{v}_1 + \frac{\partial F}{\partial u^2}\mathbf{v}_2 + \frac{\partial F}{\partial u^3}\mathbf{v}_3 \in \mathfrak{so}(3).$$

Thus from (6.17) we find the Lie–Poisson bracket on $\mathfrak{so}(3)^*$ to be

$$\begin{aligned}\{F, H\} &= u^1\left(\frac{\partial F}{\partial u^3}\frac{\partial H}{\partial u^2} - \frac{\partial F}{\partial u^2}\frac{\partial H}{\partial u^3}\right) + u^2\left(\frac{\partial F}{\partial u^1}\frac{\partial H}{\partial u^3} - \frac{\partial F}{\partial u^3}\frac{\partial H}{\partial u^1}\right)\\ &\quad + u^3\left(\frac{\partial F}{\partial u^2}\frac{\partial H}{\partial u^1} - \frac{\partial F}{\partial u^1}\frac{\partial H}{\partial u^2}\right)\\ &= -u \cdot \nabla F \times \nabla H,\end{aligned}$$

using the standard cross product on $\mathbb{R}^3$. Thus the structure matrix is

$$J(u) = \begin{bmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{bmatrix}, \qquad u \in \mathfrak{so}(3)^*.$$

Hamilton's equations corresponding to the Hamiltonian function $H(u)$ are therefore

$$\frac{du}{dt} = u \times \nabla H(u).$$

For example, if

$$H(u) = \frac{(u^1)^2}{2I_1} + \frac{(u^2)^2}{2I_2} + \frac{(u^3)^2}{2I_3},$$

where I_1, I_2, I_3 are certain constants, then Hamilton's equations become the *Euler equations* for the motion of a rigid body

$$\frac{du^1}{dt} = \frac{I_2 - I_3}{I_2 I_3} u^2 u^3, \qquad \frac{du^2}{dt} = \frac{I_3 - I_1}{I_3 I_1} u^3 u^1, \qquad \frac{du^3}{dt} = \frac{I_1 - I_2}{I_1 I_2} u^1 u^2, \tag{6.18}$$

in which (I_1, I_2, I_3) are the moments of inertia about the coordinate axes and u^1, u^2, u^3 the corresponding body angular momenta. (The angular velocities are $\omega^i = u^i/I_i$.) The Hamiltonian function is the kinetic energy of the body.

6.2. Symplectic Structures and Foliations

In order to gain a more complete understanding of the geometry underlying a general Poisson structure on a smooth manifold, we need to look more closely at the structure matrix $J(x)$ which determines the local coordinate form of the Poisson bracket. The most important invariant of this matrix is its rank. If the rank is maximal everywhere, then, as we will see, we are in the more standard situation of a "symplectic structure" on a smooth manifold, treated in most books on Hamiltonian mechanics. In the more general case of nonmaximal rank, the Poisson manifold M will be seen to be naturally foliated into symplectic submanifolds in such a way that any Hamiltonian system on M naturally restricts to any one of the symplectic submanifolds and hence, by restriction, returns us to the more classical case of Hamiltonian mechanics. However, for many problems it is more natural to remain on the larger Poisson manifold itself, especially when one is interested in the collective behaviour of systems depending on parameters, with the underlying symplectic structure varying with the parameters themselves.

The Correspondence Between One-Forms and Vector Fields

As we saw in the previous section, a Poisson structure on a manifold M sets up a correspondence between smooth functions $H: M \to \mathbb{R}$ and their associated Hamiltonian vector field $\hat{\mathbf{v}}_H$ on M. In local coordinates, this correspondence is determined by multiplication of the gradient ∇H by the structure matrix $J(x)$ determined by the Poisson bracket. This can be given a more intrinsic formulation if we recall that the coordinate-free version of the gradient of a real-valued function H is its differential dH. Thus the Poisson structure determines a correspondence between differential one-forms dH on M and their associated Hamiltonian vector fields $\hat{\mathbf{v}}_H$, which in fact extends to general one-forms:

Proposition 6.10. *Let M be a Poisson manifold and $x \in M$. Then there exists a unique linear map*

$$\mathsf{B} = \mathsf{B}|_x \colon T^*M|_x \to TM|_x,$$

from the cotangent space to M at x to the corresponding tangent space, such that for any smooth real-valued function $H \colon M \to \mathbb{R}$,

$$\mathsf{B}(dH(x)) = \hat{\mathbf{v}}_H|_x. \tag{6.19}$$

PROOF. At any point $x \in M$, the cotangent space $T^*M|_x$ is spanned by the differentials $\{dx^1, \ldots, dx^m\}$ corresponding to the local coordinate functions near x. From (6.13), we see that at $x \in M$

$$\mathsf{B}(dx^j) = \sum_{i=1}^m J^{ij}(x)\frac{\partial}{\partial x^i}\bigg|_x, \qquad j = 1, \ldots, m.$$

By linearity, for any $\omega = \sum a_j\, dx^j \in T^*M|_x$,

$$\mathsf{B}(\omega) = \sum_{i,j=1}^m J^{ij}(x) a_j \frac{\partial}{\partial x^i}\bigg|_x$$

is essentially matrix multiplication by the structure matrix $J(x)$, proving the proposition. □

Example 6.11. In the case of $\mathbb{R}^m$ with canonical coordinates (p, q, z), as in Example 6.2, if

$$\omega = \sum_{i=1}^n [a_i dp^i + b_i dq^i] + \sum_{j=1}^l c_j\, dz^j$$

is any one-form, then

$$\mathsf{B}(\omega) = \sum_{i=1}^n \left\{a_i \frac{\partial}{\partial q^i} - b_i \frac{\partial}{\partial p^i}\right\}.$$

In this particular case, the form of B does not vary from point to point. In particular, the kernel of B has the same dimension as the number of distinguished coordinates $z^1, \ldots, z^l$.

Rank of a Poisson Structure

Definition 6.12. Let M be a Poisson manifold and $x \in M$. The *rank* of M at x is the rank of the linear map $\mathsf{B}|_x \colon T^*M|_x \to TM|_x$.

In local coordinates, $\mathsf{B}|_x$ is the same as multiplication by the structure matrix $J(x)$, so the rank of M at x equals the rank of $J(x)$, independent of the choice of coordinates. Skew-symmetry of J immediately implies:

Proposition 6.13. *The rank of a Poisson manifold at any point is always an* even *integer.*

For example, the canonical Poisson structure (6.1) on $\mathbb{R}^m$, $m = 2n + l$, is of constant rank $2n$ everywhere. Later we will see that every Poisson structure of constant rank $2n$ looks locally like the canonical structure of this rank. In the case of the Lie–Poisson structure on $\mathfrak{so}(3)^*$, the rank is 2 everywhere except at the origin $u = 0$, where the rank is 0.

Since the rank of a linear mapping is determined by the dimension of its kernel, or of its range, we can compute the rank $2n$ of a Poisson manifold at a point either by looking at $\mathscr{K}|_x = \{\omega \in T^*M|_x\colon \mathsf{B}(\omega) = 0\}$, which has dimension $m - 2n$, or the image space $\mathscr{H}|_x = \{\mathbf{v} = \mathsf{B}(\omega) \in TM|_x\colon \omega \in T^*M|_x\}$, which has dimension $2n$. For instance, in the case of the canonical Poisson bracket (6.1), $\mathscr{K}|_x$ is spanned by the "distinguished differentials" $dz^1, \dots, dz^l$, while $\mathscr{H}|_x$ is spanned by the elementary Hamiltonian vector fields $\partial/\partial q^i$, $\partial/\partial p^i$ corresponding to the coordinate functions p^i, $-q^i$ respectively. The image space $\mathscr{H}|_x$ is of particular significance; it can be characterized as the span of all the Hamiltonian vector fields on M at x:

$$\mathscr{H}|_x = \{\hat{\mathbf{v}}_H|_x\colon \quad H\colon M \to \mathbb{R} \text{ is smooth}\}.$$

Symplectic Manifolds

In classical mechanics, one usually imposes an additional nondegeneracy requirement on the Poisson bracket, which leads to the more restrictive notion of a symplectic structure on a manifold.

Definition 6.14. A Poisson manifold M of dimension m is *symplectic* if its Poisson structure has maximal rank m everywhere.

In particular, according to Proposition 6.13, a symplectic manifold is necessarily even-dimensional. The canonical example is the Poisson bracket (6.1) on $\mathbb{R}^m$ in the case $m = 2n$, so there are no extra distinguished coordinates. In terms of local coordinates, a structure matrix $J(x)$ determines a symplectic structure provided it satisfies the additional nondegeneracy condition $\det J(x) \neq 0$ everywhere. In this case, the complicated nonlinear equations (6.15) describing the Jacobi identity simplify to a *linear* system of differential equations involving the entries of the inverse matrix $K(x) = [J(x)]^{-1}$.

Proposition 6.15. *A matrix $J(x)$ determines a symplectic structure on $M \subset \mathbb{R}^m$ if and only if its inverse $K(x) = [J(x)]^{-1}$ satisfies the conditions*:

(a) Skew-Symmetry:

$$K_{ij}(x) = -K_{ji}(x), \qquad i, j = 1, \dots, m,$$

(b) Closure (Jacobi Identity):

$$\partial_k K_{ij} + \partial_j K_{ki} + \partial_i K_{jk} = 0, \qquad i, j, k = 1, \dots, m, \tag{6.20}$$

everywhere.

PROOF. The equivalence of the skew-symmetry of J to that of K is elementary. To prove the equivalence of (6.20) and (6.15), we use the formula for the derivative of a matrix inverse $\partial_k K = -K \cdot \partial_k J \cdot K$, where $K = J^{-1}$. Substituting into (6.20), we find

$$\sum_{l,n=1}^{m} \{K_{il}K_{jn}\partial_k J^{ln} + K_{kl}K_{in}\partial_j J^{ln} + K_{jl}K_{kn}\partial_i J^{ln}\} = 0.$$

Multiplying by $J^{\tilde{\imath}i}J^{\tilde{\jmath}j}J^{\tilde{k}k}$, and summing over i, j, k from 1 to m, leads to (6.15) with a slightly different labelling of indices. □

Maps Between Poisson Manifolds

If M and N are Poisson manifolds, a *Poisson map* is a smooth map $\phi: M \to N$ preserving the Poisson brackets:

$$\{F \circ \phi, H \circ \phi\}_M = \{F, H\}_N \circ \phi \qquad \text{for all } F, H: N \to \mathbb{R}.$$

In the case of symplectic manifolds these are the *canonical maps* of classical mechanics. A good example is provided by the flow generated by a Hamiltonian vector field.

Proposition 6.16. *Let M be a Poisson manifold and $\hat{\mathbf{v}}_H$ a Hamiltonian vector field. For each t, the flow* $\exp(t\hat{\mathbf{v}}_H)$: *$M \to M$ determines a (local) Poisson map from M to itself.*

PROOF. Let F and P be real-valued functions, and let $\phi_t = \exp(t\hat{\mathbf{v}}_H)$. If we differentiate the Poisson condition $\{F \circ \phi_t, P \circ \phi_t\} = \{F, P\} \circ \phi_t$ with respect to t and use (1.17), we find the infinitesimal version

$$\{\hat{\mathbf{v}}_H(F), P\} + \{F, \hat{\mathbf{v}}_H(P)\} = \hat{\mathbf{v}}_H(\{F, P\})$$

at the point $\phi_t(x)$. By (6.4) this is the same as the Jacobi identity. At $t = 0$, ϕ_0 is the identity, and trivially Poisson, so a simple integration proves the Poisson condition for general t. □

For example, if $M = \mathbb{R}^2$ with canonical coordinates (p, q), then the function $H = \frac{1}{2}(p^2 + q^2)$ generates the group of rotations in the plane, determined by $\hat{\mathbf{v}}_H = p\partial_q - q\partial_p$. Thus each rotation in $\mathbb{R}^2$ is a canonical map.

Since any Hamiltonian flow preserves the Poisson bracket on M, in particular it preserves its rank.

Corollary 6.17. *If $\hat{\mathbf{v}}_H$ is a Hamiltonian vector field on a Poisson manifold M, then the rank of M at* $\exp(t\hat{\mathbf{v}}_H)x$ *is the same as the rank of M at x for any $t \in \mathbb{R}$.*

For instance, the origin in $\mathfrak{so}(3)^*$, being the only point of rank 0, is a fixed point of any Hamiltonian system with the given Lie–Poisson structure. In fact, it is easy to see that any point of rank 0 on a Poisson manifold is a fixed point for any Hamiltonian system there.

Poisson Submanifolds

Definition 6.18. A submanifold $N \subset M$ is a *Poisson submanifold* if its defining immersion $\phi\colon \tilde{N} \to M$ is a Poisson map.

An equivalent way of stating this definition is that for any pair of functions $F, H\colon M \to \mathbb{R}$ which restrict to functions $\tilde{F}, \tilde{H}\colon N \to \mathbb{R}$ on N, their Poisson bracket $\{F, H\}_M$ naturally restricts to a Poisson bracket $\{\tilde{F}, \tilde{H}\}_N$. For example, the submanifolds $\{z = c\}$ of $\mathbb{R}^m$, $m = 2n + l$ corresponding to constant values of the distinguished coordinates are easily seen to be Poisson submanifolds, with the natural reduced Poisson bracket with respect to the remaining coordinates (p, q).

If $N \subset M$ is an arbitrary submanifold then there is a simple test that will tell whether or not it can be made into a Poisson submanifold, the reduced Poisson structure, if it exists, being uniquely determined by the above remark.

Proposition 6.19. *A submanifold N of a Poisson manifold M is a Poisson submanifold if and only if $TN|_y \supset \mathscr{H}|_y$ for all $y \in N$, meaning every Hamiltonian vector field on M is everywhere tangent to N. In particular, if $TN|_y = \mathscr{H}|_y$ for all $y \in N$, N is a symplectic submanifold of M.*

PROOF. Since a Poisson bracket is determined by its local character, we can without loss of generality assume that N is a regular submanifold of M and use flat local coordinates $(y, w) = (y^1, \ldots, y^n, w^1, \ldots, w^{m-n})$ with $N = \{(y, w)\colon w = 0\}$. First suppose that N is a Poisson submanifold, and let $\tilde{H}\colon N \to \mathbb{R}$ be any smooth function. Then we can extend $\tilde{H}$ to a smooth function $H\colon M \to \mathbb{R}$ defined in a neighbourhood of N, with $\tilde{H} = H|N$. In our local coordinates, $\tilde{H} = \tilde{H}(y)$ and $H(y, w)$ is any function so that $H(y, 0) = \tilde{H}(y)$. If $\tilde{F}\colon N \to \mathbb{R}$ has a similar extension F, then by definition the Poisson bracket between $\tilde{F}$ and $\tilde{H}$ on N is obtained by restricting that of F and H to N:

$$\{\tilde{F}, \tilde{H}\}_N = \{F, H\}|N.$$

In particular, for any choice of $\tilde{F}, \tilde{H}$, the bracket $\{F, H\}|N$ cannot depend on the particular extensions F and H which are selected. Clearly, this is possible if and only if $\{F, H\}|N$ contains no partial derivatives of either F or H with respect to the normal coordinates w^i, so

$$\{F, H\}|N = \sum_{i,j} J^{ij}(y, 0)\frac{\partial F}{\partial y^i}\frac{\partial H}{\partial y^j} \equiv \sum_{i,j} \tilde{J}^{ij}(y)\frac{\partial \tilde{F}}{\partial y^i}\frac{\partial \tilde{H}}{\partial y^j}. \tag{6.21}$$

But then the Hamiltonian vector field $\hat{\mathbf{v}}_H$, when restricted to N, takes the form

$$\hat{\mathbf{v}}_H|N = \sum_{i,j} \tilde{J}^{ij}(y)\frac{\partial H}{\partial y^j}\frac{\partial}{\partial y^i}, \tag{6.22}$$

and is thus tangent to N everywhere.

Conversely, if the tangency condition $\mathscr{H}|_y \subset TN|_y$ holds for all $y \in N$, any Hamiltonian vector field, when restricted to N, must be a combination of the tangential basis vectors $\partial/\partial y^i$ only, and hence of the form (6.22). If $F(w)$ depends on w alone, then $\{F, H\} = \hat{\mathbf{v}}_H(F)$ must therefore vanish when restricted to N. In particular,

$$\{y^i, w^j\} = \{w^k, w^j\} = 0 \qquad \text{on } N \qquad \text{for all } i, j, k,$$

and hence the Poisson bracket on N takes the form (6.21) in which $\tilde{J}^{ij}(y) = J^{ij}(y, 0) = \{y^i, y^j\}|N$. The fact that the structure functions $\tilde{J}^{ij}(y)$ of the induced Poisson bracket on N satisfy the Jacobi identity easily follows from (6.15) since on restriction to N all the w-terms vanish. Thus N is a Poisson submanifold and the proposition is proved. Note that the rank of the Poisson structure on N at $y \in N$ equals the rank of the Poisson structure on M at the same point. $\square$

Example 6.20. For the Lie–Poisson structure on $\mathfrak{so}(3)^*$, the subspace $\mathscr{H}|_u$ at $u \in \mathfrak{so}(3)^*$ is spanned by the elementary Hamiltonian vectors $\hat{\mathbf{v}}_1 = u^3\partial_2 - u^2\partial_3$, $\hat{\mathbf{v}}_2 = u^1\partial_3 - u^3\partial_1$, $\hat{\mathbf{v}}_3 = u^2\partial_1 - u^1\partial_2$, ($\partial_i = \partial/\partial u^i$), corresponding to the coordinate functions u^1, u^2, u^3 respectively. If $u \neq 0$, these vectors span a two-dimensional subspace of $T\mathfrak{so}(3)^*|_u$, which coincides with the tangent space to the sphere $S^2_\rho = \{u: |u| = \rho\}$ passing through u: $\mathscr{H}|_u = TS^2_\rho|_u$, $|u| = \rho$. Proposition 6.19 therefore implies that each such sphere is a symplectic submanifold of $\mathfrak{so}(3)^*$. In terms of spherical coordinates $u^1 = \rho\cos\theta\sin\phi$, $u^2 = \rho\sin\theta\sin\phi$, $u^3 = \rho\cos\phi$ on S^2_ρ, the Poisson bracket between $\tilde{F}(\theta, \phi)$ and $\tilde{H}(\theta, \phi)$ is computed by extending them to a neighbourhood of S^2_ρ, e.g. set $F(\rho, \theta, \phi) = \tilde{F}(\theta, \phi)$, $H(\rho, \theta, \phi) = \tilde{H}(\theta, \phi)$, computing the Lie–Poisson bracket $\{F, H\}$, and then restricting to S^2_ρ. However, according to (6.10), $\{\tilde{F}, \tilde{H}\} = \{\theta, \phi\}(\tilde{F}_\theta\tilde{H}_\phi - \tilde{F}_\phi\tilde{H}_\theta)$, so we only really need compute the Lie–Poisson bracket between the spherical angles θ, ϕ:

$$\{\theta, \phi\} = -u\cdot(\nabla_u\theta \times \nabla_u\phi) = -1/(\rho\sin\phi).$$

Thus

$$\{\tilde{F}, \tilde{H}\} = \frac{-1}{\rho\sin\phi}\left(\frac{\partial\tilde{F}}{\partial\theta}\frac{\partial\tilde{H}}{\partial\phi} - \frac{\partial\tilde{F}}{\partial\phi}\frac{\partial\tilde{H}}{\partial\theta}\right)$$

is the induced Poisson bracket on $S^2_\rho \subset \mathfrak{so}(3)^*$.