SETS

LECTURE 1

Set Theory

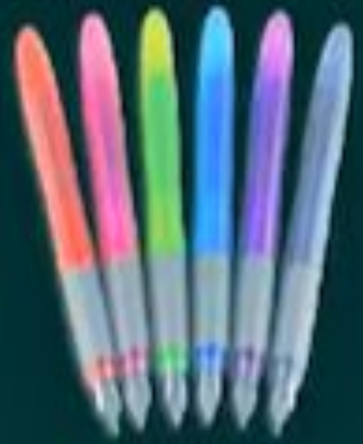
Basics of Sets

What is a set?

A set is a well-defined collection of distinct objects.

Object: An object could be anything. It can be something we can touch or see or it can be an idea or a concept.

Examples: A set of all facts learned in discrete mathematics course.
A collection of pens.
A collection of cars.
A set of odd numbers divisible by 2.
A set of vowels of English alphabet.



But, what is not a set?

"A collection of beautiful songs."

What? But why?

This is where the term well-defined came into picture.

In our example, "A set of beautiful songs"
"beautiful songs" is not well-defined. The definition of beautiful song changes from person to person.

Therefore, a set of beautiful songs is not well-defined and hence it is not a set.

More examples:

1. A collection of great people of the world.
2. A set of beautiful flowers.
3. A collection of best football players in the world.
4. A collection of most dangerous animals found in the forest.
5. A collection of the most talented boys in your class.

Distinct objects:

We can have a set with duplicate objects.
But a set with duplicate objects is similar to a set with distinct objects.

For example: $A = \{1, 2, 2, 3, 3, 3\}$
$B = \{1, 2, 3\}$

$A = B$

Eventually, we end up with a set without duplicate elements.
This is the reason why the definition

"A set is a well-defined collection of distinct objects" holds true.

Set Membership

Any object belonging to a set is called a <u>member</u> or an <u>element</u> of that set.

We will represent sets by uppercase letters and lowercase letters will be used to represent the elements of the set.

If a is an element of set A then

$a \in A$ or a is in A

If there exist an element b that does not belong to set A, then we express this fact by

$b \notin A$ or b is not in A

Set Representation

Three ways to represent a set:

1. List representation.
2. Predicate representation.
3. Missing element representation.

1. List representation:

Let us suppose we have a set A with elements 1, 2, 3, a and b.
Generally, a set is represented by listing all the elements of it. Here, set A is represented by

$$A = \{1, 2, 3, a, b\}$$

Here elements are simply listed within the pair of brackets ({})

2. Predicate representation:

In this representation, a set is defined by a predicate. This representation is more convenient then list representation.

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In this representation, a set is defined by a predicate. This representation is more convenient then list representation.

For example: $B = \{x \mid x \text{ is an odd positive integer}\}$

Let us suppose that $P(x)$ denotes "x is an odd positive integer" then

$$B = \{x \mid P(x)\}$$

If we want to tell that some element b belongs to a set B then for this $P(b)$ has to be true.

For example: $1 \in B$ because 1 is an odd positive integer.
but $2 \notin B$ because 2 is not an odd positive integer.

The sets which are usually specified by listing elements can also be specified by predicates.

For Example: $A = \{1, 2, 3, a, b\}$ is equivalent to
$\{x \mid (x = 1) \vee (x = 2) \vee (x = 3) \vee (x = a) \vee (x = b)\}$

Inclusion:

Let A and B are two sets. If every element of A is an element of B, then A is called a subset of B or A is said to be included in B.

$$A \subseteq B \text{ (A is a subset of B)}$$
or
$$B \supseteq A \text{ (B is a superset of A)}$$

Example: $A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4, 5\}$

$A \subseteq B$ but $B \supseteq A$
(Note: $B \not\subset A$)

Note: $A \subseteq B$ if and only if the quantification $\forall x(x \in A \rightarrow x \in B)$ is true.

Why?

Example: $A = \{1, 2\}$ $B = \{1, 3, 5\}$

Consider all elements of A

$1 \in A$ and $1 \in B$

$2 \in A$ but $2 \notin B$

Important properties of set inclusion:

1. Reflexivity: $A \subseteq A$

   Example: $A = \{1, 2, 3\}$

   It is true that A is itself the subset of A.

2. Transitivity: $(A \subseteq B) \land (B \subseteq C) \rightarrow (A \subseteq C)$

   Example: $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 5\}$, and $C = \{1, 2, 3, 5, 7\}$

   it is clear that $A \subseteq B$ and $B \subseteq C$

   Also, it is clear that $A \subseteq C$

$\therefore$ set inclusion is both reflexive and transitive.

Equality:

Two sets A and B are said be equal if $A \subseteq B$ and $B \subseteq A$.

$$A = B \Leftrightarrow (A \subseteq B \land B \subseteq A) \quad \text{OR} \quad A = B \Leftrightarrow \forall x(x \in A \leftrightarrow x \in B)$$

Example:

1. $A = \{1, 2, 4\}$, $B = \{1, 2, 2, 4\}$
   $A = B$
2. $A = \{\{1, 2\}, 3\}$, $B = \{1, 2, 3\}$
   $A \neq B$ because $\{1, 2\} \in A$ and $\notin B$

Important properties:

1. Reflexive: $A = A$
2. Symmetric: $A = B \rightarrow B = A$ if $A = B$ is true then $B = A$ is also true.
3. Transitive: $(A = B) \land (B = C) \rightarrow (A = C)$

Proper Subset:

A set A is said to be a proper subset of B if $A \subseteq B$ and $A \neq B$.
It is represented by $A \subset B$.

$$A \subset B \Leftrightarrow (A \subseteq B \wedge A \neq B)$$

For example: $A = \{1, 2, 4\}$
$B = \{1, 2, 4, 5\}$
then $A \subset B$

Important Properties:

Transitivity: $(A \subset B) \wedge (B \subset C) \Rightarrow (A \subset C)$

Note that proper subset is not reflexive.

Inclusion Vs Membership

Lets try to understand the difference between inclusion and membership with the help of an example.

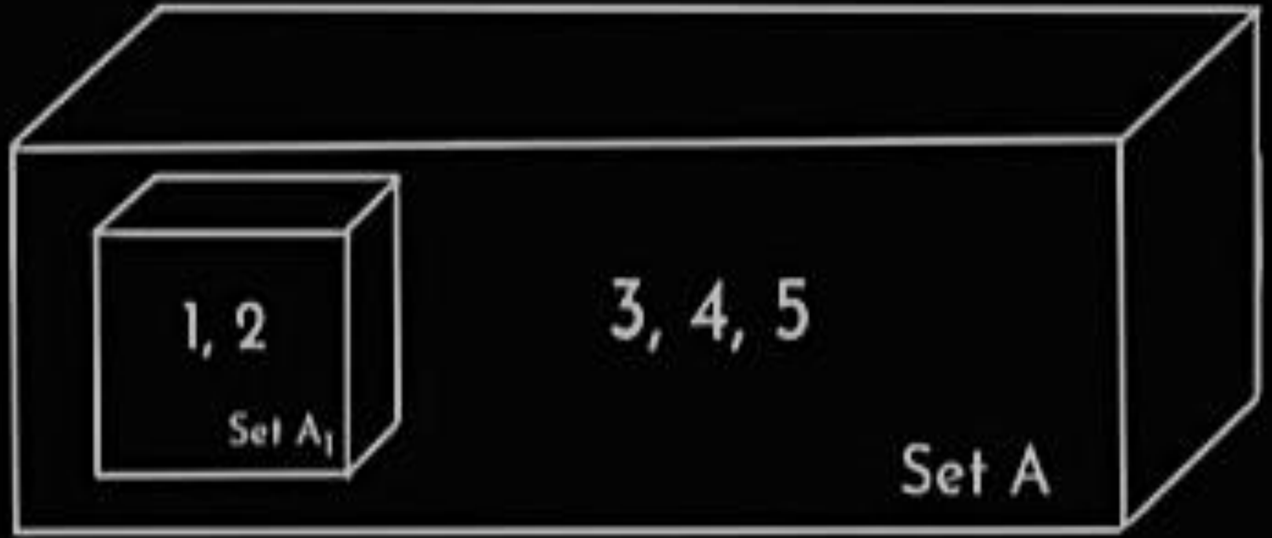
Example: Let $A = \{\{1, 2\}, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5\}$

Which of the following is true?

$1 \in B$? true.

$1 \in A$? false.

Let us assume that $\{1, 2\}$ is represented by the name "Set A_1"



Try to understand this analogy.

We have a box named "Set A." After opening the box, we can see 4 different objects. One is a box named "Set A_1" and the rest are the elements 3, 4 and 5.

So, opening the box is associated with knowing the members of the set.

So, is it true that $1 \in A$?

No. The elements of Set A are Set A_1, 3, 4, 5.

But $1 \in A_1$

More questions:

$A = \{\{1, 2\}, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5\}$

>> $\{1, 2\} \in A$?

True. $\{1, 2\}$ is a set within set A. Therefore, $\{1, 2\} \in A$.

>> $\{3, 4\} \subseteq A$?

True. Whenever it is required to answer if a particular set is a subset of a different set, see the elements of the given set and compare it with the elements of the other set.

Here, the given set is $\{3, 4\}$.

Ask this: $3 \in A$? Yes

$4 \in A$? Yes

$\therefore \{3, 4\} \subseteq A$

>> $\{1, 6\} \subseteq B$?

False. Ask yourself: $1 \in B$? Yes.

$6 \in B$? No.

$\therefore \{1, 6\} \not\subseteq B$

$A = \{\{1, 2\}, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5\}$

>> $1 \subseteq B$?

False. 1 is not a set itself.

>> $1 \in B$?

True. 1 is the element in B

$\therefore 1 \in B$

>> $\{1, 2\} \subseteq A$?

False. Ask yourself: $1 \in A$? No.

(Note: 1 belongs to set $\{1, 2\}$ contained within set A. It does not belong to A)

$2 \in A$? No.

$\therefore \{1, 2\} \not\subseteq A$

>> $\{\{1, 2\}\} \subseteq A$?

True. Ask yourself: $\{1, 2\} \in A$? Yes.

$\therefore \{\{1, 2\}\}$ is the subset of A.

>> $\{\{1, 2\}, 3, 4, 5\} \subseteq A$?

True. In fact, the given set is equal to A.

Inclusion Vs Membership
(Solved Problem)

Given $S = \{2, a, \{3\}, 4\}$ and $R = \{\{a\}, 3, 4, 1\}$. Indicate whether the following are true or false.

a) $\{a\} \in S$

False.

b) $\{a\} \in R$

Yes. Set $\{a\}$ is member of set R.

c) $\{a, 4, \{3\}\} \subseteq S$

Ask yourself: $a \in S$? Yes. Therefore, $\{a, 4, \{3\}\}$ is the subset of set S.
$4 \in S$? Yes.
$\{3\} \in S$? Yes.

d) $\{\{a\}, 1, 3, 4\} \subset R$

No. because $\{\{a\}, 1, 3, 4\} = R$ and is not a proper subset of R.

e) $R = S$

No. $2 \in S$ and $2 \notin R$, $a \in S$ and $a \notin R$, $\{3\} \in S$ and $\{3\} \notin R$ & $4 \in S$ and $4 \in R$

f) $\{a\} \subseteq S$

Ask yourself: $a \in S$? Yes.

g) $\{a\} \subseteq R$

Ask yourself: $a \in R$? No.

Universal Set, Null Set, and Singleton Set

Universal Set:

A universal set is a set which includes every set under consideration.
A universal set is represented by E.
For any predicate, P(x)

$$E = \{x \mid P(x) \vee \neg P(x)\}$$

The universal set is same as universe of discourse.

Null Set:

A set which does not contain any element is called a null set or empty set.
It is denoted by ϕ or $\{\}$.

$$\phi = \{x \mid P(x) \wedge \neg P(x)\}$$

For example: A set of all positive integers which are both even and odd.

Singleton Set:

A singleton set is a set with exactly one element.

For example: $A = \{2\}$
$B = \{\phi\}$

please note that $\{\phi\} \neq \phi$

$\{\phi\}$ consists of one element which is the null set itself while there is no element inside ϕ.

Null Set

(Solved Problem)

Determine whether the following statements are true or false.

a) $\phi \in \{\phi\}$

True. ϕ is an empty set and is also a member of set $\{\phi\}$.

b) $\phi \in \{\phi, \{\phi\}\}$

True.

c) $\{\phi\} \in \{\phi\}$

False. $\{\phi\}$ is not a member of the $\{\phi\}$ because there is only one element in the set $\{\phi\}$ which is ϕ not $\{\phi\}$.

d) $\{\phi\} \in \{\{\phi\}\}$

True. $\{\phi\}$ is the member of the set $\{\{\phi\}\}$.

e) $\{\phi\} \subset \{\phi, \{\phi\}\}$

True. Ask yourself: $\phi \in \{\phi, \{\phi\}\}$? Yes. ϕ is an element of the set $\{\phi, \{\phi\}\}$

f) $\{\{\phi\}\} \subset \{\phi, \{\phi\}\}$

True. Ask yourself: $\{\phi\} \in \{\phi, \{\phi\}\}$? Yes. $\{\phi\}$ is an element of the set $\{\phi, \{\phi\}\}$

g) $\{\{\phi\}\} \subset \{\{\phi\}, \{\phi\}\}$

False. The above statement is equivalent to $\{\{\phi\}\} \subset \{\{\phi\}\}$.