Functions

Section 2.3 of Rosen

Outline

- Definitions & terminology
  - function, domain, co-domain, image, preimage (antecedent), range, image of a set, strictly increasing, strictly decreasing, monotonic
- Properties
  - One-to-one (injective), onto (surjective), one-to-one correspondence (bijective)
  - Exercices (5)
- Inverse functions (examples)
- Operators
  - Composition, Equality
- Important functions
  - identity, absolute value, floor, ceiling, factorial

Introduction

- You have already encountered function
  - $f(x,y) = x+y$
  - $f(x) = x$
  - $f(x) = \sin(x)$
- Here we will study functions defined on <u>discrete</u> domains and ranges
- We will generalize functions to mappings
- We may not always be able to write function in a 'neat way' as above

Definition: Function

- **Definition**: A function f from a set A to a set B is an assignment of exactly one element of B to each element of A.
- We write f(a)=b if b is the unique element of B assigned by the function f to the element a $\in$ A.
- If f is a function from A to B, we write

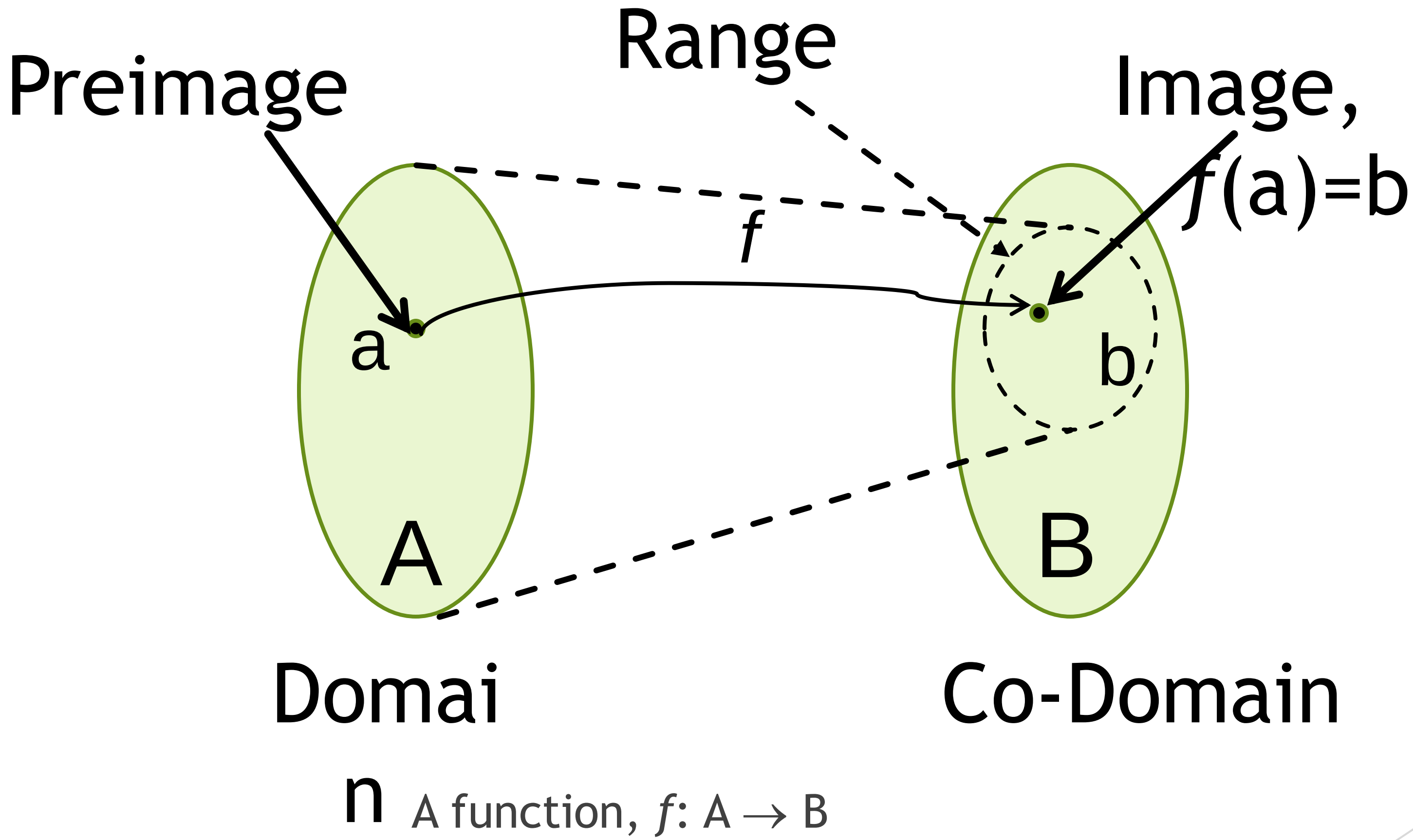
  $$f: A \rightarrow B$$

  This can be read as 'f maps A to B'
- Note the subtlety
  - Each and every element of A has a single mapping
  - Each element of B may be mapped to by several elements in A or not at all

Terminology

- Let f: A $\rightarrow$ B and f(a)=b. Then we use the following terminology:
  - A is the domain of f, denoted dom(f)
  - B is the co-domain of f
  - b is the image of a
  - a is the preimage (antecedent) of b
  - The range of f is the set of all images of elements of A, denoted rng(f)

Function: Visualization



A function, f: A $\rightarrow$ B

More Definitions (1)

- **Definition:** Let f_1 and f_2 be two functions from a set A to $\mathbb{R}$. Then f_1+f_2 and f_1f_2 are also function from A to R defined by:
  - $(f_1+f_2)(x) = f_1(x) + f_2(x)$
  - $f_1f_2(x)= f_1(x)f_2(x)$
- Example: Let $f_1(x)=x^4+2x^2+1$ and $f_2(x)=2-x^2$
  - $(f_1+f_2)(x) = x^4+2x^2+1+2-x^2 = x^4+x^2+3$
  - $f_1f_2(x) = (x^4+2x^2+1)(2-x^2)= -x^6+3x^2+2$

More Definitions (2)

- **Definition**: Let f: A$\rightarrow$ B and S $\subseteq$A. The image of the set S is the subset of B that consists of all the images of the elements of S. We denote the image of S by f(S), so that

$$f(S)=\{\, f(s) \mid \forall\, s \in S \,\}$$

- Note there that the image of S is a set and not an element.

Image of a set: Example

- Let:
  - $A = \{a_1, a_2, a_3, a_4, a_5\}$
  - $B = \{b_1, b_2, b_3, b_4, b_5\}$
  - $f=\{(a_1,b_2), (a_2,b_3), (a_3,b_3), (a_4,b_1), (a_5,b_4)\}$
  - $S=\{a_1,a_3\}$
- Draw a diagram for f
- What is the:
  - Domain, co-domain, range of f?
  - Image of S, $f(S)$?

More Definitions (3)

- **Definition**: A function f whose domain and codomain are subsets of the set of real numbers ($\mathbb{R}$) is called
  - strictly increasing if f(x)<f(y) whenever x<y and x and y are in the domain of f.
  - strictly decreasing if f(x)>f(y) whenever x<y and x and y are in the domain of f.
- A function that is increasing or decreasing is said to be monotonic

Outline

- Definitions & terminology
- Properties
  - One-to-one (injective)
  - Onto (surjective)
  - One-to-one correspondence (bijective)
  - Exercices (5)
- Inverse functions (examples)
- Operators
- Important functions

Definition: Injection

- **Definition:** A function f is said to be <u>one-to-one</u> or <u>injective</u> (or an injection) if

  $\forall$ x and y in in the domain of f, $f(x)=f(y) \Rightarrow x=y$

- Intuitively, an injection simply means that each element in the range has at most one preimage (antecedent)
- It is useful to think of the contrapositive of this definition

  $$x \neq y \Rightarrow f(x) \neq f(y)$$

Definition: Surjection

- **Definition**: A function f: A→B is called <u>onto</u> or <u>surjective</u> (or an surjection) if

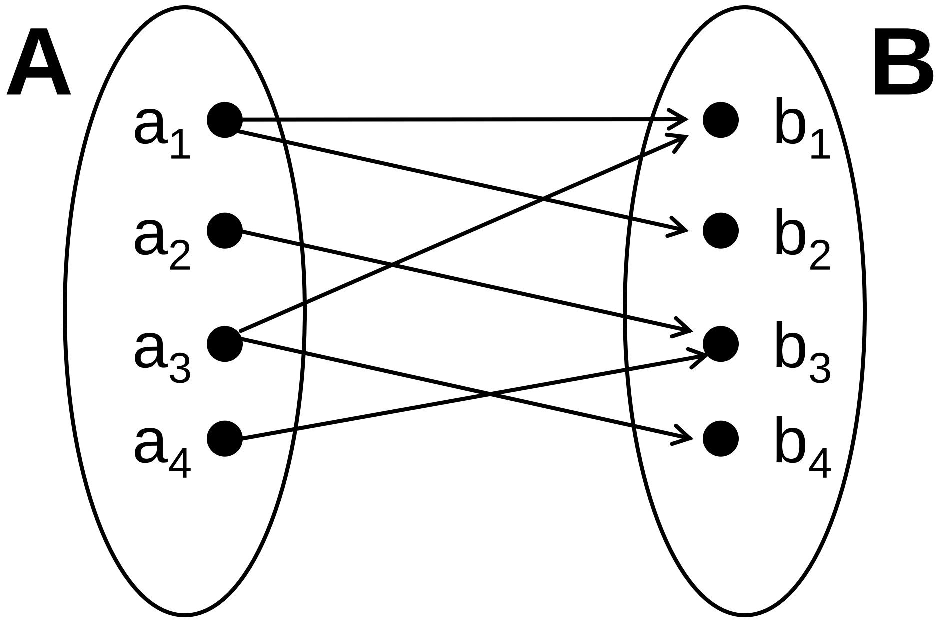
  $\forall$ b∈B, ∃ a∈A with f(a)=b

- Intuitively, a surjection means that every element in the codomain is mapped into (i.e., it is an image, has an antecedent)
- Thus, the range is the same as the codomain

Definition: Bijection

- **Definition**: A function f is a one-to-one correspondence (or a bijection), if is both one-to-one (injective) and onto (surjective)
- One-to-one correspondences are important because they endow a function with an inverse.
- They also allow us to have a concept cardinality for infinite sets
- Let's look at a few examples to develop a feel for these definitions…

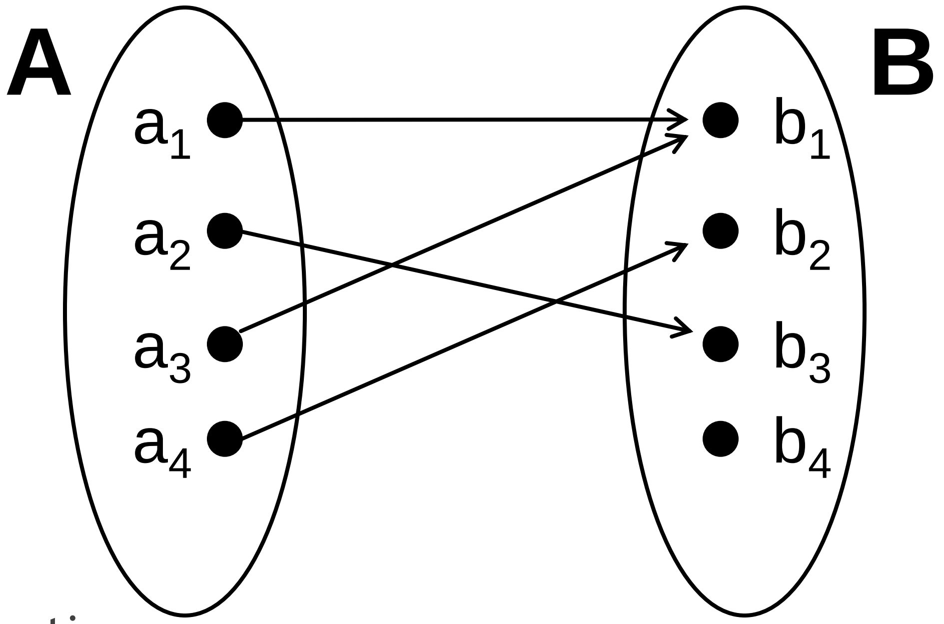
Functions: Example 1



- Is this a function? Why?

- No, because each of a_1, a_3 has two images

Functions: Example 2

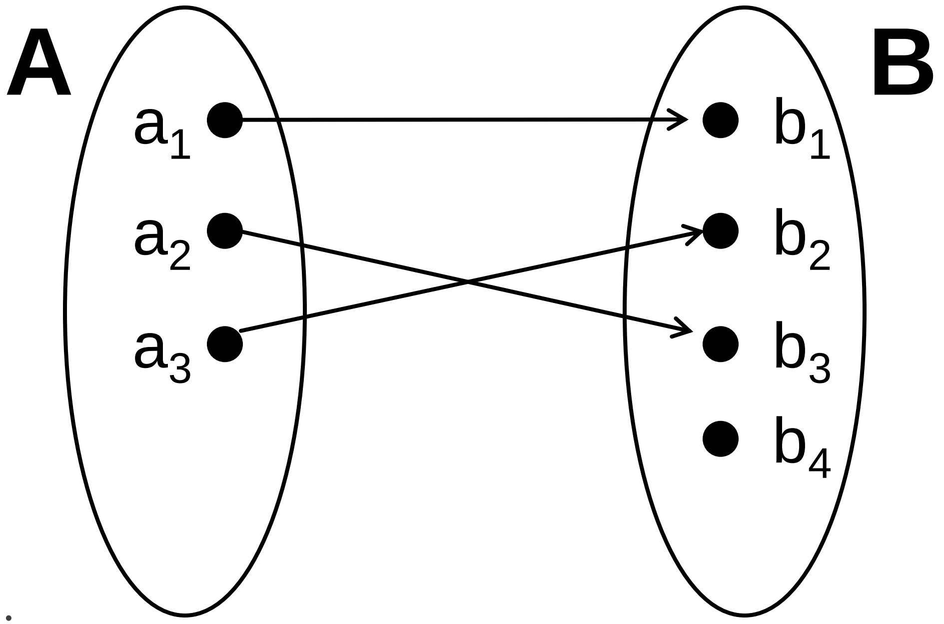


- Is this a function
  - One-to-one (injective)? Why?
  - Onto (surjective)? Why?

No, b_1 has 2 preimages

No, b_4 has no preimage

Functions: Example 3

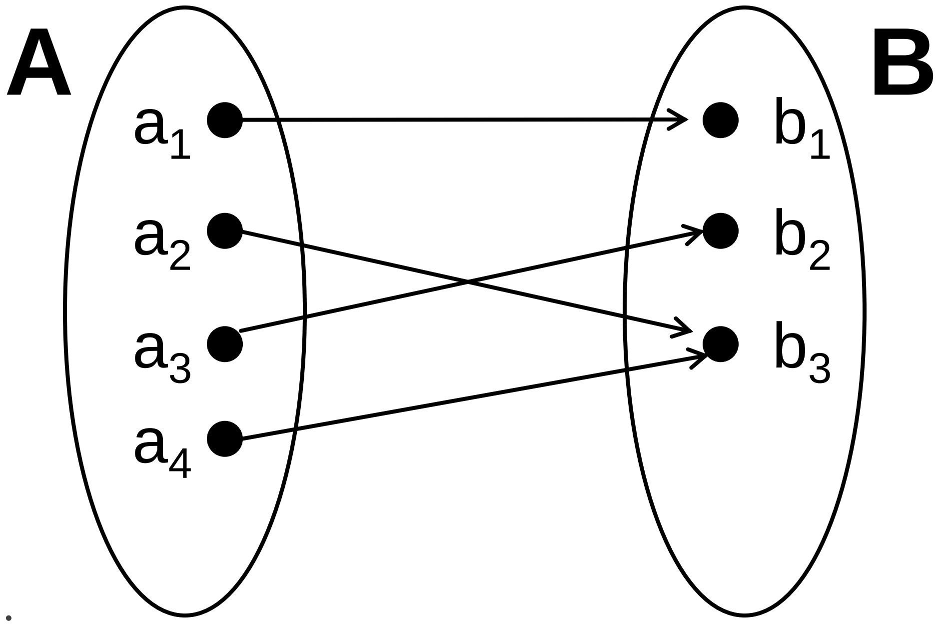


- Is this a function
  - One-to-one (injective)? Why?
  - Onto (surjective)? Why?

Yes, no b_i has 2 preimages

No, b_4 has no preimage

Functions: Example 4

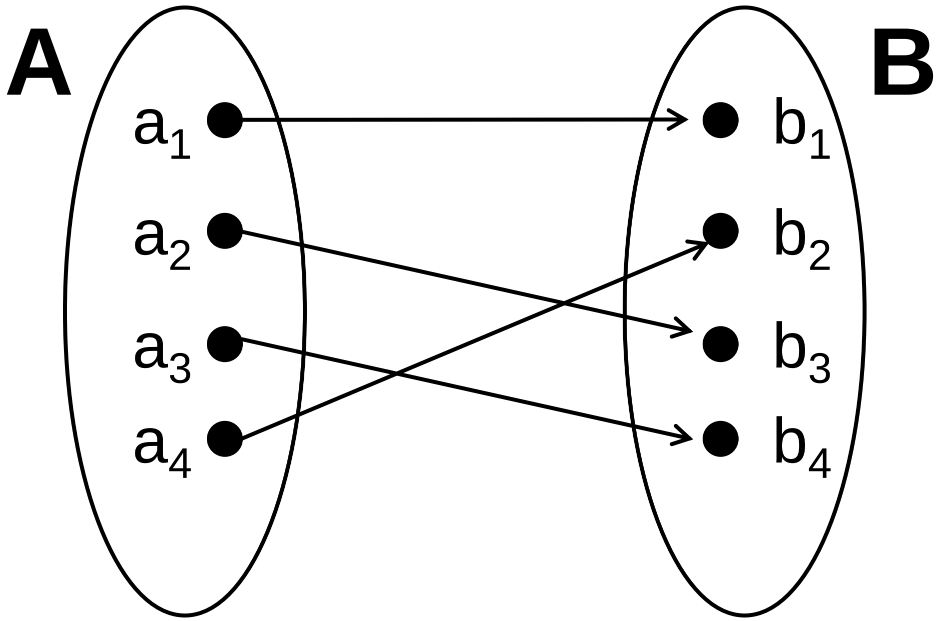


- Is this a function
  - One-to-one (injective)? Why?
  - Onto (surjective)? Why?

No, b_3 has 2 preimages

Yes, every b_i has a preimage

Functions: Example 5



- Is this a function
  - One-to-one (injective)?
  - Onto (surjective)?

Thus, it is a bijection or a one-to-one correspondence

Exercice 1

- Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x)=2x-3$$

- What is the domain, codomain, range of f?
- Is f one-to-one (injective)?
- Is f onto (surjective)?
- Clearly, $\mathrm{dom}(f)=\mathbb{Z}$. To see what the range is, note that:

$b \in \mathrm{rng}(f) \Leftrightarrow b=2a-3$, with $a \in \mathbb{Z}$

$\Leftrightarrow b=2(a-2)+1$

$\Leftrightarrow$ b is odd

Exercise 1 (cont'd)

- Thus, the range is the set of all odd integers
- Since the range and the codomain are different (i.e., rng(f) $\neq \mathbb{Z}$), we can conclude that f is not onto (surjective)
- However, f is one-to-one injective. Using simple algebra, we have:

$$f(x_1) = f(x_2) \Rightarrow 2x_1\text{-}3 = 2x_2\text{-}3 \Rightarrow x_1 = x_2 \quad \text{QED}$$

Exercise 2

- Let f be as before

$$f(x)=2x-3$$

  but now we define $f:\mathbb{N}\rightarrow\mathbb{N}$
- What is the domain and range of f?
- Is f onto (surjective)?
- Is f one-to-one (injective)?
  - By changing the domain and codomain of f, f is not even a function anymore. Indeed, $f(1)=2\cdot1-3=-1\notin\mathbb{N}$

Exercice 3

- Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by

$$f(x) = x^2 - 5x + 5$$

- Is this function
  - One-to-one?
  - Onto?

Exercice 3: Answer

- It is not one-to-one (injective)

  $f(x_1)=f(x_2) \Rightarrow x_1^2-5x_1+5=x_2^2 - 5x_2+5 \Rightarrow x_1^2 - 5x_1 = x_2^2 - 5x_2$

  $\Rightarrow x_1^2 - x_2^2 = 5x_1 - 5x_2 \Rightarrow (x_1 - x_2)(x_1 + x_2) = 5(x_1 - x_2)$

  $\Rightarrow (x_1 + x_2) = 5$

  Many $x_1,x_2 \in \mathbb{Z}$ satisfy this equality. There are thus an infinite number of solutions. In particular, $f(2)=f(3)=-1$

- It is also not onto (surjective).

  The function is a parabola with a global minimum at (5/2,-5/4). Therefore, the function fails to map to any integer less than -1

- What would happen if we changed the domain/codomain?

Exercice 4

- Let $f\!:\!\mathbb{Z}\!\rightarrow\!\mathbb{Z}$ be defined by

$$f(x) = 2x^2 + 7x$$

- Is this function
  - One-to-one (injective)?
  - Onto (surjective)?
- Again, this is a parabola, it cannot be onto (where is the global minimum?)

Exercice 4: Answer

- However, it is one-to-one! Indeed:

$f(x_1)=f(x_2) \Rightarrow 2x_1^2+7x_1=2x_2^2 + 7x_2 \Rightarrow 2x_1^2 - 2x_2^2 = 7x_2 - 7x_1$

$\Rightarrow 2(x_1 - x_2)(x_1 + x_2) = 7(x_2 - x_1) \Rightarrow 2(x_1 + x_2) = -7 \Rightarrow (x_1 + x_2) = -7$

$\Rightarrow (x_1 + x_2) = -7/2$

But $-7/2 \notin Z$. Therefore it must be the case that $x_1 = x_2$.

It follows that f is a one-to-one function.
QED

Exercise 5

- Let $f: Z \rightarrow Z$ be defined by

$$f(x) = 3x^3 - x$$

- Is this function
  - One-to-one (injective)?
  - Onto (surjective)?

Exercice 5: f is one-to-one

- To check if f is one-to-one, again we suppose that for $x_1, x_2 \in \mathbb{Z}$ we have $f(x_1)=f(x_2)$

  $f(x_1)=f(x_2) \Rightarrow 3x_1^3-x_1=3x_2^3-x_2$

  $\Rightarrow 3x_1^3 - 3x_2^3 = x_1 - x_2$

  $\Rightarrow 3\,(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = (x_1 - x_2)$

  $\Rightarrow (x_1^2 + x_1x_2 + x_2^2) = 1/3$

  which is impossible because $x_1, x_2 \in \mathbb{Z}$

  thus, f is one-to-one

Exercice 5: f is not onto

- Consider the counter example $f(a)=1$
- If this were true, we would have

  $3a^3 - a=1 \Rightarrow a(3a^2 - 1)=1$ where a and $(3a^2 - 1) \in \mathbb{Z}$
- The only time we can have the product of two integers equal to 1 is when they are both equal to 1 or -1
- Neither 1 nor -1 satisfy the above equality
  - Thus, we have identified $1 \in \mathbb{Z}$ that does not have an antecedent and f is not onto (surjective)

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  - Exercices (5)
- **Inverse functions (examples)**
- **Operators**
  - **Composition, Equality**
- Important functions
  - identity, absolute value, floor, ceiling, factorial

Inverse Functions (1)

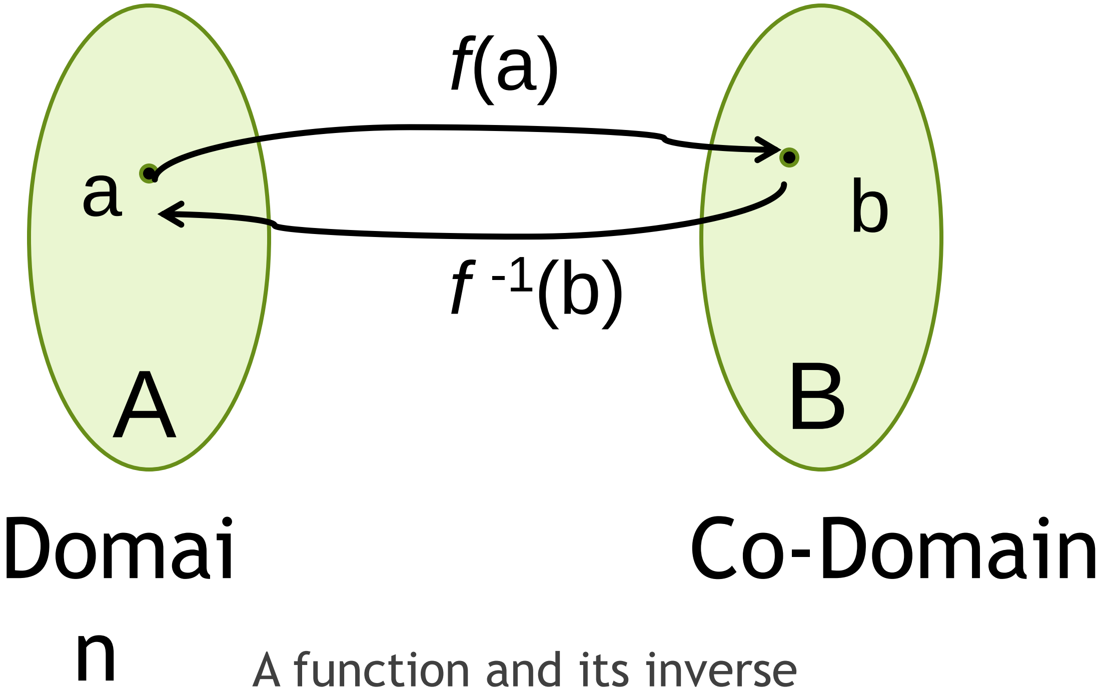
- **Definition**: Let f: A$\rightarrow$B be a bijection. The <u>inverse</u> function of f is the function that assigns to an element b$\in$B the unique element a$\in$A such that f(a)=b
- The inverse function is denote f^{-1}.
- When f is a bijection, its inverse exists and

$$f(a)=b \Leftrightarrow f^{-1}(b)=a$$

Inverse Functions (2)

- Note that by definition, a function can have an inverse if and only if it is a bijection. Thus, we say that a bijection is <u>invertible</u>
- Why must a function be bijective to have an inverse?
  - Consider the case where f is not one-to-one (not injective). This means that some element $b \in B$ has more than one antecedent in A, say a_1 and a_2. How can we define an inverse? Does $f^{-1}(b)=a_1$ or a_2?
  - Consider the case where f is not onto (not surjective). This means that there is some element $b \in B$ that does not have any preimage $a \in A$. What is then $f^{-1}(b)$?

Inverse Functions: Representation



A function and its inverse

Inverse Functions: Example 1

- Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = 2x - 3$$

- What is f^{-1}?
  1. We must verify that f is invertible, that is, is a bijection. We prove that is one-to-one (injective) and onto (surjective). It is.
  2. To find the inverse, we use the substitution
     - Let $f^{-1}(y)=x$
     - And $y=2x-3$, which we solve for x. Clearly, $x= (y+3)/2$
     - So, $f^{-1}(y)= (y+3)/2$

Inverse Functions: Example 2

- Let $f(x)=x^2$. What is f^{-1}?
- No domain/codomain has been specified.
- Say $f:\mathbb{R}\to\mathbb{R}$
  - Is f a bijection? Does its inverse exist?
    – Answer: No
- Say we specify that $f: A \to B$ where

  $A=\{x\in\mathbb{R} \mid x\leq 0\}$ and $B=\{y\in\mathbb{R} \mid y\geq 0\}$

  – Is f a bijection? Does its inverse exist?

  – Answer: Yes, the function <u>becomes</u> a bijection and thus, has an inverse

Inverse Functions: Example 2 (cont')

- To find the inverse, we let
  - $f^{-1}(y)=x$
  - $y=x^2$, which we solve for x
- Solving for x, we get $x=\pm\sqrt{y}$, but which one is it?
- Since dom(f) is all nonpositive and rng(f) is nonnegative, thus x must be nonpositive and

$$f^{-1}(y)= -\sqrt{y}$$

- From this, we see that <u>the domains/codomains are just as important to a function as the definition of the function</u> itself

Inverse Functions: Example 3

- Let $f(x)=2^x$
  - What should the domain/codomain be for this function to be a bijection?
  - What is the inverse?
- The function should be $f:\mathbb{R}\rightarrow\mathbb{R}^+$
- Let $f^{-1}(y)=x$ and $y=2^x$, solving for x we get $x=\log_2(y)$. Thus, $f^{-1}(y)=\log_2(y)$
- What happens when we include 0 in the codomain?
- What happens when restrict either sets to $\mathbb{Z}$?

Important Functions: Absolute Value

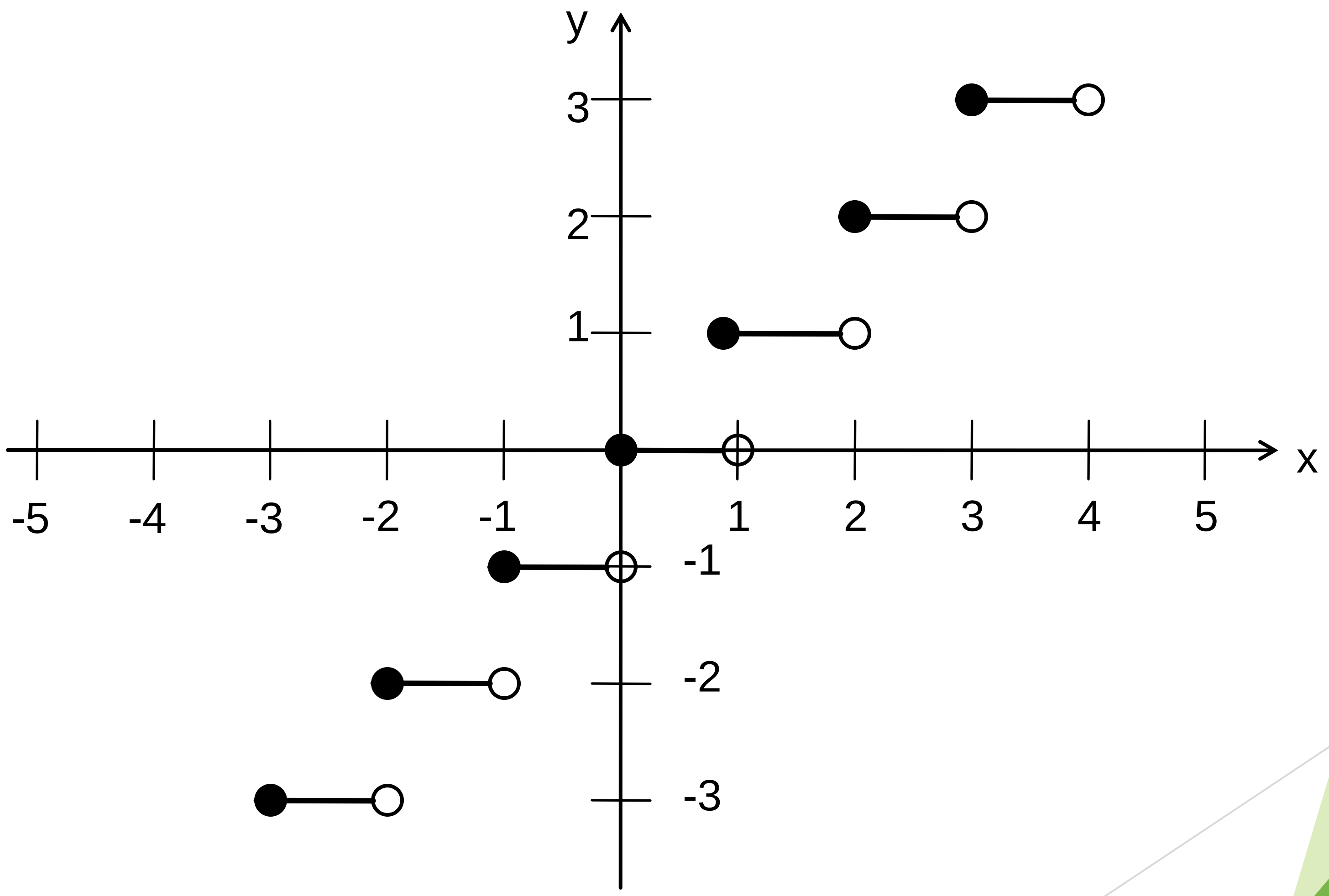
- **Definition:** The absolute value function, denoted $|x|$, f $f: \mathbb{R} \to \{y \in \mathbb{R} \mid y \geq 0\}$. Its value is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

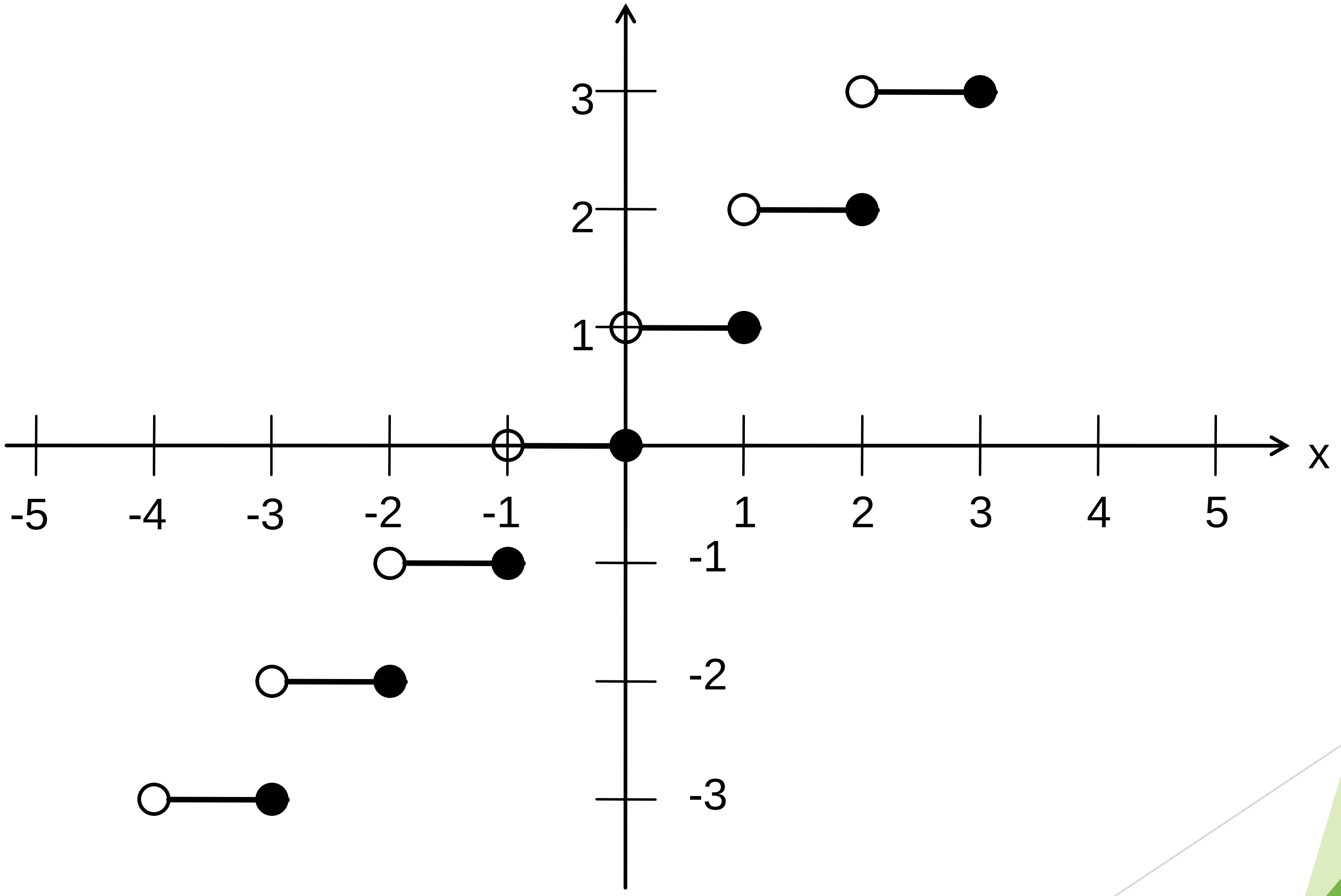
Important Functions: Floor & Ceiling

- **Definitions:**
  - The <u>floor function</u>, denoted $\lfloor x \rfloor$, is a function $R \rightarrow Z$. Its values is the <u>largest integer</u> that is less than or equal to x
  - The ceiling function, denoted $\lceil x \rceil$, is a function $R \rightarrow Z$. Its values is the <u>smallest integer</u> that is greater than or equal to x
- In LaTex: `$\lceil$`, `$\rceil$`, `$\rfloor$`, `$\lfloor$`

Important Functions: Floor



Important Functions: Ceiling



Important Function: Factorial

- The factorial function gives us the number of permutations (that is, uniquely ordered arrangements) of a collection of n objects
- **Definition**: The factorial function, denoted n!, is a function $\mathbb{N} \rightarrow \mathbb{N}^{+}$. Its value is the product of the n positive integers

$$n! = \prod_{i=1}^{i=n} i = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$$

Factorial Function & Stirling's Approximation

- The factorial function is defined on a discrete domain
- In many applications, it is useful a continuous version of the function (say if we want to differentiate it)
- To this end, we have the Stirling's formula

$$n! = \sqrt{2\pi n}\,(n/e)^n$$

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