SETS

LECTURE 3

Set Operations

(Intersection and Union with Venn Diagrams)

Definition of Intersection:

The intersection of any two sets A and B, denoted by $A \cap B$, is the set consisting of all the elements which belong to both A and B.

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

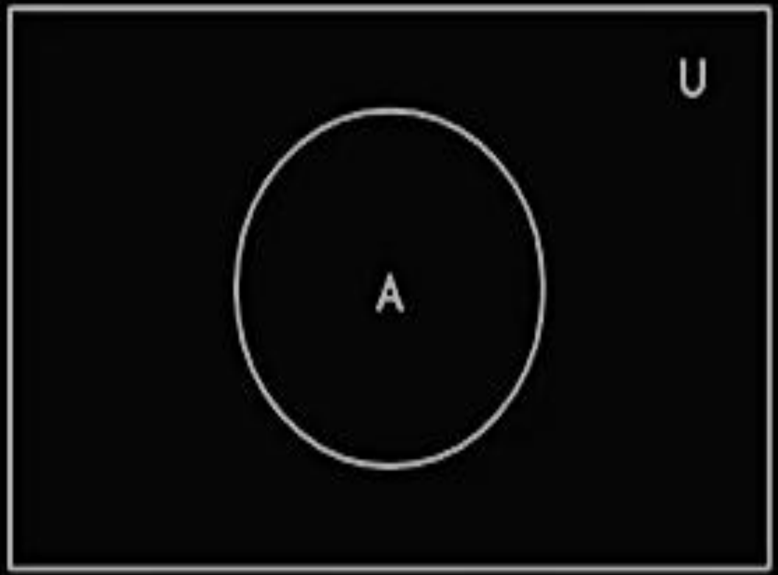
Representing Intersection of Two Sets Pictorially using Venn Diagrams

A Venn diagram is a diagram used to illustrate the logical relationship between two or more sets by using overlapping circles or other shapes.

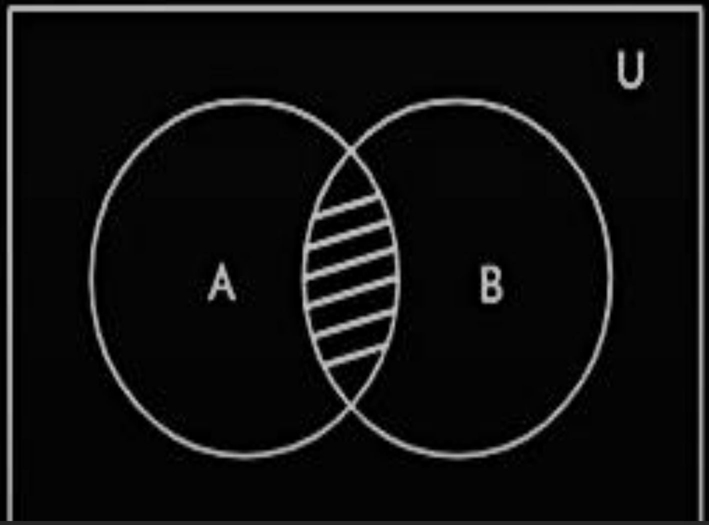
Basic Overview:

A set is usually represented by a circle and the elements of the set lies within the circle.

A universal set or universe of discourse is represented by a rectangle. Every element under consideration lies within the rectangle.



Pictorial Representation of Intersection of sets A and B



Shaded region is the common area between A and B i.e. $A \cap B$.

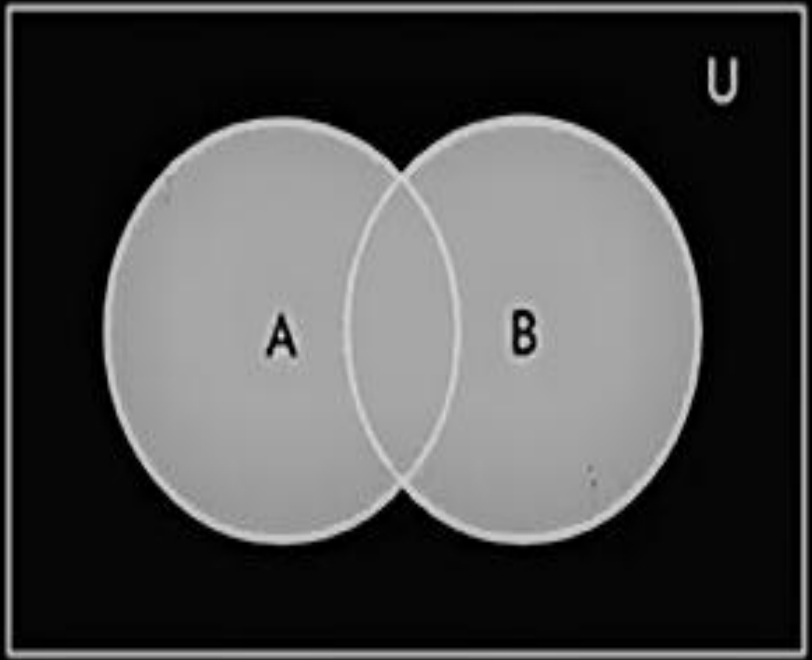
Example: Let $A = \{1, 3, 5\}$ and $B = \{1, 7, 8\}$

Definition of Union

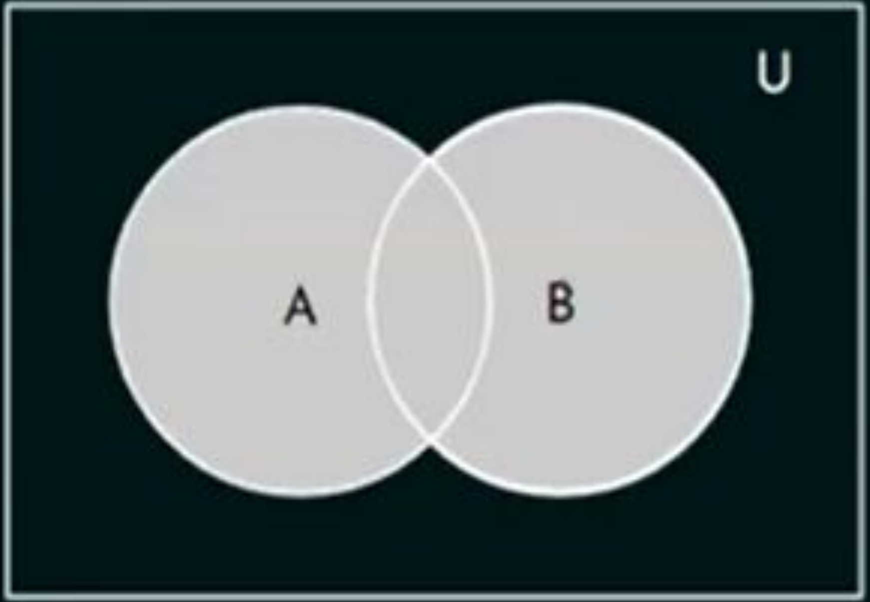
For any two sets A and B, the union of A and B, denoted by $A \cup B$, is the set of all elements which are members of the set A or set B or both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Venn Diagram of Union of Sets

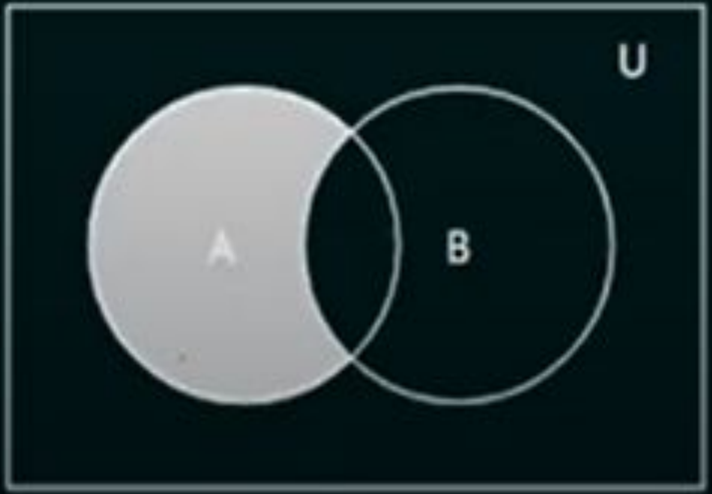


Shaded region is representing union of A and B.

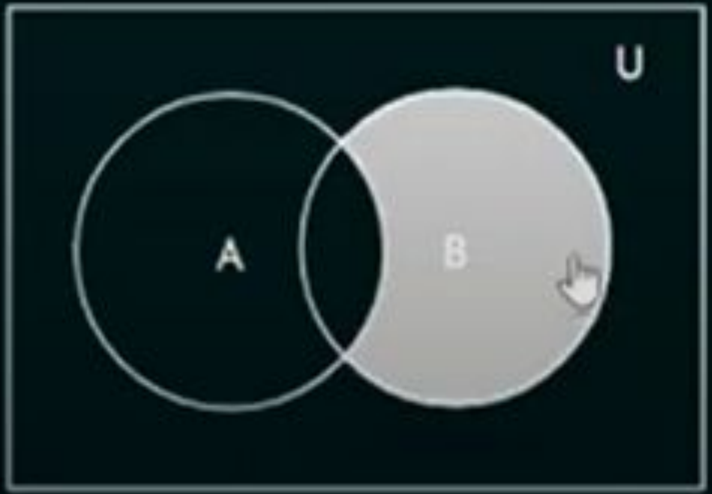


Shaded region is representing union of A and B.

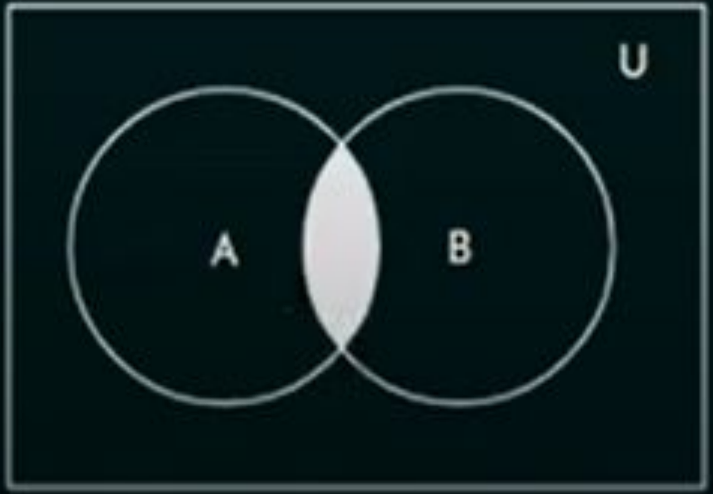
Either Circle A



or Circle B



or Both

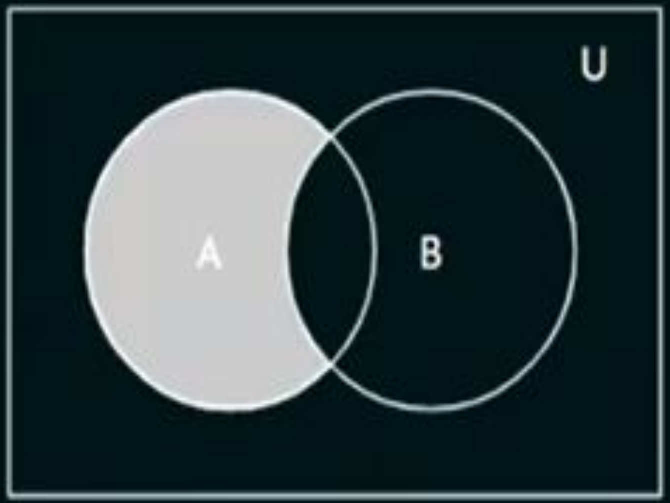


Set Difference and Complement

For any two sets A and B, the difference of A and B, denoted by A - B, is the set containing those elements that are in set A but not in set B.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Venn Diagram Representation of A - B



Shaded region represents A but not B

The difference of A and B is also called the complement of B with respect to A.

Example 1: Let say, $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$

$A - B = \{5\}$ (A but not B)

Example 2: Let say A represents a set of all computer science students at a University and B represents a set of all students who are studying discrete mathematics.

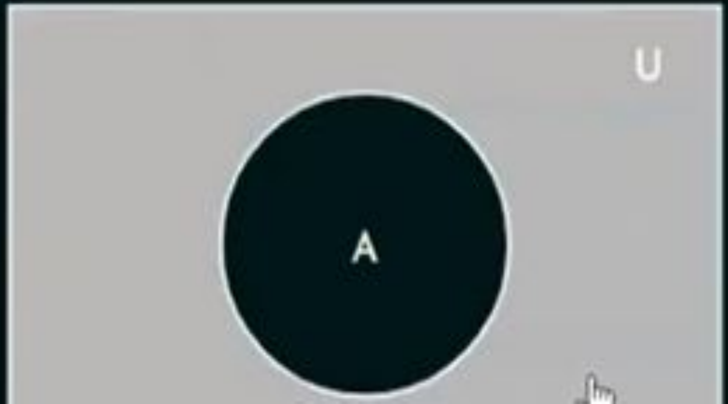
then, A - B will represent a set of all computer science students who are not studying discrete mathematics.

Complement of a Set

Let U be the Universal set.
The complement of set A, denoted by A′, is the complement of A with respect to U.
In other words, complement of set A is U - A.

$$A' = \{x \mid x \notin A\}$$

Venn Diagram for the Complement of Set A



Shaded region represents A'

Example: Let A represents a set of all positive integers greater than 10 then A' will represent a set of all positive integers less than or equal to 10.
(Note: Universal set is the set of all positive integers)

$$A' = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Set Operations
(Solved Problems)

Question 1: Let A be the set of students who live within one mile of school and B be the set of students who walk to classes. Describe the students in each of these sets.

a) $A \cup B$ b) $A \cap B$ c) $A - B$ d) $B - A$

Solution: Let say $A = \{\text{Mark, Allen, Roy}\}$
$B = \{\text{Allen, John, Mark, Henrv}\}$

a) $A \cup B$ contains all students who live within one mile of school or who walk to classes.
i.e. $A \cup B = \{\text{Mark, Allen, Roy, John, Henry}\}$

b) $A \cap B$ contains all students who live within one mile of school and who walk to classes i.e. $A \cap B = \{\text{Mark, Allen}\}$

c) $A - B$ contains all students who live within one mile of school, but who do not walk to classes i.e. $A - B = \{\text{Roy}\}$

d) $B - A$ contains all students who walk to classes, but do not live within one mile of school i.e. $B - A = \{\text{John, Henry}\}$

Question 2: Let A, B and C be non-empty sets and let $X = (A - B) - C$ and $Y = (A - C) - (B - C)$. Which one of the following is True?

(A) $X = Y$ (B) $X \subset Y$ (C) $Y \subset X$ (D) None of the mentioned

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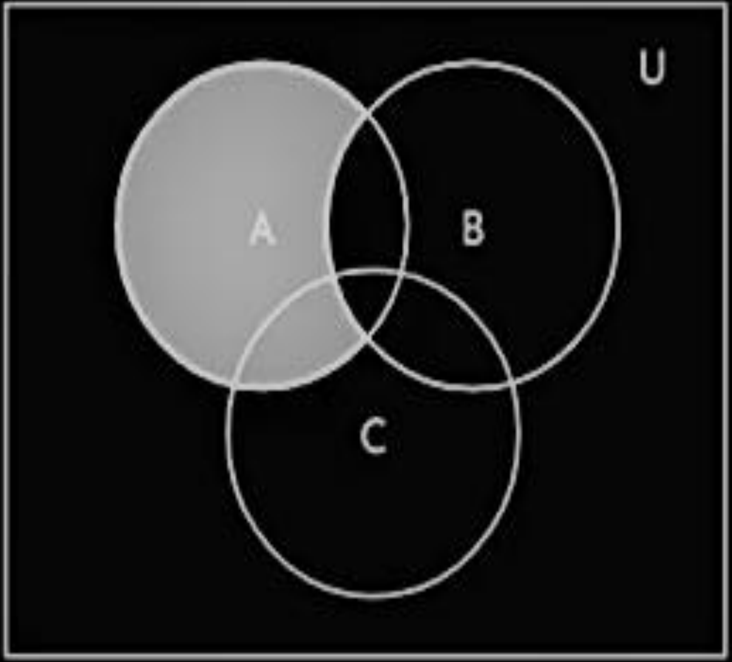
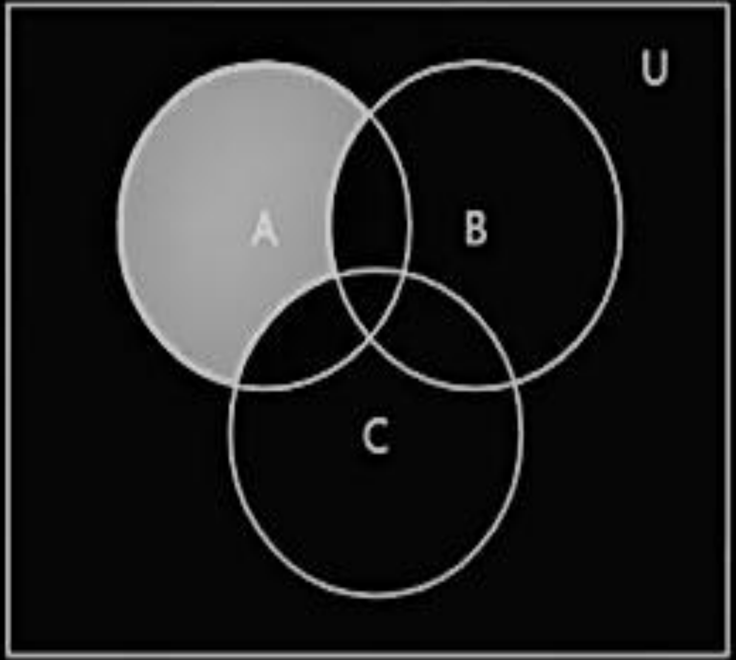
Solution:

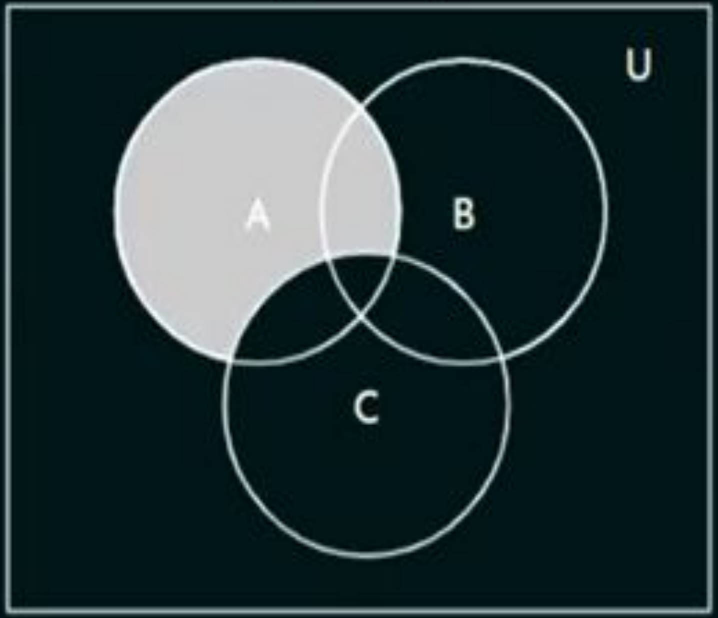
(A - B)



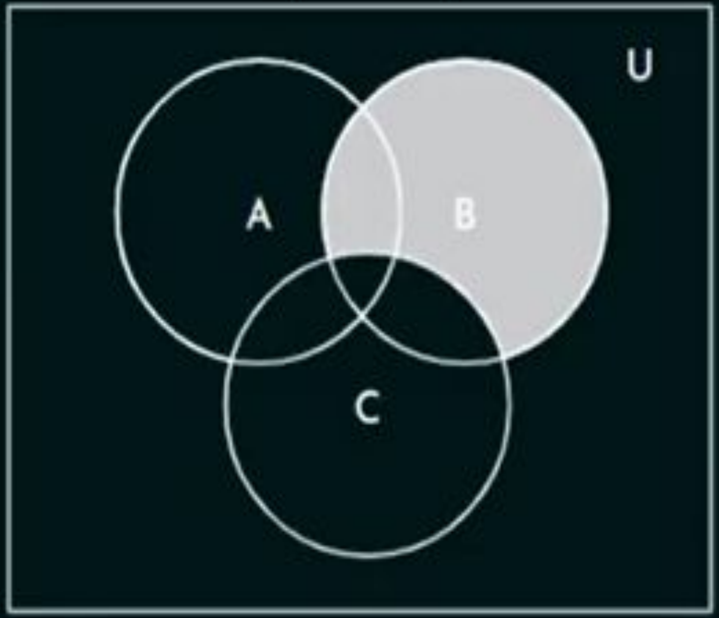
(A - B) - C



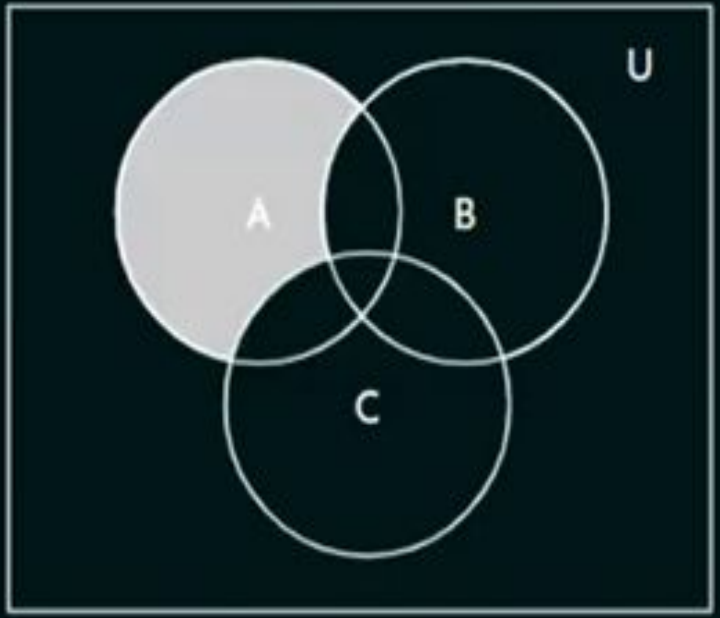
$(A - C)$



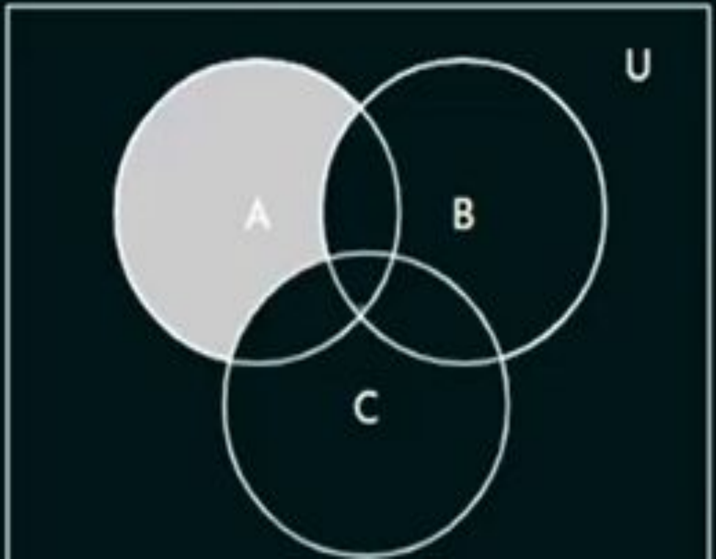
$(B - C)$



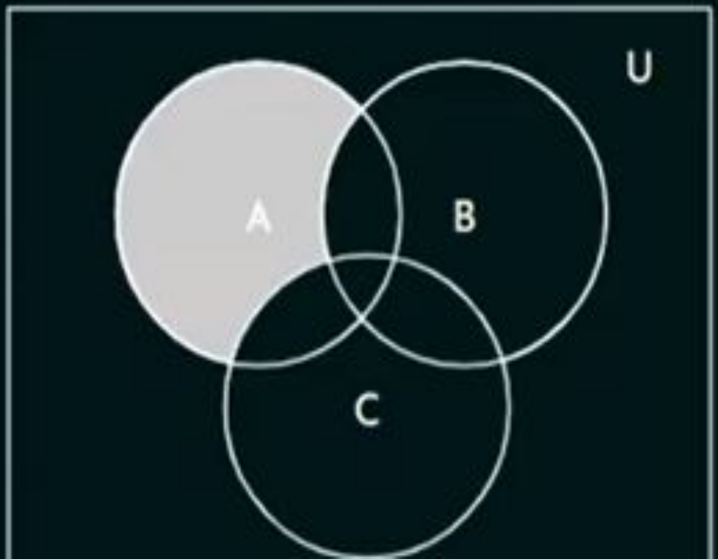
$(A - C) - (B - C)$



$(A - B) - C$



$(A - C) - (B - C)$



Therefore, $X = Y$

Alternative way: $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 7\}$, $C = \{3, 4, 6, 8\}$

$$X = (A - B) - C = \{1, 3\} - \{3, 4, 6, 8\} = \{1\}$$

$$Y = (A - C) - (B - C) = \{1, 2\} - \{2, 7\} = \{1\}$$

Therefore, $X = Y$

3.Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$.

Find a) $A \cup B$. b) $A \cap B$. c) $A - B$. d) $B - A$.

4.Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$.

Find a) $A \cup B$. b) $A \cap B$. c) $A - B$. d) $B - A$

25.Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, and
$C = \{4, 5, 6, 7, 8, 9, 10\}$.

Find a) $A \cap B \cap C$. b) $A \cup B \cup C$. c) $(A \cup B) \cap C$. d) $(A \cap B) \cup C$.

These are straightforward applications of the definitions.
a) The set of elements common to all three sets is $\{4, 6\}$.
b) The set of elements in at least one of the three sets is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
c) The set of elements in C and at the same time in at least one of A and B is $\{4, 5, 6, 8, 10\}$.
d) The set of elements either in C or in both A and B (or in both of these) is $\{0, 2, 4, 5, 6, 7, 8, 9, 10\}$.

Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets.

a) $A \cap B$
b) $A \cup B$
c) $A - B$
d) $B - A$

a) the set of students who live within one mile of school and walk to class (only students who do both of these things are in the intersection)
b) the set of students who either live within one mile of school or walk to class (or, it goes without saying, both)
c) the set of students who live within one mile of school but do not walk to class
d) the set of students who live more than a mile from school but nevertheless walk to class

Types of Relations (Part 1)

1. Reflexive Relation:

   A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.
   In other words, $\forall a((a, a) \in R)$.

   Example: Let $A = \{1, 2, 3, 4\}$
   $R_1 = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3), (4, 4)\}$
   Relation R_1 is reflexive because it contains all ordered pairs of the form (a, a) for every element $a \in A$ i.e., R_1 has $(1, 1), (2, 2), (3, 3), (4, 4)$

   $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (4, 4)\}$
   Relation R_2 is not reflexive because the ordered pair $(3, 3)$ is not in R_2.

2. Irreflexive Relation:

   A relation R on a set A is called irreflexive if $\forall a \in A, (a, a) \notin R$.

   Example: $A = \{1, 2, 3, 4\}$
   $R_3 = \{(1, 2), (2, 1), (3, 3), (4, 4)\}$ is not irreflexive because $(3, 3)$ and $(4, 4)$ is there in R_3.

   $R_4 = \{(1, 2), (2, 1)\}$ is irreflexive because $\forall a \in A, (a, a) \notin R_4$

3. Symmetric Relation:

A relation R on a set A is called symmetric if $(b, a) \in R$ holds when $(a, b) \in R$ for all $a, b \in A$

In other words, relation R on a set A is symmetric if $\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$

Example: Relation $R_5 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ is symmetric because for every $(a, b) \in R_5$ $(b, a) \in R_5$
like (1, 2) (2, 1) is in R_5.
There is no need to check for (1, 1), (2, 2).

Relation $R_6 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ is not symmetric because for (1, 2) there is no (2, 1) in R_6. Same is true for (1, 3) and (1, 4).

4. Antisymmetric Relation:

A relation R on a set A is called antisymmetric if $\forall a \forall b((a, b) \in R \wedge (b, a) \in R \rightarrow (a = b))$
Whenever we have (a, b) in R, we will never have (b, a) in R until or unless (a = b)

Example: Relation $R_7 = \{(1, 1), (2, 1)\}$ on set A is antisymmetric because (2, 1) is in R_7 but (1, 2) is not in R_7.

Types of Relations (Part II)

5. Transitive Relation:

A relation R on a set A is called transitive if $\forall a \forall b \forall c(((a, b) \in R \wedge (b, c) \in R) \rightarrow (a,c) \in R)$

Example: $A = \{1, 2, 3, 4\}$
$R_8 = \{(2, 1), (3, 1), (3, 2), (4, 4)\}$ is transitive because $(3, 2)$, $(2, 1)$, and $(3, 1)$ are there in R_8.

$R_9 = \{(2, 1), (1, 3)\}$ is not transitive as $(2, 1)$ and $(1, 3)$ are there in R_9 but there is no $(2, 3)$ in relation R_9.

6. Asymmetric Relation:

A relation R on a set A is called asymmetric if $\forall a \forall b((a, b) \in R \rightarrow (b, a) \notin R)$

Example: $A = \{1, 2, 3, 4\}$
$R_{10} = \{(1, 1), (1, 2), (1, 3)\}$ is not an asymmetric relation because of $(1, 1)$.

$R_{11} = \{(1, 2), (1, 3), (2, 3)\}$ is an asymmetric relation.

Summary

Relation	Property
1. Reflexive	$\forall a((a, a) \in R)$
2. Irreflexive	$\forall a((a, a) \notin R)$
3. Symmetric	$\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$
4. Antisymmetric	$\forall a \forall b(((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b))$
5. Asymmetric	$\forall a \forall b((a, b) \in R \rightarrow (b, a) \notin R)$
6. Transitive	$\forall a \forall b \forall c(((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$