Chapter 1: The Foundations: Logic, Sets and Functions

INTRODUCTION

- Proposition:

A **proposition** is a statement that is either *true* or *false*, but *not both*.

- Statements are declarative sentences.

- Propositional Logic:

The area of logic that deals with propositions.

- "Elephants are bigger than mice."

Is this a statement? yes

Is this a proposition? yes

What is the truth value of the proposition? true

NEGATION (Unary Operator)

Let p be a proposition. The negation of p, denoted by $\neg p$, is the statement "It is not the case that p."

- The proposition $\neg p$ is read as "not p."

- Examples
  - p:Today is Friday.
  - $\neg p$: Today is not Friday.

    or

    It is not Friday today.

- Truth tables are used to display relationships between the truth value of propositions.

- "$520 < 111$"

Is this a statement? yes

Is this a proposition? yes

What is the truth value of the proposition? false

- "y > 5"

Is this a statement? yes

Is this a proposition? no

Its truth value depends on the value of y, but this value is not specified.

- “What time is it?”

Is this a statement? no

Is this a proposition? no

A proposition has to be a statement.

- Truth table:

The Truth Table for the Negation of a Proposition.	
p	$\neg p$
T	F
F	T

- ***Logical operators*** are used to form new propositions from two or more existing propositions. The logical operators are also called ***connectives***.

LOGICAL OPERATORS (CONNECTIVES)

- Conjunction (AND)
- Disjunction (OR)
- Exclusive or (XOR)
- Implication (if – then)
- Biconditional (if and only if)

<u>CONJUNCTION: (Binary Operator)</u>

Let p and q be propositions. The *<u>conjunction</u>* of p and q, denoted by $p \wedge q$, is the proposition "p and q" which is true when both p and q are true and is false otherwise.

Examples

p:"Today is Friday."

q:"It is raining today."

$p \wedge q$: "Today is Friday and it is raining today".

The proposition is true on rainy Fridays.

- Truth table :

The Truth Table for the Conjunction of Two Propositions.		
p	q	$p \land q$
T	T	T
T	F	F
F	T	F
F	F	F

DISJUNCTION: (Binary Operator)

Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition "p or q" which is false when both p and q are false and is true otherwise.

- Note:

  inclusive or : The disjunction is true when at least one of the two propositions is true.

  - E.g. "Students who have taken calculus or computer science can take this class."

  exclusive or : The disjunction is true only when one of The proposition is true.

  - E.g. "Ice cream or pudding will be served after lunch."

Truth table :

The Truth Table for the Disjunction of Two Propositions.		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The Truth Table for the Exclusive *Or* (*XOR*) of Two Propositions.		
p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

CONDITIONAL STATEMENT:

Let p and q be propositions. The *conditional statement* $p \to q$, is the proposition "if p, then q" which is false when p is true and q is false, and true otherwise.

p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

- Example: "If I am elected, then I will lower taxes." $p \to q$

	p	q	$p \to q$
elected, lower taxes.	T	T	T
not elected, lower taxes.	F	T	T
not elected, not lower taxes.	F	F	T
elected, not lower taxes.	T	F	F

Example:

Let p be the statement "Maria learns discrete mathematics." and q the statement "Maria will find a good job." Express the statement $p \rightarrow q$ as a statement in English.

1. "If Maria learns discrete mathematics, then she will find a good job.
2. "Maria will find a good job if she learns discrete mathematics."
3. "For Maria to get a good job, it is sufficient for her to learn discrete mathematics."
4. "Maria will find a good job unless she does not learn discrete mathematics."

- If p then q
- p implies q
- q if p
- q whenever p
- p is sufficient for q
- q is necessary for p
- p only if q

Other conditional statements:

- Converse of $p \rightarrow q : q \rightarrow p$
- Contrapositive of $p \rightarrow q : \neg q \rightarrow \neg p$
- Inverse of $p \rightarrow q : \neg p \rightarrow \neg q$

- The statement and its contrapositive are equivalent.
- Converse and inverse of a statement are equivalent

BICONDITIONAL STATEMENT:

Let p and q be propositions. The *biconditional statement*

$p \leftrightarrow q$ is the proposition "p if and only if q" which is true when p and q have the same truth values, and is false otherwise.

 $p \leftrightarrow q$ has the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$

Example:

- Let p be the statement "You can take the flight" and let q be the statement "You buy a ticket."

  $p \leftrightarrow q$: "You can take the flight if and only if you buy a ticket."

The Truth Table for the Biconditional $p \leftrightarrow q$.		
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

“ p is necessary and sufficient for q”

“If p then q and conversely”

- Connectives can be used to build up complicated compound propositions involving any number of propositional variables.

Example: Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.					
p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

PRECEDENCE OF LOGICAL OPERATORS

- Parentheses are used to specify the order in which logical operators in a compound proposition are to be applied.
- To reduce the number of parentheses, the precedence order is defined for logical operators.

Precedence of Logical Operators.	
Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

TRANSLATING ENGLISH SENTENCES

Example: How can this English sentence be translated into a logical expression?

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

p: You can ride the roller coaster.

q: You are under 4 feet tall.

r: You are older than 16 years old.

$$(q \wedge \neg r) \rightarrow \neg p$$

Example: How can this English sentence be translated into a logical expression?

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

p: You can access the Internet from campus.

q: You are a computer science major.

r: You are a freshman.

$$p \rightarrow (q \vee \neg r)$$

TAUTOLOGY:

A compound proposition that is always true, no matter what the truth values of the propositions that occurs in it, is called a *tautology*.

CONTRADICTION:

A compound proposition that is always false is called a *contradiction*.

CONTINGENCY:

A compound proposition that is neither a tautology or a contradiction is called a *contingency*.

Examples of a Tautology and a Contradiction.			
p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T F	F T	T T	F F

- The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology.
  Notation $p \equiv q$

- Example: Show that $\neg p \vee q$ and $p \rightarrow q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logical Equivalences

Law	Name
$p \vee F \equiv p$ $p \wedge T \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double-negation law

Law	Name
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$(p \vee q) \wedge (p \vee r) \equiv p \vee (q \wedge r)$ $(p \wedge q) \vee (p \wedge r) \equiv p \wedge (q \vee r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws

Some useful logical Equivalences

$$p \lor \neg p \equiv T$$

$$p \land \neg p \equiv F$$

$$p \rightarrow q \equiv \neg p \lor q$$