P - NP PROBLEMS: MOTIVATION

ANALYSIS OF ALGORITHMS AND COMPLEXITY

THE $1M QUESTION

The Clay Mathematics Institute
Millenium Prize Problems

1. **Birch and Swinnerton-Dyer Conjecture**
2. **Hodge Conjecture**
3. **Navier-Stokes Equations**
4. **P vs NP**
5. **Poincaré Conjecture**
6. **Riemann Hypothesis**
7. **Yang-Mills Theory**

THE P VERSUS NP PROBLEM

Is perhaps one of the biggest open problems in computer science (and mathematics!) today.

(Even featured in one of TV shows)

But what is the P-NP problem?

Sudoku

2			3		8		5	
		3		4	5	9	8	
		8			9	7	3	4
6		7		9				
9	8						1	7
				5		6		9
3	1	9	7			2		
	4	6	5	2		8		
	2		9		3			1

3x3x3

Sudoku

2	9	4	3	7	8	1	5	6
1	7	3	6	4	5	9	8	2
5	6	8	2	1	9	7	3	4
6	5	7	1	9	2	3	4	8
9	8	2	4	3	6	5	1	7
4	3	1	8	5	7	6	2	9
3	1	9	7	8	4	2	6	5
7	4	6	5	2	1	8	9	3
8	2	5	9	6	3	4	7	1

3x3x3

Sudoku

	F		2						6			C	B	3	
	C				4	8	E	A			0		D		
D	A	8			3		2	7	F			6		5	
6			E	D	F		C		8						7
	9	3		7					A						2
E						6	F	5		8	4		3		1
C	8		1	3	9	D		0	2		E				
	D		6		5	E	B		1					0	4
9	6					1		F	3	2		0		A	
				4		A	8		D	0	9	B		2	5
2		A		0	D		5	6	C						F
5						2					A		4	8	
B						4		1		A	2	F			0
	0		7			F	3	C		D			2	9	B
		5		1			A	9	0	B				D	
	2	D	A			9						1		4	

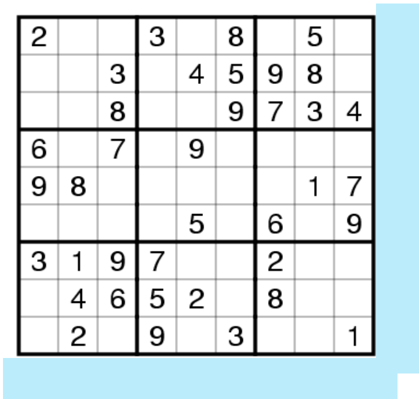
4x4x4

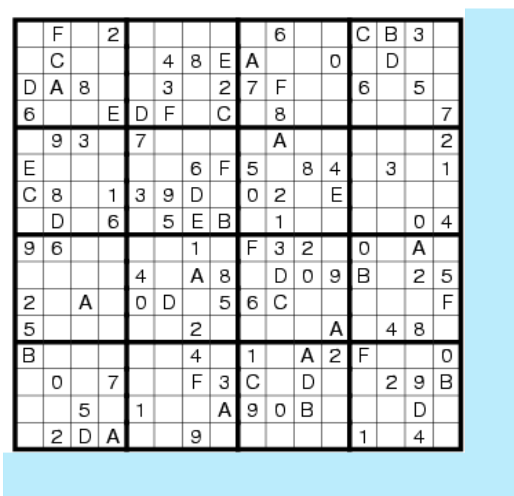
Sudoku

0	F	9	2	A	7	5	1	4	6	E	D	C	B	3	8
7	C	1	3	6	4	8	E	A	B	5	0	2	D	F	9
D	A	8	4	9	3	B	2	7	F	C	1	6	0	5	E
6	5	B	E	D	F	0	C	2	8	9	3	4	A	1	7
4	9	3	5	7	1	C	0	D	A	F	B	8	E	6	2
E	B	7	0	2	A	6	F	5	9	8	4	D	3	C	1
C	8	F	1	3	9	D	4	0	2	6	E	5	7	B	A
A	D	2	6	8	5	E	B	3	1	7	C	9	F	0	4
9	6	4	8	E	B	1	7	F	3	2	5	0	C	A	D
3	7	C	F	4	6	A	8	E	D	0	9	B	1	2	5
2	1	A	B	0	D	3	5	6	C	4	8	7	9	E	F
5	E	0	D	F	C	2	9	B	7	1	A	3	4	8	6
B	3	6	9	C	E	4	D	1	5	A	2	F	8	7	0
1	0	E	7	5	8	F	3	C	4	D	6	A	2	9	B
8	4	5	C	1	2	7	A	9	0	B	F	E	6	D	3
F	2	D	A	B	0	9	6	8	E	3	7	1	5	4	C

4x4x4

Sudoku





n x n x n

Suppose it takes you S(n) to solve n x n x n

V(n) time to verify the solution

Fact: $V(n) = O(n^2 \times n^2)$

Question: is there some constant such that

$S(n) \leq n^{constant}$?

2			3		8		5	
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6		7		9				
9	8						1	7
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	2		9		3			1

	F		2						6			C	B	3	
	C				4	8	E	A			0		D		
D	A	8			3		2	7	F			6		5	
6			E	D	F		C		8						7
	9	3		7					A						2
E						6	F	5		8	4		3		1
C	8		1	3	9	D		0	2		E				
	D		6		5	E	B		1					0	4
9	6					1		F	3	2		0		A	
				4		A	8		D	0	9	B		2	5
2		A		0	D		5	6	C						F
5						2					A		4	8	
B						4		1		A	2	F			0
	0		7			F	3	C		D			2	9	B
		5		1			A	9	0	B				D	
	2	D	A			9						1		4	

⋮

n x n x n

Sudoku

P vs NP problem

=

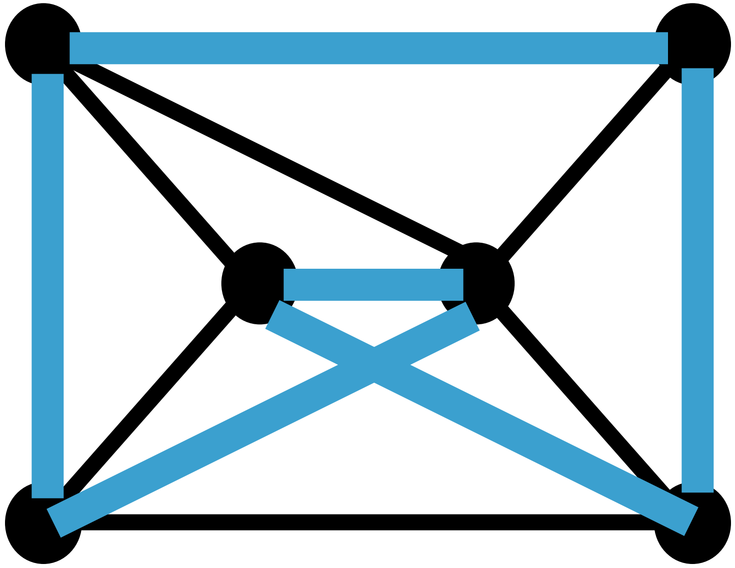
Does there exist an algorithm for n x n x n Sudoku that runs in time p(n) for some polynomial p( ) ?

THE P VERSUS NP PROBLEM (INFORMALLY)

Is proving a theorem much more difficult than checking the proof of a theorem?

Hamilton Cycle

Given a graph G = (V,E), a cycle that visits all the nodes exactly once



The Problem “HAM”

Input: Graph G = (V,E)

Output: YES if G has a Hamilton cycle

NO if G has no Hamilton cycle

The Set “HAM”

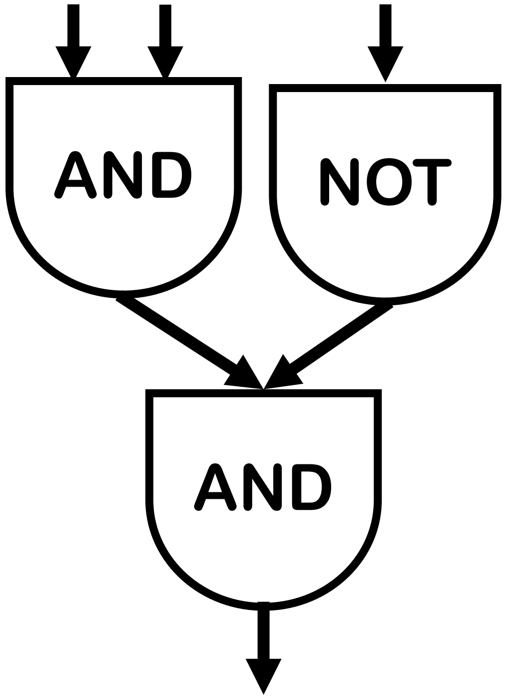
HAM = { graph G | G has a Hamilton cycle }

Circuit-Satisfiability

Input: A circuit C with one output

Output: YES if C is satisfiable

NO if C is not satisfiable



The Set “SAT”

SAT = { all satisfiable circuits C }

Sudoku

Input: n x n x n sudoku instance

Output: YES if this sudoku has a solution

NO if it does not

The Set "SUDOKU"

SUDOKU = { All solvable sudoku instances }

Decision Versus Search Problems

Decision Problem

YES/NO answers

Does G have a Hamilton cycle?

Can G be 3-colored ?

Search Problem

Find a Hamilton cycle in G if one exists, else return NO

Find a 3-coloring of G if one exists, else return NO

Reducing Search to Decision

Given an algorithm for decision Sudoku, devise an algorithm to find a solution

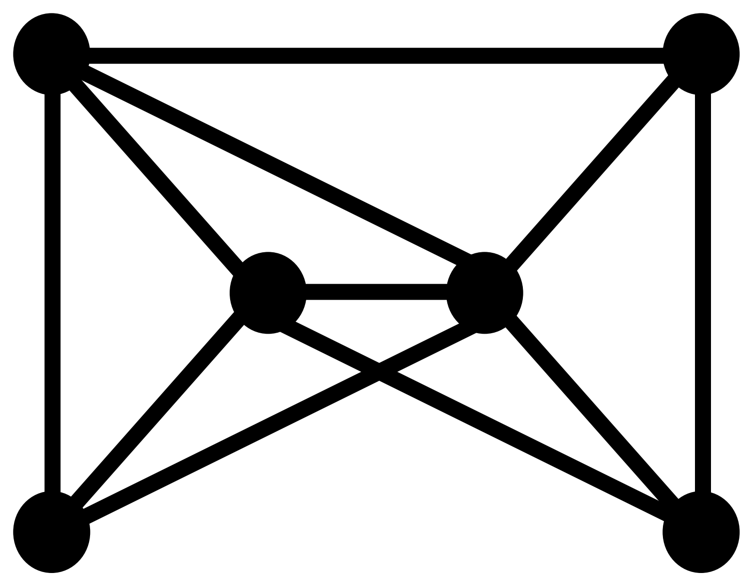
Idea:
Fill in one-by-one and use decision algorithm

2			3		8		5	
		3		4	5	9	8	
		8			9	7	3	4
6		7		9				
9	8						1	7
				5		6		9
3	1	9	7			2		
	4	6	5	2		8		
	2		9		3			1

Reducing Search to Decision

Given an algorithm for decision HAM, devise an algorithm to find a solution

Idea:
Find the edges of the cycle one by one



Decision/Search Problems

We'll study decision problems because they are almost the same (asymptotically) as their search counterparts

Polynomial Time and The Class “P” of Decision Problems

WHAT IS AN EFFICIENT ALGORITHM?

Is an $O(n)$ algorithm efficient?

How about $O(n \log n)$?

$O(n^2)$?

$O(n^{10})$?

$O(n^{\log n})$?

$O(2^n)$?

$O(n!)$?

polynomial time

$O(n^c)$ for some constant c

non-polynomial time

Does an algorithm running in $O(n^{100})$ time count as efficient?

We consider non-polynomial time algorithms to be inefficient.

And hence a necessary condition for an algorithm to be efficient is that it should run in poly-time.

Asking for a poly-time algorithm for a problem sets a (very) low bar when asking for efficient algorithms.

The question is: can we achieve even this for 3-coloring?
SAT?
Sudoku?