Rabi Oscillation and Ramsey Interferometry

Dec 30, 2025

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Pauli Operators and Basic Properties

Consider the following Pauli matrices/operators.

$$\sigma_x \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \qquad (1a)$$

$$\sigma_y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \qquad (1b)$$

$$\sigma_z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad (1c)$$

The Pauli matrices are hermitian. Also note the following where I is the identity operator.

$$\sigma_x^{\ 2} = \sigma_y^{\ 2} = \sigma_z^{\ 2} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \qquad (2a)$$

$$\sigma_x\sigma_y = -\ \sigma_y\sigma_x = i\sigma_z,\ \sigma_y\sigma_z = -\ \sigma_z\sigma_y = i\sigma_x,\ \ \sigma_z\sigma_x = -\ \sigma_x\sigma_z = i\sigma_y. \qquad (2b)$$

There is a more concise way of writing the Eqs. (2b), that is $\epsilon_{uvw}\sigma_u\sigma_v = 2i\sigma_w$, but of course at the expense of knowing what ϵ_{uvw} is. Now on, we do not need to see the 2×2 matrix representations of the Pauli matrices again. All we need to know is that the Pauli operators satisfy Eqs. (2).

Moreover, note that sometimes people use just 1 instead of I when it is clear from the context. We, for now, will keep it until we really get tired of writing an extra I in front of the numbers.

Using the vector notation, $\vec{\sigma} \equiv < \sigma_x, \sigma_y, \sigma_z >$ and $\vec{k} \equiv < k_x, k_y, k_z >$, consider the following dot product. Here $\vec{k}$ is a unit vector that is $k = \sqrt{k_x^{\ 2} + k_y^{\ 2} + k_z^{\ 2}} = 1$,

$$\left(\vec{k}.\vec{\sigma}\right)^2 = \left(k_x\sigma_x + k_y\sigma_y + k_z\sigma_z\right)^2,$$

$$\left(\vec{k}.\vec{\sigma}\right)^2 = k_x^{\ 2}\sigma_x^{\ 2} + k_y^{\ 2}\sigma_y^{\ 2} + k_z^{\ 2}\sigma_z^{\ 2} + k_xk_y\left(\sigma_x\sigma_y + \ \sigma_y\sigma_z\right) + k_yk_z\left(\sigma_y\sigma_z + \ \sigma_z\sigma_y\right) + k_zk_x\left(\sigma_z\sigma_x + \ \sigma_y\sigma_z\right).$$

Using Eqs. (2), and knowing that $\vec{k}$ is a unit vector.

$$\left(\vec{k}.\vec{\sigma}\right)^2 = I. \qquad (3)$$

Generalized Exponential Pauli Operator

Moving on, corresponding to an operator A and a single value θ we define,

$$e^{i\theta A} \equiv I + i\frac{\theta}{1!}A + i^2\frac{\theta^2}{2!}A^2 + ... + i^n\frac{\theta^n}{n!}A^n + ... \qquad (4)$$

Using Eqs. (3) and (4), consider the following where $\vec{k}$ is a unit vector.

$$e^{i\theta\vec{k}.\vec{\sigma}} = I + \frac{i\theta}{1!}\vec{k}.\vec{\sigma} - \frac{\theta^2}{2!}\left(\vec{k}.\vec{\sigma}\right)^2 + ... + \frac{i^n\theta^n}{n!}\left(\vec{k}.\vec{\sigma}\right)^n + ...,$$

$$e^{i\theta\vec{k}.\vec{\sigma}} = \left(I - \frac{\theta^2}{2!}\left(\vec{k}.\vec{\sigma}\right)^2 + \frac{\theta^4}{4!}\left(\vec{k}.\vec{\sigma}\right)^4 - ...\right) + \left(i\frac{\theta}{1!}\vec{k}.\vec{\sigma} - i\frac{\theta^3}{3!}\left(\vec{k}.\vec{\sigma}\right)^3 + i\frac{\theta^5}{5!}\left(\vec{k}.\vec{\sigma}\right)^5 - ...\right),$$

$$e^{i\theta\vec{k}.\vec{\sigma}} = \left(I - \frac{\theta^2}{2!}\left(\vec{k}.\vec{\sigma}\right)^2 + \frac{\theta^4}{4!}\left(\vec{k}.\vec{\sigma}\right)^4 - ...\right) + \left(\theta I - \frac{\theta^3}{3!}\left(\vec{k}.\vec{\sigma}\right)^2 + \frac{\theta^5}{5!}\left(\vec{k}.\vec{\sigma}\right)^4 - ...\right)i\vec{k}.\vec{\sigma},$$

$$e^{i\theta\vec{k}.\vec{\sigma}} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...\right)I + \left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...\right)i\vec{k}.\vec{\sigma}.$$

Identifying the Maclaurin series for cosine and sine,

$$e^{i\theta\vec{k}.\vec{\sigma}} = \cos\theta I + i\sin\theta\,\vec{k}.\vec{\sigma}, \quad (5)$$

We can call $e^{i\theta\vec{k}.\vec{\sigma}}$ as the *generalized exponential Pauli operator*. Consider the following,

$$\left(e^{i\theta\vec{k}.\vec{\sigma}}\right)^\dagger e^{i\theta\vec{k}.\vec{\sigma}} = \left(\cos\theta I + i\sin\theta\vec{k}.\vec{\sigma}\right)^\dagger\left(\cos\theta I + i\sin\theta\vec{k}.\vec{\sigma}\right),$$

$$\left(e^{i\theta\vec{k}.\vec{\sigma}}\right)^\dagger e^{i\theta\vec{k}.\vec{\sigma}} = \left(\cos\theta I - i\sin\theta\vec{k}.\vec{\sigma}^\dagger\right)\left(\cos\theta I + i\sin\theta\vec{k}.\vec{\sigma}\right).$$

Using the fact that the Pauli operators are hermitian,

$$\left(e^{i\theta\vec{k}.\vec{\sigma}}\right)^\dagger e^{i\theta\vec{k}.\vec{\sigma}} = \left(\cos\theta I - i\sin\theta\vec{k}.\vec{\sigma}\right)\left(\cos\theta I + i\sin\theta\vec{k}.\vec{\sigma}\right).$$

Finally, using Eq. (3),

$$\left(e^{i\theta\vec{k}.\vec{\sigma}}\right)^\dagger e^{i\theta\vec{k}.\vec{\sigma}} = I. \qquad (6)$$

Thus, the generalized exponential Pauli operator is unitary. We can define following special unitary operators by choosing the unit vector $\vec{k}$ in one of the three cartesian directions in Eq. (5),

$$U_x(\theta) \equiv e^{i\theta\,\sigma_x} = \cos\theta I + i\sin\theta\,\sigma_x, \quad (7a)$$

$$U_y(\theta) \equiv e^{i\theta\,\sigma_y} = \cos\theta I + i\sin\theta\,\sigma_y, \quad (7b)$$

$$U_z(\theta) \equiv e^{i\theta\,\sigma_z} = \cos\theta I + i\sin\theta\,\sigma_z, \quad (7c)$$

Some more properties of this generalized exponential Pauli operator are derived in the Appendix A.

Fundamental Hamiltonian of a Two-Level System

Consider a two-level system (qubit) with time-independent hamiltonian H_{static}. The two levels or states of this system are eigenstates of both the Pauli operator σ_z and the hamiltonian H_{static}. The corresponding eigenenergies are $\frac{1}{2}\hbar\omega_q$ and $1\frac{1}{2}\hbar\omega_q$ corresponding to the eigenstates $|0\rangle$ and $|1\rangle$. Here ω_q is called the *natural frequency* of the qubit. Thus, for $n = 0, 1$,

$$H_{static}|n\rangle = \left(\frac{1}{2} + n\right)\hbar\omega_q|n\rangle. \quad (8)$$

We can write the above hamiltonian in terms of the Pauli matrices as follows.

$$H_{static} = \hbar\omega_q I - \frac{1}{2}\hbar\omega_q\sigma_z. \qquad (9)$$

Consider a time dependent perturbation hamiltonian of the form,

$$H_p = \hbar\Omega\sin(\omega t + \phi)\sigma_x. \qquad (10)$$

Here, Ω is the *drive strength frequency* and ω is the *drive frequency*. The the full hamiltonian H is,

$$H = H_{static} + H_p.$$

$$H = \hbar\omega_q I - \frac{1}{2}\hbar\omega_q\sigma_z + \hbar\Omega\sin(\omega t + \phi)\sigma_x. \qquad (11a)$$

We call H the *fundamental hamiltonian*. The state of this system $|\psi\rangle$ evolves with respect to this hamiltonian as given by the Schrodinger's equation,

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle. \qquad (11b)$$

Hamiltonian Without Offset

Let's attend to the offset $\hbar\omega_q I$ in the hamiltonian first. To this end, let,

$$|\psi\rangle = e^{-i\omega_q t}|\psi_0\rangle. \qquad (12)$$

We can call Eq. (12), if we will, as *offset frame transformation*. Note the choice of the frequency in the exponent corresponding to the offset term in the hamiltonian.
Substituting Eq. (12) in Eq. (11b),

$$i\hbar\frac{\partial}{\partial t}e^{-i\omega_q t}|\psi_0\rangle = He^{-i\omega_q t}|\psi_0\rangle,$$

$$\hbar\omega_q e^{-i\omega_q t}|\psi_0\rangle + i\hbar e^{-i\omega_q t}\frac{\partial}{\partial t}|\psi_0\rangle = He^{-i\omega_q t}|\psi_0\rangle,$$

$$i\hbar\frac{\partial}{\partial t}|\psi_0\rangle = \left(H - \hbar\omega_q I\right)|\psi_0\rangle.$$

Let us define,

$$H_0 = H - \hbar\omega_q = \frac{1}{2}\hbar\omega_q\sigma_z + \hbar\Omega\sin(\omega t + \phi)\sigma_x. \qquad (13a)$$

The evolution of $|\psi_0\rangle$ is then given by the Schrodinger's equation,

$$i\hbar\frac{\partial}{\partial t}|\psi_0\rangle = H_0|\psi_0\rangle. \qquad (13b)$$

We call H_0 the zeroth order hamiltonian. Thus $|\psi_0\rangle$ evolves according to Eq. (13b) where the hamiltonian is defined in Eq. (13a). This is similar to $|\psi\rangle$ which evolves according to Eq. (11b) where the hamiltonian is defined in Eq. (11a).

Hamiltonian in the First-Order Rotating Frame

Let,

$$\theta \equiv \frac{1}{2}\omega t. \qquad (14)$$

Using the unitary operator $U_z(\theta)$ defined in Eq. (7c), let,

$$|\psi_0\rangle = e^{i\theta\,\sigma_z}|\psi_1\rangle = \left(\cos\theta I + i\sin\theta\,\sigma_z\right)|\psi_1\rangle. \qquad (15)$$

Equation (12) is called *rotating frame transformation*. Note that the wave function $|\psi_0\rangle$ and $|\psi_1\rangle$ are rotating with respect to each other at half the frequency of the driving (or perturbing) hamiltonian.
From Eq. (13b),

$$i\hbar\frac{\partial}{\partial t}\left(U_z|\psi_1\rangle\right) = H_0 U_z|\psi_1\rangle,$$
$$i\hbar\dot{U}_z|\psi_1\rangle + i\hbar U_z\frac{\partial}{\partial t}|\psi_1\rangle = -\frac{1}{2}\hbar\omega_q\sigma_z U_z|\psi_1\rangle + H_p U_z|\psi_1\rangle,$$
$$i\hbar U_z\frac{\partial}{\partial t}|\psi_1\rangle = -i\hbar\dot{U}_z|\psi_1\rangle - \frac{1}{2}\hbar\omega_q\sigma_z U_z|\psi_1\rangle + H_p U_z|\psi_1\rangle.$$

Multiplying by the conjugate of $U_z^{\dagger}$,

$$i\hbar\frac{\partial}{\partial t}|\psi_1\rangle = \left(-i\hbar U_z^{\dagger}\dot{U}_z - \frac{1}{2}\hbar\omega_q U_z^{\dagger}\sigma_z U_z + U_z^{\dagger}H_p U_z\right)|\psi_1\rangle.$$

Noting that U_z commutes with σ_z. Therefore,

$$i\hbar\frac{\partial}{\partial t}|\psi_1\rangle = \left(-i\hbar U_z^{\dagger}\dot{U}_z - \frac{1}{2}\hbar\omega_q\sigma_z + U_z^{\dagger}H_p U_z\right)|\psi_1\rangle. \quad (16)$$

The term in parentheses is the effective hamiltonian in the so-called rotating frame. Let us define,

$$H_1 = -i\hbar U_z^{\dagger}\dot{U}_z - \frac{1}{2}\hbar\omega_q\sigma_z + U_z^{\dagger}H_p U_z. \qquad (17a)$$

Then Eq. (16) can be written as follows,

$$i\hbar\frac{\partial}{\partial t}|\psi_1\rangle = H_1|\psi_1\rangle. \qquad (17b)$$

Compare Eqs. (17) with Eqs. (13) and Eqs. (11).

Let us look at some of the terms on the right side of Eq. (17a). Here, we have used Eqs. (A1) and (A3) from the Appendix.

$$U_z^{\dagger}\dot{U}_z = e^{-i\theta\sigma_z}\frac{d}{dt}e^{i\theta\sigma_z} = i\frac{1}{2}\omega\sigma_z e^{-i\theta\sigma_z}e^{i\theta\sigma_z} = i\frac{1}{2}\omega\,\sigma_z. \qquad (18a),$$
$$U_z^{\dagger}H_p U_z = \left(\cos\theta I - i\sin\theta\,\sigma_z\right)\hbar\Omega\sin(2\theta + \phi)\sigma_x\left(\cos\theta I + i\sin\theta\,\sigma_z\right),$$
$$U_z^{\dagger}H_p U_z = \hbar\Omega\sin(2\theta + \phi)\left(\cos\theta I - i\sin\theta\,\sigma_z\right)\left(\cos\theta\sigma_x + \sin\theta\,\sigma_y\right),$$
$$U_z^{\dagger}H_p U_z = \hbar\Omega\sin(2\theta + \phi)\left(\cos\theta I - i\sin\theta\,\sigma_z\right)\left(\cos\theta\sigma_x + \sin\theta\,\sigma_y\right),$$
$$U_z^{\dagger}H_p U_z = \hbar\Omega\sin(2\theta + \phi)\left(\cos^2\theta\sigma_x + \cos\theta\sin\theta\,\sigma_y - i\cos\theta\sin\theta\,\sigma_z\sigma_x - i\sin^2\theta\,\sigma_z\sigma_y\right),$$
$$U_z^{\dagger}H_p U_z = \hbar\Omega\sin(2\theta + \phi)\left(\cos^2\theta\sigma_x + \cos\theta\sin\theta\,\sigma_y + \cos\theta\sin\theta\,\sigma_y - \sin^2\theta\,\sigma_x\right),$$
$$U_z^{\dagger}H_p U_z = \hbar\Omega\sin(2\theta + \phi)\left(\cos 2\theta\sigma_x + \sin 2\theta\,\sigma_y\right),$$

Using sum and difference identities of cosine functions.

$$U_z^{\dagger}H_p U_z$$
$$= \frac{1}{2}\hbar\Omega\left[(\sin(2\theta + \phi + 2\theta) + \sin(2\theta + \phi - 2\theta))\sigma_x + (\cos(2\theta + \phi - 2\theta) - \cos(2\theta + \phi + 2\theta))\sigma_y\right],$$
$$U_z^{\dagger}H_p U_z = \frac{1}{2}\hbar\Omega\left[(\sin(4\theta + \phi) + \sin\phi)\sigma_x + (\cos\phi - \cos(4\theta + \phi))\sigma_y\right].$$

Substituting back $\theta = \frac{1}{2}\omega t$.

$$U_z^{\dagger}H_p U_z = \frac{1}{2}\hbar\Omega\left(\sin\phi\,\sigma_x + \cos\phi\,\sigma_y\right) + \frac{1}{2}\hbar\Omega\left(\sin(2\omega t + \phi)\,\sigma_x - \cos(2\omega t + \phi)\,\sigma_y\right). \; (18b)$$

Substituting Eqs. (18) in Eq. (17a),

$$H_1 = \frac{1}{2}\hbar\omega\,\sigma_z - \frac{1}{2}\hbar\omega_q\sigma_z + \frac{1}{2}\hbar\Omega\left(\sin\phi\,\sigma_x + \cos\phi\,\sigma_y\right) + \frac{1}{2}\hbar\Omega\left(\sin(2\omega t + \phi)\,\sigma_x - \cos(2\omega t + \phi)\,\sigma_y\right).$$

Let us define *detuning frequency*,

$$\Delta \equiv \omega - \omega_q. \qquad (19)$$

Then

$$H_1 = \frac{1}{2}\hbar\Omega \sin\phi\, \sigma_x + \frac{1}{2}\hbar\Omega \cos\phi\, \sigma_y + \frac{1}{2}\hbar\Delta\, \sigma_z + \frac{1}{2}\hbar\Omega\left(\sin(2\omega t + \phi)\, \sigma_x - \cos(2\omega t + \phi)\, \sigma_y\right),$$

$$H_1 = \frac{1}{2}\hbar\Omega \sin\phi\, \sigma_x + \frac{1}{2}\hbar\Omega \cos\phi\, \sigma_y + \frac{1}{2}\hbar\Delta\, \sigma_z + \frac{1}{2}\hbar\Omega\left(\sin(2\omega t + \phi)\, \sigma_x - \cos(2\omega t + \phi)\, \sigma_y\right). \ (20)$$

Let us define *Rabi frequency* (in anticipation of course) and a unit vector,

$$\Lambda \equiv \sqrt{\Omega^2 + \Delta^2}. \qquad (21)$$

$$\vec{n} \equiv < \frac{\Omega \sin\phi}{\Lambda}, \frac{\Omega \cos\phi}{\Lambda}, \frac{\Delta}{\Lambda} >. \qquad (22)$$

Then,

$$H_1 = \frac{1}{2}\hbar\Lambda\, \vec{n}.\vec{\sigma} + \frac{1}{2}\hbar\Omega\left(\sin(2\omega t + \phi)\, \sigma_x - \cos(2\omega t + \phi)\, \sigma_y\right). \qquad (23)$$

We call We call H_1 the first order hamiltonian.

Rotating Frame Approximation (RWA)

Compare the hamiltonian H_1 in Eq. (20) with H_0 in Eq. (13a). In H_0 the time dependence comes as cosine functions rotating at the rate of ω. In H_1, however, the time dependence comes as cosine functions rotating at the rate of 2ω. If we wish, we can further transform $|\psi_1\rangle$ to $|\psi_2\rangle$ and H_1 to H_2using the unitary operator $U_z(2\theta)$ such that they satisfy an equation like Eqs. (17b). Then H_2 will have a time dependence as cosine functions rotating at the rate of 4ω. We can keep going on. In general, the n^{th} order hamiltonian H_n will have time dependences as cosines of $2^n\omega t$ (need to be verified by a general theory).

Let us drop off the the terms in H_1 that are rotating at the rate of 2ω. If we had transformed to H_2 then we would have to drop off the terms rotating at the rate of 4ω. Thus, in this sense, what we are doing here is like applying the perturbation theory to the first order. Thus, to first order, the hamiltonian H_1 is given as,

Then,

$$H_1 \approx \frac{1}{2}\hbar\Lambda\, \vec{n}.\vec{\sigma}. \qquad (24)$$

Note that the approximate first order hamiltonian is independent of the phase of the sinusoidal perturbation. Also, the frequency or perturbation only affects through the tiny detuning frequency Δ that goes into the Rabi frequency Λ.

A Summary on the Way

Let us summarize our results so far. The fundamental hamiltonian is the following,

$$H = \hbar\omega_q I - \frac{1}{2}\hbar\omega_q\sigma_z + \hbar\Omega\sin(\omega t + \phi)\sigma_x. \qquad (11a)$$

We define,

$$\theta \equiv \frac{1}{2}\omega t. \qquad (14)$$

$$\Delta \equiv \omega - \omega_q. \qquad (19)$$

$$\Lambda \equiv \sqrt{\Omega^2 + \Delta^2}. \qquad (21)$$

$$\vec{n} \equiv < \frac{\Omega\sin\phi}{\sqrt{\Omega^2+\Delta^2}}, \frac{\Omega\cos\phi}{\sqrt{\Omega^2+\Delta^2}}, \frac{\Delta}{\sqrt{\Omega^2+\Delta^2}} >. \qquad (22)$$

We have,

$$H_1 \approx \frac{1}{2}\hbar\Lambda\,\vec{n}.\vec{\sigma}. \qquad (24)$$

$$i\hbar\frac{\partial}{\partial t}|\psi_1(t)\rangle = H_1|\psi_1(t)\rangle. \qquad (17b)$$

We solve $|\psi_1\rangle$ using the time-independent hamiltonian H_1. Then, the state $|\psi\rangle$ for the fundamental hamiltonian is obtained as follows.

$$|\psi_0\rangle = e^{i\theta\sigma_z}|\psi_1(t)\rangle. \qquad (15)$$

$$|\psi(t)\rangle = e^{-i\omega_q t}|\psi_0(t)\rangle. \qquad (12)$$

First-Order Calculation of the Wavefunction

Let us solve the wavefunction in Eq. (17b). Before proceeding, consider the following from Eq. (4), where A is a time independent hamiltonian.

$$\frac{\partial}{\partial t}e^{i\omega tA} = i\frac{\omega}{1!}A + 2i^2\frac{\omega^2 t}{2!}A^2 + ... + ni^n\frac{\omega^n t^{n-1}}{n!}A^n + ...,$$

$$\frac{\partial}{\partial t}e^{i\omega tA} = i\omega A\left(I + i\frac{\omega t}{1!}A + ... + i^{n-1}\frac{\omega^{n-1}t^{n-1}}{(n-1)!}A^{n-1} + ...\right),$$

$$\frac{\partial}{\partial t}e^{i\omega tA} = i\omega A e^{i\omega tA}. \qquad (25)$$

Since, after first-order RWA, our hamiltonian H_1 is time-independent, we can use the result of Eq. (25) for it. Let us show that the following form of $|\psi_1\rangle$ satisfies Eq. (17b).

$$|\psi_1(t)\rangle = e^{-i\frac{1}{\hbar}tH_1}|\psi_1(0)\rangle. \qquad (26)$$

From Eq. (17b),

$$i\hbar\frac{\partial}{\partial t}|\psi_1\rangle = i\hbar\frac{\partial}{\partial t}e^{-i\frac{1}{\hbar}tH_1}|\psi_1(0)\rangle = H_1 e^{-i\frac{1}{\hbar}tH_1}|\psi_1(0)\rangle = H_1|\psi_1(t)\rangle.$$

Thus $|\psi_1(t)\rangle$ given in Eq. (26) is the state vector, to first order be reminded. Let us simplify it further.

$$|\psi_1(t)\rangle = e^{-i\frac{1}{\hbar}t\left(\frac{1}{2}\hbar\Lambda\,\vec{n}.\vec{\sigma}\right)}|\psi_1(0)\rangle,$$

$$|\psi_1(t)\rangle = e^{-i\frac{1}{2}\Lambda t\left(\vec{n}.\vec{\sigma}\right)}|\psi_1(0)\rangle.$$

$$|\psi_1(t)\rangle = e^{-i\theta_2\left(\vec{n}.\vec{\sigma}\right)}|\psi_1(0)\rangle. \quad (27)$$

Here we have defined,

$$\theta_2 \equiv \frac{1}{2}\Lambda t. \quad (28)$$

Finally, from Eqs. (12), (15), and (27),

$$|\psi(t)\rangle = e^{-i\omega_q t}U(t)\,|\psi(0)\rangle. \quad (29a)$$

$$U(t) = e^{i\theta\,\sigma_z}e^{-i\theta_2\left(\vec{n}.\vec{\sigma}\right)}. \quad (29b)$$

Here, we have used the fact that $|\psi(0)\rangle = |\psi_0(0)\rangle = |\psi_1(0)\rangle$. Consider the unitary operator $U(t)$, defined above, a bit in detail.

$$U(t) = e^{i\theta\,\sigma_z}e^{-i\theta_2\left(\vec{n}.\vec{\sigma}\right)} = \left(\cos\theta\, I + i\sin\theta\,\sigma_z\right)\left(\cos\theta_2\, I - i\sin\theta_2\, n_x\sigma_x - i\sin\theta_2\, n_y\sigma_y - i\sin\theta_2\, n_z\sigma_z\right)$$

$$= \cos\theta\cos\theta_2\, I - i\cos\theta\sin\theta_2\, n_x\sigma_x - i\cos\theta\sin\theta_2\, n_y\sigma_y - i\cos\theta\sin\theta_2\, n_z\sigma_z$$

$$+ i\sin\theta\cos\theta_2\,\sigma_z + i\sin\theta\sin\theta_2\, n_x\sigma_y - i\sin\theta\sin\theta_2\, n_y\sigma_x + \sin\theta\sin\theta_2\, n_z I,$$

$$U(t) = \left(\cos\theta\cos\theta_2 + \sin\theta\sin\theta_2\, n_z\right)I + i\left(\sin\theta\cos\theta_2 - \cos\theta\sin\theta_2\, n_z\right)\sigma_z$$

$$- i\left(\sin\theta\, n_y + \cos\theta\, n_x\right)\sin\theta_2\sigma_x - i\left(\cos\theta n_y - \sin\theta n_x\right)\sin\theta_2\,\sigma_y. \quad (30)$$

Specially, consider the following matrix elements for this unitary operator.

$$U_{00}(t) = \left(\cos\theta\cos\theta_2 + \sin\theta\sin\theta_2\, n_z\right) + i\left(\sin\theta\cos\theta_2 - \cos\theta\sin\theta_2\, n_z\right). \quad (31a)$$

$$U_{10}(t) = \left(\cos\theta n_y - \sin\theta n_x\right)\sin\theta_2 - i\left(\sin\theta\, n_y + \cos\theta\, n_x\right)\sin\theta_2\,. \quad (31b)$$

Rabi Oscillation

Let the system be in the ground state at time $t = 0$.

$$|\psi(0)\rangle = |0\rangle.$$

Then,

$$|\psi(t)\rangle = e^{-i\omega_q t}U(t)\,|0\rangle,$$

Consider the probability of finding the system in n^{th} state after time t.

$$P_n(t) = |\langle n|\psi(t)\rangle|^2,$$

$$P_n(t) = \left|\langle n|e^{-i\omega_q t}U(t)|0\rangle\right|^2,$$

$$P_n(t) = |\langle n|U(t)|0\rangle|^2,$$

$$P_n(t) = \left|U_{n0}(t)\right|^2. \quad (32)$$

Using Eqs. (31a) and (32),

$$P_0(t) = \left(\cos\theta\cos\theta_2 + \sin\theta\sin\theta_2\, n_z\right)^2 + \left(\sin\theta\cos\theta_2 - \cos\theta\sin\theta_2\, n_z\right)^2,$$

$$P_0(t) = \left(\cos\theta\cos\theta_2\right)^2 + \left(\sin\theta\sin\theta_2\, n_z\right)^2 + \left(\sin\theta\cos\theta_2\right)^2 + \left(\cos\theta\sin\theta_2\, n_z\right)^2,$$

$$P_0(t) = \cos^2\theta_2 + n_z^{\ 2}\sin^2\theta_2. \quad (33a)$$

Using Eqs. (31a) and (32),

$$P_1(t) = \left(\sin\theta_2\right)^2\left(\left(\cos\theta n_y - \sin\theta n_x\right)^2 + \left(\sin\theta\, n_y + \cos\theta\, n_x\right)^2\right),$$

$$P_1(t) = \left(\sin\theta_2\right)^2\left(n_x^{\ 2} + n_y^{\ 2}\right),$$

$$P_1(t) = \left(n_x^{\ 2} + n_y^{\ 2}\right)\sin^2\theta_2. \quad (33b)$$

It is very satisfying to see that both probabilities add up to one. Moreover, we see that the probabilities do not depend on $\theta = \frac{1}{2}\omega t$ but rather on $\theta_2 = \frac{1}{2}\Lambda t$, where Λ is the quadrature sum of the perturbation strength frequency Ω and the detuning $\Delta \equiv \omega - \omega_q$ between the drive frequency and the 'natural' frequency of the system. Let us summarize the results below.

$$P_0(t) = 1 - \frac{\Omega^2}{\Lambda^2}\frac{1-\cos\Lambda t}{2}. \quad (34a)$$

$$P_1(t) = \frac{\Omega^2}{\Lambda^2}\frac{1-\cos\Lambda t}{2}. \quad (34b)$$

This oscillation of the probability with respect to time is called Rabi oscillation.

Procedure for the Rabi Oscillation Study

If we prepare a two-level system (qubit) in the ground state of σ_z and then turn on a sinusoidal perturbation $\hbar\Omega\sigma_x \sin(\omega t + \phi)$ for some time $0 < t < T$. Then the probability of finding the qubit in the excited state, just after the perturbation is turned off, is given as follows,

$$P(T) = \frac{\Omega^2}{\Lambda^2}\frac{1-\cos\Lambda T}{2}. \quad (35)$$

Here,

- $\Lambda = \sqrt{\Omega^2 + \Delta^2}$ is the Rabi frequency.
- Ω is the drive strength frequency.
- $\Delta = \omega - \omega_q$ is the detuning frequency.
- ω is the drive frequency.
- ω_q is the qubit's natural frequency.

We notice that greater the detuning the smaller the amplitude of the Rabi oscillation. In practice, we first fix the frequency ω near the approximate value of ω_q. Then we prepare the system in ground state, drive the system for some time T and "read-off" the discrete energy of the system which turns out to correspond to the excited level's energy with probability given by Eq. (35). We repeat this experiment many times to get one statistical point on the graph $P(T)$ vs T. We repeat this experiment for various values of T to obtain a full cycle. By curve fitting the graph, we obtain the parameters ω_q (qubit frequency) and Ω (the drive strength frequency).

Determining Π and $\Pi/2$ Pulses

If

Conflicting requirements of speed and bandwidth.

Determining Longitudinal Relaxation Time

If

Conflicting requirements of speed and band

Ramsey Interferometry

$$\mu_{jj} = \langle\psi_j|\hat{\mu}|\psi_j\rangle = -\ e\langle\psi_j|z|\psi_j\rangle\ =\ 0.$$

Then, correspondingly,

$$|\psi_r\rangle\ =\ U_z^{\dagger}\left(\frac{1}{2}\omega t\right)|\psi\rangle\ =\ \left(\cos\theta I\ -\ i\sin\theta\ \sigma_z\right)|\psi\rangle. \qquad \text{(12b)}$$

Transverse Relaxation Time

$$\mu_{jj} = \langle\psi_j|\hat{\mu}|\psi_j\rangle = -\ e\langle\psi_j|z|\psi_j\rangle\ =\ 0.$$

Appendix

Consider a time independent operator A..

$$\frac{\partial}{\partial t}e^{i\omega tA}\ =\ i\frac{\omega}{1!}A\ +\ 2i^2\frac{\omega^2 t}{2!}A^2\ +...+\ ni^n\frac{\omega^n t^{n-1}}{n!}A^n\ +...,$$

$$\frac{\partial}{\partial t}e^{i\omega tA}\ =\ i\omega A\left(I\ +\ i\frac{\omega t}{1!}A\ +...+\ i^{n-1}\frac{\omega^{n-1}t^{n-1}}{(n-1)!}A^{n-1}\ +...\right),$$

$$\frac{\partial}{\partial t}e^{i\omega tA}\ =\ i\omega Ae^{i\omega tA}. \qquad \text{(A1)}$$

$$\sigma_s = -\ ie^{i\frac{\pi}{2}\sigma_s},$$

$$e^{i\theta_1\vec{k}_1\cdot\vec{\sigma}}e^{i\theta_2\vec{k}_2\cdot\vec{\sigma}}\ =\ \left(\cos\theta_1 I\ +\ i\sin\theta_1\left(\ k_{1x}\sigma_x + k_{1y}\sigma_y + k_{1z}\sigma_z\right)\right)\left(\cos\theta_2 I\ +\ i\sin\theta_2\left(\ k_{2x}\sigma_x + k_{2y}\sigma_y + k_{2z}\sigma_z\right)\right)$$

$$=$$

$$\cos\theta_1\cos\theta_2 I\ +\ i\sin\theta_1\cos\theta_2\left(\ k_{1x}\sigma_x + k_{1y}\sigma_y + k_{1z}\sigma_z\right)$$

$$+\ i\cos\theta_1\sin\theta_2\ k_{2x}\sigma_x\ -\ \sin\theta_1\sin\theta_2\left(\ k_{1x}k_{2x} + k_{1y}k_{2x}\sigma_y\sigma_x + k_{1z}k_{2x}\sigma_z\sigma_x\right)$$

$$+\ i\cos\theta_1\sin\theta_2\ k_{2y}\sigma_y\ -\ \sin\theta_1\sin\theta_2\left(\ k_{1x}k_{2y}\sigma_x\sigma_y + k_{1y}k_{2y} + k_{1z}k_{2y}\sigma_z\sigma_y\right)$$

$$+\ i\cos\theta_1\sin\theta_2 k_{2z}\sigma_z\ -\ \sin\theta_1\sin\theta_2\left(\ k_{1x}k_{2z}\sigma_x\sigma_z + k_{1y}k_{2z}\sigma_y\sigma_z + k_{1z}k_{2z}\right)$$

$$=$$

$$\cos\theta_1\cos\theta_2 I + i\sin\theta_1\cos\theta_2\,\vec{k}_1.\vec{\sigma} + i\cos\theta_1\sin\theta_2\vec{k}_2.\vec{\sigma}$$
$$-\sin\theta_1\sin\theta_2\left(\vec{k}_1.\vec{k}_2\right)$$
$$-\sin\theta_1\sin\theta_2\left(k_{1y}k_{2x}\sigma_y\sigma_x + k_{1z}k_{2x}\sigma_z\sigma_x + k_{1x}k_{2y}\sigma_x\sigma_y + k_{1z}k_{2y}\sigma_z\sigma_y k_{1x}k_{2z}\sigma_x\sigma_z + k_{1y}k_{2z}\sigma_y\sigma_z\right)$$
$$=$$
$$\cos\theta_1\cos\theta_2 I + i\sin\theta_1\cos\theta_2\,\vec{k}_1.\vec{\sigma} + i\cos\theta_1\sin\theta_2\vec{k}_2.\vec{\sigma}$$
$$-\sin\theta_1\sin\theta_2\left(\vec{k}_1.\vec{k}_2\right)$$
$$-i\sin\theta_1\sin\theta_2\left(k_{1y}k_{2z}\sigma_x - k_{1z}k_{2y}\sigma_x + k_{1z}k_{2x}\sigma_y - k_{1x}k_{2z}\sigma_y + k_{1x}k_{2y}\sigma_z - k_{1y}k_{2x}\sigma_z\right)$$
$$=$$
$$\cos\theta_1\cos\theta_2 I + i\sin\theta_1\cos\theta_2\,\vec{k}_1.\vec{\sigma} + i\cos\theta_1\sin\theta_2\vec{k}_2.\vec{\sigma}$$
$$-\sin\theta_1\sin\theta_2\left(\vec{k}_1.\vec{k}_2\right)$$
$$-i\sin\theta_1\sin\theta_2\left(\left(k_{1y}k_{2z} - k_{1z}k_{2y}\right)\sigma_x + \left(k_{1z}k_{2x} - k_{1x}k_{2z}\right)\sigma_y + \left(k_{1x}k_{2y} - k_{1y}k_{2x}\right)\sigma_z\right)$$
$$=$$

$$e^{i\theta_1\vec{k}_1.\vec{\sigma}}e^{i\theta_2\vec{k}_2.\vec{\sigma}}$$
$$= \cos\theta_1\cos\theta_2 I + i\sin\theta_1\cos\theta_2\,\vec{k}_1.\vec{\sigma} + i\cos\theta_1\sin\theta_2\vec{k}_2.\vec{\sigma} - \sin\theta_1\sin\theta_2\left(\vec{k}_1.\vec{k}_2 I + i\left(\vec{k}_1\times\vec{k}_2\right).\vec{\sigma}\right) \quad \text{(A2)}$$

When $\vec{k}_1 = \vec{k}_2 = \vec{k}$,

$$e^{i\theta_1\vec{k}.\vec{\sigma}}e^{i\theta_2\vec{k}.\vec{\sigma}} = \cos\left(\theta_1+\theta_2\right)I + i\sin\left(\theta_1+\theta_2\right)\vec{k}.\vec{\sigma}.$$
$$e^{i\theta_1\vec{k}.\vec{\sigma}}e^{i\theta_2\vec{k}.\vec{\sigma}} = e^{i\left(\theta_1+\theta_2\right)\vec{k}.\vec{\sigma}} \quad \text{(A3)}$$

$$\sigma_s e^{i\theta\vec{k}.\vec{\sigma}} = -ie^{i\frac{\pi}{2}\sigma_s}e^{i\theta\vec{k}.\vec{\sigma}} = \cos\theta\,\sigma_s + i\sin\theta\,k_s I - \sin\theta\left(k_{s+1}\sigma_{s+1} - k_{s-1}\sigma_{s-1}\right) =$$
$$\sigma_s e^{i\theta\sigma_{s+1}} = -ie^{i\frac{\pi}{2}\sigma_s}e^{i\theta\sigma_{s+1}} = \cos\theta\,\sigma_s - \sin\theta\sigma_{s+1} = \left(\cos\theta + i\sin\theta\,\sigma_{s-1}\right)\sigma_s = e^{i\theta\sigma_{s-1}}\sigma_s$$

$$\sigma_s e^{i\theta\sigma_s} = -ie^{i\frac{\pi}{2}\sigma_s}e^{i\theta\sigma_s} = \cos\theta\,\sigma_s + i\sin\theta\,I = \left(\cos\theta + i\sin\theta\,\sigma_s\right)\sigma_s = e^{i\theta\sigma_s}\sigma_s$$
$$e^{i\theta\sigma_s}e^{i\theta\sigma_{s+1}}$$
$$= \cos\theta_1\cos\theta_2 I + i\sin\theta_1\cos\theta_2\sigma_s + i\cos\theta_1\sin\theta_2\sigma_{s+1} - i\sin\theta_1\sin\theta_2\sigma_{s+2}$$
$$= \cos\theta_2\left(\cos\theta_1 I + i\sin\theta_1\sigma_s\right) + i\sin\theta_2\left(\cos\theta_1 + \sin\theta_1\sigma_s\right)\sigma_{s+1}$$

We would like to develop a general theory next.

Consider the first-order hamiltonian,

$$H_1 = \frac{1}{2}\hbar\Lambda\,\vec{n}.\vec{\sigma} + \frac{1}{2}\hbar\Omega\left(\sin(2\omega t\ +\ \phi)\,\sigma_x - \cos(2\omega t\ +\ \phi)\,\sigma_y\right)$$

We define,

$$H_{p1} = \frac{1}{2}\hbar\Omega\left(\sin(2\omega t\ +\ \phi)\,\sigma_x - \cos(2\omega t\ +\ \phi)\,\sigma_y\right)$$

Let

$$|\psi_1\rangle = U_z(2\theta)|\psi_2\rangle = \left(\cos 2\theta I\ +\ i\sin 2\theta\,\sigma_z\right)|\psi_2\rangle. \qquad \text{(xx)}$$

Equation (12) is called *second-order rotating frame transformation*. Note that the wave function $|\psi_1\rangle$ and $|\psi_2\rangle$ are rotating with respect to each other at the frequency of the driving (or perturbing) hamiltonian.

From Eq. (17b),

$$i\hbar\frac{\partial}{\partial t}\left(U_z|\psi_2\rangle\right) = H_1U_z|\psi_2\rangle,$$

$$i\hbar\dot{U}_z|\psi_2\rangle\ +\ i\hbar U_z\frac{\partial}{\partial t}|\psi_2\rangle\ = \frac{1}{2}\hbar\Lambda\,\vec{n}.\vec{\sigma}U_z|\psi_2\rangle\ +\ H_{p1}U_z|\psi_2\rangle,$$

$$i\hbar U_z\frac{\partial}{\partial t}|\psi_1\rangle\ =-\ i\hbar\dot{U}_z|\psi_1\rangle\ -\frac{1}{2}\hbar\Lambda\,\vec{n}.\vec{\sigma}U_z|\psi_1\rangle\ +\ H_{p1}U_z|\psi_2\rangle.$$

Multiplying by the conjugate of $U_z^{\dagger}$,

$$i\hbar\frac{\partial}{\partial t}|\psi_1\rangle\ = \left(-\ i\hbar U_z^{\dagger}\dot{U}_z + \frac{1}{2}\hbar\Lambda\,U_z^{\dagger}\vec{n}.\vec{\sigma}U_z\ +\ U_z^{\dagger}H_pU_z\right)|\psi_1\rangle.$$

Noting that U_z commutes with σ_z. Therefore,

Develop general theory of successive rotating frame approximations in a two-level system
https://journals.aps.org/pra/pdf/10.1103/PhysRevA.91.013814
Generalized rotating-wave approximation for the two-qubit quantum Rabi model