Superconducting Qubit Notes

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Fundamental Equations

A capacitor is a lumped element that can hold $+\,Q$ and $-\,Q$ charges on two distinct islands. Application of charge continuity to the capacitor yields the following equation relating the current I to the charge Q.

$$I = \frac{dQ}{dt} \equiv \dot{Q}. \quad (1)$$

An inductor is a lumped element with magnetic flux, or simply flux, linking through it. Faraday's law relates the flux Φ to the voltage V (negative sign has been absorbed by the way we have defined the sense of the voltage) as follows.

$$V = \frac{d\Phi}{dt} \equiv \dot{\Phi}. \quad (2)$$

From the definitions of voltage and current, the energy E stored in a lumped element between nodes a and b, is given as follows where V is the voltage of node a with respect to node b and I is the current going in at node a and hence coming out from node b.

$$E = \int_{-\infty}^{t} VI\,dt'. \quad (3)$$

In the capacitor, Q is proportional to V. The capacitance C is the corresponding proportionality constant.

$$Q \equiv CV. \quad (4)$$

The energy in the capacitor can be written as follows.

$$EnergyCapacitor = \int_{-\infty}^{t} V\frac{dQ}{dt'}\,dt' = c\int_{-\infty}^{t} V\frac{dV}{dt'}\,dt' = \frac{C}{2}\int_{-\infty}^{t} \frac{dV^2}{dt'}\,dt = \frac{C}{2}V^2 = \frac{C}{2}\dot{\Phi}^2. \quad (5)$$

In a linear inductor, or simply inductor, the current is related to the flux as follows.

$$I \equiv \frac{\Phi}{L}. \quad (6)$$

In a superconducting Josephson Junction (JJ), the so-called non-linear inductor, the current is related to the flux as follows.

$$I \equiv I_{J0}\sin\left(2\pi\frac{\Phi}{\Phi_0}\right) = \frac{2\pi I_{J0}}{\Phi_o}\Phi + O\left(\left(2\pi\frac{\Phi}{\Phi_0}\right)^3\right). \quad (7)$$

Here, I_{J0} is the maximum current that can pass through JJ and Φ_0 is the superconducting magnetic flux quantum defined in the next section. Note that $L_J = \frac{\Phi_0}{2\pi I_{J0}}$ is the effective linear inductance of JJ.

The energy of the linear inductor can be derived as follows by substituting V from Eq. (4) and I from Eq. (6) in Eq. (3).

$$EnergyInductor = \int_{-\infty}^{t} \frac{\Phi}{L}\frac{d\Phi}{dt'}\,dt' = \frac{1}{2L}\int_{-\infty}^{t} \frac{d\Phi^2}{dt'}\,dt = \frac{1}{2L}\Phi^2. \quad (8)$$

The energy of the JJ can be derived as follows by substituting V from Eq. (4) and I from Eq. (7) in Eq. (3).

$$EnergyJJ = \int_{-\infty}^{t} I_{J0}\sin\left(2\pi\frac{\Phi}{\Phi_0}\right)\frac{d\Phi}{dt'}\,dt' = \int_{-\infty}^{t} I_{J0}\sin\left(2\pi\frac{\Phi}{\Phi_0}\right)d\Phi = \frac{I_{J0}\Phi_o}{2\pi}\cos\left(2\pi\frac{\Phi}{\Phi_0}\right) = \frac{\Phi_0^{\,2}}{4\pi^2 L_J}\cos\left(2\pi\frac{\Phi}{\Phi_0}\right). \quad (9)$$

Lagrangian and Hamiltonian

A linear resonating circuit - a capacitor in parallel with an inductor

In this case, without loss of generality, assume that node a is at voltage V, node b is ground, and the current entering at node a in these elements is respectively labelled as I_C and I_L,

$$EnergyCapacitor(kinetic) = K = \frac{C}{2}\dot{\Phi}^2.$$

$$EnergyInductor(potential) = U = \frac{1}{2L}\Phi^2.$$

Let us construct the Lagrangian as follows in terms of the generalized coordinate Φ and see if it is justified.

$$\mathcal{L}\left(\Phi, \dot{\Phi}\right) \equiv K - U = \frac{C}{2}\dot{\Phi}^2 - \frac{1}{2L}\Phi^2. \quad (10)$$

For this to be the true Lagrangian, its time integral along the evolution path of the generalized coordinate should be an extremum. This implies,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}}\right) - \frac{\partial \mathcal{L}}{\partial \Phi} = 0. \quad (11)$$

Let us see that.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}}\right) - \frac{\partial \mathcal{L}}{\partial \Phi} = \frac{d}{dt}\left(C\dot{\Phi}\right) + \frac{\Phi}{L} = \frac{d}{dt}(CV) + I_L = \frac{d}{dt}(Q) + I_L = I_C + I_L = 0.$$

The last equality holds due to Kirchhoff's current law. Thus, the Lagrangian satisfies the extremum condition. Let us find the conjugate momentum.

$$\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi} = Q. \quad (12)$$

Now let us find the hamiltonian, by applying the Legendre transformation.

$$H(\Phi, Q) = \dot{\Phi}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} - \mathcal{L}\left(\Phi, \dot{\Phi}\right) = \dot{\Phi}Q - \frac{C}{2}\dot{\Phi}^2 + \frac{1}{2L}\Phi^2 = \frac{1}{2C}Q^2 + \frac{1}{2L}\Phi^2. \quad (13)$$

A non-linear resonating circuit - a capacitor in parallel with a JJ

As before, assume that node a is at voltage V with respect to the ground node b. The current entering at node a in these elements is respectively labelled as I_C and I_J,

$$EnergyCapacitor(kinetic) = K = \frac{C}{2}\dot{\Phi}^2.$$

$$EnergyJJ(potential) = U_J = \frac{\Phi_0^2}{4\pi^2 L}\cos\left(2\pi\frac{\Phi}{\Phi_0}\right).$$

Here $L = L_J$ for the JJ.

Let us propose the following Lagrangian in terms of the generalized coordinate $\Phi_J = \Phi - \frac{1}{2}\Phi_0$.

Note that $\dot{\Phi}_J = \dot{\Phi}$.

$$\mathcal{L}_J\left(\Phi_J, \dot{\Phi}_J\right) \equiv K - U_J = \frac{C}{2}\dot{\Phi}_J^{\ 2} - \frac{\Phi_0^{\ 2}}{4\pi^2 L}\cos\left(2\pi\frac{\Phi_J + \frac{1}{2}\Phi_0}{\Phi_0}\right) = \frac{C}{2}\dot{\Phi}_J^{\ 2} + \frac{\Phi_0^{\ 2}}{4\pi^2 L}\cos\left(2\pi\frac{\Phi_J}{\Phi_0}\right). \quad (14)$$

Let us see if the extremum condition (equation of motion) is satisfied.

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}_J}{\partial\dot{\Phi}_J}\right) + \frac{\partial\mathcal{L}_J}{\partial\Phi_J} = \frac{d}{dt}\left(C\dot{\Phi}_J\right) - \frac{\Phi_0}{2\pi L}\sin\left(2\pi\frac{\Phi_J}{\Phi_0}\right) = \frac{d}{dt}\left(C\dot{\Phi}\right) - \frac{\Phi_0}{2\pi L}\sin\left(2\pi\frac{\Phi - \frac{1}{2}\Phi_0}{\Phi_0}\right) = \frac{d}{dt}\left(CV\right) + \frac{\Phi_0}{2\pi L}\sin\left(2\pi\frac{\Phi}{\Phi_0}\right)$$

$$\frac{d}{dt}Q + I_{J0}\sin\left(2\pi\frac{\Phi}{\Phi_0}\right) = I_C + I_J = 0.$$

The last equality holds due to Kirchhoff's current law, justifying the Lagrangian. The conjugate momentum is the same as before.

$$\frac{\partial\mathcal{L}_J}{\partial\dot{\Phi}_J} = C\dot{\Phi}_J = C\dot{\Phi} = Q.$$

Finally let us find the hamiltonian, by applying the Legendre transformation.

$$H_J\left(\Phi_J, Q\right) = \dot{\Phi}\frac{\partial\mathcal{L}_J}{\partial\dot{\Phi}_J} - \mathcal{L}_J\left(\Phi_J, \dot{\Phi}_J\right) = \dot{\Phi}_J Q - \frac{C}{2}\dot{\Phi}_J^{\ 2} - \frac{\Phi_o^{\ 2}}{2\pi L}\cos\left(2\pi\frac{\Phi_J}{\Phi_0}\right) = \frac{1}{2C}Q^2 - \frac{\Phi_o^{\ 2}}{2\pi L}\cos\left(2\pi\frac{\Phi_J}{\Phi_0}\right). \quad (15)$$

To fourth order, the hamiltonian can be written as follows using the Taylor’s expansion.

$$H_J\left(\Phi_J, Q\right) = \frac{1}{2C}Q^2 + \frac{1}{2L}\Phi_J^{\ 2} - \frac{1}{24}\frac{\Phi_o^{\ 2}}{2\pi L}\left(2\pi\frac{\Phi_J}{\Phi_0}\right)^4 + O\left(\left(\frac{\Phi_J}{\Phi_0}\right)^6\right),$$

$$H_J\left(\Phi_J, Q\right) = H\left(\Phi_J, Q\right) - \frac{1}{24}\frac{\Phi_o^{\ 2}}{2\pi L}\left(2\pi\frac{\Phi_J}{\Phi_0}\right)^4. \quad (16)$$

Now, as has become standard since Dirac’s time, we raise the conjugate coordinate (Φ or Φ_L) and the conjugate momentum (Q) to the pair of commuting operators status in Hilber-space for quantum mechanical treatment. The Hamiltonian correspondingly becomes the Hermitian operator with energy eigenvalues. Now onwards, these quantities are operators, without the conventional hat, in these notes.

The commutation is defined and given as follows.

$$[\Phi, Q] = \Phi Q - Q\Phi = i\hbar. \quad (17)$$

Definitions

Let us define some quantities.

Flux quanta: $\Phi_0 \equiv \frac{h}{2e} = 2.07\, mV/GHz.$ (18a)

Reduced/normalized flux: $\phi \equiv 2\pi \frac{\Phi}{\Phi_0}.$ (18b)

Excess number of Cooper pairs: $n \equiv \frac{Q}{2e}.$ (18c)

Charging energy constant: $E_C \equiv \frac{e^2}{2C}.$ (18d)

Flux energy constant for inductor: $E_L \equiv \frac{\Phi_0^{\ 2}}{4\pi^2 L}.$ (18e)

Flux energy constant for JJ: $E_L = E_J \equiv \frac{I_{J0}\Phi_0}{2\pi} = \frac{\Phi_0^{\ 2}}{4\pi^2 L_J}.$ (18f)

Linear model resonant frequency: $hf_r \equiv \sqrt{8E_c E_L}.$ (18g)

We also define zero point fluctuations in the flux and charge as follows.

$$\phi_{zpf} \equiv \frac{1}{2}\left(\frac{hf_r}{E_L/2}\right)^{1/2}, \quad (19a)$$

$$n_{zpf} \equiv \frac{1}{2}\left(\frac{hf_r}{4E_C}\right)^{1/2}, \quad (19b)$$

$$\phi_{zpf}\, n_{zpf} = \frac{1}{2}. \quad (19c)$$

Note the following useful products for quick intuition:

- Product of charging energy constant and capacitance = $E_C \times C = 19.37\, h \bullet GHz \bullet\ fF.$
- Product of flux energy constant and inductance = $E_L \times L = 163.46\, h \bullet GHz \bullet\ nH.$

Normalization of Hamiltonian

Using the definitions given in Eq. (18-19), we can normalize the hamiltonians (operators) as follows. Recall that the reduced flux ϕ and the extra Cooper pair number n are also operators.

Linear resonant circuit

The hamiltonian of the linear system (capacitor and inductor) is as follows.

$$H = \frac{1}{2C}Q^2 + \frac{1}{2L}\Phi^2,$$

$$H = \frac{4e^2}{2C}n^2 + \frac{\Phi_0^{\ 2}}{4\pi^2 2L}\phi^2,$$

$$H = 4E_c n^2 + \frac{E_L}{2}\phi^2, \quad (20a)$$

$$H = \left(\frac{4E_c}{hf_r}n^2 + \frac{E_L/2}{hf_r}\phi^2\right)hf_r ,$$

$$H = \left(\frac{1}{4n_{zpf}^2}n^2 + \frac{1}{4\phi_{zpf}^2}\phi^2\right)hf_r . \qquad (20b)$$

Non-linear resonant circuit

The hamiltonian of the non-linear system (capacitor and JJ) is as follows.

$$H_J = \left(\frac{1}{4n_{zpf}^2}n^2 + \frac{1}{4\phi_{zpf}^2}\phi^2\right)hf_r - \frac{1}{24}\frac{\Phi_o^2}{2\pi L_J}\left(2\pi\frac{\Phi_J}{\Phi_0}\right)^4,$$

$$H_J = \left(\frac{1}{4n_{zpf}^2}n^2 + \frac{1}{4\phi_{zpf}^2}\phi^2\right)hf_r - \frac{1}{24}E_J\phi^4.$$

Consider the following term.

$$-\frac{1}{24}E_J\phi^4 =- E_J\phi_{zpf}^4\frac{1}{24\phi_{zpf}^4}\phi^4 =- E_J\left(\frac{hf_r}{2E_J}\right)^2\frac{1}{24\phi_{zpf}^4}\phi^4 =- \frac{8E_cE_J}{4E_J}\frac{1}{24\phi_{zpf}^4}\phi^4 =- \frac{E_c}{12\phi_{zpf}^4}\phi^4.$$

Thus the hamiltonian for the non-linear system is as follows.

$$H_J = \left(\frac{1}{4n_{zpf}^2}n^2 + \frac{1}{4\phi_{zpf}^2}\phi^2\right)hf_r - \frac{E_c}{12\phi_{zpf}^4}\phi^4. \quad (21)$$

The commutative relationship in terms of the normalized flux and charge is given as follows,

$$[\Phi, Q] = \Phi Q - Q\Phi = i\hbar,$$

$$2e\frac{\Phi_0}{2\pi}(\phi n - n\phi) = i\hbar,$$

$$2e\frac{h}{2\pi\, 2e}(\phi n - n\phi) = i\hbar,$$

$$\hbar(\phi n - n\phi) = i\hbar,$$

$$[\phi, n] = i. \qquad (22)$$

Operator Factorization

Let us define two operators $a^\dagger$ (raising or creation) and a (lowering or annihilation) as follows.

$$n = i\, n_{zpf}(a^\dagger - a), \qquad (23a)$$

$$\phi = \phi_{zpf}(a^\dagger + a). \qquad (23b)$$

Using commutation relationship,

$$[\phi, n] = i,$$

$$\phi_{zpf}\, i\, n_{zpf}\left[(a^\dagger + a)(a^\dagger - a) - (a^\dagger - a)(a^\dagger + a)\right] = i,$$

$$\frac{i}{2}\left(a^{\dagger}a^{\dagger} + aa^{\dagger} - a^{\dagger}a - aa - a^{\dagger}a^{\dagger} + aa^{\dagger} - a^{\dagger}a + aa\right) = i,$$
$$aa^{\dagger} - a^{\dagger}a = 1,$$
$$\left[a^{\dagger}, a\right] = 1. \qquad (24)$$

From Eq. (23)

$$a^{\dagger} = \frac{1}{2\,\phi_{zpf}}\phi - \frac{i}{2\,n_{nzpf}}n, \qquad (25a)$$
$$a = \frac{1}{2\,\phi_{zpf}}\phi + \frac{i}{2\,n_{nzpf}}n. \qquad (25b)$$

Note that the raising and lowering operators, also known as ladder operators, are mutually conjugate.

Hamiltonian of Linear Resonating Circuit

Let us substitute Eq. (23) in (20b).

$$H = \left(\frac{1}{4n_{zpf}^{2}}n^{2} + \frac{1}{4\phi_{zpf}^{2}}\phi^{2}\right)hf_{r}\ ,$$
$$H = \frac{1}{4}\left(-\ (a^{\dagger} - a)(a^{\dagger} - a)\ +\ (a^{\dagger} + a)(a^{\dagger} + a)\right)hf_{r}\ ,$$
$$H = \frac{1}{4}\left(-\ a^{\dagger}a^{\dagger} + aa^{\dagger} + a^{\dagger}a - aa + a^{\dagger}a^{\dagger} + aa^{\dagger} + a^{\dagger}a + aa\right)hf_{r}\ ,$$
$$H = \frac{1}{2}\left(aa^{\dagger} + a^{\dagger}a\right)hf_{r}\ ,$$
$$H = \frac{1}{2}\left(a^{\dagger}a + 1 + a^{\dagger}a\right)hf_{r}\ ,$$
$$H = \left(a^{\dagger}a + \frac{1}{2}\right)hf_{r}. \qquad (26)$$

This is a neat expression of the hamiltonian of the linear resonating circuit. Let us look at the energy eigenstates and energy eigenvalues of this system.
Let $|n\rangle$ be an orthonormal energy eigenstate of H corresponding to eigenvalue E_{n} where n is any integer value discretely indexing the discretized energy eigenstates. Then by definition of eigenstate

$$H|n\rangle\ = E_{n}|n\rangle. \qquad (27)$$

Ground state of hamiltonian

Consider the following.

$$\langle n|H|n\rangle\ = \langle n|E_{n}|n\rangle,$$
$$hf_{r}\langle n|a^{\dagger}a|n\rangle\ = E_{n} - \frac{1}{2}hf_{r},$$
$$\langle n|a^{\dagger}a|n\rangle\ = \frac{E_{n}}{hf_{r}} - \frac{1}{2}, \qquad (28a)$$

$$\|a|n\rangle\|^2 = \frac{E_n}{hf_r} - \frac{1}{2}. \qquad (28b)$$

Since the left side of the equation is a non negative value,

$$E_n \geq \frac{1}{2}hf_r. \qquad (29)$$

Let us fix, without loss of generality, the index $n = 0$ for the lowest energy state. Then from Eq. (28b),

$$E_0 = \left(\frac{1}{2} + \|a|0\rangle\|^2\right)hf_r. \qquad (30)$$

Effect of lowering operator

Consider applying a operator on both sides of Eq. (27),

$$aH|n\rangle = aE_n|n\rangle,$$

$$\left(aa^\dagger a + \frac{1}{2}a\right)hf_r|n\rangle = E_n a|n\rangle,$$

$$\left(a^\dagger aa + a + \frac{1}{2}a\right)hf_r|n\rangle = E_n a|n\rangle,$$

$$\left(a^\dagger a + \frac{1}{2}\right)hf_r a|n\rangle = \left(E_n - hf_r\right)a|n\rangle,$$

$$Ha|n\rangle = \left(E_n - hf_r\right)a|n\rangle, \qquad (31)$$

Consider the case of ground state from Eq. (30).

$$Ha|0\rangle = \left(E_0 - hf_r\right)a|0\rangle, \qquad (32)$$

We see from Eq. (32) that if $a|0\rangle \neq 0$, then $a|0\rangle$is an eigenstate of H with an even lower eigenenergy than that of the ground state which is not tenable. Thus, we conclude the following.

$$a|0\rangle = 0. \qquad (33)$$

Using Eq. (33) in Eq. (30) we get the following.

$$E_0 = \frac{1}{2}hf_r. \qquad (34a)$$

Generally, for $n > 0$, we see from Eq. (31) that $a|n\rangle$ is also an eigenstate of H with eigenenergy $E_n - hf_r$. Let us define,

$$E_{n-1} \equiv E_n - hf_r, \quad n > 0. \qquad (34b)$$

We can summarize Eq. (34a-b) as follows.

$$E_n = \left(n + \frac{1}{2}\right)hf_r. \qquad (35)$$

Substituting Eq. (35) in (28a),

$$\langle n|a^\dagger a|n\rangle = n, \qquad (36)$$

The ket $a|n\rangle$ corresponding to the energy E_{n-1} in Eq. (31) is not normalized in general. Let us write is as follows where $\xi(n)$ is a positive real value, without loss of generality.

$$a|n\rangle = \xi(n)|n-1\rangle$$

As per definition earlier, $|n-1\rangle$ is a normalized ket. We can find $\xi(n)$ by taking the inner product of the vectors on each side of the above equation.

$$\langle n|a^\dagger a|n\rangle = \xi(n)^2,$$

Using Eq. (36),

$$\xi(n) = \sqrt{n}.$$

Thus,

$$a|n\rangle = \sqrt{n}|n - 1\rangle. \qquad (37)$$

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Effect of raising operator

Consider writing Eq. (37) as follows.

$$\sqrt{n + 1}|n\rangle = a|n + 1\rangle,$$

$$|n\rangle = \frac{1}{\sqrt{n+1}}a|n + 1\rangle.$$

Let us apply the raising operator on each side

$$a^{\dagger}|n\rangle = \frac{1}{\sqrt{n+1}}a^{\dagger}a|n + 1\rangle = \frac{1}{\sqrt{n+1}}\left(\frac{1}{hf_r}H - \frac{1}{2}\right)|n + 1\rangle = \frac{1}{\sqrt{n+1}}\left(\frac{E_{n+1}}{hf_r} - \frac{1}{2}\right)|n + 1\rangle,$$

$$a^{\dagger}|n\rangle = \frac{1}{\sqrt{n+1}}\left(n + 1 + \frac{1}{2} - \frac{1}{2}\right)|n + 1\rangle,$$

$$a^{\dagger}|n\rangle = \sqrt{n + 1}|n + 1\rangle. \qquad (38)$$

Using Eq. (37-38),

$$a^{\dagger}a|n\rangle = n|n\rangle. \qquad (39)$$

Summary for linear resonant circuit

The linear resonating circuit has the following hamiltonian.

$$H = \left(a^{\dagger}a + \frac{1}{2}\right)hf_r.$$

Its energy eigenstates are given as $|n\rangle$, for $n \geq 0$ with corresponding energy eigenvalues given as follows.

$$E_n = \left(n + \frac{1}{2}\right)hf_r.$$

Furthermore, we have the following relations.

$$a|n\rangle = \sqrt{n}|n - 1\rangle.$$

$$a^{\dagger}|n\rangle = \sqrt{n + 1}|n + 1\rangle.$$

$$a^{\dagger}a|n\rangle = n|n\rangle.$$

Hamiltonian of Non-linear Resonant Circuit

The hamiltonian for the non-linear resonator, capacitor parallel with Josephson Junction, is given in Eq. (21). Let us reexpress that as a sum of two hamiltonians.

$$H_J = H + H_1, \qquad (40a)$$

$$H = \left(\frac{1}{4n_{zpf}^2}n^2 + \frac{1}{4\phi_{zpf}^2}\phi^2\right)hf_r, \qquad (40b)$$

$$H_1 = -\frac{E_c}{12\phi_{zpf}^4}\phi^4. \qquad (40c)$$

The first part H of our Hamiltonian H_J as given in Eq. (40b) can be expressed in terms of the raising and lowering operators as before. It is the same hamiltonian as that for the linear resonant circuit except that the energy term hf_r is calculated using the effective inductance of the Josephson Junction. Thus H is the linear part of the JJ's hamiltonian and is given as follows

$$H = \left(a^\dagger a + \frac{1}{2}\right)hf_r.$$

We have calculated the eigenenergies of this hamiltonian before. Let use that as reference to to calculate the eigenenergies of the new hamiltonian H_J in the presence of the "perturbative" hamiltonian H_1. Let the eigenenergies and eigenstates of H_J be given as E_{Jn} and $|\psi_{Jn}\rangle$.

First-order perturbation theory

Consider the following hamiltonian expressed as a function of parameter λ.

$$H_J(\lambda) = H + \lambda H_1. \qquad (41)$$

Let the corresponding eigenenergies and eigenstates of this hamiltonian be given as a power series in λ as follows.

$$E_{Jn}(\lambda) = E_n + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \lambda^3 E_n^{(3)} .... \qquad (42)$$

$$|\psi_{Jn}(\lambda)\rangle = |n\rangle + \lambda|\psi^{(1)}\rangle + \lambda^2|\psi^{(2)}\rangle + \lambda^3|\psi^{(3)}\rangle + ... \qquad (43)$$

The general hamiltonian equation is given as.

$$H_J(\lambda)|\psi_{Jn}(\lambda)\rangle = E_{Jn}(\lambda)|\psi_{Jn}(\lambda)\rangle. \qquad (44)$$

Note that if we substitute $\lambda = 0$, we get,

$$H_J(0) = H. \qquad (45a)$$

$$E_{Jn}(0) = E_n. \qquad (45b)$$

$$|\psi_{Jn}(0)\rangle = |n\rangle. \qquad (45c)$$

This is the vanishing perturbation limit. On the other hand, if we substitute $\lambda = 1$, then the hamiltonian in Eq. (41) becomes the hamiltonian in question, H_J in Eq. (40a). Thus, correspondingly, the eigenenergies and eigenstates given in Eq. (42-43) become the eigenenergies and eigenstates of the hamiltonian H_j as follows.

$$H_J(1) = H_J. \quad (46a)$$

$$E_{Jn}(1) = E_{Jn} = E_n + E_n^{(1)} + E_n^{(2)} + E_n^{(3)}.... \quad (46b)$$

$$|\psi_{Jn}(1)\rangle = |\psi_{Jn}\rangle = |n\rangle + |\psi^{(1)}\rangle + |\psi^{(2)}\rangle + |\psi^{(3)}\rangle +.... \qquad (46c)$$

Let us substitute Eq. (41-43) in Eq. (44) and compare the similar powers of λ.

$$(H + \lambda H_1)\left(|n\rangle + \lambda|\psi^{(1)}\rangle + \lambda^2|\psi^{(2)}\rangle + \lambda^3|\psi^{(3)}\rangle +...\right)$$

$$= \left(E_n + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \lambda^3 E_n^{(3)} +...\right)\left(|n\rangle + \lambda|\psi^{(1)}\rangle + \lambda^2|\psi^{(2)}\rangle + \lambda^3|\psi^{(3)}\rangle +...\right)$$

Comparing the first-order terms in λ, we get the following.

$$H\lambda|\psi^{(1)} + \lambda H_1|n\rangle = \lambda E_n|\psi^{(1)}\rangle + \lambda E_n^{(1)}|n\rangle.$$

Let us take the inner product with $\langle n|$.

$$\langle n|H|\psi^{(1)} + \langle n|H_1|n\rangle = \langle n|E_n|\psi^{(1)}\rangle + \langle n|E_n^{(1)}|n\rangle,$$

$$\langle n|E_n|\psi^{(1)} + \langle n|H_1|n\rangle = \langle n|E_n|\psi^{(1)}\rangle + E_n^{(1)},$$

$$E_n^{(1)} = \langle n|H_1|n\rangle,$$

Thus to first order, using Eq. (46b),

$$E_{Jn} \approx E_n + E_n^{(1)},$$

$$E_{Jn} \approx E_n + \langle n|H_1|n\rangle. \quad (47)$$

Calculating first-order perturbation correction to eigenenergies

Let us calculate the first-order perturbation correction $\langle n|H_1|n\rangle$ to eigenenergies.

$$E_n^{(1)} = \langle n|H_1|n\rangle = \langle n| -\frac{E_c}{12\phi_{zpf}^{\ 4}}\phi^4|n\rangle = -\frac{E_c}{12}\langle n|\frac{1}{\phi_{zpf}^{\ 4}}\phi^4|n\rangle = -\frac{E_c}{12}\langle n|(a^\dagger + a)^4|n\rangle. \quad (48)$$

Let us expand the following into full 16 terms.

$$(a^\dagger + a)^4$$

$$= aa^\dagger a^\dagger a^\dagger + aa^\dagger a^\dagger a + aa^\dagger aa^\dagger + aa^\dagger aa + aaa^\dagger a^\dagger + aaa^\dagger a + aaaa^\dagger + aaaa$$

$$+ a^\dagger a^\dagger a^\dagger a^\dagger + a^\dagger a^\dagger a^\dagger a + a^\dagger a^\dagger aa^\dagger + a^\dagger a^\dagger aa + a^\dagger aa^\dagger a^\dagger + a^\dagger aa^\dagger a + a^\dagger aaa^\dagger + a^\dagger aaa.$$

The inner product in Eq. (48) will act around each of these 16 terms. The terms with unequal number of lowering and raising operators will vanish as those will amount to the inner product between different orthogonal states. This is due to the peculiar lowering and raising properties of the operators. Thus, only six terms survive.

$$\langle n|(a^{\dagger} + a)^{4}|n\rangle$$
$$= \langle n|aa^{\dagger}a^{\dagger}a|n\rangle + \langle n|aa^{\dagger}aa^{\dagger}|n\rangle + \langle n|aaa^{\dagger}a^{\dagger}|n\rangle + \langle n|a^{\dagger}a^{\dagger}aa|n\rangle + \langle n|a^{\dagger}aa^{\dagger}a|n\rangle + \langle n|a^{\dagger}aaa^{\dagger}|n\rangle.$$

Further, using Eq. (37-38),

$$\langle n|(a^{\dagger} + a)^{4}|n\rangle$$
$$= \sqrt{(n + 1)(n + 1)(n)(n)} + \sqrt{(n + 1)(n + 1)(n + 1)(n + 1)} + \sqrt{(n + 1)(n + 2)(n + 2)(n + 1)}$$
$$+ \sqrt{(n)(n - 1)(n - 1)(n)} + \sqrt{(n)(n)(n)(n)}. + \sqrt{(n)(n)(n + 1)(n + 1)},$$

$$\langle n|(a^{\dagger} + a)^{4}|n\rangle = n(n + 1) + (n + 1)^{2} + (n + 1)(n + 2) + n(n - 1) + n^{2} + n(n + 1),$$

$$\langle n|(a^{\dagger} + a)^{4}|n\rangle = (2n + 1)^{2} + (n^{2} + 3n + 2) + (n^{2} - n).$$

$$\langle n|(a^{\dagger} + a)^{4}|n\rangle = 3(2n^{2} + 2n + 1). \qquad (49)$$

Thus the first-order correction in the energy eigenvalues can be written as follows.

$$\langle n|H_{1}|n\rangle =- \frac{E_{c}}{4}(2n^{2} + 2n + 1). \qquad (50)$$

To first-order, the eigenenergies of this JJ based non-linear oscillator are given as follows.

$$E_{Jn} = \left(n + \frac{1}{2}\right)hf_{r} - \frac{E_{c}}{4}(2n^{2} + 2n + 1). \qquad (51)$$

Application to the first three levels

Specifically for the first three levels, we have:

$$E_{Jo} = \frac{1}{2}hf_{r} - \frac{1}{4}E_{c}, \qquad (52a)$$
$$E_{J1} = \frac{3}{2}hf_{r} - \frac{5}{4}E_{c}, \qquad (52b)$$
$$E_{J2} = \frac{5}{2}hf_{r} - \frac{13}{4}E_{c}. \qquad (52c)$$

The corresponding energy differences are:

$$E_{J0\to1} = hf_{r} - E_{c}, \qquad (53a)$$
$$E_{J1\to2} = hf_{r} - 2E_{c}. \qquad (53b)$$

The anharmonicity is given as:

$$\alpha \equiv E_{J1\to2} - E_{J0\to1} =- E_{c}. \qquad (54)$$

Wavefunction

Let us represent the normalized flux and charge operators, and the energy eigenstate in the function space of ϕ. Recall the position and momentum analogy. Here we are using the overhead hat symbol for the operators to avoid confusion.

$$\hat{\phi} \rightarrow \phi$$
$$\hat{n} \rightarrow -i\frac{d}{d\phi}$$
$$|n\rangle \rightarrow \psi_n(\phi)$$

Notice the following.

$$\left(\hat{\phi}\hat{n} - \hat{n}\hat{\phi}\right)|n\rangle \rightarrow -\phi i\frac{d}{d\phi}\psi_n(\phi) + i\frac{d}{d\phi}\phi\psi_n(\phi) = i\psi_n(\phi) \rightarrow i|n\rangle.$$

Thus the flux-charge commutation relationship is satisfied as expected. From Eq. (25)

$$a^\dagger = \frac{1}{2\phi_{zpf}}\phi - \frac{1}{2n_{nzpf}}\frac{d}{d\phi},$$
$$a = \frac{1}{2\phi_{zpf}}\phi + \frac{1}{2n_{nzpf}}\frac{d}{d\phi}.$$

Using Eq. (19c)

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}\phi_{zpf}}\phi - \sqrt{2}\phi_{zpf}\frac{d}{d\phi}\right),$$
$$a = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}\phi_{zpf}}\phi + \sqrt{2}\phi_{zpf}\frac{d}{d\phi}\right).$$

Let us define,

$$\xi \equiv \frac{\phi}{\sqrt{2}\phi_{zpf}} = \left(\frac{E_L}{8E_c}\right)^{1/4}\phi.$$

Using this 'supernormalized' flux coordinate,

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right), \qquad (55a)$$
$$a = \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right). \qquad (55b)$$

Using Eq. (33),

$$a|0\rangle = 0,$$
$$\xi\psi_0(\xi) + \frac{d}{d\xi}\psi_0(\xi) = 0.$$

The following normalized wavefunction is the solution of the above equation.

$$\psi_0(\xi) = \frac{1}{\pi^{1/4}}e^{-\xi^2/2}. \qquad (56)$$

From Eq. (38),

$$|n+1\rangle = \frac{1}{\sqrt{n+1}}a^\dagger|n\rangle.$$

In terms of the wavefunction of ξ,

$$\psi_{n+1}(\xi) = \frac{1}{\sqrt{2(n+1)}}\left(\xi - \frac{d}{d\xi}\right)\psi_n(\xi). \qquad (57)$$

The wavefunctions have a definite parity that is they are either even or odd and are given as follows.

$$\psi_n(\xi) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} H_n(\xi)\, e^{-\xi^2/2}. \quad (58)$$

Here, $H_n(\xi)$ are Hermite polynomials.

$$H_0(\xi) = 1.$$
$$H_1(\xi) = 2\xi.$$
$$H_2(\xi) = 4\xi^2 - 2.$$

Example

Fundamental constants:

$$e = 1.60217663 \times 10^{-19} C \mid h = 4.13\ \mu eV/GHz \mid \Phi_0 = \frac{h}{2e} = 2.06783\ \mu V/GHz.$$

Conversion factors:

$$1pH = 0.0774809\ \Phi_0 e^{-1} GHz^{-1}.$$
$$1fF = 0.00624151\ e/\mu V.$$

Consider the following example

$$C = 96.9\ fF = 0.6 e/\mu V.$$
$$L = 16.35\ nH = 0.00127\ \Phi_0 e^{-1} GHz^{-1}.$$

Then,

$$E_C = \frac{19.37\ h \bullet GHz \bullet fF}{C} = 200\ MHz\ h.$$
$$E_L = \frac{163.46\ h \bullet GHz \bullet nH}{L} = 10\ GHz\ h.$$
$$hf_r = \sqrt{8E_C E_L} = 4\ GHz\ h.$$

Let us try to see the values of different parameters in the ground-state (energy = $hf_r/2$) from the classical point of view first. Classically, we have,

$$\frac{1}{2} hf_r = \frac{1}{2C} Q^2 + \frac{1}{2L} \Phi^2,$$
$$hf_r = \frac{1}{C} Q^2 + \frac{1}{L} \Phi^2,$$

The charge flux, voltage, and current oscillate with an amplitude given as,

$$Q = \sqrt{C\, hf_r} = \sqrt{0.6\ e/\mu V \ \times 4.13\ \mu eV/GHz \ \times 4\ GHz} = 3.16\ e.$$
$$\Phi = \sqrt{L\, hf_r} = \sqrt{0.00127\ \Phi_0 e^{-1} GHz^{-1} \ \times 2e\Phi_0 \ \times 4\ GHz} = 0.10\ \Phi_0.$$
$$V = Q/C = 5.23\ \mu V.$$
$$I = \Phi/L = 12.73\ nA.$$