Lecture # 2

Topics:

- **Angle between two vectors**
- **Cross product of two vectors**

Angle Between Two Vectors

The angle between two vectors is calculated as the cosine of the angle between the two vectors. The cosine of the angle between two vectors is equal to the dot product of the individual constituents of the two vectors, divided by the product of the magnitude of the two vectors.

Let $\vec{\boldsymbol{v}} = < \boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3} >$ and $\vec{\boldsymbol{w}} == < \boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3} >$**,** then the formula for the angle between the two vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ is as follows.

$$\cos\theta = \frac{\vec{v}.\vec{w}}{|\vec{v}|\ |\vec{w}|}$$

$$\cos(\theta) = \frac{\vec{v}.\vec{w}}{\sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}\ \sqrt{(w_1)^2 + (w_2)^2 + (w_3)^2}}$$

Example: Compute the angle between the vectors $\vec{v}$ and $\vec{w}$.

$\vec{v} = \hat{\imath} + \sqrt{3}\,\hat{\jmath}, \qquad \vec{w} = \sqrt{3}\,\hat{\imath} + \hat{\jmath} - \hat{k}$

Solution: Since, we know that

$$\vec{v}.\vec{w} = |\vec{v}|\ |\vec{w}| Cos\theta$$

By rearranging, we can write

$$\cos\theta = \frac{\vec{v}.\vec{w}}{|\vec{v}|\ |\vec{w}|} - - - -(1)$$

First, we will find the dot product as

$$\vec{v}\cdot\vec{w} = \left(\hat{\imath} + \sqrt{3}\,\hat{\jmath}\right)\cdot\left(\sqrt{3}\,\hat{\imath} + \hat{\jmath} - \hat{k}\right) = (1)\cdot\left(\sqrt{3}\right) + \left(\sqrt{3}\right)\cdot(1) + (0)\cdot(1)$$

$\vec{v} \cdot \vec{w} = 2\sqrt{3}$

Now,

$$\cos\theta = \frac{\vec{v}.\vec{w}}{|\vec{v}|\,|\vec{w}|}$$

$$\cos(\theta) = \frac{2\sqrt{3}}{\sqrt{(1)^2 + \left(\sqrt{3}\right)^2 + (0)^2}\sqrt{\left(\sqrt{3}\right)^2 + (1)^2 + (-1)^2}}$$

$$\cos(\theta) = \frac{2\sqrt{3}}{\sqrt{1+3}\sqrt{3+1+1}} = \frac{2\sqrt{3}}{2\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}}$$

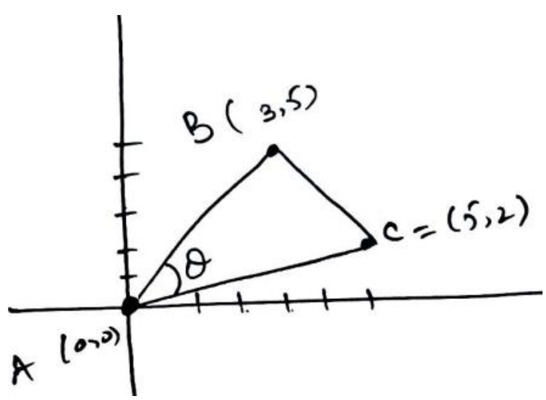
$$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{\sqrt{5}}\right) = 39.3^o$$

Finding the Angle of a Triangle

Example: Find the angle $\boldsymbol{\theta}$ in the triangle $\Delta\boldsymbol{ABC}$ determined by the vertices

$$\boldsymbol{A} = (\boldsymbol{0},\boldsymbol{0}), \boldsymbol{B} = (\boldsymbol{3},\boldsymbol{5}), \boldsymbol{C} = (\boldsymbol{5},\boldsymbol{2})$$

Solution:



The angle θ is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\overrightarrow{AB} =< 3-0, 5-0 >=< 3, 5 >= 3\hat{\imath} + 5\hat{\jmath}$

$\overrightarrow{AC} =< 5-0, 2-0 >=< 5, 2 >= 5\hat{\imath} + 2\hat{\jmath}$

$Cos\theta = \frac{\overrightarrow{AB}.\ \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{15+10}{(\sqrt{3^2+5^2})\sqrt{5^2+2^2})}$

$= \frac{25}{(\sqrt{34})\sqrt{29})}$ =35.9°

Practice Question: Compute the angle between the vectors

$\boldsymbol{\vec{v} = 2\,\hat{\imath} + 5\hat{\jmath} - 9\hat{k}}$ and $\boldsymbol{\vec{w} = 7.31\,\hat{\imath} + 8.64\hat{\jmath} + 4.25\hat{k}}$.

Food for Thought:

- What would a dot product be between two orthogonal vectors?

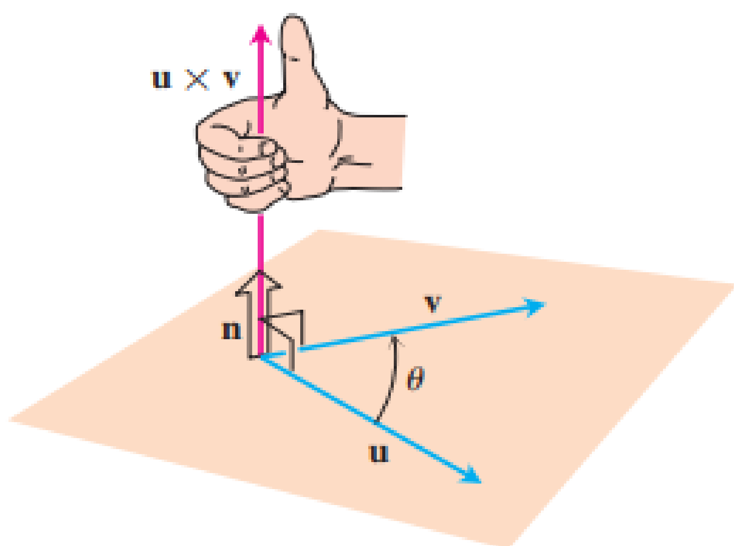
- Find any 2 vectors that are orthogonal.

Cross Product of Two vectors

Cross Product is also called a Vector Product. Cross product is a form of vector multiplication, performed between two vectors.

When two vectors are multiplied with each other and the product is also a vector quantity, then the resultant vector is called the cross product of two vectors or the vector product. The resultant vector is perpendicular to the plane containing the two given vectors.

Let $\vec{u}$ and $\vec{v}$ be two vectors, if $\vec{u}$ and $\vec{v}$ are not parallel, they determine a plane. We select a unit vector n̂ perpendicular to the plane by the right-hand rule.



This means that we choose $\hat{n}$ to be unit (normal) vector that points the way your right thumb points when your finger curl through the angle $\boldsymbol{\theta}$ from $\vec{u}$ to $\vec{v}$. The cross-product $\vec{u} \times \vec{v}$ is a vector defined as follows:

- **Geometric Definition**

$$\vec{u} \times \vec{v} = |\vec{u}|\,|\vec{v}|\,sin\,\theta\,\hat{n}$$

where $\mathbf{0} \leq \boldsymbol{\theta} \leq \boldsymbol{\pi}$ is the angle between $\vec{u}$ & $\vec{v}$ and $\hat{n}$ is the unit vector perpendicular to $\vec{u}$ & $\vec{v}$ pointing in the direction given by the right-hand rule.

- **Algebraic Definition**

$$\vec{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$$

$$\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \hat{\imath}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$\boxed{\vec{u} \times \vec{v} = \hat{\imath}(u_2v_3 - v_2u_3) - \hat{\jmath}(u_1v_3 - v_1u_3) + \hat{k}(u_1v_2 - v_1u_2)}$$

__Example 1:__

- Find the cross product of $\vec{\boldsymbol{u}} = \mathbf{2}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \mathbf{2}\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{v}} = \mathbf{3}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}$.
- Check that the cross product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$ is perpendicular to $\vec{\boldsymbol{u}}$ or not**.**
- Check that the cross product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$ is perpendicular to $\vec{\boldsymbol{v}}$ or not.

__Solution:__

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 2 & 1 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \hat{\boldsymbol{\imath}}(1-0) - \hat{\boldsymbol{\jmath}}(2+6) + \hat{\boldsymbol{k}}(0-3)$$

$$\vec{u} \times \vec{v} = \hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}}$$

To check $\vec{u} \times \vec{v}$ is perpendicular to $\vec{u}$, we will take dot product

$$\vec{u}\,.(\vec{u} \times \vec{v}\,) = \left(2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}\right).(\hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}})$$

$$\vec{u}\,.(\vec{u} \times \vec{v}\,) = 2 - 8 + 6 = 0$$

Since $\vec{u}\,.(\vec{u} \times \vec{v}\,) = 0$, represents that the cross product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$ is perpendicular to $\vec{\boldsymbol{u}}$**.**

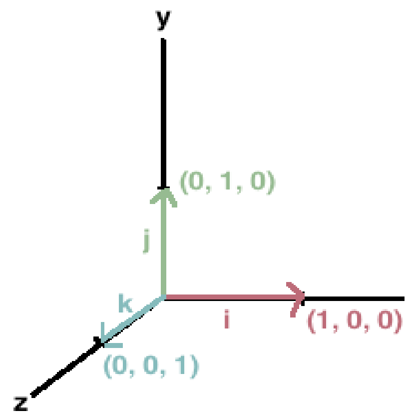
Similarly, to check $\vec{u} \times \vec{v}$ is perpendicular to $\vec{v}$, we will take dot product

$$\vec{v}\,.(\vec{u} \times \vec{v}) = \left(3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}\right).(\hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}})$$

$$\vec{v}\,.\,(\vec{u}\times\vec{v}) = 3 - 3 = 0$$

Since $\vec{v}\,.\,(\vec{u}\times\vec{v}\,) = 0$, represents that the cross product $\vec{\boldsymbol{u}}\times\vec{\boldsymbol{v}}$ is perpendicular to $\vec{\boldsymbol{v}}$**.**

Definition: Parallel vector



Two non-zero vectors $\vec{u}$ and $\vec{v}$ are **paralle**l if and only

$$\vec{\boldsymbol{u}}\times\vec{\boldsymbol{v}} = \boldsymbol{0}$$

For Example: $\hat{\imath}\times\hat{\imath} = 0,\ \hat{\jmath}\times\hat{\jmath} = 0,\ \hat{k}\times\hat{k} = 0$

Example 1:

Find $\hat{\imath}\times\hat{\jmath}$ and $\hat{\jmath}\times\hat{\imath}$.

Solution:

- For $\hat{\imath}\times\hat{\jmath}$

  As $\vec{u}\times\vec{v} = |\vec{u}|\,|\vec{v}|\,sin\,\theta\,\hat{n}$

  $$|\hat{\imath}| = 1$$

  $$|\hat{\jmath}| = 1$$

  Angle between $\hat{\imath}$ & $\hat{\jmath}$ is $\boldsymbol{\frac{\pi}{2}}$.

  By right hand rule, the vector $\hat{\imath}\times\hat{\jmath}$ is in the direction of vector $\hat{k}$ so

  $$\hat{n} = \hat{k}$$

  So,

  $$\hat{\imath}\times\hat{\jmath} = \left(|\hat{\imath}|\,|\hat{\jmath}|\,sin\left(\frac{\pi}{2}\right)\right)\hat{k}$$

  $$= (1.1.1)\,\hat{k} = \hat{k}$$

  $$\hat{\imath}\times\hat{\jmath} = \hat{k}$$

- For $\vec{\jmath}\times\vec{\imath}$

  The right-hand rule says that the direction of $\hat{\jmath}\times\hat{\imath}$ is $-\hat{k}$. So

$$\hat{j} \times \hat{\imath} = \left(|\hat{j}|\,|\hat{\imath}|\,sin\,\frac{\pi}{2}\right)(-\hat{k}) = (1.1.1)(-\hat{k})$$

$$\hat{j} \times \hat{\imath} = -\hat{k}$$

Hence proved.

<u>Practice question:</u> Prove these yourself $\boldsymbol{\hat{j} \times \hat{k} = \hat{\imath}}$, $\quad \boldsymbol{\hat{k} \times \hat{\imath} = \hat{j}}$

<u>Example 2:</u> For any vector $\vec{\boldsymbol{v}}$ find $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{v}}$.

<u>Solution:</u>

As $\vec{\boldsymbol{v}}$ is parallel to itself so $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{v}} = \mathbf{0}$.

<u>Remarks:</u>

- The dot product of unit vectors $\hat{\imath}$, $\hat{j}$ and $\hat{k}$ follows similar rules as the dot product of vectors. The angle between the same vectors is equal to 0º, and hence their dot product is equal to 1. And the angle between two perpendicular vectors is 90º, and their dot product is equal to $\mathbf{0}$.
  - $\boldsymbol{\hat{\imath}.\hat{\imath} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1}$
  - $\boldsymbol{\hat{\imath}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{\imath} = 0}$
- The cross product of unit vectors $\hat{\imath}$, $\hat{j}$ and $\hat{k}$ follows similar rules as the cross product of vectors. The angle between the same vectors is equal to $\mathbf{0^o}$, and hence their cross product is equal to $\mathbf{0}$. The angle between two perpendicular vectors is $\mathbf{90^0}$, and their cross product gives a vector, which is perpendicular to the
  - $\boldsymbol{\hat{\imath} \times \hat{\imath} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0}$
  - $\boldsymbol{\hat{\imath} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{\imath},\ \hat{k} \times \hat{\imath} = \hat{j}}$
  - $\boldsymbol{\hat{j} \times \hat{\imath} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{\imath},\ \hat{\imath} \times \hat{k} = -\hat{j}}$

For concept revision check the following links

- **https://www.cuemath.com/algebra/dot-product/**
- **https://www.cuemath.com/algebra/product-of-vectors/**

Practice Problems

Question: Find the cross product of the following vectors

1) $\vec{u} = 2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \widehat{\boldsymbol{k}}$
   $\vec{v} = \hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - \widehat{\boldsymbol{k}}$
2) $\vec{u} = -3\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} + 4\widehat{\boldsymbol{k}}$
   $\vec{v} = \hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} - \widehat{\boldsymbol{k}}$
3) $\vec{u} = 2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \widehat{\boldsymbol{k}}$
   $\vec{v} = -6\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 3\widehat{\boldsymbol{k}}$