Topics:

- **Vector-Valued Function**
- **Domain of Vector-Valued Function**
- **Derivative of Vector-Valued Function**
- **Integration of Vector-Valued Function**

Vector-Valued Function

Our first step in studying the calculus of vector-valued functions is to define what exactly a vector-valued function is. We can then look at graphs of vector-valued functions and see how they define curves in both two and three dimensions.

- **A vector valued function is a function whose domain is the set of real numbers and whose range is the set of vectors.**

Applications in Engineering and Technology:

Vector-valued functions extend calculus to functions that map from one space to another. This is crucial for understanding:

- **Curves in space** (e.g., parametrized curves)
- **Surfaces** and **volumes** in 3D spaces
- Transformations in **multivariable calculus**
- In **robotics**, vector-valued functions describe the motion of robotic arms, where each joint's position can be described by a vector.
- In **computer graphics**, they are used to model the motion and transformation of objects.

Definition

A **vector-valued function** is a function of the form

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} \quad \text{or} \quad \mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}, \tag{3.1}$$

where the **component functions** *f*, *g*, and *h*, are real-valued functions of the parameter *t*. Vector-valued functions are also written in the form

$$\mathbf{r}(t) = \langle f(t), g(t) \rangle \quad \text{or} \quad \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle. \tag{3.2}$$

In both cases, the first form of the function defines a two-dimensional vector-valued function; the second form describes a three-dimensional vector-valued function.

Example 1:

Evaluating Vector-Valued Functions and Determining Domains

For each of the following vector-valued functions, evaluate $\mathbf{r}(0)$, $\mathbf{r}\left(\frac{\pi}{2}\right)$, and $\mathbf{r}\left(\frac{2\pi}{3}\right)$. Do any of these functions have

b. To calculate each of the function values, substitute the appropriate value of t into the function:

$$\begin{aligned}
\mathbf{r}(0) &= 3\tan(0)\,\mathbf{i} + 4\sec(0)\,\mathbf{j} + 5(0)\,\mathbf{k} \\
&= 0\mathbf{i} + 4\mathbf{j} + 0\mathbf{k} = 4\mathbf{j} \\
\mathbf{r}\left(\frac{\pi}{2}\right) &= 3\tan\left(\frac{\pi}{2}\right)\mathbf{i} + 4\sec\left(\frac{\pi}{2}\right)\mathbf{j} + 5\left(\frac{\pi}{2}\right)\mathbf{k}, \text{ which does not exist} \\
\mathbf{r}\left(\frac{2\pi}{3}\right) &= 3\tan\left(\frac{2\pi}{3}\right)\mathbf{i} + 4\sec\left(\frac{2\pi}{3}\right)\mathbf{j} + 5\left(\frac{2\pi}{3}\right)\mathbf{k} \\
&= 3\left(-\sqrt{3}\right)\mathbf{i} + 4(-2)\,\mathbf{j} + \frac{10\pi}{3}\mathbf{k} \\
&= -3\sqrt{3}\mathbf{i} - 8\mathbf{j} + \frac{10\pi}{3}\mathbf{k}.
\end{aligned}$$

To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 3\tan t$, the second component function is $g(t) = 4\sec t$, and the third component function is $h(t) = 5t$. The first two functions are not defined for odd multiples of $\pi/2$, so the function is not defined

for odd multiples of $\pi/2$. Therefore, $\operatorname{dom}(\mathbf{r}(t)) = \left\{ t \,\middle|\, t \neq \frac{(2n+1)\pi}{2} \right\}$, where n is any integer.

Remark: **Whenever we are using trigonometric functions such that the angle is in real numbers or in multiples of π (e.g., $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, 4\pi$, etc.), then the calculator mode should be in Radians.**

Domain of a Vector-valued Function

The domain of a vector-valued function consists of real numbers. The domain can be all real numbers or a subset of the real numbers (natural numbers, whole numbers, integers, rational numbers, or irrational numbers).

Each real number in the domain of a vector-valued function is mapped to either a two- or a three-dimensional vector.

The domain of a vector-valued function is the set of all possible input values (typically represented as a subset of $\mathbb{R}^n$) for which the function is defined. A vector-valued function maps these inputs to vectors in $\mathbb{R}^m$.

Formula:

If a vector-valued function $\vec{r}(t)$ is represented by

$$\vec{\boldsymbol{r}}(\boldsymbol{t}) =< \boldsymbol{f}(\boldsymbol{t}), \boldsymbol{g}(\boldsymbol{t}), \boldsymbol{h}(\boldsymbol{t}) >$$

then the domain of vector-valued function $\vec{r}(t)$ is given by

$$\boldsymbol{Domain\ of}\ \vec{\boldsymbol{r}}(\boldsymbol{t}) = \boldsymbol{Domain\ of}\ \boldsymbol{f}(\boldsymbol{t}) \cap \boldsymbol{Domain\ of}\ \boldsymbol{g}(\boldsymbol{t}) \cap \boldsymbol{Domain\ of}\ \boldsymbol{h}(\boldsymbol{t})$$

In simple form, we can write

$$\boldsymbol{D}[\vec{\boldsymbol{r}}(\boldsymbol{t})] = \boldsymbol{D}[\ \boldsymbol{f}(\boldsymbol{t})] \cap \boldsymbol{D}\ [\ \boldsymbol{g}(\boldsymbol{t})] \cap \boldsymbol{D}\ [\boldsymbol{h}(\boldsymbol{t})]$$

Important Remark:

In simple words, the domain is a set/ interval of all numbers at which the function is not undefined, i.e., we do not get a math error on the calculator.

Example 1:

Consider the vector-valued function $\vec{\boldsymbol{r}}(\boldsymbol{t})$

$$\vec{r}(t) = \langle\, t, t^2, t^3 \rangle$$

a) Find the **domain** of the vector-valued function $\vec{\boldsymbol{r}}(\boldsymbol{t})$.
b) Plot the domain on real number line.

Solution: The function $\vec{r}(t)$ is a vector-valued function, meaning it outputs a vector for each value of t. Here, it has three components:

- Let the first component is $\boldsymbol{f}(\boldsymbol{t}) = \boldsymbol{t}$.
- Let the second component is $\boldsymbol{g}(\boldsymbol{t}) = \boldsymbol{t}^{\boldsymbol{2}}$.
- Let the third component is $h(t) = \boldsymbol{t}^{\boldsymbol{3}}$.

Thus, for a given value of $\boldsymbol{t}$, the output will be a vector consisting of these three values.

Since the given components are polynomial functions of variable $\boldsymbol{t}$ and the domain of polynomial function is all real numbers since any value of $\boldsymbol{t}$ can be input in the function.

Domain of $f(t) = \mathbb{R} = (-\infty, +\infty)$

Domain of $g(t) = \mathbb{R} = (-\infty, +\infty)$

Domain of $h(t) = \mathbb{R} = (-\infty, +\infty)$

Now,

Domain of $\vec{r}(t) = \boldsymbol{D}[\,\boldsymbol{f}(\boldsymbol{t})] \cap \boldsymbol{D}\,[\,\boldsymbol{g}(\boldsymbol{t})] \cap \boldsymbol{D}\,[\boldsymbol{h}(\boldsymbol{t})]$

Domain of $\vec{r}(t) = \mathbb{R} \cap \mathbb{R} \cap \mathbb{R} = (-\infty, +\infty) \cap (-\infty, +\infty) \cap (-\infty, +\infty)$

Domain of $\vec{r}(t) = \mathbb{R} = (-\infty, +\infty)$

Example 2: Find the domain of the vector-valued function given below.

$$\vec{\boldsymbol{r}}(\boldsymbol{t}) = \langle \sqrt{\boldsymbol{t}}\,, \qquad \frac{\mathbf{1}}{\boldsymbol{t}-\mathbf{1}}, \qquad \mathbf{ln}(\boldsymbol{t})\ \rangle$$

Solution: We need to analyze each component (function) separately and determine the values of $\boldsymbol{t}$ for which each function is defined.

- **First component: Let $\boldsymbol{f}(\boldsymbol{t}) = \sqrt{\boldsymbol{t}}$**

The square root function is defined only when the radicand (the expression inside the square root) is non-negative.

Therefore, we require:

$$\boldsymbol{t} \geq \mathbf{0}$$

- **Second component: Let $\boldsymbol{g}(\boldsymbol{t}) = \frac{\mathbf{1}}{\boldsymbol{t}-\mathbf{1}}$**

The reciprocal function is undefined when the denominator is zero.

Therefore, we need to exclude the value where $t - 1 = 0$.

$$t \neq 1$$

- **Third component: Let $\boldsymbol{h}(\boldsymbol{t}) = \boldsymbol{ln}\ (\boldsymbol{t})$**

The natural logarithm function is defined only for positive arguments.

Thus, we require:

$$t > 0$$

Finding domain of vector valued function:

Now we can summarize the conditions derived from each component:

- From the first component: $\boldsymbol{t} \geq \mathbf{0}$
- From the second component: $\boldsymbol{t} \neq \mathbf{1}$
- From the third component: $\boldsymbol{t} > \mathbf{0}$

Combining these conditions:

- The first condition indicates that $\boldsymbol{t}$ must be at least $\mathbf{0}$.
- The second condition excludes $\boldsymbol{t} = \mathbf{1}$.
- The third condition indicates that $\boldsymbol{t}$ must be greater than $\mathbf{0}$.

Thus, the final domain of the vector-valued function is:

$$t \in (0,1) \cup (1, \infty)$$

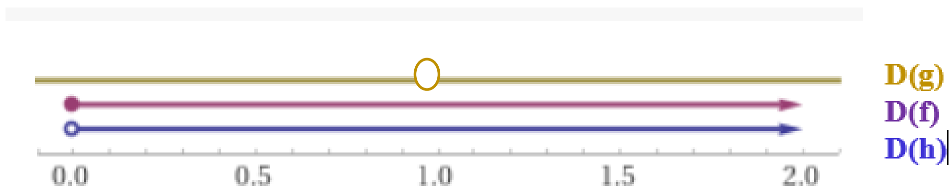
This means $\boldsymbol{t}$ can take any value greater than 0, except $\boldsymbol{t = 1}$.

Real Number Line Representation:

Now let's represent this domain on a real number line using different colors for clarity:

- **Color 1**: For the interval $(\mathbf{0}, \mathbf{1})$, where the function is defined but does not include $\boldsymbol{t = 0}$ or $\boldsymbol{t = 1}$.
- **Color 2**: For the interval $(\mathbf{1}, \infty)$, where the function is also defined but does not include $\boldsymbol{t = 1}$.
- **Open circles** at $\boldsymbol{t = 1}$ to indicate that the point is not included in the domain.

Let's plot this representation.



Example 3: Consider the vector-valued function $\vec{\boldsymbol{r}}(\boldsymbol{t})$

$$\vec{\boldsymbol{r}}(\boldsymbol{t}) = < \sqrt{\boldsymbol{2t-4}}, \quad \frac{\boldsymbol{1}}{\boldsymbol{t-5}}, \quad \ln(\boldsymbol{t-2}) >$$

a) Find the **domain** of the vector-valued function $\vec{\boldsymbol{r}}(\boldsymbol{t})$.
b) Plot the domain on real number line.

Solution: We need to analyze each component (function) separately and determine the values of $\boldsymbol{t}$ for which each function is defined.

- **First component: Let $\boldsymbol{f(t) = \sqrt{2t-4}}$**

The square root function is defined only when the radicand (the expression inside the square root) is non-negative.

Therefore, we require:

$$\boldsymbol{2t - 4 \geq 0}$$

Solving for $\boldsymbol{t}$**:**

$$\boldsymbol{2t \geq 4}$$

$$\boldsymbol{t \geq 2}$$

- **Second component: Let $\boldsymbol{g(t) = \frac{1}{t-5}}$**

The reciprocal function is undefined when the denominator is zero.

Therefore, we need to exclude the value where $t - 5 = 0$.

$$t \neq 5$$

- **Third component: Let $\boldsymbol{h(t) = ln\ (t-2)}$**

The natural logarithm function is defined only for positive arguments.

Thus, we require:

$$t - 2 > 0$$

Solving for $\boldsymbol{t}$:

$$t > 2$$

Finding domain of vector valued function:

Now we can summarize the conditions derived from each component:

- From the first component: $\boldsymbol{t} \geq \mathbf{2}$
- From the second component: $\boldsymbol{t} \neq \mathbf{5}$
- From the third component: $\boldsymbol{t} > \mathbf{2}$

Combining these conditions:

- The first condition indicates that $\boldsymbol{t}$ must be at least $\mathbf{2}$.
- The second condition excludes $\boldsymbol{t} = \mathbf{5}$.
- The third condition indicates that $\boldsymbol{t}$ must be greater than $\mathbf{2}$.

Thus, the final domain of the vector-valued function is:

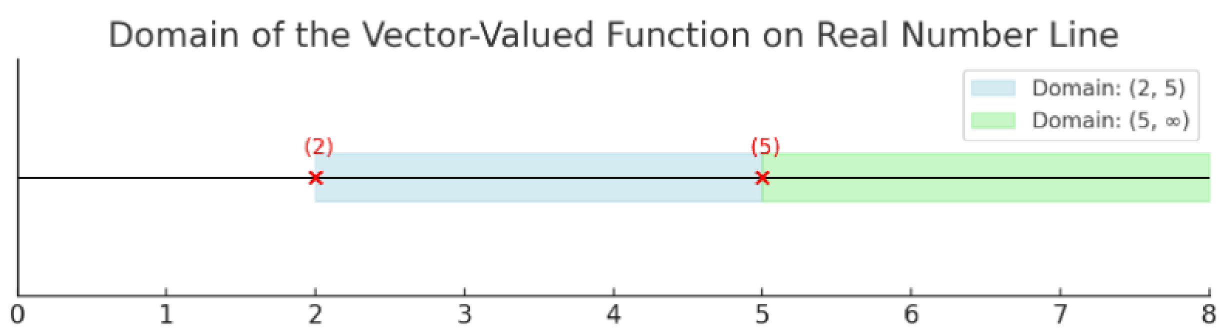
$$t \in (2,5) \cup (5, \infty)$$

Real Number Line Representation:

Now let's represent this domain on a real number line using different colors for clarity:

- **Color 1**: For the interval $(\mathbf{2}, \mathbf{5})$ (where the function is defined).
- **Color 2**: For the interval $(\mathbf{5}, \infty)$ (where the function is also defined).
- **Open circles** at $\boldsymbol{t} = \mathbf{2}$ and $\boldsymbol{t} = \mathbf{5}$ to indicate that these points are not included in the domain.

Let's plot this representation.



Representation explanation:

- The **light blue region** represents the interval $(\mathbf{2}, \mathbf{5})$ indicating where the function is defined but does not include $\boldsymbol{t} = \mathbf{2}$ or $\boldsymbol{t} = \mathbf{5}$.
- The **light green region** represents the interval $(\mathbf{5}, \infty)$, indicating where the function is defined beyond $\boldsymbol{t} = \mathbf{5}$.
- Open circles cross at $\boldsymbol{t} = \mathbf{2}$ and $\boldsymbol{t} = \mathbf{5}$ signify that these points are not included in the domain.

Example 4: Consider the vector-valued function $\vec{\boldsymbol{r}}(\boldsymbol{t})$

$$\vec{r}(t) = \langle\, t^3 \,, \ln(3-t) \,, \sqrt{t} \,\rangle$$

a) Find the **domain** of the vector-valued function $\vec{\boldsymbol{r}}(\boldsymbol{t})$.
b) Plot the domain on real number line.

Solution:

To determine the domain of the vector-valued function

$$\vec{r}(t) = \langle\, t^3 \,, \ln(3-t) \,, \sqrt{t} \,\rangle$$

we need to analyze each component of the vector function and identify any restrictions based on their domains:

First Component: Let $\boldsymbol{f(t) = t^3}$

- The function $\boldsymbol{t^3}$ is defined for all real numbers.
- Domain $f(t) = (-\infty, \infty)$.

Second Component: Let $\boldsymbol{g(t) = \ln(3-t)}$

- The natural logarithm function $\boldsymbol{\ln(t)}$ is defined for $\boldsymbol{t > 0}$.
  Therefore, for $\boldsymbol{ln(3-t)}$, we need:
  $$3 - t > 0 \implies t < 3$$
- Domain: $\boldsymbol{g(t)} = (-\infty, \boldsymbol{3})$.

Third Component: Let $\boldsymbol{h(t) = \sqrt{t}}$

- The square root function $\sqrt{\boldsymbol{t}}$ is defined for $\boldsymbol{t \geq 0}$.
  Therefore, for $\sqrt{\boldsymbol{t}}$, we need:
  $$t \geq 0$$
- Domain: $\boldsymbol{h(t)} = [\boldsymbol{0}, +\infty)$.

To find the overall domain of $\vec{r}(t)$, we must find the intersection of the domains of the individual components:

- From the first component, we have domain of $f(t) = (-\infty, \infty)$.
- From the second component, we have domain of $\boldsymbol{g(t)} = (-\infty, \boldsymbol{3})$.
- From the third component, we have domain of $\boldsymbol{h(t)} = [\boldsymbol{0}, +\infty)$.

Now,

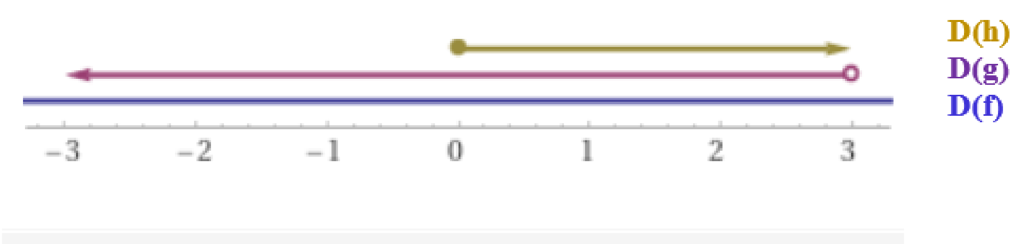
Domain of $\vec{r}(t)$ = $\boldsymbol{D}[\,\boldsymbol{f}(\boldsymbol{t})] \cap \boldsymbol{D}\,[\,\boldsymbol{g}(\boldsymbol{t})] \cap \boldsymbol{D}\,[\boldsymbol{h}(\boldsymbol{t})]$

Domain of $\vec{r}(t) = (-\infty, +\infty) \cap \ (-\infty, 3) \cap \ [0, \infty)$

Domain of $\vec{r}(t) \ = \mathbb{R} = [0,3)$

Conclusion

The domain of the vector-valued function $\vec{r}(t)$ is $[\mathbf{0}, \mathbf{3})$.



Example 5: Consider the vector-valued function $\vec{\boldsymbol{r}}(\boldsymbol{t})$

$$\vec{\boldsymbol{r}}(\boldsymbol{t}) = \langle\, \sqrt{\mathbf{4} - \boldsymbol{t}^{\mathbf{2}}}\,, \boldsymbol{e}^{-\mathbf{3}\boldsymbol{t}}, \mathbf{ln}(\boldsymbol{t} + \mathbf{1})\, \rangle$$

a) Find the **domain** of the vector-valued function $\vec{\boldsymbol{r}}(\boldsymbol{t})$.
b) Plot the domain on real number line.

Solution:

To determine the domain of the vector-valued function

$$\vec{\boldsymbol{r}}(\boldsymbol{t}) = \langle\, \sqrt{\mathbf{4} - \boldsymbol{t}^{\mathbf{2}}}\,, \boldsymbol{e}^{-\mathbf{3}\boldsymbol{t}}, \mathbf{ln}(\boldsymbol{t} + \mathbf{1})\, \rangle$$

we need to analyze each component of the vector function and identify any restrictions based on their domains:

First Component: Let $\boldsymbol{f}(\boldsymbol{t}) = \sqrt{\mathbf{4} - \boldsymbol{t}^{\mathbf{2}}}$

The square root function $\sqrt{\boldsymbol{t}}$ is defined for $\boldsymbol{t} \geq \mathbf{0}$.

- Therefore, for $\sqrt{\mathbf{4} - \boldsymbol{t}^{\mathbf{2}}}$, we need:

  $4 - t^2 \geq 0$

  $\Longrightarrow -t^2 \geq -4$

  Multiplying the inequality both sides by $(-\mathbf{1})$, we have

$\Longrightarrow t^2 \leq 4$

$\Longrightarrow \pm t \leq 2$

$\Longrightarrow -2 \leq t \leq 2$

- Domain $f(t) = [\mathbf{-2, 2}]$.

Second Component: Let $\boldsymbol{g(t)} = \boldsymbol{e^{-3t}}$

- The exponential function $\boldsymbol{e^t}$ is defined for all real numbers $\boldsymbol{t}$.
- Thus, $\boldsymbol{e^{-3t}}$ is defined for all $\boldsymbol{t}$.
- Domain: $\boldsymbol{g(t)} = (-\infty, +\infty)$.

Third Component: Let $\boldsymbol{h(t)} = \mathbf{ln}(\boldsymbol{t} + \mathbf{1})$

- The natural log function $\mathbf{ln}(\boldsymbol{t})$ is defined for $\boldsymbol{t} > \mathbf{0}$.

  Therefore, for $\mathbf{ln}(\boldsymbol{t} + \mathbf{1})$, we need:
$$t + 1 > 0 \Longrightarrow t > -1$$
- Domain: $\boldsymbol{h(t)} = (\mathbf{-1}, +\infty)$.

To find the overall domain of $\vec{r}(t)$, we must find the intersection of the domains of the individual components:

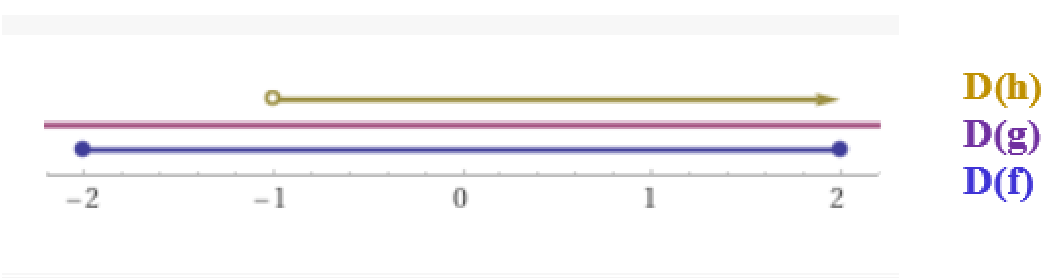
- Domain of $f(t) = [\mathbf{-2, 2}]$.
- Domain of $\boldsymbol{g(t)} = (-\infty, +\infty)$.
- Domain of $\boldsymbol{h(t)} = (\mathbf{-1}, +\infty)$.

Now,

Domain of $\vec{r}(t) = \boldsymbol{D}[\,\boldsymbol{f(t)}] \cap \boldsymbol{D}\,[\,\boldsymbol{g(t)}] \cap \boldsymbol{D}\,[\boldsymbol{h(t)}]$

Domain of $\vec{r}(t) = [-2,2] \cap (-\infty, +\infty) \cap (-1, +\infty)$

Domain of $\vec{\boldsymbol{r}}(\boldsymbol{t}) = [\mathbf{-1, 2}]$



Example 6: Consider the vector-valued function $\vec{r}(t)$

$$\vec{r}(t) = \left(\frac{t-2}{t+2}\right)\hat{\imath} + sin(t)\hat{\jmath} + \ln\,(9-t^2)\hat{k}$$

- Find the **domain** of the vector-valued function $\vec{r}(t)$.
- Plot the domain on real number line.

Solution:

To determine the domain of the vector-valued function

$$\vec{r}(t) = \left(\frac{t-2}{t+2}\right)\hat{\imath} + sin(t)\hat{\jmath} + \ln\,(9-t^2)\hat{k}$$

we need to analyze each component of the vector function and identify any restrictions based on their domains:

First Component: Let $f(t) = \frac{t-2}{t+2}$

- **This rational function is defined for all t except where the denominator is zero.**
- **We need to find when $t+2=0$.**

  $$t + 2 = 0$$
  $$\Longrightarrow t = -2$$
- So, the domain for this component is all real numbers except $t = -2$.
- Domain $f(t) = (-\infty, -2) \cup (-2, \infty)$.

  or

  Domain $f(t) = \mathbb{R} \setminus \{-2\}$

Second Component: Let $g(t) = sin(t)$

- **The sine function is defined for all real numbers.**
- Domain: $g(t) = (-\infty, +\infty)$.

Third Component: Let $h(t) = \ln\left(9-t^2\right)$

- The natural log function $\ln(t)$ is defined for $t > 0$.

  Therefore, for $\ln\left(9-t^2\right)$, we need:

  $$9 - t^2 > 0$$
  $$\Longrightarrow -t^2 > -9$$

  Multiplying the inequality both sides by (-1), we have

$\Longrightarrow t^2 \leq 9$

- $\Longrightarrow \pm t < 3$
- $\Longrightarrow -3 < t < 3$
- Domain: $\boldsymbol{h(t)} = (\boldsymbol{-3}, \boldsymbol{3})$.

To find the overall domain of $\vec{r}(t)$, we must find the intersection of the domains of the individual components:

- Domain of $f(t) = (-\infty, -2) \cup (-2, \infty)$.
- Domain of $\boldsymbol{g(t)} = (-\infty, +\infty)$.
- Domain of $\boldsymbol{h(t)} = (\boldsymbol{-3}, \boldsymbol{3})$.

Now,

Domain of $\vec{r}(t) = \boldsymbol{D}[\,\boldsymbol{f(t)}] \cap \boldsymbol{D}\,[\,\boldsymbol{g(t)}] \cap \boldsymbol{D}\,[\boldsymbol{h(t)}]$

Domain of $\vec{r}(t) = \{(-\infty, \boldsymbol{-2}) \cup (\boldsymbol{-2}, \infty)\} \cap\ (-\infty, +\infty) \cap (-3,3)$

Domain of $\boldsymbol{\vec{r}(t)} = (-3, -2) \cup (-2,3)$**.**

Practice Questions for Students:

Find the domain of the vector-valued functions.

15. Domain: $\mathbf{r}(t) = \langle t^2, \tan t, \ln t \rangle$

16. Domain: $\mathbf{r}(t) = \left\langle t^2, \sqrt{t-3}, \frac{3}{2t+1} \right\rangle$

17. Domain: $\mathbf{r}(t) = \left\langle \csc(t), \frac{1}{\sqrt{t-3}}, \ln(t-2) \right\rangle$

Let $\mathbf{r}(t) = \langle \cos t, t, \sin t \rangle$ and use it to answer the following questions.

18. For what values of t is $\mathbf{r}(t)$ continuous?

19. Sketch the graph of $\mathbf{r}(t)$.

20. Find the domain of $\mathbf{r}(t) = 2e^{-t}\,\mathbf{i} + e^{-t}\,\mathbf{j} + \ln(t-1)\mathbf{k}$.

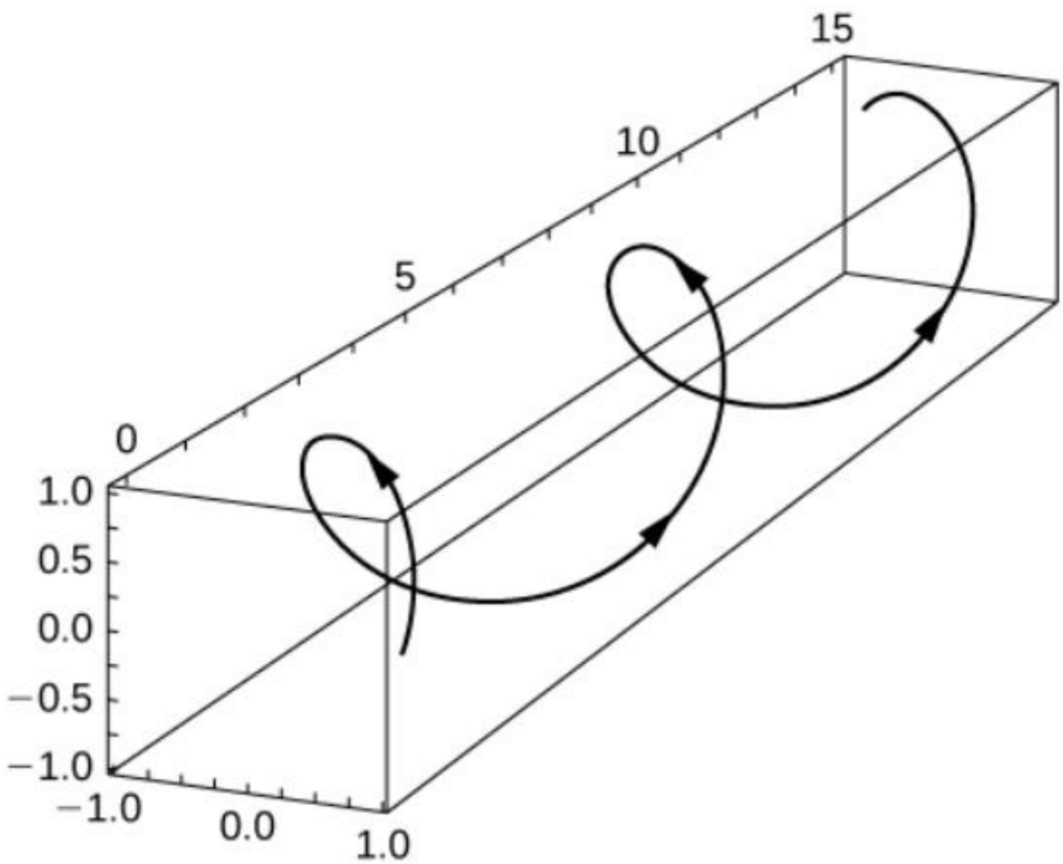
21. For what values of t is $\mathbf{r}(t) = 2e^{-t}\,\mathbf{i} + e^{-t}\,\mathbf{j} + \ln(t-1)\mathbf{k}$ continuous?

Answers of odd questions:

15. $t > 0,\ t \neq \frac{(2k+1)\pi}{2}$, where k is an integer.

17. $t > 3, t \neq n\pi$, where n is an integer.

19.



21. All t such that $t \in (1, \infty)$.

Derivative of a Vector-valued Function

The derivative of a vector-valued function is a **vector** that represents the rate of change of the function with respect to a parameter, usually $\boldsymbol{t}$. If the vector-valued function is written as $\vec{\boldsymbol{r}}(\boldsymbol{t})$, its derivative gives the velocity or direction in which the vector is changing at any given point along the curve it traces.

Definition:

Let $\vec{r}(t)$ be a vector-valued function of the form:

$$\vec{r}(t) = \langle f_1(t), f_2(t), f_3(t), \ldots \rangle,$$

where $f_1(t), f_2(t), f_3(t), \ldots$ are scalar-valued functions of t. The **derivative** of $\vec{r}(t)$ with respect to t is defined as:

$$\frac{d\vec{r}}{dt} = \left\langle \frac{df_1(t)}{dt}, \frac{df_2(t)}{dt}, \frac{df_3(t)}{dt}, \ldots \right\rangle.$$

Interpretation

The derivative of a vector-valued function is computed by differentiating each component function individually. If $\vec{r}(t)$ is a position vector, then its derivative $\frac{d\vec{r}}{dt}$ represents the **velocity vector**.

Theorem:

Differentiation of Vector-Valued Functions

Let *f*, *g*, and *h* be differentiable functions of *t*.

i. If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, then $\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}$.

ii. If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, then $\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$.

Properties of the Derivative of Vector-Valued Functions

Let **r** and **u** be differentiable vector-valued functions of *t*, let *f* be a differentiable real-valued function of *t*, and let *c* be a scalar.

i.	$\frac{d}{dt}[c\mathbf{r}(t)]$	$=$	$c\mathbf{r}'(t)$	Scalar multiple
ii.	$\frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)]$	$=$	$\mathbf{r}'(t) \pm \mathbf{u}'(t)$	Sum and difference
iii.	$\frac{d}{dt}[f(t)\,\mathbf{u}(t)]$	$=$	$f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$	Scalar product
iv.	$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)]$	$=$	$\mathbf{r}'(t) \cdot \mathbf{u}(t) + \mathbf{r}(t) \cdot \mathbf{u}'(t)$	Dot product
v.	$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)]$	$=$	$\mathbf{r}'(t) \times \mathbf{u}(t) + \mathbf{r}(t) \times \mathbf{u}'(t)$	Cross product
vi.	$\frac{d}{dt}[\mathbf{r}(f(t))]$	$=$	$\mathbf{r}'(f(t)) \cdot f'(t)$	Chain rule
vii.	If $\mathbf{r}(t) \cdot \mathbf{r}(t)$	$=$	c, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.	

Recall: Basic Formulas of Derivative from Calculus:

- $\frac{d}{dt}[c] = 0$, where c is a constant number.
- $\frac{d}{dt}[t] = 1$
- $\frac{d}{dt}[t]^n = n \times t^{n-1}$ (Power rule of t)
- $\frac{d}{dt}[f(t)]^n = n \times [f(t)]^{n-1} . \frac{d}{dt}[f(t)]$ (Power rule of $f(t)$)
- $\frac{d}{dt}[\sin(at)] = a \times \cos(at)$
- $\frac{d}{dt}[cos(at)] = -a \times \sin(at)$
- $\frac{d}{dt}[tan(at)] = -a \times \sec^2(at)$
- $\frac{d}{dt}[\ln(t)] = \frac{1}{t}$
- $\frac{d}{dt}\left[\ln\left(f(t)\right)\right] = \frac{1}{f(t)} \times \frac{d}{dt}[f(t)]$
- $\frac{d}{dt}[e^{at+b}] = a \times e^{at+b}$

Example 1: Find the derivative of vector-valued function

$$\vec{r}(t) = \langle\, t, t^2, t^3 \rangle$$

Solution:

To find the derivative of the vector-valued function

$$\vec{r}(t) = \langle t, t^2, t^3 \rangle,$$

we differentiate each component of the vector function with respect to t.

Derivative of Each Component:

1. The derivative of t with respect to t is:

$$\frac{d}{dt}(t) = 1.$$

2. The derivative of t^2 with respect to t is:

$$\frac{d}{dt}(t^2) = 2t.$$

3. The derivative of t^3 with respect to t is:

$$\frac{d}{dt}(t^3) = 3t^2.$$

Hence, the derivative of the vector-valued function $\vec{r}(t)$ is:

$$\vec{r}\,'(t) = \frac{d\vec{r}(t)}{dt} = \langle\, 1, 2t, 3t^2 \rangle$$

Example 2: Find the derivative of vector-valued function

$$\vec{\boldsymbol{r}}(\boldsymbol{t}) = (\boldsymbol{t}+\mathbf{1})\hat{\boldsymbol{\imath}} + \left(\boldsymbol{t}^{\mathbf{2}} - \mathbf{1}\right)\hat{\boldsymbol{\jmath}}$$

Solution:

To find the derivative of the vector-valued function

$$\vec{r}(t) = (t+1)\hat{i} + (t^2-1)\hat{j},$$

we differentiate each component with respect to t.

Derivative of Each Component:

1. The derivative of $(t+1)$ with respect to t is:

$$\frac{d}{dt}(t+1) = 1.$$

2. The derivative of (t^2-1) with respect to t is:

$$\frac{d}{dt}(t^2-1) = 2t.$$

Hence, the derivative of the vector-valued function $\vec{\boldsymbol{r}}(\boldsymbol{t})$ is:

$$\vec{r}\,'(t) = \frac{d\vec{r}(t)}{dt} = \hat{\boldsymbol{\imath}} + 2t\,\hat{\boldsymbol{\jmath}}$$

Example 3: Find the derivative of vector-valued function

$$\vec{\boldsymbol{r}}(\boldsymbol{t}) = \boldsymbol{e^t}\ \hat{\boldsymbol{\imath}} + \frac{\mathbf{2}}{\mathbf{9}}\boldsymbol{e^{2t}}\ \hat{\boldsymbol{\jmath}}$$

Solution:

To find the derivative of the vector-valued function

$$\vec{r}(t) = e^t\hat{i} + \frac{2}{9}e^{2t}\hat{j},$$

we differentiate each component with respect to t.

Derivative of Each Component:

1. The derivative of e^t with respect to t is:

$$\frac{d}{dt}(e^t) = e^t.$$

2. The derivative of $\frac{2}{9}e^{2t}$ with respect to t is:

$$\frac{d}{dt}\left(\frac{2}{9}e^{2t}\right) = \frac{2}{9}\cdot 2e^{2t} = \frac{4}{9}e^{2t}.$$

Hence, the derivative of the vector-valued function $\vec{\boldsymbol{r}}(\boldsymbol{t})$ is:

$$\vec{r}'(t) = \frac{d\vec{r}(t)}{dt} = e^t\hat{\boldsymbol{\imath}} + \frac{4}{9}e^{2t}\hat{\boldsymbol{\jmath}}$$

Example 4: Find the derivative of vector-valued function

$$\vec{r}(t) = \cos(2t)\,\hat{\boldsymbol{\imath}} + [\,3\,e^{2t}\sin(2t)\,]\,\hat{\boldsymbol{\jmath}}$$

Solution:

To find the derivative of the vector-valued function

$$\vec{r}(t) = \cos(2t)\hat{\imath} + \left[3e^{2t}\sin(2t)\right]\hat{\jmath},$$

we differentiate each component with respect to t.

Derivative of Each Component:

1. The derivative of $\cos(2t)$ with respect to t is:

$$\frac{d}{dt}[\cos(2t)] = -2\sin(2t).$$

2. The derivative of $3e^{2t}\sin(2t)$ with respect to t requires the product rule:

$$\frac{d}{dt}[3e^{2t}\sin(2t)] = 3\left[\frac{d}{dt}(e^{2t})\cdot\sin(2t) + e^{2t}\cdot\frac{d}{dt}(\sin(2t))\right].$$

Applying the derivatives:

$$\frac{d}{dt}(e^{2t}) = 2e^{2t}, \quad \frac{d}{dt}(\sin(2t)) = 2\cos(2t).$$

So the derivative becomes:

$$3\left[2e^{2t}\sin(2t) + e^{2t}\cdot 2\cos(2t)\right] = 6e^{2t}\sin(2t) + 6e^{2t}\cos(2t).$$

Hence, the derivative of the vector-valued function $\vec{\boldsymbol{r}}(\boldsymbol{t})$ is:

$$\vec{r}\,'(t) = \frac{d\vec{r}(t)}{dt} = -2\sin(2t)\,\hat{\boldsymbol{\imath}} + 6e^{2t}[\,\sin(2t) + \cos(2t)\,]\,\hat{\boldsymbol{\jmath}}$$

Example 5:

Using the Properties of Derivatives of Vector-Valued Functions

Given the vector-valued functions

$$\mathbf{r}(t) = (6t+8)\,\mathbf{i} + \left(4t^2+2t-3\right)\mathbf{j} + 5t\,\mathbf{k}$$

and

$$\mathbf{u}(t) = \left(t^2-3\right)\mathbf{i} + (2t+4)\,\mathbf{j} + \left(t^3-3t\right)\mathbf{k},$$

calculate each of the following derivatives using the properties of the derivative of vector-valued functions.

a. $\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)]$

b. $\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{u}'(t)\right]$

Solution

a. We have $\mathbf{r}'(t) = 6\mathbf{i} + (8t+2)\,\mathbf{j} + 5\mathbf{k}$ and $\mathbf{u}'(t) = 2t\mathbf{i} + 2\mathbf{j} + \left(3t^2-3\right)\mathbf{k}$. Therefore, according to property iv.:

$$\begin{aligned}
\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)] &= \mathbf{r}'(t)\cdot\mathbf{u}(t) + \mathbf{r}(t)\cdot\mathbf{u}'(t) \\
&= (6\mathbf{i} + (8t+2)\,\mathbf{j} + 5\mathbf{k})\cdot\left(\left(t^2-3\right)\mathbf{i} + (2t+4)\,\mathbf{j} + \left(t^3-3t\right)\mathbf{k}\right) \\
&\quad + \left((6t+8)\,\mathbf{i} + \left(4t^2+2t-3\right)\mathbf{j} + 5t\mathbf{k}\right)\cdot\left(2t\mathbf{i} + 2\mathbf{j} + \left(3t^2-3\right)\mathbf{k}\right) \\
&= 6\left(t^2-3\right) + (8t+2)(2t+4) + 5\left(t^3-3t\right) \\
&\quad + 2t(6t+8) + 2\left(4t^2+2t-3\right) + 5t\left(3t^2-3\right) \\
&= 20t^3 + 42t^2 + 26t - 16.
\end{aligned}$$

b. First, we need to adapt property v. for this problem

$$\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{u}'(t)\right] = \mathbf{u}'(t)\times\mathbf{u}'(t) + \mathbf{u}(t)\times\mathbf{u}''(t).$$

Recall that the cross product of any vector with itself is zero. Furthermore, $\mathbf{u}''(t)$ represents the second derivative of $\mathbf{u}(t)$:

$$\mathbf{u}''(t) = \frac{d}{dt}\left[\mathbf{u}'(t)\right] = \frac{d}{dt}\left[2t\mathbf{i} + 2\mathbf{j} + \left(3t^2-3\right)\mathbf{k}\right] = 2\mathbf{i} + 6t\mathbf{k}.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{u}'(t)\right] &= \mathbf{0}+\left(\left(t^2-3\right)\mathbf{i}+(2t+4)\mathbf{j}+\left(t^3-3t\right)\mathbf{k}\right)\times(2\mathbf{i}+6t\mathbf{k}) \\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ t^2-3 & 2t+4 & t^3-3t\\ 2 & 0 & 6t\end{vmatrix} \\ &= 6t(2t+4)\mathbf{i}-\left(6t\left(t^2-3\right)-2\left(t^3-3t\right)\right)\mathbf{j}-2(2t+4)\mathbf{k} \\ &= \left(12t^2+24t\right)\mathbf{i}+\left(12t-4t^3\right)\mathbf{j}-(4t+8)\mathbf{k}.\end{aligned}$$

Applications: Distance, Velocity, Speed, Acceleration and Direction

If $\vec{r}(t)$ is a **position vector** (or **displacement vector**) of a particle moving along a smooth curve in space, then

Distance: $$\text{distance} = |\vec{r}(t)|$$

Velocity: $$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$$

Speed: $$s(t) = |\vec{v}(t)|$$

Acceleration: $$\vec{a}(t) = \frac{d\,\vec{v}(t)}{dt}$$

Direction: **The unit vector $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$ is the direction of motion at time t.**

Example 1:

A person on a **hang glider** is spiralling upward due to rapidly rising air on a path having position vector $\vec{r}(t)$ given below. then find:

1) The **velocity** vector at any time t.
2) The **acceleration** vector at any time t.
3) The glider's **speed** at any time t.
4) The glider's **speed** at any time $t = 2\ sec$.

$$\vec{r}(t) = 3\cos(t)\hat{\imath} + 3\sin(t)\hat{\jmath} + t^2\hat{k}$$

Solution:

1. Velocity Vector:

The velocity vector $\vec{v}(t)$ is the derivative of the position vector $\vec{r}(t)$ with respect to t:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}.$$

Derivative of Each Component:

- The derivative of $3\cos(t)$ is:

$$\frac{d}{dt}(3\cos(t)) = -3\sin(t).$$

- The derivative of $3\sin(t)$ is:

$$\frac{d}{dt}(3\sin(t)) = 3\cos(t).$$

- The derivative of t^2 is:

$$\frac{d}{dt}(t^2) = 2t.$$

Thus, the velocity vector is:

$$\vec{v}(t) = -3\sin(t)\hat{\imath} + 3\cos(t)\hat{\jmath} + 2t\hat{k}.$$

2. Acceleration Vector:

The acceleration vector $\vec{a}(t)$ is the derivative of the velocity vector $\vec{v}(t)$ with respect to t:

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt}.$$

Derivative of Each Component:

- The derivative of $-3\sin(t)$ is:

$$\frac{d}{dt}(-3\sin(t)) = -3\cos(t).$$

- The derivative of $3\cos(t)$ is:

$$\frac{d}{dt}(3\cos(t)) = -3\sin(t).$$

- The derivative of $2t$ is:

$$\frac{d}{dt}(2t) = 2.$$

Thus, the acceleration vector is:

$$\vec{a}(t) = -3\cos(t)\hat{\imath} - 3\sin(t)\hat{\jmath} + 2\hat{k}.$$

3. Glider's Speed:

The speed of the glider is the magnitude of the velocity vector $\vec{v}(t)$. The magnitude of a vector $\vec{v}(t)$ is given by:

$$|\vec{v}(t)| = \sqrt{(-3\sin(t))^2 + (3\cos(t))^2 + (2t)^2}.$$

Simplifying each term:

- $(-3\sin(t))^2 = 9\sin^2(t)$,
- $(3\cos(t))^2 = 9\cos^2(t)$,
- $(2t)^2 = 4t^2$.

Thus, the speed is:

$$|\vec{v}(t)| = \sqrt{9\sin^2(t) + 9\cos^2(t) + 4t^2}.$$

Since $\sin^2(t) + \cos^2(t) = 1$, this simplifies to:

$$|\vec{v}(t)| = \sqrt{9 + 4t^2}.$$

4. Glider's speed at $t = 2$ *sec*:

To find the glider's speed at $t = 2$ seconds, we use the formula for the speed:

$$|\vec{v}(t)| = \sqrt{9 + 4t^2}.$$

Substituting $t = 2$:

$$|\vec{v}(2)| = \sqrt{9 + 4(2)^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

Thus, the glider's speed at $t = 2$ seconds is:

$$\boxed{5 \text{ units per second}}.$$

<u>Practice Problems</u>

Q # 1: The position vector of an object in a plane is given by $\vec{r}(t) = t^3\hat{\imath} + t^2\hat{k}$.

Find its velocity, speed and acceleration when $t = 1$.

Answer:

- The velocity vector at $t = 1$ is: $\vec{v}(1) = 3\hat{\imath} + 2\,\hat{k}$
- The speed at $t = 1$ is $|\vec{v}(1)| = \sqrt{13}$
- The acceleration vector at $t = 1$ is: $\vec{a}(1) = 6\hat{\imath} + 2\,\hat{k}$

Q # 2: Find the velocity, acceleration and speed of a particle with position vector

$$\vec{r}(t) = t^2\hat{\imath} + e^t\hat{\jmath} + te^t\hat{k}$$

Answer:

$$\vec{v}(t) = 2t\hat{i} + e^t\hat{j} + (e^t + te^t)\hat{k}.$$

$$\vec{a}(t) = 2\hat{i} + e^t\hat{j} + (2e^t + te^t)\hat{k}.$$

$$|\vec{v}(t)| = \sqrt{4t^2 + e^{2t} + e^{2t}(1 + 2t + t^2)} = \sqrt{4t^2 + e^{2t}(2 + 2t + t^2)}.$$

Motion in the Plane

In Exercises 1–4, $\mathbf{r}(t)$ is the position of a particle in the xy-plane at time t. Find an equation in x and y whose graph is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of t.

1. $\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2 - 1)\mathbf{j}, \quad t = 1$
2. $\mathbf{r}(t) = \frac{t}{t+1}\mathbf{i} + \frac{1}{t}\mathbf{j}, \quad t = -1/2$
3. $\mathbf{r}(t) = e^t\mathbf{i} + \frac{2}{9}e^{2t}\mathbf{j}, \quad t = \ln 3$
4. $\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (3\sin 2t)\mathbf{j}, \quad t = 0$

Answers:

1. $\boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j}$ **and** $\boldsymbol{a} = 2\boldsymbol{j}$ **at** $\boldsymbol{t} = 1$.
2. $\boldsymbol{v} = 4\boldsymbol{i} - 4\boldsymbol{j}$ **and** $\boldsymbol{a} = -16\boldsymbol{i} - 16\boldsymbol{j}$ **at** $\boldsymbol{t} = -\frac{1}{2}$.
3. $\boldsymbol{v} = 3\boldsymbol{i} + 4\boldsymbol{j}$ **and** $\boldsymbol{a} = 3\boldsymbol{i} + 8\boldsymbol{j}$ **at** $\boldsymbol{t} = \boldsymbol{ln}\,(3)$.
4. $\boldsymbol{v} = 6\boldsymbol{j}$ **and** $\boldsymbol{a} = -4\boldsymbol{i}$ **at** $\boldsymbol{t} = 0$.

Motion in Space

In Exercises 9–14, $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t. Write the particle's velocity at that time as the product of its speed and direction.

9. $\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2 - 1)\mathbf{j} + 2t\mathbf{k}, \quad t = 1$
10. $\mathbf{r}(t) = (1+t)\mathbf{i} + \frac{t^2}{\sqrt{2}}\mathbf{j} + \frac{t^3}{3}\mathbf{k}, \quad t = 1$
11. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 4t\mathbf{k}, \quad t = \pi/2$
12. $\mathbf{r}(t) = (\sec t)\mathbf{i} + (\tan t)\mathbf{j} + \frac{4}{3}t\mathbf{k}, \quad t = \pi/6$
13. $\mathbf{r}(t) = (2\ln(t+1))\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}, \quad t = 1$
14. $\mathbf{r}(t) = (e^{-t})\mathbf{i} + (2\cos 3t)\mathbf{j} + (2\sin 3t)\mathbf{k}, \quad t = 0$

Solutions/Answers:

9. $\mathbf{r} = (t+1)\mathbf{i} + (t^2-1)\,\mathbf{j} + 2t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j}$; Speed: $|\mathbf{v}(1)| = \sqrt{1^2 + (2(1))^2 + 2^2} = 3$; Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i}+2(1)\mathbf{j}+2\mathbf{k}}{3} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{v}(1) = 3\left(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$

10. $\mathbf{r} = (1+t)\mathbf{i} + \frac{t^2}{\sqrt{2}}\mathbf{j} + \frac{t^3}{3}\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + \frac{2t}{\sqrt{2}}\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{2}{\sqrt{2}}\mathbf{j} + 2t\mathbf{k}$; Speed: $|\mathbf{v}(1)| = \sqrt{1^2 + \left(\frac{2(1)}{\sqrt{2}}\right)^2 + (1^2)^2} = 2$; Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i} + \frac{2(1)}{\sqrt{2}}\mathbf{j} + (1^2)\mathbf{k}}{2} = \frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k} \Rightarrow \mathbf{v}(1) = 2\left(\frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k}\right)$

11. $\mathbf{r} = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (-2\cos t)\mathbf{i} - (3\sin t)\mathbf{j}$; Speed: $\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right| = \sqrt{\left(-2\sin\frac{\pi}{2}\right)^2 + \left(3\cos\frac{\pi}{2}\right)^2 + 4^2} = 2\sqrt{5}$; Direction: $\frac{\mathbf{v}\left(\frac{\pi}{2}\right)}{\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right|} = \left(-\frac{2}{2\sqrt{5}}\sin\frac{\pi}{2}\right)\mathbf{i} + \left(\frac{3}{2\sqrt{5}}\cos\frac{\pi}{2}\right)\mathbf{j} + \frac{4}{2\sqrt{5}}\mathbf{k} = -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{2}\right) = 2\sqrt{5}\left(-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}\right)$

12. $\mathbf{r} = (\sec t)\mathbf{i} + (\tan t)\mathbf{j} + \frac{4}{3}t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (\sec t\tan t)\mathbf{i} + (\sec^2 t)\,\mathbf{j} + \frac{4}{3}\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (\sec t\tan^2 t + \sec^3 t)\,\mathbf{i} + (2\sec^2 t\tan t)\,\mathbf{j}$; Speed: $\left|\mathbf{v}\left(\frac{\pi}{6}\right)\right| = \sqrt{\left(\sec\frac{\pi}{6}\tan\frac{\pi}{6}\right)^2 + \left(\sec^2\frac{\pi}{6}\right)^2 + \left(\frac{4}{3}\right)^2} = 2$; Direction: $\frac{\mathbf{v}\left(\frac{\pi}{6}\right)}{\left|\mathbf{v}\left(\frac{\pi}{6}\right)\right|} = \frac{\left(\sec\frac{\pi}{6}\tan\frac{\pi}{6}\right)\mathbf{i} + \left(\sec^2\frac{\pi}{6}\right)\mathbf{j} + \frac{4}{3}\mathbf{k}}{2} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{6}\right) = 2\left(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$

13. $\mathbf{r} = (2\ln(t+1))\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(\frac{2}{t+1}\right)\mathbf{i} + 2t\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{-2}{(t+1)^2}\right]\mathbf{i} + 2\mathbf{j} + \mathbf{k}$; Speed: $|\mathbf{v}(1)| = \sqrt{\left(\frac{2}{1+1}\right)^2 + (2(1))^2 + 1^2} = \sqrt{6}$; Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\left(\frac{2}{1+1}\right)\mathbf{i} + 2(1)\mathbf{j} + (1)\mathbf{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k} \Rightarrow \mathbf{v}(1) = \sqrt{6}\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}\right)$

14. $\mathbf{r} = (e^{-t})\,\mathbf{i} + (2\cos 3t)\mathbf{j} + (2\sin 3t)\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-e^{-t})\,\mathbf{i} - (6\sin 3t)\mathbf{j} + (6\cos 3t)\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (e^{-t})\,\mathbf{i} - (18\cos 3t)\mathbf{j} - (18\sin 3t)\mathbf{k}$; Speed: $|\mathbf{v}(0)| = \sqrt{(-e^0)^2 + [-6\sin 3(0)]^2 + [6\cos 3(0)]^2} = \sqrt{37}$; Direction: $\frac{\mathbf{v}(0)}{|\mathbf{v}(0)|} = \frac{(-e^0)\,\mathbf{i} - 6\sin 3(0)\mathbf{j} + 6\cos 3(0)\mathbf{k}}{\sqrt{37}} = -\frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k} \Rightarrow \mathbf{v}(0) = \sqrt{37}\left(-\frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k}\right)$

Integration of vector-valued function

Definition

Let f, g, and h be integrable real-valued functions over the closed interval $[a, b]$.

1. The **indefinite integral of a vector-valued function** $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$ is

$$\int [f(t)\,\mathbf{i} + g(t)\,\mathbf{j}]\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j}.$$

   The **definite integral of a vector-valued function** is

$$\int_a^b [f(t)\,\mathbf{i} + g(t)\,\mathbf{j}]\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j}.$$

2. The indefinite integral of a vector-valued function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ is

$$\int [f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} + \left[\int h(t)\,dt\right]\mathbf{k}.$$

   The definite integral of the vector-valued function is

$$\int_a^b [f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}.$$

Some important Formulas of Integration

- $\int c\,dt = ct$
- $\int t\,dt = \frac{t^2}{2}$
- $\int \sin(at)\,dt = -\frac{\cos(at)}{a}$
- $\int \cos(at)\,dt = \frac{\sin(at)}{a}$
- $\int e^{at}\,dt = \frac{e^{at}}{a}$

Example 1: (Revisiting the flight of the glider)

Suppose that we don't know the path of a glider as in the previous example, but only know its acceleration vector

$$\vec{a}(t) = -3\cos(t)\,\hat{i} - 3\sin(t)\,\hat{j} + 2\,\hat{k}$$

We also know that initially (at time $t = 0$). The glider departed from the point $(3, 0, 0)$ with velocity $\vec{v}(0) = 3\hat{j}$. Find the **glider's position** $\vec{r}(t)$ as a function of t.

Solution:

Since the acceleration vector is given as

$$\boxed{\vec{a}(t) = -3\cos(t)\,\hat{i} - 3\sin(t)\,\hat{j} + 2\,\hat{k}\_\_\_\_\_\_\_\_\_\_(1)}$$

To find the velocity vector $\vec{v}(t)$, we will take the integral of $\vec{a}(t)$, that is

$$\vec{v}(t) = \int \vec{a}(t)\,dt$$

$$\vec{v}(t) = \int [-3\cos(t)\ \hat{i} - 3\sin(t)\ \hat{j} + 2\,\hat{k}\,]dt$$

$$\boxed{\vec{v}(t) = -3\sin(t)\,\hat{i} + 3\cos(t)\,\hat{j} + 2t\hat{k} + c_1\_\_\_\_\_\_\_\_\_\_(2)}$$

where c_1 is the constant of integration. To find the value of c_1, we need an initial condition.

Putting $t = 0$ in equation (2), we have

$$\vec{v}(0) = -3\sin(0)\,\hat{i} + 3\cos(0)\,\hat{j} + 2(0)\,\hat{k} + c_1$$

$$\boxed{\vec{v}(0) = 0\,\hat{i} + 3\,\hat{j} + 0\,\hat{k} + c_1\_\_\_\_\_\_\_\_\_\_(3)}$$

Initial condition: $\vec{v}(0) = 3\hat{j}$**,**

Using this initial condition, equation (3) becomes

$$3\hat{j} = 0\,\hat{i} + 3\,\hat{j} + 0\,\hat{k} + c_1$$

$$3\hat{j} - 3\,\hat{j} = c_1$$

$$\boxed{c_1 = 0\ \_\_\_\_\_\_\_\_\_\_(4)}$$

So, putting value of equation (4) in equation (2), we get the **velocity function** $\vec{v}(t)$at any time t

$$\boxed{\vec{v}(t) = -3\sin(t)\,\hat{i} + 3\cos(t)\hat{j} + 2\,t\,\hat{k}}$$

Now, our target is to find the **displacement vector** $\vec{\boldsymbol{r}}(\boldsymbol{t})$, then we will further integrate the velocity vector.

$$\vec{\boldsymbol{r}}(\boldsymbol{t}) = \int \vec{\boldsymbol{v}}(\boldsymbol{t})\, \boldsymbol{dt}$$

$$\vec{r}(t) = \int \left[-3\sin(\mathrm{t})\,\hat{\boldsymbol{\imath}} + 3\cos(\mathrm{t})\,\hat{\boldsymbol{\jmath}} + 2\,\mathrm{t}\,\hat{\boldsymbol{k}} \right] \mathrm{dt}$$

$$\vec{r}(t) = -3\,(-\cos(\mathrm{t}))\hat{\boldsymbol{\imath}} + 3\sin(\mathrm{t})\hat{\boldsymbol{\jmath}} + 2\left(\frac{t^2}{2}\right)\hat{\boldsymbol{k}} + c_2$$

$$\boxed{\vec{r}(t) = 3\cos(\mathrm{t})\ \hat{\boldsymbol{\imath}} + 3\sin(\mathrm{t})\ \hat{\boldsymbol{\jmath}} + t^2\hat{\boldsymbol{k}} + c_2 \_\_\_\_\_\_\_\_\_\_(5)}$$

where c_2 is constant of integration, and to find its value, we need an initial condition.

Putt $\boldsymbol{t} = \mathbf{0}$ **in equation (5), we get**

$$\vec{r}(0) = 3\cos(0)\ \hat{\boldsymbol{\imath}} + 3\sin(0)\ \hat{\boldsymbol{\jmath}} + (0)^2\hat{\boldsymbol{k}} + c_2$$

$$\boxed{\vec{r}(0) = 3\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}} + c_2 \_\_\_\_\_\_\_\_\_\_(6)}$$

Now, given that: At point $(\mathbf{3}, \mathbf{0}, \mathbf{0})$, this means that

$$\vec{\boldsymbol{r}}(\mathbf{0}) = \mathbf{3}\hat{\boldsymbol{\imath}}$$

Using the initial condition in equation (6), we have

$$\vec{r}(0) = 3\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}} + c_2$$

$$3\hat{\boldsymbol{\imath}} = 3\hat{\boldsymbol{\imath}} + c_2$$

$$3\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\imath}} = c_2$$

$$\boxed{\boldsymbol{c_2} = \mathbf{0} \_\_\_\_\_\_\_\_\_\_(7)}$$

So, putting value of equation (7) in equation (5), we get the **displacement (position) vector** $\vec{\boldsymbol{r}}(\mathbf{t})$at any time $\boldsymbol{t}$

$$\boxed{\vec{\boldsymbol{r}}(\boldsymbol{t}) = \mathbf{3}\,\mathbf{cos(t)}\hat{\boldsymbol{\imath}} + \mathbf{3}\,\mathbf{sin(t)}\hat{\boldsymbol{\jmath}} + \boldsymbol{t}^{\mathbf{2}}\hat{\boldsymbol{k}}}$$

Example 2: A glider is moving in the air and we don't know the path, but only know its **acceleration vector**

$$\vec{a}(t) = t\,\hat{i} + e^{t}\,\hat{j} + e^{-t}\hat{k}$$

We also know that initially (at time $t = 0$). The glider departed from the point $(0, 1, 1)$ with velocity $\vec{v}(0) = \hat{k}$. Find the glider's position as a function of t.

Solution:

Since the **acceleration vector** $\vec{a}(t)$ is given as

$$\vec{a}(t) = t\,\hat{i} + e^{t}\,\hat{j} + e^{-t}\hat{k} \_\_\_\_\_\_\_\_ (1)$$

To find the velocity vector $\vec{v}(t)$, we will take the integral of $\vec{a}(t)$, that is

$$\vec{v}(t) = \int \vec{a}(t)\, dt$$

$$\vec{v}(t) = \int [\, t\,\hat{\boldsymbol{i}} + e^{t}\hat{\boldsymbol{j}} + e^{-t}\hat{\boldsymbol{k}}\,]\, dt$$

$$\vec{v}(t) = \frac{t^2}{2}\hat{\boldsymbol{i}} + e^{t}\hat{\boldsymbol{j}} + \frac{e^{-t}}{-1}\,\hat{\boldsymbol{k}} + c_1$$

$$\vec{v}(t) = \frac{t^2}{2}\hat{\boldsymbol{i}} + e^{t}\hat{\boldsymbol{j}} - e^{-t}\hat{\boldsymbol{k}} + c_1 \_\_\_\_\_\_\_ (2)$$

where c_1 is the constant of integration. To find the value of c_1, we need an initial condition.

Putting $t = 0$ in equation (2), we have

$$\vec{v}(0) = \frac{(0)^2}{2}\hat{\boldsymbol{i}} + e^{0}\hat{\boldsymbol{j}} - e^{-0}\hat{\boldsymbol{k}} + c_1$$

$$\vec{v}(0) = 0\,\hat{\boldsymbol{i}} + 1\hat{\boldsymbol{j}} - \frac{1}{e^0}\hat{\boldsymbol{k}} + c_1$$

$$\vec{v}(0) = 0\,\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \frac{1}{1}\hat{\boldsymbol{k}} + c_1$$

$$\vec{v}(0) = 0\,\hat{\boldsymbol{i}} + 1\hat{\boldsymbol{j}} - 1\hat{\boldsymbol{k}} + c_1$$

To find c_1, we need to use the initial condition

Initial condition: $\vec{v}(0) = \hat{k}$

$$\vec{v}(0) = 0\hat{\boldsymbol{i}} + 1\,\hat{\boldsymbol{j}} - 1\,\hat{\boldsymbol{k}} + c_1$$

$$\hat{\boldsymbol{k}} = \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + c_1$$

$$-\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + \hat{\boldsymbol{k}} = c_1$$

$$\boldsymbol{c_1} = -\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} \_\_\_\_\_\_\_\_\_\_ (3)$$

So, putting value of equation (4) in equation (2), we get the **velocity function** $\vec{\boldsymbol{v}}(\mathbf{t})$at any time $\boldsymbol{t}$

$$\vec{v}(t) = \frac{t^2}{2}\hat{\boldsymbol{i}} + e^t\hat{\boldsymbol{j}} - e^{-t}\hat{\boldsymbol{k}} + (-\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$$

$$\vec{v}(t) = \frac{t^2}{2}\hat{\boldsymbol{i}} + e^t\hat{\boldsymbol{j}} - e^{-t}\hat{\boldsymbol{k}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$$

$$\vec{v}(t) = \frac{t^2}{2}\hat{\boldsymbol{i}} + e^t\hat{\boldsymbol{j}} - \hat{\boldsymbol{j}} - e^{-t}\hat{\boldsymbol{k}} + 2\hat{\boldsymbol{k}}$$

$$\boxed{\vec{\boldsymbol{v}}(\boldsymbol{t}) = \frac{\boldsymbol{t}^2}{\mathbf{2}}\hat{\boldsymbol{i}} + (\boldsymbol{e}^{\boldsymbol{t}} - \mathbf{1})\hat{\boldsymbol{j}} - (\boldsymbol{e}^{-\boldsymbol{t}} - \mathbf{2})\hat{\boldsymbol{k}} \_\_\_\_\_\_\_\_\_\_ (\mathbf{4})}$$

Now, our target is to find the **displacement vector** $\vec{\boldsymbol{r}}(\boldsymbol{t})$, then we will further integrate the velocity vector.

$$\vec{\boldsymbol{r}}(\boldsymbol{t}) = \int \vec{\boldsymbol{v}}(\boldsymbol{t})\, \boldsymbol{dt}$$

$$\vec{r}(t) = \int \left[\frac{t^2}{2}\hat{\boldsymbol{i}} + (e^t - 1)\hat{\boldsymbol{j}} - (e^{-t} - 2)\hat{\boldsymbol{k}}\right] \mathrm{dt}$$

$$\vec{r}(t) = \frac{\mathrm{t}^{2+1}}{2(3)}\hat{\boldsymbol{i}} + (\mathrm{e}^{\mathrm{t}} - \mathrm{t})\hat{\boldsymbol{j}} - \left(\frac{\mathrm{e}^{-\mathrm{t}}}{-1} - 2.\mathrm{t}\right)\hat{\boldsymbol{k}} + c_2$$

$$\vec{r}(t) = \frac{\mathrm{t}^3}{6}\hat{\boldsymbol{i}} + (\mathrm{e}^{\mathrm{t}} - \mathrm{t})\hat{\boldsymbol{j}} - (-\mathrm{e}^{-\mathrm{t}} - 2\,\mathrm{t})\hat{\boldsymbol{k}} + c_2$$

$$\vec{r}(t) = \frac{\mathrm{t}^3}{6}\hat{\boldsymbol{i}} + (\mathrm{e}^{\mathrm{t}} - \mathrm{t})\hat{\boldsymbol{j}} + (\mathrm{e}^{-\mathrm{t}} + 2\mathrm{t})\hat{\boldsymbol{k}} + c_2 \_\_\_\_\_\_\_\_\_\_ (5)$$

where c_2 is constant of integration, and to find its value, we need an initial condition.

Putt $\boldsymbol{t} = \mathbf{0}$ **in equation (5), we get**

$$\vec{r}(t) = \frac{(0)^3}{6}\hat{\boldsymbol{i}} + (\mathrm{e}^0 - 0)\hat{\boldsymbol{j}} + (\mathrm{e}^{-0} + 2(0))\hat{\boldsymbol{k}} + c_2$$

$$\vec{r}(t) = \frac{(0)^3}{6}\hat{\boldsymbol{i}} + (\mathrm{e}^0 - 0)\hat{\boldsymbol{j}} + \left(\frac{1}{\mathrm{e}^0} + 2(0)\right)\hat{\boldsymbol{k}} + c_2$$

$$\vec{r}(t) = 0\,\hat{\boldsymbol{i}} + (1 - 0)\hat{\boldsymbol{j}} + (1 + 0)\hat{\boldsymbol{k}} + c_2$$

$$\boxed{\vec{r}(t) = 0\,\hat{\boldsymbol{i}} + 1\,\hat{\boldsymbol{j}} + 1\,\hat{\boldsymbol{k}} + c_2 \_\_\_\_\_\_\_\_ (6)}$$

Now, given that: At point $(\mathbf{0}, \mathbf{1}, \mathbf{1})$, this means that

$$\vec{\boldsymbol{r}}(\mathbf{0}) = \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$$

Using the initial condition in equation (6), we have

$$\vec{r}(0) = \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + c_2$$
$$\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} = \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + c_2$$

$$\boxed{c_2 = \mathbf{0} \_\_\_\_\_\_\_\_\_\_(7)}$$

So, putting value of equation (7) in equation (5), we get the **displacement (position) vector** $\vec{\boldsymbol{r}}(\mathbf{t})$at any time $\boldsymbol{t}$

$$\boxed{\vec{\boldsymbol{r}}(\boldsymbol{t}) = \frac{\mathbf{t}^3}{\mathbf{6}}\hat{\boldsymbol{i}} + (\mathbf{e}^{\mathbf{t}} - \mathbf{t})\hat{\boldsymbol{j}} + (\mathbf{e}^{-\mathbf{t}} + \mathbf{2t})\hat{\boldsymbol{k}}}$$

<u>Practice Problems</u>

Q # 1: Find the position vector of the particle that has the given acceleration vector

$$\vec{\boldsymbol{a}}(\boldsymbol{t}) = \boldsymbol{sin}(\boldsymbol{t})\hat{\boldsymbol{i}} + \mathbf{2cos}\,(\boldsymbol{t})\hat{\boldsymbol{j}} + \mathbf{cos}\,(\mathbf{2}\boldsymbol{t})\hat{\boldsymbol{k}}$$

with $\vec{\boldsymbol{v}}(\mathbf{0}) = \hat{\boldsymbol{i}}$ and $\vec{\boldsymbol{r}}(\mathbf{0}) = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$**.**

Q # 2: Find the position vector of the particle that has the given acceleration vector.

$$\vec{\boldsymbol{a}}(\boldsymbol{t}) = \hat{\boldsymbol{i}} + \mathbf{2}\hat{\boldsymbol{j}}$$

with $\vec{\boldsymbol{v}}(\mathbf{0}) = \hat{\boldsymbol{i}}$ and $\vec{\boldsymbol{r}}(\mathbf{0}) = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$**.**