Week 3 Lecture

Topics: Vector equations of plane in 3D

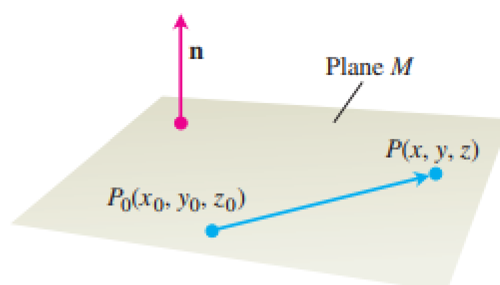
Distance from a point to a plane in 3D

Equation of a Plane in 3D (Space):

A plane in space is determined by knowing a point on the plane and its “tilt” or orientation. This “tilt” is defined by specifying a vector that is perpendicular or normal to the plane.

Suppose that a plane *M* passes through a point $P_o\,(x_o, y_o, z_o)$ and is normal to the non-zero vector $\vec{n} = A\vec{\imath} + B\vec{\jmath} + C\vec{k}$, i.e. $\vec{n}$ is a non-zero normal orthogonal vector. Then *M* is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_O P}$ is orthogonal to $\vec{n}$. Let us consider $\overrightarrow{P_O P} = \vec{u}$, such that

$\vec{u} = < x - x_o, y - y_o, z - z_o >$.



$\vec{n}$ is orthogonal to the plane thus it is orthogonal to every vector on

The plane. The vector $\vec{u}$ lies on the plane.

Thus, the dot product $\vec{n} \cdot \vec{u} = 0$.

This equation is equivalent to

$$(A\vec{\imath} + B\vec{\jmath} + C\vec{k}).[(x - x_o)\vec{\imath} + (y - y_o)\vec{\jmath} + (z - z_o)\vec{k}] = 0$$

$\Longrightarrow$ $A(x - x_o) + B(y - y_o) + C(z - z_o) = 0$

Remark:

Another form of the equation of plane.

$$Ax - Ax_o + By - By_o + Cz - Cz_o = 0$$

$$Ax + By + Cz - (Ax_o + By_o + Cz_o) = 0$$

$$Ax + By + Cz = Ax_o + By_o + Cz_o$$

$Ax + By + Cz = D$ where $D = Ax_o + By_o + Cz_o$

Example 1:

Find an equation for the plane passing through the point $\boldsymbol{P(-3, 0, 7)}$ and perpendicular to the unit normal vector $\vec{\boldsymbol{n}} = \boldsymbol{5\vec{i} + 2\vec{j} - \vec{k}}$.

Solution: The equation of plane is

$$A(x - x_o) + B(y - y_o) + C(\,z - z_o) = 0$$

$$5(x - (-3)) + 2(y - 0) + (-1)(\,z - 7) = 0$$

$5(x + 3) + 2y - z + 7 = 0$

$5x + 15 + 2y - z + 7 = 0$

$5x + 2y - z + 22 = 0$

$5x + 2y - z = -22$

Example 2: Find an equation for the plane passing through three points A (0, 0, 1), B (2, 0, 0) and C (0, 3, 0).

Solution: A vector normal to the plane is:

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & -1 \\ 3 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$

$$= \vec{i}[(0 - (-3)] - \vec{j}(-2 - 0) + \vec{k}(6 - 0)$$

$$\vec{n} = 3\vec{i} + 2\vec{j} + 6\vec{k}$$

Now the equation of plane is

$A(x - x_o) + B(y - y_o) + C(\,z - z_o) = 0$

$3(x - 0) + 2(y - 0) + 6(\,z - 1) = 0$

$$3x + 2y + 6z = 6$$

Practice questions:

Thomas Calculus Ex. 12.5: 21-26

Find equations for the planes in Exercises 21–26.

21. The plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

22. The plane through $(1, -1, 3)$ parallel to the plane

$$3x + y + z = 7$$

23. The plane through $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$

24. The plane through $(2, 4, 5)$, $(1, 5, 7)$, and $(-1, 6, 8)$

25. The plane through $P_0(2, 4, 5)$ perpendicular to the line

$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$$

26. The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A

Distance from a Point to a Plane in Space

Let $\boldsymbol{P}$ be a point on the plane, and $\boldsymbol{P}_0$ be

any point in the space, then $\vec{\boldsymbol{n}}$ is the unit

normal vector to the point $\boldsymbol{P}$. Let the vector $\overrightarrow{\boldsymbol{PP_0}}$ be denoted by the vector $\vec{u}$, i.e.

$$\overrightarrow{\boldsymbol{PP_0}} = \vec{u}$$

Now considering the shortest distance from the point $\boldsymbol{P_0}$ to the plane is perpendicular to the plane, then

$$\boldsymbol{d} = \frac{|\,\vec{\boldsymbol{u}}\ .\ \vec{\boldsymbol{n}}\,|}{|\vec{\boldsymbol{n}}|}$$

Proof:

From the figure, we can write

$$\boldsymbol{cos}\ (\boldsymbol{\theta}) = \frac{\boldsymbol{Base}}{\boldsymbol{Hypotenuse}} = \frac{\boldsymbol{d}}{|\vec{\boldsymbol{u}}\,|}$$

$$\boldsymbol{cos}\ (\boldsymbol{\theta}) = \frac{\boldsymbol{d}}{|\vec{\boldsymbol{u}}\,|}$$

$$|\vec{\boldsymbol{u}}\,|\ \boldsymbol{cos}\ (\boldsymbol{\theta}) = \boldsymbol{d}$$

$$\boldsymbol{d = |\vec{u}\,|\,cos\,(\theta)}$$

$$\boldsymbol{d = \frac{|\vec{u}\,||\vec{n}|\,cos\,(\theta)}{|\vec{n}|}}$$

$$\boldsymbol{d = \frac{|\vec{u}\ .\vec{n}|}{|\vec{n}|}}$$

Hence

$$\boxed{\boldsymbol{d = \frac{|\vec{u}\ .\vec{n}|}{|\vec{n}|}}}$$

Example:

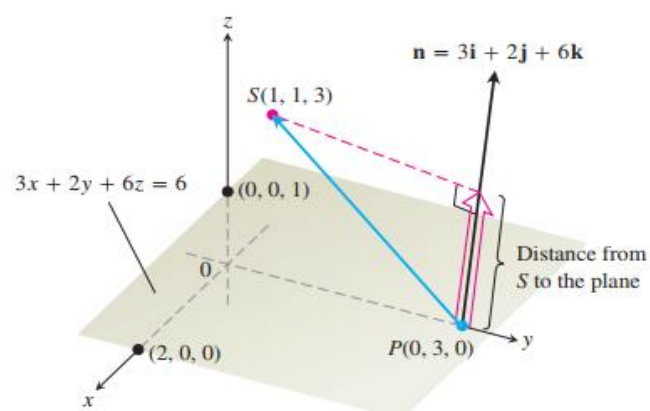
Find the distance from the point S (1, 1, 3) to the plane $3x + 2y + 6z = 6$.

Solution: $\boldsymbol{\vec{n} = 3\vec{\imath} + 2\vec{\jmath} + 6\vec{k}}$

$\boldsymbol{S = (1, 1, 3)}$

$\boldsymbol{P} = ?$

We find a point *P* in the plane and calculate the length of the vector projection of $\overrightarrow{PS}$ onto a vector $\vec{n}$ normal to the plane. The point on plane easiest to find from the plane's equation are the intercepts.



If we take P to be the y-intercept (0,3,0) then

$\overrightarrow{PS} = (1-0)\vec{\imath} + (1-3)\vec{\jmath} + (3-0)\vec{k}$

$= \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$

$|\vec{n}| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7$

The distance from S to the plane is

$$d = \frac{|\overrightarrow{PS}.\vec{n}|}{|\vec{n}|}$$

$$= \left|(\vec{\imath} - 2\vec{\jmath} + 3\vec{k}).\left(\frac{3}{7}\ \vec{\imath} + \frac{2}{7}\vec{\jmath} + \frac{6}{7}\vec{k}\right)\right| = \left|\frac{3}{7} - \frac{4}{7} + \frac{18}{7}\right| = \frac{17}{7}$$

Practice Questions:

Thomas Calculus Ex. 12.5: 39-44, 45, 46

In Exercises 39–44, find the distance from the point to the plane.

39. $(2, -3, 4)$, $x + 2y + 2z = 13$

40. $(0, 0, 0)$, $3x + 2y + 6z = 6$

41. $(0, 1, 1)$, $4y + 3z = -12$

42. $(2, 2, 3)$, $2x + y + 2z = 4$

43. $(0, -1, 0)$, $2x + y + 2z = 4$

44. $(1, 0, -1)$, $-4x + y + z = 4$

45. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.

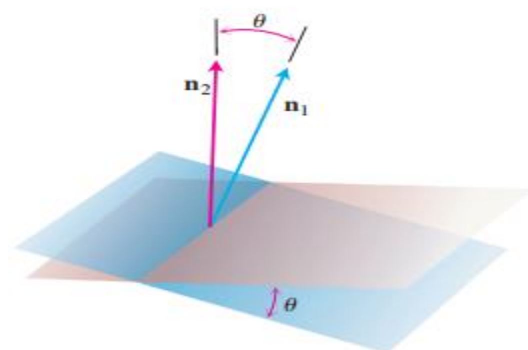
46. Find the distance from the line $x = 2 + t, y = 1 + t, z = -(1/2) - (1/2)t$ to the plane $x + 2y + 6z = 10$.

Angle between Two Planes

The angle between two intersecting planes is defined

to be the angle between their normal vectors.

$$\theta = Cos^{-1}\left(\frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right)$$



Example:

Find the angle between the planes

$\mathbf{3x - 6y - 2z = 15}$ and $\mathbf{2x + y - 2z = 5}$.

Solution:

$\vec{n_1} = 3\vec{\imath} - 6$j-$2\vec{k}$

$\vec{n_2}$=$2\vec{\imath}$+$\vec{\jmath}$-$2\vec{k}$

$\vec{n_1}$. $\vec{n_2}$=6-6+4 =4

$$|\vec{n_1}|=\sqrt{(3)^2+(-6)^2+(-2)^2}=\sqrt{9+36+4}=7$$

$$|\vec{n_2}|=\sqrt{(2)^2+(1)^2+(-2)^2}=\sqrt{4+1+4}=\sqrt{9}=3$$

So $\theta = Cos^{-1}\left(\frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right)$

$\theta = Cos^{-1}\left(\frac{4}{7*3}\right)$ = 79 degrees

Practice Questions:

Thomas Calculus Ex. 12.5: 47-52

Angles

Find the angles between the planes in Exercises 47 and 48.

47. $x + y = 1, \quad 2x + y - 2z = 2$

48. $5x + y - z = 10, \quad x - 2y + 3z = -1$

Use a calculator to find the acute angles between the planes in Exercises 49–52 to the nearest hundredth of a radian.

49. $2x + 2y + 2z = 3, \quad 2x - 2y - z = 5$

50. $x + y + z = 1, \quad z = 0$ (the xy-plane)

51. $2x + 2y - z = 3, \quad x + 2y + z = 2$

52. $4y + 3z = -12, \quad 3x + 2y + 6z = 6$