Topics:

- **Vectors equation of line in 3D**
- **Distance from a point to a line in 3D**

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Food for thought

- What is x-intercept?
- What is y-intercept?

Line in 2D (plane) and in 3D (Space)

A Line in plane (2D):

A **line** in the $\boldsymbol{xy}$**-plane** is determined when $\boldsymbol{y}$**-coordinate of a point (that is y-intercept)** on the line and the **direction of the line** (its slope or angle of inclination) are **given**. The **equation of the line** can then be written using the **point-slope form**.

Recall: Point-slope form:

For any point (x_0, y_0) on the line having slope m is given by the equation

$$y - y_0 = m(x - x_0)$$

If we fix the point, to be any point on the y-axis (y-intercept), then $x_0 = 0$ and $y_0 = c,\ \forall\ c \in \mathbb{R}$ and above equation becomes

$$y - c = mx$$

or

$$y = mx + c$$

A Line in a Space (3D):

A line $\boldsymbol{L}$ in **three-dimensional space** is determined when we know a point $\boldsymbol{P_0(x_0, y_0, z_0)}$ on $\boldsymbol{L}$ and the direction of line $\boldsymbol{L}$. In three dimensions, the direction of a line is conveniently described by a vector, so we let $\vec{\boldsymbol{v}}$ be a vector parallel to the line $\boldsymbol{L}$.

Vector Equation for a Line in space:

Let $\boldsymbol{P(x, y, z)}$ be an arbitrary point on $\boldsymbol{L}$ and let $\overrightarrow{\boldsymbol{r_0}}$ and $\vec{\boldsymbol{r}}$ be the position vectors of $\boldsymbol{P_0}$ and $\boldsymbol{P}$ (that is, they have representations $\overrightarrow{\boldsymbol{OP_0}}$ and $\overrightarrow{\boldsymbol{OP}}$). If $\vec{\boldsymbol{u}}$ is the vector with representation $\overrightarrow{\boldsymbol{P_0P}}$, as in **Figure 1**, then the Triangle Law for vector addition gives

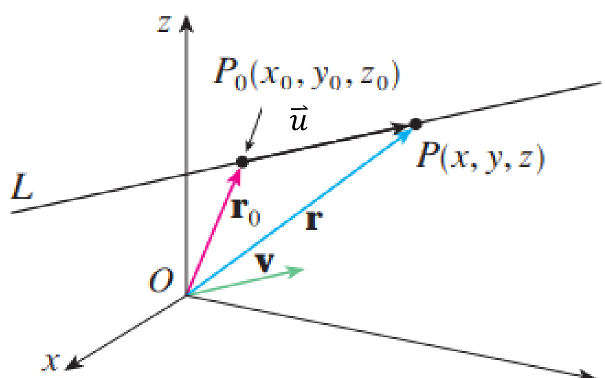


Figure 1

$$\vec{\boldsymbol{r}} = \overrightarrow{\boldsymbol{r_0}} + \vec{\boldsymbol{u}}$$

But, since $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ are parallel vectors, there is a scalar $\boldsymbol{t} \in (-\infty, +\infty)$ such that $\vec{\boldsymbol{u}} = \boldsymbol{t}\vec{\boldsymbol{v}}$. Thus

$$\vec{\boldsymbol{r}} = \overrightarrow{\boldsymbol{r_0}} + \boldsymbol{t}\vec{\boldsymbol{v}} \_\_\_\_\_\_\_\_\_\_\_\_\_\_(\mathbf{1})$$

which is a **vector equation** of a line $\boldsymbol{L}$.

Each value of the parameter $\boldsymbol{t}$ gives the position vector $\vec{\boldsymbol{r}}$ of a point on $\boldsymbol{L}$. In other words, as $\boldsymbol{t}$ varies, the line is traced out by the tip of the vector $\vec{\boldsymbol{r}}$. As **Figure 2** indicates, positive values of $\boldsymbol{t}$ correspond to points on $\boldsymbol{L}$ that lie on one side of $\boldsymbol{P_0}$.

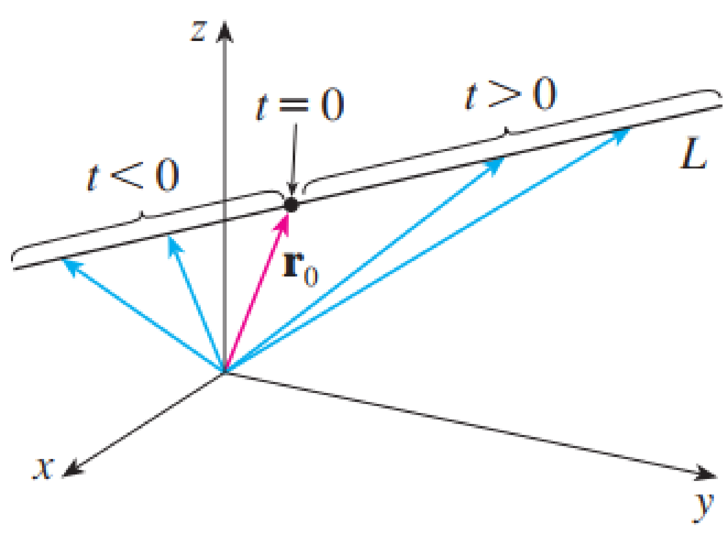


Figure 2

If the vector $\vec{v}$ that gives the direction of the line L is written in component form as $\vec{v} =< a, b, c >$, then we have $t\vec{v} =< ta, tb, tc >$.

We can also write $\vec{r} =< x, y, z >$and $\vec{r_0} =< x_0, y_0, z_0 >$, so the vector equation becomes

$$< x, y, z > = < x_0 + ta,\ y_0 + tb,\ z_0 + tc >$$

Two vectors are **equal** if and only if **corresponding components** are **equal**. Therefore, we have the **three scalar equations**:

$$\boxed{x = x_0 + ta, \qquad y = y_0 + tb, \qquad z = z_0 + tc \ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

where t $\in \mathbb{R}$.

These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} =< a, b, c >$. Each value of the parameter t gives a point (x, y, z) on L.

<u>**Parametric Equation for a Line in space**</u>

Parametric equations for a line through the point $P_0(x_0, y_0, z_0)$ and parallel to the direction vector $\vec{v} =< a, b, c >$ **are**

$$x = x_0 + ta$$

$$y = y_0 + tb$$

$$z = z_0 + tc$$

Note:

Two vectors $\vec{u}$ and $\vec{v}$ are parallel if $\vec{u} = t\,\vec{v}$, where t is scalar. We understand it with the help of an example given below.

$$\vec{u} = < 1, 2, 3 >$$

$$\vec{v} = < 2, 4, 6 >$$

Then

$$\vec{v} = \ 2 < 1, 2, 3 >$$

$$\vec{v} = 2\vec{u}$$

This implies that $\vec{\boldsymbol{u}}$ & $\vec{\boldsymbol{v}}$ are parallel.

Example 1:

Find **parametric equation** & **vector equation** for the line through the point $(\mathbf{-2}, \mathbf{0}, \mathbf{4})$ and parallel to the vector

$$\vec{\boldsymbol{v}} = \mathbf{2}\vec{\boldsymbol{\imath}} + \mathbf{4}\vec{\boldsymbol{\jmath}} - \mathbf{2}\vec{\boldsymbol{k}}$$

Solution:

Since the given point is

$$P = (x_0, y_0, z_0) = (-2,0,4)$$

and the vector is

$$\vec{v} = \mathrm{a}\vec{\imath} + \mathrm{b}\vec{\jmath} + \mathrm{c}\vec{k} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k} = 2\vec{\imath} + 4\vec{\jmath} - 2\vec{k}$$

Here we have

$$x_0 = -2, \qquad y_0 = 0, \qquad z_0 = 4$$

$$a = 2, \qquad b = 4, \qquad c = -2$$

<u>Parametric equation of line in space:</u>

$$x = x_o + ta = -2 + 2t$$

$$y = y_o + tb = 0 + 4t = 4t$$

$$z = z_o + tc = 4 - 2t$$

That is,

$$x = -2 + 2t$$

$$y = 4t$$

$$z = 4 - 2t$$

where $-\infty < \boldsymbol{t} < \infty$.

Vector Equation of Line in space:

$$\vec{r} = \vec{r}_0 + t\,\vec{v}$$

$$x\vec{\imath} + y\vec{\jmath} + z\vec{k} = (-2\vec{\imath} + 0\vec{\jmath} + 4\vec{k}) + t\,(2\vec{\imath} + 4\vec{\jmath} - 2\vec{k})\ ; \quad -\infty < t < \infty$$

$$x\vec{\imath} + y\vec{\jmath} + z\vec{k} = (-2\vec{\imath} + 0\vec{\jmath} + 4\,\vec{k}) + (2t\,\vec{\imath} + 4t\,\vec{\jmath} - 2t\,\vec{k})$$

$$x\vec{\imath} + y\vec{\jmath} + z\vec{k} = (-2 + 2t)\,\vec{\imath} + (0 + 4t)\vec{\jmath} + (4 - 2t)\,\vec{k}$$

$$x\vec{\imath} + y\vec{\jmath} + z\vec{k} = (-2 + 2t)\,\vec{\imath} + (4t)\vec{\jmath} + (4 - 2t)\,\vec{k}$$

Example 2:

Find the **parametric equation** of the line passing through origin and parallel to the vector

$$\vec{v} = 2\hat{\jmath} + \hat{k}$$

Solution:

Given that

$$\text{Origin} = P = (0,0,0) = P(x_0, y_0, z_0)$$

$$\vec{v} = 0\hat{\imath} + 2\hat{\jmath} + \hat{k} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$$

Then the parametric equations of line are

$$x = x_0 + at = 0 + 0t = 0$$

$$y = y_0 + bt = 0 + 2t = 2t$$

$$z = z_0 + ct = 0 + t = t$$

Then the required parametric equations are

$$\boldsymbol{x = 0}$$

$$\boldsymbol{y = 2t}$$

$$\boldsymbol{z = t}$$

Example 3:

Find the **parametric equation** and **vector equation** of the line through $\boldsymbol{P(-3,2,-3)}$ and $\boldsymbol{Q(1,-1,4)}$.

Solution:

$$P = (x_0, y_0, z_0) = (-3,2,-3)$$

$$Q = (x_1, y_1, z_1) = (1,-1,4)$$

The vector parallel to line is

$$\vec{v} = \overrightarrow{PQ} =< x_1 - x_0, \quad y_1 - y_0, \quad z_1 - z_0 >$$

$$\vec{v} = \overrightarrow{PQ} =< 1-(-3), \quad -1-2, \quad 4-(-3) >$$

$$\vec{v} = \overrightarrow{PQ} =< 4,-3,7 > =< a,b,c >$$

We can write the vector in standard form as

$$\boldsymbol{\vec{v} = a\vec{i} + b\vec{j} + c\vec{k} = 4\vec{i} - 3\vec{j} + 7\vec{k}}$$

Parametric equation of a line in space:

$$x = x_o + t\,v_1 = -3 + 4t$$

$$y = y_o + t\,v_2 = 2 - 3t$$

$$z = z_o + t\,v_3 = -3 + 7t$$

where $-\infty < t < \infty$.

Vector equation for a line in space:

$$xi + yj + zk = (-3i + 2j - 3k) + t\,(4i - 3j + 7k)\,,\ -\infty < t < \infty$$

$$xi + yj + zk = (-3i + 2j - 3k) + (4t\,i - 3t\,j + 7t\,k)$$

$$\boldsymbol{xi + yj + zk = (-3 + 4t)\,i + (2 - 3t)j + (-3 + 7t)k}$$

Example 4:

Find the **parametric equation** of the line passing through the point $(3, -2, 1)$ and parallel to the line having parametric equations

$$x = 1 + 2t$$

$$y = 2 - t$$

$$z = 3t$$

Solution:

Since we have given the parametric equations as

$$x = 1 + 2t = x_0 + at$$

$$y = 2 - t = y_0 + bt$$

$$z = 0 + 3t = z_0 + ct$$

This implies that

$$a = 2, \qquad b = -1, \qquad c = 3$$

which are the components of a vector $\boldsymbol{v}$**, given as**

$$\boldsymbol{v} = < \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} > = < \mathbf{2}, -\mathbf{1}, \mathbf{3} >$$

The given point is

$$P = (3, -2, 1) = P(x_0, y_0, z_0)$$

$$\boldsymbol{v} = < \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \geq < \mathbf{2}, -\mathbf{1}, \mathbf{3} >$$

That is,

$$x_0 = 3, \qquad y_0 = -2, \qquad z_0 = 1$$

$$a = 2, \qquad b = -1, \qquad c = 3$$

Then the **parametric equation** of the line passing through the point $P(3, -2, 1)$ and parallel to the line with the given parametric equations are given below:

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Putting the values, we have

$$\boldsymbol{x = 3 + 2t}$$

$$\boldsymbol{y = -2 - t}$$

$$\boldsymbol{z = 1 + 3t}$$

which are the required parametric equations of the line.

Example 5:

Find the parametric equations of the line passing through the point $(2, 3, 0)$, and perpendicular to the vectors $\vec{u} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $\vec{v} = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$.

Solution:

Since the line is perpendicular to the vectors $\vec{u}$ and $\vec{v}$, therefore using the cross product, we have

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \hat{\imath}(10 - 12) - \hat{\jmath}(5 - 9) + \hat{k}(4 - 6)$$

$$\vec{u} \times \vec{v} = -2\hat{\imath} + 4\hat{\jmath} - 2\hat{k} = a\hat{\imath} + b\hat{\jmath} - c\hat{k}$$

The given point is

$$P = (2,3,0) = P(x_0, y_0, z_0)$$

Then the parametric equations of line are

$$x = x_0 + at = 2 - 2t$$

$$y = y_0 + bt = 3 + 4t$$

$$z = z_0 + ct = 0 - 2t$$

Then the required parametric equations are

$$x = 2 - 2t$$

$$y = 3 + 4t$$

$$z = -2t$$

Practice questions:

(Textbook: Thomas Calculus 11th Edition) Ex. 12.5: 1-7, 10.

EXERCISES 12.5

Lines and Line Segments

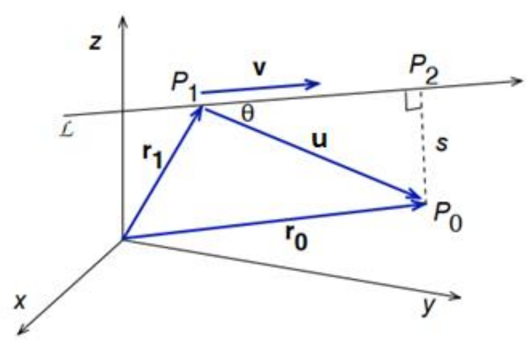
Find parametric equations for the lines in Exercises 1–12.

1. The line through the point $P(3, -4, -1)$ parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$
2. The line through $P(1, 2, -1)$ and $Q(-1, 0, 1)$
3. The line through $P(-2, 0, 3)$ and $Q(3, 5, -2)$
4. The line through $P(1, 2, 0)$ and $Q(1, 1, -1)$
5. The line through the origin parallel to the vector $2\mathbf{j} + \mathbf{k}$
6. The line through the point $(3, -2, 1)$ parallel to the line $x = 1 + 2t, y = 2 - t, z = 3t$
7. The line through $(1, 1, 1)$ parallel to the z-axis
8. The line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$
9. The line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$

10. The line through $(2, 3, 0)$ perpendicular to the vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

The Distance from a Point to a Line in Space

Let $\boldsymbol{L}$ be a line and let $\vec{\boldsymbol{v}}$ be a vector parallel to the line $\boldsymbol{L}$. This means that $\vec{\boldsymbol{v}} || \boldsymbol{L}$.



Let $\boldsymbol{P_0}$ be a point in space from which we need to measure the distance to the line $\boldsymbol{L}$, i.e., we need to measure the distance between $\boldsymbol{P_0}$ and $\boldsymbol{L}$.

We consider an arbitrary point $\boldsymbol{P_1}$ on the line $\boldsymbol{L}$ assuming it might give us a distance from the point $\boldsymbol{P_0}$.

Considering $\boldsymbol{P_0}$ and $\boldsymbol{P_1}$, we draw their **position vectors** $\vec{\boldsymbol{r}}_{\boldsymbol{0}}$ and $\vec{\boldsymbol{r}}_{\boldsymbol{1}}$ respectively from the **origin**.

We draw a vector $\vec{\boldsymbol{u}}$ at an angle $\boldsymbol{\theta}$ from $\boldsymbol{P_1}$ to $\boldsymbol{P_0}$ such that by **head to tail rule** we have,

$$\vec{\boldsymbol{r}}_{\boldsymbol{1}} + \vec{\boldsymbol{u}} = \vec{\boldsymbol{r}}_{\boldsymbol{0}}$$

This implies that

$$\vec{\boldsymbol{u}} = \vec{\boldsymbol{r}}_0 - \vec{\boldsymbol{r}}_1.$$

Now logically the shortest distance of a point from a line is the length of the perpendicular drawn from the point to the line.

Let us consider a point $\boldsymbol{P_2}$ on the line $\boldsymbol{L}$**,** such that $\boldsymbol{P_0P_2}$ is perpendicular to the line $\boldsymbol{L}$**.** i.e., $\boldsymbol{P_0P_2} \perp \boldsymbol{L}$**.** We say that $\overline{\boldsymbol{P_0P_2}} = s$.

Distance is a scalar quantity. From the figure, we can write

$$\boldsymbol{sin\,\theta} = \frac{\boldsymbol{Perpandicular}}{\boldsymbol{Hypotenuse}} = \frac{\boldsymbol{s}}{|\overrightarrow{\boldsymbol{r_0}} - \overrightarrow{\boldsymbol{r_1}}|}$$

$$\boldsymbol{sin\,\theta} = \frac{\boldsymbol{s}}{|\vec{\boldsymbol{u}}|}$$

$$|\vec{\boldsymbol{u}}|\,\boldsymbol{sin\,\theta} = \boldsymbol{s}$$

$$\boldsymbol{s} = |\vec{\boldsymbol{u}}|\,\boldsymbol{sin\,\theta}$$

Here $\boldsymbol{\theta}$ is the angle between $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$**.** $\boldsymbol{P_0P_2} \perp \boldsymbol{L}$ and $\vec{\boldsymbol{v}}||\boldsymbol{L}$ implies that $\boldsymbol{P_0P_2} \perp \vec{\boldsymbol{v}}$**.**

Now considering $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{u}}$, we have

$$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = |\vec{\boldsymbol{u}}|\,|\vec{\boldsymbol{v}}|\,\ \mathbf{sin}\,\boldsymbol{\theta}$$

We can rearrange as

$$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = |\vec{\boldsymbol{v}}||\vec{\boldsymbol{u}}|\,\boldsymbol{sin\,\theta}$$

We know $\boldsymbol{s} = |\vec{\boldsymbol{u}}|\boldsymbol{sin\,\theta}$

$$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = |\vec{\boldsymbol{v}}|\,\boldsymbol{s}$$

or

$$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = \boldsymbol{s}\,|\vec{\boldsymbol{v}}|$$

Since distance is a scalar quantity thus, we need to take the magnitude of the cross product to balance the equation.

$$|\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}| = \boldsymbol{s}\,|\vec{\boldsymbol{v}}|$$

$$\frac{|\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}|}{|\vec{\boldsymbol{v}}|} = \boldsymbol{s}$$

This implies that the distance between the line $\boldsymbol{L}$ and a point can be computed by the formula,

$$\boldsymbol{s} = \frac{|\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}|}{|\vec{\boldsymbol{v}}|}$$

where $\vec{\boldsymbol{u}} = \overrightarrow{\boldsymbol{r_0}} - \overrightarrow{\boldsymbol{r_1}}$

Example:

Find the distance from the point $\boldsymbol{S}\ (\mathbf{1}, \mathbf{1}, \mathbf{5})$ to the line:

$$\boldsymbol{L}: \begin{cases} \boldsymbol{x} = \mathbf{1} + \boldsymbol{t} \\ \boldsymbol{y} = \mathbf{3} - \boldsymbol{t} \\ \boldsymbol{z} = \mathbf{2}\boldsymbol{t} \end{cases}$$

Solution:

Since the given point is $\boldsymbol{S}(\boldsymbol{x_0}, \boldsymbol{y_0}, \boldsymbol{z_0}) = \boldsymbol{P_0}(\mathbf{1}, \mathbf{1}, \mathbf{5})$.

And the given parametric equations of line are

$$\boldsymbol{x} = \boldsymbol{x_1} + \boldsymbol{at} = \mathbf{1} + \boldsymbol{t}$$
$$\boldsymbol{y} = \boldsymbol{y_1} + \boldsymbol{bt} = \mathbf{3} - \boldsymbol{t}$$
$$\boldsymbol{z} = \boldsymbol{z_1} + \boldsymbol{ct} = \mathbf{0} + \mathbf{2}\boldsymbol{t}$$

This means that $\boldsymbol{P_1}(\boldsymbol{x_1}, \boldsymbol{y_1}, \boldsymbol{z_1}) = \boldsymbol{P_1}(\mathbf{1}, \mathbf{3}, \mathbf{0})$ and $\vec{\boldsymbol{v}} =< \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} > =< \mathbf{1}, -\mathbf{1}, \mathbf{2} >$.
Thus, the vector parallel to the line $\boldsymbol{L}$ is

$$\vec{\boldsymbol{v}} = \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \mathbf{2}\hat{\boldsymbol{k}}$$

The Line passes through the point $\boldsymbol{P_1}\ (\mathbf{1}, \mathbf{3}, \mathbf{0})$

$$\vec{u} = \overrightarrow{P_1P_0} = \overrightarrow{OP_0} - \overrightarrow{OP_1}$$

$$\vec{u} = \overrightarrow{P_1P_0} = (\hat{\imath} + \hat{\jmath} + 5\hat{k}) - (\hat{\imath} + 3\hat{\jmath} + 0\,\hat{k})$$

$$\vec{u} = (1-1)\hat{\imath} + (1-3)\hat{\jmath} + (5-0)\hat{k}$$

$$\vec{u} = 0\hat{\imath} - 2\hat{\jmath} + 5\hat{k}.$$

Now,

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \hat{\imath}\begin{vmatrix} -2 & 5 \\ -1 & 2 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 0 & 5 \\ 1 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \hat{\imath}(-4+5) - \hat{\jmath}(0-5) + \hat{k}(0+2)$$

$$\vec{u} \times \vec{v} = \hat{\imath} + 5\hat{\jmath} + 2\hat{k}$$

Taking magnitude of $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$, we have

$$|\vec{u} \times \vec{v}| = \sqrt{(1)^2 + (5)^2 + (2)^2}$$

$$|\vec{u} \times \vec{v}| = \sqrt{30}$$

Similarly, taking magnitude of vector $\vec{\boldsymbol{v}}$, we have

$$|\vec{v}| = \sqrt{(1)^2 + (-1)^2 + (2)^2} = \sqrt{6}$$

Now, apply the formula, we have

$$d = \frac{|\vec{u} \times \vec{v}|}{|\vec{v}|}$$

Putting values, we have

$$d = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{\frac{30}{6}} = \sqrt{5}$$

$$d = \sqrt{5} = 2.24$$

Hence the distance of a point $\boldsymbol{P_1}$ to the line $\boldsymbol{L}$ is **2.24** units.

Practice Questions:

Textbook: Thomas Calculus 11th Edition Ex. 12.5: 33-38

In Exercises 33–38, find the distance from the point to the line.

33. $(0, 0, 12)$; $x = 4t$, $y = -2t$, $z = 2t$

34. $(0, 0, 0)$; $x = 5 + 3t$, $y = 5 + 4t$, $z = -3 - 5t$

35. $(2, 1, 3)$; $x = 2 + 2t$, $y = 1 + 6t$, $z = 3$

36. $(2, 1, -1)$; $x = 2t$, $y = 1 + 2t$, $z = 2t$

37. $(3, -1, 4)$; $x = 4 - t$, $y = 3 + 2t$, $z = -5 + 3t$

38. $(-1, 4, 3)$; $x = 10 + 4t$, $y = -3$, $z = 4t$