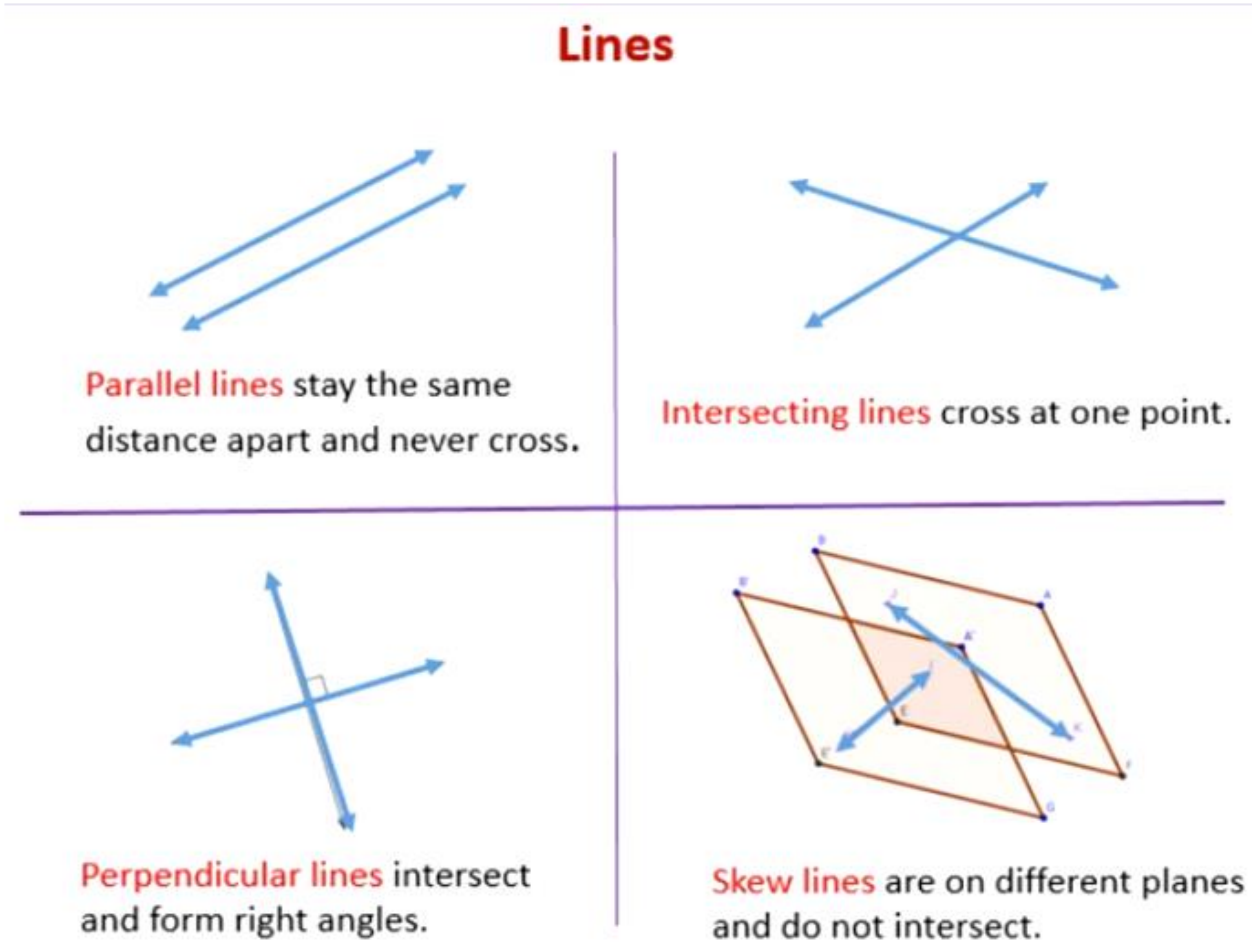
Topic: Interaction between Lines

We observe the following interaction between lines:



<u>Explanation:</u>

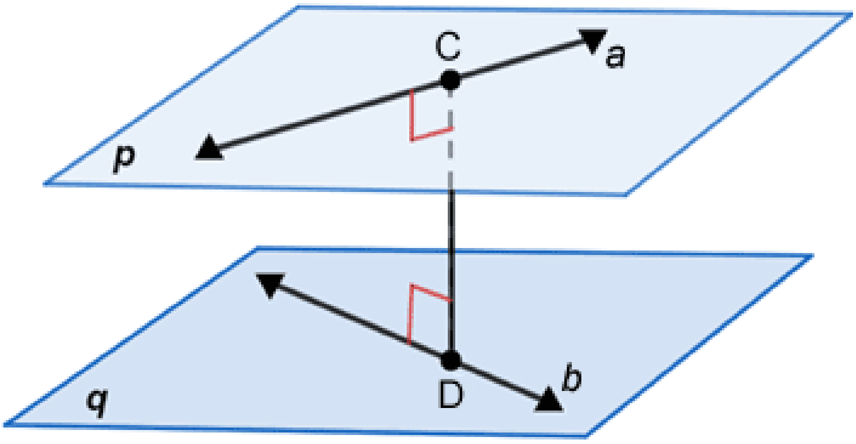
Given two lines in the two and three-dimensional plane,

- **Parallel lines** are the lines that do not intersect or meet each other at any point in a plane. They are always parallel and are at equidistant from each other. Parallel lines are non-intersecting lines
- **Perpendicular lines**, in math, are two lines that intersect each other at only one point and the angle between them is 90°, i.e. they are at right angle with each other.
- When two or more lines cross each other in a plane, they are called **intersecting lines.** The intersecting lines share a common point, which exists on all the intersecting lines, and is called the point of intersection.

The difference between intersecting lines and perpendicular lines is that **intersecting lines *meet at one, and only one point, no matter at what angle they meet,*** ***whereas*** **perpendicular lines *always meet at angle of*** 90°.

In three dimensions, a fourth case is possible. If two lines in space are not parallel, but do not intersect, then the lines are said to be skew lines. Skew lines can only exist in dimensions higher than 2D space. They have to be non-coplanar meaning that such lines exist in different planes.

- **Skew lines** that are NOT parallel, perpendicular or intersecting are skew lines. A visual representation of skew lines is given below,



For more knowledge on skew lines, refer to the link,

https://www.cuemath.com/geometry/skew-lines/

Methodology for observing the Relationship between Two Lines

Now we will check the relationship between two lines. For that we will first find the parallel vectors for each line. Consider $L_1 and\ L_2$ having parallel vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ respectively, then observe which of the following conditions is fulfilled.

- For **parallel lines,** we will check if $\overrightarrow{v_1} = k\overrightarrow{v_2}\ or\ \overrightarrow{v_2} = k\overrightarrow{v_1}$.
- For **perpendicular lines,** we will check $\overrightarrow{v_1} \cdot \overrightarrow{v_2} = 0$.
- For **intersecting lines,** we equate the parametric equations in order to find the values of the parameter variables say **t** and **s** for both lines. Upon substitution of the parameters in the parametric equations of the line we observe that both lines have the same parametric coordinates then those lines are intersecting lines. The values of the parametric coordinates $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ denote the point of intersection.

(Note if anyone of the parametric coordinates do not match then the lines will not be intersecting lines. Then we go to the last conclusion.).

- If none of the above conditions hold then the lines are **skew lines.**

Example:

a) Check whether the given three lines are **parallel lines, skew lines, perpendicular lines or intersecting lines**.

b) If the **lines are intersecting** lines, then **find** the **point of intersection** of the lines.

$$L_1: \quad x = 3 + 2t, \qquad y = -1 + 4t, \qquad z = 2 - t, \qquad -\infty < t < \infty$$

$$L_2: \quad x = 1 + 4s, \qquad y = 1 + 2s, \qquad z = -3 + 4s, \qquad -\infty < s < \infty$$

$$L_3: \quad x = 3 + 2r, \qquad y = 2 + r, \qquad z = -2 + 2r, \qquad -\infty < r < \infty$$

Solution:

Since we have three lines, so there will be three cases.

- **Case 1: We will check the interaction between line L_1 and L_2.**
- **Case 2: We will check the interaction between line L_1 and L_3.**
- **Case 3: We will check the interaction between line L_2 and L_3.**

Case 1: Interaction between line L_1 and L_2

Step 1: Check whether the lines L_1 and L_2 are parallel or not?

The corresponding vector $\vec{v_1}$ of line L_1 is given as

$$\vec{v_1} = < 2, 4, -1 > = 2i + 4j - k$$

The corresponding vector $\vec{v_2}$ of line L_2 is given as

$$\vec{v_2} = < 4, 2, 4 > = 4i + 2j + 4k$$

Since the vectors $\vec{v_1}$ and $\vec{v_2}$ are **not scalar multiple** of each other. That is $\vec{v_1} \neq k\,\vec{v_2}$. **Therefore,** $\vec{v_1}$ & $\vec{v_2}$ are **not parallel**, implies that the **lines L_1 and L_2 are not parallel.**

Note:

- **If the considered lines are not parallel, then this case will move forward. We will further check that the considered lines in this case are perpendicular, or intersecting or skew lines, or not.**
- **If the considered lines are parallel, then this case will be terminated here. There is no need to check further that the considered lines in this case are perpendicular, or intersecting or skew lines, or not.**

Step 2: Check whether the lines L_1 and L_2 are perpendicular or not?

Since $\vec{v_1} = < 2, 4, -1 > = 2i + 4j - k$ and $\vec{v_2} = < 4, 2, 4 > = 4i + 2j + 4k$, Then

$$\vec{v_1} \cdot \vec{v_2} = (2)(4) + (4)(2) + (-1)(4) = 8 + 8 - 4 = 16 - 4 = 12 \neq 0$$

$$\vec{v_1} \cdot \vec{v_2} = 12 \neq 0.$$

Since $\vec{v_1} \cdot \vec{v_2} \neq 0$, therefore, $\vec{v_1}$ & $\vec{v_2}$ are **not perpendicular**, implies that **lines L_1 and L_2 are not perpendicular.**

Note:

- **If the considered lines are not perpendicular, then this case will move forward. We will further check that the considered lines in this case are intersecting or skew lines, or not.**
- **If the considered lines are perpendicular, then it implies that the lines have a point of intersection at an angle of $90°$. So, we will find the point of intersection of the considered lines. There is no need to check further that the considered lines are skew lines, or not.**

Step 3 (a): Check whether the lines L_1 and L_2 are intersecting or not?

Sine the considered lines L_1 and L_2 are

$$L_1: \quad x = 3 + 2t, \qquad y = -1 + 4t, \qquad z = 2 - t, \qquad -\infty < t < \infty$$

$$L_2: \quad x = 1 + 4s, \qquad y = 1 + 2s, \qquad z = -3 + 4s, \qquad -\infty < s < \infty$$

- By comparing the parametric equation of x of line L_1 with parametric equation of x of line L_2, we get

$$3 + 2t = 1 + 4s \quad \Rightarrow 2t - 4s = 1 - 3 \quad \Rightarrow \mathbf{2t - 4s = -2} \ --- (1)$$

- By comparing the parametric equation of $\boldsymbol{y}$ of line $\boldsymbol{L_1}$ with parametric equation of $\boldsymbol{y}$ of line $\boldsymbol{L_2}$, we get

$$-1+4t=1+2s \quad \Longrightarrow \; 4t-2s=1+1 \quad \Longrightarrow \; \boldsymbol{4t-2s=2} \;\; ---(2)$$

- By comparing the parametric equation of $\boldsymbol{z}$ of line $\boldsymbol{L_1}$ with parametric equation of $\boldsymbol{z}$ of line $\boldsymbol{L_2}$, we get

$$2-t=-3+4s \quad \Longrightarrow \; -t-4s=-3-2 \quad \Longrightarrow \; \boldsymbol{-t-4s=-5} \;\; ---(3)$$

Note: We have 3 linear equations with 2 variables in this case. That is an over-determined system. Now, we will consider any of two equations and we will find the values of the variables s and t, then we will put these variable values to the third equation that is not considered earlier.

Now, Consider equation (1) and (2):

Multiply equation (2) by "$\mathbf{2}$" and subtract it from equation (1), we get

$$\boldsymbol{2t-4s=-2} \;\; ----(\mathbf{1})$$

$$\boldsymbol{\pm 8t \mp 4s=\pm 4} \;\; ----(\mathbf{2})$$

$$-6t \qquad = \; -6$$

$$\Longrightarrow t=\frac{-6}{-6}=1$$

$$\Longrightarrow \boldsymbol{t=1}$$

Put $\boldsymbol{t}=\mathbf{1}$ in equation $\mathbf{(1)}$ to find the value of variable $\boldsymbol{s}$, we have

As equation (1) is $\qquad \boldsymbol{2t-4s=-2} \;\; ----(\mathbf{1})$

$$\mathbf{2(1)-4s=-2} \quad \Longrightarrow \; \mathbf{2-4s=-2} \quad \Longrightarrow \mathbf{2+2=4s} \quad \Longrightarrow \mathbf{4=4s} \quad \Longrightarrow \frac{\mathbf{4}}{\mathbf{4}}=\boldsymbol{s} \quad \Longrightarrow \mathbf{1}=\boldsymbol{s}$$

This implies that

$$\boldsymbol{s=1}$$

Hence, we get the values of the variables using equation (1) and equation (2) given as

$$\boldsymbol{t=1} \quad \& \quad \boldsymbol{s=1}$$

Now, put these values in equation (3), that we didn't consider earlier, to verify the values.

Putting values of variables s and t in equation (3), we have

As equation (3) is $$-t-4s=-5 \quad ----(3)$$

$$\Rightarrow -(1)-4(1)=-5$$

$$\Rightarrow \quad -1-4=-5$$

$$\Rightarrow \quad -5=-5$$

$$L.H.S = R.H.S$$

This means that, the values are satisfied and the lines L_1 and L_2 are intersecting lines.

Note:

1) **After putting values of variables in third equation, that is not used earlier:**
    - **If $L.H.S = R.H.S$, then lines are intersecting lines.**
    - **If $L.H.S \neq R.H.S$, then lines are NOT intersecting lines, and these will be SKEW lines.**
2) **If lines are intersecting lines, then we must find the point of intersection of the lines.**

Step 3 (b): Find the point of intersection of lines

Since

$$L_1: \quad x=3+2t, \qquad y=-1+4t, \qquad z=2-t, \qquad -\infty<t<\infty$$

$$L_2: \quad x=1+4s, \qquad y=1+2s, \qquad z=-3+4s, \qquad -\infty<s<\infty$$

	Line L_1	OR	Line L_2
Parametric equation of x	$x=3+2t \Rightarrow x=3+2(1)$ $\Rightarrow x=3+2 \Rightarrow x=5$		$x=1+4s \Rightarrow x=1+4(1)$ $\Rightarrow x=1+4 \Rightarrow x=5$
Parametric equation of y	$y=-1+4t \Rightarrow y=-1+4(1)$ $\Rightarrow y=-1+4 \Rightarrow y=3$		$y=1+2s \Rightarrow y=1+2(1)$ $\Rightarrow y=1+2 \Rightarrow y=3$
Parametric equation of z	$z=2-t \Rightarrow z=2-(1)$ $\Rightarrow z=2-1 \Rightarrow z=1$		$z=-3+4s \Rightarrow z=-3+4(1)$ $\Rightarrow z=-3+4 \Rightarrow z=1$
Point of Intersection	$(x,y,z)=(5,3,1)$		$(x,y,z)=(5,3,1)$

Note: We can find the point of intersection using any of two lines and it must be same from both lines.

Case 2: Interaction between line L_1 and L_3

Step 1: Check whether the lines L_1 and L_3 are parallel or not?

The corresponding vector $\overrightarrow{v_1}$ of line L_1 is given as

$$\overrightarrow{v_1} = <2, 4, -1> = 2i + 4j - k$$

The corresponding vector $\overrightarrow{v_3}$ of line L_3 is given as

$$\overrightarrow{v_3} = <2, 1, 2> = 2i + j + 2k$$

Since the vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_3}$ are **not scalar multiple** of each other. That is $\overrightarrow{v_1} \neq k\,\overrightarrow{v_3}$. **Therefore,** $\overrightarrow{v_1}$ & $\overrightarrow{v_3}$ are **not parallel**, implies that the **lines L_1 and L_3 are not parallel.**

Note:

- **If the considered lines are not parallel, then this case will move forward. We will further check that the considered lines in this case are perpendicular, or intersecting or skew lines, or not.**
- **If the considered lines are parallel, then this case will be terminated here. There is no need to check further that the considered lines in this case are perpendicular, or intersecting or skew lines, or not.**

Step 2: Check whether the lines L_1 and L_3 are perpendicular or not?

Since $\overrightarrow{v_1} = <2, 4, -1> = 2i + 4j - k$ and $\overrightarrow{v_3} = <2, 1, 2> = 2i + j + 2k$, Then

$$\overrightarrow{v_1} \cdot \overrightarrow{v_3} = (2)(2) + (4)(1) + (-1)(2) = 4 + 4 - 2 = 8 - 2 = 6 \neq 0$$

$$\overrightarrow{v_1} \cdot \overrightarrow{v_3} = 6 \neq 0.$$

Since $\overrightarrow{v_1} \cdot \overrightarrow{v_3} \neq 0$, therefore, $\overrightarrow{v_1}$ & $\overrightarrow{v_3}$ are **not perpendicular**, implies that **lines L_1 and L_3 are not perpendicular.**

Note:

- **If the considered lines are not perpendicular, then this case will move forward. We will further check that the considered lines in this case are intersecting or skew lines, or not.**
- **If the considered lines are perpendicular, then it implies that the lines have a point of intersection at an angle of $90°$. So, we will find the point of intersection of the considered lines. There is no need to check further that the considered lines are skew lines, or not.**

<u>Step 3 (a):</u> Check whether the lines L_1 and L_3 are <u>intersecting</u> or not?

Sine the considered lines L_1 and L_3 are

$$L_1: \quad x = 3 + 2t, \qquad y = -1 + 4t, \qquad z = 2 - t, \qquad -\infty < t < \infty$$

$$L_3: \quad x = 3 + 2r, \qquad y = 2 + r, \qquad z = -2 + 2r, \qquad -\infty < r < \infty$$

- By comparing the parametric equation of x of line L_1 with parametric equation of x of line L_3, we get

$$3 + 2t = 3 + 2r \quad \Rightarrow 2t - 2r = 3 - 3 \quad \Rightarrow \mathbf{2t - 2r = 0} \ ---(4)$$

- By comparing the parametric equation of y of line L_1 with parametric equation of y of line L_3, we get

$$-1 + 4t = 2 + r \quad \Rightarrow 4t - r = 2 + 1 \quad \Rightarrow \mathbf{4t - r = 3} \ ---(5)$$

- By comparing the parametric equation of z of line L_1 with parametric equation of z of line L_3, we get

$$2 - t = -2 + 2r \quad \Rightarrow -t - 2r = -2 - 2 \quad \Rightarrow \mathbf{-t - 2r = -4} \ ---(6)$$

<u>Note:</u> We have 3 linear equations with 2 variables in this case. That is an over-determined system. Now, we will consider any of two equations and we will find the values of the variables t and r, then we will put these variable values to the third equation that is not considered earlier.

<u>Now, Consider equation (4) and (6):</u>

Subtract equation (6) from equation (4), we get

$$\mathbf{2t - 2r = 0} \ ----(4)$$

$$\mathbf{\mp t \mp 2r = \mp 4} \ ----(6)$$

$$3t \qquad = 4$$

$$\Rightarrow t = \frac{4}{3}$$

Put $t = \frac{4}{3}$ in equation **(4)** to find the value of variable r, we have

As equation (4) is $\qquad \mathbf{2t - 2r = 0} \ ----(4)$

$$2\left(\frac{4}{3}\right) - 2r = 0 \quad \Rightarrow \frac{8}{3} - 2r = 0 \quad \Rightarrow \frac{8}{3} = 2r \quad \Rightarrow \frac{8}{3(2)} = r \quad \Rightarrow \frac{8}{6} = r \quad \Rightarrow \frac{4}{3} = r$$

This implies that

$$r = \frac{4}{3}$$

Hence, we get the values of the variables using equation (4) and equation (6) given as

$$t = \frac{4}{3} \quad \& \quad r = \frac{4}{3}$$

Now, put these values in equation (5), that we didn't consider earlier, to verify the values.

Putting values of variables t and r in equation (5), we have

As equation (5) is $\quad 4t - r = 3 \quad ----(5)$

$$\Rightarrow 4\left(\frac{4}{3}\right) - \left(\frac{4}{3}\right) = 3$$

$$\Rightarrow \quad \frac{16}{3} - \frac{4}{3} = 3$$

$$\Rightarrow \quad \frac{16-4}{3} = 3$$

$$\Rightarrow \quad \frac{12}{3} = 3$$

$$\Rightarrow \quad 4 \neq 3$$

$$L.H.S \neq R.H.S$$

This means that, the values are NOT satisfied, therefore, the lines L_1 and L_3 are NOT intersecting lines. Hence, the lines L_1 and L_3 are Skew lines.

Note:

1. **After putting values of variables in third equation, that is not used earlier:**
   - **If $L.H.S = R.H.S$, then lines are intersecting lines.**
   - **If $L.H.S \neq R.H.S$, then lines are NOT intersecting lines, and these will be SKEW lines.**
2. **If lines are intersecting lines, then we must find the point of intersection of the lines.**

Case 3: Interaction between line L_2 and L_3

Step 1: Check whether the lines L_2 and L_3 are parallel or not?

The corresponding vector $\overrightarrow{v_2}$ of line $\boldsymbol{L_2}$ is given as

$$\overrightarrow{v_2} = < 4, 2, 4 > = 4\hat{\imath} + 2\hat{\jmath} + 4\,\hat{k}$$

The corresponding vector $\overrightarrow{v_3}$ of line $\boldsymbol{L_3}$ is given as

$$\overrightarrow{v_3} = < 2, 1, 2 > = 2\hat{\imath} + \hat{\jmath} + 2\,\hat{k}$$

Now,

$$\overrightarrow{v_2} = 4\hat{\imath} + 2\hat{\jmath} + 4\,\hat{k}$$

$$\overrightarrow{v_2} = 2\,(2\hat{\imath} + \hat{\jmath} + 2\,\hat{k})$$

$$\overrightarrow{v_2} = 2\,\overrightarrow{v_3}$$

Since the vectors $\overrightarrow{v_2}$ and $\overrightarrow{v_3}$ are **scalar multiple** of each other. That is $\overrightarrow{v_2} = k\,\overrightarrow{v_3}$ for $k = 2$. **Therefore,** $\overrightarrow{v_2}$ & $\overrightarrow{v_3}$ are **parallel**, implies that the **lines L_2 and L_3 are parallel.**

Note:

- **If the considered lines are not parallel, then this case will move forward. We will further check that the considered lines in this case are perpendicular, or intersecting or skew lines, or not.**
- **If the considered lines are parallel, then this case will be terminated here. There is no need to check further that the considered lines in this case are perpendicular, or intersecting or skew lines, or not.**

Practice Question for Students (Book: Thomas Calculus Ex. 12.5; Q# 62)

- Check whether the given three lines are **parallel lines, skew lines, perpendicular lines or intersecting lines**.
- If the **lines are intersecting** lines, then **find** the **point of intersection** of the lines.

$$L_1: \quad x = 1 + 2t, \qquad y = -1 - t, \qquad z = 3t, \qquad -\infty < t < \infty$$

$$L_2: \quad x = 2 - s, \qquad y = 3s, \qquad z = 1 + s, \qquad -\infty < s < \infty$$

$$L_3: \quad x = 5 + 2r, \qquad y = 1 - r, \qquad z = 8 + 3r, \qquad -\infty < r < \infty$$