

Lecture # 1

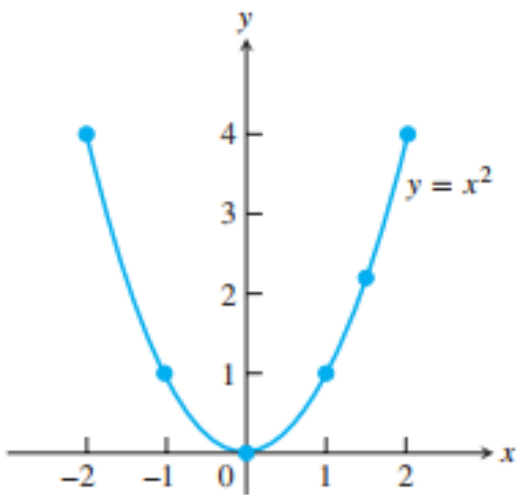
Topics:

- Introduction to Multivariate Calculus (MVC)
 - Introduction to vectors
 - Dot product of two vectors
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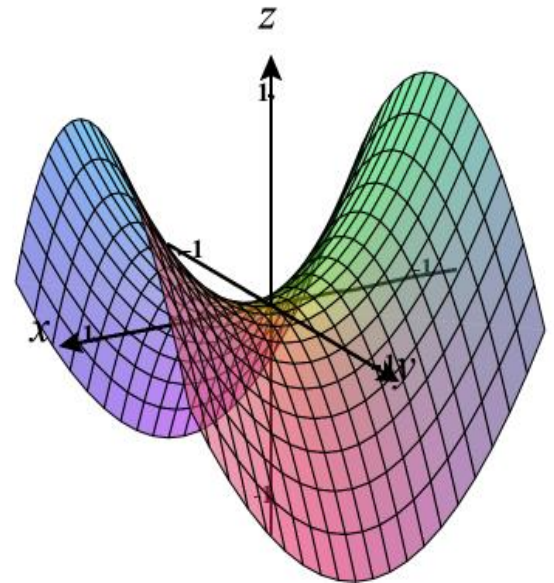
Introduction to Multivariable Calculus (MVC)

What do you understand from the word Multivariable Calculus?

Give some examples around you.



$$y = f(x) = x^2$$



$$f(x, y) = x^2 - y^2$$

Real-Life Examples:

Example 1:

Consider an airline's ticket price pattern. To avoid flying planes with many empty seats, it sells some tickets at full price and some at a discount. For a particular route, the airline's revenue, R , earned in a given time period is determined by the number of full-price tickets, x , and the number of discount tickets, y , sold. We say that R is a function of x and y , and we write

$$R = f(x, y)$$

This is just like the function notation of one-variable calculus. The variable R is the dependent variable and the variables x and y are the independent variables. The letter f stands for the **function** or **rule** that gives the value, or output, of R corresponding to given values of x and y .

The revenue, R , (in dollars) from a particular airline route is shown in table 1 as a function of the number of full-price tickets and of discount tickets sold.

Revenue from ticket sales as a function of x and y

		Number of full-price tickets, x			
		100	200	300	400
Number of discount tickets, y	200	75,000	110,000	145,000	180,000
	400	115,000	150,000	185,000	220,000
	600	155,000	190,000	225,000	260,000
	800	195,000	230,000	265,000	300,000
	1000	235,000	270,000	305,000	340,000

(Table 1)

Values of x are shown across the top, values of y are down the left side, and the corresponding values of $f(x, y)$ are in the table. For example, to find the value of $f(300, 600)$, we look in the column corresponding to $x = 300$ at the row $y = 600$, where we find the number **225,000**. Thus,

$$f(300, 600) = 225,000$$

This means that the revenue from **300** full-price tickets and **600** discount tickets is **\$225,000**.

Example 2:

Suppose you want to calculate your monthly payment on a five-year car loan; this depends on both the amount of money you borrow and the interest rate. These quantities can vary separately: the loan amount can change while the interest rate remains the same, or the interest rate can change while the loan amount remains the same. To calculate your monthly payment, you need to know both. If the monthly payment is \$ m , the loan amount is \$ L , and the interest rate is r %, then we express the fact that m is a function of L and r by writing

$$m = f(L, r)$$

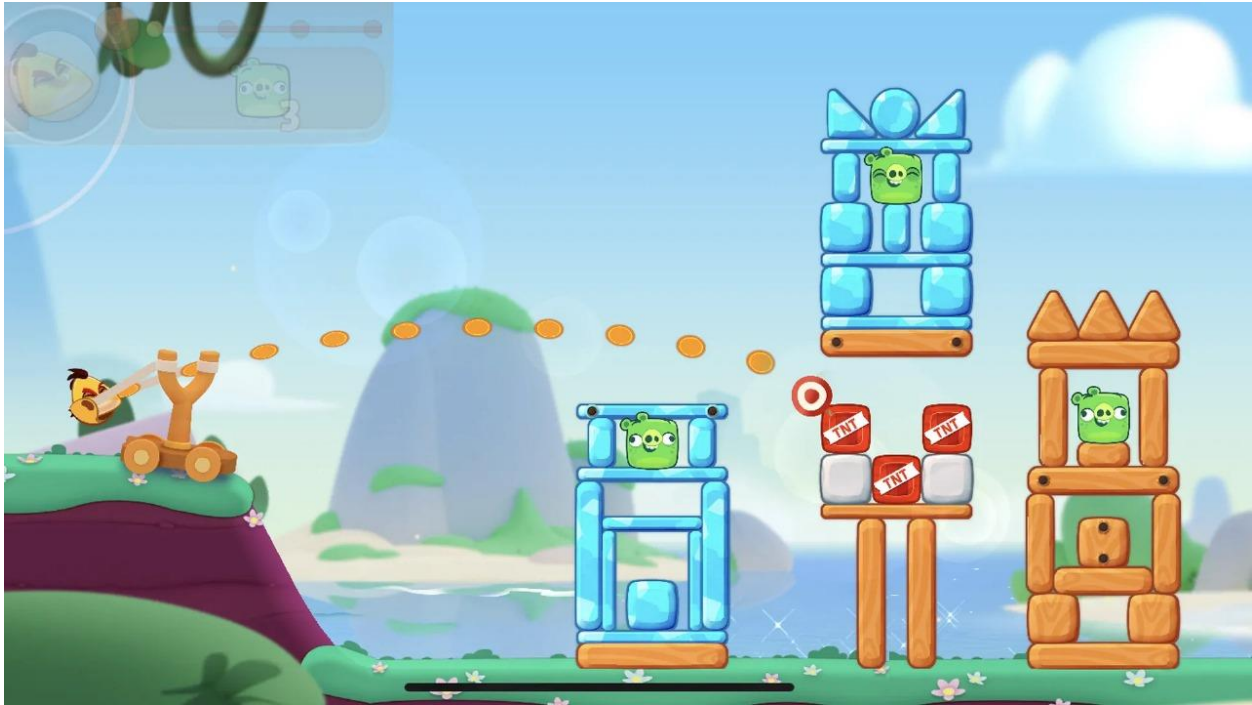
This is just like the function notation of one-variable calculus. The variable m is called the dependent variable, and the variables L and r are called the independent variables. The letter f stands for the *function* or rule that gives the value of m corresponding to given values of L and r .

Definition: Multivariable Calculus

Multivariable Calculus is the **extension of calculus** in one variable to calculus with functions of several variables.

Single Variable Function (One independent variable)	Multivariable Function (More than one independent variable)
$f(x) = x^2$	$R = f(x, y) = x + y$
For $x = 1$, $\Rightarrow f(1) = 1$ For $x = 2$, $\Rightarrow f(2) = 4$	For $x = 1$, and $y = 2$, the function is $\Rightarrow f(1, 2) = 1 + 2 = 3$

Introduction to Vectors



In Games, vectors are used for determining **positions**, **directions** and **velocities**, etc.

- The **position vectors** indicate how far the object is.
- The **velocity** vector indicates how much time it will take or how much force we should give.
- The **direction vector** indicates in which way we should apply the force.

Just for Fun:

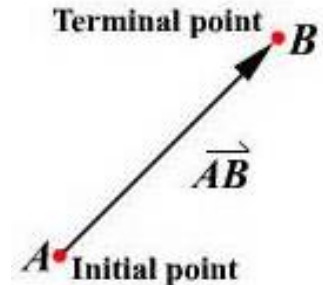
- **Zelda Game using vectors**
In the link given below, you will see the practical coding of the game Zelda using vectors and their properties.
<https://www.youtube.com/watch?v=d42w8QyPI7I>
- **Genshin Impact Gameplay**
<https://www.youtube.com/watch?v=QsHWebfAwpY>
- **Angry Birds Slingshot Stories**
<https://www.youtube.com/watch?app=desktop&v=b-GDMIWOPr0>

Vector:

The physical quantities that have magnitude, SI unit and direction is called a vector.

Examples: Displacement, Force, Acceleration, Velocity

- A vector in plane/space is directed line segment.
The directed line segment \overrightarrow{AB} has initial point A and the terminal point B.



Examples:

- Let we have points $A(x_1, y_1)$ and $B(x_2, y_2)$ in plane, then

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

- Let we have points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in space, then

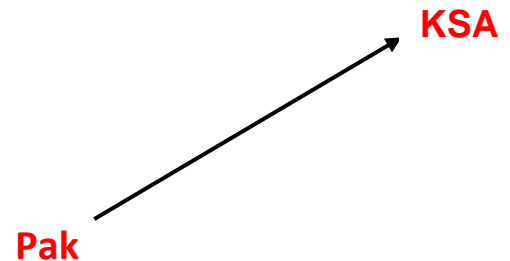
$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Displacement vector

Suppose you are a pilot and you are planning a flight from **Pakistan (Pak)** to **Kingdom of Saudi Arabia (KSA)**.

You must know two things:

- The distance to be travelled.
- The direction to go. Both these quantities together specify the displacement or displacement vector between the two countries.



Note: The **magnitude (length/norm)** of the displacement vector is the distance between the points and is represented by the length of the arrow.

Component form and Standard form of a Vector

Let \vec{v} be a three-dimensional vector with initial point at the origin $(0, 0, 0)$ and terminal point is (v_1, v_2, v_3) then the component form of \vec{v} is

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

and the standard form is

$$\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$$

Magnitude or Length of a Vector

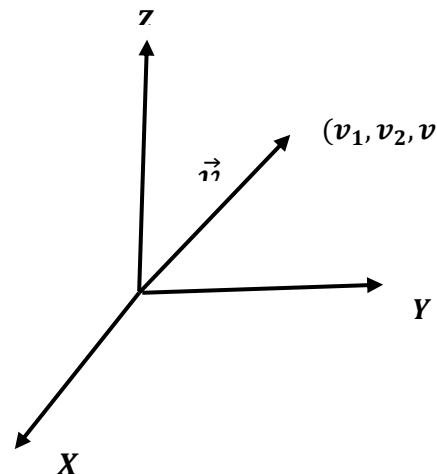
The length or magnitude of a vector

$$\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$$

is

$$|\vec{v}| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}$$

Note: It is a non-negative number.



Example 1: Component Form of a Vector and Length of a Vector

- Find the component form of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.
- Find the length of the vector \overrightarrow{PQ} .



Solution:

- The initial point P has the coordinates $(x_1, y_1, z_1) = (-3, 4, 1)$. This implies that

$$x_1 = -3, \quad y_1 = 4, \quad z_1 = 1$$

Similarly, the terminal point Q has the coordinates $(x_2, y_2, z_2) = (-5, 2, 2)$. This implies that

$$x_2 = -5, \quad y_2 = 2, \quad z_2 = 2$$

The formula to find the vector \overrightarrow{PQ} is

$$\overrightarrow{PQ} = \langle x_2 - x_1, \quad y_2 - y_1, \quad z_2 - z_1 \rangle$$

Putting values in the formula, we get

$$\overrightarrow{PQ} = \langle -5 - (-3), \quad 2 - 4, \quad 2 - 1 \rangle$$

$$\overrightarrow{PQ} = \langle -5 + 3, \quad -2, \quad 1 \rangle$$

$$\overrightarrow{PQ} = \langle -2, \quad -2, \quad 1 \rangle$$

which is the required component form of vector \overrightarrow{PQ} .

b) Since the vector \overrightarrow{PQ} is obtained in part **(a)**, that is

$$\vec{v} = \overrightarrow{PQ} = \langle -2, \quad -2, \quad 1 \rangle$$

or

$$\vec{v} = \langle v_1, v_2, v_3 \rangle = \langle -2, \quad -2, \quad 1 \rangle$$

The length or magnitude of vector $\vec{v} = \overrightarrow{PQ}$ is given by the formula

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Putting values, we get

$$|\vec{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2}$$

$$|\vec{v}| = \sqrt{4 + 4 + 1}$$

$$|\vec{v}| = \sqrt{9}$$

$$|\vec{v}| = 3$$

So, the length of vector $\vec{v} = \overrightarrow{PQ}$ is **3**.

Vector Algebra

Two principal operations involving vectors are vector addition and scalar multiplication.

Vector Addition:

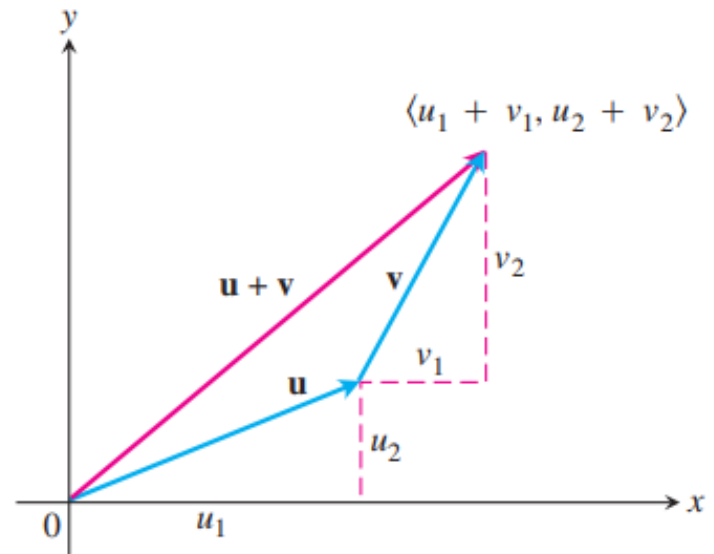
1) Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

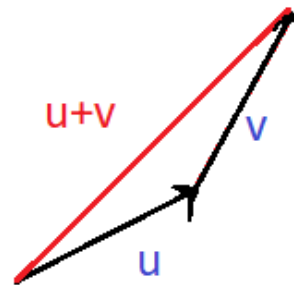
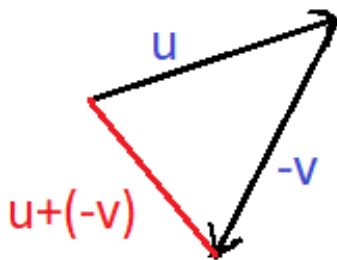
Example:

Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$

Vectors can be added geometrically using head to tail rule.



2) $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$



Question: Find $\vec{u} - \vec{w}$ when $\vec{u} = 3\hat{i} - 4.5\hat{j}$ and $\vec{w} = -2.8\hat{i} + 9\hat{k}$. Also compute the magnitude of the resultant vector.

Scalar Multiplication of Vectors

A scalar is simply a real number, it can be positive, negative or zero.

Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$

then $k\vec{v} = k \langle v_1, v_2, v_3 \rangle = \langle kv_1, kv_2, kv_3 \rangle$ where k is any scalar.

Note: If $k > 0$ then $k\vec{v}$ has the same direction as \vec{v} ; if $k < 0$ then the direction of $k\vec{v}$ is opposite to that of \vec{v} .

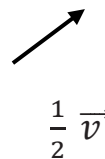
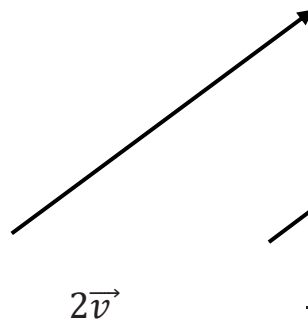
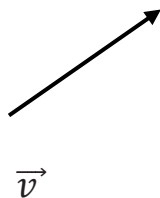
Examples:

If $\vec{v} = \langle 1, 2, 3 \rangle$,

Then,

- $2\vec{v} = \langle 2, 4, 6 \rangle$

- $\frac{1}{2}\vec{v} = \langle \frac{1}{2}, \frac{2}{2}, \frac{3}{2} \rangle$

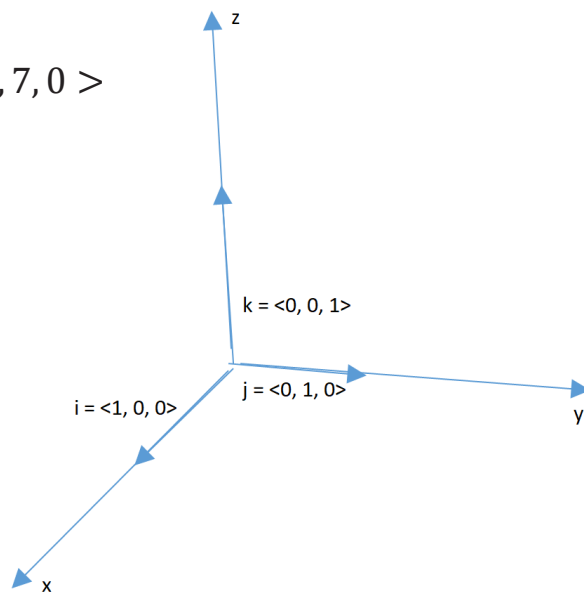


Question: Let $\vec{u} = \langle -1, 3, 1 \rangle$, and $\vec{v} = \langle 4, 7, 0 \rangle$

a. Find $2\vec{u} + 3\vec{v}$

b. $\vec{u} - \vec{v}$

c. $|\frac{1}{2}\vec{v}|$



Unit Vector

A vector \vec{v} of length 1 is called unit vector.

It is represented by \hat{v} .

- The standard unit vectors in \mathbf{R}^3 (space) are

$$\hat{i} = \langle 1, 0, 0 \rangle, \quad \hat{j} = \langle 0, 1, 0 \rangle, \quad \hat{k} = \langle 0, 0, 1 \rangle$$

- Any vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a linear combination of the standard unit vectors as follows:

$$\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle$$

$$\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$$

Example:

$$\vec{v} = \langle 2, 4, 6 \rangle = 2 \langle 1, 0, 0 \rangle + 4 \langle 0, 1, 0 \rangle + 6 \langle 0, 0, 1 \rangle$$

$$= \langle 2, 0, 0 \rangle + \langle 0, 4, 0 \rangle + \langle 0, 0, 6 \rangle$$

$$= \langle 2, 4, 6 \rangle$$

$$\vec{v} = 2\hat{i} + 4\hat{j} + 6\hat{k}.$$

Remark:

If a vector is not unit vector, then we can make it unit vector by dividing it with magnitude of vector. i.e.,

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

Example:

If $\vec{v} = \langle 1, 2, 3 \rangle$, then

$$|\vec{v}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \neq 1$$

Vector \vec{v} is not unit vector because its magnitude is not 1.

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1, 2, 3 \rangle}{\sqrt{14}} = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$$

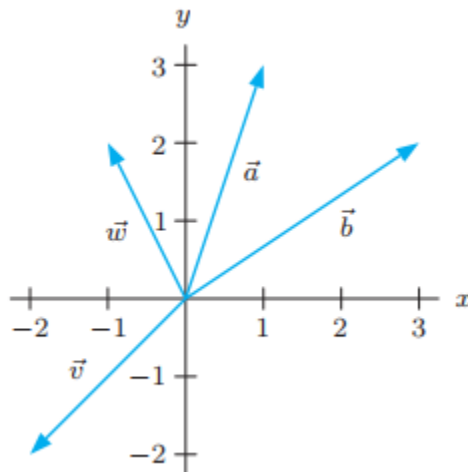
Now it is unit vector in the direction of vector \vec{v} .

Practice Problems

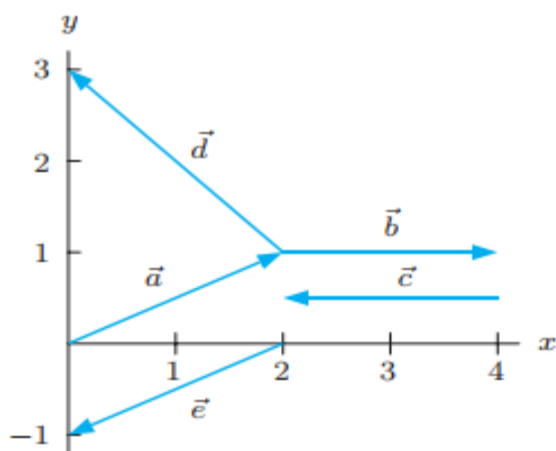
Question 1: Find the unit vector in the opposite direction to $\hat{i} - \hat{j} + \hat{k}$.

Question 2: Resolve the vectors into components:

1.



2.



3. A vector starting at the point $Q = (4, 6)$ and ending at the point $P = (1, 2)$.
4. A vector starting at the point $P = (1, 2)$ and ending at the point $Q = (4, 6)$.

Thomas Calculus; 12th Edition

Ex. 12.2: 1-9,11,12,17-22

Vectors in the Plane

In Exercises 1–8, let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -2, 5 \rangle$. Find the (a) component form and (b) magnitude (length) of the vector.

1. $3\mathbf{u}$

2. $-2\mathbf{v}$

3. $\mathbf{u} + \mathbf{v}$

4. $\mathbf{u} - \mathbf{v}$

5. $2\mathbf{u} - 3\mathbf{v}$

6. $-2\mathbf{u} + 5\mathbf{v}$

7. $\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v}$

8. $-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v}$

9. The vector \overrightarrow{PQ} , where $P = (1, 3)$ and $Q = (2, -1)$

11. The vector from the point $A = (2, 3)$ to the origin

12. The sum of \overrightarrow{AB} and \overrightarrow{CD} , where $A = (1, -1)$, $B = (2, 0)$, $C = (-1, 3)$, and $D = (-2, 2)$

Vectors in Space

In Exercises 17–22, express each vector in the form $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

17. $\overrightarrow{P_1P_2}$ if P_1 is the point $(5, 7, -1)$ and P_2 is the point $(2, 9, -2)$
18. $\overrightarrow{P_1P_2}$ if P_1 is the point $(1, 2, 0)$ and P_2 is the point $(-3, 0, 5)$
19. \overrightarrow{AB} if A is the point $(-7, -8, 1)$ and B is the point $(-10, 8, 1)$
20. \overrightarrow{AB} if A is the point $(1, 0, 3)$ and B is the point $(-1, 4, 5)$
21. $5\mathbf{u} - \mathbf{v}$ if $\mathbf{u} = \langle 1, 1, -1 \rangle$ and $\mathbf{v} = \langle 2, 0, 3 \rangle$
22. $-2\mathbf{u} + 3\mathbf{v}$ if $\mathbf{u} = \langle -1, 0, 2 \rangle$ and $\mathbf{v} = \langle 1, 1, 1 \rangle$

Dot Product of two vectors

We have seen how to add vectors. Can we multiply two vectors together?

Product of the vectors are of two types. A vector has both magnitude and direction, and based on this, the two products of vectors are,

- the dot product of two vectors, and
- the cross product of two vectors.

The dot product of two vectors is also referred to as scalar product, as the resultant value is a scalar quantity.

The cross product is called the vector product as the result is a vector, which is perpendicular to these two vectors.

The Scalar Product (Dot Product) → Produces Scalar

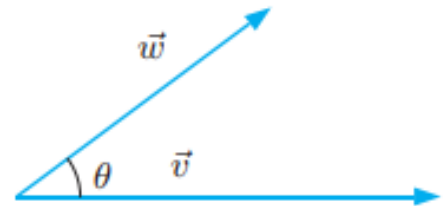
The Vector Product (Cross Product) → Produces Vector

Dot Product (Scalar Product)

Let $\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$ and $\vec{w} = w_1 \hat{i} + w_2 \hat{j} + w_3 \hat{k}$

Then the dot product of \vec{v} and \vec{w} can be defined in two ways:

- Dot Product of two vectors is equal to the product of the magnitudes of the two vectors, and the cosine of the angle between these two vectors.



Note: The dot product of two vectors is a scalar number, not a vector.

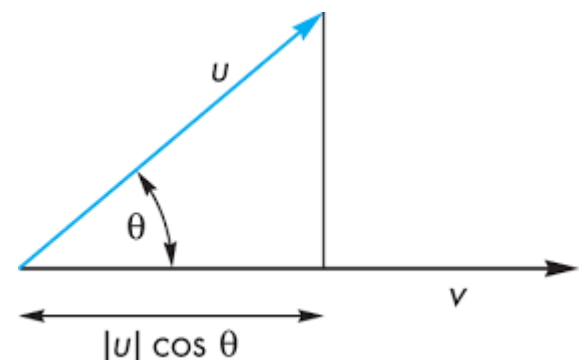
- **Geometric definition**

$$\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \theta$$

where θ is the angle between \vec{v} and \vec{w} and $0 \leq \theta \leq \pi$.

- **Algebraic definition**

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$



Example 1 Suppose $\vec{v} = \vec{i}$ and $\vec{w} = 2\vec{i} + 2\vec{j}$. Compute $\vec{v} \cdot \vec{w}$ both geometrically and algebraically.

Solution To use the geometric definition, see Figure 13.27. The angle between the vectors is $\pi/4$, or 45° , and the lengths of the vectors are given by

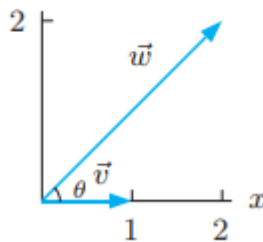
$$\|\vec{v}\| = 1 \quad \text{and} \quad \|\vec{w}\| = 2\sqrt{2}.$$

Thus,

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta = 1 \cdot 2\sqrt{2} \cos\left(\frac{\pi}{4}\right) = 2.$$

Using the algebraic definition, we get the same result:

$$\vec{v} \cdot \vec{w} = 1 \cdot 2 + 0 \cdot 2 = 2.$$



Zero Vectors

The Vectors that have **0** magnitude are called **zero vectors**, denoted by

$$\vec{0} = \langle 0, 0, 0 \rangle$$

The zero vector has **zero magnitude** and **no direction**. It is also called the **additive identity** of vectors.

Orthogonal Vectors:

Two vectors \vec{v} and \vec{w} are orthogonal if they are perpendicular, i.e., they form a right angle with each other, or the dot product of two vectors is equal to zero.

Two non-zero vectors \vec{v} and \vec{w} are perpendicular, or orthogonal, if and only if

$$\vec{v} \cdot \vec{w} = 0$$

For example:

$\hat{i} \cdot \hat{j} = 0$, means that unit vector \hat{i} and \hat{j} are orthogonal vectors

$\hat{j} \cdot \hat{k} = 0$, means that unit vector \hat{j} and \hat{k} are orthogonal vectors

$\hat{i} \cdot \hat{k} = 0$, means that unit vector \hat{i} and \hat{k} are orthogonal vectors

Remarks:

1) If we take the dot product of a vector with itself, then $\theta = 0$, and

$$\cos \theta = \cos(0) = 1$$

For example:

$$\hat{i} \cdot \hat{i} = 1, \quad \hat{j} \cdot \hat{j} = 1, \quad \hat{k} \cdot \hat{k} = 1$$

2) Magnitude and dot product are related as follows:

$$\vec{v} \cdot \vec{v} = |\vec{v}|^2$$

i.e., the dot product of the vector with itself is equal to the square of its magnitude.

Example 1: Which pairs from the following list of 3-dimensional vectors are perpendicular to one another?

$$\vec{u} = \hat{i} + 3\hat{k}, \quad \vec{v} = \hat{i} + 3\hat{j}, \quad \vec{w} = 3\hat{i} + \hat{j} - \hat{k}.$$

Solution: The geometric definition tells us that two vectors are perpendicular if and only if their dot product is zero. Since the vectors are given in components, we calculate dot products using the algebraic definition:

$$\vec{u} \cdot \vec{v} = (\hat{i} + 3\hat{j} + 0\hat{k}) \cdot (\hat{i} + 0\hat{j} + 3\hat{k}) = 1 \cdot 1 + 3 \cdot 0 + 0 \cdot 3 = 1 \neq 0$$

$$\vec{v} \cdot \vec{w} = (\hat{i} + 3\hat{j} + 0\hat{k}) \cdot (3\hat{i} + \hat{j} - \hat{k}) = 1 \cdot 3 + 3 \cdot 1 + 0 \cdot (-1) = 6 \neq 0$$

$$\vec{u} \cdot \vec{w} = (\hat{i} + 0\hat{j} + 3\hat{k}) \cdot (3\hat{i} + \hat{j} - \hat{k}) = 1 \cdot 3 + 0 \cdot 1 + 3 \cdot (-1) = 0$$

So, the only two vectors that are perpendicular (orthogonal) are \vec{u} and \vec{w} because

$$\vec{u} \cdot \vec{w} = 0$$

For concept revision check the following links

- <https://www.cuemath.com/algebra/dot-product/>
- <https://www.cuemath.com/algebra/product-of-vectors/>

Practice Problems

Question: Find the cross product of the following vectors

1) $\vec{u} = 2\hat{i} - \hat{j} - \hat{k}$

$$\vec{v} = \hat{i} + 2\hat{j} - \hat{k}$$

2) $\vec{u} = -3\hat{i} + 5\hat{j} + 4\hat{k}$

$$\vec{v} = \hat{i} - 3\hat{j} - \hat{k}$$

3) $\vec{u} = 2\hat{i} - \hat{j} - \hat{k}$

$$\vec{v} = -6\hat{i} + 3\hat{j} + 3\hat{k}$$