SOLUTIONS OF DIRECT METHODS

1.If original matrix does not have a lot of nonzero elements we can choose Direct Methods because after we apply the Direct methods precisely, we will have exact result. Also, while iterative do the same operation again and again in order to give the result Direct Methods can do that all at once.

2.

In my case ab = 18 thus our system is:

$$2x_1 + 3x_2 + x_3 = 0$$

$$20x_1 + 18x_2 + x_3 = 3$$

$$2x_1 + 18x_2 + x_3 = -16$$

Let's write our system in a matrix form:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 20 & 18 & 1 \\ 1 & 18 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 3 \\ -16 \end{bmatrix} \quad x = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \end{bmatrix}$$

Let's firstly find upper triangular matrix, in order to that firstly,we should make A_{21} to 0. To do that, we can perform this operation : $R_2 - 10R_1$ than, our matrix will become:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -12 & -9 \\ 1 & 18 & 1 \end{bmatrix}$$

Secondly,we should make A_{31} to 0. To do that, we can perform this operation : $R_3 - \frac{1}{2}R_1$ than, our matrix will become:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -12 & -9 \\ 0 & \frac{33}{2} & \frac{1}{2} \end{bmatrix}$$

Lastly,we should make A_{32} to 0. To do that, we can perform this operation : $R_3 + \frac{11}{8}R_2$ than, our matrix will become:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -12 & -9 \\ 0 & 0 & -\frac{95}{8} \end{bmatrix}$$

That's our upper triangular matrix(U), finding lower triangular matrix is easy,we just modify the identity matrix with the inverse of our coefficients in the subtraction we did in the previous stages. These are the values of our L matrix.Thus, L_{21} will be the 10, L_{31} will be the $\frac{1}{2}$ and lastly, L_{32} will be the 10, $-\frac{11}{8}$. As a result, our L and U vector is:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 10 & 1 & 0 \\ \frac{1}{2} & -\frac{11}{8} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -12 & -9 \\ 0 & 0 & -\frac{95}{8} \end{bmatrix}$$

Our system solution is Ax=b. Since, A=LU we can get this equation LUx=b. If we say Ux = y then we can divide LUx = b to the two subsystem Ly=b, Ux=y.

Let's begin with Ly = b, (y is 3x1 matrices with unknown values)

$$L.y = \begin{bmatrix} 1 & 0 & 0 \\ 10 & 1 & 0 \\ \frac{1}{2} & -\frac{11}{8} & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -16 \end{bmatrix} (b)$$

Let make calculations step by step:

Firstly, let multiply L with y matrix end find y matrix by forward substitution:

$y_1.1 + y_2.0 + y_3.0 = 0 \quad \Rightarrow \quad \text{y}_1 = 0$

$y_1.10 + y_2.1 + y_3.0 = 3 \quad \Rightarrow \quad 0.10\text{+y}_2.1 = 3,\ y_2 = 3$

$y_1.\frac{1}{2} + y_2.-\frac{11}{8} + y_3.1 = 3 \quad \Rightarrow \quad 0.\frac{1}{2} + 3.-\frac{11}{8} + y_3.1 = \text{-16} \quad \Rightarrow \quad \text{y}_3 = \text{-16} + \frac{33}{8}, \quad y_3 = -\frac{95}{8}$

Thus, our y matrix is:

$$y = \begin{bmatrix} 0 \\ 3 \\ -\frac{95}{8} \end{bmatrix}$$

We calculated y matrix, now time to calculate our x matrix,currently our equation is:

$$U.x = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -12 & -9 \\ 0 & 0 & -\frac{95}{8} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -\frac{95}{8} \end{bmatrix} (y)$$

Let make calculations step by steps:

Let multiply U with x matrix and find x matrix by backward substitution substitution:

$x_1.0 + x_2.0 + x_3.-\frac{95}{8} = -\frac{95}{8} \quad \Rightarrow \quad \text{x}_3.-\frac{95}{8} = -\frac{95}{8} \quad \Rightarrow \quad \text{x}_3 = 1$

$x_1.0 + x_2.\text{-12} + x_3.\text{-9} = 3 \quad \Rightarrow \quad \text{x}_2.\text{-12} + 1.\text{-9} = 3 \quad \Rightarrow \quad -12.\text{x}_2 = 12\ ,\ x_2 = \text{-1}$

$x_1.2 + x_2.3 + x_3.1 = 0 \quad \Rightarrow \quad 2.\text{x}_1\text{+ } 3.\text{-1} + 1.1 = 0 \quad \Rightarrow \quad \text{x}_1.2 = 2\ ,\ x_1 = 1$

As a result, we solve our equation using LU method, our solution(x) matrix is:

$$x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

3.Pseudo Code

Properties of matrix that I used:

Name:Trefethen$_{20}$
Matrix ID:2204
Number of Rows x Cols = 20 x 20
Nonzeros:158
Kind:Combinatorial Problem
Symmetric:Yes
Date:2008
Author:N.Trefethen

LUFactorization(B):
1. dimension = len(B)
2. create U matrix that is copy of B
3. create identity matrix L that has equal dimensions to B
4. for k in range of dimension -1:
5.   for i in range of k+1 and dimension:
6.   ratio = U[i,k] / U[k,k]
7.   for j in range of dimension:
8.   U[i, j] = U[i, j] - ratio*U[k, j]
9.   L[i, k] = ratio
10. return L, U

ULFactorization(A):
1. create identity matrix P
2. make identity matrix to permutation matrix by swapping
3. create matrix B which is equal to P*A*P
4. apply LU factorization to B matrix and store it's L and U matrices as bL and bU
5. U matrix of A = P*(bL)*P^T
6. L matrix of A = P*(bA)*P^T
7. return P,U,L

Explanation of Algorithm
At the beginning we create permutation matrix, using this permutation matrix and A we create matrix B which is orthogonally similar to A. Then, using LU function we get L and U vector of B.After that, in order to get L vector of A we use P*(bU)*P^T formula, same way in order to get U vector of A we use P*(bL)*P^T. Finally, we solve system using U and L matrices.

Comparing algorithm result with scipy result
Since direct methods, directly give the result if we do pricise calculations, algorithm result and scipy result are identically the same.

SOLUTIONS OF ITERATIVE METHODS

1.If our data set consist of a lot of zero elements than applying direct method is not logically.For such case, iterative methods perform better. Also, in some system we may not need exact result, if approximate results are also works for us we can again use iterative methods.

2.

2.1Jacobi Method

ab = 18 thus my system is:

$$2x_1 + 3x_2 + x_3 = 0$$
$$20x_1 + 18x_2 + x_3 = 3$$
$$x_1 + 18x_2 + x_3 = -16$$

Initial guess is : $\mathrm{P}_0 = (x_1^0, x_2^0, x_3^0) = (0, 0, -7)$

Firstly, let's rearrange the system above such that the coefficient matrix is strictly diagonally dominant:

$$20x_1 + 18x_2 + x_3 = 3$$
$$x_1 + 18x_2 + x_3 = -16$$
$$2x_1 + 3x_2 + x_3 = 0$$

We can write these equation in this form:

$$x_1 = \frac{3 - 18x_2 - x_3}{20}$$

$$x_2 = \frac{-16 - x_1 - x_3}{18}$$

$$x_3 = -2x_1 - 3x_2$$

Iteration 1 (using $\mathrm{P}_0(0, 0, -7)$) :

$x_1 = \frac{3-18x_2-x_3}{20} \quad \Rightarrow \quad \mathrm{x}_1 = \frac{3-18.0+7}{20} \quad \Rightarrow \quad \mathrm{x}_1 = 0{,}5$

$x_2 = \frac{-16-x_1-x_3}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \frac{-16-0+7}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \text{-}0{,}5$

$x_3 = -2x_1 - 3x_2 \quad \Rightarrow \quad \mathrm{x}_3 = -2.0 - 3.0 \quad \Rightarrow \quad \mathrm{x}_3 = 0$

$\mathrm{P}_1 = (0.5, -0.5, 0)$

Iteration 2 (using $\mathrm{P}_1(0.5, -0.5, 0)$) :

$x_1 = \frac{3-18x_2-x_3}{20} \quad \Rightarrow \quad \mathrm{x}_1 = \frac{3+18.(0{,}5)-0}{20} \quad \Rightarrow \quad \mathrm{x}_1 = 0{,}6$

$x_2 = \frac{-16-x_1-x_3}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \frac{-16-0{,}5-0}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \text{-}0{,}91$

$x_3 = -2x_1 - 3x_2 \quad \Rightarrow \quad \mathrm{x}_3 = -2.(0, 5) + 3.(0, 5) \quad \Rightarrow \quad \mathrm{x}_3 = 0{,}5$

$\mathrm{P}_2 = (0.6, -0.91, 0, 5)$

Iteration 3 (using $\mathrm{P}_2(0.6, -0.91, 0, 5)$) :

$x_1 = \frac{3-18x_2-x_3}{20} \quad \Rightarrow \quad \mathrm{x}_1 = \frac{3+18.(0{,}91)-0{,}5}{20} \quad \Rightarrow \quad \mathrm{x}_1 = 0{,}94$

$x_2 = \frac{-16-x_1-x_3}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \frac{-16-0{,}6-0{,}5}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \text{-}0{,}95$

$x_3 = -2x_1 - 3x_2 \quad \Rightarrow \quad \mathrm{x}_3 = -2.(0, 6) + 3.(0, 91) \quad \Rightarrow \quad \mathrm{x}_3 = 1{,}53$

$\mathrm{P}_3 = (0.94, -0.95, -1, 53)$

Iteration 4 (using $\mathrm{P}_3(0.94, -0.95, -1, 53)$) :

$x_1 = \frac{3-18x_2-x_3}{20} \quad \Rightarrow \quad \mathrm{x}_1 = \frac{3+18.(0{,}95)-1{,}53}{20} \quad \Rightarrow \quad \mathrm{x}_1 = 1{,}03$

$x_2 = \frac{-16-x_1-x_3}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \frac{-16-0{,}94-1{,}53}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \text{-}1{,}02$

$x_3 = -2x_1 - 3x_2 \quad \Rightarrow \quad \mathrm{x}_3 = -2.(0, 94) + 3.(0, 95) \quad \Rightarrow \quad \mathrm{x}_3 = 0{,}97$

$\mathrm{P}_4 = (1.03, -1.02, 0, 97)$

2.2 **Gauss-Seidel Method**

ab = 18 thus my system is:

$$2x_1 + 3x_2 + x_3 = 0$$
$$20x_1 + 18x_2 + x_3 = 3$$
$$x_1 + 18x_2 + x_3 = -16$$

Initial guess is : $\mathrm{P}_0 = (x_1^0, x_2^0, x_3^0) = (0, 0, -7)$

Firstly, let's rearrange the system above such that the coefficient matrix is strictly diagonally dominant:

$$20x_1 + 18x_2 + x_3 = 3$$

$$x_1 + 18x_2 + x_3 = -16$$

$$2x_1 + 3x_2 + x_3 = 0$$

We can write these equation in this form:

$$x_1 = \frac{3 - 18x_2 - x_3}{20}$$

$$x_2 = \frac{-16 - x_1 - x_3}{18}$$

$$x_3 = -2x_1 - 3x_2$$

Iteration 1 (using $\mathrm{P}_0(0, 0, -7)$) :

$x_1 = \frac{3-18x_2-x_3}{20} \quad \Rightarrow \quad \mathrm{x}_1 = \frac{3-18.0+7}{20} \quad \Rightarrow \quad \mathrm{x}_1 = 0{,}5$

$x_2 = \frac{-16-x_1-x_3}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \frac{-16-0{,}5+7}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \text{-}0{,}52$

$x_3 = -2x_1 - 3x_2 \quad \Rightarrow \quad \mathrm{x}_3 = -2.(0,5) - 3.(0,52) \quad \Rightarrow \quad \mathrm{x}_3 = 0{,}59$

$\mathrm{P}_1 = (0.5, -0.52, 0.59)$

Iteration 2 (using $\mathrm{P}_1(0.5, -0.52, 0.59)$) :

$x_1 = \frac{3-18x_2-x_3}{20} \quad \Rightarrow \quad \mathrm{x}_1 = \frac{3+18.(0{,}52)-0{,}59}{20} \quad \Rightarrow \quad \mathrm{x}_1 = 0{,}58$

$x_2 = \frac{-16-x_1-x_3}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \frac{-16-0{,}58-0{,}59}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \text{-}0{,}95$

$x_3 = -2x_1 - 3x_2 \quad \Rightarrow \quad \mathrm{x}_3 = -2.(0,58) + 3.(0,95) \quad \Rightarrow \quad \mathrm{x}_3 = 1{,}69$

$\mathrm{P}_2 = (0.58, -0.95, 1, 69)$

Iteration 3 (using $\mathrm{P}_2(0.58, -0.95, 1, 69)$) :

$x_1 = \frac{3-18x_2-x_3}{20} \quad \Rightarrow \quad \mathrm{x}_1 = \frac{3+18.(0{,}95)-1{,}69}{20} \quad \Rightarrow \quad \mathrm{x}_1 = 0{,}92$

$x_2 = \frac{-16-x_1-x_3}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \frac{-16-0{,}92-1{,}69}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \text{-}1{,}03$

$x_3 = -2x_1 - 3x_2 \quad \Rightarrow \quad \mathrm{x}_3 = -2.(0,92) + 3.(1,03) \quad \Rightarrow \quad \mathrm{x}_3 = 1{,}25$

$\mathrm{P}_3 = (0.92, -1.03, 1, 25)$

Iteration 4 (using $\mathrm{P}_3(0.92, -1.03, 1, 25)$) :

$x_1 = \frac{3-18x_2-x_3}{20} \quad \Rightarrow \quad \mathrm{x}_1 = \frac{3+18.(1{,}03)-1{,}25}{20} \quad \Rightarrow \quad \mathrm{x}_1 = 1{,}01$

$x_2 = \frac{-16-x_1-x_3}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \frac{-16-1{,}01-1{,}25}{18} \quad \Rightarrow \quad \mathrm{x}_2 = \text{-}1{,}01$

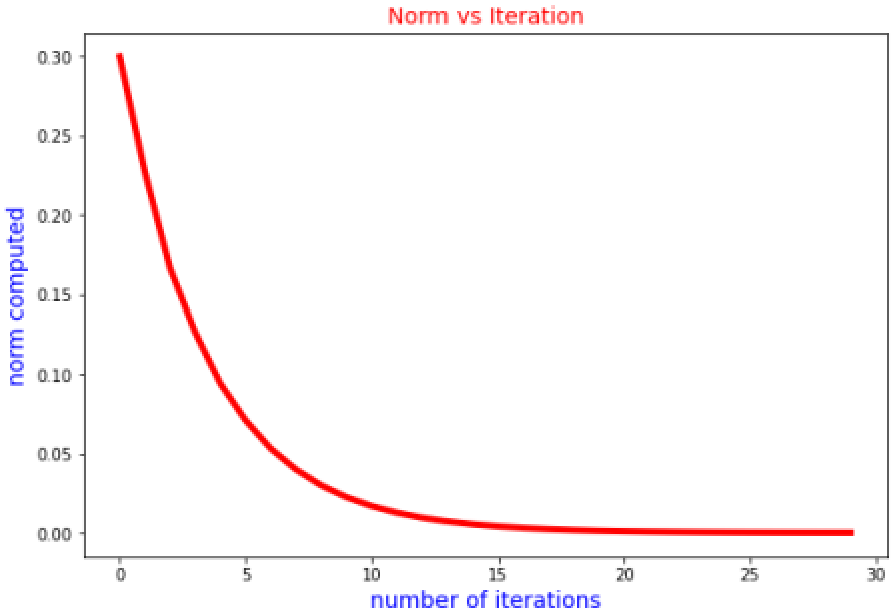
$x_3 = -2x_1 - 3x_2 \quad \Rightarrow \quad \mathrm{x}_3 = -2.(1,01) + 3.(1,01) \quad \Rightarrow \quad \mathrm{x}_3 = 1{,}01$

$\mathrm{P}_4 = (1.01, -1.01, 1, 01)$

Result

As a result, in Jakobian method in every iteration we use previous guess and the first iteration is based on the inital guess.However, in Gauss-Seidel method in every iteration we use the most latest value in order to calculate other approximations. Thus, Gauss-Seidel method give more accurate approximations than Jakobian.

Jakobian Algorithm Graphic of Results



NOTE:

I tried to plot the graph from latex but unfortunately I could not.Thus, I plotted graph my algorithm from Spyder and added it to the report as an image.I am sorry for that.