ALAMEIN INTERNATIONAL UNIVERSITY
جامعة العلمين الدولية
QUALITY ASSURANCE CENTER

Image Processing (CSE281)

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Fourier Series

Any periodic function can be expressed as an infinite series consisting of sine or cosine functions. Thus, $x(t)$ can be expressed as

$$x(t) = a_0 + a_1 \cos \omega_0 t + b_1 \sin \omega_0 t + a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t$$

$$+ a_3 \cos 3\omega_0 t + b_3 \sin 3\omega_0 t + \cdots$$

This can be written as

$$\boxed{x(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}}}$$

where

$$\omega_0 = \frac{2\pi}{T}$$

Fourier Series

- The process of determining the Fourier coefficients ***a*0, *an*,** and ***bn*** is called **Fourier analysis**.

The coefficients can be determined as follows:

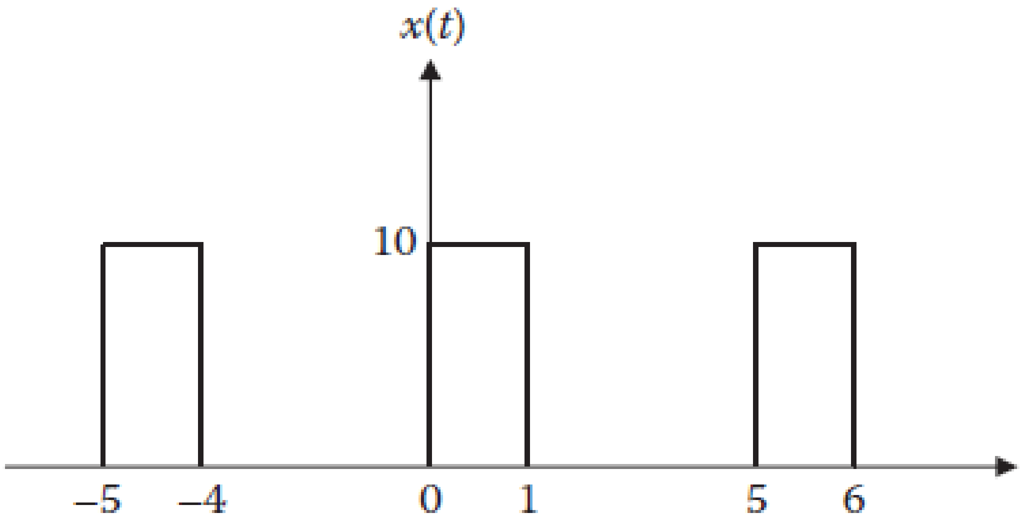
$$\boxed{a_0 = \frac{1}{T}\int_0^T x(t)dt}$$

$$\boxed{a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t \; dt}$$

$$\boxed{b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t \; dt}$$

Fourier Series

Example: ***Find the Fourier Coefficients for the following signal***



Fourier Series

Solution

The signal can be expressed as

$$x(t) = \begin{cases} 10, & 0 < t < 1 \\ 0, & 1 < t < 5 \end{cases}$$

With $T = 5$, $\omega_0 = 2\pi/5$

$$a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{5}\left[\int_0^1 10dt + \int_1^5 0dt\right] = \frac{1}{5}10 = 2$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_o t\ dt = \frac{2}{5}\int_0^1 10\cos n\omega_o t\ dt$$

$$= 4\frac{\sin n\omega_o t}{n\omega_o}\bigg|_0^1 = 4\frac{\sin n\omega_o}{n\omega_o}$$

Fourier Series

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_o t\, dt = \frac{2}{5}\int_0^1 10\sin n\omega_o t\, dt$$

$$= -4\frac{\cos n\omega_o t}{n\omega_o}\Bigg|_0^1 = 4\left[\frac{1-\cos n\omega_o}{n\omega_o}\right]$$

The Fourier series expansion of the signal is

$$x(t) = 2 + 4\sum_{n=1}^{\infty}\left(\frac{\sin n\omega_o}{n\omega_o}\cos n\omega_o t + \left[\frac{1-\cos n\omega_o}{n\omega_o}\right]\sin n\omega_o t\right)$$

Fourier Series

- The ***exponential Fourier series*** of a periodic ***signal x(t)*** is a representation of the complex exponentials at positive and negative harmonic frequencies.

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}}$$

where

$\omega_0 = 2\pi/T$ is the fundamental frequency

the coefficients c_n are given by

$$\boxed{c_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt}$$

Fourier Series

Example

Find the exponential Fourier series for $x(t) = t$, $-1 < t < 1$, with $f(t + 2n) = f(t)$ and n is an integer. Plot the amplitude and phase spectra.

Solution

Since $T = 2$, $\omega_0 = 2\pi/T = \pi$. Hence,

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\omega_0 nt}dt = \frac{1}{2}\int_{-1}^{1} te^{-jn\pi t}dt$$

Fourier Series

Using integration by parts,

$$u = t \rightarrow du = dt$$

$$dv = \frac{1}{2} e^{-jn\pi t} dt \rightarrow v = -\frac{1}{2jn\pi} e^{-jn\pi t}$$

$$c_n = -\frac{t}{2jn\pi} e^{-jn\pi t} \Big|_{-1}^{1} + \frac{1}{2jn\pi} \int_{-1}^{1} e^{-jn\pi t} dt$$

$$= \frac{j}{2n\pi} \left[e^{-jn\pi} + e^{jn\pi} \right] + \frac{1}{2n^2\pi^2(-j)^2} e^{-jn\pi t} \Big|_{-1}^{1}$$

$$= \frac{j}{n\pi} \cos n\pi - \frac{1}{2n^2\pi^2} \left(e^{-jn\pi} - e^{jn\pi} \right)$$

$$c_n = \frac{j(-1)^n}{n\pi} + \frac{2j}{2n^2\pi^2} \sin(n\pi) = \frac{j(-1)^n}{n\pi}, \quad n \neq 0$$

Fourier Series

We treat $n = 0$ as a special case.

$$c_0 = \frac{1}{2}\int_{-1}^{1} te^{-0}dt = \frac{1}{2}\frac{t^2}{2}\bigg|_{-1}^{1} = \frac{1}{4}(1-1) = 0$$

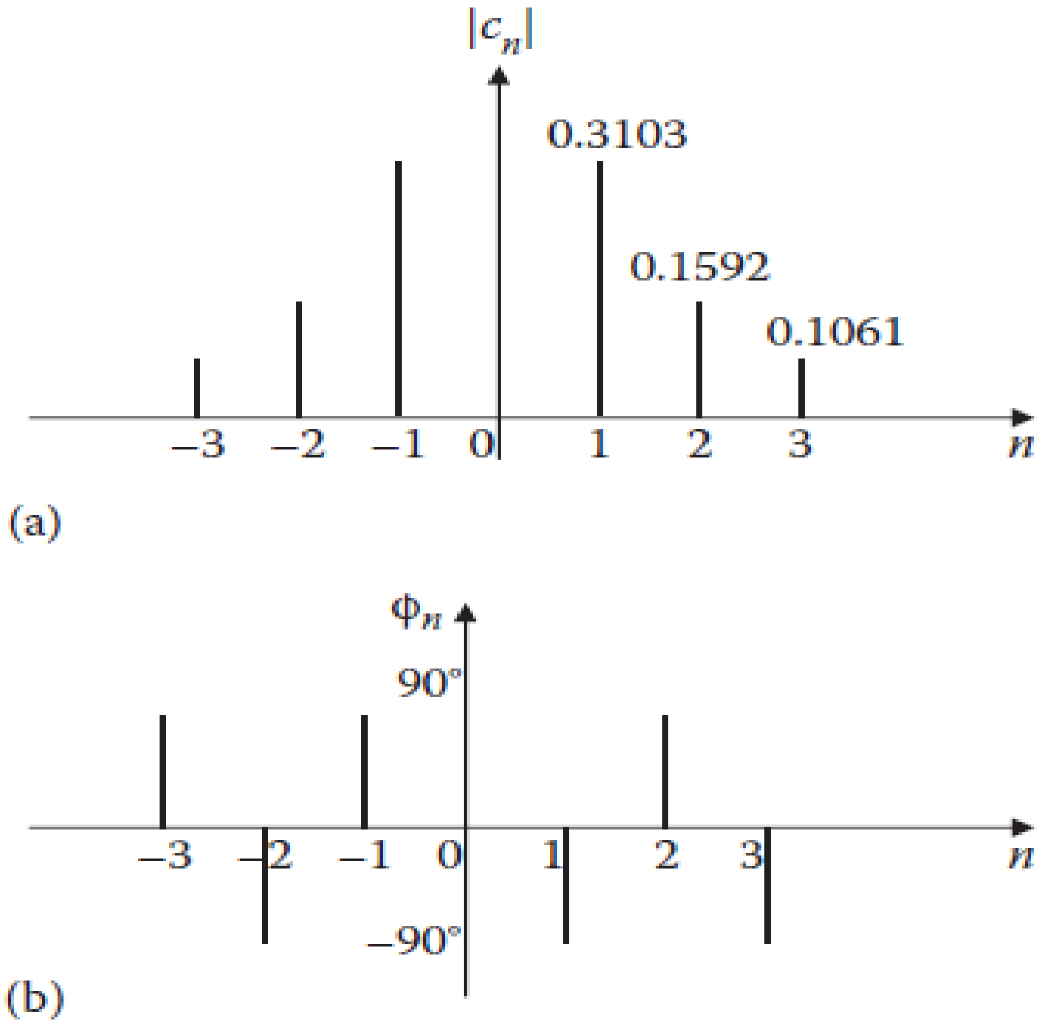
Thus,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n \frac{j}{n\pi} e^{jn\pi t}$$

Fourier Series

N	$\lvert c_n \rvert$	ϕ_n (Degrees)
–3	0.1061	90
–2	0.1592	–90
–1	0.3183	90
0	0	0
1	0.3183	–90
2	0.1592	90
3	0.1061	–90

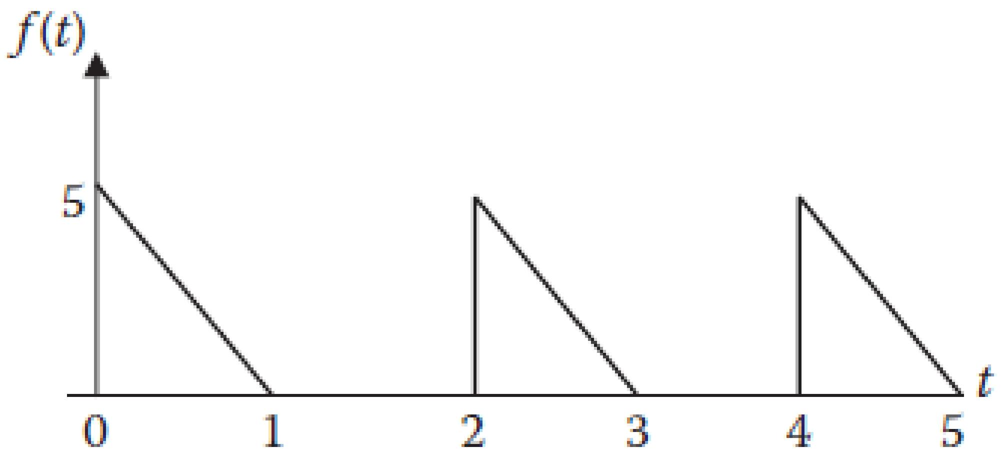
Fourier Series



Fourier Series

Example

Find the complex Fourier series of the signal in Figure



Fourier Series

$$T = 2, \quad \omega_o = 2\pi/T = \pi$$

$$f(t) = \begin{cases} 5(1-t), & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_o t}dt = \frac{1}{2}\int_0^1 5(1-t)e^{-jn\pi t}dt$$

$$= \frac{5}{2}\int_0^1 e^{-jn\pi t}dt - \frac{5}{2}\int_0^1 te^{-jn\pi t}dt = \frac{5}{2}\frac{e^{-jn\pi t}}{-jn\pi}\bigg|_0^1 - \frac{5}{2}\frac{e^{-jn\pi t}}{(-jn\pi)^2}(-jn\pi t - 1)\bigg|_0^1$$

$$= \frac{5}{2}\frac{\left[e^{-jn\pi} - 1\right]}{-jn\pi} - \frac{5}{2}\frac{e^{-jn\pi}}{-n^2\pi^2}(-jn\pi - 1) + \frac{5}{2}\frac{(-1)}{-n^2\pi^2}$$

Fourier Transform

- The ***Fourier transform*** of a signal $\boldsymbol{x(t)}$ is the integration of the product of $\boldsymbol{x(t)}$ and $\boldsymbol{e^{-jwt}}$over the interval from **−∞ to +∞.**

$$\boxed{X(\omega) = \mathcal{F}\left[x(t)\right] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt}$$

Fourier Transform

Example: **Find the Fourier Transform of (a) $\delta(t)$, (b) $e^{-at}u(t)$**

Solution

(a) For the impulse function,

$$X(\omega) = \mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = e^{-j\omega t}\Big|_{t=0} = 1$$

Fourier Transform

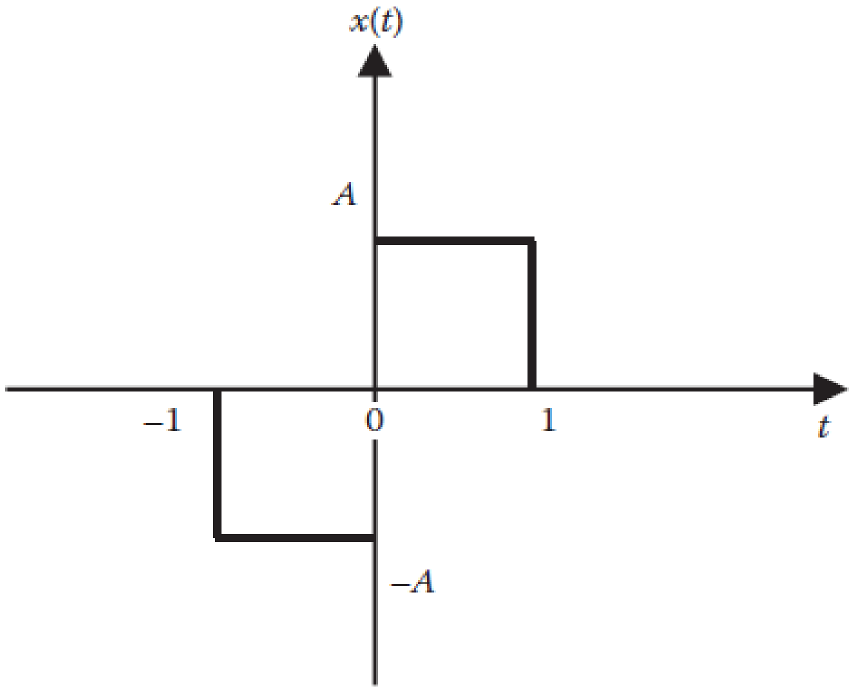
(b) Let $x(t) = e^{-at}u(t) = \begin{cases} e^{-at}, & t > 0 \\ 0, & t < 0 \end{cases}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$\mathcal{F}\left[e^{-at}u(t)\right] = X(\omega) = \frac{-1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty} = \frac{1}{a+j\omega}$$

Fourier Transform

***Example:* Find the *Fourier Transform* for the following signal**



Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-1}^{0}(-A)e^{-j\omega t}dt + \int_{0}^{1} Ae^{-j\omega t}dt$$

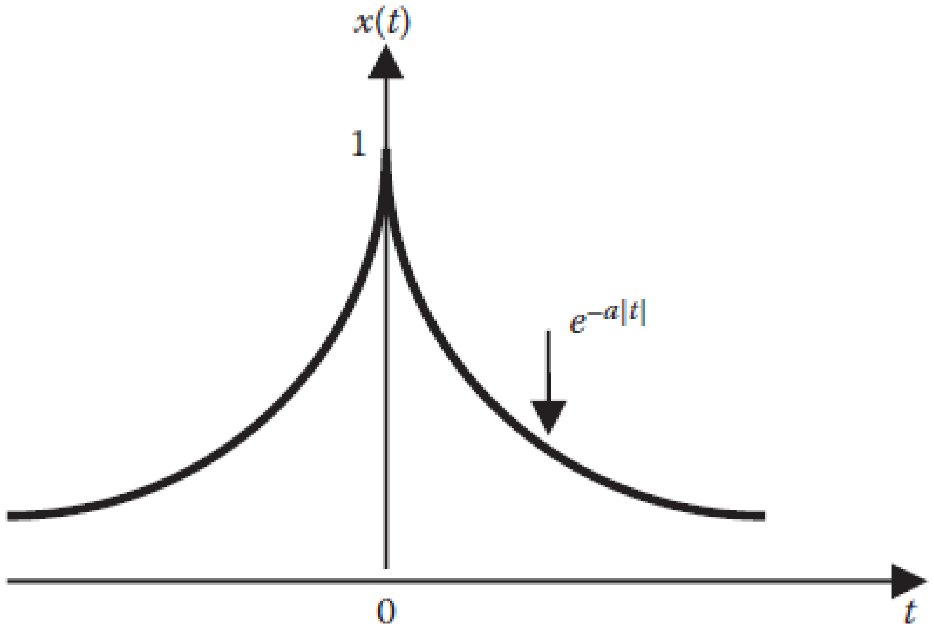
$$= \frac{A}{j\omega}e^{-j\omega t}\Big|_{-1}^{0} - \frac{A}{j\omega}e^{-j\omega t}\Big|_{0}^{1}$$

$$= \frac{-jA}{\omega}\left(1 - e^{j\omega} - e^{-j\omega} + 1\right)$$

$$= \frac{j2A}{\omega}(\cos\omega - 1)$$

Fourier Transform

Example: ***Find the Fourier Transform for the following signal***



Fourier Transform

Solution

Let $x(t) = e^{-a|t|} = \begin{cases} e^{at}, & t < 0 \\ e^{-at}, & t > 0 \end{cases}$

The Fourier transform is

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$

$$= \frac{2a}{a^2 + \omega^2}$$

Fourier Transform

Linearity

The Fourier transform is a linear transform. That is, suppose we have two functions $x_1(t)$ and $x_2(t)$, with Fourier transform given by $X_1(\omega)$ and $X_2(\omega)$, respectively, then the Fourier transform of $x_1(t)$ and $x_2(t)$ can be easily found as

$$\boxed{\mathcal{F}\left[a_1x_1(t)+a_2x_2(t)\right]=a_1X_1(\omega)+a_2X_2(\omega)}$$

Fourier Transform

Time Scaling

The scaling property of the Fourier transformation is as follows. If $\mathcal{F}(\omega)=\left[x(t)\right]$ and a is a real constant. Then

$$\boxed{\mathcal{F}\left[x(at)\right]=\frac{1}{|a|}X\left(\frac{\omega}{a}\right)}$$

Fourier Transform

Time Shifting

Another simple property of the Fourier transform is the time shifting. If $X(\omega)=\mathcal{F}\left[x(t)\right]$ and t_0 is a constant, then

$$\boxed{\mathcal{F}\left[x(t-t_0)\right]=e^{-j\omega t_0}X(\omega)}$$

Fourier Transform

FREQUENCY SHIFTING

This property forms a basis for every radio and TV transmitter. This property states that if $X(\omega) = \mathcal{F}\left[x(t)\right]$ and ω_0 is constant, then

$$\boxed{\mathcal{F}\left[x(t)e^{j\omega_0 t}\right] = X(\omega - \omega_0)}$$

Fourier Transform

Time Differentiation

If $X(\omega) = \mathcal{F}\left[x(t)\right]$, then the Fourier transform of the derivative of $x(t)$ is given by

$$\boxed{\mathcal{F}\left[x'(t)\right] = j\omega X(\omega)}$$

Fourier Transform

Frequency Differentiation

This property states that if $X(\omega) = \mathcal{F}\left[x(t)\right]$ then

$$\mathcal{F}\left[(-jt)^n x(t)\right] = \frac{d^n}{d\omega^n} X(\omega)$$

Fourier Transform

Example

A signal $x(t)$ has a Fourier transform given by

$$X(\omega) = \frac{5(1 + j\omega)}{8 - \omega^2 + 6j\omega}$$

Without finding $x(t)$, find the Fourier transform of the following:

(a) $x(t - 3)$
(b) $x(4t)$
(c) $e^{-j2t}x(t)$
(d) $x(-2t)$

Fourier Transform

We apply the relevant property for each case:

$$\text{(a)}\quad \mathcal{F}[x(t-3)] = e^{-j\omega 3}X(\omega) = \frac{5(1+j\omega)e^{-j\omega 3}}{8-\omega^2+j6\omega}$$

$$\text{(b)}\quad \mathcal{F}[x(4t)] = \frac{1}{4}X\left(\frac{\omega}{4}\right) = \frac{\frac{5}{4}(1+j\omega/4)}{8-\omega^2/16+j6\omega/4} = \frac{5(4+j\omega)}{128-\omega^2+j24\omega}$$

$$\text{(c)}\quad \mathcal{F}[e^{-j2t}x(t)] = X(\omega+2) = \frac{5\left[1+j(\omega+2)\right]}{8-(\omega+2)^2+6j(\omega+2)} = \frac{5(1+j\omega+j2)}{4-\omega^2-4\omega+6j\omega+j12}$$

$$\text{(d)}\quad \mathcal{F}\left[x(-2t)\right] = \frac{1}{2}X\left(\frac{\omega}{-2}\right) = \frac{\frac{5}{2}(1-j\omega/2)}{8-\frac{\omega^2}{4}-\frac{6j\omega}{2}} = \frac{5(2-j\omega)}{32-\omega^2-12j\omega}$$

Fourier Transform

Example: ***Find the Inverse Fourier Transform of***

$$G(\omega) = \frac{10\,j\omega}{(-j\omega + 2)(j\omega + 3)}$$

Solution

To avoid complex algebra, let $s = j\omega$. Using partial fraction,

$$G(s) = \frac{10s}{(2-s)(3+s)} = \frac{-10s}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}, \quad s = j\omega$$

$$A = (s-2)G(s)\Big|_{s=2} = \frac{-10(2)}{2+3} = -4$$

$$B = (s+3)G(s)\Big|_{s=-3} = \frac{-10(-3)}{-3-2} = -6$$

$$G(\omega) = \frac{-4}{j\omega - 2} - \frac{6}{j\omega + 3}$$

Taking the inverse Fourier transform of each term,

$$g(t) = -4e^{2t}u(-t) - 6e^{-3t}u(t)$$