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ENTROPY ESTIMATORS – IMPROVEMENTS AND COMPARISONS

Robert Wieczorkowski *, Przemysław Grzegorzewski * ◇

*Institute of Mathematics, Warsaw University of Technology,

Plac Politechniki 1, 00-661 Warsaw, Poland

◇Systems Research Institute, Polish Academy of Sciences,

ul. Newelska 6, 01-447 Warsaw, Poland

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ABSTRACT

The problem of the entropy estimation is discussed. Modifications of the well-known Vasicek's estimator and Correa's estimator are considered. An extensive simulation study for the comparison of all estimators under study is performed. It appears that the new estimators based on the bias correction and Jackknife behave better than the traditional ones.

INTRODUCTION

The entropy of a random variable was introduced by C. E. Shannon (1948) as a measure of information and uncertainty. Now the entropy is a charac-

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teristic playing a fundamental role not only in information theory and communication, but also in classification, pattern recognition, statistical physics, stochastic dynamics, statistics, etc. Just in statistics Shannon's entropy is used as a descriptive parameter (measure of dispersion), for testing normality (Vasicek (1976) and Arizono and Ohta (1989)), exponentiality (Grzegorzewski and Wieczorkowski (1998)) and uniformity (Dudewicz, van der Meulen et al. (1995)). Goodness-of -fit test for uniformity based on the sample entropy are extensively utilized in evaluating random number generators. One also know the maximum entropy principle applicable widely in problems of inference on the basis of incomplete data.

Therefore the entropy estimation seems to be an important problem, crucial for many further applications. However, there are not many papers devoted to the entropy estimation problem. The most prominent estimator of entropy, based on spacings, was proposed by Vasicek. The other entropy estimator is due to van Es (1992). Recently Correa (1995) suggested a new entropy estimator based on the local linear regression.

In this paper we show how to improve both Vasicek and Correa's estimators. We have modified them with help of the bias correction or the simple Jackknife. Then an extensive simulation study was performed. The mean square error and the bias were used as measures of the quality of estimators. It appears that our new estimators behave better than the estimators used before.

VASICEK'S, VAN ES' AND CORREA'S ESTIMATORS

For continuous random variable X with a density function f the entropy is defined as

$$H(X) \equiv H(f) = -\int_{-\infty}^{+\infty} f(x) \ln f(x)\, dx. \tag{1}$$

It may be shown easily that (1) can be expressed as

$$H(f) = \int_0^1 \ln\left(\frac{d}{dp}F^{-1}(p)\right) dp. \tag{2}$$

The best known and widely used entropy estimator was proposed by Vasicek (1976). His estimator was constructed by replacing the distribution function F in (2) by the empirical distribution function and using a difference operator instead of the differential one. The derivative $\frac{d}{dp}F^{-1}(p)$ is then estimated by a function of spacings.

Let $X_1, \dots, X_n$, $n \geq 3$, be a sample from the distribution F. Let $X_{(1)} \leq \dots \leq X_{(n)}$ denote ordered statistics from the sample $X_1, \dots, X_n$. Then Vasicek's estimator of entropy has a following form

$$V_{m,n}(X_1, \dots, X_n) = \frac{1}{n}\sum_{i=1}^{n} \ln\left(\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right), \tag{3}$$

where m is a positive integer smaller than $\frac{n}{2}$, $X_{(i)} = X_{(1)}$ for $i < 1$ and $X_{(i)} = X_{(n)}$ for $i > n$.

Vasicek proved that his estimator is consistent, i.e. $V_{m,n} \to H(f)$ as $n \to \infty$, $m \to \infty$ and $\frac{m}{n} \to 0$. Vasicek's estimator is also asymptotically normal, however Dudewicz, van der Meulen et al. (1995) showed that the convergence is not very fast.

In 1992 van Es suggested a new estimator of a form

$$E_{m,n} = \frac{1}{n-m}\sum_{i=1}^{n-m} \ln\left(\frac{n+1}{m}\left(X_{(i+m)} - X_{(i)}\right)\right) + \sum_{k=m}^{n}\frac{1}{k} + \ln m - \ln(n+1). \tag{4}$$

Van Es proved, under some conditions, consistency and asymptotic normality of this estimator.

In 1995 Correa suggested a modification of Vasicek's estimator. In estimation the density f of F in the interval $(X_{(i-m)}, X_{(i+m)})$ he used a local linear model based on $2m+1$ points: $F(X_{(j)}) = \alpha + \beta X_{(j)} + \varepsilon$, $j = m-i, \dots, m+i$. This yields a following estimator

$$C_{m,n}(X_1, \dots, X_n) = -\frac{1}{n}\sum_{i=1}^{n} \ln(b_i), \tag{5}$$

where

$$b_i = \frac{\sum_{j=i-m}^{i+m}\left(X_{(j)} - \overline{X}_{(i)}\right)(j-i)}{n \cdot \sum_{j=i-m}^{i+m}\left(X_{(j)} - \overline{X}_{(i)}\right)^2}, \tag{6}$$

$$\overline{X}_{(i)} = \frac{1}{2m+1}\sum_{j=i-m}^{i+m} X_{(j)}, \tag{7}$$

m is a positive integer smaller than $\frac{n}{2}$, $X_{(i)} = X_{(1)}$ for $i < 1$ and $X_{(i)} = X_{(n)}$ for $i > n$.

By the simulation study Correa showed that for a few distributions (standard normal, exponential with mean equal to 1 and uniform $U(0,1)$) his estimator produces smaller mean squared error than Vasicek's estimator and van Es' estimator.

ESTIMATION WITH THE BIAS CORRECTION

While proving consistency of the entropy estimator Vasicek indicated how to correct a bias correction of the $V_{m,n}$ statistic. Applying this correction we get another entropy estimator

$$\begin{aligned} V'_{m,n} &= V_{m,n} - \log(n) + \log(2m) - \left(1 - \frac{2m}{n}\right)\Psi(2m) + \Psi(n+1) \qquad (8) \\ &- \frac{2}{n}\sum_{i=1}^{m} \Psi(i+m-1) \end{aligned}$$

where $\Psi(x)$ is the digamma function defined as $\Psi(x) = \frac{d}{dx}\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. For integer arguments holds identity $\Psi(k) = \sum_{i=1}^{k-1} \frac{1}{i} - \gamma$, where γ is Euler's constant, $\gamma = 0.57721566490\ldots$.

It is surprising why neither Vasicek nor his successors haven't use $V'_{m,n}$ in their further research. Therefore we have decided to compare the properties of $V'_{m,n}$, called the Vasicek estimator with the bias correction, with other entropy estimators (including Vasicek's $V_{m,n}$). It is worth noting that a similar bias correction is used in van Es' estimator $E_{m,n}$.

JACKKNIFE ENTROPY ESTIMATION

A special technique, called Jackknife, for the purpose of bias reduction was introduced by Quenouille (1949, 1956). Let $X_1, \ldots, X_n$ be a sample from the distribution F_θ. Suppose we do not know any unbiased estimator of θ, but we know a biased estimator $\widehat{\theta}(X_1, \ldots, X_n)$. The Jackknife estimator of θ is derived from this biased estimator by leaving out one observation at a time and calculating

$$\widehat{\theta}_i = \widehat{\theta}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n). \qquad (9)$$

Denoting

$$\widehat{\theta}_{\Sigma} = \frac{1}{n}\sum_{i=1}^{n}\widehat{\theta}_i, \tag{10}$$

we obtain a bias estimate

$$\widehat{b}(\theta) = (n-1)(\widehat{\theta}_{\Sigma} - \widehat{\theta}) \tag{11}$$

and the Jackknife estimator of θ, corrected for the bias:

$$\widetilde{\theta} = \widehat{\theta} - \widehat{b}(\theta) = \widehat{\theta} - (n-1)(\widehat{\theta}_{\Sigma} - \widehat{\theta}). \tag{12}$$

In 1972 Gray and Schucany suggested, so called, the generalized Jackknife statistic

$$\theta^* = \frac{\widehat{\theta'} - R\widehat{\theta''}}{1-R}, \tag{13}$$

where $\widehat{\theta'}$ and $\widehat{\theta''}$ are biased estimators of θ, usually chosen as $\widehat{\theta'} = \widehat{\theta}$ and $\widehat{\theta''} = \widehat{\theta}_{\Sigma}$. Note that the choice of $R = \frac{n-1}{n}$ yields $\widetilde{\theta}$ as a special case.

Although the Jackknife method focuses only on reducing the bias of an estimator it may serve for variance reduction as well (see, e.g., Gray and Schucany (1972)). For more details concerning the Jackknife technique we refer the reader to Gray and Schucany (1972) and Efron (1982).

We propose to correct Correa's estimator using Jackknife formula (12) with $\widehat{\theta} = C_{m,n}$. We denote this bias corrected estimator by $C'_{m,n}$. We have chosen Correa's estimator since, according to Correa (1995) it dominates both Vasicek's and van Es' estimators.

SIMULATION STUDY

To compare the performance of the entropy estimators, an empirical sampling study was conducted. We used following entropy: Vasicek's $V_{m,n}$ estimator, Vasicek's estimator with the correction of the bias $V'_{m,n}$, van Es' $E_{m,n}$ estimator, Correa's $C_{m,n}$ estimator, and Jackknife corrected Correa's estimator $C'_{m,n}$.

We have chosen the following distributions:

- uniform distribution $U(0,1)$; in this case entropy $H(f) = 0$.

- exponential distributions $Exp(\lambda)$ with density function $f(x) = \frac{1}{\lambda}\exp(-\frac{x}{\lambda})$, $\lambda > 0$, $x > 0$; then we have $H(f) = 1 + \ln(\lambda)$. In our study we have considered $\lambda = 0.5, 1, 2, 5$.

••exponential power family of distributions $EPD(p)$ with density

$$f(x) = \frac{1}{2\Gamma(1+1/p)}\exp(-|x|^p), \quad x \in (-\infty, \infty), \quad p > 0. \tag{14}$$

For discussion of the exponential power family see, e.g., Sherman (1997) and references therein. Varying the parameter p we obtain symmetric densities with different tail behavior: the density has very heavy tails as $p \to 0$, and very light tails as $p \to \infty$ ($f(x)$ converges then to uniform distribution $U(-1,1)$). It worth noting that for $p = 2$ we get the normal distribution, namely $EPD(2) = N(0, \frac{1}{2})$, while for $p = 1$ we get the double exponential distribution. Exponential power family is interesting for comparing entropy estimators, because by straightforward calculations we can obtain simple formula for theoretical entropy

$$H(f) = \ln\left(\frac{2\Gamma(\frac{1}{p})}{p}\right) + \frac{1}{p}, \text{ for } p > 0. \tag{15}$$

In our study we used distributions with parameters $p = 0.5, 1, 2, 5$.

For each distribution 1000 samples of sizes 5, 10, 15, 20,50 and 100 were generated and the estimators of entropy and their root mean squared errors (RMSE) and bias (BIAS) were computed; each experiment was repeated 10 times and average was taken. Replications were also used to measure the precision of the estimation of RMSE and BIAS by calculation the confidence intervals – this procedure yields absolute errors for the estimated RMSE and BIAS in most cases less than 0.005 with probability greater or equal to 0.99.

The algorithm was written in C programing language and simulations were performed on Cray 6400 computer in Warsaw University of Technology. The uniform random number generator from Berdnikov et al. (1996) was used. Results of the simulations are given in Annex 2 - Tables 1 - 9. Graphs of RMSE and BIAS based on these results are given in Annex 1 - Figures 1 - 18.

CONCLUSION

Our simulation results indicate that regarding the root mean squared error (RMSE) the Vasicek estimator $V'_{m,n}$ with the bias correction is the best – this holds in 98 % cases considered in tables. Van Es' estimator $E_{m,n}$ also compares favorably with other estimators – in 93 % cases $E_{m,n}$ is better in terms of RMSE than $V_{m,n}$, $C_{m,n}$, and $C'_{m,n}$. The results show that Jackknife corrected Correa'a estimator $C'_{m,n}$ works better than $C_{m,n}$ in 42 % of cases. The traditional Vasicek's estimator $V_{m,n}$ is in general the worst.

As regards to the bias (BIAS) comparison the Vasicek estimator $V'_{m,n}$ with the bias correction and the Jackknife corrected Correa'a estimator $C'_{m,n}$ are - in general - the best. $C'_{m,n}$ has a high bias only in the case of very heavy tailed distribution. Van Es' estimator $E_{m,n}$ behaves very well only in the case of the uniform distribution. In other cases it is dominated by $V'_{m,n}$ and $C'_{m,n}$, but it is better than the usual Correa's estimator $C_{m,n}$ and usual Vasicek's estimator $V_{m,n}$. This traditional Vasicek's estimator $V_{m,n}$ has often the highest bias.

Thus, according to our simulation study, the Vasicek estimator $V'_{m,n}$ with the bias correction should be recommended to the users, because it reveals the smallest root mean squared error and a very low bias. Then we may also recommend van Es' estimator $E_{m,n}$ and the Jackknife corrected Correa'a estimator $C'_{m,n}$ (the first one has a smaller root mean squared error, while the second one has a lower bias). It is worth noting that the most popular and widely used Vasicek's estimator $V_{m,n}$.does not possess good statistical properties.

In the problem of the entropy estimation still a lot have to be done. For example, all the considered estimators depend on two parameters: m and n but the optimal choice of m according to n is still an open problem. Next, in order to reduce simultaneously bias and variance of the entropy estimator it seems to be promising to consider a statistic of a form

$$\widehat{H} = \alpha\widehat{H_1} + \beta\widehat{H}_2, \tag{16}$$

where $\widehat{H}_1$ and $\widehat{H}_2$ are biased estimators (e.g. Vasicek's estimators or Correa's estimators with or without modifications).

ANNEX 1 - Figures

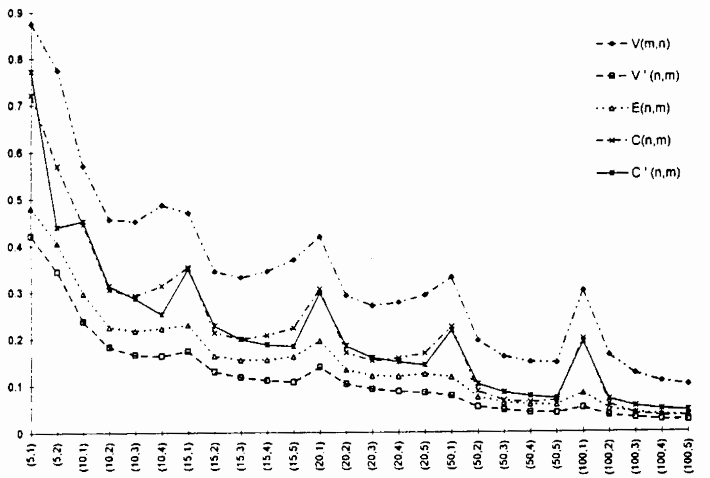

Figure 1. RMSE of the entropy estimators for the uniform distribution $U(0,1)$.

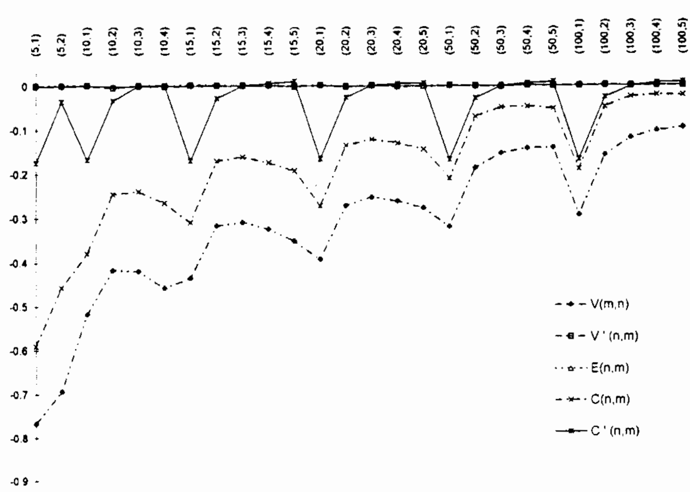

Figure 2. BIAS of the entropy estimators for the uniform distribution $U(0,1)$.

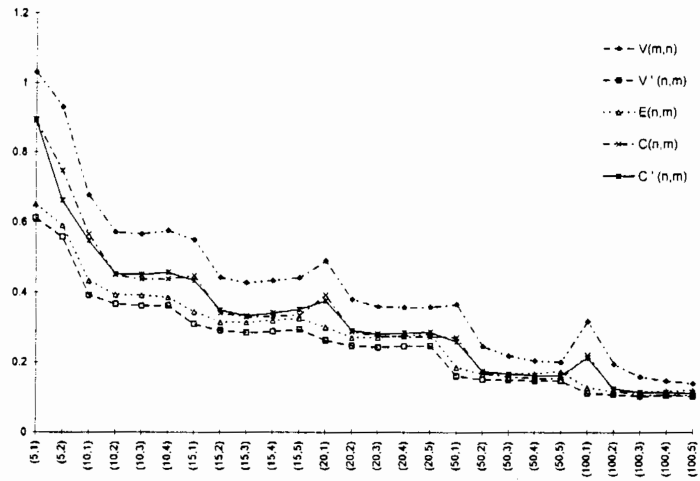

Figure 3. RMSE of the entropy estimators for the exponential distribution $Exp(\lambda)$ with mean $\lambda = 0.5$.

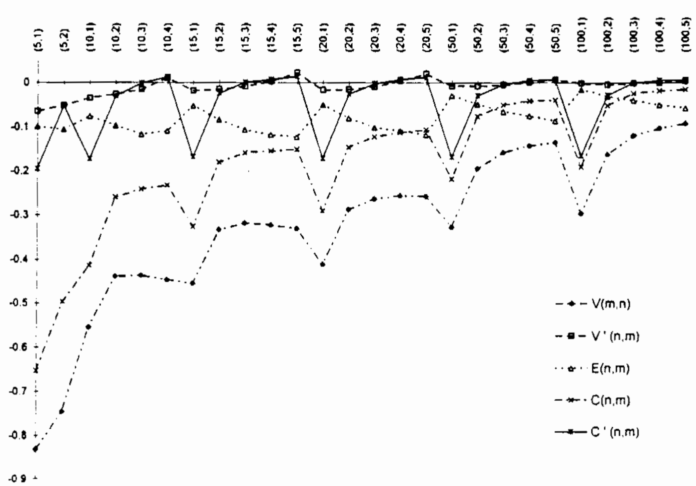

Figure 4. BIAS of the entropy estimators for the exponential distribution $Exp(\lambda)$ with mean $\lambda = 0.5$.

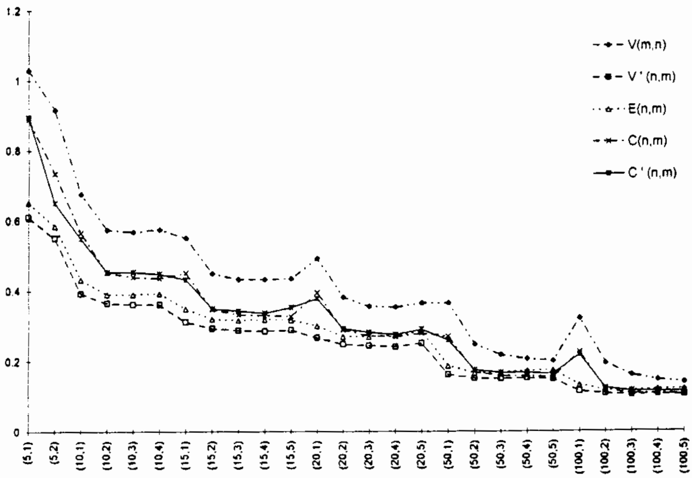

Figure 5. RMSE of the entropy estimators for the exponential distribution $Exp(\lambda)$ with mean $\lambda = 1$.

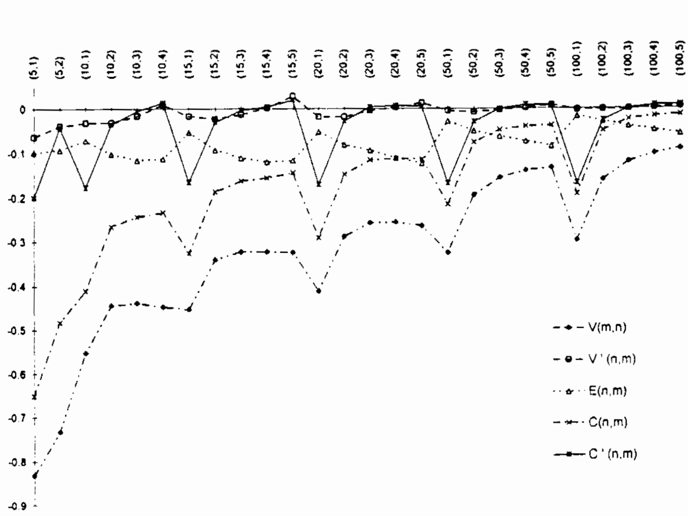

Figure 6. BIAS of the entropy estimators for the exponential distribution $Exp(\lambda)$ with mean $\lambda = 1$.

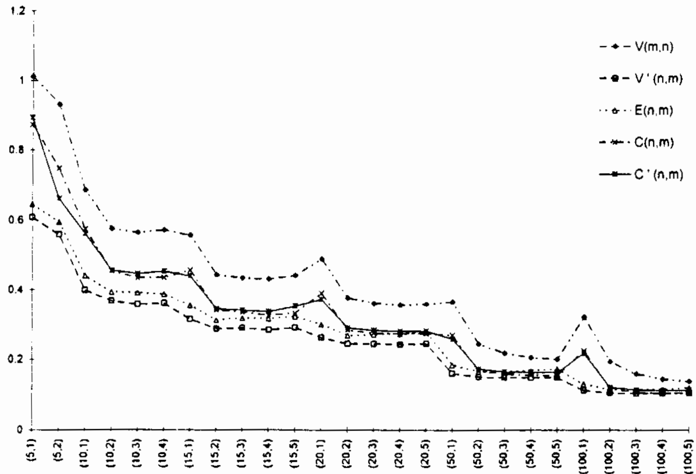

Figure 7. RMSE of the entropy estimators for the exponential distribution $Exp(\lambda)$ with mean $\lambda = 2$.

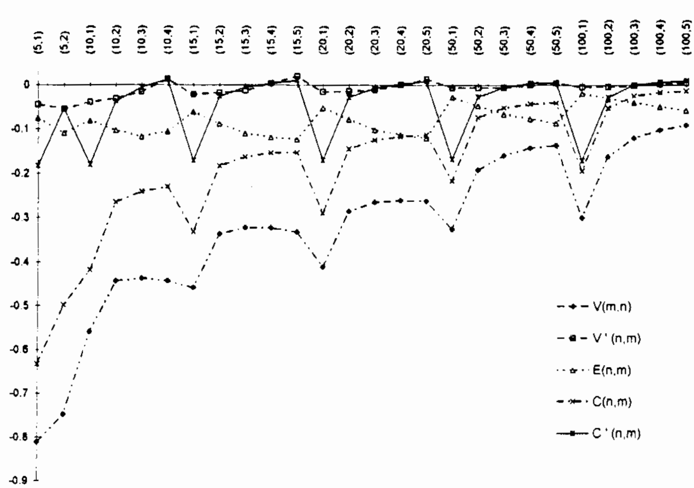

Figure 8. BIAS of the entropy estimators for the exponential distribution $Exp(\lambda)$ with mean $\lambda = 2$.

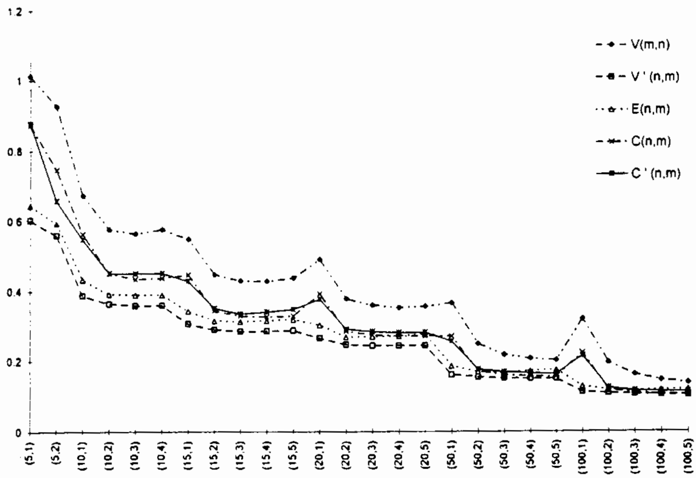

Figure 9. RMSE of the entropy estimators for the exponential distribution $Exp(\lambda)$ with mean $\lambda = 5$.

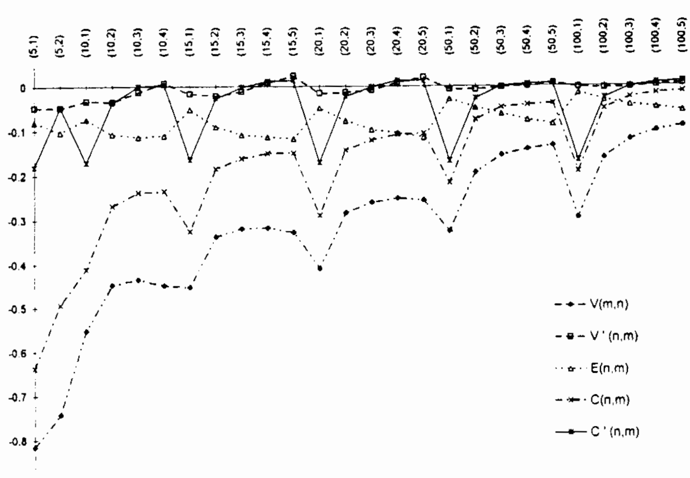

Figure 10. BIAS of the entropy estimators for the exponential distribution $Exp(\lambda)$ with mean $\lambda = 5$.

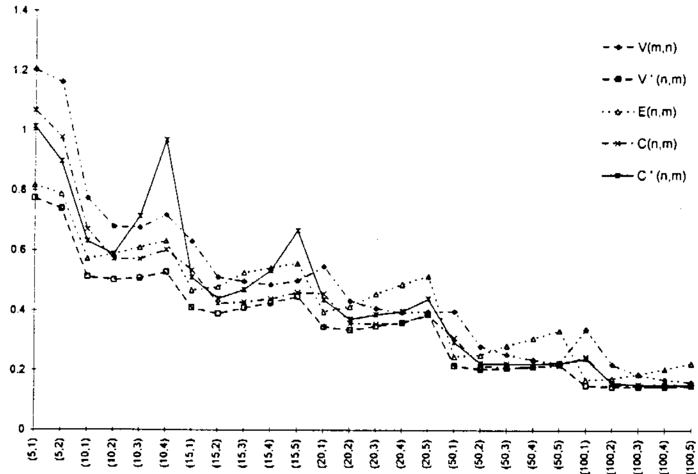

Figure 11. RMSE of the entropy estimators for the exponential power distribution $EPD(p)$ with $p = 0.5$.

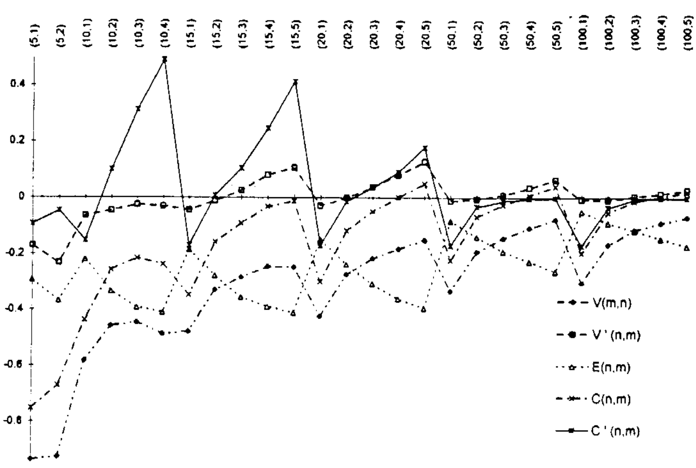

Figure 12. BIAS of the entropy estimators for the exponential power distribution $EPD(p)$ with $p = 0.5$.

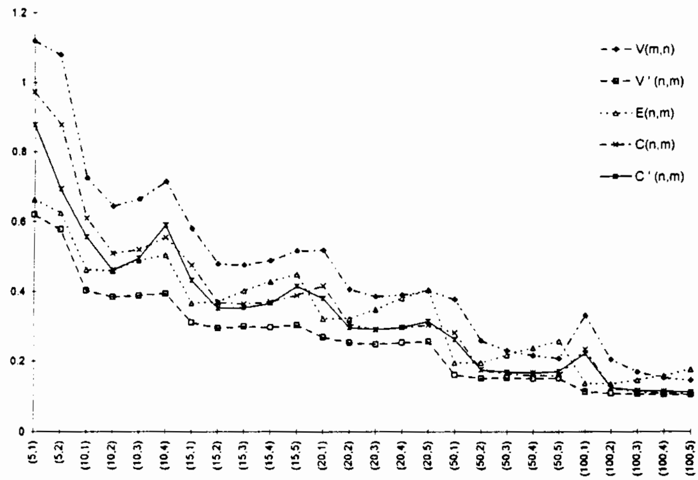

Figure 13. RMSE of the entropy estimators for the exponential power distribution $EPD(p)$ with $p = 1$.

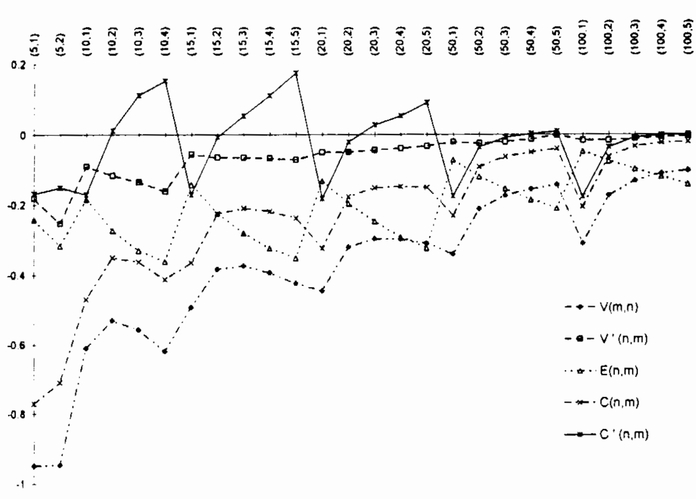

Figure 14. BIAS of the entropy estimators for the exponential power distribution $EPD(p)$ with $p = 1$.

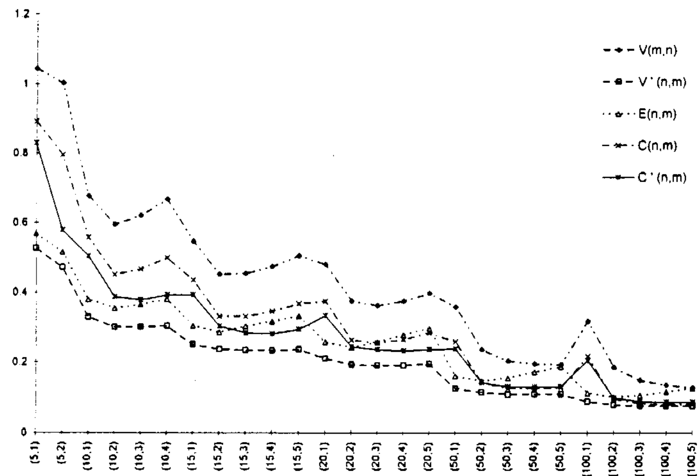

Figure 15. RMSE of the entropy estimators for the exponential power distribution $EPD(p)$ with $p = 2$, i.e.for the normal distribution $N(0, 0.5)$.

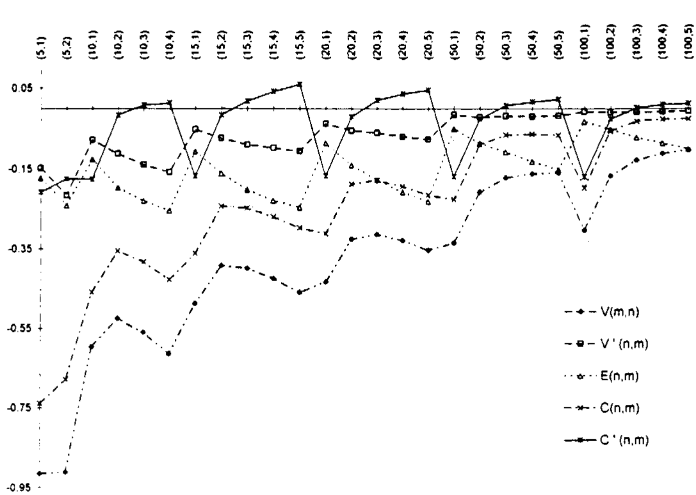

Figure 16. BIAS of the entropy estimators for the exponential power distribution $EPD(p)$ with $p = 2$, i.e.for the normal distribution $N(0, 0.5)$.

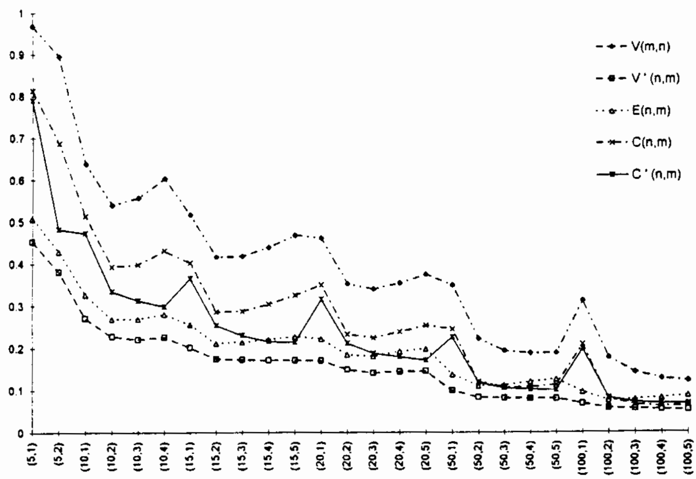

Figure 17. RMSE of the entropy estimators for the exponential power distribution $EPD(p)$ with $p = 5$.

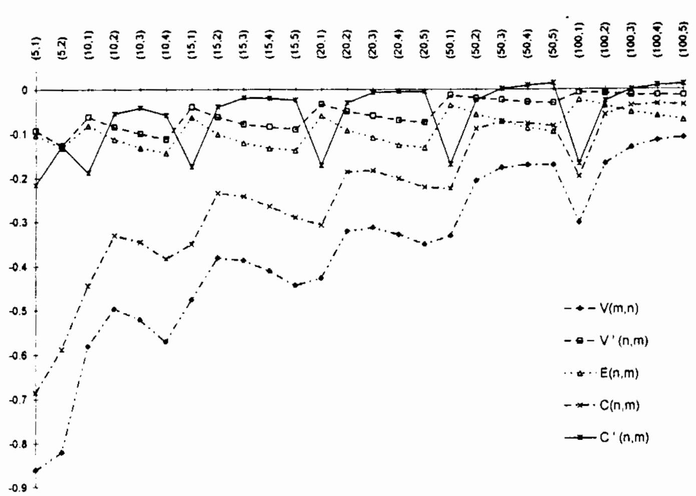

Figure 18. BIAS of the entropy estimators for the exponential power distribution $EPD(p)$ with $p = 5$.

ANNEX 2 - Tables

Table 1: RMSE and BIAS of the entropy estimators for the uniform distribution $U(0,1)$

		RMSE					BIAS				
n	m	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$
5	1	.875	.421	.481	.721	.772	-.767	.000	-.000	-.591	-.174
5	2	.774	.344	.405	.570	.440	-.693	.000	.003	-.457	-.035
10	1	.570	.237	.297	.445	.452	-.518	.001	.002	-.380	-.168
10	2	.455	.182	.225	.305	.313	-.417	-.004	-.003	-.246	-.033
10	3	.451	.165	.217	.292	.286	-.420	.001	.000	-.240	.001
10	4	.486	.163	.222	.313	.252	-.458	-.001	-.000	-.266	.003
15	1	.469	.173	.229	.353	.349	-.436	.001	.004	-.310	-.170
15	2	.343	.129	.163	.212	.227	-.318	.001	.002	-.170	-.028
15	3	.330	.117	.154	.199	.198	-.309	.001	.001	-.161	.001
15	4	.343	.110	.154	.206	.186	-.325	.001	.002	-.174	.006
15	5	.368	.107	.161	.222	.183	-.352	.000	-.001	-.193	.011
20	1	.417	.139	.195	.305	.297	-.393	.002	.003	-.273	-.166
20	2	.291	.103	.133	.169	.184	-.272	-.001	-.001	-.135	-.026
20	3	.269	.092	.120	.152	.158	-.253	.001	.001	-.121	.002
20	4	.276	.086	.119	.157	.149	-.262	-.001	-.001	-.130	.005
20	5	.291	.084	.124	.168	.142	-.278	-.001	-.000	-.144	.006
50	1	.329	.077	.117	.224	.216	-.320	.001	.001	-.211	-.168
50	2	.195	.053	.074	.087	.102	-.187	-.000	-.000	-.070	-.028
50	3	.160	.046	.062	.067	.084	-.153	-.000	-.001	-.049	-.000
50	4	.148	.041	.058	.063	.076	-.142	.001	.001	-.047	.006
50	5	.146	.040	.057	.066	.071	-.141	.000	.000	-.052	.009
100	1	.300	.051	.082	.197	.191	-.295	-.000	.001	-.190	-.168
100	2	.162	.034	.049	.058	.069	-.158	.001	.001	-.048	-.026
100	3	.123	.029	.040	.038	.054	-.119	-.000	.000	-.025	-.001
100	4	.106	.025	.036	.033	.048	-.103	.000	.000	-.021	.006
100	5	.099	.024	.034	.033	.045	-.096	.000	.000	-.022	.008

Table 2: RMSE and BIAS of the entropy estimators for the exponential distribution $Exp(\lambda)$ with mean $\lambda = .5$

		RMSE					BIAS				
n	m	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$
5	1	1.031	.612	.652	.890	.895	-.832	-.065	-.099	-.653	-.195
5	2	.929	.557	.590	.747	.662	-.745	-.052	-.106	-.496	-.052
10	1	.677	.391	.433	.566	.547	-.554	-.035	-.076	-.414	-.173
10	2	.571	.366	.393	.449	.452	-.439	-.026	-.097	-.260	-.030
10	3	.566	.361	.391	.437	.451	-.437	-.016	-.117	-.242	-.001
10	4	.575	.362	.385	.437	.457	-.447	.010	-.109	-.233	.013
15	1	.549	.308	.344	.448	.433	-.455	-.018	-.052	-.327	-.168
15	2	.442	.290	.315	.340	.349	-.334	-.015	-.084	-.180	-.024
15	3	.427	.284	.314	.328	.333	-.319	-.009	-.107	-.159	.001
15	4	.433	.288	.320	.331	.341	-.323	.003	-.118	-.155	.008
15	5	.441	.293	.325	.334	.351	-.331	.022	-.123	-.151	.014
20	1	.489	.263	.299	.391	.375	-.412	-.017	-.050	-.291	-.172
20	2	.379	.247	.271	.287	.291	-.288	-.016	-.080	-.146	-.026
20	3	.359	.243	.271	.274	.281	-.264	-.010	-.101	-.123	-.002
20	4	.356	.246	.276	.273	.284	-.257	.004	-.109	-.113	.008
20	5	.357	.247	.280	.272	.286	-.258	.020	-.118	-.107	.012
50	1	.365	.161	.186	.271	.259	-.328	-.008	-.029	-.219	-.169
50	2	.246	.151	.167	.169	.175	-.195	-.008	-.049	-.076	-.029
50	3	.219	.151	.168	.160	.168	-.158	-.005	-.065	-.050	-.003
50	4	.205	.148	.169	.154	.164	-.142	.001	-.075	-.041	.006
50	5	.200	.148	.174	.154	.163	-.135	.007	-.086	-.039	.008
100	1	.317	.112	.128	.221	.213	-.297	-.001	-.015	-.191	-.166
100	2	.195	.107	.117	.119	.125	-.163	-.005	-.031	-.052	-.028
100	3	.158	.103	.114	.106	.114	-.120	-.001	-.040	-.024	.000
100	4	.147	.105	.118	.107	.114	-.103	.000	-.050	-.018	.006
100	5	.139	.103	.120	.105	.112	-.092	.004	-.057	-.015	.008

Table 3: RMSE and BIAS of the entropy estimators for the exponential distribution $Exp(\lambda)$ with mean $\lambda = 1$

		RMSE					BIAS				
n	m	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$
5	1	1.029	.611	.652	.888	.895	-.831	-.064	-.098	-.651	-.200
5	2	.915	.550	.585	.734	.650	-.733	-.039	-.093	-.483	-.043
10	1	.675	.391	.432	.565	.548	-.552	-.032	-.072	-.411	-.177
10	2	.573	.364	.389	.450	.453	-.444	-.031	-.102	-.266	-.035
10	3	.567	.360	.389	.438	.452	-.439	-.017	-.116	-.244	-.005
10	4	.574	.360	.391	.435	.448	-.448	.009	-.113	-.234	.014
15	1	.550	.311	.348	.449	.431	-.454	-.017	-.053	-.327	-.166
15	2	.448	.292	.318	.346	.349	-.341	-.023	-.092	-.187	-.030
15	3	.431	.286	.315	.331	.342	-.323	-.013	-.111	-.163	-.003
15	4	.431	.284	.317	.328	.336	-.324	.002	-.120	-.156	.004
15	5	.433	.287	.315	.326	.352	-.325	.028	-.117	-.145	.020
20	1	.490	.265	.299	.393	.377	-.413	-.018	-.052	-.292	-.171
20	2	.380	.247	.269	.287	.292	-.289	-.018	-.081	-.148	-.027
20	3	.354	.243	.268	.270	.281	-.258	-.004	-.094	-.116	.004
20	4	.352	.240	.272	.268	.275	-.257	.003	-.111	-.113	.008
20	5	.363	.249	.283	.277	.289	-.265	.013	-.123	-.114	.005
50	1	.363	.161	.186	.269	.258	-.326	-.005	-.028	-.216	-.168
50	2	.246	.150	.165	.168	.174	-.195	-.008	-.050	-.076	-.029
50	3	.216	.149	.166	.157	.167	-.156	-.003	-.063	-.048	.000
50	4	.205	.150	.171	.156	.166	-.140	.002	-.074	-.040	.009
50	5	.199	.148	.173	.153	.163	-.134	.007	-.085	-.038	.008
100	1	.319	.113	.131	.224	.216	-.299	-.003	-.017	-.193	-.169
100	2	.192	.105	.115	.116	.122	-.161	-.002	-.029	-.050	-.026
100	3	.158	.103	.114	.106	.114	-.120	-.001	-.040	-.024	.000
100	4	.145	.104	.118	.106	.113	-.102	.002	-.049	-.017	.008
100	5	.138	.103	.119	.105	.112	-.091	.005	-.056	-.014	.009

Table 4: RMSE and BIAS of the entropy estimators for the exponential distribution $Exp(\lambda)$ with mean $\lambda = 2$

		RMSE					BIAS				
n	m	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$
5	1	1.012	.608	.646	.873	.893	-.811	-.044	-.075	-.633	-.183
5	2	.931	.558	.595	.748	.662	-.748	-.054	-.109	-.498	-.053
10	1	.685	.398	.440	.574	.561	-.559	-.039	-.081	-.418	-.180
10	2	.575	.367	.393	.453	.456	-.443	-.031	-.103	-.265	-.037
10	3	.564	.357	.391	.434	.446	-.437	-.016	-.117	-.241	-.004
10	4	.571	.361	.386	.434	.452	-.443	.014	-.105	-.230	.013
15	1	.556	.315	.355	.455	.439	-.459	-.022	-.061	-.332	-.171
15	2	.442	.287	.312	.340	.345	-.337	-.018	-.088	-.183	-.026
15	3	.433	.289	.318	.334	.341	-.323	-.013	-.110	-.163	-.004
15	4	.430	.284	.317	.327	.337	-.323	.004	-.118	-.154	.004
15	5	.441	.291	.323	.333	.353	-.333	.020	-.123	-.153	.011
20	1	.488	.263	.300	.390	.372	-.411	-.016	-.051	-.290	-.170
20	2	.376	.245	.269	.284	.291	-.286	-.014	-.077	-.145	-.028
20	3	.361	.245	.272	.275	.284	-.265	-.011	-.101	-.124	-.006
20	4	.356	.243	.273	.273	.281	-.261	.000	-.112	-.117	.003
20	5	.359	.245	.278	.273	.282	-.263	.014	-.120	-.112	.009
50	1	.364	.161	.185	.269	.259	-.326	-.006	-.027	-.217	-.168
50	2	.245	.151	.166	.168	.174	-.192	-.006	-.047	-.073	-.026
50	3	.219	.150	.168	.159	.167	-.159	-.006	-.065	-.051	-.004
50	4	.206	.149	.169	.156	.165	-.142	.000	-.076	-.042	.005
50	5	.201	.149	.174	.155	.164	-.136	.005	-.086	-.040	.006
100	1	.321	.113	.131	.225	.218	-.300	-.005	-.019	-.195	-.171
100	2	.194	.105	.115	.117	.122	-.163	-.004	-.031	-.052	-.027
100	3	.159	.103	.113	.106	.114	-.120	-.001	-.039	-.024	.001
100	4	.144	.103	.116	.104	.112	-.102	.002	-.049	-.017	.007
100	5	.138	.104	.120	.105	.113	-.091	.005	-.057	-.013	.010

Table 5: RMSE and BIAS of the entropy estimators for the exponential distribution $Exp(\lambda)$ with mean $\lambda = 5$

		RMSE					BIAS				
n	m	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$
5	1	1.013	.602	.644	.873	.879	-.815	-.048	-.080	-.637	-.181
5	2	.927	.558	.593	.746	.658	-.742	-.048	-.103	-.493	-.047
10	1	.673	.387	.433	.562	.547	-.552	-.033	-.074	-.412	-.172
10	2	.575	.363	.391	.451	.451	-.448	-.035	-.108	-.269	-.037
10	3	.563	.358	.389	.434	.450	-.435	-.013	-.114	-.239	-.000
10	4	.575	.359	.389	.436	.451	-.450	.007	-.111	-.236	.004
15	1	.547	.306	.342	.446	.428	-.454	-.017	-.052	-.327	-.165
15	2	.446	.289	.315	.343	.350	-.339	-.021	-.091	-.185	-.027
15	3	.429	.284	.312	.328	.334	-.321	-.012	-.109	-.162	-.002
15	4	.427	.285	.315	.326	.340	-.319	.008	-.114	-.151	.012
15	5	.436	.287	.318	.328	.347	-.329	.024	-.118	-.150	.013
20	1	.489	.265	.302	.391	.376	-.412	-.016	-.049	-.291	-.172
20	2	.377	.247	.269	.285	.291	-.286	-.014	-.078	-.144	-.023
20	3	.358	.244	.269	.274	.284	-.262	-.009	-.098	-.122	-.001
20	4	.351	.244	.271	.270	.281	-.253	.008	-.106	-.109	.013
20	5	.355	.244	.276	.270	.282	-.258	.020	-.116	-.107	.012
50	1	.364	.160	.186	.270	.256	-.328	-.007	-.029	-.218	-.169
50	2	.248	.154	.168	.171	.177	-.195	-.008	-.049	-.076	-.027
50	3	.217	.151	.167	.159	.169	-.156	-.003	-.063	-.048	-.000
50	4	.206	.149	.171	.156	.165	-.143	-.000	-.077	-.043	.005
50	5	.201	.149	.174	.155	.163	-.135	.006	-.086	-.039	.007
100	1	.318	.111	.128	.222	.214	-.298	-.003	-.016	-.193	-.168
100	2	.195	.107	.117	.118	.123	-.163	-.005	-.032	-.052	-.027
100	3	.161	.105	.116	.108	.115	-.122	-.003	-.043	-.026	-.002
100	4	.145	.103	.117	.105	.113	-.102	.001	-.049	-.017	.007
100	5	.137	.102	.119	.104	.111	-.091	.005	-.056	-.013	.010

Table 6: RMSE and BIAS of the entropy estimators for the exponential power distribution $EPD(p)$ with $p = .5$

		RMSE					BIAS				
n	m	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$
5	1	1.204	.775	.818	1.067	1.013	-.937	-.170	-.293	-.753	-.093
5	2	1.162	.739	.789	.977	.898	-.927	-.233	-.368	-.673	-.046
10	1	.774	.512	.573	.670	.630	-.583	-.064	-.220	-.438	-.152
10	2	.678	.501	.586	.570	.587	-.459	-.046	-.334	-.258	.099
10	3	.675	.507	.609	.570	.713	-.446	-.025	-.392	-.216	.312
10	4	.717	.527	.631	.601	.969	-.488	-.031	-.411	-.238	.490
15	1	.628	.408	.467	.534	.509	-.479	-.043	-.184	-.348	-.170
15	2	.510	.388	.478	.423	.440	-.330	-.011	-.278	-.158	.008
15	3	.496	.408	.526	.427	.469	-.284	.026	-.355	-.091	.103
15	4	.485	.425	.542	.438	.533	-.246	.080	-.389	-.032	.246
15	5	.499	.446	.557	.460	.666	-.247	.106	-.410	-.012	.413
20	1	.546	.345	.395	.456	.436	-.423	-.028	-.147	-.299	-.169
20	2	.432	.335	.413	.357	.371	-.273	-.001	-.237	-.117	-.013
20	3	.407	.347	.456	.354	.386	-.216	.037	-.306	-.048	.037
20	4	.393	.358	.488	.360	.396	-.181	.080	-.360	.001	.091
20	5	.395	.387	.515	.384	.439	-.150	.127	-.394	.049	.178
50	1	.396	.215	.246	.308	.294	-.333	-.012	-.082	-.222	-.170
50	2	.280	.204	.251	.215	.224	-.192	-.005	-.139	-.067	-.032
50	3	.252	.207	.283	.209	.222	-.144	.009	-.190	-.025	-.012
50	4	.235	.212	.307	.211	.223	-.108	.035	-.226	.008	-.004
50	5	.224	.219	.333	.217	.226	-.078	.063	-.261	.039	-.002
100	1	.337	.150	.169	.246	.239	-.302	-.006	-.048	-.196	-.170
100	2	.220	.145	.173	.154	.159	-.166	-.007	-.089	-.052	-.034
100	3	.185	.146	.188	.147	.154	-.114	.005	-.116	-.013	-.005
100	4	.170	.146	.205	.146	.153	-.089	.015	-.144	.004	-.001
100	5	.161	.148	.224	.148	.153	-.069	.027	-.170	.019	-.001

Table 7: RMSE and BIAS of the entropy estimators for the exponential power distribution $EPD(p)$ with $p = 1$

		RMSE					BIAS				
n	m	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$
5	1	1.119	.620	.663	.972	.879	-.948	-.181	-.242	-.771	-.168
5	2	1.079	.577	.625	.878	.694	-.946	-.253	-.316	-.710	-.150
10	1	.725	.402	.462	.610	.556	-.610	-.090	-.184	-.469	-.170
10	2	.644	.384	.459	.509	.461	-.530	-.117	-.273	-.350	.012
10	3	.665	.388	.489	.519	.495	-.557	-.135	-.330	-.362	.112
10	4	.716	.394	.504	.554	.590	-.619	-.162	-.361	-.412	.153
15	1	.580	.310	.366	.474	.431	-.493	-.056	-.142	-.365	-.171
15	2	.479	.294	.370	.367	.351	-.383	-.065	-.224	-.226	-.007
15	3	.475	.299	.402	.364	.353	-.374	-.065	-.280	-.210	.053
15	4	.488	.296	.429	.370	.366	-.394	-.067	-.323	-.219	.111
15	5	.516	.303	.449	.389	.415	-.424	-.071	-.351	-.238	.175
20	1	.518	.268	.321	.416	.381	-.446	-.050	-.131	-.324	-.182
20	2	.406	.253	.321	.305	.296	-.321	-.050	-.196	-.177	-.021
20	3	.385	.248	.349	.289	.291	-.297	-.044	-.247	-.151	.028
20	4	.390	.253	.381	.295	.297	-.299	-.039	-.291	-.148	.053
20	5	.401	.256	.404	.302	.314	-.310	-.032	-.323	-.150	.091
50	1	.377	.161	.196	.281	.262	-.341	-.021	-.070	-.231	-.175
50	2	.259	.151	.196	.175	.176	-.212	-.025	-.119	-.091	-.035
50	3	.230	.152	.218	.164	.170	-.173	-.020	-.154	-.062	-.007
50	4	.216	.150	.238	.159	.167	-.155	-.013	-.185	-.050	.003
50	5	.208	.151	.258	.158	.171	-.143	-.001	-.210	-.040	.010
100	1	.331	.113	.137	.234	.222	-.311	-.016	-.047	-.206	-.177
100	2	.204	.107	.135	.123	.125	-.174	-.016	-.075	-.062	-.035
100	3	.169	.106	.146	.111	.116	-.132	-.012	-.098	-.034	-.008
100	4	.152	.104	.159	.107	.114	-.111	-.008	-.120	-.024	-.002
100	5	.145	.104	.176	.106	.112	-.102	-.005	-.142	-.021	-.000

Table 8: RMSE and BIAS of the entropy estimators for the exponential power distribution $EPD(p)$ with $p = 2$, i.e. for the normal distribution $N(0, .5)$

		RMSE					BIAS				
n	m	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$
5	1	1.044	.526	.569	.892	.829	-.914	-.147	-.172	-.739	-.208
5	2	1.002	.472	.516	.795	.578	-.910	-.216	-.241	-.679	-.174
10	1	.677	.329	.379	.557	.504	-.596	-.077	-.125	-.458	-.175
10	2	.594	.300	.355	.451	.387	-.525	-.112	-.196	-.355	-.015
10	3	.620	.300	.365	.466	.378	-.560	-.139	-.230	-.382	.009
10	4	.667	.303	.379	.498	.392	-.614	-.157	-.254	-.426	.014
15	1	.545	.250	.304	.435	.392	-.487	-.051	-.105	-.361	-.168
15	2	.451	.236	.285	.330	.302	-.391	-.073	-.160	-.243	-.015
15	3	.453	.234	.302	.330	.283	-.398	-.089	-.201	-.248	.019
15	4	.473	.232	.316	.344	.280	-.424	-.097	-.229	-.270	.042
15	5	.504	.235	.331	.366	.293	-.458	-.105	-.246	-.297	.060
20	1	.479	.210	.257	.373	.333	-.432	-.037	-.085	-.312	-.167
20	2	.374	.192	.241	.263	.244	-.326	-.054	-.140	-.187	-.020
20	3	.362	.190	.257	.255	.235	-.314	-.060	-.177	-.178	.021
20	4	.374	.190	.278	.264	.232	-.329	-.069	-.207	-.193	.036
20	5	.397	.196	.297	.283	.236	-.353	-.076	-.231	-.215	.046
50	1	.358	.126	.161	.258	.238	-.335	-.015	-.050	-.226	-.169
50	2	.236	.115	.148	.143	.143	-.207	-.021	-.085	-.089	-.025
50	3	.203	.109	.156	.126	.132	-.171	-.018	-.107	-.065	.008
50	4	.195	.110	.173	.126	.131	-.162	-.019	-.131	-.063	.017
50	5	.193	.110	.187	.128	.132	-.160	-.018	-.151	-.066	.023
100	1	.316	.088	.113	.216	.205	-.304	-.008	-.032	-.198	-.170
100	2	.185	.079	.101	.096	.099	-.167	-.009	-.052	-.056	-.025
100	3	.148	.076	.106	.082	.089	-.128	-.009	-.071	-.031	.002
100	4	.134	.075	.116	.079	.087	-.111	-.008	-.086	-.026	.010
100	5	.127	.075	.126	.080	.087	-.102	-.006	-.100	-.025	.013

Table 9: RMSE and BIAS of the entropy estimators for the exponential power distribution $EPD(p)$ with $p = 5$

		RMSE					BIAS				
n	m	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$	$V_{m,n}$	$V'_{m,n}$	$E_{m,n}$	$C_{m,n}$	$C'_{m,n}$
5	1	.968	.453	.509	.813	.790	-.860	-.093	-.103	-.686	-.216
5	2	.895	.381	.429	.687	.482	-.820	-.127	-.132	-.588	-.128
10	1	.638	.271	.326	.514	.473	-.581	-.062	-.081	-.444	-.188
10	2	.540	.228	.269	.392	.334	-.497	-.085	-.113	-.331	-.054
10	3	.557	.220	.269	.397	.312	-.521	-.100	-.132	-.346	-.041
10	4	.603	.226	.280	.430	.298	-.571	-.113	-.144	-.384	-.058
15	1	.516	.201	.256	.401	.364	-.476	-.040	-.063	-.351	-.175
15	2	.415	.174	.211	.285	.254	-.382	-.063	-.101	-.235	-.039
15	3	.417	.172	.214	.287	.230	-.388	-.079	-.121	-.243	-.019
15	4	.438	.171	.220	.303	.216	-.412	-.085	-.133	-.265	-.020
15	5	.467	.171	.227	.324	.213	-.444	-.091	-.138	-.291	-.024
20	1	.460	.170	.222	.349	.315	-.428	-.033	-.060	-.308	-.172
20	2	.351	.149	.184	.232	.211	-.322	-.050	-.092	-.186	-.030
20	3	.339	.141	.180	.224	.187	-.314	-.060	-.109	-.184	-.007
20	4	.353	.144	.192	.238	.179	-.330	-.070	-.126	-.202	-.004
20	5	.373	.145	.198	.253	.171	-.352	-.075	-.132	-.222	-.005
50	1	.348	.099	.138	.244	.226	-.333	-.013	-.035	-.225	-.170
50	2	.222	.083	.110	.120	.118	-.207	-.020	-.056	-.089	-.024
50	3	.193	.081	.112	.107	.105	-.177	-.024	-.072	-.073	.002
50	4	.187	.080	.120	.108	.102	-.172	-.029	-.088	-.077	.009
50	5	.188	.080	.126	.112	.100	-.172	-.031	-.096	-.083	.014
100	1	.310	.068	.096	.208	.197	-.303	-.007	-.023	-.198	-.169
100	2	.177	.057	.075	.080	.082	-.168	-.009	-.037	-.057	-.026
100	3	.142	.055	.078	.065	.071	-.132	-.013	-.051	-.037	.000
100	4	.127	.053	.082	.061	.068	-.116	-.013	-.060	-.033	.009
100	5	.121	.052	.087	.062	.067	-.110	-.014	-.069	-.035	.012

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