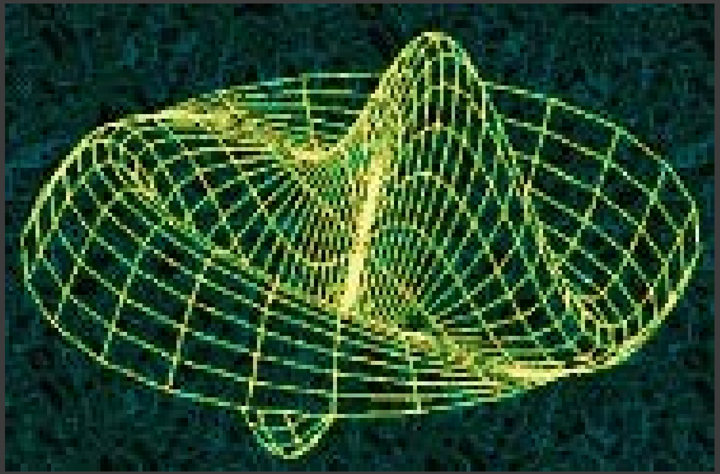
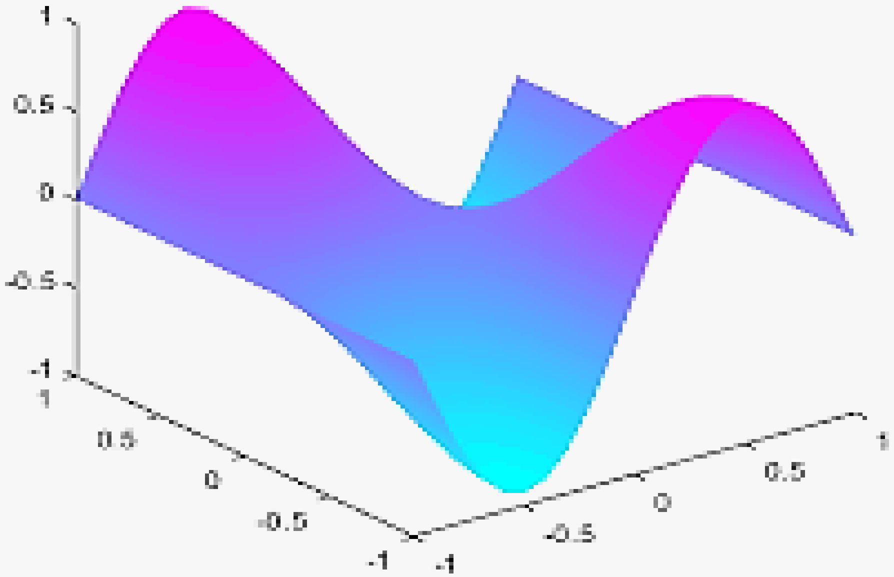
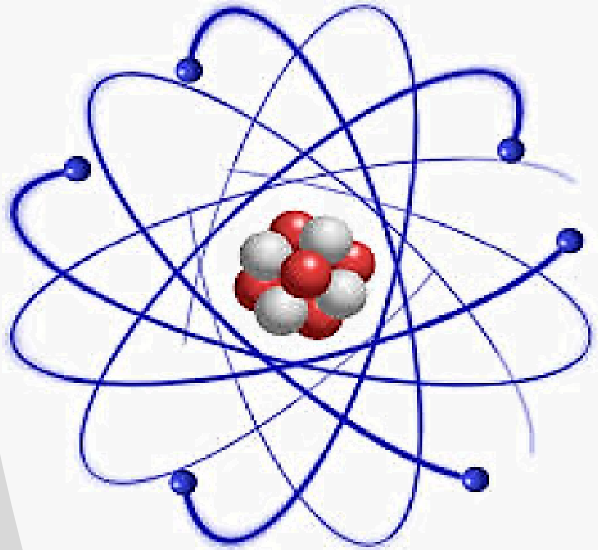
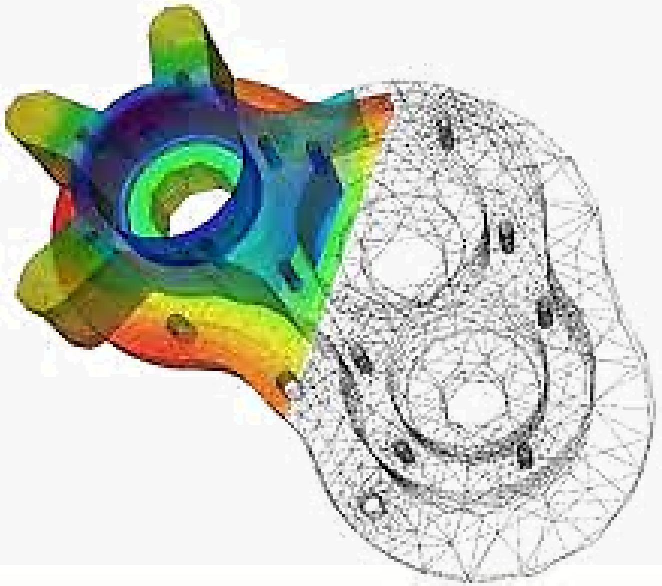




Partial Differential Equations





Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

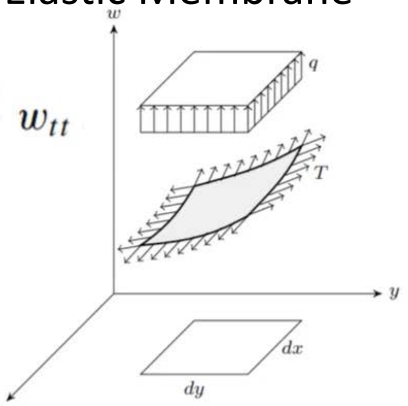
Differential Equations

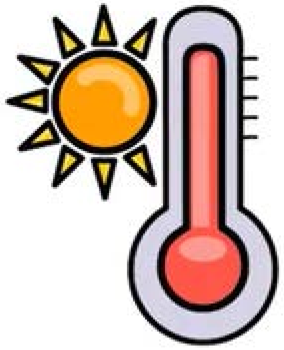
Transverse Vibrations in Elastic Membrane

$$w_{xx} + w_{yy} = -\frac{q}{T} + \frac{\rho h}{T}\, w_{tt}$$

$$w = w(x, y)$$

$$q(x, y)$$



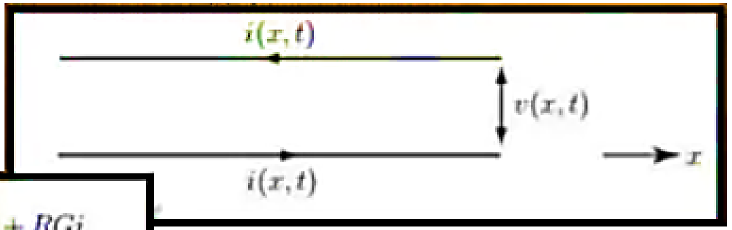


$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t)$$

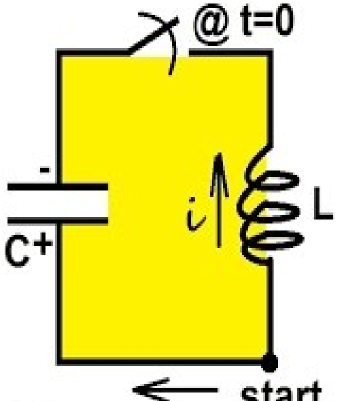
NEWTON'S LAW
OF COOLING

$$\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$$

Heat Equation for Heat Flow

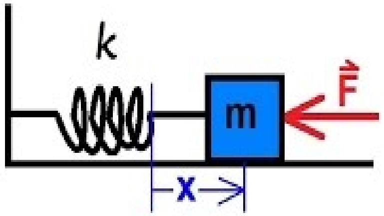


$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (RC + GL)\frac{\partial i}{\partial t} + RGi$$



$$\Sigma V = 0$$

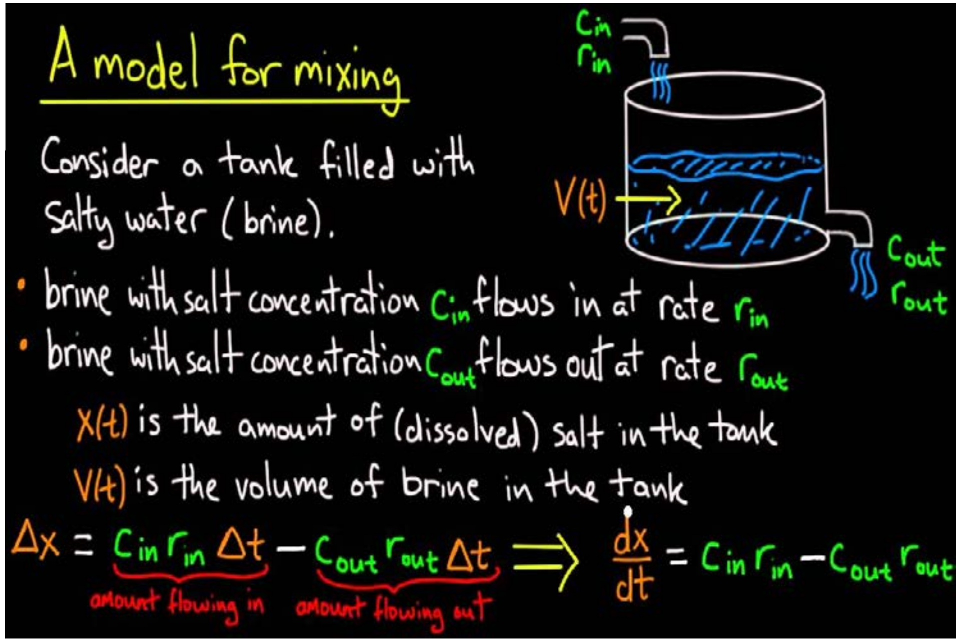
$$\frac{q}{LC} + \frac{d^2q}{dt^2} = 0$$



$$F = ma$$

$$-kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$



Differential Equations

The words differential and equation certainly suggest solving some kind of equation that contains derivatives.

Why are we studying them?

In our daily life we see several interesting phenomena. The reason why we find them interesting is because of rhythmic changes and variations that occur.

For example, everyone likes music because of a rhythmic variations in different modes of a string and symmetrical changes in beating the drums. It is not the variations which is important but the rhythmic changes in string modes, drums and singer voice that make the song beautiful.

What does it all have to do with differential equations?

The rhythmic changes and variations are represented by derivatives. Derivatives encode the information of changes and variations. A differential equation that contains derivatives interprets the phenomenon and encodes the information of physical reality about any physical phenomena.

Differential Equations

- Equations which are composed of an unknown function and its derivatives are called *differential equations*.

- An equation containing the derivative of one or more dependent variables, with respect to one or more independent variables is said to be a **differential equation** (DE).

- Differential equations (DEs) play a fundamental role in engineering because many physical phenomena are best formulated mathematically in terms of their rate of change.

$$\boxed{\frac{dv}{dt} = g - \frac{c}{m}v}$$

Differential equation describing a force balance for the falling object

$v -$ dependent variable

$t -$ independent variable

Examples of differential equations

i. $$\frac{dy}{dx} + y\cos x = \sin y$$

ii. $$\frac{d^2y}{dt^2} + ty\left(\frac{dy}{dt}\right)^2 = 0$$

iii. $$\left[1 + \left(\frac{dz}{dx}\right)^2\right]^{3/2} = \frac{d^2z}{dx^2}$$

iv. $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx$$

v. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

In order to solve these equations, we first need to determine what kind of equation we are going to deal with. There exist several types of DEs and this classification is based on various parameters. If we look at these examples, we note that two types of derivatives are involved in these DEs. We will classify a differential equation by **type**, **order**, and **linearity**.

Derivatives

Ordinary Derivatives

$$\frac{dv}{dt}$$

$\boldsymbol{v}$ is a function of one independent variable

Partial Derivatives

$$\frac{\partial u}{\partial y}$$

u is a function of more than one independent variable

Differential Equations

Ordinary Differential Equations

$$\frac{d^2v}{dt^2} + 6tv = 1$$

involve one or more

Ordinary derivatives of

unknown functions

Partial Differential Equations

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

involve one or more

partial derivatives of

unknown functions

Ordinary differential equations

Definition:

An ordinary differential equation is an equation that contains an unknown function of a single variable and its derivatives.

Examples:

1. $\dfrac{dy}{dx} = 2x + 3.$
2. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + ay = 0.$
3. $\dfrac{d^3y}{dx^3} + \left(\dfrac{dy}{dx}\right)^4 + 6y = 3.$

Here, y is dependent variable and x is independent variable, and these are ordinary differential equations.

Partial Differential Equation

Examples:

1. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

   This is a partial differential equation. For this example, u is dependent variable and x and y are independent variables.

2. $$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial t^4} = 0$$

3. $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}$$

2 and 3 are also partial differential equations. In both of these examples u is dependent variable and x and t are independent variables.

Order of Differential Equation

The order of the differential equation is order of the highest derivative in the differential equation.

Differential Equation	ORDER
$\frac{dy}{dx} = 2x + 3$	1
$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 9y = 0$	2
$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$	3

Degree of Differential Equation

The **degree** of a differential equation is power of the highest order derivative term in the differential equation.

Differential Equation	Degree
$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + ay = 0$	1
$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$	1
$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + 3 = 0$	3

Degree of Differential Equation

The degree of a differential equation is the exponent of the highest order derivative involved in the differential equation when the differential equation satisfies the following conditions:

- All of the derivatives in the equation are free from fractional powers, positive as well as negative if any.

- There is no involvement of the derivatives in any fraction.

- There shouldn’t be involvement of highest order derivative as a transcendental function (trigonometric, logarithmic or exponential, etc). The coefficient of any term containing the highest order derivative should just be a function of x, y, or some lower order derivative.

If one or more of the aforementioned conditions are not satisfied by the differential equation, it should be first reduced to the form in which it satisfies all of the above conditions. An equation has no degree or undefined degree if it is not reducible.

Examples: Degree of Differential Equation

- $\sqrt{1+(\frac{dy}{dx})^2} = y\frac{d^3y}{dx^3}$

  Since this equation involves fractional powers, we must first get rid of them. On squaring the equation, we get – $1+(\frac{dy}{dx})^2 = y^2(\frac{d^3y}{dx^3})^2$. Now, we can clearly make out that the highest order derivative is of order 3 here i.e. order of the differential equation = 3 and since its power is 2 in the equation – the degree of the differential equation = 2.

- $sin(\frac{dy}{dx}) + \frac{d^2y}{dx^2} + 3x = 0$

  Here, the highest order derivative is of order 2, and it has no involvement in any function. So, the order of the differential equation = 2, and degree = 1.

Examples: Degree of Differential Equation

- $e^{\frac{d^2y}{dx^2}} + sin(x)\frac{dy}{dx} = 1$

  Here, the highest order derivative (order = 2) has involvement in an exponential function. Note that the exponential function can be expanded as a series to bring it to a polynomial form i.e. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots..$

  Thus, the powers of the 2nd order derivative in the equation above will keep on varying as we incorporate more and more terms in the series expansion of the exponential function. Thus, the degree of the equation = Not Defined. The order of the equation = 2.

Order 2

Degree 3

$$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} + y = 4x^5$$

Order and Degree of a Differential Equation

Differential Equation	Order	Degree
$(i)\ \frac{dy}{dx} + y\cos x = \sin y$	1	1
$(ii)\ \frac{d^2y}{dt^2} + ty\left(\frac{dy}{dt}\right)^2 = 0$	2	1
$(iii)\ \left[1 + \left(\frac{dz}{dx}\right)^2\right]^{3/2} = \frac{d^3z}{dx^3}$	3	2
$(iv)\ x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx$	1	1
$(v)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	2	1

Linear & Non-Linear Differential Equation

A differential equation is linear, if

- Dependent variable and its derivatives are of degree one,
- Coefficients of a term does not depend upon dependent variable. i.e., no product of dependent variable and any of its derivatives appear.
- No transcendental function of dependent variable and its derivatives occur.

A differential equation is non-linear, if it is not linear.

In other words, a differential equation is **Linear** when the dependent variable, let's say "y" (and its derivatives) has no exponent or other function put on it.

So, **no** $y^2, y^3, \sqrt{y}, \sin y\,, \ln|y|$ etc, **just simple** $\boldsymbol{y}$

More formally a **Linear Ordinary Differential Equation** is in the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Example:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 9y = 0$$ is a **linear** ODE.

Example:

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$$

is a **non - linear** ODE because 2nd term is not of degree one.

Example:

$$x^2\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x^3$$

is **non - linear** because in 2nd term coefficient depends on *y.*

Example:

$$\frac{dy}{dx} = \sin y$$

is **non - linear** because $\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \ldots$ is a non – linear factor.

Linear DE versus Non-Linear DE

Differential Equation	Type
$(i)\ \frac{dy}{dx} + y\cos x = \sin y$	Non-Linear
$(ii)\ \frac{d^2y}{dt^2} + ty\left(\frac{dy}{dt}\right)^2 = 0$	Non-Linear
$(iii)\ \left[1 + \left(\frac{dz}{dx}\right)^2\right]^{3/2} = \frac{d^3z}{dx^3}$	Non-Linear
$(iv)\ x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx$	Linear
$(v)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	Linear

Table: Classify each differential equation

No	Differential Equations	Ordinary or Partial	Linear or nonlinear	Order	Degree	Independent variables	Dependent variables
1.	$y' = x + 6y$						
2.	$y'' = 4y + y^3$						
3.	$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} - 2y = x^3$						
4.	$y'' + 2xy' + 4y = \cos 2x$						
5.	$\frac{dy}{dx} = \frac{x^2 - 1}{y + 4}$						
6.	$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$						
7.	$\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}$						

Practice: Fill this table by using information from previous slides.

General Ordinary Differential Equation:

The most general ordinary differential equation in two variables is:

$$F(x, y, y', y'' \dots\dots) = c$$

where:

- $F(x, y, y', y'' \dots)$ is a function of x, y, y', y'' ... and so on.
- x is the independent variable.
- y is the dependent variable.
- $y', y'' \dots.$ and so on, is the first order derivative of y, second order derivative of y, and so on.
- c is some constant.

Forms of a 1st – order differential equation

1. Derivative form:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

2. Differential form:

$$(1+x)dy - ydx = 0$$

3. General form:

$$\frac{dy}{dx} = f(x, y) \qquad \text{or} \qquad F(x, y, y') = 0$$

nth – order linear differential equation

1. **nth – order linear differential equation with constant coefficients.**

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

2. **nth – order linear differential equation with variable coefficients**

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

Solution of a Differential Equation

- A ***solution*** (or ***integral***) of a differential equation is a relation between the variables, not containing derivatives, such that this relation and the derivatives obtained from it satisfy the given differential equation. This implies that a differential equation can be formed from its solution by successive differentiations and the process of algebraic operations.

- A solution of a differential equation which contains the number of arbitrary constants equal to the order of the equation is called the ***general solution***.

- Solutions obtained from the general solution by giving particular values to the constants are called ***particular solutions***.

- The graph of a particular integral is known as ***integral curve***.

Solution of a Differential Equation

A **solution** to a differential equation is a function that satisfies the equation.

Example:

Show that $x(t) = e^{-t}$ is a solution of the following differential equation:

$$\frac{dx(t)}{dt} + x(t) = 0.$$

Solution:

Given that:

$$x(t) = e^{-t}$$

$$\Longrightarrow \frac{dx(t)}{dt} = -e^{-t}$$

$$\Longrightarrow \frac{dx(t)}{dt} + x(t) = -e^{-t} + e^{-t} = 0$$

Examples

1. The 1st order differential equation

$$\frac{dy}{dx} = -\alpha y$$

has the solution: $y = ce^{-\alpha x}$, where c is an arbitrary constant.

2. The 2nd order differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

has the solution:

$$y = A\sin x + B\cos x,$$

where A and B are arbitrary constants.

Families of Solutions

Example:

$$9yy' + 4x = 0$$

Solution:

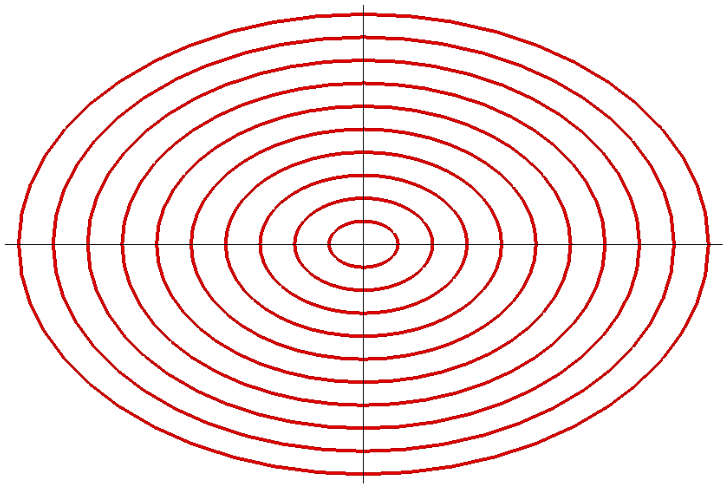
$$\int (9yy' + 4x)dx = C_1 \Rightarrow \int 9y(x)y'(x)dx + \int 4xdx = C_1$$

$$\Rightarrow \int 9ydy + 2x^2 = C_1 \Rightarrow \frac{9y^2}{2} + 2x^2 = C_1 \Rightarrow 9y^2 + 4x^2 = 2C_1$$

This yields $\frac{y^2}{4} + \frac{x^2}{9} = C$ where $C = \frac{C_1}{18}$.

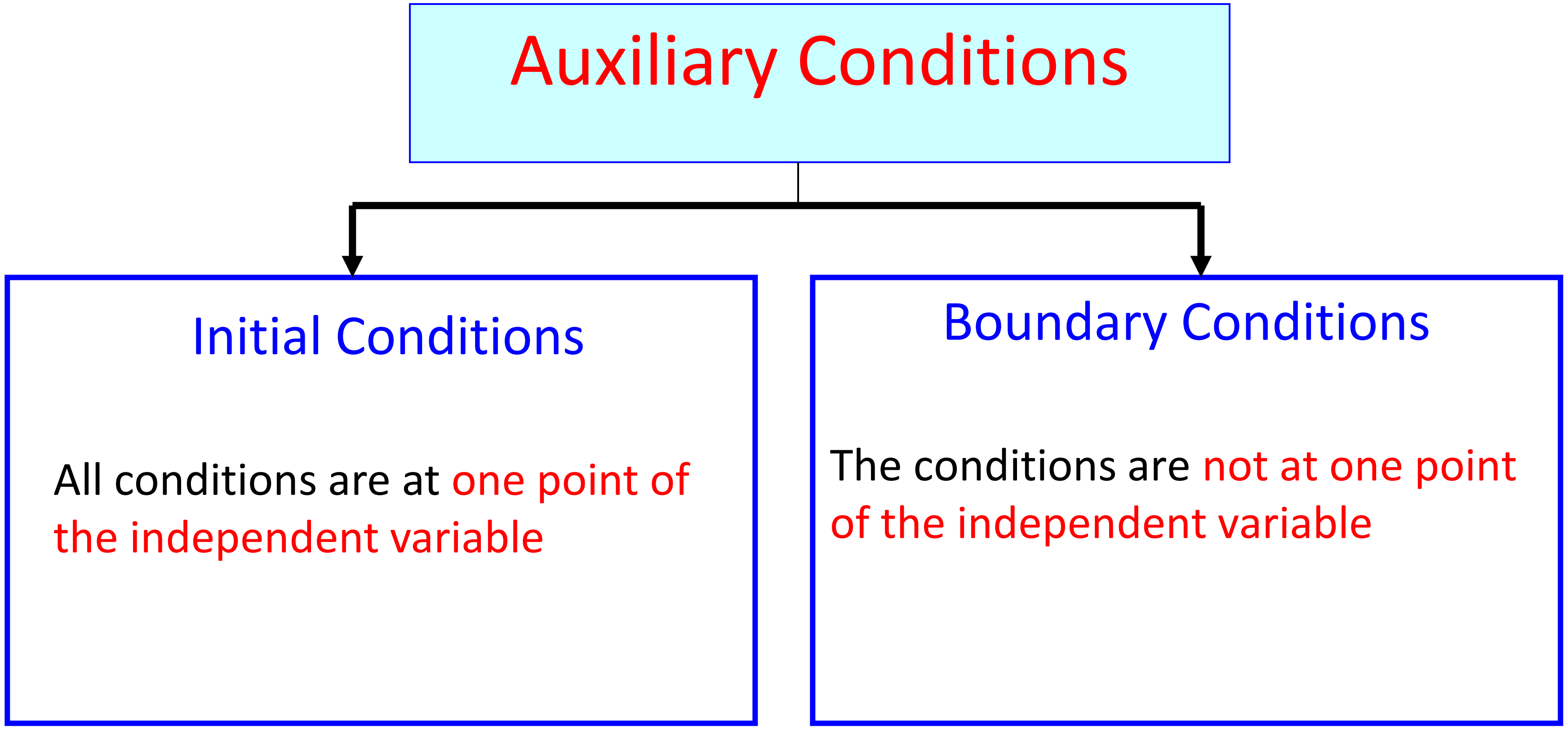
Is this a unique solution?

Observe that given any point (x_0, y_0), there is a unique solution curve of the above equation which goes through the given point.



The solution is a family of ellipses.

Auxiliary Conditions



Boundary-Value and Initial value Problems

Initial-Value Problems (IVP)

The auxiliary conditions are at one point of the independent variable

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$
$$x(0) = 1, \dot{x}(0) = 2.5$$

same

Boundary-Value Problems (BVP)

- The auxiliary conditions are not at one point of the independent variable
- More difficult to solve than initial value problems

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$
$$x(0) = 1, x(2) = 1.5$$

different

Note: Here $\dot{x}$ means first order derivative w.r.t. t