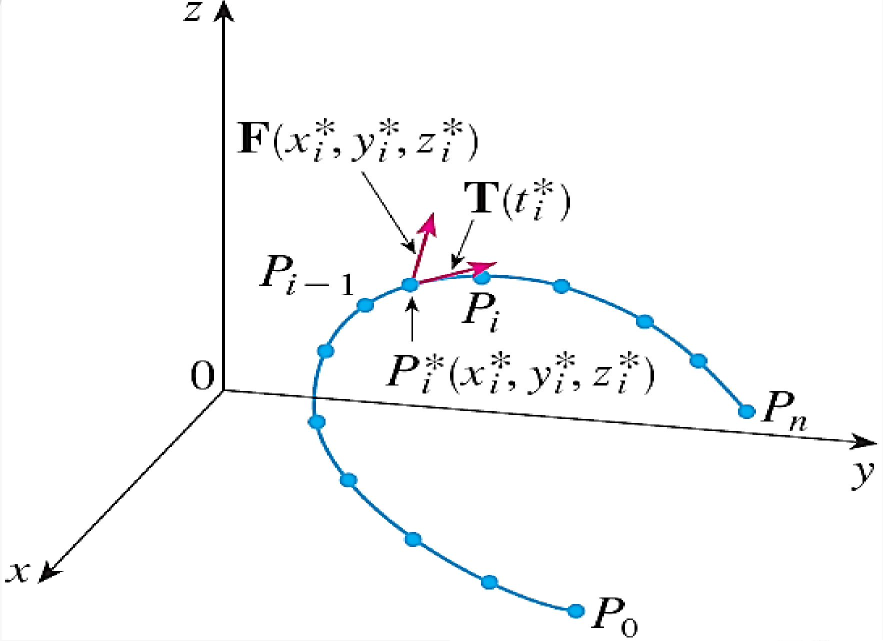
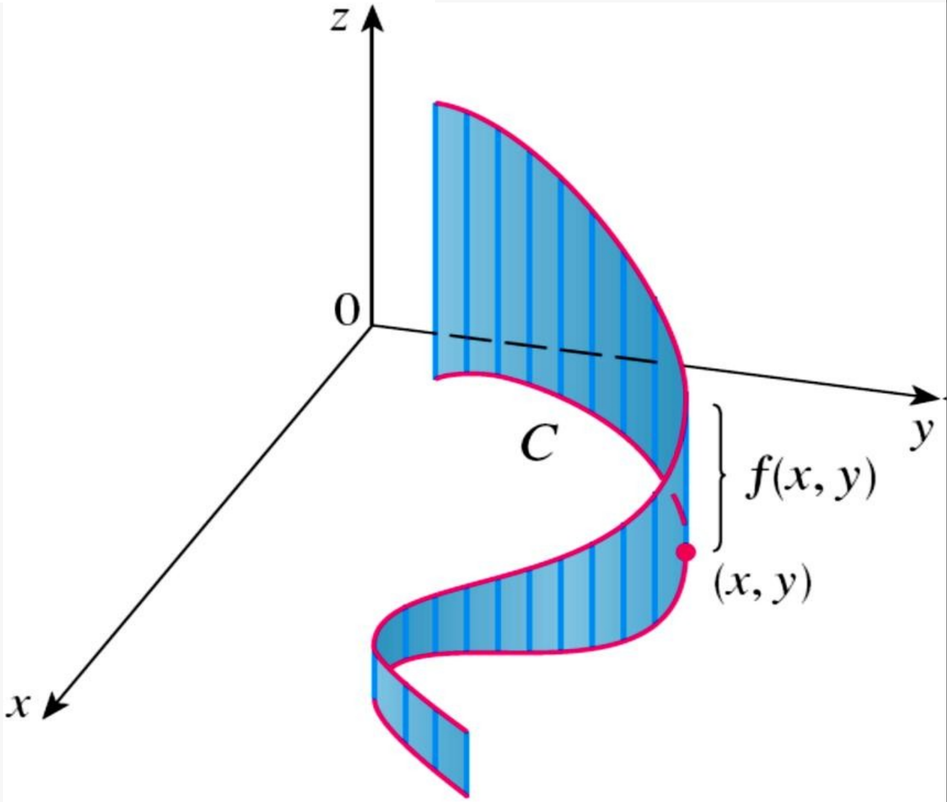
Line Integrals





Vector Calculus(MATH-243)
Instructor: Dr. Naila Amir

16

Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.2**

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

- **Chapter: 16**
  - **Section: 16.1**

Line Integrals

We now define an integral that is similar to a single integral except that, instead of integrating over an interval $[a, b]$,we integrate over a curve C. Such integrals are called **line integrals** or **path integrals**. In physics, the line integrals are used, in particular, for computations of:

- mass of a wire;
- center of mass and moments of inertia of a wire;
- electric charge
- work done by a force on an object moving in a vector field;
- magnetic field around a conductor (Ampere's Law);
- voltage generated in a loop (Faraday's Law of magnetic induction).

It is important to note that these physical ideas are also important when applied to probability density functions of two random variables.

Line Integrals of Scalar Fields (Plane curves)

We start with a plane curve C given by the parametric equations:

$$x = x(t), \qquad y = y(t); \qquad a \le t \le b. \quad (1)$$

Equivalently, C can be given by the vector equation:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = x(t)\,\mathbf{i} + y(t)\mathbf{j}.$$

We assume that C is a smooth curve. This means that $\mathrm{r}'(t)$ is continuous and $r'(t) \neq 0$. If $s(t)$ is the length of C between $r(a)$ and $r(t)$, then

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

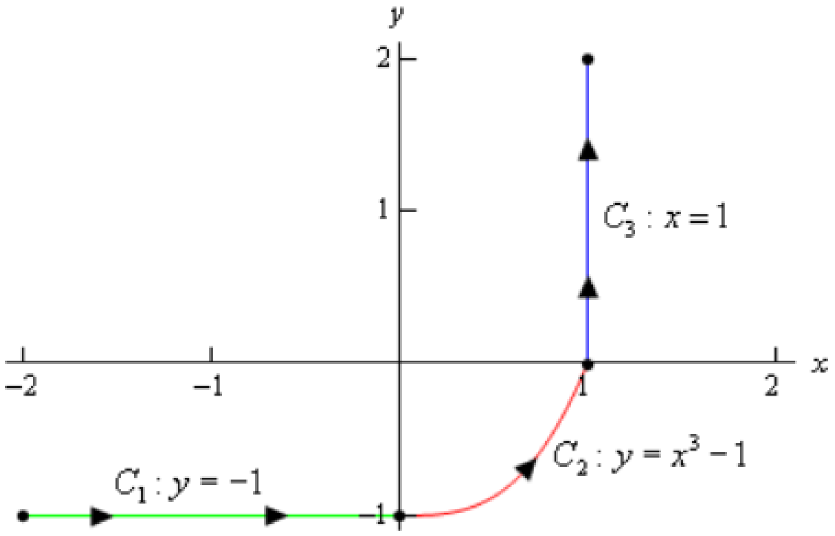
Then, this formula can be used to evaluate the line integral.

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \qquad (3)$$

The value of the line integral does not depend on the parametrization of the curve provided the curve is traversed exactly once as t increases from a to b.

Example:

Evaluate $\int\limits_C 4x^3\,ds$ where C is the curve shown below.



Solution:

First, we need to parameterize each of the curves.

$$
\begin{aligned}
C_1 &: \quad x = t,\ y = -1, && -2 \le t \le 0 \\
C_2 &: \quad x = t,\ y = t^3 - 1, && 0 \le t \le 1 \\
C_3 &: \quad x = 1,\ y = t, && 0 \le t \le 2
\end{aligned}
$$

Let us calculate the line integrals over each of these curves.

$$\int\limits_{C_1} 4x^3\,ds = \int_{-2}^{0} 4t^3\sqrt{(1)^2 + (0)^2}\,dt = \int_{-2}^{0} 4t^3\,dt = t^4\Big|_{-2}^{0} = -16$$

$$\int\limits_{C_2} 4x^3\,ds = \int_0^1 4t^3\sqrt{1 + 9t^4}\,dt = \frac{1}{9}\left(\frac{2}{3}\right)\left(1 + 9t^4\right)^{\frac{3}{2}}\Big|_0^1 = 2.268$$

Solution:

$$\int_{C_3} 4x^3\,ds = \int_0^2 4(1)^3\sqrt{(0)^2+(1)^2}\,dt = \int_0^2 4\,dt = 8$$

Finally, the line integral that we were asked to compute is,

$$\begin{aligned}\int_C 4x^3\,ds &= \int_{C_1} 4x^3\,ds + \int_{C_2} 4x^3\,ds + \int_{C_3} 4x^3\,ds \\ &= -16 + 2.268 + 8 \\ &= -5.732\end{aligned}$$

Line Integrals of Scalar Fields (Plane curves)

Any physical interpretation of a line integral:

$$\int_C f(x, y)\, ds,$$

depends on the physical interpretation of the function f. For instance, $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C.

Mass and Center of Mass

If $\rho(x,y)$ represents the density of a semicircular wire, then mass of the wire is given by:

$$M = \int_C \rho(x,y)\,ds. \qquad (5)$$

The center of mass of the wire with density function ρ is located at the point $(\bar{x},\bar{y})$, where:

$$\left.\begin{aligned} \bar{x} &= \frac{1}{M}\int_C x\rho(x,y)\,ds = \frac{M_y}{M}, \\ &\text{and} \\ \bar{y} &= \frac{1}{M}\int_C y\rho(x,y)\,ds = \frac{M_x}{M}, \end{aligned}\right\} \qquad (6)$$

where $M_x = \int_C \, y\rho(x,y)\,ds$ and $M_y = \int_C \, x\rho(x,y)\,ds$ represents first moments of wire about $x-$ and y $-$axes respectively.

Example:

A wire takes the shape of the semicircle: $x^2 + y^2 = 1, y \geq 0$, and is thicker near its base than near the top. Find the mass of the wire and center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

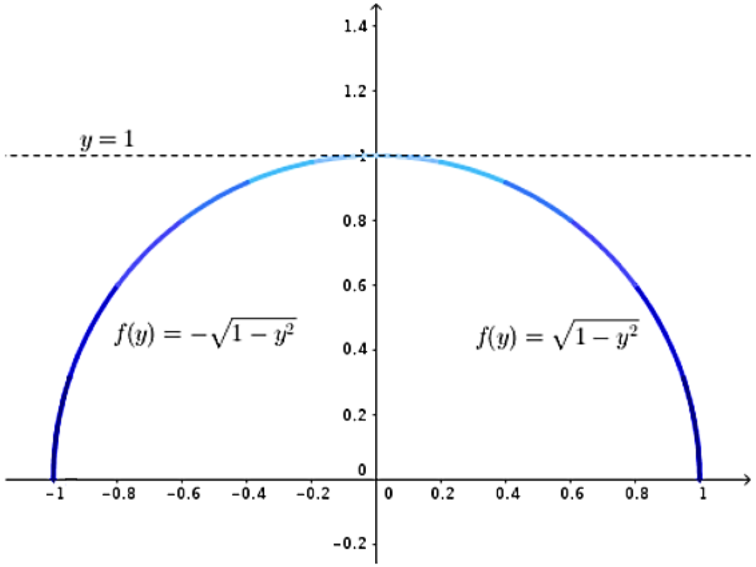
Solution:

Here we use the parametrization:

$$x = \cos t\,,\; y = \sin t\,; \qquad 0 \leq t \leq \pi.$$

Moreover, given that the linear density $\rho(x, y)$ at a point (x, y) is proportional to the distance of the point from the line $y = 1$, so

$$\begin{aligned}\rho(x,y) &= k\sqrt{(x-x)^2 + (y-1)^2}\\ &= k\sqrt{(y-1)^2}\\ &= k|y-1|\\ &= k(1-y);\ \text{since } 0 \leq y \leq 1,\end{aligned}$$



where k is a constant.

Solution:

Thus, the mass of the wire is given as:

$$M = \int_C \rho(x,y)\,ds = \int_C k(1-y)\,ds = \int_0^\pi k(1-\sin t)\,dt = k[t+\cos t]_0^\pi = k(\pi-2).$$

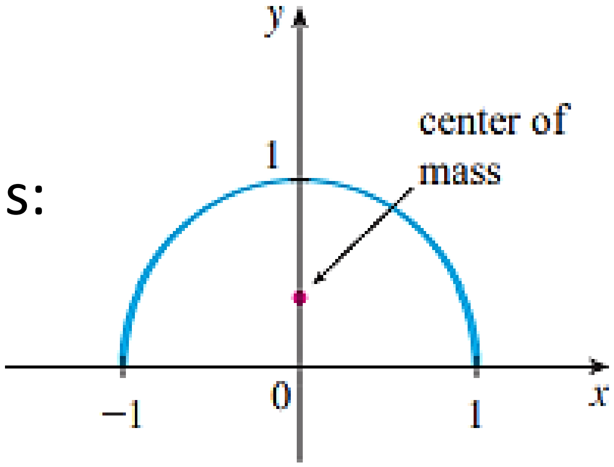
In order to determine the center of mass of the wire we proceed as:

$$\bar{y} = \frac{1}{M}\int_C y\rho(x,y)\,ds = \frac{1}{k(\pi-2)}\int_C ky(1-y)\,ds = \frac{1}{\pi-2}\int_0^\pi (\sin t - \sin^2 t)\,dt$$

$$= \frac{1}{\pi-2}\left[-\cos t - \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^\pi = \frac{4-\pi}{2(\pi-2)}.$$

Observe that due to symmetry $\bar{x}=0$, so the center of mass is given as:

$$(\bar{x},\bar{y}) = \left(0,\frac{4-\pi}{2(\pi-2)}\right) \approx (0,0.38).$$



Moments of Inertia

- A body's first moments (M_x and M_y) tell us about balance and about the torque the body exerts about different axes in a gravitational field.
- However, if the body is a rotating shaft, we are more likely to be interested in how much energy is stored in the shaft or about how much energy it will take to accelerate the shaft to a particular angular velocity.
- This is where the second moment or moment of inertia comes in.
- The mathematical difference between the ***first moments:*** M_x and M_y and the ***moments of inertia****, or* ***second moments***, I_x and I_y is that the second moments use the *squares* of the "lever-arm" distances x and y.

Moments of Inertia

Thus, if a wire with linear density $\rho(x, y)$ lies along a plane curve C, its moments of inertia about the $x-$ and $y-$axes are respectively defined as:

$$I_x = \int_C y^2 \rho(x, y)\, ds,$$

and

$$I_y = \int_C x^2 \rho(x, y)\, ds.$$

The moment I_0 (***moment of inertia about the origin, also called polar moment of inertia***), is calculated by integrating the density (mass per unit area) times the square of the distance from a representative point (*x*, *y*) to the origin. Note that: $I_0 = I_x + I_y$, or

$$I_0 = \int_C (x^2 + y^2) \rho(x, y)\, ds.$$

Example:

A wire takes the shape of the semicircle: $x^2 + y^2 = 1, y \geq 0$, and is thicker near its base than near the top. Find the first moments and moments of inertia for the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

Solution:

Here we use the parametrization:

$$x = \cos t\,,\ y = \sin t\,; \quad 0 \leq t \leq \pi.$$

Moreover, given that the linear density is proportional to its distance from the line $y = 1$, so $\rho(x, y) = k(1 - y)$ where k is a constant.

The first moment about the x –axis is given as:

$$M_x = \int_C y\rho(x, y)\, ds = \int_C ky(1 - y)\, ds = k\int_0^{\pi} (\sin t - \sin^2 t)\, dt = k\left(2 - \frac{\pi}{2}\right).$$

Solution:

The first moment about the y −axis is given as:

$$M_y = \int_C x\rho(x,y)\,ds = \int_C kx(1-y)\,ds = k\int_0^{\pi} (\cos t - \cos t \sin t)\,dt = 0.$$

The moment of inertia about the x −axis is given:

$$I_x = \int_C y^2\rho(x,y)\,ds = k\int_0^{\pi} \sin^2 t\,(1-\sin t)\,dt = k\left(\frac{\pi}{2}-\frac{4}{3}\right).$$

The moment of inertia about the y −axis is given:

$$I_y = \int_C x^2\rho(x,y)\,ds = k\int_0^{\pi} \cos^2 t\,(1-\sin t)\,dt = k\left(\frac{\pi}{2}-\frac{2}{3}\right).$$

DENSITY AND MASS (Total Charge)

Physicists also consider other types of density that can be treated in the same manner.

- For example, an electric charge is distributed over a region D and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$ at a point (x, y) in D.

Then, the **total charge** Q is given by:

$$Q = \int_C \sigma(x, y)\, ds.$$

Example:

Electric charge is distributed over a wire in the shape of the curve $y = x^3, 0 \leq x \leq 1$. The charge density $\sigma(x, y) = \sqrt{1 + 9xy}$ Coulombs per centimeter. Find the total charge Q on the wire.

Solution:

Here we use the parametrization:

$$x = t,\ y = t^3; \quad 0 \leq t \leq 1.$$

Then, the **total charge** Q is given by:

$$Q = \int_C \sigma(x, y)\, ds = \int_0^1 \left(\sqrt{1 + 9t^4}\right)\sqrt{1 + 9t^4}\, dt = \int_0^1 (1 + 9t^4)\, dt$$

$$= \left[t + \frac{9t^5}{5}\right]_0^1 = \left[1 + \frac{9(1)^5}{5}\right] - 0 = \frac{14}{5} \text{Coulombs.}$$

Line Integrals of Scalar Fields (Plane curves)

The line integrals of f along C with respect to x and y are respectively given as:

$$\int_C f(x,y)\,dx = \int_a^b f\big(x(t),y(t)\big)\,x'(t)dt, \qquad (7)$$

$$\int_C f(x,y)\,dy = \int_c^d f\big(x(t),y(t)\big)\,y'(t)dt, \qquad (8)$$

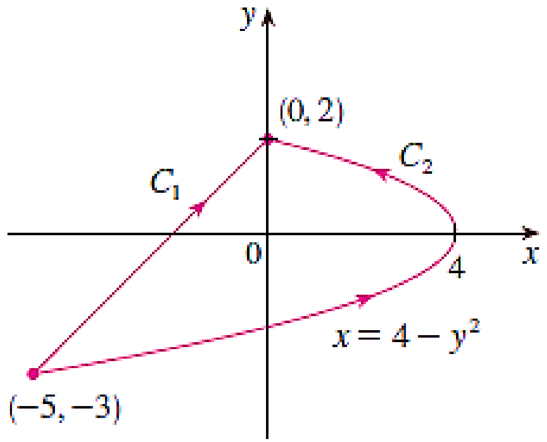
where, $x = x(t), y = y(t), dx = x'(t)dt$ and $dy = y'(t)dt$.

Note: When we want to distinguish the original line integral $\int_C f(x,y)\,ds$ from above equations we call it the line integral with respect to arc length. Moreover, it frequently happens that line integrals with respect to x and y occur together. When this happens, it's customary to abbreviate by writing:

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C [P(x,y)dx + Q(x,y)dy]. \qquad (9)$$

Example:

Evaluate



$$\int_C y^2 dx + x dy\,,$$

where (a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$ and (b) $C = C_2$ is the arc of parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$

Solution: For the present case $\mathbf{r}_0 = \langle -5,-3 \rangle$ and $\mathbf{r}_1 = \langle 0,2 \rangle$, thus, the parametric representation for the line segment is given as:

$$x = 5t - 5, \qquad y = 5t - 3; \quad 0 \le t \le 1.$$

Then $dx = 5dt$ and $dy = 5dt$. Using formula (9), we get:

$$\int_{C_1} y^2 dx + x dy = \int_0^1 (5t-3)^2 (5\,dt) + (5t-5)(5dt) = 5\int_0^1 (25t^2 - 25t + 4)\,dt = -\frac{5}{6}.$$

Solution:

(b) Since the parabola is given as a function of y, let's y as the parameter and write C_2 as:

$$x = 4 - y^2, \qquad y = y; \quad -3 \le y \le 2.$$

Then $dx = -2y\,dy$ and by using formula (9), we get:

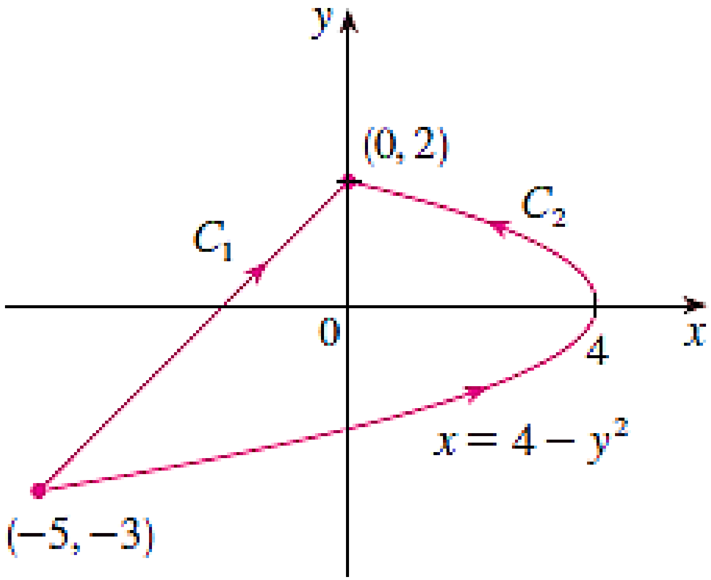
$$\int_{C_2} y^2 dx + x dy = \int_{-3}^{2} y^2(-2y\,dy) + (4 - y^2)dy = \int_{-3}^{2} (-2y^3 - y^2 + 4)\,dy = 40\frac{5}{6}.$$

Note: Alternatively, we can use t as the parameter and write C_2 as:

$$x = 4 - t^2, \qquad y = t; \quad -3 \le t \le 2.$$

Then $dx = -2y\,dy$ and $dy = dt$ by using formula (9), we get:

$$\int_{C_2} y^2 dx + x dy = \int_{-3}^{2} (-2y^3 - y^2 + 4)\,dt = 40\frac{5}{6}.$$



Curve Orientation

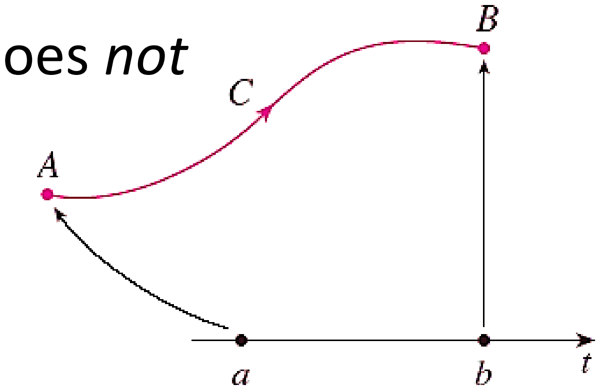
In general, a given parametrization:

$$x = x(t), \qquad y = y(t); \qquad a \leq t \leq b,$$

determines an orientation of a curve C, with the positive direction corresponding to increasing values of the parameter t. If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation, then we have:

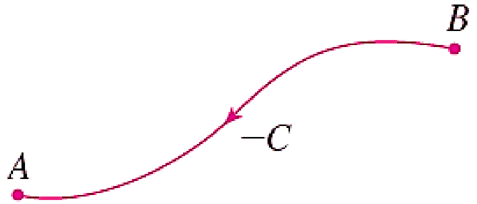
$$\int_{-C} f(x,y)\,dx = -\int_{C} f(x,y)\,dx, \qquad \int_{-C} f(x,y)\,dy = -\int_{C} f(x,y)\,dy.$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:



$$\int_{-C} f(x,y)\,dx = \int_{C} f(x,y)\,dx.$$

This is because Δs is always positive, whereas Δx and Δy change sign when we reverse the orientation of .



Line Integrals of Scalar Fields (Space curves)

We now suppose that C is a smooth space curve given by the parametric equations:

$$x = x(t), \qquad y = y(t), \qquad z = z(t); \quad a \leq t \leq b,$$

or by a vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Suppose f is a function of three variables that is continuous on some region containing C. Then, we define the line integral of f along C (w.r.t. arc length) as:

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt. \qquad (10)$$

Observe that the integrals in both Formulas (3) and (10) can be written in the more compact vector notation:

$$\int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt. \qquad (11)$$

Note: For the special case $f(x,y,z) = 1$, we get $\int_C ds = \int_a^b |\mathbf{r}'(t)|\,dt = L$, where L is the length of the curve C.

Line Integrals of Scalar Fields (Space curves)

Line integrals along C with respect to x, y, and z can also be defined. Thus, as with line integrals in the plane, we evaluate integrals of the form:

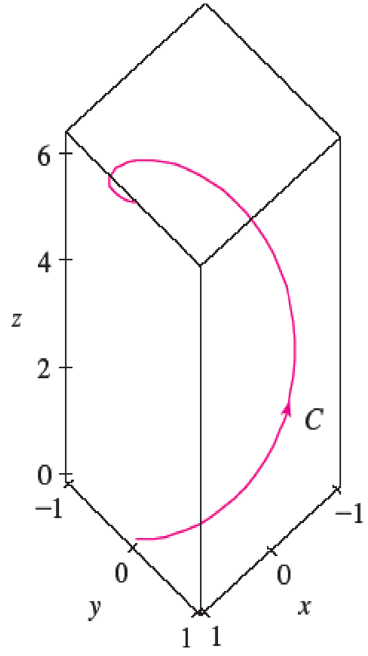
$$\int_C [P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz], \qquad (12)$$

by expressing everything: x, y, z, dx, dy, dz in terms of the parameter t.

Example:

Evaluate

$$\int_C y \sin z \, ds,$$



where C is the circular helix given by the equations:

$$x = \cos t\,, \qquad y = \sin t\,, \qquad z = t; \quad 0 \ \le \ t \ \le \ 2\pi.$$

Solution: For the present case $f(x, y, z) = y \sin z\,,$ and C is the circular helix so,

$$f(x(t), y(t), z(t)) = \sin^2 t \text{ and } \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} = \sqrt{2}.$$

Thus,

$$\int_C y \sin z \, ds = \int_0^{2\pi} \sin^2 t \sqrt{2}\, dt = \sqrt{2} \int_0^{2\pi} \left(\frac{1 - \cos 2t}{2}\right) dt = \frac{1}{\sqrt{2}} \left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} = \sqrt{2}\pi.$$

Example:

Evaluate

$$\int_C [ydx + zdy + xdz]\,,$$

where C consists of the line segment C_1 from $(2, 0, 0)$ to $(3, 4, 5)$, followed by the vertical line segment C_2 from $(3, 4, 5)$ to $(3, 4, 0)$.

Solution:

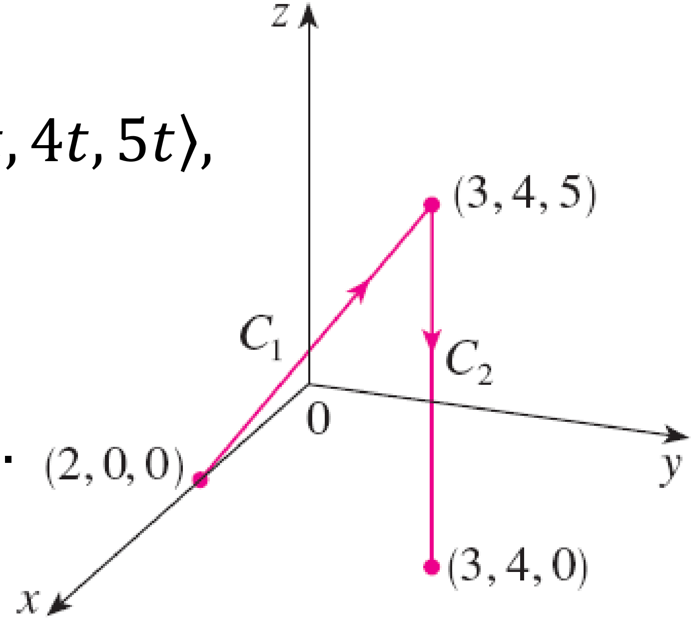
Since vector representation of a line segment is: $\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1;\;\; 0 \le t \le 1,$ therefore, C_1 can be written in the form:

$$\mathbf{r}(t) = \langle x, y, z\rangle = (1 - t)\langle 2,0,0\rangle + t\langle 3,4,5\rangle = \langle 2 + t, 4t, 5t\rangle,$$

or $$x = 2 + t,\; y = 4t,\; z = 5t;\;\; 0 \le t \le 1.$$

Likewise, C_2 can be written as:

$$x = 3, \qquad y = 4, \qquad z = 5 - 5t;\quad 0 \le t \le 1.$$



Solution:

Thus,

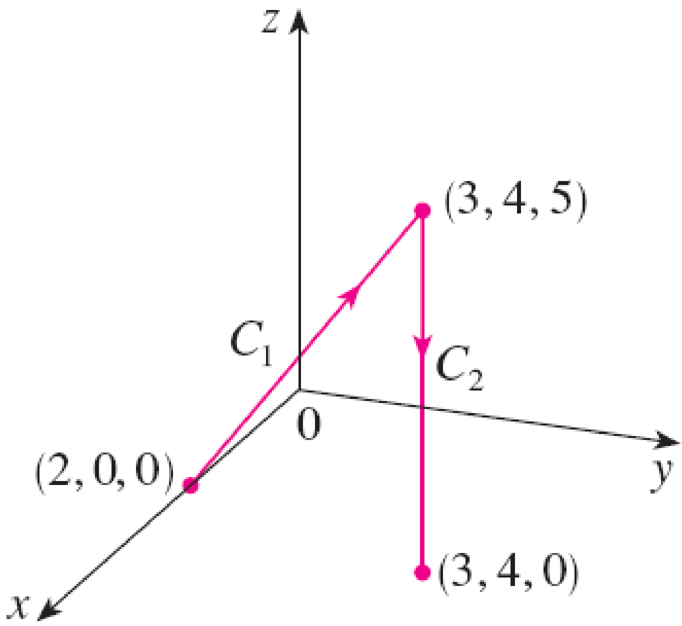
$$\int_{C_1} [ydx + zdy + xdz] = \int_0^1 (4t)dt + (5t)4dt + (2+t)5dt = \int_0^1 (10 + 29t)\,dt = 24.5,$$

and

$$\int_{C_2} [ydx + zdy + xdz] = \int_0^1 3(-5)dt = -15.$$

Therefore,

$$\int_{C} [ydx + zdy + xdz] = 24.5 - 15 = 9.5.$$



Line Integral Over a Space Curve

Physical interpretation of the line integral

$$\int_C f(x, y, z)\, ds,$$

Depends on the nature of its integrand $f(x, y, z)$. A function of three variables $f\ (x, y, z)$ can be interpreted as a scalar field that varies at each point (x, y, z).

Examples:

- Pressure: $P = P(x, y, z)$
- Temperature: $T = T(x, y, z)$
- Density: $\rho = \rho(x, y, z)$ density of an object occupying a region E in space.

Mass, Center of Mass and Moments

Let $\rho(x, y, z)$ represent the density function of a solid object that occupies the region E in units of mass per unit volume, at any given point (x, y, z), then mass of the wire is given by:

$$M = \int_C \rho(x, y, z)\, ds.$$

The center of mass of the wire with density function ρ is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \text{and} \qquad \bar{z} = \frac{M_{xy}}{M},$$

where:

$$M_{yz} = \int_C x\rho(x, y, z)\, ds, \qquad M_{xz} = \int_C y\rho(x, y, z)\, ds \quad \text{and} \quad M_{xy} = \int_C z\rho(x, y, z)\, ds,$$

represents first moments about the coordinate planes.

Moments of Inertia

the ***moments of inertia***, or ***second moments***, I_x, I_y and I_z about the coordinate axes are defined as:

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\, ds,$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\, ds,$$

and

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\, ds.$$

Moment of inertia about the origin is defined as:

$$2I_0 = 2\int_C (x^2 + y^2 + z^2)\rho(x, y, z)\, ds = I_x + I_y + I_z.$$

Total Electric Charge

The total electric charge on a solid object occupying a region E and having charge density $\rho(x, y, z)$ is given by:

$$Q = \int_C \rho(x, y, z)\, ds.$$

Centroids of Geometric Figures

When the density of a solid object is constant i.e., $\rho(x, y, z) = 1$ the center of mass is called the **centroid** of the object.

Example:

Find the mass, first moments and center of mass of a wire in the shape of the helix:

$$x = t, y = \cos t\,, z = \sin t\,;\quad 0 \leq t \leq 2\pi,$$

if the density at any point is equal to the square of the distance from the origin.

Solution:

For the present case:

$$x = t, \qquad y = \cos t\,, \qquad z = \sin t\,; \qquad 0 \leq t \leq 2\pi.$$

Moreover, given that the density $\rho(x, y, z)$ at a point (x, y, z) is equal to the square of the distance from the origin so

$$\rho(x, y, z) = x^2 + y^2 + z^2.$$

Thus, the mass of the given wire is given as:

$$M = \int_C \rho(x, y, z)\, ds = \int_0^{2\pi} (t^2 + 1)\left(\sqrt{2}\right) dt = \sqrt{2}\left[\frac{t^3}{3} + t\right]_0^{2\pi} = \sqrt{2}\left[\frac{8\pi^3}{3} + 2\pi\right].$$

Solution:

Thus, the first moments of the wire are given as:

$$M_{yz} = \int_C x\rho(x,y,z)\,ds = \int_0^{2\pi} \sqrt{2}t(t^2+1)\,dt = \sqrt{2}\left[\frac{t^4}{4}+\frac{t^2}{2}\right]_0^{2\pi} = 2\sqrt{2}\pi^2(1+2\pi^2).$$

$$M_{xz} = \int_C y\rho(x,y,z)\,ds = \int_0^{2\pi} \sqrt{2}\cos t\,(t^2+1)\,dt = 0.$$

$$\text{and} \quad M_{xy} = \int_C z\rho(x,y,z)\,ds = \int_0^{2\pi} \sqrt{2}\sin t\,(t^2+1)\,dt = 0.$$

In order to determine the center of mass of the given wire, we proceed as:

$$\bar{x} = \frac{M_{yz}}{M} = \frac{2\sqrt{2}\pi^2(1+2\pi^2)}{\sqrt{2}\left[\frac{8\pi^3}{3}+2\pi\right]} = \frac{3\pi(1+2\pi^2)}{4\pi^2+3}, \qquad \bar{y} = \frac{M_{xz}}{M} = 0, \qquad \text{and} \qquad \bar{z} = \frac{M_{xy}}{M} = 0,$$

Thus, the center of the mass is given as:

$$(\bar{x},\bar{y},\bar{z}) = \left(\frac{3\pi(1+2\pi^2)}{4\pi^2+3}, 0,0\right).$$

Practice Questions

Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.

Chapter: 16

Exercise-16.2: Q – 1 to 16, Q – 33 to 38.

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 16

Exercise-16.1: Q – 1 to 42.