Gauss' Law-I & II

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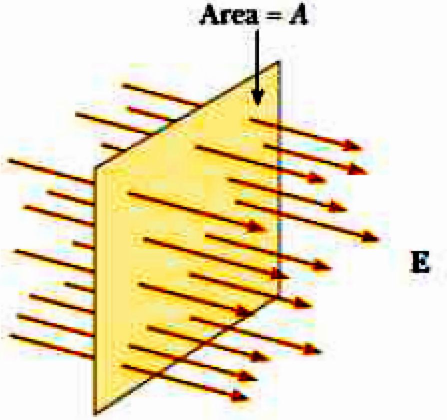
Electric Flux

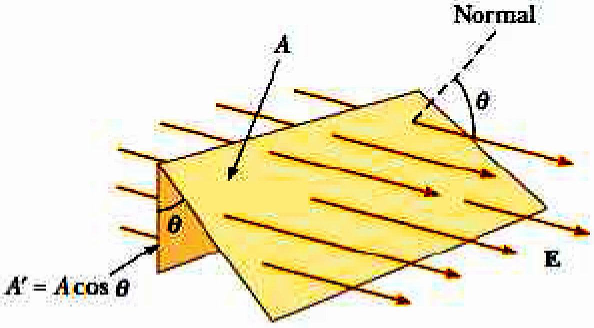
Electric flux is proportional to the number of electric field lines passing through some surface.

$$\boxed{\Phi \propto N}$$



If area is flat and field is uniform, electric flux is defined as

$$\boxed{\Phi_E = \vec{E} \bullet \vec{A}}$$



If area is curved and field is non-uniform,

$$\Delta\Phi_E = \vec{E}_i \bullet \Delta\vec{A}_i$$

Approximate Flux

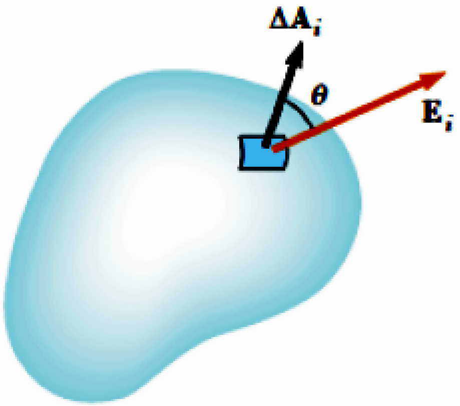
$$\Phi_E = \sum \vec{E} \bullet \Delta\vec{A}$$

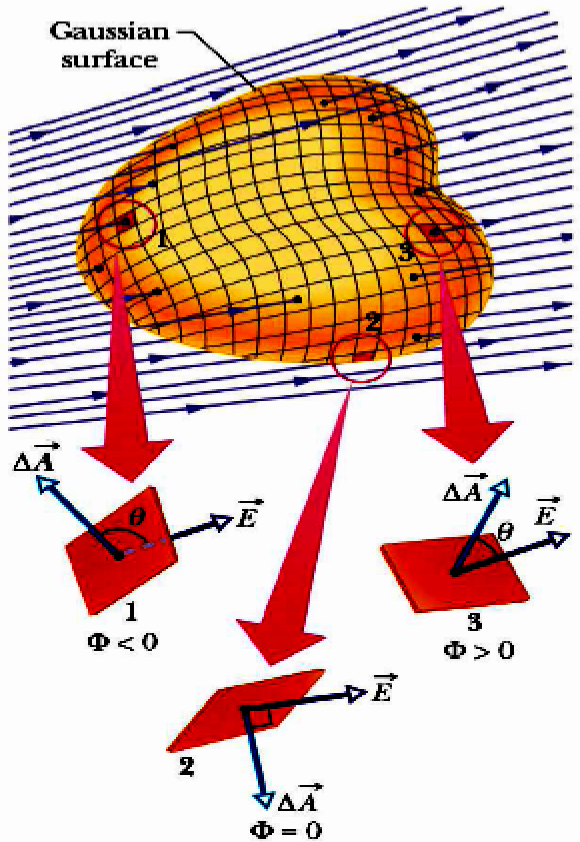
Exact Flux

$$\Phi_E = \lim_{\Delta A \to 0} \sum \vec{E} \bullet \Delta\vec{A}$$

$$\boxed{\Phi_E = \oint \vec{E} \bullet d\vec{A}}$$

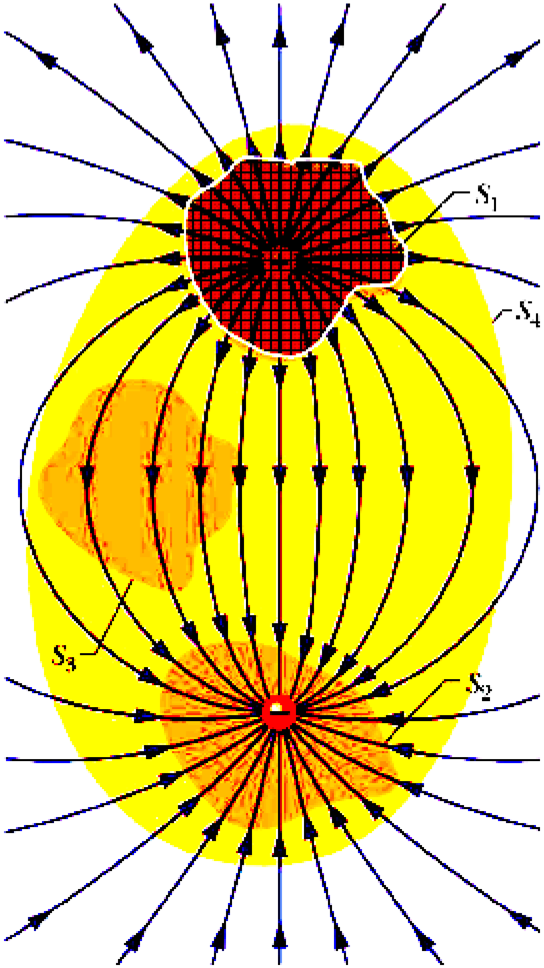
When $\theta < 90^\circ$, the flux is positive (field lines leaving the surface), and when $\theta > 90^\circ$, the flux is negative (field lines entering the surface).



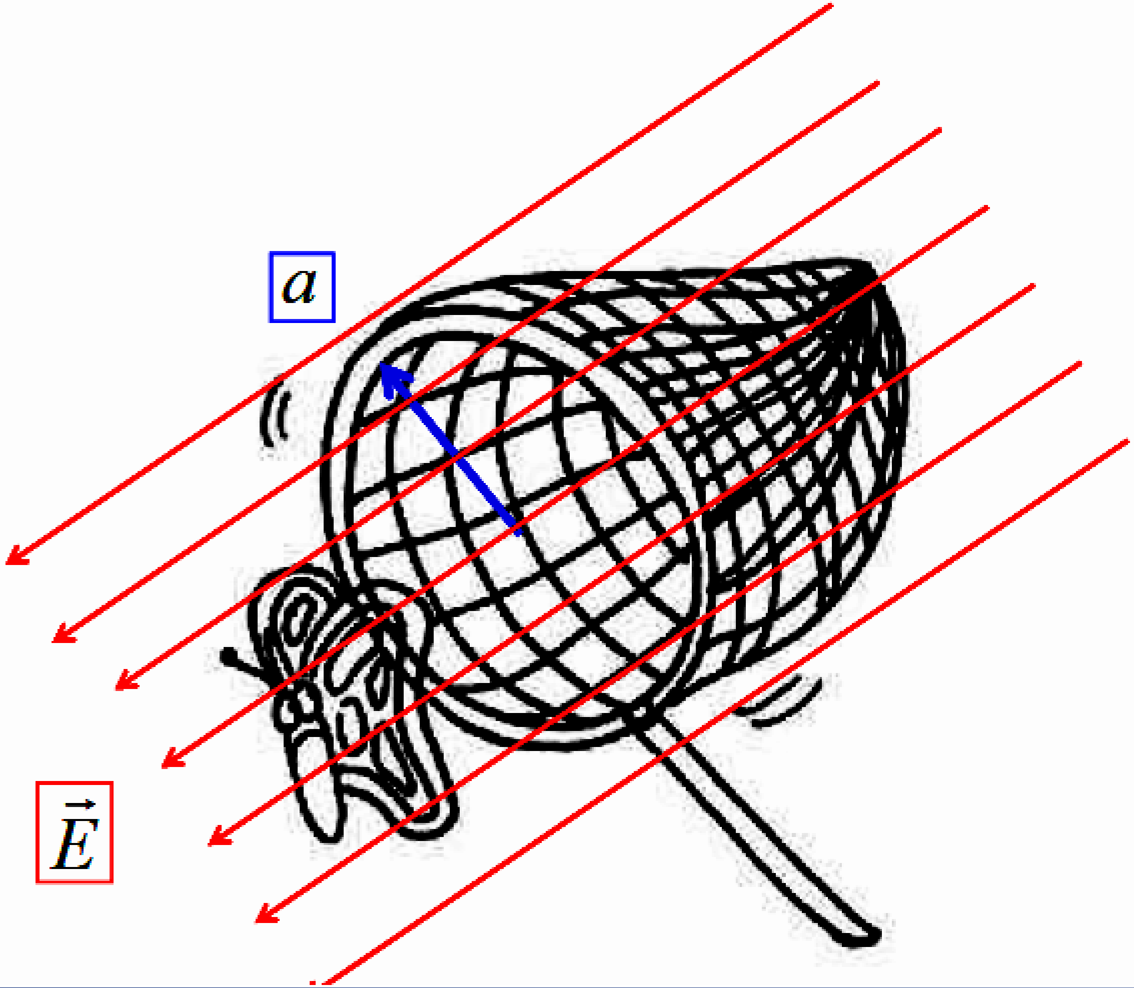


Consider a dipole with equal positive and negative charges. Imagine four surfaces S_1, S_2, S_3, S_4, as shown.

- S_1 encloses the positive charge. Note that the field is everywhere outward, so the flux is positive.
- S_2 encloses the negative charge. Note that the field is everywhere inward, so the flux through the surface is negative.
- S_3 encloses no charge. The flux through the surface is negative at the upper part, and positive at the lower part, but these cancel, and there is no net flux through the surface.
- S_4 encloses both charges. Again there is no net charge enclosed, so there is equal flux going out and coming in— no net flux through the surface.



A butterfly net is in the uniform electric field E as shown in the figure below. The rim, a circle of radius a, is aligned perpendicular to the field. Find the electric flux through the netting.



Net flux through the net will be the sum of flux through the rim and that through the netting

$$\Phi_{Net} = \Phi_{Rim} + \Phi_{Netting}$$

As the number of field line entering the netting are same to that of leaving the rim. So net flux through the butterfly net must be zero :

$$\Phi_{Rim} + \Phi_{Netting} = 0 \qquad (1)$$

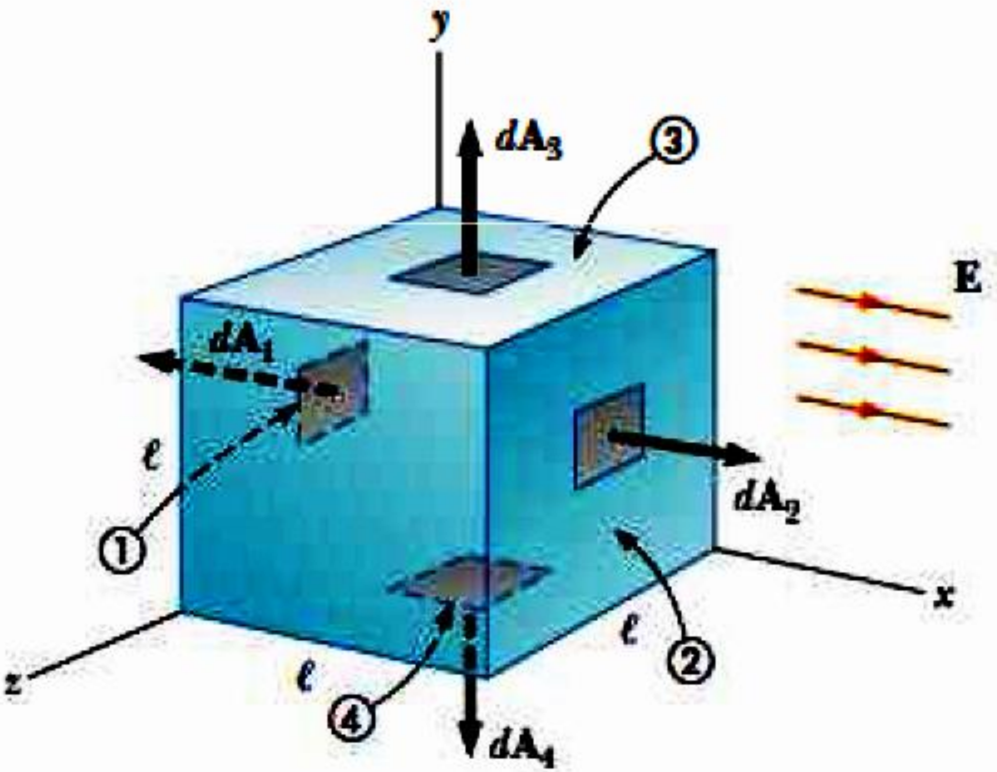
Flux through the rim is

$$\Phi_{Rim} = \vec{E} \bullet \vec{A} = EA = \pi a^2 E$$

From eq.(1), flux through the netting will be

$$\Phi_{Netting} = -\pi a^2 E$$

Consider a uniform electric field E oriented in the *x direction*. A cube of edge length l is oriented as shown in Figure. (a) Find the flux through the right face (2) of the cube. (b) Find the net electric flux through the cube.



(a) Electric flux through the face right face, 2, is

$$\Phi_2 = \vec{E} \bullet \vec{A} = EA\cos 0 = EA = El^2$$

(b) The net flux is the sum of the fluxes through all faces of the cube. First, note that the flux through four of the faces (3, 4, and the unnumbered ones may be 5 and 6) is zero because E is perpendicular to area A on these faces.

$$\Phi_3 = \Phi_4 = \Phi_5 = \Phi_6 = 0$$

Electric flux through the face I is

$$\Phi_1 = \vec{E} \bullet \vec{A} = EA\cos 180 = -EA = -El^2$$

Net electric flux through the cube

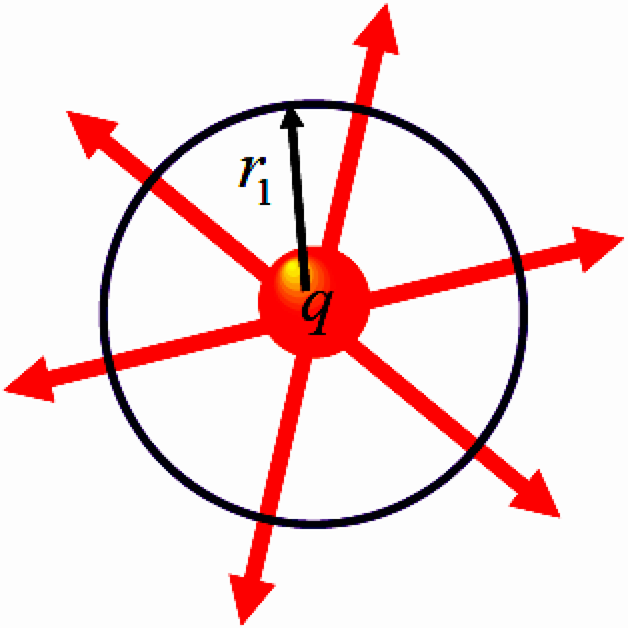
$$\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6 = 0$$

It should be like that because number of field line entering the cube are to that of leaving. So net flux is zero.

Flux Through a Closed Surface

The electric field around a point charge at a distance r1 is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathbf{r}_1^{\,2}}$$



Thus the flux through a sphere of radius r1 is

$$\Phi_1 = \oint \vec{E} \bullet d\vec{A} = \oint E dA$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathbf{r}_1^{\,2}} \times 4\pi\mathbf{r}_1^{\,2}$$

$$\Phi_1 = \frac{q}{\varepsilon_0}$$

Now we change the radius of sphere

The electric field around a point charge at a distance r2 is

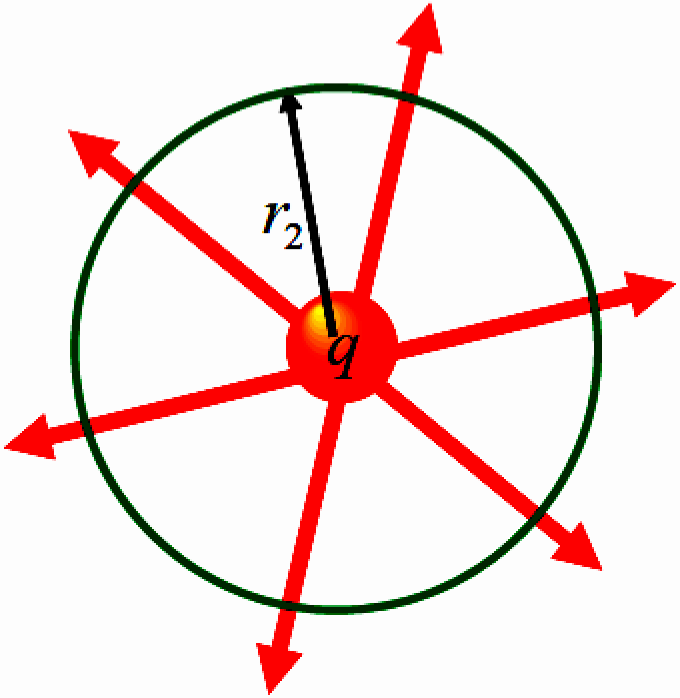
$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathbf{r}_2^{\;2}}$$

Thus the flux through a sphere of radius r2 is

$$\Phi_2 = \oint \vec{E} \bullet d\vec{A} = \oint E dA$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathbf{r}_2^{\;2}} \times 4\pi\mathbf{r}_2^{\;2}$$

$$\Phi_2 = \frac{q}{\varepsilon_0}$$



The flux is the same as before

$$\Phi_2 = \Phi_1 = \frac{q}{\varepsilon_0}$$

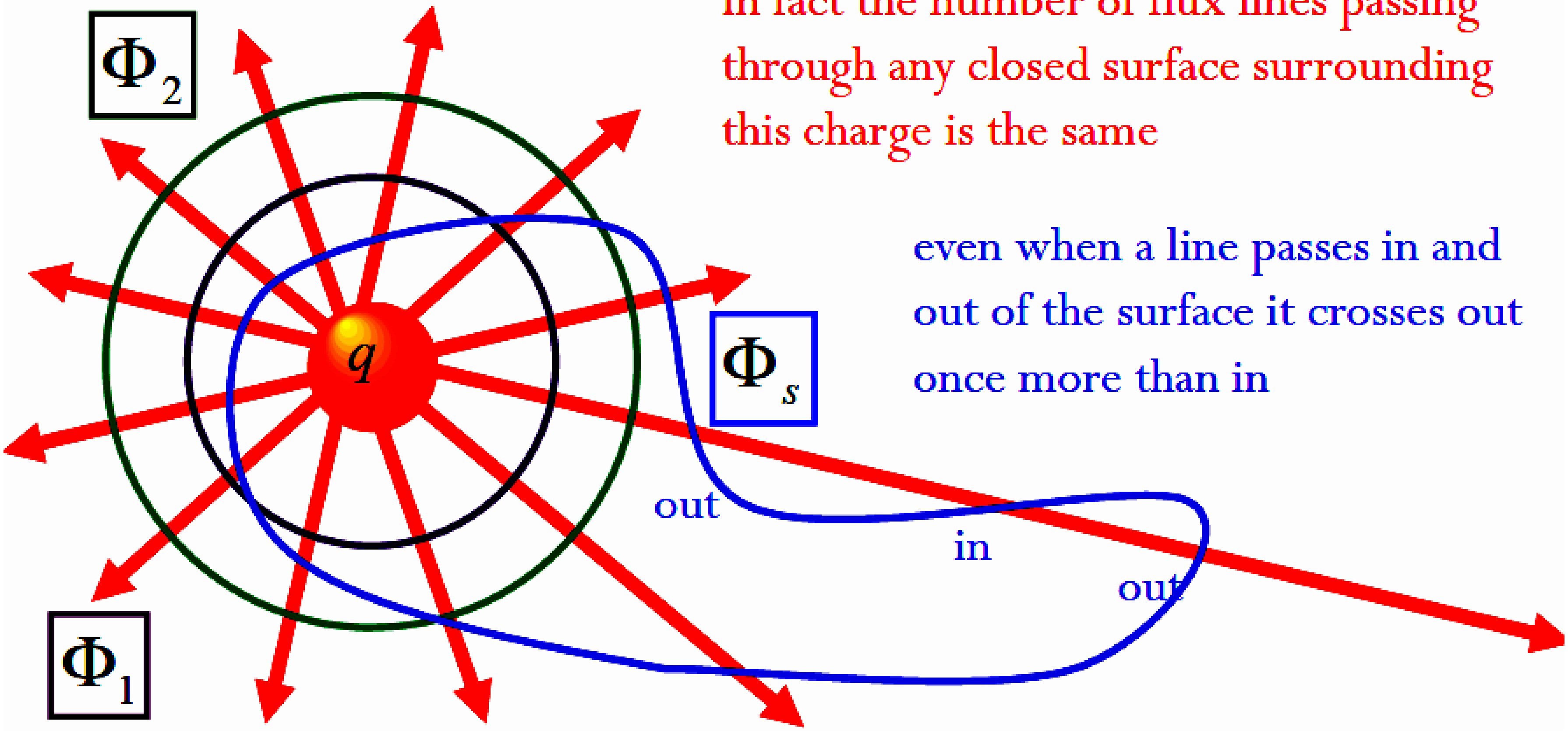
Just what we would expect because the number of field lines passing through each sphere is the same

$$\boxed{\Phi \propto N}$$

$$\boxed{\Phi_S = \Phi_2 = \Phi_1 = \frac{q}{\varepsilon_0}}$$

In fact the number of flux lines passing through any closed surface surrounding this charge is the same

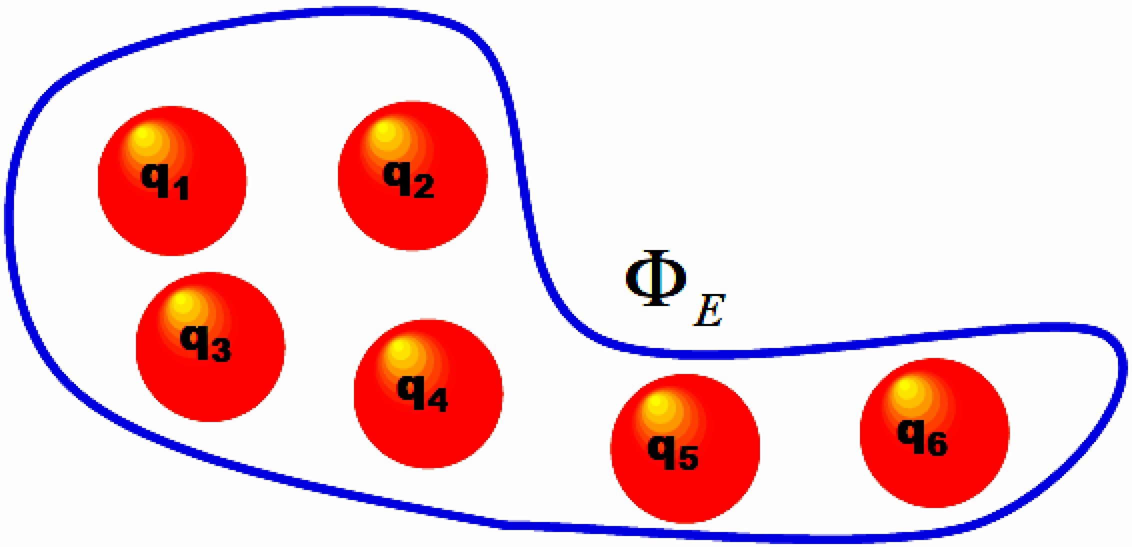
even when a line passes in and out of the surface it crosses out once more than in



Principle of Superposition:

Since the flux is related to the number of field lines passing through a surface, the total flux is

$$\Phi_E = \frac{q_1}{\varepsilon_0} + \frac{q_2}{\varepsilon_0} + \dots\dots\dots\dots$$

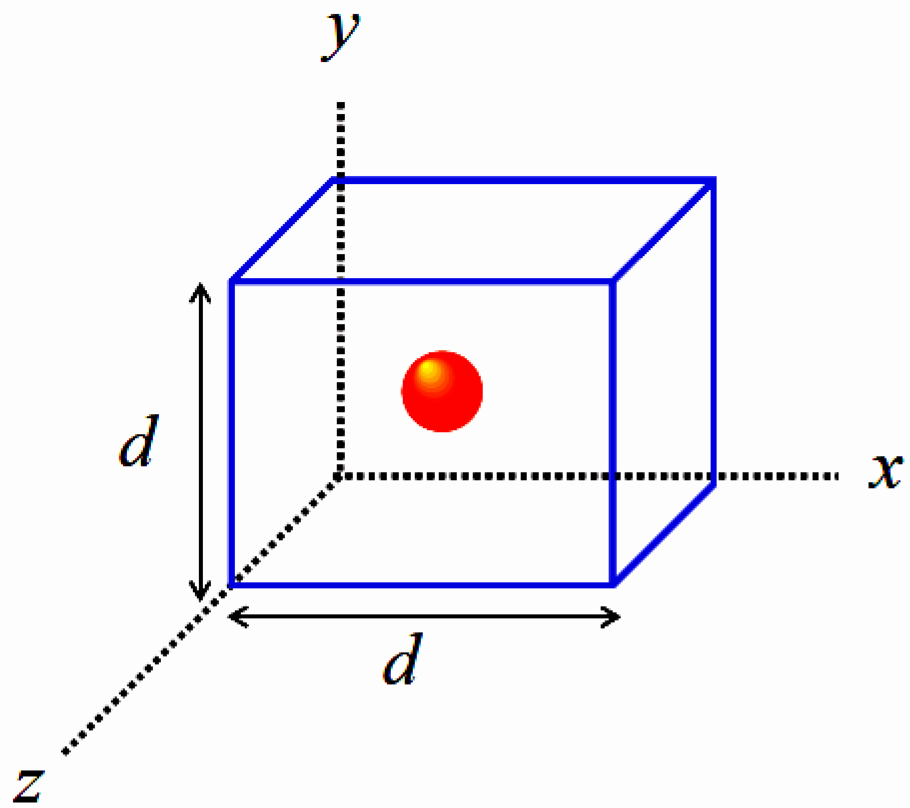


$$\boxed{\Phi_E = \frac{q_{enclosed}}{\varepsilon_0}}$$

Gauss's Law

For any closed surface

A point charge q is placed at the center of a cube of edge length d. (a) Find the flux through the whole cube. (b) What will be the flux through each face of cube?



(a) Electric flux through any closed surface enclosing charge q is

$$\Phi_E = \frac{q}{\varepsilon_\circ}$$

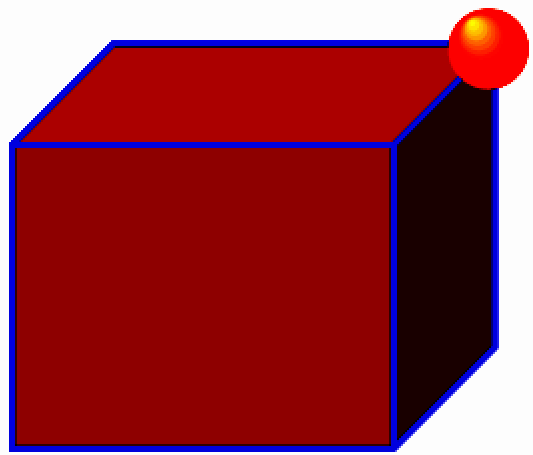
So the total flux through the cube enclosing charge q would be

$$\Phi_{cube} = \frac{q}{\varepsilon_\circ}$$

(b) Since the charge is in the center of the cube, we expect that the flux through any side would be the same, or 1/6 of the total ux. Hence the flux through each face will be

$$\Phi_{face} = \frac{q}{6\varepsilon_\circ}$$

A point charge q is placed at one corner of a cube of edge a. What is the flux through each of the cube faces?



There are eight cubes which can be wrapped around the charge. Total flux through theses eight cubes will be

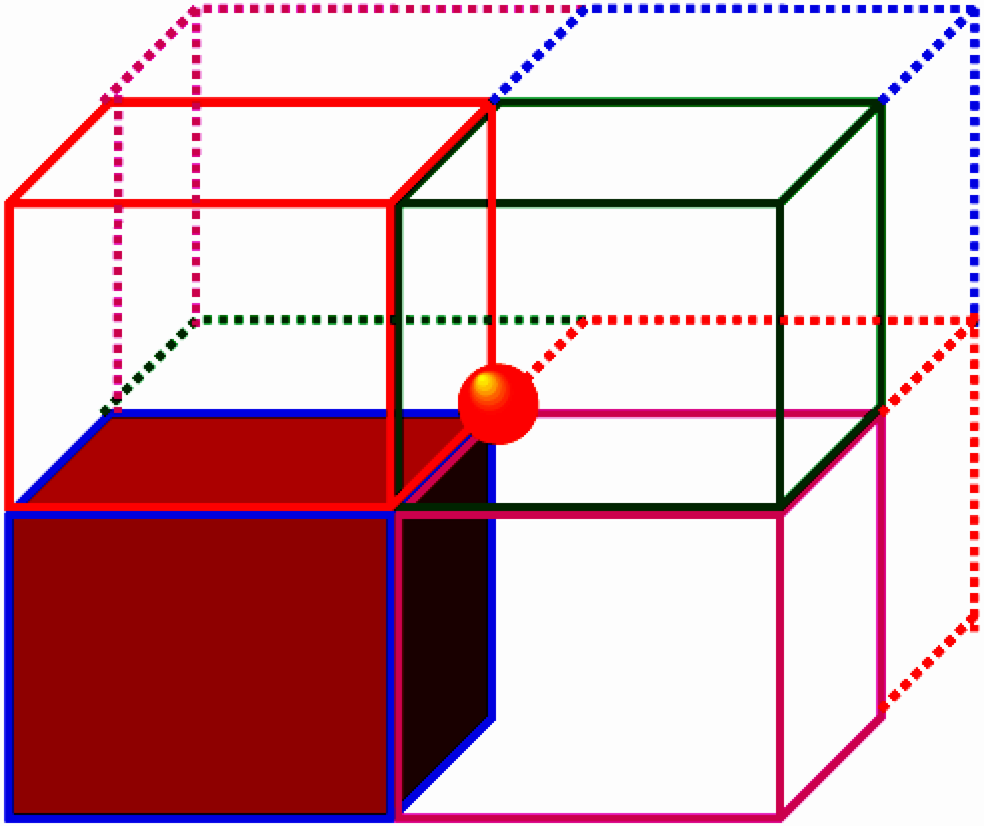
$$\Phi_E = \frac{q}{\varepsilon_\circ}$$

For faces which touch the charge the electric field is parallel to the surface, so the flux would be zero.

So net flux will be through 24 faces opposite the charge.

Hence flux through each face opposite to charge will be

$$\Phi_{face} = \frac{q}{24\varepsilon_\circ}$$



Gauss' Law

The electric flux through a closed surface is proportional to the charge enclosed:

$$\Phi_E = \oint \vec{E}.d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

Gauss's Law is used to calculate electric field due to enclosed charge if symmetry exists.

Gauss's Law does not tell us anything new, it is NOT a new law of physics, but another way of expressing Coulomb's Law

Gauss's Law is sometimes easier to use than Coulomb's Law, especially if there is lots of symmetry in the problem

How to apply Gauss' Law

Area element of Gaussian Surface

Charge enclosed by Gaussian Surface

Gaussian Surface

$$\oint \vec{E}.d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

Electric field due to enclosed charge at every point of Gaussian Surface

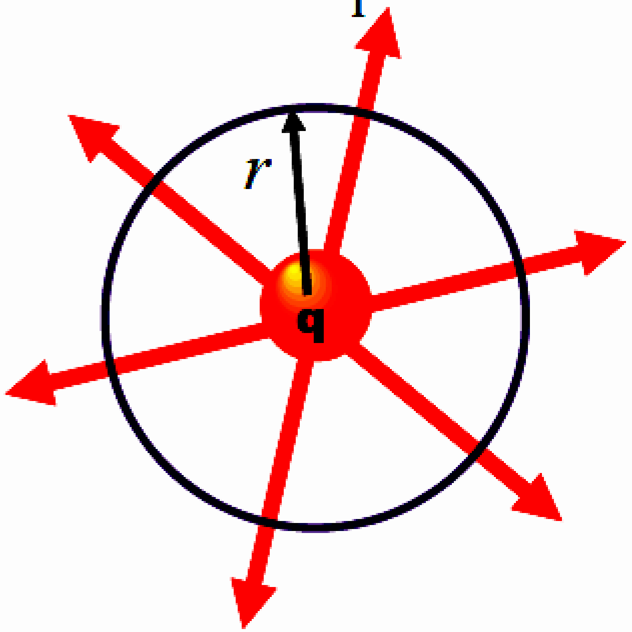
Permittivity of free space

Gaussian Surface must be chosen according to symmetry of enclosed charge

Electric Field due to a point charge

consider spherical Gaussian surface of radius r centred on a point charge q.

$$\oint \vec{E}.d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$$



By symmetry **E** is ⊥ to surface and is constant at all the points of Gaussian surface

$$E\oint dA = \frac{q}{\varepsilon_o}$$

$$E \times 4\pi r^2 = \frac{q}{\varepsilon_o} \quad \Rightarrow \quad E = \frac{q}{4\pi\varepsilon_o r^2}$$

Review: Point Charges

- The Electric Field is related to Coulomb's Force by

$$\vec{E} = \vec{F} / q_{\circ}$$

- Thus knowing the field we can calculate the force on a charge or vice versa.
- Using superposition we find

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{Q_i}{|\mathbf{r}_i|^2} \hat{\mathbf{r}}_i \qquad \vec{F} = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_{\circ} Q_i}{|\mathbf{r}_i|^2} \hat{\mathbf{r}}_i$$

But when these point charges are placed at every point of a surface (Continuous charge distributions) then it will be useful to deal with charge densities instead of charges, i.e., we shift from summation to integral.

Volume Charge Density: 3D

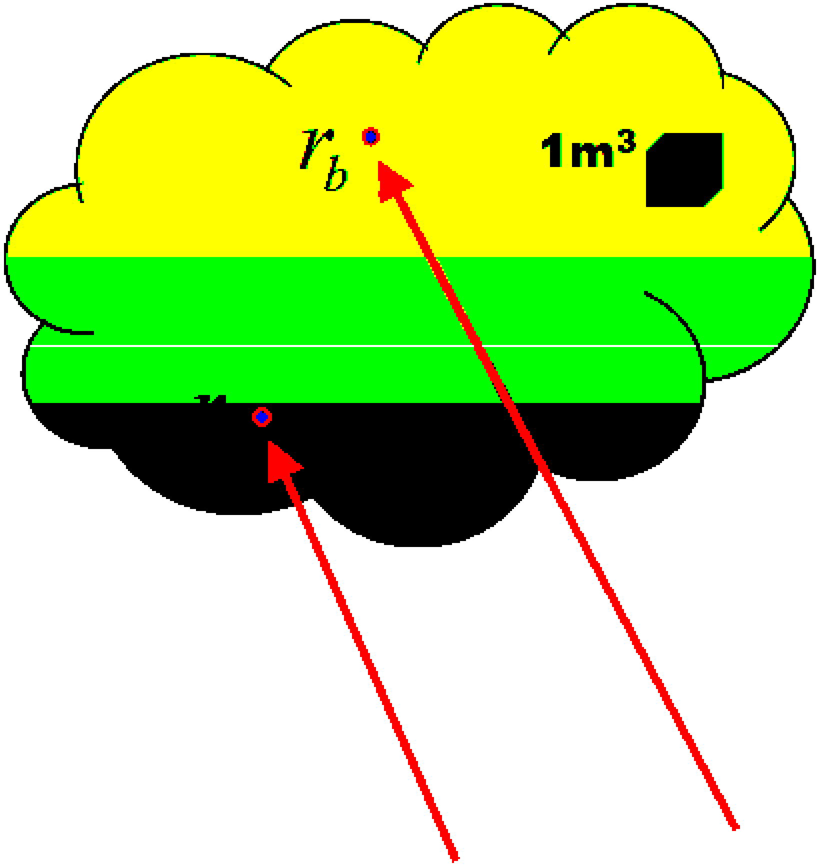
$$\rho = \frac{dq}{dV}$$

$$q = \int \rho dV$$

Uniform charge density

$$\rho = \frac{q}{V}$$

$$q = \rho V$$

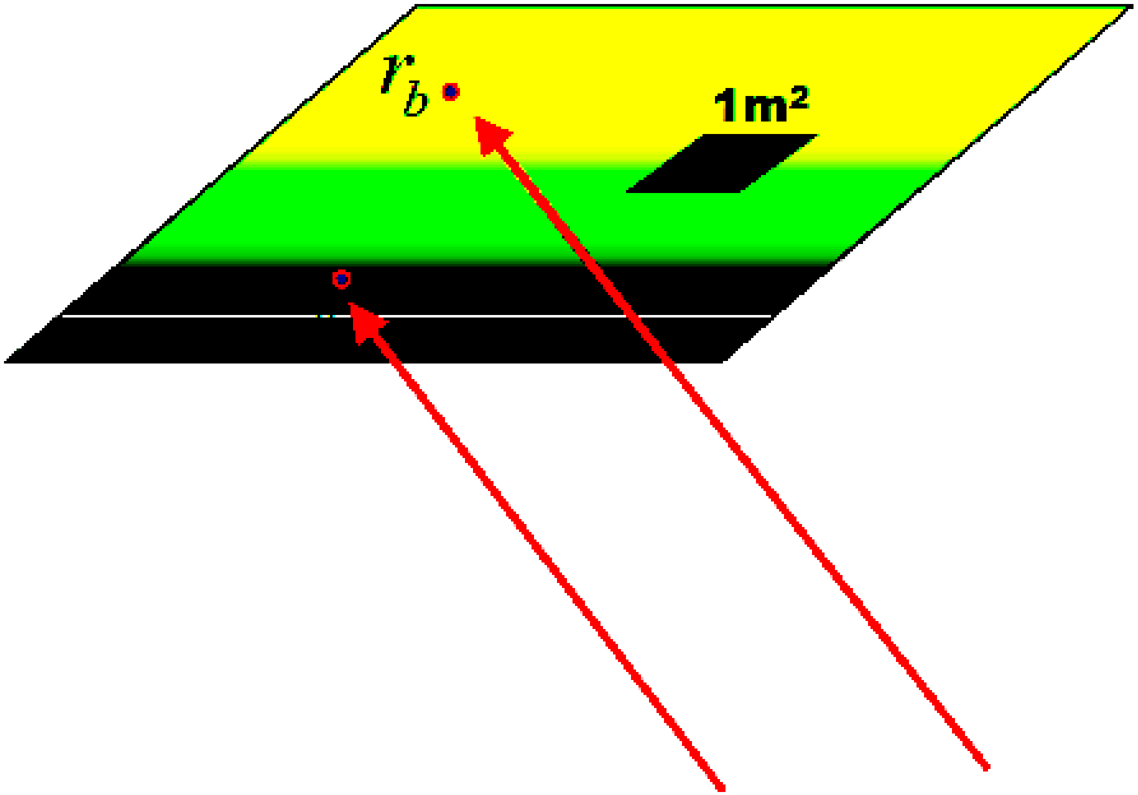


$$\rho(r_a) > \rho(r_b)$$

Surface Charge Density: 2D

$$\sigma = \frac{dq}{dA}$$

$$q = \int \sigma dA$$



Uniform charge density

$$\sigma = \frac{q}{A}$$

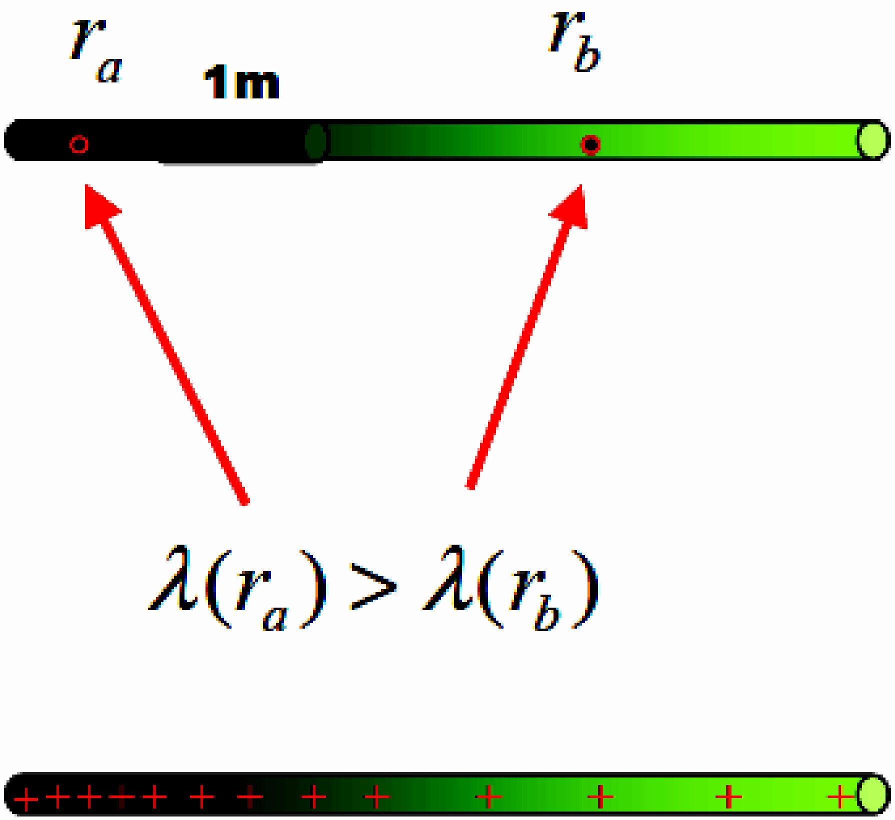
$$q = \sigma A$$

$$\sigma(r_a) > \sigma(r_b)$$

Linear Charge Density: 1D

$$\lambda = \frac{dq}{dL}$$

$$q = \int \lambda dL$$



Uniform charge density

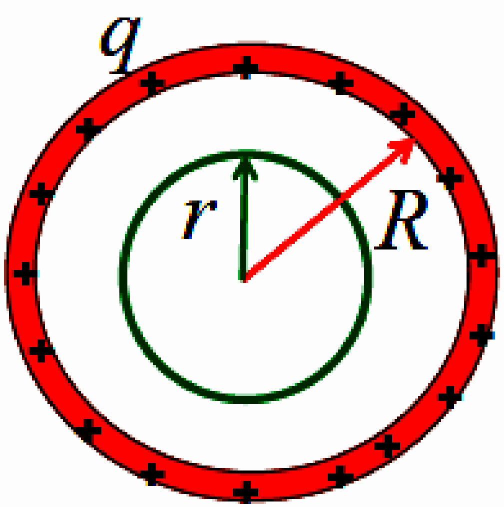
$$\lambda = \frac{q}{L}$$

$$q = \lambda L$$

Electric Field due to a charged spherical shell

Consider an insulating spherical shell of radius R having a total positive charge q.

Inside shell $r < R$



consider spherical Gaussian surface of radius r concentric with charged shell of radius R.

$$\oint \vec{E}.d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$$

$$\oint \vec{E}.d\vec{A} = 0 \qquad \therefore q_{enc} = 0$$

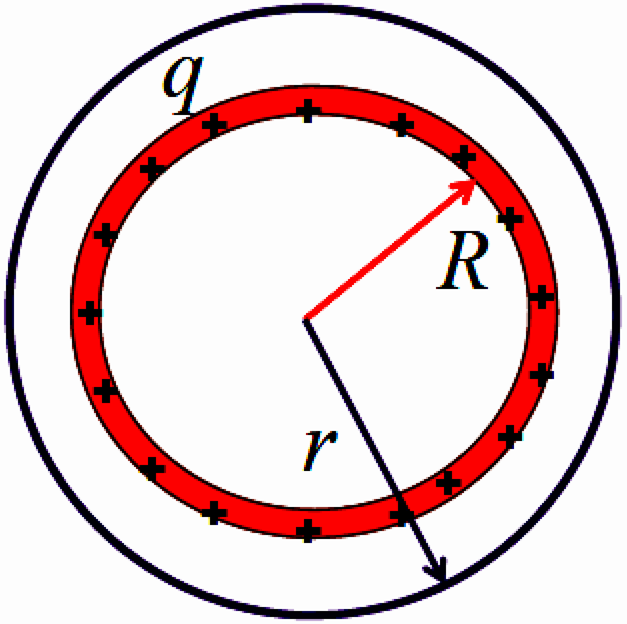
$$\boxed{E = 0}$$

If a charged particle is located inside a shell of uniform charge, there is no net electric force on the particle from the shell.

Outside shell $r > R$

consider spherical Gaussian surface of radius r concentric with charged shell of radius R.

$$\oint \vec{E}.d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$$

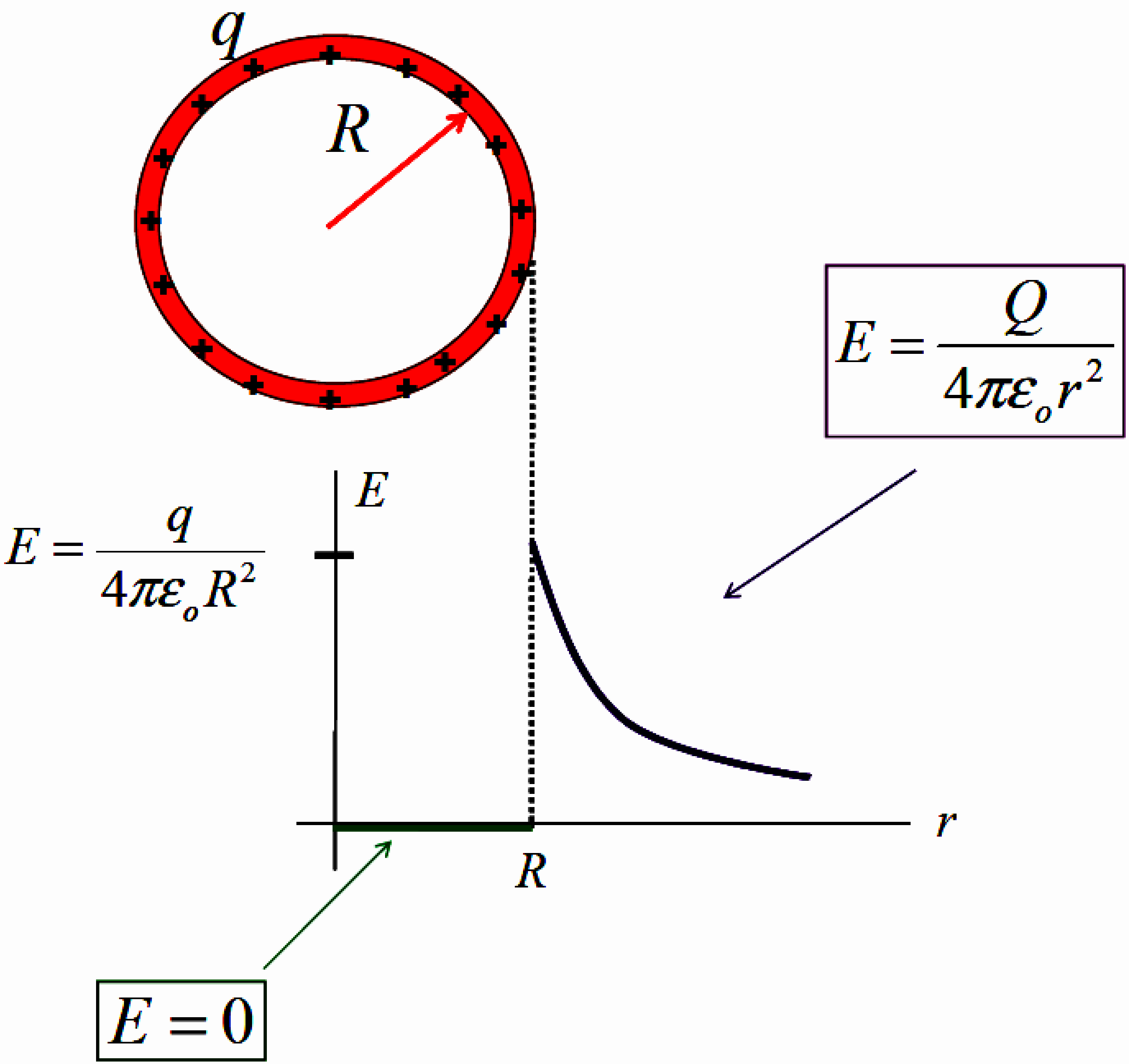


By symmetry **E** is ⊥ to surface and is constant at all the points of Gaussian surface

$$E\oint dA = \frac{q}{\varepsilon_o} \qquad \therefore q_{enc} = q$$

$$E \times 4\pi r^2 = \frac{q}{\varepsilon_o} \quad \Rightarrow \quad \boxed{E = \frac{q}{4\pi\varepsilon_o r^2}}$$

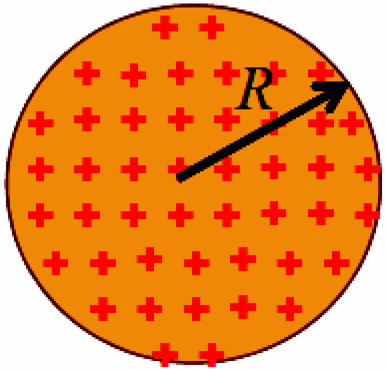
A shell of uniform charge attracts and repels a charged particle that is outside the shell as if all shell's charge were concentrated at the center.


q
R
$E = \frac{Q}{4\pi\varepsilon_o r^2}$
E
$E = \frac{q}{4\pi\varepsilon_o R^2}$
r
R
$E = 0$

E due to a spherically symmetric charge distribution

Consider an insulating solid sphere of radius R having a uniform volume charge density ρ and carries a total positive charge Q. Its volume charge density is

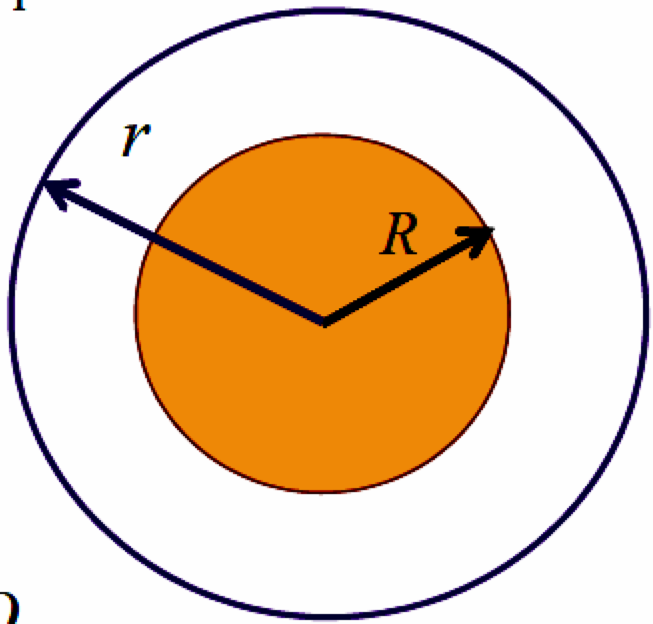
$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} \quad \mathrm{C/m^3}$$



By symmetry, the electric field due to this charge distribution will be everywhere radial from the center of the sphere.

Outside sphere

Consider a spherical Gaussian surface of radius r concentric with spherical charged distribution of radius R such that $r > R$.



$$\oint \vec{E}.d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$$

By symmetry **E** is ⊥ to surface and is constant at all the points of Gaussian surface

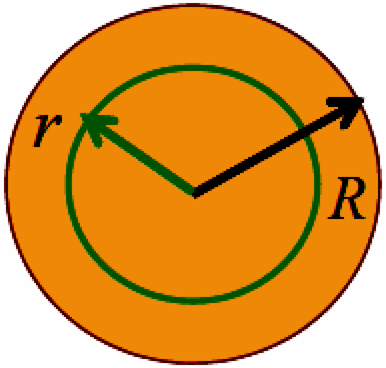
$$E\oint dA = \frac{Q}{\varepsilon_o} \qquad \therefore q_{enc} = Q$$

$$E \times 4\pi r^2 = \frac{Q}{\varepsilon_o} \quad \Rightarrow \quad \boxed{E = \frac{Q}{4\pi\varepsilon_o r^2}}$$

For a uniformly charged sphere, the field in the region external to the sphere is equivalent to that of a point charge located at the center of the sphere.

Inside Sphere

Consider a spherical Gaussian surface of radius r concentric with spherical charged distribution of radius R such that $r < R$.



$$\oint \vec{E}.d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$$

Volume of Gaussian Surface

By symmetry **E** is $\perp$ to surface and is constant at all the points of Gaussian surface

$$E\oint dA = \frac{\rho 4\pi r^3 / 3}{\varepsilon_o}$$

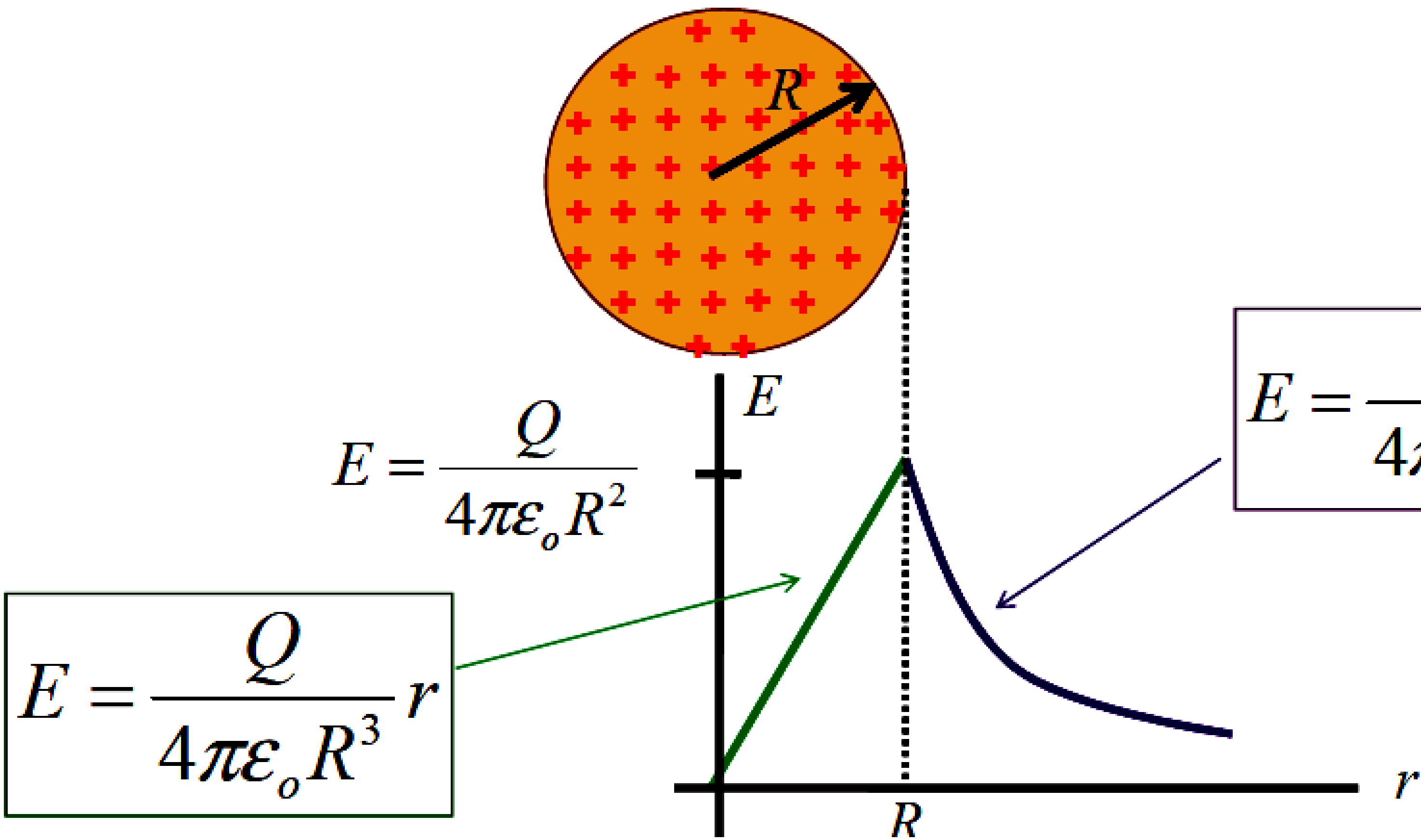
$$q_{enc} = \rho V'$$

$$= \rho 4\pi r^3 / 3$$

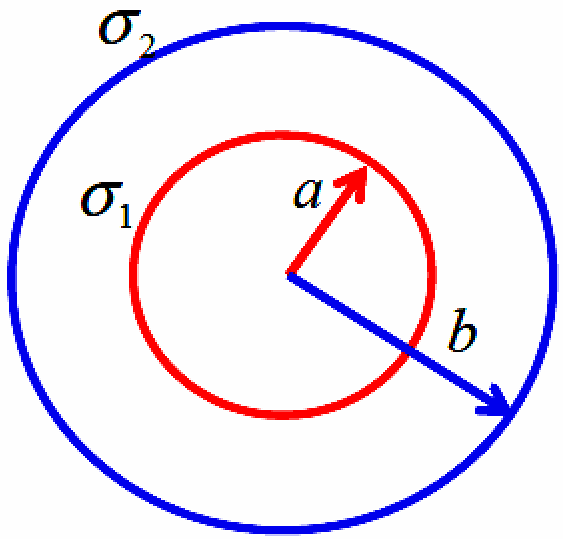
$$E \times 4\pi r^2 = \frac{\rho 4\pi r^3}{3\varepsilon_o} \Rightarrow \boxed{E = \frac{\rho}{3\varepsilon_o} r}$$

$$\boxed{E = \frac{Q}{4\pi\varepsilon_o R^3} r} \qquad \therefore \rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

It shows that $E \longrightarrow 0$ as $r \longrightarrow 0$.

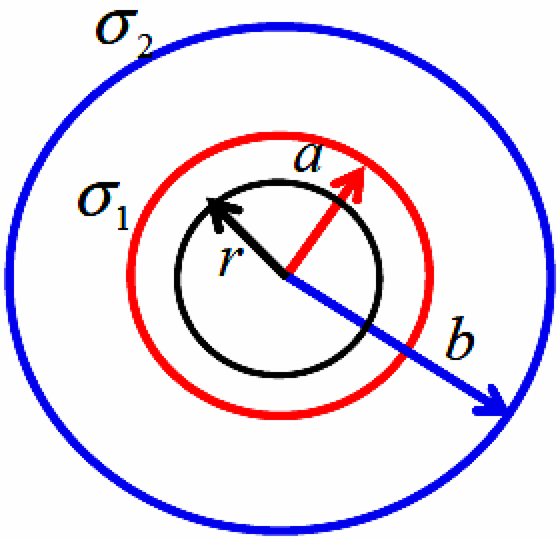


Two charged, thin, concentric spherical shells of radii a and b ($a < b$) contain uniform surface charge densities σ_1 and σ_2, respectively. Determine the electric field for (a) $r < a$, (b) $a < r < b$, (c) $r > b$.



(a) r < a

Consider a spherical Gaussian surface of radius r such that r < a. According to Gauss's law



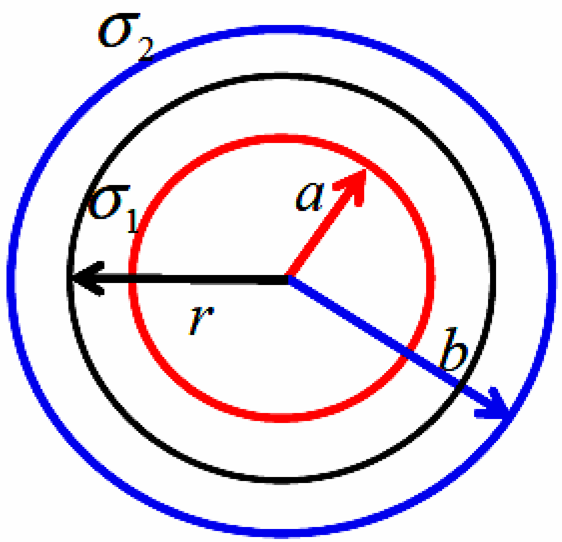
$$\oint \vec{E}.d\vec{A} = q_{enc} / \varepsilon_o$$

As charge enclosed is zero, so

$$E = 0$$

(b) a < r < b

Consider a spherical Gaussian surface of radius r such that a<r <b. According to Gauss's law



$$\oint \vec{E}.d\vec{A} = q_{enc} / \varepsilon_o$$

As electric field always remains perpendicular to the surface of shell and enclosed charge is

$$q_{enc} = \sigma_1 \times 4\pi a^2$$

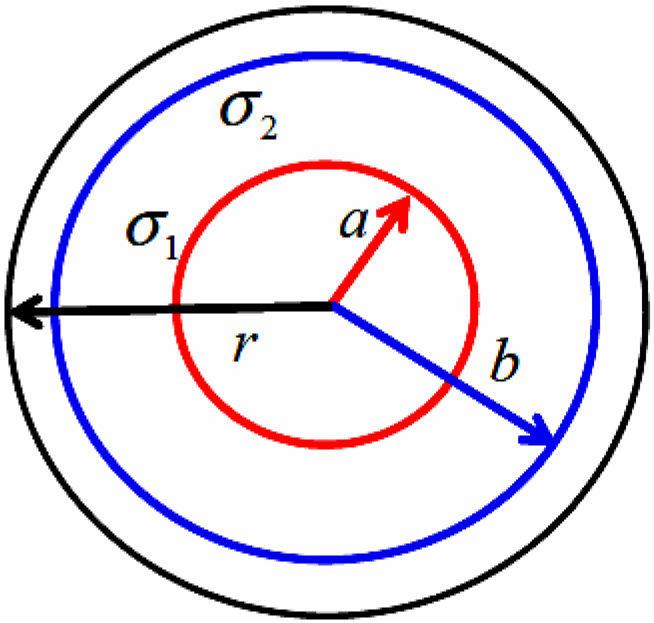
$$E\oint dA = (\sigma_1 \times 4\pi a^2) / \varepsilon_o$$

$$E \times 4\pi r^2 = 4\pi a^2 \sigma_1 / \varepsilon_o \quad \Rightarrow \quad E = \frac{\sigma_1 a^2}{\varepsilon_o r^2}$$

(c) r > b

Consider a spherical Gaussian surface of radius r such that r >b. According to Gauss's law

$$\oint \vec{E}.d\vec{A} = q_{enc} / \varepsilon_o$$



As electric field always remains perpendicular to the surface of shell and enclosed charge is

$$q_{enc} = \sigma_1 \times 4\pi a^2 + \sigma_2 \times 4\pi b^2$$

$$E\oint dA = (\sigma_1 \times 4\pi a^2 + \sigma_2 \times 4\pi b^2) / \varepsilon_o$$

$$E \times 4\pi r^2 = (4\pi a^2 \sigma_1 + 4\pi b^2 \sigma_2) / \varepsilon_o \Rightarrow E = \frac{\sigma_1 a^2 + \sigma_2 b^2}{\varepsilon_o r^2}$$

E due to Line of Charges

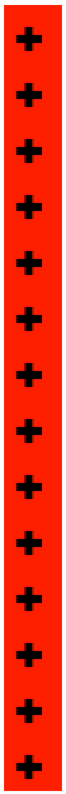
Consider a line of positive charge of infinite length and constant charge per unit length λ.

- By symmetry, E must be directed outward from the line and must have the same magnitude at all points equidistant from the plane.

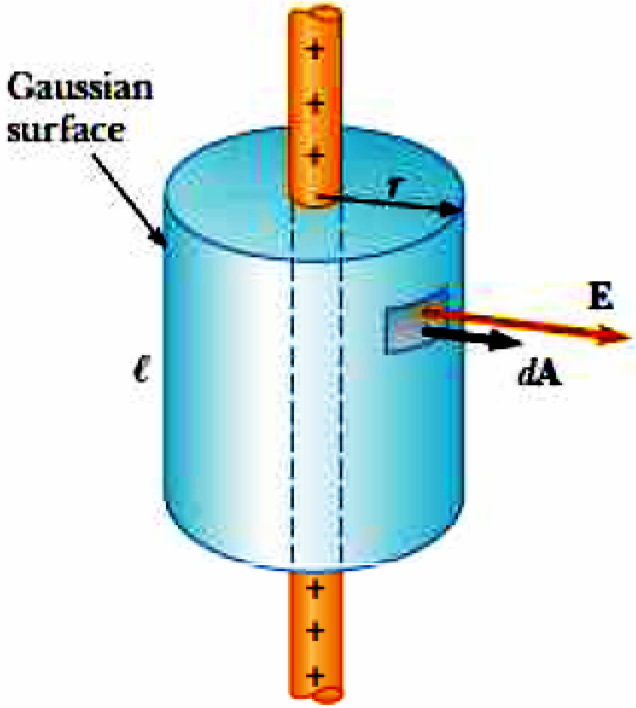
To find electric field at a distance r from line, let's consider a cylindrical Gaussian surface of radius r and length L that is coaxial with the line charge.

According to Gauss' law

$$\oint \vec{E}.d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$$



- For flat ends of Gaussian surface —the electric field is perpendicular to vector area everywhere on the surface—there is no contribution to the surface integral from this surface.
- For curved part, electric field is constant at all points and is parallel to area element. we will restrict our attention to only the curved surface of the cylinder..
- The enclosed charge will be λL.



$$E\oint dA = \frac{q_{enc}}{\varepsilon_o}$$

$$E \times 2\pi r L = \frac{\lambda L}{\varepsilon_o} \Rightarrow \boxed{E = \frac{\lambda}{2\pi\varepsilon_o r}}$$

Note:

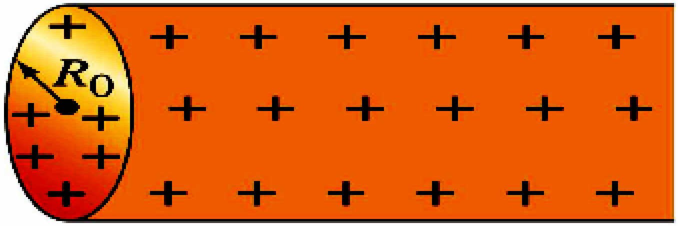
- If the line charge in this example were of finite length, it does not possess sufficient symmetry for us to make use of Gauss's law. This is because
- the magnitude of the electric field is no longer constant over the Gaussian surface—the field near the ends of the line would be different from that far from the ends.
- the electric field is not perpendicular to the Gaussian surface at all points—the field vectors near the ends would have a component parallel to the line.

When there is insufficient symmetry in the charge distribution, as in this situation, it is necessary to use Coulomb's law to calculate E.

E due to a Cylindrically Symmetric Charged Distribution

Consider an insulating solid cylinder of radius R and length L such that R<<L. Suppose it has a uniform volume charge density ρ and carries a total positive charge Q. Its volume charge density is

$$\rho = \frac{Q}{\pi R^2 L} \quad \mathrm{C/m^3}$$

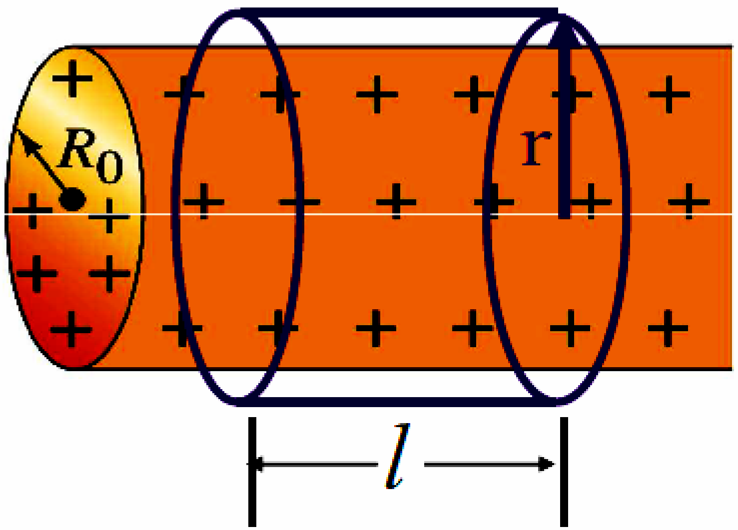


- By symmetry, the electric field due to this charge distribution will be everywhere radial from the axis of the cylinder.

Outside Cylinder

Consider a cylindrical Gaussian surface of radius r and length l concentric with cylindrical charged distribution such that $r > R$ and $l << L$.

- For flat ends of Gaussian surface —the electric field is perpendicular to vector area everywhere on the surface—there is no contribution to the surface integral from this surface.
- For curved part, electric field is constant at all points and is parallel to area element. we will restrict our attention to only the curved surface of the cylinder..
- The enclosed charge will be



$$q_{enc} = \rho V'$$

Volume of the region enclosing charge

$$= \rho \pi R^2 l$$

According to Gauss' law $$\oint \vec{E}.d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$$

$$E\oint dA = \frac{\rho\pi R^2 l}{\varepsilon_o}$$

$$E \times 2\pi r l = \frac{\rho\pi R^2 l}{\varepsilon_o} \Rightarrow \boxed{E = \frac{\rho R^2}{2\varepsilon_o r}}$$

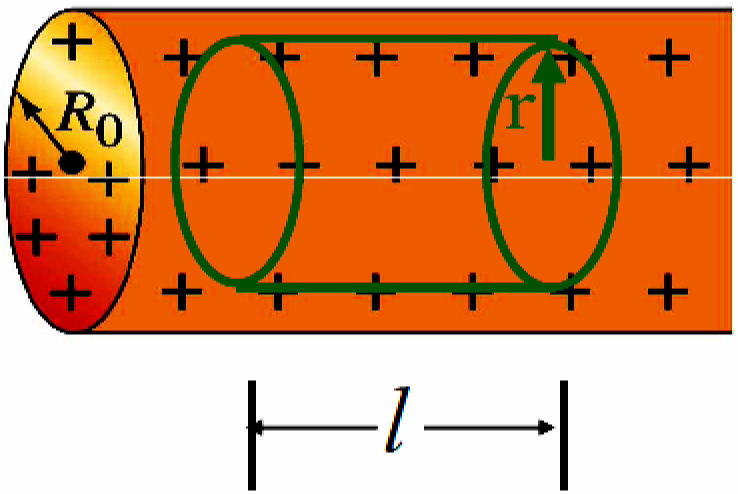
$$\boxed{E = \frac{Q}{2\pi\varepsilon_o r L}} \qquad \therefore \rho = \frac{Q}{\pi R^2 L}$$

For a uniformly charged cylinder, the field in the region external to the cylinder is equivalent to that of a line of charge located at the axis of the cylinder. $\boxed{E = \lambda / 2\pi\varepsilon_o r}$

Inside cylinder

Consider a cylindrical Gaussian surface of radius r and length l concentric with cylindrical charged distribution such that $r < R$ and $l << L$.

- For flat ends of Gaussian surface —the electric field is perpendicular to vector area everywhere on the surface—there is no contribution to the surface integral from this surface.
- For curved part, electric field is constant at all points and is parallel to area element. we will restrict our attention to only the curved surface of the cylinder..
- The enclosed charge will be



$$q_{enc} = \rho V'$$

Volume of the region enclosing charge

$$= \rho \pi r^2 l$$

According to Gauss' law $\oint \vec{E}.d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$

$$E\oint dA = \frac{\rho \pi r^2 l}{\varepsilon_o}$$

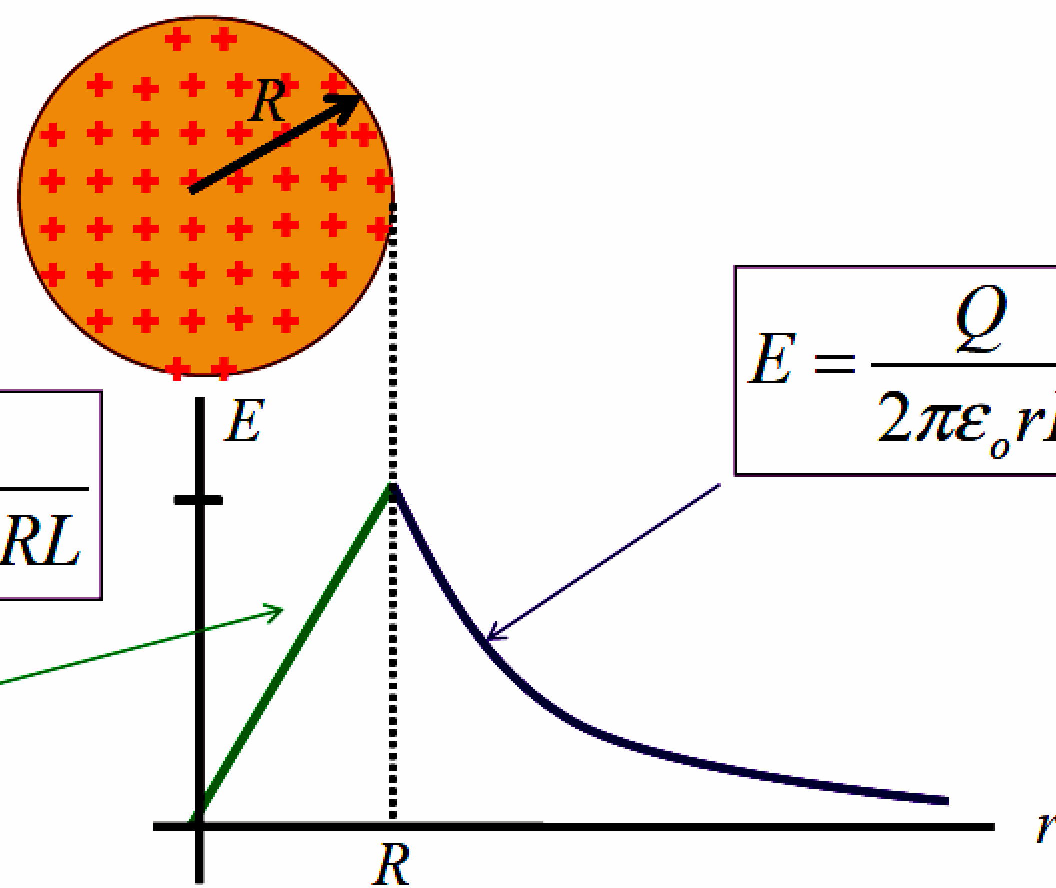
$$E \times 2\pi r l = \frac{\rho \pi r^2 l}{\varepsilon_o} \Rightarrow \boxed{E = \frac{\rho}{2\varepsilon_o} r}$$

$$\boxed{E = \frac{Q}{2\pi \varepsilon_o R^2 L} r} \qquad \therefore \rho = \frac{Q}{\pi R^2 L}$$

It shows that E $\rightarrow$ 0 as r $\rightarrow$ 0.

For a uniformly charged cylinder, the field in the region internal to the cylinder varies with distance as proportional to r.

Cross sectional view of
cylindrical charge distribution



$$E = \frac{Q}{2\pi\varepsilon_o RL}$$

$$E = \frac{Q}{2\pi\varepsilon_o rL}$$

$$E = \frac{Q}{2\pi\varepsilon_o R^2 L} r$$