Z- TRANSFORM

Introduction

- Developed Laplace transform as an extension of the CT Fourier transform.
- Laplace transform can be applied to a broader class of signals than the Fourier transform.
- In this lecture we develop the *z*-transform as the discrete-time counterpart of the Laplace transform.

Z - Transform

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\ &= \left[\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \right] e^{st} \\ &= \underbrace{H(s)}_{\text{eigenvalue}} \; \underbrace{e^{st}}_{\text{eigenfunction}} \end{aligned}$$

$$x[n] = z^{n} \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m] z^{n-m} \\ &= \left[\sum_{m=-\infty}^{\infty} h[m] z^{-m} \right] z^{n} \\ &= \underbrace{H(z)}_{\text{eigenvalue}} \; \underbrace{z^{n}}_{\text{eigenfunction}} \end{aligned}$$

$$\boxed{\begin{aligned} H(s) &= \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \quad -- \ \text{CT} \\ H(z) &= \sum_{n=-\infty}^{\infty} h[n] z^{-n} \quad -- \ \text{DT} \end{aligned}}$$

Z - Transform

- For a discrete-time LTI system with impulse response $h[n]$, the response $y[n]$ of the system to a complex exponential of the form z^n is:

$$y[n] = H(z)\, z^n$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

- For the case $z = e^{j\omega}$ with ω real, the equation above corresponds to the discrete-time FT of $h[n]$. In the more general case it is called the $z-$transform, and is defined as:

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

z is a complex variable

- and we refer to the $z-$transform pair as:

$$\boxed{x[n] \xleftrightarrow{z} X(z)}$$

Z - Transform

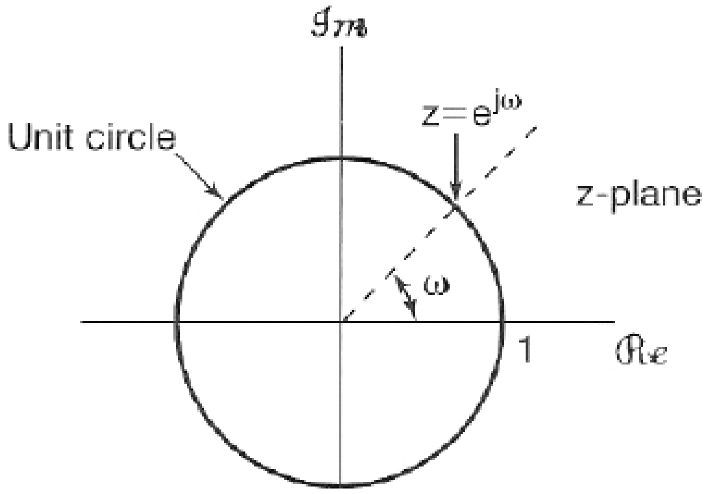
- To explore the relationships between the $z-$transform and the DTFT we express the complex variable z in polar form as:

$$z = re^{j\omega}$$

- with r the magnitude of z and ω the angle of z. We now express the $z-$transform as

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\omega}\right)^{-n} = \sum_{n=-\infty}^{\infty} \left\{x[n]r^{-n}\right\}\left(e^{-j\omega}\right)^{n}$$

- i.e., as the FT of the sequence $x[n]$ multiplied by a real exponential r^{-n}



Note that for $r=1$, or equivalently $|z|=1$, we have the relation:

$$X(z)\big|_{z=e^{j\omega}} = X(e^{j\omega}) = \mathcal{F}\left\{x[n]\right\}$$

i.e., the z-transform is the same as the Fourier transform.

Z – Transform and ROC

$z = re^{j\omega} \qquad , r = |z|$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\omega}\right)^{-n} = \sum_{n=-\infty}^{\infty}\left(x[n]r^{-n}\right)e^{-j\omega n}$$

$$= \mathcal{F}\{x[n]r^{-n}\}$$

- $\text{ROC} = \left\{ z = re^{j\omega} \text{ at which } \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty \right\}$

  — depends only on $r = |z|$, just like the ROC in s-plane only depends on $Re\{s\}$

- Unit circle ($r = 1$) in the ROC $\Rightarrow$ DTFT $X(e^{j\omega})$ exists

Example - 1

- Find the z-transform and plot the ROC

$$x[n] = a^n u[n] \text{ - right-sided}$$

Example - 1

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

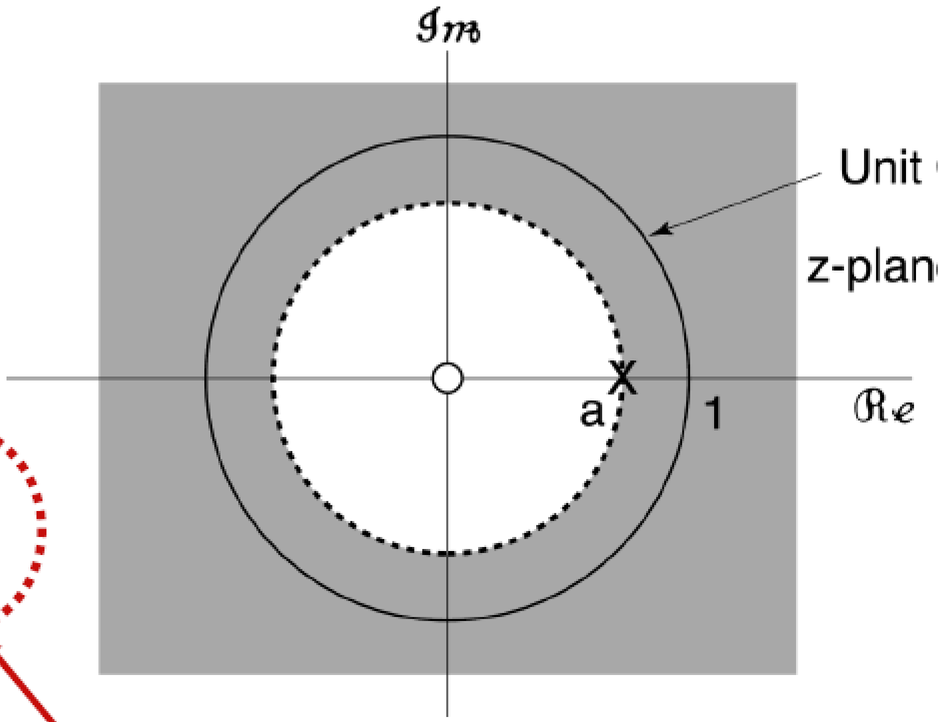
$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

If $|az^{-1}| < 1$, i.e., $|z| > |a|$

That is, ROC $|z| > |a|$, outside a circle

This form to find pole and zero locations



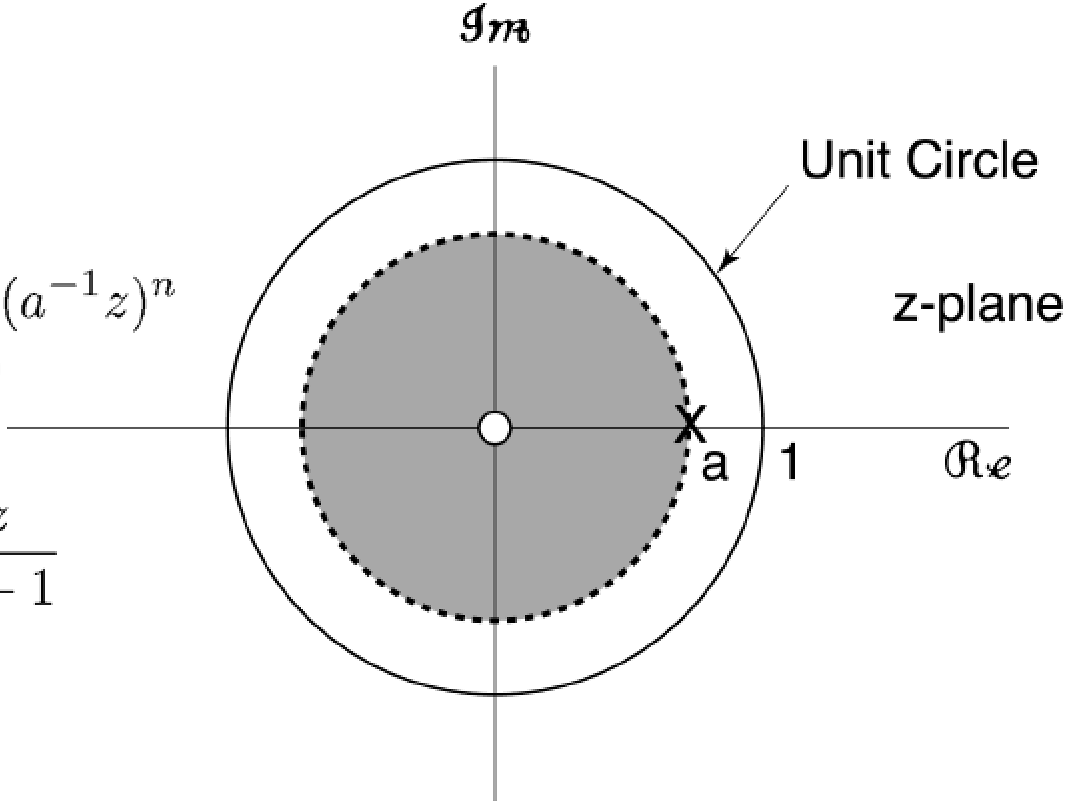
Example - 2

- Find the z-transform and plot the ROC

$$x[n] = -a^n u[-n-1] \text{ - left-sided}$$

Example - 2

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \{-a^n u[-n-1] z^{-n}\} \\
&= -\sum_{n=-\infty}^{-1} a^n z^{-n} \\
&= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n \\
&= 1 - \frac{1}{1 - a^{-1} z} = \frac{a^{-1} z}{a^{-1} z - 1} \\
&= \frac{z}{z-a},
\end{aligned}
$$



If $|a^{-1}z| < 1$, $i.e., |z| < |a|$

Same $X(z)$ as in **Example .1**, but different ROC.

Example - 3

- Consider a signal that is the sum of two real exponentials:

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

Example - 3

$$X(z)=\sum_{n=-\infty}^{\infty}\left\{7\left(\frac{1}{3}\right)^{n}u[n]-6\left(\frac{1}{2}\right)^{n}u[n]\right\}z^{-n}$$

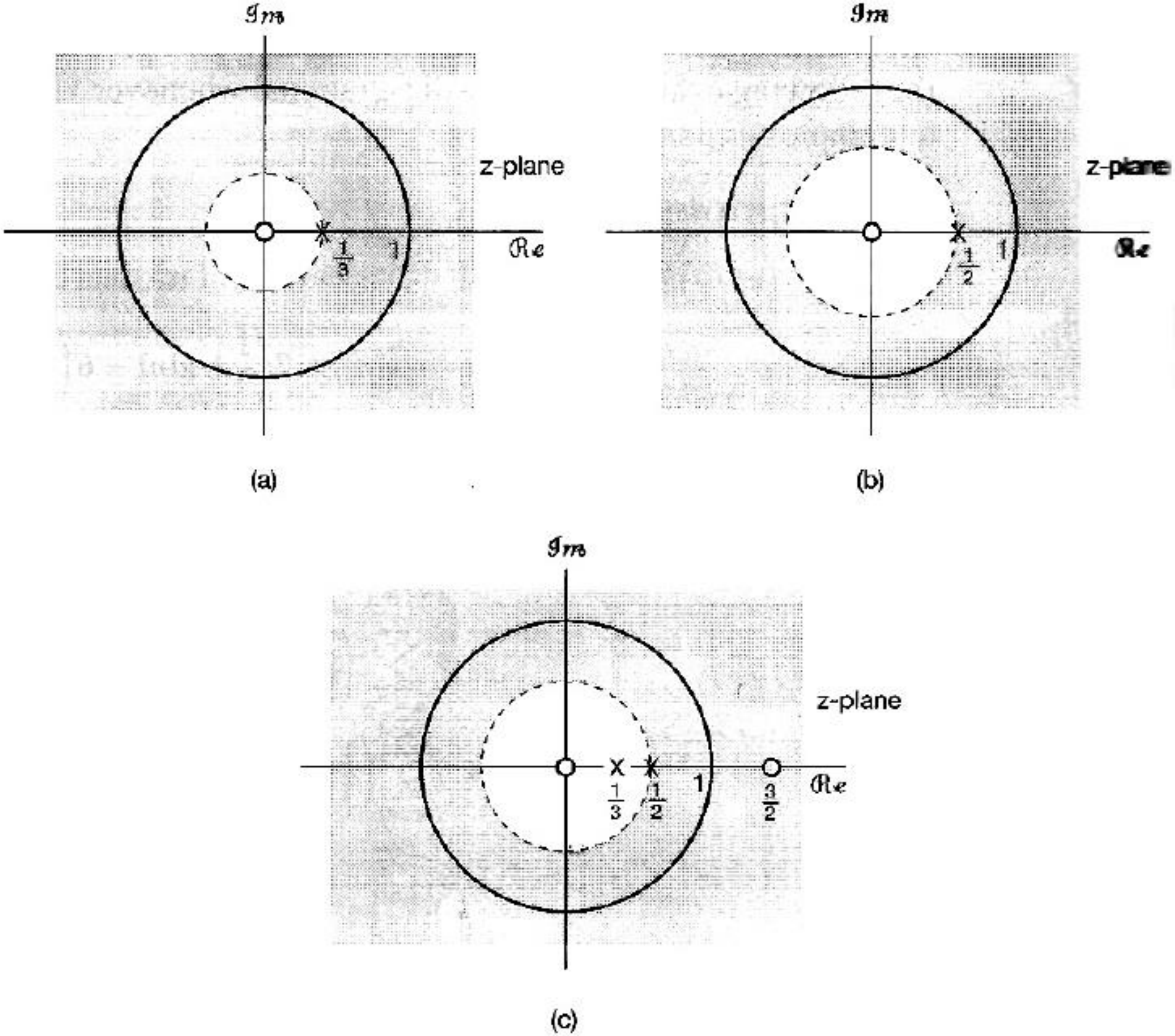
$$=7\sum_{n=0}^{\infty}\left(\frac{1}{3}z^{-1}\right)^{n}-6\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^{n}$$

$$=\frac{7}{1-\frac{1}{3}z^{-1}}-\frac{6}{1-\frac{1}{2}z^{-1}}$$

$$=\frac{1-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}=\frac{z\left(z-\frac{3}{2}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)};$$

$$|z|>\max(1/3,\ 1/2)\Rightarrow|z|>1/2$$

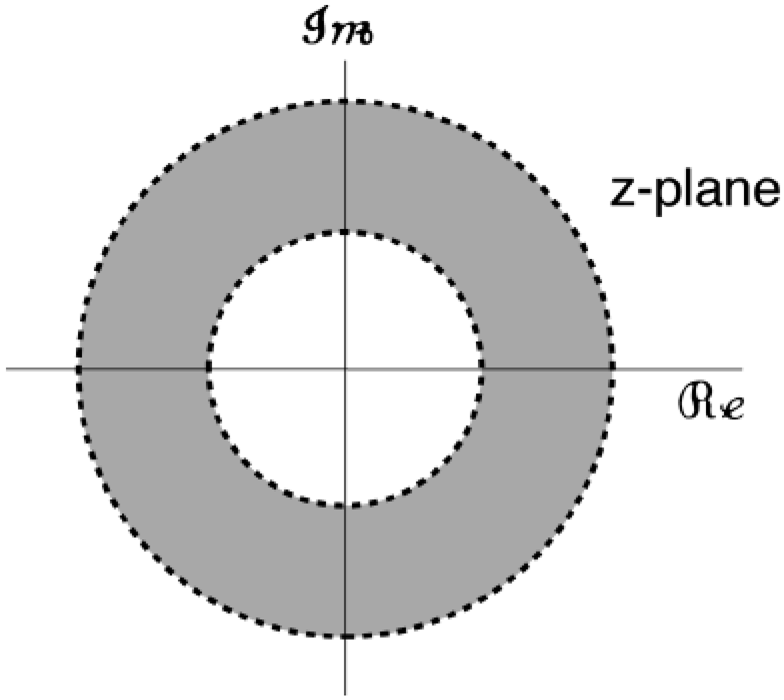
Example - 3



Im
z-plane
Re
$\frac{1}{3}$
1
(a)
Im
z-plane
Re
$\frac{1}{2}$
1
(b)
Im
z-plane
Re
$\frac{1}{3}$
$\frac{1}{2}$
1
$\frac{3}{2}$
(c)

Properties of ROC

Property (1) -- The ROC of $X(z)$ consists of a ring in the *z*-plane centered about the origin (equivalent to a vertical strip in the *s*-plane)



Property (2) -- The ROC does *not* contain any poles (same as in *LT*).

Properties of ROC

Property (3) -- If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly at $z = 0$ and/or $z = \infty$.

Why?

$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}$$

Examples: CT counterpart

$\delta[n] \longleftrightarrow 1 \quad \text{ROC all } z$	$\delta(t) \longleftrightarrow 1 \quad \text{ROC all } s$
$\delta[n-1] \longleftrightarrow z^{-1} \quad \text{ROC } z \neq 0$	$\delta(t-T) \longleftrightarrow e^{-sT} \quad \Re e\{s\} \neq -\infty$
$\delta[n+1] \longleftrightarrow z \quad \text{ROC } z \neq \infty$	$\delta(t+T) \longleftrightarrow e^{sT} \quad \Re e\{s\} \neq \infty$

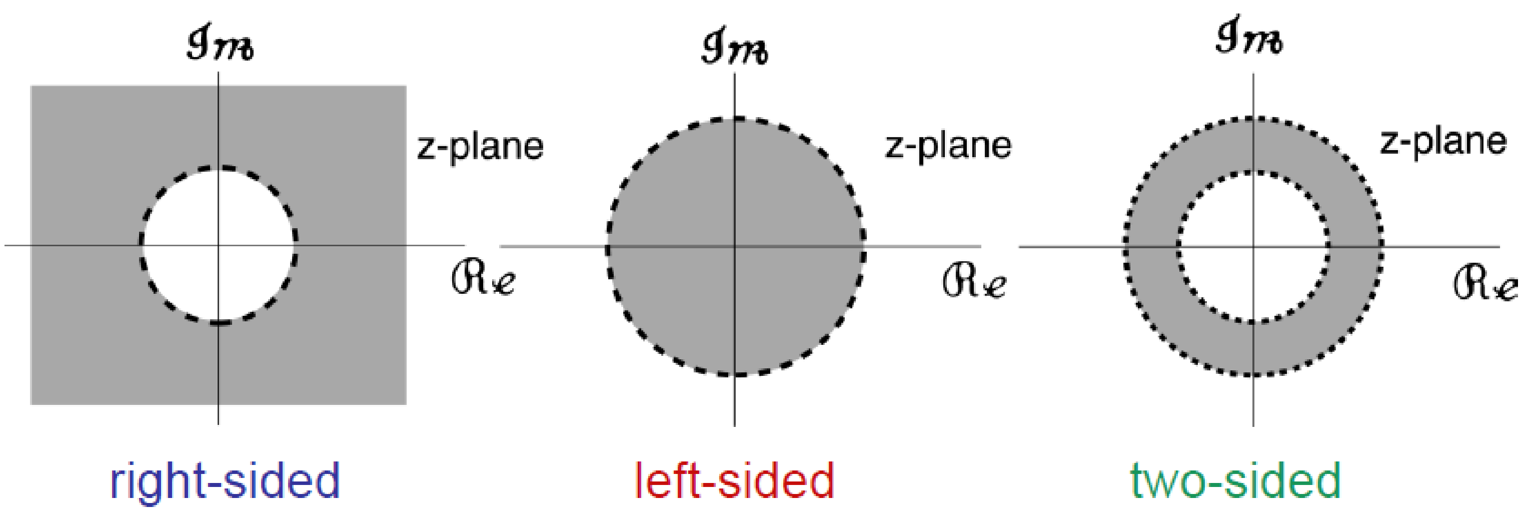
Properties of ROC

Property (4) -- If $x[n]$ is a right-sided sequence, and if $|z| = r_o$ is in the ROC, then all finite values of z for which $|z| > r_o$ are also in the ROC.

Property (5) -- If $x[n]$ is a left-sided sequence, and if $|z| = r_o$ is in the ROC, then all finite values of z for which $0 < |z| < r_o$ are also in the ROC.

Property (6) -- If $x[n]$ is two-sided, and if $|z| = r_o$ is in the ROC, then the ROC consists of a ring in the *z*-plane including the circle $|z| = r_o$.

What types of signals do the following ROC correspond to?



Properties of ROC

Property (7) -- If the z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or it extends to infinity.

Property (8) -- If the z-transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is right-sided, then the ROC is the region in the z-plane outside the outermost pole -- i.e., outside the circle of radius equal to the largest magnitude of the poles of $X(z)$. Furthermore, if $x[n]$ is causal (i.e., right-sided and equal to 0 for $n < 0$), then the ROC also includes $z = \infty$.

Property (9) -- If the z-transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is left-sided, then the ROC is the region in the z-plane inside the innermost pole -- i.e., inside the circle of radius equal to the smallest magnitude of the poles of $X(z)$ other than any at $z = 0$ and extending inward to and possibly including $z = 0$. In particular, if $x[n]$ is anti-causal (i.e., left-sided and equal to 0 for $n > 0$), then the ROC also includes $z = 0$.

END