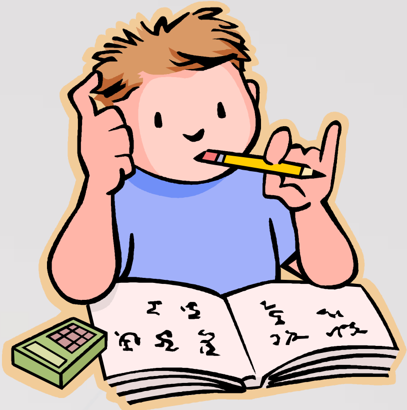
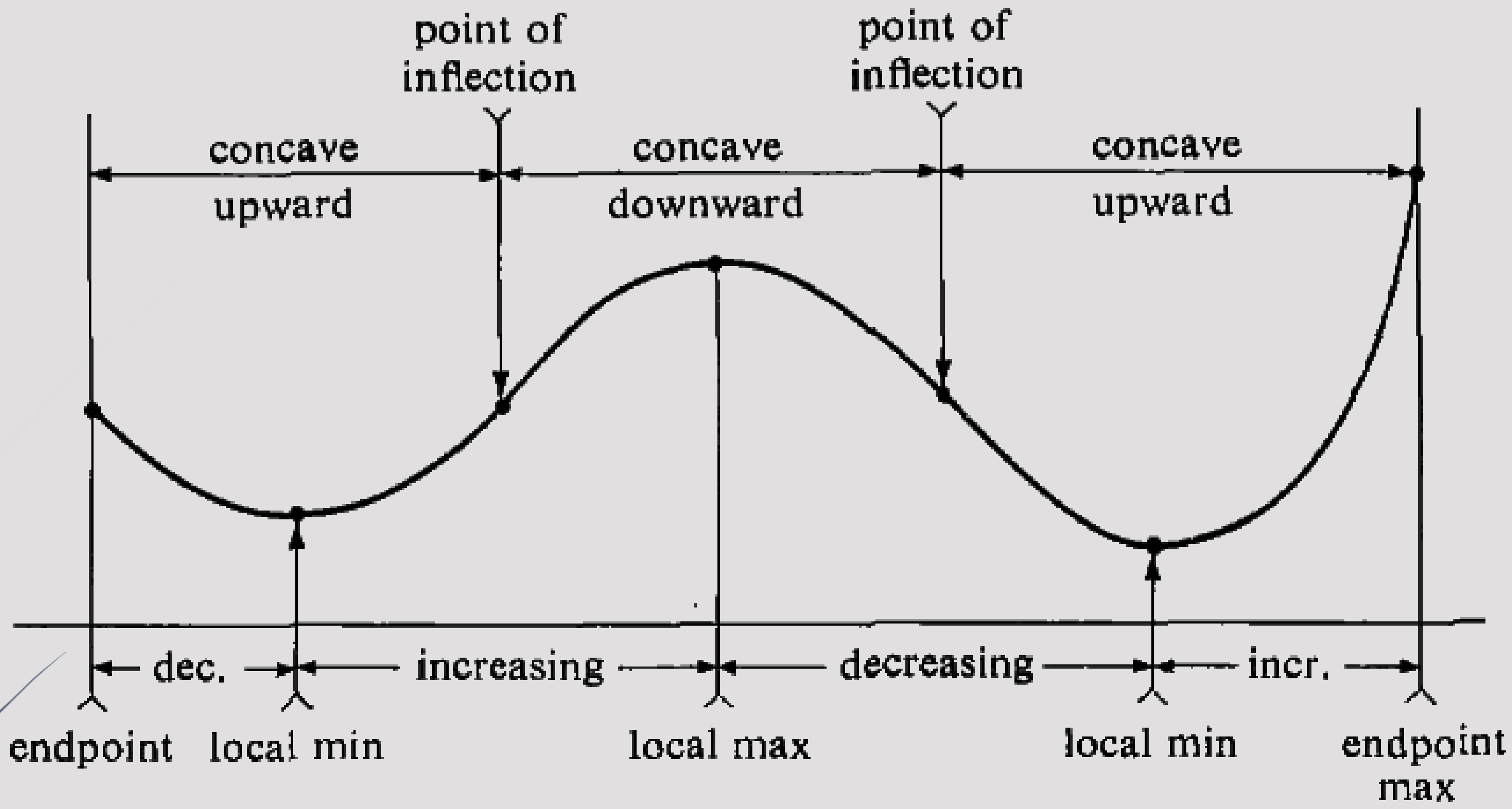


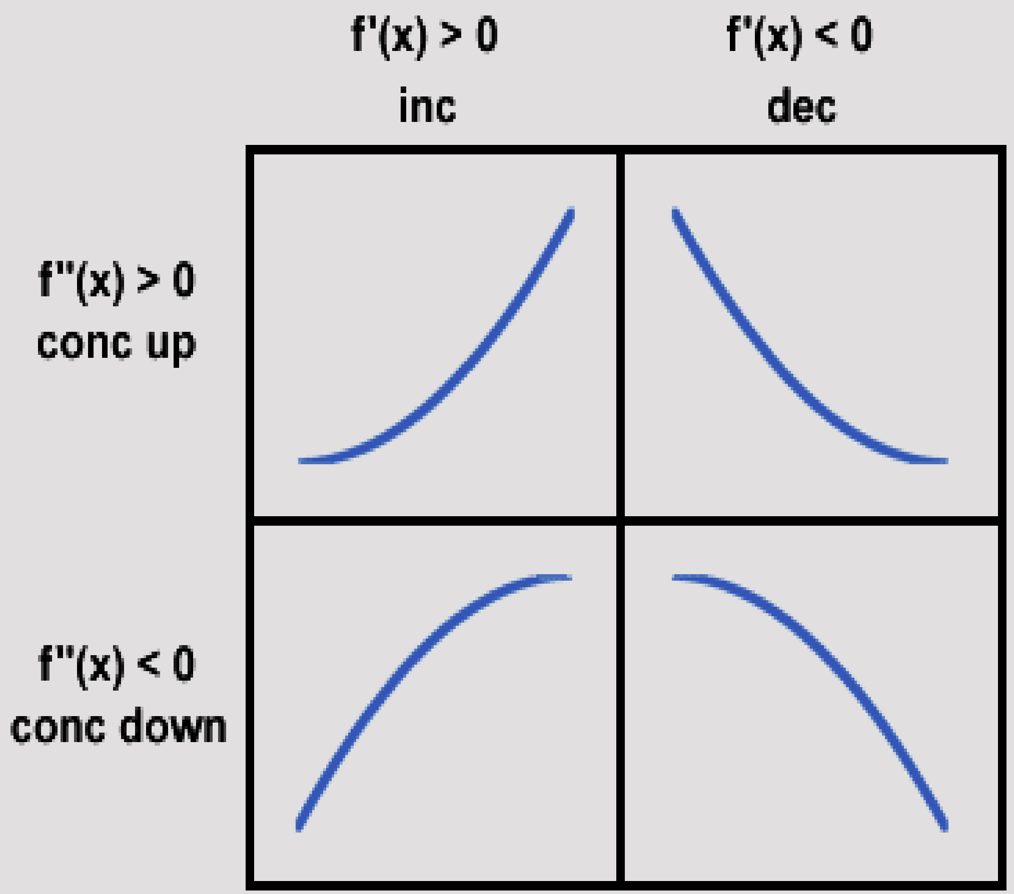
Applications of Derivatives

Calculus & Analytical Geometry MATH- 101

Instructor: Dr. Naila Amir (SEECS, NUST)



Concavity and Curve Sketching



Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 4
  - Sections: 4.4

Example :

Graph the function:

$$f(x) = x\sqrt{4-x}.$$

Solution:

Step 1. Domain: $(-\infty, 4]$ or $x \le 4$

Symmetry: $f(x)$ is neither even nor odd $\Rightarrow$ NO symmetry

Step 2. First and second derivative:

✓ $f'(x) = (8-3x)/(2\sqrt{4-x})$

$$f''(x) = \frac{3x-16}{4(4-x)^{3/2}}$$

Step 3 & 4. Critical points, rise and fall:

$f'(x) = \dfrac{8-3x}{2\sqrt{4-x}}$

$f\left(\frac{8}{3}\right) = \frac{16}{3\sqrt{3}}$

$\left(\frac{8}{3}, \frac{16}{3\sqrt{3}}\right)$

Critical points:

$8/3$

Intervals	$(-\infty, 8/3)$	$(8/3, 4)$
Sign of $f'(x)$	$+$	$-$
Behavior of $f(x)$	$\nearrow$	$\searrow$

$\nearrow \searrow$ max

$f'(0) = \dfrac{8-0}{2\sqrt{4-0}} = \dfrac{8}{2\times 2} > 0$

$f'(3) = 8-9/2\sqrt{4-3} = -\frac{1}{2} < 0$

$f'(x) = 0$

$\Rightarrow \dfrac{8-3x}{2\sqrt{4-x}} = 0$

$\Rightarrow 8-3x = 0$

$\Rightarrow x = \frac{8}{3} \approx 2.67$

Since

$x = \frac{8}{3} \in (-\infty, 4]$

So $x = \frac{8}{3}$ is a CP

$f'(x)$ is undefined at $x = 4$

Step 5. Concavity and points of inflection:

$f''(x) = \dfrac{3x - 16}{4(4-x)^{3/2}}$

Points of inflection:

NO PI

Intervals	$(-\infty, 4)$
Sign of $f''(x)$	$-$
Behavior of $f(x)$	CD

$f''(0) = \dfrac{-16}{4(4)^{3/2}} < 0$

$f''(x) = 0$

$\Rightarrow 3x - 16 = 0$

$\Rightarrow x = \dfrac{16}{3} \approx 5.3$

Since

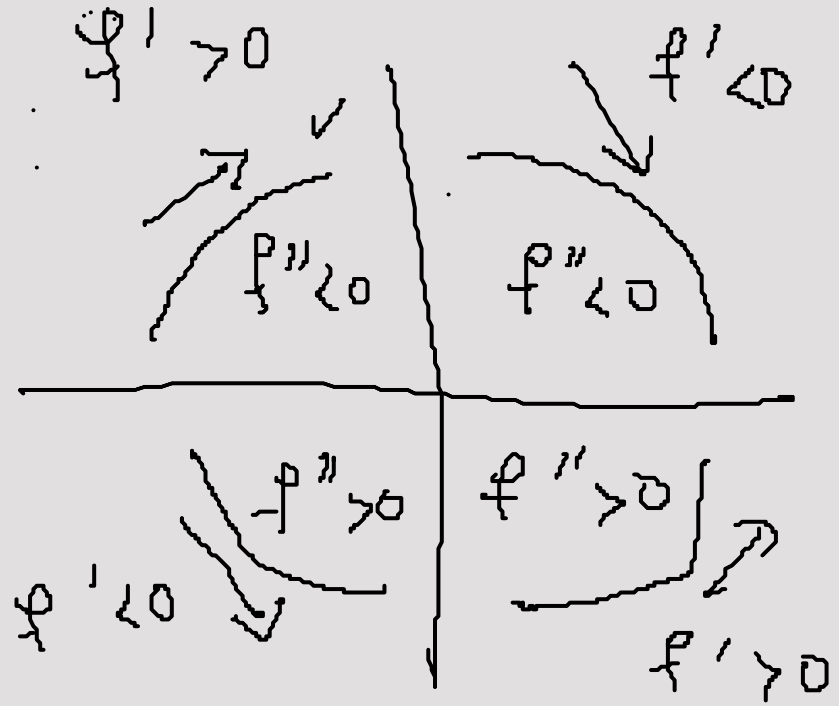
$5.3 \notin (-\infty, 4]$

so $x = \dfrac{16}{3}$ is not P.I

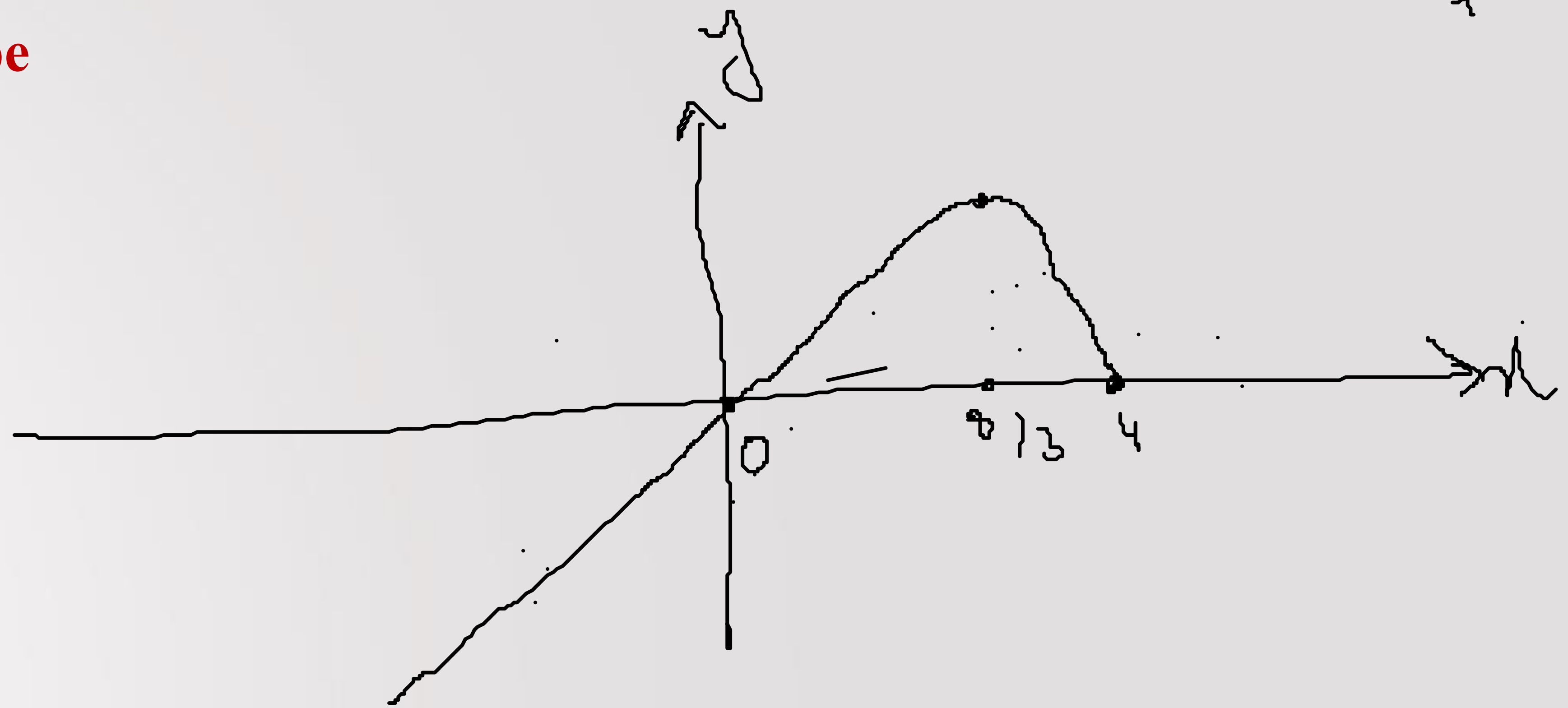
$f''(x)$ is undefined for $x = 4$

Step 6. Summarize the information from step 4 and 5 and sketch a general graph.

Intervals	$(-\infty, 8/3)$	$(8/3, 4)$
Sign of $f'(x)$	$\nearrow$ +	− $\searrow$
Sign of $f''(x)$	CD −	− CD
Behavior of $f(x)$		



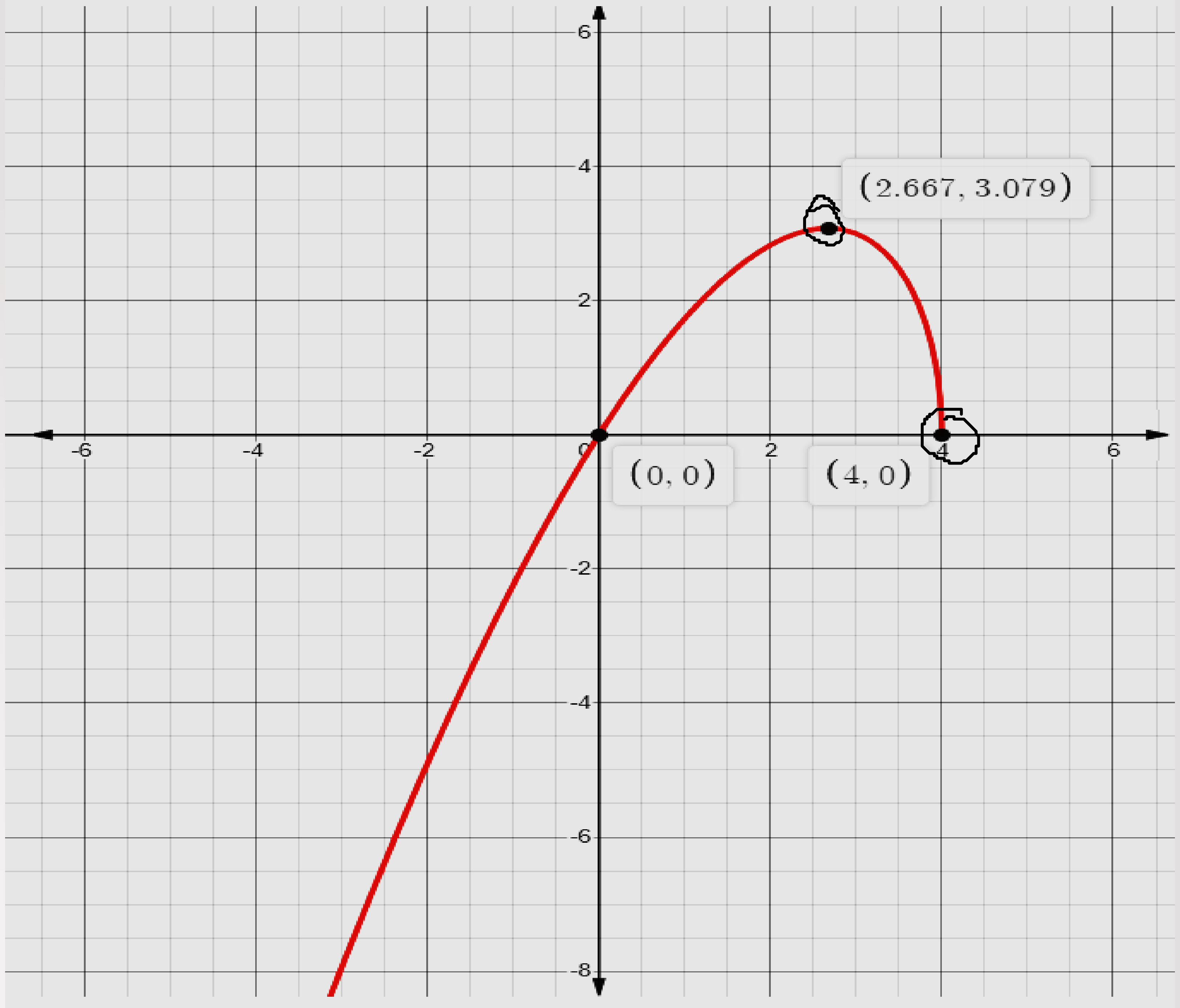
General shape



Step 7. Identify asymptotes (if any). Plot the curve's intercepts (if convenient) and the points where $f'(x)$ and $f''(x)$ are zero. Indicate any local extreme values and inflection points. Use the general shape in step 6 as a guide to sketch the curve.

Intercepts	Asymptotes
y-intercept $x=0 \Rightarrow y=0$ $(0,0)$ x-intercept $y=0 \Rightarrow x\sqrt{4-x}=0$ $\Rightarrow x=0, x=4$ $(0,0)$ & $(4,0)$	No Asymptotes

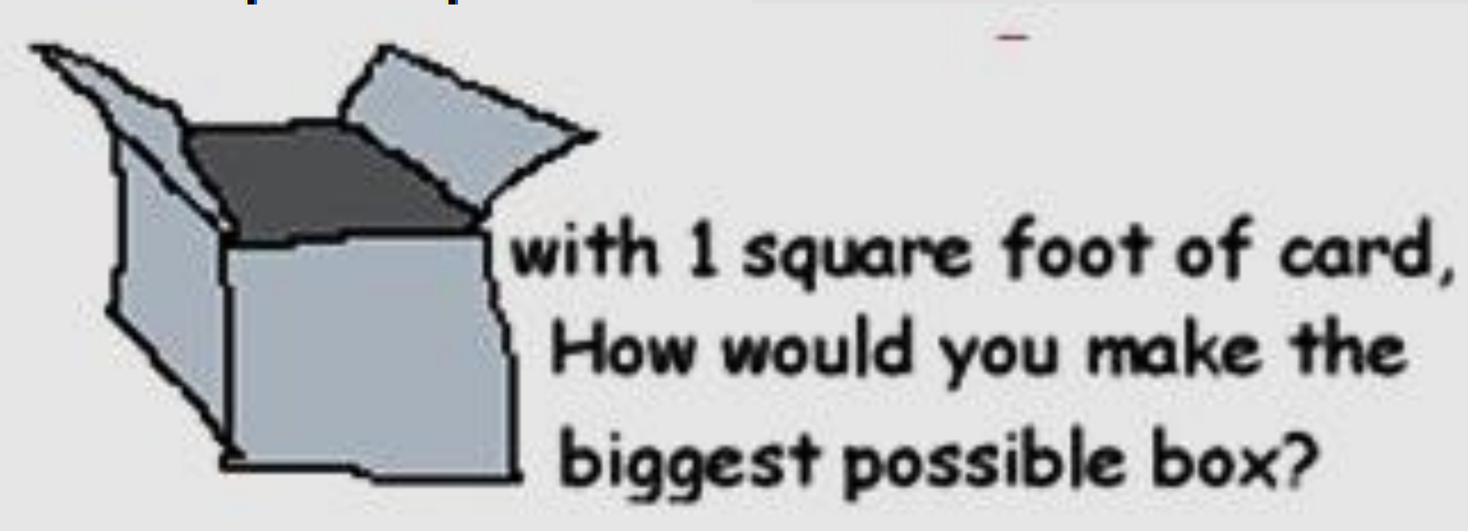
Step 7. Identify asymptotes (if any). Plot the curve's intercepts (if convenient) and the points where $f'(x)$ and $f''(x)$ are zero. Indicate any local extreme values and inflection points. Use the general shape in step 6 as a guide to sketch the curve.



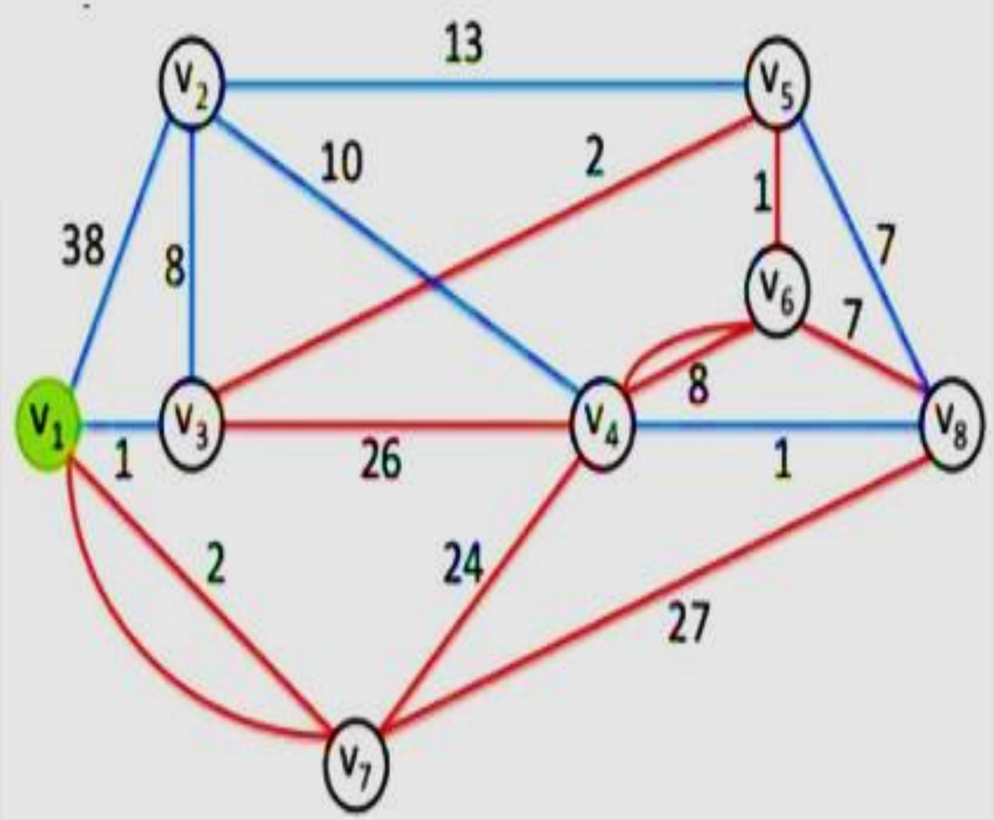
Applied Optimization Problems

We know about max and min …

How can we use those principles???



The aim for **the min-max k-Chinese postman problem (MM k-CPP)** is to minimize the length of the longest tour of k-Chinese postman tour.



Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 4
    - Sections: 4.5

Applied Optimization Problems

- To optimize something means to maximize or minimize some aspect of it.
- One common application of derivatives is calculating the minimum or maximum value of a function. For example,
  - companies often want to minimize production costs or maximize revenue.
  - In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume.
  - A traveler wants to minimize transportation time.
- We are interested to show how to set up these types of minimization and maximization problems and solve them by using the tools developed in the previous lectures.

Applied Optimization Problems

- We will be mainly interested in solving problems such as:
  - Maximizing areas, volumes, and profits.
  - Minimizing distances, times, and costs.
- In solving such practical problems, the greatest challenge is often to convert the word problem into a mathematical optimization problem—by setting up the function that is to be maximized or minimized.

Solving Applied Optimization Problems

1. Assign symbols to all given quantities and quantities to be determined.

   $A = xy$ ✓

2. Write a primary equation for the quantity to be maximized or minimized.

   $P = 2x + 2y$ ✓

3. Reduce the primary equation to one having a single independent variable. This may involve the use of secondary equation.
4. Determine the domain. Make sure it makes sense.
5. Determine the max or min by differentiation.

Secondary Eq

$P = 2x + y = 2400\,ft \longrightarrow 2$

$\Rightarrow \boxed{y = 2400 - 2x} \longrightarrow 3$

Example:

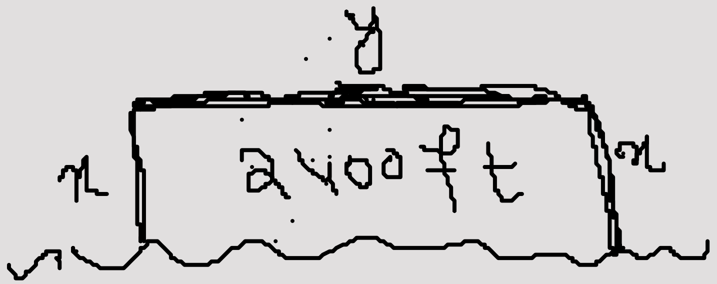
A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river.

What are the dimensions of the field that has the largest area?

Primary Eq $A = xy \longrightarrow 1$

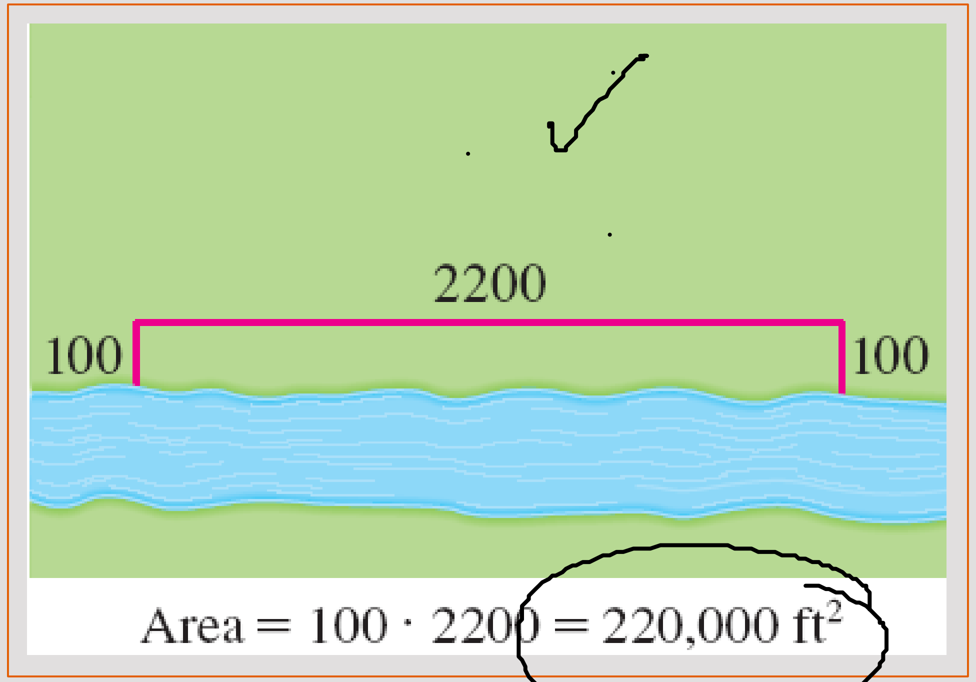
Solution:

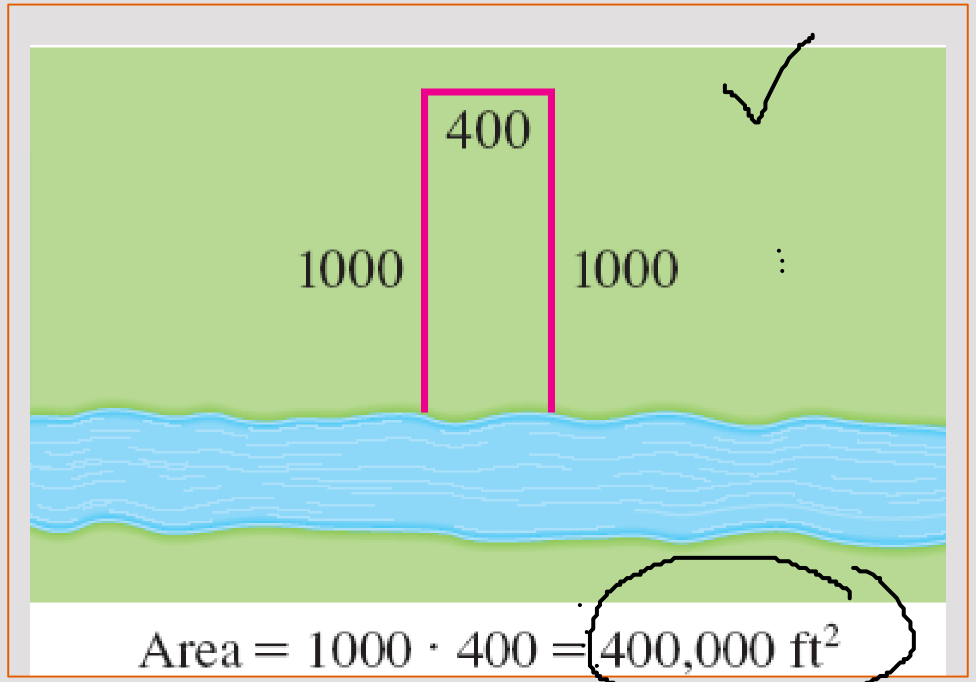
In order to get a feeling for what is happening in the problem, let's experiment with some special cases.

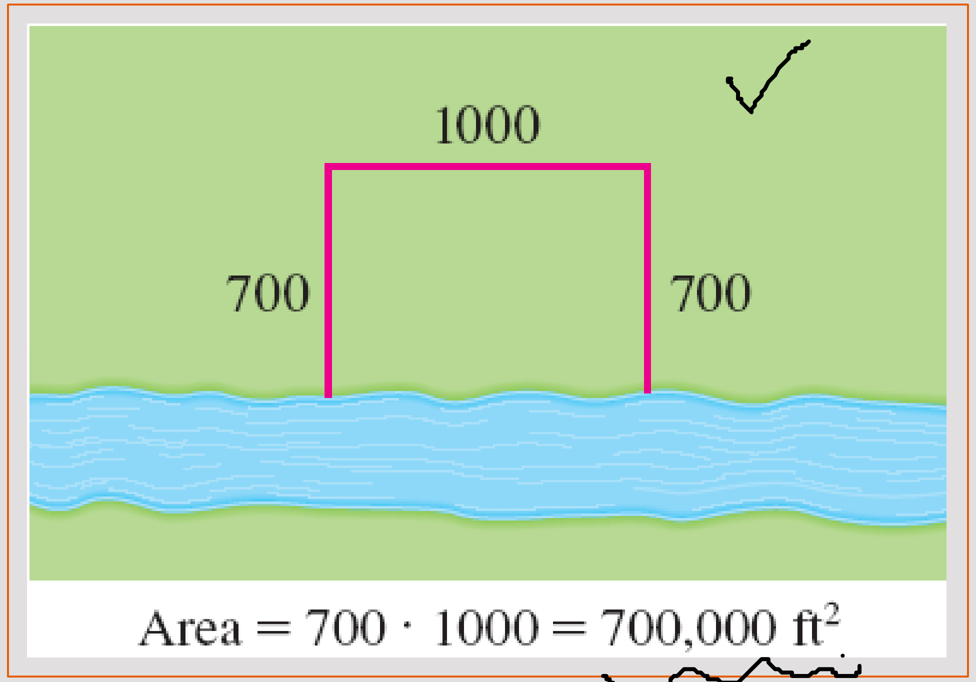


Solution:

Here are three possible ways of laying out the 2400 ft of fencing.

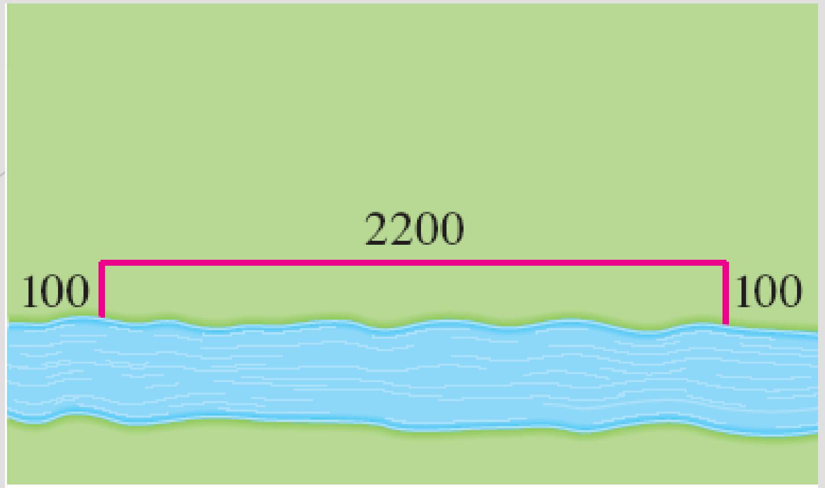




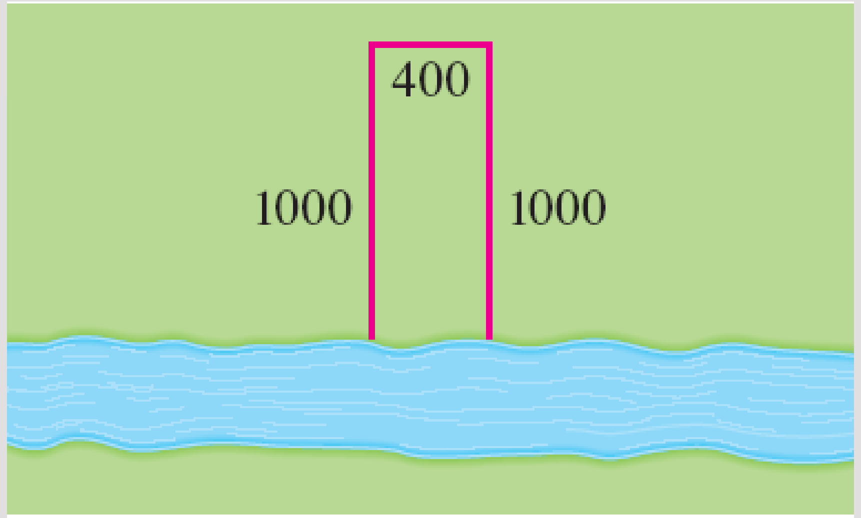


- We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas.

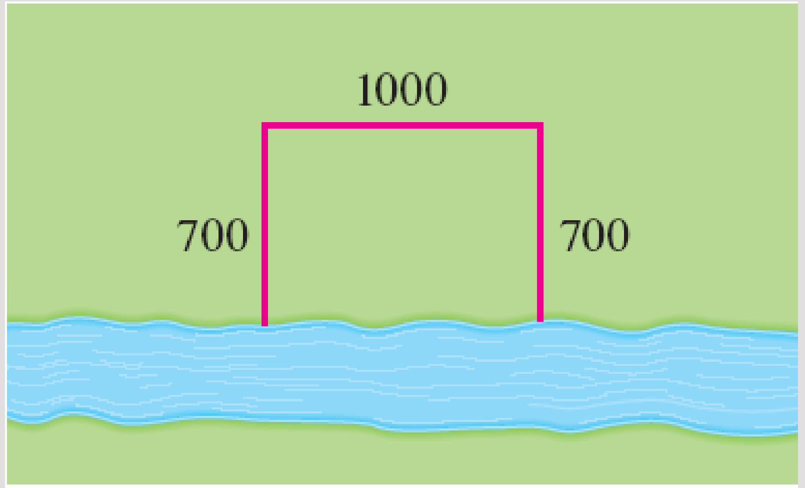
- It seems plausible that there is some intermediate configuration that produces the largest area.



Area = 100 · 2200 = 220,000 ft²



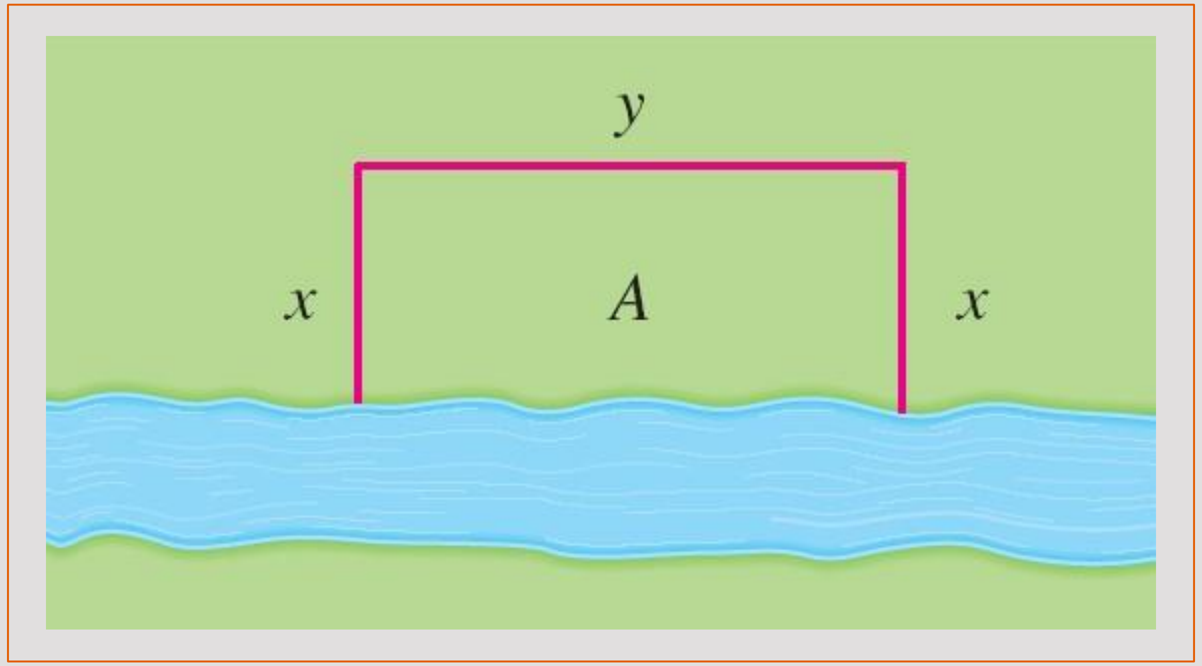
Area = 1000 · 400 = 400,000 ft²



Area = 700 · 1000 = 700,000 ft²

- We wish to maximize the area A of the rectangle.
- Let x and y be the depth and width of the rectangle (in feet).
- Then, we express A in terms of x and y:

$$A = xy.$$



We want to express A as a function of just one variable.

- So, we eliminate y by expressing it in terms of x.

$A = xy$

- To do this, we use the given information that the total length of the fencing is $2400\,ft$.

- Thus, $2x + y = 2400 \Rightarrow y = 2400 - 2x.$

$A \geq 0$

$x(2400 - 2x) \geq 0$

- This gives:

$$A = x(2400 - 2x) = 2400x - 2x^2.$$

$x \geq 0$

or

$2400 - 2x \geq 0$

- Note that $x \geq 0$ and $x \leq 1200$ (otherwise $A < 0$).

So, the function that we wish to maximize is:

$$A(x) = 2400x - 2x^2; \qquad 0 \le x \le 1200.$$

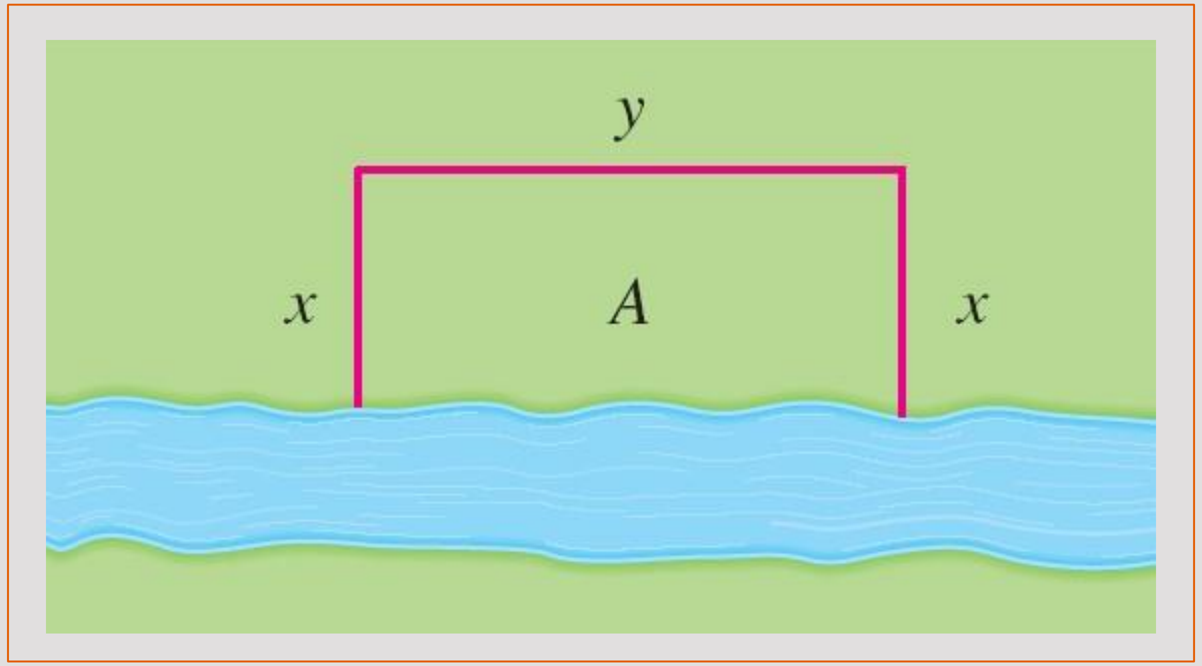
- The derivative is: $A'(x) = 2400 - 4x$.
- So, to find the critical numbers, we solve: $A'(x) = 2400 - 4x = 0$
- This gives: $x = 600$.
- The maximum value of A must occur either at that critical number or at an endpoint of the interval.
- Now $A(0) = 0$; $A(600) = 720{,}000$; and $A(1200) = 0$.
- So, the maximum value is:

$$A(600) = 720{,}000.$$

$x = 600$

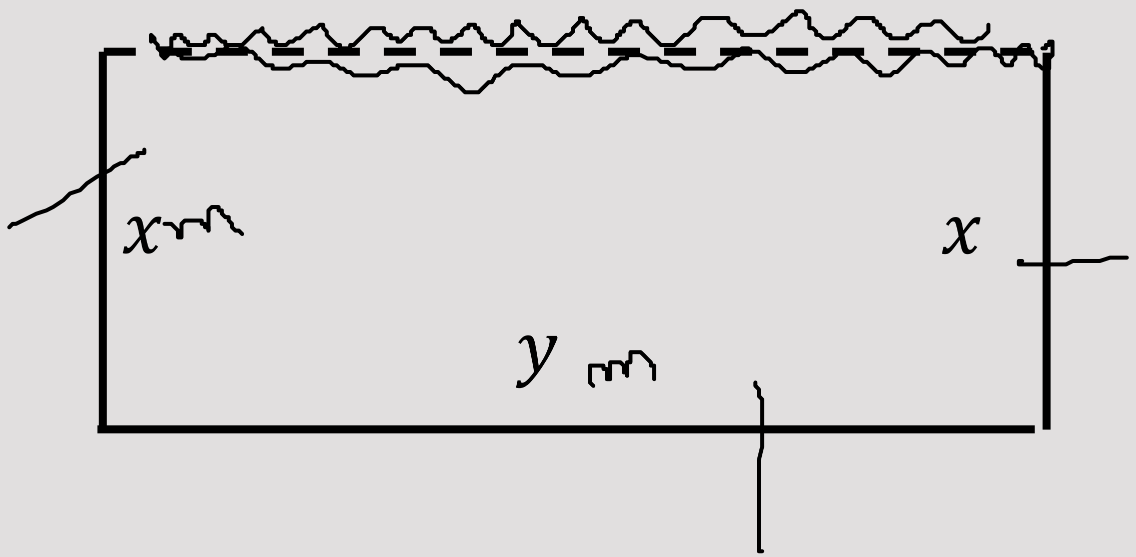
$y = 2400 - 2x$

- Thus, for the maximum area the rectangular field should be:
  - $600\ ft$ deep ✓
  - $1200\ ft$ wide ✓



Example:

A rectangular plot is to be bounded on one side by a straight river and enclosed on the other three sides by a fence. With $800\ m$ of fence at your disposal, what is the ***largest*** **area** that can be enclosed?



$A = xy \rightarrow$ Prim

$P = 2x + y \rightarrow$ Sec

800m

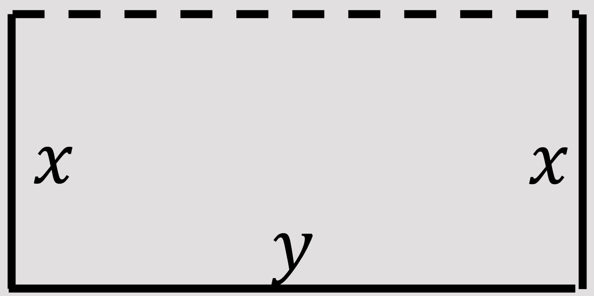
Solution:

$A \geq 0$

$x(-2x+800) \geq 0$

$x \geq 0$ or $-2x \geq -800$

$\Rightarrow x \leq 400$



For the present case:

$$2x + y = 800 \Longrightarrow y = -2x + 800$$

and area is given as:

$$A = xy \Longrightarrow A = x(-2x + 800) = -2x^2 + 800x.$$

Domain $[0, 400]$

In order to obtain largest area, we proceed as:

$$A'(x) = -4x + 800. \checkmark$$

For critical points we use:

$$A'(x) = 0 \Longrightarrow -4x + 800 = 0 \Longrightarrow x = 200.$$

$f'(c) = 0$

$f''(c) < 0$ max

Note that:

$$\boxed{A''(x) = -4.}$$

And at $x = 200$, $A''(x) = -4 < 0$. Therefore, by second derivative test we conclude that there exist a maximum value at $x = 200$. Moreover,

$$A = -2(200)^2 + 800(200) = \boxed{80000.}$$

min $f'(c) = 0$, $f''(c) > 0$

Thus, the largest area that can be enclose is 80000. m^2

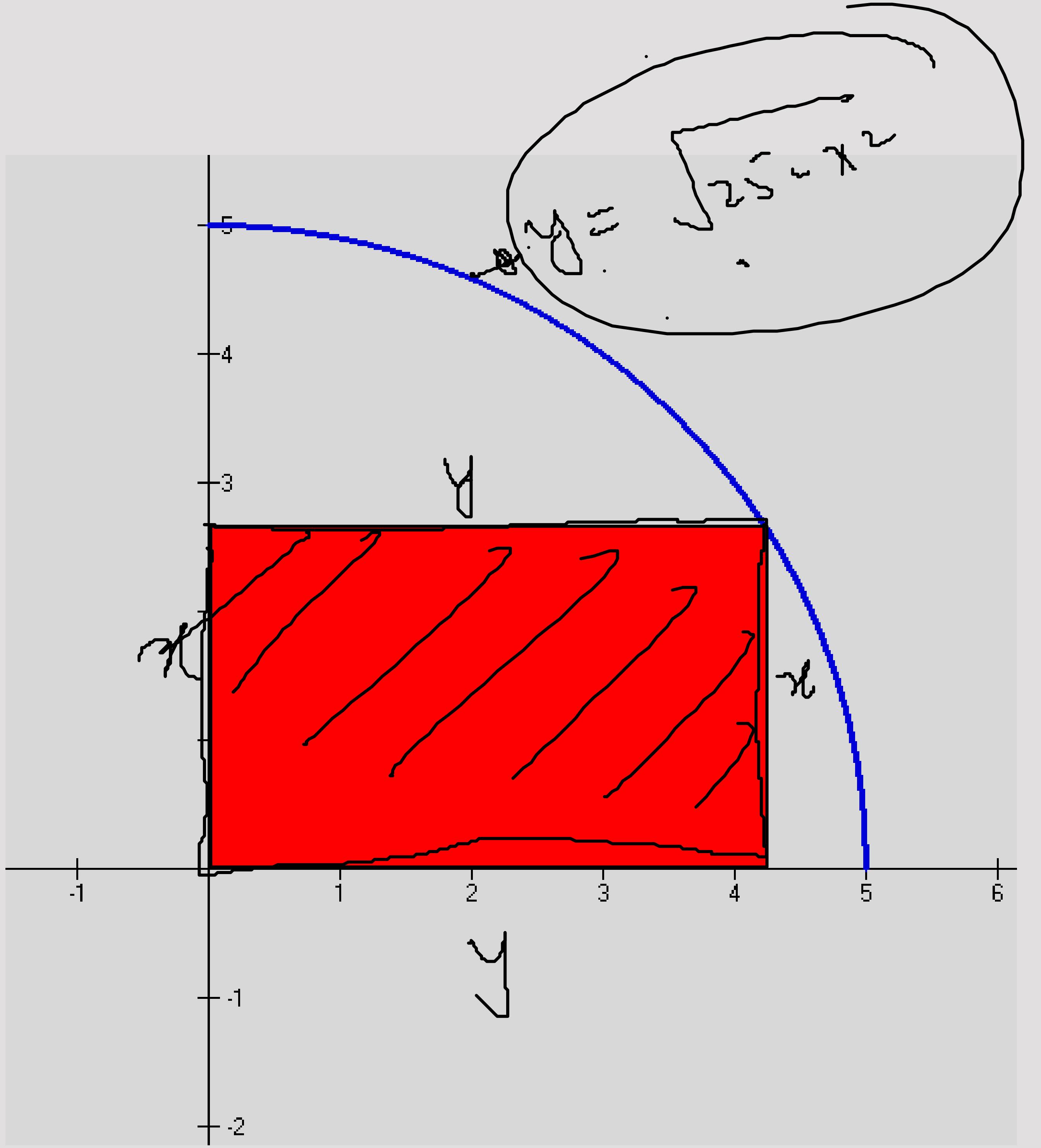
Example:

The graphs of $y = \sqrt{25 - x^2}$, $x = 0$ and $y = 0$ bound a region in the first quadrant.

Find the dimensions of the rectangle of maximum perimeter that can be inscribed in this region.

$$\boxed{P = 2x + 2y}$$

$$x^2 + y^2 = 25$$



Solution:

For the present case:

$$P = 2x + 2y \Rightarrow P = 2x + 2\sqrt{25 - x^2}. \quad [\because y = \sqrt{25 - x^2}]$$

$x = \frac{5}{\sqrt{2}}$

In order to obtain maximum perimeter, we proceed as:

$$P'(x) = 2 + \frac{-2x}{\sqrt{25 - x^2}}.$$

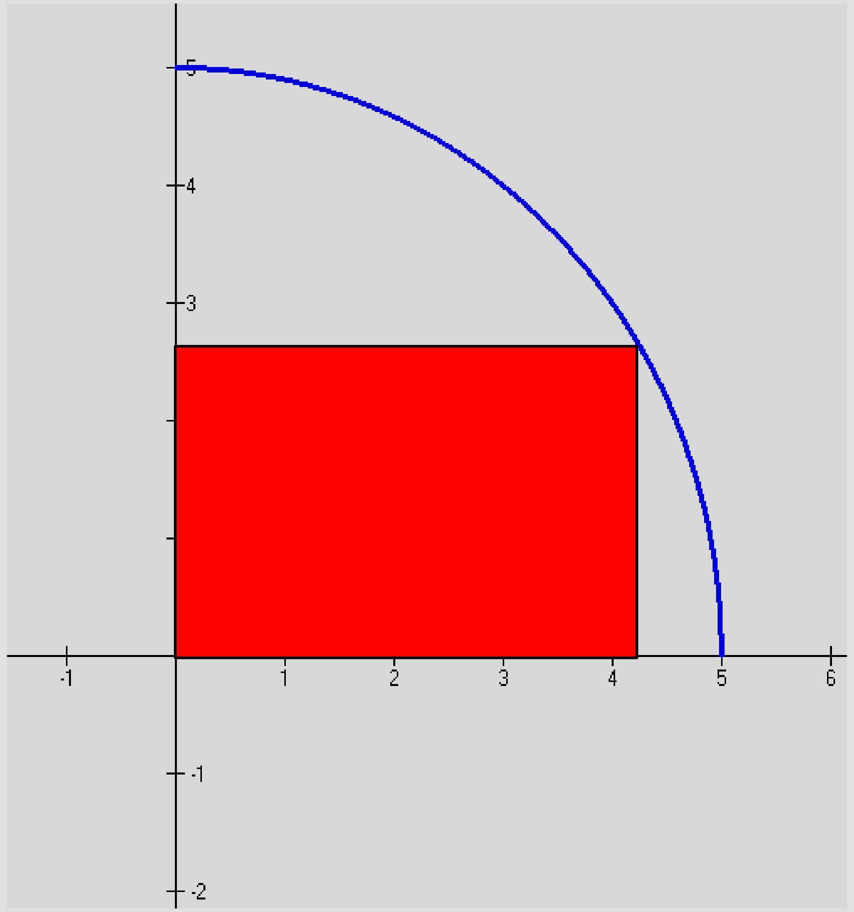
Critical points:

$$P'(x) = 0 \Rightarrow 2 + \frac{-2x}{\sqrt{25 - x^2}} = 0 \Rightarrow x = \pm\frac{5}{\sqrt{2}}.$$

and

$$P'(x) \text{ is undefined at } x = \pm 5.$$

But the only critical point is: $x = \frac{5}{\sqrt{2}}$. (Why???)



Solution:

Note that:

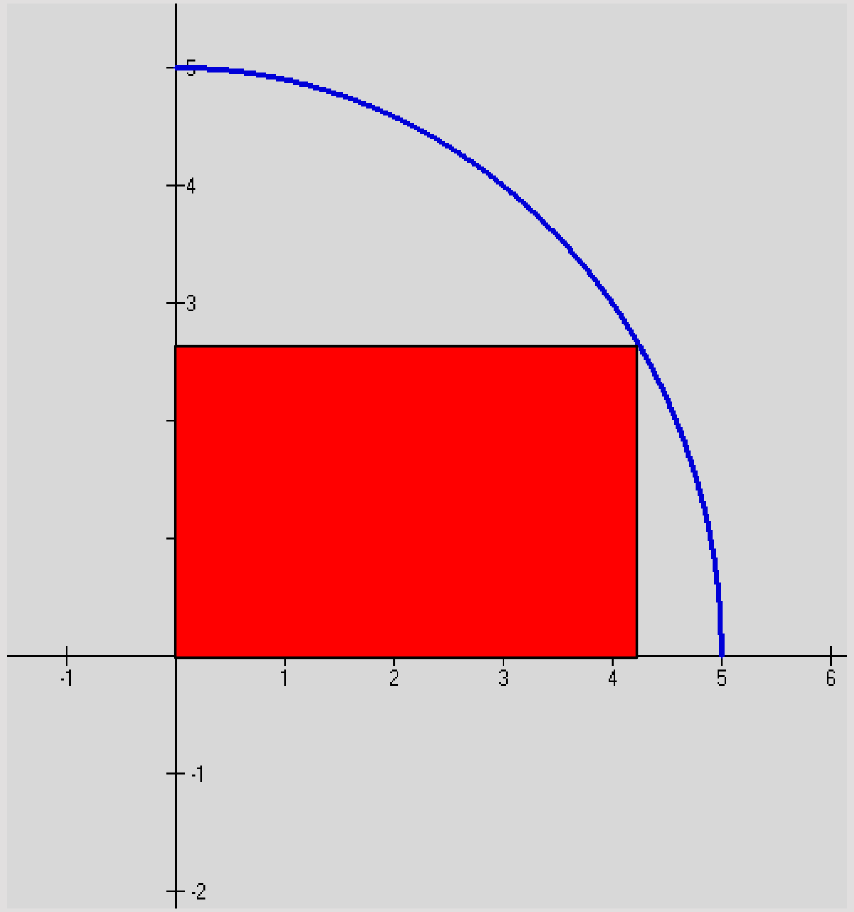
max

$+$ $-$

$P'(x) > 0$ $\quad \frac{5}{\sqrt{2}} \quad$ $P'(x) < 0$

$x = \frac{5}{\sqrt{2}}$

$y = \sqrt{25 - x^2}$



Since P' changes sign from $+\text{ve}$ to $-\text{ve}$, therefore, there exists maximum value at $x = \frac{5}{\sqrt{2}}$.

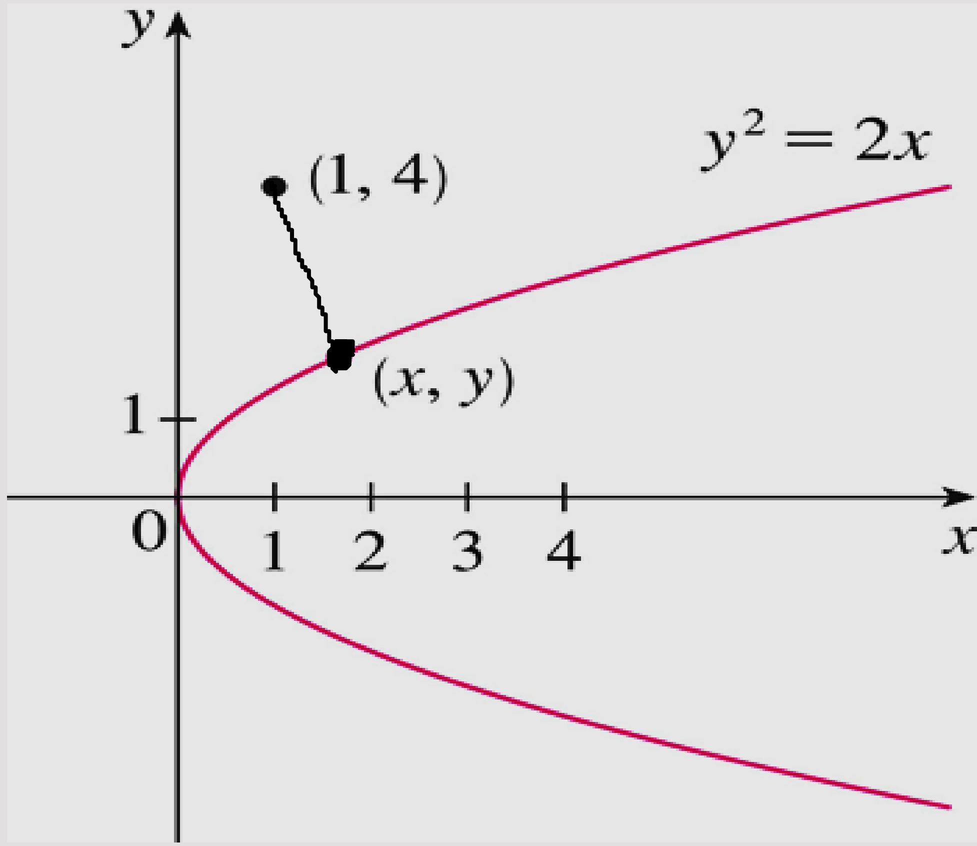
Thus, the dimensions of the rectangle of maximum perimeter that can be inscribed in the given region are: $\boxed{\frac{5}{\sqrt{2}} \text{ and } \frac{5}{\sqrt{2}}}$.

Example:

Find the point on the parabola:

$$y^2 = 2x$$

that is closest to the point $(1, 4)$.



Solution:

The distance between the point $(1, 4)$ and the point (x, y) is:

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

However, if the point (x, y) lies on the parabola, then

$$\boxed{x = \frac{y^2}{2}.}$$

Thus, the expression for d becomes:

$$d = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2}.$$

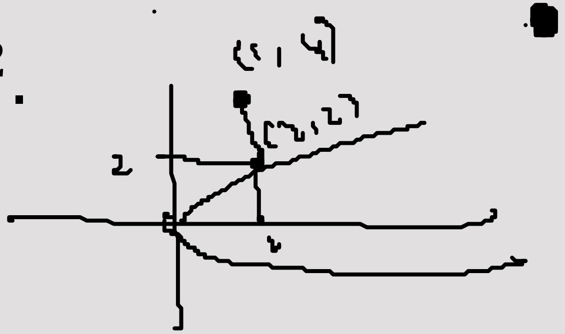
Note: Alternatively, we could have substituted $y = \sqrt{2x}$ to get d in terms of x alone.

- Note that the minimum of d occurs at the same point as the minimum of d^2.
- However, d^2 is easier to work with therefore, instead of minimizing $\underline{d}$, we minimize its square:

$$\underline{d^2} = \underline{f(y)} = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2.$$

min $\bullet$ — d

d^2



- Critical points:

$$f'(y) = 2\left(\frac{y^2}{2} - 1\right)(y) + 2(y-4)(1) = y^3 - 2y + 2y - 8 = y^3 - 8.$$

(1, 4)

$$f'(y) = 0 \Rightarrow y^3 - 8 = 0 \Rightarrow y^3 = 8 \Rightarrow \boxed{y = 2.}$$

$y - 2$

$x = \frac{y^2}{2} = 2$

- Observe that

$-$ $\searrow$ $\nearrow$ $+$

$f'(x) < 0$ 2 $f'(x) > 0$

min

Thus, by first derivative test, there exists a minimum value at $y = 2$. Thus, the point on $y^2 = 2x$ closest to $(1,4)$ is $(2,2)$.