CONTINUOUS TIME FOURIER TRANSFORM (CTFT)

Continuous Time Fourier Transform

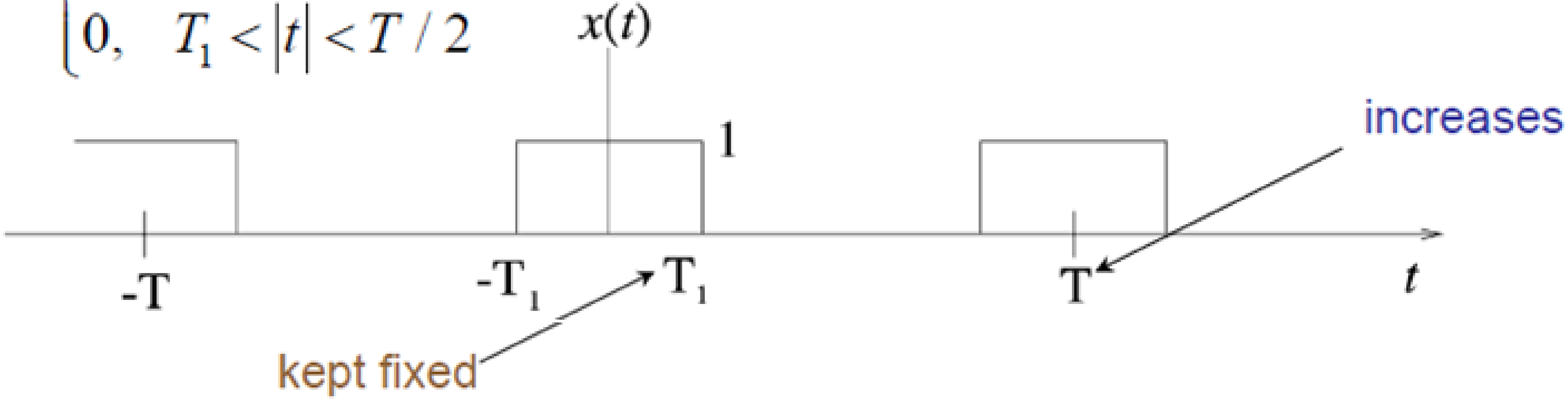
- $x(t)$ - an aperiodic signal
  - view it as the limit of a periodic signal as $T \to \infty$
- For a periodic signal, the harmonic components are spaced $\omega_0 = 2\pi/T$ apart ...
- As $T \to \infty$, $\omega_0 \to 0$, and harmonic components are spaced closer and closer in frequency

$$\Downarrow$$

Fourier series → Fourier integral

Example – Square Wave

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



$$a_k = \frac{\sin k\omega_o T_1}{k\pi}$$

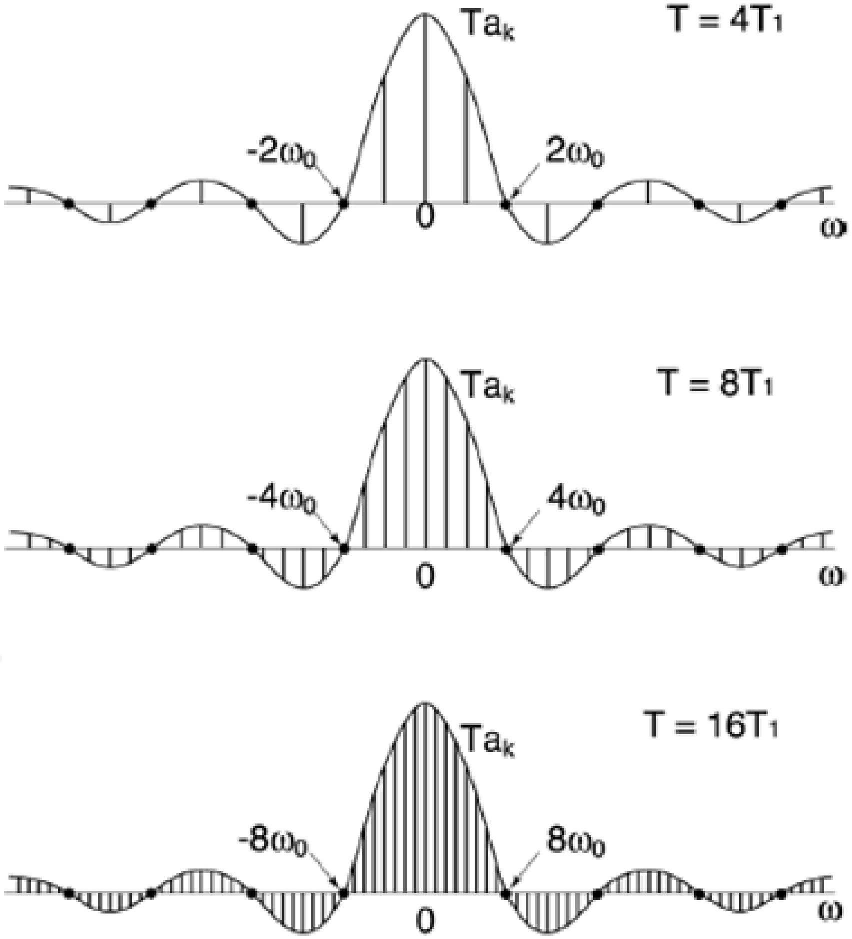
Example – Square Wave

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

$$\Downarrow$$

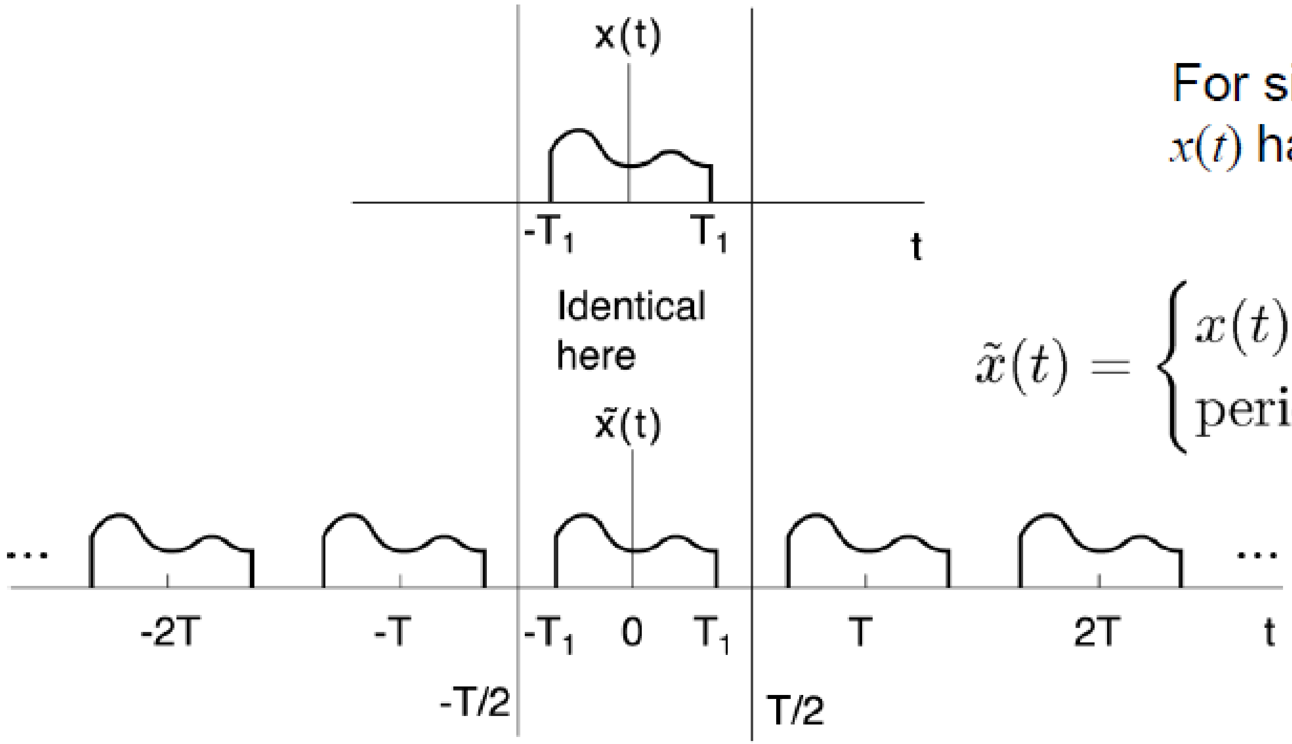
$$Ta_k = \left.\frac{2\sin\omega T_1}{\omega}\right|_{\omega = k\omega_0}$$

Note: Envelope is independent of T



Discrete frequency points become denser in ω as T increases

Synthesis and Analysis Equations



For simplicity, assume $x(t)$ has a finite duration.

$$\tilde{x}(t) = \begin{cases} x(t), & -\frac{T}{2} < t < \frac{T}{2} \\ \text{periodic}, & |t| > \frac{T}{2} \end{cases}$$

$$\text{As } T \to \infty,\ \tilde{x}(t) = x(t) \text{ for all } t$$

Synthesis and Analysis Equations

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \left(\omega_0 = \frac{2\pi}{T}\right)$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt$$

$\uparrow$

$\tilde{x}(t) = x(t)$ in this interval

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt \qquad (1)$$

If we define

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

then Eq.(1) $\Rightarrow$

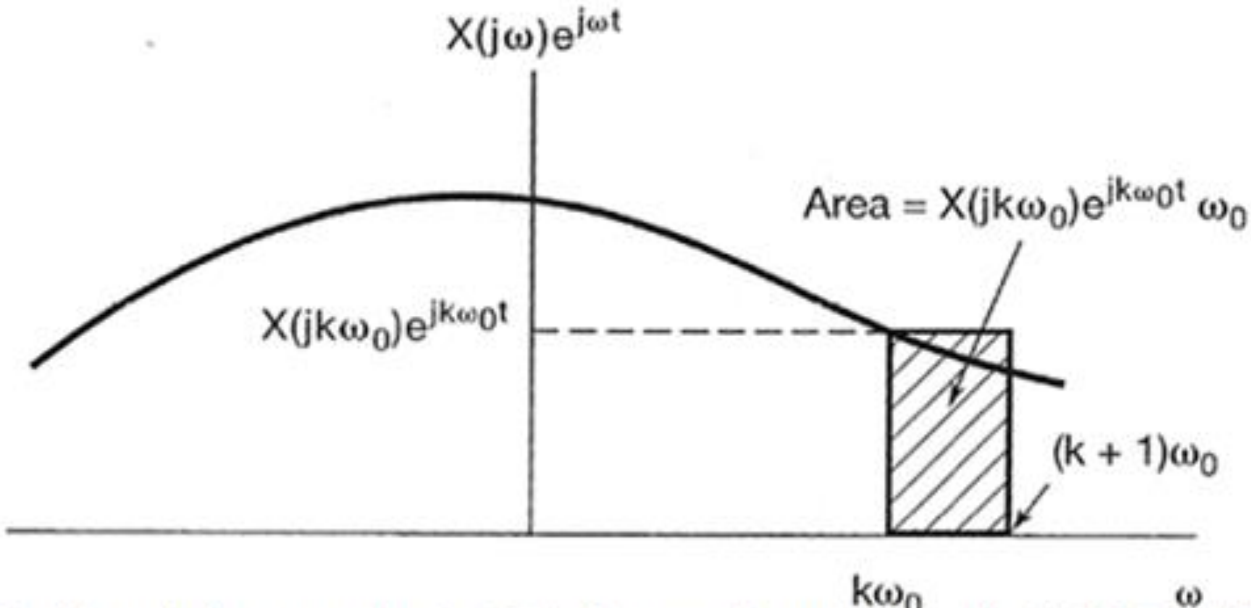
$$a_k = \frac{X(jk\omega_0)}{T}$$

Synthesis and Analysis Equations

Thus, for $-\frac{T}{2} < t < \frac{T}{2}$ $\quad x(t) = \tilde{x}(t) \quad = \quad \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T} X(jk\omega_0)}_{a_k} e^{jk\omega_0 t}$

$$= \quad \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \omega_0 X(jk\omega_0) e^{jk\omega_0 t}$$

$$\Downarrow$$



Each term in the summation for $\tilde{x}(t)$ is the area of a rectangle of height $X(jk\omega_0)e^{jk\omega_0 t}$ and width ω_0. As $\omega_0 \to 0$, the summation converges to the integral of $X(j\omega)e^{j\omega t}$.

Synthesis and Analysis Equations

Thus, for $-\frac{T}{2} < t < \frac{T}{2}$

$$x(t) = \tilde{x}(t) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T} X(jk\omega_0)}_{a_k} e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \omega_0 X(jk\omega_0) e^{jk\omega_0 t}$$

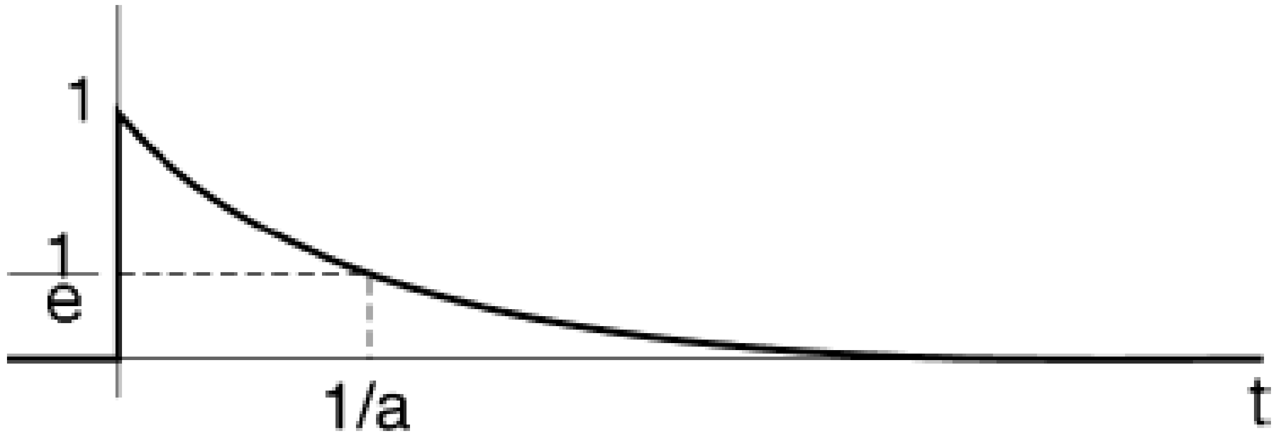
$$\Downarrow$$

As $T \to \infty$, $\sum \omega_0 \to \int d\omega$, we get the CT Fourier Transform pair

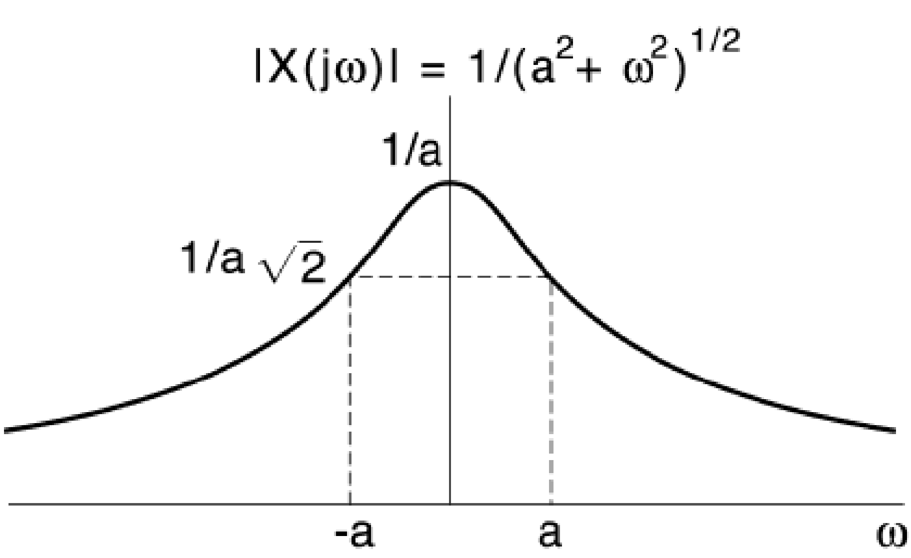
$$\boxed{\begin{array}{ll} x(t) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega & \text{Synthesis equation} \\ X(j\omega) = \displaystyle\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt & \text{Analysis equation} \end{array}}$$

Problem-1 - Decaying Exponential

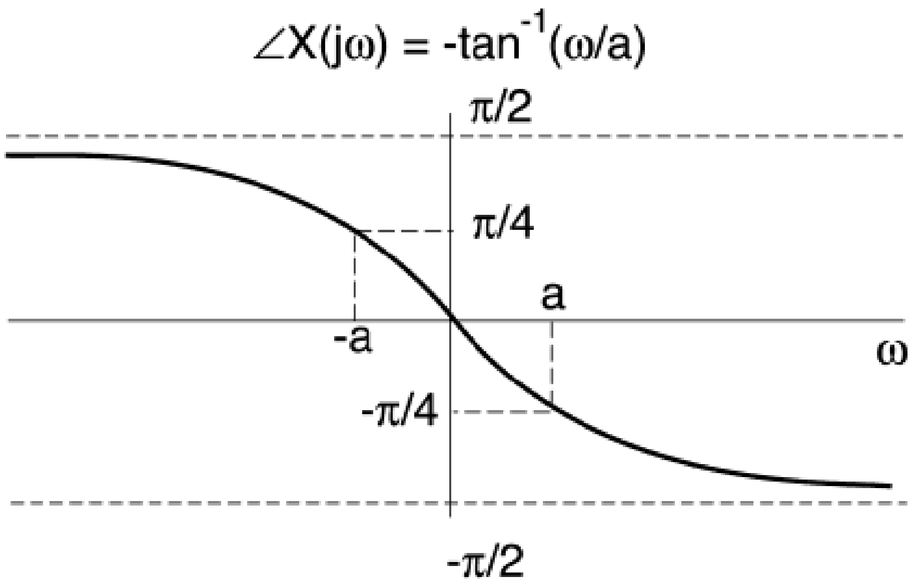
$$x(t) \quad = \quad e^{-at}u(t), a > 0$$



Problem-1 - Decaying Exponential



Even Symmetry

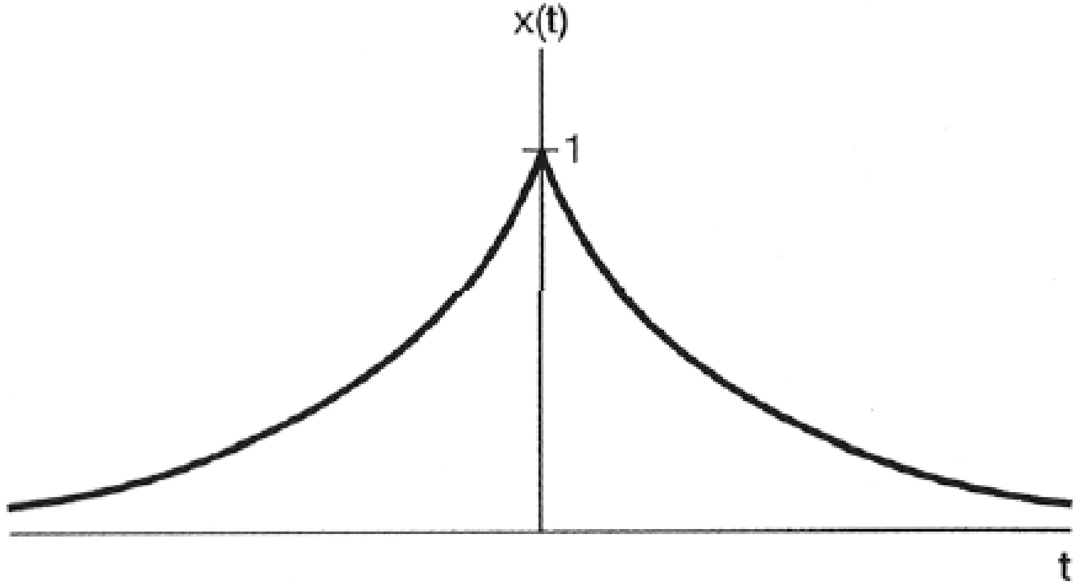


Odd Symmetry

Problem-2: Two-Sided Decaying Exponential

- Consider the signal:

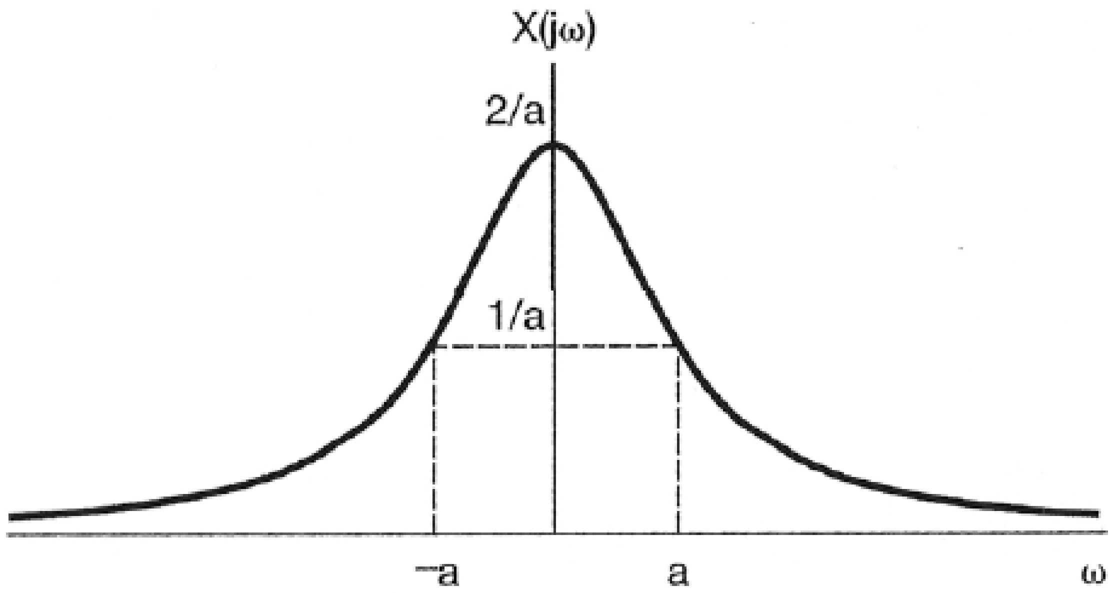
$$x(t) = e^{-a|t|}, \quad a > 0$$



Even Symmetry

Problem-2: Two-Sided Decaying Exponential

In this case $X(j\omega)$ is real.



Even Symmetry

Convergence of Fourier Transform

- Dirichlet conditions for convergence of Fourier Transform

1. $x(t)$ be absolutely integrable:

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

2. $x(t)$ have a finite number of maxima and minima within any finite interval

3. $x(t)$ have a finite number of discontinuities within any finite interval. Furthermore each of these discontinuities must be finite.

$\Rightarrow$ Absolutely integrable signals that are continuous or that have a finite number of discontinuities have Fourier Transforms.

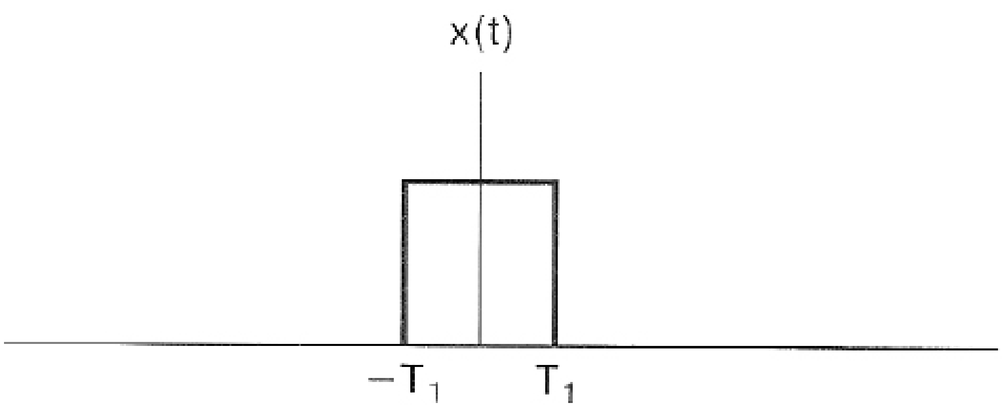
Square Pulse In Time Domain

- Consider the rectangular pulse signal:

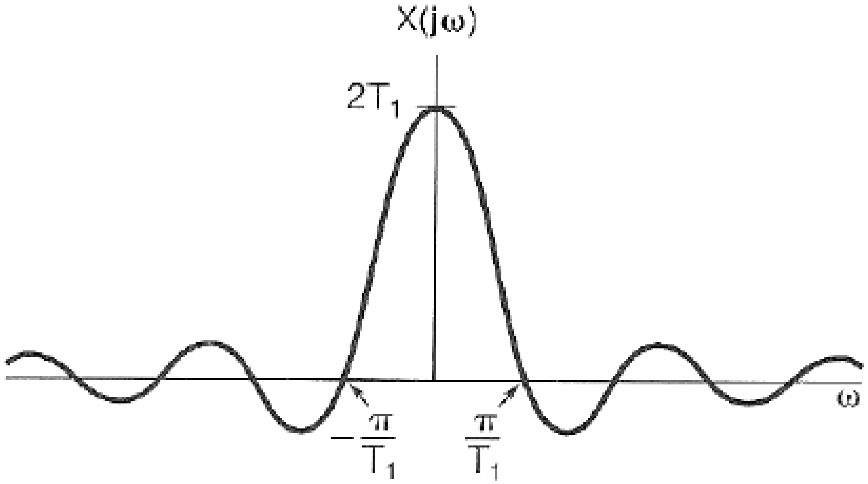
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

- The Fourier transform of this signal is:

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt = 2\frac{\sin(\omega T_1)}{\omega}$$



(a)



(b)

Square Pulse In Frequency Domain

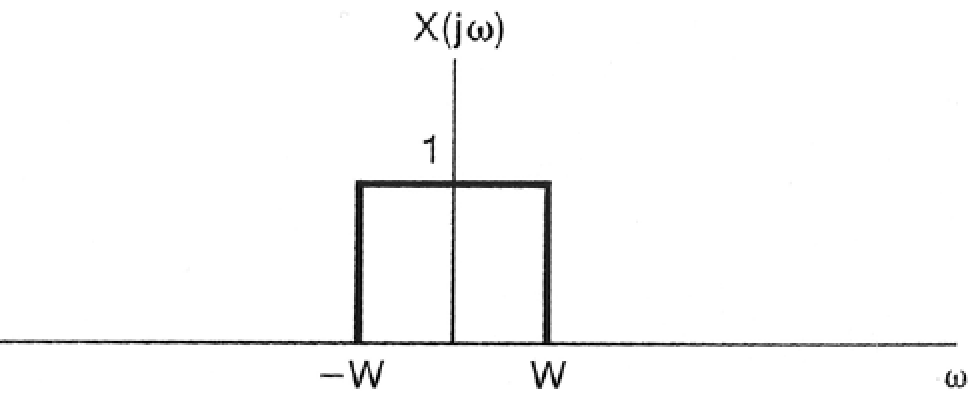
- Consider signal $x(t)$ with Fourier transform:

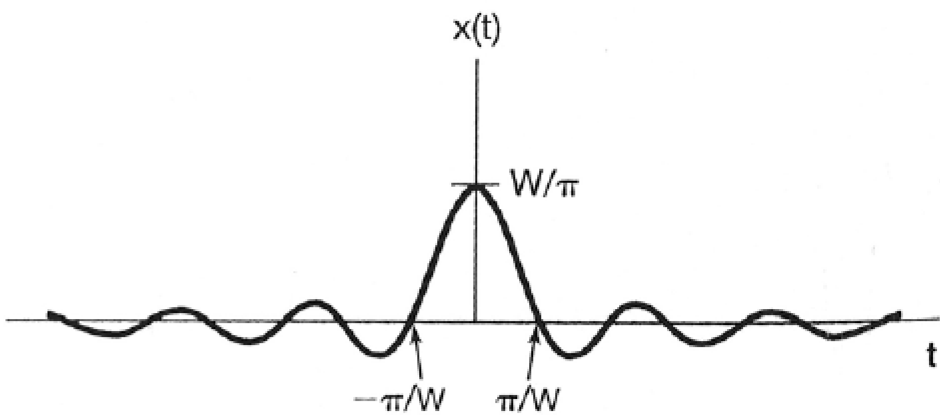
$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

- Then the signal is computed as:

$$x(t) = \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t} d\omega = \frac{\sin(Wt)}{\pi t}$$

Note duality between square pulse in time and square pulse in frequency





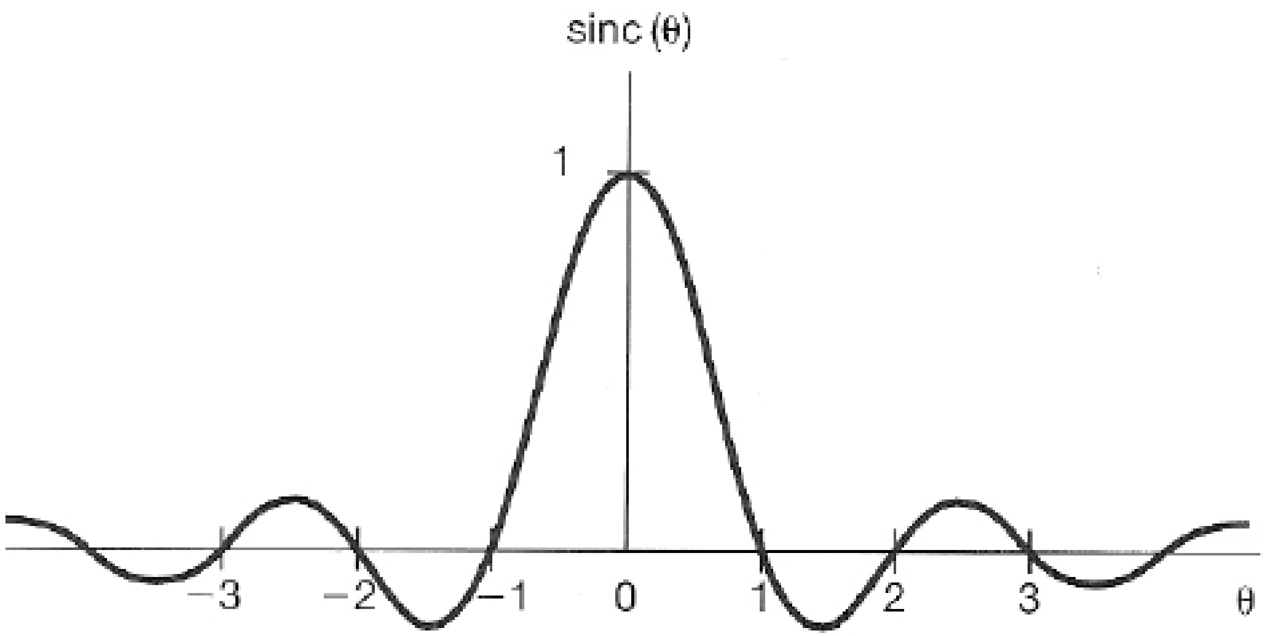
Sinc Functions

- Functions of the form:

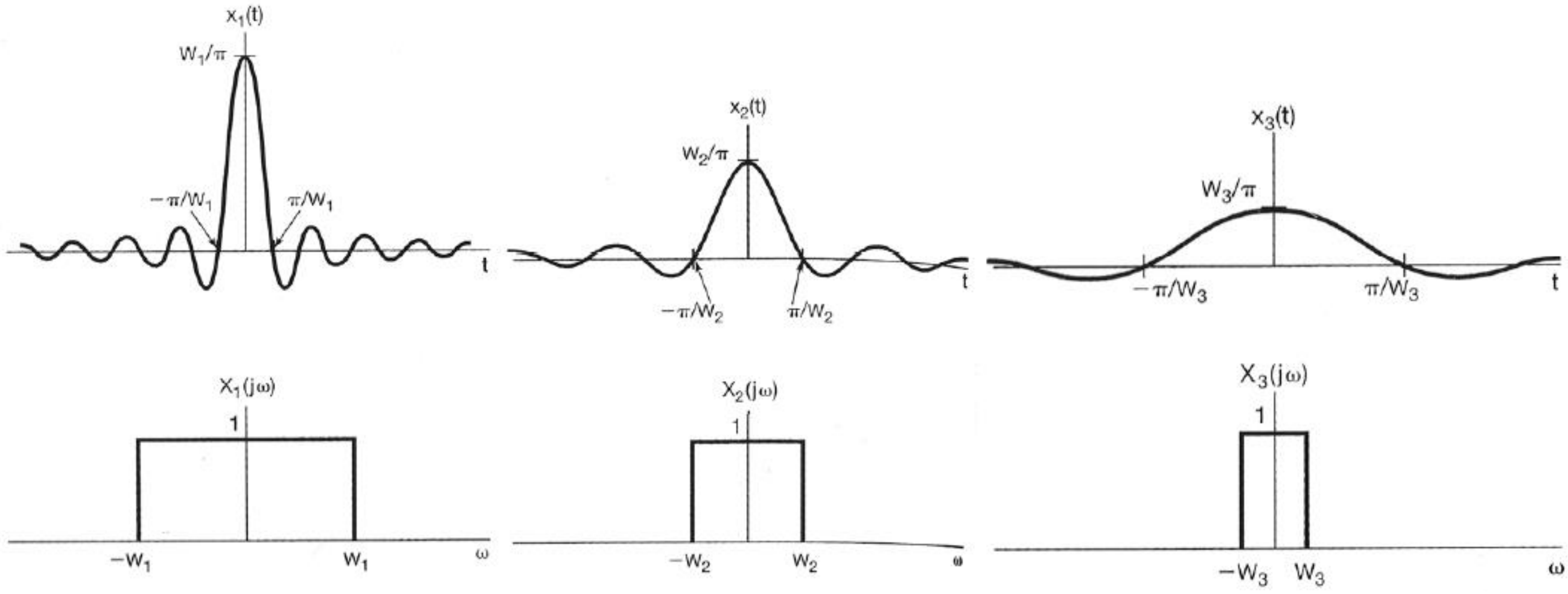
$$\frac{\sin(\omega T)}{\omega} \text{ or } \frac{\sin(Wt)}{\pi t}$$

occur often in Fourier analysis and in LTI system analysis and are referred to as *sinc functions*, and are of the form :

$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$



Uncertainty Principle



As width in frequency (W) becomes smaller, width in time ($2\pi/W$) becomes larger—with a constant product (uncertainty)

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