Complex Functions And Mappings

COMPLEX FUNCTIONS:

Note that we have defined the complex number z to be of the form $z = x + iy$.

Let us assume a function $f(z)$ having a domain of complex numbers ($z \in \mathbb{C}$). The range of the function would also (generally) be a set of complex numbers (i.e. $f(z) \in \mathbb{C}$).

say $f(z) = u + iv$, where $u, v \in \mathbb{R}$

but $z = z(x, y)$ [z is a complex valued function of x and y]

$\Rightarrow$ $u = u(x, y)$ [u is a real valued function of x and y]

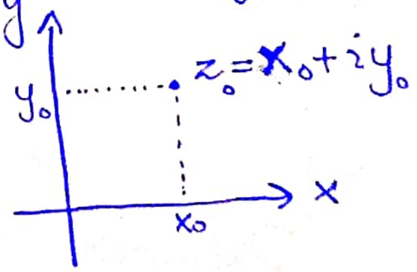
and $v = v(x, y)$ [v is a real valued function of x and y]

$\Rightarrow f(z) = u(x, y) + iv(x, y)$

MAPPINGS:

Note that z is represented in the x-y plane.

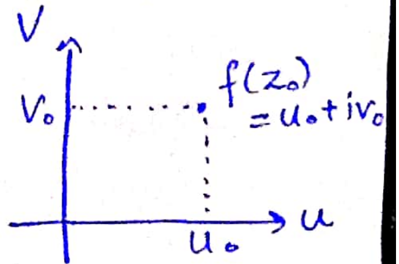
Our goal is to somehow graphically represent the complex function $f(z)$.



Let us apply an analogy:

if z is represented in the x-y plane and $z = x + iy$

then since $f(z) = u + iv$, it must be represented in the u-v plane.

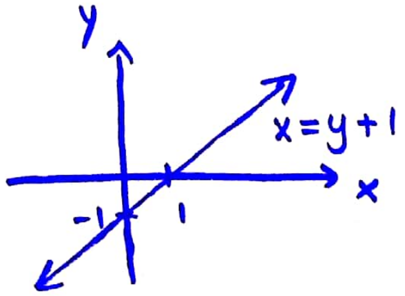


Now, note that $z_0, z_1, z_2, \ldots$ in the x-y plane map to $f(z_0), f(z_1), f(z_2), \ldots$ in the u-v plane.

Suppose that we know the relation between x and y which defines the domain of $f(z)$ i.e. the set of points z in the x-y plane.
Further note that we can find both u and v in terms of x and y by substituting $z = x+iy$ ~~if~~ in the definition of $f(z)$ and simplifying it to the form $f(z) = u(x,y) + iv(x,y)$.
We can then, using the known relation between x and y, find a relation between u and v and thus plot range of $f(z)$ on the u-v plane.

e.g: Domain of $f(z)$:

$\Rightarrow$ all of the points $z = x+iy$ in x-y plane such that $x = y+1$.



Definition of $f(z)$: $f(z) = 2z$

Plot the range of $f(z)$ in the u-v plane.

$f(z) = 2z \quad ; \quad z = x+iy$

$f(z) = 2(x+iy)$

$= 2x + i(2y)$

but $f(z) = u + iv$

$\Rightarrow u = 2x$ and $v = 2y$

Now, we know that $x = y+1$

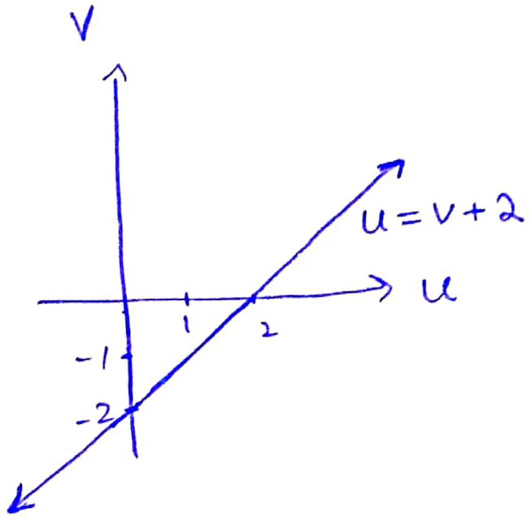
$\Rightarrow \quad u = 2(y+1)$

$\quad\quad u = 2y + 2$

but $\quad v = 2y$

$\Rightarrow \quad \boxed{u = v + 2}$

Range of $f(z)$:



NOTE:

This u-v plane is also called the w-plane like the x-y plane is called the z-plane.

In the above example, we can say that for $w = f(z)$, **f maps z onto w** or

z is mapped onto w by f.

* The above example is of a complex linear mapping $w = az + b$ with $a \neq 0$. It can distort the size of a figure in the complex plane, but it cannot alter the basic shape of the figure.

*** detailed discussion of complex linear mapping is done later on in this handout.**

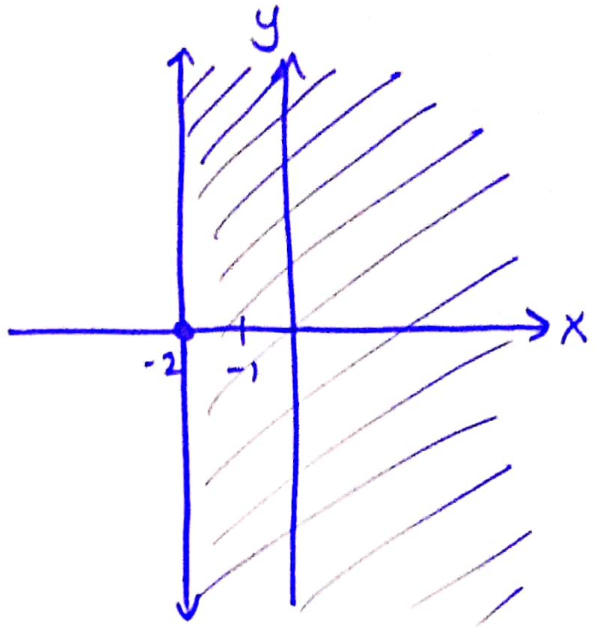
e.g: Image of half plane under $w = iz$

Domain of $f(z)$:

z such that

$x \geq -2$ and

$-\infty < y < \infty$



Definition of $f(z)$:

$w = f(z) = iz$

Range of $f(z)$

= Image of domain of $f(z)$ under $w = f(z) = iz$.

= ?

$f(z) = iz \quad ; \quad z = x + iy$

$f(z) = i(x + iy)$

$= xi + i^2 y$

$= -y + ix$

but $f(z) = u + iv$

$\Rightarrow u = -y \quad , \quad v = x$

Now, since $x \geq -2$

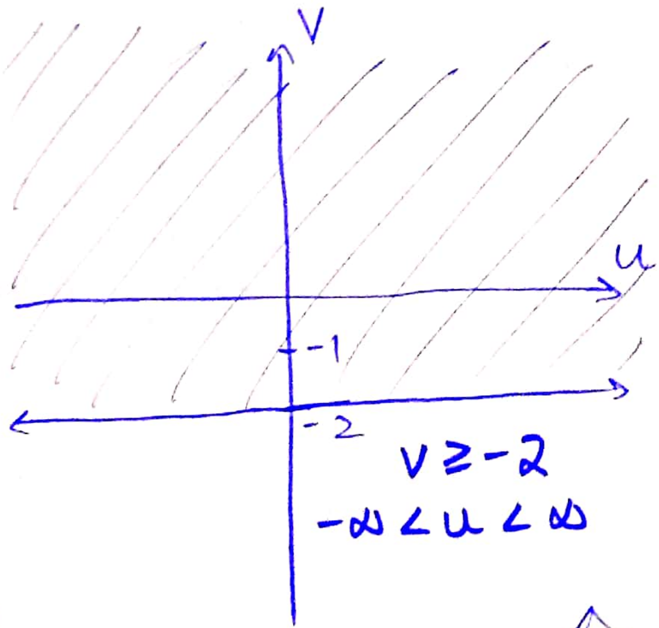
$\Rightarrow \boxed{v \geq -2}$

and since $-\infty < y < \infty$

$\Rightarrow +\infty > -y > -\infty$

or $-\infty < -y < \infty$

or $\boxed{-\infty < u < \infty}$



$v \geq -2$

$-\infty < u < \infty$

Thus Range of $f(z)$

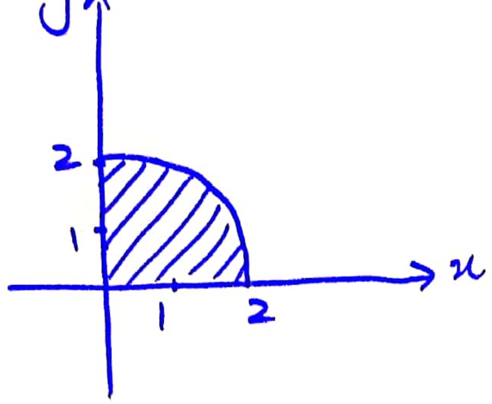
Uptil now we were discussing things in the cartesian plane. Let us now create an analogy of our discussion for the polar coordinates.

$$z = re^{i\theta}, \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

$$\Rightarrow f(z) = Re^{i\phi}, \quad R^2 = u^2 + v^2, \quad \tan\phi = \frac{v}{u}$$

$\mapsto$ parameterized in r, θ

e.g: Image of a <u>Parametric Curve</u> bounding a region under $w = z^2$



Domain of $f(z)$:

$z = re^{i\theta}$ such that

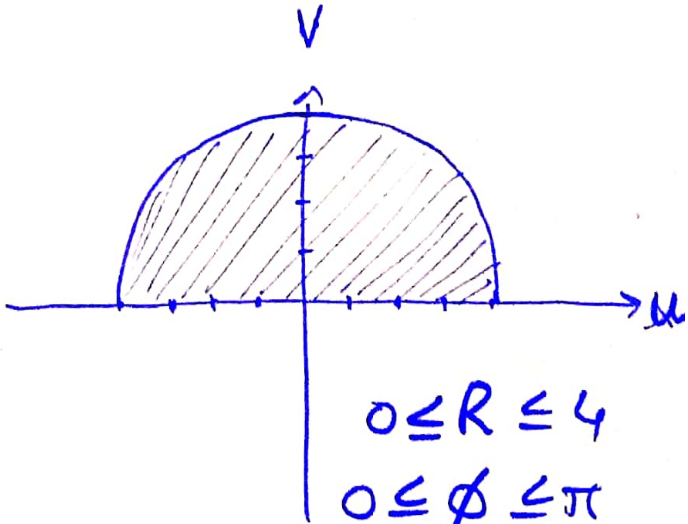
$0 \le r \le 2, \quad 0 \le \theta \le \frac{\pi}{2}$

Definition of $f(z)$:

$$w = f(z) = z^2.$$

Range of $f(z) = ?$

$$f(z) = z^2$$
$$= (re^{i\theta})^2$$
$$= r^2 e^{i2\theta}$$



$0 \le R \le 4$

$0 \le \phi \le \pi$

Range of $f(z)$ ⇑

but $f(z) = Re^{i\phi}$

$$\Rightarrow R = r^2 \;\; \& \;\; \phi = 2\theta$$

but $0 \le r \le 2 \;\; \& \;\; 0 \le \theta \le \frac{\pi}{2}$

$\Rightarrow 0 \le r^2 \le 4 \qquad 0 \le 2\theta \le \pi$

or $0 \le R \le 4 \qquad 0 \le \phi \le \pi$

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Linear Mappings:

TRANSLATIONS:

$T(z) = z + b$ where $b \in \mathbb{C}$ is a complex constant

Now $z = x + iy$

$b = x_0 + iy_0$

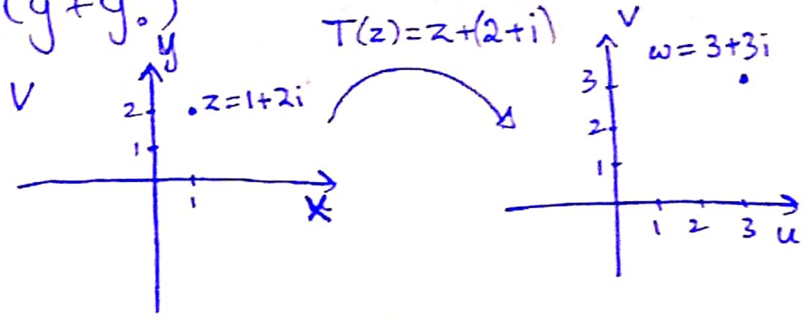
$T(z) = x + iy + x_0 + iy_0$

$= x + x_0 + i(y + y_0)$

but $T(z) = u + iv$

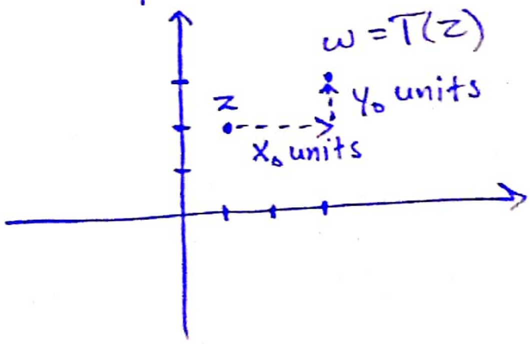
$u = x + x_0$

$v = y + y_0$



$\Downarrow$

(representing both z and w in same plane)



ROTATIONS:

$R(z) = az$ where $a \in \mathbb{C}$ and $|a| = 1$

discussing in polar coordinates:

$z = re^{i\theta}$

$a = e^{i\theta_{rotation}}$ $\longrightarrow$ check $|a| = \cos^2\theta_{rot} + \sin^2\theta_{rot} = 1$

$R(z) = e^{i\theta_{rot}} re^{i\theta}$

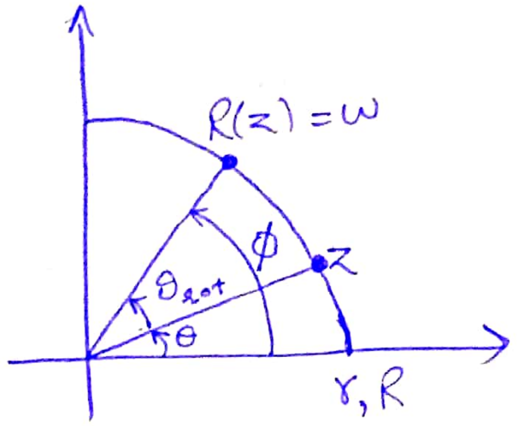
$= re^{i(\theta + \theta_{rot})}$

but $R(z) = Re^{i\phi}$

$\Rightarrow R = r$

$\phi = \theta + \theta_{rot}$

(representing both z and w in the same plane)



MAGNIFICATIONS:

$M(z) = az$ where $a \in \mathbb{R}$ and $a > 0$

Discussing in polar coordinates:

$z = re^{i\theta}$

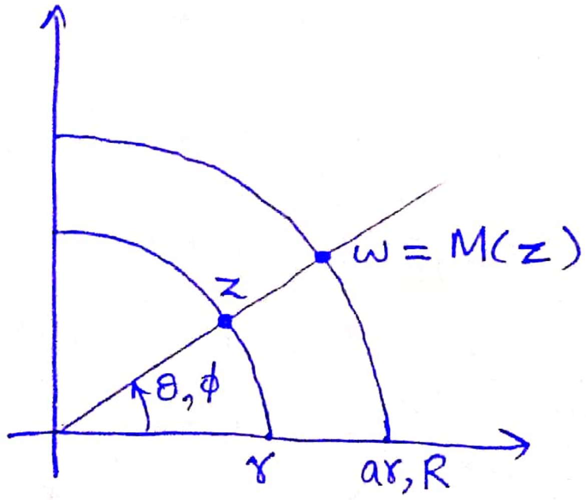
$M(z) = (ar)e^{i\theta}$

but $M(z) = Re^{i\phi}$

$\Rightarrow R = ar$

$\phi = \theta$

(representing both z and w in the same plane)



Generally, image of a point z_0 under a Linear Mapping:

$$f(z) = az + b = \underbrace{|a|}_{\text{magnification}} \underbrace{\left(\frac{a}{|a|}\right)}_{\text{rotation}} z + \underbrace{b}_{\text{translation}}$$

where $a, b \in \mathbb{C}$.

(i) rotating z_0 through an angle of $\text{Arg}(a)$ about the origin

(ii) magnifying the result by $|a|$, and

(iii) translating the result by b.

e.g: Find the image of a rectangle with vertices $-1+i$, $1+i$, $1+2i$ and $-1+2i$ under the linear mapping $f(z) = 4iz + 2 + 3i$

$f(z) = 4iz + 2 + 3i$

$a = 4i$

$|a| = \sqrt{0^2 + 4^2} = 4$

$b = 2 + 3i$

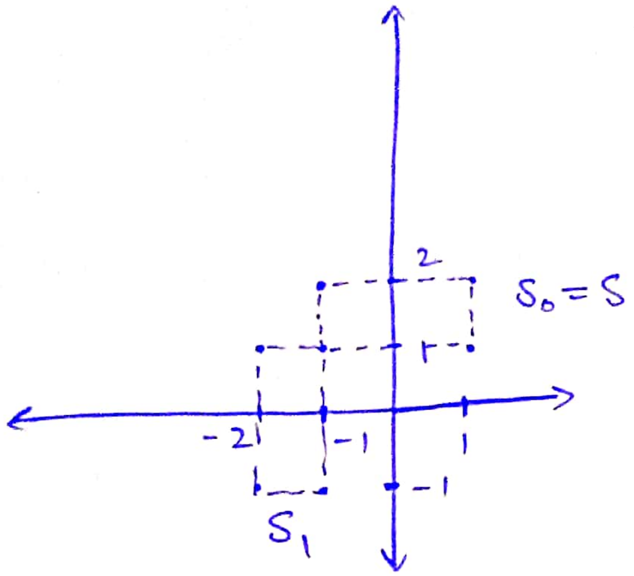
$$f(z) = |a| \cdot \frac{a}{|a|} \cdot z + b$$

$$= 4 \cdot \frac{4i}{4} \cdot z + b$$

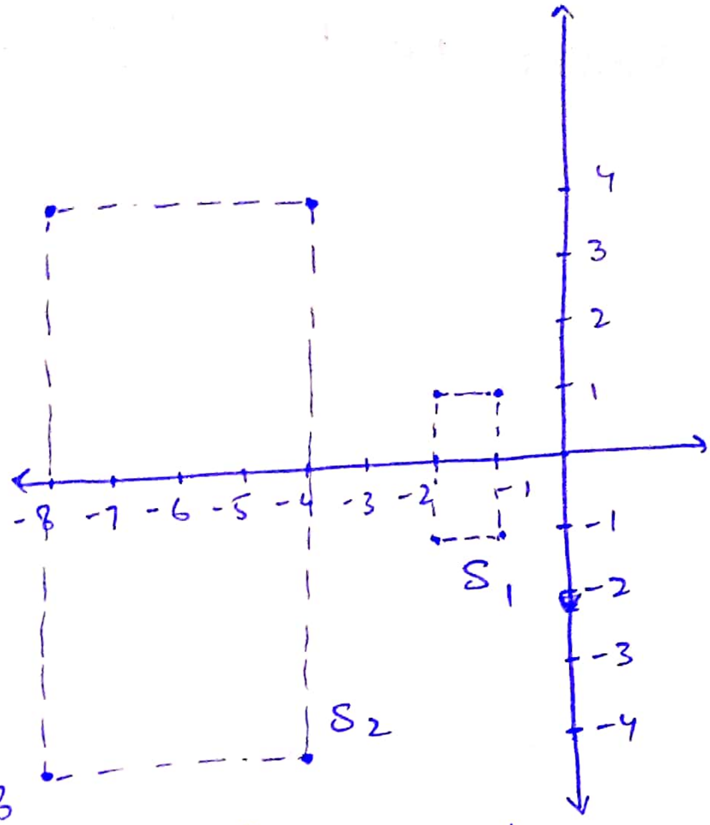
$$= (4)(i) z + b$$

$$= 4e^{i\pi/2} z + b$$

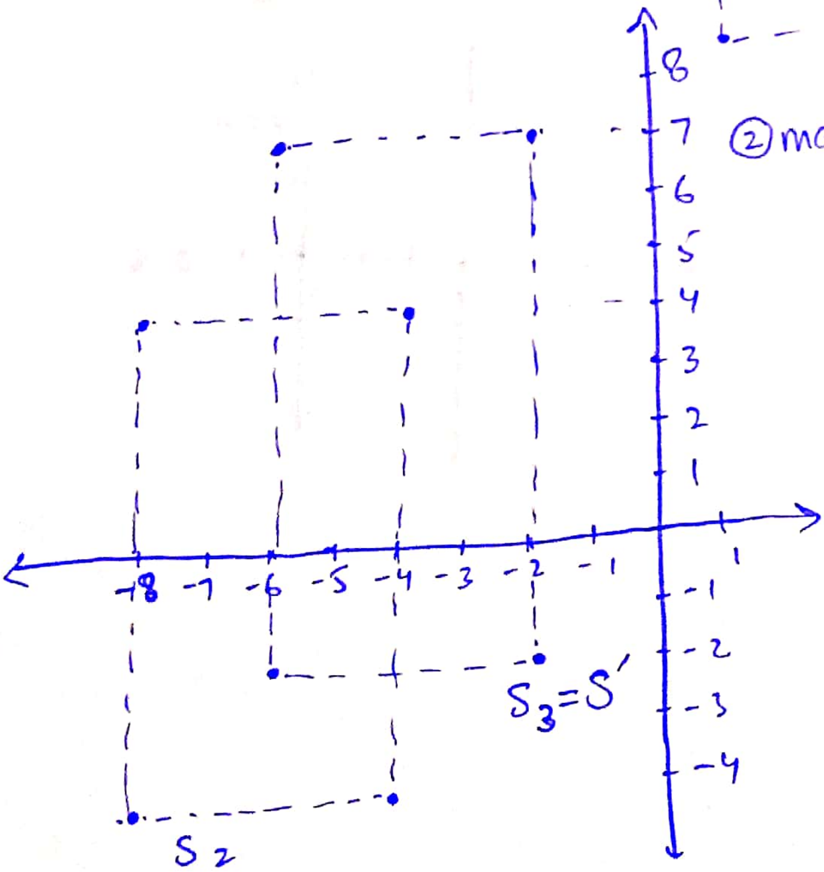
$$= \underbrace{4}_{\text{magnification}} \underbrace{e^{i\pi/2}}_{\text{rotation}} z + \underbrace{2+3i}_{\text{translation}}$$



① rotation by $\pi/2$



② magnification by 4.



$$S \xrightarrow[\text{transformation}]{\text{linear}} S'$$

③ Translation by $2+3i$

Power Function: (z^n)

$f(z) = z^n$.

$z = re^{i\theta}$

$\Rightarrow f(z) = r^n e^{in\theta}$

but $f(z) = Re^{i\phi}$

$\Rightarrow R = r^n, \ \phi = n\theta$.

$z^{1/n}$

$f(z) = z^{1/n}$

$z = re^{i\theta}$

$\Rightarrow f(z) = r^{1/n} e^{i\theta/n}$

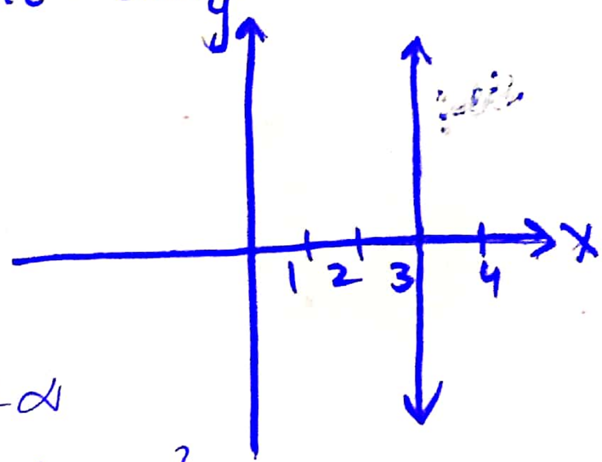
but $f(z) = Re^{i\phi}$

$\Rightarrow R = r^{1/n}$,

$\phi = \theta/n$.

one e.g. is already explained for $w = z^2$ (previously in this handout)

e.g. Image of a vertical line under $w = z^2$



Domain of $f(z)$:

$z = x + iy$ such that

$x = 3$

$-\infty \leq y < \infty$

or $z = 3 + iy$; $-\infty < y < \infty$

Definition of $f(z)$: $w = f(z) = z^2$

Range of $f(z) = ?$

$f(z) = z^2 = (x+iy)^2 = x^2 + i^2y^2 + 2xyi$

$= x^2 - y^2 + 2xyi$

but $f(z) = u + vi$

$\Rightarrow u = x^2 - y^2, \quad v = 2xy$

but $x = 3$

$\Rightarrow u = 9 - y^2, \quad v = 6y$

or $y = \frac{v}{6}$

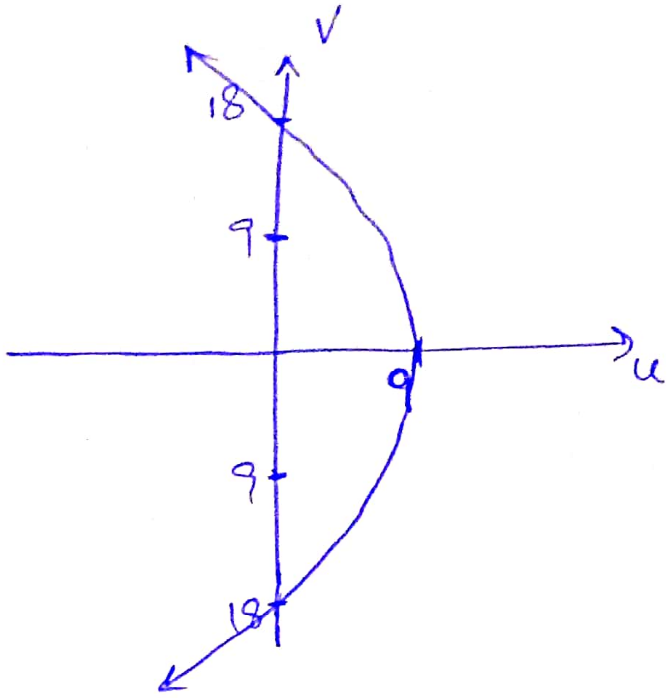
$\Rightarrow u = 9 - \frac{v^2}{36}$

$\Rightarrow 36u = 324 - v^2$

$v^2 = -36u + 324$

$v^2 = -36(u-9)$

(equation of parabola)



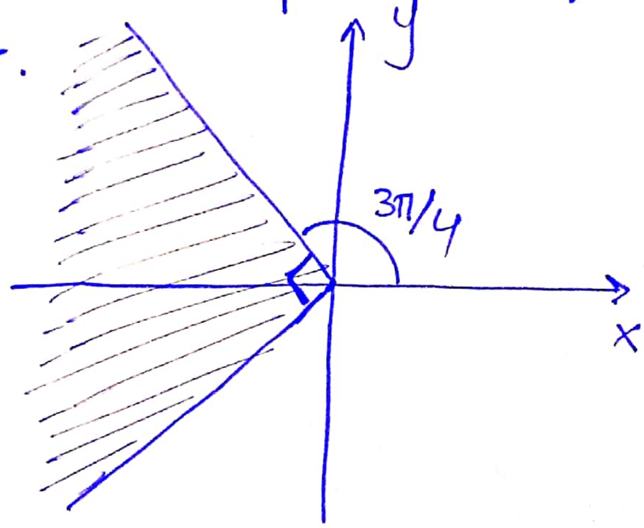
when $v = 0$,

$u = 9$

when $u = 0$

$v = \pm 18$

e.g: Find the image of the region shown in the following figure under the principal square root function $w = z^{1/2}$.



Principal Square Root Function

$z^{1/n} = \sqrt[n]{|z|}\, e^{i\frac{Arg(z)}{n}}$

$\Rightarrow z^{1/2} = \sqrt{|z|}\, e^{i\frac{Arg(z)}{2}}$

Now, $-\pi < Arg(z) \leq \pi$

Domain of $f(z)$:

$z = re^{i\theta}$ such that $0 \leq r < \infty$ and

$-\pi < \theta \leq -\frac{3\pi}{4}$

union $\frac{3\pi}{4} \leq \theta \leq \pi$

Definition of $f(z)$:

$w = f(z) = z^{1/2}$.

Range of $f(z) = ?$

$$f(z) = z^{1/2} \quad ; \quad z = re^{i\theta}$$

$$= r^{1/2} e^{i\theta/2}$$

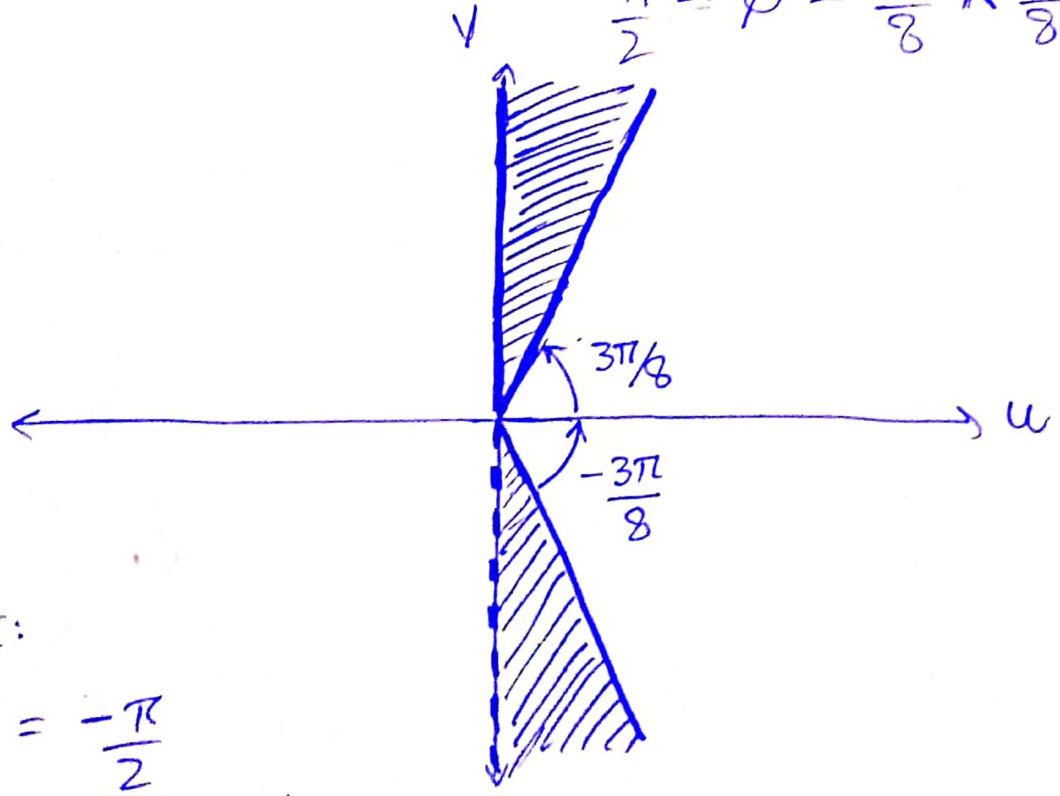
but $f(z) = Re^{i\phi}$

$\Rightarrow$ $R = r^{1/2}$, $\phi = \dfrac{\theta}{2}$

$\because$ $0 \le r < \infty$, $-\pi < \theta \le -\dfrac{3\pi}{4} \wedge \dfrac{3\pi}{4} \le \theta \le \pi$

$\Rightarrow$ $0 \le r^{1/2} < \infty$, $-\dfrac{\pi}{2} < \dfrac{\theta}{2} \le -\dfrac{3\pi}{8} \wedge \dfrac{3\pi}{8} \le \dfrac{\theta}{2} \le \dfrac{\pi}{2}$

$\Rightarrow$ $0 \le R < \infty$, $-\dfrac{\pi}{2} < \phi \le -\dfrac{3\pi}{8} \wedge \dfrac{3\pi}{8} \le \phi \le \dfrac{\pi}{2}$.



NOTE:

$\phi = -\dfrac{\pi}{2}$

is excluded.

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Reciprocal Function:

$$f(z) = \frac{1}{z}$$

$$z = re^{i\theta}$$

$$\Rightarrow f(z) = \frac{1}{re^{i\theta}}$$

$$= \frac{1}{r} e^{i(-\theta)}$$

but $f(z) = Re^{i\phi}$

thus $R = \frac{1}{r}$, $\phi = -\theta$

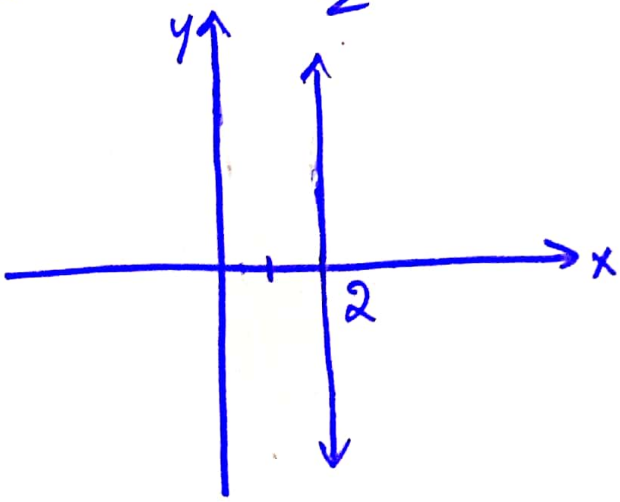
e.g: Image of a line under $w = \frac{1}{z}$

Domain of $f(z)$:

$z = x + iy$ such that

$x = 2$ and

$-\infty < y < \infty$



Definition of $f(z)$:

$$w = f(z) = \frac{1}{z}$$

Range of $f(z) = ?$

$$f(z) = \frac{1}{z} = \frac{1}{x+iy}$$

$$= \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

but $f(z) = u + iv$

$\Rightarrow u = \dfrac{x}{x^2+y^2}$, $v = \dfrac{-y}{x^2+y^2}$

but $x = 2$

$\Rightarrow u = \dfrac{2}{4+y^2}$ — ① , $v = \dfrac{-y}{4+y^2}$ — ②

from ① $4+y^2 = \dfrac{2}{u}$ and putting in ②

$v = \dfrac{-y}{2/u} \Rightarrow v = -\dfrac{yu}{2}$

or $y = -\dfrac{2v}{u}$ — ③

putting ③ in ①:

$u = \dfrac{2}{4+\left(-\dfrac{2v}{u}\right)^2}$

$u = \dfrac{2}{4+\dfrac{4v^2}{u^2}}$

$u = \dfrac{2}{4\left(1+\dfrac{v^2}{u^2}\right)}$

$u = \dfrac{1}{2\left(1+\dfrac{v^2}{u^2}\right)}$

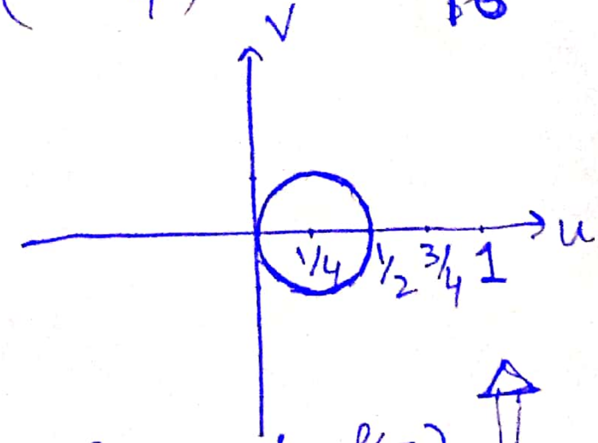
$\Rightarrow u + \dfrac{v^2}{u} = \dfrac{1}{2}$

$\Rightarrow u^2+v^2 = \dfrac{u}{2}$

$u^2 - \dfrac{u}{2} + v^2 = 0$

$u^2 - 2(u)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + v^2 = \left(\frac{1}{4}\right)^2$

$\left(u - \frac{1}{4}\right)^2 + v^2 = \frac{1}{16}$



Range of $f(z)$ ⇑

Exponential Function:

$$f(z) = e^z = e^x \cos y + i e^x \sin y$$

for $z = x + iy$.

$$f(z) = e^z$$
$$= e^{x+iy}$$
$$= e^x . e^{iy}$$

but $f(z) = Re^{i\phi}$

$\Rightarrow R = e^x$ and $\phi = y$ (in radians)

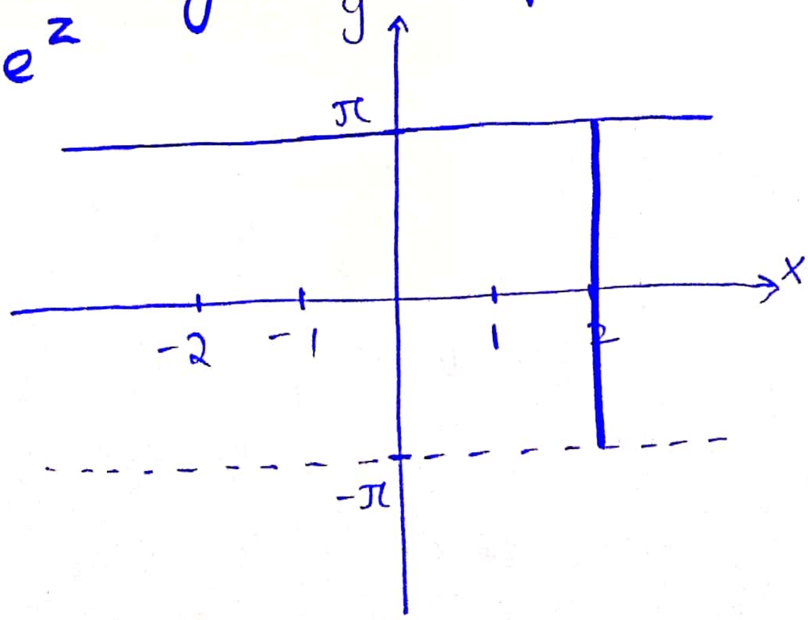
↳ fundamentally, max range of ϕ is:

$$-\pi < \phi \leq \pi$$

Thus the fundamental region in z-plane that can be mapped onto w-plane is:

$$-\infty < x < \infty, \quad -\pi < y \leq \pi.$$

e.g. Image of vertical line segment in fundamental region under $w = e^z$



Domain of $f(z)$:

$z = x + iy$ such that

$x = 2$

$-\pi < y \leq \pi$

Definition of $f(z)$:

$w = f(z) = e^z$.

Range of $f(z) = ?$

$f(z) = e^z = e^{x+iy}$

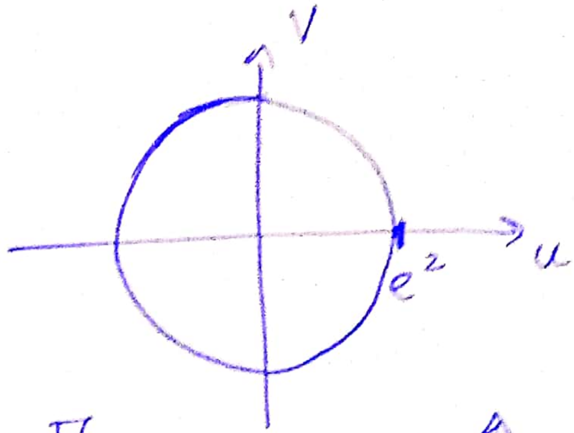
$= e^x e^{iy}$

but $f(z) = Re^{i\phi}$

$R = e^x, \quad \phi = y$

$R = e^2, \quad -\pi < \phi \le \pi$

$\therefore x = 2 \qquad \therefore -\pi < y \le \pi$



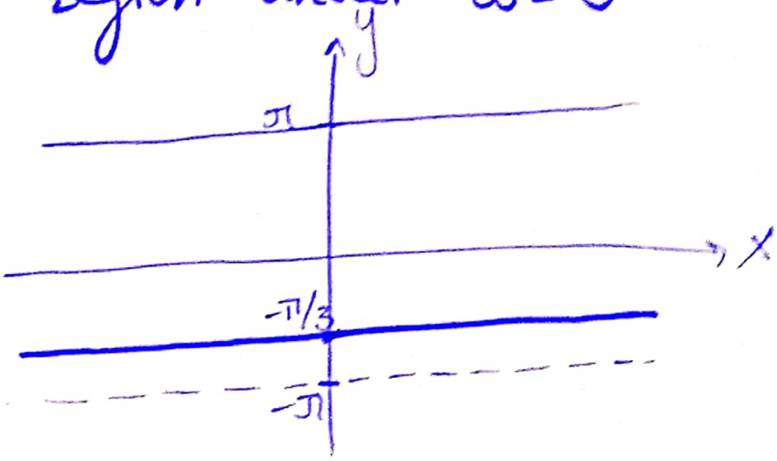
Range of $f(z)$

e.g: Image of a horizontal line segment in fundamental region under $w = e^z$

Domain of $f(z)$:

$z = x + iy$ such that

$-\infty < x < \infty$ and $y = -\frac{\pi}{3}$



Definition of $f(z)$:

$w = f(z) = e^z$

Range of $f(z)$:

$f(z) = e^z = e^{x+iy}$

$= e^x e^{iy}$

but $f(z) = Re^{i\phi}$

$R = e^x, \quad \phi = y$

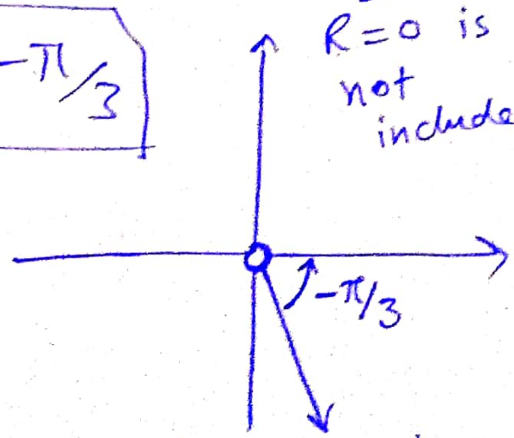
$\because y = -\pi/3, \quad \boxed{\phi = -\pi/3}$

Now $-\infty < x < \infty$

$\Rightarrow 0 < e^x < \infty$

$\Rightarrow \boxed{0 < R < \infty}$

NOTE: $R = 0$ is not included.



Range of $f(z)$

Complex Logarithmic Function:

$$f(z) = \ln z$$

$$z = re^{i\theta}, \qquad |z| > 0$$

$$\Rightarrow f(z) = \ln(re^{i\theta})$$

$$= \ln r + \ln e^{i\theta}$$

$$= \ln r + i\theta$$

where $\theta = Arg(z)$

$$f(z) = \ln z = \ln r + i\theta$$

$\hookrightarrow$ principal logarithmic function

generally,

for $z = re^{i(\theta + 2n\pi)}$ $\qquad n = 0, \pm 1, \pm 2, \dots$

$$f(z) = \ln z = \ln(re^{i(\theta + 2n\pi)})$$

$$= \ln r + \ln e^{i(\theta + 2n\pi)}$$

$$= \ln r + i(\theta + 2n\pi)$$

where $\theta + 2n\pi = arg(z)$

Considering

$$f(z) = \ln r + i\theta$$

but $f(z) = u + iv$

$$\Rightarrow u = \ln r, \qquad v = \theta$$

or $u = \ln|z|, \qquad v = Arg(z)$

$\because 0 < |z| < \infty$, $\quad \because -\pi < Arg(z) \leq \pi$

$\Rightarrow -\infty < u < \infty$, $\quad \Rightarrow -\pi < v \leq \pi$.

e.g: (mappings)

① $w = \ln z$ maps the circle $|z| = r$ onto the vertical line segment $u = \ln r$, $-\pi < v \leq \pi$.

② $w = \ln z$ maps the ray $Arg(z) = \theta$ onto the horizontal line $v = \theta$, $-\infty < u < \infty$.

Complex Powers:

(mappings not discussed for such functions)

$f(z) = z^{\alpha}$ where $\alpha \in \mathbb{C}$

$= e^{\ln(z^{\alpha})}$

$= e^{\alpha \ln z}$

$\Rightarrow z^{\alpha} = e^{\alpha \ln z}$ → multivalued function

Trignometric Functions:

$\sin z = \sin x \cosh y + i \cos x \sinh y$

$\cos z = \cos x \cosh y - i \sin x \sinh y$

$\boxed{\cos(iy) = \cosh y \quad \sin(iy) = i\sinh y}$

∵ for $f(z) = \sin z$; $z = x + iy$

$= \sin(x + iy)$

$= \sin x \cos(iy) + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$

but $f(z) = u + iv$

$\Rightarrow u = \sin x \cosh y$, $v = \cos x \sinh y$

for $f(z) = \cos z$; $z = x + iy$

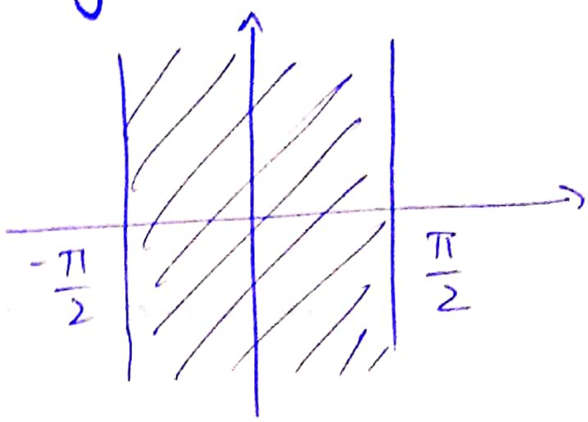
$= \cos(x + iy) = \cos x \cos(iy) - \sin x \sin iy$

$= \cos x \cosh y - i \sin x \sinh y$

but $\boxed{f(z) = u + iv}$

$\Rightarrow u = \cos x \cosh y$, $v = -\sin x \sinh y$

e.g Describe the image of the region $-\frac{\pi}{2} \le x \le \pi/2$, $-\infty < y < \infty$, under $w = \sin z$



$f(z) = \sin z$
$= \sin x \cosh y + i \cos x \sinh y$

$\Rightarrow u = \sin x \cosh y$
$v = \cos x \sinh y$

$\because \cosh^2\alpha - \sinh^2\alpha = 1$

$$\therefore \frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1 \quad \text{—} (1)$$

(1) gives us the set of equations of hyperbola for the set of vertical lines $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ except for $x = \pm\frac{\pi}{2}, 0$.

for $x = \frac{\pi}{2}$	for $x = -\frac{\pi}{2}$	for $x = 0$
$u = \cosh y$	$u = -\cosh y$	$u = 0$
$\Rightarrow u \ge 1$	$\Rightarrow u \le -1$	
$\because \cosh y \ge 1$	$\because \cosh y \ge 1$	
for $-\infty < y < \infty$	for $-\infty < y < \infty$	

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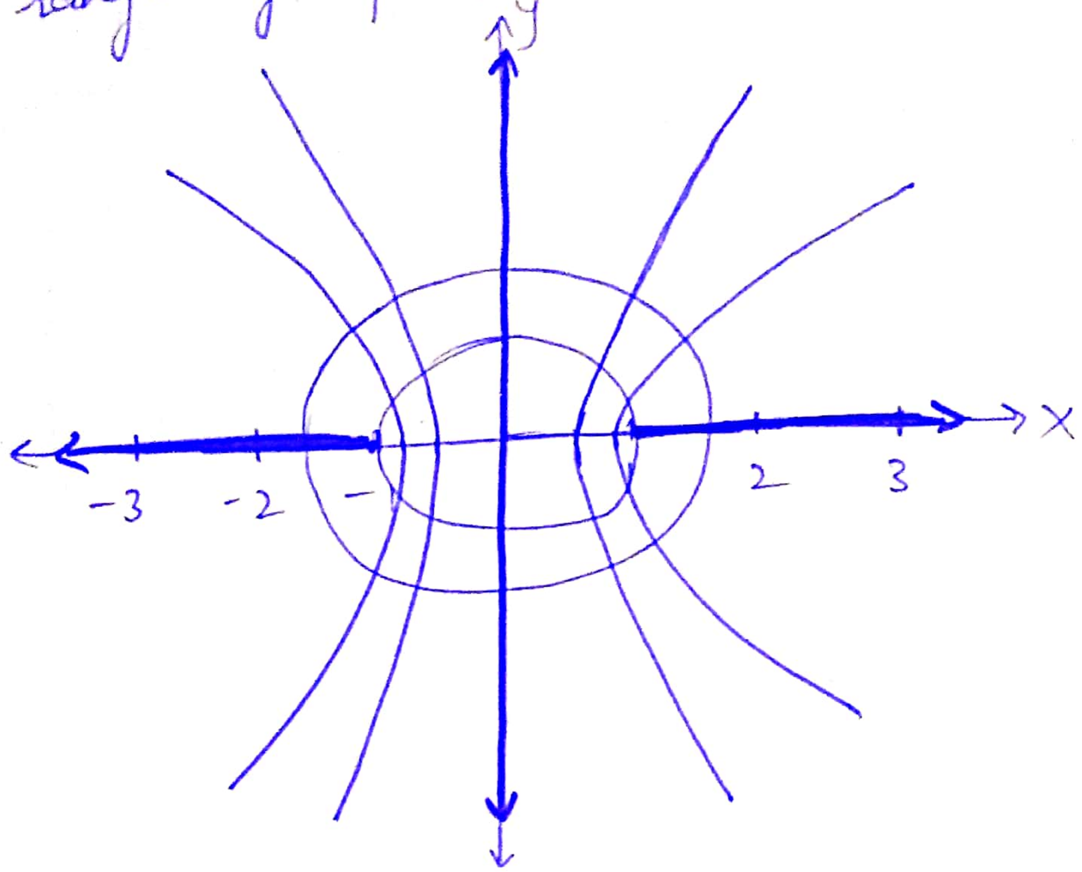
The range of $f(z)$ is thus:



Note that here the ellipses are mapped by the horizontal lines in the z-plane.

$\because \cos^2 x + \sin^2 x = 1$

$$\frac{u^2}{\sinh y^2} + \frac{v^2}{\cosh y^2} = 1$$

where $-\infty < y < \infty$

equation of ellipses formed for $-\infty < y < \infty$

BILINEAR / MÖBIUS TRANSFORMATION:

Let $a,b,c,d \in \mathbb{C}$, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc \neq 0$.

We define a bilinear transformation or a Möbius Transformation $T: \mathbb{C} \to \mathbb{C}$ as:

for $c \neq 0$,

$$T(z) = \begin{cases} \dfrac{az+b}{cz+d}, & z \in \mathbb{C} \wedge z \neq -\dfrac{d}{c} \\ \dfrac{a}{c}, & z = \infty \\ \infty, & z = -\dfrac{d}{c} \end{cases}.$$

and for $c = 0$,

$$T(z) = \begin{cases} \dfrac{az+b}{d}, & z \in \mathbb{C} \\ \infty, & z = \infty \end{cases}$$

in general we write a bilinear transformation as $w = T(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$ without any ambiguity

NOTE:

The inverse of a bilinear transformation is also a bilinear transformation.

$$w = T(z) = \frac{az+b}{cz+d}, \quad T^{-1}(w) = T^{-1}[T(z)]$$

$$z = T^{-1}(w) = \frac{-dw+b}{cw-a}$$

Thus, $z = \dfrac{-dw+b}{cw-a}$ is also a bilinear transformation except at $w = a/c$.

Cross Ratio:

If there are four points z_1, z_2, z_3, z_4 taken in order, then the ratio $\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross ratio of z_1, z_2, z_3, z_4.

A bilinear transformation preserves cross ratio of four points

i.e. $$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

e.g: Find the bilinear transformation that maps the points $z = 1, i, -1$ into the points $w = i, 0, -i$. Hence find the image of $|z| < 1$.

$$w = \frac{az+b}{cz+d}$$

here we have four unknows ; a, b, c, d

but are given with only three known values of z mapped on w, thus, dividing the numerator and denominator by d:

$$w = \frac{\frac{a}{d}z + \frac{b}{d}}{\frac{c}{d}z + 1}$$ say $\frac{a}{d} = p$, $\frac{b}{d} = q$, $\frac{c}{d} = r$

then $$w = \frac{pz+q}{rz+1}$$

Now we have 3 unknows : p, q, r

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$z=1 \Rightarrow w=i$:

$$i=\frac{p+q}{r+1}, \quad p+q-ir-i=0 \quad \text{— (1)}$$

$z=i \Rightarrow w=0$:

$$0=\frac{pi+q}{ri+1}, \quad pi+q=0 \quad \text{— (2)}$$

and $z=-1 \Rightarrow w=-i$:

$$-i=\frac{(-p+q)}{(-r+1)}, \quad -p+q-ir+i=0 \quad \text{— (3)}$$

from (2): $q=-pi$

(1) becomes: $(1-i)p-ir-i=0$ — (4)

(3) becomes: $(-1-i)p-ir+i=0$ — (5)

subtracting (5) from (4):

$$(2)p-(2i)=0$$

$$\boxed{p=i}$$

$$\Rightarrow \quad q=-i\cdot i=(-1)(-1)$$

$$\boxed{q=1}$$

from (1):

$$1+\cancel{i}-ir-\cancel{i}=0$$

$$\cancel{-}ir=\cancel{-}1$$

$$r=\frac{1}{i}$$

$$\boxed{r=-i}$$

$\Rightarrow \quad w = \dfrac{pz+q}{rz+1}$ becomes:

$$\boxed{w = \frac{iz+1}{1-iz}}$$

B Image of $|z|<1$ — ?

ition

s

$|z|<1$

$\Rightarrow \sqrt{x^2+y^2} < 1$

or $x^2+y^2<1$

Now $w = u+iv$ & $z = x+iy$

)

z)

$$u+iv = \frac{i(x+iy)+1}{1-i(x+iy)}$$

$$u+iv = \frac{xi-y+1}{1-xi+y}$$

$$= \frac{(1-y)+ix}{(1+y)-ix} \times \frac{(1+y)+ix}{(1+y)+ix}$$

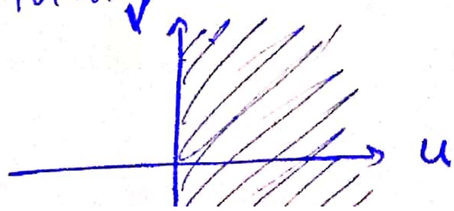
$$= \frac{1-y^2-x^2+2ix}{(1+y)^2+x^2}$$

$$u = \frac{1-y^2-x^2}{(1+y)^2+x^2}, \quad v = \frac{2x}{(1+y)^2+x^2}$$

Note : $x^2+y^2<1 \Rightarrow 1-y^2-x^2>0$

$\Rightarrow \boxed{u>0}$, also note that $-\infty < x < \infty$

$\Rightarrow \boxed{-\infty < v < \infty}$.



E.g: Find the bilinear transformation which maps the points $z = 0, -1, i$ onto $w = i, 0, \infty$. Also find image of unit circle.

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

here we will use this cross-ratio preservation formula because one of the values of w is ∞ and thus we cannot construct 3 equations from the formula of möbius transformation.

rearranging: $$\frac{(w-w_1)(w_2/w_3-1)}{(w_1-w_2)(1-w/w_3)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$\because$ $w_3 = \infty$,

$$\frac{w_2}{w_3} = 0 \ \& \ \frac{w}{w_3} = 0.$$

$$\Rightarrow \frac{(w-i)(-1)}{(i-0)(1)} = \frac{(z-0)(-1-i)}{(0+1)(i-z)}$$

$$\frac{i-w}{i} = \frac{(-z-iz)}{i-z}$$

$$1-\frac{w}{i} = \frac{(-z-iz)}{i-z}$$

$$w = \frac{+i(+z+iz)}{i-z} + i$$

$$= \frac{i\cancel{z} - z + i^2 - \cancel{z}\,i}{i-z}$$

$$= \frac{\cancel{-}(z+1)}{\cancel{-}(z-i)}$$

$$\boxed{w = \frac{z+1}{z-i}}$$

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$|z| = 1 \rightarrow$ unit circle

if $w = \dfrac{z+1}{z-i}$ then $\boxed{z = \dfrac{+iw+1}{w-1}}$

$$|z| = 1 \Rightarrow \left|\frac{iw+1}{w-1}\right| = 1$$

$$|iw+1| = |w-1|$$

$$|i(u+iv)+1| = |u+iv-1|$$

$$|(1-v)+iu| = |(u-1)+iv|$$

$$\sqrt{(1-v)^2+u^2} = \sqrt{(u-1)^2+v^2}$$

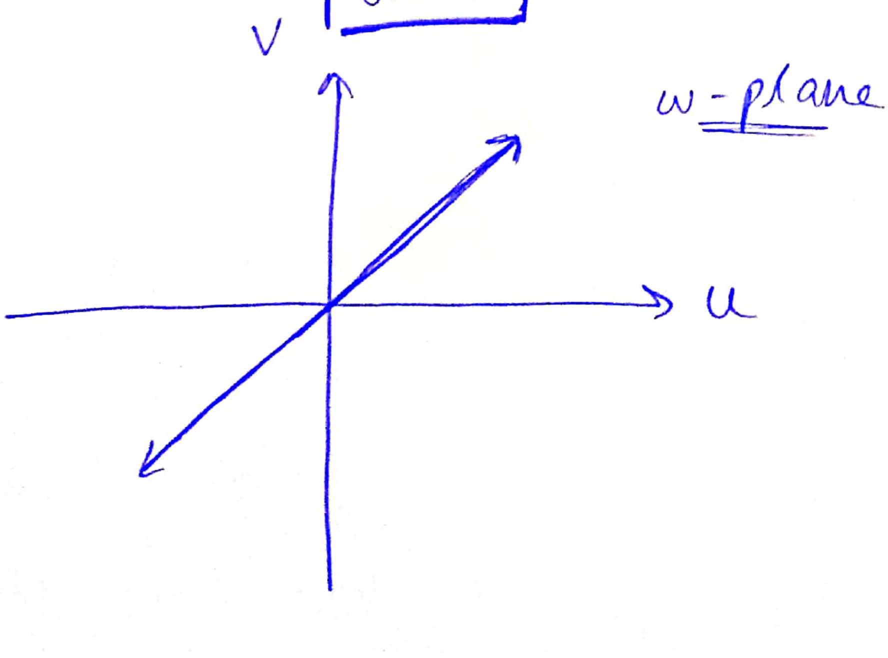
$$(1-v^2)^2+u^2 = (u-1)^2+v^2$$

$$\cancel{1}-2v+\cancel{v^2}+\cancel{u^2} = \cancel{u^2}-2u+\cancel{1}+\cancel{v^2}$$

$$-2v+2u = 0$$

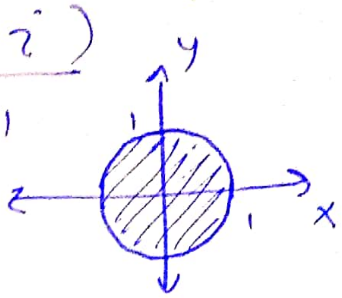
$$u-v = 0$$

$$\boxed{u=v}$$



e.g: Show that the circle $|z|=1$ is transformed onto the real axis of the w plane by $w = i\left(\frac{1-z}{1+z}\right)$. The interior of the circle transforms to upper half or lower half?

$$w = i\left(\frac{1-z}{1+z}\right) = \frac{(-iz+i)}{1+z} = \frac{(-iz+i)}{z+1}$$



$$\Rightarrow \quad z = \frac{(-w+i)}{w+i} = \frac{i-w}{i+w}$$

$$|z| \leq 1 \quad , \quad \Rightarrow \left|\frac{i-w}{i+w}\right| \leq 1 \quad , \quad |i-w| \leq |i+w|$$

$w = u + iv$

$$\Rightarrow |i-u-iv| \leq |i+u+iv|$$

$$|(-u)+(1-v)i| \leq |u+(1+v)i|$$

$$\cancel{u^2} + (1-v)^2 \leq \cancel{u^2} + (1+v)^2$$

$$\cancel{1} - 2v + \cancel{v^2} \leq \cancel{1} + 2v + \cancel{v^2}$$

$$0 \leq 4v$$

$$v \geq 0$$

$\Rightarrow$ The interior of the circle transforms to the upper half.