Fourier transform & Convolution:

In view of the duality between time and frequency domains, there are two convolution results involving the Fourier transform.

Convolution in time: Suppose that $\mathcal{F}[f(t)] = F(j\omega)$, $\mathcal{F}[g(t)] = G(j\omega)$

and $y(t) = f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau$

$$\mathcal{F}[y(t)] = Y(j\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} f(\tau) \left[\int_{-\infty}^{+\infty} e^{-j\omega t}\, g(t-\tau)\, dt \right] d\tau .$$

Let $t-\tau = u \Rightarrow t = u+\tau$, $dt = du$

$$Y(j\omega) = \int_{-\infty}^{+\infty} f(\tau) \left[\int_{-\infty}^{\infty} g(u)\, e^{-j\omega(u+\tau)}\, du \right] d\tau$$

$$= \int_{-\infty}^{+\infty} f(\tau)\, e^{-j\omega\tau}\, d\tau \int_{-\infty}^{+\infty} g(u)\, e^{-j\omega u}\, du = F(j\omega)\, G(j\omega)$$

hence, $\mathcal{F}[f(t) * g(t)] = F(j\omega)\, G(j\omega)$.

Convolution in frequency: $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega)\, e^{j\omega t}\, d\omega$, $g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(j\omega)\, e^{j\omega t}\, d\omega$

$$F(j\omega) * G(j\omega) = \int_{-\infty}^{+\infty} F(y)\, G(\omega - y)\, dy$$

$$\mathcal{F}^{-1}[F(\omega) * G(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} F(y)\, G(\omega-y)\, dy \right] e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(y) \left[\int_{-\infty}^{+\infty} G(\omega-y)\, e^{j\omega t}\, d\omega \right] dy \qquad \begin{array}{l} \omega - y = u \\ \omega = y + u \end{array}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(y) \left[\int_{-\infty}^{+\infty} G(u)\, e^{j(y+u)t}\, du \right] dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(y)\, e^{jyt}\, dy \int_{-\infty}^{+\infty} G(u)\, e^{jut}\, du = 2\pi\, f(t)\, g(t)$$

giving, $\mathcal{F}[f(t)g(t)] = \frac{1}{2\pi} F(j\omega) * G(j\omega)$.

Ex:- A relaxed causal LTI system is described by the following ordinary differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = \frac{dx}{dt} \quad \text{—— (i)}$$

Where $x(t)$ is the input and $y(t)$ is the output. Using Fourier transform techniques, determine the output $y(t)$ when the input $x(t) = u(t)$, the unit step signal.

Sol:- Taking Fourier transform of (i), we get

$$(j\omega)^2 Y(j\omega) + 3(j\omega)Y(j\omega) + 2Y(j\omega) = (j\omega)X(j\omega)$$

or

$$\frac{Y(j\omega)}{X(j\omega)} = H(j\omega) = \frac{j\omega}{(j\omega)^2 + 3j\omega + 2} \quad \text{—— (ii)}$$

$$Y(j\omega) = \left[\frac{j\omega}{(j\omega+2)(j\omega+1)}\right] X(\omega) \quad \text{—— (iii)}$$

$$x(t) = u(t) \longleftrightarrow X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega} \quad \text{—— (iv)}$$

Using (iv) in (iii), we have

$$Y(\omega) = \left[\frac{j\omega}{(j\omega+2)(j\omega+1)}\right]\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$$

$$= \frac{1}{(j\omega+2)(j\omega+1)} \quad \text{—— (v)}$$

The second term is zero.

$$Y(\omega) = \frac{1}{j\omega+1} - \frac{1}{j\omega+2} \quad \text{—— (vi)}$$

By taking inverse Fourier transform on both sides of (vi), we have

$$y(t) = \left[e^{-t} - e^{-2t}\right]u(t).$$

Energy & Power Signals:-

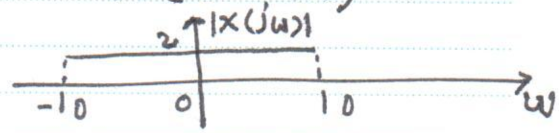
The idea of the "size" of a signal is crucial to many applications. It is nice to know, for example, how much electricity can be used in a defibrillator without ill effects or the amount of the signal driving a set of headphones. So, we use the concept of energy and power.
Defibrillation is the process of applying a controlled shock to allow restoration of the normal rythm in a serious cardic arrest.

$$E = \int_{-\infty}^{+\infty} [f(t)]^2 dt \,, \quad P_{ave} = \lim_{T \to \infty} \int_{-T/2}^{T/2} [f(t)]^2 dt$$

Using Parseval's relation, $E = \int_{-\infty}^{+\infty} [f(t)]^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(j\omega)|^2 d\omega$.

Ex:- Determine the energy of the signal $f(t) = \dfrac{2 \sin 10(t - \frac{\pi}{10})}{10(t - \pi/10)}$

$$F(j\omega) = 2\, \text{Rect}\left(\frac{\omega}{20}\right) e^{-\frac{j\omega\pi}{10}}$$



$$E = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(j\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-10}^{10} (2)^2 d\omega = \frac{1}{2\pi}(4)(20) = \frac{40}{\pi} \, J.$$

There are important signals $f(t)$, defined in general for $-\infty < t < \infty$, for which the integral $\int_{-\infty}^{+\infty} [f(t)]^2 dt$ is either unbounded (i.e, it becomes infinite) or does not converge to a finite limit. For such signals we calculate average power P of the signal.

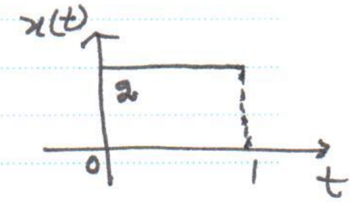
$f(t) = \cos(\omega_0 t)$, $E = \int_{-\infty}^{+\infty} \cos^2 \omega_0 t \, dt$ is unbounded.

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \cos^2 \omega_0 t \, dt = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{1 + \cos(2\omega_0 t)}{2} dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \left[\frac{t}{2} + \frac{1}{2\omega_0} \sin(2\omega_0 t) \right]_{-T/2}^{T/2} = 1/2$$

Thus while the signal has unbounded energy associated with it, its power content is $1/2$.

Signals whose associated energy is finite are sometimes called energy signals, while those whose associated energy is unbounded but whose total power is finite are known as power signals.



<u>Ex:</u> For the CTS (Continuous time signal) shown in figure, determine whether it is an energy signal or power signal?

$$E = \int_{-\infty}^{+\infty} x^2(t)\,dt = \int_0^1 4\,dt = 4, \quad P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)\,dt = \lim_{T\to\infty}\frac{1}{T}[4] = 0$$

Hence, $x(t)$ is an energy signal and not a power signal.

<u>Exercise:</u> Determine the values of power and energy, state whether each of the following signals is a power or energy signal.

(i) $x(t) = e^{-2t}u(t)$ (ii) $x(t) = \cos^2 t$ (iii) $x(t) = 2\,\mathrm{rect}(t/4)$

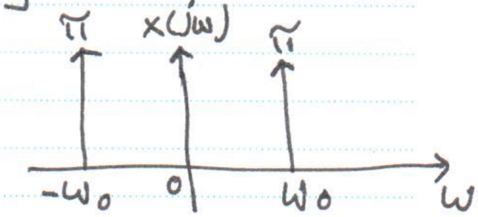
(iv) $x(t) = e^{j(2t+\pi/6)}$ (v) $x(t) = \cos(3\pi t) + 2\sin\left(\frac{2\pi}{3}\right)$

(vi) $x(t) = \begin{cases} t, & t > 0 \\ 0, & t < 0 \end{cases}$ (vii) $x(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$.

Modulation:- Fourier transform is widely used in Communication systems. Some of the fundamental idea is modulation (Amplitude, frequency etc).

Recall the FT of the complex exponential is

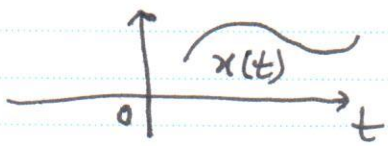
$$\mathcal{F}\left[e^{j\omega_0 t}\right] = 2\pi\,\delta(\omega-\omega_0)$$



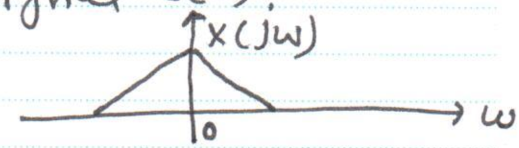
Let $x(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

$$X(j\omega) = \mathcal{F}[x(t)] = \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$

Now let us consider a generic signal $x(t)$.



of spectrum $X(j\omega)$



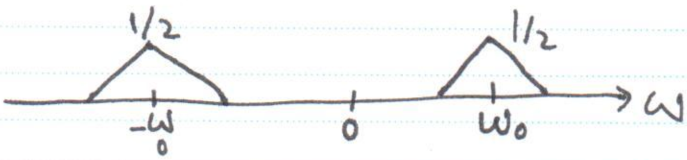
and let us multiply this by $\cos(\omega_0 t)$

$$\mathcal{F}[x(t)\cos(\omega_0 t)] = X(j\omega) * \left[\pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)\right]$$

$$= X(j\omega) * \pi\frac{1}{2\pi}\delta(\omega-\omega_0) + X(j\omega) * \pi\cdot\frac{1}{2\pi}\delta(\omega+\omega_0)$$

$$= \frac{1}{2}X(j(\omega-\omega_0)) + \frac{1}{2}X(j(\omega+\omega_0))$$

So, the effect is that of creating two replicas of the spectrum of $x(t)$ and to shift them in frequency, placing them at $\omega = \pm\omega_0$ each scaled by a factor $1/2$.



This operation is called modulation.

EX:: The message signal $x(t) = 5 + 2\cos 100t \cos 200t$ modulates a carrier signal $\cos(1000t)$ to produce the AM signal, $y(t) = x(t)\cos(1000t)$. Determine & sketch the spectrum of (a) $X(jw)$ (b) $Y(jw)$.

Solution:: $\cos 200t \cos 100t = \frac{1}{2}[\cos(200t+100t) + \cos(200t-100t)]$

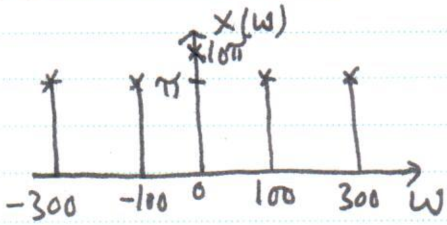
$$x(t) = 5 + \cos 300t + \cos 100t$$

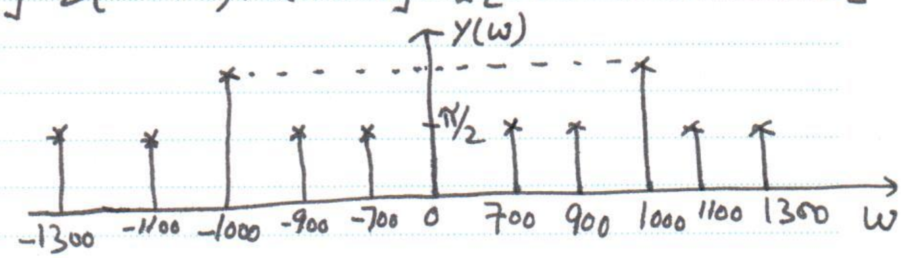
(a) $X(jw) = \mathcal{F}[x(t)] = (5)(2\pi\delta(w)) + \pi[\delta(w-100)+\delta(w+100)] + \pi[\delta(w-300)+\delta(w-300)]$

(b) $y(t) = x(t)\cos(1000t)$, $\mathcal{F}\{y(t)\} = Y(jw) = \frac{1}{2}[X(w-1000) + X(w+1000)]$

$Y(jw) = 5\pi[\delta(w-1000)+\delta(w+1000)] + \frac{\pi}{2}[\delta(w-1100)+\delta(w-900)]$

$+\frac{\pi}{2}[\delta(w+1100)+\delta(w+900)] + \frac{\pi}{2}[\delta(w-700)+\delta(w-1300)] + \frac{\pi}{2}[\delta(w+700)+\delta(w+1300)]$





EX:: Let $f(t) = 10\cos t$, $g(t) = 2\delta(t+4)$, $y = f * g$, Calculate $Y(w)$.

Sol:: $f(t) * g(t) = \int_{-\infty}^{+\infty} 10\cos(t-\tau) \cdot 2\delta(\tau+4)\,d\tau = 20\cos(t-(-4))$

$f * g = 20\cos(t+4)$, $Y(w) = \mathcal{F}[y(t)] = \mathcal{F}[20\cos(t+4)]$

$Y(w) = 20[\pi\delta(w+1) + \pi\delta(w-1)]\, e^{4jw}$ ——— (i)

OR Convolution theorem:

$F(jw) = \mathcal{F}\{f(t)\} = \mathcal{F}[10\cos t] = 10[\pi\delta(w-1) + \pi\delta(w+1)]$

$G(jw) = \mathcal{F}[g(t)] = \mathcal{F}[2\delta(t+4)] = 2e^{4jw}$

$Y(jw) = F(jw) \times G(jw) = 10[\pi\delta(w-1) + \pi\delta(w+1)] \cdot 2e^{4jw}$ same as (i).

Please bring hard copy of these notes in class 04-01-2017.

Q. Determine & sketch the magnitude & phase spectra of the signal $x(t)$ given by:

$$x(t) = \frac{2\sin(20\pi(t - 1/100))}{\pi(t - \frac{1}{100})} \cos(200\pi t).$$

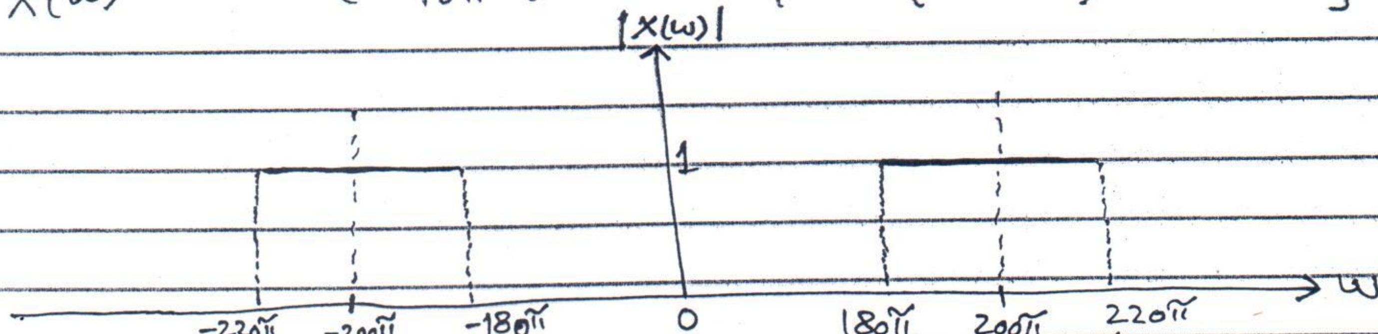
$$x(t) = x_1(t)\cos(200\pi t), \text{ where } x_1(t) = x_2(t - \tfrac{1}{100}).$$

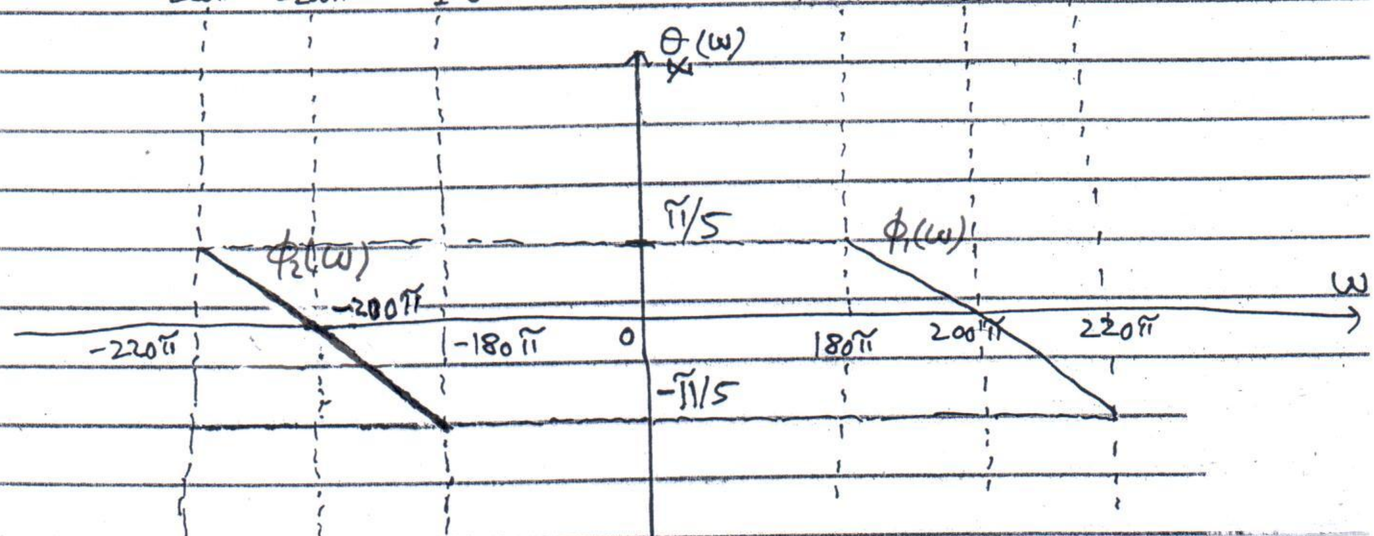
where $x_2(t) = \dfrac{2\sin(20\pi t)}{20\pi t} \longleftrightarrow X_2(\omega) = 2\,\mathrm{rect}\left(\dfrac{\omega}{40\pi}\right)$

$$\Rightarrow X_1(\omega) = X_2(\omega)\, e^{-j\omega(\frac{1}{100})} = 2\,\mathrm{rect}\left(\frac{\omega}{40\pi}\right) e^{-j(\frac{\omega}{100})}.$$

$$x(t) = x_1(t)\cos(200\pi t) \longleftrightarrow X(\omega) = \frac{1}{2}\left[X(\omega - 200\pi) + X(\omega + 200\pi)\right]$$

$$X(\omega) = \left[\mathrm{rect}\left(\frac{\omega - 200\pi}{40\pi}\right) e^{-j(\frac{\omega - 200\pi}{100})} + \mathrm{rect}\left(\frac{\omega + 200\pi}{40\pi}\right) e^{-j(\frac{\omega + 200\pi}{100})}\right].$$





$\phi_1(\omega) = \dfrac{-\omega + 200\pi}{100}$

When $\omega = 180\pi$, $\phi_1 = \dfrac{-180\pi + 200\pi}{100} = \dfrac{\pi}{5}$

When $\omega = 200\pi$, $\phi_1 = \dfrac{-200\pi + 200\pi}{100} = 0$

When $\omega = 220\pi$, $\phi_1 = \dfrac{-220\pi + 200\pi}{100} = -\dfrac{\pi}{5}$

$\phi_2(\omega) = -\dfrac{(\omega + 200\pi)}{100}$

When $\omega = -180\pi$, $\phi_2 = -\dfrac{(180\pi + 200\pi)}{100} = \dfrac{-20}{100}\pi = -\dfrac{\pi}{5}$

When $\omega = -200\pi$, $\phi_2 = -\dfrac{(-200\pi + 200\pi)}{100} = 0$

When $\omega = -220\pi$, $\phi = -(-220\pi + 200\pi) - \pi/5$

NUST School of Electrical Engineering & Computer Science
Complex Variables and transforms-

Q-1. Let $f(t) = \cos(2t-3)$, $g(t) = U(t)$, $U(t)$ be the unit-step function. Calculate $\mathcal{F}[f(t)g(t)]$ using Convolution in the frequency domain.

Q-2. For a function $f(t)$, $F(jw) = \left[\mathrm{Sinc}\left(\frac{w}{2}\right)\right]^2$, find $f(t)$. [Hint: Convolution].

Q-3. Consider the signal $f(t) = \begin{cases} 3, & 0 \le t < 2 \\ -5, & 2 \le t < 4. \end{cases}$

Express $f(t)$ using rect(t) function, compute the Fourier transform of $f(t)$, using the properties and the rect function.

Q-4. Use the integral definition of Convolution to evaluate $y(t)$, where $y = x * h$ and $x(t) = e^{-(t-2)} U(t-2)$, $h(t) = \begin{cases} 1, & -1 < t < 3 \\ 0, & \text{otherwise}. \end{cases}$

Q-5. Consider the functions $f(t) = \begin{cases} 1, & -2 \le t < 0 \\ -2, & 0 \le t < 3 \\ 0, & \text{otherwise} \end{cases}$, $g(t) = \begin{cases} 1, & -1 \le t < 0 \\ 0, & \text{otherwise}. \end{cases}$

and $h(t) = (f * g)(t)$. Use integral & graphical approach to Calculate $h(t)$. Find $H(jw)$.

Q-6. Determine whether the following signals are energy signals, power signals, or neither.

(a) $x(t) = e^{-at} U(t)$, $a > 0$ (b) $x(t) = A\cos(w_0 t + \theta)$, (c) $x(t) = t\,U(t)$.

NUST School of Electrical Engineering & Computer Science

MATH 232 - Complex Variables & transforms -

Solution Manual

Q-1 $f(t) = \cos(2t-3)$, $g(t) = U(t)$. $F(j\omega) = \pi e^{-\frac{3\omega}{2}j}\left[\delta(\omega+2)+\delta(\omega-2)\right]$

$G(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$.

$$F[f(t)g(t)] = \frac{1}{2\pi} F(j\omega) * G(j\omega) = \frac{1}{2\pi}\pi e^{-\frac{3\omega}{2}j}\left[\delta(\omega+2)+\delta(\omega-2)\right] * \left(\frac{1}{j\omega}+\pi\delta(\omega)\right)$$

$$= 0.5\, e^{-\frac{3\omega}{2}j}\delta(\omega+2) * \frac{1}{j\omega} + 0.5\pi e^{-\frac{3\omega}{2}j}\delta(\omega+2) * \delta(\omega)$$

$$+ 0.5\, e^{-\frac{3\omega}{2}j}\delta(\omega-2) * \frac{1}{j\omega} + 0.5\pi e^{-\frac{3\omega}{2}j}\delta(\omega-2) * \delta(\omega)$$

$$= 0.5\, e^{-\frac{3}{2}(-2)j}\frac{1}{j(\omega+2)} + 0.5\pi e^{-\frac{3j}{2}(-2)}\delta(\omega+2) + 0.5\, e^{-\frac{3}{2}(2)j}\frac{1}{j(\omega-2)} + 0.5\pi e^{-\frac{3j}{2}(2)}\delta(\omega-2)$$

$$= \frac{0.5\, e^{3j} j(\omega-2) + 0.5\, e^{-3j}(\omega+2)}{j(\omega+2)(\omega-2)} + 0.5\pi\delta(\omega+2)e^{3j} + 0.5\pi e^{-3j}\delta(\omega-2).$$

Thus,

$$F\left(\cos(2t-3)\,U(t)\right) = \frac{j\omega\cos(3)+2\sin(3)}{4-\omega^2} + 0.5\pi\left(e^{3j}\delta(\omega+2)+e^{-3j}\delta(\omega-2)\right).$$

Q-2 $F(j\omega) = \left[\operatorname{sinc}\left(\frac{\omega}{2}\right)\right]^2 \Rightarrow f(t) \longleftrightarrow \operatorname{sinc}\left(\frac{\omega}{2}\right)\operatorname{sinc}(\omega/2)$

we know that $\operatorname{rect}(t) \longleftrightarrow \operatorname{sinc}\left(\frac{\omega}{2}\right)$, $f(t) = \operatorname{rect}(t) * \operatorname{rect}(t)$

$$f(t) = \int_{-\infty}^{+\infty}\operatorname{rect}(t)\operatorname{rect}(t-u)\,du = \int_{t-1/2}^{t+1/2}\operatorname{rect}(u)\,du = \begin{cases} 0, & -\infty < t \le -1 \\ t+1, & -1 < t \le 0 \\ 1-t, & 0 < t \le 1 \\ 0, & 1 < t < \infty. \end{cases}$$

Q-3

$$f(t) = 3\operatorname{rect}\left(\frac{t-1}{2}\right) - 5\operatorname{rect}\left(\frac{t-3}{2}\right)$$

we know that $\operatorname{rect}(t) \longleftrightarrow \operatorname{sinc}\left(\frac{\omega}{2}\right)$. Using linearity, time shifting and scaling,

$$F(j\omega) = 3\left\{2e^{-j\omega}\operatorname{sinc}\left(\frac{2\omega}{2}\right)\right\} - 5\left\{2e^{-j3\omega}\operatorname{sinc}\left(\frac{2\omega}{2}\right)\right\}$$

$$= 2\left(3e^{-j\omega} - 5e^{-j3\omega}\right)\operatorname{sinc}(\omega), \quad -\infty < \omega < +\infty.$$

M T W T F S Date

Q-4 $y(t) = (x*h)(t) = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)\,d\tau = \int_{-\infty}^{+\infty} e^{-(t-\tau-2)} u(t-\tau-2)h(\tau)\,d\tau$

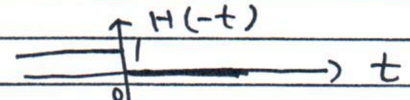
$= \int_{-\infty}^{t-2} e^{-(t-\tau-2)} h(\tau)\,d\tau$. Using the definition of h we have

if $t-2<-1$, ie, $t<1$, then $h(\tau)=0$, and so $y(t)=0$.

if $-1<t-2<3$, ie, $1<t<5$, $y(t) = \int_{-1}^{t-2} e^{-(t-\tau-2)}\,d\tau = 1-e^{1-t}$

if $t-2>3$, ie, $t>5$. $y(t) = \int_{-1}^{3} e^{-(t-\tau-2)}\,d\tau = e^{t-2}\left[e^{3}-e^{1}\right]$

So, $y(t) = \begin{cases} 0, & t<1 \\ 1-e^{1-t}, & 1<t<5 \\ e^{t-2}(e^{3}-e^{1}), & t>5 \end{cases}$



$h(-t) = \begin{cases} 0, & -t<0 \\ 1, & -t\geq 0 \end{cases} = \begin{cases} 0, & t>0 \\ 1, & t\leq 0. \end{cases}$

Q-6

(a). $E = \int_{-\infty}^{+\infty} [x(t)]^2 dt = \int_0^{\infty} e^{-2at}\,dt = \left[\frac{e^{-2at}}{-2a}\right]_0^{\infty} = \frac{-1}{2a}[0-1] = \frac{1}{2a}$ Energy signal

(b). $E = \int_{-\infty}^{+\infty} [x(t)]^2 dt$ is not finite. Power $= \lim_{T\to\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} A^2\cos^2(\omega_0 t+\theta)\,dt\right]$

$= \lim_{T\to\infty}\left[\frac{1}{T}A^2\int_{-T/2}^{T/2}\left(\frac{1+\cos 2(\omega_0 t+\theta)}{2}\right)dt\right] = \frac{A^2}{2}\lim_{T\to\infty}\left[t+\frac{\sin 2(\omega_0 t+\theta)}{2\omega_0}\Big|_{-T/2}^{T/2}\right]$

$= \frac{A^2}{2}\lim_{T\to\infty}\left[\left(\frac{T}{2}+\frac{\sin 2(\omega_0\frac{T}{2}+\theta)}{2\omega_0}\right)-\left(\frac{-T}{2}+\frac{\sin(\omega_0\cdot -T/2+\theta)}{2\omega_0}\right)\right]$

$= \frac{A^2}{2}\lim_{T\to\infty}\frac{1}{T}\left(T/2-(-T/2)\right) = A^2/2$, Power signal.

(c). $E = \int_0^{\infty} t^2 dt = \infty$, $P = \lim_{T\to\infty}\frac{1}{T}\int_0^{T/2} t^2 dt] = \lim_{T\to\infty}\frac{1}{T}\left(\frac{t^3}{3}\right)_0^{T/2} = \infty$

$x(t)$ is neither Energy nor Power signal.