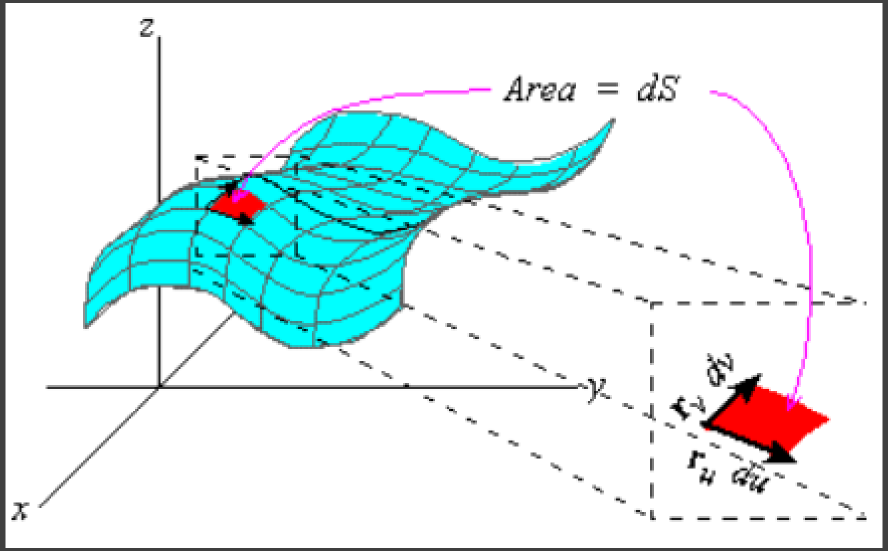
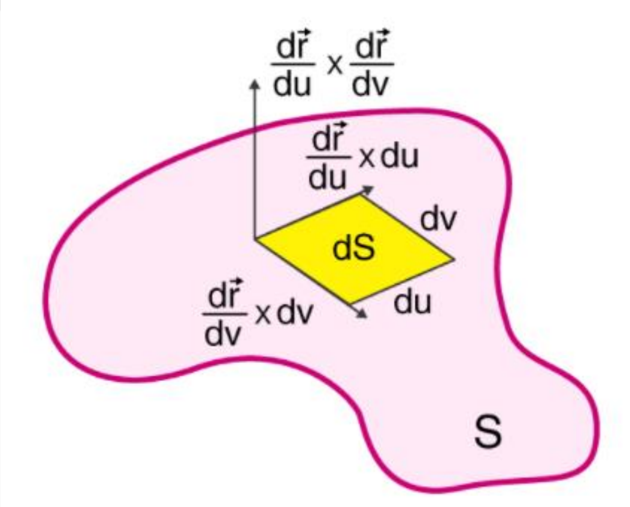


Surface Integrals

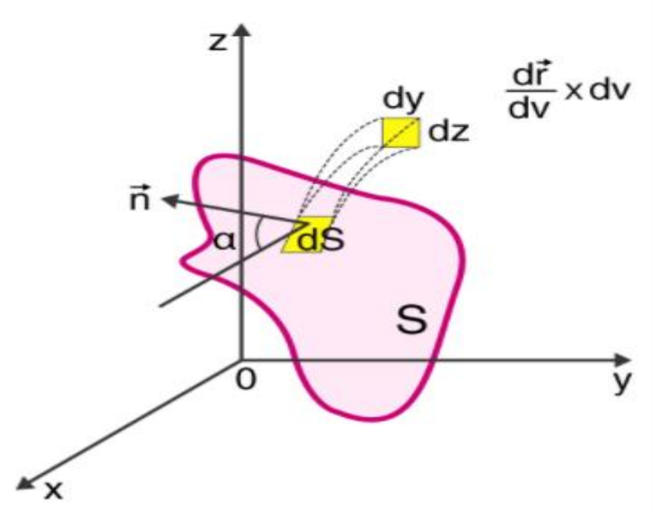
Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir



Surface Integral of Scalar Field

$$\iint_S f(x,y,z)dS = \iint_{D(u,v)} f[x(u,v),y(u,v),z(u,v)].\left|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right| dudv$$



Surface Integral of Vector Field

- If the surface "S" oriented is outward, then the surface integral of the vector field is given as:

$$\iint_S F(x,y,z).dS = \iint_S F(x,y,z).ndS = \iint_{D(u,v)} F[x(u,v),y(u,v),z(u,v))].\left[\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right] dudv.$$

- If the surface "S" oriented is inward, then the surface integral of the vector field is given as:

$$\iint_S F(x,y,z).dS = \iint_S F(x,y,z).ndS = \iint_{D(u,v)} F[x(u,v),y(u,v),z(u,v))].\left[\frac{\partial r}{\partial v}\times\frac{\partial r}{\partial u}\right] dudv$$

Where dS = ndS is known as the vector element of the surface.

16

Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.7**

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

- **Chapter: 16**
  - **Section: 16.6**

Surface Integrals for Scalar Fields

- The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length.
- Suppose f is a function of three variables whose domain includes a surface S.
- We will define the surface integral of f over S such that the value of the surface integral is equal to the surface area of S in the case where $f(x, y, z) = 1$.
- If S is a smooth surface defined parametrically as:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}; \qquad (u, v) \in D,$$

and $f(x, y, z)$ is a continuous function defined on S, then the **integral of f over S** is:

$$\iint_S f(x, y, z)dS = \iint_D f(\mathbf{r}(u, v))|\mathbf{r}_u \times \mathbf{r}_v|dA.$$

- When using this formula, remember that $f(\mathbf{r}(u, v))$ is evaluated by writing $x = x(u, v)$, $y = y(u, v), z = z(u, v)$ in the formula for $f(x, y, z)$. Moreover, observe that:

$$\iint_S 1dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|dA = A(S).$$

Graphs of a Function

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations:

$$x = x, \qquad y = y, \qquad z = g(x, y).$$

So, we have:

$$\iint_S f(x, y, z)\,dS = \iint_D f(x, y, g(x, y))\sqrt{1 + [z_x]^2 + [z_y]^2}\,dA.$$

Similar formulas apply when it is more convenient to project S onto the yz −plane or xy −plane.

Example:

Evaluate

$$\iint_S y\,dS,$$

where S is the surface $z = x + y^2, 0 \le x \le 1, 0 \le y \le 2.$

Solution:

For the present case we have: $x = x,\ y = y,\ z = x + y^2$. Thus,

$$\iint_S y\,dS = \iint_D y\sqrt{1 + [z_x]^2 + [z_y]^2}\,dA = \int_0^1 \int_0^2 y\sqrt{1 + [1]^2 + [2y]^2}\,dy\,dx = \frac{13\sqrt{2}}{3}.$$

Applications

Surface integrals have applications similar to those for the integrals we have previously considered. For example, suppose a thin sheet (say, of aluminum foil) has the shape of a surface S with the density (mass per unit area) at the point (x, y, z) as $\rho(x, y, z)$. Then, the **total mass** of the sheet is:

$$M = \iint_S \rho(x, y, z)dS\,.$$

The **center of mass** is: $(\bar{x}, \bar{y}, \bar{z})$, where,

$$\bar{x} = \frac{1}{M}\iint_S x\rho(x, y, z)dS\,, \qquad \bar{y} = \frac{1}{M}\iint_S y\rho(x, y, z)dS\,, \qquad \bar{z} = \frac{1}{M}\iint_S z\rho(x, y, z)dS\,.$$

First moments and **moments of inertia** can also be defined as before.

Example:

Find the center of mass of a thin hemispherical shell of radius a and constant density $\rho = c$.

Solution:

We model the shell with the hemisphere: $g(x, y, z) = x^2 + y^2 + z^2 = a^2,\ z \geq 0$.

The symmetry of the surface about the z −axis tells us that $\bar{x} = \bar{y} = 0$. It remains only to find $\bar{z}$ from the formula $\bar{z} = \frac{M_{xy}}{M}$. The mass of the shell is:

$$M = \iint_S \rho(x, y, z) dS = c \iint_S dS = cA(S) = 2\pi a^2 c.$$

For the present case:

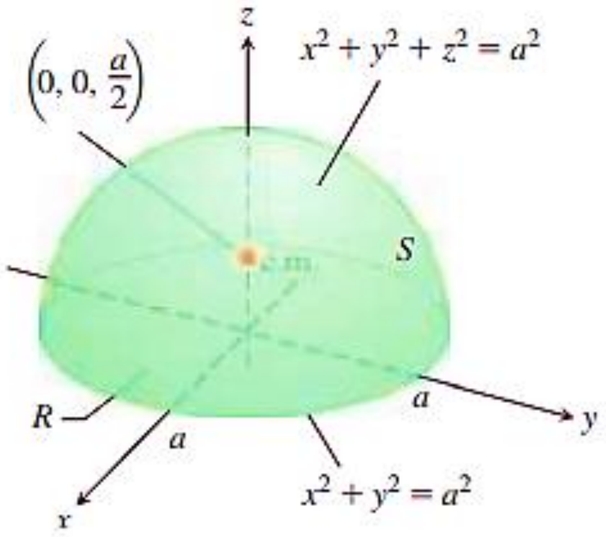
$$\mathbf{r}(\varphi, \theta) = \langle a \sin\varphi \cos\theta\, , a \sin\varphi \sin\theta\, , a \cos\varphi \rangle,$$

where the parameter domain is:

$$D \ = \ \{(\varphi, \theta) | 0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi\ \}.$$

Moreover,

$$\mathbf{r}_\varphi \times \mathbf{r}_\theta = \langle a^2 \sin^2\varphi \cos\theta\, , a^2 \sin^2\varphi \sin\theta\, , a^2 \sin\varphi \cos\varphi \rangle.$$



The center of mass of a thin hemispherical shell of constant density lies on the axis of symmetry halfway from the base to the top.

Solution:

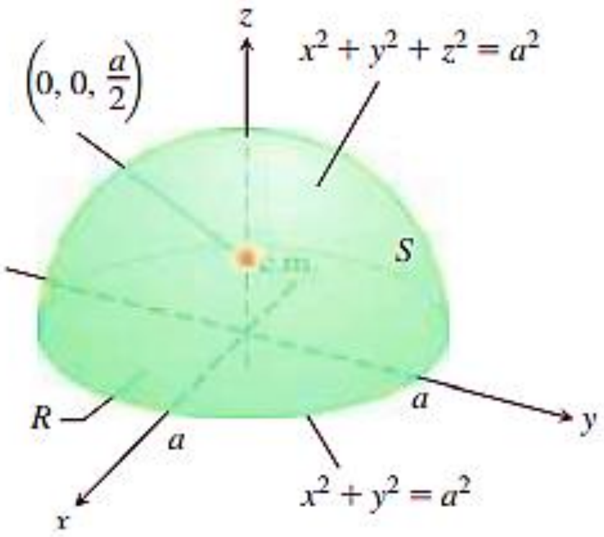
and

$$\left|\mathbf{r}_\varphi \times \mathbf{r}_\theta\right| = a^2 \sin\varphi\,.$$

Thus,

$$M_{xy} = \iint_S z\rho(x,y,z)dS = c\iint_S z\,dS = c\int_0^{2\pi}\int_0^{\pi/2} a^3\cos\varphi\sin\varphi\,d\varphi d\theta$$

$$= \frac{ca^3}{2}\int_0^{2\pi}\int_0^{\pi/2}\sin(2\varphi)\,d\varphi d\theta = ca^3\pi,$$



and

$$\bar{z} = \frac{M_{xy}}{M} = \frac{ca^3\pi}{2\pi a^2 c} = \frac{a}{2}.$$

Thus, the center of mass of a thin hemispherical shell of radius a and constant density ρ is given as:

$$(\bar{x},\bar{y},\bar{z}) = \left(0,0,\frac{a}{2}\right).$$

Piecewise Smooth Surfaces

If S is a piecewise smooth surface—a finite union of smooth surfaces $S_1, S_2, S_3, \dots, S_n$ that intersect only along their boundaries—then the surface integral of f over S is defined by:

$$\iint_S f(x,y,z)dS = \iint_{S_1} f(x,y,z)dS + \iint_{S_2} f(x,y,z)dS + \cdots + \iint_{S_n} f(x,y,z)dS.$$

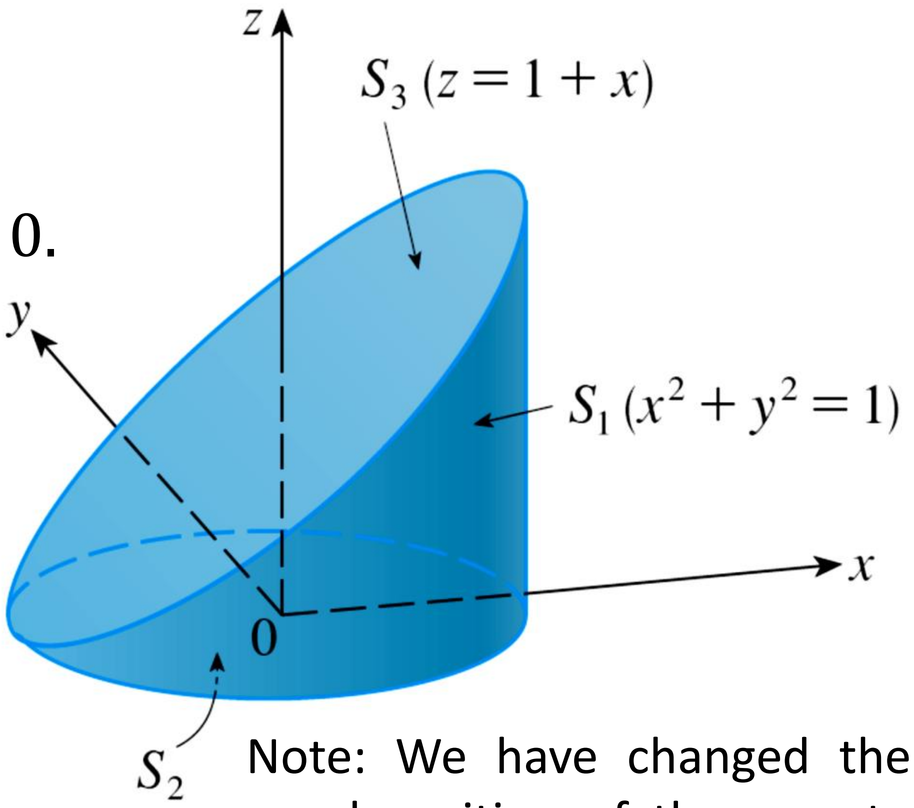
Example:

Evaluate

$$\iint_S z\,dS,$$

where S is the surface whose:

- Sides S_1 are given by the cylinder $x^2 + y^2 = 1$.
- Bottom S_2 is the disk $x^2 + y^2 \leq 1$ in the plane $z = 0$.
- Top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.



Note: We have changed the usual position of the axes to get a better look at S.

Solution:

For S_1, we use θ and z as parameters and write its parametric equations as:

$$x = \cos\theta, \qquad y = \sin\theta, \qquad z = z,$$

where: $0 \le \theta \le 2\pi,\ 0 \le z \le 1 + x = 1 + \cos\theta.$

Therefore,

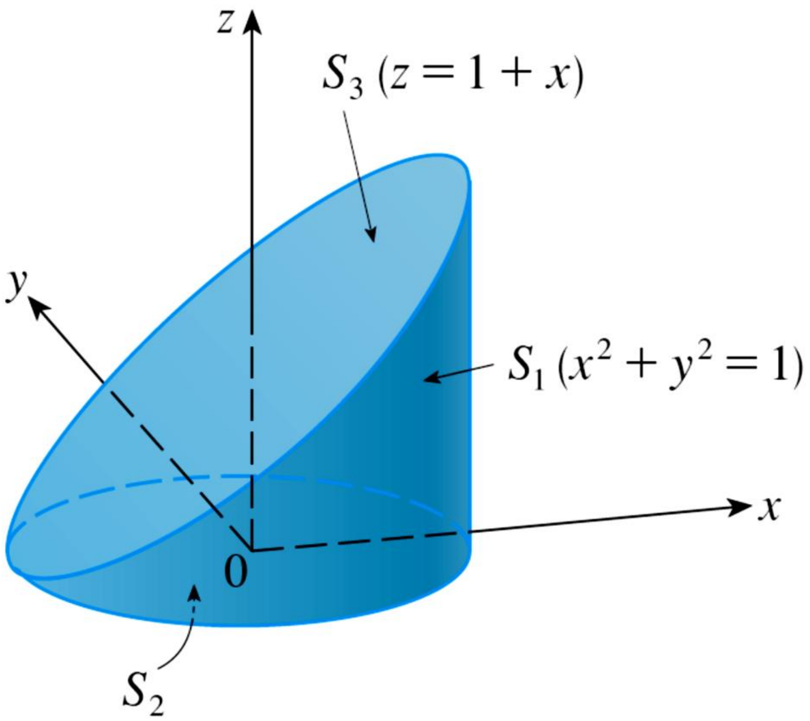
$$\iint_{S_1} z\,dS = \iint_D z\,|\mathbf{r}_\theta \times \mathbf{r}_z|\,dA$$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta, \quad \text{where: } |\mathbf{r}_\theta \times \mathbf{r}_z| = 1$$

$$= \int_0^{2\pi} \tfrac{1}{2}(1+\cos\theta)^2\,d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi}\left[1 + 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$= \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3\pi}{2}$$



Solution:

Since S_2 lies in the plane $z = 0$, we have:

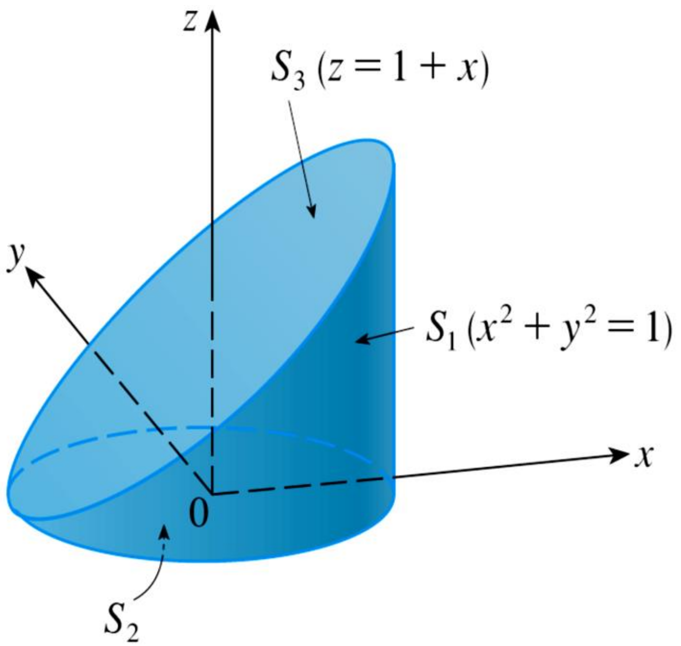
$$\iint_{S_2} z dS = \iint_{S_2} 0 dS = 0.$$

S_3 lies above the unit disk D and is part of the plane $z = 1 + x$. So, taking $z = 1 + x$ and converting to polar coordinates, we have the following result.

$$\iint_{S_3} z\, dS = \iint_{D} (1+x)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \int_0^{2\pi}\int_0^1 (1+r\cos\theta)\sqrt{1+1+0}\, r\, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 (r+r^2\cos\theta)\, dr\, d\theta = \sqrt{2}\,\pi$$



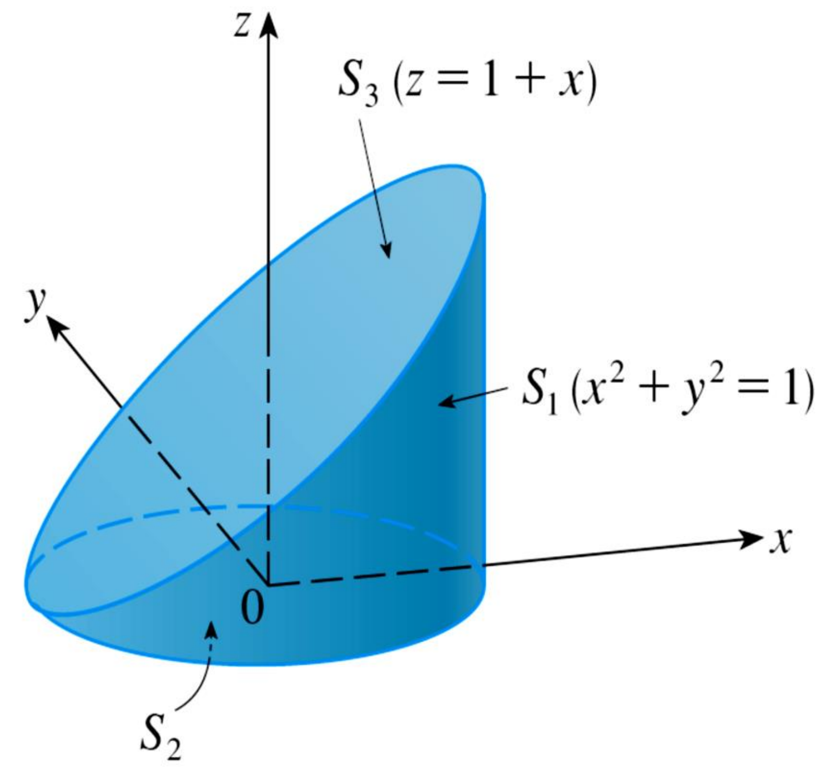
Example:

Therefore,

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS$$

$$= \frac{3\pi}{2} + 0 + \sqrt{2}\,\pi$$

$$= \left(\tfrac{3}{2} + \sqrt{2}\right)\pi$$



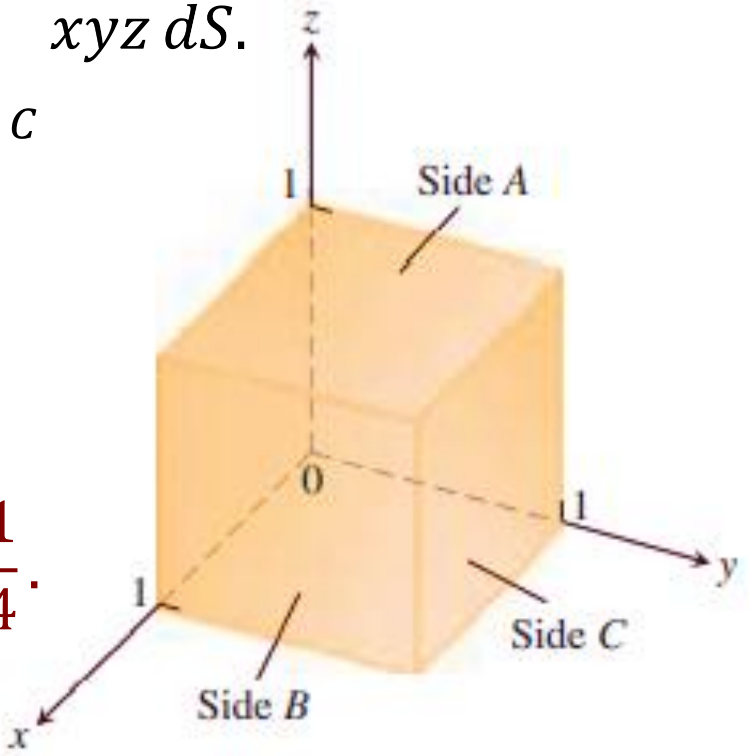
Example:

Integrate $f(x,y,z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1, y = 1$, and $z = 1$.

Solution:

We integrate xyz over each of the sides and add the results. Since $xyz = 0$ on the sides that lie in the coordinate planes, the integral over the surface of the cube reduces to:

$$\iint_S xyz\,dS = \iint_{\text{Side } A} xyz\,dS + \iint_{\text{Side } B} xyz\,dS + \iint_{\text{Side } C} xyz\,dS.$$



Side A is the surface $z = 1$ over the square region $R_{xy}: 0 \leq x \leq 1$, $0 \leq y \leq 1$, in the xy −plane. For this surface and region,

$$\iint_{\text{Side } A} xyz\,dS = \iint_{R_{xy}} xy\sqrt{1 + [0]^2 + [0]^2}\,dA = \int_0^1 \int_0^1 xy\,\sqrt{1}\,dx\,dy = \frac{1}{4}.$$

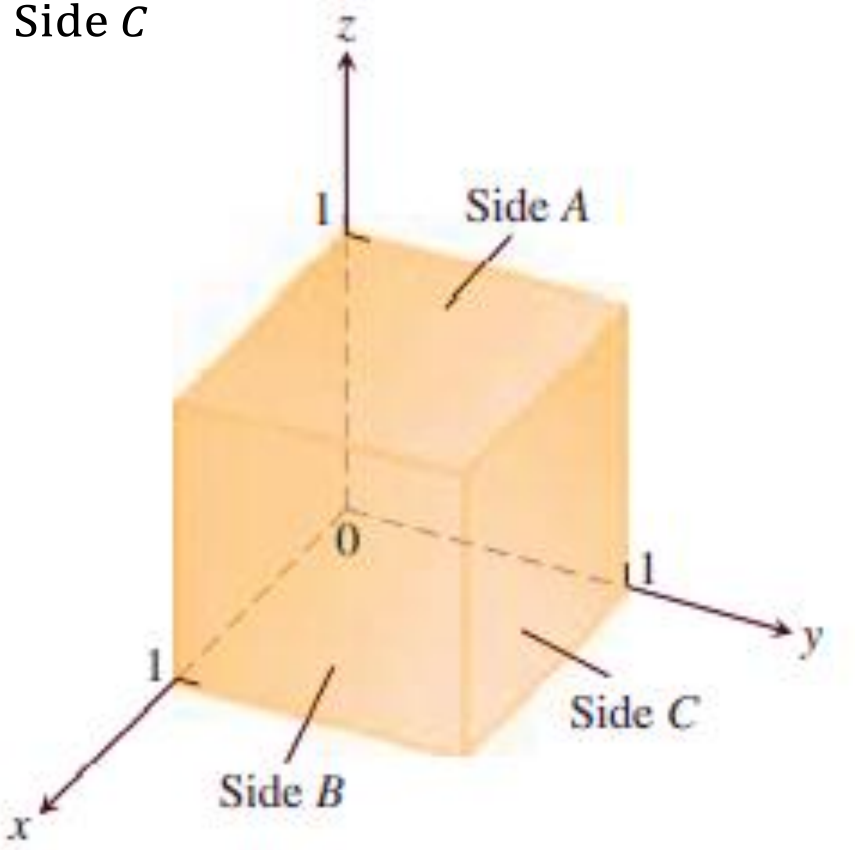
Solution:

Using symmetry, the integrals of xyz over sides B and C are also $\frac{1}{4}$. Hence,

$$\iint_S xyz\,dS = \iint_{\text{Side } A} xyz\,dS + \iint_{\text{Side } B} xyz\,dS + \iint_{\text{Side } C} xyz\,dS$$

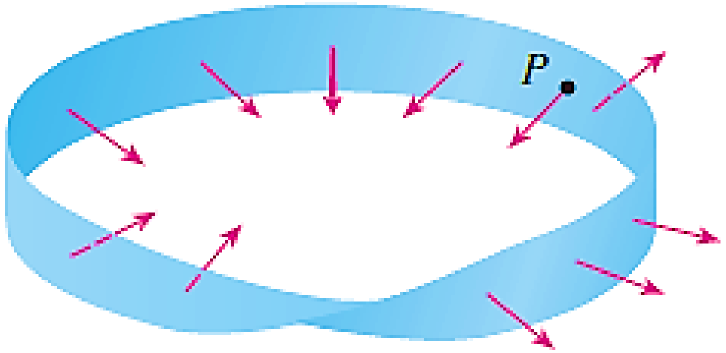
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

$$= \frac{3}{4}.$$

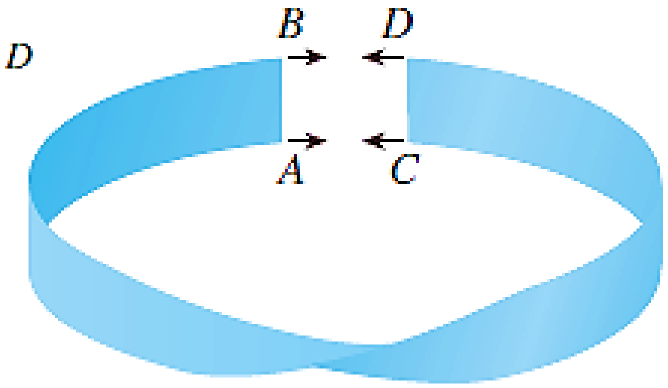


Oriented Surface & Orientation

To define surface integrals of **vector fields**, we need to rule out non-orientable surfaces such as the Möbius strip shown in the figure below. We can construct one for ourself by taking a long rectangular strip of paper, giving it a half-twist, and taping the short edges together. If an ant were to crawl along the Möbius strip starting at a point , it would end up on the “other side” of the strip (that is, with its upper side pointing in the opposite direction). Then, if the ant continued to crawl in the same direction, it would end up back at the same point without ever having crossed an edge. Therefore, a Möbius strip really has only one side. From now on we consider only orientable (two-sided) surfaces.



A Möbius strip



Constructing a Möbius strip

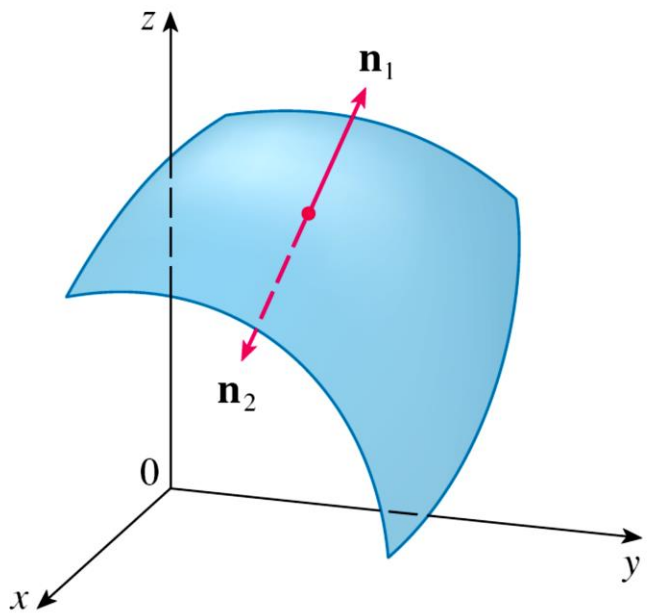
Oriented Surface & Orientation

We start with a surface S that has a tangent plane at every point (x, y, z) on S (except at any boundary point).

There are two unit normal vectors: $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at (x, y, z).

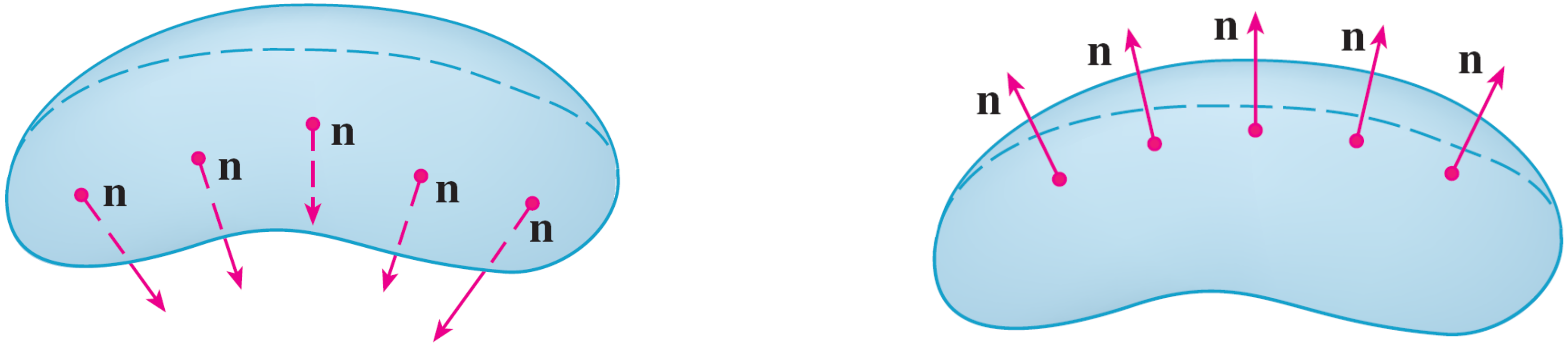
If it is possible to choose a unit normal vector $\mathbf{n}$ at every such point (x, y, z) so that $\mathbf{n}$ varies continuously over S, then S is called an **oriented surface**.

The given choice of $\mathbf{n}$ provides S with an orientation.



Oriented Surface & Orientation

There are two possible orientations for any orientable (two-sided) surface.



If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector:

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

This gives the **upward orientation** of the surface. The opposite orientation is given by $-\mathbf{n}$.

Oriented Surface & Orientation

For a surface $z = g(x, y)$ given as the graph of g, we use:

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1 \rangle,$$

to associate with the surface a natural orientation given by the unit normal vector:

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{\left|\mathbf{r}_x \times \mathbf{r}_y\right|} = \frac{\langle -g_x, -g_y, 1 \rangle}{\sqrt{1 + [g_x]^2 + [g_y]^2}}.$$

As the $\mathbf{k}$ −component is positive, this gives the **upward orientation** of the surface.

Oriented Surface & Orientation

For instance, a parametric representation of the sphere of radius a:

$$x^2 + y^2 + z^2 = a^2,$$

is given by a vector function:

$$\mathbf{r}(\varphi, \theta) = \langle a \sin\varphi \cos\theta\,, a \sin\varphi \sin\theta\,, a \cos\varphi \rangle,$$

where the parameter domain is:

$$D \;=\; \{(\varphi, \theta) | 0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi\,\}.$$

Moreover,

$$\mathbf{r}_\varphi \times \mathbf{r}_\theta = \langle a^2 \sin^2\varphi \cos\theta\,, a^2 \sin^2\varphi \sin\theta\,, a^2 \sin\varphi \cos\varphi \rangle.$$

and

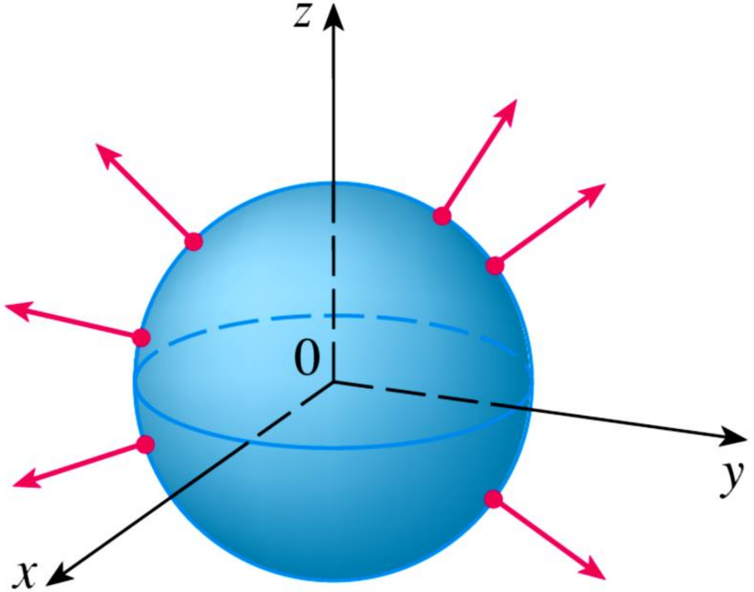
$$\left|\mathbf{r}_\varphi \times \mathbf{r}_\theta\right| = a^2 \sin\varphi\,.$$

Thus, the orientation induced by $\mathbf{r}(\varphi, \theta)$ is defined by the unit normal vector:

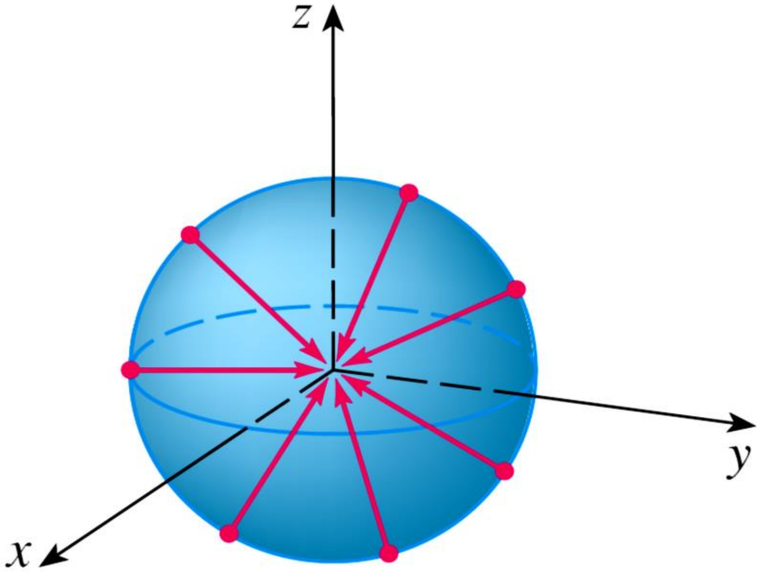
$$\mathbf{n} = \frac{\mathbf{r}_\varphi \times \mathbf{r}_\theta}{\left|\mathbf{r}_\varphi \times \mathbf{r}_\theta\right|} = \frac{\langle a^2 \sin^2\varphi \cos\theta\,, a^2 \sin^2\varphi \sin\theta\,, a^2 \sin\varphi \cos\varphi \rangle}{a^2 \sin\varphi} = \frac{1}{a}\mathbf{r}(\varphi, \theta).$$

Oriented Surface & Orientation

Observe that $\mathbf{n}$ points in the same direction as the position vector, that is, outward from the sphere (figure a). The opposite (inward) orientation would have been obtained (figure 9) if we had reversed the order of the parameters because $\mathbf{r}_\theta \times \mathbf{r}_\varphi = -(\mathbf{r}_\varphi \times \mathbf{r}_\theta)$.



(Figure: a)



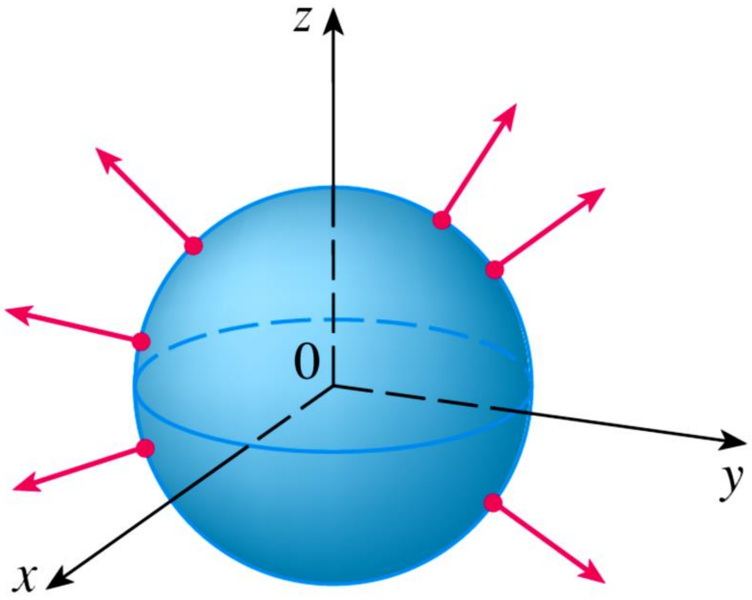
(Figure: b)

Oriented Surface & Orientation

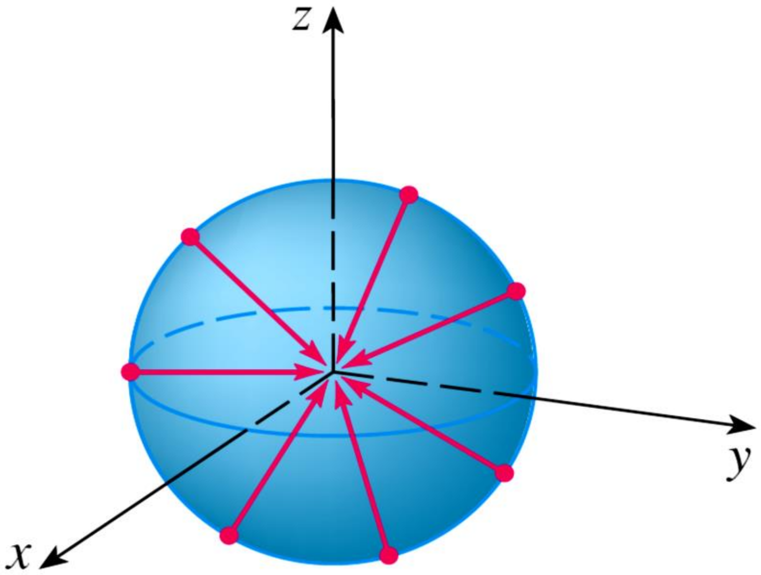
For a closed surface—a surface that is the boundary of a solid region E—the convention is that:

- The **positive orientation** is the one for which the normal vectors point outward from E. (Figure: a)

- Inward-pointing normal vectors give the **negative orientation**. (Figure: b)



(Figure: a)



(Figure: b)

Surface Integrals of Vector Fields

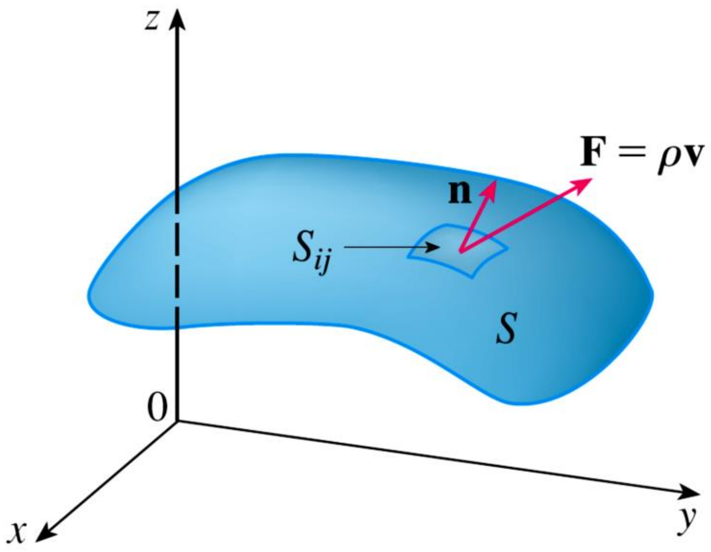
Suppose that S is an oriented surface with unit normal vector $\mathbf{n}$. Then, imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing through S. Think of S as an imaginary surface that doesn't impede the fluid flow—like a fishing net across a stream. Then, the rate of flow (mass per unit time) per unit area is $\rho\mathbf{v}$. If we divide S into small patches S_{ij}, then S_{ij} is nearly planar. So, we can approximate the mass of fluid crossing S_{ij} in the direction of the normal $\mathbf{n}$ per unit time by the quantity:

$$(\rho\mathbf{v} \cdot \mathbf{n})A(S_{ij}),$$

where $\rho, \mathbf{v}$, and $\mathbf{n}$ are evaluated at some point on S_{ij} (recall that the component of the vector $\rho\mathbf{v}$ in the direction of the unit vector $\mathbf{n}$ is $\rho\mathbf{v} \cdot \mathbf{n}$).

Summing these quantities and taking the limit, we get the surface integral of the function $\rho\mathbf{v} \cdot \mathbf{n}$ over S:

$$\iint_S \rho(x, y, z)\mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z)\, dS. \qquad (*)$$



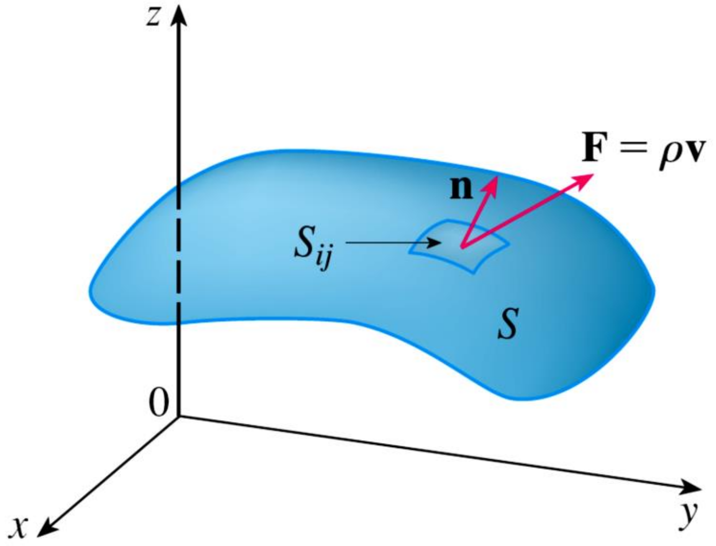
This is interpreted physically as the rate of flow through S.

Surface Integrals of Vector Fields

If we write $\mathbf{F} = \rho\mathbf{v}$, then $\mathbf{F}$ is also a vector field on $\mathbb{R}^3$. Then, the integral $(*)$ takes the form:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \mathbf{F}\,d\boldsymbol{S} = \iint_D \left[\mathbf{F}\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}\right]|\mathbf{r}_u\times\mathbf{r}_v|\,dA = \iint_D [\mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)]\,dA,$$

where D is the parameter domain. It is called the **surface integral** (or **flux integral**) of $\mathbf{F}$ over S, where $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$.



Example: Flux Integral

Find the flux of the vector field: $\mathbf{F}(x,y,z) = \langle z, y, x\rangle$ across the sphere: $x^2 + y^2 + z^2 = 1$.

Solution:

Using the parametric representation of sphere, we get:

$$\mathbf{r}(\varphi,\theta) = \langle \sin\varphi\cos\theta\,, \sin\varphi\sin\theta\,, \cos\varphi\rangle,$$

where, $0 \le \varphi \le \pi, 0 \le \theta \le 2\pi$, we have:

$$\mathbf{F}\big(\mathbf{r}(\varphi,\theta)\big) = \langle \cos\varphi\,, \sin\varphi\sin\theta\,, \sin\varphi\cos\theta\rangle.$$

Now:

$$\mathbf{r}_\varphi \times \mathbf{r}_\theta = \langle \sin^2\varphi\cos\theta\,, \sin^2\varphi\sin\theta\,, \sin\varphi\cos\varphi\rangle.$$

Therefore,

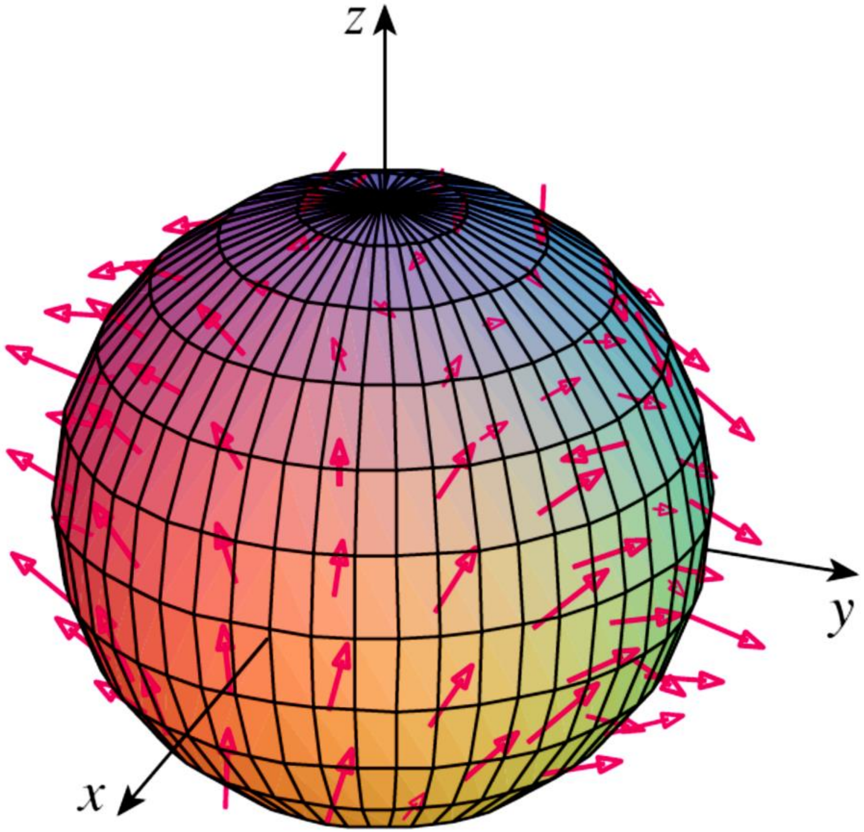
$$\mathbf{F}\big(\mathbf{r}(\varphi,\theta)\big)\cdot\big(\mathbf{r}_\varphi \times \mathbf{r}_\theta\big) = 2\cos\varphi\sin^2\varphi\cos\theta + \sin^3\varphi\sin^2\theta\,.$$

Thus, the flux of the vector field is given as:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D [\mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)]\,dA = \int_0^{2\pi}\int_0^{\pi}[2\cos\varphi\sin^2\varphi\cos\theta + \sin^3\varphi\sin^2\theta]d\varphi d\theta = \frac{4\pi}{3}.$$

Example: Flux Integral

- The figure shows the vector field $\mathbf{F}$ at points on the unit sphere.
- If, for instance, the vector field $\mathbf{F}$ is a velocity field describing the flow of a fluid with density 1, then the answer, $4\pi/3$, represents: the rate of flow through the unit sphere in units of mass per unit time.



Surface Integrals of Vector Fields

In the case of a surface S given by a graph of the function $z = g(x, y)$, we can think of x and y as parameters and write:

$$\mathbf{F} \cdot \mathbf{n} = \frac{\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)}{|\mathbf{r}_x \times \mathbf{r}_y|} = \frac{\langle P, Q, R \rangle \cdot \langle -g_x, -g_y, 1 \rangle}{\sqrt{1 + [g_x]^2 + [g_y]^2}}$$

So that:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_D [-P g_x - Q g_y + R] \, dA.$$

Note:

- This formula assumes the upward orientation of S.
- For a downward orientation, we multiply by -1.
- Similar formulas can be worked out if S is given by $y = h(x, z)$ or $x = k(y, z)$.

Example:

Evaluate:

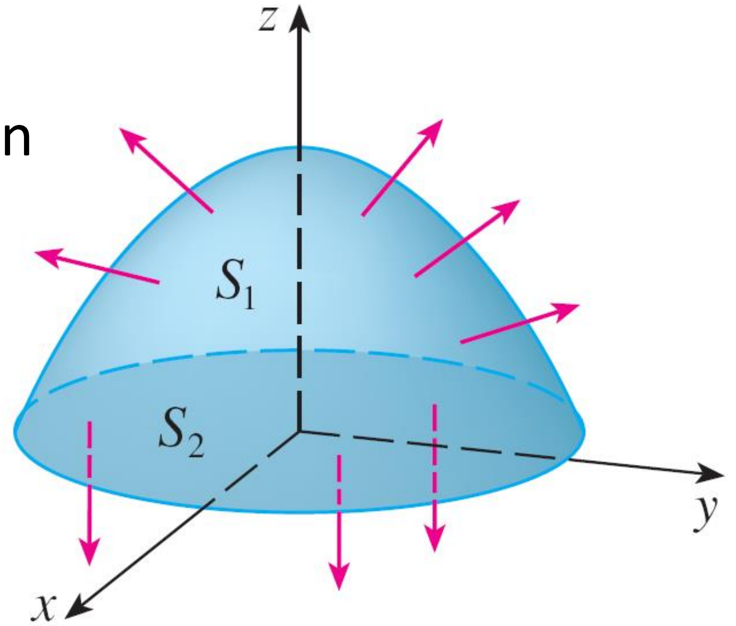
$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region *E* enclosed by the paraboloid $z = g(x, y) = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution:

S consists of a parabolic top surface S_1 and a circular bottom surface S_2. Since S is a closed surface, we use the convention of positive (outward) orientation.

This means that S_1 is oriented upward. Moreover, *D* is the projection of S_1 on the xy −plane, namely, the disk: $x^2 + y^2 \leq 1$.



Solution:

On S_1, $P(x, y, z) = y,\ Q(x, y, z) = x,\ R(x, y, z) = z = 1 - x^2 - y^2$.

Also, $z = g(x, y) = 1 - x^2 - y^2$, so:

$$\frac{\partial g}{\partial x} = -2x, \qquad \frac{\partial g}{\partial y} = -2y.$$

So, we have:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \left[-P g_x - Q g_y + R\right] dA = \iint_D [2xy + 2xy + 1 - x^2 - y^2]\, dA$$

$$= \iint_D [4xy + 1 - x^2 - y^2]\, dA = \int_0^{2\pi} \int_0^1 [1 - r^2 + 4r^2 \cos\theta \sin\theta]\, r dr d\theta = \frac{\pi}{2}.$$

Solution:

The disk S_2 is oriented downward. So, its unit normal vector is $\mathbf{n} = -\mathbf{k}.$ Moreover, $z = 0$ on S_2, thus we have:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \mathbf{F} \cdot (-\mathbf{k})\, dS = \iint_D [-z]\, dA = \iint_D [0]\, dA = 0.$$

Finally, we compute $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ as the sum of the surface integrals of $\mathbf{F}$ over the pieces S_1 and S_2, as:

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS \\ &= \frac{\pi}{2} + 0, \\ &= \frac{\pi}{2}.\end{aligned}$$

Applications

- Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations.
- For instance, if $\mathbf{E}$ is an electric field, the surface integral

$$\iint_S \mathbf{E} \cdot \mathbf{n}\, dS$$

is called the **electric flux** of **E** through the surface *S*.

- One of the important laws of electrostatics is **Gauss's Law**, which says that the net charge enclosed by a closed surface S is:

$$Q = \mathcal{E}_0 \iint_S \mathbf{E} \cdot \mathbf{n}\, dS$$

where $\mathcal{E}_0 \approx\ 8.8542 \times 10^{-12}\ \mathrm{C}^2/\mathrm{N} \cdot \mathrm{m}^2$ is a constant (called the permittivity of free space) that depends on the units used.

Applications

Another application occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then, the heat flow is defined as the vector field:

$$\mathbf{F} = -K\,\nabla u,$$

where K is an experimentally determined constant called the conductivity of the substance. Then, the rate of heat flow across the surface S in the body is given by the surface integral:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = -K\iint_S \nabla u\cdot\mathbf{n}\,dS.$$

Example:

The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

Solution:

Taking the center of the ball to be at the origin, we have:

$$u(x, y, z) = C(x^2 + y^2 + z^2)$$

where C is the proportionality constant.

Then, the heat flow is:

$$\mathbf{F}(x, y, z) = -K\,\nabla u = -KC(2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k})$$

where K is the conductivity of the metal.

Solution:

Instead of using the usual parametrization of the sphere, we observe that the outward unit normal to the sphere $x^2 + y^2 + z^2 = a^2$ at the point (x, y, z) is:

$$\mathbf{n} = \frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

Thus,

$$\mathbf{F} \cdot \mathbf{n} = -\frac{2KC}{a}(x^2 + y^2 + z^2).$$

However, on S, we have: $x^2 + y^2 + z^2 = a^2$ and $\mathbf{F} \cdot \mathbf{n} = -2aKC.$ Thus, the rate of heat flow across S is:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -2aKC \iint_S dS = -2aKCA(S) = -2aKC(4\pi a^2) = -8KC\pi a^3.$$

Practice Questions

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

Chapter: 16

Exercise-16.7: Q – 5 to 30, Q – 33 to 47.

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 16

Exercise-16.6: Q – 1 to 46.