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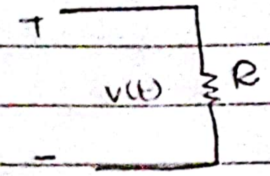
Q:- Let $v(t) = 3 - 4\cos(3t + \pi/3) - 5\sin(4t - \pi/3)$ be the input signal to the circuit shown below:

(a) Determine the average power delivered to $1\,\Omega$ resistor using Parseval's theorem.



(b) Plot 1-sided and 2-sided discrete frequency spectra.

Sol (a): Method 1 Using CFS.

$$v(t) = 3 - 4\cos(3t + \pi/3) - 5\sin(4t - \pi/3)$$

$$= 3 - 4\left[\frac{e^{i(3t+\pi/3)} + e^{-i(3t+\pi/3)}}{2}\right] - 5\left[\frac{e^{i(4t-\pi/3)} - e^{-i(4t-\pi/3)}}{2i}\right]$$

$$= 3 - 2\left[e^{i3t}e^{i\pi/3} + e^{-i3t}\cdot e^{-i\pi/3}\right] - \frac{5}{2i}\left[e^{i4t}\cdot e^{-i\pi/3} - e^{-i4t}\cdot e^{+i\pi/3}\right]$$

$$= 3 - 2\left[e^{i3t}(\cos\pi/3 + i\sin\pi/3) + e^{-i3t}(\cos\pi/3 - i\sin\pi/3)\right]$$
$$+ \frac{5i}{2}\left[e^{i4t}(\cos\pi/3 - i\sin\pi/3) - e^{-i4t}(\cos\pi/3 + i\sin\pi/3)\right]$$

$$= 3 - 2\left[\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)e^{i3t} + \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)e^{-i3t}\right]$$
$$+ \frac{5i}{2}\left[\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)e^{i4t} - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)e^{-i4t}\right]$$

$$= 3 + \left[(-1 - i\sqrt{3})e^{i3t} + (-1 + i\sqrt{3})e^{-i3t}\right] + \left[\left(\frac{5\sqrt{3}}{4} + \frac{5}{4}i\right)e^{i4t} + \left(\frac{5\sqrt{3}}{4} - \frac{5}{4}i\right)e^{-i4t}\right] \rightarrow (1)$$

We know that general CFS of any periodic function $v(t)$ is given by:

$$V(t) = c_0 + \sum_{k=1}^{\infty}\left[c_k e^{ik\omega x} + c_{-k}e^{-ik\omega x}\right] = c_0 + \left[c_1 e^{i\omega x} + c_{-1}e^{-i\omega x}\right]$$
$$+ \left[c_2 e^{2i\omega x} + c_{-2}e^{-2i\omega x}\right] + \left[c_3 e^{3i\omega x} + c_{-3}e^{-3i\omega x}\right] + \dots \rightarrow (2)$$

Comparing (1) and (2) we get

$c_0 = 3$

$c_1 = c_{-1} = 0$; $c_2 = c_{-2} = 0$

$c_3 = -1 - i\sqrt{3}$; $c_{-3} = \overline{c_3} = -1 + i\sqrt{3}$

$c_4 = \frac{5\sqrt{3}}{4} + i\frac{5}{4}$; $c_{-4} = \overline{c_4} = \frac{5\sqrt{3}}{4} - i\frac{5}{4}$

$c_k = c_{-k} = 0 \quad \forall \quad k \geq 5.$

Parseval's theorem states that "If $f(t)$ is a periodic function with period T then

$$P_{avg} = \frac{1}{T}\int_{x_0}^{x_0+T} [f(t)]^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \longrightarrow (3)$$

where c_k are the complex Fourier coefficients.

Using (3) we have

$$P_{avg} = \sum_{k=-\infty}^{\infty} |c_k|^2$$

$$= |c_0|^2 + |c_3|^2 + |c_{-3}|^2 + |c_4|^2 + |c_{-4}|^2 \rightarrow (4)$$

Now

$|c_0| = |3| = 3$

$|c_3| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2 = |c_{-3}|$

$|c_4| = \sqrt{\left(\frac{5\sqrt{3}}{4}\right)^2 + \left(\frac{5}{4}\right)^2} = \sqrt{\frac{25}{16}(3+1)} = \sqrt{\frac{25}{16}(4)} = \sqrt{\frac{25}{4}} = \frac{5}{2} = |c_{-4}|$

Using above information in (4) we get

$$P_{avg} = (3)^2 + (2)^2 + (2)^2 + \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2$$

$$= 9 + 4 + 4 + \frac{25}{4} + \frac{25}{4}$$

$$= \frac{59}{2} = 29.5 \text{ Watts}$$

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Method 2 Using TFS

$$V(t) = 3 - 4\cos\left(3t + \frac{\pi}{3}\right) - 5\sin\left(4t - \frac{\pi}{3}\right)$$

$$= 3 - 4\left[\cos 3t \cos\frac{\pi}{3} - \sin 3t \sin\frac{\pi}{3}\right] - 5\left[\sin 4t \cos\frac{\pi}{3} - \cos 4t \sin\frac{\pi}{3}\right]$$

$$= 3 - 4\left[\frac{1}{2}\cos 3t - \frac{\sqrt{3}}{2}\sin 3t\right] - 5\left[\frac{1}{2}\sin 4t - \frac{\sqrt{3}}{2}\cos 4t\right]$$

$$= 3 + \left[-2\cos 3t + 2\sqrt{3}\sin 3t\right] + \left[-\frac{5}{2}\sin 4t + \frac{5}{2}\sqrt{3}\cos 4t\right] \rightarrow (1)$$

We know that TFS of a periodic function $v(t)$ is given by:

$$V(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \rightarrow (2)$$

From (1) and (2)

$$\frac{a_0}{2} = 3 \Rightarrow a_0 = 6$$

$$a_1 = a_2 = b_1 = b_2 = 0$$

$$a_3 = -2 \; ; \; b_3 = 2\sqrt{3} \; ; \; b_4 = -\frac{5}{2} \; ; \; a_4 = \frac{5}{2}\sqrt{3}$$

$$a_k = b_k = 0 \quad \forall \; k \geq 5.$$

Using parseval relation for real Fourier coefficient we have

$$P_{avg} = \frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right)$$

$$= \frac{\overset{a_0}{(6)^2}}{4} + \frac{1}{2}\left[\left\{\overset{a_3}{(-2)^2} + \overset{b_3}{(2\sqrt{3})^2}\right\} + \left\{\overset{b_4}{\left(-\frac{5}{2}\right)^2} + \overset{a_4}{\left(\frac{5}{2}\sqrt{3}\right)^2}\right\}\right]$$

$$= \frac{36}{4} + \frac{1}{2}\left[4 + 12 + \frac{25}{4} + \frac{75}{4}\right] = \frac{36}{4} + \frac{41}{2} = \frac{59}{2} \text{ Watts}$$

Method 3:- Using compact form (using cosine wave)

Note
$\varphi_K \in (-\pi, \pi]$
$-\pi < \varphi_2 \leq \pi$

$$V(t) = 3 - 4\cos\left(3t + \frac{\pi}{3}\right) - 5\sin\left(4t - \frac{\pi}{3}\right)$$

$$= 3 + 4\cos\left(3t + \frac{\pi}{3} - \pi\right) - 5\cos\left(4t - \frac{\pi}{3} - \frac{\pi}{2}\right)$$

$\left[\because -\cos\theta = \cos(\theta - \pi) \quad \sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)\right]$

$$= 3 + 4\cos\left(3t - \frac{2\pi}{3}\right) + 5\cos\left(4t - \frac{5\pi}{6} + \pi\right) \quad [\text{using } -\cos\theta = \cos(\theta + \pi)]$$

$$= 3 + 4\cos\left(3t - \frac{2\pi}{3}\right) + 5\cos\left(4t + \frac{\pi}{6}\right) \longrightarrow (1)$$

Note:- Alternatively we can use $-\sin\theta = \cos\left(\theta + \frac{\pi}{2}\right)$

so that $-5\sin\left(4t - \frac{\pi}{3}\right) = \cos\left(4t - \frac{\pi}{3} + \frac{\pi}{3}\right)$

$= \cos\left(4t + \frac{\pi}{6}\right)$.

we know that amplitude phase form of FS of Fourier series of $v(t)$ is given by

$$V(t) = A_0 + \sum_{K \geq 1}^{\infty} A_K \cos(K\omega t + \varphi_K) \longrightarrow (2)$$

From (1) & (2)

$$A_0 = 3 \; ; \; A_1 = A_2 = 0 \; ; \; A_3 = 4 \; ; \; A_4 = 5, \; A_K = 0 \; \forall K \geq 5$$

Also $A_0 = \frac{a_0}{2}$ and $A_K = \sqrt{a_K^2 + b_K^2} \Rightarrow A_K^2 = a_K^2 + b_K^2$

Thus Parseval's relation in this case becomes

$$P_{av} = A_0^2 + \frac{1}{2}\sum_{K \geq 1}^{\infty} A_K^2.$$

$\left[\because P_{av} = \frac{a_0^2}{4} + \frac{1}{2}\sum_{K=1}^{\infty}(a_K^2 + b_K^2)\right.$

$$= (3)^2 + \frac{1}{2}\left[\underbrace{(4)^2}_{A_3^2} + \underbrace{(5)^2}_{A_4^2}\right]$$

$$= 9 + \frac{1}{2}[16 + 25] = \frac{59}{2} = 29.5 \text{ Watts}$$

Observation

- If θ is $-ve$ we use $-\cos\theta = \cos(\theta + \pi)$
- If θ is $+ve$ we use $-\cos\theta = \cos(\theta - \pi)$

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Method 4: Using compact form (using sine wave) $(-\pi < \phi_k \le \pi)$

$$V(t) = 3 - 4\cos\left(3t + \frac{\pi}{3}\right) - 5\sin\left(4t - \frac{\pi}{3}\right)$$

$$= 3 + 4\sin\left(3t + \frac{\pi}{3} - \frac{\pi}{2}\right) + 5\sin\left(4t - \frac{\pi}{3} + \pi\right)$$

[using $-\cos\theta = \sin(\theta - \pi/2)$, $-\sin\theta = \sin(\theta + \pi)$]

$$= 3 + 4\sin\left(3t - \frac{\pi}{6}\right) + 5\sin\left(4t + \frac{2\pi}{3}\right)$$

Comparing above with

$$V(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin(k\omega t + \phi_k).$$

we get

$A_0 = 3$, $A_3 = 4$, $A_4 = 5$, $A_k = 0$ $k \ge 5$

$A_1 = A_2 = 0$

Using Parseval's relation

$$P_{avg} = A_0^2 + \frac{1}{2}\sum_{k=1}^{\infty} A_k^2$$

$$= (3)^2 + \frac{1}{2}\left[(4)^2 + (5)^2\right] = 9 + \frac{1}{2}\left[16 + 25\right] = \frac{59}{2} = 29.5 \text{ W}$$

Observation:

- If θ is -ve use $-\sin\theta = \sin(\theta + \pi)$
- If θ is +ve use $-\sin\theta = \sin(\theta - \pi)$

Method 5: Direct Method (without using Parseval's relation)

We know that

$$P_{avg} = \frac{1}{T}\int_{x_0}^{x_0+T} [f(x)]^2 \, dx.$$

Given that $V(t) = 3 - 4\cos\left(3t - \frac{\pi}{3}\right) - 5\sin\left(4t - \frac{\pi}{3}\right)$

Note that we are given 3rd and 4th harmonics. So $\omega_3 = 3$ and $\omega_4 = 4$

of ω_3 and ω_4

L.c.f = M.C.F $\angle$ = 1 (minimum common factor or least common factor)

Thus, fundamental frequency $= \omega = 1$ rad/sec

Fundamental period $= T = \frac{2\pi}{\omega} \Rightarrow T = 2\pi$ sec.

Alternatively, we can calculate time period by using

$$V(t+T) = 3 - 4\cos(3(t+T) + \pi/3) - 5\sin(4(t+T) - \pi/3)$$

$$= 3 - 4\cos[(3t + \pi/3) + 3T] - 5\sin[(4t - \pi/3) + 4T]$$

Let $3T = 2\pi N_1$ and $4T = 2\pi N_2$

$\Rightarrow T = \frac{2\pi}{3} N_1 \rightarrow ①$ and $T = \frac{2\pi}{4} N_2 \rightarrow ②$

From ① & ②

$$\frac{2\pi}{3} N_1 = \frac{2\pi}{4} N_2 \Rightarrow \frac{N_1}{N_2} = \frac{3}{4}$$

so $T = \frac{2\pi}{3}(3) = \frac{2\pi}{4}(4) = 2\pi$

so time period is $T = 2\pi$ sec

and fundamental frequency is $= \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$ rad/sec.

Now

$$P_{avg} = \frac{1}{T}\int_{x_0}^{x_0+T} [V(t)]^2 dt$$

where $x_0 = 0$, $x_0 + T = 2\pi$, $T = 2\pi$

Thus

$$P_{avg} = \frac{1}{2\pi}\int_0^{2\pi} \left[3 - 4\cos\left(3t + \frac{\pi}{3}\right) - 5\sin(4t - \pi/3)\right]^2 dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \Big[9 + 16\cos^2(3t+\pi/3) + 25\sin^2(4t - \pi/3) - 24\cos(3t + \pi/3) - 30\sin(4t - \pi/3) + 40\sin(4t - \pi/3)\cos(3t + \pi/3)\Big] dt \rightarrow (*)$$

we know that

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta \Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

So, (*) can be written as

$$P_{avg} = \frac{1}{2\pi}\int_0^{2\pi} \Big\{9 + 16\left(\frac{1 + \cos 2(3t + \pi/3)}{2}\right) + \frac{25}{2}\left(1 - \cos 2(4t - \pi/3)\right) - 24\cos\left(3t + \frac{\pi}{3}\right) - 30\sin\left(4t - \frac{\pi}{3}\right) + 40\sin\left(4t - \frac{\pi}{3}\right)\cos\left(3t + \frac{\pi}{3}\right)\Big] dt$$

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$$\Rightarrow P_{avg} = \frac{1}{2\pi}\int_0^{2\pi}\begin{bmatrix} 9 + 8 + 8\cos(6t + 2\pi/3) + (25/2) - (25/2)\cos(8t - 2\pi/3) \\ - 30\sin(4t - \pi/3) - 24\cos(3t + \pi/3) + 40\sin(4t - \pi/3)\cos(3t + \pi/3) \end{bmatrix} dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\begin{bmatrix} \frac{59}{2} + 8\cos\left(6t + \frac{2\pi}{3}\right) - \frac{25}{2}\cos\left(8t - \frac{2\pi}{3}\right) - 30\sin\left(4t - \frac{\pi}{3}\right) - 24\cos\left(3t + \frac{\pi}{3}\right) \\ + \frac{40}{2}\left[\sin\left(3t + \cancel{\pi/3} + 4t - \cancel{\pi/3}\right) - \sin\left(3t + \pi/3 - 4t + \pi/3\right)\right] \end{bmatrix} dt$$

Using $\cos\alpha \sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$.

$$\Rightarrow P_{avg} = \frac{1}{2\pi}\int_0^{2\pi}\begin{bmatrix} \frac{59}{2} + 8\cos\left(6t + \frac{2\pi}{3}\right) - \frac{25}{2}\cos\left(8t - \frac{2\pi}{3}\right) - 30\sin\left(4t - \frac{\pi}{3}\right) - 24\cos\left(3t + \frac{\pi}{3}\right) \\ + 20\sin(7t) - 20\sin\left(-t + \frac{2\pi}{3}\right) \end{bmatrix} dt$$

$$= \frac{1}{2\pi}\begin{bmatrix} \frac{59t}{2} + \frac{8\sin(6t + 2\pi/3)}{6} - \frac{25}{2}\frac{\sin(8t - 2\pi/3)}{8} + \frac{30\sin(4t - \pi/3)}{4} \\ - \frac{24\sin(3t + \pi/3)}{3} + \frac{20\cos(7t)}{7} + \frac{20\cos(-t + 2\pi/3)}{-1} \end{bmatrix}_0^{2\pi}$$

$$= \frac{1}{2\pi}\begin{bmatrix} \frac{59}{2}(2\pi - 0) + \frac{8}{6}\left\{\sin\left(12\pi + \frac{2\pi}{3}\right) - \sin\left(\frac{2\pi}{3}\right)\right\} - \frac{25}{16}\left\{\sin\left(16\pi - \frac{2\pi}{3}\right) - \sin\left(-\frac{2\pi}{3}\right)\right\} \\ + \frac{30}{4}\left\{\sin\left(8\pi - \frac{\pi}{3}\right) - \sin\left(-\frac{\pi}{3}\right)\right\} - \frac{24}{3}\left\{\sin\left(6\pi + \frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\right\} - \frac{20}{7}\left\{\cos 14\pi - \cos 0\right\} \\ - \frac{20}{1}\left\{\cos\left(-2\pi + \frac{2\pi}{3}\right) - \cos(2\pi/3)\right\} \end{bmatrix}$$

$$= \frac{1}{2\pi}\begin{bmatrix} 59\pi + \frac{4}{3}\left\{\cancel{\frac{\sqrt{3}}{2}} - \cancel{\frac{\sqrt{3}}{2}}\right\} - \frac{25}{16}\left\{-\cancel{\frac{\sqrt{3}}{2}} + \cancel{\frac{\sqrt{3}}{2}}\right\} + \frac{15}{2}\left\{-\cancel{\frac{\sqrt{3}}{2}} + \cancel{\frac{\sqrt{3}}{2}}\right\} \\ - 8\left\{\cancel{\frac{\sqrt{3}}{2}} - \cancel{\frac{\sqrt{3}}{2}}\right\} - \frac{20}{7}\left\{1 - 1\right\} - 20\left\{\cancel{-\tfrac{1}{2}} + \cancel{\tfrac{1}{2}}\right\} \end{bmatrix}$$

$$= \frac{1}{2\cancel{\pi}}(59\cancel{\pi}) = \frac{59}{2} = 29.5 \text{ Watts}.$$

(b) 1-sided frequency spectra.

For 1-sided frequency spectra, we consider

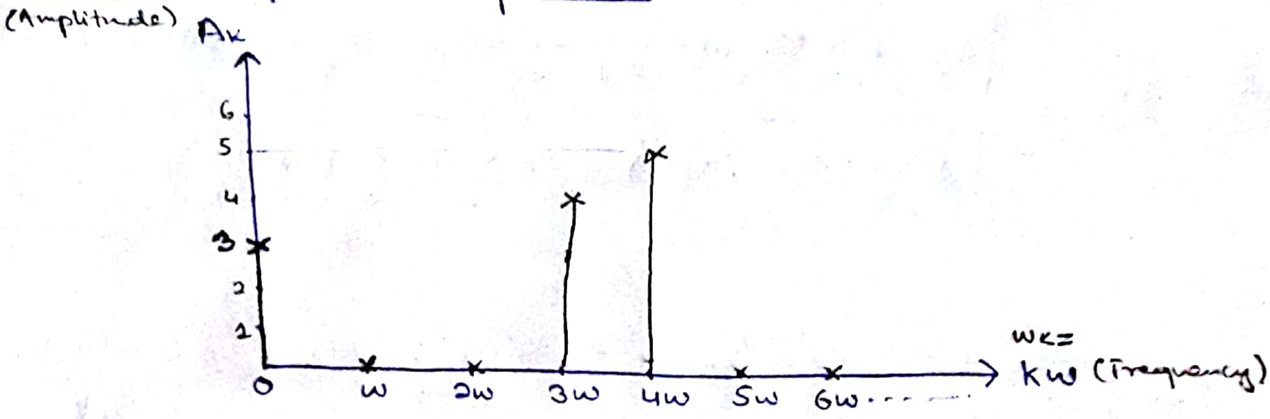
$V(t) = 3 - 4\cos(3t + \pi/3) - 5\sin(4t - \pi/3)$

$= 3 + 4\cos(3t - \frac{2\pi}{3}) + 5\cos(4t + \frac{\pi}{6})$ [As done in part a method 3.]

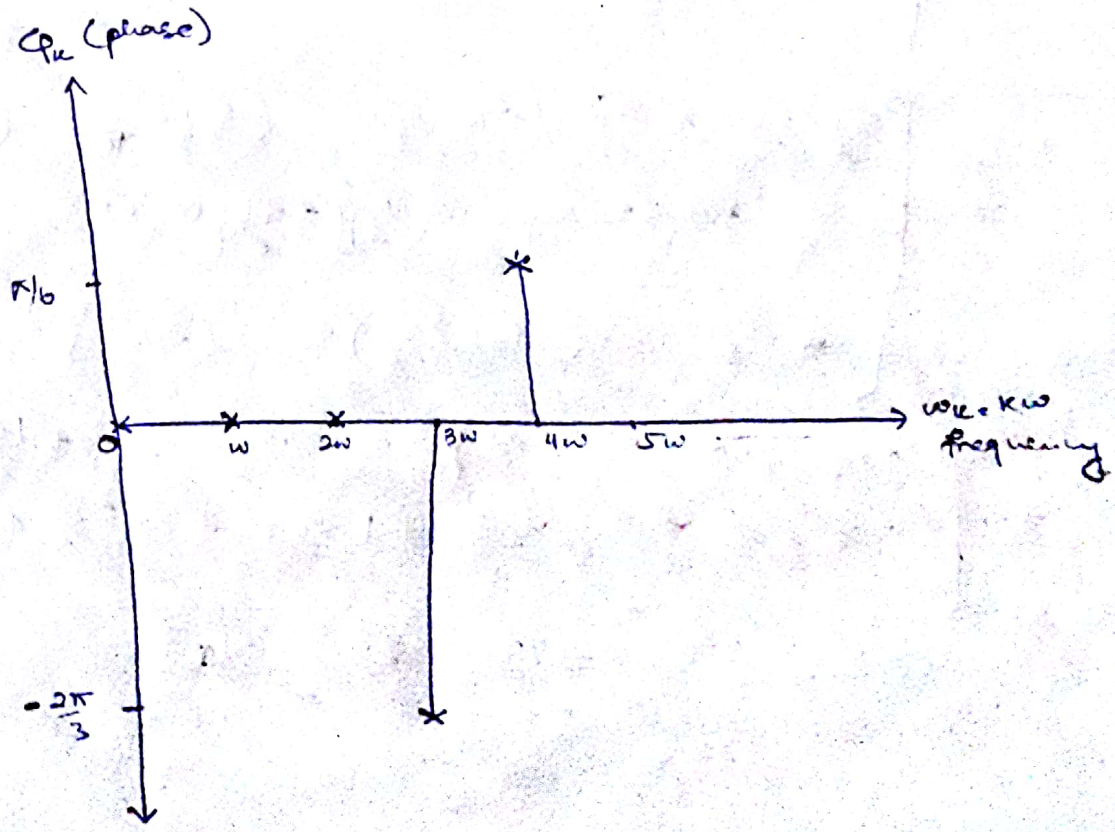
Identifying $A_0 = 3$, $A_3 = 4$, $A_4 = 5$
$A_1 = A_2 = 0$, $A_k = 0 \quad \forall k \geq 5$

and $\varphi_0 = 0$, $\varphi_3 = -2\pi/3$, $\varphi_4 = \pi/6$
$\varphi_1 = \varphi_2 = 0$, $\varphi_k = 0 \quad \forall k \geq 5$.

1-sided amplitude spectrum:



1-sided phase spectrum



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2-sided frequency Spectrum.

In order to plot 2-sided frequency spectrum we need complex Fourier coefficients. For this we consider,

$$v(t) = 3 - 4\cos(3t + \pi/3) - 5\sin(4t - \pi/3)$$

$$= 3 + 4\cos(3t - 2\pi/3) + 5\cos(4t + \pi/6) \quad \text{[using compact form]}$$

$$= 3 + 4\left[\frac{e^{i(3t - 2\pi/3)} + e^{-i(3t - 2\pi/3)}}{2}\right] + 5\left[\frac{e^{i(4t + \pi/6)} + e^{-i(4t + \pi/6)}}{2}\right]$$

$$= 3 + 2\left[e^{i3t} \cdot e^{-i2\pi/3} + e^{-i3t} \cdot e^{i2\pi/3}\right] + \frac{5}{2}\left[e^{i4t} \cdot e^{i\pi/6} + e^{-i4t} e^{-i\pi/6}\right]$$

From above we see that

$$c_0 = 3$$

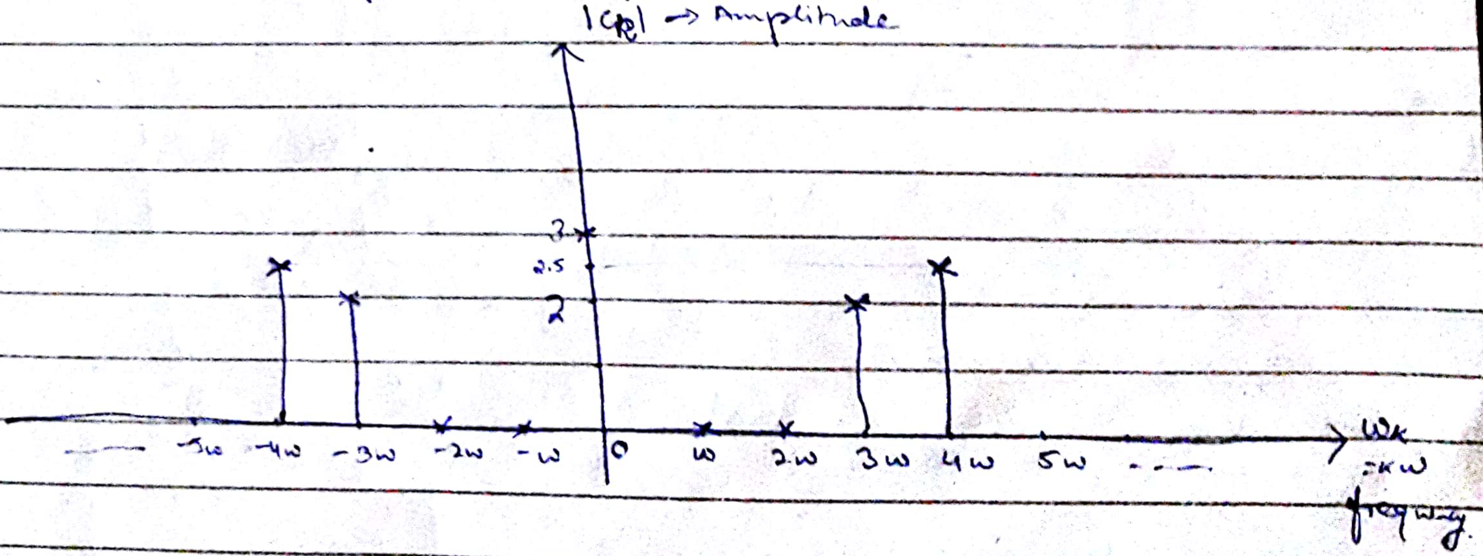
$$c_3 = c_{-3} = 2 \Rightarrow |c_3| = |c_{-3}| = 2$$

$$c_4 = c_{-4} = \frac{5}{2} \Rightarrow |c_4| = |c_{-4}| = 5/2 = 2.5$$

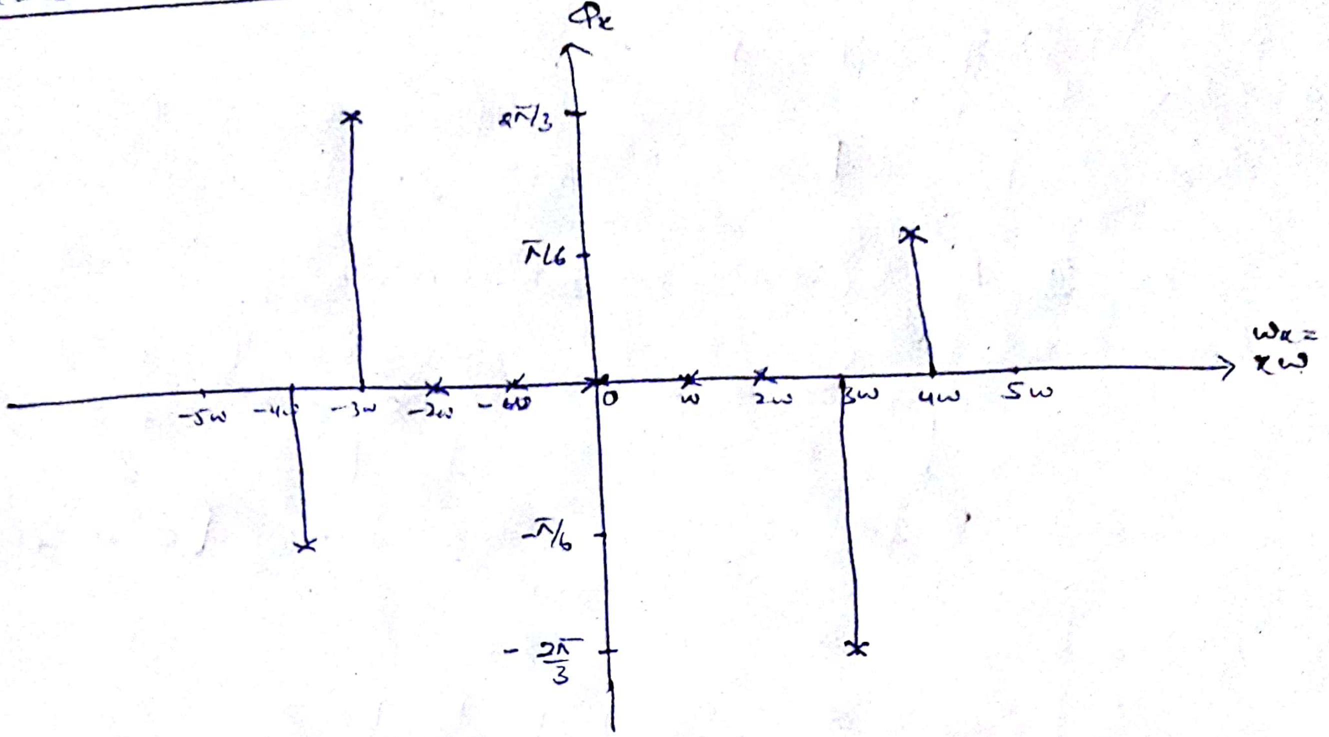
$$\varphi_0 = 0, \quad \varphi_1 = \varphi_{-1} = \varphi_2 = \varphi_{-2} = 0, \quad \varphi_{+3} = -\frac{2\pi}{3}, \quad \varphi_{-3} = \frac{2\pi}{3}$$

$$\varphi_4 = \pi/6, \quad \varphi_{-4} = -\pi/6$$

2-sided Amplitude Spectrum:



2-sided phase spectrum



Alternative method of obtaining 2-sided spectrum.

Using method 1 of part (a)

$$v(t) = 3 - 4\cos\left(3t + \frac{\pi}{3}\right) - 5\sin\left(4t - \frac{\pi}{3}\right)$$

can be written as

$$v(t) = 3 + \left[(-1 - i\sqrt{3})\, e^{i3t} + (-1 + i\sqrt{3})\, e^{-i3t}\right]$$
$$+ \left[\left(\frac{5\sqrt{3}}{4} + \frac{5}{4}i\right) e^{i4t} + \left(\frac{5\sqrt{3}}{4} - \frac{5}{4}i\right) e^{-i4t}\right]$$

From above we observe that

$C_0 = 3 \Rightarrow |C_0| = 3$ and $\arg(C_0) = \arg(3) = 0$

$C_{-1} = C_1 = 0 \Rightarrow |C_{-1}| = |C_1| = 0$ and $\arg(C_{-1}) = \arg(C_1) = 0$

Similarly $C_{-2} = C_2 = 0$ and $C_k = C_{-k} = 0 \;\forall\; k \geq 5$.

$C_{+3} = -1 - i\sqrt{3} \Rightarrow |C_3| = 2$ and $\arg(C_3) = \arg(-1 - i\sqrt{3})$ (III Q)

$= -\pi + \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right)$

$= -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$

$C_{-3} = -1 + i\sqrt{3} \Rightarrow |C_{-3}| = 2$ and $\arg(C_{-3}) = \arg(-1 + i\sqrt{3}) \leftarrow$ (II Q)

$= \pi + \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

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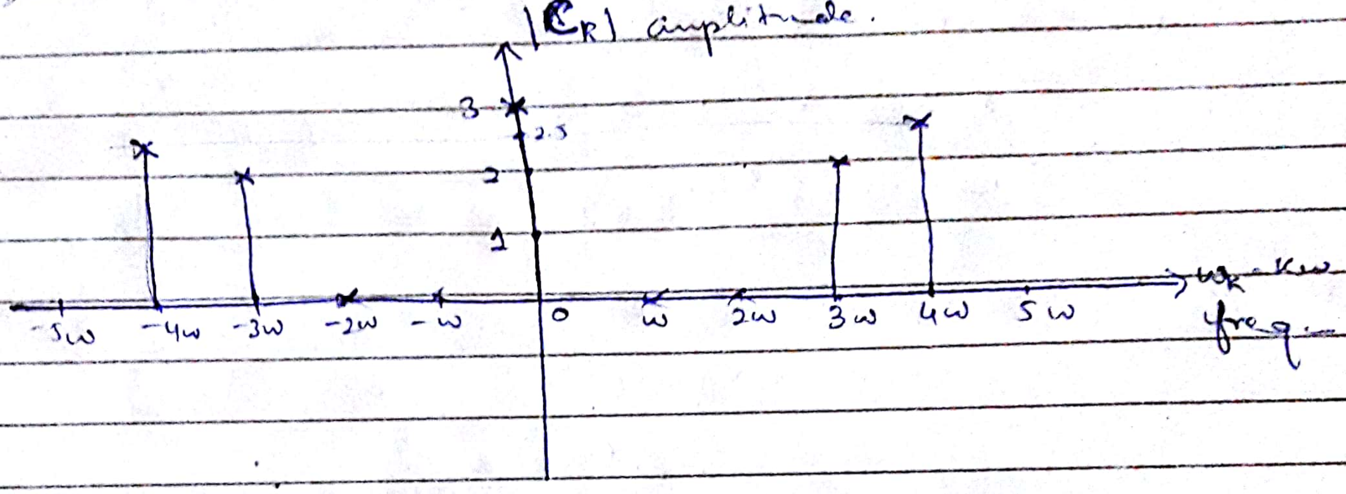
$C_4 = \frac{5\sqrt{3}}{4} + \frac{5i}{4} \Rightarrow |C_4| = \frac{5}{2}$ and $\arg(C_4) = \arg\left(\frac{5\sqrt{3}}{4} + i\frac{5}{4}\right)$ (I Q)

$= \tan^{-1}\left(\frac{5/4}{5\sqrt{3}/4}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

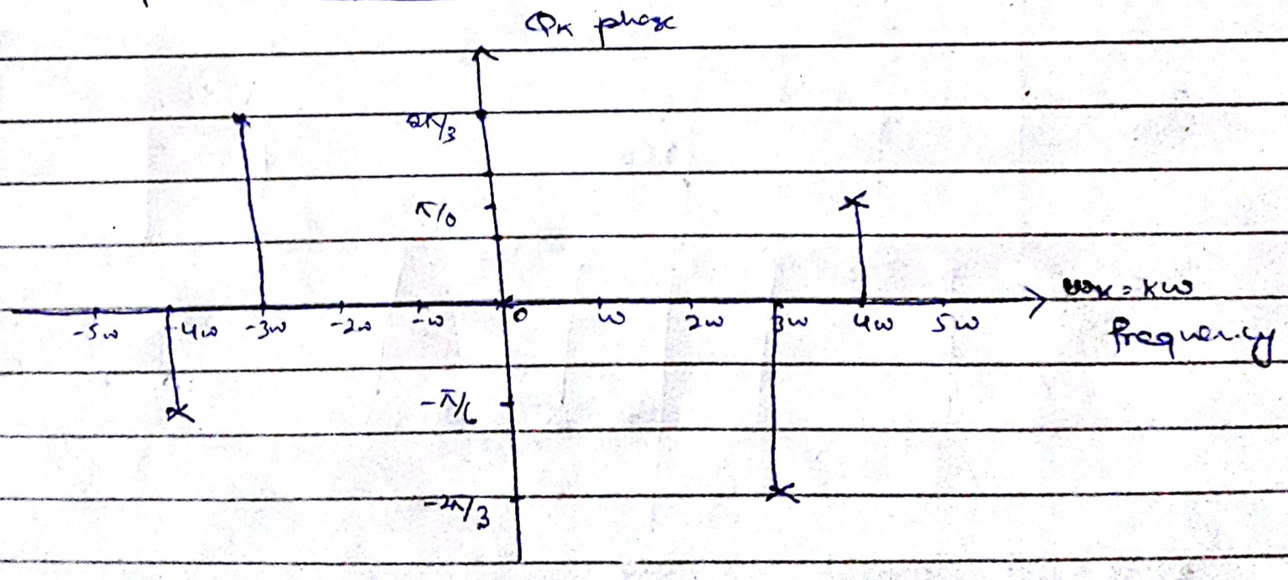
(IV Q)

$C_{-4} = \frac{5\sqrt{3}}{4} - \frac{i5}{4} \Rightarrow |C_{-4}| = \frac{5}{2}$ and $\arg\left(\frac{5\sqrt{3}}{4} - \frac{i5}{4}\right) = \tan^{-1}\left(\frac{-5/4}{+5\sqrt{3}/4}\right) = \frac{-\pi}{6}$

Thus, we have the 2-sided amplitude spectrum as:



2-sided phase spectrum:



Q2:- (a) Suppose we have a Fourier series $f(x) = \sin(100x) + \cos x$. Find its complex Fourier coefficients.

(b) Plot one-sided and two-sided discrete frequency spectrum.

(a) Given that $f(x) = \sin(100x) + \cos x$.

This can be written as

$$f(x) = \frac{e^{ix}+e^{-ix}}{2} + \frac{e^{i100x}-e^{-i100x}}{2i}$$

$$= \left[\frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}\right] - \frac{i}{2}\left[e^{i100x} - e^{-i100x}\right] \quad \left[\because -i = \frac{1}{i}\right]$$

$$= \left[\frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}\right] + \left[\left(\frac{-i}{2}\right)e^{i100x} + \left(\frac{i}{2}\right)e^{-i100x}\right]$$

For the present case we have

$$C_1 = \frac{1}{2}\ ;\ C_{-1} = \frac{1}{2}$$

$$C_{100} = -\frac{i}{2}\ ;\ C_{-100} = \frac{i}{2}$$

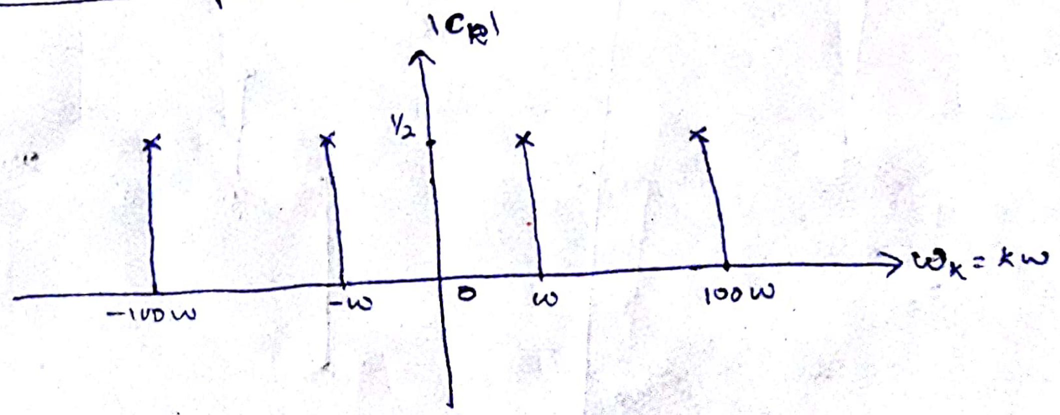
(b) $|C_1| = \frac{1}{2}$ and $\arg(C_1) = \arg\left(\frac{1}{2}\right) = 0$

$|C_{-1}| = \frac{1}{2}$ and $\arg(C_{-1}) = 0$

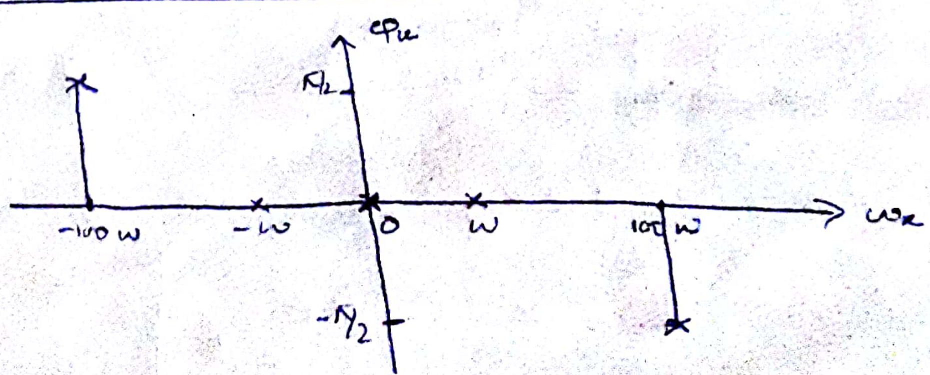
$|C_{100}| = \frac{1}{2}$ and $\arg(C_{100}) = -\frac{\pi}{2}$

$|C_{-100}| = \frac{1}{2}$ and $\arg(C_{-100}) = \frac{\pi}{2}$

<u>2-sided amplitude spectrum:</u>



<u>2-sided phase spectrum</u>



Date:

1-sided frequency spectrum

Consider

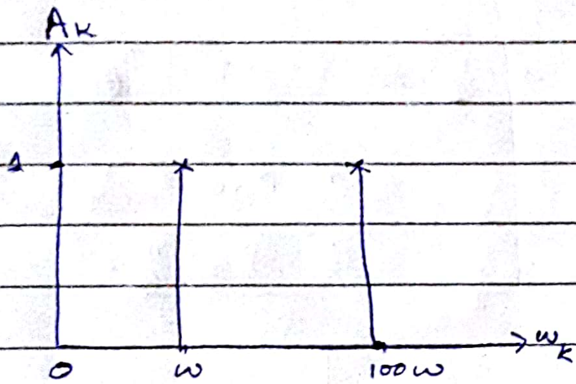
$$f(x) = \underbrace{\sin(100x)}_{100^{th}\ \text{harmonic}} + \underbrace{\cos x}_{1^{st}\ \text{harmonic}}.$$

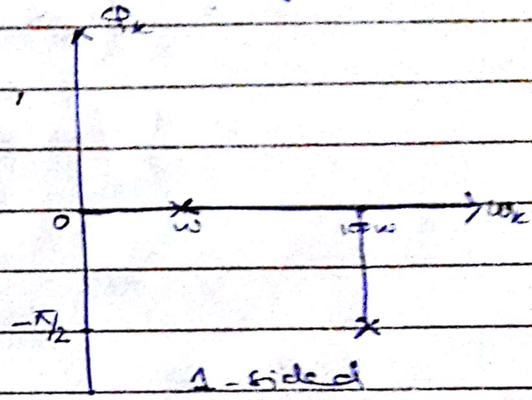
For present case $a_1 = 1,\ b_1 = 0$

$a_{100} = 0,\ b_{100} = 1.$

$A_k^2 = a_k^2 + b_k^2 \Rightarrow A_1 = 1$ and $A_{100} = 1.$

$\varphi_k = \tan^{-1}\left(\frac{-b_k}{a_k}\right) \Rightarrow \varphi_1 = \tan^{-1}\left(\frac{0}{1}\right) = 0$ and $\varphi_{100} = \tan^{-1}\left(\frac{1}{0}\right) = -\frac{\pi}{2}$





1-sided phase spectrum.

one-sided amplitude spectrum.

Alternative method for finding 1-sided frequency spectrum

$$f(x) = \cos x + \sin(100x)$$

$$= \cos x + \cos(100x - \pi/2) \qquad \left[\because \cos(\theta - \tfrac{\pi}{2}) = \sin\theta\right]$$

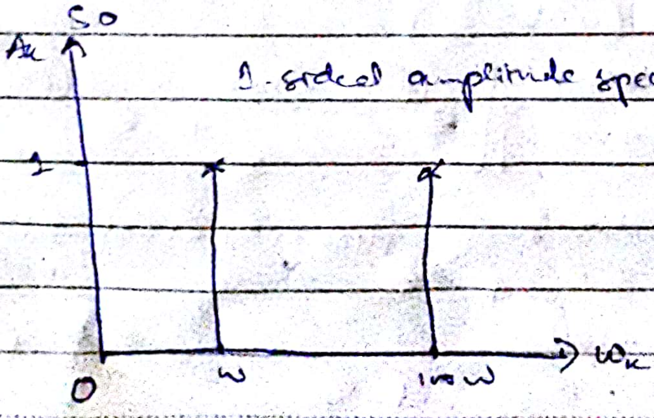
$$= \cos(x + 0) + \cos(100x - \pi/2) \qquad \left[f(x) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega x + \varphi_k)\right]$$

Thus, $A_1 = 1$, $A_{100} = 1$

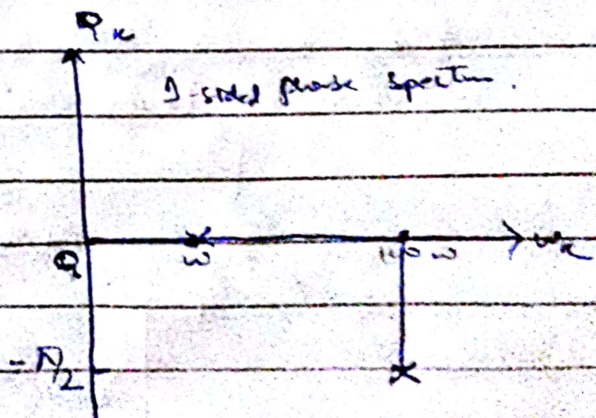
$\varphi_1 = 0$, $\varphi_{100} = -\pi/2$

So

1-sided amplitude spectrum.



1-sided phase spectrum.



Q3: Let $f(x) = \cos(4\pi t - \pi/3)\cos(6\pi t + \pi/4)$.

Plot 1-sided and 2-sided frequency spectrum.

Sol: $f(x) = \cos\left(4\pi t + \frac{\pi}{3}\right)\cos\left(6\pi t - \frac{\pi}{6}\right)$

$$= \frac{1}{2}\left[\cos\left(4\pi t + \frac{\pi}{3} + 6\pi t - \frac{\pi}{6}\right) + \cos\left(4\pi t + \frac{\pi}{3} - 6\pi t + \frac{\pi}{6}\right)\right]$$

$\left[\because \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]\right]$

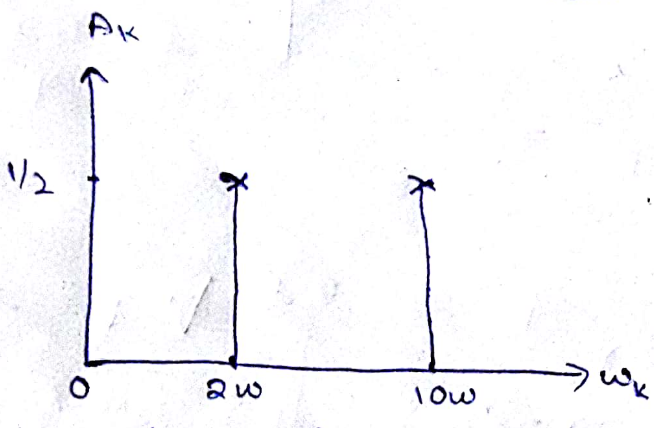
$$= \frac{1}{2}\left[\cos\left(10\pi t + \frac{\pi}{6}\right) + \cos\left(-2\pi t + \frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(10\pi t + \frac{\pi}{6}\right) + \cos\left(2\pi t - \frac{\pi}{2}\right)\right] \quad [\because \cos(-\theta) = \cos\theta]$$

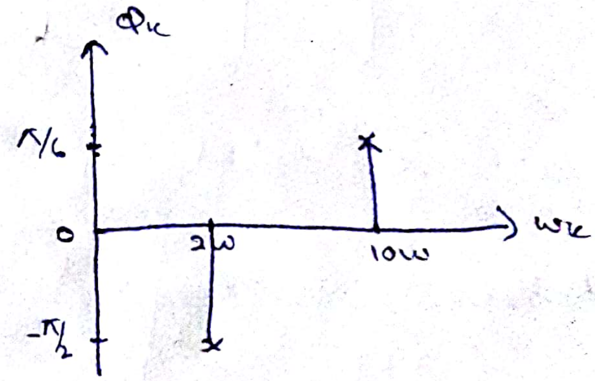
$$= \frac{1}{2}\cos\left(\underline{2\pi t} - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(\underline{10\pi t} + \frac{\pi}{6}\right)$$

For present case $A_2 = \frac{1}{2}$ and $A_{10} = \frac{1}{2}$

$\varphi_2 = -\frac{\pi}{2}$ and $\varphi_{10} = \frac{\pi}{6}$.



1-sided amplitude spectrum.



1-sided phase spectrum.

In order to plot 2-sided frequency spectrum we need complex Fourier coefficients.

$$f(x) = \cos\left(4\pi t + \frac{\pi}{3}\right)\cos\left(6\pi t - \frac{\pi}{6}\right)$$

$$= \left[\frac{e^{i(4\pi t + \pi/3)} + e^{-i(4\pi t + \pi/3)}}{2}\right]\left[\frac{e^{i(6\pi t - \pi/6)} + e^{-i(6\pi t - \pi/6)}}{2}\right]$$

$$= \frac{1}{4}\left\{ e^{i4\pi t + i\frac{\pi}{3} + i6\pi t - i\frac{\pi}{6}} + e^{i4\pi t + i\pi/3 - i6\pi t + i\pi/6} + e^{-i4\pi t - i\pi/3 + i6\pi t - i\frac{\pi}{6}} + e^{-i4\pi t - i\frac{\pi}{3} - i6\pi t + i\frac{\pi}{6}} \right.$$

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$$\Rightarrow f(x) = \frac{1}{4}\left[e^{i10\pi t + i\pi/6} + e^{-2i\pi t + i\pi/2} + e^{+2i\pi t - i\pi/2} + e^{-10i\pi t - i\pi/6} \right]$$

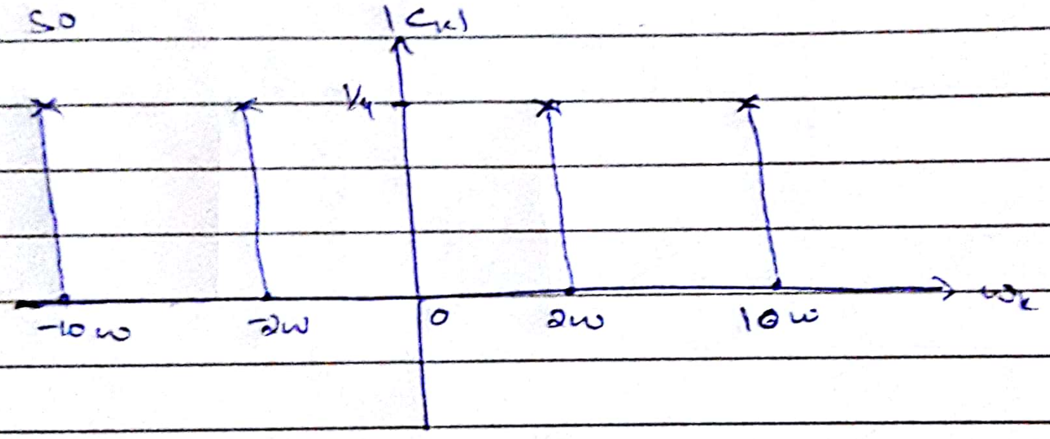
$$= \frac{1}{4}\left[e^{i2\pi t} \cdot e^{-i\pi/2} + e^{-i2\pi t} \cdot e^{i\pi/2} \right] + \frac{1}{4}\left[e^{i10\pi t} \cdot e^{i\pi/6} + e^{-i10\pi t} \cdot e^{-i\pi/6} \right]$$

From above we observe that

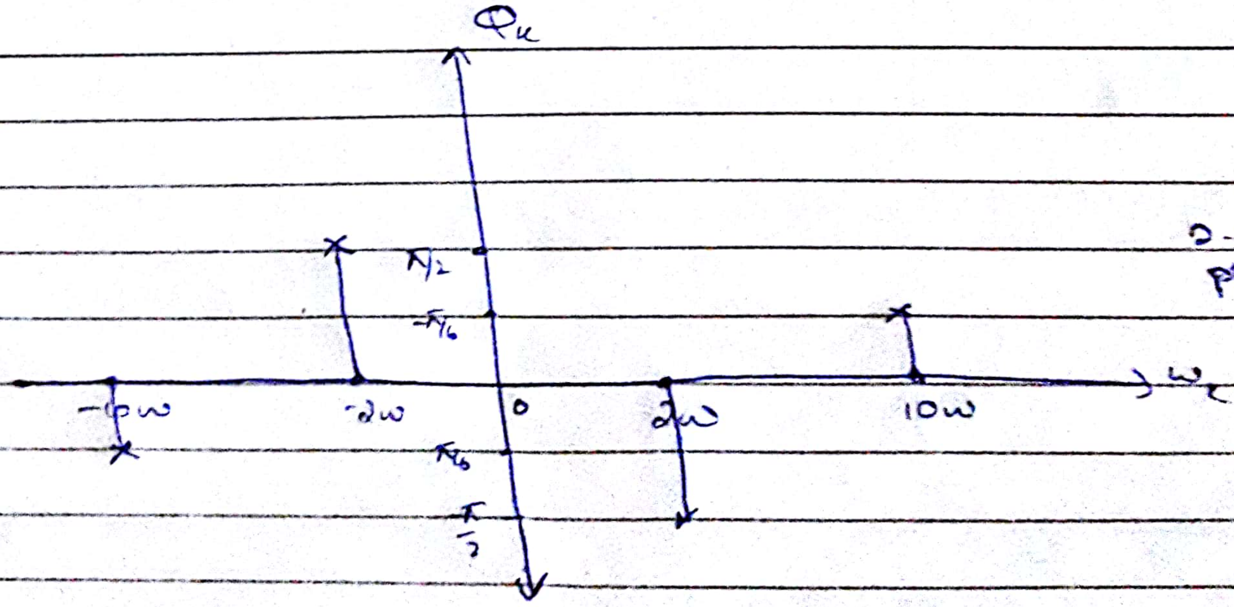
$C_2 = \frac{1}{4}$, $\varphi_2 = -\frac{\pi}{2}$ | $C_{10} = \frac{1}{4}$, $\varphi_{10} = \frac{\pi}{6}$

$C_{-2} = \frac{1}{4}$, $\varphi_{-2} = \frac{\pi}{2}$ | $C_{-10} = \frac{1}{4}$, $\varphi_{-10} = -\frac{\pi}{6}$

So



2-sided Amplitude Spectrum



2-sided phase spectrum