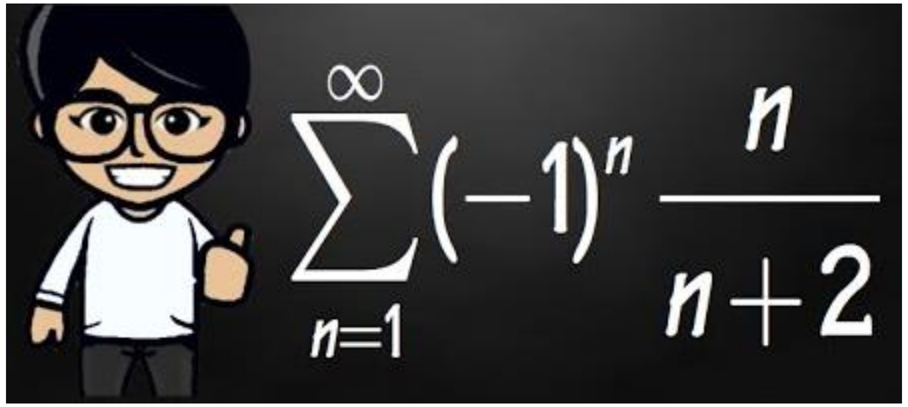
Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

Chapter: 11 (11.6)

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

Chapter: 11 (11.5)

Alternating Series, Absolute and Conditional Convergence



Calculus & Analytical Geometry MATH-101
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Alternating Series

A series in which the terms are alternately positive and negative is known as an **alternating series**. It is customary to express an alternating series in one of the following forms:

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1}a_n + \cdots$$

or $$\sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^{n}a_n + \cdots$$

with $a_n > 0, \ \forall\, n.$

Example

Below are some examples of alternating series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \qquad (1)$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^{n}4}{2^{n}} + \cdots \qquad (2)$$

$$1 - 2 + 3 - 4 + 5 - \cdots + (-1)^{n+1}n + \cdots \qquad (3)$$

The Alternating Series Test

The series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} b_n\,; \quad \text{where} \;\; a_n = (-1)^{n+1} b_n \;\text{ and }\; b_n \geq 0 \;\; \forall\, n,$$

converges if the following two conditions are satisfied:

1. $\lim_{n\to\infty} b_n = 0.$

and

2. $\{b_n\}$ is a nonincreasing (or decreasing) sequence.

Example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \qquad (1)$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^{n}4}{2^{n}} + \cdots \qquad (2)$$

$$1 - 2 + 3 - 4 + 5 - \cdots + (-1)^{n+1}n + \cdots \qquad (3)$$

- Series in (1), called the **alternating harmonic series**, satisfies all the conditions of the alternating series test, therefore it is convergent.
- Series in (2) is a geometric series with ratio $r = -1/2$. Since the series also satisfy all the conditions of the alternating series test, therefore, it is convergent.
- Series in (3) is a divergent series because the n^{th} term does not approach zero.

Absolute Convergence

A convergent alternating series

$$\sum_{n=1}^{\infty} a_n ,$$

converges absolutely if the corresponding series of absolute values

$$\sum_{n=1}^{\infty} |a_n| ,$$

converges.

Example:

The geometric series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots ,$ converges absolutely because the corresponding series of absolute values $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, converges.

Conditional Convergence

An alternating series

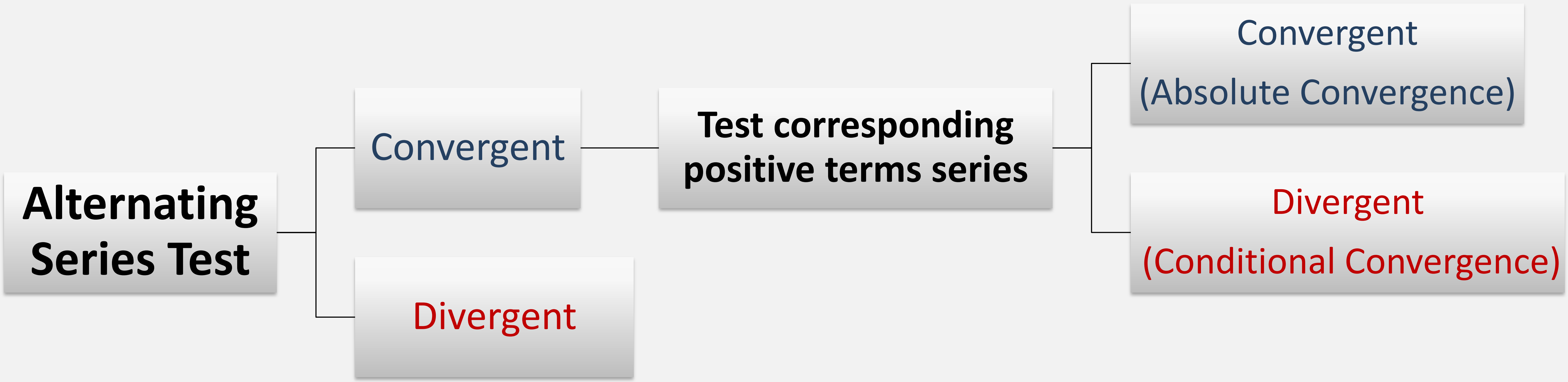
$$\sum_{n=1}^{\infty} a_n \,,$$

is said to be **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

Example:

The alternating harmonic series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$, converges but the corresponding series of absolute values $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$, is divergent. Therefore, the alternating harmonic series converges conditionally.

Absolute/Conditional Convergence



The Absolute Convergence Test

If

$$\sum_{n=1}^{\infty} |a_n| \, ,$$

converges, then

$$\sum_{n=1}^{\infty} a_n \, ,$$

converges. In other words, every absolutely convergent series is always convergent.

Note: This test is useful to investigate the convergence/divergence of those series which are neither positive term series nor alternating series.

Example

Let $\sum_{n=1}^{\infty} a_n$ be the series:

$$\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}-\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}-\frac{1}{2^7}-\frac{1}{2^8}+\cdots$$

where the signs of the terms vary in pairs as indicated and where $|a_n| = 1/2^n$.

Determine whether the given series converges or diverges.

Solution

Note that the given series is neither alternating nor geometric nor positive term series, so none of the earlier tests can be applied to test the convergence of this series. Let us consider the corresponding series of absolute values:

$$\sum_{n=1}^{\infty} |a_n| = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \cdots + \frac{1}{2^n} + \cdots.$$

This is a geometric series with $a = 1/2$ and $r = 1/2$. Since $1/2 < 1$, so $\sum |a_n|$ is a convergent series. Thus, the given series $\sum a_n$ is absolutely convergent and hence, by the absolute convergence test it is convergent.

Example

Test the following series for convergence and divergence:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Solution:

For $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots + \frac{(-1)^{n+1}}{n^2} + \cdots$ the corresponding series of absolute values is:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

which is a convergent series. Thus, the given series is absolutely convergent and hence, by the absolute convergent test it is convergent.

Observation

From the preceding discussion we observe that an arbitrary series may be classified in exactly *one* of the following ways:

1) absolutely convergent
2) conditionally convergent
3) divergent

Of course, for positive-term series we only need to determine convergence or divergence.

Example

Investigate the behavior of the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots + \frac{(-1)^{n-1}}{n^p} + \cdots , \qquad p > 0.$$

This series is commonly known as alternating p −series.

Solution:

Observe that for $p > 0$, the sequence $\{1/n^p\}$ is a decreasing sequence because $1/n^p > 1/(n+1)^p \ \forall\, n$. Moreover,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1/n^p = 0.$$

Therefore, by alternating series test the given series is convergent.

Solution

Now we investigate the absolute/conditional convergence of the alternating p −series. For that we consider the corresponding positive term series:

$$\sum_{n=1}^{\infty} |a_n| = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots$$

which is an ordinary p −series that converges for $p > 1$ and diverges if $p \leq 1$. Thus, for $p > 1$ the alternating p −series converges absolutely while for $0 < p \leq 1$, the given series is conditionally convergent.

The Ratio Test for Absolute Convergence

Let $\sum a_n$ be a series of non-zero terms, and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

Then

1. If $L < 1$, the series is absolutely convergent.
2. If $L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, the series is divergent.
3. If $L = 1$, the test is inconclusive. Apply a different test; the series may be absolutely convergent, conditionally convergent or divergent.

The Root Test for Absolute Convergence

Let $\sum a_n$ be a series of non-zero terms, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

Then

1. If $L < 1,$ the series is absolutely convergent.
2. If $L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty,$ the series is divergent.
3. If $L = 1,$ the test is inconclusive. Apply a different test; the series may be absolutely convergent, conditionally convergent or divergent.

Example

Determine whether the following series is absolutely convergent, conditionally convergent or divergent:

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^2+4}{2^n}.$$

Solution:

Using the ratio test we have:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2+4}{2^{n+1}}.\frac{2^n}{n^2+4}\right| = \frac{1}{2}\lim_{n\to\infty}\left|\frac{n^2+2n+5}{n^2+4}\right| = \frac{1}{2} < 1.$$

Hence, by ratio test the series is absolutely convergent.

Practice Questions

Test the following series for absolute/conditional convergence or divergence.

1. $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1}$$
2. $$\sum_{n=1}^{\infty} \frac{\cos n\,\pi}{\sqrt{n\pi}}$$
3. $$\sum_{n=1}^{\infty} (-1)^n \left(\frac{n+2}{5n+3}\right)^n$$

Practice Questions

Determine the values of x for which the given series (i) converges absolutely (ii) converges conditionally (iii) diverges.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n(n+1)}$$

Summary

We now have a variety of tests that can be used to investigate a series for convergence or divergence. The following summary may be helpful in deciding which test to apply.

Test	Series	Convergence or Divergence	Comments
n^{th}-term	$\sum a_n$	Diverges if $\lim\limits_{n\to\infty} a_n \neq 0$	Inconclusive if $\lim\limits_{n\to\infty} a_n = 0$
Geometric Series	$\sum\limits_{n=1}^{\infty} ar^{n-1}$	(i) Converges with sum $S = \dfrac{a}{1-r}$ if $\lvert r\rvert < 1$ (ii) Diverges if $\lvert r\rvert \geq 1$	Useful for comparison tests if the n^{th} term a_n of a series is similar to ar^{n-1}
p-series	$\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$	(i) Converges if $p > 1$ (ii) Diverges if $p \leq 1$	Useful for comparison tests if the n^{th} term a_n of a series is similar to $\dfrac{1}{n^p}$

Summary

Test	Series	Convergence or Divergence	Comments
Integral	$\sum_{n=1}^{\infty} a_n$ $a_n = f(n)$	(i) Converges if $\int_1^\infty f(x)dx$ converges (ii) Diverges if $\int_1^\infty f(x)dx$ diverges	The function f obtained from $a_n = f(n)$ must be continuous, positive, decreasing, integrable.
Comparison	$\sum a_n$, $\sum b_n$ $a_n > 0, b_n > 0$	(i) if $\sum b_n$ converges and $a_n \leq b_n$ for every n, then $\sum a_n$ converges. (ii) If $\sum b_n$ diverges and $a_n \geq b_n$ for every n, then $\sum a_n$ diverges. (iii) If $\lim_{n\to\infty}(a_n/b_n) = c$ for some positive real number c, then both series converge or both diverge	The comparison series $\sum b_n$ is often a geometric series or a p-series. To find b_n in (iii) consider only the terms of a_n that have the greatest effect on the magnitude.
Ratio	$\sum a_n$	If $\lim_{n\to\infty} \lvert\frac{a_{n+1}}{a_n}\rvert = L$, the series (i) converges (absolutely) if $L < 1$ (ii) diverges is $L > 1 (or\ \infty)$	Inconclusive if $L = 1$. Useful of a_n involves factorials or n^{th} powers. If $a_n > 0$ for every n, disregard the absolute value.

Summary

Test	Series	Convergence or Divergence	Comments
Root	$\sum a_n$	If $\lim\limits_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L, (or\ \infty)$, the series (i)converges (absolutely) if $L < 1$ (ii) diverges is $L > 1(or\ \infty)$	Inconclusive if $L = 1$ Useful if a_n involves n^{th} powers If $a_n > 0$ for every n, disregard the absolute value.
Alternating Series	$\sum(-1)^n a_n$ $a_n > 0$	Converges if $a_k \geq a_{k+1}$ for every k and $\lim\limits_{n\to\infty} a_n = 0$	Applicable only to an alternating series.
$\sum \lvert a_n\rvert$	$\sum a_n$	If $\sum \lvert a_n\rvert$ converges, then $\sum a_n$ converges.	Useful for series that contain both positive and negative terms.

Practice Questions

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- Exercise: 11.6

  Q # 1 to Q # 44

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 11.5

  Q # 1 to Q # 32