FOURIER SERIES REPRESENTATION OF PERIODIC SIGNALS

Fourier Series Representation

- LTI systems are based on representing signals as linear combinations of shifted impulses
- An alternative representation for signals and LTI systems - linear combinations of a set of complex exponentials
- The resulting representations are known as CT and DT Fourier series and transforms
- First we focus on CT and DT periodic signals, with a representation known as Fourier series
- In later lectures we extend the analysis to the Fourier transform representation of aperiodic, finite energy signals

Set of Basic Signals

- What are the desirable characteristics of a set of basic signals?

1. We can represent large and useful classes of signals using these building blocks
2. The response of LTI systems to these basic signals is particularly simple and useful

Set of Basic Signals

- Both of the desirable properties are provided by the set of complex exponential signals in continuous and discrete time

- Signals of the form e^{st} in continuous time And z^n in discrete time

- Where s and z are complex numbers.

Continuous Time: e^{st}

Discrete Time: z^n

- **Previous focus:** Unit samples and impulses

- **New Focus:** Eigenfunctions of all LTI systems

Response of LTI Systems to Complex Exponentials

- Definition: For every operator, there is a set of functions which after the operator is applied, produces the same function multiplied by a constant factor

- Such a function is called the eigenfunction of the operator, and the constant factor is called its corresponding eigenvalue

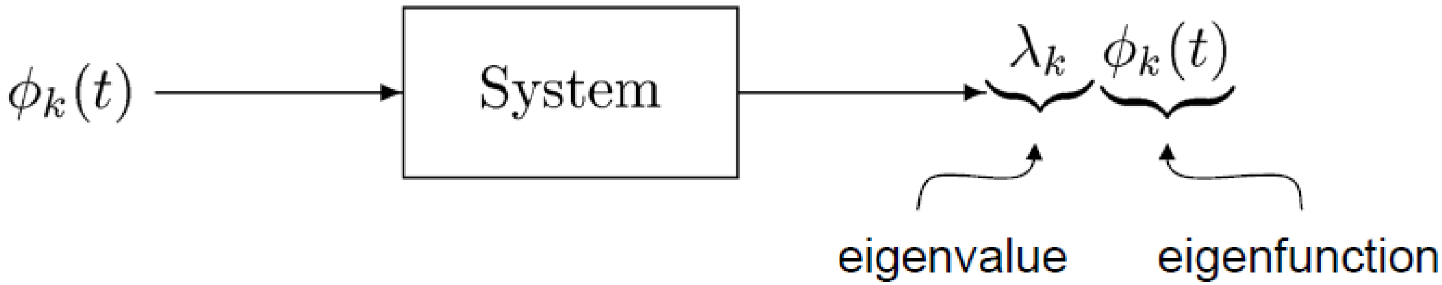
- An eigenvalue is just a number: Real or complex

- The response of an LTI system to a complex exponential input is the same complex exponential with only a change of amplitude, that is:

$$\boxed{\begin{array}{l}\text{Continuous Time: } e^{st} \rightarrow H(s)e^{st} \\ \text{Discrete Time: } z^{n} \rightarrow H(z)z^{n}\end{array}}$$

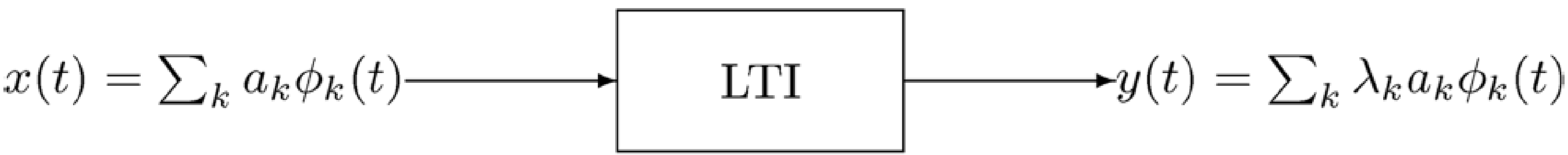
Eigenfunctions $\Phi_k(t)$ and their Properties

(Focus on CT systems now, but results apply to DT systems as well.)



Eigenfunction in => same function out with a "gain"

From the superposition property of LTI systems:



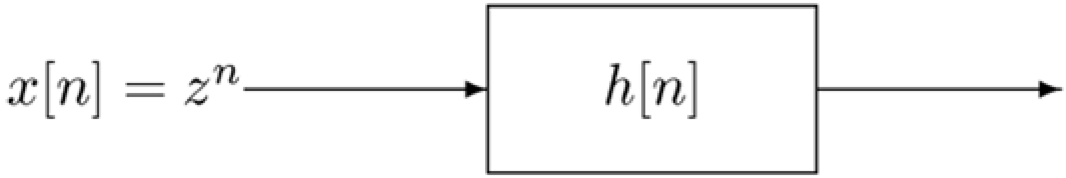
Now the task of finding response of LTI systems is to determine λ_k.

Complex Exponentials as Eigenfunctions of LTI Systems

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\
&= \left[\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \right] e^{st} \\
&= \underbrace{H(s)}_{\text{eigenvalue}} \underbrace{e^{st}}_{\text{eigenfunction}}
\end{aligned}$$

Complex Exponentials as Eigenfunctions of LTI Systems



$$y[n] = \sum_{m=-\infty}^{\infty} h[m] z^{n-m}$$

$$= \left[\sum_{m=-\infty}^{\infty} h[m] z^{-m} \right] z^n$$

$$= \underbrace{H(z)}_{\text{eigenvalue}} \; \underbrace{z^n}_{\text{eigenfunction}}$$

$$\boxed{\begin{aligned} H(s) &= \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \quad -- \text{ CT} \\ H(z) &= \sum_{n=-\infty}^{\infty} h[n] z^{-n} \quad -- \text{ DT} \end{aligned}}$$

Decomposing Signals in Terms of Eigenfunctions

- Consider a combination of three complex exponentials:

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

- For an LTI system, because of the eigenfunction property, the system response to each component separately is:

$$a_1 e^{s_1 t} \rightarrow a_1 H(s_1) e^{s_1 t}$$
$$a_2 e^{s_2 t} \rightarrow a_2 H(s_2) e^{s_2 t}$$
$$a_3 e^{s_3 t} \rightarrow a_3 H(s_3) e^{s_3 t}$$

- and from superposition, the resulting output is of the form:

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

Eigenvalues and Eigenfunctions

- Consider an LTI system with input-output relationship:

  $$y(t) = x(t-3)$$

- Show that the complex exponential $x(t) = e^{j2t}$ $(s = j2)$ is an eigenfunction and find its eigenvalue

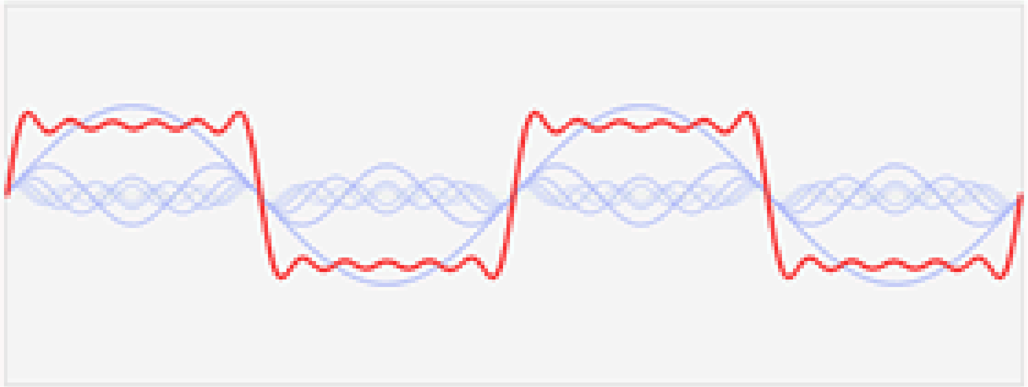
What is a Fourier Series

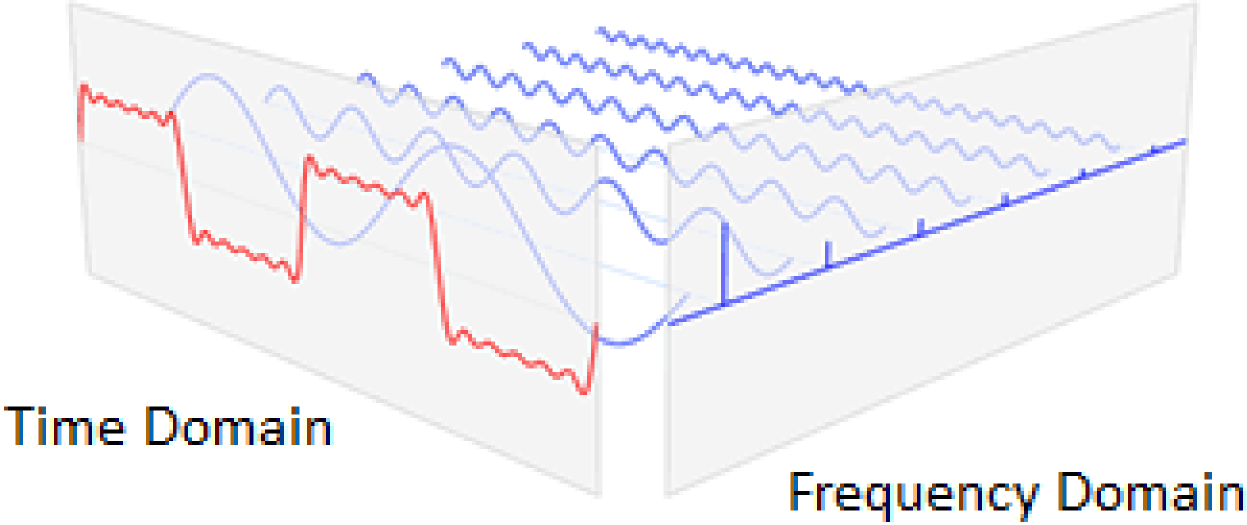
- A Fourier series is an expansion of a periodic function in terms of an infinite sum of exponential terms
- The computation and study of Fourier series is known as harmonic analysis
- Harmonic analysis is extremely useful as a way to break up an arbitrary periodic function into a set of simple terms
- The set of simple terms can be solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical

What is a Fourier Series

- Fourier analysis relates the function's time domain, shown in red, to the function's frequency domain, shown in blue

Approximation of a Square Wave





How Do We Find the Fourier Coefficients

First, for simple periodic signals consisting of a few sinusoidal terms

$$\text{Ex: } x(t) = \cos 4\pi t + 2\sin 8\pi t$$

Euler's relation (memorize!)

$$= \frac{1}{2}\left[e^{j4\pi t} + e^{-j4\pi t}\right] + \frac{2}{2j}\left[e^{j8\pi t} - e^{-j8\pi t}\right]$$

$$\omega_0 = 4\pi \qquad T = \frac{2\pi}{\omega_0} = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$a_0 = 0$ – no dc component

$$a_1 = \frac{1}{2}$$

$$a_{-1} = \frac{1}{2}$$

$$a_2 = \frac{1}{j}$$

$$a_{-2} = -\frac{1}{j}$$

$$a_3 = 0$$

$$a_{-3} = 0$$

$$\vdots$$

How Do We Find the Fourier Coefficients

Suppose $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$ (Given $x(t)$, how find a_k?)

1) multiply by $e^{-jn\omega_0 t}$
2) integrate over one period

$$\Downarrow$$

$$\begin{aligned}
\int_T x(t)e^{-jn\omega_0 t}dt &= \int_T \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) e^{-jn\omega_0 t}dt \\
&= \sum_{k=-\infty}^{\infty} a_k \left(\int_T e^{j(k-n)\omega_0 t}dt \right)
\end{aligned}$$

(Here $\int_T$ denotes integral over *any* interval of length T (one period).)
Next, note that

$$\begin{aligned}
\int_T e^{j(k-n)\omega_0 t}dt &= \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases} \\
&= T\delta[k-n] \quad \text{Orthogonality}
\end{aligned}$$

$$\Downarrow$$

How Do We Find the Fourier Coefficients

$$\int_T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \left(\int_T e^{j(k-n)\omega_0 t}dt\right) = \sum_{k=-\infty}^{\infty} a_k \cdot T\delta[k-n]$$

$$\int_T x(t)e^{-jn\omega_0 t}dt = a_n T$$

$$\Downarrow$$

CT Fourier Series Pair $\left(\omega_0 = \frac{2\pi}{T}\right)$

$$\boxed{\begin{array}{ll} x(t) = \displaystyle\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} & \text{(Synthesis equation)} \\ a_k = \displaystyle\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt & \text{(Analysis equation)} \end{array}}$$

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