EE-381 Robotics-1
UG ELECTIVE

Lecture 6

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Inverse Kinematics (IK)

- How to compute the position of each joint given the end –effector pose?
- How to generate smooth paths/trajectories for the end-effector?

Inverse Kinematics (IK)

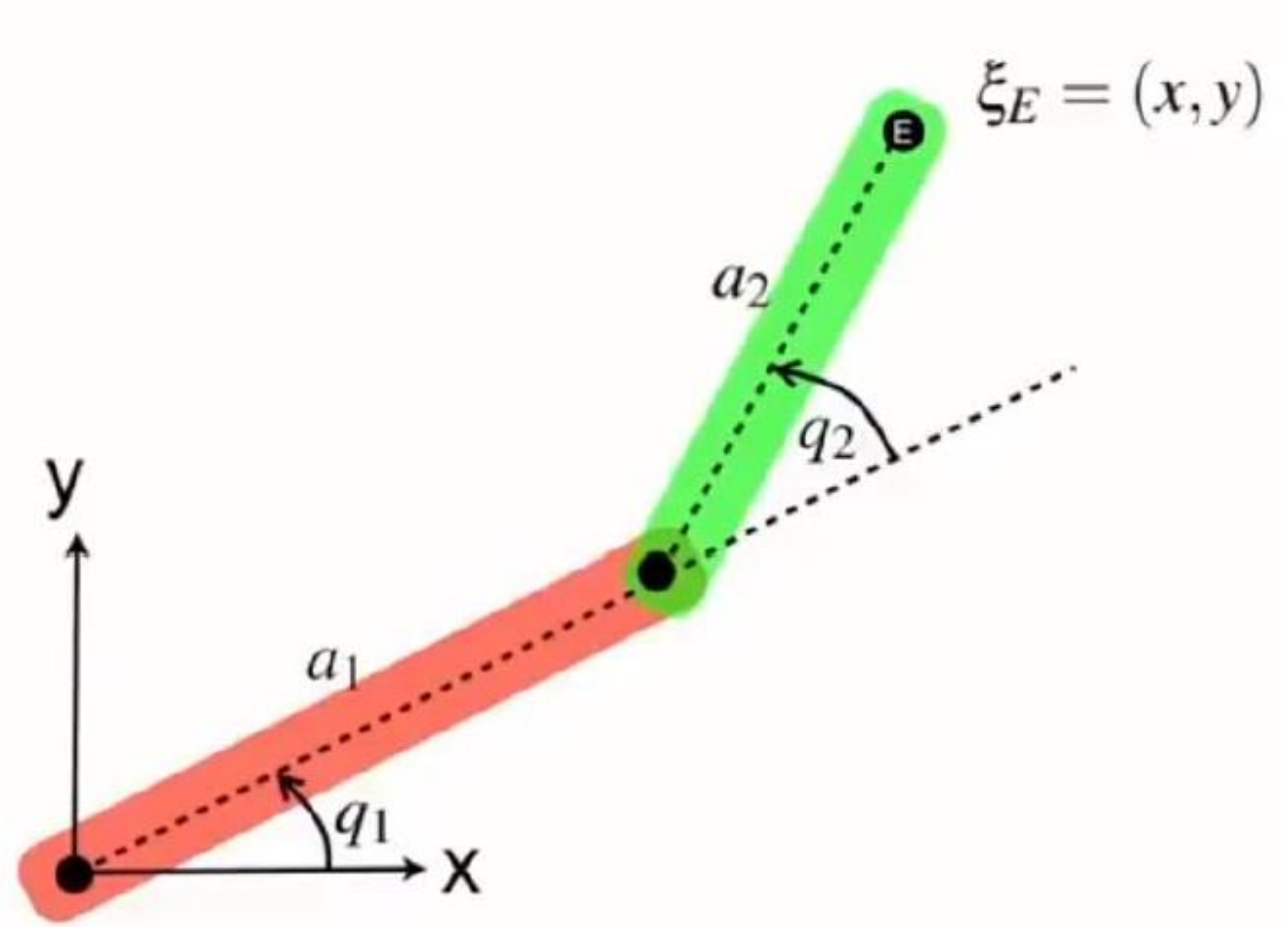
- What joint angles to set to achieve a certain end-effector pose.

$$\xi_N = \mathcal{K}(\boldsymbol{q})$$

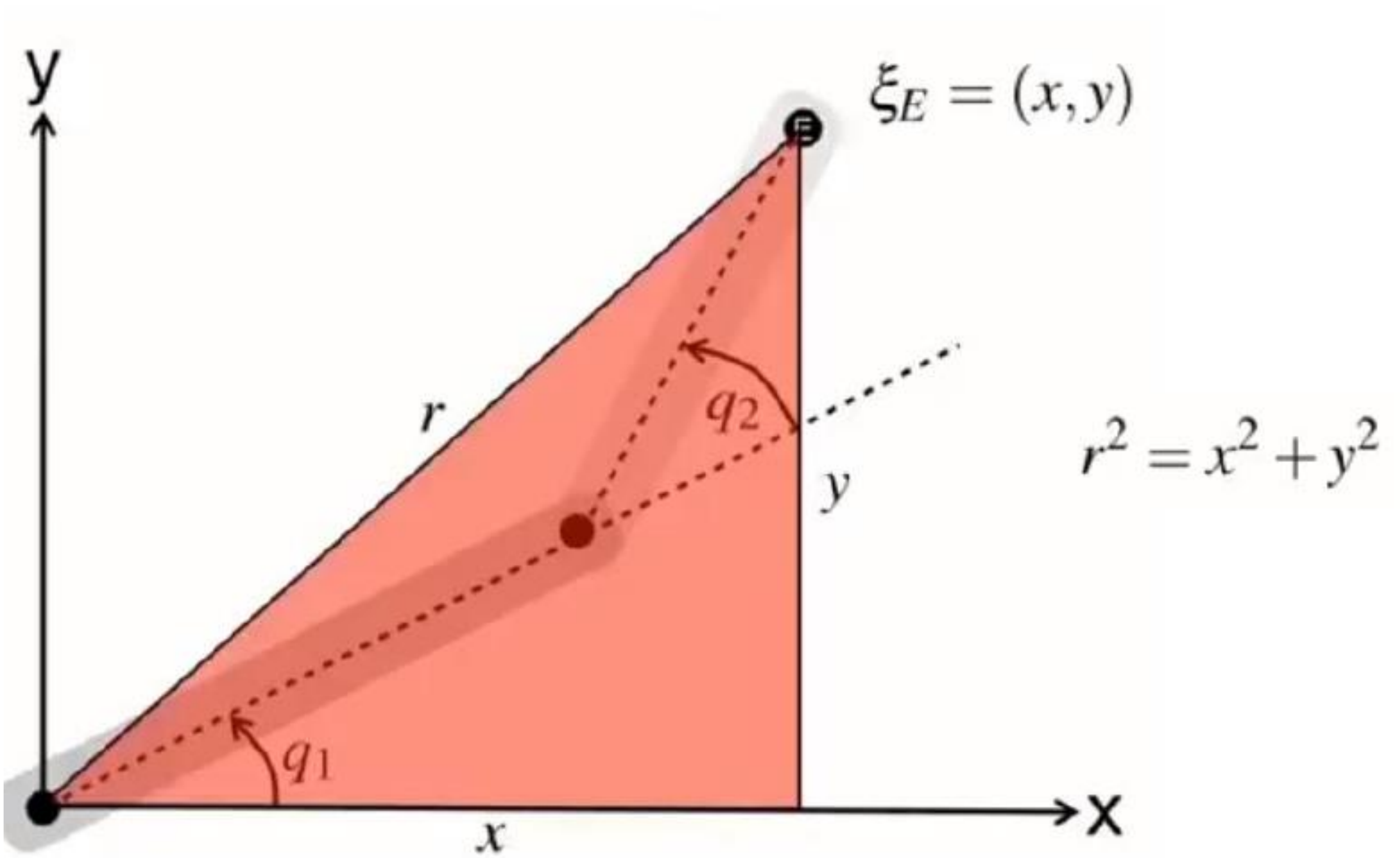
$$\boldsymbol{q} = \{q_j, j \in [1 \ldots N]\}$$

$$\boldsymbol{q} = \mathcal{K}^{-1}(\xi_N)$$

IK for 2 Joint Arm- Geometric Approach



IK for 2 Joint Arm- Geometric Approach



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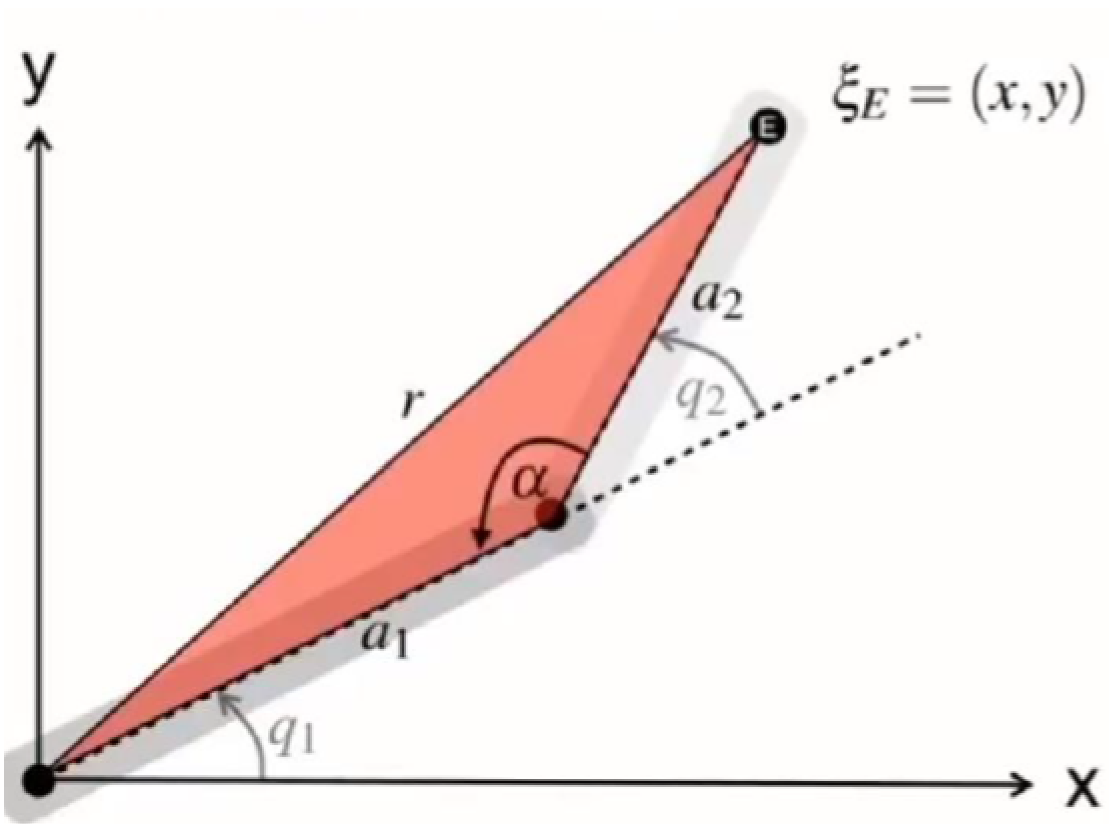
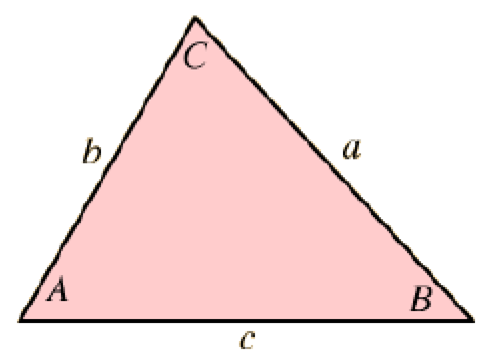


***We need to find angle alpha so we apply cosine rule**

$$c^2 = a^2 + b^2 - 2ab\cos C$$



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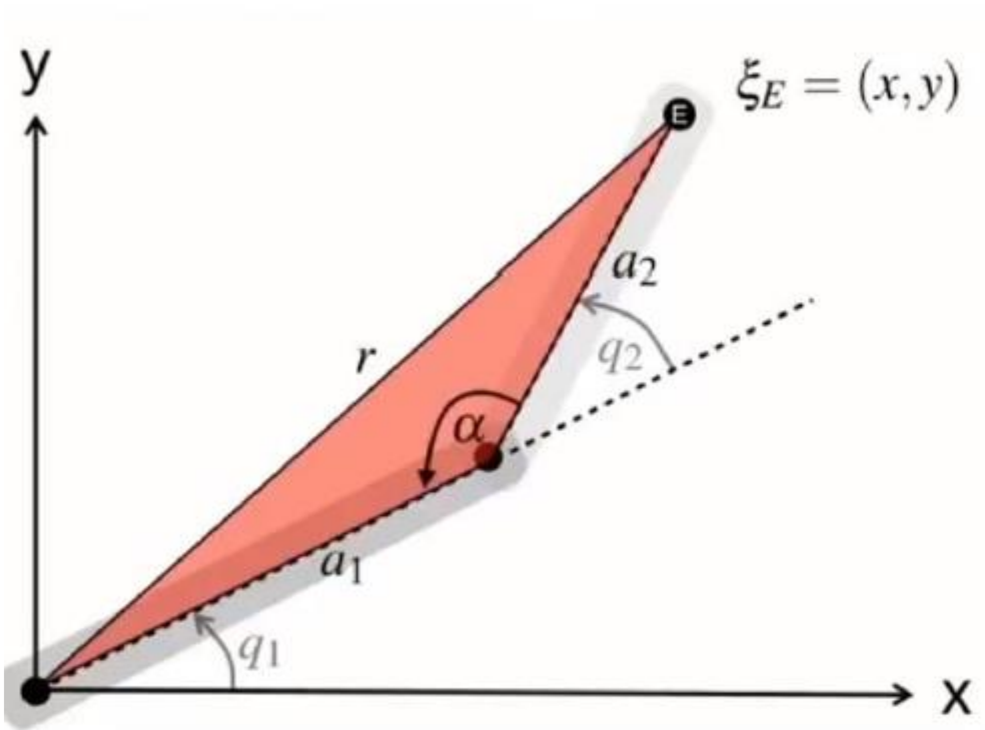


$$r^2 = x^2 + y^2$$

$$r^2 = a_1^2 + a_2^2 - 2a_1 a_2 \cos\alpha$$

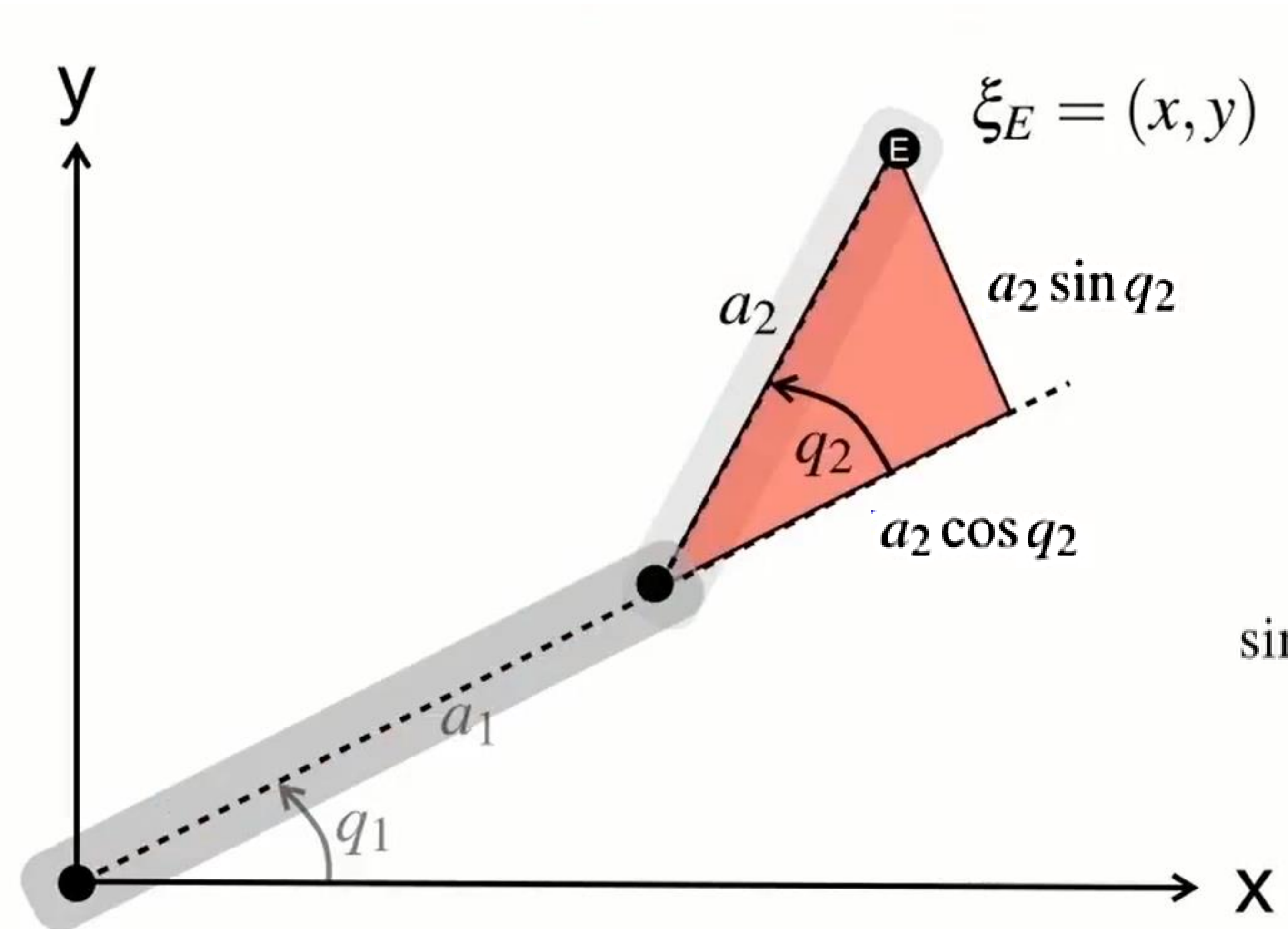
$$\cos\alpha = \frac{a_1^2 + a_2^2 - r^2}{2a_1 a_2} = \frac{a_1^2 + a_2^2 - x^2 - y^2}{2a_1 a_2}$$

$$q_2 = \pi - \alpha$$

$$\cos q_2 = -\cos\alpha$$

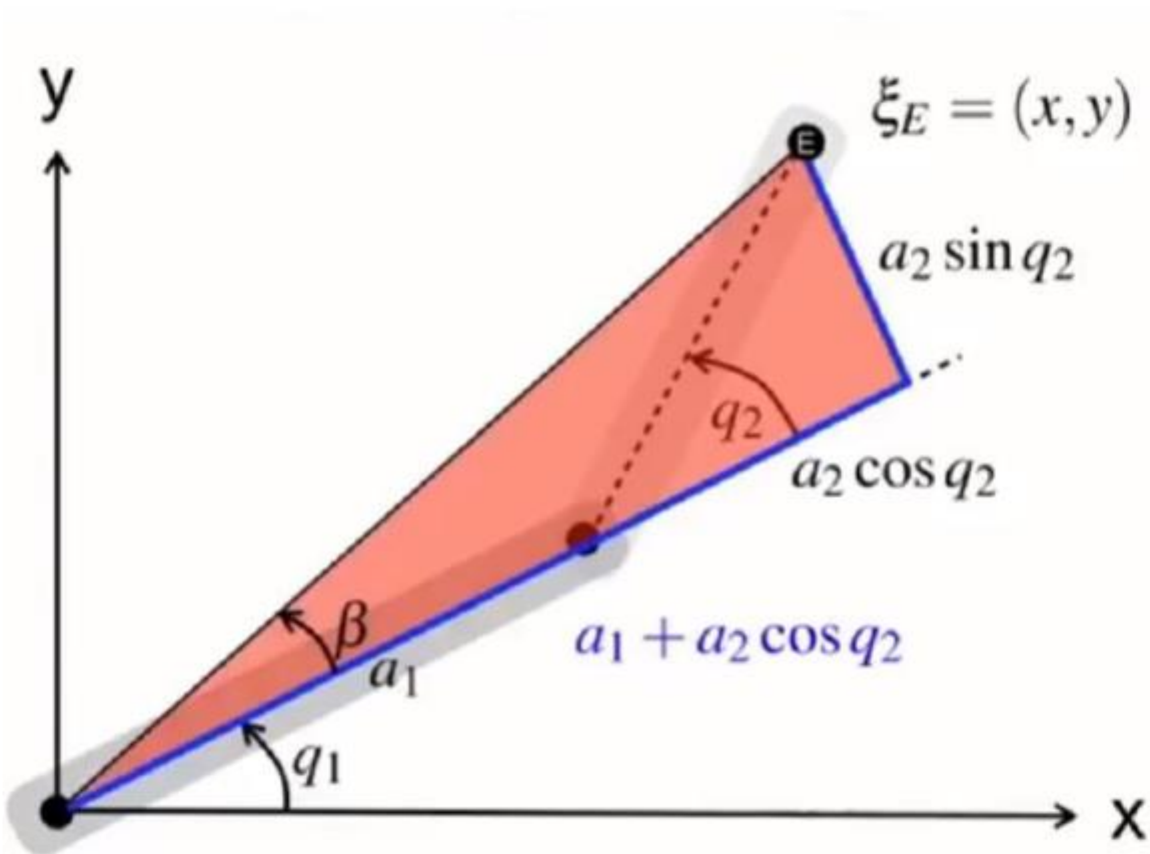
$$\cos q_2 = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

IK for 2 Joint Arm- Geometric Approach



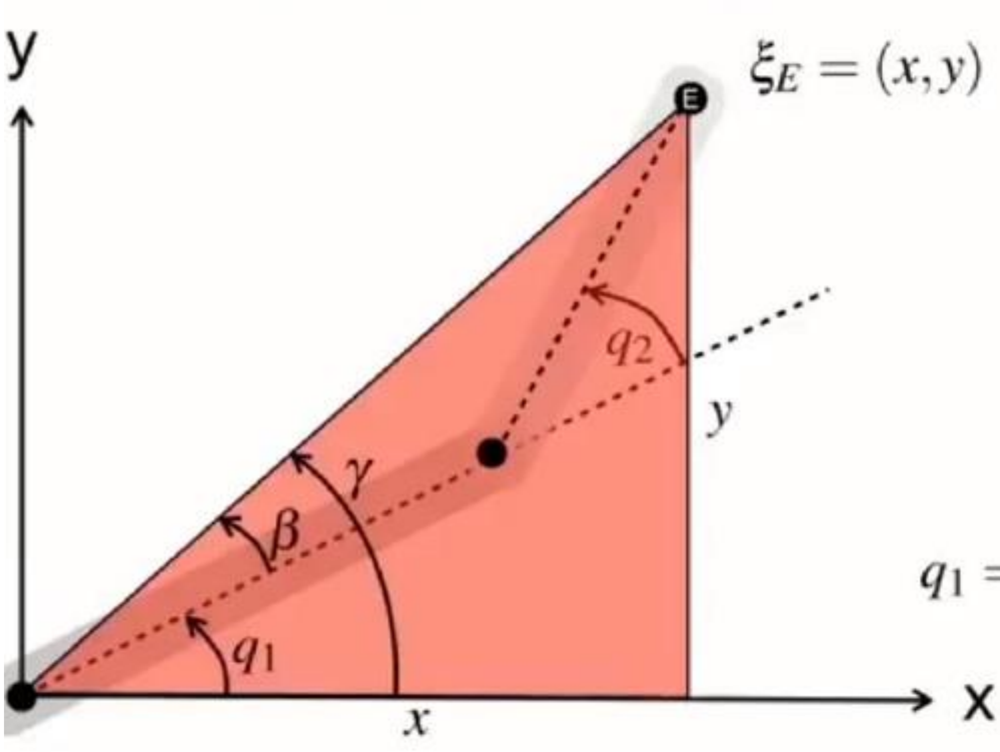
$$\sin q_2 = \sqrt{1 - \cos q_2^2}$$

IK for 2 Joint Arm- Geometric Approach



$$\beta = \tan^{-1} \frac{a_2 \sin q_2}{a_1 + a_2 \cos q_2}$$

IK for 2 Joint Arm- Geometric Approach

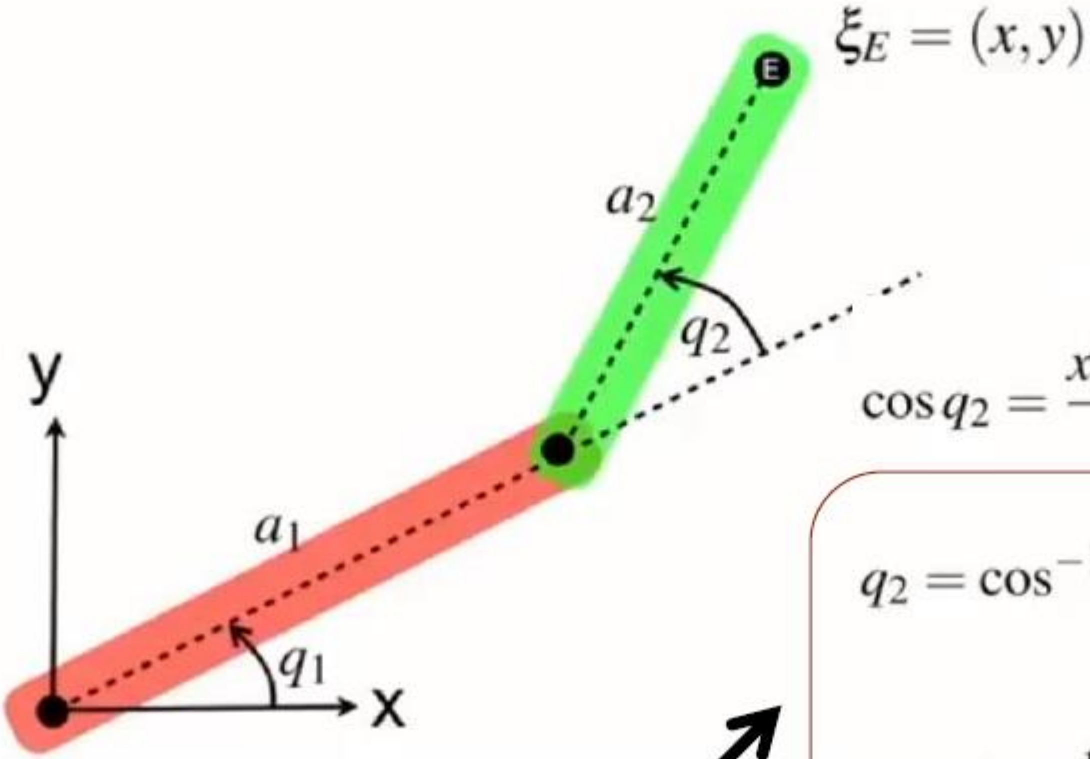


$$\gamma = \tan^{-1}\frac{y}{x}$$

$$q_1 = \gamma - \beta$$

$$q_1 = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{a_2 \sin q_2}{a_1 + a_2 \cos q_2}$$

IK for 2 Joint Arm- Geometric Approach



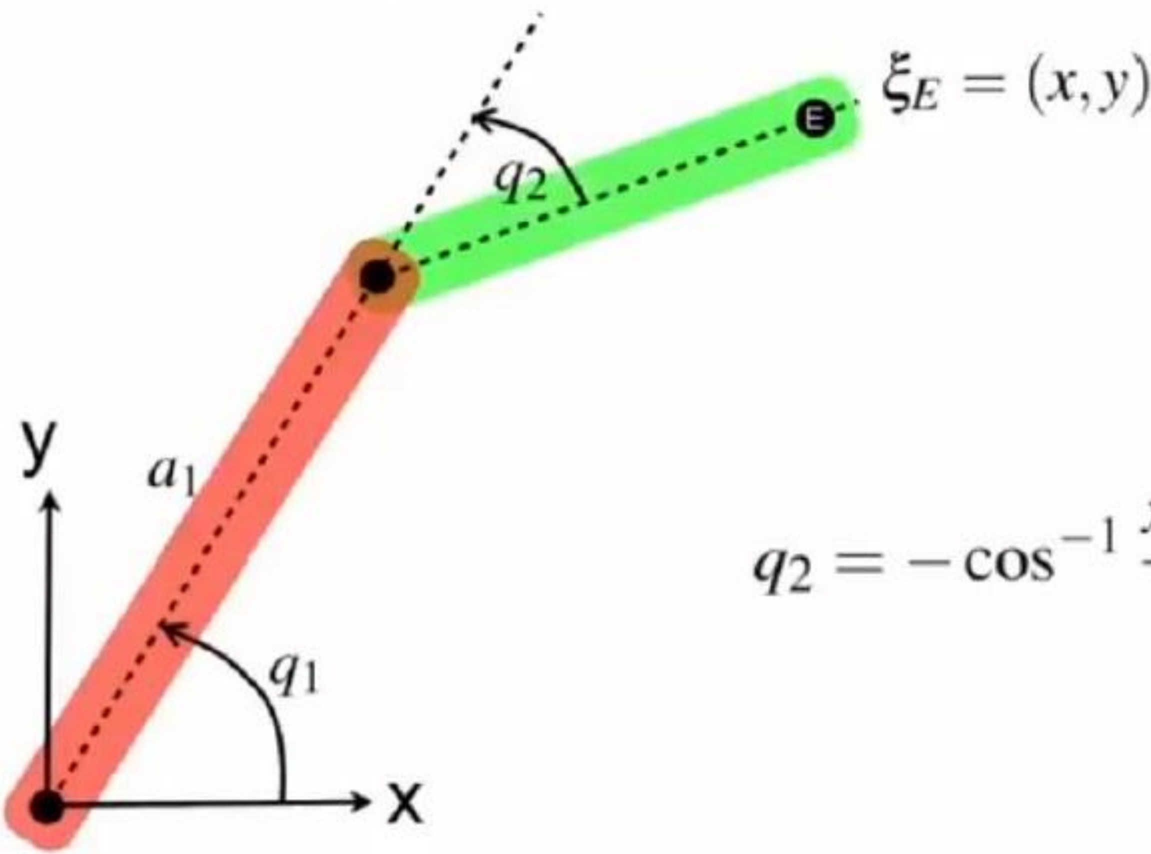
$$\cos q_2 = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2}$$

Inverse Kinematic Solution

$$q_2 = \cos^{-1} \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2}$$

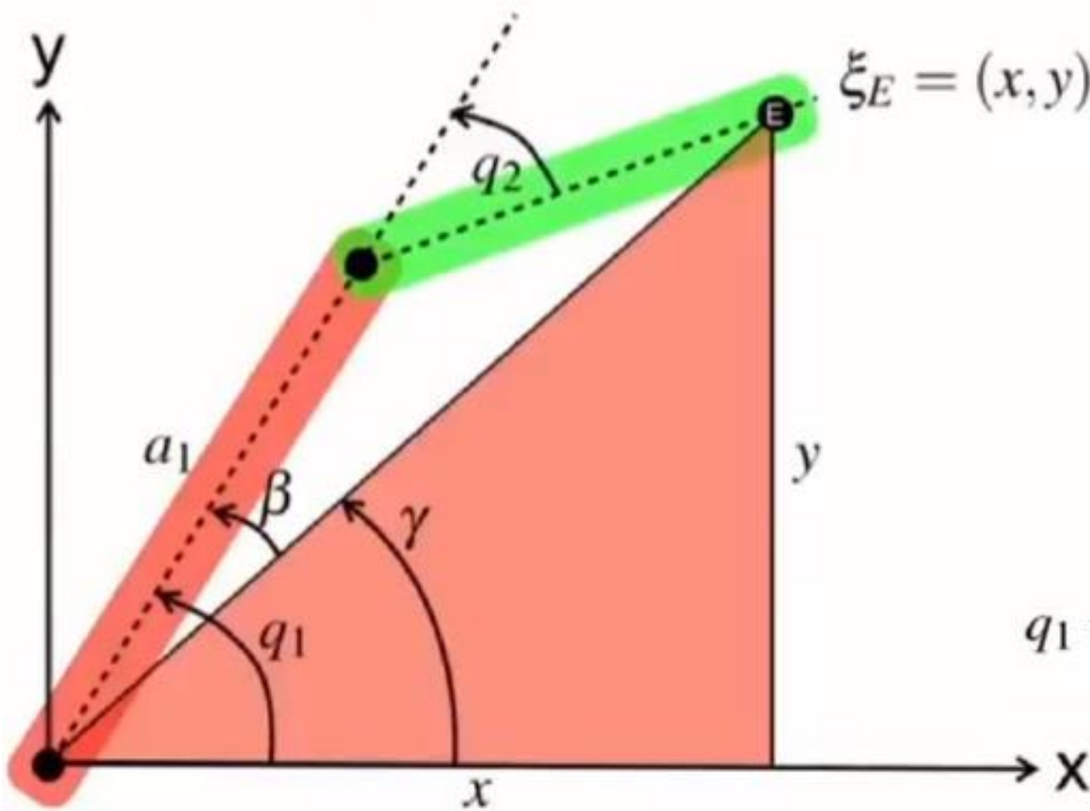
$$q_1 = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{a_2 \sin q_2}{a_1 + a_2 \cos q_2}$$

IK for 2 Joint Arm- Geometric Approach



$$q_2 = -\cos^{-1} \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

IK for 2 Joint Arm- Geometric Approach

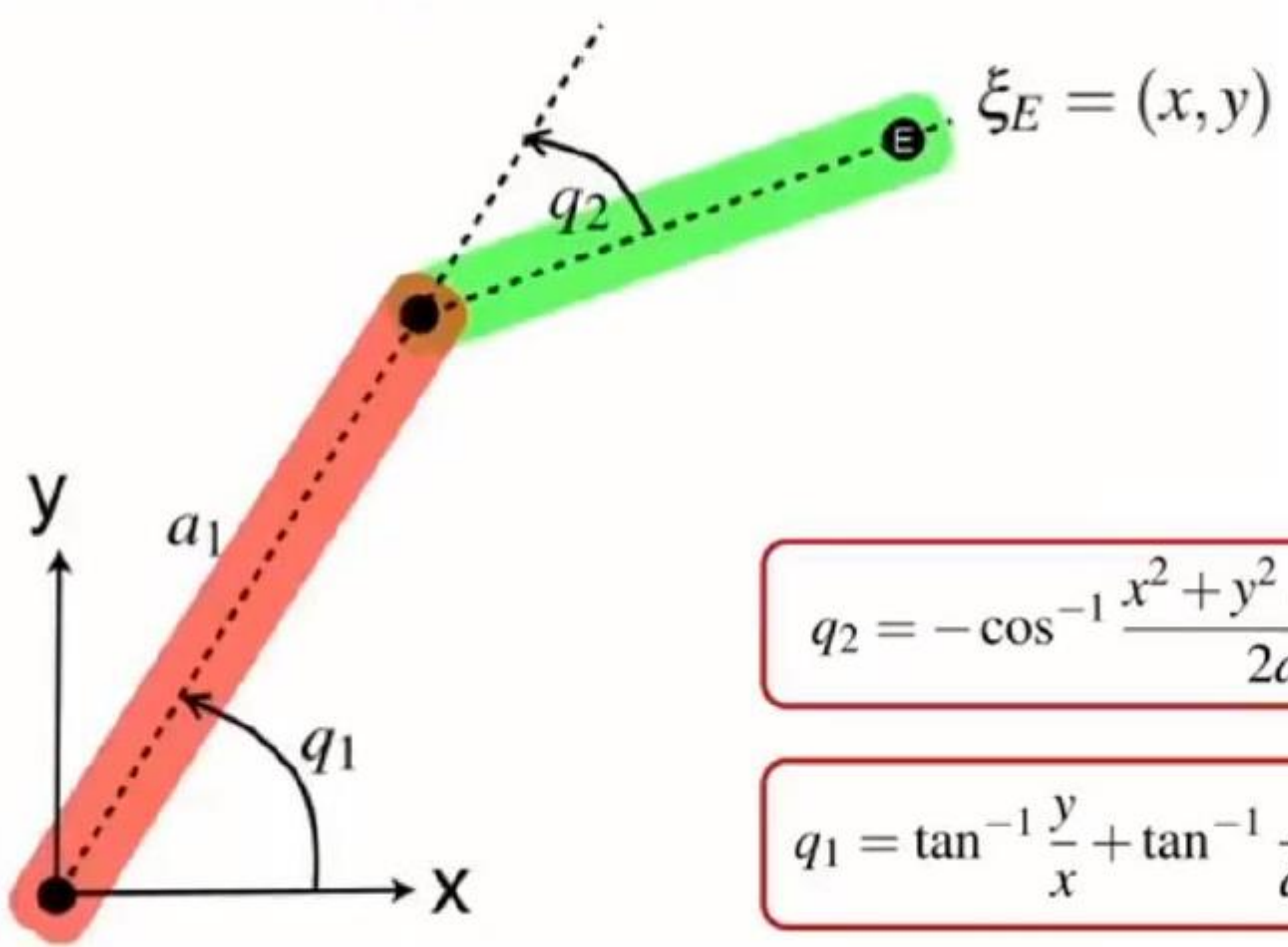


$$\gamma = \tan^{-1} \frac{y}{x}$$

$$q_1 = \gamma + \beta$$

$$q_1 = \tan^{-1} \frac{y}{x} + \tan^{-1} \frac{a_2 \sin q_2}{a_1 + a_2 \cos q_2}$$

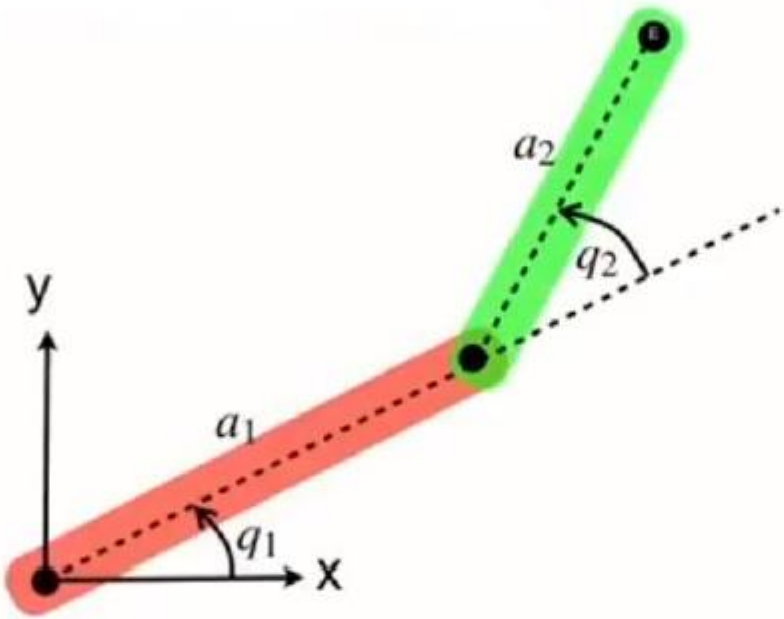
IK for 2 Joint Arm- Geometric Approach



$$q_2 = -\cos^{-1}\frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

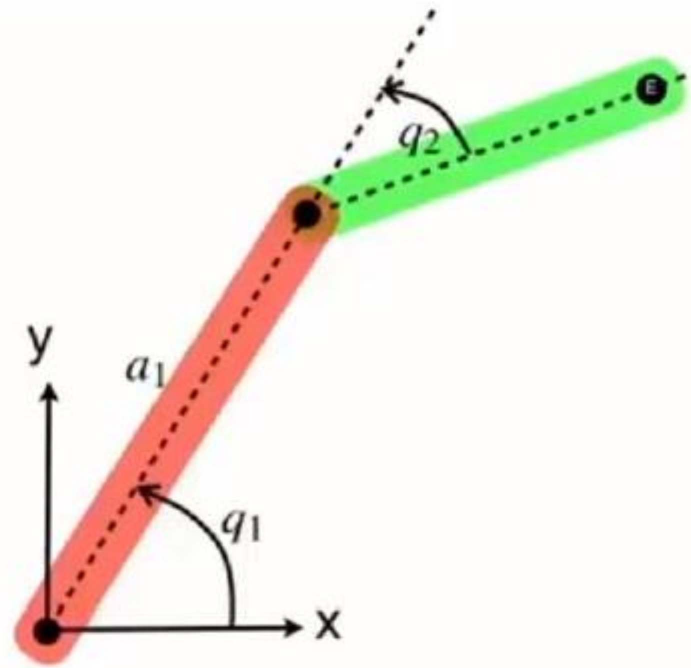
$$q_1 = \tan^{-1}\frac{y}{x} + \tan^{-1}\frac{a_2 \sin q_2}{a_1 + a_2 \cos q_2}$$

IK for 2 Joint Arm- Geometric Approach



$$q_2 = \cos^{-1}\frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2}$$

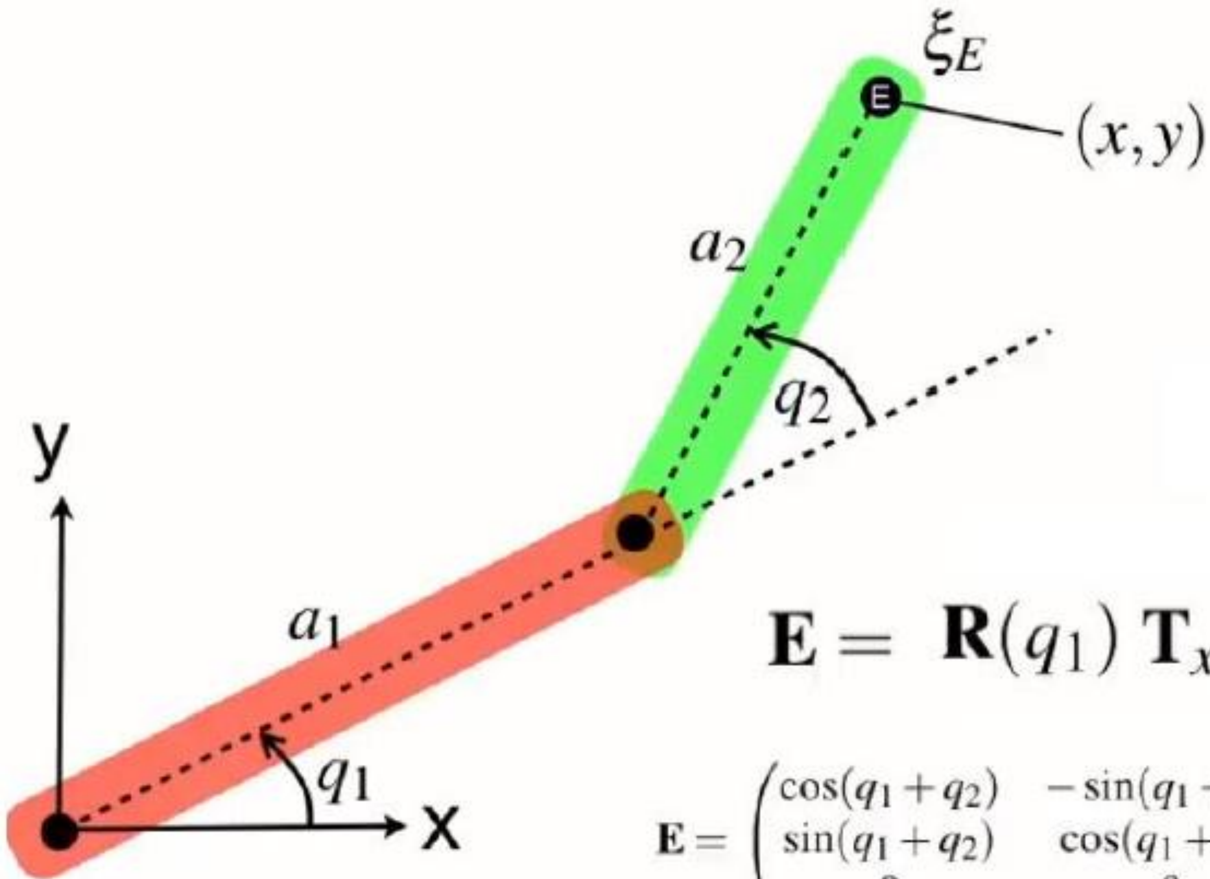
$$q_1 = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{a_2\sin q_2}{a_1+a_2\cos q_2}$$



$$q_2 = -\cos^{-1}\frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2}$$

$$q_1 = \tan^{-1}\frac{y}{x} + \tan^{-1}\frac{a_2\sin q_2}{a_1+a_2\cos q_2}$$

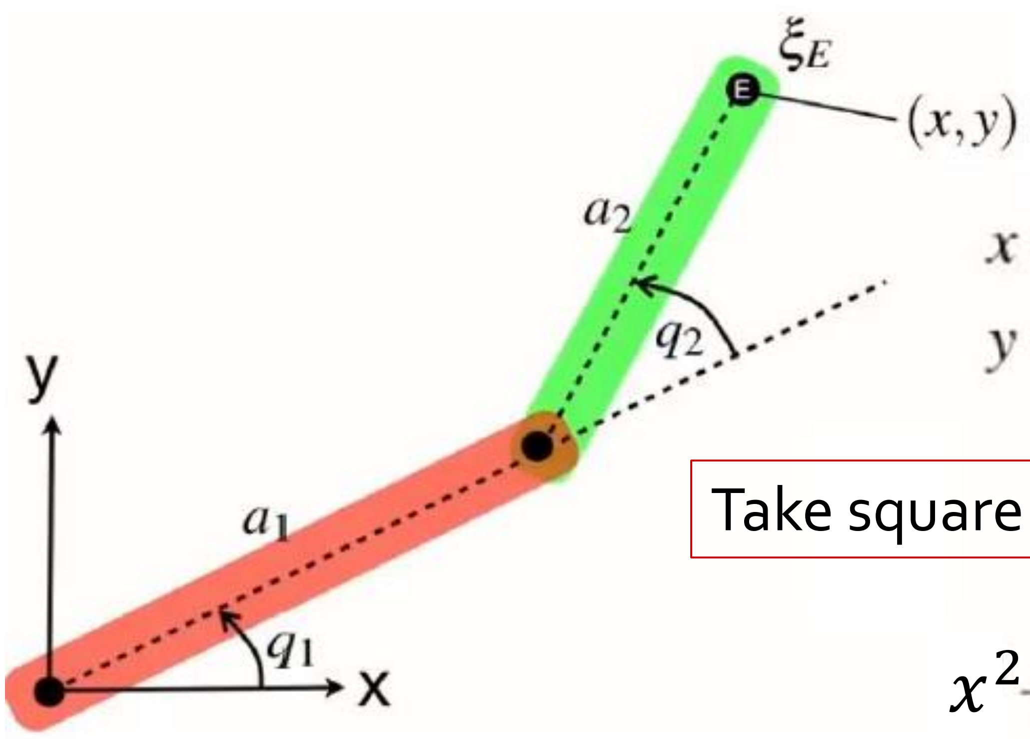
IK for 2 Joint Arm- **Analytic Approach**



$$\mathbf{E} = \mathbf{R}(q_1)\,\mathbf{T}_x(a_1)\,\mathbf{R}(q_2)\,\mathbf{T}_x(a_2)$$

$$\mathbf{E} = \begin{pmatrix} \cos(q_1+q_2) & -\sin(q_1+q_2) & a_2\cos(q_1+q_2)+a_1\cos q_1 \\ \sin(q_1+q_2) & \cos(q_1+q_2) & a_2\sin(q_1+q_2)+a_1\sin q_1 \\ 0 & 0 & 1 \end{pmatrix}$$

IK for 2 Joint Arm- Analytical Approach



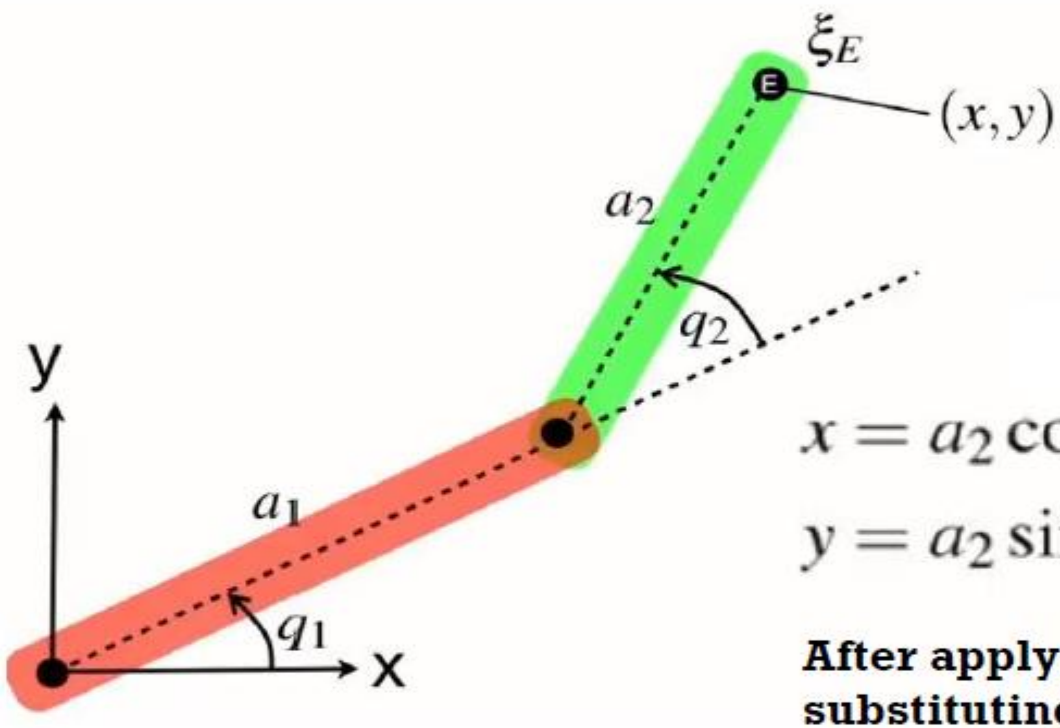
$$x = a_2 \cos(q_1 + q_2) + a_1 \cos q_1$$
$$y = a_2 \sin(q_1 + q_2) + a_1 \sin q_1$$

Take square and then add

$$x^2 + y^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos q_2$$

$$\cos q_2 = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

IK for 2 Joint Arm- Analytical Approach

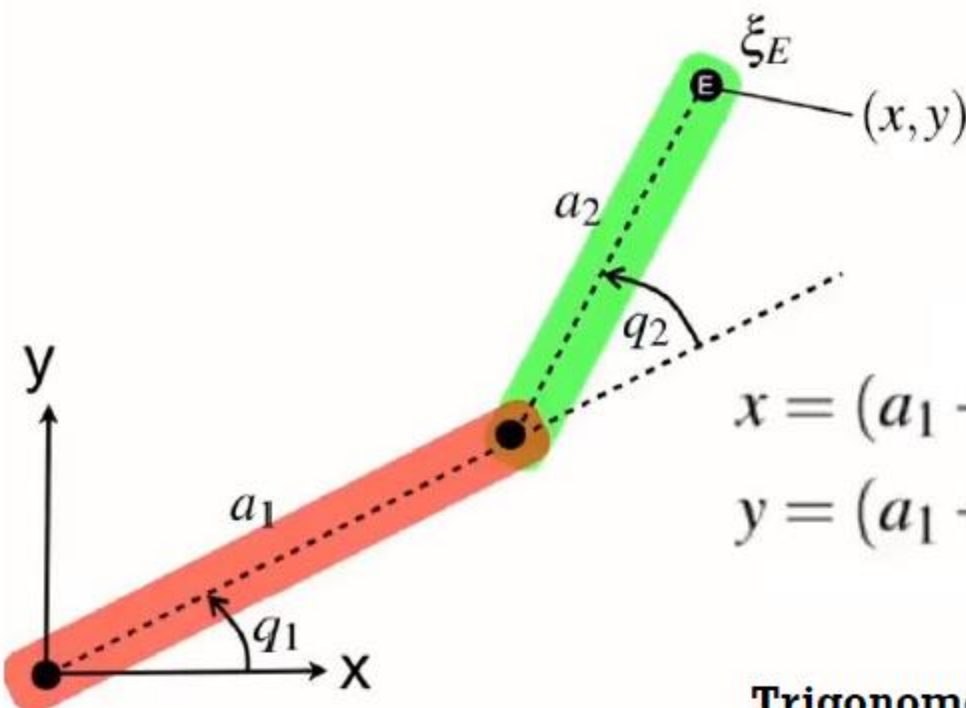


$$x = a_2 \cos(q_1 + q_2) + a_1 \cos q_1$$
$$y = a_2 \sin(q_1 + q_2) + a_1 \sin q_1$$

After applying sum of a angle identities and substituting cosq2 by C2 and sinq2 by S2

$$x = (a_1 + a_2 C_2) \cos q_1 - a_2 S_2 \sin q_1$$
$$y = (a_1 + a_2 C_2) \sin q_1 + a_2 S_2 \cos q_1$$

IK for 2 Joint Arm- Analytical Approach

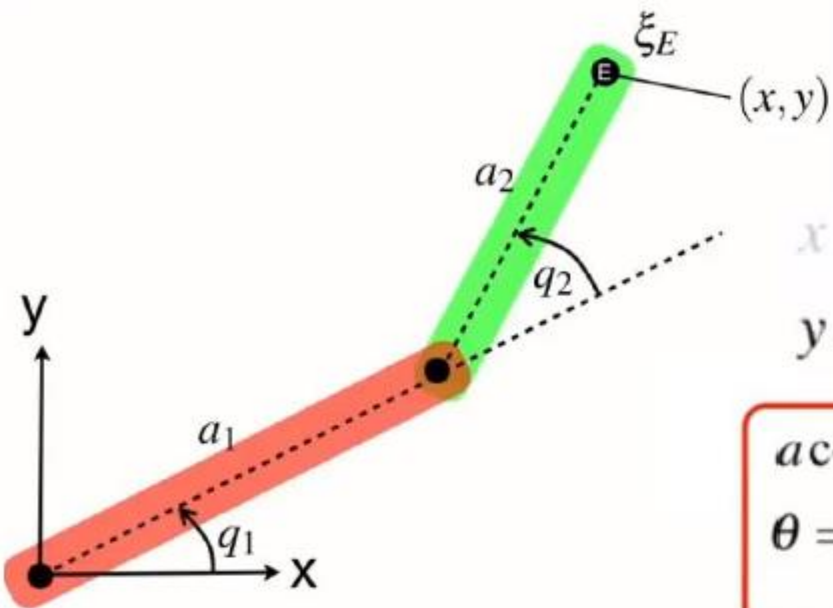


$$x = (a_1 + a_2 C_2)\cos q_1 - a_2 S_2 \sin q_1$$
$$y = (a_1 + a_2 C_2)\sin q_1 + a_2 S_2 \cos q_1$$

Trigonometric Solution:

$$a\cos\theta + b\sin\theta = c$$
$$\theta = \tan^{-1}\frac{c}{\pm\sqrt{a^2 + b^2 - c^2}} - \tan^{-1}\frac{a}{b}$$

IK for 2 Joint Arm- **Analytical Approach**



$$x = (a_1 + a_2 C_2)\cos q_1 - a_2 S_2 \sin q_1$$

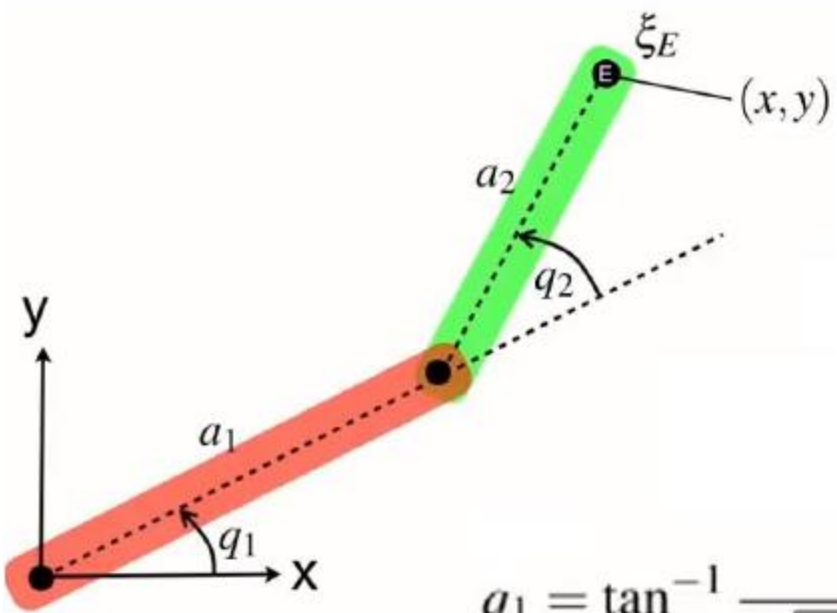
$$y = (a_1 + a_2 C_2)\sin q_1 + a_2 S_2 \cos q_1$$

$$\boxed{\begin{aligned} &a\cos\theta + b\sin\theta = c \\ &\theta = \tan^{-1}\frac{c}{\pm\sqrt{a^2+b^2-c^2}} - \tan^{-1}\frac{a}{b} \end{aligned}}$$

$$a = a_2 S_2,\, b = a_1 + a_2 C_2,\, c = y$$

$$q_1 = \tan^{-1}\frac{y}{\sqrt{a_1^2 + a_2^2 + 2a_1 a_2 C_2 - y^2}} - \tan^{-1}\frac{a_2 S_2}{a_1 + a_2 C_2}$$

IK for 2 Joint Arm- Analytical Approach



$$q_1 = \tan^{-1} \frac{y}{\sqrt{a_1^2 + a_2^2 + 2a_1 a_2 C_2 - y^2}} - \tan^{-1} \frac{a_2 S_2}{a_1 + a_2 C_2}$$

$$x^2 + y^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos q_2$$

$$q_1 = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{a_2 S_2}{a_1 + a_2 C_2}$$