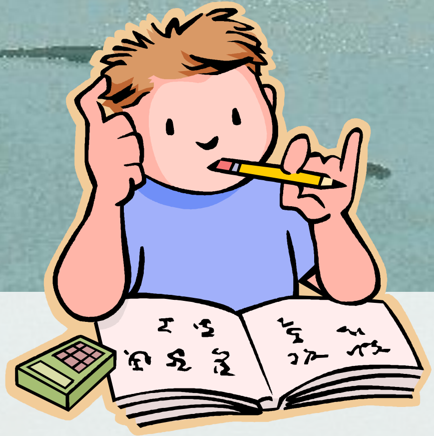
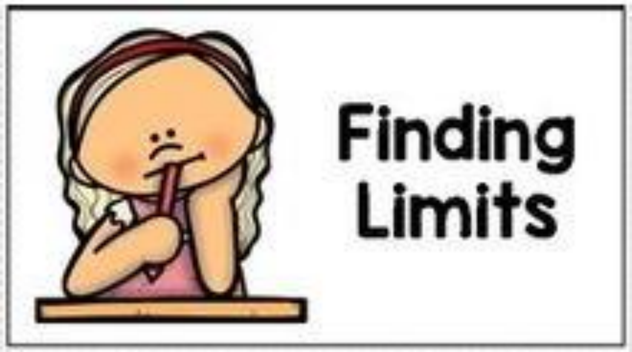


Limits



Calculus & Analytical Geometry
MATH- 101

Instructor: Dr. Naila Amir
(SEECS, NUST)

- An Introduction To Limits
- One-Sided Limits
- Laws for Calculating Limits
- **Limits Involving Infinity**
  - **Infinity as a Limit**
  - **Limit at infinity**

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 2
    - Sections: 2.5

Asymptotes

- An **asymptote** of a curve is a line or curve to which the curve converges. In other words, the curve and its asymptote get infinitely close, but they never meet at infinity.
- In most cases, the asymptote(s) of a curve can be found by taking the limit of a value where the function is not defined.
- Asymptotes are generally straight lines, unless mentioned otherwise.
- Asymptotes can be broadly classified into three categories: horizontal, vertical and oblique.

Vertical Asymptotes

- If $f(x)$ approaches infinity (or negative infinity) as x approaches a from the left or the right, then the line $x = a$ is a vertical asymptote of the graph of f.
- A function may have any number of vertical asymptotes.

Horizontal Asymptotes

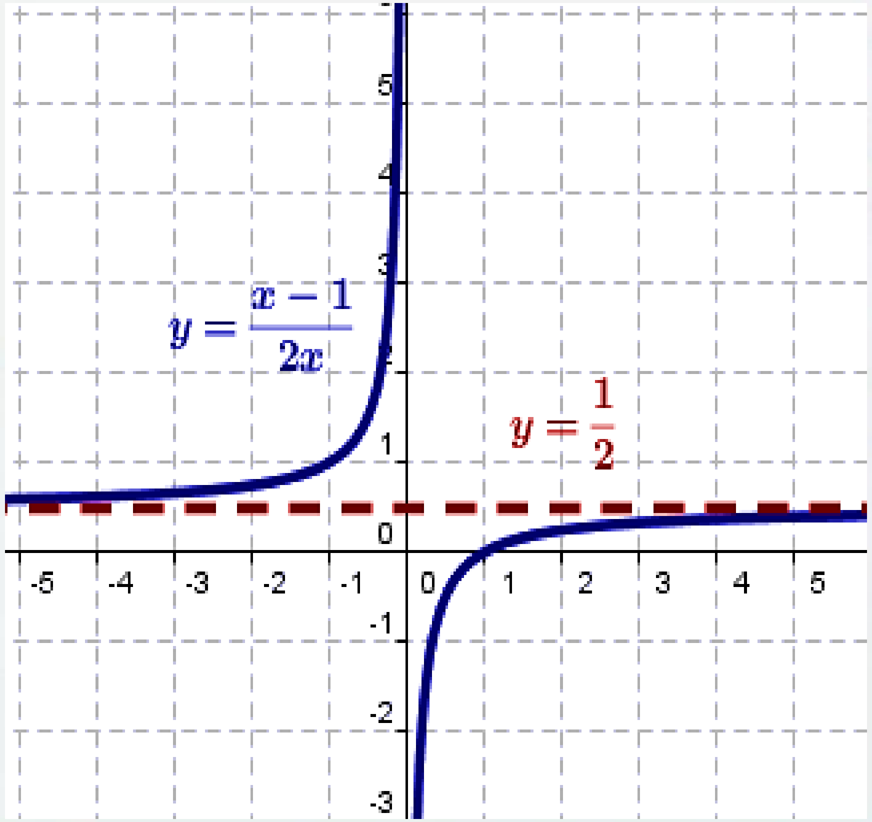
- The line $y = L$ is a horizontal asymptote of $f(x)$ if

$$\lim_{x\to\infty} f(x) = L$$

or

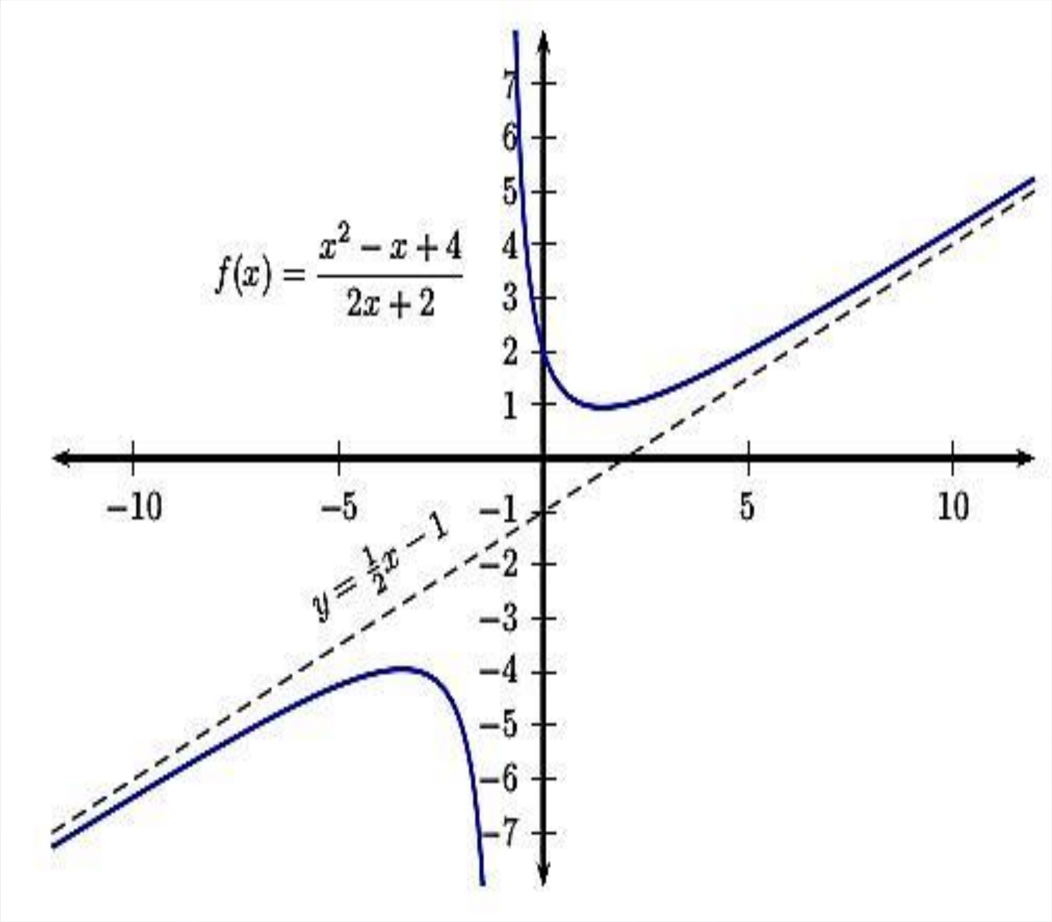
$$\lim_{x\to-\infty} f(x) = L$$

- Notice that a function can have at most two HORIZONTAL asymptotes (Why?)



Oblique Asymptotes

If the degree of the numerator of a rational function is one greater than the degree of the denominator, the graph has an **oblique (slanted) asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \infty$



Example: Determine

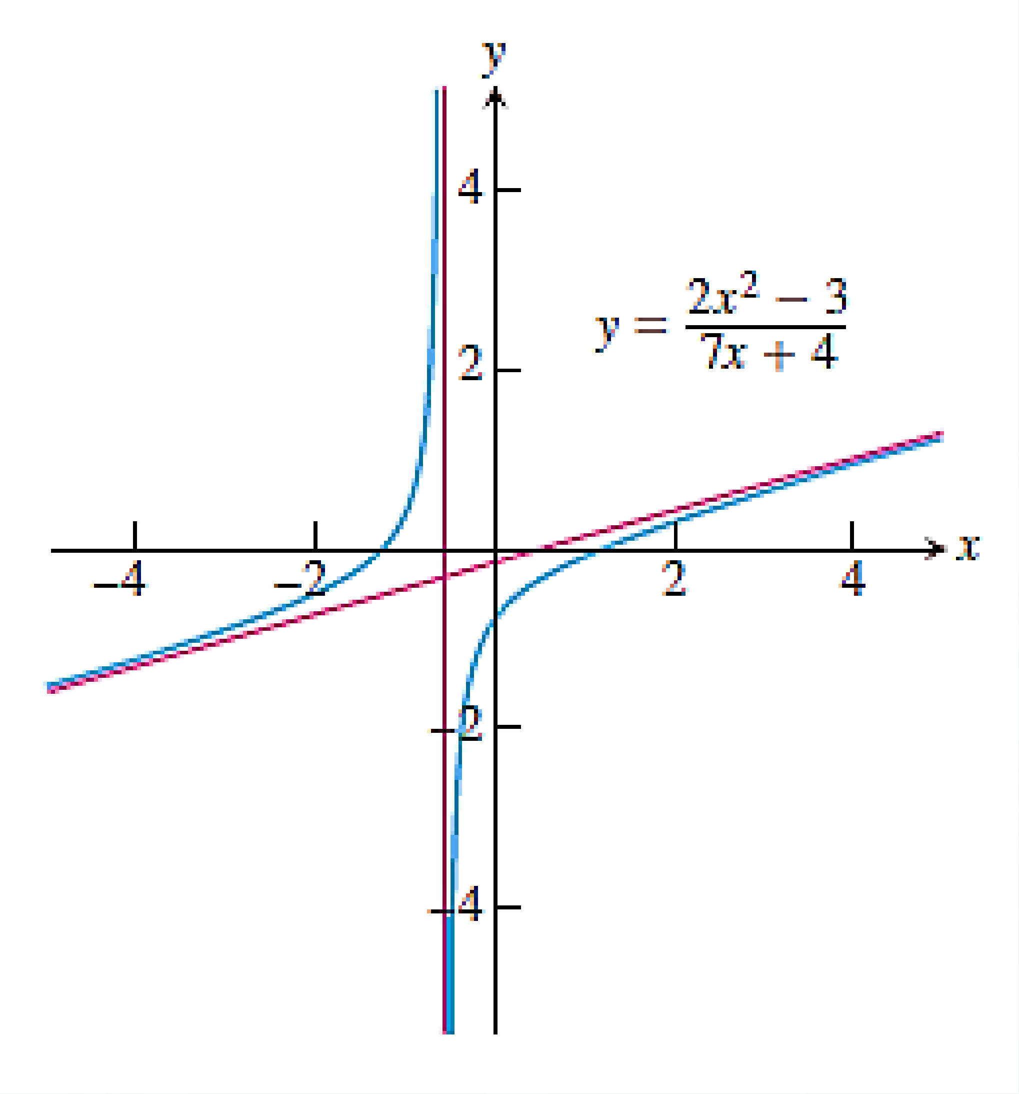
$$\lim_{x\to\infty}\frac{2x^2-3}{7x+4}.$$

Solution:

$$\lim_{x\to\infty}\frac{2x^2-3}{7x+4}$$

$$=\lim_{x\to\infty}\left[\left(\frac{2}{7}x-\frac{8}{49}\right)+\left(\frac{-115}{49(7x+4)}\right)\right]$$

$$=\infty.$$



$y=\frac{2}{7}x-\frac{8}{49}$ is an oblique asymptote.

Example: Evaluate $\lim_{x\to\infty}\left(\dfrac{4x^2-5x+21}{7x^3+5x^2-10x+1}\right)$

Solution: $$\lim_{x\to\infty}\left(\frac{4x^2-5x+21}{7x^3+5x^2-10x+1}\right)=\lim_{x\to\infty}\left(\frac{\frac{4x^2}{x^3}-\frac{5x}{x^3}+\frac{21}{x^3}}{\frac{7x^3}{x^3}+\frac{5x^2}{x^3}-\frac{10x}{x^3}+\frac{1}{x^3}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{4}{x}-\frac{5}{x^2}+\frac{21}{x^3}}{7+\frac{5}{x}-\frac{10}{x^2}+\frac{1}{x^3}}\right)$$

$$=\frac{0}{7}=0.$$

$y=0$ is horizontal asymptote.

Example: Evaluate $\lim_{x\to\infty}\left(\frac{x^2+2x-4}{12x+31}\right)$. Moreover, determine Vertical/horizontal/oblique asymptotes (if any).

Solution:

$$\lim_{x\to\infty}\left(\frac{x^2+2x-4}{12x+31}\right)=\lim_{x\to\infty}\left(\frac{\frac{x^2}{x}+\frac{2x}{x}-\frac{4}{x}}{\frac{12x}{x}+\frac{31}{x}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{x+2-\frac{4}{x}}{12+\frac{31}{x}}\right)$$

$$=\frac{\infty+2}{12}=\infty$$

Solution: By definition, the line $y = L$ is a horizontal asymptote of $f(x)$ if

$$\lim_{x\to\infty} f(x) = L.$$

But for the present case

$$\lim_{x\to\infty} f(x) = \infty.$$

Thus, for the given function no horizontal asymptote exist.

In order to determine oblique asymptotes we are going to perform long division and rewrite the given function as:

$$\frac{x^2+2x-4}{12x+31} = \left(\frac{1}{12}x - \frac{7}{144}\right) + \left(\frac{-359/144}{12x+31}\right)$$

Note that:

$$\lim_{x\to\infty}\left(\frac{x^2+2x-4}{12x+31}\right) = \lim_{x\to\infty}\left[\left(\frac{1}{12}x - \frac{7}{144}\right) + \left(\frac{-359/144}{12x+31}\right)\right] = \infty$$

Here $y = \frac{1}{12}x - \frac{7}{144}$ is an oblique asymptote for the given function.

Solution: By definition if $f(x)$ approaches infinity (or negative infinity) as x approaches a from the left or the right, then the line $x = a$ is a vertical asymptote of the graph of f. In other words, vertical asymptote can be determined by considering those points where the function $f(x)$ is undefined.

For the present case:

$$f(x) = \frac{x^2 + 2x - 4}{12x + 31},$$

is undefined at $x = -\frac{31}{12}$. Thus, $x = -\frac{31}{12}$ is vertical asymptote for the given function.

Example: Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ where

$$f(x)=5+\frac{10}{1+e^{-.25x}}.$$

Solution:

As $x\to\infty$ the values of $e^{-.25x}$ get

arbitrarily close to 0 so

$$\lim_{x\to\infty} f(x)=5+\frac{10}{1+0}=15.$$

As $x \to -\infty$ the values of $e^{-.25x}$ get

arbitrarily large so

$$\lim_{x \to -\infty} f(x) = 5 + \frac{10}{\infty} = 5 + 0 = 5.$$

Note: $y = 15$ and $y = 5$ are horizontal asymptotes.

Guidelines for Finding Limits at $\pm\infty$ of Rational Functions

1. If the degree of the numerator is __less than__ the degree of the denominator, then the limit of the rational function is __0__.
2. If the degree of the numerator is __equal to__ the degree of the denominator, then the limit of the rational function is the __the ratio of the leading coefficients__ .
3. If the degree of the numerator is __greater than__ the degree of the denominator, then the limit of the rational function __is infinite__.

Using the Guidelines…

$$\lim_{x\to\infty}\frac{x^3-2x+3}{x^4+3x^2+x} = 0$$

$$\lim_{x\to\infty}\frac{2x^3+9x+3}{x^3+7x^2+1} = 2$$

$$\lim_{x\to\infty}\frac{6x^8-12x-17}{18x^8-13x^2+24} = \frac{1}{3}$$

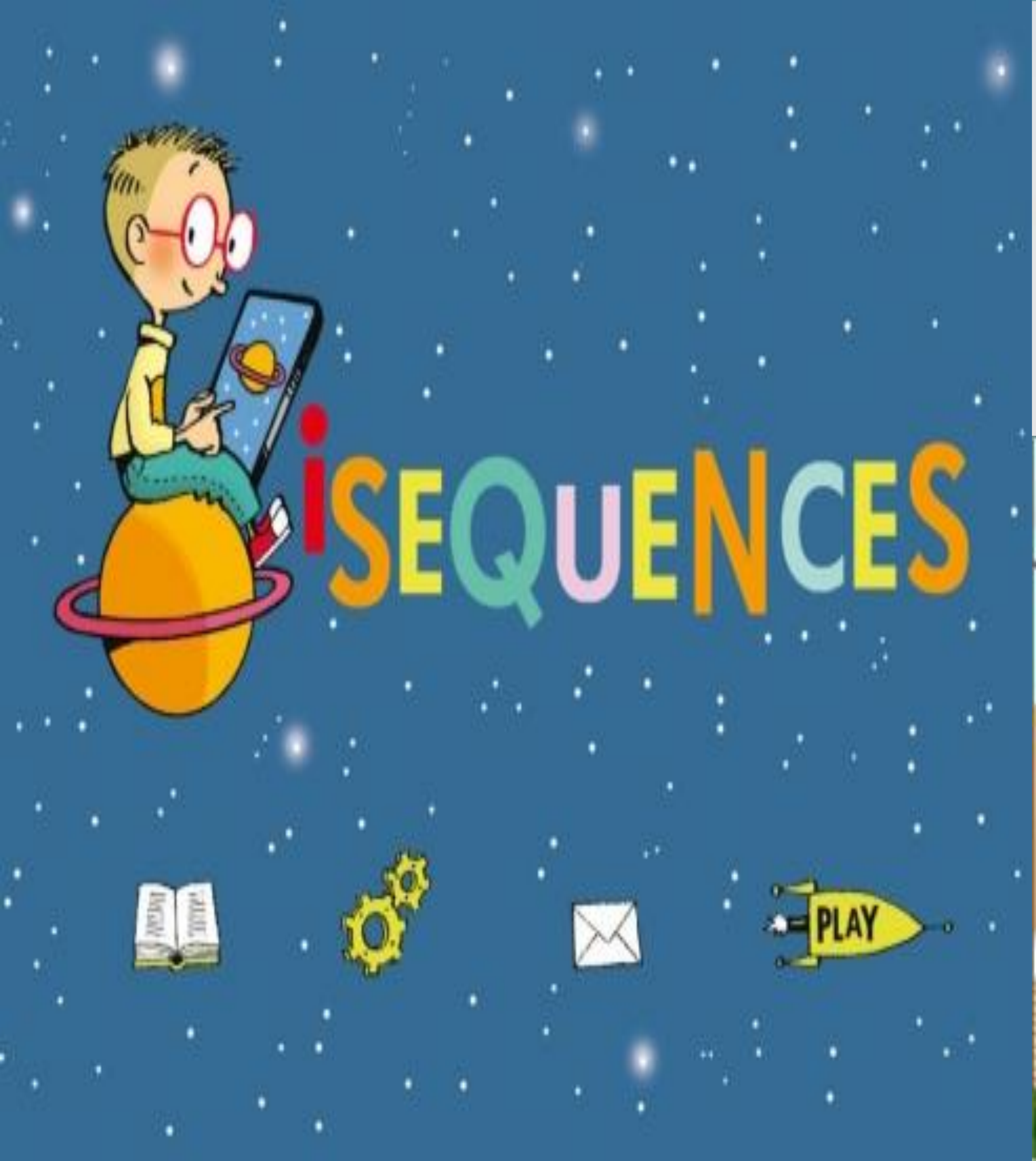
$$\lim_{x\to\infty}\frac{x^5-5x-6}{x^3+2x^2+8} = \infty$$

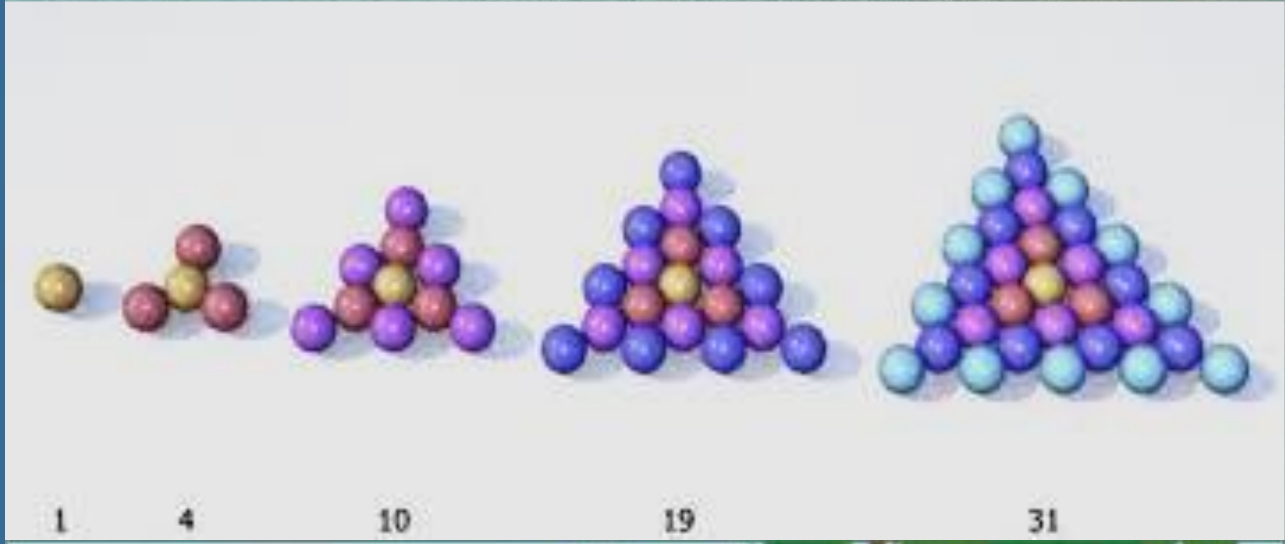
Practice Questions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 2
  - Exercise: 2.5

    Q # 1 – 26, 47 – 50


iSEQUENCES
PLAY

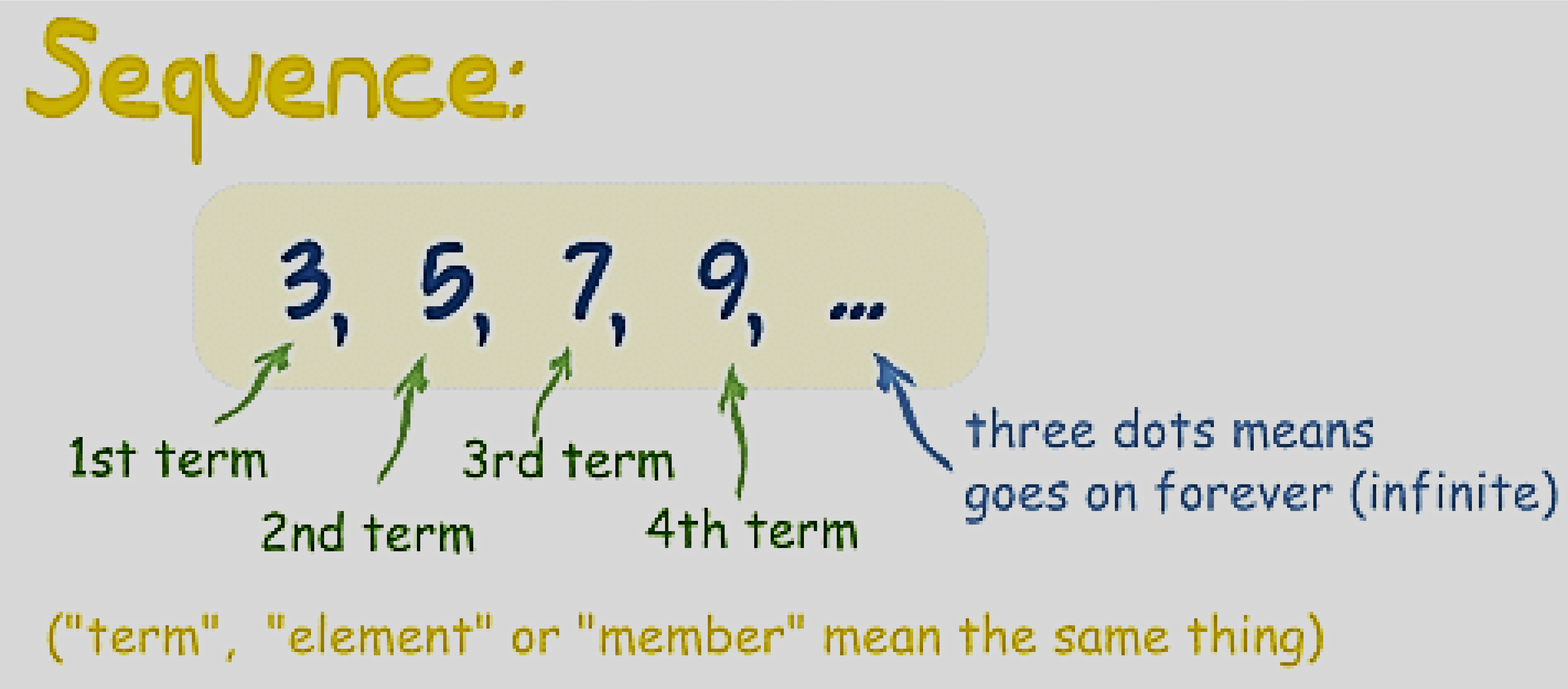

1
4
10
19
31


362
372
382
402
15
GO

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 11
  - Sections: 11.1

A sequence can be thought as a list of things (usually numbers) that are in a definite order.



Sequence as a function

- A real sequence is a function $a: \mathbb{N} \to \mathbb{R}$, whose domain is a subset of the natural numbers N and whose range is a subset of R.
- If Dom a is a finite subset of $\mathbb{N}$, the sequence is called a finite sequence otherwise it is said to be an infinite sequence.

- The values of the sequence $a\colon \mathbb{N} \to \mathbb{R}$, are usually written as

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

instead of $a(1), a(2), \ldots., a(n), \ldots$ at the points $1,2, \ldots, n, \ldots$ of its domain $\mathbb{N}$.

- Each of the following are equivalent ways of denoting a sequence.

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\},\ \ \{a_n\},\ \ \{a_n\}_{n=1}^{\infty}$$

Examples

1. $\{n\} = \{1,2,3,4,\ldots,n,\ldots\}$

2. $\left\{\dfrac{n}{n+1}\right\} = \left\{\dfrac{1}{2},\dfrac{2}{3},\dfrac{3}{4},\ldots,\dfrac{n}{n+1},\ldots\right\}$

3. $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\} = \left\{-\dfrac{2}{3},\dfrac{3}{9},-\dfrac{4}{27},\ldots,\dfrac{(-1)^n(n+1)}{3^n},\ldots\right\}$

4. $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty} = \left\{1,\ \dfrac{1}{2},\ \dfrac{1}{3},\ \dfrac{1}{4},\ \dfrac{1}{5},\ldots,\dfrac{1}{n},\ldots\right\}$

5. $\left\{\dfrac{n}{3^{n-1}}\right\}_{n=2}^{\infty} = \left\{\dfrac{2}{3},\ \dfrac{3}{9},\ \dfrac{4}{27},\ \dfrac{5}{81},\ldots,\dfrac{n}{3^{n-1}},\ldots\right\}$

6. $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty} = \left\{0,1,\sqrt{2},\sqrt{3},\ldots,\sqrt{n-3},\ldots\right\}$

Limit of a sequence

- A sequence $\{a_n\}$ has the limit L if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon, \quad whenever \;\; n > N$$

- We write

$$\lim_{n\to\infty} a_n = L \quad or \quad a_n \to L \; as \; n \to \infty$$

Convergence/Divergence

- If $\lim_{n\to\infty} a_n$ exists we say that the sequence **converges**.
  - Note that for the sequence to converge, the limit must be finite
- If the sequence does not converge we will say that it **diverges.**
  - Note that a sequence diverges if it approaches to infinity or if the sequence does not approach to anything

The limit laws

- If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and *c* is a constant, then

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n,$$

$$\lim_{n\to\infty}(c \cdot a_n) = c \cdot \lim_{n\to\infty} a_n,$$

$$\lim_{n\to\infty} c = c$$

The limit laws

$$\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \; if \;\; \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty}(a_n{}^p) = (\lim_{n\to\infty} a_n)^p, \; if \;\; p > 0 \; and \;\; a_n > 0$$

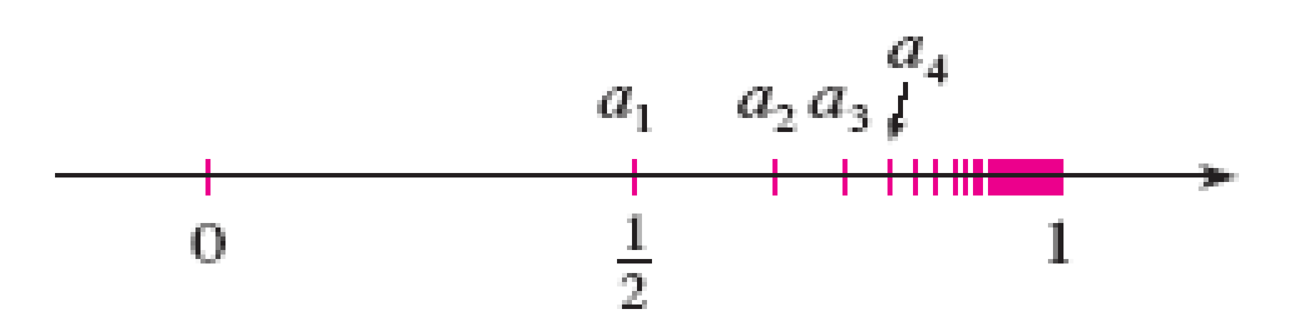
LIMIT OF THE SEQUENCE

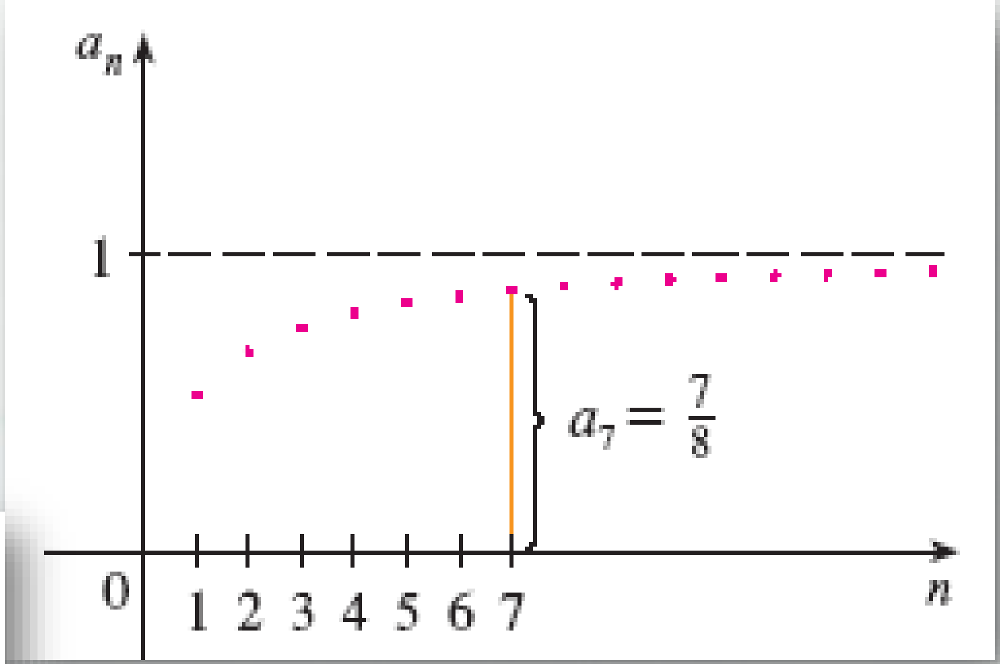
$$\left\{\frac{n}{n+1}\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots\right\}$$

as $n \to \infty$, $\quad a_n = \frac{n}{n+1} \to 1$

i.e., $\quad \lim_{n\to\infty} \frac{n}{n+1} = 1.$

The sequence $\{a_n\} = \left\{\frac{n}{n+1}\right\}$, converges to 1.





EXAMPLES

Let $a_n = \frac{n^2+n+1}{3n^2+1}$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+n+1}{3n^2+1}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{3+\frac{1}{n^2}}$$

$$= \frac{1}{3}.$$

EXAMPLES

Let $a_n = \frac{n^2+1}{n+1}$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{n+1}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{\frac{1}{n}+\frac{1}{n^2}}$$

$$= \infty.$$

EXAMPLES

(a) $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty}\frac{1}{n} = -1 \cdot 0 = 0$ Constant Multiple Rule

(b) $\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n} = 1 - 0 = 1$ Difference Rule

(c) $\lim_{n\to\infty}\frac{5}{n^2} = 5 \cdot \lim_{n\to\infty}\frac{1}{n} \cdot \lim_{n\to\infty}\frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$ Product Rule

(d) $\lim_{n\to\infty}\frac{4-7n^6}{n^6+3} = \lim_{n\to\infty}\frac{(4/n^6)-7}{1+(3/n^6)} = \frac{0-7}{1+0} = -7.$ Sum and Quotient Rules