Communication Systems
EE-351

Lectures 2 and 3

Signals and Systems

- **Signal** is set of information or data
- **System** is an entity that process a set of signals (inputs) to yield another set of signals (outputs)

Energy of a signal:

- Consider a signal x(t), energy of a signal is defined as:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Example:

$$x(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$E_x = \int_0^{\infty} |e^{-t}|^2 \, dt = \int_0^{\infty} e^{-2t} dt = \frac{1}{2}$$

- For a signal x(t), if $E_x$is finite, i.e., $E_x < \infty$, then x(t) is termed as an **energy signal**.

Signal Power

Power of a signal:

- power of a signal x(t) is defined as:

$$P_x = \lim_{\tilde{T}\to\infty} \frac{1}{\tilde{T}} \int_{-\tilde{T}/2}^{\tilde{T}/2} |x(t)|^2 dt$$

$$= \lim_{\tilde{T}\to\infty} \frac{energy\ in\ a\ window\ of\ size\ \tilde{T}}{\tilde{T}}$$

- If P_x is finite, i.e., $P_x < \infty$, then x(t) is termed as a **power signal**.

Power of an energy signal

- If x(t) is an energy signal,

$$P_x = \lim_{\tilde{T}\to\infty} \frac{1}{\tilde{T}} \int_{-\tilde{T}/2}^{\tilde{T}/2} |x(t)|^2 dt$$
$$\leq \lim_{\tilde{T}\to\infty} \frac{1}{\tilde{T}} \int_{-\infty}^{\infty} |x(t)|^2 dt$$
$$= \lim_{\tilde{T}\to\infty} \frac{E_x}{\tilde{T}} = 0$$

- Power of an energy signal, P_x

  $P_x \leq 0$ from above inequality

  $P_x \geq 0$ since P_x is non-negative quantity

  only possibility is that P_x=0

i.e., power of an energy signal is zero.

Energy of a power signal

Now, look at the energy of a power signal,

If x(t) is a power signal,

- Energy in a window of size $\tilde{T} \approx P_x . \tilde{T}$ comes from ($P_x = \lim_{\tilde{T}\to\infty} \frac{energy\ in\ a\ window\ of\ size\ \tilde{T}}{\tilde{T}}$)

Total energy = $\lim_{\tilde{T}\to\infty}$ (Energy in a window size $\tilde{T}$)

= $\lim_{\tilde{T}\to\infty} P_x . \tilde{T}$ = ∞ as P_x is constant, $\tilde{T}$ tends to inf.

Therefore, energy of a power signal is ∞.

What kind of signal is a power signal?

Periodic Signals:

x(t) is periodic with time period, T

If $x(t) = x(t + kT) \quad \forall t, \forall k \in \mathbb{Z}$

- Shifting a signal t + an integer value times T, and the signal remains unchanged — is a periodic signal
- Sinusoidal signals are periodic signals: $\sin(2\pi f t)$ is a periodic signal,

$$\sin\big(2\pi f(t + kT)\big) = \sin(2\pi f t + 2\pi k f T) = \sin(2\pi f T + k2\pi)$$
$$= \sin(2\pi f t)$$

where fT=1

T=fundamental period of the sinusoidal signal

Power of a periodic signal

Let, T= time period

$$P_x = \lim_{\tilde{T}\to\infty} \frac{1}{\tilde{T}} \int_{-\tilde{T}/2}^{\tilde{T}/2} |x(t)|^2 dt$$

If $\tilde{T}$=mT, naturally, as m tends to infinity, $\tilde{T}$ tends to inf.

$$= \lim_{m\to\infty} \frac{1}{mT} \underbrace{\int_{-mT/2}^{mT/2} |x(t)|^2 dt}$$

energy in m periods = m times energy in a single period, i.e.

$$= \lim_{m\to\infty} \frac{1}{mT} m \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$P_x = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

- P_x is the power of the periodic signal, i.e., (energy in a single period T)/T

Example

- $x(t) = Acos(2\pi Ft)$, where T=1/F

$$P_x = \frac{1}{T}\int_{-T/2}^{T/2} A^2 cos^2(2\pi Ft)dt = \frac{A^2}{T}\int_{-T/2}^{T/2} \frac{1+\cos(4\pi Ft)}{2}$$

$$= \frac{A^2}{T}\frac{1}{2}\mathrm{T} + \frac{A^2}{T}\frac{1}{8\pi F}[\sin\left(\frac{4\pi FT}{2}\right) - \sin\left(4\pi F\left(-\frac{T}{2}\right)\right)] = \frac{A^2}{2}$$

- Hence, power of $Acos(2\pi Ft + \emptyset)$= $\frac{A^2}{2}$, does not depend on phase

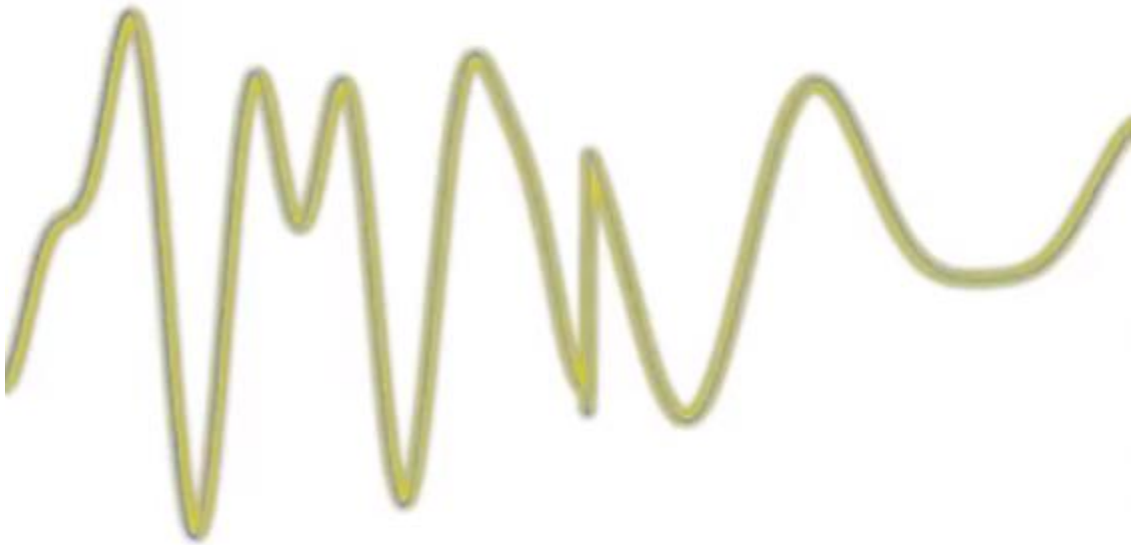
Frequency domain representation of signals

- One of the fundamental tools available for communication, also termed as the spectrum of the signal:
  - Fourier series (discrete in freq.)—define for periodic signals
  - Fourier Transform (continuous in freq.)—define for aperiodic continuous signals

What does Fourier’s theory actually mean?

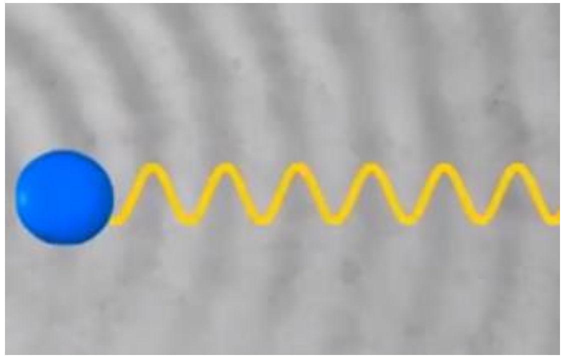
- “Any function of a variable, whether continuous or discontinuous, can be expanded in a series of sines of multiples of the variable.”

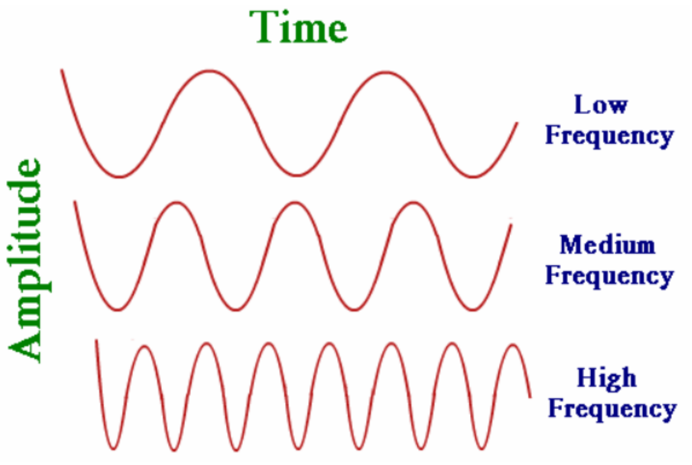
Joseph Fourier



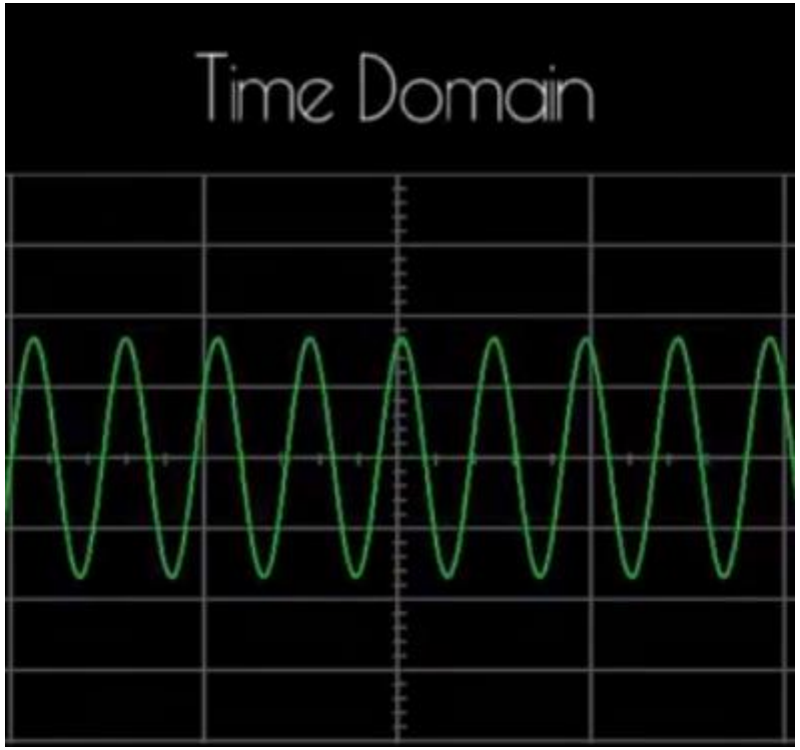
Example of sound signal:

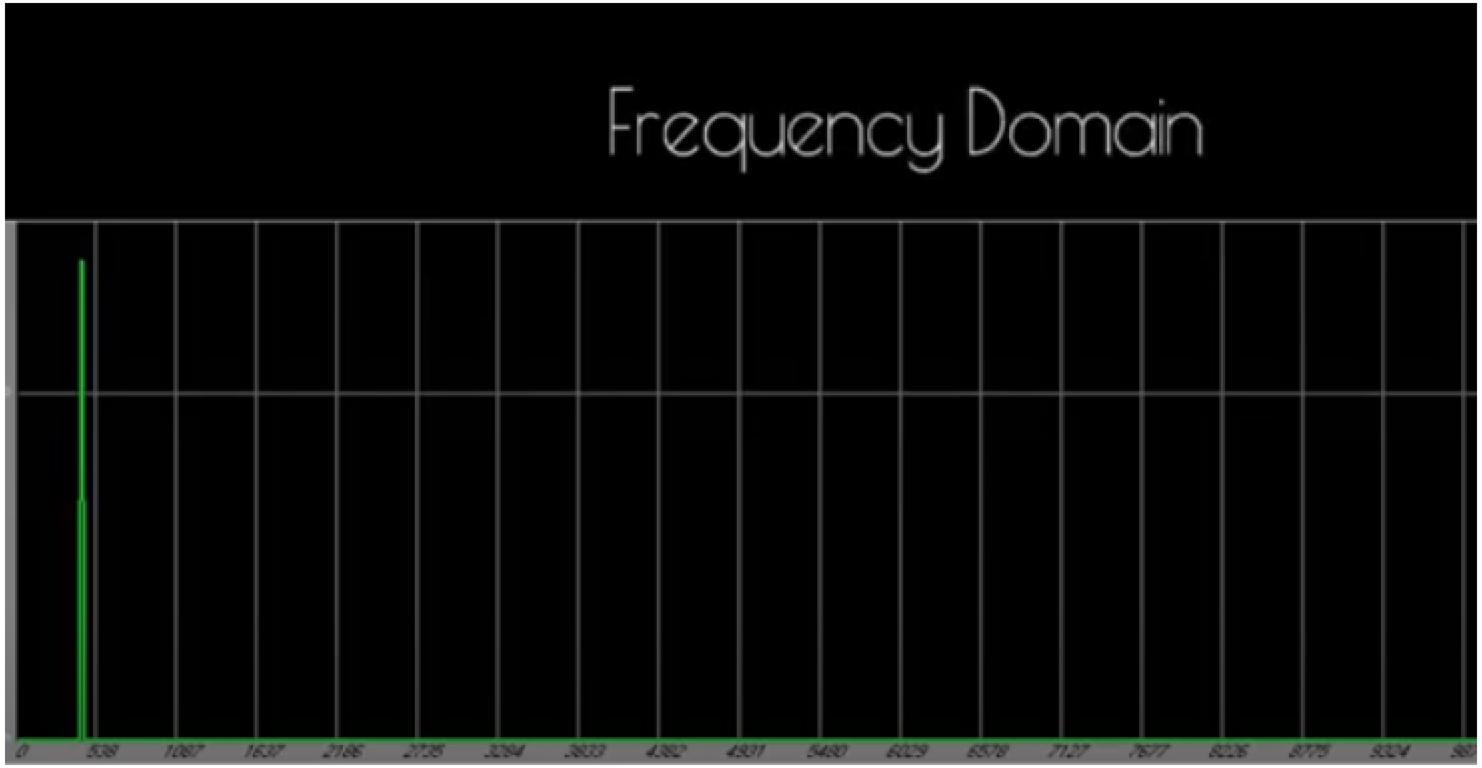
- Sine wave
- What properties of this sine wave can we change?
  - Amplitude or loudness
  - Frequency or pitch
  - Phase



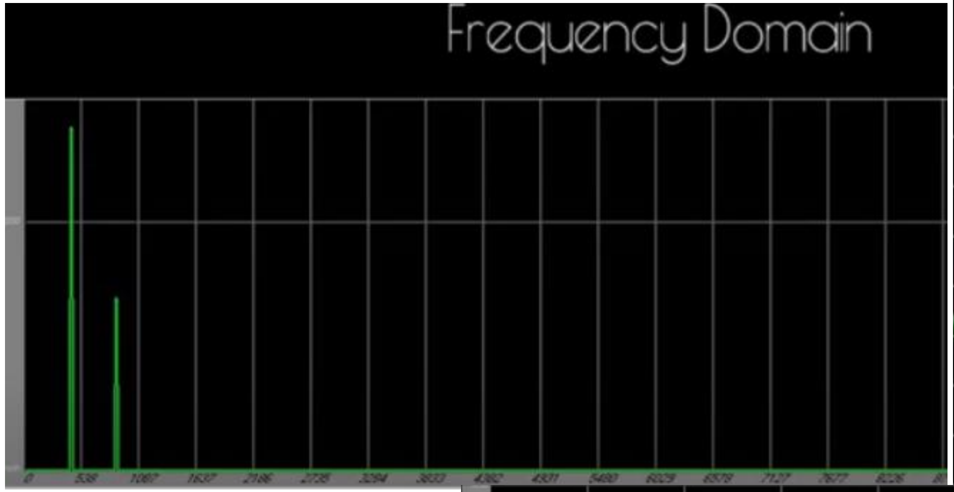


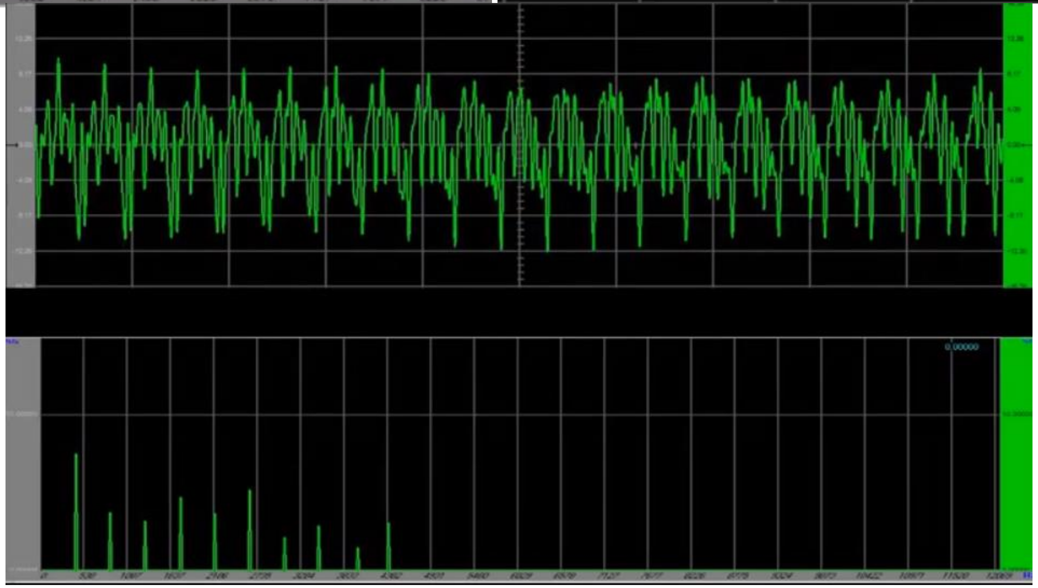
Single sine wave:





Add another sine wave





- “Any function (sound signal) of a variable (time) can be expanded in a series of sines of multiples of that variable.”

- In other words, sound is actually a whole load of sine waves at different frequencies and amplitudes all added together.

- This series of sine waves is known as Fourier series.

Frequency domain representation of signals

Consider x(t) to be a periodic signal with period T,

Fourier series of x(t) is given as:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_o t}$$

where $F_o = \frac{1}{T}$ fundamental freq. of periodic signal x(t), $e^{j2\pi k F_o t} = cos2\pi k F_o t + jsin2\pi k F_o t$ complex sinusoid, c_k= Kth discrete Fourier series coefficient or coefficient of the kth harmonic in the linear combination

Coefficients of Discrete Fourier Series (DFS):

$$\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi l F_o t}dt = c_l$$

lth coefficient of DFS

$$=\frac{1}{T}\int_{-T/2}^{T/2}\left(\sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_o t}\right)e^{-j2\pi l F_o t}dt$$

$$=\sum_{k=-\infty}^{\infty} c_k \underbrace{\frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi(k-l)F_o t}dt}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi(k-l)F_o t}dt$$

- If $k = l, \frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi(k-l)F_o t}dt = \frac{1}{T}\int_{-T/2}^{T/2} 1.\,dt = \frac{1}{T}.\,T = 1$

Coefficients of Discrete Fourier Series:

- If $k \neq l$, $\frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi(k-l)F_o t}dt = \frac{1}{T}\frac{1}{j2\pi(k-l)F_o}\underbrace{\left(e^{j\pi(k-l)} - e^{-j\pi(k-l)}\right)}_{\text{Sinusoid evaluated at phase}}$

$$\pi(k-l)\ \&\ -\pi(k-l)$$

- The diff. between two phases is $2\pi(k-l)$ i.e., integer multiple of 2π, these two complex sinusoids are equal, hence the difference = 0

$$= \frac{1}{T}\frac{1}{j2\pi(k-l)F_o}\left(e^{j\pi(k-l)} - e^{-j\pi(k-l)}\right) = 0$$

Hence, $\frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi(k-l)F_o t}dt = \begin{cases} 1\ if\ k = l \\ 0\ if\ k \neq l \end{cases}$

Coefficients of Discrete Fourier Series:

Orthogonal property:

$$\frac{1}{T}\int_{-T/2}^{T/2} x(t)y^*(t)dt = 0$$

$$\frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi kF_ot}(e^{j2\pi lF_ot})^*dt = 0$$

$$\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi lF_ot}dt = \sum_{k=-\infty}^{\infty} c_k \frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi(k-l)F_ot}dt = \sum_{k=-\infty}^{\infty} c_k.\delta(k-l)$$

Delta function $\delta(0) = 1$

$\delta(any\ other\ integer\ n) = 0$

$if\ n \neq 0$

Fourier Series for a periodic signal

- Example: Consider a periodic stream of pulse with width d= T/4 with amplitude A, and time period T.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_o t}$$

Also,

$$c_l = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j2\pi l F_o t} dt$$

Therefore, for the above particular signal,

$$c_l = \frac{1}{T}\int_{0}^{T/4} A e^{-j2\pi l F_o t} dt$$

Fourier Series for a periodic signal

If $l = 0$, $c_l = \frac{A}{T}\int_0^{T/4} 1.dt = \frac{A}{T}\frac{T}{4} = \frac{A}{4}$

- This Fourier coefficient corresponding to $l = 0$, also known as DC coefficient corresponds to the 0 freq. Rest of them corresponds to $l \neq 0$ that are AC coefficients.

If $l \neq 0$, $c_l = \frac{1}{T}\int_0^{T/4} Ae^{-j2\pi lF_o t}dt = \frac{A}{T}\frac{e^{-j2\pi lF_o T/4} - 1}{-j2\pi lF_o} = \frac{A}{j2\pi l}\left(1 - e^{-j\pi l/2}\right)$

Taking $e^{-j\pi l/4}$ common,

$=\frac{A}{j2\pi l}e^{-j\pi l/4}\left(e^{j\pi l/4} - e^{-j\pi l/4}\right) = \frac{A}{j2\pi l}e^{-\frac{j\pi l}{4}}(2jsin\left(\frac{\pi l}{4}\right)) = \frac{A}{\pi l}e^{-\frac{j\pi l}{4}}sin\frac{\pi l}{4}$

Fourier Series for a periodic signal

$$c_l = \frac{A}{\pi l}\sin(\frac{\pi l}{4})e^{-\frac{j\pi l}{4}}$$

Hence,

$$c_l = \begin{cases} \frac{A}{4}, & l = 0 \\ \frac{A}{\pi l}\sin\left(\frac{\pi l}{4}\right)e^{-\frac{j\pi l}{4}}, & l \neq 0 \end{cases}$$

For magnitude only,

$|c_l| = \left|\frac{A}{\pi l}\sin\left(\frac{\pi l}{4}\right)e^{-\frac{j\pi l}{4}}\right| = \left|\frac{A}{\pi l}\sin\left(\frac{\pi l}{4}\right)\right|$ magnitude spectrum

Power of a periodic signal

For a periodic signal x(t), the power is defined as:

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_o t}$$

$$x^*(t) = \left(\sum_{m=-\infty}^{\infty} c_m e^{j2\pi mF_o t}\right)^* = \sum_{m=-\infty}^{\infty} {c_m}^* e^{-j2\pi mF_o t}$$

P = $\frac{1}{T}\int_{-T/2}^{T/2} x(t).\, x^*(t)\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \left(\sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_o t}\right)\left(\sum_{m=-\infty}^{\infty} {c_m}^* e^{-j2\pi mF_o t}\right) dt$

Power of a periodic signal

$$= \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_k {c_m}^* e^{j2\pi(k-m)F_o t} dt$$

$$= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_k {c_m}^* \frac{1}{T} \int_{-T/2}^{T/2} e^{j2\pi(k-m)F_o t} \, dt = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_k {c_m}^* \delta(k-m)$$

The only term in these double summation that will survive corresponding to $k = m$

Therefore, $c_k {c_m}^*$will be $c_k {c_k}^*$which is $|c_k|^2$

$$P = \sum_{k=-\infty}^{\infty} c_k {c_k}^* = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Power of a periodic signal

$$\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2 dt = \sum_{k=-\infty}^{\infty}|c_k|^2$$

- ✓ $\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2 dt$ power in time domain
- ✓ $\sum_{k=-\infty}^{\infty}|c_k|^2$ power in frequency domain, also known as the **Parseval's theorem**