Applications of Derivatives

Calculus & Analytical Geometry MATH- 101

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Objectives

- Extreme Values of functions.
- Rolle's theorem ✓

→ f is cts $[a,b]$
→ f is differentiable (a,b)
→ $f(a) = f(b)$

$\exists$ at least one $c \in (a,b)$ s.t. $f'(c) = 0$

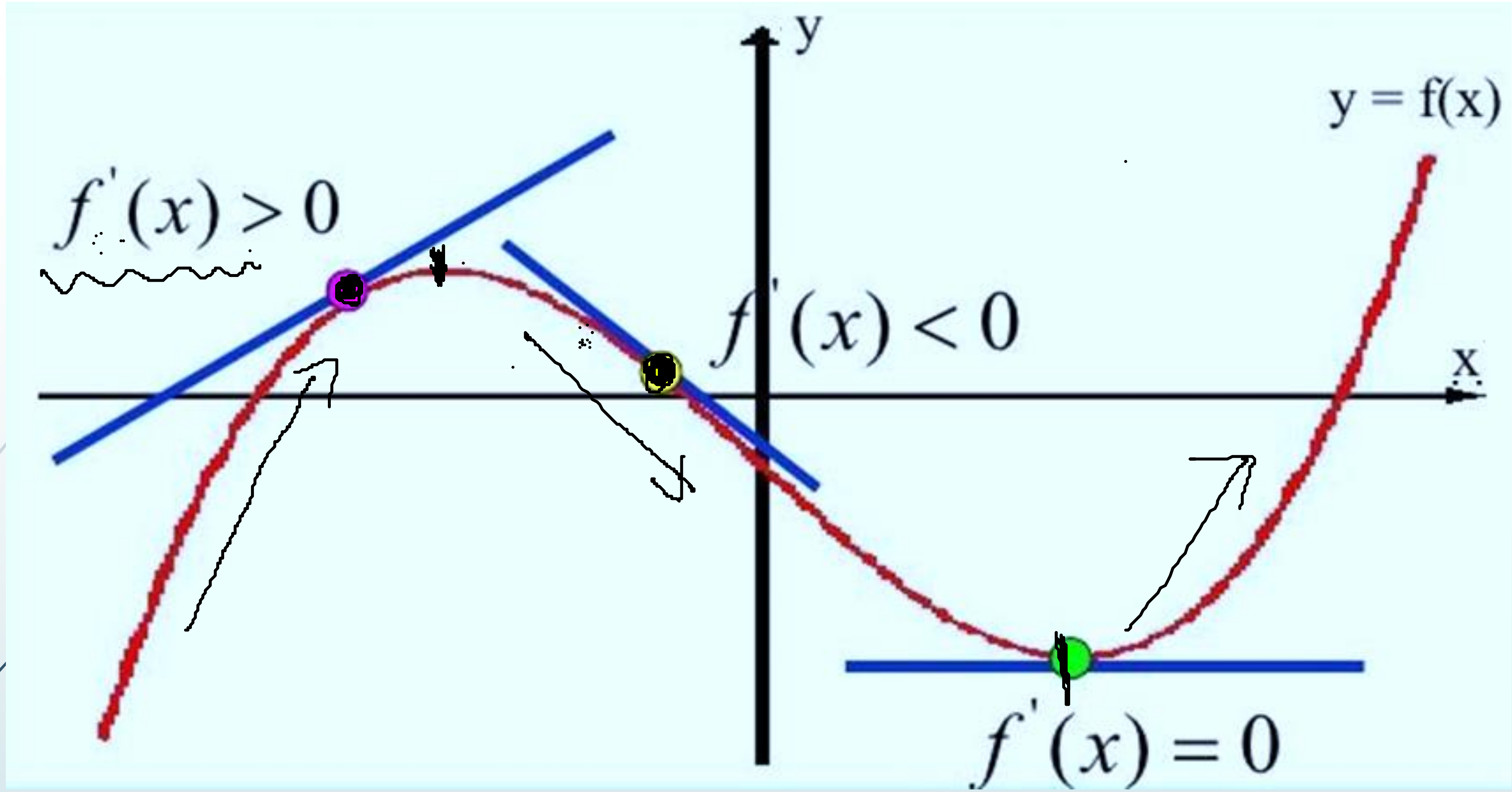
- The Mean Value theorem. ✓

→ f is cts on $[a,b]$ → f is diff on (a,b)

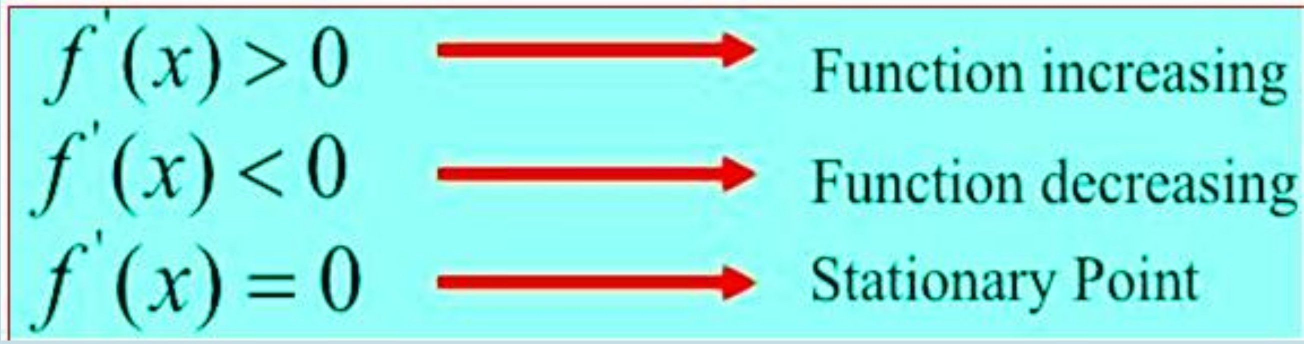
→ $\exists$ at least one $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- Monotonic Functions and The First Derivative Test ✓



Increasing and Decreasing Functions and the First Derivative Test



Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 4
  - Sections: 4.3

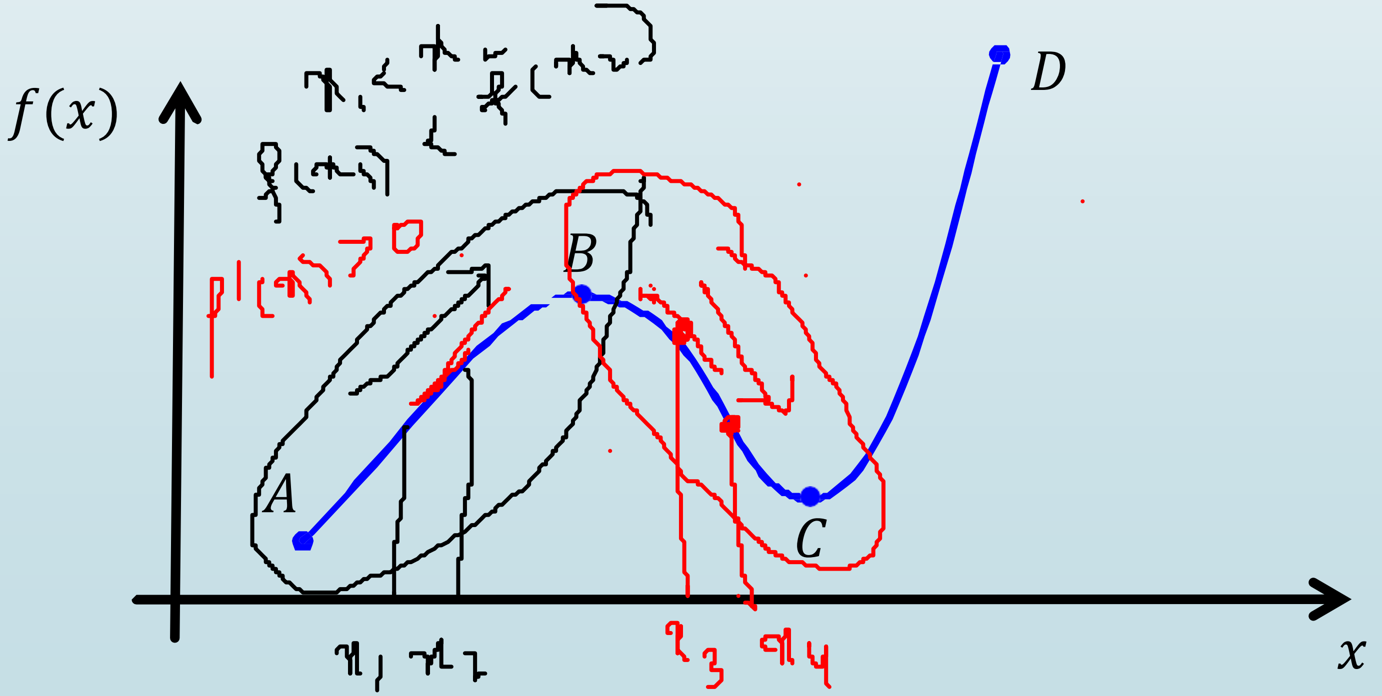
Increasing and Decreasing Functions

A function $f(x)$ is **strictly increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

A function $f(x)$ is **strictly decreasing** on an interval I if $f(x_1) > f(x_2)$ Whenever $x_1 < x_2$.

Slope of Tangent line

$f'(x) > 0$

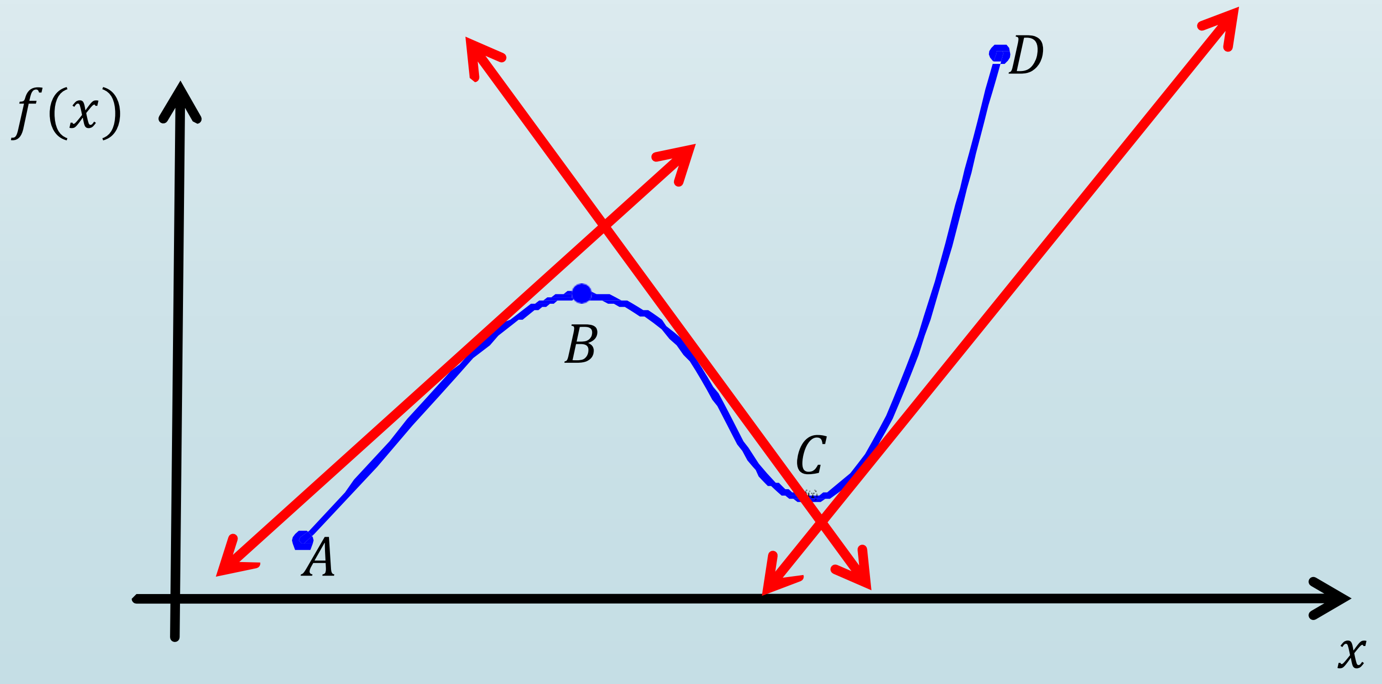


$x_3 < x_4$

$f(x_3) > f(x_4)$

How the Derivative is connected to Increasing/Decreasing Functions

- When the function is increasing, what is the sign (+ or –) of the slopes of the tangent lines? POSITIVE Slope ✓
- When the function is decreasing, what is the sign (+ or –) of the slopes of the tangent lines? NEGATIVE Slope



First Derivative Test for Increasing and Decreasing Functions

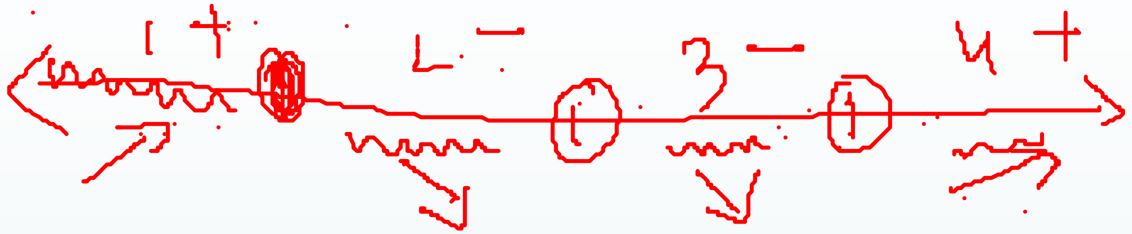
Let $f(x)$ be differentiable on the open interval (a, b)

If $f'(x) > 0$ for each value of x in an interval (a, b), then $f(x)$ is increasing on (a, b).

If $f'(x) < 0$ for each value of x in an interval (a, b), then $f(x)$ is decreasing on (a, b).

If $f'(x) = 0$ for each value of x in an interval (a, b), then $f(x)$ is constant on (a, b).

Procedure for finding intervals on which a function is increasing or decreasing



If $f(x)$ is a continuous function on an open interval (a,b). To find the open intervals on which *f* is increasing or decreasing:

1. Find the critical points of $f(x)$ in (a,b).
2. Make a sign chart: The critical points, divide the $x-$axis into intervals. Test the sign $(+ \text{ or } -)$ of the **derivative** inside each of these intervals.
3. If $f'(x) > 0$ in an interval, then $f(x)$ is increasing in that interval.
4. If $f'(x) < 0$ in an interval, then $f(x)$ is decreasing in that interval.

Critical Points

$c \in (a,b) \longrightarrow f'(c) = 0$ ✓ Static

Singular points ← $f'(c)$ is undefined

$\searrow$ $(-\infty,-1) \cup (0,2)$

Example:

$\nearrow$ $(-1,0) \cup (2,\infty)$

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution: $f(x)$ is continuous and Domain of $f(x)$ is the set of all Real numbers.

1. Find the critical points: Calculate the derivative and determine where the derivative is 0 or undefined

$$f'(x) = 12x^3 - 12x^2 - 24x = 0$$

$$\Rightarrow 12x(x^2 - x - 2) = 0$$

$$\Rightarrow 12x(x-2)(x+1) = 0$$

$$\Rightarrow x = 0, 2, -1$$

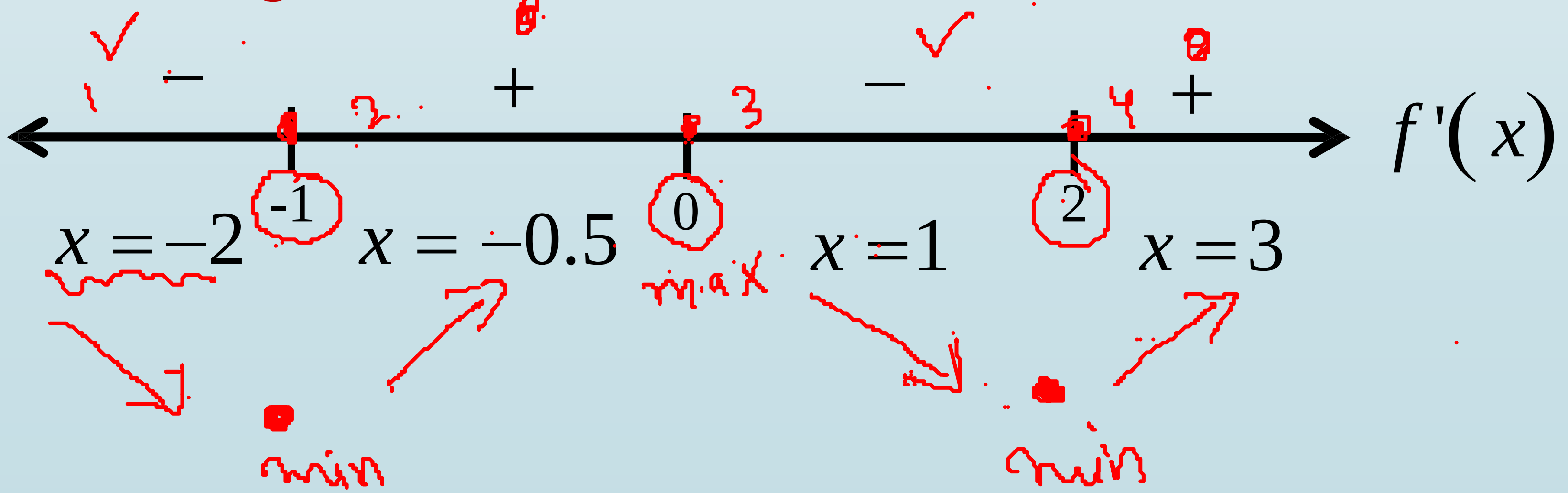
1 → $(-\infty,-1)$

2 → $(-1,0)$

3 → $(0,2)$

4 → $(2,\infty)$

2. Find the sign of the derivative on each interval:



$f'(-2) = -96 < 0$

$f'(-0.5) = 7.5 > 0$

$f'(1) = -24 < 0$

$f'(3) = 144 > 0$

- The function is increasing on:

$$(-1,0) \cup (2,\infty)$$ ✓

because the **first** derivative is positive on this interval.

- The function is decreasing on:

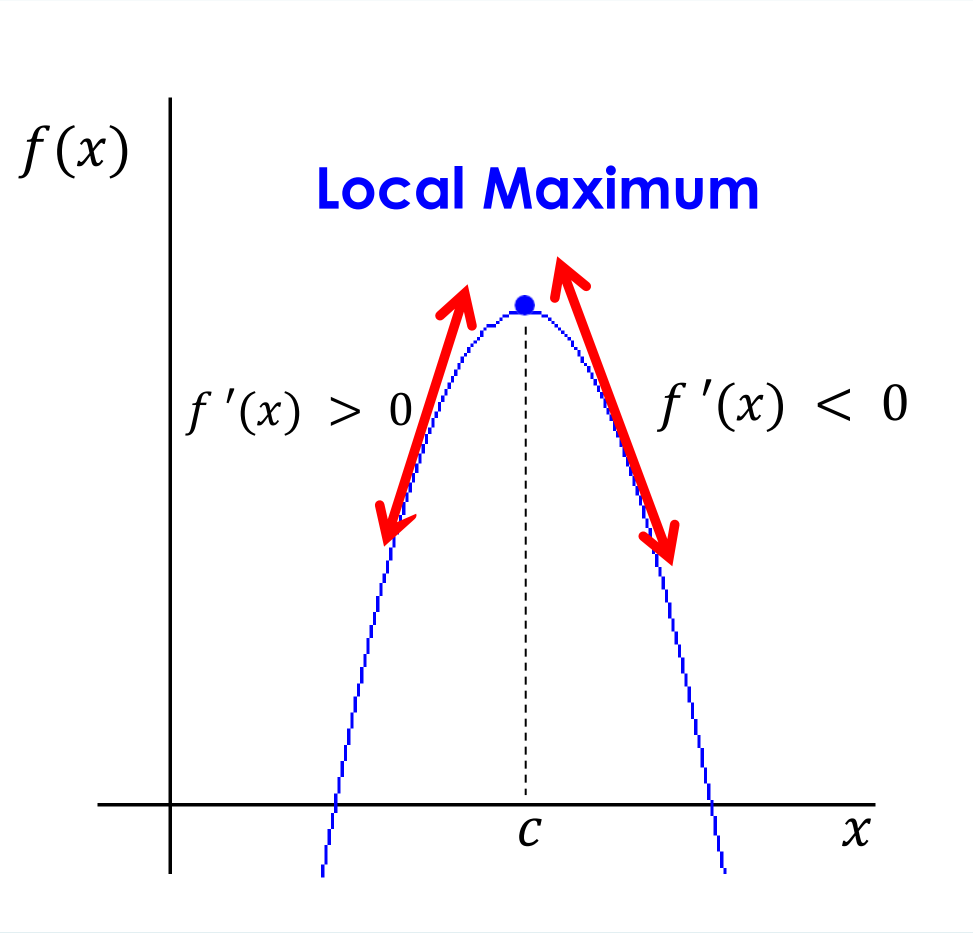
$$(-\infty,-1) \cup (0,2)$$ ✓

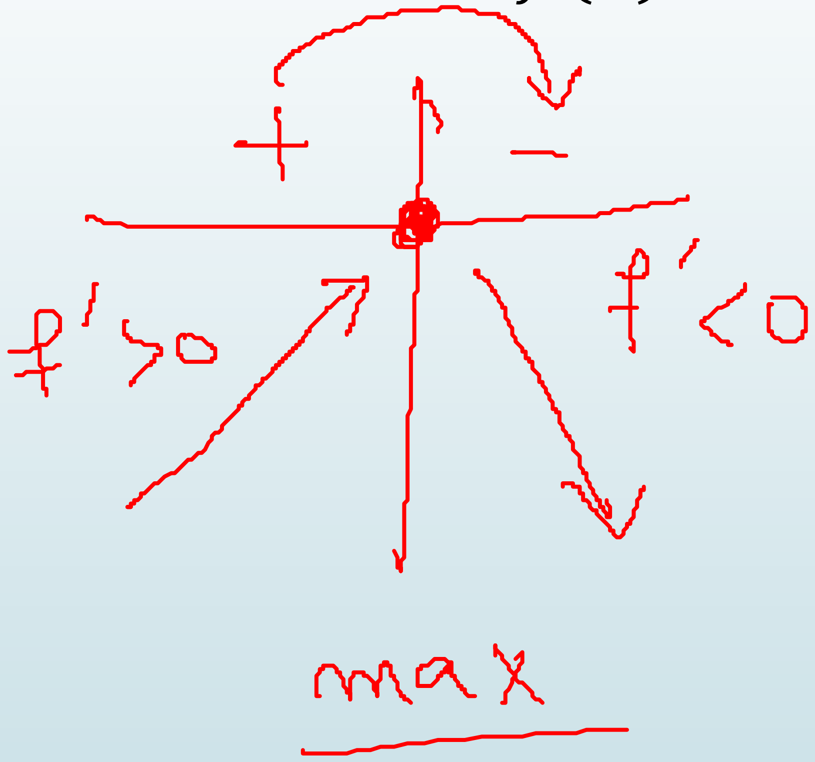
because the **first** derivative is negative on this interval.

The First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function $f(x)$.

a) If $f'(x)$ changes sign from positive to negative at c, then $f(x)$ has a local maximum at c.

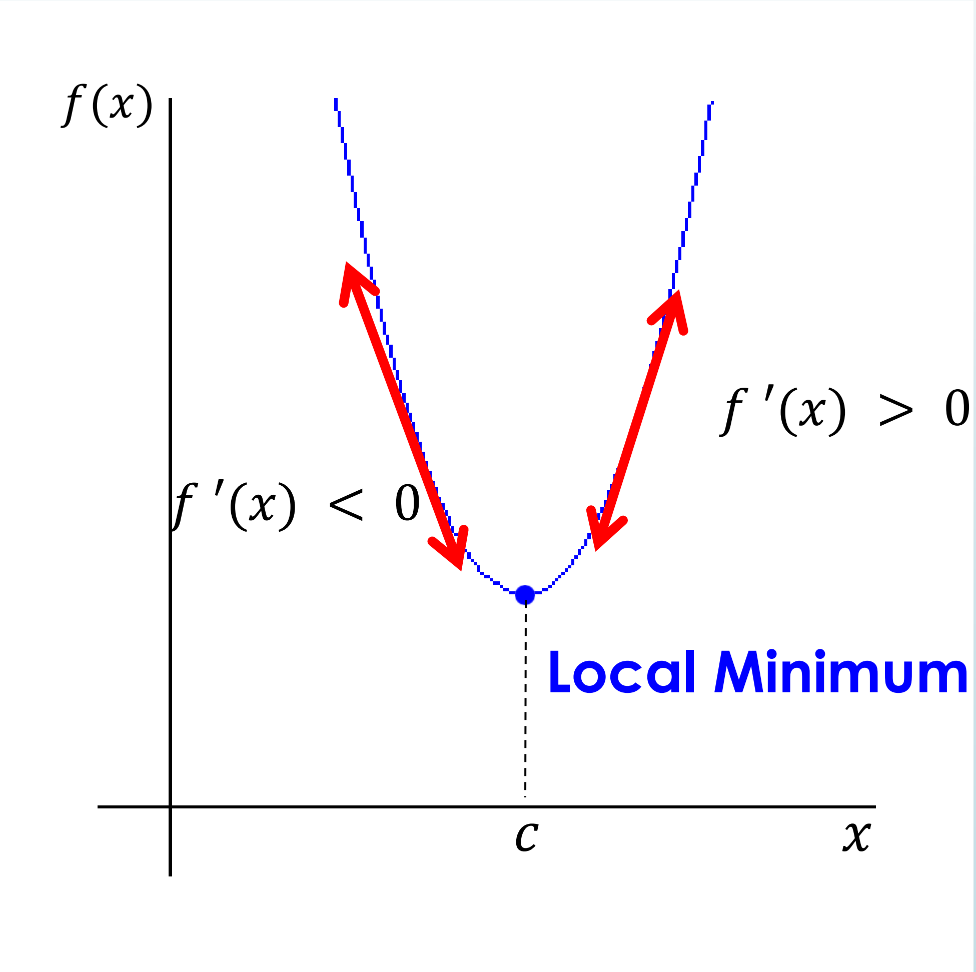




The First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function $f(x)$.

b) If $f'(x)$ changes sign from negative to positive at c, then $f(x)$ has a local minimum at c.



+ • +

or

− • − ??

− +

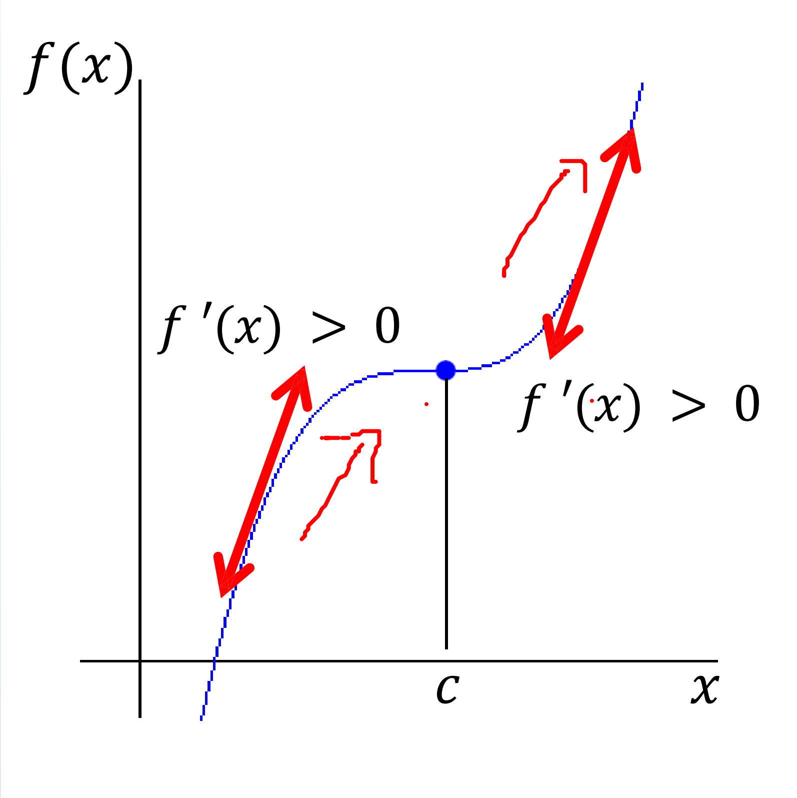
$f' < 0$ $f' > 0$

min

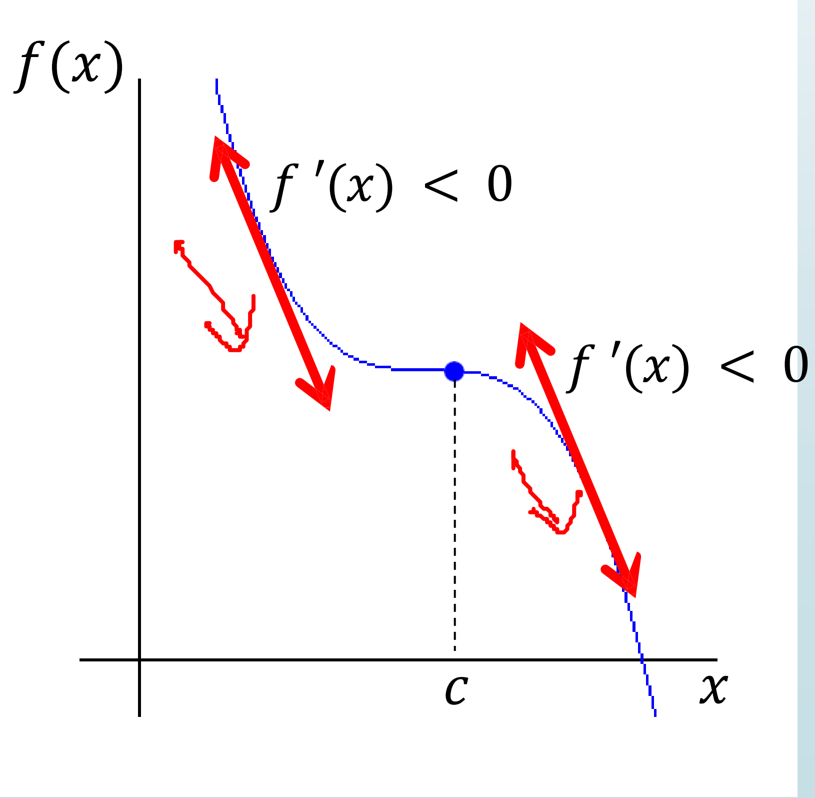
The First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function $f(x)$.

c) If $f'(x)$ does not changes sign at c (i.e., $f'(x)$ is positive on both sides of c or it is negative on both sides), then $f(x)$ has no local maximum or minimum at c. minimum at c.



No Local Maximum or Minimum



+ +

OR

− −

No Extreme value

The First Derivative Test

Determine the sign of the derivative of $f(x)$ to the left and right of the critical point.

left	right	conclusion
$+$	$-$	$f(c)$ is a relative maximum
$-$	$+$	$f(c)$ is a relative minimum
No change		No relative extremum

a: Right

$f' < 0$ — max

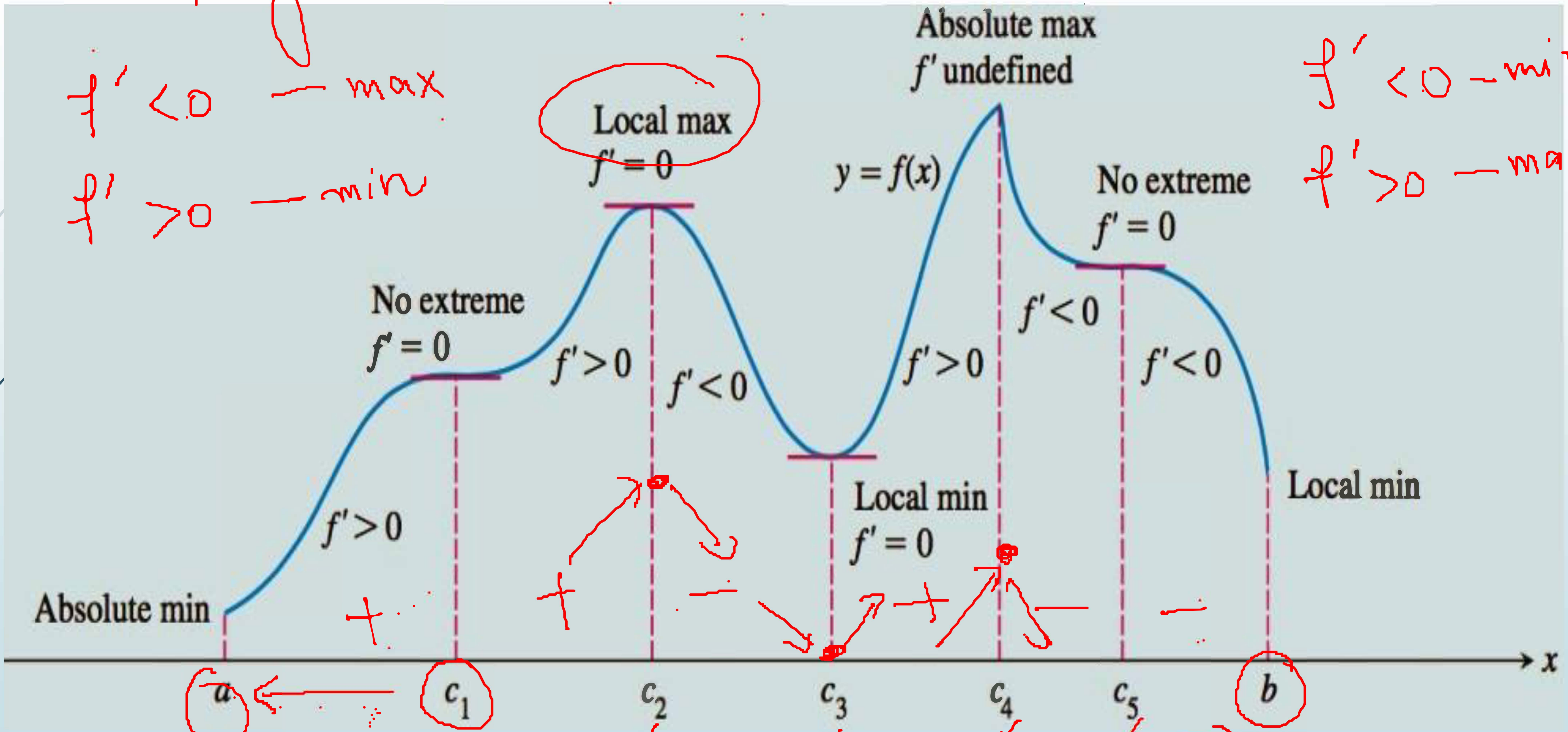
$f' > 0$ — min

[a, b]

b: Left

$f' < 0$ — min

$f' > 0$ — max



Example: Find all the relative extrema of

① → $(-\infty, 0)$
② → $(0, 4)$
③ → $(4, \infty)$

$$f(x) = x^3 - 6x^2 + 1$$

$$\Rightarrow f'(x) = 3x^2 - 12x = 0$$

Stationary points: $x = 0, 4$

Singular points: None

Relative max.
$f(0) = 1$

Relative min.
$f(4) = -31$

① max ② ③

Sign of f'	$+$	0	$-$	0	$+$		
		0		4			
Behavior of f	$\nearrow$		$\searrow$		$\nearrow$		

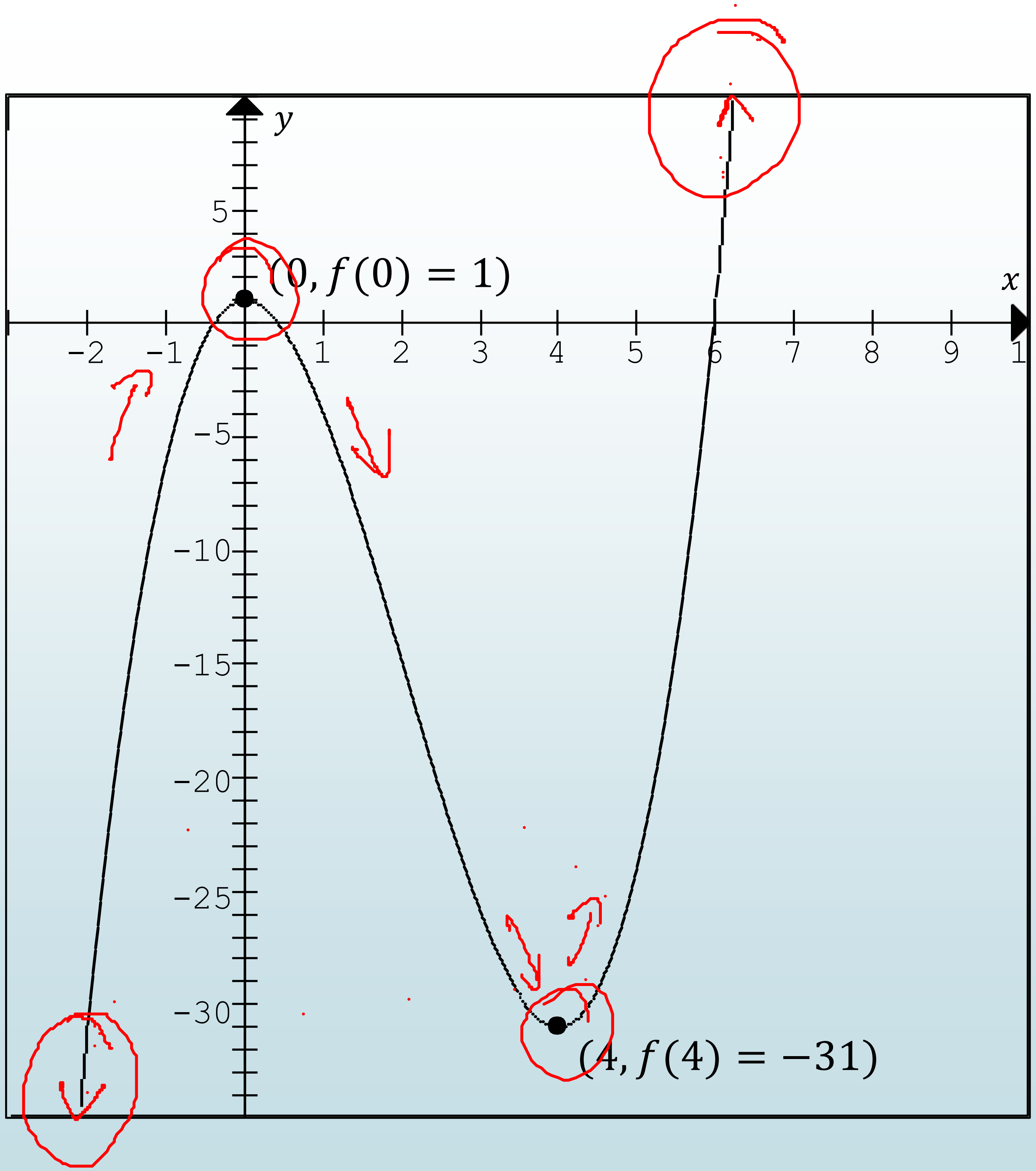
min

① → $x = -0.5$

$f'(-0.5) = \frac{3}{4} + 6 > 0$

↗ $(-\infty, 0) \cup (4, \infty)$

↘ $(0, 4)$



f is cts and $(-\infty, \infty)$

No absolute Extreme

Example:

Find all the relative extrema of

$$f(x) = \sqrt[3]{x^3 - 3x}$$

$$\Longrightarrow f'(x) = \frac{x^2 - 1}{\sqrt[3]{x^3 - 3x}}$$

Stationary points: $x = \pm 1$

Singular points: $x = 0, \pm\sqrt{3}$

Stationary points: $x = \pm 1$

Singular points: $x = 0, \pm\sqrt{3}$

$x = 1$ and

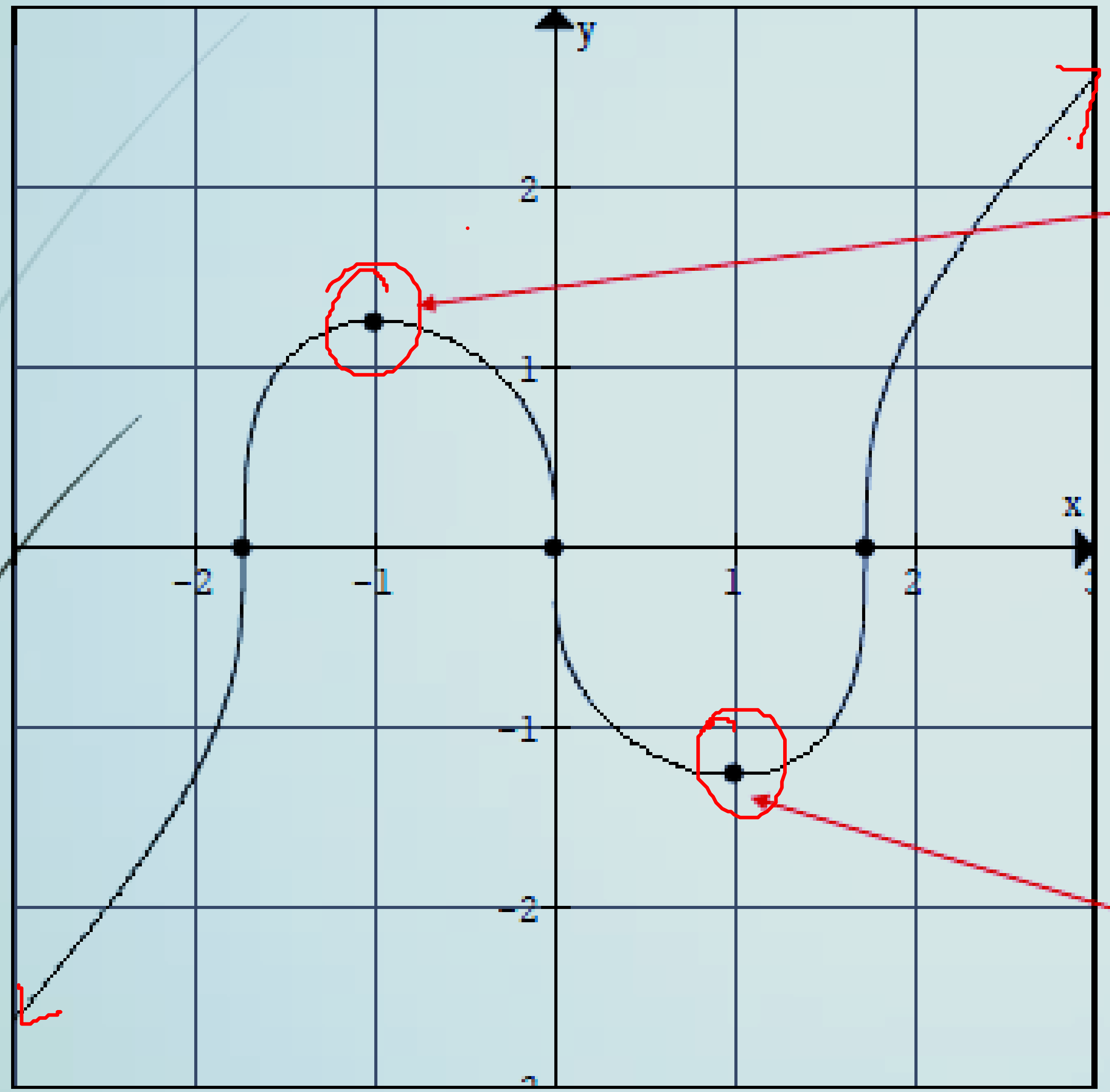
$x = -1$

$f(x)$

$\boxed{\text{Relative max. } f(-1) = \sqrt[3]{2}}$ ✓

$\boxed{\text{Relative min. } f(1) = -\sqrt[3]{2}}$ ✓

				max							
	1		2		3		4		5		6
✓ Sign of f'	+	ND	+	0	−	ND	−	0	+	ND	+
		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
Behavior of f	↗		↗		↘		↘		↗		↗
		No				No		min		No	



Local max. $f(-1) = \sqrt[3]{2}$

$$f(x) = \sqrt[3]{x^3 - 3x}$$

Local min. $f(1) = -\sqrt[3]{2}$

Domain Not a Closed Interval

Example: Find the absolute extrema of $f(x) = \frac{1}{(x-2)}$ on $[3, \infty)$

Solution:

$$f(x) = \frac{1}{(x-2)}$$

$$\Longrightarrow \; f'(x) = \frac{-1}{(x-2)^2}$$

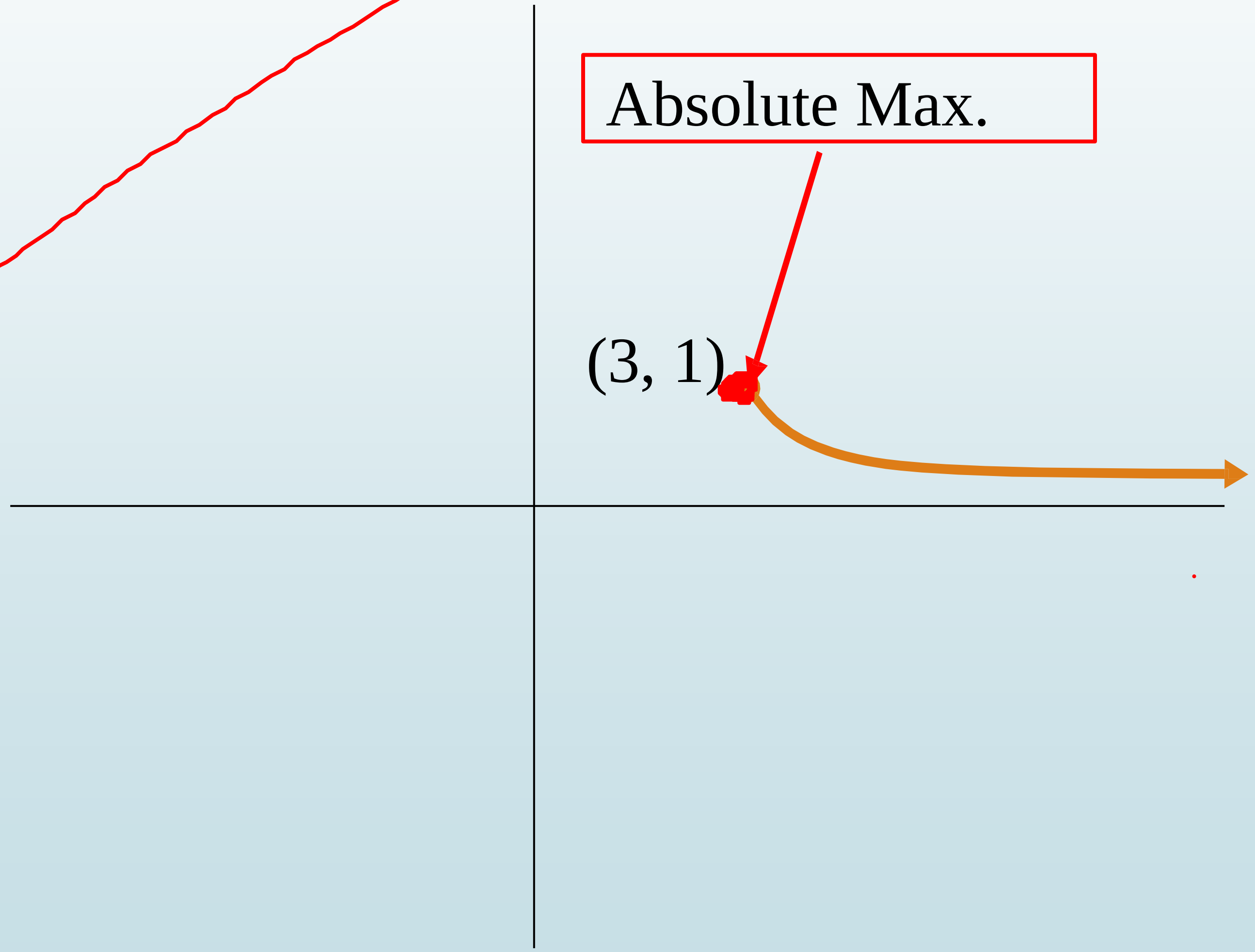
Singular point: $x = 2$ (Not a critical point)

At end point: $x = 3$

$$f'(3) = \frac{-1}{(3-2)^2} < 0 \quad \text{Decreasing}$$

and

$$f(3) = 1 \quad \text{Absolute Max.}$$



Practice Questions

Book: Thomas Calculus (11th Edition) by Georg B.Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter: 4**

  - **Exercise: 4.3**

    Q # 1 – 36.