16

Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
    - **Section: 16.3**

Book: Thomas’ Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

- **Chapter: 16**
    - **Section: 16.2, 16.3**

Fundamental Theorem of Line Integrals for Conservative Field

Let C be a piecewise smooth curve lying in an open region R and given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \le t \le b.$$

If $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative in R, and M and N are continuous in R, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a))$$

where f is a potential function of $\mathbf{F}$. That is, $\mathbf{F}(x, y) = \nabla f(x, y)$.

If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \le t \le b$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))$$

where f is a potential function of $\mathbf{F}$.

A curve is called **closed** if its terminal point coincides with its initial point, that is, $\mathbf{r}(b) = \mathbf{r}(a)$.

Equivalent Conditions

Let $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ have continuous first partial derivatives in an open connected region R, and let C be a piecewise smooth curve in R. The following conditions are equivalent.

1. $\mathbf{F}$ is conservative. That is, $\mathbf{F} = \nabla f$ for some function f.
2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.
3. $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every *closed* curve C in R.

EXAMPLE Find the work done by the conservative field

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \nabla f, \quad \text{where} \quad f(x, y, z) = xyz,$$

in moving an object along any smooth curve C joining the point $A(-1, 3, 9)$ to $B(1, 6, -4)$.

Solution With $f(x, y, z) = xyz$, we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_A^B \nabla f \cdot d\mathbf{r} \\
&= f(B) - f(A) \\
&= xyz\big|_{(1,6,-4)} - xyz\big|_{(-1,3,9)} \\
&= (1)(6)(-4) - (-1)(3)(9) \\
&= -24 + 27 = 3.
\end{aligned}$$

Flow Integrals and Circulation for Velocity Fields

Instead of being a force field, suppose that **F** represents the velocity field of a fluid flowing through a region in space (a tidal basin or the turbine chamber of a hydroelectric generator, for example). Under these circumstances, the integral of **F**. **T** along a curve in the region gives the **fluid's flow along the curve.**

DEFINITIONS Flow Integral, Circulation

If $\mathbf{r}(t)$ is a smooth curve in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $t = a$ to $t = b$ is

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

The integral in this case is called a **flow integral**. If the curve is a closed loop, the flow is called the **circulation** around the curve.

Example: Finding Circulation Around a Circle

Find the circulation of the field $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le 2\pi$.

Solution:

On the circle, $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{j}$, and

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

Then

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + \sin^2 t + \cos^2 t$$

gives

$$\text{Circulation} = \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{2\pi} (1 - \sin t \cos t)\, dt$$

$$= \left[t - \frac{\sin^2 t}{2} \right]_0^{2\pi} = 2\pi.$$

Flux Across a Plane Curve

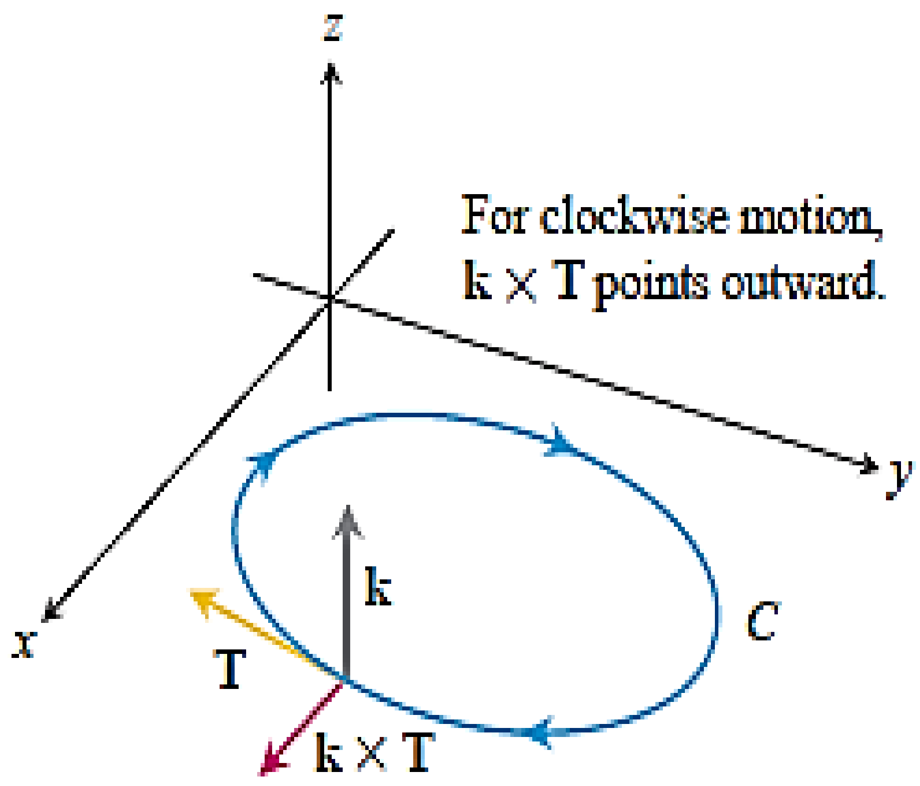
To find the rate at which a fluid is entering or leaving a region enclosed by a smooth curve C in the xy −plane, we calculate the line integral over C of $\mathbf{F}.\mathbf{n}$, the scalar component of the fluid's velocity field in the direction of the curve's outward-pointing normal vector. The value of this integral is the ***flux*** of $\mathbf{F}$ across C.

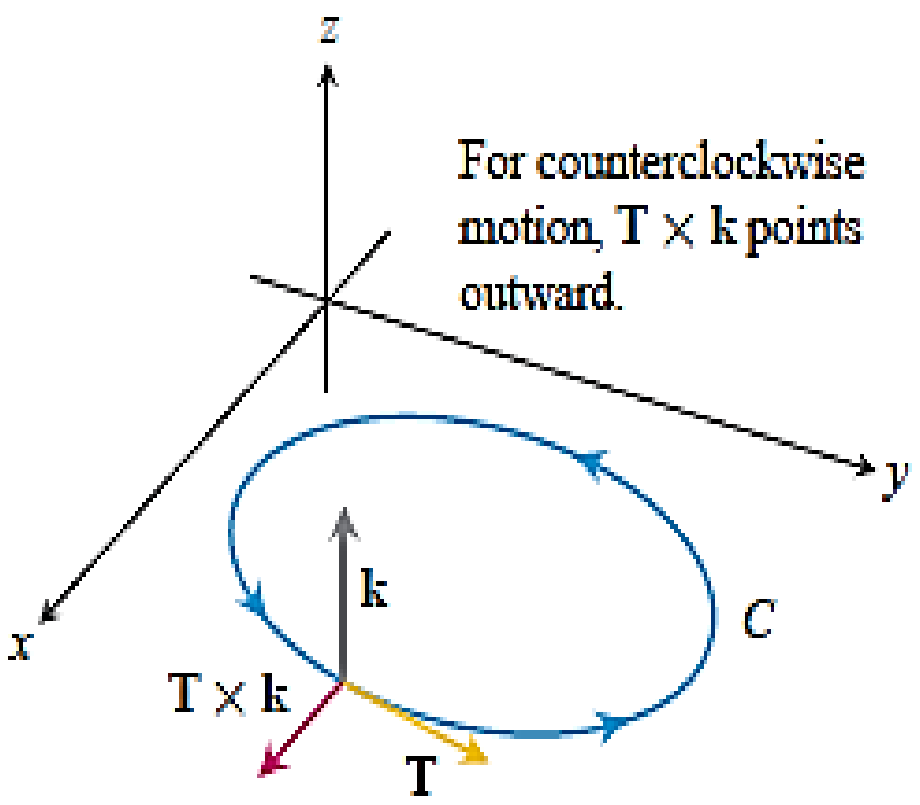
DEFINITION **Flux Across a Closed Curve in the Plane**

If C is a smooth closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\mathbf{F}$ across C is

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \mathbf{n}\, ds.$$

Notice the difference between flux and circulation. **Flux** is the integral of the **normal component** of **F**; **circulation** is the integral of the **tangential component** of **F**.





To find an outward unit normal vector for a smooth curve C in the xy −plane that is traversed counterclockwise as t increases, we take $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. For clockwise motion, we take $\mathbf{n} = \mathbf{k} \times \mathbf{T}$.

Flux Across a Plane Curve

We begin with a smooth parametrization: $x = g(t), \quad y = h(t); \quad a \leq t \leq b$.
In terms of components,

$$\mathbf{n} = \mathbf{T} \times \mathbf{k} = \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} \right) \times \mathbf{k} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

If $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, then

$$\mathbf{F} \cdot \mathbf{n} = M(x, y)\frac{dy}{ds} - N(x, y)\frac{dx}{ds}.$$

Hence,

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds} \right) ds = \oint_C M\,dy - N\,dx.$$

Calculating Flux Across a Smooth Closed Plane Curve

$$(\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C) = \oint_C M\,dy - N\,dx$$

The integral can be evaluated from any smooth parametrization $x = g(t), y = h(t)$, $a \leq t \leq b$, that traces C counterclockwise exactly once.

Example: Finding Flux Across a Circle

Find the flux of $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ across the circle $x^2 + y^2 = 1$ in the xy-plane.

Solution:

The parametrization $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le 2\pi$, traces the circle counterclockwise exactly once. We can therefore use this parametrization.

$$M = x - y = \cos t - \sin t, \qquad dy = d(\sin t) = \cos t\, dt$$

$$N = x = \cos t, \qquad dx = d(\cos t) = -\sin t\, dt,$$

We find

$$\text{Flux} = \int_C M\,dy - N\,dx = \int_0^{2\pi} (\cos^2 t - \sin t \cos t + \cos t \sin t)\, dt$$

$$= \int_0^{2\pi} \cos^2 t\, dt = \int_0^{2\pi} \frac{1 + \cos 2t}{2}\, dt = \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} = \pi.$$

The flux of $\mathbf{F}$ across the circle is π. Since the answer is positive, the net flow across the curve is outward. A net inward flow would have given a negative flux.

Exact Differential Forms

It is often convenient to express work and circulation integrals in the differential form

$$\int_C M\,dx + N\,dy + P\,dz$$

Such line integrals are relatively easy to evaluate if $M\,dx + N\,dy + P\,dz$ is the total differential of a function f and C is any path joining the two points from A to B. For then

$$\begin{aligned}\int_C M\,dx + N\,dy + P\,dz &= \int_C \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \int_A^B \nabla f \cdot d\mathbf{r} \\ &= f(B) - f(A).\end{aligned}$$

Thus,

$$\int_A^B df = f(B) - f(A),$$

just as with differentiable functions of a single variable.

DEFINITION Any expression $M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz$ is a **differential form.** A differential form is **exact** on a domain D in space if

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

for some scalar function f throughout D.

Notice that if $M\,dx + N\,dy + P\,dz = df$ on D, then $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the gradient field of f on D. Conversely, if $\mathbf{F} = \nabla f$, then the form $M\,dx + N\,dy + P\,dz$ is exact. The test for the form being exact is therefore the same as the test for $\mathbf{F}$ being conservative.

Component Test for Exactness of $M\,dx + N\,dy + P\,dz$

The differential form $M\,dx + N\,dy + P\,dz$ is exact on an open simply connected domain if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

This is equivalent to saying that the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative.

EXAMPLE Show that $y\,dx + x\,dy + 4\,dz$ is exact and evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz$$

over any path from $(1, 1, 1)$ to $(2, 3, -1)$.

Solution We let $M = y$, $N = x$, $P = 4$ and apply the Test for Exactness:

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}.$$

These equalities tell us that $y\,dx + x\,dy + 4\,dz$ is exact, so

$$y\,dx + x\,dy + 4\,dz = df$$

for some function f, and the integral's value is $f(2, 3, -1) - f(1, 1, 1)$.

We find f up to a constant by integrating the equations

$$\frac{\partial f}{\partial x} = y, \qquad \frac{\partial f}{\partial y} = x, \qquad \frac{\partial f}{\partial z} = 4.$$

From the first equation we get

$$f(x, y, z) = xy + g(y, z).$$

The second equation tells us that

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x, \qquad \text{or} \qquad \frac{\partial g}{\partial y} = 0.$$

Hence, g is a function of z alone, and

$$f(x, y, z) = xy + h(z).$$

The third equation tells us that

$$\frac{\partial f}{\partial z} = 0 + \frac{dh}{dz} = 4, \qquad \text{or} \qquad h(z) = 4z + C.$$

Therefore,

$$f(x, y, z) = xy + 4z + C.$$

The value of the line integral is independent of the path taken from $(1, 1, 1)$ to $(2, 3, -1)$, and equals

$$f(2, 3, -1) - f(1, 1, 1) = 2 + C - (5 + C) = -3.$$

Practice Questions

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Chapter: 16

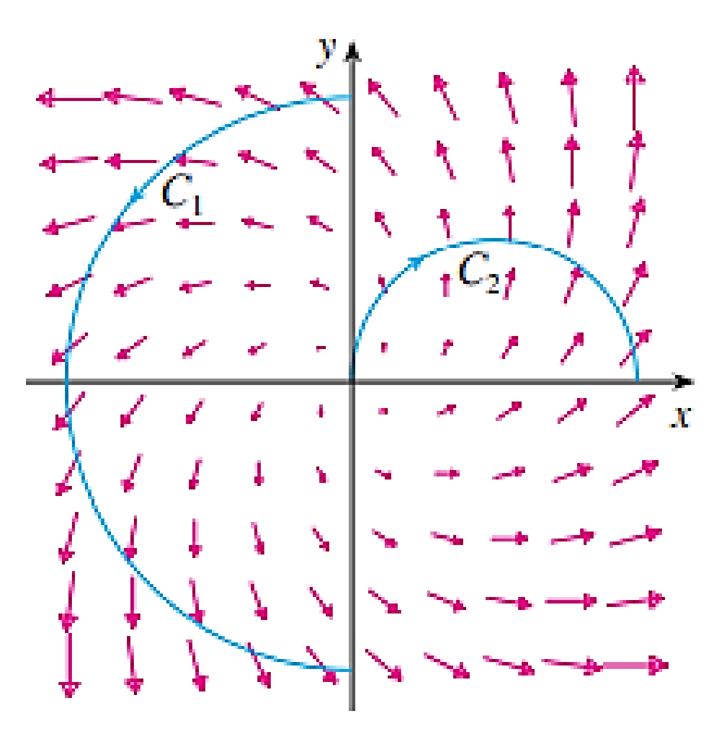
Exercise-16.3: Q – 3 to 22, Q – 27 to 28.

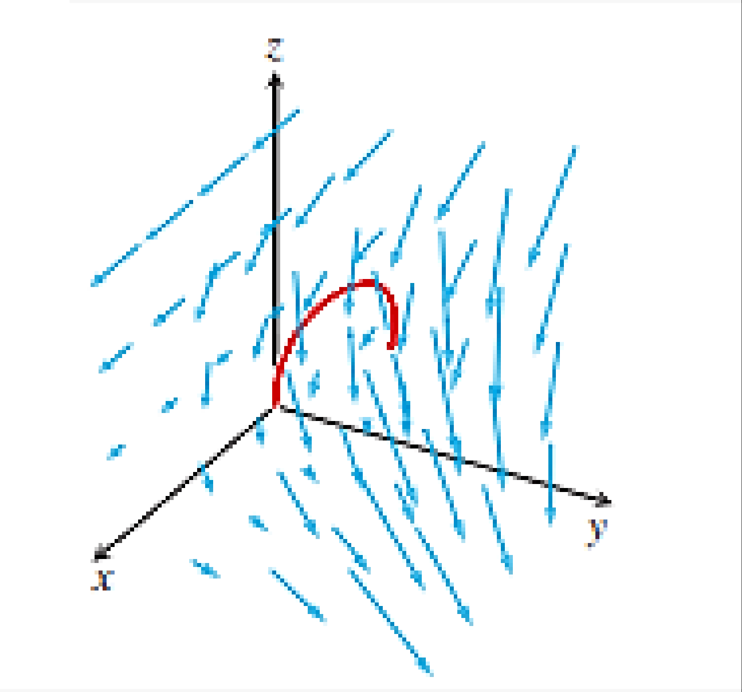
Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 16

Exercise-16.2: Q – 27 to 46, Q – 55 to 62.
Exercise-16.3: Q – 1 to 32.

Green's Theorem





Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

16

Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.4**

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- **Chapter: 16**
  - **Section: 16.4**

Green's Theorem in the Plane

We know that every line integral:

$$\int_C Mdx + Ndy,$$

can be written as a flow integral:

$$\int_C \mathbf{F}.\mathbf{T}ds,$$

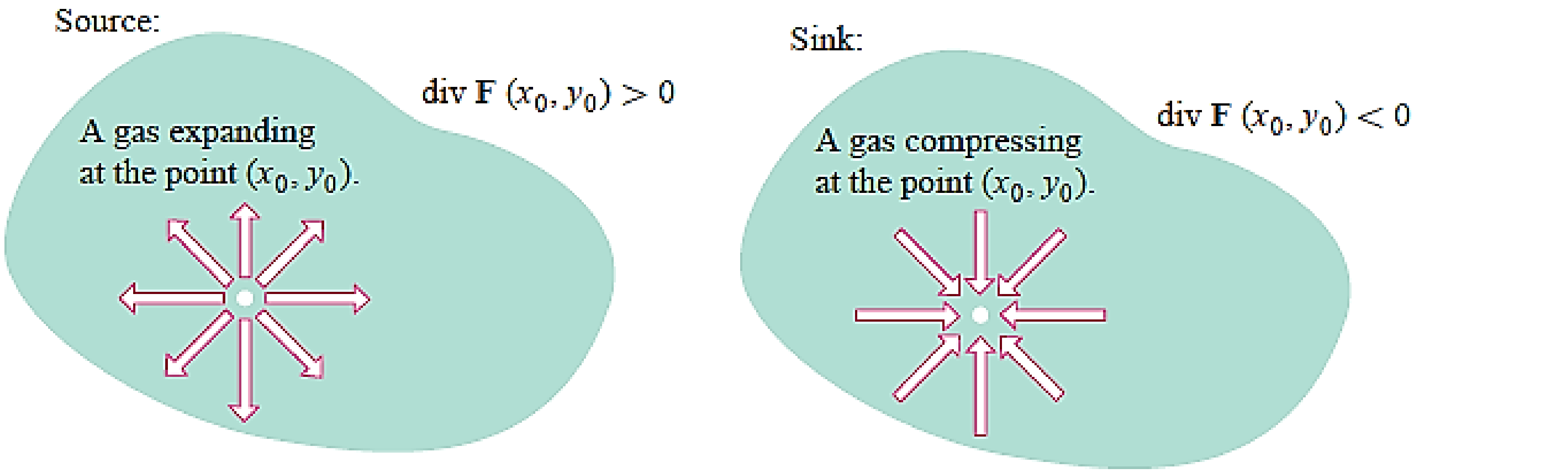
If the integral is independent of path, so the field $\mathbf{F}$ is conservative (over a domain satisfying the basic assumptions), we can evaluate the integral easily from a potential function for the field. We now consider how to evaluate the integral if it is ***not*** associated with a conservative vector field, but is a flow or flux integral across a closed curve in the xy −plane. The means for doing so is a result known as **Green's Theorem**, which converts the line integral into a double integral over the region enclosed by the path. We frame our discussion in terms of velocity fields of fluid flows because they are easy to picture. However, Green's Theorem applies to any vector field satisfying certain mathematical conditions. For Green's Theorem we need two ideas: **Flux Density and Circulation Density.**

Divergence (Flux Density)

DEFINITION Divergence (Flux Density)

The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$



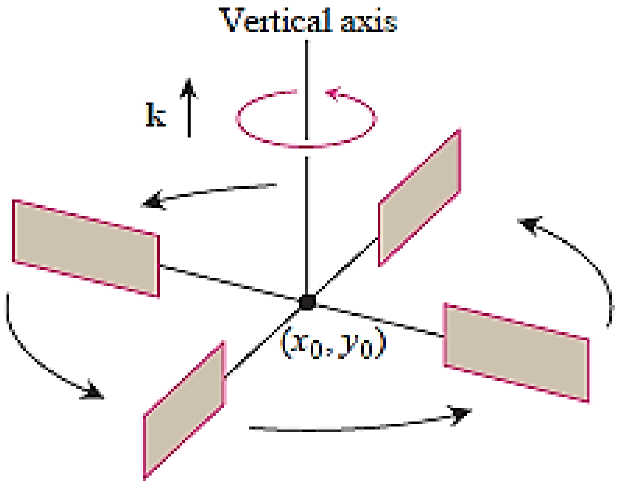
If a gas is expanding at a point the lines of flow have positive divergence; if the gas is compressing, the divergence is negative.

Spin Around an Axis: The k-Component of Curl

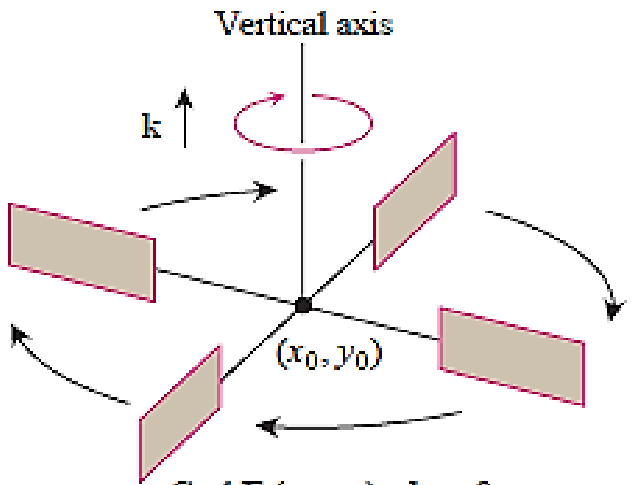
DEFINITION k-Component of Curl (Circulation Density)

The k-component of the curl (circulation density) of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$



Curl $\mathbf{F}$ $(x_0, y_0) \cdot \mathbf{k} > 0$
Counterclockwise circulation



Curl $\mathbf{F}$ $(x_0, y_0) \cdot \mathbf{k} < 0$
Clockwise circulation

If water is moving about a region in the xy −plane in a thin layer, then the $\boldsymbol{k}$ −component of the curl at a point (x_0, y_0) gives a way to measure how fast and in what direction a small paddle wheel spins if it is put into the water at (x_0, y_0) with its axis perpendicular to the plane, parallel to $\boldsymbol{k}$. Looking downward onto the xy −plane, it spins counterclockwise when $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ is positive and clockwise when the $\boldsymbol{k}$ −component is negative.

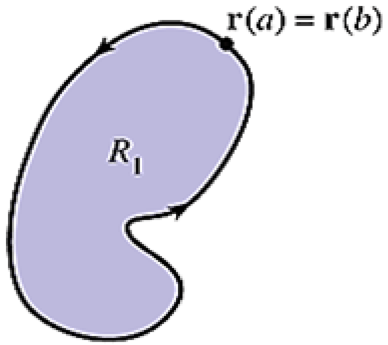
Green's Theorem

Let R be a simply connected region with a piecewise smooth boundary C, oriented counterclockwise (that is, C is traversed *once* so that the region R always lies to the *left*). If M and N have continuous partial derivatives in an open region containing R, then

$$\int_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA.$$

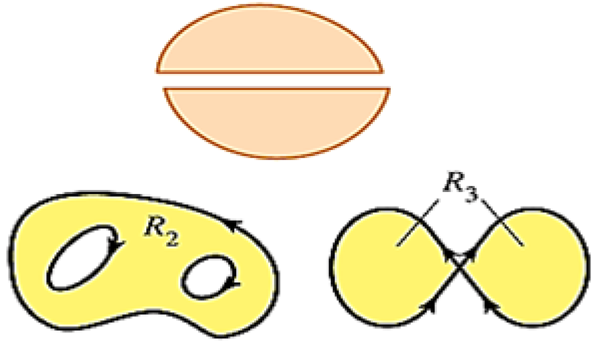
or

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$



Simply connected

One form of Green's Theorem tells us how **circulation density** can be used to calculate the line integral for flow in the xy −plane. Second form of the theorem tells us how we can calculate the flux integral, which gives the flow across the boundary, from **flux density.**



Not simply connected

Two Forms for Green's Theorem

In first form, Green's Theorem says that under suitable conditions the outward flux of a vector field across a simple closed curve in the plane equals the double integral of the divergence of the field over the region enclosed by the curve.

Green's Theorem (Flux-Divergence or Normal Form)

The outward flux of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ across a simple closed curve C equals the double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{Outward flux}} = \oint_C M\, dy - N\, dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy}_{\text{Divergence integral}}$$

$$= \iint_R \operatorname{div} \mathbf{F}\, dA.$$

Two Forms for Green's Theorem

In second form, Green's Theorem says that the counterclockwise circulation of a vector field around a simple closed curve is the double integral of the $\mathbf{k}$ −component of the curl of the field over the region enclosed by the curve.

Green's Theorem (Circulation-Curl or Tangential Form)

The counterclockwise circulation of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ around a simple closed curve C in the plane equals the double integral of $(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M\,dx + N\,dy}_{\text{Counterclockwise circulation}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}_{\text{Curl integral}}$$

$$= \iint_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA.$$

Note:

The two forms of Green's Theorem are equivalent. Applying first form to the field $\mathbf{G}_1 = \langle N, -M \rangle$, gives second form, and applying second form to $\mathbf{G}_2 = \langle -N, M \rangle$ provides us with the first form.

Evaluating a Line Integral Using Green's Theorem

- If we construct a closed curve C by piecing together a number of different curves end to end, the process of evaluating a line integral over C can be lengthy because there are so many different integrals to evaluate.
- If C bounds a region R to which Green's Theorem applies, however, we can use Green's Theorem to change the line integral around C into one double integral over R.
- Before working some examples there are some alternate notations that we need to acknowledge. When working with a line integral in which the path satisfies the condition of Green's theorem, we will often denote the line integral as,

$$\oint_C P\,dx + Q\,dy \qquad \text{or} \qquad \oint\limits_C P\,dx + Q\,dy$$

- Both of these notations do assume that C satisfies the conditions of Green's theorem so be careful in using them.
- Also, sometimes the curve C is not thought of as a separate curve but instead as the boundary of some region R and in these cases we may see C denoted as ∂R.

Example: Evaluating a Line Integral Using Green's Theorem

Evaluate the integral:

$$\oint_C xy\,dy - y^2\,dx$$

where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

Solution:

We can use either form of Green's Theorem to change the line integral into a double integral over the square.

1. **With the Normal form:** Taking $M = xy, N = y^2$, and C and R as the square's boundary and interior gives:

$$\oint_C xy\,dy - y^2\,dx = \iint_R (y + 2y)\,dxdy = \int_0^1 \int_0^1 3y\,dxdy$$

$$= \int_0^1 [3xy]_{x=0}^{x=1}\,dy = \int_0^1 3y\,dy = \left[\frac{3}{2}y^2\right]_{y=0}^{y=1} = \frac{3}{2}.$$

Solution:

2. **With the Tangential form:**

Taking $M = -y^2$ and $N = xy$ we get:

$$\oint_C -y^2\, dx + xy\, dy = \iint_R [y - (-2y)]\, dxdy = \iint_R [y + 2y]\, dxdy = \frac{3}{2}.$$

Example: Finding Outward Flux

Calculate the outward flux of the field $\mathbf{F}(x, y) = \langle x, y^2 \rangle$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

Solution:

With $M = x, N = y^2$, C the square and R the square's interior we have:

$$\text{Flux} = \oint_C \mathbf{F}.\mathbf{n}\, ds = \oint_C x\, dy - y^2\, dx = \iint_R \left(M_x + N_y\right) dxdy = \int_{-1}^{1} \int_{-1}^{1} (1 + 2y)\, dxdy = 4.$$