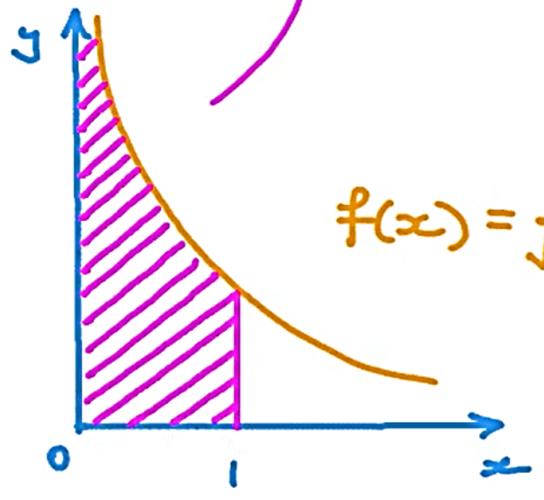
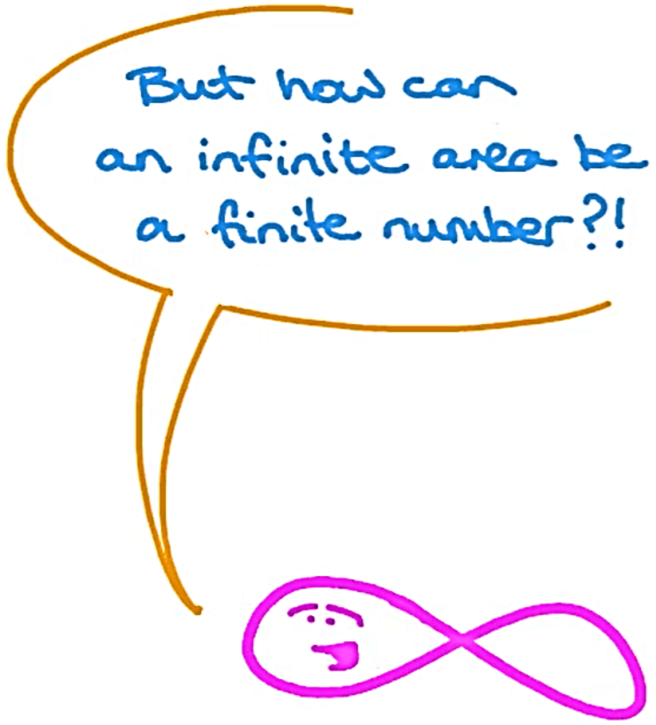
$$\int_0^1 f(x)\,dx \neq \infty$$

?!?





$$\int_a^b f(x)\,dx = \lim_{t\to 0^+}\int_t^b f(x)\,dx$$

case 1

→ One or both limits of integration are $\pm\infty$

$$\int_a^{\infty} f(x)\,dx \qquad \int_{-\infty}^{b} f(x)\,dx \qquad \int_{-\infty}^{\infty} f(x)\,dx$$

case 2

→ The integrand $f(x)$ has a discontinuity on $[a, b]$

$$\int_a^b f(x)\,dx$$

Improper Integrals

Calculus & Analytical Geometry MATH-101

Instructor: Dr. Naila Amir (SEECS, NUST)

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter:** 8
  - **Section:** 8.8

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- **Chapter:** 10
  - **Section:** 10.3, 10.4

Improper Integrals

- The definition of a definite integral:

$$\int_a^b f(x)\,dx,$$

requires that the interval $[a, b]$ be finite, and that $f(x)$ be continuous on $[a, b]$.

- In practice, we may encounter problems that fail to meet one or both of these conditions. So, we extend the concept of a definite integral to the case where the interval is infinite and also to the case where $f(x)$ has an infinite discontinuity in $[a, b]$.

- Integrals that possess either property are said to be **improper integrals.**

- Improper integrals play an important role when investigating the convergence of certain infinite series.

Improper Integral

TYPE-1:
Infinite Limits of Integration

Example

$$\int_{1}^{\infty} \frac{1}{x^2}\, dx$$

TYPE-2:
Discontinuous Integrand

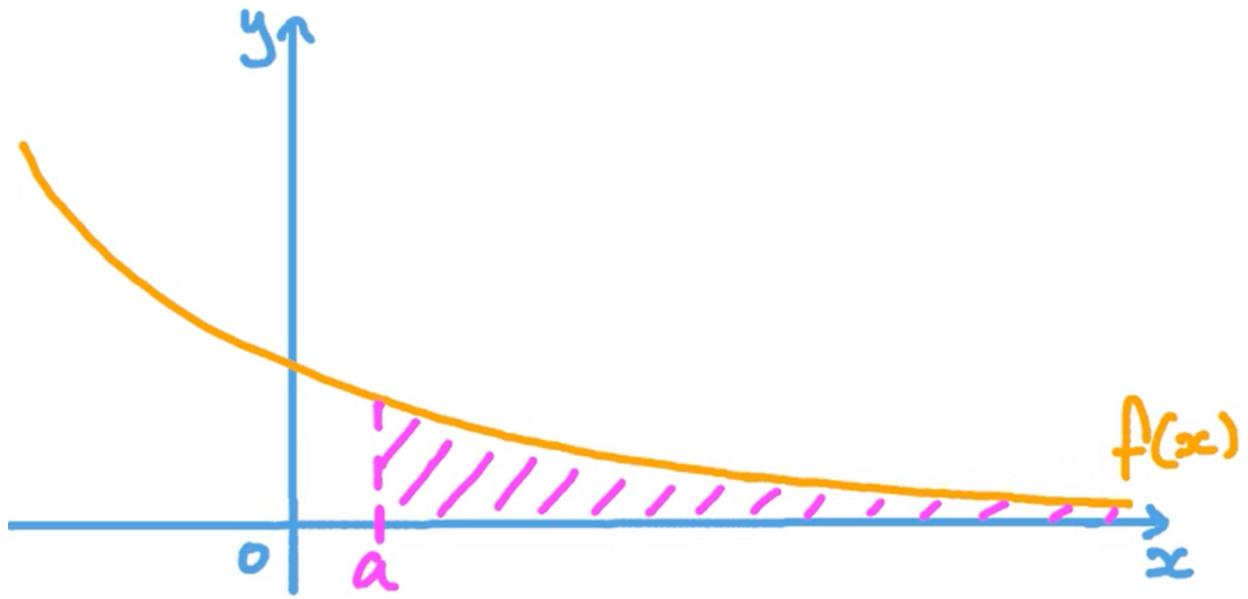
Example

$$\int_{-1}^{1} \frac{1}{x^2}\, dx$$

Type 1: Improper Integrals with Infinite Limits of Integration

- $\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$, where $t > a$.
- $\int_{-\infty}^{a} f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx$, where $t < a$.
- $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{s\to-\infty} \int_s^c f(x)\,dx + \lim_{t\to\infty} \int_c^t f(x)\,dx$, where c is any real number.

- An integral with an infinite bound converges if the limit exists, and diverges if it does not.



$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \underbrace{\int_a^t f(x)\,dx}, \quad \text{for } t > a$$

$\int_a^t f(x)\,dx$ exists

$\lim_{t\to\infty} \int_a^t f(x)\,dx$ exists

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 1:

If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\left(\int_a^b f(x)\,dx\right)$$

Example

$$\int_1^\infty \frac{1}{x^2}\,dx$$

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 1:

If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \left(\int_{a}^{b} f(x)\,dx \right)$$

Example

$$\int_{-\infty}^{0} xe^{x}\,dx$$

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 1:

If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

where c is any real number.

Example

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 1

- The improper integrals:

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\left(\int_a^b f(x)\,dx\right)$$

and

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\left(\int_a^b f(x)\,dx\right)$$

are **convergent** if the corresponding **limit exists** and **divergent** if the **limit does not exist.**

- The improper integral:

$$\int_{-\infty}^{\infty} f(x)\,dx$$

is convergent only if both improper integrals

$$\int_c^{\infty} f(x)\,dx \quad \text{and} \quad \int_{-\infty}^{c} f(x)\,dx$$

are convergent, otherwise it diverges.

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 1

- Any of the integrals:

$$\int_a^{\infty} f(x)\,dx\,, \qquad \int_{-\infty}^{b} f(x)\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\,dx$$

can be interpreted as an **area** if $f(x) \geq 0$ on the interval of integration.

- If $f(x) \geq 0$ and the improper integral **converges**, we say the area under the curve is **finite** and is equal to the value of the limit. However, If $f(x) \geq 0$ and the improper integral **diverges**, we say the area under the curve is **infinite**.

Example:

Evaluate

$$\int_{1}^{\infty} (1-x)\cdot e^{-x}\,dx.$$

Solution:

Note that:

$$\int (1-x)\cdot e^{-x}\,dx = -e^{-x}\cdot(1-x) - \int e^{-x}dx = x\cdot e^{-x} + C$$

Now apply the definition of an improper integral

$$\int_{1}^{\infty} (1-x)\cdot e^{-x}\,dx = \lim_{b\to\infty}\left[\frac{x}{e^{x}}\right]_{1}^{b} = \left(\lim_{b\to\infty}\frac{b}{e^{b}}\right) - \frac{1}{e} \qquad (*)$$

$\left(\frac{\infty}{\infty}\right)$ form

Solution:

Now using L'Hopital rule for the first term, we get:

$$\lim_{b\to\infty} \frac{b}{e^b} = \lim_{b\to\infty} \frac{1}{e^b} = 0$$

Therefore, equation $(*)$ becomes:

$$\int_1^\infty (1-x)\cdot e^{-x}\,dx = -\frac{1}{e}$$

Example:

Evaluate

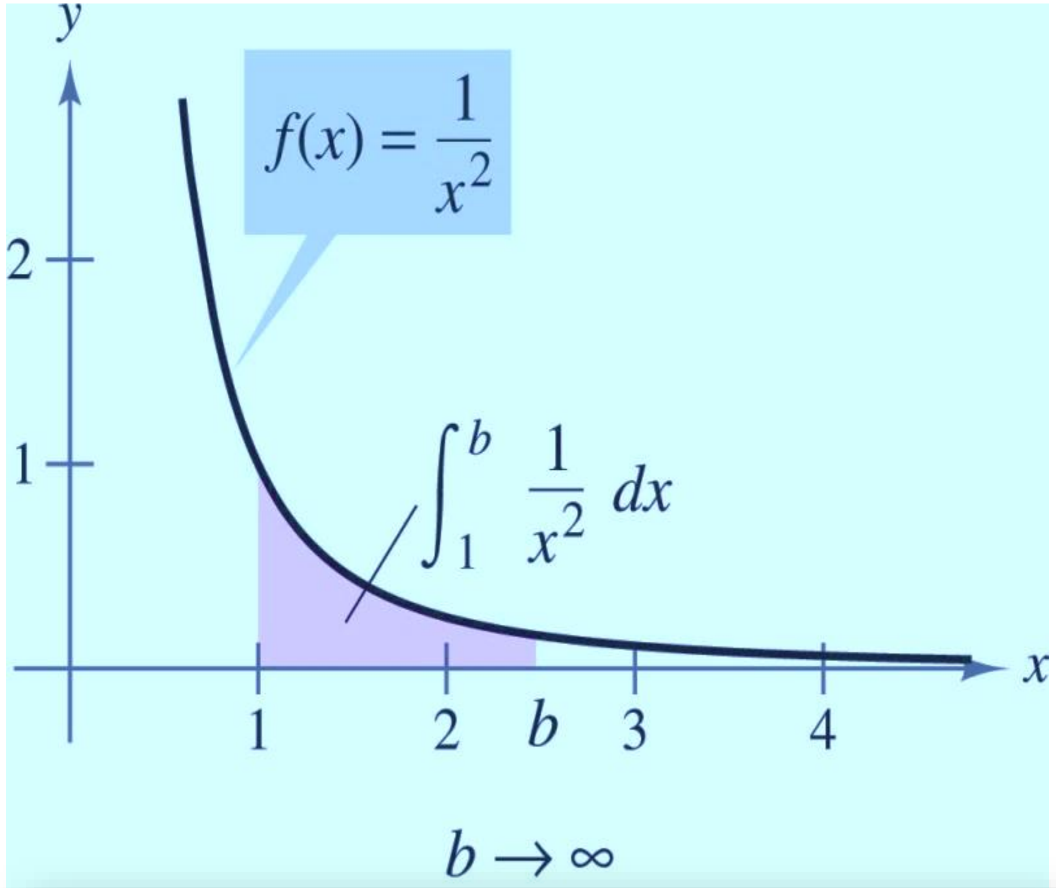
$$\int_1^{\infty} \frac{dx}{x^2}.$$

Solution:

Note that:

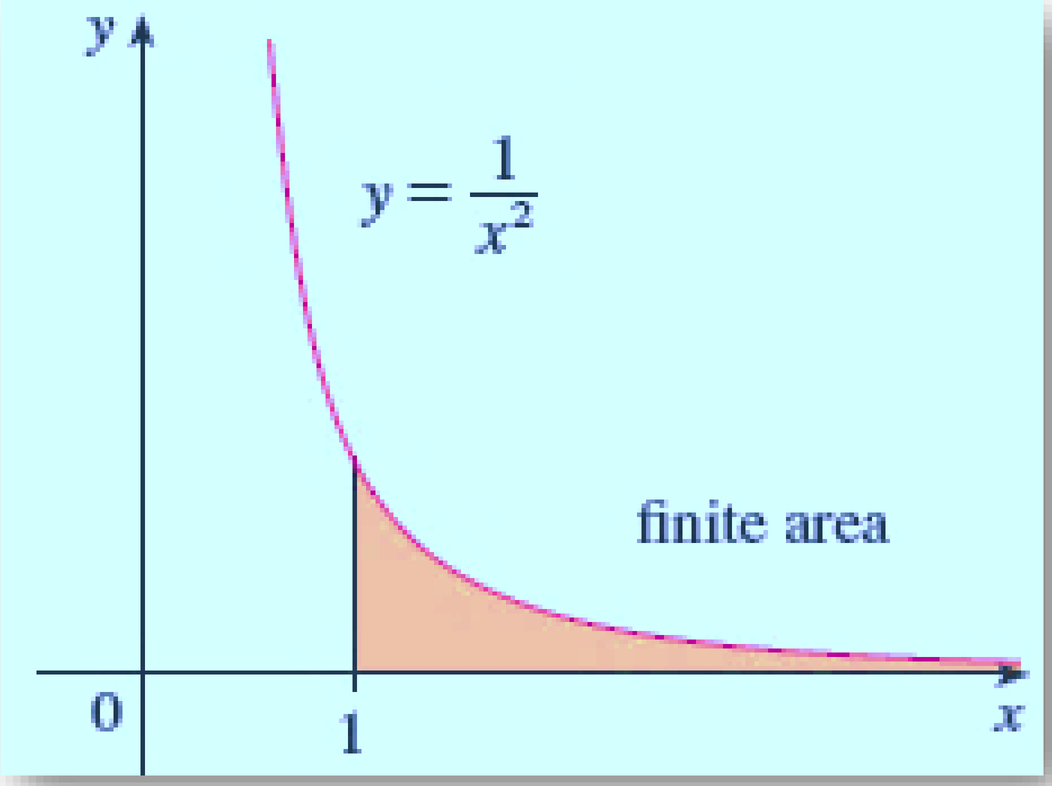
$$\int_1^{b} \frac{dx}{x^2} = -\frac{1}{x}\Big]_1^b = -\frac{1}{b} + 1 \;= 1 - \frac{1}{b}.$$

This can be interpreted as the area of the shaded region in the figure:



Now, taking the limit as $b \to \infty$

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x^2} = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1$$



Thus, the improper integral converges, and the area has finite value 1.

Example:

Evaluate

$$\int_1^\infty \frac{dx}{x}.$$

Solution:

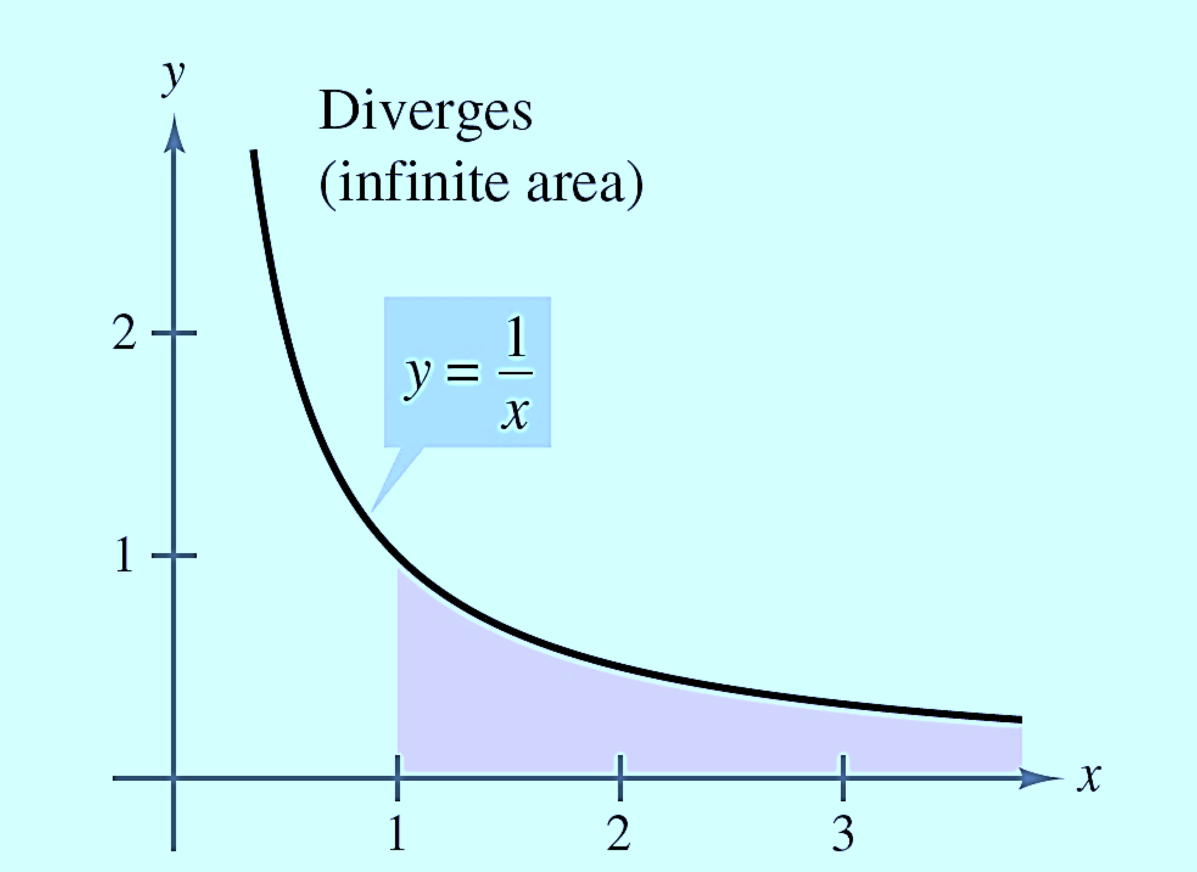
Note that:

$$\int_1^b \frac{dx}{x} = \ln x\Big]_1^b = \ln b - 0.$$

Now, taking the limit as $b \to \infty$

$$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} (\ln b - 0) = \infty.$$

Thus, the improper integral diverges, and the unbounded region has an infinite area.



This unbounded region has an infinite area.

Conclusion:

- Geometrically, this says that although the curves $y = 1/x^2$ and $y = 1/x$ look very similar for $x > 0$, the region under $y = 1/x^2$ to the right of $x = 1$ has a finite area whereas the corresponding region under $y = 1/x$ has infinite area.

- Note that both $1/x^2$ and $1/x$ approach 0 as $x \to \infty$ but $1/x^2$ approaches 0 faster than $1/x$. The values of $1/x$ don't decrease fast enough for its integral to have a finite value.

We summarize this as follows:

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ is convergent if } p > 1 \text{ and divergent if } p \leq 1.$$

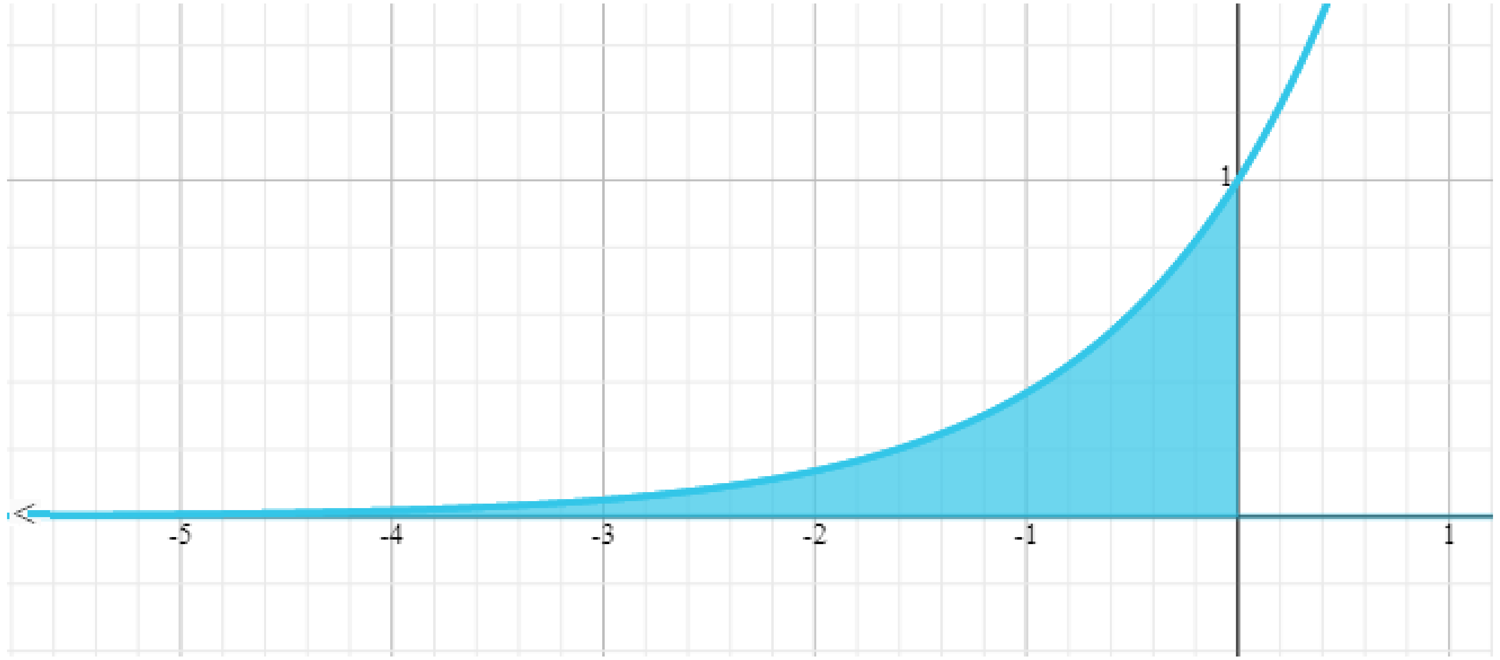
Example:

Evaluate

$$\int_{-\infty}^{0} e^{x}\,dx.$$

Solution:

$$\int_{-\infty}^{0} e^{x}\,dx = \lim_{a\to-\infty}\int_{a}^{0} e^{x}\,dx$$



$$= \lim_{a\to-\infty}\left[e^{x}\Big|_{a}^{0}\right]$$

$$= \lim_{a\to-\infty}\left[e^{0}-e^{a}\right]$$

$$= 1-e^{-\infty} = 1.$$

Here, the improper integral converges, and the area has finite value 1.

Example:

Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx\,.$$

Solution:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx\,.$$

Now

$$\int_{-\infty}^{0} \frac{1}{x^2+1}\,dx = \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{x^2+1}\,dx = \lim_{a\to-\infty} \left[\tan^{-1} x\Big|_a^0\right]$$

$$= \lim_{a\to-\infty} [\tan^{-1}(0) - \tan^{-1}(a)]$$

$$= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

Solution:

Now

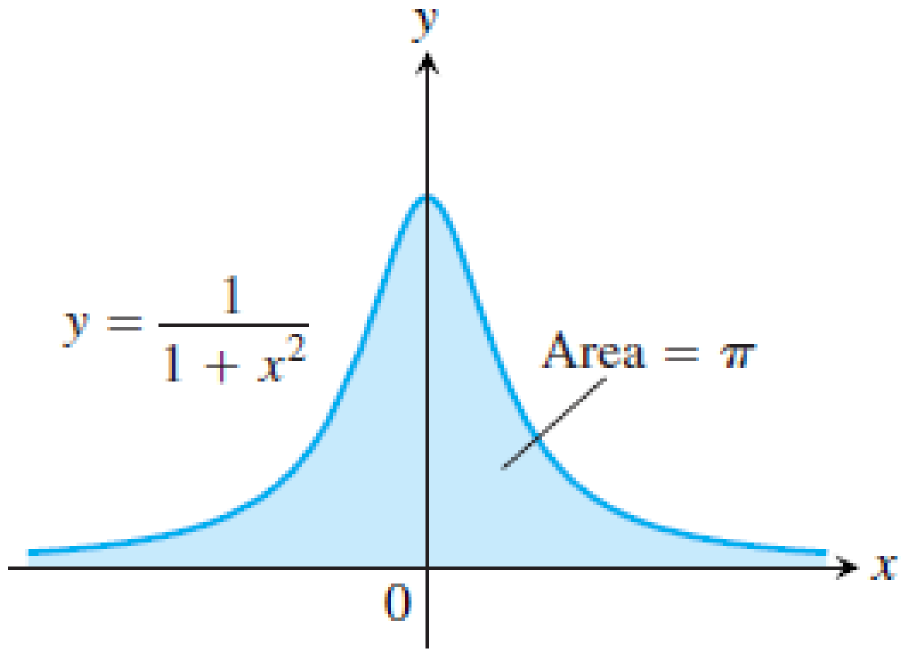
$$\int_0^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_0^{b} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \left[\tan^{-1} x\Big|_0^b\right]$$

$$= \lim_{b\to\infty} [\tan^{-1}(b) - \tan^{-1}(0)]$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$



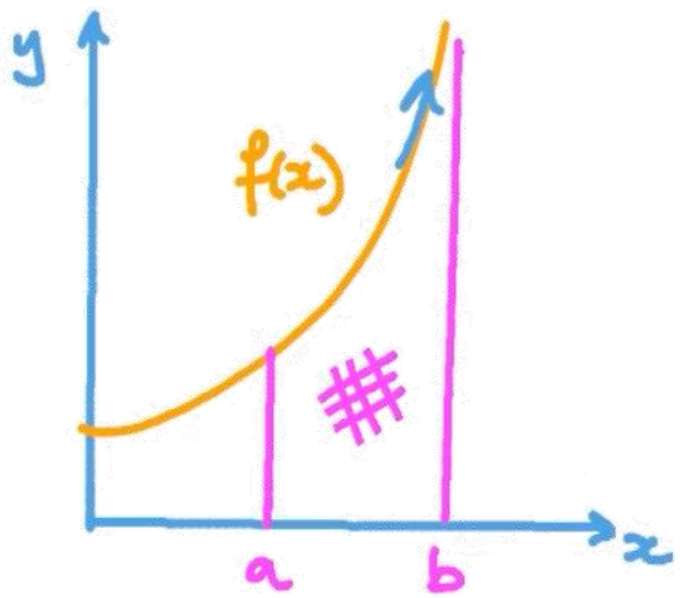
Since $1/x^2+1 > 0$, so the improper integral can be interpreted as the (finite) area beneath the curve and above the x –axis

Type 2: Integrals in which the integrands become infinite within the intervals of Integration.

If f is continuous on $[a, b)$ and has a discontinuity at b, then

$$\boxed{\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx}$$

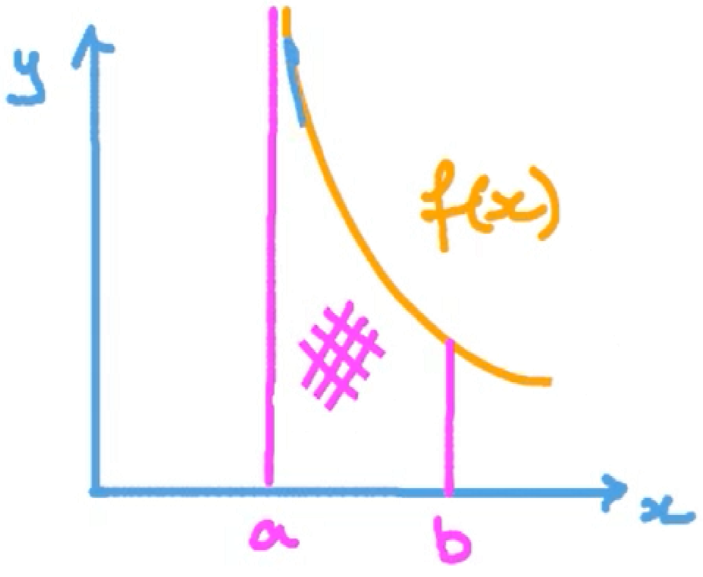
if the limit exists and is finite.



If f is continuous on $(a, b]$ and has a discontinuity at a, then

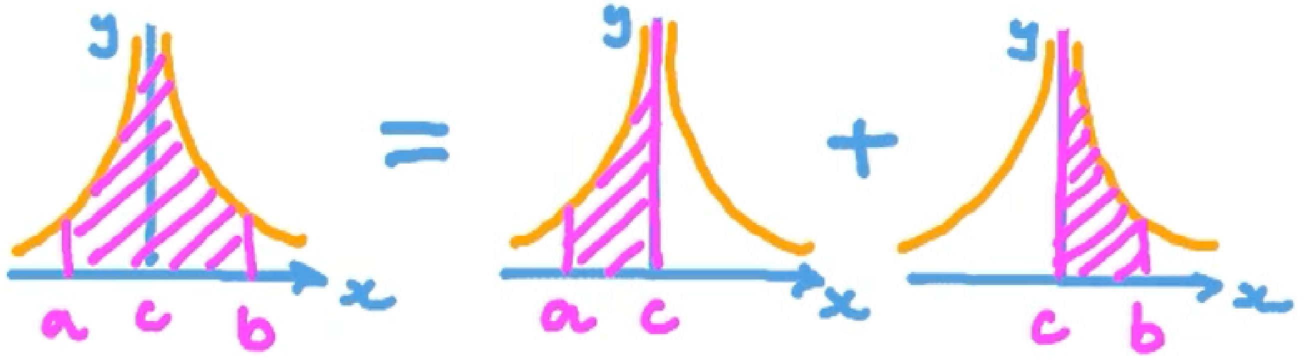
$$\boxed{\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx}$$

if the limit exists and is finite.



If f has a discontinuity at c where $a < c < b$ ... and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define:

$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$



Example:

Evaluate

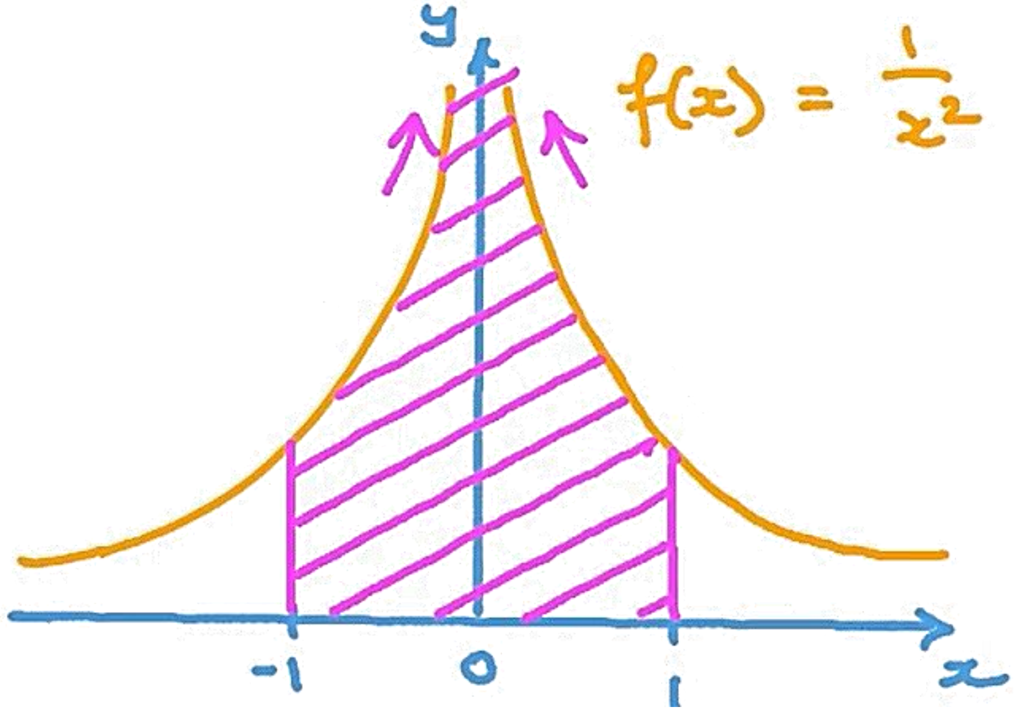
$$\int_{-1}^{1} \frac{1}{x^2}\, dx\,.$$

Solution:

The standard tool that we might use to evaluate the given integral is the fundamental theorem of calculus. Thus,

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg|_{-1}^{1} = \left[-\frac{1}{1}\right] - \left[-\frac{1}{(-1)}\right] = -1 - 1 = -2.$$

Now, if we look at the graph of $f(x) = \frac{1}{x^2}$, we see that it has an infinite discontinuity at $x = 0$. Moreover, the area under the curve appears to be entirely above the x-axis. So it should be positive. But our integral is equal to a negative number. This means our calculation is wrong. With this integral and others like it, the infinite discontinuity is causing us problems.



Caution!!!!!

- From now on, whenever we encounter an integral:
$$\int_a^b f(x)\,dx,$$
we must examine the function $f(x)$ on $[a, b]$ and then decide if the integral is an ordinary definite integral or an improper integral because this type of integral may not appear to be improper at first glance since there are no signs of obvious infinities appearing in the integral limits.

- If $f(x)$ is continuous on $[a, b]$, it will be a proper, an ordinary integral. Otherwise, if the function is discontinuous on $[a, b]$, then the integral will be an **improper integral**.

- We cannot use standard techniques to evaluate this integral and we need some additional tools.

- Let us introduce some new concepts, which will allow us to evaluate some integrals where an infinite discontinuity exists.

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 2:

If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+}\left(\int_c^b f(x)\,dx\right).$$

Example

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$$

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 2:

If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \left(\int_a^c f(x)\,dx \right).$$

Examples:

$$\int_0^{\pi/2} \sec x \; dx \quad \text{and} \quad \int_0^1 \frac{1}{1-x}\,dx$$

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 2:

If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example

$$\int_{-1}^{1} \frac{1}{x^2}\,dx$$

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 2

- The improper integrals:

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\left(\int_c^b f(x)\,dx\right)$$

and

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\left(\int_a^c f(x)\,dx\right)$$

are **convergent** if the corresponding **limit exists** and **divergent** if the **limit does not exist.**

- The improper integral on the left side of the equation:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,,$$

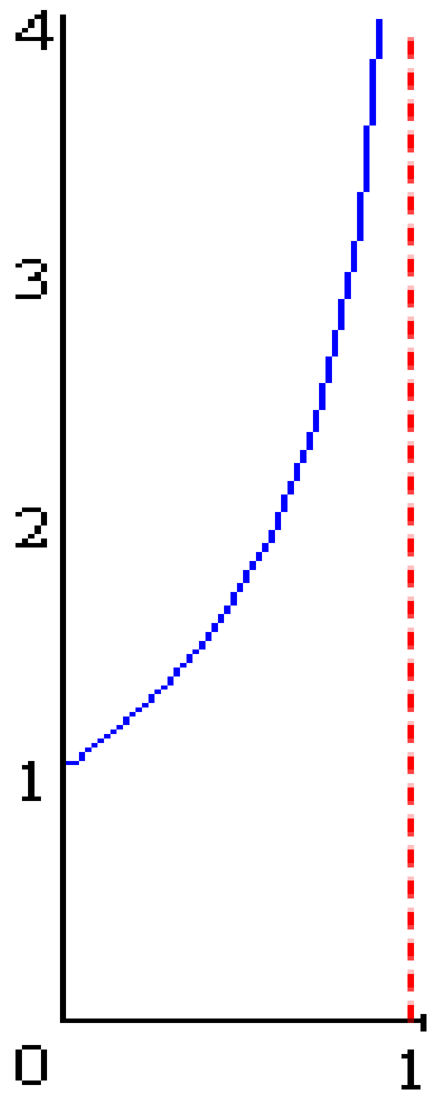
converges if *both* integrals on the right side converge; otherwise, it diverges.

Example:

Evaluate

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\, dx .$$

The function is undefined at $x = 1$.



Solution:

We can define the given integral as:

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\, dx = \lim_{b\to 1^-} \int_0^b \sqrt{\frac{1+x}{1-x}}\, dx$$

First, we solve the corresponding indefinite integral as:

$$\int \frac{\sqrt{1+x}}{\sqrt{1-x}}\, dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \frac{\sqrt{1+x}}{\sqrt{1+x}}\, dx = \int \frac{1+x}{\sqrt{1-x^2}}\, dx$$

Solution:

$$\Rightarrow \int \frac{\sqrt{1+x}}{\sqrt{1-x}}\,dx = \underbrace{\int \frac{1}{\sqrt{1-x^2}}\,dx}_{I_1} + \underbrace{\int \frac{x}{\sqrt{1-x^2}}\,dx}_{I_2}. \qquad \text{(I)}$$

Now

$$I_1 = \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x\,. \qquad \text{(II)}$$

For evaluating I_2, let $u = 1 - x^2 \Rightarrow du = -2x\,dx$ and $-\frac{1}{2}du = x\,dx$. Thus,

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,dx = -\frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{1/2}\right) = -\sqrt{u} = -\sqrt{1-x^2} \qquad \text{(III)}$$

Using (II) and (III) in (I), we get:

$$\int \frac{\sqrt{1+x}}{\sqrt{1-x}}\,dx = \arcsin x - \sqrt{1-x^2} + C.$$

Solution:

Thus,

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx = \lim_{b\to 1^-} \int_0^b \sqrt{\frac{1+x}{1-x}}\,dx$$

$$= \lim_{b\to 1^-} \left[\arcsin x - \sqrt{1-x^2} \Big|_0^b \right]$$

$$= \lim_{b\to 1^-} \left[\left(\arcsin b - \sqrt{1-b^2} \right) - \left(\arcsin(0) - \sqrt{1-(0)^2} \right) \right]$$

$$= \lim_{b\to 1^-} \left[\left(\arcsin b - \sqrt{1-b^2} \right) \right] + 1$$

$$= \frac{\pi}{2} + 1$$

This integral converges because it approaches a finite value.

Example:

Evaluate

$$\int_0^1 \frac{dx}{x}.$$

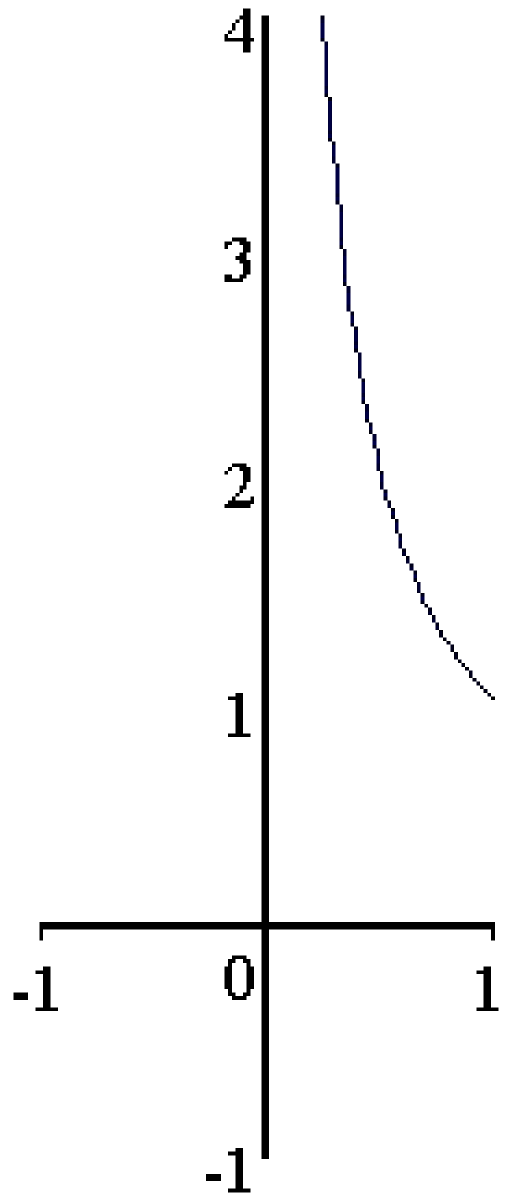
The function is undefined at $x = 0$.

Solution:

We can define the given integral as:

$$\int_0^1 \frac{1}{x} dx = \lim_{b \to 0^+} \int_b^1 \frac{1}{x}\, dx = \lim_{b \to 0^+} \left[\ln x \Big|_b^1 \right]$$

$$= \lim_{b \to 0^+} \; [\ln 1 - \ln b] = \lim_{b \to 0^+} \left[\ln \frac{1}{b} \right] = \infty$$



This integral diverges.

Example:

Evaluate

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

The function is undefined at $x = 1$.

Solution:

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 (x-1)^{-2/3}\ dx + \int_1^3 (x-1)^{-2/3}\ dx$$

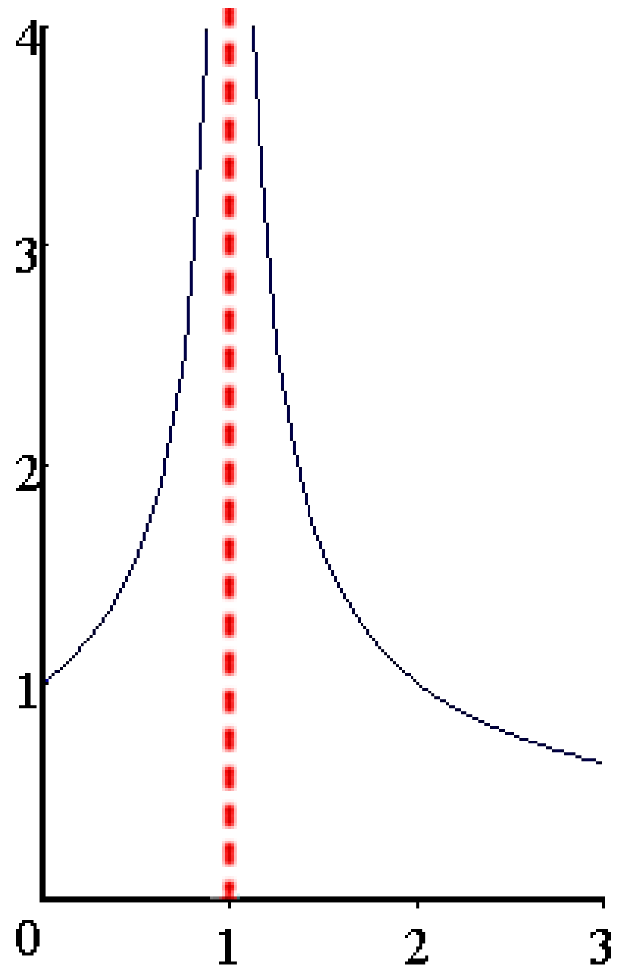
We can define the given integral as:

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b (x-1)^{-2/3}\ dx + \lim_{c\to 1^+} \int_c^3 (x-1)^{-2/3}\ dx$$

$$= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big|_0^b + \lim_{c\to 1^+} 3(x-1)^{1/3}\Big|_c^3$$

$$= \lim_{b\to 1^-}\left[3(b-1)^{1/3} - 3(-1)^{1/3}\right] + \lim_{c\to 1^+}\left[3\cdot 2^{1/3} - 3(c-1)^{1/3}\right]$$

$$= 3 + 3\sqrt[3]{2}\,.$$



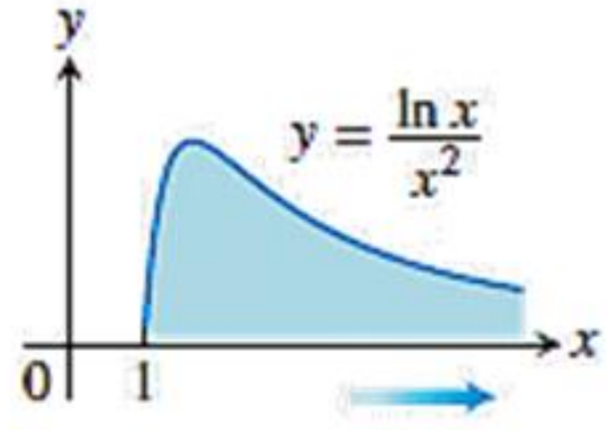
This integral converges because it approaches a finite value.

Summary

Infinite limits of integration: Type 1

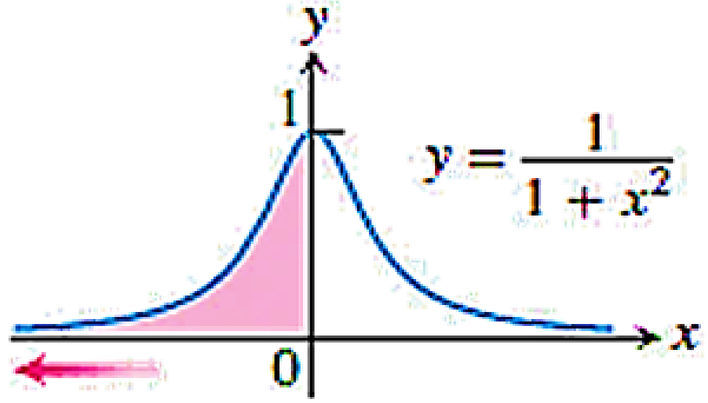
1. Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^2}\,dx$$



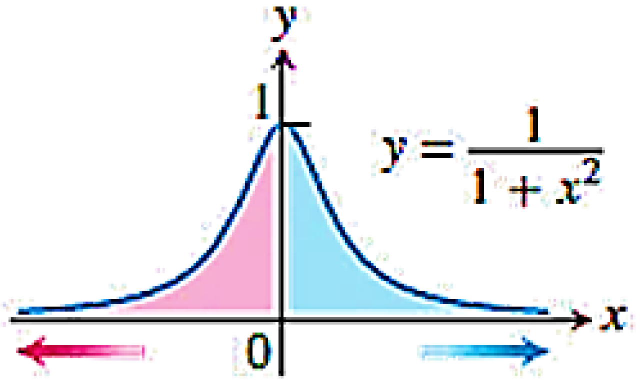
2. Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^{0} \frac{dx}{1+x^2}$$



3. Both limits

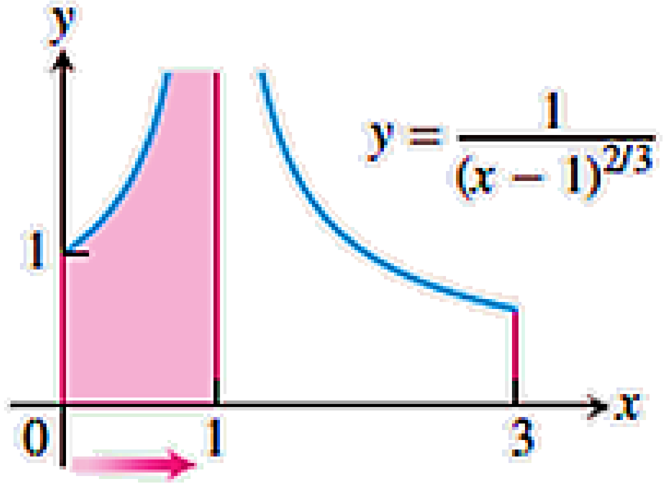
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_b^{0} \frac{dx}{1+x^2} + \lim_{c\to\infty} \int_0^{c} \frac{dx}{1+x^2}$$



Integrand becomes infinite: Type 2

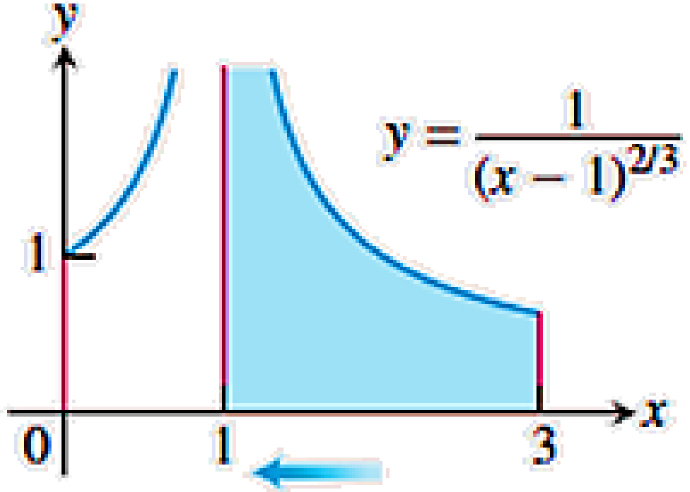
4. Upper endpoint

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$



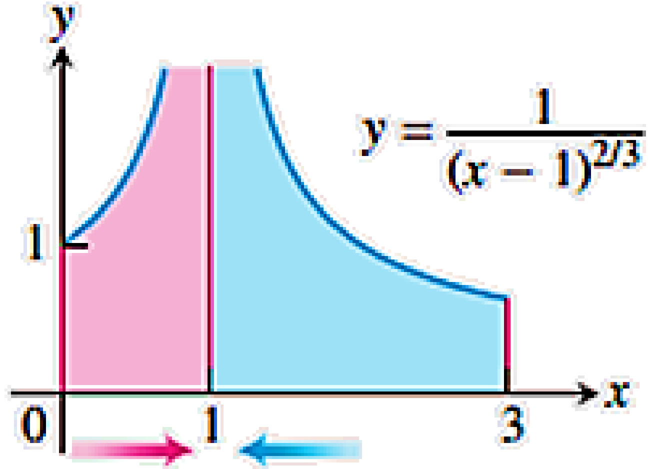
5. Lower endpoint

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{d\to 1^+} \int_d^3 \frac{dx}{(x-1)^{2/3}}$$



6. Interior point

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$



Practice Questions

Evaluate the following improper integrals and determine whether these integrals are convergent or they are divergent.

1. $\int_0^\infty e^{-2x} \cos 2x \, dx.$ $\left(Ans:\ \frac{1}{4}\right)$

2. $\int_{-\infty}^0 \frac{dx}{(2x-1)^3}.$ $\left(Ans:\ \frac{-1}{4}\right)$

3. $\int_{-\infty}^\infty \frac{x}{\sqrt{x^2+2}} dx.$ $(Ans:\ Divergent)$

4. $\int_{-\infty}^\infty \frac{x}{x^4+1} dx..$ $(Ans:\ 0)$

Practice Questions

Evaluate the following improper integrals and determine whether these integrals are convergent or they are divergent.

5. $\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx.$ $\left(Ans\colon \frac{\pi}{2}\right)$

6. $\int_{-1}^8 \frac{1}{x^{1/3}}\, dx\,.$ $\left(Ans\colon \frac{9}{2}\right)$

7. $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx.$ $(Ans\colon 2(e-1)\,)$

8. $\int_0^5 \frac{1}{x^2+2x-3}\, dx.$ $(Ans\colon Divergent)$

Practice Questions

Book: Thomas Calculus (11th Edition) by Georg B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Exercise: 8.8

  Q # 1 to Q # 34, Q # 65, Q # 66.

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 10.3

  Q # 1 to Q # 24.
- Exercise: 10.4

  Q # 1 to Q # 30.