EE-381 Robotics-1

UG ELECTIVE COURSE

Lecture 3

Dr. Hafsa Iqbal
Department of Electrical Engineering,
School of Electrical Engineering and Computer Science,
National University of Sciences and Technology,
Pakistan

Last Lecture

Enrollment Code: **983675410**

- Robot Configurations
- Robot Programming/training
- Translation and rotation

Pose

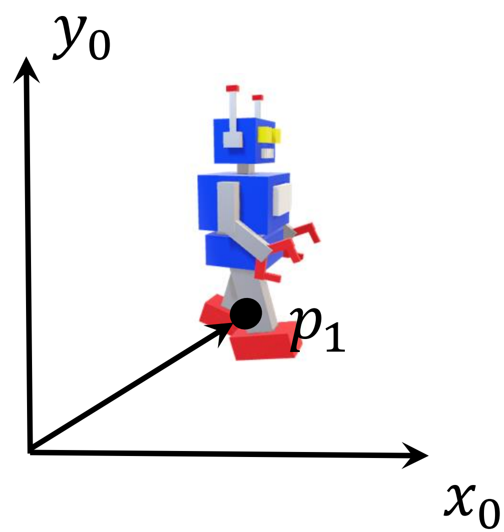
- How to define the pose of an object in space?
- **Pose**: combination of **position and orientation**

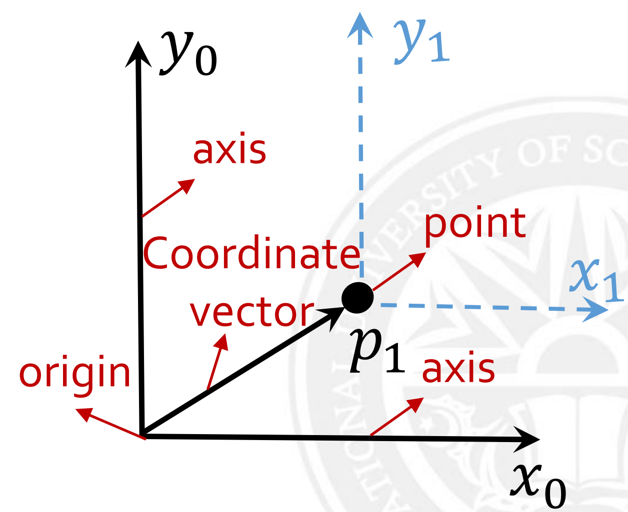
- A point in space?

- Coordinate frame/ Cartesian coordinate system?

Pose

- ***Convention***: Attach the coordinate frame to the object. It enables us to <u>describe</u> the pose of the object with respect to reference/universal coordinate frame.
- **Assumption**: Object has rigid body
- What should be the required dimension to define the pose of an object?





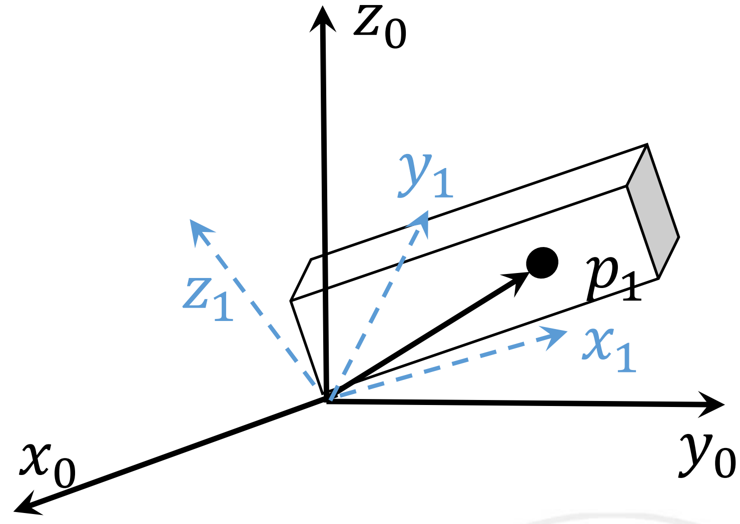
Pose: Position

- **Position**: we can locate any point in space with 3D position vector

$$p_1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

or

$$p_1 = ux_1 + vy_1 + wz_1$$



Pose: Position

- Project the point p_1 on reference frame $\{0\}$

$$p_0 = (ux_1 + vy_1 + wz_1).\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} (ux_1 + vy_1 + wz_1).x_0 \\ (ux_1 + vy_1 + wz_1).y_0 \\ (ux_1 + vy_1 + wz_1).z_0 \end{bmatrix}$$

$$p_0 = \begin{bmatrix} ux_1.x_0 + vy_1.x_0 + wz_1.x_0 \\ ux_1.y_0 + vy_1.y_0 + wz_1.y_0 \\ ux_1.z_0 + vy_1.z_0 + wz_1.z_0 \end{bmatrix}$$

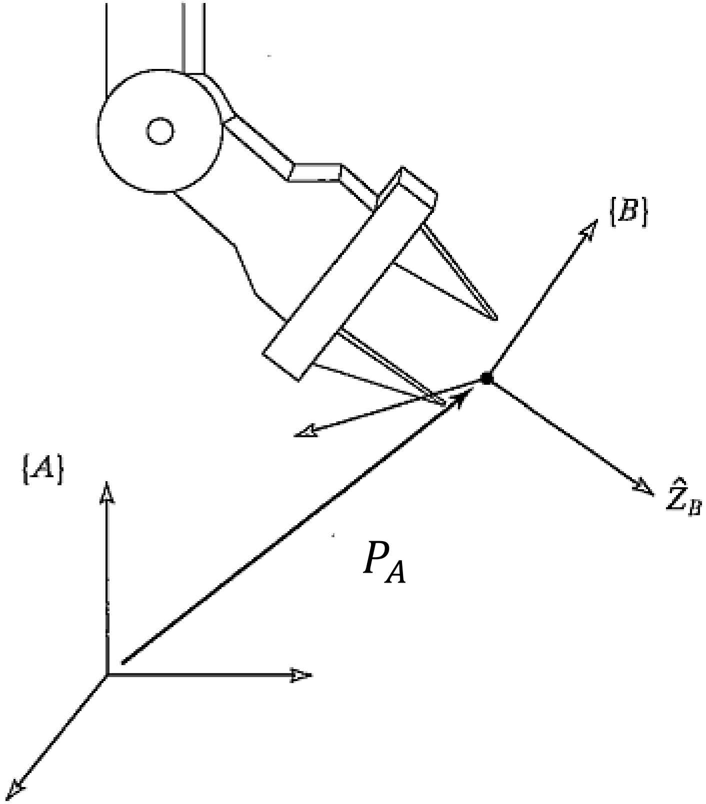
$$p_0 = \underbrace{\begin{bmatrix} x_1.x_0 + y_1.x_0 + z_1.x_0 \\ x_1.y_0 + y_1.y_0 + z_1.y_0 \\ x_1.z_0 + y_1.z_0 + z_1.z_0 \end{bmatrix}}_{R_1^0} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{p_1}$$

$$p_0 = R_1^0 p_1$$

Pose: Rotation

- To describe the orientation of a body, we attach a coordinate system to the body and then give a description of this coordinate system relative to the reference system.

- $R_B^A = \begin{bmatrix} x_B.x_A & y_B.x_A & z_B.x_A \\ x_B.y_A & y_B.y_A & z_B.y_A \\ x_B.z_A & y_B.z_A & z_B.z_A \end{bmatrix}$



Pose

- **Position** of point are described with vectors

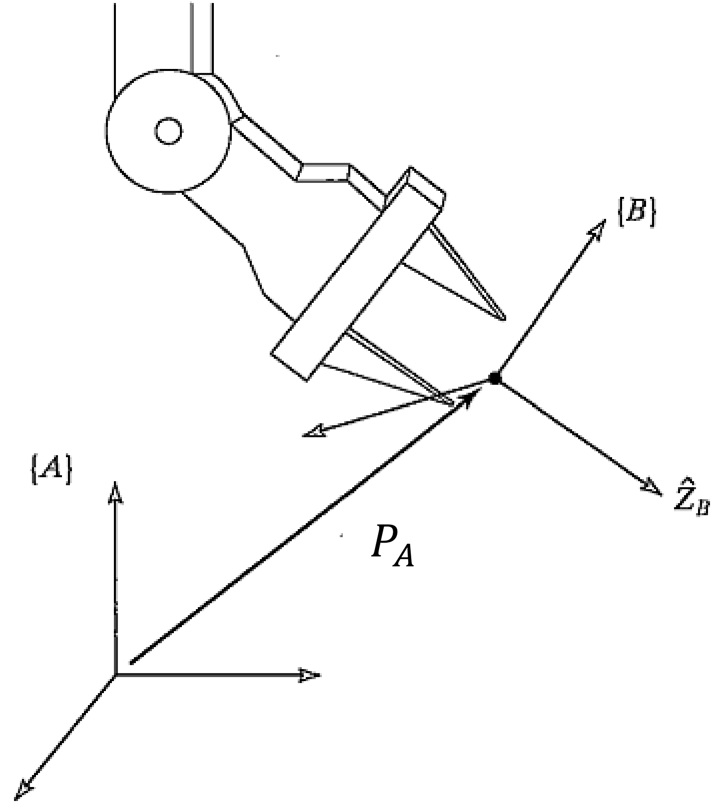
- **Orientation** of bodies are described with an attached coordinate system using **Rotation matrix**

Frame Description

- The information needed to completely specify the whereabouts of the manipulator hand is a position and an orientation

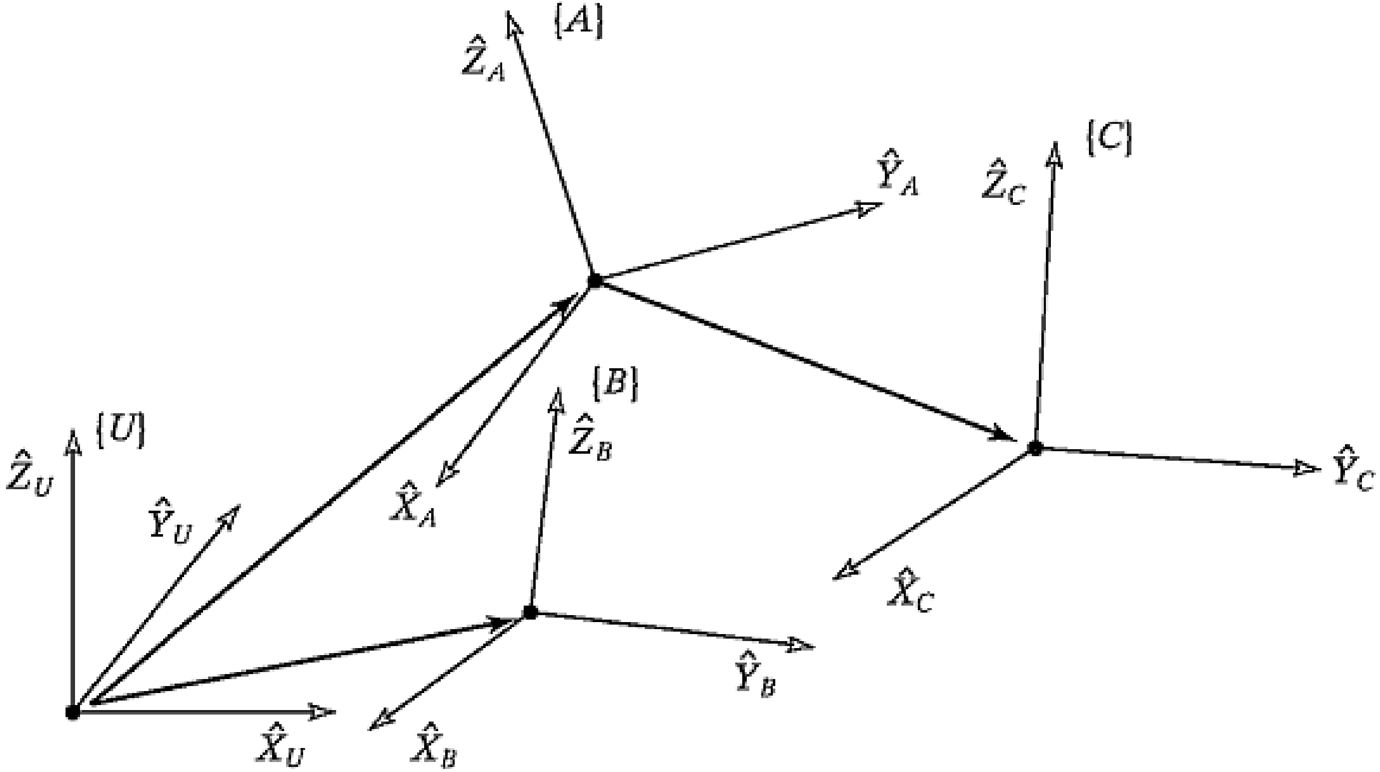
- Position and orientation of frame

$$\{B\} = \{R_B^A, d_B^A\}$$



Frame Description

- Compound rotations



Homogeneous Representation

- Translation represented by a vector d
  - vector addition
- Rotation represented by a matrix R
  - matrix-matrix and matrix-vector multiplication
- Convenient to have a uniform representation of translation and rotation.

- Obviously vector addition will not work for rotation.

- Can we use matrix multiplication to represent translation?

Mappings

Changing descriptions from Frame to Frame

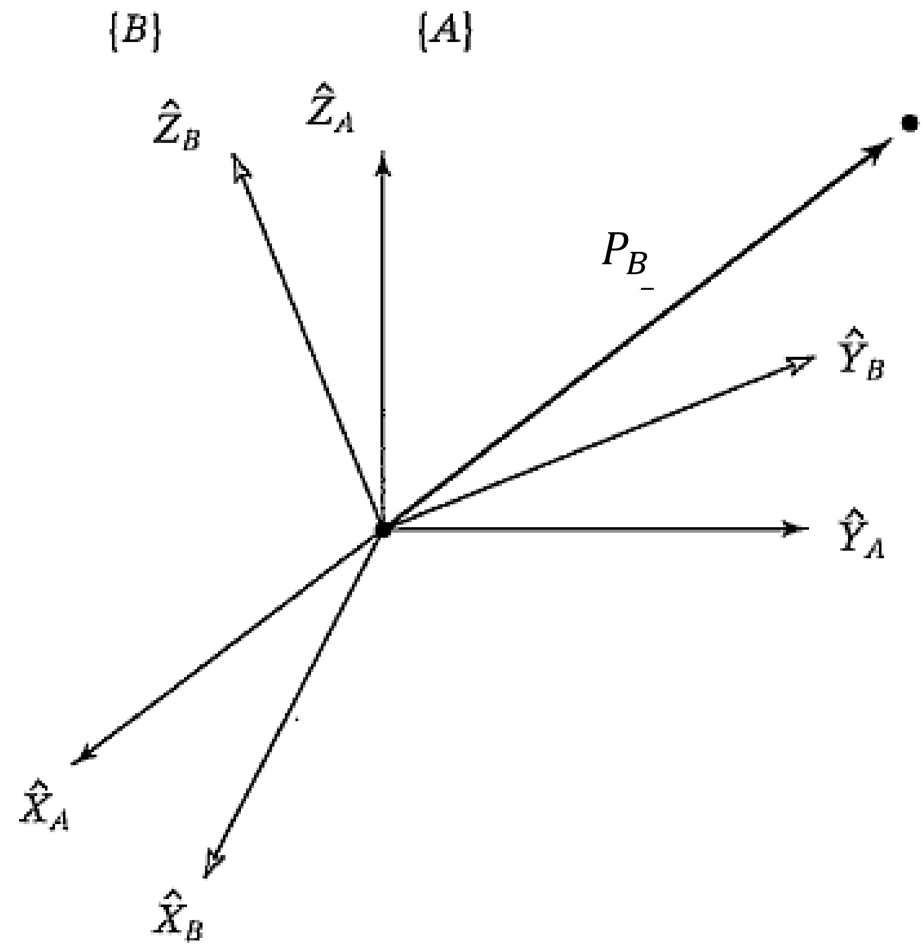
- In robotics, we are concerned with expressing the same quantity in terms of various reference coordinate systems

- We now consider the mathematics of mapping in order to change descriptions from frame to frame.

Mappings

- **Case 1**: Mappings involving rotated frames

$$P_A = R_B^A P_B$$

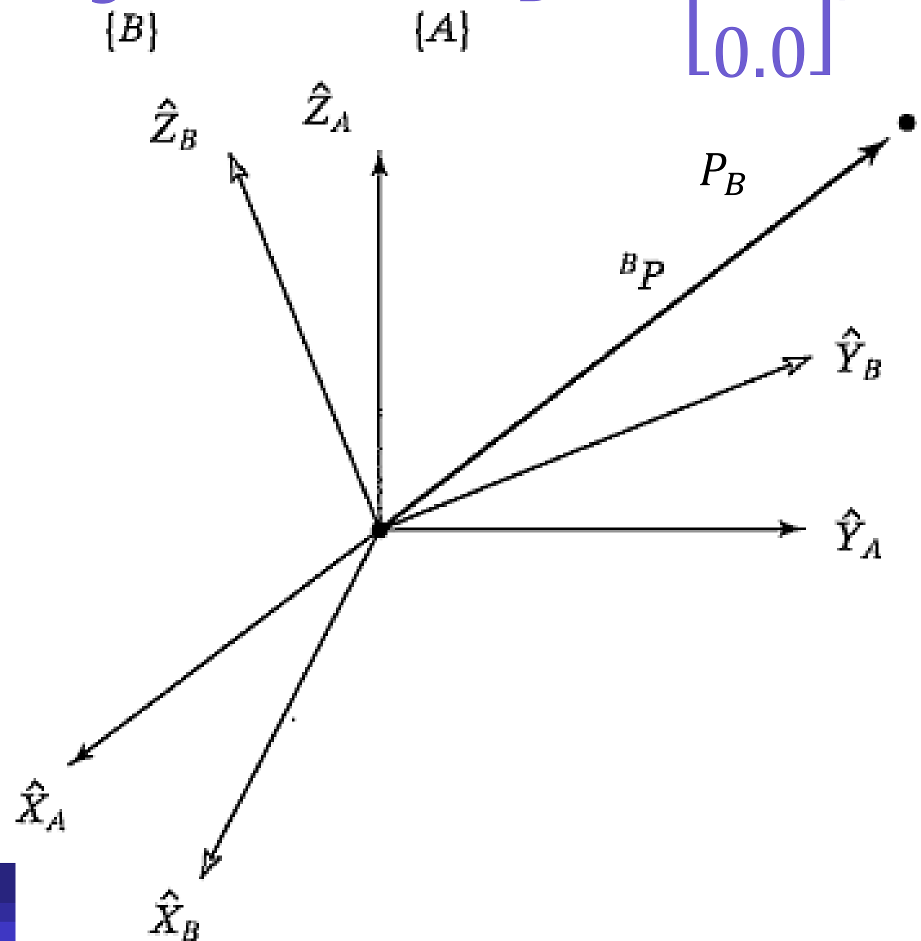


Mappings

Example: Figure shows a frame {B} that is rotated relative to frame {A} about Z by 30 degrees. Given P_B is $\begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix}$, Find P_A?

- **Solution:**

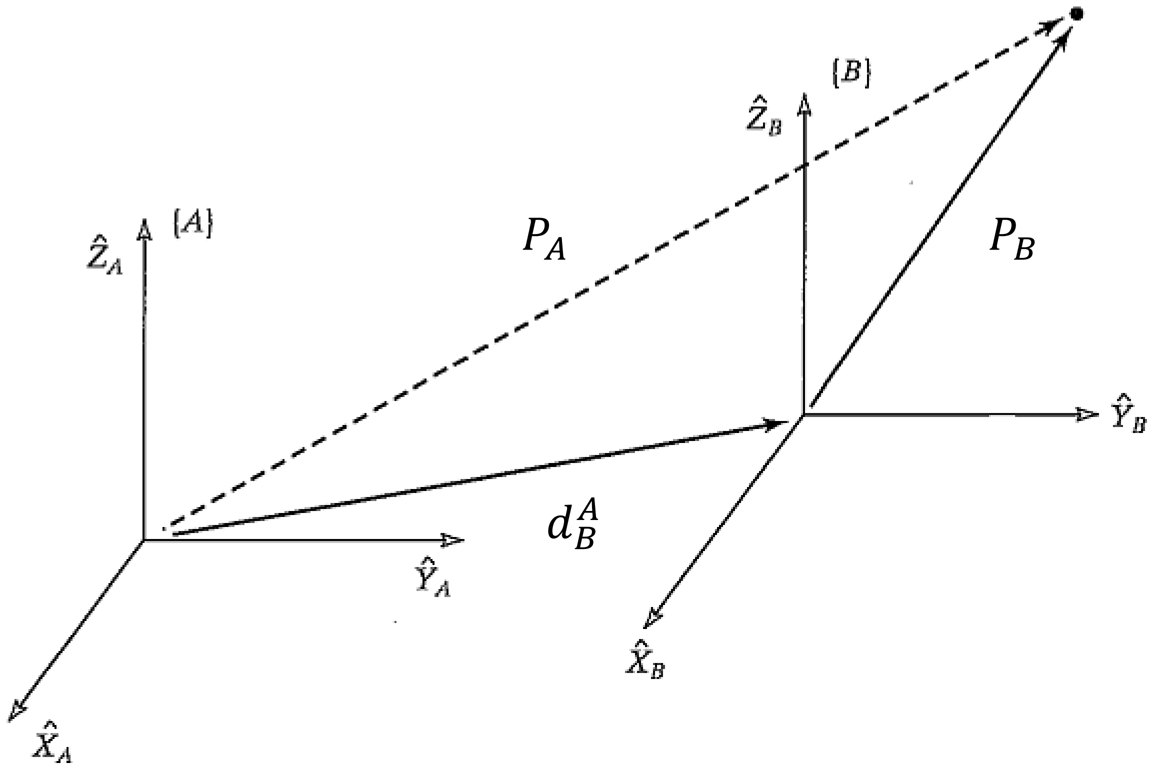
$$P_A = R_B^A P_B$$



Mappings

- **Case 2**: Mappings involving translated frame

$$P_A = d_B^A + P_B$$



Mappings

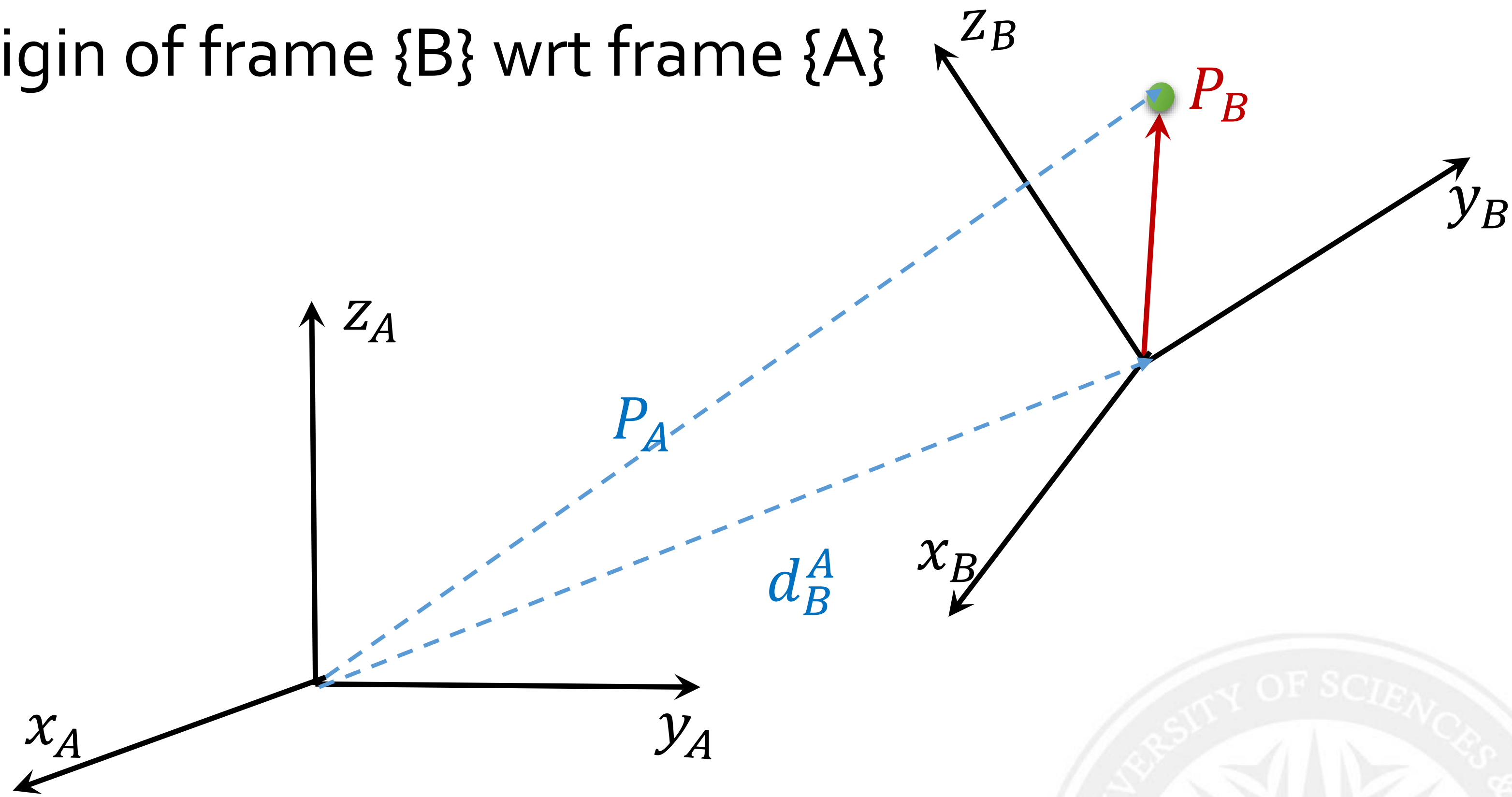
- **Case 2**: Mappings involving general frame

- Very often, we know the description of a vector with respect to some frame {B}, and we would like to know its description with respect to another frame, {A}. We now consider the general case of mapping.

$$\boxed{P_A = R_B^A P_B + d_B^A}$$

Mappings

- d_B^A: origin of frame {B} wrt frame {A}



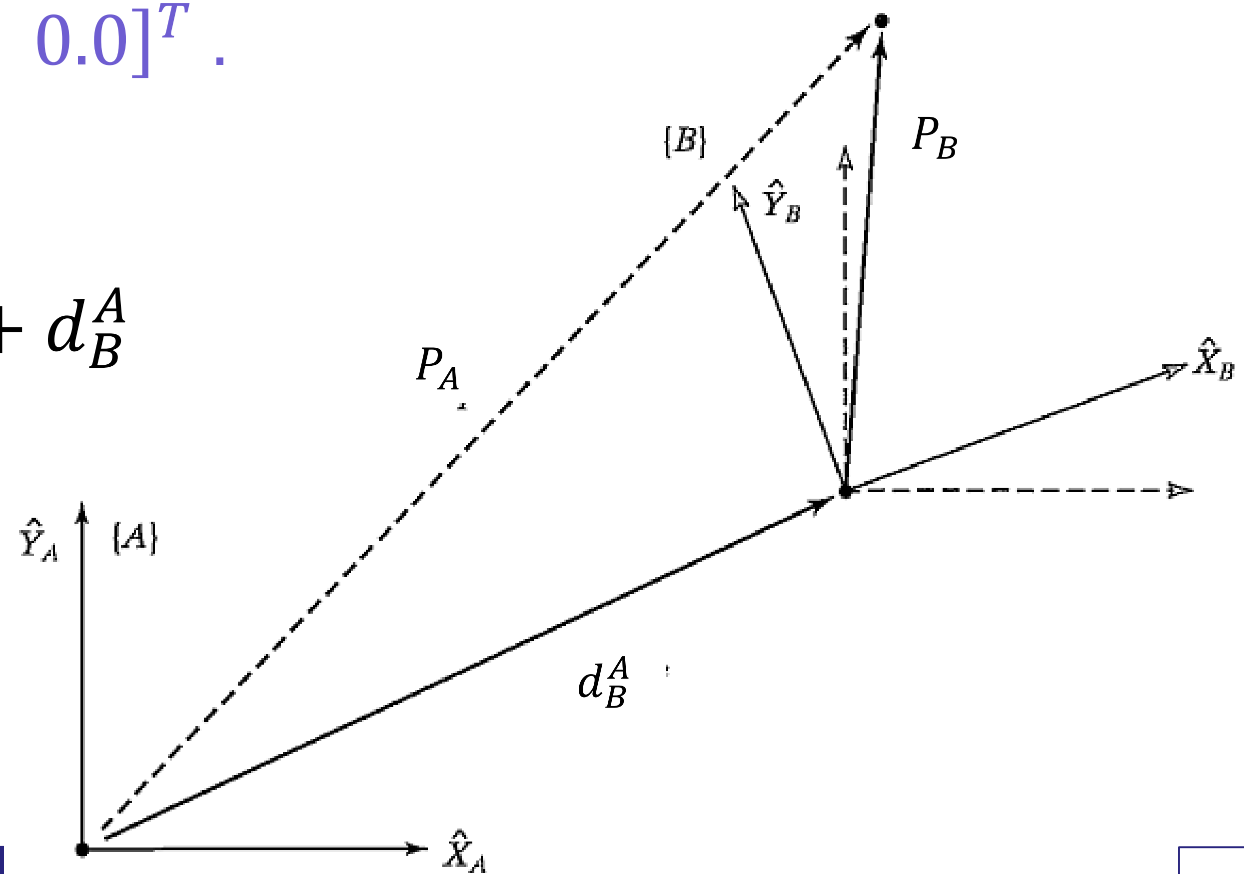
- Point P of frame {B} in frame {A}: $P_A = R_B^A P^B + d_B^A$

Mappings

Example: Figure shows a frame {B}, which is rotated relative to frame {A} about Z by 30 degrees, translated 10 units in X_A, and translated 5 units in Y_A. Find P_A, where $P_B = [3.0, \quad 7.0\ , \quad 0.0]^T$.

- **Solution:**

$$P_A = R_B^A P_B + d_B^A$$



Homogeneous Representation

- The compact form of $P_A = R_B^A P_B + d_B^A$

$$\underbrace{\begin{bmatrix} P_A \\ 1 \end{bmatrix}}_{\tilde{P}_A} = \underbrace{\begin{bmatrix} R_B^A & d_B^A \\ 0 & 1 \end{bmatrix}}_{T_B^A} \underbrace{\begin{bmatrix} P_B \\ 1 \end{bmatrix}}_{\tilde{P}_B}$$

- $\tilde{P}_{\mathrm{B}}$ and $\tilde{P}_A$ are called homogeneous coordinates
- T_B^A are called Homogeneous transformation matrix
- It represent the position and orientation (pose) of a frame with respect to another frame

Homogeneous Representation

- It represent the position and orientation (pose) of a frame with respect to another frame

$$T = \begin{bmatrix} \boxed{\begin{matrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{matrix}} & \boxed{\begin{matrix} d_x \\ d_y \\ d_z \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

- R can be derived from the perspective of projective geometry, i.e., dot product.
- Pure transformations

$$T = \underbrace{\begin{bmatrix} \boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}}_{\text{Pure rotation}} \qquad T = \underbrace{\begin{bmatrix} \boldsymbol{I} & \boldsymbol{d} \\ \boldsymbol{0} & 1 \end{bmatrix}}_{\text{Pure translation}}$$

Pure transformations

Pure rotation transformations

$$Rot_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\,sin\theta & 0 \\ 0 & sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot_y(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -sin\theta & 0 & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot_z(\theta) = \begin{bmatrix} cos\theta & -\,sin\theta & 0 & 0 \\ sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Pure Transformations

Pure translation transformations

$$Trans_x(d) = \begin{bmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Trans_y(d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Trans_z(d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Composition

- Composition of transformations

- When a transformation is applied with respect to the **fixed** frame:
  - A **pre-multiplication** is used

- When a transformation is applied with respect to the **mobile** frame (**current new**)
  - A **post-multiplication** is used

Example 1

- A frame {A} is rotated 90^o about x-axis, and then it is translated a vector $(6,-2,10)$ with respect to the **fixed** (initial) frame. Find the homogeneous transformation that describes {B} with respect to {A}.
- **Solution**

$$T_B^A = Trans(6,-2,10)Rot_x(90^o)$$

pre-multiplication

Example 2

- Find the homogeneous transformations matrix that represents a rotation of an angle α about the x −axis, followed by a translation of b units along the **new** x-axis, followed by a translation of d units along the **new** z-axis, followed by a rotation of an angle θ about the **new** z-axis
- **Solution**

$$T_B^A = Rot_x(\alpha)Trans_x(b)Trans_z(d)Rot_z(\theta)$$

post-multiplication

Summary: Composition

- Decomposition in pure transformations
  - Any homogeneous transformation can be decomposed in 2 components:

$$T = \begin{bmatrix} R_B^A & d_B^A \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & 0 \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & d \\ \boldsymbol{0} & 1 \end{bmatrix}$$

- Interpretation of the (de)composition of T

1. **Interpretation 1:(pre-multiplication) (rigid frame)**
   - First, it applies a translation of $\boldsymbol{d}$ units
   - Then it applies a rotation $\boldsymbol{R}$ with respect to the **fixed** (initial) frame
2. **Interpretation 2: (post-multiplication) (moving frame)**
   - First, it applies a rotation $\boldsymbol{R}$
   - Then it applies a translation of $\boldsymbol{d}$ units with respect to the new frame

Advantages

- Homogeneous transformation represent the pose (position + orientation) of a frame with respect to another frame
- It change the reference frame in which a point is represented (using a linear relation):

$$\tilde{P}^A = T_B^A \tilde{P}^B$$

  - Note: the point must be represented using homogeneous coordinates (its notation uses)

- It apply a transformation (rotation + translation) to a point in the same reference frame

Example 2

- A frame {A} is rotated 90^o about x, and then it is translated a vector $(6,-2,10)$ with respect to the fixed (initial) frame. Consider a point $P=(-5,2,-12)$ with respect to the new frame {B}. Determine the coordinates of that point with respect to the initial frame.

Solution

pre-multiplication

- Homogeneous transformation

$$T_B^A = Trans(6,-2,10)Rot_x(90^0)$$

- Point after transformation ? $\tilde{P}^A = T_B^A \tilde{P}^B$

Example 3

- A frame {A} is translated a vector $(6, -2, 10)$ and then it is rotated 90^o about x-axis of the fixed (initial) frame. Consider a point $P = (-5, 2, -12)$ with respect to the new frame {B}. Find the coordinates of that point with respect to the initial frame.

Solution **pre-multiplication**

- Homogeneous transformation

$$T_B^A = Rot_x(90^0)Trans(6, -2, 10)$$

- Transformed point? $\tilde{P}^A = T_B^A \tilde{P}^B$

Inverse Transformation

- Inverse of a homogeneous transformation:

$$T = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ \boldsymbol{0} & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{d} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

Why?

$$P^A = \mathrm{d}_B^A + \mathbf{R}_B^A P^B \xrightarrow{\text{Solve for } P^{\mathrm{B}}} P^{\mathrm{B}} = (R_B^A)^T\, P^{\mathrm{A}} - \mathrm{d}_B^A (R_B^A)^T$$

$$\tilde{P}^A = T_B^A\, \tilde{P}^{\mathrm{B}} \xrightarrow{\text{Solve for } P^{\mathrm{B}}} \tilde{P}^{\mathrm{B}} = (T_B^A)^{-1}\, \tilde{P}^A = T_A^B\, \tilde{P}^A$$

Note that $(T_B^A)^{-1} = T_A^B$

- Product of homogeneous transformations:

$$T_1 = \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{d}_1 \\ \boldsymbol{0} & 1 \end{bmatrix}, T_2 = \begin{bmatrix} \boldsymbol{R}_2 & \boldsymbol{d}_2 \\ \boldsymbol{0} & 1 \end{bmatrix} \Rightarrow T_1 T_2 = \begin{bmatrix} \boldsymbol{R}_1\boldsymbol{R}_2 & \boldsymbol{R}_1\boldsymbol{d}_2 + \boldsymbol{d}_1 \\ \boldsymbol{0} & 1 \end{bmatrix}$$

Compound Transformations

Example: A frame {A} is translated a vector $(6, -2, 10)$ and then it is rotated 90^o about x-axis of the fixed (initial) frame. Thus, we have a description of T_B^A. Find, T_A^B

Solution

- Homogeneous transformation

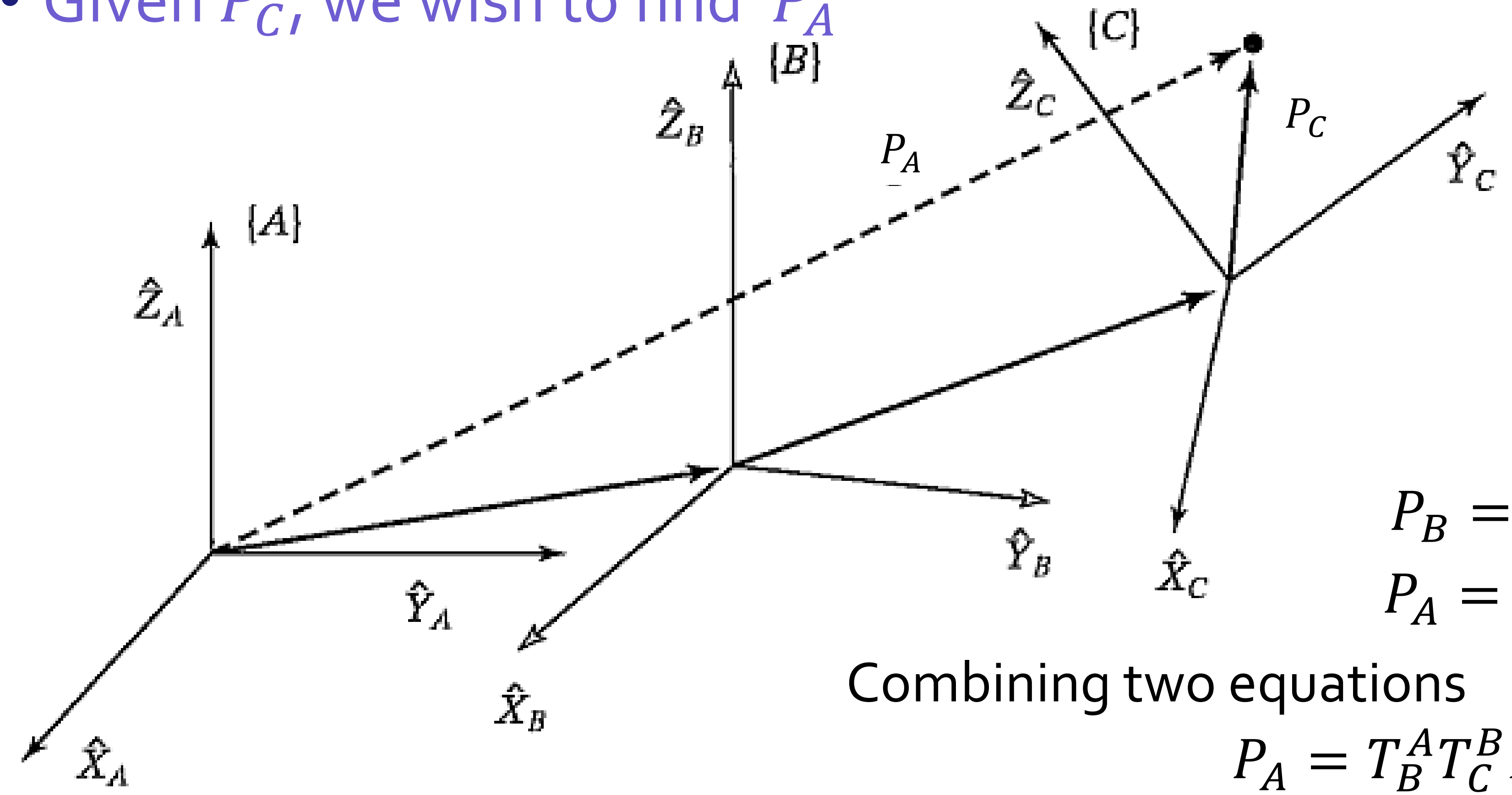
$$T_B^A = Rot_x(90^0)Trans(6, -2, 10)$$

- T_A^B?

$$(T_B^A)^{-1} = T_A^B$$

Compound Transformations

- Given P_C, we wish to find P_A



$$P_B = T_C^B P^C$$

$$P_A = T_B^A P^B$$

Combining two equations

$$P_A = T_B^A T_C^B P^C$$

We could define

$$T_C^A = T_B^A T_C^B$$

Transforming Equations

- We can express frame {D} as product of different transformations in different ways
- $T_D^U = T_A^U T_D^A$
- $T_D^U = T_B^U T_C^B T_D^C$

Equate both equations

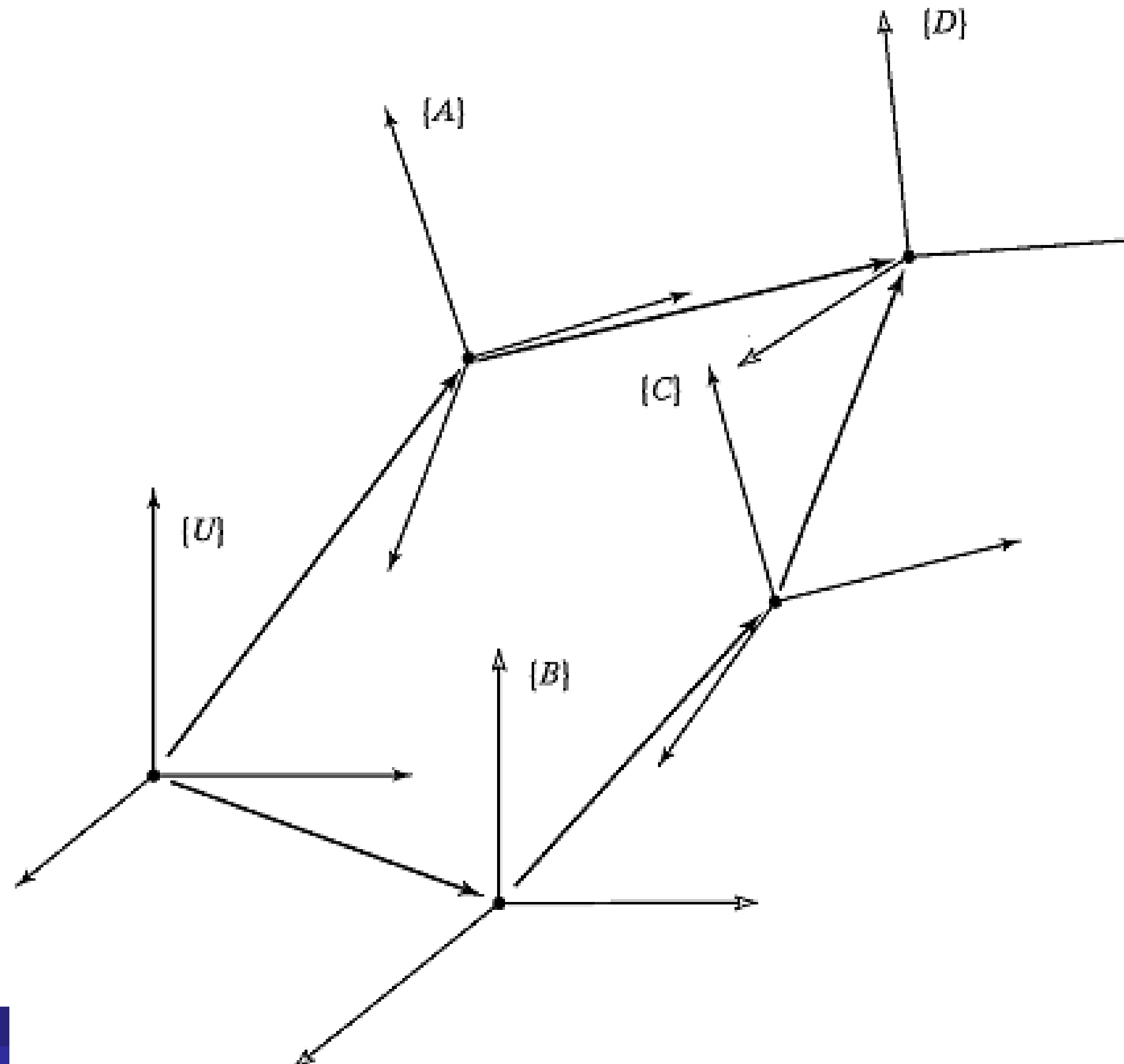
- $T_B^U T_C^B T_D^C = T_A^U T_D^A$

- It can be used to find

unknown transforms.

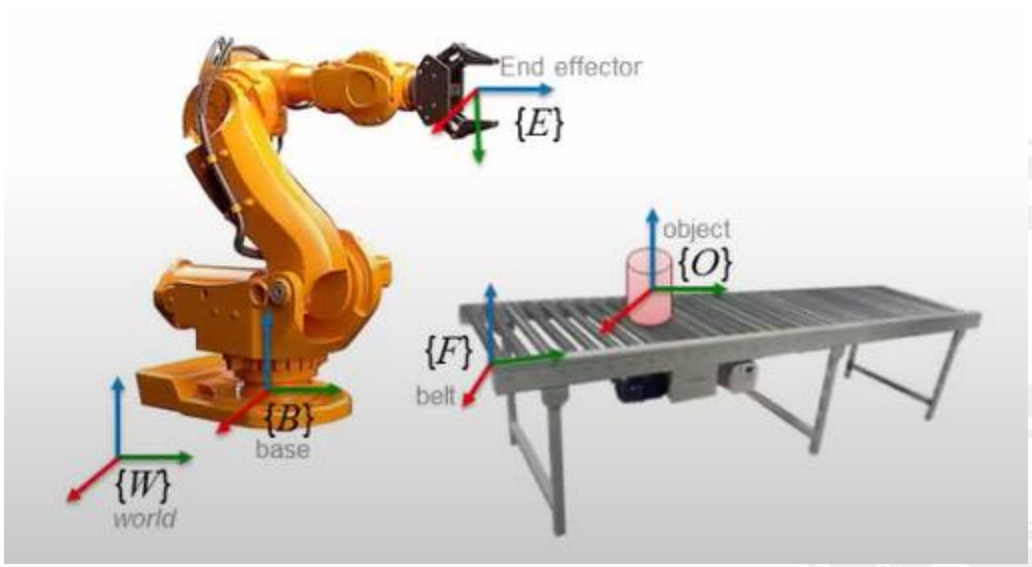
For example T_C^B

$T_C^B = \left(T_B^U\right)^{-1} T_A^U T_D^A \left(T_D^C\right)^{-1}$



Example

- Consider that the transformations of the belt and of the robot base with respect to a reference frame {W} are known. The transformation of the object with respect to the belt, as well as the transformation of the end effector with respect to the robot base are also known.
  a) Find the pose of the object with respect to the base of the robot
  b) Find the pose of the object with respect to the end effector



Example

- Consider that the transformations of the belt and of the robot base with respect to a reference frame {W} are known. The transformation of the object with respect to the belt, as well as the transformation of the end effector with respect to the robot base are also known.
  a) Find the pose of the object with respect to the base of the robot
  b) Find the pose of the object with respect to the end effector

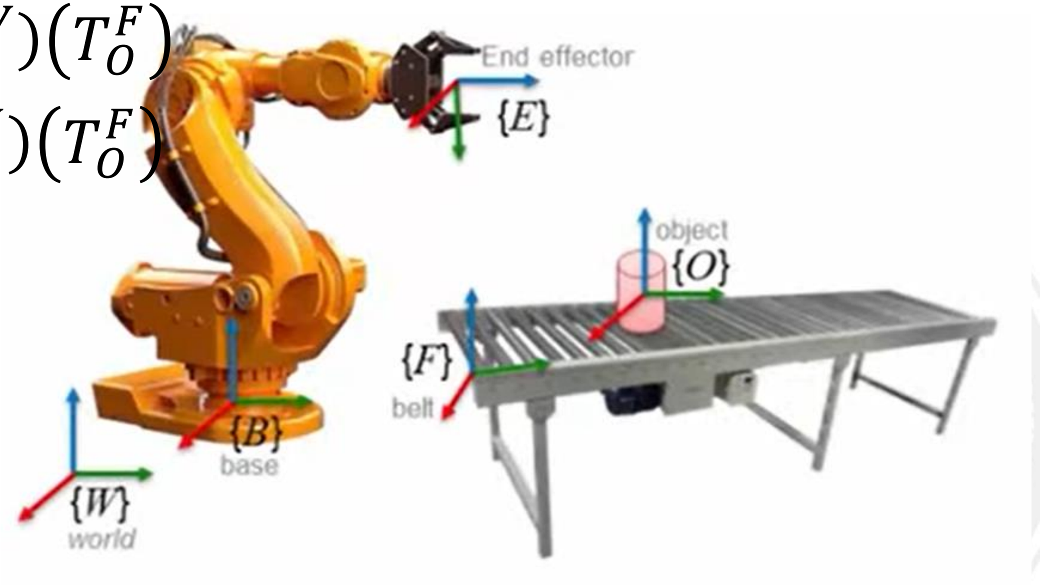
- **Solution**
- Known transformations: $T_F^W, T_B^W, T_O^F, T_E^B,$
  a) Desired pose (in terms of the known transformations): T_O^B

$$T_O^B = (T_W^B)(T_O^W) = (T_B^W)^{-1}\left((T_F^W)(T_O^F)\right)$$

Example

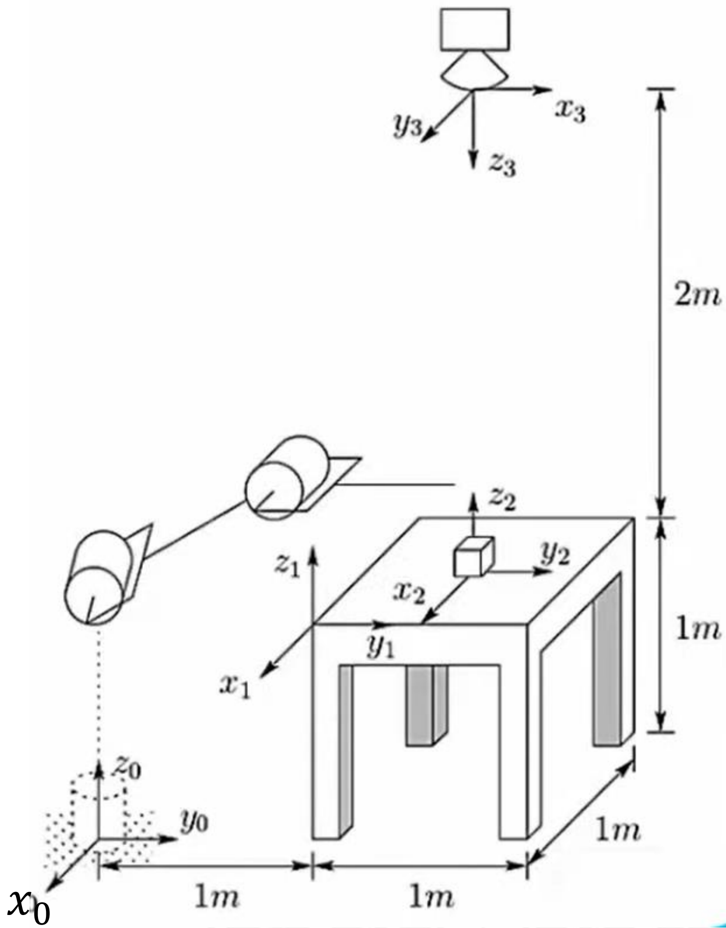
b) Desired pose (in terms of the known transformations):T_O^E

- Known transformations: $T_F^W, T_B^W, T_O^F, T_E^B,$
  - $T_O^E = (T_W^E)(T_O^W)$
  - $= (T_E^W)^{-1}(T_F^W)(T_O^F)$
  - $= \left((T_B^W)(T_E^B)\right)^{-1}(T_F^W)(T_O^F)$
  - $= (T_E^B)^{-1}(T_B^W)^{-1}(T_F^W)(T_O^F)$



Example

- The figure shows a robot whose base is 1m away from the base of the table. The table is 1m height and its surface is a square. Frame {1} is fixed on a corner of the table. A 20cm cube is located on the middle of the table, and it has frame {2} attached to its center. A camera is located 2m above the table, just over the cube, and it has frame {3} attached to it.
  - Find the homogeneous transformations that relate each of these frames with the base system {0}.
  - Find the homogeneous transformations that relates the cube frame {2} wrt the camera frame {3}.



Robot Modeling and Control, Chapter 2, problem 37

Example

- Solution

a) By inspection, the homogeneous transformations that relate each of the frames wrt the base frame {0} are:

$$
{}^0T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad
{}^0T_2 = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad
{}^0T_3 = \begin{bmatrix} 0 & 1 & 0 & -0.5 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

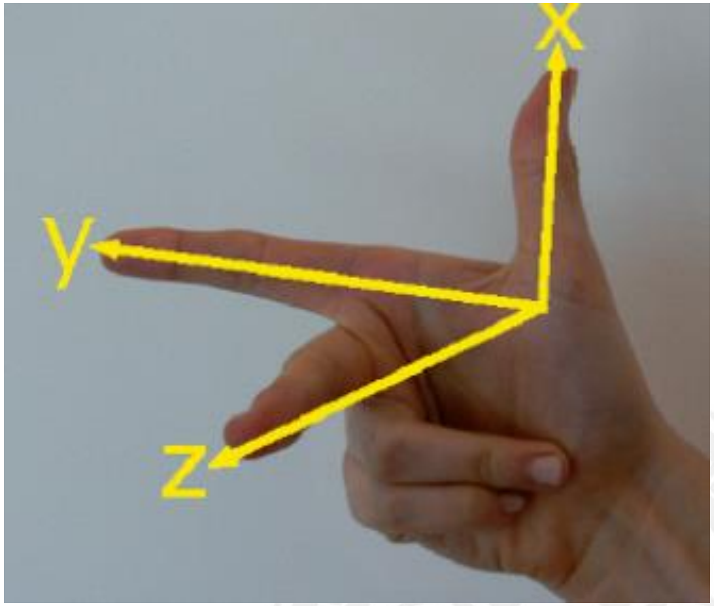
b) Using the composition of transformations:

$$
{}^3T_2 = {}^3T_0\,{}^0T_2 = {}^0T_3^{-1}\,{}^0T_2
$$

$$
= \begin{bmatrix} 0 & 1 & 0 & -1.5 \\ 1 & 0 & 0 & 0.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Orientation in 3D

- Rotation can be represented in different ways such as: orthonormal rotation matrices, Euler and Cardan angles, rotation axis and angle, and unit quaternions.



- All can be represented as vectors or matrices

Orthonormal Rotation Matrix

- We can represent the orientation of a coordinate frame by its unit vectors expressed in terms of the reference coordinate frame. Each unit vector has three elements and they form the columns of a 3 x 3 *orthonormal matrix* R_B^A

$$\begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} = R_B^A \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}$$

- Which transforms the description of a vector defined with respect to frame {*B*} to a vector with respect to {*A*}.

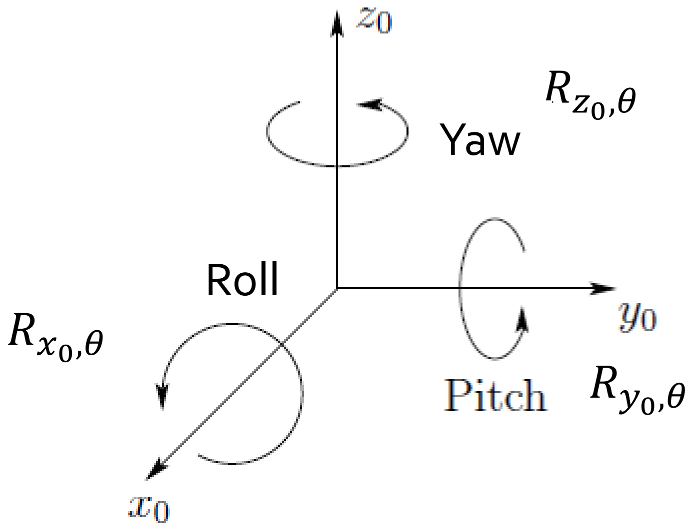
Orthonormal Rotation Matrices

- Orthonormal rotation matrices are;

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

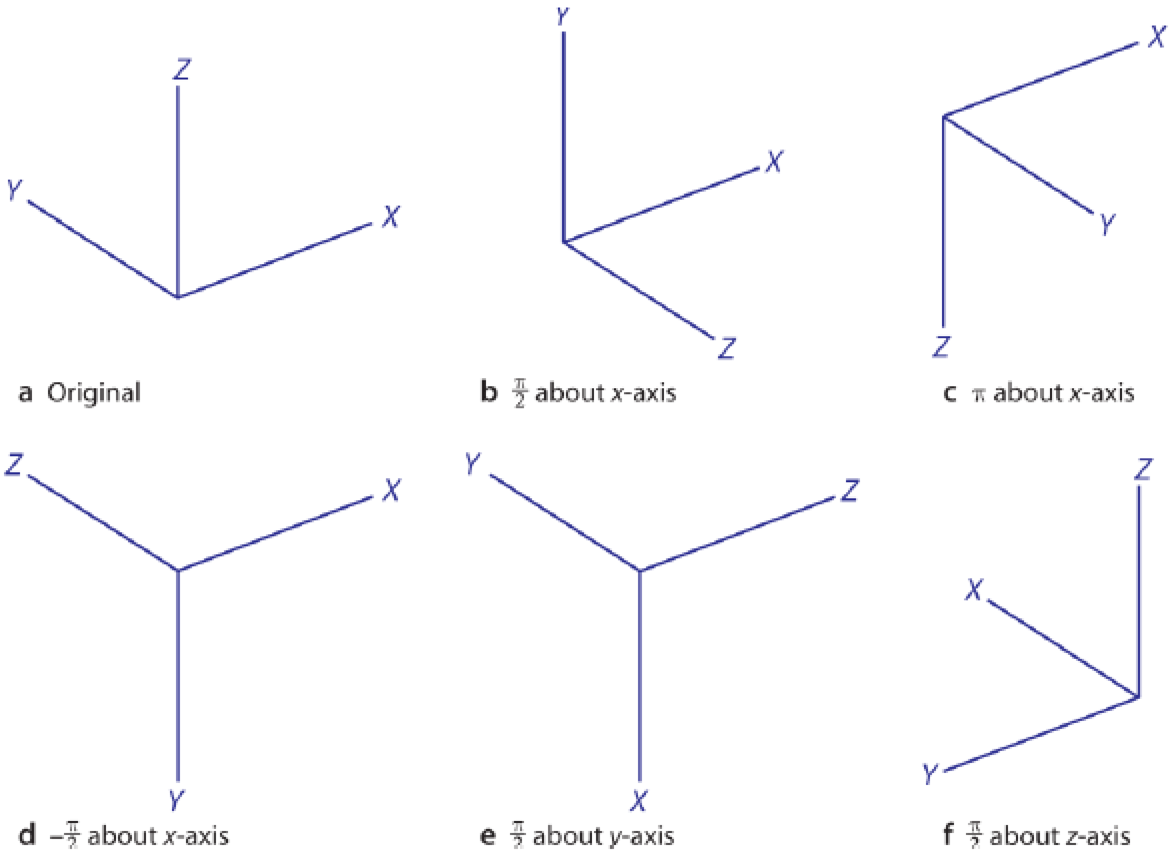
$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



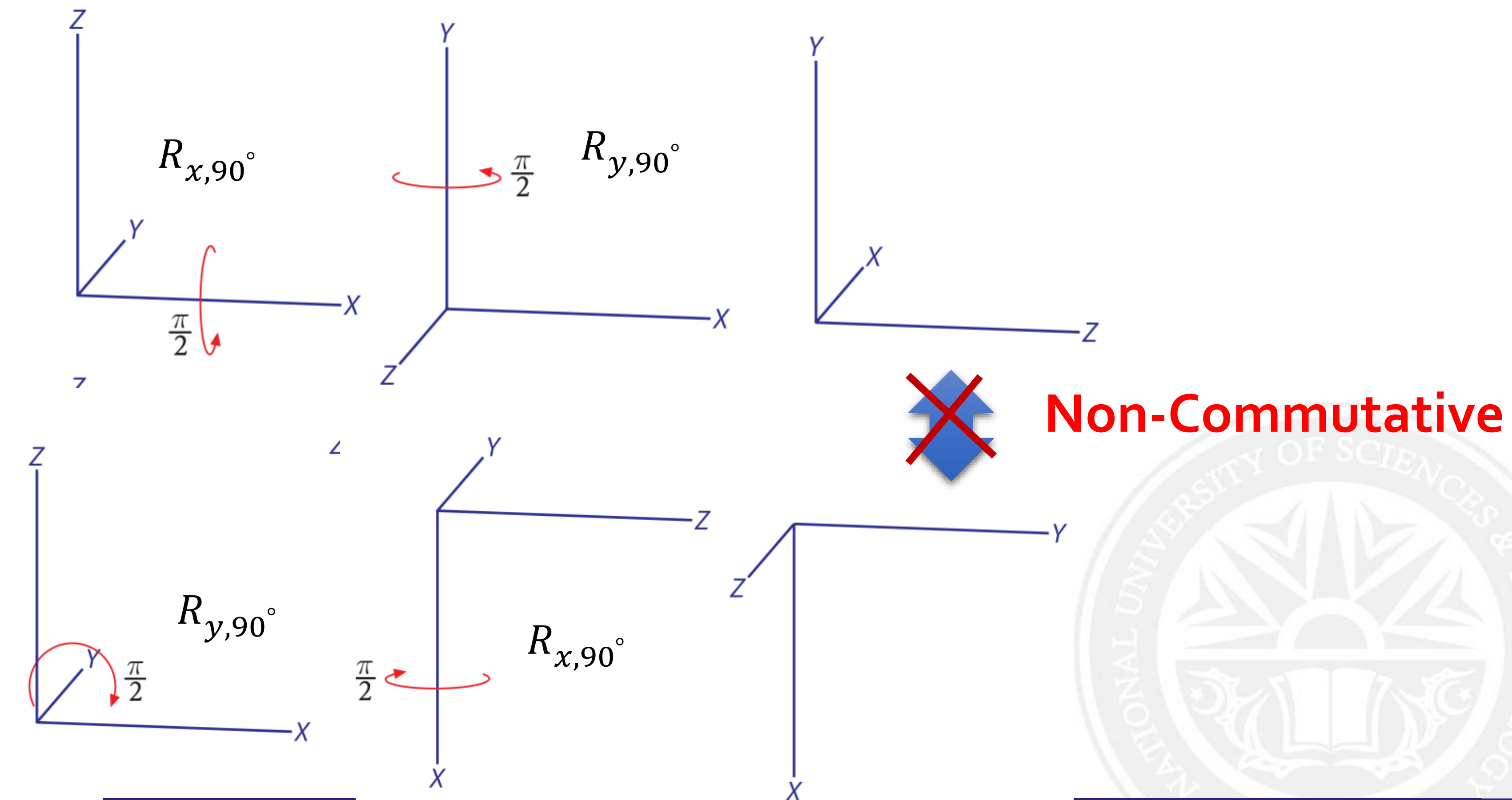
Euler's Rotation Theorem

- Euler's rotation theorem states that any rotation can be considers as a sequences of rotations about different coordinate axis



Euler's Rotation Theorem

- Orientation and Pose estimation is not simple. Slight variation results in posture change



Three-Angle Representations

- We recall that Euler's rotation theorem states that *any* rotation can be represented by *not more than three* rotations about coordinate axis.
- Euler's rotation theorem requires successive rotation about three axis such that no two successive rotations are about the same axis.
- There are two classes of rotation sequence: Eulerian and Cardanian, named after Euler and Cardano respectively,
  - Euler: XYX, XZX, YXY, YZY, ZXZ, or ZYZ
  - Cardanian: XYZ, XZY, YZX, YXZ, ZXY, or ZYX

Euler's Angle

- Euler-Angles: Most common sequence: ZYZ

Yaw-pitch-yaw

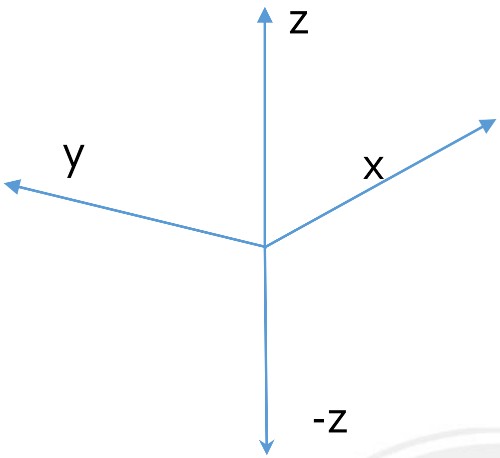
$$R = R_z(\phi)R_y(\theta)R_z(\psi) \qquad \text{or} \qquad R = R_{z,\phi}R_{y,\theta}R_{z,\psi}$$

The Euler angles are the 3-vector $\Gamma = (\phi, \theta, \psi)$

- Cardan Angles: roll-pitch-yaw sequence as XYZ or ZYX depending on if it is a mobile robot or a robotic arm.

Three-Angle Representations

- Convention for vehicles (ships, aircraft and cars): x-axis points in the forward direction and z-axis points either up or down.



- Convention for robot gripper: the *z*-axis points forward and the *x*-axis is either up or down. This leads to the XYZ angle sequence

Roll-pitch-yaw

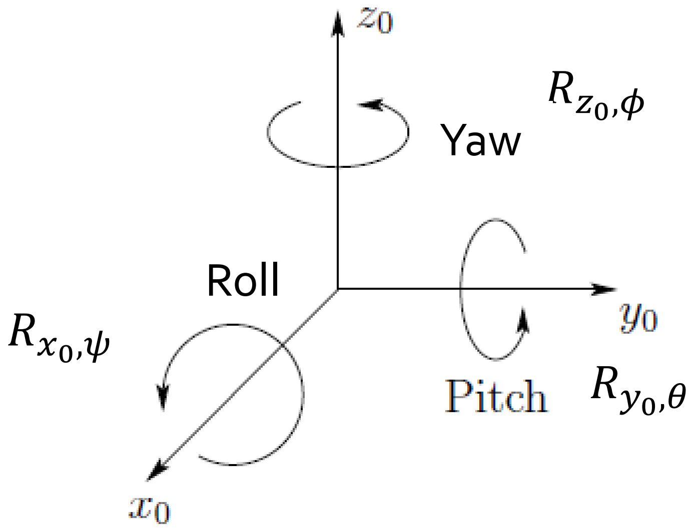
Roll, Pitch, Yaw Angles

- As we studied following rotation before

$$R_{x,\psi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\phi} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Roll, Pitch, Yaw Angles

- roll-pitch-yaw transformation will be

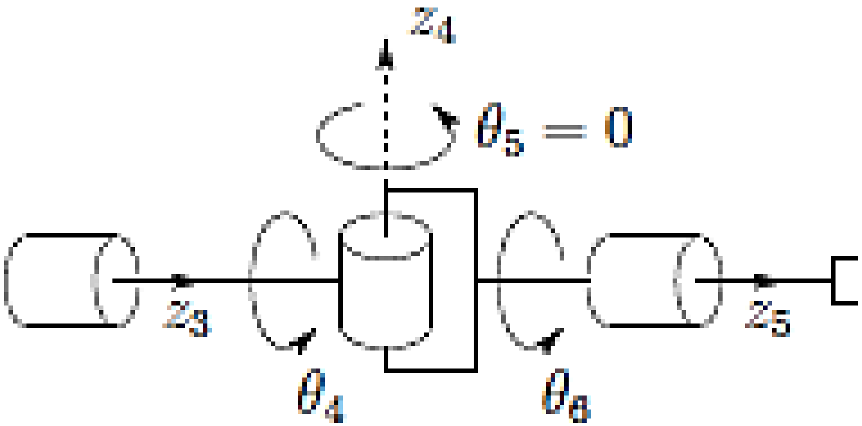
$$\begin{aligned} R_1^0 &= R_{z,\phi}R_{y,\theta}R_{x,\psi} \\ &= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \end{aligned}$$

- Similarly pitch-yaw-roll can also be computed as

$$R_1^0 = R_{y,\theta}R_{x,\psi}\,R_{z,\phi}$$

Singularity

- When two of the axes become aligned, the system loses a degree of freedom



- when the axis of first and third joint aligned then their rotation makes the motion of second joint equal to zero.
- The singularity occurs when the axes of two of the joints are aligned, and the third joint loses its ability to rotate, resulting in a loss of one degree of freedom.