NATIONAL UNIVERSITY OF SCIENCES & TECHNOLOGY

Linear Algebra and ODE (MATH-121)

Assignment # 2

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Class: BEE-12C

Semester: 2nd

Dated: 2/03/2021

CMS ID: 345834

Assignment 2

① : Find a set of vectors spanning the null space of:

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 3 & 6 & -2 \\ -2 & 1 & 2 & 2 \\ 0 & -2 & -4 & 0 \end{bmatrix}$$

To satisfy $Ax = 0$ for Null space;

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 3 & 6 & -2 \\ -2 & 1 & 2 & 2 \\ 0 & -2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix is;

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ 2 & 3 & 6 & -2 & 0 \\ -2 & 1 & 2 & 2 & 0 \\ 0 & -2 & -4 & 0 & 0 \end{array}\right]$$

∴ We find echelon form

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 3 & 6 & 0 & 0 \\ 0 & -2 & -4 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 2R_1 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 3R_2 \\ R_4 + 2R_2 \end{array}$$

In equations,

$$x_1 - x_4 = 0 \quad \Rightarrow \quad x_1 = x_4$$
$$x_2 + 2x_3 = 0 \quad \Rightarrow \quad x_2 = -2x_3$$

We can express x_1 and x_2 in terms of free variables x_3 and x_4.

Let $\boxed{\begin{array}{l} x_3 = p \,;\, x_2 = -2p \\ x_4 = t \,;\, x_1 = t \end{array}}$ $\therefore$ p, t are any real numbers

Hence, the Null Space of A is,

$$N(A) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -2p \\ p \\ t \end{bmatrix}$$

In terms of linear combination,

$$N(A) = \begin{bmatrix} t \\ 0 \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} 0 \\ -2p \\ p \\ 0 \end{bmatrix} = \underline{t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + p \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}$$

Hence, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ span N(A).

$$S_{(N(A))} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(2) Do the polynomials

$p(t) = t^3 + 2t + 1$, $q(t) = t^2 - t + 2$, $r(t) = t^3 + 2$

$s(t) = -t^3 + t^2 - 5t + 2$ span P_3?

$$P_3 = at^3 + bt^2 + ct + dt^0$$

To prove;

$$\alpha_1 p + \alpha_2 q + \alpha_3 r + \alpha_4 s = P_3$$

$$\Rightarrow \quad \alpha_1(t^3+2t+1) + \alpha_2(t^2-t+2) + \alpha_3(t^3+2) + \alpha_4(-t^3+t^2-5t+2) = P_3$$

$$\Rightarrow \quad t^3(\alpha_1+\alpha_3-\alpha_4) + t^2(0\alpha_1+\alpha_2+\alpha_4) + t(2\alpha_1-\alpha_2-5\alpha_4) + t^0(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4) = P_3$$

In matrin / augmented form,

$$\left[\begin{array}{cccc:c} 1 & 0 & 1 & -1 & a \\ 0 & 1 & 0 & 1 & b \\ 2 & -1 & 0 & -5 & c \\ 1 & 2 & 2 & 2 & d \end{array}\right]$$

Vectors span P_3 if they have a consistent system of equations;

$$\begin{array}{cl} \left[\begin{array}{cccc:c} 1 & 0 & 1 & -1 & a \\ 0 & 1 & 0 & 1 & b \\ 0 & -1 & -2 & -3 & c-2a \\ 0 & 2 & 1 & 3 & d-a \end{array}\right] & \begin{array}{l} \\ \\ R_3-2R_1 \\ R_4-R_1 \end{array} \end{array}$$

$$\begin{array}{cl} \left[\begin{array}{cccc:c} 1 & 0 & 1 & -1 & a \\ 0 & 1 & 0 & 1 & b \\ 0 & 0 & -2 & -2 & c-2a+b \\ 0 & 0 & 1 & 1 & d-a-2b \end{array}\right] & \begin{array}{l} \\ \\ R_3+R_2 \\ R_4-2R_2 \end{array} \end{array}$$

$$\begin{array}{cl} \left[\begin{array}{cccc:c} 1 & 0 & 1 & -1 & a \\ 0 & 1 & 0 & 1 & b \\ 0 & 0 & 2 & -2 & -c+2a-b \\ 0 & 0 & 2 & 2 & 2d-2a-4b \end{array}\right] & \begin{array}{l} \\ \\ -1R_3 \\ 2R_4 \end{array} \end{array}$$

$$\begin{array}{cl} \left[\begin{array}{cccc:c} 1 & 0 & 1 & -1 & a \\ 0 & 1 & 0 & 1 & b \\ 0 & 0 & 2 & 2 & -c+2a-b \\ 0 & 0 & 0 & 0 & -4a-3b+c+2d \end{array}\right] & \begin{array}{l} \\ \\ \\ R_4-R_3 \end{array} \end{array}$$

Which shows that the system is <u>inconsistent</u>; Hence, p, q, r and s $\boxed{\text{do not span}}$ P_3 for any values of α_1, α_2, α_3 and α_4.

③ Linear Dependence;

$p(t) = 2t^2 + t + 1 \,,\ q(t) = 3t^2 + t - 5 \,,\ r(t) = t + 13$

$\Rightarrow \quad \alpha_1 p + \alpha_2 q + \alpha_3 = 0$;

Above equation should have a non-trivial solution for vectors to be linearly dependent.

$$\alpha_1(2t^2 + t + 1) + \alpha_2(3t^2 + t - 5) + \alpha_3(t + 13) = 0 \quad (1)$$

$$t^2(2\alpha_1 + 3\alpha_2) + t(\alpha_1 + \alpha_2 + \alpha_3) + t^0(\alpha_1 - 5\alpha_2 + 13\alpha_3) = 0$$

Augmented Matrix;

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -5 & 13 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -13 & 26 & 0 \end{array}\right] \begin{array}{l} \\ R_2(2) - R_1 \\ R_3(2) - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array}$$

$$2\alpha_1 + 3\alpha_2 = 0$$

$$-\alpha_2 + 2\alpha_3 = 0$$

Let $\boxed{\alpha_3 = t}$; $\quad \therefore t$ is any real number

$$\boxed{\begin{array}{l} \alpha_2 = 2t \\ \alpha_1 = -3t \end{array}}$$

Putting α_1, α_2 and α_3 in ①

$$-3t(2t^2+t+1) + 2t(3t^2+t-5) + t(t+13) = 0$$

$$-\cancel{6t^3} - 3t^2 - 3t + \cancel{6t^3} + 2t^2 - 10t = -t^2 - 13t$$

$$-t^2 - 13t = -t^2 - 13t$$

Hence,

$$-\alpha_3 r = \alpha_1 p + \alpha_2 q$$

or

$$\boxed{\alpha_3 r = (-1)(\alpha_1 p + \alpha_2 q)}$$