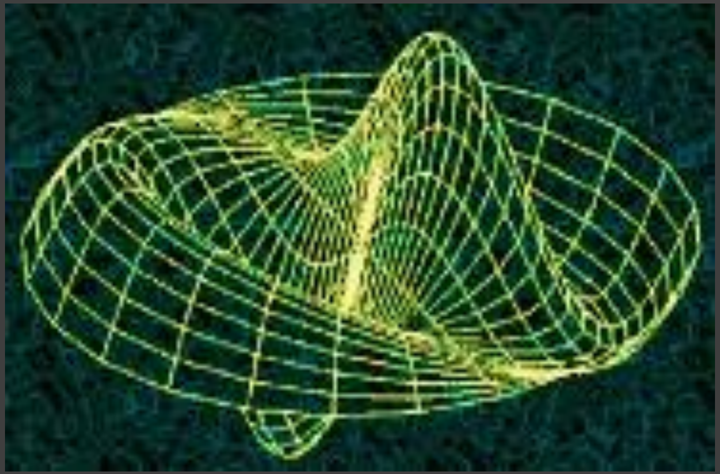
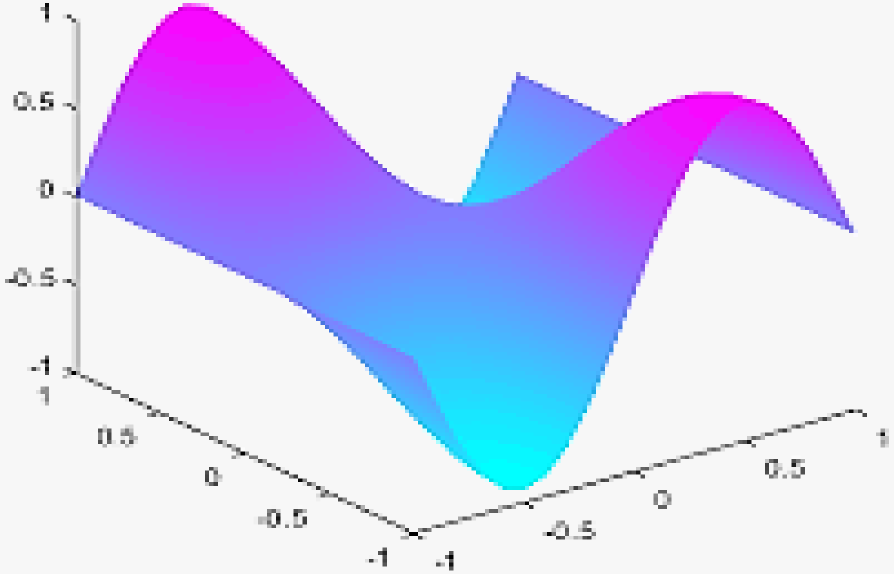
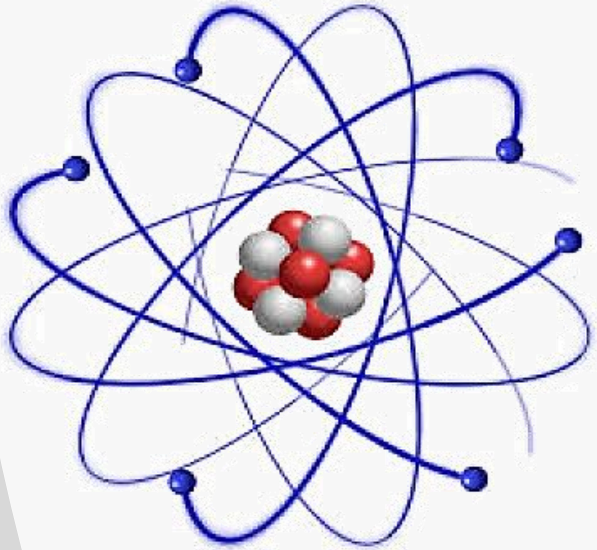
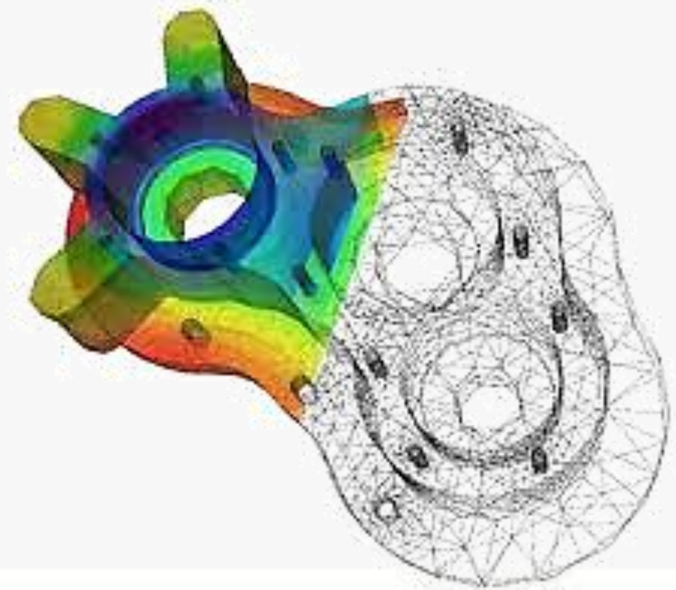




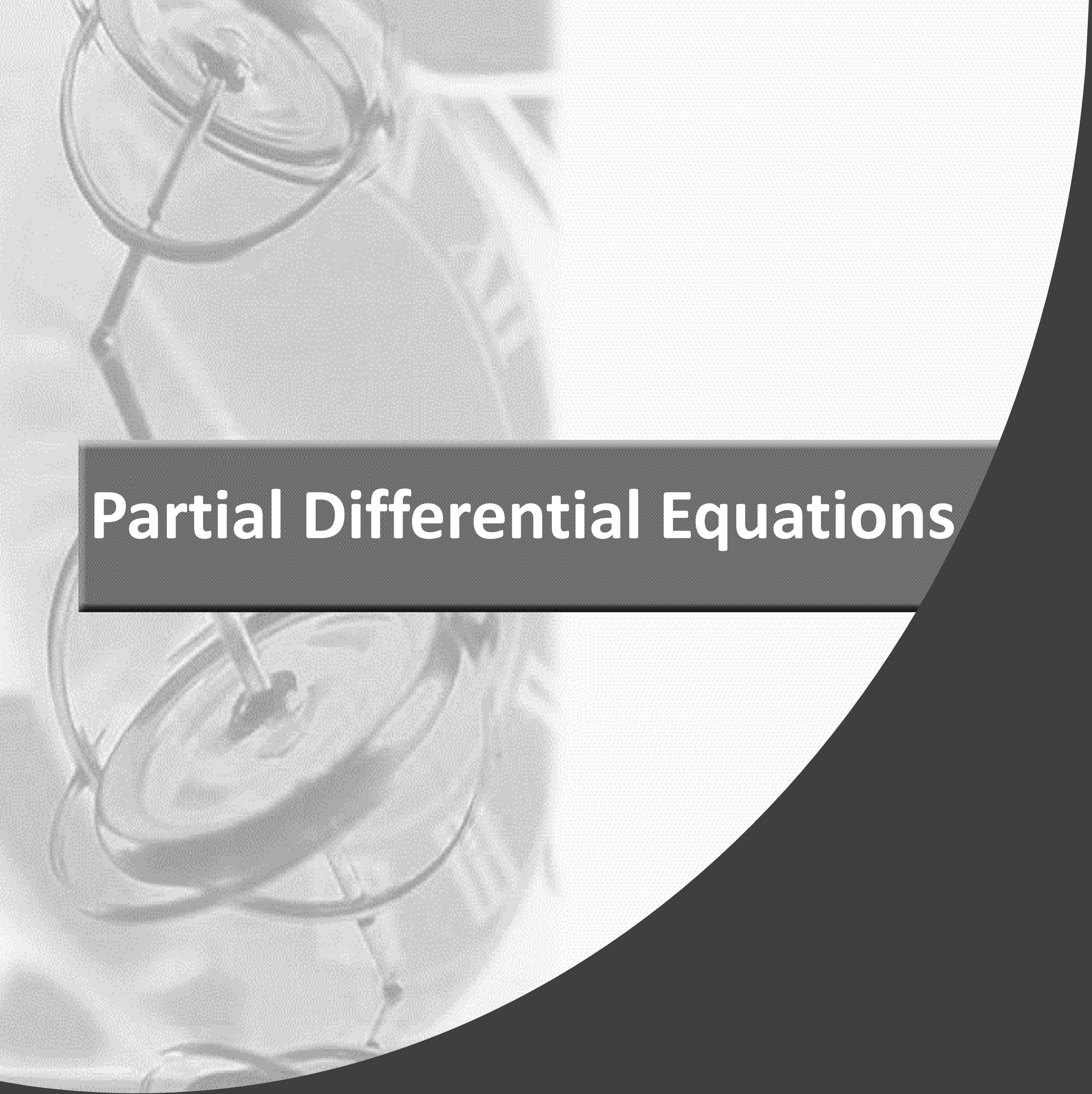
Partial Differential Equations





Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir



Partial Differential Equations

Book: Applied Partial Differential Equations With Fourier series and boundary value problems by Richard Haberman

- Chapter: 2
  - Sections: 2.5

The Laplacian in Various Coordinate Systems

- The two-dimensional Laplacian and its higher dimensional versions are of paramount importance in applications.

- They appear, for example, in the wave and heat equations, and also in Laplace's equation.

- So far, we solved these equations over rectangular and box shaped regions. To extend our applications to regions such as the disk, the sphere or the cylinder, it is to our advantage to use new coordinate systems in which the region and its boundary have simple expressions.

- For example, for problems over a disk we change to polar coordinates, where the equation of a circle centered at the origin reduces to $r = a$. Similarly, problems over spheres are simplified by a change to spherical coordinates.

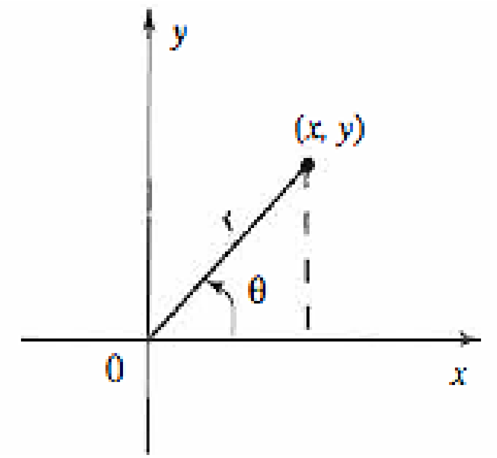
- For later applications, we will express the Laplacian in various coordinate systems.

The Laplacian in Polar Coordinate System

If we want to solve a partial differential equation (PDE) on the domain whose shape is a 2D disk, it is much more convenient to represent the solution in terms of the polar coordinate system than in terms of the usual Cartesian coordinate system. For example, the behavior of the drum surface when we hit it by a stick would be best described by the solution of the wave equation in the polar coordinate system. Let us now derive the Laplacian and the Laplace's equation in the polar coordinate system. Recall the relationship between rectangular and polar coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x.$$



For $\theta \in (-\pi, \pi]$, we have:

$$\theta = \arctan\left(\frac{y}{x}\right) + k\pi,$$

where $k = 0$ if $x > 0$ and $y > 0$ or $x > 0$ and $y < 0$; or $k = 1$ if $x < 0$ and $y \geq 0$; or $k = -1$ if $x < 0$ and $y < 0$. Also, if $x = 0$, then $\theta = \pi/2$ if $y > 0$ and $\theta = -\pi/2$ if $y < 0$.)

The Laplacian in Polar Coordinate System

Differentiating $r^2 = x^2 + y^2$ with respect to x, we obtain:

$$2r\frac{\partial r}{\partial x} = 2x \Longrightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.$$

Differentiating again with respect to x and simplifying, we obtain:

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{y^2}{r^3}.$$

Similarly,

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2},$$

and

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{r^2}\right) = \frac{2y}{r^3}\frac{\partial r}{\partial x} = \frac{2xy}{r^4}.$$

Differentiating now with respect to y, we obtain in a similar way

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} \quad \text{and} \quad \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}, \qquad \frac{\partial^2 \theta}{\partial y^2} = -\frac{2xy}{r^4}.$$

The Laplacian in Polar Coordinate System

From the relations we get the following interesting identities:

1. $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{y^2}{r^3} + \dfrac{x^2}{r^3} = \dfrac{1}{r}.$

2. $\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2} = \dfrac{2xy}{r^4} - \dfrac{2xy}{r^4} = 0.$

3. $\dfrac{\partial \theta}{\partial x}\dfrac{\partial r}{\partial x} + \dfrac{\partial \theta}{\partial y}\dfrac{\partial r}{\partial y} = \left(\dfrac{-y}{r^2}\right)\left(\dfrac{x}{r}\right) + \left(\dfrac{x}{r^2}\right)\left(\dfrac{y}{r}\right) = 0.$

4. $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = \dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} = 1.$

We are now ready to change the Laplacian $\nabla^2 u = u_{xx} + u_{yy}$ from Cartesian coordinates to polar coordinates. Since u is function of x and y which are functions of r and θ so using the chain rule in two dimensions, we have:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}.$$

The Laplacian in Polar Coordinate System

Applying the product rule for differentiation and the chain rule, we obtain:

(I)

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} \\
&= \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial \theta}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} \\
&\quad + \left(\frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta^2}\frac{\partial \theta}{\partial x}\right)\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} \\
&= \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial \theta}{\partial x}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} \\
&\quad + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}.
\end{aligned}$$

Changing x to y, we obtain:

(II)

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial \theta}{\partial y}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2}.$$

The Laplacian in Polar Coordinate System

Adding (I) and (II) and using the identities $(1)-(4)$ we get the polar form of the Laplacian as:

$$\nabla^2 u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}, \qquad (\mathrm{III})$$

And the Laplace equation in two-dimensions in terms of polar coordinates takes the form:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

The Laplacian in Cylindrical Coordinates

If u is a function of three variables x, y, and z, the Laplacian is:

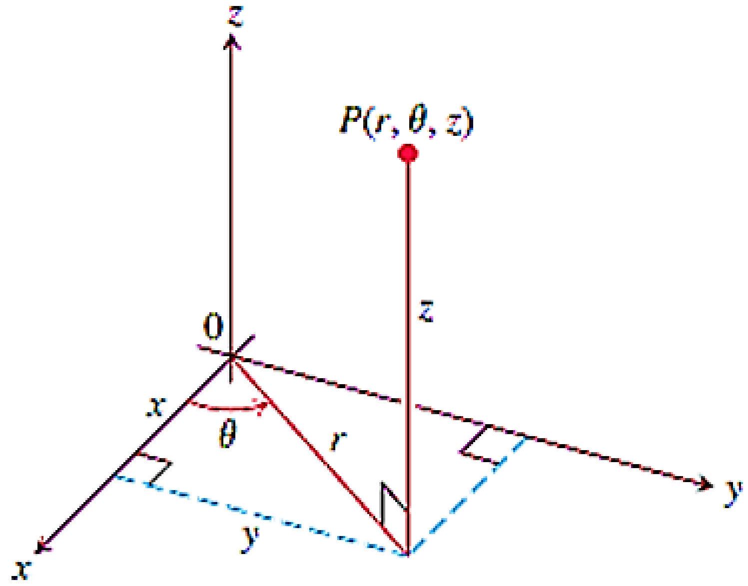
$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz}.$$

The relationships between rectangular and cylindrical coordinates are given as:

$$x = r\cos\theta\,, \qquad y = r\sin\theta\,, \qquad z = z,$$

where we now use r and θ to denote polar coordinates in the xy −plane. The **cylindrical form of the Laplacian** is now evident from (III) and is written as:

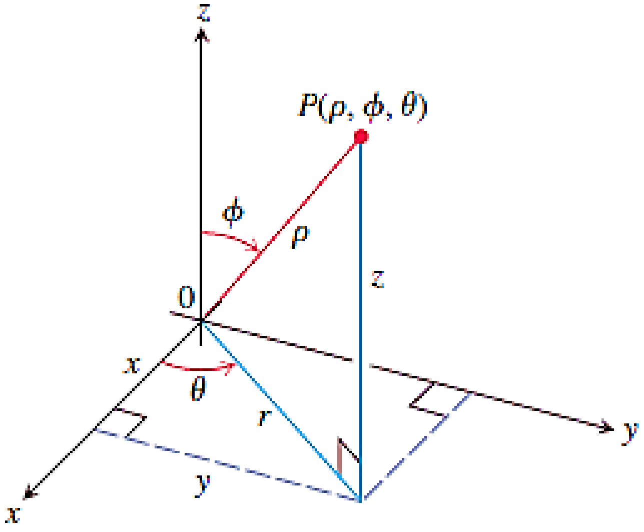
$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}. \qquad \text{(IV)}$$



The Laplacian in Spherical Coordinates

Our designated point (x, y, z) on the sphere is indicated by the three spherical coordinates (ρ, θ, ϕ), where:

$$x = r\cos\theta = \rho\sin\phi\cos\theta\,,$$
$$y = r\sin\theta = \rho\sin\phi\sin\theta\,,$$
$$z = \rho\cos\phi\,.$$
$$\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2,$$
$$\theta = \arctan(y/x)\,,$$
$$\phi = \arccos(z/\rho) \quad \text{or} \quad \phi = \arctan(r/z)\,.$$



Our goal is to express $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$ in terms of r, θ, and ϕ. From the polar form of the Laplacian (III), we have:

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}. \qquad \text{(V)}$$

Observe that the relations:

$$z = \rho\cos\phi \quad \text{and} \quad r = \rho\sin\phi\,,$$

are analogous to those between polar and rectangular coordinates.

The Laplacian in Spherical Coordinates

So, by using again the polar form of the Laplacian with z and r (in place of x and y), we get from (III):

$$u_{zz} + u_{rr} = u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi}. \quad \text{(VI)}$$

Adding u_{zz} to (V) and using (VI) we get:

$$u_{xx} + u_{yy} + u_{zz} = u_{\rho\rho} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi}. \qquad \text{(VII)}$$

It remains to express u_r in spherical coordinates. Note that:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial r} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial r} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial r} \qquad \text{(VIII)}$$

Since,

$$\phi = \arctan(r/z) \Longrightarrow \frac{\partial \phi}{\partial r} = \frac{1}{1+\frac{r^2}{z^2}}\left(\frac{1}{z}\right) \Longrightarrow \frac{\partial \phi}{\partial r} = \frac{z}{z^2+r^2} = \frac{\rho\cos\phi}{\rho^2} \Longrightarrow \frac{\partial \phi}{\partial r} = \frac{\cos\phi}{\rho},$$

and

$$r = \rho\sin\phi \Longrightarrow 1 = \frac{\partial \rho}{\partial r}\sin\phi + \rho\cos\phi\frac{\partial \phi}{\partial r} = \frac{\partial \rho}{\partial r}\sin\phi + \cos^2\phi\,.$$

The Laplacian in Spherical Coordinates

Hence,

$$\frac{\partial\rho}{\partial r}=\frac{1-\cos^2\phi}{\sin\phi}=\frac{\sin^2\phi}{\sin\phi}=\sin\phi \Longrightarrow \frac{\partial\rho}{\partial r}=\frac{r}{\rho}.$$

Moreover, note that since r and θ are polar coordinates in xy −plane, so $\frac{\partial\theta}{\partial r}=0$. Using values of $\frac{\partial\rho}{\partial r}, \frac{\partial\theta}{\partial r}$ and $\frac{\partial\phi}{\partial r}$ in (VIII) we get:

$$\frac{\partial u}{\partial r}=\frac{r}{\rho}\frac{\partial u}{\partial\rho}+\frac{\cos\phi}{\rho}\frac{\partial u}{\partial\phi} \qquad (\mathrm{IX})$$

Using (IX) in (VII) we obtain:

$$\nabla^2 u=u_{xx}+u_{yy}+u_{zz}=u_{\rho\rho}+\frac{1}{r}\left(\frac{r}{\rho}u_\rho+\frac{\cos\phi}{\rho}u_\phi\right)+\frac{1}{r^2}u_{\theta\theta}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\phi\phi}.$$

Since $r=\rho\sin\phi$, therefore, the **Laplacian in spherical coordinates** is given as:

$$\Longrightarrow \nabla^2 u=u_{\rho\rho}+\frac{2}{\rho}u_\rho+\frac{1}{\rho^2}\left(\csc^2\phi\, u_{\theta\theta}+\cot\phi\, u_\phi+u_{\phi\phi}\right). \qquad (\mathrm{X})$$

Example:

Use spherical coordinates to compute the Laplacian of:

$$f(x,y,z) = \ln(x^2+y^2+z^2)\,; \quad (x,y,z) \neq (0,0,0).$$

Solution:

In spherical coordinates, we have:

$$f\,(\rho,\theta,\phi) = \ln\rho^2 = 2\ln\rho\,.$$

Since f is independent of θ and ϕ, all partial derivatives in these variables are zero. Thus, from (X) we get:

$$\nabla^2 f = f_{\rho\rho} + \frac{2}{\rho} f_\rho = -\frac{2}{\rho^2} + \frac{2}{\rho}\left(\frac{2}{\rho}\right) = \frac{2}{\rho^2}.$$

Practice:

- In Exercises $1-8$, compute the Laplacian in an appropriate coordinate system and decide if the given function satisfies Laplace's equation $\nabla^2 u = 0$. The appropriate dimension is indicated by the number of variables.

1. $u(x,y) = \frac{x}{x^2+y^2}$.
2. $u(x,y) = \tan^{-1}(\frac{y}{x})$.
3. $u(x,y) = \frac{1}{\sqrt{x^2+y^2}}$.
4. $u(x,y,z) = \frac{z}{\sqrt{x^2+y^2}}$.
5. $u(x,y,z) = (x^2+y^2+z^2)^{3/2}$.
6. $u(x,y) = \ln(x^2+y^2)$.
7. $u(x,y,z) = (x^2+y^2+z^2)^{-1/2}$.
8. $u(x,y) = \tan^{-1}(\frac{y}{x})\frac{y}{x^2+y^2}$.

- Show that if $u(r,\theta,\phi)$ depends only on r, then the Laplacian takes the form:

$$\nabla^2 u = u_{rr} + \frac{2}{r}u_r.$$

Moreover, determine what is the form of the Laplacian if the function u depends only on r and θ?

Laplace's Equation for a Circular Disk

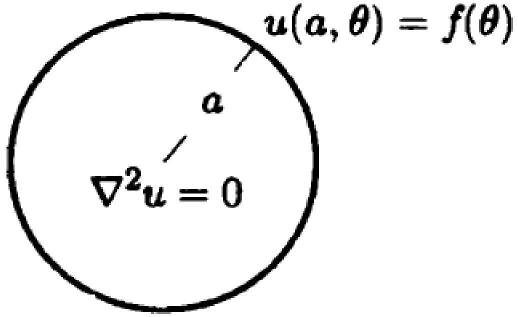
The steady-state (or time independent) temperature distribution in a circular plate of radius a, with prescribed temperature at the boundary, satisfies the two-dimensional Laplace equation (in polar coordinates):

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0; \qquad 0 < r < a, \qquad -\pi < \theta < \pi, \qquad (1)$$

and the boundary condition:

$$u(a,\theta) = f(\theta); \quad -\pi < \theta < \pi. \qquad (2)$$

Note that f is necessarily 2π $-$periodic. Equations (1) and (2) describe a Dirichlet problem over a disk of radius a.



Laplace's Equation for a Circular Disk

- Apparently, it seems that we cannot use separation of variables because there are no homogeneous subsidiary conditions.
- However, the introduction of polar coordinates requires some discussion that will illuminate the use of the method of separation of variables.
- If we solve Laplace's equation on a rectangle $0 < x < L, 0 < y < H$, then conditions are necessary at the endpoints of definition of the variables, $x = 0, L$ and $y = 0, H$.
- Fortunately, these coincide with the physical boundaries.
- Mathematically, we need conditions at the endpoints of the coordinate system: $r = 0, a$ and $\theta = \pm\pi$. Note that the limits on θ are somewhat arbitrary here and are chosen for convenience here. Any set of limits that covers the complete disk will work, however as we'll see with these limits, we will get another familiar boundary value problem arising. The best choice here is often not known until the separation of variables is done. At that point you can go back and make your choices.
- Here, only $r = a$ corresponds to a physical boundary. Thus, we need conditions motivated by considerations of the physical problem at $r = 0$ and at $\theta = \pm\pi$.

Laplace's Equation for a Circular Disk

- Note that Laplace's equation in terms of polar coordinates is singular at $r = 0$ (*i.e.,* we get division by zero). However, we know from physical considerations that the temperature must remain finite everywhere in the disk and so let's impose the condition that:

$$|u(0,\theta)| < \infty$$

  This may seem like an odd condition, and it definitely doesn't conform to the other boundary conditions that we've seen to this point, but it will work out for us as we'll see.

- This problem is similar to the circular wire situation. So, for boundary conditions for θ we'll do something similar to what we did for the 1-D head equation on a thin ring. The two limits on θ are really just different sides of a line in the disk means $\theta = -\pi$ corresponds to the same point as $\theta = \pi$. So, we can use the periodic conditions here. In other words, we say that the temperature is continuous there and the heat flow in the θ −direction is continuous, which imply:

$$u(r,-\pi) = u(r,\pi) \quad \text{and} \quad u_\theta(r,-\pi) = u_\theta(r,\pi)$$

- Above equations are called periodicity conditions. Note that these conditions are all linear and homogeneous. This problem is thus suited for the method of separation of variables.

Laplace's Equation for a Circular Disk

Thus, we are required to solve the following BVP:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0; \qquad 0 < r < a, \qquad -\pi < \theta < \pi, \qquad (1)$$

$$|u(0,\theta)| < \infty; \quad u(a,\theta) = f(\theta), \qquad (2)$$

$$u(r,-\pi) = u(r,\pi); \qquad u_\theta(r,-\pi) = u_\theta(r,\pi) \qquad (3)$$

Following the method of separation of variables, we will look for product solutions of of the form:

$$u(r,\theta) = R(r)\Theta(\theta). \qquad (4)$$

Plugging this into (1), (2) & (3) and simplifying, we obtain:

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0, \qquad (5)$$

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0. \qquad (6)$$

and

$$\Theta(-\pi) = \Theta(\pi), \qquad \Theta_\theta(-\pi) = \Theta_\theta(\pi) \quad \text{and} \quad |R(0)| < \infty. \qquad (7)$$

Laplace's Equation for a Circular Disk

Let us first solve the BVP:

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \qquad (6)$$

$$\Theta(-\pi) = \Theta(\pi), \qquad \Theta_\theta(-\pi) = \Theta_\theta(\pi).$$

The eigenvalues and corresponding eigenfunctions (”circular harmonics”) for this problem are given as:

$$\lambda = k^2; \qquad \text{for} \quad k = 0,1,2,\dots,$$

and

$$\Theta_k(\theta) = \begin{cases} a_0; & \text{for} \quad k = 0, \\ a_k\cos(k\theta) + b_k\sin(k\theta); & \text{for} \quad k = 1,2,\dots, \end{cases} \qquad (8)$$

respectively. For these values of λ, equation (5) takes form:

$$r^2R''(r) + rR'(r) - k^2R(r) = 0 \Longrightarrow (r^2D^2 + rD - k^2)R(r) = 0, \qquad (9)$$

where$k = 0,1,2,\dots,$ and $D = \frac{d}{dr}$. This is a 2nd order **Cauchy-Euler differential equation**.

Laplace's Equation for a Circular Disk

In order to solve ODE in (9), let us consider: $r = e^t$ or $t = \ln|r|$, and $\Delta = \frac{d}{dt}$, so that:

$$rD = \Delta$$
$$r^2D^2 = \Delta(\Delta - 1)$$

Using above in (9) we get:

$$[\Delta(\Delta - 1) - \Delta - k^2]R(t) = 0 \Longrightarrow [\Delta^2 - k^2]R(t) = 0.$$

This is a second order linear homogeneous ODE which possesses a general solution of the form:

$$R(t) = \begin{cases} c_{01} + c_{02}t; & \text{for} \quad k = 0, \\ c_1 e^{kt} + c_2 e^{-kt}; & \text{for} \quad k = 1,2,\ldots, \end{cases}$$

Since $r = e^t$ or $t = \ln|r|$, so we get:

$$R(r) = \begin{cases} c_{01} + c_{02}\ln|r|; & \text{for} \quad k = 0, \\ c_1 r^k + c_2 r^{-k}; & \text{for} \quad k = 1,2,\ldots, \end{cases} \qquad (10)$$

Laplace's Equation for a Circular Disk

Since $|R(0)| < \infty$, Each of the solutions above will have $R(r) \to \infty$ as $r \to 0$. Therefore, in order to meet this boundary condition, we must have $c_{02} = c_2 = 0$. Thus, the general solution (10) reduces to the following form:

$$R_k(r) = \begin{cases} c_{01}; & \text{for} \quad k = 0, \\ c_1 r^k; & \text{for} \quad k = 1,2,\dots, \end{cases} \qquad (11)$$

Using (8) and (11) we get:

$$u_k(r,\theta) = R_k(r)\Theta_k(r) = \begin{cases} A_0; & \text{for} \quad k = 0, \\ [A_k\cos(k\theta) + B_k\sin(k\theta)]r^k; & \text{for} \quad k = 1,2,\dots, \end{cases}$$

where $A_0 = a_0 c_{01},\ A_k = a_k c_1$ and $B_k = b_k c_1$. Superposing these solutions, we get the general form of the solution as:

$$u(r,\theta) = \sum_{k=1}^{\infty} u_k(r,\theta) = A_0 + \sum_{k=1}^{\infty} [A_k\cos(k\theta) + B_k\sin(k\theta)]r^k. \qquad (12)$$

Laplace's Equation for a Circular Disk

Applying our final boundary condition $u(a,\theta) = f(\theta)$, we get:

$$f(\theta) = A_0 + \sum_{k=1}^{\infty} [A_k \cos(k\theta) + B_k \sin(k\theta)] a^k \,. \qquad (13)$$

This is a trigonometric Fourier series for $f(\theta)$ on the interval $-\pi \le \theta \le \pi$, where the Fourier coefficients A_0, A_k and B_k are given as:

$$\left.\begin{aligned}
A_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta, \\
A_k &= \frac{1}{a^k \pi} \int_{-\pi}^{\pi} f(\theta) \cos(k\theta)\, d\theta\,, \qquad (k = 1,2,3,\ldots), \\
&\text{and} \\
B_k &= \frac{1}{a^k \pi} \int_{-\pi}^{\pi} f(\theta) \sin(k\theta)\, d\theta\,. \qquad (k = 1,2,3,\ldots).
\end{aligned}\right\} \qquad (14)$$

Thus, solution of the Laplace equation in a circular ring is given by equation (12) where coefficients can be determined by using (14).

Three-dimensional Laplace's equation and Dirichlet problems.

Steady-state temperature problems inside a **three-dimensional** solid lead to Dirichlet problems associated with Laplace's equation in three dimensions:

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0.$$

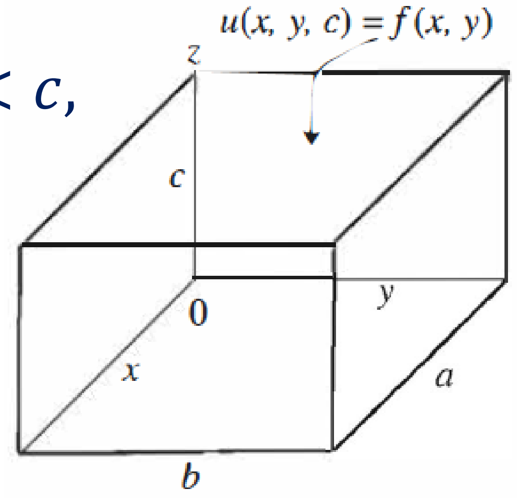
Consider such a problem as described in accompanying figure. The boundary values for u are 0 except on the upper horizontal face where we have: $u(x, y, c) = f(x, y)$. For the present case the BVP is given as:

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0; \quad 0 < x < a, \quad 0 < y < b, \quad 0 < z < c, \qquad (1)$$

$$u(0, y, z) = u(a, y, z); \quad 0 < y < b, \quad 0 < z < c,$$

$$u(x, 0, z) = u(x, b, z); \quad 0 < x < a, \quad 0 < z < c,$$

$$u(x, y, 0) = 0; \;\; u(x, y, c) = f(x, y); \quad 0 < y < b, \quad 0 < z < c,$$



We look for solutions that are expressible as products of a function of x alone, $X(x)$, a function of y alone, $Y(y)$, and a function of z alone, $Z(z)$, i.e.,

$$u(x, y, z) = X(x)Y(y)Z(z). \qquad (2)$$

Three-dimensional Laplace's equation and Dirichlet problems.

Introducing (2) in (1) we get:

$$X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0.$$

Dividing both sides of above equation by $X(x)Y(y)Z(z)$, we get:

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0.$$

Now the first term is a function of the independent variable x only, the second term a function of the independent variable y only and the last term a function only of the independent variable z. Since x, y, and z are independent of each other, each of the three terms in the equation must be constant and their sum equal to zero. We will set the first term equal to α, the second term equal to β, and the third term equal to $-(\alpha + \beta)$. Thus, on simplification we get following three ODEs along with boundary conditions:

$$X''(x) - \alpha X(x) = 0, \qquad Y''(y) - \beta Y(y) = 0, \qquad Z''(z) + (\alpha + \beta)z(z) = 0,$$

$$X(0) = X(a) = 0, \qquad Y(0) = Y(b) = 0, \qquad Z(0) = 0.$$

Three-dimensional Laplace's equation and Dirichlet problems.

Let us consider first BVP that is given as:

$$X''(x) - \alpha X(x) = 0, \qquad X(0) = X(a) = 0.$$

For $\alpha = -\mu^2$, the eigenvalues and corresponding eigenfunctions for this problem are given as:

$$\mu = \frac{m\pi}{a}; \qquad \text{for} \quad m = 1,2, \dots,$$

and

$$X_m(x) = A_m \sin\left(\frac{m\pi x}{a}\right); \qquad \text{for} \quad m = 1,2, \dots, \qquad (3)$$

respectively. Next we consider first BVP that is given as:

$$Y''(y) - \beta Y(y) = 0, \qquad Y(0) = Y(b) = 0.$$

For $\beta = -\nu^2$ The eigenfunctions for this problem are given as:

$$Y_n(y) = B_n \sin\left(\frac{n\pi y}{b}\right); \qquad \text{for} \quad n = 1,2, \dots, \qquad (4)$$

where $\nu = \frac{n\pi}{b}$; for $m = 1,2, \dots$, are the corresponding eigenvalues.

Three-dimensional Laplace's equation and Dirichlet problems.

Using values of α and β in the last ODE involving z as variable we get:

$$Z''(z) + (\alpha + \beta)z(z) = 0 \Longrightarrow Z''(z) - (\mu^2 + \nu^2)Z(z) = 0,$$

$$\Longrightarrow Z''(z) - \lambda_{mn} Z(z) = 0; \;\; m, n = 1,2, \dots .$$

where $\lambda_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$; $m, n = 1,2, \dots,$ are the eigenvalues for this problem and the corresponding eigenfunctions are given as:

$$Z_{mn}(z) = C_{mn} \sinh(\lambda_{mn} z); \qquad \text{for} \quad m, n = 1,2, \dots, \tag{5}$$

From $(3), (4)$ and (5) we get:

$$u_{mn}(x, y, z) = E_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sinh(\lambda_{mn} z); \qquad m, n = 1,2, \dots,$$

where $E_{mn} = A_m B_n C_{mn}$ are arbitrary constants. Superposition of these solutions provides us with the general solution:

$$u(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sinh(\lambda_{mn} z). \tag{6}$$

Three-dimensional Laplace's equation and Dirichlet problems.

Using the boundary condition $u(x, y, c) = f(x, y)$ we get:

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sinh(\lambda_{mn} c) \,.$$

To meet this last requirement, we choose the coefficients $E_{mn} \sinh(\lambda_{mn} c)$ to be the coefficients of double Fourier sine of $f(x, y)$ defined for all $0 < x < a,\ 0 < y < b$. Thus, it follows that:

$$E_{mn} = \frac{4}{ab \sinh(\lambda_{mn} c)} \int_0^b \int_0^a f(x, y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dxdy; \quad m, n = 1,2,3, \dots . \qquad (7)$$

Thus, the general solution of the Laplace equation in 3D is given by (6) where coefficients can be determined by (7).