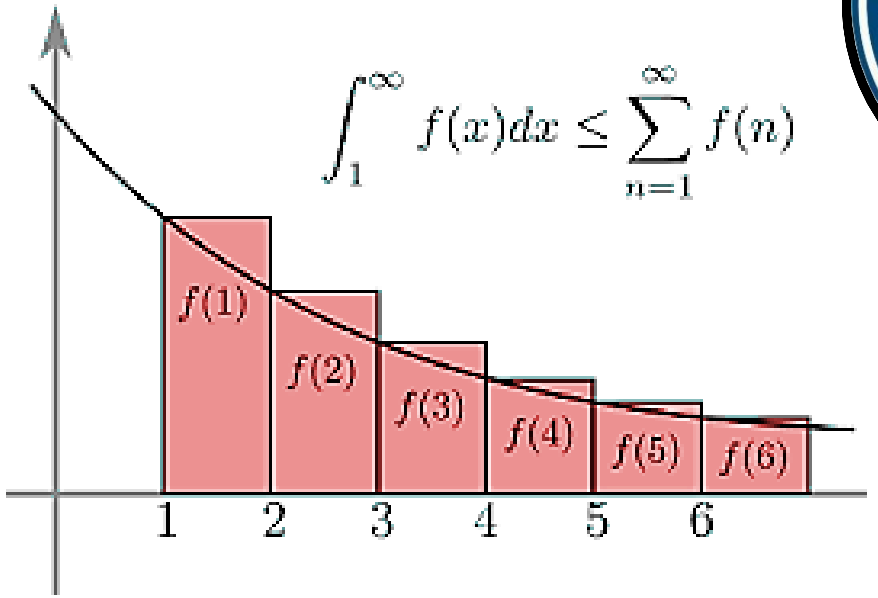
Infinite Series

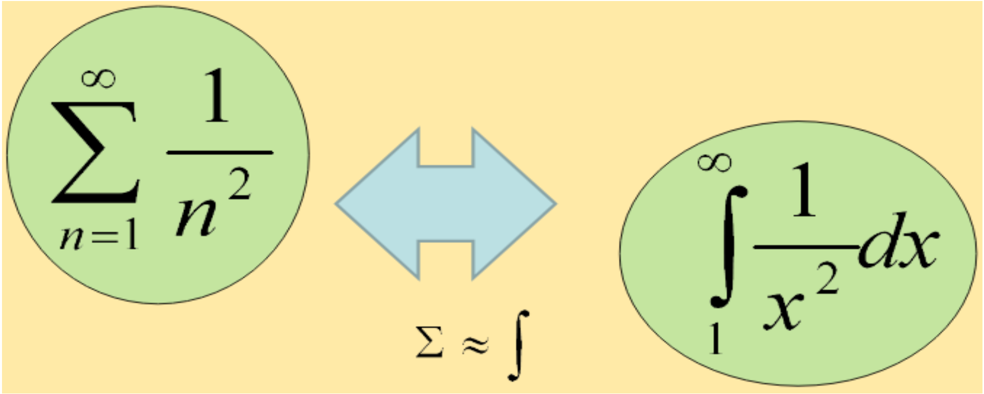
Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

Chapter: 11 (11.2, 11.3)

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

Chapter: 11 (11.2, 11.3)





Calculus & Analytical Geometry MATH-101

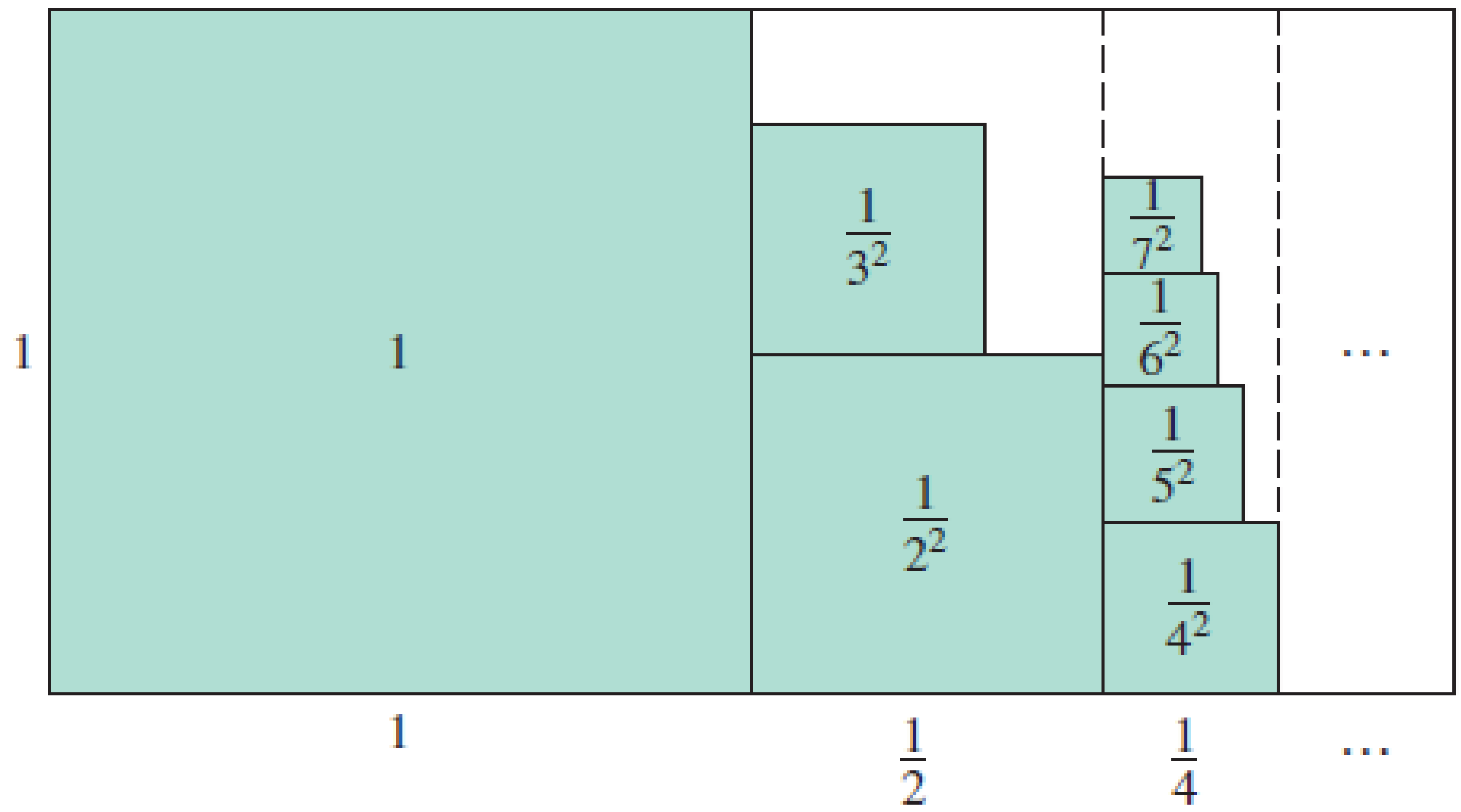
Instructor: Dr. Naila Amir (SEECS, NUST)

Infinite Series

Section: 11.2

Infinite Series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$



Infinite Series

- Let $\{a_n\}$ be a sequence then the sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots,$$

is known as an **infinite series** (or simply series).

- The sum of the first n terms $S_n = \sum_{i=1}^{n} a_i$, is called the $\boldsymbol{n^{th}}$ **partial sum** and the sequence $\{S_n\}$ is the sequence of partial sums.
- If the sequence $\{S_n\}$ converges to L, we say that the series $\sum_{n=1}^{\infty} a_n$ **converges** to L (or sum of the series is L) and we write

$$\sum_{n=1}^{\infty} a_n = L.$$

- If the sequence $\{S_n\}$ diverges, we say that the series $\sum_{n=1}^{\infty} a_n$ **diverges.**

Geometric Series

- A series of the form:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots,$$

where a and r are fixed real numbers such that $a, r \neq 0$, is called a geometric series.

Here "a" is the first term of the series and "r" is the common ratio.

- Geometric series is divergent if $|r| = 1$ and $|r| > 1$, however this series converges when $|r| < 1$ and its value is:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Combining Series

Whenever we have two convergent series, we can add them term by term, subtract them term by term, or multiply them by constants to make new convergent series.

Theorem:

If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are convergent series, then

1. Sum and Difference Rule:

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = A \pm B.$$

2. Constant Multiple Rule:

$$\sum_{n=1}^{\infty} k a_n = k \sum_{n=1}^{\infty} a_n = kA.$$

Example

Determine the sum of the following series:

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$

Difference Rule

$$= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/6)}$$

Geometric series with $a = 1$ and $r = 1/2, 1/6$

$$= 2 - \frac{6}{5}$$

$$= \frac{4}{5}$$

Example

Determine the sum of the following series.

(b)

$$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n}$$ Constant Multiple Rule

$$= 4\left(\frac{1}{1-(1/2)}\right)$$ Geometric series with $a = 1, r = 1/2$

$$= 8$$

Remarks

- We can add a finite number of terms to a series or delete a finite number of terms without altering the series convergence or divergence, although in the case of the convergence this will usually change the sum.

- As long as we preserve the order of its term, we can reindex any series without altering its convergence.

Divergent Series

One reason that a series may fail to converge is that its terms don't become small. For example, the series

$$\sum_{n=1}^{\infty} \frac{n+1}{n},$$

diverges because the partial sums eventually outgrow every preassigned number. Each term is greater than 1, so the sum of n terms is greater than n.

Important Theorems

Theorem 1:

$$\text{If} \sum_{n=1}^{\infty} a_n \text{ converges, then } a_n \to 0.$$

Proof: We have:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$
$$S_{n-1} = a_1 + a_2 + a_3 + \cdots + a_{n-1}$$

So that

$$S_n - S_{n-1} = a_n. \qquad (1)$$

Since the series converges, so $\lim_{n\to\infty} S_n$ exists. Let $\lim_{n\to\infty} S_n = S$. Now $n-1 \to \infty$ as $n \to \infty$, so, we have $\lim_{n\to\infty} S_{n-1} = S$. Thus, from (1) we get:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

Important Theorems

Theorem 1:

$$\text{If} \sum_{n=1}^{\infty} a_n \text{ converges}, \text{then } a_n \to 0.$$

Remark:

This theorem states only a necessary condition for convergence, i.e., for a convergent series its n^{th} term must converge to zero. However, its converse is not true, i.e., the n^{th} term approaching zero does not imply convergence of the series,

$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \underset{\nLeftarrow}{\Rightarrow} \quad \lim_{n\to\infty} a_n = 0.$$

Example: $a_n \to 0$ but the Series Diverges

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

commonly known as harmonic series. Note that:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

But one can easily show that the harmonic series is a divergent series.

Important Theorems

Theorem 2: The n^{th} term test for divergence

$$\sum_{n=1}^{\infty} a_n \text{ diverges if } \lim_{n \to \infty} a_n \text{ fails to exist or is different from zero.}$$

Examples

$$(a) \quad \sum_{n=1}^{\infty} n^2 \quad \text{diverges because } n^2 \to \infty.$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{n+1}{n} \quad \text{diverges because } \frac{n+1}{n} \to 1 \text{ as } n \to \infty.$$

$$(c) \quad \sum_{n=1}^{\infty} (-1)^{n+1} \quad \text{diverges because } \lim_{n\to\infty} (-1)^{n+1} \text{ does not exist.}$$

$$(d) \quad \sum_{n=1}^{\infty} \frac{-n}{2n+5} \quad \text{diverges because } \lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0.$$

Positive term Series

A series

$$\sum_{n=1}^{\infty} a_n \, ,$$

is said to be a positive term series (or a series of positive terms) if $a_n > 0$ for all positive integers n.

Note:

If $\sum_{n=1}^{\infty} a_n$ is a positive term series, then its sequence of partial sums $\{S_n\}$ is monotonically increasing and the series converges if $\{S_n\}$ is bounded.

Series known to converge or diverge

1. A geometric series with $|r| < 1$ converges.
2. A geometric series with $|r| > 1$ diverges.
3. A geometric series with $|r| = 1$ diverges.
4. A repeating decimal converges.
5. Telescoping series converge.
6. The harmonic series is divergent.

Practice Questions

Test the following series for convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$

2. $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right)$

3. $\sum_{n=1}^{\infty} (1.075)^n$

4. $\sum_{n=1}^{\infty} \frac{4}{9^n}$

Practice Questions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Exercise: 11.2

  Q # 1 to Q # 58

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 11.2

  Q # 1 to Q # 48

Convergence/Divergence of a series

- In order to examine the convergence or divergence of an infinite series

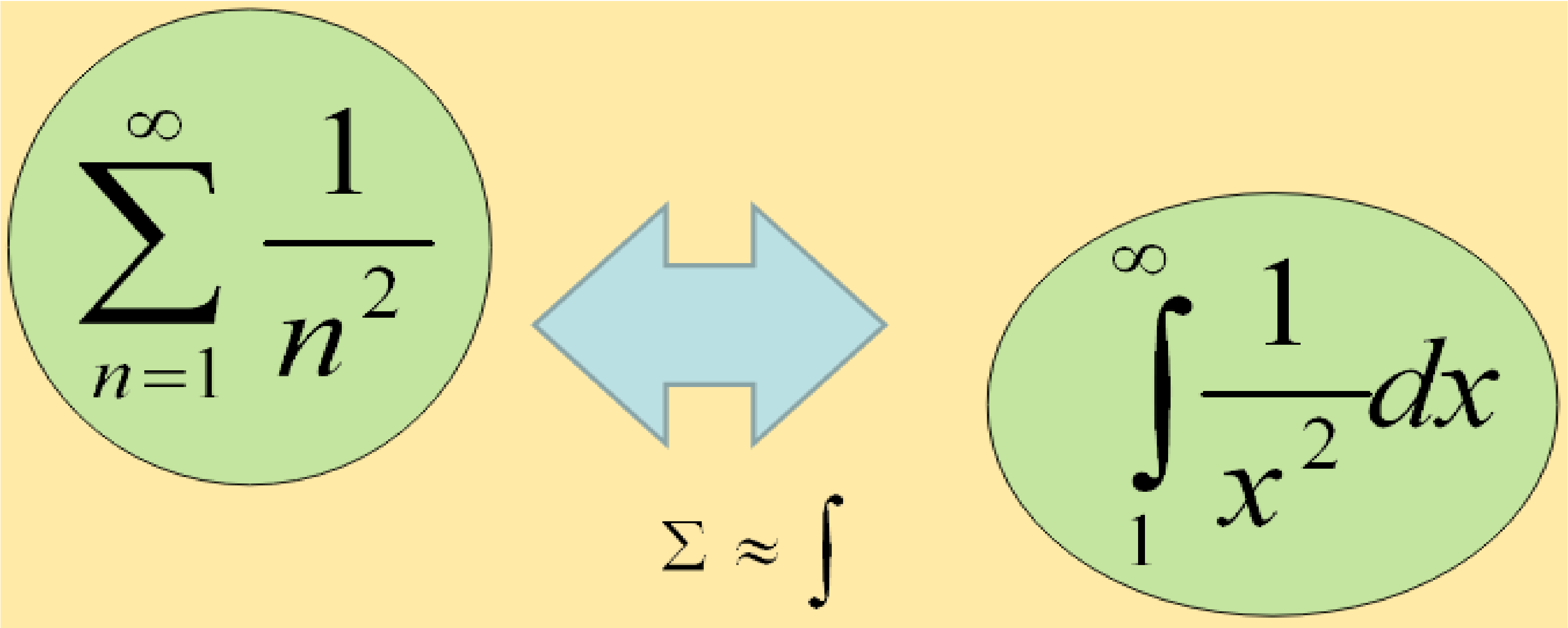
$$\sum_{n=1}^{\infty} a_n ,$$

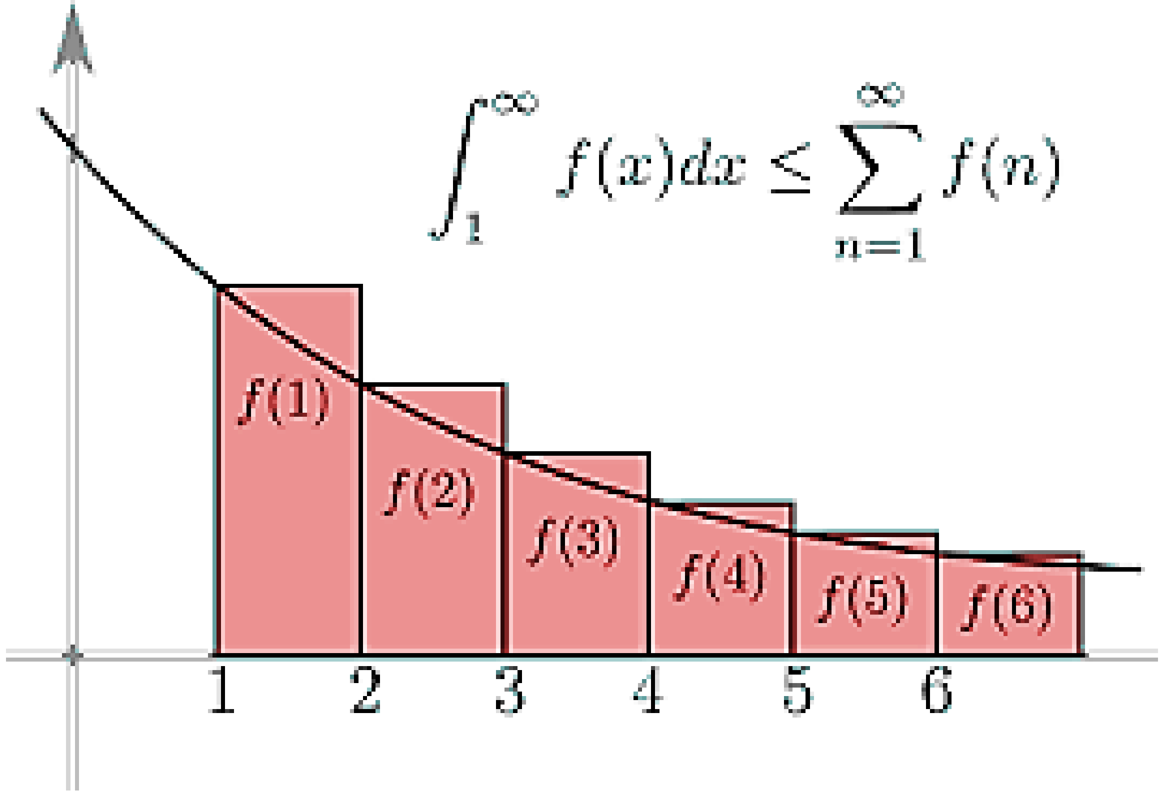
we need the n^{th} partial sum: $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ of the series. If the sequence of these partial sums $\{S_n\}$ converges to L, then the series is convergent, and sum of the series is L. If $\{S_n\}$ diverges, then the series diverges.

- But, for most of the series, it is often impossible to find an explicit formula for S_n. However, there exist several tests in literature to test the convergence or divergence of a series that employ the n^{th} term a_n. **But these tests just provide us the information about the convergence or divergence of the series, they do not give us the sum of a convergent series.**

The Integral Test

Section: 11.3





The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, nonincreasing function of x for all $x \geq N$ (N a positive integer). Then

1. the series $\displaystyle\sum_{n=N}^{\infty} a_n$ converges if the integral $\displaystyle\int_N^{\infty} f(x)\,dx$ converges,

2. the series $\displaystyle\sum_{n=N}^{\infty} a_n$ diverges if the integral $\displaystyle\int_N^{\infty} f(x)\,dx$ diverges,

Example

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

converges or diverges?

Solution:

Here $a_n = f(n) = \frac{n}{n^2+1}$. Thus, $f(x) = \frac{x}{x^2+1}$. The given function is positive and continuous for all $x \geq 0$. Also,

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0; \quad \text{for} \quad x > 1.$$

Thus, $f(x)$ is a decreasing function for $x > 1$ and we can apply integral test.

Using the Integral test

—

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \frac{1}{2} \int_1^{b} \frac{2x}{x^2+1}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2}\left[\ln(x^2+1)\right]_1^b = \frac{1}{2}\lim_{b\to\infty}\left[\ln(b^2+1) - \ln 2\right]$$

$$= \infty.$$

The improper integral diverges.

Thus, the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

Example

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges or diverges?

Solution:

Here $a_n = f(n) = \frac{1}{n^2+1}$. Thus, $f(x) = \frac{1}{x^2+1}$. The given function is positive and continuous. Also,

$$f'(x) = \frac{-2x}{(x^2+1)^2} < 0; \quad \text{for} \quad x > 1.$$

Thus, $f(x)$ is a decreasing function for $x > 1$ and we can apply integral test.

Using the Integral test

$$\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2+1}\,dx$$

$$= \lim_{b\to\infty}[\arctan x]_1^b = \lim_{b\to\infty}[\arctan b - \arctan 1]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

The improper integral converges.

Thus, the series $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ converges.

Harmonic series and p-series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called a $\boldsymbol{p}$ **−series**. A p −series converges if $p > 1$ and diverges if $p < 1$ or $p = 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \frac{1}{n} + \ldots$$

is called the harmonic series and it diverges since $p = 1$.

Example

Shows that the $\boldsymbol{p}$ **−series**

$$\sum_{n=1}^{\infty} \frac{1}{n^P} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^P} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$ and diverges if $p \leq 1$.

Solution:

If $\boldsymbol{p > 1}$, then $f(x) = \frac{1}{x^p}$ is a positive decreasing function of x. Since,

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b\to\infty} \int_1^{b} x^{-p} dx = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b = \frac{1}{1-p} \lim_{b\to\infty} \left[\frac{1}{b^{p-1}} - 1\right] = \frac{1}{p-1},$$

Thus, the series converges by the integral test.

Example

Solution:

If $\boldsymbol{p < 1}$, then $1 - p > 0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b = \frac{1}{1-p}\lim_{b\to\infty}\left[b^{1-p} - 1\right] = \infty.$$

Thus, the series diverges by the integral test.

If $\boldsymbol{p = 1}$, then we have

$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}[\ln x]_1^b = \lim_{b\to\infty}[\ln b - \ln 1] = \infty.$$

Thus, the series diverges by the integral test.

Practice Questions

Test the following series for convergence or divergence.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n}}{1 + e^{2n}}$

Practice Questions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Exercise: 11.3

  Q # 1 to Q # 30

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 11.3

  Q # 1 to Q # 12