# PROPERTIES OF Z-TRANSFORM

# Properties of z-Transform

(1) Linearity:

$$x_1[n] \xleftrightarrow{z} X_1(z); \qquad \text{ROC} = R_1$$

$$x_2[n] \xleftrightarrow{z} X_2(z); \qquad \text{ROC} = R_2$$

$$\boxed{ax_1[n] + bx_2[n] \xleftrightarrow{z} aX_1(z) + bX_2(z); \quad \text{ROC} = R_1 \cap R_2}$$

(2) Scaling in the $z$-Domain:

$$x[n] \xleftrightarrow{z} X(z); \qquad \text{ROC} = R$$

$$\boxed{z_0^n x[n] \xleftrightarrow{z} X\left(\frac{z}{z_0}\right); \qquad \text{ROC} = |z_0| R}$$

special case: $z_0 = e^{j\omega_0}$, $e^{j\omega_0 n} x[n] \xleftrightarrow{z} X(e^{-j\omega_0} z); \qquad \text{ROC} = R$

# Properties of z-Transform

(3) Time Reversal:

$$x[n] \overset{z}{\longleftrightarrow} X(z); \qquad \text{ROC} = R$$

$$\boxed{x[-n] \overset{z}{\longleftrightarrow} X\left(\frac{1}{z}\right); \qquad \text{ROC} = 1/R}$$

(4) Time Shifting $\boxed{x[n-n_0] \longleftrightarrow z^{-n_0} X(z),}$

The rationality of $X(z)$ unchanged, *different* from LT. ROC unchanged except for the possible addition or deletion of the origin or infinity

$n_o > 0 \Rightarrow \text{ROC } z \neq 0$ (maybe)

$n_o < 0 \Rightarrow \text{ROC } z \neq \infty$ (maybe)

# Properties of z-Transform

(5) *z*-Domain Differentiation $\boxed{nx[n] \longleftrightarrow -z\dfrac{dX(z)}{dz}, \quad \text{same ROC}}$

Derivation:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$\frac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} nx[n]z^{-n-1}$$

$$-z\frac{dX(z)}{dx} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

# Properties of z-Transform

(6) Time Expansion: process of inserting zeros between samples

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

$$x[n] \xleftrightarrow{z} X(z), \qquad \text{ROC} = R$$

$$\boxed{x_{(k)}[n] \xleftrightarrow{z} X(z^k), \qquad \text{ROC} = R^{1/k}}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(z^k) = \sum_{n=-\infty}^{\infty} x[n] \left(z^k\right)^{-n} = \sum_{n=-\infty}^{\infty} x[n] z^{-kn}$$

(7) Conjugation:

$$x[n] \xleftrightarrow{z} X(z), \qquad \text{ROC} = R$$

$$\boxed{x^*[n] \xleftrightarrow{z} X^*(z^*), \qquad \text{ROC} = R}$$

# Convolution Property

- If

$$x_1[n] \xleftrightarrow{z} X_1(z), \quad ROC = R_1$$

$$x_2[n] \xleftrightarrow{z} X_2(z), \quad ROC = R_2$$

- then

$$\boxed{x_1[n] * x_2[n] \xleftrightarrow{z} X_1(z) \cdot X_2(z), \quad ROC = R_1 \cap R_2}$$

- The ROC of $X_1(z) \cdot X_2(z)$ includes the intersection of $R_1$ and $R_2$ and may be larger if pole-zero cancellation occurs in the product.

# Convolution Property and System Functions

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

$Y(z) = H(z)X(z)$ , ROC at least the intersection of the ROCs of $H(z)$ and $X(z)$, can be bigger if there is pole/zero cancellation. *e.g.*

$$\begin{aligned} H(z) &= \frac{1}{z-a}, \quad |z| > a \\ X(z) &= z-a, \quad z \neq \infty \\ Y(z) &= 1 \quad \text{ROC all } z \end{aligned}$$

$$\boxed{H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}} \quad \text{— The System Function}$$

$H(z)$ + ROC tells us everything about system

# Problem-1

- Consider an LTI system for which

$$y[n] = h[n] * x[n]$$

- with

$$h[n] = \delta[n] - \delta[n-1]$$

- If the z-transform of $x[n]$ is $X(z)$, determine the z-transform of $y[n]$.

# Problem-1

- Note that

$$\delta[n]-\delta[n-1] \stackrel{z}{\longleftrightarrow} 1-z^{-1} \quad ROC: \text{ entire } z-\text{plane, except origin}$$

- Note zero at $z=1$ and

$$x[n] \stackrel{z}{\longleftrightarrow} X(z) \quad ROC = R$$

- giving

$$y[n] \stackrel{z}{\longleftrightarrow} (1-z^{-1})X(z) \quad ROC = R$$

- with the possible deletion of $z=0$ and/or addition of $z=1$.

- Note that for this system we get

$$y[n]=[\delta[n]-\delta[n-1]]*x[n]=x[n]-x[n-1]$$

- Thus we get a first difference of the sequence $x[n]$

# Integration Property

- Consider the inverse of first differencing, namely accumulation or summation.
- Let $w[n]$ be the running sum of $x[n]$

$$w[n] = \sum_{k=-\infty}^{n} x[k] = u[n] * x[n]$$

- giving

$$w[n] = \sum_{k=-\infty}^{n} x[k] \overset{z}{\longleftrightarrow} \frac{1}{1-z^{-1}} X(z) \quad ROC: \text{intersection of } R \text{ with } |z| > 1$$

- This is the integration property of DT $z-$transforms

# Problem-2

- Consider the $z$-transform

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|$$

- determine inverse $z$-transform):

# Problem-2

- Using the differentiation property we get:

$$nx[n] \xleftrightarrow{z} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}}, \quad |z| > |a|$$

- We recognize that:

$$a(-a)^n u[n] \xleftrightarrow{z} \frac{a}{1+az^{-1}}, \quad |z| > |a|$$

- Using the time shifting property we get:

$$a(-a)^{n-1} u[n-1] \xleftrightarrow{z} \frac{az^{-1}}{1+az^{-1}}, \quad |z| > |a|$$

$$x[n] = \frac{-(-a)^n}{n} u[n-1]$$

# Problem-3

- Consider determining the inverse $z-$transform for

$$X(z) = \frac{az^{-1}}{\left(1 - az^{-1}\right)^2}, \qquad |z| > |a|$$

# Problem-3

- From earlier examples

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{\left(1-az^{-1}\right)}, \quad |z|>|a|$$

- and hence

$$na^n u[n] \overset{z}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{\left(1-az^{-1}\right)}\right)=\frac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad |z|>|a|$$

END