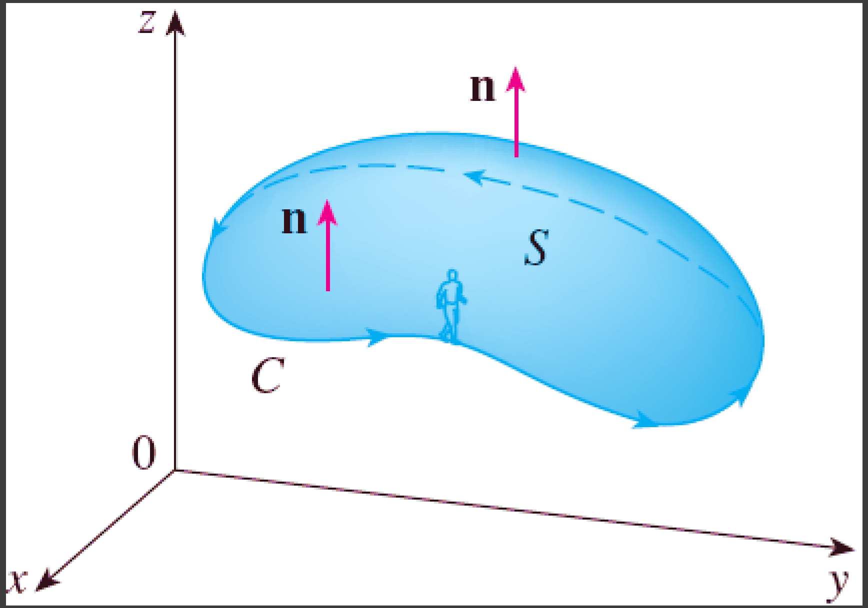
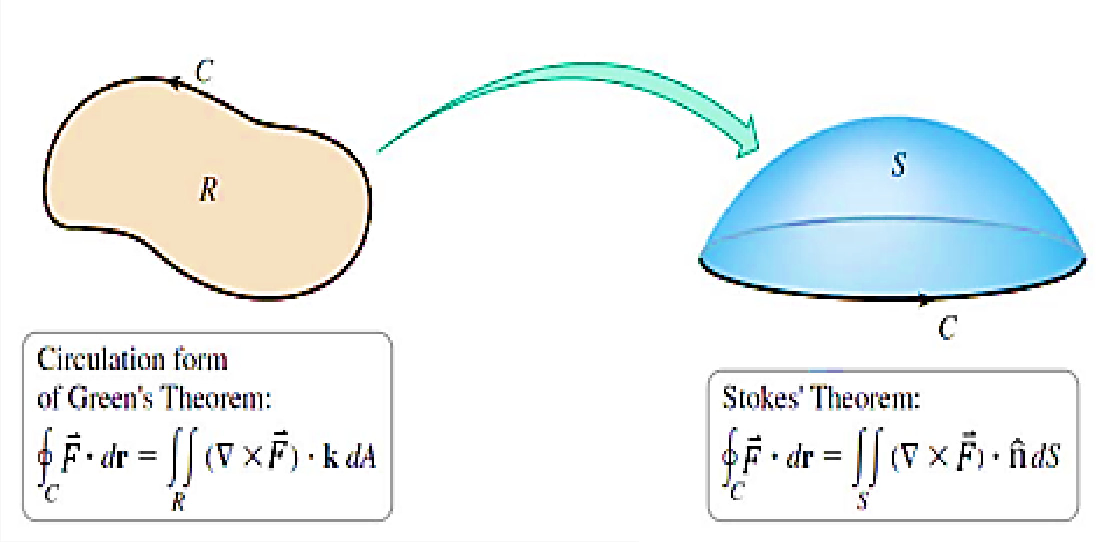
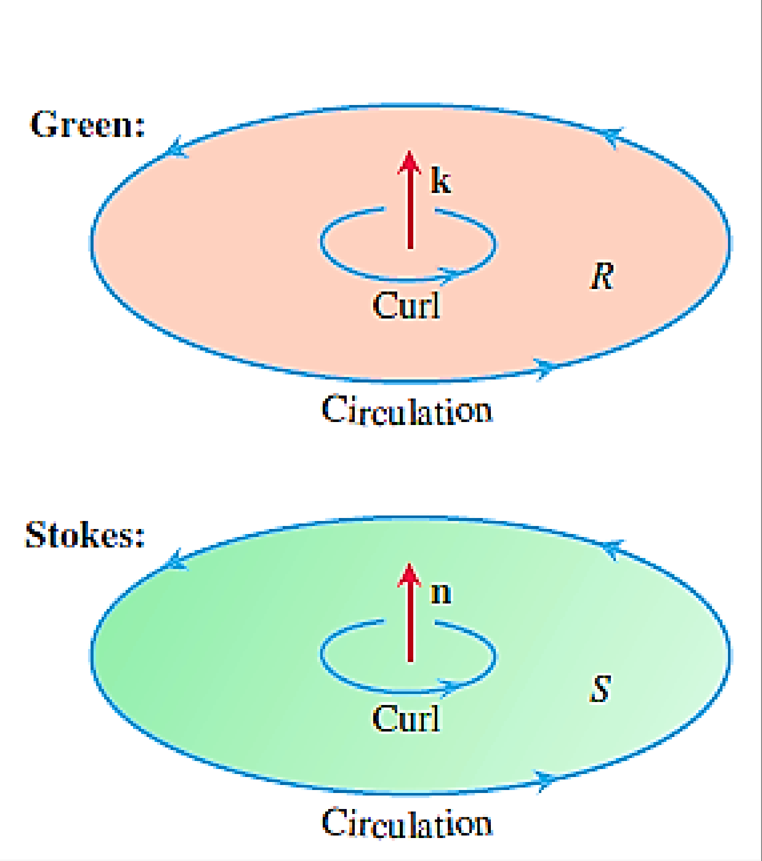




Stokes' Theorem



Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

16

Vector Calculus

Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.8**

Book: Thomas’ Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

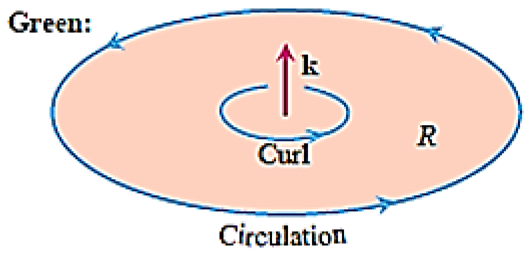
- **Chapter: 16**
  - **Section: 16.7**

Stokes' Theorem & Green's Theorem

For a vector field $\mathbf{F}$ and a smooth surface S the Green's theorem can be generalized to higher dimensions as follows:

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA$$

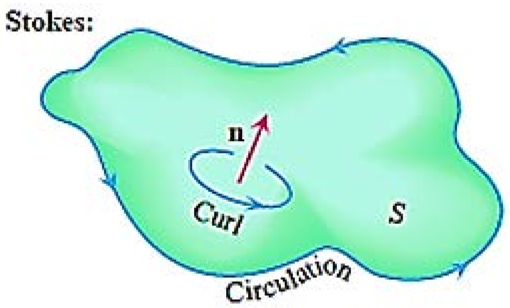
Circulation in R



$$\downarrow$$

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS$$

Circulation in 3D



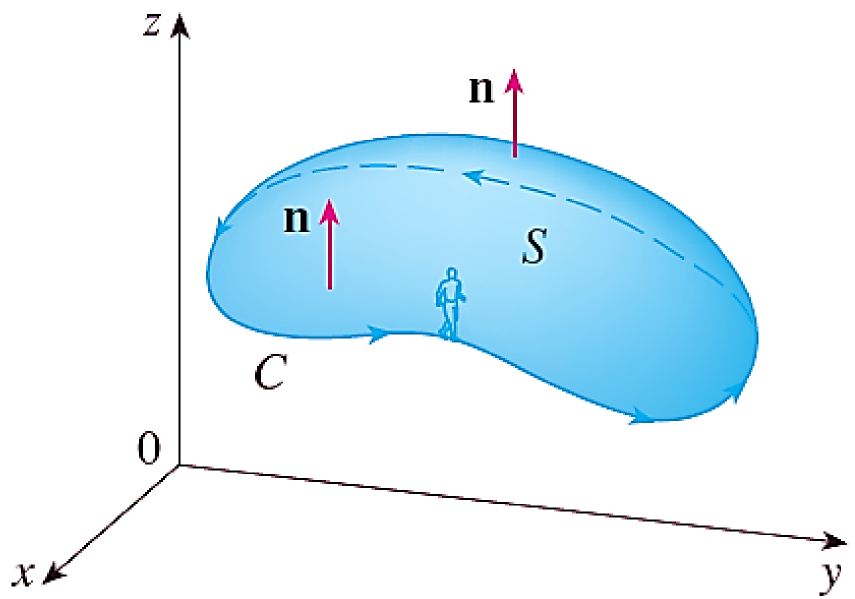
- Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve.
- Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (a space curve).

Stokes' Theorem

Let S be an oriented piecewise-smooth surface bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then,

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS \quad \text{or} \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S},$$

where $\mathbf{n}$ is the unit normal vector at any point of S drawn in the sense in which a right-handed screw would advance when rotated in the sense of the description of C.



Stokes' Theorem

Thus, Stokes' Theorem says:

> The line integral around the boundary curve of S of the tangential component of $\mathbf{F}$ is equal to the surface integral of the normal component of the curl of $\mathbf{F}$.

Consider the special case where the surface S is **flat** and lies in the xy −plane with upward orientation. Then, the unit normal is $\mathbf{k}$**,** the surface integral becomes a double integral, and the Stokes' Theorem becomes:

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dA.$$

This is precisely the vector form of the Green's Theorem. Thus, we see that Green's Theorem is really a special case of Stokes' Theorem.

Example:

Evaluate

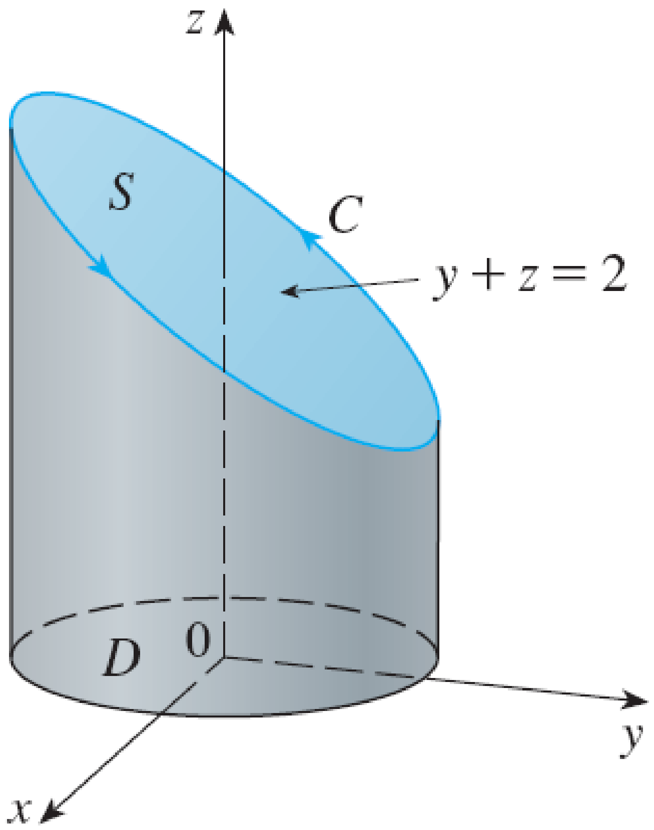
$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F}(x, y, z) = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

Solution:

(Orient C to be counterclockwise when viewed from above.)

In order to use Stokes' Theorem we first compute:



$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix},$$

$$= (1 + 2y)\mathbf{k}.$$

Solution:

Consider the elliptical region S in the plane $y + z = 2$ that is bounded by C. If we orient S upward, C has the induced positive orientation and we have $\mathbf{n} = \mathbf{k}$. Moreover, the projection D of S on the xy −plane is the disk $x^2 + y^2 \leq 1$. Thus, we have:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D (1 + 2y) dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r \sin\theta) r dr d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3} \sin\theta \right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3} \sin\theta \right) d\theta$$

$$= \frac{1}{2}(2\pi) + 0 = \pi.$$

Example: Finding Circulation

Find the circulation of the field $\mathbf{F}(x, y, z) = (x^2 - y)\,\mathbf{i} + 4z\,\mathbf{j} + x^2\mathbf{k}$ around the curve C in which the plane $z = 2$ meets the cone $z = \sqrt{x^2 + y^2}$ counterclockwise as viewed from above.

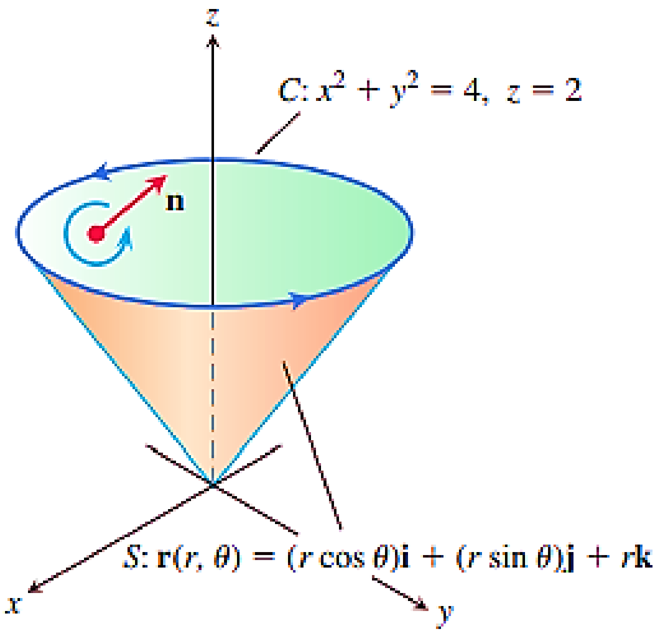
Solution:

Stokes' Theorem enables us to find the circulation by integrating over the surface of the cone. Traversing C in the anti-clockwise direction viewed from above corresponds to taking the *inner* normal $\mathbf{n}$ to the cone, the normal with a positive *z*-component. We parametrize the cone as:

$$x = x, \qquad y = y, \qquad z = \sqrt{x^2 + y^2},$$

i.e., $\mathbf{r}(x, y) = \left\langle x, y, \sqrt{x^2 + y^2} \right\rangle$. We then have:

$$\mathbf{r}_x \times \mathbf{r}_y = \left\langle \frac{-x}{\sqrt{x^2 + y^2}}, \frac{-y}{\sqrt{x^2 + y^2}}, 1 \right\rangle,$$



Note: Alternatively, we can parametrize the cone as:

$$\mathbf{r}(r, \theta) = \langle r\cos\theta\,, r\sin\theta\,, r \rangle.$$

Solution:

Now,

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y & 4z & x^2 \end{vmatrix} = \langle -4, -2x, 1 \rangle.$$

Therefore,

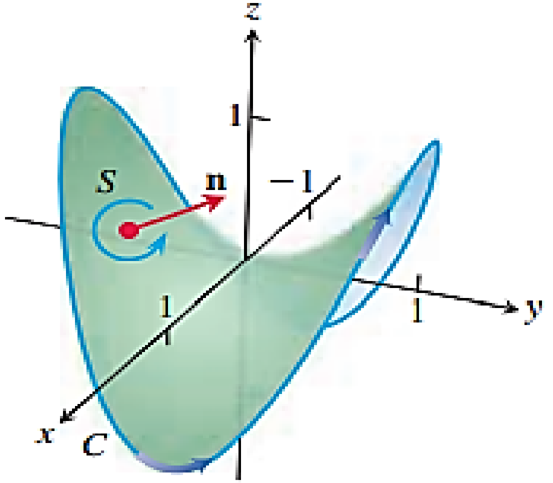
$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S \text{curl}\mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \left[\text{curl}\mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}\right] |\mathbf{r}_x \times \mathbf{r}_y| d\mathrm{A}$$

$$= \iint_D [\text{curl}\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)] d\mathrm{A} = \iint_D \left[\langle -4, -2x, 1 \rangle \cdot \left\langle \frac{-x}{\sqrt{x^2 + y^2}}, \frac{-y}{\sqrt{x^2 + y^2}}, 1 \right\rangle\right] d\mathrm{A}$$

$$= \iint_D \left(\frac{4x + 2xy}{\sqrt{x^2 + y^2}} + 1\right) dA = \int_0^{2\pi} \int_0^2 (4\cos\theta + r\sin 2\,\theta + 1) r\,dr\,d\theta = 4\pi.$$

Example: Finding Circulation

Find a parametrization for the surface S formed by the part of the hyperbolic paraboloid: $z = y^2 - x^2$ lying inside the cylinder of radius one around the z −axis and for the boundary curve C of S. Then verify Stokes' Theorem for S using the normal having positive $\mathbf{k}$ −component and the vector field $\mathbf{F} = \langle y, -x, x^2 \rangle$.



Solution:

As the unit circle is traversed counterclockwise in the xy −plane, the z −coordinate of the surface with the curve C as boundary is given by $y^2 - x^2$. A parametrization of C is given by:

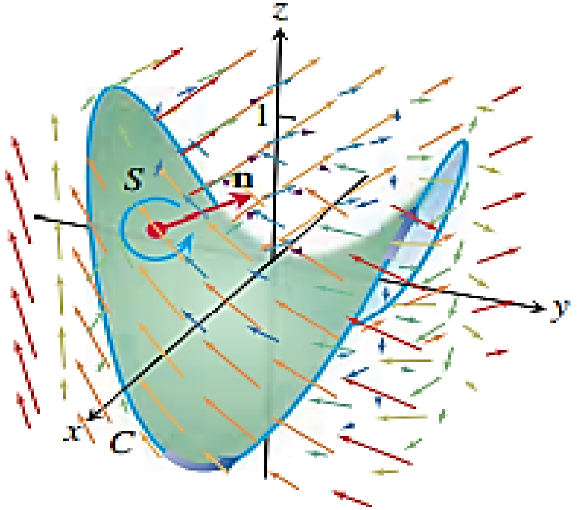
$$\mathbf{r}(t) = \langle \cos t\,, \sin t\,, \sin^2 t - \cos^2 t \rangle; \quad 0 \le t \le 2\pi.$$

We then have:

$$\mathbf{r}'(t) = \langle -\sin t\,, \cos t\,, 4 \sin t \cos t \rangle.$$

and

$$\mathbf{F}\big(\mathbf{r}(t)\big) = \langle \sin t\,, -\cos t\,, \cos^2 t \rangle.$$



The counterclockwise circulation along C is given as:

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^{2\pi} \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} [-1 + 4 \sin t \cos^3 t]\, dt = -2\pi.$$

Solution:

We now compute the same quantity by integrating $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}$ over the surface S. A parametrization of S is given as:

$$\mathbf{r}(r,\theta) = \langle r\cos\theta\,, r\sin\theta\,, r^2(\sin^2\theta - \cos^2\theta)\rangle;\;\; 0 \le r \le 1 \;\;\text{and}\;\; 0 \le \theta \le 2\pi.$$

We then have:

$$\mathbf{r}_r = \langle \cos\theta\,, \sin\theta\,, 2r(\sin^2\theta - \cos^2\theta)\rangle,$$

$$\mathbf{r}_\theta = \langle -r\sin\theta\,, r\cos\theta\,, 4r^2\sin\theta\cos\theta\rangle,$$

$$\mathbf{r}_r \times \mathbf{r}_\theta = \langle 2r^2\cos\theta\,, -2r^2\sin\theta\,, r\rangle,$$

$$\nabla\times\mathbf{F} = \langle 0, -2x, -2\rangle = \langle 0, -2r\cos\theta\,, -2\rangle$$

and $$\nabla\times\mathbf{F}\cdot(\mathbf{r}_r\times\mathbf{r}_\theta) = 4r^3\sin\theta\cos\theta - 2r.$$

Thus, the surface integral of $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}$ over S is given as:

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_D [\nabla\times\mathbf{F}\cdot(\mathbf{r}_r\times\mathbf{r}_\theta)]dA = \int_0^{2\pi}\int_0^1 [4r^3\sin\theta\cos\theta - 2r]drd\theta = -2\pi.$$

So, the surface integral of $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}$ over S equals the counterclockwise circulation of $\mathbf{F}$ along C, as asserted by Stokes' Theorem.

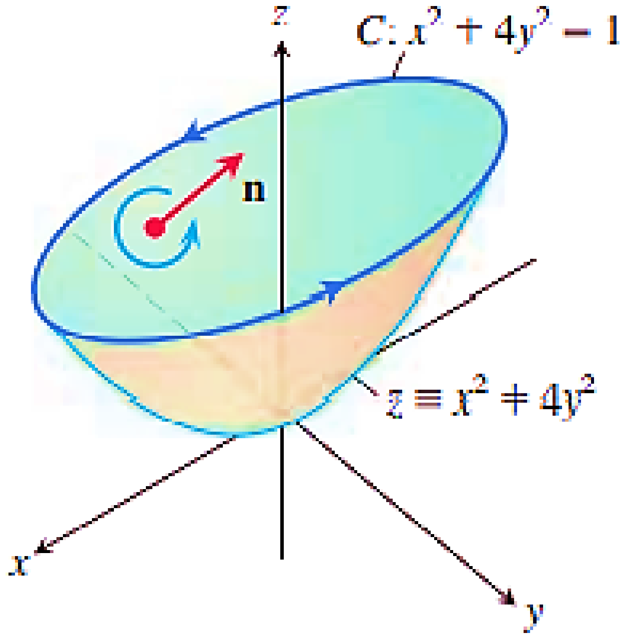
Example: Finding Circulation

Let the surface S be the ellipitical paraboloid $z = x^2 + 4y^2$ lying beneath the plane $z = 1$. We define the orientation of S by taking the inner normal vector $\mathbf{n}$ to the surface, which is the normal having a positive $\mathbf{k}$ −component. Find the flux of $\nabla \times \mathbf{F}$ across S in the direction $\mathbf{n}$ for the vector field $\mathbf{F} = \langle y, -xz, xz^2 \rangle$.

Solution:

We use Stokes' Theorem to calculate the curl integral by finding the equivalent counterclockwise circulation of $\mathbf{F}$ around the curve of intersection C of the paraboloid $z = x^2 + 4y^2$ and the plane $z = 1$. The orientation of S is consistent with traversing C in a counterclockwise direction around the z −axis. The curve C is the ellipse $x^2 + 4y^2 = 1$ in the plane $z = 1$. We can parametrize the ellipse by:

$$\mathbf{r}(t) = \left\langle \cos t\,, \frac{\sin t}{2}, 1 \right\rangle; \;\; 0 \le t \le 2\pi.$$



We then have:

$$\mathbf{r}'(t) = \left\langle -\sin t\,, \frac{\cos t}{2}, 0 \right\rangle.$$

and

$$\mathbf{F}(\mathbf{r}(t)) = \left\langle \frac{\sin t}{2}, -\cos t\,, \cos t \right\rangle.$$

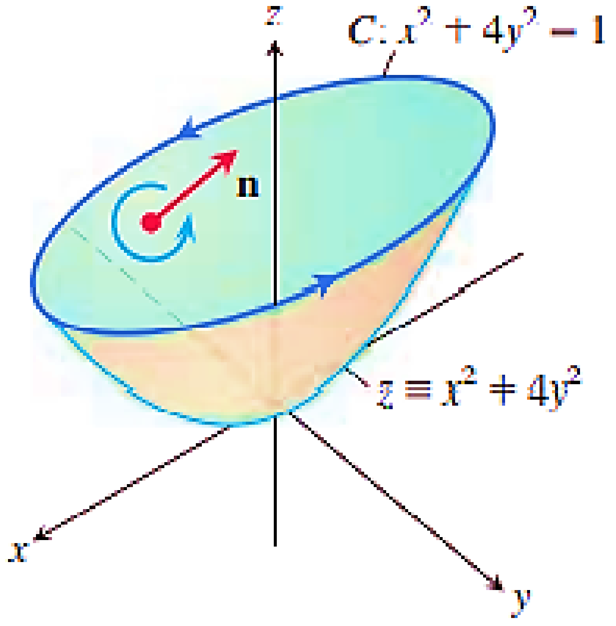
Solution:

The counterclockwise circulation of $\mathbf{F}$ along C is given as:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)dt = \int_0^{2\pi} \left[-\frac{1}{2}\right] dt = -\pi.$$

Thus, using Stokes' Theorem, the flux of $\nabla \times \mathbf{F}$ across S in the direction $\mathbf{n}$ for the vector field $\mathbf{F}$ comes out to be:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = -\pi.$$

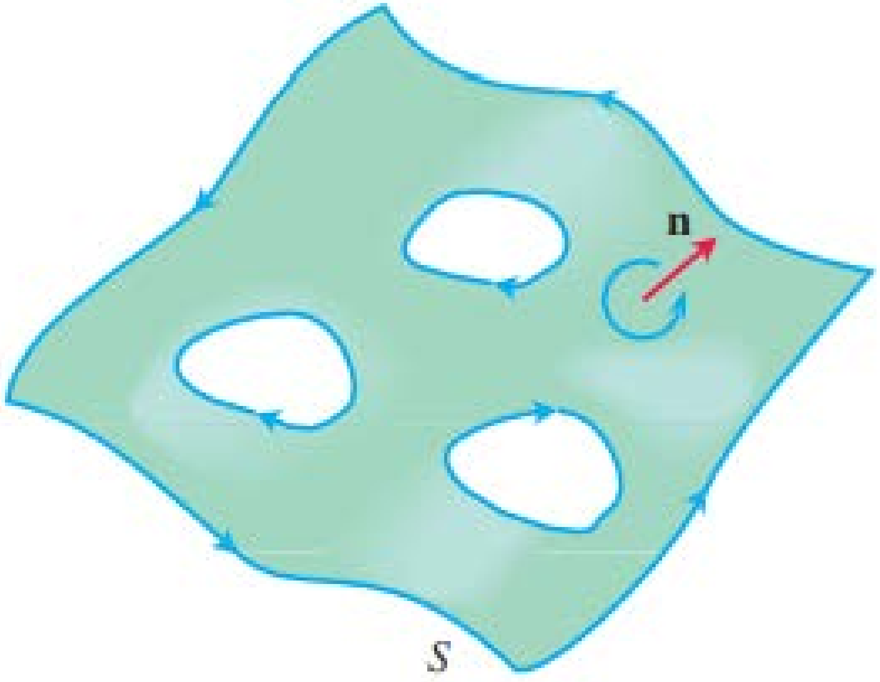


Stokes' Theorem for Surfaces with Holes

- Stokes' Theorem holds for an oriented surface S that has one or more holes.

- The surface integral over S of the normal component of $\nabla \times \mathbf{F}$ equals the sum of the line integrals around all the boundary curves of the tangential component of $\mathbf{F}$, where the curves are to be traced in the direction induced by the orientation of S.

- For such surfaces, the theorem is unchanged, but C is considered as a union of simple closed curves.



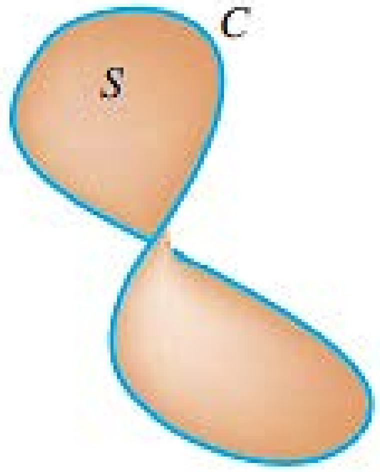
Stokes' Theorem also holds for oriented surfaces with holes. Consistent with the orientation of S, the outer curve is traversed counterclockwise around $\mathbf{n}$ and the inner curves surrounding the holes are traversed clockwise.

Conservative Fields and Stokes' Theorem

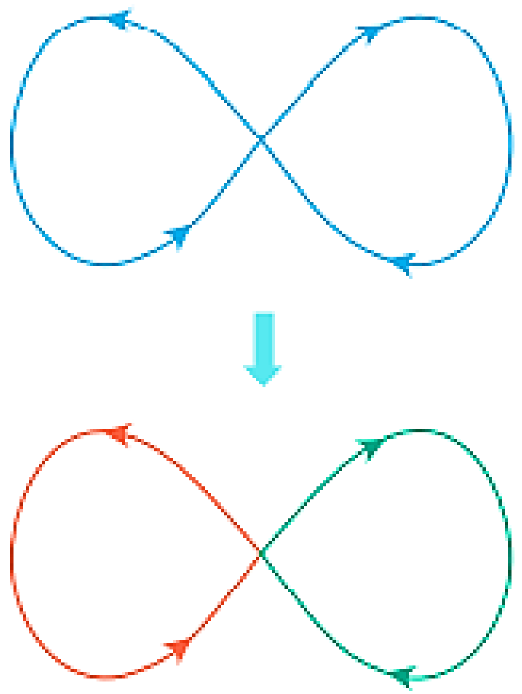
- A field $\mathbf{F}$ being conservative in an open region D in space is equivalent to the integral of $\mathbf{F}$ around every closed loop in D being zero.
- This, in turn, is equivalent in simply connected open regions to saying that $\nabla \times \mathbf{F} = \mathbf{0}$ (which gives a test for determining if $\mathbf{F}$ is conservative for such regions).

Theorem: If $\nabla \times \mathbf{F} = \mathbf{0}$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 0.$$



(a)



(b)

(a) In a simply connected open region in space, a simple closed curve C is the boundary of a smooth surface S. (b) Smooth curves that cross themselves can be divided into loops to which Stokes' Theorem applies.

Conservative Fields and Stokes' Theorem

The following diagram summarizes the results for conservative fields defined on connected, simply connected open regions. For such regions, the four statements are equivalent to each other.

$$\begin{array}{ccc}
\mathbf{F} \text{ conservative on } D & \Longleftrightarrow & \mathbf{F} = \nabla f \text{ on } D \\
\Updownarrow & & \Downarrow \ \begin{array}{l}\text{Vector identity} \\ \text{(continuous second} \\ \text{partial derivatives)}\end{array} \\
\begin{array}{c}\oint_C \mathbf{F}\cdot d\mathbf{r} = 0 \\ \text{over any closed} \\ \text{path in } D\end{array} & \underset{\begin{array}{c}\text{Domain's simple} \\ \text{connectivity and} \\ \text{Stokes' Theorem}\end{array}}{\Longleftarrow} & \nabla \times \mathbf{F} = \mathbf{0} \text{ throughout } D
\end{array}$$

Practice Questions

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

Chapter: 16

Exercise-16.8: Q – 2 to 15, Q – 17.

Book: Thomas’ Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 16

Exercise-16.7: Q – 7 to 24.