CARTESIAN AND CYLINDRICAL COORDINATES

Coordinate Systems

- In order to describe the spatial variations of the physical quantities, we must be able to define all points uniquely in space in a suitable manner

- Requires using an appropriate coordinate system

- Considerable amount of work and time may be saved by choosing a coordinate system that best fits a given problem

- A hard problem in one coordinate system may turn out to be easy in another system

- Three best-known coordinate systems: the Cartesian, Cylindrical and Spherical

Coordinate Systems

- The concepts demonstrated in Cartesian coordinates are equally applicable to other systems of coordinates

- For example, procedure for finding dot or cross product of two vectors in a cylindrical system is the same as that used in the Cartesian system

- Sometimes, it is necessary to transform points and vectors from one coordinate system to another

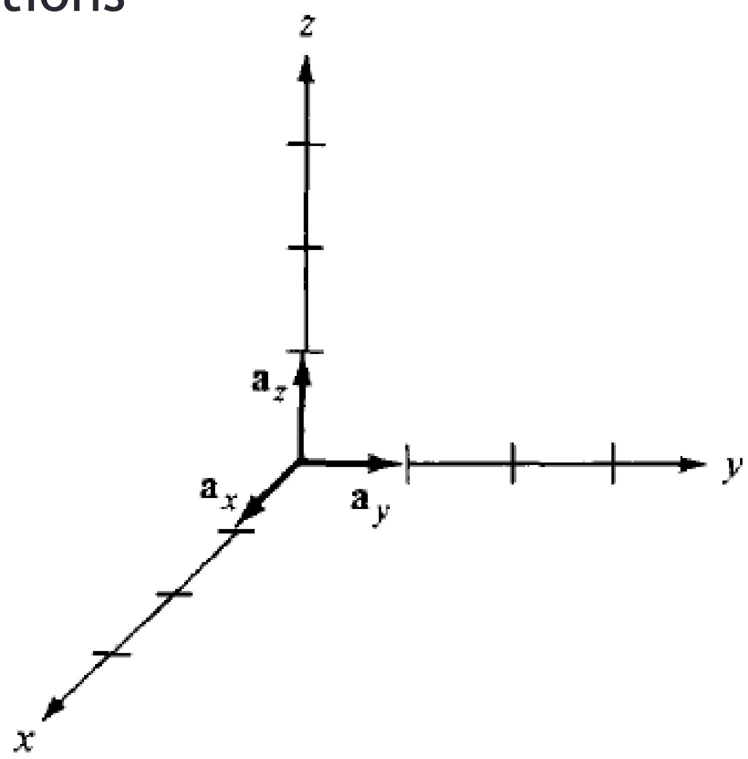
- The techniques for doing this will be presented and illustrated with examples

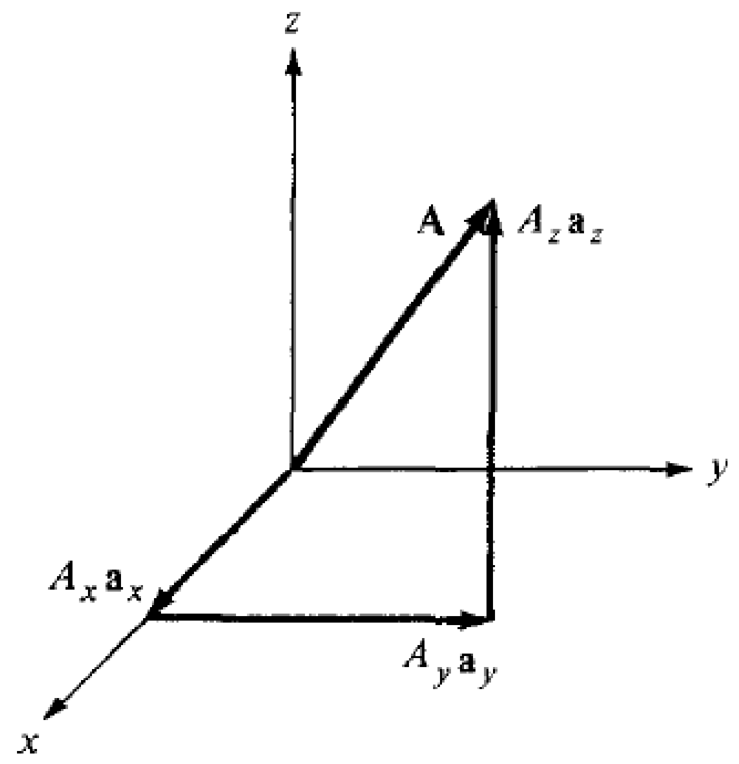
Cartesian Coordinates (X,Y,Z)

- A vector **A** in Cartesian (also known as rectangular) coordinates can be written as:

$$(A_x, A_y, A_z) \qquad \text{or} \qquad A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

- where $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are unit vectors along the x-, y-, and z-directions





Cylindrical Coordinates (ρ,Φ,z)

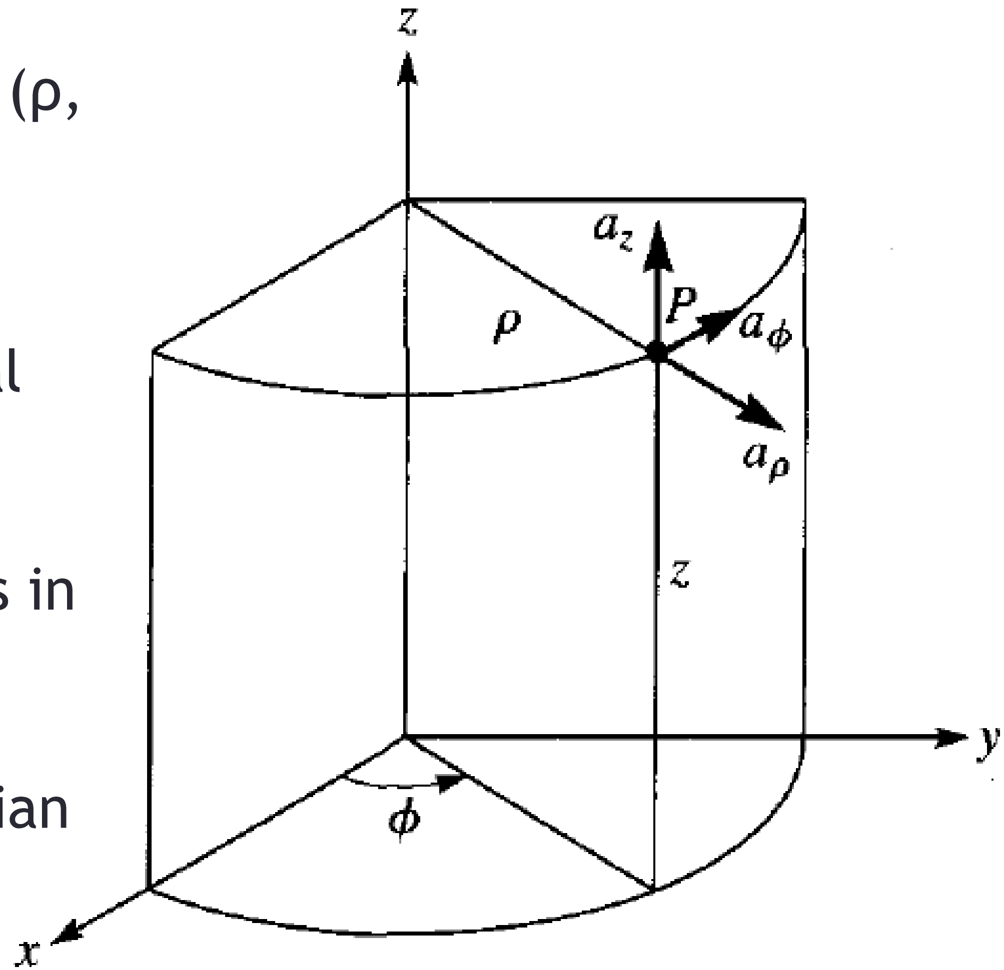
- Very convenient when dealing with problems having cylindrical symmetry

- A point P in cylindrical coordinates is represented as (ρ, Φ, z) as shown in the figure:

- ρ is the radius of the cylinder passing through P or the radial distance from the z-axis

- Φ is measured from the x-axis in the xy-plane

- z is the same as in the Cartesian system



Cylindrical Coordinates (ρ, Φ, z)

- The ranges of the variables are:

$$0 \leq \rho < \infty$$

$$0 \leq \phi < 2\pi$$

$$-\infty < z < \infty$$

- A vector A in cylindrical coordinates can be written as:

$$(A_\rho, A_\phi, A_z) \qquad \text{or} \qquad A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z$$

- where $\mathbf{a}_\rho$, $\mathbf{a}_\Phi$, and $\mathbf{a}_z$ are unit vectors in the ρ, Φ and z directions

- For example, if a force of 10 N acts on a particle in a circular motion, the force may be represented as $\mathbf{F} = 10\mathbf{a}_\Phi$ N

- The magnitude of **A** is:

$$|\mathbf{A}| = (A_\rho^2 + A_\phi^2 + A_z^2)^{1/2}$$

Cylindrical Coordinates (ρ,Φ,z)

- Notice that the unit vectors $\mathbf{a}_\rho$, $\mathbf{a}_\Phi$, and $\mathbf{a}_z$ are mutually perpendicular because our coordinate systems are orthogonal

- $\mathbf{a}_\rho$ points in the direction of increasing ρ, $\mathbf{a}_\Phi$ in the direction of increasing Φ, and $\mathbf{a}_z$ in the positive z-direction, so we have:

$$\mathbf{a}_\rho \cdot \mathbf{a}_\rho = \mathbf{a}_\phi \cdot \mathbf{a}_\phi = \mathbf{a}_z \cdot \mathbf{a}_z = 1$$

$$\mathbf{a}_\rho \cdot \mathbf{a}_\phi = \mathbf{a}_\phi \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_\rho = 0$$

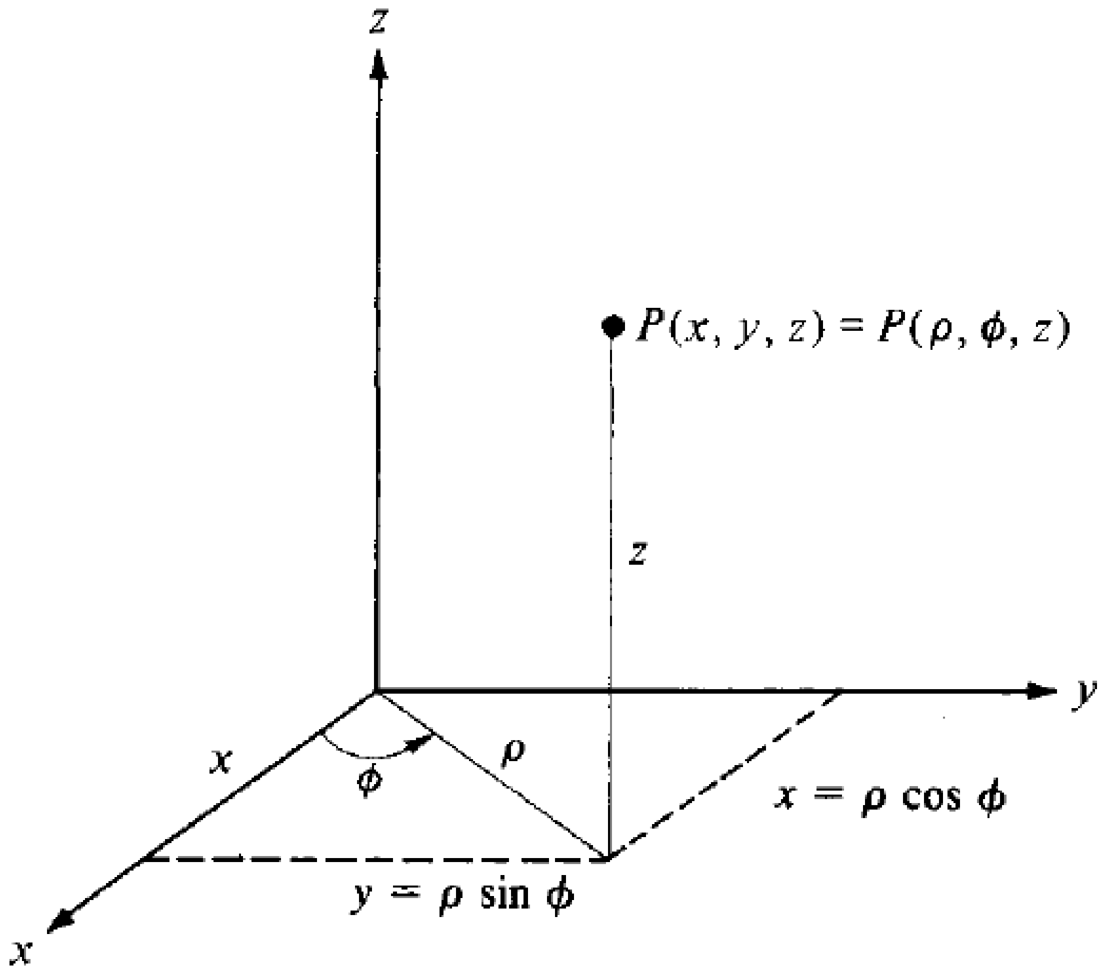
$$\mathbf{a}_\rho \times \mathbf{a}_\phi = \mathbf{a}_z$$

$$\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_\rho$$

$$\mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi$$

Point Transformations

- Relationship between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system (ρ, Φ, z) are easily obtained from figure shown:



Point Transformations

- For transforming a point from Cartesian (x, y, z) to Cylindrical (ρ, *Φ*, z) coordinates:

$$\rho = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}, \qquad z = z$$

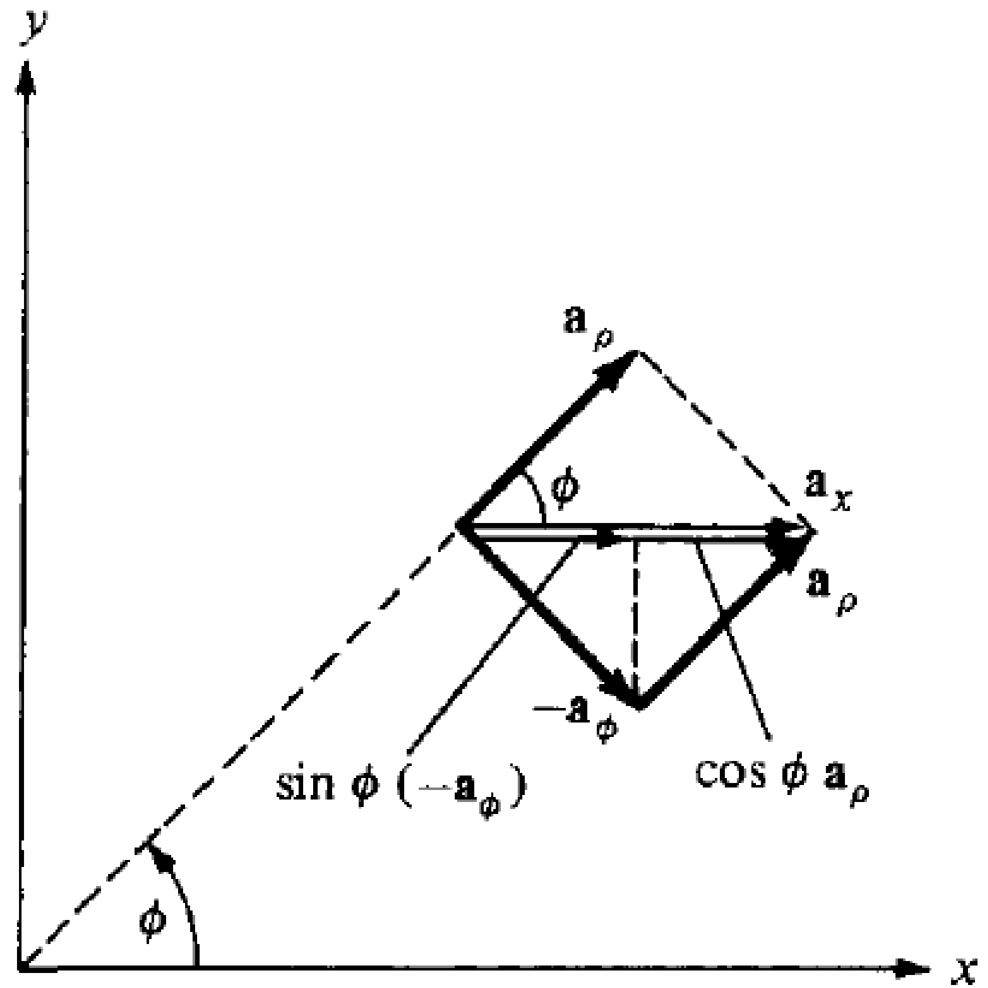
- For transforming a point from Cylindrical (ρ, *Φ*, z) to Cartesian (x, y, z) coordinates:

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z$$

Unit Vector Transformations

- The relationships between ($\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$) and ($\mathbf{a}_\rho$, $\mathbf{a}_\phi$, $\mathbf{a}_z$) are obtained geometrically from figure below showing cylindrical components of $\mathbf{a}_x$

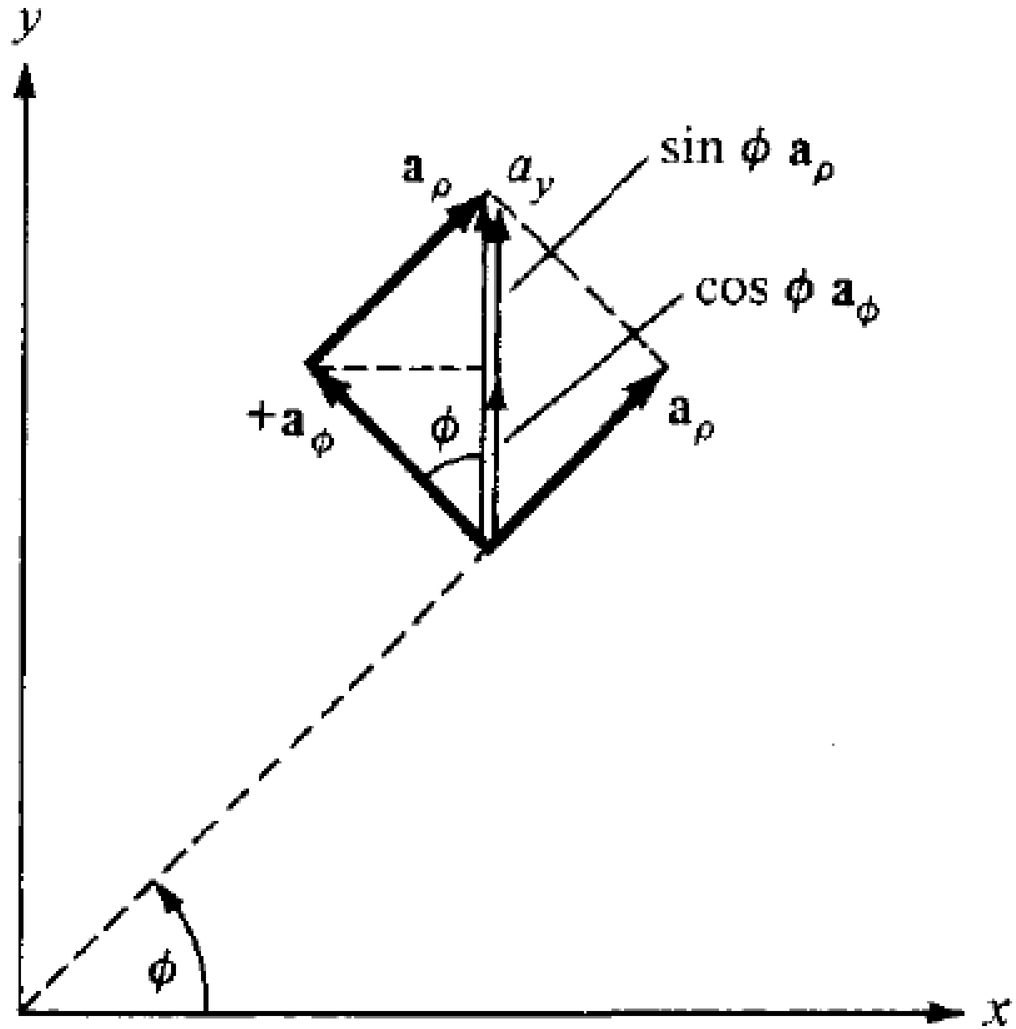
$$\mathbf{a}_x = \cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi$$



Unit Vector Transformations

- The relationships between ($\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$) and ($\mathbf{a}_\rho$, $\mathbf{a}_\phi$, $\mathbf{a}_z$) are obtained geometrically from figure below showing cylindrical components of $\mathbf{a}_y$

$$\mathbf{a}_y = \sin\phi\,\mathbf{a}_\rho + \cos\phi\,\mathbf{a}_\phi$$



Unit Vector Transformations

- In summary, we have:

$$\mathbf{a}_x = \cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_y = \sin\phi\,\mathbf{a}_\rho + \cos\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_z = \mathbf{a}_z$$

OR

$$\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$$

$$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$$

$$\mathbf{a}_z = \mathbf{a}_z$$

Vector Transformations

- Finally, the relationship between (Ax, Ay, Az) and (Aρ, A_ϕ, Az) are obtained by simply substituting the unit vector transformations into the equation below:

$$A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

- After collecting terms, we get:

$$\mathbf{A} = (A_x \cos\phi + A_y \sin\phi)\mathbf{a}_\rho + (-A_x \sin\phi + A_y \cos\phi)\mathbf{a}_\phi + A_z\mathbf{a}_z$$

OR

$$A_\rho = A_x \cos\phi + A_y \sin\phi$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

$$A_z = A_z$$

Vector Transformations

- The transformations may be written in matrix form as:

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

AND

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Problem-1

a) Convert point P(0, - 4 , 3) from Cartesian to cylindrical coordinates

b) Evaluate **Q** at P in Cartesian and cylindrical coordinate systems

$$\mathbf{Q} = \frac{\sqrt{x^2 + y^2}\,\mathbf{a}_x}{\sqrt{x^2 + y^2 + z^2}} - \frac{yz\,\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}$$