EE-381 Robotics-1
UG ELECTIVE

Lecture 10

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Time and Motion

- How pose changes over time?
- Linear and Angular velocity of rigid bodies/joints
- Motion of a Manipulator

Basic-Linear Velocity

- Derivative of a vector

$$V_Q^B = \frac{d}{dt}Q^B = \lim_{\Delta t \to 0} \frac{Q^B(t+\Delta t) - Q^B(t)}{\Delta t}$$

- The velocity of a position vector can be thought of as the linear velocity of the point in space represented by the position vector.
- It is important to indicate the frame in which the vector is differentiated.
- Velocity vector when expressed in terms of frame {A}

$$\left(V_Q^B\right)^A = \left(\frac{d}{dt}Q^B\right)^A$$

Basic-Linear Velocity

Frame wrt which position vector is differentiated

Frame wrt which resulting velocity vector is expressed

$$\left(V_Q^B\right)^A = \left(\frac{d}{dt} Q^B\right)^A$$

Leading superscript can be omitted when expressed in terms of itself

$$\left(V_Q^B\right)^B = V_Q^B$$

We can always remove the outer, leading superscript by explicitly including the rotation matrix that accomplishes the change in reference frame

$$\left(V_Q^B\right)^A = R_B^A \; V_Q^B$$

Basic-Linear Velocity

- Generally, the velocity of the origin of a frame is considered relative to some understood universe reference frame

$$v_C = V_{CORG}^{U}$$

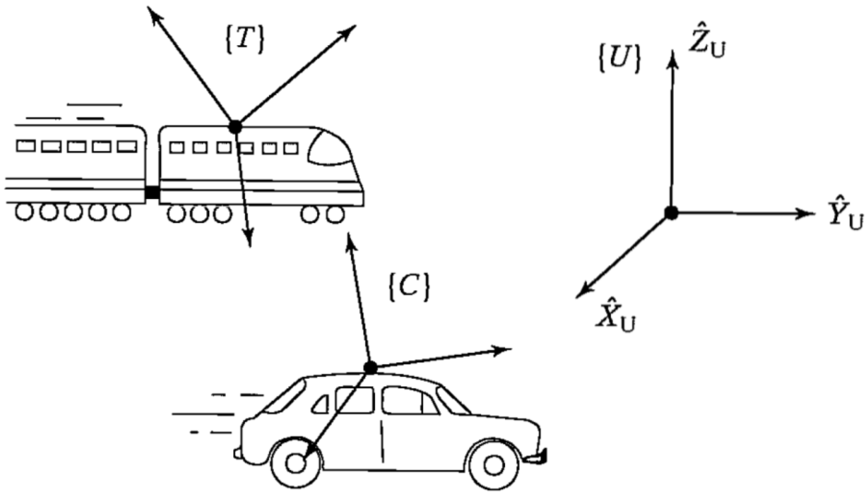
- We use the notation to refer to the velocity of the origin of frame {C}

- What is v_C^A?

Velocity of the origin of frame {C} expressed in terms of frame {A} (though differentiation was done relative to {U})

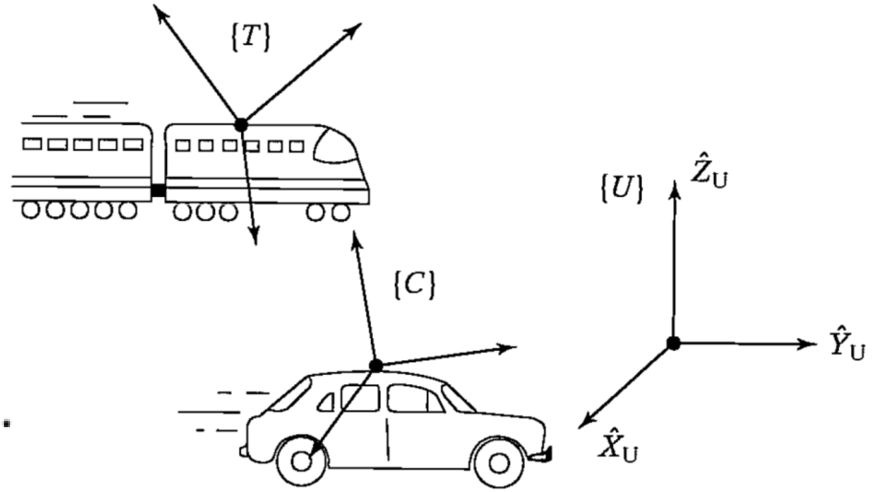
Example

- Figure shows a fixed universe frame, {U}, a frame attached to a train travelling at 100mph, {T}, and a frame attached to a car travelling at 30mph, {C}. Both vehicles are heading in the $\hat{X}$ direction of {U}. The rotation matrices, R_T^U and R_C^U, are known and constant



Example

- What is $\left(\frac{d}{dt}P_{CORG}^{U}\right)^{U}$?



$$\frac{{}^{U}d}{dt}\,{}^{U}P_{CORG} = {}^{U}V_{CORG} = v_C = 30\hat{X}.$$

- What is ${}^{C}({}^{U}V_{TORG})$?

$$^{C}({}^{U}V_{TORG}) = {}^{C}v_T = {}^{C}_{U}R v_T = {}^{C}_{U}R(100\hat{X}) = {}^{U}_{C}R^{-1}\,100\hat{X}.$$

- What is ${}^{C}({}^{T}V_{CORG})$?

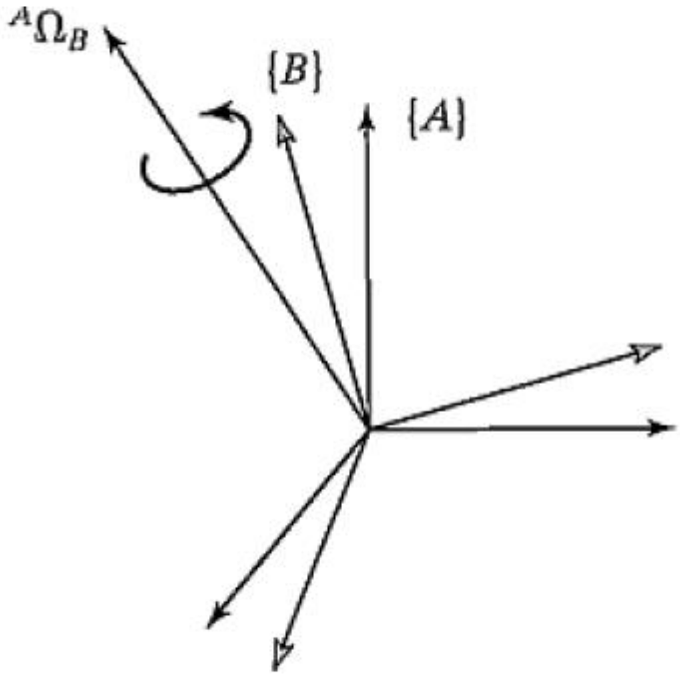
$$^{C}({}^{T}V_{CORG}) = {}^{C}_{T}R\,{}^{T}V_{CORG} = -{}^{U}_{C}R^{-1}\,{}^{U}_{T}R\,70\hat{X}.$$

Basics-Angular Velocity

- Angular velocity vector –symbol Ω
- **Linear velocity** describes an attribute of a **point**,
- **Angular velocity** describes an attribute of a **body**.
- We always attach a frame to the bodies, therefore angular velocity describes rotational motion of a frame

Basics-Angular Velocity

- Ω_B^A describes the rotation of frame B relative to frame A



- Physically, at any instant, the direction of Ω_B^A indicates the instantaneous axis of rotation of {B} relative to {A}, and the magnitude of Ω_B^A indicates the speed of rotation.

Basics-Angular Velocity

- An angular velocity vector may be expressed in any coordinate system, and so another leading superscript may be added; for example, $\left(\Omega_B^A\right)^C$ is the angular velocity of frame {B} relative to {A} expressed in terms of frame{C}
- **Simplified notation**: angular velocity of frame (C) relative to some understood reference frame, (U)

$$\omega_C = {}^U\Omega_C$$

- The angular velocity of frame (C) expressed in terms of (A) (though the angular velocity is with respect to (U)).

$${}^A\omega_C$$

Linear and Angular Velocities of Rigid Bodies

- We focus on motion of rigid body with regards to velocity

- We therefore extend the notions of translations and orientations described earlier to the time-varying case

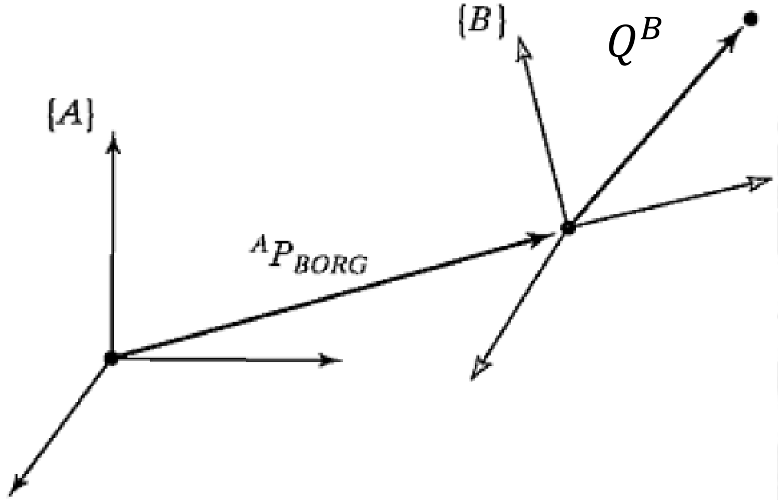
- We attach a coordinate system to any body that we wish to describe. Then, motion of rigid bodies can be equivalently studied as the motion of frames relative to one another.

Linear and Angular Velocities of Rigid Bodies

Linear Velocity

- Consider a frame {B} attached to a rigid body. We wish to describe the motion of {B} relative to frame {A}. For this time instant assume no change in orientation of B relative to A i.e. motion of Q is due to P^{A}_{BORG} or Q^{B} changing in time

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\, {}^{B}V_{Q}$$



Skew-Symmetric Matrices

- A matrix is skew symmetric iff;

$$S + S^T = 0$$

- A result from linear algebra states that for any orthonormal matrix R, there exists a skew matrix S such that;

$$S = \dot{R}R^T$$

A skew-symmetric matrix of 3D is specified by three parameters s $= [s_1, s_2, s_3]^{\mathrm{T}}$ as;

$$S = \begin{bmatrix} 0 & -s_1 & s_2 \\ s_1 & 0 & -s_3 \\ -s_2 & s_3 & 0 \end{bmatrix}$$

Linear and Angular Velocities of Rigid Bodies

- We can derive an interesting relationship between the derivative of an orthonormal matrix and a certain skew-symmetric matrix as follows. For any n x n orthonormal matrix, R, we have

$$RR^T = I_n$$

n x n identity matrix

- Differentiating by product rule

$$\dot{R}R^T + R\dot{R}^T = 0_n$$

- Using the commutative property
- Let

$$\dot{R}R^T + (\dot{R}R^T)^T = 0_n.$$

$$S = \dot{R}R^T \qquad S + S^T = 0_n$$

Linear and Angular Velocities of Rigid Bodies

Velocity of a point due to rotating reference frame

- Consider a fixed vector P^B unchanging with respect to frame (B). Its description in another frame {A} is given as

$$^{A}P = {}^{A}_{B}R\,{}^{B}P$$

If frame {B} is rotating (i.e., the derivative ${}^{A}_{B}\dot{R}$ is non zero), then ^{A}P will be changing even though ^{B}P is constant; that is,

$$^{A}\dot{P} = {}^{A}_{B}\dot{R}\,{}^{B}P \qquad \text{or}$$

$$^{A}V_{P} = {}^{A}_{B}\dot{R}\,{}^{B}P$$

Substituting for ^{B}P

$$^{A}V_{P} = {}^{A}_{B}\dot{R}\,{}^{A}_{B}R^{-1}\,{}^{A}P.$$

$$^{A}V_{P} = {}^{A}_{B}S\,{}^{A}P.$$

The skew symmetric matrix we have introduced is called the **angular-velocity matrix**

Linear and Angular Velocities of Rigid Bodies

Skew Symmetric Matrices and Vector Cross-Product

- If we assign the elements in a skew-symmetric matrix S as

$$S = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$$

- and define the 3 x 1 column vector

$$\Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$$

Angular Velocity Vector

- then it is easily verified that

$$SP = \Omega \times P$$

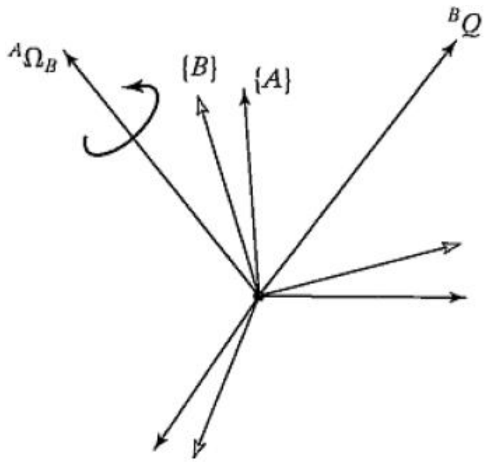
- where P is any vector, and x is the vector cross-product, Hence

$${}^{A}V_{P} = {}^{A}_{B}S\,{}^{A}P \longrightarrow {}^{A}V_{P} = {}^{A}\Omega_{B} \times {}^{A}P$$

Linear and Angular Velocities of Rigid Bodies

Rotational Velocity

- Let us consider two frames with coincident origins and with zero linear relative velocity; their origins will remain coincident for all time.
- The orientation of frame {B} with respect to frame {A} is changing in time
- Rotational velocity of {B} relative to {A} is described by a vector called Ω_B^A

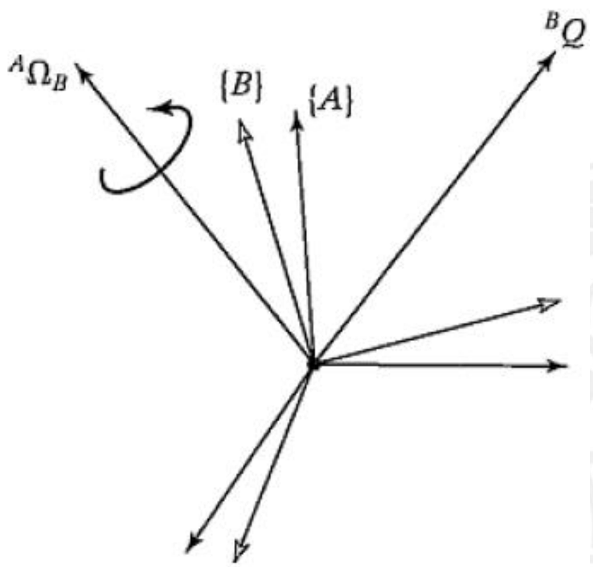


Linear and Angular Velocities of Rigid Bodies

- Let us consider that the vector Q is constant as viewed from frame {B}; that is

$$^{B}V_{Q} = 0$$

- Even though it is constant relative to {B}, it is clear that point Q will have a velocity as seen from {A} that is due to the rotational velocity Ω_B^A



Linear and Angular Velocities of Rigid Bodies

- Rotational Velocity

$$^{A}V_{Q} = {}^{A}\Omega_{B} \times {}^{A}Q$$

- In the general case, the vector Q could also be changing with respect to frame {B}, so, adding this component, we have

$$^{A}V_{Q} = {}^{A}({}^{B}V_{Q}) + {}^{A}\Omega_{B} \times {}^{A}Q$$

$$^{A}V_{Q} = {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q.$$

Linear velocity of point Q in {B} Rotational velocity of Q in {B}

Linear and Angular Velocities of Rigid Bodies

- Linear + Rotational Velocity

$$^{A}V_{Q} = \boxed{^{A}V_{BORG}} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q$$

Linear velocity of origin of {B}

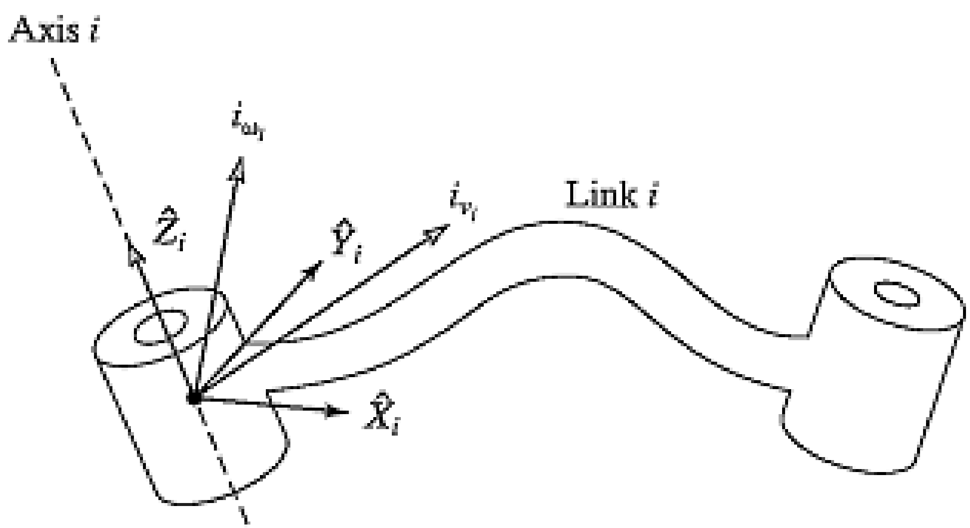
the derivative of a vector in a moving frame as seen from a stationary frame.

- Mathematical insight of Angular Velocity

$$^{A}V_{Q} = {}^{A}\Omega_{B} \times {}^{A}Q$$

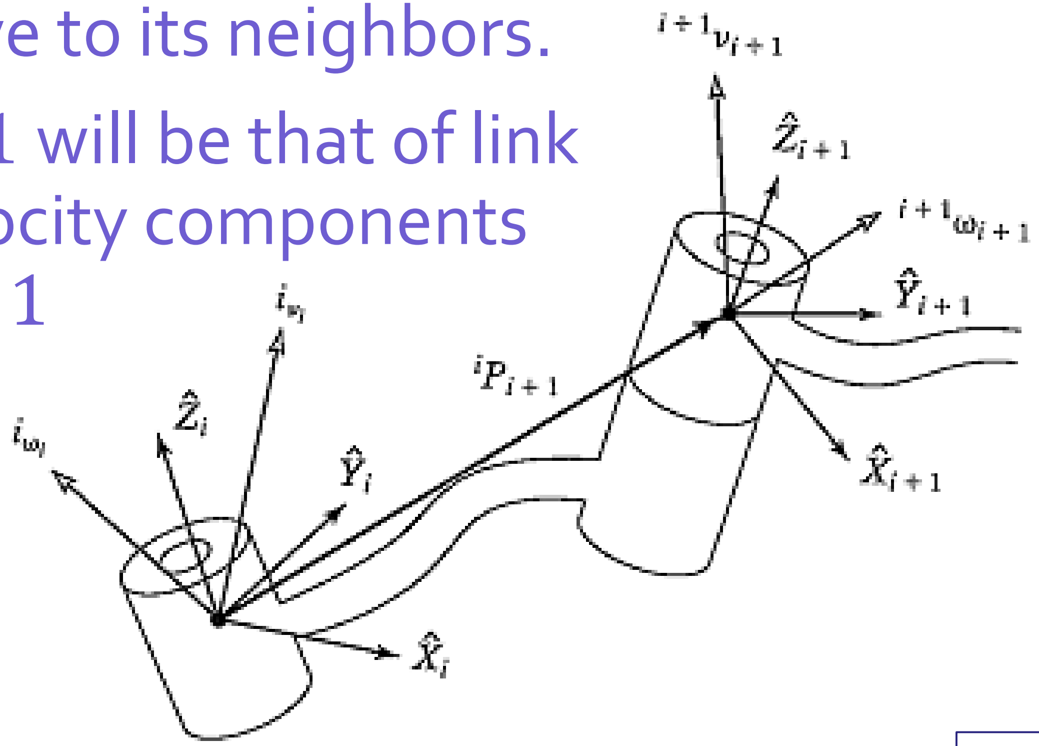
Motion of the Links of the Robot

- Link frame $\{0\}$ is our reference frame
- v_i is the linear velocity and ω_i is the angular velocity of frame $\{i\}$
- At any instant, each link of a robot in motion has some linear and angular velocity



Velocity Propagation from Link to Link

- We now consider the problem of calculating the linear and angular velocities of the links of a robot.
- A manipulator is a chain of bodies, each one capable of motion relative to its neighbors.
- The velocity of link $i + 1$ will be that of link i, plus whatever new velocity components were added by joint $i + 1$

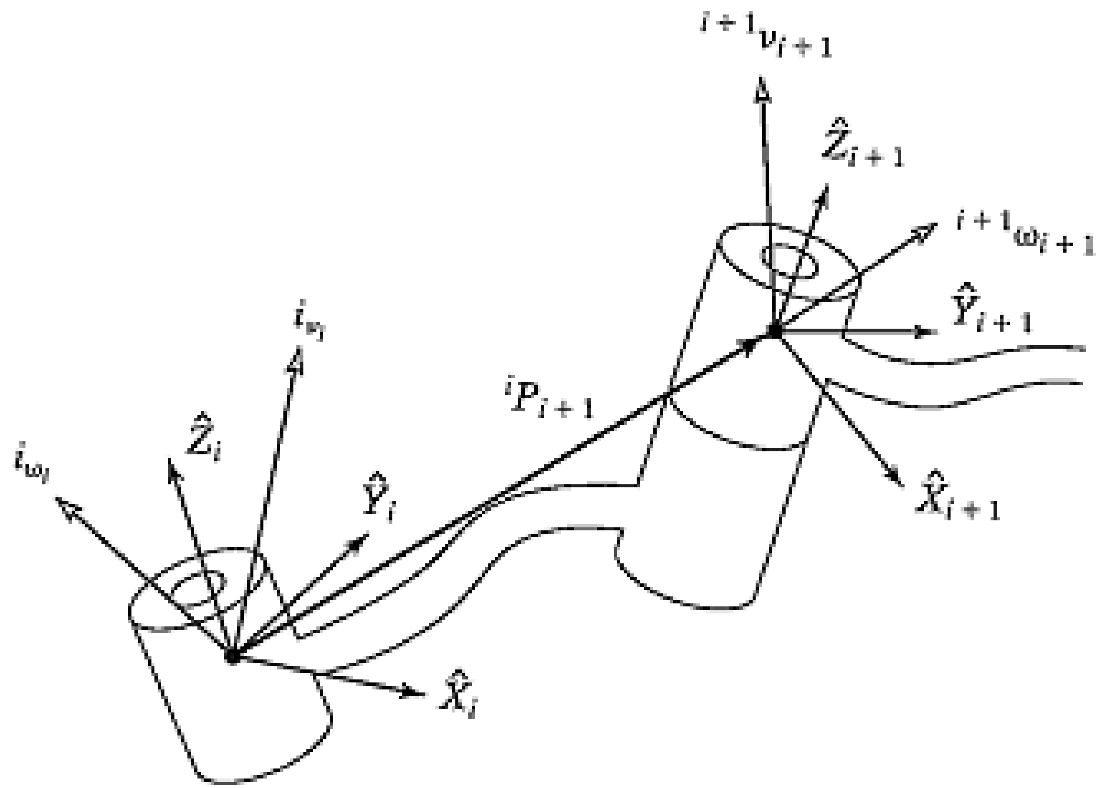


Velocity Propagation from Link to Link

- Rotational velocities can be added when both ω vectors are written with respect to the same frame. Therefore, the angular velocity of link $i+1$ is the same as that of link i plus a new component caused by rotational velocity at joint $i+1$

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_{i} + {}^{i}_{i+1}R\,\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}. $$

$$ \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$



Velocity Propagation from Link to Link

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_{i} + {}^{i}_{i+1}R\, \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}. $$

- By premultiplying both sides by R_i^{i+1}, we can find the description of the angular velocity of link $i + 1$ with respect to frame $\{i + 1\}$

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_{i} + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} $$

- Remember that <u>linear velocity is associated with a point</u>, but <u>angular velocity is associated with a body</u>. Hence, the term "velocity of a link" here means the <u>linear velocity of the origin of the link frame</u> and the <u>rotational velocity of the link</u>

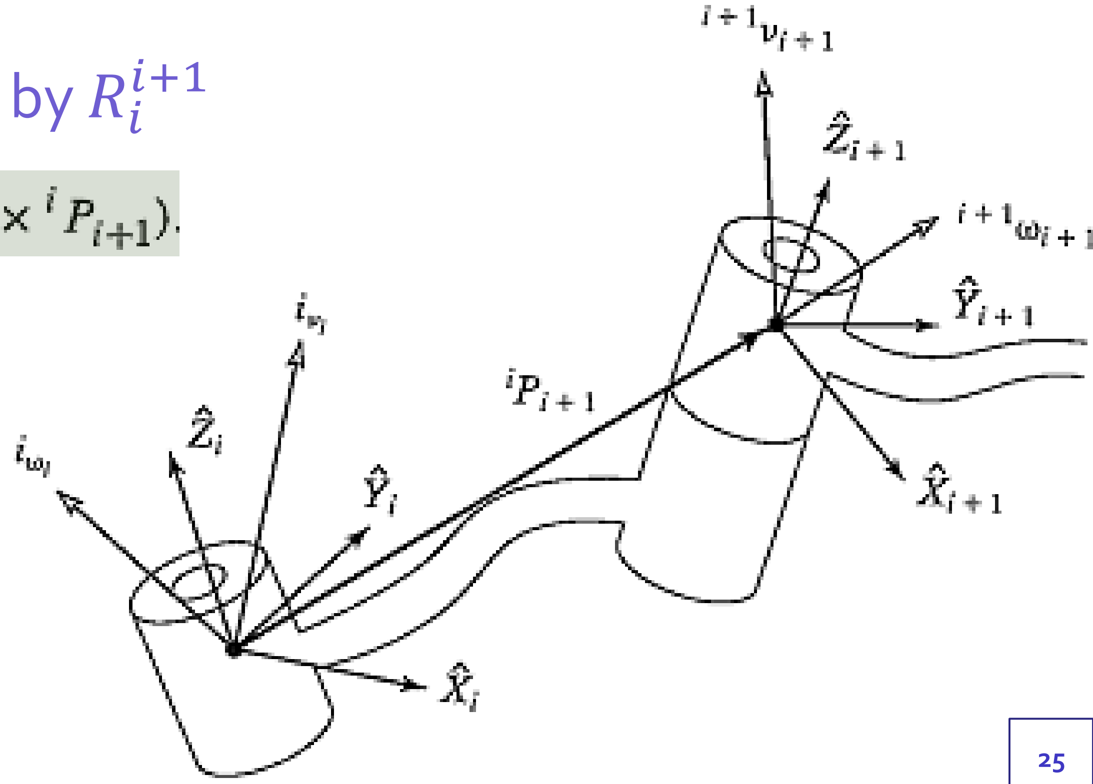
Velocity Propagation from Link to Link

- The linear velocity of the origin of frame $\{i + 1\}$ is the same as that of the origin of frame $\{i\}$ plus a new component caused by rotational velocity of link i.

$$^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}$$

By premultiplying both sides by R_i^{i+1}

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}).$$



Velocity Propagation from Link to Link

- Results for Revolute Joints

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

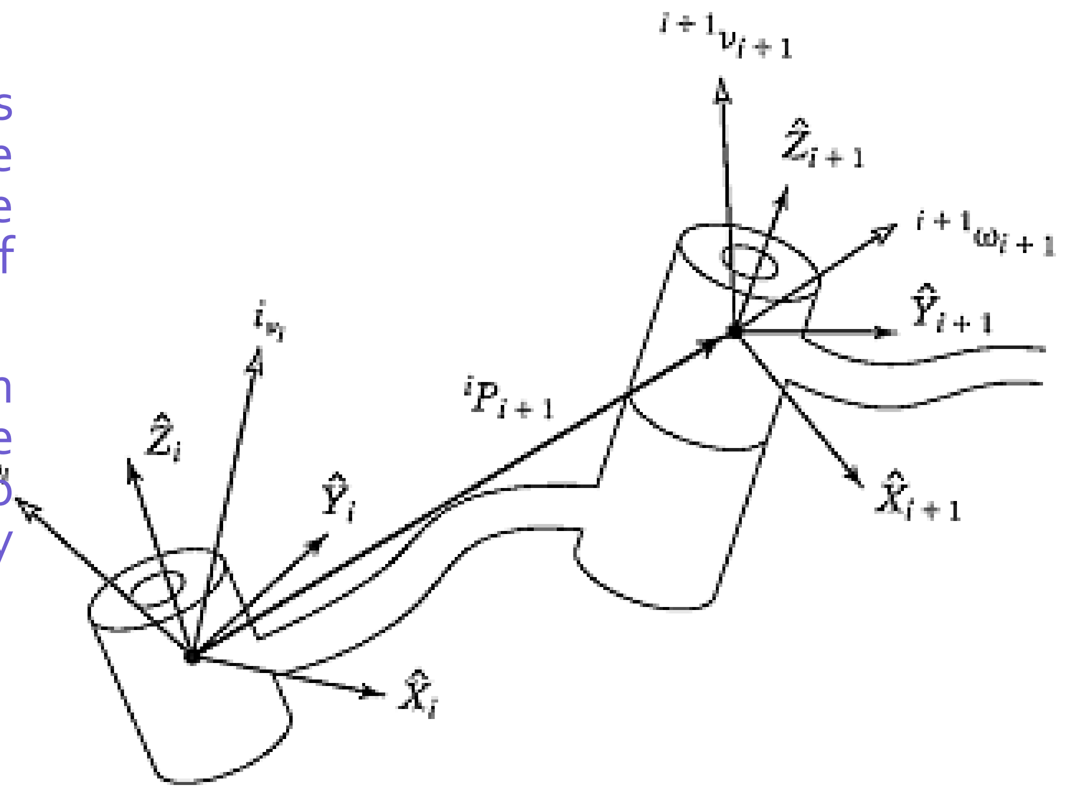
$$^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}).$$

- Results for Prismatic Joints

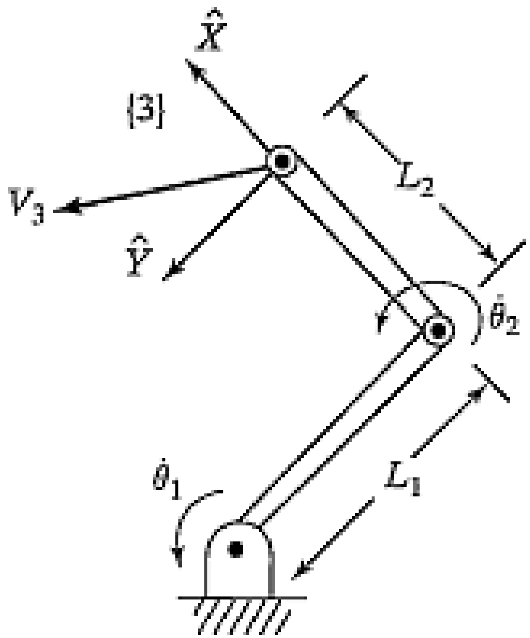
$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i,$$

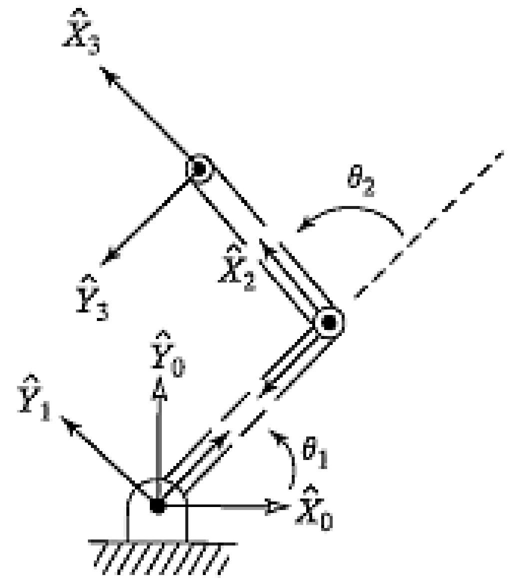
$$^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}.$$

- Applying these equations successively from link to link, we can compute $^{N}\omega_{N}$ and $^{N}v_{N}$ the rotational and linear velocities of the last link
- If the velocities are desired in terms of the base coordinate system, they can be rotated into base coordinates by multiplication with $^{0}R_{N}$



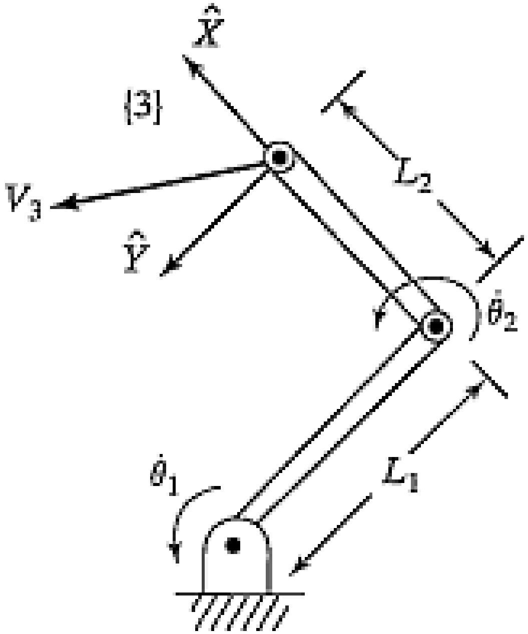
Example





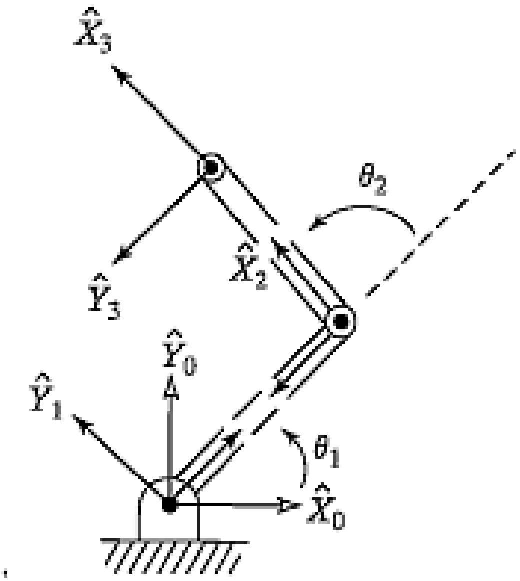
- A two-link manipulator with rotational joints is shown. Calculate the velocity of the tip of the arm as a function of joint rates. Give the answer in two forms—in terms of frame {3} and also in terms of frame {0}

Example



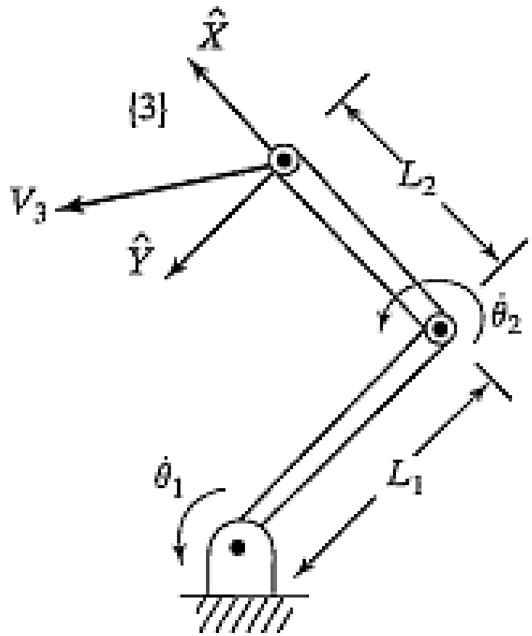
$$ {}^{i+1}v_{i+1} = {}^{i+1}_{\ \ i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) $$

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}. $$



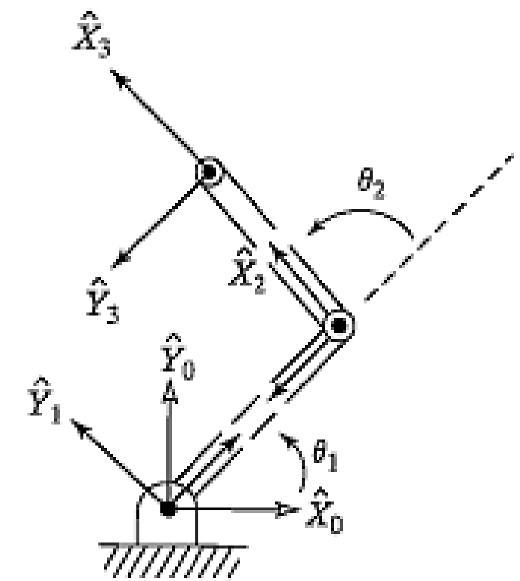
$$ {}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Example



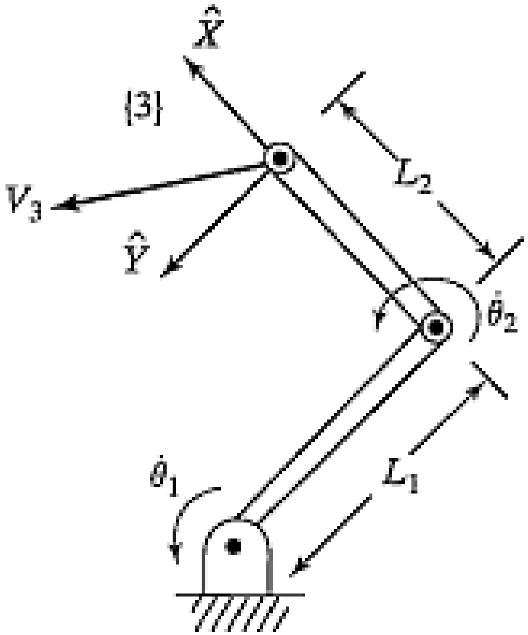
$$ {}^{i+1}v_{i+1} = {}^{i+1}_{\ \ i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) $$

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}. $$



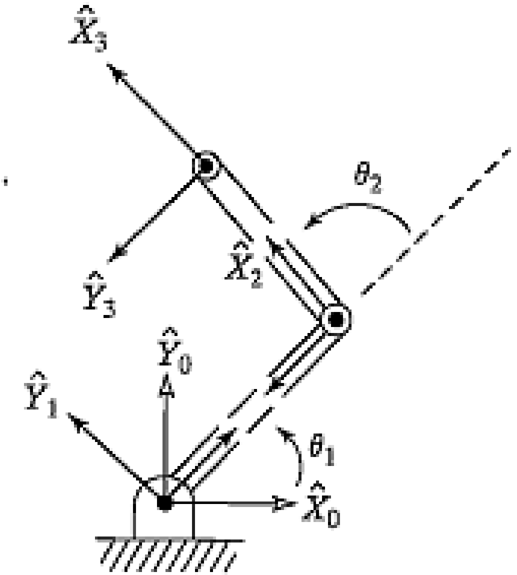
$$ {}^{1}\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \qquad {}^{1}v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, $$

Example



$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}.$$

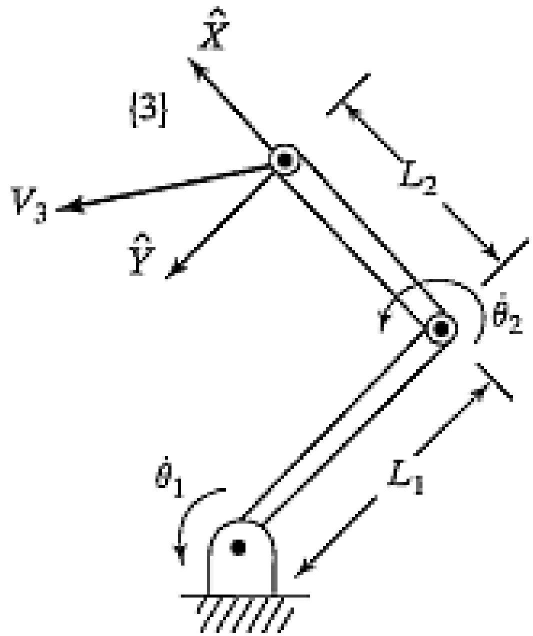
$$^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1})$$



$$^{2}\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

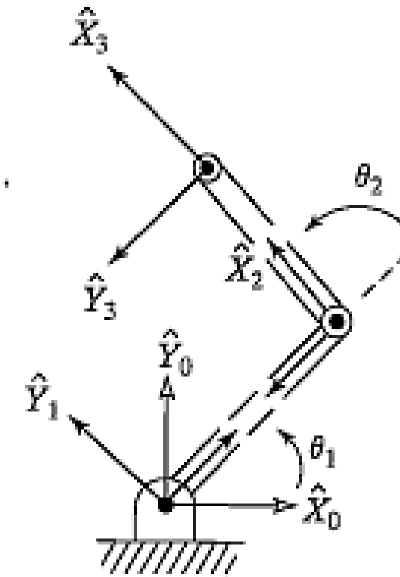
$$^{2}v_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix}.$$

Example



$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}.$$

$$^{i+1}\upsilon_{i+1} = {}^{i+1}_{i}R({}^{i}\upsilon_i + {}^{i}\omega_i \times {}^{i}P_{i+1})$$

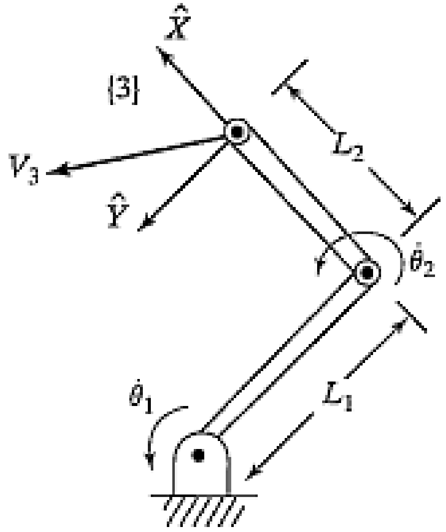


$$^{3}\omega_3 = {}^{2}\omega_2$$

$$^{3}\upsilon_3 = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

Example

- To find these velocities with respect to the nonmoving base frame, we rotate them with the rotation matrix R_3^0 which is,

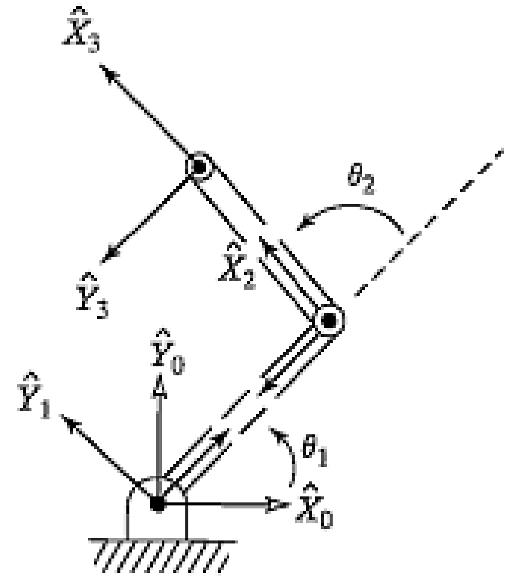


$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}.$$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1})$$

$$^{0}_{3}R = {}^{0}_{1}R \; {}^{1}_{2}R \; {}^{2}_{3}R = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{0}v_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$



Jacobians

- The Jacobian is a multidimensional form of the derivative.
- Suppose, for example, that we have six functions, each of which is a function of six independent variables:

$$
\begin{aligned}
y_1 &= f_1(x_1, x_2, x_3, x_4, x_5, x_6), \\
y_2 &= f_2(x_1, x_2, x_3, x_4, x_5, x_6), \\
&\vdots \\
y_6 &= f_6(x_1, x_2, x_3, x_4, x_5, x_6).
\end{aligned}
$$

Jacobians

- We could also use vector notation to write these equations:

$$Y = F(X)$$

- To calculate the differentials of y_i as a function of differentials of xi, we simply use the chain rule to calculate, and we get

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_1}{\partial x_6}\delta x_6,$$

$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_2}{\partial x_6}\delta x_6,$$

$$\vdots$$

$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_6}{\partial x_6}\delta x_6,$$

Chain Rule

$$F'(x) = f'(g(x))g'(x)$$

Vector Notation

$$\delta Y = \frac{\partial F}{\partial X}\delta X$$

Jacobians

- Vector Notation

$$\delta Y = \frac{\partial F}{\partial X}\delta X$$

- The 6 x 6 matrix of partial derivatives is what we call the Jacobian, J.

$$\delta Y = J(X)\delta X$$

dividing both sides by the differential time element

$$\dot{Y} = J(X)\dot{X}$$

the Jacobian maps velocities in X to those in Y

- At any particular instant, X has a certain value, and J(X) is a linear transformation. At each new time instant, x has changed, and therefore, so has the linear transformation. Jacobians are time-varying linear transformations

Jacobians

- In the field of robotics, we generally use Jacobians that relate joint velocities to Cartesian velocities of the tip of the arm—for example,

$$^{0}\nu = {}^{0}J(\Theta)\dot{\Theta},$$

- Θ is the vector of joint angles and ν is the vector of cartesian velocities
- Note that, for any given configuration of the manipulator, joint rates are related to velocity of the tip in a linear fashion, yet this is only an instantaneous relationship—in the next instant, the Jacobian has changed slightly

Jacobians

- For the general case of a six-jointed robot, the Jacobian is 6 x 6, Θ is 6 x 1, and ${}^0v$ is 6 x 1. This 6 x 1 Cartesian velocity vector is 3 x 1 linear velocity vector and 3 x 1 rotational velocity vector stacked together:

$$ {}^0\nu = \begin{bmatrix} {}^0v \\ {}^0\omega \end{bmatrix} $$

- Jacobians of any dimension (including non square) can be defined
  - Rows → Number of degree of freedom in Cartesian space
  - Columns → Number of joints of the manipulator space

Jacobians

Previous Example

$$ {}^{3}v_3 = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} \qquad {}^{0}v_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} $$

$$ {}^{3}J(\Theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix} \qquad {}^{0}J(\Theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} $$

Jacobians

- Given a Jacobian written in frame {B}, that is

$$\begin{bmatrix} {}^B v \\ {}^B \omega \end{bmatrix} = {}^B \nu = {}^B J(\Theta)\dot{\Theta},$$

- To express the Jacobian in another frame,{A}

$$\begin{bmatrix} {}^A v \\ {}^A \omega \end{bmatrix} = \left[\begin{array}{c|c} {}^A_B R & 0 \\ \hline 0 & {}^A_B R \end{array}\right] \begin{bmatrix} {}^B v \\ {}^B \omega \end{bmatrix} \longrightarrow \begin{bmatrix} {}^A v \\ {}^A \omega \end{bmatrix} = \left[\begin{array}{c|c} {}^A_B R & 0 \\ \hline 0 & {}^A_B R \end{array}\right] {}^B J(\Theta)\dot{\Theta}.$$

- Therefore

$${}^A J(\Theta) = \left[\begin{array}{c|c} {}^A_B R & 0 \\ \hline 0 & {}^A_B R \end{array}\right] {}^B J(\Theta)$$

Jacobians and Singularities

- To achieve a desired velocity, we usually require inverse of jacobian for required joint rates

$$\dot{\Theta} = J^{-1}(\Theta)\nu$$

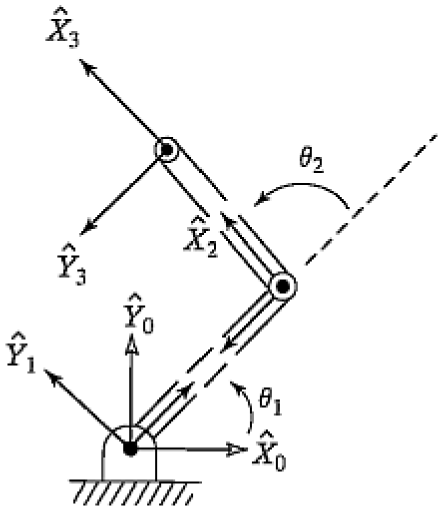
- Is Jacobian matrix invertible i.e. non-singular?
- Furthermore: Is the Jacobian invertible for all values of Θ? If not, where is it not invertible?

Jacobians and Singularities

- Most manipulators have values of Θ where the Jacobian becomes singular. Such locations are called singularities of the mechanism or singularities for short. All manipulators have singularities at the boundary of their workspace, and most have singularities inside their workspace.
  - **Workspace-boundary singularities** occur when the manipulator is fully stretched out or folded back on itself in such a way that the end-effector is at or very near the boundary of the workspace.
  - **Workspace-interior singularities** occur away from the workspace boundary;they generally are caused by a lining up of two or more joint axes.

Example

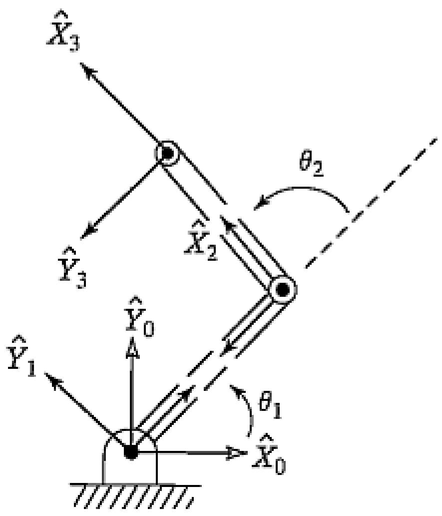
- Where are the singularities of the simple two-link arm ?
- What is the physical explanation of the singularities?
- Are they workspace-boundary singularities or workspace-interior singularities?



Example

- To find the singular points of a mechanism, we must examine the determinant of its Jacobian. Where the determinant is equal to zero, the Jacobian has lost full rank and is singular

$$DET[J(\Theta)] = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix} = l_1 l_2 s_2 = 0.$$



- Singularity of the mechanism exists when θ_2 is 0 or 180 degrees.
- When $\theta_2 = 0$, the arm is stretched straight out. In this configuration, motion of the end-effector is possible along only one Cartesian direction (the one perpendicular to the arm). Therefore, the mechanism has lost one degree of freedom.

Note

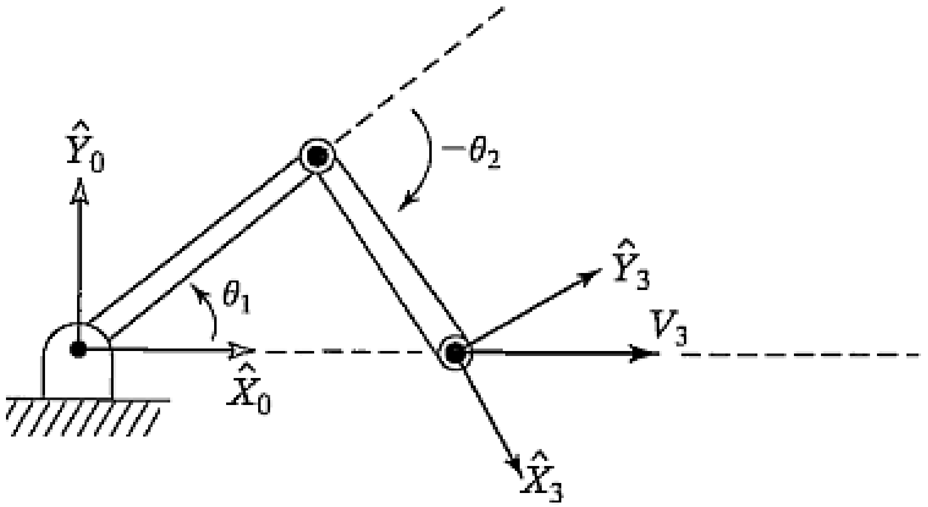
- The danger in applying the equation below in a robot control system is that, at a singular point, the inverse Jacobian blows up! This results in joint rates approaching infinity as the singularity is approached.

$$\dot{\Theta} = J^{-1}(\Theta)\nu$$

Example

- Consider the two-link robot as it is moving its end-effector along the X axis at 1.0 m/s. Show that joint rates are reasonable when far from a singularity, but that, as a singularity is approached at $\Theta_2 = 0$, joint rates tend to infinity.

$$^{0}J(\Theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$



Example

- We start by calculating the inverse of the Jacobian written in {0}

$$^{0}J^{-1}(\Theta) = \frac{1}{l_1 l_2 s_2} \begin{bmatrix} l_2 c_{12} & l_2 s_{12} \\ -l_1 c_1 - l_2 c_{12} & -l_1 s_1 - l_2 s_{12} \end{bmatrix}.$$

For a velocity of 1 m/s in the X direction, we can calculate joint rates as a function of manipulator configuration:

$$\dot{\Theta} = J^{-1}(\Theta)\nu$$

$$\dot{\theta}_1 = \frac{c_{12}}{l_1 s_2},$$

$$\dot{\theta}_2 = -\frac{c_1}{l_2 s_2} - \frac{c_{12}}{l_1 s_2}$$

Clearly, as the arm stretches out toward $\theta_2 = 0$, both joint rates go to infinity.

Static Forces in Manipulators

- Typically, the robot is pushing on something in the environment with the chain's free end (the end-effector) or is perhaps supporting a load at the hand.

- **Goal:** We wish to solve for the joint torques that must be acting to keep the system in static equilibrium.

- The chainlike nature of a manipulator leads us quite naturally to consider how forces and moments "propagate" from one link to the next.

Static Forces in Manipulators

- A three Step Process
  - Lock all the joints so that the manipulator becomes a structure.
  - Consider each link in this structure and write a force-moment balance relationship in terms of the link frames.
  - Finally, we compute what static torque must be acting about the joint axis in order for the manipulator to be in static equilibrium.
- In this way, we solve for the set of joint torques needed to support a static load acting at the end-effector.

Static Forces in Manipulators

- Assumptions for now (we will relax them later): No forces on links due to gravity.

- The static forces and torques we are considering at the joints are those caused by a static force or torque (or both) acting on the last link for example, as when the manipulator has its end-effector in contact with the environment.

Static Forces in Manipulators

- Notations

  f_i = force exerted on link i by link $i-1$,

  n_i = moment/torque exerted on link i by link $i-1$.

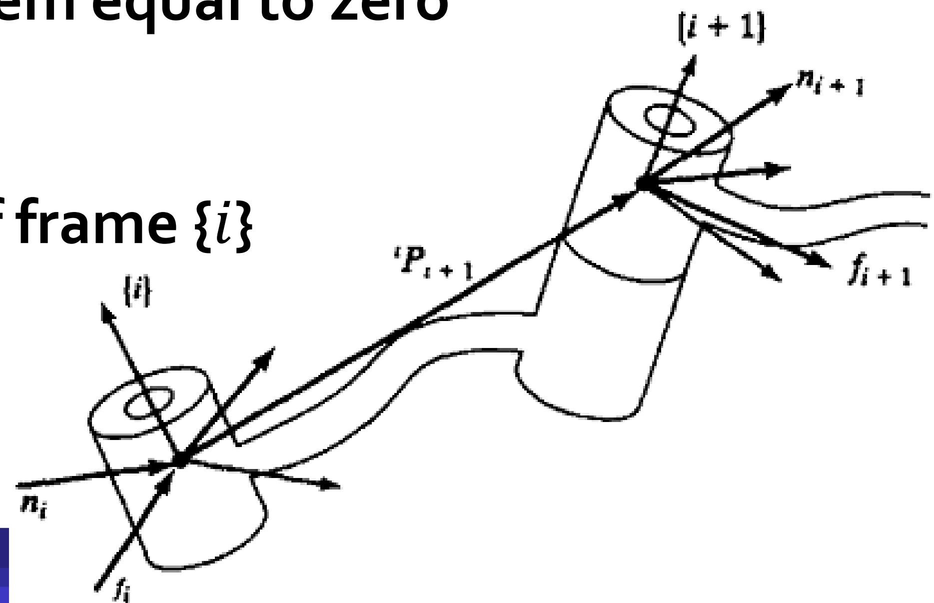
- Figure below shows forces and moments (excluding gravity) acting on link i.

Summing the forces and setting them equal to zero

$$^{i}f_i - {}^{i}f_{i+1} = 0.$$

Summing moments about origin of frame $\{i\}$

$$^{i}n_i - {}^{i}n_{i+1} - {}^{i}P_{i+1} \times {}^{i}f_{i+1} = 0$$



Static Forces in Manipulators

- The result can be written as

$$ {}^{i}f_i - {}^{i}f_{i+1} = 0. \qquad {}^{i}n_i - {}^{i}n_{i+1} - {}^{i}P_{i+1} \times {}^{i}f_{i+1} = 0 $$

$$ {}^{i}f_i = {}^{i}f_{i+1}, \qquad {}^{i}n_i = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1}. $$

OR

$$ {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}, $$

$$ {}^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i. $$

Static Forces in Manipulators

- What torques are needed at the joints in order to balance the reaction forces and moments acting on the links?

Torque: rotational motion, causing objects to rotate around an axis.

$$\tau_i = {}^i n_i^T \, {}^i\hat{Z}_i .$$

the dot product of the joint-axis vector with the moment vector acting on the link is computed

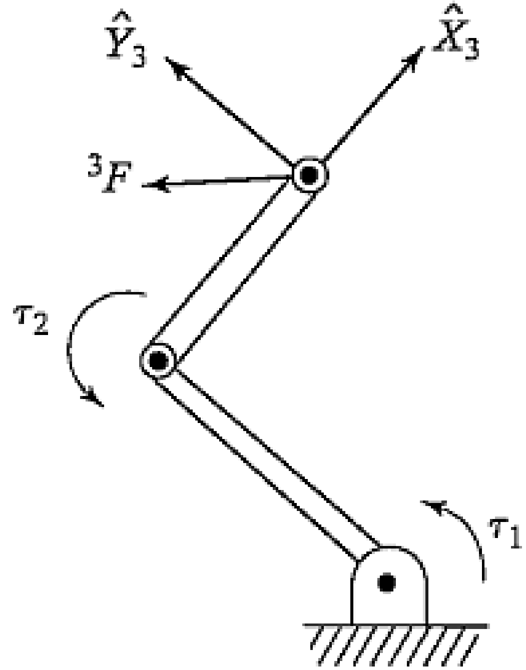
Moment: turning effect of a force, encompassing both rotational and general forces.

In the case that joint i is prismatic, we compute the joint actuator force as

$$\tau_i = {}^i f_i^T \, {}^i\hat{Z}_i .$$

Example

- The two-link manipulator is applying a force vector ^3F with its end-effector. (Consider this force to be acting at the origin of {3}.) Find the required joint torques as a function of configuration and of the applied force.



Using our results

$$^i f_i = {}^{i}_{i+1}R \, {}^{i+1} f_{i+1},$$

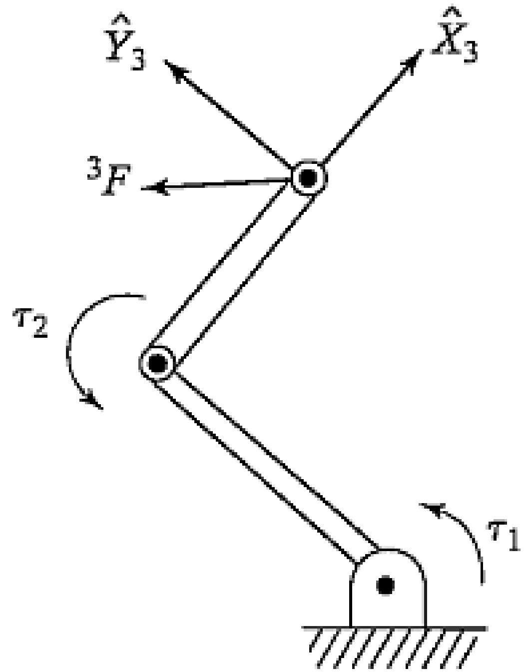
$$^i n_i = {}^{i}_{i+1}R \, {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_i.$$

Example

Using our results

$$^{i}f_i = {}^{\;\,i}_{i+1}R\; {}^{i+1}f_{i+1},$$

$$^{i}n_i = {}^{\;\,i}_{i+1}R\; {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i.$$



$$^{2}f_2 = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}$$

$$^{2}n_2 = l_2 \hat{X}_2 \times \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix}$$

Example

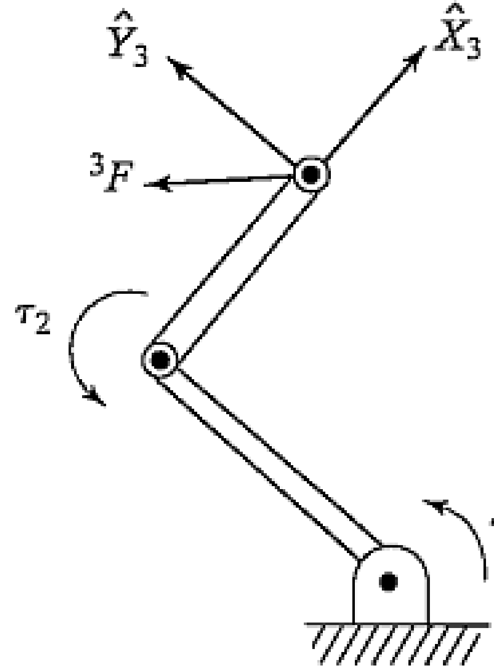
Using our results

$$ {}^{i}f_i = {}^{i}_{i+1}R \, {}^{i+1}f_{i+1}, $$

$$ {}^{i}n_i = {}^{i}_{i+1}R \, {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i. $$

$$ {}^{1}f_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 f_x - s_2 f_y \\ s_2 f_x + c_2 f_y \\ 0 \end{bmatrix} $$

$$ {}^{1}n_1 = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} + l_1 \hat{X}_1 \times {}^{1}f_1 = \begin{bmatrix} 0 \\ 0 \\ l_1 s_2 f_x + l_1 c_2 f_y + l_2 f_y \end{bmatrix} $$



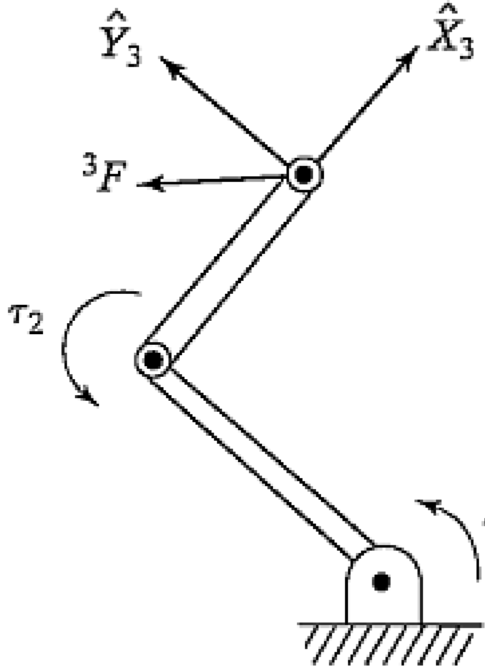
Example

Using our results

$$\tau_i = {}^i n_i^T \, {}^i\hat{Z}_i.$$

$$\tau_1 = l_1 s_2 f_x + (l_2 + l_1 c_2) f_y,$$

$$\tau_2 = l_2 f_y.$$



$$\tau = \begin{bmatrix} l_1 s_2 & l_2 + l_1 c_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

$${}^3J(\Theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix}$$

Note that this matrix is transpose of Jacobian we found earlier

Jacobians in the Force Domain

- We have found joint torques that will exactly balance forces at the hand in the static situation.

- When forces act on a mechanism, work (in the technical sense) is done if the mechanism moves through a displacement.

- **Work** is defined as a force acting through a distance and is a scalar with units of energy.

Jacobians in the Force Domain

- If we assume a infinitesimal displacement for our static case (relaxing stringent static definition). we can equate the work done in Cartesian terms with the work done in joint-space terms.

$$\mathcal{F} \cdot \delta \chi = \tau \cdot \delta \Theta$$

where $\mathcal{F}$ is a 6×1 Cartesian force-moment vector acting at the end-effector, $\delta \chi$ is a 6×1 infinitesimal Cartesian displacement of the end-effector, τ is a 6×1 vector of torques at the joints, and $\delta \Theta$ is a 6×1 vector of infinitesimal joint displacements.

Jacobians in the Force Domain

$$\mathcal{F} \cdot \delta\chi = \tau \cdot \delta\Theta$$

$$\mathcal{F}^T \delta\chi = \tau^T \delta\Theta .$$

The definition of the Jacobian is

$$\delta\chi = J\delta\Theta,$$

so we may write

$$\mathcal{F}^T J\delta\theta = \tau^T \delta\Theta,$$

which must hold for all $\delta\Theta$; hence, we have

$$\mathcal{F}^T J = \tau^T .$$

Transposing both sides yields this result:

$$\tau = J^T \mathcal{F}.$$

Hence, the Jacobian transpose maps Cartesian forces acting at the hand into equivalent joint torques.

Jacobians in the Force Domain

- When the Jacobian is written with respect to frame {0}, then force vectors written in {0} can be transformed, as is made clear by the following notation

$$\tau = {}^{0}J^{T}\, {}^{0}\mathcal{F}.$$

- When the Jacobian loses full rank, there are certain directions in which the end-effector cannot exert static forces even if desired. Thus, singularities manifest themselves in the force domain as well as in the position domain.