MCS [Assignment 3]

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- Problem 1. $s(t) = \mathrm{Re}\{s_l(t)\, e^{j2\pi f_c t}\}$, $s_l(t) = \sum_n b_n\, p(t - nT_s)$

$b_n \in \{-3, -1, 1, 3\}$

PSD := $S(f) = \frac{1}{2}\left[S_l(f - f_c) + S_l(-f - f_c)\right]$ — i

where $S_l(f) = \dfrac{E\{|b_n|^2\}}{2T_s} |P(f)|^2$ — ii

Note : $A = 1$ when we compare $s_l(t)$ with a generic complex envelope $g(t) = A \sum_n x_n\, p(t - nT_s)$

- $p(t)$ is rectangular pulse with unit height and width T_s ;

» $P(f) = \int_{-\infty}^{\infty} p(t)\, e^{-j2\pi f t}\, dt = \int_0^{T_s} e^{-j2\pi f t}\, dt$

$= \left[\frac{1}{-j2\pi f} \cdot e^{-j2\pi f t}\right]_0^{T_s}$

$= \dfrac{e^{-j2\pi f T_s} - 1}{-j2\pi f} = \dfrac{e^{-j\pi f T_s}\left[e^{-j\pi f T_s} - e^{j\pi f T_s}\right]}{-j2\pi f}$

$= \dfrac{e^{-j\pi f T_s}}{\pi f} (\sin(\pi f T_s))$

$= T_s\, e^{-j\pi f T_s}\, \mathrm{sinc}(f T_s)$ $\quad \because \mathrm{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$

» $|P(f)|^2 = T_s^2\, \mathrm{sinc}^2(f T_s)$ $\quad \because |e^{j\theta}| = 1$

Now, we find $E\{|b_n|^2\}$

• Since b_n elements are distributed with equal probability :

$$\gg E\{|b_n|^2\} = (-3)^2(0.25) + (-1)^2(0.25) + (1)^2(0.25) \ldots$$
$$+ (3)^2(0.25)$$
$$= 5$$

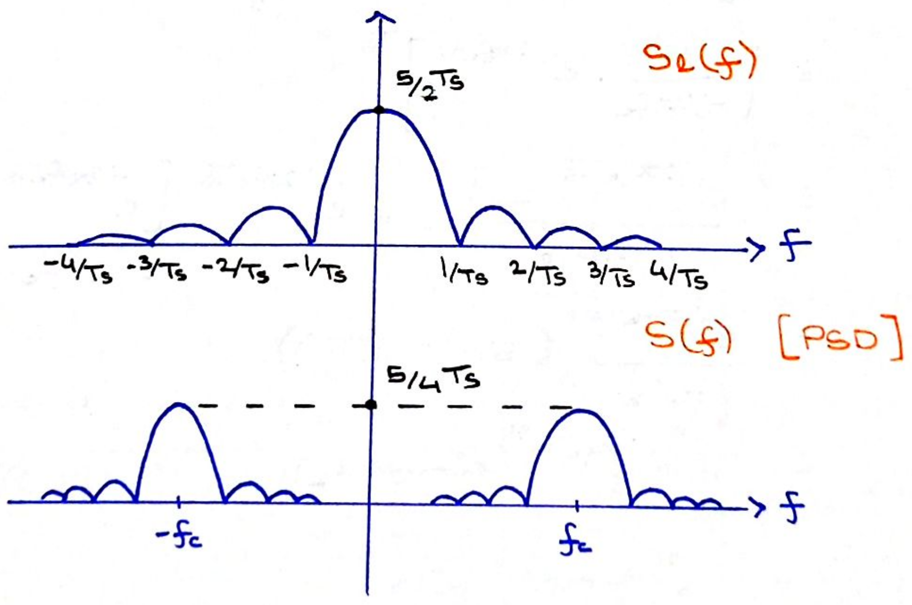
Substituting in ii ;

$$\gg S_\ell(f) = \frac{5}{2T_s} \cdot \left[T_s^2 \operatorname{sinc}^2(f\,T_s)\right]$$
$$= \underline{\frac{5}{2} T_s \operatorname{sinc}^2(f\,T_s)}$$

Subsequently, i becomes :

$$\gg S(f) = \frac{1}{2}\left[\frac{5}{2} T_s \operatorname{sinc}^2((f-f_c)T_s) + \frac{5}{2} T_s \operatorname{sinc}^2((-f-f_c)T_s)\right]$$
$$= \underline{\frac{5}{4} T_s \left[\operatorname{sinc}^2((f-f_c)T_s) + \operatorname{sinc}^2((-f-f_c)T_s)\right]}$$

Sketch (Not drawn to scale)

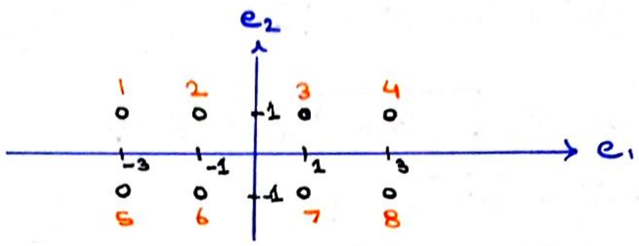


- Problem 2 | Figure

a) Energy of each symbol & average symbol energy

- Energy of a symbol is $\langle x, x\rangle = |x|^2 = \int_0^{T_s} x^2(t)\,dt$

Let the constellation labelling be :



» Energy ($\mathcal{E}$) of 1 : $\left[\sqrt{(-3)^2+(1)^2}\right]^2 = \boxed{10}$

$\mathcal{E}$ of 2 : $\left[\sqrt{(-1)^2+(1)^2}\right]^2 = \boxed{2}$

- From symmetry ;

$\mathcal{E}$ of 4, 5, 8 = $\mathcal{E}$ of 1 = $\boxed{10}$

$\mathcal{E}$ of 3, 6, 7 = $\mathcal{E}$ of 2 = $\boxed{2}$.

- Average Energy ($\bar{\mathcal{E}}$) $= \dfrac{\sum \mathcal{E}_n}{\text{len}(\bar{S})}$, $n \in \{1, \dots, 8\}$, $\bar{S} \triangleq$ set of symbols

$$\bar{\mathcal{E}} = \frac{4(10) + 4(2)}{8}$$

$$\boxed{\bar{\mathcal{E}} = 6}$$

b) Union Bound

$$P(\text{symbol error}) = \frac{1}{M} \sum_{i}^{M} \sum_{\substack{j \\ i \neq j}}^{M} Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right) \quad - i$$

- We only need to find distances from i symbol 1 and 2 and the rest can be deduced from symmetry.

» $i = 1$: $d_{12} = d_{15}^{T} = \sqrt{(-1+3)^2 + 0^2} = 2$

transposed

$$d_{13} = \sqrt{(1+3)^2 + 0^2} = 4$$

$$d_{14} = \sqrt{(3+3)^2 + 0^2} = 6$$

$$d_{16} = \sqrt{(-1+3)^2 + (-1-1)^2} = \sqrt{8}$$

$$d_{17} = \sqrt{(1+3)^2 + (-1-1)^2} = \sqrt{20}$$

$$d_{18} = \sqrt{(3+3)^2 + (-1-1)^2} = \sqrt{40}$$

» $i = 2$: $d_{21} = \sqrt{(-3+1)^2 + 0^2} = 2$

$$d_{23} = \sqrt{(1+1)^2 + 0^2} = 2$$

$$d_{24} = \sqrt{(3+1)^2 + 0^2} = 4$$

$$d_{25} = \sqrt{(-3+1)^2 + (-1-1)^2} = \sqrt{8}$$

$$d_{26} = \sqrt{(-1+1)^2 + (-1-1)^2} = 2$$

$$d_{27} = \sqrt{(1+1)^2 + (-1-1)^2} = \sqrt{8}$$

$$d_{28} = \sqrt{(3+1)^2 + (-1-1)^2} = \sqrt{20}$$

- From symmetry, case $i = 1$ is equal to $i = 4 = 5 = 8$
  case $i = 2$ is equal to $i = 3 = 6 = 7$

- i can be expanded as :

$$P(\text{symbol error}) = \frac{1}{8}\Bigg[4\Bigg(Q\left(\frac{2}{\sqrt{2N_0}}\right) + Q\left(\frac{4}{\sqrt{2N_0}}\right) + Q\left(\frac{6}{\sqrt{2N_0}}\right) + Q\left(\frac{2}{\sqrt{2N_0}}\right) \ldots$$
$$+ Q\left(\frac{\sqrt{8}}{\sqrt{2N_0}}\right) + Q\left(\frac{\sqrt{20}}{\sqrt{2N_0}}\right) + Q\left(\frac{\sqrt{40}}{\sqrt{2N_0}}\right)\Bigg) \ldots$$
$$+ 4\Bigg(Q\left(\frac{2}{\sqrt{2N_0}}\right) + Q\left(\frac{2}{\sqrt{2N_0}}\right) + Q\left(\frac{4}{\sqrt{2N_0}}\right) + Q\left(\frac{\sqrt{8}}{\sqrt{2N_0}}\right) \ldots$$
$$+ Q\left(\frac{2}{\sqrt{2N_0}}\right) + Q\left(\frac{\sqrt{8}}{\sqrt{2N_0}}\right) + Q\left(\frac{\sqrt{20}}{\sqrt{2N_0}}\right)\Bigg)\Bigg]$$
$$= \frac{1}{8}\Bigg[20\,Q\left(\frac{2}{\sqrt{2N_0}}\right) + 8\,Q\left(\frac{4}{\sqrt{2N_0}}\right) + 4\,Q\left(\frac{6}{\sqrt{2N_0}}\right) \ldots$$
$$+ 12\,Q\left(\frac{\sqrt{8}}{\sqrt{2N_0}}\right) + 8\,Q\left(\frac{\sqrt{20}}{\sqrt{2N_0}}\right) + 4\,Q\left(\frac{\sqrt{40}}{\sqrt{2N_0}}\right)\Bigg]$$

c) Symbol Waveform

The arrow marks the 4^{th} symbol ;

- S_4 can be represented as a linear combination of the basis functions.

» $S_4 = 3e_1 + e_2$

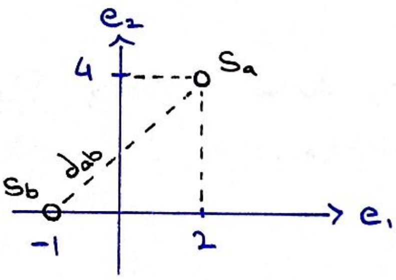
e_1 and e_2 of QPSK are as follows : {for $0 \le t \le T_s$}

- $e_1(t) = \sqrt{2/T_s}\,\cos(2\pi f_c t)$ -ive
- $e_2(t) = \sqrt{2/T_s}\,\cos(2\pi f_c t + \pi/2) = \sqrt{2/T_s}\,\sin(2\pi f_c t)$

Substituting,

» $$S_4 = \boxed{\frac{3\sqrt{2}}{\sqrt{T_s}}\cos(2\pi f_c t) - \sqrt{\frac{2}{T_s}}\sin(2\pi f_c t) \quad , \; 0 \le t \le T_s}$$

o Problem 3 | $N_o/2 = 25/16$



a) BER

$$P(\text{bit error}) = Q\left(\frac{d_{ab}}{\sqrt{2N_o}}\right) = Q\left(\frac{d_{ab}}{2\sqrt{N_o}/\sqrt{2}}\right) = Q\left(\frac{1}{2}\cdot\frac{d_{ab}}{\sqrt{N_o/2}}\right)$$

$$= Q\left(\frac{2}{5}\,d_{ab}\right)$$

$\because \sqrt{N_o/2} = 5/4$

» $d_{ab} = \sqrt{(2+1)^2 + (4-0)^2} = 5$

- Substituting,

$P(\text{bit error}) = Q(2)$

- From Q-table,

$P(\text{bit error}) = \boxed{0.02275 = 2.275\ \%}$

b) Symbol Expression

Like P2 (c), S_a can be represented as a linear combination of the basis functions.

- e_1 and e_2 are given to be: {for $0 < t < T_s$}

- $e_1(t) = \sqrt{\frac{2}{T_s}} \cos(2\pi f_c t)$
- $e_2(t) = K\sqrt{\frac{2}{T_s}} \cos\left(\frac{2\pi}{T_s} t\right) \cos(2\pi f_c t)$

» $S_a = 2e_1 + 4e_2 , \quad 0 < t < T_s$

$$= \boxed{\frac{2\sqrt{2}}{T_s} \cos(2\pi f_c t) + 4K\sqrt{\frac{2}{T_s}} \cos\left(\frac{2\pi}{T_s} t\right)\cos(2\pi f_c t), \ldots \quad 0 < t < T_s}$$

- Problem 4 & 5 done using Python and attached with this ensemble.

- Problem 6 | $f_c = 10$ MHz

$T_s = 1/f_c = 0.1\ \mu s$

The given pulse is not a Nyquist pulse as, when multiplied with an impulse train of period T_s, the resulting is a combination of impulses and not a single central impulse $\delta(t)$.

- Mathematically,

$$p(t) \sum_n \delta(t - nT_s) = \delta(t) - k\,\delta(t - T_s) - k\,\delta(t + T_s)$$

where k is magnitude of $p(t)$ at $t - T_s \rightarrow (-0.2, 0.2)\ \mu s$

which violates Nyquist condition.

$$p(t) \sum_n \delta(t - nT_s) = \delta(t)$$