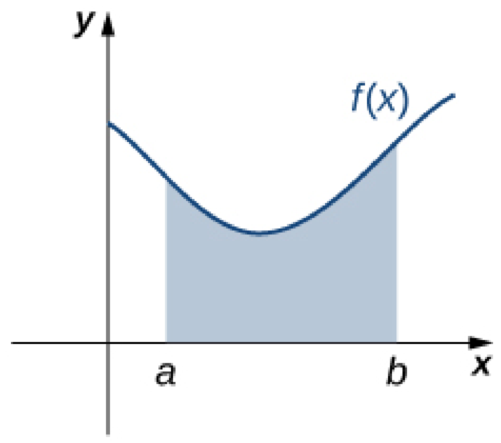
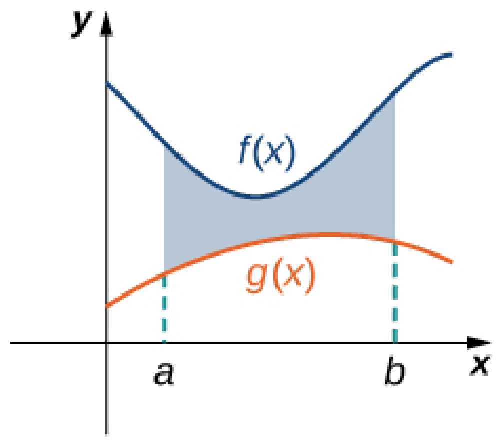
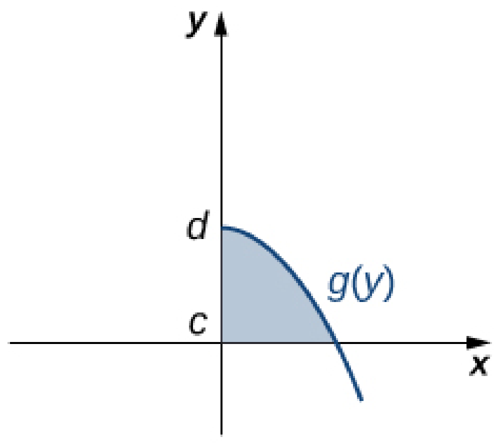
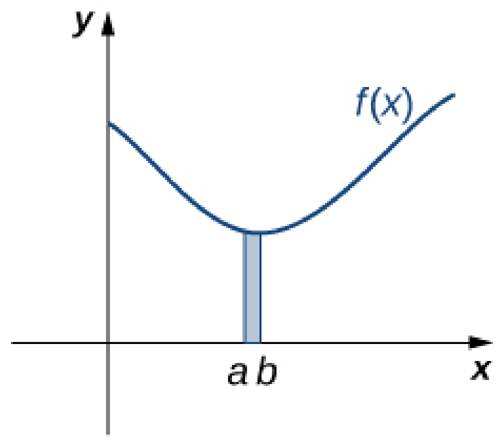
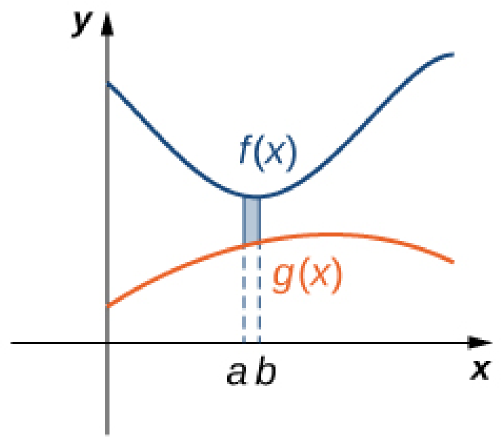
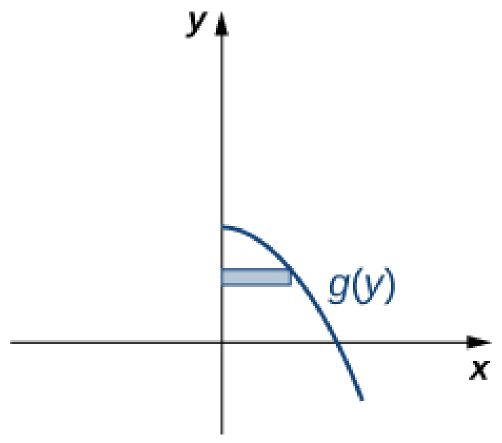
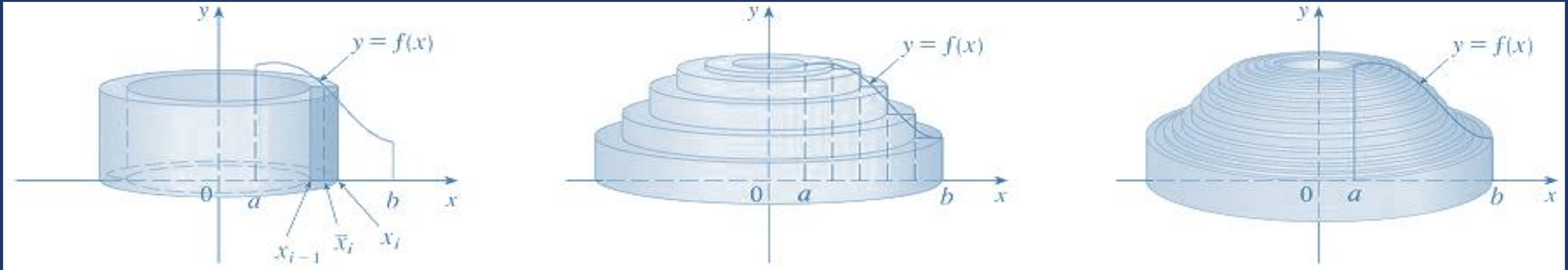
Applications of Integration

Calculus & Analytical Geometry MATH-101
Instructor: Dr. Naila Amir (SEECS, NUST)

Volume of solids of revolution:

Compare	Disk Method	Washer Method	Shell Method
Volume formula	$V = \int_a^b \pi[f(x)]^2\,dx$	$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$	$V = \int_c^d 2\pi y\, g(y)\,dy$
Solid	No cavity in the center	Cavity in the center	With or without a cavity in the center
Interval to partition	$[a, b]$ on x-axis	$[a, b]$ on x-axis	$[c, d]$ on y-axis
Rectangle	Vertical	Vertical	Horizontal
Typical region			
Typical element			

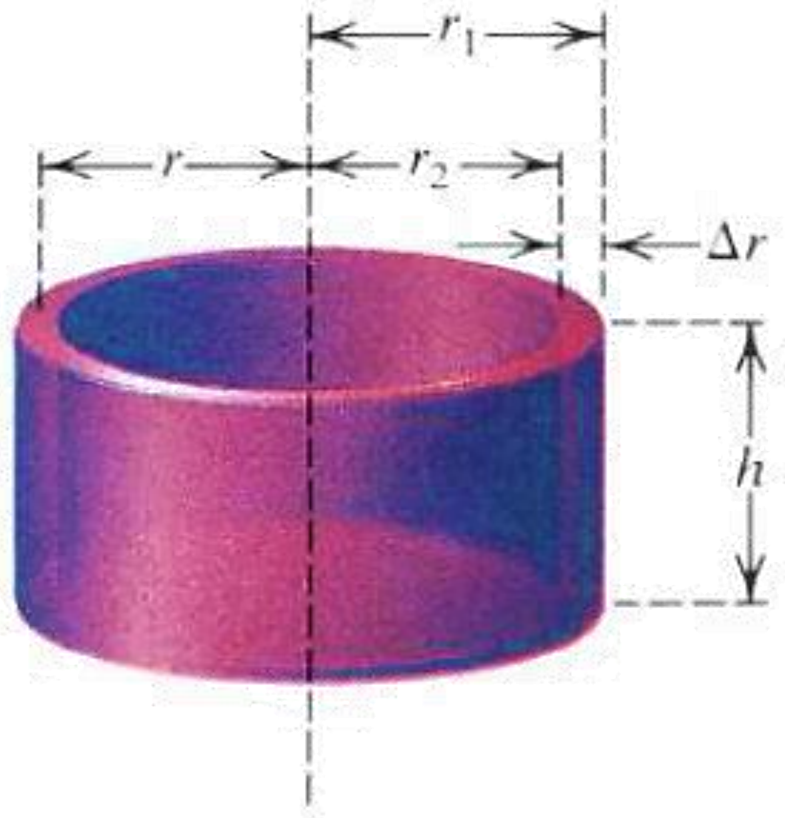


Volumes by Cylindrical Shells

- **Book:** Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano
  Chapter: 6 (Section: 6.2)
- **Book:** Calculus (5th Edition) by Swokowski, Olinick and Pence
  Chapter: 6 (Section: 6.3)

VOLUMES BY CYLINDRICAL SHELLS

- In the previous lecture we found volumes of solids of revolution by using circular disks or washers.
- For certain types of solids, it is convenient to use hollow circular cylinders- that is, thin cylindrical shells of the type illustrated in figure, where r_1 is the ***outer radius***, r_2 is the ***inner radius***, h is the ***altitude***, and $\Delta r = r_1 - r_2$ is the ***thickness*** of the shell. The **average radius** of the shell is given as: $r = \frac{1}{2}(r_1 + r_2)$.



- We can find the volume of the shell by subtracting the volume $\pi r_2^2 h$ of the inner cylinder from the volume $\pi r_1^2 h$ of the outer cylinder as:

$$\pi r_1^2 h - \pi r_2^2 h = \pi(r_1^2 - r_2^2)h$$
$$= 2\pi . \frac{1}{2}(r_1 + r_2)h(r_1 - r_2) = 2\pi r h \Delta r,$$

which gives us the following general rule:

$$V = 2\pi(average\ radius)(altitude)(thickness).$$

VOLUMES BY CYLINDRICAL SHELLS

- For disks or washers, we defined the volume of a solid S as the definite integral:

$$V = \int_a^b A(x)\,dx,$$

where $A(x)$ is an integrable cross-sectional area of S from $x = a$ to $x = b$.

- The area $A(x)$ was obtained by slicing through the solid with a plane perpendicular to the x −axis.
- Now, we use the same integral definition for volume, but obtain the area by slicing through the solid in a different way.
- Now we will slice through the solid using circular cylinders of increasing radii, like cookie cutters.

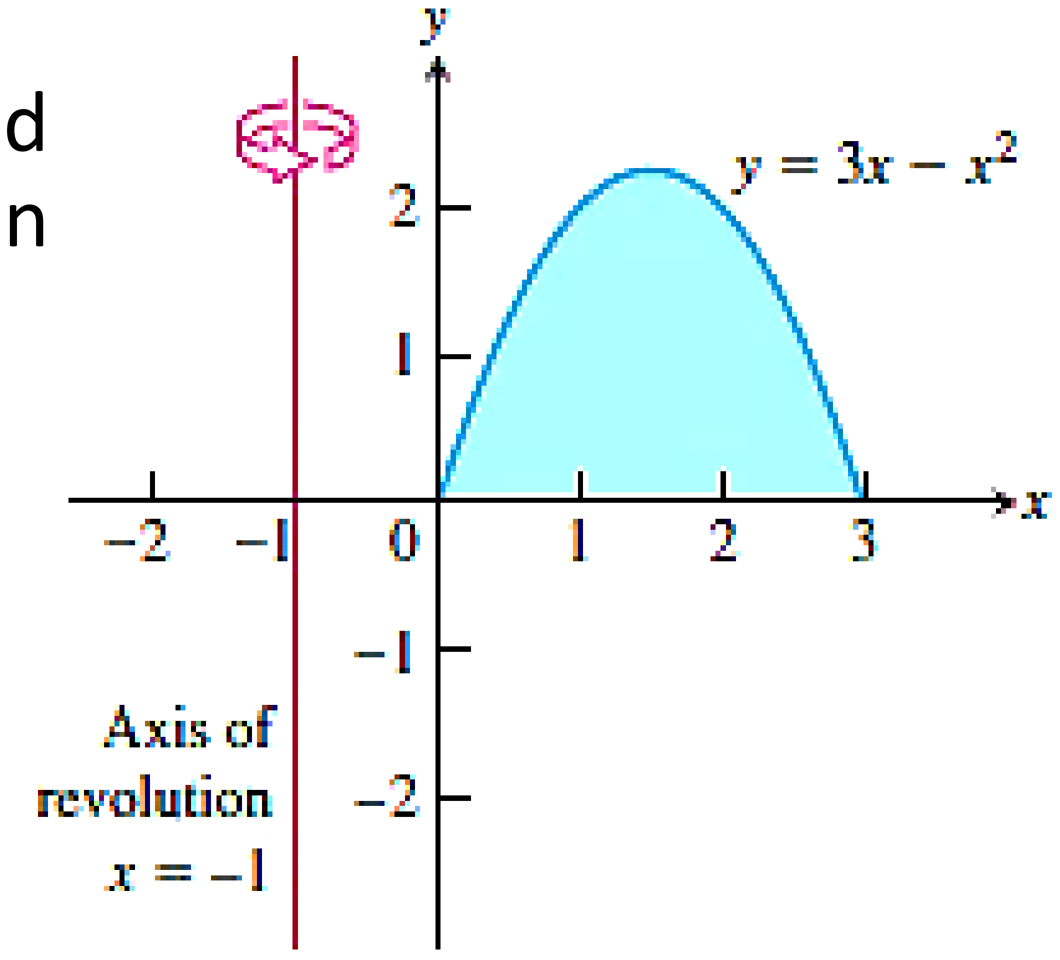
VOLUMES BY CYLINDRICAL SHELLS

- We will slice straight down through the solid perpendicular to the x –axis, with the axis of the cylinder parallel to the y –axis.
- The vertical axis of each cylinder is the same line, but the radii of the cylinders increase with each slice. In this way the solid S is sliced up into thin cylindrical shells of constant thickness that grow outward from their common axis, like circular tree rings.
- Unrolling a cylindrical shell shows that its volume is approximately that of a rectangular slab with area $A(x)$ and thickness Δx.
- This allows us to apply the same integral definition for volume as before.
- Before describing the method in general, let’s look at an example to gain some insight.

VOLUME OF SOLID OF REVOLUTION

Example:

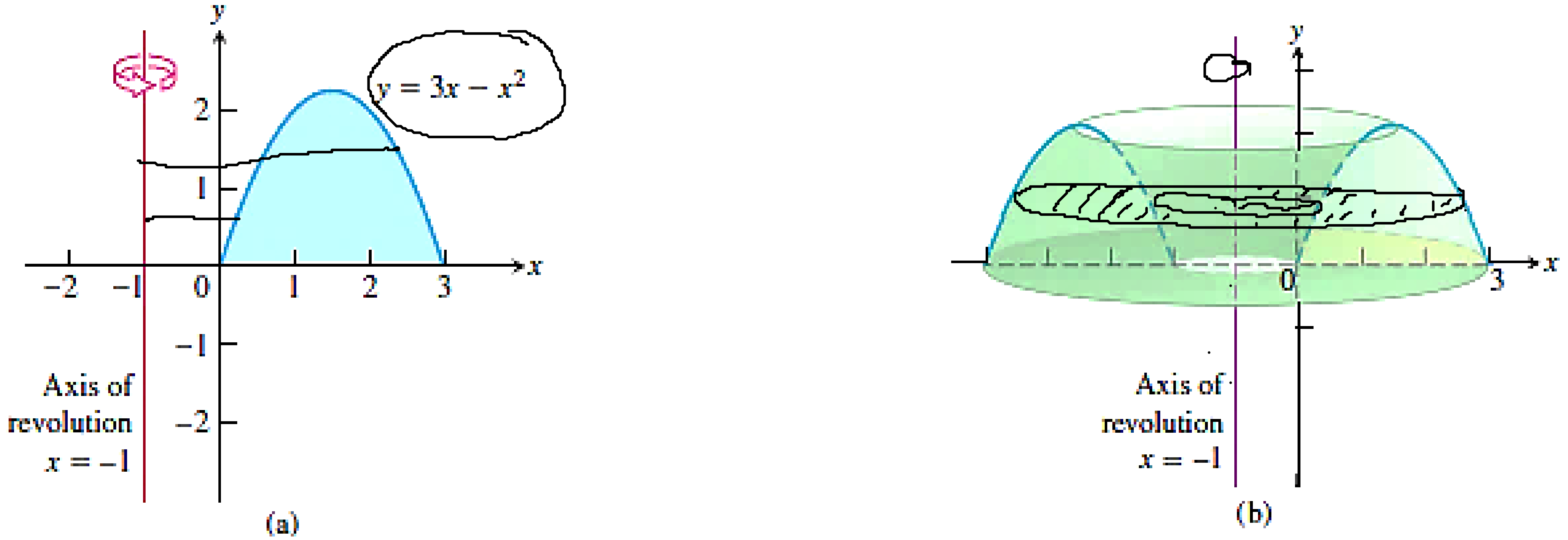
Let's consider the problem of finding the volume of the solid obtained by rotating about the vertical line $x = -1$, the region bounded by the curve $y = 3x - x^2$ and x −axis.



(a)

VOLUME OF SOLID OF REVOLUTION

- If we slice perpendicular to the vertical line $x = -1$, we get a washer.
- However, to compute the inner radius and the outer radius of the washer, we would have to solve the given equation for x in terms of y.



VOLUME OF SOLID OF REVOLUTION

Sol :-

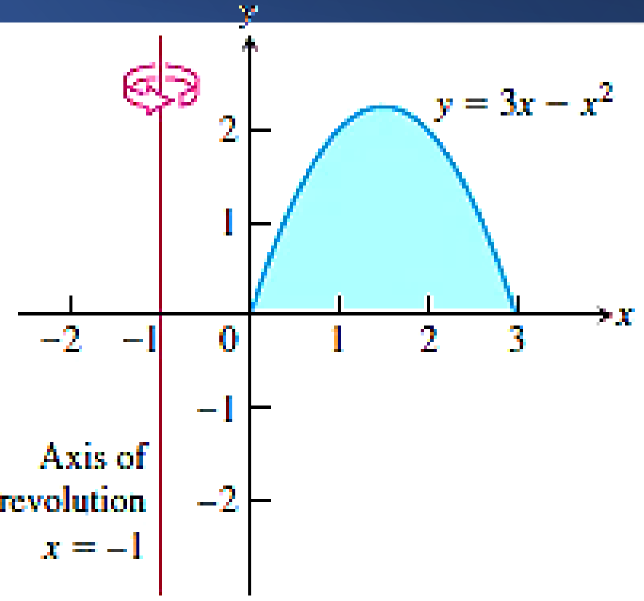
$$y = 3x - x^2 \Rightarrow x^2 - 3x = -y$$

$$\Rightarrow \underbrace{x^2 - 2(x)\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2} - \underline{\left(\frac{3}{2}\right)^2} = -y$$

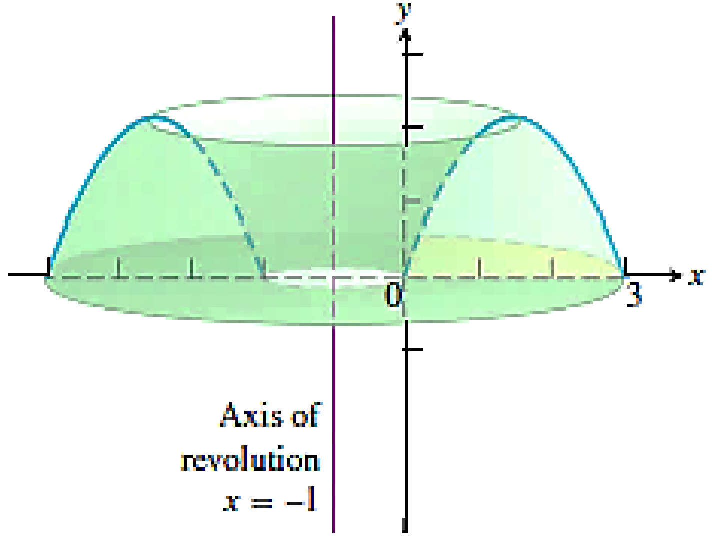
$$\Rightarrow \left(x - \frac{3}{2}\right)^2 = \frac{9}{4} - y$$

$$\Rightarrow x - \frac{3}{2} = \pm\sqrt{\frac{9}{4} - y}$$

$$\Rightarrow x = \frac{3}{2} \pm \sqrt{\frac{9}{4} - y}$$



(a)



(b)

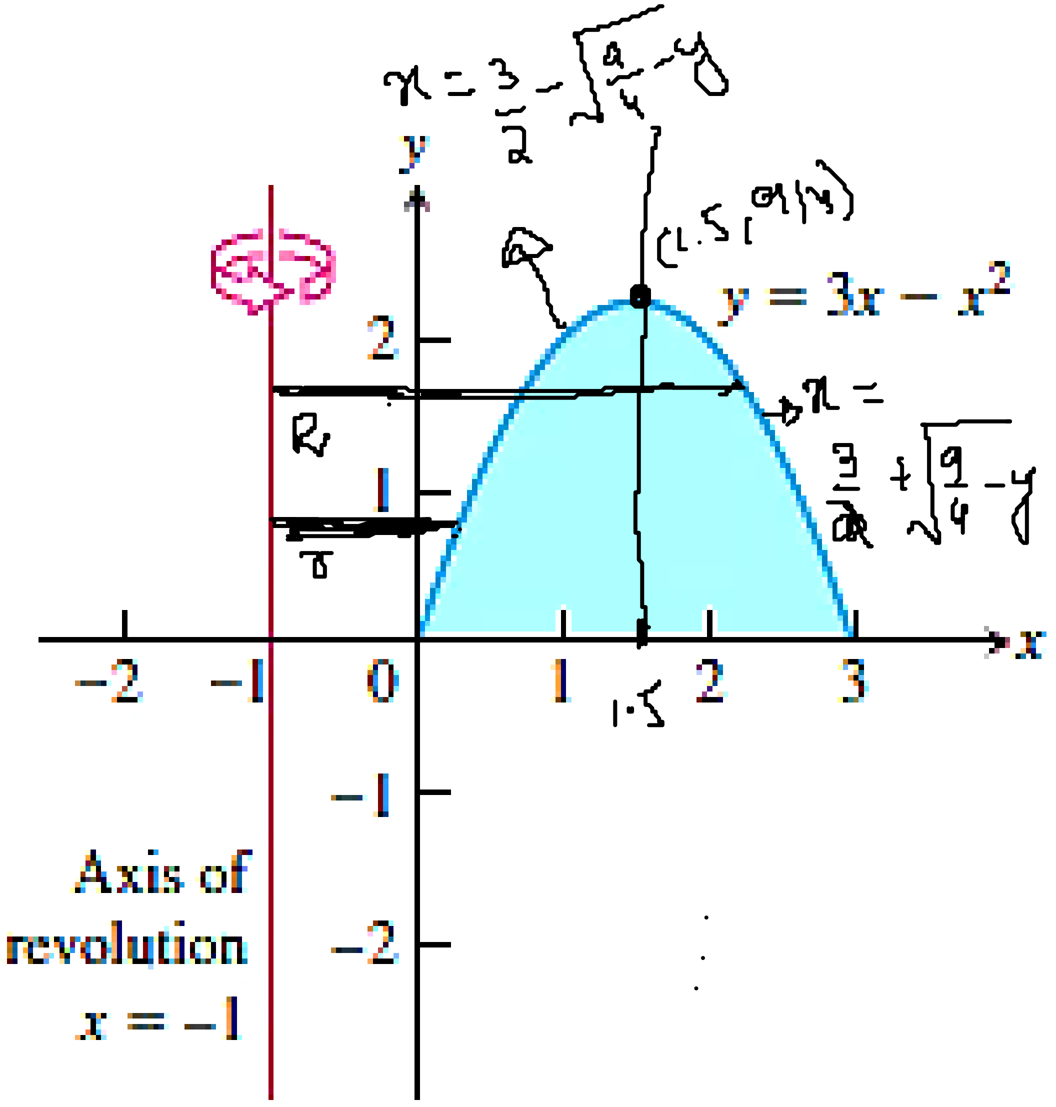
VOLUME OF SOLID OF REVOLUTION

R = outer radius $= \frac{3}{2} + \sqrt{\frac{9}{4} - y} - (-1)$

$= \frac{3}{2} + \sqrt{\frac{9}{4} - y} + 1$

r = Inner radius $= \frac{3}{2} - \sqrt{\frac{9}{4} - y} + 1$

$$V = \pi \int_0^{9/4} \left[R^2(y) - r^2(y)\right] dy$$



(a)

VOLUME OF SOLID OF REVOLUTION

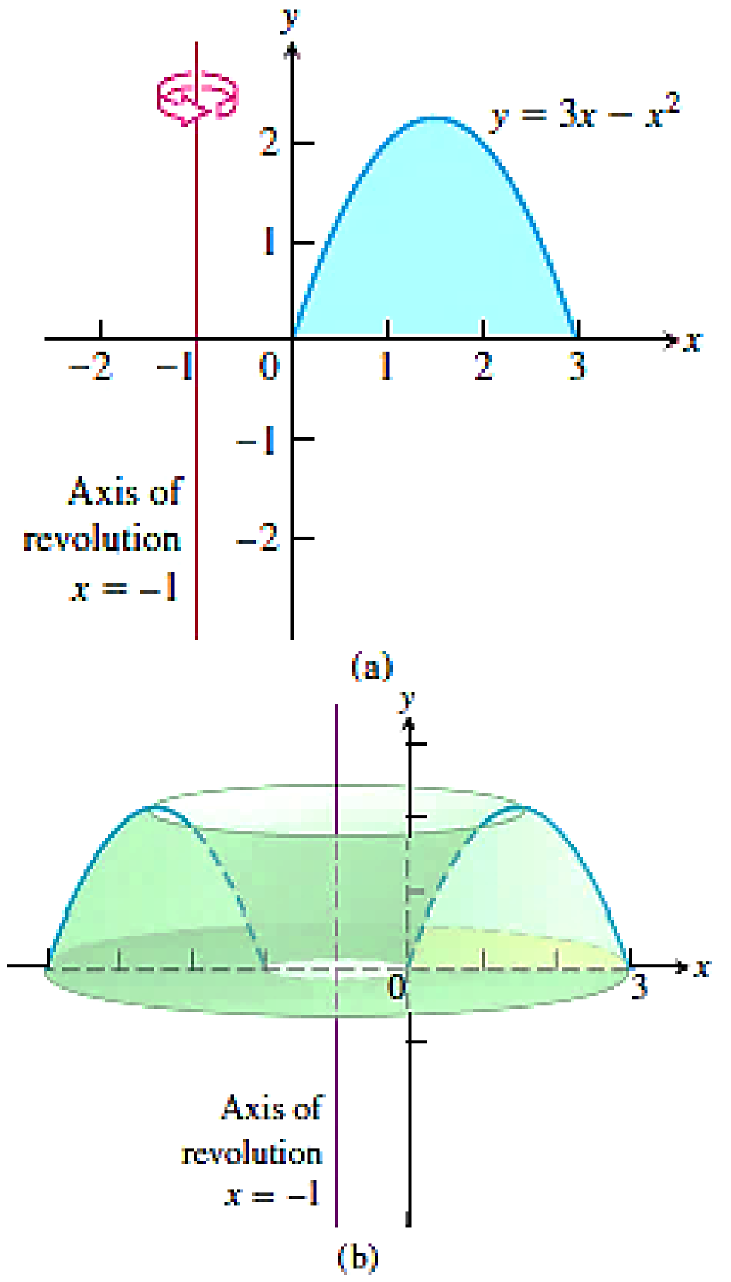
$$V = \pi \int_0^{9/4} \left(10\sqrt{\frac{9}{4} - y}\right) dy$$

$$= 10\pi \int_0^{9/4} \left(\sqrt{\frac{9}{4} - y}\right) dy$$

$$= 10\pi \left(-\frac{2}{3}\right)\left(\frac{9}{4} - y\right)^{3/2} \Bigg|_0^{9/4}$$

$$= -\frac{20}{3}\pi \left[\left(\cancel{\frac{9}{4} - \frac{9}{4}}\right)^{3/2} - \left(\frac{9}{4} - 0\right)^{3/2}\right] = \frac{20}{3}\pi \left(\frac{9}{4}\right)^{3/2}$$

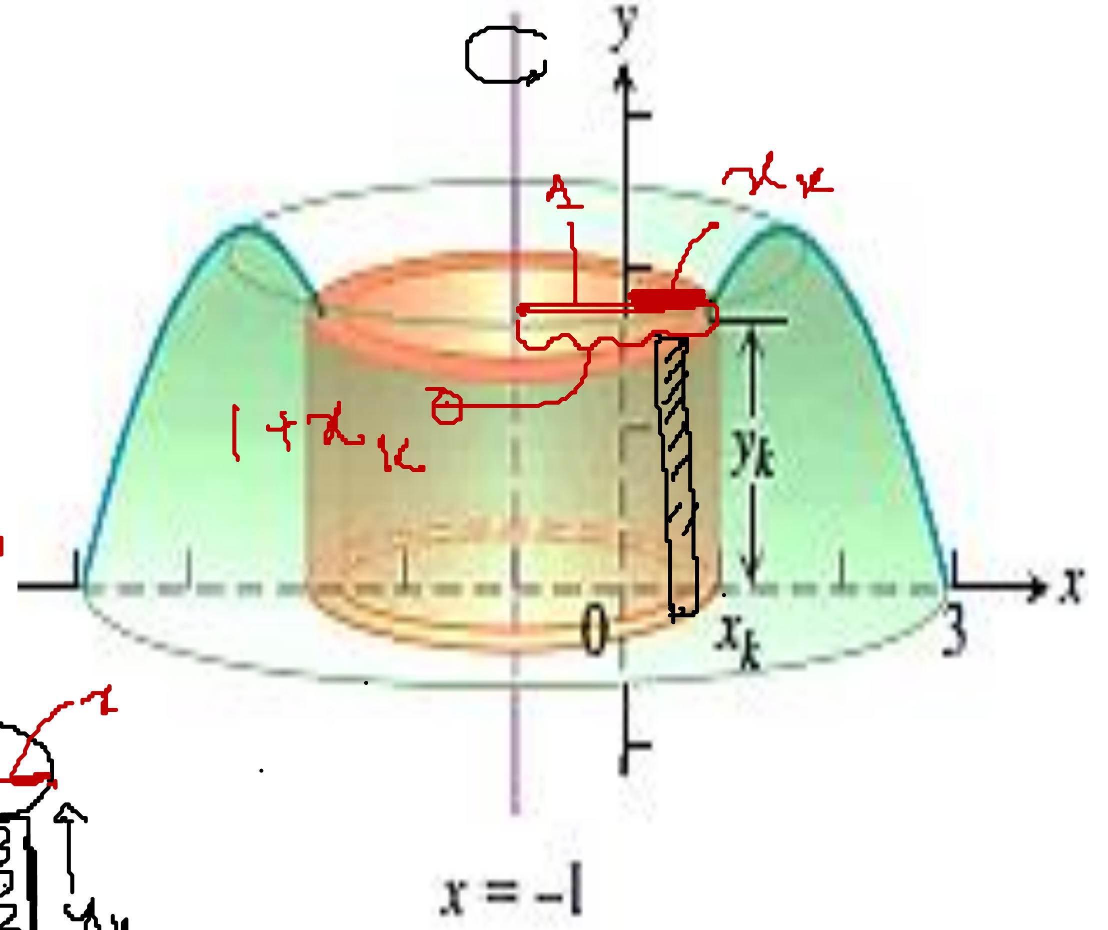
$$= \frac{45}{2}\pi$$



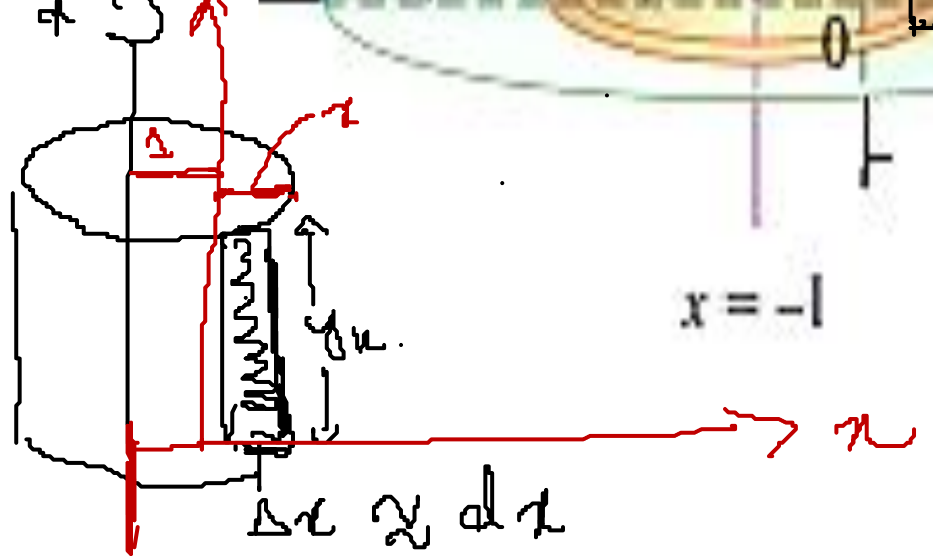
VOLUMES BY CYLINDRICAL SHELLS

Fortunately, there is a method—the method of cylindrical shells—that is easier to use in such a case.

- Instead of rotating a horizontal strip of thickness Δy we rotate a vertical strip of thickness Δx.
- This rotation produces a cylindrical shell of height y_k above a point x_k within the base of the vertical strip, radius $1 + x_k$ and of thickness Δx.

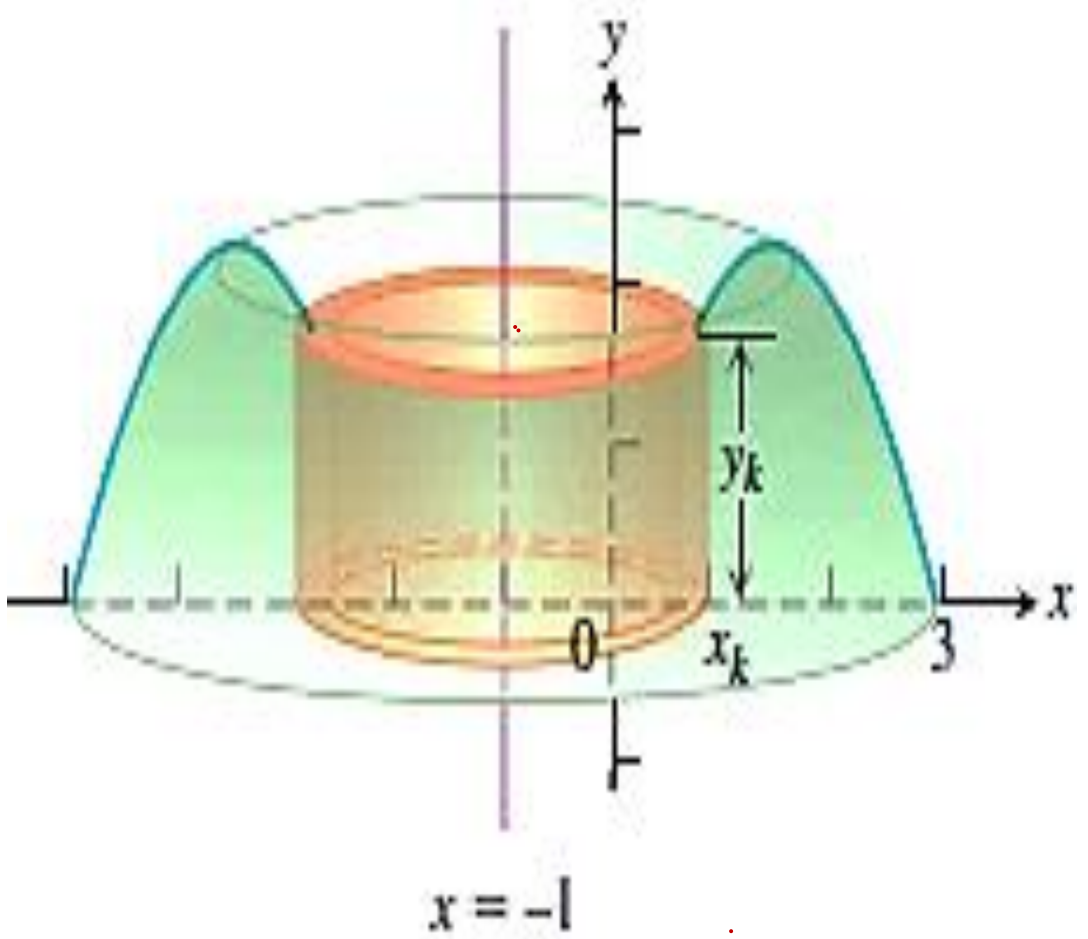


$r = 1 + x_k$



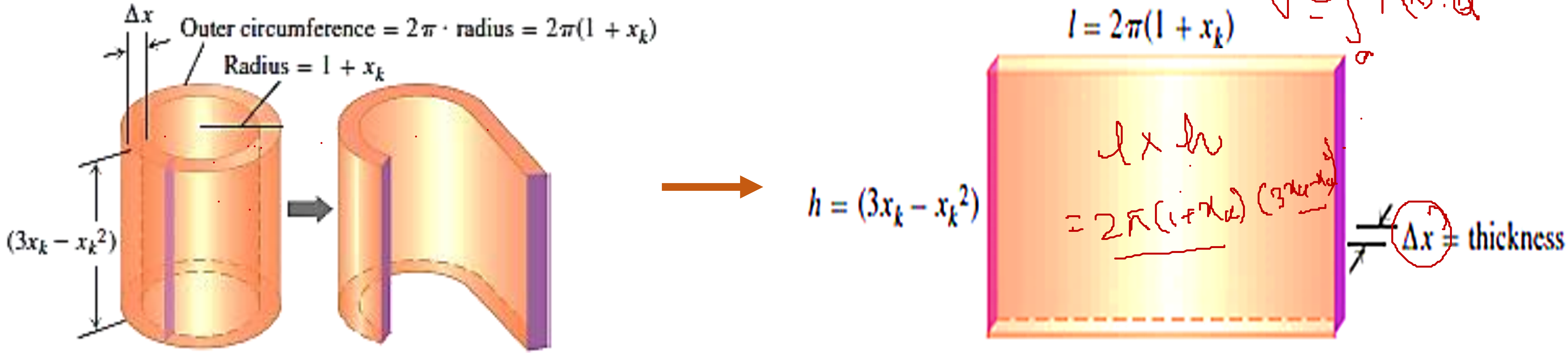
VOLUMES BY CYLINDRICAL SHELLS

- An example of a cylindrical shell is shown as the orange-shaded region in the figure.
- We can think of the cylindrical shell shown in the figure as approximating a slice of the solid obtained by cutting straight down through it, parallel to the axis of revolution, all the way around close to the inside hole.
- We then cut another cylindrical slice around the enlarged hole, then another, and so on, obtaining n cylinders.
- The radii of the cylinders gradually increase, and the heights of the cylinders follow the contour of the parabola.



VOLUMES BY CYLINDRICAL SHELLS

- Each slice is having thickness Δx, its radius is approximately $1 + x_k$ and its height is approximately $3x_k - x_k^2$.
- If we unroll the cylinder and flatten it out, it becomes a rectangular slab with thickness Δx.
- The outer circumference of the kth cylinder is $2\pi \times \text{radius} = 2\pi(1 + x_k)$ and this gives us the length of the rolled-out rectangular slab.



$V = \int_0^b A(x).dx$

$l \times h$

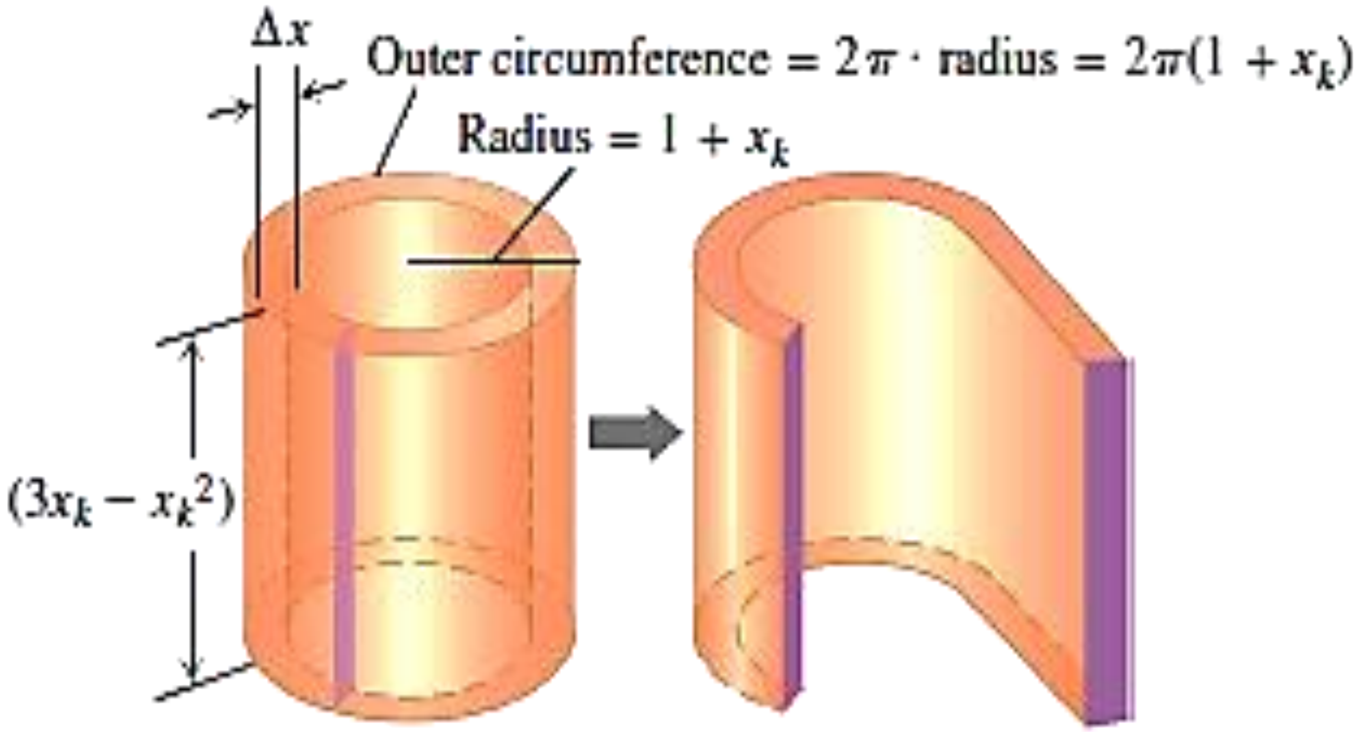
$= 2\pi(1+x_k)(3x_k - x_k^2)$

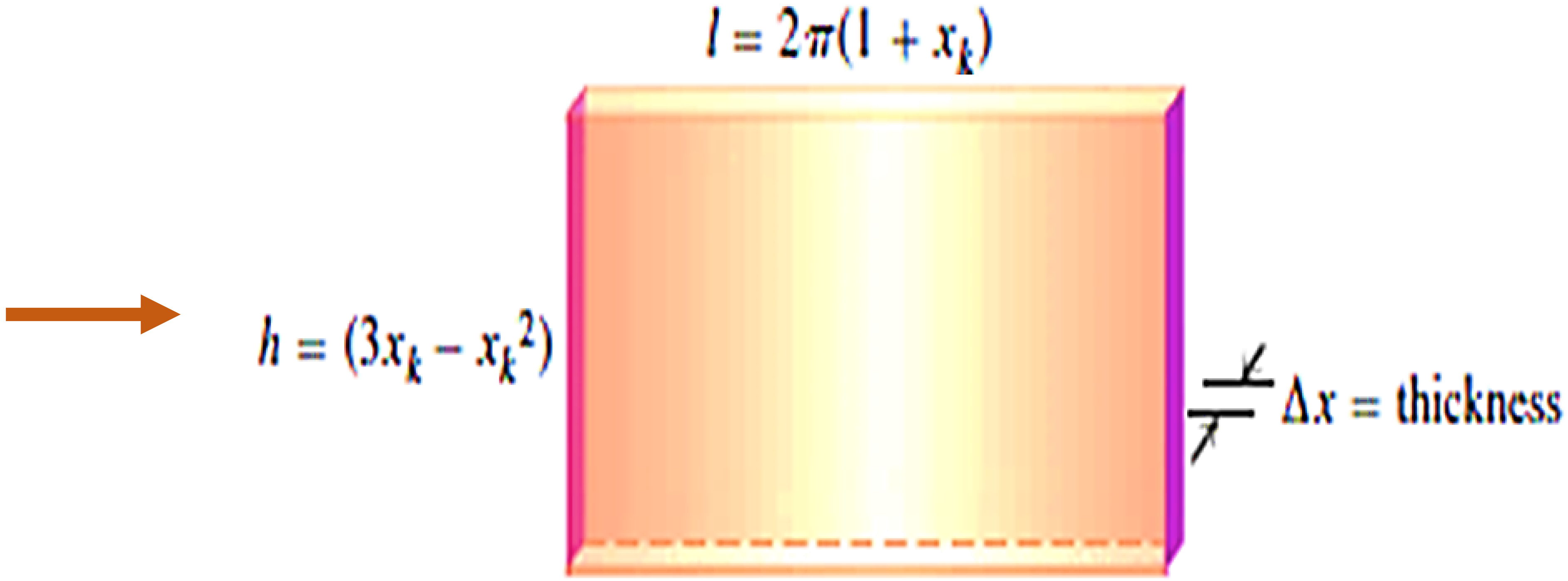
VOLUMES BY CYLINDRICAL SHELLS

Volume is approximated by that of a rectangular solid:

$$\Delta V_k = \text{circumference} \times \text{height} \times \text{thickness}$$

$$= 2\pi(1 + x_k).\left(3x_k - x_k^2\right).\Delta x.$$





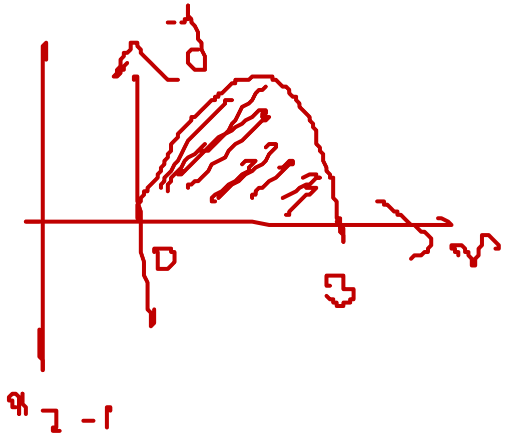
Summing together the volumes of the individual cylindrical shells over the interval $[0, 3]$ gives the Riemann sum:

$$\sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi(x_k + 1)(3x_k - x_k^2)\Delta x.$$

$$\int_a^b A(x)\,dx$$

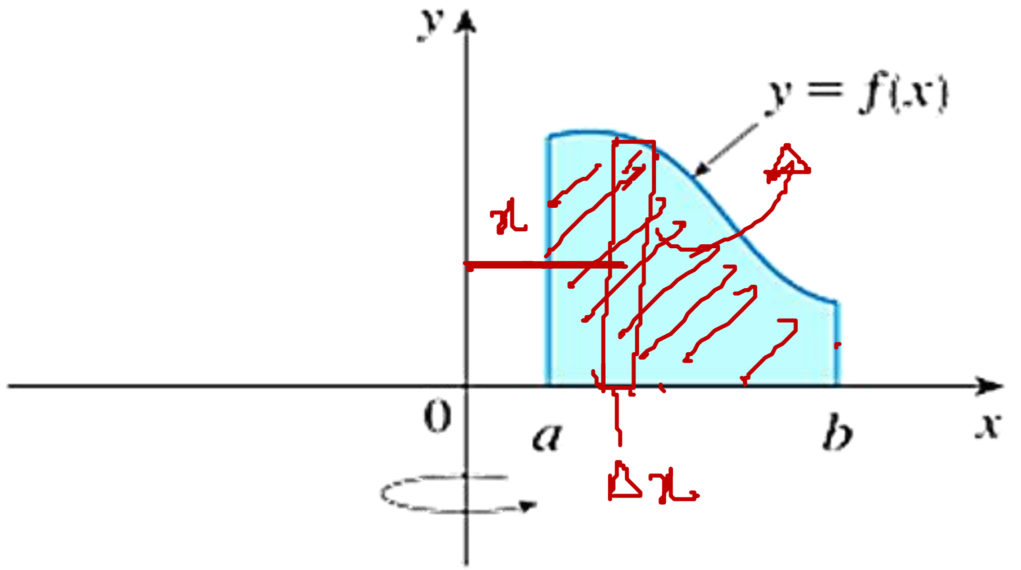
Taking the limit as $n \to \infty$ gives the volume integral:

$$V = \int_0^3 \underbrace{2\pi(1 + x)(3x - x^2)}_{A(x)}\underbrace{dx}_{\Delta x} = 2\pi \int_0^3 (2x^2 + 3x - x^3)dx = \frac{45}{2}\pi.$$

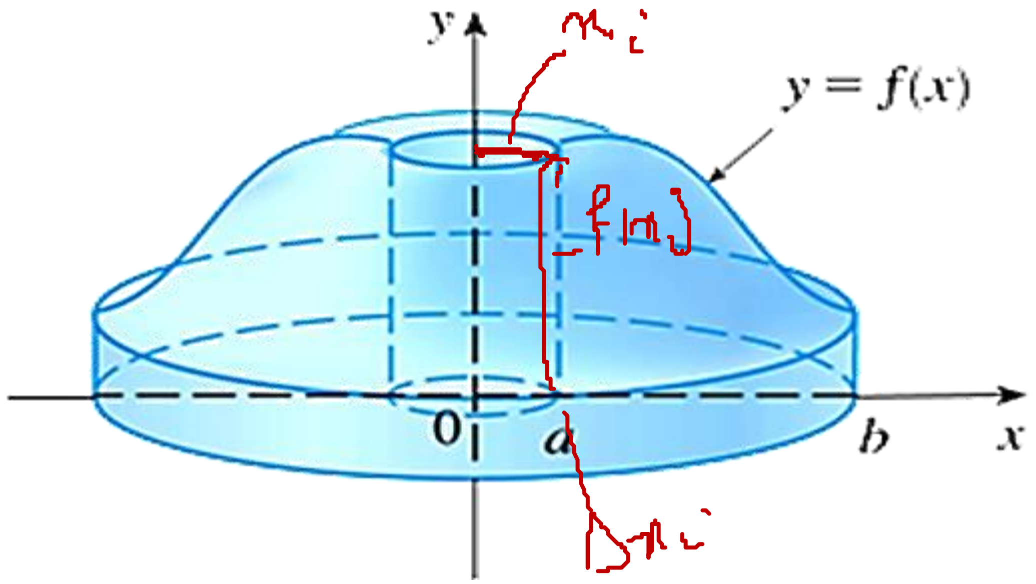


CYLINDRICAL SHELLS METHOD

- Let S be the solid obtained by rotating about the y −axis the region bounded by the curve $y = f(x)$ [where $f(x) \geq 0$], $y = 0$, $x = a$ and $x = b$, where $b > a \geq 0$.

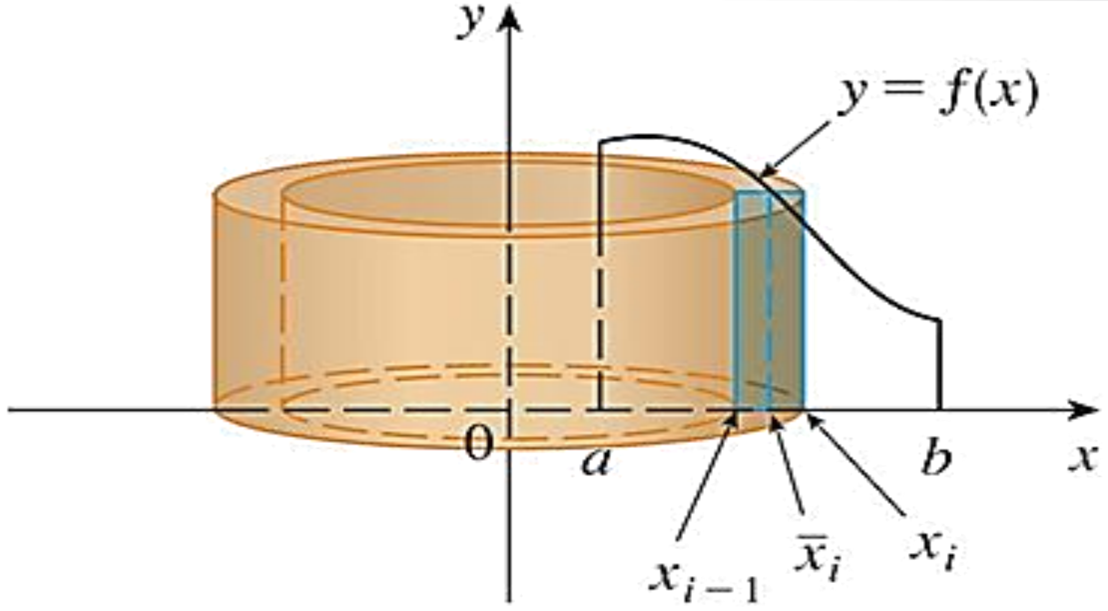


- Divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width Δx and let be $\bar{x}_i$ the midpoint of the ith subinterval.

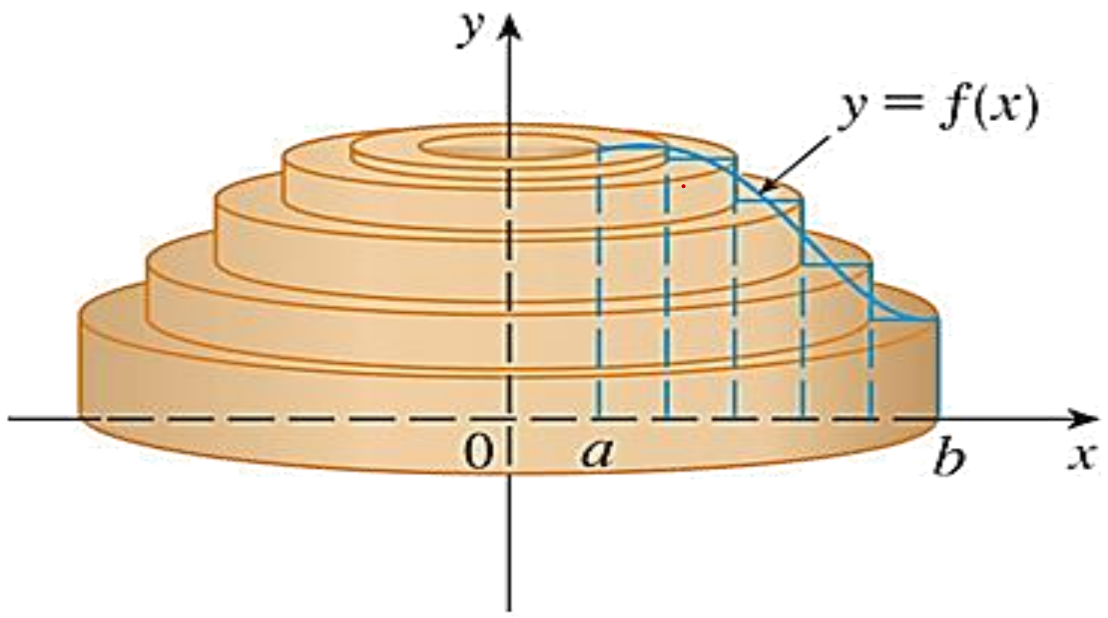


CYLINDRICAL SHELLS METHOD

- The rectangle with base $[x_{i-1}, x_i]$ And height $f(\overline{x_i})$ is rotated about the y −axis.



- The result is a cylindrical shell with average radius $\bar{x}_i$, height $f(\bar{x}_i)$, and thickness Δx.



- Thus, the volume of a cylindrical shell is given by:

$$V_i = (2\pi\bar{x}_i)[f(\bar{x}_i)]\Delta x. \checkmark$$

CYLINDRICAL SHELLS METHOD

So, an approximation to the volume V of S is given by the sum of the volumes of these shells:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi\bar{x}_i\, f(\bar{x}_i)\Delta x$$

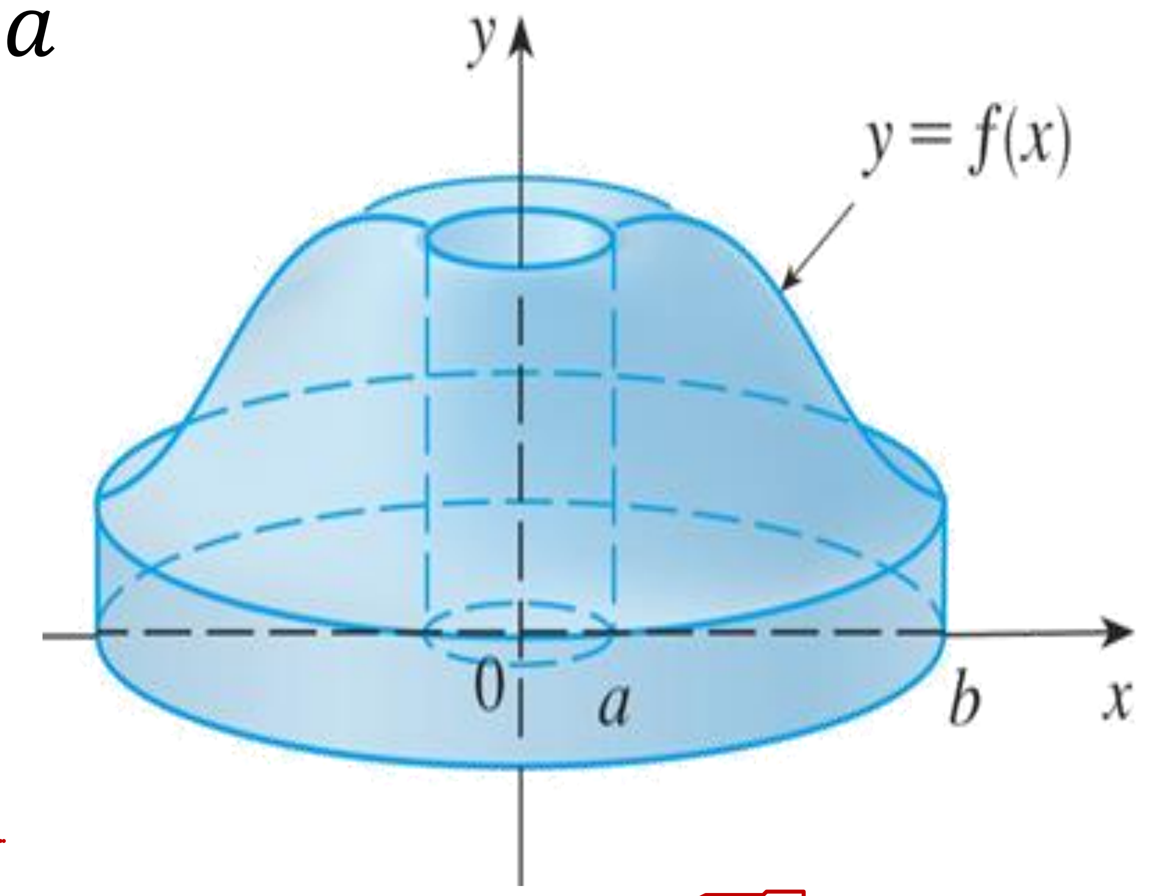
Taking the limit as $n \to \infty$ gives the volume integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\bar{x}_i\, f(\bar{x}_i)\Delta x = \int_a^b 2\pi x f(x) dx$$

CYLINDRICAL SHELLS METHOD

Thus, The volume of the solid obtained by rotating about the y −axis the region under the curve $y = f(x)$ from a to b, is:

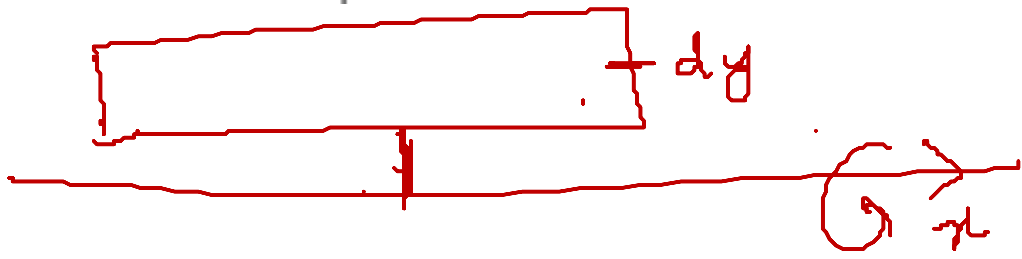
$$V = \int_a^b 2\pi x f(x)dx$$



where $0 \leq a < b$.

① $\boxed{x\text{-axis}}$

$$V = \int_c^d 2\pi y f(y) dy$$

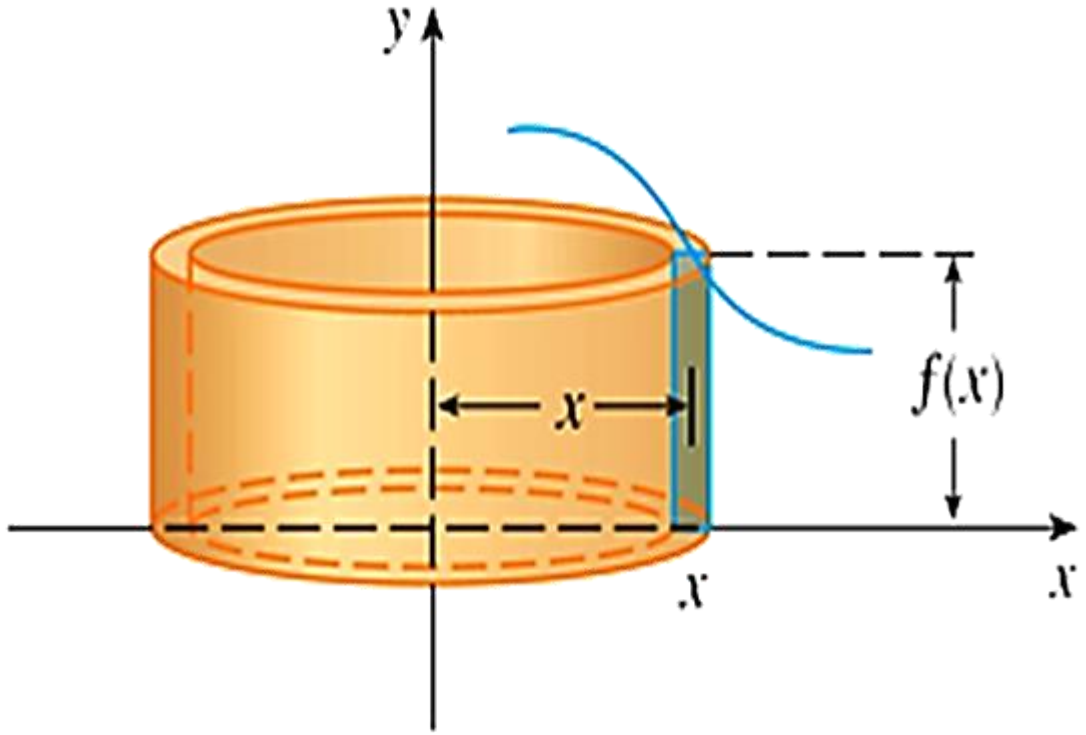


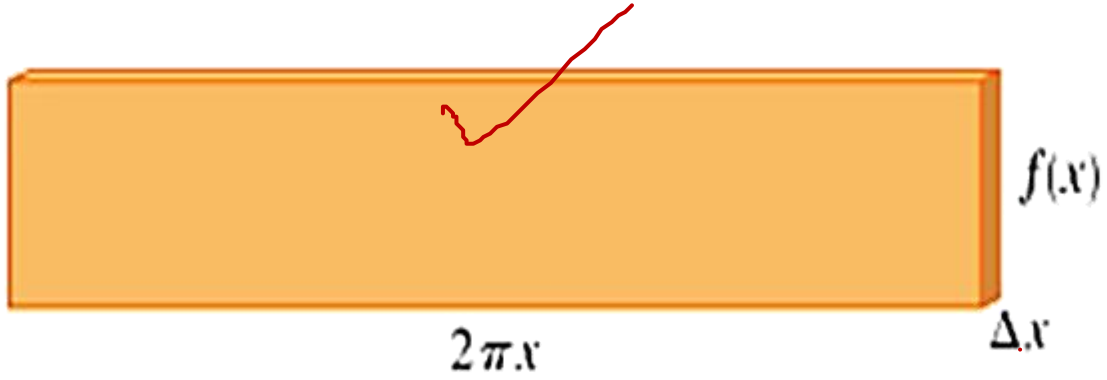
CYLINDRICAL SHELLS METHOD

Here's the best way to remember the formula.

Think of a typical shell, cut and flattened, with radius x, circumference $2\pi x$, height $f(x)$, and thickness Δx or dx, then the volume is given as:

$$V = \int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \underbrace{[f(x)]}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$





Example

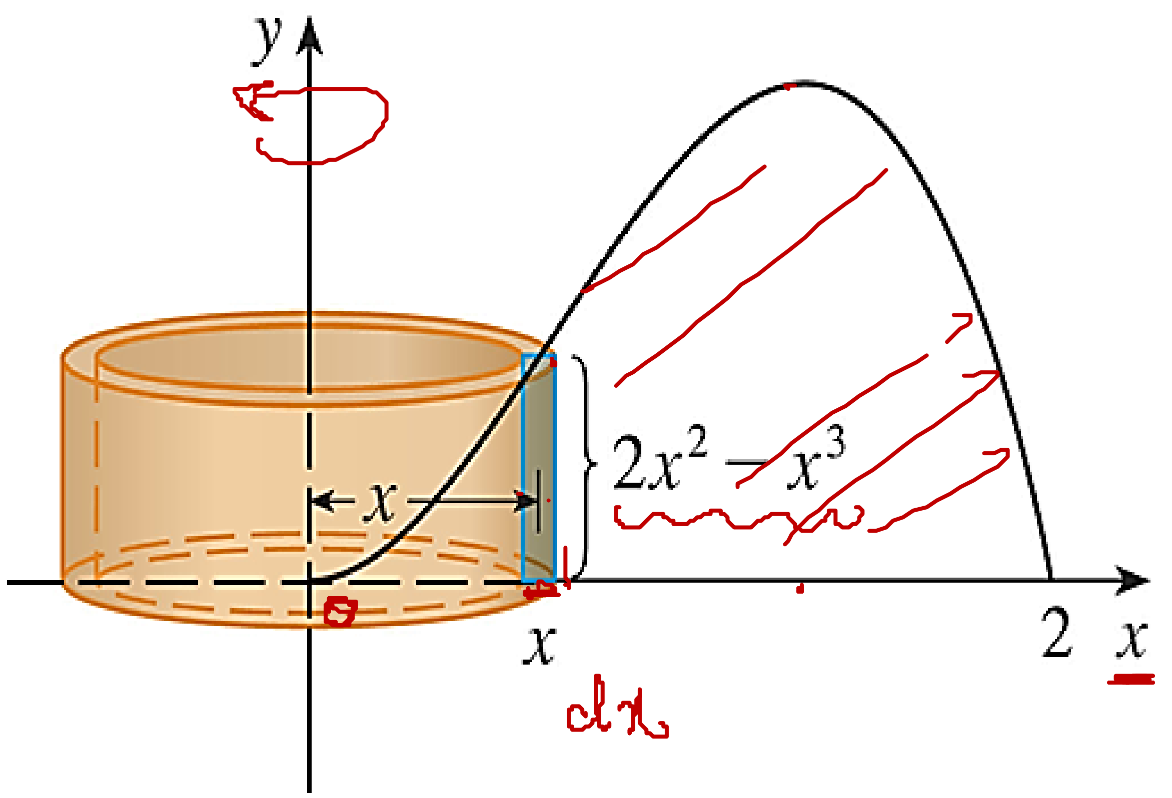
Find the volume of the solid obtained by revolving about the y −axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Solution:

We see that a typical shell has radius x, circumference $2\pi x$, and height $f(x) = 2x^2 - x^3$. So, by the shell method, the volume is:

$$V = \int_0^2 (2\pi x)(2x^2 - x^3)\,dx = 2\pi \int_0^2 (2x^3 - x^4)dx$$

$$= 2\pi \left[\frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^2 = 2\pi\left(8 - \frac{32}{5}\right) = \frac{16}{5}\pi.$$



Example

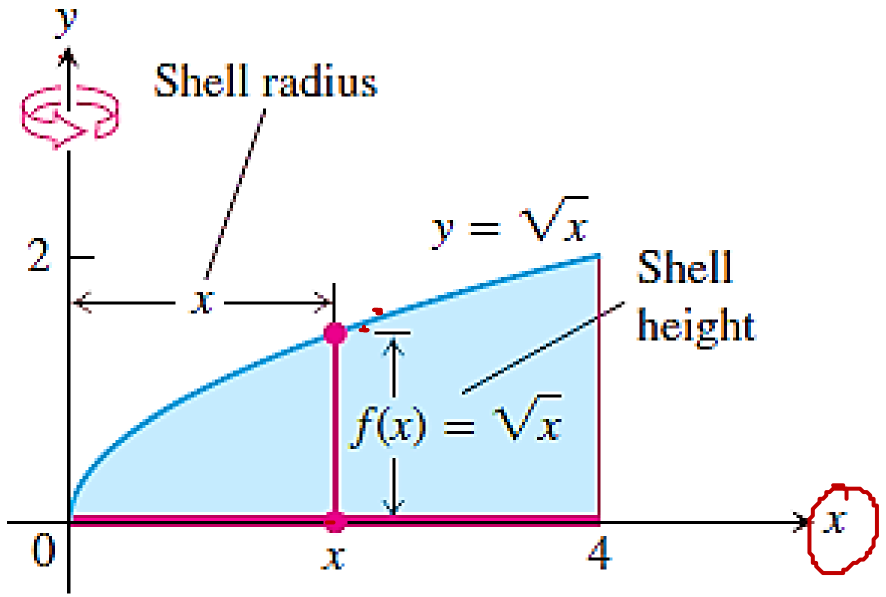
The region bounded by the curve $y = \sqrt{x}$, the x −axis and the line $x = 4$ is revolved about the y −axis to generate a solid. Determine the volume of the resulting solid.

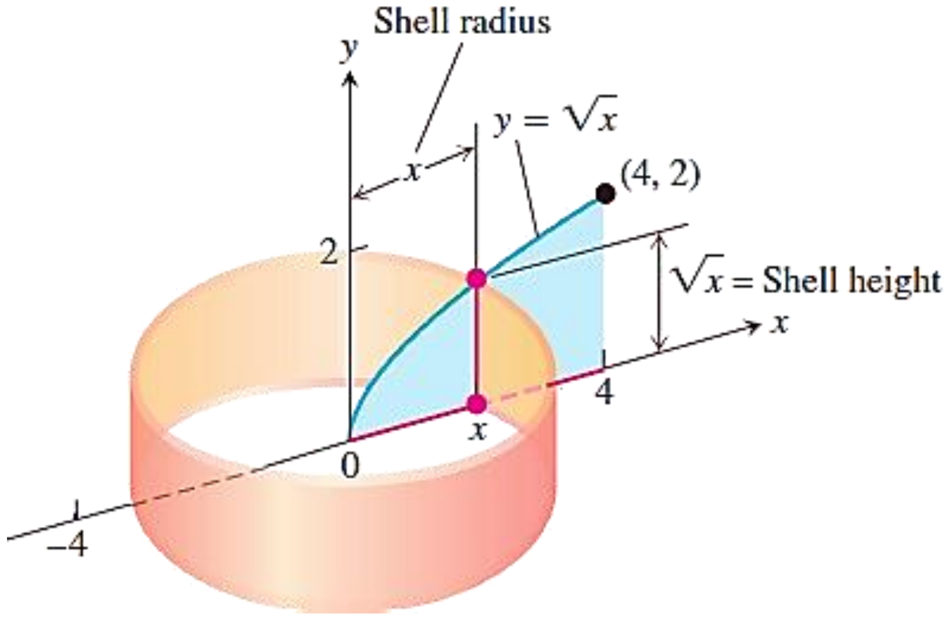
Solution:

We see that the shell has radius x, circumference $2\pi x$, and height $\sqrt{x}$. So, by the shell method, the volume is given as:

$$V = \int_0^4 \overset{C}{(2\pi x)}\overset{H}{(\sqrt{x})}\,\overset{T}{dx} = 2\pi \int_0^4 x^{3/2}\,dx$$

$$= 2\pi \left[\frac{x^{5/2}}{5/2}\right]_0^4 = \frac{128}{5}\pi.$$





Example

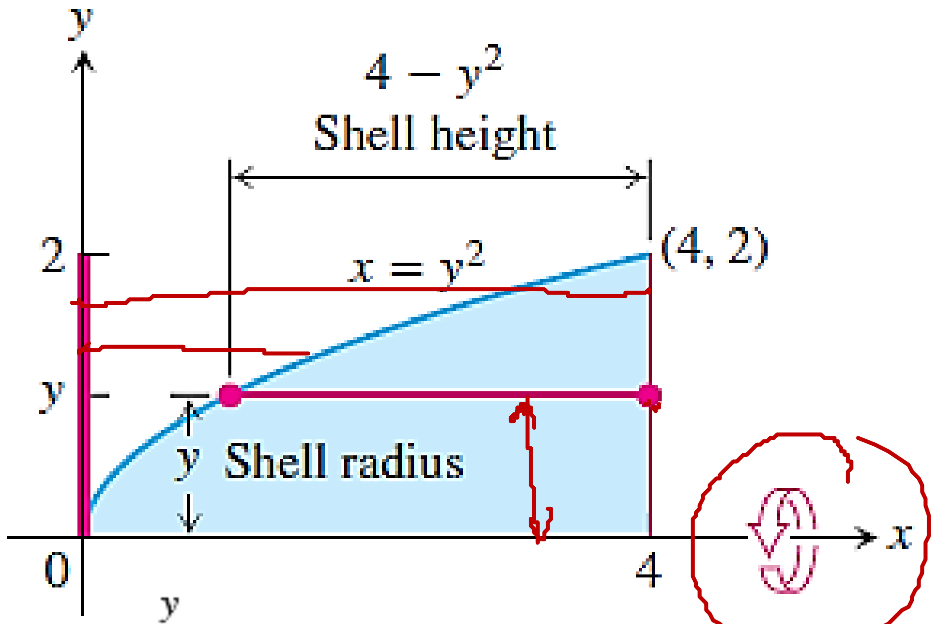
The region bounded by the curve $y = \sqrt{x}$, the x –axis and the line $x = 4$ is revolved about the x –axis to generate a solid. Find the volume of the resulting solid.

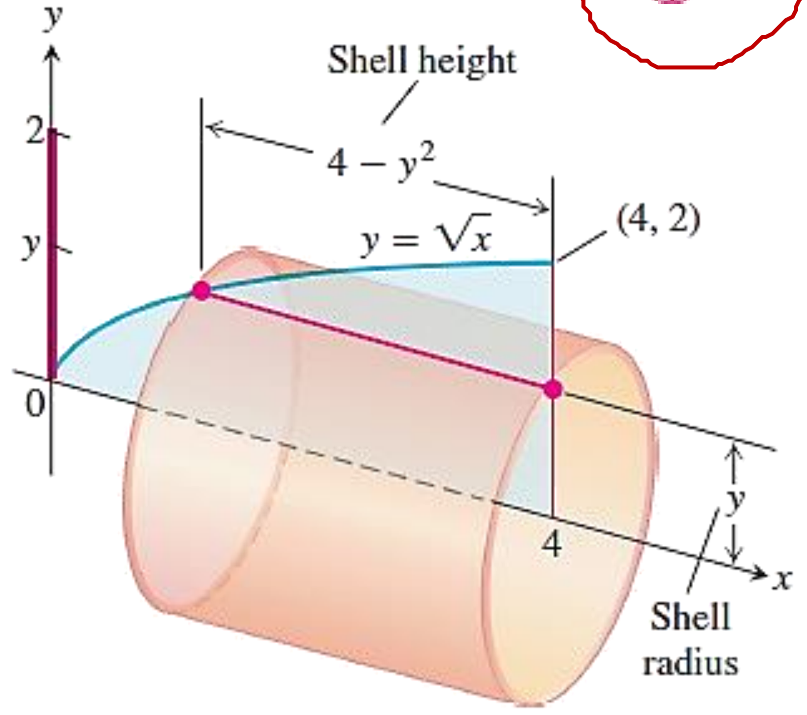
Solution:

We see that the shell has radius y, circumference $2\pi y$, and height $4 - y^2$. So, by the shell method, the volume is given as:

$$V = \int_0^2 (2\pi y)(4 - y^2)\, dy = 2\pi \int_0^2 (4y - y^3)\, dy$$

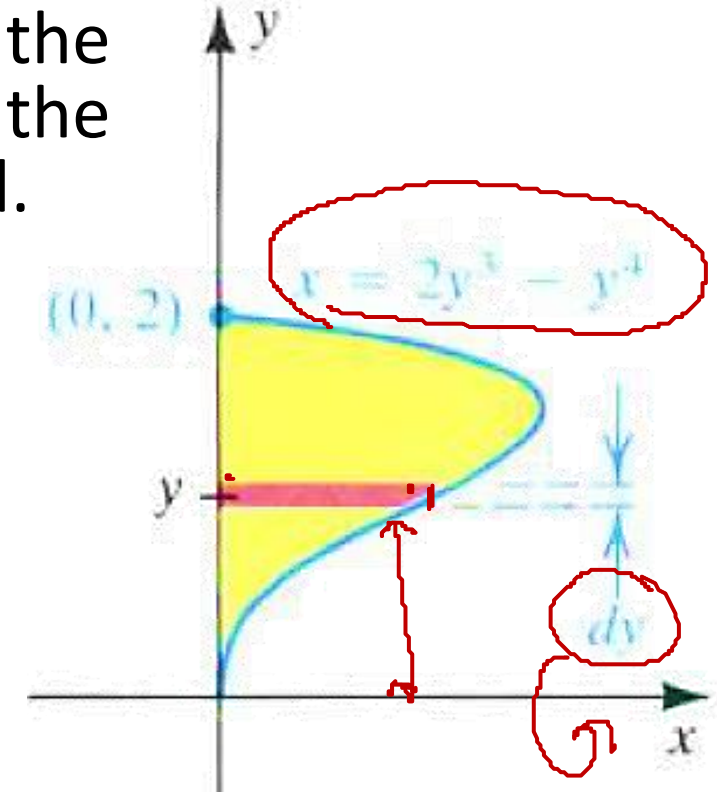
$$= 2\pi \left[4\frac{y^2}{2} - \frac{y^4}{4} \right]_0^2 = 8\pi.$$

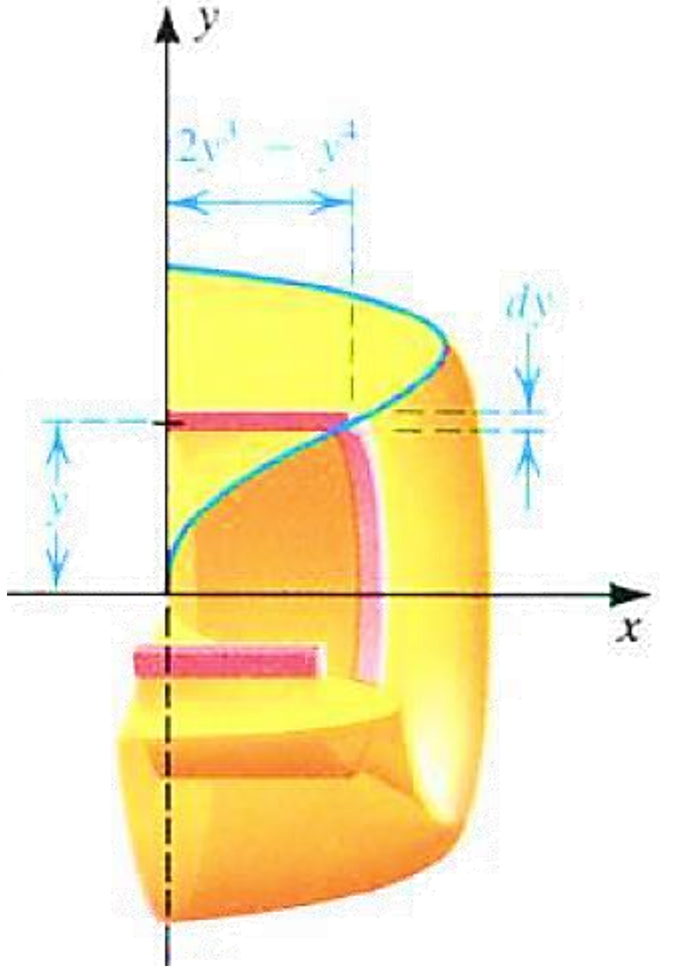




Example

The region in the first quadrant bounded by the graph of the equation $x = 2y^3 - y^4$ and the y −axis is revolved about the x −axis. Set up the integral for the volume of the resulting solid.





Solution:

For the present case:

Thickness of shell: dy

Height: $2y^3 - y^4$

Radius: y

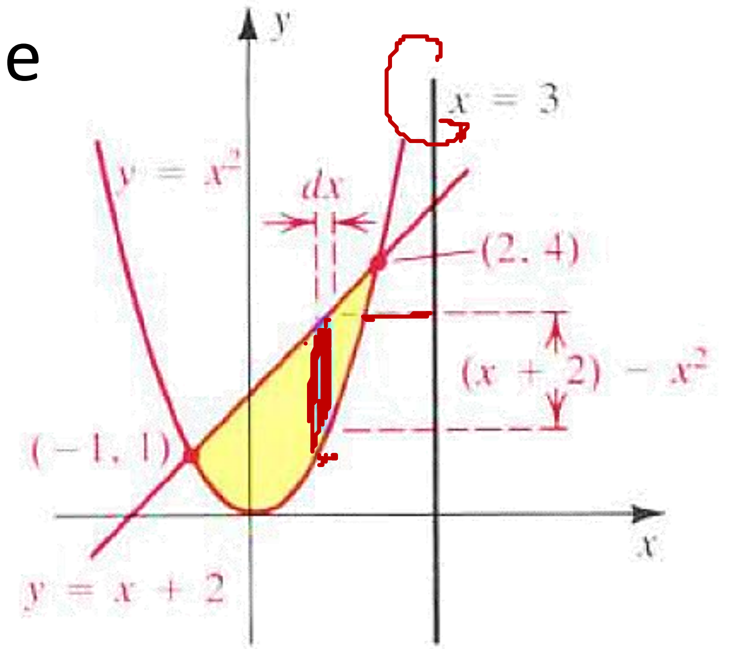
Circumference: $2\pi y$

Thus, the volume of solid is given as:

$$V = \int_0^2 2\pi y[2y^3 - y^4]\, dy.$$

Example

The region bounded by the graphs of $y = x^2$ and $y = x + 2$ is revolved about the line $x = 3$. Set up the integral for the volume of the resulting solid.



Solution:

For the present case:

Thickness of shell: dx

Height: $(x + 2) - x^2$

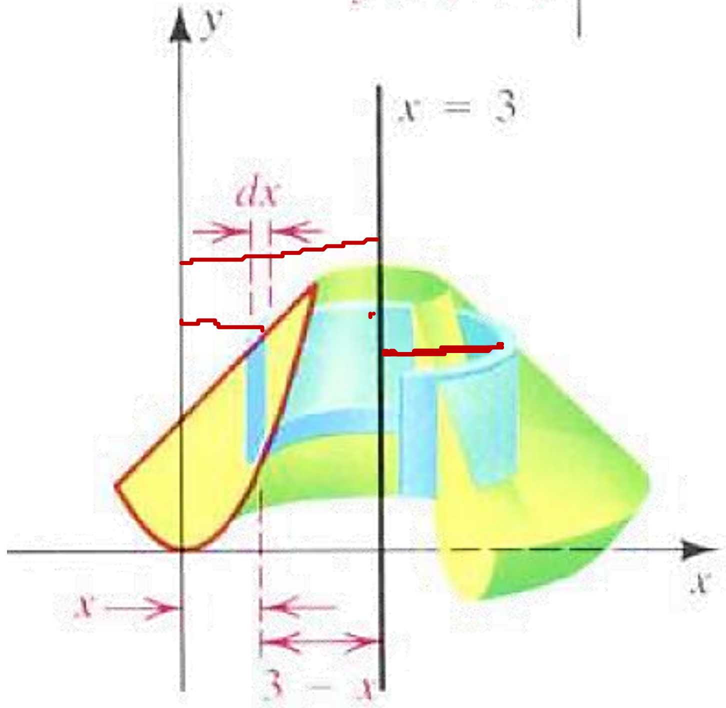
Radius: $3 - x$ ✓

Circumference: $2\pi(3 - x)$

Thus, the volume of solid is given as:

$$V = \int_{-1}^{2} 2\pi(3 - x)[x + 2 - x^2]\, dx.$$

✓



For the sake of comparison following table may be helpful:

Method	Axis of revolution	Variable of integration	Formula
Disks	The x − axis	x	$V = \pi \int_a^b [f(x)]^2 \, dx$
	The y −axis	y	$V = \pi \int_c^d [f(y)]^2 \, dy$
Cylindrical Shells	The x − axis	y	$V = 2\pi \int_c^d y f(y) \, dy$
	The y −axis	x	$V = 2\pi \int_a^b x f(x) \, dx$

Practice Questions

1. Determine the volume of the solid of revolution generated by revolving the region bounded by $y = x^2 - 1$ from $x = 1$ to $x = 3$ about y −axis.
2. Determine the volume of the solid of revolution generated by revolving the region bounded by $y = x^{1/3}$ from $x = 0$ to $x = 8$ about x −axis.
3. Determine the volume of the solid of revolution generated by revolving about x −axis, the region bounded by the graphs of $y = x, y = 2 - x$ and the x −axis.
4. Determine the volume of the solid of revolution generated by revolving about x −axis, the region bounded by $y = 4x - x^2$ and the x −axis.
5. Determine the volume of the solid of revolution generated by revolving about x −axis, the region bounded by $y = 2 - x^2$ and $y = x^2$.
6. Determine the volume of the solid of revolution generated by revolving the region bounded by $y = \sqrt{x - 1}, y = 0$ and $x = 10$ about the line $y = 5$.

Practice Questions

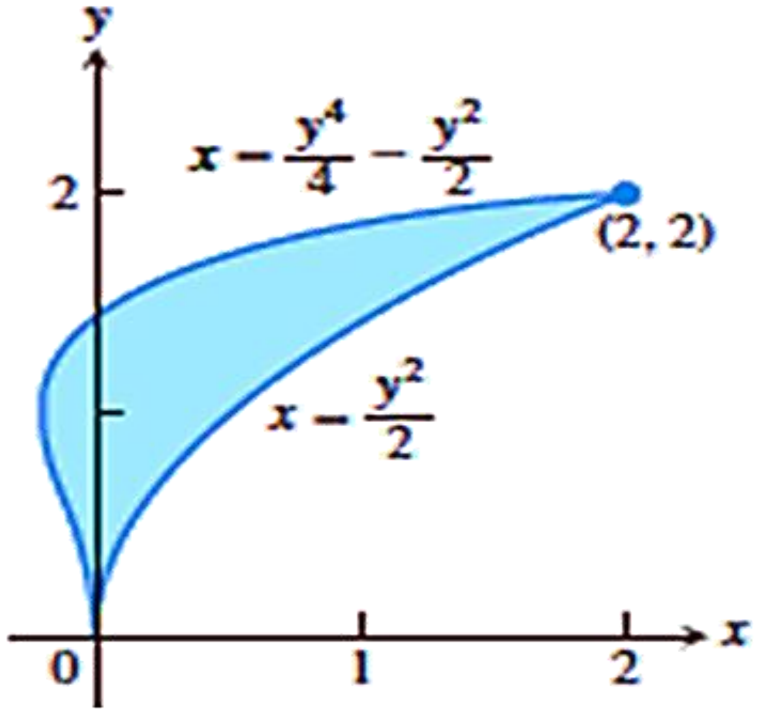
7. Use the washer method to find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.
8. Use the shell method to find the volume of the solid generated by revolving the shaded regions about the indicated axes.

a. The x-axis

b. The line $y = 2$

c. The line $y = 5$

d. The line $y = -5/8$



Practice Questions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Exercise: 6.2

  Q # 1 to Q # 36

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 6.3

  Q # 1 to Q # 30