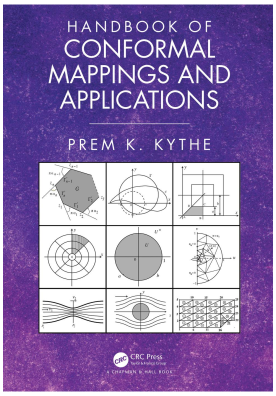

Handbook of Conformal Mappings and Applications

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Linear and Bilinear Transformations

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3

Linear and Bilinear Transformations

The central problem in the theory of conformal mapping is to determine a function f which maps a given region $D \subset \mathbb{C}$ conformally onto another region $G \subset \mathbb{C}$. The function f does not always exist, and it is not always uniquely determined. The Riemann mapping theorem (§2.6, Theorem 2.15) guarantees the existence and uniqueness of a conformal map of D onto the unit disk U under certain specific conditions. First, we will introduce definitions of certain curves, and some elementary mappings, before we will study linear and bilinear transformations.

3.1 Definition of Certain Curves

Some curves are defined as follows:

3.1.1 Line, or straight line, has the equation $lx + my + p = 0$, where l, m, p are real, and $l^2 + m^2 > 0$. Then $\Re\{\bar{\lambda} z\} = p$, or $\Im\{\mu z\} = p$, where $\lambda = l + i\,m \neq 0$, and $\mu = m + i\,l = i\,\bar{\lambda}$.

The *distance* of a point z_0 from the line $lx + my + p = 0$ is given by $\left|\dfrac{\Re\{\bar{\lambda} z_0\} - p}{\lambda}\right|$.

The angle between the x-axis and the line $lx+my+p = 0$ is given by $\arg\{\lambda + \frac{\pi}{2} + n\pi\}$, $n = 0, \pm 1, \pm 2, \dots$.

3.1.2 Circle *in the extended sense* means a circle or a straight line. There is no distinction between circles and lines in the theory of bilinear transformations. However, a circle in the z-plan is defined by $|z - z_0| = r$ with center at z_0 and radius r; in the w-plane it is defined by $|w - w_0| = R$ with center at w_0 and radius R.

3.1.3 Ellipse has the equation $|z - z_1| + |z - z_2| = k$, $k > |z_1 - z_2|$; its foci are z_1 and z_2; major axis is k; eccentricity is $\dfrac{|z_1 - z_2|}{k}$; exterior of the ellipse is defined as $|z - z_1| + |z - z_2| > k$, and interior as $|z - z_1| + |z - z_2| < k$.

3.1.4 Hyperbola has the equation $|z - z_1| + |z - z_2| = \pm k$, $0 < k < |z_1 - z_2|$, where the plus or minus sign is used according to whether the branch is nearer to the focus z_2 or it is nearer to the focus z_1. The real axis is k; eccentricity $\dfrac{|z_1 - z_2|}{k}$; equations of asymptotes are

$$\left\{ (\bar{z}_1 - \bar{z}_2)\, e^{\pm\theta}\Big(z - \frac{z_1 + z_2}{2}\Big)\right\} = 0, \quad \cos\theta = \frac{k}{|z_1 - z_2|};$$

the exterior, not containing foci is defined by $\big||z - z_1| - |z - z_2|\big| < k$.

3.1.5 Rectangular Hyperbola is the hyperbola of §3.1.4 if $k = \dfrac{|z_1 - z_2|}{\sqrt{2}}$. Moreover, if $z_2 = -z_1$ (i.e., the origin is the center of the curve), then the equation of the rectangular hyperbola is

$$\Re\Big\{\frac{z^2}{z_1^2}\Big\} = \frac{1}{2},$$

with foci $\pm z_1$. The exterior of the hyperbola is defined by

$$-\infty \le \Re\Big\{\frac{z^2}{z_1^2}\Big\} < \frac{1}{2}.$$

Equations of asymptotes are (three forms):

$$\Re\Big\{\frac{z^2}{z_1^2}\Big\} = 0;\ \Re\Big\{\frac{z}{z_1}\Big\} = \pm\Im\Big\{\frac{z}{z_1}\Big\} = \frac{1}{2};\ \Re\Big\{\frac{z(1 \pm i)}{z_1}\Big\} = 0.$$

3.1.6 Parabola has the equation $|z - z_0| = |k - \Re\{\lambda z\}||\lambda|^{-1}$, $\lambda \neq 0$, $k \gtrless \Re\{\lambda z_0\}$. It has focus at $z = z_0$; directrix $\Re\{\lambda z\} = k$; latus rectum $2|k - \Re\{\lambda z_0\}||\lambda|^{-1}$; and vertex $z = z_0 + \dfrac{k - \Re\{\lambda z_0\}}{2\lambda|}$.

3.1.7 Cassini's ovals or Cassinians, presented in Figure 3.1, are defined by $|z-z_1||z-z_2| = \alpha$, $\alpha > 0$; its foci are z_1, z_2. For $\alpha = \frac{1}{4}|z_1 - z_2|^2 = 1$, the curve is a *lemniscate*.

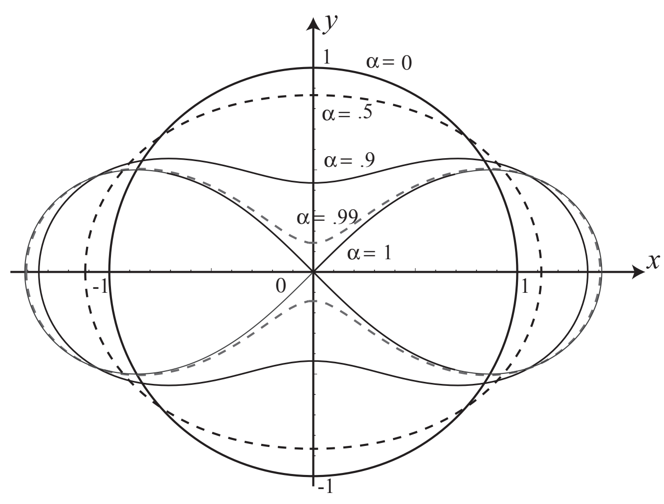

Figure 3.1 Cassini's ovals and lemniscate.

Cassini's ovals are plotted using the definition with rectangular coordinates:

$$F(x, y, \alpha) = \left[(x+\alpha)^2 + y^2\right]\left[(x-\alpha)^2 + y^2\right] = 1$$

for values of $\alpha = 0, 0.5, 0.9, 0.99$, and 1. For $\alpha = 0$ the curve becomes the unit circle. The region does not remain simply connected for $\alpha = 1$; it is a doubly connected region and its boundary curve is called the *lemniscate*.

3.1.8 Cardioid and Limaçons. The cardioid, defined by $r = 2a(1+\cos\theta)$, is the curve in Figure 3.2(a) that is described by a point P of a circle of radius a as it rolls on the outside of a fixed circle of radius a. It is a special case of the limaçon of Pascal, which has the equation $r = b + a\cos\theta$. As described in Figure 3.2(b), the line $0Q$ joining the origin to any point Q on a circle of diameter a passes through 0; then the limaçon is the locus of all points P such that $PQ = b$. Figure 3.2(b) represents the case when $2a > b$, and Figure 3.2(c) the case when $2a < b$. If $b \geq 2a$, the curve is convex. If $b = a$, the limaçon becomes the cardioid.

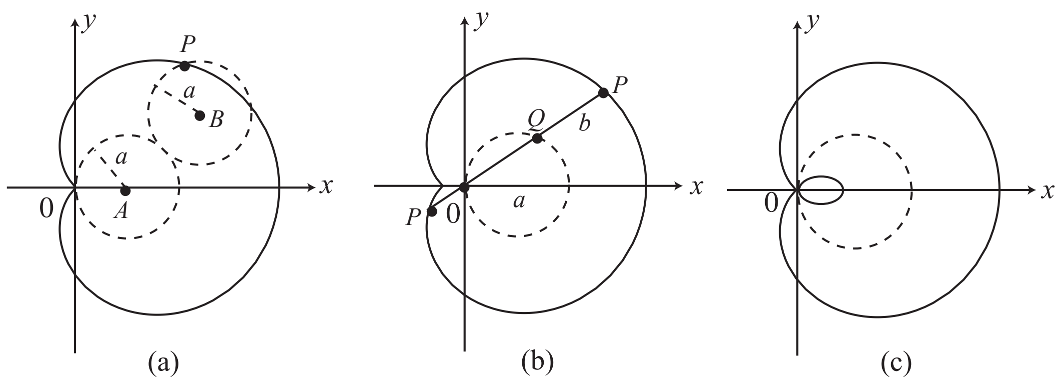

Figure 3.2 Cardioid and limaçons.

3.2 Bilinear Transformations

The *bilinear (linear-fractional, or Möbius) transformation* is of the form

$$w = f(z) = \frac{az+b}{cz+d}, \tag{3.2.1}$$

where a, b, c, d are complex constants such that $ad - bc \neq 0$ (otherwise the function $f(z)$ would be identically constant). If $c = 0$ and $d = 1$, or if $a = 0$, $d = 0$ and $b = c$, then the function (3.2.1) reduces to a linear transformation $w = az + b$, or an inversion $w = \dfrac{1}{z}$, respectively. The transformation (3.2.1) is also written as

$$w = \frac{a}{c} + \frac{bc - ad}{c(cz+d)}, \tag{3.2.2}$$

that can be viewed as composed of the following three successive functions:

$$z_1 = cz + d, \quad z_2 = \frac{1}{z_1}, \quad w = \frac{a}{c} + \frac{bc - ad}{c}\, z_2,$$

which shows that the mapping (3.2.1) is a linear transformation, followed by an inversion which is followed by another linear transformation. The bilinear transformation (3.2.1) maps the extended z-plane conformally onto the extended w-plane such that the pole at

$z = -d/c$ is mapped into the point $w = \infty$. The inverse transformation

$$z = f^{-1}(w) = \frac{b - dw}{-a + cw} \tag{3.2.3}$$

is also bilinear defined on the extended w-plane, and maps it conformally onto the extended z-plane such that the pole at $w = a/c$ is mapped into the point $z = \infty$. Note that $f'(z) = \dfrac{ad - bc}{(cz + d)^2} \neq 0$; also $[f^{-1}(w)]' = \dfrac{-ad + bc}{(cw - a)^2} \neq 0$. A bilinear transformation carries circles into circles (in the extended sense), as explained in §3.1.

Example 3.1. The bilinear transformation that maps the square of side 2, centered at the origin of the z-plane onto the parallelogram shown in Figure 3.3 is given by $w = \dfrac{az + b}{cz + d}$ such that the points $1 + i$, $1 - i$, $-1 - i$, and $-1 + i$ are mapped onto the points $1 - 2i$, $1 + 2i$, $\dfrac{1 + 2i}{5}$, and $-\dfrac{1 + 2i}{5}$, respectively, as shown in Figure 3.3. ■

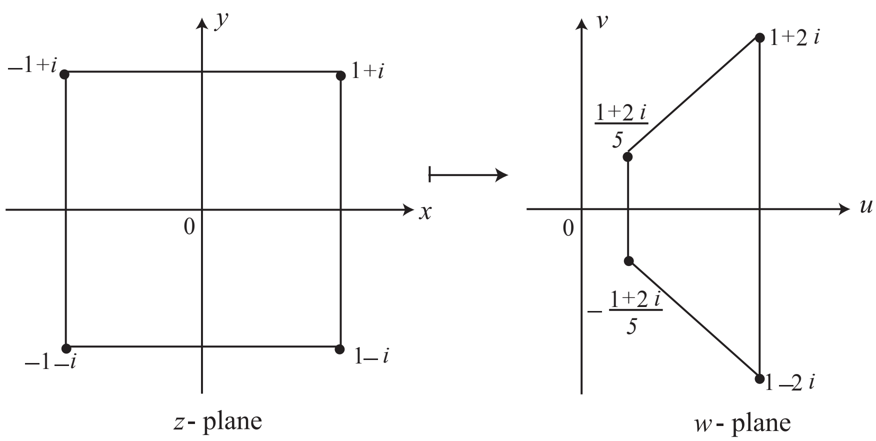

Figure 3.3 Conformal and isogonal mappings.

Example 3.2. The transformation that maps the half-plane $\Im\{z\} > \Re\{z\}$ onto the interior of the circle $|w - 1| = 3$ is

$$w = \frac{(1 + 3i)z + 4i}{z + i}.$$

To prove it, let us regard the given half-plane as the 'interior' of the 'circle' through ∞ defined by the line $\Im\{z\} = \Re\{z\}$. The three points in clockwise order are $z_1 = \infty, z_2 = 0$, and $z_3 = -1-i$. The three points on the circle $|w-1| = 3$ in clockwise order are $w_1 = 1+3i$, $w_2 = 4$, and $w_3 = -2$. The linear fractional transformation that maps three points in order is defined by Eq (3.2.1), where from the images of ∞ and 0 we have $a = 1 + 3i$ and $b/c = 4$, i.e., $b = 4i$ and $c = i$. Substituting these into Eq (3.2.1), we get the desired mapping.

Since $z_0 = i$ is in the half-plane $\Im\{z\} > \Re\{z\}$, its image under the above map is $w_0 = 2.5 + 1.5\,i$; thus, as $w_0 - 1 = 1.5 + 1.5\,i$ has a modulus $1.5 \times \sqrt{2} \approx 2.12132 < 3$, we find that w_0 is in the interior of $|w - 1| = 3$ (see Figure 3.4). ■

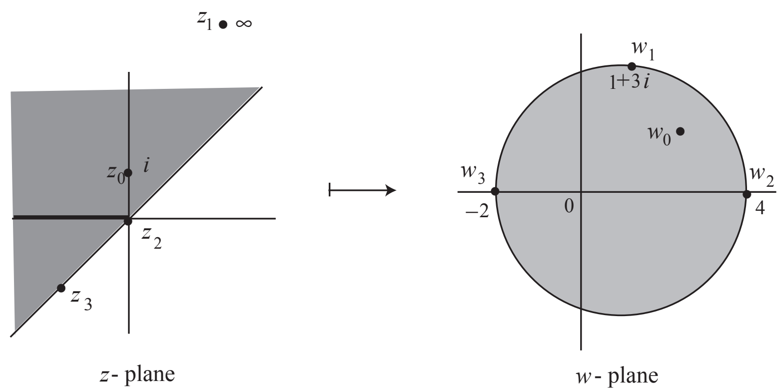

Figure 3.4 $w = \dfrac{(1+3i)z+4i}{z+i}$.

3.2.1 Fixed Points. The fixed points of a mapping are those points at which $w = z$. For example, the mapping $w = z^3$ has a fixed point $z = w = 1$. Let F_1 and F_2 be the *fixed points* of the bilinear transformation (3.2.1), $ad - bc \neq 0$. Then F_1 and F_2 are the roots of the equation

$$cF^2 + (d-a)F - b = 0.$$

We will ignore the *identical transformation* $w = z$, and consider the following cases:

(i) If $F_1 = F_2 = \infty$, we have $w = z + b$, $b \neq 0$, p, q real. Then (i) any line $\Im\{\bar{b}z\} = p$ is mapped onto itself, but onto no other 'circle' (in the extended sense); and (ii) the line $\Re\{\bar{b}z\} = q$ is mapped onto $\Re\{\bar{b}w\} = q - |b|^2$.

(ii) If $F_1 = F_2 \neq \infty$, we have

$$\frac{1}{w-F} = \frac{1}{z-F} + k,$$

where

$$c \neq 0,\ F = \frac{a-d}{2c} = \frac{2b}{d-a},\ k = \frac{2c}{a+d}.$$

The transformation is 'parabolic'. The straight line $\Im\{k(z-r)\} = 0$ and every circle touching this straight line at F is mapped onto itself; no other circle has this property. The set of 'circles' orthogonal to the above set is, as a whole, mapped onto itself.

(iii) If $F_1 \neq F_2 = \infty$, we have $w = F_1 = \alpha\,(z - F_1)$, where $\alpha = a/d$, $\alpha \neq 0, 1$, $c = 0$, and $F_1 = \dfrac{b}{d-a}$. Let D denote the set of concentric circles centered at F_1, and E denote the set of straight lines through F_1. Then the entire set D is mapped onto itself; so is E. None of the 'circles' other than those stated below is mapped onto itself:

(a) α real, but $\alpha \neq -1$: then every line of E is mapped onto itself;

(b) $\alpha = -1$: then every circle of D, and every line of E, is mapped onto itself;

(c) $|\alpha| = 1$, but $\alpha \neq -1$: then every circle of D is mapped onto itself;

(d) α not real, $|\alpha| \neq 1$: then no circle or line is mapped onto itself (this is known as the *loxodromic* transformation).

(iv) If $F_1 \neq F_2$, $F_1 \neq \infty$, $F_2 \neq \infty$, we have

$$\frac{w - F_1}{w - F_2} = \alpha\, \frac{z - F_1}{z - F_2},$$

where

$$\alpha = \frac{cF_2 + d}{cF_1 + d} = \frac{\left(a + d - \sqrt{(a-d)^2 + 4bc}\right)^2}{4(ad - bc)}.$$

Let E denote a pencil of 'circles' through F_1 and F_2, and let D denote the set of 'circles' orthogonal to the circles of E (co-axial system). Then the results are the same as in (iii) above, but replace circle or straight line there by 'circle'.

The fixed points should, however, not be confused with the *involutory* transformation for which the mapping $w = f(z)$ has the same form as the mapping $z = f^{-1}(w)$. An example is the mapping $w = \dfrac{1}{z}$, or the mapping $w = \dfrac{z+1}{z-1}$.

3.2.2 Linear Transformation. $w = az + b$, $a \neq 0$. The special cases are as follows:

(a) $w = az$: (i) if $a > 0$, then there is *magnification* by ratio a centered at 0; (ii) if $a = e^{i\tau}$, τ real, then there is *rotation* by the angle τ about 0; and (iii) if $a = m\, e^{i\tau}$, $m > 0$, and τ real, then there is *rotation* by the angle τ about 0, followed by a *magnification* of ratio m centered at 0.

(b) $w = z + b$: There is a *translation* by the vector $\overline{0b}$.

(c) $w = az + b$, $a \neq 1$: Then $F_0 = \dfrac{b}{1-a}, a = me^{i\tau}, m > 0, \tau$ real. There is *rotation* about F_0 by the angle τ, followed by *magnification* by ratio m centered at F_0.

The correspondence of the above linear transformations between the z-plane and the w-plane is as follows:

(i) The line $x = p$ is mapped onto the line $\Re\{\dfrac{w-b}{a}\} = p$.

(ii) The line $y = q$ is mapped onto the line $\Im\{\dfrac{w-b}{a}\} = q$.

(iii) The line $lx + my = p$ is mapped onto the line $\Re\{(l - im)\dfrac{w-b}{a}\} = p$.

(iv) The line $\Re\{az + b\} = p$ is mapped onto the line $u = P$.

(v) The line $\Im\{az + b\} = q$ is mapped onto the line $v = Q$.

(vi) The line $\Re\{(l - im)(az + b) = p$ is mapped onto the line $lu + mv = P$.

(vii) The circle $|z - z_0| = r$ is mapped onto the circle $|w - (aw_0 + b)\,| = |a|R$.

(vii) The circle $\left|z - \dfrac{w_0 - b}{a}\right| = \dfrac{r}{|a|}$ is mapped onto the circle $|w - w_0| = R$.

3.2.3 Composition of Bilinear Transformations. We will consider the general bilinear transformation (3.2.1) with $ad - bc \neq 0$. Note that if $ad - bc = 0$, then we have $w =$ const. The general bilinear transformation (3.2.1) consists of the following three successive transformations:

(i) translation of z: $z_1 = z + d/c$;

(ii) inversion of z_1: $z_2 = 1/z_1$; and

(iii) translation, rotation, and scale change of z_2: $w = \dfrac{a}{c} - \dfrac{ad - bc}{c^2} z_2$.

Map 3.1. If $a = -d$, the transformation is *involutory*, i.e., w and z may be interchanged. The magnification is $\left|\dfrac{dw}{dz}\right| = \left|\dfrac{ad - bc}{(cz + d)^2}\right|$, or

$$|f'(z)|^2 = \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x}\Big)^2 = \Big(\frac{\partial u}{\partial y}\Big)^2 + \Big(\frac{\partial v}{\partial y}\Big)^2.$$

The scale or magnification factor m is then defined as $m = |dw/dz| = |f'(z)|$, where $m = |m|$, and $M = 1/m$. The critical points are $z = -d/c$ and $z = \infty$; at these points the transformation is *not* conformal in the usual sense. The details of the transformation are as follows:

(i) The points $z = 0;\ -b/a;\ z_\infty = -d/c$ are mapped onto the points $w = b/d;\ 0;\ \infty$.

(ii) The point at infinity $z = \infty$ is mapped onto the point $w_\infty = a/c$.

Map 3.2. The bilinear transformation $w = f(z)$ that transforms the three points z_k into the points w_k, $k = 1, 2, 3$, can be expressed as (Saff and Snider [1976:329])

$$\begin{vmatrix} 1 & z & w & zw \\ 1 & z_1 & w_1 & z_1 w_1 \\ 1 & z_2 & w_2 & z_2 w_2 \\ 1 & z_3 & w_3 & z_3 w_3 \end{vmatrix} = 0.$$

Map 3.3. Let z be fixed and let $\Re\{z\} \ge 0$. Define a sequence of bilinear transformations

$$T_0(w) = \frac{a_0}{z + a_0 + b_1 + w}, \quad T_k(w) = \frac{a_k}{z + b_{k+1} + w}, \quad k = 1, \dots, n-1,$$

such that each $a_j > 0$ is real, and each b_j is zero or purely imaginary for $j = 0, 1, 2, \dots, n-1$. We use induction to prove that the chain of transformations defined by $\zeta = S(w) = (T_0 \circ \cdots \circ T_{n-2} \circ T_{n-1})(w)$ maps the half-plane $\Re\{w\} > 0$ onto a region contained in the disk $|\zeta - 1/2| < 1/2$. Thus, using the *Wallis criterion*, if $P(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n$ is a polynomial of degree $n > 0$ with complex coefficients $c_k = p_k + i\, q_k$, $k = 1, 2, \dots, n$, if $Q(z) = p_1 z^{n-1} + i\, q_2 z^{n-2} + p_3 z^{n-3} + i\, q_4 z^{n-4} + \cdots$, and if $Q(z)/P(z)$ can be written as a continued fraction, then we have $Q(z)/P(z) = (T_0 \circ T_{n-2} \cdots \circ T_{n-1})(z)$ (Saff and Snider [1976:330]).

Theorem 3.1. (Schwarzian derivative) (i) *The Schwarzian derivative (also called the Schwarz differential operator), defined by*

$$\{w, z\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2,$$

is invariant under bilinear transformations. (Nehari [1952:199]; Wen [1992:76]).

(ii) *The Schwarzian derivative of the function $w = w(z)$ that maps the upper half-plane $\Im\{z\} > 0$ conformally onto a circular triangle with interior angles $\alpha\pi$, $\beta\pi$ and $\gamma\pi$ is given by*

$$\{w, z\} = \frac{1-\alpha^2}{2z^2} + \frac{1-\beta^2}{2(1-z)^2} + \frac{1-\alpha^2-\beta^2+\gamma^2}{2z(1-z)}.$$

(Nevanlinna and Paatero [1969:342]).

3.3 Cross-Ratio

A cross-ratio between four distinct finite points z_1, z_2, z_3, z_4 is defined by

$$(z_1, z_2, z_3, z_4) = \frac{z_1-z_2}{z_1-z_4} \cdot \frac{z_3-z_4}{z_3-z_2}. \tag{3.3.1}$$

If z_2, z_3, or z_4 is a point at infinity, then (3.3.1) reduces to

$$\frac{z_3-z_4}{z_1-z_4}, \quad \frac{z_1-z_2}{z_1-z_4}, \quad \text{or} \quad \frac{z_1-z_2}{z_3-z_2},$$

respectively. The cross-ratio (z, z_1, z_2, z_3) is invariant under bilinear transformations.

Theorem 3.2. *A bilinear transformation is uniquely defined by a correspondence of the cross-ratios, i.e.,*

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)\,, \tag{3.3.2}$$

which maps any three distinct points z_1, z_2, z_3 in the extended z-plane into three prescribed points w_1, w_2, w_3 in the extended w-plane. The cross-ratio (z, z_1, z_2, z_3) is the image of z under a bilinear transformation that maps three distinct points z_1, z_2, z_3 into $0, 1, \infty$.

3.3.1 Symmetric Points.

The points z and z^* are said to be *symmetric* with respect to a circle C (in the extended sense) through three distinct points z_1, z_2, z_3 iff

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}. \tag{3.3.3}$$

The mapping that carries z into z^* is called a *reflection* with respect to C. Two reflections obviously yield a bilinear transformation.

If C is a straight line, then we choose $z_3 = \infty$, and the condition for symmetry (3.3.3) gives

$$\frac{z^*-z_1}{z_1-z_2} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_1-\bar{z}_2}. \tag{3.3.4}$$

Let z_2 be any finite point on the line C. Then, since $|z^*-z_1| = |z-z_1|$, the points z and z^* are equidistant from the line C. Moreover, since $\Im\{\frac{z^*-z_1}{z_1-z_2}\} = -\Im\{\frac{z-z_1}{z_1-z_2}\}$, the line C is the perpendicular bisector of the line segment joining z and z^*. If C is the circle $|z-a| = R$, then

$$\begin{aligned}
\overline{(z, z_1, z_2, z_3)} &= \overline{(z-a, z_1-a, z_2-a, z_3-a)} \\
&= \left(\bar{z}-\bar{a}, \frac{R^2}{z_1-a}, \frac{R^2}{z_2-a}, \frac{R^2}{z_3-a}\right) \\
&= \left(\frac{R^2}{\bar{z}-\bar{a}}, z_1-a, z_2-a, z_3-a\right) \\
&= \left(\frac{R^2}{\bar{z}-\bar{a}} + a, z_1, z_2, z_3\right).
\end{aligned}$$

Hence, in view of (3.3.3), we find that the points z and $z^* = \dfrac{R^2}{\bar{z} - \bar{a}} + a$ are symmetric, i.e.,

$$(z^* - a)(\bar{z} - \bar{a}) = R^2. \tag{3.3.5}$$

Note that $|z^* - a||z - a| = R^2$; also, since $\dfrac{z^* - a}{z - a} > 0$, the points z and z^* are on the same ray from the point a (Figure 3.5). Also, the point symmetric to a is ∞.

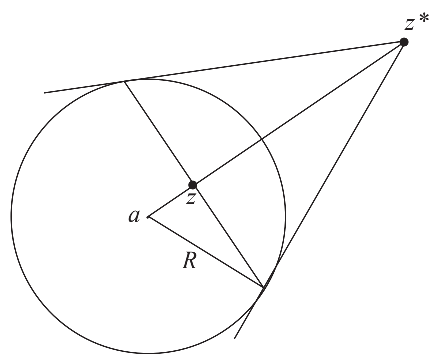

Figure 3.5 Symmetry with respect to a circle.

A GENERALIZATION of this result is as follows: If Γ denotes an analytic Jordan curve with parametric equation $z = \gamma(s)$, $s_1 < s < s_2$, then for any point z *sufficiently close* to Γ, the point

$$z^* = \gamma\, \overline{(\gamma^{-1}(z))} \tag{3.3.6}$$

defines a symmetric point of z with respect to Γ (Sansone and Gerretsen, [1960: 103]; Papamichael, Warby and Hough [1986]). Some examples are as follows:
(i) If Γ is the circle $x^2 + y^2 = a^2/9$, then $z^* = a^2/9\bar{z}$.
(ii) If Γ is the ellipse $\dfrac{(x + a/2)^2}{a^2} + y^2 = 1$, then

$$z^* = -\frac{a}{2} + \frac{(a^2+1)(\bar{z} + a/2) + 2ia\sqrt{a^1 - 1 - (\bar{z} + a/2)}}{a^2 - a}.$$

(iii) If Γ_1 is a cardioid defined by $z = \gamma(s) = \left(\dfrac{1}{2} + \cos\dfrac{s}{2}\right) e^{is}$, $-\pi < s \le \pi$, then from (3.3.6) we cannot write an explicit expression for symmetric points with respect to Γ_1. However, for any real t,

$$\gamma\,(\pm\, it) = \left(\frac{1}{2} + \cosh\frac{t}{2}\right) e^{\mp t}$$

defines two real symmetric points with respect to Γ_1, provided the parameter t satisfies the equation $\gamma(it)\,\gamma(-it) = a^2$, i.e., $\dfrac{1}{2} + \cosh\dfrac{t}{2} = a$ which has the roots

$$t = 2\,\cosh^{-1}(a - 1/2) = \pm 2\,\log\,\rho, \tag{3.3.7}$$

where $\rho = a - \frac{1}{2} + \sqrt{\left(a - \frac{1}{2}\right)^2 - 1}$.

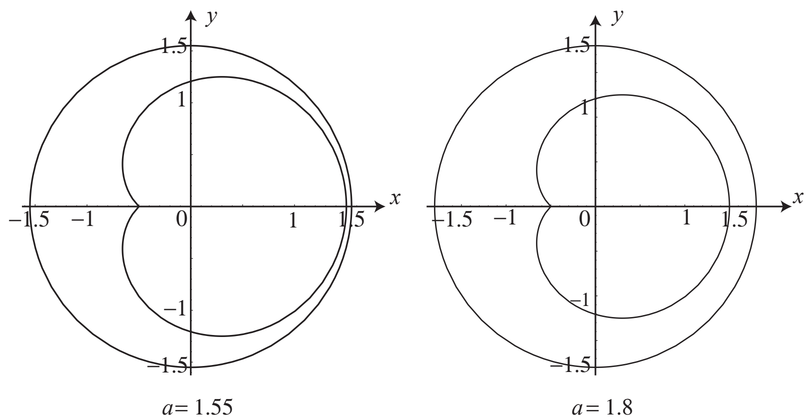

Figure 3.6 Cardioid inside a circle.

(iv) If a doubly connected region Ω is bounded outside by the circle $\Gamma_2 = \{z : |z| = a,\ a > 1.5\}$ and inside by the cardioid Γ_1 defined above in (iii) (see Figure 3.6), then it follows from (3.3.7) that there is one pair of real common symmetric points $\zeta_1 \in \operatorname{Int}(\Gamma_1)$ and $\zeta_2 \in \operatorname{Ext}(\Gamma_2)$ such that $\zeta_1 = a/\rho^2$ and $\zeta_2 = a\rho^2$, where ρ is defined in (3.3.7).

3.3.2 Symmetry Principle. The *symmetry principle* states that if a bilinear transformation maps a circle C_1 onto a circle C_2, then it maps any pair of symmetric points with respect to C_1 into a pair of symmetric points with respect to C_2. This means that bilinear transformations preserve symmetry.

A practical application of the symmetry principle is to find bilinear transformations which map a circle C_1 onto a circle C_2. We already know that the transformation (3.3.2) can always be determined by requiring that the three points $z_1, z_2, z_3 \in C_1$ map onto three points $w_1, w_2, w_3 \in C_2$. But a bilinear transformation is also determined if a point $z_1 \in C_1$ should map into a point $w_1 \in C_2$ and a point $z_2 \notin C_1$ should map into a point $w_2 \notin C_2$. Then, by the symmetry principle, the point z_2^* which is symmetric to z_2 with respect to C_1 is mapped into the point w_2^* which is symmetric to w_2 with respect to C_2, and then the desired bilinear transformation is given by

$$(w, w_1, w_2, w_2^*) = (z, z_1, z_2, z_2^*)\,. \tag{3.3.8}$$

Let U^+ denote the region $|z| < 1$, U^- the region $|z| > 1$, and let $C = \{|z| = 1\}$ be their common boundary (Figure 3.7). Let $f(z)$ be a function defined on U^+. Then this function can be related to the function $f^*(z)$ defined in U^- in the same manner as in (2.6.3) for half-planes, except that now the conjugate complex points are replaced by points inverse with respect to the circle C according to the relation (3.3.5). Thus,

$$f^*(z) = \overline{f\left(\frac{1}{\bar z}\right)} = \bar f\left(\frac{1}{z}\right). \tag{3.3.9}$$

This relation is symmetrical, i.e.,

$$f(z) = \overline{f^*\left(\frac{1}{\bar z}\right)}, \quad (f^*(z))^* = f(z). \tag{3.3.10}$$

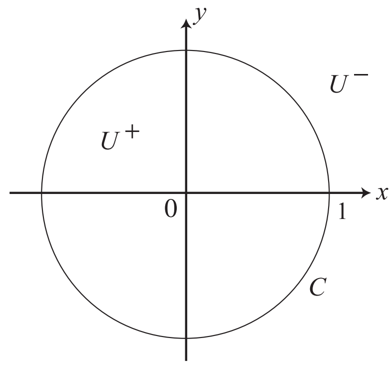

Figure 3.7 Unit disk.

If $f(z)$ is regular or meromorphic in U^+, then $f^*(z)$ is regular or meromorphic in U^-. Also, if $f(z)$ is a rational function defined by (2.6.4), then

$$f^*(z) = \frac{\bar{a}_n\, z^{-n} + \bar{a}_{n-1}\, z^{-n+1} + \cdots + \bar{a}_0}{\bar{b}_m\, z^{-m} + \bar{b}_{-m+1}\, z^{m-1} + \cdots + \bar{b}_0}. \tag{3.3.11}$$

Moreover, if $f(z)$ has the power series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k\, z^k \quad z \in U^+,$$

then

$$f^*(z) = \sum_{k=-\infty}^{\infty} \bar{a}_k\, z^{-k}, \quad z \in U^-. \tag{3.3.12}$$

If $f(z)$ has a zero (pole) of order k at $z = \infty$ ($z = 0$), so does $f^*(z)$. Let us assume that $f(z)$ approaches a definite limit value $f^+(t)$ as $z \to t \in C$ from U^+. Then $f^{*-}(t)$ exists and

$$f^{*-}(t) = \bar{f}^-\left(\frac{1}{t}\right) = \overline{f^+(t)}, \tag{3.3.13}$$

because $1/\bar{z} \to t$ from U^+ as $z \to t$ from U^-, and hence $f^*(z) = \bar{f}\left(\frac{1}{z}\right) = \overline{f\left(\frac{1}{\bar{z}}\right)} \to \overline{f^+(t)}$. If $f(z)$ is regular in U^+ except possibly at infinity and continuous on C from the left, then let $F(z)$ be sectionally regular and be defined by

$$F(z) = \begin{cases} f(z) & \text{for } z \in U^+, \\ f^*(z) & \text{for } z \in U^-. \end{cases} \tag{3.3.14}$$

Then, $F^*(z) = F(z)$, and, as in (2.6.7),

$$F^-(t) = \overline{F^+(t)}, \quad F^+(t) = \overline{F^-(t)}. \tag{3.3.15}$$

Moreover, in view of the Schwarz reflection principle, if $\Im\{f^+(t)\} = 0$ on some part of the circle C, then $f^*(z)$ is the analytic continuation of $f(z)$ through this part of C.

3.3.3 Special Cases. We will discuss the following special cases of the transformations (3.2.1) and (3.2.3).

Map 3.4. (Linear transformation $w = az + b$, where $a = a_1 + ia_2$ and $b = b_1 + ib_2$. Then $w = u + iv = (\alpha_1 + ia_2)(x + iy) + (b_1 + b_2) = (a_1x - a_2y + b_1) + i(a_1y + a_2x + b_2)$. The magnification factor and the clockwise rotation are

$$m = \frac{|dw|}{|dx|} = |a| = \sqrt{a_1^2 + a_2^2}, \tag{3.3.16}$$

$$\alpha = \arg\left\{\frac{dw}{dz}\right\} = \arg\{m\} = \arctan\left(\frac{a_2}{a_1}\right). \tag{3.3.17}$$

Let $a = 1 + i$ and $b = -\frac{1}{2} - i\frac{3}{2}$. Then $m = \sqrt{2}$ and $\alpha = \pi/4$. The rectangular lines $u = k_u$ and $v = v_k$ are represented in the z-plane by the rectangular lines $y = x - \left(\frac{1}{2} + k_u\right)$ and $y = -x + \left(\frac{3}{2} + k_v\right)$, respectively. The transformation is presented in Figure 3.8.

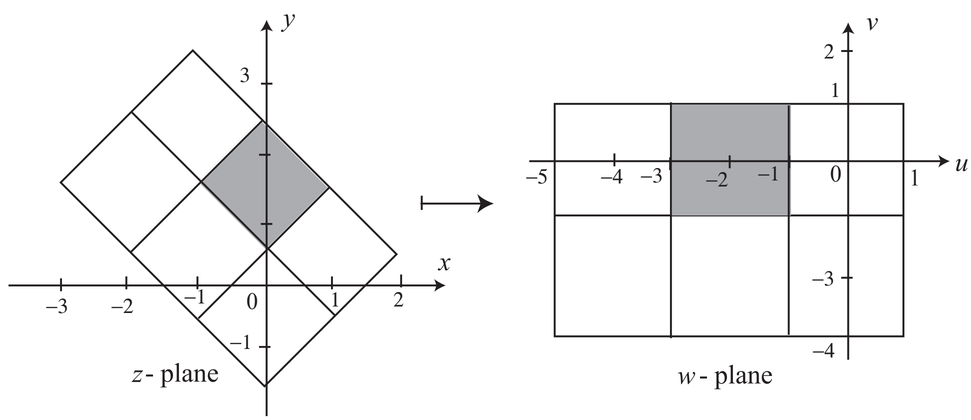

Figure 3.8 Linear transformation $w = az + b$.

In general, the mapping $w = az + b$ translates by b, rotates by $\arg\{a\}$ and magnifies (or contracts) by $|a|$.

Example 3.3. Consider the transformation $w = (1+i)z + (1+2i) = \sqrt{2}\, e^{i\pi/4}z + 1 + 2i$ in the z-plane. It maps the rectangle bounded by the lines AB: $x = 0$; BC: $y = 0$; CD: $x = 2$; and DA: $y = 1$ in the z-plane onto the rectangle bounded by the lines A'B': $u + v = 3$; B'C': $u - v = -1$; C'D': $u + v = 7$; and D'A': $u - v = -3$, respectively, such that the points $z = (0, 1), (0, 0), (2, 0), (2, 2)$ are mapped onto the points $w = (0, 3), (1, 2), (3, 4), (2, 5)$, respectively. The rectangle ABCD is translated by $(1 + 2i)$, rotated by an angle $\pi/4$ in the counterclockwise direction, and contracted by $\sqrt{2}$. ■

Map 3.5. The transformation $w = 2z - 2i$ maps the circle $|z - i| = 1$ onto the circle $|w| = 2$. The transformation that maps the disk $|z - i| = 1$ onto the exterior of the circle $|w| = 2$ is $w = \dfrac{2}{z - i}$. In fact, let $w = g(z) = \dfrac{az + b}{cz + d}$. Choose $g(i) = \infty$, so $g(z) = \dfrac{az + b}{z - i}$ without loss of generality. Take three points on the circle $|z - i| = 1$ as $0, 1 + i$, and $2i$; then $g(0) = ib, g(1 + i) = a(1 + i)$, and $g(2i) = 2a - ib$. Since all these three points must lie on $|w| = 2$, the simplest choice is $a = 0$ and $b = 2$. Then $g(z) = \dfrac{2}{z - i}$.

Map 3.6. The inversion $w = \dfrac{1}{z}$ is an involutory transformation with fixed points $F_1 = 1$ and $F_2 = -1$. The mapping from z-plane to w-plane, or inversely, is presented in Figure 3.9, where the radius $\overline{0A}$ is 1; the angle $\angle z0\bar{z}$ is bisected by the real axis; $a\bar{z} : \bar{z}B = Aw : Bw$; p, q and r are real; and the points $\bar{z}$ and w are inverse with respect to the unit circle.

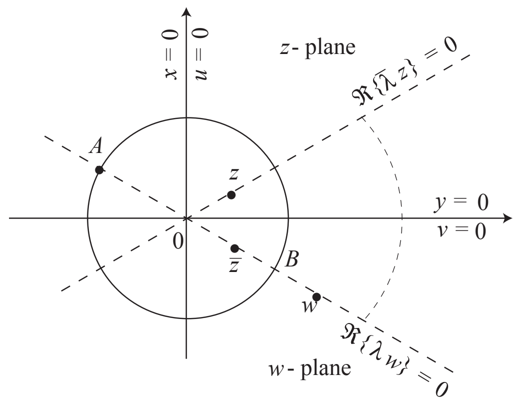

Figure 3.9 Transformation $w = \dfrac{1}{z}$.

There are five special cases under the inversion $w = \dfrac{1}{z}$. They are:

(a) The interior (exterior) of the circle of D in the z-plane is mapped onto the interior (exterior) of the same circle, with points $z_\infty = 0; \infty; i; -i$ in the z-plane mapped into the points $\infty; w_\infty - 0; -i; i$, respectively, in the w-plane.

(b) The line $\Re\{\bar{\lambda} z\} = 0$ in the z-plane is mapped onto the line $\Re\{\lambda w\} = 0$ in the w-plane; and the circle $|z| = r$ in the z-plane is mapped onto the circle $|w| = 1/|z|$ in the w-plane. These two cases are geometrically obvious and no figures are needed for them.

The remaining three cases are presented in Figure 3.10.; details, with $z = p, q, r, r_p, r_q$ real, and $w = P, Q, R, R_p, R_q$ real, are as follows:

(c) The line $\Re\{\bar{\lambda} z\} = 0$ is mapped onto the circle

$$\left| w - \frac{\bar{\lambda}}{2p} \right| = \left| \frac{\lambda}{2p} \right|.$$

See Figure 3.10(c).

(d) The circle $|z - z_0| = |z_0| \neq 0$ is mapped onto the line $\Re\{z_0\} = \frac{1}{2}$.

See Figure 3.10(d).

(e) The circle $|z - z_0| = r \neq |z_0|$ is mapped onto the circle

$$\left| w - \frac{\bar{z}_0}{|z_0|^2 - r^2} \right| = \frac{r}{\big||z_0|^2 - r^2\big|}.$$

See Figure 3.10(e).

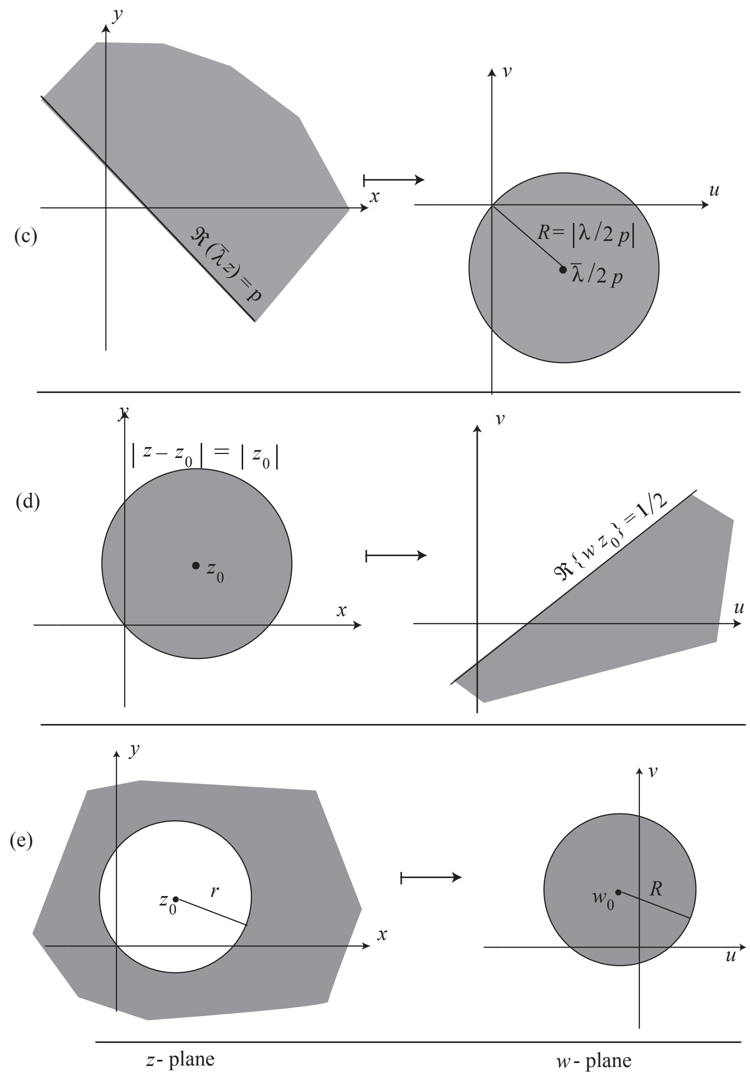

Figure 3.10 Transformation $w = \dfrac{1}{z}$.

Map 3.7. The transformation $w = \dfrac{1}{z}$, or $z = \dfrac{1}{w}$, which maps the interior of a circle in the z-plane onto the exterior of the circle in the w-plane and conversely, is shown in Figure 3.11.

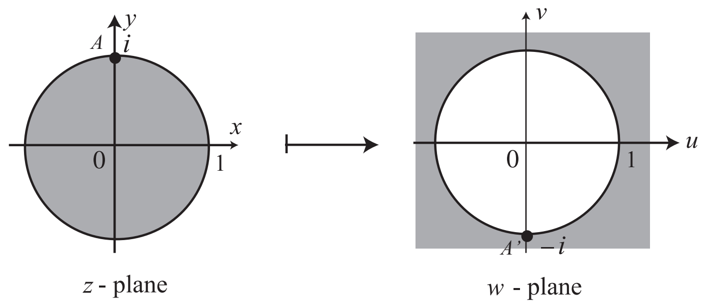

Figure 3.11 Circle in the z-plane onto the exterior of a circle.

Since

$$w = u + iv = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}\, i, \quad z = x + iy = \frac{u}{u^2 + v^2} - \frac{v}{u^2 + v^2}\, i,$$

we get

$$u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}.$$

A line through the origin in the z-plane is mapped onto a line through the origin in the w-plane. However, a line not passing through the origin in the z-plane is mapped onto a circle in the w-plane which passes through the origin, as the following example shows.

Example 3.4. The equation $ax + by + c$, $c \neq 0$ represents a straight line in the z-plane. Under the inversion map $w = 1/z$, this line is mapped onto a circle in the w-plane which passes through the origin. The details are as follows: The mapped curve is

$$\frac{au}{u^2 + v^2} - \frac{bu}{u^2 + v^2} + c = 0.$$

Thus, $au - bv + c\left(u^2 + v^2\right) = 0$. Dividing by c, we obtain

$$u^2 + v^2 + \frac{a}{c} u = \frac{b}{c} v = 0,$$

which is the equation of the required circle. ■

Map 3.8. The transformation $w = \dfrac{1}{z}$, which maps the interior of the circle $|z - 1| = 1$ in the z-plane onto the right plane $u \geq \frac{1}{2}$, $-\infty < v < \infty$ in the w-plane, is shown in Figure 3.12.

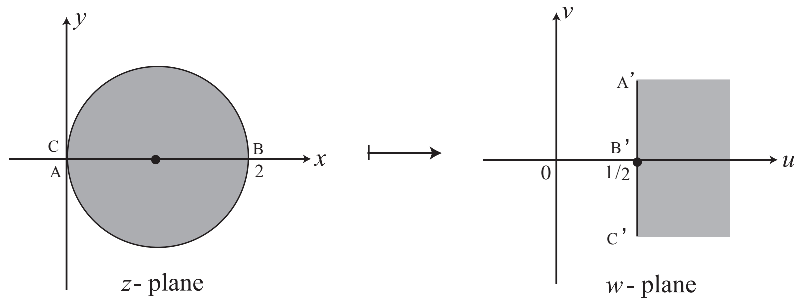

Figure 3.12 Circle in the z-plane onto a rectangle.

Map 3.9. The transformation that maps the exterior of the unit circle in the z-plane onto the interior of the unit circle in the w-plane is defined by

$$w = 1/z,$$

and is presented in Figure 3.13.

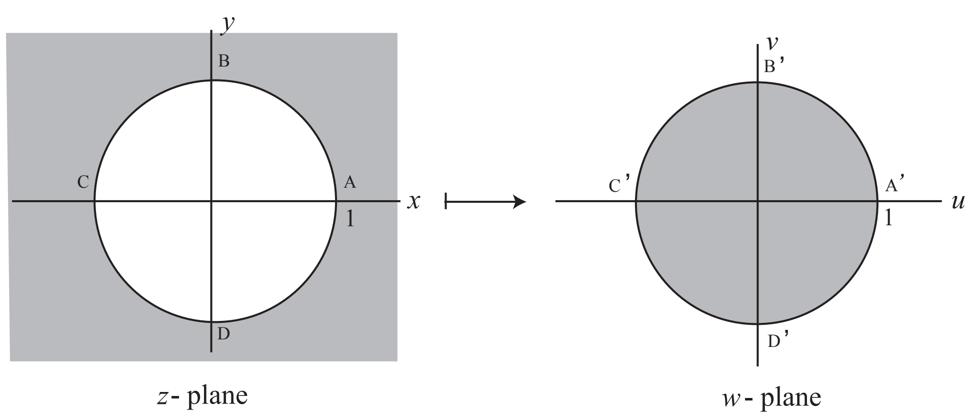

Figure 3.13 Exterior of the unit circle onto the interior of the unit circle.

Map 3.10. The transformation $w = k/z$ is known as the *reciprocal* or the *inverse transformation* for real k. We have $w = R\,e^{i\Theta} = k\left(r\,e^{i\theta}\right)^{-1} = \dfrac{k}{r}\,e^{-i\theta}$, which gives $R = k/r$, and $\Theta = -\theta$. The fixed points are: $F_{1,2} = \pm\sqrt{k}$, and the critical points are $z = 0, \infty$. It is a useful transformation in electric transmission through a single wire, discussed in Example 22.4 with $k = 3600$. Two particular cases for $k = -\alpha$, $\alpha > 0$, and $k = 1$ are given in Maps 3.6, 3.7, 3.8, and 3.9, respectively.

Map 3.11. The transformation $w = a/z$ maps the exterior of a circle of radius a: $x^2 + y^2 \geq a^2$, in the z-plane onto the unit disk $|w| \leq 1$ in the w-plane.

Map 3.12. The transformation $w = z + \dfrac{1}{z}$, which maps the exterior of the upper semi-circle in the z-plane onto the upper-half of the w-plane, is presented in Figure 3.14.

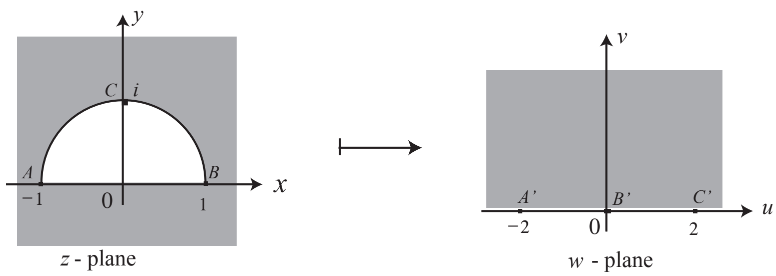

Figure 3.14 Exterior of a circle in the z-plane onto the upper-half of the w-plane.

Map 3.13. The transformation $w = z+\dfrac{1}{z}$, or $z = \dfrac{w+\sqrt{w^2-1}}{2}$, which maps the interior of the upper semi-circle in the z-plane onto the lower-half of the w-plane, is presented in Figure 3.15.

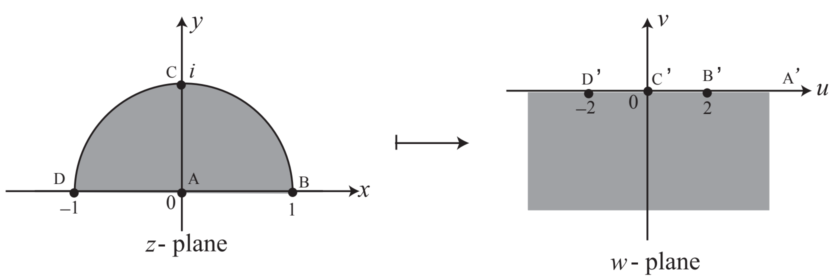

Figure 3.15 Interior of the upper half circle onto the lower-half of the w-plane.

Map 3.14. The transformation $w = z + \dfrac{1}{z}$, which maps the annular region between two upper semi-circles in the z-plane onto the upper-half of the ellipse in the w-plane, where the ellipse (B'C'D') is defined by

$$\left(\frac{ku}{k^2+1}\right)^2 + \left(\frac{kv}{k^2-1}\right)^2 = 1,$$

is presented in Figure 3.16.

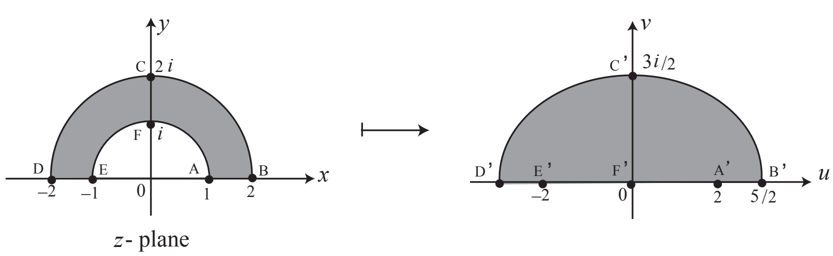

Figure 3.16 Annulus between two semi-circles onto the upper-half of the ellipse.

Map 3.15. If $a = c = 1, d = -b = \alpha$, then $w = \dfrac{z-\alpha}{z+\alpha}$. For this bilinear transformation the above three successive cases are presented in Figure 3.17; these three cases may be treated as separate transformations (see cases 3.11 and 3.44 also).

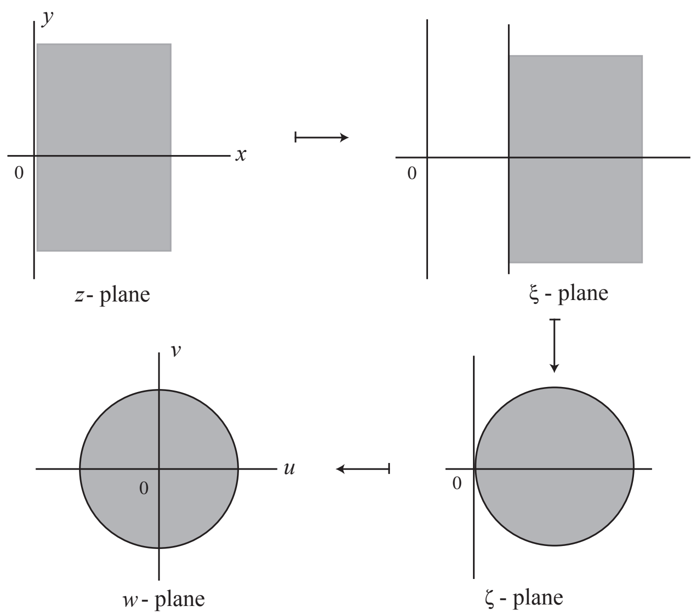

Figure 3.17 $w = \dfrac{z-\alpha}{z+\alpha}$.

In Figure 3.17, we start in the z-plane and translate the right half-plane by α in the ξ-plane so that $\xi = z + \alpha$; the right half-plane in the ξ-plane is mapped onto the circle in the ζ-plane by $\zeta = 1/\xi = 1/(z+\alpha)$; finally, the circle in the ζ-plane is mapped onto the circle in the w-plane so that $w = 1 + 2\alpha\zeta = 1 - \dfrac{2\alpha}{z+\alpha} = \dfrac{z-\alpha}{z\alpha}$.

We can also select suitable parameters for this transformation; for example, given two circles, marked I with radius r' and marked II with radius r'', such that the distance between their centers is h, and the distance of their centers as c_1 and c_2 from the origin of the coordinates (see Figure 3.18), we have the case of translation along the real axis, since

$$w = \frac{z-\alpha}{z+\alpha} = \frac{x+iy-\alpha}{x+iy+\alpha} = Re^{i\Theta}, \tag{3.3.18}$$

where

$$R = \left|\frac{z-\alpha}{z+\alpha}\right|, \quad R^2 = \frac{(x-\alpha)^2+y^2}{(x+\alpha)^2+y^2} = \frac{x^2+y^2+\alpha^2-2\alpha x}{x^2+y^2+\alpha^2+2\alpha x}. \tag{3.3.19}$$

The equation of circle I from Figure 3.18 is $(x-c_1)^2 + (y^2 = (r')^2$, or $x^2+y^2 = (r')^2 - c_1^2 + 2xc_1$. Then the radius R' of the map of circle I is

$$R'^2 = \frac{r'^2 - c_1^2 + 2xc_1 + \alpha^2 - 2\alpha x}{r'^2 - c_1^2 + 2xc_1 + \alpha^2 + 2\alpha x} = \frac{r'^2 - c_1^2 + \alpha^2 + 2x(c_1-\alpha)}{r'^2 - c_1^2 + \alpha^2 + 2x(c_1+\alpha)}. \tag{3.3.20}$$

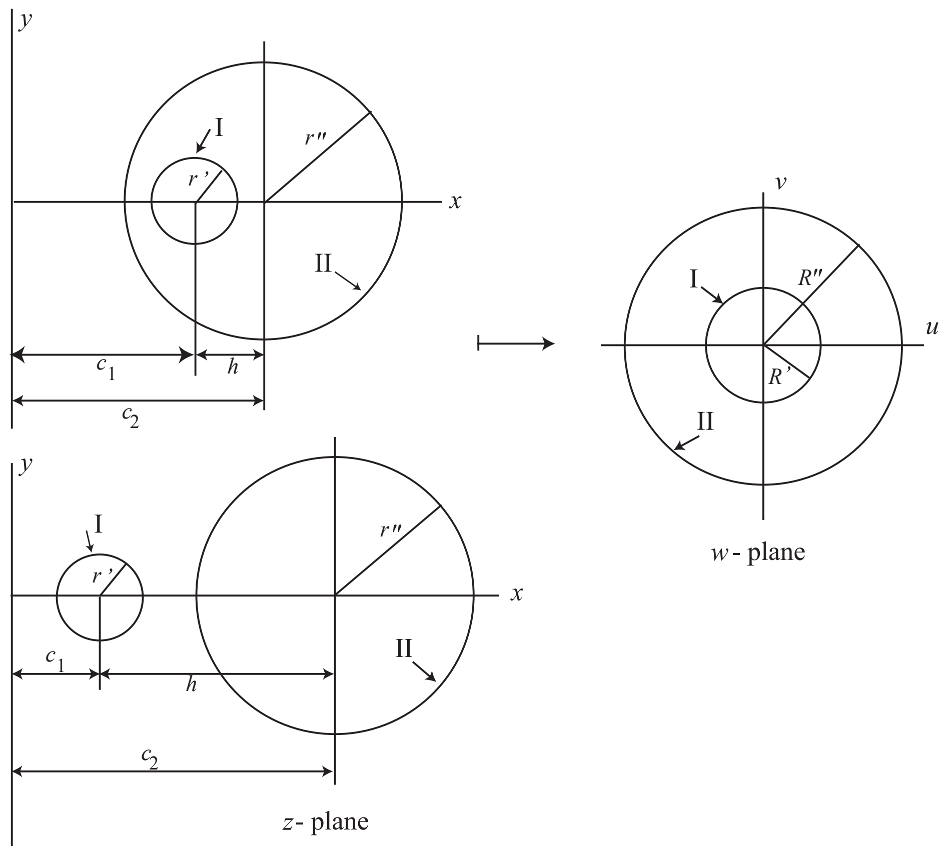

Figure 3.18 $w = \dfrac{w-\alpha}{w+\alpha}$, Two circles.

If we select α and c_1 such that the relation $\alpha^2 = c_1^2 - r'^2$ is satisfied, then R' becomes independent of x, i.e.,

$$R' = \left(\frac{c_1-\alpha}{c_1+\alpha}\right)^{1/2} = \frac{r'}{c_1+\alpha}, \quad r' = \{(c_1-\alpha)(c_1+\alpha)\}^{1/2}. \tag{3.3.21}$$

The circle II can be treated similarly by selecting $\alpha_2^2 = c_2^2 - r''^2$, thus finally giving

$$R'' = \frac{r''}{c_2+\alpha}. \tag{3.3.22}$$

If we substitute $h = c_2 - c_1$, then this transformation can be written as

$$w = \frac{z-\alpha}{z+\alpha} \quad \text{where } \alpha^2 = c_1^2 - r'^2, \tag{3.3.23}$$

and

$$c_1 = \frac{r''^2 - r'^2 - h^2}{2h}, \;\; c_2 = \frac{r''^2 - r'^2 + h^2}{2h} = c_1 + h, \;\; R' = \frac{r'}{c_1+\alpha}, \;\; R'' = \frac{r''}{c_2+\alpha}. \;\; \blacksquare$$

A critical point for this transformation is $z = -\alpha$, and the inverse of this transformation is of the same type, i.e., $z = \alpha \dfrac{1+w}{1-w}$.

The general form of the bilinear transformation (3.2.1) includes as special cases all the intermediate transformations which are discussed below.

Map 3.16. The bilinear transformations $w = \dfrac{z+1}{z-1}$, $z = \dfrac{w+1}{w-1}$, are involutory, with p, q real, and the fixed-points $F_{1,2} = 1 \pm \sqrt{2}$. The interior of the circle of D is mapped onto the interior of the same circle; however, the exterior of the circle of E is mapped onto the interior of the same circle (see Figure 3.19).

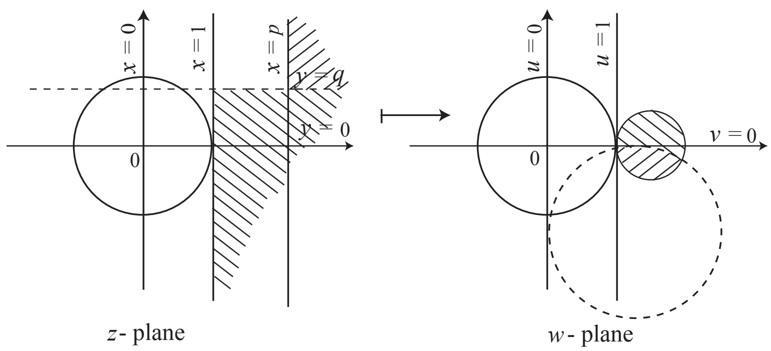

Figure 3.19 Transformation $w = \dfrac{z+1}{z-1}$.

Other details for some particular cases, with $z = p, q$ real and $w = P, Q$ real, are as follows:

(a) The points $z = 0; z_\infty = 1; \infty$ are mapped onto the points $w = -1; \infty; w_\infty = 1$.

(b) The half-plane $x \geq 0$ is mapped onto the region $|w| \geq 1$.

(c) The half-plane $y \geq 0$ is mapped onto the half-plane $v \leq 0$.

(d) The region $|z| \leq 1$ is mapped onto the half-plane $u \leq 0$.

(e) The line $x = 1$ in D is mapped onto the line $u = 1$.

(f) The line $x = p$, $p \neq 1$ is mapped onto the circle $\left| w - \dfrac{p}{p-1} \right| = \dfrac{1}{|p-1|}$.

(g) The line $y = q, q \neq 0$ is mapped onto the circle $\left| w - \dfrac{q-1}{q} \right| = \dfrac{1}{|q|}$.

Map 3.17. The transformation $w = \dfrac{1+z}{1-z} = e^{i\pi} \dfrac{z+1}{z-1}$, is the inverse of the Map 3.16. It maps the upper half-plane $\Im\{z\} > 0$ onto the unit disk $B(0, 1)$.

Map 3.18. The transformation $w = \dfrac{1+z}{1-z}$, or $z = -\dfrac{1-w}{1+w}$ maps the unit upper semi-circle in the z-plane onto the first quadrant of the w-plane, such that the points $z = -1, 0, 1$

map into the points $w = 0, 1, -1$, respectively, as shown in Figure 3.20.

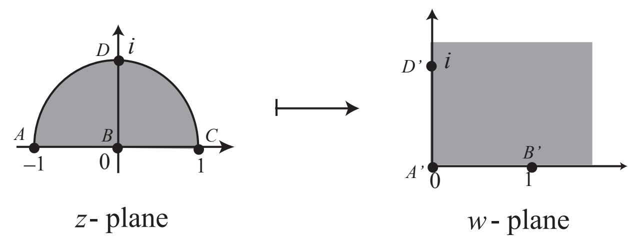

Figure 3.20 Upper semi-circle in the z-plane onto the first quadrant of the w-plane.

Map 3.19. The transformation $w = \dfrac{z-1}{z+1}$, or $z = \dfrac{1+w}{1-w}$ maps the right-half of the z-plane onto the unit circle, such that the points $z = 0, i, -i$ map into the points $w = -1, i, -i$, respectively. This map is shown in Figure 3.21(a).

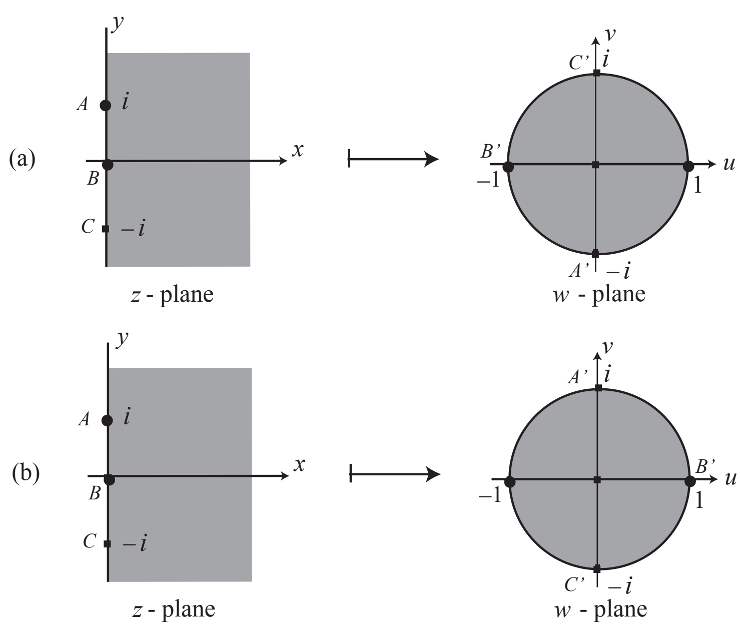

Figure 3.21 Right-half z-plane onto the unit circle.

Example 3.5. To find the linear fractional transformation that maps the interior of the circle $|z - i| = 2$ onto the exterior of the circle $|w - 1| = 3$, we need only three points on the first circle in clockwise order and three on the second circle in counterclockwise order. Let $z_1 = -i, z_2 = -2 + i$, and $z_3 = 3i$ and $w_1 = 4$, $w_2 = 1 + 3i$, and $w_3 = -2$. Then we get

$$\frac{(w-4)(3+3i)}{(w+2)(-3+3i)} = \frac{(z+i)(-2-2i)}{(z-3i)(-2+2i)},$$

or

$$w = \frac{z-7i}{z-i}.$$

The center of the first circle is $z = i$, which, under this transformation, is mapped onto the point $w = \infty$, and thus it lies outside the second circle. ■

Map 3.20. The transformation $w = \dfrac{1-z}{1+z} = e^{i\pi}\dfrac{z-1}{z+1}$, is the inverse of the Map 3.18. Thus, this transformation, or its inverse $z = \dfrac{1-w}{1+w}$, also maps the right-half of the z-plane onto the unit circle such that the points $z = 0, i, -i$ map into the points $w = 1, -i, i$, respectively, as in Figure 3.21(b).

Map 3.21. The transformation $w = \dfrac{1-z}{1+z}$ maps the exterior of the unit circle $D : |z| > 1$ onto the region $G : \Re\{w\} < 0$, which is the left-half of the w-plane. To see this, first consider the left of the regions D and G, and choose any three points on the unit circle that give negative (clockwise) orientation, say, $z_1 = 1, z_2 = -i, z_3 = -1$. Similarly the three points $w_1 = 0, w_2 = i, w_3 = \infty$ in the left region of G. Then the bilinear transformation is given by

$$\frac{w-0}{i-0} = \frac{(z-1)(-i+1)}{(z+1)(-i-1)},$$

which simplifies to the above transformation.

Map 3.22. The bilinear transformation $w = \dfrac{z-i}{z+i}$, $z = i\dfrac{1+w}{1-w}$, has the fixed-points $F_{1,2} = \frac{1}{2}(1-i)\left(1 \pm \sqrt{3}\right)$, and p, q, P, Q real. Two particular cases (A) and (B) are presented in Figure 3.22 and Figure 3.23, respectively.

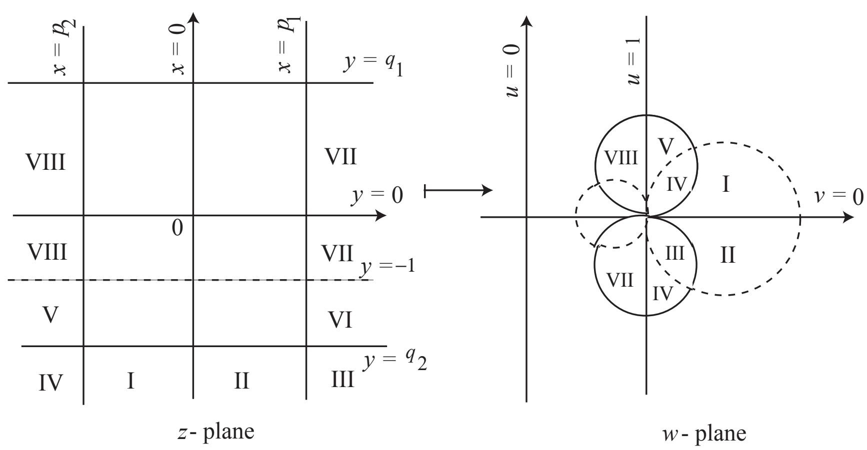

Figure 3.22 Transformation $w = \dfrac{z-1}{z+1}$ (Case A).

The details of the case A, with $z = p, q$ real and $w = P, Q$ real, are as follows:

(a) The points $z = 0, i, -i; z_\infty = \infty$ are mapped onto the points $w = -1, 0, \infty; w_\infty = 1$.

(b) The half-plane $x > 0$ is mapped onto the half-plane $v < 0$.

(c) The half-plane $y > 0$ is mapped onto the region $|w| < 1$.

(d) The region $|z| < 1$ is mapped onto the half-plane $u < 0$.

(e) The line $x - y = 1$ is mapped onto the line $u - v = 1$.

(f) The line $x = p$, $p \neq 0$ in D is mapped onto the circle $\left|w - \left(1 - \frac{1}{p}\right)\right| = \frac{1}{|p|}$.

(g) The line $y = -1$ is mapped onto the line $u = 1$.

(h) The line $y = q, q \neq -1$ is mapped onto the circle $\left|w - \frac{q}{q+1}\right| = \frac{1}{|q+1|}$.

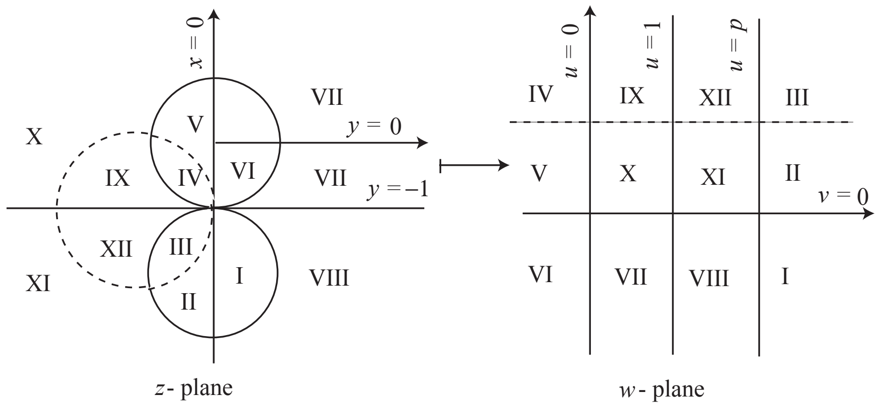

Figure 3.23 Transformation $w = \dfrac{z-1}{z+1}$ (Case B).

The details of the case B, with $z = p, q$ real and $w = P, Q$ real, are as follows:

(a) The line $y = -1$ is mapped onto the line $u = 1$.

(b) The circle $\left|z - \frac{i\,p}{1-p}\right| = \frac{1}{|1-p|}$ is mapped onto the line $u = P$, $P \neq 1$.

(c) The circle $\left|z - \left(-i - \frac{1}{q}\right)\right| = \frac{1}{|q|}$ is mapped onto the line $v = Q$, $Q \neq 0$.

Map 3.23. The transformation $w = \frac{z-i}{z+i}$, or $z = i\frac{1+w}{1-w}$, which maps the upper-half z-plane onto the circle, is presented in Figure 3.24.

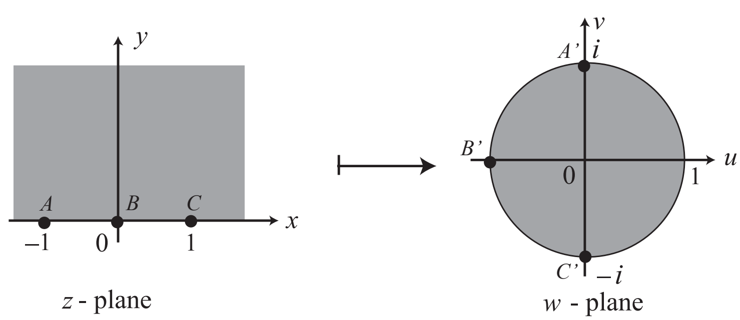

Figure 3.24 Upper-half z-plane onto a circle.

Map 3.24. The bilinear transformation $w = \frac{i+z}{i-z}$ is a composition of the following three transformations: (i) $w = -\frac{1}{s}$ (Map 3.6); (ii) $s = \frac{\zeta - i}{\zeta + i}$ (Map 3.23); and (iii) $\zeta = z$

(translation). It maps the upper half-plane $\Im\{z\} >: 0$ in the z-plane onto the unit disk $|w < 1$ in the w-plane (see Figure 3.25).

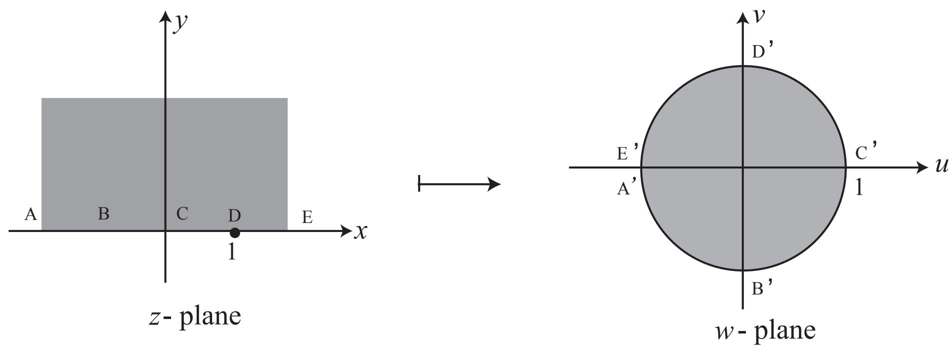

Figure 3.25 Upper half-plane onto the unit disk.

Map 3.25. The transformation

$$w(z) = i\frac{1-z}{1+z}$$

maps the unit disk onto the upper half-plane $\Im\{w\} > 0$.

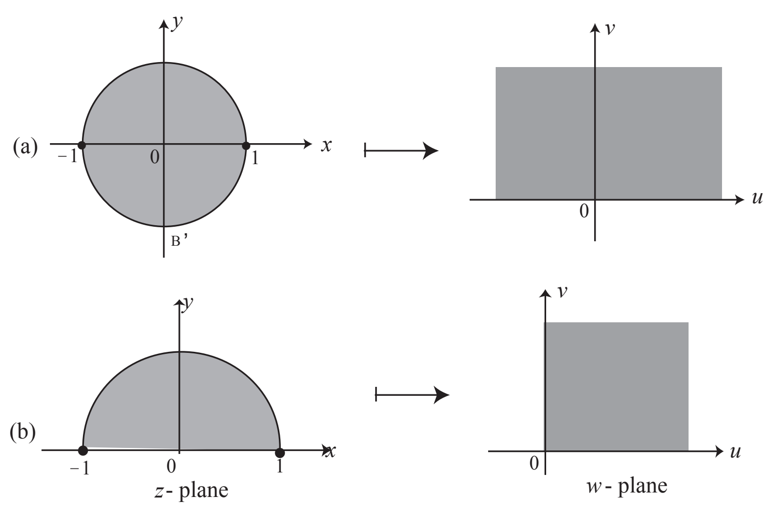

Figure 3.26 $w = i\dfrac{1-z}{1+z}$.

Since the lower boundary of the semi-disk, i.e., the interval $[-1, 1]$, is perpendicular to the upper semi-circle at the point 1, and since w is conformal at $z = 1$, the images of the interval $[-1, 1]$ and the semi-circle intersect at right angle. Since they both pass through the point -1, which is mapped onto ∞, we conclude that these images are perpendicular lines. Thus, using $w(0) = i$ and $w(i) = 1$, the interval $[-1, 1]$ in the z-plane is mapped onto the upper half of the imaginary v-axis, and the semi-circle is mapped on the right half of the real u-axis. Finally, testing the image of an interior point, say $i/2$, we find that $w(i/2) = 4/5 + 3i/5$, which is a point in the first quadrant. Hence, the upper semi-disk in the z-plane is mapped onto the first quadrant $\{u > 0, v > 0\}$ of the w-plane, since the boundary is mapped onto boundary and interior points onto interior points (Figure 3.26).

Map 3.26. The transformation

$$w = -i\frac{z+i}{z-i}, \quad \text{or} \quad z = i\frac{w-i}{w+i},$$

maps the points $z = i, -i, 0$, respectively, onto the points $w = \infty, 0$, and a point with $v > 0$. Thus, this map guarantees that the circle in the z-plane transforms into a straight line, which passes through the origin $w = 0$ and the interior of the circle is mapped onto the upper half of the w-plane.

Map 3.27. The bilinear transformation $w = f(z) = \dfrac{z}{2z-8}$ maps the region Int (Γ), where $\Gamma = \{|z-2| = 2\}$, conformally onto the region $\Re\{w\} < 0$, such that the point $z = 2$ goes into $w = -1/2$ (Saff and Snider [1976:317]).

Map 3.28. The bilinear transformations $w = \dfrac{z}{z-1}$, $z = \dfrac{w}{w-1}$ are involutory, with $z_\infty = 1, w_\infty = 1$, the fixed-points $F_1 = 0, F_2 = 2$, and p, q real. The interior (exterior) of circles of D and E map onto exterior (interior) of circles. The details of different particular cases, with $z = p, q$ real and $w = P, Q$ real, are as follows:

(a) The half-plane $y \geq 0$ is mapped onto the half-plane $v \leq 0$.

(b) The half-plane $x \geq 0$ is mapped onto the region $\left|w - \frac{1}{2}\right| \geq \frac{1}{2}$.

(c) The region $|x| \leq 1$ is mapped onto the half-plane $u \leq \frac{1}{2}$.

(d) The half-plane $x \leq \frac{1}{2}$ is mapped onto the region $|w| \leq 1$.

(e) The line $x = 1$ is mapped onto the line $u = 1$, passing through $w = 1$.

(f) The line $x = p$, $p \neq 1$ is mapped onto the circle $\left|w - \dfrac{2p-1}{2(p-1)}\right| = \dfrac{1}{|2p-2|}$ passing through $w = 1$.

(g) The line $y = q$ is mapped onto the circle $\left|w - (1 - \dfrac{1}{2q})\right| = \dfrac{1}{2|q|}$ passing through $w = 1$.

Map 3.29. The function $w = f(z) = \dfrac{z}{z-(1+i)}$ maps the lens-shaped region bounded by the circles $|z-1| = 1$ and $|z-i| = 1$ onto the region bounded by the rays $\arg\{w\} = 3\pi/4$ and $\arg\{w\} = 5\pi/4$ such that $f(0) = 0$ and $f(1+i) = \infty$.

Map 3.30. The transformation $w = \dfrac{z}{iz+2}$ maps the disk $|z-i| = 1$ onto the real line. To see this, since the points $z = 0, 1+i, 2i$ lie on the given circle in the z-plane and since the real line in the w-plane passes through the points $w = 0, 1, \infty$, so we choose $h(z)$ such that $h(0) = 0, h(1+i) = 1$, and $h(2i) = \infty$. Then $h(z) = \dfrac{z}{iz+2}$. Since $h(i) = i$, so h maps the disk $|z-i| < 1$ onto the upper half-plane.

Map 3.31. Circle onto circle. The transformation is

$$w - w_0 = \Re\Big\{ e^{i\tau} \frac{z - z_0 - r\alpha}{\bar{\alpha} z - \alpha z_0 - r} \Big\}, \tag{3.3.24}$$

where τ is real, $|\alpha| \neq 1$, and is presented in Figure 3.27.

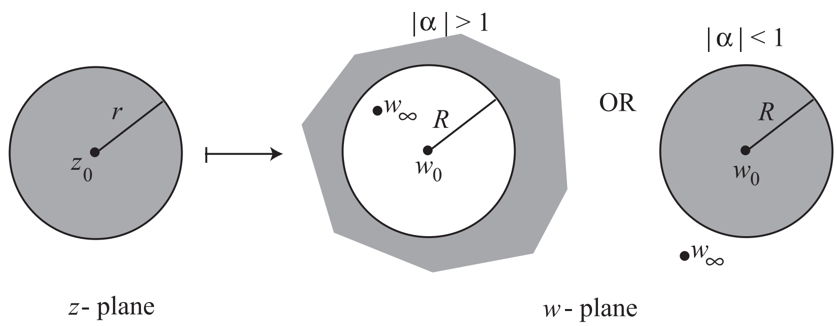

Figure 3.27 Circle onto circle.

Map 3.32. Mapping of the unit circle onto itself:

$$w = e^{i\tau} \frac{z - \alpha}{\bar{\alpha} z - 1}, \quad \tau \text{ real and } |\tau| \neq 1. \tag{3.3.25}$$

Map 3.33. The transformation $w = e^{i\theta} \dfrac{z - z_0}{1 - \bar{z}_0 z}$, which maps a circle onto the circle in the w-plane such that the point z_0 maps into the point $w = 0$, is shown in Figure 3.28.

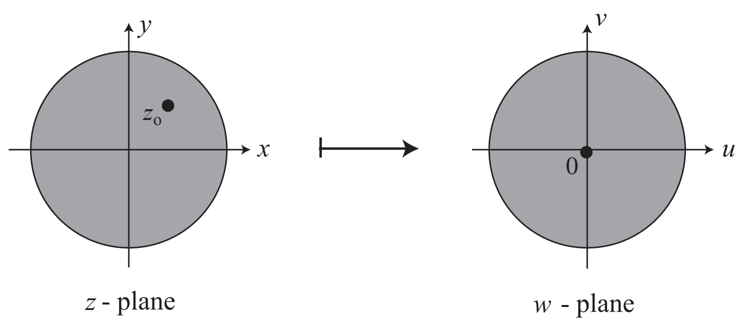

Figure 3.28 Circle onto a circle.

Map 3.34. The function $f(z) = \rho\, e^{i\alpha} \dfrac{z - z_1}{z - z_2}$ maps the region enclosed by two arcs onto the region enclosed by the sector shown in Figure 3.29, such that a fixed point $\zeta \neq z_{1,2}$ on Γ_1 goes into a point ω on the u-axis, where $\alpha = \arg \left\{ \dfrac{\zeta - z_1}{\zeta - z_2} \right\}$ and $\rho = \omega \left| \dfrac{\zeta - z_1}{\zeta - z_2} \right|$ (Pennisi [1963:321]).

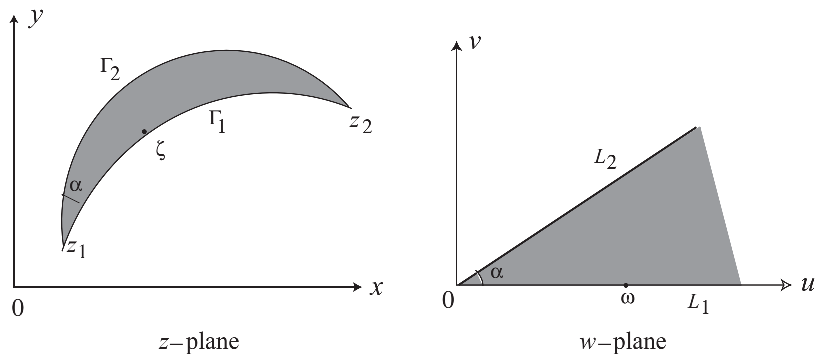

Figure 3.29 Crescent-shaped region.

Map 3.35. The transformation $w = \dfrac{1}{1-z}$, $z = \dfrac{w-1}{w}$, with $z_\infty = 1; w_\infty = 0$; the fixed points $F_1 = \frac{1}{2} + \frac{i}{2}\sqrt{3}$, and $F_2 = \dfrac{1}{F_1}$, and p, q, P, Q are real. The interior of the circle in the z-plane is mapped onto the interior of the circle in the w-plane. The details of the transformation, with $z = p, q$ real and $w = P, Q$ real, are as follows:

(a) The half-plane $y \geq 0$ is mapped onto the half-plane $v \geq 0$.

(b) The half-plane $x \geq 0$ is mapped onto the region $\left|w - \frac{1}{2}\right| \geq \frac{1}{2}$.

(c) The region $|z| \leq 1$ is mapped onto the half-plane $u \geq \frac{1}{2}$.

(d) The region $|z - 1| \leq 1$ is mapped onto the region $|w| \geq 1$.

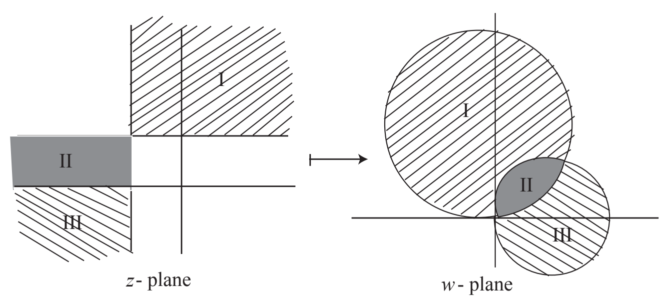

Figure 3.30 Transformation $w = \dfrac{1}{1-z}$.

There are two special cases, one presented in Figure 3.30, and the other in Figure 3.31; their respective details, with $z = p, q$ real and $w = P, Q$ real, are as follows:

For Figure 3.30:

(a) The line $x = 1$ is mapped onto the line $u = 0$.

(b) The line $x = p \neq 1$ is mapped onto the circle $\left|w - \dfrac{1}{2(1-p)}\right| = \dfrac{1}{2|1-p|}$.

(c) The line $y = q \neq 0$ is mapped onto the circle $\left|w - \dfrac{1}{2q}\right| = \dfrac{1}{2|q|}$.

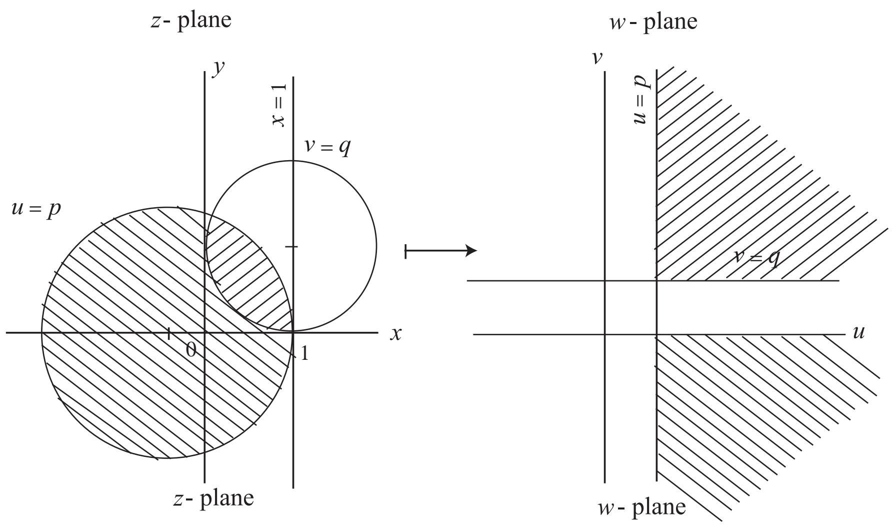

Figure 3.31 Transformation $w = \dfrac{1}{1-z}$.

For Figure 3.31:

(a) The circle $\left|z - \left(1 - \dfrac{1}{2p}\right)\right| = \dfrac{1}{2|p|}$ is mapped onto the line $u = P \neq 0$.

(ib) The circle $\left|z - \left(1 + \dfrac{1}{2q}\right)\right| = \dfrac{1}{2|q|}$ is mapped onto the line $v = Q \neq 0$.

Map 3.36. The transformation $w = \dfrac{1}{z} - a$, a real, maps the line $y = 0$ and the circle $\left|z + \frac{1}{2}i\right| = \frac{1}{2}$ onto the lines $v = 0$ and $v = 1$, respectively.

Map 3.37. The transformation $w = \dfrac{2i\,r}{z} - a$, a real, maps the line $x = 0$ and the circle $|z - r| = r > 0$ onto the lines $v = 1$ and $v = 1$, respectively. ■

Map 3.38. The transformation is

$$w = \frac{r_2}{r_2 - r_1}\Big(\frac{2r_1}{z} + i\Big). \tag{3.3.26}$$

It is presented in Figure 3.32.

The details of the transformation for this case with $z = p, q$ real and $w = P, Q$ real, are as follows:

(a) The points $z = 0$; ∞; $2ir_1$ are mapped onto the points $w = \infty$; $\dfrac{ir_2}{r_2 - r_1}$; 0.

(b) The circle $|z - ir_1| = r_1$ is mapped onto the line $v = 0$.

(c) The circle $|z - ir_2| = r_2$ is mapped onto the line $v = 1$.

(d) The line $x = 0$ is mapped onto the line $u = 0$.

(e) The line $y = 0$ is mapped onto the line $v = \dfrac{r_2}{r_2 - r_1}$.

(f) The point $z_3 = ir_1\left(1 - e^{i\theta}\right)$ is mapped onto the point $w_3 = \dfrac{r_2}{r_2 - r_1} \cot \dfrac{\theta}{2}$.

(g) The small circle touching both circles and passing through z_3 is mapped onto the circle $\left|w + w_3 - \frac{1}{2}i\right| = \frac{1}{2}$.

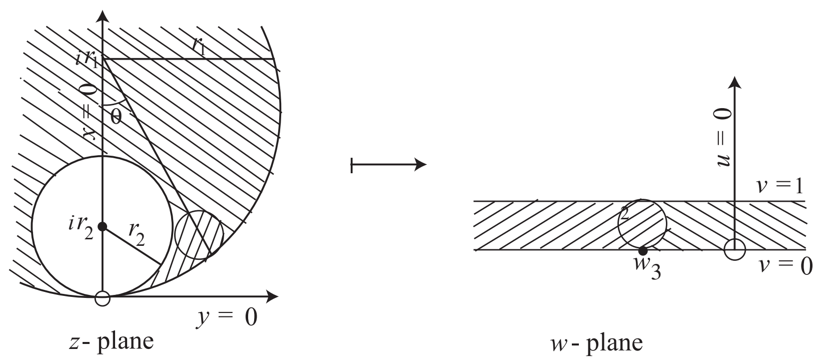

Figure 3.32 Two circles in outer contact.

Map 3.39. The transformation

$$w = \frac{i\,r_2}{r_1 + r_2}\left(1 - \frac{2r_1}{z}\right) \tag{3.3.27}$$

is presented in Figure 3.33.

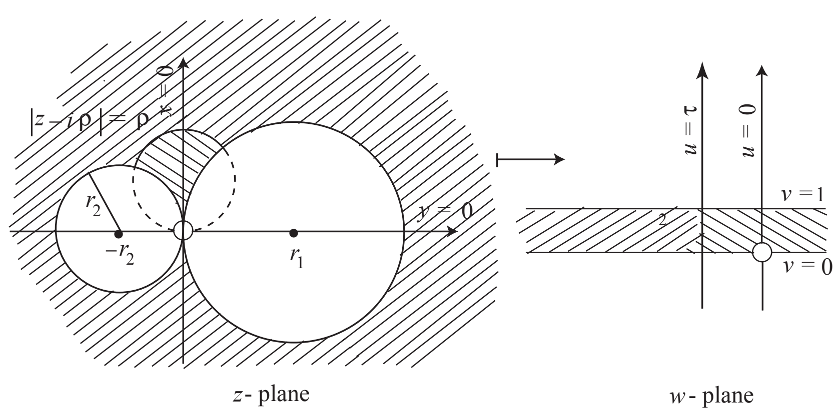

Figure 3.33 Two circles in outer contact.

The details of the transformation for this case with $z = p, q$ real and $w = P, Q$ real, are as follows:

(a) The points $z = 0;\ \infty;\ 2r_1$ are mapped onto the points $w = \infty;\ \dfrac{ir_2}{r_1 + r_2};\ 0$.

(b) The circle $|z - r_1| = r_1$ is mapped onto the line $v = 0$.

(c) The circle $|z + r_2| = r_2$ is mapped onto the line $v = 1$.

(d) The circle $|z - i\rho| = r_2$ is mapped onto the line $u = \tau$, where $\tau = -\dfrac{r_1 r_2}{\rho(r_1 + r_2)}$.

(e) The lines $x = 0$; $y = 0$ are mapped onto the lines $v = \dfrac{r_2}{r_2 - r_1}$; $u = 0$.

Map 3.40. The transformation that maps the upper-half of the unit circle in the z-plane onto the upper half-plane $\Im\{w\} > 0$ is defined by

$$w = \left(\frac{1+z}{1-z}\right)^2,$$

and is presented in Figure 3.34.

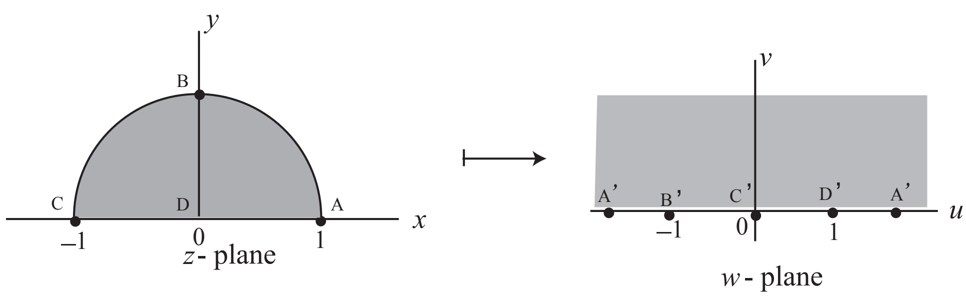

Figure 3.34 Upper-half of the unit circle onto the upper half-plane.

Map 3.41. The transformation that maps the upper half plane $\Im\{z\} > 0$ onto a round corner in the w-plane is $w = (z+1)^\alpha + (z-1)^\alpha$, $1 < \alpha < 2$; it is presented in Figure 3.35. The transformation maps the line segment $y = 0, -1 \le x \le 1$ in the z-plane onto the part $-2^\alpha \cos(\alpha 1)\pi \le u \le 2^\alpha$, $-2^\alpha \sin(\alpha - 1)\pi \le v \le 0$, i.e., the part DCB, of $(v\cos(\alpha\pi) - u\sin(\alpha\pi))^{1/\alpha} + (-v)^{1/\alpha} = 2\left(-\sin(\alpha\pi)\right)^{1/\alpha}$. Note that for $\alpha = 3\pi/2$, the curve is the asteroid $u^{2/3} + (-v)^{2/3} = 2$.

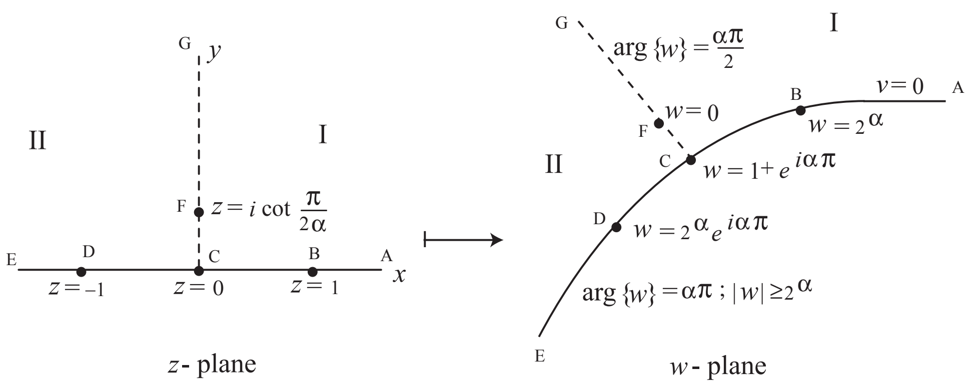

Figure 3.35 $\Im\{z\} > 0$ onto a round corner.

Map 3.42. The transformation that maps the sector of angle π/m of a circle in the z-plane onto the upper half-plane $\Im\{w\} > 0$ is defined by

$$w = \left(\frac{1+z^m}{1-z^m}\right)^2,$$

and is presented in Figure 3.36.

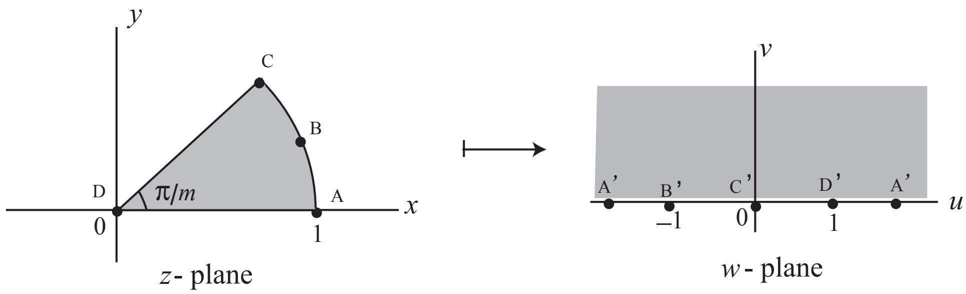

Figure 3.36 Sector of angle π/m of a circle onto the upper half-plane.

Map 3.43. The transformation

$$w = \frac{iz^2+1}{iz^2-1}, \tag{3.3.28}$$

conformally maps the upper half-plane $U = \Im\{z > 0\}$ onto the unit disk $C(0,1)$. To prove it, first note that the transformation $z = g(w) = \dfrac{w-1}{w+1}$ (Map 3.17) maps the right half-plane $\Re\{w\} > 0$ onto $B(0,1)$. Also, multiplication by $i = e^{i\pi/2}$, with $z = h(w) = iw$ rotates the complex plane by $\pi/2$ and thus maps the right half-plane $C(0, \frac{1}{2})$ onto the upper half-plane U, while its inverse $w = h^{-1}(z) = -iz$ will map U to the right half-plane. Hence, to map the upper half-plane U onto the unit disk, the composition of these two mappings gives the required map:

$$w = g \circ h^{-1}(z) = \frac{-iz-1}{-iz+1} = \frac{iz+1}{iz-1}. \tag{3.3.29}$$

Similarly, we already know (see Maps 4.6) that the square map $w = z^2$ maps the upper right quadrant $\{0 < \arg\{z\} < \pi/2\}$ onto the upper half-plane U. Thus, composition of this map with the previously constructed map, where we replace z by w in (3.3.29), we obtain the final required mapping (3.3.28).

Map 3.44. The transformation that maps the lens-shaped region of angle π/m, where this region is bounded by two circular arcs, in the z-plane onto the upper half-plane $\Im\{w\} > 0$ is defined by

$$w = e^{2mi\,\mathrm{arccot}(p)} \left(\frac{z+1}{z-1}\right)^m, \tag{3.3.30}$$

and is presented in Figure 3.37.

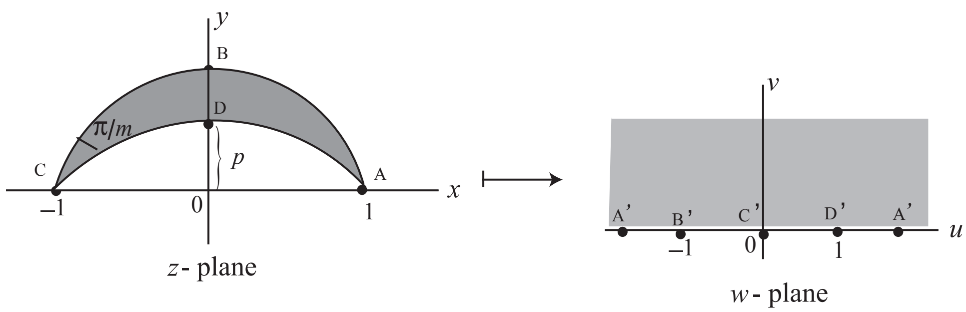

Figure 3.37 Lens-shaped region of angle π/m onto the upper half-plane.

Map 3.45. The transformation $w = \dfrac{z-a}{az-1}$, where $a > 1$, $-1 < x_2 < x_1 < 1$, $R_0 > 1$, and

$$a = \frac{1 + x_1 x_2 + \sqrt{(1-x_1^2)(1-x_2^2)}}{x_1 + x_2},$$
$$R_0 = \frac{1 - x_1 x_2 + \sqrt{(1-x_1^2)(1-x_2^2)}}{x_1 + x_2}, \tag{3.3.31}$$

maps the region between the two circles in the z-plane onto the annular region in the w-plane, as shown in Figure 3.38.

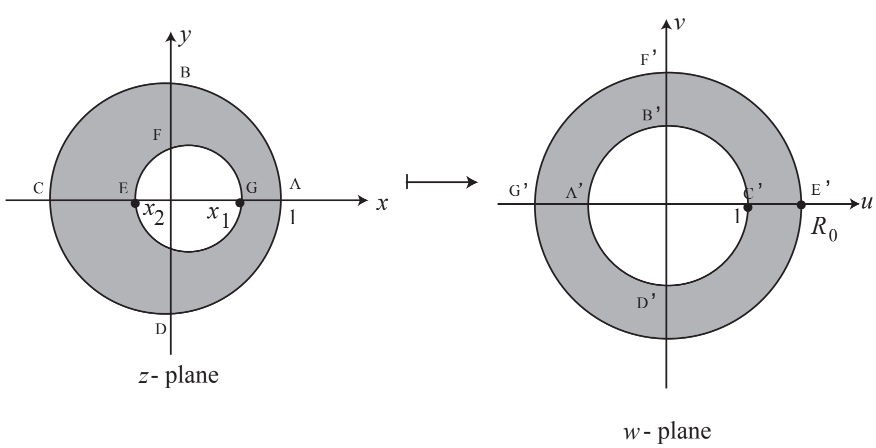

Figure 3.38 Map 3.45.

Map 3.46. The transformation $w = \dfrac{z-a}{az-1}$, where a and R_0 are the same as in (3.3.31), with $x_2 < a < x_1$, and $0 < R_0 < 1$ when $1 < x_2 < x_1$, maps the region exterior to the two circles in the z-plane onto the annular region in the w-plane, as shown in Figure 3.39.

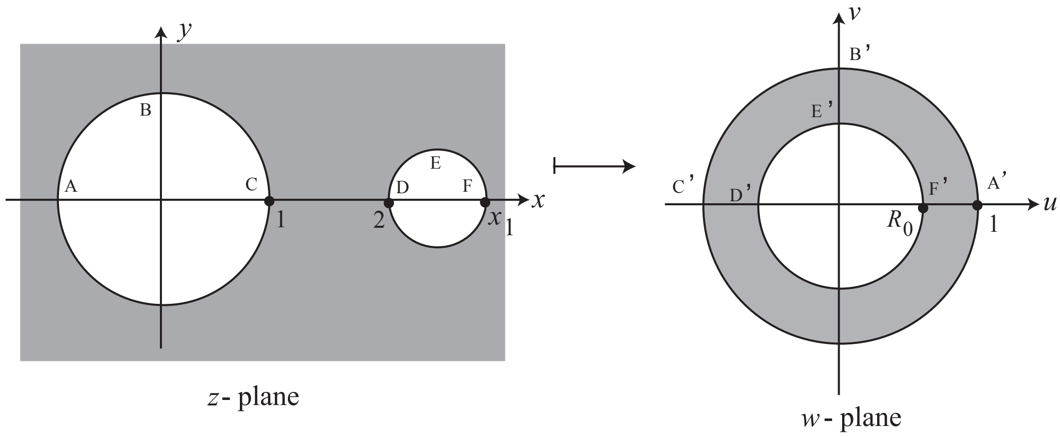

Figure 3.39 Map 3.46.

Map 3.47. The transformation $w = e^{i\lambda}\,\dfrac{z - z_0}{z + \bar z_0}$, where λ is a real number, maps the upper half-plane $-\infty < x < \infty, 0 \le y < \infty$, in the z-plane onto the unit disk $|w| \le 1$ in the w-plane.

Map 3.48. The transformation $w = e^{i\lambda}\,\dfrac{z - z_0}{1 - \bar z_0 z}$, where λ is a real number, maps the unit disk $x^2 + y^2 \le 1$, in the z-plane onto the unit disk $|w| \le 1$ in the w-plane.

Map 3.49. The transformation $w = e^{i\beta}\,\dfrac{z - \alpha}{\bar\alpha z - 1}$ for some $|\alpha| < 1$, $-\pi < \beta < \pi$, maps the unit disk onto itself.

Map 3.50. Cassini's ovals. There are four cases:

Map 3.50a. Either part of the interior of Cassini's ovals $|z - z_1||z - z_2| = C$, with foci $z_1 = \pm\sqrt{w_1}$, is mapped onto the interior of the circle $|w - w_1| = C$, $C \le |w_1|$. This case is presented in Figure 3.40, where $w_1$is taken as a *real* and positive number.

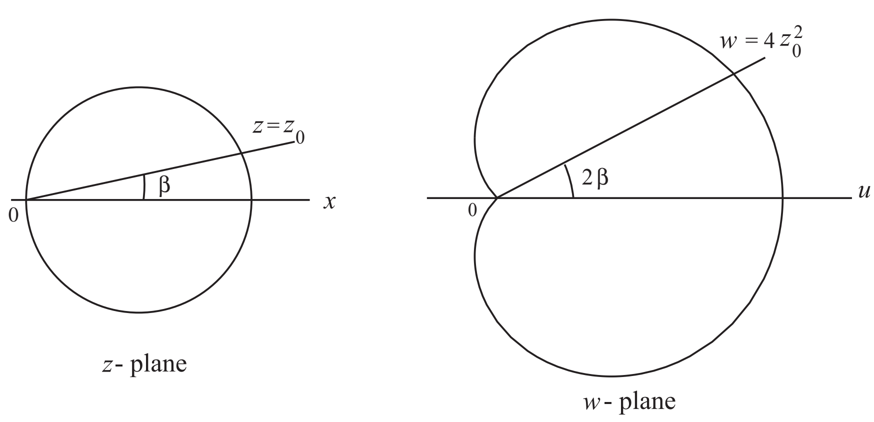

Figure 3.40 Cassini's ovals, Map 3.50a.

Map 3.50b. Cassini's ovals $|z - z_1||z - z_2| = C$, with foci $z_1 = \pm\sqrt{w_1}$, is mapped onto the circle $|w - w_1| = C$, $C > |w_1|$, counted twice.

Map 3.50c. The part $x > 0$ of the interior of Cassini's ovals, and the part $x < 0$ of the interior of Cassini's ovals, both are mapped onto the interior of the circle cut from $w = 0$ to $w' = \Re\{w_1\} = \left(C^2 - \Im\{w_1\}^2\right)^{1/2}$, i.e., point a. This case is presented in Figure 3.41.

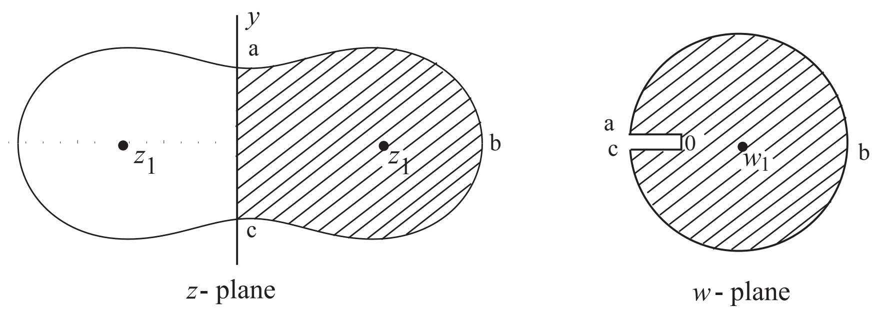

Figure 3.41 Cassini's ovals, Map 3.50c.

Map 3.50d. (Generalized Cassini's Ovals) The details of the transformation are as follows:

(a) Each part of $|z - z_1||z - z_2| \cdots |z - z_n| = C$, with foci $z_j = e^{2i\pi j/n}\sqrt[n]{w_0}$, $j = 1, 2, \ldots, n$, is mapped onto the circle $|w - w_0| = C$, $0 < C \le |w_0|$.

(b) Closed curve $|z - z_1||z - z_2| \cdots |z - z_n| = C$ (which is approximately circle for large C) is mapped onto the circle $|w - w_0| = C$, counted n times, $0 < |w_0| < C$.

(c) Region bounded by this curve and the lines $z = r\,e^{i\theta}$, $z = r\,e^{i(\theta + 2\theta/n}$, θ fixed, is mapped onto the interior of $|w - w_0| = C$, $C > |w_0|$, cut from 0 to the point at which the line $w = r^n\,e^{in\theta}$ meets the circle.

This transformation for $n = 4$ is presented in Figure 3.42.

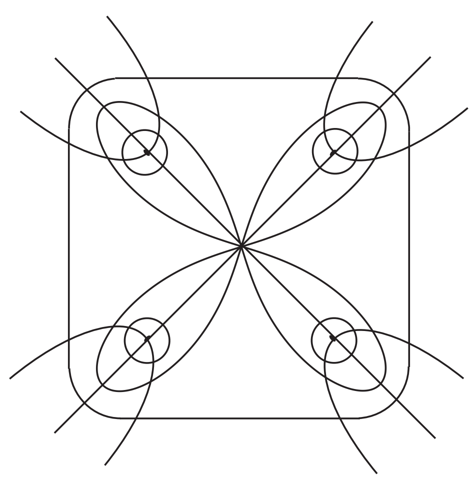

Figure 3.42 Cassini's ovals.

Map 3.51. Cardioid and Limaçon. The mapping from z-plane onto the w-plane is presented in Figure 3.43.

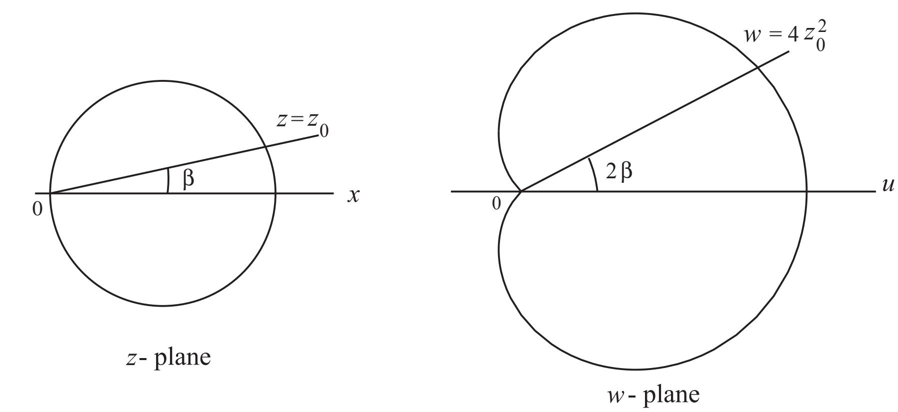

Figure 3.43 Circle onto cardioid.

The details of this transformation are as follows:

(a) The circles $\left\{ \begin{array}{l} |z - z_0| = |z_0| \\ |z + z_0| = |z_0| \end{array} \right\}$ are mapped onto the cardioid $w = 2z_0^2(1 + \cos\theta)\, e^{i\theta}$, $-\pi \le \theta < \pi$.

(b) The circles $\left\{ \begin{array}{l} |z - z_0| = c \\ |z + z_0| = c \end{array} \right\}$, $c > 0, z_0 \neq 0$, are mapped onto the limaçon $w - w_1 = 2c(z_0 + c\cos\theta)\, e^{i\theta}$, $w_1 = z_0^2 - c^2$, $-\pi \le \theta < \pi$.

Map 3.52. Cardioid and Generalized Cardioids ($\alpha > 1$), and Lemniscates ($0 < \alpha < 1$) (Figure 3.44).

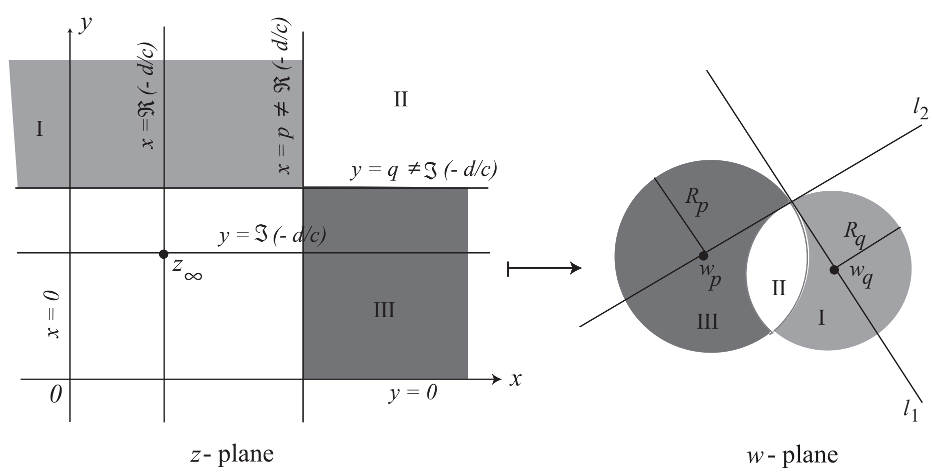

Figure 3.44 Generalized cardioid.

The details of this transformation are as follows:

The line $\Re\{e^{-i\theta}\, z\} = s$, $s > 0$ is mapped onto each part of $R^\beta = s^{-1}\cos(\beta\theta + \phi)$, $|\theta - \alpha(2k\pi - \phi| \le \alpha\pi/2$, $k = 0, \pm 1, \pm 2, \ldots$; the curves are closed, with a cusp at 0, with tangents $w = R\, e^{i\alpha\left(k\pi + \frac{\pi}{2} - \phi\right)}$, $k = 0, \pm 1, \pm 2, \ldots$; there are n parts if $\alpha^{-1} = n \in \mathbb{Z}$.

3.4 Straight Lines and Circles

We will consider different cases of linear and bilinear transformations that map given elements, like points, lines, circles, onto prescribed points, lines, and circles, using the cross-ratio. This classification of linear and bilinear transformations is based on Kober [1957:2-32]. A consequence of the Riemann mapping theorem is that simply connected domains D_z and D_w can always be mapped conformally onto each other. Three parameters may be chosen at will, based on the elements in w corresponding to similar elements in z, such as points, circles, rectangles, and so on. It is often convenient to start with a line, or a circle, or a rectangle (or square) in the w plane and create an image in the z-plane by applying various mapping functions $z = g(w)$, where $g(w) = f^{-1}(w)$ are the inverse of the 'forward' mapping function $w = f(z)$. Often a unit circle is used as an intermediate image in constructing sequential mappings. Some examples are as follows.

Map 3.53. Lines parallel to the axes. $z_\infty = -d/c$, $w_\infty = a/c$ (see Figure 3.45).

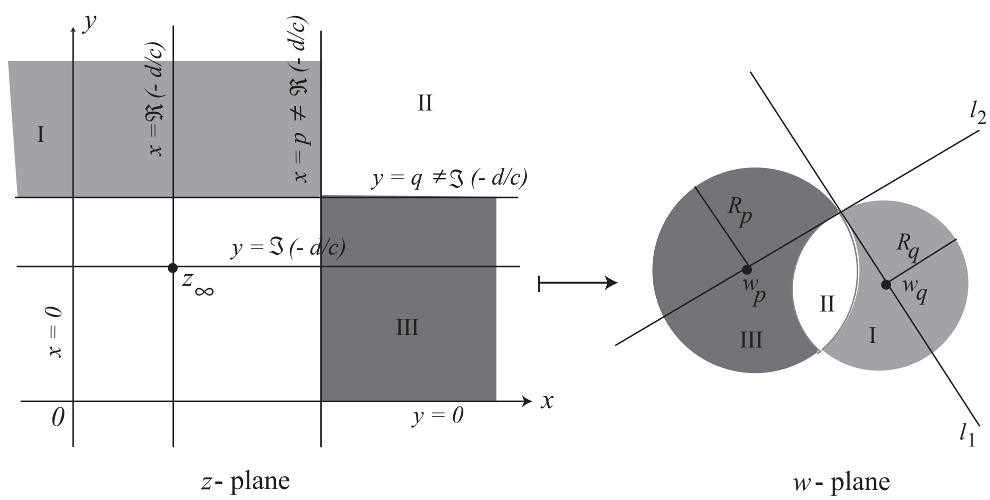

Figure 3.45 Lines parallel to axes (direct map).

Different cases of this transformation, with $z = p, q, r, r_p, r_q$ real and $w = P, Q, R, R_p, R_q$ real, are as follows:

(a) The line $x = -\Re\{d/c\}$ is mapped onto the line $l_1 : \Re\big\{c\left(\bar{a}\bar{d} - \bar{b}\bar{c}\right)(cw - a)\big\} = 0$.

(b) The line $x = p$, $p \neq -\Re\{d/c\}$ is mapped onto the circle $|w - w_p| - R_p$, where $w_p = \dfrac{a\bar{c}p + \frac{1}{2}\left(a\bar{d} + b\bar{c}\right)}{p|c|^2 + \Re\{c\bar{d}\}}$, and $R_p = \left|\dfrac{ad - bc}{2p|c|^2 + 2\Re\{\bar{c}d\}}\right|$.

(c) The line $y = -\Im\{d/c\}$ is mapped onto the line l_2: $\Im\{c\left(\bar{a}\bar{d} - \bar{b}\bar{c}\right)(cw - a)\} = 0$.

(d) The line $y = q$, $q \neq -\Im\{d/c\}$ is mapped onto the circle $|w - w_q| = R_q$, where $w_q = \dfrac{a\bar{c}p + \frac{1}{2}i\,\left(a\bar{d} - b\bar{c}\right)}{q|c|^2 + \Im\{\bar{c}d\}}$, and $R_q = \left|\dfrac{ad - bc}{2q|c|^2 + 2\Im\{\bar{c}d\}}\right|$.

The inverse mapping is presented in Figure 3.46.

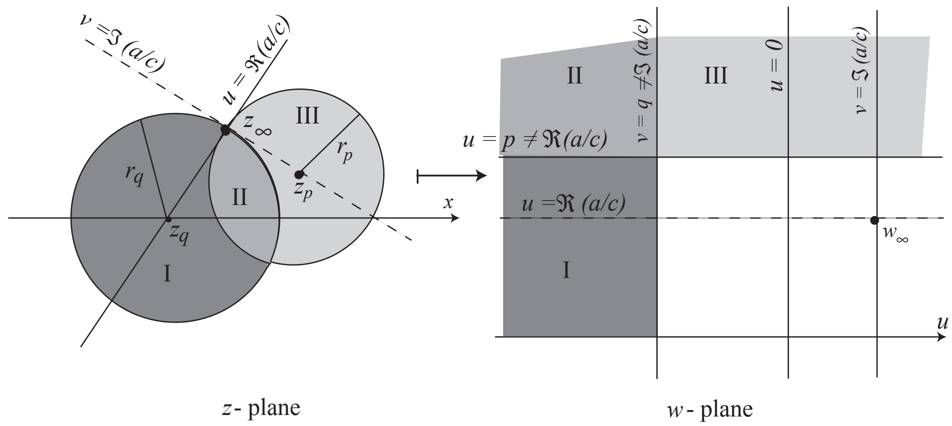

Figure 3.46 Lines parallel to axes (inverse map).

The details of different cases with $z = p, q, r, r_p, r_q$ real and $w = P, Q, R, R_p, R_q$ real, are as follows:

(a) The line $\Re\{c\left(\bar{a}\bar{d} - \bar{b}\bar{c}\right)(cz + d)\} = 0$ is mapped onto the line $u = \Re\{a/c\}$.

(b) The circle with center $z_p = \dfrac{\frac{1}{2}\left(\bar{a}d + b\bar{c}\right) - \bar{c}dp}{p|c|^2 - \Re\{a\bar{c}\}}$ and radius $r_p = \left|\dfrac{ad - bc}{2P|c|^2 - 2\Re\{a\bar{c}\}}\right|$, is mapped onto the line $u = P,\ P \neq \Re\{a/c\}$.

(c) The line $\Im\{c\left(\bar{a}\bar{d} - \bar{b}\bar{c}\right)(cz + d)\} = 0$ is mapped onto the line $v = \Im\{a/c\}$.

(d) The circle with center $z_q = \dfrac{\frac{1}{2}i\left(\bar{a}d - b\bar{c}\right) - \bar{c}dq}{q|c|^2 - 2\Im\{a\bar{c}\}}$ and radius $r_q = \left|\dfrac{ad - bc}{2q|c|^2 - 2\Im\{a\bar{c}\}}\right|$, is mapped onto the line $v = Q,\ Q \neq \Im\{a/c\}$.

Map 3.54. Other lines and circles. $z_\infty = -d/c$, $w_\infty = a/c$ (see Figure 3.47.)
The details of the transformation with $z = p, q, r, r_p, r_q$ real and $w = P, Q, R, R_p, R_q$ real, are as follows:

(1a, b.) The line $\Re\{\bar{\lambda}z\} = p$ passing through z_∞ is mapped onto the line $\Re\{\bar{\Lambda}w\} = P$ passing through w_∞.

(2.) The line $\Re\{\bar{\lambda}z\} = p$ not passing through z_∞ is mapped onto the circle $|w - w_0| = R$ passing through w_∞.

(3.) The circle $|z - z_0| = r$ passing through z_∞ is mapped onto the line $\Re\{\bar{\Lambda}w\} = P$ not passing through w_∞.

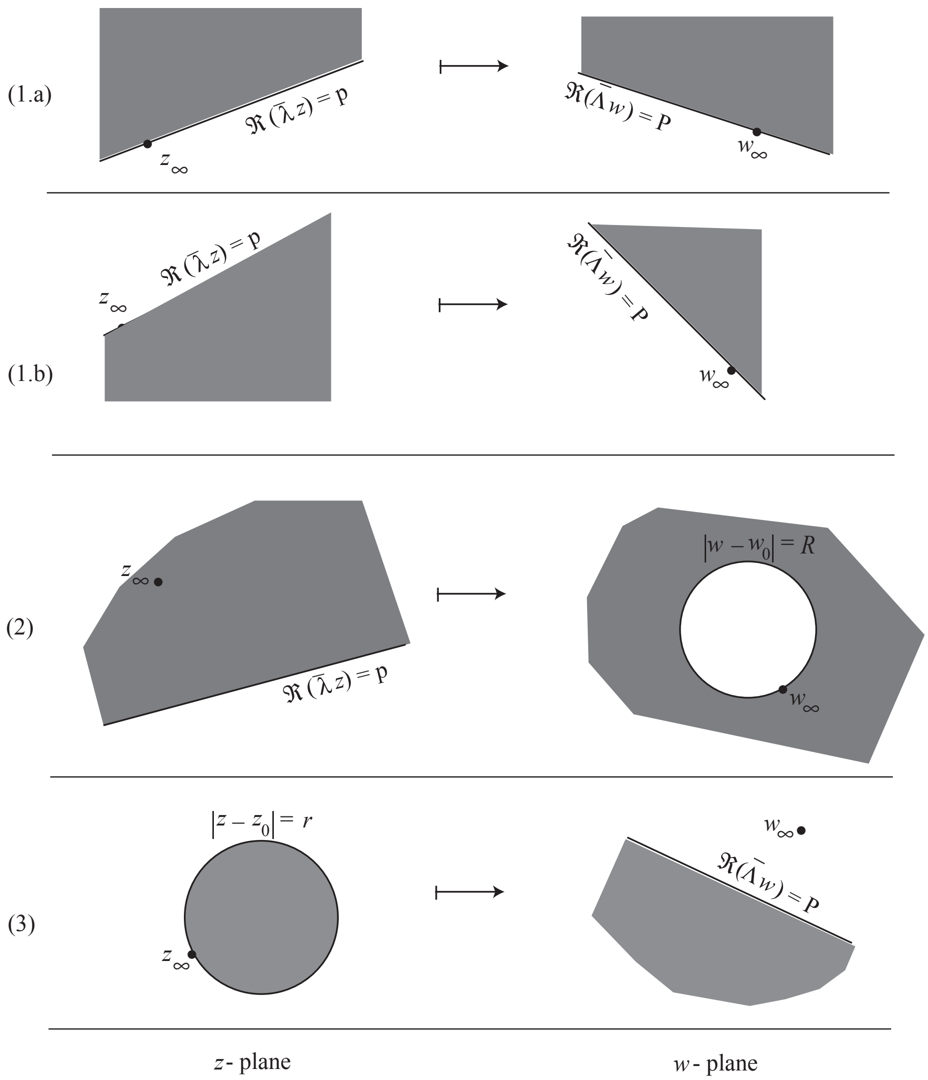

Figure 3.47 Other lines and circles (direct map).

Map 3.55. The circle $|z_0 + d/c| = r$ not passing through z_∞ is mapped onto the circle $|w - w_0| = R$ not passing through w_∞ (see Figure 3.48).

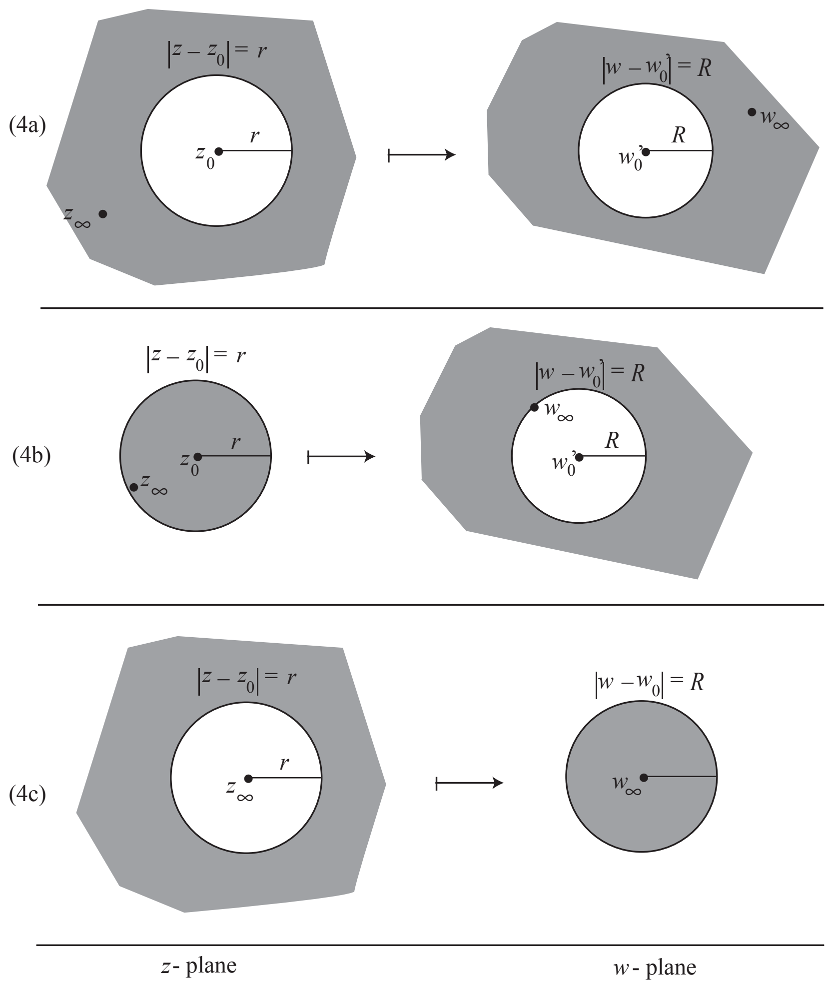

Figure 3.48 Circles not passing through z_∞.

The details of the transformation with $z = p, q, r, r_p, r_q$ real and $w = P, Q, R, R_p, R_q$ real, are as follows, where

$$w_0' = \frac{(az_0 + b)(\bar{a}\bar{z}_0 + \bar{d}) - a\bar{c}r^2}{|cz_0 + d|^2 - |c|^2 r^2}, \quad R = \frac{r|ad - bd|}{|cz_0 + d|^2 - |c|^2 r^2}.$$

(a) The circle $|z - z_0| = r$ not passing through z_∞ is mapped onto the circle $|w - w_0'| = R$ not passing through w_∞.

(b) The circle $|cz_0 + d| > |c|r$ is mapped onto the circle $|w - w_0| = R$, where $w_0' = \dfrac{az_0 + b - \bar{c}R^2 s}{cz_0 + d}$, $R = r|s|$, and $s = \dfrac{ad - bc}{|cz_0 + d|^2 - |cr|^2}$.

(c) The circle $0 < |cz_0 + d| < |c|r$ is mapped onto the circle $|w - w_0| = R$, where w_0', and R, as in (4a).

(d) The point $z_0 - d/c = z_\infty$ is mapped onto the point $w_0 = a/c = w_\infty$; $R = \left|\dfrac{ad - bc}{c^2 r}\right|$.

Map 3.56. THREE POINTS ONTO THREE POINTS. There are five cases (a)-(e) discussed below. Using the cross-ratio (3.3.1), the transformation is defined by (3.3.2). Then

Map 56a. This is defined by

$$\frac{w-w_1}{w-w_2}\frac{w_3-w_2}{w_3-w_1}=\frac{z-z_1}{z-z_2}\cdot\frac{z_3-z_2}{z_3-z_1}, \tag{3.4.1}$$

where z_1, z_2, z_3 are given and finite, and w_1, w_2, w_3 are finite. This is presented in Figure 3.49(a).

Map 3.56b. The transformation is of the form $\{w, w_1, w_2, w_3\} = \{z, z_1, z_2, \infty\}$, i.e.,

$$\frac{w-w_1}{w-w_2}\frac{w_3-w_2}{w_3-w_1}=\frac{z-z_1}{z-z_2}, \tag{3.4.2}$$

where the three given points are z_1, z_2, ∞ and w_1, w_2, w_3. This case is presented in Figure 3.49(b).

Map 3.56c. The transformation is of the form

$$\frac{w-w_3}{w_3-w_1}=\frac{z_2-z_1}{z-z_2}, \tag{3.4.3}$$

where the three given points are z_1, z_2, ∞ and w_1, ∞, w_3. This case is presented in Figure 3.49(c).

Map 3.56d. The transformation is of the form

$$\frac{w-w_1}{w_1-w_2}=\frac{z-z_1}{z_1-z_2}, \tag{3.4.4}$$

where the three given points are z_1, z_2, ∞ and w_1, w_2, ∞. This case is presented in Figure 3.49(d).

Map 3.56e. The transformation is of the form

$$\frac{w-w_1}{w-w_2}=\frac{z-z_1}{z-z_2}\frac{z_3-z_2}{z_3-z_1}, \tag{3.4.5}$$

where the three given points are z_1, z_2, z_3 and w_1, w_2, ∞. This case is presented in Figure 3.49(e).

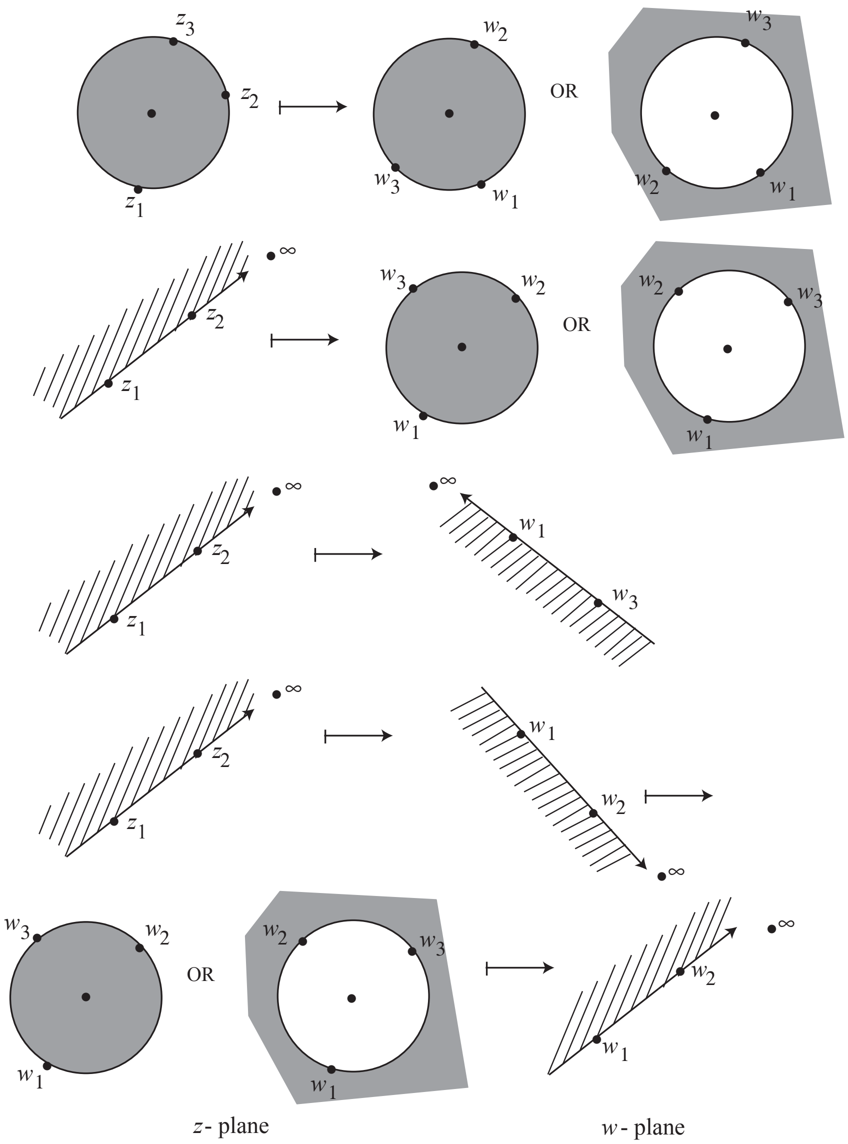

Figure 3.49 Three points onto three points.

Map 3.57. Straight line onto straight line. The transformation is

$$\bar{\lambda}w = P + \frac{a\left(\bar{\lambda}z - p\right) - i\,b}{i\,c\left(\bar{\lambda}z - p\right) + d}, \tag{3.4.6}$$

where λ, p, Λ and P are preassigned. This transformation is presented in Figure 3.50(a). In the case of the mapping of a line onto itself, the above holds except that $\lambda = \Lambda$ and $p = P$, i.e., $\Re\{\bar{\lambda}z\} = p$ is the same as $\Re\{\bar{\Lambda}w\} = P$ (Figure 50(b)).

The details for the case 12a, with $z = p, q$ real and $w = P, Q$ real, are as follows:

If $ad - bd > 0$, then
(a) the half-plane I is mapped onto the half-plane I; and
(b) the half-plane II is mapped onto the region II.
If $ad - bc < 0$, then
(c) the half-plane I is mapped onto the half-plane II.

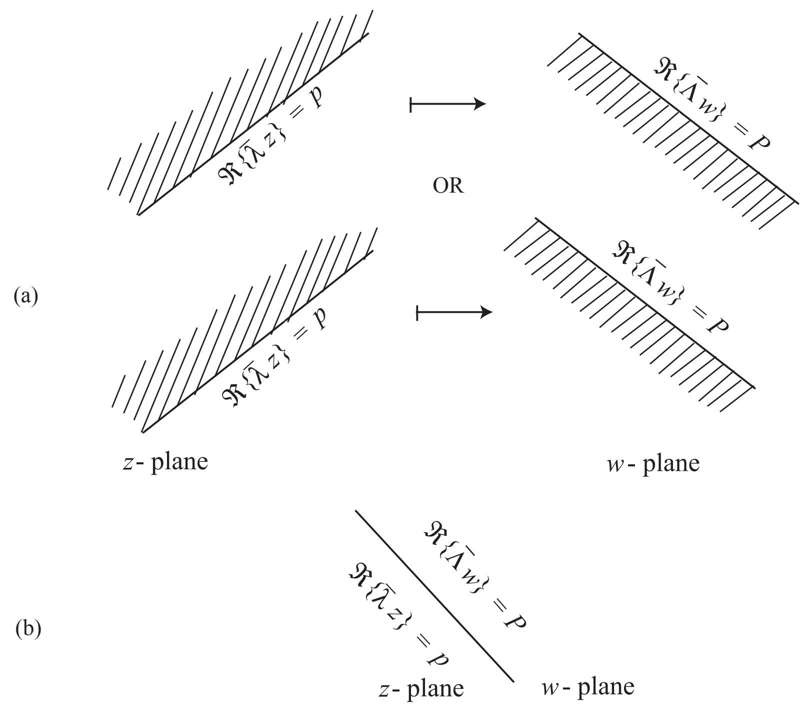

Figure 3.50 Straight line onto straight line.

Map 3.58. Angle onto itself, with arms interchanged. Given, z_0, $\alpha \neq 0$ $(-\pi < \alpha < \pi)$, and τ, where α and tau are real, the case is presented in Figure 3.51. For involutory transformation $(w - w_0)(z - z_0) = b\,e^{i(\alpha+2\tau)}$, $b > 0$, the regions I and II are mapped onto the regions I and II, respectively; but for $b < 0$, the region I is mapped onto the region II. Other details for this case, with $z = p, q$ real and $w = P, Q$ real, are as follows:

The points $z_0; z_1 = \infty; z_2 = z_0 + ae^{i\tau}$, where a is real, $a > 0$ in Figure 3.51, are mapped onto the points $w_0 = \infty; w_1 = z_0; w_2 = z_0 + \dfrac{b}{a}\,e^{i(\alpha+\tau)}$.

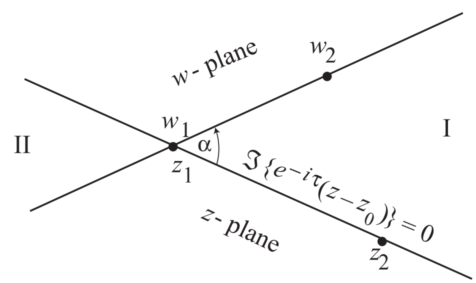

Figure 3.51 Angle onto itself.

Map 3.59. Straight line onto circle. The transformation is

$$w - w_0 = \Re\left\{e^{i\tau}\,\frac{\bar{\lambda}z - p + \beta}{\bar{\lambda}z - p - \beta}\right\}, \quad \tau \text{ real; } \Re\{\beta\} \neq 0, \tag{3.4.7}$$

given in Figure 3.52.

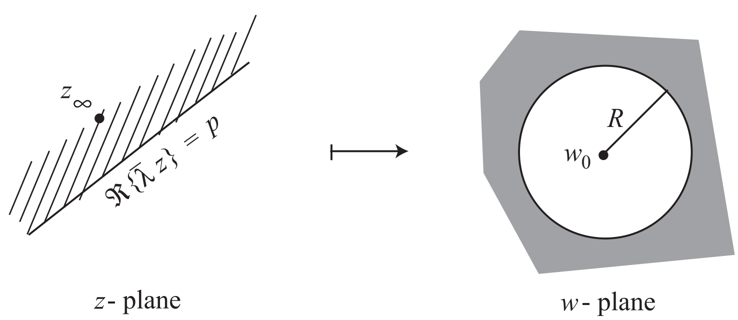

Figure 3.52 Straight line onto circle.

Map 3.60. Circle onto straight line. The transformation is

$$\bar{\Lambda}w = P + i\,a + b\frac{z - z_0 + r\,e^{i\tau}}{z - z_0 - r\,e^{i\tau}}, \tag{3.4.8}$$

where a, b, τ are real, $b \neq 0$, given in Figure 3.53.

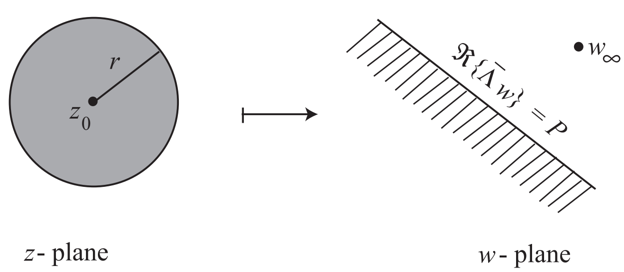

Figure 3.53 Circle onto straight line.

Map 3.61. Circle and line in-contact mapped onto two parallel lines. Given $z_0, z_2, \Lambda, P_1, P_2$, the transformation is

$$\bar{\Lambda} w = i\,a + P_1 + \frac{2\,(z_2 - z_0)\,(P_2 - P_1)}{z - z_0}, \tag{3.4.9}$$

where a is real, and $r = |z_2 - z_0|, \lambda = z_2 - z_0$, and is presented in Figure 3.54.

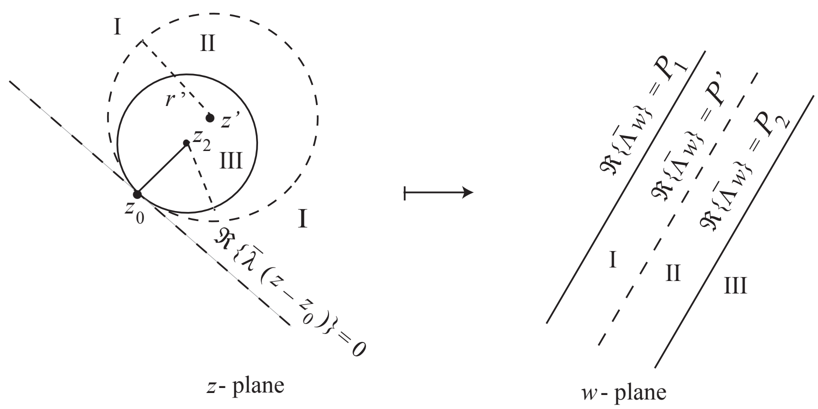

Figure 3.54 Circle onto circle.

The details of the transformation with $z = p, q$ real and $w = P, Q$ real, are as follows:
(a) The line $\Re\{\bar{\lambda} z\} = \Re\{\bar{\lambda} z_0\}$ is mapped onto the line $\Re\{\bar{\Lambda} w\} = P_1$.
(b) The circle $|z - z_2| = r$ is mapped onto the line $\Re\{\bar{\Lambda} w\} = P_2$.
(c) The circle $|z - z'| = r'$ is mapped onto the line $\Re\{\bar{\Lambda} w\} = P' = P_1 + \dfrac{r}{r'}\,(p_2 - p_1)$.

Map 3.62. Two circles in-contact mapped onto two parallel lines. (Inner Contact) Given $r_1 > 0$, $r_2 > 0$, z_1, z_2, and a real, the transformation is

$$\bar{\Lambda} w = P_1 + i\,a + \gamma \frac{z + z_0 - 2z_1}{z_0 - z}, \tag{3.4.10}$$

where $\gamma = r_2 \dfrac{P_2 - P_1}{r_1 - r_2}$. The details are provided in Figure 3.55 and where $z_0 = \dfrac{r_1 z_2 - r_2 z_1}{r_1 - r_2}$.

The details of the transformation with $z = p, q$ real and $w = P, Q$ real, are as follows:
(a) The circle $|z - z_1| = r_1$ is mapped onto the line $\Re\{\bar{\Lambda} w\} = P_1$.
(b) The circle $|z - z_2| = r_2$ is mapped onto the line $\Re\{\bar{\Lambda} w\} = P_2$.
(c) The circle $|z - z'| = r'$ touching at z_0 is mapped onto the line $\Re\{\bar{\Lambda} w\} = P'$, where $P' = P_1 + (P_2 - P_1)\dfrac{r_2}{r_1}\dfrac{r' - r_1}{r_2 - r_1}$.

(d) The line $\Re\{\bar{\lambda}\,(z - z_0)\} = 0$ touching at z_0, where $\lambda = z_1 - z_2$, is mapped onto the line $\Re\{\bar{\Lambda} w\} = P_0$, where $P_0 = \dfrac{P_1 r_1 - P_2 r_2}{r_1 - r_2}$.

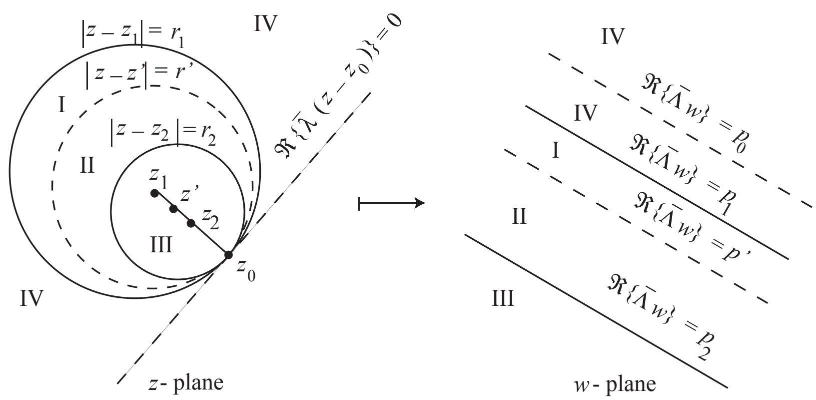

Figure 3.55 Two circles in contact mapped onto two parallel lines (inner contact).

Map 3.63. Two circles in outer contact mapped onto parallel lines. (Outer Contact) Given $|z_1 - z_2| = r_1 + r_2$, transformation is

$$\bar{\Lambda} w = P_1 + i\,a + \gamma \frac{z + z_0 - 2z_1}{z_0 - z}, \tag{3.4.11}$$

where a is real, and $\gamma = r_2 \dfrac{P_1 - P_2}{r_1 + r_2}$. This case is presented in Figure 3.56, where $z_0 = \dfrac{r_1 z_2 + r_2 z_1}{r_1 + r_2}$.

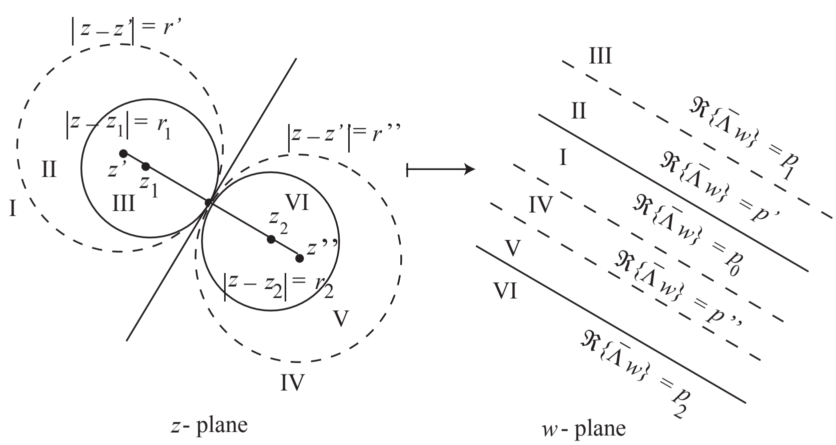

Figure 3.56 Two circles in outer contact mapped onto two parallel lines (outer contact).

The details of the transformation with $z = p, q$ real and $w = P, Q$ real, are as follows:

(a) The circle $|z - z_1| = r_1$ is mapped onto the line $\Re\{\bar{\Lambda} w\} = P_1$.

(b) The circle $|z - z_2| = r_2$ is mapped onto the line $\Re\{\bar{\Lambda} w\} = P_2$.

(c) circle $|z - z'| = r'$ touching at z_0 is mapped onto the line $\Re\{\bar{\Lambda} w\} = P'$, where $P' =$

$P_2 + (P_1 - P_2)\dfrac{r_1}{r'}\dfrac{r' + r_2}{r_1 + r_2}$.

(d) The circle $|z - z''| = r''$ touching at z_0, where $\lambda = z_1 - z_2$, is mapped onto the line $\Re\{\bar{\Lambda}w\} + P''$, where $P'' = P_1 + (P_2 - p_1)\dfrac{r_2}{r''}\dfrac{r'' + r_1}{r_1 + r_2}$.

(e) The line $\Re\{\bar{\lambda}(z - z_0)\} = 0$ touching at z_0, where $\lambda = z_1 - z_2$, is mapped onto the line $\Re\{\bar{\Lambda}w\} = P_0$, where $P_0' = \dfrac{P_1 r_1 + P_2 r_2}{r_1 + r_2}$.

Map 3.64. Two 'circles', intersecting at two points, mapped onto two intersecting straight lines. Given z_1; $r_1 > 0$; z_2; $r_2 > 0$ and $\alpha = \beta$, i.e., $\dfrac{\bar{\Lambda}_1}{\bar{\Lambda}_2}\dfrac{z_0 - z_1}{z_0 - z)2}$ is real, the transformation is presented in Figure 3.57, where $|r_1 - r_2| < |z_1 - z_2| < r_1 + r_2$, and z_0, w_0 are the points of intersection of the z- and w-plane, respectively.

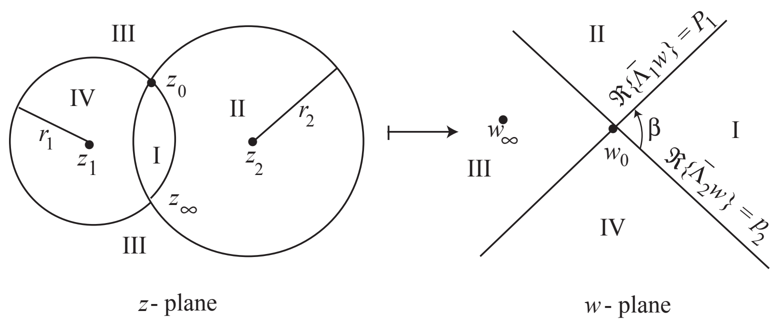

Figure 3.57 Two intersecting 'circles'.

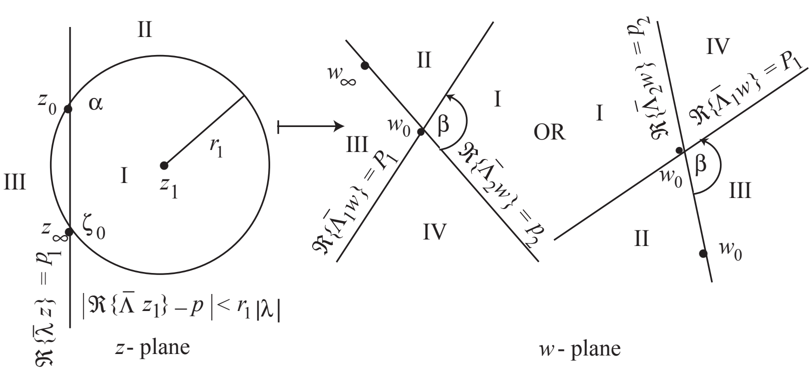

Figure 3.58 Two intersecting 'circles', alternative case.

An alternative situation is considered in Figure 3.58, where z_0 and ζ_0 are the points of intersection of the two 'circles' in the z-plane; w_0 is the point of intersection of $\Re\{\Lambda_1 w\} = P_1$ and $\Re\{\Lambda_1 w\} = P_2$. The required transformation is

$$w = w_0 + K\Lambda_1 \frac{(z - z_0)(z_1 - \zeta_0)}{(z - \zeta_0)(z_0 - \zeta_0)}, \quad K \neq 0 \text{ real.} \tag{3.4.12}$$

The details of the alternative case with $z = p, q$ real and $w = P, Q$ real, are as follows:

(a) The points $z = 0;\ \infty;\ 2r_1$ is mapped onto the points $w = \infty;\ \dfrac{ir_2}{r_1 + r_2};\ 0$.

(b) The circle $|z - z_1| = r_1$ is mapped onto the line $\Re\{\bar\Lambda_1 w\} = P_1$.

(c) The circle $|z - z_2| = r_2$ is mapped onto the line $\Re\{\bar\Lambda_2 w\} = P_2$.

(d) The circle $|z + r_2| = r_2$ is mapped onto the line $\Re\{\bar\Lambda_2 w\} = P_2$, or line $\Re\{\bar\lambda z\} = p$.

(e) The points $z = z_0; \zeta_0; z = \infty$ are mapped onto the points $w = w_0;\ \infty;\ w_\infty = w_0 + K\Lambda_1 \dfrac{z_1 - \zeta_0}{z_0 - \zeta_0}$.

(f) The circle passing through z_0, ζ_0 is mapped onto the lines passing through w_0.

(g) The circle orthogonal to $|z - z_0| = r_1$, and to $|z - z_2| = r_2$ or to $\Re\{\bar\lambda z\} = p$ is mapped onto the circle with center w_0.

Map 3.65. The transformation presented in Figure 3.59 is

$$w = \frac{K}{\sigma} \frac{z(i\,\sigma - z_1)}{z - i\,\sigma}, \quad \sigma > 0, K > 0. \tag{3.4.13}$$

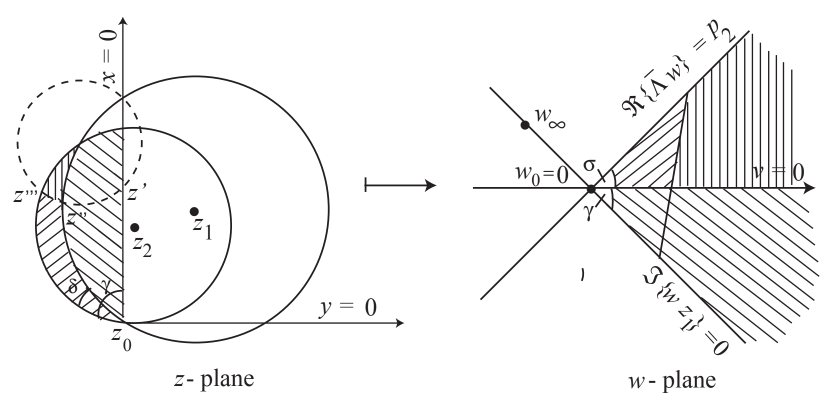

Figure 3.59 Two intersecting 'circles'.

The details of the transformation, with $z = p, q$ real and $w = P, Q$ real, are as follows:

(a) The points $\zeta_0 = i\sigma; z_0 = 0; z'; z''; z'''$ are mapped onto the points $w = \infty; w_0 = 0; w'; w''; w'''$.

(b) The point $z = \infty$ is mapped onto the $w_\infty = \dfrac{K}{\sigma}(i\,\sigma - z_1) = -\dfrac{K}{\sigma}\bar z_1$.

(c) The circle $|z - z_1| = r_1$, $\Im\{z_1\} = \frac{1}{2}\sigma$ is mapped onto the line $v = 0$.

(d) The circle $|z - z_2| = r_2$, $\Im\{z_2\} = \frac{1}{2}\sigma$ is mapped onto the line $\Re\{\bar\Lambda_2 w\} = P_2$.

(e) The line $x = 0$ is mapped onto the line $\Im\{wz_1\} = 0$.

(f) The circle (z', z'', z''', ζ_0) is mapped onto the line (w', w'', w''').

(g) The curvilinear triangle $\{z_0, z', z''\}$ is mapped onto the right triangle $\{w, w', w''\}$.

(h) The curvilinear triangle $\{z_0, z'', z'''\}$, each with sum of angles, is mapped onto the right triangle $\{w_0, w'', w'''\}$.

Map 3.66. CIRCLES AND STRAIGHT LINE, WITHOUT COMMON POINT, ONTO TWO CONCENTRIC CIRCLES. The transformation, presented in Figure 3.60, is

$$w - w_1 = R_1\, e^{i\tau} \frac{(z - z_0)|\lambda| - \lambda(A + \sigma)}{(z - z_0)|\lambda| - \lambda(A - \sigma)}, \tag{3.4.14}$$

where τ is real, $A = \dfrac{p - \Re\{\bar{\lambda} z_0\}}{|\lambda|}$, and $\sigma = \sqrt{A^2 - \rho^2} > 0$ or $\sigma = -\sqrt{A^2 - \rho^2}$.

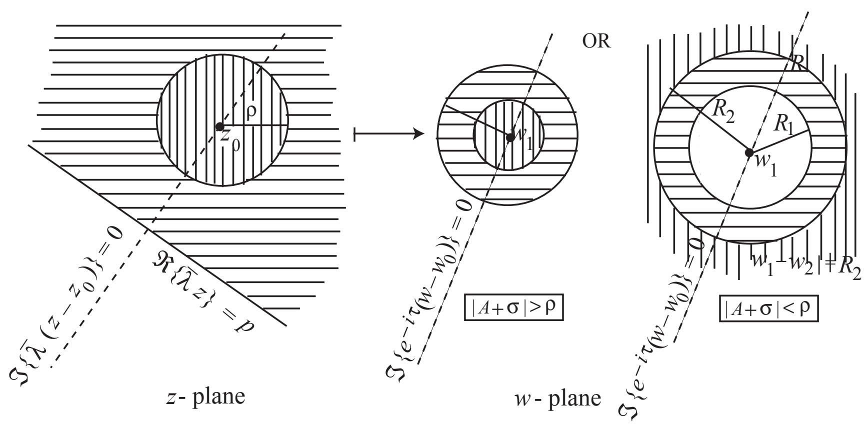

Figure 3.60 Circles and straight lines.

The details of the transformation, with $z = p, q$ real and $w = P, Q$ real, are as follows:
(a) The line $\Re\{\bar{\lambda} z\} = p$ is mapped onto the circle $|w - w_0| = R_1$.
(b) The circle $|z - z_0| = \rho$ is mapped onto the circle $|w - w_1| = R_2$.
(c) The line $\Im\{\bar{\lambda}(z - z_0)\} = 0$ is mapped onto the line $\Im\{e^{-i\tau}(w - w_1)\} = 0$.

Map 3.67. TWO CIRCLES, WITHOUT COMMON POINT, ONTO TWO CONCENTRIC CIRCLES. Given $z_1, z_2, r_1 > 0, r_2 > 0$, $z_1 \neq z_2$, and $w_0, R_1 > 0, R_2 > 0$, and $\dfrac{R_2}{R_1} = \dfrac{r_2}{r_1}\left|\dfrac{t}{s - t}\right|$, where s, t are the real roots of the equations $st = r_1^2$, and $(d - s)(d - t) = r_2^2$, $d = |z_2 - z_1| > 0$, the transformation is

$$w - w_0 = t\frac{R_1}{r_1}\, e^{i\theta}\, \frac{d(z - z_1) - s(z_2 - z_1)}{d(z - z_1) - t(z_2 - z_1)}, \quad \theta \text{ real.} \tag{3.4.15}$$

This transformation is presented Figure 3.61.

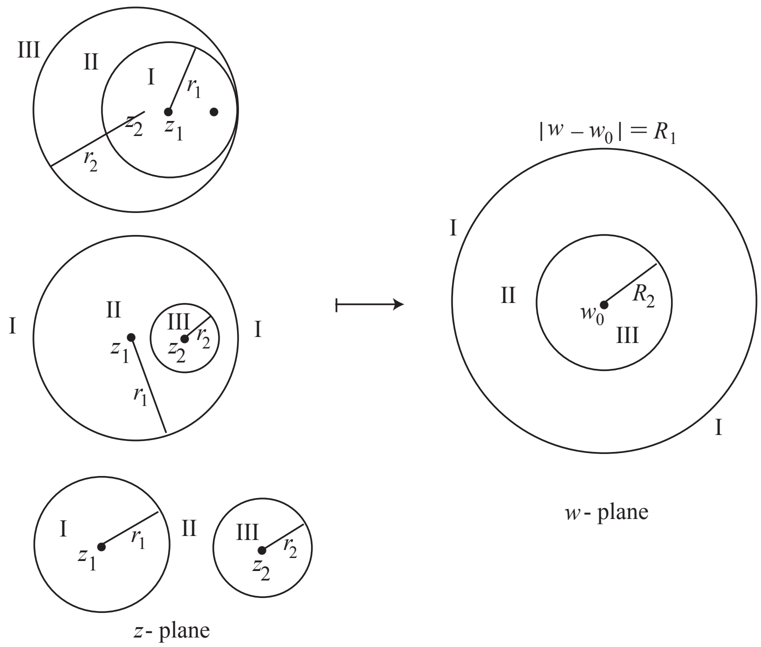

Figure 3.61 Transformation (3.2.35).

The details of this transformation, with $z = p, q$ real and $w = P, Q$ real, are as follows:
(a) The circle $|z - z_1| = r_1$ is mapped onto the circle $|w - w_0| = R_1$.
(b) The circle $|z - z_2| = r_2$ is mapped onto the circle $|w - w_0| = R_2$.
(c) The radical axis of these circles is mapped onto the circle $|w - w_0| = R_1|t|/r_1$.
(d) The point $z_\infty = z_1 + (t/d)(z_2 - z_1)$ is mapped onto the point $w = \infty$.
(e) The point $z_0 = z_1 + (s/d)(z_2 - z_1)$ is mapped onto the point $w = w_0$.
(f) Any 'circle' passing through z_∞ and z_0 is mapped onto the line passing through w_0.

Map 3.68. The transformation, presented in Figure 3.62, is

$$w = R_1 \frac{z - i\sigma}{z + i\sigma}, \tag{3.4.16}$$

where $\sigma = \sqrt{k^2 - \rho^2}$, $0 < \rho < k$.

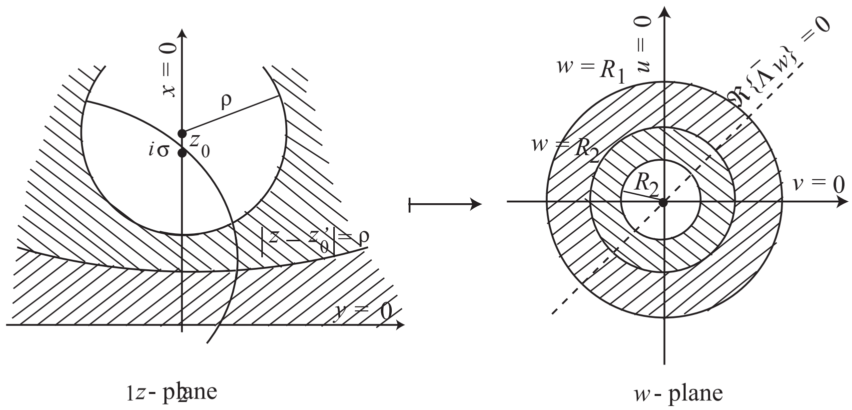

Figure 3.62 Transformation (3.2.33).

The details of the transformation, with $z = p, q$ real and $w = P, Q$ real, are as follows:

(a) The points $z = 0; \infty; i\sigma; -i\sigma$ are mapped onto the points $w = -R_1; R_1; 0; \infty$, respectively.

(b) The line $x = 0$ is mapped onto the line $v = 0$.

(c) The line $y = 0$ is mapped onto the circle $|w| = R_1$.

(d) The circle $|z - ik| = \rho'$, where $z_0' = i\sigma \dfrac{R_1^2 + R^2}{R_1^2 - R^2}$, $\rho' = \dfrac{2\sigma R_1 R}{|R_1^2 - R^2|}$, and $R_1 \neq R$, is mapped onto the circle $|w| = R_2 = \dfrac{R_1 \rho}{|k + \sigma|}$.

(e) The circle $|z| = \sigma$ is mapped onto the line $u = 0$.

(f) The circle $\left|z - \sigma \dfrac{\Im\{\Lambda\}}{\Re\{\Lambda\}}\right| = \sigma \left|\dfrac{\Lambda}{\Re\{\lambda\}}\right|$, where $\Re\{\Lambda\} \neq 0$, intersecting $x = 0$ at $\pm i\sigma$, is mapped onto the line $\Re\{\bar{\Lambda} w\} = 0$.

Map 3.69. The transformation

$$w = \frac{tR_1}{r_1} \frac{z - s}{z - t}, \tag{3.4.17}$$

where s and t are roots of the equations $st = r_1^2$ and $(z_2 - s)(z_2 - t) = r_2^2$, is presented in Figure 3.63.

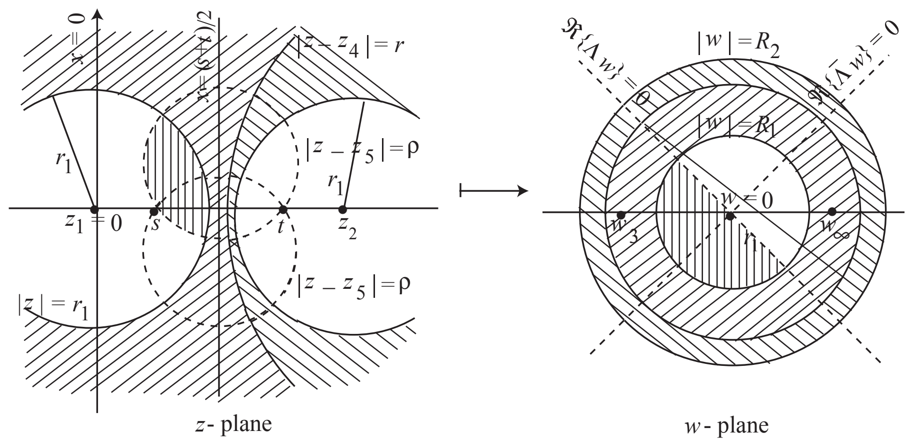

Figure 3.63 Transformation (3.4.17).

The details of this transformation, with $z = p, q$ real and $w = P, Q$ real, are as follows:

(a) The circles $|z - z_{1,2}| = r_{1,2}$, $z_1 = 0, z_2 > 0$ are mapped onto the circles $|w| = R_{1,2}$; $\dfrac{R_2}{R_1} = \dfrac{r_2}{r_1}\dfrac{t}{r_2 - t}$.

(b) The points $z_0 = s; z_\infty = t; z = \infty$; and $z_3 = \dfrac{s+t}{2}$ are mapped onto the points $w_0 = 0, \infty; w_\infty = \dfrac{tR_1}{r_1}$; and $w_3 = -\dfrac{tR_1}{r_1}$.

(c) The circle $|z - z_0| = r$, where $z_4 = r_1^2 \dfrac{R_1^2 - R^2}{tR_1^2 - sR_1^2}$ and $r = \dfrac{r_1 R_1 R|t - s|}{|sR_1^2 - tR_1^2|}$, is mapped onto the circle $|w| = R \neq \infty$.

(d) The line $x = \dfrac{s+t}{2}$ is mapped onto the circle $|w| = \dfrac{R_1 t}{r_1} = w_\infty$.

(e) The circle $|z - z_5| = \rho$, where $z_5 = \dfrac{t\Lambda + s\bar{\Lambda}}{2\Re\{\Lambda\}}$ and $\rho = \left|\dfrac{\Lambda(s-t)}{2\Re\{\Lambda\}}\right|$, $\Re\{\Lambda\} \neq 0$ is mapped onto the line $\Re\{\bar{\Lambda} w\} = 0$.

(f) The circle $|z - \bar{z}_5| = \rho$ is mapped onto the line $\Re\{\Lambda w\} = 0$.

(g) The circle $y = 0$ is mapped onto the line $v = 0$.

(h) The curvilinear rectangular quadrilateral formed by $|z| = r_1, |z - z_2| = r_2$ between $|w| = R_1$, $|w| = R$ and $|z - z_5| \leq \rho$, is mapped onto the half-ring left of $\Re\{\bar{\Lambda}\} = 0$.

(i) The region $|z - z_2| \leq r_2$ is mapped onto the region $|w| \geq R_2$.

3.5 Ellipses and Hyperbolas

For ellipses and hyperbolas, we will set t, t', and τ fixed, $0 < t < t' < 1$, and $0 < \tau < \pi/2$; also $l = t + 1/t$, $l' = t' + 1/t'$, and $\sigma = \arg\{k/\alpha\}$. The ellipse has been defined in §3.1.3, and the hyperbolas in §3.1.4.

Map 3.70. In the case of ellipses, we find that (i) the regions $|z| < |k/\alpha| t$ and $|z| > |k/\alpha| t^{-1}$ are both mapped onto the region $|w + 2k| + |w - 2k| > 2l|k|$ which is the exterior of the ellipses; (ii) the annular regions $t|k/\alpha| < |z| < |k/\alpha|$ and $t^{-1}|k/\alpha| > |z| > |k/\alpha|$ are both mapped onto the region $2l|k| > |w+2k| + |w-2k| > 4|k|$ which is the exterior of the same ellipses, excluding the segment joining the foci; and (iii) the annular regions $t|k/\alpha| < |z| < t'|k/\alpha|$ and $t^{-1}|k/\alpha| > |z| > (1/t')|k/\alpha|$ are both mapped onto the region $2l|k| > |w+2k| + |w-2k| > 2l'|k|$ which is the annular region bounded by two confocal ellipses.

Map 3.71. The transformation $z = \frac{1}{2}\left[(a-b)w + \dfrac{a+b}{w}\right]$ maps the exterior of an ellipse with semi-axes a and b: $(x/a)^2 + (y/b)^2 \geq 1$, in the z-plane onto the unit disk $|w| \leq 1$ in the w-plane.

Map 3.72. In the case of hyperbolas, we find that (i) the half-lines $\arg\{z\} = \sigma + \tau$ and $\arg\{z\} = \sigma - \tau$ are both mapped onto the line $|w+2k| - |w-2k| = 4|k|\cos\tau$ which is one branch of the hyperbola, with asymptotes $\arg\{w\} = \arg\{k \pm \tau\}$; (ii) the half-lines $\arg\{z\} = \sigma+\tau+\pi$ and $\arg\{z\} = \sigma-\tau+\pi$ are both mapped onto the line $|w+2k|-|w-2k| = -4|k|\cos\tau$ which is the other branch of the hyperbola; (iii) the half-lines $\arg\{z\} = \sigma + \pi/2$ and $\arg\{z\} = \sigma - \pi/2$ are both mapped onto the line $\Re\{\bar{k}w\} = 0$; (iv) the half-line $\arg\{z\} = \sigma$ is mapped onto the half-line $|w| > 2|k|$, $\arg\{w/k\} = 0$, counted twice; and (v) the half-line $\arg\{z\} = \sigma + \pi$ is mapped onto the half-line $|w| > 2|k|$, $\arg\{w/k\} = \pi$, counted twice. These properties are presented in Figure 3.64.

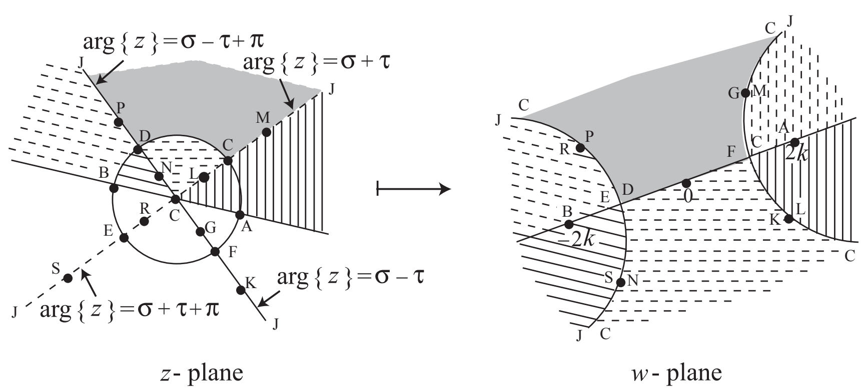

Figure 3.64 Mapping of different regions onto different regions of a hyperbola.

References used: Ahlfors [1966], Boas [1987], Carrier, Krook and Pearson [1966], Ivanov and Trubetskov [1995], Kantorovich and Krylov [1958], Kober [1957], Kythe [1996], Nehari [1952], Nevanlinna and Paatero [1969], Pennisi et al. [1963], Papamichael, Warby and Hough [1986], Saff and Snider [1976], Schinzinger and Laura [2003], Silverman [1967], Wen [1992].