Problem #1 (10 points): Verify if the function $f(x,y) = \sin x \cosh y + i \cos x \sinh y$ satisfies the Cauchy-Riemann conditions. If it does, find the associated analytic function $f(z)$.

Solution: Let $f(x,y) = u(x,y) + iv(x,y)$ where u and v are real. Then $u = \sin x \cosh y$ and $v = \cos x \sinh y$ s.t.

$$u_x = \cos x \cosh y = v_y, \qquad v_x = -\sin x \sinh y = -u_y,$$

i.e. CR conditions hold.

$$f(z) = \frac{(e^{ix} - e^{-ix})(e^y + e^{-y})}{4i} + i\frac{(e^{ix} + e^{-ix})(e^y - e^{-y})}{4} =$$

$$= \frac{e^{ix}e^{-y} - e^{-ix}e^y}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \sin z.$$

Problem #2 (20 points): Given the imaginary part, $v(x,y)$, of an analytic function, $f(z) = u(x,y) + iv(x,y)$, find the real part, $u(x,y)$, and the analytic function.

(a) $v(x,y) = x(c-y)$, where c is constant.
(b) $v(x,y) = \frac{x}{x^2+y^2}$.

Solution:

(a) $v(x,y) = x(c-y)$, where c is constant.

$$v_x = c - y = -u_y \quad \Longrightarrow \quad u = -cy + y^2/2 + f(x),$$

$$v_y = -x = u_x \quad \Longrightarrow \quad u = -x^2/2 + g(y),$$

therefore

$$u = (y^2 - x^2)/2 - cy + \text{ const.},$$

$$f = (y^2 - x^2)/2 - cy + \text{ const.} + ix(c-y) =$$

$$= -(x+iy)^2/2 + ic(x+iy) + \text{ const.},$$

i.e.

$$f(z) = -z^2/2 + icz + \text{ const.}.$$

(b) $v(x,y) = \frac{x}{x^2+y^2}$, i.e. $v(r,\theta) = \frac{\cos\theta}{r}$.

$$v_r = -\frac{\cos\theta}{r^2} = -\frac{u_\theta}{r} \quad \Longrightarrow \quad u = \frac{\sin\theta}{r} + f(r)$$

$$v_\theta = -\frac{\sin\theta}{r} = r u_r \quad \Longrightarrow \quad u = \frac{\sin\theta}{r} + g(\theta)$$

therefore

$$u = \frac{\sin\theta}{r} + \text{ const.},$$

$$f = \frac{\sin\theta}{r} + \text{ const.} + i\frac{\cos\theta}{r} =$$

$$= i\frac{\cos\theta - i\sin\theta}{r} + \text{ const.} = i\frac{e^{-i\theta}}{r} + \text{ const.} =$$

$$= i\frac{\bar{z}}{z\bar{z}} + \text{ const.} = \frac{i}{z} + \text{ const.}$$

Problem #3 (15 points): Determine where the following functions are analytic; find singular points.

(a) $\frac{1}{z^3+1}$.
(b) $\sec z$.
(c) $\exp(\cos^2 z)$.

Solution:

(a) $\frac{1}{z^3+1}$. It is analytic everywhere except for roots of equation $z^3 + 1 = 0$, which are s.t.

$$z^3 = r^3 e^{3i\theta} = -1 = 1 \cdot e^{\pi i + 2\pi i k}$$

$$\Longrightarrow \; r = 1, \quad \theta = \frac{\pi(1+2k)}{3}, k \in \mathbb{Z},$$

i.e. different singular points are

$$z = -1, \quad z = e^{\pi i/3}, \quad z = e^{5\pi i/3}.$$

(b) $\sec z$.

$$\sec z = \frac{1}{\cos z},$$

a ratio of functions analytic in the whole $\mathbb{C}$, so it is analytic except for points where $\cos z = 0$, i.e. $z = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$.

(c) $\exp(\cos^2 z)$. It is analytic everywhere in $\mathbb{C}$, being a composition of analytic functions, i.e. entire.

Problem #4 (15 points): Show that the real and imaginary parts of a twice-differentiable function $f(\bar{z})$ satisfy Laplace's equation. Show that $f(\bar{z})$ is nowhere analytic unless it is constant.

Solution: Let $f(\bar{z}) = f(x - iy) = u(x,y) + iv(x,y)$ where u and v are real. Then

$$f_x = f'(\bar{z}) = u_x + iv_x, \qquad f_y = -if'(\bar{z}) = u_y + iv_y.$$

Thus,

$$f'(\bar{z}) = u_x + iv_x = i(u_y + iv_y),$$

which implies

$$u_x = -v_y, \qquad v_x = u_y.$$

Differentiating these relations, one gets

$$u_{xx} + u_{yy} = -v_{xy} + v_{yx} = 0, \qquad v_{xx} + v_{yy} = u_{xy} - u_{yx} = 0.$$

For analyticity, f must satisfy CR relations $u_x = v_y$, $v_x = -u_y$. But then $u_x = v_y = -v_y = 0$ etc., i.e. $u_x = u_y = v_x = v_y = 0$, which means u and v are constants and so is f.

Problem #5 (15 points): Consider the following complex potential

$$\Omega(z) = -\frac{k}{2\pi z}, \qquad k \in \mathbb{R},$$

referred to as a *doublet*. Calculate the corresponding velocity potential, stream function, and velocity field. Sketch the stream function (streamlines).

Solution:

$$\Omega(z) = -\frac{k}{2\pi z} = u + iv = -\frac{k(x - iy)}{2\pi(x^2 + y^2)},$$

i.e. the velocity potential is

$$u(x,y) = -\frac{kx}{2\pi(x^2 + y^2)},$$

the stream function is

$$v(x,y) = \frac{ky}{2\pi(x^2 + y^2)},$$

and the velocity field is given by 2-vector

$$\left(u_x = v_y = \frac{k(x^2 - y^2)}{2\pi(x^2 + y^2)^2}, u_y = -v_x = \frac{kxy}{\pi(x^2 + y^2)^2} \right).$$

Extra-Credit Problem #6 (? points): Consider the complex analytic function, $\Omega(z) = \phi(x,y) + i\psi(x,y)$, in a domain D. Let us transform from z to w using $w = f(z)$, $w = u + iv$, where $f(z)$ is analytic in D, with the corresponding domain in the w plane, D'. Establish the following:

$$\frac{\partial\phi}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial\phi}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial\phi}{\partial v},$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}\frac{\partial\phi}{\partial u} - \frac{\partial^2 u}{\partial x\partial y}\frac{\partial\phi}{\partial v} + \left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\phi}{\partial u^2} -$$

$$-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2\phi}{\partial u\partial v} + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2\phi}{\partial v^2}.$$

Also find the corresponding formulae for $\partial\phi/\partial y$ and $\partial^2\phi/\partial y^2$. Recall that $f'(z) = \partial u/\partial x - i\partial u/\partial y$, and $u(x,y)$ satisfies LaplaceÕs equation in the domain D. Show that

$$\nabla^2_{x,y}\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} =$$

$$= (u_x^2 + u_y^2)\left(\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right) = |f'(z)|^2\nabla^2_{u,v}\phi.$$

Consequently, we find that if ϕ satisfies LaplaceÕs equation $\nabla^2_{x,y}\phi = 0$ in the domain D, then so long as $f'(z) \neq 0$ in D it also satisfies LaplaceÕs equation $\nabla^2_{u,v}\phi = 0$ in domain D'.

Solution: Considering ϕ as function of u and v, we find

$$\frac{\partial\phi}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial\phi}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial\phi}{\partial v},$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial\phi}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial\phi}{\partial v},$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial\phi}{\partial x} =$$

$$= \frac{\partial^2 u}{\partial x^2}\frac{\partial\phi}{\partial u} + \left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\phi}{\partial u^2} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2\phi}{\partial u\partial v} +$$

$$+\frac{\partial^2 v}{\partial x^2}\frac{\partial\phi}{\partial v} + \left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2\phi}{\partial v^2} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2\phi}{\partial u\partial v} =$$

$$= \frac{\partial^2 u}{\partial x^2}\frac{\partial\phi}{\partial u} - \frac{\partial^2 u}{\partial x\partial y}\frac{\partial\phi}{\partial v} + \left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\phi}{\partial u^2} -$$

$$-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2\phi}{\partial u\partial v} + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2\phi}{\partial v^2},$$

where in the last equality CR condition $v_x = -u_y$ is used. Similarly,

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial\phi}{\partial y} =$$

$$= \frac{\partial^2 u}{\partial y^2}\frac{\partial\phi}{\partial u} + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2\phi}{\partial u^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2\phi}{\partial u\partial v} +$$

$$+\frac{\partial^2 v}{\partial y^2}\frac{\partial\phi}{\partial v} + \left(\frac{\partial v}{\partial y}\right)^2\frac{\partial^2\phi}{\partial v^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2\phi}{\partial u\partial v} =$$

$$= \frac{\partial^2 u}{\partial y^2}\frac{\partial\phi}{\partial u} + \frac{\partial^2 u}{\partial x\partial y}\frac{\partial\phi}{\partial v} + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2\phi}{\partial u^2} +$$

$$+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2\phi}{\partial u\partial v}+\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\phi}{\partial v^2},$$

where $v_y = u_x$ is used in the last equality. Adding up the two derived expressions for second derivatives, after cancellations, the last formulas follow when we also use that $u_{xx}+u_{yy}=0$ and $|f'(z)|^2 = |u_x - iu_y|^2 = u_x^2 + u_y^2$.

Problem #7 (20 points): Find the location of the branch points and discuss possible branch cuts for the following functions:

(a) $(z-i)^{1/3}$
(b) $\log\frac{1}{z-2}$

Solution:

(a) Let $z-i=\epsilon e^{i\theta_p}$ which is a circular contour centered at $z=i$. We have just a power function in terms of $\zeta = z-i$, so $z=i$ and $z=\infty$ are branch points. Any line connecting $z=\infty$ and $z=i$ is a branch cut, e.g. $\{z=iy|y\in[1,+\infty)\}$ is as good as any. There are 3 distinct branches.

(b) $\log\frac{1}{z-2}=-\log(z-2)$. Again this is $-\log z$ but with shifted origin. So the branch points are $z=2$ and $z=\infty$. A branch cut must connect the branch points, it can be e.g. $\{z=x|x\in[2,+\infty)\}$ or $\{z=x|x\in(-\infty,2]\}$.

Problem #8 (10 points): Solve for all values of z: $4+2e^{z+i}=2$.

Solution:

$$4+2e^{z+i}=2 \qquad \Leftrightarrow \qquad e^{z+i}=-1=e^{i\pi+2\pi in},\ n\in\mathbb{Z},$$

therefore

$$z+i=i\pi+2\pi in \qquad \Leftrightarrow \qquad z=i(\pi-1+2\pi n),\ n\in\mathbb{Z}.$$

Problem #9 (15 points): Derive $\tan^{-1}z=\frac{1}{2i}\log\frac{i-z}{i+z}$ and then find $\frac{d}{dz}\tan^{-1}z$.

Solution: One needs to find $w=f(z)$ such that $z=\tan w$. Then

$$z=\frac{\sin w}{\cos w}=\frac{e^{iw}-e^{-iw}}{i(e^{iw}+e^{-iw})}.$$

Let $\zeta=e^{iw}$, then $e^{-iw}=1/\zeta$. Substituting these into the above equation, we find

$$iz(\zeta+1/\zeta)=\zeta-1/\zeta$$

or

$$(1-iz)\zeta^2=1+iz \qquad \Leftrightarrow \qquad \zeta^2=\frac{i-z}{i+z},$$

i.e.

$$e^{2iw}=\frac{i}{i+z} \qquad \Leftrightarrow \qquad w=\frac{1}{2i}\log\frac{i-z}{i+z}.$$

Then

$$\frac{d}{dz}\tan^{-1}z=w'(z)=\frac{1}{2i}\left(-\frac{1}{i-z}-\frac{1}{i+z}\right)=$$

$$=-\frac{1}{(i-z)(i+z)}=\frac{1}{z^2+1},$$

as in the real case (as should be).

Problem #10 (15 points): Consider the complex velocity potential $\Omega(z)=-ik\log(z-z_0)$, where k is real. Find the corresponding velocity potential and stream function. Show that the velocity is purely circumferential relative to the point $z=z_0$, being counterclockwise if $k>0$. Sketch the flow configuration. The strength of this flow, called a point vortex, is defined to be $M=\oint_C V_\theta\,ds$, where V_θ is the velocity in the circumferential direction and ds is the increment of arc length in the direction tangent to the circle C. Show that $M=2\pi k$.

Solution: Let $\Omega(x,y)=\phi(x,y)+i\psi(x,y)$. Since $\log(z?z_0)=\log|z-z_0|+i\theta$, where θ is the angle between the line connecting z_0 and z and positive x direction. Then the velocity potential $\phi=k\theta$ and the stream function $\psi=-k\log r$, where $r=|z-z_0|=\sqrt{(x-x_0)^2+(y-y_0)^2}$ and $\theta=\tan^{-1}\frac{y-y_0}{x-x_0}$. For the components of the velocity field V we get

$$V_r=\frac{\partial\phi}{\partial r}=0, \qquad V_\theta=\frac{1}{r}\frac{\partial\phi}{\partial\theta}=\frac{k}{r},$$

so we have only nonzero V_θ component which means that the velocity is purely circumferential relative to the point $z=z_0$ and $\text{sign}(V_\theta)=\text{sign}(k)$ means it is counterclockwise if $k>0$. To compute M, let C be a circle of radius R around z_0. Then

$$M=\oint_C V_\theta\,ds=\int_0^{2\pi}\frac{k}{R}\cdot R\,d\theta=2\pi k.$$

Problem #11 (20 points): Show that the solution to Laplace equation $\nabla^2T=\partial^2T/\partial u^2+\partial^2T/\partial v^2=0$ in the region $-\infty<u<\infty$, $v>0$, with the boundary conditions $T(u,0)=T_0$ if $u>0$ and $T(u,0)=-T_0$ if $u<0$, is given by

$$T(u,v)=T_0\left(1-\frac{2}{\pi}\tan^{-1}\frac{v}{u}\right).$$

Solution: From the text we have solutions to Laplace's equation,

$$\begin{aligned}\Omega(z)&=A\log w+iB\\&=A\log(re^{i\theta})+iB\\&=A\log r+i\underbrace{(A\theta+B)}_{\psi(\theta)}\end{aligned}$$

and so $\psi(\theta)$ satisfies Laplace's equation where $w=re^{i\theta}$, $r=\sqrt{u^2+v^2}$ and $\theta=\tan^{-1}(v/u)$. Now, apply the boundary conditions. At $\theta=0$, we have $\psi(0)=B=T_0$ and at $\psi(\pi)=A\pi+T_0=-T_0$ and so $A=-2T_0/\pi$. Therefore,

$$\begin{aligned}T(u,v)&=\psi(u,v)\\&=A\theta+B\\&=\frac{-2T_0}{\pi}\tan^{-1}(v/u)+T_0\\&=T_0\left(1-\frac{2}{\pi}\tan^{-1}\frac{v}{u}\right)\end{aligned}$$

Extra-Credit Problem #12 (? points):

(a) The above.

(b) Now we'll use this result to solve Laplace's equation in $|z|<1$ with the boundary conditions

$$T(r=1,\theta)=\begin{cases}T_0, & 0<\theta<\pi\\ -T_0, & \pi<\theta<2\pi\end{cases}.$$

Show that the transformation

$$w=i\left(\frac{1-z}{1+z}\right)\qquad z=\frac{i-w}{i+w}$$

maps
- $|z|\le 1$ to the upper-half w-plane ($w=u+iv$ and $v\ge 0$),
- $r=1$, $0<\theta<\pi$ onto $v=0$, $u<0$, and
- $r=1$, $\pi<\theta<2\pi$ onto $v=0$, $u>0$.

(c) Use the result in part (b) and the mapping function to show that the solution of the boundary value problem in the circle is given by

$$\begin{aligned}T(x,y)&=T_0\left[1-\frac{2}{\pi}\cot^{-1}\left(\frac{2y}{1-(x^2+y^2)}\right)\right]\\&=T_0\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{1-(x^2+y^2)}{2y}\right)\right]\end{aligned}$$

or, in polar coordinates,

$$\begin{aligned}T(r,\theta)&=T_0\left[1-\frac{2}{\pi}\cot^{-1}\left(\frac{2r\sin\theta}{1-r^2}\right)\right]\\&=T_0\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{1-r^2}{2r\sin\theta}\right)\right].\end{aligned}$$

Solution:

(a) see the previous problem.

(b) One could do this in polar or Cartesian coordinates or staying in $(z,\bar{z})$. We do this in Cartesian.

$$\begin{aligned}w&=i\left(\frac{1-z}{1+z}\right)\\&=i\left(\frac{1-(x+iy)}{1+(x+iy)}\right)\frac{(1+x)-iy}{(1+x)-iy}\\&=i\left(\frac{(1-x)(1+x)-iy(1-x)-iy(1+x)-y^2}{(1+x)^2+y^2}\right)\\&=i\left(\frac{1-x^2-iy-iy-y^2}{(1+x)^2+y^2}\right)\\&=\frac{2y}{(1+x)^2+y^2}+i\frac{1-(x^2+y^2)}{(1+x)^2+y^2}\end{aligned}$$

For u and v we have

$$\begin{aligned}u(x,y)&=\frac{2y}{(1+x)^2+y^2}\\v(x,y)&=\frac{1-(x^2+y^2)}{(1+x)^2+y^2}\end{aligned}$$

For $|z|\le 1$ we have $x^2+y^2\le 1$ and we see clearly that $v\ge 0$ and since $y\in\mathbb{R}$ it follows $u\in\mathbb{R}$.

For $r=1$, $x^2+y^2=1$ and $v(x,y)=0$. Now, using $y=r\sin\theta$ we can say

$$\begin{aligned}&y>0\Longleftrightarrow 0<\theta<\pi,\text{ and}\\&y<0\Longleftrightarrow \pi<\theta<2\pi,\end{aligned}$$

it is the case that

$$\begin{aligned}&u\in(0,\infty)\Longleftrightarrow 0<\theta<\pi,\text{ and}\\&u\in(-\infty,0)\Longleftrightarrow \pi<\theta<2\pi,\end{aligned}$$

(c) Plug in for u and v from part (b) to see

$$\begin{aligned}\frac{v}{u}&=\frac{\frac{1-(x^2+y^2)}{(1+x)^2+y^2}}{\frac{2y}{(1+x)^2+y^2}}\\&=\frac{1-(x^2+y^2)}{2y}\\&=\frac{1-r^2}{2r\sin\theta}\end{aligned}$$

and the result follows.

Problem #13 (30 points): Find the location of the branch points and discuss a branch cut structure associated with the function:

(a) $f(z) = \frac{z-1}{z}$
(b) $f(z) = \log(z^2-3)$
(c) $f(z) = e^{\sqrt{z^2-1}}$
(d) $f(z) = (z^2-1)^{1/3}$
(e) $f(z) = \tan^{-1} z = \frac{1}{2i}\log\frac{i-z}{i+z}$

Solution:

(a) $f(z) = \frac{z-1}{z}$. This is a rational function singular at $z=0$ but single-valued, so no branch points.
(b) $f(z) = \log(z^2-3)$. Here z^2-3 is entire single-valued function so the only branch points are those where $z^2-3=0$ or $z^2-3=\infty$. Thus, there are three branch points, $z=\pm\sqrt{3}$ and $z=\infty$. A branch cut must make sure there is no possibility going around any single of them, in this case it must connect all three points. E.g. consider a cut on real axis $\{z = x | x \in [-3, +\infty)\}$.
(c) $f(z) = e^{\sqrt{z^2-1}}$. Since function e^z is entire, the only *possible* branch points are those of $\sqrt{z^2-1}$, i.e. $z=\pm 1$ and $z=\infty$. However, doing the circle argument $z-1 = r_1 e^{i\theta_1}$, $z+1 = r_2 e^{i\theta_2}$, $\theta_1 \to \theta_1 + 2\pi$, $\theta_2 \to \theta_2 + 2\pi$, one sees that $z=\infty$ is not a branch point since $e^{(2\pi i + 2\pi i)/2} = 1$ (consider $\sqrt{z^2-1} = e^{\frac{1}{2}\log(z-1)+\frac{1}{2}\log(z+1)}$), which corresponds to encircling *both* $z=1$ and $z=-1$, equivalent to encircling just $z=\infty$. Thus, $z=\infty$ is not a branch point even for $\sqrt{z^2-1}$. But $z=\pm 1$ are branch points, and a branch cut connecting them is $\{z = x | x \in [-1,1]\}$.
(d) $f(z) = (z^2-1)^{1/3}$. By definition, we have $f(z) = e^{\frac{1}{3}\log(z^2-1)}$ and the only possible branch points are that of $\log(z^2-1)$, i.e. $z=\pm 1$ and $z=\infty$. By doing circle argument $z-1 = r_1 e^{i\theta_1}$, $z+1 = r_2 e^{i\theta_2}$, $\theta_1 \to \theta_1 + 2\pi$, $\theta_2 \to \theta_2 + 2\pi$, one sees that all three points are indeed branch points. A branch cut must connect them, e.g. a possible branch cut is $\{z = x | x \in [-1, +\infty)\}$ on the real axis.
(e) $f(z) = \tan^{-1} z = \frac{1}{2i}\log\frac{i-z}{i+z}$. This is (up to a constant) log of rational function, so the branch points are those where $\frac{i-z}{i+z} = 0$ or ∞, i.e. there are two branch points $z=\pm i$. As for $z=\infty$, it is not a branch point since $\lim_{z\to\infty}\frac{i-z}{i+z} = -1 \neq 0, \infty$. A branch cut must connect the two points, so a possible one is interval $[-i, i]$ on the imaginary axis.

Problem #14 (15 points): Consider the complex velocity potential

$$\Omega(z, z_0) = \frac{M}{2\pi}[\log(z-z_0) - \log z]$$

for $M>0$, which corresponds to a source at $z=z_0$ and a sink at $z=0$. Find the corresponding velocity potential and stream function. Let $M = k/|z_0|$, $z_0 = |z_0|e^{i\theta_0}$, and show that

$$\Omega(z, z_0) = -\frac{k}{2\pi}\left(\frac{\log z - \log(z-z_0)}{z_0}\right)\frac{z_0}{|z_0|}.$$

Take the limit as $z_0 \to 0$ to obtain

$$\Omega(z) = \lim_{z_0\to 0}\Omega(z, z_0) = -\frac{ke^{i\theta_0}}{2\pi}\frac{1}{z}.$$

This is called a "doublet" with strength k. The angle θ_0 specifies the direction along which the source/sink coalesces. Find the velocity potential and the stream function of the doublet, and sketch the flow.

Solution:

$$\Omega(z, z_0) = \frac{M}{2\pi}[\log|z-z_0| - \log|z| + i(\arg(z-z_0) - \arg(z))],$$

so the velocity potential

$$\phi = \operatorname{Re}\Omega = \frac{M}{2\pi}\log\frac{|z-z_0|}{|z|}$$

and the stream function

$$\psi = \operatorname{Im}\Omega = \frac{M}{2\pi}\left(\arg(z-z_0) - \arg(z)\right).$$

The first formula to show is obtained by just substituting $M = k/|z_0|$ into the definition of $\Omega(z, z_0)$ and multiplying and dividing it by z_0. Then, since $z_0/|z_0| = e^{i\theta_0}$,

$$\Omega(z) = \lim_{z_0\to 0}\Omega(z, z_0) =$$

$$= -\frac{ke^{i\theta_0}}{2\pi}\lim_{z_0\to 0}\frac{\log z - \log(z-z_0)}{z_0} =$$

$$= -\frac{ke^{i\theta_0}}{2\pi}\frac{1}{z},$$

the last equality being true by the definition of the derivative considering z_0 as variable and z as constant. Let $z = re^{i\theta}$, then

$$\Omega(z) = -\frac{ke^{i\theta_0}}{2\pi}\frac{1}{z} = -\frac{ke^{-i(\theta-\theta_0)}}{2\pi r} = $$
$$= -\frac{k(\cos(\theta-\theta_0) - i\sin(\theta-\theta_0))}{2\pi r},$$

i.e. (since $\Omega = \phi + i\psi$), we get the velocity potential ϕ and the stream function ψ,

$$\phi(r,\theta) = -\frac{k\cos(\theta-\theta_0)}{2\pi r}, \qquad \psi(r,\theta) = \frac{k\sin(\theta-\theta_0)}{2\pi r}.$$