Derivatives

Calculus & Analytical Geometry MATH- 101

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Topics to be covered

- Definition of derivative.
- Geometric interpretation of the derivative.
- Alternate form of the derivative.
- Differentiable on an interval; one-sided derivatives.
- When does a function *not* have a derivative at a point?
- Differentiability by function type.
- Differentiation rules.
- Derivatives of some common functions.
- Second and higher order derivatives.
- Glossary.

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 3
  - Sections: 3.1,3.2, 3.4

Definition of Derivative

- The derivative is the formula which gives the slope of the tangent line at any point x for $f(x)$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \operatorname{Lim}_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

- Note: the limit must exist
  - No hole
  - No jump
  - No sharp corner

Differentiable on an Interval; One-Sided Derivatives

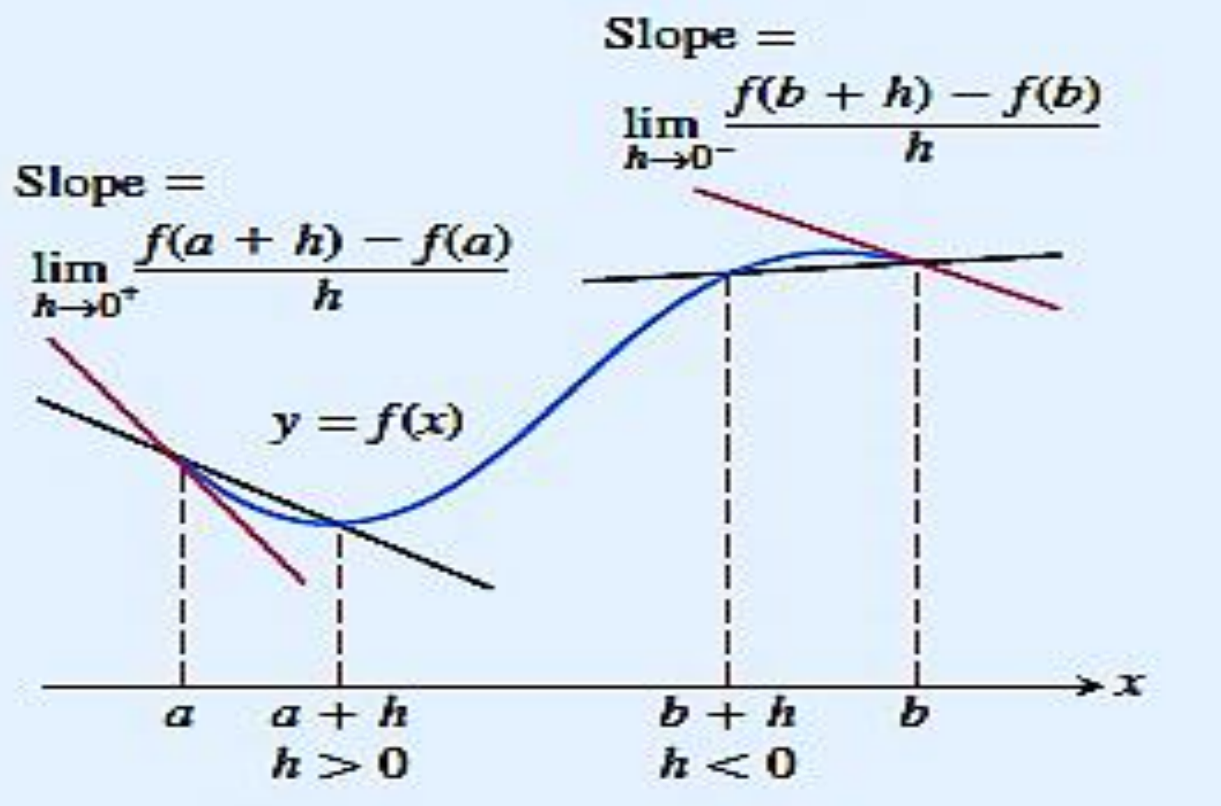
- A function $y = f(x)$ is **differentiable** on an **open interval** (finite or infinite) if it has a derivative at each point of the interval.

- A function $y = f(x)$ differentiable on a **closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits:

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \text{Right} - \text{hand derivative at } a$$

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \text{Left} - \text{hand derivative at } b$$

exist at the end points.



Derivatives at endpoints are one-sided limits.

Alternative Definition of One-Sided Derivatives

A function $y = f(x)$ differentiable on a **closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits:

Right − hand derivative at a

$$f'(a) = \lim_{\Delta x \to 0^+} \frac{f(a+\Delta x) - f(a)}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

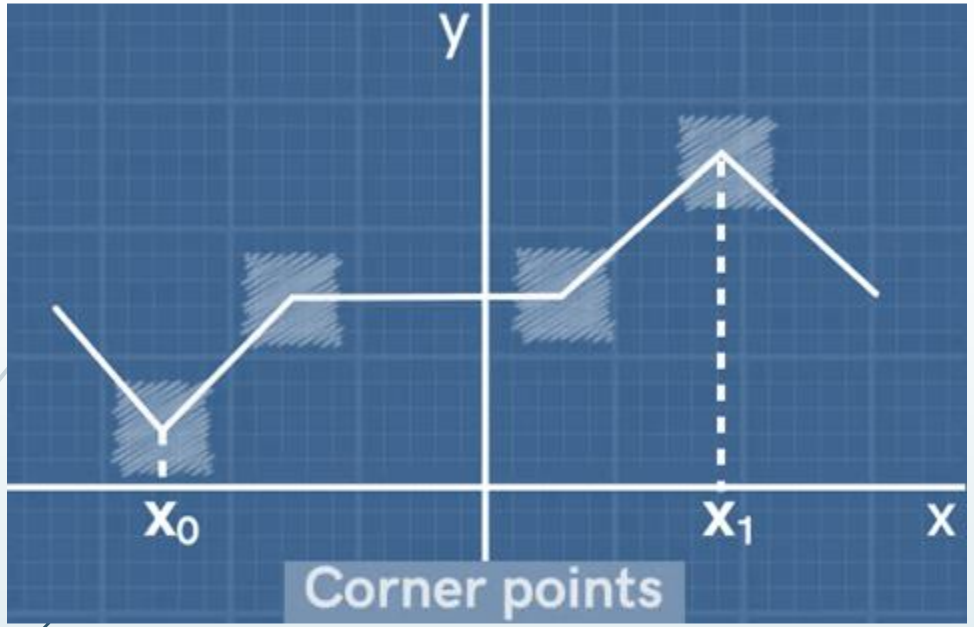
$$f'(b) = \lim_{\Delta x \to 0^-} \frac{f(b+\Delta x) - f(b)}{\Delta x} = \lim_{h \to 0} \frac{f(b-h) - f(b)}{h}$$

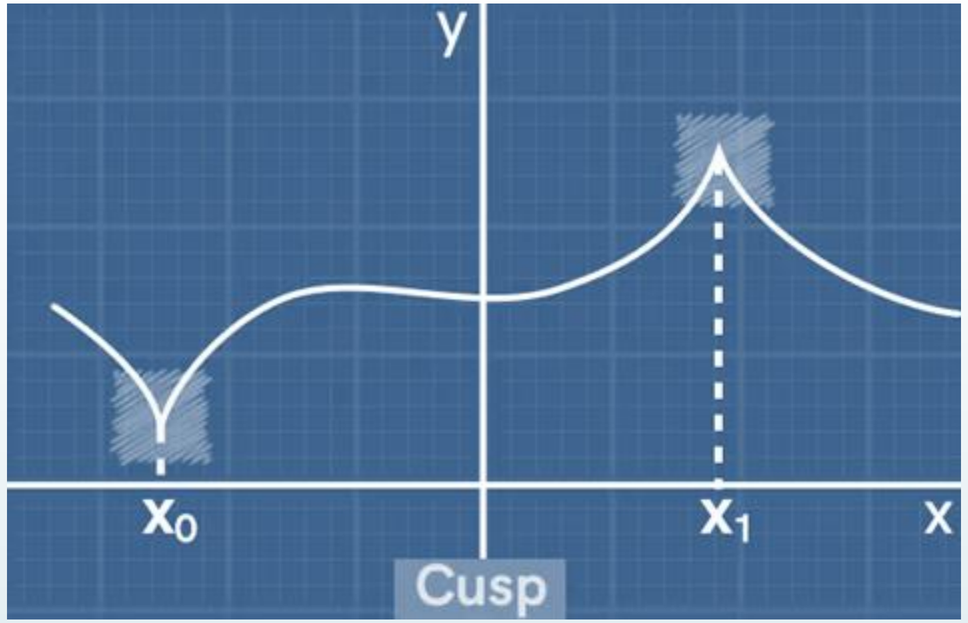
Provided $h > 0$

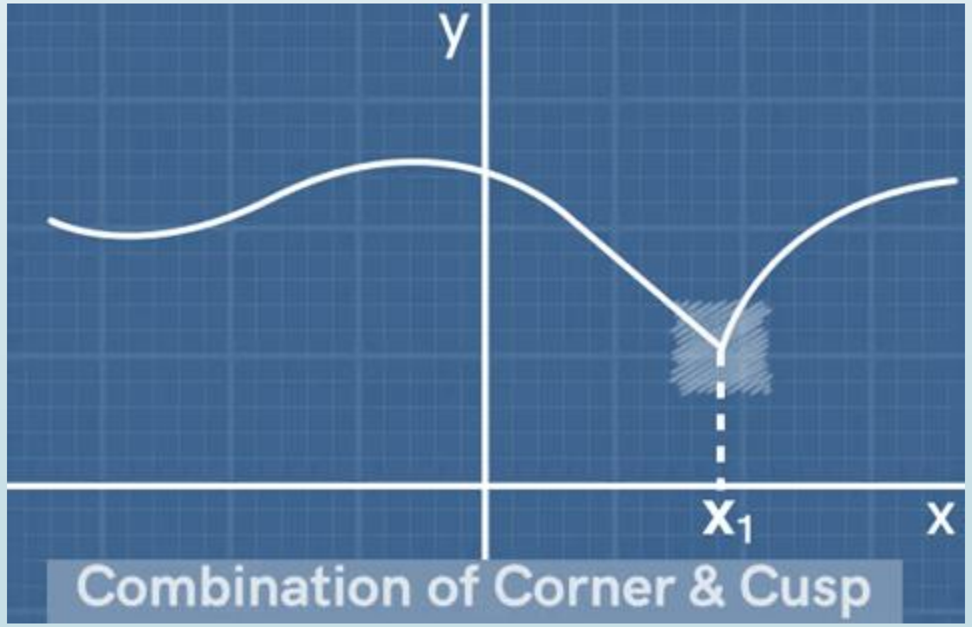
Left − hand derivative at b

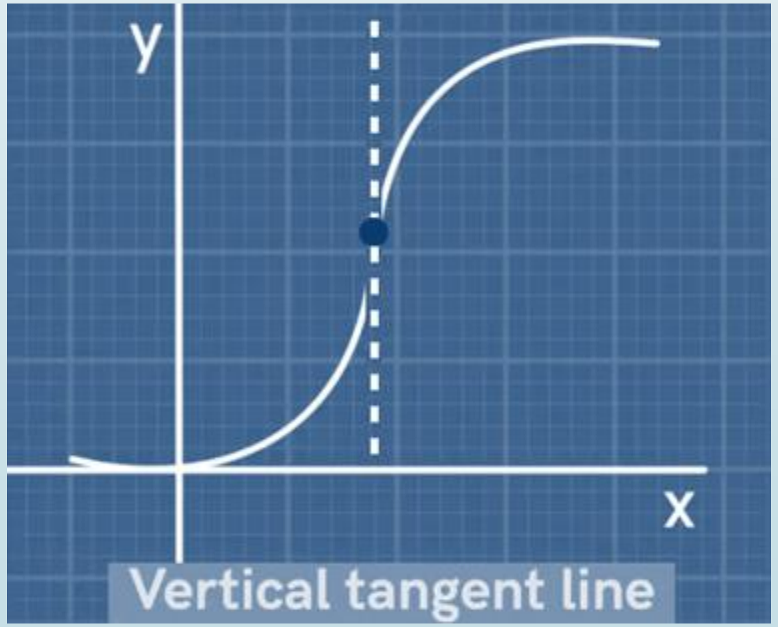
exist at the end points.

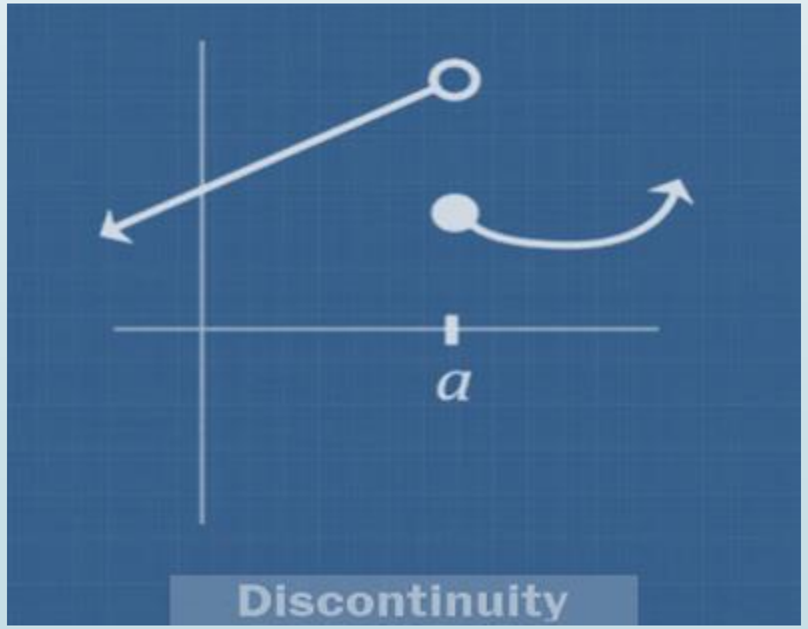
When Does a Derivative Not Exist?





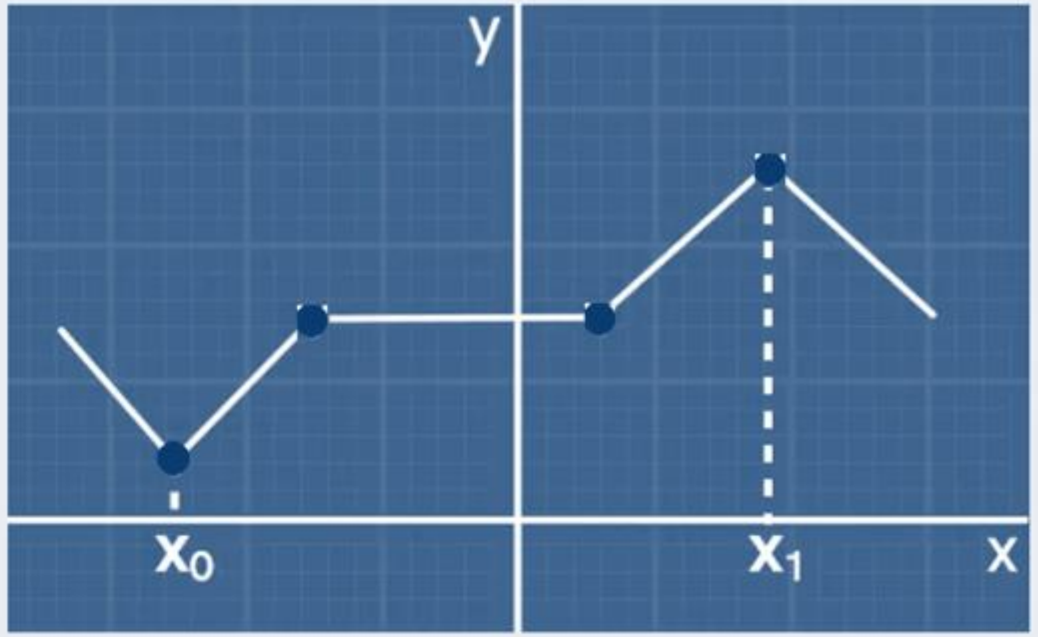






When Does a Derivative Not Exist?

1. Corner points



- $\Delta x > 0$

$$\lim_{\Delta x \to 0^+} \frac{f(x + \Delta x) - f(x)}{\Delta x} \to a$$

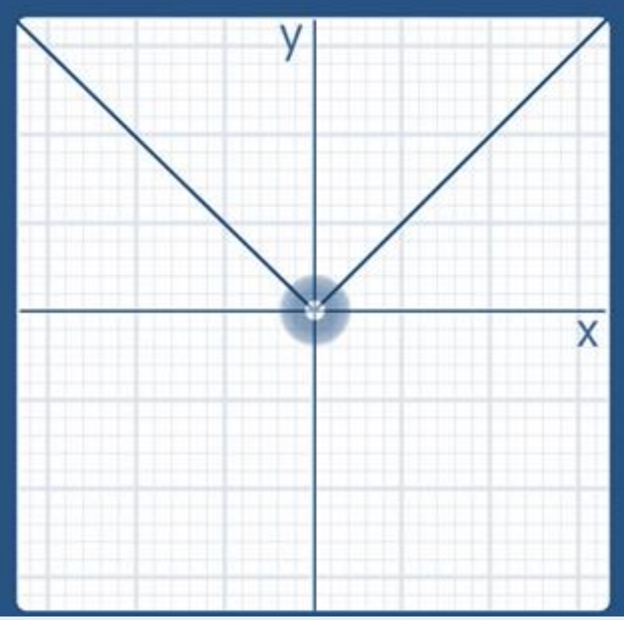
- $\Delta x < 0$

$$\lim_{\Delta x \to 0^-} \frac{f(x + \Delta x) - f(x)}{\Delta x} \to b$$

$a \neq b$

Example:

Absolute value or Modulus function : $y = f(x) = |x| \rightarrow \begin{cases} y = x & \text{if } x \geq 0 \\ = -x & \text{if } x < 0 \end{cases}$



Derivative at x = 0 ?

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$$

1. $\Delta x = h > 0$

Average rate of change $= \dfrac{f(0+h) - f(0)}{h}$

$= \dfrac{h - 0}{h}$

$= 1 \rightarrow$ Independent of h

$\Delta x = h \rightarrow 0$

Average rate of change $= 1$

$$1 = \lim_{\Delta x \to 0^+} \frac{f(0+\Delta x) - f(0)}{\Delta x}$$

2. $\Delta x = -h < 0$

Average rate of change $= \dfrac{f(0-h) - f(0)}{-h}$

$= \dfrac{h - 0}{-h}$

$= -1 \rightarrow$ Independent of h

$\Delta x = -h \rightarrow 0$

Average rate of change $= -1$

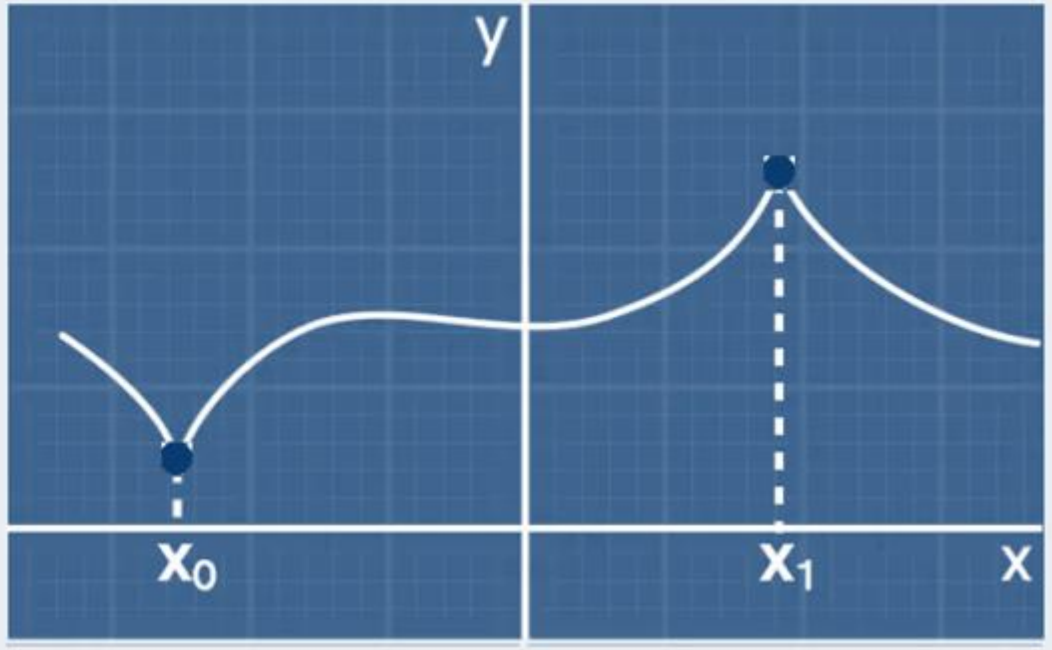
$$-1 = \lim_{\Delta x \to 0^-} \frac{f(0+\Delta x) - f(0)}{\Delta x}$$

$f'(0) = \lim_{\Delta x \to 0} \dfrac{f(0+\Delta x) - f(0)}{\Delta x}$ does not exist

$$1 = \lim_{\Delta x \to 0^+} \frac{f(0+\Delta x) - f(0)}{\Delta x} \neq \lim_{\Delta x \to 0^-} \frac{f(0+\Delta x) - f(0)}{\Delta x} = -1$$

When Does a Derivative Not Exist?

2. Cusp



- $\Delta x > 0$

$$\lim_{\Delta x \to 0^+} \frac{g(x+\Delta x) - g(x)}{\Delta x} \to +\infty \text{ or } -\infty$$

- $\Delta x < 0$

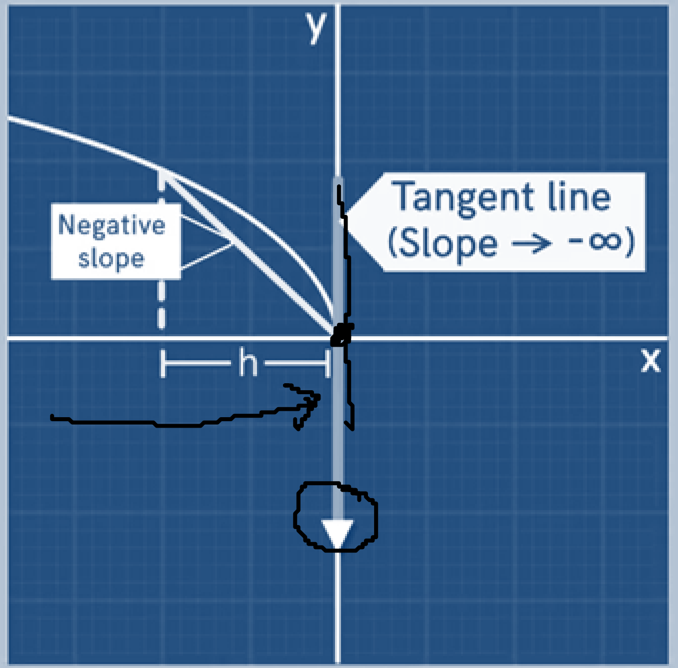
$$\lim_{\Delta x \to 0^-} \frac{g(x+\Delta x) - g(x)}{\Delta x} \to -\infty \text{ or } +\infty$$

Example:

Derivative at x = 0 ⟶ Does not exist

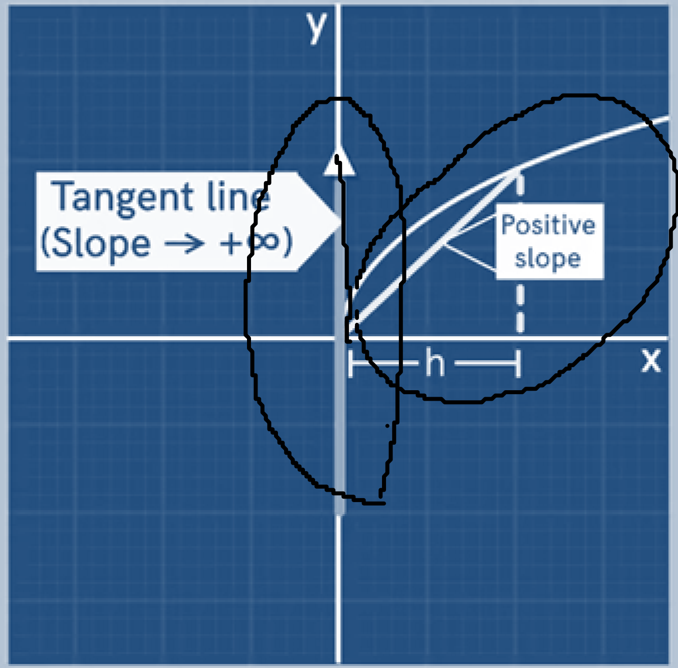
$$\frac{\Delta x}{0} \to \text{undefined}$$

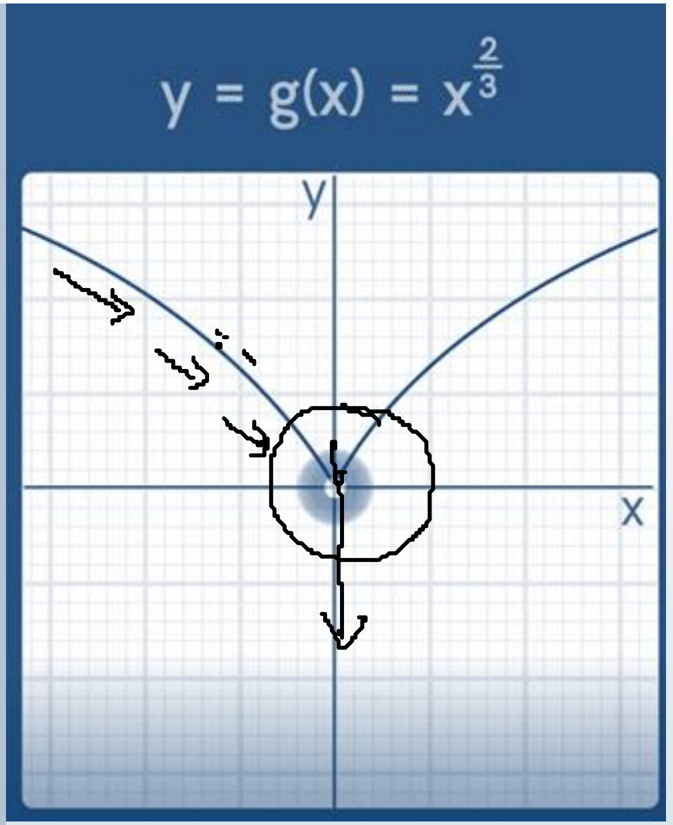
$$y = g(x) = x^{\frac{2}{3}}$$



Slope of a vertical line is undefined

Two different tangent lines at x = 0





$y = x^{5/7}$

$y = (x-2)^{2/3}$ ✓

2 $\Delta x = -h < 0$

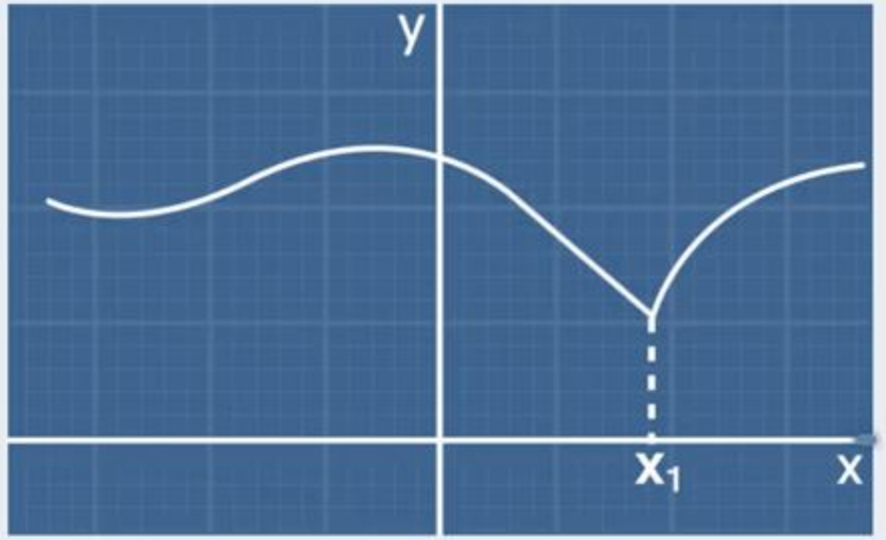
$$\lim_{\Delta x \to 0^-} \frac{g(0-h) - g(0)}{-h} = \frac{-1}{h^{1/3}} \to -\infty$$ ✓

1 $\Delta x = h > 0$

$$\lim_{\Delta x \to 0^+} \frac{g(0+h) - g(0)}{h} = \frac{1}{h^{1/3}} \to +\infty$$ ✓

When Does a Derivative Not Exist?

3. Combination of Corner & Cusp



- $\Delta x > 0$

$$\lim_{\Delta x \to 0^+} \frac{g(x+\Delta x) - g(x)}{\Delta x} \to \pm\infty \text{ or finite number}$$

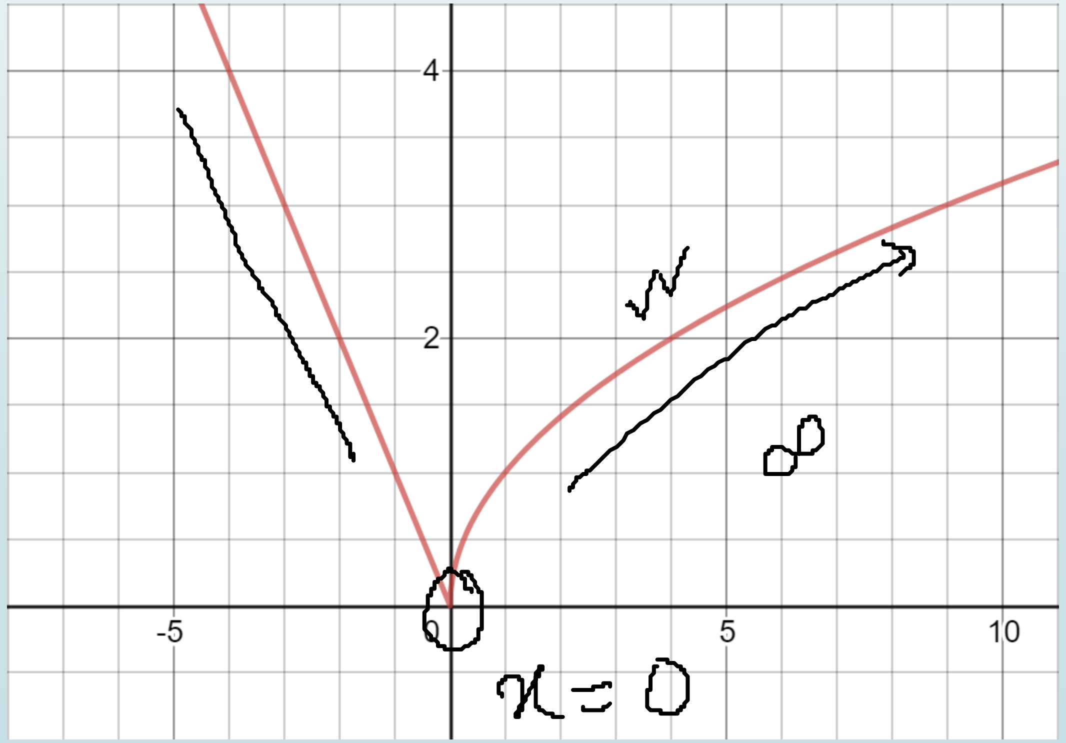
- $\Delta x < 0$

$$\lim_{\Delta x \to 0^-} \frac{g(x+\Delta x) - g(x)}{\Delta x} \to \text{finite number or } \pm\infty$$

Example:

$$f(x) = \begin{cases} \sqrt{x}; & x \geq 0 \\ -x; & x < 0 \end{cases}$$

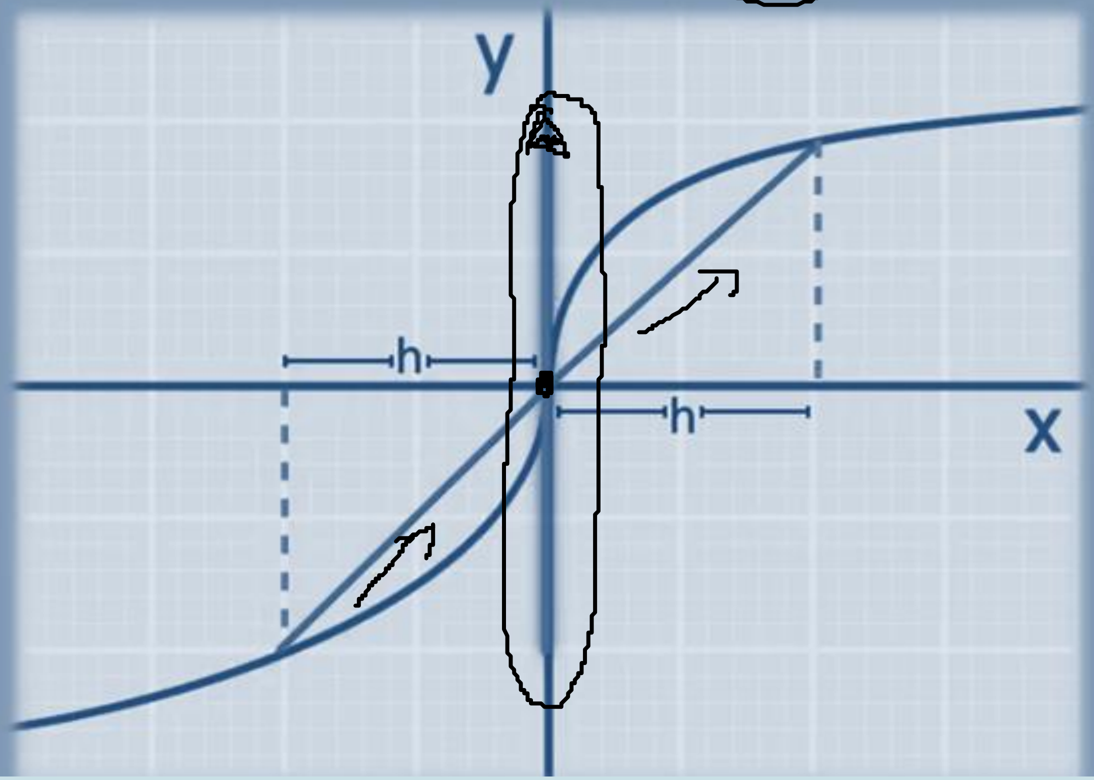
is not differentiable at $x = 0$.



When Does a Derivative Not Exist?

Derivative at x = 0 (does not exist)

$$y = z(x) = x^{1/3}$$



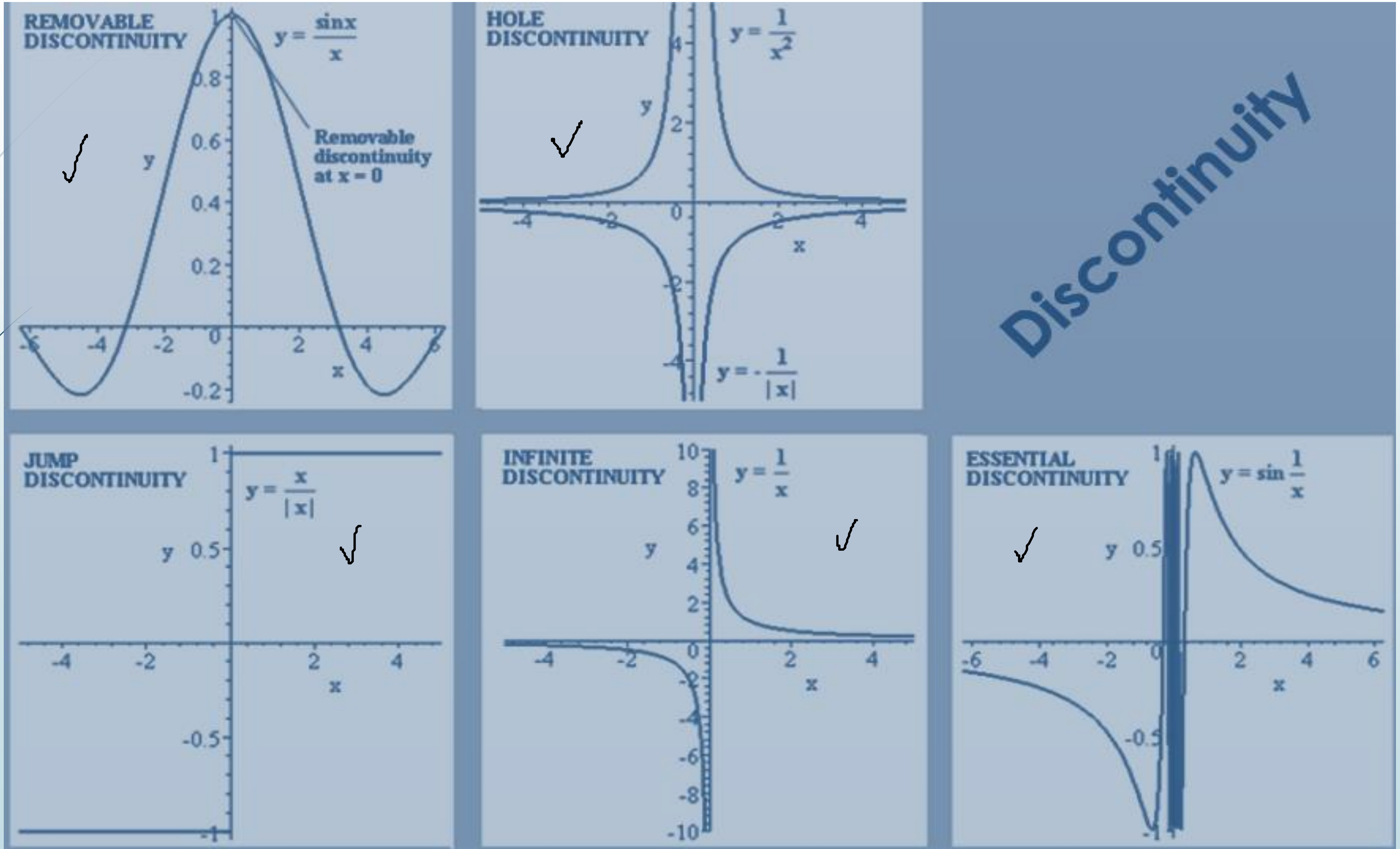
1. $\Delta x = h > 0$

$$\lim_{\Delta x \to 0^+} \frac{z(0+h) - z(0)}{h} = \frac{1}{h^{2/3}} \to +\infty$$

2. $\Delta x = -h < 0$

$$\lim_{\Delta x \to 0^-} \frac{z(0+h) - z(0)}{-h} = \frac{1}{h^{2/3}} \to +\infty$$

When Does a Derivative Not Exist?



Differentiation Rules

Rules	Function	Derivative
Constant	c	0
Multiplication by constant	cf	cf'
Power Rule	x^n	nx^{n-1}
Sum Rule	$f+g$	$f'+g'$
Difference Rule	$f-g$	$f'-g'$
Product Rule	fg	$f'g+fg'$
Quotient Rule	f/g	$(f'g-fg')/g^2$

Derivatives of some common functions

Common Functions	Function	Derivative
Constant	c	0
Line	ax	a
Square	x^2	$2x$ ✓
Square Root	$\sqrt{x}$	$(\frac{1}{2})x^{-\frac{1}{2}}$ ✓
Exponential	e^x	e^x
	a^x	$\ln(a)\ a^x$
Logarithms	$\ln(x)$	$1/x$
	$\log_a(x)$	$1\ /\ (x \ln(a))$

Derivatives of some common functions

Common Functions	Function	Derivative
Trigonometric (x is in radians)	$\sin(x)$	$\cos(x)$ ✓
	$\cos(x)$	$-\sin(x)$
	$\tan(x)$	$\sec^2(x)$
Inverse Trigonometric	$\sin^{-1}(x)$	$1/\sqrt{(1-x^2)}$
	$\cos^{-1}(x)$	$-1/\sqrt{(1-x^2)}$
	$\tan^{-1}(x)$	$1/(1+x^2)$

Examples:

Determine the derivatives of the following functions:

1. $y = \sqrt[3]{x^2}(2x - x^2)$.
2. $y = (2x - 7)^{-1}(x + 5)$.
3. $y = \frac{\cos x}{1 - sinx}$.

Example:

Determine the derivative of $y = \sqrt[3]{x^2}(2x - x^2)$.

Solution:

$$y = (x)^{2/3}(2x - x^2)$$

$$y' = \frac{dy}{dx} = \frac{d}{dx}\left[\underbrace{x^{2/3}}_{f}\,\underbrace{(2x - x^2)}_{g}\right]$$

$$= \frac{d}{dx}\left(x^{2/3}\right).(2x - x^2) + x^{2/3}\frac{d}{dx}(2x - x^2)$$

$$= \frac{2}{3}x^{-1/3}(2x - x^2) + x^{2/3}(2 - 2x)$$

$$= \frac{2x(2 - x)}{3\sqrt[3]{x}} + 2(1 - x)\sqrt[3]{x^2} \quad ✓$$

Example:

Determine the derivative of $y = \dfrac{\cos x \longrightarrow u}{1 - sinx \longrightarrow v}$.

Solution:

$$y = \frac{\cos x}{1 - \sin x}$$

$$y' = \frac{dy}{dx} = \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$$

$$= \frac{(1 - \cancel{\sin x})}{(1-\sin x)^{\cancel{2}}} = \frac{1}{1-\sin x}$$

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

" ' " derivative w.r.t "x"

$$y = \cos x \cdot [1-\sin x]^{-1} \qquad \left[(fg)' = f'g + fg' \right]$$

$$y' = (-\sin x)(1-\sin x)^{-1} + (\cos x) \cdot \left(\frac{\cos x}{(1-\sin x)^2} \right)$$

$$= \frac{-\sin x}{1-\sin x} + \frac{\cos^2 x}{(1-\sin x)^2}$$

$$= \frac{-\sin x(1-\sin x) + \cos^2 x}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$$

$$= \frac{1-\sin x}{(1-\sin x)^2} = \frac{1}{1-\sin x}$$

$$\frac{d}{dx}(1-\sin x)^{-1}$$

$$= -(1-\sin x)^{-2} \cdot \frac{d}{dx}(1-\sin x)$$

$$= \frac{-1(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{\cos x}{(1-\sin x)^2}$$

Second- and Higher-Order Derivatives

Let $y = f(x)$ be differentiable on some interval $[a\,, b]$. The derivative $y' = \frac{dy}{dx} = f'(x) = \frac{df}{dx}$ of $f(x)$ is also a function and it may also possess derivative in $[a\,, b]$. If we apply the definition of derivative to $f'(x)$, the resulting limit, (if it exists) is called **second derivative** of $y = f(x)$ and is denoted by:

$$y'' = f''(x).$$

Thus,

$$y'' = \underline{f''(x)} = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}. \checkmark$$

$$= \left(f'(x) \right)'$$

$f(x) = x^5 + 2x + 3$

$f'(x) = \underline{5x^4 + 2}$

$(f'(x))' = f''$

$= 20x^3$

$f''' = (f'')' = 60x^2$

Second- and Higher-Order Derivatives

Continuing in this way, we can evaluate the third, fourth and higher derivatives of $f(x)$ whenever they exist. The successive derivatives of $y = f(x)$ are denoted by:

$$y', y'', y''', y^{(4)}, \dots, y^{(n)}$$

$y^{(n)} \neq y^n \leftarrow$ n^{th} degree or power of y

n^{th} derivative of y

Or $$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \dots, \frac{d^ny}{dx^n}$$

Or $$f'(x), f''(x), f'''(x), f^{(4)}(x), \dots, f^{(n)}(x)$$

Or $D^1y, D^2y, D^3y, D^4y, \dots, D^ny$, where $D^1 = D = \frac{d}{dx}$

The Second Derivative

Consider the following function

$$f(x) = 5x^3 - 3x^2 + 10x - 5 \checkmark$$

By differentiating this function, we get

$$f'(x) = 15x^2 - 6x + 10$$

This is a function and so it can be differentiated. Thus, we have

$$f''(x) = \left(f'(x)\right)' = 30x - 6$$

This is called the second derivative of the given function $f(x)$.

The Higher-order Derivative

Again,

$$f''(x) = (f'(x))' = 30x - 6$$

This is a function as so we can differentiate it again. This will be called the third derivative

$$f'''(x) = (f''(x))' = 30 \checkmark$$

Continuing,

$$f^{(n)}(x) = 0$$

$$f^{(4)}(x) = (f'''(x))' = 0$$

$$\forall\, n \geq 4$$

Note that since the given function $f(x)$ is a cubic polynomial so fourth and all other higher-order derivatives will be zero.

Example: Finding Higher Derivatives

Let $y = \frac{1}{x}$, then

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2} = -\frac{1!}{x^2}. \checkmark$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-x^{-2}) = 2x^{-3} = \frac{2}{x^3} = \frac{2!}{x^3}.$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}(2x^{-3}) = -6x^{-4} = -\frac{6}{x^4} = -\frac{3!}{x^4}.$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}(-6x^{-4}) = 24x^{-5} = \frac{24}{x^5} = \frac{4!}{x^5}. \checkmark$$

$\vdots$

$$\checkmark \boxed{\frac{d^ny}{dx^n} = \frac{(-1)^n n!}{x^{n+1}}.} \rightarrow n=1$$

Example: Finding Higher Derivatives $f(x) = \sin x$

$$f(x) = \sin x$$

$$f'(x) = \cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$f''(x) = -\sin x = \sin\left(x + 2\left(\frac{\pi}{2}\right)\right)$$

$$f'''(x) = -\cos x = \sin\left(x + 3\left(\frac{\pi}{2}\right)\right)$$

$$f^{(4)}(x) = \sin x = \sin\left(x + 4\left(\frac{\pi}{2}\right)\right)$$

$$f^{(5)}(x) = \cos x$$

$$\vdots$$

$$\boxed{f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)}$$

Practice

Determine the n^{th} −order derivatives of the following functions:

1. $f(x) = \cos x$ ✓
2. $f(x) = e^x$ ✓
3. $f(x) = \ln x$ ✓

Glossary

a) $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ or $\frac{\Delta f(x)}{\Delta x}$ represents:

1) the Rate of Change of $f(x)$ = change of $f(x)$ /change of x.

2) Average Speed when independent variable is time.

3) The slope of the secant line passing though the points

$$A = \big(x, f(x)\big) \ \& \ B = (x + \Delta x, f(x + \Delta x)).$$

b) $f'(x)$, is used for the value of

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

By "differentiating $f(x)$" we mean "get the derivative of $y = f(x)$". The notation for the derivative of $y = f(x)$ is $f'(x)$ or df/dx or dy/dx.

Glossary

c) $f'(c)$, the derivative of $f(x)$ evaluating at $x = c$, represents the:

1) Instantaneous rate of change of $f(x)$ at $x = c$.
2) Instantaneous speed at $x = c$ when x represents time.
3) Slope of the tangent line to the graph $y = f(x)$ at $x = c$.
4) Marginal value of $f(x)$ in Business.

Practice Questions

Book: Thomas Calculus (11th Edition) by Georg B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter: 3**
  - **Exercise: 3.1**
    Q # 1 – 30, 35 – 44
  - **Exercise: 3.2**
    Q # 1 – 38, 50, 54 – 56
  - **Exercise: 3.4**
    Q # 1 – 34, 39 – 50