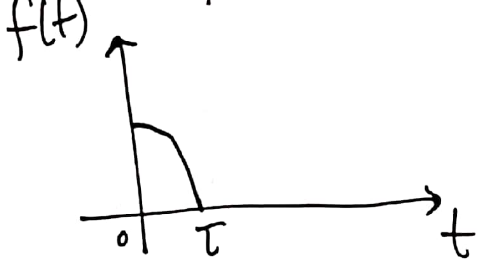
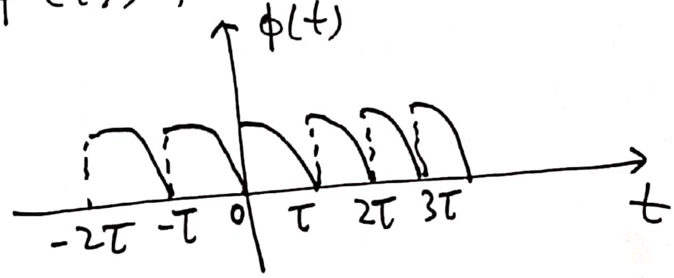
Functions defined over a finite interval

For F.S, $f(t)$ must be periodic. However, if $f(t)$ is defined only over a finite time interval $0 \le t \le \tau$, we can define $\phi(t)$ which is the periodic extension of $f(t)$ to obtain full-range series as

$$\phi(t) = f(t), \quad (0 < t < \tau)$$

$$\phi(t+\tau) = \phi(t), \text{ for all } t.$$





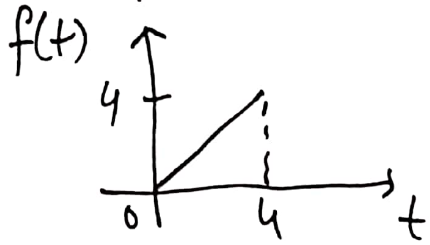
We expand $\phi(t)$, which is periodic into Fourier series. Since, within the interval $0 \le t \le \tau$, $\phi(t)$ is identical to $f(t)$. It follows that this F.S expansion of $\phi(t)$ will be representative of $f(t)$ within this interval.

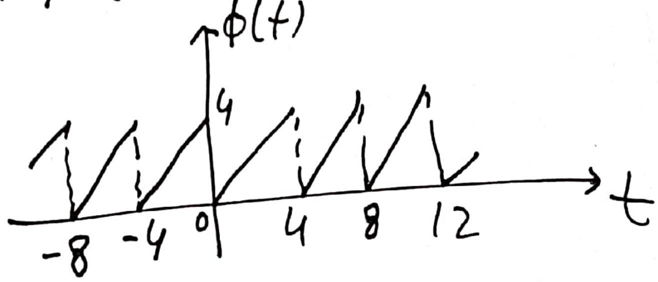
Ex: Find a full range F.S. expansion of $f(t) = t$ valid in the finite interval $0 < t < 4$.

Sol: Define the periodic function $\phi(t)$ by

$$\phi(t) = f(t) = t \quad (0 < t < 4)$$

$$\phi(t+4) = \phi(t), \text{ for all } t.$$





$$a_0 = \frac{1}{2}\int_0^4 f(t)\,dt = 4$$

$$a_n = \frac{1}{2}\int_0^4 f(t)\cos\left(\tfrac{1}{2}n\pi t\right)dt = 0$$

$$b_n = \frac{1}{2}\int_0^4 f(t)\sin\left(\tfrac{1}{2}n\pi t\right)dt$$

$$= -\frac{4}{n\pi}.$$

$$\phi(t) = 2 - \sum_1^\infty \frac{4}{n\pi}\sin\left(\tfrac{1}{2}n\pi t\right)$$

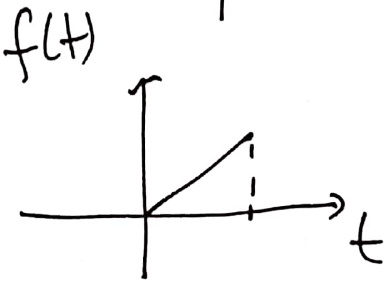
$$f(t) = t = 2 - \frac{4}{\pi}\sum_{n=1}^\infty \frac{1}{n}\sin\left(\tfrac{1}{2}n\pi t\right) \quad (0 < t < 4).$$

Half-range Cosine & Sine Series:-

even periodic extension

$$\phi(t) = \begin{cases} f(t), & 0 < t < \tau \\ f(-t), & -\tau < t < 0 \end{cases}$$

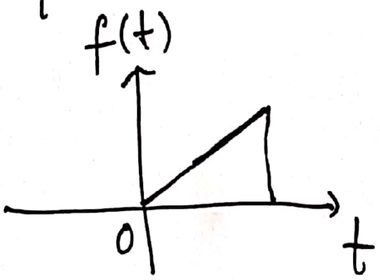
$$\phi(t+2\tau) = \phi(t), \text{ for all } t.$$

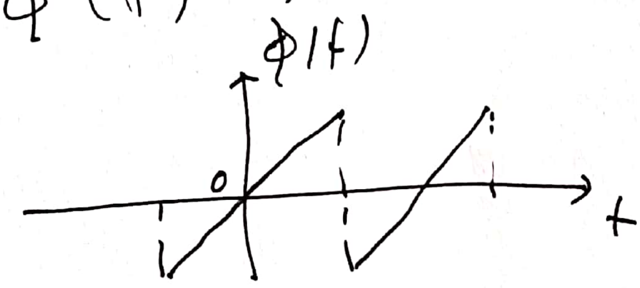


odd periodic extension:

$$\phi(t) = \begin{cases} f(t), & (0 < t < \tau) \\ -f(-t), & (-\tau < t < 0) \end{cases}$$

$$\phi(t+2\tau) = \phi(t) \; ,$$

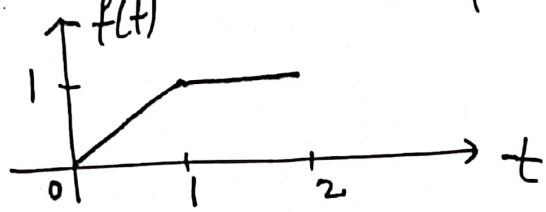




Ex: Let $f(t) = \begin{cases} t, & 0 < t < 1 \\ 1, & 1 < t < 2 \end{cases}$

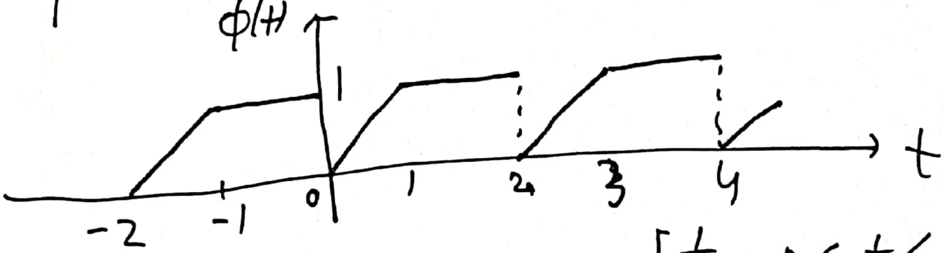
Give full range, half-range even and odd periodic extension of $f(t)$.

Sol:



Full periodic extension: $\phi(t) = \begin{cases} t, & 0 < t < 1 \\ 1, & 1 < t < 2 \end{cases}$

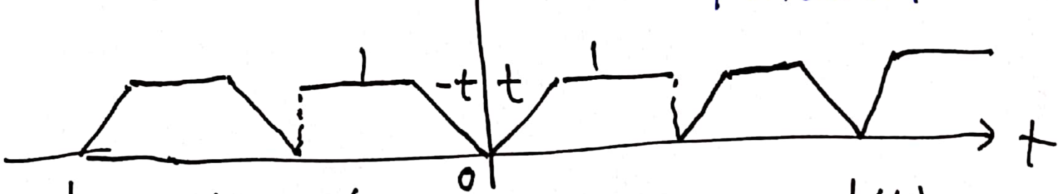
$\phi(t+2) = \phi(t)$ Period = 2



Even periodic extension: $\phi(t) = \begin{cases} t, & 0 < t < 1 \\ 1, & 1 < t < 2 \\ -t, & -1 < t < 0 \\ 1, & -2 < t < 1. \end{cases}$

$\phi(t+4) = \phi(t)$ $\phi(t)$

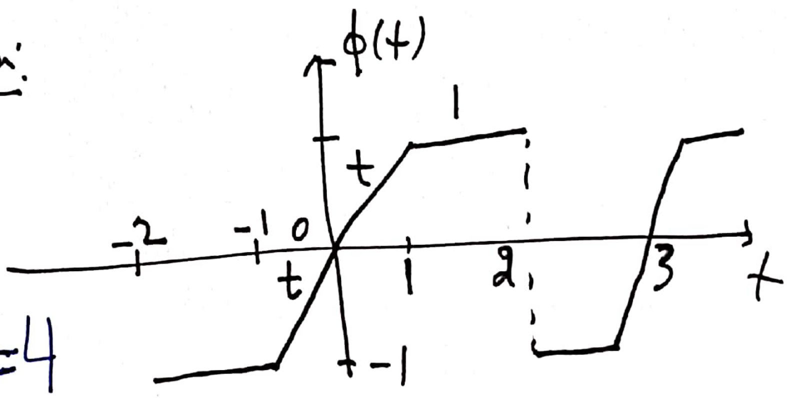
Period = 4



Odd periodic extension:

$\phi(t) = \begin{cases} -1, & -2 < t < -1, \\ t, & -1 < t < 1, \\ 1, & 1 < t < 2. \end{cases}$

$\phi(t+4) = \phi(t)$. Period = 4



F.S.25

Discrete frequency spectra

In expressing a periodic function $f(t)$ by its fourier series expansion, we are decomposing the function into its harmonic or frequency components. If $f(t)$ is periodic of period T then it has frequency components at frequencies $\omega_n = n\omega_0 = n\left(\frac{2\pi}{T}\right)$, $n = 1, 2, 3, \ldots$ where ω_0 is the frequency of the parent function $f(t)$.

A Fourier series may therefore be interpreted as constituting a frequency spectrum of the periodic function $f(t)$, and provides an altenative representation of the function to its time-domain waveform.

A plot of amplitude against angular frequency is called the amplitude spectrum, while that of phase against angular frequency is called the phase spectrum. These spectra (amplitude & phase) only occur at discrete frequency ω_n, are referred as discrete frequency spectra.

one sided discrete fourier spectra based on TFs:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= \frac{1}{2}a_0 + a_1 \cos\omega t + a_2 \cos 2\omega t + \dots + a_n \cos n\omega t + \dots$$
$$+ b_1 \sin\omega t + b_2 \sin 2\omega t + \dots + b_n \sin n\omega t + \dots$$

$$= \frac{1}{2}a_0 + (a_1 \cos\omega t + b_1 \sin\omega t) + (a_2 \cos 2\omega t + b_2 \sin 2\omega t)$$
$$+ \dots + (a_n \cos n\omega t + b_n \sin n\omega t) + \dots \quad \text{(i)}$$

Consider, $a_1 \cos\omega t + b_1 \sin\omega t$

Suppose, $a_1 = A_1 \cos\phi_1$, $-b_1 = A_1 \sin\phi_1$

then, $a_1 \cos\omega t - (-b_1 \sin\omega t)$

$$= A_1[\cos\omega t \cos\phi_1 - \sin\omega t \sin\phi_1]$$

$$= A_1 \cos(\omega t + \phi_1), \text{ where}$$

$$A_1 = \sqrt{a_1^2 + b_1^2}, \quad \phi_1 = \tan^{-1}\left(-\frac{b_1}{a_1}\right)$$

In general, (i) Could be expressed as

$$f(t) = A_0 + A_1 \cos(\omega t + \phi_1) + A_2 \cos(2\omega t + \phi_2)$$
$$+ A_3 \cos(3\omega t + \phi_3) + \dots + A_n \cos(n\omega t + \phi_n) + \dots$$

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t + \phi_n) \quad \text{— (ii)}$$

where, $A_n = \sqrt{a_n^2 + b_n^2}$, $\phi_n = \tan^{-1}(-b_n/a_n)$

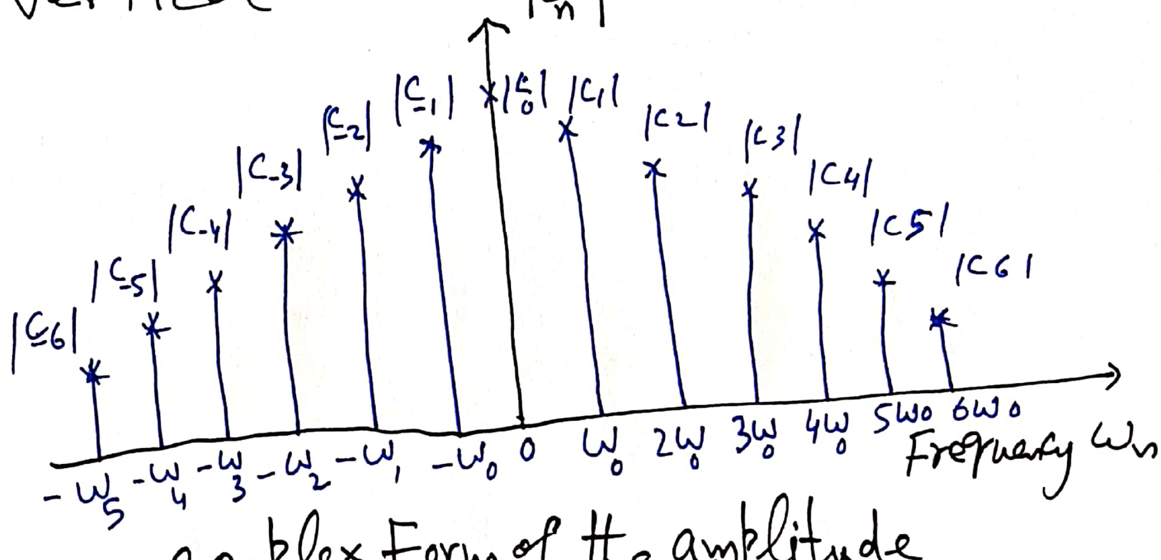
(ii) is called the Compact/Quadrature form of the fourier series.

A_n & ϕ_n are the amplitude & phase related to the nth harmonic. Plot of A_n & ϕ_n gives 1-sided discrete Fourier spectra.

Two sided spectra: The complex fourier Coefficients are $c_n = |c_n| e^{j\phi_n}$, $(n = 0, \pm1, \pm2, \ldots)$

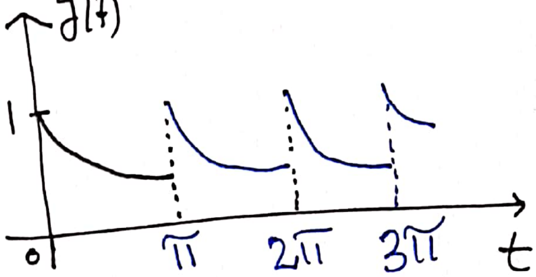
in which $|c_n|$ & ϕ_n denote the magnitude and argument of c_n respectively. The amplitude spectrum is the plot of $|c_n|$ against ω_n & the phase spectrum that of ϕ_n against ω_n.

Since, $|c_n^*| = |c_{-n}| = |c_n|$, the amplitude spectrum will be symmetrical about the vertical axis.



Complex Form of the amplitude spectrum

EX: Let $g(t) = e^{-t/2}$, $0 \le t \le \pi$, $g(t+\pi) = g(t)$ be a periodic function. Sketch graph of $g(t)$. obtain TFS, Compact TFS & sketch 1-sided discrete spectra. Find CFS (Complex form of FS) & sketch 2-sided Fourier spectra.



Sol: Here $T_0 = \pi$, $\omega_0 = \frac{2\pi}{\pi} = 2$.

$$g(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(2nt) + b_n \sin(2nt)\right].$$

where, $a_0 = \frac{1}{\pi}\int_0^{\pi} e^{-t/2}\,dt = \frac{1}{\pi}\left[\frac{e^{-t/2}}{-1/2}\right]_0^{\pi} = 0.504.$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\cos(2nt)\,dt = 0.504\left(\frac{2}{1+16n^2}\right)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\sin(2nt)\,dt = 0.504\left(\frac{8n}{1+16n^2}\right)$$

TFS: $0.504\left[1 + \sum_{n=1}^{\infty}\frac{2}{1+16n^2}\left(\cos(2nt) + 4n\sin(2nt)\right)\right].$

The Compact Form: $A_0 = a_0 = 0.504$

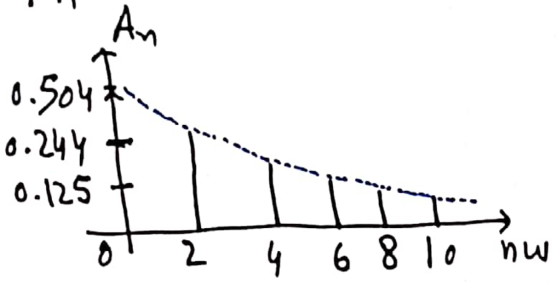
$$A_n = \sqrt{a_n^2 + b_n^2} = (0.504)\sqrt{\frac{4}{(1+16n^2)^2} + \frac{64n^2}{(1+16n^2)^2}}$$

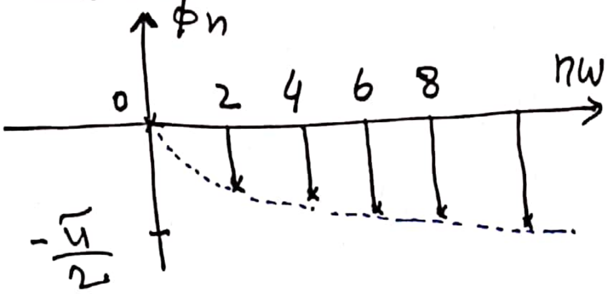
$$A_n = (0.504)\left(\frac{2}{\sqrt{1+16n^2}}\right).$$

$$\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right) = \tan^{-1}(-4n) = -\tan^{-1}(4n).$$

$$g(t) = 0.504 + 0.504\sum_{n=1}^{\infty}\frac{2}{\sqrt{1+16n^2}}\cos(2nt - \tan^{-1}4n).$$

n	0	1	2	3	4
C_n	0.504	0.244	0.125	0.084	0.063
ϕ_n	0	-75.96	-85.24	-86.42	-87.14





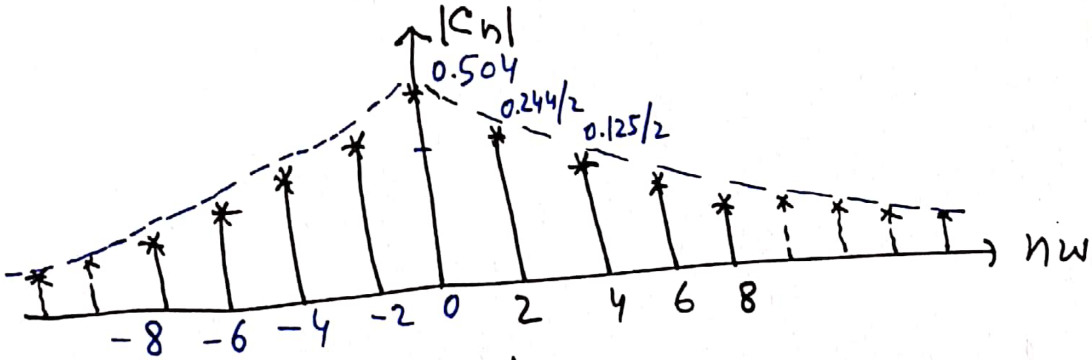
Complex Form of Fourier Series:-

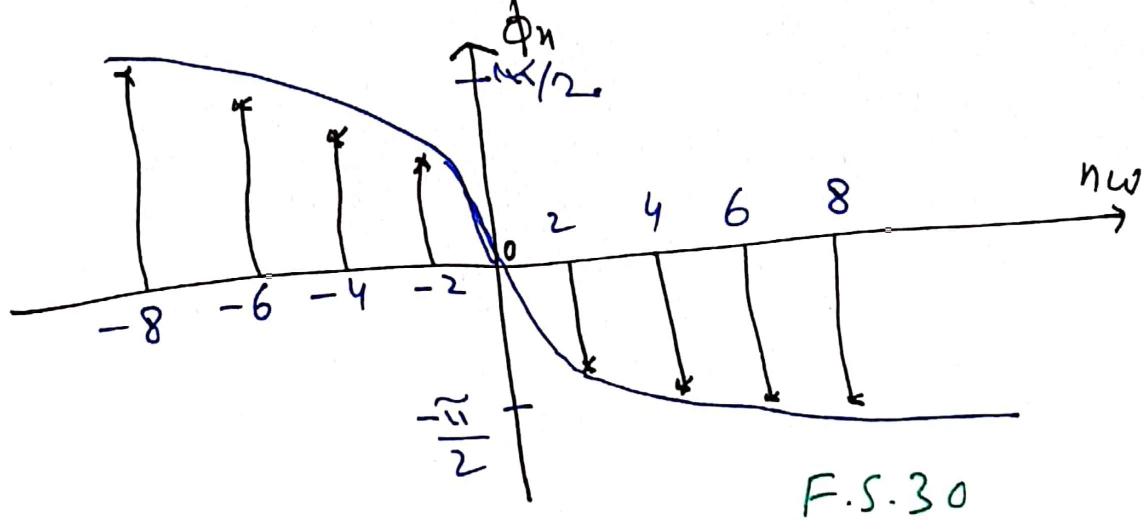
$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2nt}, \quad \text{As } T_0 = \pi, \omega_0 = \frac{2\pi}{\pi} = 2.$$

$$\text{where, } C_n = \frac{1}{\pi}\int_0^{\pi} e^{-\frac{t}{2}} e^{-j2nt}\, dt = \frac{-1}{\pi(1+2jn)} e^{-(\frac{1}{2}+j2n)t}\Big|_0^{\pi}.$$

$$C_n = 0.504\left(\frac{1}{1+4nj}\right)$$

$$g(t) = 0.504 \sum_{n=-\infty}^{\infty} \frac{1}{1+j4n} e^{j2nt}$$





F.S.30

Ex: A periodic function $f(t) = \begin{cases} 0, & -2<t<0 \\ 1, & 0<t<4 \end{cases}$,

$f(t+4) = f(t)$

for all t, has the Complex F.S.

$$f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{j}{2n\pi}\left[(-1)^n - 1\right] e^{jn\frac{\pi}{2}}.$$

Using this obtain TFS. write down C_n, $|C_n|$ (the magnitude of C_n) & ϕ_n related to the phase of the nth harmonic.

Solution: $a_0 = 2C_0 = 1$, $a_n - jb_n = \frac{j}{n\pi}\left[(-1)^n - 1\right]$

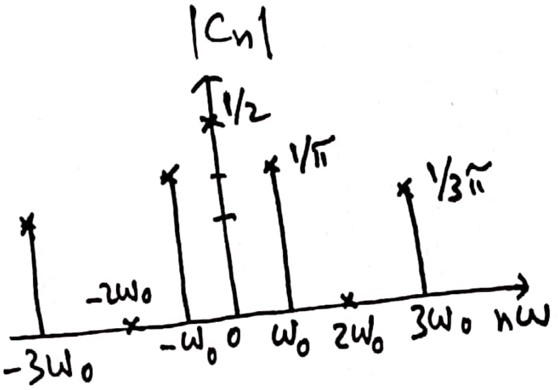
$a_n + jb_n = \frac{-j}{n\pi}\left[(-1)^n - 1\right]$, giving $a_n = 0$, $b_n = \frac{1}{n\pi}\left[1-(-1)^n\right]$

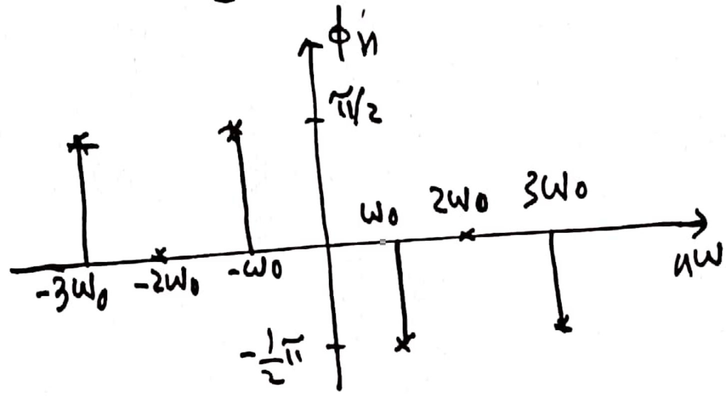
$b_n = \begin{cases} 0, & n \text{ is even} \\ \frac{2}{n\pi}, & n \text{ is odd.} \end{cases}$ TFS is

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)} \sin\left(\frac{(2n-1)\pi}{2} t\right).$$

$$C_n = \frac{j}{2n\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & n = \pm 2, \pm 4, \ldots \text{(even)} \\ \frac{-j}{n\pi}, & n = \pm 1, \pm 3, \pm 5, \ldots \text{(odd)} \end{cases}$$

$$|C_n| = \begin{cases} 0, & n = \pm 2, \pm 4, \ldots \\ 1/n\pi, & n = 1, 3, 5, \ldots \\ -1/n\pi, & n = -1, -3, -5, \ldots \end{cases} \qquad \phi_n = \arg(C_n) = \begin{cases} 0, & n = \pm 2, \pm 4, \ldots \\ -\pi/2, & n = 1, 3, 5, \ldots \\ \pi/2, & n = -1, -3, -5, \ldots \end{cases}$$





Ex:- A periodic signal is given by $g(t) = 3 - 2\sin(3t - \frac{\pi}{3}) - 2\cos(4t + \pi/3)$. Express the periodic signal in a (a) TFS (b) Compact FS (c) Complex or exponential Fourier series (d) sketch the 1-sided spectra, and (e) sketch the 2-sided spectra.

$\frac{2\pi}{3}, \quad \frac{2\pi}{4} = \frac{\pi}{2}$

$\frac{2\pi}{3} N_1 = \frac{\pi}{2} N_2$

$\frac{2}{3} N_1 = \frac{1}{2} N_2$

$\frac{N_1}{N_2} = \frac{3}{4}$

$P = \frac{2\pi}{3}(3) = \frac{\pi}{2}(4) = 2\pi$

The signal is periodic of period 2π and $\omega = 1$.

$\sin(3t - \pi/3) = \sin 3t \cos\frac{\pi}{3} - \cos 3t \sin \pi/3$

$= \frac{1}{2}\sin 3t - \frac{\sqrt{3}}{2}\cos 3t$

$\cos(4t + \pi/3) = \cos 4t \cos \pi/3 - \sin 4t \sin \pi/3$

$= \frac{1}{2}\cos 4t - \frac{\sqrt{3}}{2}\sin 4t$.

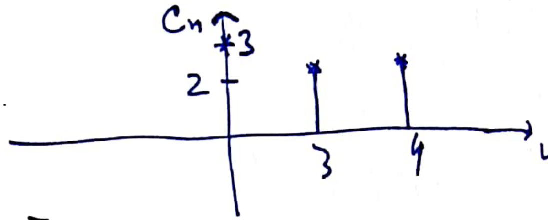
$g(t) = 3 + 2(\frac{\sqrt{3}}{2})\cos 3t - 2(\frac{1}{2})\cos 4t$

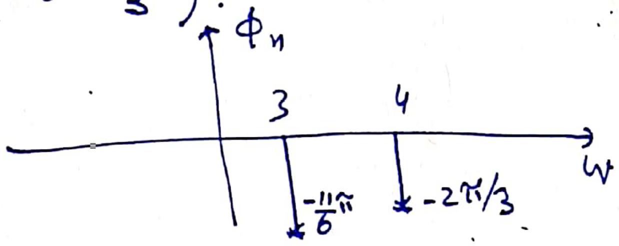
$+ [-2(\frac{1}{2})\sin 3t + 2(\frac{\sqrt{3}}{2}\sin 4t)]$

$= \underset{a_0}{3} + \underset{a_3}{\sqrt{3}}\cos 3t \underset{a_4}{-}\cos 4t \underset{b_3}{-}\sin 3t + \underset{b_4}{\sqrt{3}}\sin 4t$.

(b) $g(t) = 3 + 2\cos(3t - \frac{\pi}{3} - \frac{3\pi}{2}) + 2\cos(4t + \frac{\pi}{3} - \pi)$

$= 3 + 2\cos(3t - \frac{11}{6}\pi) + 2\cos(4t - \frac{2\pi}{3})$.

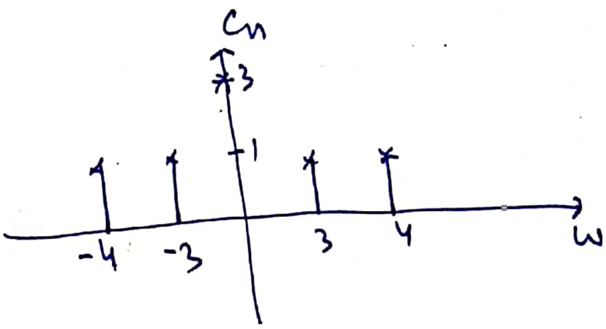


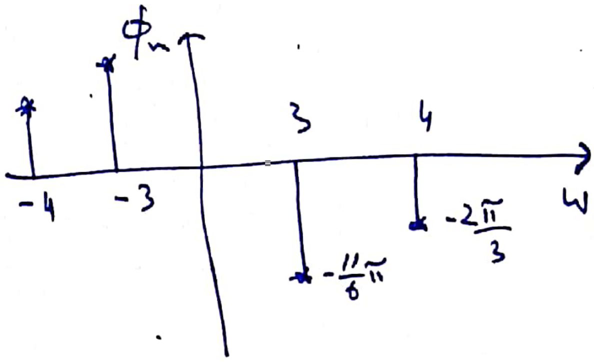


Complex Fourier series:

$g(t) = 3 + 2\left(\frac{e^{j(3t - \frac{11}{6}\pi)} + e^{-j(3t - \frac{11}{6}\pi)}}{2}\right) + 2\left(\frac{e^{j(4t - \frac{2\pi}{3})} + e^{-j(4t - 2\pi/3)}}{2}\right)$

$= e^{j\frac{2\pi}{3}} e^{-4jt} + e^{j\frac{11}{6}\pi} e^{-j3t} + 3 + e^{-\frac{11}{6}\pi j} e^{j3t} + e^{-j\frac{2\pi}{3}} e^{j4t}$.





$f(t) = 2\sin(4\pi t + \frac{\pi}{3}) + \sin(6\pi t - \frac{\pi}{4}) - \cos(8\pi t + 1)$.

Find Fundamental time period & frequency, Complex Fourier series, Trigonometric Fourier series, Compact form of Fourier. Sketch discrete Fourier spectra.

$\omega_1 = 4\pi,\ \omega_2 = 6\pi,\ \omega_3 = 8\pi,\ \omega_0 = 2\pi\ (HCF),\ T = \frac{2\pi}{2\pi} = 1$ (Second)

or $\frac{2\pi}{4\pi} N_1 = \frac{2\pi}{6\pi} N_2 = \frac{2\pi}{8\pi} N_3,\ T = \frac{1}{2} N_1 = \frac{1}{3} N_2 = \frac{1}{4} N_3,\ T = 1,\ \omega = \frac{2\pi}{1} = 2\pi$ rad/Sec.

$$f(t) = 2\left[\frac{e^{j(4\pi t + \pi/3)} - e^{-j(4\pi t + \pi/3)}}{2j}\right] + 2\left[\frac{e^{j(6\pi t - \pi/4)} - e^{-j(6\pi t - \pi/4)}}{2j}\right] - \left[\frac{e^{j(8\pi t + 1)} + e^{-j(8\pi t + 1)}}{2}\right]$$

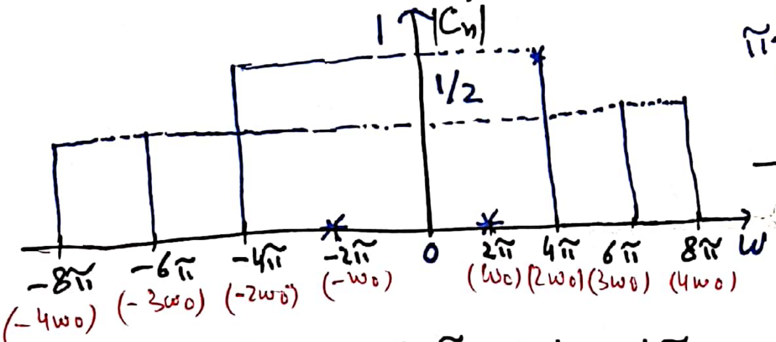
$$= e^{j(4\pi t + \pi/3)} e^{-j\pi/2} - e^{-j(4\pi t + \pi/3)} e^{j\pi/2} + \frac{1}{2} e^{-j\pi/2} e^{j(6\pi t - \pi/4)} - \frac{1}{2} e^{-j\pi/2} e^{-j(6\pi t - \pi/4)} - \frac{1}{2} e^{j(8\pi t + 1)} - \frac{1}{2} e^{-j(8\pi t + 1)}$$

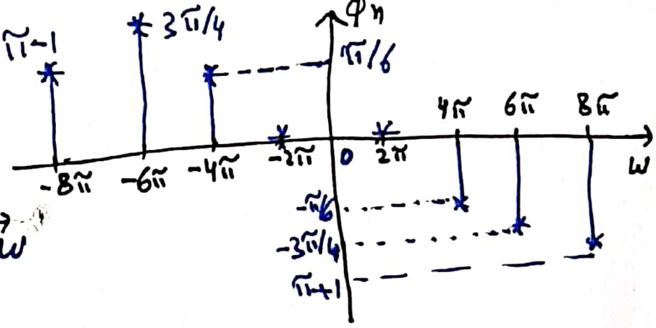
$$= e^{-j\pi/6} e^{j4\pi t} + e^{j\pi} e^{-j\pi/3} e^{-j\pi/2} e^{-4j\pi t} + \frac{1}{2} e^{-j\pi/2} e^{-j\pi/4} e^{j6\pi t} + \frac{1}{2} e^{j\pi} e^{-j\pi/2} e^{j\pi/4} e^{-j6\pi t} + \frac{1}{2} e^{j\pi} e^{j} e^{j8\pi t} + \frac{1}{2} e^{j\pi} e^{-j} e^{-j8\pi t}$$

$$= \frac{1}{2} e^{j\pi} e^{-j} e^{-j8\pi t} + \frac{1}{2} e^{j3\pi/4} e^{-j6\pi t} + e^{j\pi/6} e^{-4j\pi t} + e^{-j\pi/6} e^{j4\pi t} + \frac{1}{2} e^{-j3\pi/4} e^{j6\pi t} + \frac{1}{2} e^{j\pi} e^{j} e^{j8\pi t}.$$

$D_2 = e^{-j\pi/6}$, $D_3 = \frac{1}{2} e^{-j3\pi/4}$

$\omega_0 = 2\pi,\ 4\omega = 8\pi,\ D_{-4} = \frac{1}{2} e^{(\pi-1)j},\ D_{-3} = \frac{1}{2} e^{j3\pi/4},\ D_{-2} = e^{+j\pi/6},\ D_4 = \frac{1}{2} e^{j(\pi+1)}$.





TFS: $f(t) = 2\left(\sin 4\pi t \cos\frac{\pi}{3} + \cos 4\pi t \sin\frac{\pi}{3}\right) + \sin(6\pi t)\cos(\frac{\pi}{4}) - \cos(6\pi t)\sin(\frac{\pi}{4})$
$- \cos 8\pi t \cos 1 + \sin 8\pi t \sin 1$

$= (2)(\frac{1}{2})\sin 4\pi t + (2)(\frac{\sqrt{3}}{2})\cos 4\pi t + (\frac{1}{\sqrt{2}})\sin(6\pi t) - \frac{1}{\sqrt{2}}\cos(6\pi t) - \cos 1 \cos 8\pi t + \sin 1 \sin 8\pi t.$

$= \sin 4\pi t + \sqrt{3}\cos 4\pi t + \frac{1}{\sqrt{2}}\sin(6\pi t) - \frac{1}{\sqrt{2}}\cos(6\pi t) - \cos 1 \cos 8\pi t + \sin 1 \sin 8\pi t$

So, $b_2 = 1,\ a_2 = \sqrt{3},\ b_3 = \frac{1}{\sqrt{2}},\ a_3 = \frac{-1}{\sqrt{2}},\ a_4 = -\cos 1,\ b_4 = \sin 1$.

F.S.33