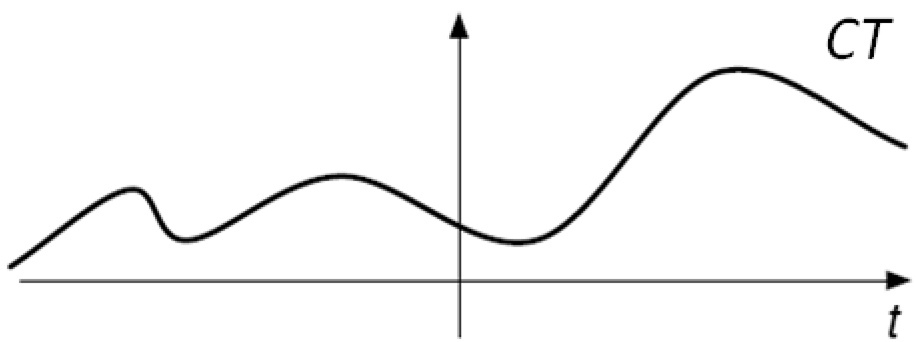
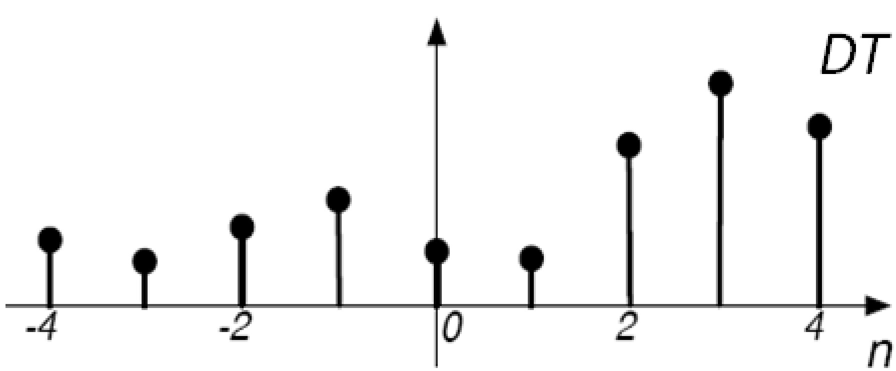
LINEAR CONSTANT COEFFICIENT DIFFERENTIAL EQUATION (LCCDE)

System Characterization



$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

Linear Constant Coefficient
Differential Equation



Linear Constant Coefficient Difference Equation

$$y[n] = x[n] + y[n-1]$$

LTI Systems Described by LCCDEs

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Repeated use of differentiation property: $\frac{d}{dt} \leftrightarrow s, \quad \frac{d^k}{dt^k} \leftrightarrow s^k$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$\Downarrow$$

$$Y(s) = H(s)X(s)$$

where $H(s) = \underbrace{\frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}}_{\text{Rational}}$

$\longleftarrow$ roots of numerator $\Rightarrow$ *zeros*

$\longleftarrow$ roots of denominator $\Rightarrow$ *poles*

ROC =? Depends on:

1) Locations of *all* poles.
2) Boundary conditions, *i.e.* right-, left-, two-sided signals.

LCCDEs – Example-1

- For an LTI system with the input and output of the form:

$$x(t) = e^{-3t}u(t)$$

$$y(t) = \left[e^{-t} - e^{-2t}\right]u(t)$$

- Find the LCCDE

LCCDEs – Example-1

we can determine the system function as:

$$X(s)=\frac{1}{s+3};\ \text{ROC } \operatorname{Re}\{s\}>-3$$

$$Y(s)=\frac{1}{(s+1)(s+2)};\ \text{ROC } \operatorname{Re}\{s\}>-1$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{s+3}{(s+1)(s+2)}=\frac{s+3}{s^2+3s+2};\ \text{ROC } \operatorname{Re}\{s\}>-1$$

- System is stable (roots in left-half plane), and causal, with differential equation of the form:

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=\frac{dx(t)}{dt}+3x(t)$$

LCCDEs – Example-2

- Consider an LTI system with the input and output relationship of the form:

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

- Find the impulse response of the system

LCCDEs – Example-2

- Taking Laplace Transforms of both sides gives:

$$sY(s) + 3Y(s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+3}$$

- Need additional system information to specify ROC; e.g., causal system

$\Rightarrow$ ROC $\mathrm{Re}\{s\} > -3$

$$h(t) = e^{-3t}u(t)$$

- or that the system is anti-causal $\Rightarrow$ ROC $\mathrm{Re}\{s\} < -3$

$$h(t) = -e^{-3t}u(-t)$$

LCCDEs – Example-3

- Consider a stable and causal system with impulse response $h(t)$ and system function $H(s)$.
- Suppose $H(s)$ is rational, contains a pole at $s=-2$, and does not have a zero at the origin.
- The location of all other poles and zeros is unknown.
- For each of the following statements, determine whether you can definitely say it is true, it is false, or that there is insufficient information.

(a) $\mathcal{F}\left\{h(t)e^{3t}\right\}$ converges

(b) $\int_{-\infty}^{\infty} h(t)dt = 0.$

(c) $th(t)$ is the impulse response of a causal and stable system.

(d) $\frac{dh(t)}{dt}$ contains at least one pole in its Laplace Transform

(e) $h(t)$ has finite duration.

(f) $H(s)=H(-s)$

(g) $\lim_{s\to\infty} H(s)=2$

LCCDEs – Example-3

(a) Statement (a) is false since $\mathcal{F}\left\{h(t)e^{3t}\right\}$ corresponds to the value of the Laplace transform of $h(t)$ at $s = -3$. If this converges, it implies that $s = -3$ is in the ROC. A causal and stable system must always have its ROC to the right of all of its poles. However $s = -3$ is not to the right of the pole at $s = -2$.

LCCDEs – Example-3

(b) Statement (b) is false because it is equivalent to stating that $H(0)=0$ which contradicts the fact that $H(s)$ does not have a zero at the origin.

LCCDEs – Example-3

(c) Statement (c) is true. The Laplace Transform of $th(t)$ has the same ROC as that of $H(s)$. This ROC includes the $j\omega-$axis and therefore the corresponding system is stable. Also $h(t)=0$ for $t<0$ implies that $th(t)=0$ for $t<0$. Thus $th(t)$ represents the impulse response of a causal system.

LCCDEs – Example-3

(d) Statement (d) is true. The system $dh(t)/dt$ has the Laplace transform $sH(s)$. The multiplication by s does not eliminate the pole at $s=-2$.

(e) Statement (e) is false. If $h(t)$ is of finite duration, then if its Laplace transform has any points in its ROC, the ROC must be the entire $s-$plane. However this is not consistent with $H(s)$ having a pole at $s=-2$.

LCCDEs – Example-3

(f) Statement (f) is false. If it were true, then since $H(s)$ has a pole at $s=-2$, it must also have a pole at $s=2$. This is inconsistent with the fact that all the poles of a causal and stable system must be in the left half of the $s-$plane.

(g) The truth of statement (g) cannot be determined with the given information. The statement requires that the degree of the numerator and denominator of $H(s)$ be equal, and we have insufficient information about $H(s)$ to determine whether this is the case.

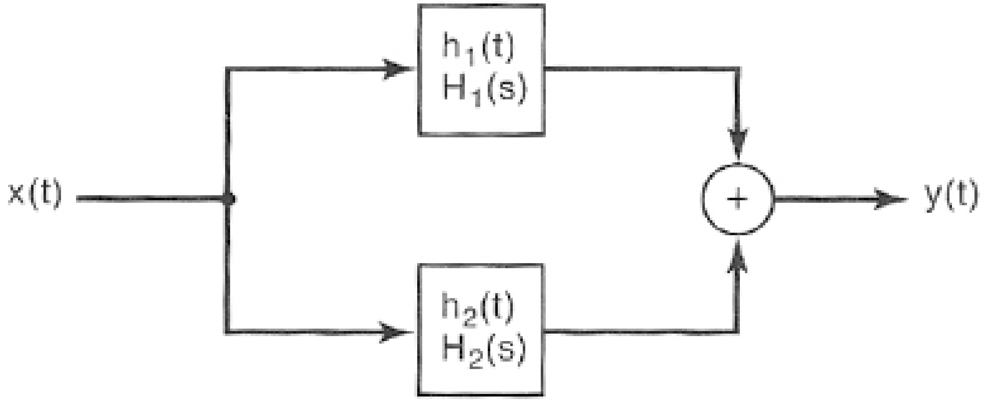
System Function Algebra

- Consider the parallel interconnection of two systems. The impulse response of the overall system is:

$$h(t) = h_1(t) + h_2(t)$$

with Laplace transform

$$H(s) = H_1(s) + H_2(s)$$



(a)

Parallel Interconnection

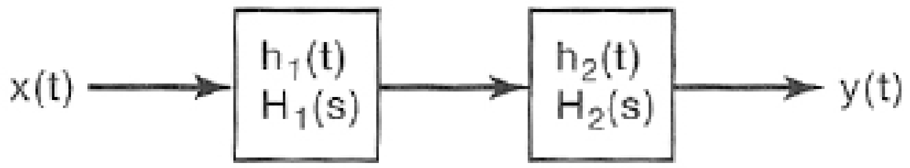
System Function Algebra

- Consider the series interconnection of two systems. The impulse response of the overall system is:

$$h(t) = h_1(t) * h_2(t)$$

with Laplace transform

$$H(s) = H_1(s) H_2(s)$$

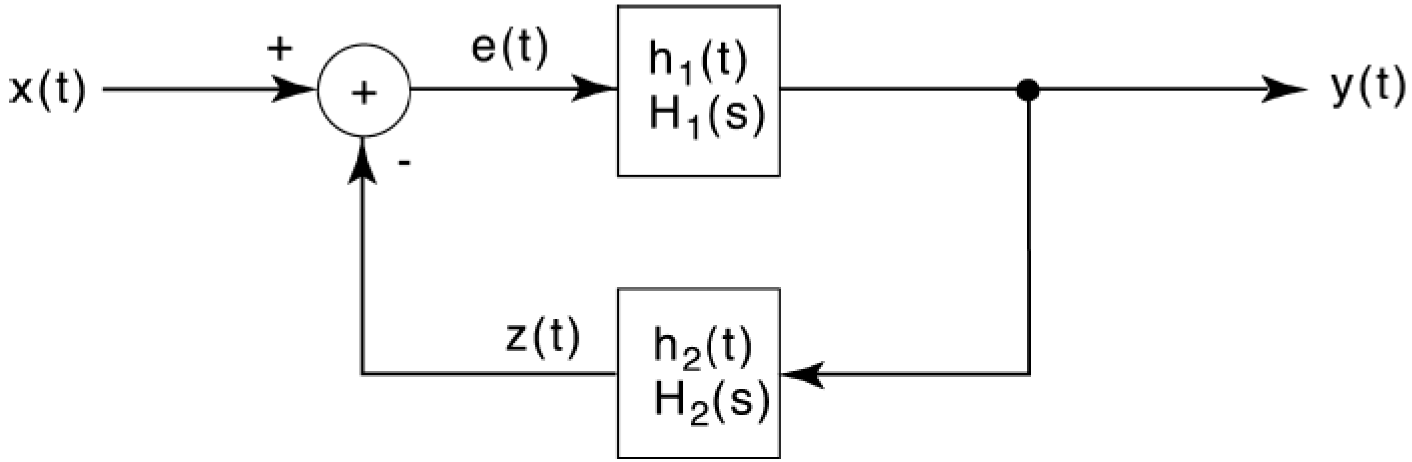


(b)

Series Interconnection

System Function Algebra

Example: A basic feedback system consisting of ***causal*** blocks



$$E(s) = X(s) - Z(s) = X(s) - H_2(s)Y(s)$$

$$Y(s) = H_1(s)E(s) = H_1(s)[X(s) - H_2(s)Y(s)]$$

$$\Downarrow$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

ROC: Determined by the roots of $1+H_1(s)H_2(s)$, instead of $H_1(s)$

System Function Algebra – Example-4

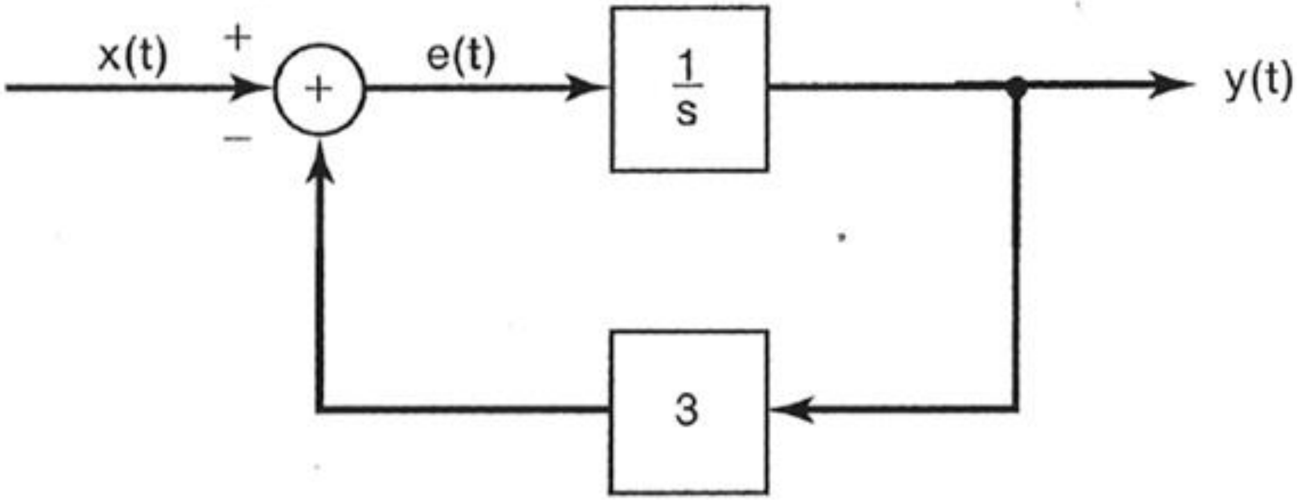
- Consider the causal LTI system with system function:

$$H(s) = \frac{1}{s+3}$$

- Determine the block diagram for this system.
- Also, determine the differential equation for this system

System Function Algebra – Example-4

$$H(s)=\frac{Y(s)}{X(s)}=\frac{(1/s)}{1+\left(\frac{1}{s}\right)3}=\frac{1}{s+3}$$



$$\frac{dy(t)}{dt}+3y(t)=x(t)$$

END