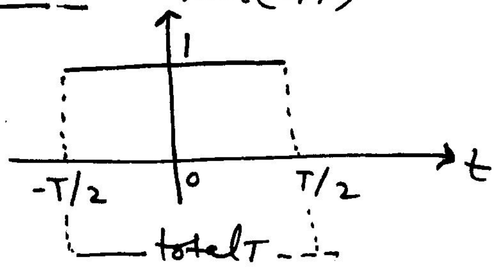
Fourier transform of some useful functions:-

rect(t/T)



Rectangular pulse:

$$\text{rect}\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| < T/2 \\ 0, & \text{otherwise} \end{cases}$$

Here T is the width of the pulse.

So, rect(t) is a pulse of width 1.

$$F[f(t)] = F(j\omega) = \int_{-\infty}^{+\infty} \text{rect}(t/T)\, e^{-j\omega t}\, dt = \int_{-T/2}^{T/2} (1)\, e^{-j\omega t}\, dt$$

$$= \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T/2}^{T/2} = \frac{1}{j\omega}\left[e^{j\omega T/2} - e^{-j\omega T/2}\right] = \frac{1}{j\omega}\left[2j \sin\frac{\omega T}{2}\right)$$

$$= \frac{T \sin(\omega T/2)}{\omega T/2} = T\, \text{sinc}(\omega T/2).$$

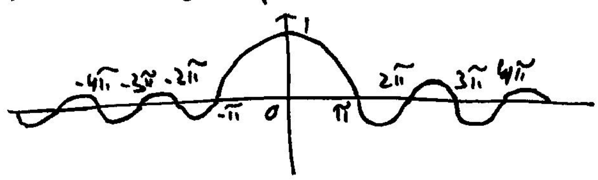
Thus the Fourier transform pair is

$$\text{rect}(t/T) \longleftrightarrow T\, \text{sinc}(\omega T/2)$$

pulse width — pulse width — half-pulse width

Sinc function (sampling function): The function $\frac{\sin x}{x}$ aries frequently in analysis of signals, we denote it by sinc x. It is also known as the filtering or interpolating function.

$\text{sinc}\, x = \frac{\sin x}{x}$, sinc x is an even function, $\text{sinc}(x) = 0$ for $x = \pm n\pi$



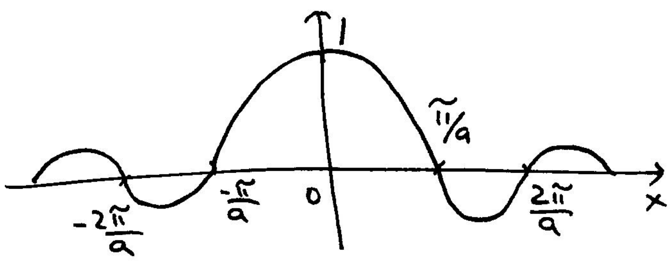
In general, $\text{sinc}(ax) = \frac{\sin ax}{ax}$

$\text{sinc}(ax) = 0$

for $ax = \pm n\pi$

or $x = \pm \frac{n\pi}{a}$

where $n = 0, 1, 2, 3, \ldots$



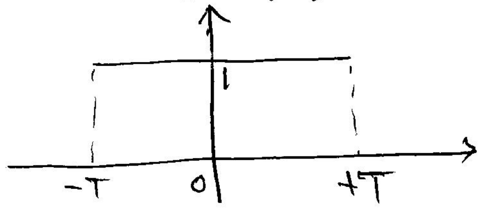
$$\text{rect}(t) \longleftrightarrow \text{Sinc}(\omega/2)$$

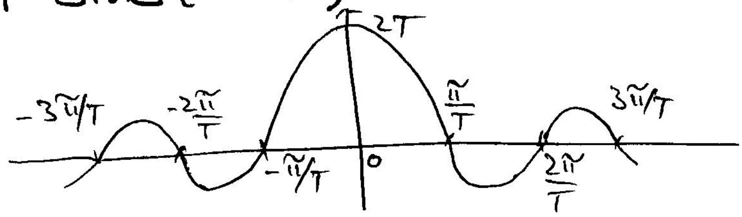
and $\text{rect}\left(\frac{t}{2T}\right) \longleftrightarrow 2\cancel{A}T\,\text{Sinc}(\omega T)$

and $A\,\text{rect}\left(\frac{t}{2T}\right) \longleftrightarrow 2AT\,\text{Sinc}(\omega T)$

where, $A\,\text{rect}\left(\frac{t}{2T}\right) = \begin{cases} A, & |t| \le T \\ 0, & |t| > T \text{ (otherwise)}. \end{cases}$

$$\text{rect}\left(\frac{t}{2T}\right) \longleftrightarrow 2T\,\text{Sinc}(\omega T)$$





<u>Duality &/or Symmetry Property</u>: The duality property states that if $f(t) \longleftrightarrow F(\omega)$

then $F(t) \longleftrightarrow 2\pi f(-\omega)$

<u>Ex</u>:- Find the Fourier transform of e^{-jat} & e^{jat}.

$$F[\delta(t-a)] = \int_{-\infty}^{+\infty} \delta(t-a)\, e^{-j\omega t}\, dt = e^{-ja\omega}$$

Also, $F[\delta(t+a)] = \int_{-\infty}^{+\infty} \delta(t+a)\, e^{-j\omega t}\, dt = e^{ja\omega}$

Using duality, $F[e^{-jat}] = 2\pi\delta(-\omega-a) = 2\pi\delta(\omega+a)$.

Similarly, $F[e^{jat}] = 2\pi\delta(\omega - a)$.

<u>Fourier transform of Sinat & Cosat</u>:

$$F[\text{Cos}\,at] = F\left[\frac{e^{jat}+e^{-jat}}{2}\right] = \frac{2\pi}{2}\left[\delta(\omega-a) + \delta(\omega+a)\right]$$

Also, $F[\text{Sin}\,at] = F\left[\frac{e^{jat}-e^{-jat}}{2j}\right] = \frac{1}{2j}2\pi\left[\delta(\omega-a) - \delta(\omega+a)\right]$

$= -j\pi\left[\delta(\omega-a) - \delta(\omega+a)\right] = j\pi\left[\delta(\omega+a) - \delta(\omega-a)\right].$

Fourier transform of sinc(at) using duality/symmetry:-

We know that $\text{rect}\left(\frac{t}{2T}\right) \longleftrightarrow 2T\,\text{sinc}(\omega T)$

or $\frac{1}{2T}\,\text{rect}\left(\frac{t}{2T}\right) \longleftrightarrow \text{sinc}(\omega T)$

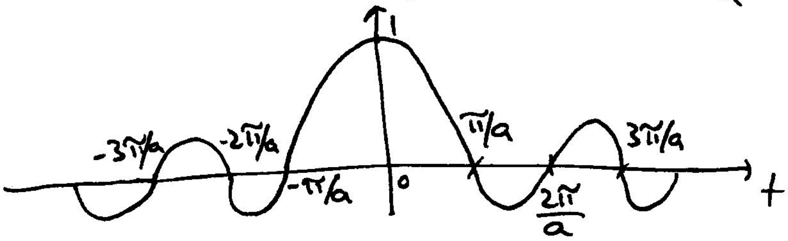
Duality

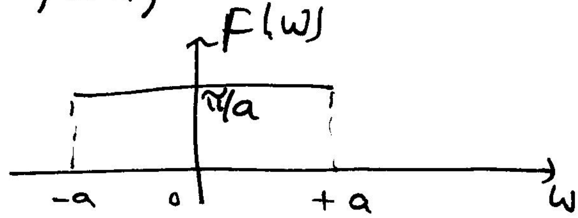
Now, $\text{sinc}(tT) \longleftrightarrow 2\pi\left\{\frac{1}{2T}\,\text{rect}\left(\frac{-\omega}{2T}\right)\right\}$

$= \frac{\pi}{T}\,\text{rect}\left(\frac{\omega}{2T}\right)$, rect(t) is an even function

By replacing 'T' with 'a', we get

$\text{sinc}(at) \longleftrightarrow \frac{\pi}{a}\,\text{rect}(\omega/2a)$





We summarize: $\text{rect}\left(\frac{t}{T}\right) \longleftrightarrow T\,\text{sinc}\left(\frac{\omega T}{2}\right)$

and $\text{sinc}(at) \longleftrightarrow \frac{\pi}{a}\,\text{rect}\left(\frac{\omega}{2a}\right)$.

Conclusion: we have $\frac{1}{T}\,\text{rect}(t/T) \longleftrightarrow \text{sinc}(\omega T/2)$

Compare

For $\text{sinc}(at) \longleftrightarrow$?

Comparison gives, $a = T/2 \Rightarrow T = 2a$, so using duality we have

$\text{sinc}(at) \longleftrightarrow 2\pi\left[\frac{1}{2a}\,\text{rect}(-\omega/2a)\right] = \frac{\pi}{a}\,\text{rect}(\omega/2a)$.

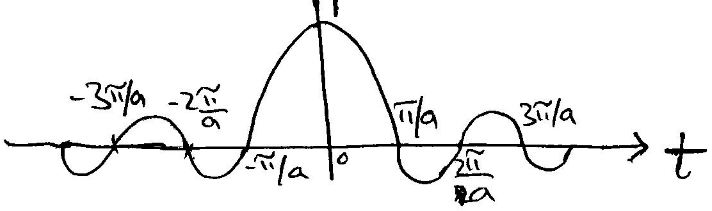
Also, $\frac{a}{\pi}\,\text{sinc}(at) \longleftrightarrow \text{rect}\left(\frac{\omega}{2a}\right)$ Rect(t) & sinc(t) are even functions.

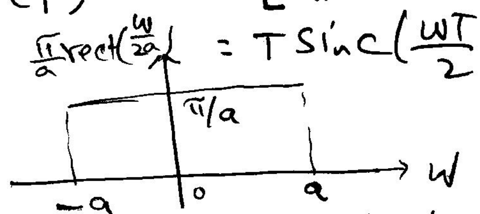
Compare

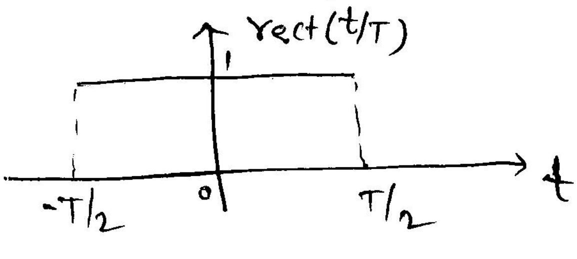
$\text{rect}(t/T) \longleftrightarrow$?

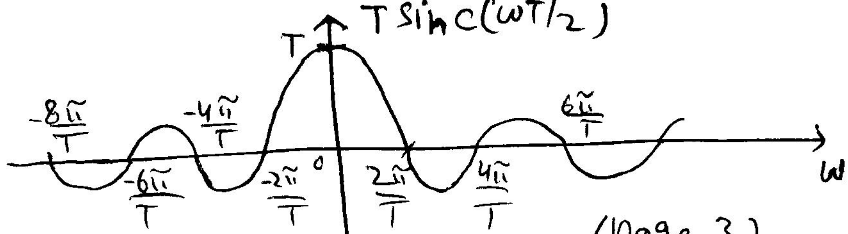
$T = 2a \Rightarrow a = T/2$, using duality, $\text{rect}\left(\frac{t}{T}\right) \longleftrightarrow 2\pi\left[\frac{T/2}{\pi}\,\text{sinc}\left(-\frac{\omega T}{2}\right)\right]$

$= T\,\text{sinc}\left(\frac{\omega T}{2}\right)$

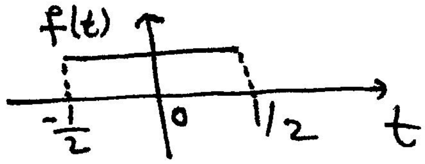








Normalized rectangular pulse:



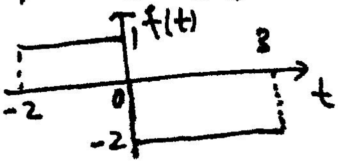
$$f(t) = \mathrm{rect}(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

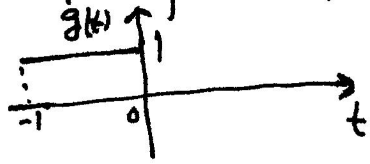
$$\mathcal{F}[f(t)] = F(j\omega) = \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right) = \frac{\sin(\omega/2)}{\omega/2} = \mathrm{sinc}(\omega/2)$$

Consider the functions

Ex-1 $f(t) = \begin{cases} 1, & -2 \le t < 0 \\ -2, & 0 \le t < 3 \\ 0, & \text{otherwise.} \end{cases}$, $g(t) = \begin{cases} 1, & -1 \le t < 0 \\ 0, & \text{otherwise} \end{cases}$

Express $f(t)$ and $g(t)$ in the form of $\mathrm{rect}(t)$.

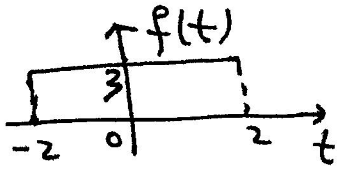




$$g(t) = \mathrm{rect}(t + 1/2)$$

$$f(t) = \mathrm{rect}\left(\tfrac{1}{2}(t+1)\right) - 2\,\mathrm{rect}\left(\tfrac{1}{3}\left(t - \tfrac{3}{2}\right)\right)$$

Relationship between rect(t) and unit step function:



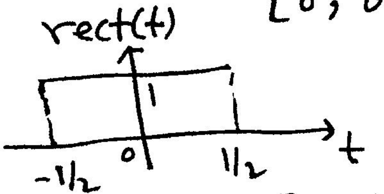
$$f(t) = \begin{cases} 3, & -2 \le t \le 2 \\ 0, & \text{otherwise} \end{cases} \qquad f(t) = 3[u(t+2) - u(t-2)] = 3\,\mathrm{rect}\left(\tfrac{1}{4}t\right).$$

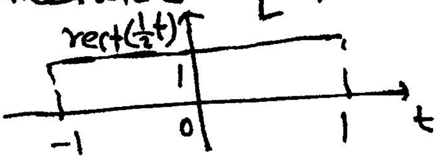
Explanation: $\mathrm{rect}(t) = \begin{cases} 1, & -\frac{1}{2} < t < 1/2 \\ 0, & \text{otherwise} \end{cases}$, $\mathrm{rect}\left(\frac{1}{4}t\right) = \begin{cases} 1, & -\frac{1}{2} < \frac{1}{4}t < 1/4 \\ 0, & \text{otherwise.} \end{cases}$

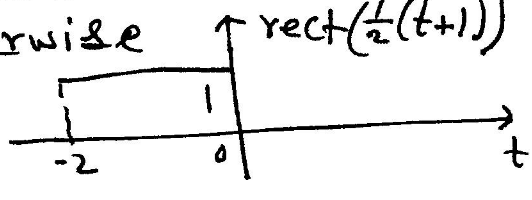
$\mathrm{rect}\left(\frac{1}{4}t\right) = \begin{cases} 1, & -2 < t < 2 \\ 0, & \text{otherwise} \end{cases}$, $3\,\mathrm{rect}\left(\frac{1}{4}t\right) = \begin{cases} 3, & -2 < t < 2 \\ 0, & \text{otherwise.} \end{cases}$

Explanation Ex-1: $\mathrm{rect}\left(\frac{1}{2}t\right) = \begin{cases} 1, & -\frac{1}{2} < \frac{1}{2}t < 1/2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & -1 < t < 1 \\ 0, & \text{otherwise} \end{cases}$

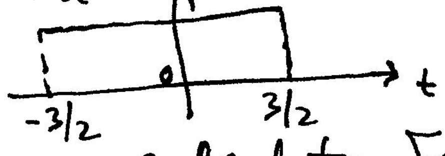
$$\mathrm{rect}\left(\tfrac{1}{2}(t+1)\right) = \begin{cases} 1, & -1 < t+1 < 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & -2 < t < 0 \\ 0, & \text{otherwise} \end{cases}$$

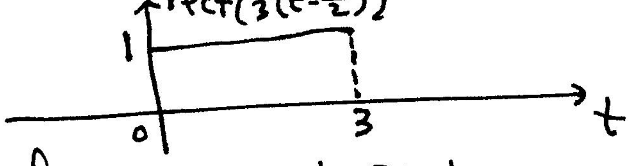






Also, $\mathrm{rect}\left(\frac{1}{3}t\right) = \begin{cases} 1, & -\frac{1}{2} < \frac{1}{3}t < 1/2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & -\frac{3}{2} < t < \frac{3}{2} \\ 0, & \text{otherwise} \end{cases}$, $\mathrm{rect}\left(\frac{1}{3}\left(t - \frac{3}{2}\right)\right) = \begin{cases} 1, & -\frac{3}{2} < t - \frac{3}{2} < \frac{3}{2} \\ 0, & \text{otherwise.} \end{cases}$





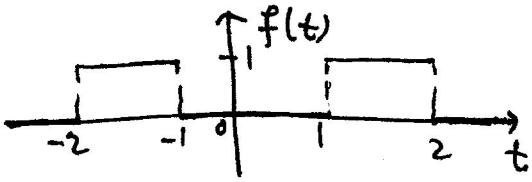
Now, we calculate Fourier transform of $f(t)$ in Ex-1.

$$F(j\omega) = \int_{-2}^{0} e^{-j\omega t}\,dt - 2\int_{0}^{3} e^{-j\omega t}\,dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-2}^{0} - 2\left.\frac{e^{-j\omega t}}{-j\omega}\right|_{0}^{3}$$

$$= \frac{-1}{j\omega}\left(1 - e^{2j\omega}\right) + \frac{2}{j\omega}\left(e^{-3j\omega} - 1\right) = -\frac{1}{j\omega} + \frac{1}{j\omega}e^{2j\omega} + \frac{2}{j\omega}e^{-3j\omega} - \frac{2}{j\omega}.$$

Ex:- Let $f(t)=\begin{cases}1, & -2<t<-1\\ 0, & -1<t<1\\ 1, & 1<t<2\\ 0, & \text{otherwise}\end{cases}$

i). Express $f(t)$ using $\text{rect}(t)$.
ii). Calculate $F(jw)$



$$f(t)=\text{rect}\left(t+\tfrac{3}{2}\right)+\text{rect}\left(t-\tfrac{3}{2}\right)$$

$$F(w)=\mathcal{F}[f(t)]=\text{sinc}(w/2)\,e^{\frac{3}{2}jw}+\text{sinc}(w/2)\,e^{-\frac{3}{2}jw}$$

$$=2\,\text{sinc}(w/2)\left(\frac{e^{3/2\,jw}+e^{-\frac{3}{2}jw}}{2}\right)=2\,\text{sinc}(w/2)\left(\cos\tfrac{3}{2}w\right)$$

$$=\frac{2}{w}\left(2\cos\left(\tfrac{3}{2}w\right)\sin\left(\tfrac{w}{2}\right)\right)=\frac{2}{w}\left(\sin\left(\tfrac{3}{2}w+\tfrac{w}{2}\right)-\sin\left(\tfrac{3}{2}w-\tfrac{w}{2}\right)\right)$$

$$=\frac{2}{w}\left(\sin 2w-\sin w\right)=4\,\frac{\sin 2w}{2w}-2\,\frac{\sin w}{w}=4\,\text{sinc}(2w)-2\,\text{sinc}(w).$$

By definition,

$$F(w)=\int_{-2}^{-1}e^{-jwt}\,dt+\int_{1}^{2}e^{-jwt}\,dt=\left.\frac{e^{-jwt}}{-jw}\right|_{-2}^{-1}+\left.\frac{e^{-jwt}}{-jw}\right|_{1}^{2}$$

$$=\frac{-1}{jw}\left(e^{jw}-e^{2jw}+e^{-2jw}-e^{-jw}\right)=\frac{1}{jw}\left(e^{2jw}-e^{-2jw}\right)-\frac{1}{jw}\left(e^{jw}-e^{-jw}\right)$$

$$=\frac{2}{w}\left(\sin 2w-\sin w\right)=4\,\frac{\sin 2w}{2w}-2\,\frac{\sin w}{w}=4\,\text{sinc}(2w)-2\,\text{sinc}(w).$$

Exercise:

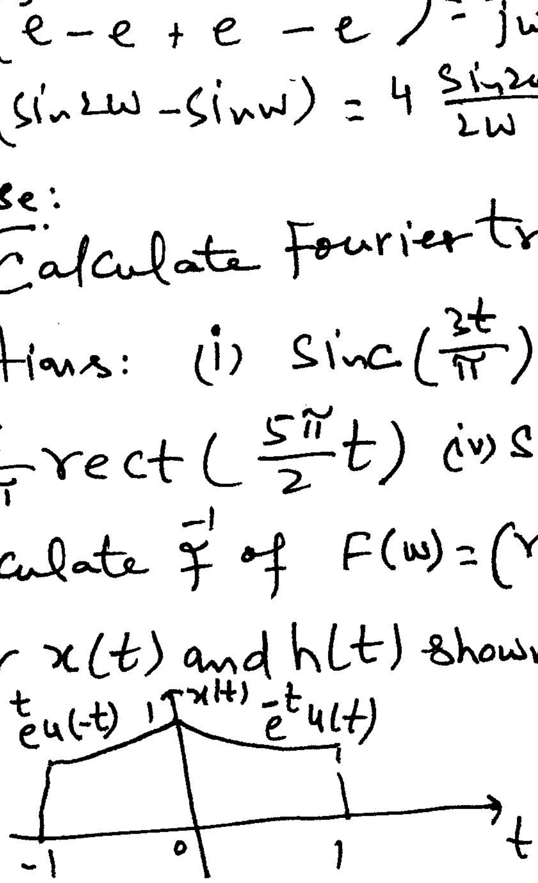
Q-1 Calculate Fourier transform of the following functions: (i) $\text{sinc}\left(\frac{3t}{\pi}\right)$ (ii) $\text{sinc}^2\left(\frac{2t-4}{3}\right)$

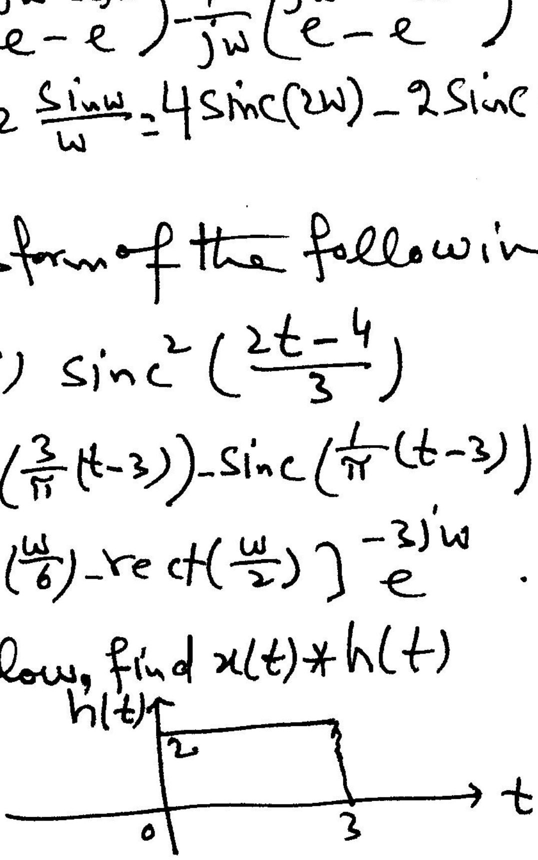
(iii) $\frac{5}{2\pi}\text{rect}\left(\frac{5\pi}{2}t\right)$ (iv) $\text{sinc}\left(\frac{3}{\pi}(t-3)\right)-\text{sinc}\left(\frac{1}{\pi}(t-3)\right)$

Q-2: calculate $\mathcal{F}^{-1}$ of $F(w)=\left[\text{rect}\left(\frac{w}{6}\right)-\text{rect}\left(\frac{w}{2}\right)\right]e^{-3jw}$.

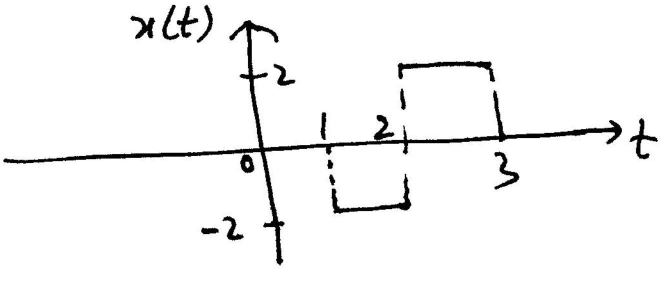
Q-3 For $x(t)$ and $h(t)$ shown below, find $x(t)*h(t)$

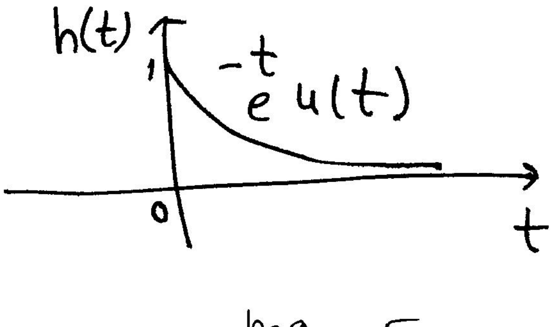
i)





ii)





Convolution $y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$.

Note:

1. "t" is not the variable of integration.
2. We must evaluate the above integral for each value of t for $-\infty < t < \infty$.
3. What the convolution integral tell us:

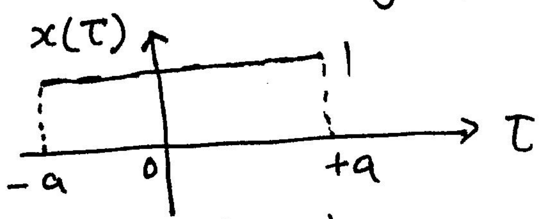
a) Reflect $h(\tau)$, the impulse response to obtain $h(-\tau)$. This is the position for $t=0$.

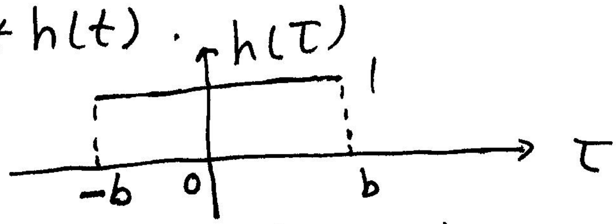
b) For every value of t, $h(t-\tau) = h(-(\tau - t))$ is a time-shifted version of $h(-\tau)$.

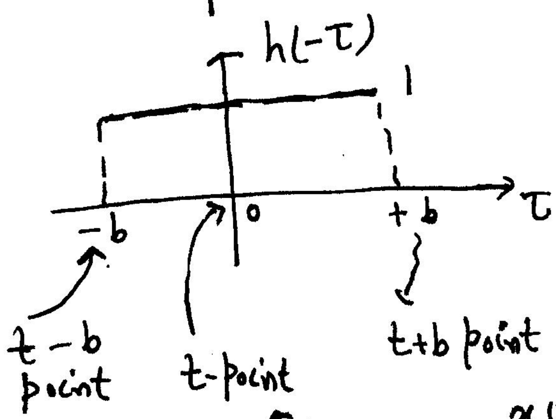
c) multiply by $x(\tau)$ and find the non-zero area.

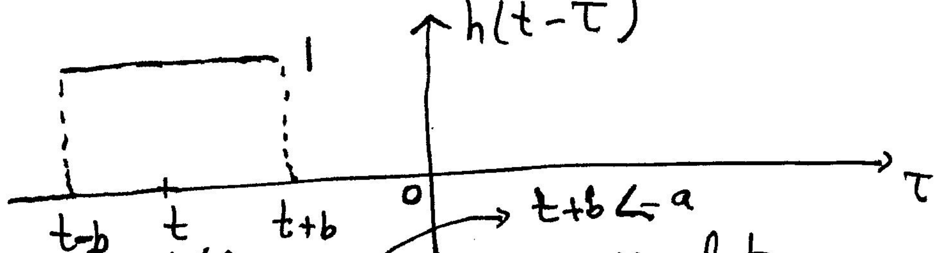
Ex:- $x(t) = \text{rect}\left(\frac{t}{2a}\right)$, $h(t) = \text{Rect}\left(\frac{t}{2b}\right)$, $a > b > 0$.

Determine $y(t) = x(t) * h(t)$.



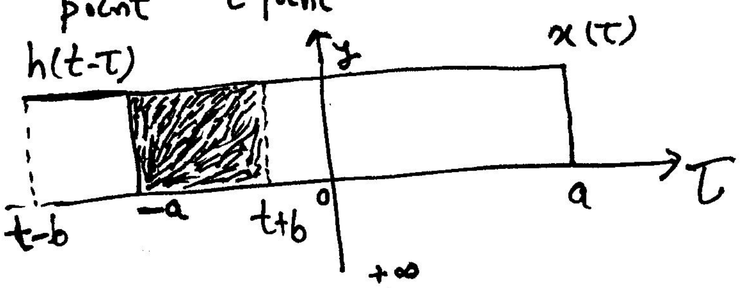






$\rightarrow t+b < -a$

$\underbrace{t+b<-a}$ for $t < -a-b$, No overlap between $x(\tau)$ and $h(t-\tau)$, $y(t) = 0$,



$t+b > -a \rightarrow t > -a-b$

$t-b < -a \rightarrow t < -a+b$

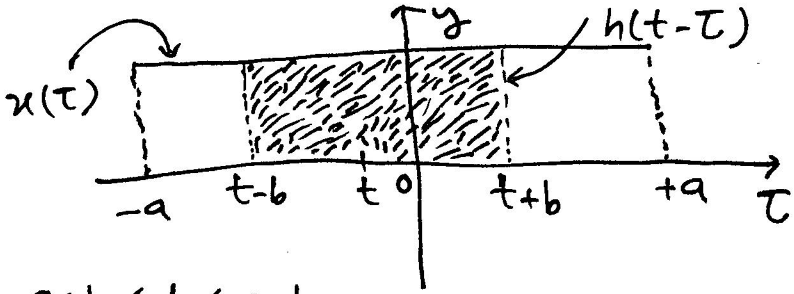
$-a-b < t < -a+b$.

$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-a}^{t+b} (1)(1)\, d\tau = t+b+a$,

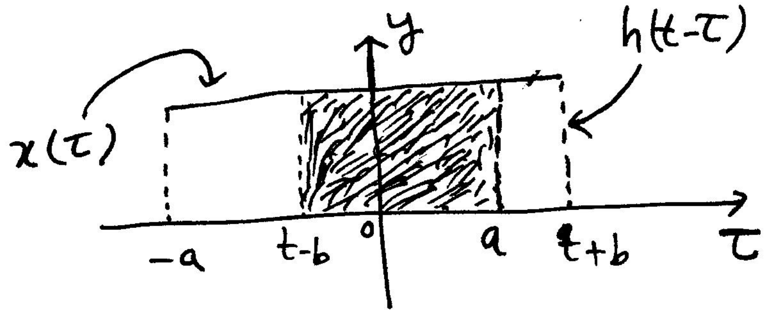
$-a-b < t < -a+b$.

Now: $t+b < a \rightarrow t < a-b$

$\therefore t-b \geq -a \rightarrow t \geq -a+b$



$$y(t) = \int_{t-b}^{t+b} (1)(1)\,dt = 2b \quad , \quad -a+b \leq t < a-b \, .$$



$t+b \geq a \rightarrow t \geq a-b$

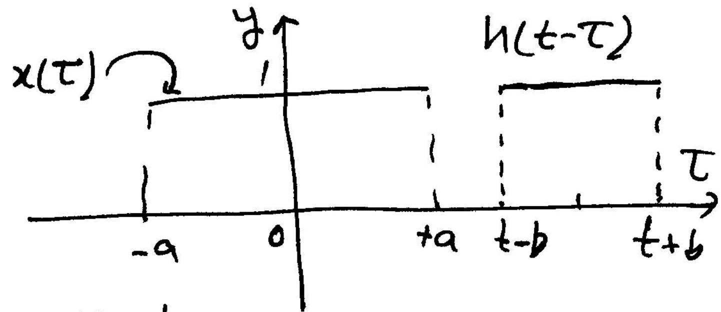
$t-b < a \rightarrow t < a+b$

$a-b \leq t < a+b$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau = \int_{t-b}^{a} (1)(1)\,d\tau = -t+a+b \, , \quad a-b \leq t < a+b \, .$$

$t-b \geq a \rightarrow t \geq a+b,$

$y(t) = 0$.



Let us combine to write the output

$$y(t) = \begin{cases} 0 & t < -a-b \\ t+b+a & -a-b \leq t < -a+b \\ 2b & -a+b \leq t < a-b \\ -t+b+a & a-b \leq t < a+b \\ 0 & t \geq a+b \, . \end{cases}$$