INTEGRATION

Calculus & Analytical Geometry MATH-101

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Techniques of Integration

- Substitution Rule
- Integration by Parts
- Integration of Rational
- Integration of Irrational Functions
- Trigonometric Substitution
- Trigonometric Integrals

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter:** 8
  - **Section:** 8.4

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- **Chapter: 9**
  - **Section:** 9.2

Sine and Cosine Integrals

- In this section, we will see how to use trigonometric identities to integrate certain combinations of trigonometric functions.
- We start with powers of sine and cosine.

$\int \sin^{100} x \cos^{97} x \, dx$

Example: Evaluate

$$\int \cos^3 x \, dx. \checkmark$$

Solution:

- Simply substituting $u = \cos x$ isn't helpful, since then $du = -\sin x \, dx$.
- In order to integrate powers of cosine, we would need an extra $\sin x$ factor.
- Similarly, a power of *sine* would require an extra $\cos x$ factor.
- Thus, here we can separate one cosine factor and convert the remaining $\cos^2 x$ factor to an expression involving sine using the identity: $\sin^2 x + \cos^2 x = 1$, i.e.,

$u = \sin x$

$du = \cos x \, dx$

$$\cos^3 x = \cos^2 x . \cos x = (1 - \sin^2 x) \cos x$$

Sine and Cosine Integrals

Solution:

- We can then evaluate the integral by substituting $u = \sin x$ and $du = \cos x\, dx$.

$$\int \cos^3 x\, dx = \int \cos^2 x \cdot \cos x\, dx$$

$$= \int (1 - \sin^2 x) \cos x\, dx$$

$$= \int (1 - u^2) du = u - \frac{1}{3}u^3 + C$$

$$\Longrightarrow \int \cos^3 x\, dx = \sin x - \frac{1}{3}\sin^3 x + C.$$

- In general, we try to write an integrand involving powers of sine and cosine in a form where we have only one sine factor. The remainder of the expression can be in terms of cosine.
- We could also try only one cosine factor. The remainder of the expression can be in terms of sine.

Sine and Cosine Integrals

Example: Evaluate

$$\int \sin^5 x \cos^2 x \, dx.$$

Solution:

- We could convert $\cos^2 x$ to $1 - \sin^2 x$. However, we would be left with an expression in terms of $\sin x$ with no extra $\cos x$ factor.
- Instead, we separate a single sine factor and rewrite the remaining $\sin^4 x$ factor in terms of $\cos x$. So, we have:

$$\sin^5 x \cos^2 x = (\sin^2 x)^2 \cos^2 x \sin x = (1 - \cos^2 x)^2 \cos^2 x \sin x \, .$$

- Substituting $u = \cos x$, we have $du = -\sin x \, dx$. So, $\Rightarrow$ sin dx = −du

$$\int \sin^5 x \cos^2 x \, dx = \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx$$

$$= \int (1 - u^2)^2 u^2 \, (-du) = -\int (u^2 - 2u^4 + u^6) du = -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C$$

$$\Rightarrow \int \sin^5 x \cos^2 x \, dx = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C.$$

Sine and Cosine Integrals

- In the preceding examples, an odd power of sine or cosine enabled us to separate a single factor and convert the remaining even power.
- If the integrand contains even powers of both sine and cosine, this strategy fails.
- In that case, we can take advantage of the following half-angle identities:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

Example: Evaluate

$$\int \sin^2 x \, dx.$$

If we write $\sin^2 x = 1 - \cos^2 x$, the integral is not simple to evaluate. However, using the half-angle formula for $\sin^2 x$ we have:

$$\int \sin^2 x \, dx = \frac{1}{2} \int (1 - \cos 2x) dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C.$$

Sine and Cosine Integrals

- To summarize, we list guidelines to follow when evaluating integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

if m is odd

if n is also odd.

where $m \geq 0$ and $n \geq 0$ are integers.

- If the power of cosine is odd $(n = 2k + 1)$, save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine as:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx = \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx. \quad (A)$$

Then, substitute $u = \sin x$.

- If the power of sine is odd $(m = 2k + 1)$, save one sine factor and Use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine as:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx = \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx. \quad (B)$$

Then, substitute $u = \cos x$.

Sine and Cosine Integrals

- Note that, if the powers of both sine and cosine are odd, either (A) or (B) can be used.
- If the powers of both sine and cosine are even, use the half-angle identities:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \text{ and } \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Sometimes, it is helpful to use the identity:

$$\sin x \cos x = \frac{1}{2}\sin 2x.$$

m is even and n is also even

Tangent and Sec Integrals

- We can use a similar strategy to evaluate integrals of the form:

$$\int \tan^m x \sec^n x \, dx, \quad \checkmark$$

$u = \tan x$

$du = \sec^2 x \, dx$

where $m \geq 0$ and $n \geq 0$ are integers.

- As

$$\frac{d}{dx}(\tan x) = \sec^2 x,$$

we can separate a $\sec^2 x$ factor. Then, we convert the remaining (even) power of secant to an expression involving tangent using the identity $\sec^2 x = 1 + \tan^2 x$.

- Alternately, as

$u = \sec x$ ✓

$du = (\sec x \tan x) \, dx$

$$\frac{d}{dx}(\sec x) = \sec x \tan x,$$

we can separate a $\sec x \tan x$ factor and convert the remaining (even) power of tangent to secant.

Tangent and Sec Integrals

Example: Evaluate

$$\int \boxed{\tan^6 x} \sec^4 x \, dx,$$

$u = \tan x$

$du = \sec^2 x \, dx$

Solution:

If we separate one $\sec^2 x$ factor, we can express the remaining $\sec^2 x$ factor in terms of tangent using the identity: $\sec^2 x = 1 + \tan^2 x$. Then, we can evaluate the integral by substituting $u = \tan x$ so that $du = \sec^2 x \, dx$. Thus,

$$\int \tan^6 x \sec^4 x \, dx = \int \tan^6 x \sec^2 x \sec^2 x \, dx = \int \tan^6 x \, (1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int u^6 (1 + u^2) \, du = \int (u^6 + u^8) \, du = \frac{u^7}{7} + \frac{u^9}{9} + C.$$

$$\Rightarrow \int \tan^6 x \sec^4 x \, dx = \frac{1}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C.$$

Tangent and Sec Integrals

Example: Evaluate

$u = \sec x$

$du = (\sec x \tan x)\, dx$

$$\int \tan^3 x \sec^7 x\, dx,$$

Solution:

If we separate one $\sec^2 x$ factor, we are left with a $\sec^5 x$ factor that can not be easily converted to tangent. However, if we separate a $\sec x \tan x$ factor, we can convert the remaining power of tangent to an expression involving only secant. Here we can use the identity: $\tan^2 x = 1 - \sec^2 x$. We can then evaluate the integral by substituting $u = \sec x$ so that $du = \sec x \tan x \; dx$. Thus,

$$\int \tan^3 x \sec^7 x\, dx = \int \tan^2 x \sec^6 x\, (\sec x \tan x)\; dx = \int (\sec^2 x - 1)\sec^6 x\, (\sec x \tan x) dx$$

$$= \int (u^2 - 1)u^6\, du = \int (u^8 - u^6)\, du = \frac{u^9}{9} - \frac{u^7}{7} + C.$$

$$\Rightarrow \int \tan^5 x \sec^7 x\, dx = \frac{1}{9}\sec^9 x - \frac{1}{7}\sec^7 x + C.$$

Tangent and Sec Integrals

- To summarize, we list guidelines to follow when evaluating integrals of the form:

$$\int \tan^m x \sec^n x \, dx,$$

where $m \geq 0$ and $n \geq 0$ are integers.

- If the power of secant is even ($n = 2k, k \geq 2$) save $\sec^2 x$. Then, use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$ as:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx = \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then, substitute $u = \tan x$.

- If the power of tangent is odd ($m = 2k + 1$), save $\sec x \tan x$. Then, use $\tan^2 x = 1 - \sec^2 x$ to express the remaining factors in terms of $\sec x$ as:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx$$

Then, substitute $u = \sec x$. ✓

Other Integrals

- If an even power of tangent appears with an odd power of secant, There is no standard method of evaluation. However, it is useful to express the integrand completely in terms of $\sec x$. We possibly use integration by parts.

- For other cases, we don't the guidelines are not as clear-cut. We may need to use:
  - Identities
  - Integration by parts
  - A little ingenuity

$\int \tan^m x \sec^n x \, dx$

- Integrals of the form

$$\int \cot^m x \csc^n x \, dx,$$

Practice

can be found by similar methods.

Other Integrals

- Finally, if an integrand has one of the forms:

$$\cos mx \cos nx \,, \sin mx \sin nx \text{ or } \sin mx \cos nx \,,$$

we use a product-to-sum formula to evaluate the given integral.

	Integral	Identity
a	$\int \sin mx \cos nx \, dx$	$\sin A \cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$ ✓
b	$\int \sin mx \sin nx \, dx$	$\sin A \sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$ ✓
c	$\int \cos mx \cos nx \, dx$	$\cos A \cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$ ✓

Practice Questions

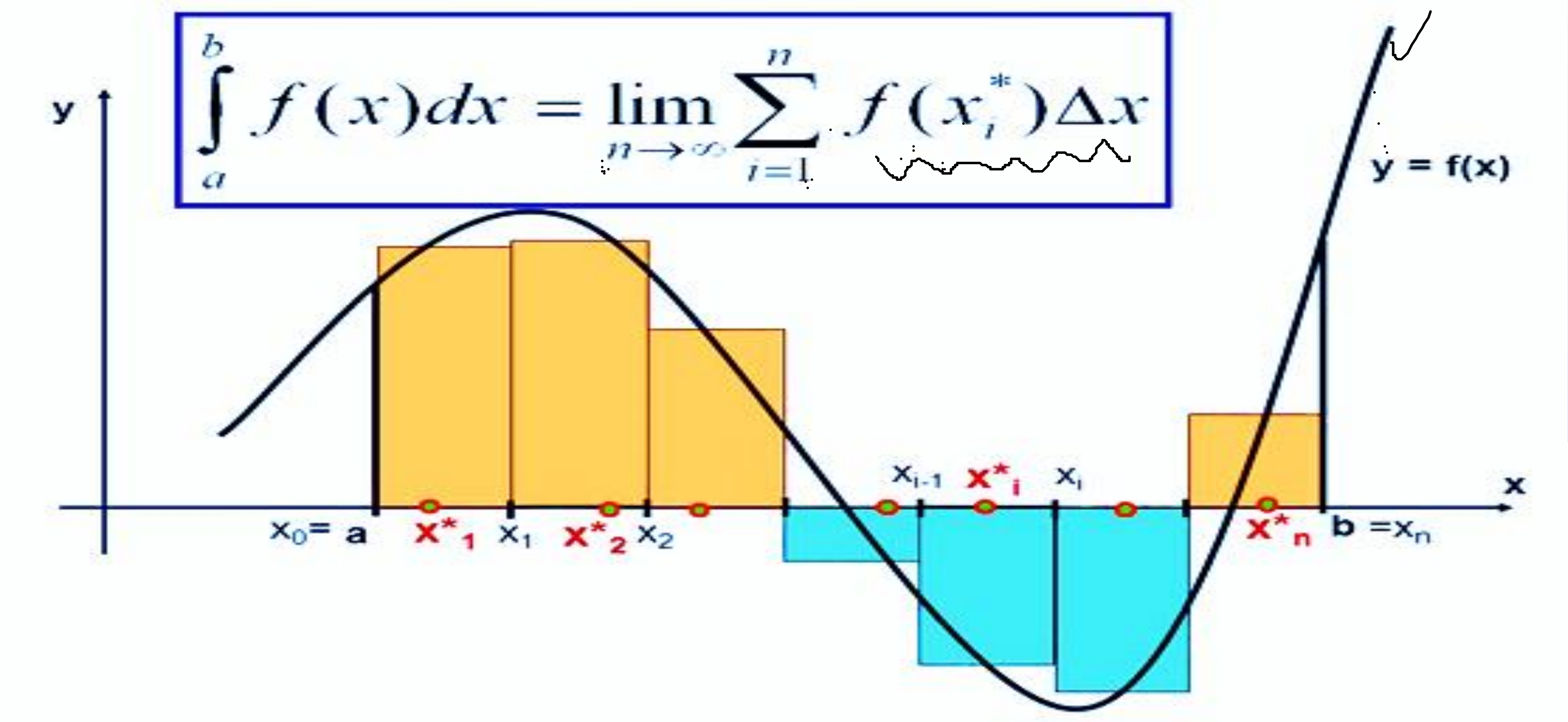
Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

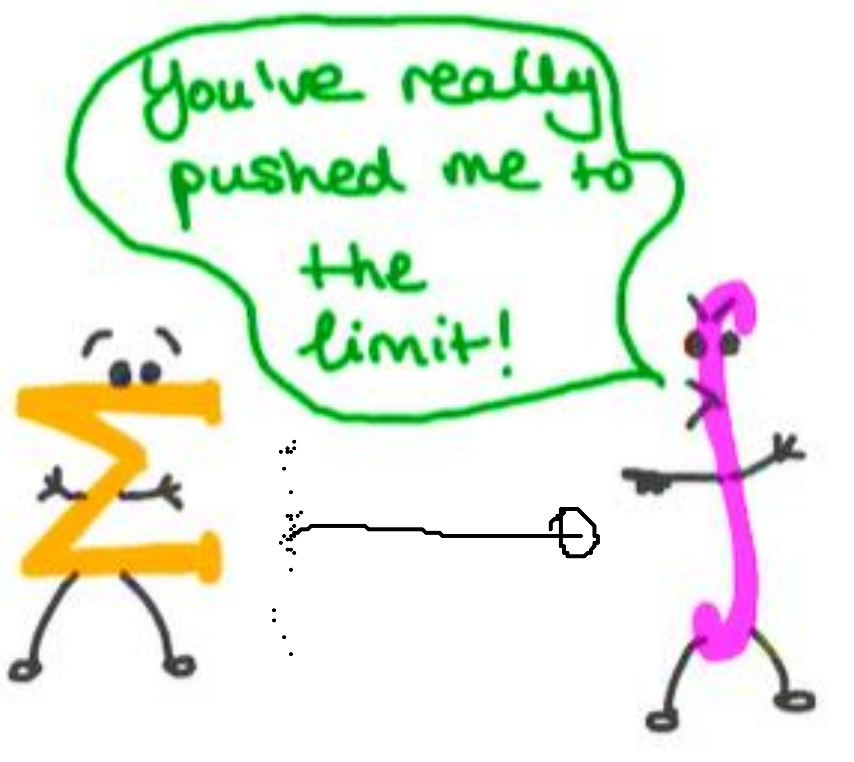
- **Chapter:** 9
  - Exercise: 9.2

    Q # 1 to Q # 18, Q # 24 to Q # 30.

Riemann Sums & Definite Integrals

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$





Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

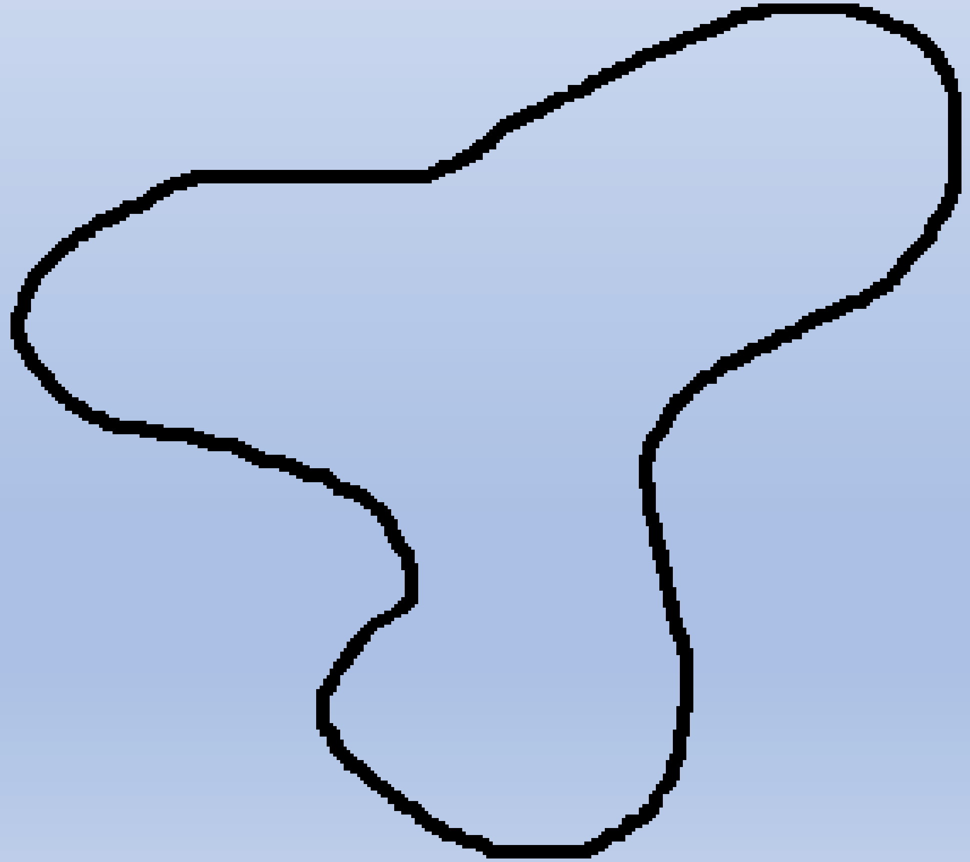
- **Chapter:** 5
  - **Section:** 5.3, 5.4

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- **Chapter:** 5
  - **Section:** 5.4, 5.5, 5.6

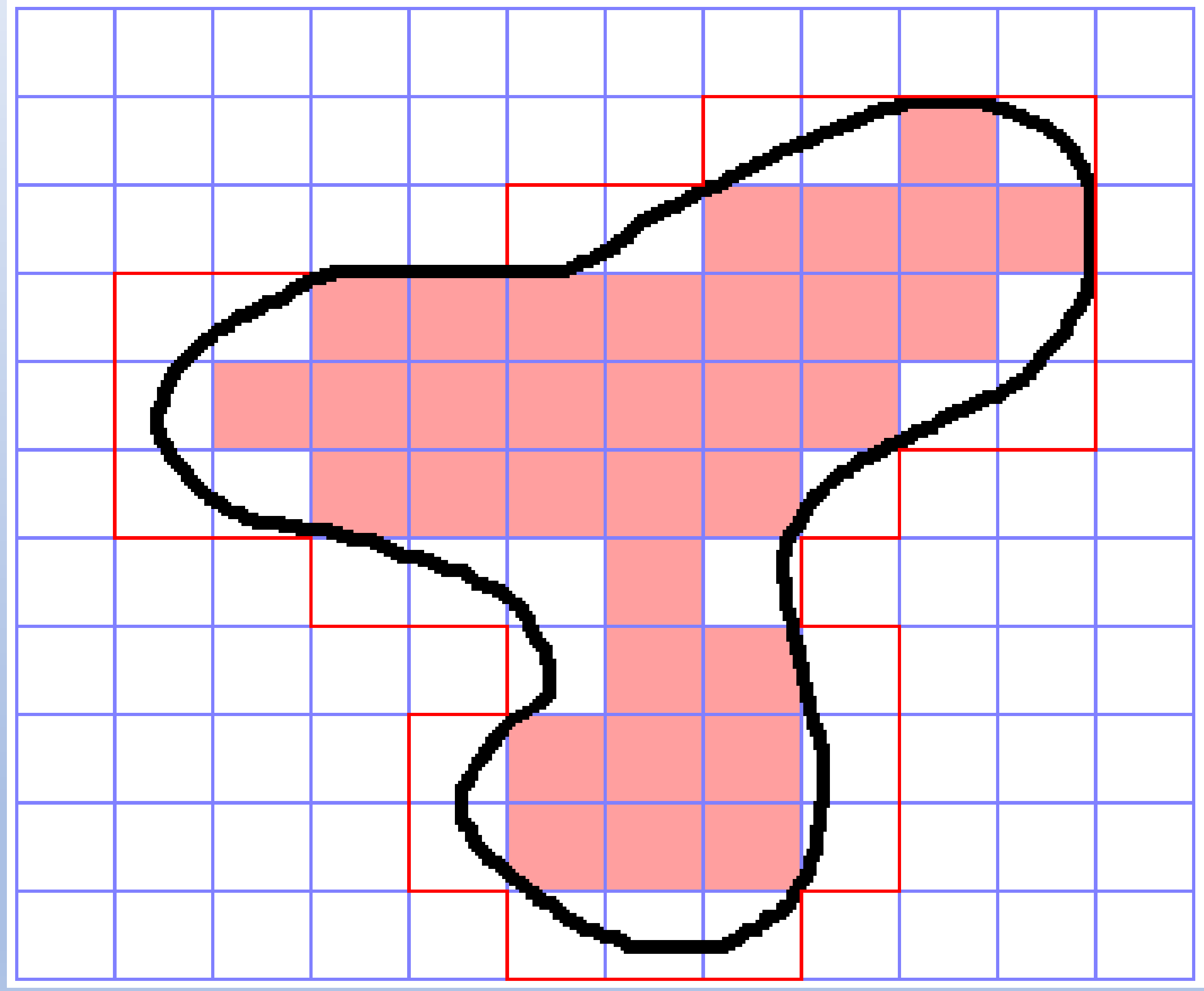
The area under a curve

Let us first consider the irregular shape shown below.



The area under a curve

We can find an approximation by placing a grid of squares over it.

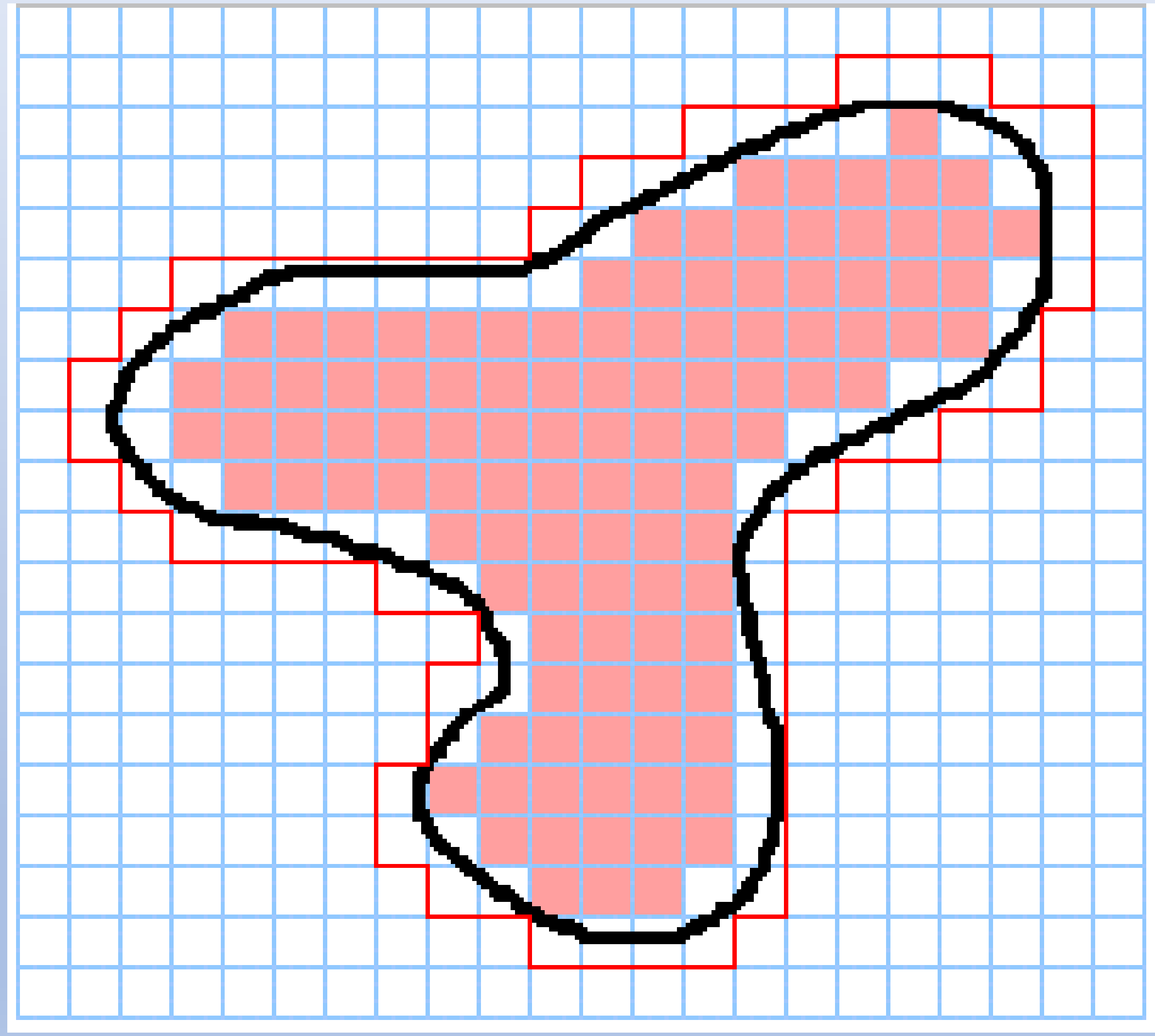


By counting squares,
$A > 33$ and $A < 60$
i.e. $33 < A < 60$

The area under a curve

By taking a finer 'mesh' of squares we could obtain a better approximation for A.

We now study another way of approximating to A, using rectangles, in which A can be determined by a limit process.

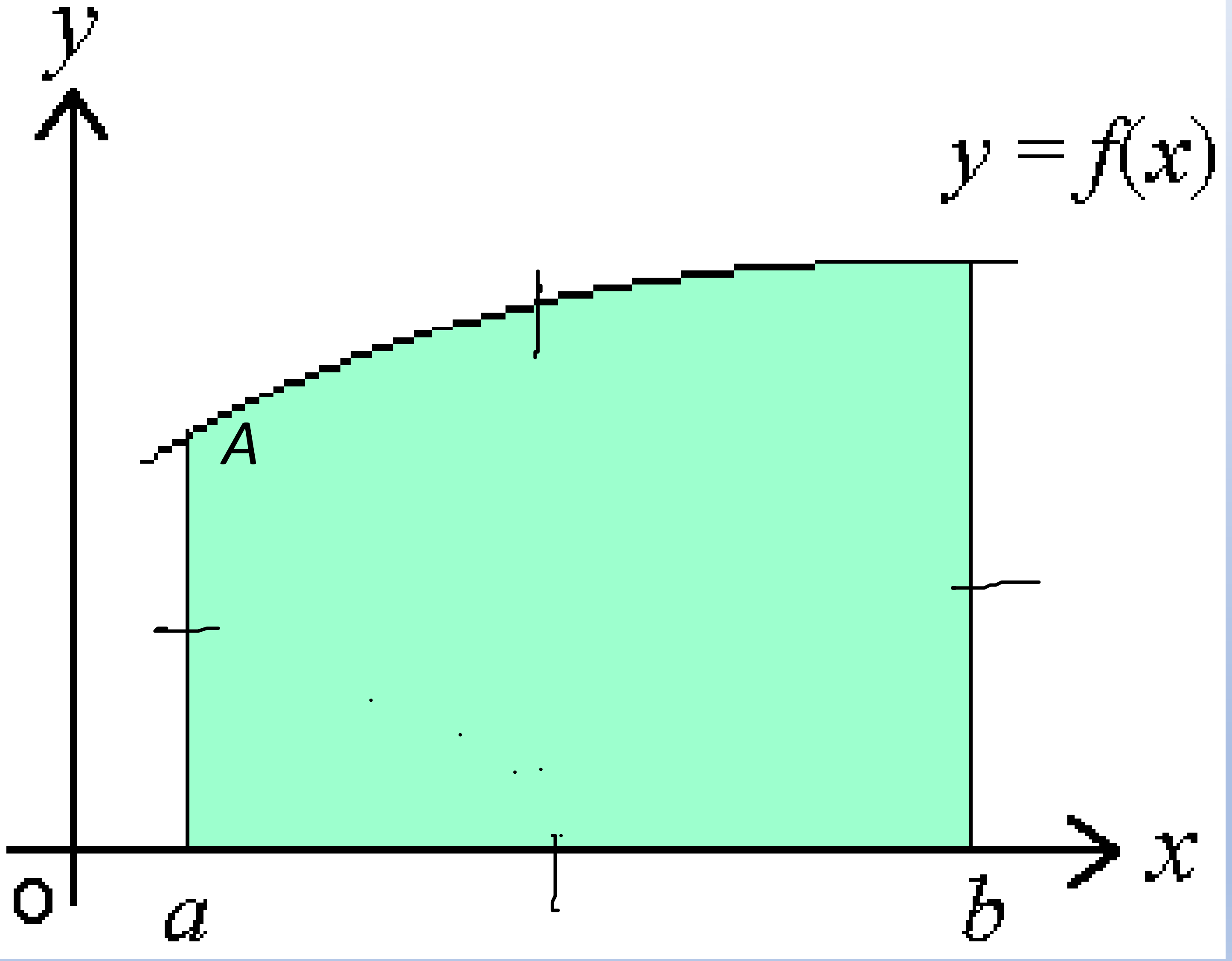


The area under a curve

$[a, b]$

The accompanying figure shows part of the curve $y = f(x)$ from $x = a$ to $x = b$.

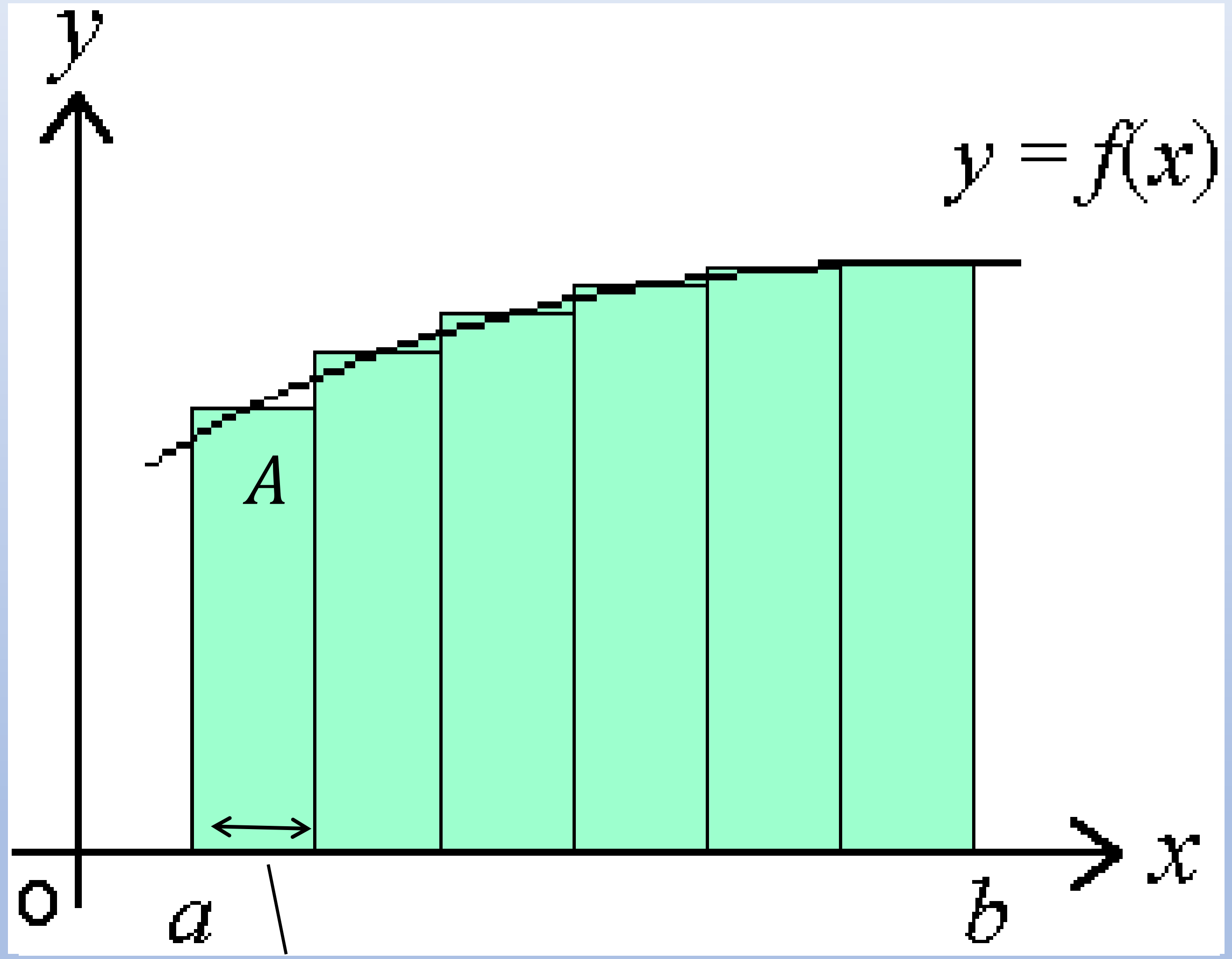
We will find an expression for the area A bounded by the curve, the x −axis, and the lines $x = a$ and $x = b$.



The area under a curve

The interval $[a, b]$ is divided into n sections of equal width, Δx.

The n rectangles are then drawn to approximate the area A under the curve.



Δx

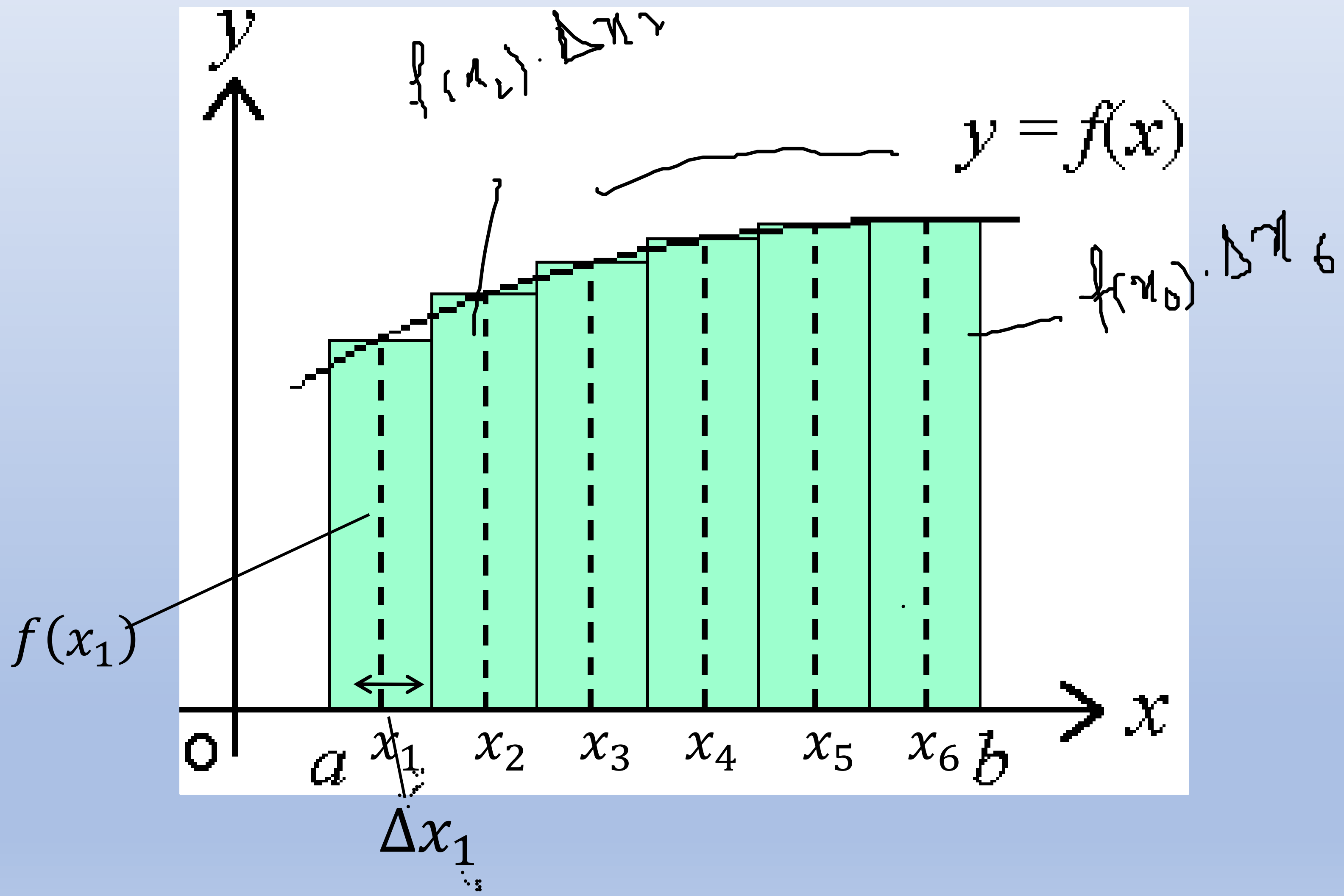
The area under a curve

Dashed lines represent the height of each rectangle.

The position of each line is given by an x −coordinate, x_n.

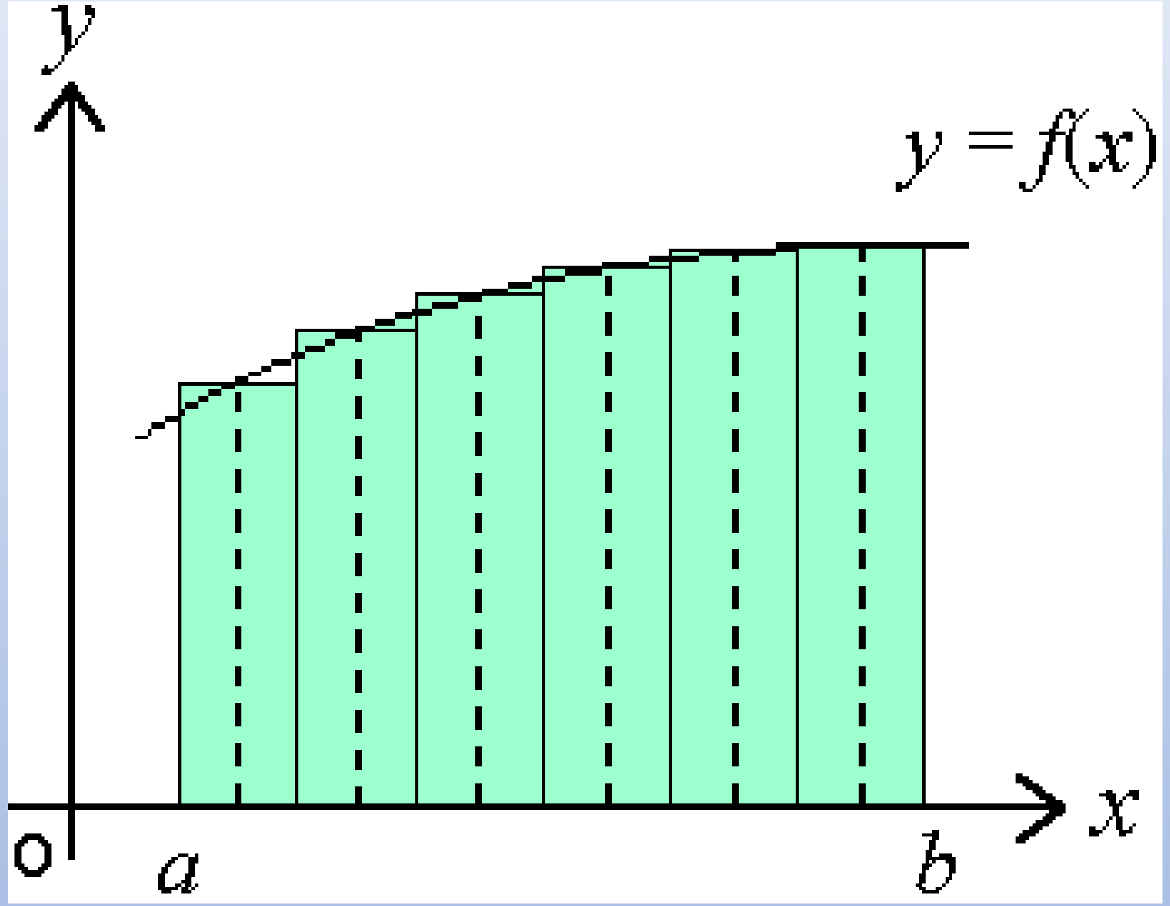
The first rectangle has height $f(x_1)$ and breadth Δx_1

Thus, the area of the first rectangle $= f(x_1).\Delta x_1$



The area under a curve

An approximation for the area under the curve, between $x = a$ to $x = b$, can be found by summing the areas of the rectangles.



$$A \approx f(x_1).\Delta x_1 + f(x_2).\Delta x_2 + f(x_3).\Delta x_3 + f(x_4).\Delta x_4 + f(x_5).\Delta x_5 + f(x_6).\Delta x_6$$

Riemann Sum

Using the Greek letter Σ (sigma) to denote 'the sum of', we have

$$A \approx \sum_{i=1}^{i=6} f(x_i).\Delta x_i.$$

$A \approx f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_6)\Delta x_6$

More generally, for n number of rectangles, we have

$$\boxed{A \approx \sum_{i=1}^{i=n} f(x_i).\Delta x_i = S.}$$

This sum is called Riemann Sum for $f(x)$ on the interval $[a,b]$.

Riemann Sum

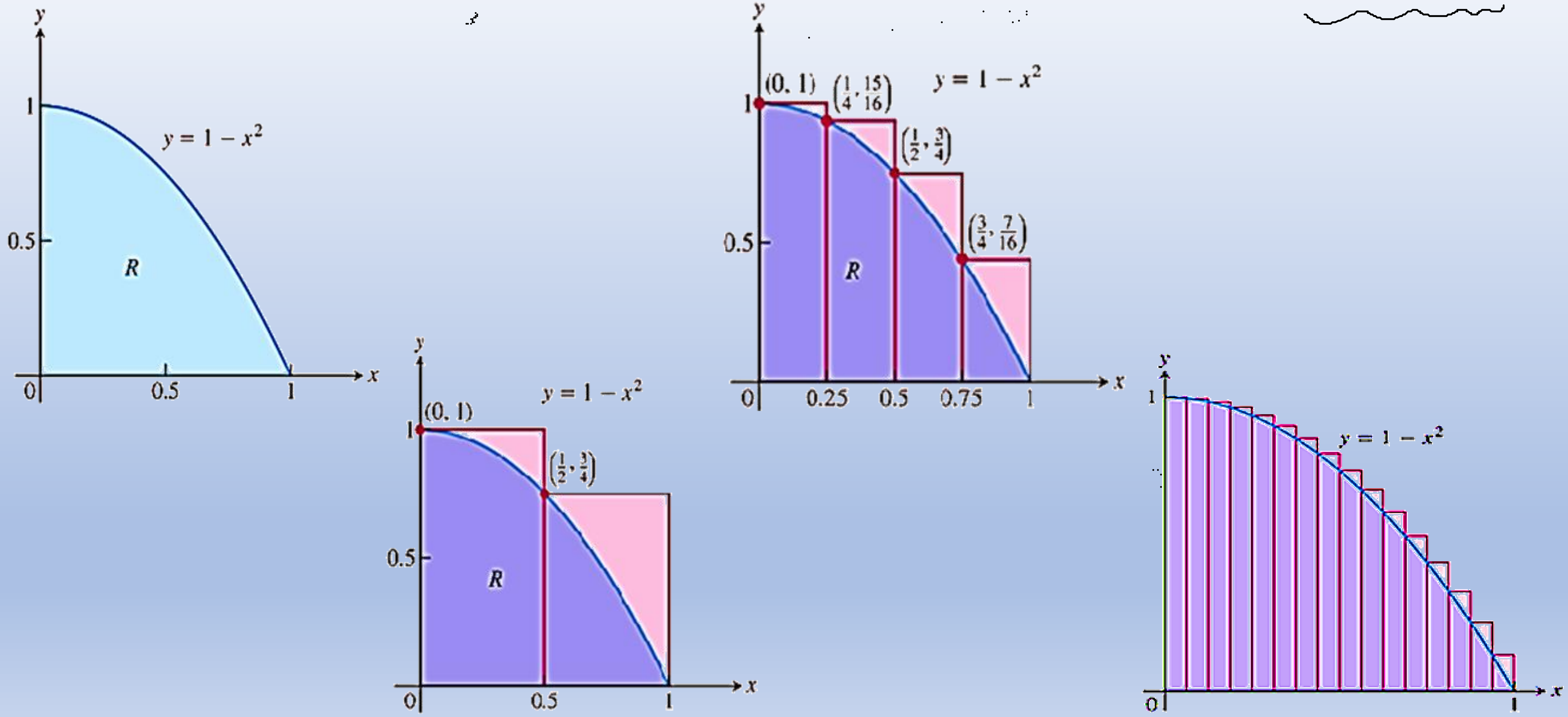
By increasing the number n rectangles, we decrease their width Δx, i.e., as $n \to \infty$, $\Delta x \to 0$.

So, we define:

$$A = \lim_{n \longrightarrow \infty} \sum_{i=1}^{i=n} f(x_i) \Delta x_i.$$

provided the limit exists.

As the number of rectangles increased, the approximation of the area under the curve approaches a value.



Definite Integral

The definite integral from a to b, represented by:

$$\int_a^b f(x)\,dx$$

is the number to which all Riemann sums tend as the number of rectangles approaches infinity ($n \to \infty$) and as the width of all rectangles tends to zero ($\Delta x \to 0$):

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i\,.$$

Note: The function $f(x)$ must be continuous on the interval $[a, b]$.

Definite Integral

upper limit of integration

Integration Symbol

$$\underbrace{\int_a^b f(x)}_{\text{integrand}}\, dx$$

lower limit of integration

variable of integration (dummy variable)

Note: that the value of a definite integral is a *real number,* not a family of antiderivatives, as was the case for indefinite integrals.

Properties of the Definite Integral

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ $a=b$

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then

$m \le f(x) \le M$

$$\underset{m}{\min f} \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$

Piecewise
fn

$[a, b]$

$a \longrightarrow b$

$\int_a^b f(x)\,dx$

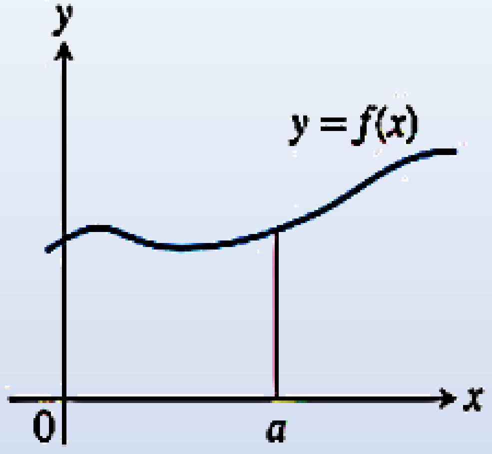
b —— a

$\int_b^a f(x)\,dx$

m — min

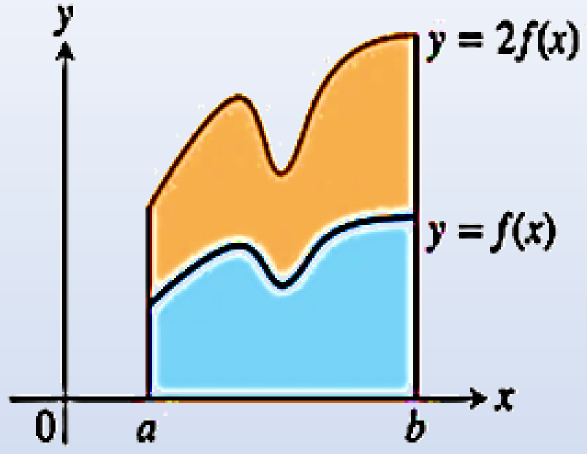
M — max

Geometric Interpretations of the Properties of the Definite Integral



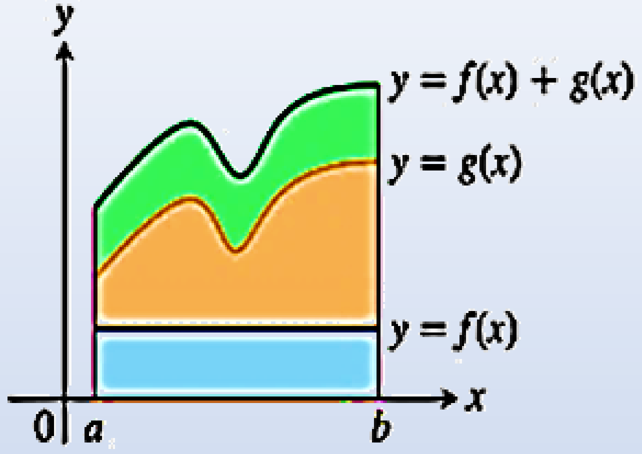
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



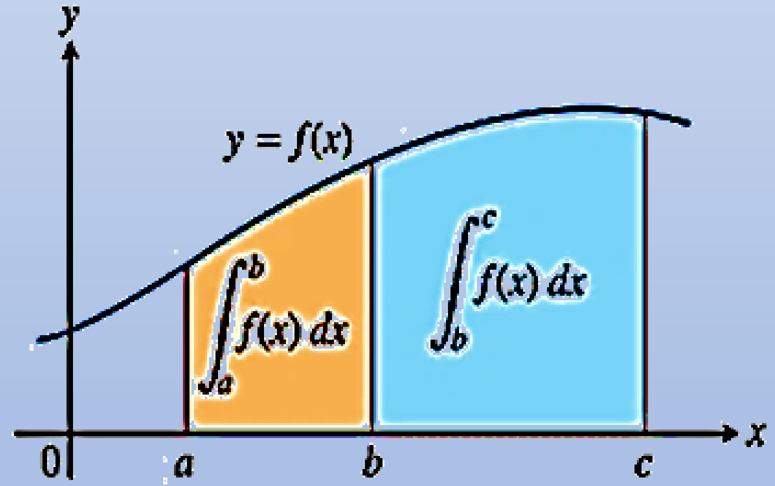
(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



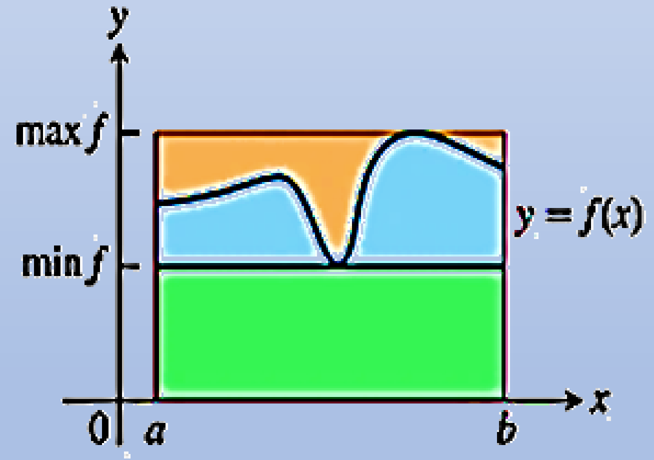
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



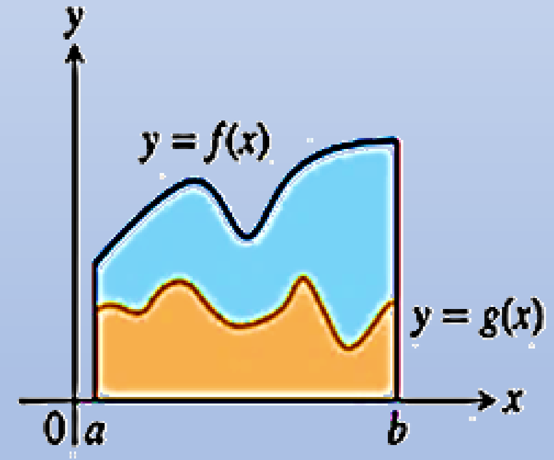
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx$$
$$\le \max f \cdot (b - a)$$



(f) *Domination:*

$$f(x) \ge g(x) \text{ on } [a, b]$$
$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

Using the Properties of the Definite Integral

Given: $\int_1^3 f(x)dx = 6$ $\quad \int_3^7 f(x)dx = 9$ $\quad \int_1^3 g(x)dx = -4$

$$\int_1^3 3f(x)dx = 3\int_1^3 f(x)dx = 3(6) = 18$$

$$\int_1^3 (2f(x) - 4g(x))dx = 2\int_1^3 f(x)dx - 4\int_1^3 g(x)dx = 2(6) - 4(-4) = 28$$

$$\int_1^7 f(x)dx = \int_1^3 f(x)dx + \int_3^7 f(x)dx = 6 + 9 = 15$$

$$\int_3^1 f(x)dx = -\int_1^3 f(x)dx = -6$$

The Fundamental Theorem of Calculus

If $f(x)$ is continuous at every point in $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Examples:

1. $$\int_1^5 5x\,dx = \frac{5x^2}{2}\Bigg|_1^5 = \frac{5(5)^2}{2} - \frac{5(1)^2}{2} = \frac{125}{2} - \frac{5}{2} = \frac{120}{2} = 60.$$

2. $$\int_{\pi/6}^{2\pi/3} \sin x\,dx = -\cos x\Big|_{\pi/6}^{2\pi/3} = -\cos\left(\frac{2\pi}{3}\right) - \left[-\cos\left(\frac{\pi}{6}\right)\right] = -\left(\frac{-1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right) = \frac{1+\sqrt{3}}{2} \approx 1.366$$