PROPERTIES OF LAPLACE TRANSFORM

Linearity

- Many parallel properties of the CTFT, but for Laplace transforms we need to determine implications for the ROC

- For example:

<u>Linearity</u>

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(s) + bX_2(s)$$

ROC at least the intersection of ROCs of $X_1(s)$ and $X_2(s)$

ROC can be bigger (due to pole-zero cancellation)

E.g. $\quad x_1(t) = x_2(t)$ and $a = -b$

Then $\quad ax_1(t) + bx_2(t) = 0 \longrightarrow X(s) = 0$

$\Rightarrow$ ROC entire s-plane

Time Shifting

- If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC: R1$$

- then

$$\boxed{x(t-t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0} X(s), \quad ROC: R1}$$

$$x(t-T) \longleftrightarrow e^{-sT} X(s), \text{ same ROC as } X(s)$$

Time Shifting

$$\frac{e^{3s}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad ?$$

Time Shifting

$$\frac{e^{-sT}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad e^{-2t}u(t)|_{t\to t-T}$$

$$\downarrow T = -3$$

$$\frac{e^{3s}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad e^{-2(t+3)}u(t+3)$$

Shift in s-domain

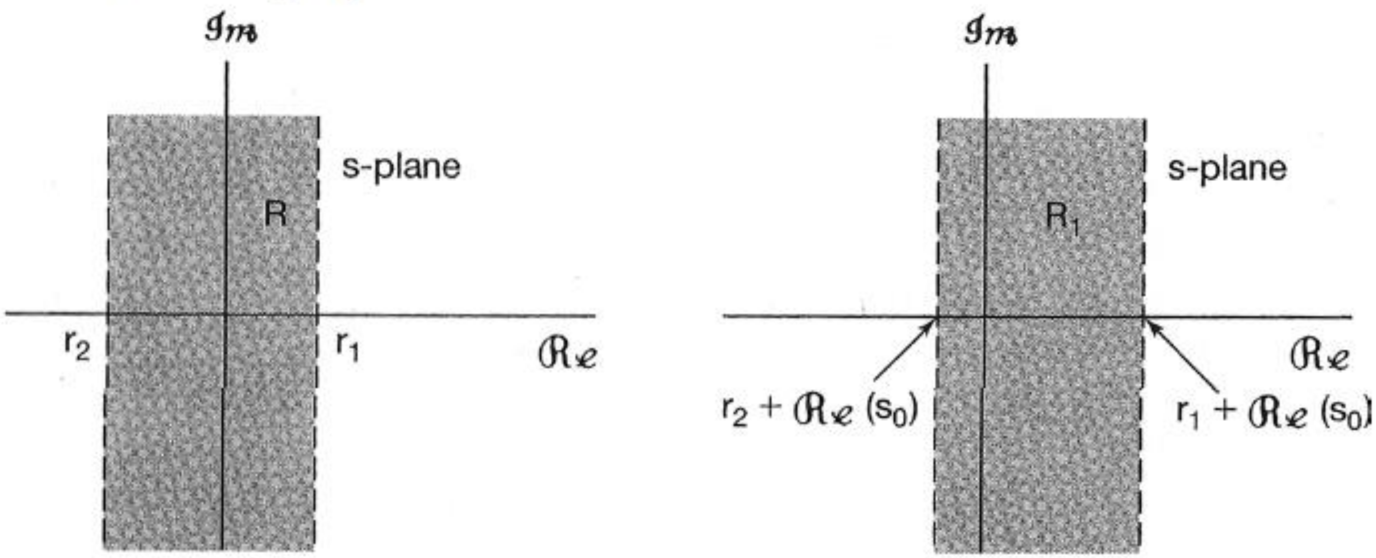
- If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad ROC: R$$

- then

$$\boxed{e^{s_0 t}x(t) \xleftrightarrow{\mathcal{L}} X(s-s_0), \quad ROC: R+\mathrm{Re}\{s_0\}}$$

- ROC associated with $X(s-s_0)$ is that of $X(s)$, shifted by $\mathrm{Re}\{s_0\}$



Shift in s-domain

$$x(t) \xleftrightarrow{\mathcal{LT}} X(s); \quad \text{with ROC=}R$$

$$e^{s_0 t} x(t) \xleftrightarrow{\mathcal{LT}} X(s - s_0); \quad \text{with ROC=}R + \mathrm{Re}\{s_0\}$$

- Special Case: $s_0 = j\omega_0$ gives:

$$e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{LT}} X(s - j\omega_0); \quad \text{with ROC=}R$$

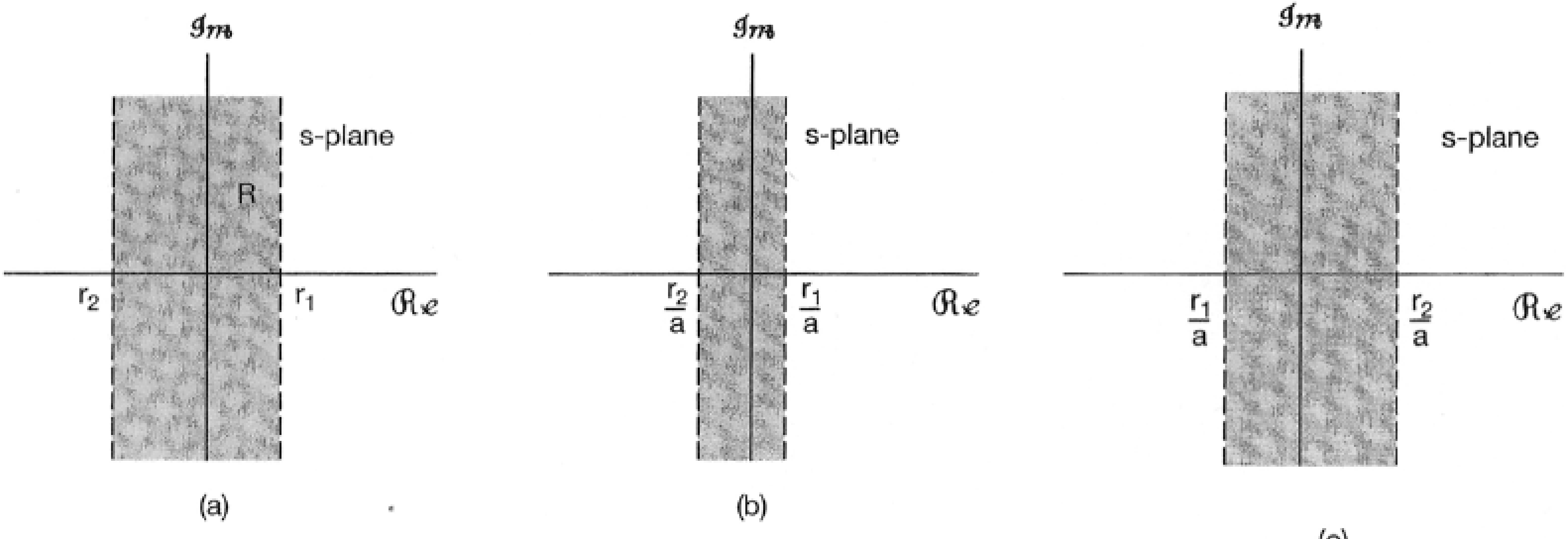
Time Scaling

$$x(t) \xleftrightarrow{\mathcal{L}} X(s); \quad \text{with ROC} = R$$

$$\boxed{x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right); \quad \text{with ROC } R_1 = \frac{R}{a}}$$

- special case: $a = -1$

$$x(-t) \xleftrightarrow{\mathcal{L}} X(-s); \quad \text{with ROC } R_1 = -R$$



ROC of $X(s)$ ROC of $(1/|a|)X(s/a)$, $a>1$ ROC of $(1/|a|)X(s/a)$, $0>a>-1$

Conjugation Property

- If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \text{ with } ROC = R$$

- then:

$$x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*), \text{ with } ROC = R$$

- and:

$$X(s) = X^*(s^*), \quad \text{when } x(t) \text{ is real}$$

- Consequently, if $x(t)$ is real and if $X(s)$ has a pole or zero at $s = s_0$, then $X(s)$ also has a pole or zero at the complex conjugate point $s = s_0^*$.

- Example: $X(s)$ has a pole at $s = 1 + 3j$; it must also have a pole at $s = 1 - 3j$

Convolution Property

- If

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s), \quad ROC = R_1$$

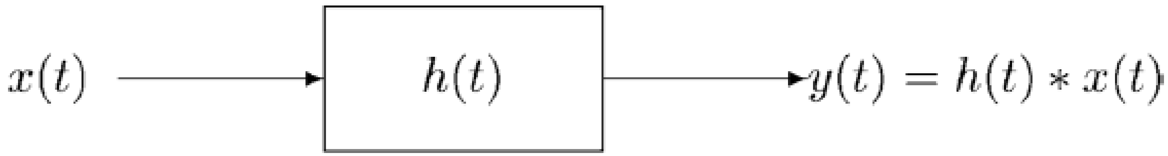
$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s), \quad ROC = R_2$$

- Then

$$\boxed{x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)X_2(s), \quad ROC = R_1 \cap R_2}$$

- The ROC of $X_1(s)X_2(s)$ includes the intersection of the $ROCs$ of $X_1(s)$ and $X_2(s)$ and can be larger if pole-zero cancellation occurs in the product.

Convolution Property



For $\quad x(t) \longleftrightarrow X(s), y(t) \longleftrightarrow Y(s), h(t) \longleftrightarrow H(s)$

Then

$$Y(s) = H(s) \cdot X(s)$$

- ROC of $Y(s) = H(s)X(s)$: at least the overlap of the ROCs of $H(s)$ & $X(s)$

Convolution Property

- ROC could be empty if there is no overlap between the two ROCs
  e.g.
  $$x(t) = e^{t}u(t), \text{ and } \quad h(t) = -e^{-t}u(-t)$$
- ROC could be larger than the overlap of the two. e.g.

Example:

$$X_1(s) = \frac{s+1}{s+2}, \quad \mathrm{Re}\{s\} > -2; \quad X_2(s) = \frac{s+2}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

$X_1(s) \cdot X_2(s) = 1$, ROC entire s-plane

Time Domain Differentiation

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}ds, \quad \frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} sX(s)e^{st}ds$$

$$\Downarrow$$

$$\boxed{\frac{dx(t)}{dt} \longleftrightarrow sX(s), \text{ with ROC containing the ROC of } X(s)}$$

ROC could be bigger than the ROC of $X(s)$, if there is pole-zero cancellation. e.g.,

$$\begin{aligned} x(t) &= u(t) &\leftrightarrow \quad & \tfrac{1}{s}, & \Re e\{s\} > 0 \\ \frac{dx(t)}{dt} &= \delta(t) &\leftrightarrow \quad & 1 = s \cdot \tfrac{1}{s} & \text{ROC} = \text{entire s-plane} \end{aligned}$$

s-Domain Differentiation

s-Domain Differentiation

$$\boxed{-tx(t) \leftrightarrow \frac{dX(s)}{ds}, \text{ with same ROC as } X(s)}$$

(Derivation is
similar to $\frac{d}{dt} \leftrightarrow s$)

E.g., $\quad te^{-at}u(t) \leftrightarrow \ ?$

s-Domain Differentiation

$$\text{E.g.,}\qquad te^{-at}u(t) \leftrightarrow -\frac{d}{ds}\left[\frac{1}{s+a}\right] = \frac{1}{(s+a)^2},$$

$$\Re e\{s\} > -a$$

Time Domain Integration

- If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad ROC = R$$

- then

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s), \quad ROC = R \cap \{\mathrm{Re}\{s\} > 0\}}$$

- Inverse of differentiation property

- Property can be derived from convolution property, i.e.,

$$\int_{-\infty}^{t} x(\tau)d\tau = u(t) * x(t) \xleftrightarrow{\mathcal{L}} U(s)X(s) = \frac{1}{s}X(s)$$

END