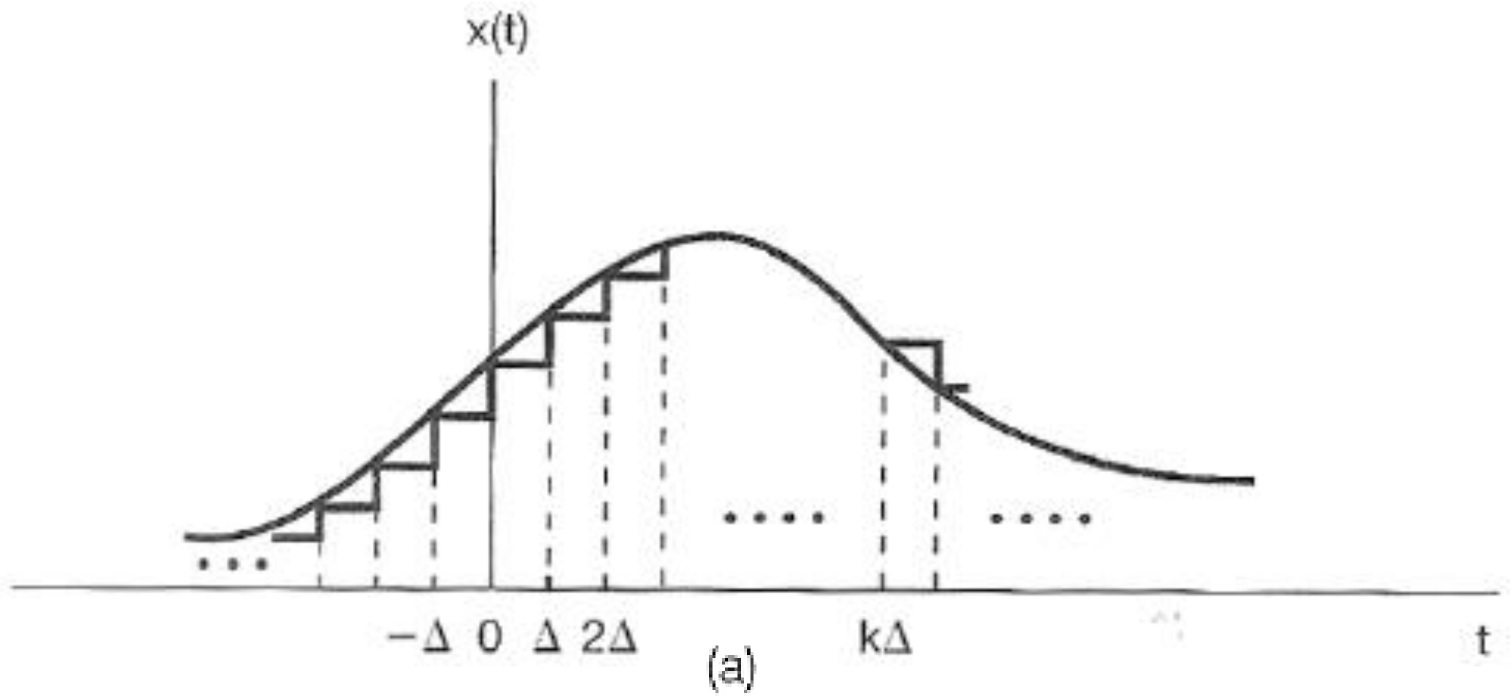
CT CONVOLUTION

CT Signals as Sum of Impulses

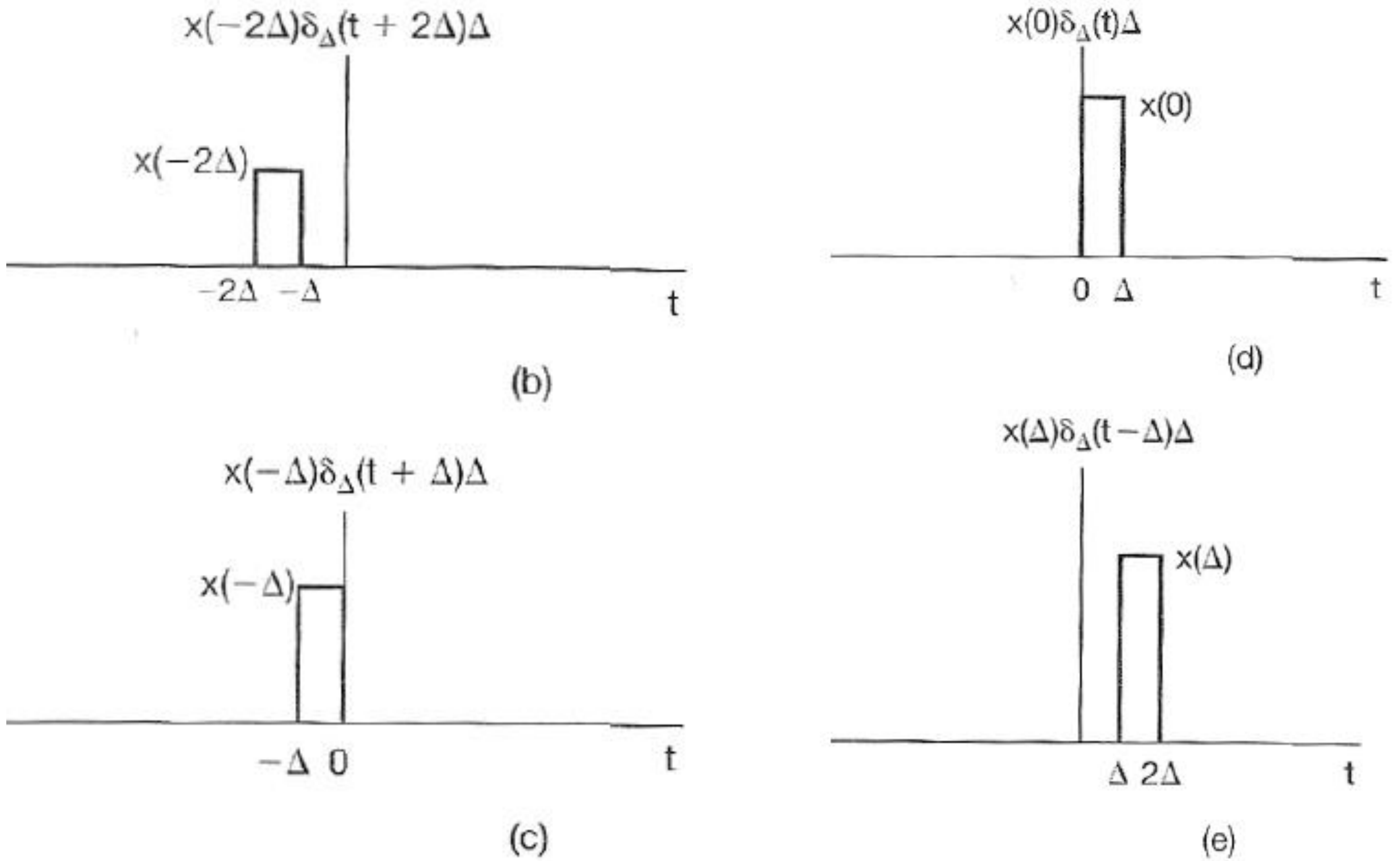
- Consider the staircase approximation, $\hat{x}(t)$, to a CT signal, $x(t)$, as shown below:



- It may be seen that the CT signal can be approximately expressed as a linear combination of delayed pulses

CT Signals as Sum of Impulses

- The approximation expressed as a linear combination of delayed pulses is shown in parts: (b)-(e)



CT Signals as Sum of Impulses

- The unit impulse can be written as:

$$\delta_\Delta(t) = \begin{cases} \dfrac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & otherwise \end{cases}$$

- Since $\Delta\delta_\Delta(t)$ has unit amplitude we get:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

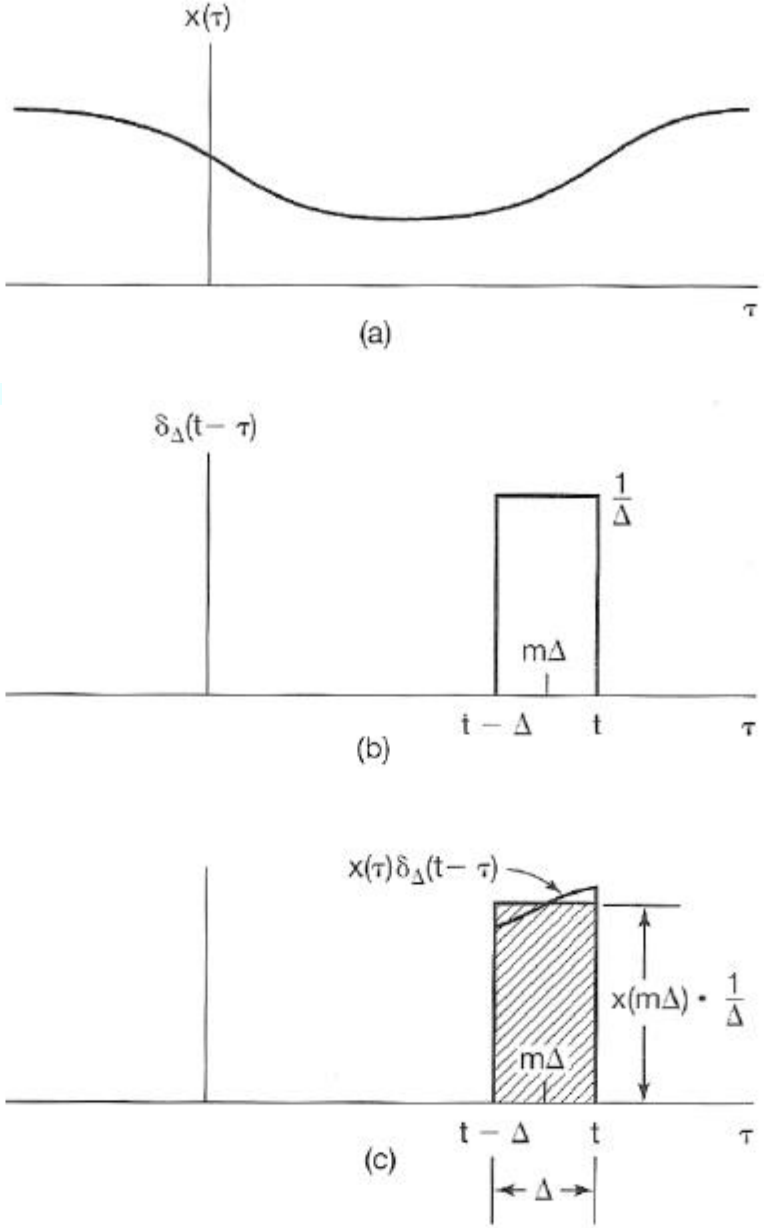
- For any value of t, only one term in sum is non-zero. As Δ approaches 0, the approximation becomes better and in the limit equals $x(t)$:

$$x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

CT Signals as Sum of Impulses

- As $\Delta \to 0$ the summation approaches an integral.
- Consider signals (shown in figure) $x(\tau)$, $\delta_\Delta(t-\tau)$ and their product. The shaded region has area that approaches the area under $x(\tau)\delta_\Delta(t-\tau)$ as $\Delta \to 0$
- Can show that $x(t)$ equals the limit as $\Delta \to 0$ of the area under $x(\tau)\delta_\Delta(t-\tau)$.
- Moreover, the limit as $\Delta \to 0$ of $\delta_\Delta(t)$ is the unit impulse function $\delta(t)$; consequently

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$ "sifting" property of CT impulse



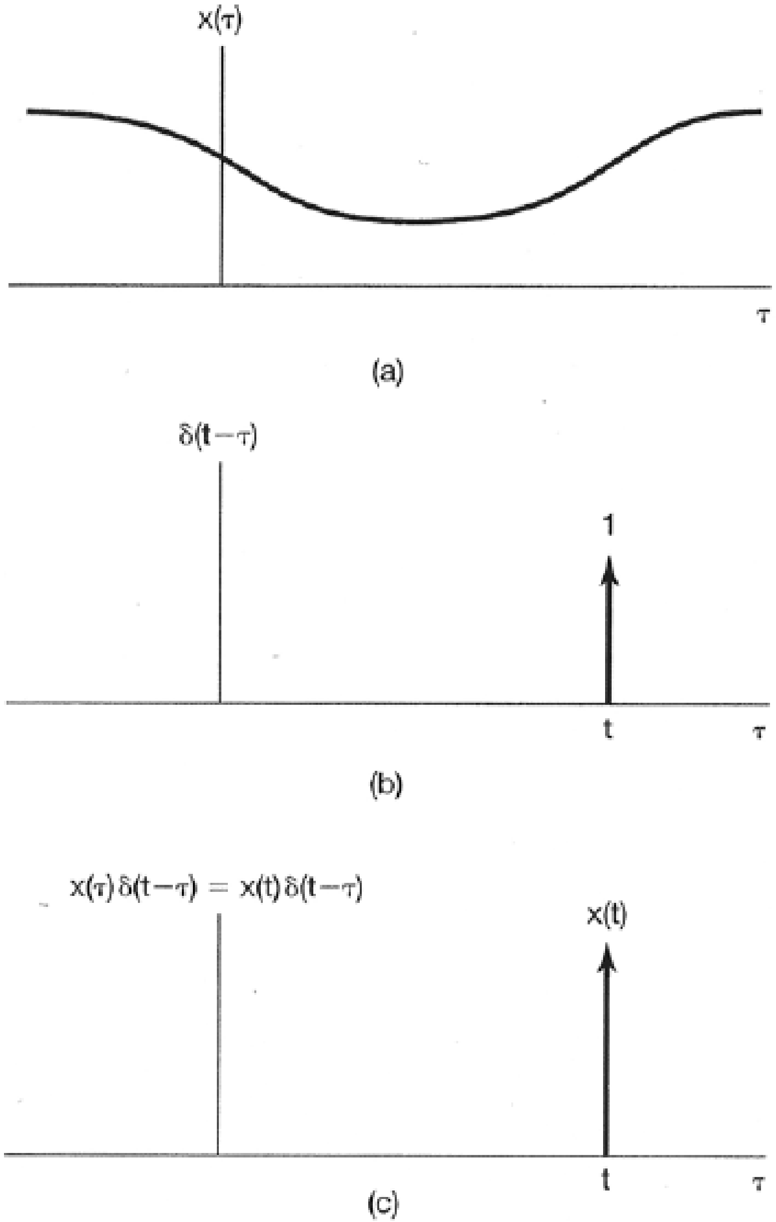
CT Signals as Sum of Impulses

- The signal $\delta(t-\tau)$ viewed as a function of τ with t fixed, is a unit impulse located at τ = t

- Thus the signal $x(\tau)\delta(t-\tau)$ is a scaled impulse at τ = t with an area equal to the value of *x(t)*

- The integral of this signal from $\tau = -\infty \; to \; \tau = \infty$ equals *x(t)* as below:

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau = x(t)$$



CT Step Signal - Example

- For the example of a CT step function, $x(t) = u(t)$ the sifting property becomes

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\,d\tau = \int_{0}^{\infty} \delta(t-\tau)\,d\tau$$

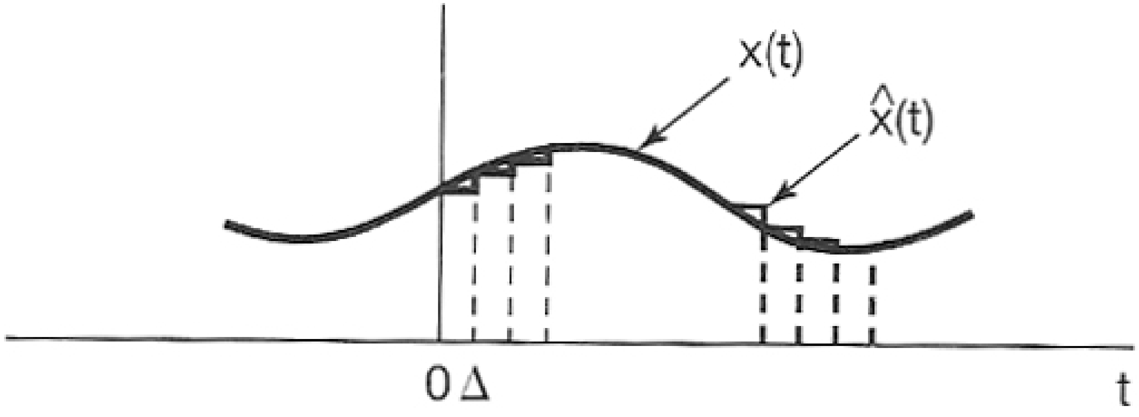
CT Convolution Integral

- The approximation represents the signal $\hat{x}(t)$ as a sum of the scaled and shifted versions of the basic pulse signal $\delta_\Delta(t)$.
- Thus the response $\hat{y}(t)$ of an LTI system to this signal will be the superposition of the responses to the scaled and shifted versions of $\delta_\Delta(t)$.
- We define $\hat{h}_{k\Delta}(t)$ as the response of a linear system to the input $\delta_\Delta(t-k\Delta)$.
- From superposition we get:

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\hat{h}_{k\Delta}(t)\Delta$$

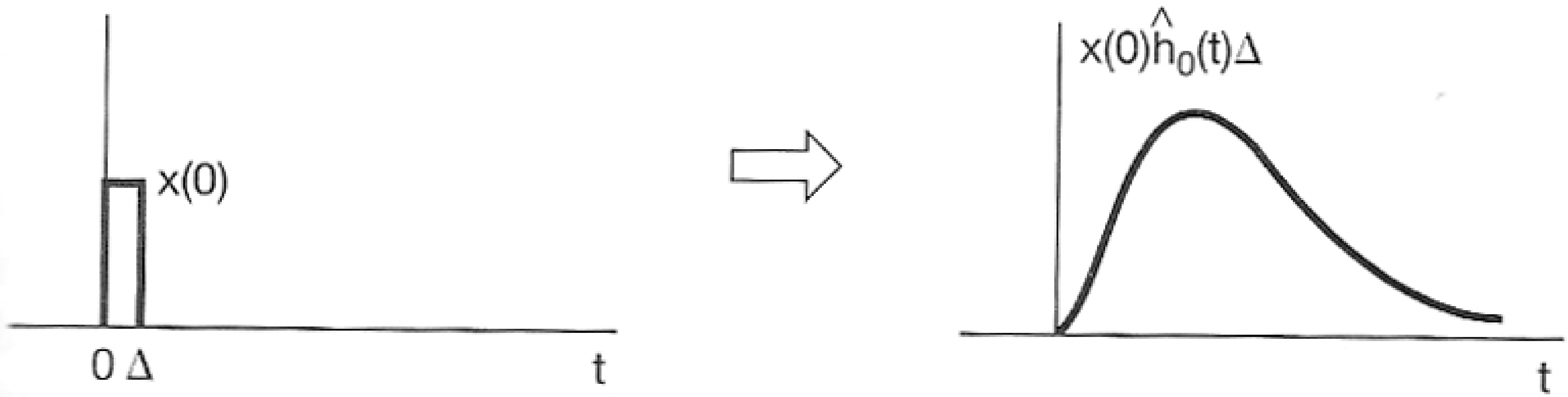
CT Convolution

- Part (a): the input $x(t)$ and its approximation $\hat{x}(t)$.



(a)

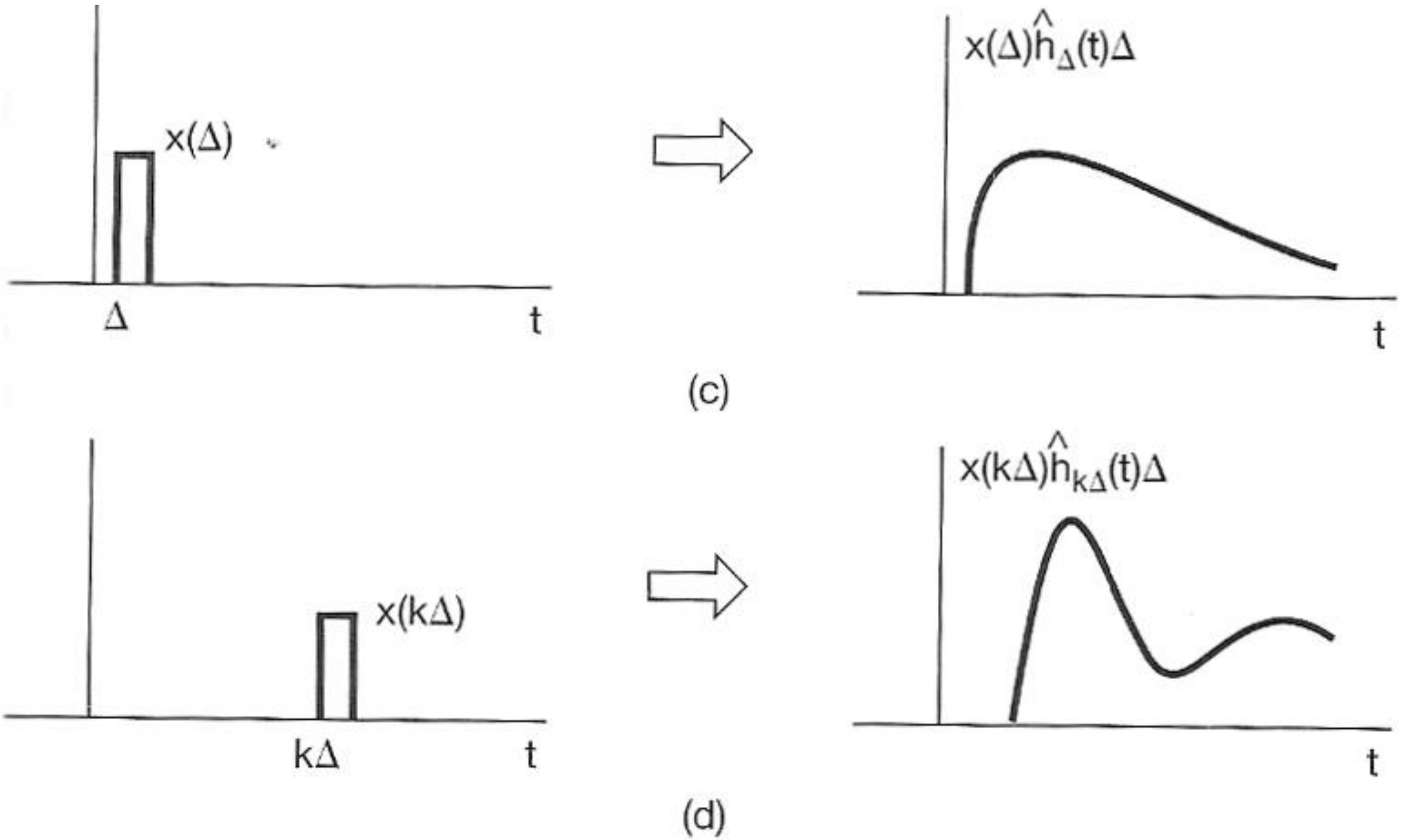
- Part (b): response of the system to the weighted pulse at *t* = *0*



(b)

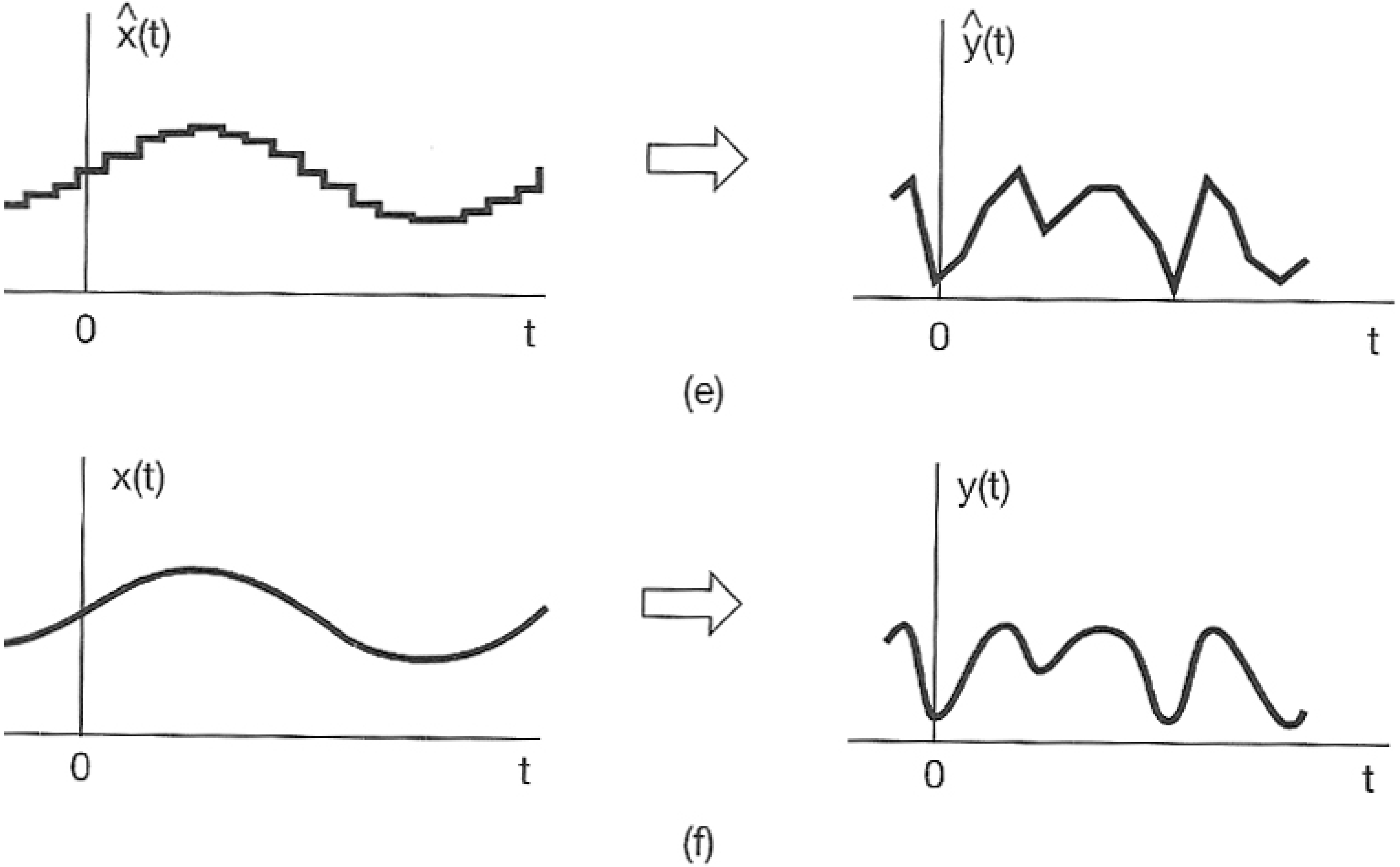
CT Convolution

- Part (c)-(d): response of the system to the weighted pulses at $t = k\Delta$



CT Convolution

- Part (e): The input approximation, $\hat{x}(t)$, and the output approximation, $\hat{y}(t)$.
- Part (f): The output $y(t)$ corresponding to the input $x(t)$.



CT Convolution

What happens when Δ becomes vanishingly small?

$$\hat{x}(t) \xrightarrow{\Delta \to 0} x(t)$$

$$\hat{y}(t) \xrightarrow{\Delta \to 0} y(t)$$

- and the output can be expressed as

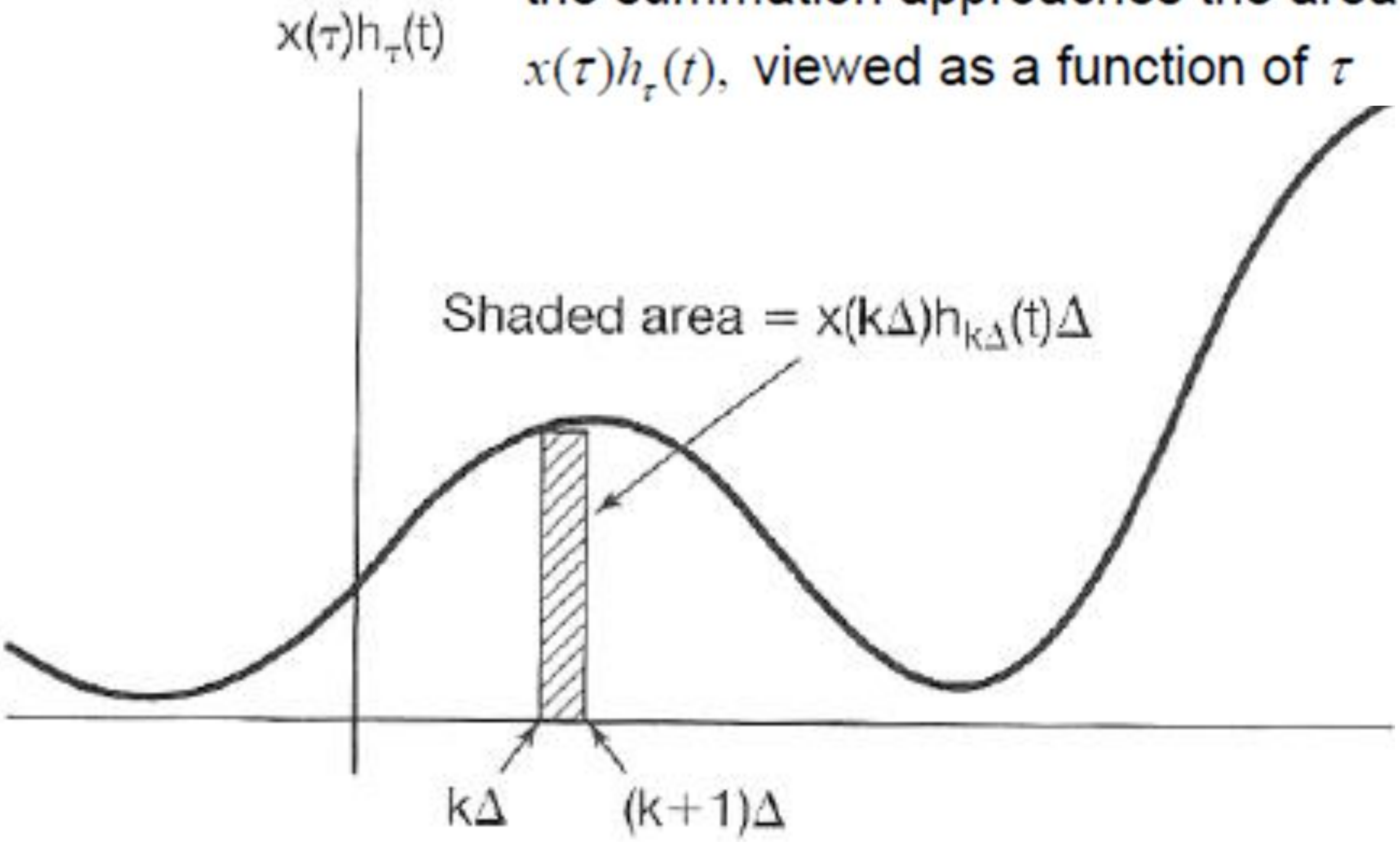
$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta) h_{k\Delta}(t) \Delta$$

- which becomes an integral as $\Delta \to 0$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h_{\tau}(t) d\tau$$

CT Convolution

- The shaded rectangle in the figure represents one term in the summation, and as $\Delta \to 0$ the summation approaches the area under $x(\tau)h_\tau(t)$, viewed as a function of τ



CT Convolution

- When the system is time-invariant, then $h_\tau(t) = h(t-\tau)$, the response of the LTI system to the unit impulse $\delta(t-\tau)$, and the integral becomes:

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau;} \qquad \boxed{y(t) = x(t) * h(t)}$$

- which is called the *convolution* or *superposition integral*

As in discrete time systems, a continuous-time LTI system is completely characterized by its impulse response – i.e., by its response to a single elementary signal, the unit impulse δ(t).

CT Convolution - Procedure

- The procedure for evaluating the convolution integral is quite similar to that for its discrete-time counterpart, the convolution sum.
- For any value of t, the output $y(t)$ is a weighted integral of the input
- To evaluate this integral for a specific value of t, we first obtain the signal $h(t-\tau)$ (regarded as a function of τ with t fixed) from $h(\tau)$ by a reflection about the origin and a shift to the right by t if $t>0$ or a shift to the left by $|\tau|$ for $t<0$.
- We next multiply together the signals $x(\tau)$ and $h(t-\tau)$, and $y(t)$ is obtained by integrating the resulting product from $\tau=-\infty$ to $\tau=\infty$.

CT Convolution - Problem

- Let $x(t)$ be the input to an LTI system with unit impulse response, $h(t)$, where:

$$x(t) = e^{-at}\,u(t), \quad a > 0$$

$$h(t) = u(t)$$

Find $\quad y(t) = x(t) * h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

END