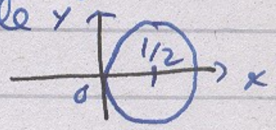
Ex: Find the curve or region in the complex plane represented by each of the following equations or inequalities:

(a) $Re\left(\frac{1}{z}\right) = 2$. writing $z = x+iy$, $\frac{1}{z} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} + i\,\frac{-y}{x^2+y^2}$

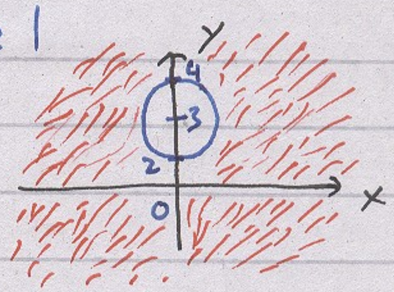
giving $Re\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2} = 2 \quad \Rightarrow \quad x^2+y^2 = 2x$, completing square we have $\left(x-\frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$. The locus represents the circle centered at $\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$.



(b) $|z-3i| \geq 1$. writing $z = x+iy$, we have

$$|x+iy-3i| \geq 1 \quad \Rightarrow \quad |x+(y-3)i| \geq 1$$

$$x^2 + (y-3)^2 \geq 1$$



(c) $$\left[|z+1|\right] * \left[|z-1|\right] = 1 \qquad \text{—— (i)}$$

The curve is symmetric with respect to both the x- and y- axes since the equation is invariant if we change z to $\bar{z}$ and $-\bar{z}$, respectively.

(i) $\Rightarrow \quad |z^2-1| = 1 \quad \Rightarrow \quad |r^2(\cos 2\theta + i \sin 2\theta) - 1| = 1$

$$(r^2 \cos 2\theta - 1)^2 + (r^2 \sin 2\theta)^2 = 1$$

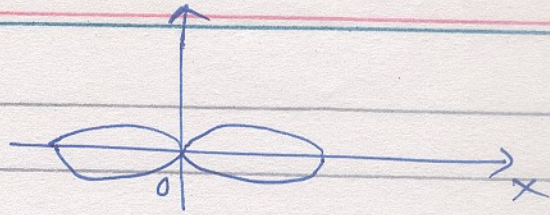
$$r^4 \cos^2 2\theta - 2r^2 \cos 2\theta + 1 + r^4 \sin^2 2\theta = 1$$

$$r^4 - 2r^2 \cos 2\theta = 0 \quad \Rightarrow \quad r^2 = 2\cos 2\theta$$

This is a standard form of a lemniscate.

[Complex Variables 01]

$$|z| + \mathrm{Re}(z) \le 1.$$



Graph of part

$$\sqrt{x^2+y^2} + x \le 1$$

$$\sqrt{x^2+y^2} \le 1 - x \Rightarrow x^2+y^2 \le 1 - 2x + x^2 \Rightarrow y^2 \le 1 - 2x.$$

The inequality represents the region on and inside the parabola $y^2 = 1 - 2x$.

[Complex Variables 02]

(d) $$0 < \mathrm{Arg}\,\frac{z-j}{z+j} < \frac{\pi}{2}.$$

$$\frac{z-j}{z+j} = \frac{(x^2+y^2-1) + j(-2x)}{x^2+(y+1)^2}.$$

One observes that $\mathrm{Arg}\,\frac{z-j}{z+j} \in \left(0, \frac{\pi}{2}\right)$ if and only if $\frac{z-j}{z+j}$ has both positive real and imaginary parts, i.e., $x < 0$ and $x^2+y^2-1 > 0 \Rightarrow x^2+y^2 > 1$.

Therefore, $0 < \mathrm{Arg}\,\frac{z-j}{z+j} < \frac{\pi}{2}$ represents the region exterior to the unit circle $|z| = 1$ and in the left half-plane.

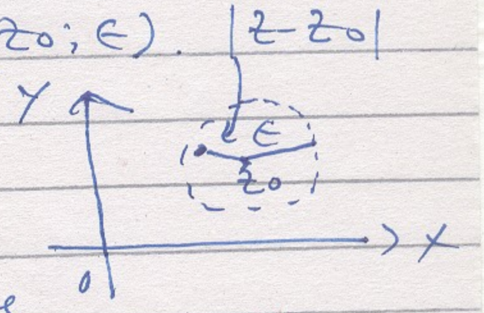
*1 The distance between the two points representing z_1 & z_2 in the Complex plane is

$$|z_1 - z_2| = |(x_1-x_2) + j(y_1-y_2)| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.$$

For example, the locus of points z in the Complex plane defined by the relation $|z - z_0| = r$, z_0 is Complex and r is real, represents a circle centered at z_0 and with radius r.

Some definitions:-

The set of points z such that $|z - z_0| < \epsilon$, where $z_0 \in \mathbb{C}$, $\epsilon \in \mathbb{R}$, contains points that are inside the circle centered at z_0 and with radius ϵ. We call it a neighborhood of z_0 and denote it by $N(z_0; \epsilon)$.



A point z_0 is said to be an interior point of a set S whenever there is some neighborhood of z_0 that contains only points of S; it is called an exterior point of S when there exists a neighborhood of it containing no points of S. If z_0 is neither of these, it is a boundary point of S. A boundary point is, therefore, a point all of whose neighborhoods contain atleast one point in S and atleast one point not in S. The totality of all boundary points is called boundary of S. The circle $|z| = 1$, for instance, is the boundary of each of the sets

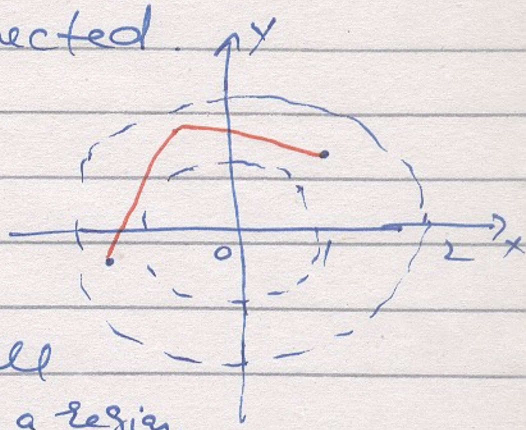
$$|z| < 1 \quad \text{and} \quad |z| \leq 1. \qquad \text{(i)}$$

A set is open if it contains none of its boundary points. A set is closed if it contains all of its boundary points, and the closure of a set S is the closed set consisting of all points in S together with boundary of S. Note that first set in (i) is open and second is its closure. Some sets are, of course, neither open nor closed. For example, the punctured disk $0 < |z| \leq 1$ is neither open nor closed.

An open set S is connected if each pair of points z_1 and z_2 in it can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in S.

[Complex variables 03]

The open set $|z|<1$ is connected. The annulus $1<|z|<2$ is, of course, open and it is also connected.



A nonempty set (open) that is connected is called a domain. Note that any neighborhood is a domain. A domain together with some, none, or all of its boundary points is referred to as a region.

Problems:-

1. Which of the following sets are connected set? Which of the following are domain?

(a). The set $A \cup B$ and set $A \cap B$, where A consists of the points given by $|z-j|<1$ while B is the set of points $|z-1|<1$.

(b) The set $C \cup D$, where C consists of the points for which $|z| \leq 1$ while B is given by $Re\, z \geq 1$.

2. Represent graphically the set of values z for which

$$\left|\frac{z-3}{z+3}\right| < 2.$$

3. If A, B and C are the point sets defined by $|z+j|<3$, $|z|<5$, $|z+1|<4$, represent graphically each of the following:

$A \cap B \cap C$, $A \cup B \cup C$, $A \cap B \cup C$, $(A \cup B) \cap (B \cup C)$.

4. Find the region in the complex plane that is represented by

$$0 < Arg\, \frac{z-1}{z+1} < \frac{\pi}{4}.$$

5. Let S be the open set consisting of all points z such that $|z|<1$ or $|z-2|<1$. Check if S is connected?

[Complex variables etc]

EX:- Let $w = e^{i\frac{2\pi}{3}}$ and define $f(z) = wz$. what type of geometric transformation is f (find $|f(z)|$ and $\arg(f(z))$ interms of $|z|$ and $\arg(z)$)?

$$|f(z)| = |wz| = |z\, e^{i\frac{2\pi}{3}}| = |z|\,|e^{i\frac{2\pi}{3}}| = |z|$$

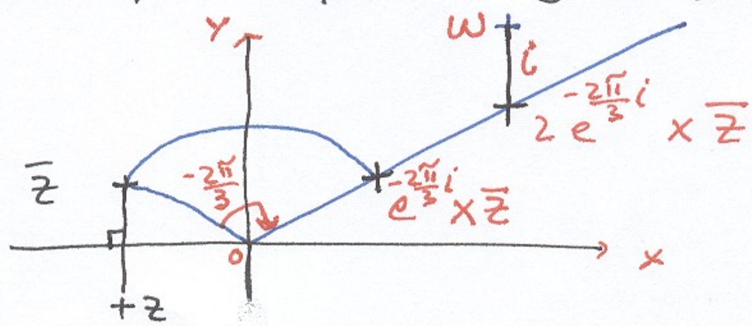
$$z = re^{i\theta}, \qquad \theta = Arg(z) + 2k\pi, \quad k = 0, \pm1, \pm2, \ldots$$

$$f(z) = wz = \left(e^{i\frac{2\pi}{3}}\right)\left(re^{i\theta}\right) = r\, e^{i\left(\arg z + \frac{2\pi}{3}\right)}$$

hence, $\arg(f(z)) = \arg(z) + \frac{2\pi}{3}$.

Geometrically it gives rotation in the counterclockwise direction by $\frac{2\pi}{3}$ rad.

The number $z \in \mathbb{C}$ is represented in the following diagram. Construct the point representing $w = 2e^{-\frac{2\pi}{3}i} \times \bar{z} + i$.



EX:- Solve: $z^2 - 2z + (1+2i) = 0$. ——— (1)

$(z-1)^2 + 2i = 0 \Rightarrow (z-1)^2 = -2i$, Now we calculate square root of $-2i$.

Let $w = -2i$, $|w| = 2$, $Arg\, w = -\frac{\pi}{2}$, $w = (2)^{1/2}\left[\cos\frac{-\frac{\pi}{2}+2k\pi}{2} + i\sin\frac{2k\pi - \pi/2}{2}\right]$

$k = 0, 1.$

$k=0$, $w_0 = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 1 - i$

$k=1$, $w_1 = \sqrt{2}\left[\cos\frac{2\pi - \pi/2}{2} + i\sin\frac{2\pi - \pi/2}{2}\right] = \sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\left(\frac{3\pi}{4}\right)\right] = -1 + i$.

Hence, $z_0 = 1 + w_0 = 1 + 1 - i = 2 - i$ & $z_1 = 1 + w_1 = 1 + (-1+i) = i$ are solutions of (1).

or using quadratic formula, $z = \frac{-(-2) \pm \sqrt{4 - 4(1)(1+2i)}}{2(1)} = 1 \pm \sqrt{-2i}$

rest procedure is same as above.

Check: $z = i$, $(i)^2 - 2(i) + (1+2i) = -1 - 2i + 1 + 2i = 0$

$z = 2-i$, $(2-i)^2 - 2(2-i) + (1+2i) = 4 - 1 - 4i - 4 + 2i + 1 + 2i = 0$.

[Complex Variables 05]