Communication Systems
EE-351

Lectures 17 and 18

Frequency Modulation:

- FM wave is a nonlinear function of a modulating wave.
- Problem: This property makes **the spectral analysis of FM more complex** than AM wave.
- How then can we tackle the spectral analysis of FM wave:
- Two ways (two-stage spectral analysis) to answer this question:
  - First, simple case of single-tone modulation as we discussed in last lecture, the **narrow-band FM wave**.
  - Second is the more general case, also for single-tone modulation, but with **wide-band wave**.

Objective of this analysis:

- The objective of doing so is to establish a relationship between the transmission bandwidth of an FM wave and the message bandwidth.

- We will subsequently see, the **two-stage spectral analysis** provides us with enough insight to propose a useful solution to the problem.

First Stage: Generating Narrowband FM:

This approximation ($s(t) = A_c \cos(2\pi F_c t) - A_c \sin(2\pi F_c t)\ \beta \sin(2\pi F_m t)$) can be employed for narrowband FM generation.

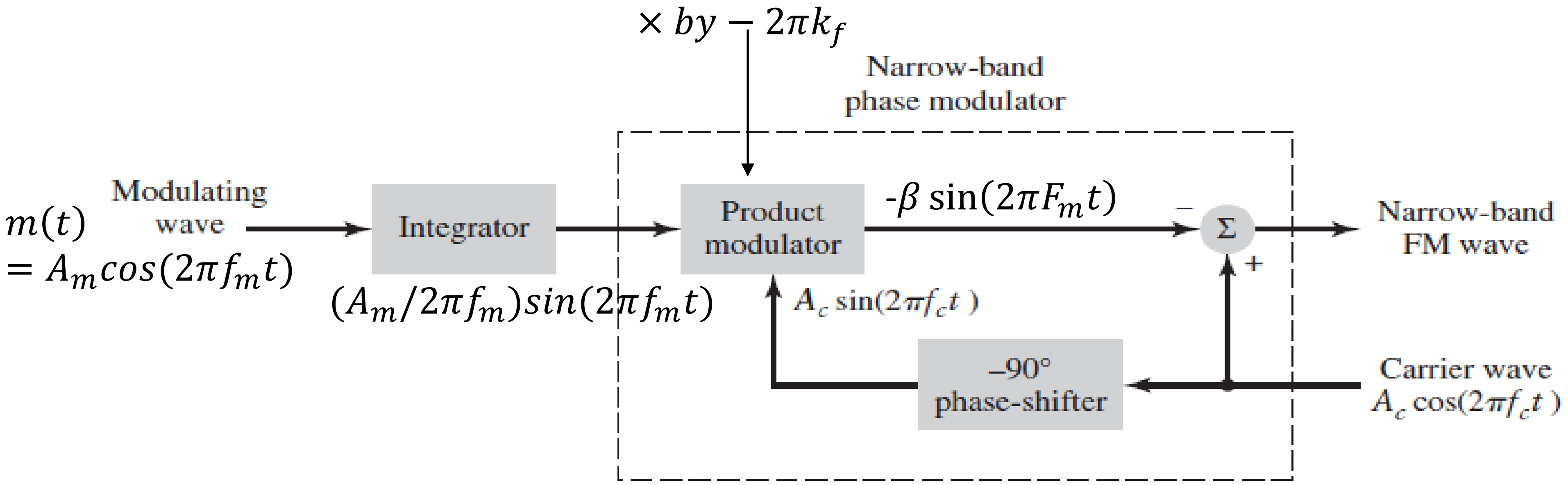

FIGURE 4.4 Block diagram of an indirect method for generating a narrow-band FM wave.

First Stage: Conclusion:

- Ideally, an **FM wave has a constant envelope** and, for the case of a sinusoidal modulating signal of frequency f_m, the angle $\theta_i(t)$ is also sinusoidal with the same frequency.

First Stage: Conclusion:

- But the modulated wave produced by the narrow-band modulator of Fig. 4.4 differs from this ideal condition in two fundamental respects:
  - The envelope contains a ***residual* amplitude modulation that varies with time**.
  - The angle $\theta_i(t)$ contains ***harmonic distortion*** in the form of third- and higher order harmonics of the modulation frequency, f_m.

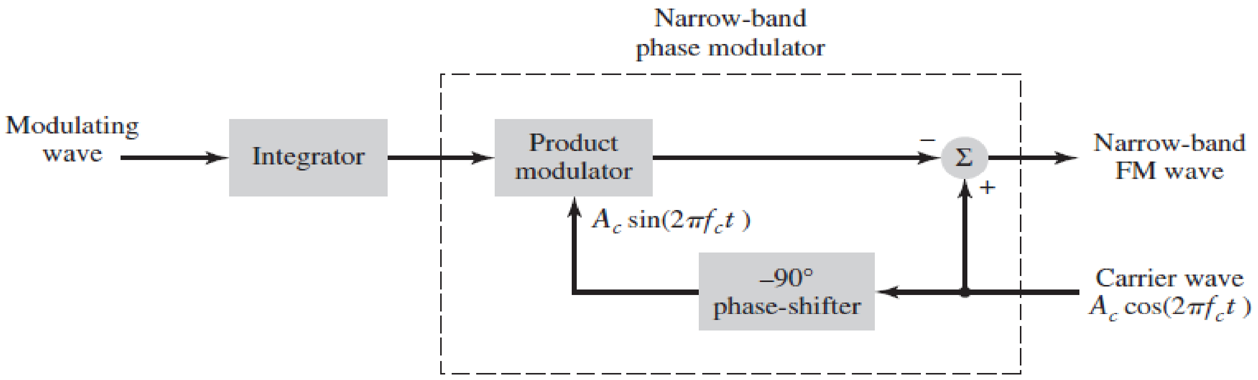

FIGURE 4.4 Block diagram of an indirect method for generating a narrow-band FM wave.

First Stage: Conclusion:

$$s(t) = A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t)$$

- We may expand the modulated wave into <u>three frequency components</u>:

$$s(t) = A_c \cos(2\pi f_c t) + \frac{1}{2}\beta A_c \{\cos[2\pi(f_c + f_m)t] - \cos[2\pi(f_c - f_m)t]\}$$

This expression is somewhat similar to the corresponding one defining an AM wave, which is reproduced from Example 3.1 of Chapter 3 as follows:

$$s(t) = A_c \cos(2\pi f_c t) + \frac{1}{2}\mu A_c \{\cos[2\pi(f_c + f_m)t] + \cos[2\pi(f_c - f_m)t]\}$$

The basic difference between an AM wave and a narrow-band FM wave is that the <u>algebraic sign of the lower side-frequency in the narrow-band FM is reversed.</u>

Nevertheless, a narrow-band FM wave requires <u>essentially the same transmission bandwidth</u> (i.e., $2f_m$ for sinusoidal modulation) as the AM wave.

Generation of Frequency Modulated Signal:

- Cartesian representation of band-pass signals (baseband representation/complex envelope) is well-suited for linear modulation.

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$

- However, for nonlinear modulation, the polar representation is well-suited.

$$s(t) = a(t)\cos[2\pi f_c t + \phi(t)]$$

Drill Problem 4.3:

- Show that the polar representation of $s(t)$ is exactly equivalent to its cartesian representation:

$$a(t) = [s_I^2(t) + s_Q^2(t)]^{\frac{1}{2}}$$

$$\phi(t) = \tan^{-1}\left[\frac{s_Q(t)}{s_I(t)}\right]$$

Drill Problem 4.4:

- Consider the narrow-band FM wave approximately defined by Eq.

$$s(t) = A_c \cos(2\pi F_c t) - \beta A_c \sin(2\pi F_c t) \sin(2\pi F_m t)$$

- Building on Problem 4.3, do the following:
- Determine the envelope of this modulated wave. What is the ratio of the maximum to the minimum value of this envelope?

$$\frac{A_{\max}}{A_{\min}} \approx \left(1 + \frac{1}{2}\beta^2\right)$$

Drill Problem 4.4:

- Determine the average power of the narrow-band FM wave, expressed as a percentage of the average power of the unmodulated carrier wave.

$$s(t) = A_c \cos(2\pi f_c t) + \frac{1}{2}\beta A_c\{\cos[2\pi(f_c + f_m)t] - \cos[2\pi(f_c - f_m)t]\}$$

$$\frac{P_{avg}}{P_c} = 1 + \frac{\beta^2}{2}$$

- Part (c) : Try yourself.

Stage 2: Generating Wideband FM:

- We now determine the spectrum of the single-tone FM wave defined by the exact formula in Eq:

$$s(t)_{FM} = A_c \cos(2\pi f_c \mathrm{t} + \beta \sin(2\pi f_m t))$$

(non periodic function of time)

For $\beta > 1$

- In general, such an FM wave produced by a sinusoidal modulating wave is a **periodic function of time t** only when the carrier frequency, f_c is an integral multiple of the modulation frequency, f_m.
- How can we simplify the **spectral analysis of the wide-band** FM wave defined in the above Eq.
- The answer lies in using the **complex baseband representation** of a modulated (i.e., bandpass) signal.

Generating Wideband FM:

- Assume that the carrier frequency, f_c is large enough (compared to the bandwidth of the FM wave).

$$s(t) = A_c \cos\theta$$
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$\cos\theta$ = real part of $e^{j\theta}$

$$s(t) = A_c Re\left|e^{j(2\pi f_c t + \beta \sin(2\pi f_m t))}\right|$$
$$= A_c Re\left|e^{j2\pi f_c t} e^{j\beta sin(2\pi f_m t)}\right|$$

$\tilde{s}(t) = A_c e^{j\beta sin(2\pi f_m t)}$ is the complex envelope of the FM wave $s(t)$

Generating Wideband FM:

- The important point to note from $A_c e^{j\beta sin(2\pi f_m t)}$ is that unlike the original FM wave $s(t)$, the complex envelope is a periodic function of time with a fundamental frequency equal to the modulation frequency, f_m.
- Check periodicity:

Generating Wideband FM:

- We may therefore expand $\tilde{s}(t)$ $[\tilde{s}(t) = A_c e^{j\beta sin(2\pi f_m t)}]$ in the form of a complex Fourier series as follows:

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_m t}$$

where the complex Fourier coefficient is

$$c_n = f_m \int_{-1/(2f_m)}^{1/(2f_m)} \tilde{s}(t) e^{(-j2\pi n f_m t)} dt$$

$$c_n = f_m A_c \int_{-1/(2f_m)}^{1/(2f_m)} e^{[j\beta sin(2\pi f_m t) - j2\pi n f_m t]} dt$$

Generating Wideband FM:

- Change variable t into θ,

$$2\pi f_m t = \theta$$

- Taking derivative,

$$2\pi f_m dt = d\theta$$

$$dt = \frac{d\theta}{2\pi f_m}$$

- Limits become,

$$t = -\frac{1}{2f_m} \Longrightarrow \theta = -\pi$$

$$t = \frac{1}{2f_m} \Longrightarrow \theta = \pi$$

$$c_n = f_m A_c \int_{-\pi}^{\pi} e^{[j\beta sin\theta - jn\theta]} \frac{d\theta}{2\pi f_m}$$

Generating Wideband FM:

$$c_n = \frac{A_c}{2\pi}\int_{-\pi}^{\pi} e^{j[\beta sin\theta - n\theta]}\, d\theta$$

- The integral on the right-hand side of Eq except for the carrier amplitude A_c, is referred to as the **nth order Bessel function** of the first kind and argument β.
- This function is commonly denoted by the symbol $J_n(\beta)$, so we may write:

$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j[\beta sin\theta - n\theta]}\, d\theta$$

$$c_n = A_c J_n(\beta)$$

Generating Wideband FM:

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_m t}$$

Put $c_n = A_c J_n(\beta)$

$$\tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t}$$

Generating Wideband FM:

- Also,

$$s(t) = A_c Re\left| e^{j(2\pi f_c t + \beta\ \sin(2\pi f_m t))} \right|$$

$$\tilde{s}(t) = A_c e^{j\beta sin(2\pi f_m t)}$$

Replace it with,

$$\tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t}$$

$$s(t) = A_c Re\left| \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi(f_c + n f_m)t} \right|$$

Generating Wideband FM:

$$s(t) = A_c Re\left|\sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi(f_c+nf_m)t}\right|$$

The carrier amplitude is a constant and may therefore be taken outside the real-time operator Re[.]. Moreover, we may interchange the order of summation and real-part operation, as they are both linear operators. Accordingly, we may rewrite Eq in the simplified form:

$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos 2\pi(f_c + nf_m)t$$

The Fourier series expansion of the single-tone FM signal.

Generating Wideband FM:

- The discrete spectrum is obtained by taking the Fourier transforms of both sides of $s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos 2\pi(f_c + nf_m)t$

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right]$$

Properties of Bessel Function:

1. $J_n(x)$ decreases as n increases

$$J_0(x) > J_1(x) > J_2(x) \dots$$

2. $J_{-n}(x) = (-1)^n J_n(x)$

$$J_{-1}(x) = (-1)^1 J_1(x) = -J_1(x)$$
$$J_{-2}(x) = (-1)^2 J_2(x) = J_2(x)$$
$$J_{-n}(x) = \begin{cases} J_n(x); n\ even \\ -J_n(x); n\ odd \end{cases}$$

3. $\sum_{n=-\infty}^{\infty} J_n^2(x) = 1$

Carson’s Rule of Bandwidth:

- Consider significant amplitude only.

$\beta + 1$ sidebands are significant