INTEGRATION

Calculus & Analytical Geometry MATH-101

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Techniques of Integration

- Substitution Rule
- Integration by Parts
- Integration of Rational
- Integration of Irrational Functions ✓
- Trigonometric Substitution
- Trigonometric Integrals ✓

① Non repeated linear
② Repeated lin
③ Non-repeated quadr
④ Repeated quadr

$$\frac{p(x)}{q(x)} \; ; q(x) \neq 0$$

→ deg $p(x) <$ deg $q(x)$

→ factorize $q(x)$

Integration of Irrational Functions

- An algebraic function involving one or more radicals of polynomials is called an irrational function. Integrals of irrational functions usually contain linear, quadratic or linear fractional expressions under the radical sign.

- Integration of irrational functions is more difficult than rational functions. However, there are some particular types that can be reduced to rational forms by suitable substitutions.

- Case 1: Integrals Involving radicals of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$

  These kinds of integrals can be evaluated with the help of trigonometric substitutions.

Integration of Irrational Functions

- Case 2: Integrals Involving Fractional Powers of x

  To integrate a function that contains only one irrational expression of the form $x^{m/n}$, we make the substitution $u = x^{1/n}$. If an irrational function contains more than one rational power of x, we use the substitution $u = x^{1/n}$, where n is the least common multiple (LCM) of the denominators of all fractional powers of x.

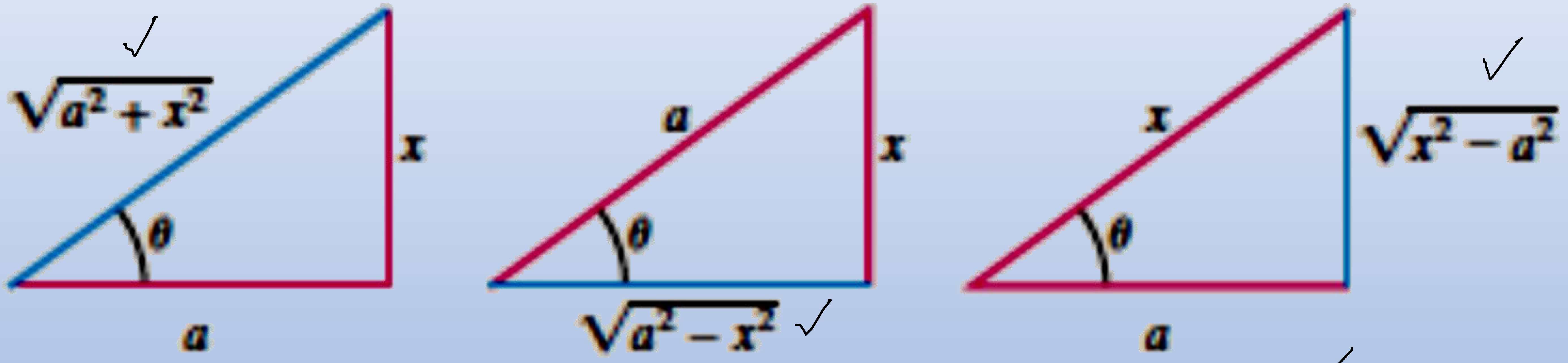
- Case 3: Integrals Involving $\left(\frac{ax+b}{cx+d}\right)^{1/n}$

  These types of integrals can be integrated using the substitution $u = \left(\frac{ax+b}{cx+d}\right)^{1/n}$, where a, b, c, d are real numbers.

- Case 4: Integrals Involving Quadratic Expressions

  We can use the technique of completing the square to deal with such integrals.

Trigonometric Substitutions



$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\lvert\sec\theta\rvert$	$\sqrt{a^2-x^2} = a\lvert\cos\theta\rvert$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter:** 8
  - **Section:** 8.5 ✓

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- **Chapter: 9**
  - **Section:** 9.3 ✓

Trigonometric Substitution

- A number of practical problems require us to integrate algebraic functions that contain an expression of the form:

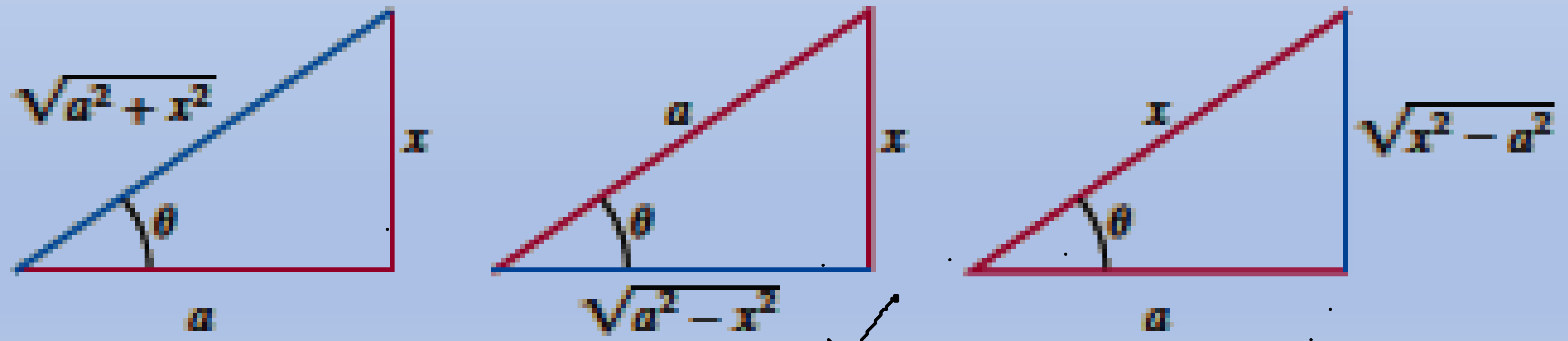
$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2} \quad \text{or} \quad \sqrt{x^2 - a^2}.$$

- Sometimes, the best way to perform the integration is to make a trigonometric substitution that gets rid of the root sign.

- In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities.

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

- They come from the reference right triangles.



$x = a\tan\theta$ $\qquad$ $x = a\sin\theta$ $\qquad$ $x = a\sec\theta$

$\sqrt{a^2+x^2} = a|\sec\theta|$ $\qquad$ $\sqrt{a^2-x^2} = a|\cos\theta|$ $\qquad$ $\sqrt{x^2-a^2} = a|\tan\theta|$

Example:

Evaluate

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx.$$

$\sqrt{a^2 - x^2}$

$x = a \sin\theta$

$a = 3$

Solution:

Let $x = 3\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 3\cos\theta\ d\theta$ and

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3|\cos\theta|.$$

(Note that $\cos\theta \ge 0$ because $-\pi/2 \le \theta \le \pi/2$.) Thus, the given integral becomes:

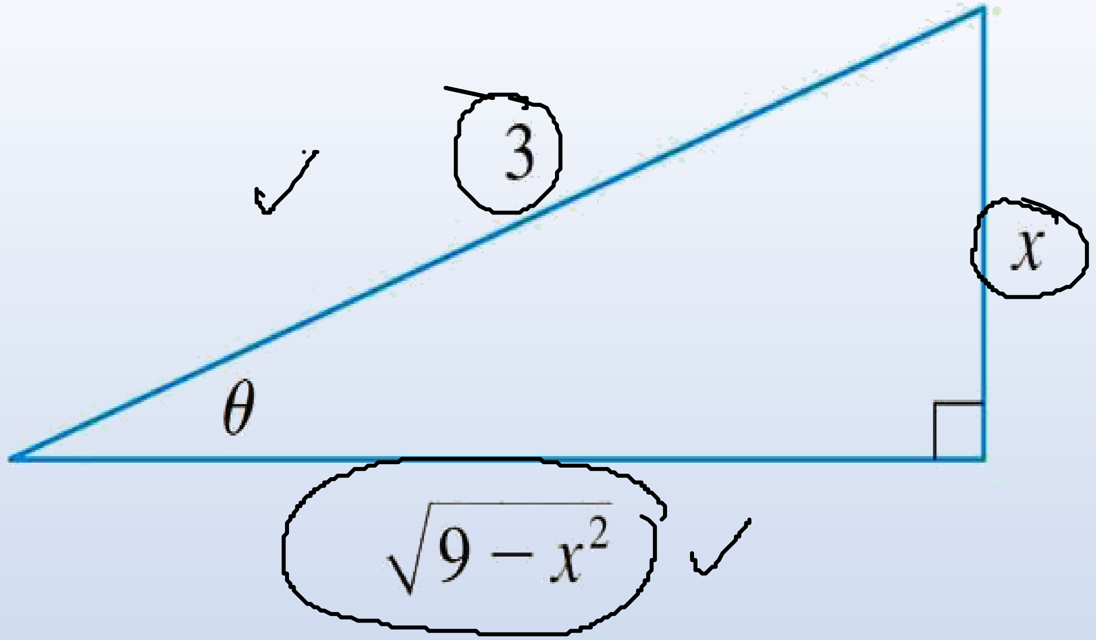
$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}.3\cos\theta\,d\theta = \int \frac{\cancel{9}\cos^2\theta}{\cancel{9}\sin^2\theta}\,d\theta. = \int \cot^2\theta\, d\theta$$

Solution:

$$\Rightarrow \int \frac{9\cos^2\theta}{9\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta$$

$$= \int (\operatorname{cosec}^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C$$



Since this is an indefinite integral, we must return to the original variable x. This can be done by expressing $\cot\theta$ in terms of $\sin\theta = x/3$ and $\cos\theta = \frac{\sqrt{9-x^2}}{3}$. Thus,

$$\cot\theta = \frac{\sqrt{9-x^2}}{x}.$$

$\sin^{-1}\theta \neq (\sin\theta)^{-1}$

Since $\sin\theta = x/3$, we have:

arc sin θ

$$\theta = \arcsin\left(\frac{x}{3}\right) = \sin^{-1}\left(\frac{x}{3}\right)$$

and so

$$\boxed{\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C.}$$

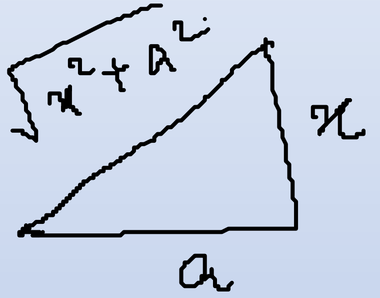
Example:

$\sqrt{x^2+a^2}$

Evaluate

$x = a\tan\theta$

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$$



Solution:

Let $x = 2\tan\theta$, where $-\pi/2 < \theta < \pi/2$. Then $dx = 2\sec^2\theta\, d\theta$ and

$a=4$

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta+4} = \sqrt{4(\tan^2\theta+1)} = \sqrt{4\sec^2\theta} = 2|\sec\theta|.$$

(Note that $\sec\theta > 0$ because $-\pi/2 < \theta < \pi/2$.) Thus, the given integral becomes:

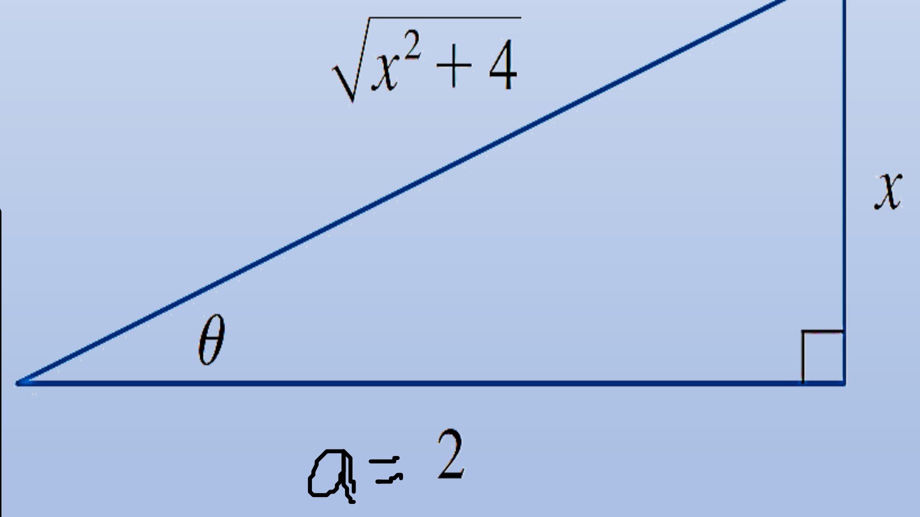
$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{\cancel{2}\sec^{\cancel{2}}\theta}{4\tan^2\theta\,\cancel{2}\cancel{\sec\theta}}\,d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta.$$

Solution:

$$\Longrightarrow \frac{1}{4}\int \frac{1}{\cancel{\cos\theta}}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = -\frac{1}{4\sin\theta} + C.$$

In order to return to the original variable x, we consider the following right-angled triangle. Note that:

$$\sin\theta = \frac{x}{\sqrt{x^2+4}} \Longrightarrow \frac{1}{\sin\theta} = \frac{\sqrt{x^2+4}}{x}$$



Thus,

$$\boxed{\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = -\frac{\sqrt{x^2+4}}{4x} + C.}$$

Example:

Evaluate

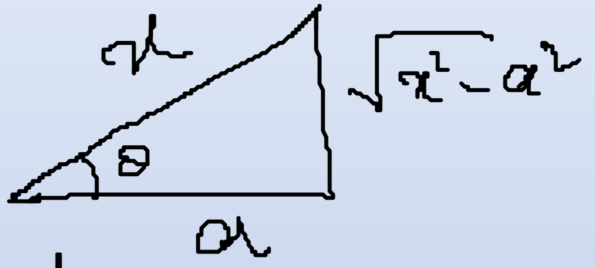
$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx; \quad a > 0.$$

$\sec\theta = \frac{x}{a}$

$\Rightarrow \cos\theta = \frac{a}{x}$

$\sqrt{x^2 - a^2}$

$x = a\sec\theta$



Solution:

Let $x = a\sec\theta$, where $0 < \theta < \frac{\pi}{2}$ or $\pi < \theta < \frac{3\pi}{2}$. Then $dx = a\sec\theta\tan\theta\,d\theta$ and

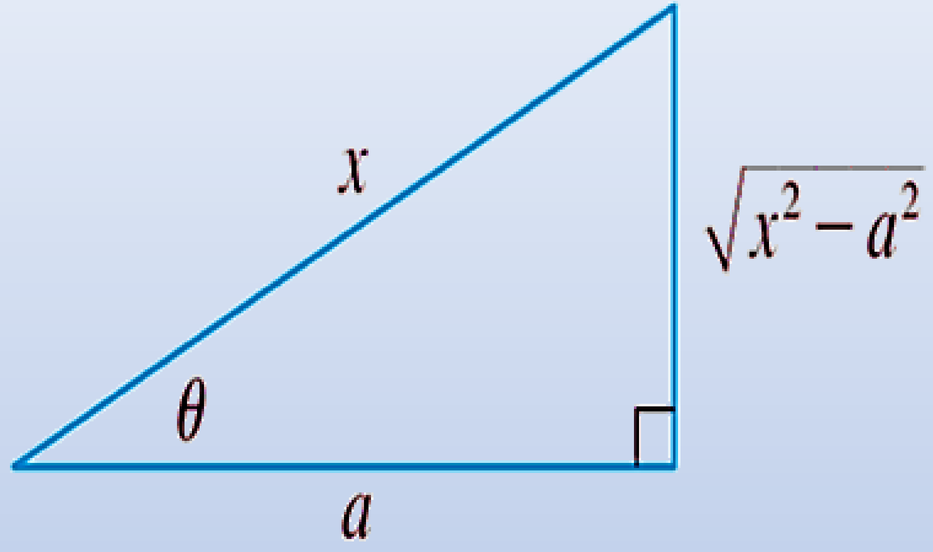
$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta\,.$$

Thus:

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \int \frac{\cancel{a}\sec\theta\,\cancel{\tan\theta}}{\cancel{a}\,\cancel{\tan\theta}\,.}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C.$$

Solution:

In order to return to the original variable x, we consider the following right-angled triangle. Note that:



$$\sin\theta = \frac{\sqrt{x^2-a^2}}{x},$$

and

$$\cos\theta = \frac{a}{x}.$$

Thus,

$$\sec\theta = \frac{1}{\cos\theta} = \frac{x}{a} \text{ and } \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{x^2-a^2}}{a}$$

Hence,

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C = \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C.$$

Practice Questions

Book: Thomas Calculus (11th Edition) by Georg B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Exercise: 8.5
  Q # 1 to Q # 28, Q # 37 to Q # 40.

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 9.3
  Q # 1 to Q # 32.

Integration of Irrational Functions
Cases: 2, 3 & 4

Case 2: Integrals Involving Fractional Powers of x

To integrate a function that contains only one irrational expression of the form $x^{m/n}$, we make the substitution $u = x^{1/n}$. If an irrational function contains more than one rational power of x, we use the substitution $u = x^{1/n}$, where n is the least common multiple (LCM) of the denominators of all fractional powers of x.

Case 3: Integrals Involving $\left(\frac{ax+b}{cx+d}\right)^{1/n}$

These types of integrals can be integrated using the substitution $u = \left(\frac{ax+b}{cx+d}\right)^{1/n}$, where a, b, c, d are real numbers.

Case 4: Integrals Involving Quadratic Expressions

We can use the technique of completing the square to deal with such integrals.

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- **Chapter: 9**
  - **Section:** 9.5, 9.6

Example:

Evaluate

$$\int \frac{1}{x-\sqrt{x}}\,dx.$$

$$2\int \frac{1}{u-1}\,du$$

Solution:

Let $u = \sqrt{x}$, then $x = u^2$ and $dx = 2\,u\,du$. Thus,

$$\int \frac{1}{x-\sqrt{x}}\,dx = \int \frac{1}{u^2-u}(2u)\,du = \int \frac{2}{u-1}\,du = 2\ln|u-1| + C.$$

$$\Rightarrow \int \frac{1}{x-\sqrt{x}}\,dx = 2\ln|\sqrt{x}-1| + C.$$

Example:

Evaluate

$$\int \frac{x + x^{2/3} + x^{1/6}}{x(1 + x^{1/3})}\, dx.$$

3 , 6

$n = 6$

$u = x^{1/n}$

LCM = 6

$u = x^{1/6}$

Solution:

Let $u = x^{1/6}$, then $x = u^6$ and $dx = 6u^5 du$. Thus,

$$\int \frac{x + x^{2/3} + x^{1/6}}{x(1 + x^{1/3})}\, dx = \int \left[\frac{u^6 + u^4 + u}{\cancel{u^6}(1 + u^2)}\right](6\cancel{u^5})\, du = 6\int \left[\frac{u^5 + u^3 + 1}{1 + u^2}\right] du.$$

$$\Rightarrow 6\int \left[u^3 + \frac{1}{1 + u^2}\right] du = 6\left[\frac{u^4}{4} + \arctan u\right] + C.$$

$$\frac{u^5 + u^3}{1 + u^2} + \frac{1}{1 + u^2} = \frac{u^3\cancel{(1+u^2)}}{\cancel{1+u^2}} + \frac{1}{1+u^2} = u^3 + \frac{1}{1+u^2}$$

Thus,

$$\boxed{\int \frac{x + x^{2/3} + x^{1/6}}{x(1 + x^{1/3})}\, dx = \frac{3}{2}x^{2/3} + 6\arctan\left(x^{1/6}\right) + C.}$$

Example:

Evaluate

$$\int \frac{x}{\sqrt{x+1}}\,dx.$$

Solution:

Let $u = \sqrt{x+1}$, then $x = u^2 - 1$ and $dx = 2\,u\,du$. Thus,

$$\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{u^2-1}{\cancel{u}}(2\cancel{u})du = 2\int (u^2-1)\,du = \frac{2u^3}{3} - 2u + C.$$

$$\Rightarrow \int \frac{x}{\sqrt{x+1}}\,dx = \frac{2(x+1)^{3/2}}{3} - 2(x+1)^{1/2} + C.$$

Example: $x+2 = \frac{4}{u^3+1} \Rightarrow x = \frac{4}{u^3+1} - 2 = \frac{4-2u^3-2}{u^3+1} = \frac{2-2u^3}{u^3+1} = 2(1-u^3)/1+u^3$

Evaluate

$$\int \frac{2}{(2-x)^2}\left(\frac{2-x}{2+x}\right)^{1/3} dx.$$

$\left(\frac{ax+b}{cx+d}\right)^{1/n} = u$

Solution:

Let $u = \left(\frac{2-x}{2+x}\right)^{1/3} \Longrightarrow \frac{2-x}{2+x} = u^3$. Then

$\frac{2-x}{2+x} = u^3 \Rightarrow \frac{2-x-2+2}{2+x} = u^3 \Rightarrow \frac{4-(x+2)}{x+2} = u^3 \Rightarrow \frac{4}{x+2} - 1 = u^3 \Rightarrow \frac{4}{x+2} = u^3+1 \Rightarrow \frac{x+2}{4} = \frac{1}{u^3+1}$

$$x = \frac{2(1-u^3)}{1+u^3},\quad 2-x = \frac{4u^3}{1+u^3} \text{ and } dx = \frac{-12u^2}{(1+u^3)^2}du.$$

Thus,

$$\int \frac{2}{(2-x)^2}\left(\frac{2-x}{2+x}\right)^{1/3} dx = \int\left[\frac{2\cancel{(1+u^3)^2}}{(4u^3)^2}.u.\frac{(-12)u^2}{\cancel{(1+u^3)^2}}\right]du = \frac{-3}{2}\int\left[\frac{1}{u^3}\right]du$$

Solution:

$$\frac{-3}{2}\int\left[\frac{1}{u^3}\right]du=\frac{-3}{2}\frac{u^{-2}}{(-2)}=\frac{3}{4u^2}+C.$$

Making use of $u = \left(\frac{2-x}{2+x}\right)^{1/3}$ in above equation, we get:

$$\int\frac{2}{(2-x)^2}\left(\frac{2-x}{2+x}\right)^{1/3}dx=\frac{3}{4}\left(\frac{2+x}{2-x}\right)^{2/3}+C.$$

Example:

Evaluate

$$\int \frac{2x-1}{x^2-6x+13}\,dx.$$

$x^2 - 6x + 13$

$= \underbrace{(x)^2 - 2(x)(3) + 9}_{(x-3)^2} - 9 + 13$

$= (x-3)^2 + 4$

Solution:

Note that $x^2 - 6x + 13 = x^2 - 6x + 9 - 9 + 13 = (x-3)^2 + 4$. Thus,

$$\int \frac{2x-1}{x^2-6x+13}\,dx = \int \frac{2x-1}{(x-3)^2+4}\,dx.$$

Let $u = x - 3 \Longrightarrow x = u + 3$ and $du = dx$. Then

$$\int \frac{2x-1}{x^2-6x+13}\,dx = \int \frac{2(u+3)-1}{u^2+4}\,du = \int \frac{2u+5}{u^2+4}\,du.$$

Solution:

$$\int \frac{2u+5}{u^2+4}\,du = \int \frac{2u}{u^2+4}\,du + 5\int \frac{1}{u^2+4}\,du$$

$$= \ln|u^2+4| + \frac{5}{2}\arctan\left(\frac{u}{2}\right) + C.$$

$\int \frac{1}{1+x^2}\,dx$, arctan x

Show??

Since $u = x - 3$, so above equation can be written as:

$$\int \frac{2x-1}{x^2-6x+13}\,dx = \ln|(x-3)^2+4| + \frac{5}{2}\arctan\left(\frac{x-3}{2}\right) + C.$$

Example:

Evaluate

$$\int \frac{1}{\sqrt{x^2+8x+25}}\,dx.$$

Solution:

Note that $x^2+8x+25 = x^2+8x+16-16+25 = (x+4)^2+9$. Thus,

$$\int \frac{1}{\sqrt{x^2+8x+25}}\,dx = \int \frac{1}{\sqrt{(x+4)^2+9}}\,dx.$$

$\sqrt{x^2+a^2}$

$\Rightarrow x = a\tan\theta$

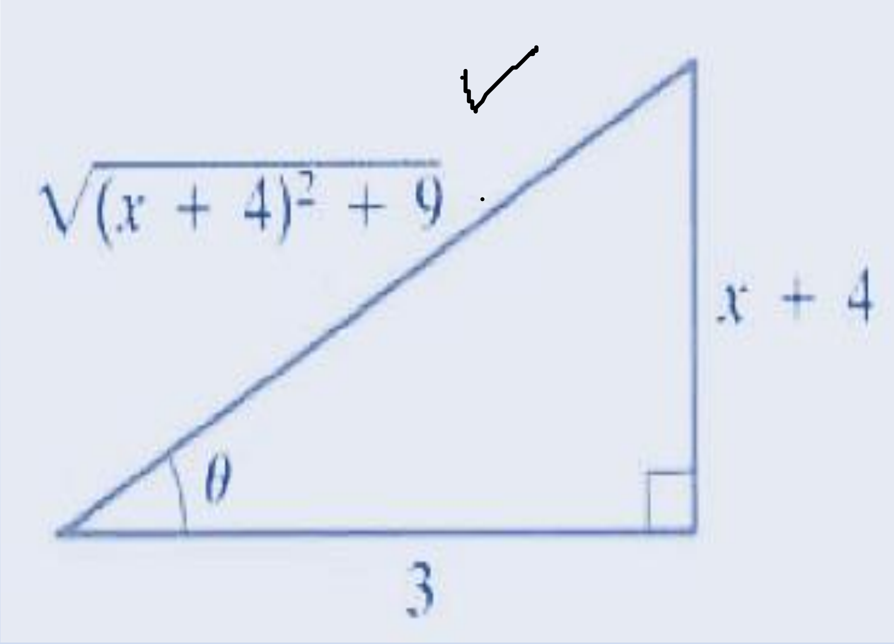
Let $x+4 = 3\tan\theta \Longrightarrow dx = 3\sec^2\theta\; d\theta$ and

$$\int \frac{1}{\sqrt{x^2+8x+25}}\,dx = \int \frac{3\sec^2\theta}{\sqrt{9\tan^2\theta+9}}\,d\theta.$$

Solution:

$$\int \frac{3\sec^2\theta}{\sqrt{9(\tan^2\theta+1)}}\,d\theta = \int \frac{\cancel{3}\,\sec^{\cancel{2}}\theta}{\cancel{3}\,\cancel{\sec}\,\theta}\,d\theta = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C.$$



To return to the variable x, we use the right-angled triangle and get:

$$\sin\theta = \frac{x+4}{\sqrt{(x+4)^2+9}} \quad \text{and} \quad \cos\theta = \frac{3}{\sqrt{(x+4)^2+9}}.$$

So that, $\sec\theta = \frac{\sqrt{(x+4)^2+9}}{3}$ and $\tan\theta = \frac{x+4}{3}$. Thus,

$$\int \frac{1}{\sqrt{x^2+8x+25}}\,dx = \ln\left|\frac{\sqrt{(x+4)^2+9}}{3} + \frac{x+4}{3}\right| + C.$$

$$= \ln\left|x+4+\sqrt{(x+4)^2+9}\right| - \ln|3| + C = \ln\left|x+4+\sqrt{(x+4)^2+9}\right| + K.$$

Where $K = C - \ln|3|$.

Practice Questions

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 9.5

  Q # 1 to Q # 20.

- Exercise: 9.6

  Q # 1 to Q # 20.