DISCRETE TIME (DT) FOURIER SERIES

DT Sinusoids Frequency & Rate of Oscillation

- for the CT signal $x(t) = e^{j\omega_0 t}$ we have the following two properties:
  1. the larger the magnitude of ω_0, the higher the rate of oscillation in the signal
  2. $e^{j\omega_0 t}$ is periodic for any value of ω_0

DT Sinusoids Frequency & Rate of Oscillation

- for the DT signal $x[n] = e^{j\omega_0 n}$ these properties don't hold for the following reason:

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$$

- thus the exponential at frequency $\omega_0 + 2\pi$ is the same as the exponential at frequency $\omega_0 \Rightarrow$ we only need to consider the frequency interval $-\pi \leq \omega < \pi$
- a DT sinusoid, $e^{j\omega_0 n}$, is periodic of period N only when:

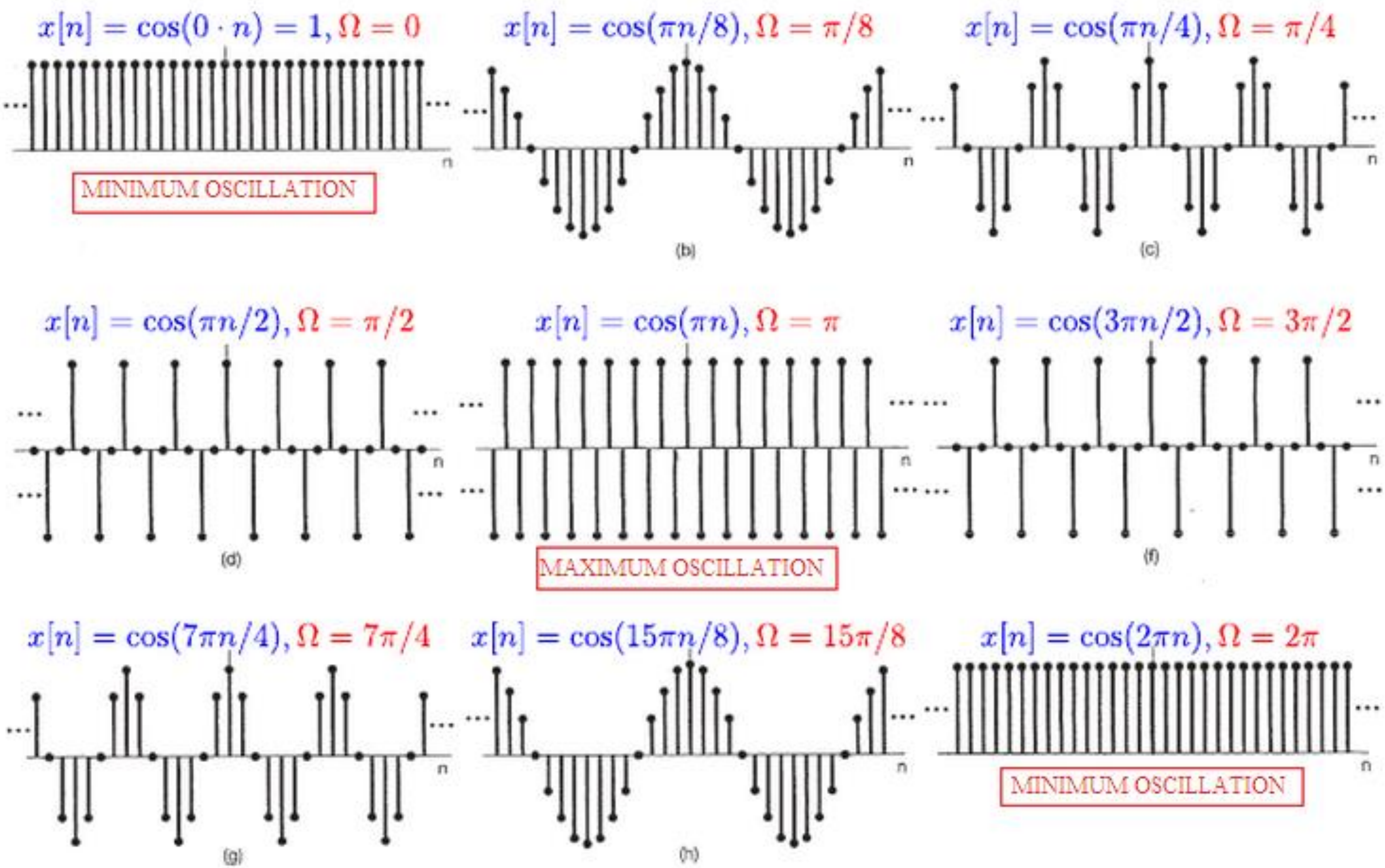
$$e^{j\omega_0 (n+N)} = e^{j\omega_0 n} \Rightarrow e^{j\omega_0 N} = 1$$

$\omega_0 N = 2\pi m$ for some integer m

$$\frac{\omega_0}{2\pi} = \frac{m}{N}$$

- if the above condition is not met, the DT sinusoid is not periodic

DT Sinusoids Frequency & Rate of Oscillation



DTFS for Periodic Signals

- $x[n]$ - periodic with fundamental period N, fundamental frequency

$$x[n+N] = x[n] \quad \text{and} \quad \omega_0 = \frac{2\pi}{N}$$

- There are only *N different signals* in the set of discrete-time complex exponential signals

- There are only N distinct signals of this form

$$e^{j(k+N)\omega_0 n} = e^{jk\omega_0 n} e^{j\overbrace{N\omega_0 n}^{2\pi n}} = e^{jk\omega_0 n}$$

- Only $e^{j\omega n}$ which are periodic with period N will appear in the *FS*

DTFS for Periodic Signals

- Remember – FS uses harmonically related complex exponentials with fundamental frequencies that are integer multiples of the fundamental frequency of the periodic signal to be represented

- Since the exponential sequences are distinct only over a range of N successive values of k, the FS summation may be written as:

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

— This is a *finite* series

DTFS for Periodic Signals

Any DT periodic signal has a Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$\Downarrow$$

$$x[0] = \sum_{k=<N>} a_k$$

$$x[1] = \sum_{k=<N>} a_k e^{jk\omega_0}$$

$$x[2] = \sum_{k=<N>} a_k e^{j2k\omega_0}$$

$$\vdots$$

$$x[N-1] = \sum_{k=<N>} a_k e^{j(N-1)k\omega_0}$$

N equations for N unknowns, $a_0, a_1, \cdots, a_{N-1}$

DTFS for Periodic Signals

So, from $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$

multiply both sides by $e^{-jm\omega_0 n}$

and then $\sum_{n=<N>}$

$$\sum_{n=<N>} x[n]e^{-jm\omega_0 n} = \sum_{n=<N>} \left(\sum_{k=<N>} a_k e^{jk\omega_0 n} \right) e^{-jm\omega_0 n}$$

$$= \sum_{k=<N>} a_k \underbrace{\left(\sum_{n=<N>} e^{j(k-m)\omega_0 n} \right)}_{=N\delta[k-m] - \text{ orthogonality}}$$

$$= N a_m$$

DT Fourier Series Pair

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} \qquad \text{(Analysis equaiton)}$$

Note: It is convenient to think of a_k as being defined for *all* integers k. So:

1) $a_{k+N} = a_k$ — Special property of DT Fourier Coefficients

(k gives the location of the coefficient on the frequency axis)

2) We only use N consecutive values of a_k in the synthesis equation. (Since $x[n]$ is periodic, it is specified by N numbers, either in the time or frequency domain)

DTFS Example

$$x[n] = \sin(\omega_0 n)$$

$$\omega_0 = \frac{2\pi}{N} \; (N \text{ an integer})$$

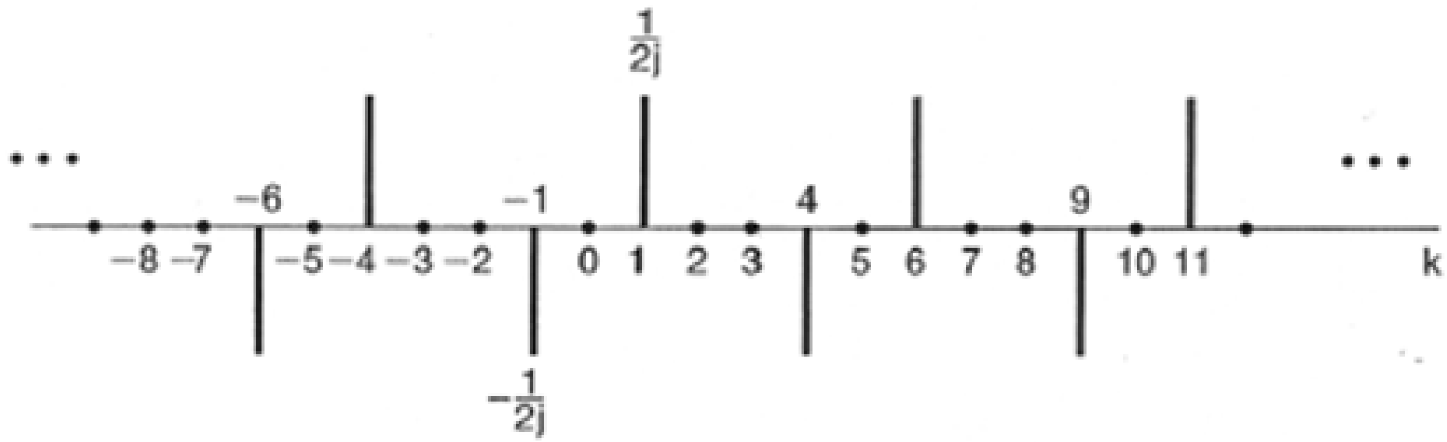
$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}$$

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j}, \; (a_k = 0,\, k \neq \pm 1)$$

$$a_{N+1} = a_{kN+1} = a_1$$

DTFS Example

Period N=5



Fourier coefficients for $x[n]=\sin[(2\pi/5)n]$

DTFS Example

$$\begin{aligned} x[n] &= \cos(\pi n/8) + \cos(\pi n/4 + \pi/4) \\ &\quad - \text{ periodic with period } N = 16 \Rightarrow \omega_0 = \pi/8 \end{aligned}$$

DTFS Example

$x[n] = \cos(\omega_0 n) + \cos(2\omega_0 n + \pi/4)$; first and second harmonics

$$x[n] = \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right] + \frac{1}{2}\left[e^{j\pi/4}e^{j2\omega_0 n} + e^{-j\pi/4}e^{-j2\omega_0 n}\right]$$

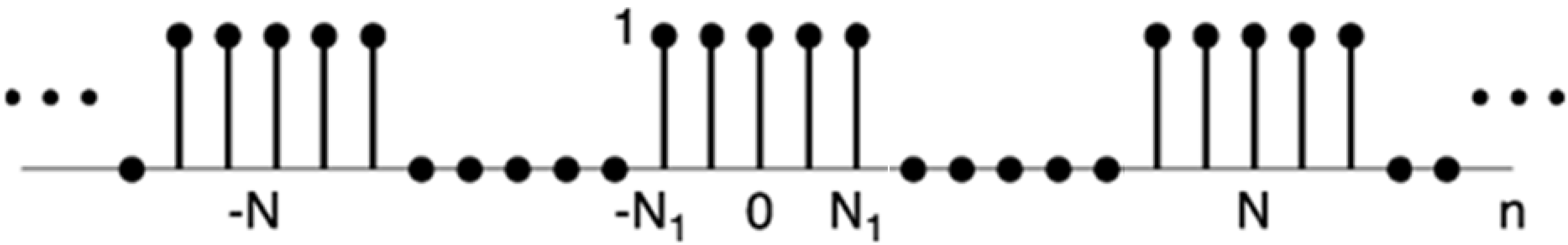
$$\Downarrow$$

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1/2 \\ a_{-1} &= 1/2 \\ a_2 &= e^{j\pi/4}/2 \\ a_{-2} &= e^{-j\pi/4}/2 \\ a_3 &= 0 \\ a_{-3} &= 0 \\ &\vdots \end{aligned}$$

$$a_{15} = a_{-1+16} = a_{-1} = 1/2$$

$$a_{66} = a_{2+4\times 16} = a_2 = e^{j\pi/4}/2$$

DTFS Square Wave

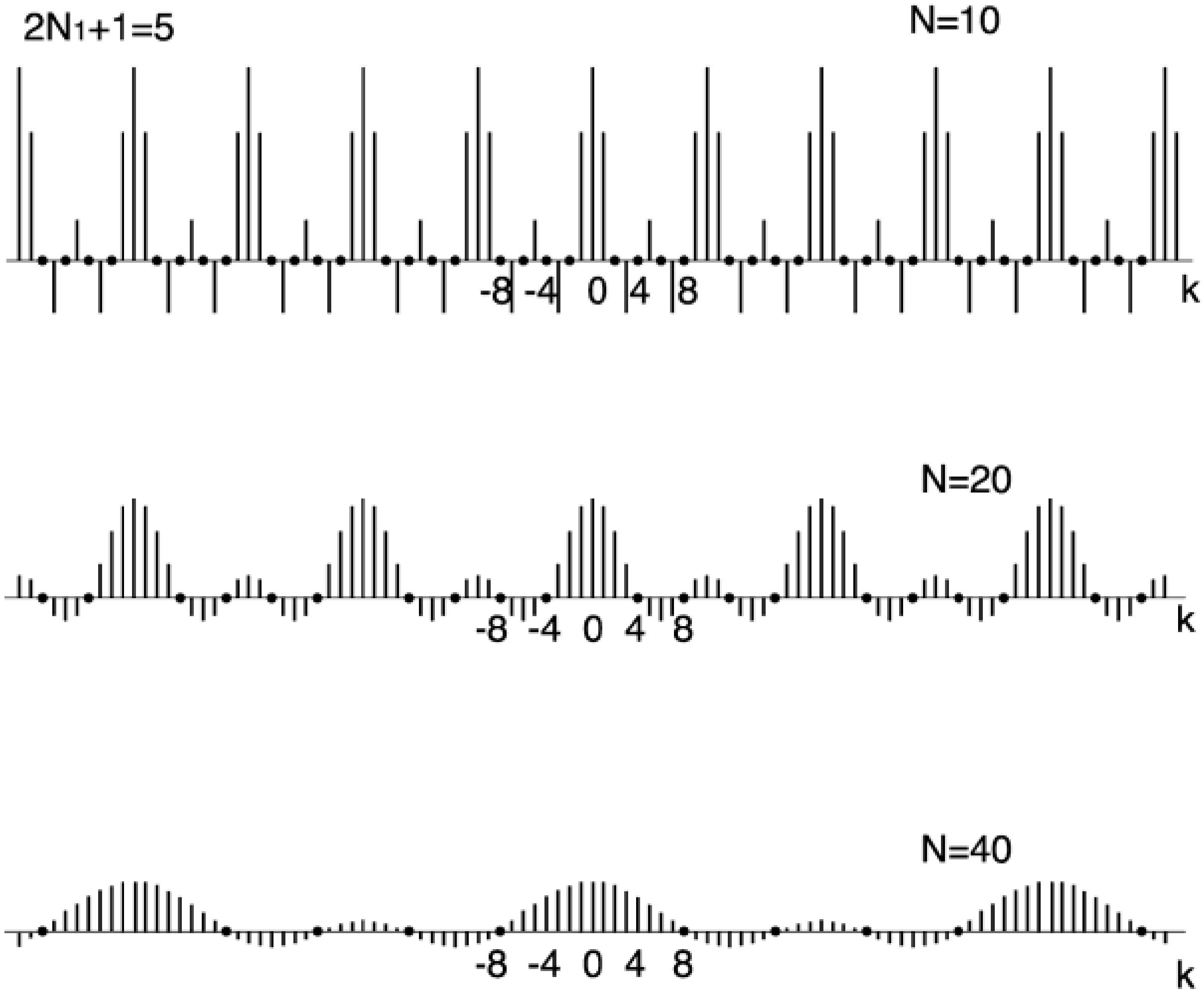


$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} \qquad \text{(Analysis equaiton)}$$

DTFS Square Wave

$$a_k = \frac{1}{N}\frac{\sin\left[2\pi k(N_1 + 1/2)/N\right]}{\sin(\pi k/N)}$$



DTFS Square Wave Convergence

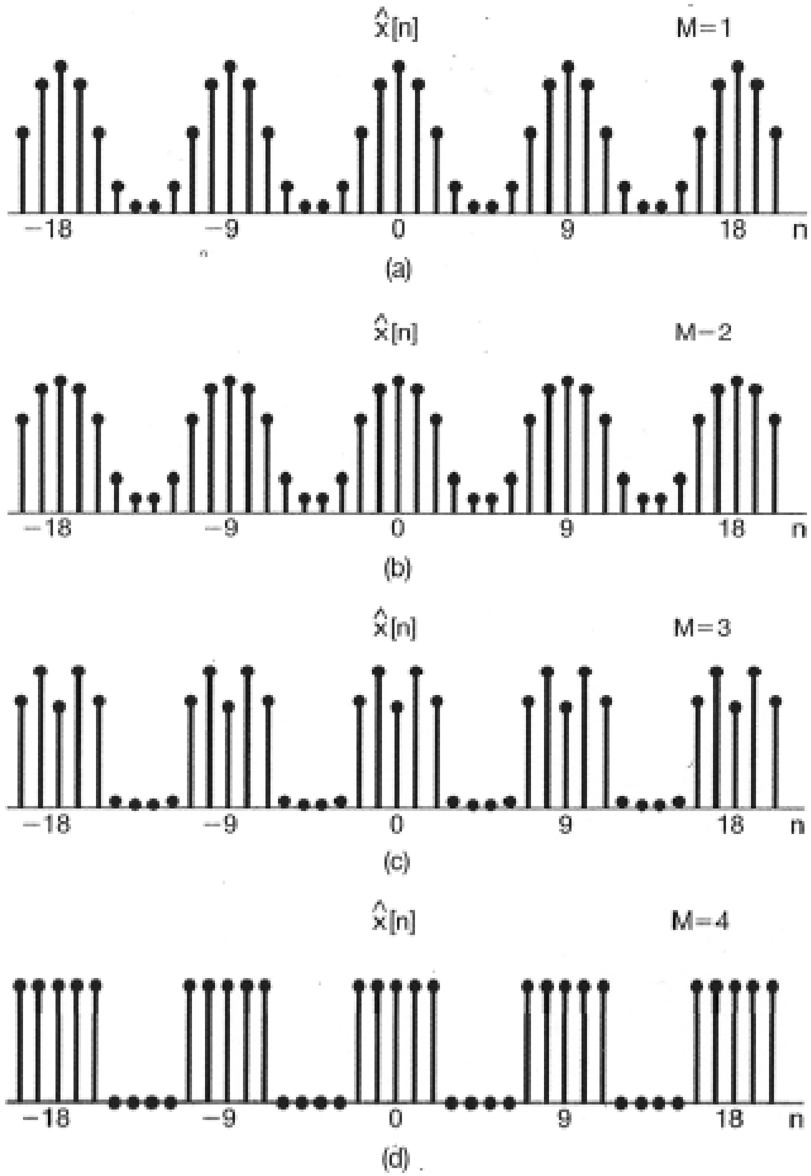
$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(2\pi/N)n}$$

N=9 square wave;

$2N_1+1=5$;

M=# Terms in partial sum

No Gibbs phenomenon – convergence to ideal square wave in finite number of terms of summation.



END