Lecture-9 & 10

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Energy in SHM

According to work-energy theorem

$$\Delta U = -W$$

If we stretch the spring from its equilibrium position x = 0 (U(0)=0) to some final position x (U(x)), the potential energy stored in the mass-spring system is

$$U(x) - U(0) = -\int_0^x F_s dx = -\int_0^x (-kx) dx$$

$$U(x) = \frac{1}{2}kx^2 \qquad \therefore U(0) = 0$$

Whether spring is stretched or compressed by displacement x, potential energy stored in the system is $\frac{1}{2}kx^2$

As we have seen, for harmonic oscillator

$$x(t) = A\cos(\omega t + \phi)$$

$$v(t) = -\omega A\sin(\omega t + \phi)$$

So

$$U = \frac{1}{2}kx^2$$

$$= \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

Potential energy is stored in the Spring due to compression

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\omega^2 A^2\sin^2(\omega t + \phi)$$

$$= \frac{1}{2}kA^2\sin^2(\omega t + \phi)$$

Kinetic energy is stored in the mass due to motion

$$\therefore \omega^2 = \frac{k}{m}$$

- Both, K and U, oscillates with time and have a maximum value of $\frac{1}{2}kA^2$
- During the motion both, K and U vary between zero and maximum value.
- At maximum displacement, K is zero but U has maximum value $\frac{1}{2}kA^2$
- At the equilibrium position (x = 0), potential energy is zero but the Kinetic energy has maximum value $\frac{1}{2}kA^2$
- At other positions, sum of K and U is $\frac{1}{2}kA^2$.
- This constant is the total energy E

The total energy of the oscillator remains constant at every time

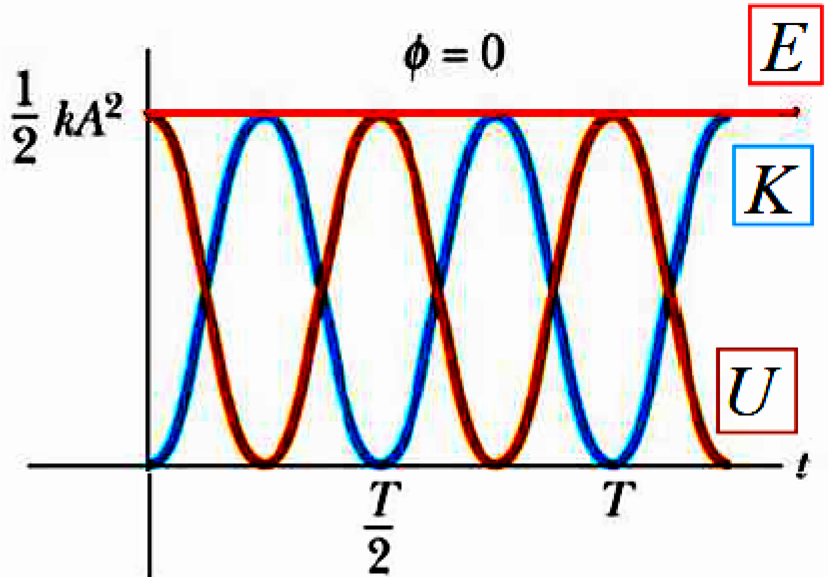
$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

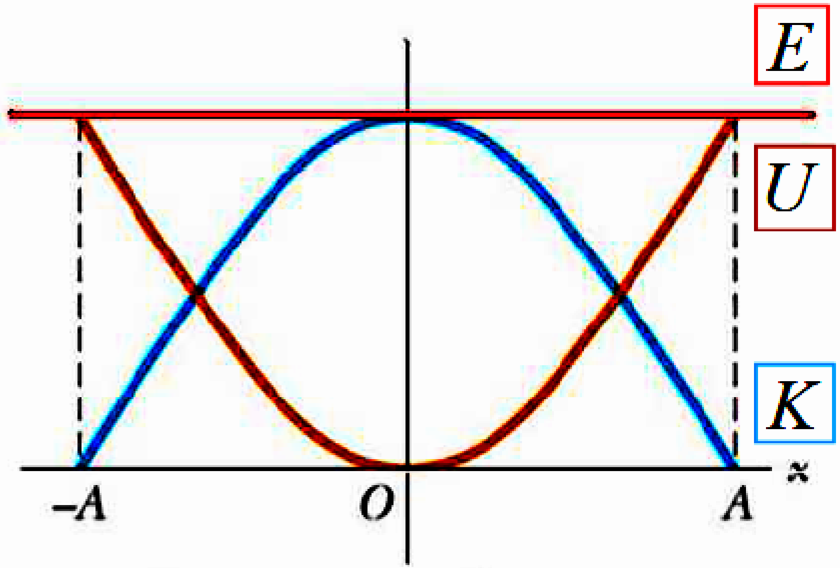
$$= \frac{1}{2}kA^2 \sin^2(\omega t + \phi) + \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$

$$= \frac{1}{2}kA^2$$

Energy versus time at $\phi=0$

Energy versus displacement





$$U = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

$$U = \frac{1}{2}kx^2$$

$$K = \frac{1}{2}kA^2\sin^2(\omega t + \phi)$$

$$K = \frac{1}{2}mv^2$$

$$E = \frac{1}{2}kA^2$$

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- Finally, we can use the principle of conservation of energy to obtain the velocity for an arbitrary displacement by expressing the total energy at some arbitrary position x as

$$K + U = E$$

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} = \pm\omega\sqrt{(A^2 - x^2)}$$

It substantiates the fact that the speed is a maximum at equilibrium position x = 0 and is zero at the turning points $x = \pm A$.

Consider a simple harmonic oscillator, mass attached to a spring with spring constant k,

(a) When the displacement is one half the amplitude, what fraction of the total energy is kinetic and what fraction is potential?

(b) At what displacement is the energy half kinetic and half potential?

(a) When the displacement is one half the amplitude, what fraction of the total energy is kinetic and what fraction is potential?

Since $$E = \frac{1}{2}kA^2$$

So $$U = \frac{1}{2}kx^2 = \frac{1}{2}k\left(\frac{A}{2}\right)^2 = \frac{1}{4}\left(\frac{1}{2}kA^2\right) = \frac{1}{4}E$$

$$K = E - U = E - \frac{1}{4}E = \frac{3}{4}E$$

The energy is 25% potential and 75% kinetic.

(b) At what displacement is the energy half kinetic and half potential?

$$U = E/2 \Rightarrow \frac{1}{2}kx^2 = \frac{1}{4}kA^2 \Rightarrow x = \frac{1}{\sqrt{2}}A$$

Energy will be half kinetic and half potential at $x = A/\sqrt{2}$

A block of mass M= 4 kg is suspended from a spring with a force constant k=5.38N/m. A m=50 g bullet is fired into the block from below with a speed of V_0=150m/s and comes to rest in the block. Find the amplitude of the resulting SHM.

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Let block/bullet system moves with speed V. By law of conservation of linear momentum

$$mV_\circ = (m+M)V \Rightarrow V = \frac{mV_\circ}{(m+M)}$$

Note: This is the velocity of system at zero displacement

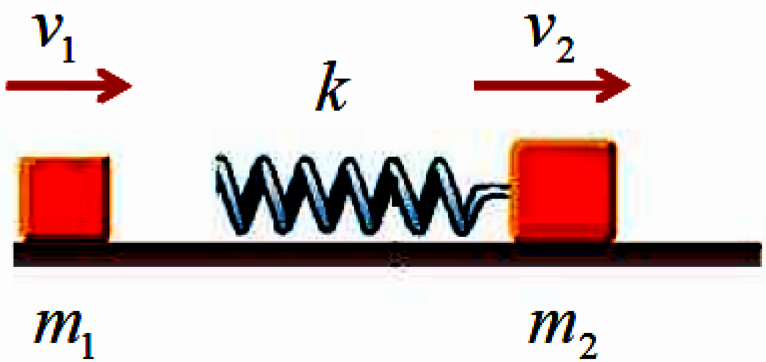
Now amplitude can be found by conservation of energy of oscillator,

$$\frac{1}{2}kA^2 = \frac{1}{2}(m+M)V^2 \qquad \therefore E = K$$

$$A = \sqrt{\frac{(m+M)}{k}}V$$

$$A = \sqrt{\frac{(m+M)}{k}}\frac{mV_\circ}{(m+M)} = \frac{mV_\circ}{\sqrt{k(m+M)}} = \frac{(0.5)(150)}{\sqrt{(538)(0.5+4)}} =$$

A block of mass m_1= 1.88 kg slides along a frictionless table with a speed of 10.3m/s. Directly in front of it , and moving in the same direction, is a block of mass m_2=4.92kg moving at 3.27 m/s. A massless spring with a spring constant k=11.2N/cm is attached to the back side of mass m_2 as shown. When the blocks collide, what is the maximum compression in the spring?



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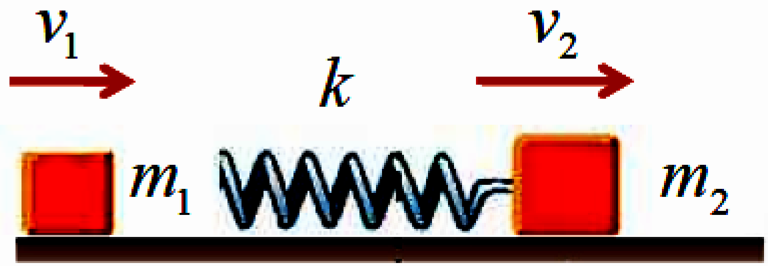
At the moment of maximum compression, the two block will move as one. Let their speed is V So by law of conservation of mom.



$$(m_1+m_2)V = m_1v_1+m_2v_2 \Rightarrow V = \frac{m_1v_1+m_2v_2}{m_1+m_2} = 5.21m/s$$

Loss in Kinetic energy is

$$\Delta K = \frac{1}{2}(m_1+m_2)V^2 - \frac{1}{2}m_1v_1^2 - \frac{1}{2}m_2v_2^2 = -33.7J$$

By law of conservation of energy

$$\Delta U = -\Delta K$$

$$U_f - U_i = -\Delta K$$

$$\frac{1}{2}kx^2 = 33.7J$$

$$\therefore U_i = 0$$

$$\therefore U_f = \frac{1}{2}kx^2$$

So maximum compression is

$$x = \sqrt{\frac{2(33.7J)}{k}} = \sqrt{\frac{2(33.7J)}{1120N/m}} = 0.24m$$

A 7.94 kg mass is resting on a spring. The spring is compressed by 10.2cm by the mass.

(a) Calculate the force constant of the spring.

(b) The mass is pushed down an additional 28.6cm and released. How much potential energy is stored in the spring just before mass is released?

(c)How high above this new (lowest) position will the mass rise?

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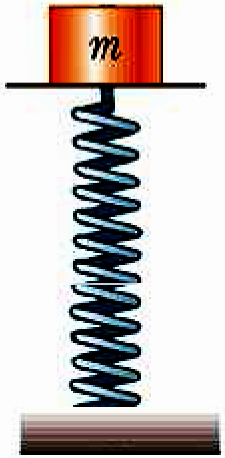


(a) $$kx = mg$$

$$k = \frac{mg}{x} = \frac{(7.94)(9.8)}{0.102} = 764 N/m$$

(b) Maximum compression in the spring is x= 10.2cm+28.6cm so potential energy stored in the spring will be

$$U = \frac{1}{2}kx^2$$

$$U = \frac{1}{2}(764)(0.286 + 0.102)^2 = 57.5J$$

(c) By law of conservation of energy

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}mv_f^2 + mgy_f + \frac{1}{2}kx_f^2 = \frac{1}{2}mv_i^2 + mgy_i + \frac{1}{2}kx_i^2$$

$$\frac{1}{2}m(0)^2 + mgh + \frac{1}{2}k(0)^2 = \frac{1}{2}m(0)^2 + mg(0) + \frac{1}{2}kx_i^2$$

$$mgh = \frac{1}{2}kx_i^2$$

$$h = \frac{k}{2mg}x_i^2 = \frac{764}{2(7.94)(9.81)}(0.286 + 0.102)^2 = 0.73m$$

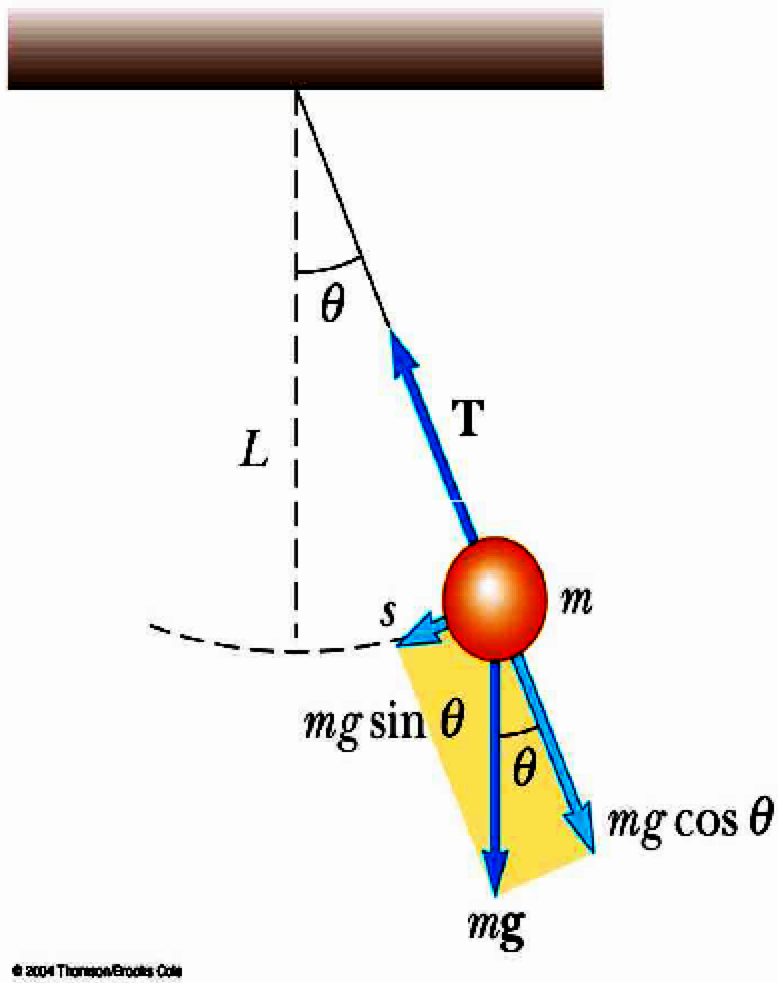
Pendulums

- When we were discussing the energy in a simple harmonic oscillator (mass attached to the spring), we talked about the 'springiness' of the system as storing the potential energy.
- But when we talk about a regular pendulum there is nothing 'springy' – so where is the potential energy stored?
- We have seen, the harmonic oscillator (mass attached to the spring) executes SHM under the effect of Hook's Law.
- The pendulum bob is clearly oscillating as it moves back and forth – but is it exhibiting SHM?
- To answer these questions, let's have a little chat!

The Simple Pendulum

- A simple pendulum is an idealized body consisting of a particle suspended by a light inextensible cord.
- When pulled to one side of its equilibrium position and then released, the pendulum swings in a vertical plane under the influence of gravity.
- Hence, the potential energy in a simple pendulum is stored in raising the bob up against the gravitational force.
- The motion of pendulum is periodic and oscillatory.
- We wish to determine the period of the pendulum.

- The weight mg is resolved into a radial component of magnitude mgcosθ and a tangential component of magnitude mgsinθ.
- The radial component mgcosθ supplies the necessary centripetal force to keep pendulum moving in circular path.
- The tangential component mgsinθ supplies the necessary restoring force to keep the pendulum motion oscillatory.



So the restoring force is:

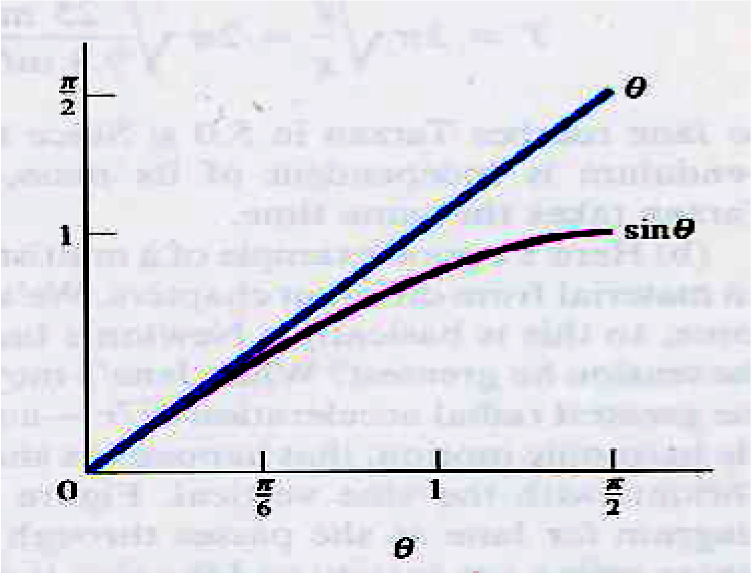
$$F = -mg \sin\theta$$

- The minus sign indicates the the restoring force is directed towards equilibrium; a condition for oscillatory motion.
- Wait a minute! Resorting force is not proportional to the angular displacement θ but to sin θ instead. The motion of simple pendulum is not SHM therefore!
- However , If we assume that the angle θ is small, then

$$\sin\theta \approx \theta$$

$$F = -mg\theta$$

From fig. $s = L\theta$



So $$F = -mg(s/L) = -(mg/L)s$$

- For small displacements, the restoring force is proportional to the displacement and is directed towards the equilibrium position. This is exactly the criterion for SHM.
- Thus, a simple pendulum executes simple harmonic motion only for small displacements

From above equation $k = mg/L$ $\qquad \therefore F = -kx$

Period of the pendulum is then

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/L}} = 2\pi\sqrt{\frac{L}{g}}}$$

Note: Period of the pendulum is independent of mass

Note: This analysis is allowed only for small θ, Why? What if θ is large?

If a pendulum has a period of 1 sec at the equator, what would be its period at the south pole?

As time period for pendulum is

$$T \propto \sqrt{\frac{1}{g}}$$

If g_e is the gravitational constant at equator, g_p is the gravitational constant at pole, T_e=1 sec is the time period of pendulum at the equator and the time period T_p at the pole then we have

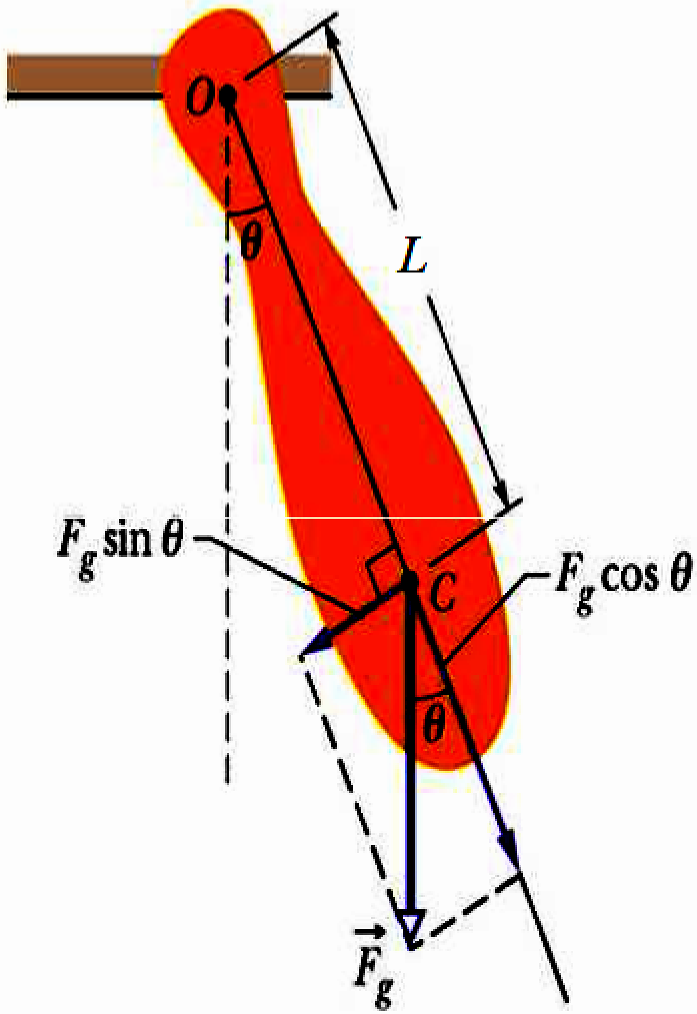
$$\frac{T_p}{T_e} \propto \frac{\sqrt{1/g_p}}{\sqrt{1/g_e}}$$

$$T_p = T_e\sqrt{g_e/g_p} = 1\times\sqrt{9.78/9.84} =$$ 

The Physical Pendulum

- Any rigid body mounted so that it can swing in a vertical plane about some axis passing through it is called a physical pendulum
- All real pendulums are physical pendulum
- In physical pendulum the mass is not all concentrated in the bob but it lies at the center of mass of the body (C in the diagram) which is a distance L from the pivot point O.

- The weight F_g is resolved into a radial component of magnitude $F_g \cos\theta$ and a tangential component of magnitude $F_g \sin\theta$
- The radial component supplies the necessary centripetal force to keep pendulum moving in circular path.
- The tangential component supplies the necessary restoring torque to keep the pendulum motion oscillatory.



So the restoring torque is:

$$\tau = -LF_g \sin\theta$$

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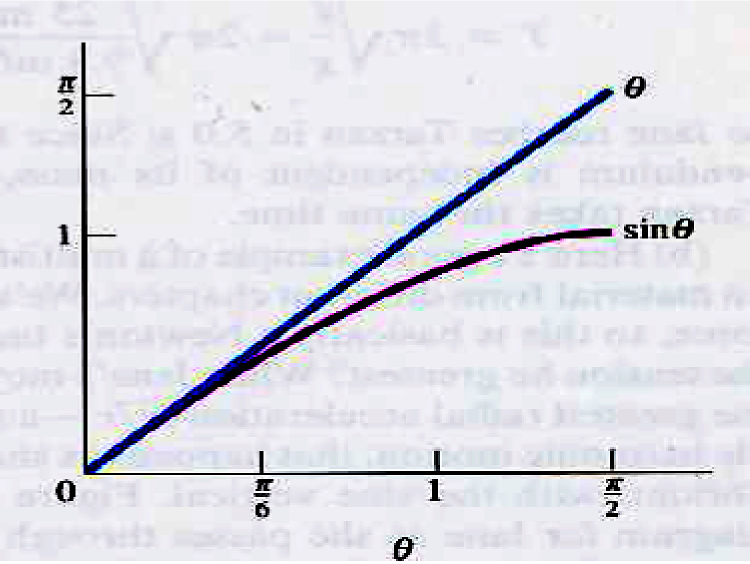


- For small displacements, the restoring force is proportional to the displacement and is directed towards the equilibrium position. This is exactly the criterion for SHM.
- Thus, a simple pendulum executes simple harmonic motion only for small displacements

From above equation $k = mgL$ $\therefore \tau = -k\theta$

Period of the physical pendulum is then

$$\boxed{T = 2\pi\sqrt{\frac{I}{k}} = 2\pi\sqrt{\frac{I}{mgL}}}$$

The Physical vs. Simple Pendulum

The period of a pendulum is given by:

$$T = 2\pi\sqrt{\frac{I}{mgL}}$$

If all of the mass of the pendulum is concentrated in the bob, then $I = mL^2$ and we get:

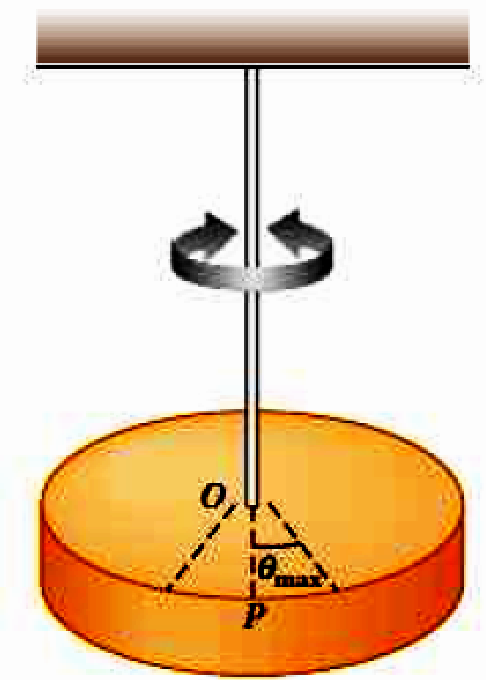
$$T = 2\pi\sqrt{\frac{I}{mgL}} = 2\pi\sqrt{\frac{mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}$$

$\boxed{\text{Simple pendulum is a special case of physical pendulum}}$

Torsional Pendulum

- A torsional pendulum consists of a rigid object suspended by a wire attached to a rigid support.
- The object oscillates about the wire *with an amplitude* θ.

When the object is twisted through some angle θ, the twisted wire exerts on the object a restoring torque that is proportional to the angular position. That is,



$$\tau = -\kappa\theta$$

Where κ is torsion constant.

$$I\alpha = -\kappa\theta \quad \Rightarrow \quad I\frac{d^2\theta}{dt^2} = -\kappa\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta$$

This is the equation of motion for a simple harmonic oscillator, with

$$\omega^2 = \frac{\kappa}{I}$$

$$\therefore \frac{d^2\theta}{dt^2} = -\omega^2\theta$$

So time period is

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\kappa}}}$$

There is no small-angle restriction in this situation as long as the elastic limit of the wire is not exceeded.