DETERMINING SYSTEM PROPERTIES

System Function of an LTI System

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$h(t) \quad \longleftrightarrow \quad H(s) \text{ – the system function}$$

The system function characterizes the system

$$\Downarrow$$

System properties correspond to properties of $H(s)$ and its ROC

A first example:

$$\text{System is stable} \Leftrightarrow \int_{-\infty}^{\infty} |h(t)| dt < \infty \quad \Leftrightarrow \quad \begin{array}{l}\text{ROC of } H(s) \\ \text{includes the } j\omega \text{ axis}\end{array}$$

LTI Systems - Causality

- For a causal LTI system, the impulse response is zero for $t < 0$ and thus is right-sided

The ROC associated with the system function for a causal system is a right-half plane

If *H(s)* is rational, then we can determine whether the system is causal simply by checking to see if its ROC is a right-half plane.

For a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole.

LTI Systems Causality – Example 1

- Consider a system with impulse response

$$h(t) = e^{-t}u(t)$$

- Since $h(t) = 0$ for $t < 0$, this system is causal, with system function

$$H(s) = \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

- The system function is rational, and the ROC is to the right of the right-most pole, consistent with the statement that causality for systems with rational system functions is equivalent to the ROC being to the right of the right-most pole.

LTI Systems Causality – Example 2

- Consider a system with impulse response

$$h(t) = e^{-|t|}$$

LTI Systems Causality – Example 2

- Since $h(t) \neq 0$ for $t < 0$, this system is not causal.

The system function is:

$$H(s) = \frac{-2}{s^2 - 1} = \frac{-2}{(s+1)(s-1)}, \quad -1 < \mathrm{Re}\{s\} < +1$$

- Thus $H(s)$ is rational and has an ROC that is not

to the right of the rightmost pole $(s = 1)$, consistent with the fact that the system is not causal.

LTI Systems - Stability

- Stability of an LTI system is equivalent to its impulse response being absolutely integrable, in which case the FT of the impulse response converges.
- Since the Fourier transform of a signal equals the Laplace transform evaluated along the $j\omega$ axis, we have the following:

> An LTI system is stable, if and only if the ROC of its system function, $H(s)$, includes the $j\omega$-axis (i.e., $\mathrm{Re}\{s\} = 0$)

LTI Systems - Stability

- For one very important class of systems, stability can be characterized very simply in terms of the locations of the poles.
- Consider a causal LTI system with a rational system function $H(s)$.
- Since the system is causal, the ROC is to the right of the rightmost pole.
- For this system to be stable, the rightmost pole of $H(s)$ must be to the left of the $j\omega$-axis, i.e.,

> A causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane -- i.e., all of the poles have negative real parts.

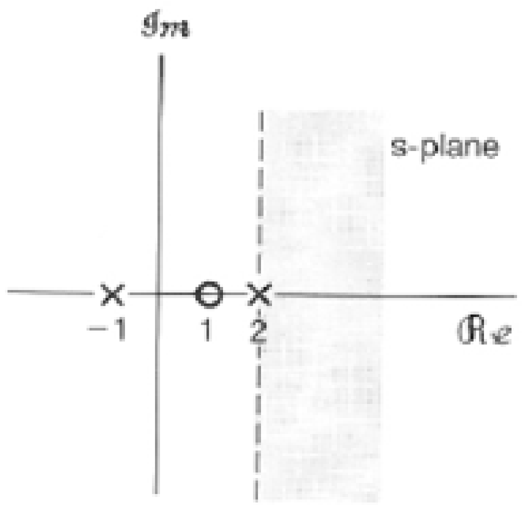
LTI Systems - Stability

- Consider an LTI system with system function:

$$H(s) = \frac{s-1}{(s+1)(s-2)}$$

- If the system is Causal, determine the time domain impulse response

LTI Systems - Stability



$$h(t) = \left(\frac{2}{3} e^{-t} + \frac{1}{3} e^{2t} \right) u(t)$$

(Using Partial Fractions and Laplace Transform table)

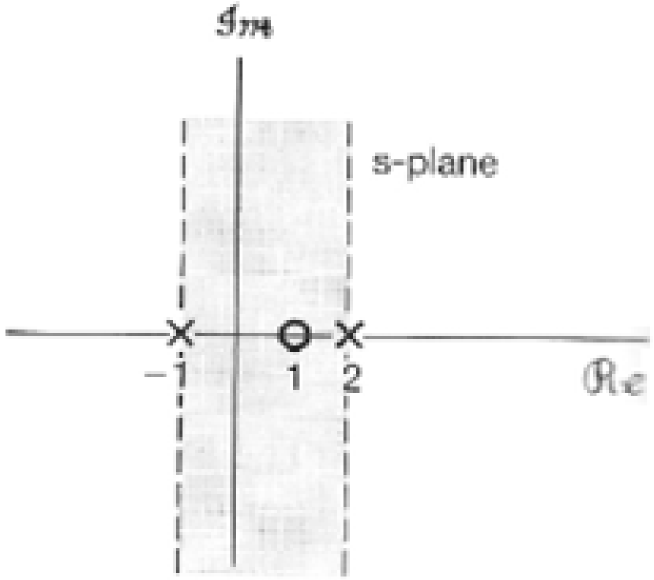
LTI Systems - Stability

- Consider an LTI system with system function:

$$H(s) = \frac{s-1}{(s+1)(s-2)}$$

- If the system is Stable, determine the time domain impulse response

LTI Systems - Stability



$$h(t) = \left(\frac{2}{3}e^{-t}u(t)\right) - \left(\frac{1}{3}e^{2t}u(-t)\right)$$

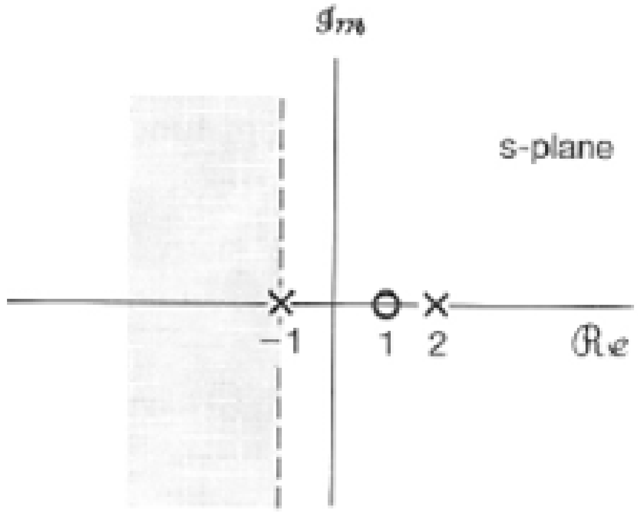
LTI Systems - Stability

- Consider an LTI system with system function:

$$H(s) = \frac{s-1}{(s+1)(s-2)}$$

- If the system is non-causal and un-stable, determine the time domain impulse response

LTI Systems - Stability



$$h(t) = -\left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(-t)$$

LTI Systems – Stability/Causality

- Consider a system with impulse response

$$h(t) = e^{-t}u(t)$$

- Since $h(t) = 0$ for $t < 0$, this system is causal, with system function

$$H(s) = \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

- The impulse response is absolutely integrable, and thus the system is stable with a pole at $s = -1$ which is in the left-half of the s-plane.

- Now consider the causal system with impulse response

$$h(t) = e^{2t}u(t)$$

- This system is unstable, since $h(t)$ is not absolutely integrable. Also we see that the system has a pole at $s = 2$ in the right half of the s-plane.

Laplace Transform Pairs

Signal	Transform	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$

Signal	Transform	ROC
$\delta(t-T)$	e^{-sT}	All s
$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

END