Applications of Derivatives

Calculus & Analytical Geometry MATH- 101
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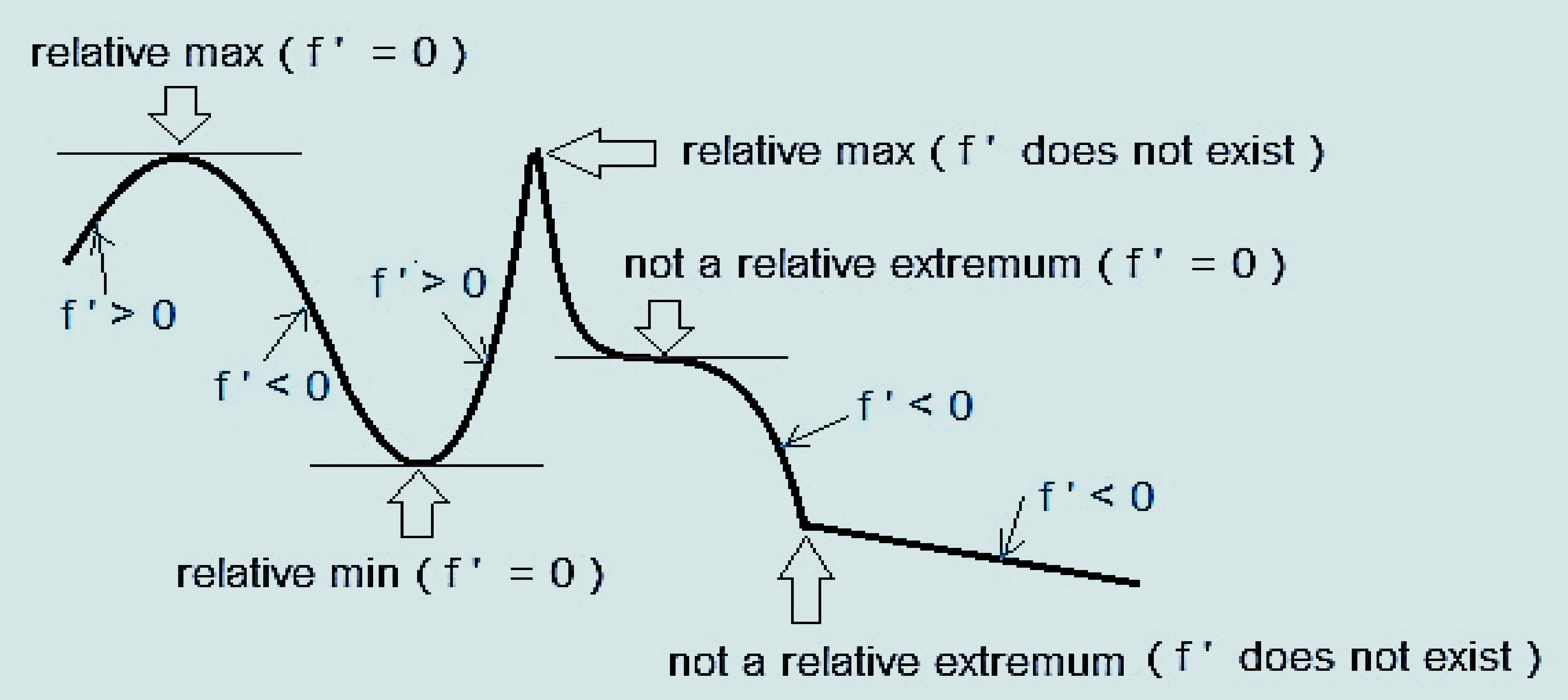
Objectives

- Extreme Values of functions.
- Rolle's theorem
- The Mean Value theorem.
- Monotonic Functions and The First Derivative Test

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 4
    - Sections: 4.1, 4.2, 4.3

4.1



Extreme Values of Functions

Extreme Values of Functions

- Extreme Values of a function occur when the function changes from increasing to decreasing or from decreasing to increasing.
- In particular, we have two types of minimum or maximum values.
- We say that $f(x)$ has an **absolute (or global) maximum** at $x = c$ if $f(x) \leq f(c)$ for every x in the domain we are working on.
- We say that $f(x)$ has a **relative (or local) maximum** at $x = c$ if $f(x) \leq f(c)$ for every x in some open interval around $x = c$.
- We say that $f(x)$ has an **absolute (or global) minimum** at $x = c$ if $f(x) \geq f(c)$ for every x in the domain we are working on.
- We say that $f(x)$ has a **relative (or local) minimum** at $x = c$ if $f(x) \geq f(c)$ for every x in some open interval around $x = c$.

Critical Points

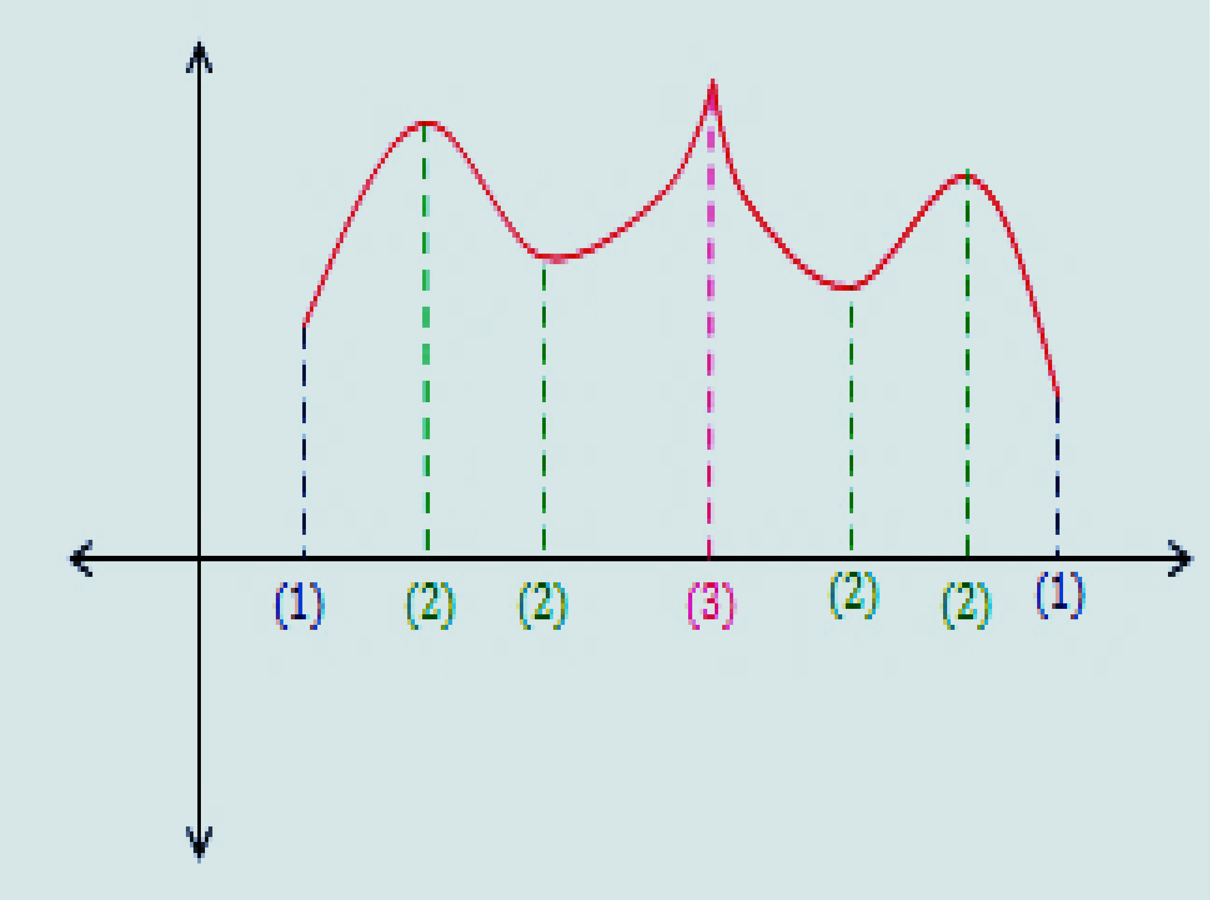
- An interior point of the domain of a function f where f' is zero **(stationary point)** or undefined **(singular point)** is a **critical point**.

- Hence the only domain points where a function f can possibly have an extreme value (local or global) are:

  (1) *Endpoints of an interval.*

  (2) *Stationary Points*: $f'(c) = 0.$

  (3) *Singular Points*: $f'(c)$ *does not exist.*



Finding absolute extrema on $[a, b]$

1. Find all critical numbers for $f(x)$ in (a, b).
2. Evaluate $f(x)$ for all critical numbers in (a, b).
3. Evaluate $f(x)$ for the endpoints a and b of the interval $[a, b]$.
4. The largest value found in steps 2 and 3 is the absolute maximum for f on the interval $[a, b]$ and the smallest value found is the absolute minimum for f on $[a, b]$.

EXAMPLE :

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2,3]$.

Solution: We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values. The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are:

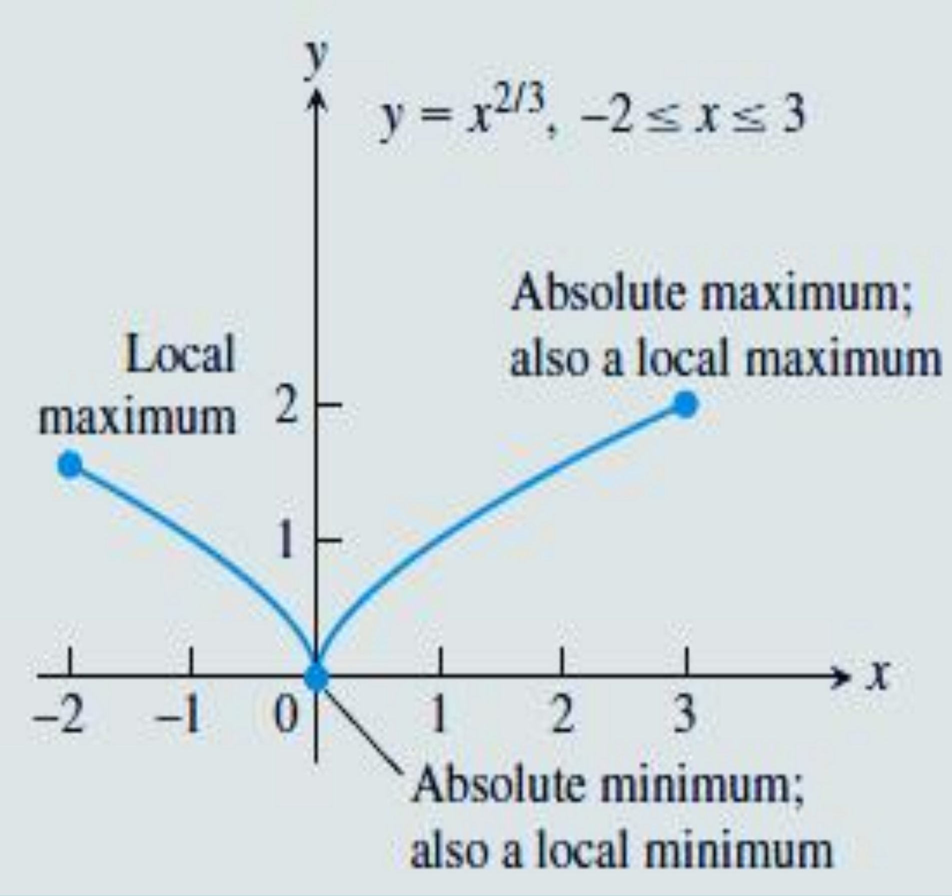
Critical point value:

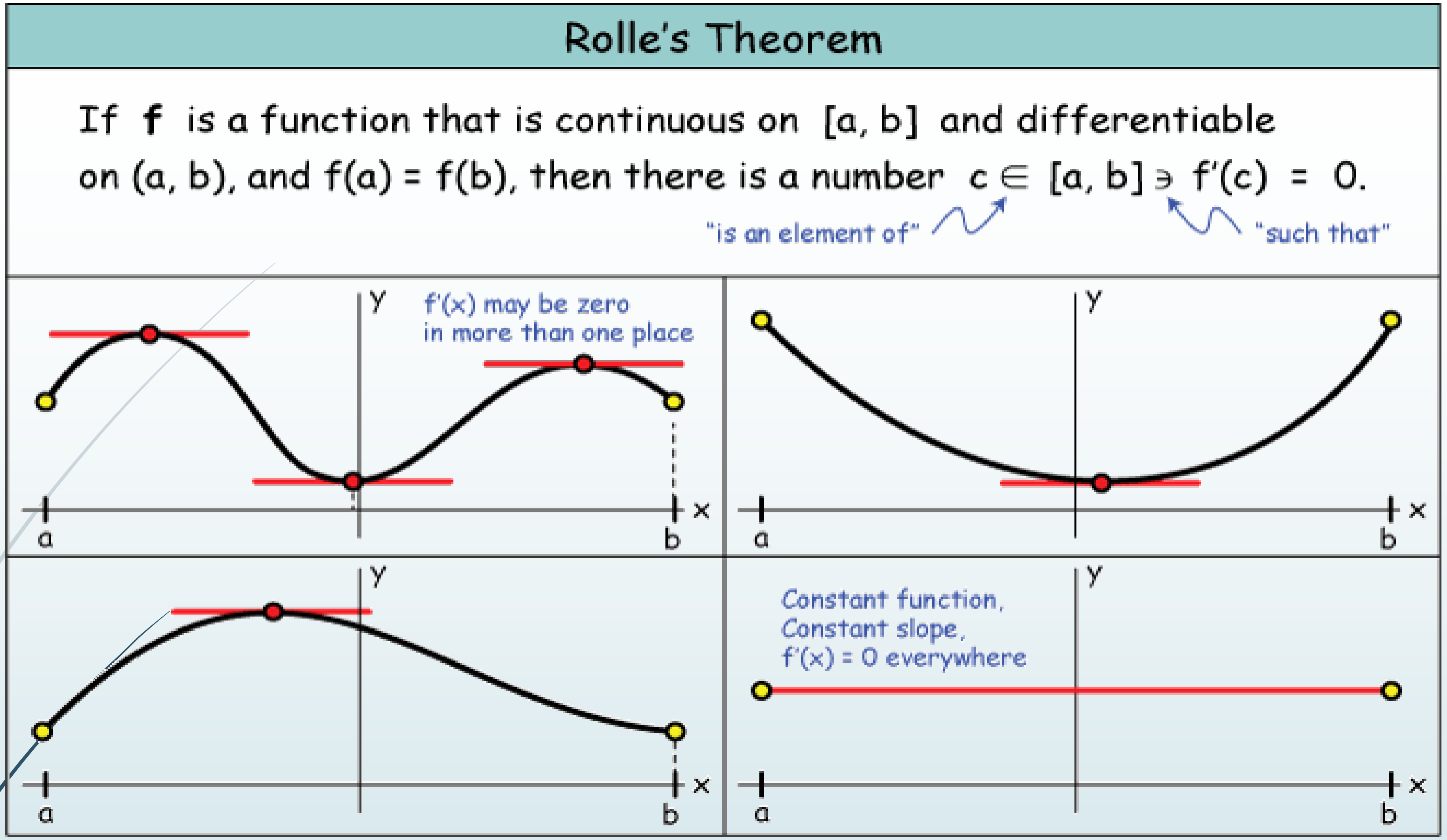
$$f(0) = 0, \quad \text{(Absolute minimum)}$$

Endpoint values:

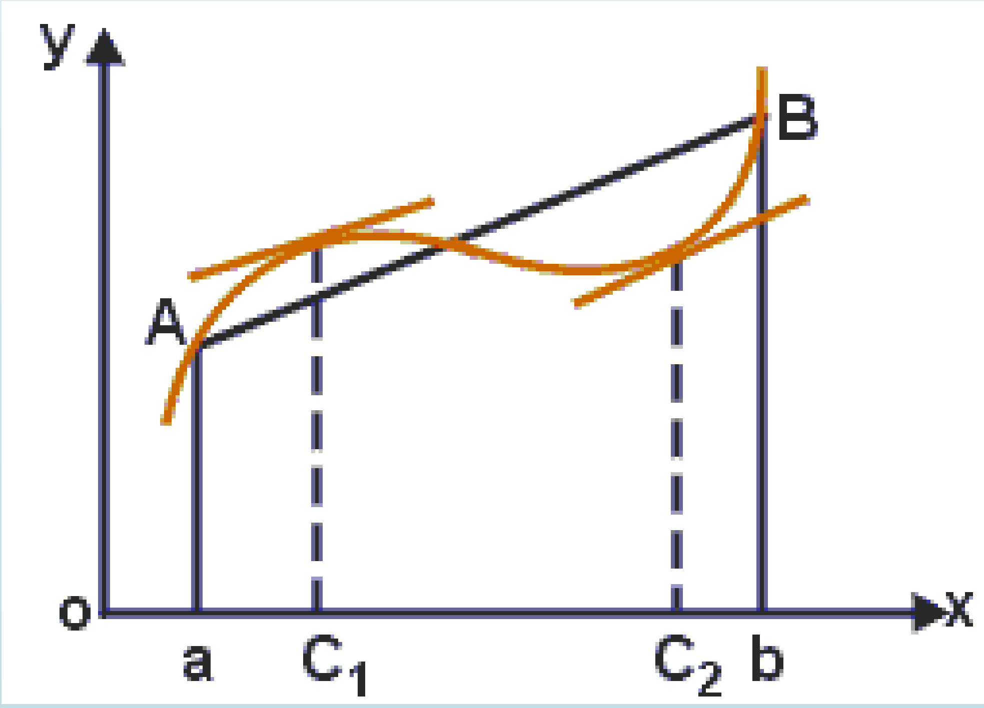
$$f(-2) = (-2)^{2/3} = \sqrt[3]{4}$$

$$f(3) = (3)^{2/3} = \sqrt[3]{9}. \quad \text{(Absolute maximum)}$$





4.2 Rolle's Theorem Mean Value Theorem



Rolle's Theorem

Recall theTheorem on Local Extrema

If $f(c)$ is a local extremum, then either f is not differentiable at c or $f'(c) = 0$.
We will use this to prove

Rolle's Theorem

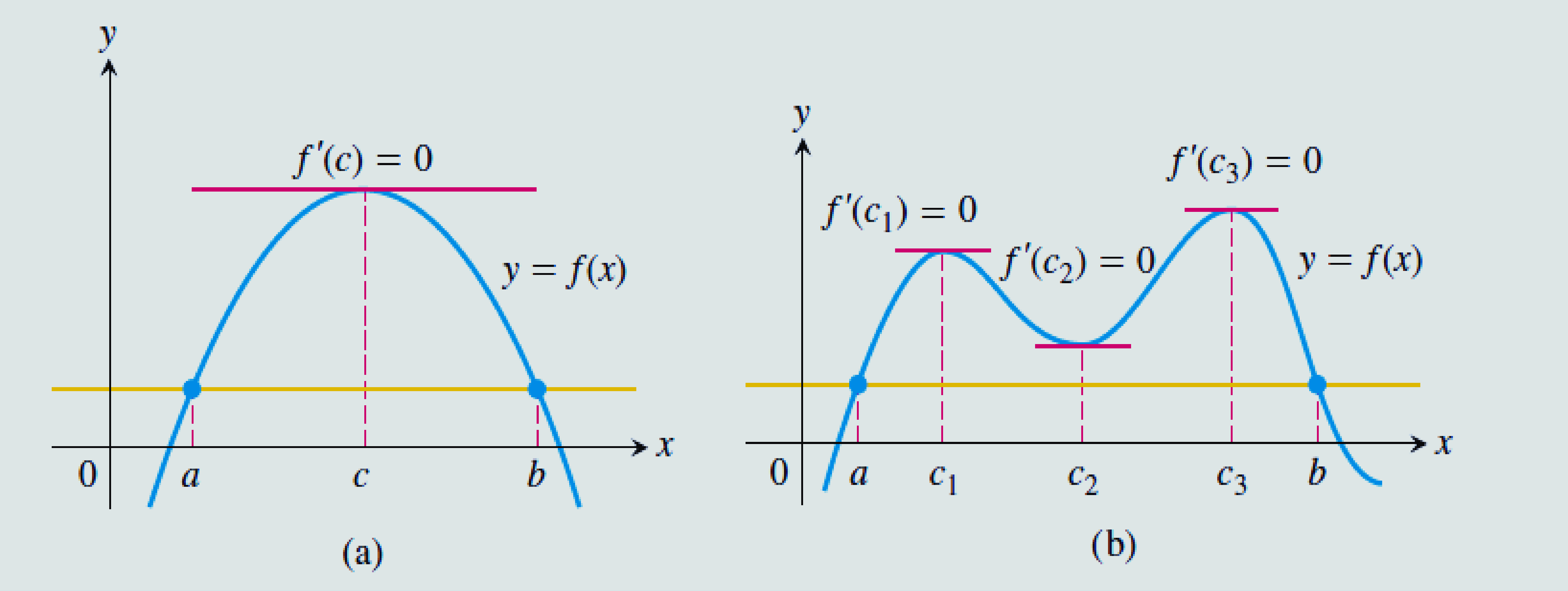
Let $a < b$.

1. If $f(x)$ is continuous on the closed interval $[a, b]$,
2. If $f(x)$ is differentiable on the open interval (a, b),
3. and $f(a) = f(b)$,

   then there is a c in (a, b) with $f'(c) = 0$.

That is, under these hypotheses, $f(x)$ has a horizontal tangent somewhere between a and b.

Rolle’s Theorem



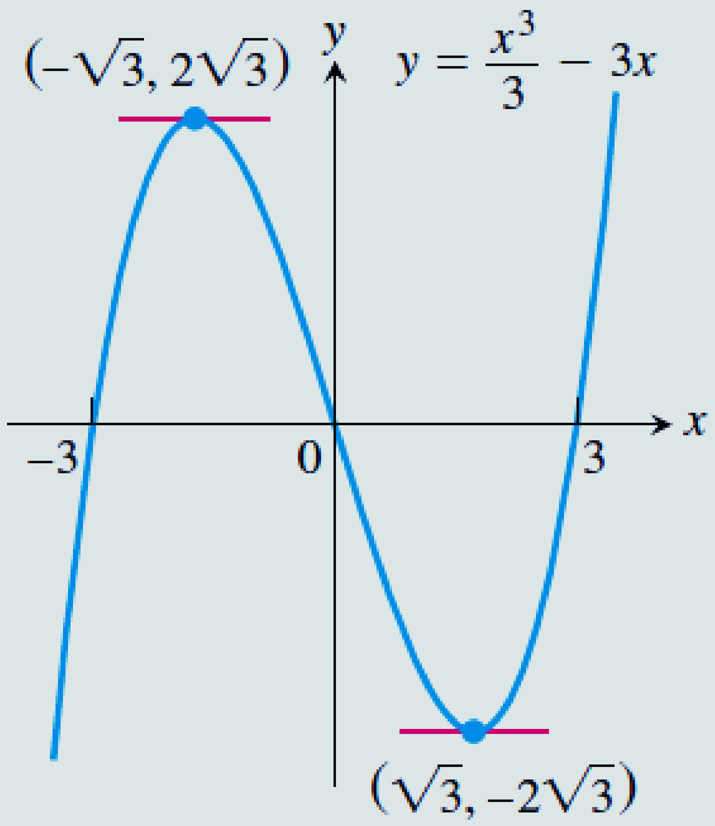
Rolle’s Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

EXAMPLE: Horizontal tangents of a cubic polynomial

The polynomial function

$$f(x) = \frac{x^3}{3} - 3x$$

is continuous at every point of $[-3, 3]$ and is differentiable at every point of $(-3, 3)$. Since $f(-3) = f(3) = 0$, Rolle's Theorem says that f' must be zero at least once in the open interval between $a = -3$ and $b = 3$. In fact, $f'(x) = x^2 - 3$ is zero twice in this interval, once at $x = -\sqrt{3}$ and again at $x = \sqrt{3}$.



Mean Value Theorem

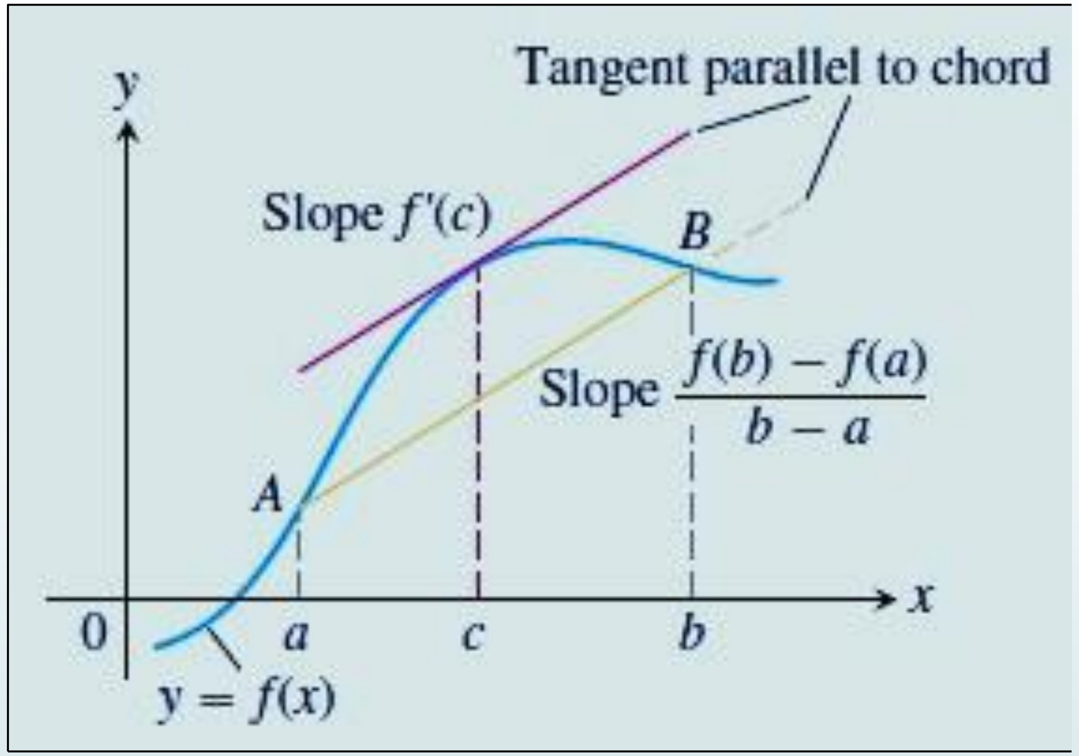
Let $a < b$.

1. If $f(x)$ is continuous on the closed interval $[a, b]$,
2. If $f(x)$ is differentiable on the open interval (a, b)

then there is a c in (a, b) with

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

That is, under appropriate smoothness conditions the slope of the curve at some point between a and b is the same as the slope of the line joining $(a, f(a))$ to $(b, f(b))$.



Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

Example:

We illustrate The Mean Value Theorem by considering $f(x) = x^3$ on the interval [1,3]. f is a polynomial and so continuous everywhere. For any x we see that $f'(x) = 3x^2$. So f is continuous on $[1, 3]$ and differentiable on $(1, 3)$. Thus,

$$\frac{f(b) - f(a)}{b - a} = f'(c) \Longrightarrow 3c^2 = \frac{f(3) - f(1)}{3 - 1} = \frac{27 - 1}{2} = 13.$$

So we seek a c in [1,3] with $3c^2 = 13$.

$$\Longrightarrow 3c^2 = 13 \Longrightarrow c^2 = \frac{13}{3} \Longrightarrow c = \pm\sqrt{\frac{13}{3}}.$$

But $-\sqrt{\frac{13}{3}} \notin [1,3]$. Thus, $c = \sqrt{\frac{13}{3}}$.

A Physical Interpretation

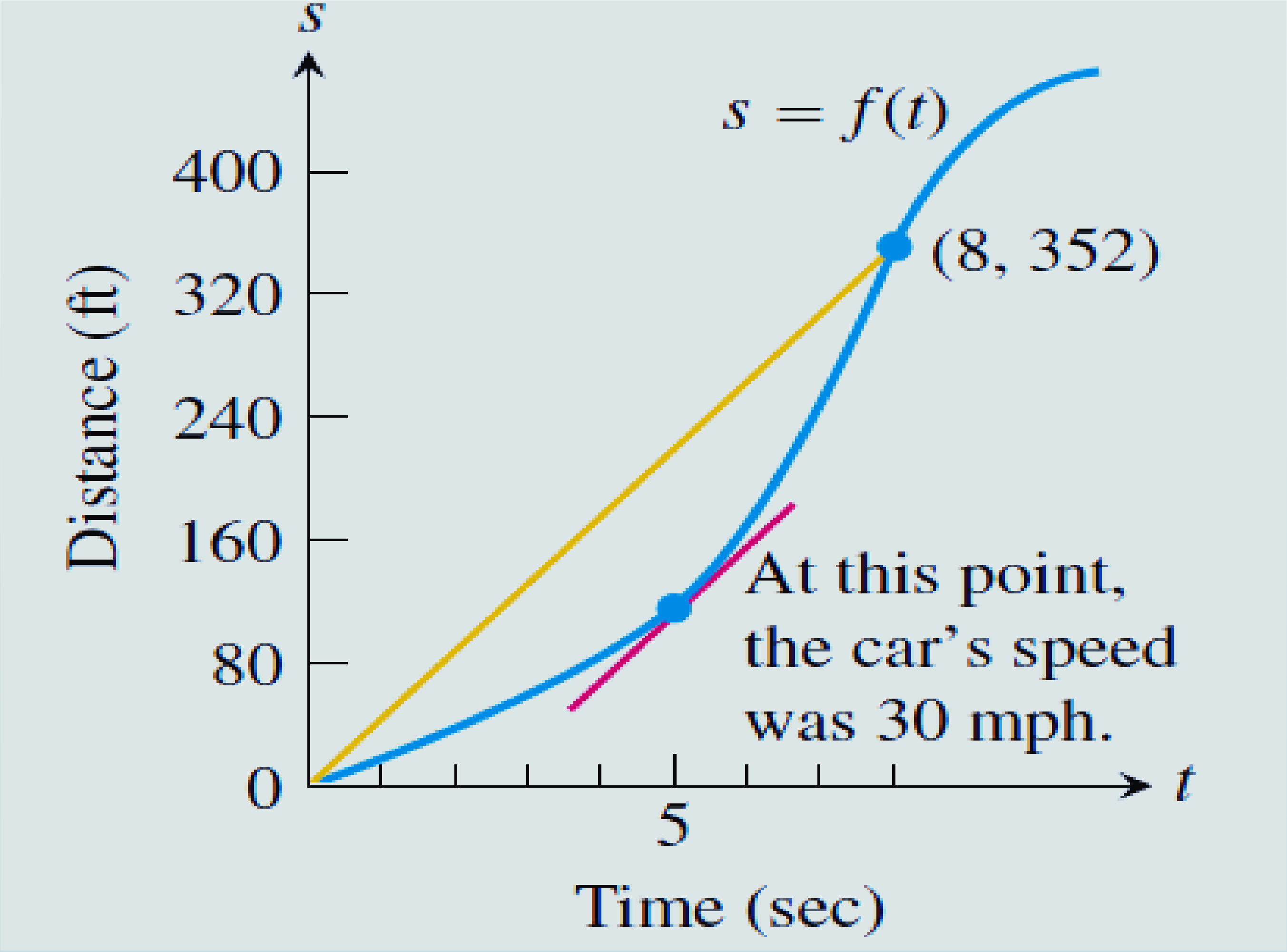
If we think of the number

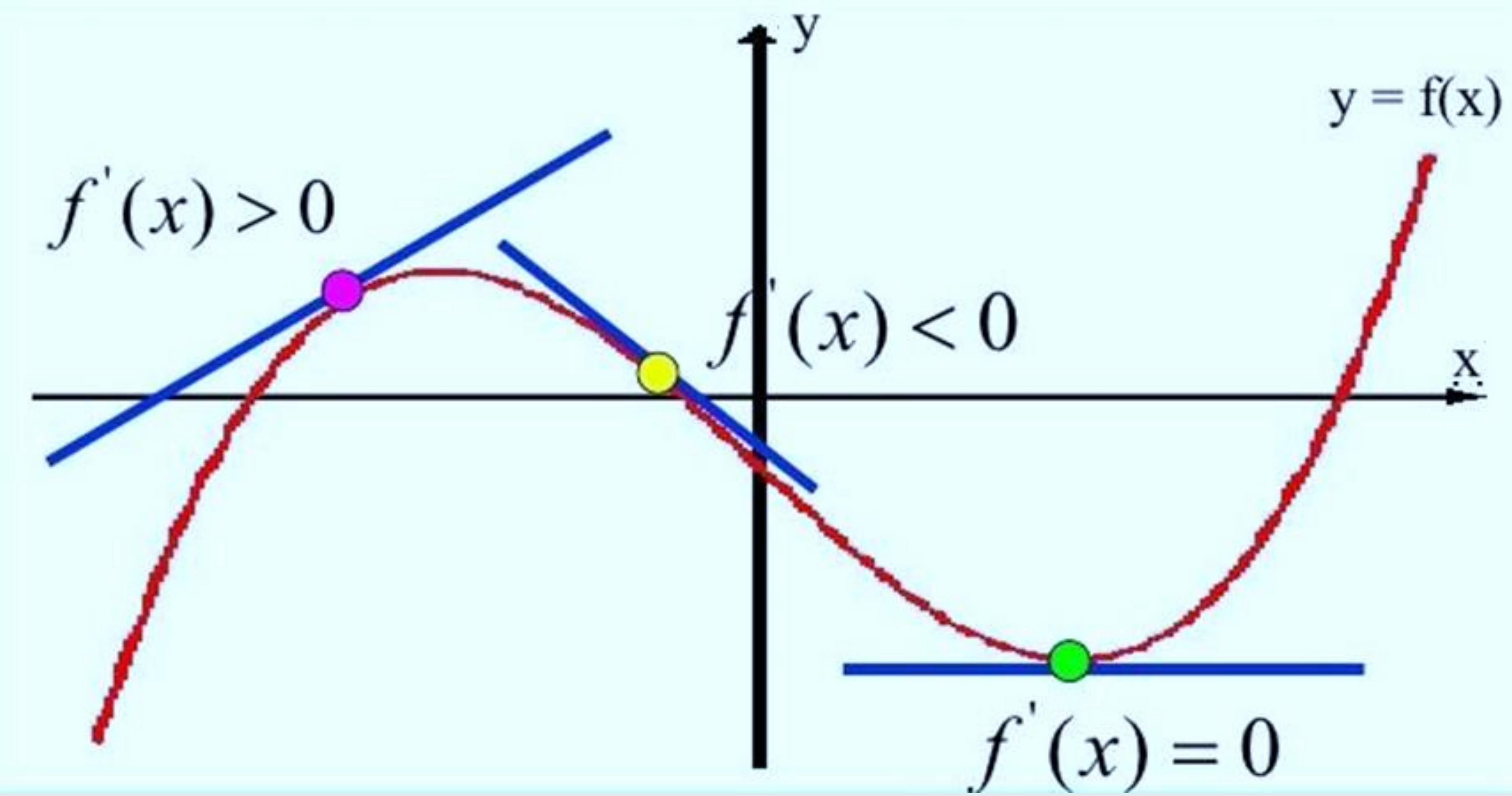
$$\frac{f(b) - f(a)}{b - a}$$

as the average change in $f(x)$ over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

Example

If a car accelerating from zero takes $8\,\text{sec}$ to go $352\,ft$, its average velocity for the $8-\text{sec}$ interval is $\frac{352}{8} = 44\frac{ft}{sec}$. At some point during the acceleration, the Mean Value Theorem says, the speedometer must read exactly $30\,mph$ $\left(44\frac{ft}{sec}\right)$.





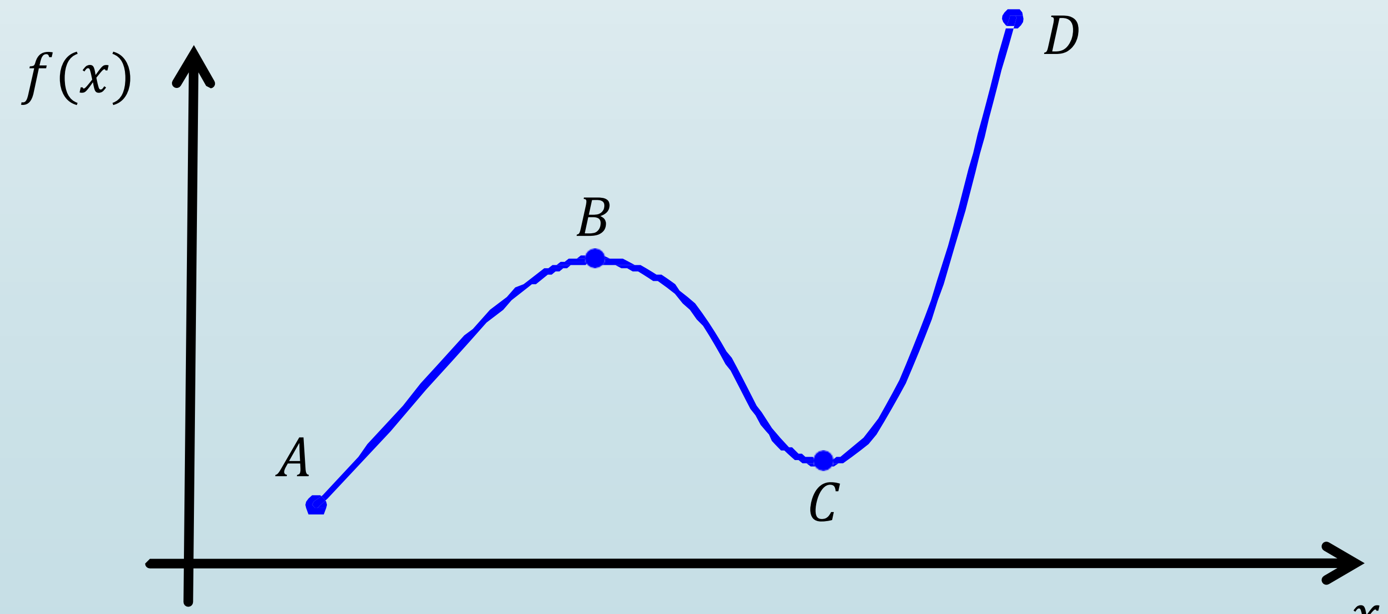
$f'(x) > 0$	$\longrightarrow$	Function increasing
$f'(x) < 0$	$\longrightarrow$	Function decreasing
$f'(x) = 0$	$\longrightarrow$	Stationary Point

4.3 Increasing and Decreasing Functions and the First Derivative Test

Increasing and Decreasing Functions

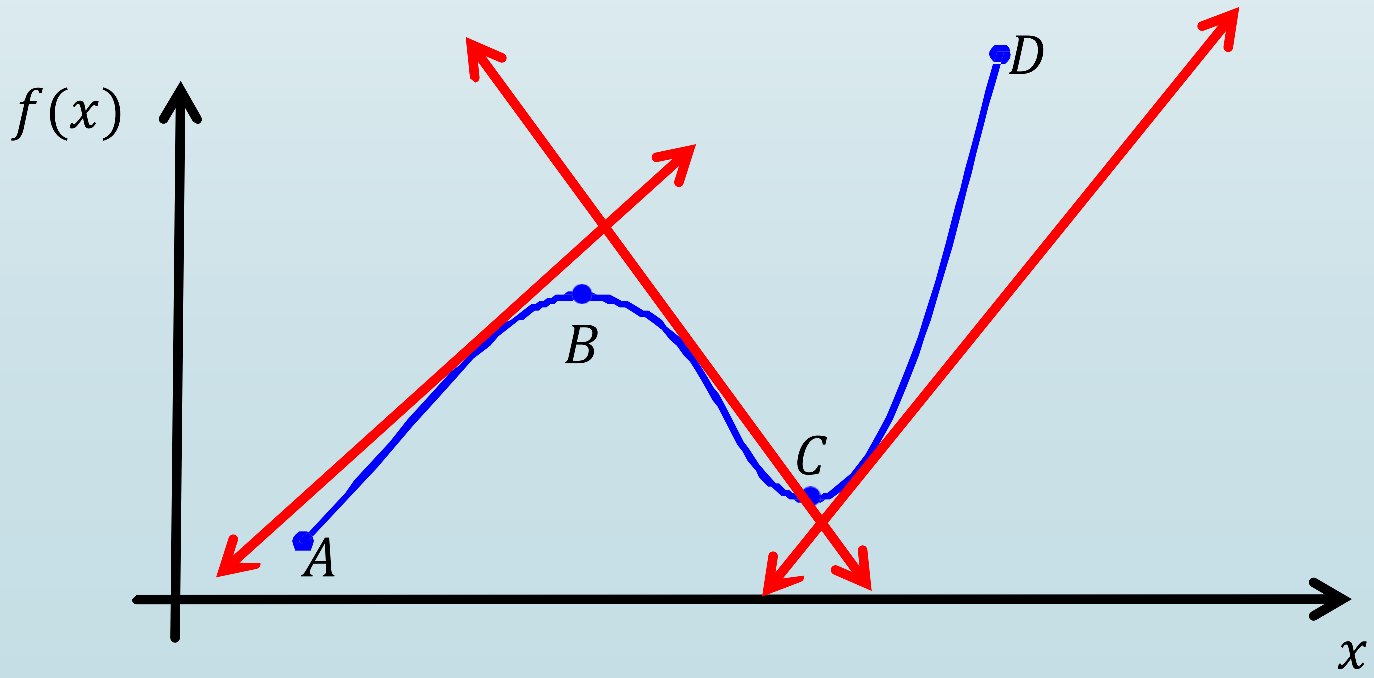
A function $f(x)$ is **strictly increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

A function $f(x)$ is **strictly decreasing** on an interval I if $f(x_1) > f(x_2)$ Whenever $x_1 < x_2$.



How the Derivative is connected to Increasing/Decreasing Functions

- When the function is increasing, what is the sign $(+ \text{ or } -)$ of the slopes of the tangent lines? POSITIVE Slope
- When the function is decreasing, what is the sign $(+ \text{ or } -)$ of the slopes of the tangent lines? NEGATIVE Slope



First Derivative Test for Increasing and Decreasing Functions

Let $f(x)$ be differentiable on the open interval (a, b)

If $f'(x) > 0$ for each value of x in an interval (a, b), then $f(x)$ is increasing on (a, b).

If $f'(x) < 0$ for each value of x in an interval (a, b), then $f(x)$ is decreasing on (a, b).

If $f'(x) = 0$ for each value of x in an interval (a, b), then $f(x)$ is constant on (a, b).

Procedure for finding intervals on which a function is increasing or decreasing

If $f(x)$ is a continuous function on an open interval (a, b). To find the open intervals on which *f* is increasing or decreasing:

1. Find the critical points of $f(x)$ in (a, b).
2. Make a sign chart: The critical points, divide the $x-$axis into intervals. Test the sign $(+ \text{ or } -)$ of the **derivative** inside each of these intervals.
3. If $f'(x) > 0$ in an interval, then $f(x)$ is increasing in that interval.
4. If $f'(x) < 0$ in an interval, then $f(x)$ is decreasing in that interval.

Example:

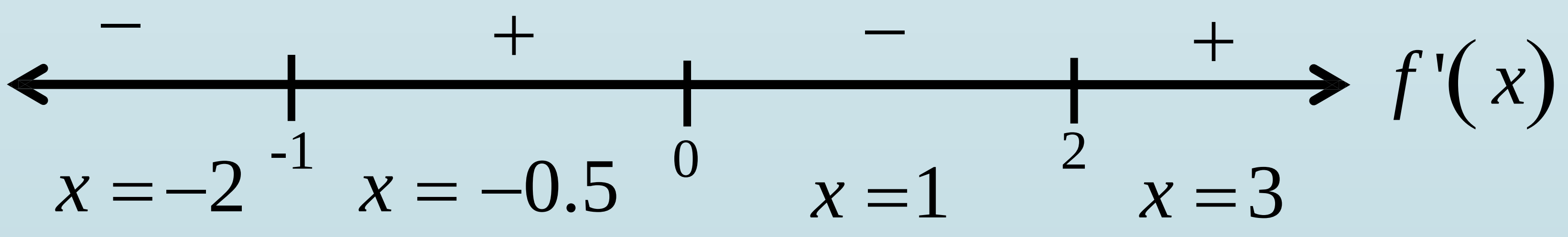
Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution: $f(x)$ is continuous and Domain of $f(x)$ is the set of all Real numbers.

1. Find the critical points: Calculate the derivative and determine where the derivative is 0 or undefined

$$f'(x) = 12x^3 - 12x^2 - 24x = 0$$
$$\Rightarrow 12x(x^2 - x - 2) = 0$$
$$\Rightarrow 12x(x-2)(x+1) = 0$$
$$\Rightarrow x = 0, 2, -1$$

2. Find the sign of the derivative on each interval:



$$f'(-2) = -96$$
$$f'(-0.5) = 7.5$$
$$f'(1) = -24$$
$$f'(3) = 144$$

- The function is increasing on:

$$(-1,0) \cup (2, \infty)$$

because the **first** derivative is positive on this interval.

- The function is decreasing on:

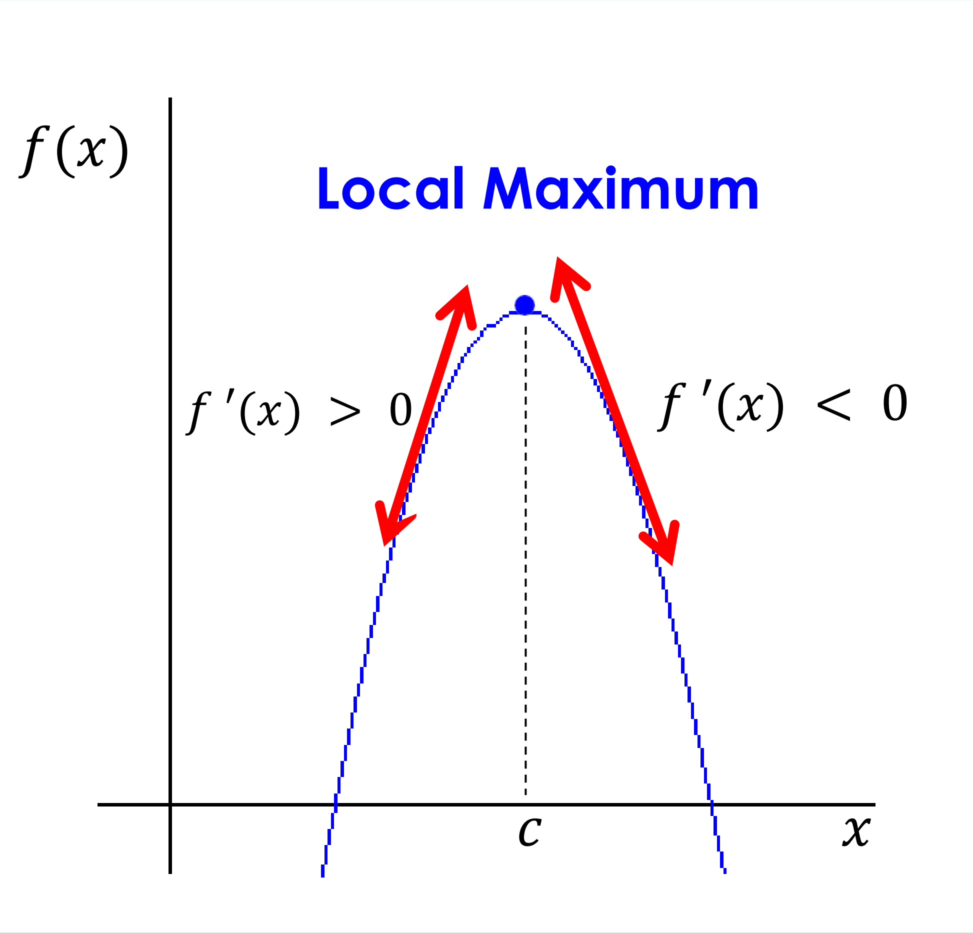
$$(-\infty, -1) \cup (0,2)$$

because the **first** derivative is negative on this interval.

The First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function $f(x)$.

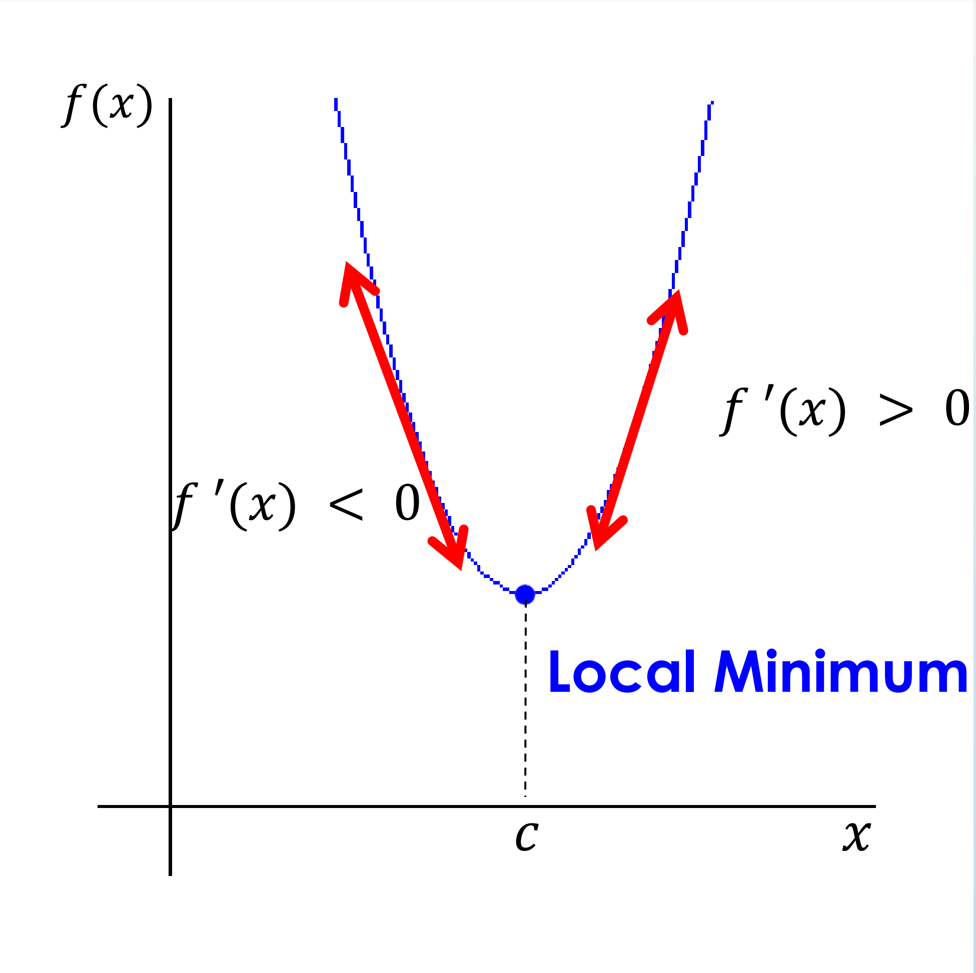
a) If $f\,'(x)$ changes sign from positive to negative at c, then $f(x)$ has a local maximum at c.



The First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function $f(x)$.

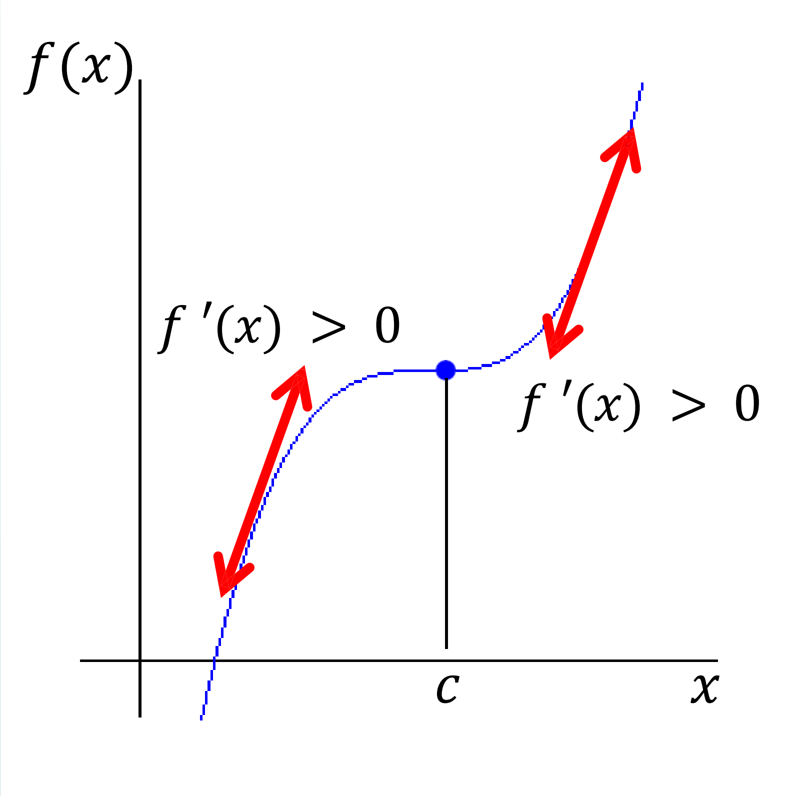
b) If $f'(x)$ changes sign from negative to positive at c, then $f(x)$ has a local minimum at c.



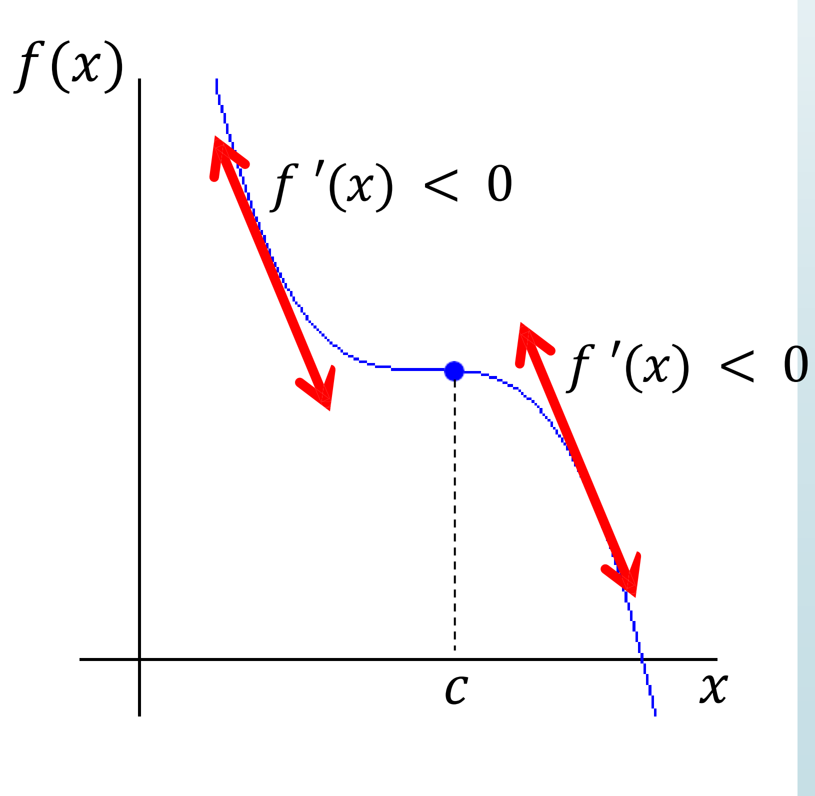
The First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function $f(x)$.

c) If $f'(x)$ does not changes sign at c (i.e., $f'(x)$ is positive on both sides of c or it is negative on both sides), then $f(x)$ has no local maximum or minimum at c. minimum at c.



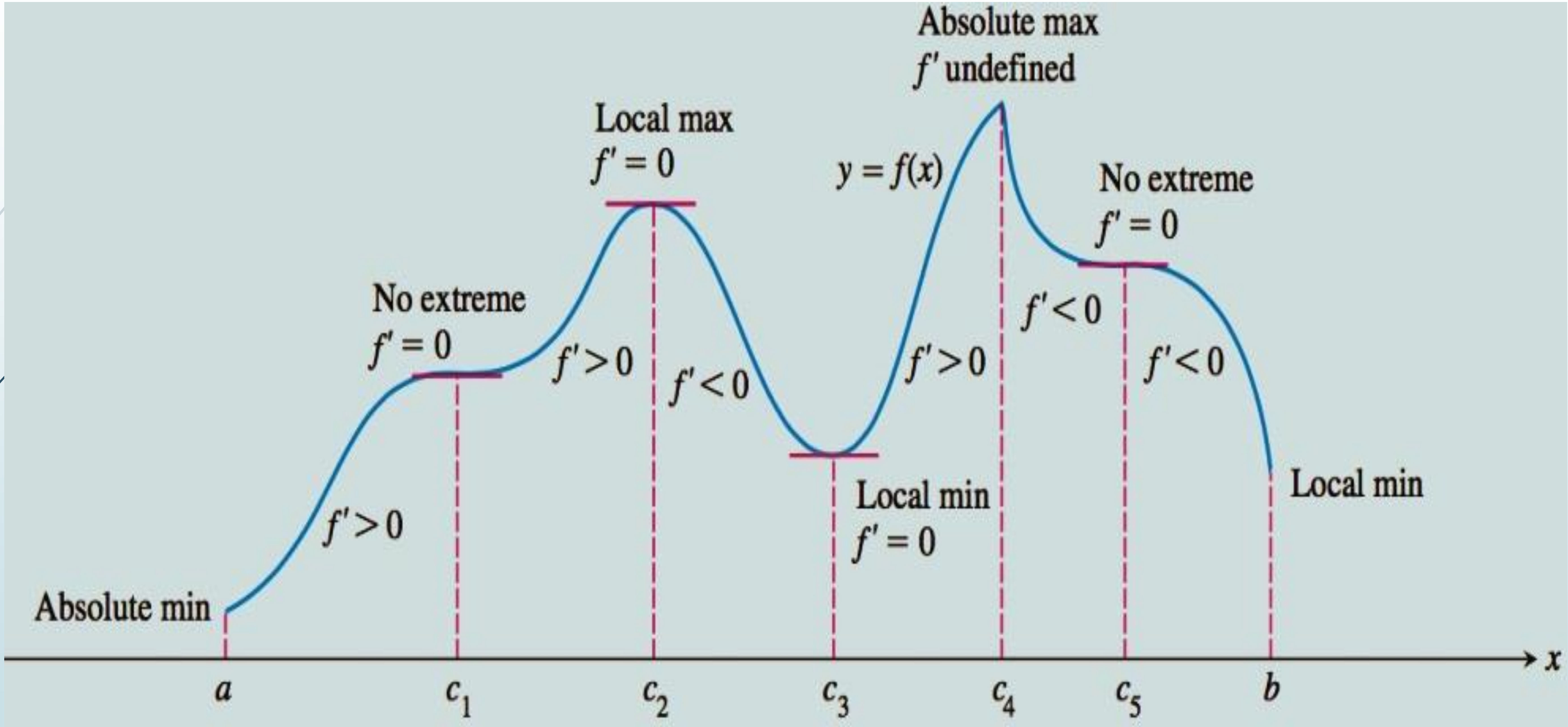
No Local Maximum or Minimum



The First Derivative Test

Determine the sign of the derivative of $f(x)$ to the left and right of the critical point.

left	right	conclusion
$+$	$-$	$f(c)$ is a relative maximum
$-$	$+$	$f(c)$ is a relative minimum
No change		No relative extremum



Absolute max
f′ undefined
Local max
f′ = 0
y = f(x)
No extreme
f′ = 0
No extreme
f′ = 0
f′ > 0
f′ < 0
f′ < 0
f′ > 0
f′ < 0
f′ > 0
Local min
f′ = 0
Local min
Absolute min
x
a
c1
c2
c3
c4
c5
b

Example: Find all the relative extrema of

$$f(x) = x^3 - 6x^2 + 1$$

$$\Longrightarrow \; f'(x) = 3x^2 - 12x = 0$$

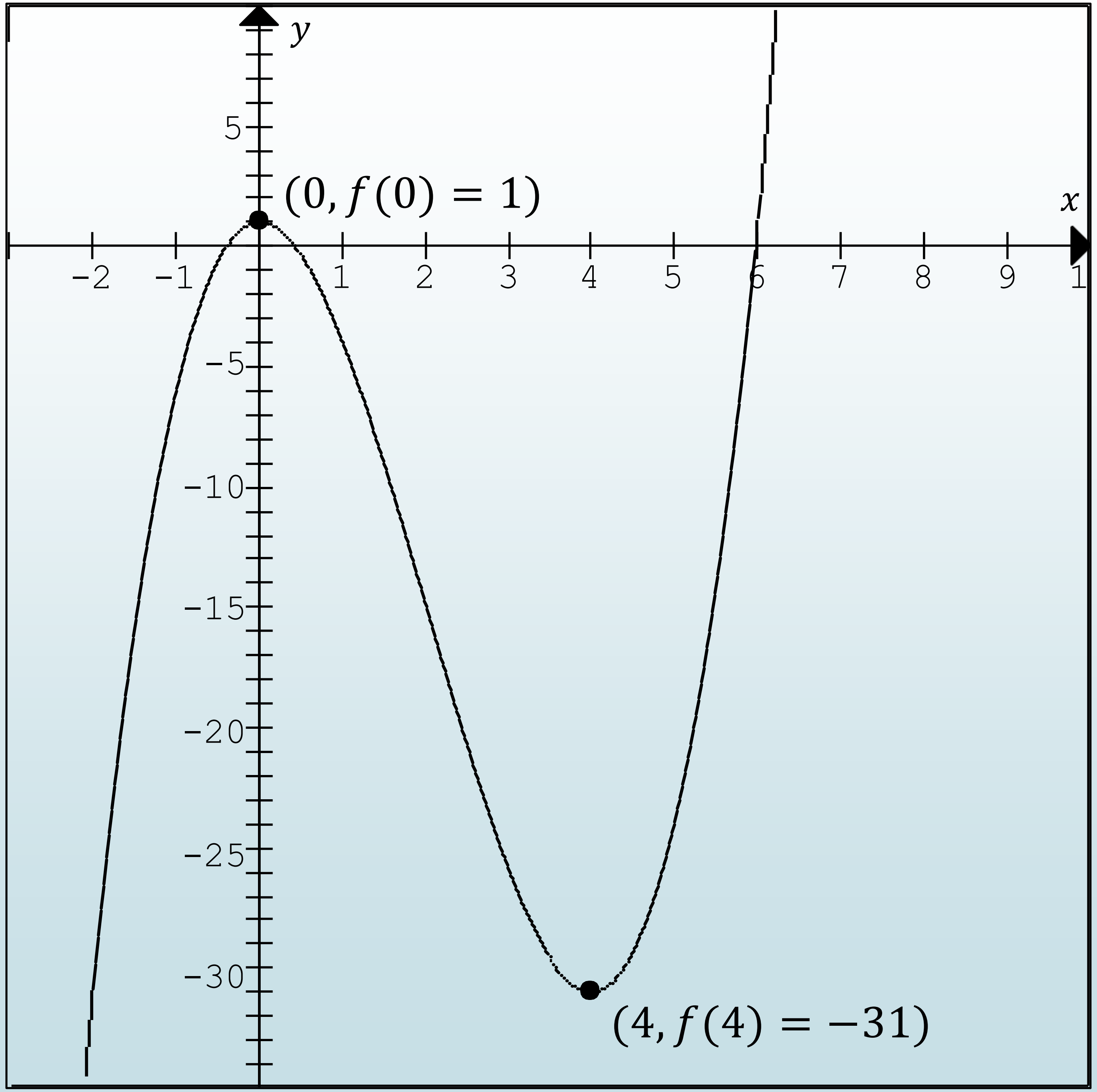
Stationary points: $x = 0,4$

Singular points: None

Relative max.
$f(0) = 1$

Relative min.
$f(4) = -31$

		0		4	
f'	$+$	0	$-$	0	$+$
f	$\nearrow$		$\searrow$		$\nearrow$



y
x
5
−5
−10
−15
−20
−25
−30
−2
−1
1
2
3
4
5
6
7
8
9
1
$(0, f(0) = 1)$
$(4, f(4) = -31)$

Example:

Find all the relative extrema of

$$f(x) = \sqrt[3]{x^3 - 3x}$$

$$\Longrightarrow \ f'(x) = \frac{x^2 - 1}{\sqrt[3]{x^3 - 3x}}$$

Stationary points: $x = \pm 1$

Singular points: $x = 0, \pm\sqrt{3}$

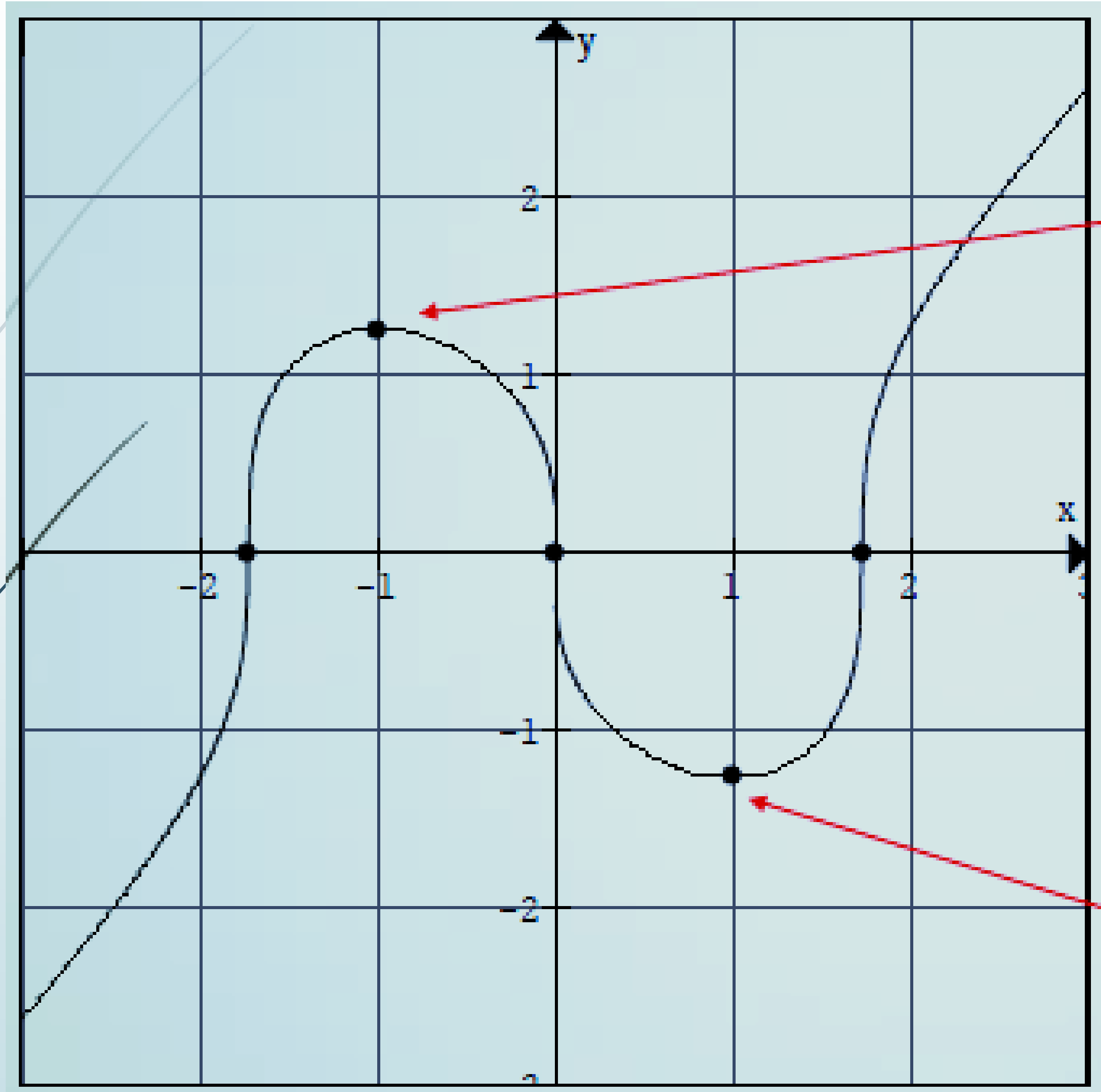
Stationary points: $x = \pm 1$

Singular points: $x = 0, \pm\sqrt{3}$

Relative max.
$f(-1) = \sqrt[3]{2}$

Relative min.
$f(1) = -\sqrt[3]{2}$

f'	+	ND	+	0	−	ND	−	0	+	ND	+
		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f	↗		↗		↘		↘		↗		↗



Local max. $f(-1) = \sqrt[3]{2}$

$$f(x) = \sqrt[3]{x^3 - 3x}$$

Local min. $f(1) = -\sqrt[3]{2}$

Domain Not a Closed Interval

Example: Find the absolute extrema of $f(x) = \frac{1}{(x-2)}$ on $[3, \infty)$

Solution:

$$f(x) = \frac{1}{(x-2)}$$

$$\Longrightarrow \; f'(x) = \frac{-1}{(x-2)^2}$$

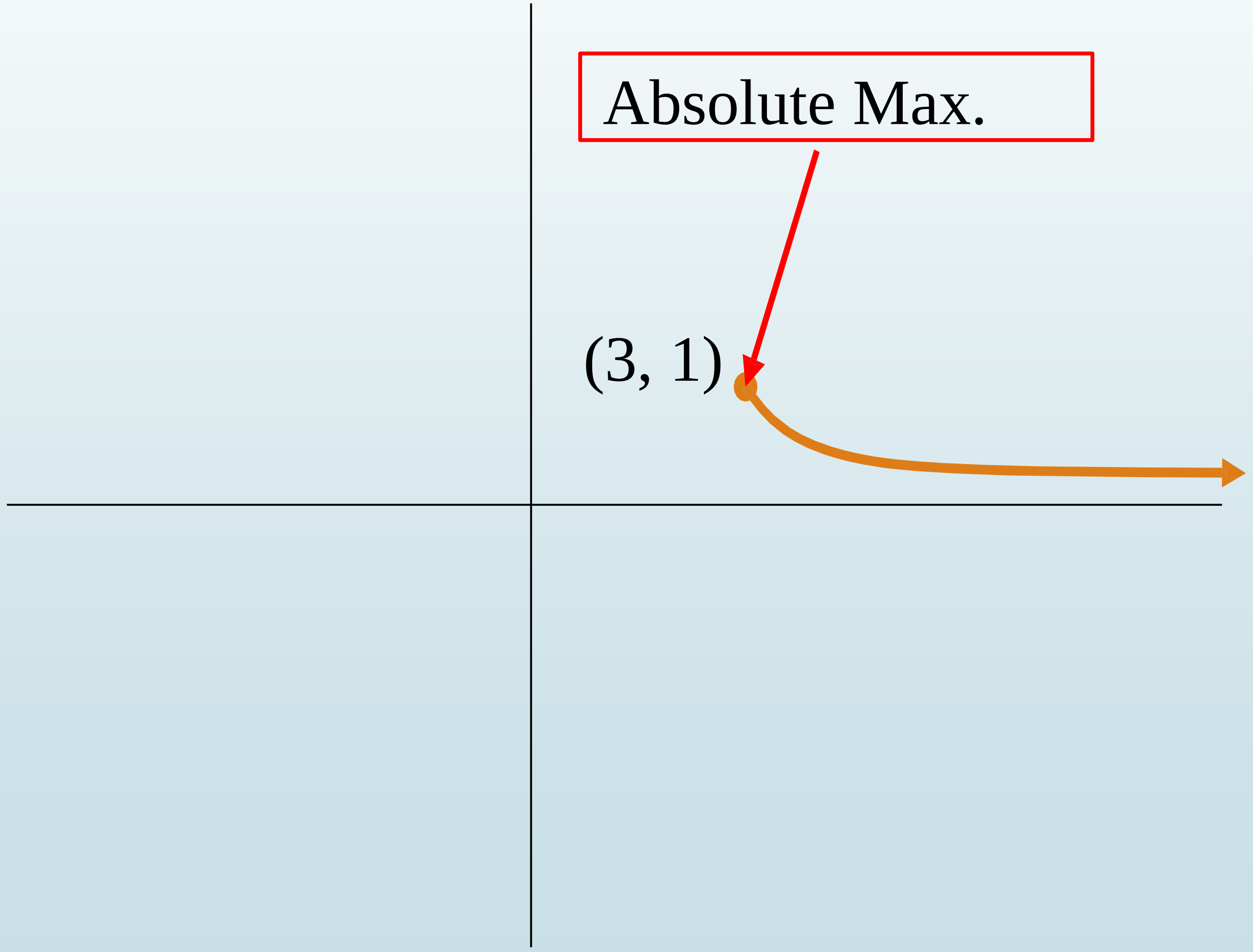
Singular point: $x = 2$ (Not a critical point)

At end point: $x = 3$

$$f'(3) = \frac{-1}{(3-2)^2} < 0 \qquad \text{Decreasing}$$

and

$$f(3) = 1 \qquad \text{Absolute Max.}$$



Practice Questions

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- **Chapter: 4**
  - **Exercise: 4.2**

    Q # 1 – 11.
  - **Exercise: 4.3**

    Q # 1 – 36.