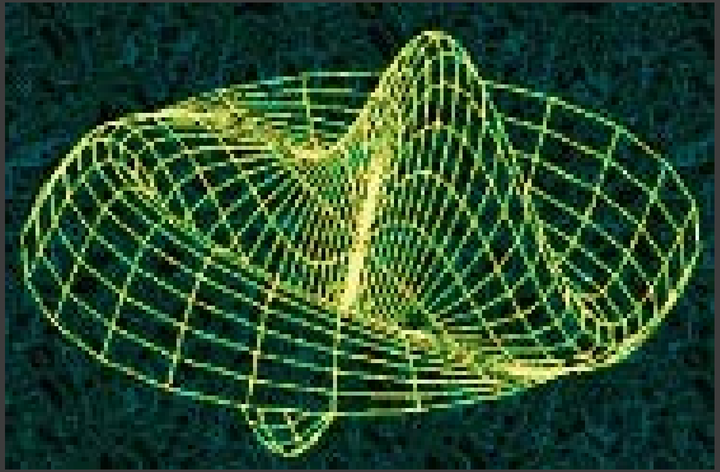
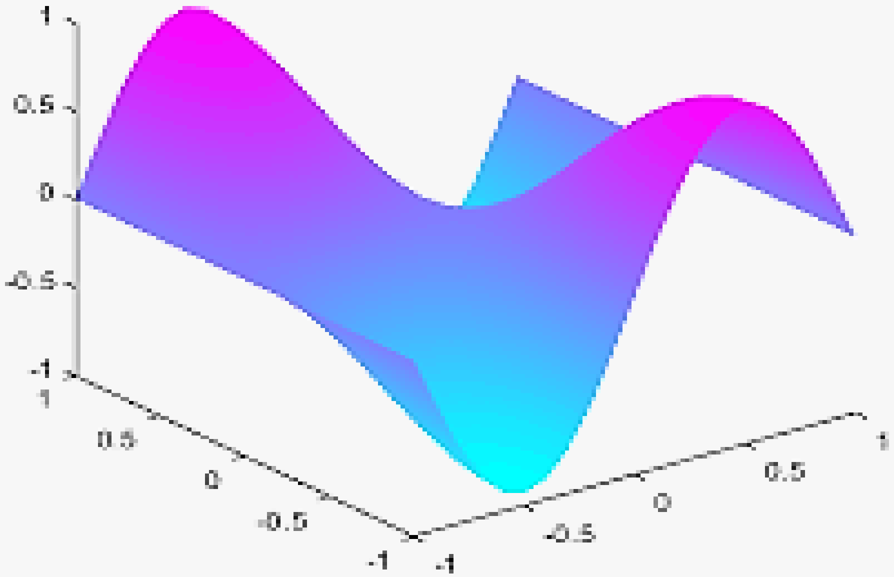
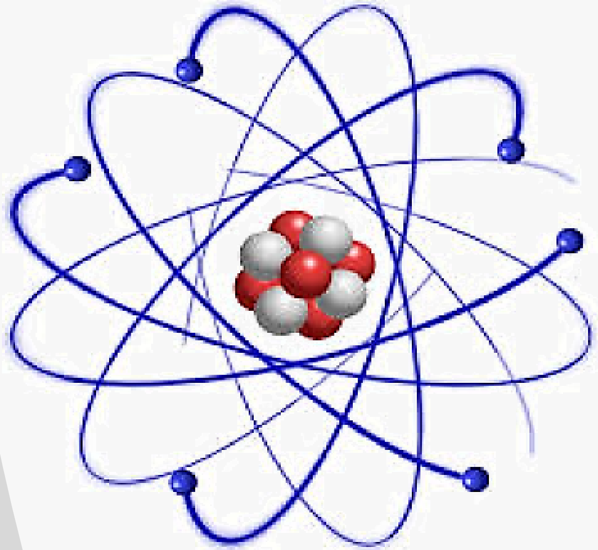
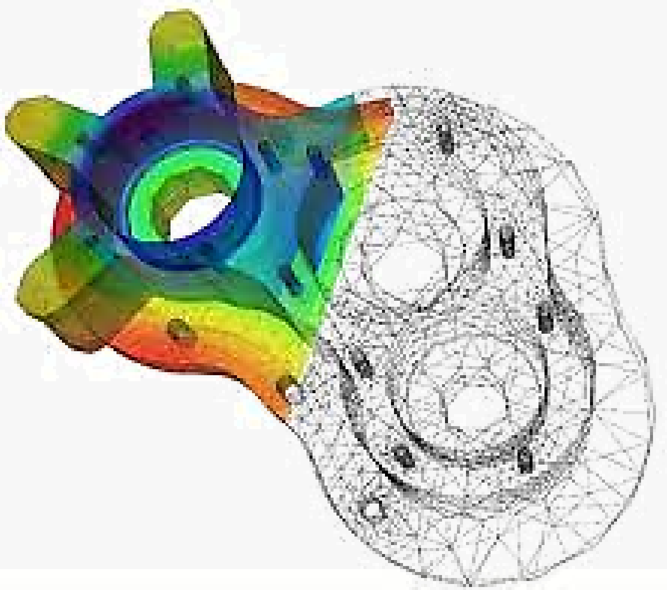


Partial Differential Equations







Vector Calculus(MATH-243)
Instructor: Dr. Naila Amir

Partial Differential Equations

Book: Applied Partial Differential Equations With Fourier series and boundary value problems by Richard Haberman

- Chapter: 2
  - Sections: 2.5

Laplace's Equation in Rectangular Coordinates

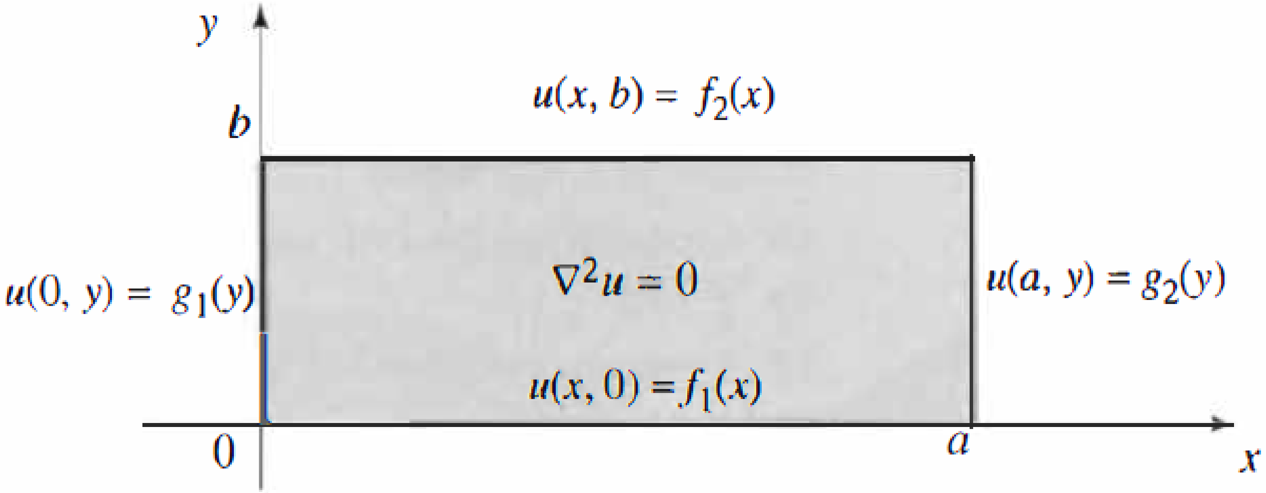
We have already seen that the steady-state temperature distributions associated with the one-dimensional heat equation:

$$u_t = c^2 u_{xx}\,, \qquad 0 < x < \pi, \qquad t > 0\,,$$

satisfy, since $u_{xx} = 0$ steady-state solutions are time independent. The equation $u_{xx} = 0$ is easily solved and yields only linear solutions $u(x) = c_1\,x + c_2$. For steady-state or time independent problems in two-dimensions over a rectangle $a \times b$, we consider the equation:

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a, \qquad 0 < y < b.$$

This equation is known as **Laplace's equation** in two variables and is obtained by setting the time derivative equal to zero in the heat equation in two-dimensions.



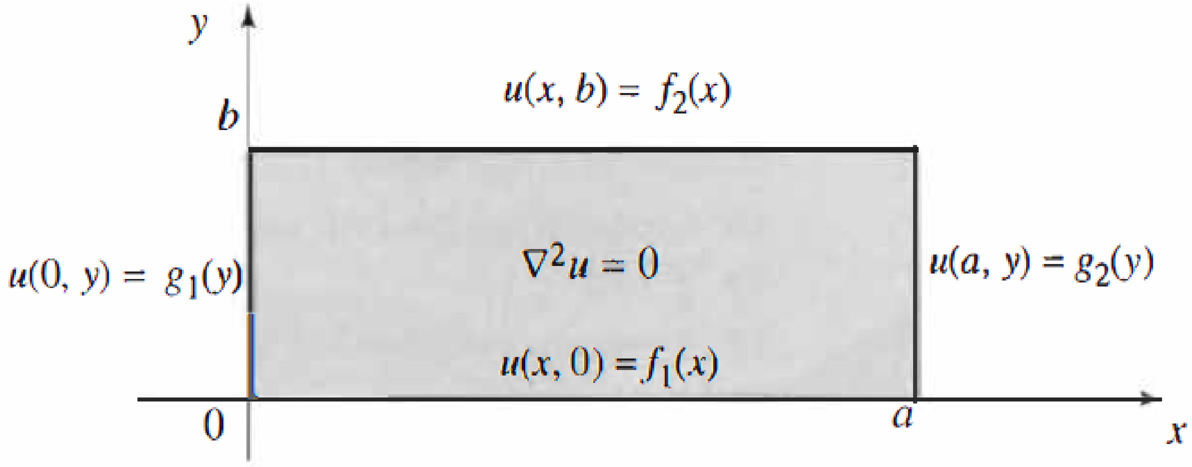
Laplace's Equation in Rectangular Coordinates

Laplace's equation has a wide variety of solutions. In a typical problem, the solution that we seek will be determined by the given boundary conditions. More specifically, we impose the boundary conditions:

$$u(x,0) = f_1(x), \qquad u(x,b) = f_2(x), \qquad 0 < x < a,$$

$$u(0,y) = g_1(y), \qquad u(a,y) = g_2(y), \qquad 0 < y < b.$$

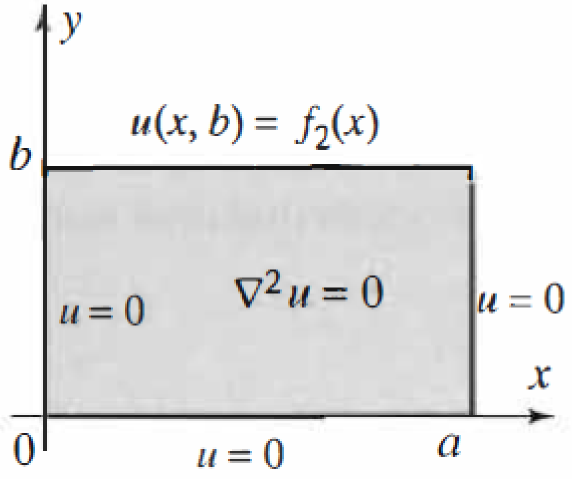
A problem consisting of Laplace's equation on a region in the plane together with specified boundary values is called a **Dirichlet problem.** Thus, the problem we described above is a Dirichlet problem on a rectangle. Rather than attacking this problem **in** its full generality, we will start by solving the special case when f_1, g_1 and g_2 are all zero.



A Dirichlet problem on a rectangle

Solve the boundary value problem described in the accompanying figure using the method of separation of variables.

Solution:



We are required to determine solution of the BVP:

(1) $\nabla^2 u = u_{xx} + u_{yy} = 0, \qquad 0 < x < a, \qquad 0 < y < b.$

(2) $u(x, 0) = 0, \qquad u(x, b) = f_2(x), \qquad 0 < x < a,$

(3) $u(0, y) = 0, \qquad u(a, y) = 0, \qquad 0 < y < b.$

We begin by looking for product solution $u(x, y) = X(x)Y(y)$. Substituting into (1) and using the separation method, we arrive at the equations:

$$X''(x) + \alpha\, X(x) = 0, \qquad (4)$$

$$Y''(y) - \alpha\, Y(y) = 0, \qquad (5)$$

where α is the separation constant, with the boundary conditions:

$$X(0) = 0, \qquad X(a) = 0, \qquad \text{and} \qquad Y(0) = 0. \qquad (6)$$

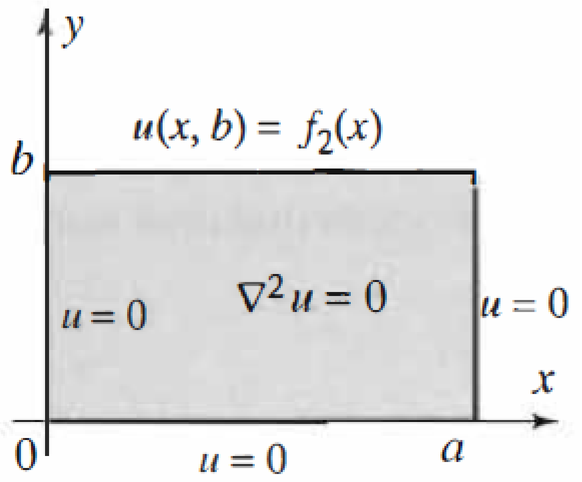
Solution: A Dirichlet problem on a rectangle

The solution of the Dirichlet problem described in the figure is therefore given by:

$$u(x,y) = \sum_{k=1}^{\infty} u_k(x,y) = \sum_{k=1}^{\infty} E_k \sin\left(\frac{k\pi x}{a}\right) \sinh\left(\frac{k\pi y}{a}\right), \qquad (*)$$

where the constants E_k can be determined as:

$$E_k = \frac{2}{a \sinh\left(\frac{k\pi b}{a}\right)} \int_0^a f_2(x) \sin\left(\frac{k\pi x}{a}\right) dx, \qquad k = 1,2,3, \dots . \qquad (**)$$



Example: Steady-state temperature in a square plate

Determine the steady-state temperature distribution in a 1×1 square plate where one side is held at $100°$ and the other three sides are held at $0°$.

Solution:

We are required to determine solution of the BVP:

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \qquad 0 < x < 1, \qquad 0 < y < 1.$$

$$u(x,0) = 0, \qquad u(x,1) = 100, \qquad 0 < x < 1,$$

$$u(0,y) = 0, \qquad u(1,y) = 0, \qquad 0 < y < 1.$$

For the present case $a = b = 1$ and $f_2(x) = 100$. Thus, from $(*)$ and $(**)$ we have:

$$u(x,y) = \sum_{k=1}^{\infty} E_k \sin(k\pi x)\ \sinh(k\pi y)\,,$$

and

$$E_k = \frac{200}{\sinh(k\pi)} \int_0^1 \sin(k\pi x)\,dx = \frac{200(1-\cos k\pi)}{k\pi \sinh(k\pi)}.$$

Solution: Steady-state temperature in a square plate

Simplifying, we find the required solution as:

$$u(x,y) = \frac{400}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)\pi x \sinh(2k-1)\pi y}{(2k-1)\sinh(2k-1)\pi}.$$

Note that when $y = 1$, this reduces to the Fourier sine series expansion of 100, matching the boundary condition. Also, if $0 < y < 1$, the ratio of the hyperbolic sines decays exponentially with k and hence leads to rapid convergence of the series.

Unifying Power of Methods. Electrostatics, Elasticity

- The Laplace equation $\nabla^2 u = u_{xx} + u_{yy} = 0$ also governs the electrostatic potential of electrical charges in any region that is free of these charges. Thus, our steady-state heat problem can also be interpreted as an electrostatic potential problem. Then $(*), (**)$ is the potential in the rectangle R when the upper side of R is at potential and the other three sides are grounded.

- In the steady-state case, the two-dimensional wave equation:
$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right),$$
also reduces to the Laplace equation $\nabla^2 u = u_{xx} + u_{yy} = 0$. In that case $(*), (**)$ is the displacement of a rectangular elastic membrane (rubber sheet, drumhead) that is fixed along its boundary, with three sides lying in the xy −plane and the fourth side given the displacement.

- This is an impressive demonstration of the ***unifying power*** of mathematics. It illustrates that ***entirely different physical systems may have the same mathematical model*** and can thus be treated by the same mathematical methods.

Dirichlet Problem in a Rectangle

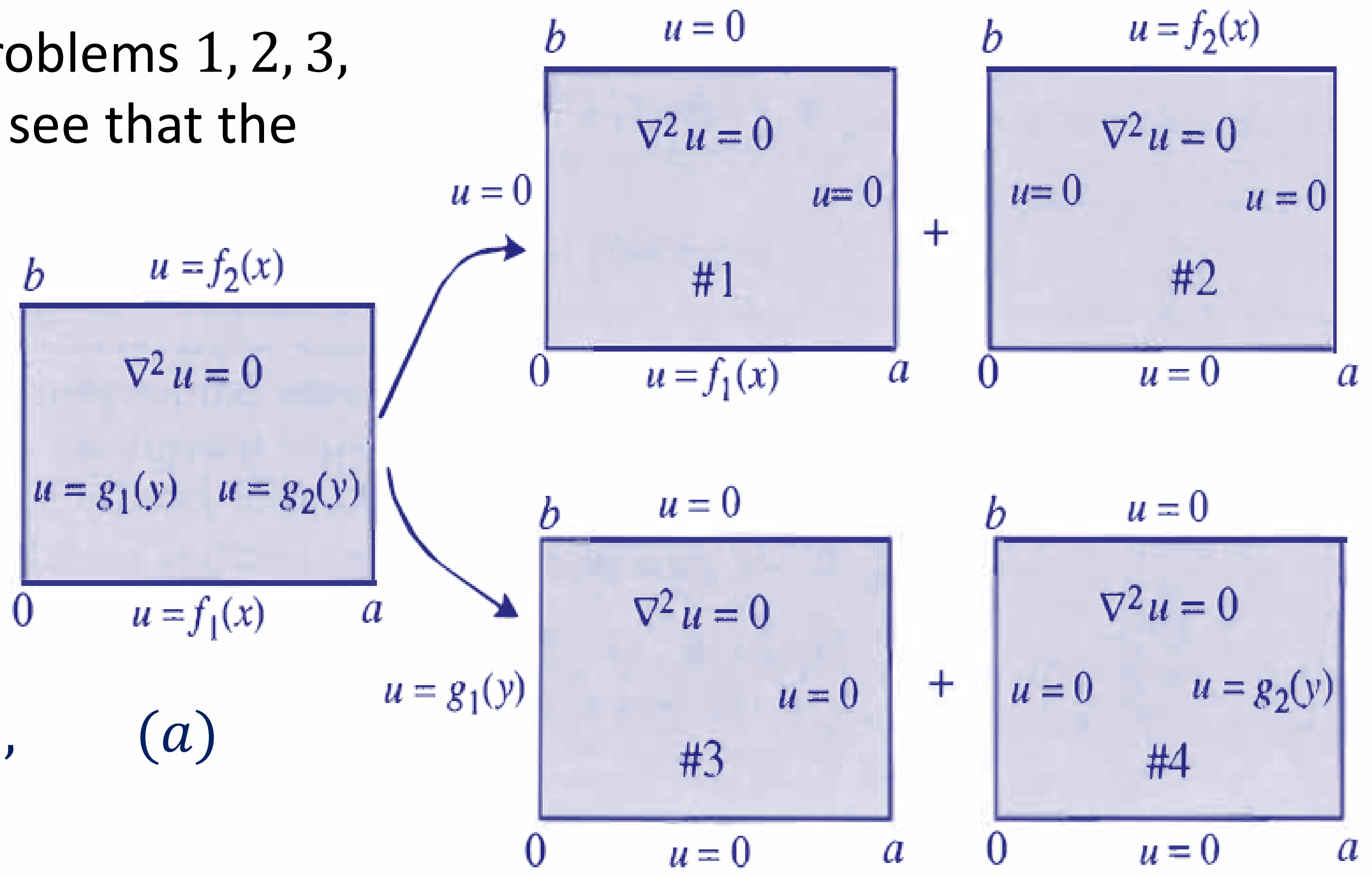
We now return to the general problem. It turns out that this problem can be solved by using the solution to example discussed earlier in which one boundary condition was non-homogeneous while rest were homogeneous. The trick is to decompose the original problem into four subproblems, as described in the figure below. Linearity is used here to decompose the Dirichlet problem into the "sum" of four simpler Dirichlet subproblems.

Let u_1, u_2, u_3, u_4 be the solutions of subproblems 1, 2, 3, 4, respectively. By direct computation, we see that the function:

$$u = u_1 + u_2 + u_3 + u_4,$$

is the solution to the problem. Thus, we need only determine u_1, u_2, u_3, u_4. The function u_2 is already computed in previous example. We have:

$$u_2(x, y) = \sum_{k=1}^{\infty} B_k \sin\left(\frac{k\pi x}{a}\right) \sinh\left(\frac{k\pi y}{a}\right), \qquad (a)$$



Dirichlet Problem in a Rectangle

where,

$$B_k = \frac{2}{a \sinh\left(\frac{k\pi b}{a}\right)} \int_0^a f_2(x) \sin\left(\frac{k\pi x}{a}\right) dx, \qquad k = 1,2,3,\ldots. \qquad (a')$$

The other solutions can be found analogously. In particular, u_4 is the same as u_2 except that a and b are interchanged, as are x and y. Thus,

$$u_4(x,y) = \sum_{k=1}^{\infty} D_k \sin\left(\frac{k\pi y}{b}\right) \sinh\left(\frac{k\pi x}{b}\right), \qquad (b)$$

where,

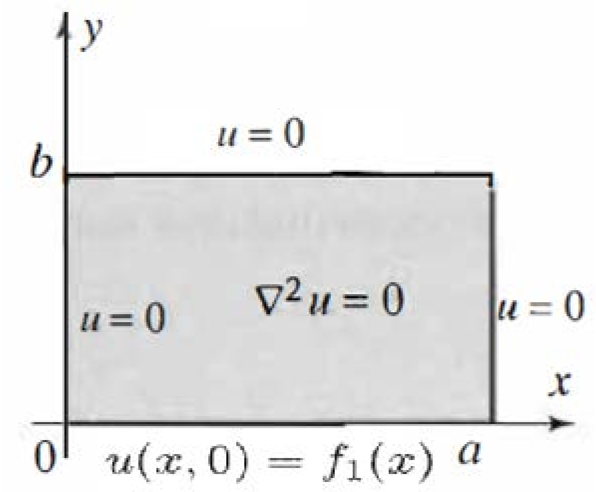
$$D_k = \frac{2}{b \sinh\left(\frac{k\pi a}{b}\right)} \int_0^b g_2(y) \sin\left(\frac{k\pi y}{b}\right) dy, \qquad k = 1,2,3,\ldots. \qquad (b')$$

Similarly, we can determine the other two solutions u_1 and u_3.

Dirichlet Problem in a Rectangle

Let us solve the boundary value problem described in the accompanying figure (subproblem -1) using the method of separation of variables.

We are required to determine solution of the BVP:



(1) $\nabla^2 u = u_{xx} + u_{yy} = 0, \qquad 0 < x < a, \qquad 0 < y < b.$

(2) $u(x,0) = f_1(x), \qquad u(x,b) = 0, \qquad 0 < x < a,$

(3) $u(0,y) = 0, \qquad u(a,y) = 0, \qquad 0 < y < b.$

We begin by looking for product solution $u(x,y) = X(x)Y(y)$. Substituting into (1) and using the separation method, we arrive at the equations:

$$X''(x) + \alpha\, X(x) = 0, \qquad (4)$$

$$Y''(y) - \alpha\, Y(y) = 0, \qquad (5)$$

where α is the separation constant, with the boundary conditions:

$$X(0) = 0, \qquad X(a) = 0, \qquad \text{and} \qquad Y(b) = 0. \qquad (6)$$

Dirichlet Problem in a Rectangle

Let us consider the boundary value problem in X that is given as:

$$X''(x) + \alpha\, X(x) = 0,$$

$$X(0) = 0, \qquad X(a) = 0$$

We can check that the values $\alpha \le 0$, lead to trivial solutions only. For $\alpha = \lambda^2 > 0$, we obtain the solutions:

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x\,.$$

Imposing the boundary conditions on X forces $c_1 = 0$, and

$$\lambda = \lambda_k = \frac{k\pi}{a} \quad or \quad \alpha = \left(\frac{k\pi}{a}\right)^2, \qquad k = 1,2,3, \ldots$$

and hence,

$$X_k(x) = c_k \sin\left(\frac{k\pi x}{a}\right), \qquad k = 1,2,3, \ldots \quad (7)$$

Dirichlet Problem in a Rectangle

Let us now consider (5), which on using $\alpha = \lambda_k^2 = \left(\frac{k\pi}{a}\right)^2$ takes the form:

$$Y''(y) - \left(\frac{k\pi}{a}\right)^2 Y(y) = 0. \quad (5')$$

The general solution of $(5')$ is a linear combination of exponentials or a linear combination of hyperbolic functions. For the present case it is convenient to express its general solution in terms of hyperbolic functions instead of exponential functions. Thus, the general solution of $(5')$ is given as:

$$Y(y) = Y_k(y) = d_k \cosh\left(\frac{k\pi y}{a}\right) + e_k \sinh\left(\frac{k\pi y}{a}\right). \quad (8)$$

Using the boundary condition $Y(b) = 0$ in (8) we get:

$$e_k = -d_k \coth\left(\frac{k\pi b}{a}\right).$$

Thus, (8) can be written as:

$$Y_k(y) = d_k \cosh\left(\frac{k\pi y}{a}\right) - d_k \coth\left(\frac{k\pi b}{a}\right) \sinh\left(\frac{k\pi y}{a}\right). \quad (8')$$

Dirichlet Problem in a Rectangle

Equation $(8')$ can be further simplified as:

$$Y_k(y) = \frac{d_k}{\sinh\left(\frac{k\pi b}{a}\right)}\left[\sinh\left(\frac{k\pi b}{a}\right)\cosh\left(\frac{k\pi y}{a}\right) - \cosh\left(\frac{k\pi b}{a}\right)\sinh\left(\frac{k\pi y}{a}\right)\right].$$

Since $\sinh(\alpha-\beta) = \sinh\alpha\cosh\beta - \cosh\alpha\sinh\beta$, so choosing $\alpha = \frac{k\pi b}{a}$ and $\beta = \frac{k\pi y}{a}$, above equation takes the form:

$$Y_k(y) = F_k \sinh\left(\frac{k\pi(b-y)}{a}\right); \quad \text{where} \quad F_k = \frac{d_k}{\sinh\left(\frac{k\pi b}{a}\right)}, \qquad k = 1,2,3,\ldots . \quad (9)$$

From (7) and (9) we obtain:

$$u_k(x,y) = X_k(x)Y_k(y) = A_k \sin\left(\frac{k\pi x}{a}\right)\sinh\left(\frac{k\pi(b-y)}{a}\right); \quad \text{where} \quad A_k = c_kF_k, \qquad (10)$$

as the **eigenfunctions** of our problem corresponding to eigenvalues $\lambda_k = \frac{k\pi}{a}$.

Dirichlet Problem in a Rectangle

Superposing these solutions (10), we get the general form of the solution as:

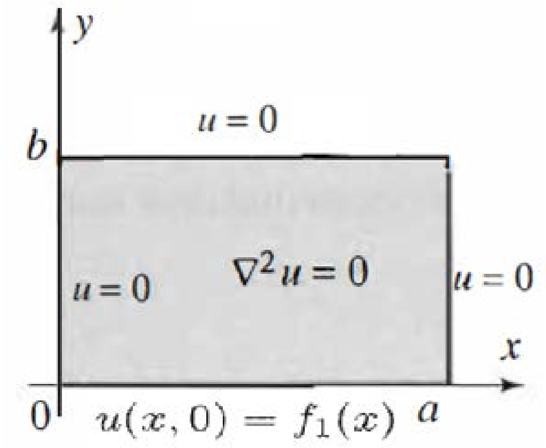
$$u(x,y) = \sum_{k=1}^{\infty} u_k(x,y) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{a}\right) \sinh\left(\frac{k\pi y}{a}\right), \qquad (11)$$

where the constants E_k are to be chosen. Thus, by using $u(x,0) = f_1(x)$ we get:

$$f_1(x) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{a}\right) \sinh\left(\frac{k\pi b}{a}\right). \qquad (12)$$

To meet this last requirement, we choose the coefficients $E_k \sinh\frac{k\pi b}{a}$ to be the Fourier sine coefficients of $f_2(x)$ on the interval $0 < x < a$. Thus, it follows that:

$$A_k = \frac{2}{a \sinh\left(\frac{k\pi b}{a}\right)} \int_0^a f_2(x) \sin\left(\frac{k\pi x}{a}\right) dx, \qquad k = 1,2,3,\dots. \qquad (13)$$



The solution of the Dirichlet problem described in the figure is therefore given by (11) with coefficients determine by (13).

Dirichlet Problem in a Rectangle

Thus, for subproblem-1, we have:

$$u_1(x,y) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{a}\right) \sinh\left(\frac{k\pi(b-y)}{a}\right), \qquad (c)$$

where,

$$A_k = \frac{2}{a \sinh\left(\frac{k\pi b}{a}\right)} \int_0^a f_1(x) \sin\left(\frac{k\pi x}{a}\right) dx, \qquad k = 1,2,3,\dots, \qquad (c')$$

In particular, u_3 is the same as u_1 except that a and b are interchanged, as are x and y. Thus,

$$u_3(x,y) = \sum_{k=1}^{\infty} C_k \sin\left(\frac{k\pi y}{b}\right) \sinh\left(\frac{k\pi(a-x)}{b}\right), \qquad (d)$$

where,

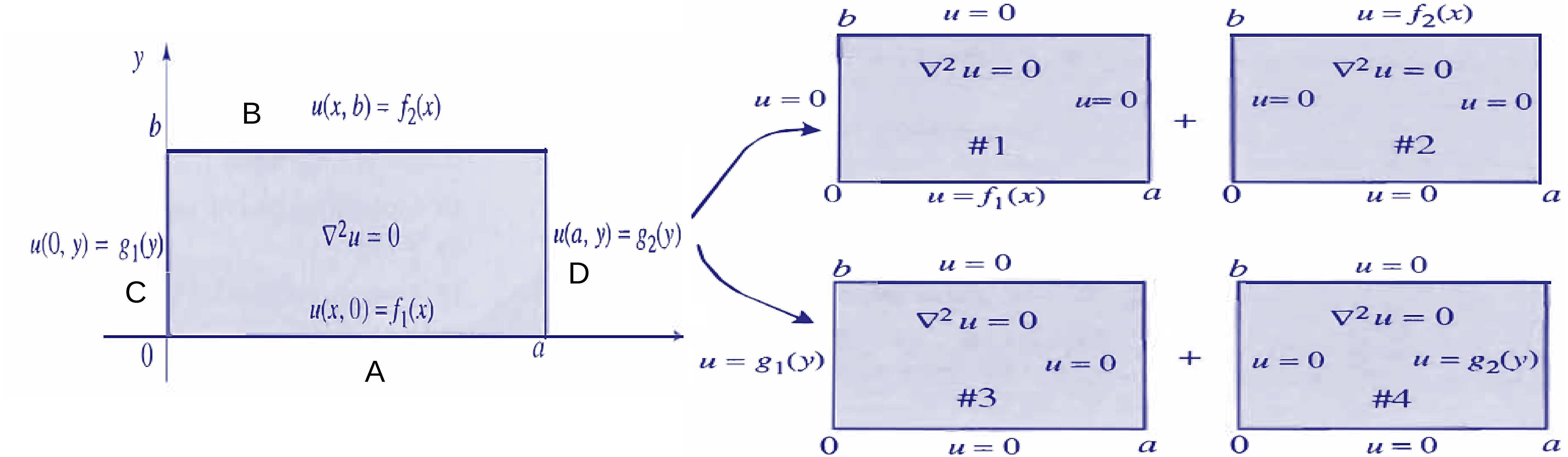
$$C_k = \frac{2}{b \sinh\left(\frac{k\pi a}{b}\right)} \int_0^b g_1(y) \sin\left(\frac{k\pi y}{b}\right) dy, \qquad k = 1,2,3,\dots. \qquad (d')$$

Dirichlet Problem in a Rectangle

Thus, the solution of the Dirichlet problem in a rectangle shown in figure is given as:

$$
\begin{aligned}
u(x,y) &= \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{a}\right)\sinh\left(\frac{k\pi(b-y)}{a}\right) + \sum_{k=1}^{\infty} B_k \sin\left(\frac{k\pi x}{a}\right)\sinh\left(\frac{k\pi y}{a}\right) \\
&+ \sum_{k=1}^{\infty} C_k \sin\left(\frac{k\pi y}{b}\right)\sinh\left(\frac{k\pi(a-x)}{b}\right) + \sum_{k=1}^{\infty} D_k \sin\left(\frac{k\pi y}{b}\right)\sinh\left(\frac{k\pi x}{b}\right),
\end{aligned}
$$

where the coefficients A_k, B_k, C_k and D_k are given by $(a') - (d')$.



The Laplacian in Various Coordinate Systems

- The two-dimensional Laplacian and its higher dimensional versions are of paramount importance in applications.

- They appear, for example, in the wave and heat equations, and also in Laplace's equation.

- So far, we solved these equations over rectangular and box shaped regions. To extend our applications to regions such as the disk, the sphere or the cylinder, it is to our advantage to use new coordinate systems in which the region and its boundary have simple expressions.

- For example, for problems over a disk we change to polar coordinates, where the equation of a circle centered at the origin reduces to $r = a$. Similarly, problems over spheres are simplified by a change to spherical coordinates.

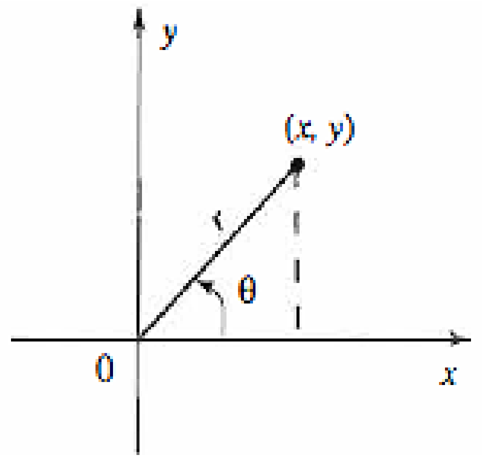
- For later applications, we will express the Laplacian in various coordinate systems.

The Laplacian in Polar Coordinate System

If we want to solve a partial differential equation (PDE) on the domain whose shape is a 2D disk, it is much more convenient to represent the solution in terms of the polar coordinate system than in terms of the usual Cartesian coordinate system. For example, the behavior of the drum surface when we hit it by a stick would be best described by the solution of the wave equation in the polar coordinate system. Let us now derive the Laplacian and the Laplace's equation in the polar coordinate system. Recall the relationship between rectangular and polar coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x.$$



For $\theta \in (-\pi, \pi]$, we have:

$$\theta = \arctan\left(\frac{y}{x}\right) + k\pi,$$

where $k = 0$ if $x > 0$ and $y > 0$ or $x > 0$ and $y < 0$; or $k = 1$ if $x < 0$ and $y \geq 0$; or $k = -1$ if $x < 0$ and $y < 0$. Also, if $x = 0$, then $\theta = \pi/2$ if $y > 0$ and $\theta = -\pi/2$ if $y < 0$.)

The Laplacian in Polar Coordinate System

Differentiating $r^2 = x^2 + y^2$ with respect to x, we obtain:

$$2r\frac{\partial r}{\partial x} = 2x \Longrightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.$$

Differentiating again with respect to x and simplifying, we obtain:

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{y^2}{r^3}.$$

Similarly,

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2},$$

and

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{r^2}\right) = \frac{2y}{r^3}\frac{\partial r}{\partial x} = \frac{2xy}{r^4}.$$

Differentiating now with respect to y, we obtain in a similar way

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} \quad \text{and} \quad \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}, \qquad \frac{\partial^2 \theta}{\partial y^2} = -\frac{2xy}{r^4}.$$

The Laplacian in Polar Coordinate System

From the relations we get the following interesting identities:

1. $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{y^2}{r^3} + \dfrac{x^2}{r^3} = \dfrac{1}{r}.$

2. $\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2} = \dfrac{2xy}{r^4} - \dfrac{2xy}{r^4} = 0.$

3. $\dfrac{\partial \theta}{\partial x}\dfrac{\partial r}{\partial x} + \dfrac{\partial \theta}{\partial y}\dfrac{\partial r}{\partial y} = \left(\dfrac{-y}{r^2}\right)\left(\dfrac{x}{r}\right) + \left(\dfrac{x}{r^2}\right)\left(\dfrac{y}{r}\right) = 0.$

4. $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = \dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} = 1.$

We are now ready to change the Laplacian $\nabla^2 u = u_{xx} + u_{yy}$ from Cartesian coordinates to polar coordinates. Since u is function of x and y which are functions of r and θ so using the chain rule in two dimensions, we have:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}.$$

The Laplacian in Polar Coordinate System

Applying the product rule for differentiation and the chain rule, we obtain:

(I)

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} \\
&= \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial \theta}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} \\
&\quad + \left(\frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta^2}\frac{\partial \theta}{\partial x}\right)\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} \\
&= \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial \theta}{\partial x}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} \\
&\quad + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}.
\end{aligned}
$$

Changing x to y, we obtain:

(II)

$$
\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial \theta}{\partial y}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2}.
$$