Practice Question lecture # 36

Q:- Heat conduction in a thin circular ring.

$$\frac{\partial u}{\partial t} = c^2 \cdot \frac{\partial^2 u}{\partial x^2} \; ; \quad -L < x < L, \quad t > 0 \rightarrow (1)$$

$$\left. \begin{aligned} u(-L,t) &= u(L,t) \\ u_x(-L,t) &= u_x(L,t), \quad t > 0 \end{aligned} \right\} \rightarrow (2)$$

$$u(x,0) = f(x) \; ; \quad -L < x < L \rightarrow (3)$$

Solution:- Consider

$$u(x,t) = X(x)\, T(t) \rightarrow (4)$$

so that eq (1) takes the form:

$$X(x)\, T'(t) = c^2 \, X''(x)\, T(t)$$

$$\Rightarrow \quad \frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = K$$

where K is a separation constant. This leads to following two ODE's:

$$T'(t) - c^2 K T(t) = 0 \rightarrow (5)$$

and $$X''(x) - K\, X(x) = 0 \rightarrow (6)$$

Using (4) in (2) we get

$$X(-L) = X(L) \rightarrow (7)$$

and $$X'(-L) = X(L) \rightarrow (8)$$

Let us consider the B.V.P involving x first that consists of an ODE with two boundary conditions given as:

$$X''(x) - K\, X(x) = 0 \rightarrow (6)$$

$$X(-L) = X(L) \rightarrow (7)$$

$$X'(-L) = X(L) \rightarrow (8)$$

Case 1:- $K = 0$ (6) reduces to $X''(x) = 0$

$\Rightarrow \quad X'(x) = c_1 \longrightarrow (9^*)$

$\Rightarrow \quad X(x) = c_1 x + c_2 \longrightarrow (9)$

Using (7) in (9) we get

$X(-L) = X(L) \Rightarrow c_1(-L) + c_2 = c_1(L) + c_2$

$\Rightarrow -c_1 L = c_1 L$

$\Rightarrow c_1 L + c_1 L = 0$

$\Rightarrow 2c_1 L = 0$

$\Rightarrow \boxed{c_1 = 0} \quad \because L \neq 0$

Thus (9) reduces to

$X(x) = c_2$

Using (8) in (9^*)

$X'(-L) = X'(L)$

$\Rightarrow \quad c_1 = c_1 \quad$ but $c_1 = 0$.

Alternative

$X(x) = c_2$

$\Rightarrow X'(x) = 0$

Thus $X'(-L) = X'(L)$ is not going to contribute and $\boxed{X(x) = c_2}$

Thus for $\underline{K = 0}$

$\boxed{X(x) = c_2}$; where c_2 is an arbitrary constant $\rightarrow (10)$

<u>Case 2</u> :- $\underline{K > 0}$

Let $K = \alpha^2$, so that (6) becomes

$X'' - \alpha^2 X = 0$

$\Rightarrow (D^2 - \alpha^2) X = 0$

Auxilliary equation is

$D^2 - \alpha^2 = 0$

$\Rightarrow D^2 = \alpha^2$

$\Rightarrow D = \pm \alpha$

$\Rightarrow X(x) = c_3 e^{-\alpha x} + c_4 e^{\alpha x}$

or we can express general solution in terms of hyperbolic functions as well. In that case

$$X(x) = c_3 \cosh(\alpha x) + c_4 \sinh(\alpha x) \longrightarrow (11)$$

We will continue with the hyperbolic form of general solution given in (11). Now using (7) we get

$$X(-L) = X(L) \Rightarrow c_3 \cosh(-\alpha L) + c_4 \sinh(-\alpha L) = c_3 \cosh(\alpha L) + c_4 \sinh(\alpha L)$$

$$\Rightarrow \cancel{c_3 \cosh(\alpha L)} - c_4 \sinh(\alpha L) = \cancel{c_3 \cosh(\alpha L)} + c_4 \sinh(\alpha L)$$

$$\Rightarrow 2 c_4 \sinh(\alpha L) = 0$$

$$\Rightarrow c_4 = 0 \quad \because \sinh(\alpha L) \neq 0$$

Thus,

$$X(x) = c_3 \cosh(\alpha x).$$

Now $X'(x) = c_3 \alpha \sinh(\alpha x)$

Using (8) we get

$$X'(-L) = X'(L) \Rightarrow -c_3 \alpha \sinh(\alpha L) = c_3 \alpha \sinh(\alpha L)$$

$$\Rightarrow 2 c_3 \alpha \sinh(\alpha L) = 0$$

$$\Rightarrow c_3 = 0 \quad \because \alpha \neq 0 \ \& \ \sinh(\alpha L) \neq 0$$

Thus, $X(x) = 0$, a trivial solution which is of no interest.

Case 3: $K < 0$

Let $K = -\alpha^2$, so that (6) takes the form

$$X'' + \alpha^2 X = 0$$

$$\Rightarrow (D^2 + \alpha^2) X = 0$$

$$D^2 + \alpha^2 = 0 \Rightarrow D = \pm i\alpha$$

$\Rightarrow$ The general solution is given as:

$$X(x) = c_5 \cos(\alpha x) + c_6 \sin(\alpha x) \Rightarrow X'(x) = -c_5 \alpha \sin(\alpha x) + \alpha c_6 \cos(\alpha x)$$

$$X(-L) = X(L) \Rightarrow c_5 \cancel{\cos(\alpha L)} - c_6 \sin(\alpha L) = c_5 \cancel{\cos(\alpha L)} + c_6 \sin(\alpha L)$$

$$\Rightarrow 2 c_6 \sin(\alpha L) = 0 \Rightarrow c_6 \sin \alpha L = 0 \rightarrow (12)$$

Before analyzing (12) let us consider (8) first

$X'(-L) = X'(L) \Rightarrow +c_5 \lambda \sin(\lambda L) + c_6 \cancel{\lambda \cos(\lambda L)} = -c_5 \lambda \sin(\lambda L) + \cancel{c_6 \lambda \cos(\lambda L)}$

$\Rightarrow 2c_5 \lambda \sin(\lambda L) = 0$

$\Rightarrow c_5 \sin(\lambda L) = 0 \rightarrow (13) \; [\because \lambda \neq 0]$

From (12) and (13) we observe that if $\sin(\lambda L) \neq 0$ then $c_5 = 0$ and $c_6 = 0$ which leads to a trivial solution. Thus, for a non-trivial solution we should have

$$\sin(\lambda L) = 0$$

$$\Rightarrow \lambda L = n\pi \; ; \; n = 1, 2, \ldots$$

$$\Rightarrow \lambda = \frac{n\pi}{L} \; , \; n = 1, 2, \ldots$$

Thus, c_5 and c_6 are both arbitrary and

$$X(x) = c_5 \cos\left(\frac{n\pi}{L} x\right) + c_6 \sin\left(\frac{n\pi}{L} x\right).$$

~~H~~Hence $\cos\left(\frac{n\pi}{L} x\right)$ and $\sin\left(\frac{n\pi}{L} x\right)$ both are eigen functions corresponding to the eigen values $\lambda = n\pi/L$, $n = 1, 2, \ldots$

Therefore

$$X_n(x) = a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right) \rightarrow (14)$$

~~[illegible] eq (5) [illegible] takes [illegible] form~~

~~[illegible]~~

Let us consider eq (5)

$$T'(t) - c^2 K T(t) = 0$$

$$\Rightarrow T'(t) = c^2 K T(t)$$

$$\Rightarrow \frac{dT}{dt} = c^2 K T$$

$\Rightarrow \quad \frac{dT(t)}{T(t)} = c^2 K \, dt$

Integrating both sides of above we get

$$T(t) = C_7 \, e^{c^2 K t} \longrightarrow (15)$$

For $K = 0$

$$T(t) = C_7$$

and for $K = -\lambda^2 = -(n\pi/L)^2$

$$T(t) = C_n \, e^{-c^2 (\frac{n\pi}{L})^2 t}$$

In other words,

$$T(t) = \begin{cases} C_7 & ; \; n = 0 \\ C_n e^{-(\frac{cn\pi}{L})^2 t} & ; \; n = 1, 2, 3, \ldots \end{cases} \longrightarrow (16)$$

Since $u(x,t) = X(t)\,T(t)$, so using (10), (14) and (16)

$$U_n(x,t) = \begin{cases} C_2 C_7 & ; \; n = 0 \\ \left[a_n \cos(\frac{n\pi}{L}x) + b_n \sin\frac{n\pi}{L}\right] C_n e^{-(\frac{cn\pi}{L})^2 t} & ; \; n = 1, 2, \ldots \end{cases}$$

OR

$$U_n(x,t) = \begin{cases} A_0 & ; \; n = 0 \\ \left[A_n \cos(\frac{n\pi}{L}x) + B_n \sin(\frac{n\pi}{L}x)\right] e^{-(\frac{cn\pi}{L})^2 t} & ; \; n = 1, 2, \ldots \end{cases}$$

Due to Superposition Principle we get

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos(\frac{n\pi}{L}x) + B_n \sin(\frac{n\pi}{L}x)\right] e^{-(\frac{cn\pi}{L})^2 t} \rightarrow (17)$$

Using (3) in (17) we get

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos(\frac{n\pi}{L}x) + B_n \sin(\frac{n\pi}{L}x)\right]$$

which is a trigonometric Fourier series of periodic function $f(x)$ and A_0, A_n, B_n are Fourier coefficients that can be determined by using following formulas;

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \qquad \rightarrow (18)$$

and $$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Eq. (17) along with (18) provides us with the solution of the given boundary value problem.

Practice Question/lecture # 41

Q:- A. Solve for the motion of a string of length $L = \pi/2$ if $c = 1$ and the initial displacement and velocity are given by $f(x) = 0$ and $g(x) = x\cos x$ resp.

Sol:- (Note:- Write the whole working shown in lecture 40 for a wave equation by replacing $L = \pi/2$, $c = 1$, $f(x) = 0$, $g(x) = x\cos x$.)

From lecture 40 we have seen that

$$u(x,t) = \sum_{n=1}^{\infty}\left[C_n\cos(\lambda_n t) + C_n^*\sin(\lambda_n t)\right]\sin\left(\frac{n\pi}{L}x\right) \qquad \rightarrow (4)$$

where $\lambda_n = \frac{cn\pi}{L}$ and

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \qquad \rightarrow (5)$$

$$C_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad \rightarrow (6)$$

Using $L = \pi/2$, $c = 1$, $f(x) = 0$ and $g(x) = x\cos x$

in (4), (5) and (6) we get

$$u(x,t) = \sum_{n\geq 1}^{\infty} \left[c_n \cos(\lambda_n t) + c_n^* \sin(\lambda_n t)\right] \sin\left(\frac{n\pi x}{\pi/2}\right) \rightarrow \text{(I)}$$

where $\lambda_n = \dfrac{cn\pi}{L} = \dfrac{(1)(n)(\pi)}{(\pi/2)} = 2n,$

$c_n = 0 \quad \because f(x) = 0$

and

$$c_n^* = \frac{2}{n\pi} \int_0^{\pi/2} x\cos x \sin(2nx)\, dx \rightarrow (*)$$

Since, $x\cos x$ is odd, its Fourier sine series on $0 < x < \pi/2$ is identical to it's Fourier series on $-\pi/2 < x < \pi/2$. Alternatively, we can solve it directly.

We know that

$$\cos a \sin b = \frac{1}{2}\left[\sin(a+b) - \sin(a-b)\right]$$

$$\Rightarrow \cos x \sin(2nx) = \frac{1}{2}\left[\sin(x + 2nx) - \sin(x - 2nx)\right]$$

$(a = x, \quad b = 2nx)$

$$\Rightarrow \cos x \sin(2nx) = \frac{1}{2}\left[\sin(2n+1)x + \sin(2n-1)x\right]$$

So

$$\int x\cos x \sin(2nx)\, dx = \frac{1}{2}\int \left[x\sin(2n+1)x + x\sin(2n-1)x\right] dx \rightarrow (**)$$

Since $\int u \sin u\, du = -u\cos u + \int \cos u\, du = \sin u - u\cos u + C.$

So, ~~[illegible]~~ we have:

$$\int x \sin(2n+1)x \, dx = c_1 + \left[\sin(2n+1)x - (2n+1)x\cos(2n+1)x\right]\frac{1}{(2n+1)^2}$$

and

$$\int x \sin(2n-1)x \, dx = c_2 + \left[\sin(2n-1)x - (2n-1)x\cos(2n-1)x\right]\frac{1}{(2n-1)^2}$$

and (**) becomes

$$\int x\cos x \sin(2nx)\, dx = \frac{1}{2}\left[\frac{\sin(2n+1)x - (2n+1)x\cos(2n+1)x}{(2n+1)^2} + \frac{\sin(2n-1)x - (2n-1)x\cos(2n-1)x}{(2n-1)^2}\right]$$

$$+ c_3 \rightarrow (***)$$

Using *** in (*) we have

$$c_n^* = \frac{\cancel{2}}{n\pi}\left[\frac{1}{\cancel{2}}\left\{\left|\frac{\sin(2n+1)x - (2n+1)x\cos(2n+1)x}{(2n+1)^2}\right|_0^{\pi/2} + \left|\frac{\sin(2n-1)x - (2n-1)x\cos(2n-1)x}{(2n-1)^2}\right|_0^{\pi/2}\right\}\right]$$

$$= \frac{1}{n\pi}\left[\frac{\sin(2n+1)\pi/2 - (2n+1)\pi/2\cos(2n+1)\pi/2}{(2n+1)^2} + \frac{\sin(2n-1)\pi/2 - (2n-1)x\cos(2n-1)\pi/2}{(2n-1)^2}\right]$$

$$= \frac{1}{n\pi}\left[\frac{(-1)^n}{(2n+1)^2} + \frac{(-1)^{n+1}}{(2n-1)^2}\right] \quad \left[\because \cos(2n+1)\pi/2 = 0 = \cos(2n-1)\pi/2,\ \sin(2n+1)\pi/2 = (-1)^n \text{ \& } \sin(2n-1)\pi/2 = (-1)^{n+1}\right.$$

$$= \frac{(-1)^n}{n\pi}\left[\frac{1}{(2n+1)^2} - \frac{1}{(2n-1)^2}\right]$$

$$= \frac{(-1)^n}{n\pi}\left[\frac{(2n-1)^2 - (2n+1)^2}{(2n+1)^2(2n-1)^2}\right]$$

$$= \frac{(-1)^n}{n\pi}\left[\frac{\cancel{4n^2} + \cancel{1} - 4n - \cancel{4n^2} - \cancel{1} - 4n}{(2n+1)^2(2n-1)^2}\right]$$

$$= (-1)^{n+1}\left[\frac{8}{\pi(2n+1)^2(2n-1)^2}\right] = \frac{8(-1)^{n+1}}{\pi\left[(2n+1)(2n-1)\right]^2}$$

$$= \frac{8}{\pi}\left[\frac{(-1)^{n+1}}{[4n^2-1]^2}\right].$$

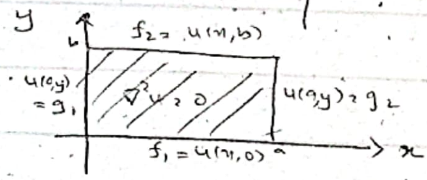
Thus, (I) becomes

$$u(x,t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{[4n^2-1]^2} \sin(2n t) \sin(2n x).$$

Q: B (1) and (2) Solve as previous question by replacing $L=1$, $c^2=1$, $g(x)=0$ and $f(x) = K\sin(3\pi x)$ for (1) and $f(x) = K[\sin(\pi x) - \frac{1}{2}\sin(2\pi x)]$ for (2).

Q: C: Repeat same process of part A with $L=1$, c^2, $g(x)=0$ and $f(x)$ given in question as a piecewise function.

QD:- Solve the Dirichlet problem in the following



(4) $a=1$, $b=2$, $f_2(x) = x$, $f_1 = g_1 = g_2 = 0$.

Sol:- See detailed working of example solved in lecture 37 for a dirichlet problem on a rectangle given as:

(1) ← $\nabla^2 u = u_{xx} + u_{yy} = 0$; $0<x<a$; $0<y<b$

(2) ← $u(x,0) = 0$, $u(x,b) = f_2$ $\quad 0<x<a$

(3) ← $u(0,y) = 0$; $u(a,y) = 0$ $\quad 0<y<b$.

whose solution was determined as;

$$u(x,y) = \sum_{K=1}^{\infty} E_K \sin\left(\frac{K\pi x}{a}\right) \sinh\left(\frac{K\pi y}{a}\right)$$

where

$$E_K = \frac{2}{a \sinh\left(\frac{K\pi b}{a}\right)} \int_0^a f_2(x) \sin\left(\frac{K\pi x}{a}\right) dx \; ; \quad K = 1, 2, \ldots$$

for exam

You are required to show all working however

I'm directly using the results determined here.

Now for present case

$a = 1,\ b = 2,\ f_2 = x$

$f_1 = g_1 = g_2 = 0$, so we get

$$u(x,y) = \sum_{k=1}^{\infty} E_k \sin\left(\frac{k\pi}{1}x\right) \sinh\left(\frac{k\pi}{1}y\right)$$

$$\Rightarrow u(x,y) = \sum_{k=1}^{\infty} E_k \sin(k\pi x) \sinh(k\pi y)$$

where

$$E_k = \frac{2}{\sinh(2k\pi)} \int_0^1 x \sin(k\pi x)\, dx \quad \left[\because f_2(x) = x,\ a = 1,\ b = 2\right]$$

$$\Rightarrow E_k = \frac{2}{\sinh(2k\pi)} \left[-\frac{x\cos(k\pi x)}{k\pi}\Big|_0^1 + \frac{1}{k\pi}\int_0^1 \cos(k\pi x)\, dx \right]$$

$$= \frac{2}{\sinh(2k\pi)} \left[-\frac{\cos(k\pi)}{k\pi} + \frac{1}{(k\pi)^2} \left| \sin(k\pi x) \right|_0^1 \right]$$

$$= \frac{2}{\sinh(2k\pi)} \left[\frac{(-1)(-1)^k}{k\pi} + \frac{1}{(k\pi)^2} \left[\cancel{\sin(k\pi)} - \cancel{\sin(0)} \right] \right]$$

$$= \frac{2(-1)^{k+1}}{(k\pi)\sinh(2k\pi)}.$$

so

$$u(x,y) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{(k\pi)\sinh(2k\pi)} \sin(k\pi x)\sinh(k\pi y).$$

Q: D: (2 — 6) solve on similar lines as previous question and examples solved in class.

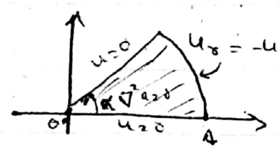
Q E:- Solve the Laplace's Equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \longrightarrow (1)$$

over the wedge region $0 < r < 1$ and $0 < \theta < \alpha$, supplied with boundary conditions:

$$u(r,0) = u(r,\alpha) = 0 \quad \rightarrow (2)$$

$$u_r(1,\theta) = -u(1,\theta) - \theta. \rightarrow (3)$$



Sol.- Here one condition is not given but it is understood from the statement of problem that

$$|u(0,\theta)| < \infty, \longrightarrow (4)$$

this means that the temperature must remain finite everywhere in the disk. Let us consider

$$u(r,\theta) = R(r)\,\Theta(\theta) \longrightarrow (5)$$

Using (5) in (1) we get

$$R''(r)\,\Theta(\theta) + \frac{1}{r}R'(r)\,\Theta(\theta) + \frac{1}{r^2}R(r)\,\Theta''(\theta) = 0$$

$$\Rightarrow r^2 R''(r)\,\Theta(\theta) + r R'(r)\,\Theta(\theta) + R(r)\,\Theta''(\theta) = 0$$

$$\Rightarrow \frac{r^2 R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

[Dividing both sides of above by $R(r)\,\Theta(\theta)$]

$$\Rightarrow r^2 \frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = K$$

where K is a separation constant.

From here we get a set of ODES that is given as:

$$\Theta''(\theta) + K\,\Theta(\theta) = 0 \rightarrow (6)$$

and $r^2 R''(r) + r R'(r) - K R(r) = 0 \rightarrow (7)$

Using (5) in (2) we get

$$\Theta(0) = 0 \text{ and } \Theta(\alpha) = 0 \rightarrow (8)$$

Using (5) in (4) we get

$$|R(0)| < \infty \text{ or } R(r) \rightarrow \infty \text{ as } r \rightarrow 0 \rightarrow (8')$$

Eq. (6) together with (8) provides us with a boundary value problem involving "θ" as variable. Let us solve this BVP first.

$$\Theta''(\theta) + K\,\Theta(\theta) = 0 \rightarrow (6)$$

$$\Theta(0) = 0, \quad \Theta(\alpha) = 0 \rightarrow (8)$$

This boundary value problem posses trivial solutions if $\underline{\underline{K=0}}$ and $\underline{\underline{K<0}}$. However if $\underline{K>0}$, this boundary value problem possesses a non-trivial solution which can be determined as follows:

Let $K = \lambda^2$, so that (6) becomes

$$\Theta''(\theta) + \lambda^2\,\Theta(\theta) = 0$$

$$\Rightarrow [D^2 + \lambda^2]\,\Theta(\theta) = 0 \quad \left[\text{where } D = \frac{d}{d\theta}\right]$$

This is a second order homogeneous ODE. Characteristic (or auxilliary) equation of this ODE is given as:

$$\phi^2 + \lambda^2 = 0$$

$$\Rightarrow \phi = \pm i\lambda \quad ; \quad \text{Complex roots}$$

Thus the general solution of this ODE is given as:

$$\Theta(\theta) = c_1 \cos(\lambda\theta) + c_2 \sin(\lambda\theta)$$

Now $\Theta(0) = 0 \Rightarrow \boxed{c_1 = 0}$

Thus

$$\Theta(\theta) = c_2 \sin(\lambda\theta).$$

Now $\Theta(\alpha) = 0 \Rightarrow c_2 \sin(\lambda\alpha) = 0$

$\Rightarrow \sin(\lambda\alpha) = 0$ for $c_2 \neq 0$ for non-trivial solution.

$\Rightarrow \lambda\alpha = n\pi \;;\; n = 1, 2, \ldots$

$\Rightarrow \boxed{\lambda = \frac{n\pi}{\alpha}} \;;\; n = 1, 2, \ldots$

Thus, $\Theta_n(\theta) = c_n \sin\left(\frac{n\pi\theta}{\alpha}\right) \;;\; n = 1, 2, \ldots$ (9)

means $\sin\left(\frac{n\pi}{\alpha}\right)$ are eigenfunctions corresponding to eigen values $\lambda = \frac{n\pi}{\alpha}$. Let us now consider eq (7) which is actually a Cauchy Euler's equation. For $K = \lambda^2$ this equation can be rewritten as:

$$r^2 R''(r) + r R'(r) - \lambda^2 R(r) = 0$$

Let $\frac{dR}{dr} = DR$ where $D = \frac{d}{dr}$, then above equation takes the form

$$[r^2 D^2 + rD - \lambda^2] R(r) = 0 \longrightarrow (10)$$

let $r = e^t \Rightarrow t = \ln|r|$. If we let

$\Delta = \frac{d}{dt}$ then we have

$$rD = \Delta$$

$$r^2 D^2 = \Delta(\Delta - 1)$$

Thus (10) can be expressed as

$$[\Delta(\Delta-1) + \Delta - \lambda^2] R(t) = 0$$

$$\Rightarrow [\Delta^2 - \cancel{\Delta} + \cancel{\Delta} - \lambda^2] R(t) = 0$$

$$\Rightarrow [\Delta^2 - \lambda^2] R(t) = 0$$

This is a second order homogeous ODE whose characteristic equation is given as

$$\Delta^2 - \lambda^2 = 0$$

$$\Rightarrow \Delta^2 = \lambda^2$$

$$\Rightarrow \Delta = \pm \lambda \quad [\text{real distinct roots}]$$

Thus, general solution is given as

$$R(t) = C_3 e^{-\lambda t} + C_4 e^{\lambda t}$$

since $r = e^t$ or $t = \ln|r|$ so we have

$$\Rightarrow R(r) = C_3 r^{-\lambda} + C_4 r^{\lambda} \longrightarrow (11)$$

By using (8^*) i.e. $|R(0)| < \infty$ in (11) we see that each of solutions above will have $R(r) \to \infty$ as $r \to 0$. Therefore, in order to meet this boundary condition we must have $c_3 = 0$. Thus (11) reduces to

$$R(r) = c_4 r^{\lambda}$$

Since $\lambda = \frac{n\pi}{\alpha}$; $n = 1, 2, \_\_\_$ so we can write above equation as:

$$R_n(r) = d_n r^{(n\pi/\alpha)} \quad ; \; n = 1, 2, \_ \longrightarrow (12)$$

Since $u(r,\theta) = R(r)\,\Theta(\theta)$ so using (9) and (12) we have

$$u_n(r,\theta) = R_n(r)\,\Theta_n(\theta) = \left[c_n \sin\left(\frac{n\pi\theta}{\alpha}\right)\right]\left[d_n r^{\left(\frac{n\pi}{\alpha}\right)}\right]$$

$$\Rightarrow u_n(r,\theta) = A_n \sin\left(\frac{n\pi\theta}{\alpha}\right) r^{(n\pi/\alpha)}, \quad \text{where } A_n = c_n d_n.$$

Superposing these solutions, we get the solution of given PDE (1) as:

$$u(r,\theta) = \sum_{n=1}^{\infty} u_n(r,\theta) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi\theta}{\alpha}\right) r^{(n\pi/\alpha)} \longrightarrow (13)$$

Differentiating both sides of (13) partially w.r.t "r" we get

$$u_r(r,\theta) = \sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{\alpha}\right) r^{\left(\frac{n\pi}{\alpha}-1\right)} \sin\left(\frac{n\pi\theta}{\alpha}\right) \longrightarrow (14)$$

Using the initial condition (3)

$$u_r(1,\theta) = -u(1,\theta) - \theta$$

we get

$$\sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{\alpha}\right) \sin\left(\frac{n\pi\theta}{\alpha}\right) = -\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi\theta}{\alpha}\right) - \theta \quad [\because r = 1]$$

$$\Rightarrow -\theta = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\alpha}\theta\right)\left(\frac{n\pi}{\alpha}\right) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\alpha}\right)$$

$$\Rightarrow -\theta = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\alpha}\theta\right)\left[\frac{n\pi}{\alpha}+1\right]$$ [Using properties of series $\sum_i a_i + \sum_i b_i = \sum_i (a_i+b_i)$]

$$\Rightarrow -\theta = \sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{\alpha}+1\right) \sin\left(\frac{n\pi\theta}{\alpha}\right) \; ; \quad 0 < \theta < \alpha$$

This is half range Fourier sine series of $-\theta$ over the interval $0 < \theta < \alpha$ and the Fourier coefficients $A_n\left(\frac{n\pi}{\alpha}+1\right)$ can be determined as:

$$A_n\left(\frac{n\pi}{\alpha}+1\right) = \frac{2}{\alpha}\int_0^{\alpha} (-\theta)\sin\left(\frac{n\pi}{\alpha}\theta\right) d\theta$$

$$\Rightarrow A_n = \frac{-2}{\alpha\left(\frac{n\pi}{\alpha}+1\right)} \int_0^{\alpha} \theta \sin\left(\frac{n\pi}{\alpha}\theta\right) d\theta$$

$$\Rightarrow A_n = \frac{-2}{(n\pi+\alpha)}\left[\left.\frac{-\theta\cos\left(\frac{n\pi}{\alpha}\theta\right)}{\frac{n\pi}{\alpha}}\right|_0^{\alpha} + \frac{1}{\frac{n\pi}{\alpha}}\int_0^{\alpha}\cos\left(\frac{n\pi}{\alpha}\theta\right) d\theta\right]$$

$$\Rightarrow A_n = \frac{-2}{(n\pi+\alpha)}\left[\frac{-\alpha\cos(n\pi)+0}{\left(\frac{n\pi}{\alpha}\right)} + \frac{\alpha}{n\pi}\left|\frac{\sin\left(\frac{n\pi}{\alpha}\theta\right)}{\frac{n\pi}{\alpha}}\right|_0^{\alpha}\right]$$

$$\Rightarrow A_n = \frac{-2}{(n\pi+\alpha)}\left[\frac{-\alpha^2 \overbrace{\cos(n\pi)}^{(-1)^n}}{n\pi} + \frac{\alpha^2}{n^2\pi^2}\left(\overbrace{\sin(n\pi)}^{0} - \overbrace{\sin(0)}^{0}\right)\right]$$

$$\Rightarrow \boxed{A_n = \frac{+2\alpha^2(-1)^n}{n\pi(n\pi+\alpha)}}$$

Thus,

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{2\alpha^2(-1)^n}{n\pi(\alpha+n\pi)} \, r^{n\pi/\alpha} \sin\left(\frac{n\pi}{\alpha}\theta\right)$$

$$\Rightarrow \boxed{u(r,\theta) = \frac{2\alpha^2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n(\alpha+n\pi)} \, r^{n\pi/\alpha} \sin\left(\frac{n\pi}{\alpha}\theta\right);}$$

$0 < \theta < \alpha$ and $0 < r < 1$.