Applications of Derivatives

Calculus & Analytical Geometry MATH- 101

Instructor: Dr. Naila Amir (SEECS, NUST)

Evaluate: $\lim_{x\to 0^+}[\cos(2x)]^{1/x^2}$ (1^∞) As $x \to 0^+$, $\cos 2x \to 1$

$\frac{1}{x^2} \to \infty$

$$y = [\cos(2x)]^{1/x^2}$$

$$\ln y = \ln [\cos(2x)]^{1/x^2}$$

$$= \frac{1}{x^2} \ln [\cos(2x)]$$

$$\lim_{x\to 0^+} [\ln y] = \lim_{x\to 0^+} \left[\left(\frac{1}{x^2}\right) \cdot \ln(\cos(2x)) \right] \quad (\infty \cdot 0)$$

$$= \lim_{x\to 0^+} \left[\frac{\ln(\cos(2x))}{x^2} \right] \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0^+}[\ln y] = \lim_{x\to 0^+}\left[\frac{\frac{1}{\cos(2x)}(-\sin(2x))(2)}{2x}\right]$$

$$= \lim_{x\to 0^+}\left[\frac{-\cancel{2}\tan(2x)}{\cancel{2}x}\right]$$

$$= \lim_{x\to 0^+}\left[\frac{-\tan(2x)}{x}\right] \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0^+}\left[\frac{-\sec^2(2x)\,(2)}{1}\right]$$

$$= \lim_{x\to 0^+}\left[\frac{-2}{\cos^2(2x)}\right] \quad \left[\because \sec\theta = \frac{1}{\cos\theta}\right]$$

$$= -2$$

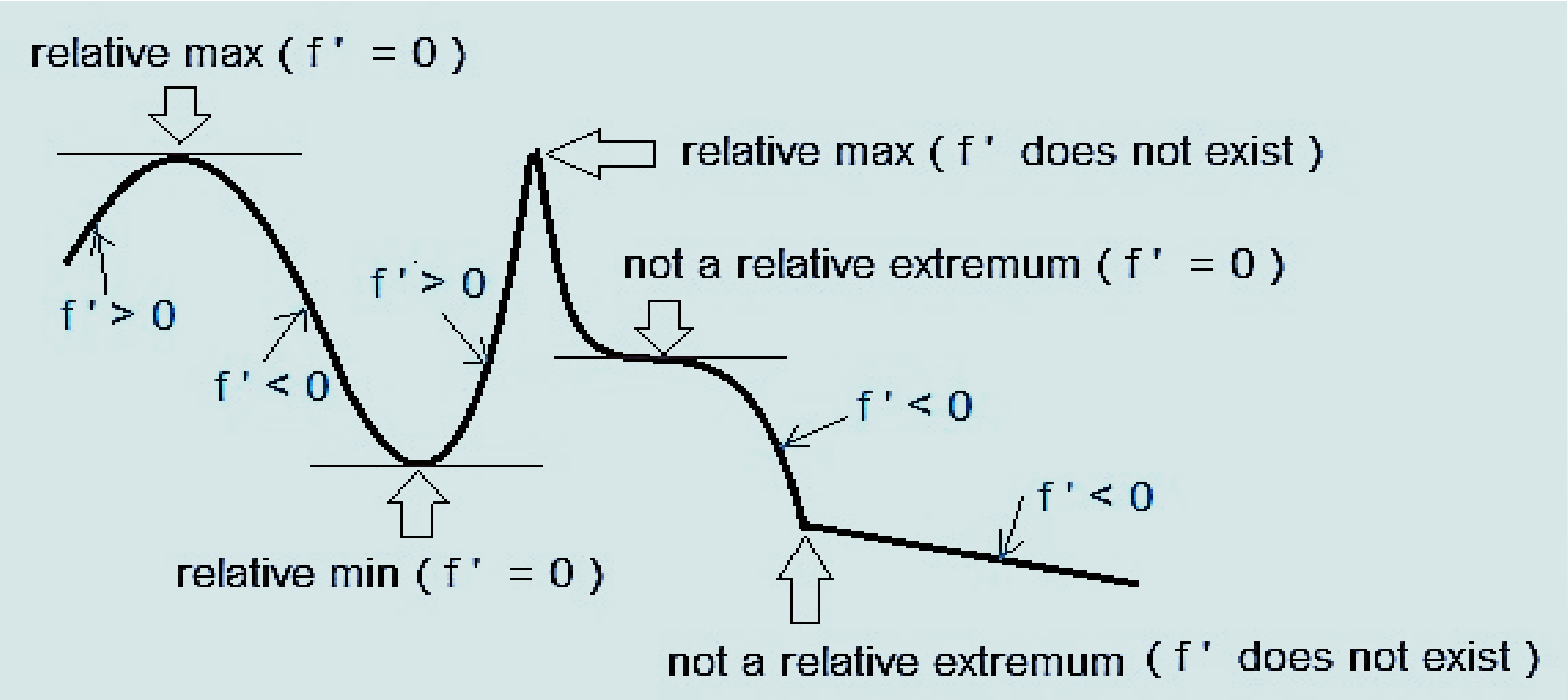
$$y = e^{\ln y}$$

$$\lim_{x \to 0^+} [y] = \lim_{x \to 0^+} \left[e^{\ln y} \right]$$

$$= e^{\lim_{x \to 0^+} [\ln y]}$$

$$= e$$

$$\Rightarrow \lim_{x \to 0^+} \left[[\cos(2x)]^{1/x^2} \right] = e^{-2} \checkmark$$



4.1

Extreme Values of Functions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 4
  - Sections: 4.1

Extreme Values of Functions

max

min

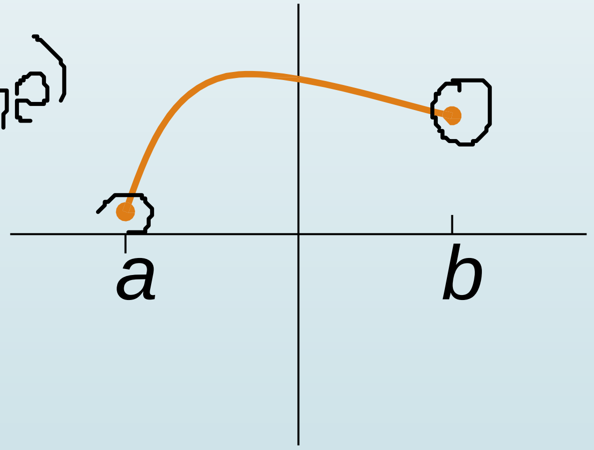
- Extreme Values of a function occur when the function changes from increasing to decreasing or from decreasing to increasing.
- In particular, we have two types of minimum or maximum values.
- We say that $f(x)$ has an **absolute (or global) maximum** at $x = c$ if $f(x) \leq f(c)$ for every x in the domain we are working on.
- We say that $f(x)$ has a **relative (or local) maximum** at $x = c$ if $f(x) \leq f(c)$ for every x in some open interval around $x = c$.
- We say that $f(x)$ has an **absolute (or global) minimum** at $x = c$ if $f(x) \geq f(c)$ for every x in the domain we are working on.
- We say that $f(x)$ has a **relative (or local) minimum** at $x = c$ if $f(x) \geq f(c)$ for every x in some open interval around $x = c$.

Extreme Value Theorem ✓

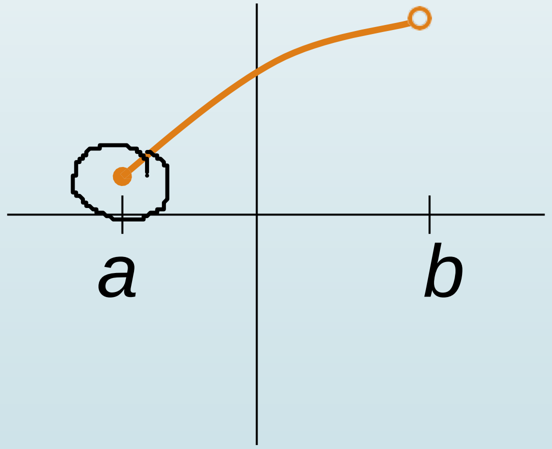
If a function f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum and absolute minimum on $[a, b]$. Each extremum occurs at a critical number or at an endpoint.

Abs. max

Abs. min

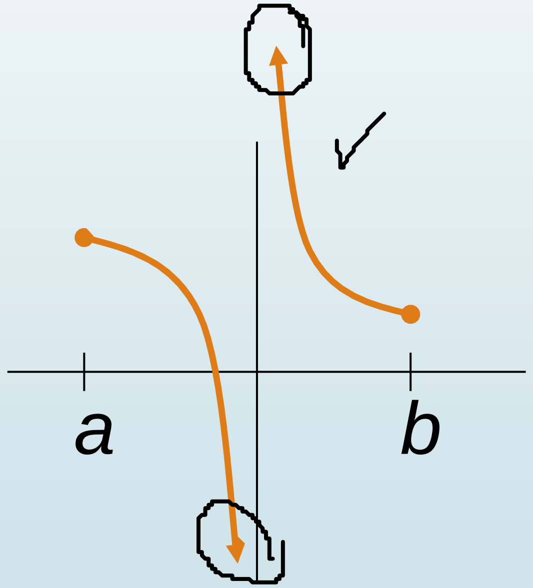


Attains absolute max. and min.



Attains absolute min. but no max.

Open Interval



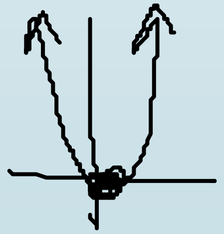
No absolute min. and no absolute max.

Not continuous

x^2 $[-2, 2]$

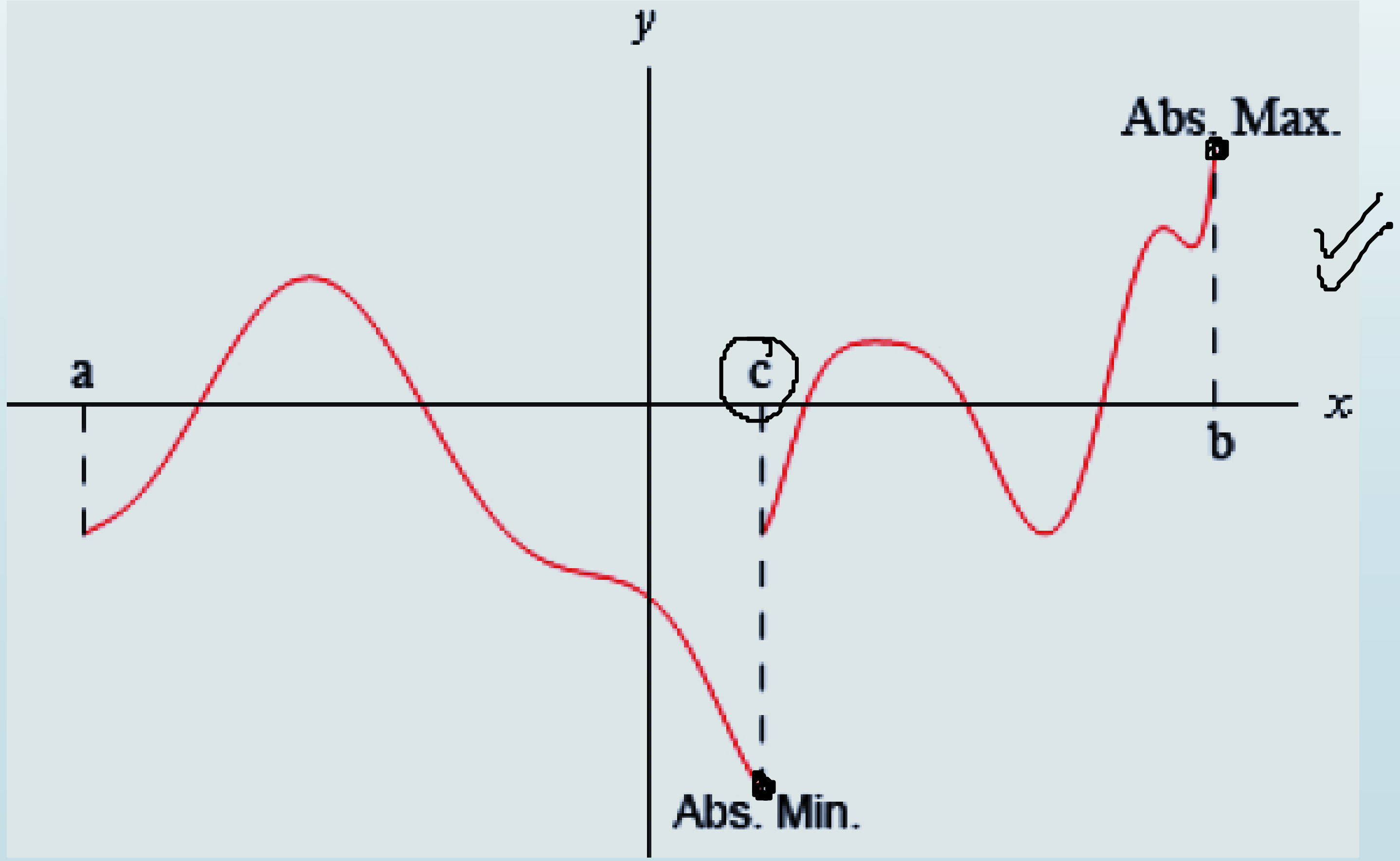
0

x^2 $(-\infty, \infty)$



Example:

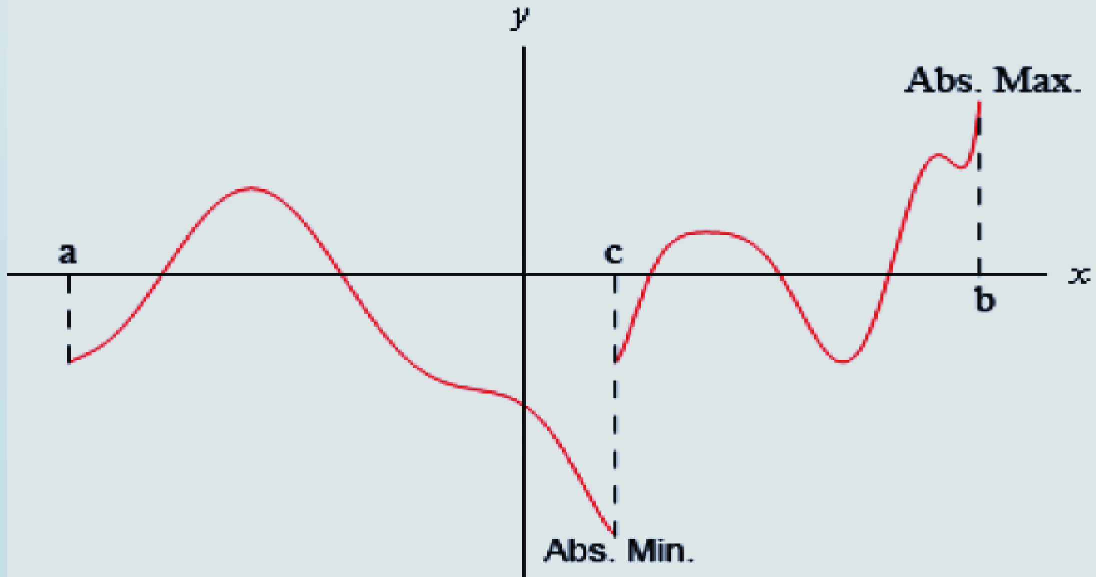
- We should also point out that just because a function is not continuous at a point that doesn't mean that it won't have both absolute extrema in an interval that contains that point.
- Below is the graph of a function that is not continuous at a point in the given interval and yet has both absolute extrema.



Observations:

- This graph is not continuous at $x = c$, yet it does have both an absolute maximum ($x = b$) and an absolute minimum ($x = c$).
- The point of all this is that we need to be careful to only use the Extreme Value Theorem when the conditions of the theorem are met and not misinterpret the results if the conditions aren't met.
- In order to use the Extreme Value Theorem we must have an interval and the function must be continuous on that interval.
- If we don't have an interval and/or the function isn't continuous on the interval then the function may or may not have absolute extrema.

EVT $\rightarrow \begin{cases} \text{cts} \\ [\ \ ] \end{cases}$

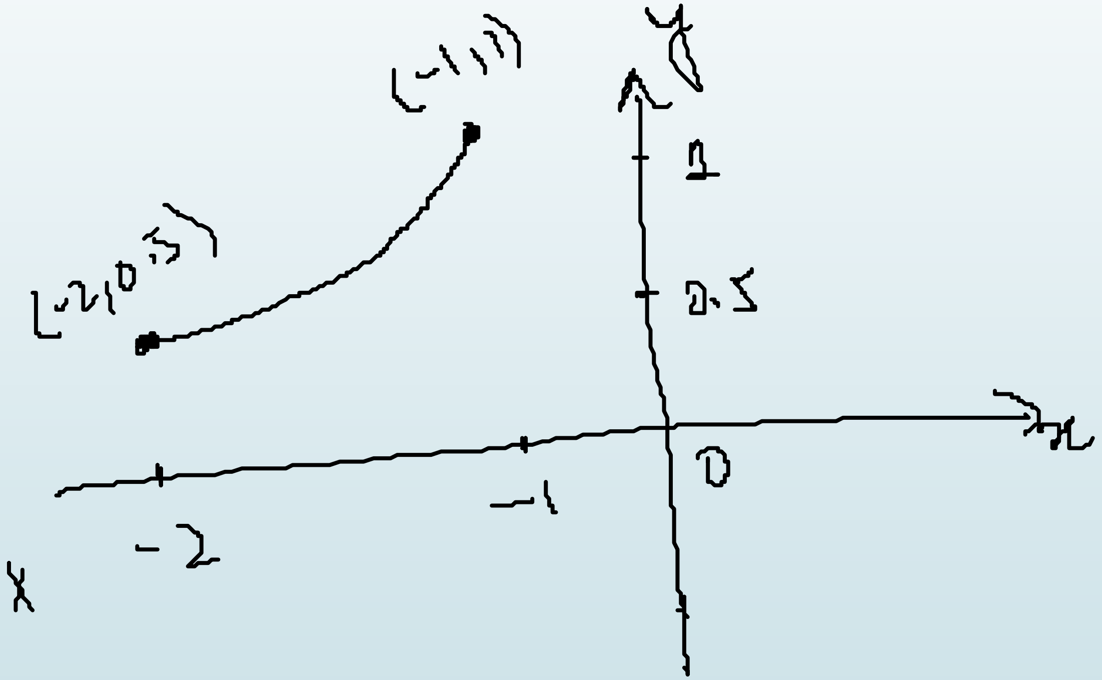


Graph the function given below and calculate any absolute extreme values, if they exist. Moreover, plot them on the graph and state the coordinates

$$f(x) = -\frac{1}{x} \qquad -2 \leq x \leq -1.$$

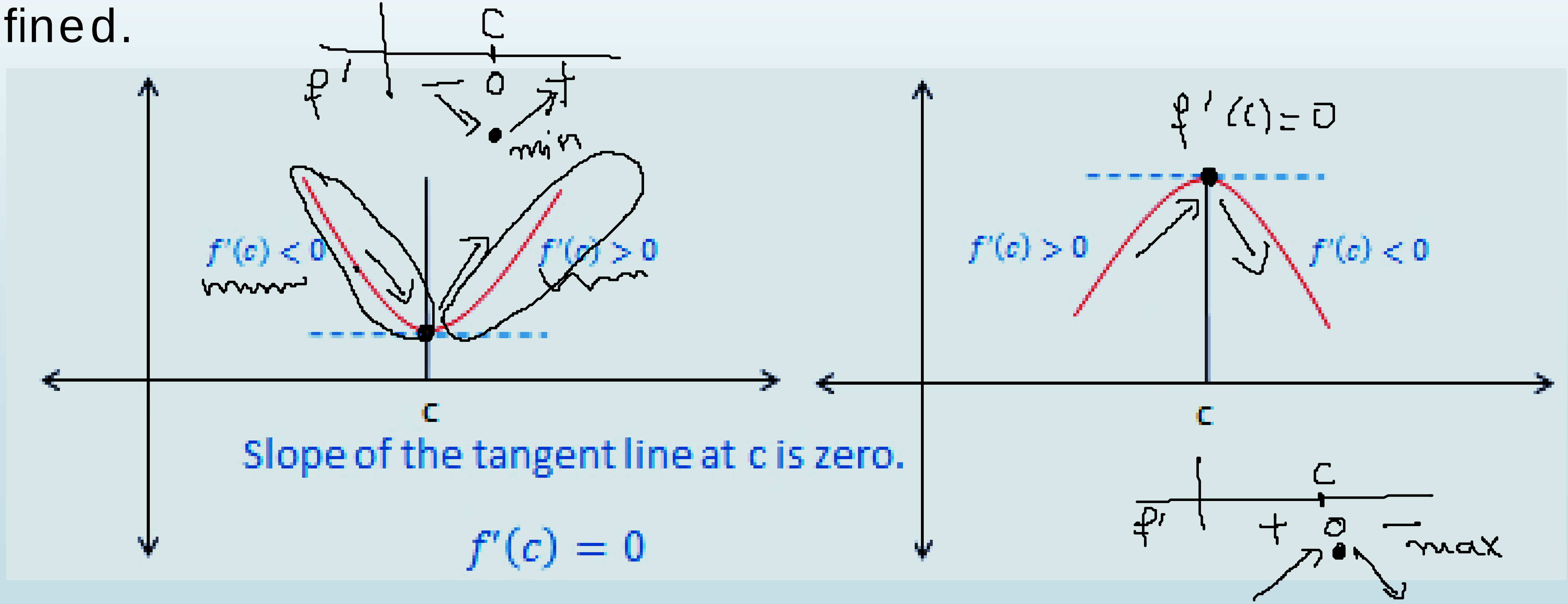
$f(-2) = 0.5$ Abs. min

$f(-1) = 1$ Abs. max



The First Derivative Theorem for Local Extreme Values

- If a function has a local maximum or minimum value at a point c in the domain and the derivative is defined at that point, then $f'(c) = 0$.
- Theorem says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined.



$f'(x) = 0$

Critical Points

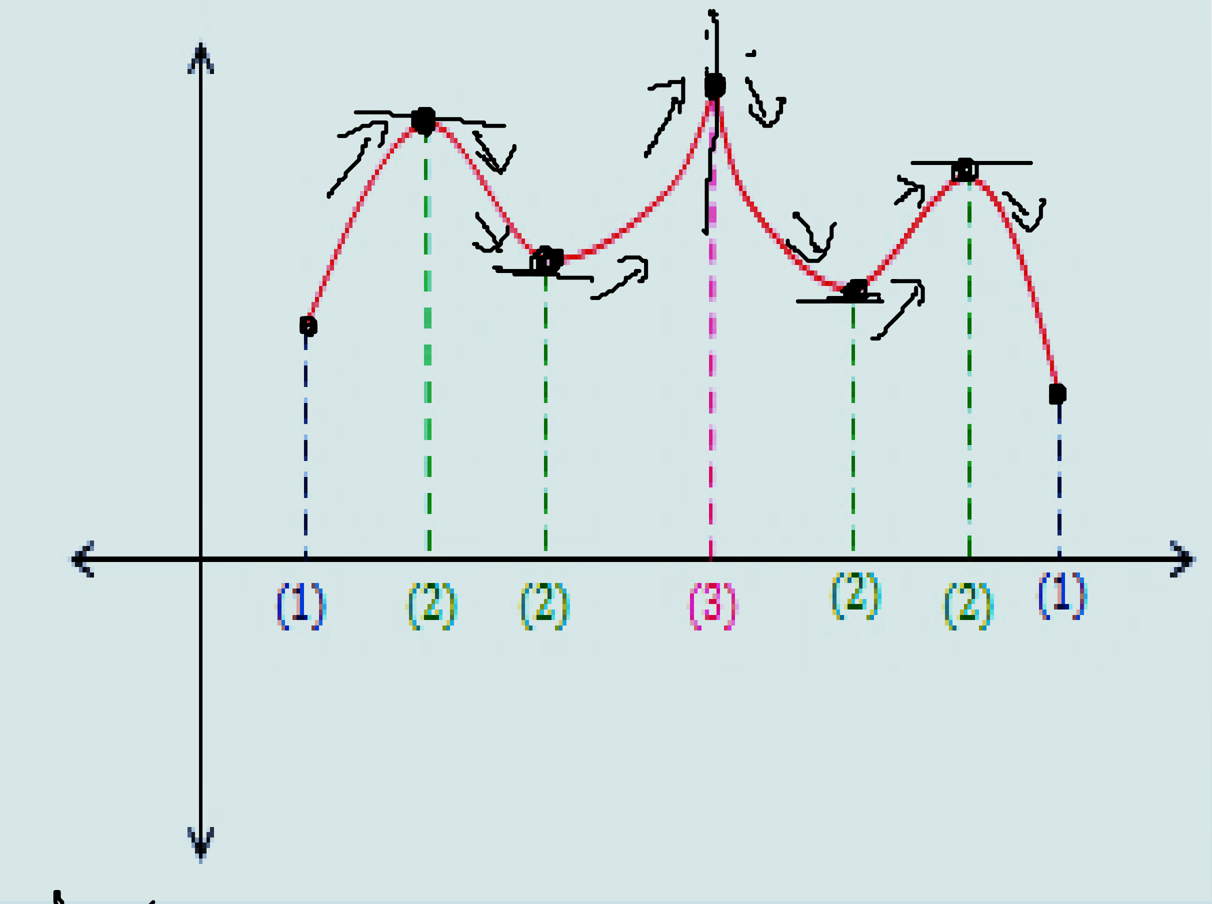
- An interior point of the domain of a function f where f' is zero **(stationary point)** or undefined **(singular point)** is a **critical point**.

- Hence the only domain points where a function f can possibly have an extreme value (local or global) are:

  √ (1) *Endpoints of an interval.*

  √ (2) *Stationary Points*: $f'(c) = 0$.

  √ (3) *Singular Points*: *$f'(c)$ does not exist.*



Critical points

Stationary

Singular

√ $f'(c) = 0$

$f'(x)$ is undefind

Note:

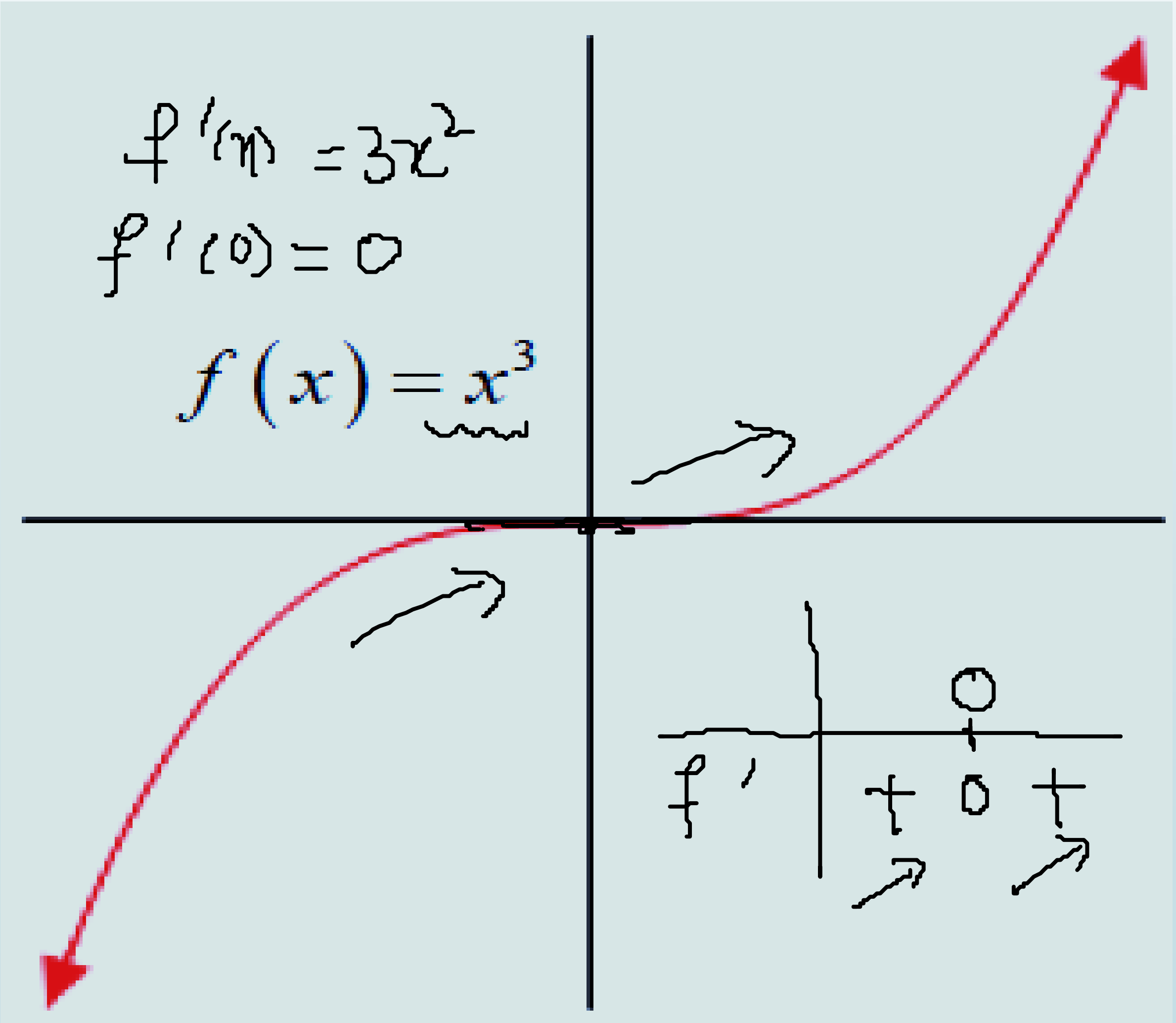
- Be careful not to misuse "The First Derivative Theorem for Local Extreme Values" because its converse is false.
- A differentiable function may have a critical point at $x = c$ without having a local extreme value there. This means that not every critical number correspond to a local maximum or local minimum. We use "local extrema" to refer to either a max or a min.

For example, the function $f(x) = x^3$, has a critical point at the origin and zero value there, but the function has no relative extrema and no absolute extrema.

If local extrem $\Rightarrow f'(c) = 0$

derivative is defined at $x = c$

$\not\Leftarrow$

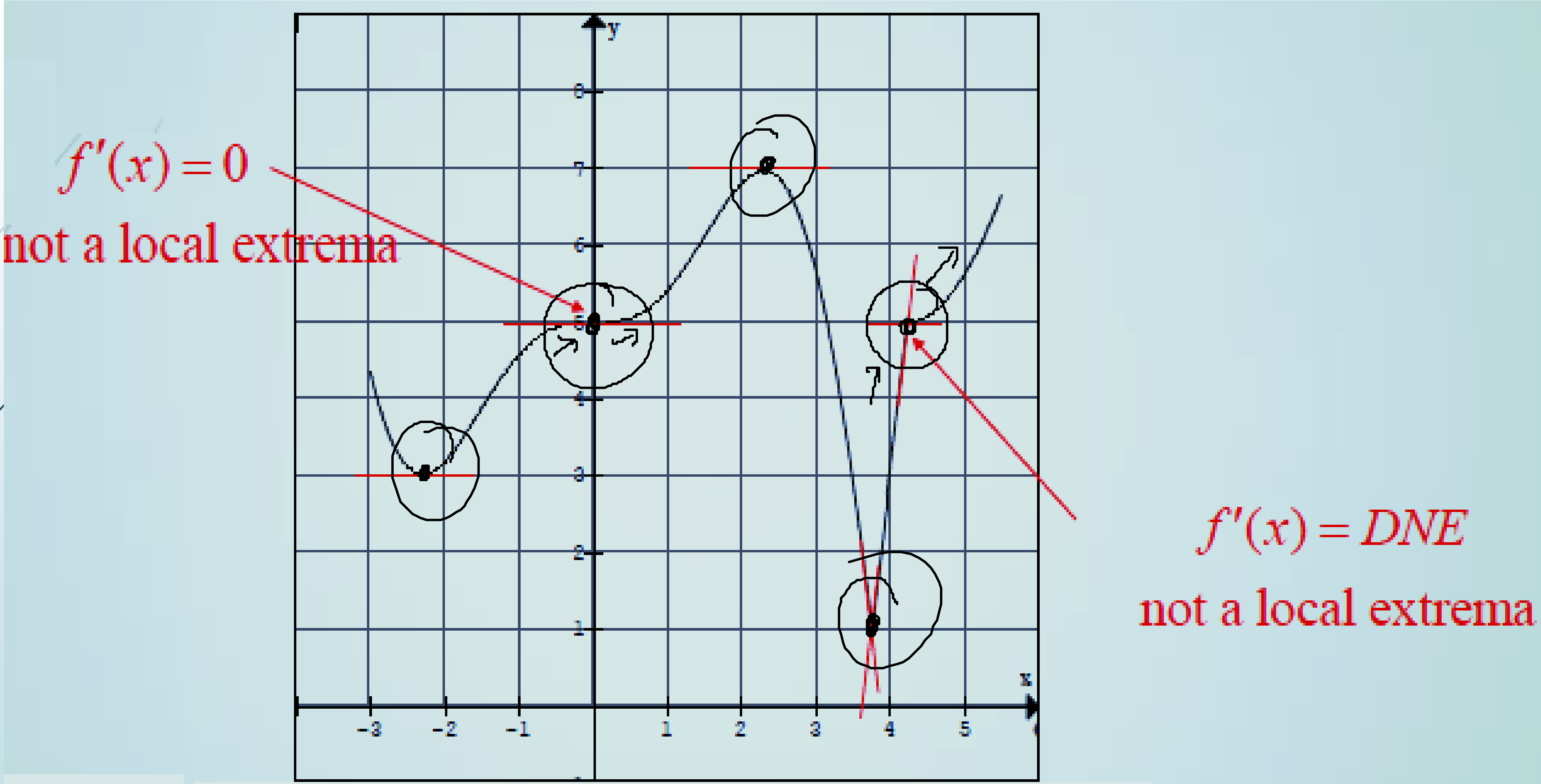


Fermat's Theorem:

$\boxed{\text{If a function } f(x) \text{ has a local maximum or minimum at } x = c\text{, then } x = c \text{ is a critical number of } f(x).}$

Note that the theorem does not say that at every critical number the function has a local maximum or local minimum

Example



Two critical points of $f(x)$ that do not correspond to local extrema

Candidates for Relative Extrema

1. **Stationary points:** any x such that x is in the *domain* of f and $f'(x) = 0$.

2. **Singular points:** any x such that x is in the *domain* of f and $f'(x)$ is undefined.

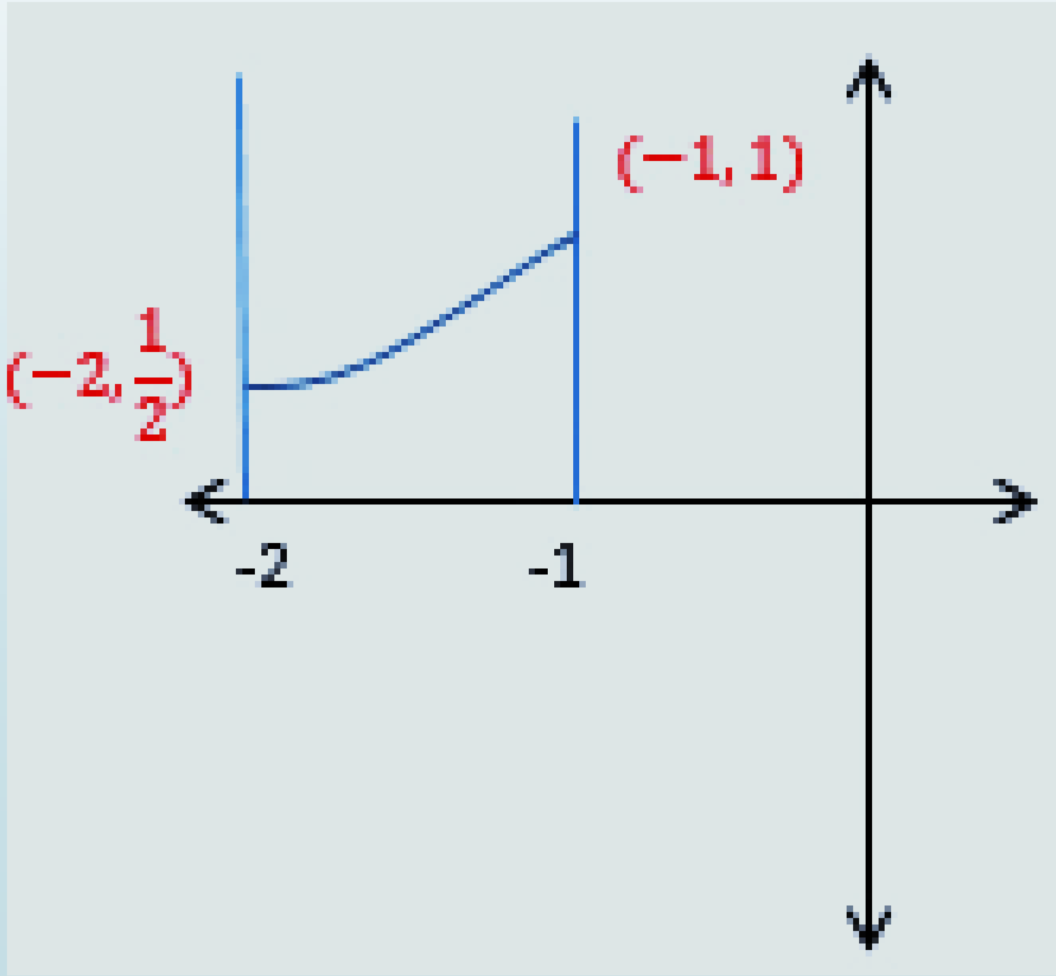
3. **End points** of an interval.

Remark: notice that not every critical number correspond to a local maximum or local minimum. We use "local extrema" to refer to either a max or a min.

Example:

Graph the function given below and calculate any absolute extreme values, if they exist. Moreover, plot them on the graph and state the coordinates.

$$f(x) = -\frac{1}{x} \qquad -2 \leq x \leq -1$$



$f(x) = -x^{-1}$

$f'(x) = x^{-2} = \frac{1}{x^2}$

$f'(x) \neq 0$

$f'(x)$ *is undefined at* $x = 0$

$x = 0$ *is not a critical point because not in* $[-2, -1]$

End points

$x = -2, -1$

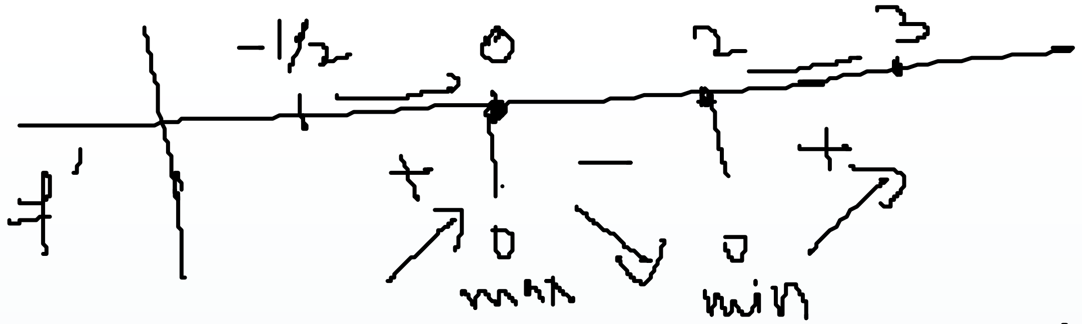
$f(-2) = \frac{1}{2}$ Absolute minimum

$f(-1) = 1$ Absolute maximum

Finding absolute extrema on $[a, b]$

1. Find all critical numbers for $f(x)$ in (a, b).
2. Evaluate $f(x)$ for all critical numbers in (a, b).
3. Evaluate $f(x)$ for the endpoints a and b of the interval $[a, b]$.
4. The largest value found in steps 2 and 3 is the absolute maximum for f on the interval $[a, b]$ and the smallest value found is the absolute minimum for f on $[a, b]$.

Example



Find the absolute extrema of $f(x) = x^3 - 3x^2$ on $\left[-\frac{1}{2}, 3\right]$.

For the present case: $f'(x) = 3x^2 - 6x = 3x(x-2)$.

Critical values of $f(x)$ inside the interval $(-1/2, 3)$ are: $x = 0, 2$.

Evaluate

$f(0) = 0$ ✓ ← Absolute Max.

$f(2) = -4$ ← Absolute Min.

$f\left(-\frac{1}{2}\right) = -\frac{7}{8}$

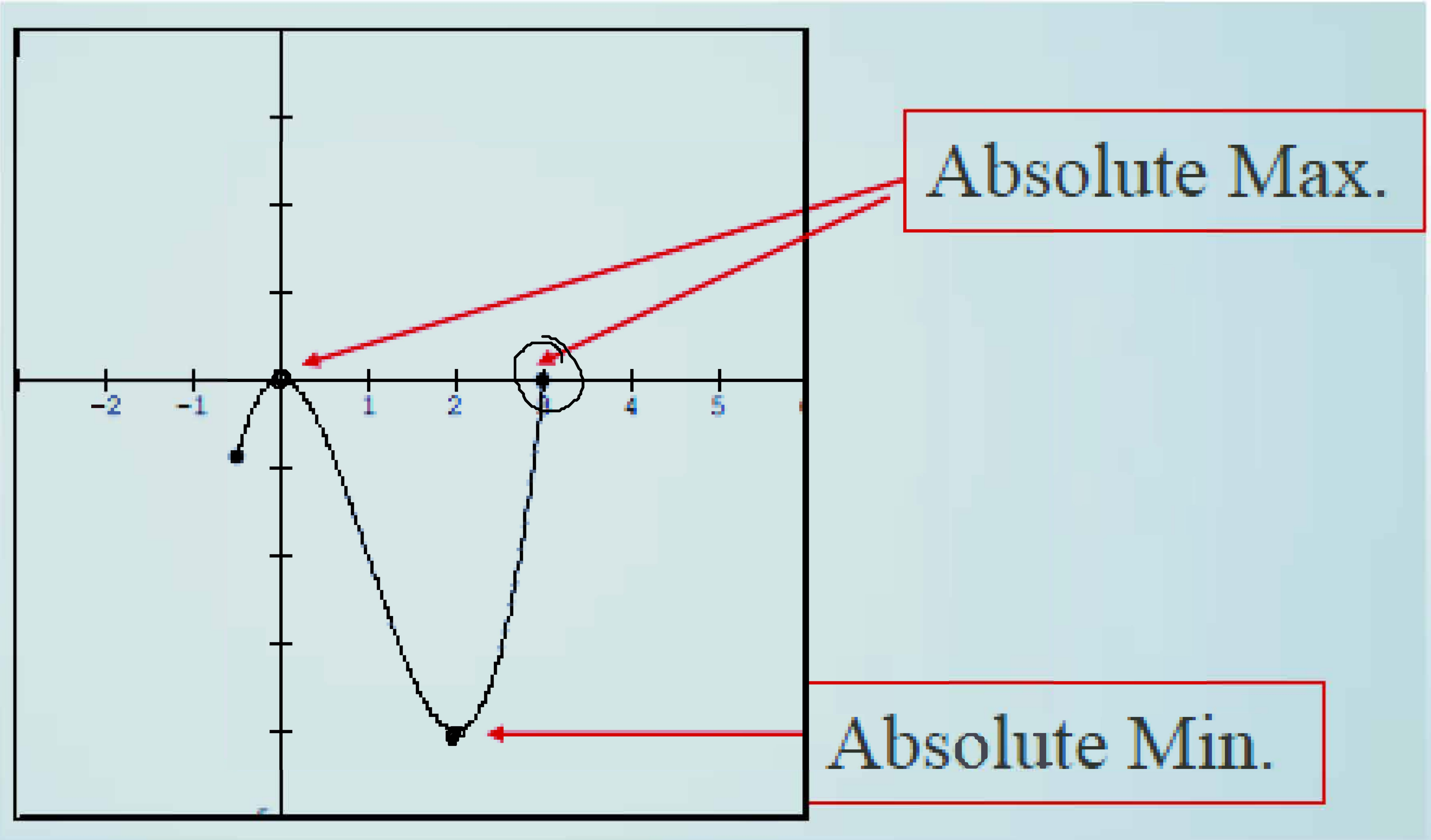
$f(3) = 0$ ✓ ← Absolute Max.

$f'(x) = 0$

$3x(x-2) = 0$

$\Rightarrow \boxed{x = 0, 2}$

Stationary



Absolute Max.
Absolute Min.
−2
−1
1
2
4
5

Example

Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.

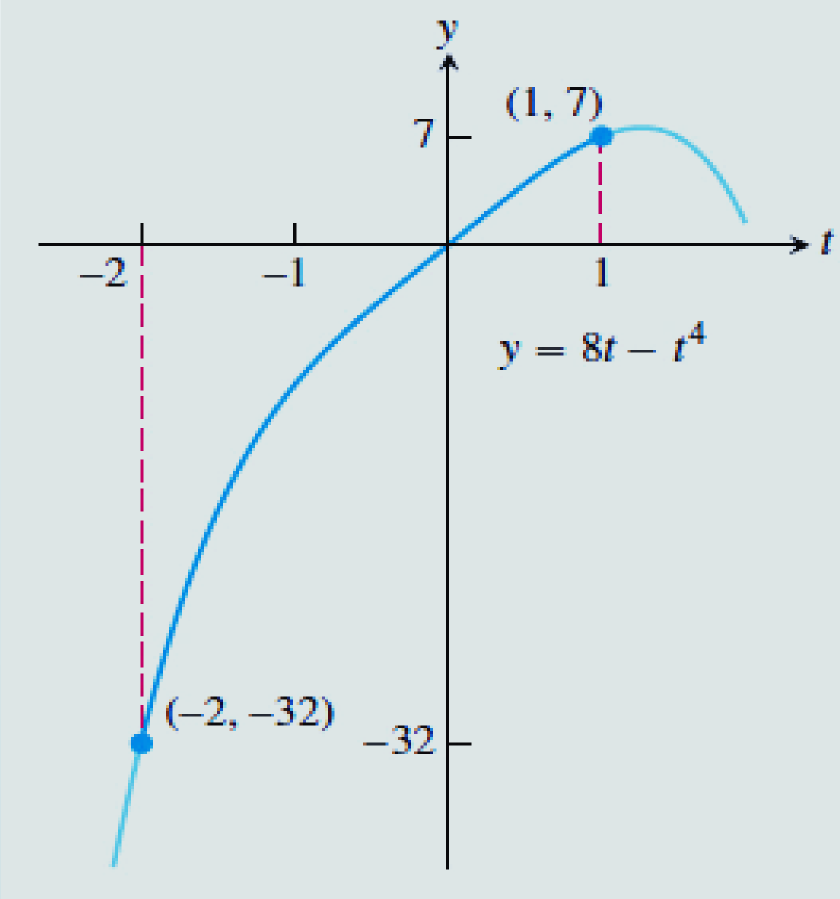
Solution:

The function is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$. Solving this equation gives

$$8 - 4t^3 = 0, \qquad \text{or} \qquad t = (2)^{3/2} > 1,$$

a point not in the given domain.

Therefore, the function's absolute extrema occur at the endpoints, $g(-2) = -32$ (absolute minimum), and $g(1) = 7$ (absolute maximum).



EXAMPLE :

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2,3]$.

Solution: We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values. The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

Singular

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are:

Critical point value:

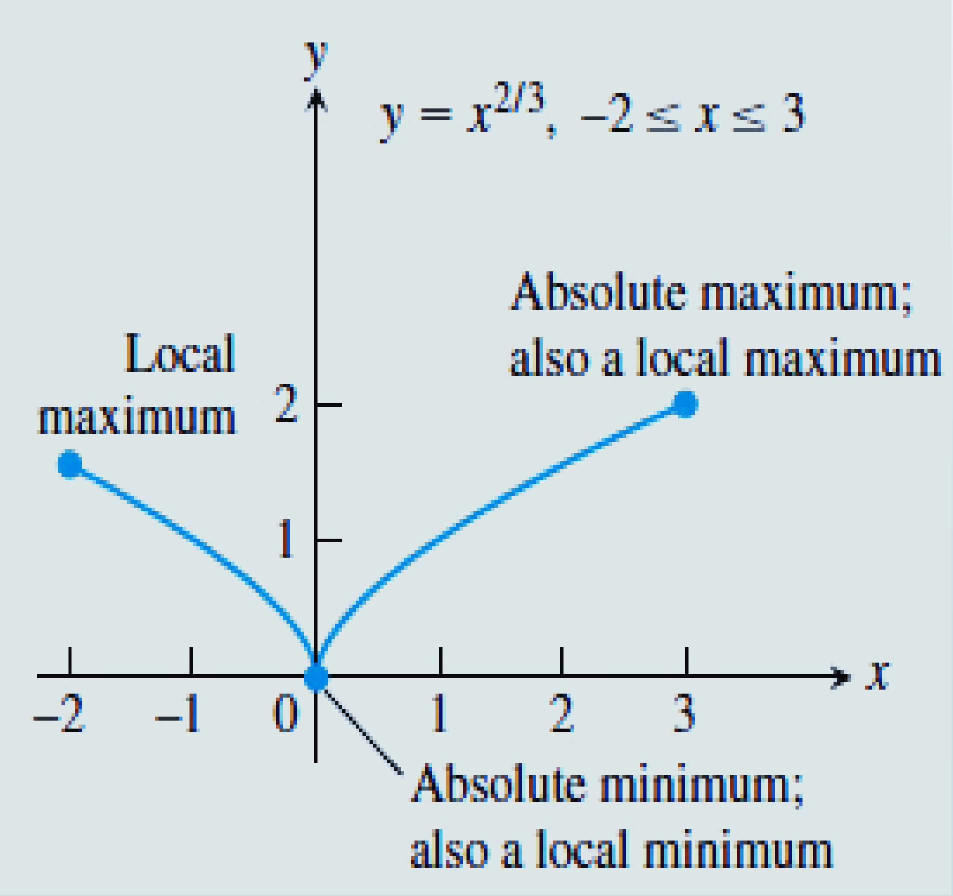
$$f(0) = 0, \quad \text{(Absolute minimum)}$$

min

Endpoint values:

$$f(-2) = (-2)^{2/3} = \sqrt[3]{4}$$

$$f(3) = (3)^{2/3} = \sqrt[3]{9} \quad \text{(Absolute maximum)}$$



Practice Questions

Book: Thomas Calculus (11th Edition) by Georg B.Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter: 4**
  - **Exercise: 4.1**

    Q # 1 – 54.