ELECTRIC FIELDS DUE TO CONTINUOUS CHARGE DISTRIBUTIONS - VOLUME CHARGE

Volume Charge

- A volume charge may be visualized by a region of space with a large number of charges separated by very small distances

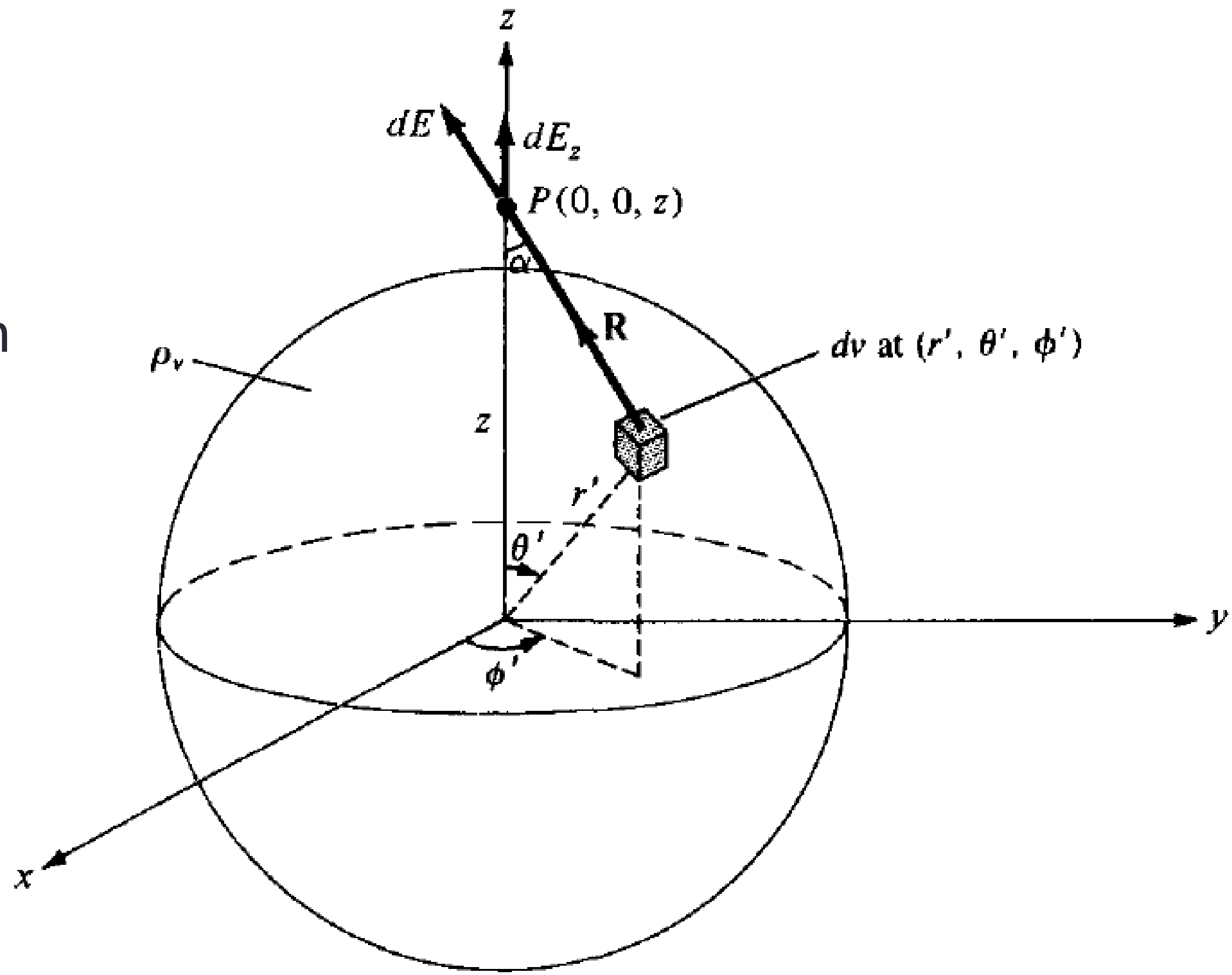
- We can replace this distribution of very small particles with a smooth continuous distribution described by a *volume charge density*

- We denote the volume charge density by ρ_v, having the units of coulombs per cubic meter (C/m^3)

Volume Charge

- A volume charge distribution with uniform charge density ρ_v is shown in figure

- We choose a volume of the shape of a cube



Volume Charge

- The charge dQ associated with the elemental volume dv is

$$dQ = \rho_v \, dv$$

- The total charge is given as:

$$Q = \int \rho_v \, dv = \rho_v \int dv$$

- Here, elemental volume dv depends upon the shape of the volume charge

- From figure, we have the electric field due to dv at $P(0,0,z)$ as:

$$d\boldsymbol{E} = \frac{\rho_v dv}{4\pi\epsilon_o R^2} \boldsymbol{a_R}$$

Volume Charge

- From the figure, $\mathbf{a}_R$ may be written as:

$$\mathbf{a}_R = \cos\alpha\, \mathbf{a}_z - \sin\alpha\, \mathbf{a}_\rho$$

- Due to the symmetry of the charge distribution, the contributions to E_x or E_y add up to zero

- We are left with only E_z, given by:

$$E_z = \mathbf{E} \cdot \mathbf{a}_z = \int dE \cos\alpha = \frac{\rho_v}{4\pi\varepsilon_o} \int \frac{dv \cos\alpha}{R^2}$$

- We need to derive expressions for dv, R^2, and $cos\alpha$

$$dv = r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi'$$

Volume Charge

- Applying the cosine rule to the figure, we have:

$$R^2 = z^2 + r'^2 - 2zr' \cos\theta'$$

$$r'^2 = z^2 + R^2 - 2zR \cos\alpha$$

- It is convenient to evaluate the integral in terms of R and r'

- Hence, we express $\cos\theta'$, $\cos\alpha$, and $\sin\theta'\, d\theta'$ in terms of R and r', that is:

$$\cos\alpha = \frac{z^2 + R^2 - r'^2}{2zR}$$

$$\cos\theta' = \frac{z^2 + r'^2 - R^2}{2zr'}$$

Volume Charge

- Differentiating the above equation with respect to θ' keeping z and r' fixed, we obtain:

$$\sin\theta'\, d\theta' = \frac{R\, dR}{z\, r'}$$

- Substituting values in the integral, we get:

$$E_z = \frac{\rho_v}{4\pi\varepsilon_o} \int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r'^2 \frac{R\, dR}{zr'}\, dr' \frac{z^2 + R^2 - r'^2}{2zR} \frac{1}{R^2}$$

$$= \frac{\rho_v 2\pi}{8\pi\varepsilon_o z^2} \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r' \left[1 + \frac{z^2 - r'^2}{R^2}\right] dR\, dr'$$

$$= \frac{\rho_v \pi}{4\pi\varepsilon_o z^2} \int_0^a r' \left[R - \frac{(z^2 - r'^2)}{R}\right]_{z-r'}^{z+r'} dr'$$

$$= \frac{\rho_v \pi}{4\pi\varepsilon_o z^2} \int_0^a 4r'^2\, dr' = \frac{1}{4\pi\varepsilon_o} \frac{1}{z^2} \left(\frac{4}{3}\pi a^3 \rho_v\right)$$

Volume Charge

- Earlier, the total charge was given as:

$$Q = \int \rho_v \, dv = \rho_v \int dv$$

- If we assume that the whole volume charge has a spherical volume with radius a, then:

$$Q = \rho_v \frac{4\pi a^3}{3}$$

- Using this value in the equation for electric field, we get:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o z^2}\,\mathbf{a}_z$$

Volume Charge

- Due to the symmetry of the charge distribution, the electric field at $P(r,\theta,\Phi)$ is readily obtained as:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\,\mathbf{a}_r$$

- It may be observed that the above equation is identical to the electric field at the same point due to a point charge Q located at the origin or the center of the spherical charge distribution

Problem-1

- A charge distribution is given by the following density:

$$\rho_v = \begin{cases} \dfrac{\rho_0 r}{R}, & 0 \le r \le R \\ 0, & r > R \end{cases}$$

- Determine electric field **E** at distance r > R and plot the magnitude of the electric field versus distance r.

Problem-2

- Calculate the total charge within each of the indicated volume:

a) $0 \le \rho \le 0.1, 0 \le \emptyset \le \pi, 2 \le z \le 4;\ \rho_v = \rho^2 z^2 sin 0.6\emptyset$

b) $Universe;\ \rho_v = \dfrac{e^{-2r}}{r^2}$