Inversion and reflection:- $w = \frac{1}{z}$

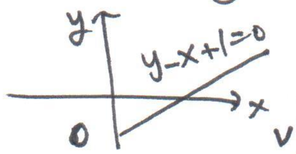
If $z = re^{i\theta}$ and $w = Re^{i\phi}$, putting these values we get
$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$, on equating $R = \frac{1}{r}$ and $\phi = -\theta$

Thus the point $P(r,\theta)$ in the z-plane is mapped onto the point $P'(\frac{1}{r}, -\theta)$ in the w-plane. Hence the transformation is an inversion of z and followed by reflection into the real axis.

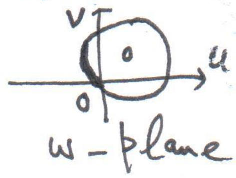
The points inside the unit circle ($|z|=1$) map onto points outside it, and points outside the unit circle into points inside it.

$w = \frac{1}{z} = \frac{1}{x+iy}$, $u = \frac{x}{x^2+y^2}$, $v = -\frac{y}{x^2+y^2}$. This shows that a circle in the z-plane transforms to another circle in the w-plane. But a circle through origin transforms into a straight line.

Ex:- under the transformation $w = \frac{1}{z}$, the image of the straight line $y-x+1=0$ is.



$$\frac{-v}{u^2+v^2} - \frac{u}{v^2+v^2} + 1 = 0 \Rightarrow u^2+v^2-u-v=0$$



This is equation of a circle with center at $(\frac{1}{2}, \frac{1}{2})$ and radius $1/\sqrt{2}$.

Ex:- Find the image of $|z-3i| = 3$ under the mapping $w = 1/z$.

$w = \frac{1}{z} \Rightarrow z = \frac{1}{w} \Rightarrow x+iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$, $x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$.

The given curve is $|z-3i| = 3 \Rightarrow x^2+(y-3)^2 = 9$

or $\frac{u^2}{(u^2+v^2)^2} + \left(-\frac{v}{u^2+v^2} - 3\right)^2 = 9 \Rightarrow u^2+v^2+9u^4+9v^4+6u^2v+6v^3+18u^2v^2 = 9u^4+18u^2v^2+9v^4$

$\Rightarrow u^2+v^2+6v(u^2+v^2) = 0 \Rightarrow (u^2+v^2)(6v+1) = 0 \Rightarrow 6v+1 = 0$ is the equation of the image.

Second method: $\left|\frac{1}{w} - 3i\right| = 3 \Rightarrow |1-3iw| = 3|w| \Rightarrow |1-3i(u+iv)| = 3|u+iv|$

$\Rightarrow (1+3v)^2+9u^2 = 9(u^2+v^2) \Rightarrow 1+6v+9v^2+9u^2 = 9(u^2+v^2) \Rightarrow 1+6v = 0$

Third Method: $|z-3i| = 3 \Rightarrow z-3i = 3e^{i\theta} \Rightarrow z = 3i+3e^{i\theta}$.

$$w = \frac{1}{z} = \frac{1}{3i+3e^{i\theta}} \Rightarrow 3w = \frac{1}{i+e^{i\theta}} \Rightarrow 3(u+iv) = \frac{1}{i+\cos\theta+i\sin\theta}$$

$$3u+3iv = \frac{\cos\theta - i(1+\sin\theta)}{\cos^2\theta+(1+\sin\theta)^2} \Rightarrow 3v = -\frac{1+\sin\theta}{2+2\sin\theta} = -\frac{1}{2} \Rightarrow 6v+1 = 0.$$

1. Show that under the transformation $w = \frac{1}{z}$, the image of the hyperbola $x^2 - y^2 = 1$ is the lemniscate $R^2 = \cos 2\phi$.

2. What is the region of the w-plane into ways the rectangular region in the z-plane bounded by the lines $x=0, y=0, x=1$ and $y=2$ is mapped under the transformation $w = z + (2-i)$?

Region bounded by $u=2, V=-1, u=3$ and $V=1$.

Bilinear Transformation (mobius transformation):

$$w = \frac{az+b}{cz+d}, \quad ad - bc \neq 0 \quad \text{——— (i)}$$

is known as bilinear transformation. (i) $\Rightarrow z = \frac{-dw+b}{cw-a}$ ——— (ii)

This is also bilinear except $w = a/c$. From (i), every point of z-plane is mapped into unique point in the w-plane except $z = -d/c$. From (ii), every point of w-plane is mapped into unique point in z-plane except $w = a/c$.

Cross-Ratio: If there are four points z_1, z_2, z_3, z_4 taken in order, then the ratio $\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross ratio of z_1, z_2, z_3, z_4.

A bilinear transformation preserves cross ratio of four points.

means $$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

Ex:- Find the bilinear transformation which maps the points $z = 1, i, -1$ into the points $w = i, 0, -i$. Hence find the image of $|z|<1$. Let the required transformation be $w = \frac{az+b}{cz+d}$.

or $w = \frac{pz+q}{rz+1}$ ——— (i) $\quad p = a/d, q = b/d, r = c/d$.

on substituting the values of z and corresponding values of w in (i),

$i = \frac{p+q}{r+1}$ $\quad$ Solving for p, q, r, $p = i, q = 1, r = -i$

$0 = pi + q / ri + 1$ $\quad$ $w = \frac{iz+1}{-iz+1} \Rightarrow u + iv = \frac{i(x+iy)+1}{-i(x+iy)+1} = \frac{-x^2-y^2+1+2ix}{x^2+(y+1)^2}$

$-i = -p+q / -r+1$ $\quad$ $u = -\frac{x^2-y^2+1}{x^2+(y+1)^2}$, $|z|<1 \Rightarrow x^2+y^2<1 \Rightarrow 1-x^2-y^2>0$

$u>0$ is the image.

Ex:- Find the bilinear transformation which maps the points $z=0,-1,i$ onto $w=i,0,\infty$. Also find the image of the unit circle.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad \text{——— (i)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$ on putting $z=0,-1,i$ into $w=i,0,\infty$ respectively

$$\frac{(w-i)(-1)}{(-1)(0-i)}=\frac{(z-0)(-1-i)}{(z-i)(-1-0)} \Rightarrow w=\frac{z+1}{z-i} \text{ —(ii)}, \quad z=\frac{iw+1}{w-1} \text{ —(iii)}$$

And $|z|=1 \Rightarrow \left|\frac{iw+1}{w-1}\right|=1 \Rightarrow |1+iw|=|w-1| \Rightarrow |1+i(u+iv)|=|u+iv-1|$

$\Rightarrow (1-v)^2+u^2=(u-1)^2+v^2 \Rightarrow u-v=0 \Rightarrow v=u$.

Ex:- Show that the transformation $w=i\frac{1-z}{1+z}$ transforms the circle $|z|=1$ onto the real axis of the w-plane and the interior of the circle into the upper half of the w-plane.

$$w=i\left(\frac{1-z}{1+z}\right) \Rightarrow u+iv=\frac{1-(x+iy)}{1+x+iy}=\frac{2y+i(1-x^2-y^2)}{(1+x)^2+y^2}$$

$u(x,y)=\frac{2y}{(1+x)^2+y^2}$ —(i), $v(x,y)=\frac{1-(x^2+y^2)}{(1+x)^2+y^2}$ —(ii), when $|z|=1 \Rightarrow x^2+y^2=1 \Rightarrow v=0$.

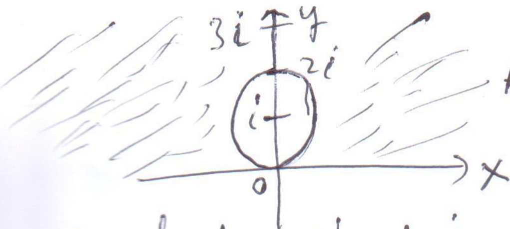
Now the equation of the interior of the circle is $x^2+y^2<1$.

$\frac{(i)}{(ii)} \Rightarrow \frac{u}{v}=\frac{2y}{1-(x^2+y^2)} \Rightarrow x^2+y^2=\frac{1-2vy}{u}$, $1-\frac{2vy}{u}<1$ ($\because x^2+y^2<1$)

$-\frac{2vy}{u}<0$, $2vy>0 \Rightarrow v>0$. $v>0$ is the equation of the upper half plane.

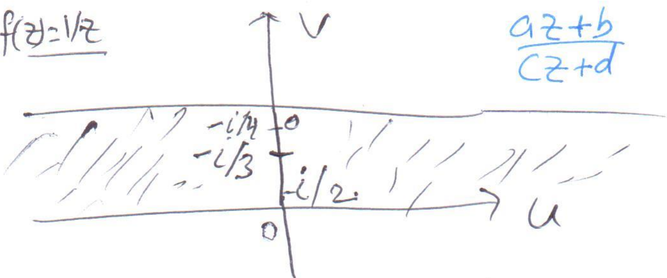
1- Show that $w=\frac{i-z}{i+z}$ maps the real axis of the z-plane into the circle $|w|=1$ and (ii) the half plane $y>0$ into the interior of the unit circle $|w|<1$ in the w-plane.

2. Show that the transformation $w=3-z/z-2$ transforms the circle with center $(5/2,0)$ and radius $1/2$ in the z-plane into the imaginary axis in the w-plane and the interior of the circle into the right half of the plane.



Inversion $w=f(z)=1/z$

An explanation



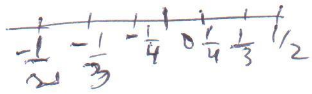
$\frac{az+b}{cz+d}$

Image of $|z-i|>1$ is $-\frac{1}{2}\le v\le 0$.

$z=2i$, $x=0, y=2$, The image is $u(x,y)=\frac{x}{x^2+y^2}$, $v(x,y)=\frac{-y}{x^2+y^2}$

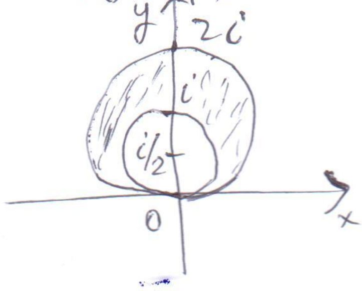
$u=0$, $v=\frac{-2}{0^2+2^2}=-\frac{1}{2}$

$z=3i$, $x=0, y=3$, $u=0$, $v=\frac{-3}{0^2+3^2}=\frac{-1}{3}$

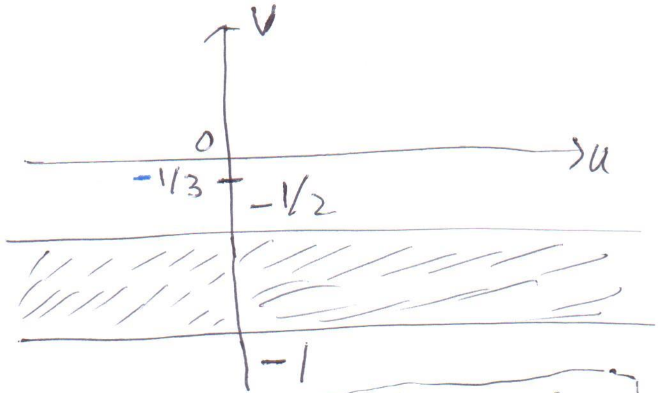


$z=4i$, $x=0, y=4$, $u=0$, $v=\frac{-4}{0^2+4^2}=\frac{-1}{4}$

Now,



Image



Images of various Points:

$z=i/2$, $x=0, y=\frac{1}{2}$, $u=0$, $v=\frac{-1/2}{0^2+(\frac{1}{2})^2}=\frac{-1}{1/2}=-2$

$z=i$, $x=0, y=1$, $u=0$, $v=\frac{-1}{0^2+1^2}=-1$

$z=\frac{3}{2}i$, $x=0, y=\frac{3}{2}$, $u=0$, $v=\frac{-3/2}{0^2+(3/2)^2}=\frac{-1}{3/2}=\frac{-2}{3}$

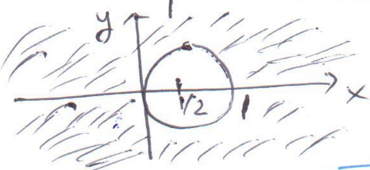
$z=2i$, $x=0, y=2$, $u=0$, $v=\frac{-2}{0^2+2^2}=-1/2$

Note: For boundaries of Circles in the form of horizontal/vertical lines use standard procedure. Points for images (as this handout) is for strip location.

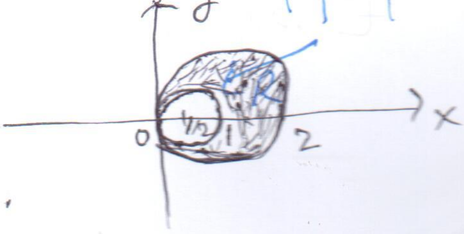
$z=3i$, $x=0, y=3$, $u=0$, $v=\frac{-3}{0^2+3^2}=-\frac{1}{3}$, which is not part in the w-plane.

Hence, image of the region between two circles with Centeres, i.e, $|z-\frac{i}{2}|>1$ & $|z-i|<2$ is the strip $-1\le v\le -1/2$.

Exercise: Try/repeat above for i) $|z-\frac{1}{2}|>\frac{1}{2}$



ii) Two Cylinders: Note: In this Case(s) The boundaries will be mapped to $u=$ Constants while regions to strips in the form $a\le u\le b$, a & b Constants.



The Bilinear or Fractional transformation:

$w = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha\delta - \beta\gamma \neq 0$, is bilinear or fractional

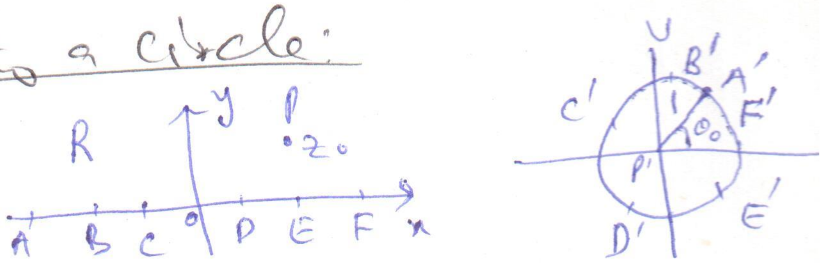
translation, rotation, stretching, and inversion.

Maps three distinct points of the z-plane into three distinct points of the w-plane, one of which may be infinity.

for four distinct points $\dfrac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$ is called the Cross ratio of z_1, z_2, z_3, z_4.

mapping of a half plane onto a circle:

$w = e^{i\theta_0}\left(\dfrac{z - z_0}{z - \bar{z}_0}\right)$



$w = \dfrac{az + b}{cz + d}$, where $ad - bc \neq 0$ —— (1)

If $c = 0$, $w = az + b$, reduces as a linear transformation.

Also if a, b & d are zero, we get inversion transformation.

(1) can also be expressed as $z = \dfrac{-dw + b}{cw - a}$ —— (2)

(2) is inverse of (1) and we see that it too is a bilinear transformation

Notice that $z = -d/c$ when $w \to \infty$

$w = a/c$ when $z \to \infty$.

$w = \dfrac{az + b}{cz + d} = \dfrac{a}{c}\,\dfrac{z + b/a}{z + d/c} = \dfrac{a}{c}\cdot\dfrac{z + \frac{d}{c} + \left(\frac{b}{a} - d/c\right)}{z + d/c}$

$w = \dfrac{a}{c} + \dfrac{bc - ad}{c^2}\cdot\dfrac{1}{z + d/c}$

$w = A + B\cdot\dfrac{1}{z + D}$ —— (3), $\dfrac{a}{c} = A$, $\dfrac{bc - ad}{c^2} = B$, $d/c = D$.

i) translate the region by the vector D.

ii) Apply the inversion transformation.

(iii) magnify the region by factor B.

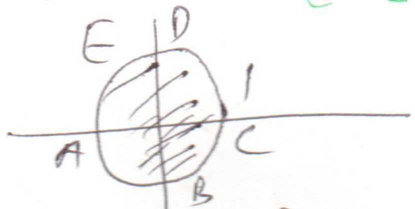
(iv) Rotate the region through the angle arg(B).

(V) Finally, translate the region through the vector A.

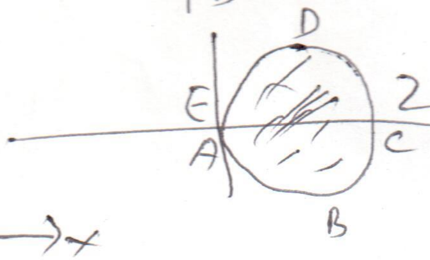
Since all these five steps preserve the circle the general bilinear transformation preserves the circle

EX: $|z| \leq 1$ under $w = -i\left(\frac{z-1}{z+1}\right)$ $= -i\left(\frac{z+1-2}{z+1}\right)$ $= -i\left(1-\frac{2}{z+1}\right)$

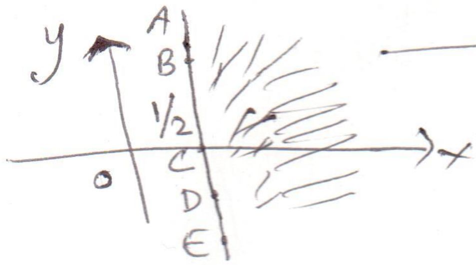
$$w = -i + 2i \cdot \frac{1}{z+1}$$



i) translation

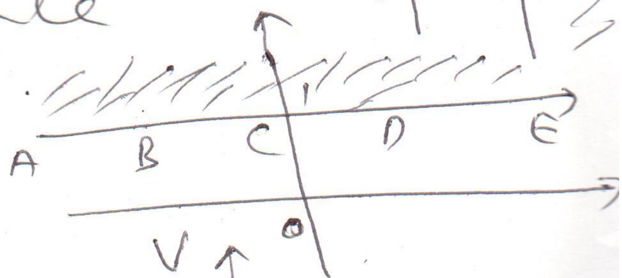


ii) The inversion yields



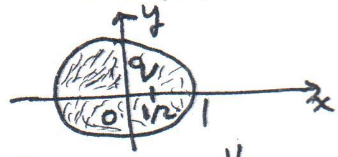
(iii) The magnification by $|2i| = 2$ gives

iv) The rotation through the angle $\arg(2i) = \pi/2$ yields

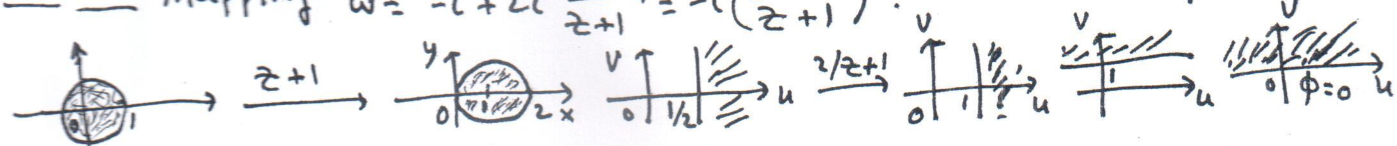


Finally the translation by the vector '$-i$' gives

A line of charge 'q' per unit length is located at $z=\frac{1}{2}$ while $|z|=1$ is a grounded conductor. Find the complex potential inside $|z|\le 1$ and the electrostatic potential.



Solution: mapping $w=-i+2i\frac{1}{z+1}=-i\left(\frac{z-1}{z+1}\right)$.



Line of charge $z=\frac{1}{2}$, $w=-i\left(\frac{1/2-1}{1/2+1}\right)=i/3$. Same is the case if we place an imaginary line of charge '$-q$' per unit length at $w=-i/3$, so the charge will be zero at ground conductor.

Complex potential: $F(z)=-2q\ln(w-i/3)+2q\ln|w+i/3|$.

$$=2q\ln\left|\frac{w+i/3}{w-i/3}\right|=2q\ln\left|\frac{-i\left(\frac{z-1}{z+1}\right)+i/3}{-i\left(\frac{z-1}{z+1}\right)-i/3}\right|$$

$$F(z)=2q\ln\left|\frac{z-2}{2z-1}\right| \quad \text{(After simplification).}$$

Electrostatic potential: $\phi(x,y)=\mathrm{Re}\left[2q\ln\left|\frac{(x+iy)-2}{2(x+iy)-1}\right|\right]$

$$\phi(x,y)=2q\ln\left|\frac{(x-2)^2+y^2}{(2x-1)^2+4y^2}\right|.$$

Verification of mapping: $w=-i\left(\frac{z-1}{z+1}\right)$, $z=\frac{i-w}{i+w}$

$|z|\le 1 \Rightarrow \left|\frac{i-w}{i+w}\right|\le 1 \Rightarrow |i-w|\le|i+w|$

$|i-(u+iv)|\le|i+(u+iv)| \Rightarrow |-u+i(1-v)|\le|u+i(1+v)|$

$\Rightarrow u^2+(1-v)^2\le u^2+(1+v)^2 \Rightarrow \cancel{u^2}+\cancel{1}-2v+\cancel{v^2}\le\cancel{u^2}+\cancel{1}+2v+\cancel{v^2}$

$4v\ge 0 \Rightarrow v\ge 0$ (upper half plane)