National University of Sciences & Technology
School of Electrical Engineering and Computer Science
Department of Basic Sciences

MATH-101: Calculus and Analytical Geometry (3+0): BEE2k20-ABC Fall 2020

Assignment 2	
CLO: 1 (Understand the concept of limit, continuity and derivative with its application to find extrema)	
Maximum Marks: 40	Instructor: Dr. Naila Amir
Announcement Date: 23rd November 2020	Due Date: 1st December 2020

Instructions:

- Understanding the question is part of the assignment and copying is not allowed.
- Express your answer in the most simplified form. Direct calculations using calculator are not allowed, you need to show the detail of your work to get the maximum marks.
- This is an individual assignment.
- Assignment must be handwritten and properly scanned in a single pdf file. These two pages must be part of every assignment.
- Assignment must be properly tagged and is required to be submitted on MS teams.
- Assignment is not acceptable after deadline.

Tasks: Attempt all questions.

Students Name	CMS Id.	Section
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Total Marks	**Marks Obtained**	**Weight in 10**
40 Marks		

Q - 1: [15 marks]

For the function $f(x) = \frac{2x^3+x^2-8x-4}{x^2-3x+2}$

a) Determine the following:

1. Domain of $f(x)$.
2. x & y −intercepts (if any).
3. Vertical/horizontal/oblique asymptotes (if any).
4. Holes (if any).

b) By using the information in part (a), graph the given function.

Q - 2: [10 marks]

Evaluate the following limits:

a) $\lim\limits_{y \to x} \frac{y^{2/3}-x^{2/3}}{y-x}$.

b) $\lim\limits_{x \to \pi} \frac{\tan(\sin x)}{\sin x}$.

c) $\lim\limits_{x \to 0} f(x), \quad \text{where} \quad f(x) = \begin{cases} (1+x)^{1/x}; & x \neq 0 \\ 1; & x = 0 \end{cases}$

d) $\lim\limits_{x \to -1} x^3 \left\lfloor \frac{1}{x} \right\rfloor$. (Floor Function)

Q - 3: [15 marks]

Let $f(x) = x^2$ and $g(x) = \begin{cases} -4, & x \leq 0 \\ |x-4|, & x > 0 \end{cases}$

a) Determine fog and gof.

b) Graph the functions fog and gof.

c) Determine whether fog and gof are continuous at $x = 0$. If not continuous then what type of discontinuity exists at this point?

Q.1

$$f(x) = \frac{2x^3 + x^2 - 8x - 4}{x^2 - 3x + 2}$$

a)

(1.) Domain of $f(x)$

$$f(x) = \frac{x^2(2x+1) - 4(2x+1)}{(x-2)(x-1)}$$

$$= \frac{(2x+1)(x^2-4)}{(x-2)(x-1)} = \frac{(2x+1)(x-2)(x+2)}{(x-2)(x-1)}$$

Domain of $f(x) = \mathbb{R} - [1, 2]$

$$= (-\infty, 1) \cup (1, 2) \cup (2, \infty)$$

(2.) x & y intercepts

For x intercept; $y = 0$

$$\Rightarrow \frac{(2x+1)(x+2)}{(x-1)} = 0$$

$$(2x+1)(x+2) = 0$$

x-intercepts are -2 & $-1/2$

For y intercept; $x = 0$

$$\Rightarrow \frac{(2(0)+1)(0+2)}{(-1)} = y$$

$$-2 = y$$

y-intercepts are -2

(3) Asymptote

Vertical Asymptotes:

- Denominator $= 0$

$x - 1 = 0$

$x = 1$ Vertical Asymptote

- Horizontal Asymptote:

  Since the power of numerator is greater than denominator, there is no horizontal asymptote.

- Oblique Asymptote:

  Using long division

$$
\begin{array}{r|l}
 & 2x + 7 \\
x^2 - 3x + 2 & \cancel{2x^3} + x^2 - 8x - 4 \\
 & -\cancel{2x^3} \pm 6x^2 - 4x + 0 \\
\hline
 & \quad \cancel{7x^2} - 12x - 4 \\
 & \quad \pm \cancel{7x^2} \mp 21x \pm 14 \\
\hline
 & \qquad 9x - 18
\end{array}
$$

- We get, $\dfrac{9x-18}{x^2-3x+2} + (2x+7)$

  Rational expression approaches 0 as $x \to \infty$.

  $y = 2x + 7$ Oblique Asymptote

- Holes:

  Setting common factor $(x-2)$ equal to zero then solving for y.

  $x - 2 = 0 \quad \to \quad x = 2$

  $y = \dfrac{(2x+1)(x+2)}{(x-1)} \Rightarrow \dfrac{(2(2)+1)(2+2)}{(2-1)}$

  $\dfrac{20}{1} = y$

  $y = 20$

  The hole is at $(2, 20)$

b) Graph:

$$y = \frac{2x^3 + x^2 - 8x - 4}{x^2 - 3x + 2}$$

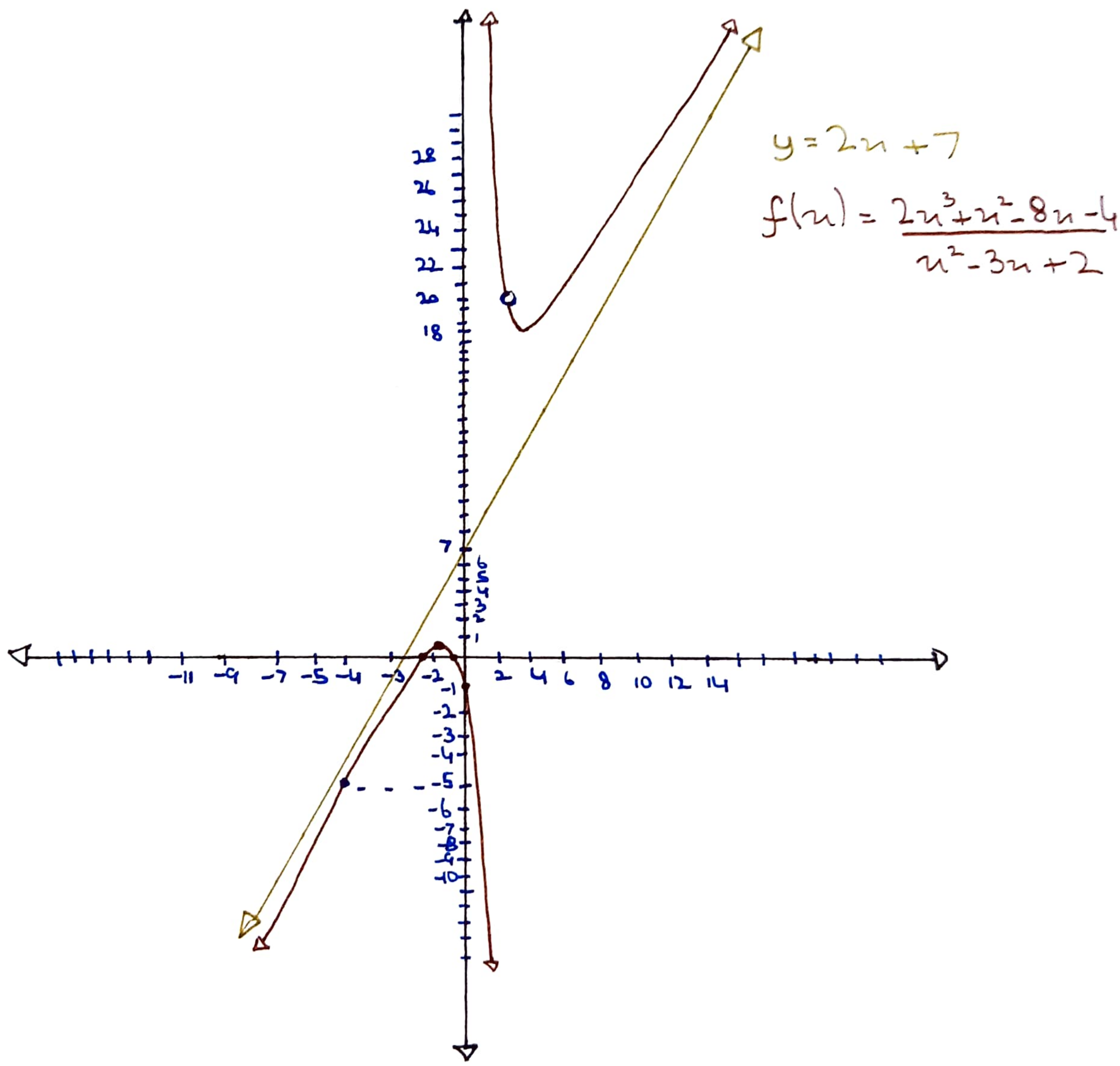
x	-8	-4	0	$+0.5$	$+4$	2
y	-10	-2	-2	-10	18	Und.

Asymptotes:

$x = 1$

$y = 2x + 7$

x	-3	-2	-1	0	1	2
y	1	3	5	7	9	11



Q2

a) $\lim_{y\to x} \dfrac{y^{2/3} - x^{2/3}}{y-x}$

$\lim_{y\to x} \dfrac{y^{(1/3)2} - x^{(1/3)2}}{y^{(3)(1/3)} - x^{(3)(1/3)}}$

$\lim_{y\to x} \dfrac{(y^{1/3} - x^{1/3})(y^{1/3} + x^{1/3})}{(y^{1/3})^3 - (x^{1/3})^3}$

$\lim_{y\to x} \dfrac{\cancel{(y^{1/3} - x^{1/3})}(y^{1/3} + x^{1/3})}{\cancel{(y^{1/3} - x^{1/3})}(y^{2/3} + y^{1/3}x^{1/3} + x^{2/3})}$

$\lim_{y\to x} \dfrac{(y^{1/3} + x^{1/3})}{(y^{2/3} + y^{1/3}x^{1/3} + x^{2/3})}$

$\dfrac{(x^{1/3} + x^{1/3})}{(x^{2/3} + x^{2/3} + x^{2/3})}$

$\dfrac{2x^{1/3}}{3x^{2/3}}$

$\dfrac{2}{3} x^{1/3 - 2/3}$

$\boxed{\dfrac{2}{3x^{1/3}}}$

b) $\lim_{x\to \pi} \dfrac{\tan(\sin x)}{\sin(x)}$

Let $y = \sin x$ As $x \to \pi$, $y \to 0$

$\lim_{y\to 0} \dfrac{\tan(y)}{y}$

$\lim_{y\to 0} \dfrac{\sin(y)}{\cos(y)} \cdot \dfrac{1}{y}$

$\lim_{y\to 0} \dfrac{\sin(y)}{y} \cdot \dfrac{1}{\cos(y)}$

$$\lim_{y\to 0} \frac{\sin y}{y} \cdot \lim_{y\to 0} \frac{1}{\cos y}$$

$$(1) \cdot \lim_{y\to 0} \frac{1}{\cos y}$$

$$(1) \cdot \frac{1}{\cos(0)}$$

$$\boxed{1}$$

c) $\lim_{x\to 0} f(x)$, $f(x) = \begin{cases} (1+x)^{1/x} & ; x \neq 0 \\ 1 & ; x = 0 \end{cases}$

Let $(1+x)^{1/x} = y$

$$\ln(y) = \ln(1+x)^{1/x}$$

$$\ln(y) = \frac{1}{x} \ln(1+x)$$

Applying limits,

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \ln(y)$$

From L'Hopital Rule;

$$\lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = \ln(y)$$

$$\ln(y) = 1$$

$$y = e^1$$

$$y = e$$

$$\boxed{\lim_{x\to 0} f(x) = e}$$

Alternatively, without L'Hopital's Rule

$y = (1+x)^{1/x}$

Using Binomial Expansion

$$\left(1 + \left(\frac{1}{x}\right)x + \frac{(1/x)(1/x - 1)}{2!}x^2 + \dots\right)$$

$$\left(1 + 1 + \frac{1}{2!}(1/x)(1/x - 1)\,x^{\cancel{2}} + \dots\right)$$

$$1 + 1 + \frac{1}{2!}\,\frac{1}{\cancel{x}}(1-x)\cancel{x} \dots$$

$$1 + 1 + \frac{1}{2!}(1-x) \dots$$

Applying limits

$$y = \lim_{x \to 0}\left(1 + 1 + \frac{1}{2!}(1-x) \dots\right)$$

$$= 1 + 1 + \frac{1}{2!} + \dots$$

As,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\boxed{y = e^1 = e}$$

Hence,

$$\lim_{x \to 0} f(x) = e$$

d) $\lim_{x \to -1} x^3 \lfloor \frac{1}{x} \rfloor$

Approaching from right

$$\lim_{x \to -1^+} x^3 \lfloor \tfrac{1}{x} \rfloor$$

$$(-1)^3(-2) = 2$$

$\lim_{x \to -1^-} x^3 \lfloor \frac{1}{x} \rfloor$

$(-1)^3 (1/-1.001)$

$(-1)^3 (-1)$

$\boxed{1}$

Q3

$f(x) = x^2 \quad , \quad g(x) = \begin{cases} -4 & x \le 0 \\ |x-4| & x > 0 \end{cases}$

ⓐ $f \circ g(x) = f(g(x)) = \begin{cases} 16 & x \le 0 \\ (|x-4|)^2 & x > 0 \end{cases}$

$g \circ f(x) = g(f(x)) = \begin{cases} -4 & x = 0 \\ |x^2 - 4| & x > 0 \end{cases}$

ⓒ Continuity at $x = 0$

For $f \circ g(x)$:-

$\lim_{x \to 0^-} g(x) = -4 \quad \& \quad \lim_{x \to 0^+} g(x) = 4$

Hence, two sided limit does not exist.

$g(x)$ is not continuos at $x = 0$.

$\Downarrow$

$f \circ g(x)$ is not continuous at $x = 0$

For $g \circ f(x)$:-

$\lim_{x \to 0^-} f(x) = 0 \quad \& \quad \lim_{x \to 0^+} f(x) = 0$

$f(x)$ is continuous at $x = 0$

Now, $\lim_{x \to 0} [g(f(0))]$

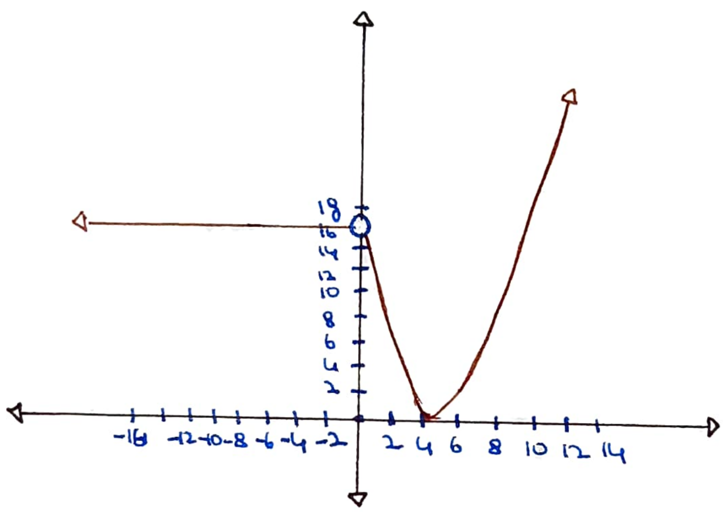
$\lim_{x \to 0^-} g(x) = -4 \quad \& \quad \lim_{x \to 0^+} g(x) = 4$

Hence, $g(f(x))$ is discontinuous at $x = 0$

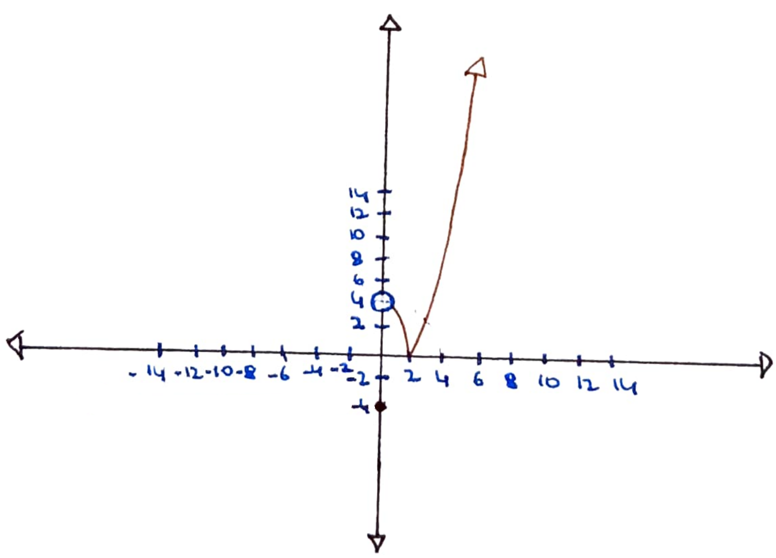
b) Graph

$$f\circ g(x) = \begin{cases} 16 & x \le 0 \\ (|x-4|)^2 & x > 0 \end{cases}$$

x	-3	-2	-1	0	1	2	3
16	16	16	16	16			
$(\lvert x-4\rvert)^2$					9	4	1



$$g\circ f(x) = \begin{cases} -4 & x = 0 \\ |x^2-4| & x > 0 \end{cases}$$



x	-3	-2	-1	0	2	4
-4	-4	-4	-4	-4		
$\lvert x^2-4\rvert$					0	12