Fourier Series:- A French mathematical Physicist, Fourier (1766-1830) used such trigonometric series in his investigations into the theory of heat, and they appeared throughout his study. However, Fourier did not invent Fourier series. Daniel Bernoulli and Euler used such series while investigating problems concerning vibrating strings and astronomy. It turns out that expressing a function as a series of trigonometric functions (Sine and Cosine) is sometime more advantageous than expanding it as a Power Series. In particular, astronomical phenomena are usually periodic as are heart beats, tides and vibrating strings, so it makes sense to express them in terms of periodic functions.

The integral formulas that define the coefficients a_0, a_n, and b_n were discovered by Euler in 1777. Today, Fourier series, the Fourier integral and Fourier transform constitute a branch of mathematical analysis that is valuable in the study of wave phenomena.

Periodic Functions:- A periodic function is any function for which

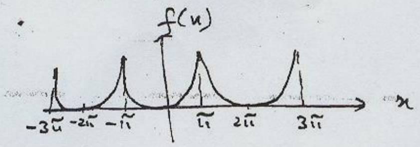
$$f(x+p) = f(x) \text{ ; for all } x .$$

The smallest constant p that satisfies this condition is called the period of the function. By iteration of this, we have

$$f(x+np) = f(x), \quad n = 0, \pm1, \pm2, \ldots$$

Ex:- $f(x) = c$, i.e., constant function is a periodic function of a period p for any value of p.

Ex:- $f(x) = x^2$; $-\pi \le x \le \pi$;
$f(x+2\pi) = f(x)$



Ex:- Find the period of the function $f(x) = \cos 2x$.

We know that $\cos(\theta + 2\pi m) = \cos\theta$. So, $\cos\theta$ is periodic function of period 2π $(m=1)$; Now, $\cos 2(x+p) = \cos 2x$

$\cos(2x+2p) = \cos 2x$; As, $\cos(\theta + 2\pi m) = \cos\theta$;

$\Rightarrow 2p = 2\pi m \Rightarrow p = \pi m$; when $m=1$; we get

$p = \pi$; which is the required period for $\cos 2x$.

Ex:- Find the period of the function: $f(t) = \tan(\pi t)$.

$f(t+p) = f(t) \Rightarrow \tan(\pi(t+p)) = \tan(\pi t)$ i.e, $\tan(\pi t + \pi p) = \tan(\pi t)$.

But, $\tan(x)$ has period π; hence, $\pi p = \pi$; hence, $p = 1$.

Ex:- If $f(t) = \sin(\pi t)$, $f(t+p) = f(t) \Rightarrow \sin(\pi(t+p)) = \sin(\pi t)$

i.e, $\sin(\pi t + \pi p) = \sin(\pi t)$; But $\sin(x)$ has period 2π; so,

$\pi p = 2\pi \Rightarrow \sin(\pi t)$ has period $p = 2$.

Ex:- Find the period of the function:

$$f(t) = \cos\left(\frac{t}{3}\right) + \cos(t/4).$$

By definition,

$$f(t+p) = \cos\left(\frac{t+p}{3}\right) + \cos\left(\frac{t+p}{4}\right) = \cos\frac{t}{3} + \cos\frac{t}{4}.$$

Since, $\cos(t + 2\pi m) = \cos t$; for any integer m, we see that

$\frac{p}{3} = 2\pi m$ and $\frac{p}{4} = 2\pi n$, where m and n are integers. Therefore, $p = 6\pi m = 8\pi n$; when $m = 4$ and $n = 3$, we obtain the smallest value of p; hence, $p = 24\pi$.

Exercises:- Calculate the periods of the following functions:

(1). $f(t) = \cos\left(\frac{\pi t}{2}\right)$ (2). $\sin\left(\frac{\pi t}{4}\right)$.

(3). $f(t) = \sin\left(\frac{2\pi t}{k}\right)$ (4). $f(t) = \cos(nkt)$.

(5). $f(t) = \sin(t/2k)$ (6). $\cos(\pi t)$.

(7). $f(t) = \sin\left(\frac{t}{4}\right) + \sin\left(\frac{t}{6}\right)$

(8). $f(t) = \sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{3}\right)$.

Let, $f(t)$ be a periodic function over $d \le t \le d+T$; where T is the period of the function; $f(t+T)=f(t)$, the Fourier Series expansion of $f(t)$ is

$$f(t) = \frac{1}{2}a_0 + \sum_{n\geq 1}^{\infty} a_n \cos n\omega t + \sum_{n \geq 1}^{\infty} b_n \sin n\omega t \; ; \; T = 2\pi/\omega . \qquad \text{(1)}$$

we have to find values of a_0, a_n and b_n.

Following results are to be noted:

$$\int_d^{d+T} \cos n\omega t\, dt = \begin{cases} 0, & n \neq 0 \\ T, & n = 0 \end{cases} \; ; \quad \int_d^{d+T} \sin n\omega t\, dt = \left[0, \text{ for all } n. \right.$$

$$\int_d^{d+T} \sin m\omega t \sin n\omega t\, dt = \begin{cases} 0, & m \neq n \\ \frac{1}{2}T, & m = n \neq 0 \end{cases}$$

$$\int_d^{d+T} \cos m\omega t \cos n\omega t\, dt = \begin{cases} 0, & m \neq n \\ \frac{1}{2}T, & m = n \neq 0 . \end{cases}$$

$$\int_d^{d+T} \cos m\omega t \sin n\omega t\, dt = 0, \text{ for all } m \text{ and } n .$$

integrating equation (1), in the interval $d \le t \le d+T$.

$$\int_d^{d+T} f(t)\, dt = \frac{1}{2}a_0 \int_d^{d+T} dt + \sum_{n \geq 1}^{\infty} \left(a_n \int_d^{d+T} \cos n\omega t\, dt + b_n \int_d^{d+T} \sin n\omega t\, dt \right)$$

$$= \frac{1}{2}a_0(T) + \sum_{n\geq 1}^{\infty} a_n(0) + b_n(0)$$

$$\int_d^{d+T} f(t)\, dt = \frac{1}{2}T a_0 .$$

$$\Rightarrow \quad \frac{1}{2}a_0 = \frac{1}{T}\int_d^{d+T} f(t)\, dt .$$

$$a_0 = \frac{2}{T}\int_d^{d+T} f(t)\, dt .$$

To obtain the Fourier Coefficient $a_n (n \neq 0)$, we multiply equation (i) throughout by $\cos m\omega t$ and integrate with respect to t over the interval $d \le t \le d+T$, giving,

$$\int_d^{d+T} f(t)\cos m\omega t\, dt = \frac{1}{2}a_0 \int_d^{d+T} \cos m\omega t\, dt + \sum_{n=1}^{\infty} a_n \int_d^{d+T} \cos n\omega t \cos m\omega t\, dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_d^{d+T} \cos m\omega t \sin n\omega t\, dt.$$

we find that, when $m \neq 0$, the only non-zero integral on the right-hand side is the one that occurs in the first summation when $n = m$, i.e, we have

$$\int_d^{d+T} f(t)\cos m\omega t\, dt = a_m \int_d^{d+T} \cos m\omega t \cos m\omega t\, dt = \frac{1}{2}a_m T$$

giving; $$a_m = \frac{2}{T}\int_d^{d+T} f(t)\cos m\omega t\, dt$$ or

$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos n\omega t\, dt.$$

The value of a_0, calculated before may be obtained by taking $n=0$, so,

$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos n\omega t\, dt \; ; \; n = 0, 1, 2, \ldots$$

This explains why the constant term in Fourier series expansion was taken as $\frac{1}{2}a_0$ instead of a_0, since this ensures compatibility of the results for a_0 and a_n. Although a_0 and a_n satisfy the same formula it is usualy safer to work them out separately.

Finally, to obtain the Fourier Coefficients b_n, we multiply throughout by $\sin m\omega t$ and integrate with respect to t over $d \le t \le d+T$,

$$\int_d^{d+T} f(t)\sin m\omega t\,dt = \frac{1}{2}a_0\int_d^{d+T}\sin m\omega t\,dt + \sum_{n=1}^{\infty}\left(a_n\int_d^{d+T}\sin m\omega t\cos n\omega t\,dt\right)$$

$$+ \sum_{n=1}^{\infty} b_n\int_d^{d+T}\sin m\omega t\sin n\omega t\,dt$$

we find that the only non-zero integral on the right hand side is the one that occurs in the second summation when $m=n$,

i.e, $$\int_d^{d+T} f(t)\sin n\omega t\,dt = b_n\int_d^{d+T}\sin n\omega t\sin n\omega t\,dt = \frac{1}{2}b_n T.$$

giving, $$b_n = \frac{2}{T}\int_d^{d+T} f(t)\sin n\omega t\,dt \quad (n=1,2,3,\ldots)$$

Especially, in more practical case, if the function $f(t)$ is periodic of period 2π, then, $\omega=1$, and the series becomes,

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nt + \sum_{n=1}^{\infty} b_n\sin nt$$; with the

fourier Coefficients a_n and b_n becomes,

$$a_n = \frac{1}{\pi}\int_d^{d+2\pi} f(t)\cos nt\,dt \quad (n=0,1,2,\ldots)$$

$$b_n = \frac{1}{\pi}\int_d^{d+2\pi} f(t)\sin nt\,dt \quad (n=1,2,\ldots).$$

Functions defined over a finite interval:- one of the requirement of Fourier Series is that the function to be expanded be periodic. Therefore, a function $f(t)$ is not periodic cannot have a Fourier Series representation. However, we can obtain a Fourier Series expansion that represents a non-periodic function $f(t)$ that is defined only over a finite time interval $0 \le t \le \tau$. This is a facility that is used frequently to solve problems in practice, particularly boundary value problem involving partial differential equations, such as the consideration of heat flow along a bar or the vibrations of a string. Various forms of Fourier Series representations of $f(t)$, valid only in the interval $0 \le t \le \tau$, are possible, including series consisting of cosine terms only or series consisting of sine terms only.

Full-Range Series:- suppose the function $f(t)$ is defined only over the finite time interval $0 \le t \le \tau$. Then, to obtain a full-range Fourier Series representation of $f(t)$ (that is a series consisting of both cosine and sine terms), we define the periodic extension $\phi(t)$ of $f(t)$ by

$$\phi(t) = f(t) \quad (0 < t < \tau); \quad \phi(t+\tau) = \phi(t).$$

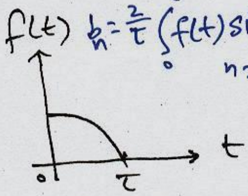
$\omega_n = n\omega_0 = n\frac{2\pi}{\tau}$

The Graphical representation is shown below.

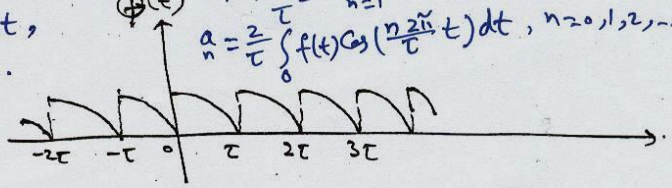
$$\phi(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n t) + b_n \sin(\omega_n t),$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} f(t)\sin\left(\frac{n2\pi}{\tau}t\right)dt, \quad n = 1, 2, 3, \ldots$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} f(t)\cos\left(\frac{n2\pi}{\tau}t\right)dt, \quad n = 0, 1, 2, \ldots$$



(a) Graph of a function defined only over only $0 \le t \le \tau$

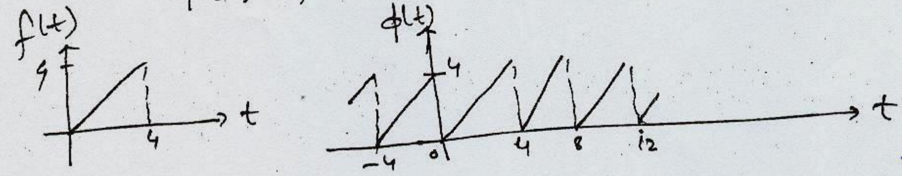


(b). Periodic extension of $f(t)$.

Ex: Find a full-range Fourier Series expansion of $f(t)$ valid in the finite interval $0<t<4$. Draw graph of both $f(t)$ and periodic extension of $f(t)$ [i.e, $\phi(t)$].

Define the periodic function $\phi(t)$ by;

$$\phi(t) = f(t) = t \quad (0<t<4); \qquad \phi(t+4) = \phi(t).$$



$T = 4$

$\omega = \frac{2\pi}{4} = \frac{\pi}{2}$

$\frac{2}{T} = \frac{2}{4} = \frac{1}{2}$

$$a_0 = \frac{1}{2}\int_0^4 f(t)\,dt = \frac{1}{2}\int_0^4 t\,dt = 4.$$

$$a_n = \frac{1}{2}\int_0^4 f(t)\cos\tfrac{1}{2}n\pi t\,dt \quad ; \quad n = 1, 2, \ldots$$

$$a_n = \frac{1}{2}\int_0^4 t\cos\tfrac{1}{2}n\pi t\,dt$$; which an integration by parts gives

$$= \frac{1}{2}\left[\frac{2t}{n\pi}\sin\tfrac{1}{2}n\pi t + \frac{4}{(n\pi)^2}\cos\tfrac{1}{2}n\pi t\right]_0^4 = 0.$$

and

$$b_n = \frac{1}{2}\int_0^4 f(t)\sin\tfrac{1}{2}n\pi t\,dt = \frac{1}{2}\int_0^4 t\sin\tfrac{1}{2}n\pi t\,dt$$

$$= \frac{1}{2}\left[\frac{-2t}{n\pi}\cos\tfrac{1}{2}n\pi t + \frac{4}{(n\pi)^2}\sin\tfrac{1}{2}n\pi t\right]_0^4 = \frac{-4}{n\pi}.$$

Thus, the Fourier Series expansion of $\phi(t)$ is

$$\phi(t) = 2 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\tfrac{1}{2}n\pi t.$$

Since, $\phi(t) = f(t)$ for $0<t<4$, it follows that Fourier Series representation of $f(t)$ within this interval, so that,

$$f(t) = t = 2 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\tfrac{1}{2}n\pi t. \qquad (21)$$

Half-Range Cosine and Sine Series:- Rather than develop the periodic extension $\phi(t)$ of $f(t)$ as to obtain a full-range series, it is possible to formulate periodic extensions that are either even or odd functions, so that the resulting Fourier series of the extended periodic functions consists either of Cosine terms only 'or' Sine terms only.

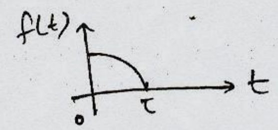
For a function $f(t)$ defined only over the finite interval $0 \le t \le \tau$. Its even periodic extension $F(t)$ is the even periodic function is

$$F(t) = \begin{cases} f(t) \; ; & 0 < t < \tau \\ f(-t) \; ; & -\tau < t < 0 \end{cases} \quad ; \; f(t+2\tau) = f(t) \; ;$$

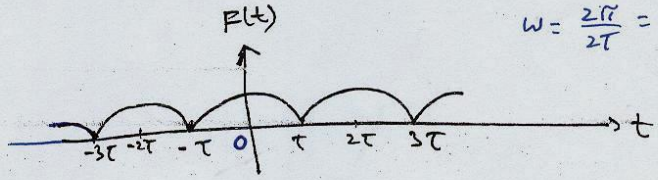
$T = 2\tau$

$\omega = \frac{2\pi}{2\tau} = \frac{\pi}{\tau}$.

Graphically :



A function $f(t)$



Even periodic extension $F(t)$

So, we have Fourier Half-Range Cosine Series as:

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{\tau} \; ; \text{ where,}$$

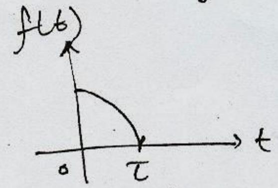
$\frac{2}{T}, \frac{4}{T} = \frac{4}{2\tau} = \frac{2}{\tau}$.

$$a_n = \frac{2}{\tau} \int_0^{\tau} f(t) \cos \frac{n\pi}{\tau} t \, dt ; \quad n = 0, 1, 2, \ldots$$

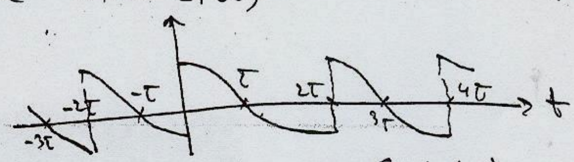
For a function $f(t)$ defined only over the finite interval $0 \le t \le \tau$, its odd periodic extension $G(t)$ is the odd periodic function defined by

$$G(t) = \begin{cases} f(t) \; ; & 0 < t < \tau \\ -f(-t) \; ; & -\tau < t < 0 \end{cases} \quad ; \; G(t+2\tau) = G(t)$$

Graphically, we see that G(t)



A function f(t)

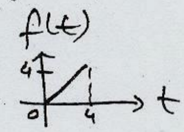


odd periodic extension G(t)

EX:- For the function $f(t)=t$ defined only in the interval $0<t<4$

Obtain: (a). a half-range Cosine Series Expansion.

(b). a half-range Sine Series Expansion.

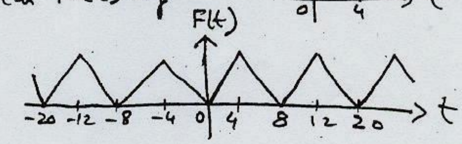


Solution:- Define the periodic function $F(t)$ by

(a).

$$F(t)=\begin{cases} f(t)=t\ ; & 0<t<4 \\ f(-t)=-t\ ; & -4<t<0 \end{cases}$$

$F(t+8)=F(t)$.



Then, since $F(t)$ is an even periodic function with period 8, it has a convergent Fourier Cosine Series expansion; $T=4$.

$$a_0=\frac{2}{4}\int_0^4 f(t)\,dt=\frac{1}{2}\int_0^4 t\,dt=\frac{1}{2}\left[\frac{t^2}{2}\right]_0^4=4.$$

$$a_n=\frac{2}{4}\int_0^4 f(t)\cos\tfrac{1}{4}n\pi t\,dt=\frac{1}{2}\int_0^4 t\cos\tfrac{1}{4}n\pi t\,dt \quad (;\ n=1,2,3,\ldots;$$

$$=\frac{1}{2}\left[\frac{4t}{n\pi}\sin\tfrac{1}{4}n\pi t+\frac{16}{(n\pi)^2}\cos\tfrac{1}{4}n\pi t\right]_0^4.$$

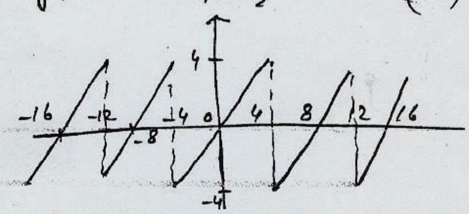
$$=\frac{8}{(n\pi)^2}\left(\cos n\pi-1\right)=\frac{8}{(n\pi)^2}\left[(-1)^n-1\right].$$

$$a_n=\begin{cases} 0\ ; & n \text{ is even.} \\ \dfrac{-16}{(n\pi)^2}\ ; & n \text{ is odd.} \end{cases}$$

hence,
$$F(t)=2-\frac{16}{\pi^2}\sum_{\substack{n=1\\(n\text{ odd})}}^{\infty}\frac{1}{n^2}\cos\tfrac{1}{4}n\pi t$$

or
$$f(t)=2-\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\tfrac{1}{4}(2n-1)\pi t\ ; \quad 0<t<4.$$

(b). Define the periodic function $G(t)$ by; — (24)



$$G(t) = \begin{cases} f(t) = t \,; & 0<t<4 \\ -f(-t) = -(-t) = t \,; & -4<t<0. \end{cases}$$

$$G(t+8) = G(t).$$

Then, since, $G(t)$ is an odd periodic function with period 8. it has convergent Fourier sine series; Taking $T=4$, we have

$$b_n = \frac{2}{4}\int_0^4 f(t)\sin\tfrac{1}{4}n\pi t\,dt = \frac{1}{2}\int_0^4 t\sin\tfrac{1}{4}n\pi t\,dt \,; \quad n=1,2,\ldots$$

$$= \frac{1}{2}\left[-\frac{4t}{n\pi}\cos\tfrac{1}{4}n\pi t + \frac{16}{(n\pi)^2}\sin\tfrac{1}{4}n\pi t\right]_0^4 .$$

$$= -\frac{8}{n\pi}\cos n\pi = -\frac{8}{n\pi}(-1)^n = \frac{8}{n\pi}(-1)^{n+1}.$$

Thus, Fourier series expansion of $G(t)$ is

$$G(t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\tfrac{1}{4}n\pi t \,;$$

Since, $G(t) = f(t)$ for $0<t<4$, it follows that this Fourier series is representative of $f(t)$ within this interval. Thus, the half-range Fourier sine series expansion is

$$f(t) = t = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\tfrac{1}{4}n\pi t \,; 0<t<4.$$

<u>Exercises</u>:- (1). Show that the half-range Fourier sine series expansion of the function $f(t)=1$, valid for $0<t<\pi$, is

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{2n-1} \,; \quad 0<t<\pi.$$

(2). Determine the half-range cosine series expansion of the function, $f(t) = 2t-1$, valid for $0<t<1$. Sketch the graphs.

(3). The function, $f(t) = 1-t^2$ is to be represented by a Fourier series expansion over the finite interval $0<t<1$. obtain a suitable (a). full-range series expansion, (b). half-range sine series expansion. (c). half-range cosine series expansion.

(4). A function $f(t)$ is defined by : $f(t) = \pi t - t^2$; $0 \le t \le \pi$, and is to be represented by either a half-range Fourier Sine Series or a half-range Fourier Cosine Series. Find both of these Series.

(5). A function $f(t)$ is defined by $0 \le t \le \pi$; by

$$f(t) = \begin{cases} \sin t ; & 0 \le t \le \frac{1}{2}\pi \\ 0 ; & \frac{1}{2}\pi \le t \le \pi . \end{cases}$$

Find a half-range Series expansion of $f(t)$ on this interval.

(6). Find the Fourier Series expansion of the function $f(t)$ valid for $-1 < t < 1$, where,

$$f(t) = \begin{cases} 1 ; & -1 < t < 0 \\ \cos \pi t ; & 0 < t < 1 . \end{cases}$$

Sketch the even extension of the given function ; find the Fourier Cosine Series expansion:

(7). $f(x) = x^2$; $0 \le x < 1$

(8). $$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 2, & 1 \le x < 2 . \end{cases}$$

(9). $f(x) = 1 - x$, $0 \le x < \pi$.

Sketch the odd extension of the given function ; find Half-range Fourier sine series expansion:

(10). $f(x) = x^2$; $0 \le x < 1$;

(11). $$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 2 ; & 1 \le x < 2 . \end{cases}$$

(12). $f(x) = 1 - x$; $0 \le x < \pi$.