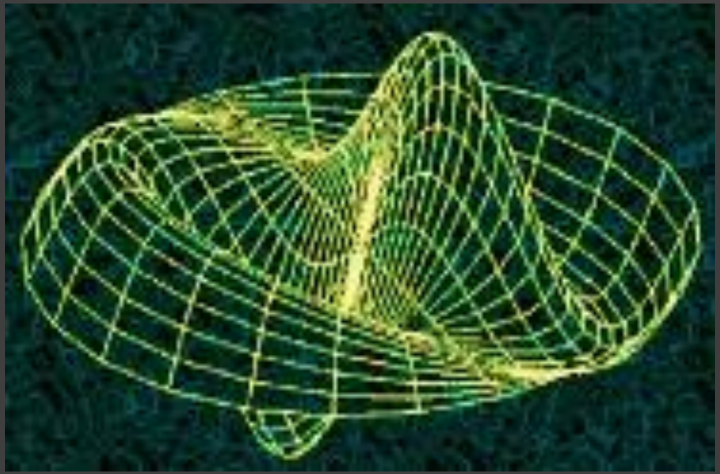
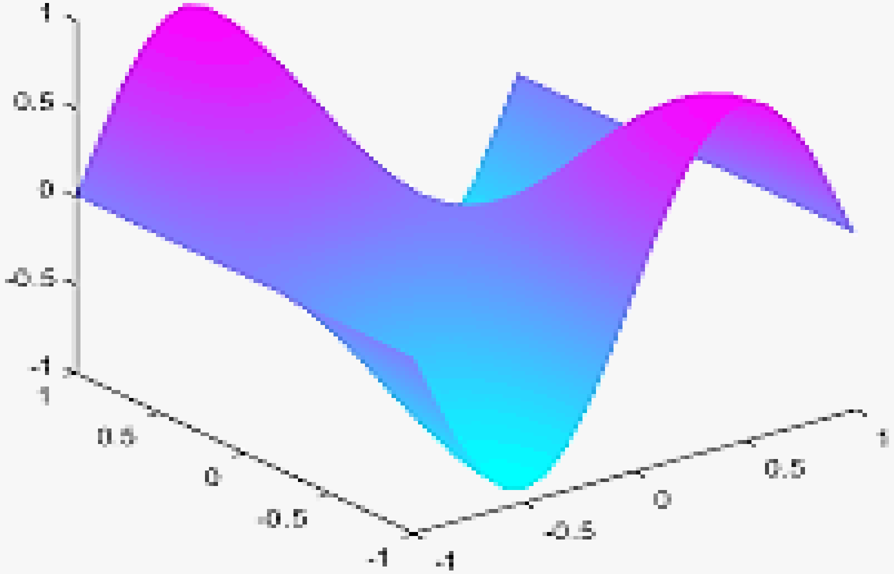
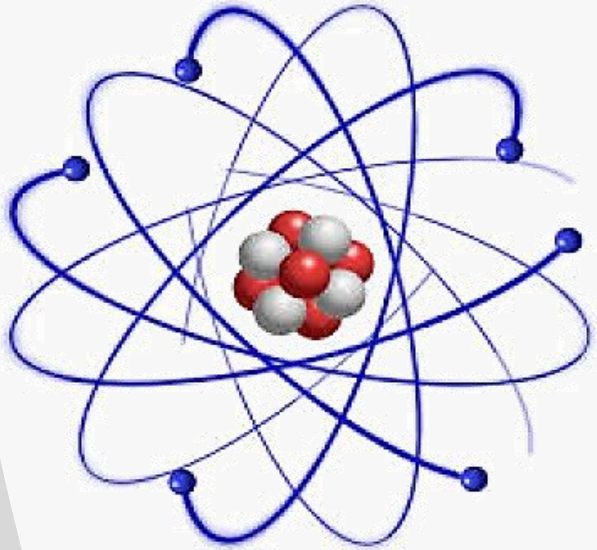
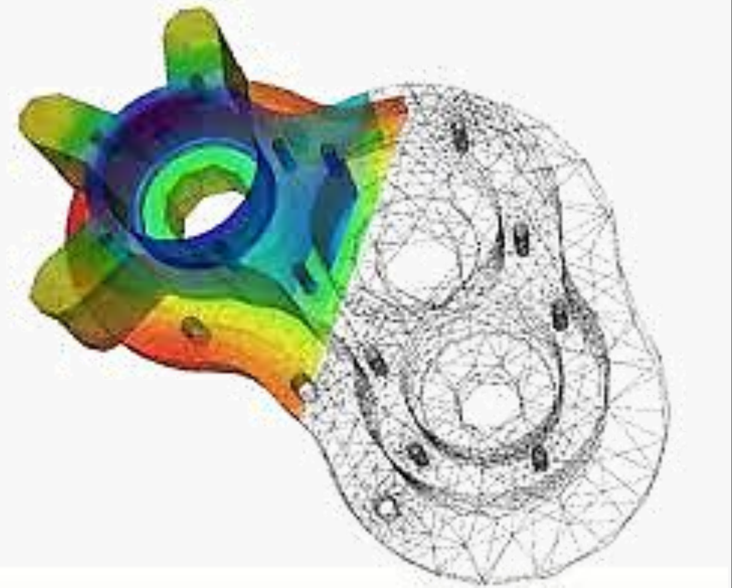




Partial Differential Equations





Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

Partial Differential Equations

Book: Advanced Engineering Mathematics (9th Edition) by Ervin Kreyszig

- Chapter: 12
  - Sections: 12.2, 12.3

Book: Linear Partial Differential Equations for Scientists and Engineers (4th Edition) by Lokenath Debnath

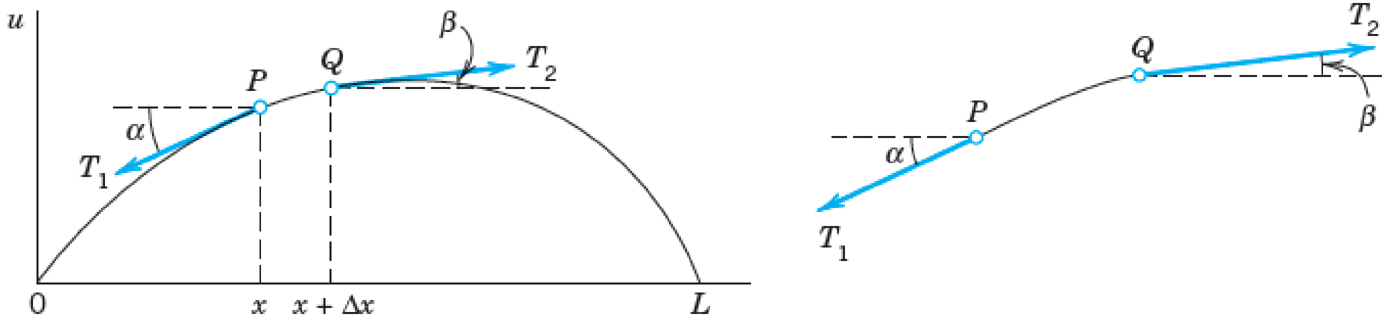
- Chapter: 3, 7
  - Sections: 3.2, 7.3

Book: Applied Partial Differential Equations With Fourier series and boundary value problems by Richard Haberman

- Chapter: 4
  - Sections: 4.1 - 4.4

The Vibrating String: Wave Equation

- One of the most important problems in mathematical physics is the vibration of a stretched string. Simplicity and frequent occurrence in many branches of mathematical physics make it a classic example in the theory of partial differential equations.
- We want to derive the PDE modeling small transverse vibrations of an elastic string, such as a violin string.
- We place the string along the x −axis, stretch it to length L, and fasten it at the ends $x = 0$ and $x = L$. We then distort the string, and at some instant, say $t = 0$, we release it and allow it to vibrate.
- The problem is to determine the vibrations of the string, that is, to find its deflection $u(x, t)$ at any point x and at any time $t > 0$. $u(x, t)$ will be the solution of a PDE that is the model of our physical system to be derived.
- This PDE should not be too complicated, so that we can solve it. Thus, we consider some reasonable simplifying assumptions.



The Vibrating String: Wave Equation

Physical Assumptions:

1. The mass of the string per unit length is constant (“homogeneous string”). The string is perfectly elastic and does not offer any resistance to bending.

2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.

3. The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.

Under these assumptions we may expect solutions $u(x, t)$ that describe the physical reality sufficiently well. The model of the vibrating string will consist of a PDE (“wave equation”) and additional conditions. To obtain the PDE, we consider the ***forces acting on a small portion of the string.***

The Vibrating String: Wave Equation

Since the string offers no resistance to bending, the tension is tangential to the curve of the string at each point. Let T_1 and T_2 be the tension at the endpoints P and Q of that portion. Since the points of the string move vertically, there is no motion in the horizontal direction. Hence the horizontal components of the tension must be constant. We thus obtain:

$$T_1 \cos\alpha = T_2 \cos\beta = T = \text{const.} \qquad (1)$$

In the vertical direction we have two forces, namely, the vertical components $-T_1 \sin\alpha$ and $T_2 \sin\beta$ of T_1 and T_2: here the minus sign appears because the component at P is directed downward. By **Newton's second law** the resultant of these two forces is equal to the mass $\rho\Delta x$ of the portion times the acceleration u_{tt}, evaluated at some point between x and $x + \Delta x$; here ρ is the mass of the undeflected string per unit length, and Δx is the length of the portion of the undeflected string. Hence:

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\Delta x u_{tt}.$$

Using (1), we can divide this relation by $T_2 \cos\beta = T_1 \cos\alpha = T$ to obtain:

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho\Delta x}{T} u_{tt} \Longrightarrow \tan\beta - \tan\alpha = \frac{\rho\Delta x}{T} u_{tt}. \qquad (2)$$

The Vibrating String: Wave Equation

Now $\tan\alpha$ and $\tan\beta$ are slopes of the string at x and $x+\Delta x$, i.e.,

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)\Big|_{x} \qquad \text{and} \qquad \tan\beta = \left(\frac{\partial u}{\partial x}\right)\Big|_{x+\Delta x}.$$

Here we are using partial derivatives for slopes instead of total derivatives because u is function of x and t both. Now dividing (2) by Δx and using values of $\tan\alpha$ and $\tan\beta$, we obtain:

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)\Big|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)\Big|_{x}\right] = \frac{\rho}{T}u_{tt} \Longrightarrow \frac{u_x(x+\Delta x)-u_x(x)}{\Delta x} = \frac{\rho}{T}u_{tt}.$$

If $\Delta x \longrightarrow 0$, we obtain the linear PDE:

$$\Longrightarrow u_{xx} = \frac{\rho}{T}u_{tt} \Longrightarrow u_{tt} = c^2 u_{xx}; \qquad \text{where} \quad c^2 = \frac{T}{\rho}. \qquad (3)$$

This is called the **one-dimensional wave equation**. We see that it is homogeneous and of the second order. The physical constant T/ρ is denoted by c^2 (instead of c) to indicate that this constant is *positive,* a fact that will be essential to the form of the solutions. “One-dimensional” means that the equation involves only one space variable x.

The Vibrating String: Wave Equation

We now need to complete the model by adding additional conditions and then solving the resulting model. The model of a vibrating elastic string (a violin string, for instance) consists of the **one-dimensional wave equation:**

$$\Longrightarrow u_{tt} = c^2 u_{xx}; \qquad \text{where} \quad c^2 = \frac{T}{\rho}. \qquad (3)$$

for the unknown deflection $u(x,t)$ of the string, a PDE that we have just obtained, and some ***additional conditions***, which we shall now derive. Since the string is fastened at the ends $x = 0$ and $x = L$, we have the two **boundary conditions:**

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t \geq 0 \qquad (4)$$

Furthermore, the form of the motion of the string will depend on its *initial deflection* (deflection at time $t = 0$), call it $f(x)$ and on its *initial velocity* (velocity at $t = 0$), call it $g(x)$. We thus have the two **initial conditions:**

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 \leq x \leq L \qquad (5)$$

We now require to find a solution of the PDE (3) satisfying the conditions (4) and (5).

The Vibrating String: Wave Equation

We shall do this in three steps, as follows:

Step 1. By the "**method of separating variables**" or *product method,* setting:

$$u(x,t) = X(x)T(t), \qquad (6)$$

we obtain from (3) two ODEs, one for $X(x)$ and the other one for $T(t)$.

Step 2. We determine solutions of these ODEs that satisfy the boundary conditions (4).

Step 3. Finally, using **Fourier series**, we compose the solutions found in Step 2 to obtain a solution of (3) satisfying both (4) and (5), that is, the solution of our model of the vibrating string.

String Vibrations

Substitute the solution (6) into the wave equation (3) to obtain:

$$X(x)T''(t) = c^2X''(x)T(t) \qquad (7),$$

where $(\ )'$ corresponds to differentiation with respect to either x or $t \Longrightarrow X' = \frac{dX}{dx}$ and $T' = \frac{dT}{dt}$. Equation (7) can be rewritten as:

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = k,$$

where k must be a constant since the left–hand side of the equality is a function of t only and the other side of the equality is a function of x only. Therefore, we obtain two ordinary differential equations:

$$T''(t) - kc^2T(t) = 0, \qquad (9)$$

and

$$X''(x) - kX(x) = 0. \qquad (10)$$

String Vibrations

Let us first consider the equation (10) involving $X(x)$ that is given as:

$$X''(x) - kX(x) = 0. \qquad (10)$$

If k is equal to zero, the general solution of above equation is:

$$X(x) = Ax + B. \qquad (11)$$

Using (7) the boundary conditions (4) take the form:

$$u(0,t) = X(0)T(t) = 0 \quad \text{and} \quad u(L,t) = X(L)T(t) = 0$$

mean that $X(0) = X(L) = 0$ for all t. Using these boundary conditions in (11) we get:

$$A = B = 0,$$

This means that $X(x) = 0$, a trivial solution, which is of no interest.

String Vibrations

If k is a positive number, the general solution of (10) is given as:

$$X(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}. \qquad (12)$$

Using the boundary conditions $X(0) = X(L) = 0$ in (12) we get:

$$X(0) = A + B = 0$$

and

$$X(L) = Ae^{\sqrt{k}L} + Be^{-\sqrt{k}L} = 0$$

$$\Longrightarrow A = B = 0,$$

This means that $X(x) = 0$, a trivial solution. Therefore, in order to get $\text{non} - \text{trivial}$ solution k must be a **negative number**.

String Vibrations

Assuming $k = -p^2$, a negative number, we get

$$X'' + p^2 X = 0$$

$$\Longrightarrow X(x) = A\cos(px) + B\sin(px)$$

$B.C.'s \Longrightarrow X(0) = 0 = A$, and $X(L) = B\sin(pL) = 0$. For non-trivial solution $B \neq 0$.

Thus,

$$pL = n\pi \Longrightarrow p = \frac{n\pi}{L}; \qquad n = 1,2,3,\ldots$$

and

$$X_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right); \qquad n = 1,2,3,\ldots. \quad (13)$$

String Vibrations

X_n are eigenfunctions of the equation $X''(x) + p^2 X(x) = 0$, and $p_n = \dfrac{n\pi}{L}$; $n = 1,2,3,\ldots$ are of the corresponding eigenvalues of the equation. Let us consider the other equation:

$$T'' + c^2 p^2 T = T'' + \left(\frac{nc\pi}{L}\right)^2 T = 0.$$

Define $\lambda_n = \dfrac{nc\pi}{L} \Rightarrow T'' + \lambda_n^2 T = 0$, The general solution is:

$$T_n(t) = B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t). \qquad (14)$$

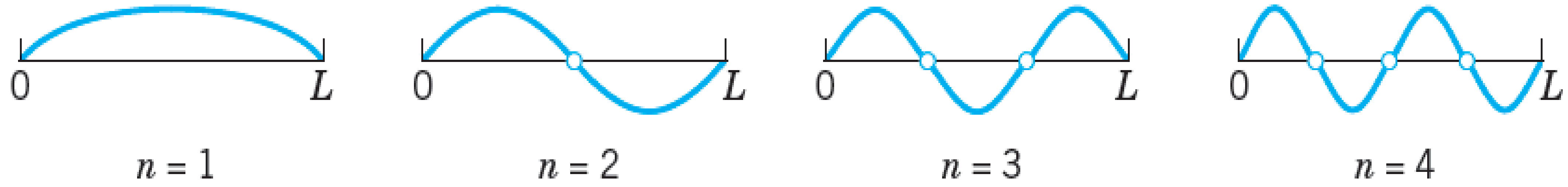
Using (13) and (14), we get:

$$u_n(x,t) = [C_n \cos(\lambda_n t) + C_n^* \sin(\lambda_n t)] \sin\left(\frac{n\pi x}{L}\right); \quad n = 1,2,3 \ldots \quad (15)$$

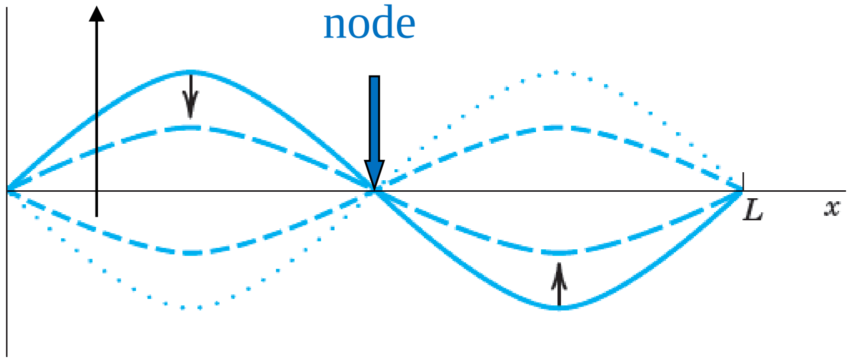
where $C_n = B_n A_n$ and $C_n^* = B_n^* A_n$ are arbitrary constants. These functions are called the **eigenfunctions**, or *characteristic functions*, and the values $\lambda_n = \frac{nc\pi}{L}$ are called the **eigenvalues**, or *characteristic values*, of the vibrating string. The set $\{\lambda_1, \lambda_2, \ldots\}$ is called the **spectrum**.

String Vibrations

Note that each $u_n(x,t)$ represents a harmonic motion having the **frequency** $\frac{\lambda_n}{2\pi} = \frac{cn}{L}$ cycles per unit time. This motion is called the nth **normal mode** of the string. The first normal mode is known as the *fundamental mode* $(n = 1)$ and the others are known as *overtones*. Since in (15) $\sin\left(\frac{n\pi x}{L}\right) = 0$ at $x = \frac{L}{n}, \frac{2L}{n}, \dots, \frac{(n-1)L}{n}$, so the nth normal mode has $n - 1$ nodes, that is, points of the string that do not move (in addition to the fixed endpoints).



The accompanying figure shows the second normal mode for various values of *t*. At any instant, the string has the form of a sine wave. When the left part of the string is moving down, the other half is moving up, and conversely. For the other modes the situation is similar.



Second normal mode for various values of t

String Vibrations

The eigenfunctions (15) satisfy the wave equation (3) and the boundary conditions (4) (string fixed at the ends). A single $u_n(x,t)$ will generally not satisfy the initial conditions (5). But since the wave equation (3) is linear and homogeneous, it follows from principle of superposition that the sum of solutions $u_n(x,t)$ is a solution of (3). To obtain a solution that also satisfies the initial conditions (5), we consider the infinite series:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} [C_n \cos(\lambda_n t) + C_n^* \sin(\lambda_n t)] \sin\left(\frac{n\pi x}{L}\right). \quad (16)$$

Using (5) in (16) we get:

$$u(x,0) = f(x) \Longrightarrow \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = f(x).$$

Hence, we must choose C_n so that $u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$ becomes the **Fourier sine series** of $f(x)$. Thus,

$$C_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \qquad (17)$$

String Vibrations

Differentiating both sides of (16) partially with respect to "t" we get:

$$u_t(x,t) = \sum_{n=1}^{\infty} [-\lambda_n C_n \sin(\lambda_n t) + \lambda_n C_n^* \cos(\lambda_n t)] \sin\left(\frac{n\pi x}{L}\right). \quad (18)$$

Using (5) in (18) we get:

$$u_t(x,0) = g(x) \Longrightarrow \sum_{n=1}^{\infty} \lambda_n C_n^* \sin\left(\frac{n\pi x}{L}\right) = g(x).$$

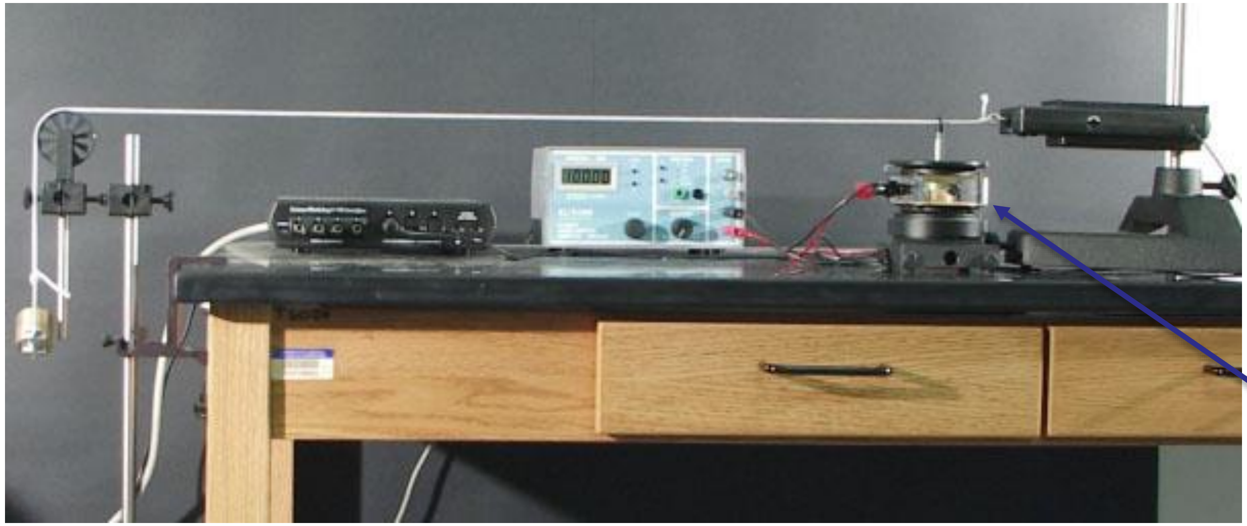
Hence, we must choose $\lambda_n C_n^*$ so that for $t = 0$ the derivative u_t becomes the Fourier sine series of $g(x)$. Thus,

$$\lambda_n C_n^* = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \Longrightarrow C_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx, \qquad (19)$$

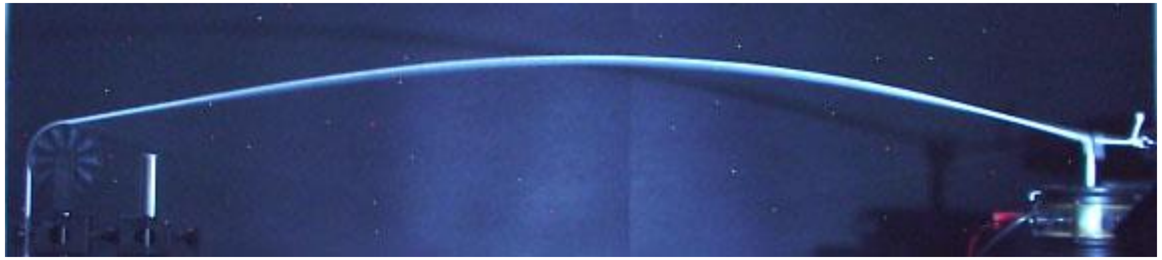
where $\lambda_n = \frac{cn\pi}{L}$. Thus, (16) with coefficients (17) and (19) is a solution of (3) that satisfies all the conditions in (4) and (5), provided the series (16) converges.

String vibration setup

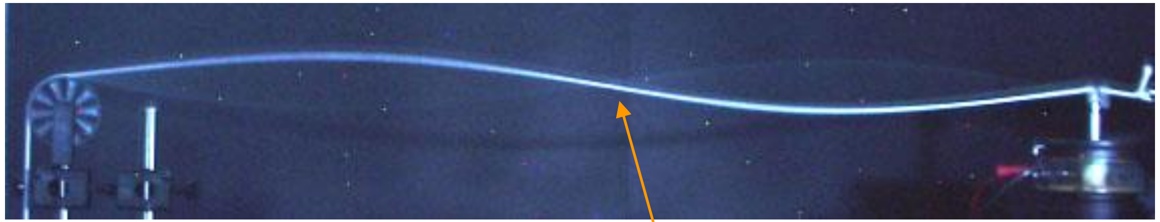
Hanging mass to
adjust tension



Loudspeaker with adjustable
frequency control



Fundamental mode



Second normal mode
With one node

Node points: not moving

Third normal mode
with two nodes

Practice Questions

Book: Advanced Engineering Mathematics (9th Edition) by Ervin Kreyszig

Chapter: 12

Exercise – 12.3: Q – 5 to 13, Q – 15 to 18.