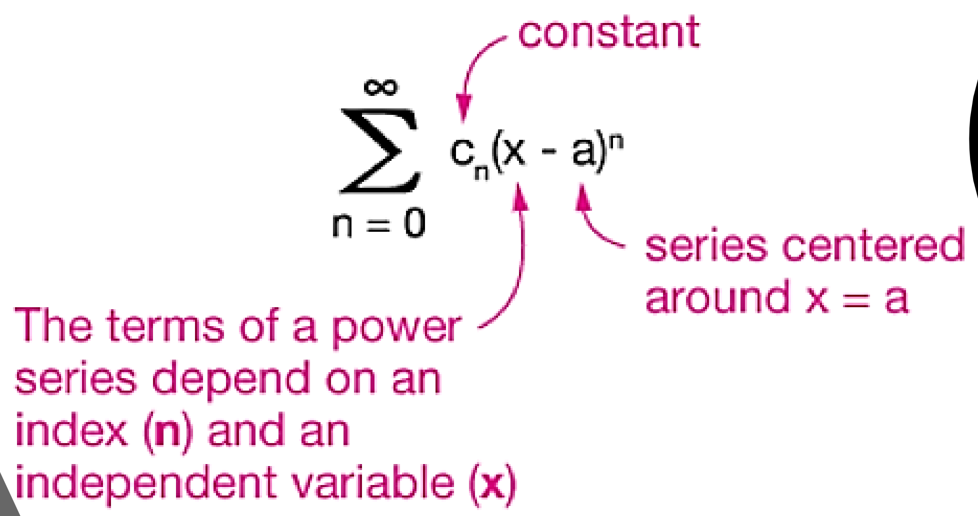
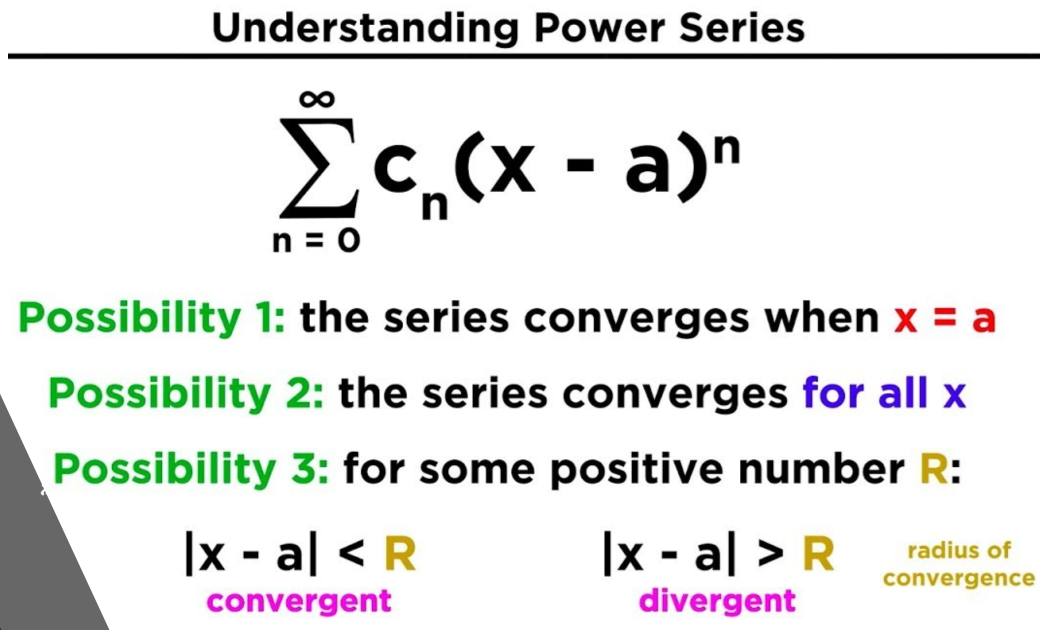
Power Series





Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

Chapter: 11 (11.7)

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

Chapter: 11 (11.6)

Calculus & Analytical Geometry MATH-101

Instructor: Dr. Naila Amir (SEECS, NUST)

Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \qquad (1)$$

where:

- x is a variable.
- c_n are constants called the coefficients of the series.

For each fixed *x*, the power series is a series of constants that we can test for convergence or divergence. A power series may converge for some values of *x* and diverge for other values of *x*.

Power Series

The sum of the power series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

is a function:

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

whose domain is the set of all x for which the series converges. Notice that $f(x)$ resembles a polynomial. The only difference is that $f(x)$ has infinitely many terms while a polynomial has finite terms.

Example

Consider the power series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

If we take $c_n = 1$ for all n, the power series becomes the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots$$

which **converges** when – $1 < x < 1$ and **diverges** when $|x| \geq 1$. Thus, for $|x| < 1$, we have:

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots$$

Power Series

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots \quad (2)$$

is called any of the following:

- A power series in $(x - a)$.
- A power series centered at a.
- A power series about a.

If $x=0$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

Note that (2) can be reduced to a power series of form (1) by using $x - a = y$. Moreover, when $x = a$, all the terms are 0 for $n \geq 1$. So, the power series in (2) always converges when $x = a$.

Convergence Of Power Series

- If a numerical value of x is substituted in (1), then the power series becomes a series of constant terms. The behavior (convergence/divergence) of such series can be determined easily by means of the tests that we have already discussed.

- In order to check that for which values of x the power series converges, we use the Ratio test for absolute convergence or the root test for absolute convergence.

Example

For what values of x is the power series

$$\sum_{n=0}^{\infty} n!\,x^n = 0! + 1!\,x + 2!\,x^2 + \cdots$$

If $x = 0$

$c_0 = 0! = 1$

is convergent?

Solution:

For the present case: $a_n = n!\,x^n$. If $x \neq 0$, we have:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = |x|\lim_{n\to\infty}(n+1) = \infty.$$

By the Ratio Test, the series diverges when $x \neq 0$. Thus, the given series converges only when $x = 0$.

Example

For what values of x is the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$$

$(-\infty, \infty)$

is convergent?

Solution: For the present case: $a_n = \frac{x^n}{(2n)!}$. We have:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(2n+2)!} \frac{(2n)!}{x^n}\right| = |x| \lim_{n\to\infty} \left|\frac{(2n)!}{(2n+2)(2n+1)(2n)!}\right| = 0.$$

By the Ratio Test, the series converges absolutely for all values of x.

Example

For what values of x is the power series

$$\sum_{n=0}^{\infty}(nx)^{2n} = \overset{c_0}{\boxed{0}} + x^2 + 2^4x^4 + \cdots$$

$\{0\}$

is convergent?

Solution:

For the present case: $a_n = (nx)^{2n}$. If $x \neq 0$, we have:

$$\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}|(nx)^{2n}|^{1/n} = |x|^2 \lim_{n\to\infty}(n^2) = \infty.$$

By the Root Test, the series diverges for all non-zero values of x and the series converges for $x = 0$

Example

For what values of x is the power series

$$\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$$

Converges absolutely, converges conditionally and diverges?

Solution:

For the present case: $a_n = \frac{x^n}{\ln n}$. Then:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{x^n}\right| = |x|. \lim_{n\to\infty} \left(\frac{\ln n}{\ln(n+1)}\right)$$

$$= |x|. \lim_{n\to\infty} \left(\frac{1/n}{1/(n+1)}\right) = |x|. \lim_{n\to\infty} \left(\frac{n+1}{n}\right) = |x|.$$

Example

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x|. \; = L$$

By the Ratio Test, the given series is:

- Absolutely convergent when $|x| < 1$. Now $|x| < 1 \Leftrightarrow -1 < x < 1$. Thus, the series converges when $-1 < x < 1$.
- Divergent when $|x| > 1$.
- The Ratio Test fails when $|x| = 1$. So, we must consider $x = -1$ and $x = 1$ separately.

Example

If we put $x = 1$ in the series, it becomes

$\ln n$ —— n^{small}

$$\sum_{n=2}^{\infty} \frac{x^n}{\ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

Note that if we choose $b_n = \frac{1}{n}$, then we have:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\ln n}{1/n} = \lim_{n\to\infty} \frac{n}{\ln n} = \infty.$$

Since $\sum_{n=2}^{\infty} b_n$ is divergent so by limit comparison test $\sum_{n=2}^{\infty} a_n$ is also divergent.

Example

If we put $x = -1$ in the series, it becomes

$$\sum_{n=2}^{\infty} \frac{x^n}{\ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

(i) $a_n = \frac{1}{\ln n} \to 0$ as $n \to \infty$

(ii) $\ln n < \ln(n+1)$

which is an alternating series. By alternating series test, this series is convergent. The corresponding positive terms series is:

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}.$$

$\frac{1}{\ln n} > \frac{1}{\ln(n+1)}$

$a_n > a_{n+1}$

which is a divergent series (shown when $x = 1$). Thus, the given power series converges conditionally when $x = -1$.

Conclusion

the power series:

$$\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$$

$R = 1$

- is Absolutely Convergent when $|x| < 1$. Now $|x| < 1 \Leftrightarrow -1 < x < 1$.
- is Conditionally Convergent when $x = -1$, and,
- is Divergent when $|x| \geq 1$.

$[-1, 1)$ IOC

Thus, the power series converges when $-1 \leq x < 1$ and diverges when $|x| \geq 1$.

Example

For what values of x is the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

is convergent?

Solution:

For the present case: $a_n = \frac{(x-3)^n}{n}$. Then:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$

$$= |x-3|. \lim_{n\to\infty} \left(\frac{n}{n+1}\right) = |x-3|.$$

Example

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-3|.$$

By the Ratio Test, the given series is:

- Absolutely convergent, when $|x - 3| < 1$. Now
$$|x-3| < 1 \Leftrightarrow -1 < x-3 < 1 \Leftrightarrow 2 < x < 4.$$
Thus, the series converges when $2 < x < 4$.

- Divergent when $|x - 3| > 1$, i.e., the power series diverges when $x < 2$ or $x > 4$.

- The Ratio Test gives no information when $|x - 3| = 1$. So, we must consider $x = 2$ and $x = 4$ separately.

Example

If we put $x = 4$ in the series, it becomes

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

$\boxed{[2, 4)}$

the harmonic series, which is divergent.

If we put $x = 2$, the series is

$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which converges by the Alternating Series Test.

Thus, the given series converges for $2 \le x < 4$.

Bessel Function

- The main use of a power series is that it provides a way to represent some of the most important functions that arise in mathematics, physics, and chemistry.

- In particular, the sum of the power series

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

is called a Bessel function of order 0, after the German astronomer Friedrich Bessel (1784–1846). These functions first arose when Bessel solved Kepler's equation for describing planetary motion.

Example

Find the domain of the Bessel function of order 0, defined by:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Solution:

For the present case: $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$. Then:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{2^{2n+2}[(n+1)!]^2} . \frac{2^{2n}(n!)^2}{x^{2n}}\right| = \frac{x^2}{4} . \lim_{n\to\infty} \left(\frac{1}{(n+1)^2}\right) = 0.$$

Thus, by the Ratio Test, the given series converges for all values of x. In other words, the domain of the Bessel function $J_0(x)$ is: $(-\infty, \infty)$.

Observation

In the series we have seen so far, the set of values of x for which the series is convergent has always turned out to be one of the following:

- A finite interval, e.g., $2 \leq x < 4$.
- An infinite interval $(-\infty, \infty)$,
- A collapsed interval, e.g., $[0, 0] = \{0\}$.

Theorem

For a given power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

$a = 0$

$\boxed{|x| < R}$

exactly one of the following three possibilities is true:

I. The series converges only when $x = a$.

ROC

II. The series converges for all x.

III. There is a positive number R such that the series converges if $\boxed{|x-a| < R}$ and diverges if $|x-a| > R$.

Radius Of Convergence & Interval Of Convergence

- The number R in (III) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case I and $R = \infty$ in case II.

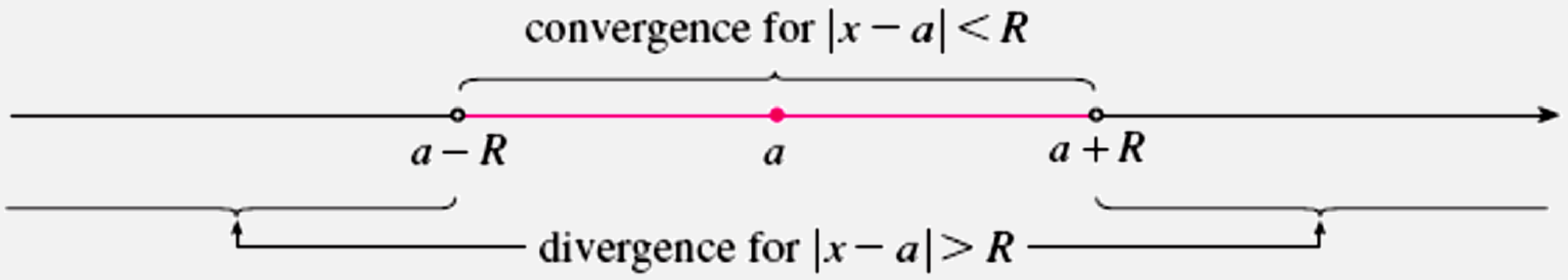
- The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case I, the interval consists of just a single point a. In case II, the interval is $(-\infty, \infty)$. In case III, the interval of convergence is given by the inequality $|x - a| < R$ that can be rewritten as $a - R < x < a + R$.

- When x is an endpoint of the interval, that is, $x = a \pm R$, anything can happen: The series might converge at one or both endpoints. It might diverge at both endpoints.

Radius Of Convergence & Interval Of Convergence

Thus, in case III, there are four possibilities for the interval of convergence:

1. $(a - R, a + R)$
2. $(a - R, a + R]$
3. $[a - R, a + R)$
4. $[a - R, a + R]$



Examples

Series	Radius of convergence	Interval of convergence
$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

How to Test a Power Series for Convergence

- In general, the Ratio Test or the Root Test are used to determine the radius of convergence R.

- The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence. So, the endpoints must be checked with some other test.

Example

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}.$$

Solution:

For the present case: $a_n = \frac{n(x+2)^n}{3^{n+1}}$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}}.\frac{3^{n+1}}{n(x+2)^n}\right|$$

$$= \frac{|x+2|}{3}.\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{|x+2|}{3}.$$

Example

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+2|}{3}.$$

$|x - a| < R$

By the Ratio Test, the given series is:

- Convergent, when $\frac{|x+2|}{3} < 1$ or $|x+2| < \boxed{3.}$
- Divergent, when $\frac{|x+2|}{3} > 1$ or $|x+2| > 3$,
- Thus, the **radius of convergence** is $R = 3$.

Note that: $|x+2| < 3 \Leftrightarrow -3 < x+2 < 3 \Leftrightarrow -5 < x < 1$. So, we test the series at the endpoints –5 and 1.

Example

- When $x = -5$, the series becomes:

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n n \,.$$

  This is a divergent alternating series since nth term does not approach 0 as $n \to \infty$.

- If we put $x = 1$, the series is given as:

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} n$$

  which is a divergent series by nth term test for divergence.

Thus, the given power series converges only when $-5 < x < 1$. Thus, the **interval of convergence** is: $(-5,1)$.

Example

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}.$$

Solution:

For the present case: $a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$, so

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}x^{n+1}}{\sqrt{n+2}} . \frac{\sqrt{n+1}}{3^n x^n}\right|$$

$$= 3|x|. \lim_{n\to\infty} \left(\frac{\sqrt{n+1}}{\sqrt{n+2}}\right) = 3|x|.$$

Example

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \underline{3|x|}.$$

$|x| < R$

By the Ratio Test, the given series is:

- Convergent, when $3|x| < 1$ or $|x| < 1/3$.
- Divergent, when $3|x| > 1$ or $|x| > 1/3$,
- Thus, the **radius of convergence** is $R = 1/3$.

Note that: $|x| < 1/3 \Leftrightarrow -1/3 < x < 1/3 \Leftrightarrow -1/3 < x < 1/3$. So, we need to test the series at the endpoints $-1/3$ and $1/3$.

Example

- When $x = -1/3$, the series becomes:

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-3)^n (-1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}.$$

  This is a p −series with $p = \frac{1}{2} < 1$, so it is a divergent series.

- If we put $x = 1/3$, the series is given as:

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-3)^n (1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}.$$

  which is an alternating series. This series is convergent by alternating series test.

Thus, the given power series converges when $-1/3 < x \leq 1/3$. Hence, the **interval of convergence** is: $(-1/3, 1/3]$.

Practice Questions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Exercise: 11.7

  Q # 1 to Q # 38

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 11.6

  Q # 1 to Q # 46