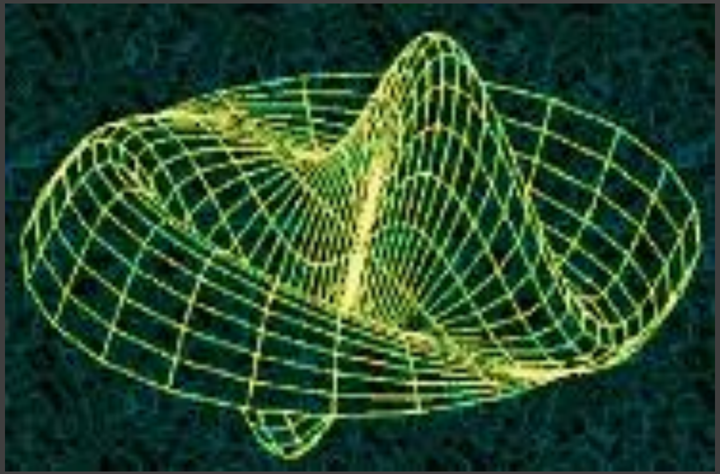
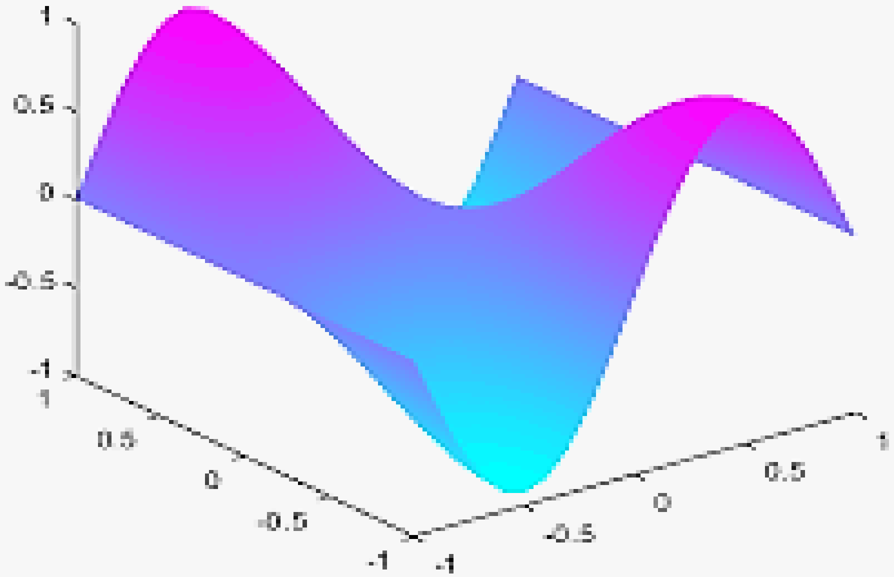
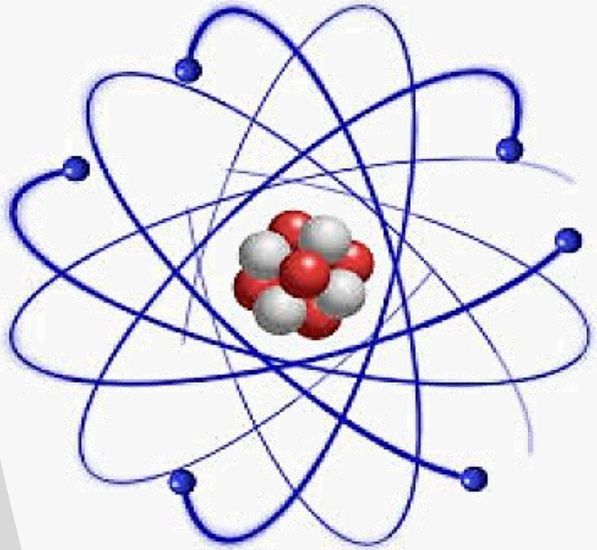
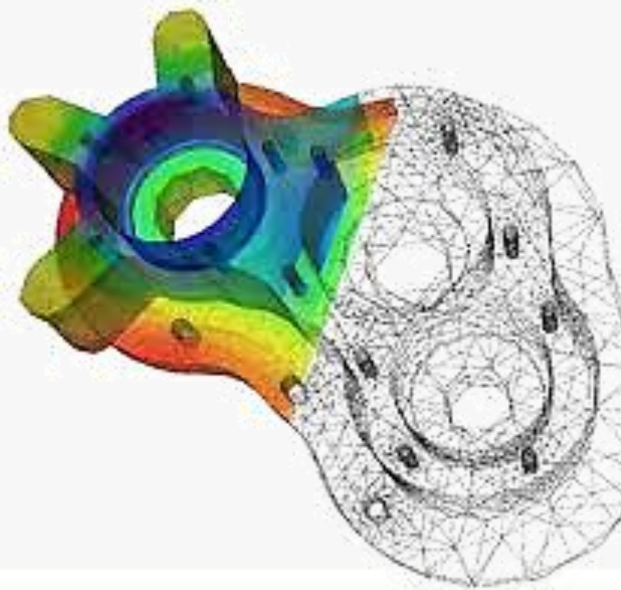


Partial Differential Equations







Vector Calculus(MATH-243)
Instructor: Dr. Naila Amir

Question # 1

Consider the heat boundary value problem:

$$u_t = u_{xx}; \quad 0 < x < 1, \qquad t > 0,$$
$$u_x(0,t) = -u(0,t), \qquad u_x(1,t) = -u(1,t), \qquad t > 0,$$
$$u(x,0) = x; \qquad 0 < x < 1$$

This models a heat problem in a bar that is losing heat at its ends at the rate proportional to the temperature of the endpoints. Show that the temperature $u(x,t)$ in the bar is given as:

$$u(x,t) = A_0 e^{-x} e^{t} + \sum_{n=1}^{\infty} A_n[(n\pi)\cos(n\pi x) - \sin(n\pi x)]e^{n^2\pi^2 t},$$

where $A_0 = \frac{2e(e-2)}{e^2-1}$ and $A_n = \frac{2}{1+n^2\pi^2}\left[\frac{2(-1)^n-1}{n\pi}\right]; n = 1,2,\ldots$

[Hint: Using $u(x,0) = f(x)$ and the orthogonality relation $\int_0^1 X_i(x)X_j(x)\,dx = 0$, if $i \neq j$ we get: $A_0 = \frac{2e^2}{e^2-1}\int_0^1 f(x)e^{-x}dx$ and $A_n = \frac{2}{1+n^2\pi^2}\int_0^1 f(x)X_n(x)dx\,; n = 1,2,\ldots$]

Question # 2

A quantum-mechanical particle on the line with an infinite potential outside the interval $(0, L)$ (particle in a box) is described by the Schrödinger equation:

$$u_t = iu_{xx}; \quad 0 < x < L, \qquad t > 0,$$

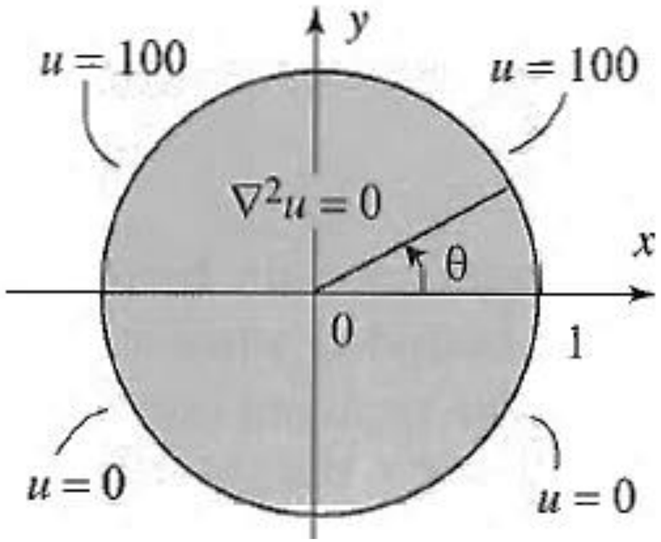
with Dirichlet conditions $u(0,t) = 0$ and $u(L,t) = 0$ at the ends. Use separation of variables to find a representation formula for $u(x,t)$ as a series.

Question # 3

The steady state temperature in a disk of radius 1 is described by a two-dimensional Laplace equation in polar coordinates as:

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0; \quad 0 < r < 1; 0 < \theta < 2\pi.$$

Determine the solution when the upper half of the circumference is kept at 100° and the lower half is kept at 0°.



Question # 4

(The hammer blow) Let $u(x,0) \equiv 0$ and $u_t(x,0) = g(x)$, where

$$g(x) = \begin{cases} 1, & |x| < 3 \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $u(x,t)$ provides solution to the one-dimensional wave equation:

$$u_{tt} = 4u_{xx}.$$

Using d' Alembert's solution, determine $u(x,t)$ and sketch the string profile at time $t = 3/4$.

Question # 5

A square membrane with $a = 1$, $b = 1$, and $c = 1/\pi$, is placed in the xy −plane. The edges of the membrane are held fixed, and the membrane is stretched into a shape modeled by the function:

$$f(x, y) = xy(x-1)(y-1), 0 < x < 1, 0 < y < 1.$$

Suppose that the membrane starts to vibrate from rest. Determine the position of each point on the membrane for $t > 0$. (hint: $g(x, y) = 0 \Longrightarrow B^*_{mn} = 0$)