Integral forms of Maxwell's Equation

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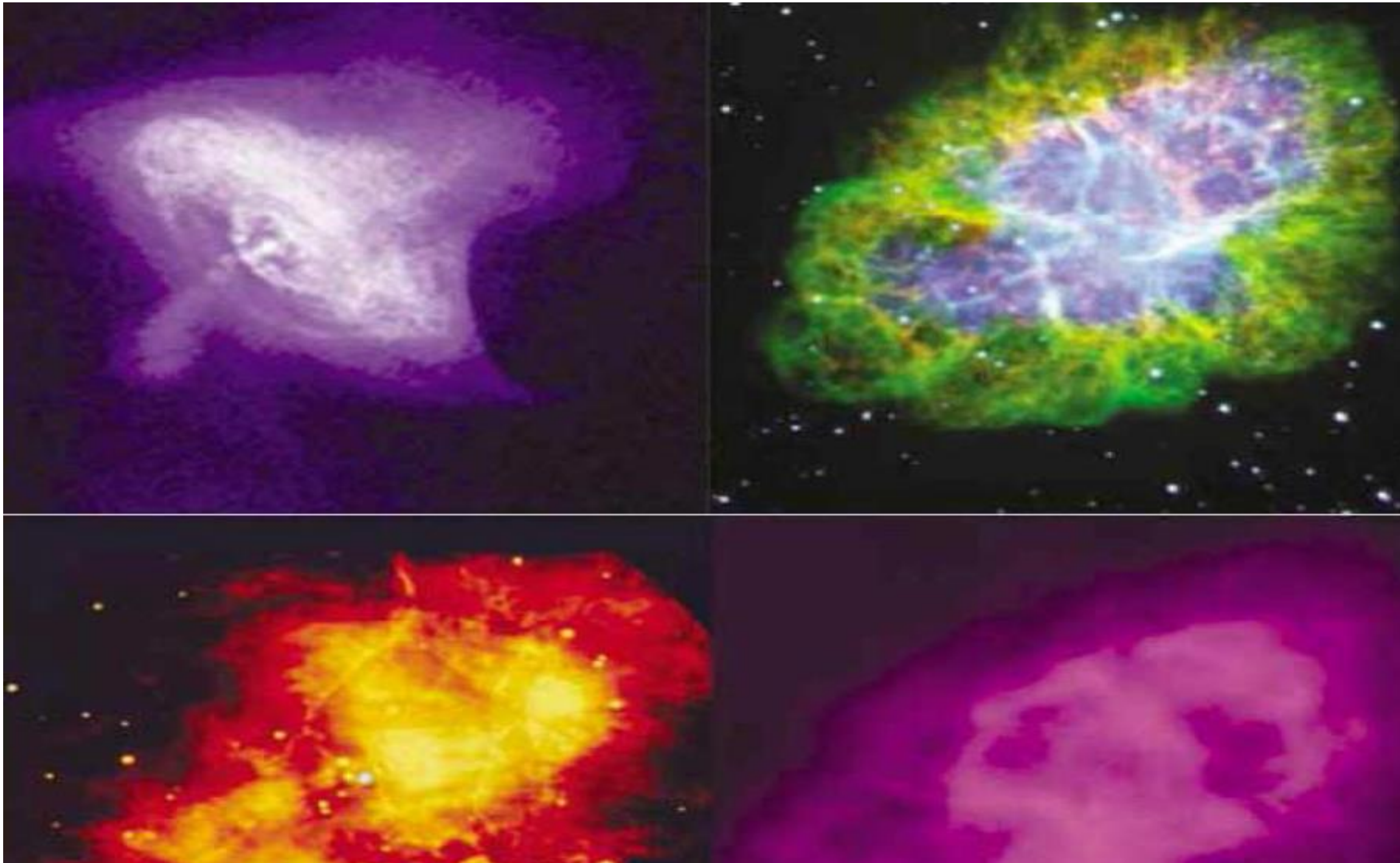


Electromagnetic waves cover a broad spectrum of wavelengths, with waves in various wavelength ranges having distinct properties. These images of the Crab Nebula show different structure for observations made with waves of various wavelengths. The images (clockwise starting from the upper left) were taken with x-rays, visible light, radio waves, and infrared waves.

Maxwell's Observations

- We have seen that charges in motion produce magnetic fields.
- Magnetic field due to current carrying conductor can be found by using Ampere's law

$$\oint \vec{B} \bullet d\vec{l} = \mu_\circ i_{conduction}$$

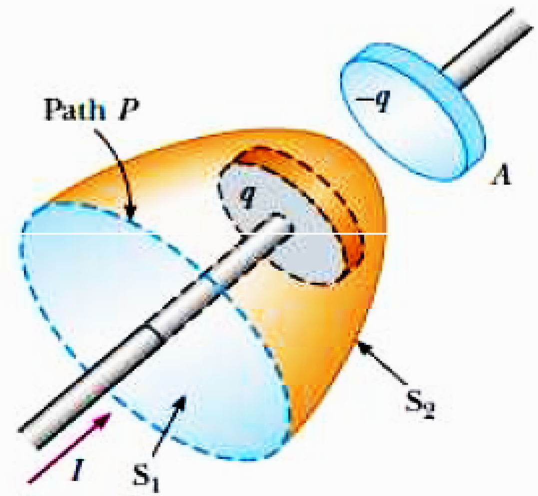
conduction current refers to the current carried by the wire,

- Maxwell recognized the limitation that Ampere's law in this form is valid only if any electric fields present are constant in time (Steady currents).
- Maxwell modified Ampere's law to include time-varying electric fields.

Now consider the two surfaces S1 and S2 in Figure below, bounded by the same path *P*. When the path P is considered as bounding S1,

$$\oint \vec{B} \bullet d\vec{l} = \mu_\circ I \qquad \rightarrow B \neq 0 \text{ at P}$$

because the conduction current I passes through S1. When the path is considered as bounding S2,



$$\oint \vec{B} \bullet d\vec{l} = 0 \qquad \rightarrow B=0 \text{ at P}$$

because no conduction current passes through S2.

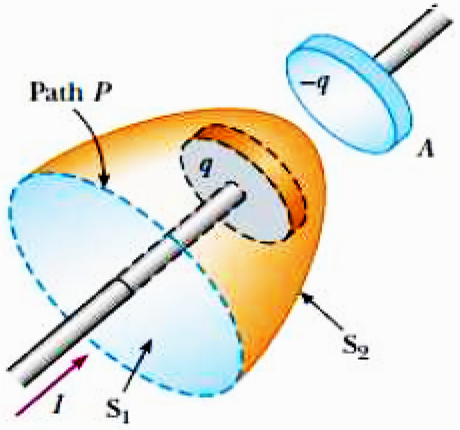
Thus, we have a contradictory situation that arises from the discontinuity of the current!

Maxwell solved this problem by postulating an additional term on the right side of Ampere's law, which includes a factor called the *displacement current* I_d defined as

$$I_d = \varepsilon_\circ \frac{d\Phi_E}{dt}$$

We can understand the meaning of this expression by referring to Figure shown below. If q is the charge on the plates at any instant, the electric flux through S2 is simply

$$\Phi_E = \frac{q}{\varepsilon_\circ}$$



Hence, the displacement current through S2 is

$$I_d = \varepsilon_\circ \frac{d\Phi_E}{dt} = \frac{dq}{dt}$$

That is, the displacement current I_d through S2 is precisely equal to the conduction current I through S1!

- As the capacitor is being charged (or discharged), the changing electric field between the plates may be considered equivalent to a current that acts as a continuation of the conduction current in the wire.
- When the expression for the displacement current is added to the conduction current on the right side of Ampere's law, the difficulty represented in case of capacitor is resolved.
- No matter which surface bounded by the path P is chosen, either a conduction current or a displacement current passes through it.
- With this new term I_d, we can express the general form of Ampere's law (sometimes called the ***Ampere–Maxwell law***) as

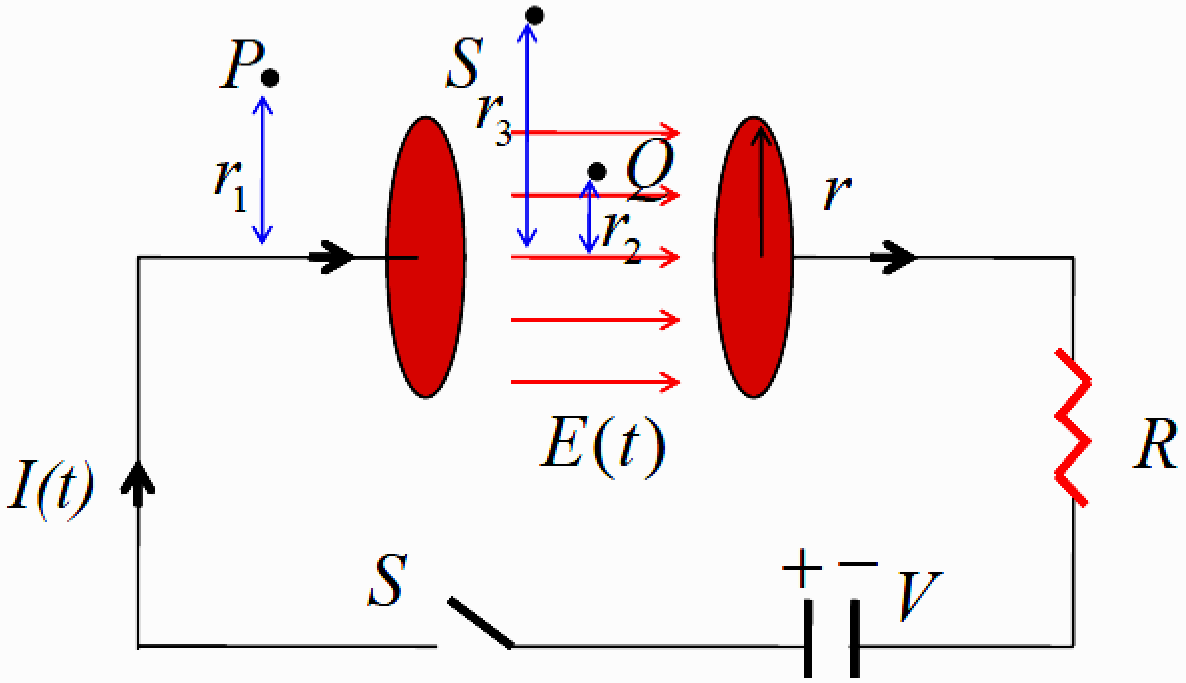
$$\boxed{\oint \vec{B} \bullet d\vec{l} = \mu_\circ (I + I_d) = \underbrace{\mu_\circ I}_{\text{Conduction current}} + \underbrace{\mu_\circ \varepsilon_\circ \frac{d\Phi_E}{dt}}_{\text{Time varying } E}}$$

Magnetic fields can be produced both by conduction currents and by time-varying electric fields.

A parallel plate capacitor with circular plates of radius r is being charged as shown below. Current I(t) in the wires and electric field E(t) between the plates varies with time as

$$I(t) = he^{-t/RC} \qquad E(t) = k(1 - e^{-t/RC})$$

Where h and k are constant, R is the resistance and C is the capacitance of capacitor. Find the magnetic field at points P,Q and S.



Magnetic field at point P

Consider an Amperian loop of radius r1

$$\oint \vec{B} \bullet d\vec{l} = \mu_\circ I$$

$$B(2\pi r_1) = \mu_\circ h e^{-t/RC}$$

$$B = \frac{\mu_\circ h}{2\pi r_1} e^{-t/RC}$$

Magnetic field at point Q

Consider an Amperian loop of radius r2 such that r2 < r

$$\oint \vec{B} \bullet d\vec{l} = \mu_\circ \varepsilon_\circ \frac{d\Phi_E}{dt}$$

$$B\oint dl = \mu_\circ \varepsilon_\circ \frac{dEA}{dt} = \mu_\circ \varepsilon_\circ A \frac{dE}{dt}$$

$$B(2\pi r_2) = \mu_\circ \varepsilon_\circ (\pi r_2{}^2) \frac{d}{dt}\left(k(1 - e^{-t/RC})\right)$$

$$B = \frac{\mu_\circ \varepsilon_\circ r_2}{2RC} k e^{-t/RC}$$

Magnetic field at point S

Consider an Amperian loop of radius r3 such that r3 > r

$$\oint \vec{B} \bullet d\vec{l} = \mu_\circ \varepsilon_\circ \frac{d\Phi_E}{dt}$$

$$B\oint dl = \mu_\circ \varepsilon_\circ \frac{dEA}{dt} = \mu_\circ \varepsilon_\circ A \frac{dE}{dt}$$

$$B(2\pi r_3) = \mu_\circ \varepsilon_\circ (\pi r^2) \frac{d}{dt}\left(k(1 - e^{-t/RC})\right)$$

$$B = \frac{\mu_\circ \varepsilon_\circ r^2}{2 r_3 RC} k e^{-t/RC}$$

Maxwell's Equation
Static EM Fields

Gauss's Law	**Electric field due to charges**	$\int \vec{E} \bullet d\vec{a} = \frac{q_{enc}}{\varepsilon_\circ}$
Gauss's Law *(Magnetism)*	**Nonexistence of magnetic monopole**	$\int \vec{B} \bullet d\vec{a} = 0$
No Name	**Conservativeness of electrostatic field**	$\oint \vec{E} \bullet d\vec{l} = 0$
Ampere's Law	**Magnetic field due to currents**	$\oint \vec{\mathrm{B}} \bullet d\vec{l} = \mu_\circ i$

Maxwell's Equation
Varying EM Fields

Gauss's Law	**Electric field due to charges**	$\int \vec{E} \bullet d\vec{a} = \frac{q_{enc}}{\varepsilon_\circ}$
Gauss's Law *(Magnetism)*	**Nonexistence of magnetic monopole**	$\int \vec{B} \bullet d\vec{a} = 0$
Faraday's Law	**Electric Field due to Changing magnetic flux**	$\oint \vec{E} \bullet d\vec{l} = -\frac{d\Phi_B}{dt}$
Ampere-Maxwell's Law	**Magnetic field due to currents and Changing electric flux**	$\oint \vec{\mathrm{B}} \bullet d\vec{l} = \mu_\circ i + \varepsilon_\circ \frac{d\Phi_E}{dt}$

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