Assignment 1

MATH - 351

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Q. : Rate of convergence of Newton - Raphson Method

$$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad , \quad n = 0, 1, 2 \dots \qquad (i)$$

- Suppose errors $\varepsilon$ as:

$$\left.\begin{aligned} \varepsilon_{n+1} &= x_{n+1} - \alpha \\ \varepsilon_n &= x_n - \alpha \end{aligned}\right\} \text{ where } \alpha \text{ is the true root}$$

$\Rightarrow$ (i) becomes after substitution:

$$\Rightarrow \varepsilon_{n+1} + \cancel{\alpha} = \varepsilon_n + \cancel{\alpha} - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \varepsilon_{n+1} = \varepsilon_n - \underbrace{\overbrace{\frac{f(x_n)}{f'(x_n)}}}_{\text{Expanding using Taylor's}}$$

$$\therefore f(x) = f(\alpha) + f'(\alpha)(x - \alpha) + \frac{1}{2!} f''(\alpha)(x - \alpha)^2 \dots$$

$$\Rightarrow \varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha) + f'(\alpha)\varepsilon_n + 1/2!\, f''(\alpha)\varepsilon_n^2 + \dots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \dots}$$

$$\Rightarrow \varepsilon_{n+1} = \frac{\left[\begin{array}{l} \cancel{\varepsilon_n f'(\alpha)} + \varepsilon_n^2 f''(\alpha) - f(\alpha) - \cancel{f'(\alpha)\varepsilon_n} - \dots \\ \dots \ \frac{1}{2}!\, f''(\alpha)\varepsilon_n^2 - \dots \end{array}\right]}{f'(\alpha) + \varepsilon_n f''(\alpha) + \dots}$$

As $f(\alpha) = 0$ ; And neglecting higher powers of $\varepsilon_n$, we get:

$$\Rightarrow \varepsilon_{n+1} \approx \frac{\varepsilon_n^2 f''(\alpha) - \frac{1}{2!}\varepsilon_n^2 f''(\alpha)}{f'(\alpha) + \cancel{\varepsilon_n f''(\alpha)}}$$

$$\Rightarrow \boxed{\varepsilon_{n+1} \approx \frac{1}{2} \cdot \frac{f''(\alpha)}{f'(\alpha)} \varepsilon_n^2}$$

neglecting term as $f'(\alpha) >> \varepsilon_n f''(\alpha)$

- Comparing with general form:

$$\therefore \varepsilon_{n+1} = \varepsilon_n^k (A)$$

$$\Rightarrow \boxed{A = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)}}$$

$$\Rightarrow \boxed{k = 2}$$

**Q₂: Rate of convergence of Secant Method**

$$\Rightarrow x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})}, \quad n = 1, 2 \ldots$$

- <u>Suppose errors $\varepsilon$ as:</u>

$$\left.\begin{aligned} \varepsilon_{n+1} &= x_{n+1} - \alpha \\ \varepsilon_n &= x_n - \alpha \\ \varepsilon_{n-1} &= x_{n-1} - \alpha \end{aligned}\right\}$$ where $\alpha$ is the true root

$$\Rightarrow \varepsilon_{n+1} = \varepsilon_n - \frac{(\varepsilon_n - \varepsilon_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})}$$

$$\Rightarrow \varepsilon_{n+1} = \frac{\cancel{\varepsilon_n f(x_n)} - \varepsilon_n f(x_{n-1}) - \cancel{\varepsilon_n f(x_n)} + \varepsilon_{n-1} f(x_n)}{f(x_n) - f(x_{n-1})}$$

$$\Rightarrow \varepsilon_{n+1} = \frac{\varepsilon_{n-1} f(x_n) - \varepsilon_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$\Rightarrow \varepsilon_{n+1} = \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right)\left(\frac{\varepsilon_{n-1} f(x_n) - \varepsilon_n f(x_{n-1})}{x_n - x_{n-1}}\right)$$

$$\Rightarrow \varepsilon_{n+1} = \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right)\left(\frac{\overbrace{f(x_{n-1})/\varepsilon_{n-1}}^{\mathrm{I}} - \overbrace{f(x_n)/\varepsilon_n}^{\mathrm{II}}}{x_n - x_{n-1}}\right)\varepsilon_n \varepsilon_{n-1}$$

$\Rightarrow$ I Expansion:

$$\frac{f(x_{n-1})}{\varepsilon_{n-1}} = f'(\alpha) + \frac{1}{2!} f''(\alpha)\varepsilon_{n-1} + \dots$$

$\Rightarrow$ II Expansion:

$$\frac{f(x_n)}{\varepsilon_n} = f'(\alpha) + \frac{1}{2!} f''(\alpha)\varepsilon_n + \dots$$

$$\Rightarrow \frac{f(x_{n-1})}{\varepsilon_{n-1}} - \frac{f(x_n)}{\varepsilon_n} = \frac{1}{2!} f''(\alpha)(\varepsilon_{n-1} - \varepsilon_n)$$

$$\Rightarrow \varepsilon_{n+1} = \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right)\left(\frac{\frac{1}{2} f''(\alpha)(\varepsilon_{n-1} - \varepsilon_n)}{x_n - x_{n-1}}\right)\varepsilon_n \varepsilon_{n-1}$$

$$\Rightarrow \varepsilon_{n+1} = -\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right)\frac{1}{2} f''(\alpha)\,\varepsilon_n \varepsilon_{n-1}$$

can neglect - sign due to error nature

As $\varepsilon_{n-1} - \varepsilon_n = x_{n-1} - x_n$

$$\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \approx f'(\alpha)$$ from Taylor's expansion

$$\Rightarrow \varepsilon_{n+1} = \left(\frac{f'(\alpha) f''(\alpha)}{2}\right)\varepsilon_n \varepsilon_{n-1}$$

$$\Rightarrow \varepsilon_{n+1} = M \varepsilon_n \varepsilon_{n-1}$$ ; where $M = \frac{1}{2} f'(\alpha) f''(\alpha)$

- Comparing with general form:

$\therefore \varepsilon_{n+1} = A \varepsilon_n^k$ and $\varepsilon_n = A \varepsilon_{n-1}^k$

$\varepsilon_{n-1} = A^{-1} \varepsilon_n^{1/k}$

$$\Rightarrow \varepsilon_n^k \propto \varepsilon_n \varepsilon_{n-1}$$

$$\Rightarrow \varepsilon_n^k \propto \varepsilon_n \varepsilon_n^{1/k} \Rightarrow \varepsilon_n^k \propto \varepsilon_n^{(k+1)/k}$$

From proportionality;

$$\Rightarrow k = \frac{k+1}{k}$$

$$\Rightarrow k^2 - k - 1 = 0$$

$$\therefore k = 1.618 \quad \text{or} \quad -0.618$$

As order of convergence $> 0$; $k = 1.618$

$$\Rightarrow \boxed{\varepsilon_{n+1} = M \varepsilon_n^{1.618}}, \text{ where } M = \tfrac{1}{2}(f'(\alpha))^{-1} f''(\alpha)$$

$$\text{or } M = \frac{1}{2f'(\alpha)} f''(\alpha)$$

$Q_{1/2}$: Rate of convergence of Regula Falsi method

$$\Rightarrow x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n), \quad n = 1, 2 \ldots$$

which is the same form as that of Secant method, however, with the difference being that it is a bracketed method. Following the same expansions as in $Q_{1/1}$: Secant Method, we obtain:

$$\Rightarrow \varepsilon_{n+1} = M \varepsilon_n \varepsilon_{n-1} \; ; \text{ where } M = \frac{1}{2f'(\alpha)} f''(\alpha)$$

As Regula Falsi is bracketed, and as we assumed $f(\alpha) = 0$, it implies we have a root in bracketed interval $(a, b)$; and that either $a$ or $b$ remains fixed and the other point varies with iteration.

Assuming $a$ to be fixed, then:

$$\Rightarrow \quad \varepsilon_{n+1} = M \varepsilon_n \varepsilon_0 \quad ; \text{ where } M = \frac{1}{2f'(\alpha)} f''(\alpha)$$

and $\varepsilon_0 = a - \alpha$ ; where $\alpha$ is the true root

is now independent of $n$

$$\Rightarrow \quad \varepsilon_{n+1} = M' \varepsilon_n \quad ; \text{ where } M' = M \varepsilon_0$$

Comparing with general form:

$$\therefore \quad \varepsilon_{n+1} = A \varepsilon_n^k$$

$$\Rightarrow \boxed{A = \frac{1}{2f'(\alpha)} f''(\alpha)(a - \alpha)}$$

$$\Rightarrow \boxed{k = 1}$$

: Regula Falsi is linear