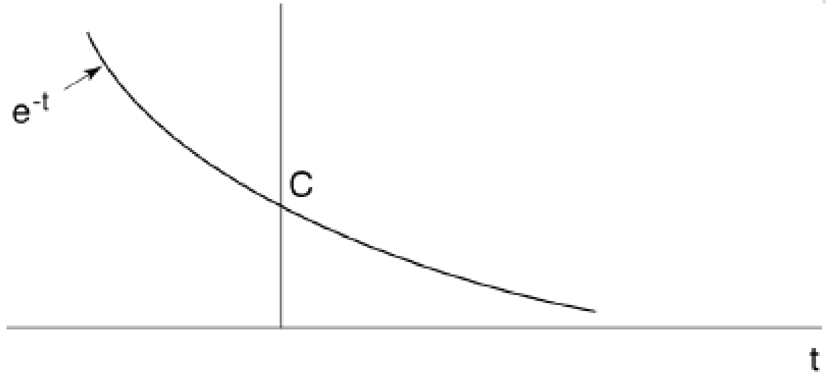
PROPERTIES OF ROCS

Laplace Transforms and ROCs

- Some signals do not have Laplace Transforms (have no ROC)

(a) $x(t) = Ce^{-t}$ for all t since $\displaystyle\int_{-\infty}^{\infty} \underbrace{|x(t)e^{-\sigma t}|}_{\| \atop Ce^{-(\sigma+1)t}} dt = \infty$ for all σ



(b) $x(t) = e^{j\omega_0 t}$ for all t FT: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt = \int_{-\infty}^{\infty} e^{-\sigma t} dt = \infty \text{ for all } \sigma$$

$X(s)$ is defined only in ROC;
we don't allow impulses in Laplace Transforms

Laplace Transforms and ROCs

- *Property 1*: The ROC of $X(s)$ consists of strips parallel to the $j\omega-$axis in the $s-$ plane

- The validity of this property stems from the fact that the ROC of $X(s)$ consists of those values of $s=\sigma+j\omega$ for which the FT of $x(t)e^{-\sigma t}$ converges.

- Thus the ROC of the Laplace transform of $x(t)$ consists of those values of s for which $x(t)e^{-\sigma t}$ is absolutely integrable:

$$\int_{-\infty}^{\infty} |\, x(t)\,|\, e^{-\sigma t} dt < \infty$$

- Property 1 follows since this condition depends only on σ

Laplace Transforms and ROCs

$\boxed{\text{• } \textit{Property 2:} \text{ For rational Laplace transforms, the ROC does not contain any poles}}$

- Since $X(s)$ is infinite at a pole, the absolute integrable condition clearly does not converge at a pole, and thus the ROC cannot contain values of s that are poles.

$\boxed{\text{• } \textit{Property 3:} \text{ If } x(t) \text{ is of finite duration and is absolutely integrable, then the ROC is the entire } s-\text{plane}}$

Finite Duration Signal - Example

- Consider the finite duration signal:

$$x(t) = \begin{cases} e^{-at}, & 0 < t < T \\ 0, & otherwise \end{cases}$$

$$X(s) = \int_0^T e^{-at} e^{-st} dt = \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right]; \quad \text{all } s$$

- What happens at $s = -a$? (Use L'hopital's rule)

$$\lim_{s \to -a} X(s) = \lim_{s \to -a} \left[\frac{\frac{d}{ds}\left(1 - e^{-(s+a)T}\right)}{\frac{d}{ds}(s+a)} \right] = \lim_{s \to -a} T e^{-aT} e^{-sT} = T$$

L'hopital's Rule

- Given two functions of the form $f(x)$ and $g(x)$ with the properties:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$$

- and we wish to evaluate the limit:

$$\lim_{x\to 0} \frac{f(x)}{g(x)}$$

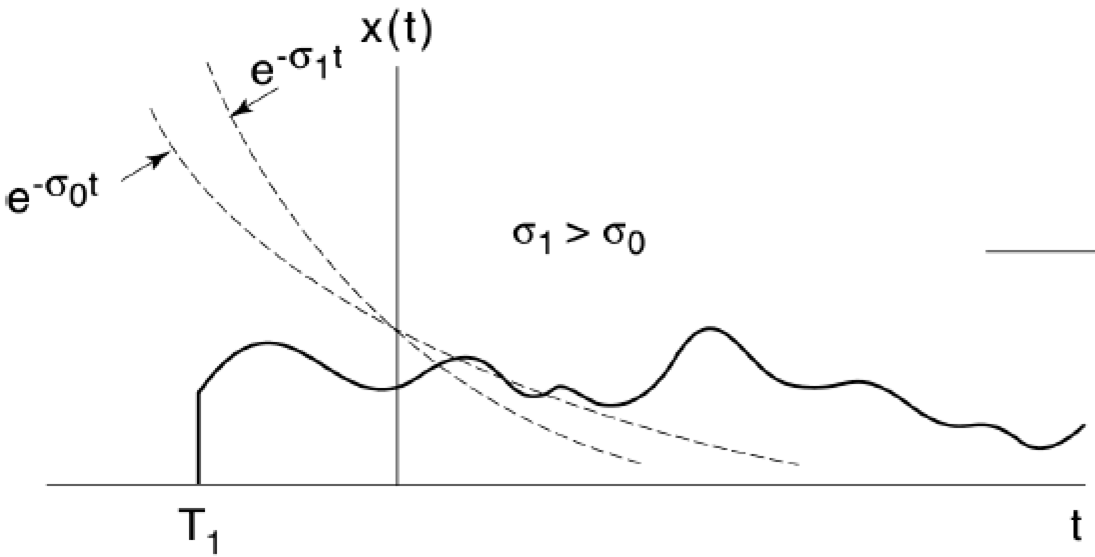
- l'Hopital's rule states that'

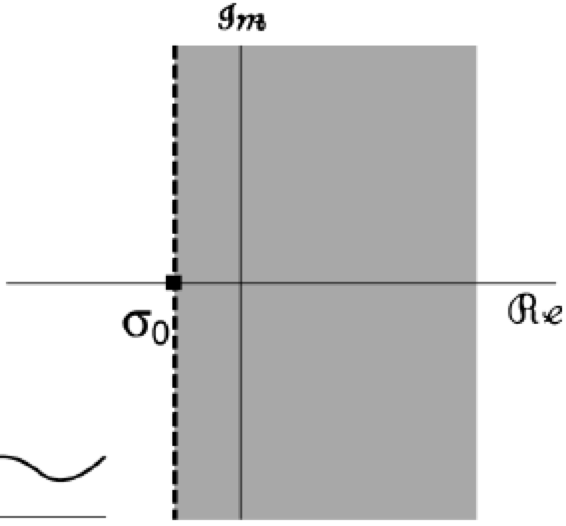
$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

Laplace Transforms and ROCs

- *Property 4 :* If $x(t)$ is right-sided, and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathrm{Re}\{s\} > \sigma_0$ will also be in the ROC

- A right-sided signal is a signal for which $x(t) = 0$ prior to some finite time T_1





Laplace Transforms and ROCs

• *Property 5 :* If $x(t)$ is left-sided, and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathrm{Re}\{s\} < \sigma_0$ will also be in the ROC

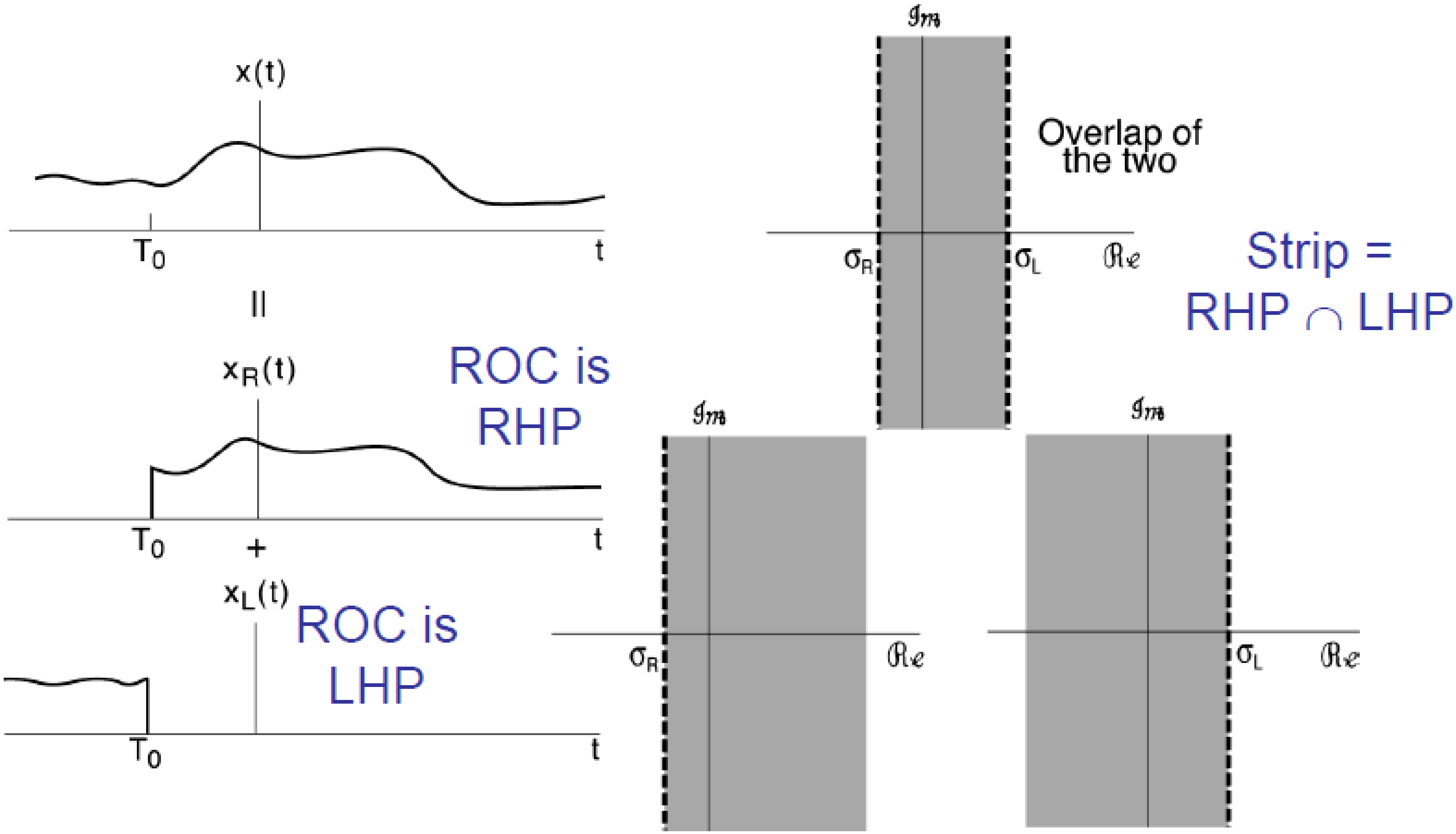
• A left-sided signal is a signal for which $x(t) = 0$ after some finite time T_2

• *Property 6 :* If $x(t)$ is two-sided, and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\mathrm{Re}\{s\} = \sigma_0$

• A two-sided signal is a signal that is of infinite extent for both $t > 0$ and $t < 0$

Laplace Transforms and ROCs

6) If $x(t)$ is two-sided and if the line $Re(s) = \sigma_o$ is in the ROC, then the ROC consists of a strip in the s-plane that includes the line $Re(s) = \sigma_o$.

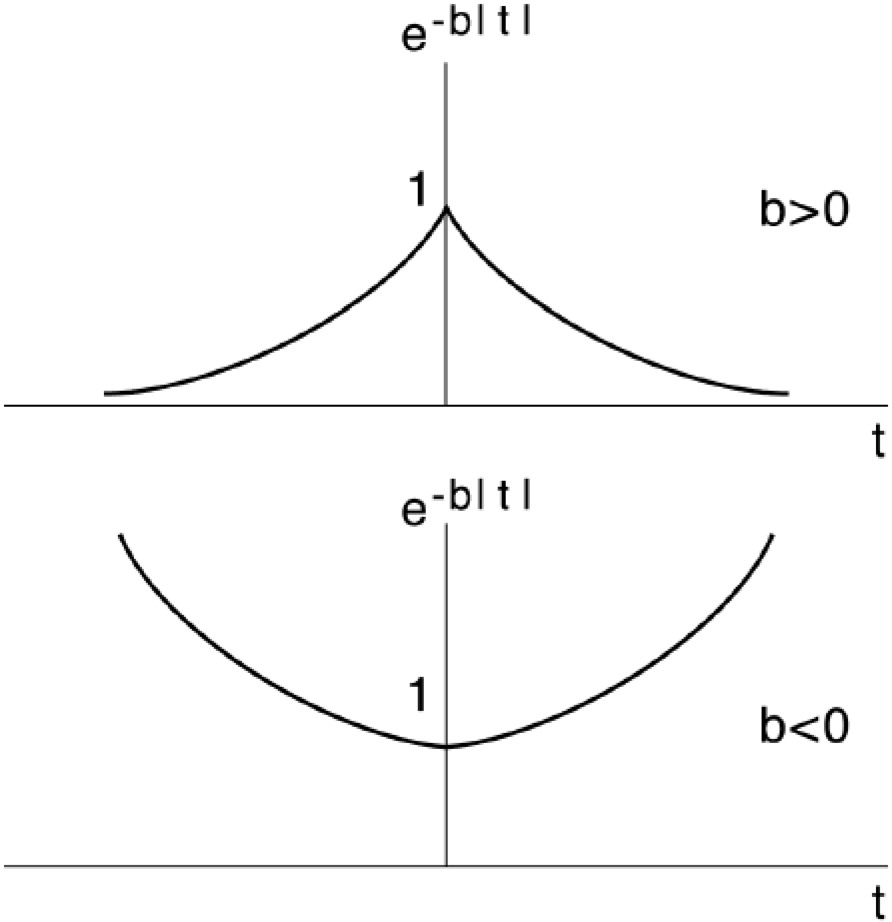


Laplace Transforms and ROCs

$$x(t) = e^{-b|t|}$$

$x(t) = x_L(t) + x_R(t)$ -- Left-Right Decomposition

$x(t) = e^{-bt}u(t) + e^{+bt}u(-t)$



- Multiply by $e^{\sigma t}$ and product will be integrable

- No choice of $e^{\sigma t}$ will dampen both sides

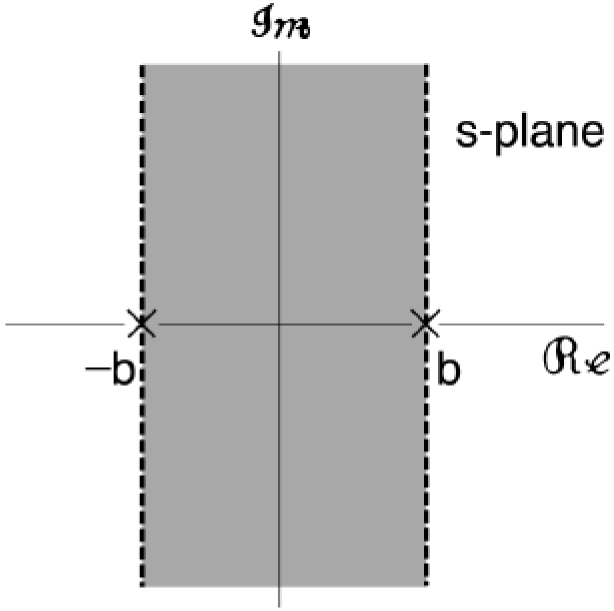
Laplace Transforms and ROCs

$$x(t) \quad = \quad e^{bt}u(-t) \quad + \quad e^{-bt}u(t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$-\frac{1}{s-b}, \; \Re e\{s\} < b \qquad \frac{1}{s+b}, \; \Re e\{s\} > -b$$

Overlap if $b > 0 \Rightarrow X(s) = \frac{-2b}{s^2-b^2}$, with ROC:



What if $b < 0$? $\Rightarrow$ No overlap $\Rightarrow$ No Laplace Transform

Laplace Transforms and ROCs

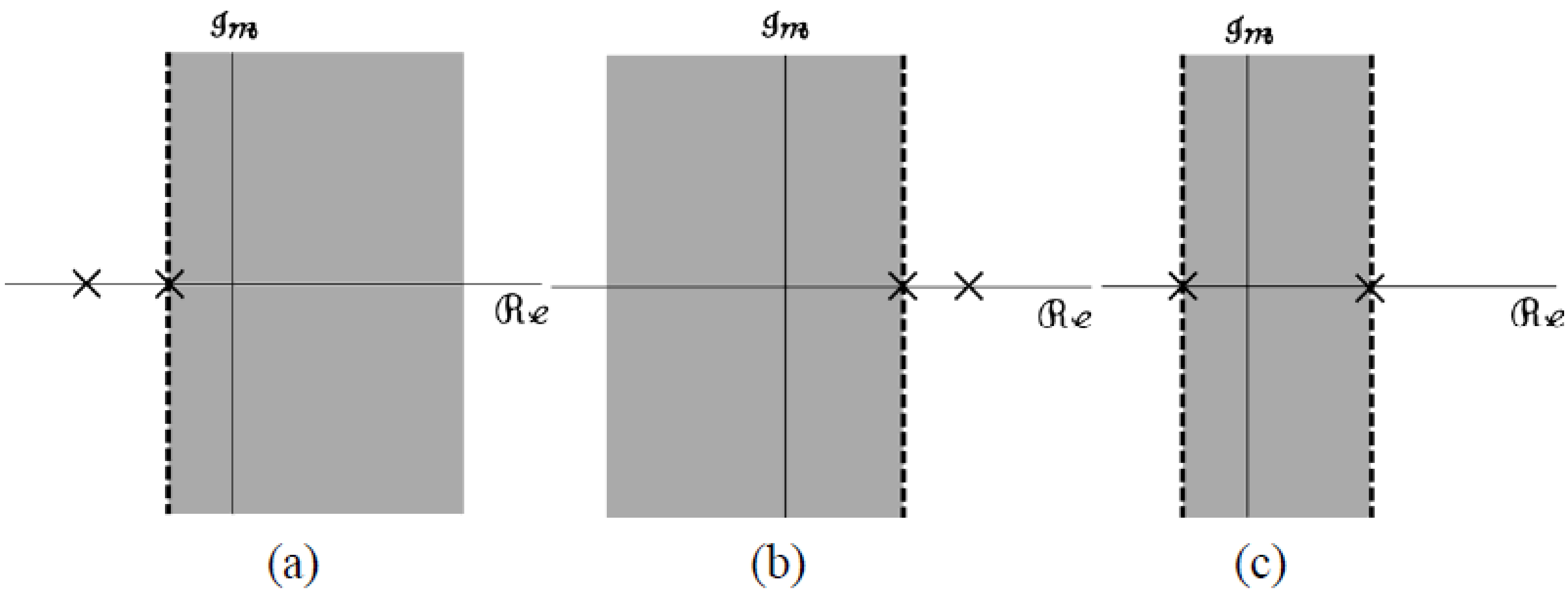
- *Property 7 :* If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC

- *Property 8 :* If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right-sided, the ROC is the region in the s-plane to the right of the rightmost pole, If $x(t)$ is left-sided, the ROC is the region in the s-plane to the left of the leftmost pole.

Laplace Transforms and ROCs

7) If $X(s)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

8) Suppose $X(s)$ is rational, then
   (a) If $x(t)$ is right-sided, the ROC is to the right of the rightmost pole.
   (b) If $x(t)$ is left-sided, the ROC is to the left of the leftmost pole.



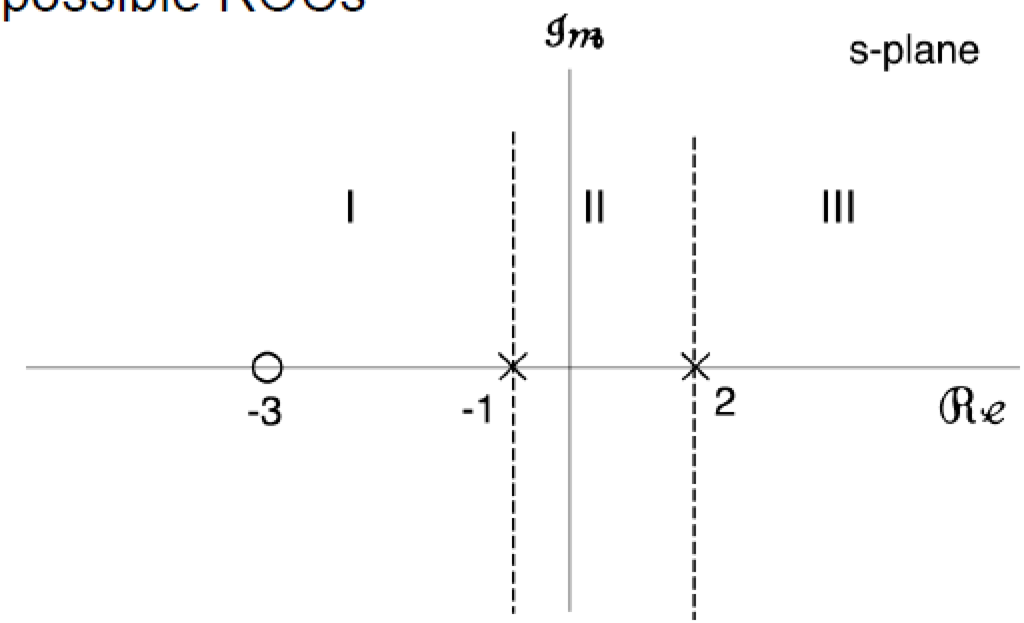
9) If ROC of $X(s)$ includes the $j\omega$-axis, then *FT* of $x(t)$ exists.

Laplace Transforms and ROCs

9) If ROC of $X(s)$ includes the $j\omega$-axis, then *FT* of $x(t)$ exists.

Example: $$X(s) = \frac{(s+3)}{(s+1)(s-2)}$$

Three possible ROCs



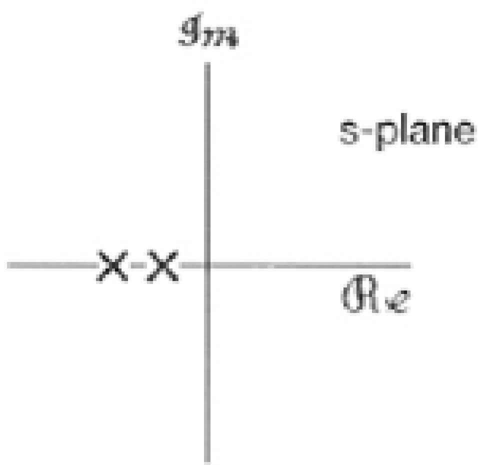
			Fourier Transform exists?
$x(t)$ is right-sided	ROC:	III	No
$x(t)$ is left-sided	ROC:	I	No
$x(t)$ extends for all time	ROC:	II	Yes

Laplace Transforms and ROCs

- Let

$$X(s) = \frac{1}{(s+1)(s+2)}$$

- with the associated pole-zero pattern shown in the figure, part (a).

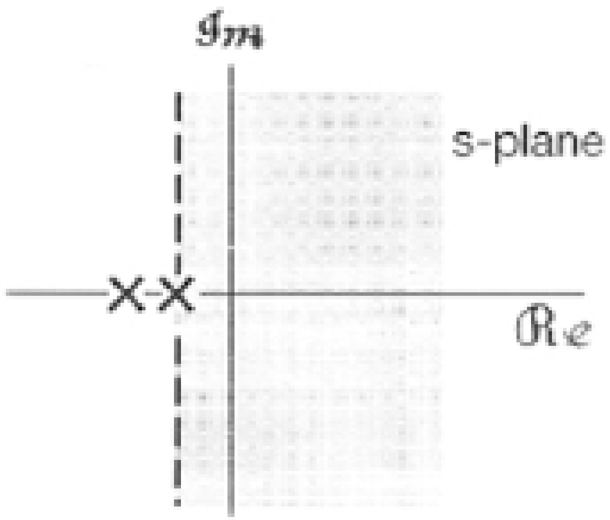


(a)

- There are three possible ROCs, corresponding to three distinct signals

Laplace Transforms and ROCs

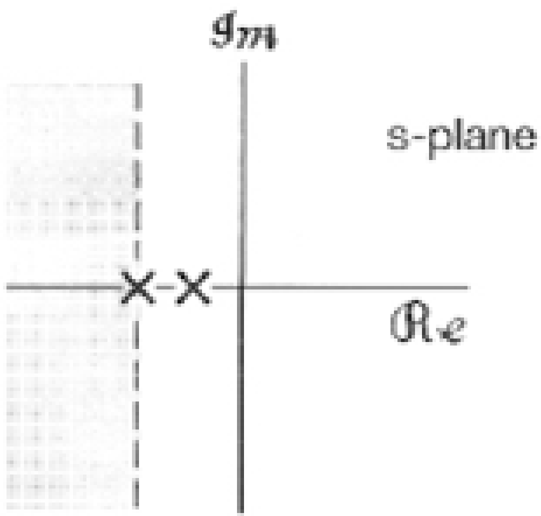
- Figure part(b) corresponds to a right-sided signal with a valid FT;



(b)

Laplace Transforms and ROCs

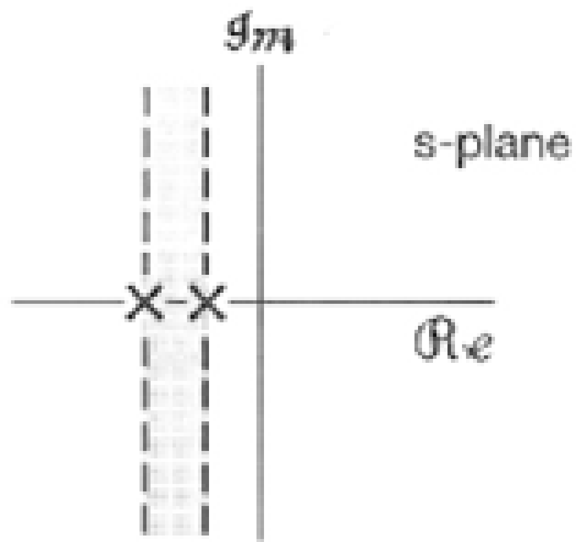
- Figure part(c) corresponds to a left-sided signal with no FT;



(c)

Laplace Transforms and ROCs

- Figure part(d) corresponds to a two-sided signal with no FT;



(d)

END