Complex Variables and Transforms

Identifying Poles and Calculating Residue

→ We know that the pole is a type of singularity (isolated)

with the help of which we can determine the the highest order of the z of the Laurent part if we expand the $f(z)$ about that pole.

Example $f(z) = \frac{1}{z-1}$ So at $z_0 = 1$ we have a pole of order 1 (also called as simple pole).

→ Just see where the denominator is going to zero and check its order (degree).

Example $f(z) = \frac{1}{(z-1)(z+2)^2}$ Now we have 2 poles at $z_0 = 1$ (order 1) and $z_0 = -2$ of order

Example $f(z) = \frac{\sin z}{z^2}$ → Now Apparently it seems that at $z_0 = 0$ A pole of order 2, but thats NOT t

If rewrite $\frac{\sin z}{z \cdot z}$, we can see that $\frac{\sin z}{z}$ at $z_0 =$ gives 1. (Means that 1 degree is canceled out wit $\sin z$). So we are left with $\frac{1}{z}$ so only orde 1 exists.

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So total order = ... $= 3 - 1 = 2$

Similarly $\frac{\sin^2 z}{z^2}$ at $z_0 = 0$ we have No pole exists -

Special CASES

→ Correct determination of the poles and its order is very important because these will be used to determine Residue (Discussed Later) and then Evaluate the complex integrals -

→ Although we know how to determine pole but Still Sometimes we get Unlucky and end up with wrong calculated Order -

→ There is a simple procedure from which we ca...

NOTE: → The point at which Numerator becomes zero is called as zeros of the Numerator -

Example $\frac{z^2-1}{(z+5)}$ we have zeros at $z_0 = \pm 1$

→ Now If at any pole $z = z_0$ of order m, then if Numerator also becomes zero at same point $z = z_0$, then it will be calcened out a order of pole -

Example $\frac{\sin z}{z^3}$ → Zero at $z_0 = 0$
Pole at $z_0 = 0$

So total order = Pole order - Zero Order
$= 3 - 1 = 2$ order pole -

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Correctly find the order of the pole -

let suppose that the $f(z)$ is a function

Like $$f(z) = \frac{g(z)}{(z-z_0)^N}$$

→ N is the correct order of pole z_0 -

→ Suppose we Guess M order, then to check whether M is correct order and M=N

we calculate

$$\lim_{z \to z_0} (z-z_0)^M f(z)$$

Now there are three possibilities:-

(i) $M > N$, means we multiply by too high power of $(z-z_0)$ due to which we will get zero all the times - (at $z=z_0$)

(ii) $M < N$, means that $(z-z_0)$ factor(s) will still be present in denominator and we will get ∞ each time -

(iii) $M = N$, if we guessed the right pole, and we will always get the finite answer $= L$ -

→ So Using this simple Procedure, we can Check our Correctness of the order Selection of Pole -

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Special CASES

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→ There is a simple procedure from which we can

Example ① $f(z) = \frac{1}{\cos z}$, → It is clear that pole at $z_0 = \pm \pi/2, \pm 3\pi/2 \ldots\ldots$ of order 1 -

→ To Check the formula correctness, let suppose order is $2 \Rightarrow m=2$

Putting in formula :-

$$\lim_{z \to \pi/2} \frac{(z-\pi/2)^2}{\cos z} \qquad \left(\tfrac{0}{0} \text{ form}\right)$$

→ So Apply L-Hobital

$$\lim_{z \to \pi/2} \frac{2(z-\pi/2)}{-\sin z} = \lim \frac{-2(z-\pi/2)}{\sin z}$$

Apply Limit $= \frac{-2(0)}{1} = 0$ → which shows that $M > N$ and order is less than 2 -

→ So Now Checking for order 1 -

$M=1$

$$\lim_{z_0 \to \pi/2} \frac{(z-\pi/2)}{\cos z} \Rightarrow \frac{0}{0} \text{ form, So L-Hobital}$$

$$\lim_{z_0 \to \pi/2} \frac{1}{-\sin z} = -1 \quad \rightarrow \text{So finite Answer}$$

→ So we came to know that 1 is correct order -

Example 2

$f(z) = \dfrac{z}{\sin^2 z}$ → It is clear that $f(z)$ has zero at $z_0 = 0$ (order 1) and pole of order 2 at $z_0 = 0$

→ So 1 Degree will be canceled, so total order of $f(z)$ at 0 is 1 (simple pole)

→ This can be proved by the same procedure.

$M = 1$

So $\lim\limits_{z \to 0} \dfrac{(z-0)(z)}{\sin^2 z} = \dfrac{z^2}{\sin^2 z} \quad \left(\dfrac{0}{0}\right)$

$\overset{H}{=} \lim\limits_{z \to 0} \dfrac{2z}{2\sin z \cos z}$ → Again $\left(\dfrac{0}{0}\right)$

$\overset{H}{=} \lim\limits_{z \to 0} \dfrac{2}{2\cos^2 z - 2\sin^2 z} = -2$ → ~~So~~ finit answer

→ So the order was correct → Simple pole.

RESIDUES

→ If we Expand the $f(z)$ [any function having a Laurent Part] about its pole z_0; the coefficient of the term $\dfrac{1}{(z-z_0)^1}$ is called the Residue of that $f(z)$.

Example find Residue of $e^{1/z}$

Solution

→ the Expansion of $e^{1/z} = 1 + \frac{1}{z} + \frac{(1/z)^2}{2!} \ldots\ldots$

So Coefficient of $\frac{1}{z}$ is 1, So Residue is 1 for this function -

Generalized Formula For Residue

Let $f(z)$ be a function having a N order pole at z_0, then the formula to Calculate its Residue at $z=z_0$ is given as:-

$$\text{Residue at } z=z_0 = \lim_{z \to z_0} \frac{1}{(N-1)!} \cdot \frac{d^{(N-1)}}{dz^{(N-1)}} \left[(z-z_0)^N \cdot f(z) \right]$$

→ We Must use the correct order of pole before finding the residue -

→ while finding Residue, if we come with $\frac{0}{0}$ or form, always Apply L-Hobital Rule -

IMPORTANCE → the Residue helps us to calculate the Complex Integral by well known theorem Called as Cauchy Residue Theorem which States that :-

$$\oint f(z)\,dz = [\text{Sum of all Residues of Poles inside the curve} \times 2\pi i]$$

Examples

(i) find the all poles and then calculate its Respective Residues.

(a) $f(z) = \dfrac{1}{\sin z \cos z}$

(b) $f(z) = \tan z$

(c) $f(z) = \dfrac{1}{(\sinh z)^2}$

(d) $f(z) = \dfrac{\sin z}{(\cos z^3 - 1)}$

(e) $f(z) = \dfrac{z^2}{\sin z}$

This Handout: (Page 1-7) Courtesy to M. Uzair BEE 9-B.

$f(t) = 2\sin(4\pi t + \frac{\pi}{3}) + \sin(6\pi t - \frac{\pi}{4}) - \cos(8\pi t + 1)$.

Find Fundamental time period & frequency, Complex Fourier series, Trigonometric Fourier series, Compact form of Fourier. Sketch discrete Fourier spectra.

$\omega_1 = 4\pi,\ \omega_2 = 6\pi,\ \omega_3 = 8\pi,\ \omega_0 = 2\pi$ (MCF), $T = \frac{2\pi}{2\pi} = 1$ (Second)

or $\frac{2\pi}{4\pi}N_1 = \frac{2\pi}{6\pi}N_2 = \frac{2\pi}{8\pi}N_3$, $T = \frac{1}{2}N_1 = \frac{1}{3}N_2 = \frac{1}{4}N_3$, $T = 1$, $\omega = \frac{2\pi}{1} = 2\pi$ rad/s

$$f(t) = 2\left[\frac{e^{j(4\pi t+\pi/3)} - e^{-j(4\pi t+\pi/3)}}{2j}\right] + 2\left[\frac{e^{j(6\pi t-\pi/4)} - e^{-j(6\pi t-\pi/4)}}{2j}\right] - \left[\frac{e^{j(8\pi t+1)} + e^{-j(8\pi t+1)}}{2}\right]$$

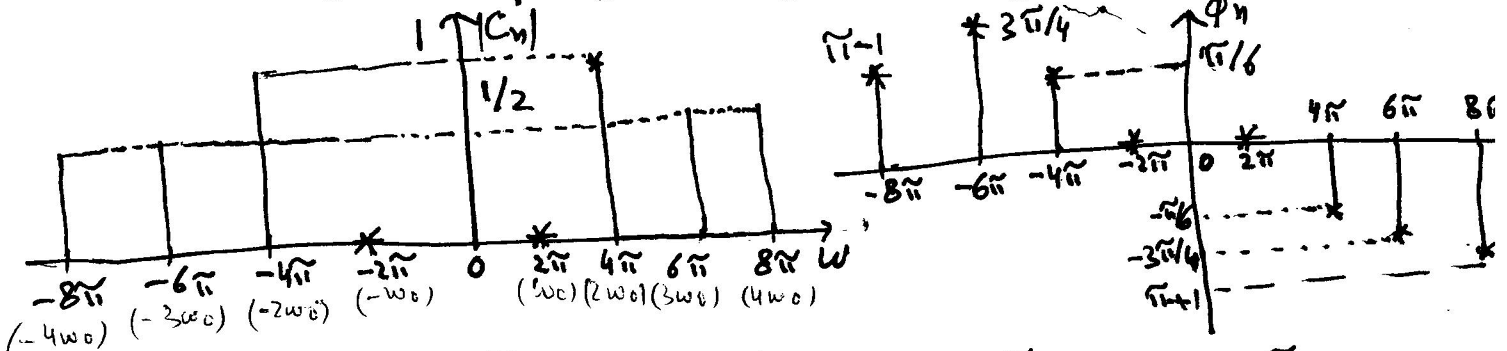
$$= e^{j(4\pi t+\pi/3)}e^{-j\pi/2} - e^{-j(4\pi t+\pi/3)}e^{j\pi/2} + \frac{1}{2}e^{-j\pi/2}e^{j(6\pi t-\pi/4)} - \frac{1}{2}e^{-j\pi/2}e^{-j(6\pi t-\pi/4)}$$
$$- \frac{1}{2}e^{j(8\pi t+1)} - \frac{1}{2}e^{-j(8\pi t+1)}$$

$$= e^{-j\pi/6}e^{j4\pi t} + e^{j\pi}e^{-j\pi/3}e^{-j\pi/2}e^{-4j\pi t} + \frac{1}{2}e^{-j\pi/2}e^{-j\pi/4}e^{j6\pi t} + \frac{1}{2}e^{j\pi}e^{-j\pi/2}e^{j\pi/4}e^{-j6\pi t}$$
$$+ \frac{1}{2}e^{j\pi}e^{j}e^{j8\pi t} + \frac{1}{2}e^{j\pi}e^{-j}e^{-j8\pi t}.$$

$$= \frac{1}{2}e^{j\pi}e^{-j}e^{-j8\pi t} + \frac{1}{2}e^{j3\pi/4}e^{-j6\pi t} + e^{j\pi/6}e^{-4j\pi t} + e^{-j\pi/6}e^{j4\pi t}$$
$$+ \frac{1}{2}e^{-j3\pi/4}e^{j6\pi t} + \frac{1}{2}e^{j\pi}e^{j}e^{j8\pi t}.$$

$D_2 = e^{-j\pi/6}$, $D_3 = \frac{1}{2}e^{-j3\pi/4}$, $D_4 = \frac{1}{2}e^{j(\pi+1)}$.

$\omega_0 = 2\pi$, $4\omega = 8\pi$, $D_{-4} = \frac{1}{2}e^{(\pi-1)j}$, $D_{-3} = \frac{1}{2}e^{j3\pi/4}$, $D_{-2} = e^{+j\pi/6}$,



TFS: $f(t) = 2\left(\sin 4\pi t \cos\frac{\pi}{3} + \cos 4\pi t \sin\frac{\pi}{3}\right) + \sin(6\pi t)\cos(\frac{\pi}{4}) - \cos(6\pi t)\sin(\frac{\pi}{4})$
$- \cos 8\pi t \cos 1 + \sin 8\pi t \sin 1$

$= (2)(\frac{1}{2})\sin 4\pi t + (2)(\frac{\sqrt{3}}{2})\cos 4\pi t + (\frac{1}{\sqrt{2}})\sin(6\pi t) - \frac{1}{\sqrt{2}}\cos(6\pi t) - \cos 1 \cos 8\pi t + \sin 1 \sin 8\pi t$

$= \sin 4\pi t + \sqrt{3}\cos 4\pi t + \frac{1}{\sqrt{2}}\sin(6\pi t) - \frac{1}{\sqrt{2}}\cos(6\pi t) - \cos 1 \cos 8\pi t + \sin 1 \sin 8\pi t$

So, $b_2 = 1$, $a_2 = \sqrt{3}$, $b_3 = \frac{1}{\sqrt{2}}$, $a_3 = \frac{-1}{\sqrt{2}}$, $a_4 = -\cos 1$, $b_4 = \sin 1$.

A periodic function $f(t) = \begin{cases} 0, & -2 < t < 0 \\ 1, & 0 < t < 2 \end{cases}$, $f(t+4) = f(t)$.

for all t, called the square wave has the complex form of the Fourier series

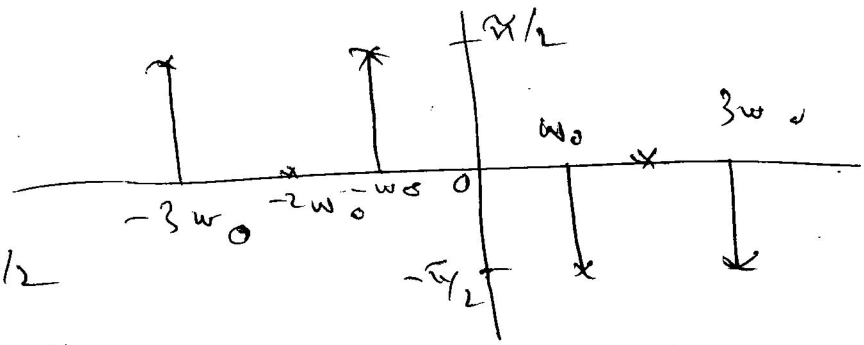
$$f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{j}{2n\pi}\left[(-1)^n - 1\right] e^{jn\frac{\pi}{2}t}.$$

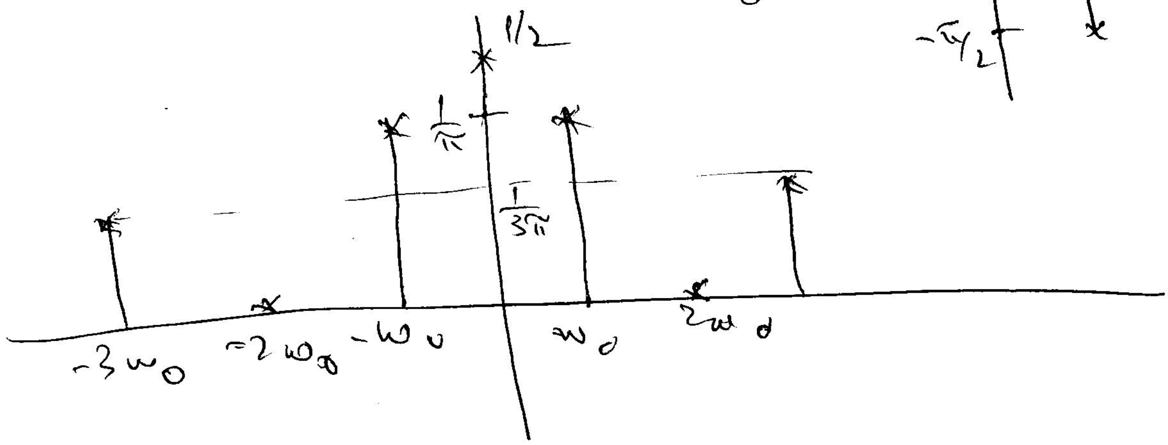
Write an expression for $|C_n|$ (the magnitude of C_n) and ϕ_n related to the phase of the nth harmonic. Sketch discrete frequency spectra.

$$C_n = \frac{j}{2\pi n}\left[(-1)^n - 1\right] = \begin{cases} 0, & n = \pm 2, \pm 4, \ldots \\ \frac{-j}{n\pi}, & n = \pm 1, \pm 3, \pm 5, \ldots \end{cases}$$

$$|C_n| = \begin{cases} 0, & n = \pm 2, \pm 4, \ldots \\ \frac{1}{n\pi}, & n = 1, 3, 5, \ldots \\ -\frac{1}{n\pi}, & n = -1, -3, -5, \ldots \end{cases}$$

$$\phi_n = \arg(C_n) = \begin{cases} 0, & n = \pm 2, \pm 4, \ldots \\ -\pi/2, & n = 1, 3, 5, \ldots \\ \frac{\pi}{2}, & n = -1, -3, -5, \ldots \end{cases}$$





EX:- A periodic voltage V(t) (in volts) period 5 ms and specified by

$$V(t)=\begin{cases}60, & 0<t<5/4\,\text{ms}\\ 0, & \frac{5}{4}\text{ms}<t<5\,\text{ms}\end{cases} \qquad V(t+5\,\text{ms})=V(t)$$

is applied across the terminals of a 15 Ω resistor. (a) obtain expression for the Coefficients c_n of the complex Fourier series representation of V (t) and write down the values of the first five non-zero terms.
(b) Calculate the power associated with each of the first five non-zero terms of the Fourier expansion. (c) calculate the total power delivered to the 15 Ω resistor. (d) what is the percentage of the total power delivered to the resistor by the first five non-zero terms of F.

Sol:- $c_n=\frac{1}{5}\int_0^{5/4}60\,e^{-j\frac{2n\pi}{5}t}\,dt=12\left[\frac{-5}{2n\pi j}e^{-j\frac{2n\pi}{5}t}\right]_0^{5/4}=\frac{30}{jn\pi}\left[1-e^{-jn\pi/2}\right], n\neq$

$c_0=\frac{1}{5}(60)(5/4)=15$. First five non-zero terms are $c_0=15$, $c_1=\frac{30}{\pi j}(1+j)$

$c_2=\frac{30}{j\pi}=-\frac{30}{\pi}j$, $c_3=\frac{10}{j\pi}(1-j)=\frac{10}{\pi}(-1-j)$, $c_4=0$, $c_5=\frac{6}{j\pi}(1+j)=\frac{6}{\pi}(1-j)$

Power associated with the first five non-zero terms, $P_0=\frac{1}{15}[15^2]=15$W

$P_1=\frac{1}{15}[2|c_1|^2]=\frac{2}{15}(13.5)^2=24.3$W, $P_2=\frac{1}{15}[2|c_2|^2]=\frac{1}{15}(9.55)^2=12.16$W,

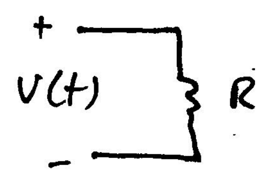
$P_3=\frac{1}{15}[2|c_3|^2]=\frac{2}{15}(4.5)^2=2.70$W, $P_4=0$, $P_5=\frac{1}{15}[2|c_5|^2]=\frac{2}{15}(2.70)^2=0.97$W

The total power delivered by first five non-zero terms $=P_0+\dots+P_5=55.13$W

Total power delivered to 15 Ω is $P=\frac{1}{15}\left[\frac{1}{5}\int_0^{5/4}(60)^2dt\right]=\frac{1}{15}\left(\frac{1}{5}\right)(60^2)\left(\frac{5}{4}\right)=60$

% of total power delivered by the first five non-zero terms is $\frac{55.13}{60}\times100\%=91.9\%$

Problems:- 1. Let $V(t)=\sin(4\pi t)$ be the input signal to the circuit shown. Find the average power delivered to the resistor $R=1\,\Omega$.



2. Let $V(t)=3-5\sin(4t-\frac{\pi}{3})-4\cos(3t+\pi/3)$ be the input signal to the circuit shown. Find the average power delivered to the resistor $R=1\,\Omega$ by using both sides of Parseval's theorem.

3. A periodic function f(t), of period 2π, is defined in the range $-\pi<t<\pi$ by $f(t)=\sin\frac{1}{2}t$. Show that the complex form of Fourier series expansion for f(t) is $f(t)=\sum_{n=-\infty}^{+\infty}\frac{j\,4n(-1)^n}{\pi(4n^2-1)}e^{jnt}$. Sketch 2-sided discrete Fourier spectra and obtain trigonometric Fourier series
[Ref/Keep this question: Example 7.18, page 610, Glyn James at LMS].