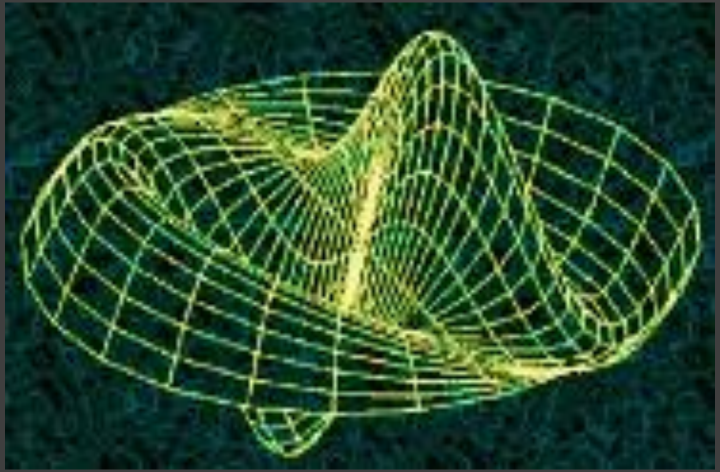
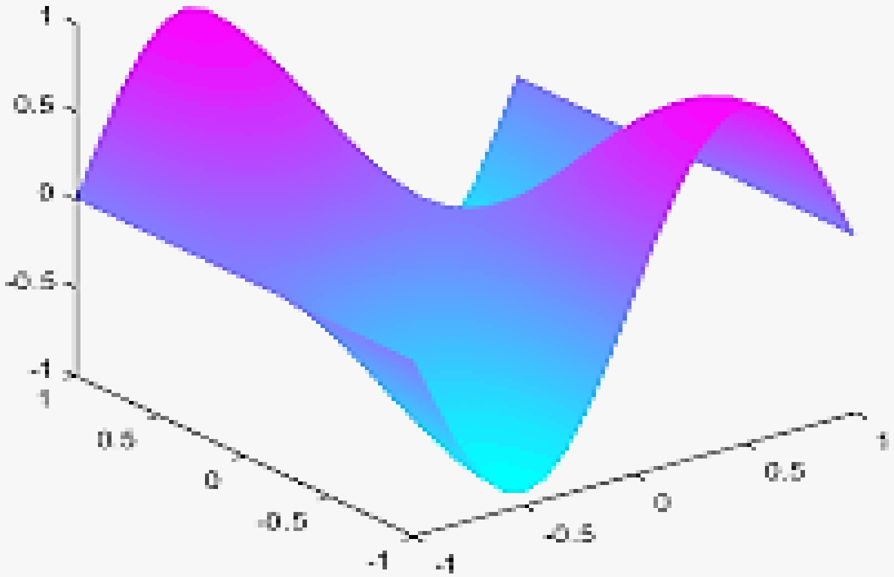
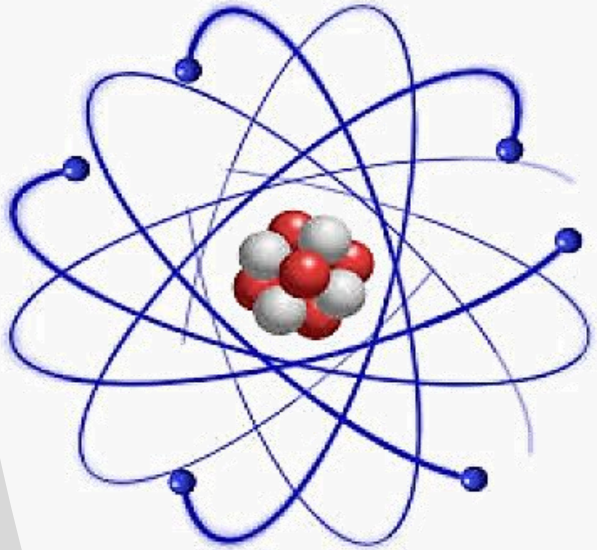
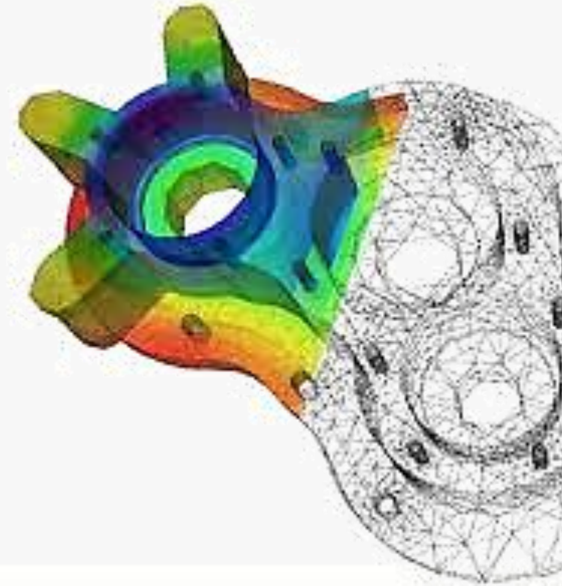


Partial Differential Equations

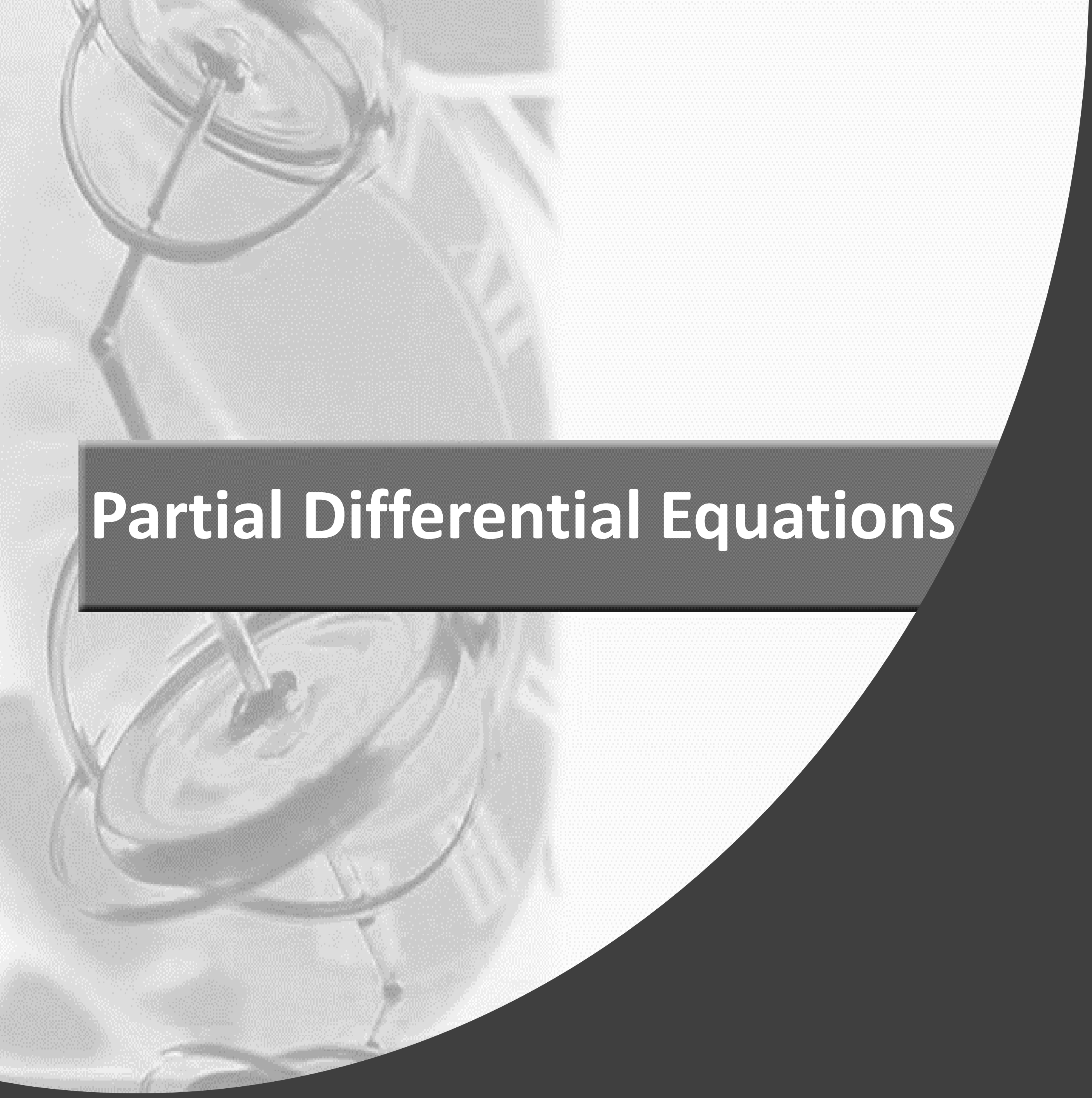






Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir



Partial Differential Equations

Book: Advanced Engineering Mathematics (9th Edition) by Ervin Kreyszig

- Chapter: 12
  - Sections: 12.6

Book: Applied Partial Differential Equations With Fourier series and boundary value problems by Richard Haberman

- Chapter: 2
  - Sections: 2.5

Steady-State Temperature Distribution

- For general boundary conditions, since the steady-state solution is independent oft, we must have $\partial u/\partial t \ = \ 0$.

- Substituting this in the PDE for one-dimensional heat equation, we see that the steady-state distribution satisfies the differential equation $u_{xx} = 0$, or $\frac{d^2y}{dx^2} = 0$ because u, the steady-state solution, is a function of x only.

- The general solution of this simple differential equation is $u(x) = Ax + B$, where A and B are constants that are determined using the boundary conditions.

Example: Steady-State Solution

Describe the steady-state solution in a bar of length L with one end kept at temperature T_1 and the other at temperature T_2 . Assume that the lateral surface is insulated and that there are no internal sources of heat.

Solution:

For the present case we have $u(0) = T_1$ and $u(L) = T_2$. Hence, from the fact that:

$$u(x) = Ax + B,$$

it follows that $B = T_1$ and $AL + T_1 = T_2$. Solving for A, we get:

$$A = \frac{T_2 - T_1}{L}$$

and so

$$u(x) = \left(\frac{T_2 - T_1}{L}\right) x + T_1.$$

Thus, the graph of the steady-state solution is a straight line that goes through the given boundary values T_1 at 0 and T_2 at L. We next illustrate how steady-state solutions can be used to solve certain nonhomogeneous boundary value problems.

Non-zero (Non-homogeneous) Boundary Conditions

Consider the heat boundary value problem:

$$
\text{(I)} \quad \begin{cases} \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}, 0 < x < L, \qquad t > 0, \\ u(0,t) = T_1 \text{ and } u(L,t) = T_2, \quad t > 0, \quad \text{(Boundary conditions)} \\ u(x,0) = f(x), \quad 0 < x < L. \quad \text{(Initial condition)} \end{cases}
$$

The problem is nonhomogeneous when T_1 and T_2 are not both zero. If we try to solve it in this case using the method of separation of variables, we will encounter difficulties because of the boundary conditions. As we now show, the problem can be reduced to the zero-ends case by subtracting and then adding the steady-state solution. We begin by finding the steady-state solution, $g(x)$, corresponding to the boundary conditions (2). From previous example, we have:

$$
g(x) = \left(\frac{T_2 - T_1}{L}\right) x + T_1.
$$

Non-zero (Non-homogeneous) Boundary Conditions

Now suppose that $u(x,t)$ is a solution of the BVP $(*)$ and let $v(x,t)$ be defined by the equation: $v(x,t) = u(x,t) - g(x)$. It is easy to verify that $v(x,t)$ is solution of the boundary value problem:

$$
\text{(II)} \quad \begin{cases} \dfrac{\partial v}{\partial t} = c^2 \dfrac{\partial^2 v}{\partial x^2}, \quad 0 < x < L, \qquad t > 0, \\ v(0,t) = 0 \;\; \text{and} \;\; v(L,t) = 0, \;\; t > 0, \;\; \text{(Boundary conditions)} \\ v(x,0) = f(x) - g(x), \;\; 0 < x < L. \;\; \text{(Initial condition)} \end{cases}
$$

By using method of separation of variables we get:

$$
v(x,t) = \sum_{k=1}^{\infty} E_k \sin\left(\frac{k\pi x}{L}\right) e^{-\lambda_k^2 t}, \qquad \text{(III)}
$$

where $\lambda_k^2 = \left(\frac{ck\pi}{L}\right)^2$ and $E_k = \frac{2}{L}\int_0^L [f(x) - g(x)] \sin\left(\frac{k\pi x}{L}\right) dx$ (IV). Thus, the solution of the BVP (I) is given as:

$$
u(x,t) = v(x,t) + g(x). \quad \text{(V)}
$$

Example: A non-homogeneous BVP

Determine solution of the BVP:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}\ , 0 < x < \pi, \qquad t > 0\ ,$$

$u(0,t) = 0$ and $u(\pi,t) = 100,\ \ t > 0\ ,\ \ $ (Boundary conditions)

$u(x,0) = 100\ ,\ \ 0 < x < \pi.\ \ $ (Initial condition)

Solution:

For the present case: $T_1 = 0$ and $T_2 = 100$, thus,

$$g(x) = \left(\frac{T_2 - T_1}{L}\right)x + T_1 = \left(\frac{100-0}{\pi}\right)x + 0 = \frac{100}{\pi}x$$

and

$$f(x) - g(x) = 100 - \frac{100}{\pi}x.$$

Solution: A non-homogeneous BVP

We now determine the coefficients in the series solution $v(x,t)$ in (III). Using (IV) we get:

$$E_k = \frac{2}{\pi}\int_0^{\pi}\left[100 - \frac{100}{\pi}x\right]\sin(kx)\,dx = \frac{200}{k\pi}.$$

and $\lambda_k^2 = k^2$. Using above in (III) we get:

$$v(x,t) = \frac{200}{\pi}\sum_{k=1}^{\infty}\frac{\sin(kx)}{k}e^{-k^2 t},$$

Thus, (V) takes the form:

$$u(x,t) = \frac{100}{\pi}x + \frac{200}{\pi}\sum_{k=1}^{\infty}\frac{\sin(kx)}{k}e^{-k^2 t}.$$

Two-Dimensional Heat Equation

We end discussion of heat equation by stating the solution of the two-dimensional heat equation with homogeneous boundary conditions. This two-dimensional heat problem models the distribution of temperature in a thin rectangular plate with insulated faces, edges kept at zero temperature, and with an initial temperature distribution $f(x, y)$. The solution of the problem is based on the separation of variables technique. The BVP corresponding to two-dimensional heat equation is given as:

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad 0 < x < a, \qquad 0 < y < b, \qquad t > 0,$$

$$\left.\begin{aligned} u(0, y, t) = u(a, y, t) = 0, &\quad 0 < y < b, \qquad t > 0, \\ u(x, 0, t) = u(x, b, t) = 0, &\quad 0 < x < a, \qquad t > 0, \end{aligned}\right\} \text{(Boundary conditions)}$$

$$u(x, y, 0) = f(x, y), 0 < x < a, \qquad 0 < y < b. \quad \text{(Initial condition)}$$

Two-Dimensional Heat Equation

The solution of this BVP for two-dimensional heat equation is based on the separation of variables technique and is given as:

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} A_{mk} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{k\pi y}{b}\right) e^{-\lambda_{mk}^2 t},$$

where,

$$\lambda_{mk}^2 = (c\pi)^2\left(\frac{m^2}{a^2} + \frac{k^2}{b^2}\right)$$

and

$$A_{mk} = \frac{4}{ab}\int_0^b\int_0^a f(x,y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{k\pi y}{b}\right) dxdy; \qquad m,k = 1,2,3,\ldots$$

Example: Two-Dimensional Heat Equation

Solve the heat problem in a square plate with $a = b = 1$, and $c = \frac{1}{\pi}$. Assume that the edges are kept at zero temperature and the initial temperature distribution is $u(x, y, 0) = 100°$.

Solution:

For the present case we have:

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} A_{mk}\sin(m\pi x)\sin(k\pi y)\, e^{-\lambda_{mk}^2 t},$$

where, $\lambda_{mk}^2 = m^2 + k^2$ and

$$A_{mk} = 400\int_0^1\int_0^1 \sin(m\pi x)\sin(k\pi y)\, dxdy = \frac{400}{\pi^2}\frac{[1-(-1)^m][1-(-1)^k]}{mk}; \qquad m, k = 1,2,3, \ldots$$

Since $A_{mk} = 0$ if either m or n is even, we get:

$$u(x,y,t) = \frac{1600}{\pi^2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin(2m-1)\pi x\sin(2k-1)\pi y}{(2m-1)(2k-1)}\, e^{-\lambda_{(2m-1)(2k-1)}^2 t}.$$

Laplace's Equation in Rectangular Coordinates

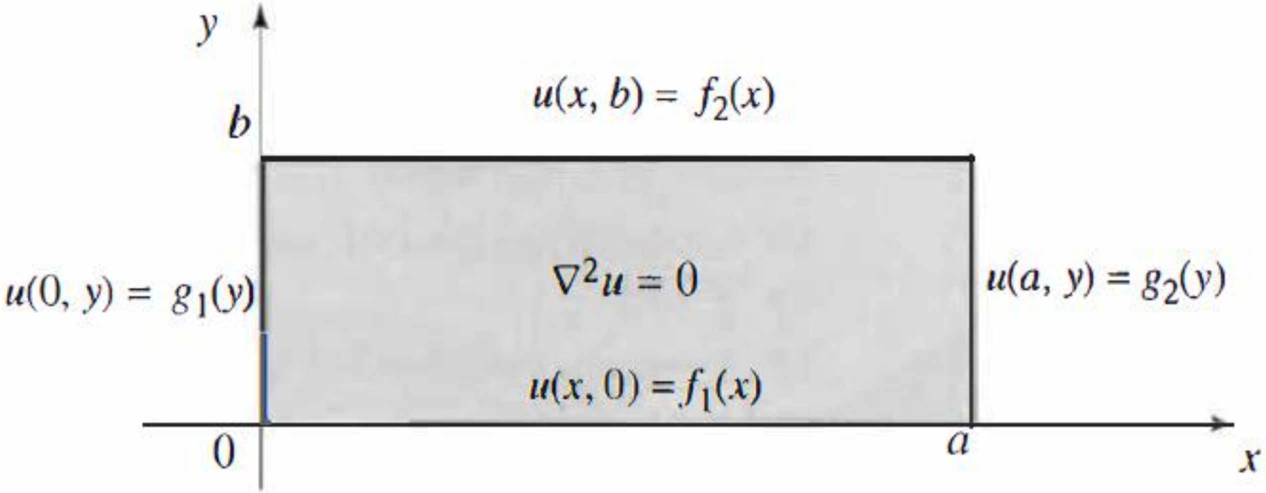
We have already seen that the steady-state temperature distributions associated with the one-dimensional heat equation:

$$u_t = c^2 u_{xx}\,, \qquad 0 < x < \pi, \qquad t > 0\,,$$

satisfy, since $u_{xx} = 0$ steady-state solutions are time independent. The equation $u_{xx} = 0$ is easily solved and yields only linear solutions $u(x) = c_1\, x + c_2$. For steady-state or time independent problems in two-dimensions over a rectangle $a \times b$, we consider the equation:

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a, \qquad 0 < y < b.$$

This equation is known as **Laplace's equation** in two variables and is obtained by setting the time derivative equal to zero in the heat equation in two-dimensions.



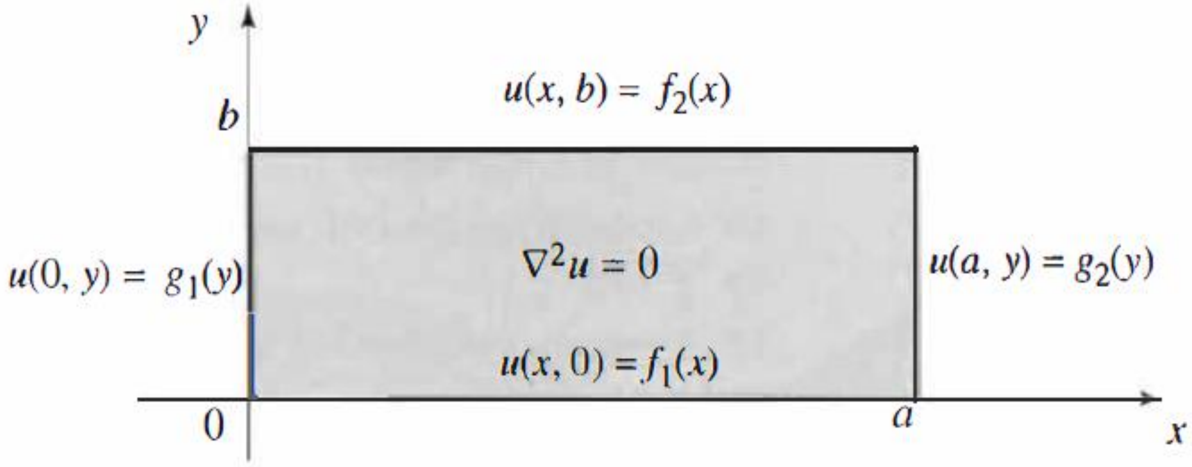
Laplace's Equation in Rectangular Coordinates

Laplace's equation has a wide variety of solutions. In a typical problem, the solution that we seek will be determined by the given boundary conditions. More specifically, we impose the boundary conditions:

$$u(x,0) = f_1(x), \qquad u(x,b) = f_2(x), \qquad 0 < x < a,$$

$$u(0,y) = g_1(y), \qquad u(a,y) = g_2(y), \qquad 0 < y < b.$$

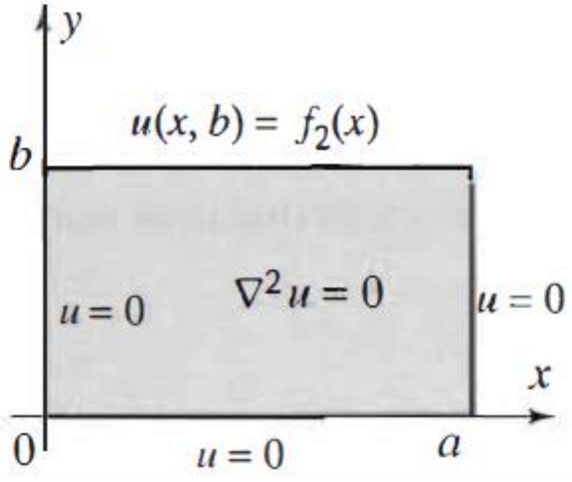
A problem consisting of Laplace's equation on a region in the plane together with specified boundary values is called a **Dirichlet problem.** Thus, the problem we described above is a Dirichlet problem on a rectangle. Rather than attacking this problem **in** its full generality, we will start by solving the special case when f_1, g_1 and g_2 are all zero.



Example: A Dirichlet problem on a rectangle

Solve the boundary value problem described in the accompanying figure using the method of separation of variables.

Solution:



We are required to determine solution of the BVP:

(1) $\nabla^2 u = u_{xx} + u_{yy} = 0, \qquad 0 < x < a, \qquad 0 < y < b.$

(2) $u(x,0) = 0, \qquad u(x,b) = f_2(x), \qquad 0 < x < a,$

(3) $u(0,y) = 0, \qquad u(a,y) = 0, \qquad 0 < y < b.$

We begin by looking for product solution $u(x,y) = X(x)Y(y)$. Substituting into (1) and using the separation method, we arrive at the equations:

$$X''(x) + \alpha\, X(x) = 0, \qquad (4)$$

$$Y''(y) - \alpha\, Y(y) = 0, \qquad (5)$$

where α is the separation constant, with the boundary conditions:

$$X(0) = 0, \qquad X(a) = 0, \qquad \text{and} \qquad Y(0) = 0. \qquad (6)$$

Solution: A Dirichlet problem on a rectangle

For the boundary value problem in X, we can check that the values $\alpha \leq 0$, lead to trivial solutions only. For $\alpha = \lambda^2 > 0$, we obtain the solutions:

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x \,.$$

Imposing the boundary conditions on X forces $c_1 = 0$, and

$$\lambda = \lambda_k = \frac{k\pi}{a} \quad or \quad \alpha = \left(\frac{k\pi}{a}\right)^2, \qquad k = 1,2,3,\ldots$$

and hence,

$$X_k(x) = c_k \sin\left(\frac{k\pi x}{a}\right), \qquad k = 1,2,3,\ldots \quad (7)$$

Now consider (5), which on using $\alpha = \lambda_k^2 = \left(\frac{k\pi}{a}\right)^2$ takes the form:

$$Y''(y) - \left(\frac{k\pi}{a}\right)^2 Y(y) = 0. \quad (5')$$

The general solution of $(5')$ can be written as:

$$Y(y) = Y_k(y) = A_k e^{k\pi y/a} + B_k e^{-k\pi y/a}. \quad (8)$$

Solution: A Dirichlet problem on a rectangle

Using the boundary condition $Y(0) = 0$ in (7) we get $B_n = -A_n$. This gives:

$$Y_k(y) = A_k\left(e^{k\pi y/a} - e^{-k\pi y/a}\right) = 2A_k \sinh\frac{k\pi y}{a}. \quad (9)$$

From (7) and (9) we obtain:

$$u_k(x,y) = X_k(x)Y_k(y) = E_k \sin\left(\frac{k\pi x}{a}\right)\sinh\left(\frac{k\pi y}{a}\right); \quad \text{where} \quad E_k = 2c_kA_k, \qquad (10)$$

as the **eigenfunctions** of our problem corresponding to eigenvalues $\lambda_k = \frac{k\pi}{a}$. Superposing these solutions (10), we get the general form of the solution as:

$$u(x,y) = \sum_{k=1}^{\infty} u_k(x,y) = \sum_{k=1}^{\infty} E_k \sin\left(\frac{k\pi x}{a}\right)\sinh\left(\frac{k\pi y}{a}\right), \qquad (11)$$

where the constants E_k are to be chosen, if possible, so that the non-homogeneous boundary condition is satisfied. Thus, by using $u(x,b) = f_2(x)$ we get:

$$f_2(x) = \sum_{k=1}^{\infty} E_k \sin\left(\frac{k\pi x}{a}\right)\sinh\left(\frac{k\pi b}{a}\right). \qquad (12)$$

Solution: A Dirichlet problem on a rectangle

To meet this last requirement, we choose the coefficients $E_k \sinh \frac{k\pi b}{a}$ to be the Fourier sine coefficients of $f_2(x)$ on the interval $0 < x < a$. Thus, it follows that:

$$E_k = \frac{2}{a \sinh\left(\frac{k\pi b}{a}\right)} \int_0^a f_2(x) \sin\left(\frac{k\pi x}{a}\right) dx, \qquad k = 1,2,3,\dots. \qquad (13)$$

The solution of the Dirichlet problem described in the figure is therefore given by (11) with coefficients determine by (13).