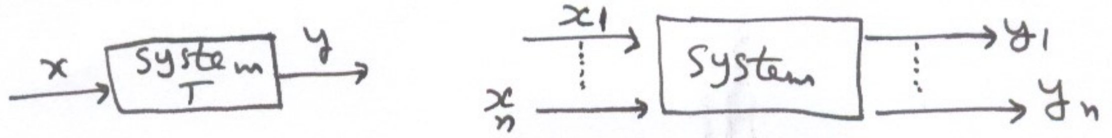
System representation: A system is a mathematical model of a physical process that relates the input (or excitation) signal to the output (response) signal.

The system is viewed as a transformation $y = Tx$



If the input and output signals x and y are continuous-time signals, then the system is called a continuous time system. If the operator T satisfies the following two conditions, then T is called a linear operator and the system represented by a linear operator T is called a linear system.

Given that $T(x_1) = y_1$ and $T(x_2) = y_2$, then $T(x_1 + x_2) = y_1 + y_2$

$T(\alpha x) = \alpha y$, for any signals x_1 & x_2, x and any scalar α.

Due to the second condition, a zero input yields a zero output.

A system is called time-invariant if a time shift (delay or advance) in the input signal causes the same time shift in the output signal. Thus, for CT system, the system is time-invariant if $T\{f(t-\tau)\} = y(t-\tau)$.

If the system is linear and also time-invariant, then it is called a LTI system.

The frequency response of Continuous LTI systems:

The output $y(t)$ of a continuous-time LTI system equals the convolution of the input $x(t)$ with the impulse response $h(t)$;

ie, $y(t) = x(t) * h(t)$

Applying the convolution property, $Y(\omega) = X(\omega) H(\omega)$

$H(\omega) = Y(\omega)/X(\omega)$, the function $H(\omega)$ is called the frequency response of the system, giving magnitude & phase response of the system.

Differentiation property:

$$\mathcal{F}[f'(t)] = \int_{-\infty}^{+\infty} \underset{\text{II}}{f'(t)} \underset{\text{I}}{e^{-j\omega t}} \, dt$$

$$= f(t) e^{-j\omega t} \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(t)(-j\omega) e^{-j\omega t} \, dt$$

$\mathcal{F}[f'(t)] = j\omega F(j\omega)$, In general, we have $\mathcal{F}[f^{(n)}(t)] = (j\omega)^n F(j\omega)$.

EX:- Consider a CT LTI system described by

$$\frac{dy}{dt} + 2y = x(t) \quad \text{——— (i)}$$

Using the FT, find the output $y(t)$ when the input signal is $x(t) = e^{-t} u(t)$, $u(t)$ is the unit-step function.

Taking the Fourier transform of (i), we have

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega)$$

$$Y(\omega)[2 + j\omega] = \frac{1}{1+j\omega}$$

$$Y(\omega) = \frac{1}{(1+j\omega)(2+j\omega)} = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

Therefore, output is $y(t) = (e^{-t} - e^{-2t}) u(t)$.

EX:- A relaxed, causal LTI system is described by the following ODE, where $x(t)$ is the input and $y(t)$ is the output:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = \frac{dx}{dt} \quad \text{— (i)},$$

using FT techniques, determine the output $y(t)$ when the input $x(t) = u(t)$, the unit step signal.

Sol:

$$(j\omega)^2 Y(\omega) + 3j\omega Y(\omega) + 2Y(\omega) = j\omega X(\omega)$$

$$Y(\omega) = \left[\frac{j\omega}{(j\omega)^2 + 3j\omega + 2}\right] X(\omega), \quad X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$Y(\omega) = \left[\frac{j\omega}{(j\omega+2)(j\omega+1)}\right]\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$$

$$= \frac{j\omega}{(j\omega+2)(j\omega+1)}\left(\frac{1}{j\omega}\right) = \frac{1}{j\omega+1} - \frac{1}{j\omega+2}$$

output is $y(t) = [e^{-t} - e^{-2t}] u(t)$.

Q. Let $f(t) = \cos(w_0 t - 3)$, $g(t) = H(t)$, $H(t)$ is Heaviside unit-step function, Calculate $\mathcal{F}[f(t)g(t)]$ using Convolution theorem in frequency

$$\mathcal{F}[f(t)g(t)] = \frac{1}{2\pi} F(jw) * G(jw); \quad F(jw) = \mathcal{F}[f(t)], \quad G(jw) = \mathcal{F}[g(t)]$$

$$f(t) = \cos(w_0 t - 3), \quad g(t) = H(t).$$

$$F(jw) = \pi e^{-\frac{3jw}{w_0}} \left(\delta(w + w_0) + \delta(w - w_0) \right).$$

$$G(jw) = \frac{1}{jw} + \pi \delta(w).$$

$$\mathcal{F}[f(t)g(t)] = \frac{1}{2\pi} F(jw) * G(jw).$$

$$\frac{1}{2\pi} F(jw) * G(jw) = \frac{1}{2\pi} \pi e^{-\frac{3jw}{w_0}} \left[\delta(w+w_0) + \delta(w-w_0) \right] * \left(\frac{1}{jw} + \pi \delta(w) \right)$$

$$= 0.5\, e^{-\frac{3jw}{w_0}} \delta(w+w_0) * \frac{1}{jw} + 0.5\pi e^{-\frac{3jw}{w_0}} \delta(w+w_0) * \delta(w)$$

$$+ 0.5\, e^{-\frac{3jw}{w_0}} \delta(w-w_0) * \frac{1}{jw} + 0.5\pi e^{-\frac{3jw}{w_0}} \delta(w-w_0) * \delta(w)$$

$$= 0.5\, e^{-\frac{3jw_0}{w_0}} \delta(w+w_0) * \frac{1}{jw} + 0.5\pi e^{-\frac{3jw_0}{w_0}} \delta(w+w_0) * \delta(w)$$

$$+ 0.5\, e^{-\frac{3jw_0}{w_0}} \delta(w-w_0) * \frac{1}{jw} + 0.5\pi e^{-\frac{3jw_0}{w_0}} \delta(w-w_0) * \delta(w)$$

$$= 0.5\, e^{3j} \delta(w+w_0) * \frac{1}{jw} + 0.5\pi e^{3j} \delta(w+w_0)$$

$$+ 0.5\, e^{-3j} \delta(w-w_0) * \frac{1}{jw} + 0.5\pi e^{-3j} \delta(w-w_0).$$

$$= \frac{0.5\, e^{3j}\, j(w-w_0) + 0.5\, e^{-3j}\, j(w+w_0)}{j(w+w_0)(w-w_0)}$$

$$+ 0.5\pi e^{3j} \delta(w+w_0) + 0.5\pi e^{-3j} \delta(w-w_0)$$

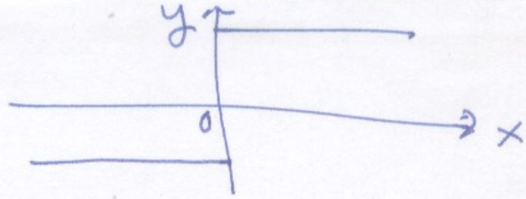
$$= \frac{jw\cos(3) + w_0 \sin(3)}{w_0^2 - w^2} + 0.5\, e^{3j} \delta(w+w_0) + 0.5\, e^{-3j} \delta(w-w_0)$$

Thus, the final answer is given by

$$\mathcal{F}[\cos(w_0 t - 3) U(t)] = 0.5\, e^{3j} \delta(w+w_0) + 0.5\, e^{-3j} \delta(w-w_0) + \frac{jw\cos(3) + w_0 \sin(3)}{w_0^2 - w^2}$$

Result: $\delta(t-t_1) * \delta(t+t_2) = \delta(t - t_1 + t_2)$.

$$\mathrm{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$



Fourier transform of unit-step function

Consider, $e^{-\epsilon |t|}\, \mathrm{sign}(t)$.

$$\mathcal{F}(\mathrm{sgn}(t)) = \lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{0} (-1)\, e^{-\epsilon(-t)} \cdot e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-\epsilon(t)}\, e^{-j\omega t}\, dt \right]$$

$$= \lim_{\epsilon \to 0} \left\{ -\int_{-\infty}^{0} e^{(\epsilon - j\omega)t}\, dt + \int_{0}^{\infty} e^{-(\epsilon + j\omega)t}\, dt \right]$$

$$= \lim_{\epsilon \to 0} \left\{ \left. \frac{e^{-(\epsilon + j\omega)t}}{-(\epsilon + j\omega)} \right|_{0}^{\infty} - \left. \frac{e^{(\epsilon - j\omega)t}}{(\epsilon - j\omega)} \right|_{-\infty}^{0} \right.$$

$$= \lim_{\epsilon \to 0} \left\{ \left(0 - \frac{1}{-(\epsilon + j\omega)}\right) - \left(\frac{1}{\epsilon - j\omega} - 0\right) \right]$$

$$= \lim_{\epsilon \to 0} \left\{ \frac{1}{\epsilon + j\omega} - \frac{1}{\epsilon + j\omega} \right\}$$

If $\omega \neq 0$, $\displaystyle \lim_{\epsilon \to 0} \frac{\cancel{\epsilon} - j\omega - \cancel{\epsilon} - j\omega}{(\epsilon + j\omega)(\epsilon - j\omega)} = \frac{-2j\omega}{\epsilon^2 + \omega^2} = \frac{-2j\omega}{\omega^2} = \frac{-2j}{\omega} = \frac{2}{j\omega}$

If $\omega = 0$, $\displaystyle \lim_{\epsilon \to 0} \left(\cancel{\frac{1}{\epsilon}} - \cancel{\frac{1}{\epsilon}} \right) = 0$.

$$\mathcal{F}[\mathrm{sgn}(t)] = \begin{cases} \dfrac{2}{j\omega}, & \omega \neq 0 \\ 0, & \omega = 0. \end{cases}$$

$$u(t) = \frac{1}{2}\left[1 + \mathrm{sgn}(t)\right]$$

$$\mathcal{F}[u(t)] = \frac{1}{2}\mathcal{F}[1] + \frac{1}{2}\mathcal{F}[\mathrm{sgn}(t)]$$

$$= \frac{1}{2}\left(2\pi\delta(\omega)\right) + \frac{1}{2}\left(\frac{2}{j\omega}\right)$$

$$\mathcal{F}[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}.$$

NUST School of Electrical Engineering & Computer Science

MATH 232 - Complex Variables & transforms - Problem Sheet (Fourier Analysis)

Q-1. Find and sketch $x_1(t) * x_2(t)$ for the following cases:

(a) $x_1(t) = rect\left(\frac{t}{2} - 0.5\right)$, $x_2(t) = rect(t-0.5) - rect(t-1.5)$

(b) $x_1(t) = -t[u(t+1) - u(t)]$, $x_2(t) = u(t)$, (c) $x_1(t) = e^{-t}u(t)$, $x_2(t) = rect\left(\frac{t}{3} - 0.5\right)$

(d) $x_1(t) = \sin t\,[u(t) - u(t-2\pi)]$, $x_2(t) = rect\left(\frac{t}{2\pi} - 0.5\right)$.

Q-2. Find and sketch the magnitudes and phases of the Fourier transform of the signals below.

(a) $x(t) = rect(t-2)$ (b) $x(t) = 5\,Sinc(2(t-1))$ (c) $x(t) = 5\,\delta(3t-12)$.

(d) $x(t) = 10\cos(400\pi t)$ (e) $x(t) = 2\,rec(t)\cos(20\pi t)$.

Q-3

Show that $\delta(t-t_1) * \delta(t-t_2) = \delta(t-(t_1+t_2))$.

Q-4 Show that $\mathcal{F}[u(t)] = \pi\delta(w) + \frac{1}{jw}$, where $u(t)$ is the unit-step function.

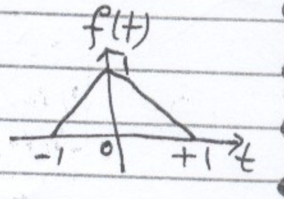
Q-5

Let $f(t) = \cos(2t-3)$, $g(t) = u(t)$, $u(t)$ is the unit-step function.

Calculate $\mathcal{F}[f(t)g(t)]$ using convolution theorem in the frequency domain $\mathcal{F}[f(t)g(t)] = \frac{1}{2\pi} F(jw) * G(jw)$, where

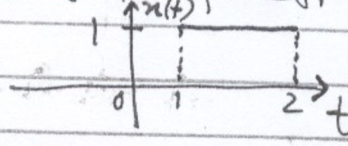
$F(jw) = \mathcal{F}[f(t)]$, $G(jw) = \mathcal{F}[g(t)]$.

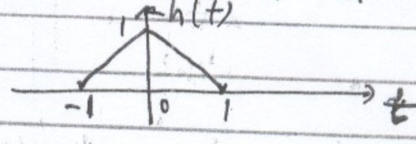
Q-6 Calculate the Fourier transform of the triangular pulse shown in figure.



Q-7

Rectangular & triangular functions are shown in figure:





Calculate $x(t) * h(t)$ and show that $x(t)*h(t) = h(t)*x(t)$.

Q-8 A Continuous-time signal $f(t)$ has the following spectrum:

a) $F(jw) = \dfrac{e^{-jwt_0}}{j(w+w_0)} + \pi\,\delta(w+w_0)\,e^{jw_0t_0}$

b) $F(jw) = \dfrac{e^{jwt_0}}{j(w-w_0)} + \pi\,\delta(w-w_0)\,e^{jw_0t_0}$

where w_0 and t_0 are constant positive real values.

Evaluate $f(t)$ [Inverse Fourier transform].