EX: Evaluate $\int_C (2z)\,dz$, where C is the straight line segment from $1+i$ to $2+3i$.

$C:\ z(t) = (1+i) + t[(2+3i)-(1+i)],\ 0 \le t \le 1.$
$= (1+i) + t(1+2i)$

$\int_C f(z)\,dz = \int_0^1 2[(1+i)+t(1+2i)](1+2i)\,dt$

$(1+i)(1+2i) = 1+2i+i-2 = -1+3i$

$= 2(1+2i)\int_0^1 [(1+i)+t(1+2i)]\,dt$

$(1+2i)^2 = 1-4+4i = -3+4i$

$= 2(1+2i)\left[(1+i)t + (1+2i)\frac{t^2}{2}\right]_0^1$

$= 2(1+2i)\left[(1+i)+\frac{1}{2}(1+2i)\right] = 2(1+2i)(1+i) + (1+2i)^2$

$= 2(-1+3i)+(-3+4i) = -2+6i-3+4i = -5+10i$

Now, $\int_C 2z\,dz = \left[z^2\right]_{1+i}^{2+3i} = (2+3i)^2 - (1+i)^2$

$= (4-9+12i) - (\cancel{1}-\cancel{1}+2i) = -5+12i-2i = -5+10i$

Note: $\int_C (2\bar{z})\,dz$, where we take C as above.

$\int_C f(z)\,dz = \int_0^1 2[(1-i)+t(1-2i)](1+2i)\,dt$

$= 2(1+2i)\left[(1-i)+\frac{1}{2}(1-2i)\right] = 2(1+2i)(1-i) + (1-2i)(1+2i)$

$= 2(3+i)+5 = 11+i$

$(1-i)(1+2i)$
$1-i+2i+2$
$= 3+i$

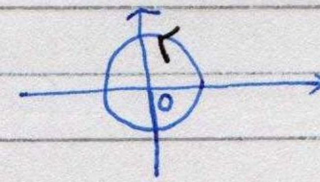
on the other hand,

$\int_C (2\bar{z})\,dz = \left[\bar{z}^2\right]_{1+i}^{2+3i} = (2-3i)^2 - (1-i)^2$

$= (4+9-12i) - (\cancel{1}-\cancel{1}-2i) = 13-12i+2i = 13-10i.$

EX:- $\int_C 2z\,dz$, where C is a unit circle traversed in Counterclockwise direction.

$C: \; z(t) = e^{it}, \; 0 \le t \le 2\pi$



$$\int_C 2z\,dz = 2\int_0^{2\pi} e^{it} \cdot i e^{it}\,dt$$

$$= 2i\int_0^{2\pi} e^{2it}\,dt = 2i\left[\frac{e^{2it}}{2i}\right]_0^{2\pi} = \frac{2i}{2i}\left[e^{4\pi i} - 1\right] = 0$$

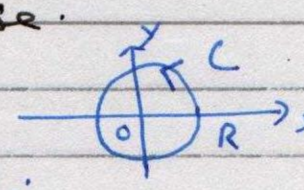
on the other hand: $\int_C 2\bar{z}\,dz$, C as above.

$$\int_0^{2\pi} 2e^{-it} \cdot ie^{it}\,dt = 2i\int_0^{2\pi} dt = 2i(2\pi) = 4\pi i \neq 0.$$

In the first case, $\int_C 2z\,dz = \left[z^2\right]_A^B = 0$, As A & B are same being closed curve.

EX:- $\int_C \frac{1}{z}\,dz$, where C is a circle centered at 0 and oriented Counterclockwise.

Parametrize C: $z = Re^{i\theta}, \; 0 \le \theta \le 2\pi$



$$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{Re^{i\theta}} Re^{i\theta}(i\,d\theta) = \int_0^{2\pi} i\,d\theta = 2\pi i$$

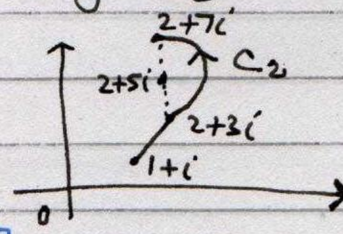
on the other hand,

$$\int_C \frac{1}{z^2}\,dz = \int_{\theta=0}^{\theta=2\pi} \frac{1}{R^2 e^{2i\theta}} Re^{i\theta}(i\,d\theta) = \frac{i}{R}\int_0^{2\pi} e^{-i\theta}\,d\theta = 0.$$

In general, $\int_C \frac{1}{z^n}\,dz = 0$, for all integers $n \neq 1$

When $n = 1$, we have $\int_C \frac{1}{z}\,dz = 2\pi i$

EX:- Evaluate $\int_C (2z+1)\,dz$, where C is C_1 followed by C_2.



C_2: hemicircle centered at $2+5i$.

C_1: Line segment between $1+i$ & $2+3i$.

Parametrize C_1: $z(t)=(1+i)+t\left[(2+3i)-(1+i)\right]$, $0\le t\le 1$.

$$z(t)=(1+i)+t(1+2i),\quad 0\le t\le 1.$$

$$\int_{C_1}(2z+1)\,dz=\int_{t=0}^{t=1}\left\{2\left((1+i)+t(1+2i)\right)+1\right\}(1+2i)\,dt.$$

$$=\quad -4+12i.$$

Parametrize C_2: $z(t)=(2+5i)+2e^{i\theta}$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$; $10i+20i^2=-20+10i$

$$\int_{C_2}(2z+1)\,dz=\int_{\theta=-\pi/2}^{\theta=\pi/2}\left\{2\left(2+5i+2e^{i\theta}\right)+1\right\}2ie^{i\theta}\,d\theta$$

$$(4+10i+4e^{i\theta})\,2ie^{i\theta}+2ie^{i\theta}$$
$$=(4+10i)2ie^{i\theta}+8ie^{2i\theta}+2ie^{i\theta}$$
$$=2ie^{i\theta}(1+4+10i)+8ie^{2i\theta}$$

$$=\int_{-\pi/2}^{\pi/2}\left[(5+10i)\,2i\,e^{i\theta}+8i\,e^{2i\theta}\right]d\theta$$

$$=-20+10i\left.\frac{e^{i\theta}}{i}\right|_{-\pi/2}^{\pi/2}+8i\left.\frac{e^{2i\theta}}{2i}\right|_{-\pi/2}^{\pi/2}$$

$$\frac{i}{i}\times\frac{1}{i}(-20+10i)=i(20-10i)=10-20i$$

$$=10-20i\left(i-(-i)\right)+4\left(-1-(-1)\right)$$

$$=\quad -40+20i$$

$$e^{i\pi/2}=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$$
$$e^{i\pi}=\cos\pi+i\sin\pi$$

$$\int_C f(z)\,dz=(-4+12i)+(-40+20i)=-44+32i$$

$$2i(10-20i)=20i+40$$

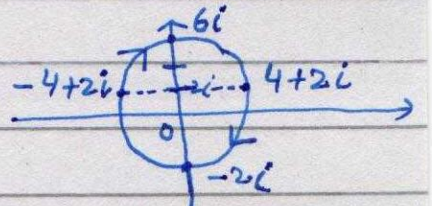
OR $F'(z)=2z+1$, $F(z)=z^2+z$.

$$\int_C(2z+1)\,dz=F(\text{End point})-F(\text{Start point})$$
$$=F(2+7i)-F(1+i)$$
$$=\left[(2+7i)^2+(2+7i)\right]-\left[(1+i)^2+(1+i)\right]=-44+32i$$

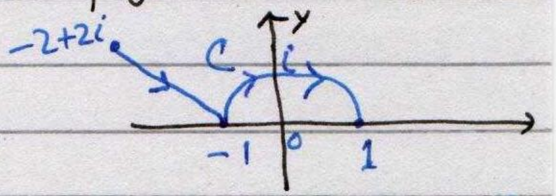
Exercises:-

(a) Parametrize the circle $|z-2i|=4$ traversed once in the clockwise direction starting from the point $z=4+2i$.

$$C:\ z(t)=4e^{-it}+2i,\ 0\le t\le 2\pi.$$



(b). Parametrize the contour indicated in figure. Also give a parametrization for the opposite

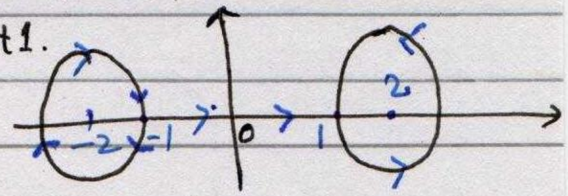


$$C:\quad z(t)=\begin{cases} -2+2i+(1-2i)t, & 0\le t\le 1 \\ e^{-\pi i t}, & 1\le t\le 2.\end{cases}$$

$[-1-(-2+2i)]$
$= -1+2-2i$
$= 1-2i$

$-C$ is parametrized by $z(-t)$, $-2\le t\le 0$.

(c). Parametrize the bar bell-shaped contour shown in figure; it has initial point -1 and terminal point 1.



$$z(t)=\begin{cases} -2+e^{-i\pi t}, & -2\le t\le 0 \\ -1+2t, & 0\le t\le 1 \\ 2+e^{\pi i t}, & 1\le t\le 3.\end{cases}$$

(d) Interpreting t as a time and parametrization $z(t)$, $a\le t\le b$ as the position function of a moving particle, give the physical meaning of the following quantities.

(i) $z'(t)$ is the velocity of the moving particle at time t.

(ii) $|z'(t)|$ is the instantaneous speed of the particle at time t.

(iii) $|z'(t)dt|$ is an increment of the distance travelled by the particle during the time interval dt.

(iv) $\int_a^b |z'(t)|\,dt$ is the distance traveled by the particle as it moved along the contour from $z(a)$ to $z(b)$.