Assignment # 1

Q1: Given that

$$y^2 - x^2 - 6y - z + 9 = 0$$

$$\Rightarrow y^2 - 6y + 9 - x^2 - z = 0$$

$$\Rightarrow (y)^2 - 2(3)(y) + (3)^2 - x^2 - z = 0$$

$$\Rightarrow (y-3)^2 - x^2 - z = 0 \longrightarrow (*)$$

$\Rightarrow$ $f(x, y, z) = 0$; with one linear and two quadratic variables having opposite signature.

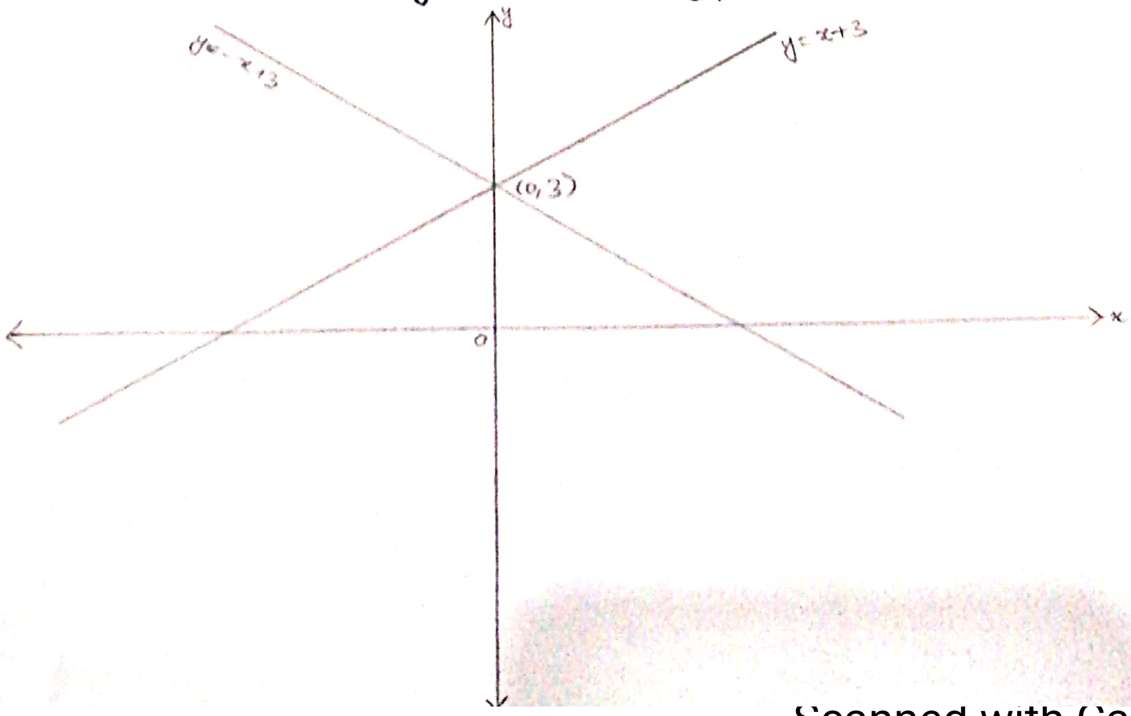
Equation $(*)$ represents a hyperbolic paraboloid with axis along z-axis and vertex at $(0, 3, 0)$.

(1) xy - trace : $z = 0$

Equation of trace:

$$(y-3)^2 - x^2 = 0 \; [\text{using } z = 0 \text{ in } (*)]$$

$$\Rightarrow y = \pm x + 3 \; [\text{pair of intersecting lines}]$$



(2) Trace on plane parallel to xy-plane: $z = c > 0$

Let $c = 1$

Equation of trace:

$$(y-3)^2 - x^2 = 1 \quad [\text{using } z = c = 1 \text{ in } *]$$

This represents a hyperbola with major axis along y-axis. (Fig (a))

Note: For other values of $z = c > 1$ we will get hyperbolas.

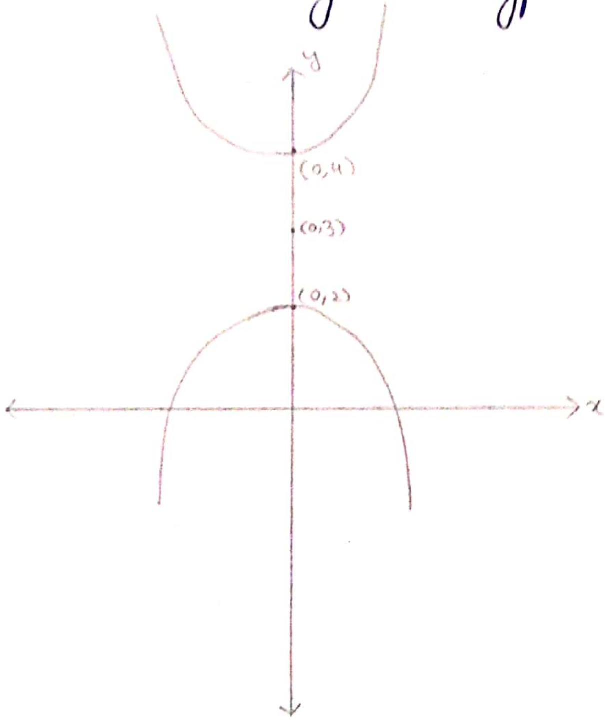


Fig (a)

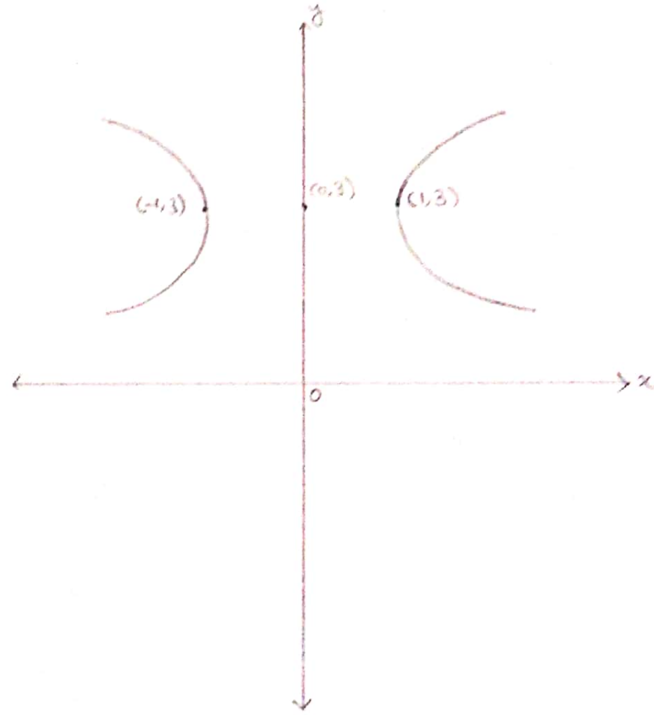


Fig (b)

(3) Trace on plane parallel to xy-plane: $z = c < 0$

Let $c = -1$

Equation of trace:

$$(y-3)^2 - x^2 = -1$$

or $$x^2 - (y-3)^2 = 1$$

This represents a hyperbola with major axis along x-axis. (Fig (b))

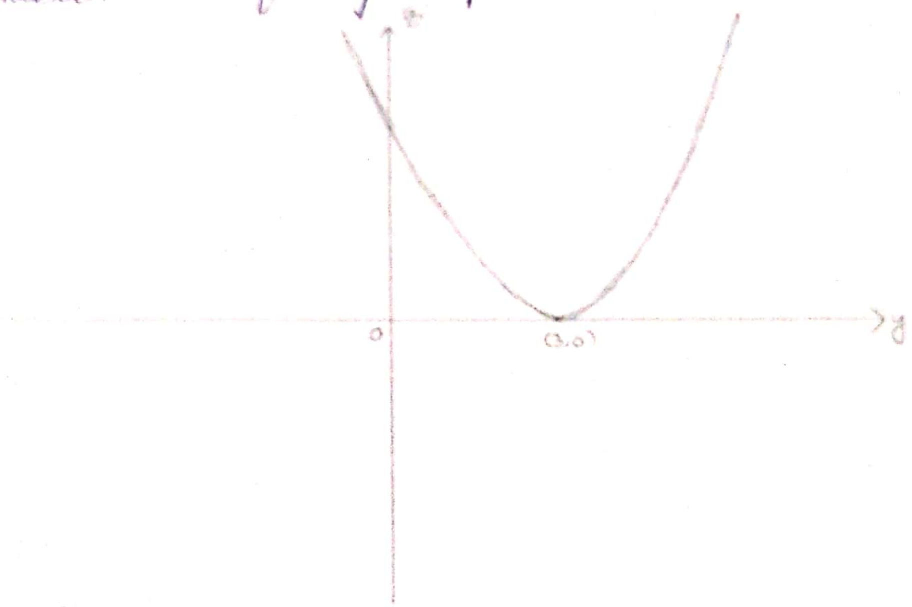
Note: For other values of $z = c < -1$ we will get hyperbolas.

(4) yz- trace : $x=0$

Equation of trace:

$$(y-3)^2 = z. \quad [x=0 \text{ in eq. } \circledast]$$

This represents a parabola with vertex at $(3,0)$ in yz-plane. In 3D the vertex is $(0,3,0)$ and parabola is facing upwards.



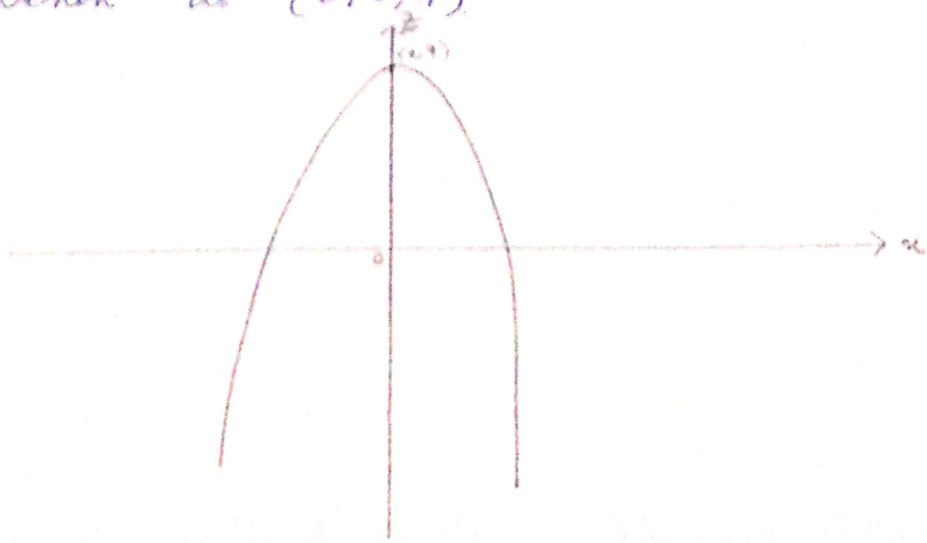
(5) xz- trace : $y=0$

Equation of trace:

$$(0-3)^2 - x^2 = z \quad [\text{use } y=0 \text{ in eq } \circledast]$$

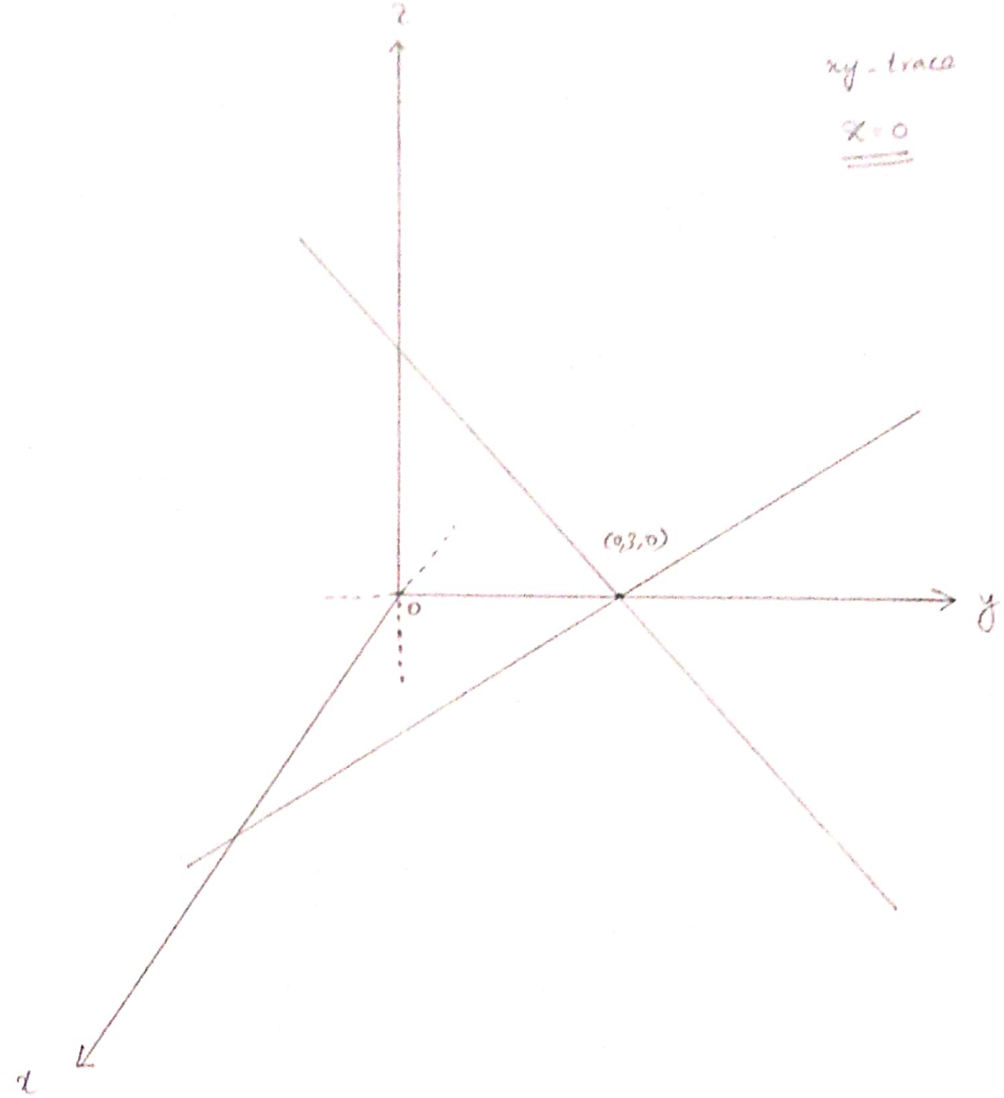
$$\Rightarrow \quad 9 - x^2 = z \Rightarrow x^2 = 9 - z.$$

This represents a parabola with vertex $(0,9)$ in xz-plane facing downwards. In 3D the vertex is $(0,0,9)$

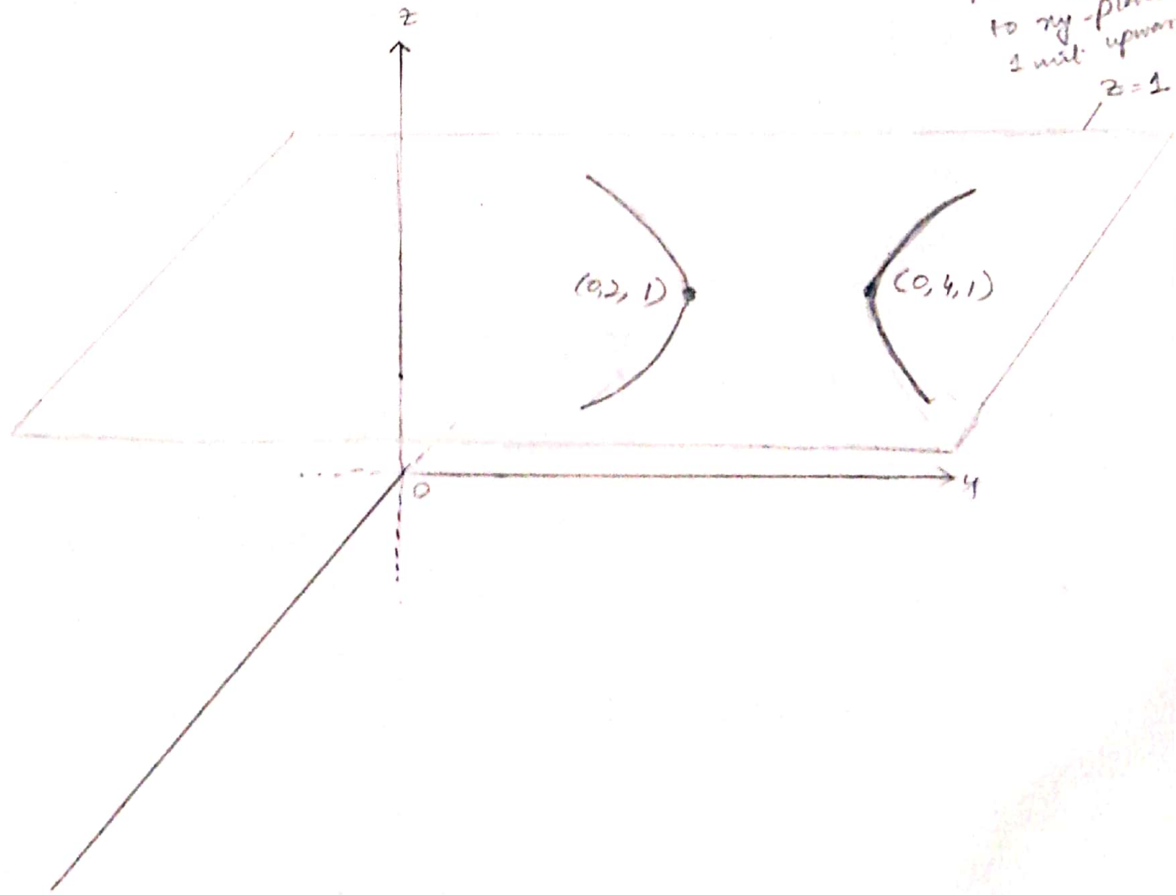


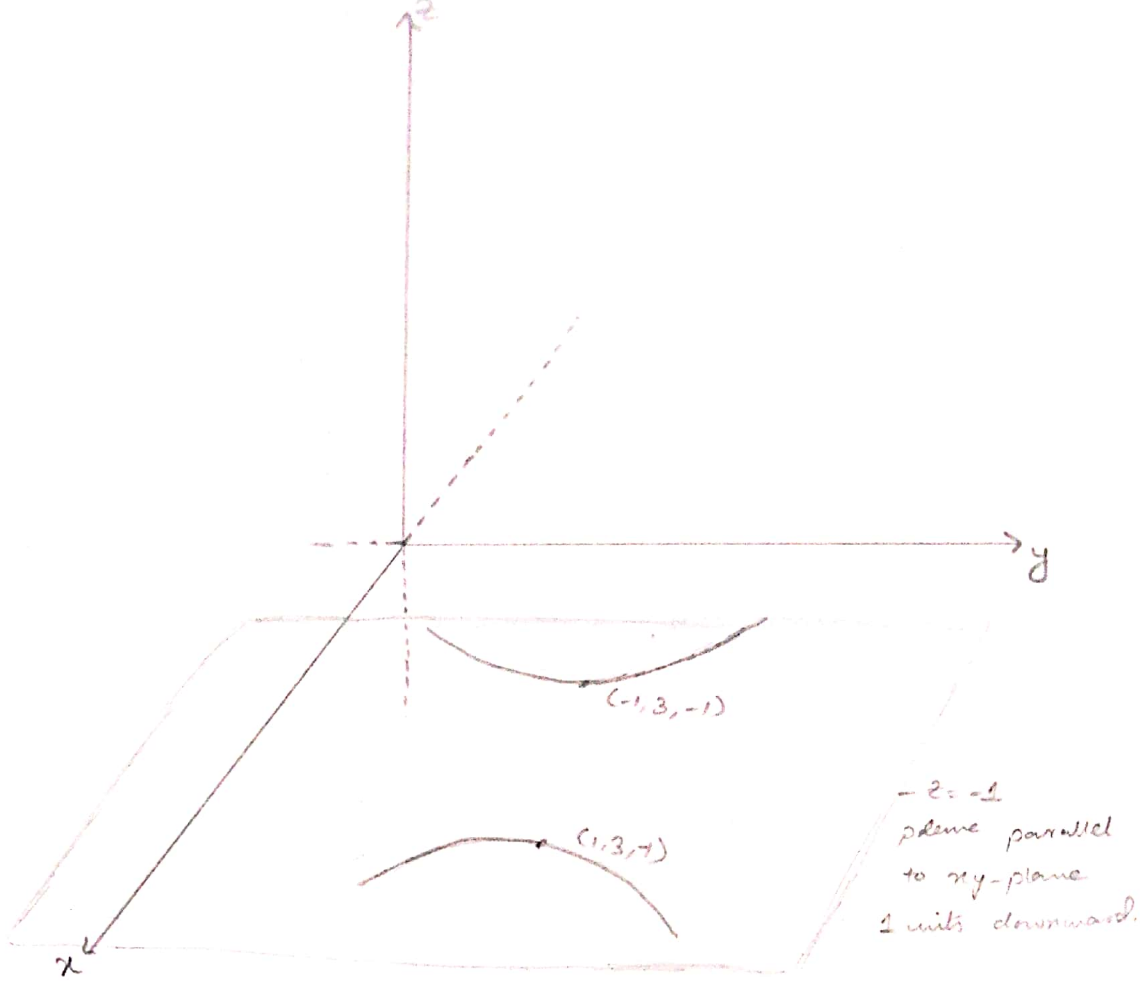
xy - trace

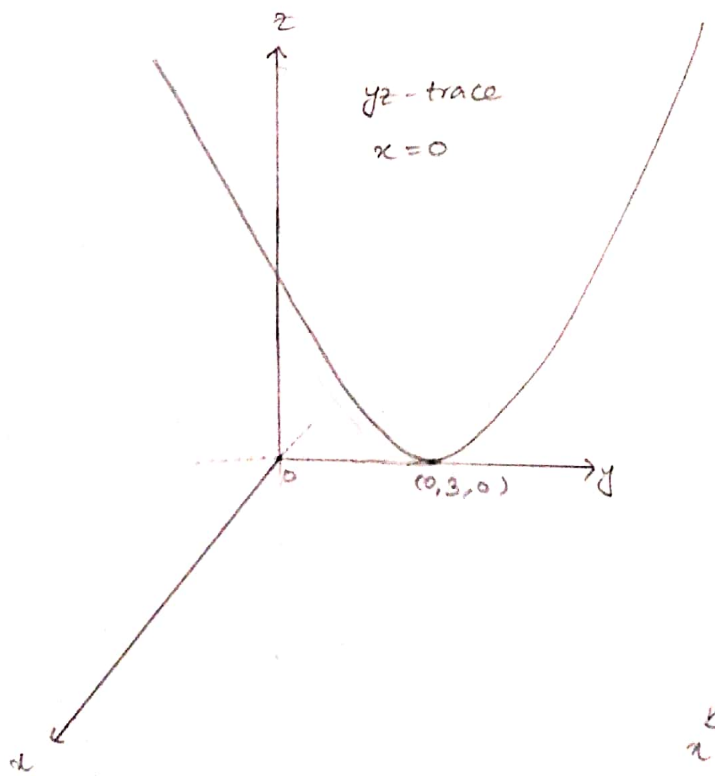
$x = 0$

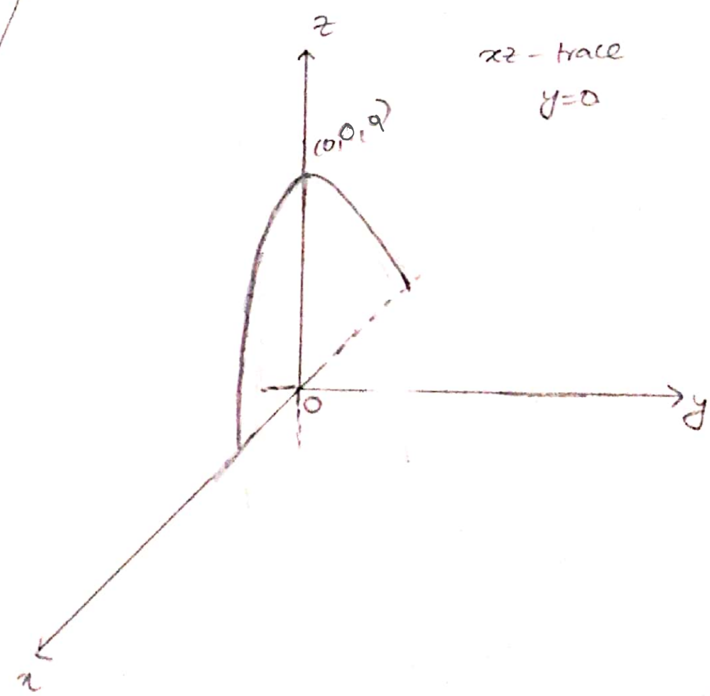


plane parallel to xy-plane 1 unit upward




z
y
x
(-1,3,-1)
(1,3,-1)
z = -1
plane parallel
to xy-plane
1 units downward.


z
yz - trace
x = 0
0
(0,3,0)
y
x


z
xz - trace
y=0
(0,0,9)
0
y
x

Q2. (a)

The line through A B is:

$x = 1+t, \; y = -t, \; z = -1+5t \quad ; t \in \mathbb{R}$

[Note: here we have used $(x_0, y_0, z_0) = (1, 0, -1)$ and $\langle a, b, c \rangle = \langle 1, -1, 5 \rangle$]

The line through C + D is parallel to the line through A + B and is given as:

$L_1: \; x = 1+t, \; y = 2-t, \; z = 3+5t \quad ; \; t \in \mathbb{R}. \rightarrow (1)$

The line through B + C is:

$x = 1, \; y = 2+2s, \; z = 3+4s \quad ; \; s \in \mathbb{R}$

[Note: here we have used $(x_0, y_0, z_0) = (1, 2, 3)$ and $\langle a, b, c \rangle = \langle 0, 2, 4 \rangle$]

The line through A + D is parallel to the line through B + C and is given as:

$L_2: \; x = 2, \; y = -1+2s, \; z = 4+4s \quad ; \; s \in \mathbb{R}. \rightarrow (2)$

The lines L_1 and L_2 intersect each other at point D (2, 1, 8) where $t = 1$ and $s = 1$.

[Note: For finding point of intersection D of L_1 + L_2 we equate values of x, y and z in (1) + (2) and determine s and t which are unknown constants.]

Thus, coordinates of D are: (2, 1, 8).

(b) In order to determine the cosine of the interior angle at B we need vectors $\vec{BA}$ and $\vec{BC}$. Let "θ" be interior angle at B, then

$$\cos\theta = \frac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}||\vec{BC}|} = \frac{\langle 1,-1,5\rangle \cdot \langle 0,2,4\rangle}{\sqrt{27}\ \sqrt{20}}$$

$$= \frac{0-2+20}{\sqrt{9\times3\times5\times4}} = \frac{18}{6\sqrt{15}} = \frac{3}{\sqrt{15}}.$$

Thus, $\boxed{\cos\theta = \frac{3}{\sqrt{15}}.}$

(c) Area of parallelogram $= |\vec{BA}\times\vec{BC}|$

$$\vec{BA}\times\vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 5 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= \hat{i}\,[-4-10] - \hat{j}\,[4-0] + \hat{k}\,[2-0]$$

$$= \langle -14, -4, 2\rangle$$

$$|\vec{BA}\times\vec{BC}| = \sqrt{(-14)^2+(-4)^2+(2)^2} = \sqrt{216} = 6\sqrt{6}$$

Thus, area of parallelogram $= 6\sqrt{6}$.

(d) In order to determine equation of plane we need a point and a normal vector. For the present case the normal vector to the plane is given as:

$$\vec{n} = \vec{BA}\times\vec{BC} = \langle -14,-4,2\rangle$$

and the point is: $(1,0,-1)$.

Thus, the equation of plane is given as:

$$-14(x-1) - 4(y-0) + 2(z+1) = 0$$

$$\Rightarrow -14x + 14 - 4y + 2z + 2 = 0$$

$$\Rightarrow -2\,[7x + 2y - z - 7 - 1] = 0$$

$$\Rightarrow 7x + 2y - z - 8 = 0 \Rightarrow \boxed{7x + 2y - z = 8}.$$

Q3:- (a) $\vec{r}(t) = \langle \sec t, \tan t, \frac{4}{3}t \rangle$; $t = \pi/6$

$\vec{v}(t) =$ velocity $= \vec{r}'(t) = \langle \sec t \tan t, \sec^2 t, \frac{4}{3} \rangle$

$\vec{a}(t) =$ acceleration $= \vec{r}''(t) = \langle \sec t \tan^2 t + \sec^3 t, 2\sec^2 t \tan t, 0 \rangle$

speed $= |\vec{r}'(t)|_{t=\pi/6} = |\vec{v}(t)|_{t=\pi/6}$

$= |\vec{v}(\frac{\pi}{6})| = \sqrt{\sec^2(\frac{\pi}{6}) \tan^2(\frac{\pi}{6}) + \sec^4(\frac{\pi}{6}) + \frac{16}{9}}$

$= 2$

Direction of motion at given value of $t = \dfrac{\vec{v}(\pi/6)}{|\vec{v}(\pi/6)|}$

$= \dfrac{\langle \sec(\pi/6)\tan(\pi/6), \sec^2(\pi/6), 4/3 \rangle}{2}$

$= \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$

Particle's velocity at given t as the product of speed and direction $= (2)\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$

$= \langle \frac{2}{3}, \frac{4}{3}, \frac{4}{3} \rangle$

(b) $\vec{r}(t) = \langle e^{-t}, 2\cos(3t), 2\sin(3t) \rangle$; $t = \pi/6$.

$\vec{v}(t) = \vec{r}'(t) = \langle -e^{-t}, -6\sin(3t), 6\cos(3t) \rangle$

$\vec{a}(t) = \vec{r}''(t) = \langle e^{-t}, -18\cos(3t), -18\sin(3t) \rangle$

Speed $= |\vec{v}(\frac{\pi}{6})| = \sqrt{[-e^{-\pi/6}]^2 + [-6\sin(\frac{3\pi}{6})]^2 + [6\cos(\frac{3\pi}{6})]^2}$

$= \sqrt{e^{-\pi/3} + 36\sin^2(\frac{\pi}{2}) + 36\cos^2(\frac{\pi}{2})}$

$= \sqrt{36 + e^{-\pi/3}} \approx 6.03$

Direction of motion $= \dfrac{\vec{v}(\pi/6)}{|\vec{v}(\pi/6)|} = \dfrac{\langle -e^{-\pi/6}, -6\sin(\pi/2), 6\cos(\pi/2) \rangle}{\sqrt{36 + e^{-\pi/3}}}$

$\Rightarrow$ Direction of motion $= \dfrac{\langle -0.59, -6, 0\rangle}{6.03}$

Velocity as a product $= (6.03)\dfrac{\langle -0.59, -6, 0\rangle}{(6.03)}$

$= \langle -0.59, -6, 0\rangle$

Q4.(a) Surface : $z = x^2 + y^2 \longrightarrow (1)$

Plane : $5x - 6y + z - 8 = 0 \longrightarrow (2)$

Using (1) in (2), we get

$$5x - 6y + x^2 + y^2 - 8 = 0$$

$$\Rightarrow x^2 + 5x + y^2 - 6y - 8 = 0$$

$$\Rightarrow x^2 + 2(x)\left(\frac{5}{2}\right) + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + y^2 - 2(y)(3) + (3)^2 - (3)^2 - 8 = 0$$

$$\Rightarrow \left(x + \frac{5}{2}\right)^2 + (y - 3)^2 - \frac{25}{4} - 9 - 8 = 0$$

$$\Rightarrow \left(x + \frac{5}{2}\right)^2 + (y-3)^2 = \frac{93}{4}$$

Thus, the curve of intersection of the surface with the plane is a circle:

$$\left(x + \frac{5}{2}\right)^2 + (y-3)^2 = \frac{93}{4}$$

with center at $\left(-\frac{5}{2}, 3\right)$ and radius $\frac{\sqrt{93}}{2}$.

Parametrization of circle:

$$x = -\frac{5}{2} + \frac{\sqrt{93}}{2}\cos t \;;\; y = 3 + \frac{\sqrt{93}}{2}\sin t.$$

and $z = x^2 + y^2 \Rightarrow z = \left(-\frac{5}{2} + \frac{\sqrt{93}}{2}\cos t\right)^2 + \left(3 + \frac{\sqrt{93}}{2}\sin t\right)^2$,

$0 \le t \le 2\pi$.

Thus, the required vector function is:

$$\vec{r}(t) = \left\langle -\frac{5}{2} + \frac{\sqrt{93}}{2}\cos t,\ 3 + \frac{\sqrt{93}}{2}\sin t,\ \left(-\frac{5}{2} + \frac{\sqrt{93}}{2}\cos t\right)^2 + \left(3 + \frac{\sqrt{93}}{2}\sin t\right)^2 \right\rangle;$$

$t \in [0, 2\pi]$.

(b) Cylinder: $x^2 + y^2 = 9$

hyperbolic paraboloid: $z = xy$.

The projection of cylinder onto the xy-plane is the circle:

$$x^2 + y^2 = 9 \ ; \quad z = 0,$$

So we can write:

$$x = 3\cos t, \quad y = 3\sin t$$

where $0 \le t \le 2\pi$.

From the equation of hyperbolic paraboloid we have:

$$z = (3\cos t)(3\sin t)$$

$$\Rightarrow z = 9\cos t \sin t = 9\sin t \cos t = \frac{9}{2}\sin(2t)$$

Thus, the parametric equations for the curve of intersection are:

$x = 3\cos t$, $y = 3\sin t$, $z = \frac{9}{2}\sin(2t)$ and the corresponding vector equation is:

$$\vec{r}(t) = \left\langle 3\cos t,\ 3\sin t,\ \frac{9}{2}\sin(2t) \right\rangle,$$

where $t \in [0, 2\pi]$.

Q5:- (a) $\operatorname{cosec}\varphi = 2\cos\theta + 4\sin\theta$.

Solution. Given equation is in spherical coordinates and we are required to convert This in Cartesian coordinates. Consider the equation:

$$\operatorname{cosec}\varphi = 2\cos\theta + 4\sin\theta$$

$$\Rightarrow \frac{1}{\sin\varphi} = 2\cos\theta + 4\sin\theta$$

$$\Rightarrow 1 = 2\sin\varphi\cos\theta + 4\sin\varphi\sin\theta$$

$$\Rightarrow \rho = 2\rho\sin\varphi\cos\theta + 4\rho\sin\varphi\sin\theta$$

$$\Rightarrow \sqrt{x^2+y^2+z^2} = 2x + 4y \quad \left[\begin{array}{l} \because \rho^2 = x^2+y^2+z^2 \\ x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \end{array}\right]$$

$$\Rightarrow x^2+y^2+z^2 = (2x+4y)^2.$$

(b) $\rho - \cos\varphi = 2 + \cos^2\varphi$.

Solution. $\rho - \cos\varphi = 2 + \cos^2\varphi$

$$\Rightarrow \sqrt{x^2+y^2+z^2} - \frac{z}{\rho} = 2 + \frac{z^2}{\rho^2} \quad \left[\begin{array}{l} \because \rho^2 = x^2+y^2+z^2 \\ z = \rho\cos\varphi \Rightarrow \cos\varphi = \frac{z}{\rho} \end{array}\right]$$

$$\Rightarrow \sqrt{x^2+y^2+z^2} - \frac{z}{\sqrt{x^2+y^2+z^2}} - \frac{z^2}{x^2+y^2+z^2} = 2$$

$$\Rightarrow \frac{(x^2+y^2+z^2)^{3/2} - z(x^2+y^2+z^2)^{1/2} - z^2}{x^2+y^2+z^2} = 2$$

$$\Rightarrow (x^2+y^2+z^2)^{3/2} - z\sqrt{x^2+y^2+z^2} - z^2 = 2(x^2+y^2+z^2)$$

$$\Rightarrow (x^2+y^2+z^2)^{3/2} - z\sqrt{x^2+y^2+z^2} - 2x^2 - 2y^2 - 3z^2 = 0$$