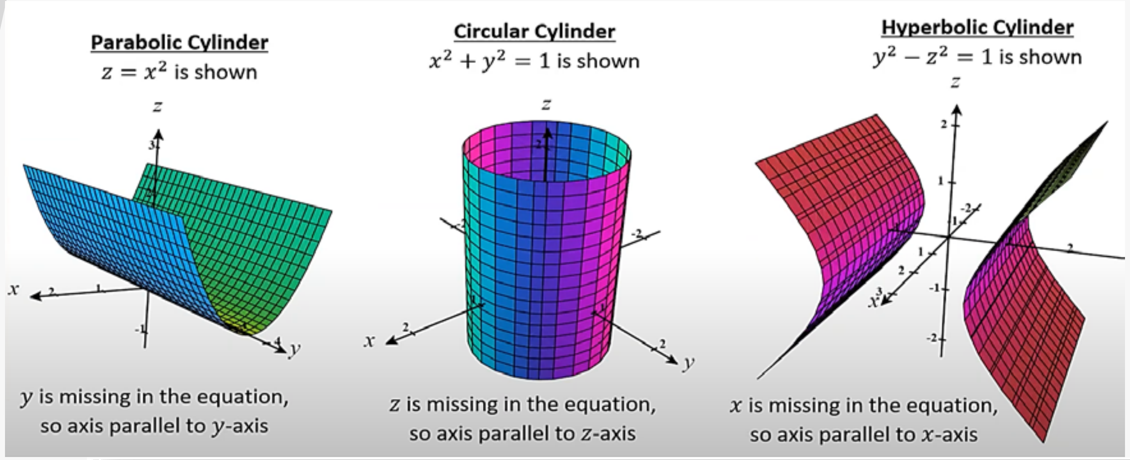
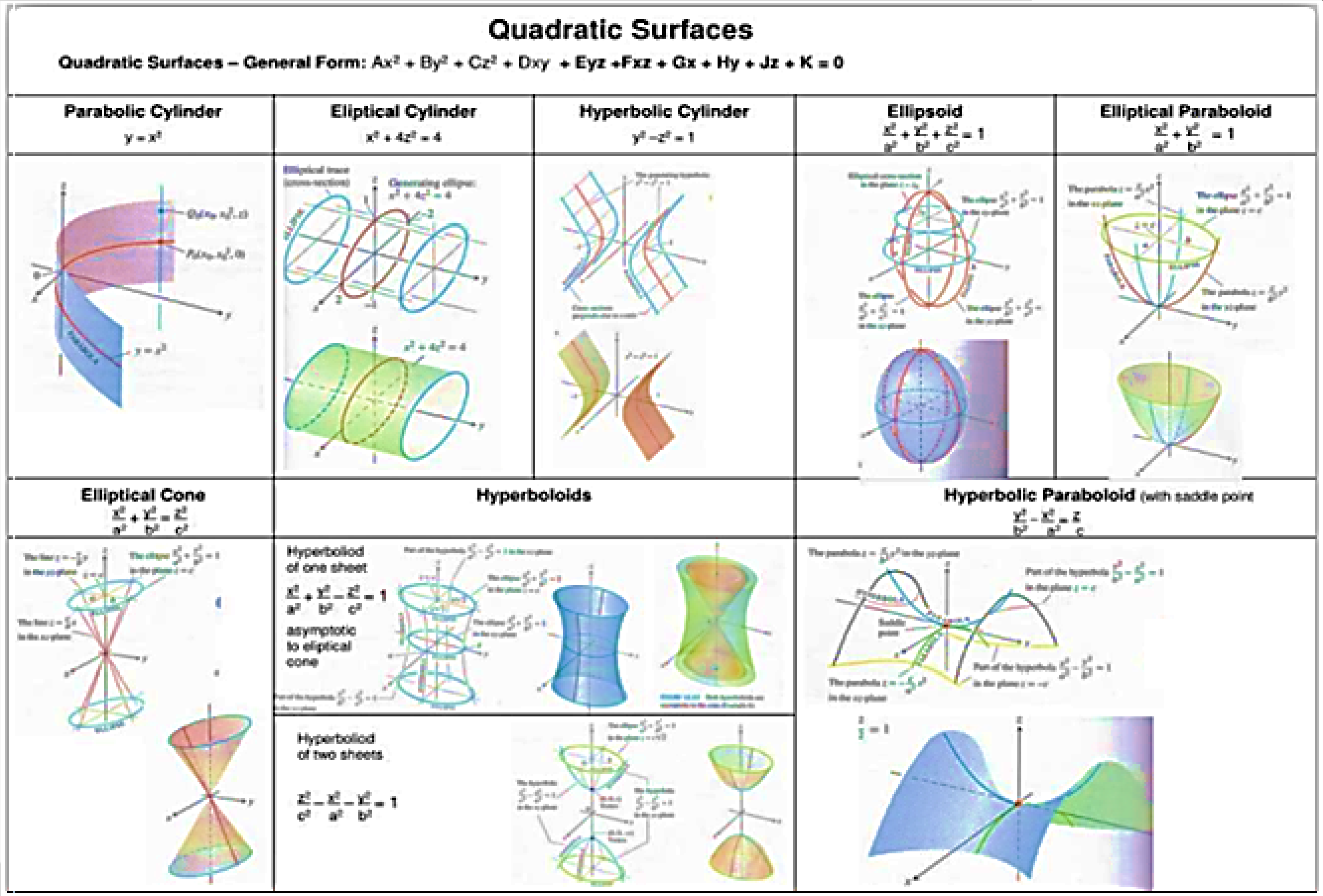
Cylinders & Quadric Surfaces





Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

Vectors And
The Geometry Of Space

Book: Thomas’ Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Chapter: 12 , Section: 12.6

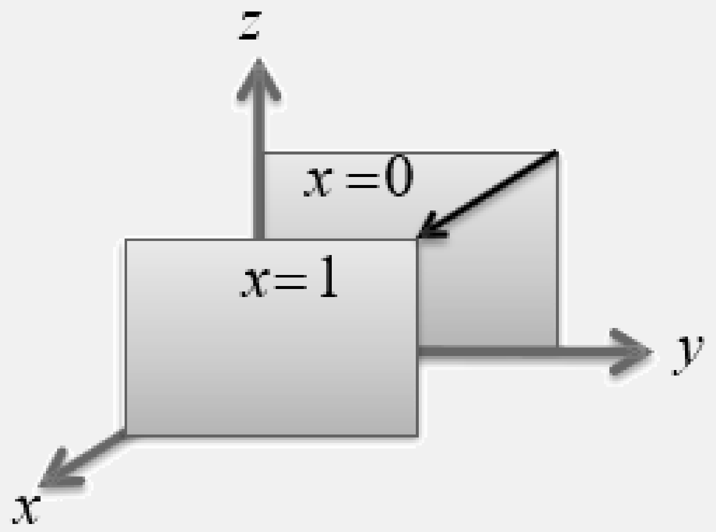
Book: Calculus Early Transcendentals (6th Edition) By James Stewart.
Chapter: 12 , Section: 12.6

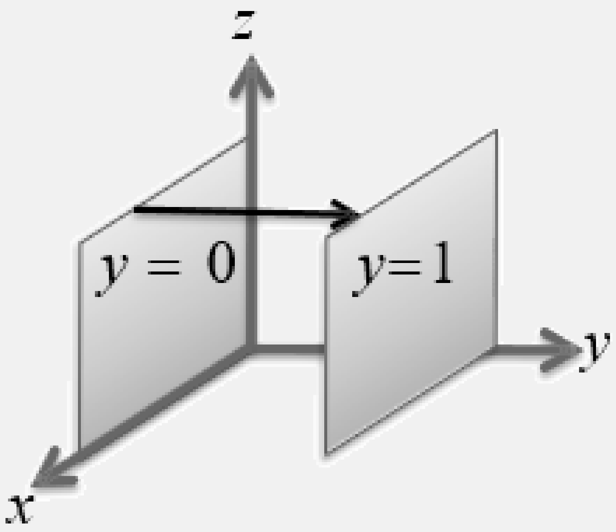
Book: Calculus (5th Edition) by Swokowski, Olinick and Pence
Chapter: 14, Section: 14.6

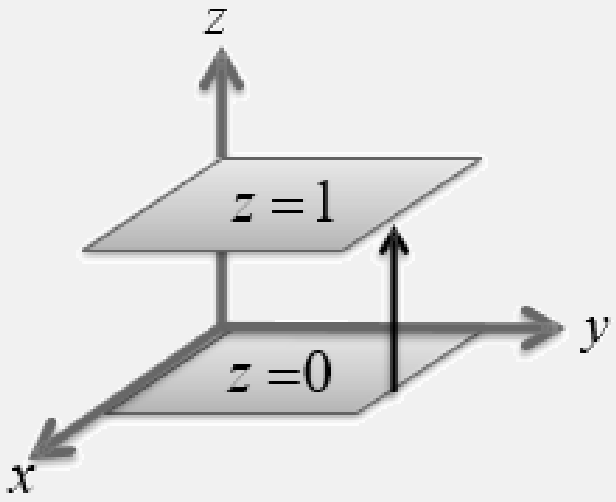
Traces

To sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes.

These curves are called traces (or cross-sections) of the surface.

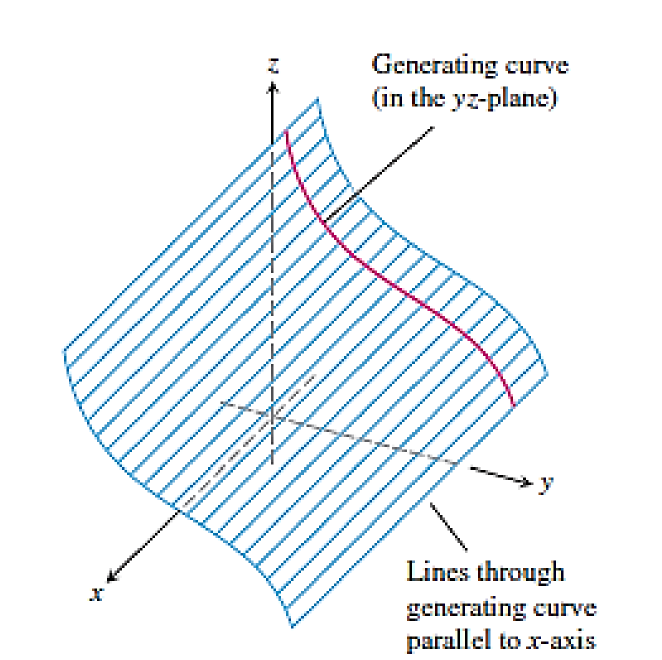






Cylinder

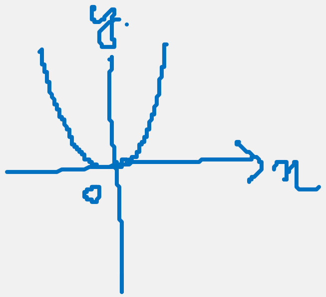
- A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass-through a given plane curve.
- Alternatively, we can say that a **cylinder** is a surface that is generated by moving a straight line along a given planar curve while holding the line parallel to a given fixed line. The curve is called a **generating curve** for the cylinder.
- In solid geometry, where *cylinder* means *circular cylinder*, the generating curves are circles, but now we allow generating curves of any kind.



A cylinder and generating curve.

Example: Parabola in 3D:

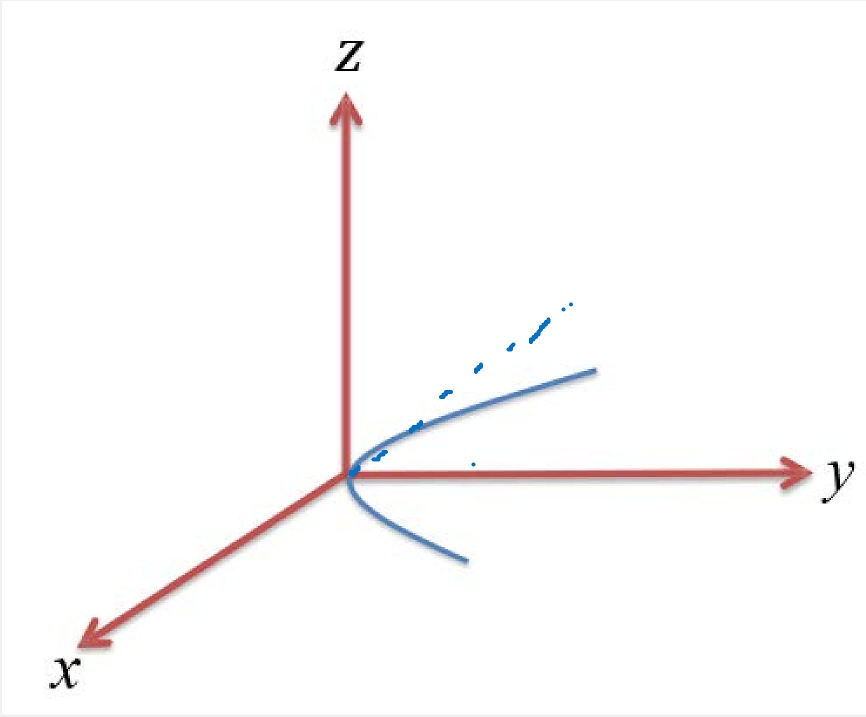
We use basic functions from calculus – 1 and extend them in 3D.



Suppose we have a simple parabola that is to be sketched in 3D.

$$y = x^2; \quad 0 < z < 1.$$

Step – 1: Sketch the parabola on xy –plane

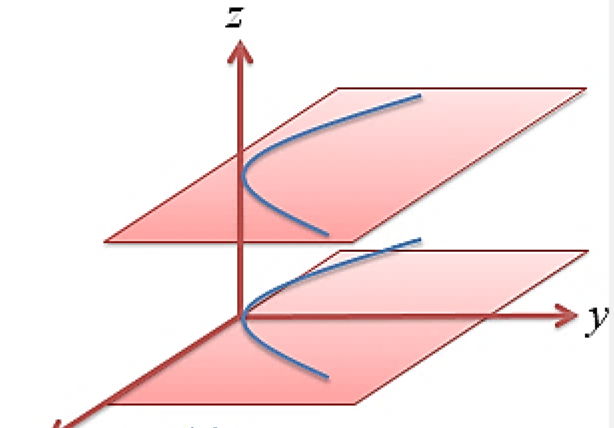


Example: Parabola in 3D:

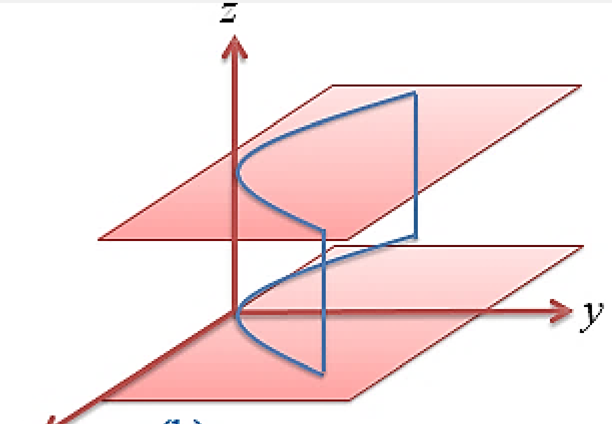
Step – 2: Sketch a similar parabola on a plane parallel to xy −plane. (a)

Step – 3: Join all edges uniformly. (b)

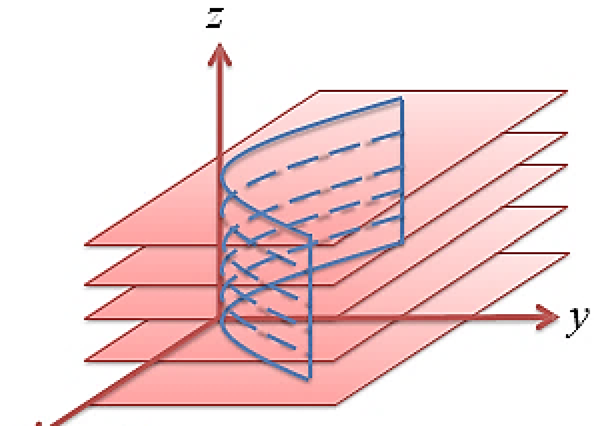
Step – 4: Identify the surface. (c)



(a)



(b)



(c)

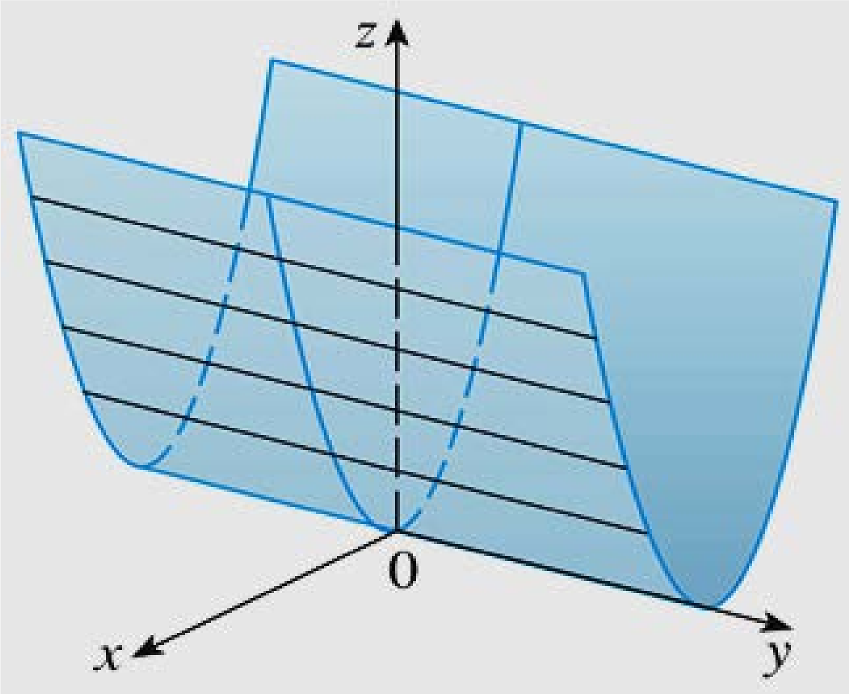
Example: Sketch the graph of the surface $z = x^2$

- Notice that the equation of the graph, $z = x^2$, doesn't involve y. This means that any vertical plane with equation $y = k$ (parallel to the xz −plane) intersects the graph in a curve with equation $z = x^2$.

- So, these vertical traces are parabolas.

- The graph is a surface, called a parabolic cylinder, made up of infinitely many shifted copies of the same parabola.

- Here, the rulings of the cylinder are parallel to the y −axis.



Cylinders

In previous examples, notice that the variables z and y were missing from the equations of the cylinder.

- In both cases, we get surfaces whose rulings are parallel to one of the coordinate axes.
- If one of the variables x, y, or z is missing from the equation of a surface, then the surface is a cylinder.

Example:

Identify and sketch the following surfaces:

a. $x^2 + y^2 = 1.$

b. $y^2 + z^2 = 1.$

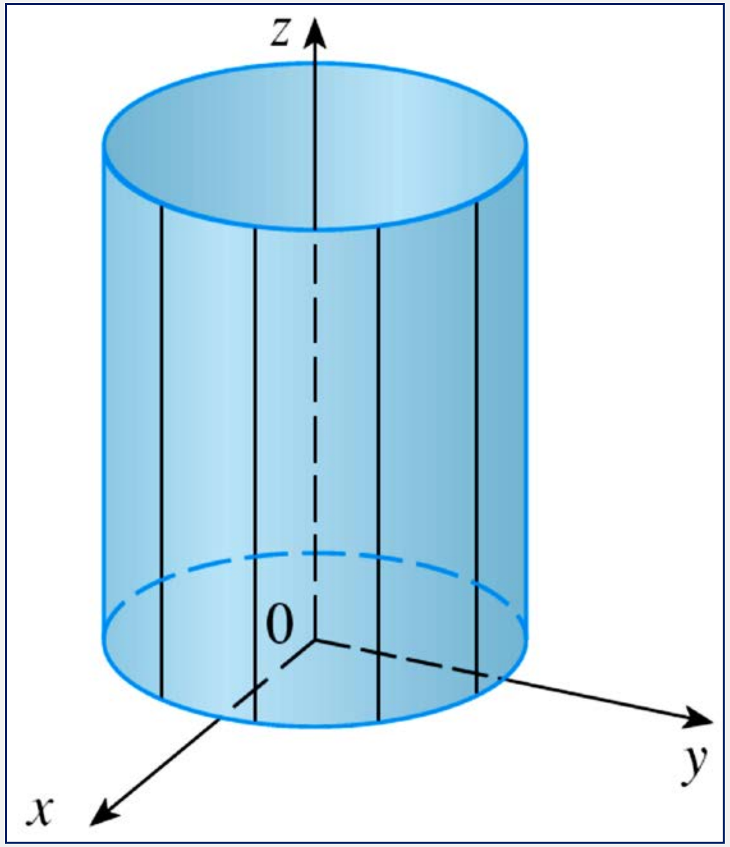
Example:

Solution: (a)

Consider the equation:

$$x^2 + y^2 = 1.$$

- Here, z is missing and the equations $x^2 + y^2 = 1$, $z = k$ represent a circle with radius 1 in the plane $z = k$.
- Thus, the surface $x^2 + y^2 = 1$ is a circular cylinder whose axis is the z −axis.
- Here, the rulings are vertical lines.



Example:

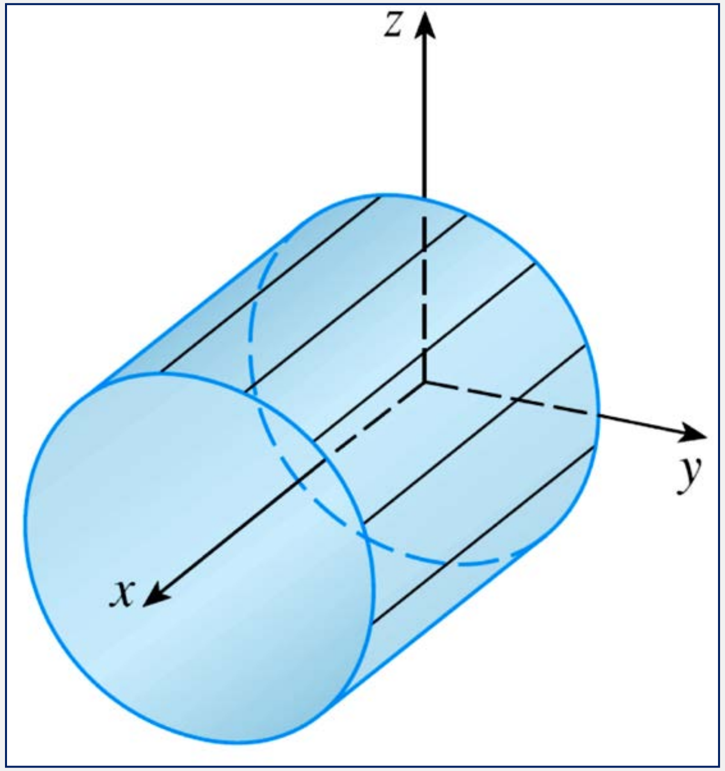
Solution: (b)

Consider the equation:

$$y^2 + z^2 = 1.$$

- In this case, x is missing and the surface is a circular cylinder whose axis is the x −axis.

- It is obtained by taking the circle $y^2 + z^2 = 1$, $x = 0$ in the yz −plane, and moving it parallel to the x −axis.

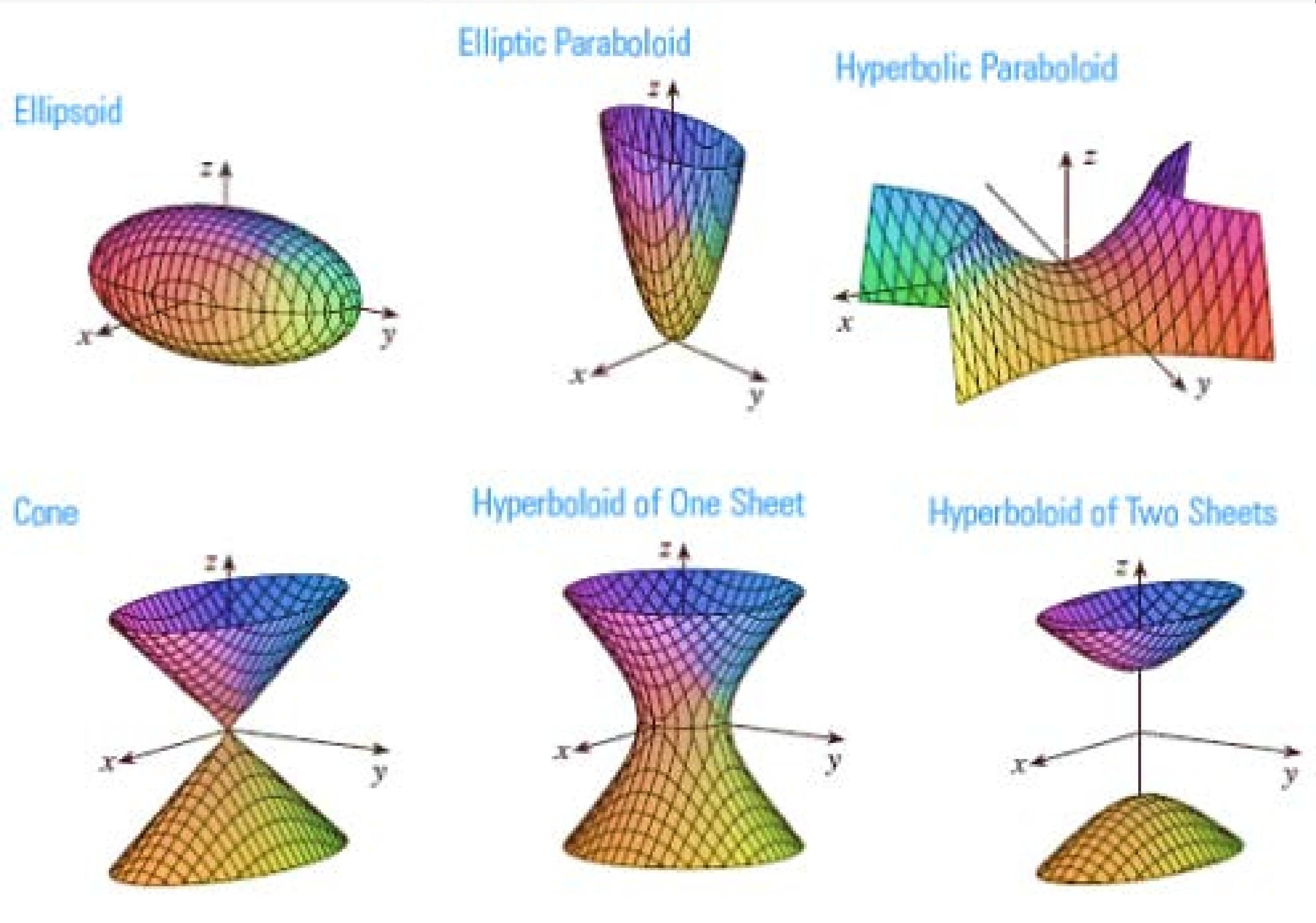


Note:

When we are dealing with surfaces, it is important to recognize that an equation like $x^2 + y^2 = 1$ represents a cylinder and not a circle.

- The trace of the cylinder $x^2 + y^2 = 1$ in the xy −plane is the circle with equations: $x^2 + y^2 = 1, z = 0$.

Quadric Surfaces



Quadric Surface

A quadric surface is a graph in space of a second-degree equation in x, y and z. The general form of the equation is:

$Ax^2 + By^2 + Cxy + Dx + Fy + E = 0$

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Jz + K = 0,$$

where $A, \ldots, K$ are all constants.

It is not hard to realize that the solution of the above equation is a surface in 3D, in general.

What are the solutions of above equation???

The answer can be found for different possible combinations of constants. Interestingly it leads to the classification of several basic surfaces in 3D. Now we see how to sketch few basic surfaces.

Quadric Surface

Sketching:

We study these surfaces with a simple observation that a surface is generated by infinite family of curves that lie on it. Normally a surface can be drawn by tracing out a few curves and joining them together to form it. To do this we need to formally define a procedure which is known as sketching the traces.

Trace:

A trace of a surface S is a geometrical curve which one obtain on a plane that intersects the surface. Basic traces are found on the fundamental planes:

- xy – plane $\quad \boldsymbol{z = 0}$
- plane parallel to xy – plane $\quad \boldsymbol{z = 1}$
- yz – plane $\quad \boldsymbol{x = 0}$
- plane parallel to yz – plane $\quad \boldsymbol{x = 1}$
- xz – plane $\quad \boldsymbol{y = 0}$
- plane parallel to xz – plane $\quad \boldsymbol{y = 1}$

Ellipsoid

Problem: Identify the surface by sketching out traces on different coordinate planes.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

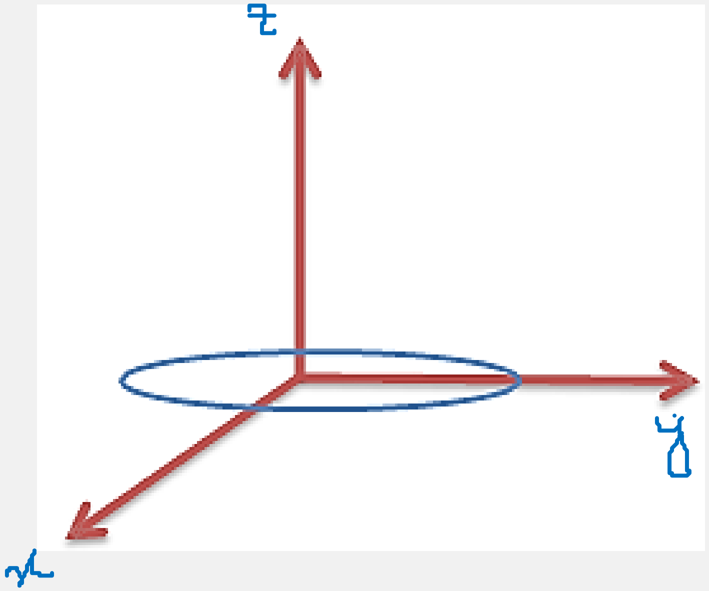
Solution: We note that xy − trace $(z = 0)$ is an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Similarly for trace $(z = 1)$ we get:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{1}{c^2},$$

which is an ellipse of shorter width.

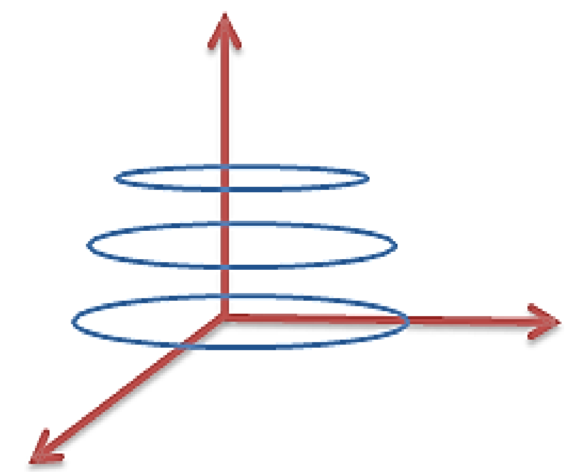


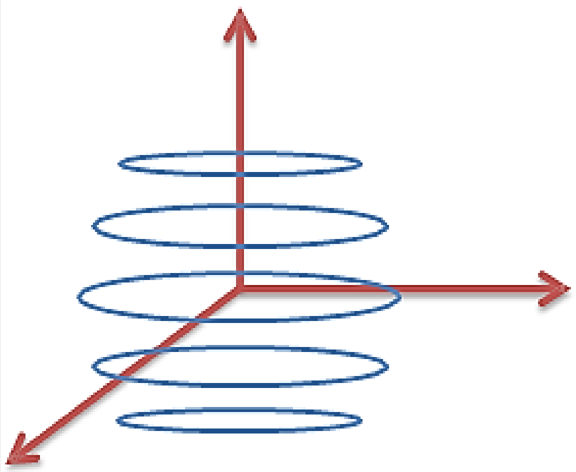
Ellipsoid

Solution: Similarly, for $z = k$, where $k = 2, 3, \ldots$ such that $k < c$ we will get:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}.$$

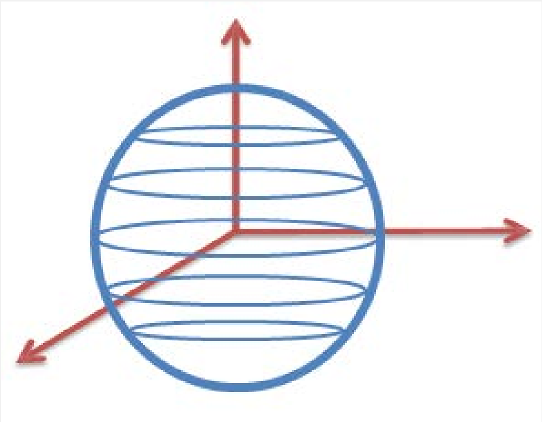
Even if we take negative values of z we get the same pattern below xy −plane . Thus, traces on plane parallel to xy − plane, $z = \pm k$ such that $k < c$ are all ellipse.





Ellipsoid

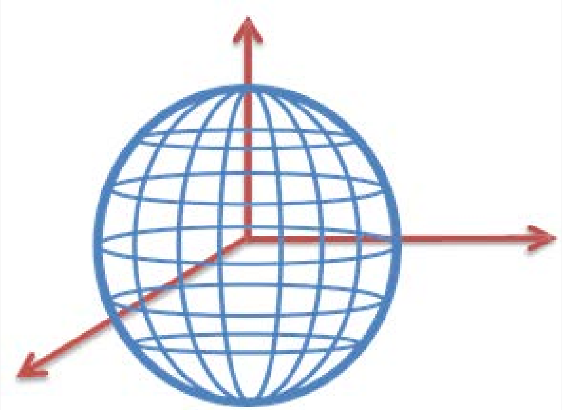
Solution: If we join the edges, we get the required surface just using sequence of one type of traces.



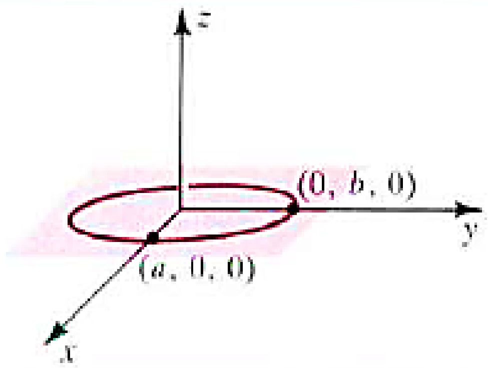
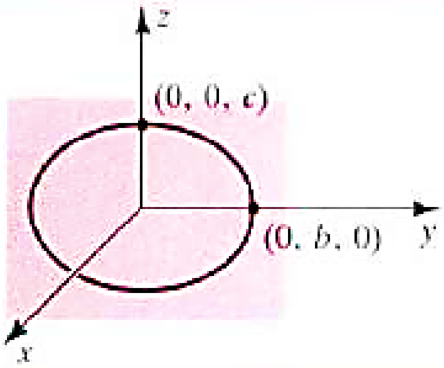
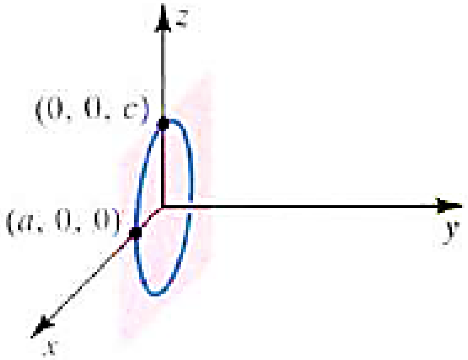
However, if we take other traces, we can easily determine that each trace result into an ellipse.

$yz-$Trace, $x=0$; $\quad \dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$ Ellipse

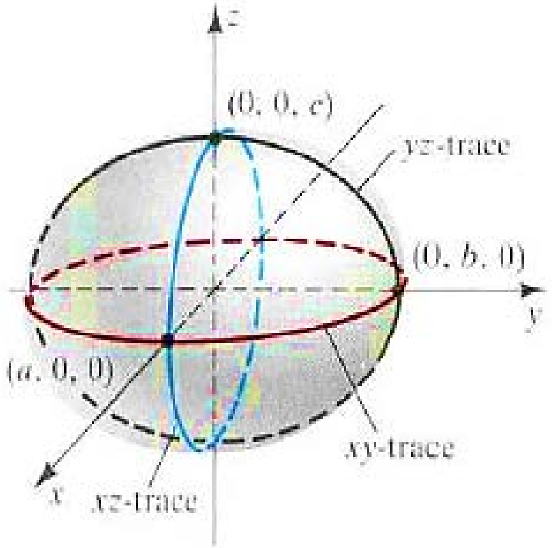
$zx-$Trace, $y=0$; $\quad \dfrac{x^2}{a^2}+\dfrac{z^2}{c^2}=1.$ Ellipse



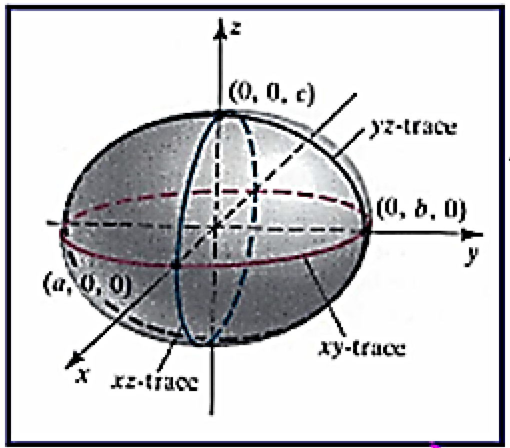
Ellipsoid

<table>
<tr><th>Trace</th><th>Equation of trace</th><th>Description of trace</th><th>Sketch of trace</th></tr>
<tr><td>xy-trace</td><td>$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$</td><td>Ellipse</td><td>
</td></tr>
<tr><td>yz-trace</td><td>$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$</td><td>Ellipse</td><td>
</td></tr>
<tr><td>xz-trace</td><td>$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$</td><td>Ellipse</td><td>
</td></tr>
</table>

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



Ellipsoid



Observations

- Equation $\boxed{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1}$
- Form $f(x,y,z)=1$ with
  - all variables quadratic
  - all variables with positive coefficient

- Centre at $(0,0,0)$
- x –intercept: $\pm a$
- y –intercept: $\pm b$
- z –intercept: $\pm c$

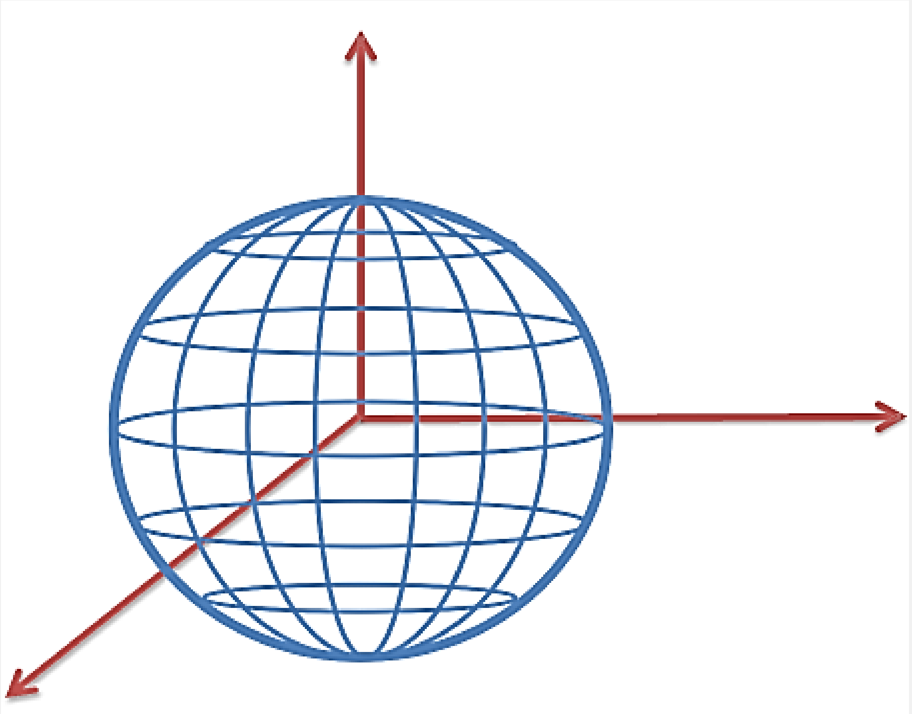
Example Describe the surface $4x^2+4y^2+z^2+8y-4z=-4$

Sphere

If $a = b = c$, then the surface is a sphere:

$$x^2 + y^2 + z^2 = a^2,$$

whose traces are circles in different planes.

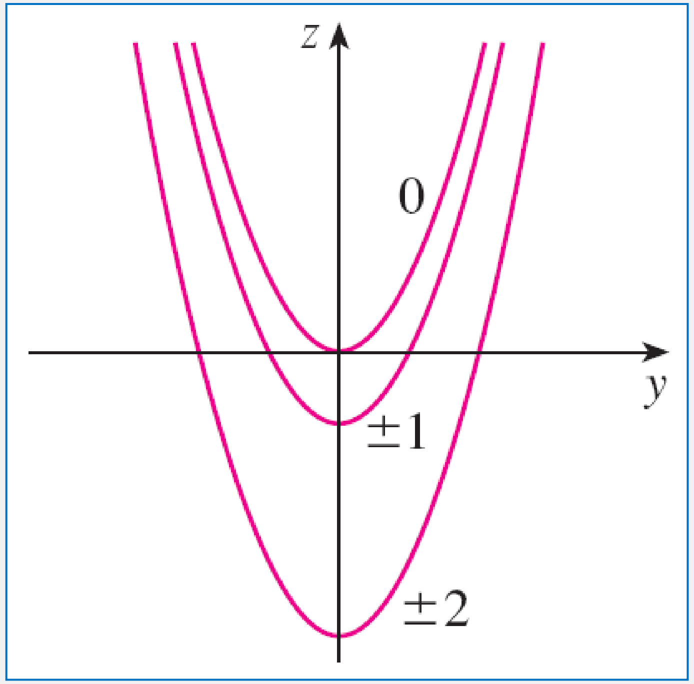


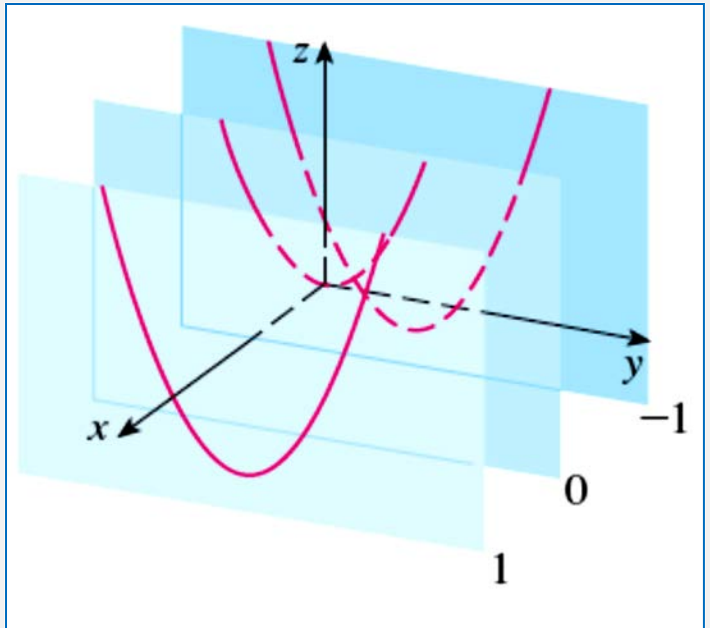
Quadric Surfaces

Problem: Identify and sketch the surface

$$y^2 - x^2 = z.$$

Solution: The traces in the vertical planes $x = k$ are the parabolas $y^2 = z + k^2$, which open upward.

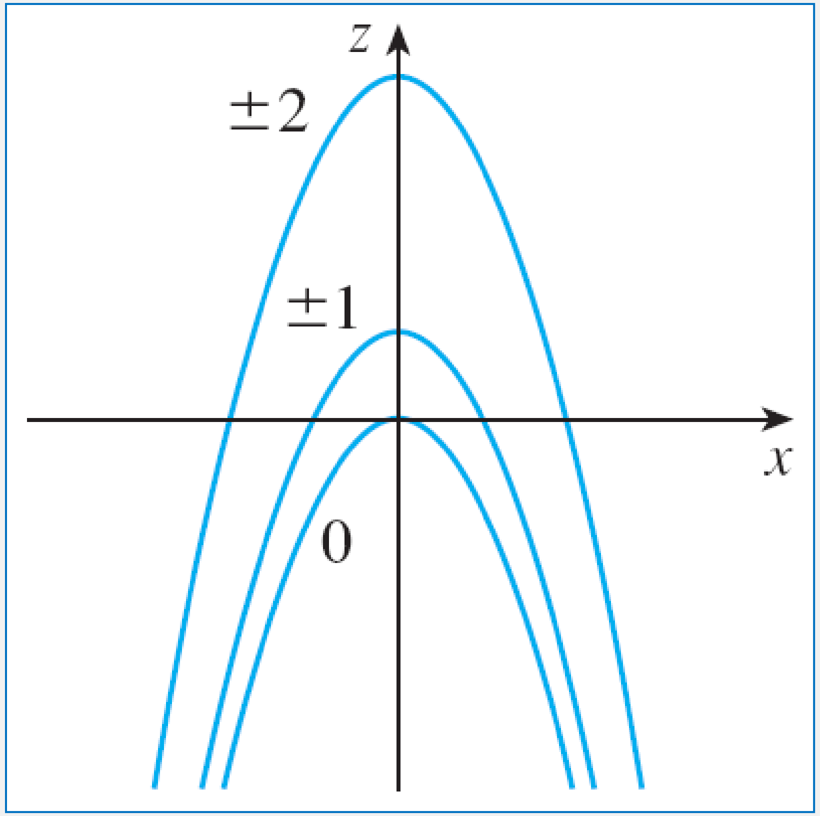


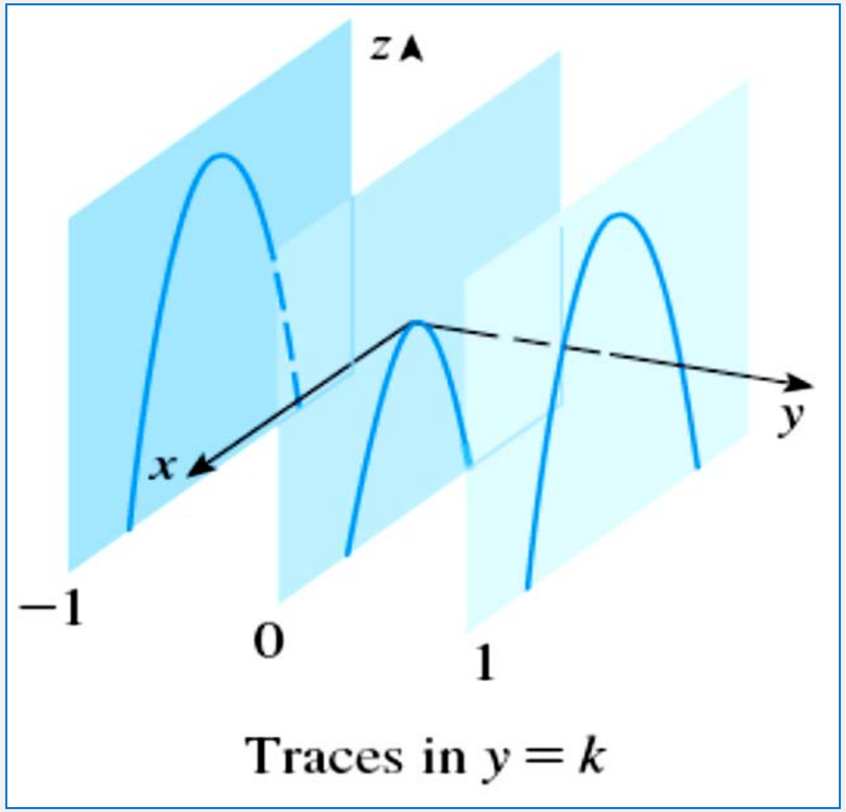


Traces in $x = k$

Quadric Surfaces

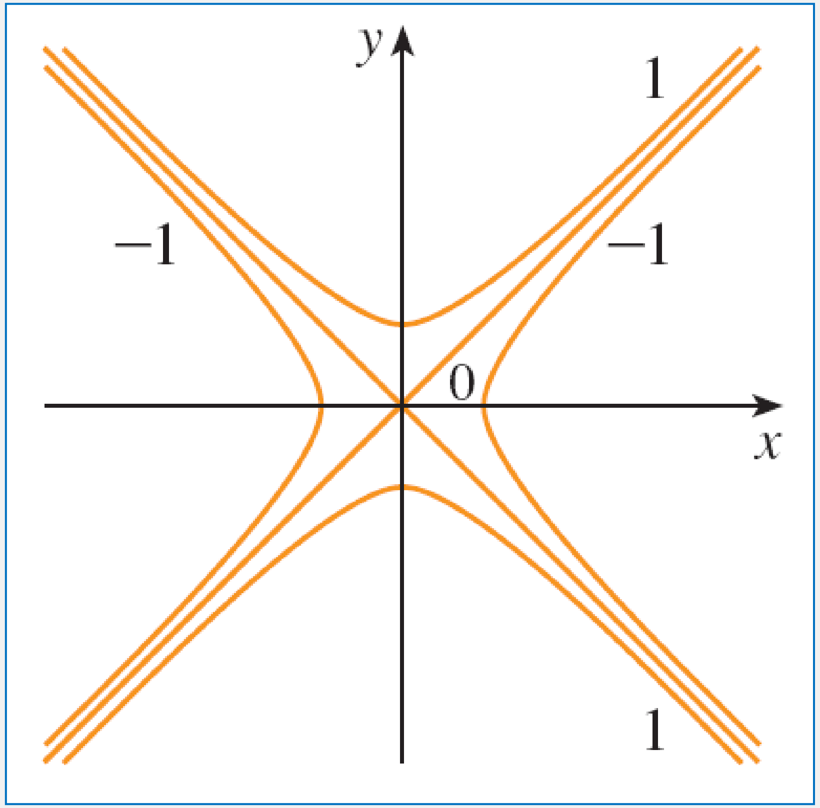
Solution: The traces in planes $y = k$ are the parabolas $x^2 = k^2 - z$, which open downward.

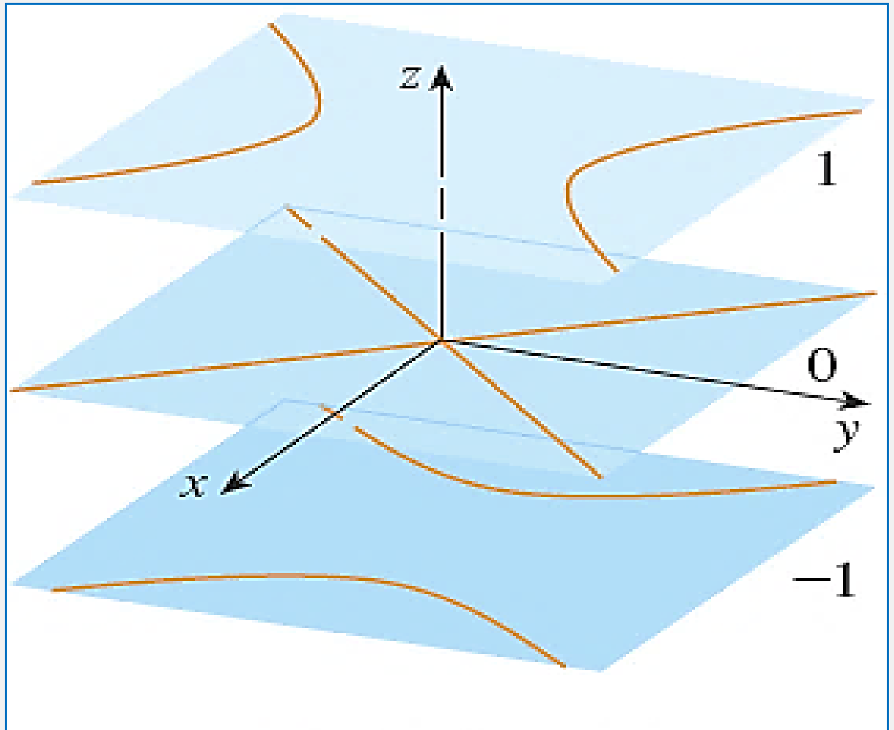




Quadric Surfaces

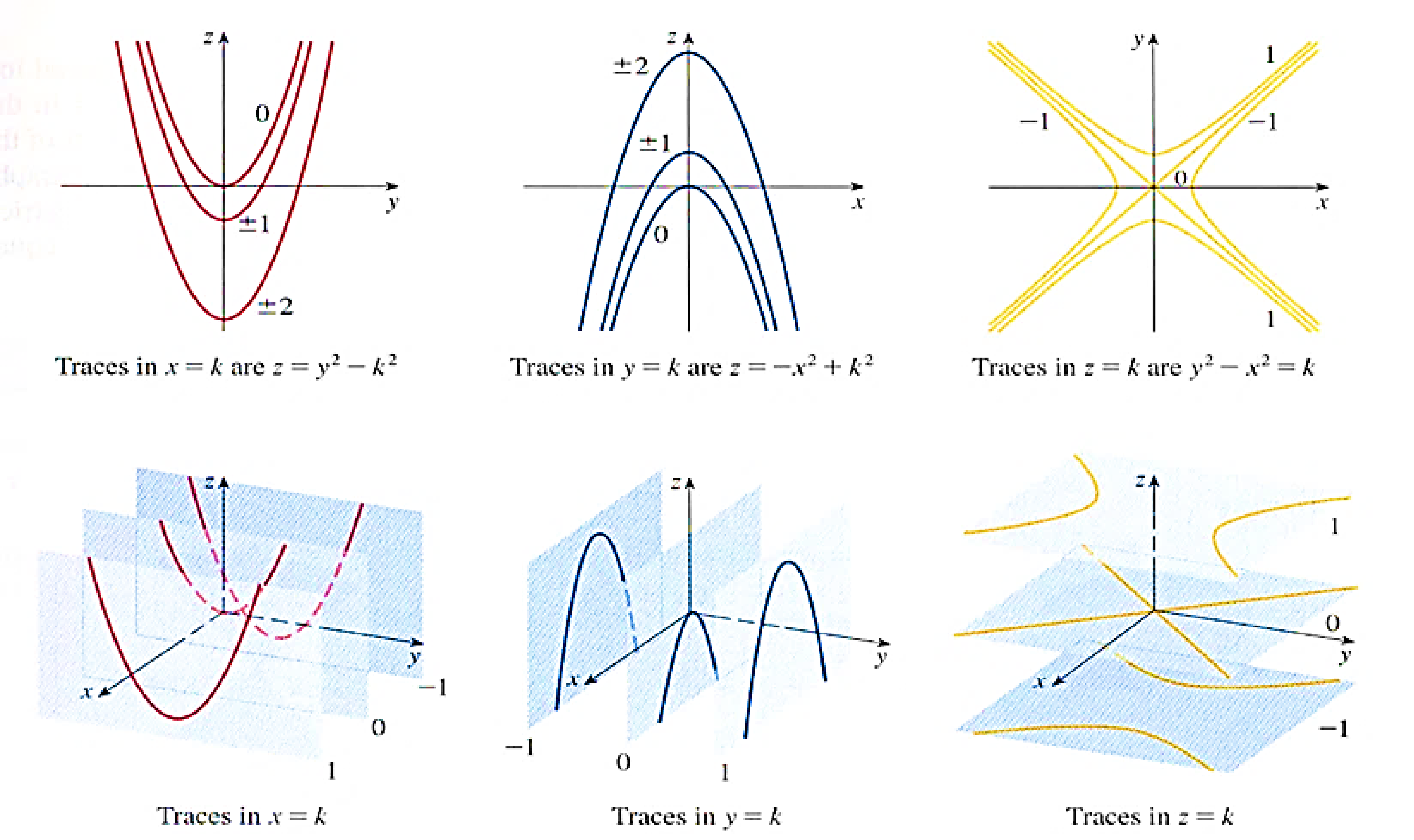
Solution: The horizontal traces $z = k$, are $y^2 - x^2 = k$, a family of hyperbolas.





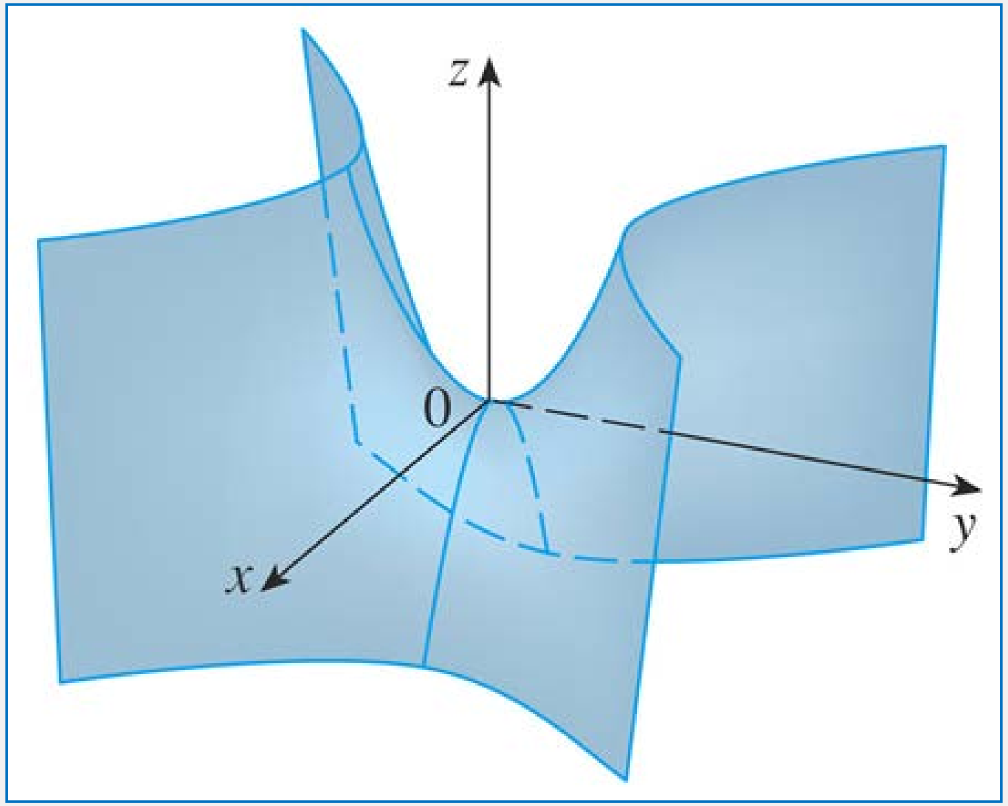
Traces in $z = k$

Quadric Surfaces



Hyperbolic Paraboloid

Here, we fit together the traces from the previous figure to form the surface: $z = y^2 - x^2$, which is commonly known as hyperbolic paraboloid.

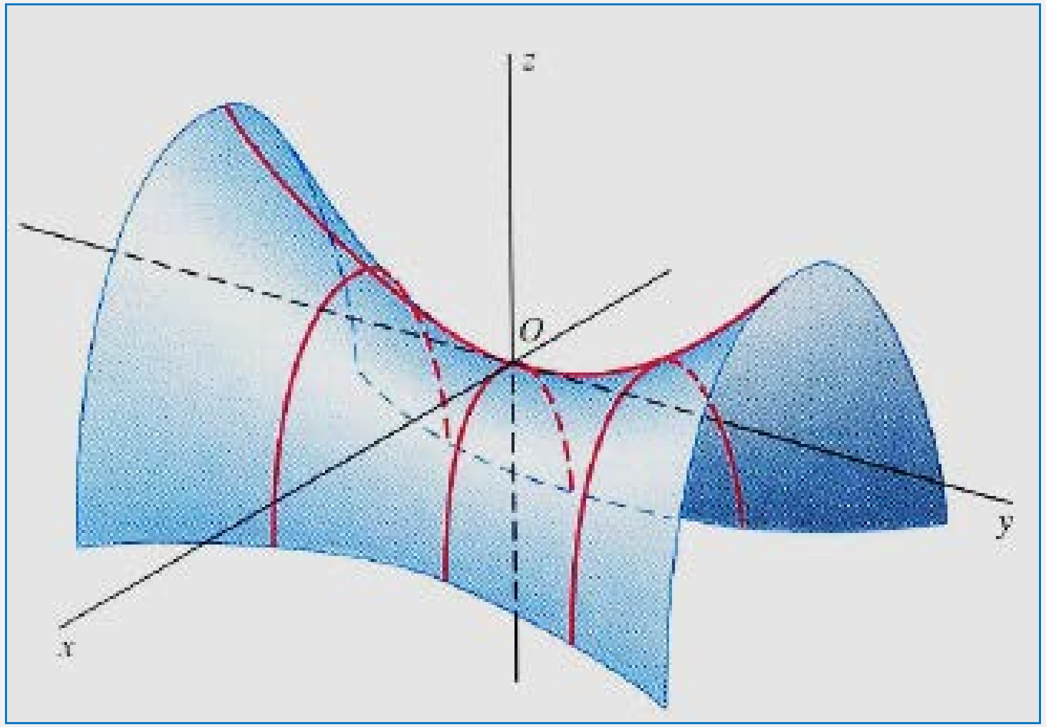


Hyperbolic Paraboloid

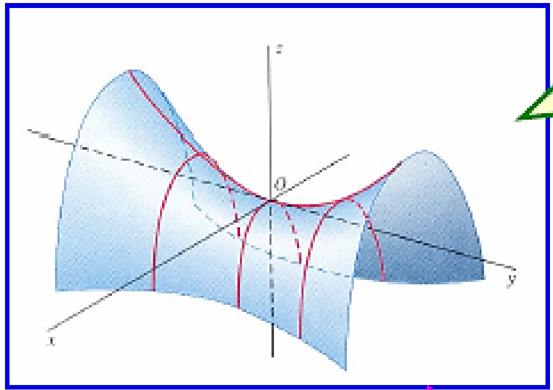
In general, we have, $\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}; c > 0.$

where the traces are:

- traces in planes parallel to $yz -$ plane are parabolas opening upwards.
- traces in planes parallel to $xz -$ plane are parabolas opening downwards.
- traces in planes parallel to $xy -$ plane are hyperbolas.



Hyperbolic Paraboloid



Observations

- Equation $\boxed{z = \frac{y^2}{b^2} - \frac{x^2}{a^2}}$ with
  - one linear variable
  - two quadratic variables with opposite signs

- Such surfaces are like a horse saddle
- To get an idea about orientation
  - imagine the horse saddle and the axis along which the horse would stand
  - e.g. in above figure the horse would stand along Y-axis

Example Sketch the surface $z = \frac{x^2}{9} - \frac{y^2}{4}$.

Elliptic Cone

Problem: Identify the surface by sketching out traces on different coordinate planes.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

Solution: The coordinate with negative coefficient, z −axis, is the axis of cone.

$xy -$ Trace, $z=0$; $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=0 \quad$ origin

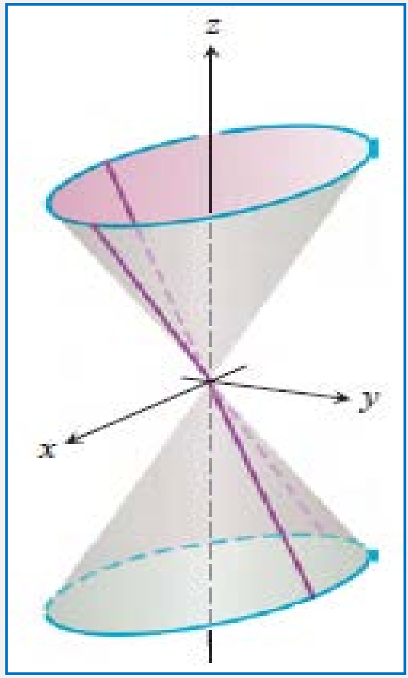
$yz -$ Trace, $x=0$; $\quad z=\pm\left(\frac{c}{b}\right)y \quad$ Straight lines intersecting at origin

$zx -$ Trace, $y=0$; $\quad z=\pm\left(\frac{c}{a}\right)x \quad$ Straight lines intersecting at origin

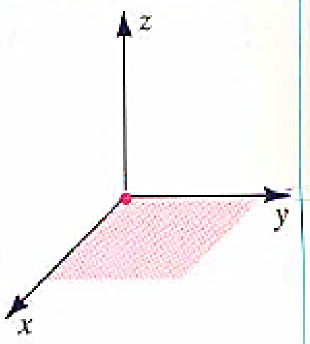
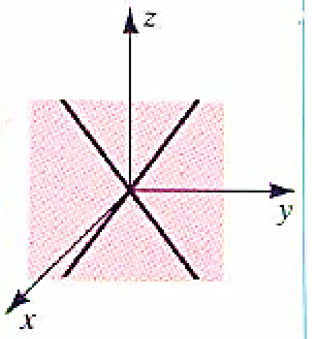
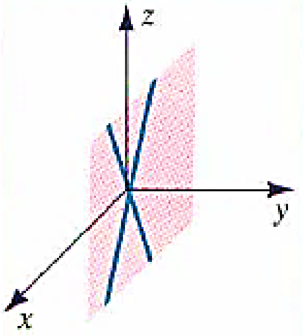
Trace in the plane $z=k$, parallel to xy −plane,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}$$

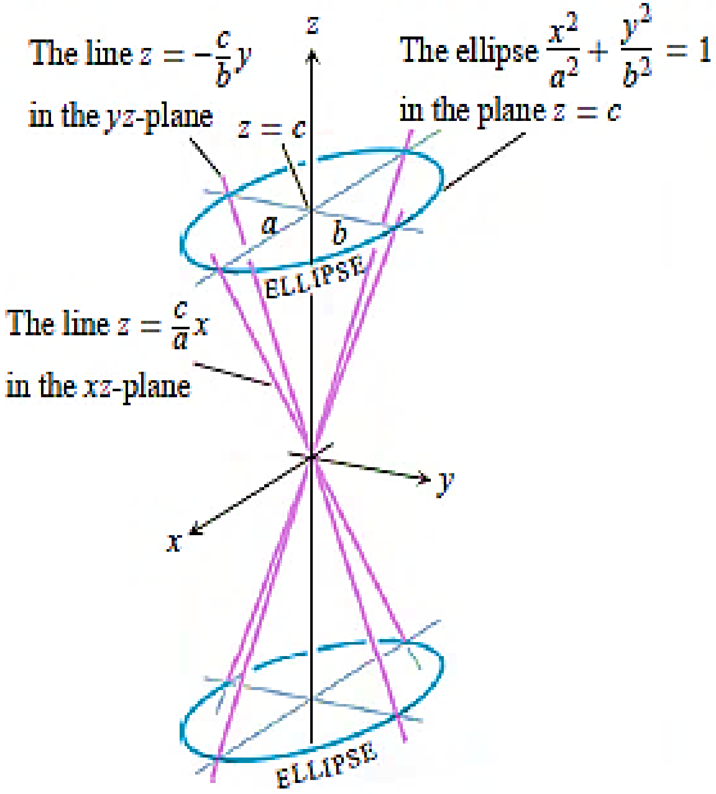
Ellipse



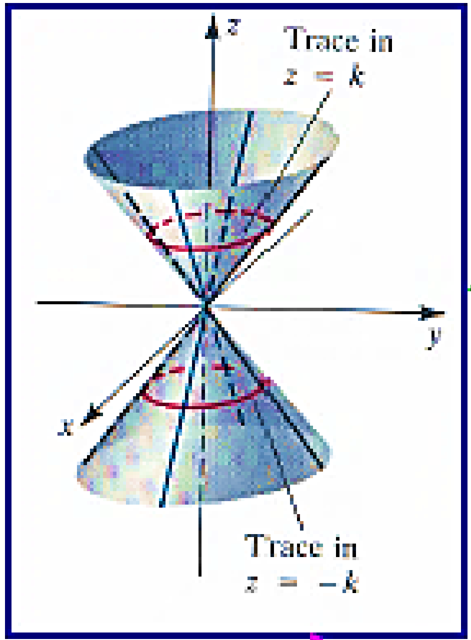
Elliptic Cone

Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$	Origin	
yz-trace	$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	Two intersecting lines	
xz-trace	$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 0$	Two intersecting lines	

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$



Elliptic Cone



Observations

- Equation $\boxed{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=0}$
- Form $f(x,y,z)=0$ with
  - all variables quadratic
  - sign of coefficient of one variable different from other two.

- Axis of symmetry corresponds to variable with different sign
- Opens along axis of symmetry

Example Sketch the surface $x=\sqrt{y^2+z^2}$.

Hyperboloid of One Sheet

Problem: Identify the surface by sketching out traces on different coordinate planes.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Solution: The coordinate with negative coefficient, z −axis, is the axis of hyperboloid.

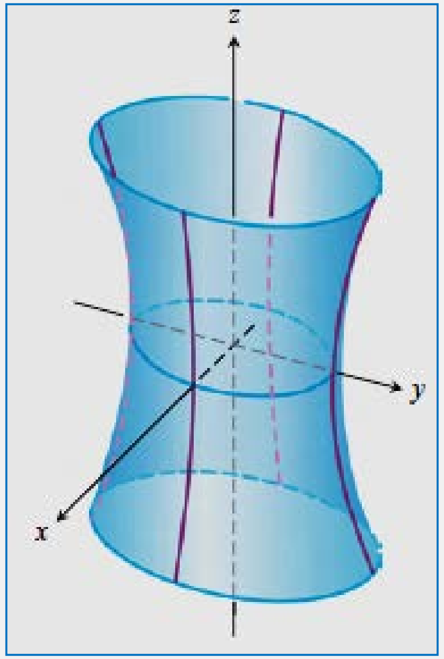
xy − Trace, $z = 0$; $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ Ellipse

yz − Trace, $x = 0$; $\quad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$ Hyperbola

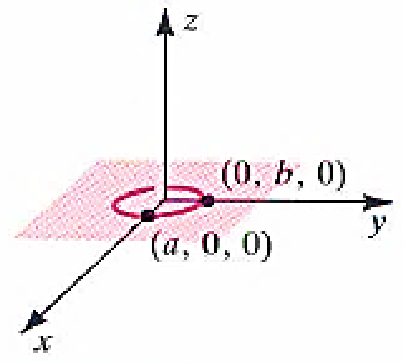
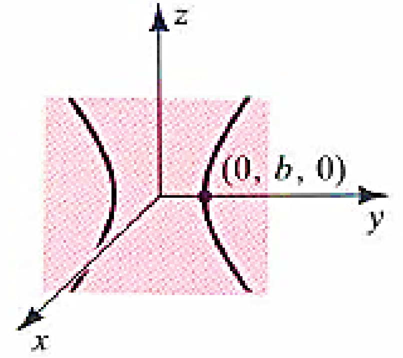
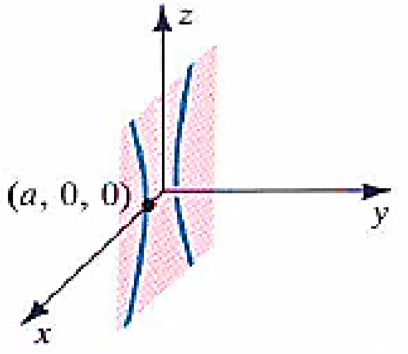
zx − Trace, $y = 0$; $\quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1.$ Hyperbola

Trace on the plane $z = \pm k$, parallel to the xy −plane,

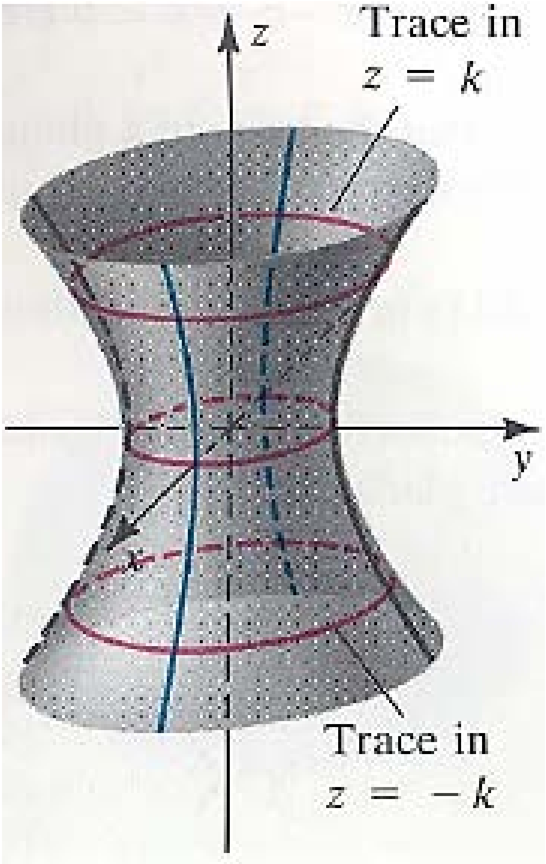
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}.$$ Ellipse



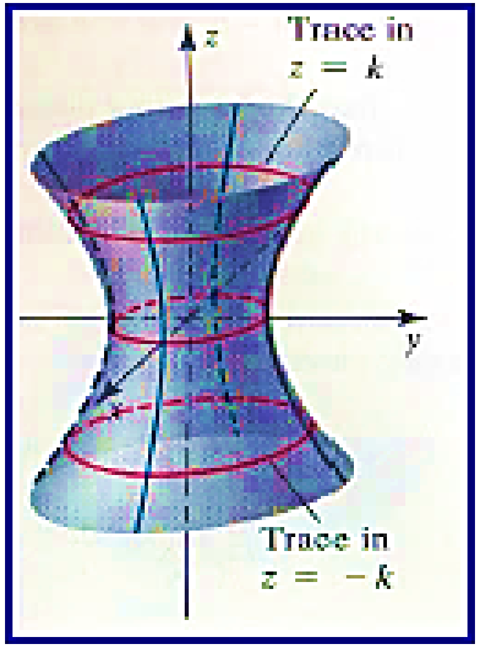
Hyperboloid of one sheet

Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	Ellipse	
yz-trace	$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Hyperbola	
xz-trace	$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$	Hyperbola	

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



Hyperboloid of one sheet



Observations

- Equation $\boxed{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1}$
- Form $f(x,y,z)=1$ with
  - all variables quadratic
  - one variable with negative coefficient

- Axis of symmetry corresponds to variable with negative coefficient
- Opens along axis of symmetry

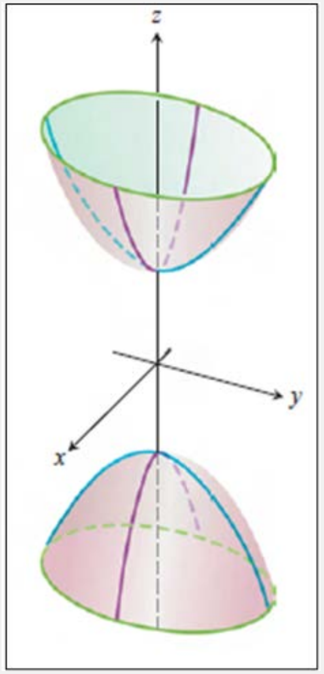
Example Sketch the graph of $16x^2-9y^2+36z^2=144$.

Hyperboloid of Two Sheets

Problem: Identify the surface by sketching out traces on different coordinate planes.

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Solution: The coordinate with positive coefficient, z −axis, is the axis of hyperboloid.



$xy-$Trace, $z=0$; $\quad -\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$ None

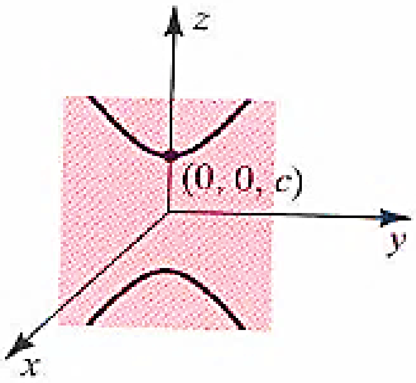
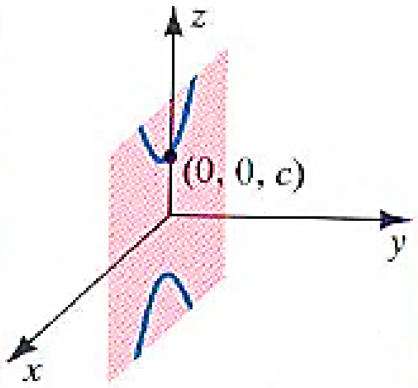
$yz-$Trace, $x=0$; $\quad -\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$ Hyperbola

$zx-$Trace, $y=0$; $\quad -\frac{x^2}{a^2}+\frac{z^2}{c^2}=1.$ Hyperbola

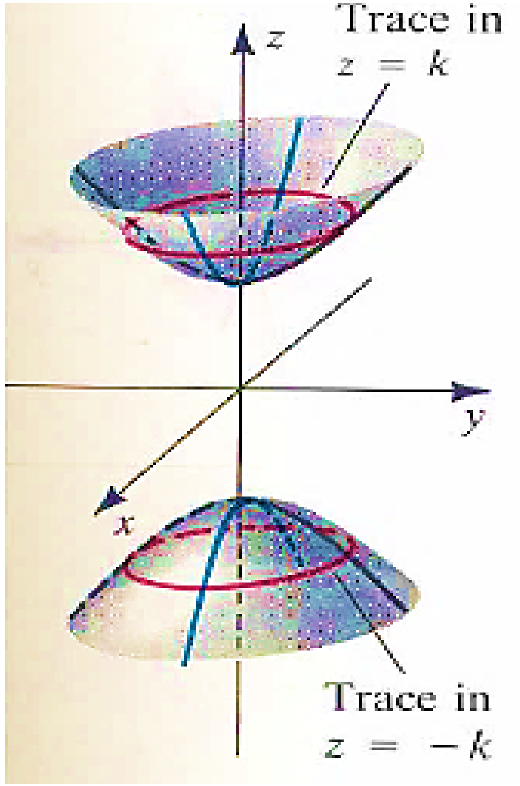
Trace on plane parallel to $xy-$ plane, $z=\pm k$ such that $k>c$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1+\frac{k^2}{c^2}.$$ Ellipse

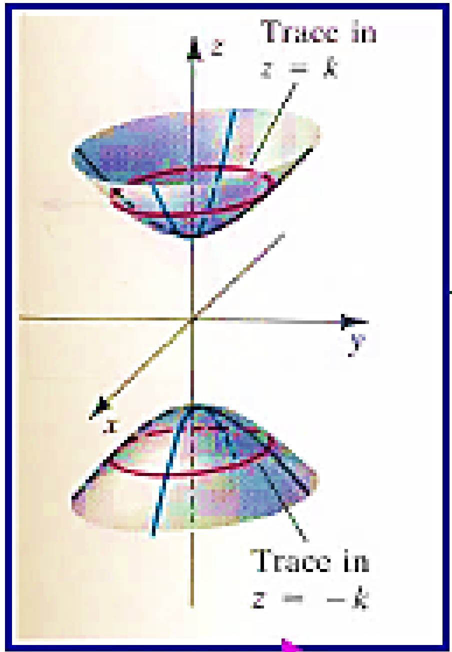
Hyperboloid of two sheets

Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	None	No graph
yz-trace	$-\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Hyperbola	
xz-trace	$-\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$	Hyperbola	

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



Hyperboloid of two sheets



Observations

- Equation $\boxed{-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$
- Form $f(x, y, z) = 1$ with
  - all variables quadratic
  - two variable with negative coefficient

- Axis of symmetry corresponds to variable with positive coefficient
- Opens along axis of symmetry

Example Identify the surface given by $x^2 - 4y^2 - 2z^2 = 4$.

Elliptic Paraboloid

Problem: Identify the surface by sketching out traces on different coordinate planes.

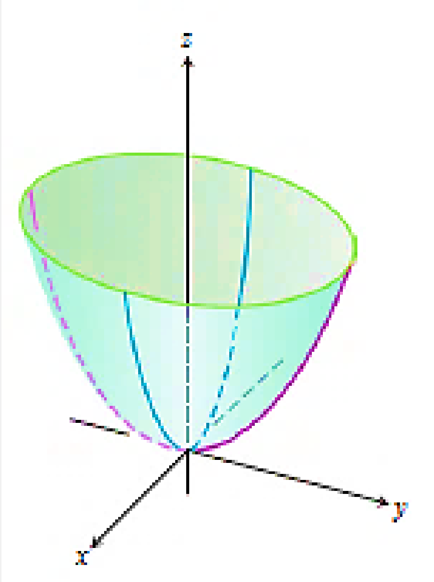
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}.$$

Solution: The coordinate with power 1, z –axis, is the axis of elliptic Paraboloid. Paraboloid will lie above xy –plane if c is positive and below xy –plane if c is negative.

xy – Trace, $z=0$; $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=0$ origin

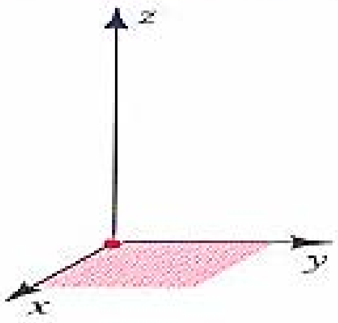
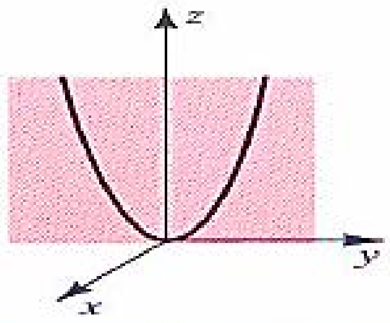
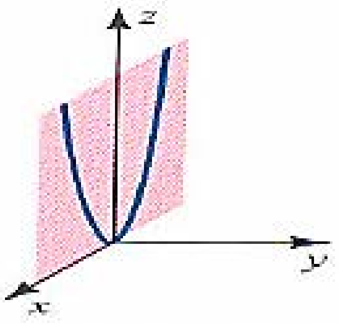
yz – Trace, $x=0$; $\quad z=c\frac{y^2}{b^2}$ Parabola

zx – Trace, $y=0$; $\quad z=c\frac{x^2}{a^2}$ Parabola

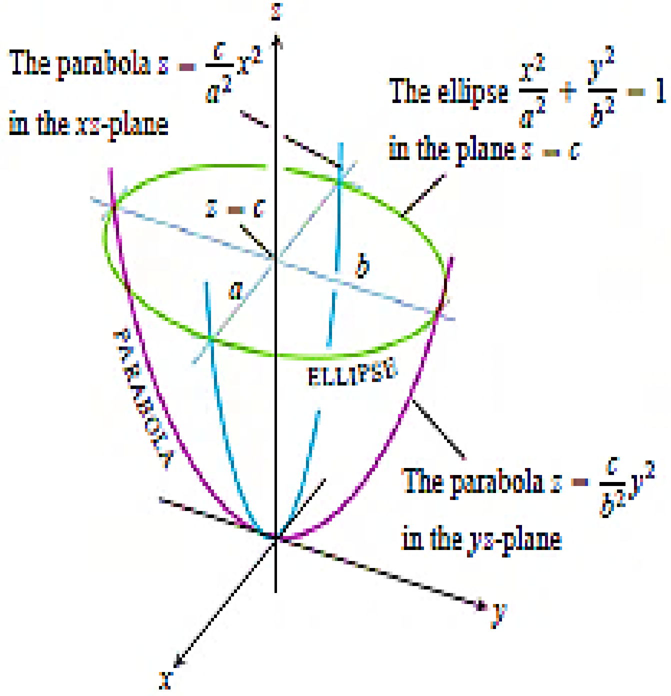


Trace on plane parallel to xy –plane, $z=k$ is an ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k}{c}$.

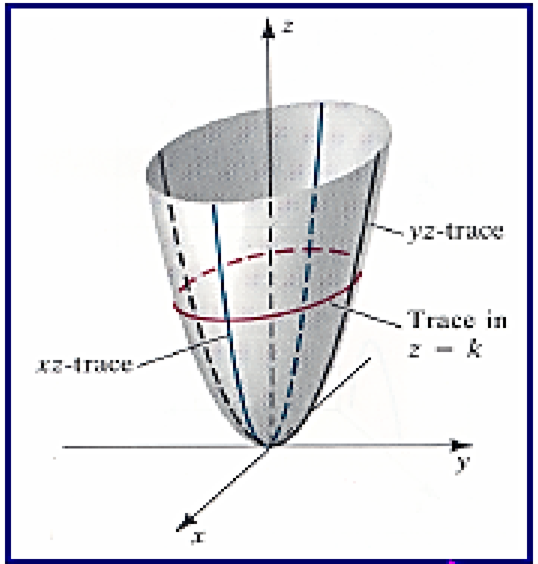
Elliptic Paraboloid

Trace	Description of trace	Sketch of trace
xy-trace	Origin	
yz-trace	Parabola	
xz-trace	Parabola	

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$



Elliptic Paraboloid



Observations

- Equation $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = z}$ with
  - one linear variable
  - two quadratic variables with same sign

- Axis of symmetry corresponds to variable which is linear

Example Describe the surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

Elliptic Paraboloid

Paraboloid with axis along z –axis:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

Paraboloid with axis along x –axis:

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x}{a}$$

Paraboloid with axis along y –axis:

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = \frac{y}{b}$$

Techniques For Identifying And Sketching Quadric Surfaces

- Bring the given equation in one of the standard forms
- Compare with standard equations to identify the surface
- Use the graph of standard form to complete the sketch

Complete the square (if needed)

Use the idea of translation of surfaces (when needed)

Translation of Surfaces

A change from (x, y, z) to $(x-h, y-k, z-l)$ means the surface has translated

- h units along X-axis
- k units along Y-axis
- l units along Z-axis

Note: h, k, l may be negative

Example:

$\frac{(x-1)^2}{a^2} + \frac{(y+2)^2}{b^2} + \frac{z^2}{c^2} = 1$ means an ellipsoid with centre at $(1, -2, 0)$.

Summary of Ideas: Cylinders and Quadric Surfaces

- **Cylinders** are surfaces that are created from parallel lines (called **rulings**). They come in more shapes than the "soda can" shape we are most familiar with. They are created when on variable is allowed to be anything.
- **Quadrics** are 3-dimensional analogs of *conics*. They are
  - Spheres
  - Elliptic Paraboloids
  - Cones
  - Hyperboloids of one sheet
  - Hyperboloids of two sheets
  - Hyperbolic Paraboloid (or Saddle)
- To graph these, we must figure out the center and the orientation.

Practice Questions

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 12

Exercise-12.6: Q – 1 to 44.