Gradient, Divergence & Curl

Vector Calculus(MATH-243)

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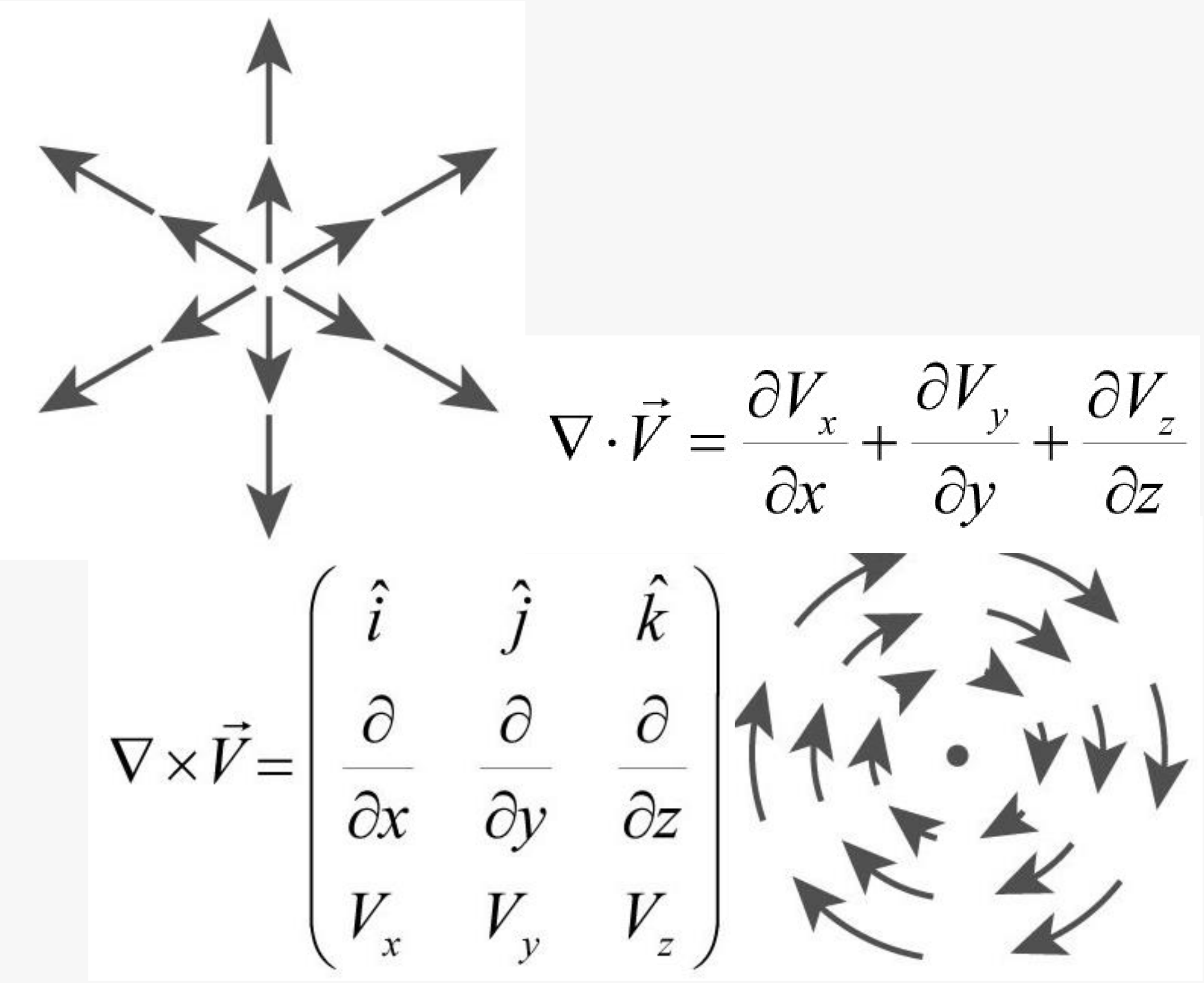
DIVERGENCE & CURL OF A VECTOR FIELD



$$\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\nabla \times \vec{V} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{pmatrix}$$

16

Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.5**

Gradient

A gradient is a vector differential operator on a scalar field like temperature. Every point in space having a specific temperature. The gradient is a differential operator that gives us a vector field, which in every point shows us in what direction in $2-$ or $3-$ dimensional space the field of values is ***increasing* the fastest**. By moving in the opposite direction of the gradient you are seeing the fastest decline of the scalar value, say temperature. By moving perpendicularly to the gradient, we are staying at the same scalar value. So, gradient operates on a scalar field to give a vector field.

Definition:

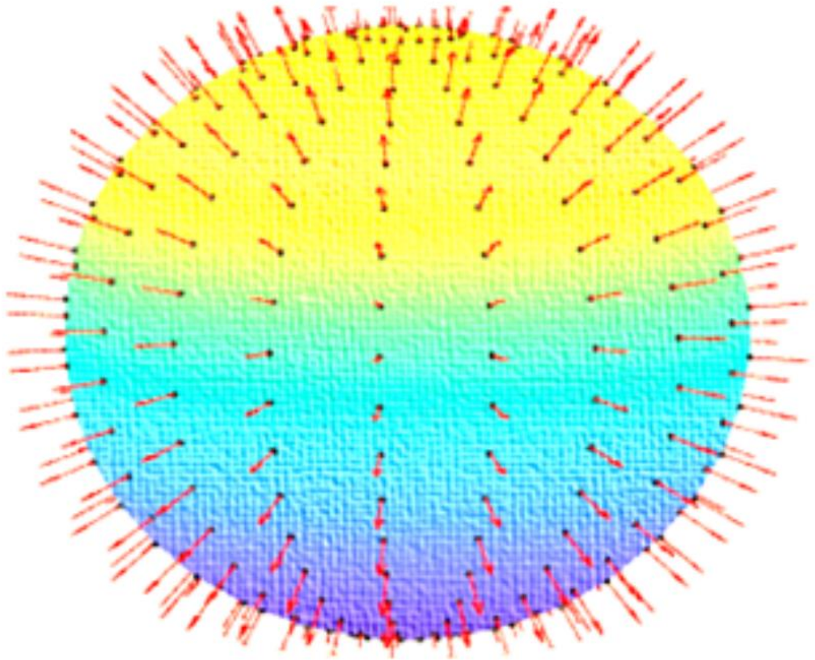
Let $f(x, y, z)$ be a differentiable scalar function on a region in $\mathbb{R}^3$. Let $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ (the "del" or "nebla" operator), then the **gradient** of the function f is defined as:

$$\operatorname{grad} f = \nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle,$$

which is a vector quantity. This is the reason we say that grad f is a **vector field**.

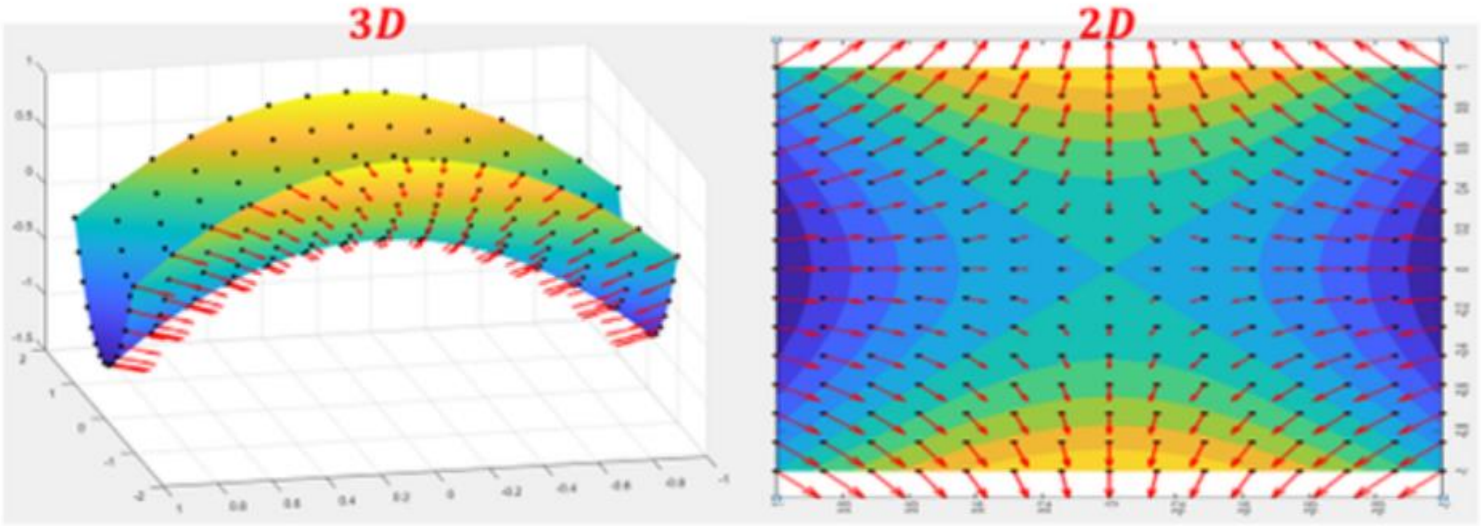
Physical Significance of Gradient

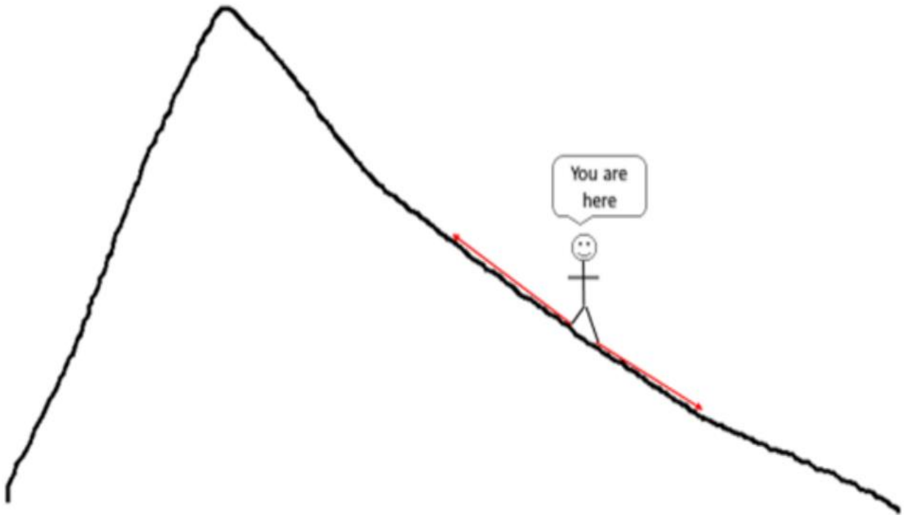
The **gradient** is a fancy word for derivative, or the rate of change of a function. If we imagine standing at a point in the input space of f, the vector ∇f tells us in which direction we should travel to increase the value of f most rapidly. Thus, the gradient at any location points in the direction of **greatest increase** of a function. The gradient does **not** give us the coordinates of where to go; it gives us the **direction to move** to increase our temperature. Consider a ball. Now take any point on the ball and imagine a vector acting perpendicular to the ball on that point. That is a gradient vector in 3D. Now imagine vectors acting on all points of the ball. The red arrows perpendicular to the surface of the ball are the gradients (in 3D) of various points on the ball.



Physical Significance of Gradient

If we apply gradient function to a 2D structure, the gradients will be **tangential to the surface**. For a better understanding of gradient representation in 2D, consider that we are climbing a mountain. We are at a certain point on the mountain. Now, we have to move in a certain direction to gain or lose altitude (shown in red arrows). These vector lines represent the gradients at our point of location in 2D. If we will represent these gradients in 3D form, they will be perpendicular to the surface of the mountain.





Note: In 3D form, gradients are **surface normal** to particular points. In 2D format, gradient tangent represent the direction of steepest descent or ascent.

Divergence

Divergence is a differential operator that acts on a vector field to give a scalar field, so the opposite of gradient. If we have a vector field like a gravitational field or an electric field, by taking the divergence we get the scalar field that is proportional to the mass or charge density "causing" that field. Everywhere in space where we *do* have gravitational or electric fields, but where *no* mass or charge is located, the divergence is zero.

Definition:

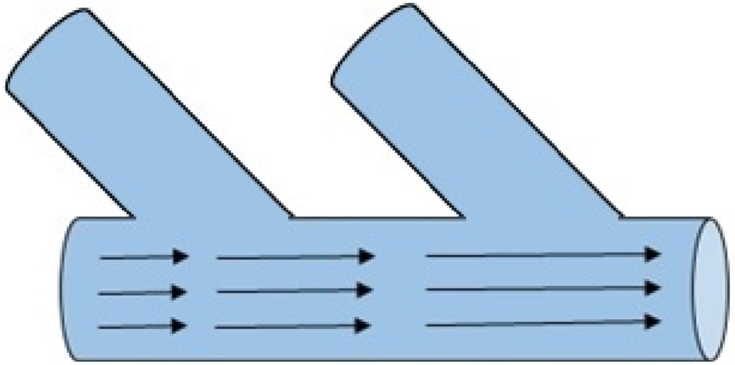
Let $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ be a differentiable vector field on a region in $\mathbb{R}^3$. Let $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ (the "del" or "nebla" operator), then the **divergence** of the vector field is defined as:

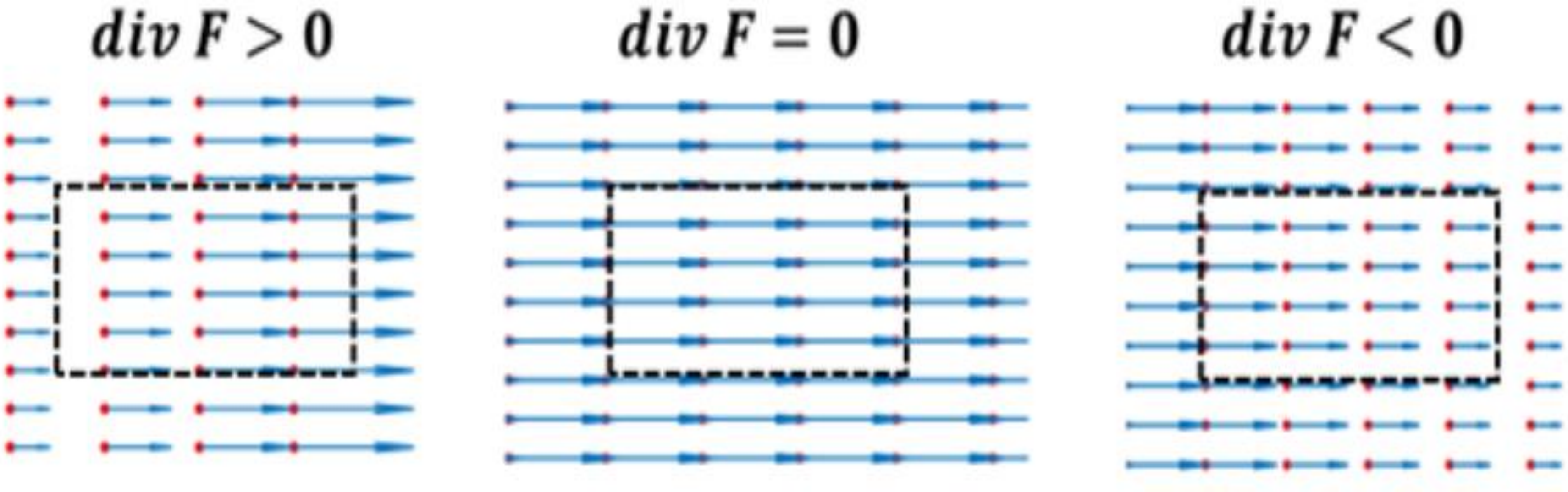
$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

which is a scalar quantity. This is the reason we say that $\operatorname{div}\mathbf{F}$ is a **scalar field**.

Physical meaning of Divergence

Consider water flowing through a large pipe. Now, it has smaller pipes joined to it. Hence, as the water flows, more water is added along the way by the smaller pipes. Hence, the mass flow rate increases as the water flows. In another case, consider that there is a leakage in the pipe. Hence the mass flow rate decreases as it flows. This change in the flow rate through the pipe, whether it increases or decreases, is called as **divergence**. Divergence denotes only the magnitude of change and so, it is a scalar quantity. It does not have a direction. When the initial flow rate is less than the final flow rate, **divergence is positive** (divergence > 0). If the two quantities are same, **divergence is zero**. If the initial flow rate is greater than the final flow rate **divergence is negative** (divergence <0).





Curl

The curl is a differential operator acting on a vector field to give another vector field. Curl alludes to something round or rotation, which is why it is also called rotation, abbreviated by rot. The curl measures the net boost that an element affected by the vector field that the curl operator acts upon would get when going in a small closed loop in a specific plane. This boost yields the curl vector component perpendicular to the specific plane that the local small loop, for which the boost is measured, lies in.

Definition:

Let $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ be a vector field on $\mathbb{R}^3$ and assume that the partial derivatives of P, Q and R all exist. Let $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ (the "del" or "nebla" operator), then the **curl** of the vector field is defined as:

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle,$$

which is a vector quantity. This is the reason we say that curl $\mathbf{F}$ is a **vector field**.

Physical meaning of Curl

Imagine pouring water in a cup. The water won't just low linearly but rather, as it reaches the end of the cup, it will flow in a rotational motion before settling in the cup. Or consider water draining down the sink, it will swirl in a rotational motion before going out. If we plot this rotational flow of water as vectors and measure it, it will denote the **curl**. Curl is a *measure of how much a vector field circulates or rotates about a given point*. Curl gives the measure of angular velocity of an object. If curl is zero, it means the object is not rotating. When the flow is counter-clockwise, curl is considered to be positive and when it is clock-wise, curl is negative.

Potential

- In nature we observe fundamental forces which we describe via fields.

- A **field** describes the force that a particle affected by the field will undergo at every point in space within the field.

- When a force is exerted on a particle over a distance this requires energy; thus, we can think of **potential** *as the energy arising from fields that causes motion and the expenditure of work on particles in the field*.

- Thus, potential means the ability of a body, or the energy stored in a body to do some work.

- In physics, a potential may refer to the scalar potential or to the vector potential. In either case, it is a field defined in space, from which many important physical properties may be derived. Leading examples are the gravitational potential and the electric potential, from which the motion of gravitating or electrically charged bodies may be obtained.

Potential (Examples)

- If we hold a baseball off the edge of a tall building it has gravitational potential energy associated with the earth’s mass. The earth produces a gravitational field which will exert a force on any other mass with a magnitude depending on its distance from the centre of the earth. Thus, it exerts a force on the baseball which acts as it travels towards the ground. The work done in this process is equal to the difference in potential the baseball has in the earth’s gravitational field between the top of the building and the ground.

- An electron in a vacuum will produce an electric field in all directions. Another charged particle will experience a force whose magnitude will depend on its location in the field. A positive charge will experience a force directed towards the electron and the work done in its motion towards the electron is equal to the potential at the point it started (if the potential at the electron is defined as 0).

Potential Function and Conservative Vector Field

- A **potential function** for a vector field $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ is a scalar function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$. A vector field $\mathbf{F}$ is **conservative** if it has a potential function f. Not every vector field is the gradient of some function. However electrostatic and gravitational fields are. The reason such fields are called *conservative* is that they model forces of physical systems in which energy is conserved.
- A region R in $\mathbb{R}^2$ or $\mathbb{R}^3$ is **connected** if every pair of points in R can be connected with a continuous curve that lies entirely within R. A region R is **simply connected** if every simple (not self-crossing) closed (forms a loop) curve lying entirely in R can be continuously contracted to a point (in essence, this means the region *doesn't have any holes*).
- An easy way to check whether a vector field is conservative without finding the potential function is the following:

  Theorem: Let $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ be a vector field where P, Q, and R have continuous first partial derivatives. Suppose $\mathbf{F}$ is defined on a connected and simply connected region D in $\mathbb{R}^3$. Then, $\mathbf{F}$ is conservative on D if and only if $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$ or $\langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 0,0,0\rangle$ i.e.,

  $$R_y = Q_z, \qquad P_z = R_x \qquad \text{and} \qquad Q_x = P_y.$$

  For $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ in $\mathbb{R}^2$, we just need $Q_x = P_y$.

Procedure for finding Potential Function

Given a vector field $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$, such that $\mathbf{F} = \nabla f$. In order to find a potential function f for $\mathbf{F}$ we proceed as:

1. Take $\int f_x \, dx$. We get $f = \int f_x \, dx + g(y, z)$, where $f_x = P$ and $g(y, z)$ is a function of y and z, an "integration constant" for our multivariable function f.
2. Take f_y and compare with Q (they should be equal) to solve for $g_y(y, z)$.
3. Take $\int g_y(y, z) \, dy$. We get $g(y, z) = \int g_y(y, z) \, dy + h(z)$, where $h(z)$ is a function of z that we're treating as an "integration constant" for our multivariable function g.
4. Take f_z and compare with R to solve for $h'(z)$ and therefore $h(z)$ (up to a constant).
5. Putting all these pieces together completely solves for the potential function f.

Practice:

For the vector field

$$\mathbf{F}(x, y, z) = \langle 12x^2, \cos y \cos z\,, 1 - \sin y \sin z \rangle$$

determine the following:

a) The divergence of $\mathbf{F}$.

b) The curl of $\mathbf{F}$.

c) Is the given field a conservative field? If yes, then determine the corresponding potential function.

Laplace Operator

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f. If f is a function of three variables, we have:

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

This expression occurs so often that we abbreviate it as $\nabla^2 f$.

The operator $\nabla^2 = \nabla \cdot \nabla$ is called the **Laplace operator** due to its relation to Laplace's equation:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Vector Differential Identities

If $\boldsymbol{F}$ and $\boldsymbol{G}$ are vector fields and φ and ψ are scalar fields then

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\varphi) = \nabla^2\varphi$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) = 0$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\varphi) = 0$$

$$\boldsymbol{\nabla}(\varphi\psi) = \varphi\boldsymbol{\nabla}\psi + \psi\boldsymbol{\nabla}\varphi$$

$$\boldsymbol{\nabla}\cdot(\varphi\boldsymbol{F}) = \varphi\boldsymbol{\nabla}\cdot\boldsymbol{F} + \boldsymbol{F}\cdot\boldsymbol{\nabla}\varphi$$

$$\boldsymbol{\nabla}\times(\varphi\boldsymbol{F}) = \varphi\boldsymbol{\nabla}\times\boldsymbol{F} + \boldsymbol{\nabla}\varphi\times\boldsymbol{F}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F}$$

$$\boldsymbol{\nabla}(\boldsymbol{F}\cdot\boldsymbol{G}) = \boldsymbol{F}\times(\boldsymbol{\nabla}\times\boldsymbol{G}) + \boldsymbol{G}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) + (\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{G} + (\boldsymbol{G}\cdot\boldsymbol{\nabla})\boldsymbol{F}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{G}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) - \boldsymbol{F}\cdot(\boldsymbol{\nabla}\times\boldsymbol{G})$$

$$\boldsymbol{\nabla}\times(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{F}(\boldsymbol{\nabla}\cdot\boldsymbol{G}) - \boldsymbol{G}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) + (\boldsymbol{G}\cdot\boldsymbol{\nabla})\boldsymbol{F} - (\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{G}$$

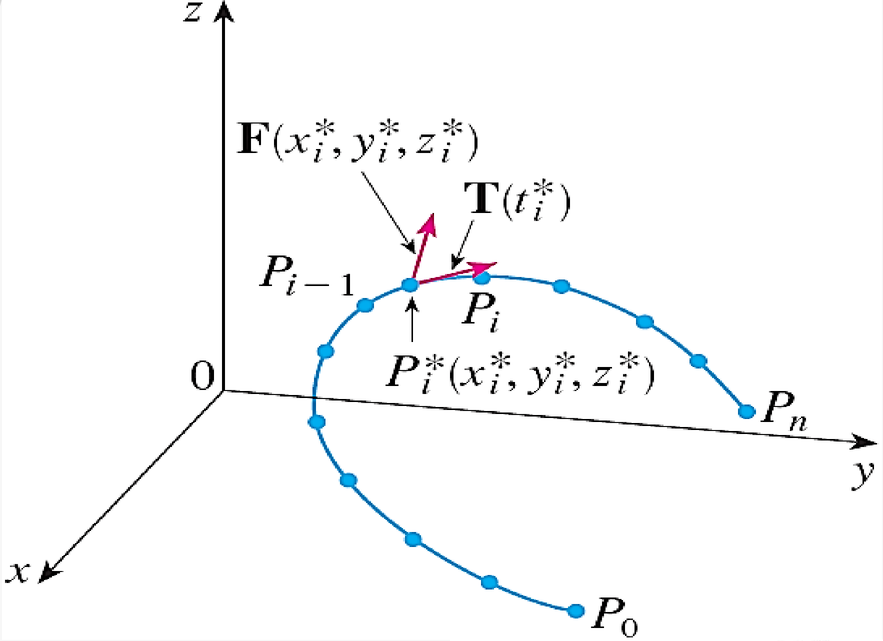
Practice Questions

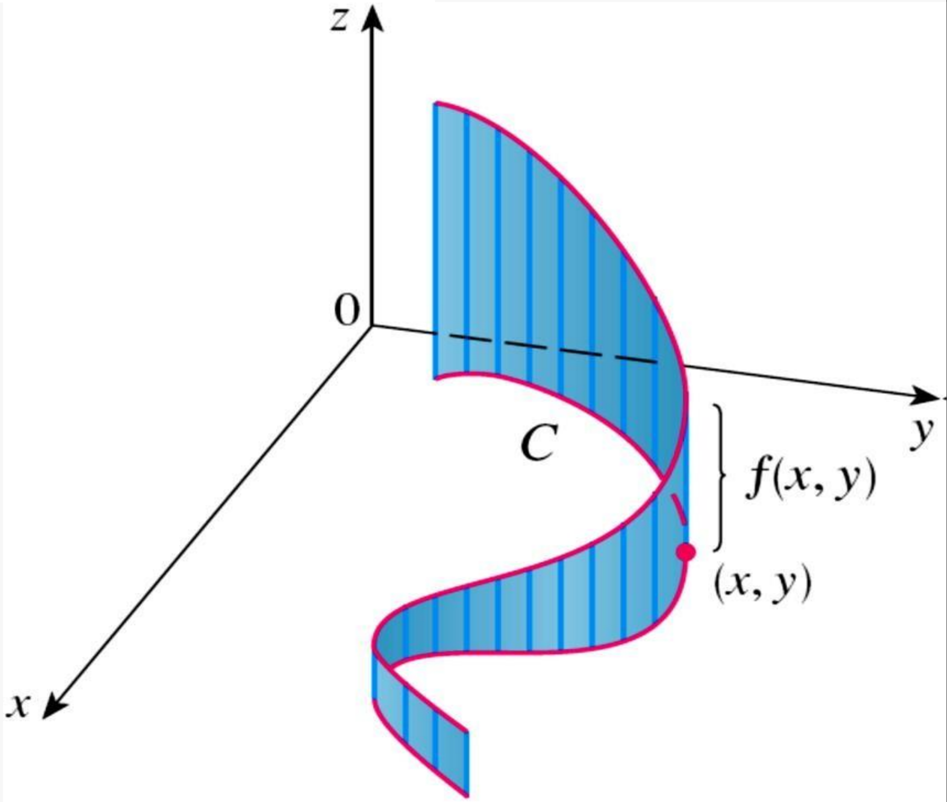
Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

Chapter: 16

Exercise-16.5: Q – 1 to 18, Q – 23 to 32.

Line Integrals





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Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.2**

Line Integrals

- We now define an integral that is similar to a single integral except that, instead of integrating over an interval $[a, b]$,we integrate over a curve C.
- Such integrals are called **line integrals** or **path integrals**. However, “curve integrals” would be better terminology.
- These integrals were introduced in the early 19th century to solve problems involving:
  - Fluid flow
  - Forces
  - Electricity
  - Magnetism

Line Integrals of Scalar Fields

We start with a plane curve C given by the parametric equations:

$$x = x(t), \qquad y = y(t); \qquad a \leq t \leq b. \quad (1)$$

Equivalently, C can be given by the vector equation:

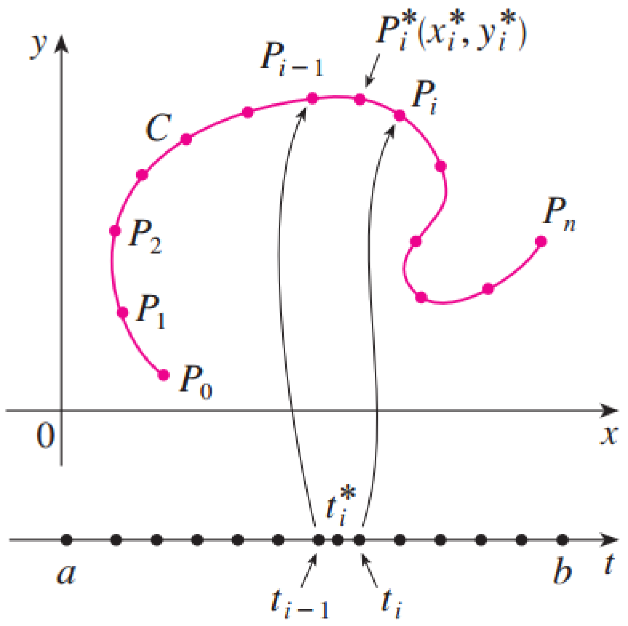
$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = x(t)\,\mathbf{i} + y(t)\mathbf{j}.$$

We assume that C is a smooth curve. This means that $\mathrm{r}'(t)$ is continuous and $r'(t) \neq 0$.

If we divide the interval $[a, b]$ into n subintervals of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into subarcs with lengths $\Delta s_1, \Delta s_2, \dots, \Delta s_n$. We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc. (This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.) Now if $f(x, y)$ is any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum:

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i\,,$$

which is similar to a Riemann sum. Then we take the limit of these sums and make the following definition by analogy with a single integral.



Line Integrals of Scalar Fields

If $f(x,y)$ is defined on a smooth curve C given by (1), the **line integral of f along C** is:

$$\int_C f(x,y)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*,y_i^*)\Delta s_i\,, \qquad (2)$$

provided this limit exists. If $s(t)$ is the length of C between $r(a)$ and $r(t)$, then

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Then, this formula can be used to evaluate the line integral.

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt. \qquad (3)$$

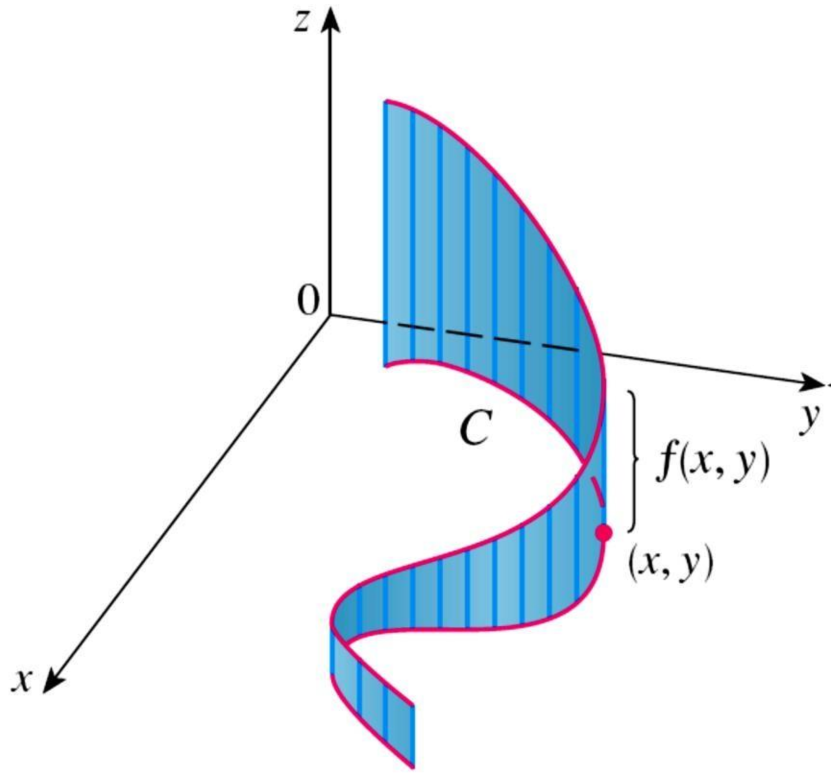
The value of the line integral does not depend on the parametrization of the curve provided the curve is traversed exactly once as t increases from a to b.

Line Integrals of Scalar Fields

Just as for an ordinary single integral, we can interpret the line integral of a positive function as an area. In fact, if $f(x, y) \geq 0$, then

$$\int_C f(x, y)\, ds,$$

represents the area of one side of the "fence" or the curved "curtain" below the surface and above C.

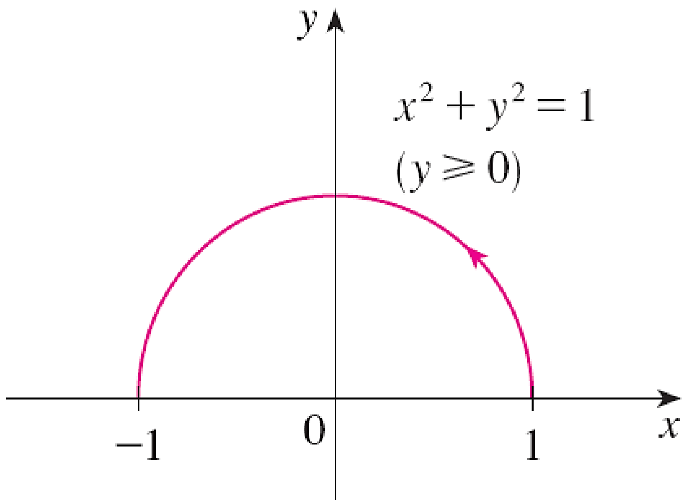


Example:

Evaluate

$$\int_C (2 + x^2 y)\, ds,$$

where C is the upper half of the unit circle $x^2 + y^2 = 1$.



Solution:

The unit circle can be parameterized by means of the equations:

$$x = \cos t\,, \qquad y = \sin t\,. \quad (*)$$

Also, the upper half of the circle is described by the parameter interval: $0 \le t \le \pi$. For the present case we have $f(x, y) = 2 + x^2 y$. Using $(*)$, we get:

$$f(x(t), y(t)) = 2 + \cos^2 t \sin t\,, \qquad \frac{dx}{dt} = -\sin t \;\text{ and }\; \frac{dy}{dt} = \cos t.$$

Solution:

Thus, by using formula (3):

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt,$$

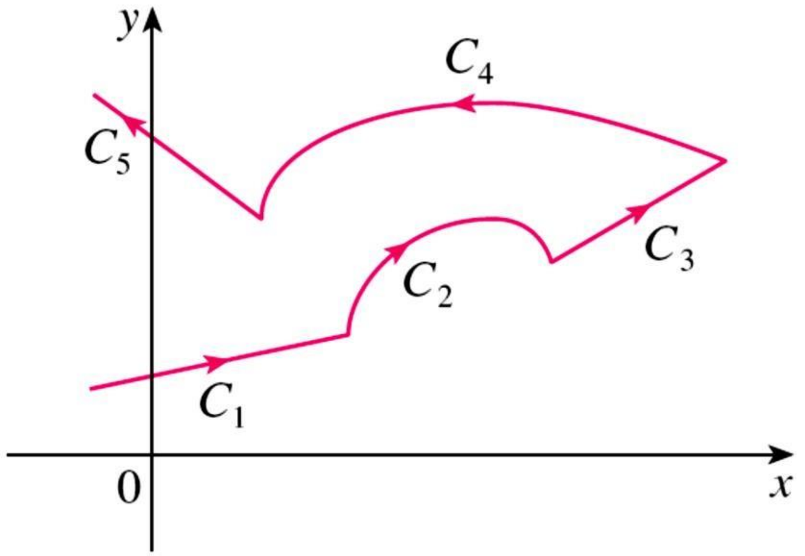
we get:

$$\begin{aligned}
\int_C \left(2+x^2y\right)ds &= \int_0^\pi \left(2+\cos^2 t\sin t\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^\pi \left(2+\cos^2 t\sin t\right)\sqrt{\sin^2 t+\cos^2 t}\,dt \\
&= \int_0^\pi \left(2+\cos^2 t\sin t\right)dt \\
&= \left[2t-\frac{\cos^3 t}{3}\right]_0^\pi = 2\pi+\tfrac{2}{3}
\end{aligned}$$

Piecewise-smooth Curve

Now, let C be a piecewise-smooth curve. That is, C is a union of a finite number of smooth curves $C_1, C_2, \dots, C_n$, where the initial point of C_{i+1} is the terminal point of C_i.Then, we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C as:

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds. \quad (4)$$



Example:

Evaluate

$$\int_C (2x)\, ds,$$

where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

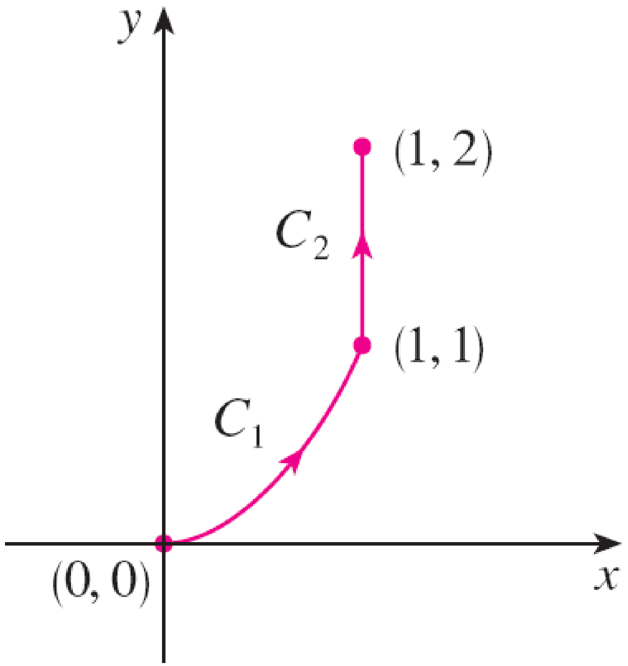
Solution:

Parametric equations for C_1 are:

$$x = t, \qquad y = t^2; \quad 0 \le t \le 1$$

and parametric equations for C_2 are:

$$x = 1, \qquad y = 1 + t; \quad 0 \le t \le 1.$$



Solution:

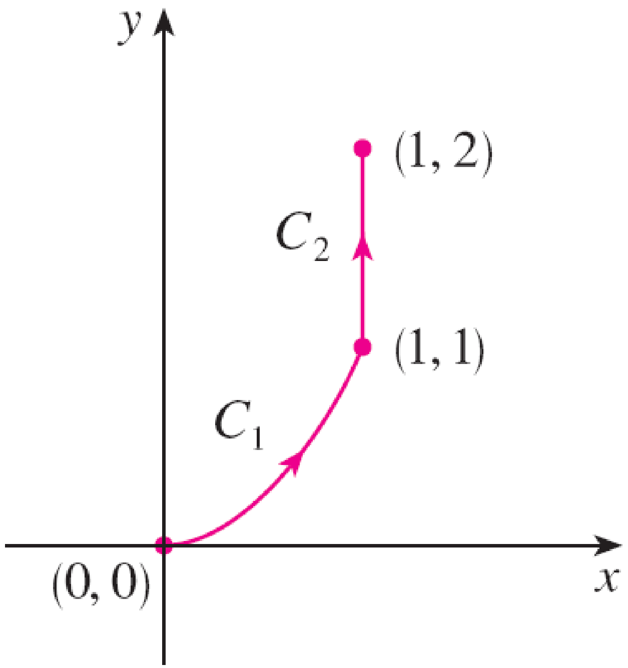
Therefore, for C_1 we have:

$$\int_{C_1} 2x\,ds = \int_0^1 2t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^1 2t\sqrt{1+4t^2}\,dt$$

$$= \tfrac{1}{4}\cdot\tfrac{2}{3}\left(1+4t^2\right)^{3/2}\Big]_0^1$$

$$= \frac{5\sqrt{5}-1}{6}$$



Solution:

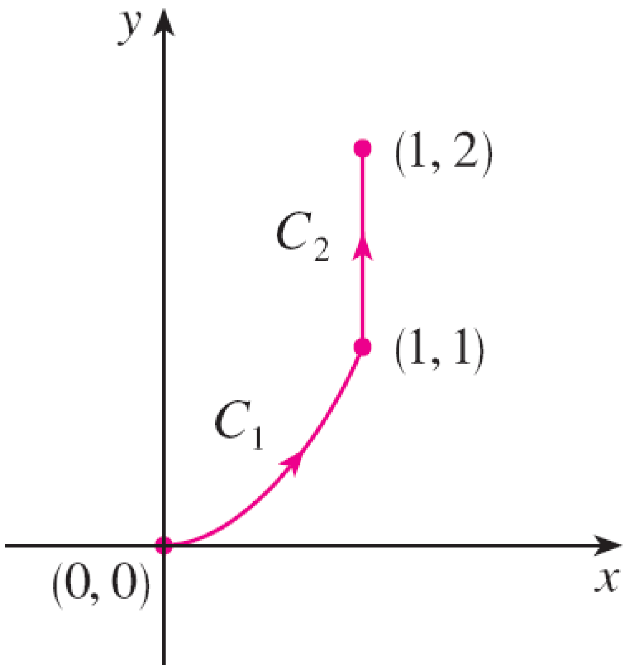
and for C_2 we have:

$$\int_{C_2} 2x\,ds = \int_0^1 2(1)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 2\,dt = 2$$

Thus,

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$$

$$= \frac{5\sqrt{5}-1}{6} + 2$$



Line Integrals of Scalar Fields

Any physical interpretation of a line integral:

$$\int_C f(x, y)\, ds,$$

depends on the physical interpretation of the function f. For instance, $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C.

Mass and Center of Mass

If $\rho(x, y)$ represents the density of a semicircular wire, then mass of the wire is given by:

$$M = \int_C \rho(x, y)\, ds. \qquad (5)$$

The center of mass of the wire with density function ρ is located at the point $(\bar{x}, \bar{y})$, where:

$$\left.\begin{aligned} \bar{x} &= \frac{1}{M}\int_C x\rho(x, y)\, ds, \\ \text{and} \qquad \bar{y} &= \frac{1}{M}\int_C y\rho(x, y)\, ds. \end{aligned}\right\} \qquad (6)$$

Example:

A wire takes the shape of the semicircle: $x^2 + y^2 = 1, y \geq 0$, and is thicker near its base than near the top. Find the mass of the wire and center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

Solution:

Here we use the parametrization:

$$x = \cos t\,,\ y = \sin t\,; \quad 0 \leq t \leq \pi.$$

Moreover, given that the linear density is proportional to its distance from the line $y = 1$, so $\rho(x, y) = k(1 - y)$ where k is a constant. Thus, the mass of the wire is given as:

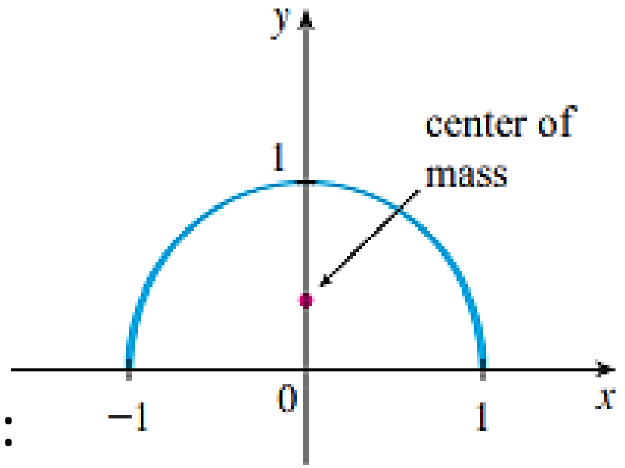
$$M = \int_C \rho(x, y)\, ds = \int_C k(1 - y)\, ds = \int_0^\pi k(1 - \sin t)\, dt = k[t + \cos t]_0^\pi = k(\pi - 2).$$

Solution:

In order to determine the center of mass of the wire we proceed as:

$$\bar{y} = \frac{1}{M}\int_C y\rho(x,y)\,ds = \frac{1}{k(\pi-2)}\int_C ky(1-y)\,ds = \frac{1}{\pi-2}\int_0^{\pi}(\sin t - \sin^2 t)\,dt$$

$$= \frac{1}{\pi-2}\left[-\cos t - \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{\pi} = \frac{4-\pi}{2(\pi-2)}.$$



Observe that due to symmetry $\bar{x} = 0$, so the center of mass is given as:

$$(\bar{x},\bar{y}) = \left(0,\frac{4-\pi}{2(\pi-2)}\right) \approx (0{,}0.38).$$