Applications of Derivatives

Calculus & Analytical Geometry MATH- 101

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L'hopital's Rule

$$\lim_{x\to a} \frac{f(x)}{g(x)} \longrightarrow \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

$g'(x) \neq 0$

$0 \cdot \infty$ or $\infty \cdot 0$

$$\lim_{x\to a} [f(x) \cdot g(x)]$$

$$f(x)\,g(x) = \frac{f(x)}{1/g(x)}$$

$$\text{or } f(x)\,g(x) = \frac{g(x)}{1/f(x)} \longrightarrow \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

L'hopital's

L'hopital's Rule

$\infty - \infty$

$\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$

Indeterminate power

1. power

ln

exp

2. product

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

$0 \cdot \infty$

$$\left.\begin{aligned} f(x) &\to 0 \\ g(x) &\to \infty \end{aligned}\right\} x \to a$$

$$f(x)\,g(x) = \frac{f(x)}{1/g(x)}$$

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{1/g(x)} \left(\frac{0}{0}\right)$$

$$\lim_{x\to a} \frac{g(x)}{1/f(x)} = \frac{\infty}{\infty}$$

Using L'hopital's rule evaluate the following limit:

$$\lim_{x\to 1^+}\left[(x-1)\tan\left(\frac{\pi}{2}x\right)\right] \to \left[0 \cdot (-\infty)\right]$$

Sol:- As $x \to 1^+$, $(x-1) \to 0$, $\tan\left(\frac{\pi}{2}x\right) \to (-\infty)$

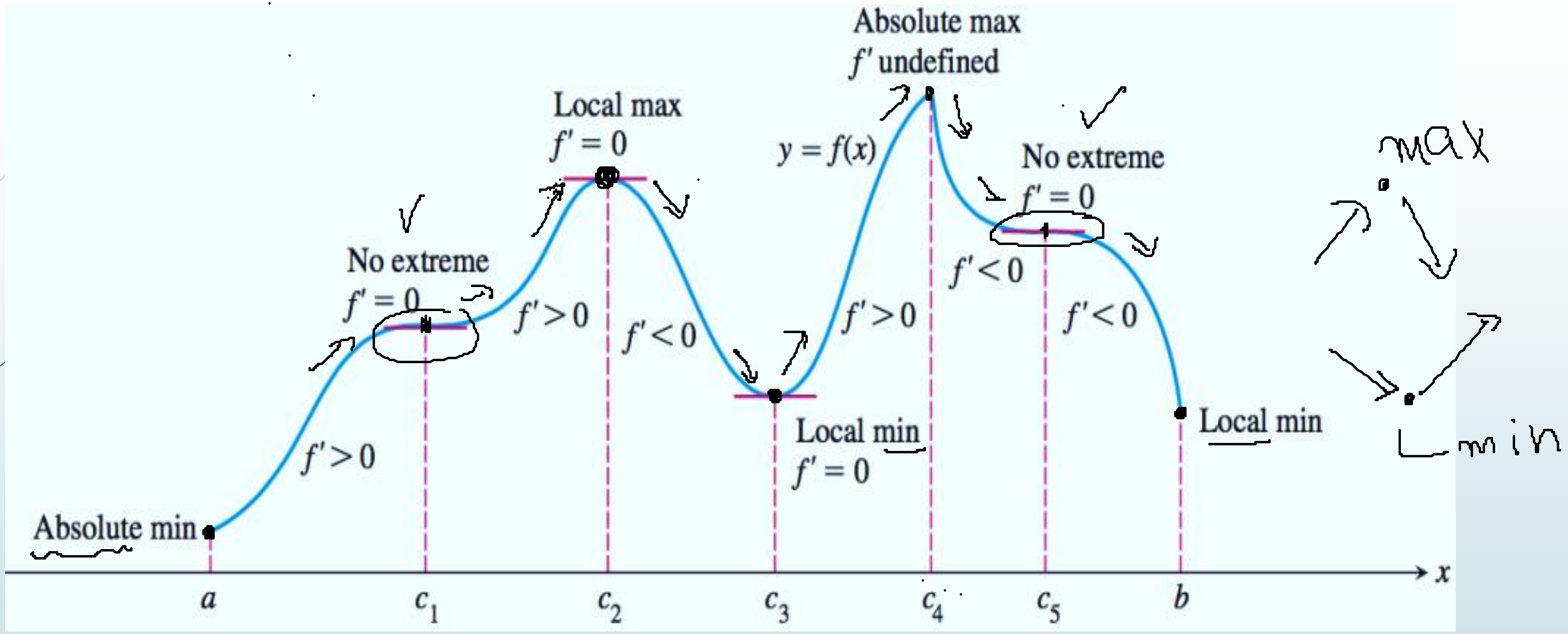
$$\lim_{x\to 1^+}\left[(x-1)\tan\left(\frac{\pi}{2}(x)\right)\right] = \lim_{x\to 1^+}\left[\frac{x-1}{1/\tan(\pi/2\, x)}\right]$$

$$= \lim_{x\to 1^+}\left[\frac{x-1}{\cot\left(\frac{\pi}{2}x\right)}\right] \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1^+}\left[\frac{1}{-\frac{\pi}{2}\csc^2\left(\frac{\pi}{2}x\right)}\right]$$

$$= \frac{1}{-\pi/2} = -\frac{2}{\pi}$$

Practice: Using L'hopital's rule evaluate: $\lim_{x\to 0^+}\left[[\cos(2x)]^{1/x^2}\right]$.



Extreme Values of Functions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

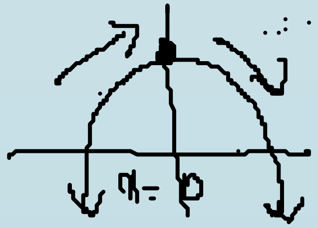
- Chapter: 4
  - Sections: 4.1

Extreme Values of Functions

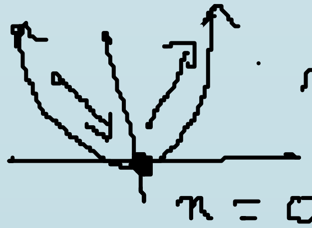
Extrema

min max

- Extreme Values of a function occur when the function changes from increasing to decreasing or from decreasing to increasing.
- An important practical problem for which differentiation can often provide quick and easy answers is that of finding the extreme values, that is maximum and minimum values of a continuous function.
- Once we can do this, we can solve a variety of ***optimization problems*** in which we find the optimal (best) way to do something in a given situation.
- While we can all visualize the minimum and maximum values of a function we want to be a little more specific in our work here. In particular we want to differentiate between two types of minimum or maximum values.



$f(0)$ is max



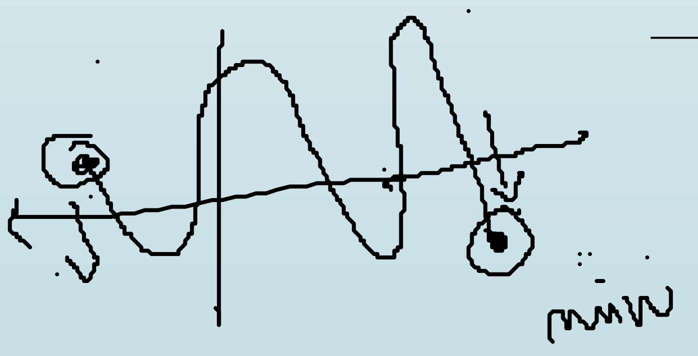
x^2

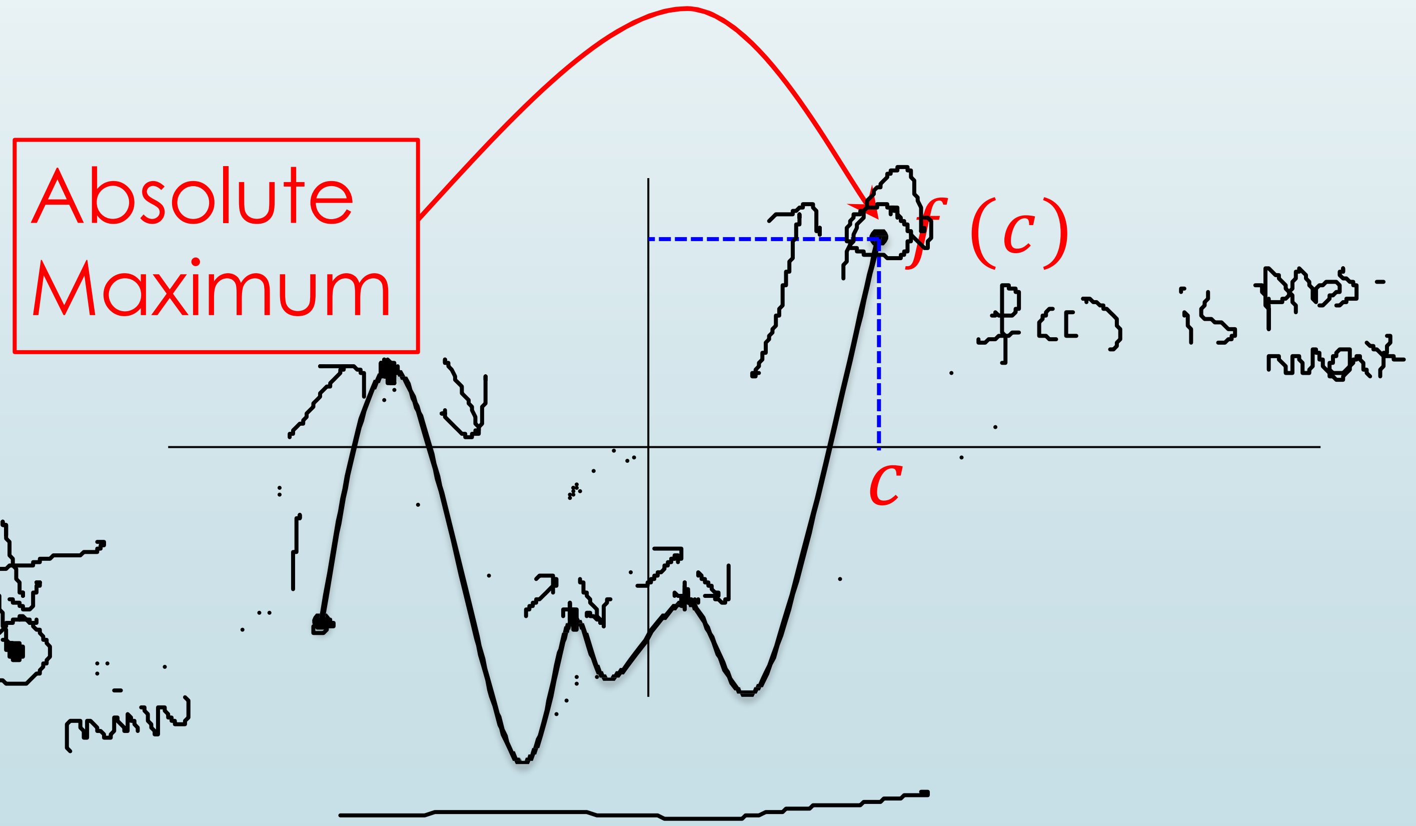
$f(0)$ min

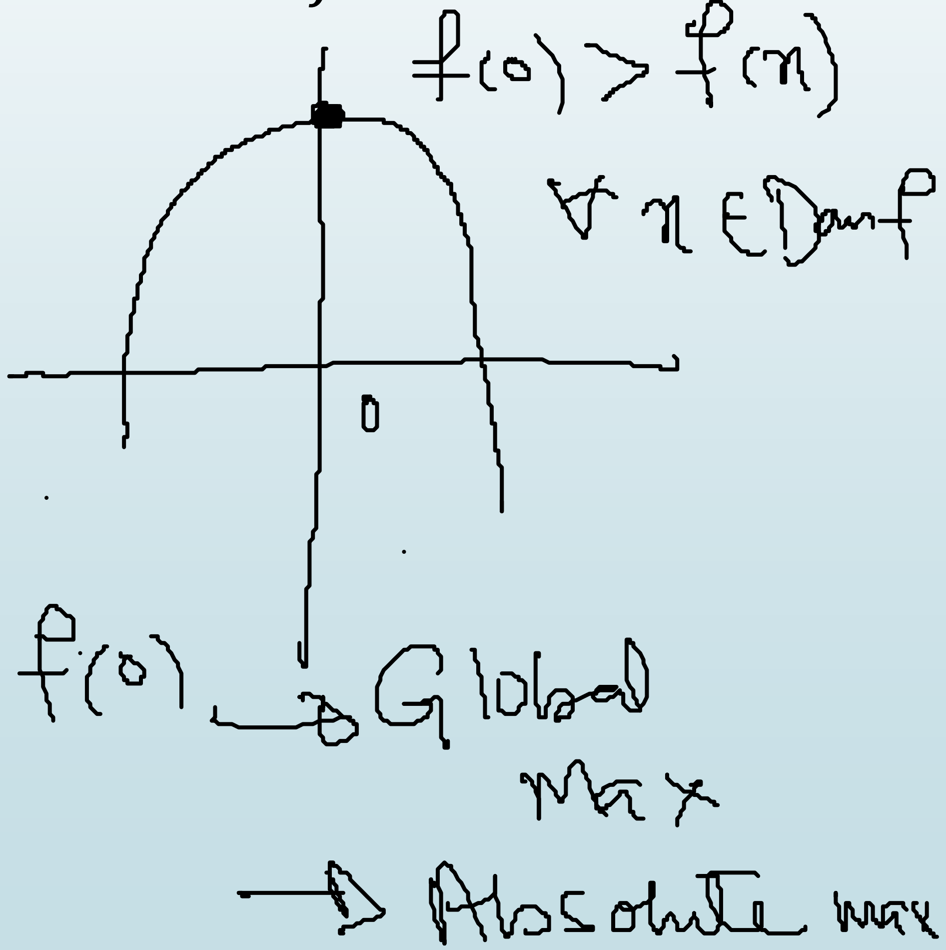
Absolute (global) maximum

A function f has an ***absolute (global) maximum*** at $x = c$ if $f(x) \leq f(c)$ **for all** x in the domain D of f.

The number $f\,(c)$ is called the ***absolute maximum*** value of f in D.



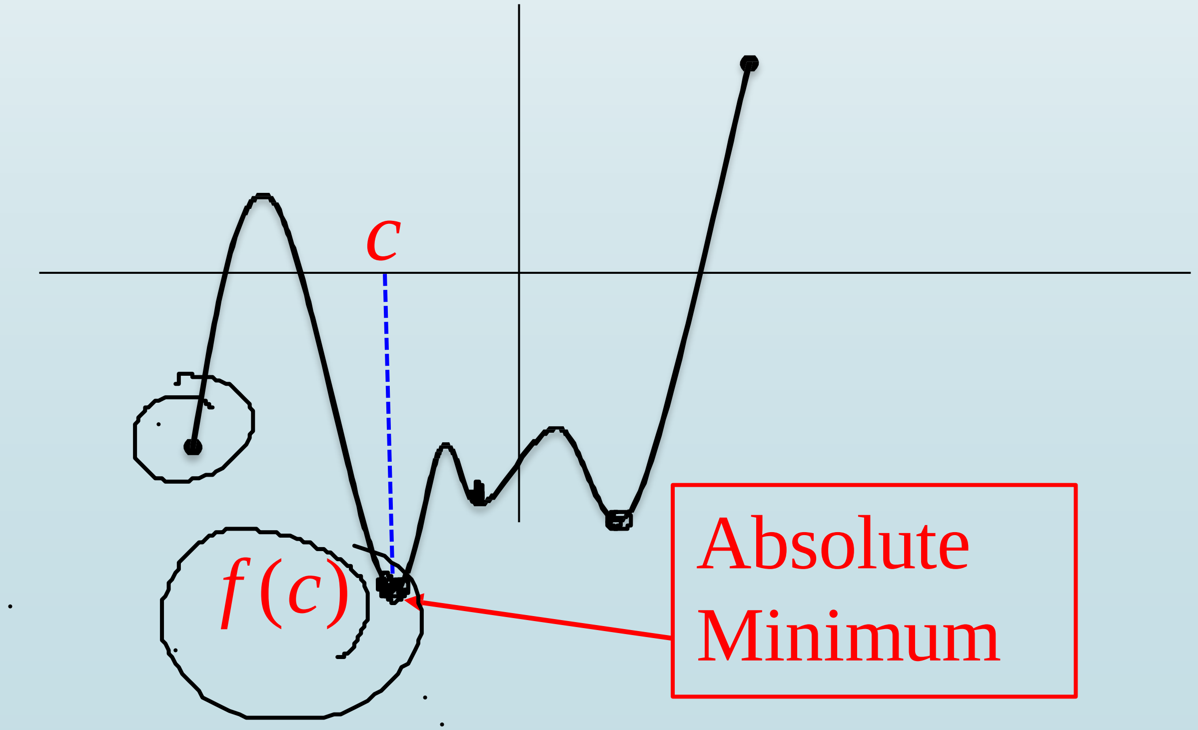




Absolute (global) minimum

A function f has an ***absolute (global) minimum*** at $x = c$ if $f(c) \le f(x)$ ***for all*** x in the domain D of f.

The number $f(c)$ is called the ***absolute minimum*** value of f in D.

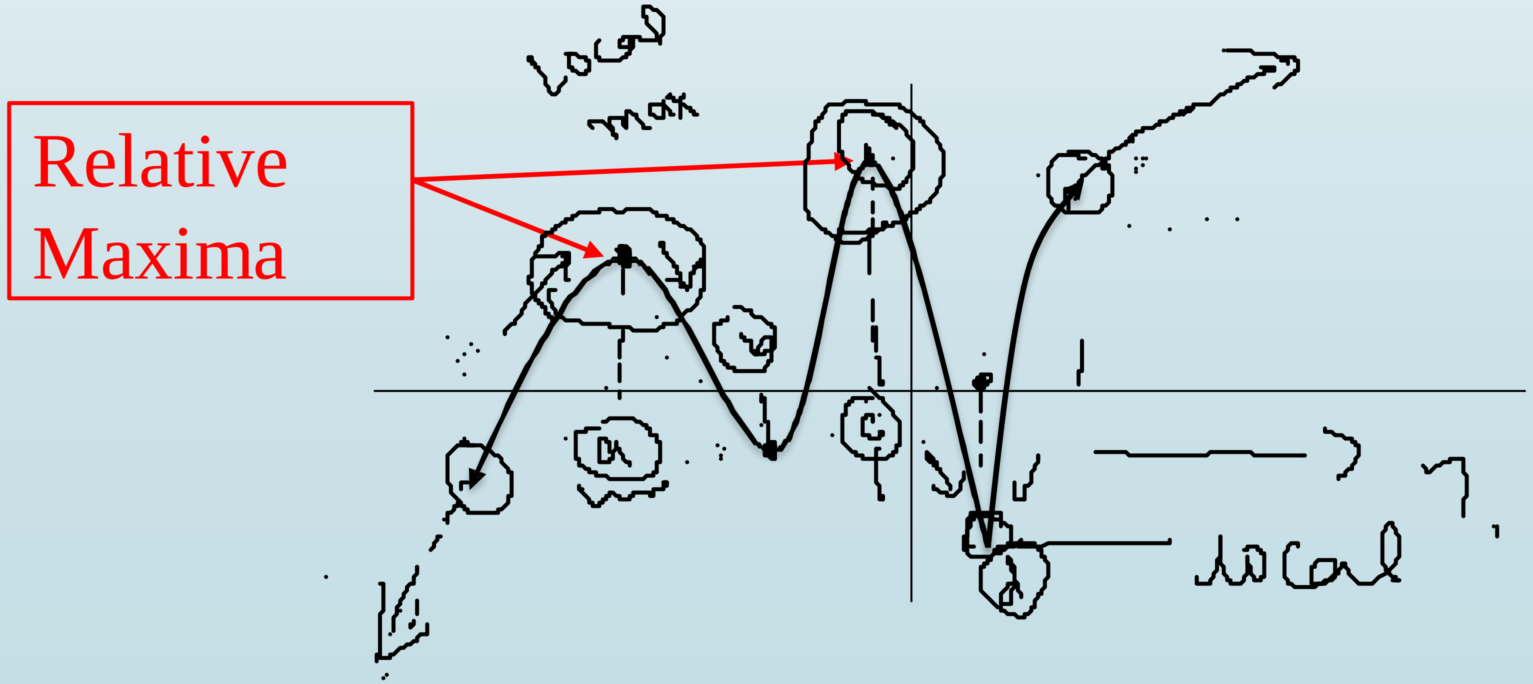


Relative (local) maximum

A function f has a ***relative (local) maximum*** at $x = c$ if there exists an open interval (r, s) containing c such that $f(x) \leq f(c)$ for all $r < x < s$.

No Abs Max

No Abs min

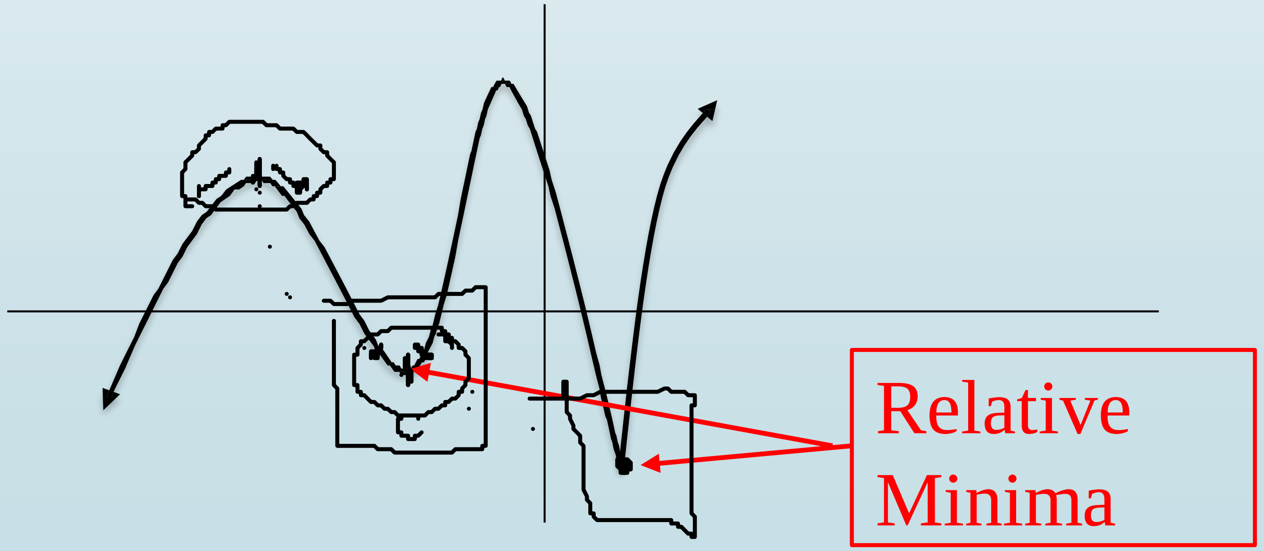


Relative (local) minimum

A function f has a ***relative (local) minimum*** at $x = c$ if there exists an open interval (r, s) containing c such that $f(c) \leq f(x)$ for all $r < x < s$.

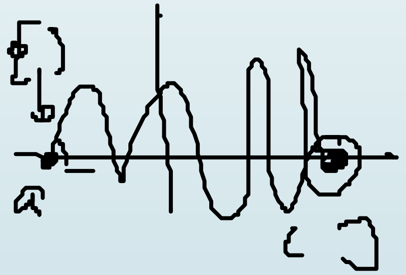
Absolute → Entire Domain

Local open interval



Extreme Values of Functions

- We say that $f(x)$ has an **absolute (or global) maximum** at $x = c$ if $f(x) \leq f(c)$ for every x in the domain we are working on.
- We say that $f(x)$ has a **relative (or local) maximum** at $x = c$ if $f(x) \leq f(c)$ for every x in some open interval around $x = c$.
- We say that $f(x)$ has an **absolute (or global) minimum** at $x = c$ if $f(x) \geq f(c)$ for every x in the domain we are working on.
- We say that $f(x)$ has a **relative (or local) minimum** at $x = c$ if $f(x) \geq f(c)$ for every x in some open interval around $x = c$.



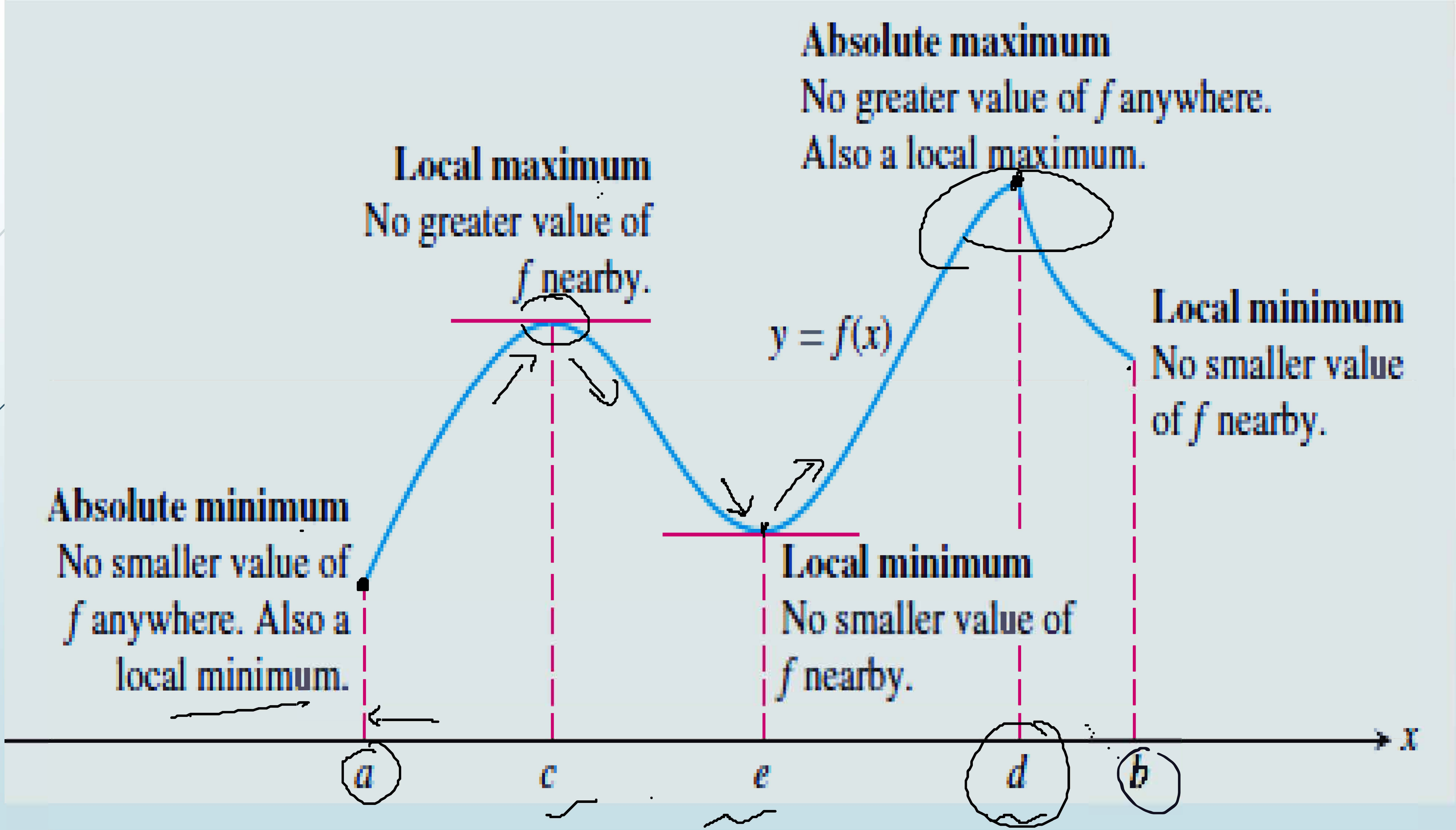
Note: When we say an "open interval around $x = c$" we mean that we can find some interval , not including the endpoints, such that $a < c < b$. We can extend the definitions of local extrema to the endpoints of intervals by defining f to have a **local maximum** or **local minimum** value *at an endpoint* c if the appropriate inequality holds for all x in some half-open interval in its domain containing c. [ ) ( ]

Subtle difference between absolute and relative

- We will have an absolute maximum (or minimum) at $x = c$ provided $f(c)$ is the largest (or smallest) value that the function will ever take on the domain that we are working on.
- A relative maximum or minimum is slightly different. All that's required for a point to be a relative maximum or minimum is for that point to be a maximum or minimum in some interval of x's around $x = c$. There may be larger or smaller values of the function at some other place, but relative to $x = c$, or local to $x = c$, $f(c)$ is larger or smaller than all the other function values that are near it.
- An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, a list of all local maxima will automatically include the absolute maximum if there is one. Similarly, a list of all local minima will include the absolute minimum if there is one.

Abs max $\longrightarrow$ local

Every local max is not Abs



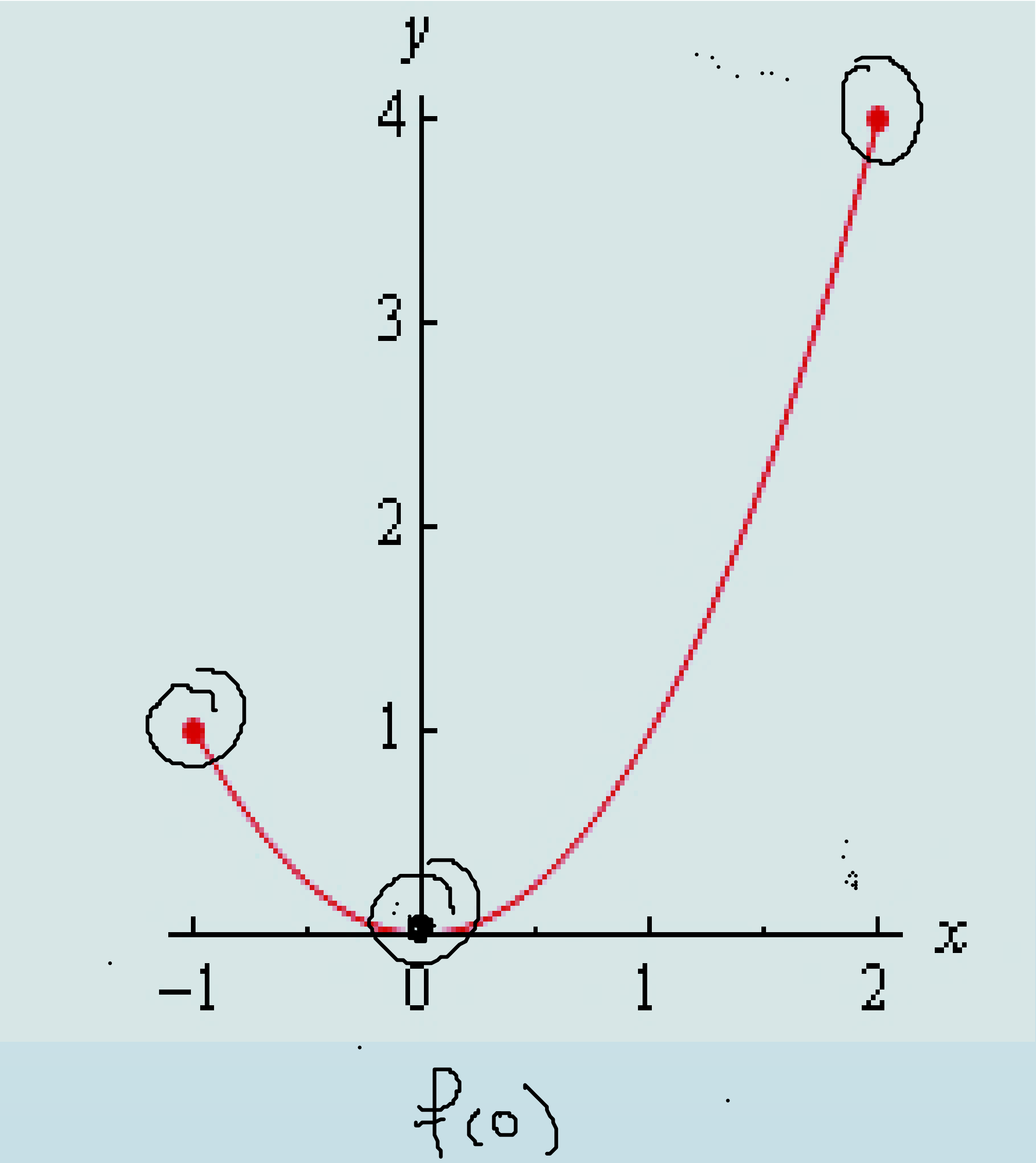
Absolute maximum
No greater value of f anywhere.
Also a local maximum.
Local maximum
No greater value of f nearby.
y = f(x)
Local minimum
No smaller value of f nearby.
Absolute minimum
No smaller value of f anywhere. Also a local minimum.
Local minimum
No smaller value of f nearby.
a
c
e
d
b
x

Example:

Identify the absolute extrema and relative extrema for

$$f(x) = x^2 \quad \text{on} \quad [-1,2].$$

- A relative and absolute minimum of zero at $x = 0$.
- A relative and an absolute maximum of four at $x = 2$.
- A relative maximum of one at $x = -1$.

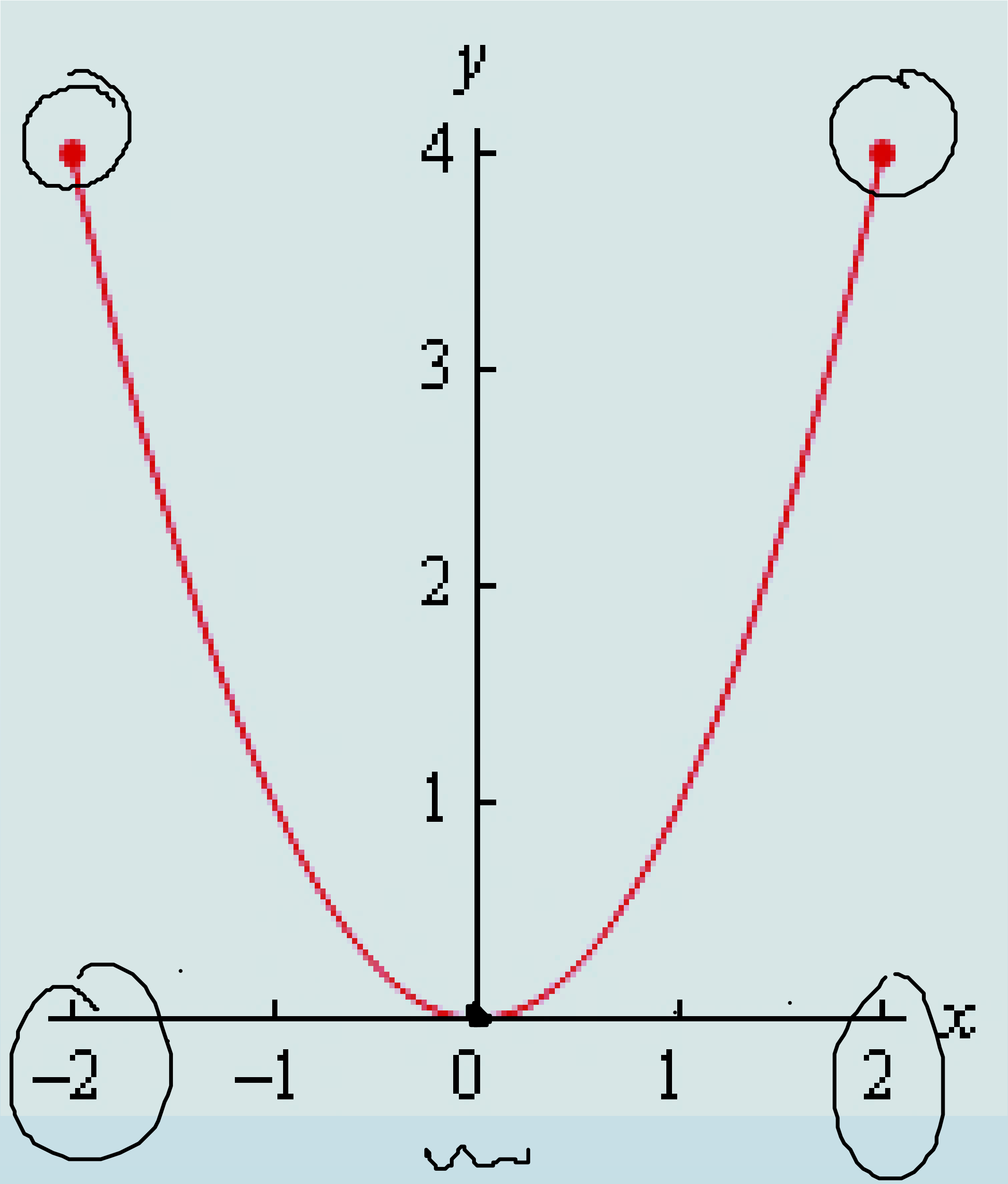


Example:

Identify the absolute extrema and relative extrema for

$$f(x) = x^2 \quad \text{on} \quad [-2,2].$$

- A relative and absolute minimum of zero at $x = 0$.
- A relative and an absolute maximum of four at $x = 2$ and $x = -2$.

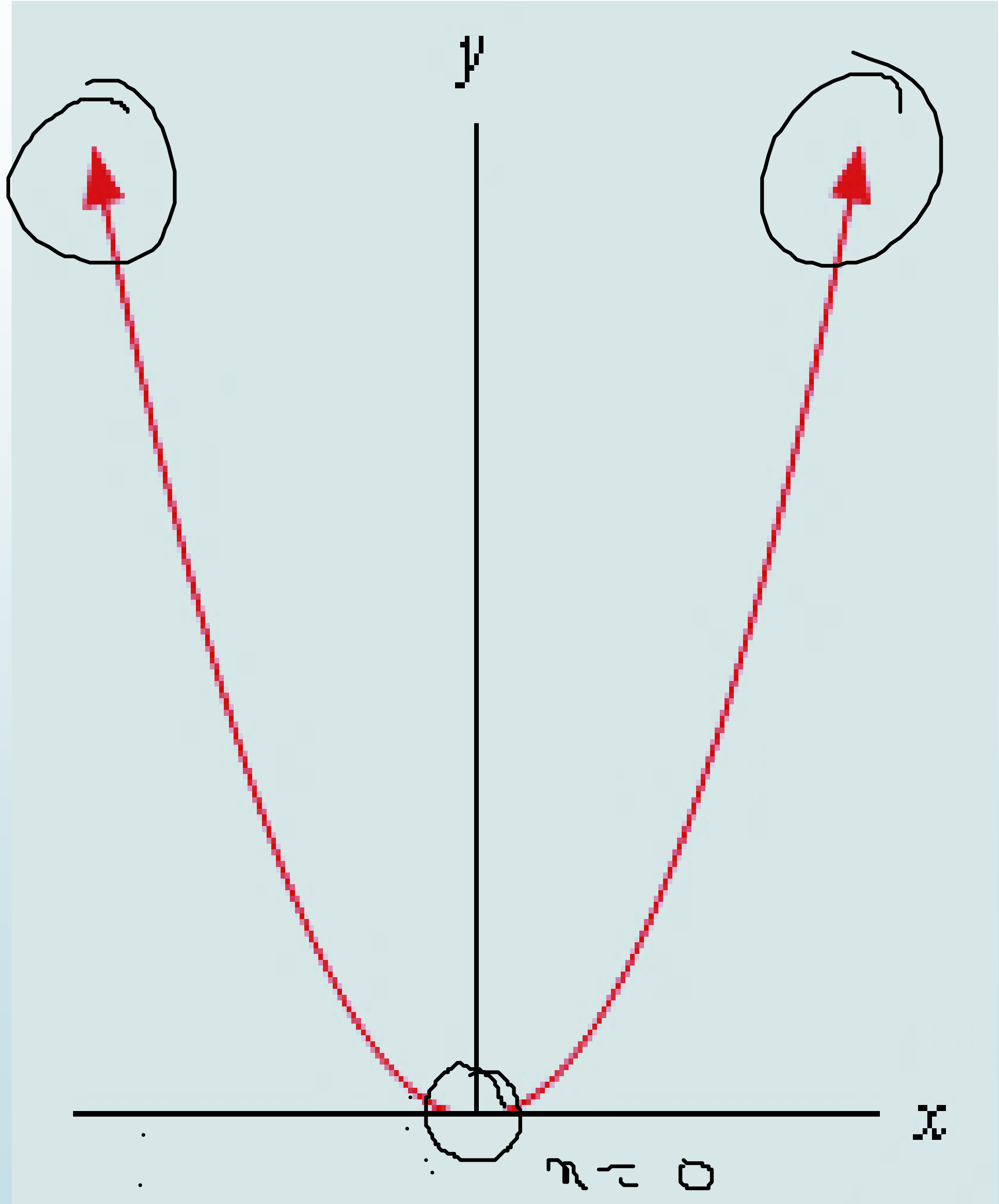


Example:

Identify the absolute extrema and relative extrema for

$$f(x) = x^2 \quad \text{on} \quad (-\infty, \infty).$$

- A relative and absolute minimum of zero at $x = 0$.
- In this case the graph doesn't stop increasing at either end and so there are no maximums of any kind (relative or absolute) for this function.

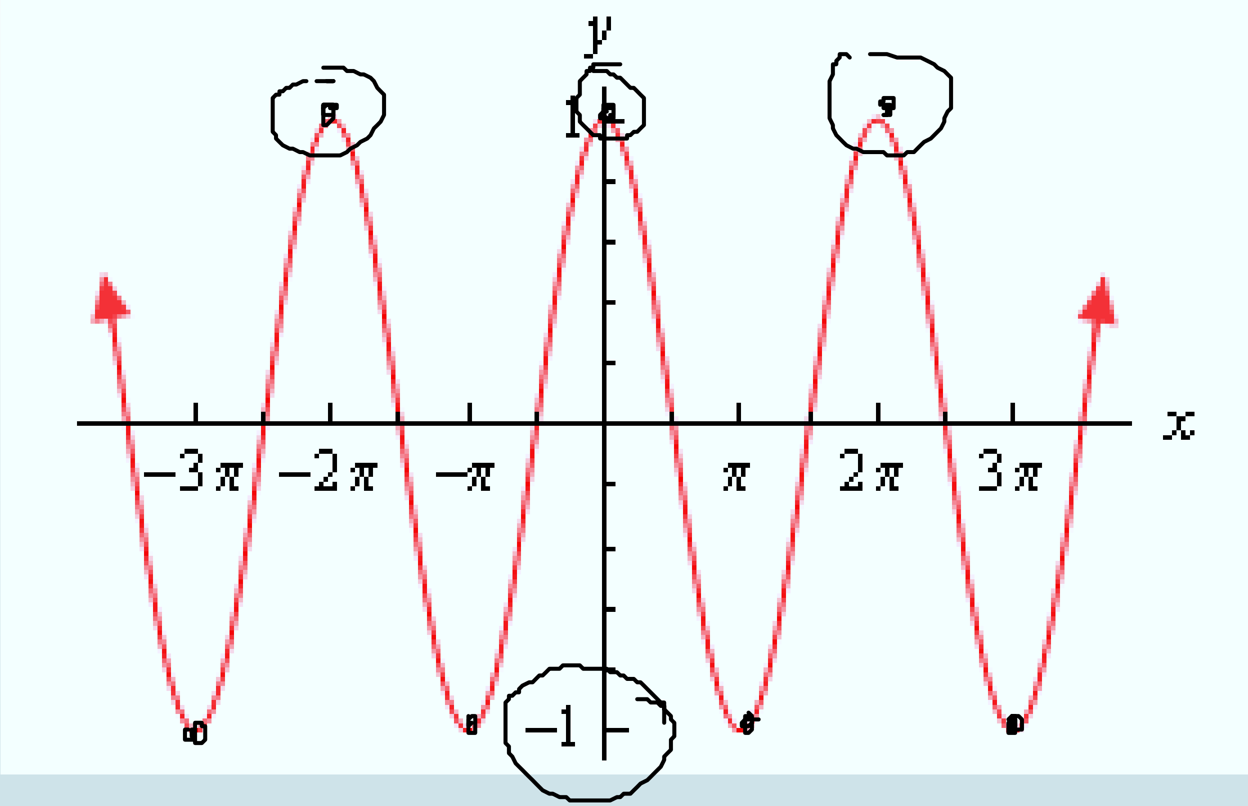


Example:

Identify the absolute extrema and relative extrema for

$$f(x) = \cos x \quad \text{on} \quad (-\infty, \infty).$$

- A relative and absolute minimum of -1 at $x = \pm\pi, \pm 3\pi, \ldots$.
- A relative and absolute maximum of 1 at $x = 0, \pm 2\pi, \pm 4\pi, \ldots$.

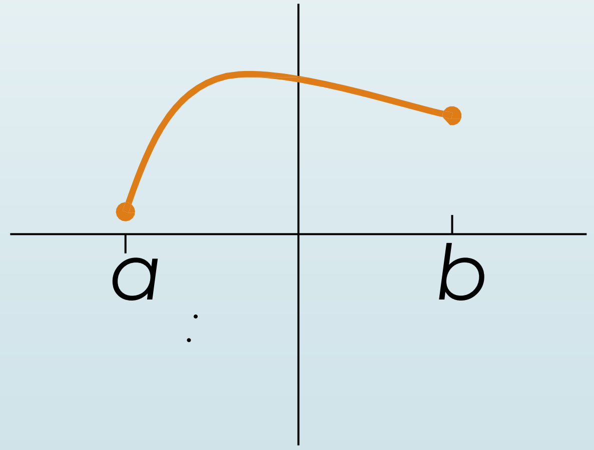


The Extreme Value Theorem

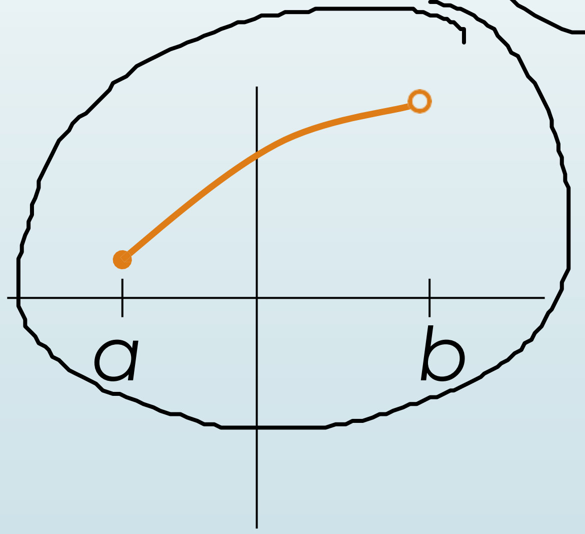
- Note that all the functions considered in examples were continuous functions.
- Next notice that every time we restricted the domain to a closed interval (*i.e.* the interval contains its end points) we got absolute maximums and absolute minimums.
- Finally, when we did not restrict the domain, we may or may not get both an absolute maximum and an absolute minimum.
- These observations lead us the following theorem.

Extreme Value Theorem

If a function f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum and absolute minimum on $[a, b]$. Each extremum occurs at a critical number or at an endpoint.

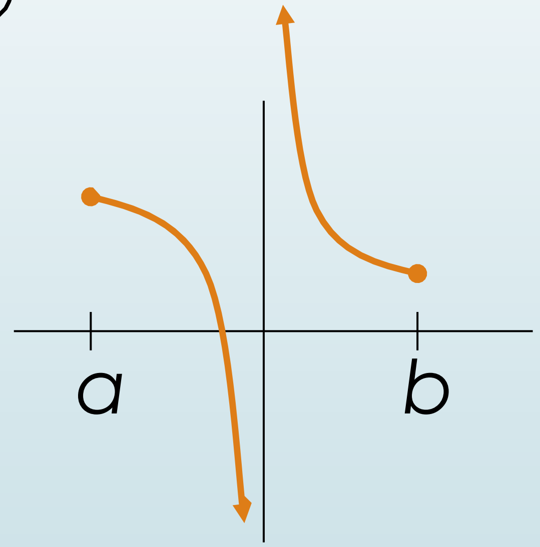


Attains absolute max. and min.



Attains absolute min. but no max.

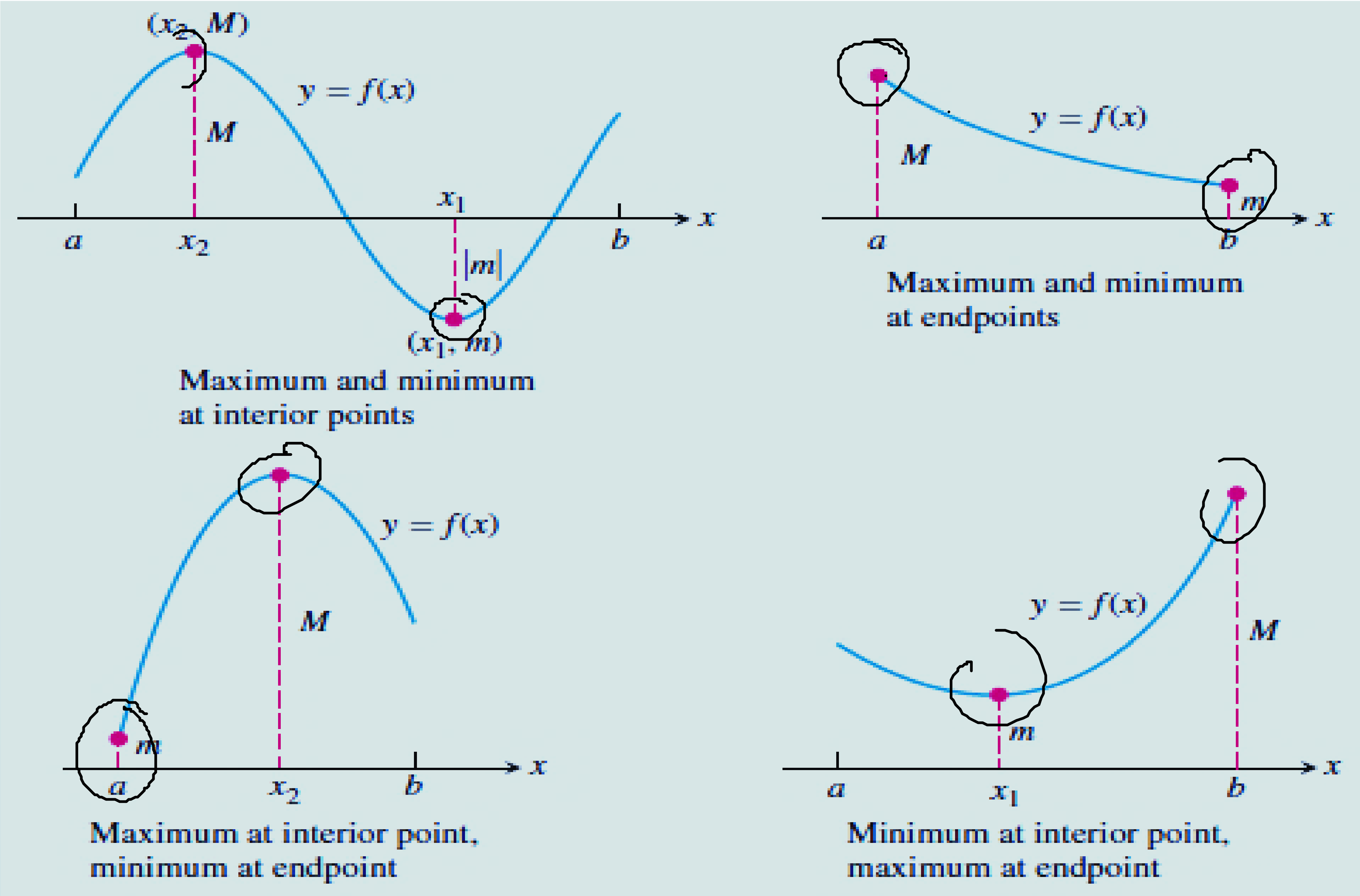
Open Interval



No absolute min. and no absolute max.

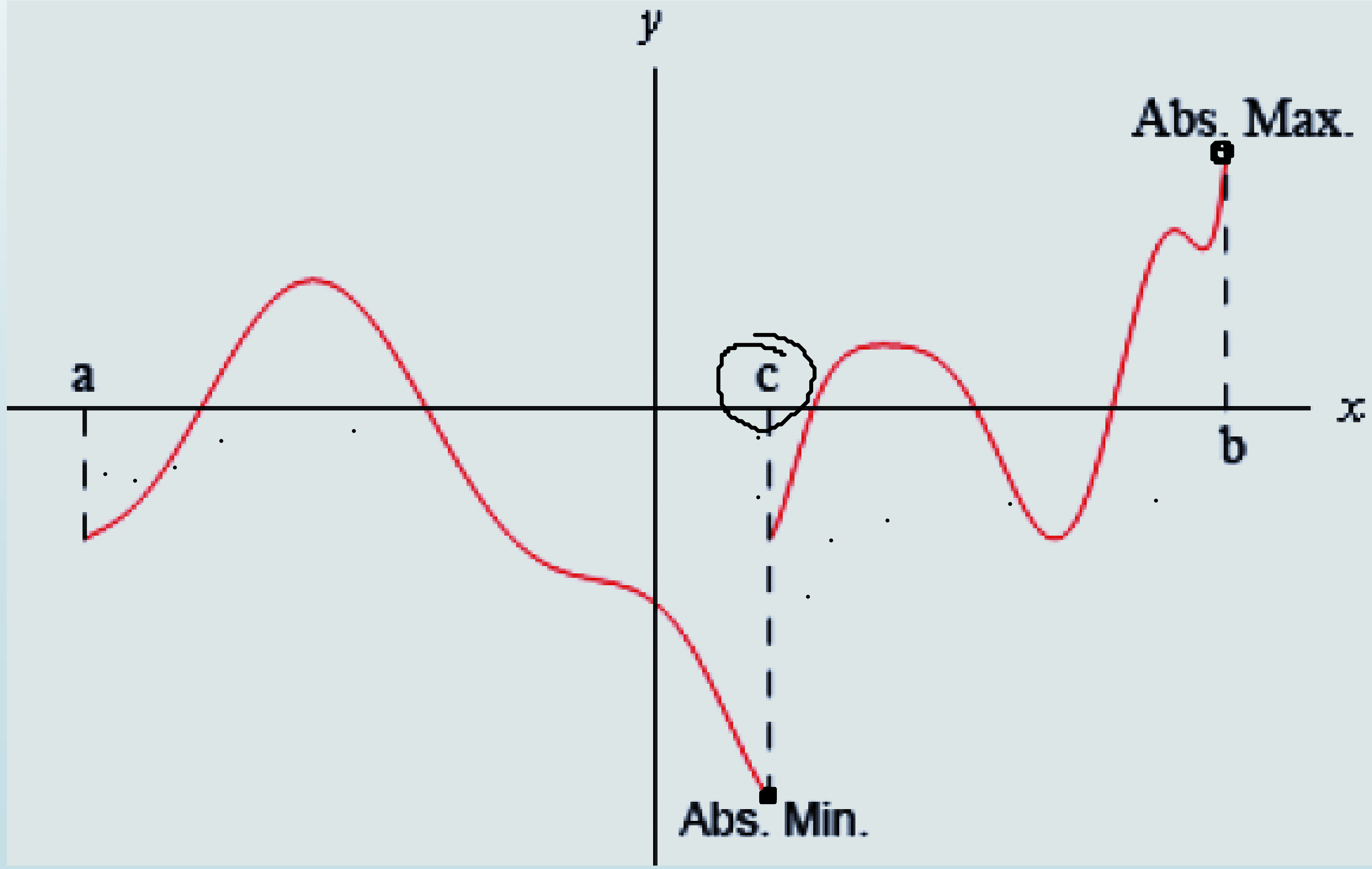
Not continuous

Some possibilities for a continuous function’s maximum and minimum on a closed interval $[a, b]$.



Example:

- We should also point out that just because a function is not continuous at a point that doesn’t mean that it won’t have both absolute extrema in an interval that contains that point.
- Below is the graph of a function that is not continuous at a point in the given interval and yet has both absolute extrema.



Observations:

- This graph is not continuous at $x = c$, yet it does have both an absolute maximum ($x = b$) and an absolute minimum ($x = c$).
- The point of all this is that we need to be careful to only use the Extreme Value Theorem when the conditions of the theorem are met and not misinterpret the results if the conditions aren't met.
- In order to use the Extreme Value Theorem we must have an interval and the function must be continuous on that interval.
- If we don't have an interval and/or the function isn't continuous on the interval then the function may or may not have absolute extrema.

X EVT

CG X

[a, b] X