Cylindrical & Spherical Coordinates

Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

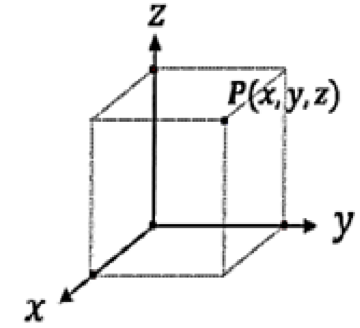
3-D Coordinate Systems

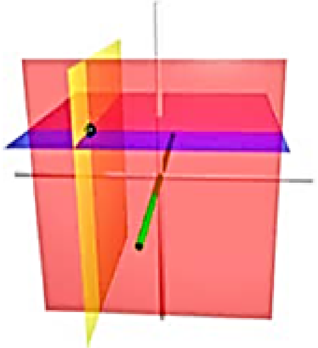
1. Cartesian Coordinates

   Or

   Rectangular Coordinates

   $P\ (x, y, z)$

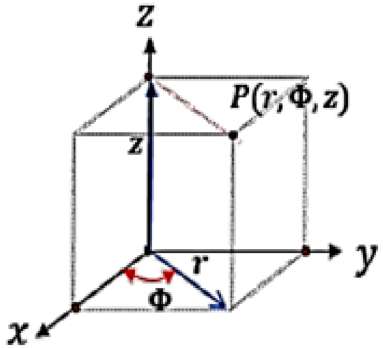


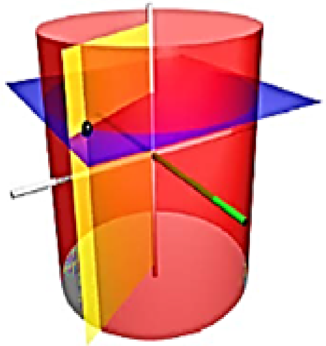


2. Cylindrical Coordinates

   $P\ (r, \Phi, z)$

$$\boxed{\begin{aligned} x &= r\cos\Phi, \\ y &= r\sin\Phi, \\ z &= z. \end{aligned}}$$

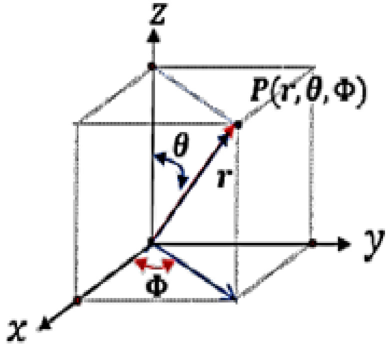


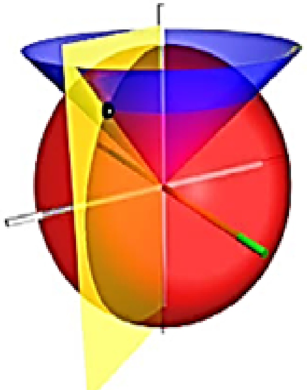


3. Spherical Coordinates

   $P\ (r, \theta, \Phi)$

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\Phi\,, \\ y &= r\sin\theta\sin\Phi, \\ z &= r\cos\theta\,. \end{aligned}}$$





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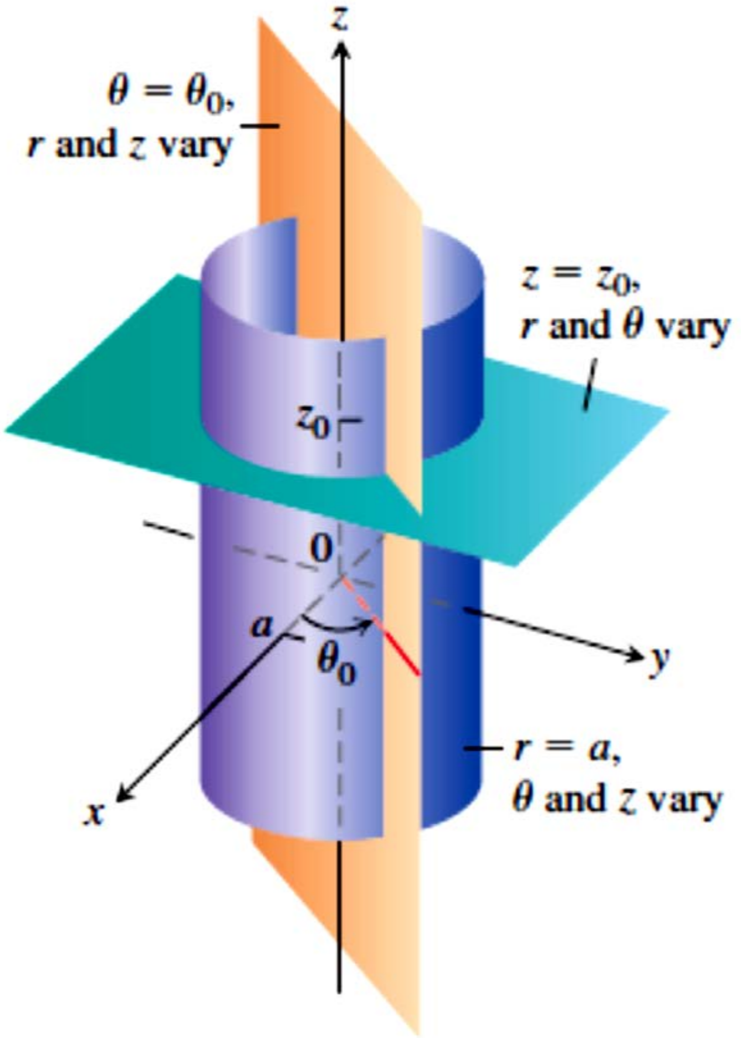
Vectors And
The Geometry Of Space

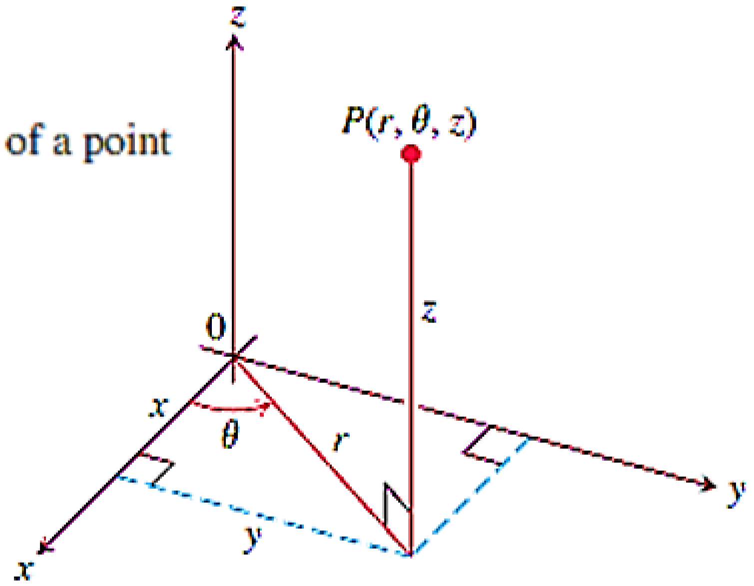
Book: Thomas’ Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Chapter: 15 , Section: 15.7

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.
Chapter: 15 , Section: 15.7, 15.8

Introduction to Cylindrical Coordinate Systems



The cylindrical coordinates of a point in space are r, θ, and z.



Cylindrical to rectangular

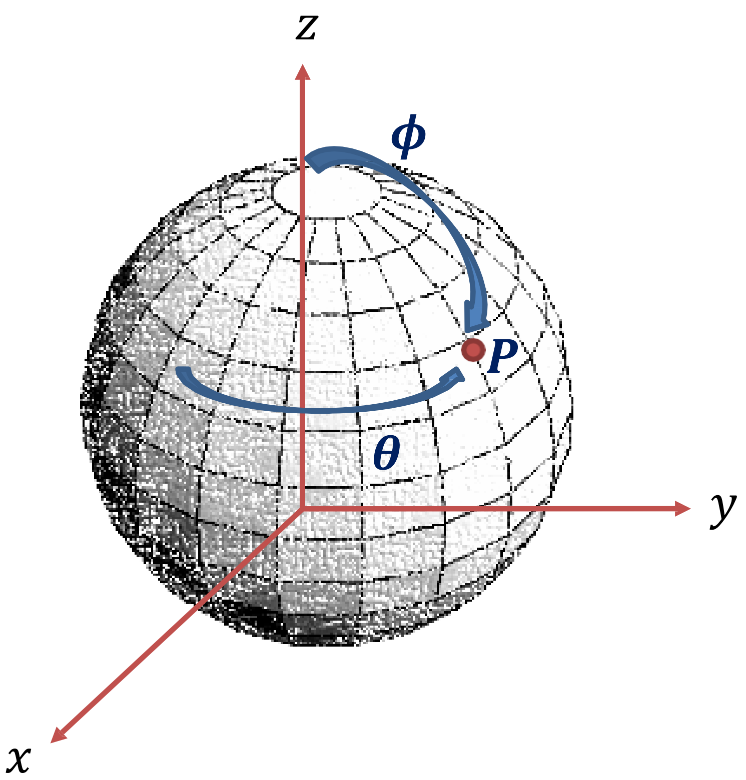
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
where, $0 \le \theta \le 2\pi$.

Rectangular to Cylindrical

$r^2 = x^2 + y^2$
$\tan\theta = \dfrac{y}{x}$
$z = z$

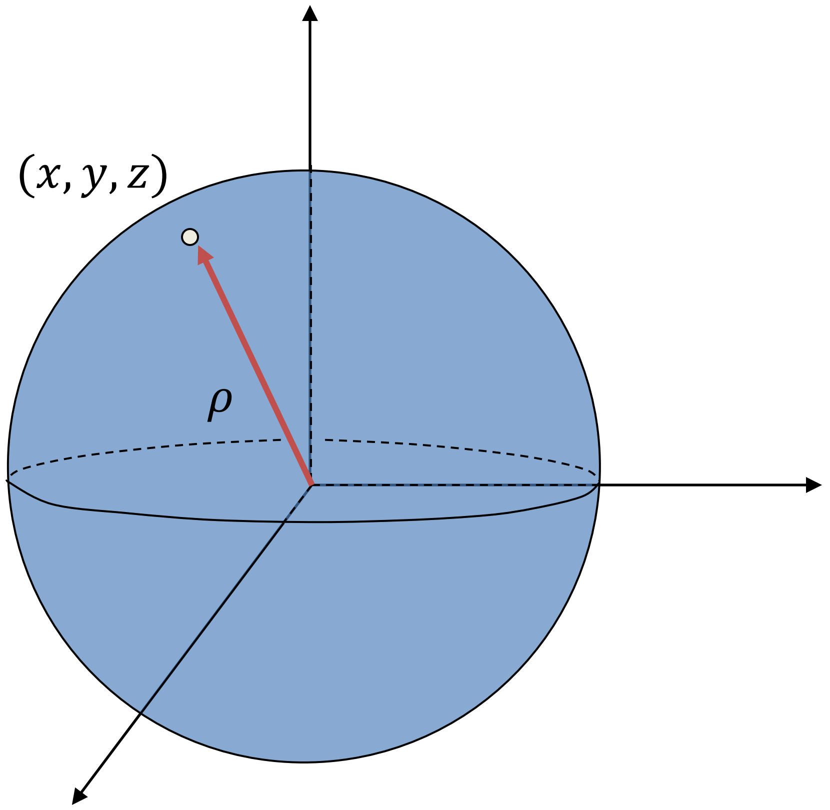
Spherical Coordinate System

We are now in a position to naturally ***generalize*** the concept of coordinate systems further in a three-dimensional space by replacing the whole idea of rectangular boxes and cylinders with ***spheres*** to approach a particular point in space.



Now the sense of ***approaching*** a particular point is changed entirely because we reach point P via some sphere of fixed radius and angle.

Representing 3D points in Spherical Coordinates

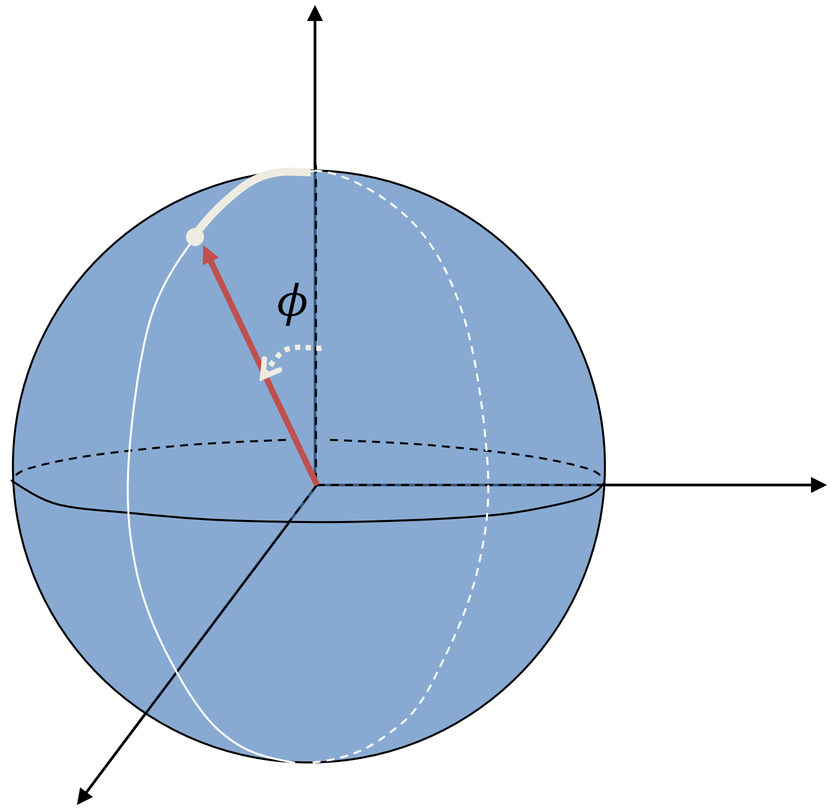


- We start with a point (x, y, z) given in rectangular coordinates.

- Then, measuring its distance ρ from the origin, we locate it on a sphere of radius ρ (distance from origin to the point) centered at the origin.

- We use a method similar to the method used to measure *latitude* and *longitude* on the surface of the Earth.

Representing 3D points in Spherical Coordinates



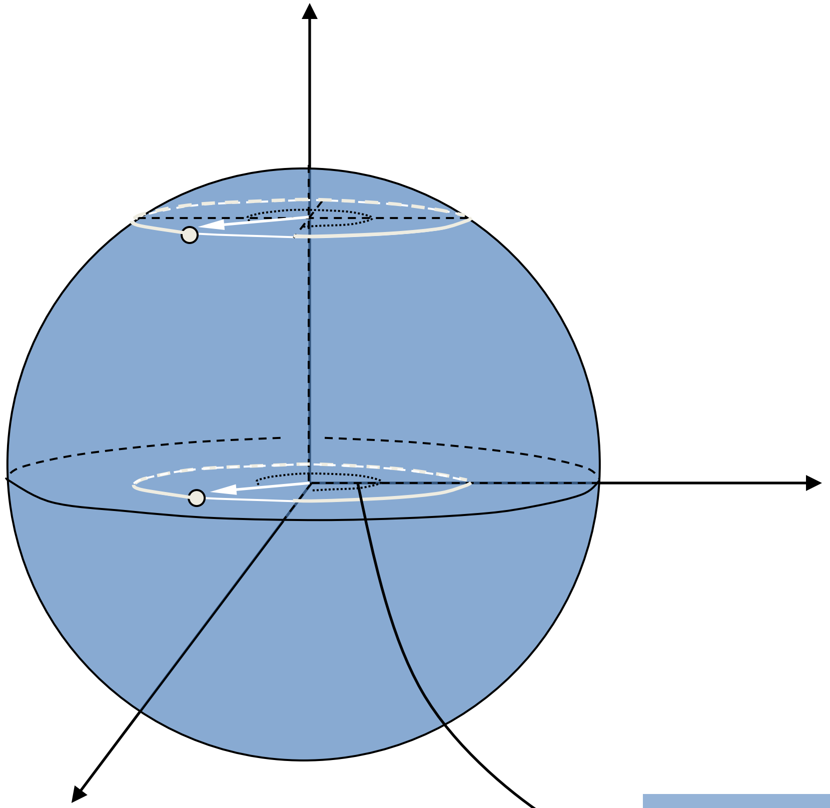
We measure the *longitude* angle starting at the “north pole” in the plane given by the great circle.

This angle is called ϕ. This is the angle between positive $z-$axis and the line segment joining the point and origin. The range of this angle is:

$$0 \leq \phi \leq \pi.$$

Note: all angles are measured in radians, as always.

Representing 3D points in Spherical Coordinates



- We measure the *latitude* angle on the latitude circle, starting at the positive $x-$axis and rotating towards the positive y $-$axis.

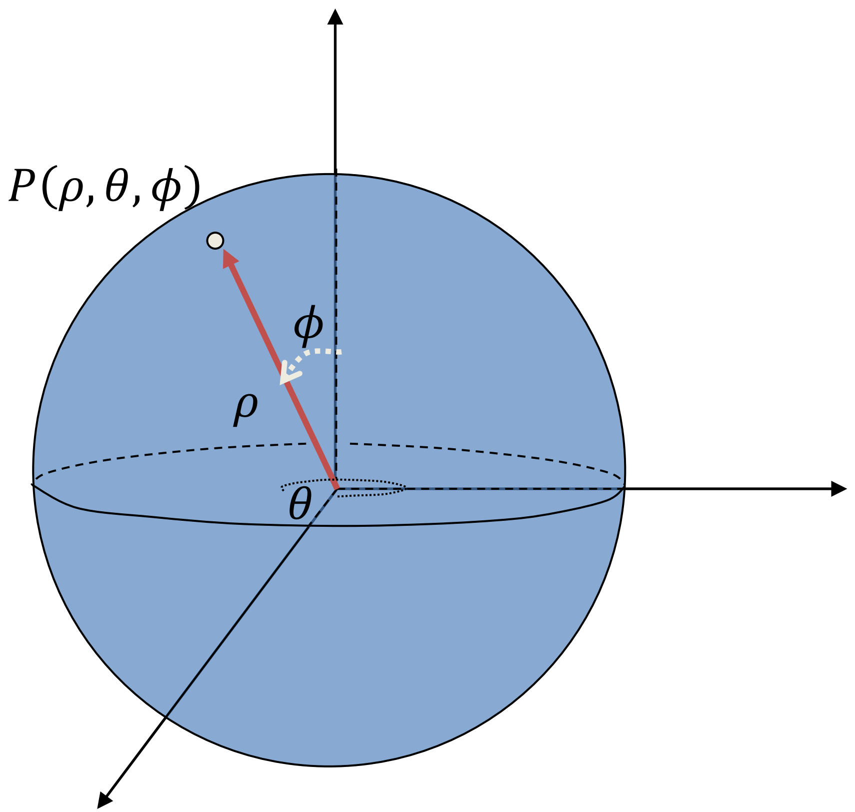
- The range of the angle is:

$$0 \leq \theta \leq 2\pi.$$

Angle is called θ.

Note that this is the same angle as the θ in cylindrical coordinates!

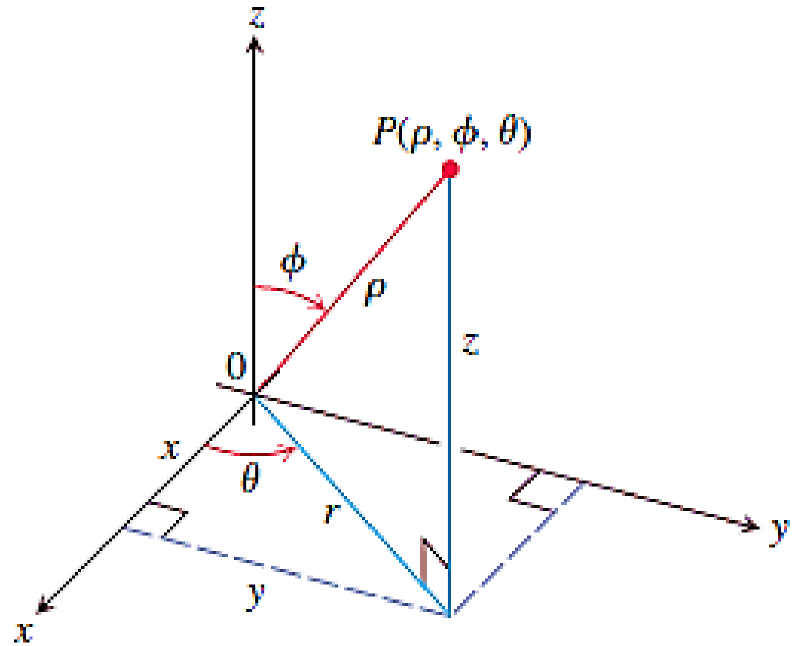
Representing 3D points in Spherical Coordinates



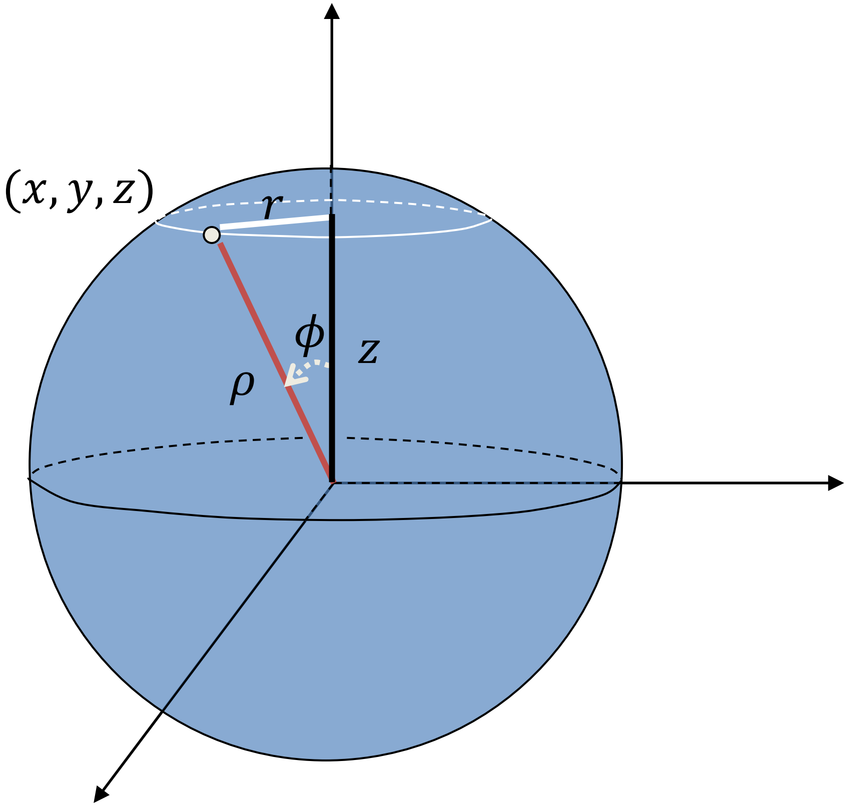
Our designated point on the sphere is indicated by the three spherical coordinates:

$$(\rho, \theta, \phi),$$

- ρ is the is the distance from P to the origin $(\rho \geq 0)$.
- ϕ is the angle that $\overrightarrow{OP}$ makes with the positive z −axis $(0 \leq \phi \leq \pi)$.
- θ is the angle from cylindrical coordinates $(0 \leq \theta \leq 2\pi)$.



Conversion Between Rectangular and Spherical Coordinates



First note that if r is the usual cylindrical coordinate for (x, y, z)

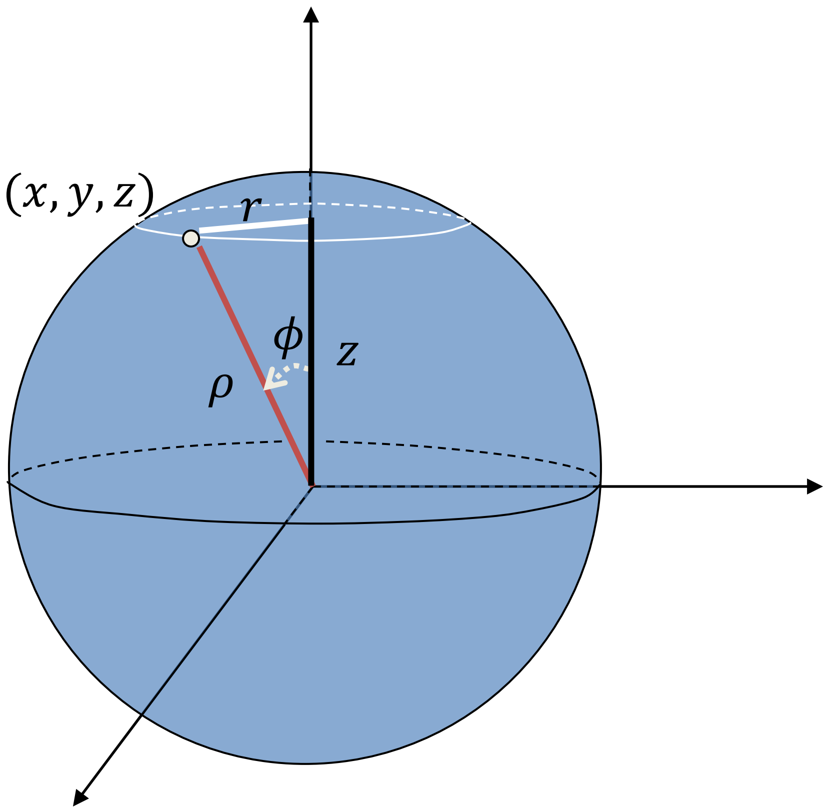
we have a right triangle with:

- acute angle ϕ
- hypotenuse ρ and
- legs r and z.

It follows that:

$$\sin(\phi) = \frac{r}{\rho} \quad \cos(\phi) = \frac{z}{\rho} \quad \tan(\phi) = \frac{r}{z}.$$

Conversion Between Rectangular and Spherical Coordinates



Spherical to rectangular:

$$x = r\cos(\theta) = \rho\sin(\phi)\cos(\theta),$$

$$y = r\sin(\theta) = \rho\sin(\phi)\sin(\theta),$$

$$z = \rho\cos(\phi).$$

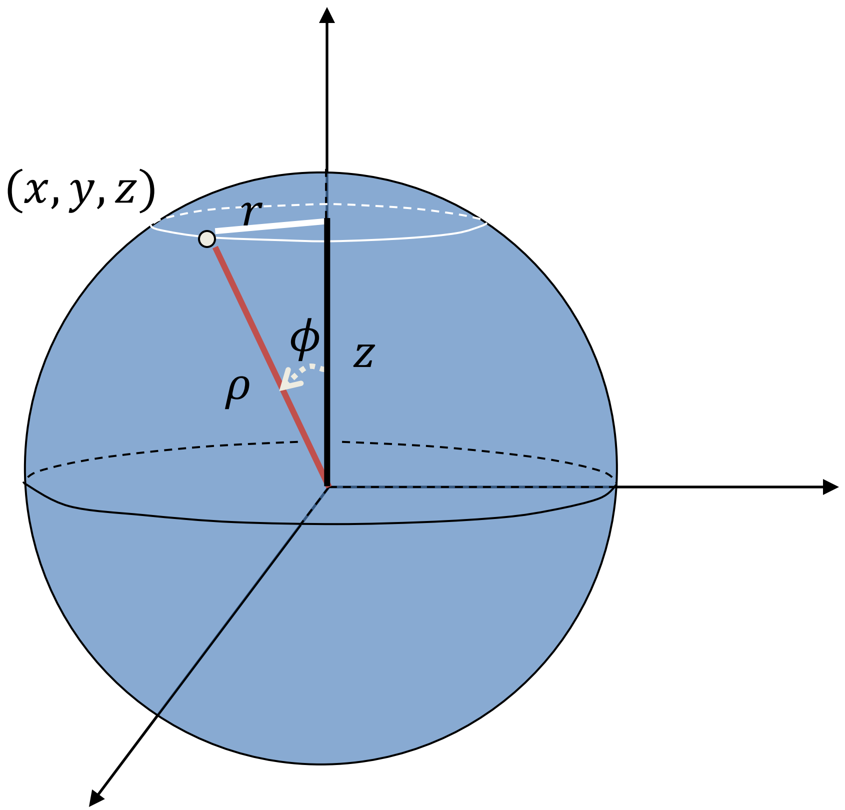
where:

$$0 \leq \rho < \infty,$$

$$0 \leq \phi \leq \pi,$$

$$0 \leq \theta \leq 2\pi.$$

Conversion Between Rectangular and Spherical Coordinates



Rectangular to Spherical:

$$\rho = \sqrt{x^2 + y^2 + z^2},$$

$$\tan(\theta) = \frac{y}{x},$$

$$\tan(\phi) = \frac{r}{z} = \frac{\sqrt{x^2 + y^2}}{z},$$

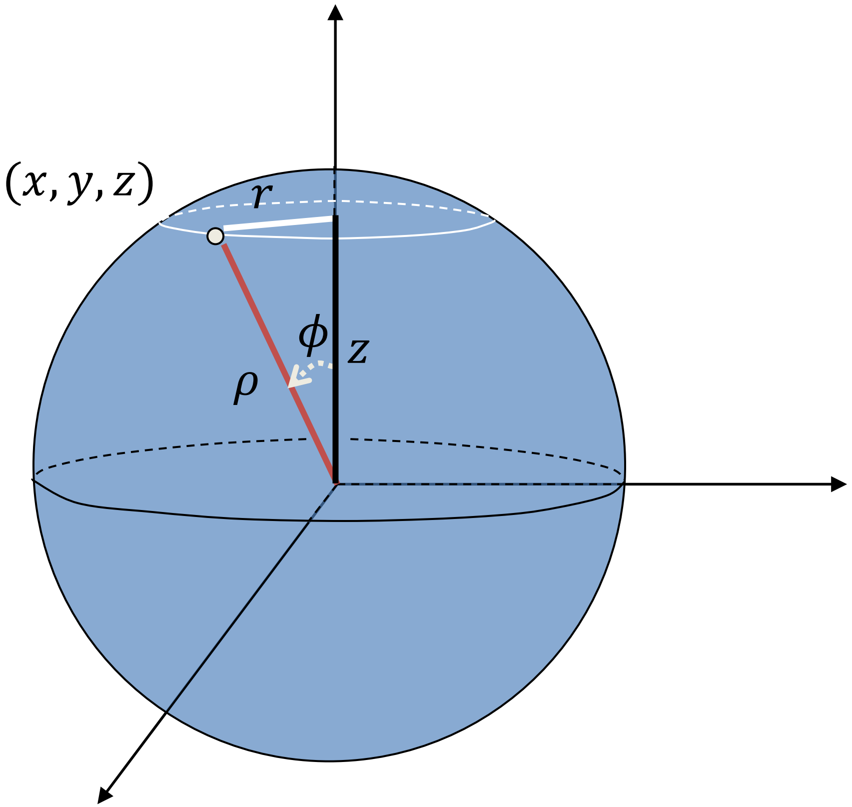
$$\cos(\phi) = \frac{z}{\rho} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

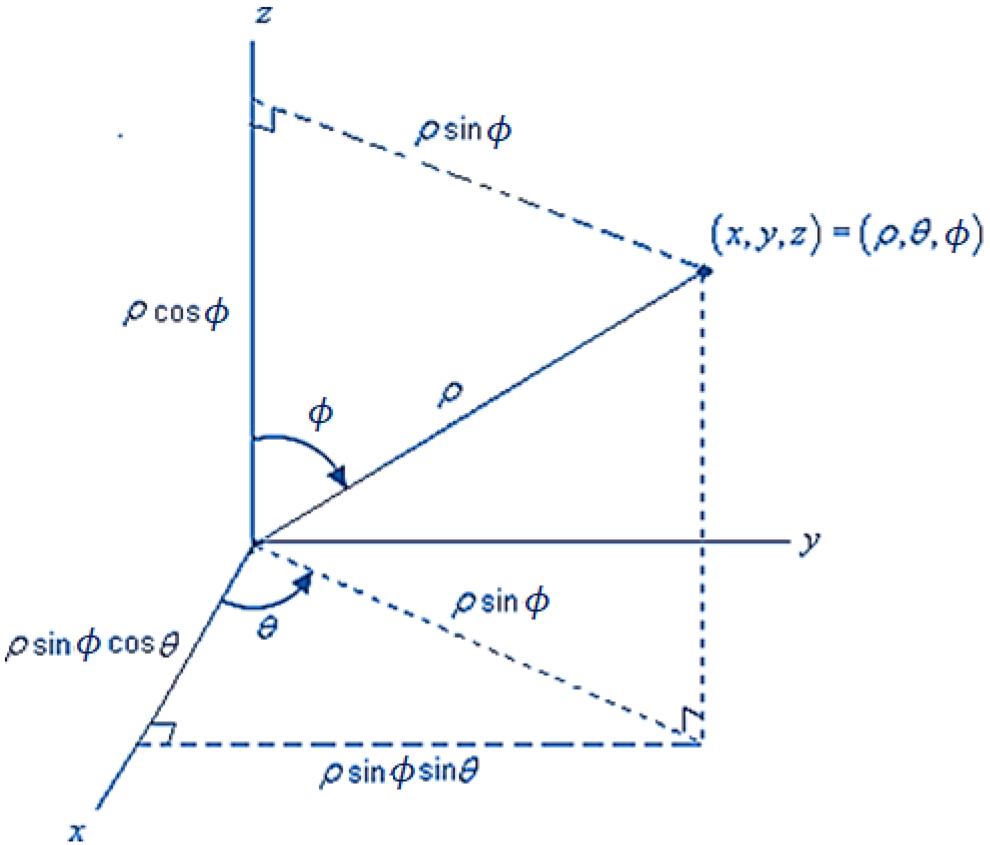
Conversion Between Rectangular and Spherical Coordinates

$$\boxed{x = \rho \sin\phi \cos\theta\,, \qquad y = \rho \sin\phi \sin\theta\,, \qquad z = \rho\cos\phi\,.}$$

$$\boxed{\rho^2 = x^2 + y^2 + z^2, \qquad \theta = \arctan(y/x)\,, \qquad \phi = \arccos(z/\rho)\,.}$$

where: $0 \le \rho < \infty, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$

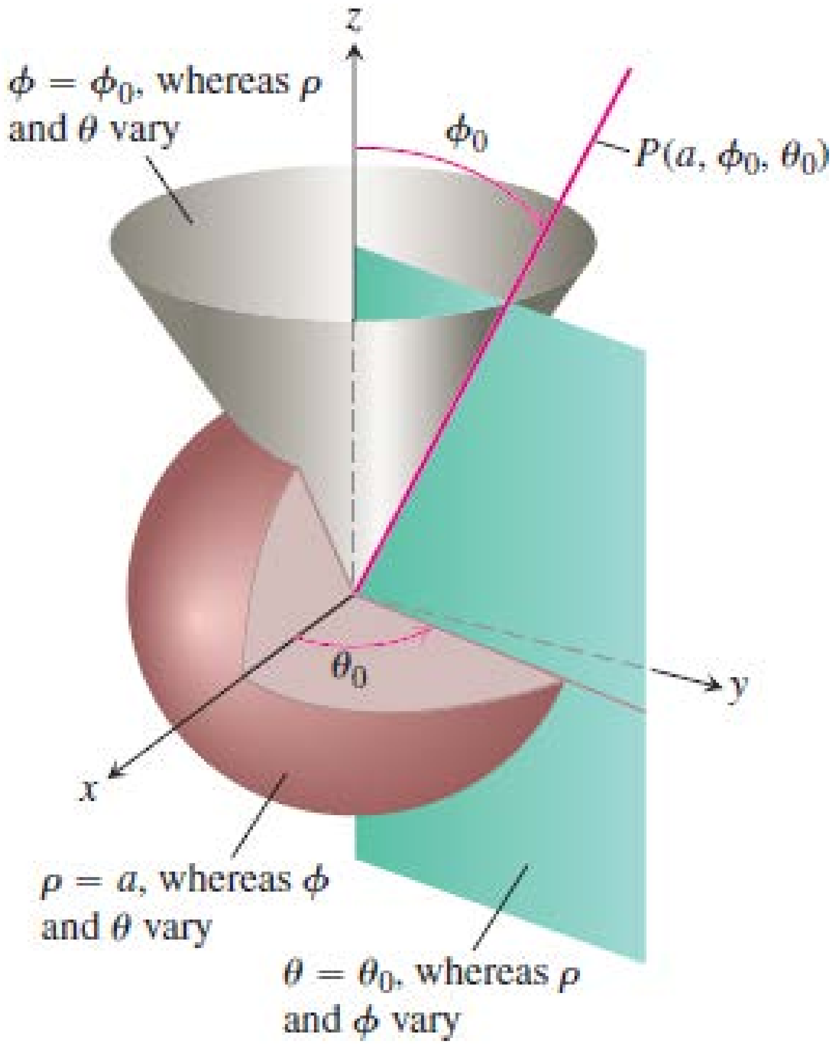




Spherical Coordinate System

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

- the sphere with center at the origin and radius a has the simple equation $\rho = a$ (constant). This is the reason for the name "spherical" coordinates.
- The graph of the equation $\theta = \theta_0$(constant) is a vertical half-plane.
- The equation $\phi = \phi_0$(constant) represents a half-cone with the z −axis as its axis.



Constant-coordinate equations in spherical coordinates yield spheres, single cones, and half-planes.

Example:

Plot the point $(2, \pi/4, \pi/3)$ and find its rectangular coordinates.

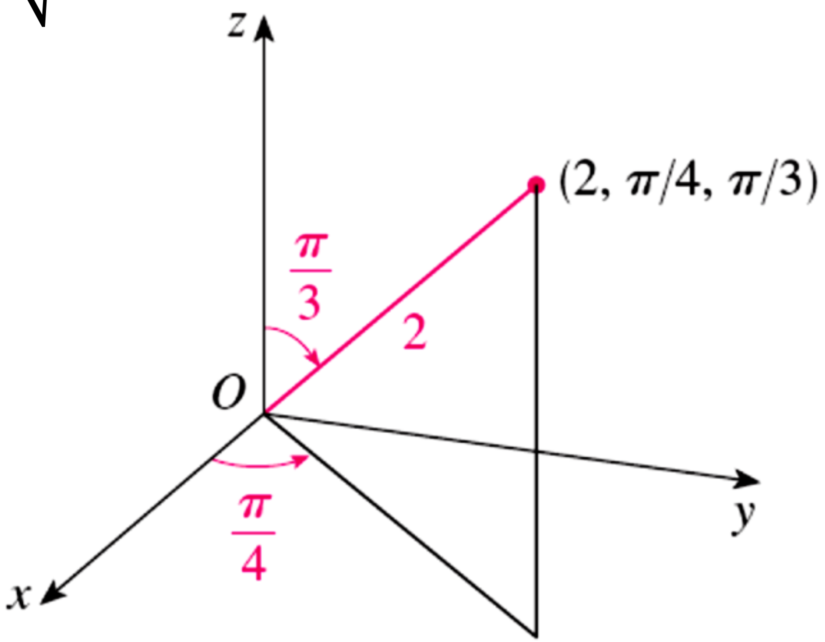
Solution:

Since,

$$x = \rho \sin\phi \cos\theta = 2 \sin\frac{\pi}{3} \cos\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin\phi \sin\theta = 2 \sin\frac{\pi}{3} \sin\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos\phi = 2 \cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$



Thus, the point $\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$, in rectangular coordinates is:

$$\left(\sqrt{3/2}, \sqrt{3/2}, 1\right).$$

Example:

Find spherical coordinates for the point $(0,2\sqrt{3},-2)$.

Solution: We know that:

$$\rho^2 = x^2 + y^2 + z^2, \quad \theta = \arctan(y/x)\,, \ \phi = \arccos(z/\rho)\,.$$

Therefore,

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = 4$$
$$\cos\phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$$
$$\tan\theta = \frac{y}{x} = \frac{2\sqrt{3}}{0} \Rightarrow \theta = \frac{\pi}{2}.$$

Thus, the spherical coordinates of the given point are:

$$\left(4, \frac{\pi}{2}, \frac{2\pi}{3}\right).$$

Example:

Find a spherical coordinate equation for the sphere: $x^2 + y^2 + (z-1)^2 = 1$.

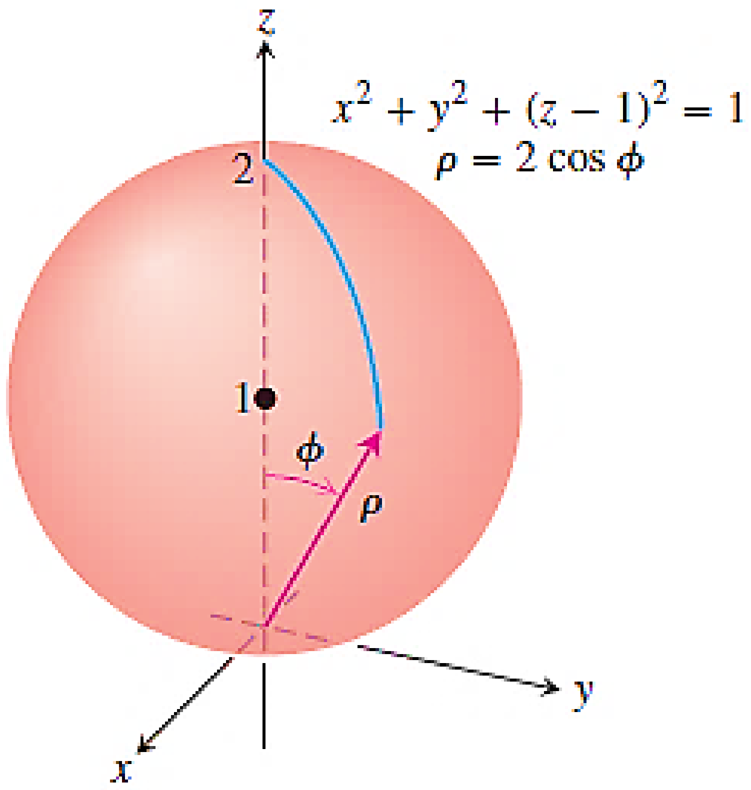
Solution:

We know that:

$$x = \rho \sin\phi \cos\theta\,, \qquad y = \rho \sin\phi \sin\theta\,, \qquad z = \rho\cos\phi\,.$$

Therefore,

$$\begin{aligned}
&x^2 + y^2 + (z-1)^2 = 1\\
&\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + (\rho\cos\phi - 1)^2 = 1\\
&\rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1\\
&\rho^2(\sin^2\phi + \cos^2\phi) = 2\rho\cos\phi\\
&\rho = 2\cos\phi\,.
\end{aligned}$$



Example:

Find a spherical coordinate of the cone: $z = \sqrt{x^2 + y^2}$.

Solution:

We know that:

$$x = \rho \sin\phi \cos\theta\,, \qquad y = \rho \sin\phi \sin\theta\,, \qquad z = \rho\cos\phi\,.$$

Therefore,

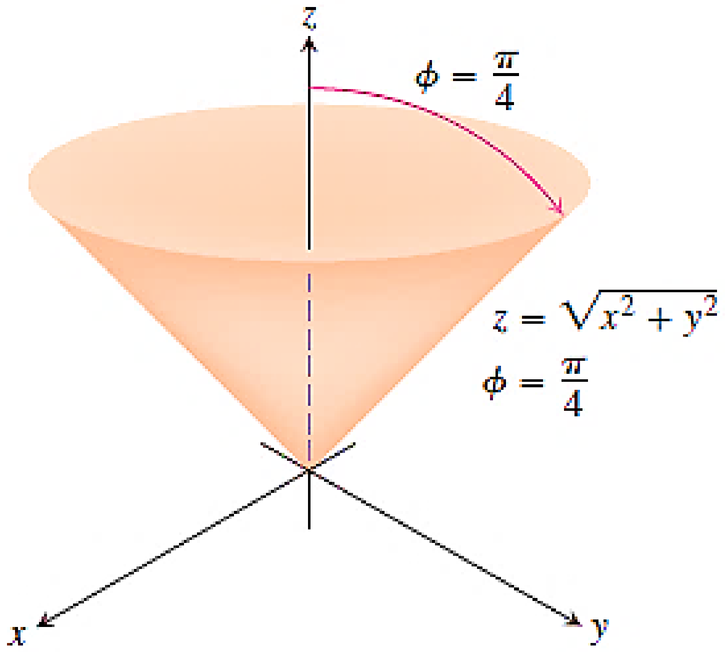
$$z = \sqrt{x^2 + y^2}$$

$$\rho\cos\phi = \sqrt{\rho^2 \sin^2\phi}$$

$$\rho\cos\phi = \rho\sin\phi \qquad \rho \ge 0,\ \sin\phi \ge 0$$

$$\cos\phi = \sin\phi$$

$$\phi = \frac{\pi}{4}. \qquad 0 \le \phi \le \pi$$



Coordinate Conversion Formulas in Triple Integrals

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

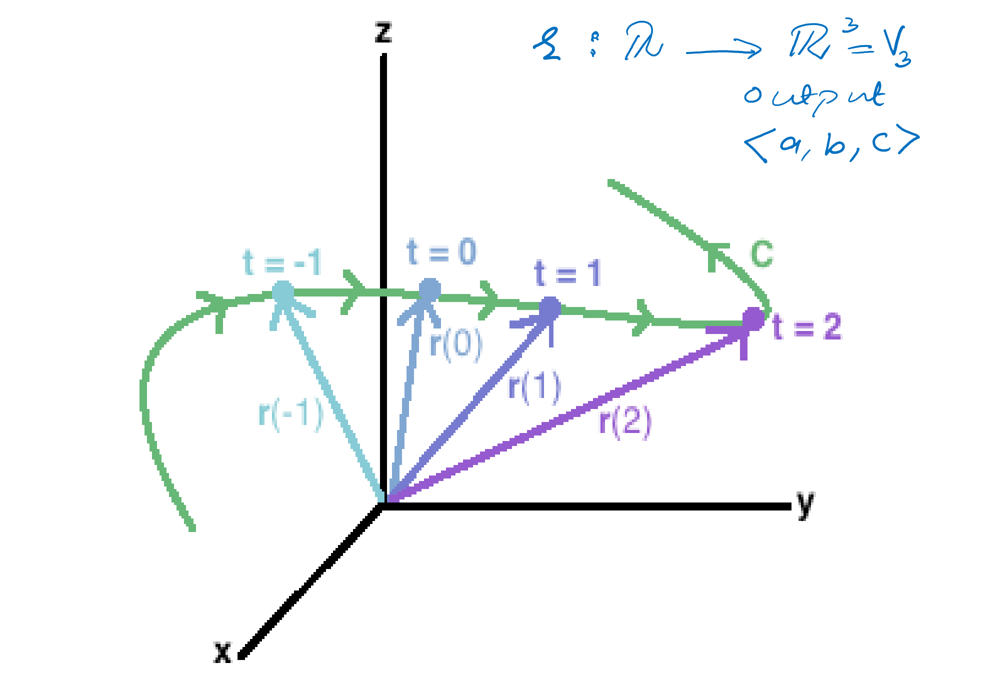
Practice Questions

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 15

Exercise-15.7: Q – 13 to 22.

Vector Valued Functions & Space Curves



A curve C in three-dimensions represents by a vector-valued function r(t), where sample values t=-1, t=0, t=1, and t=2 are arbitrarily plotted.

$f : \mathbb{R} \to \mathbb{R}$ $f(x) = y \in \mathbb{R}$ → real valued function

$s : \mathbb{N} \to \mathbb{R}$ → sequence

- The functions that we have been using so far have been real-valued functions.
- We now study functions whose values are vectors, because such functions are needed to describe curves and surfaces in space.
- We will also use vector-valued functions to describe the motion of objects through space.
- In particular, we will use them to derive Kepler's laws of planetary motion.

vector valued fn

output was vectors

$T : V \to W$ $T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$

linear transformation

$a, b \in \mathbb{R}$

$\vec{u}, \vec{v} \in V$

CVT $\begin{cases} f : \mathbb{C} \to \mathbb{C} & f(z) = w \in \mathbb{C} \\ f : \mathbb{R} \to \mathbb{C} & \end{cases}$ → complex valued fn

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Vectors And The Geometry Of Space

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 13 , Section: 13.1

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

Chapter: 13 , Section: 13.1

Vector Function

- In general, a function is a rule that assigns to each element in the domain an element in the range.

- A **vector-valued function**, or **vector function,** is simply a function whose:
  - Domain is a set of real numbers.
  - Range is a set of vectors.

- We are most interested in vector functions $\mathbf{r}$ whose values are three-dimensional (3-D) vectors.

- This means that, for every number t in the domain of $\mathbf{r}$, there is a unique vector in V_3 denoted by $\mathbf{r}(t)$.

$V_3 = \mathbb{R}^3 \longrightarrow$ vector space

$\langle a, b, c \rangle$ elements of

Component Functions

If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f, g, and h which are real-valued functions, are called the component functions of $\mathbf{r}$. We can write:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

We usually use the letter t to denote the independent variable because it represents time in most applications of vector functions.

Example:

If

$$\mathbf{r}(t) = \langle t^3, \ln(3 - t), \sqrt{t} \rangle$$

then the component functions are:

$$f(t) = t^3, \qquad g(t) = \ln(3 - t), \qquad h(t) = \sqrt{t}.$$

By our usual convention, the domain of $\mathbf{r}$ consists of all values of t for which the expression for $\mathbf{r}(t)$ is defined. The expressions $t^3, \ln(3 - t)$, and $\sqrt{t}$ are all defined when $3 - t > 0$ and $t \geq 0$. Therefore, the domain of $\mathbf{r}$ is the interval $[0, 3)$.

Limit of a Vector Function

The limit of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions as follows:

Definition:

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

provided the limits of the component functions exist.

Note: Limits of vector functions obey the same rules as limits of real-valued functions.

Example:

Find $\lim_{t\to\infty} \mathbf{r}(t)$, where

$$\mathbf{r}(t) = (\arctan t)\mathbf{i} + e^{-2t}\mathbf{j} + \frac{\ln t}{t}\mathbf{k}.$$

Solution:

We know that: $\lim_{t\to a} \mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$. Thus,

$$\lim_{t\to\infty} \mathbf{r}(t) = \left[\underbrace{\lim_{t\to\infty} (\arctan t)}_{= \pi/2}\right]\mathbf{i} + \left[\lim_{t\to\infty} \cancel{e^{-2t}}^{0}\right]\mathbf{j} + \left[\lim_{t\to\infty} \cancel{\frac{\ln t}{t}}^{0}\right]\mathbf{k}.$$

$$\Longrightarrow \lim_{t\to\infty} \mathbf{r}(t) = \frac{\pi}{2}\mathbf{i}.$$

$$\lim_{t\to\infty} (\arctan t) = \frac{\pi}{2}$$

$$\lim_{t\to\infty} (e^{-2t}) = 0$$

L'hopital's rule

$$\lim_{t\to\infty} \left(\frac{\ln t}{t}\right) = \lim_{t\to\infty} \left[\frac{1/t}{1}\right] = 0$$