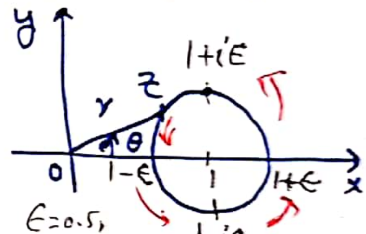
Multivalued functions & Branches: In the definition of an analytic function, one of the conditions imposed was the function is single valued. However, so far we have discussed few functions which are multivalued. Examples include the complex logarithmic function, complex power, argument of a complex number, and a complex root. Nevertheless, it happens that the properties of analytic functions can still be applied to these and other multivalued functions of a complex variable provided that suitable care is taken. This care amounts to identify the "branch points" of the multivalued function under consideration. If z is varied in such a way that its path in the Argand diagram forms a closed diagram (curve) that encloses a branch point, then, in general, $f(z)$ will not return its original value. 1

For definiteness, let us consider the multivalued function $f(z) = z^{1/2}$ and express z as $z = re^{i\theta}$. Consider figure (i), it is clear that, as the point z transverses any closed contour C that does not enclose the origin, θ will return to its original value after one complete circuit. Due to the reason, the argument θ, as viewed from $z=0$, goes up a bit, down a bit, then back up to where it started (See red arrows). However, for any closed contour C that does enclose the origin (figure ii), after one circuit $\theta \rightarrow \theta + 2\pi$. Thus, for the function $f(z) = z^{1/2}$, after one circuit,



$\epsilon = 0.5$,

$Arg(1.5) = 0$,

$Arg(1-0.5) = Arg(0.5) = 0$

$Arg(1+\frac{1}{2}i) = \tan^{-1}(\frac{1/2}{1})$ increased.

Fig. ii

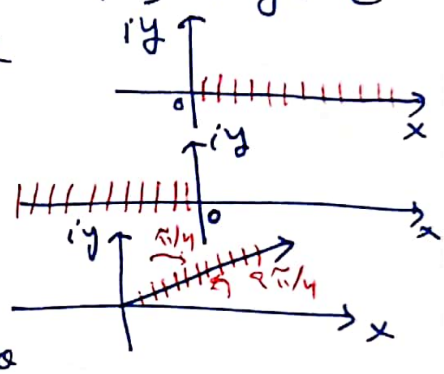
$$r^{1/2} e^{i\theta/2} \longrightarrow r^{1/2} e^{i\left(\frac{\theta+2\pi}{2}\right)} = -\sqrt{r}\, e^{i\theta/2} \quad \left(e^{i\pi} = \cos\pi + i\sin\pi = -1\right)$$

In other words, the value of the function $f(z)$ changes around the closed loop enclosing the origin; in this case $f(z) \rightarrow -f(z)$. Thus, $z=0$ is a branch point of the function $f(z) = z^{1/2}$. It is important to note that if any closed contour enclosing the origin is traversed twice then $f(z) = z^{1/2}$ returns to its original value. For some other functions e.g. $z^{1/3}$, number of loops to return original value will be different (three loops). $\log z$ also have branch point at $z=0$ but original value is never recovered.

In order that $f(z)$ may be treated as a single-valued function we define a branch cut in the Argand diagram. A "branch cut" is a line or curve in the complex plane and may be treated as an artificial barrier that we must not cross. Branch cuts are positioned in such a way that we are prevented from making a complete circuit around any one branch point, and so the function in question remains single valued.

For the function $f(z) = z^{1/2}$, we may take as a branch cut any curve starting at the origin $z=0$ and extending out to $|z| = \infty$ in any direction, since all such curves would equally well prevent us from making a closed loop around the branch point at origin. It is usual, however, to take the cut along the real or imaginary axis. If $0 \leq \theta < 2\pi$, the cut is along the positive real axis. If we consider the principal argument $-\pi < \theta \leq \pi$, the cut is along the negative real axis.

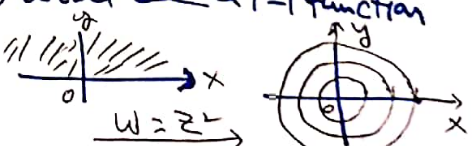
For $\frac{\pi}{4} \leq \theta < \frac{9\pi}{4}$, cut is along the ray



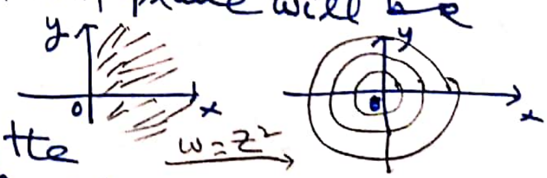
Square of square root function:-

Consider $w = f(z) = z^2$, $z = re^{i\theta}$, $w = f(z) = r^2 e^{2i\theta}$.

If $0 \leq \theta < 2\pi$, then $0 \leq \theta < 4\pi$. In this case, $f(z) = z^2$ will not be with one output, i.e, not a 1-1 function. However, if we restrict $0 \leq \theta < \pi \Rightarrow 0 \leq 2\theta < 2\pi$. In this case $f(z)$ will be a 1-1 function and the upper half-plane will be mapped to $\mathbb{C}$.



Similarly if we consider $-\pi < \theta \leq \pi$, then same problem will arise. In this case, if we restrict $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$, then we have $-\pi < 2\theta \leq \pi$, i.e, the right half-plane will be mapped to the whole complex plane and the function will be 1-1 in this restricted domain.



First Case: $z = \pm 1$, both will be mapped to the positive real axis. So we exclude negative real axis from the domain to make 1-1.

Second Case: $z = \pm i$ both are mapped to -1. So we exclude negative imaginary axis " " .

What happen when we take $f^{-1}(z)$?:

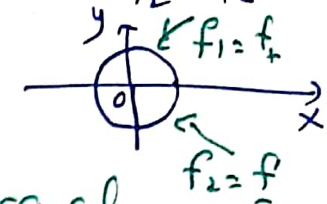
$$w = f^{-1}(z) = z^{1/2} \quad (\text{for } f(z) = z^2).$$

Principal branch: $w = f(z) = \sqrt{r}\, e^{i\theta/2}$, $0 < \theta \leq 2\pi$, thus $0 < \frac{\theta}{2} \leq \pi$. (3)

If we calculate $\sqrt{i} = 1\, e^{i(\pi/2)/2} = e^{i\pi/4} = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$.

we get only one root but if we consider, $\sqrt{i} = \sqrt{1}\, e^{i(\frac{\pi}{2}+2\pi)/2} = e^{i\frac{5\pi}{4}} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$.

$w = f_1 = \sqrt{r}\, e^{i\theta/2}$, $w = f_2 = \sqrt{r}\, e^{i\left(\frac{\theta+2\pi}{2}\right)} = \sqrt{r}\, e^{i\theta/2} e^{i\pi} = -\sqrt{r}\, e^{i\theta/2} = -f_1$



In case of principal argument, $-\pi < \theta \leq \pi \Rightarrow -\frac{\pi}{2} < 2\theta \leq \frac{\pi}{2}$

$f(z) = \sqrt{r}\, e^{i\theta/2}$, $-\pi < \theta \leq \pi$, is called the principal branch of the square root function.

Ex:- An Analytic branch of $z^{1/3}$ is defined on the cut plane domain:

$D = \left\{ z = r e^{i\theta},\ r > 0,\ \frac{\pi}{4} < \theta < \frac{9\pi}{4} \right\}$ such that i is mapped to $(i)^{1/3} = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$, i.e, an analytic function $f(z)$ is defined on the domain D such that $[f(z)]^3 = z$ for every $z \in D$ & $f(i) = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$. Using this information compute $f(-i)$.

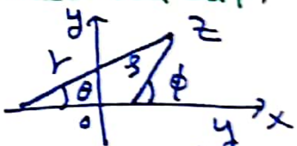
Sol:- $f(re^{i\theta}) = r^{1/3} e^{i\left(\frac{\theta + 2\pi k}{3}\right)}$, $k = 0, 1, 2$. Given that $i = e^{i\frac{\pi}{2}}$, $\arg\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 5\pi/6$.

we have, $\frac{\frac{\pi}{2} + 2k\pi}{3} = \frac{5\pi}{6} \Rightarrow k = 1$. Hence, the branch is

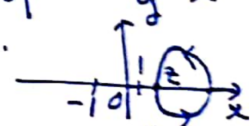
$w = f_1(z) = (r)^{1/3} e^{i\left(\frac{\theta+2\pi}{3}\right)}$, so we have $f(-i) = (1)^{1/3} e^{i\left(\frac{3\pi/2 + 2\pi}{3}\right)} = e^{i\left(\frac{7\pi}{6}\right)} = -\frac{\sqrt{3}}{2} - i\frac{1}{2}$.

Ex:- $f(z) = \sqrt{z^2 - 1} = (z+1)^{1/2} (z-1)^{1/2}$. Show that $z = -1$ & $z = 1$ are branch points. Find & sketch the branch cut.

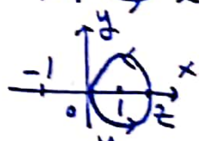
Write $z + 1 = r e^{i\theta}$ & $z - 1 = s e^{i\phi}$, so that $f(z) = \sqrt{rs}\, e^{\frac{i}{2}(\theta+\phi)}$.

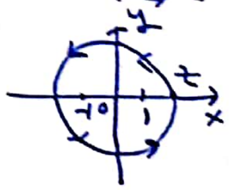


Let us try doing some walks. In each case, we will start at the point z on the real axis and travel anticlockwise. Let us decide that $\arg z = 0$, and that the value of $f(z)$ is real & positive, i.e, $\arg[f(z)] = 0$.



In the first picture, our walk does not go around either 1 or -1. The argument θ & ϕ both go down a bit at first, then back up to 0 as we return to the real axis, then both go up a bit. And when we finally return to z, both θ & ϕ have returned to their starting values. Hence, f has also returned to its starting values, and there is no branching behaviour.





In the second picture, our walk goes around 1 but not -1. The argument θ, as viewed from -1, goes up a bit, down a bit, then back up to where it started. However, ϕ, as viewed from 1, goes up & up, eventually increasing by 2π when we return to z. Since, $\theta \to \theta$ & $\phi \to \phi \to \phi + 2\pi$.

we have $e^{\frac{i}{2}(\theta+\phi)} \longrightarrow e^{\frac{i}{2}(\theta+\phi+2\pi)} = e^{\frac{i}{2}(\theta+\phi)} \cdot e^{i\pi} = -e^{\frac{i}{2}(\theta+\phi)} \Rightarrow f(z) \to -f(z)$.

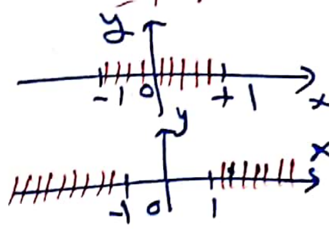
So, '1' is a branch point of f.

...arly, if we walk around -1 but not 1, then $f(z)\to -f(z)$, & so $z=-1$ is also a branch point of f. In the third picture, our walk goes around both 1 & -1. This time both $\theta\to\theta+2\pi$, $\phi\to\phi+2\pi$,

$$e^{\frac{i}{2}(\theta+\phi)} \longrightarrow e^{\frac{i}{2}(\theta+2\pi+\phi+2\pi)} \longrightarrow e^{\frac{i}{2}(\theta+\phi)}\cdot e^{i4\pi} = e^{\frac{i}{2}(\theta+\phi)},\quad f(z)\to f(z)$$

$e^{i4\pi} = \cos 4\pi + i\sin 4\pi = 1$.

How do we cut the plane to prevent branching?
The cut which joins the two points will be sufficient.
Possible cuts are shown in the figure.



Problems:-

(i). Let $f_2(z) = r^{1/3} e^{\frac{i}{3}(\theta+2\pi)}$, $r>0$, $-\pi<\theta<\pi$. Show that f_2 is a branch of the multiple valued cube root function $w=f(z)=z^{1/3}$.

(ii). Consider the multivalued function $f(z) = (z^2+1)^{1/2}$. Find the branch points and a suitable branch cut.

(iii). Consider the multivalued function $f(z) = z^{1/3}$, $z\in\mathbb{C}\setminus[0,0)$. Suppose that we chose the branch such that $f(i) = e^{i5\pi/6}$. Compute $f(-2)$.

(iv). Consider the multivalued function $f(z) = [(1-z)^3 z]^{1/4}$. Show that $z=0$ & $z=1$ are branch points of the function. Is $z=\infty$, a branch point?

(v). Consider the multiple-valued function $F(z) = (z-1+i)^{1/2}$. What is the branch point of F? explain. Explicitly define two distinct branches of f_1 & f_2 of F. State the branch cut. Zill Q-55, page 132.

(vi). Consider a branch of $z^{1/2}$ that is analytic in the domain consisting of the z-plane less the points on the branch cut $y=0$, $x\le 0$. When $z=4$, the multivalued function $z^{1/2}$ equals $+2$ or -2. Suppose for our branch $z^{1/2}=2$ when $z=4$. What value does this branch assume when $z = 9\left[-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$?

(vii). Suppose a branch cut of the ftn. $f(z) = z^{-1/4}(z^2+1)$ assumes negative real values for $y=0$, $x>0$. There is a branch cut along the line $x=y$, $y\ge 0$. What values does this function assume at these points?

i). $z=-1$ (ii) $z=2i$ (iii) $z=-1-i$.

Multivalued ftns.

$w = f(z)$

z $\quad$ $f_1(z)$, $f_2(z)$ $\quad$ 5 $\quad$ 1

which have two or more than values of w for some or all values of z in given domain.

$$w = f(z) = z^{1/2} = (re^{i\theta})^{1/2} = \sqrt{r}\left\{e^{i(\theta+2n\pi)}\right\}^{1/2}$$

$$f(z) = r^{1/2} e^{i(n\pi+\theta/2)} = \sqrt{r}\, e^{i\theta/2} e^{in\pi}$$

$$w_1 = f_1(z) = \sqrt{r}\, e^{i\theta/2}, \quad f_2(z) = \sqrt{r}\, e^{i\theta/2}.$$

Branch:- multivalued ftns are Collection of single valued ftns. $\quad z \rightarrow \boxed{w} \rightarrow f_1(z),\ f_2(z)$

If $f(z)$ is a multivalued ftn then branch is a single valued ftn which is defined in the subdomain of $f(z)$.

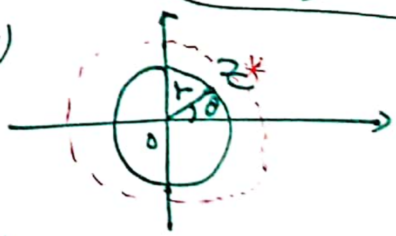
$$z^{1/3} = r^{1/3} e^{i\left(\frac{\theta+2n\pi}{3}\right)} = r^{1/3} e^{i\theta/3} e^{\frac{i2n\pi}{3}}$$

$$e^{i2n\pi/3} = 1, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i.$$

Branch Point:-

$$f(z) = z^{1/2}.$$

polar form: $f(z) = z^{1/2} = (re^{i\theta})^{1/2} = f_1(z)$



$z=0$, $\quad f(z) = z^{1/2} = \left[re^{i(\theta+2\pi)}\right]^{1/2}$

$$= r^{1/2} e^{i\theta/2} e^{\pi i} = -r^{1/2} e^{i\theta/2} = f_2(z)$$

Branch Point is a point ($z=0$) about which when we make a rotation the value of the function changes.

$$f(z) = -r^{1/2} e^{i(\theta+2\pi)/2} = -r^{1/2} e^{i\theta/2} e^{\pi i} = f_1(z)$$

$2\pi \rightarrow$ point $z \rightarrow f_1(z) \rightarrow f_2(z)$ $\quad f_1(z) \rightleftarrows f_2(z)$.

$4\pi \rightarrow$ " " $\rightarrow f_2(z) \rightarrow f_1(z)$

B. P is a point about which values of multivalued function $f(z)$ are interchanged when z describes a closed path round that point.

Branch point is a type of singularity.

B.P. is a point where fn is discontinuous.

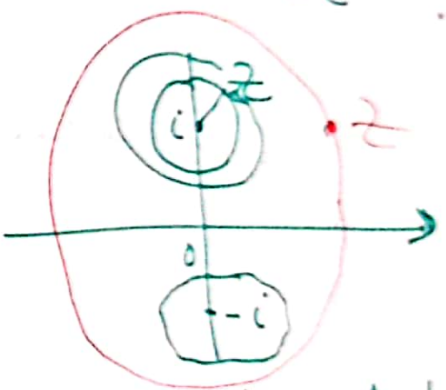
$f(z) = z^{1/2}$, $f'(z) = \frac{1}{2\sqrt{z}}$, $z=0$, a branch point.

$f(z) = (z^2+1)^{1/2}$, $f'(z) = \frac{2z}{(z^2+1)^{1/2}}$, $z^2+1=0$ $\Rightarrow z = \pm i$.

$(z^2+1)^{1/2} = \{(z+i)(z-i)\}^{1/2}$ $z+i = r_1 e^{i\theta_1}$, $z-i = r_2 e^{i\theta_2}$

$= \sqrt{r_1 r_2}\, e^{i\frac{\theta_1}{2}} \cdot e^{i\frac{\theta_2}{2}}$

$= \sqrt{r_1 r_2}\, e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}$



$\alpha_1 = \theta_1$, $\alpha_2 = \theta_1 + 2\pi$

$f(z) = \sqrt{r_1 r_2}\, e^{i\left(\frac{\theta_1+2\pi}{2}\right)} e^{i\theta_2/2}$

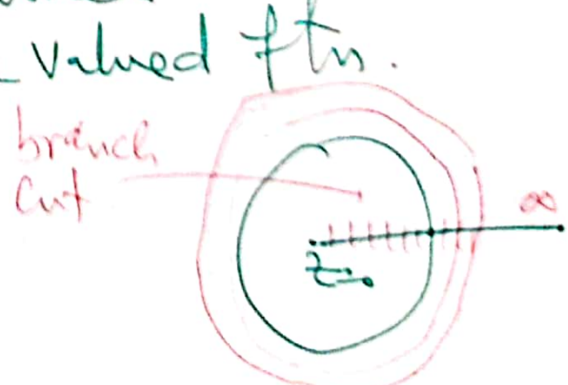
$= -\sqrt{r_1 r_2}\, e^{i\theta_1/2} e^{i\theta_2/2}$. $z = i$, rotate

both points, $\theta_1 \to \theta_1 + 2\pi$, $\theta_2 \to \theta_2 + 2\pi$

$f(z) = \sqrt{r_1 r_2}\, e^{i\theta_1/2} e^{i\theta_2/2}$ reduces

Branch cut $\to$ line/curve which ~~restrict~~ the multivalued fn into single valued fn.

$z^{1/2}$, $-\pi \leq \theta < \pi$.



$0 \leq \theta < 2\pi$