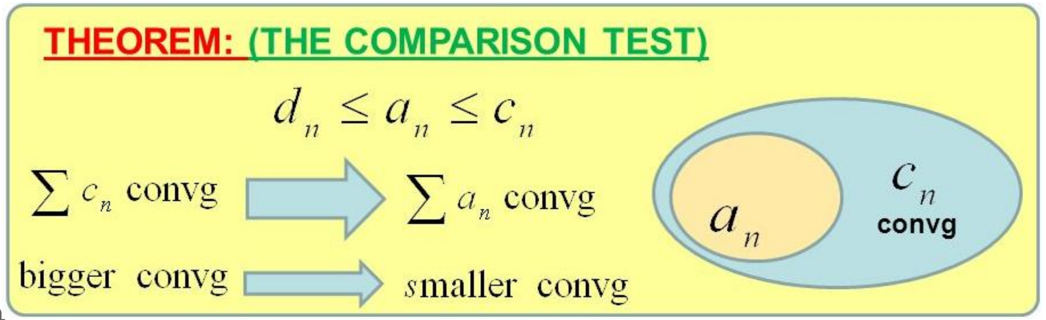
Infinite Series

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

Chapter: 11 (11.4, 11.5)

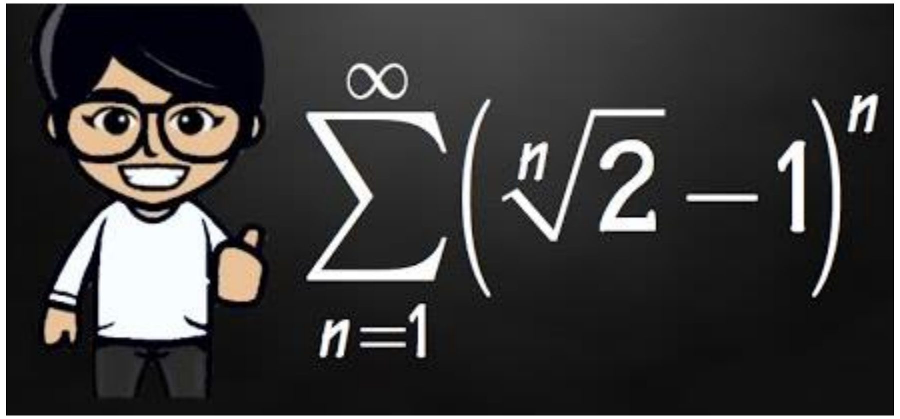
Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

Chapter: 11 (11.3, 11.4)



Ratio Test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

Root Test $\lim_{n\to\infty} \sqrt[n]{a_n}$



Calculus & Analytical Geometry MATH-101
Instructor: Dr. Naila Amir (SEECS, NUST)

Convergence/Divergence of a series

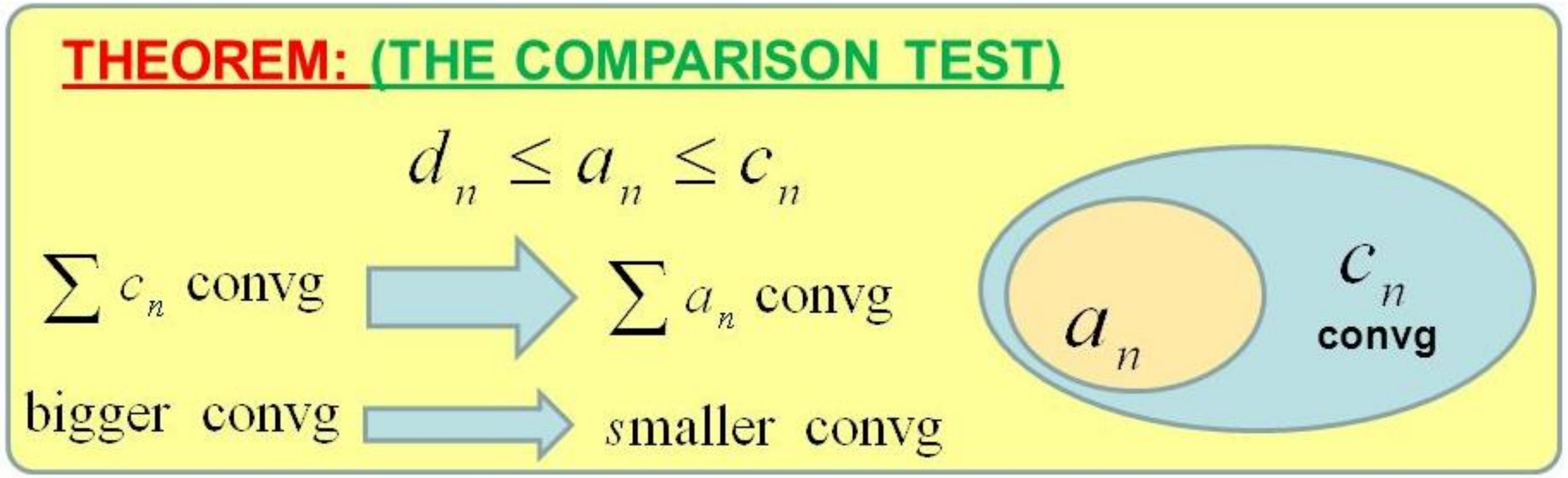
- In order to examine the convergence or divergence of an infinite series

$$\sum_{n=1}^{\infty} a_n \, ,$$

we need the n^{th} partial sum: $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ of the series. If the sequence of these partial sums $\{S_n\}$ converges to L, then the series is convergent, and sum of the series is L. If $\{S_n\}$ diverges, then the series diverges.

- But, for most of the series, it is often impossible to find an explicit formula for S_n. However, there exist several tests in literature to test the convergence or divergence of a series that employ the n^{th} term a_n. **But these tests just provide us the information about the convergence or divergence of the series, they do not give us the sum of a convergent series.**

The Comparison Tests



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Chapter: 11 (11.4)

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Chapter: 11 (11.3)

The Basic Comparison Test

Let $\sum a_n$ and $\sum b_n$ be positive-term series.

1. If $\sum b_n$ converges and $a_n \leq b_n$ for every positive integer n,

   then $\sum a_n$ converges.

2. If $\sum b_n$ diverges and $a_n \geq b_n$ for every positive integer n,

   then $\sum a_n$ diverges.

Example

Test the following series for convergence and divergence:

$$\sum_{n=2}^{\infty} \frac{3n}{n^2 - 2}$$

$b_n = \frac{1}{n}$

Solution: For every $n \geq 2$,

$$\frac{3n}{n^2 - 2} > \frac{3n}{n^2} = 3\left(\frac{1}{n}\right).$$

$a_n > b_n$

Since $3\sum_{n=2}^{\infty} \frac{1}{n}$ is a *divergent* series so by basic comparison test the given series is divergent.

Example

Test the following series for convergence and divergence:

$$\sum_{n=1}^{\infty} \frac{5n}{2n^3 + n^2 + 1}$$

$a_n < \frac{b_n}{c}$

Solution: For every $n \geq 1$,

b_n

$$\frac{5n}{2n^3 + n^2 + 1} < \frac{5n}{2n^3} = \frac{5}{2}\left(\frac{1}{n^2}\right).$$

Since $\frac{5}{2}\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2}}$ is a *convergent* series so by basic comparison test the given series is convergent.

$p = 2 > 1$

Euler's series

The Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be positive-term series.

1. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c > 0$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

3. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Example

Test the following series for convergence and divergence:

$$\sum_{n=2}^{\infty} \frac{3n}{n^2 - 2}.$$

Solution:

Here $a_n = \dfrac{3n}{n^2 - 2}$. Let $b_n = \dfrac{1}{n}$, so that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\frac{3n}{n^2 - 2}}{\frac{1}{n}} \right) = \lim_{n\to\infty} \left(\frac{3n}{n^2 - 2} \times n \right) = 3 > 0$$

Since $\sum b_n$ is divergent so $\sum a_n$ is also divergent.

Example

Test the following series for convergence and divergence:

$$\sum_{n=1}^{\infty} \frac{5n}{2n^3 + n^2 + 1}.$$

Solution:

Here $a_n = \dfrac{5n}{2n^3 + n^2 + 1}$. Let $b_n = \dfrac{1}{n^2}$, so that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{5n^3}{2n^3 + n^2 + 1} \right) = \frac{5}{2} > 0$$

Since $\sum b_n$ is convergent so $\sum a_n$ is also convergent.

Example

Test the following series for convergence and divergence:

$$\sum_{n=2}^{\infty} \frac{(1 + n \ln n)}{(n^2 + 5)}.$$

Solution:

Here $a_n = \dfrac{(1 + n \ln n)}{(n^2 + 5)}$. For large n, we expect a_n to behave like $\dfrac{n \ln n}{n^2} = \dfrac{\ln n}{n}$,

which is greater than $\dfrac{1}{n}$ $for\ n \geq 3$, so we take $b_n = \dfrac{1}{n}$, so that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n + n^2 \ln n}{n^2 + 5} \right) = \infty$$

So, by limit comparison test $\sum a_n$ is divergent.

Practice Questions

Test the following series for convergence or divergence.

1. $$\sum_{n=1}^{\infty} \frac{1}{n^{1/2} + n^{3/2}}$$

2. $$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$$

3. $$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$$

4. $$\sum_{n=1}^{\infty} \frac{1}{n!}$$

$$\frac{1}{n!} < \underbrace{\frac{1}{2^{n-1}}}_{b_n} \quad \forall n \geq 3$$

Practice Questions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Exercise: 11.4

  Q # 1 to Q # 36

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 11.3

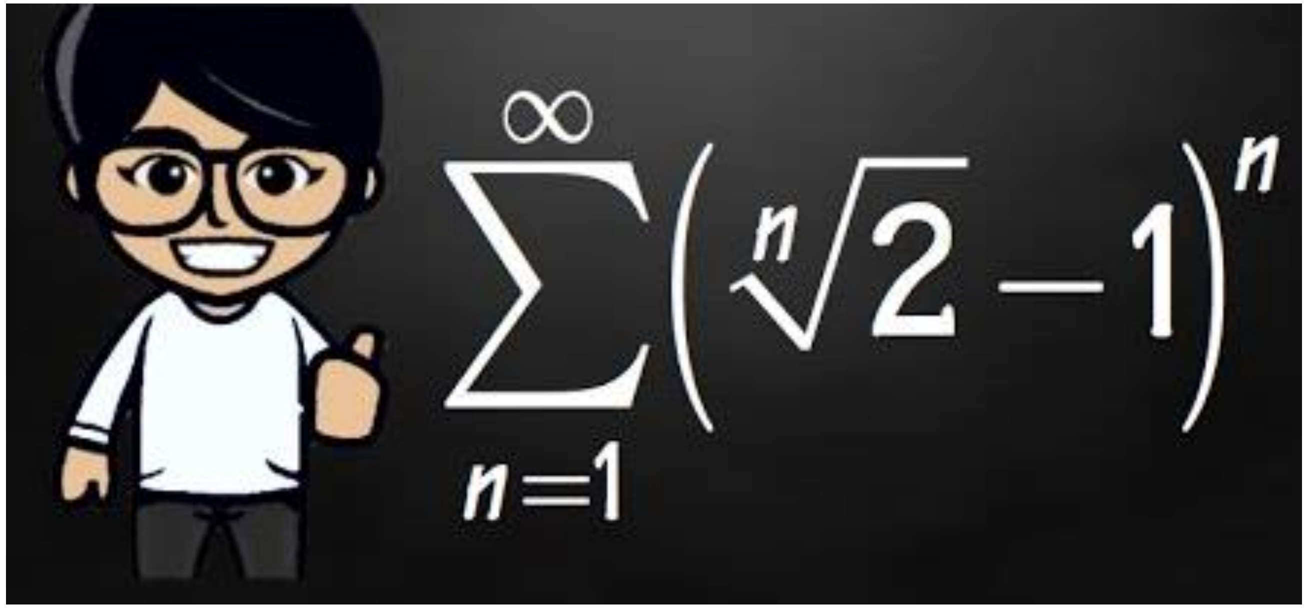
  Q # 13 to Q # 46

Ratio Test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

Root Test $\lim_{n\to\infty} \sqrt[n]{a_n}$

The Ratio and Root Tests

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n^8}$$



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Chapter: 11 (11.5)

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

Chapter: 11 (11.4)

The Ratio Test (D’ Alembert’s Test)

Let $\sum a_n$ be a positive-term series and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

Then

1. the series converges if $L < 1$,

2. the series diverges if $L > 1$ or L is infinite,

3. the test is inconclusive if $L = 1$.

Example

Investigate the convergence/divergence of the following series

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}.$$

Solution:

Here $a_n = \dfrac{2^n + 5}{3^n}$ and $a_{n+1} = \dfrac{2^{n+1} + 5}{3^{n+1}}$. Thus,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{2^{n+1} + 5}{3^{n+1}} . \frac{3^n}{2^n + 5} \right) = \lim_{n\to\infty} \left(\frac{1}{3} . \frac{2 + 5.2^{-n}}{1 + 5.2^{-n}} \right) = \frac{1}{3} \lim_{n\to\infty} \left(\frac{2 + 5.2^{-n}}{1 + 5.2^{-n}} \right) = \frac{2}{3}.$$

Since, $L = \dfrac{2}{3} < 1$ so by ratio test $\sum a_n$ is convergent.

The Ratio Test (D' Alembert's Test)

- Test the following series for convergence and divergence:

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}.$$

$(n+1)! = (n+1)\,n!$

Solution:

Here $a_n = \dfrac{(2n)!}{(n!)^2}$ and $a_{n+1} = \dfrac{(2n+2)!}{[(n+1)!]^2}$. Thus,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{(2n+2)!}{[(n+1)!]^2} . \frac{(n!)^2}{(2n)!} \right) = \lim_{n\to\infty} \left(\frac{(2n+2)(2n+1)(2n)!}{[(n+1).n!]^2} . \frac{(n!)^2}{(2n)!} \right)$$

$$= \lim_{n\to\infty} \left(\frac{(2n+2)(2n+1)(2n)!}{(n+1)^2(n!)^2} . \frac{(n!)^2}{(2n)!} \right) = \lim_{n\to\infty} \left(\frac{2(n+1)(2n+1)}{(n+1)^2} \right) = 2 \lim_{n\to\infty} \left(\frac{2n+1}{n+1} \right)$$

$$= 2(2) = 4.$$

Since, $L = 4 > 1$ so by ratio test $\sum a_n$ is divergent.

The Root Test

Let $\sum a_n$ be a positive-term series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L.$$

Then

1. the series converges if $L < 1$,
2. the series diverges if $L > 1$ or L is infinite,
3. the test is inconclusive if $L = 1$.

Example

- Investigate the convergence/divergence of the following series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Solution:

Here $a_n = \dfrac{n^2}{2^n}$ and $\sqrt[n]{a_n} = \left(\dfrac{n^2}{2^n}\right)^{1/n} = \dfrac{(n^2)^{1/n}}{(2^n)^{1/n}} = \dfrac{(n)^{2/n}}{2}$. Thus,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{(n)^{2/n}}{2}\right) = \frac{1}{2} \lim_{n\to\infty} \left((n)^{2/n}\right) = \frac{1}{2}.$$

Since, $L = \dfrac{1}{2} < 1$ so by root test $\sum a_n$ is convergent.

Example

- Investigate the convergence/divergence of the following series

$$\sum_{n=1}^{\infty}\left(\frac{1}{1+n}\right)^{n}.$$

Solution:

Here $a_n = \left(\frac{1}{1+n}\right)^{n}$ and $\sqrt[n]{a_n} = \left(\left(\frac{1}{1+n}\right)^{n}\right)^{1/n} = \frac{1}{1+n}$. Thus,

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\left(\frac{1}{1+n}\right) = 0.$$

Since, $L = 0 < 1$ so by root test $\sum a_n$ is convergent.

Example

- Investigate the convergence/divergence of the following series

$$\sum_{n=2}^{\infty} \left(\frac{n}{\ln n}\right)^n.$$

Solution:

Here $a_n = \left(\frac{n}{\ln n}\right)^n$ and $\sqrt[n]{a_n} = \left(\left(\frac{n}{\ln n}\right)^n\right)^{1/n} = \frac{n}{\ln n}$. Thus,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n}{\ln n}\right) = \lim_{n\to\infty} \left(\frac{1}{1/n}\right) = \infty.$$

So by root test $\sum a_n$ is divergent.

Practice Questions

Determine whether the following series converges or diverges?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{(\ln n)^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(n^2)!}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)(3^n+1)}{4^n+1}$

Practice Questions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Exercise: 11.5

  Q # 1 to Q # 44

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 11.4

  Q # 13 to Q # 40