Complex Integrals

i) Complex Line Integral

In MVC, we have:

Let $\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$ be a vector field, $C: \bar{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, $a \le t \le b$ be a parametric curve. Then $d\bar{r} = dx\hat{i} + dy\hat{j} = \bar{r}'(t)dt$.

$$\int_C \bar{F} \cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(t)) \cdot \vec{r}'(t)dt.$$

More explicitly,

$$\int_C P dx + Q dy = \int_a^b \left(P(x(t),y(t))x'(t) + Q(x(t),y(t))\underbrace{\,}_{y'(t)}\right) dt.$$

Now suppose $f(z) = u(x,y) + jv(x,y)$ be a complex function, and C as a curve in the complex plane, still defined by $\begin{cases} x(t) \\ y(t) \end{cases}$, i.e, $z = z(t) = x(t) + j(y(t))$.

Then
$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

More explicitly using $u(x,y), v(x,y)$: $z = x + jy \Rightarrow dz = dx + jdy$.

$$\int_C f(z)dz = \int_C (u(x,y) + jv(x,y))(dx + jdy)$$

$$= \int_C u(x,y)dx - v(x,y)dy + j\int_C v(x,y)dx + u(x,y)dy.$$

[complex integrals 1]

Contours:-

We know $C: z(t) = x(t) + j\,y(t),\ a \le t \le b$.

$$z'(t) = x'(t) + j\,y'(t).$$

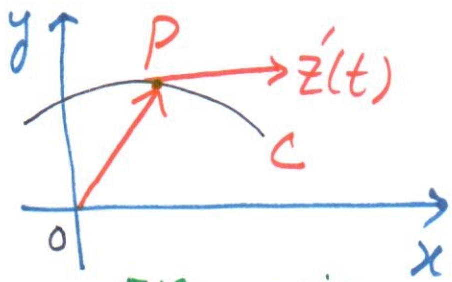
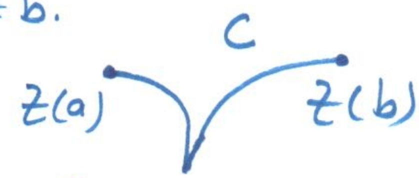


Figure (i)

We say a curve in the complex plane is smooth if $z'(t)$ is continuous and never vanishes in the interval $a \le t \le b$.

A smooth curve can have no sharp corners or cusps.



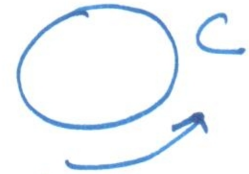
Curve C is not smooth.

Figure (ii)

A piecewise smooth curve C has a continuously turning tangent, except possibly at the points where the component smooth curves $C_1, C_2, \ldots, C_n$ are joined together.

In complex analysis, a piecewise smooth curve C is called a contour or path.

The positive direction on a contour C to be the direction on the curve corresponding to increasing value of the parameter t.



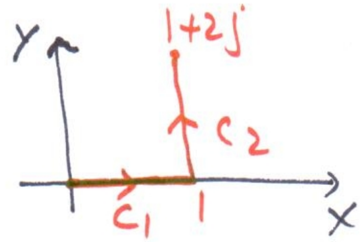
positive direction

It is also said that the curve C has positive orientation.

In the case of a simple closed curve C, the positive direction roughly corresponds to the counterclockwise direction.

Ex:- Integrate $f(z) = Re(z)$ along the curve c:

we parametrize c_1, c_2:



$c_1: z = z(t) = t + 0j, \quad 0 \leq t \leq 1, \quad z'(t) = 1$

$c_2: z = z(t) = 1 + tj, \quad 0 \leq t \leq 2, \quad z'(t) = j.$

$$\therefore \int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz$$

$$= \int_0^1 Re(t+0j)\cdot 1\,dt + \int_0^2 Re(1+tj)\cdot j\,dt$$

$$= \int_0^1 t\,dt + \int_0^2 j\,dt = \frac{1}{2} + 2j.$$

A continuous curve (an arc) C in the complex plane is defined by $C: z(t) = x(t) + j\,y(t) \quad (t \in [a,b],\ a < b) \quad (1)$

where $x(t)$ and $y(t)$ are real-valued, continuous functions of the real variable t.

For a parametrized curve C defined by (1), the point $z(a)$ is called the initial point of C and $z(b)$ the terminal point of C. If the initial and terminal points coincide i.e, $z(a) = z(b)$, then C is said to be a closed curve. If $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$, so that C do not intersect itself, the curve is said to be simple. (Complex

A closed curve $C: z(t), t \in [a,b]$, that is simple in the interval with the possible exception that $z(a) = z(b)$ is said to be a simple closed curve or Jordan curve.

[Complex integrals 04]

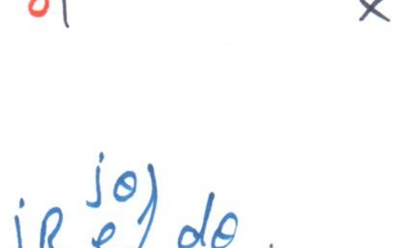
Simple closed

not simple closed.

simply connected domain
(Interior of a simple closed curve)

Not simply connected
(have holes).
multiply connected

EX:- Evaluate $I = \int_C (z - z_0)^n \, dz$, $n \in \mathbb{Z}$, and C is the circle of radius R, in the complex plane, centered at z_0, and oriented counterclockwise.



we use the angle θ to parametrize C

$$C: \quad z - z_0 = R e^{j\theta} \quad (0 \le \theta \le 2\pi)$$

$$z = z_0 + R e^{j\theta} \quad (0 \le \theta \le 2\pi)$$

$$\frac{dz}{d\theta} = jR e^{j\theta}$$

$$I = \int (z - z_0)^n \, dz = \int^{2\pi} \left(R e^{j\theta}\right)^n \left(jR e^{j\theta}\right) d\theta .$$

$$I = j R^{n+1} \int_0^{2\pi} e^{j(n+1)\theta} \, d\theta .$$

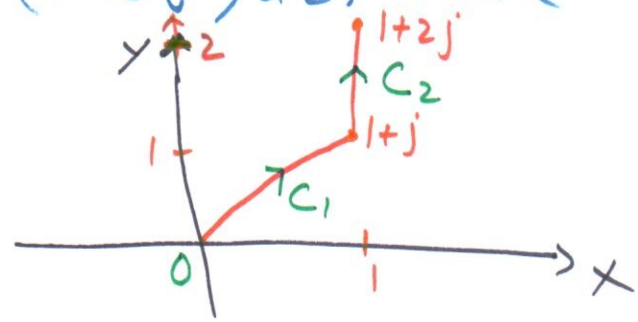
Case i: $n \neq -1 \Rightarrow n+1 \neq 0$. $I = \dfrac{1}{j(n+1)} e^{j(n+1)\theta} \Big|_0^{2\pi}$

$$= \frac{1}{j(n+1)} \left[e^{j(n+1)2\pi} - e^{j(0)} \right] = \frac{1}{j(n+1)} [1-1] = 0 .$$

Case ii: $n = -1 \Rightarrow n+1 = 0$. $I = j \displaystyle\int_0^{2\pi} 1 \, d\theta = 2\pi j$

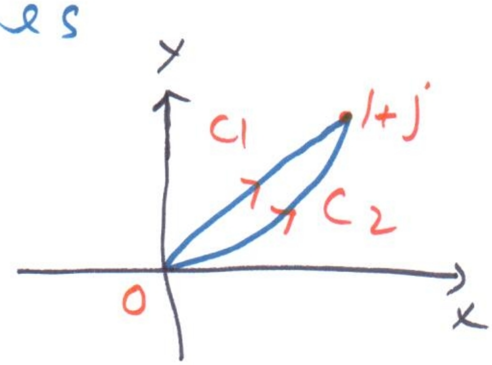
Conclusion: $\displaystyle\int_C (z-z_0)^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi j, & n = -1 . \end{cases}$

class activity: (i). Evaluate $\displaystyle\int_C (x^2 + j y^2) dz$, where C is a piecewise smooth curve shown in figure.



$$\int_C (x^2 + j y^2) dz = \int_{C_1} (x^2 + j y^2) dz + \int_{C_2} (x^2 + j y^2) dz$$
$$= \frac{2}{3} j + \left(\frac{-7}{3} + j \right) .$$

(ii). Find $\displaystyle\int_C |z|^2 dz$ along the curves



(a) $C = C_1$: $z_1(t) = t + jt \ (0 \leq t \leq 1)$,

(b) $C = C_2$: $z_2(t) = t^2 + jt \ (0 \leq t \leq 1)$.

observe that $\displaystyle\underbrace{\int_{C_1} |z|^2 dz}_{\left(\frac{2}{3} + \frac{2}{3} j\right)} \neq \underbrace{\int_{C_2} |z|^2 dz}_{\left(\frac{5}{6} + \frac{8}{15} j\right)}$.

Despite the fact that the straight line C_1 and the parabola C_2 have the same initial and terminal points

Cauchy's integral theorem:- Suppose $f(z)$ is analytic in a simply connected domain D, then for any piecewise smooth simple closed path C in D,

$$\int_C f(z)\,dz = 0 .$$

EX:- For any piecewise smooth closed curve C, we have $\int_C e^z dz = 0$, $\int_C \sin z\, dz = 0$, $\int_C (z-z_0)^n dz = 0 \quad (n = 0, 1, 2, \dots)$

EX:- Let C be the unit circle. Then although $f(z) = \dfrac{1}{(z+2)^2}$ is not analytic $z = -2$, it is analytic in the unit disk. So $\int_C \dfrac{1}{(z+2)^2} dz = 0$.

Note: The theorem may fail if $f(z)$ is not analytic, e.g., $C: z(t) = e^{jt}$, $0 \le t \le 2\pi$ (unit circle)

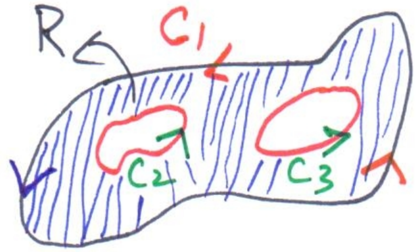
then $\int_C \bar{z}\, dz = \int_0^{2\pi} e^{-jt} \cdot j e^{jt} dt$

$$= j \int_0^{2\pi} dt = 2\pi j \neq 0 .$$

[complex integrals 06]

Cauchy's theorem for multiply connected domain:-

Let R be a closed region whose boundary consists of finitely many simple closed contours C_j oriented so that the points in R lie to the left of the boundary.

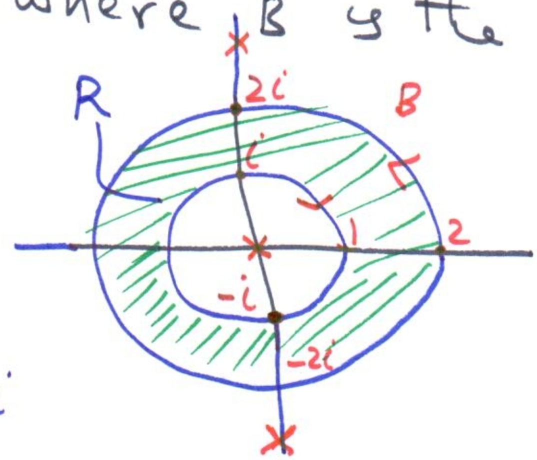


Let $B = \cup C_j$ and suppose f is analytic on R. Then

$$\int_B f(z)\,dz = 0.$$

As you walk around the contour the region R is on your left.

EX:- Evaluate $\displaystyle\int_B \frac{1}{z^2(z^2+9)}\,dz$ where B is the boundary of the annular region R.



$f(z) = \dfrac{1}{z^2(z^2+9)}$ is analytic everywhere except $z = 0, \pm 3i$

so $f(z)$ is analytic on region R since $0, \pm 3i$ are not in the region R.

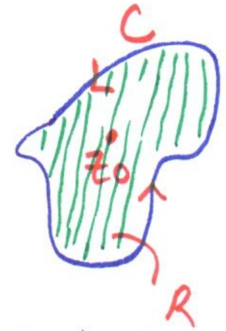
Hence by Cauchy's theorem for multiply connected domains

$$\int_B f(z)\,dz = 0.$$

[Complex integrals 07]

The cauchy integral formual:-

Suppose $f(z)$ is analytic everywhere within and on a simple closed Contour C (with Counterclockwise orientation).



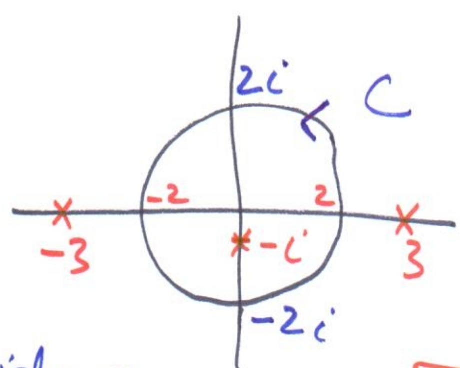
Let z_0 be any point interior the Countor C.

Then $$\int_C \frac{f(z_0)}{z-z_0}\,dz = 2\pi i\,[f(z_0)].$$

EX:- Let c be the circle $|z|=2$ (with Counter-clockwise orientation). find $\int \frac{z}{(9-z^2)(z+i)}\,dz$.

$$f(z) = \frac{z}{9-z^2} \text{ is}$$

analytic everywhere except at $z = \pm 3$. So $f(z)$ is analytic on and inside c.



So by the cauchy integral formula,

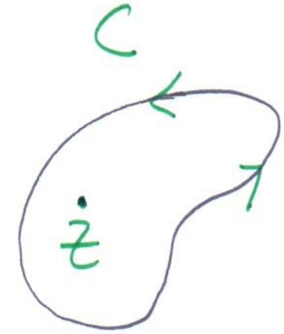
$$\int_c \frac{z}{(9-z^2)(z+i)}\,dz = \int_c \frac{f(z)}{z+i}\,dz$$

$$= \int_c \frac{f(z)}{z-(-i)} = 2\pi i\,[f(-i)]$$

$$= 2\pi i\left(\frac{-i}{9-(-i)^2}\right) = \frac{2\pi}{10} = \frac{\pi}{5}.$$

[complex integrals 08]

Derivatives of Analytic Functions:-



Let f be analytic on a simple closed contour and on it's interior.
Let z be any point interior to C.
Then by the Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z}\, ds \,, \quad \text{and}$$

$$f'(z) = \frac{1}{2\pi i} \frac{\partial}{\partial z} \int_C \frac{f(s)}{s-z}\, ds$$

$$= \frac{1}{2\pi i} \int_C \frac{\partial}{\partial z}\left(\frac{f(s)}{s-z} \right) ds$$

$$= \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2}\, ds \,.$$

Similarly,

$$f''(z) = \frac{1}{2\pi i} \int_C \frac{\partial}{\partial z}\left[\frac{f(s)}{(s-z)^2} \right] ds$$

$$= \frac{2}{2\pi i} \int_C \frac{f(s)}{(s-z)^3}\, dz \,.$$

[Complex integrals 09]

This procedure could be summarized in a result.

Result: Suppose f is analytic at a point z_0 and suppose $f(z) = u(x,y) + j\,v(x,y)$.

Then i) There is a neighborhood of z_0 for which the derivative of $f(z)$ to all orders exist and are analytic at this point;

ii) The functions $u(x,y)$ and $v(x,y)$ have continuous partial derivatives at (x_0, y_0);

iii) Let C be a simple closed contour (with positive orientation). Suppose $f(z)$ is analytic on C and interior of C and z_0 is inside C. Then

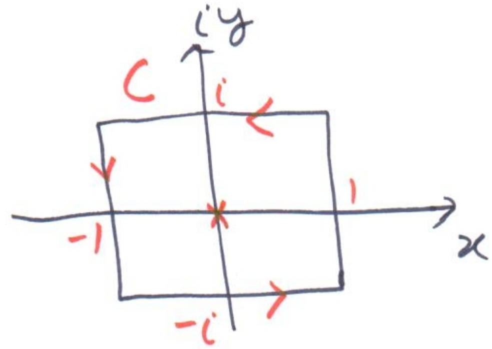
$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$$\Rightarrow \boxed{\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = 2\pi i \left[\frac{f^{(n)}(z_0)}{n!} \right]}$$

where $f^{(n)}(z_0)$ is the nth-order derivative of $f(z)$ at z_0.

Generalized Cauchy integral formula.

EX: Find $\int_C \frac{\sinh z}{z^4}\,dz$ where C is the square below.



$$f(z) = \sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right)$$

is an entire (Analytic in the entire complex plane).

Now, $$f^{(3)}(0) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-0)^4}\,dz$$

Since 0 is inside the Contour.

Hence, $$\int_C \frac{\sinh z}{z^4}\,dz = \frac{2\pi i}{3!}f^{(3)}(0) = \frac{\pi i}{3}f^{(3)}(0).$$

Now, $f'(z) = \cosh z$, $f''(z) = \sinh z$, $f'''(z) = \cosh z$

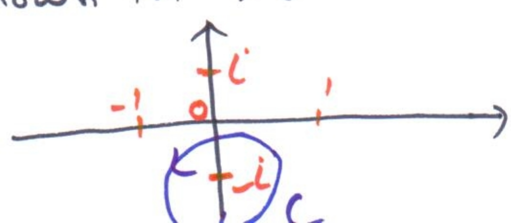
& $f^{(3)}(0) = \cosh 0 = 1$.

Hence, $$\int_C \frac{\sinh z}{z^4}\,dz = \frac{\pi i}{3}.$$

Class activity:- Evaluate $\int_C \frac{z^4 - 3z^2 + 6}{(z+i)^3}\,dz$

where C is the Contour shown in figure.

[Complex integrals !!]



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Complex Variables & transforms - BEE3CD - Problem Sheet No. 03

Q-1 Evaluate the Contour integral $\oint_C \frac{2z}{(z-1)(z+2)(z+i)} dz$

where C is a Contour that includes the three points $z=1$, $z=-2$ and $z=-i$.

Q-2 Let C be the circle $|z|=5$ traversed once Counterclockwise.

Evaluate $\int_C \frac{z^2+z-1}{z-4} dz$.

Q-3. Use Cauchy integral formula and Generalized Cauchy integral formula to evaluate the following integrals:

(a) $\int_C \frac{z^2+1}{z} dz$ (b) $\int \frac{z^3}{z^2+6z+5} dz$

(c) $\int_C \frac{\sin z}{z^2+1} dz$ (d) $\int \frac{z\, e^{z^2+z}}{(z-1)^3} dz$

where C is the circle $|z|=4$, positively oriented.

Q-4 Using Cauchy's integral formula compute the integral:

$$\int_{|z|=2} \frac{(z+1)}{(z-1)^2(z-3)} dz.$$