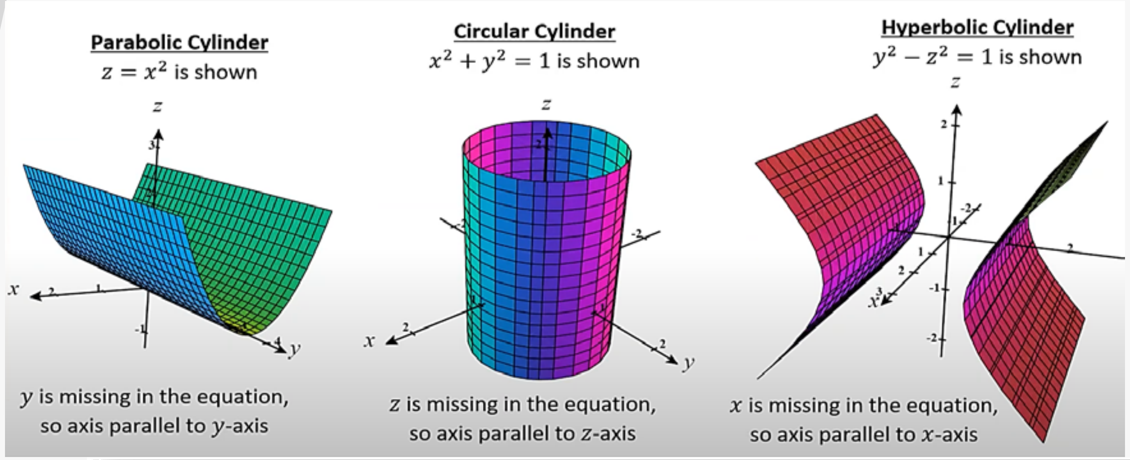
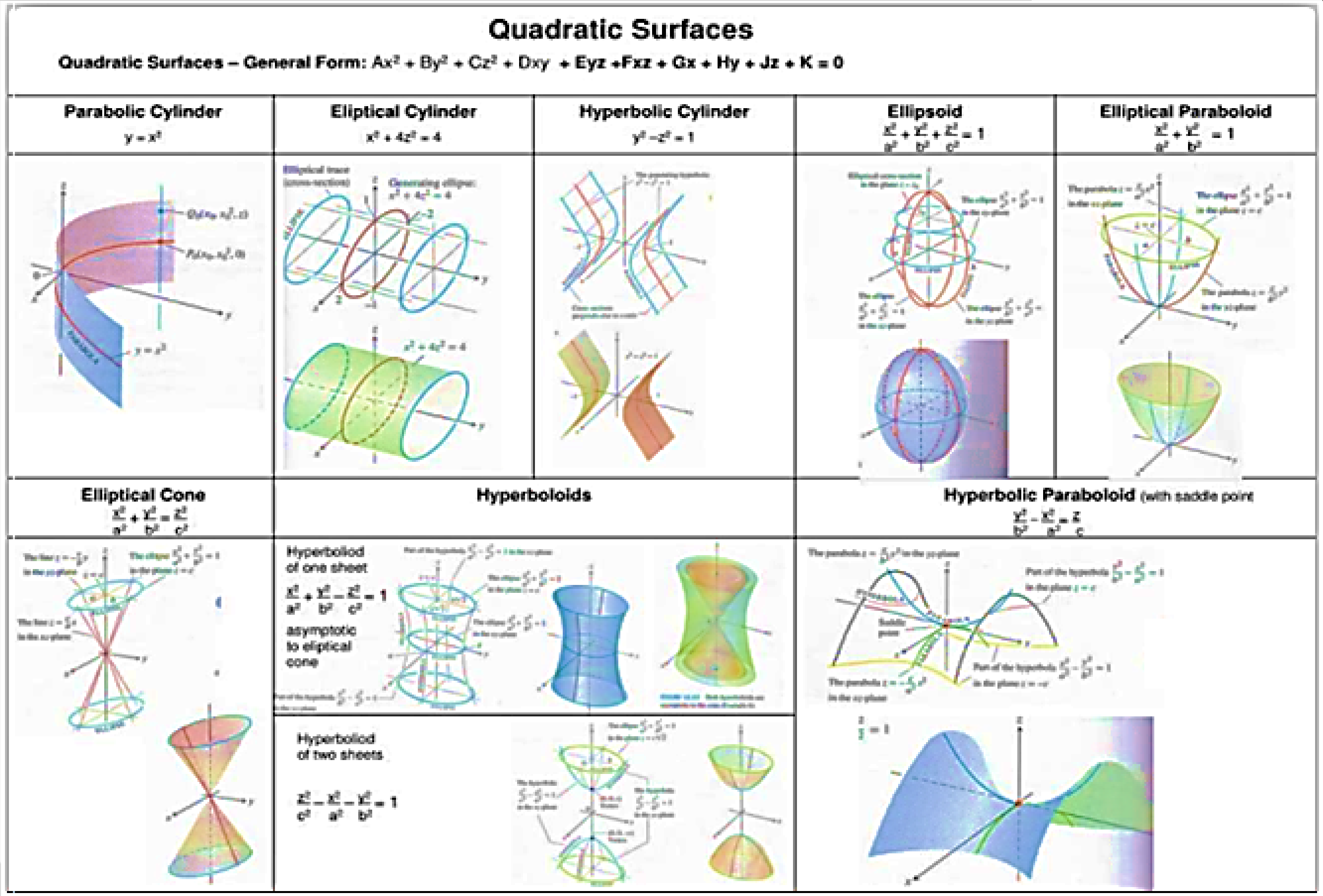
Cylinders & Quadric Surfaces





Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

Vectors And
The Geometry Of Space

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Chapter: 12 , Section: 12.6

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.
Chapter: 12 , Section: 12.6

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence
Chapter: 14, Section: 14.6

Practice Question

Problem: Identify the surface by sketching out traces on different coordinate planes.

$$x^2 + 2z^2 - 6x - y + 10 = 0.$$

Solution: In order to identify and sketch the given surface first we are going to transform the equation into the standard form by completing the square. For this we proceed as:

$$x^2 - 2(x)(3) + 9 - 9 + 2z^2 - y + 10 = 0.$$
$$\Longrightarrow (x-3)^2 + 2z^2 - y + 1 = 0.$$
$$\Longrightarrow (x-3)^2 + 2z^2 = y - 1.$$
$$\Longrightarrow \frac{(x-3)^2}{2} + z^2 = \frac{1}{2}(y-1).$$

This is an elliptic paraboloid. The axis of the paraboloid is parallel to the y −axis, with vertex at the point $(3,1,0)$.

Elliptic paraboloid

Solution: The trace in the xy −plane: $z = 0$ is given as:

$$(x-3)^2 = y - 1.$$

This is a parabola in xy −plane with vertex at $(3,1)$. While this is representing a parabolic cylinder in 3D with vertex at $(3,1,0)$.

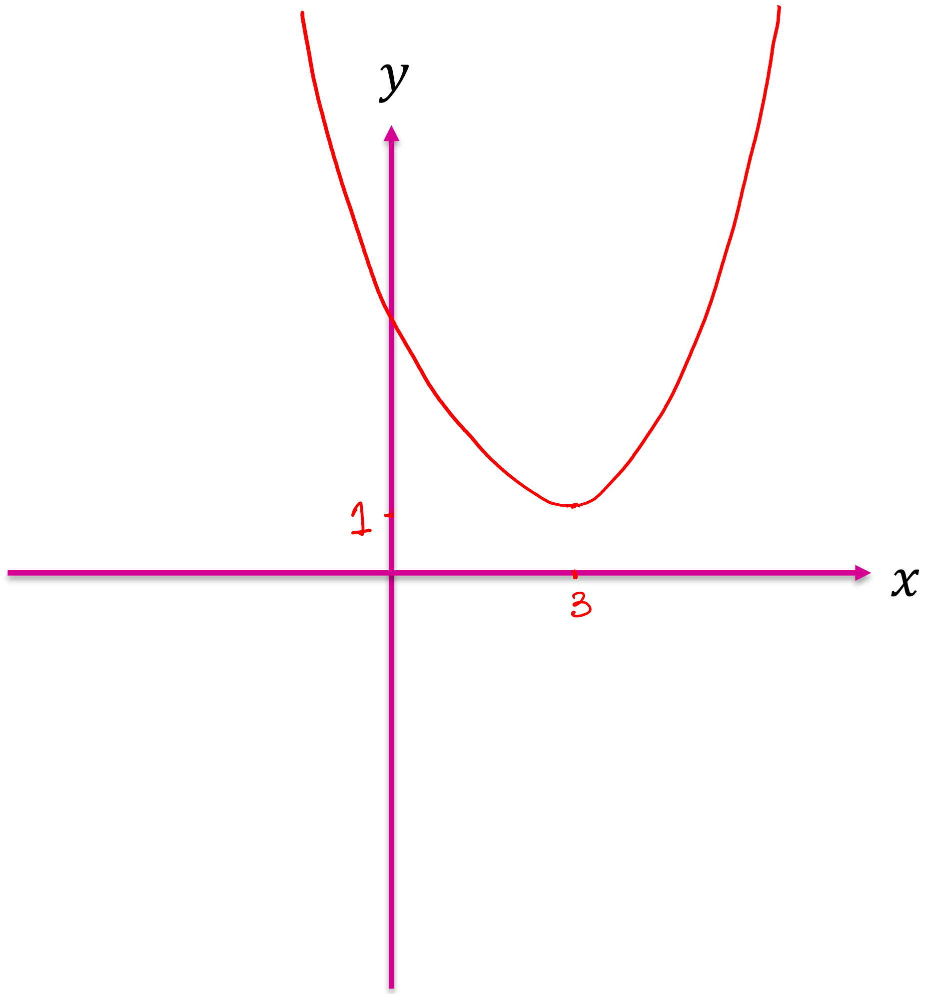
For $z = k$, we will get:

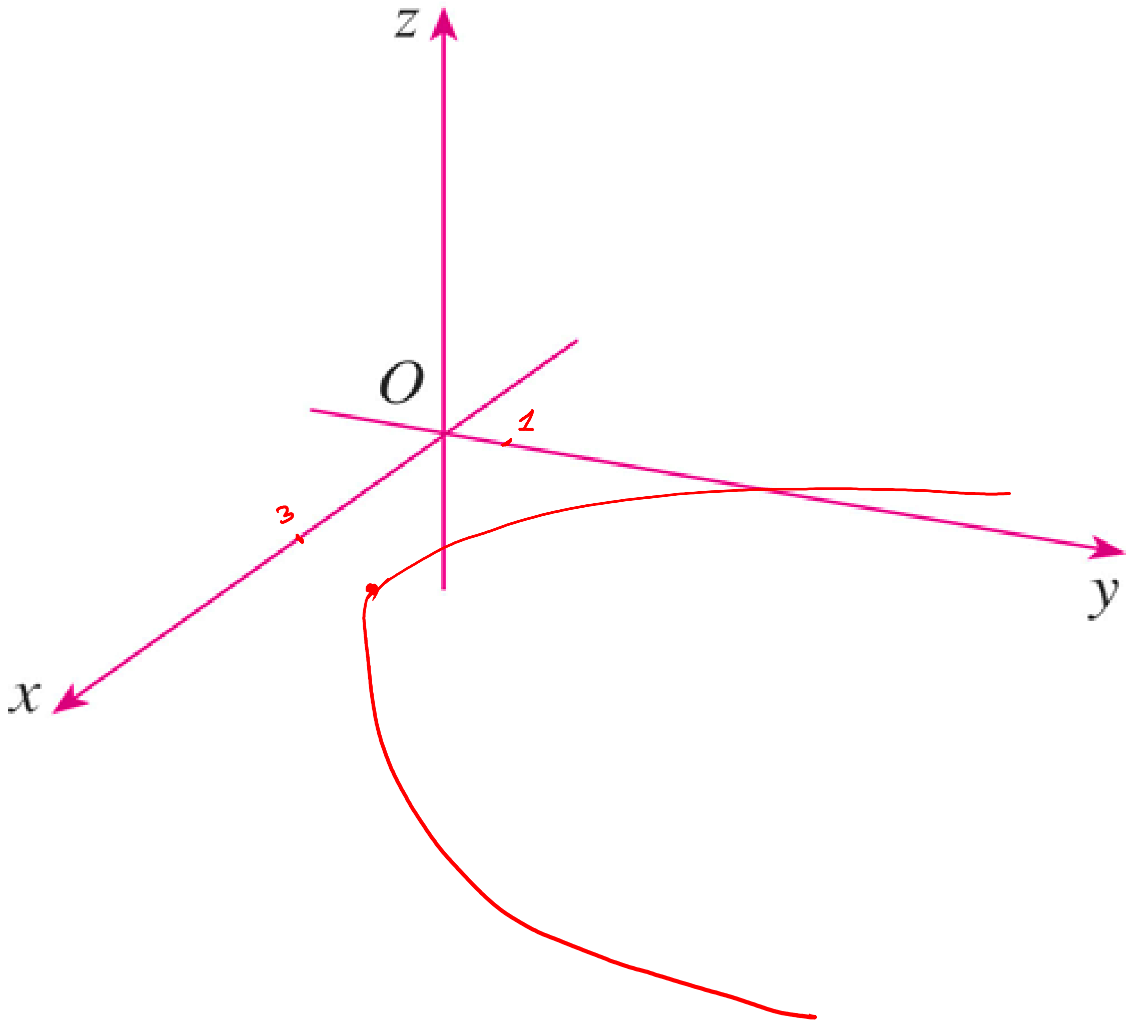
$$(x-3)^2 = y - (1 + 2k^2),$$

which is going to provide us with parabolas in xy −plane for different values of k.

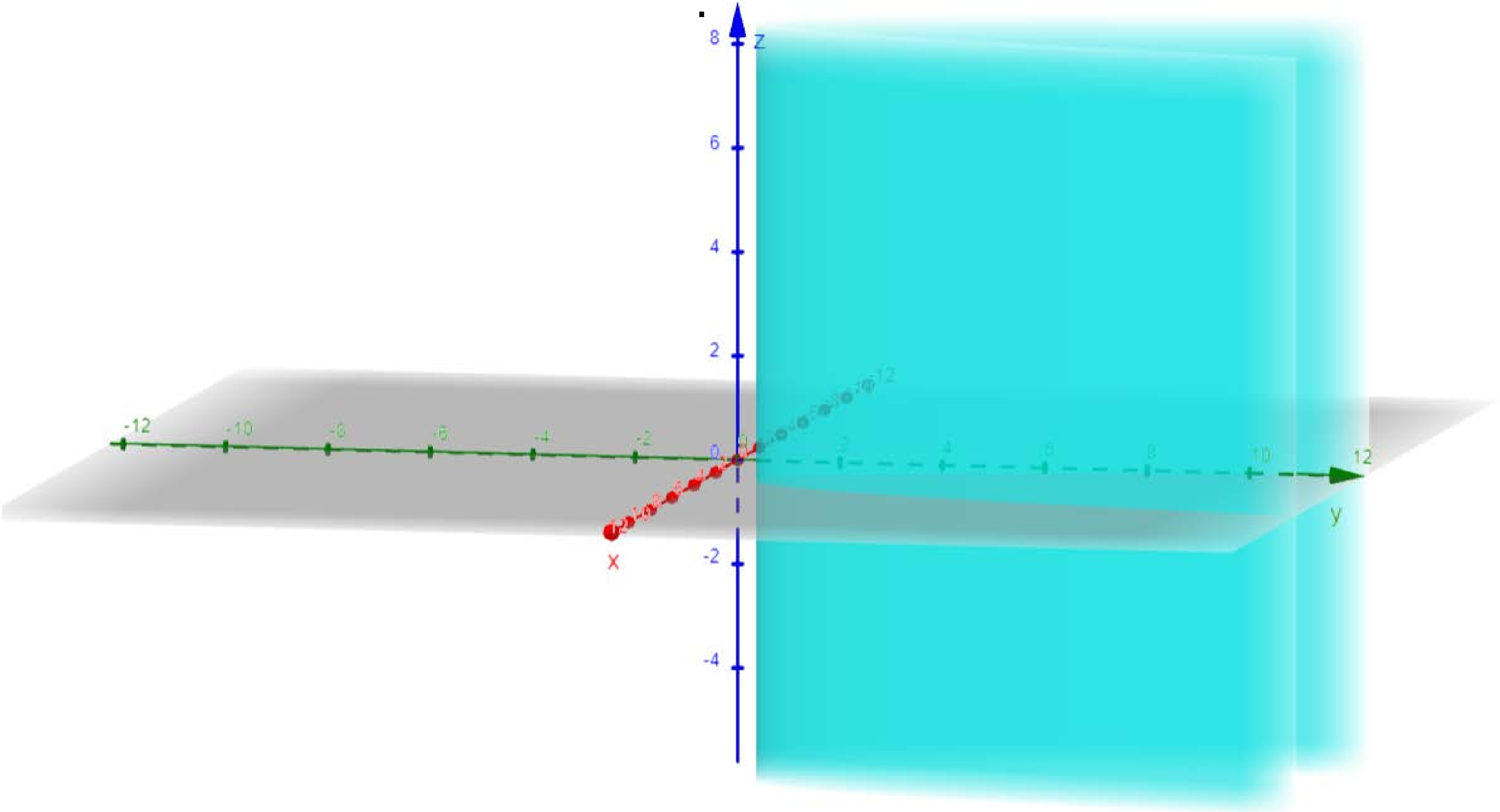
Solution: The trace in the xy −plane: $z = 0$:

$$(x-3)^2 = y - 1.$$





Solution: 3D plot of the trace: $(x-3)^2 = y-1\,; z = 0$ is a parabolic cylinder with vertex at $(3,1,0)$.



Elliptic paraboloid

Solution: The trace in the yz −plane: $x = 0$ is given as:

$$z^2 = \frac{1}{2}(y - 10).$$

This is a parabola in yz −plane with vertex at $(10,0)$. While this is representing a parabolic cylinder in 3D with vertex at $(0,10,0)$.

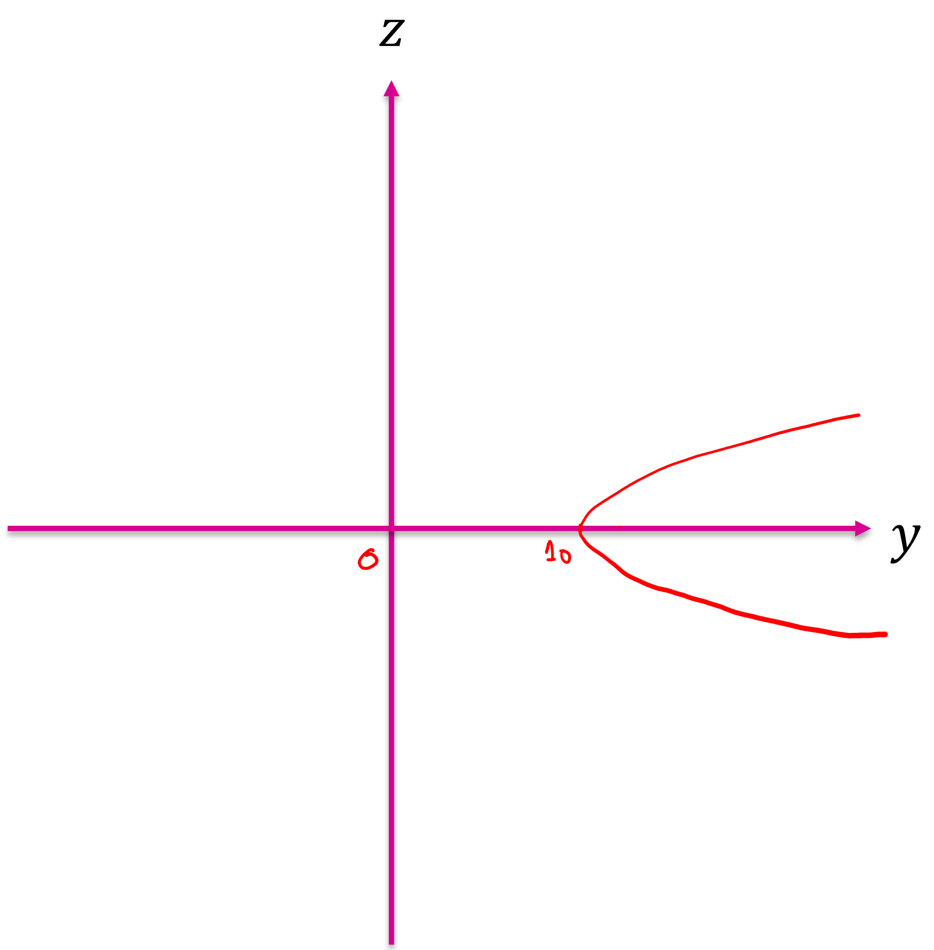
For $x = k$, we will get:

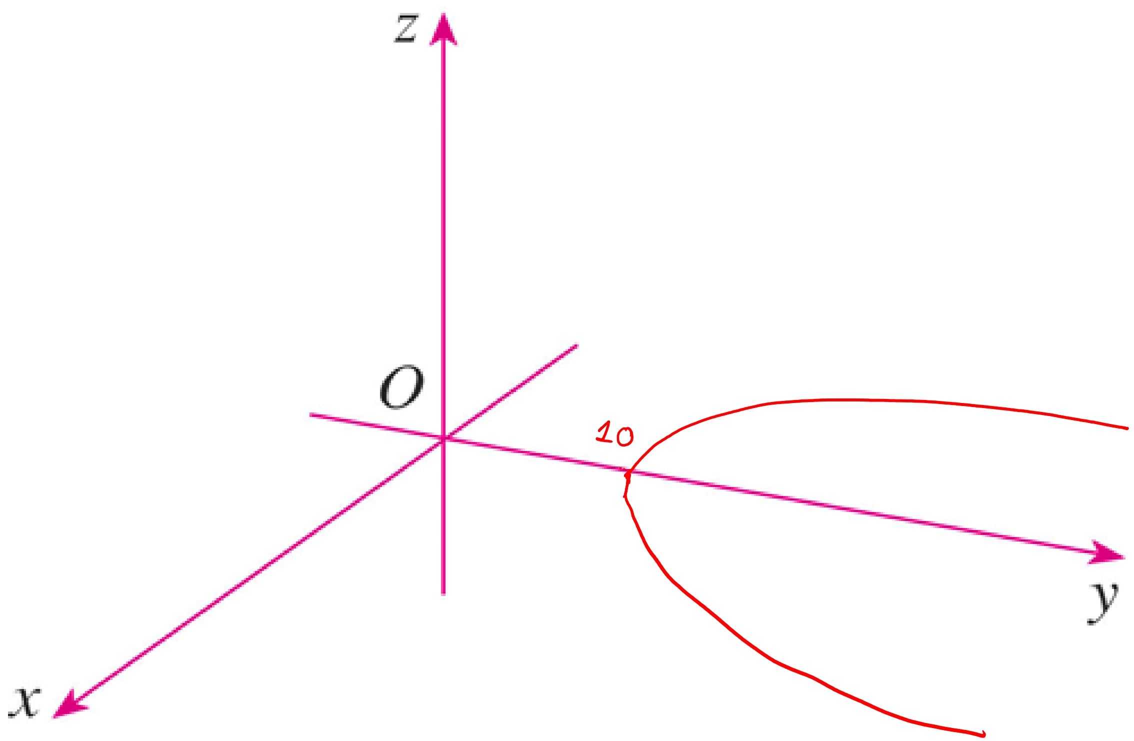
$$2z^2 = y - [1 + (k - 3)^2],$$

which is going to provide us with parabolas in yz −plane for different values of k.

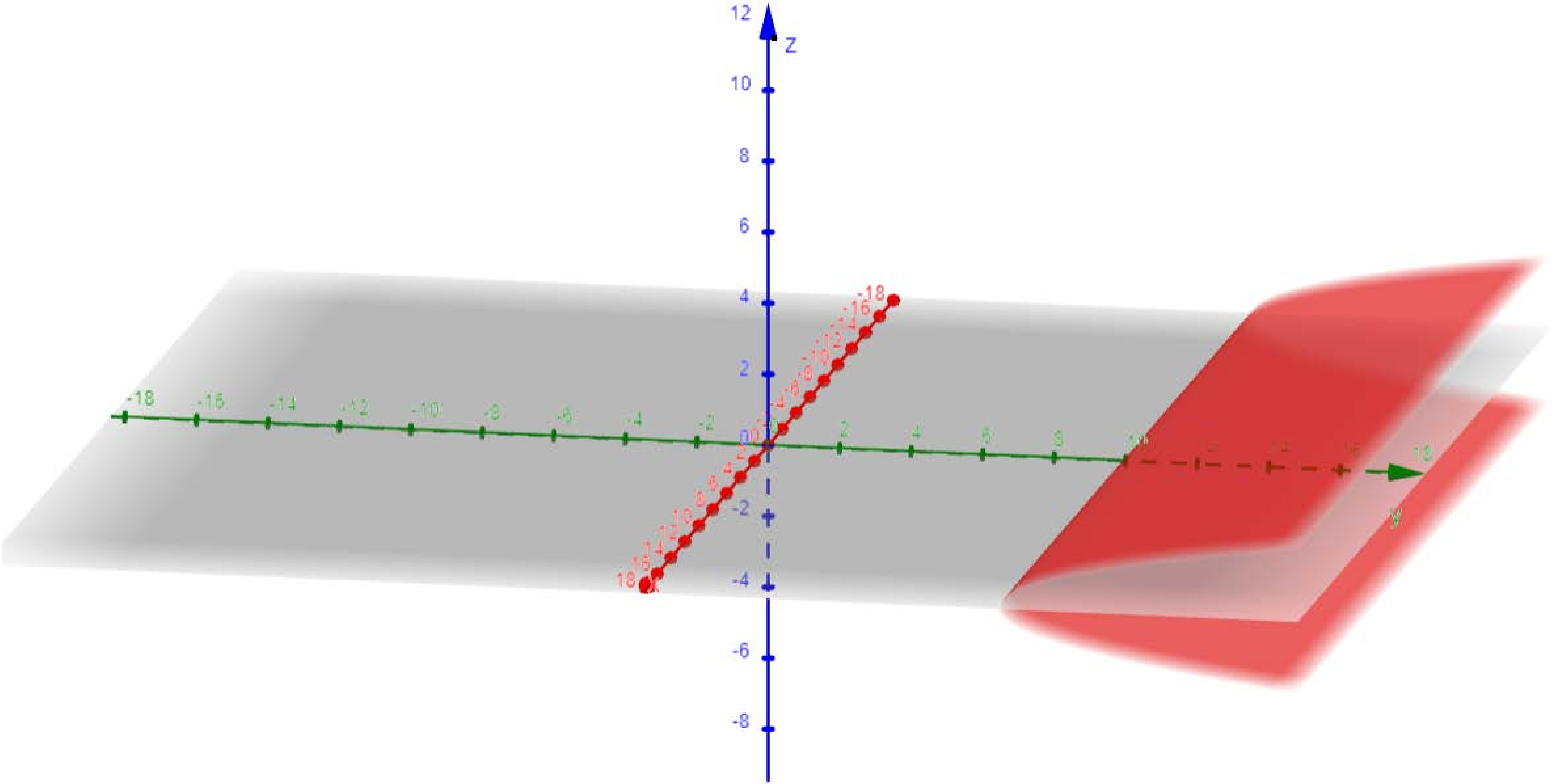
Solution: The trace in the yz −plane: $x = 0$:

$$z^2 = \frac{1}{2}(y - 10).$$





Solution: 3D plot of the trace: $z^2 = \frac{1}{2}(y - 10); x = 0$ is a parabolic cylinder with vertex at $(0,10,0)$.



Elliptic paraboloid

Solution: The trace in the xz −plane: $y = 0$ is given as:

$$(x-3)^2 + 2z^2 = -1.$$

No graph is possible for above equation.

For $y = 1,$ we will get:

$$(x-3)^2 + 2z^2 = 0.$$

which is a point $(3,0)$ in a plane parallel to xz −plane.

For $y = k > 1,$ we will get:

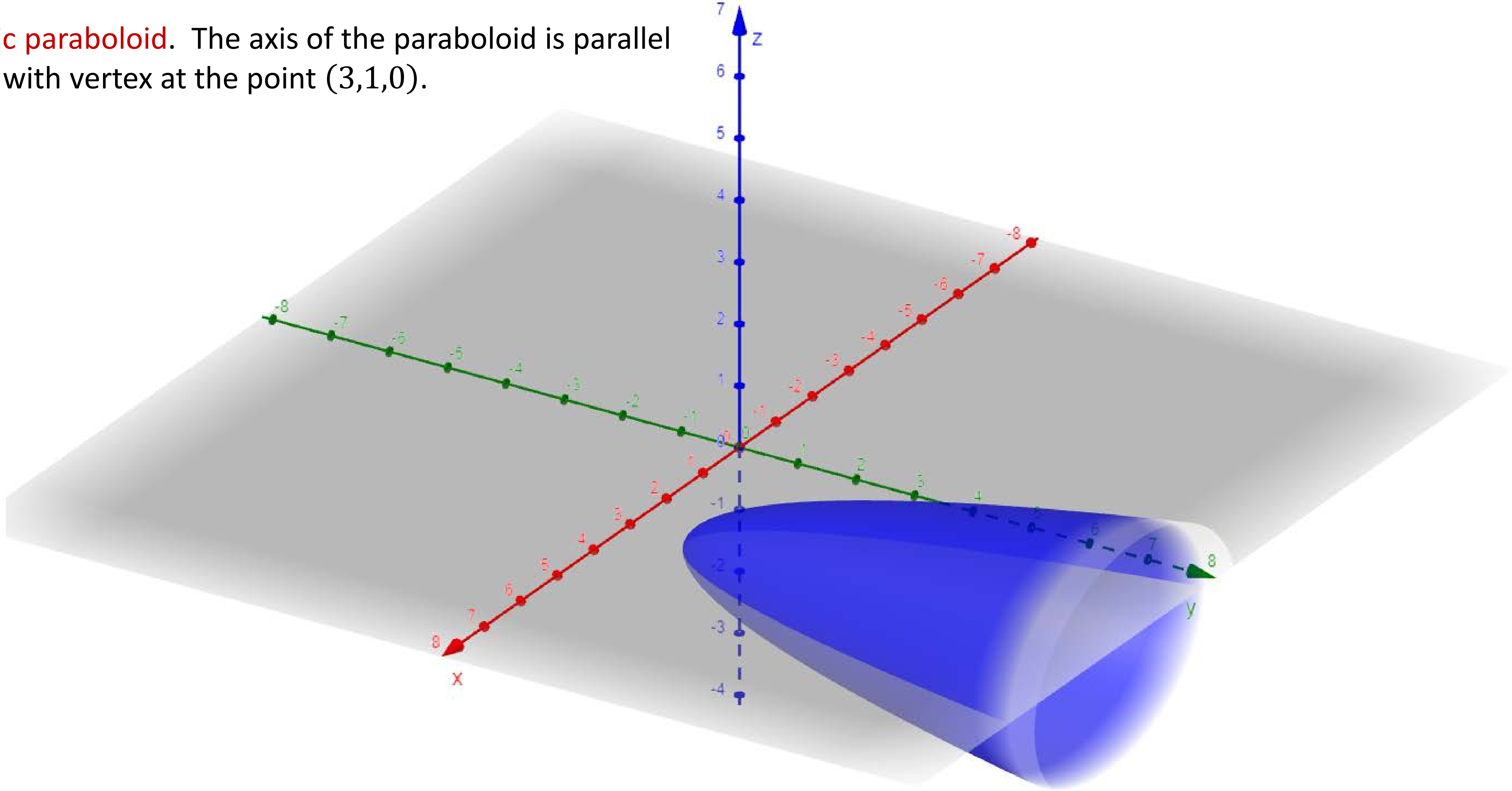
$$(x-3)^2 + 2z^2 = k - 1 \Longrightarrow \frac{(x-3)^2}{k-1} + \frac{z^2}{(k-1)/2} = 1.$$

[Ellipse in planes parallel to xz −plane]

Solution: Elliptic Paraboloid

$$\frac{(x-3)^2}{2} + z^2 = \frac{1}{2}(y-1).$$

This is an elliptic paraboloid. The axis of the paraboloid is parallel to the y −axis, with vertex at the point $(3,1,0)$.



Cylindrical & Spherical Coordinates

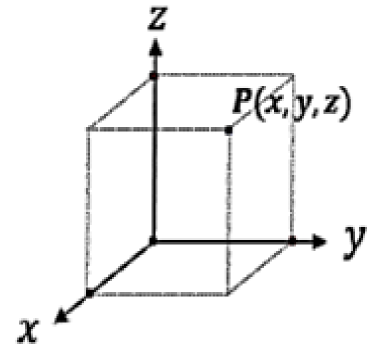
3-D Coordinate Systems

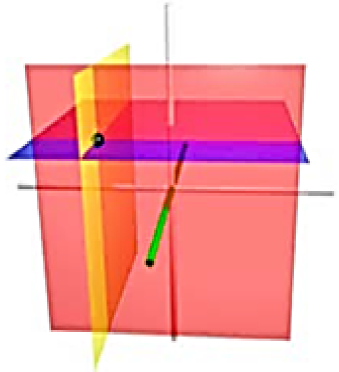
1. Cartesian Coordinates

Or

Rectangular Coordinates

$P\ (x, y, z)$

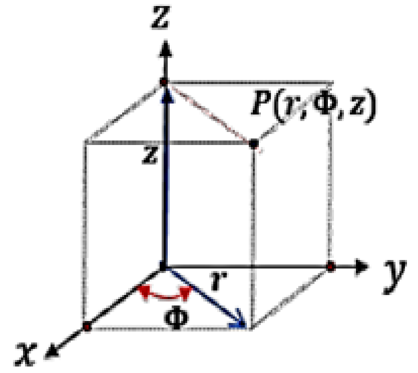


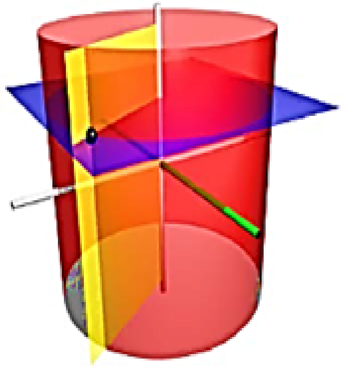


2. Cylindrical Coordinates

$P\ (r, \Phi, z)$

$$\boxed{\begin{aligned} x &= r\cos\Phi, \\ y &= r\sin\Phi, \\ z &= z. \end{aligned}}$$

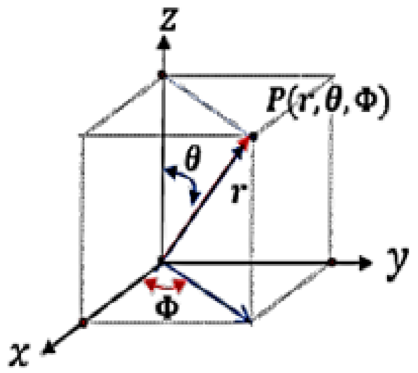


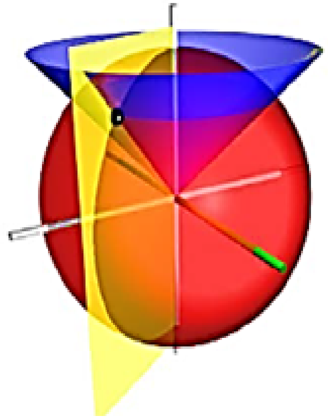


3. Spherical Coordinates

$P\ (r, \theta, \Phi)$

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\Phi, \\ y &= r\sin\theta\sin\Phi, \\ z &= r\cos\theta. \end{aligned}}$$





15

Vectors And
The Geometry Of Space

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Chapter: 15 , Section: 15.7

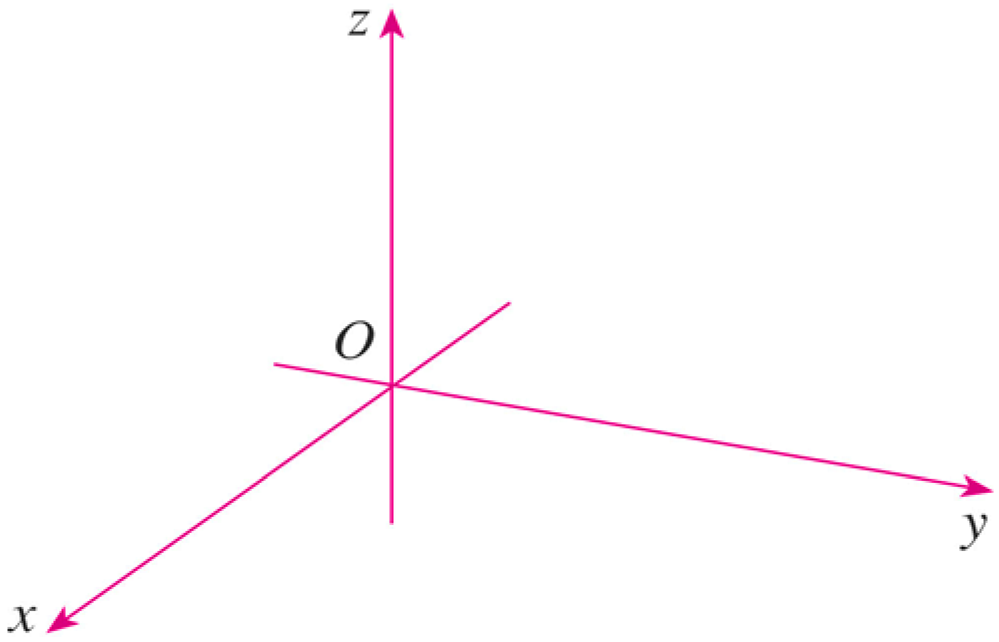
Book: Calculus Early Transcendentals (6th Edition) By James Stewart.
Chapter: 15 , Section: 15.7, 15.8

Introduction

- The laws of electromagnetic are invariant with coordinate system.
- When we solve problems in electromagnetic. For example, finding Electric Field at a point, we desire the fields.
- The position is expressed in terms of Co-ordinates.
- The co-ordinates are specified by the co-ordinate System.
- Most Common Co-ordinate systems are:
    - Cartesian Or Rectangular Coordinate system
    - Cylindrical Coordinate system
    - Spherical Coordinate system

Limitation of Cartesian System

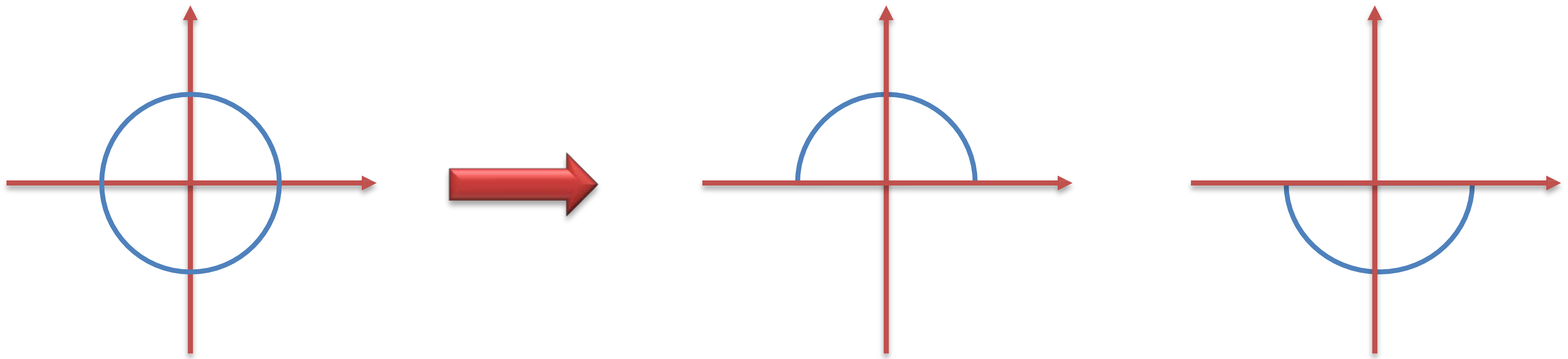
So far, we have studied Cartesian coordinate system and sketched the graph of several multivariable functions in it.



However it is observed that certain objects are difficult to sketch in this coordinate system. The same problem even arise in 2D Cartesian plane.

Limitation of Cartesian System

For example: a circle in Cartesian plane has to be cut into two parts in order to treat it as a function of one variable $y = f(x)$.



Similarly, it is difficult to work with the equation $x^2 + y^2 = r^2$ for doing calculus on circle. Imagine if we were to calculate the derivative of y with respect to x and vice versa. Further the integration of function y with respect to x will not so simple either.

Solution: Identify some symmetry in the given objects and use appropriate coordinate system.

Limitation of Cartesian System

It is observed that certain objects carry an inherited ***symmetry*** which can help us to introduce a coordinate system that not only simplifies the representation of functions but also help us to visualize the function easily.

What geometric object correspond to the following equation?

$$x^2 + y^2 = \sqrt{x^2 + y^2} + x.$$

It is not at all trivial to sketch above function in xy −plane. It is even harder to do calculus with this function.

We didn't realize that it is in fact a **heart !!!**

Polar Coordinate System

How do we know that its heart?

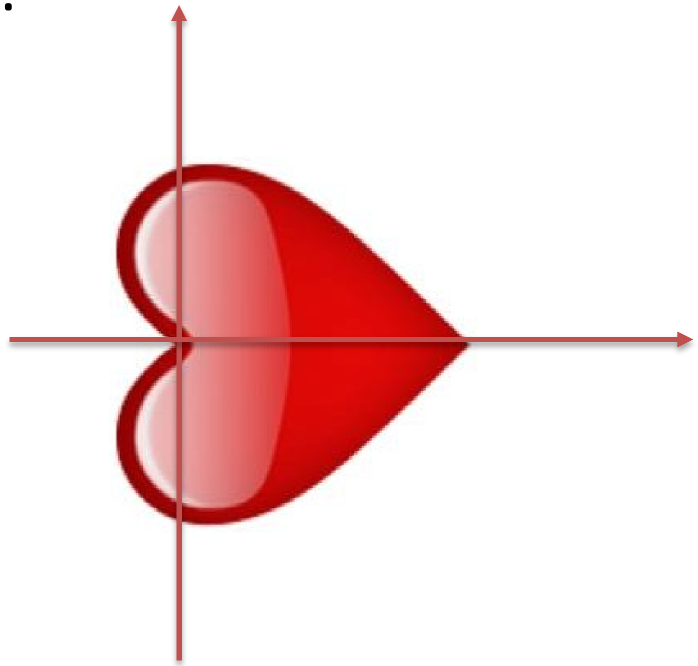
Answer: By simply bringing the polar coordinate system:

$$x = r\cos\theta\,, \qquad\qquad y = r\sin\theta,$$

the equation takes the form:

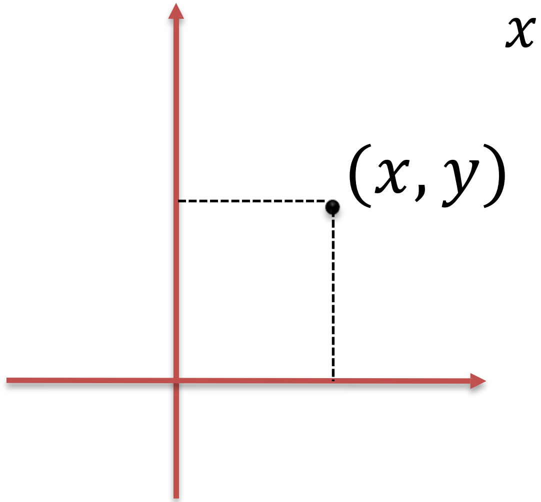
$$r = 1 + \cos\theta\,.$$

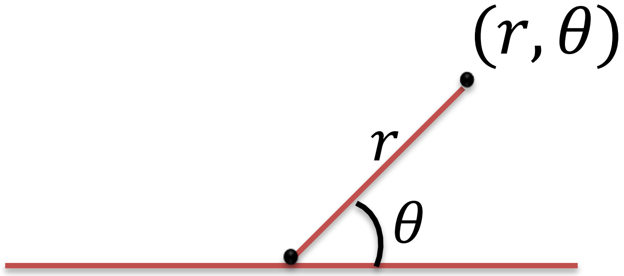
heart is ***not*** as complicated to visualize in a new coordinate system as it was in the Cartesian coordinate system !!!



Polar Coordinate System

$$x = r\cos\theta\,, \qquad y = r\sin\theta,$$





$$r^2 = x^2 + y^2, \qquad \theta = \arctan(y/x)$$

Sense: In order to get to some point in a plane we need a stick that can be rotated at an angle to get us there, unlike the Cartesian coordinates which requires two perpendicular lines to be drawn. Each point (x, y) can be approached using (r, θ).

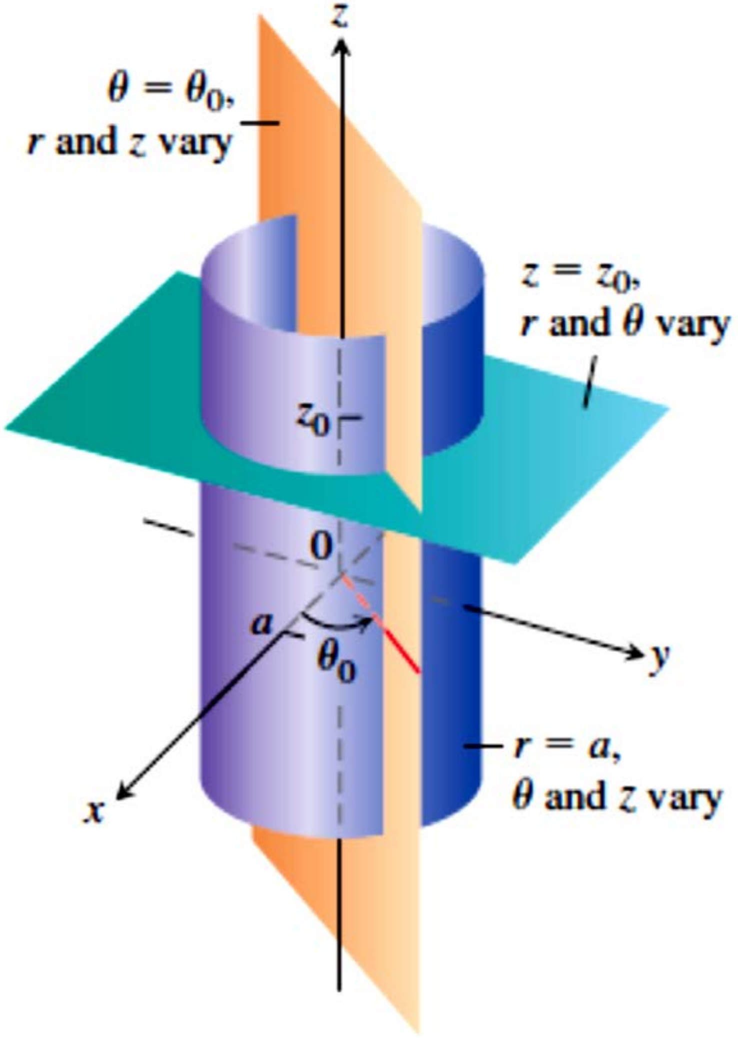
Polar Coordinate System

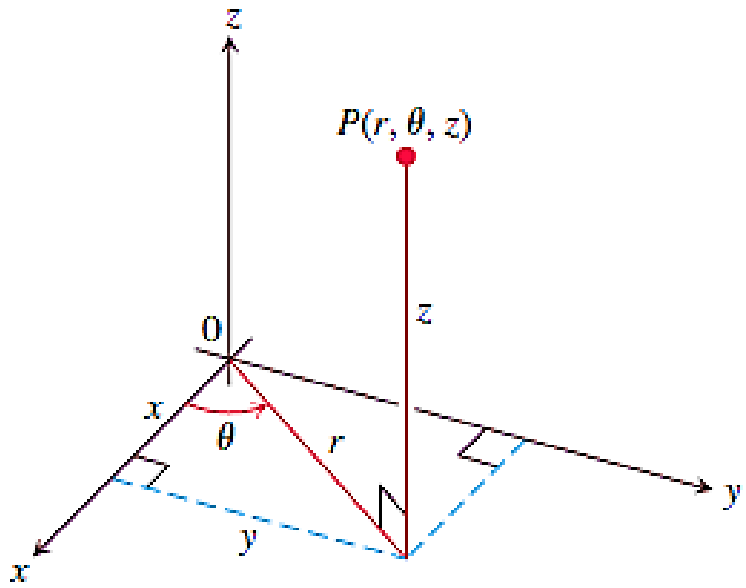
Conclusion:

In calculus, the change of coordinate systems help us to:

- sketch complicated functions (especially implicit)
- understand geometry
- comprehend underlying symmetry
- calculate derivatives and integration easily
- avoid expression swelling

Introduction to Cylindrical Coordinate Systems





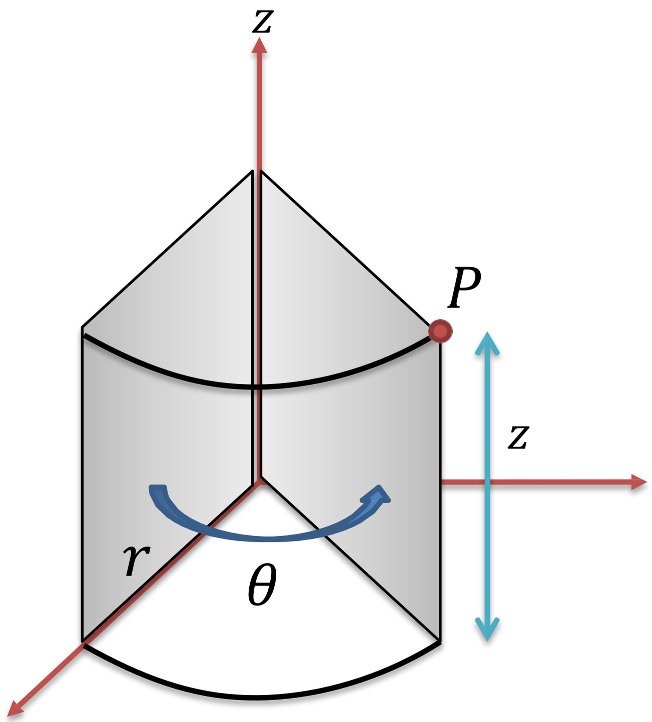
The cylindrical coordinates of a point in space are r, θ, and z.

$P\ (r, \theta, z)$

$$\boxed{\begin{aligned} x &= r\cos\theta\,, \\ y &= r\sin\theta\,, \\ z &= z. \end{aligned}}$$

Cylindrical Coordinate System

We are now in a position to naturally ***generalize*** polar coordinates in a three-dimensional space. For example: just make the third variable z as arbitrary to find a triplet (r, θ, z) which identify a point in space.



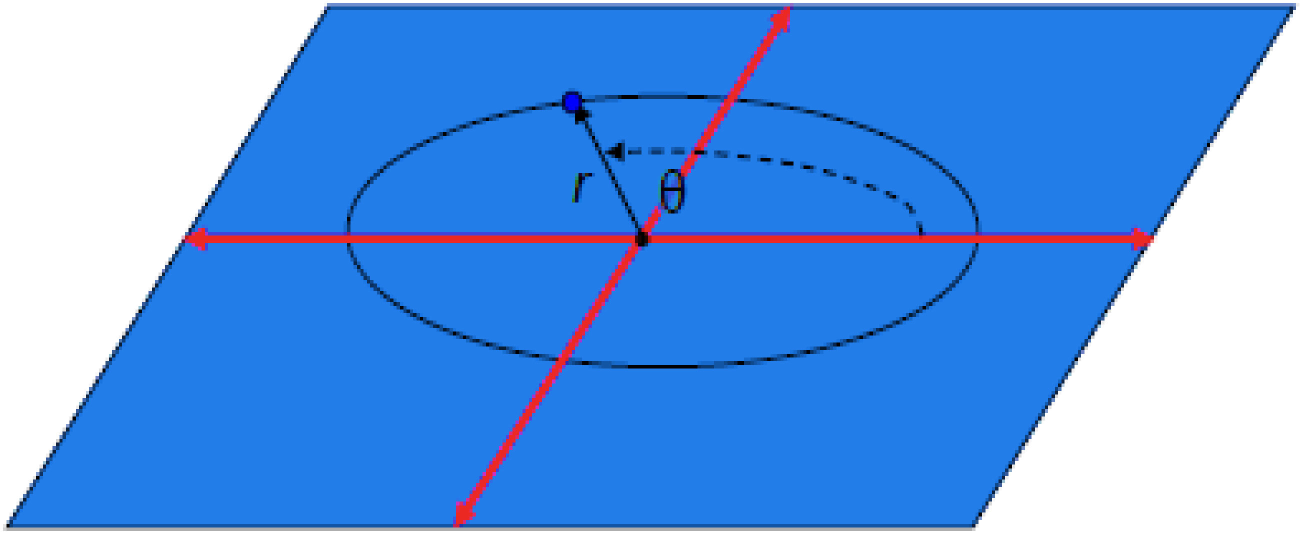
Now the sense of ***approaching*** a particular point is changed entirely because we reach point P via some cylinder of fixed radius and angle.

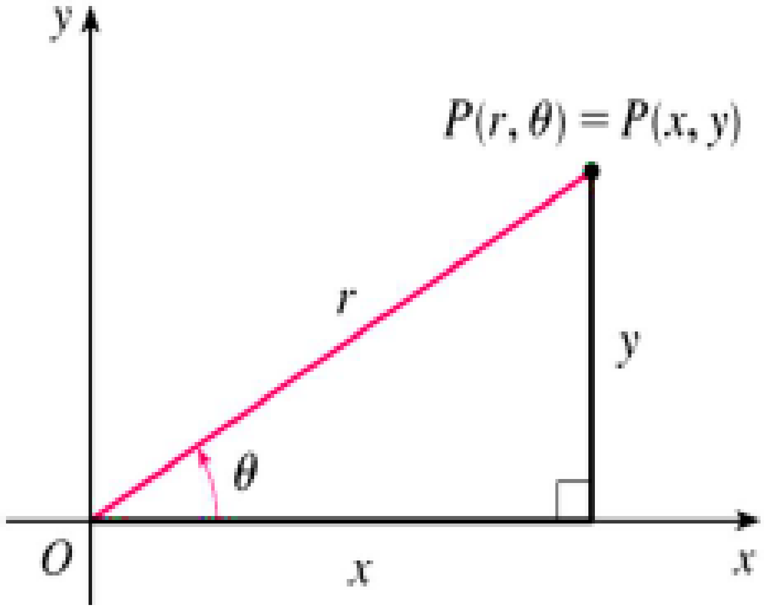
Representing 3D points in Cylindrical Coordinates.

Recall polar representations in the plane. If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then

$$x = r\cos\theta\,, \qquad y = r\sin\theta\,,$$

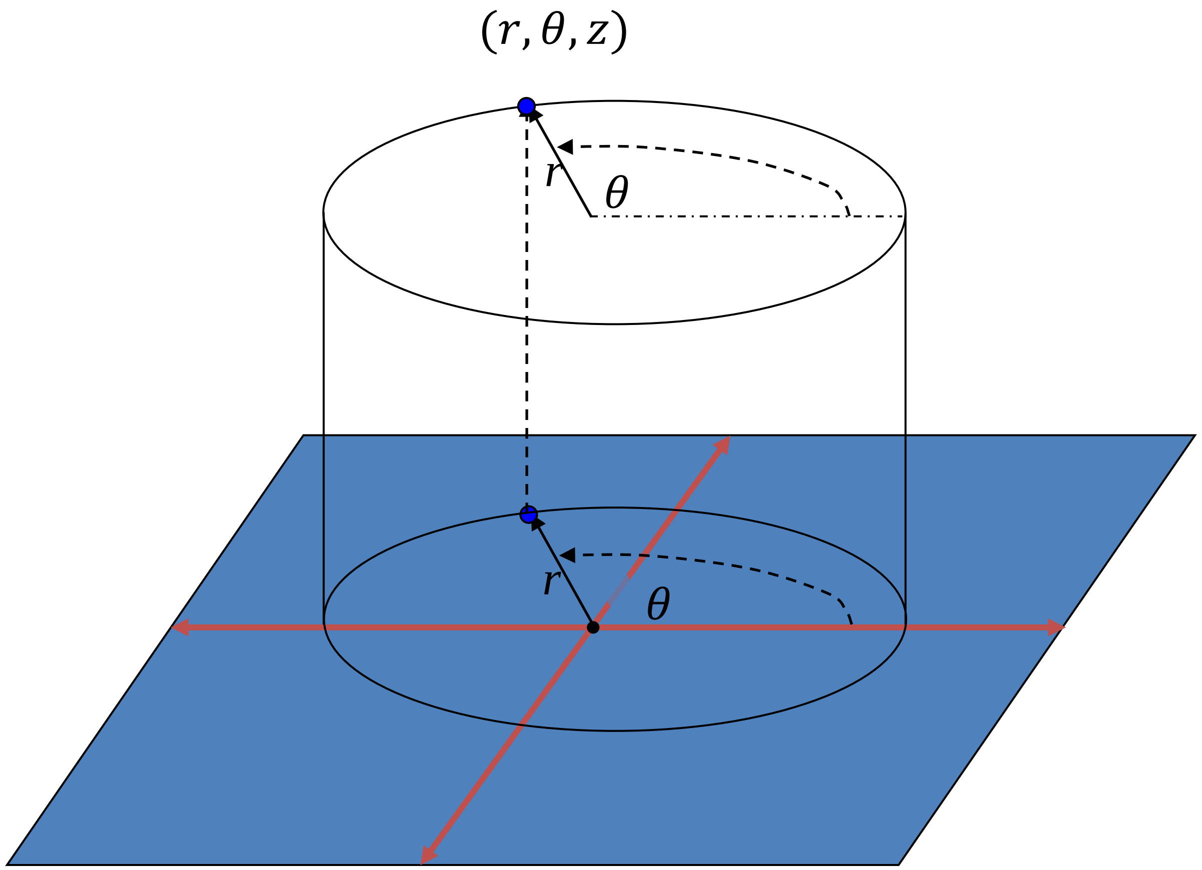
$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}.$$





Representing 3D points in Cylindrical Coordinates.

Cylindrical coordinates just adds a z −coordinate to the polar coordinates (r, θ).



Cylindrical to rectangular

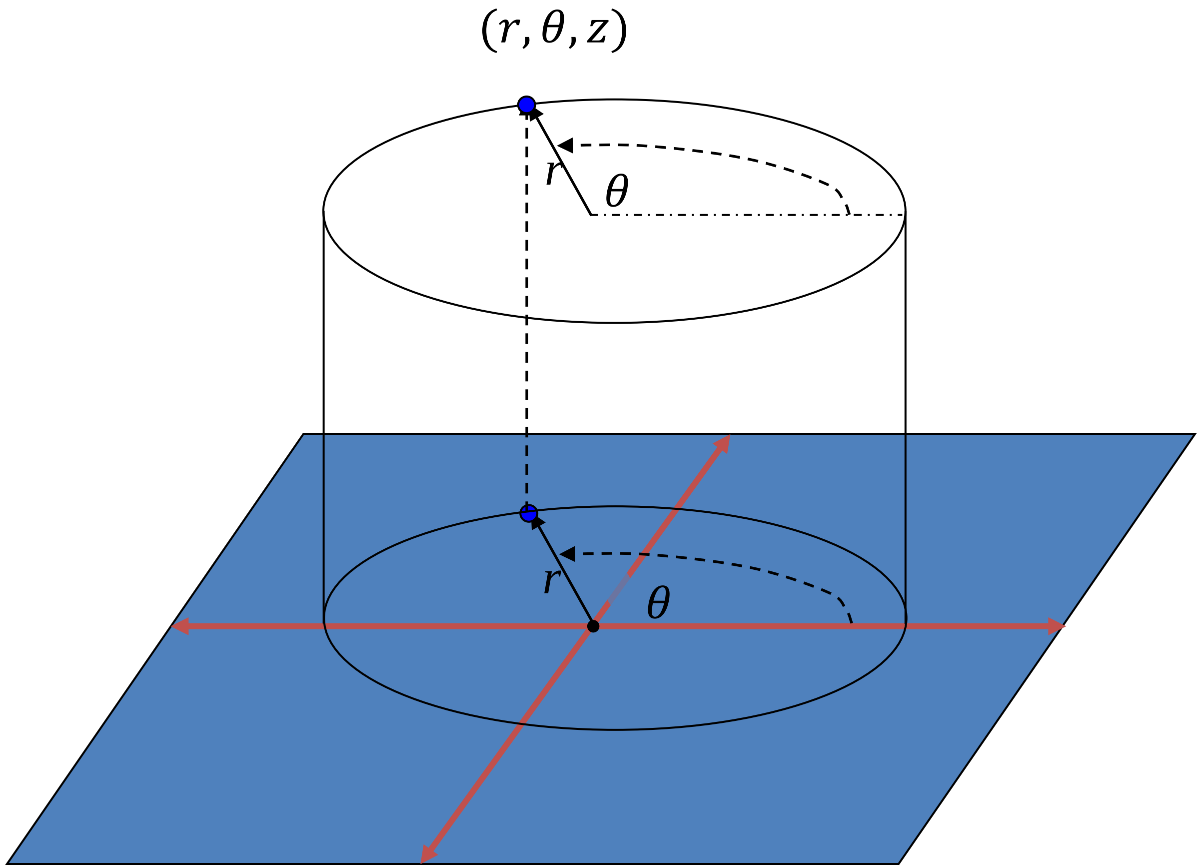
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$\text{where, } 0 \le \theta \le 2\pi.$$

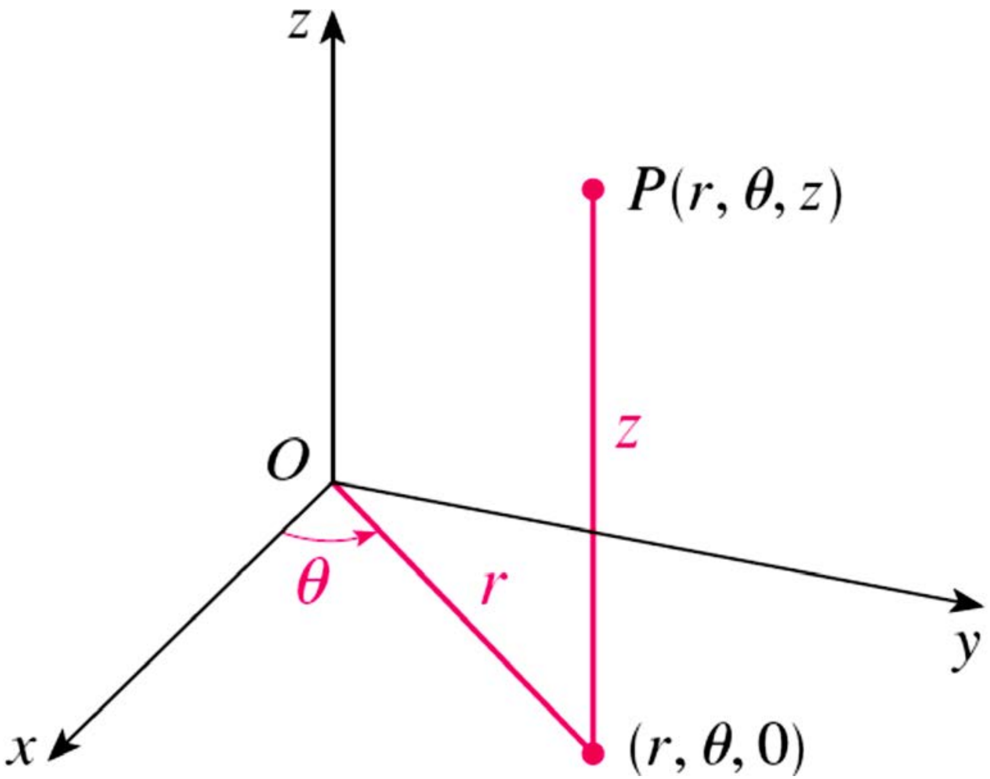
Rectangular to Cylindrical

$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}$$
$$z = z$$

Representing 3D points in Cylindrical Coordinates.

Thus, in the cylindrical coordinate system, a point P in three-dimensional (3-D) space is represented by the ordered triple (r, θ, z) where: r and θ are polar coordinates of the projection of P onto the xy–plane and z is the directed distance from the xy –plane to P.

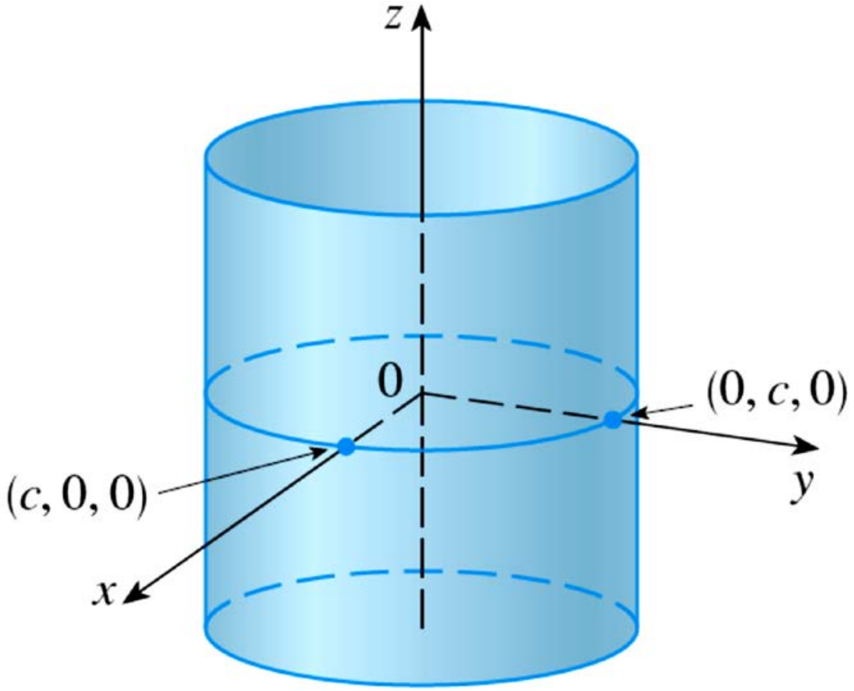




Cylindrical Coordinates.

In cylindrical coordinates, this cylinder has the very simple equation $r = c$. This is the reason for the name "cylindrical" coordinates.

Notice that a circle becomes a cylinder in 3D. Therefore, it would be very useful to use a cylindrical coordinate system in situations where cylindrical symmetry is present, and the z –axis is chosen to coincide with this axis of symmetry. . For example, fluid flows across pipelines (mostly cylindrical), electron orbiting around nucleus, planets circular orbits around sun, the galactic structure of galaxies.



Example:

a. Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.

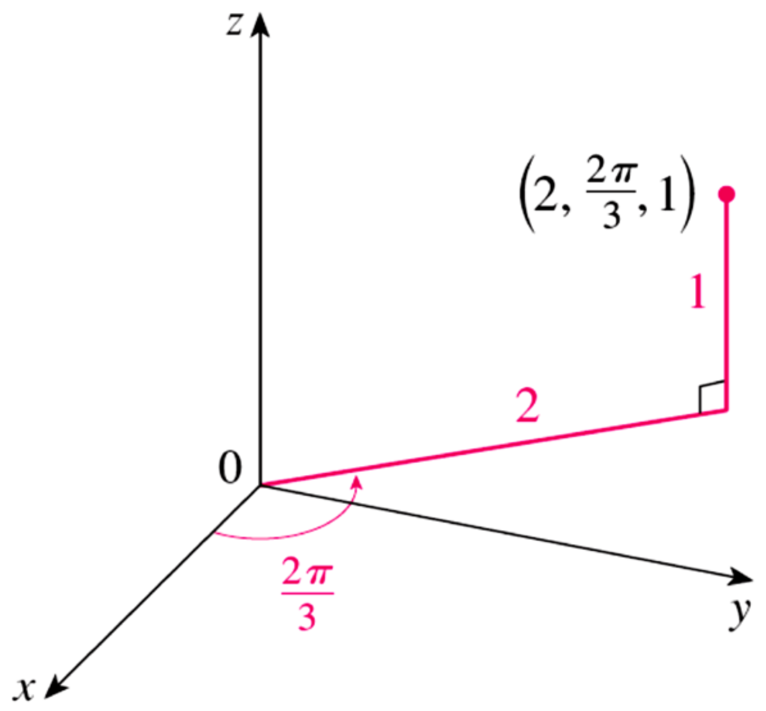
b. Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Solution (a): The point with cylindrical coordinates $(2, 2\pi/3, 1)$ is plotted in the accompanying figure. Corresponding rectangular coordinates are:

$$x = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1,$$

$$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3},$$

$$z = 1.$$



Thus, the point is $(-1, \sqrt{3}, 1)$ in rectangular coordinates.

Solution (b):

In order to determine the cylindrical coordinates of the point with rectangular coordinates (3, –3, –7), we proceed as:

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2},$$

$$z = -7,$$

$$\tan\theta = \frac{-3}{3} = -1, \text{so } \theta = \frac{7\pi}{4} \text{ or } \theta = \frac{-\pi}{4}.$$

As with polar coordinates, there are infinitely many choices. Therefore, one set of cylindrical coordinates is:

$$\left(3\sqrt{2}, -\frac{\pi}{4}, -7\right).$$

Another is:

$$\left(3\sqrt{2}, \frac{7\pi}{4}, -7\right).$$

Example:

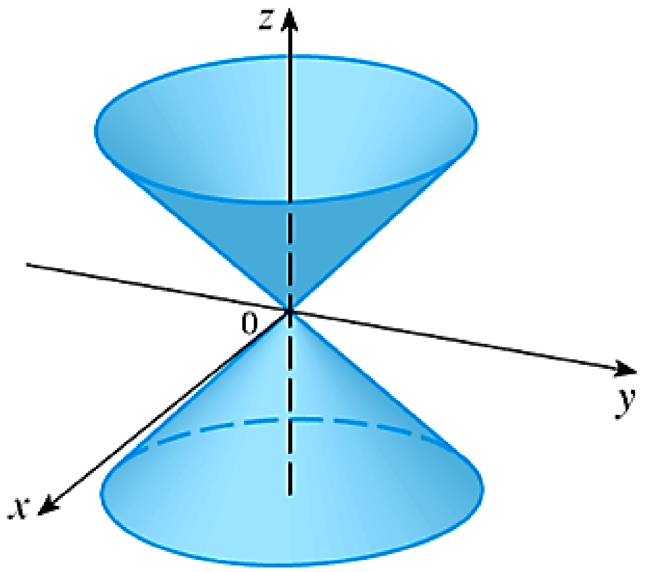
Describe the surface whose equation in cylindrical coordinates is $z = r$.

Solution:

The equation says that the z −value, or height, of each point on the surface is the same as r, the distance from the point to the z −axis. Since θ doesn't appear, it can vary. So, any horizontal trace in the plane $z = k\ (k > 0)$ is a circle of radius k. These traces suggest the surface is a **cone**. This prediction can be confirmed by converting the equation into rectangular coordinates. We recognize the equation:

$$z^2 = x^2 + y^2,$$

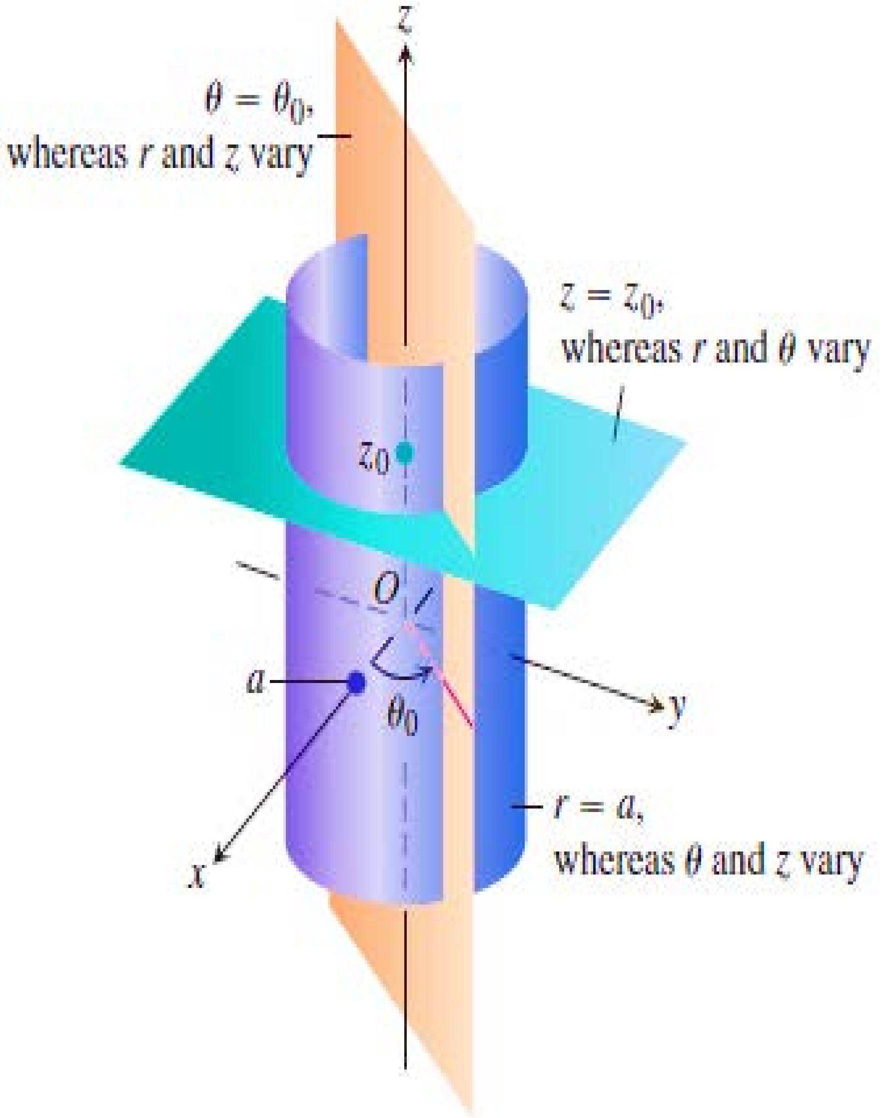
as a circular cone whose axis is the z −axis.



Cylindrical Coordinate System

In cylindrical coordinates,

- The equation $r = a$ describes not just a circle in the xy −plane but an entire cylinder about the z −axis.
- The z −axis is given by $r = 0$.
- The equation $\theta = \theta_0$ describes the plane that contains the z −axis and makes an angle θ_0 with the positive x −axis.
- And the equation $z = z_0$ describes a plane perpendicular to the z −axis



Constant-coordinate equations in cylindrical coordinates yield cylinders & planes.

Cylindrical Coordinate System

Practice Questions:

In the following questions sketch each graph and convert the given coordinate system into another.

1. $r = \sin\theta\,;\;\; 0 \leq z \leq 1.$
2. $y = x; 0 \leq z \leq 1.$
3. $y = 1; 0 \leq z \leq 1.$
4. $r = 1;\; \theta = \frac{\pi}{4}.$
5. $r = 1; 0 \leq \theta \leq \frac{\pi}{4}.$
6. Convert a sphere of radius 1 centered at origin in cylindrical coordinates.

Practice Questions

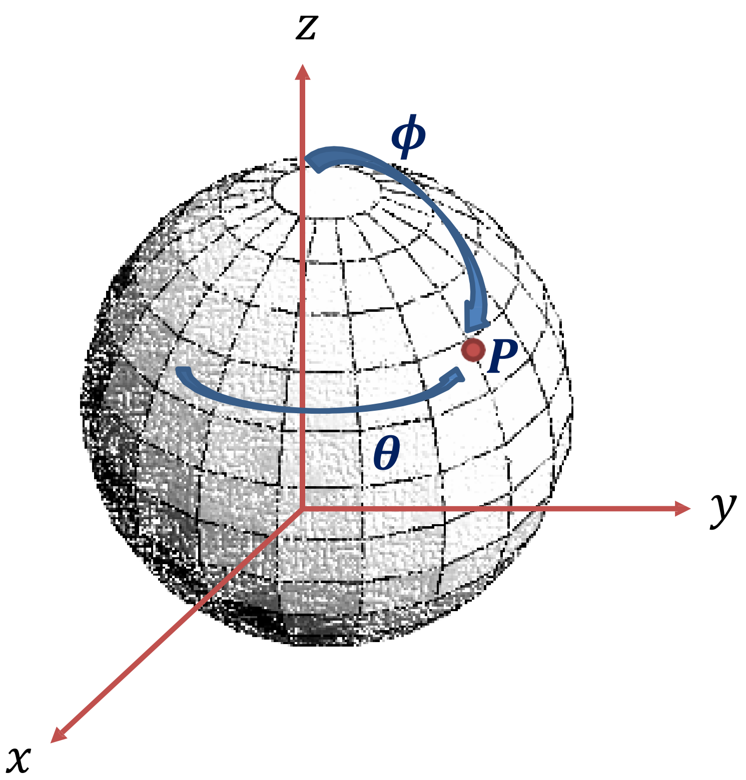
Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 15

Exercise-15.7: Q – 1 to 12.

Spherical Coordinate System

We are now in a position to naturally ***generalize*** the concept of coordinate systems further in a three-dimensional space by replacing the whole idea of rectangular boxes and cylinders with ***spheres*** to approach a particular point in space.



Now the sense of ***approaching*** a particular point is changed entirely because we reach point P via some sphere of fixed radius and angle.

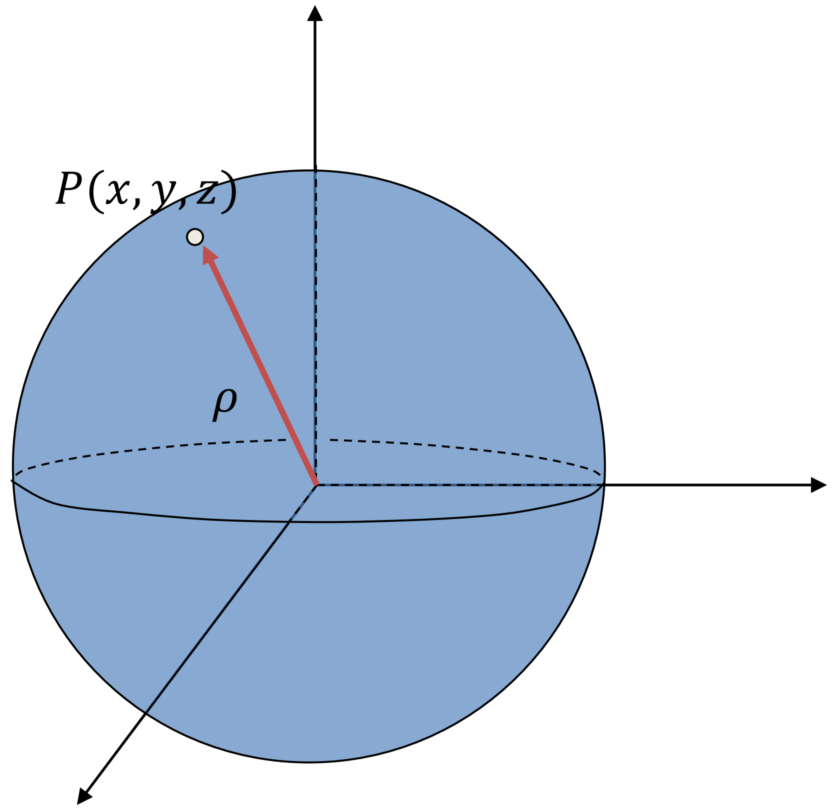
Representing 3D points in Spherical Coordinates

Spherical Coordinates are the 3D analog of polar representations in the plane.

We divide 3-dimensional space into:

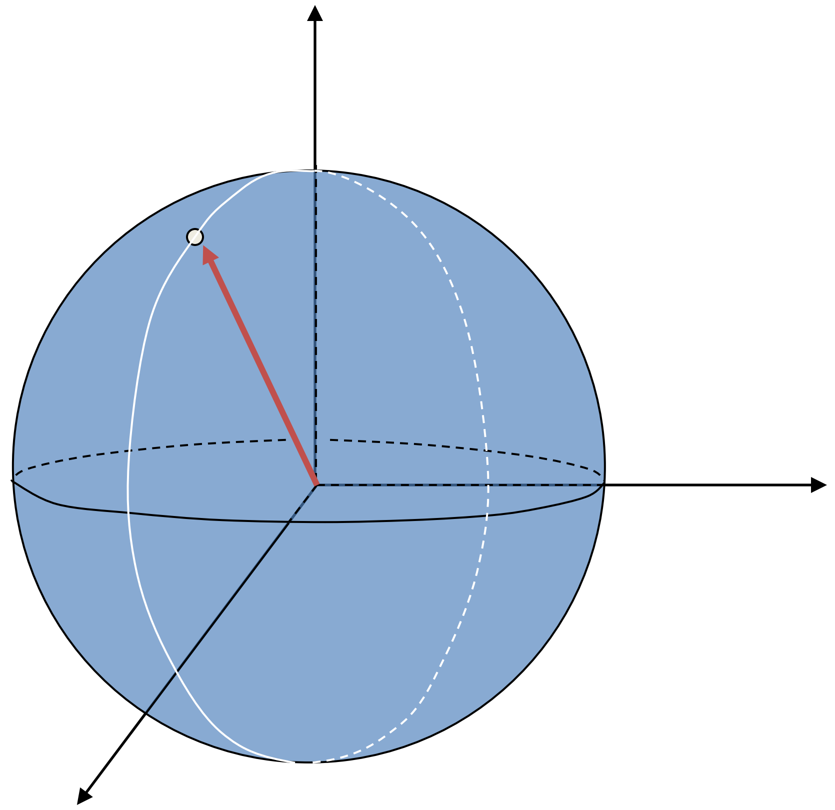
1. a set of concentric spheres centered at the origin.
2. rays emanating outward from the origin

Representing 3D points in Spherical Coordinates



- We start with a point $P(x, y, z)$ given in rectangular coordinates.
- Then, measuring its distance ρ from the origin, we locate it on a sphere of radius ρ (distance from origin to the point) centered at the origin.
- Next, we have to find a way to describe its location on the sphere.

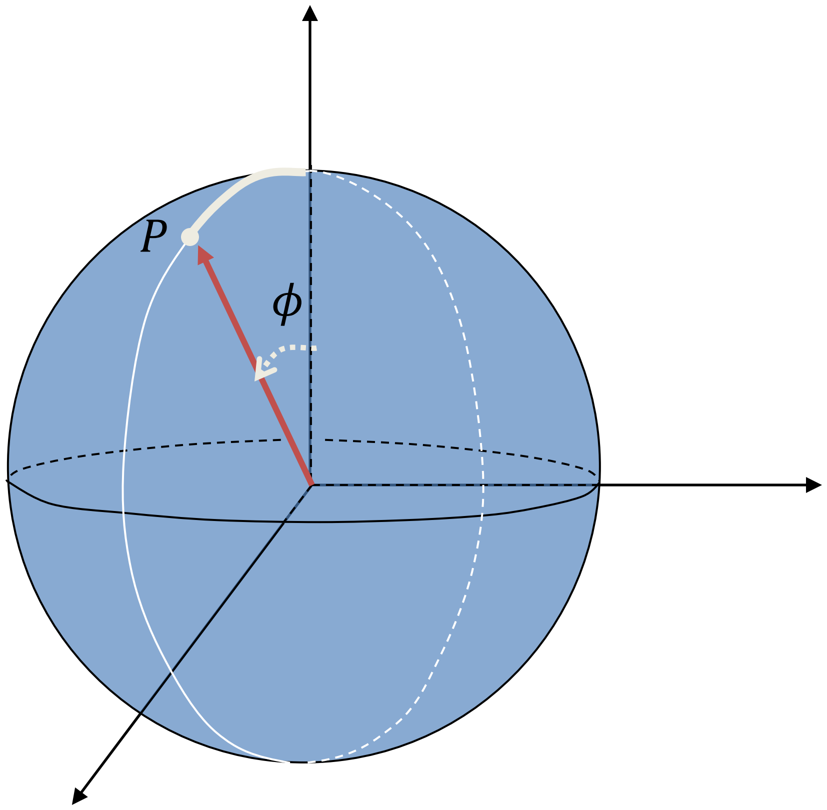
Representing 3D points in Spherical Coordinates



- We use a method similar to the method used to measure *latitude* and *longitude* on the surface of the Earth.

- We find the great circle that goes through the “north pole,” the “south pole,” and the point.

Representing 3D points in Spherical Coordinates



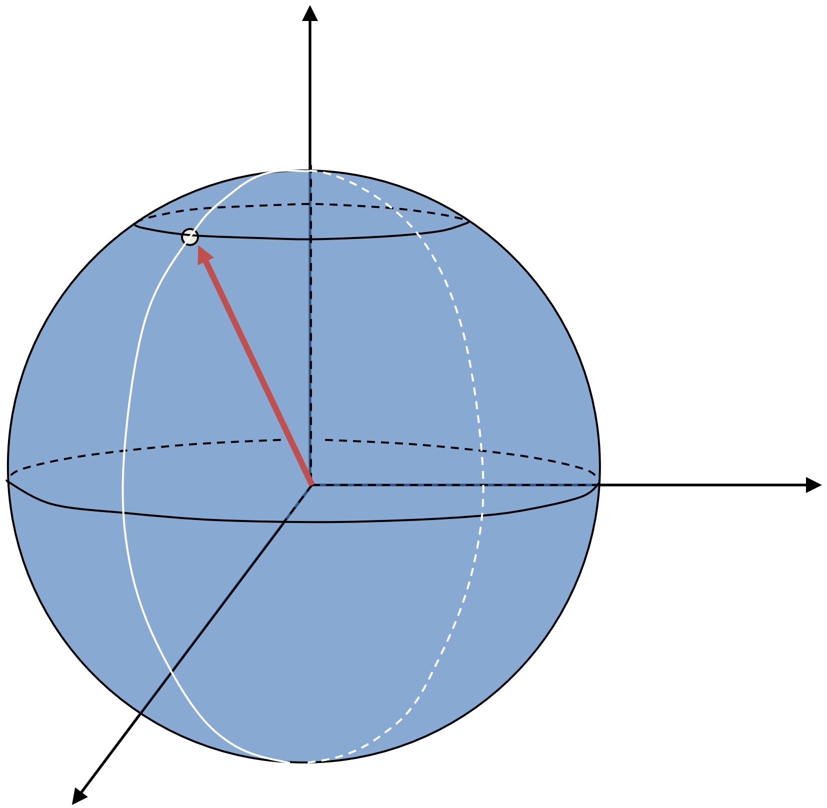
We measure the *longitude* or *polar* angle starting at the "north pole" in the plane given by the great circle.

This angle is called ϕ. This is the angle that $\overrightarrow{OP}$ makes with the positive z −axis. The range of this angle is:

$$0 \leq \phi \leq \pi.$$

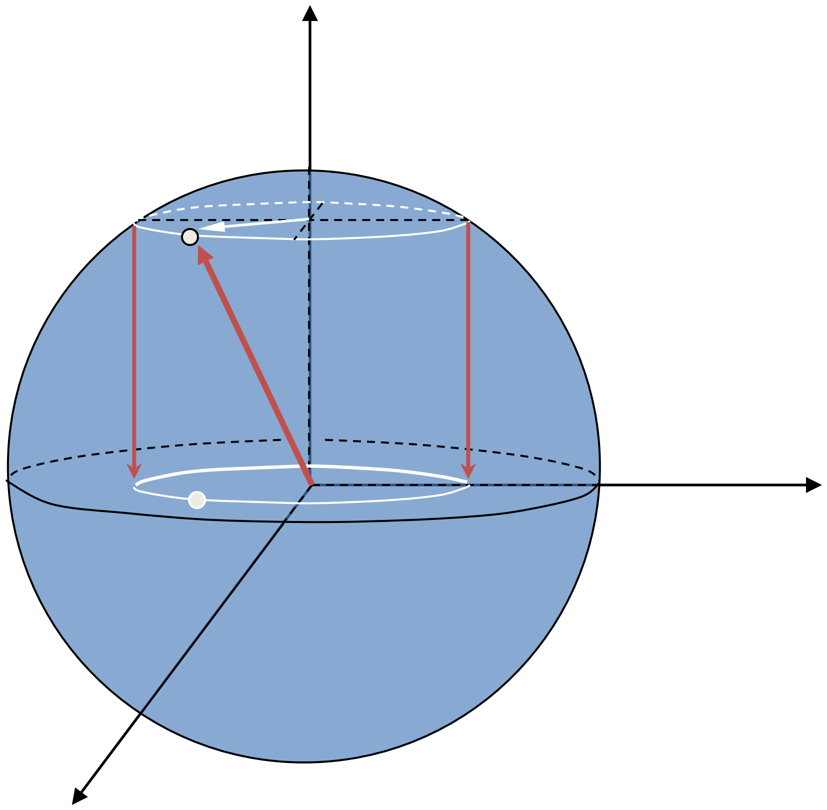
Note: all angles are measured in radians, as always.

Representing 3D points in Spherical Coordinates



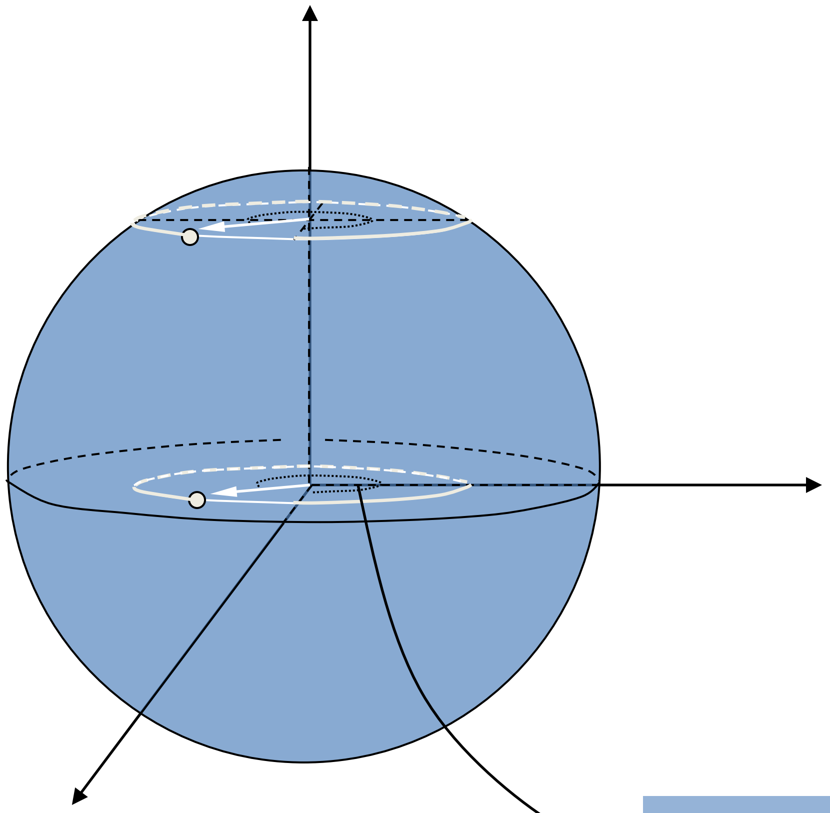
Next, we draw a horizontal circle on the sphere that passes through the point.

Representing 3D points in Spherical Coordinates



And “drop it down” onto the xy −plane.

Representing 3D points in Spherical Coordinates



Angle is called θ.

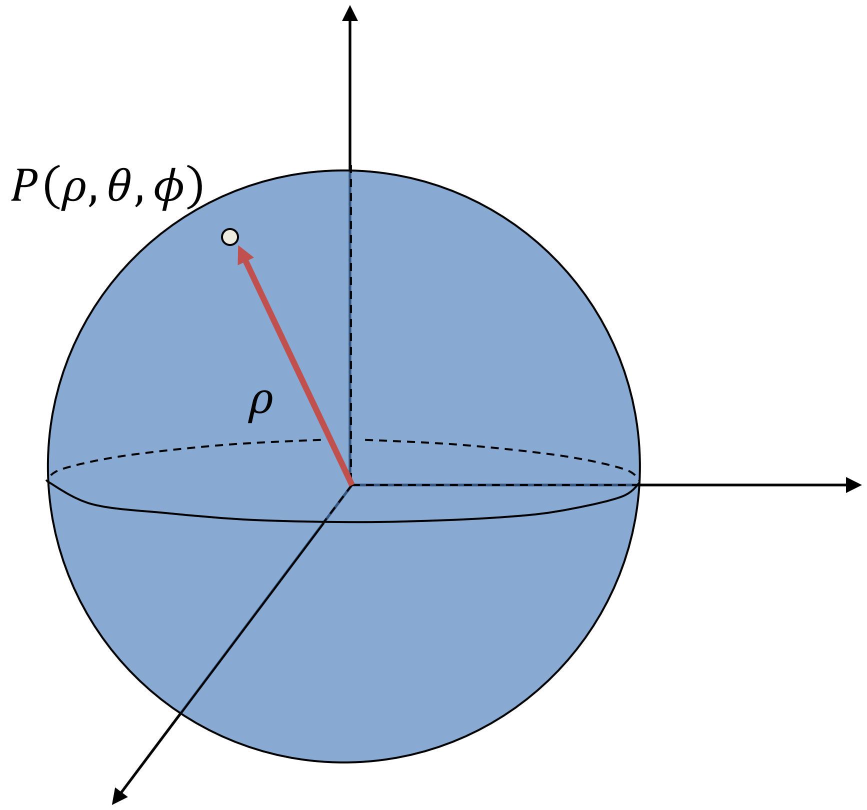
- We measure the *latitude* or *azimuthal* angle on the latitude circle, starting at the positive $x-$ axis and rotating towards the positive $y-$axis.

- The range of the angle is:

$$0 \leq \theta \leq 2\pi.$$

Note that this is the same angle as the θ in cylindrical coordinates!

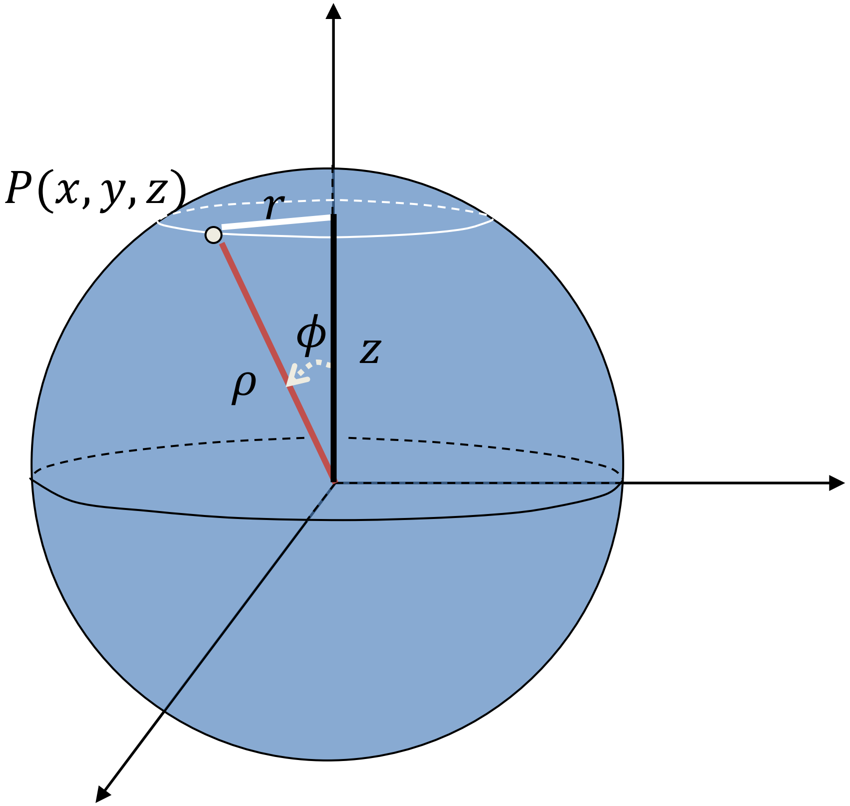
Representing 3D points in Spherical Coordinates



- Our designated point on the sphere is indicated by the three spherical coordinates: $P(\rho, \theta, \phi)$---(radial distance, azimuthal angle, polar angle).

- Please note that this notation is not at all standard and varies from author to author and discipline to discipline. (In particular, physicists often use ϕ to refer to the azimuthal angle and θ refer to the polar angle.)

Conversion Between Rectangular and Spherical Coordinates



First note that if r is the usual cylindrical coordinate for $P(x,y,z)$

we have a right triangle with:

- acute angle ϕ
- hypotenuse ρ and
- legs r and z.

It follows that:

$$\sin(\phi) = \frac{r}{\rho} \quad \cos(\phi) = \frac{z}{\rho} \quad \tan(\phi) = \frac{r}{z}.$$