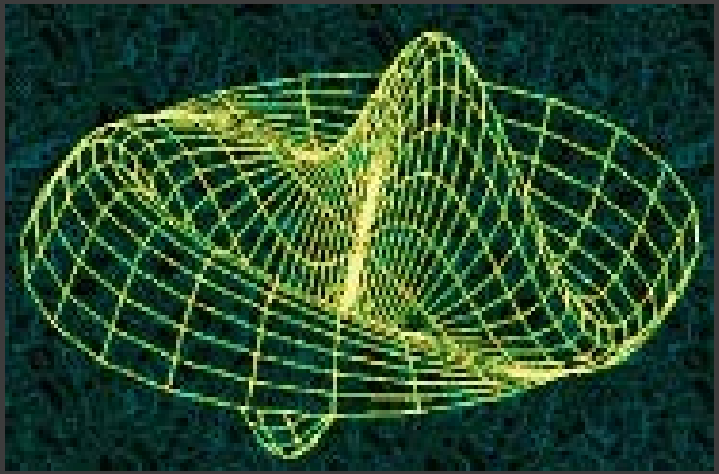
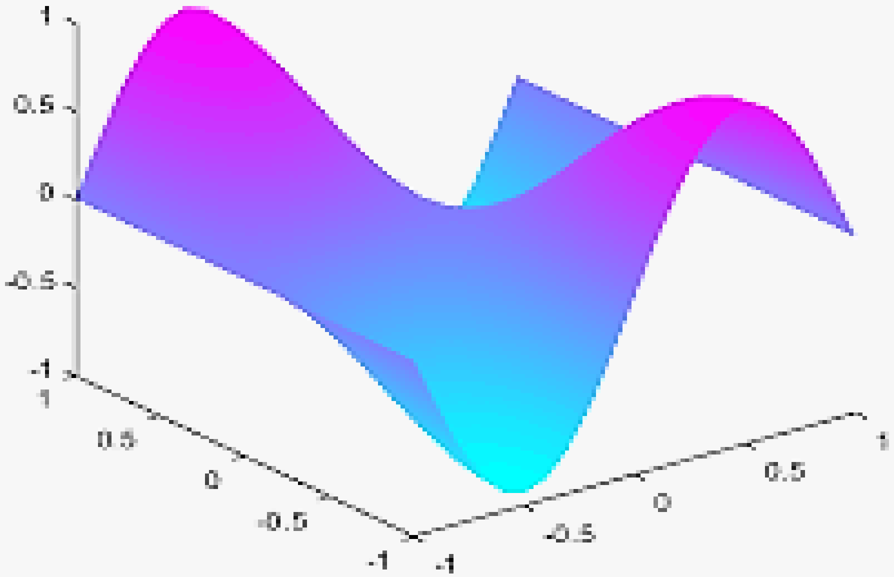
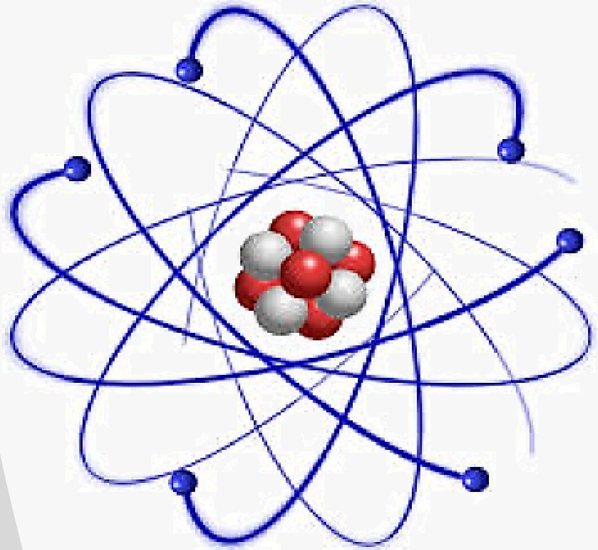
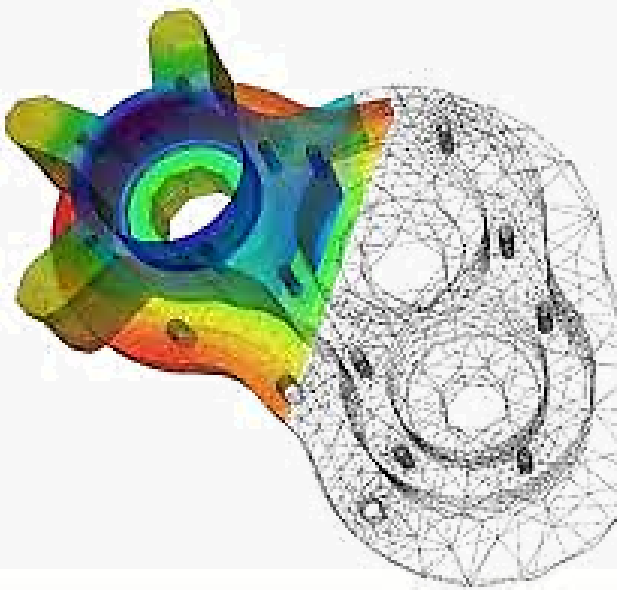


Partial Differential Equations







Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

Differential Equations

- Equations which are composed of an unknown function and its derivatives are called *differential equations*.

- An equation containing the derivative of one or more dependent variables, with respect to one or more independent variables is said to be a **differential equation** (DE).

- Differential equations (DEs) play a fundamental role in engineering because many physical phenomena are best formulated mathematically in terms of their rate of change.

$$\boxed{\frac{dv}{dt} = g - \frac{c}{m}v}$$

Differential equation describing a force balance for the falling object

v – dependent variable

t – independent variable

Families of Solutions

Example:

$$9yy' + 4x = 0$$

Solution:

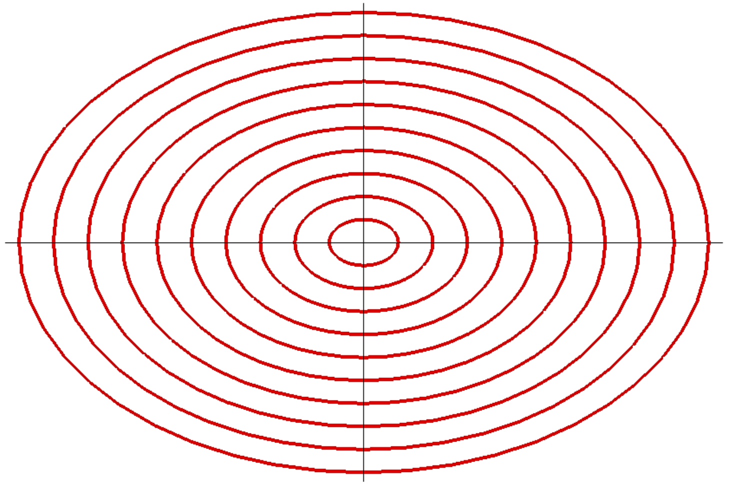
$$\int (9yy' + 4x)dx = C_1 \Rightarrow \int 9y(x)y'(x)dx + \int 4xdx = C_1$$

$$\Rightarrow \int 9ydy + 2x^2 = C_1 \Rightarrow \frac{9y^2}{2} + 2x^2 = C_1 \Rightarrow 9y^2 + 4x^2 = 2C_1$$

This yields $\dfrac{y^2}{4} + \dfrac{x^2}{9} = C$ where $C = \dfrac{C_1}{18}$.

Is this a unique solution?

Observe that given any point (x_0, y_0), there is a unique solution curve of the above equation which goes through the given point.



The solution is a family of ellipses.

Uniqueness of a Solution

In order to uniquely specify a solution to an n^{th} order differential equation we need n conditions.

$$\frac{d^2x(t)}{dt^2} + 4x(t) = 0$$

Second order ODE

$$x(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

General solution of given ODE

$$\boxed{\begin{aligned} x(0) &= a \\ \dot{x}(0) &= b \end{aligned}}$$

Two conditions are needed to uniquely specify the solution

Auxiliary Conditions

Auxiliary Conditions

Initial Conditions

All conditions are at one point of the independent variable.

Boundary Conditions

The conditions are at different point of the independent variable.

Boundary-Value and Initial value Problems

Initial-Value Problems (IVP)

The auxiliary conditions are at one point of the independent variable

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$
$$x(0) = 1, \dot{x}(0) = 2.5$$

same

Boundary-Value Problems (BVP)

- The auxiliary conditions are not at one point of the independent variable
- More difficult to solve than initial value problems

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$
$$x(0) = 1, x(2) = 1.5$$

different

Note: Here $\dot{x}$ means first order derivative w.r.t. t

Formation Of Ordinary Differential Equations

Given a relation:

$$F(x, y, c_1, c_2, \dots, c_n) = 0, \qquad (1)$$

between the variables x, y and containing n constants $c_1, c_2, \dots, c_n$, it is always possible to form a differential equation of order n such that the given relation (1) is the general solution of the equation.

This is done by differentiating (1) n times thereby obtaining n equations and then eliminating the n constants from the original relation and the n derived equations.

Formation Of Ordinary Differential Equations

Example: Determine the ordinary differential equation for the given function:

$$y = A\sin x + B\cos x.$$

Solution: The required differential equation can be determined by eliminating the two constants from the relation:

$$y = A\sin x + B\cos x \qquad (1)$$

Differentiating (1) twice w.r.t "x", we get:

$$\frac{dy}{dx} = A\cos x - B\sin x$$

and

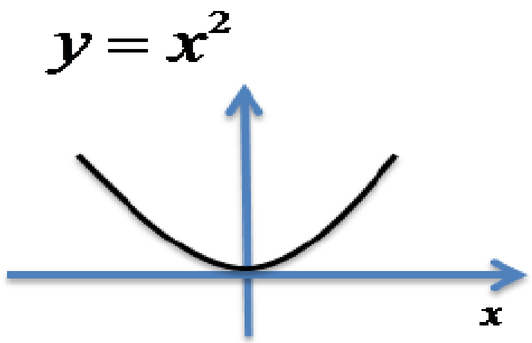
$$\frac{d^2y}{dx^2} = -A\sin x - B\cos x = -y \quad [using\ (1)]$$

Thus, the required differential equation is:

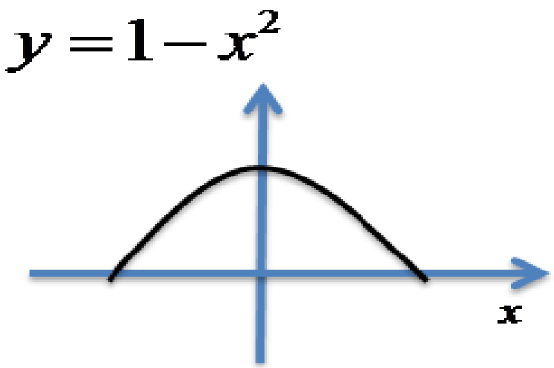
$$\frac{d^2y}{dx^2} + y = 0.$$

Formation Of Ordinary Differential Equations

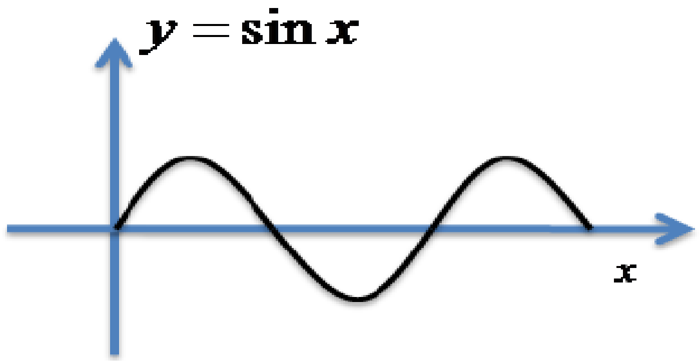
On the similar lines, we can obtain the ODEs (ordinary differential equations) of the following well-studied functions.



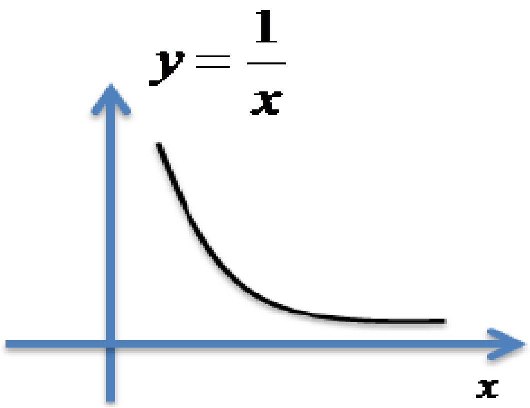
ODE $\boxed{y'' = 2}$



ODE $\boxed{y'' = -2}$



ODE $\boxed{y'' = -y}$



ODE $\boxed{y'' = 2y^3}$

Partial Differential Equations

The ordinary differential equations (ODEs) basically carry the information of how some change occurs in a physical phenomenon.

Does it tell the complete story?

NO !!!

The reason is very simple. The changes that we observe around us normally occur in a three dimensional space in which we live in. Therefore, a physical phenomenon is better represented by equations that contain functions which are of more than one variables, like $u = u(x, y)$ or $u = u(x, y, z)$. However, we are familiar with this fact that for such functions, change is represented by partial derivatives instead of ordinary derivatives.

Partial differential equations

- A **partial derivative** of a function of several variables expresses how fast the function changes when one of its variables is changed, the others being held constant.
- A **partial differential equation** (**PDE**) is an equation which imposes relations between the various partial derivatives of a multivariable function. The function is often thought of as an "unknown“, which is required to be determined.
- Partial differential equations are present everywhere in mathematically-oriented scientific fields, such as physics and engineering. For instance, they are foundational in the modern scientific understanding of sound, heat, diffusion, electrostatics, electrodynamics, thermodynamics, fluid dynamics, elasticity, general relativity, and quantum mechanics (Schrodinger equation, Pauli equation, etc.).
- Ordinary differential equations form a subclass of partial differential equations, corresponding to functions of a single variable.
- Stochastic partial differential equations and nonlocal equations are particularly widely studied extensions of the "PDE" notion. More classical topics, on which there is still much active research going on, include elliptic and parabolic partial differential equations, fluid mechanics, Boltzmann equations, and dispersive partial differential equations.

Partial Differential Equations

A partial differential equation (PDE) is an equation that contains partial derivatives (e.g., $u_x, u_y, u_{xx}, u_{yy}, \ldots$) in it.

For a function of two variables $u = u(x, y)$, a general first-order PDE can be expressed as:

$$F(x, y, u, u_x, u_y) = 0$$

Similarly, a general second-order PDE can be written as

$$F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0$$

In this course we will focus on those PDEs which are of at most second-order.

Example: $$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$

Examples

1. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

   This is a partial differential equation. For this example, u is dependent variable and x and y are independent variables.

2. $$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial t^4} = 0$$

3. $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}$$

   2 and 3 are also partial differential equations. In both of these examples u is dependent variable and x and t are independent variables.

Notation

It is a common practice to use subscript notation in writing partial differential equations.

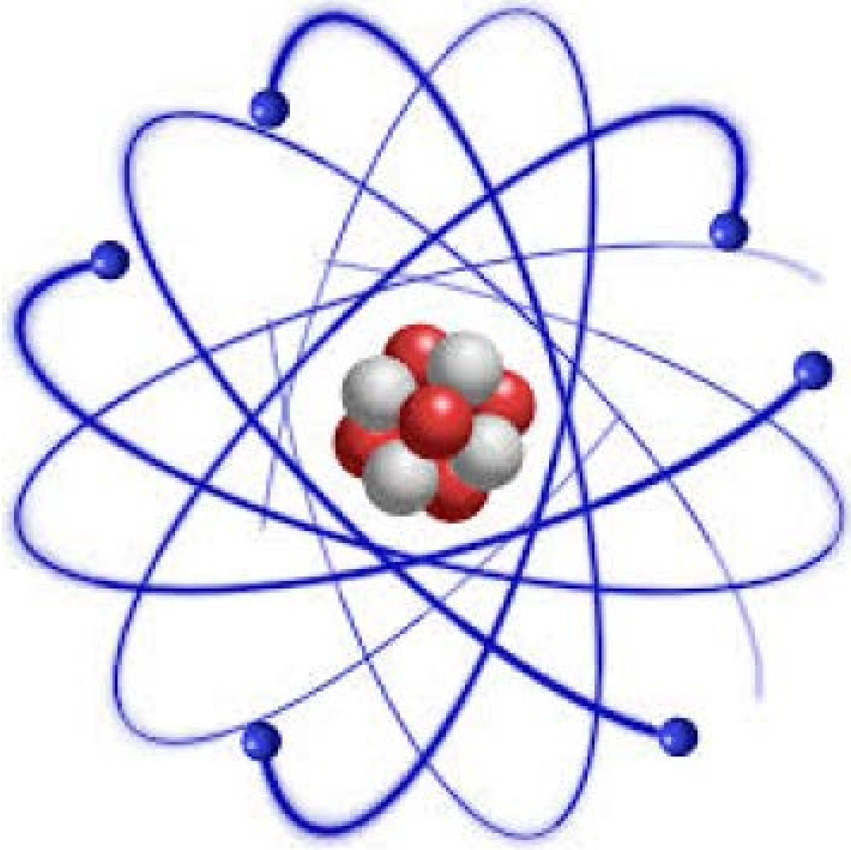
For example: the Laplace Equation in three dimensional space:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

can be written as:

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0.$$

Partial Differential Equations Everywhere!!!



The equation that describes the evolution of the state $\psi(x,t)$, of a quantum particle is a PDE that due to E. Schrödinger is given as:

$$i\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2}.$$

Partial Differential Equations Everywhere!!!

The equation that describes the dynamics of our universe is a system of PDEs and in a matrix form are compiled as:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu},$$

due to A. Einstein.

Order of Partial Differential Equation

The order of the differential equation is order of the highest derivative in the differential equation.

Partial Differential Equation	**Order**
$x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = nx$	1
$2u_{xx} + 2u_{xt}u_t + 3u_t = 0$	2
$2u_{xx} + u_{xt} + 3u_{tt} + 4u_x + \cos(2t) = 0$	2

Linear/Non-linear PDEs

A PDE is linear if it is linear in the unknown function and its derivatives.

Examples of linear PDEs:

$$2u_{xx} + 1u_{xt} + 3u_{tt} + 4u_x + \cos(2t) = 0$$

$$2u_{xx} - 3u_t + 4u_x = 0$$

Examples of Non–linear PDEs:

$$2u_{xx} + \underline{(u_{xt})^2} + 3u_{tt} = 0$$

$$\underline{\sqrt{u_{xx}}} + 2u_{xt} + 3u_t = 0$$

$$2u_{xx} + 2\underline{u_{xt}u_t} + 3u_t = 0$$

Quasi-linear PDE

A partial differential equation is said to be **quasi-linear** if it is linear with respect to all the highest order derivatives of the unknown function. In other words, a PDE is known as a **quasi-linear** PDE if all the terms with highest order derivatives of dependent variables occur linearly, that is the coefficients of such terms are functions of only lower order derivatives of the dependent variables. However, terms with lower order derivatives can occur in any manner.

Examples:

1. $u_{xx} + u_{xt} + 3u_{tt} + \cos(2t) = 0$ (Linear PDE)
2. $u_x u_{xx} + u_y u_{yy} = 0$ (Quasi–linear PDE)
3. $2u_{xx} + u_{xt} + 3u_{tt} + 4(u_x)^2 + \cos(2t) = 0$ (Quasi–linear PDE)
4. $2(u_{xx})^2 - 3u_t + 4u_x = 0$ (Neither linear nor quasi–linear PDE)

Homogeneous/Non-homogeneous PDE

If all the terms of a partial differential equation contains the dependent variable or its partial derivatives, then such a partial differential equation is called a **homogeneous PDE** otherwise the it is known as a **non-homogeneous PDE**. In simple words, if an equation can be written in the form $\mathfrak{D}u = 0$ then it is a homogeneous PDE while an equation of the form: $\mathfrak{D}u = f \neq 0$, then it is a non-homogeneous PDE.

Examples:

1. $u_{xx} + u_{xt} + 3u_{tt} + \cos(2t) = 0$ (Non−homogeneous PDE)
2. $u_x u_{xx} + u_y u_{yy} = x^2 + y$ (Non−homogeneous PDE)
3. $2u_{xx} + u_{xt} + 3u_{tt} + 4(u_x)^2 + \cos(2t) = 0$ (Non−homogeneous PDE)
4. $2(u_{xx})^2 - 3u_t + 4u_x = 0$ (Homogeneous PDE)

Examples

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial t} - T \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\left(x^2 + y^2\right)\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x \partial y} - 3u = 0$$

$$ux\frac{\partial^2 u}{\partial x^2} + u^2 xy \frac{\partial^2 u}{\partial x \partial y} + uy \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u^3 = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$$

Practice: Classify the above PDEs on the basis of the following parameters: order, linear/quasi-linear/non-linear and homogeneous/non-homogeneous.

Significance of Second order PDE

- PDEs are used to model many systems in many different fields of science and engineering.
- Many physical problems from fluid mechanics, heat transfer, rigid body dynamics and elasticity are modelled by second order PDE’s.
- In some problems fourth order PDE's do arise, however, as we split higher order ordinary differential equations into system of first order equations, it is also a common practice to split a 4th order PDE into two second order PDE 's along with the necessary boundary and initial conditions and solve them together.
- Hence in solution of PDE's, understanding the methods to solve second order PDE is very important to solve real world problems.

Important Examples:

- Laplace Equation
- Heat Equation
- Wave Equation

Laplace Equation

$$\frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} + \frac{\partial^2 u(x,y,z)}{\partial z^2} = 0$$

Laplace equation is:

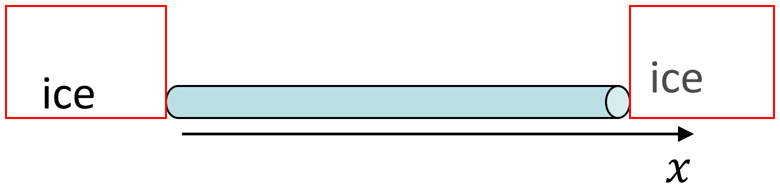
- Used to describe the steady state distribution of heat in a body.
- Also used to describe the steady state distribution of electrical charge in a body.

Heat Equation

$$\frac{\partial u(x,y,z,t)}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

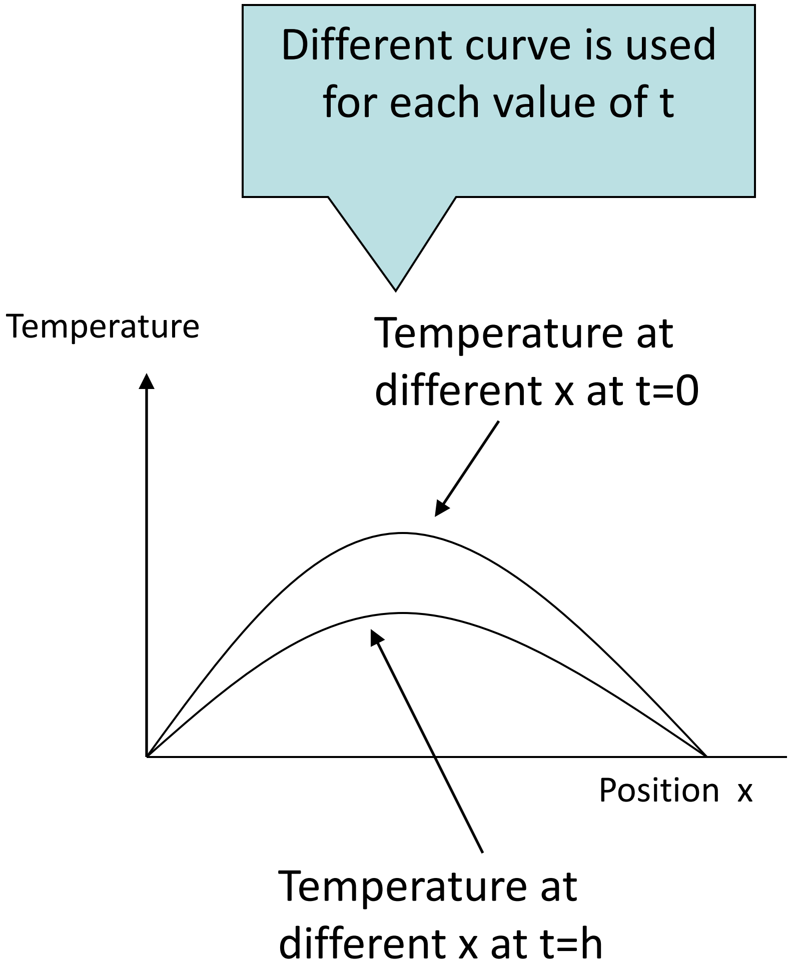
The function $u(x,y,z,t)$ is used to represent the temperature at time t in a physical body at a point with coordinates (x,y,z). For the present case: α is the thermal diffusivity. It is sufficient to consider the case $\alpha = 1$.

Heat Equation



Thin metal rod insulated everywhere except at the edges. At $t = 0$ the rod is placed in ice.

$$\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{\partial T(x,t)}{\partial t} = 0$$
$$T(0,t) = T(1,t) = 0$$
$$T(x,0) = \sin(\pi x)$$



$T(x,t)$ is used to represent the temperature at time t at the point x of the thin rod.

Wave Equation

$$\frac{\partial^2 u(x,y,z,t)}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

The function $u(x,y,z,t)$ is used to represent the displacement at time t of a particle whose position at rest is (x,y,z). The constant c represents the propagation speed of the wave.

Classification of Linear Second Order PDEs:

- Linear second order PDEs are important sets of equations that are used to model many systems in many different fields of science and engineering.

- Classification is important because:
  - Each category relates to specific engineering problems.
  - Different approaches are used to solve these categories.

Consider the following second order equation in two independent variables:

$$A\frac{\partial^2 u(x,y)}{\partial x^2} + B\frac{\partial^2 u(x,y)}{\partial x \partial y} + C\frac{\partial^2 u(x,y)}{\partial y^2} + D\frac{\partial u(x,y)}{\partial x} + E\frac{\partial u(x,y)}{\partial y} + Fu(x,y) = G \quad (*)$$

where A, B, C, D, E & F are all functions of $x, y, u, \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ only so that the equation is at most quasi-linear.

Classification of Linear Second Order PDEs:

Equation $(*)$ can be rewritten in more general form as:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + H = 0, \qquad (**)$$

where A, B, C are functions of x and y, while H is a function of x, y, u, u_x and u_y only so that the equation is at most quasi-linear. Then $(*)$ or $(**)$ are classified into three classes of PDEs depending on the discriminant, $B^2 - 4AC$ as:

- **Elliptic** if $B^2 - 4AC < 0$,
- **Parabolic** if $B^2 - 4AC = 0$,
- **Hyperbolic** if $B^2 - 4AC > 0$.

Example:

Consider the Laplace equation that is given is:

$$\frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} = 0. \qquad (1)$$

For the present case:

$$A = 1, B = 0, C = 1 \Longrightarrow B^2 - 4AC < 0.$$

This means that the **Laplace equation is elliptic**. One possible solution of this PDE is given as:

$$u(x,y) = e^x \sin y\,. \qquad (2)$$

Check:

$$u_x = e^x \sin y\,, \qquad u_{xx} = e^x \sin y$$
$$u_y = e^x \cos y\,, \qquad u_{yy} = -e^x \sin y$$
$$\Longrightarrow u_{xx} + u_{yy} = 0$$

Thus, (2) is a solution of (1).

Example:

Consider the heat equation that is given is:

$$\alpha \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0. \qquad (1)$$

For the present case:

$$A = \alpha,\ B = 0,\ C = 0 \Longrightarrow B^2 - 4AC = 0.$$

This means that the **heat equation is parabolic**.

Similarly, if we consider the wave equation:

$$c^2 \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x,t)}{\partial t^2} = 0,$$

We observe that

$$A = c^2 > 0,\ B = 0,\ C = -1 \Longrightarrow B^2 - 4AC > 0,$$

Thus, we conclude that the **wave equation is hyperbolic**.