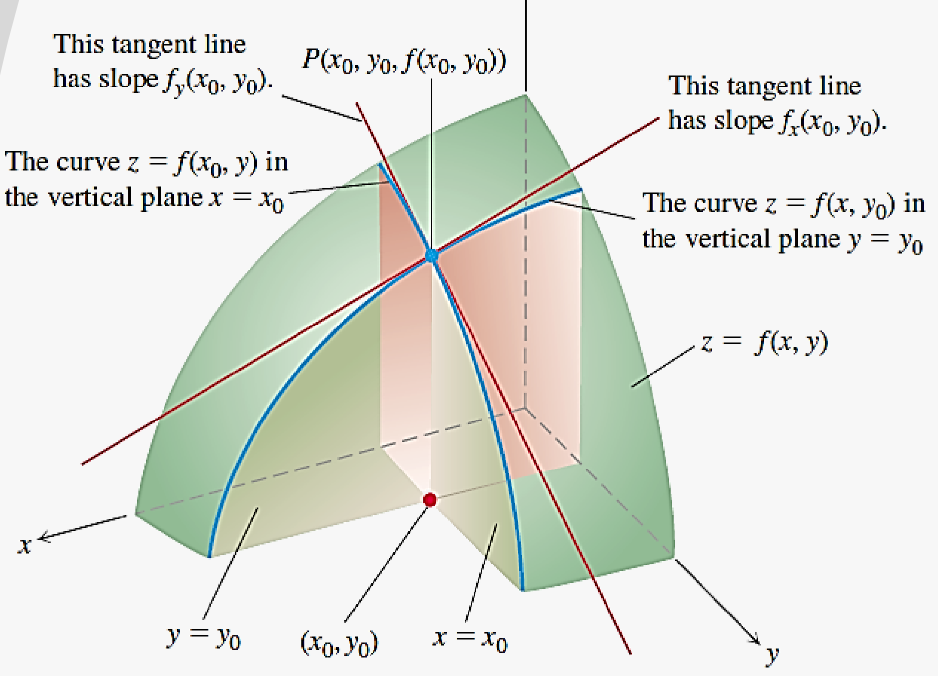
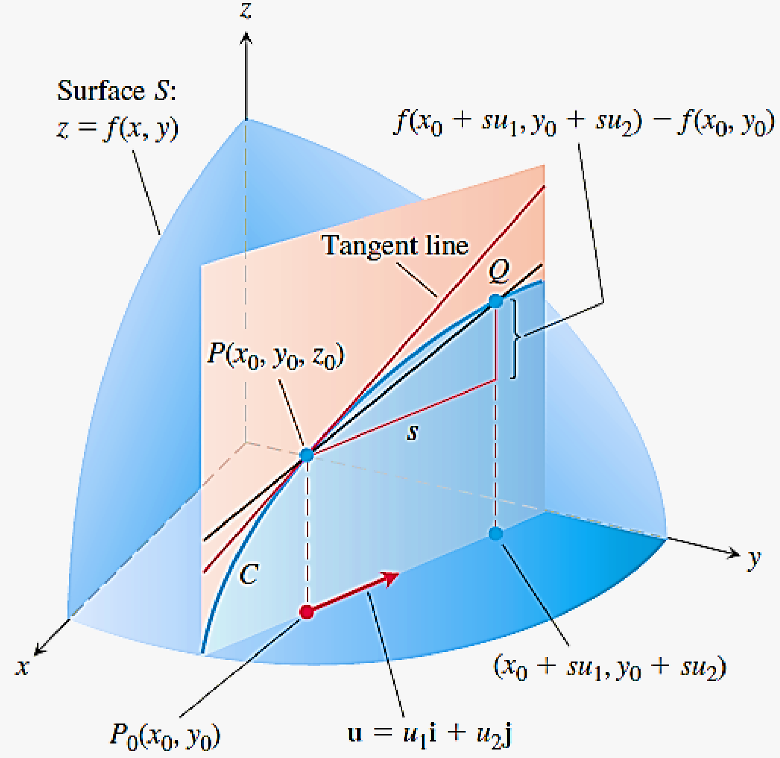
Partial Derivatives & Directional Derivatives





Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

14

Partial Derivatives

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Chapter: 14 , Section: 14.3, 14.5

Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.
Chapter: 14 , Section: 14.3, 14.6

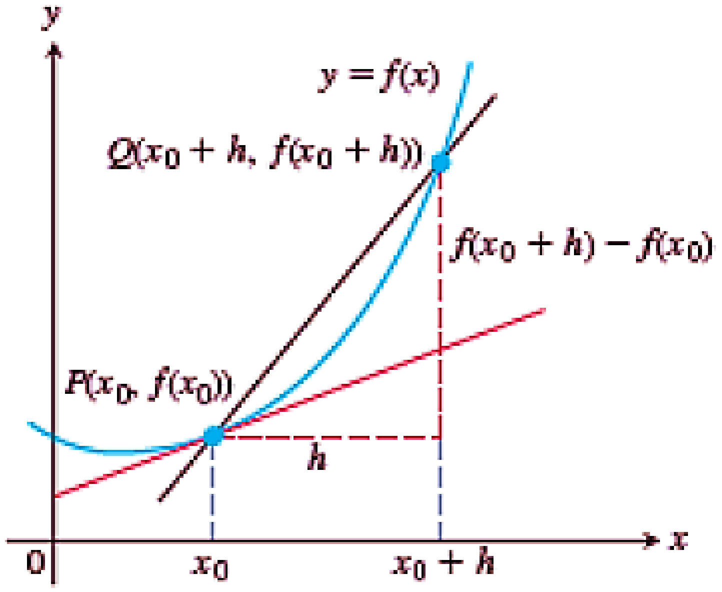
Derivatives of a Function of Single Variable:

The derivative is the formula which gives the slope of the tangent line at any point x_0 for $f(x)$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exist. There is no hole, no jump and no sharp corner. The following are all interpretations for the limit of the difference quotient:

- The slope of the graph of $y = f(x)$ at $x = x_0$.
- The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$.
- The rate of change of $f(x)$ with respect to x at $x = x_0$.
- The derivative $f'(x_0)$ at $x = x_0$.



Partial Derivatives of a Function of Two Variables:

Suppose we have a multi-variable function of two variables $z = f(x, y)$, defined in domain D of xy $-$plane. Therefore, our function f depends on x and y, both. Now if we want to take derivative of f , then we have two options: either to take the derivative with respect to x or with respect to y. If f is a function of two variables, its partial derivatives are the functions $\frac{\partial f}{\partial x} = f_x$ and $\frac{\partial f}{\partial y} = f_y$ defined by:

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h},$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h},$$

provided the limit exists.

Notations for Partial Derivatives

There are many alternative notations for partial derivatives. If $z = f(x, y)$, we write:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Partial Derivatives of a Function at a point:

If f is a function of two variables, its partial derivative with respect to x at a point (x_0, y_0) is given as:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

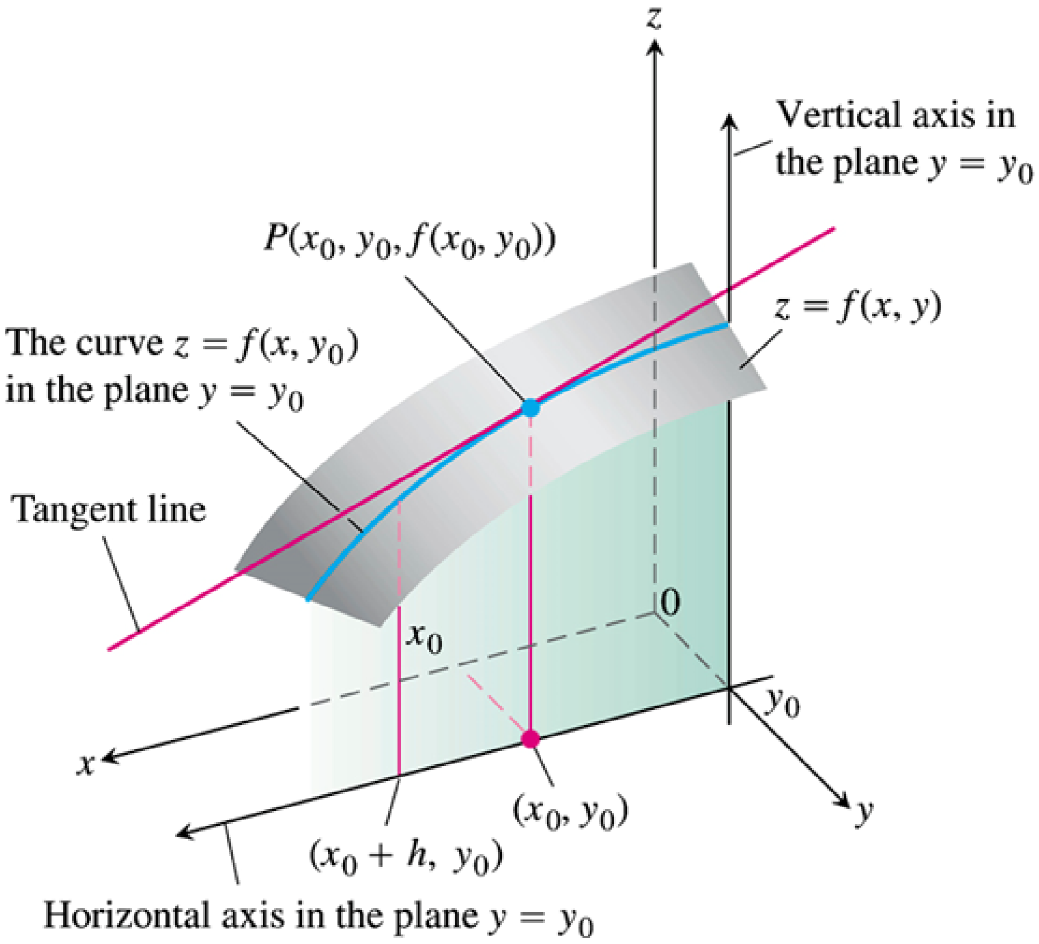
If f is a function of two variables, its partial derivative with respect to y at a point (x_0, y_0) is:

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

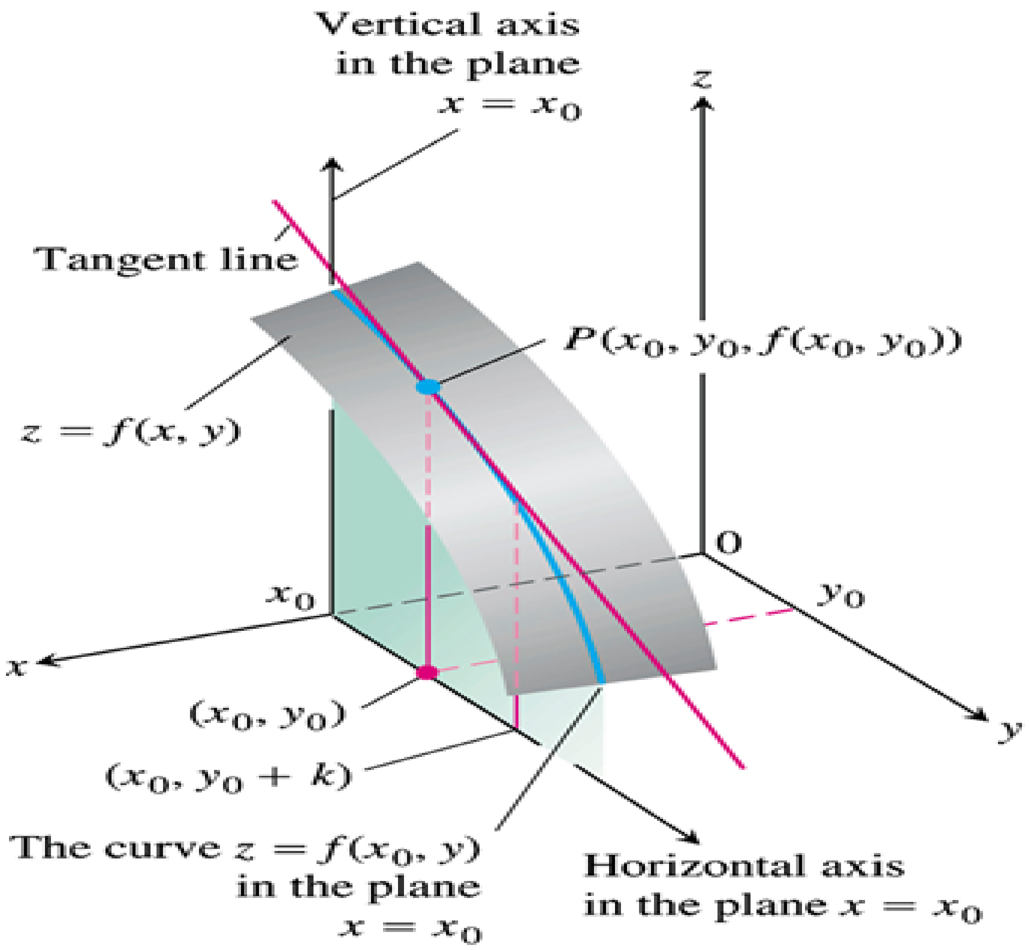
provided the limit exists.

The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0)

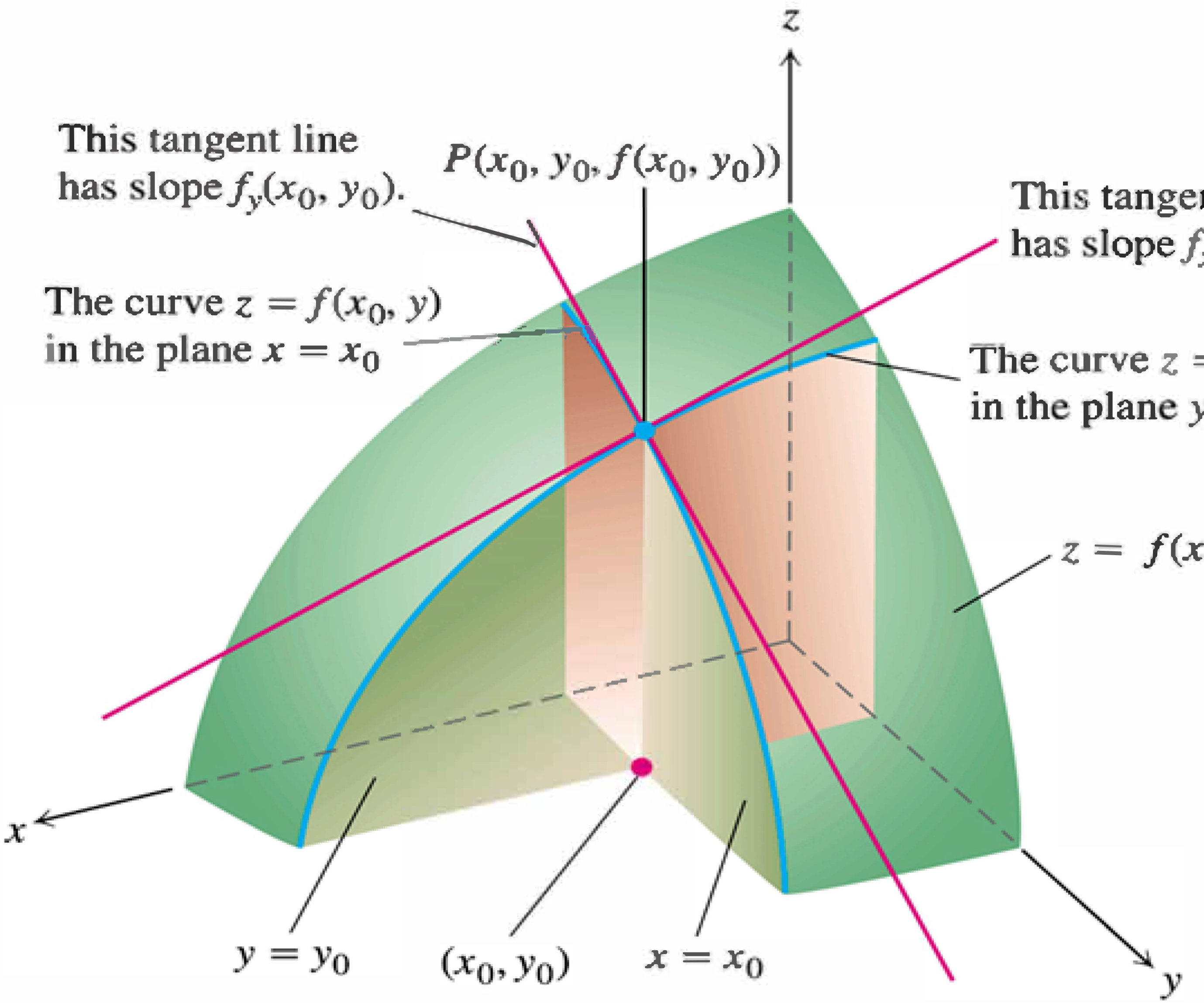
The slope of the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the value of the partial derivative of f with respect to x at (x_0, y_0). The tangent line to the curve at P is the line in the plane $y = y_0$ that passes through P with this slope. The partial derivative $f_x = \frac{\partial f}{\partial x}$ at (x_0, y_0) gives the rate of change of f with respect to x when y is held fixed at the value y_0.



The partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0)



The slope of the curve $z = f(x_0, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the vertical plane $x = x_0$ is the value of the partial derivative of f with respect to y at (x_0, y_0). The tangent line to the curve at P is the line in the plane $x = x_0$ that passes through P with this slope. The partial derivative $f_y = \frac{\partial f}{\partial y}$ at (x_0, y_0) gives the rate of change of f with respect to y when x is held fixed at the value x_0.



The tangent lines at the point $(x_0, y_0, f(x_0, y_0))$ determine a plane that, in this picture at least, appears to be tangent to the surface.

Example:

If $z = f\ (x, y) = 4 - x^2 - 2y^2$, determine $f_x(1,1), f_y(1,1)$ and interpret these numbers as slopes.

$z = 4 - x^2 - 2y^2$

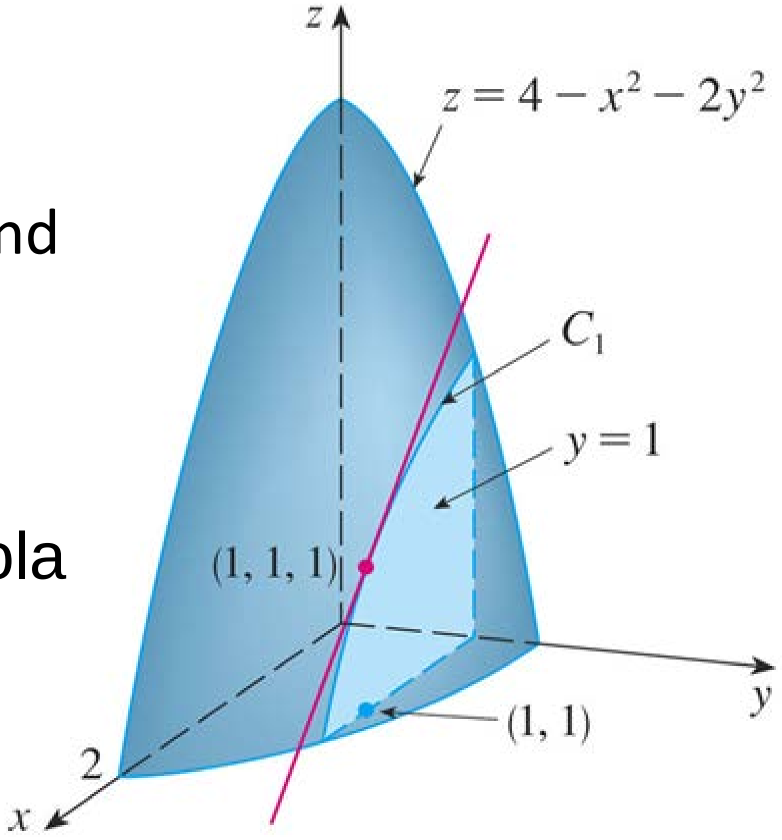
$\Rightarrow \quad x^2 + 2y^2 + (z - 4) = 0$

$f(x, y, z) = 0$

Solution: For the present case:

$$f_x(x, y) = -2x \Longrightarrow f_x(1,1) = -2,$$

$$f_y(x, y) = -4y \Longrightarrow f_y(1,1) = -4.$$



The graph of $z = f\ (x, y) = 4 - x^2 - 2y^2$ is the paraboloid and the vertical plane $y = 1$ intersects it in the parabola:

$$z = 2 - x^2, \qquad y = 1.$$

We label it C_1. The *slope of the tangent line* to this parabola at the point $(1,1,1)$ is given by:

$$f_x(1,1) = -2.$$

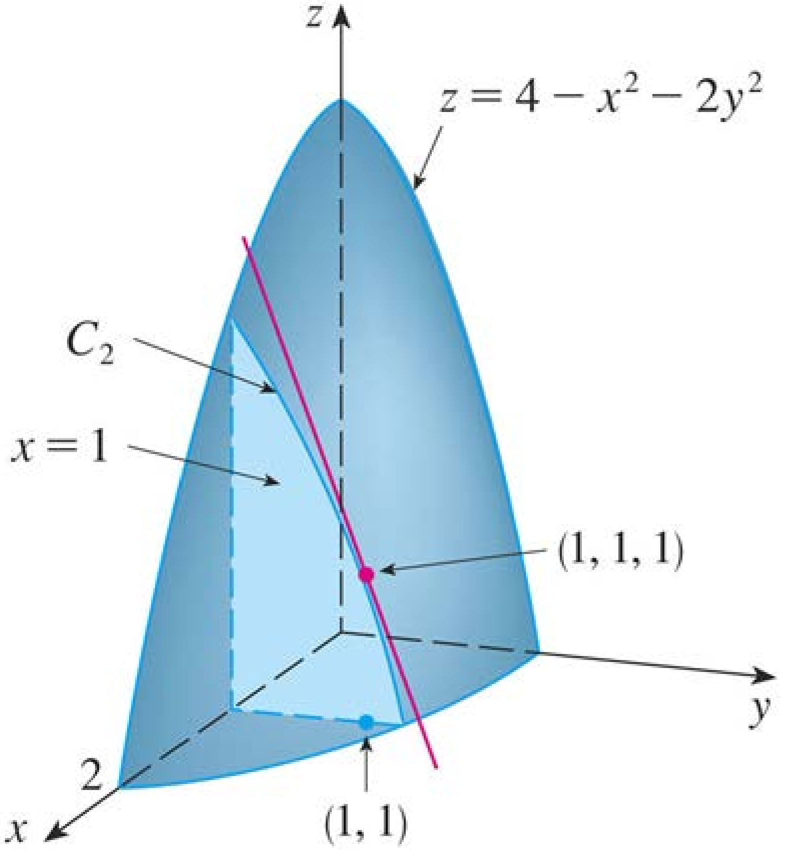
Solution:

Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola:

$$z = 3 - 2y^2, \qquad x = 1.$$

The *slope of the tangent line* to this parabola at the point $(1,1,1)$ is given by:

$$f_y(1,1) = -4.$$

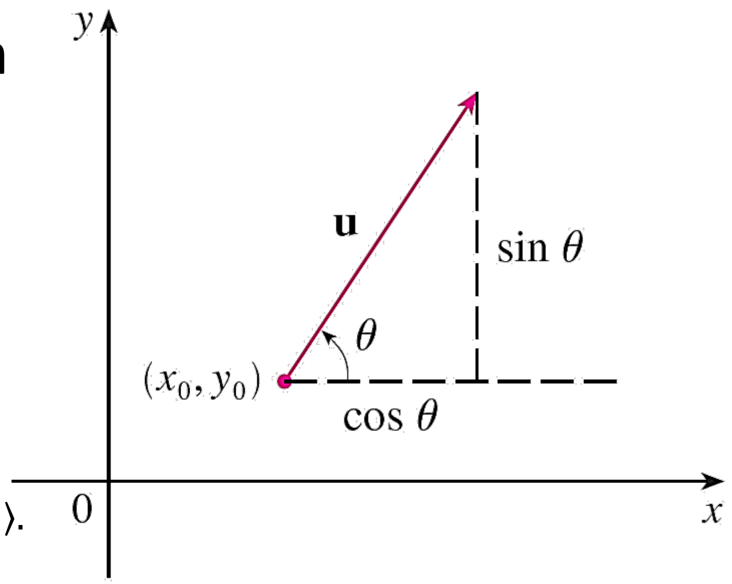


Directional Derivatives

Our objective is to introduce a type of derivative, called a *directional derivative*, that enables us to find the rate of change of a function of two or more variables in any direction. Recall that if $z = f\ (x, y)$, then the partial derivatives f_x and f_y are defined as:

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$
$$f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}, \qquad \text{(I)}$$

and represent the rates of change of z in the $x-$ and $y-$directions, that is, in the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$. Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b\rangle$



A unit vector $\mathbf{u} = \langle a, b\rangle = \langle \cos\theta\,, \sin\theta\rangle$.

Directional Derivatives

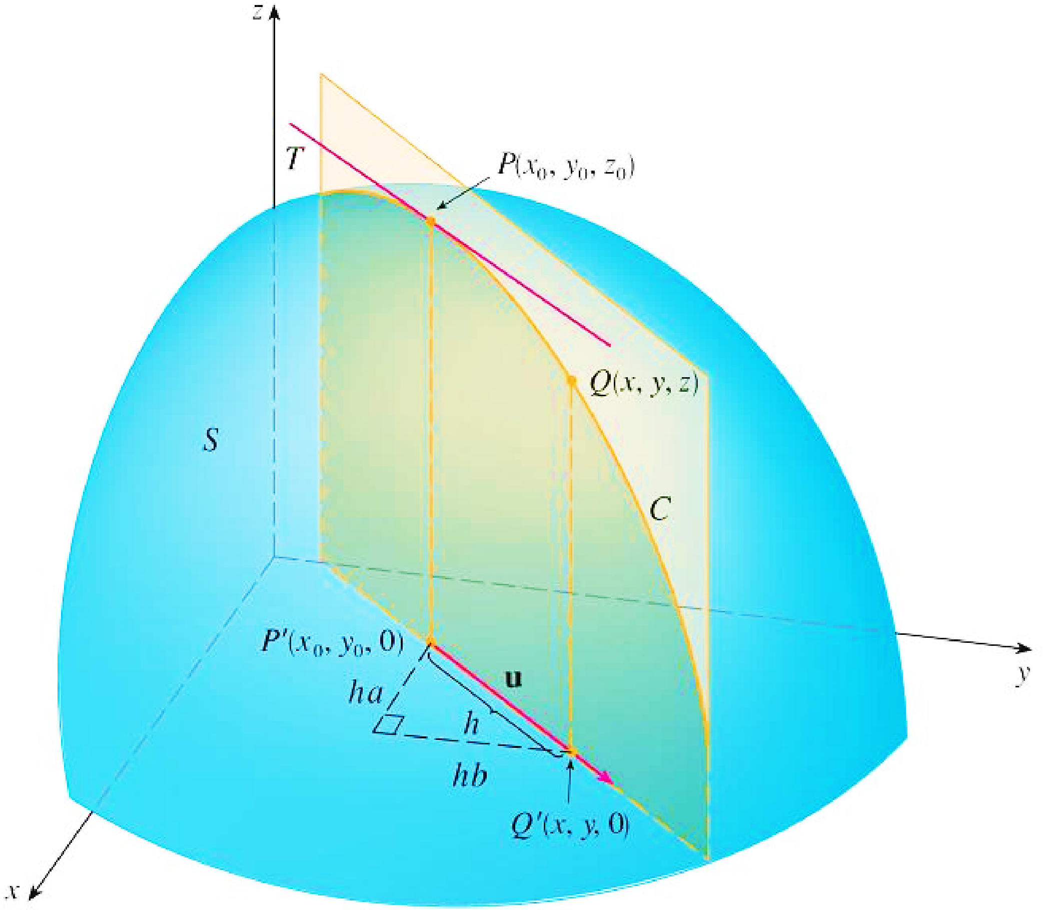
To do this we consider the surface S with the equation $z = f\,(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P in the direction of $\mathbf{u}$ intersects S in a curve C. The slope of the tangent line T to C at the point P is the rate of change of z in the direction of $\mathbf{u}$.

If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy −plane, then the vector $\overrightarrow{P'Q'}$ is parallel to $\mathbf{u}$ and so:

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle,$$

for some scalar h. Therefore $x - x_0 = ha$ and $y - y_0 = hb$. Thus,

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$



Directional Derivatives

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of $\mathbf{u}$, which is called the directional derivative of f in the direction of $\mathbf{u}$. Thus, the **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u}$ is given as:

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}, \qquad \text{(II)}$$

provided the limit exists. Equation (II) represents the derivative of f at the point (x_0, y_0) in the direction of $\mathbf{u}$. By comparing (II) with (I), we see that if $\mathbf{u} = \mathbf{i} = \langle 1,0 \rangle$, then $D_{\mathbf{i}} f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0,1 \rangle$, then $D_{\mathbf{j}} f = f_x$. In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

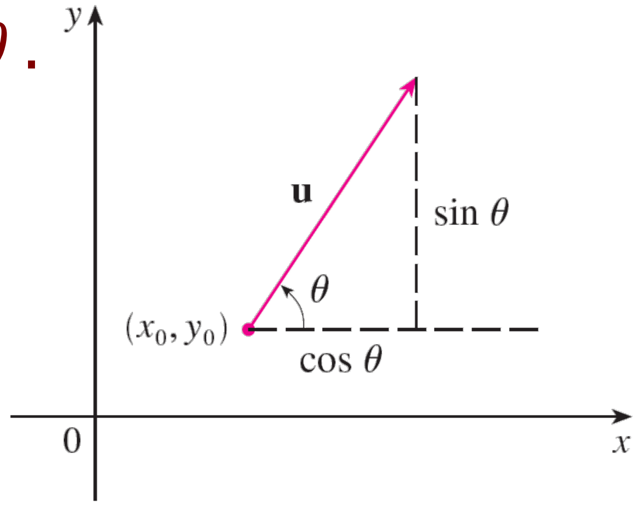
Directional Derivatives

We now develop an efficient formula to calculate the directional derivative for a differentiable function f. We begin with the line $x = x_0 + ha, y = y_0 + hb$, through $P(x_0, y_0)$ parametrized with the arc length parameter h increasing in the direction of the unit vector $\mathbf{u} = \langle a, b \rangle$. Then

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x –axis, then we can write $\mathbf{u} = \langle \cos\theta\,, \sin\theta \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos\theta + f_y(x, y) \sin\theta\,.$$



A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta\,, \sin\theta \rangle$.

Example:

Find the directional derivative $D_{\mathbf{u}}f(x,y)$ if

$$z = f\ (x,y) = x^3 - 3xy + 4y^2,$$

and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. Moreover, determine $D_{\mathbf{u}}f(1,2)$.

Solution:

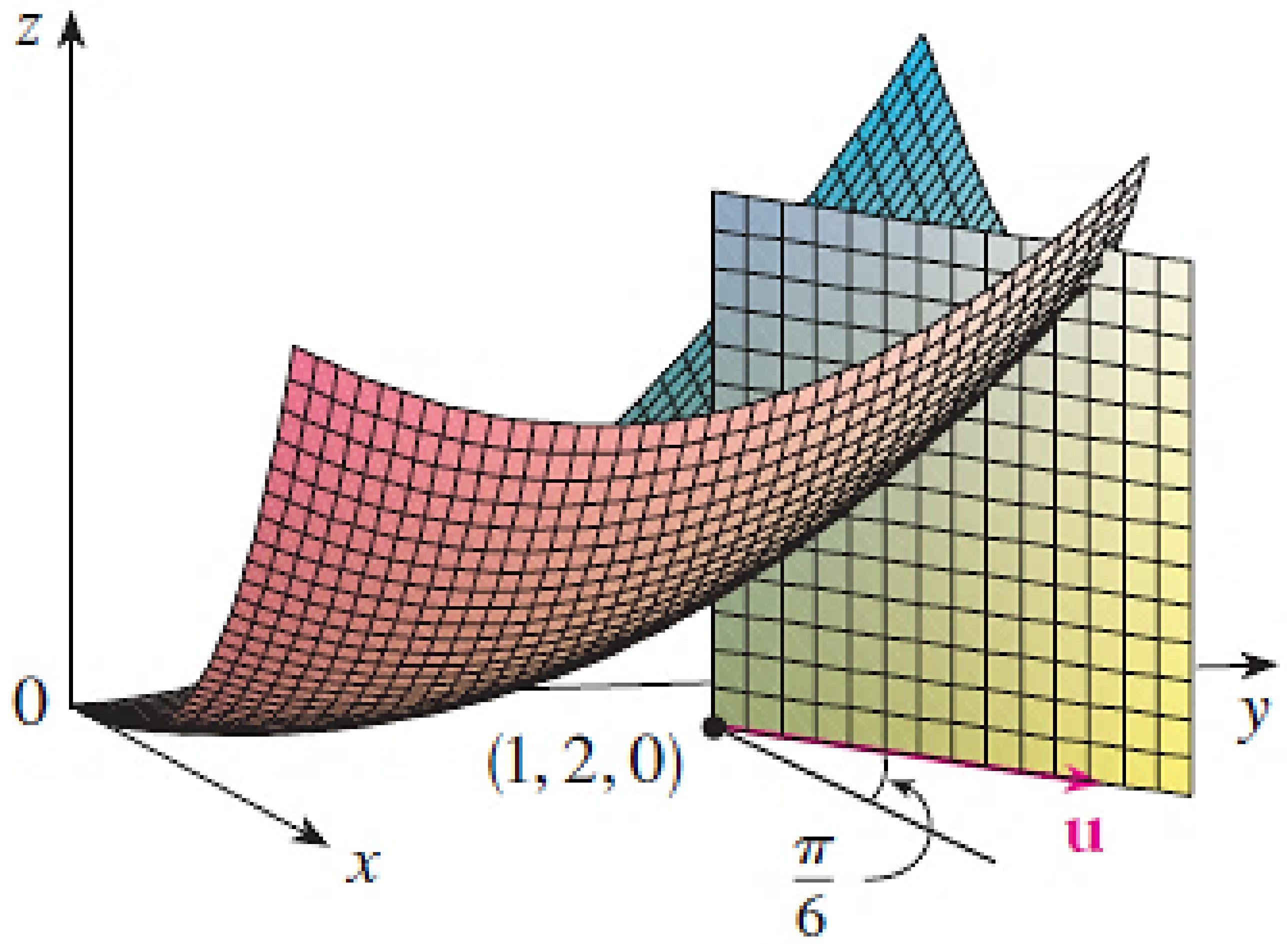
For the present case:

$$\begin{aligned} D_{\mathbf{u}}f(x,y) &= f_x(x,y)\cos\frac{\pi}{6} + f_y(x,y)\sin\frac{\pi}{6} \\ &= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2} \\ &= \frac{1}{2}\left[3\sqrt{3}\ x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right] \end{aligned}$$

Therefore,

$$D_{\mathbf{u}}f(1,2) = \frac{1}{2}\left[3\sqrt{3}\ (1)^2 - 3(1) + \left(8 - 3\sqrt{3}\right)(2)\right] = \frac{13 - 3\sqrt{3}}{2}.$$

Solution:



The directional derivative $D_{\mathbf{u}}f(1,2)$ represents the rate of change of z in the direction of $\mathbf{u}$. This is the slope of the tangent line to the curve of intersection of the surface $z = f\ (x, y) = x^3 - 3xy + 4y^2$ and the vertical plane through $(1,2,0)$ in the direction of $\mathbf{u}$.

The Gradient Vector

Note that the directional derivative of a differentiable function can be written as the dot product of two vectors:

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$$

$$= \left\langle f_x(x,y), f_y(x,y) \right\rangle \cdot \langle a, b \rangle$$

$$= \left\langle f_x(x,y), f_y(x,y) \right\rangle \cdot \mathbf{u}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well. So, we give it a special name: **the *gradient* of** f and a special notation: $\mathbf{grad}\ f$ or ∇f, which is read "del f ".

The Gradient Vector

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

With the notation for the gradient vector, we can rewrite the directional derivative of a differentiable function f as:

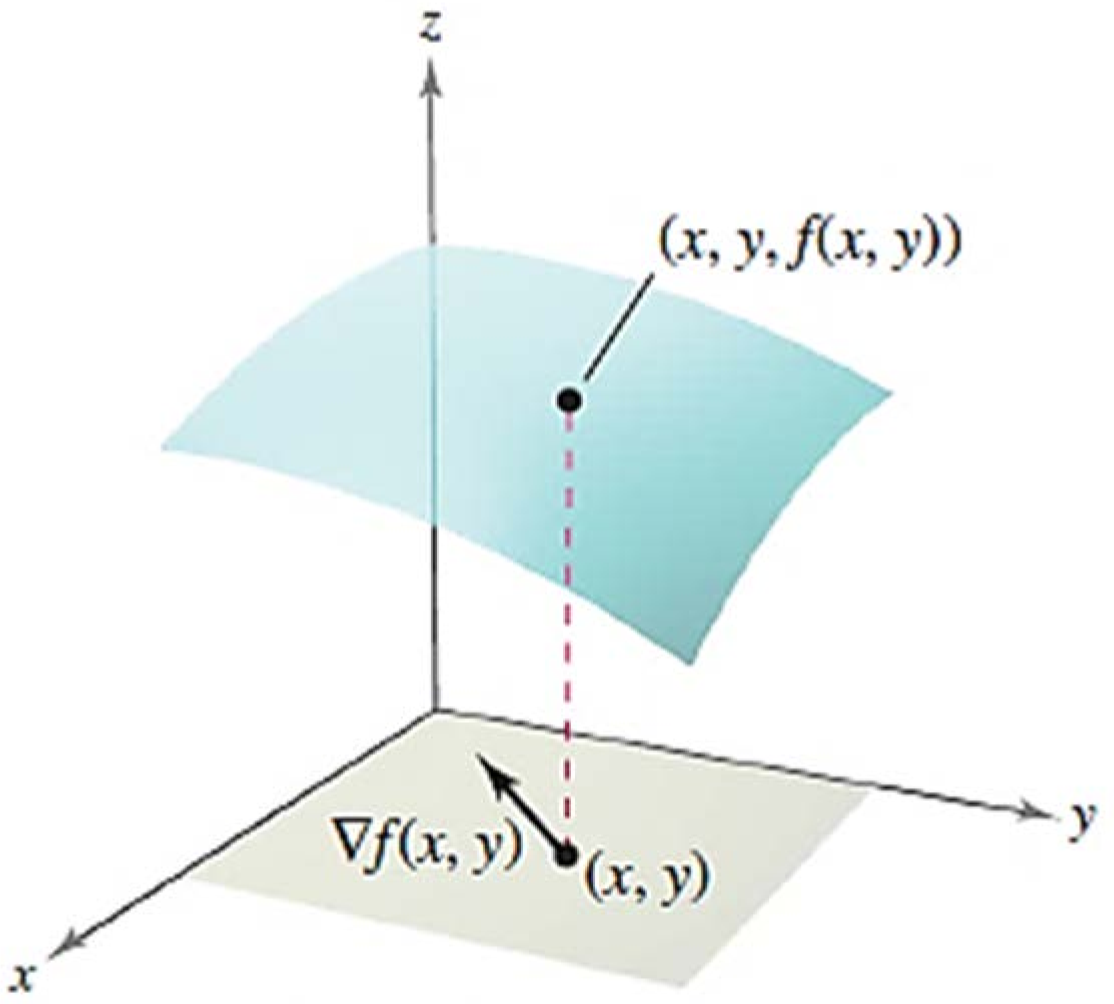
$$D_{\mathbf{u}} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \cdot \mathbf{u} = \nabla f(x,y) \cdot \mathbf{u}.$$

This expresses the directional derivative in the direction of a unit vector $\mathbf{u}$ as the scalar projection of the gradient vector onto $\mathbf{u}$. Using properties of dot product, we have:

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta\,,$$

where θ is the angle between the vectors $\mathbf{u}$ and ∇f.

The Gradient of a Function of Two Variables



The gradient of f is a vector in the xy –plane.

Properties of the Directional derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$, which means that $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is:
$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$
2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is:
$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|.$$
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and
$$D_{\mathbf{u}}f = |\nabla f| \cos\left(\frac{\pi}{2}\right) = 0.$$

Example:

Find the directions in which $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$

(a) increases most rapidly at the point $(1, 1)$, and

(b) decreases most rapidly at $(1, 1)$.

(c) What are the directions of zero change in f at $(1, 1)$?

Solution:

(a) The function increases most rapidly in the direction of f at $(1, 1)$. The gradient there is:

$$\nabla f\Big|_{(1,1)} = \langle x, y\rangle\Big|_{(1,1)} = \langle 1,1\rangle.$$

Its direction is:

$$\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1,1\rangle.$$

Solution:

(b) The function decreases most rapidly in the direction of $-\nabla f$ at $(1,1)$, which is:

$$-\mathbf{u} = \frac{-1}{\sqrt{2}}\langle 1,1\rangle.$$

(c) The directions of zero change at $(1,1)$ are the directions orthogonal to ∇f:

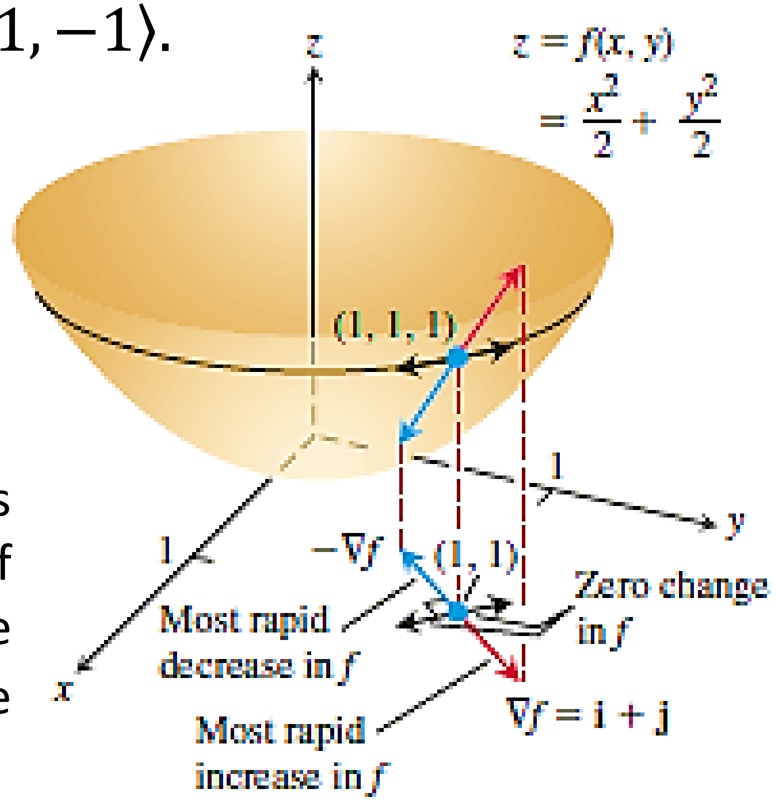
$$\mathbf{n} = \frac{1}{\sqrt{2}}\langle -1,1\rangle \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\langle 1,-1\rangle.$$

$\vec{\nabla} f\big|_{(1,1)} = \langle 1, 1\rangle$

$\langle -1,1\rangle \cdot \langle 1,1\rangle = 0$

or $\langle 1,-1\rangle \cdot \langle 1,1\rangle = 0$

The direction in which $f(x,y)$ increases most rapidly at $(1,1)$ is the direction of $\nabla f|_{(1,1)} = \langle 1,1\rangle$. It corresponds to the direction of steepest ascent on the surface at $(1,1,1)$.



Example:

Determine the gradient of the function:

$$f\,(x,y) = \sin x + e^{xy}.$$

Moreover, compute the gradient vector at $(0,1)$.

Solution:

If $f\,(x,y) = \sin x + e^{xy}$, then

$$\nabla f(x,y) = \left\langle f_x(x,y), f_y(x,y)\right\rangle = \langle \cos x + ye^{xy}, xe^{xy}\rangle,$$

and

$$\nabla f(0,1) = \langle 2,0\rangle.$$

Example:

Determine the gradient of the function:

$$f\,(x,y) = \sin(xy) + x^3 e^{y^2}.$$

Moreover, compute the gradient vector at $(0,1)$.

Solution:

$$f_x = y\cos(xy) + 3x^2 e^{y^2}.$$

$$f_y = x\cos(xy) + (2y)x^3 e^{y^2}.$$

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle y\cos(xy) + 3x^2 e^{y^2},\ x\cos(xy) + 2x^3 y e^{y^2} \rangle$$

Thus,

$$\nabla f(0,1) = \langle (1)\cos(0) + 3(0)e^{1^2},\ (0)\cos(0) + 2(0)(1)e^{1^2} \rangle$$

$$= \langle 1, 0 \rangle$$

Algebraic Rules for Gradient:

Let f and g be any functions of several variables and k is any constant then following rules are valid:

1. Constant multiple rule: $\nabla(kf) = k\nabla f$.
2. Sum rule: $\nabla(f+g) = \nabla f + \nabla g$.
3. Difference rule: $\nabla(f-g) = \nabla f - \nabla g$.
4. Product rule: $\nabla(fg) = f\nabla g + g\nabla f$.
5. Quotient rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

EXAMPLE Illustrating the Gradient Rules

We illustrate the rules with $f(x, y) = x - y \qquad g(x, y) = 3y$

$$\nabla f = \mathbf{i} - \mathbf{j} \qquad \nabla g = 3\mathbf{j}.$$

$\nabla f = \langle 1, -1 \rangle \qquad \nabla g = \langle 0, 3 \rangle$

We have

1. $\nabla(2f) = \nabla(2x - 2y) = 2\mathbf{i} - 2\mathbf{j} = 2\nabla f$
2. $\nabla(f + g) = \nabla(x + 2y) = \mathbf{i} + 2\mathbf{j} = \nabla f + \nabla g$
3. $\nabla(f - g) = \nabla(x - 4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$
4. $$\begin{aligned}\nabla(fg) &= \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j} \\ &= 3y(\mathbf{i} - \mathbf{j}) + 3y\mathbf{j} + (3x - 6y)\mathbf{j} \\ &= 3y(\mathbf{i} - \mathbf{j}) + (x - y)3\mathbf{j} = g\nabla f + f\nabla g\end{aligned}$$
5. $$\begin{aligned}\nabla\left(\frac{f}{g}\right) &= \nabla\left(\frac{x - y}{3y}\right) = \nabla\left(\frac{x}{3y} - \frac{1}{3}\right) \\ &= \frac{1}{3y}\mathbf{i} - \frac{x}{3y^2}\mathbf{j} = \frac{3y\mathbf{i} - 3x\mathbf{j}}{9y^2} \\ &= \frac{3y(\mathbf{i} - \mathbf{j}) - (x - y)3\mathbf{j}}{9y^2} = \frac{g\nabla f - f\nabla g}{g^2}.\end{aligned}$$

Example:

Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2\,,-1)$ in the direction of the vector $\mathbf{v} = \langle 2,5\rangle$.

Solution:

We first compute the gradient vector at $(2, -1)$ that is given as:

$$\nabla f(x, y) = \langle 2xy^3, 3x^2y^2 - 4\rangle \Longrightarrow \nabla f(2, -1) = \langle -4,8\rangle.$$

Note that $\mathbf{v}$ is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of $\mathbf{v}$ is given as:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{29}}\langle 2,5\rangle.$$

Therefore,

$$D_{\mathbf{u}}f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = \langle -4,8\rangle \cdot \frac{1}{\sqrt{29}}\langle 2,5\rangle = \frac{32}{\sqrt{29}}.$$

Functions of Three Variables

For functions of three variables, we can define directional derivatives in a similar manner. Again, $D_{\mathbf{u}}f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ and is given as:

$$\begin{aligned} D_{\mathbf{u}}f(x, y, z) &= f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c \\ &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle \cdot \mathbf{u}. \end{aligned}$$

For a function f of three variables, the **gradient vector**, denoted by ∇f or $\mathbf{grad}\ f$, is:

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Then, just as with functions of two variables, the formula for the directional derivative can be rewritten as:

$$D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.$$

Example:

If $f(x, y, z) = x\sin(yz)$, find the gradient of f and the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution:

The gradient of f is:

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle,$$

$$= \langle \sin(yz), xz\cos(yz), xy\cos(yz)\rangle.$$

At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0,0,3\rangle = 3\mathbf{k}$. The unit vector in the direction of the vector: $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is given as:

$$\mathbf{u} = \frac{1}{\sqrt{6}}\langle 1,2,-1\rangle = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}.$$

Therefore,

$$D_{\mathbf{u}}f(1,3,0) = \nabla f(1,3,0) \cdot \mathbf{u} = 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\right) = -\frac{3}{\sqrt{6}} = -\sqrt{\frac{3}{2}}.$$