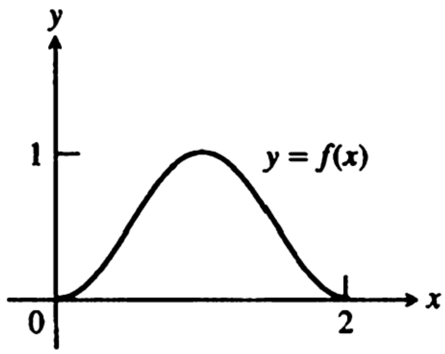
National University of Sciences & Technology
School of Electrical Engineering and Computer Science
Department of Basic Sciences

MATH-101: Calculus and Analytical Geometry (3+0): BEE2k20-ABC Fall 2020

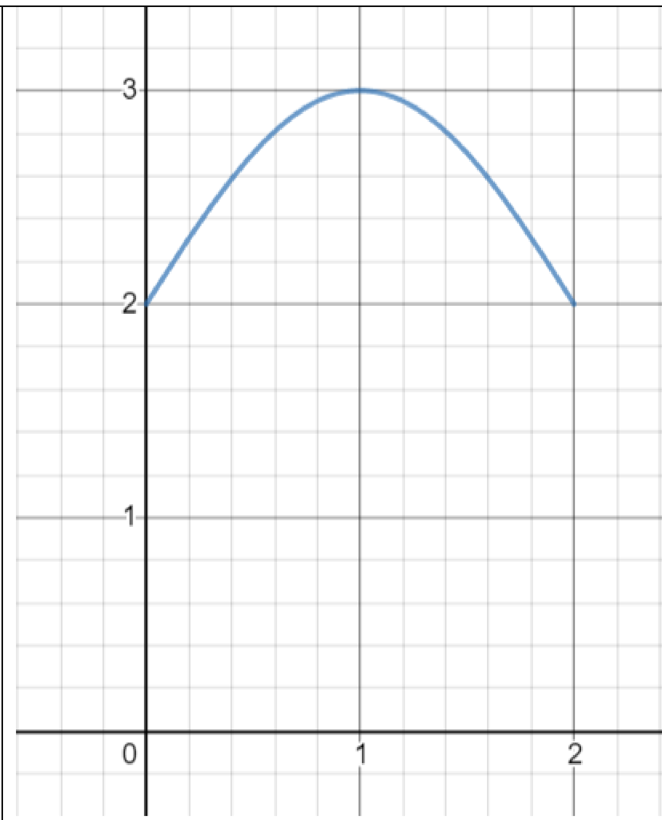
Master Solution (Assignment # 1)
CLO: 1 (Understand the concept of limit, continuity and derivative with its application to find extrema)

Q - 1: [CLO-1: 30 marks]

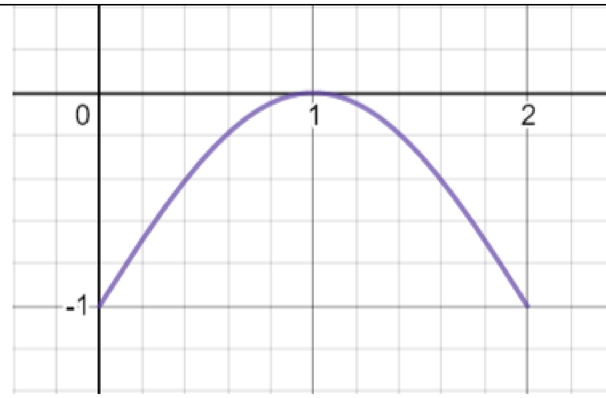
The accompanying figure shows the graph of a function $f\ (x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions and sketch their graphs by clearly mentioning the type of transformation used in each case:



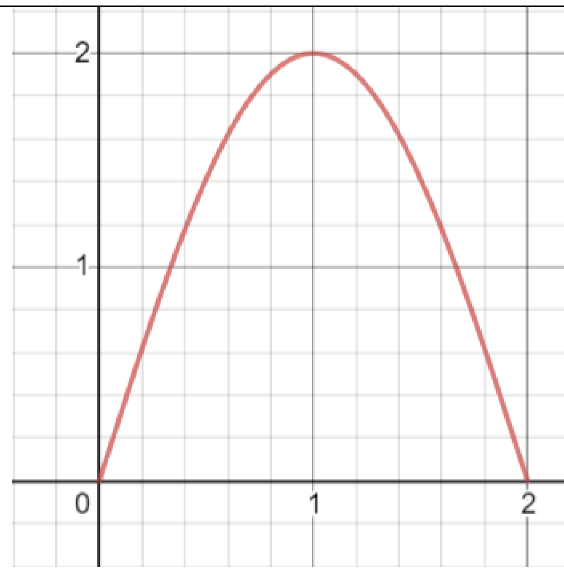
1) $f(x) + 2$

Vertical Shift 2 units up Domain = [0,2] Range = [2,3]	

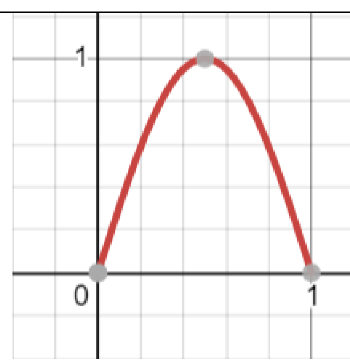
2) $f(x) - 1$

Vertical Shift 1 units down Domain = $[0,2]$ Range = $[-1,0]$	

3) $2f\ (x)$

Vertical Stretch by a factor of 2 Domain = $[0,2]$ Range = $[0,2]$	

4) $f\ (2x)$

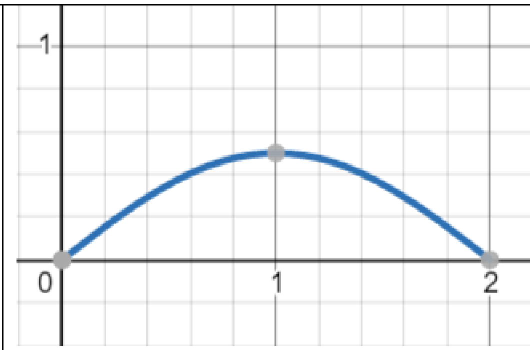
Horizontal compress by a factor of 2 Domain = $[0,1]$ Range = $[0,1]$	

5) $\frac{1}{2}f\ (x)$

Vertical compress by a factor of 2

Domain = $[0,2]$

Range = $\left[0,\frac{1}{2}\right]$

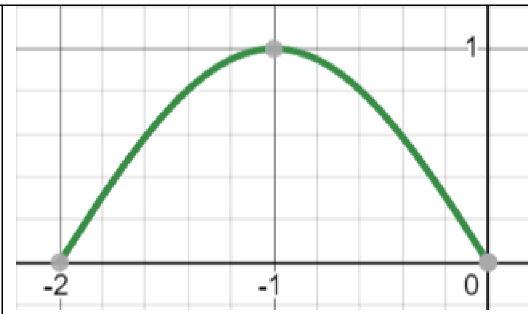


6) $f(x+2)$

Horizontal shift 2 units left

Domain = $[-2,0]$

Range = $[0,1]$

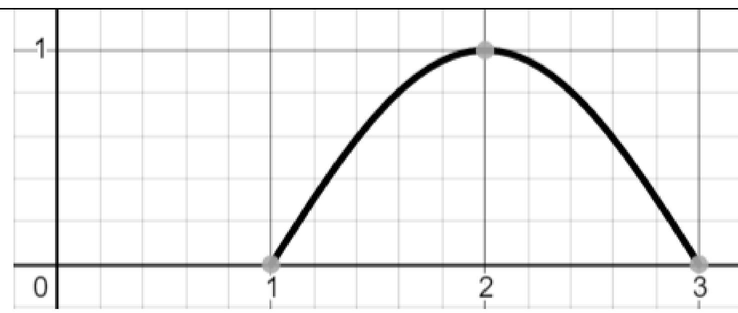


7) $f\ (x\ -\ 1)$

Horizontal shift 1 units right

Domain = $[1,3]$

Range = $[0,1]$

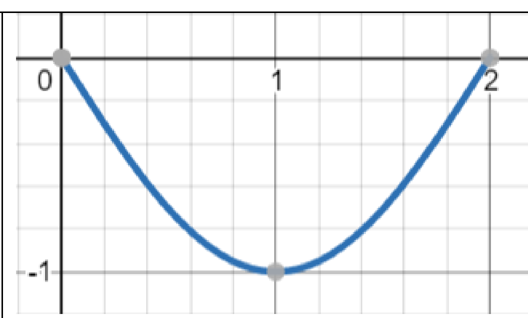


8) $-f\ (x)$

Reflection about x –axis

Domain = $[0,2]$

Range = $[-1,0]$

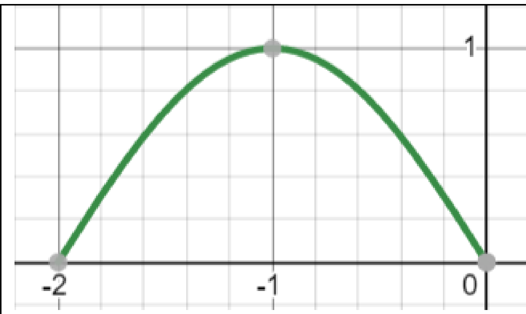


9) $f(-x)$

Reflection about y −axis

Domain = $[-2,0]$

Range = $[0,1]$

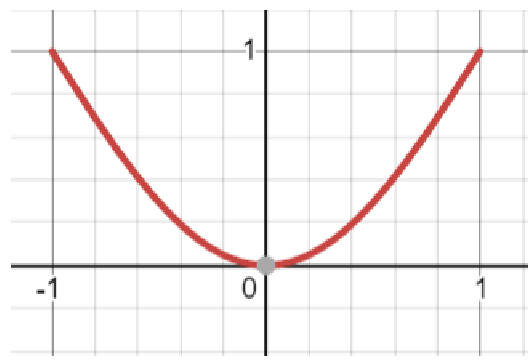


10) $-f\ (x\ +\ 1)\ +\ 1$

Reflection about x −axis followed by horizontal shift 1 units left by vertical shift 1 units up.

Domain = $[-1,1]$

Range = $[0,1]$



Q - 2: [CLO-1: 20 marks]

Draw graphs and determine domain and range of the following functions.

1) $f(x) = |x| + |x - 1|$

We know that:

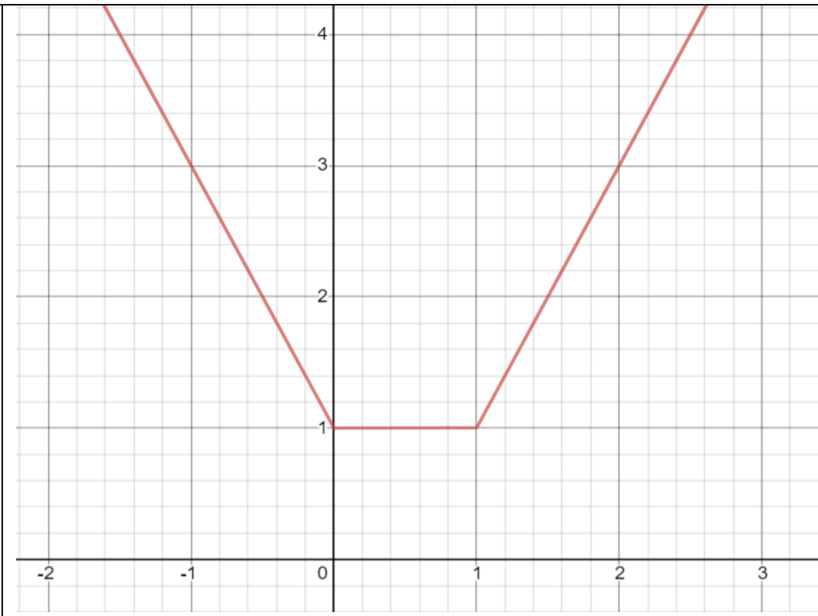
$$|x| = \begin{cases} x; & \text{if } x \geq 0 \\ -x; & \text{if } x < 0 \end{cases} \quad \text{and} \quad |x-1| = \begin{cases} x-1; & \text{if } x \geq 1 \\ -x+1; & \text{if } x < 1 \end{cases}$$

Addition of these two functions yield:

$$f(x) = |x| + |x-1| = \begin{cases} x + (x-1) = 2x - 1; & \text{if } x \geq 0 \\ -x + (-x+1) = -2x + 1; & \text{if } x < 0 \end{cases}$$

Domain = $(-\infty, \infty)$

Range = $[1, \infty)$



2) $f(x) = x - \lfloor x \rfloor$

We know that:

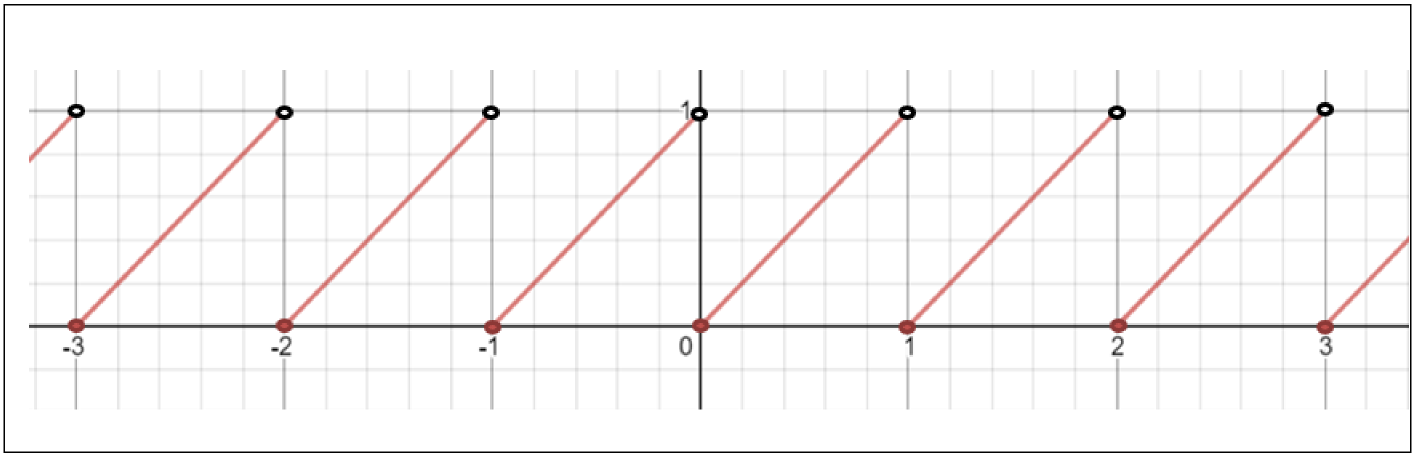
$$\lfloor x \rfloor = \begin{cases} 0; & \text{if } 0 \leq x < 1 \\ 1; & \text{if } 1 \leq x < 2 \\ 2; & \text{if } 2 \leq x < 3 \\ \vdots & \\ -1; & \text{if } -1 \leq x < 0 \\ -2; & \text{if } -2 \leq x < -1 \end{cases}$$

Note that if x is an integer then $\lfloor x \rfloor = x$. Thus, when x is an integer whether positive or negative we get $f(x) = x - \lfloor x \rfloor = x - x = 0$. But if x is not an integer then in that case $f(x) \neq 0$. For example, if we take

- $x = 2.5$ then $f(x) = 2.5 - \lfloor 2.5 \rfloor = 2.5 - 2 = 0.5$
- $x = -2.5$ then $f(x) = -2.5 - \lfloor -2.5 \rfloor = -2.5 + 3 = 0.5$
- $x = 1.7$ then $f(x) = 1.7 - \lfloor 1.7 \rfloor = 1.7 - 1 = 0.7$
- $x = -1.7$ then $f(x) = -1.7 - \lfloor -1.7 \rfloor = -1.7 + 2 = 0.3$
- $x = 0.4$ then $f(x) = 0.4 - \lfloor 0.4 \rfloor = 0.4 - 0 = 0.4$
- $x = -0.4$ then $f(x) = -0.4 - \lfloor -0.4 \rfloor = -0.4 + 1 = 0.6$

Thus,

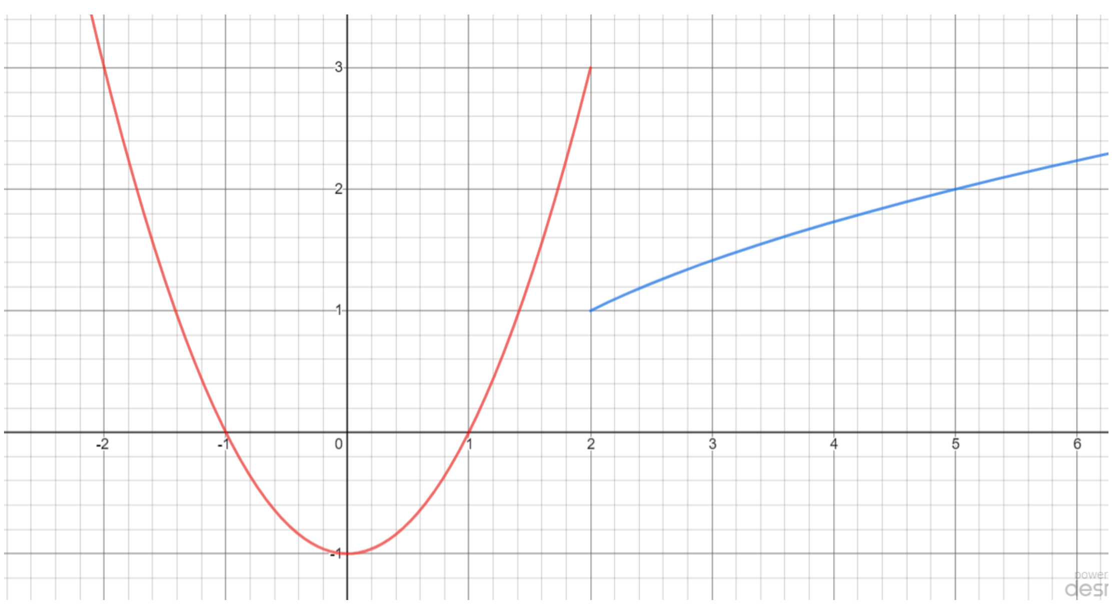
Domain = $(-\infty, \infty)$ and Range = $[0,1)$



3) $f(x) = \begin{cases} x^2 - 1, & x \leq 2 \\ \sqrt{x-1}, & x > 2 \end{cases}$

Note that $x^2 - 1$; $x \leq 2$, is a shifted parabola and domain of this piece of function is given as: $(-\infty, 2]$. The other piece of the function is given as: $\sqrt{x-1}$; $x > 2$. Domain of this piece is given as: $[2, \infty)$. Thus, the domain of the function $f(x)$ is given by the combined domain of all the pieces. Thus, Domain = $(-\infty, 2] \cup [2, \infty) = (-\infty, \infty)$.

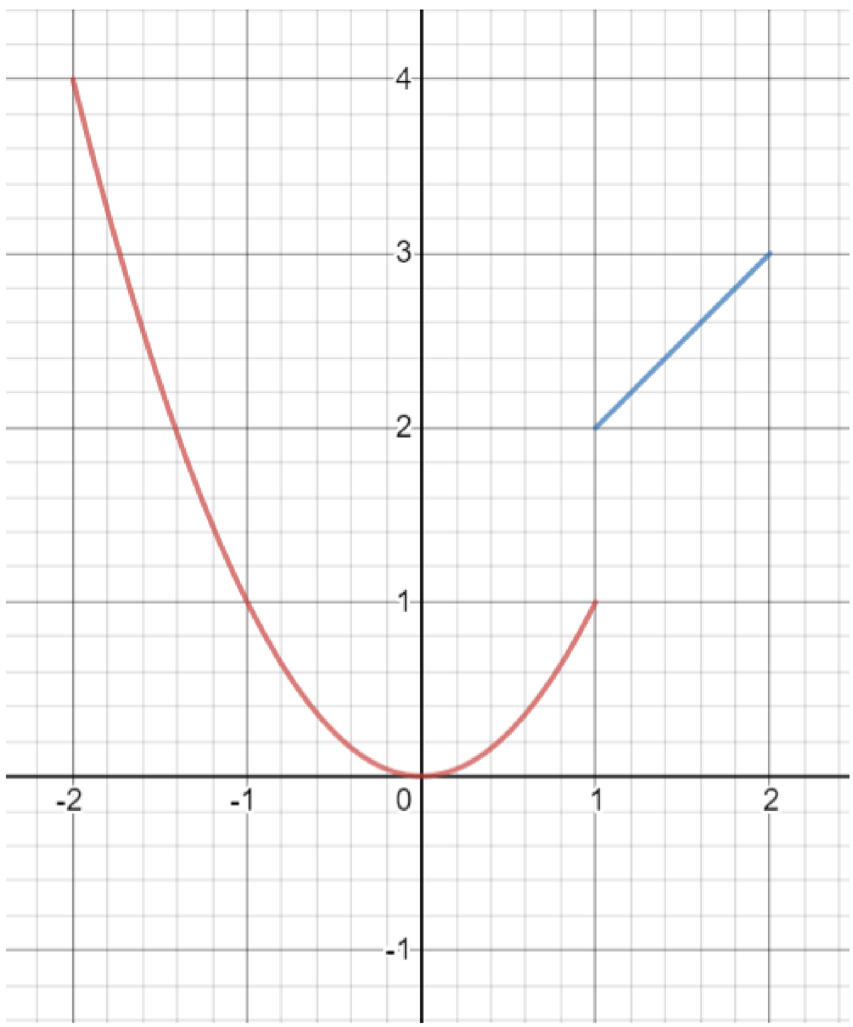
Similarly note that range of $x^2 - 1$; $x \leq 2$ is given as: $[-1, \infty)$ and the range of $\sqrt{x-1}$; $x > 2$ is given as: $[1, \infty)$. Thus, the range of the function $f(x)$ is given as: Range = $[-1, \infty) \cup [1, \infty) = [-1, \infty)$.



4) $f(x) = \begin{cases} x^2, & -2 \leq x < 1 \\ x+1, & 1 \leq x \leq 2 \end{cases}$

Note that the domain of $x^2;\ -2 \leq x < 1$ is given as: $[-2,1)$ and the domain of $x+1;\ 1 \leq x \leq 2$ is given as: $[1,2]$. Thus, the domain of the function $f(x)$ is given by the combined domain of all the pieces. Thus, Domain = $[-2,1) \cup [1,2] = [-2,2]$.

Similarly note that range of $x^2;\ -2 \leq x < 1$ is given as: $[0,4]$ and the range of $x+1;\ 1 \leq x \leq 2$ is given as: $[2,3]$. Thus, the range of the function $f(x)$ is given as: Range = $[0,4] \cup [2,3] = [0,4]$.



Q - 3: [CLO-1: 20 marks]

Determine the formulas and domain for the functions $(f+g)(x)$, $(fg)(x)$, $\left(\frac{f}{g}\right)(x)$, $(fog)(x)$, and $(gof)(x)$, where

1) $f(x) = \frac{1}{\sqrt{4-x^2}}$ and $g(x) = \sqrt{x^2-1}$

Domain of $f(x) = (-2,2)$ and Domain of $g(x) = (-\infty,-1] \cup [1,\infty)$

- $(f+g)(x) = \frac{1}{\sqrt{4-x^2}} + \sqrt{x^2-1}$

  Domain of $(f+g)(x) = (-2,-1] \cup [1,2)$

- $(fg)(x) = \frac{\sqrt{x^2-1}}{\sqrt{4-x^2}} = \sqrt{\frac{x^2-1}{4-x^2}}$

  Domain of $(fg)(x) = (-2,-1] \cup [1,2)$

- $\left(\frac{f}{g}\right)(x) = \frac{1}{(\sqrt{4-x^2})(\sqrt{x^2-1})} = \frac{1}{\sqrt{(4-x^2)(x^2-1)}}$

  Domain of $\left(\frac{f}{g}\right)(x) = (-2,-1) \cup (1,2)$

- $(fog)(x) = f(g(x)) = \frac{1}{\sqrt{4-\left(\sqrt{x^2-1}\right)^2}} = \frac{1}{\sqrt{5-x^2}}$

  Domain of $(fog)(x) = \left(-\sqrt{5},-1\right] \cup [1,\sqrt{5})$

- $(gof)(x) = g(f(x)) = \sqrt{\left(\frac{1}{\sqrt{4-x^2}}\right)^2 - 1} = \sqrt{\frac{1}{4-x^2} - 1} = \sqrt{\frac{x^2-3}{4-x^2}}$

  Domain of $(gof)(x) = \left(-2,-\sqrt{3}\right] \cup [\sqrt{3},2)$

Note: For detailed working see last three pages

2) $f(x) = x^2 + 3$ and $g(x) = \sqrt{x^2-3}$

Domain of $f(x) = (-\infty,\infty)$ and
Domain of $g(x) = \left(-\infty,-\sqrt{3}\right] \cup [\sqrt{3},\infty)$

- $(f+g)(x) = \frac{1}{\sqrt{4-x^2}} + \sqrt{x^2-1}$

  Domain of $(f+g)(x) = \left(-\infty, -\sqrt{3}\right] \cup [\sqrt{3}, \infty)$

- $(fg)(x) = \frac{1}{\sqrt{4-x^2}} + \sqrt{x^2-1}$

  Domain of $(fg)(x) = \left(-\infty, -\sqrt{3}\right] \cup [\sqrt{3}, \infty)$

- $\left(\frac{f}{g}\right)(x) = \frac{1}{\sqrt{4-x^2}} + \sqrt{x^2-1}$

  Domain of $\left(\frac{f}{g}\right)(x) = (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

- $(fog)(x) = \frac{1}{\sqrt{4-x^2}} + \sqrt{x^2-1}$

  Domain of $(fog)(x) = \left(-\infty, -\sqrt{3}\right] \cup [\sqrt{3}, \infty)$

- $(gof)(x) = \frac{1}{\sqrt{4-x^2}} + \sqrt{x^2-1}$

  Domain of $(gof)(x) = (-\infty, \infty)$

Note: For detailed working see last three pages

Q3 (1) $f(x) = \frac{1}{\sqrt{4-x^2}}$ and $g(x) = \sqrt{x^2-1}$

Sol:- Domain of $f(x) = (-2, 2)$
Domain of $g(x) = (-\infty, -1] \cup [1, \infty)$

$\rightarrow (f+g)(x) = \frac{1}{\sqrt{4-x^2}} + \sqrt{x^2-1}$

Domain is given by intersection of domains of the function $f(x)$ and $g(x)$. Thus

Domain of $(f+g)(x) = (-2, -1] \cup [1, 2)$

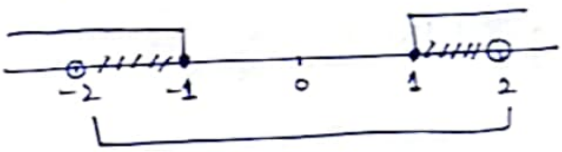
$\rightarrow (f \cdot g)(x) = \frac{\sqrt{x^2-1}}{\sqrt{4-x^2}} = \sqrt{\frac{x^2-1}{4-x^2}}$

Domain of $(f \cdot g)(x) = (-2, -1] \cup [1, 2)$

because $\frac{x^2-1}{4-x^2} \geq 0 \Rightarrow x^2 - 1 \geq 0 \Rightarrow x^2 \geq 1$ or $|x| \geq 1$
$\Rightarrow (-\infty, -1] \cup [1, \infty)$

and $4 - x^2 \neq 0 \Rightarrow 4 \neq x^2 \Rightarrow x \neq \pm 2$
also $\sqrt{4-x^2}$ is possible only if
$4 - x^2 \geq 0 \Rightarrow x^2 \leq 4$ or $|x| \leq 2$ $\Bigg\} \Rightarrow (-2, 2)$

Combinig all above we get



Domain $= (-2, -1] \cup [1, 2)$

$\rightarrow (f/g)(x) = \frac{1}{\sqrt{(4-x^2)(x^2-1)}}$

We need $(4-x^2)(x^2-1) > 0$
$\Rightarrow 4 - x^2 > 0$ or $(x^2-1) > 0$
$\Rightarrow +x^2 < 4$ or $x^2 > 1$
or $|x| < 2$ or $|x| > 1$

Thus, Domain of $\left(\frac{f}{g}\right)(x) = (-2, -1) \cup (1, 2)$

$\rightarrow (f \circ g)(x) = f(g(x)) = \frac{1}{\sqrt{4 - (\sqrt{x^2-1})^2}} = \frac{1}{\sqrt{4-x^2+1}} = \frac{1}{\sqrt{5-x^2}}$

Here $5 - x^2 > 0$ or $5 > x^2$ or $|x| < \sqrt{5}$

For finding domain of a composite function

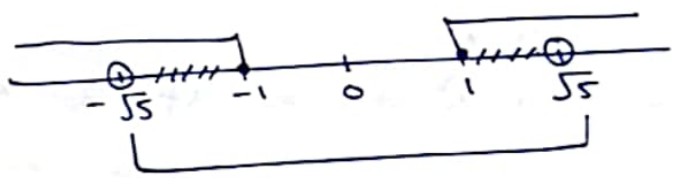
① Find domain of the input (inner) function (A common mistake to skip this step) & outer function.

② Find domain of the new function after performing the composition. This means that we need to find those inputs, x, in the domain of g for which $g(x)$ is in the domain of f. We exclude those inputs, x, from the domain of g for which $g(x)$ is not in the domain of f. The resulting set will be domain of $(f\circ g)(x)$.

Since domain of $g(x)$ is $(-\infty, -1] \cup [1, \infty)$

Thus

Domain of $(f\circ g)(x) = (-\sqrt{5}, -1] \cup [1, \sqrt{5})$

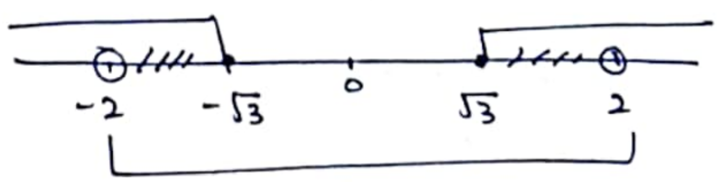


$$\to (g\circ f)(x) = g(f(x)) = \sqrt{\left(\frac{1}{\sqrt{4-x^2}}\right)^2 - 1} = \sqrt{\frac{1}{4-x^2} - 1} = \sqrt{\frac{1-4+x^2}{4-x^2}}$$

$$\Rightarrow (g\circ f)(x) = \sqrt{\frac{x^2-3}{4-x^2}}$$

$4 - x^2 > 0 \Rightarrow x^2 < 4$ or $|x| < 2$

and $x^2 - 3 \geq 0 \Rightarrow x^2 \geq 3$ or $|x| \geq \sqrt{3}$



Domain of $(g\circ f)(x) = (-2, -\sqrt{3}] \cup [\sqrt{3}, 2)$

Q3(2) $f(x) = x^2+3$ and $g(x) = \sqrt{x^2-3}$

Sol: Domain of $x^2+3 = (-\infty, \infty)$

Domain of $\sqrt{x^2-3} = (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$

$[\because x^2-3 \geq 0$ or $x^2 \geq 3$ or $|x| \geq \sqrt{3}]$

$\Rightarrow (f+g)(x) = (x^2+3) + \sqrt{x^2-3}$

Domain of $(f+g)(x) = (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$ [Intersection of domain of f & g]

$\Rightarrow (f \cdot g)(x) = (x^2+3)\sqrt{x^2-3}$

Domain $= (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$

$\Rightarrow (f/g)(x) = \dfrac{x^2+3}{\sqrt{x^2-3}}$

Domain $= (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$ $[\because x^2-3>0 \Rightarrow |x| > \sqrt{3}]$

$\Rightarrow (f \circ g)(x) = f(g(x))$

$= (\sqrt{x^2-3})^{\cancel{2}} + 3$

$= x^2 - \cancel{3} + \cancel{3}$

$= x^2$

Domain $= (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$

[$\because$ the inner function is defined only

for $x \in (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$

$\Rightarrow (g \circ f)(x) = \sqrt{(x^2+3)^2 - 3}$

$= \sqrt{x^4 + 6x + 9 - 3}$

$= \sqrt{x^4 + 6x + 6}$

Domain $= (-\infty, \infty)$