Chapter-8 Discrete Fourier Transform (DFT)

Summary

1. The aim of this chapter is to study a variant of DTFT called DFT. It starts with the theory of Discrete Fourier Series.

2. Any analogue signal can be represented by either its mathematical expression or sum of weighted exponentials (sinusoids) i.e., Fourier Series. Similarly, any discrete signal can be represented by either it's expression or it's own version of sum of weighted exponentials i.e., Discrete Fourier Series.

3. The Fourier Series pair for analogue signals is given as

   $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \qquad a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi}{T}kt}dt$$

   similarly, the Discrete Fourier Series Pair for discrete signal is given as

   $$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{X}[k]e^{j\frac{2\pi}{N}kn} \qquad \tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j\frac{2\pi}{N}kn}$$

   These relationships either depend of T or N i.e., the period of signals so both Fourier series assume that the signal to be evaluated is periodic. The series coefficients are also periodic with either T or N.
   So, all signals that are periodic naturally or can be made periodic artificially (for finite duration signals and assuming their length to be one period) can be represented as Fourier series. The signals that are not periodic and are not of finite duration can be cut down into pieces and each piece can be imagined a new finite duration signal that can be made periodic and hence can be represented by Fourier series. So, all signals can be represented by Fourier series.

4. Discrete Fourier Transform (DFT) is just equals to $\tilde{X}[k]$ but just for one period and considered zero beyond that period N. So DFT is not periodic although, there is no harm in drawing it and assuming it to be periodic with period N. It's your choice which literature you follow. You text book takes it to be non-periodic. Now as we mentioned in point-3 that all signals can be implemented using DFS so all signals can have DFT.

5. DFS coefficients or DFT are the samples of DTFT with samples taken after every $\frac{2\pi}{N}$ interval. DTFT is periodic with 2π and DFT with N, so there are N samples in the interval $[0 - 2\pi]$ of DTFT. And what should be N is up to you to choose. It can be changed by padding zeros after each period of signals if the signal is

naturally periodic or simply pad zeros after the signal ends if the signal is of finite length. Large value of N will result in more refined and high-resolution DFS or DFT. Since you are padding just zeros, so you are not introducing any new information to the signal, this is just a trick to make signal's DFT more visible. Please refer to lecture notes for examples to elaborate this idea.

6. DTFT of periodic signals don't exist for all ω as they are not absolute summable. We had to assume some DTFT expression first and find it's IDTFT to prove that the expression is indeed the DTFT of that signal. Now after studying DFT, that general expression of DTFT in terms of DFT can be expressed as

$$X(e^{j\omega}) = \sum_{k=0}^{N-1} X[k]\delta\left(\omega - \frac{2\pi k}{N}\right)$$

so just find out DFT, $X[k]$ and plug it into the above equation to find the DTFT of any periodic sequence. See the proof in the lecture slides. Here DTFT is in the form of impulses as we mentioned in Chapter-2.

7. Next thing we learned in this chapter is periodic convolution or circular convolution. This is the convolution for periodic signals and is different from conventional linear convolution. You already know how to find circular convolution but it is important to mention that for given two signals, there can be multiple results of circular convolutions and all are valid. It depends on what value of N you choose for both signals before doing circular convolution. For example:

$x_1[n] = [1\ 2\ 3\ 4]$, $x_2[n] = [5\ 6]$ and we want to circularly convolve these two signals. One way is to make the length of both signals equal by padding zeros to the shorter signal in the end i.e., $x_2[n] = [5\ 6\ 0\ 0]$, the answer of convolution is $y[n] = [29\ 16\ 27\ 38]$.

But in the same problem if I assume $x_1[n] = [1\ 2\ 3\ 4\ 0]$, $x_2[n] = [5\ 6\ 0\ 0\ 0]$ then the result of circular convolution will be $y[n] = [5\ 16\ 27\ 38\ 24]$. So, let's avoid any confusion and stick to one solution i.e., whenever you are asked to find circular convolution of two signal of unequal lengths, just follow first method. With different N, the convolution result will be different and this is called Modulo-N circular convolution.

8. The frequency domain representation of signals is extremely important tool for signal processing and is used in all fields of engineering where signal processing is used e.g., telecom, microwave/antenna design engineering, machine learning etc. That is, you want to see the behavior of any signal or

system at specific frequency. Normally all signals processing approaches in DSP, machine learning, DIP utilize frequency dependent features of signals for various tasks so studying DTFT, z-transform or DFT (and other frequency dependent transforms) is very important.
The motivation behind studying DFT is to efficiently implement DTFT in computers because DTFT in computationally very expensive but DFT works only for one period $[0 \; to \; N-1]$. Moreover, we have some algorithms like Fast Fourier Transform (FFT) to efficiently implement DFT and it drastically reduces the DFT computations.

9. LTI systems, which are the backbone of signal processing, are represented by linear convolution only i.e., $y[n] = x[n] * h[n]$ or $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$. Now that we have studied circular convolution and DFT for our ease of calculations i.e., $y[n] = x[n] \circledast h[n]$ or $Y[k] = X[k]H[k]$, we cannot ignore LTI systems. There must be some way to represent LTI systems with circular convolution or DFT. The trick to do that is simple. The circular convolution of two signals will be equal to linear convolution if we pad zeros on both signals of length $L \; and \; P$ to make their lengths equal to $L + P - 1$ and then do circular convolution. The result will be equal to linear convolution. Note that, circular convolution of two signals in time is multiplication of their DFTs in frequency domain.

10. Lastly, we studied block convolution i.e., if input signal $x[n]$ to the LTI system $h[n]$ is very long (or coming in real time), we break down that signal into pieces and use circular convolution to get the output $y[n]$ using overlap add and overlap save methods.

Practice Examples

Just follow, understand and redo the examples of the lecture slides. These are enough to give to insight about the significance of this chapter. AND
8.2, 8.3, 8.7, .814, 8.23