Chapter2: Boolean Algebra and Logic Gates

Lecture1- Introduction to Boolean Algebra

Engr. Arshad Nazir, Asst Prof
Dept of Electrical Engineering
SEECS

Chapter Contents

Definition of Boolean Algebra
Basic Theorems and Properties of Boolean Algebra
Boolean Functions
Canonical and Standard Forms
Other Logic Operations
Digital Logic Gates
Integrated Circuits

Objectives

- Introduction to Boolean algebra
- Study basic rules and theorems of Boolean algebra and their applications in digital circuits

Boolean algebra

George Boole, 1815–1864. Born to working-class parents and unable to afford a formal education, Boole taught himself mathematics and joined the faculty of Queen's College in Ireland. He wrote *An Investigation of the Laws of Thought* (1854), which introduced binary variables and the three fundamental logic operations: AND, OR, and NOT (image courtesy of the American Institute of Physics).

- Boolean algebra is the basic mathematics needed for the study of logic design of digital systems.
- In 1854, George Boole an English mathematician in his famous book "*Investigation on the Laws of Thought*" gave the concept of logical algebra today known as Boolean algebra.
- Boolean algebra is a convenient and systematic way of expressing and analyzing the operation of logic circuits.

Boolean algebra

Claude Shannon

• *Master Thesis 1938: A Symbolic Analysis of Relays and Switching Circuits*
• *Realized Boole's algebra could be used to simplify telephone relay networks*

- Claude Shannon was the first to apply Boole's work to the analysis of relays and switching circuits in 1938.
- In 1904, Huntington gave a set of postulates that form the basis of formal definition of Boolean algebra.

Set Notations

- Mathematical systems can be defined with:
  - A set of elements; A set of elements is any collection of objects having a common property.
  - A set of operators; A binary operator defined on a set S of elements is a rule that assigns to each pair of elements from S a unique element from S.
  - A number of unproved axioms or postulates that form the basic assumptions from which it is possible to deduce the rules, theorems and properties of the system.
- The following notations are being used in this class:
  - $x \in S$ indicates that x is an element of the set S.
  - $y \notin S$ indicates that y is not an element of the set S.
  - A = {1, 2, 3, 4} indicates that set A exists with a finite number of elements (1, 2, 3, 4).

Basic Postulates

- The postulates of a mathematical system form the basic assumptions from which it is possible to deduce the rules, theorems, and properties of the systems. Some common postulates to formulate various algebraic structures are:
  - Closure. A set S is closed w.r.t a binary operator if, for every pair of elements of S, the binary operator specifies a rule for obtaining a unique element of S.

    For example, the set of natural numbers N={1, 2, 3, 4,.....} is closed w.r.t binary operator + by the rules of arithmetic addition, but not w.r.t binary operator – by the rules of arithmetic subtraction.
  - Associative Law. A binary operator * on a set S is said to be associative when:

    $$(x * y) * z = x * (y * z)$$
  - Commutative Law. A binary operator * on set S is said to be commutative when:

    $$x * y = y * x$$
  - Identity Element. A set is said to have an identity element with respect to a binary operation * on S if there exists an element $e \in S$ with the property that

    $$e * x = x * e = x \text{ for every } x \in S$$

    For example, the element 0 is an identity element w.r.t the binary operator + on the set of integers $I = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$, since

    $$x + 0 = 0 + x = x \text{ for any } x \in I$$

    The set of natural numbers N, has no identity element, since 0 is excluded.

Basic Postulates

- Inverse. A set S having the identity element e w.r.t a binary operator $*$ is said to have an inverse whenever, for every $x \in S$, there exists an element $y \in S$ such that

$$x * y = e$$

Example: In the set of integers $I = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$, and the binary operator +, with $e = 0$, the inverse of an element a is (-a), a +(-a) = 0.

  - The additive inverse of element a is –a and it defines subtraction, since a + (–a) = 0. Multiplicative inverse of a is 1/a and defines division, since a.1/a = 1

- Distributive Law. If $*$ and $\cdot$ are two binary operators on a set S, $*$ is said to be distributive over $\cdot$ when

$$x * (y \cdot z) = (x * y) \cdot (x * z)$$

Note: + and · are binary operators. Binary operator + defines addition and binary operator · defines multiplication

Two-value Boolean algebra is defined by the ,the set of two elements B={0, 1}, the operators of AND (·) and OR (+) and Huntington Postulates are satisfied

Huntington Postulates

- In 1904 E. V. Huntington formulated a number of postulates that give us formal definition of Boolean algebra. Boolean algebra is an algebraic structure defined by a set of elements, B , together with two binary operators, + and $\cdot$, provided that the following (Huntington) postulates are satisfied:
  1. Closure.
     a) with respect to the binary operation OR (+); $c=x + y$
     b) with respect to the binary operation AND ($\cdot$); $c=x \cdot y$
  2. Identity.
     a) with respect to OR (+) is 0:
        $x + 0 = 0 + x = x$, for $x = 1$ or $x = 0$
     b) with respect to AND ($\cdot$) is 1:
        $x \cdot 1 = 1 \cdot x = x$, for $x = 1$ or $x =0$
  3. Commutative Law.
     a) With respect to OR (+):
        $x + y = y + x$
     b) With respect to AND ($\cdot$):
        $x \cdot y = y \cdot x$

Huntington Postulates Continued…

4. Distributive Law.

   a) with respect to the binary operation OR (+):

      $x + (y \cdot z) = (x + y) \cdot (x + z)$ + is distributive over $\cdot$

   b) with respect to the binary operation AND (·):

      $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ $\cdot$ is distributive over +

5. Complement. For every element x, that belongs to B, there also exists an element x´ (complement of x) such that:

   a) $x + x' = 1$, for $x = 1$ or $x = 0$

   b) $x \cdot x' = 0$, for $x = 1$ or $x = 0$

6. Membership.

   There exists at least two elements, x and y, of the set such that

   $x \neq y$.

   $0 \neq 1$

Notes on Huntington Postulates

Comparing Boolean algebra with arithmetic and ordinary

algebra, we note the following differences:

- The associative law is not listed but it can be derived from the existing postulates for both + and . operations.
- The distributive law of + over . i.e.,

  $$x+(y . z) = (x + y) . (x + z)$$

  is valid for Boolean algebra but not for ordinary algebra.
- Boolean algebra doesn't have inverses (additive or multiplicative) therefore there are no operations related to subtraction or division.
- Postulate 5 defines an operator called complement that is not available in ordinary algebra.
- Boolean algebra deals with only two elements, 0 and 1

Operator Tables

- A two-valued Boolean algebra is defined on a set of two elements B={0,1}, with rules for the two binary operators + and . as shown in the following operator tables:

- AND Operation

x	y	x·y
0	0	0
0	1	0
1	0	0
1	1	1

- OR Operation

x	y	x+y
0	0	0
0	1	1
1	0	1
1	1	1

- NOT Operation

x	x'
0	1
1	0

Proving the Distributive Law

x	y	z	$y + z$	$x \cdot (y + z)$	$x \cdot y$	$x \cdot z$	$(x \cdot y) + (x \cdot z)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

Duality

- The duality principle states that every algebraic expression deducible from the postulates of Boolean algebra remains valid if the operators and identity elements are interchanged.
  - ➢ The Huntington postulates have been listed in pairs and designed as part (a) and part (b).
  - ➢ If the dual of an algebraic equation is required, we interchange the OR and AND operators and replace 1's by 0's and 0's by 1's.
  - ➢ Example:

x	y	x·y
0	0	0
0	1	0
1	0	0
1	1	1

x	y	x+y
0	0	0
0	1	1
1	0	1
1	1	1

Postulates and Theorems of Boolean Algebra

P2	(a) $x+0=x$	(b) $x.1 = x$
P5	(a) $x+x' = 1$	(b) $x.x' = 0$
T1, idempotent	(a) $x + x = x$	(b) $x. x = x$
T2	(a) $x + 1 = 1$	(b) $x .0 = 0$
T3, involution	$(x')' = x$	
P3, commutative	(a) $x+y = y+x$	(b) $x.y = y.x$
T4, Associative	(a) $x+(y+z)=(x+y)+z$	(b) $x.(y.z) =(x.y).z$
P4, distributive	(a) $x(y+z)=xy+xz$	(b) $x+yz=(x+y)(x+z)$
T5, DeMorgan	(a) $(x+y)' =x'.y'$	(b) $(xy)'=x'+y'$
T6, absorption	(a) $x+x.y = x$	(b) $x.(x+y) =x$

Proving Theorem 1(a) Idempotent Law

$x + x = x$

$x + x = (x + x) \cdot 1$	By postulate:	2(b)
$= (x + x) \cdot (x + x')$		5(a)
$= x + x \cdot x'$		4(b)
$= x + 0$		5(b)
$= x$		2(a)

Proving Theorem 1(b) Idempotent Law

$x \cdot x = x$

$x \cdot x = x\,x + 0$	By postulate:	2(a)
$= x\,x + xx'$		5(b)
$= x\,(x + x')$		4(a)
$= x \cdot 1$		5(a)
$= x$		2(b)

Proving Theorem 2(a)

$x + 1 = 1$

$x + 1 = 1 \cdot (x + 1)$	By postulate:	2(b)
$= (x + x')(x + 1)$		5(a)
$= x + x' 1$		4(b)
$= x + x'$		2(b)
$= 1$		5(a)

- Theorem 2(b) can be proved by duality:

  $x \cdot 0 = 0$

Proving Theorem 6(a) Absorption

$x + x \cdot y = x$

LHS	$= x \cdot 1 + x \cdot y$	by postulate:	2(b)
	$= x \cdot (1 + y)$		4(a)
	$= x \cdot (y + 1)$		3(a)
	$= x \cdot 1$	by theorem:	2(a)
	$= x$	by postulate:	2(b)

Proving Theorem 6(b) Absorption

$x \cdot (x + y) = x$

LHS	$= (x + 0) \cdot (x + y)$	by postulate:	2(a)
	$= x + 0 \cdot y$		4(b)
	$= x + y \cdot 0$		3(b)
	$= x + 0$	by theorem:	2(b)
	$= x$	by postulate:	2(a)

Operator Precedence

- The operator precedence for evaluating Boolean expressions is:
  1. Parentheses
  2. NOT
  3. AND
  4. OR
- In other words, expressions inside the parentheses must be evaluated before all other operations. The next operation that holds precedence is the complement, and then follows the AND and, finally, the OR.
- As an example, consider the truth table for one of the DeMorgan's theorems

Truth Tables to Verify DeMorgan's Theorem

A)	X	Y	$X+Y$	$\overline{X+Y}$	B)	X	Y	$\overline{X}$	$\overline{Y}$	$\overline{X} \cdot \overline{Y}$
	0	0	0	1		0	0	1	1	1
	0	1	1	0		0	1	1	0	0
	1	0	1	0		1	0	0	1	0
	1	1	1	0		1	1	0	0	0

The End