Rotational Motion
and
Moment of Inertia

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Radian: One radian is the angle subtended by an arc length equal to the radius of the arc.

$$\theta = \frac{s}{r}$$

$$2\pi \; rad = 360^\circ$$

$$1^\circ = \pi / 180 \, \text{rad}$$

Angular Displacement:

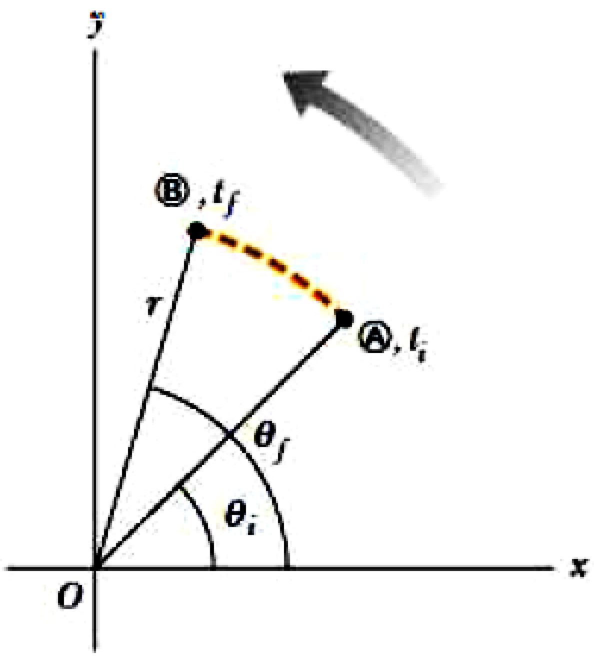
$$\Delta\theta = \theta_f - \theta_i$$

Angular Velocity:

$$\omega = \Delta\theta / \Delta t$$

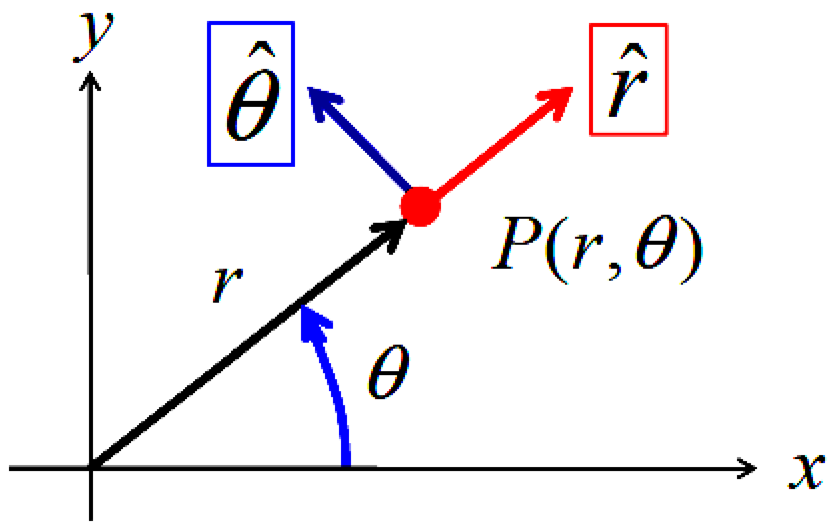
Angular Acceleration:

$$\alpha = \Delta\omega / \Delta t$$



Polar Coordinates

A useful 2-D coordinate system for the study of physical systems, being at rest or in motion, having polar symmetries.



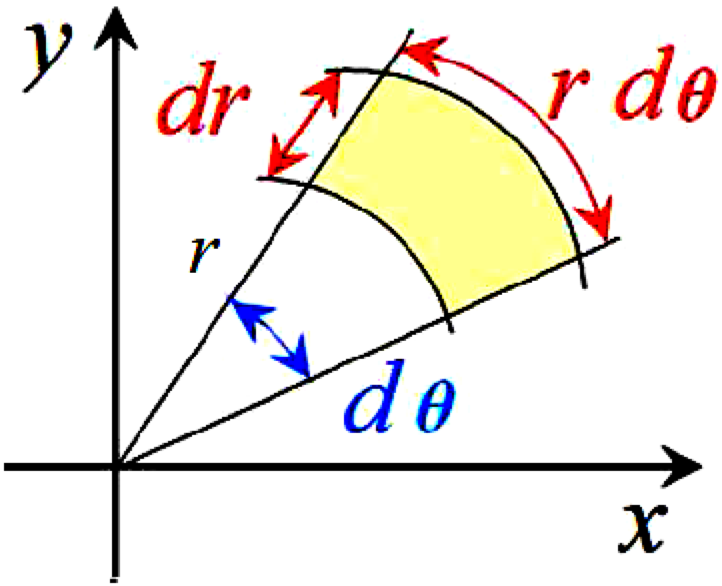
$$0 \le r < \infty$$
$$0 \le \theta \le 2\pi$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$
$$\theta = \tan^{-1}(y/x)$$

$$\boxed{\vec{ds} = dr\hat{r} + rd\theta\hat{\theta}}$$

$$\boxed{da = rdrd\theta}$$



What is common and what is different between unit vectors in Cartesian coordinates $\hat{i}$ & $\hat{j}$ and polar coordinates $\hat{r}$ & $\hat{\theta}$??

****In 2-D Cartesian coordinates, line element and area are:** $\boxed{\vec{ds} = dx\hat{i} + dy\hat{j}}$ $\boxed{da = dxdy}$

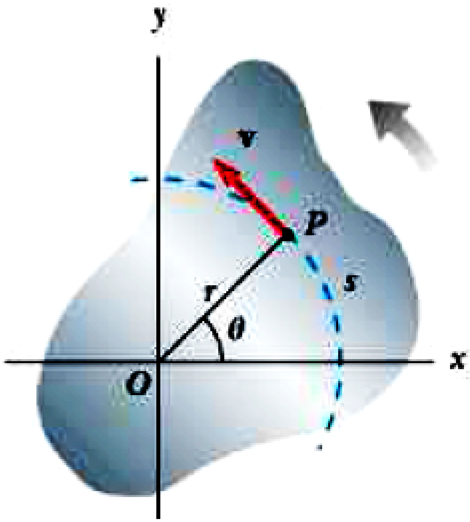
Constant Angular Acceleration Review

$$\omega_f = \omega_i + \alpha t$$

$$\theta = \omega_i t + \frac{1}{2}\alpha t^2$$

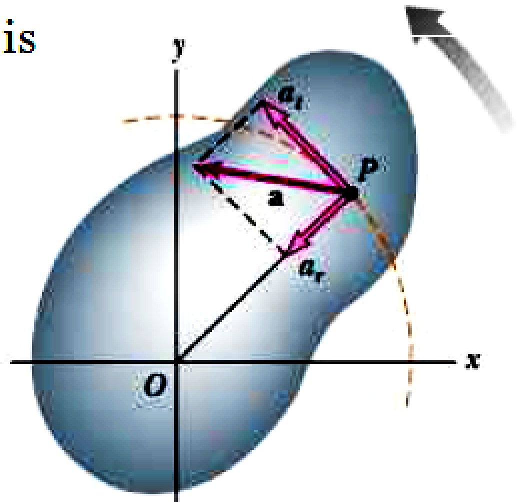
$$2\alpha\theta = \omega_f^2 - \omega_i^2$$

When a rigid object is rotating about a fixed axis, every particle on the object rotates through the same angle in a given time interval and has the same angular speed and the same angular acceleration. That is, the quantities θ, ω and α characterize the rotational motion of the entire rigid object as well as individual particles in the object. Using these quantities, we can greatly simplify the analysis of rigid-object rotation.



$$\boxed{v_t = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega}$$

$$\boxed{a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha}$$



The total linear acceleration vector at the point P is

$$\vec{a} = a_r\hat{r} + a_t\hat{\theta}$$

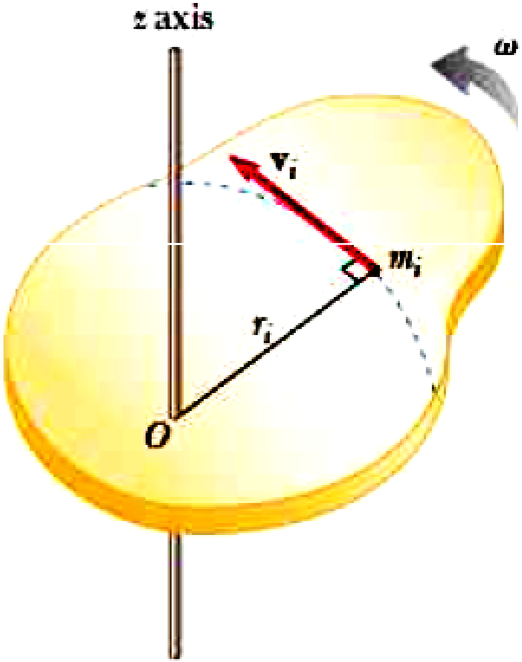
where the magnitude of $\mathbf{a_r}$ is the centripetal acceleration $\mathbf{a_c}$.

$$a_r = a_c = \frac{v^2}{r} = r\omega^2$$

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{r^2\omega^4 + r^2\alpha^2} = r\sqrt{\omega^4 + \alpha^2}$$

Let us consider an object as a collection of particles and assume that it rotates about a fixed z axis with an angular speed ω. Let's consider a particle of mass m_i located at a distance r_i from axis of rotation. Each such particle has kinetic energy determined by its mass and tangential speed.

$$K_i = \frac{1}{2} m_i v_i^2$$



The *total kinetic* energy of the rotating rigid object is the sum of the kinetic energies of the individual particles:

$$K_R = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

$$K_R = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

Where $\sum_i m_i r_i^2 = I$ is called moment of inertia. So

$$K_R = \frac{1}{2} I \omega^2$$

- The moment of inertia is a measure of the resistance of an object to changes in its rotational motion, just as mass is a measure of the tendency of an object to resist changes in its linear motion.

- There is one major difference between mass and moment of inertia. Mass is an inherent property of an object. The moment of inertia of an object depends on your choice of rotation axis. Thus, there is no single value of the moment of inertia for an object. There is a minimum value of the moment of inertia, which is that calculated about an axis passing through the center of mass of the object.

Moment of Inertia

We can evaluate the moment of inertia of an extended rigid object by imagining the object to be divided into many small volume elements, each of which has mass mi→ 0.

$$I = \lim_{m_i \to 0} \sum_i r_i^2 m_i = \int r^2 dm$$

Above expression for moment of inertia can be written in terms of mass density rather than mass:

$$\boxed{\lambda = m / L} \qquad \boxed{\sigma = m / A} \qquad \boxed{\rho = m / V}$$

$$\boxed{dm = \lambda dL} \qquad \boxed{dm = \sigma dA} \qquad \boxed{dm = \rho dV}$$

$$\boxed{I = \int r^2 \lambda dL} \qquad \boxed{I = \int r^2 \sigma dA} \qquad \boxed{I = \int r^2 \rho dV}$$

Dynamics

$$\boxed{\frac{d^2\theta}{dt^2} = \tau / I} \Rightarrow \boxed{\theta(t)}$$

$$\boxed{\omega = \frac{d\theta}{dt}}$$

$$\boxed{p = mv_t = m\frac{ds}{dt} = m\frac{d(r\theta)}{dt} = mr\omega}$$

$$\boxed{P.E = Ig\theta(t)}$$

$$\boxed{\therefore r \perp v_t}$$

$$\boxed{K.E_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2}$$

$$\boxed{L = r \times mv_t = r * mr\omega = mr^2\omega = I\omega}$$

So the most important and basic parameters of rotation are θ and I.

$$\boxed{\frac{dL}{dt} = \tau_{ext}}$$

Statics

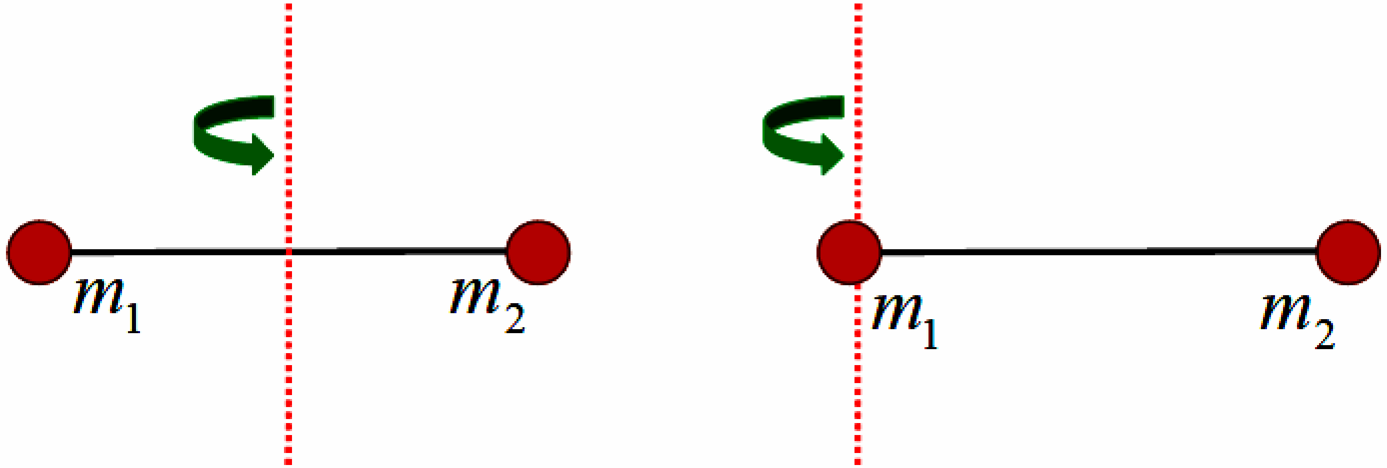
$$\boxed{\sum \vec{\tau} = 0}$$

- Translational kinetic energy of an object is the energy associated with its motion through space.
- An object rotating about a fixed axis remains stationary in space, so there is no translational kinetic energy associated with translational motion.
- The individual particles making up the rotating object, however, are moving through space—they follow circular paths. Consequently, there should be kinetic energy associated with rotational motion, called rotational kinetic energy.

Consider a rigid body consisting of two particles of mass m1=m2=m connected by massless rod of length L as shown in figure below.

(a) What is rotational inertia of body about an axis passing through the COM and perpendicular to rod?

(b) What is rotational inertia of body about an axis passing through the left end and perpendicular to rod?



(a)

$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2$$

$$= m(L/2)^2 + m(L/2)^2$$

$$= \frac{1}{2} mL^2$$

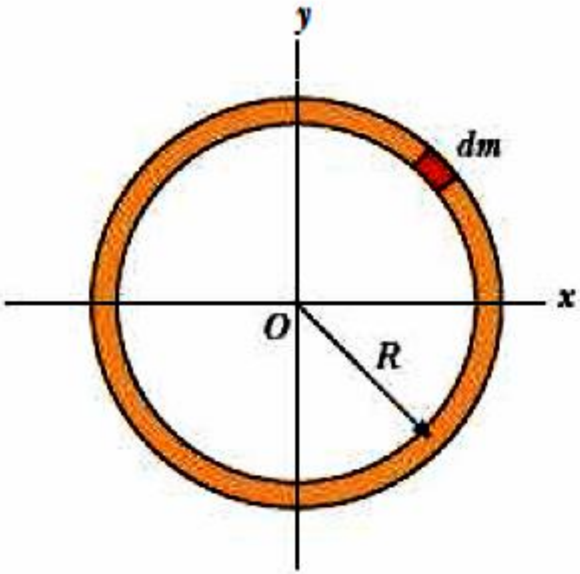
(b)

$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2$$

$$= m(0)^2 + m(L)^2$$

$$= mL^2$$

Find the moment of inertia of a uniform thin hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center



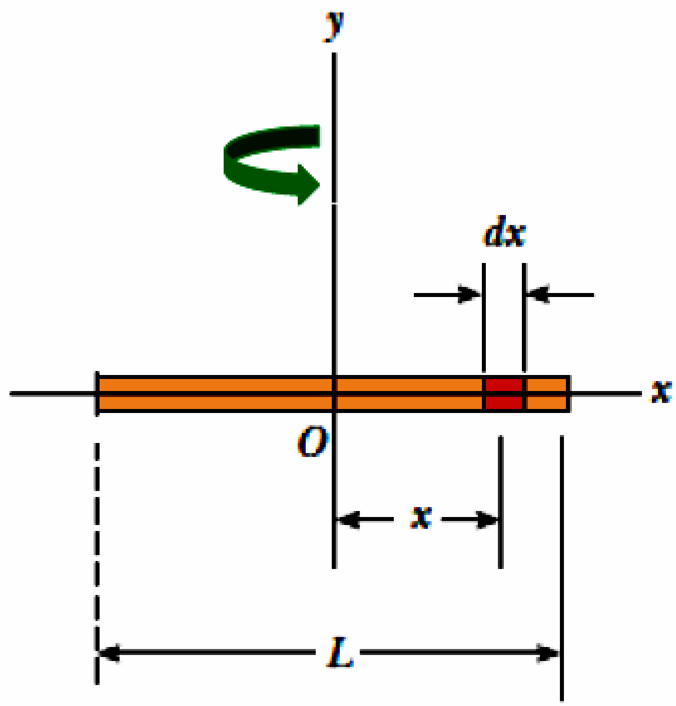
$$I = \int r^2 dm$$

Because the hoop is thin, all mass elements dm are the same distance r = R from the axis, so

$$I = R^2 \int dm = R^2 M$$

Note that this moment of inertia is the same as that of a single particle of mass M located a distance R from the axis of rotation.

Calculate the moment of inertia of a uniform rigid rod of length L and mass m about an axis perpendicular to the rod (the y axis) and passing through its center of mass.



Here $$I = \int r^2 dm = \int r^2 \lambda dL$$

$$\lambda = m / L \qquad dL = dx \qquad r = x$$

$$I = \int_{-L/2}^{L/2} x^2 \frac{m}{L} dx$$

$$= \frac{m}{L}\left(\frac{x^3}{3}\right)_{-L/2}^{L/2}$$

$$= \frac{1}{12} mL^2$$

Attention! About face! Compute an order-of-magnitude estimate for the moment of inertia of your body as you stand tall and turn about a vertical axis through the top of your head and the point halfway between your ankles. In your solution state the quantities you measure or estimate and their values.

Model your body as a cylinder of mass 60.0 kg and circumference 75.0 cm. Then its radius is

$$\frac{0.750 \text{ m}}{2\pi} = 0.120 \text{ m}$$

and its moment of inertia is

$$\frac{1}{2}MR^2 = \frac{1}{2}(60.0 \text{ kg})(0.120 \text{ m})^2 = 0.432 \text{ kg}\cdot\text{m}^2 \sim \boxed{10^0 \text{ kg}\cdot\text{m}^2 = 1 \text{ kg}\cdot\text{m}^2}.$$

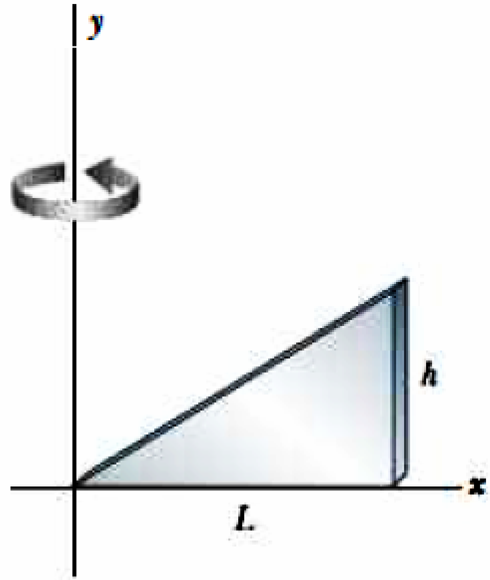
A uniform thin solid door has height 2.20 m, width 0.870 m, and mass 23.0 kg. Find its moment of inertia for rotation on its hinges. Is any piece of data unnecessary?

Every particle in the door could be slid straight down into a high-density rod across its bottom, without changing the particle's distance from the rotation axis of the door. Thus, a rod 0.870 m long with mass 23.0 kg, pivoted about one end, has the same rotational inertia as the door:

$$I = \frac{1}{3}ML^2 = \frac{1}{3}(23.0\text{ kg})(0.870\text{ m})^2 = \boxed{5.80\text{ kg}\cdot\text{m}^2}.$$

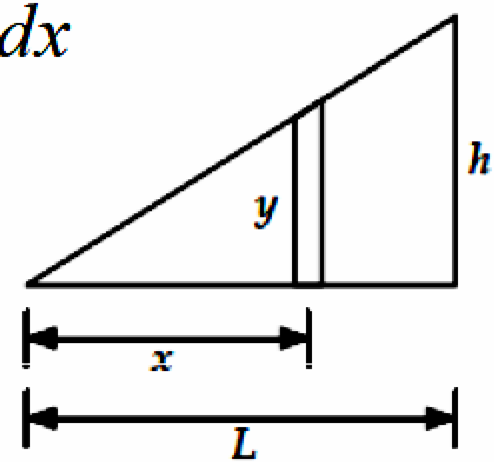
The $\boxed{\text{height of the door is unnecessary}}$ data.

Calculate the moment of inertia of a thin plate, in the shape of a right triangle, about an axis that passes through one end of the hypotenuse and is parallel to the opposite leg of the triangle, as in Figure. Let m represent the mass of the triangle and L the length of the base of the triangle perpendicular to the axis of rotation. Let h represent the height of the triangle and **w the thickness of the plate, much smaller than L or h.**



Let's consider a small strip of height y at a distance x from the axis of rotation

$$I = \int r^2 dm = \int r^2 \rho dV = \int_0^L x^2 \rho y w dx$$



From similar triangles

$$\frac{y}{x} = \frac{h}{L} \Rightarrow y = \frac{xh}{L}$$

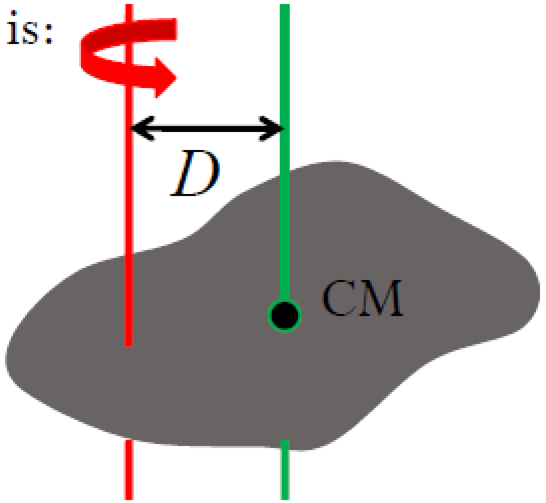
$$\rho = \frac{m}{V} = \frac{m}{hLw/2} = \frac{2m}{hLw}$$

So

$$I = \int_0^L x^2 \left(\frac{2m}{hLw}\right)\left(\frac{xh}{L}\right) w dx = \frac{2m}{L^2}\int_0^L x^3 dx = \frac{mL^2}{2}$$

Parallel Axis Theorem

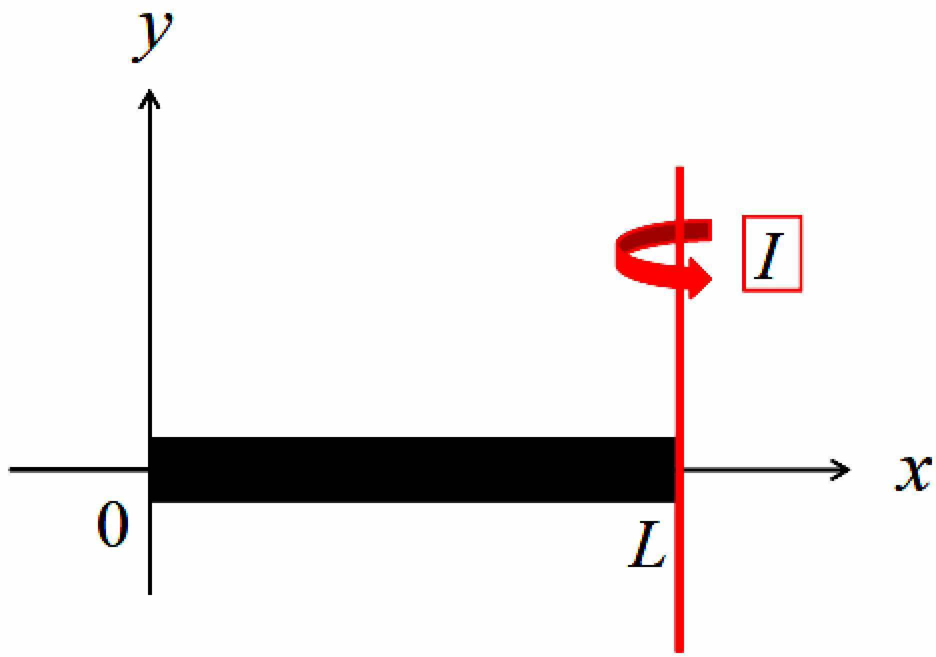
- The moments of inertia of rigid objects with simple geometry (high symmetry) are relatively easy to calculate provided the rotation axis coincides with an axis of symmetry (axis passing through COM).
- The calculation of moments of inertia about an arbitrary axis can be cumbersome, however, even for a highly symmetric object.
- Fortunately, use of parallel-axis theorem often simplifies the calculation. Suppose the moment of inertia about an axis through the center of mass of an object is I_{CM}. Then moment of inertia about any axis parallel to and a distance D away from this axis is:



$$\boxed{I = I_{cm} + MD^2}$$

Where M is mass of object

Calculate the moment of inertia of a uniform rigid rod of length L and mass m about an axis perpendicular to the rod (parallel to y axis) and passing through its one end as shown.



Moment of inertia of rod about an axis perpendicular to the rod and passing through its Cm is

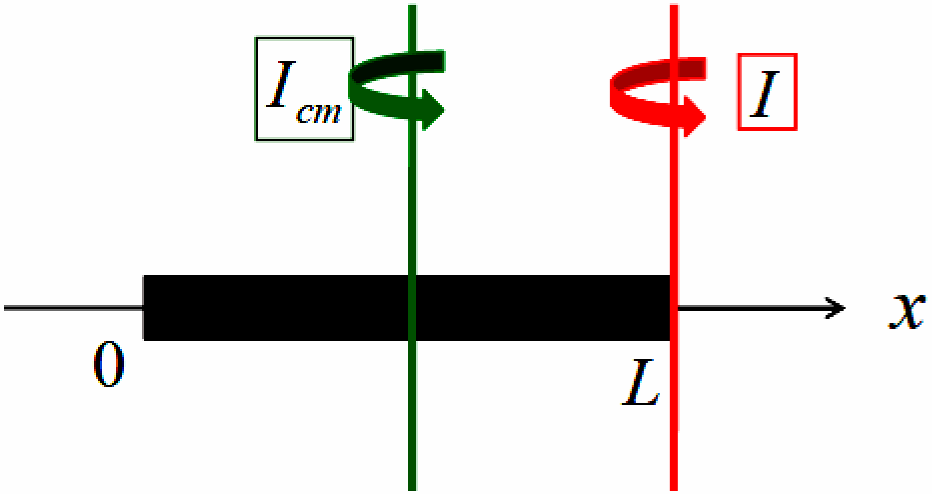
$$I_{cm} = \frac{1}{12} m L^2$$

Moment of inertia of rod about required axis would be

$$I = I_{cm} + m(L/2)^2$$

$$I = \frac{1}{12} m L^2 + \frac{1}{4} m L^2$$

$$I = \frac{1}{3} m L^2$$



The density of the Earth, at any distance r from its center, is approximatelv

$$\rho = [14.2 - 11.6(r/R)] \times 10^3 \text{ kg/m}^3$$

where R is the radius of the Earth. Show that this density leads to a moment of inertia $I = 0.330MR^2$ about an axis through the center, where M is the mass of the Earth.

For a spherical shell $dI = \frac{2}{3} dmr^2 = \frac{2}{3}\left[\left(4\pi r^2 dr\right)\rho\right]r^2$

$$I = \int dI = \int \frac{2}{3}\left(4\pi r^2\right) r^2 \rho(r)\,dr$$

$$I = \int_0^R \frac{2}{3}\left(4\pi r^4\right)\left(14.2 - 11.6\frac{r}{R}\right)\left(10^3\ \text{kg/m}^3\right)dr$$

$$= \left(\frac{2}{3}\right)4\pi\left(14.2\times 10^3\right)\frac{R^5}{5} - \left(\frac{2}{3}\right)4\pi\left(11.6\times 10^3\right)\frac{R^5}{6}$$

$$I = \frac{8\pi}{3}\left(10^3\right)R^5\left(\frac{14.2}{5} - \frac{11.6}{6}\right)$$

$$M = \int dm = \int_0^R 4\pi r^2\left(14.2 - 11.6\frac{r}{R}\right)10^3\,dr$$

$$= 4\pi\times 10^3\left(\frac{14.2}{3} - \frac{11.6}{4}\right)R^3$$

$$\frac{I}{MR^2} = \frac{(8\pi/3)\left(10^3\right)R^5(14.2/5 - 11.6/6)}{4\pi\times 10^3 R^3 R^2 (14.2/3 - 11.6/4)} = \frac{2}{3}\left(\frac{.907}{1.83}\right) = 0.330$$

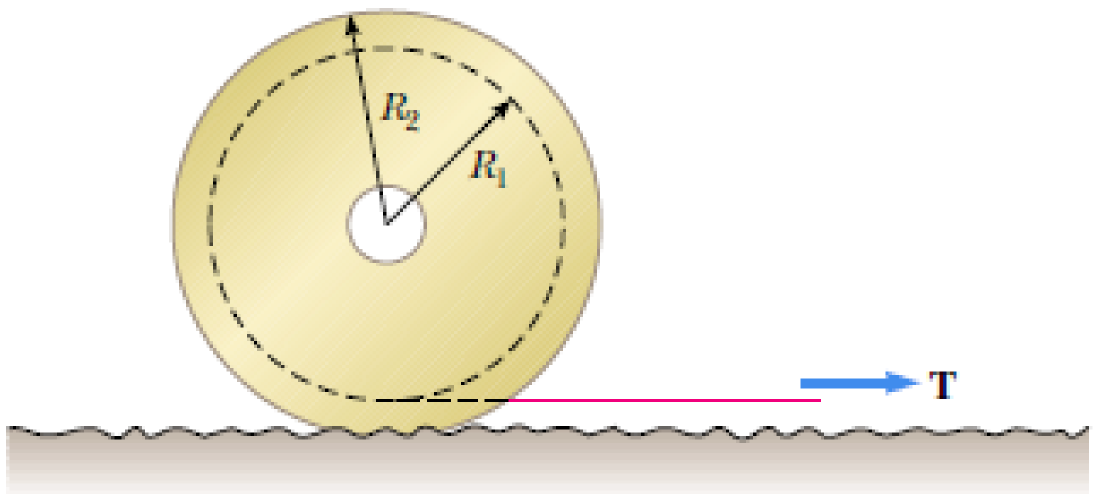
$$\therefore I = \boxed{0.330MR^2}$$

NUST

A spool of thread consists of a cylinder of radius R_1 with end caps of radius R_2 as in the end view shown in Figure. The mass of the spool, including the thread, is m and its moment of inertia about an axis through its center is I. The spool is placed on a rough horizontal surface so that it rolls without slipping when a force **T** acting to the right is applied to the free end of the thread. Show that the magnitude of the friction force exerted by the surface on the spool is given by

$$f = \left(\frac{I + mR_1R_2}{I + mR_2^2} \right) T$$

Determine the direction of the force of friction.



$\sum F_x = ma_x$ reads $-f + T = ma$. If we take torques around the center of mass, we can use $\sum \tau = I\alpha$, which reads $+fR_2 - TR_1 = I\alpha$. For rolling without slipping, $\alpha = \frac{a}{R_2}$. By substitution,

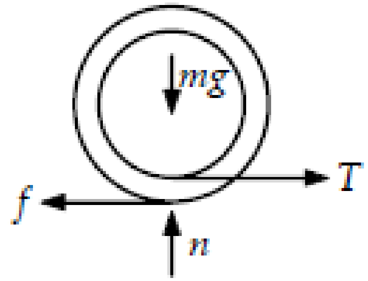


FIG. P10.90

$$fR_2 - TR_1 = \frac{Ia}{R_2} = \frac{I}{R_2 m}(T - f)$$

$$fR_2^2 m - TR_1 R_2 m = IT - If$$

$$f\left(I + mR_2^2\right) = T(I + mR_1 R_2)$$

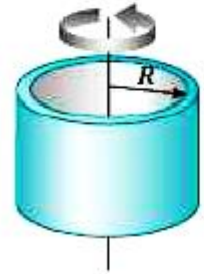
$$f = \left(\frac{I + mR_1 R_2}{I + mR_2^2}\right)T$$

Since the answer is positive, the friction force is confirmed to be $\boxed{\text{to the left.}}$

Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

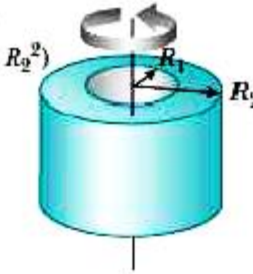
Hoop or thin cylindrical shell

$I_{CM} = MR^2$



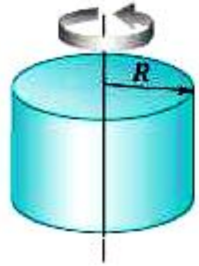
Hollow cylinder

$I_{CM} = \frac{1}{2}M(R_1^{\,2} + R_2^{\,2})$



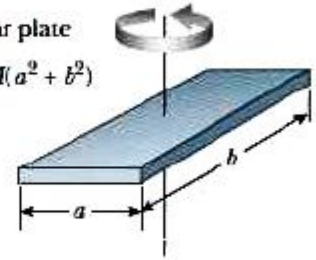
Solid cylinder or disk

$I_{CM} = \frac{1}{2}MR^2$



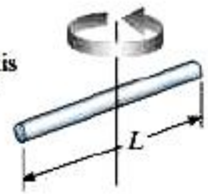
Rectangular plate

$I_{CM} = \frac{1}{12}M(a^2 + b^2)$



Long thin rod with rotation axis through center

$I_{CM} = \frac{1}{12}ML^2$

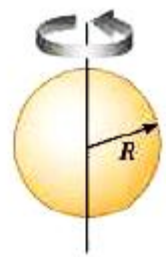


Long thin rod with rotation axis through end

$I = \frac{1}{3}ML^2$

Solid sphere

$I_{CM} = \frac{2}{5}MR^2$



Thin spherical shell

$I_{CM} = \frac{2}{3}MR^2$