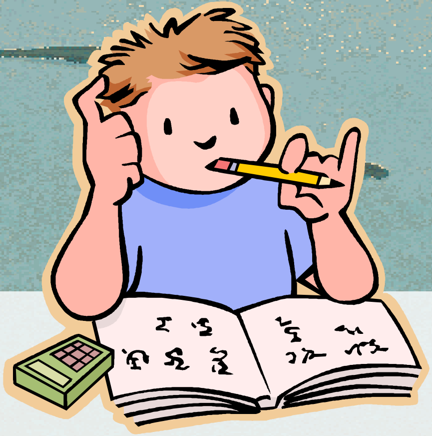
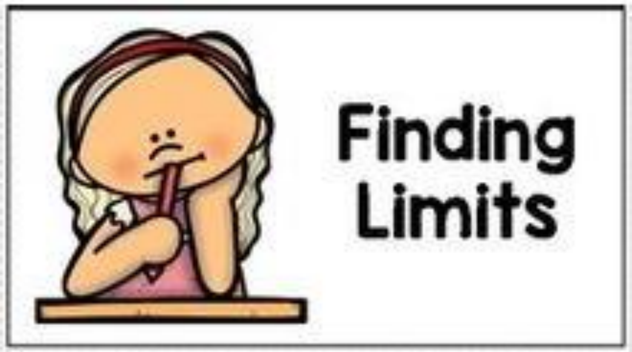


Limits



Calculus & Analytical Geometry
MATH- 101

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(SEECS, NUST)

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 2
    - Sections: 2.2, 2.4

- **An Introduction To Limits**
- **One-Sided Limits**
- **Laws for Calculating Limits**
- Limits Involving Infinity
  - **Infinity as a Limit**
  - Limit at infinity

One-Sided Limits

Limits of the form

$$\lim_{x \to a} f(x) = L$$

are called **two-sided limits** since the values of x get close to a from both the right and left sides of a.

Limits which consider values of x on only one side of a are called **one-sided limits**.

The **right-hand limit,**

$$\lim_{x \to a^+} f(x) = L$$

is read "the limit of $f(x)$ as x approaches a from the right is L."

As x gets closer and closer to a from the right $(x > a)$, the values of $f(x)$ get closer and closer to L.

The **left-hand limit,**

$$\lim_{x \to a^-} f(x) = L$$

is read "the limit of $f(x)$ as x approaches a from the left is L."

As x gets closer and closer to a from the right ($x < a$), the values of $f(x)$ get closer and closer to L.

- One-sided limits are related to limits in the following way.

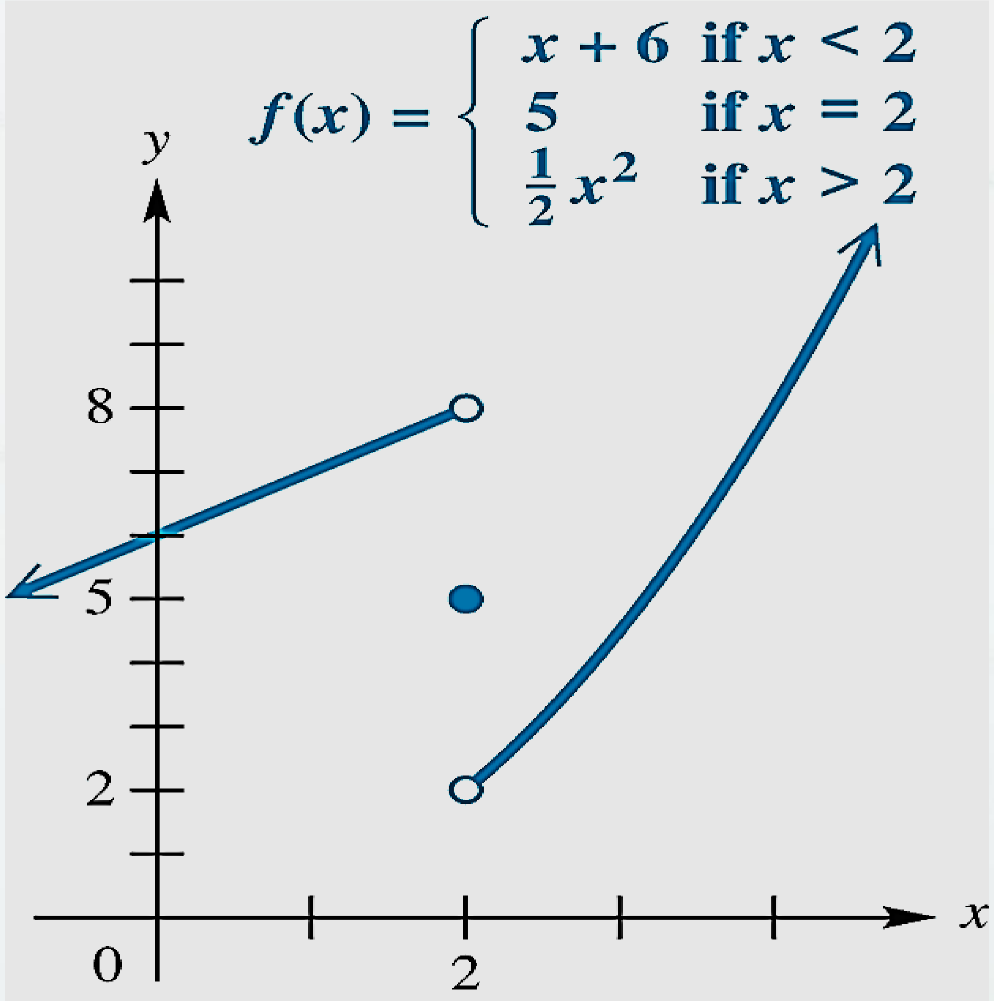
THEOREM

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

Example: Find $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$ where

$$f(x) = \begin{cases} x+6 & \text{if } \; x<2 \\ 5 & \text{if } \; x=2 \\ \dfrac{1}{2}x^2 & \text{if } \; x>2 \end{cases}$$



Solution: In order to evaluate $\lim_{x \to 2^+} f(x)$, we make use of the formula

$$f(x) = \frac{1}{2}x^2.$$

In the limit $\lim_{x \to 2^-} f(x)$, where $x < 2$, use

$$f(x) = x + 6.$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{1}{2}x^2 = \frac{1}{2}\left(2^2\right) = 2.$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x+6) = 2+6 = 8.$$

$$\lim_{x \to 2} f(x) \text{ doesn't exist since } \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x).$$

Example: For the given graph, determine the following:

1. $\lim\limits_{x\to 1^-} f(x),\ \lim\limits_{x\to 1^+} f(x),\ \lim\limits_{x\to 1} f(x),\ f(1).$

2. $\lim\limits_{x\to 6^-} f(x),\ \lim\limits_{x\to 6^+} f(x),\ \lim\limits_{x\to 6} f(x),\ f(6).$

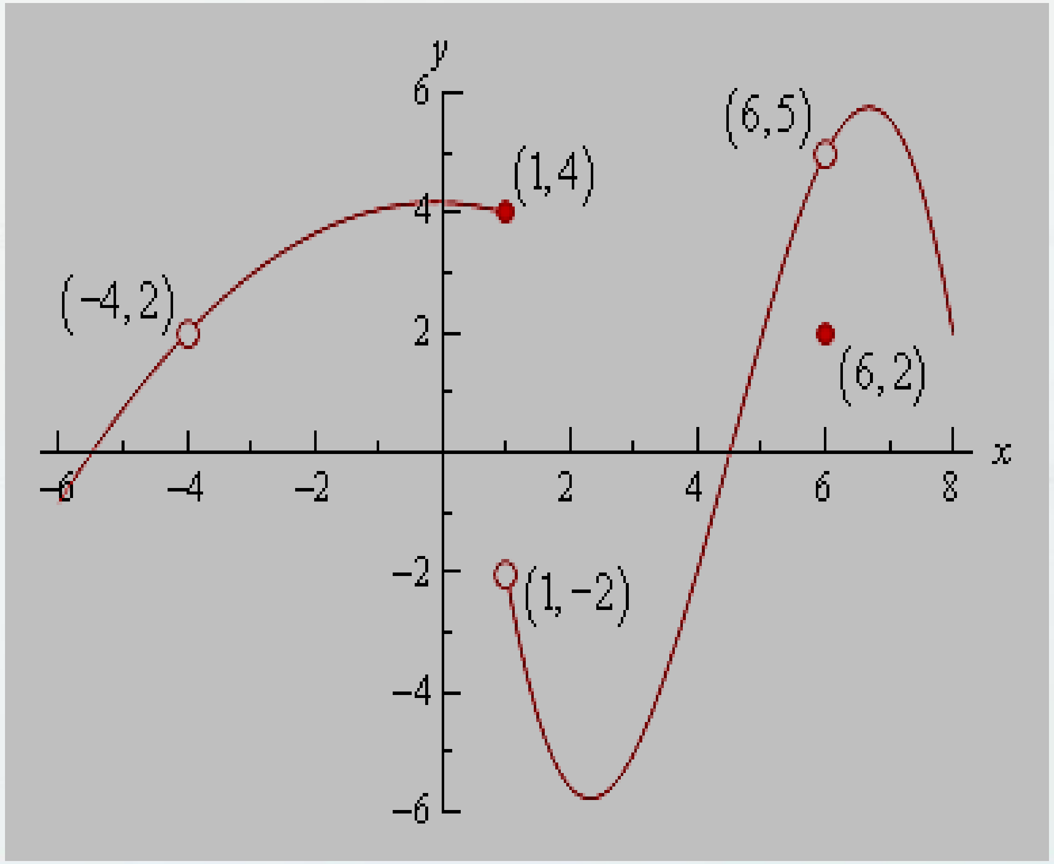
Solution:

1. $\lim\limits_{x\to 1^-} f(x) = 4,$

   $\lim\limits_{x\to 1^+} f(x) = -2,$

   $\lim\limits_{x\to 1} f(x)$ does not exist,

   $f(1) = 4.$

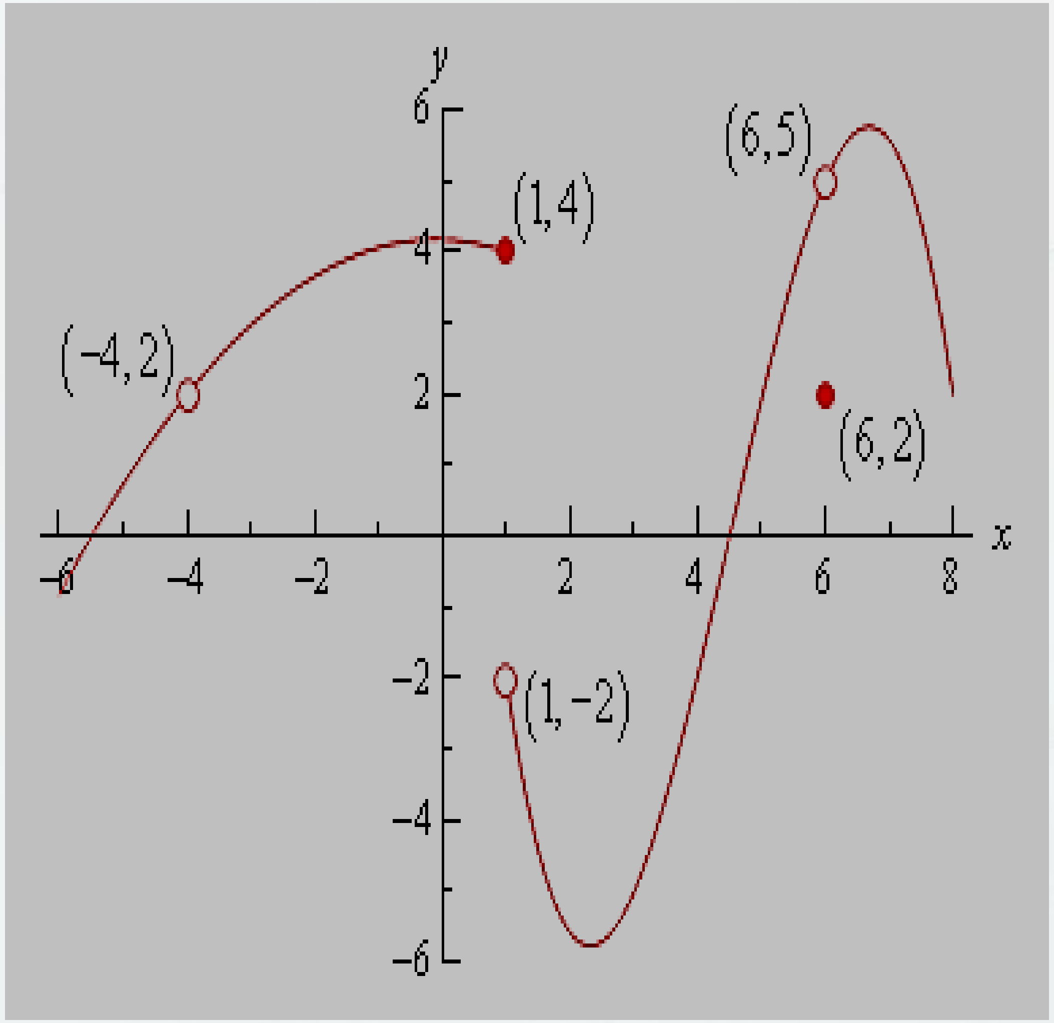


2. $\lim_{x \to 6^-} f(x) = 5,$

$\lim_{x \to 6^+} f(x) = 5,$

$\lim_{x \to 6} f(x) = 5,$

$f(6) = 2.$



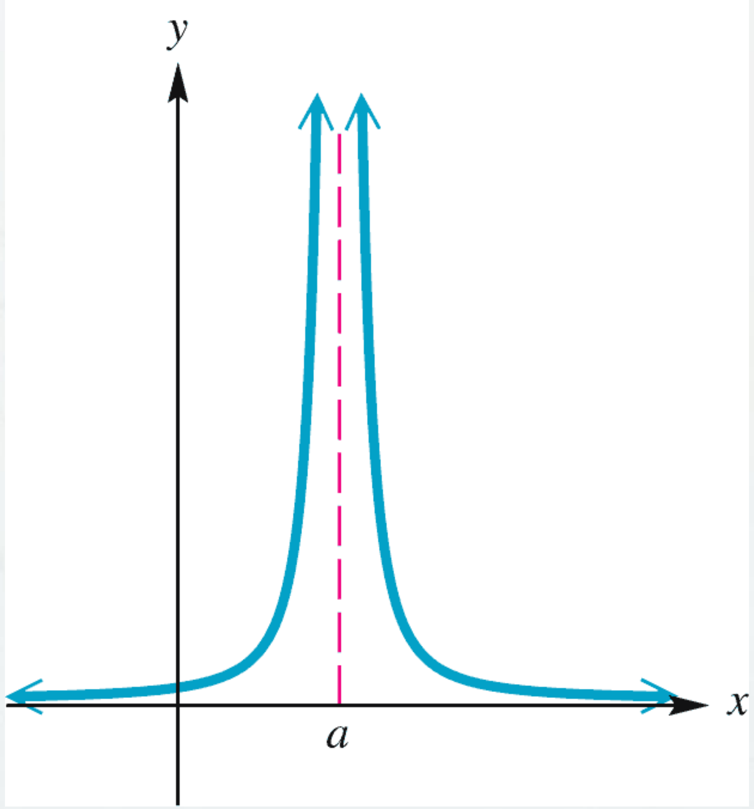
Exercise: Determine

$\lim_{x \to -4^-} f(x),\quad \lim_{x \to -4^+} f(x),$

$\lim_{x \to -4} f(x),\quad f(-4).$

Infinity as a Limit

A function may increase without bound as x gets closer and closer to a from the right.



The right-hand limit does not exist but the behavior is described by writing

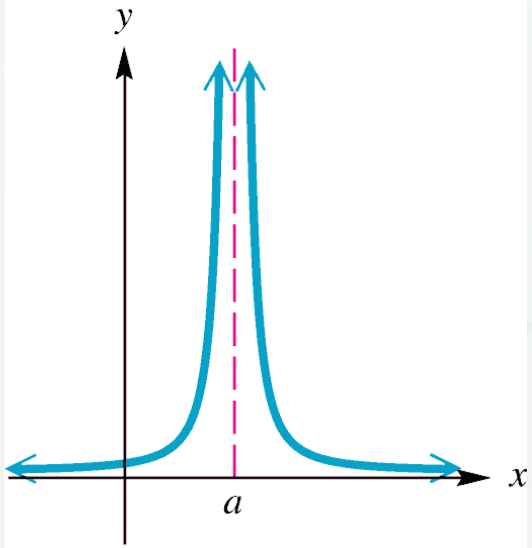
$$\lim_{x \to a^+} f(x) = \infty$$

If the values of $f(x)$ decrease without bound, write

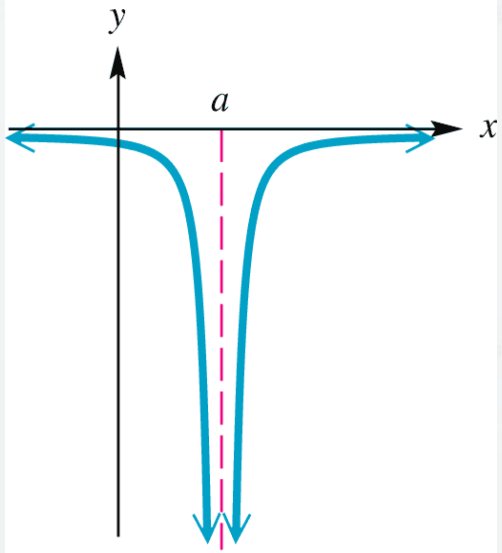
$$\lim_{x \to a^+} f(x) = -\infty$$

The notation is similar for left-handed limits.

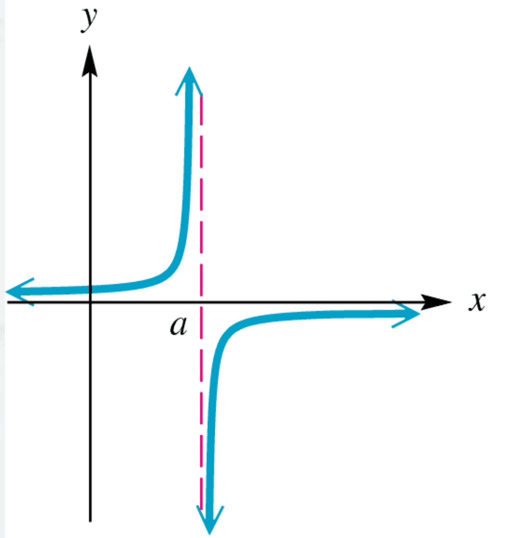
Infinity as a Limit



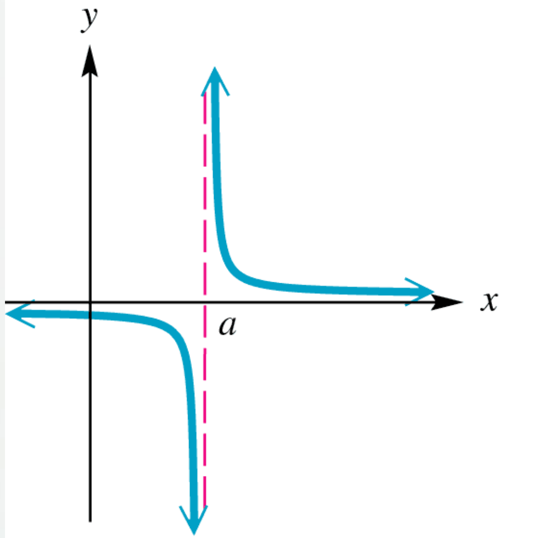
$$\lim_{x \to a^-} f(x) = \infty$$
$$\lim_{x \to a^+} f(x) = \infty$$
$$\lim_{x \to a} f(x) = \infty$$



$$\lim_{x \to a^-} f(x) = -\infty$$
$$\lim_{x \to a^+} f(x) = -\infty$$
$$\lim_{x \to a} f(x) = -\infty$$



$$\lim_{x \to a^-} f(x) = \infty$$
$$\lim_{x \to a^+} f(x) = -\infty$$
$\lim_{x \to a} f(x)$ does not exist.



$$\lim_{x \to a^-} f(x) = -\infty$$
$$\lim_{x \to a^+} f(x) = \infty$$
$\lim_{x \to a} f(x)$ does not exist.

Example: Find $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$ where

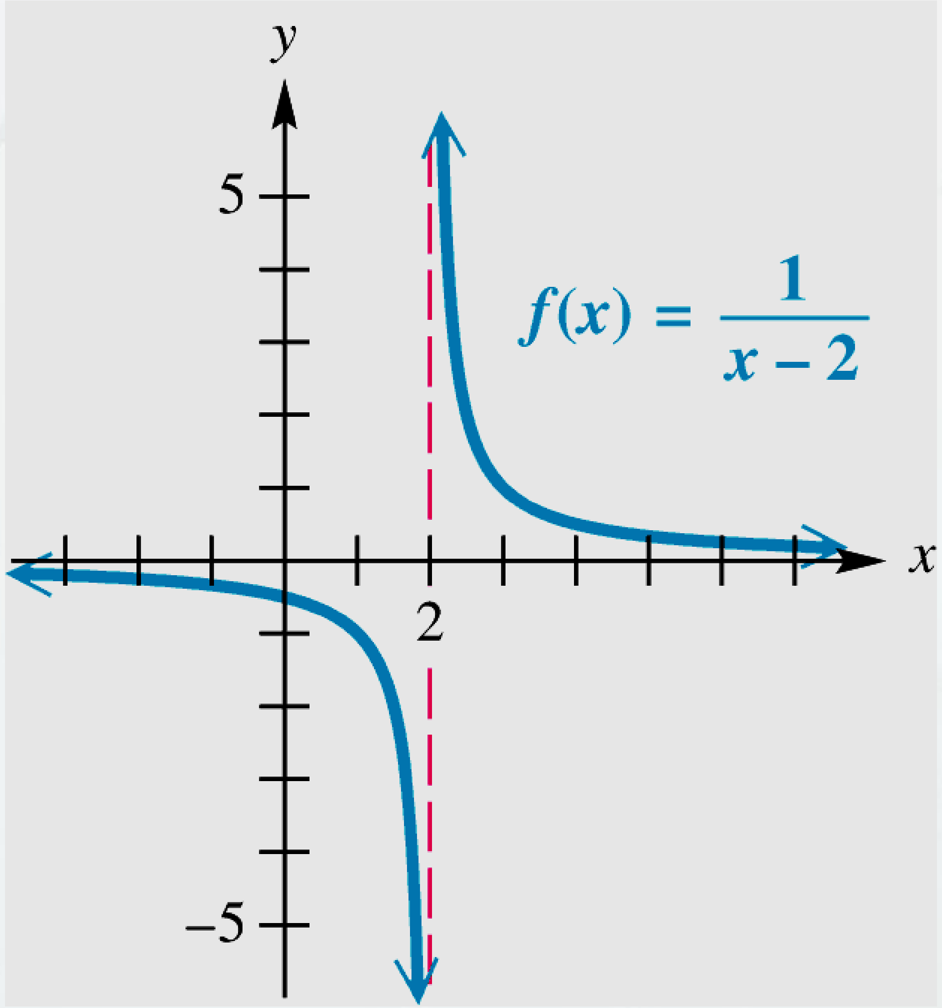
$$f(x) = \frac{1}{x-2}.$$

Solution:

$$\lim_{x \to 2^+} f(x) = \infty$$

and

$$\lim_{x \to 2^-} f(x) = -\infty.$$



Rules for Limits

1. **Constant rule** If k is a constant real number, $\lim_{x \to a} k = k.$

2. **Limit of identity function** $\lim_{x \to a} x = a.$

Rules for Limits (Continued)

For the following rules, we assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist.

3. Sum and difference rules

$$\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x).$$

Rules for Limits (Continued)

4. Product Rule

$$\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x).$$

5. Quotient Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

provided $\lim_{x\to a} g(x) \neq 0.$

Rules for Limits (Continued)

6. Polynomial rule If $p(x)$ defines a polynomial function, then

$$\lim_{x\to a} p(x) = p(a).$$

7. Rational function rule If $f(x)$ defines a rational function $f(x) = \dfrac{p(x)}{q(x)}$

where $p(x)$ and $q(x)$ are polynomials with $q(a) \neq 0$ then

$$\lim_{x\to a} f(x) = f(a).$$

8. Power rule For any real number k,

$$\lim_{x\to a}[f(x)]^k = \left[\lim_{x\to a} f(x)\right]^k$$

provided this limit exists.

Rules for Limits (Continued)

9. Exponent rule For any real number $b > 0$,

$$\lim_{x \to a} b^{f(x)} = b^{\lim\limits_{x \to a} f(x)}.$$

10. Logarithm rule For any real number $b > 0$ with $b \neq 1$,

$$\lim_{x \to a} \left[\log_b f(x)\right] = \log_b \left[\lim_{x \to a} f(x)\right]$$

provided that $\lim\limits_{x \to a} f(x) > 0.$

Rules for Limits (Continued)

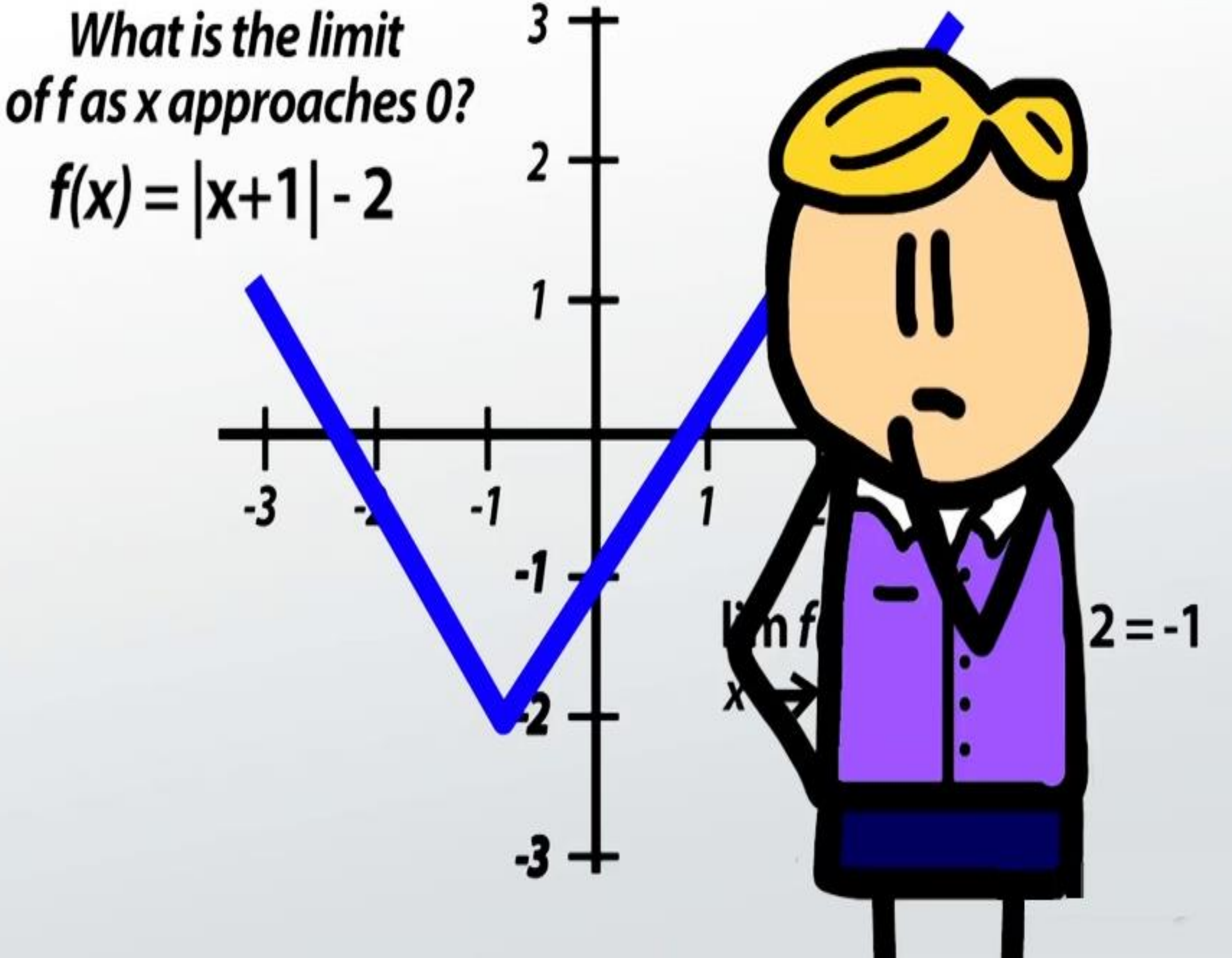
11. The Sandwich Theorem (Squeeze Theorem or Pinching Theorem)

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing a, except possibly at $x = a$ itself. Suppose also that

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L.$$

Then $\lim_{x \to a} f(x) = L$.

Examples



Example: Find $\lim_{x\to 4}(3+2x)$.

Solution: $\lim_{x\to 4}(3+2x)=\lim_{x\to 4}3+\lim_{x\to 4}2x$

$$=3+\lim_{x\to 4}2\cdot\lim_{x\to 4}x$$

$$=3+2\cdot 4$$

$$=11.$$

Example: Find $\lim\limits_{x \to 5} 3x^2$.

Solution:

$$\lim_{x \to 5} 3x^2 = \lim_{x \to 5} 3 \cdot \lim_{x \to 5} x^2$$

$$= 3 \cdot \lim_{x \to 5} x^2$$

$$= 3 \cdot \lim_{x \to 5} x \cdot \lim_{x \to 5} x$$

$$= 3 \cdot 5 \cdot 5$$

$$= 75.$$

Note:

For any polynomial function of the form $f(x) = kx^n$,

$$\lim_{x \to a} f(x) = k \cdot a^n = f(a).$$

Example: Find

$$\lim_{x\to 2}(4x^3-6x+1).$$

Solution: $\lim_{x\to 2}(4x^3-6x+1)=\lim_{x\to 2}4x^3-\lim_{x\to 2}6x+\lim_{x\to 2}1$

$$=4\cdot 2^3-6\cdot 2+1$$

$$=21$$

Example: Find

$$\lim_{x\to 1}\frac{x^2+2x-3}{x^2-3x+2}.$$

Solution: Since

$$\frac{x^2+2x-3}{x^2-3x+2}=\frac{(x+3)(x-1)}{(x-2)(x-1)}=\frac{x+3}{x-2}$$

Therefore,

$$\lim_{x\to 1}\frac{x^2+2x-3}{x^2-3x+2}=\lim_{x\to 1}\frac{x+3}{x-2}=\frac{1+3}{1-2}=-4$$

Example: Suppose that the inequalities

$$\frac{1}{2}-\frac{x^2}{24} \le \frac{1-\cos x}{x^2} \le \frac{1}{2}$$

hold for values of x close to zero. What does this tell you about

$$\boxed{\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)?}$$

Solution:

$$\lim_{x\to 0}\left(\frac{1}{2}-\frac{x^2}{24}\right)=\lim_{x\to 0}\left(\frac{1}{2}\right)-\lim_{x\to 0}\left(\frac{x^2}{24}\right)=\frac{1}{2} \quad \text{and} \quad \lim_{x\to 0}\left(\frac{1}{2}\right)=\frac{1}{2}.$$

By using sandwich theorem

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)=\frac{1}{2}.$$

Example: Evaluate

$$\lim_{x\to 1^-} \frac{\sqrt{2x}\,(x-1)}{|x-1|}$$

Solution:

$$\lim_{x\to 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|}$$

$$= \lim_{x\to 1^-} \frac{\sqrt{2x}(x-1)}{-(x-1)} \qquad \left[\because \quad |x-1| = \begin{cases} x-1, & x \ge 1 \\ -(x-1), & x < 1 \end{cases}\right]$$

$$= \lim_{x\to 1^-} \sqrt{2x}$$

$$= -\sqrt{2}.$$

Exercise: Evaluate the following:

1. $\lim_{x\to 1} \lfloor 2x \rfloor (x-1) = ???$

2. $f(x) = \begin{cases} \cos x, & x \le 0, \\ 1-x, & x > 0. \end{cases}$

   $\lim_{x\to 0} f(x) = ???$

3. $f(x) = \begin{cases} x+2, & x \le -1, \\ ax^2, & x > -1. \end{cases}$

   Determine a provided $\lim_{x\to -1} f(x)$ exists.

Practice Questions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 2
  - Exercise: 2.2

    Q # 1 - 48