Complex Differentiability:- Suppose the complex function $f(z)$ is defined in a neighborhood of a point z_0. The derivative of f at z_0, denoted by $f'(z_0)$ is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

provided this limit exists.

EX:- Find the derivative of $f(z) = z^2 - 5z$.

Sol:-

$$f(z+\Delta z) = (z+\Delta z)^2 - 5(z+\Delta z)$$

$$= z^2 + 2z\Delta z + (\Delta z)^2 - 5z - 5\Delta z.$$

$$f(z+\Delta z) - f(z) = 2z\Delta z + (\Delta z)^2 - 5\Delta z.$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{2z\Delta z + (\Delta z)^2 - 5\Delta z}{\Delta z}.$$

$$= \lim_{\Delta z \to 0} (2z + \Delta z - 5) = 2z - 5.$$

[Complex Variables 12]

Ex:- Let $f(z) = |z|^2$. Discuss differentiability of $f(z)$.

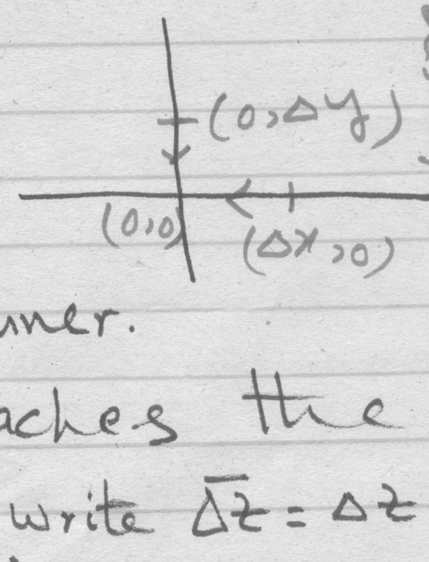
Let $f(z+\Delta z) - f(z) = \Delta w$.

Consider, $\dfrac{\Delta w}{\Delta z} = \dfrac{|z+\Delta z|^2 - |z|^2}{\Delta z}$ $\left(\begin{array}{l} z\bar{z} = |z|^2 \\ \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \end{array} \right)$.

$$= \frac{(z+\Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \bar{z} + \overline{\Delta z} + z\,\frac{\overline{\Delta z}}{\Delta z}.$$

If the limit of $\dfrac{\Delta w}{\Delta z}$ exists, it may be found by letting the point $\Delta z = (\Delta x, \Delta y)$ approach the origin in the Δz plane in any manner.



In particular, when Δz approaches the origin horizontally. We may write $\overline{\Delta z} = \Delta z$.

Hence, if the **limit** of $\dfrac{\Delta w}{\Delta z}$ exists, its value must be $\bar{z} + z$.

[Complex variables 13]

However, when Δz approaches the origin vertically, $\overline{\Delta z} = -\Delta z$, we find that the limit must be $\bar{z} - z$.

Since, limit is always unique, it follows $\bar{z} + z = \bar{z} - z \Rightarrow z = 0$, if $\frac{dw}{dz}$ to be exists.

This shows that a function can be differentiable at a certain point, but nowhere else in any neighboorhood of that point. [Complex Variables] 14

Cauchy - Riemann equations: The derivative of a complex variable is defined by

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad \text{--- (1)}$$

Since $z = x + jy$, $\Delta z = \Delta x + j\Delta y$, (1) becomes

$$f'(z) = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{f(x + \Delta x, y + \Delta y) - f(x, y)}{\Delta x + j\Delta y}$$

If $f'(z)$ is unique, the limit must be independent of how Δz approaches zero. i.e,

$$f'(z) = \lim_{\Delta x \to 0} \left[\lim_{\Delta y \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\Delta x + j\Delta y} \right)$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \frac{\partial f}{\partial x} \quad \text{— (ii)}$$

[Complex variables 15]

Also,

$$f'(z) = \lim_{\Delta y \to 0} \left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\Delta x + j\Delta y} \right)$$

$$= \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{j\Delta y} = -j\frac{\partial f}{\partial y} \quad \text{— (iii)}$$

Because $f'(z)$ has meanings only if limits in equations (ii) & (iii) are same, $f(z)$ must satisfy

$$\frac{\partial f}{\partial x} = -j\frac{\partial f}{\partial y} \quad \text{— (iv)}$$

It is also convenient to express (iv) in terms of the real and imaginary parts of $f(z)$.

$f(z) = u(x,y) + jv(x,y)$; (iv) $\Rightarrow \frac{\partial}{\partial x}(u+jv) = -j\frac{\partial}{\partial y}(u+jv)$

From equality of real & imaginary parts;

$$\left.\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right\} \text{— (v)}$$

Conditions expressed by (V) are the CR Conditions expressed in terms of the real & imaginary parts of $f(z)$.

<u>EX</u>: Let $f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & z \neq 0, \\ 0, & z = 0. \end{cases}$

Show that (a) The real and imaginary parts of f satisfy the C-R equations at $(0,0)$; (b) $f'(0)$ does not exits.

<u>Sol</u>: By Computing the real and imaginary parts of f, we have

$$u(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0); \end{cases}$$

$$v(x,y) = \begin{cases} \dfrac{y^3 - 3x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

[Complex Variables /6]

Next, Compute partial derivatives at $(0,0)$ by definition:

$$u_x(0,0) = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x - 0} = \lim_{x \to 0} \frac{\frac{x^3}{x^2} - 0}{x} = 1,$$

$$u_y(0,0) = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y - 0} = \lim_{y \to 0} \frac{0}{y} = 0.$$

Similarly, $v_x(0,0) = 0$ and $v_y(0,0) = 1$. Thus, the C-R equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at $(0,0)$.

Now, By definition of derivative,

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{(\bar{z})^2}{z^2}.$$

[Complex Variables 17]

Easy to show that this limit DNE by showing that the limits are different in two directions: $z = x \longrightarrow 0$ and $z = x + jx \longrightarrow 0$.

The last example showed that the C-R equations are the necessary conditions for the differentiability of $f(z)$. For $f(z)$ to be differentiable: The partials (derivatives) satisfy the CR equations and are continuous there.

(In the last example, partials are not continuous at $(0,0)$.)

Expression for $f'(z)$: If $f(z)$ is differentiable, using equation (ii), (In deriving CR equations)

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(u(x,y) + jv(x,y)\right)$$

$$= u_x + j\,V_x$$

or equivalently equation (iii),

$$f'(z) = -j\frac{\partial f}{\partial y} = -j\frac{\partial}{\partial y}\left[u(x,y) + jv(x,y)\right]$$

$$= \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y}.$$

<u>EX:</u> Let $f(z) = z^2$, writing $z = x + jy$

$$f = (x+jy)^2 = (x^2 - y^2) + j(2xy).$$

$$u(x,y) = x^2 - y^2, \quad V(x,y) = 2xy$$

$$u_x = 2x, \quad V_y = 2x \quad \text{so,} \quad u_x = V_y$$

$$u_y = -2y; \quad V_x = 2y \quad \& \quad u_y = -V_x.$$

CR equations are satisfied for all x, y.
Moreover, $u, v \in C^2$ (first order partials are continuous)
Hence, $f(z)$ is differentiable. It is legitimate to write $f'(z) = u_x + jV_x = 2x + j(2y) = 2(x+jy)$

$$= 2z.$$

[Complex Variables 18]

EX:- Let $f(z) = x^3 + j(1-y)^3$. Find the Point (set of points): where CR equations are satisfied. $f(z)$ is differentiable, and hence write expression for $f'(z)$ for those points.

Sol:- Here $u(x,y) = x^3$ and $v(x,y) = (1-y)^3$.

$$u_x = 3x^2, \quad u_y = 0 = v_x, \quad v_y = -3(1-y)^2.$$

Now we first try to find points at which satisfy C-R equations: From $u_x = v_y$ and $u_y = -v_x$, we have $x^2 + (1-y)^2 = 0$ and so $x = 0$ and $y = 1$. Thus C-R equations are satisfied only when $z = 0 + 1 \cdot j = j$. We also note that the first order partial derivatives u_x, u_y, v_x and v_y are Continuous in the neighborhood of j. So $f(z)$ is differentiable only at $z = j$. Hence we have

$$f'(z) = u_x + j v_x = 3x^2 \quad \text{only when } z = j.$$

[Complex variables 19]

C-R equations in polar Coordinates:- There are many ways to derive C-R equations in polar coordinates, like using chain rule with C-R equations in Cartesian coordinates. One can derive by definition as:

[Complex variables 20]

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}, \quad z = r e^{i\theta}.$$

$$= \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} \frac{f(r+\Delta r, \theta+\Delta\theta) - f(r,\theta)}{(r+\Delta r)e^{i(\theta+\Delta\theta)} - r e^{i\theta}}.$$

Letting $\Delta\theta = 0$, we have

$$\lim_{\Delta r \to 0} \frac{f(r+\Delta r, \theta) - f(r,\theta)}{(r+\Delta r)e^{i\theta} - r e^{i\theta}} = e^{-i\theta} \lim_{\Delta r \to 0} \frac{f(r+\Delta r,\theta) - f(r,\theta)}{\Delta r}.$$

$$= e^{-i\theta} \frac{\partial f}{\partial r} \longrightarrow \text{(vi)}$$

Letting $\Delta r = 0$, we have

$$\lim_{\Delta\theta \to 0} \frac{f(r, \theta+\Delta\theta) - f(r,\theta)}{r e^{i(\theta+\Delta\theta)} - r e^{i\theta}} = \lim_{\Delta\theta \to 0} \frac{1}{r e^{i\theta}} \frac{f(r,\theta+\Delta\theta) - f(r,\theta)}{(e^{i\Delta\theta} - 1)}$$

Noting that $\lim_{\Delta\theta \to 0} \frac{e^{i\Delta\theta} - 1}{\Delta\theta} = i$, the last expression becomes

$$\frac{1}{i(\varrho e^{i\theta})} \lim_{\Delta\theta \to 0} \frac{f(\varrho, \theta+\Delta\theta) - f(\varrho,\theta)}{\Delta\theta} = \frac{1}{i\varrho} e^{-i\theta} \frac{\partial f}{\partial\theta} \quad \text{—— (vii)}$$

In component form,

$$\text{(vi)} \Rightarrow \quad e^{-i\theta} \frac{\partial f}{\partial \varrho} = e^{-i\theta} \frac{\partial}{\partial \varrho}\left(u(\varrho,\theta) + jv(\varrho,\theta)\right)$$

$$= e^{-i\theta}\left(\frac{\partial u}{\partial \varrho} + i \frac{\partial v}{\partial \varrho}\right) \quad \text{—— (viii)}$$

$$\text{(vii)} \Rightarrow \quad -i \frac{e^{-i\theta}}{\varrho}\left[\frac{\partial}{\partial\theta}\left(u(\varrho,\theta) + jv(\varrho,\theta)\right)\right]$$

$$= \frac{e^{-i\theta}}{\varrho}\left(\frac{\partial v}{\partial\theta} - i \frac{\partial u}{\partial\theta}\right). \quad \text{—— (ix)}$$

[Complex Variables 21]

The real part of $\frac{df}{dz}$ can be written as either $e^{-i\theta} \frac{\partial u}{\partial \varrho}$ or $\frac{e^{-i\theta}}{\varrho} \frac{\partial v}{\partial\theta}$ (due to (viii) & (ix)).

Multiplying both of these expressions by $e^{i\theta}$ yields

$$\frac{\partial u}{\partial \varrho} = \frac{1}{\varrho} \frac{\partial v}{\partial\theta} \quad \text{—— (x)}$$

Similarly, the imaginary part of $\frac{df}{dz}$ can be written as $e^{-i\theta}\frac{\partial v}{\partial r}$ "or" $-\frac{1}{r}e^{-i\theta}\frac{\partial u}{\partial \theta}$.

Multiplying $-e^{i\theta}$ yields

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad \text{——— (XI)}$$

$$u_r = \frac{1}{r}V_\theta \;,\; \frac{1}{r}u_\theta = -V_r$$

C-R equations in polar coordinates.

Result: If $f(re^{i\theta}) = u(r,\theta) + iV(r,\theta)$ is a complex function written in polar coordinates r,θ then Cauchy Riemann equations are written $u_\theta = -rV_r$ & $ru_r = V_\theta$. If the C-R equations hold and all the polar components and partials are continuous on an open disk about z_0 then $f'(z_0)$ exists and

$$f'(z) = e^{-i\theta}(u_r + jV_r) \quad \left(\text{due to (VI)}\right).$$

{Complex Variables 22}

EX: Use Cauchy-Riemann equations in polar form to determine where in the complex plane the following function is differentiable.

(a). $$f(z) = z^{-2}, \quad z \neq 0.$$

$$f(z) = z^{-2} = \frac{1}{r^2} e^{-2j\theta} = \frac{1}{r^2}(\cos 2\theta - j \sin 2\theta).$$

$$u(r,\theta) = \frac{1}{r^2}\cos 2\theta, \quad v(r,\theta) = -\frac{1}{r^2}\sin 2\theta.$$

$$u_r = -2r^{-3}\cos 2\theta, \quad v_r = -2r^{-3}\sin 2\theta$$

$$u_\theta = -2r^{-2}\sin 2\theta, \quad v_\theta = -2r^{-2}\cos 2\theta.$$

$$u_\theta = -r v_r \quad \text{and} \quad v_\theta = r u_r.$$

All partials are continuous for all $z \neq 0$.

$$f'(z) = e^{-j\theta}(u_r + j v_r) = -2r^{-3}e^{-3j\theta} = -2z^{-3}.$$

(b). $$f(z) = r^2 \sin 2\theta - j r^2 \cos 2\theta.$$

$$u(r,\theta) = r^2 \sin 2\theta, \quad v(r,\theta) = -r^2 \cos 2\theta.$$

$$u_r = 2r\sin 2\theta, \quad v_r = -2r\cos 2\theta$$

$$u_\theta = 2r^2\cos 2\theta, \quad v_\theta = 2r^2 \sin 2\theta$$

$$f(z) = -jr^2(\cos 2\theta + j\sin 2\theta)$$

$$= -jz^2.$$

[complex variables 23]

$r u_r = v_\theta$, $-r v_r = u_\theta$, All partials are continuous. Hence, $f(z)$ is differentiable and

$$f'(z) = e^{-j\theta} \left(u_r + j v_r \right)$$

$$= e^{-j\theta} \left[2r \sin 2\theta + j (2 r \cos 2\theta) \right]$$

$$= -2rj \left(\cos 2\theta + j \sin 2\theta \right) e^{-j\theta} = -2rj\, e^{j\theta} = -2jz .$$

[Complex Variable 24]

<u>Analytic Functions</u>:- Even though the requirement of differentiability is a stringent demand, there is a class of functions that is of great importance whose members even satisfy more severe requirements. These functions are called Analytic functions.

A Complex function $w = f(z)$ is said to be analytic at a point z_0 if $f(z)$ is differentiable at z_0 and at every point in some neighborhood of z_0.

<u>EX</u>:- $f(z) = |z|^2$. we have check earlier that $f(z)$ is differentiable at $z = 0$, but it's not analytic at $z=0$ because there exists no ngbrhood of $z=0$ throughout which $f(z)$ is differentiable.

Harmonic Functions:-

A real valued function u of two variables x & y is said to be harmonic in a given domain D, if throught the domain, it has continuous partial derivatives of the first and second order and satisfies the Laplace equation:

$$u_{xx} + u_{yy} = 0 .$$

Harmonic functions play an important role in Science and Engineering, the temparature $T(x,y)$ in thin plates lying in the XY-plane are often harmonic.

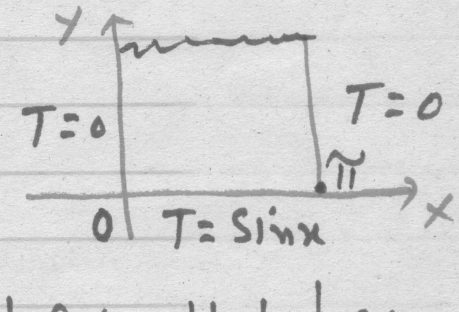
For example it is easy to verify that the function $T(x,y) = e^{-y} \sin x$ is harmonic in any domain of the XY-plane and, in particular in the semi-infinite vertical strip $0 < x < \pi$, $y > 0$.

If $T(x,y)$ is the temparature function,

$$T_{xx} + T_{yy} = 0,$$

$$T(0,y) = 0, \quad T(\pi,y) = 0,$$

$$T(x,0) = \sin x, \quad \lim_{y \to \infty} T(x,y) = 0$$



Temparature in the thin homogenuous plate that has no heat sources or sinks.

[Complex Variables 25]

Result: If a function $f(z)$ is analytic in a domain D, then its component function $u(x,y)$, $v(x,y)$ are harmonic in D.

Since $f(z) = u(x,y) + jv(x,y)$ is analytic, $u(x,y)$ and $v(x,y)$ satisfy C-R equations

$$u_x = v_y \quad , \quad u_y = -v_x$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

[Complex Variables 26]

$(u,v) \in C^2$, means $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y \partial x}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} + \left(-\frac{\partial^2 v}{\partial x \partial y}\right) = 0$$

which shows $u(x,y)$ is harmonic.

Similarly, it can be shown that $v(x,y)$ is harmonic.

Definition:- If two functions $u(x,y)$ and $v(x,y)$ are harmonic in a domain D and their first order partial derivatives satisfy the C-R equations through D, $v(x,y)$ is said to be a harmonic conjugate of u. The meaning of the word conjugate here is, of course, different from $\bar{z}$ (conjugate of the complex number z).

If a harmonic function $u(x,y)$ is given/known, we can calculate its harmonic conjugate $v(x,y)$ and hence write the analytic function $f(z)$ as $f(z) = u(x,y) + j\,v(x,y)$.

Ex:- Let $u(x,y) = y^3 - 3x^2y$. Check if $u(x,y)$ is harmonic? If so, find its harmonic conjugate and hence write the analytic function $f(z) = u + jv$.

$u(x,y) = y^3 - 3x^2y$, $u_x = -6xy$, $u_{xx} = -6y$

$u_y = 3y^2 - 3x^2$, $u_{yy} = 6y$; $u_{xx} + u_{yy} = -6y + 6y = 0$.

Hence, $u(x,y)$ is harmonic.

Using C-R equations

$$\frac{\partial v}{\partial y} = -6xy \quad , \quad \frac{\partial v}{\partial x} = -(3y^2 - 3x^2)$$

$$v(x,y) = -3xy^2 + g(x) \Rightarrow \frac{\partial v}{\partial x} = -3y^2 + g'(x)$$

So, $-3y^2 + g'(x) = 3x^2 - 3y^2$

$g'(x) = 3x^2 \Rightarrow g(x) = x^3 + c$.

Hence, the function $v(x,y) = x^3 + c$ is a harmonic conjugate of $u(x,y)$.

The corresponding analytic function is

$$f(z) = (y^3 - 3x^2y) + j(x^3 - 3xy^2 + c) = j(z^3 + c) \quad \Bigg| \; x = \frac{z + \bar{z}}{2}, \; y = \frac{z - \bar{z}}{2j}$$

[Complex Variables 27]

Laplace Equation in polar Coordinates:

Transform the Laplace equation into polar Coordinates, and show that

$u(r,\theta) = e^{-\theta}\cos(\ln r)$, $r>0$, $0<\theta<2\pi$ is harmonic. Also find the harmonic Conjugate of $u(r,\theta)$ and write the Corresponding analytic function.

we know that the C-R equations in polar form are $r\dfrac{\partial u}{\partial r} = \dfrac{\partial v}{\partial \theta}$, $\dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r}$.

Differentiating these equations partially with respect to r and θ respectively, we have

[Complex Variables 28]

$$\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 v}{\partial \theta \partial r}$$

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r \partial \theta}.$$

$$r\frac{\partial u}{\partial r} + r^2\frac{\partial^2 u}{\partial r^2} = r\frac{\partial^2 v}{\partial \theta \partial r}$$

adding last two equations and using $\dfrac{\partial^2 v}{\partial \theta \partial r} = \dfrac{\partial^2 v}{\partial r \partial \theta}$ we get

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

Laplace equation in polar coordinates.

Now, $u(r,\theta) = e^{-\theta}\cos(\ln r)$,

$$\frac{\partial u}{\partial r} = -\frac{e^{-\theta}\sin(\ln r)}{r} \Rightarrow r\frac{\partial u}{\partial r} = -e^{-\theta}\sin(\ln r)$$

Again differentiating w.r.t 'r',

$$\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2} = -\frac{e^{-\theta}\cos(\ln r)}{r}$$

$$\Rightarrow r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} = -e^{-\theta}\cos(\ln r)$$

and $\frac{\partial u}{\partial \theta} = -e^{-\theta}\cos(\ln r)$, $\frac{\partial^2 u}{\partial \theta^2} = e^{-\theta}\cos(\ln r)$

hence, $r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} = -\frac{\partial^2 u}{\partial \theta^2}$

$$\Rightarrow r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Hence, $u(r,\theta)$ is harmonic.

Construction of $v(r,\theta)$: using C-R equations

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

$$\frac{\partial v}{\partial \theta} = r\,\frac{\partial u}{\partial r} = -e^{-\theta}\sin(\ln r)$$

$$\Rightarrow \quad v(r,\theta) = e^{-\theta}\sin(\ln r) + g(r)$$

Differentiating last equation w.r.t 'r':

$$\frac{\partial v}{\partial r} = \frac{e^{-\theta}\cos(\ln r)}{r} + g'(r)$$

$$\Rightarrow \quad -\frac{\partial u/\partial \theta}{r} = \frac{e^{-\theta}\cos(\ln r)}{r} + g'(r)$$

Since, $$\frac{\partial u}{\partial \theta} = -e^{-\theta}\cos(\ln r)$$

$$\frac{e^{-\theta}\cos(\ln r)}{r} = \frac{e^{-\theta}\cos(\ln r)}{r} + g'(r)$$

$$\Rightarrow \quad g'(r) = 0 \quad \Rightarrow \quad g(r) = c.$$

$$v(r,\theta) = e^{-\theta}\sin(\ln r) + c.$$

Hence, the corresponding analytic function is

$$f(z) = \left[e^{-\theta}\cos(\ln r)\right] + j\left[e^{-\theta}\sin(\ln r) + c\right].$$