Z- TRANSFORM CAUSALITY, STABILITY AND LCCDES

Causality

$h[n]$ right-sided $\Rightarrow$ ROC is the exterior of a circle *possibly* including $z = \infty$:

$$H(z) = \sum_{n=N_1}^{\infty} h[n]z^{-n}$$

If $N_1 < 0$, then the term $h[N_1]z^{-N_1} \rightarrow \infty$ at $z = \infty$
$\Rightarrow$ ROC outside a circle, but does *not* include ∞.

Causal $\Leftrightarrow N_1 \geq 0$

No z^m terms with $m > 0$
$\Rightarrow z = \infty \in$ ROC

$\Downarrow$

A DT LTI system with system function $H(z)$ is causal $\Leftrightarrow$ the ROC of $H(z)$ is the exterior of a circle *including* $z = \infty$

Causality for Rational System Functions

$$H(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

$$\Downarrow \text{ No poles at } \infty, \text{ if } M \le N$$

A DT LTI system with rational system function $H(z)$ is causal

$\Leftrightarrow$ (a) the ROC is the exterior of a circle outside the outermost pole;

and (b) if we write $H(z)$ as a ratio of polynomials

$$H(z) = \frac{N(z)}{D(z)}$$

then

$$\text{degree } N(z) \le \text{ degree } D(z)$$

Causality for Rational System Functions

A discrete-time LTI system with rational system function *H(z)* is causal if and only if:

(a) the ROC is the exterior of a circle outside the outermost pole;

(b) with *H(z)* expressed as a ratio of polynomials in *z*, the order of the numerator cannot be greater than the order of the denominator

Example

- Consider a system with system function of the form:

$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + (1/4)z + (1/8)}$$

- Without even knowing the ROC for this system, we can conclude that the system is NOT causal, because the numerator of $H(z)$ is of higher order than the denominator.

Causality - Example

- Consider a system with system function

$$H(z) = \frac{1}{1-(1/2)z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

- Is the system right-sided or left-sided?
- Is the system Causal?
- Determine the time domain impulse response of the system.

Causality - Example

- Since the ROC for this system function is the exterior of a circle outside the outermost pole $(z = 2)$ we know that the impulse response is right-sided.
- To determine if the system is causal, we need only check the other condition for causality, namely that $H(z)$, when expressed as a ratio of polynomials in z, has numerator degree no larger than the denominator.

For this example we have:

$$H(z) = \frac{2-(5/2)z^{-1}}{\left(1-(1/2)z^{-1}\right)\left(1-2z^{-1}\right)} = \frac{2z^2-(5/2)z}{z^2-(5/2)z+1}$$

- so that the numerator and denominator of $H(z)$ are both of degree two, and the system is causal.

- We can find the impulse response for this system as:

$$h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n\right]u[n]$$

- since $h[n] = 0$ for $n < 0$, we see that the system is causal.

Stability

- LTI System Stable $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty \Leftrightarrow$ ROC of $H(z)$ includes the unit circle $|z| = 1$

$\Rightarrow$ Frequency Response $H(e^{j\omega})$ (DTFT of $h[n]$) exists.

A causal LTI system with rational system function is stable $\Leftrightarrow$ all poles are inside the unit circle, i.e. have magnitudes < 1

An LTI system is stable, if and only if the ROC of its system function *H(z)* includes the unit circle, |z|=1.

Stability - Example

- Consider the system with system function

$$H(z) = \frac{1}{1-(1/2)z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

- Is the system stable?
- What should be the limits of the ROC to make the system stable?
- Determine the time domain impulse response of the stable system.

Stability - Example

- Since the associated ROC is the region $|z|>2$, which does not include the unit circle, the system is not stable (equivalently we see that the impulse response is not absolutely summable).
- Now consider the same system but one whose ROC is the region $1/2<|z|<2$, then the ROC does contain the unit circle, so that the corresponding system is non-causal, but stable.
- In this case the system impulse response is:

$$h[n]=\left(\frac{1}{2}\right)^{n}u[n]-2^{n}u[-n-1]$$

- which is absolutely summable.
- A third possible choice of ROC for the same system function is the region $|z|<1/2$
- The corresponding system is neither causal (since the ROC is not outside the outermost pole) nor stable (since the ROC does not include the unit circle).

- In this case the system impulse response is:

$$h[n]=-\left[\left(\frac{1}{2}\right)^{n}+2^{n}\right]u[-n-1]$$

DT LTI Systems Described by LCCDEs

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Use the time-shift property

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Downarrow$$

$$Y(z) = H(z)X(z)$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$ — Rational

ROC: Depends on Boundary Conditions, left-, right-, or two-sided.

For Causal Systems $\Rightarrow$ ROC is outside the outermost pole

DT LCCDE – Example:1

- Consider an LTI system with input-output difference equation:

$$y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{3}x[n-1]$$

- Determine the System Function and Impulse response of the system

DT LCCDE – Example:1

$$Y(z)-\frac{1}{2}z^{-1}Y(z)=X(z)+\frac{1}{3}z^{-1}X(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

Case I: ROC $|z|>1/2$, we solve for $h[n]$ as:

$$h[n]=\left(\frac{1}{2}\right)^{n}u[n]+\frac{1}{3}\left(\frac{1}{2}\right)^{n-1}u[n-1]$$; right-sided, causal and stable

Case II: ROC $|z|<1/2$, we solve for $h[n]$ as:

$$h[n]=-\left(\frac{1}{2}\right)^{n}u[-n-1]-\frac{1}{3}\left(\frac{1}{2}\right)^{n-1}u[-n]$$; left-sided, anti-causal and unstable

END