Simple Harmonic Motion

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Oscillations

- Each day we encounter many kinds of mechanical oscillations, *i.e., swinging pendulum of clock, vibrating guitar string and vibrating air molecules that transmit sound waves.*
- In addition to these mechanical oscillations, we can have also electromagnetic oscillations, *such as electrons surging back and forth in circuits that are responsible for transmitting and receiving radio or TV signals.*
- These oscillating systems – whether mechanical, electromagnetic or other types - have a common mathematical formulation and are most easily expressed in terms of *sine and cosine functions*.

Restoring Force

- Imagine an oscillating system, such as the pendulum of a clock or the mass attached to the spring. What must be the properties of the force that produces such oscillations?
- If you displace a pendulum in one direction from its equilibrium position, the force (which is due to gravity) pushes it back towards equilibrium. If you displace it in other direction, the force still acts towards equilibrium position.
- No matter what the direction of the displacement, the force always acts in a direction to restore the system to its equilibrium position. Such a force is called a restoring force.
- Restoring force is responsible for periodic motion, harmonic motion, oscillation, or vibration.

- Amplitude

Amplitude is the magnitude of the maximum displacement form equilibrium position.

- Period T

For any object in simple harmonic motion, the time required to complete one cycle is the *period T*.

$$T = \frac{2\pi}{\omega}$$

Where ω is called angular frequency.

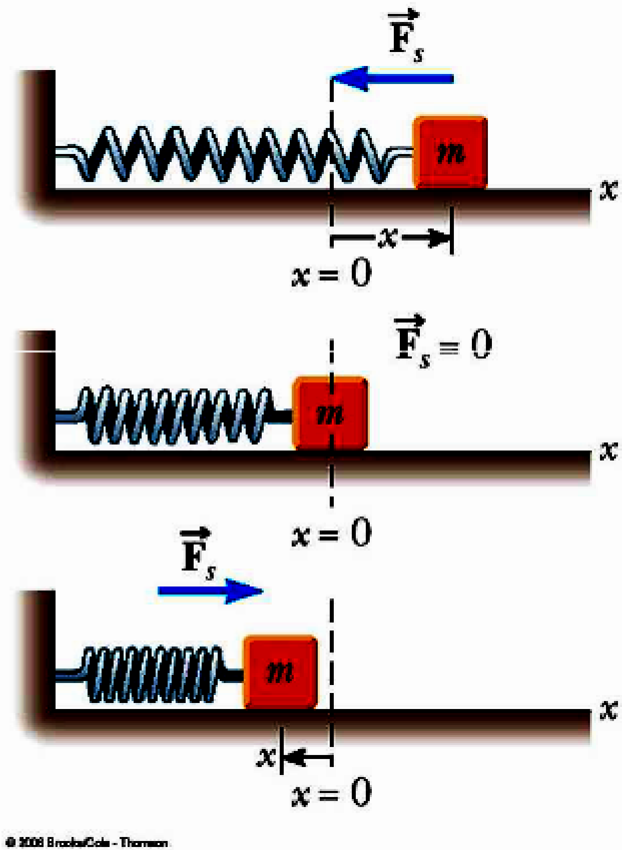
- Frequency f

The *frequency f* of the simple harmonic motion is the number of cycles of the motion per second.

$$f = \frac{1}{T} \quad \Rightarrow \quad \omega = 2\pi f$$

Simple Harmonic Motion

- A very special kind of motion occurs when the force acting on a body is proportional to the displacement of the body from some equilibrium position.
- If this force is always directed toward the equilibrium position, repetitive back-and-forth motion occurs about this position. Such motion is called simple harmonic motion (SHM).



Hook's Law

Consider a mass m attached the a spring. The restoring force of an ideal spring is given by,

$$F = -kx$$

where k is the spring constant and x is the displacement of the spring from its unstrained length. The minus sign indicates that the restoring force always points in a direction opposite to the displacement of the spring. SI unit of k = N/m.

$$ma = -kx$$

$$a = -\frac{k}{m}x$$

An object moves with **simple harmonic motion** whenever its acceleration is proportional to its displacement from some equilibrium position and is oppositely directed (towards equilibrium position).

Simple Harmonic Oscillator

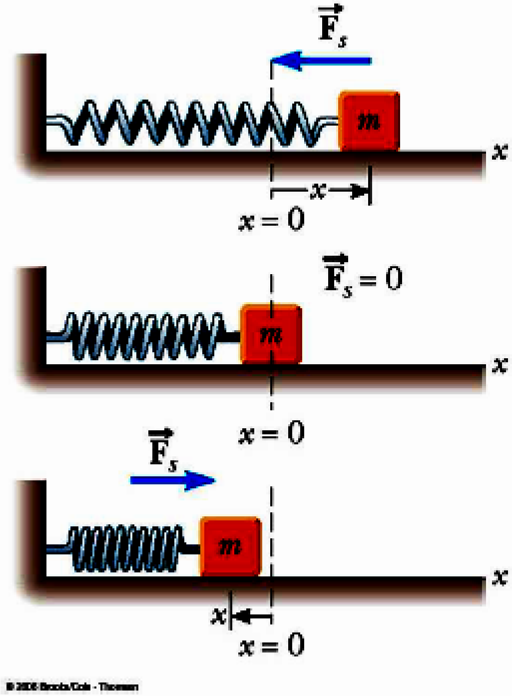
According to Hook's law $$F = -kx$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Let's set $\boxed{\frac{k}{m} = \omega^2}$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$



The general solution of this ODE is $\boxed{x(t) = A\cos(\omega t + \phi)}$

where A is amplitude, $(\omega t+\phi)$ is phase, ω is angular frequency and ϕ is phase constant

General Solution of ODE

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad x = e^{rt}$$

$$r^2 e^{rt} + \frac{k}{m} e^{rt} = 0$$

$$e^{rt}\underline{\left(r^2 + \frac{k}{m}\right)} = 0$$

$$r^2 + \frac{k}{m} = 0$$

$$r = \pm\sqrt{-\frac{k}{m}}$$

General Solution of ODE

$$r^2 + \frac{K}{m} = 0$$

$$r = \pm\sqrt{-\frac{K}{m}} = \pm i\sqrt{\frac{K}{m}}$$

$$r = \pm i w$$

$$x(t) = C_1 e^{+iwt} + C_2 e^{-iwt}$$

General Solution of ODE

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

$$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$$

$$x(t) = C_1\left[\cos(\omega t) + i\cdot\sin(\omega t)\right] + C_2\left[\cos(\omega t) - i\sin(\omega t)\right]$$

General Solution of ODE

$$C_2\left[\cos(\omega t) - i\sin(\omega t)\right]$$

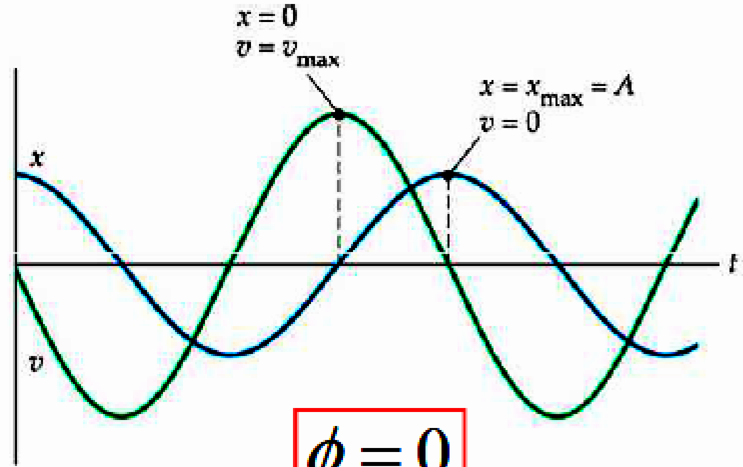
$$x(t) = (C_1 + C_2)\cos(\omega t) + i(C_1 - C_2)\sin(\omega t)$$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$\omega = 2\pi f \quad \Rightarrow \quad f = \frac{1}{2\pi}\omega \quad \Rightarrow \quad \boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

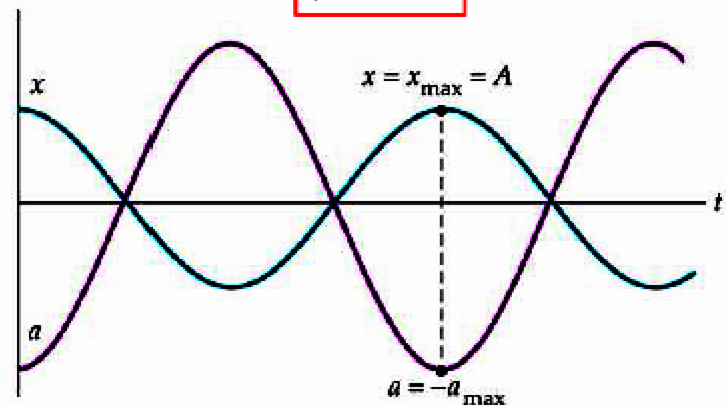
$$T = \frac{1}{f} \quad \Rightarrow \quad \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

$$\boxed{x(t) = A\cos(\omega t + \phi)}$$



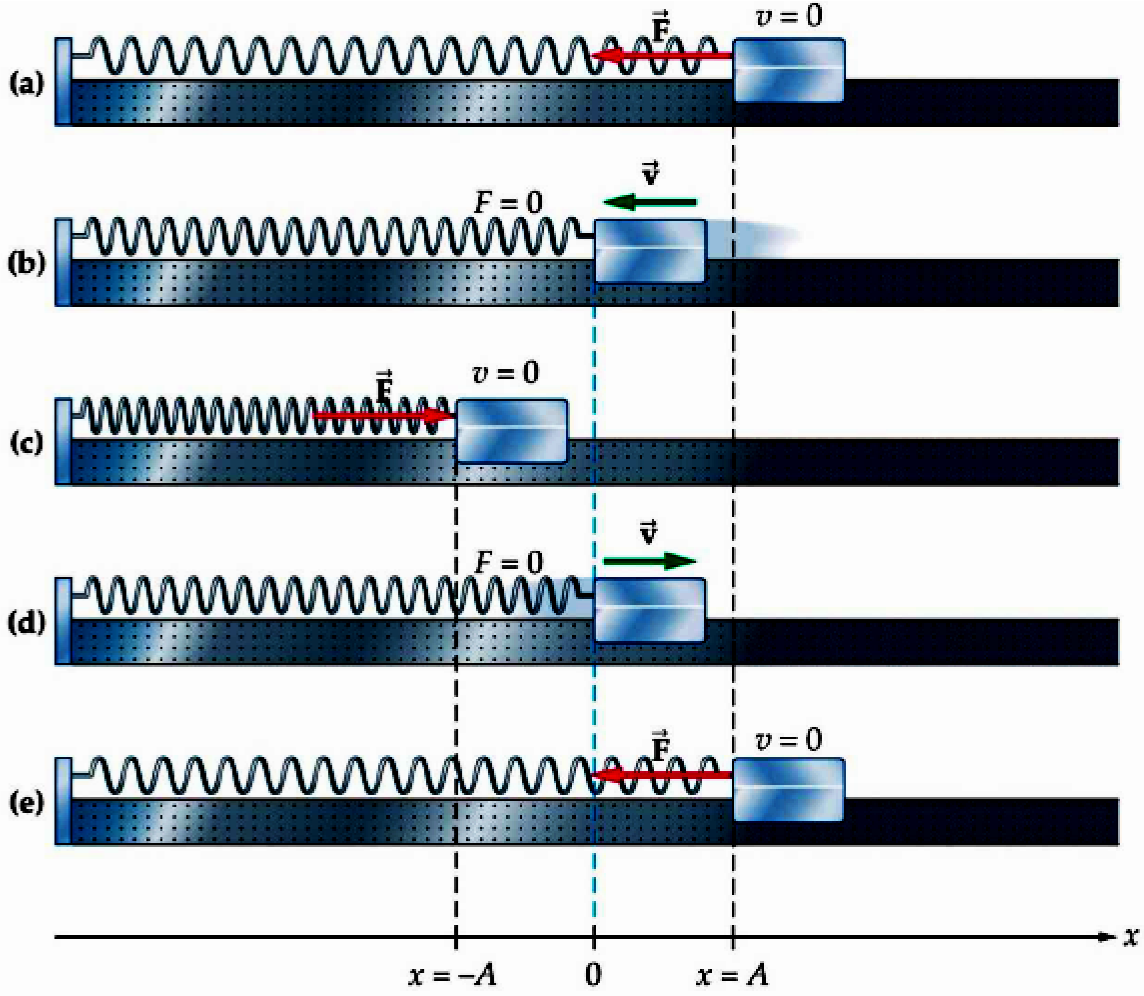
$$\boxed{\phi = 0}$$

$$\boxed{v(t) = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)}$$



$$\boxed{a(t) = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi)}$$

Simple Harmonic Oscillator



$x = A$; $v = 0$; $a = -a_{max}$

$x = 0$; $v = -v_{max}$; $a = 0$

$x = -A$; $v = 0$; $a = a_{max}$

$x = 0$; $v = v_{max}$; $a = 0$

$x = A$; $v = 0$; $a = -a_{max}$

To Emphasize This!

THROW THESE AWAY FOR HARMONIC OSCILLATOR!

$$v_f = v_i + at \qquad x_f = x_i + v_i t + \frac{1}{2} a t^2$$

The acceleration is not constant!

$$v_f^{\,2} = v_i^{\,2} + 2a\Delta x \qquad x_f = x_i + \bar{v} t$$

THESE ARE *WRONG* AND WILL GIVE

YOU *WRONG* ANSWERS!!

A 680g block is fastened to a spring of spring constant k=65N/m. The block is pulled a distance x = 11cm from its equilibrium position x=0 on a frictionless surface and released from rest at t=0

(a) What are the angular frequency, the frequency and the period of the motion

(b) What is the maximum speed of oscillating block

(c) What is the maximum acceleration of oscillating block

(a) What are the angular frequency, the frequency and the period of the motion

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65N/m}{0.68kg}} = 9.78\,rad/\sec$$

$$f = \frac{\omega}{2\pi} = \frac{9.78\,rad/\sec}{2\pi} = 1.6Hz$$

$$T = \frac{1}{f} = \frac{1}{1.6Hz} = 0.64\sec$$

(b) What is the maximum speed of oscillating block

As velocity of oscillator is

$$v = -\omega A \sin(\omega t + \phi)$$

So maximum speed of the block will be

$$v_{\max} = \omega A = (9.78)(0.11) = 1.1 m/s$$

(c) What is the maximum acceleration of oscillating block

As acceleration of oscillator is

$$a = -\omega^2 A \cos(\omega t + \phi)$$

So maximum acceleration will be

$$a_{\max} = \omega^2 A = (9.78)^2 (0.11) = 11 m/s^2$$

A 1 kg object stretches a vertical spring by 2cm. If the object is set into oscillation, find the period of the motion.

A 1 kg object stretches a vertical spring by 2cm. If the object is set into oscillation, find the period of the motion.

As $$kx = mg$$

$$k = \frac{mg}{x} = \frac{(1kg)(9.8m/s^2)}{2\times10^{-2}m} = 490N/m$$

So $$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1kg}{490N/m}} = 0.28\sec$$

A 2.14 kg object hangs from the spring. A 325g body hung below the object stretches the spring 1.80cm farther. If 325 g body is removed and the object is set into oscillation, find the period of the motion.

A 2.14 kg object hangs from the spring. A 325g body hung below the object stretches the spring 1.80cm farther. If 325 g body is removed and the object is set into oscillation, find the period of the motion.

Here $$kx = mg$$

$$k = \frac{mg}{x} = \frac{(0.325kg)(9.8m/s^2)}{1.8\times10^{-2}m} = 177N/m$$

So $$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2.14kg}{177N/m}} = 0.69\,\text{sec}$$

A 2.14 kg object hangs from the spring. A 325g body hung below the object stretches the spring 1.80cm farther. If the object is set into oscillation, find the period of the motion.

A 2.14 kg object hangs from the spring. A 325g body hung below the object stretches the spring 1.80cm farther. If the object is set into oscillation, find the period of the motion.

Here $kx = mg$

$$k = \frac{mg}{x} = \frac{(0.325kg)(9.8m/s^2)}{1.8\times10^{-2}m} = 177N/m$$

So $$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2.14kg + 0.325}{177N/m}} = 0.087\sec$$

A 680g block is fastened to a spring of spring constant k=65N/m. The block is pulled a distance x = 11cm from its equilibrium position x=0 on a frictionless surface and released from rest at t=0

Q. What is the phase constant ϕ for the motion?

As displacement of oscillator as a function of time is

$$x = A\cos(\omega t + \phi)$$

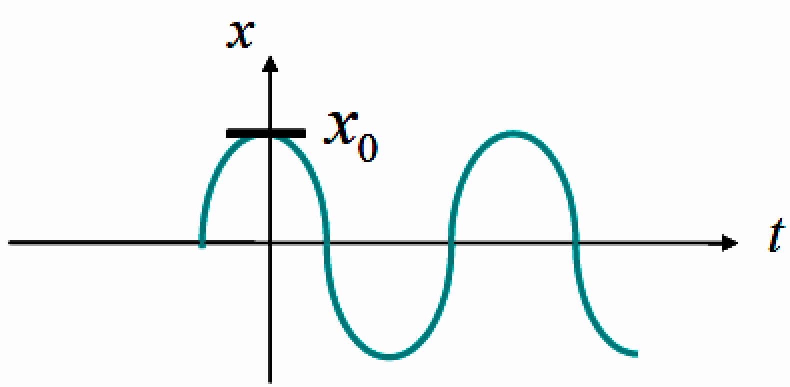
Here, it is given that at t = 0, x = A=11cm. so

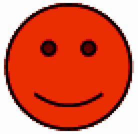
$$A = A\cos(\omega 0 + \phi)$$

$$1 = \cos\phi$$

$$\phi = 0$$

$$\boxed{x(t) = A\cos(\omega t)}$$

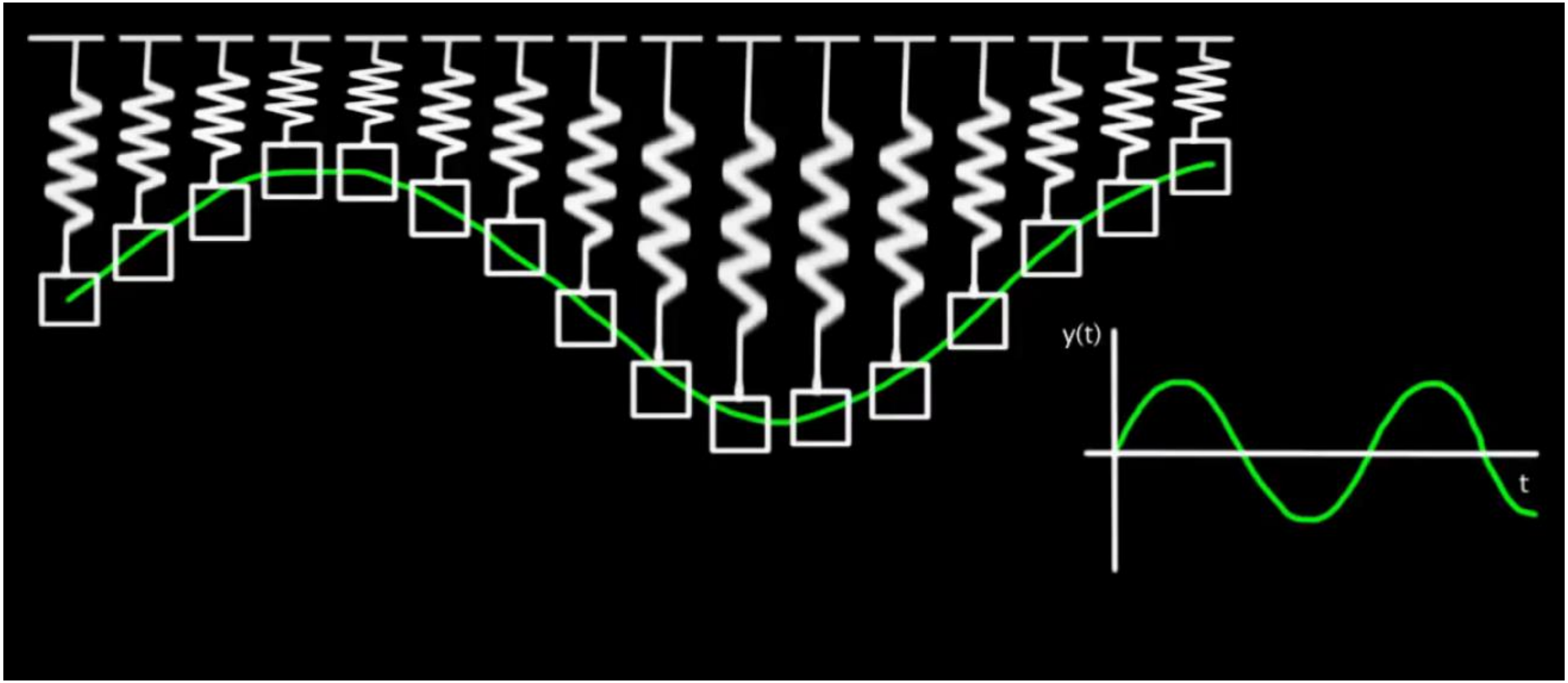


Perfectly fine... 

Phase Constant

- There is no problem as for as angular frequency, linear frequency and time period of SHM are concerned. OK?
- What do you think of phase constant ϕ?
- Its time to dig out more about phase constant ϕ.
- Let's see another problem, same problem with bit modification

Phase Constant



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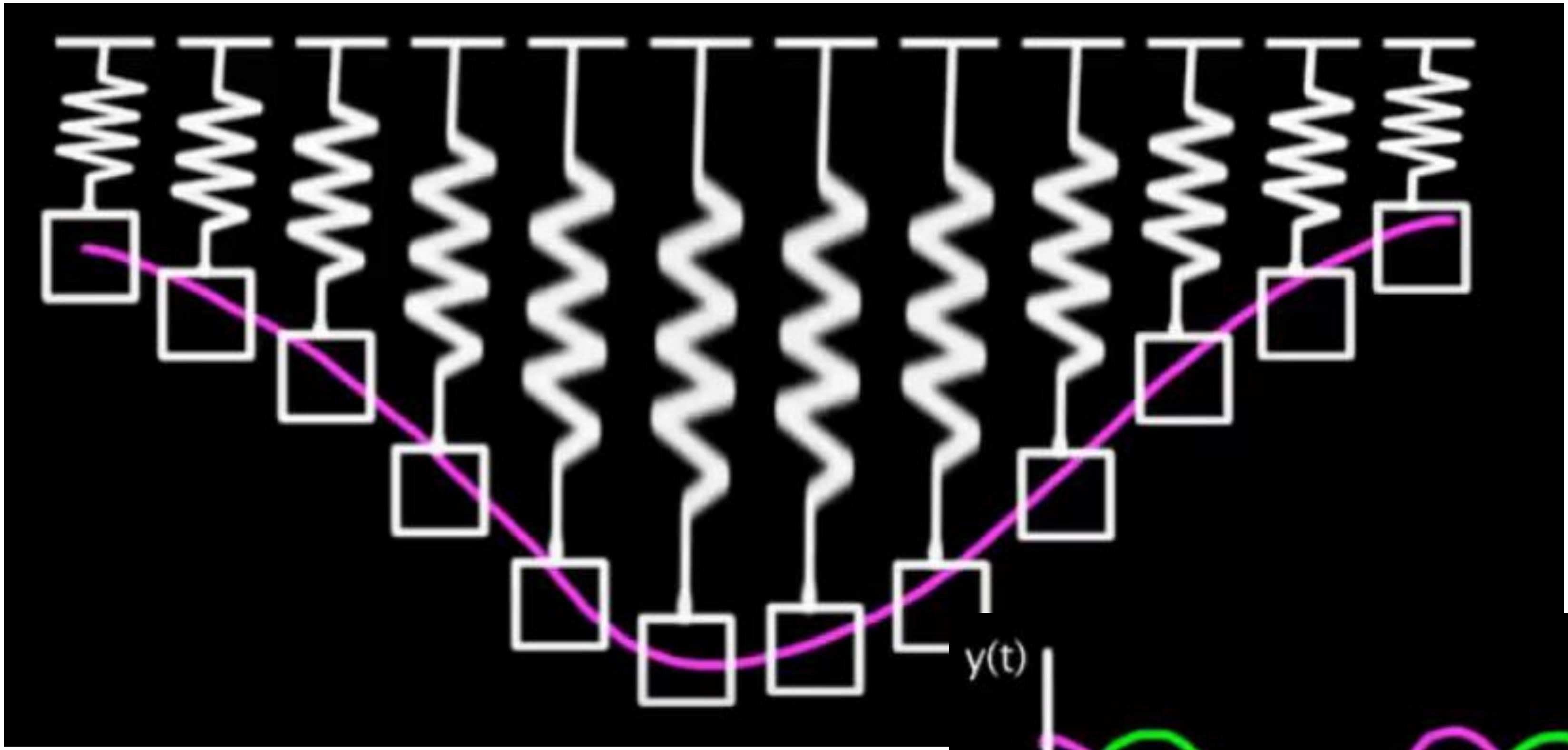
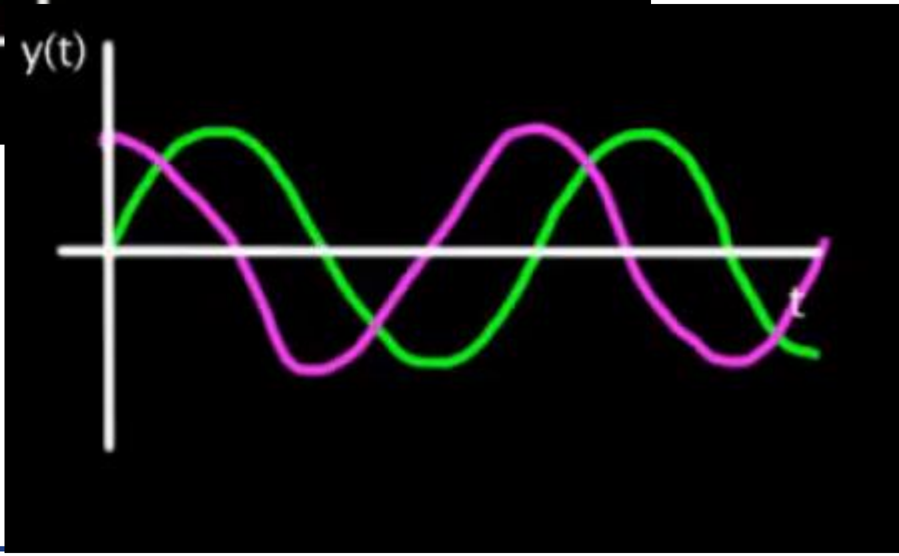
Phase Constant





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A 680g block is fastened to a spring of spring constant k=65N/m. The block is pulled a distance x = -11cm from its equilibrium position x=0 on a frictionless surface and released from rest at t=0

Q. What is the phase constant ϕ for the motion?

As displacement of oscillator as a function of time is

$$x = A\cos(\omega t + \phi)$$

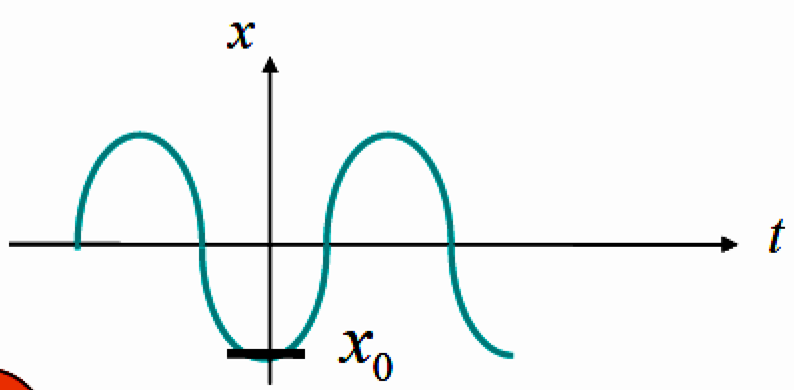
Here, it is given that at t = 0, x = -A=-11cm. so

$$-A = A\cos(\omega 0 + \phi)$$

$$-1 = \cos\phi$$

$$\phi = \pi$$

$$\boxed{x(t) = A\cos(\omega t + \pi)}$$



Perfectly fine again.......

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A 680g block is fastened to a spring of spring constant k=65N/m and set into oscillation. At time t = 0, this oscillating mass-spring system has displacement x=A/2.

Q. What is the phase constant ϕ for the motion?

Let's start as before!!!!

As displacement of oscillator as a function of time is

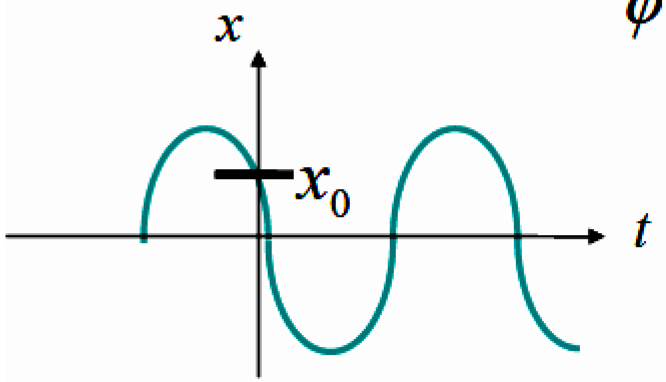
$$x = A\cos(\omega t + \phi)$$

Here, it is given that at t = 0, x = A/2. so

$$A/2 = A\cos(\omega 0 + \phi)$$

$$1/2 = \cos\phi$$

$$\phi = \pm\pi/3$$

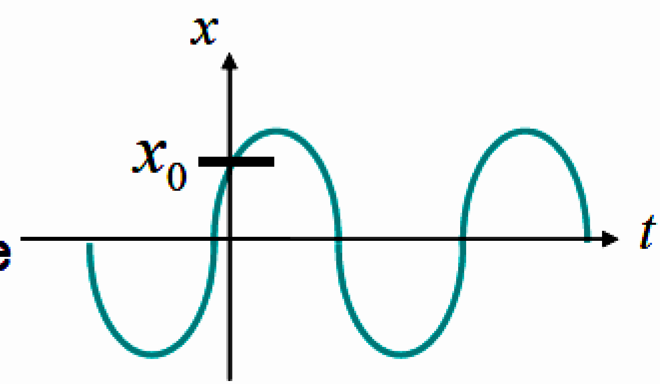


$$x = A\cos(\omega t + \pi/3)$$

Which will be the correct one?

Do I need extra information?

Yes!



$$x = A\cos(\omega t - \pi/3)$$

A 680g block is fastened to a spring of spring constant k=65N/m and set into oscillation. At time t = 0, this oscillating mass-spring system has displacement x=A/2 and is moving toward X=A (velocity is positive).

Q. What is the phase constant ϕ for the motion?

$$\boxed{\phi = -\pi / 3}$$

A 680g block is fastened to a spring of spring constant k=65N/m and set into oscillation. At time t = 0, this oscillating mass-spring system has displacement x=A/2 and is moving toward X=0 (velocity is negative).

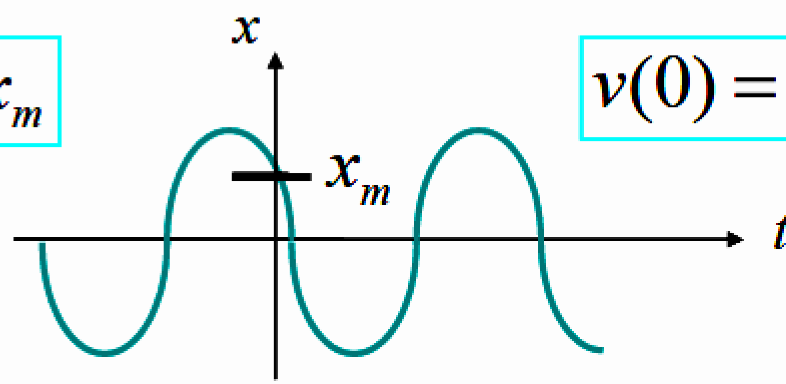
Q. What is the phase constant ϕ for the motion?

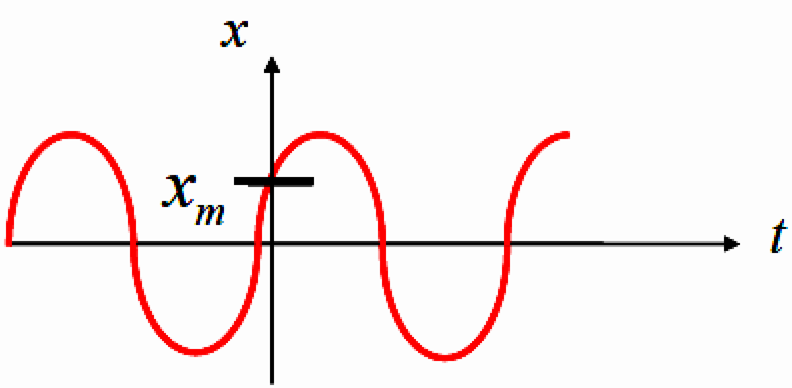
$$\boxed{\phi = \pi / 3}$$

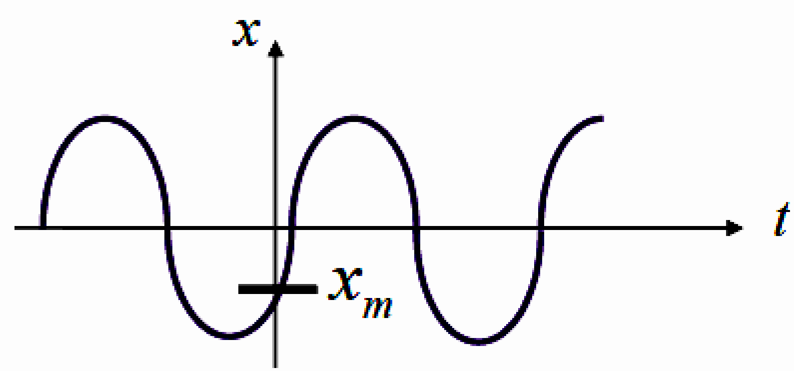
- The phase constant ϕ specifies the oscillator parameters at $t = 0$ or we can determine phase constant ϕ from initial position and initial velocity of the oscillator.

$x(0) = x_m$

$v(0) = -Ve$







$x(0) = x_m$ $\quad v(0) = +Ve$

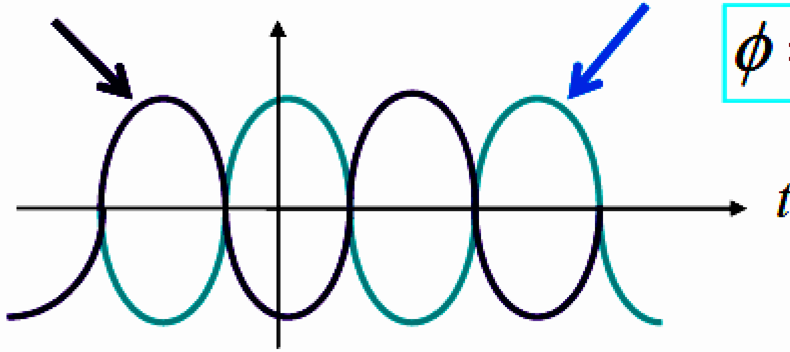
$x(0) = -x_m$ $\quad v(0) = +Ve$

$$x(t) = -A\cos(\omega t)$$

$$\phi = \pi$$

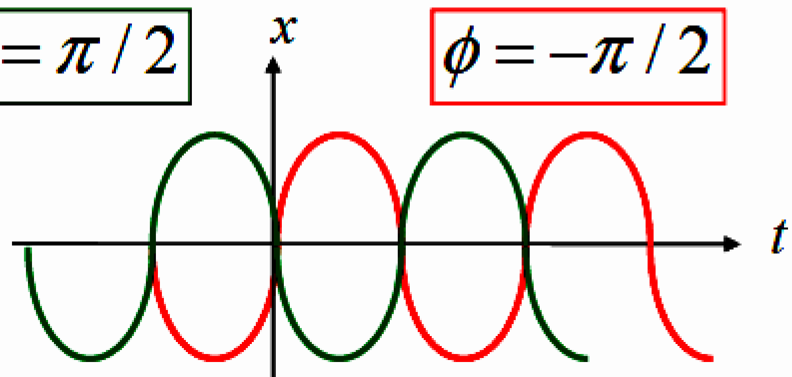
$$x(t) = A\cos(\omega t)$$

$$\phi = 0$$



$$\phi = \pi / 2$$

$$\phi = -\pi / 2$$

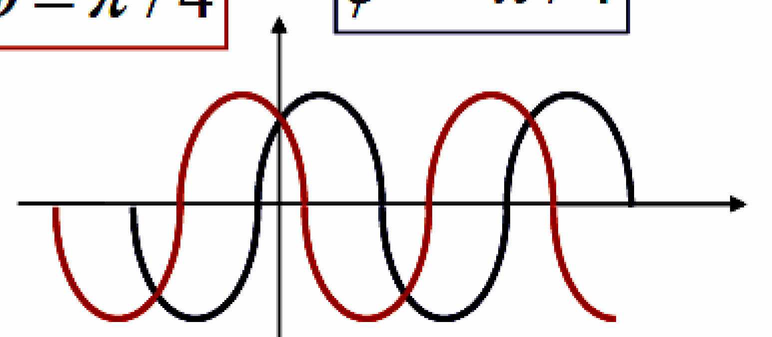


$$x(t) = A\cos(\omega t - \pi / 2)$$

$$x(t) = A\cos(\omega t + \pi / 2)$$

$$\phi = \pi / 4$$

$$\phi = -\pi / 4$$



$$x(t) = A\cos(\omega t - \pi / 4)$$

$$x(t) = A\cos(\omega t + \pi / 4)$$

Graphs-General

- The graphs show (general case)

(a) Displacement x(t) as a function of time

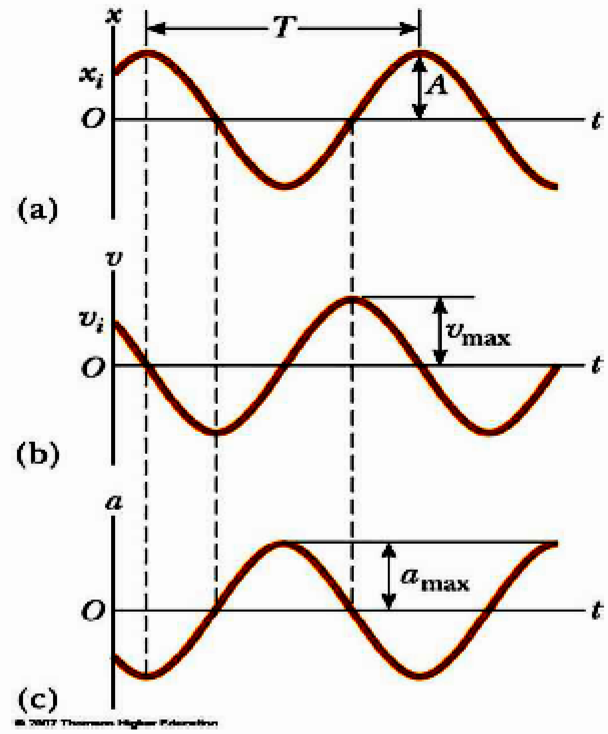
$$\boxed{x(t) = A\cos(\omega t + \phi)}$$

(b) Velocity v(t) as a function of time

$$\boxed{v(t) = -\omega A\sin(\omega t + \phi)}$$

(c) Acceleration a(t) as a function of time

$$\boxed{a(t) = -\omega^2 A\cos(\omega t + \phi)}$$



The velocity is **90º** out of phase with the displacement and the acceleration is **180º** out of phase with the displacement

At time t = 0, an oscillating mass-spring system has displacement 10cm, velocity -12m/s and acceleration -20m/sec square. What is the Phase angle?

At time t = 0, an oscillating mass-spring system has displacement 10cm, velocity -12m/s and acceleration -20m/sec square. What is the Phase angle?

For harmonic oscillator

$$x(t) = A\cos(\omega t + \phi)$$

$$v(t) = -\omega A\sin(\omega t + \phi)$$

$$a(t) = -\omega^2 A\cos(\omega t + \phi)$$

At t = 0

$$x(0) = A\cos\phi \qquad (1)$$

$$v(0) = -\omega A\sin\phi \qquad (2)$$

$$a(0) = -\omega^2 A\cos\phi \qquad (3)$$

From (1) and (3)

$$\boxed{\omega^2 = -\frac{a(0)}{x(0)} = \frac{20}{0.1}}$$

$$\omega^2 = \frac{20}{0.1} \quad \Rightarrow \quad \omega \approx 14/s$$

From (1) and (2)

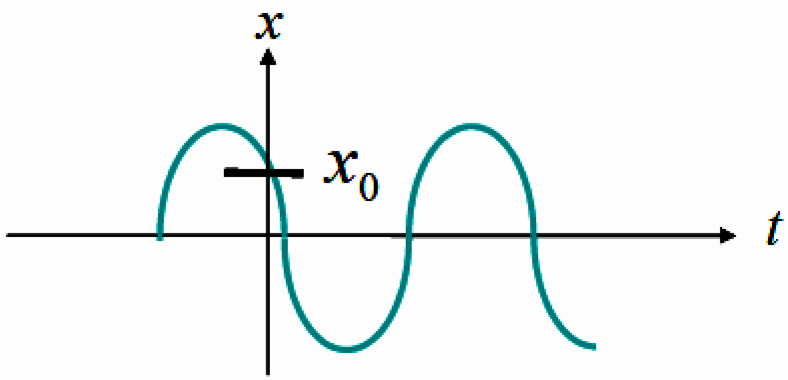
$$\omega \frac{\sin\phi}{\cos\phi} = -\frac{v(0)}{x(0)}$$

$$\boxed{\tan\phi = -\frac{1}{\omega}\frac{v(0)}{x(0)}}$$

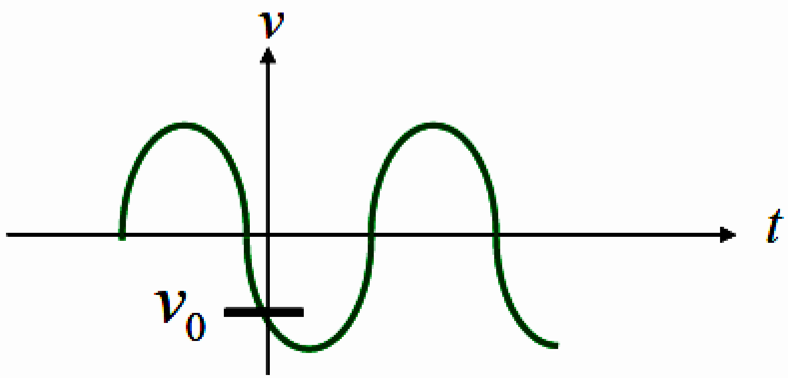
$$\tan\phi = -\frac{1}{14}\frac{-12}{0.1} = 8.57$$

$$\phi = \tan^{-1}(8.57) \approx 83^\circ$$

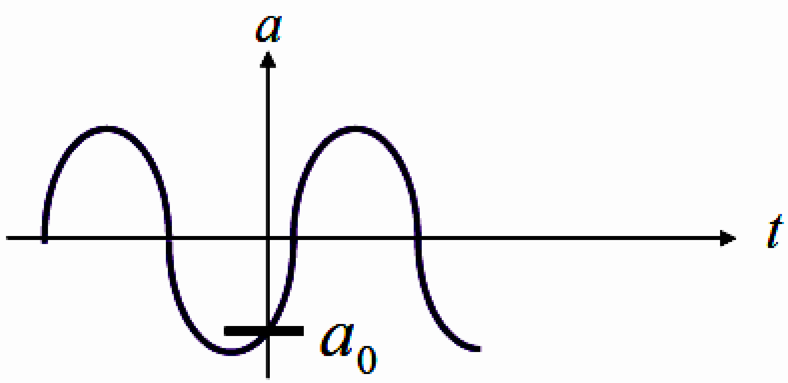
$$\boxed{x(t) = A\cos(\omega t + 83)}$$



$$\boxed{v(t) = -\omega A\sin(\omega t + 83)}$$



$$\boxed{a(t) = -A\omega^2\cos(\omega t + 83)}$$



At time t = 0, an oscillating mass-spring system has displacement 10cm, velocity 12m/s and acceleration -20m/sec square. What is the Phase angle?

At time t = 0, an oscillating mass-spring system has displacement 10cm, velocity 12m/s and acceleration -20m/sec square. What is the Phase angle?

For harmonic oscillator

$$x(t) = A\cos(\omega t + \phi)$$

$$v(t) = -\omega A\sin(\omega t + \phi)$$

$$a(t) = -\omega^2 A\cos(\omega t + \phi)$$

At t = 0

$$x(0) = A\cos\phi \qquad (1)$$

$$v(0) = -\omega A\sin\phi \qquad (2)$$

$$a(0) = -\omega^2 A\cos\phi \qquad (3)$$

From (1) and (3) $$\boxed{\omega^2 = -\frac{a(0)}{x(0)} = \frac{20}{0.1}}$$

$$\omega^2 = \frac{20}{0.1} \quad \Rightarrow \quad \omega = 14/s$$

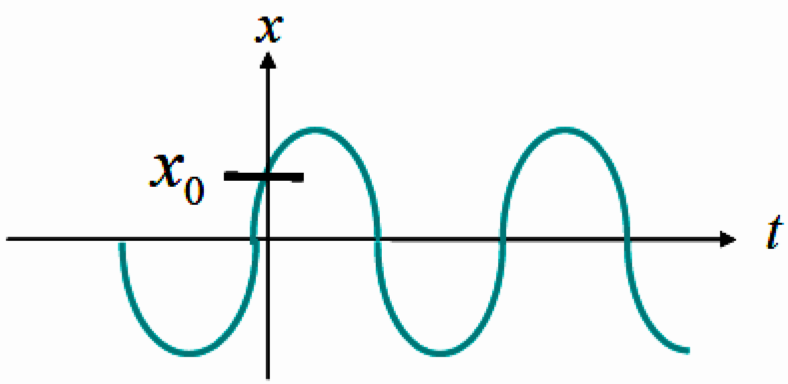
From (1) and (2) $$\omega \frac{\sin\phi}{\cos\phi} = -\frac{v(0)}{x(0)}$$

$$\boxed{\tan\phi = -\frac{1}{\omega}\frac{v(0)}{x(0)}}$$

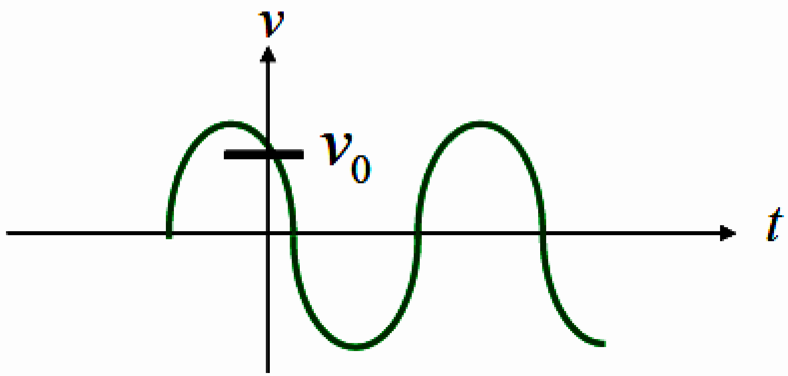
$$\tan\phi = -\frac{1}{14}\frac{12}{0.1} = -8.57$$

$$\phi = \tan^{-1}(-8.57) \approx -83^\circ$$

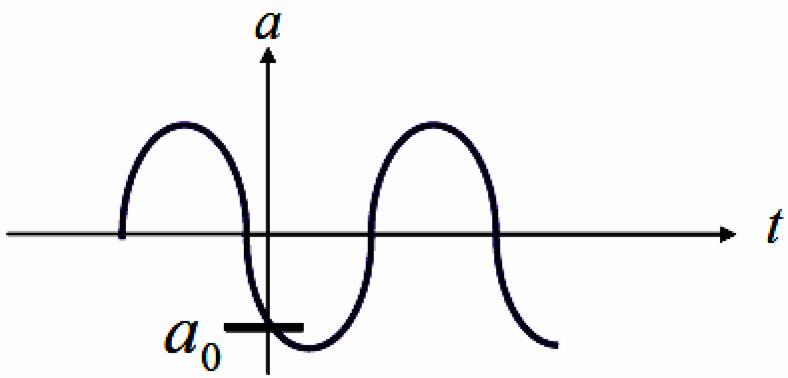
$$x(t) = A\cos(\omega t - 83)$$



$$v(t) = -\omega A\sin(\omega t - 83)$$



$$a(t) = -A\omega^2\cos(\omega t - 83)$$



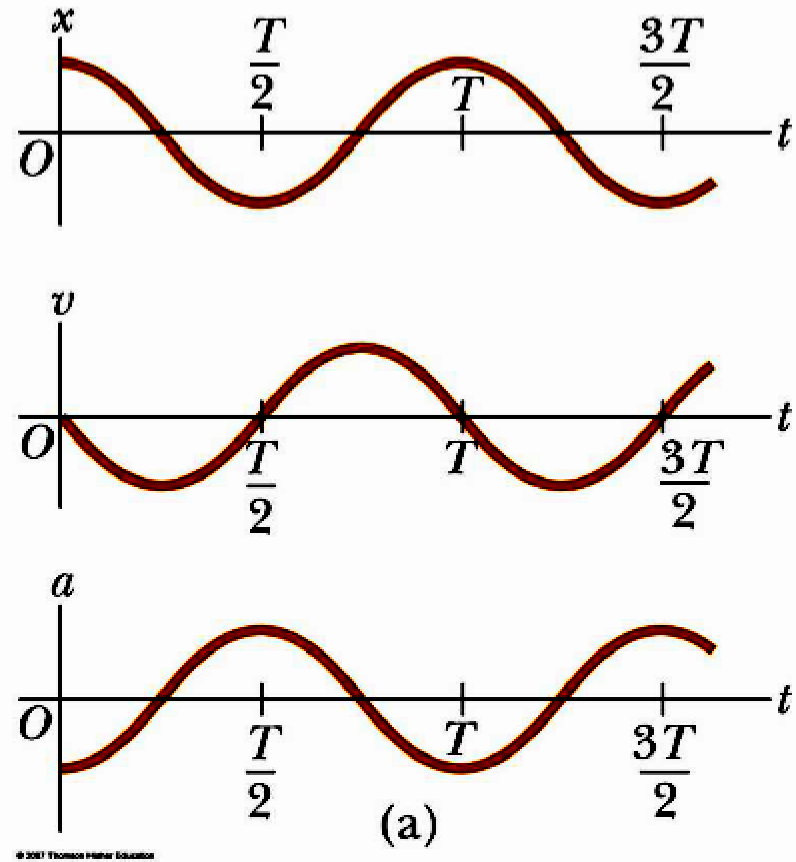
Initial conditions at $t = 0$: $x(0) = A; v(0) = 0$

This means $\phi = 0$

$$\boxed{x(t) = A\cos(\omega t)}$$

$$\boxed{v(t) = -\omega A\sin(\omega t)}$$

$$\boxed{a(t) = -\omega^2 A\cos(\omega t)}$$

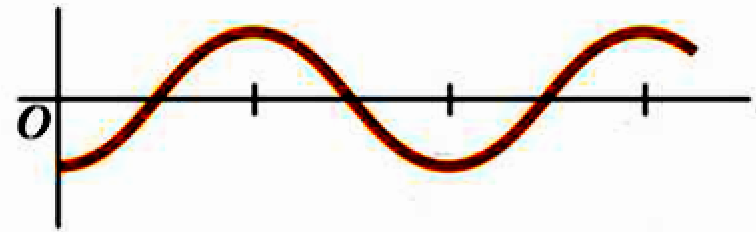


$$\tan\phi = -\frac{1}{\omega}\frac{v(0)}{x(0)} = -\frac{1}{\omega}\frac{0}{A} = 0 \quad \Rightarrow \quad \phi = 0$$

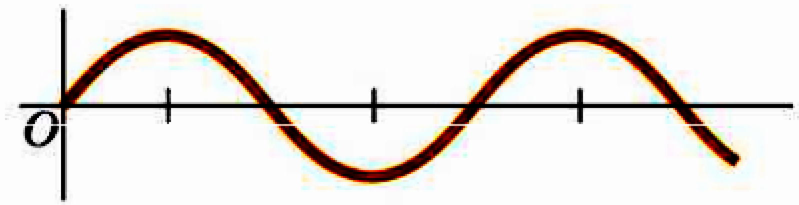
Initial conditions at **t = 0: x(0) = -A; v(0) = 0**

This means $\phi = \pi$

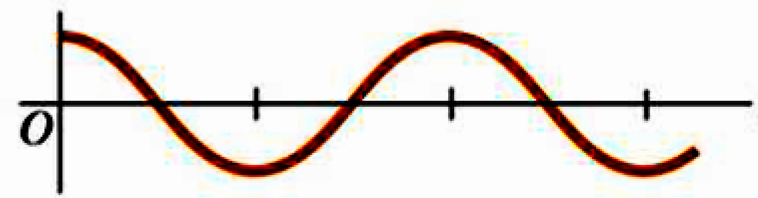
$$\boxed{x(t) = -A\cos(\omega t)}$$



$$\boxed{v(t) = \omega A\sin(\omega t)}$$



$$\boxed{a(t) = \omega^2 A\cos(\omega t)}$$



$$\tan\phi = -\frac{1}{\omega}\frac{v(0)}{x(0)} = -\frac{1}{\omega}\frac{0}{-A} = 0 \quad \Rightarrow \quad \phi = \pi$$

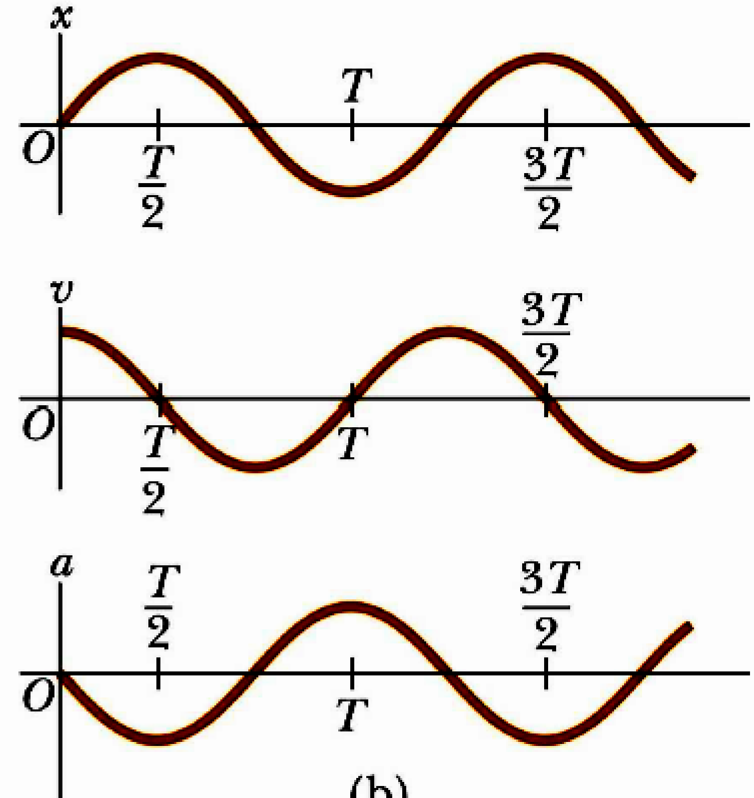
Initial conditions at **t = 0: x(0) = 0; v(0) = v**

This means **$\phi = -\pi/2$**

$$\boxed{x(t) = A\sin(\omega t)}$$

$$\boxed{v(t) = \omega A\cos(\omega t)}$$

$$\boxed{a(t) = -\omega^2 A\sin(\omega t)}$$



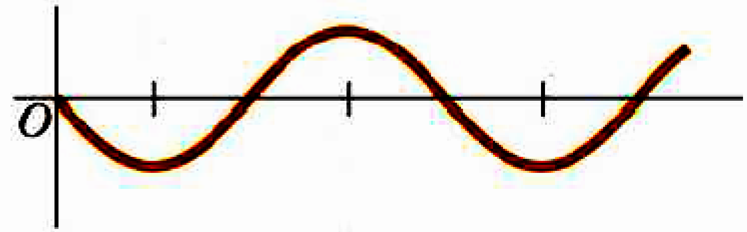
(b)

$$\tan\phi = -\frac{1}{\omega}\frac{v(0)}{x(0)} = -\frac{1}{\omega}\frac{v}{0} = -\infty \quad \Rightarrow \quad \phi = -\pi/2$$

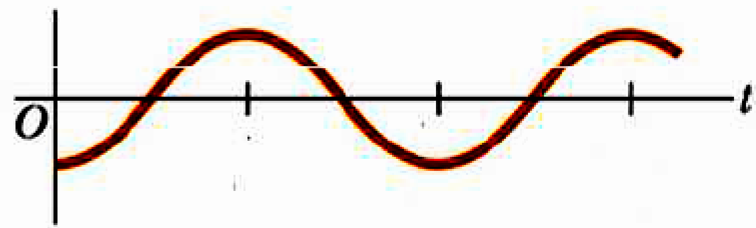
Initial conditions at **t = 0: x(0) = 0; v(0) = -v**

This means $\phi = \pi/2$

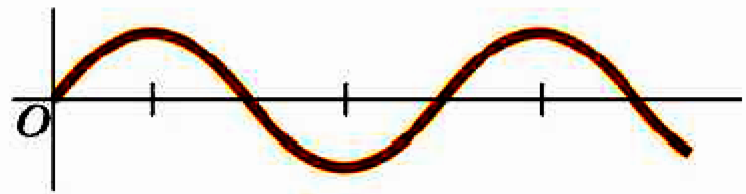
$$\boxed{x(t) = -A\sin(\omega t)}$$



$$\boxed{v(t) = -\omega A\cos(\omega t)}$$



$$\boxed{a(t) = \omega^2 A\sin(\omega t)}$$



$$\tan\phi = -\frac{1}{\omega}\frac{v(0)}{x(0)} = -\frac{1}{\omega}\frac{-v}{0} = \infty \quad \Rightarrow \quad \phi = \pi/2$$