Solution of difference equations using z-transform:

Modeling of Physical System

Continuous time (Differential equation)

Discrete time (Difference equation)

Simple example: $x(n+1) = 3x(n)$

$x(1) = 3x(0)$, $x(2) = 3x(1) = 3^2 x(0)$

$x(3) = 3x(2) = 3^3 x(0), \ldots, x(n) = 3^n x(0)$, is the solution,

where $x(0)$ is the initial condition.

Newton's Law of Cooling:

If $t(0)$ is the initial temparature, $t(n)$ is the temparature after 'n' steps, s is the temparature of surroundings, then

$$t(n+1) - t(n) = k(s - t(n)) \quad \text{—— (i)}$$

where $k > 0$ is a constant.

Example: Suppose $t(0) = 90°c$, $s = 20°c$, temparature of the surroundings. $k = 0.1$. Solve equation (i) with these values.

Sol:- $t(n+1) - t(n) = 0.1(20 - t(n))$, $t(0) = 90$.

$$t(n+1) - 0.9\,t(n) = 2\,u(n) \quad \text{—— (ii)}$$

$u(n)$ is the unit-step sequence.

Taking z-transform of (ii), $Z[t(n)] = T(z)$,

$$zT(z) - z\,t(0) - 0.9\,T(z) = 2\frac{z}{z-1}.$$

$$T(z)[z - 0.9] = 2\frac{z}{z-1} + 90z$$

$$T(z) = 2\frac{z}{(z-1)(z-0.9)} + \frac{90z}{z-0.9} \quad \text{—— (iii)}$$

[z.transform 13]

- Find Inverse Z-transform of $T(z)$ to get $t(n)$.

Note:
$\frac{z}{z-1}$ ✓
$\frac{1}{z-1}$ ✗

Partial fraction: $\frac{2z}{(z-1)(z-0.9)} = A\frac{z}{z-1} + B\frac{z}{z-0.9}$

$A = 20$, $B = -20$.

$$T(z) = 20\frac{z}{z-1} - 20\frac{z}{z-0.9} + \frac{90z}{z-0.9}$$

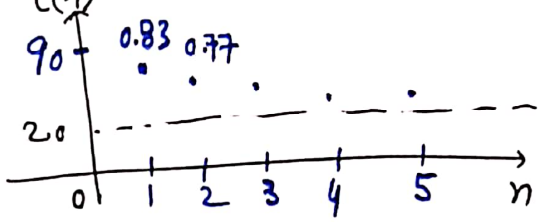
$$= 20\frac{z}{z-1} + 70\frac{z}{z-0.9}$$

$$\mathcal{Z}[a^n] = \frac{z}{z-a}.$$

$$t(n) = 20\,u(n) + 70(0.9)^n$$

$n=0$, $t(0) = 20 + 70(0.9)^0 = 90$

$n \to \infty$, $t(n) \to 20$



Fibonacci numbers: $F(n+2) = F(n+1) + F(n)$
with $F(0) = F(1) = 1$;

So, $F(2)$, $F(3) = 3$, $F(4) = 5$, $F(5) = 8, \ldots$

Mathematics of Finance:

Present value of an annuity after n periods, $x(n)$, obeys

$$x(n+1) = \frac{P + x(n)}{1 + r}$$

P = Payment, r is the interest rate.

[z transform 14]

Scanned with Car

Multiplication by 'n' (or differentiation in z) property:

Show that $n\,x[n] \longleftrightarrow -z\frac{d}{dz}[X(z)]$.

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\,z^{-n}$$

Differentiating both sides with respect to z, we have

$$\frac{d}{dz}[X(z)] = \sum_{n=-\infty}^{\infty} -n\,x[n]\,z^{-n-1}$$

and

$$-z\frac{d\,X(z)}{dz} = \sum_{n=-\infty}^{\infty} \{n\,x[n]\}\,z^{-n} = \mathcal{Z}\{n\,x[n]\}$$

Thus, we conclude that $n\,x[n] \longleftrightarrow -z\frac{d}{dz}[X(z)]$.

Example: Find the z-transform of each of the following sequences:

(a). $x[n] = n\,a^{n}u[n]$. —— (i)

$$a^{n}u[n] \longleftrightarrow \frac{z}{z-a}, \quad |z| > |a|, \quad \text{—— (ii)}$$

Using the multiplication by n property, we get

$$n\,a^{n}u[n] \longleftrightarrow z\frac{d}{dz}\left(\frac{z}{z-a}\right) = \frac{az}{(z-a)^2}, \quad |z|>|a|. \quad \text{—— (iii)}$$

(b). $x[n] = n\,a^{n-1}u[n]$.

$$n\,a^{n-1}u[n] \longleftrightarrow \frac{d}{da}\left(\frac{z}{z-a}\right) = \frac{z}{(z-a)^2}, \quad |z|>|a|. \quad \text{—— (iv)}$$

Note that dividing both sides of equation (iii) by a, we obtain equation (iv).

Practice: Consider the following difference equation

$$y(n) = a\,y(n-1) + x(n).$$

Calculate the transfer function $H(z) = \frac{Y(z)}{X(z)}$, and find its poles & zeros. Find h(n) with ROC.

[z transform 15]