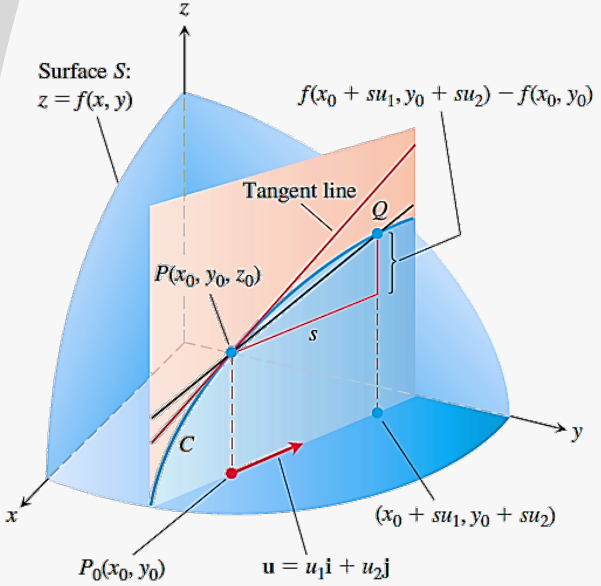
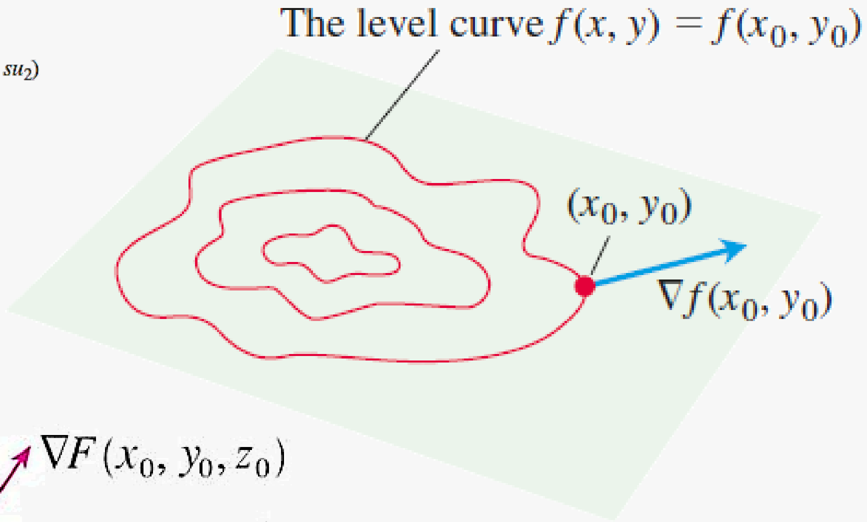
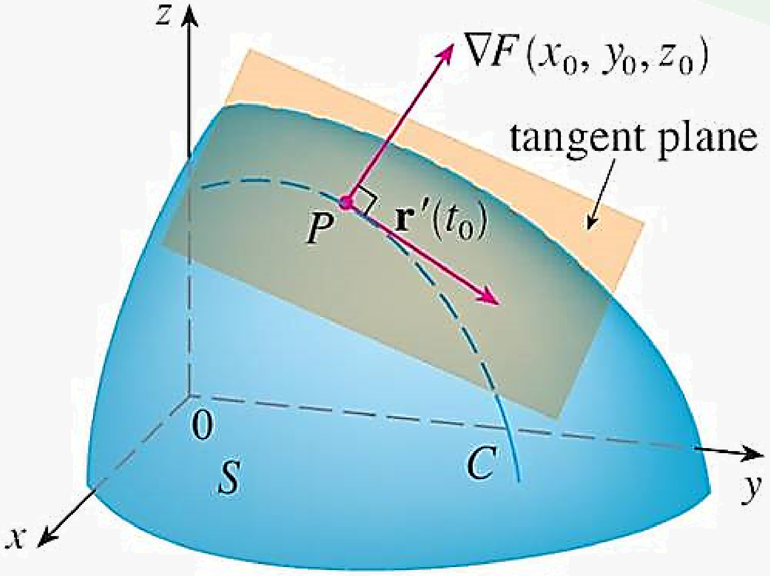
Tangent Planes & Normal Lines







Vector Calculus(MATH-243)
Instructor: Dr. Naila Amir

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Partial Derivatives

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Chapter: 14 , Section: 14.5, 14.6

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.
Chapter: 14 , Section: 14.1, 14.6

Gradients to Level Curves

If $z = f\,(x, y)$ is surface then $f(x, y) = k$ is called the ***level curve*** of $f(x, y)$ of level k and is the intersection of the horizontal plane $z = k$ and the surface $z = f\,(x, y)$. If $f(x, y)$ is a differentiable function of two variables then the level curve $f(x, y) = c$ can be parametrized by $\mathbf{r}(t) = \langle g(t), h(t) \rangle$, i.e.,

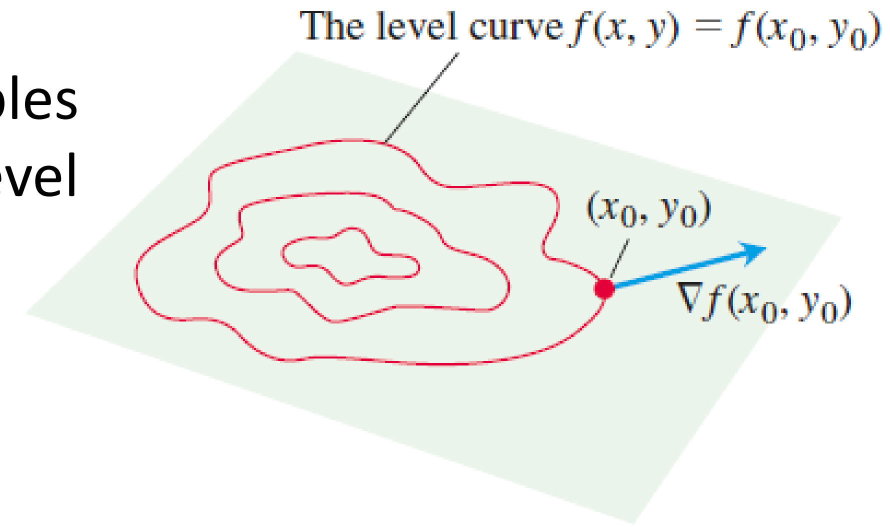
$$f\big(g(t), h(t)\big) = c.$$

Differentiating both sides of this equation with respect to t leads to the equations:

$$\nabla f \cdot \frac{dr}{dt} = 0 \Longrightarrow \nabla f \cdot \mathbf{r}'(t) = 0. \qquad (*)$$

Equation $(*)$ says that ∇f is orthogonal to the tangent vector $\mathbf{r}'(t)$, so it is orthogonal to the tangent line to the curve $f(x, y) = c$ at a point (x_0, y_0).

Thus, the gradient of a differentiable function of two variables at a point (x_0, y_0) is always normal to the function's level curve through that point.



Tangent line & Normal Line to a Level Curve

- At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0).
- This observation enables us to find equations for tangent lines to level curves. They are the lines normal to the gradients.
- The line through a point (x_0, y_0) normal to a nonzero vector $\mathbf{N} = \langle A, B \rangle$ has the equation:

$$A(x - x_0) + B(y - y_0) = 0.$$

- If $\mathbf{N}$ is the gradient $\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$ and this gradient is a nonzero vector, then the equation for tangent line to level curve at (x_0, y_0) is given by:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

$$\Longrightarrow f_y(x_0, y_0)(y - y_0) = -f_x(x_0, y_0)(x - x_0),$$

$$\Longrightarrow y - y_0 = -f_x(x_0, y_0)/f_y(x_0, y_0)\,(x - x_0).$$

- Thus, the normal of the level curve at (x_0, y_0) is given as:

$$y - y_0 = f_y(x_0, y_0)/f_x(x_0, y_0)\,(x - x_0).$$

Tangent line & Normal Line to a Level Curve

- For a differentiable of single variable: $y = f(x)$, the tangent line is given as:
$$y - f(x_0) = f'(x_0)(x - x_0).$$
Actually, this comes from the linear approximation: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$. The normal vector of this line is $(f'(x_0), -1)$.
- The linear approximation for a differentiable function of two variables $z = f(x, y)$, is:
$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$
- When we move on level curve $f(x, y) = f(x_0, y_0)$ we get the tangent line of the level curve at (x_0, y_0) as:
$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0,$$
$$\Longrightarrow f_y(x_0, y_0)(y - y_0) = -f_x(x_0, y_0)(x - x_0),$$
$$\Longrightarrow y - y_0 = -\frac{f_x(x_0, y_0)}{f_y(x_0, y_0)}(x - x_0).$$
- Thus, the normal of the level curve at (x_0, y_0) is given as:
$$y - y_0 = \frac{f_y(x_0, y_0)}{f_x(x_0, y_0)}(x - x_0).$$

Example:

Find an equation for the tangent to the ellipse:

$$\frac{x^2}{4} + y^2 = 2,$$

at the point $(-2, 1)$.

Solution: The ellipse is a level curve of the function:

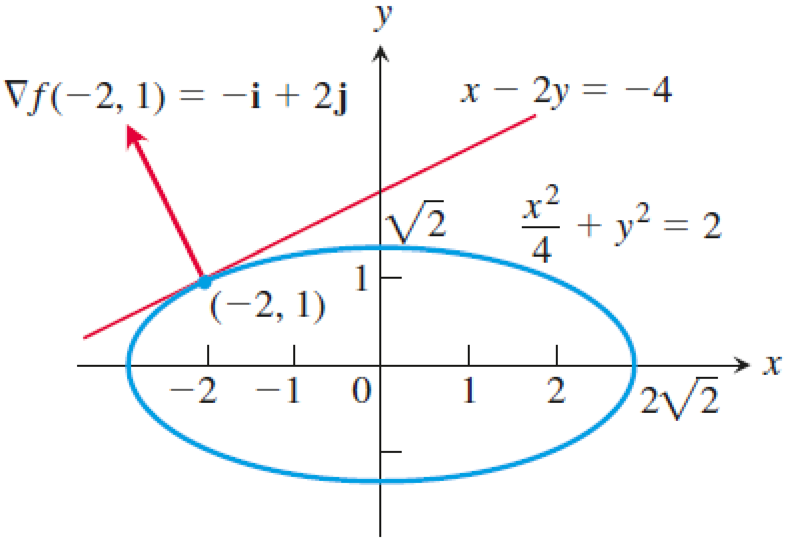
$$f(x, y) = \frac{x^2}{4} + y^2.$$

The gradient of f at $(-2,1)$ is:

$$\nabla f(-2,1) = \left\langle \frac{x}{2}, 2y \right\rangle \Big|_{(-2,1)} = \langle -1,2 \rangle.$$

Because this gradient vector is nonzero, the tangent to the ellipse at $(-2,1)$ is the line:

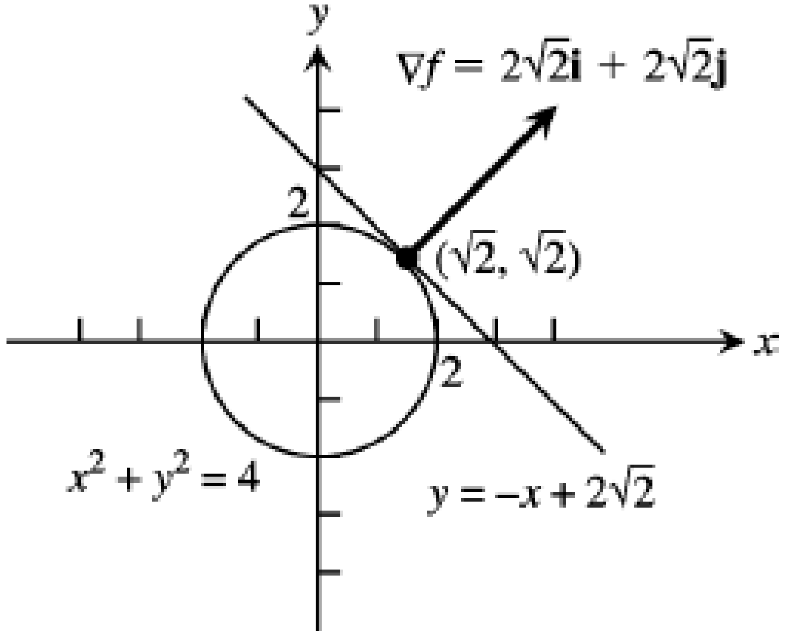
$$(-1)(x+2) + (2)(y-1) = 0 \Longrightarrow x - 2y = -4.$$



We can find the tangent to the ellipse $\frac{x^2}{4} + y^2 = 2$ by treating the ellipse as a level curve of the function $f(x, y) = \frac{x^2}{4} + y^2$.

Example:

sketch the curve $x^2 + y^2 = 4$, together with ∇f and the tangent line at the point $\left(\sqrt{2}, \sqrt{2}\right)$. Then write an equation for the tangent line.



Solution: The circle is a level curve of the function:

$$f(x, y) = x^2 + y^2.$$

The gradient of f at $\left(\sqrt{2}, \sqrt{2}\right)$ is:

$$\nabla f\left(\sqrt{2}, \sqrt{2}\right) = \langle 2x, 2y \rangle\Big|_{(\sqrt{2}, \sqrt{2})} = \left\langle 2\sqrt{2}, 2\sqrt{2} \right\rangle.$$

Because this gradient vector is nonzero, the tangent to the circle at $(\sqrt{2}, \sqrt{2})$ is the line:

$$y - \sqrt{2} = \left(-\frac{2\sqrt{2}}{2\sqrt{2}}\right)\left(x - \sqrt{2}\right) \Rightarrow y - \sqrt{2} = \sqrt{2} - x \Rightarrow x + y = 2\sqrt{2}.$$

Example:

Consider the curve $x^2 - xy + y^2 = 7$. Determine ∇f and equation for the tangent line at the point $(-1,2)$.

Solution: The given curve is a level curve of the function:

$$f(x,y) = x^2 - xy + y^2.$$

The gradient of f at $(-1,2)$ is:

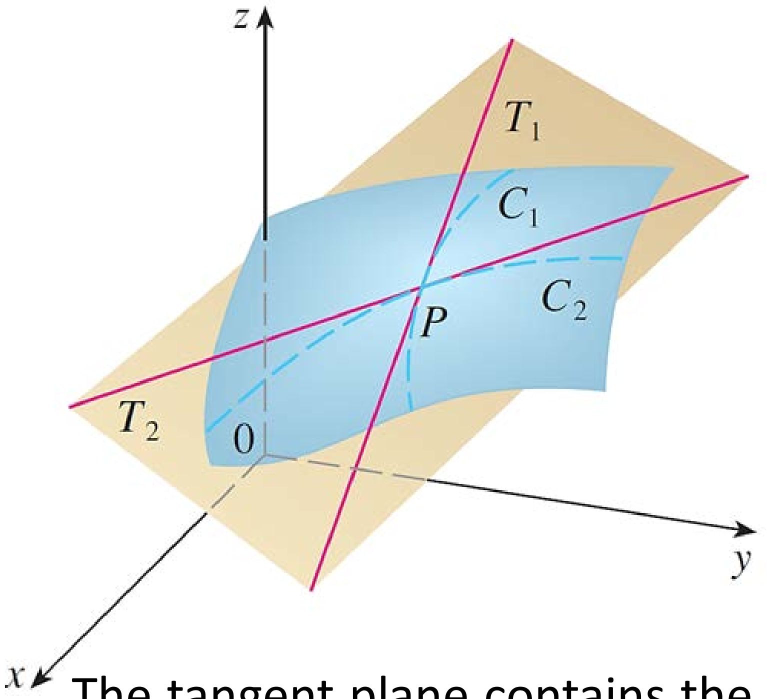
$$\nabla f(-1,2) = \langle 2x - y, -x + 2y \rangle\Big|_{(-1,2)} = \langle -4,5 \rangle.$$

Because this gradient vector is nonzero, the tangent to the level curve at $(-1,2)$ is the line:

$$y - 2 = \left(-\frac{(-4)}{5}\right)(x+1) \Longrightarrow 5(y-2) = 4(x+1) \Longrightarrow 4x - 5y + 14 = 0.$$

Tangent Planes

- Consider a surface S with equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S.
- Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S.
- The point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.
- Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.
- If C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane.
- Therefore, we can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P.



The tangent plane contains the tangent lines T_1 and T_2.

Tangent Planes

- The tangent plane at P is the plane that most closely approximates the surface S near the point P.
- We know that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

- By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form:

$$z - z_0 = a(x - x_0) + b(y - y_0). \qquad (1)$$

- If (1) represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1. Setting $y = y_0$ in (1) we get:

$$z - z_0 = a(x - x_0) \qquad \text{where} \qquad y = y_0,$$

and we recognize this as the equation (in point-slope form) of a line with slope a. But we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$. Therefore, $a = f_x(x_0, y_0)$.

Tangent Planes and Normal line

- Similarly, putting $x = x_0$ in (1), we get:

$$z - z_0 = b(y - y_0),$$

which must represent the tangent line T_2, so $b = f_y(x_0, y_0)$.

- Suppose that $f(x, y)$ has continuous partial derivatives. An equation of the **tangent plane** to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is given as:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \qquad (2)$$

- Note the similarity between the equation of a tangent plane and the equation of a tangent line:

$$y - y_0 = f'(x_0)(x - x_0).$$

- The **normal line** to the surface $z = f(x, y)$ at P is the line passing through P and perpendicular to the tangent plane. Its direction is given by the gradient, and its symmetric equations are:

$$\frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = z - z_0. \qquad (3)$$

Example:

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution:

Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \Longrightarrow f_x(1,1) = 4.$$

$$f_y(x, y) = 2y \Longrightarrow f_y(1,1) = 2.$$

Then (2) gives the equation of the tangent plane at $(1, 1, 3)$ as:

$$z - 3 = 4(x - 1) + 2(y - 1),$$

or

$$z = 4x + 2y - 3.$$

Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on the surface S and passes through the point P. Recall that the curve C is described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Let t_0 be the parameter value corresponding to P; that is, $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. Since C lies on S, any point $(x(t), y(t), z(t))$ must satisfy the equation of S, that is,

$$F\big(x(t), y(t), z(t)\big) = k. \qquad (4)$$

If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation (3) as follows:

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

$$\Longrightarrow \nabla F \cdot \mathbf{r}'(t) = 0. \qquad (5)$$

Tangent Planes to Level Surfaces

In particular, when $t = t_0$ we have:

$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0. \qquad (6)$$

Equation (6) says that *the gradient vector at* P, $\nabla F(x_0, y_0, z_0)$, *is perpendicular to the tangent vector* $\mathbf{r}'(t_0)$ *to any curve* C *on* S *that passes through* P. If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. Using the standard equation of a plane, we can write the equation of this tangent plane as:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0. \qquad (7)$$