APPLICATIONS OF GAUSS LAW

Gauss's Law

- We will now consider how we may use the Gauss's law below:

$$\Psi = \oint d\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

$$= \text{Total charge enclosed } Q = \int \rho_v \, dv$$

- The procedure for applying Gauss's law to calculate the electric field involves first knowing whether symmetry exists

- Once symmetric charge distribution exists, we construct a mathematical closed surface (known as a Gaussian surface) around the source of electric field

Gauss's Law

- The solution to the Gauss's law equation is easy if we are able to choose a closed surface which satisfies two conditions:

1. **D***s* is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_s.d\mathbf{S}$ becomes either $D_s dS$ or zero, respectively
2. On that portion of the closed surface for which $\mathbf{D}_s.d\mathbf{S}$ is not zero, *Ds* = constant

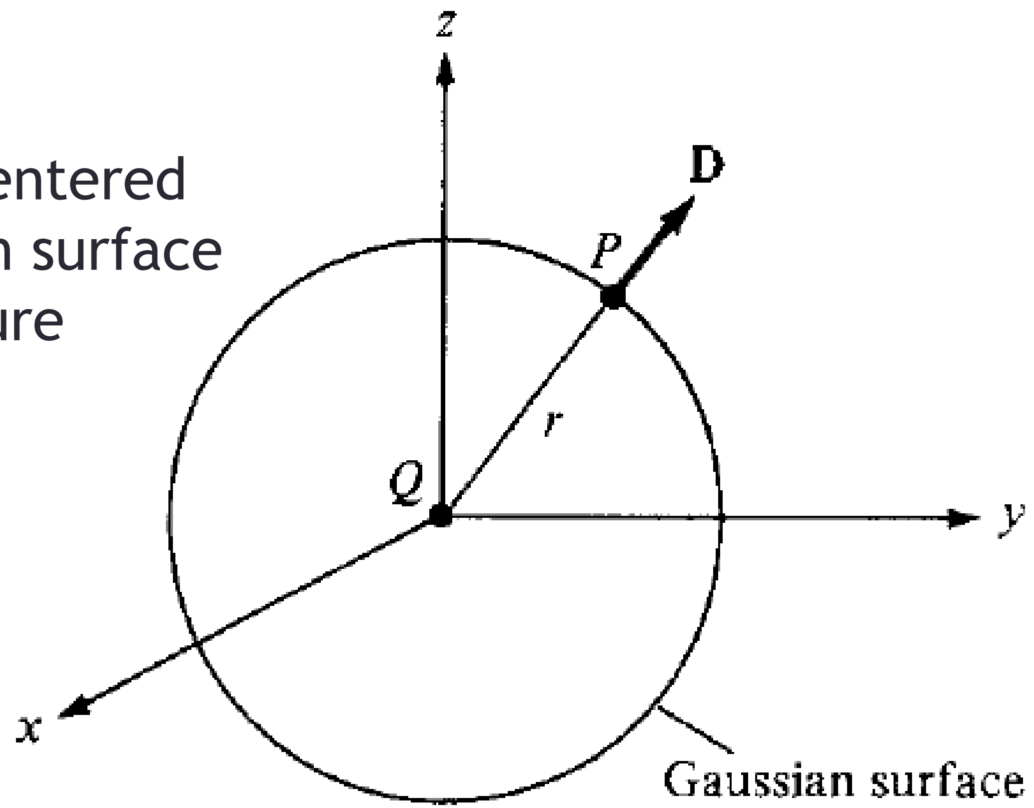
- We will now apply Gauss's law to the four types of charged sources, namely point, line, surface and volume charge

A Point Charge

- Suppose a point charge *Q* is located at the origin

- To determine **D** at a point *P*, it is easy to see that choosing a spherical surface containing *P* will satisfy symmetry conditions

- Thus, a spherical surface centered at the origin is the Gaussian surface in this case as shown in figure



A Point Charge

- Since **D** is everywhere normal to the Gaussian surface, that is, **D** = $D_r\mathbf{a}_r$, applying Gauss's law ($\psi = Q_{enclosed}$) gives:

$$Q = \oint \mathbf{D} \cdot d\mathbf{S} = D_r \oint dS = D_r\, 4\pi r^2$$

- Where $\int dS = \int_0^{2\pi}\int_0^{\pi} r^2 sin\theta d\theta d\emptyset = 4\pi r^2$ is the surface area of the Gaussian surface

- Thus:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

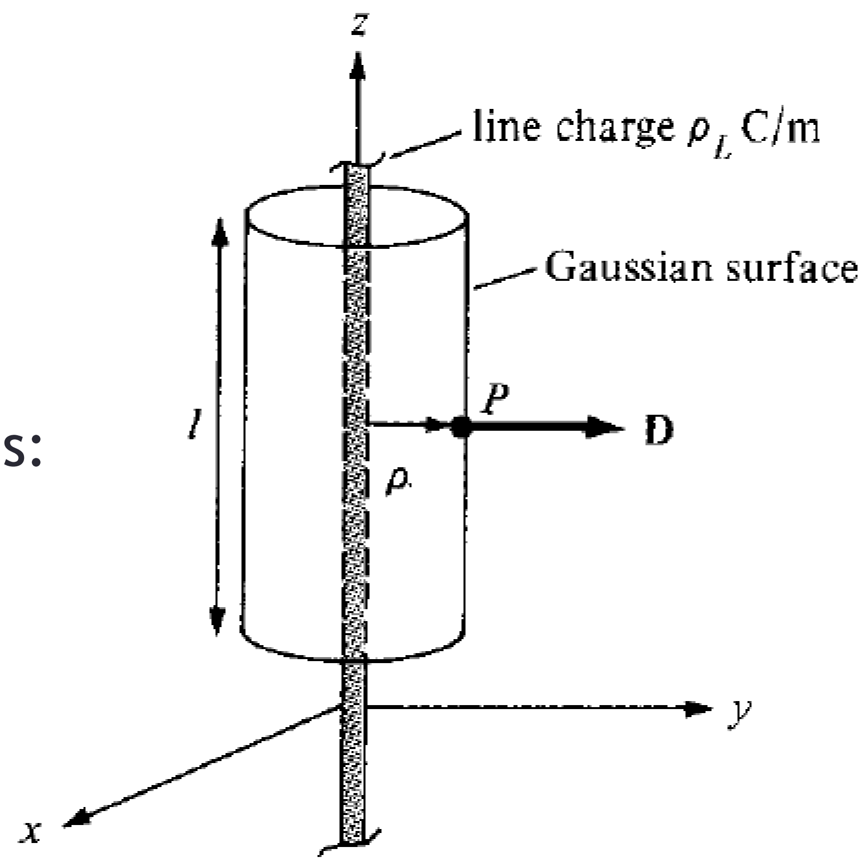
- Which is the same result obtained previously

Infinite Line Charge

- Suppose the infinite line of uniform charge ρ_L C/m lies along the z-axis as shown in figure

- To determine **D** at a point *P*, we choose a cylindrical surface containing *P* to satisfy symmetry condition

- **D** is constant on and normal to the cylindrical Gaussian surface; that is: $\mathbf{D} = D_\rho \mathbf{a}_\rho$



Infinite Line Charge

- If we apply Gauss's law to an arbitrary length l of the line:

$$\rho_L \ell = Q = \oint \mathbf{D} \cdot d\mathbf{S} = D_\rho \oint dS = D_\rho 2\pi\rho\ell$$

- Where $\int dS = 2\pi\rho l$ is the surface area of the Gaussian surface

- Note that $\int \mathbf{D}.\,d\mathbf{S}$ evaluated on the top and bottom surfaces of the cylinder is zero since **D** has no z-component; meaning that **D** is tangential to those surfaces

- Thus:

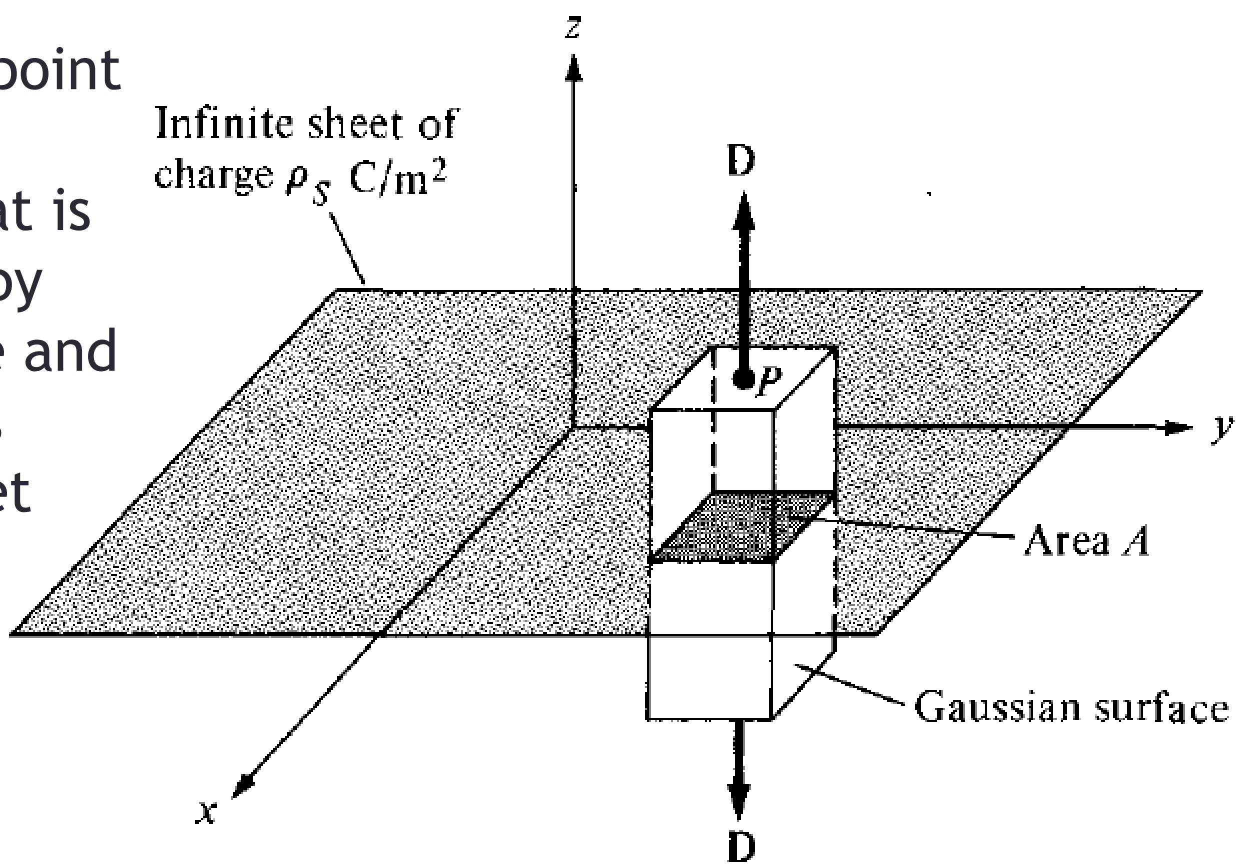
$$\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho$$

- Which is the same result obtained previously

Infinite Sheet of Charge

- Consider the infinite sheet of uniform charge ρ_s C/m^2 lying on the z = 0 plane

- To determine **D** at point *P*, we choose a rectangular box that is cut symmetrically by the sheet of charge and has two of its faces parallel to the sheet



Infinite Sheet of Charge

- As **D** is normal to the sheet, **D** = $D_z\mathbf{a}_z$, and applying Gauss's law gives:

$$\rho_S \int dS = Q = \oint \mathbf{D} \cdot d\mathbf{S} = D_z \left[\int_{\text{top}} dS + \int_{\text{bottom}} dS \right]$$

- Note that **D.*d*S** evaluated on the sides of the box is zero because **D** has no components along $\mathbf{a}_x$ and $\mathbf{a}_y$

- If the top and bottom area of the box each has area A, the above equation becomes:

$$\rho_S A = D_z (A + A)$$

- And thus:

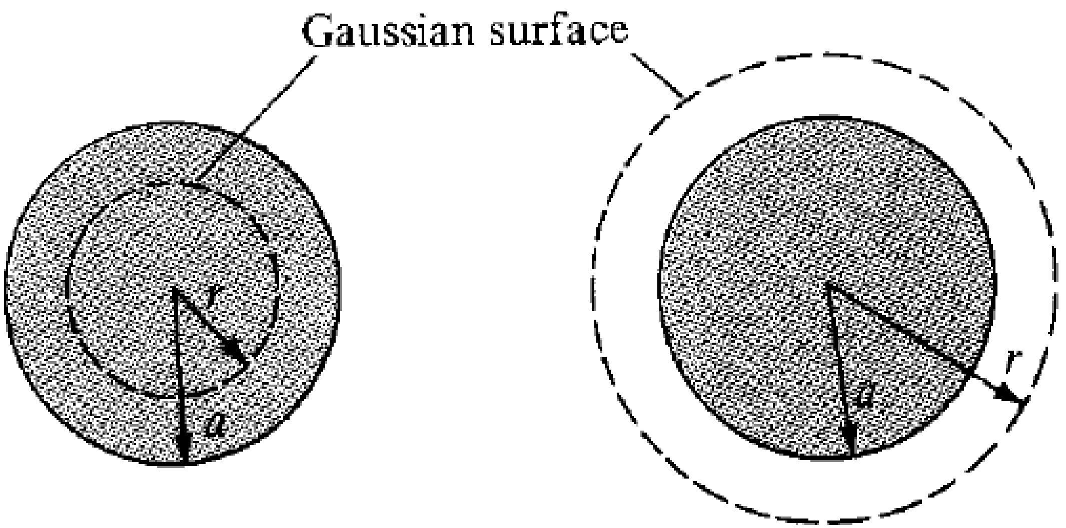
$$\mathbf{D} = \frac{\rho_S}{2} \mathbf{a}_z$$

- Or:

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_o} = \frac{\rho_S}{2\varepsilon_o} \mathbf{a}_z$$

Uniformly Charged Sphere

- Consider a sphere of radius a with a uniform charge ρ_v C/m^3
- To determine **D** everywhere, we construct Gaussian surfaces for cases $r \leq a$ and $r \geq a$ separately
- Since the charge has spherical symmetry, it is obvious that a spherical surface is an appropriate Gaussian surface



(a) (b)

Uniformly Charged Sphere

- For $r \leq a$, the total charge enclosed by the spherical surface of radius r, as shown in figure (a), is:

$$Q_{\text{enc}} = \int \rho_v \, dv = \rho_v \int dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_v \frac{4}{3} \pi r^3$$

and

$$\Psi = \oint \mathbf{D} \cdot d\mathbf{S} = D_r \oint dS = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r 4\pi r^2$$

Uniformly Charged Sphere

- Hence, $\psi = Q_{enc}$ gives:

$$D_r 4\pi r^2 = \frac{4\pi r^3}{3}\rho_v$$

- Or:

$$\mathbf{D} = \frac{r}{3}\rho_v \, \mathbf{a}_r \qquad 0 < r \leqslant a$$

- For $r \geq a$, the Gaussian surface is shown in figure (b)

- The charge enclosed by the surface is the entire charge in this case, that is:

$$Q_{\text{enc}} = \int \rho_v \, dv = \rho_v \int dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_v \frac{4}{3}\pi a^3$$

Uniformly Charged Sphere

- We have the total flux as:

$$\Psi = \oint \mathbf{D} \cdot d\mathbf{S} = D_r 4\pi r^2$$

- Hence from the previous two equations, we have:

$$D_r 4\pi r^2 = \frac{4}{3}\pi a^3 \rho_v$$

- Or:

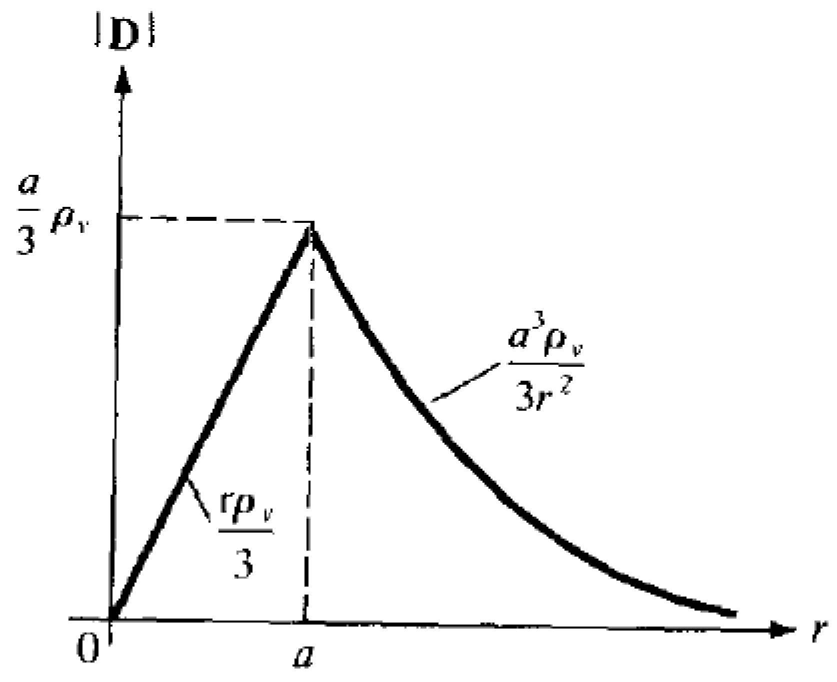
$$\mathbf{D} = \frac{a^3}{3r^2}\rho_v \mathbf{a}_r \qquad r \geqslant a$$

Uniformly Charged Sphere

- Thus **D** everywhere is given as:

$$\mathbf{D} = \begin{cases} \dfrac{r}{3}\rho_v\,\mathbf{a}_r & 0 < r \leqslant a \\[2ex] \dfrac{a^3}{3r^2}\rho_v\,\mathbf{a}_r & r \geqslant a \end{cases}$$

- The sketch of |**D**| versus distance from the center of the sphere is shown:



Problem-1

- The electric field density in a region is given by $\vec{\mathbf{D}} = 10\overrightarrow{\boldsymbol{a_r}} + 5\overrightarrow{\boldsymbol{a_\theta}} + 3\overrightarrow{\boldsymbol{a_\emptyset}}$ nC/m^2. Calculate the electric flux passing through the surface bounded by the region $z \geq 0\ and\ x^2 + y^2 + z^2 = 36$.

Problem-2

- In a rectangular coordinate system in free space:

$$\mathbf{D} = y^2z^3\boldsymbol{a_x} + 2xyz^3\boldsymbol{a_y} + 3xy^2z^2\boldsymbol{a_z}\ \text{pC/m}^2$$

a. Find the electric flux in the given surface in a direction away from the origin: $x = 3; 0 \leq y \leq 2; 0 \leq z \leq 1$

b. Find the magnitude of the electric field intensity at P(3,2,1)