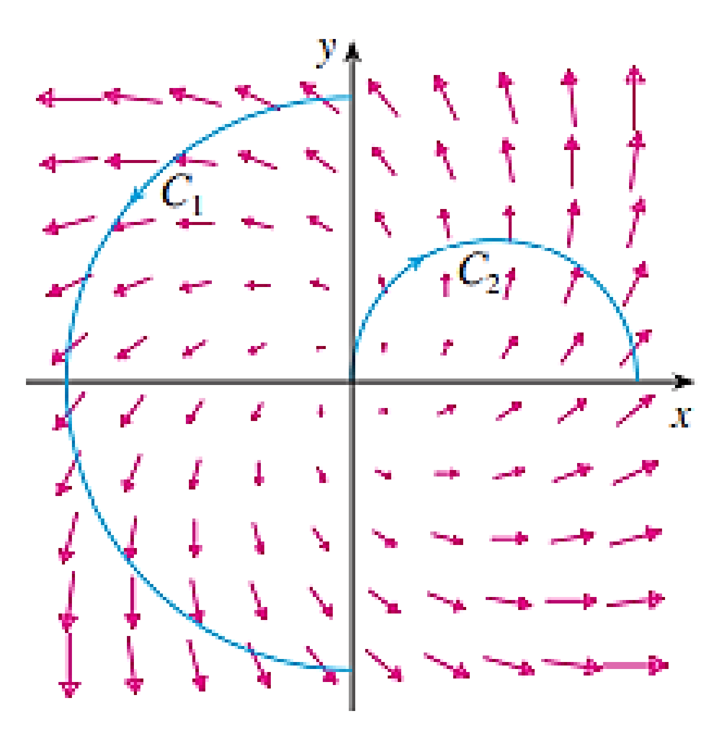
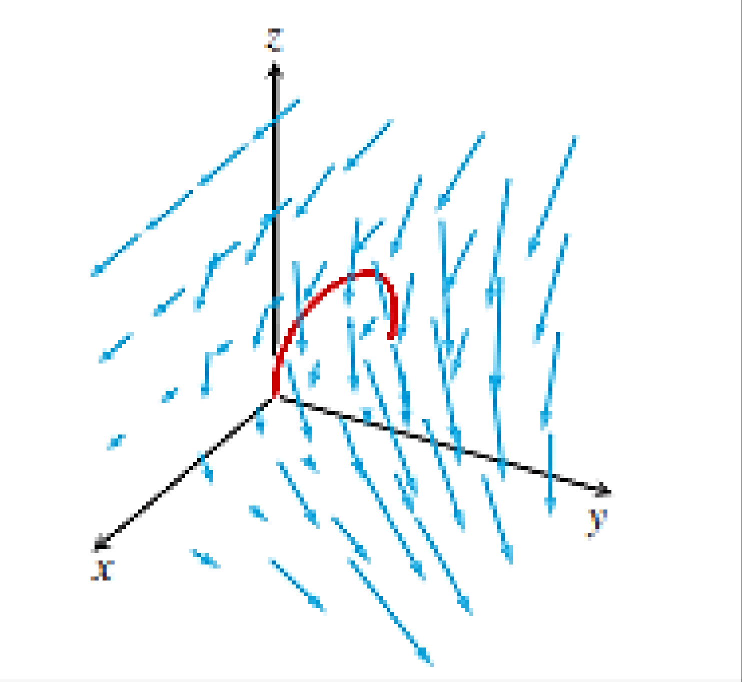
Green's Theorem





Vector Calculus(MATH-243)

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16

Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.4**

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

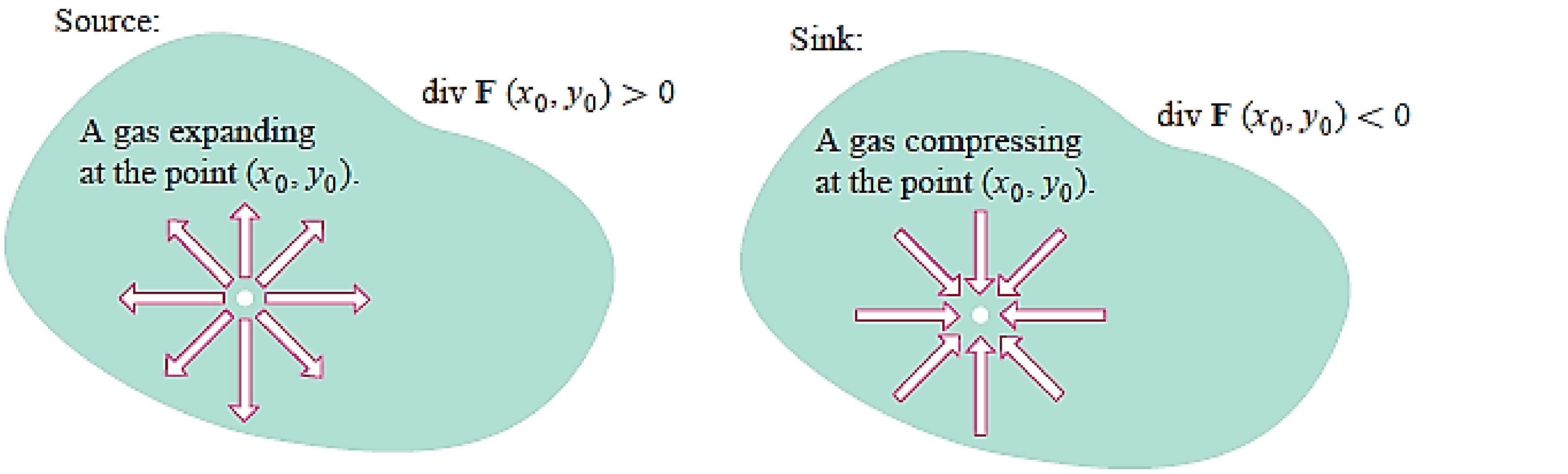
- **Chapter: 16**
  - **Section: 16.4**

Divergence (Flux Density)

DEFINITION Divergence (Flux Density)

The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$



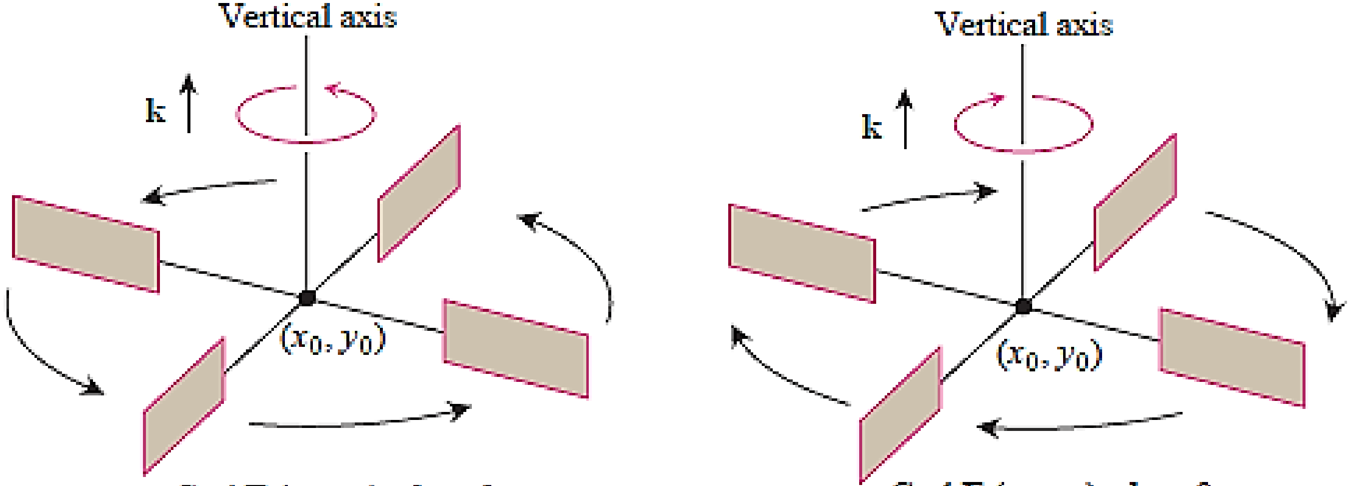
If a gas is expanding at a point the lines of flow have positive divergence; if the gas is compressing, the divergence is negative.

Spin Around an Axis: The k-Component of Curl

DEFINITION k-Component of Curl (Circulation Density)

The k-component of the curl (circulation density) of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$



Curl $\mathbf{F}(x_0, y_0) \cdot \mathbf{k} > 0$
Counterclockwise circulation

Curl $\mathbf{F}(x_0, y_0) \cdot \mathbf{k} < 0$
Clockwise circulation

If water is moving about a region in the xy −plane in a thin layer, then the $\boldsymbol{k}$ −component of the curl at a point (x_0, y_0) gives a way to measure how fast and in what direction a small paddle wheel spins if it is put into the water at (x_0, y_0) with its axis perpendicular to the plane, parallel to $\boldsymbol{k}$. Looking downward onto the xy −plane, it spins counterclockwise when $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ is positive and clockwise when the $\boldsymbol{k}$ −component is negative.

Two Forms for Green's Theorem

In first form, Green's Theorem says that under suitable conditions the **outward flux** of a vector field across a simple closed curve in the plane equals the double integral of the divergence of the field over the region enclosed by the curve.

Green's Theorem (Flux-Divergence or Normal Form)

The outward flux of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ across a simple closed curve C equals the double integral of $\operatorname{div} \mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{Outward flux}} = \oint_C M\,dy - N\,dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy}_{\text{Divergence integral}}$$

$$= \iint_R \operatorname{div} \mathbf{F}\, dA.$$

Note: if $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ is a parameterization of a curve C traversed counterclockwise and the velocity vector is: $\mathbf{v}(t) = \langle x'(t), y'(t) \rangle$ then $\mathbf{n}(t) = \mathbf{T} \times \mathbf{k} = \langle y'(t), -x'(t) \rangle$ is outward unit normal vector for a smooth curve C in the xy −plane. We rotate $\mathbf{T}$ clockwise 90 degrees to obtain $\mathbf{n}$. We say that $\mathbf{n}$ is the **normal component** of the field $\mathbf{F}$.

Two Forms for Green's Theorem

In second form, Green's Theorem says that the **counterclockwise circulation** of a vector field around a simple closed curve is the double integral of the $\mathbf{k}$ −component of the curl of the field over the region enclosed by the curve.

Green's Theorem (Circulation-Curl or Tangential Form)

The counterclockwise circulation of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ around a simple closed curve C in the plane equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M\, dx + N\, dy}_{\text{Counterclockwise circulation}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy}_{\text{Curl integral}}$$

$$= \iint_R (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dA.$$

Note:

The two forms of Green's Theorem are equivalent. Applying first form to the field $\mathbf{G}_1 = \langle N, -M \rangle$, gives second form, and applying second form to $\mathbf{G}_2 = \langle -N, M \rangle$ provides us with the first form.

Example: Finding Outward Flux

Calculate the outward flux of the field $\mathbf{F}(x, y) = \langle x, y^2 \rangle$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

Solution:

With $M = x, N = y^2$, C the square and R the square's interior we have:

$$\text{Flux} = \oint_C \mathbf{F}.\mathbf{n}\, ds = \oint_C x\, dy - y^2\, dx = \iint_R \left(M_x + N_y\right) dxdy = \int_{-1}^{1}\int_{-1}^{1} (1 + 2y)\, dxdy = 4.$$

Example:

Evaluate

$$\oint_C y^3\,dx - x^3 dy$$

where C is the positively oriented circle of radius 2 centered at the origin.

Solution:

With $M = y^3, N = -x^3$, C the circle of radius 2 centered at the origin and R is the disk of radius 2 centered at the origin, we have:

$$\oint_C y^3\,dx - x^3 dy = \iint_R \left(N_x - M_y\right) dxdy = \iint_R (-3x^2 - 3y^2)\,dxdy$$

Since R is a disk, it seems like the best way to solve this integral is to use polar coordinates. Thus, we have

$$\oint_C y^3\,dx - x^3 dy = -3\iint_R (x^2 + y^2)\,dxdy = -3\int_0^{2\pi}\int_0^2 (r^2)\,r dr d\theta = -24\pi.$$

Application of Green's theorem: An area formula

Typically, we use **Green’s theorem** as an alternative way to calculate a line integral:

$$\oint_C \mathbf{F}.\,ds\,,$$

Where $\mathbf{F} = \langle M, N \rangle$. Instead of calculating the above line integral directly, we calculate the double integral:

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

We can determine the area of a region R with the following double integral.

$$\text{Area of } R = A = \iint_R dA.$$

Let’s think of this double integral as the result of using Green’s Theorem. In other words, let’s assume that:

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

Application of Green's theorem: An area formula

There are several possibilities:

$M = 0$	$M = -y$	$M = -\dfrac{y}{2}$
$N = x$	$N = 0$	$N = \dfrac{x}{2}$

Then Green's Theorem gives the following formulas for the area of region R:

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x dy - y dx\,.$$

where C is the boundary of the region R.

Example:

Find the area enclosed by the ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Solution:

The ellipse has parametric equations:

$$x = a\cos t \quad \text{and} \quad y = b\sin t\,,$$

where $0 \le t \le 2\pi$. Using the area formula, we have:

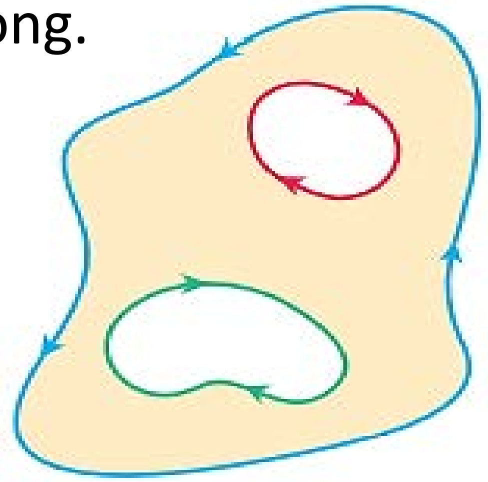
$$A = \frac{1}{2}\oint_C xdy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)dt - (b\sin t)(-a\sin t)dt$$

$$= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab.$$

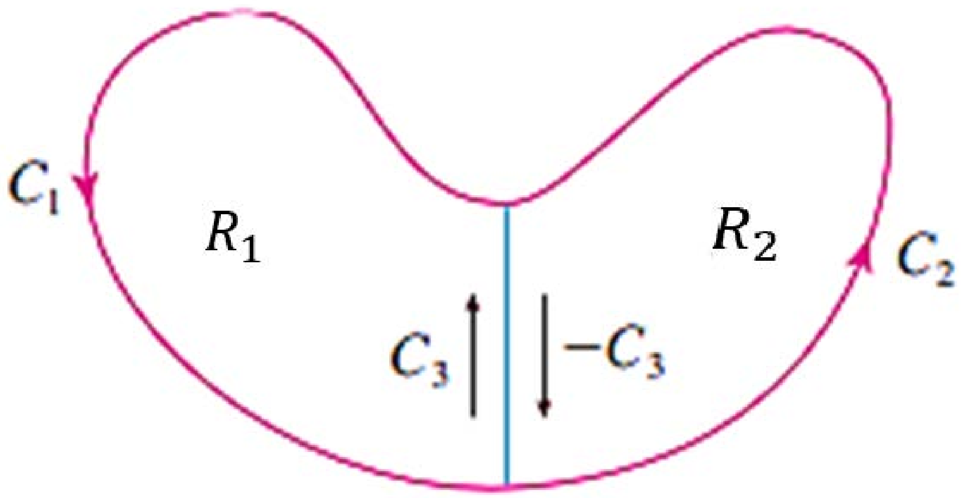
Remarks

- Green's theorem works **only** for the case where C is a **simple closed curve**. If C is an open curve, please *don't even think* about using Green's theorem.
- Green's theorem can be used only for vector fields in two dimensions. It **cannot** be used for vector fields in three dimensions, even if C is a closed path.
- We can extend Green's theorem to the case where the region is a **finite union of nonoverlapping simple regions**.
- Green's Theorem can also be extended to apply to regions with **holes**, that is, regions that are not simply-connected (or multiply-connected), as long as the bounding curves are smooth, simple, and closed and we integrate over each component of the boundary in the direction that keeps the region on our immediate left as we go along.



Green's Theorem for finite union of nonoverlapping simple regions

So far, we have seen Green's Theorem only for the case where the region R is simple. So, Green's theorem, as stated, will not work on regions that have holes in them. However, many regions do have holes in them. So, let's see how we can deal with those kinds of regions. We start by considering the regions which do not have holes but the region R is a finite union of simple regions. The arguments that we're going to go through will be similar to those that we need for regions with holes in them, except it will be a little easier to deal with and write down. Let us consider the region shown in the accompanying figure. We can write $R = R_1 \cup R_2$, where R_1 and R_2 are both simple. The boundary of R_1 is $C_1 \cup C_3$ and the boundary of R_2 is $C_2 \cup (-C_3)$.



Green's Theorem for finite union of nonoverlapping simple regions

So, applying Green's Theorem to R_1 and R_2 separately, we get:

$$\oint_{C_1 \cup C_3} [Mdx + Ndy] = \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA, \qquad \text{and} \qquad \oint_{C_2 \cup (-C_3)} [Mdx + Ndy] = \iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

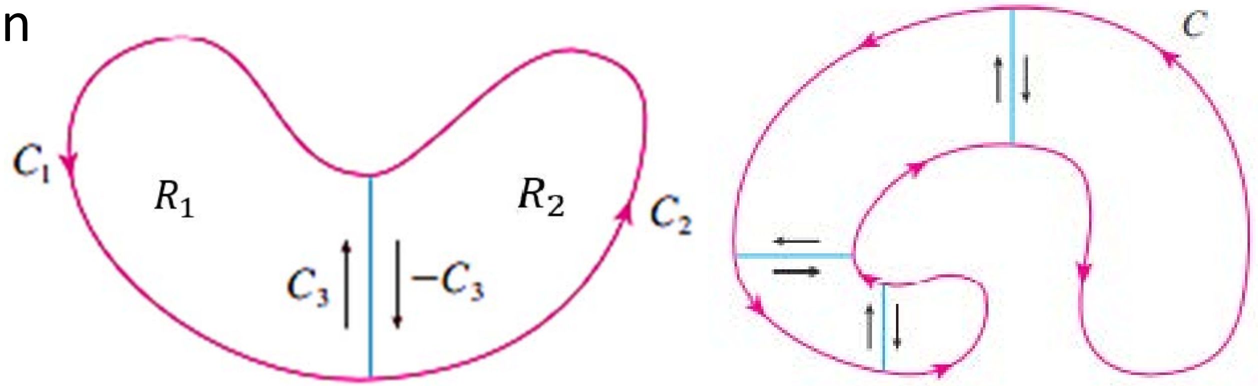
Addition of these two equations results in cancelation of the line integrals along C_3 and $-C_3$ and we get:

$$\oint_{C} [Mdx + Ndy] = \iint_{R} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA,$$

which is the Green's theorem for the region $R = R_1 \cup R_2$, with boundary:

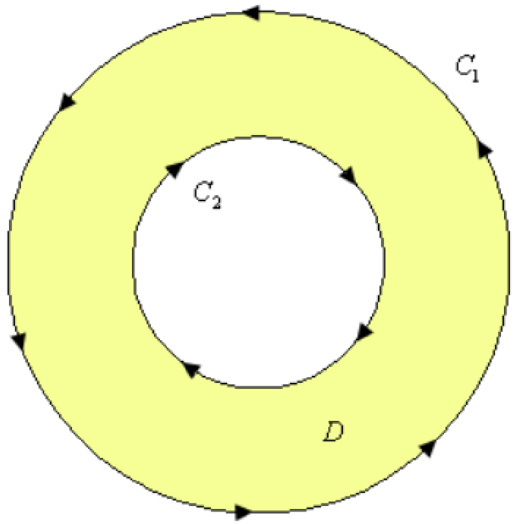
$$C = [C_1 \cup C_3] \cup [C_2 \cup (-C_3)] = C_1 \cup C_2.$$

The same sort of argument allows us to establish Green's Theorem for any finite union of non-overlapping simple regions. This idea will help us in dealing with regions that have holes in them.



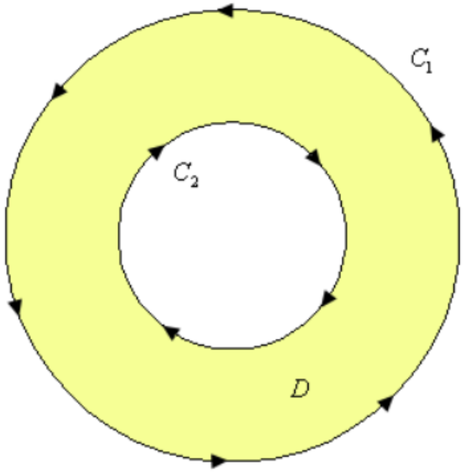
Green's Theorem for Multiply Connected Domains

- Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected.
- For this let us consider a ring as shown in figure below. Observe that the boundary of the region of ring consists of two simple closed curves C_1 and C_2.
- We assume that these boundary curves are oriented so that the region is always on the left as the curve is traversed.
- Thus, the positive direction is counterclockwise for the outer curve but clockwise for the inner curve.
- Note that the curve C_2 seems to violate the original definition of positive orientation. We originally said that a curve had a positive orientation if it was traversed in a counter-clockwise direction. However, this was only for regions that do not have holes. For the boundary of the hole this definition won't work, and we need to resort to the second definition that as we traverse the path following the positive orientation the region R must always be on the left.



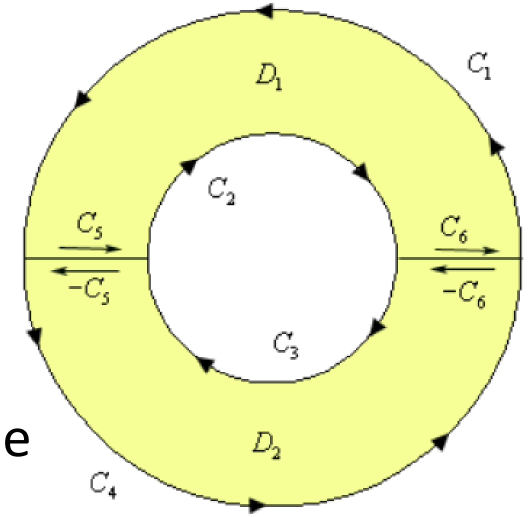
Green’s Theorem for Multiply Connected Domains

- Now, since this region has a hole in it, we will apparently not be able to use Green’s theorem on any line integral with the curve $C = C_1 \cup C_2$.
- However, if we cut the disk in half and rename all the various portions of the curves, we get the sketch in the accompanying figure.
- The boundary of the upper portion D_1 of the disk is: $C_1 \cup C_2 \cup C_5 \cup C_6$ and the boundary on the lower portion D_2 of the disk is: $C_3 \cup C_4 \cup (-C_5) \cup (-C_6)$.
- Note that we can use Green’s theorem on each of these new regions since they don’t have any holes in them. This means that we can do the following:



$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_{D_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA + \iint_{D_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \oint_{C_1 \cup C_2 \cup C_5 \cup C_6} [M dx + N dy] + \oint_{C_3 \cup C_4 \cup (-C_5) \cup (-C_6)} [M dx + N dy]$$



- Now, we can break up the line integrals into line integrals on each piece of the boundary.

Green's Theorem for Multiply Connected Domains

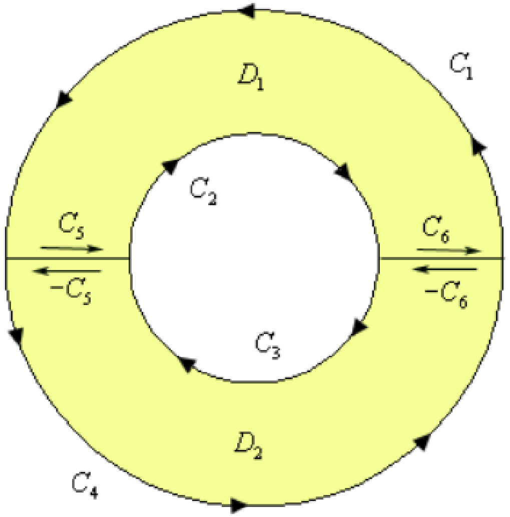
- Note that that boundaries that have the same curve, but opposite direction will cancel. Thus, we get:

$$\iint\limits_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint\limits_{C_1 \cup C_2 \cup C_5 \cup C_6} [Mdx + Ndy] + \oint\limits_{C_3 \cup C_4 \cup (-C_5) \cup (-C_6)} [Mdx + Ndy]$$

$$= \int\limits_{C_1} [Mdx + Ndy] + \int\limits_{C_2} [Mdx + Ndy] + \int\limits_{C_3} [Mdx + Ndy] + \int\limits_{C_4} [Mdx + Ndy].$$

- At this point we can add the line integrals back up as follows:

$$\iint\limits_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint\limits_{C_1 \cup C_2 \cup C_3 \cup C_4} [Mdx + Ndy] = \oint\limits_{C} [Mdx + Ndy].$$



- The conclusion of all of this is that we could have just used Green's theorem on the disk from the start even though there is a hole in it. This will be true in general for regions that have holes in them.

Example:

Evaluate

$$\oint_C y^3\,dx - x^3 dy$$

where C is the union of two circles of radius 2 and radius 1 centered at the origin with positive orientation.

Solution:

For the present case, the region D is the region between these two circles of radius 2 and radius 1 centered at the origin. With $M = y^3$ and $N = -x^3$, we have:

$$\oint_C y^3\,dx - x^3 dy = -3\iint_R (x^2 + y^2)\,dxdy = -3\int_0^{2\pi}\int_1^2 (r^2)\,rdrd\theta = -\frac{45\pi}{2}.$$

Example:

If $\mathbf{F}(x,y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$, show that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi,$$

for every positively oriented simple closed path that encloses the origin

Solution:

Since C is an arbitrary closed path that encloses the origin, it's difficult to compute the given integral directly. So, let's consider a counterclockwise-oriented circle C' with center the origin and radius a, where a is chosen to be small enough that lies inside C'. Let D be the region bounded by C and C'. Then its positively oriented boundary is $C \cup (-C')$ and so the general version of Green's Theorem gives:

$$\oint_C [Mdx + Ndy] + \oint_{-C'} [Mdx + Ndy] = \oint_C [Mdx + Ndy] - \oint_{C'} [Mdx + Ndy] =$$

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_D \left(\frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2} \right) dA = 0$$

Solution:

Therefore:

$$\oint_C [Mdx + Ndy] = \oint_{C'} [Mdx + Ndy] \Longrightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C'} \mathbf{F} \cdot d\mathbf{r}\,.$$

We now easily compute this last integral using the parametrization given by:

$$\mathbf{r}(t) = \langle \mathrm{a}\cos t\,, a\sin t \rangle; \qquad 0 \le t \le 2\pi.$$

Thus,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} dt = 2\pi.$$

Practice Questions

Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.

Chapter: 16

Exercise-16.4: Q – 1 to 14, Q – 17 to 19.

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 16

Exercise-16.4: Q – 1 to 38.