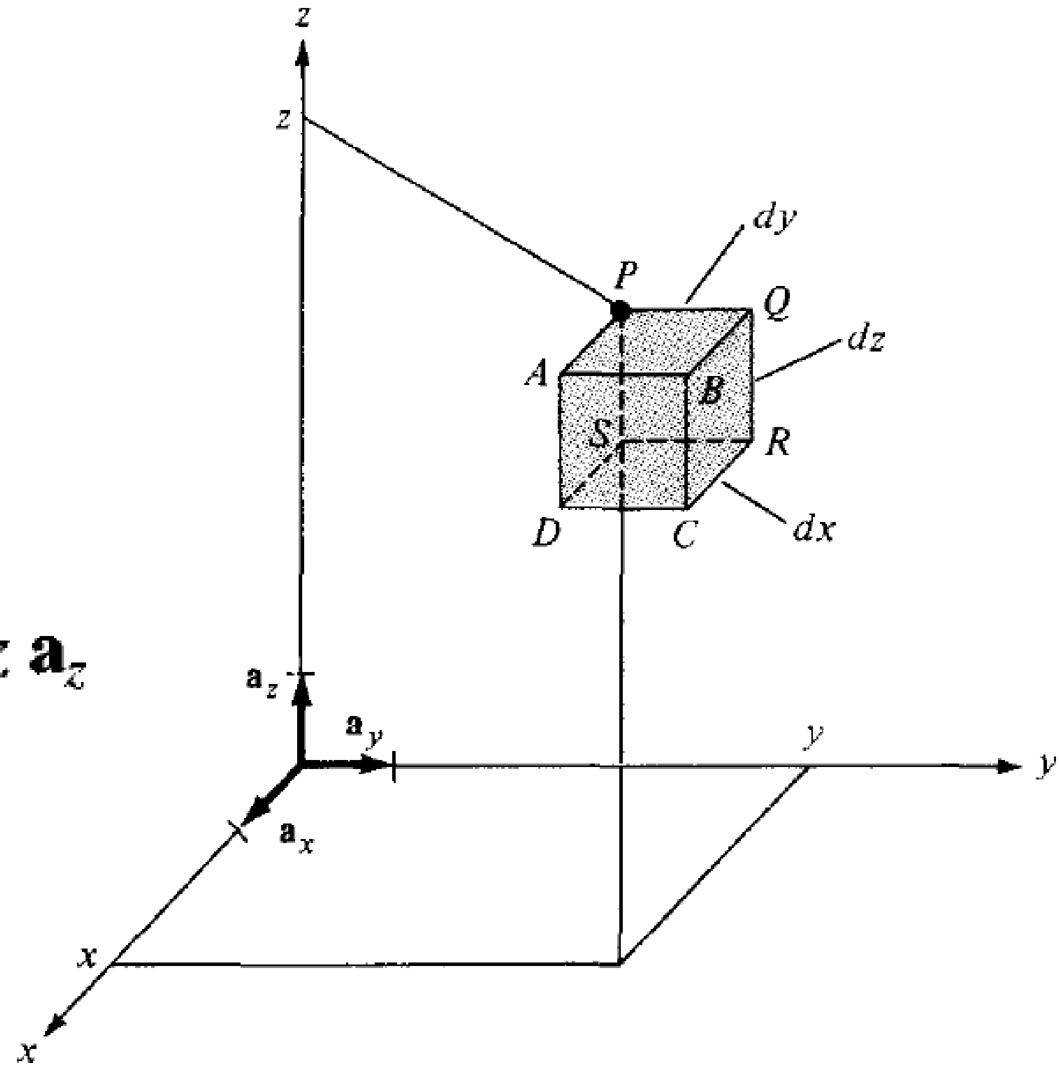
# VECTOR CALCULUS I

# Differential Elements – Cartesian Coordinates

- Differential elements in length, area, and volume are useful in vector calculus

- Differential displacement or length is given by:

$$d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$$



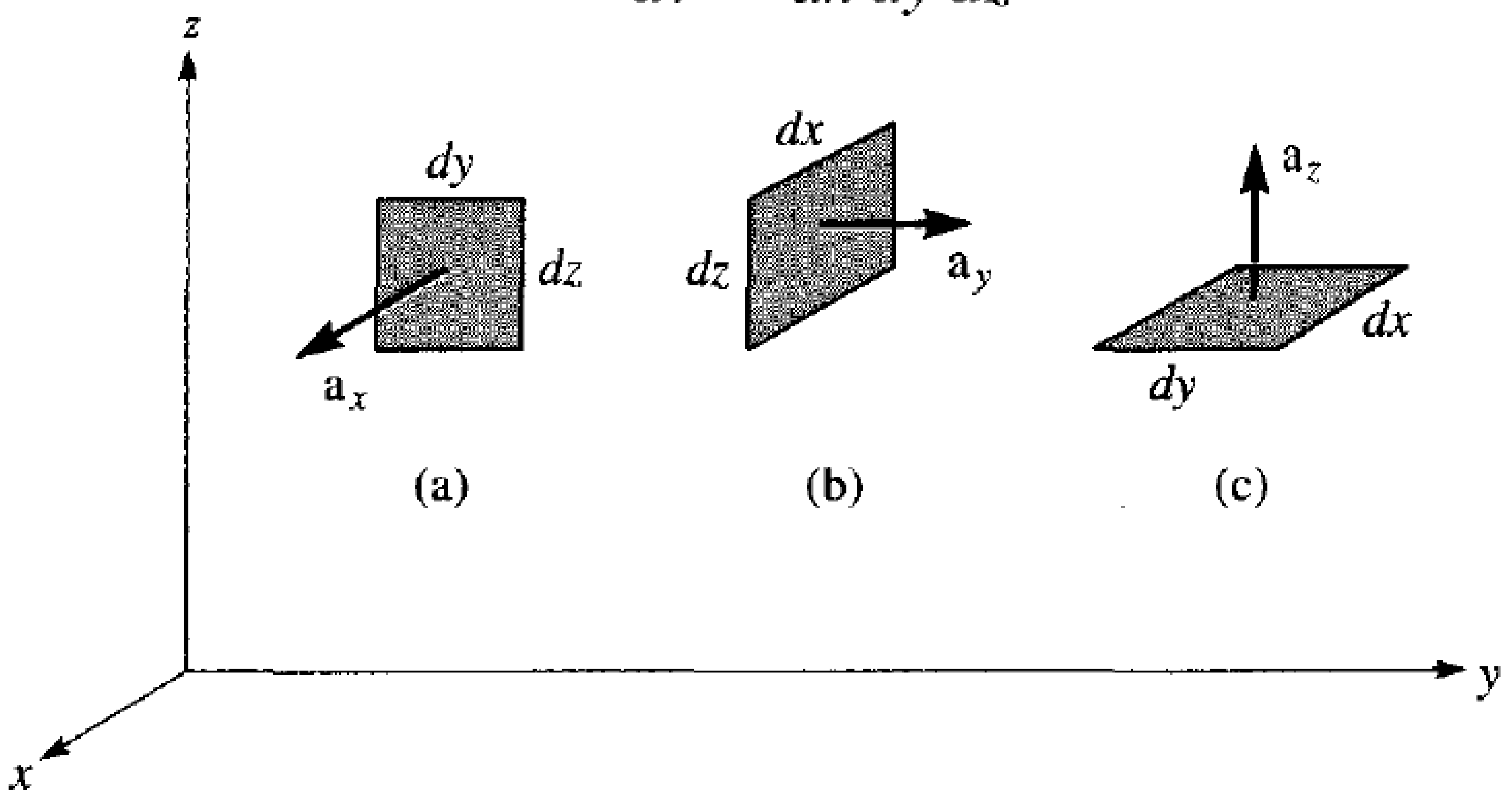
# Differential Elements – Cartesian Coordinates

- Differential normal areas are given by:

$$
\begin{aligned}
d\mathbf{S} &= dy\,dz\,\mathbf{a}_x \\
&\quad dx\,dz\,\mathbf{a}_y \\
&\quad dz\,dy\,\mathbf{a}_z
\end{aligned}
$$

- Differential volume is given by:

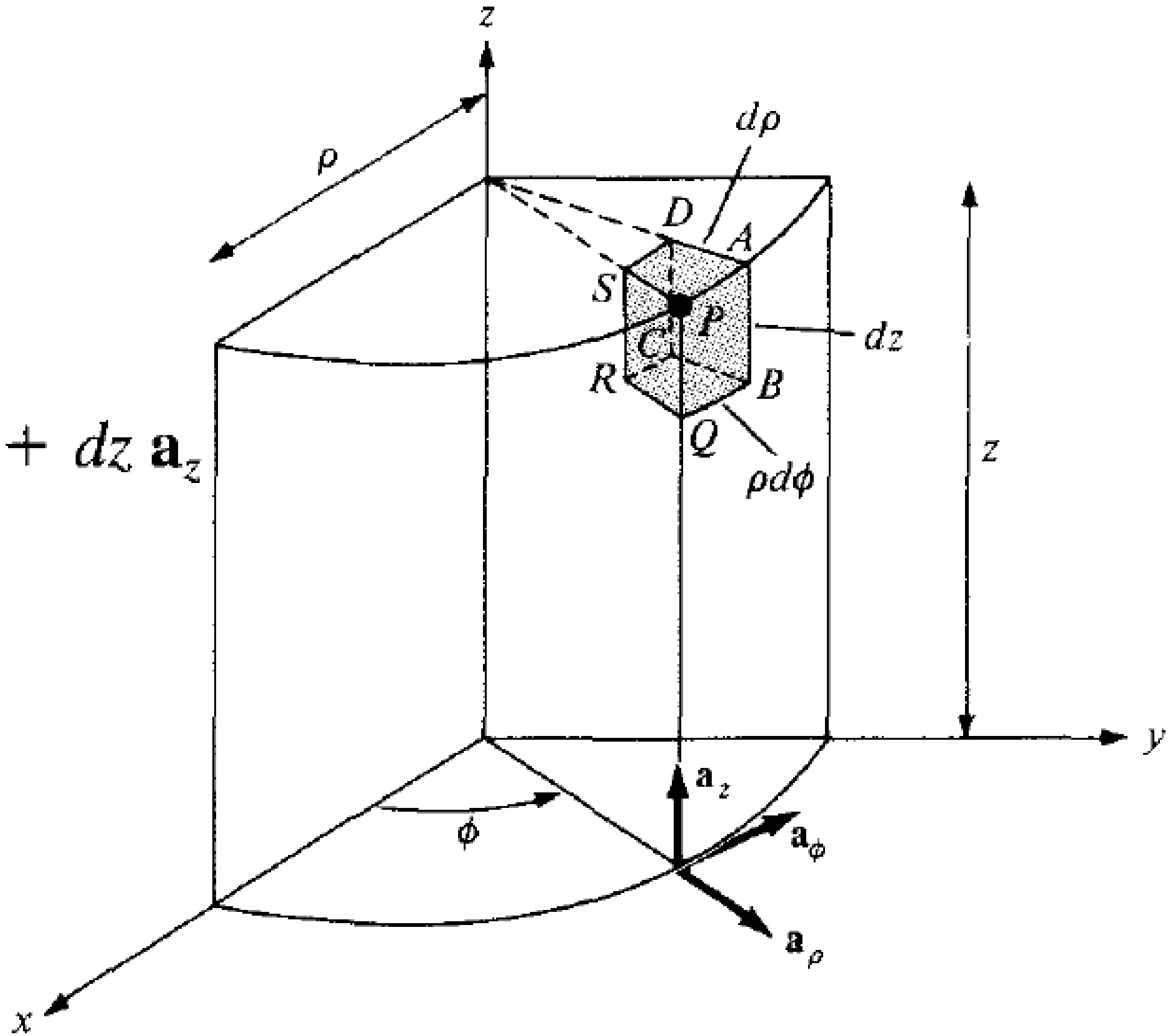
$$dv = dx\,dy\,dz$$



# Differential Elements –Cylindrical Coordinates

- Differential displacement or length is given as:

$$d\mathbf{l} = d\rho\, \mathbf{a}_\rho + \rho\, d\phi\, \mathbf{a}_\phi + dz\, \mathbf{a}_z$$



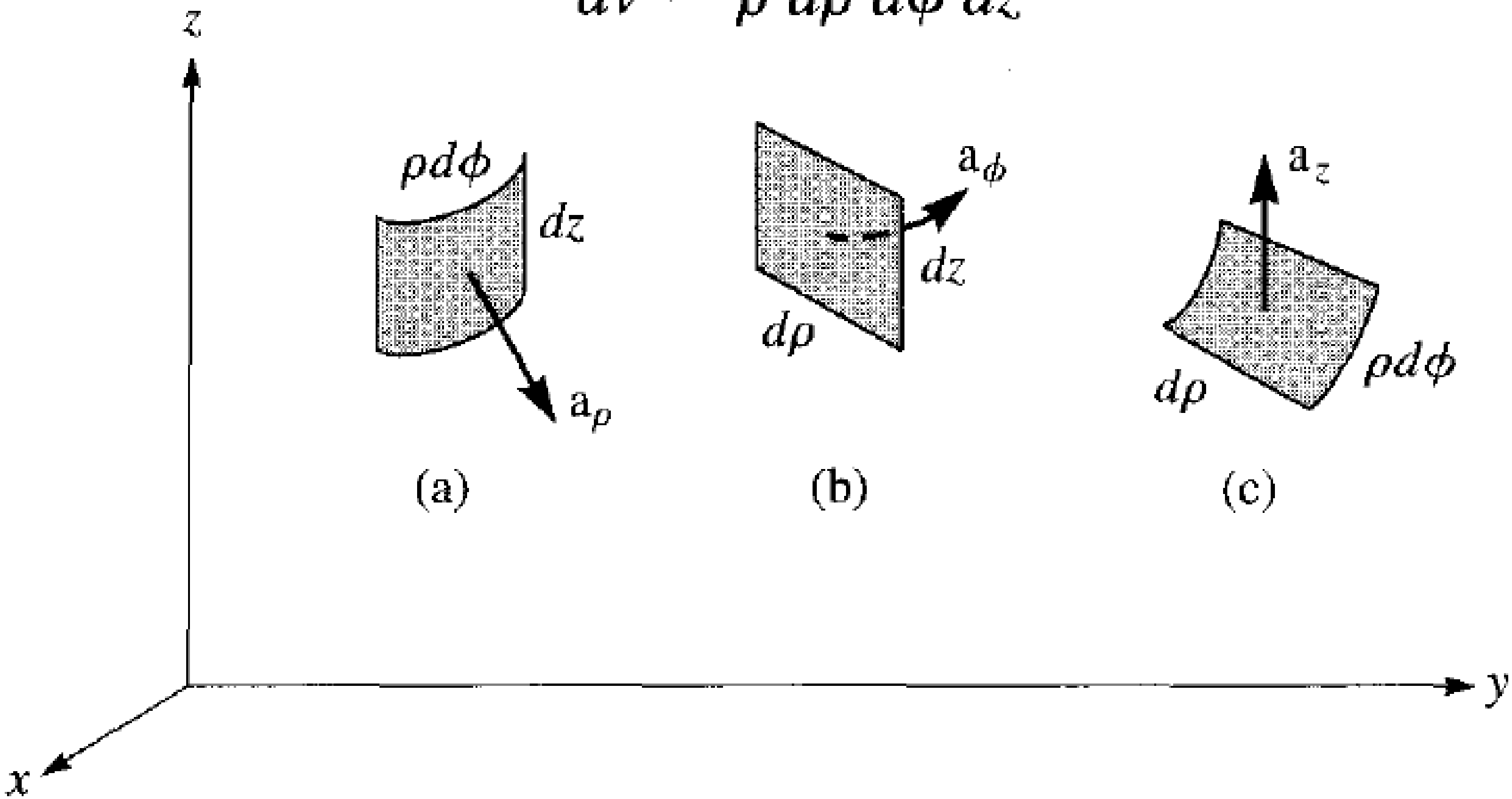
# Differential Elements –Cylindrical Coordinates

- Differential normal areas are given as:

$$d\mathbf{S} = \rho\, d\phi\, dz\, \mathbf{a}_\rho$$
$$d\rho\, dz\, \mathbf{a}_\phi$$
$$\rho\, d\phi\, d\rho\, \mathbf{a}_z$$

- Differential volume is given as:

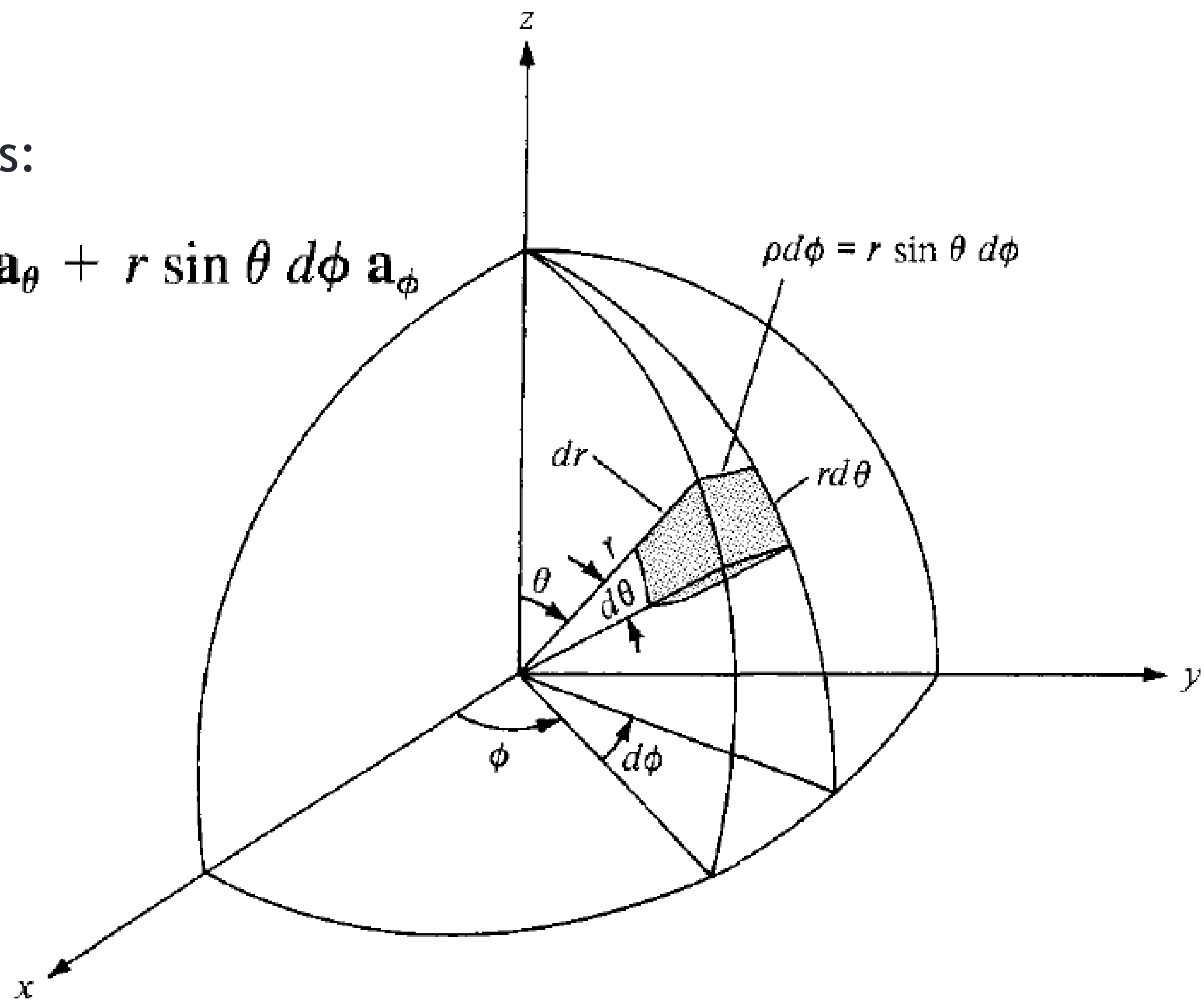
$$dv = \rho\, d\rho\, d\phi\, dz$$



# Differential Elements – Spherical Coordinates

- Differential displacement or length is given as:

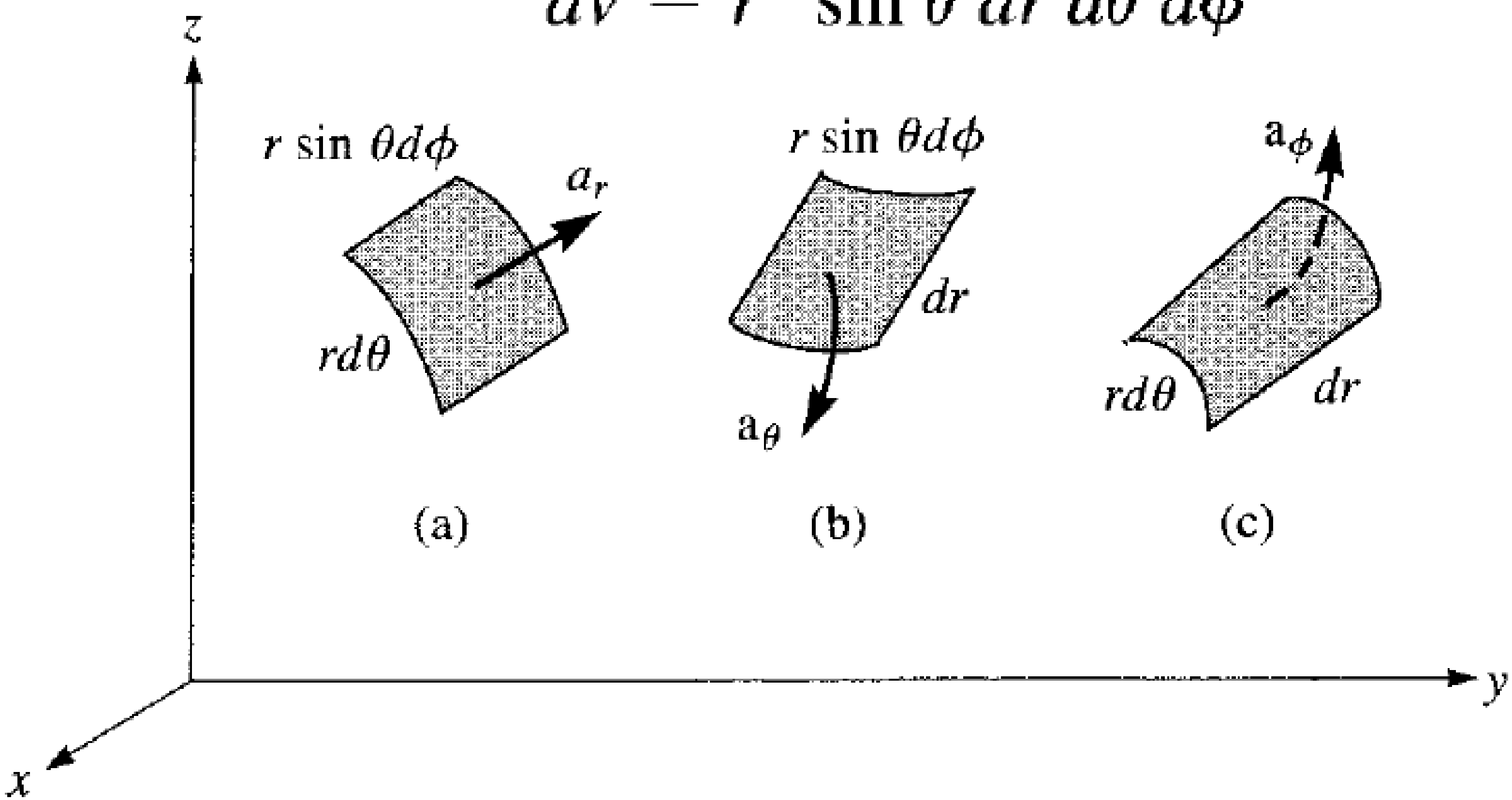
$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$$



# Differential Elements – Spherical Coordinates

- Differential normal areas are:
$$dS = r^2 \sin\theta \, d\theta \, d\phi \, \mathbf{a}_r$$
$$r \sin\theta \, dr \, d\phi \, \mathbf{a}_\theta$$
$$r \, dr \, d\theta \, \mathbf{a}_\phi$$

- Differential volume is:
$$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$



# Differential Elements

- These differential elements are very important as they will be referred to again and again throughout the course

- The student is encouraged to learn to derive them from the figures

# Line Integral

- Concept of integration will now be extended to cases when the integrand involves a vector

- By a line we mean the **path in space**

- The **line integral of A** along curve *L* is the integral of the tangential component of **A** along curve *L*:

$$\int_L \mathbf{A} \cdot d\mathbf{l} = \int_a^b |\mathbf{A}| \cos\theta \, dl$$

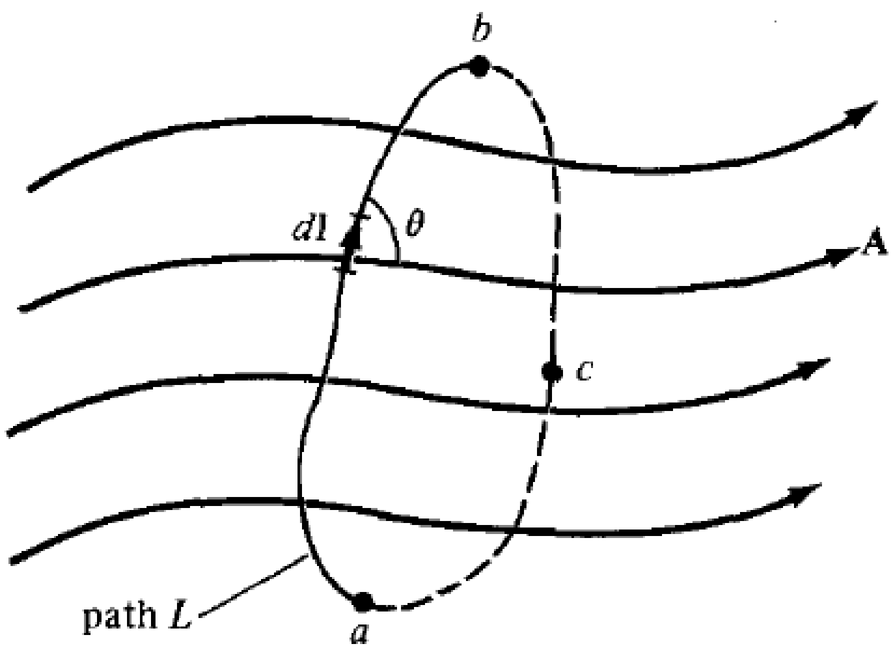
- Also called line integral of **A** around *L (shown in figure on next slide)*

# Line Integral

- If the path of integration is a closed curve – the line integral becomes a closed contour integral

$$\oint_L \mathbf{A} \cdot d\mathbf{l}$$

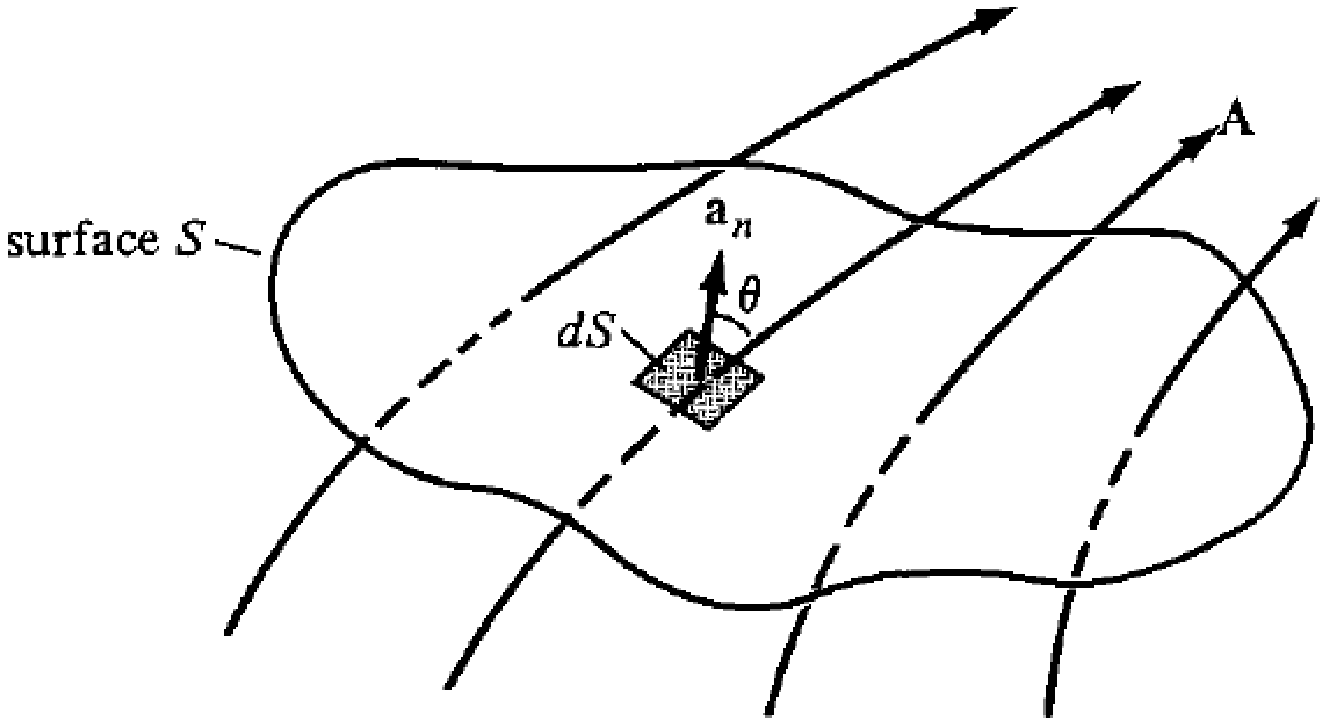
- This is called the circulation of A around *L*



# Surface Integral

➢Given a vector field **A**, continuous in a region containing the smooth surface *S*, the surface integral or the flux of **A** through *S* is:

$$\Psi = \int_S |\mathbf{A}| \cos\theta \, dS = \int_S \mathbf{A} \cdot \mathbf{a}_n \, dS = \int_S \mathbf{A} \cdot d\mathbf{S}$$



# Volume Integral

- A closed line path defines an open surface whereas a closed surface defines a volume

- If the scalar $\rho_v$ is the volume density of a certain quantity, then the volume integral of $\rho_v$ over the volume $v$ is:

$$\int_v \rho_v \, dv$$

- The physical meaning of a line, surface, or volume integral depends on the nature of the physical quantity represented by **A** or $\rho_v$

- For example, line integral of an electric field around a closed loop is equal to the voltage generated in that loop

# DEL Operator

- The del operator, written as $\nabla$, is a vector differential operator
- In Cartesian coordinates:

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

- This vector differential operator, otherwise known as the gradient operator, is not a vector in itself
- But when it operates on a scalar function, it results in a vector
- The del operator will be used in defining different quantities in subsequent sections

# DEL Operator – Cylindrical Coordinate System

- We have:

$$\rho = \sqrt{x^2 + y^2}, \qquad \tan\phi = \frac{y}{x}$$

- Hence:

$$\frac{\partial}{\partial x} = \cos\phi \frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \sin\phi \frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}$$

- By substitution:

$$\nabla = \mathbf{a}_\rho \frac{\partial}{\partial \rho} + \mathbf{a}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \mathbf{a}_z \frac{\partial}{\partial z}$$

# DEL Operator – Spherical Coordinate System

- We have:

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \tan\theta = \frac{\sqrt{x^2 + y^2}}{z}, \qquad \tan\phi = \frac{y}{x}$$

- Hence:

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}$$

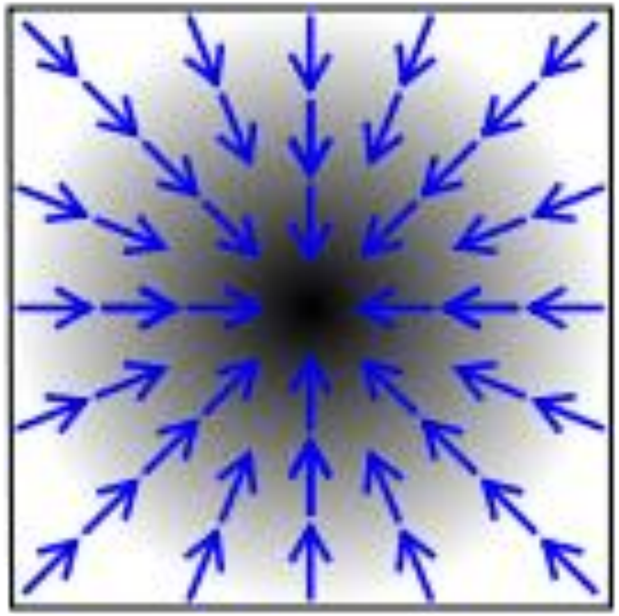
$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

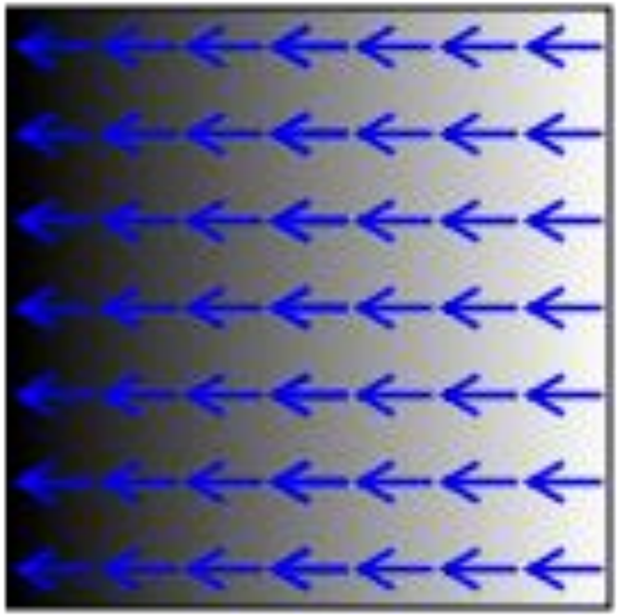
- By substitution:

$$\nabla = \mathbf{a}_r\frac{\partial}{\partial r} + \mathbf{a}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{a}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

# Gradient of a Scalar

- The gradient of a scalar field $V$ is a vector that represents both the magnitude and the direction of the **maximum** space rate of increase of $V$





# Gradient of a Scalar

- In Cartesian coordinates, we have:

$$\text{grad } V = \nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

- For Cylindrical coordinates:

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z$$

- For Spherical coordinates:

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r \sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

# Fundamental Properties of $\nabla V$

- Magnitude of $\nabla V$ equals the maximum rate of change in $V$ per unit distance

- $\nabla V$ points in the direction of the maximum rate of change in $V$

- $\nabla V$ at any point is perpendicular to the constant $V$ surface that passes through that point

- The projection (or component) of $\nabla V$ in the direction of a unit vector **a** is $\nabla V \bullet \mathbf{a}$ and is called the directional derivative of $V$ along **a** - Rate of change of $V$ in the direction of **a**

- If **A** = $\nabla V$, $V$ is said to be the scalar potential of **A**

# Problem-1

- The surfaces $\mathrm{r} = 2, r = 4, \theta = 30°, \theta = 50°, \emptyset = 20°, and\ \emptyset = 60°$ define a closed surface as shown below. Find the area BCGF.