Solution Practice Problems

Lecture #34.

Q1:- Solve the initial-value problem

$$y^2 u_x^2 + x^2 u_y^2 = (xyu)^2 \quad ; \quad u(x,0) = 3e^{(x^2/4)}$$

Solution: Consider

$$u(x,y) = f(x)\,g(y) \neq 0$$

Using this solution in given PDE we get

$$y^2[f'(x)g(y)]^2 + x^2[f(x)\,g'(y)]^2 = [xy\,f(x)\,g(x)]^2$$

$$\Rightarrow y^2[f'(x)]^2 g^2 + x^2 f^2 [g'(y)]^2 = x^2 y^2 f^2 g^2$$

$$\Rightarrow \frac{\cancel{y^2}[f'(x)]^2 \cancel{g^2}}{x^2 \cancel{y^2} f^2 \cancel{g^2}} + \frac{\cancel{x^2}\cancel{f^2}[g'(y)]^2}{\cancel{x^2} y^2 \cancel{f^2} g^2} = 1$$

$$\Rightarrow \frac{1}{x^2}\left[\frac{f'(x)}{f(x)}\right]^2 + \frac{1}{y^2}\left[\frac{g'(y)}{g(y)}\right]^2 = 1$$

or

$$\frac{1}{x^2}\left[\frac{f'(x)}{f(x)}\right]^2 = 1 - \frac{1}{y^2}\left[\frac{g'(y)}{g(y)}\right]^2 = \lambda^2$$

where λ^2 is separation constant. Thus,

$$\frac{1}{x}\left[\frac{f'(x)}{f(x)}\right] = \lambda \quad \text{and} \quad 1 - \frac{1}{y^2}\left[\frac{g'(y)}{g(y)}\right]^2 = \lambda^2$$

$$\Rightarrow f'(x) = x\lambda f(x) \to ① \quad \text{and} \quad 1-\lambda^2 = \frac{1}{y^2}\left[\frac{g'(y)}{g(y)}\right]^2$$

$$\Rightarrow \frac{g'(y)}{y\,g(y)} = \sqrt{1-\lambda^2}$$

$$\Rightarrow g'(y) = y\,g(y)\sqrt{1-\lambda^2} \to ②$$

From ①

$$f'(x) = x\lambda f(x)$$

$$\Rightarrow \frac{df(x)}{dx} = x\lambda f(x) \Rightarrow \frac{d\,f(x)}{f(x)} = x\lambda\,dx$$

$$\Rightarrow \ln|f(x)| = \frac{\lambda x^2}{2} + \ln|A|$$

$\Rightarrow \ln|f(x)| = \alpha \frac{x^2}{2} \ln e + \ln|A|$

$\Rightarrow \ln|f(x)| = \ln\left|A e^{\frac{\alpha x^2}{2}}\right|$

$\Rightarrow f(x) = A e^{\alpha x^2/2} \longrightarrow (3)$ (A is arbitrary constant)

From (2)

$g'(y) = y\, g(y) \sqrt{1-\alpha^2}$

$\Rightarrow \frac{dg(y)}{dy} = y\, g(y) \sqrt{1-\alpha^2}$

$\Rightarrow \frac{dg(y)}{g(y)} = y\sqrt{1-\alpha^2}\, dy$

$\Rightarrow \ln|g(y)| = \frac{y^2}{2}\sqrt{1-\alpha^2} + \ln|B|$

$\Rightarrow \ln|g(y)| = \frac{y^2}{2}\sqrt{1-\alpha^2} \ln e + \ln|B|$

$\Rightarrow g(y) = B e^{\frac{y^2\sqrt{1-\alpha^2}}{2}} \longrightarrow (4)$ (B is arbitrary constant)

From (3) & (4)

$u(x,y) = C \exp(\alpha x^2/2) \exp(y^2\sqrt{1-\alpha^2}/2)$

where $C = AB$

$\Rightarrow u(x,y) = C \exp\left[\frac{\alpha x^2}{2} + \frac{y^2}{2}\sqrt{1-\alpha^2}\right] \longrightarrow (5)$

Given that $u(x,0) = 3e^{(x^2/4)}$

Using this initial condition in (5) we get

$3\exp\left[\frac{x^2}{4}\right] = C \exp\left[\frac{\alpha x^2}{2} + 0\right]$

$\Rightarrow 3\exp\left[\frac{x^2}{4}\right] = C\exp\left[\frac{\alpha x^2}{2}\right]$

Comparison of both sides yield

$C = 3$ and $\alpha = 1/2$.

Thus solution (5) takes the form

$u(x,y) = 3\exp\left[\frac{x^2}{4} + \frac{y^2}{2}\sqrt{1-\frac{1}{4}}\right]$

$\Rightarrow u(x,y) = 3\exp\left[\frac{1}{4}\left(x^2 + y^2\sqrt{3}\right)\right]$

Q2:- Use the separation of variables $u(x,y) = f(x) + g(y)$ to solve the equation:

$$u_x^2 + u_y^2 = 1.$$

Sol:- For the present case:

$$u(x,y) = f(x) + g(y)$$

$$\Rightarrow u_x = f'(x) \quad \text{and} \quad g'(y) = u_y.$$

Thus,

$$u_x^2 + u_y^2 = [f'(x)]^2 + [g'(y)]^2 = 1$$

$$\Rightarrow [f'(x)]^2 = [1 - [g'(y)]^2] = \lambda^2$$

where λ^2 is a separation constant. Thus, we obtain

$$f'(x) = \lambda \quad \text{and} \quad 1 - [g'(y)]^2 = \lambda^2 \Rightarrow [g'(y)]^2 = 1 - \lambda^2$$

$$\Rightarrow f(x) = \lambda x + A \quad \text{and} \quad g(y) = y\sqrt{1-\lambda^2} + B$$

where A and B are arbitrary constants. Thus, the solution of the given PDE is:

$$u(x,y) = \lambda x + y\sqrt{1-\lambda^2} + C,$$

where $C = A + B$ is an arbitrary constant.

Q3:- Use the separation of variables $u(x,y) = f(x) + g(y)$ to solve the equation.

$$u_x^2 + u_y + x^2 = 0$$

Sol.- $u(x,y) = f(x) + g(y)$

$$\Rightarrow u_x = f'(x) \quad \text{and} \quad u_y = g'(y)$$

Thus,

$u_x^2 + u_y + x^2 = 0$ takes the form

$$[f'(x)]^2 + g'(y) + x^2 = 0 \Rightarrow [f'(x)]^2 + x^2 = -g'(y) = \lambda^2$$

where λ^2 is the separation constant. Thus,

$$[f'(x)]^2 + x^2 = \lambda^2 \quad \text{and} \quad -g'(y) = \lambda^2.$$

Now

$$[f'(x)]^2 + x^2 = \lambda^2$$

$$\Rightarrow f'(x) = \sqrt{\lambda^2 - x^2} \Rightarrow f(x) = \int \sqrt{\lambda^2 - x^2}\, dx + A.$$

Let $x = \alpha \sin t \Rightarrow dx = \alpha \cos t \, dt$

Thus, $f(x) = \int \sqrt{\alpha^2 - x^2}\, dx + A = \int \sqrt{\alpha^2 - \alpha^2 \sin^2 t}\, (\alpha \cos t)\, dt + A$

$$= \alpha^2 \int [\sqrt{1 - \sin^2 t}\, \cos t]\, dt + A = \alpha^2 \int [\cos^2 t]\, dt + A.$$

$$= \alpha^2 \int \left[\frac{1 + \cos 2t}{2}\right] dt + A$$

$$= \frac{\alpha^2}{2}\left[t + \frac{\sin 2t}{2} + \frac{2A}{\alpha^2}\right]$$

$$= \frac{\alpha^2}{2}\left[\sin^{-1}\left(\frac{x}{\alpha}\right) + \frac{\cancel{2}\sin t \cos t}{\cancel{2}}\right] + A$$

$$= \frac{\alpha^2}{2}\left[\sin^{-1}\left(\frac{x}{\alpha}\right) + \frac{x}{\alpha}\sqrt{1 - \frac{x^2}{\alpha^2}}\right] + A$$

$$= \frac{\alpha^2}{2}\left[\frac{x}{\alpha}\sqrt{\frac{\alpha^2 - x^2}{\alpha^2}} + \sin^{-1}\left(\frac{x}{\alpha}\right)\right] + A$$

$$\Rightarrow f(x) = \frac{x}{2}\sqrt{\alpha^2 - x^2} + \frac{\alpha^2}{2}\sin^{-1}\left(\frac{x}{\alpha}\right) + A.$$

and $g'(y) = -\alpha^2$

$$\Rightarrow g(y) = -\alpha^2 y + B.$$

Thus,

$$u(x,y) = \frac{x}{2}\sqrt{\alpha^2 - x^2} + \frac{\alpha^2}{2}\sin^{-1}\left(\frac{x}{\alpha}\right) - \alpha^2 y + C$$

where $C = A + B$ is an arbitrary constant.