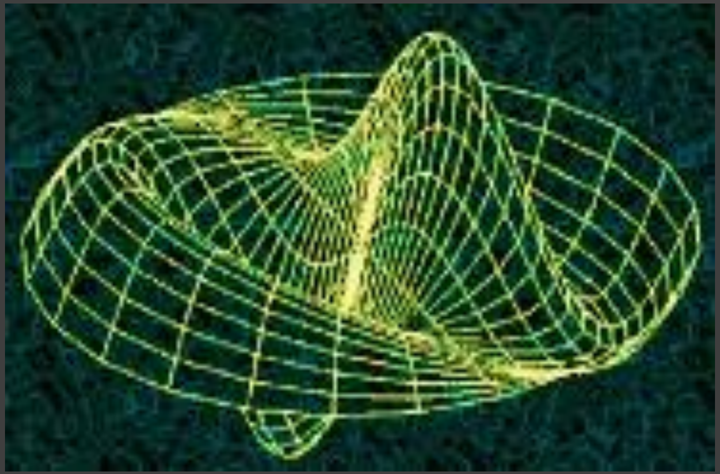
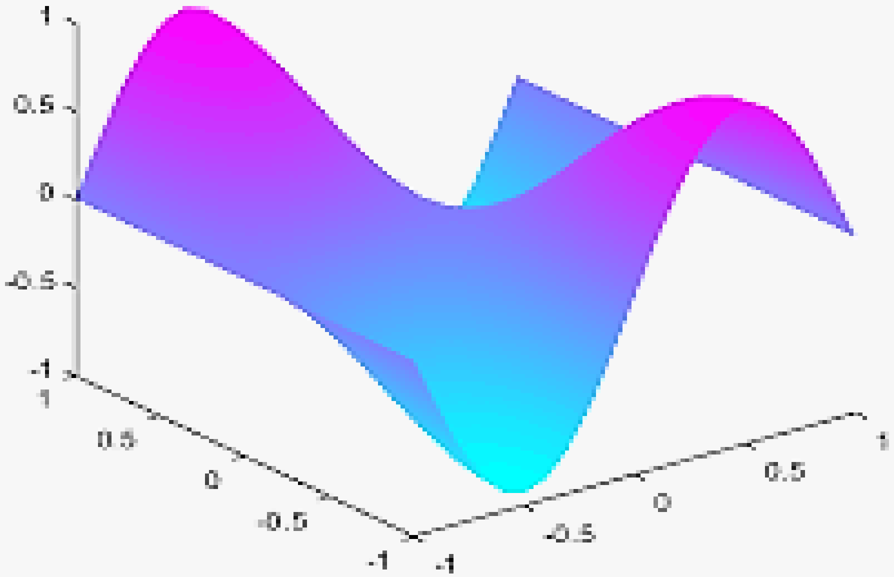
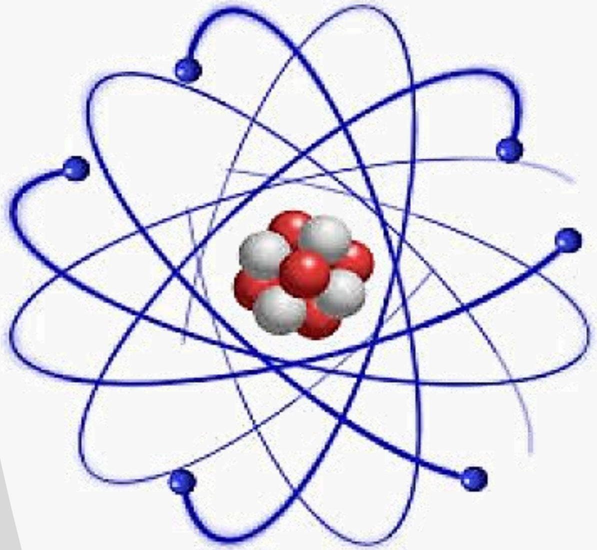
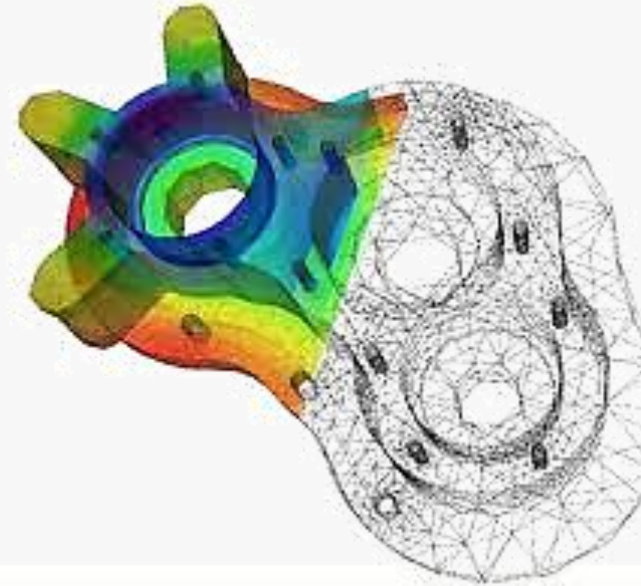


Partial Differential Equations







Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

Partial Differential Equations

Book: Linear Partial Differential Equations for Scientists and Engineers (4^{th} Edition) by Lokenath Debnath

- Chapter: 1, 2
  - Sections: 1.2, 1.3, 1.6, 2.1, 2.7

Book: Advanced Engineering Mathematics (9^{th} Edition) by Ervin Kreyszig

- Chapter: 12
  - Sections: 12.1

Partial Differential Equations

A **partial differential equation** (PDE) is an equation that contains partial derivatives (e.g., $u_x, u_y, u_{xx}, u_{yy}, \ldots$) in it.

For a function of two variables $u = u(x, y)$, a general first-order PDE can be expressed as:

$$F(x, y, u, u_x, u_y) = 0$$

Similarly, a general second-order PDE can be written as

$$F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0$$

In this course we will focus on those PDEs which are of at most second-order.

Example: $\dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial u(x,t)}{\partial t}$

Classification of Linear Second Order PDEs:

- Linear second order PDEs are important sets of equations that are used to model many systems in many different fields of science and engineering.

- Classification is important because:
    - Each category relates to specific engineering problems.
    - Different approaches are used to solve these categories.

Consider the following second order equation in two independent variables:

$$A\frac{\partial^2 u(x,y)}{\partial x^2} + B\frac{\partial^2 u(x,y)}{\partial x \partial y} + C\frac{\partial^2 u(x,y)}{\partial y^2} + D\frac{\partial u(x,y)}{\partial x} + E\frac{\partial u(x,y)}{\partial y} + Fu(x,y) = G \quad (*)$$

where A, B, C, D, E & F are all functions of $x, y, u, \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ only and G is function of x and y only, so that the equation is at most quasi-linear. Note that if G is identically zero, the equation is said to be homogeneous; otherwise, it is non-homogeneous.

Classification of Linear Second Order PDEs:

Equation $(*)$ can be rewritten in more general form as:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + H = 0, \qquad (**)$$

where A, B, C are functions of x and y, while H is a function of x, y, u, u_x and u_y only so that the equation is at most quasi-linear. Then $(*)$ or $(**)$ are classified into three classes of PDEs depending on the discriminant, $B^2 - 4AC$ as:

- **Elliptic** if $B^2 - 4AC < 0$,
- **Parabolic** if $B^2 - 4AC = 0$,
- **Hyperbolic** if $B^2 - 4AC > 0$.

Laplace Equation (Elliptic)

$$\frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} + \frac{\partial^2 u(x,y,z)}{\partial z^2} = 0$$

Heat Equation (Parabolic)

$$\frac{\partial u(x,y,z,t)}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Wave Equation (Hyperbolic)

$$\frac{\partial^2 u(x,y,z,t)}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Important Second-Order PDEs

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ *One-dimensional wave equation*

(2) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ *One-dimensional heat equation*

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ *Two-dimensional Laplace equation*

(4) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$ *Two-dimensional Poisson equation*

(5) $$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$ *Two-dimensional wave equation*

(6) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$ *Three-dimensional Laplace equation*

Here c is a positive constant, t is time, x, y, z are Cartesian coordinates, and *dimension* is the number of these coordinates in the equation.

Solutions of PDEs

- A **solution** of a PDE in some region R of the space of the independent variables is a function that has all the partial derivatives appearing in the PDE in some domain D containing R and satisfies the PDE everywhere in R.

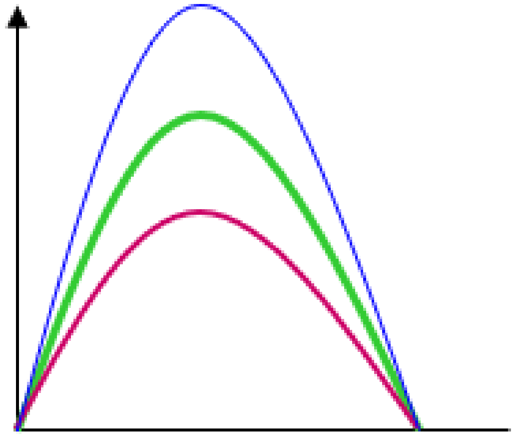
- In other words, the solution $u = u(x, y)$ of a partial differential equation is a function defined in a domain D, of xy −plane that satisfies the equation identically.

- **Note:** All PDEs have infinitely many solutions !!!

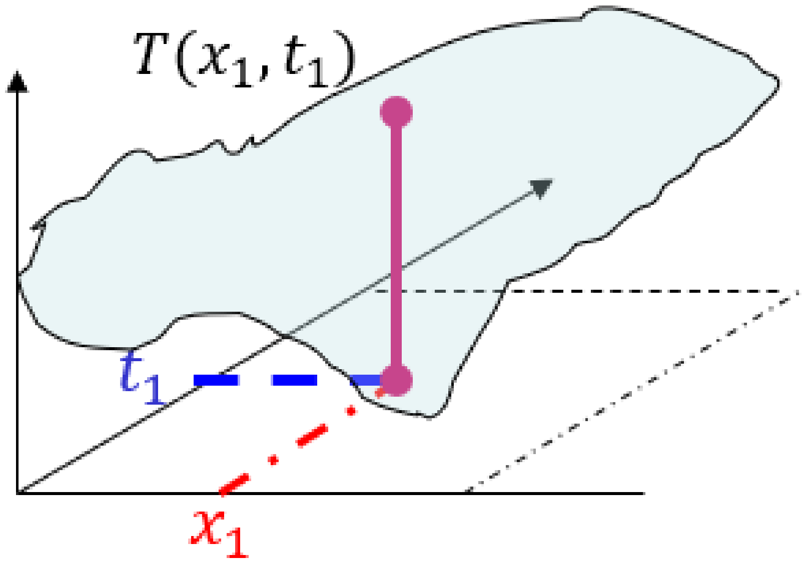
- We normally obtain the solution of a second-order PDE by looking at its type and comparing it with one of the equation (wave, heat or Laplace).

Representing the Solution of a PDE (Two Independent Variables)

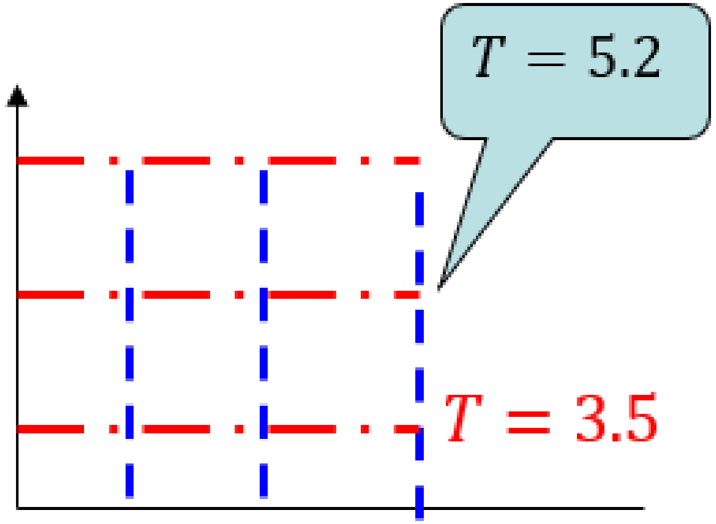
Three main ways to represent the solution



Different curves are used for different values of one of the independent variable



Three dimensional plot of the function $T(x, t)$



The axis represent the independent variables. The value of the function is displayed at grid points

Solutions of PDEs

- The general solution of a linear ordinary differential equation of nth order is a family of functions depending on n independent arbitrary constants.
- In the case of partial differential equations, the general solution depends on arbitrary functions rather than on arbitrary constants.
- To illustrate this, consider the equation

$$u_{xy} = 0.$$

If we integrate this equation with respect to y, we obtain:

$$u_x(x, y) = f(x).$$

A second integration with respect to x yields

$$u(x, y) = g(x) + h(y),$$

where $g(x)$ and $h(y)$ are arbitrary functions.

- Suppose u is a function of three variables, x, y, and z. Then, for the equation:

$$u_{yy} = 2,$$

one can easily find the general solution: $u(x, y, z) = y^2 + yf(x, z) + g(x, z)$, where f and g are arbitrary functions of two variables x and z.

Solutions of PDEs

- Recall that in the case of ODEs, the first task is to find the general solution, and then a particular solution is determined by finding the values of arbitrary constants from the prescribed conditions.
- But, for PDEs, selecting a particular solution satisfying the supplementary conditions from the general solution of a PDE may be as difficult as, or even more difficult than, the problem of finding the general solution itself.
- This is so because the general solution of a PDE involves arbitrary functions; the specialization of such a solution to the particular form which satisfies supplementary conditions requires the determination of these arbitrary functions, rather than merely the determination of constants.
- For linear homogeneous ODEs of order n, a linear combination of n linearly independent solutions is a solution (Superposition Principle). Unfortunately, this is not true, in general, in the case of partial differential equations.
- This is due to the fact that the solution space of every homogeneous linear PDE is infinite dimensional. Thus, we generally prefer to directly determine the particular solution of a partial differential equation satisfying prescribed supplementary conditions.

Mathematical Problems

- A problem consists of finding an unknown function of a partial differential equation satisfying appropriate supplementary conditions.
- These conditions may be initial conditions (I.C.) and/or boundary conditions (B.C.).
- For example, the PDE

$$u_t - u_{xx} = 0; \quad 0 < x < l, \qquad t > 0,$$

with

$$\begin{array}{lll} \text{I. C.} & u\,(x,0) = \sin x, & 0 \leq x \leq l, t > 0, \\ \text{B. C.} & u\,(0,t) = 0, & t \geq 0, \\ \text{B. C.} & u\,(l,t) = 0, & t \geq 0, \end{array}$$

constitutes a problem which consists of a PDE and three supplementary conditions. The equation describes **the heat conduction in a rod of length $\boldsymbol{l}$**. The last two conditions are called the **boundary conditions** which describe the function at two prescribed boundary points. The first condition is known as the **initial condition** which prescribes the unknown function $u(x,t)$ throughout the given region at some initial time t, in this case $t = 0$.

- This problem is known as the **initial boundary-value problem (IBVP).**

Mathematical Problems

- Mathematically speaking, the time and the space coordinates are regarded as independent variables. In this respect, the initial condition is merely a point prescribed on the t-axis and the boundary conditions are prescribed, in this case, as two points on the x −axis.
- Initial conditions are usually prescribed at a certain time $t = t_0$ or $t = 0$, but it is not customary to consider the other end point of a given time interval.
- In many cases, in addition to prescribing the unknown function, other conditions such as their derivatives are specified on the boundary and/or at time t_0 .
- A mathematical problem is said to be **well-posed** if it satisfies the following requirements:
    1. **Existence:** There is at least one solution.
    2. **Uniqueness:** There is at most one solution.
    3. **Continuity:** The solution depends continuously on the data.
- The first requirement is an obvious logical condition, but we must keep in mind that we cannot simply state that the mathematical problem has a solution just because the physical problem has a solution. We may be erroneously developing a mathematical model, say, consisting of a partial differential equation whose solution may not exist at all.

Mathematical Problems

- The same can be said about the uniqueness requirement. In order to really reflect the physical problem that has a unique solution, the mathematical problem must have a unique solution.
- For physical problems, it is not sufficient to know that the problem has a unique solution. Hence the last requirement is not only useful but also essential. If the solution is to have physical significance, a small change in the initial data must produce a small change in the solution. The data in a physical problem are normally obtained from experiment and are approximated in order to solve the problem by numerical or approximate methods. It is essential to know that the process of making an approximation to the data produces only a small change in the solution.

Fundamental Theorem on Superposition

- We know that if an ODE is linear and homogeneous, then from known solutions we can obtain further solutions by superposition.
- For PDEs the situation is quite similar. For a PDE we have the following theorem:

Theorem:

*If u_1 and u_2 are solutions of a **linear homogeneous PDE** in some region R, i.e., $L[u] = 0$, where L is a linear operator then:*

$$u = c_1 u_1 + c_2 u_2,$$

with any constants c_1 and c_2, is also a solution of that PDE in the region R.

Note:

$\partial/\partial t$ and $\partial^2/\partial x^2$ are examples of linear operators since they satisfy the definition of linear operator $[L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)]$. Any linear combination of linear operators is also a linear operator. For example: $\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$ is a linear operator.

Formation of PDEs

Partial Differential Equation can be formed either by **elimination of arbitrary constants** or by the **elimination of arbitrary functions** from a relation involving three or more variables

Formation of PDE by eliminating arbitrary constants

Let us consider the functional relation:

$$f(x, y, z, a, b) = 0. \qquad (1)$$

where a and b are arbitrary constant to be eliminated and z is a function of x and y. Differentiating (1) partially with respect to x and y, we get:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0. \qquad (2)$$

The set of three equations (1) and (2) involves two arbitrary parameters a and b. In general, these two parameters can be eliminated from this set to obtain a first-order equation of the form:

$$F\,(x, y, z, z_x, z_y) = 0. \qquad (3)$$

Thus, the system of surfaces (1) gives rise to a first-order partial differential equation (3). In other words, an equation of the form (1) containing two arbitrary parameters is called a **complete solution** or a **complete integral** of equation (3).

Example:

Form the PDE by eliminating a and b from $z = (x^2 + a^2)(y^2 + b^2)$.

Solution:

Given that $z = (x^2 + a^2)(y^2 + b^2)$. $\quad (1)$

Differentiating (1) partially with respect to x we get:

$$\frac{\partial z}{\partial x} = z_x = (2x)(y^2 + b^2) \Longrightarrow \frac{z_x}{2x} = y^2 + b^2. \qquad (2)$$

Differentiating (1) partially with respect to y, we get:

$$\frac{\partial z}{\partial y} = z_y = (x^2 + a^2)(2y) \Longrightarrow \frac{z_y}{2y} = x^2 + a^2. \qquad (3)$$

Using (2) and (3) in (1), we get:

$$z = \frac{z_x}{2x} . \frac{z_y}{2y} \Longrightarrow 4xyz = z_x z_y.$$

Example:

Show that a family of spheres of fixed radius r given by the equation:

$$x^2 + y^2 + (z - c)^2 = r^2, \qquad \text{where } c \text{ is some constant}$$

satisfies the first-order linear partial differential equation $yz_x - xz_y = 0$.

Solution:

Given that: $x^2 + y^2 + (z - c)^2 = r^2. \qquad (1)$

Differentiating (1) partially with respect to x and y we get:

$$x + z_x(z - c) = 0 \quad \text{and} \quad y + z_y(z - c) = 0. \qquad (2)$$

Eliminating the arbitrary constant c from these equations, we obtain the first-order, linear partial differential equation:

$$yz_x - xz_y = 0.$$

Example: Formation of PDE by eliminating arbitrary functions

Form the partial differential equation for all surfaces of revolution with the z −axis as the axis of symmetry that satisfy the equation:

$$z = f(x^2 + y^2), \qquad (1)$$

where f is an arbitrary function.

Solution:

Differentiating (1) partially with respect to x we get:

$$z_x = (2x)f'(x^2 + y^2). \qquad (2)$$

Differentiating (1) partially with respect to y we get:

$$z_y = (2y)f'(x^2 + y^2). \qquad (3)$$

Dividing (2) by (3) we get:

$$\frac{z_x}{z_y} = \frac{x}{y} \Longrightarrow yz_x - xz_y = 0.$$

Example: Formation of PDE by eliminating arbitrary functions

Find Partial Differential by eliminating two arbitrary functions from:

$$z = yf(x) + xg(y), \qquad (1)$$

where f and g are arbitrary functions.

Solution:

Differentiating (1) partially with respect to x we get:

$$z_x = yf'(x) + g(y). \qquad (2)$$

Differentiating (1) partially with respect to y we get:

$$z_y = f(x) + xg'(y). \qquad (3)$$

Differentiating (2) partially with respect to y we get:

$$z_{xy} = f'(x) + g'(y) \qquad (4)$$

By using $x \times (2) + y \times (3)$ and later (1) and (4) we get:

$$\begin{aligned} xz_x + yz_y &= xyf'(x) + xg(y) + yf(x) + xyg'(y) \\ &= [xg(y) + yf(x)] + xy[f'(x) + g'(y)] = z + xyz_{xy}. \end{aligned}$$

Thus, $xz_x + yz_y = z + xyz_{xy}$ is the required PDE.