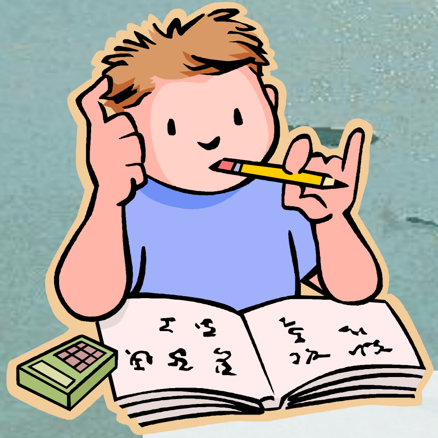


Continuity

Calculus & Analytical Geometry
MATH- 101

Instructor: Dr. Naila Amir
(SEECS, NUST)

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 2
    - Sections: 2.6

Objectives

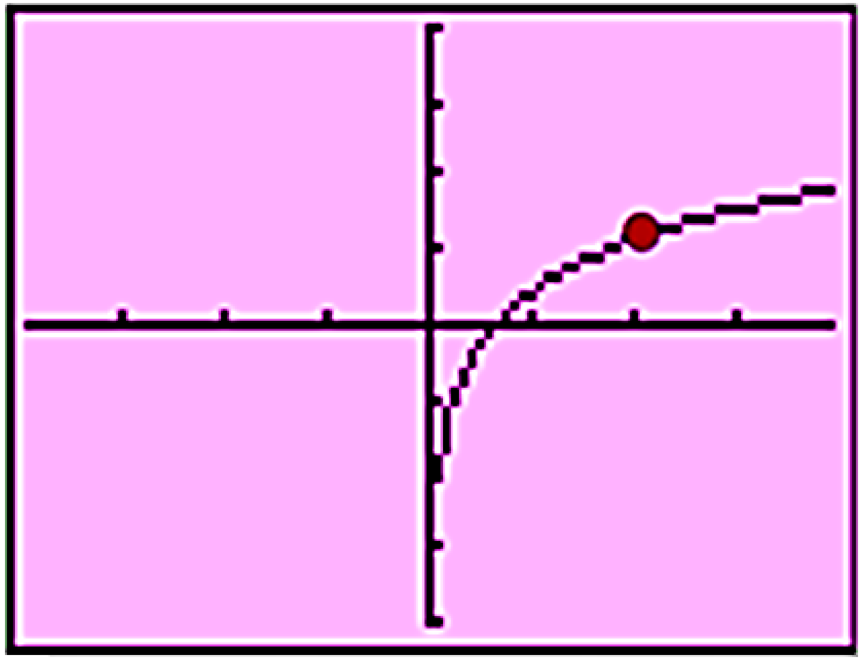
- Determine continuity at a point and continuity on open and closed intervals.

- Use properties of continuity.

- Understand and use the Intermediate Value Theorem.

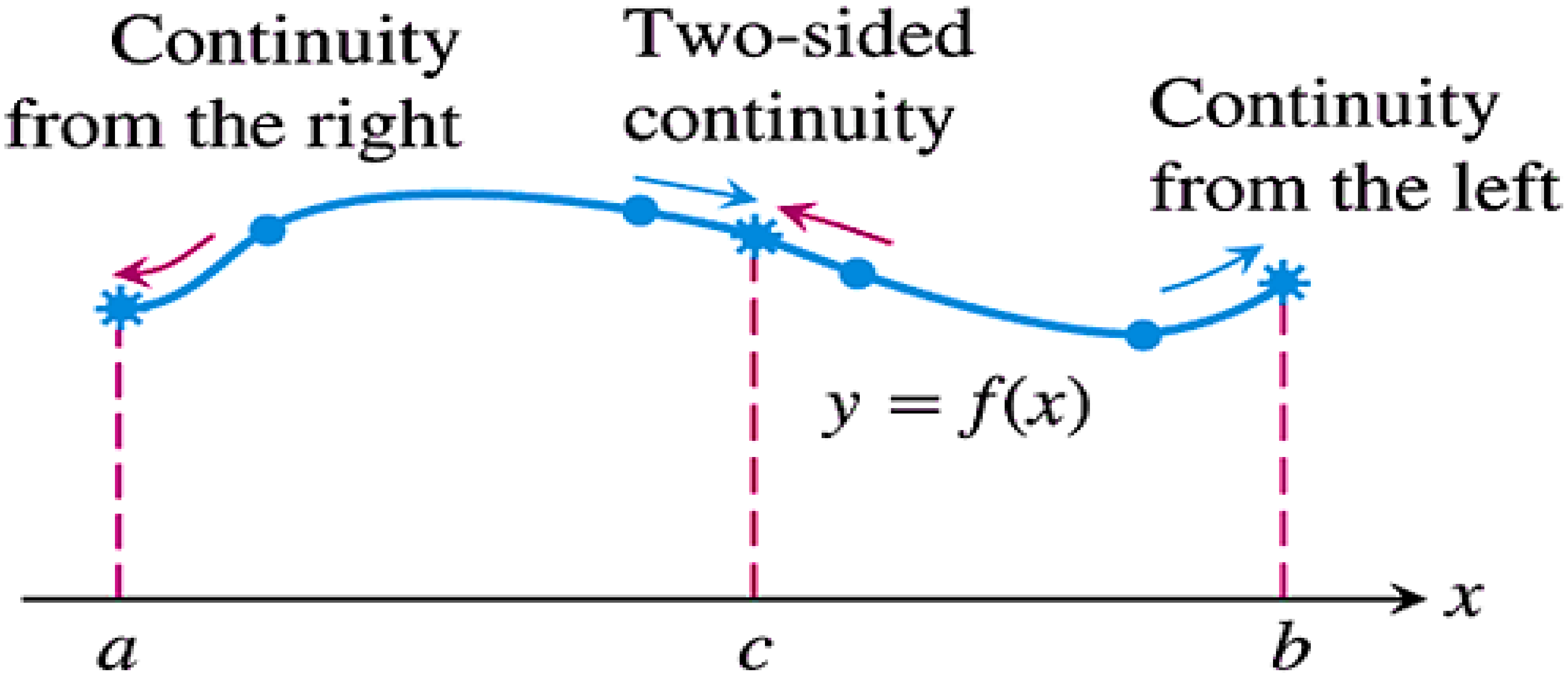
DEFINITION Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point** c of its domain if $\lim_{x \to c} f(x) = f(c)$.



Endpoint: A function $y = f(x)$ is **continuous at a left endpoint** a or is **continuous at a right endpoint** b of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$



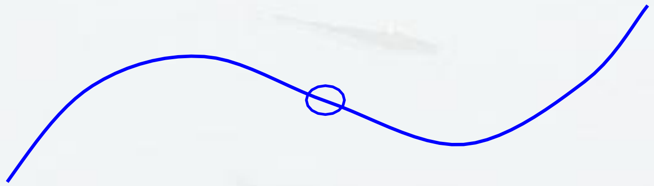
Continuity at points a, b, and c.

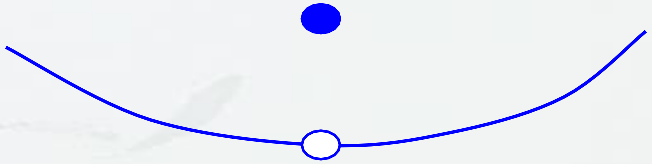
Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (*c* lies in the domain of *f*)
2. $\lim_{x \to c} f(x)$ exists (*f* has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

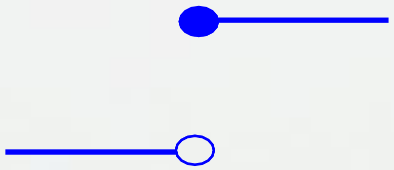
Removable Discontinuities:



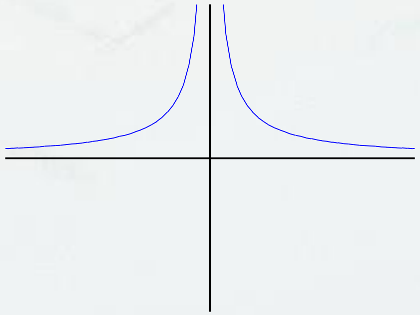


(We can fill the hole.)

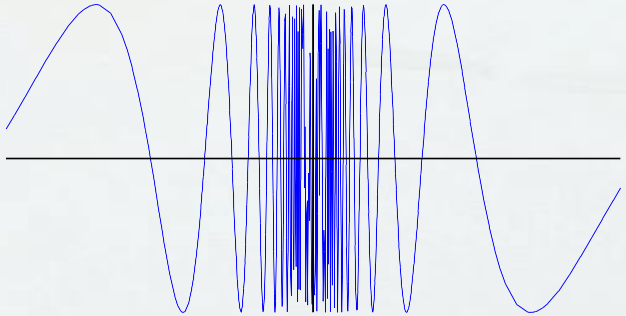
Nonremovable Discontinuities:



jump

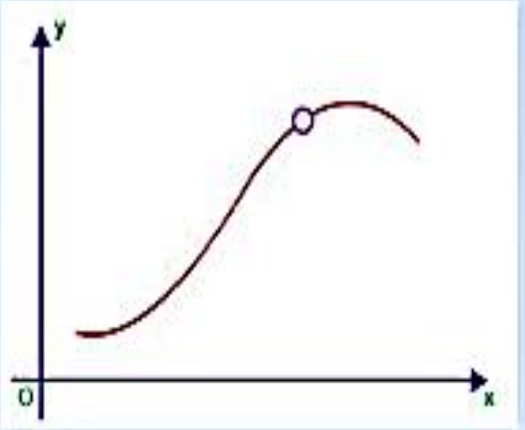
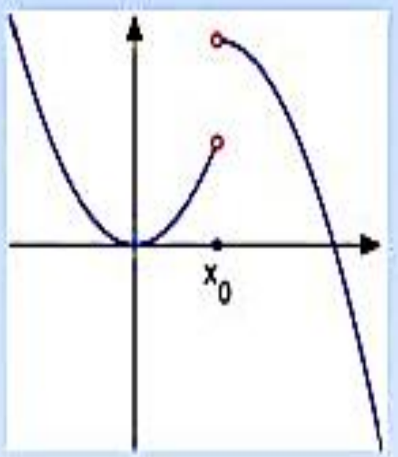
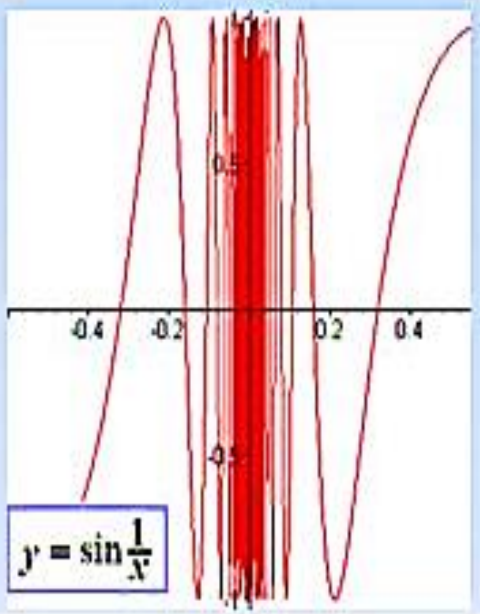
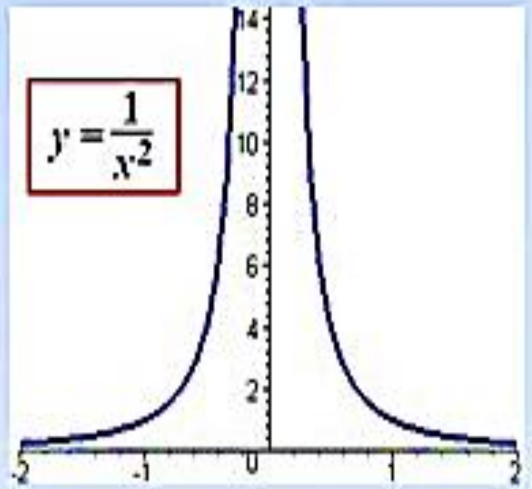


Infinite
(Essential)



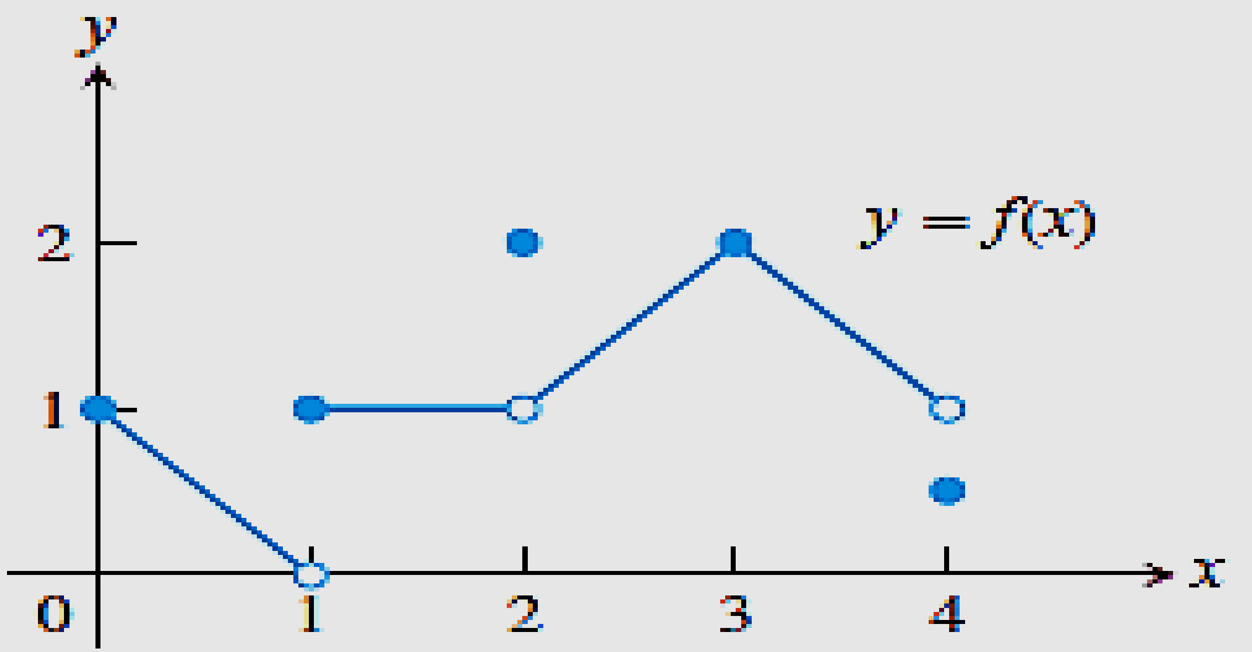
oscillating

Types of Discontinuities

Removable	Jump	Oscillating	Infinite
			

Example

Consider the function $y = f(x)$, whose domain is the closed interval $[0, 4]$.
Discuss the continuity of $f(x)$ at $x = 0, 1, 2, 3$ and 4.



The function is continuous on $[0, 4]$ except at $x = 1$, $x = 2$, and $x = 4$

Example

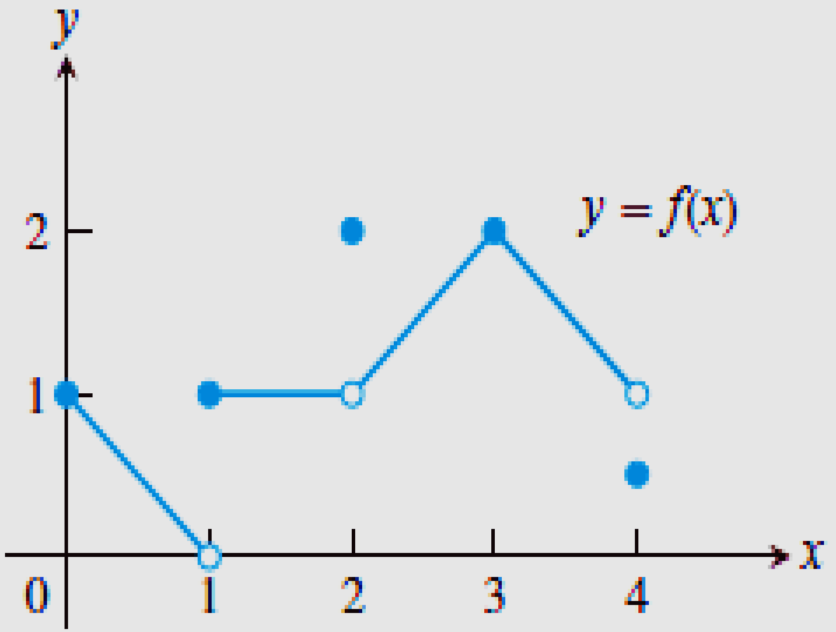
Solution:

Points at which f is continuous:

At $x = 0$, $\lim_{x \to 0^+} f(x) = f(0)$.

At $x = 3$, $\lim_{x \to 3} f(x) = f(3)$.

At $0 < c < 4$, $c \neq 1, 2$, $\lim_{x \to c} f(x) = f(c)$.



Points at which f is discontinuous:

At $x = 1$, $\lim_{x \to 1} f(x)$ does not exist.

At $x = 2$, $\lim_{x \to 2} f(x) = 1$, but $1 \neq f(2)$.

At $x = 4$, $\lim_{x \to 4^-} f(x) = 1$, but $1 \neq f(4)$.

At $c < 0$, $c > 4$, these points are not in the domain of f.

Continuous Extension to a Point

A function (such as a rational function) may have a limit even at a point where its denominator is zero. If $f(c)$ is not defined, but $\lim_{x \to c} f(x) = L$ exists, we can dene a new function $F(x)$ by the rule:

$$F(x) = \begin{cases} f(x); & \text{if } x \text{ is in the domain of } f \\ L; & \text{if } x = c \end{cases}$$

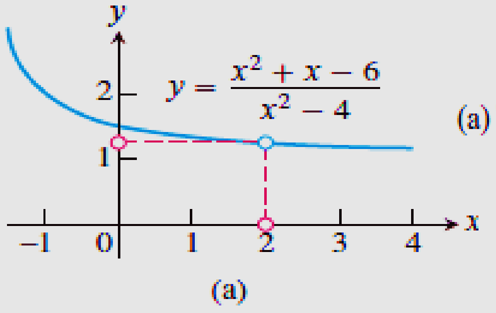
The function $F(x)$ is continuous at $x = c$. It is called the continuous extension of f to $x = c$. For a rational function $f(x)$, continuous extensions are usually found by canceling common factors.

Example

Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}$$

has a continuous extension to $x = 2$, and find that extension.



(a) The graph of $f(x)$

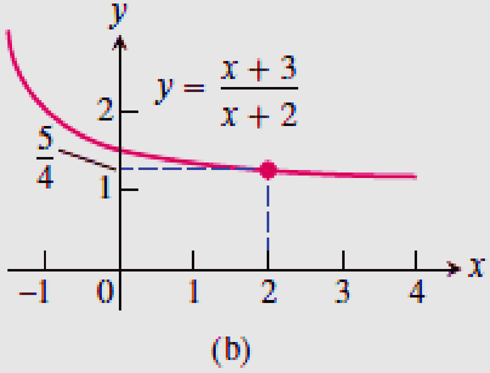
Solution:

Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x - 2)(x + 3)}{(x - 2)(x + 2)} = \frac{x + 3}{x + 2}.$$

The new function

$$F(x) = \frac{x + 3}{x + 2}$$



(b) the graph of its continuous extension $F(x)$

is equal to $f(x)$ for $x \neq 2$, but is continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} f(x) = \frac{5}{4}.$$

Places to test for continuity

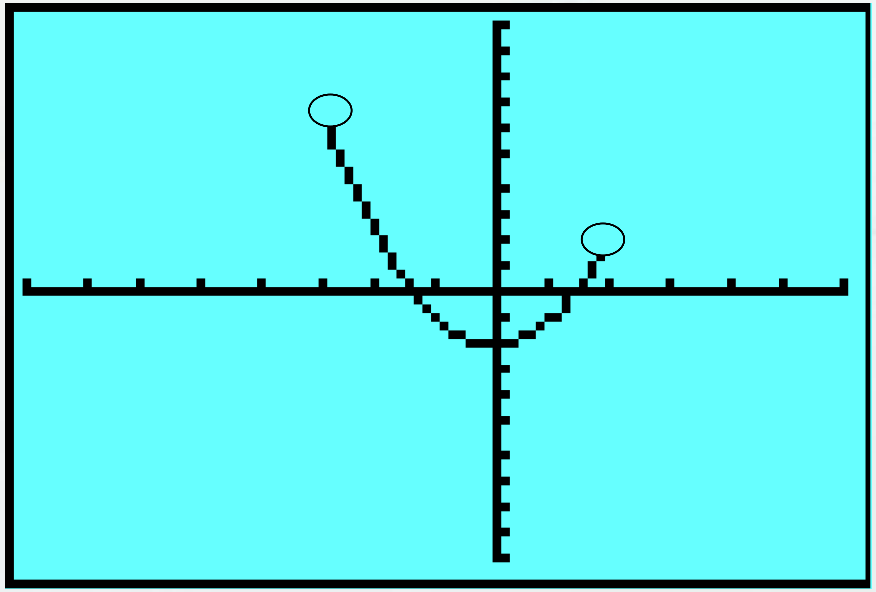
- Rational Expression
  - Values that make denominator = 0
- Piecewise Functions
  - Changes in interval
- Absolute Value Functions
  - Use piecewise definition and test changes in interval
- Step Functions
  - Test jumps from 1 step to next.

Continuity on an open interval

A function is continuous on **an open interval** (a, b) if it is continuous on each point in the interval. A function that is continuous on the entire real line **is every where continuous**.

Example:

$f(x)$ is continuous on $(-3,2)$.



Continuity on a closed interval

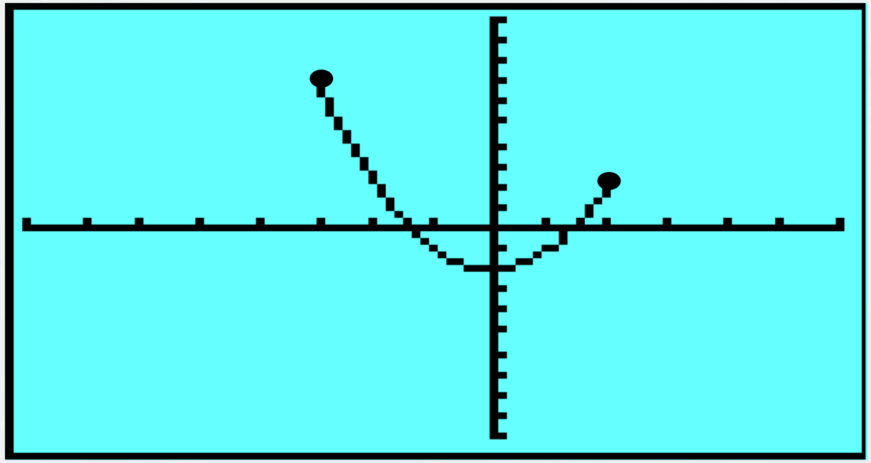
The concept of a one-sided limit allows us to extend the definition of continuity to closed intervals. A function $f(x)$ is continuous on the closed interval $[a, b]$ if it is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a) \text{ and } \lim_{x\to b^-} f(x) = f(b),$$

i.e., the function is continuous from the right at a and continuous from the left at b.

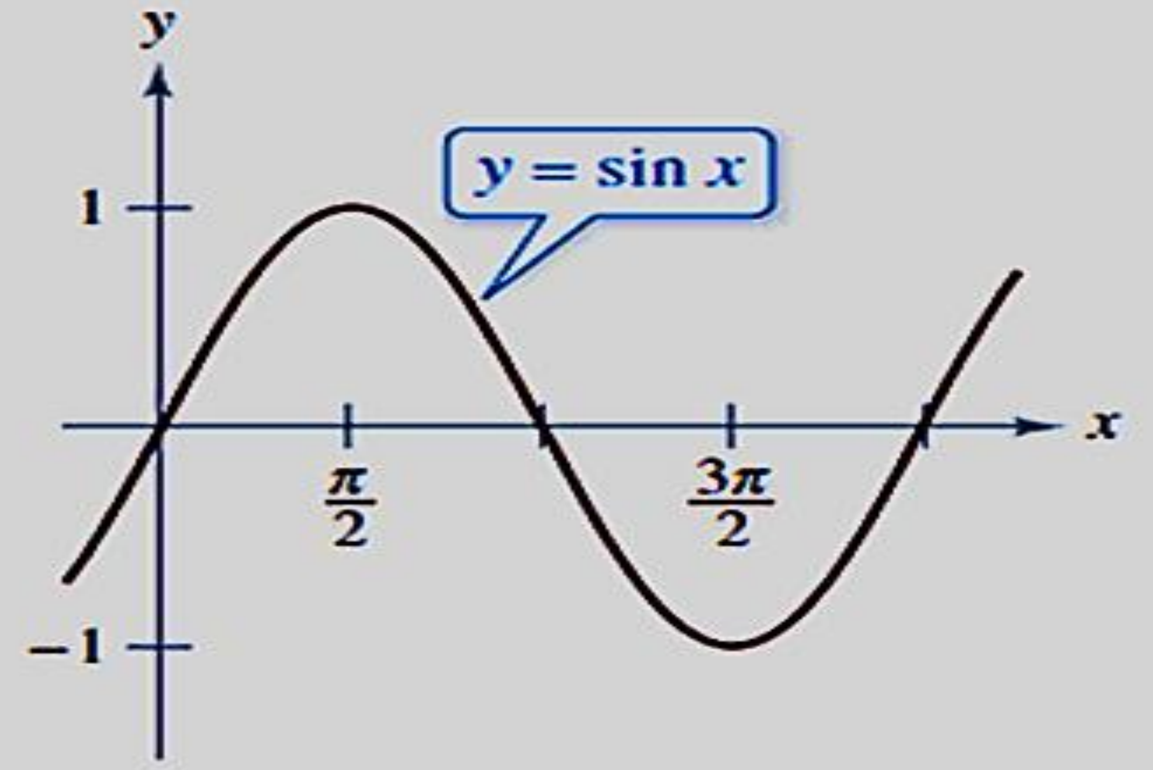
Example:

$f(x)$ is continuous on $[-3,2]$.



Example

The domain of the function $y = \sin x$ is the set of all real numbers. $f(x)$ is continuous on its entire domain, as shown in figure.



Continuous on entire real number line

Example

Discuss the continuity of $f(x) = \sqrt{1 - x^2}$.

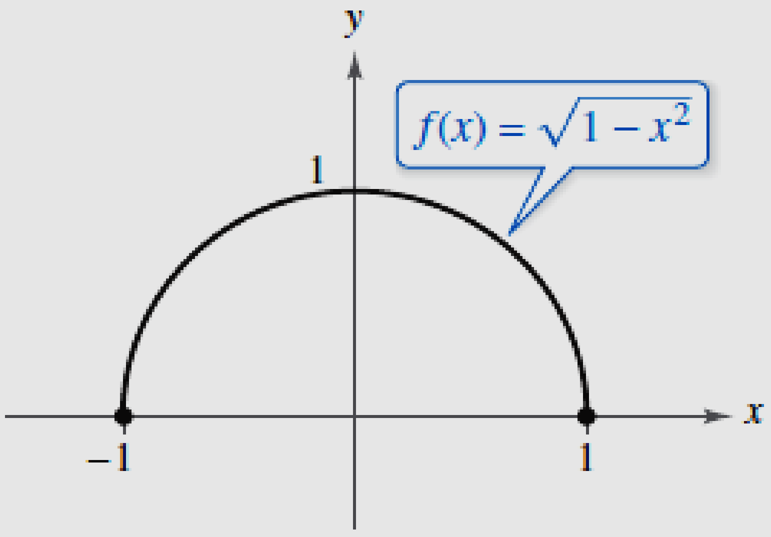
Solution:

The domain of $f(x)$ is the closed interval $[-1, 1]$. At all points in the open interval $(-1, 1)$, the given function is continuous. Moreover,

$$\lim_{x \to -1^+} \sqrt{1 - x^2} = 0 = f(-1) \qquad \text{Continuous from the right}$$

and

$$\lim_{x \to 1^-} \sqrt{1 - x^2} = 0 = f(1) \qquad \text{Continuous from the left}$$



f is continuous on $[-1, 1]$.

This implies that $f(x)$ is continuous on the closed interval $[-1, 1]$.

Continuity by Function Type

The list below summarizes the functions we have studied so far that are continuous at every point in their domains.

1. Polynomial: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0$
2. Rational: $r(x) = \dfrac{p(x)}{q(x)}, \quad q(x) \neq 0$
3. Radical: $f(x) = \sqrt[n]{x}$
4. Trigonometric: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$

With this summary, we can conclude that a wide variety of elementary functions are continuous at every point in their domains.

Continuity by Function Type

- Polynomials are continuous everywhere.
- Rational functions and other trigonometric functions are continuous except at the values of x, where their denominators equal zero.
  - “Removable” discontinuity if factoring and canceling “removes” the zero in the denominator.
  - “Non-removable” otherwise.
- For piecewise functions, find the values of $f(x)$ at the value of x separating the regions of the function.
  - If the values of $f(x)$ are equal, the function is continuous.
  - Otherwise, there is a (non-removable) discontinuity at this point.

Example

1. The function $f(x) = |x|$ is continuous at every value of x.
   - If $x > 0$, we have $f(x) = x$, a polynomial.
   - If $x < 0$, we have $f(x) = -x$, another polynomial.
   - Finally, at the origin,

$$\lim_{x \to 0} |x| = |0| = 0.$$

2. The function $f(x) = 1/x$ is a continuous function on its entire domain because it is continuous at every point of its domain. It has a point of discontinuity at $x = 0$ but $x = 0$ does not belong to domain of $f(x)$.

Properties of Continuous Functions

If $f(x)$ and $g(x)$ are functions, continuous at $x = c$, then

- $s \cdot f(x)$ is continuous (where s is a constant)
- $f(x) \pm g(x)$ is continuous
- $f(x) \cdot g(x)$ is continuous
- $\dfrac{f(x)}{g(x)}$ is continuous
- $f(g(x))$ is continuous

Applying Properties of Continuous Functions

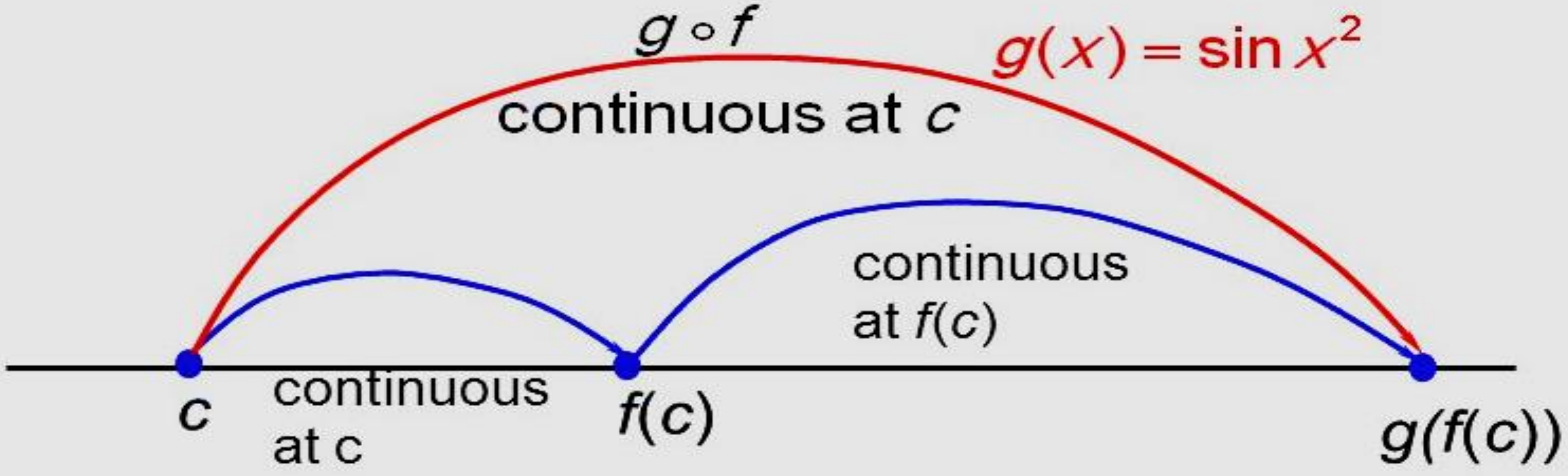
By using properties of continuous functions, it follows that each of the functions below is continuous at every point in its domain.

$$f(x) = x + \sin x, \quad f(x) = 3 \tan x, \quad f(x) = \frac{x^2 + 1}{\cos x}$$

Properties of Continuous Functions

Theorem **Composite of Continuous Functions**

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.



$f(x) = x^2$ $\quad$ $g(f(x)) = \sin x^2$

Continuity of a composition function

Let $f(x) = x^2 + 1$ and $g(x) = \cos x$. Discuss the continuity of the composite functions $(f \circ g)(x)$ and $(g \circ f)(x)$.

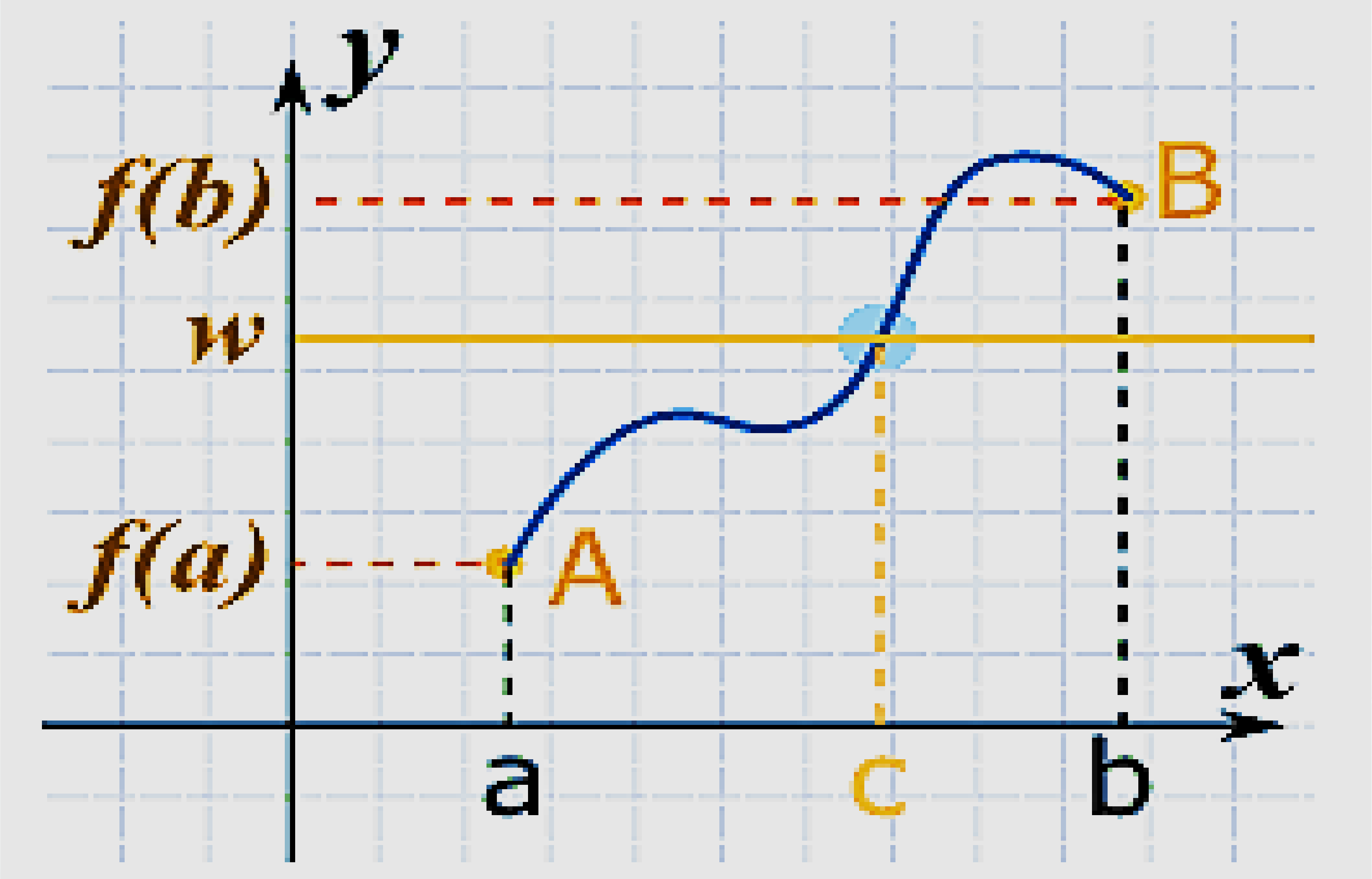
Solution:

Since both $f(x) = x^2 + 1$ and $g(x) = \cos x$ are continuous on $(-\infty, \infty)$. Therefore, both

$$(f \circ g)(x) = \cos^2 x + 1, \text{ and}$$
$$(g \circ f)(x) = \cos(x^2 + 1)$$

are continuous on $(-\infty, \infty)$.

The Intermediate Value Theorem



The Intermediate Value Theorem

Following theorem is an important theorem concerning the behavior of functions that are continuous on a closed interval.

THEOREM **Intermediate Value Theorem**

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

The Intermediate Value Theorem

- The Intermediate Value Theorem tells us that at least one number c exists, but it does not provide a method for finding c. Such theorems are called **existence theorems.**

- A proof of this theorem is based on a property of real numbers called *completeness*.

- The Intermediate Value Theorem states that for a continuous function $f(x)$, if x takes on all values between a and b, $f(x)$ must take on all values between $f(a)$ and $f(b)$.

The Intermediate Value Theorem

- As an example of the application of the Intermediate Value Theorem, consider a person's height. A girl is 5 feet tall on her thirteenth birthday and 5 feet 7 inches tall on her fourteenth birthday.

- Then, for any height h between 5 feet and 5 feet 7 inches, there must have been a time t when her height was exactly h.

- This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.

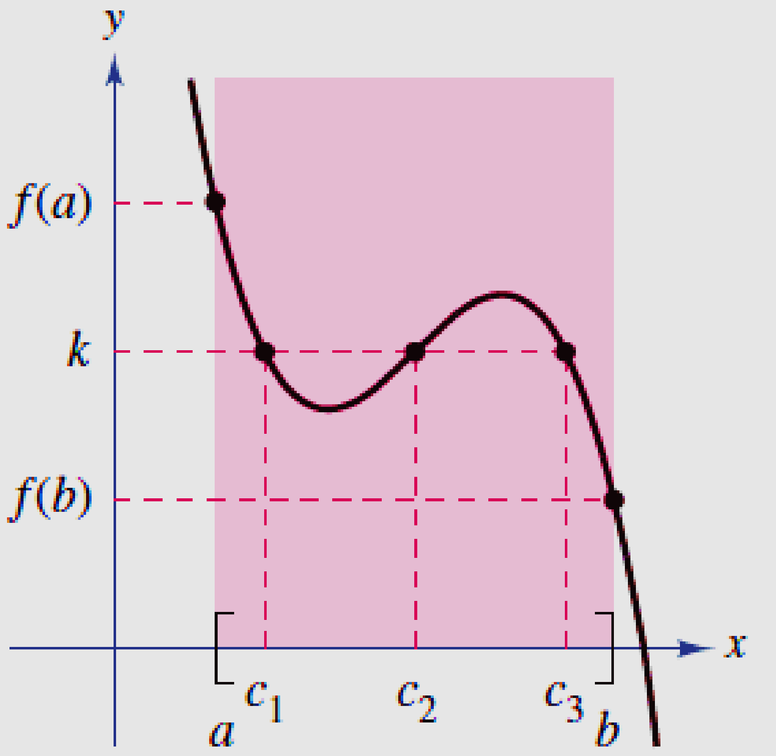
Example

- Let's consider $y = x^2 + 1$ for x-values between 1 and 5.
- Note that, $y = x^2 + 1$ is a smooth curve that has no rips, tears, or holes in it, so we call it continuous.
- If we put $x = 1$ into $y = x^2 + 1$, it will produce $y = 2$ and if we use $x = 5$, then we get $y = 26$.
- Thus the Intermediate Value Theorem will guarantee that the function $y = x^2 + 1$ will produce all of the real numbers between 2 and 26.
- Furthermore, the Intermediate Value Theorem guarantees that these y-values will be produced by numbers chosen for x between 1 and 5.

The Intermediate Value Theorem

The Intermediate Value Theorem guarantees the existence of *at least one* number c in the closed interval $[a, b]$.

There may, of course, be more than one number c such that $f(c) = k$, as shown in figure.

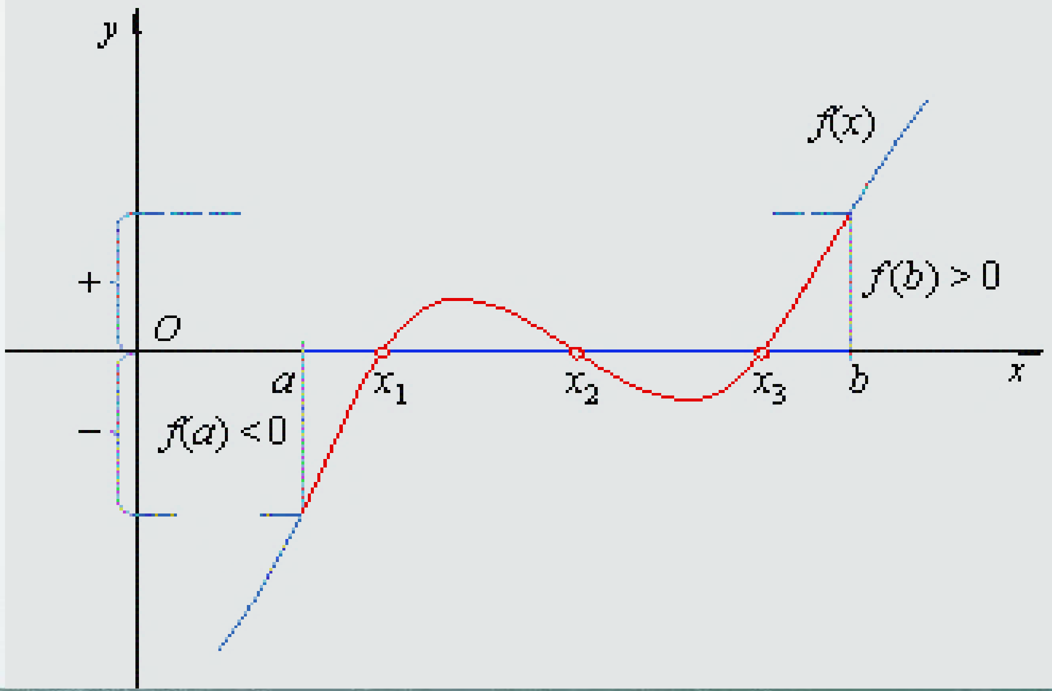


f is continuous on $[a, b]$.
[There exist three c's such that $f(c) = k$.]

Bolzano Theorem: A special case of IVT

The Intermediate Value Theorem (IVT) can often be used to locate the zeros of a function that is continuous on a closed interval.

Specifically, if $f(x)$ is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, the Intermediate Value Theorem guarantees the existence of at least one zero of $f(x)$ in the closed interval $[a, b]$.



Example

Use the Intermediate Value Theorem to show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval $[0, 1]$.

Solution:

Note that $f(x)$ is continuous on the closed interval $[0, 1]$.

Since

$$f(0) = (0)^3 + 2(0) - 1 = -1 \quad \text{and} \quad f(1) = (1)^3 + 2(1) - 1 = 2$$

it follows that $f(0) < 0$ and $f(1) > 0$.

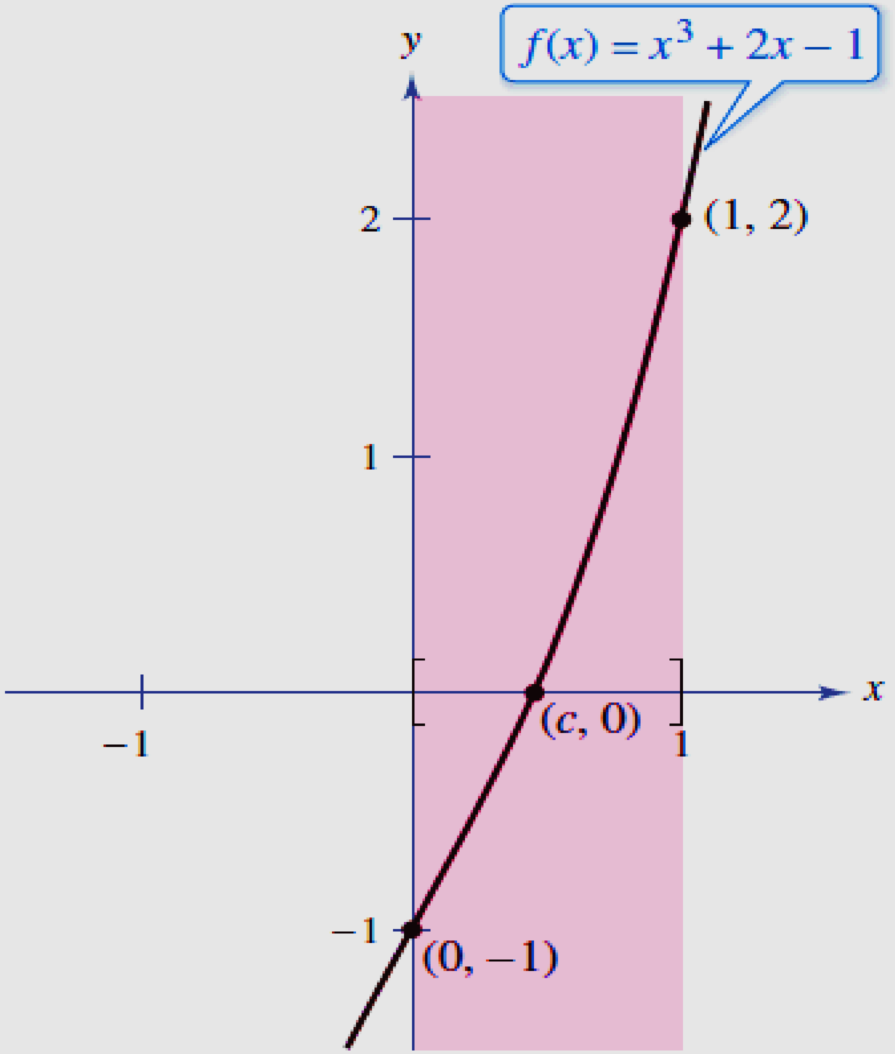
Example

We can therefore apply the Intermediate Value Theorem to conclude that there must be some c in $[0, 1]$ such that

$$f(c) = 0,$$

i.e.,

$f(x)$ has a zero in the closed interval $[0,1]$.



f is continuous on $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$.

Practice Questions

Q#1: Discuss the continuity of the following functions:

1. $f(x) = \begin{cases} \frac{x-4}{\sqrt{x}-2}; & x \geq 0 \text{ and } x \neq 4 \\ 4; & x = 4 \end{cases}$ at $x = 4.$

2. $f(x) = \begin{cases} \frac{x^2-a^2}{x-a}; & 0 \leq x < a \\ a; & x = a \\ 2a; & x > a \end{cases}$ at $x = a.$

3. $f(x) = 2^{1/x}$ at $x = 0.$

4. $f(x) = \begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1}; & x \neq 0 \\ 0; & x = 0 \end{cases}$ at $x = 0.$

Practice Questions

Q#1: Discuss the continuity of the following functions:

5. $f(x) = \begin{cases} \dfrac{\sin 3x}{\sin 2x}; & x \neq 0 \\ 2/3; & x = 0 \end{cases}$ at $x = 0$.

6. $f(x) = x - |x|$ at $x = 1$.

7. $f(x) = \begin{cases} (1+x)^{1/x}; & x \neq 0 \\ 1; & x = 0 \end{cases}$ at $x = 0$.

Q#2: Let $f(x) = x^2$ and $g(x) = \begin{cases} -4; & x \leq 0 \\ |x-4|; & x > 0 \end{cases}$

Determine whether fog and gof are continuous at $x = 0$. If not continuous then what type of discontinuity exists at this point?

Practice Questions

Q#3: Show that the function $f(x) = \begin{cases} x; & \text{if } x \text{ is irrational} \\ 1-x; & \text{if } x \text{ is rational} \end{cases}$ is continuous at $x = 1/2$.

Q#4: Find the constant "c", provided the function $f(x) = \begin{cases} \frac{1-\sqrt{x}}{x-1}; & 0 \le x < 1 \\ c; & x = 1 \end{cases}$

is continuous for all $x \in [0,1]$.

Q#5: Determine the constants "a" and "b", such that the function:

$$f(x) = \begin{cases} x^3; & x < -1 \\ ax + b; & -1 \le x < 1 \\ x^2 + 2; & x \ge 1 \end{cases}$$

is continuous for all $x \in \mathbb{R}$.

Practice Questions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

– Chapter: 2

- Exercise: 2.6

  Q # 1 – 40