Transformations
CS-477 Computer Vision

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1. Geometric Transformations
2. 2D Geometric Transformations

- What is transformations?
  - To define a point, we have to define a coordinate system, OR
  - Transformations are functions that convert points from one coordinate system to another
  - This is also called
    - spatial transformation,
    - geometric transformation,
    - warp
  - Examples: Translation, rotation, scaling, shear etc.

Formal definition

"Geometric transformations refer to the processes of altering the position, orientation, or scale of objects or points in a geometric space."

These transformations can occur in various dimensions, including 2D and 3D, and can involve transformations from one dimension to another.

Types

- **2D-to-2D (image-to-image)**
  - This involves transforming objects or points from one 2D plane to another 2D plane.
  - Common 2D-2D transformations include translation (shifting), rotation, scaling (resizing), shearing (skewing), and reflection.

2D-to-2D


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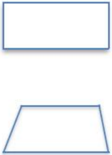
80 RUNS IN 84 BALLS
LAHORE 57-3 6 v QUETTA 136-9

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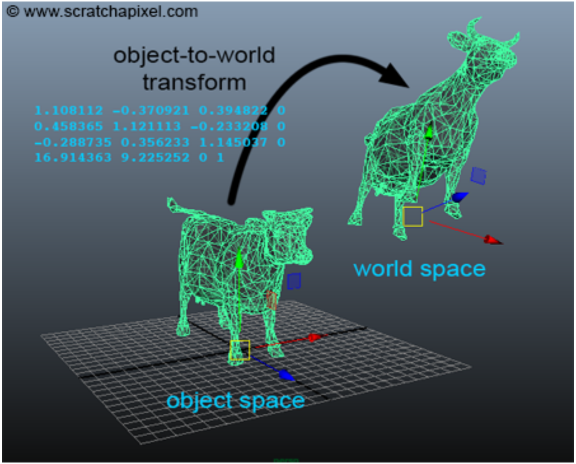
- **3D-to-3D (world-to-world)**
  - This involves transforming objects or points within a 3D space.
  - Common 3D-3D transformations include 3D translation, 3D rotation, 3D scaling, and more complex operations like 3D affine transformations.

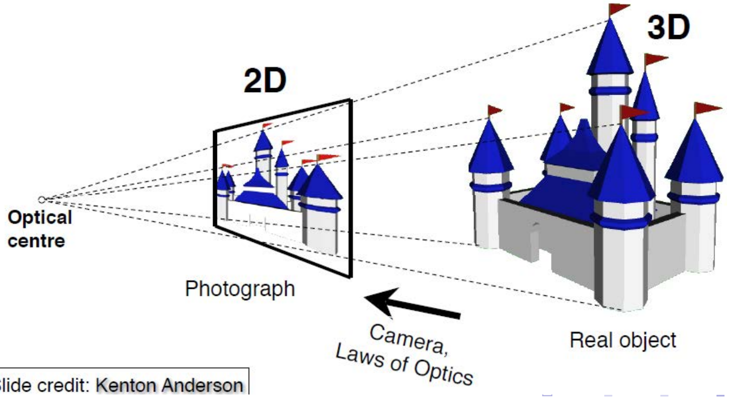
3D-to-3D


© www.scratchapixel.com
object-to-world
transform
1.108112 -0.370921 0.394822 0
0.458365 1.121113 -0.233208 0
-0.288735 0.356233 1.145037 0
16.914363 9.225252 0 1
world space
object space

- **3D-to-2D (camera model)**
  - Points from a 3D space are projected onto a 2D plane.
  - This is commonly used in computer graphics, computer vision, and engineering applications when rendering 3D scenes onto 2D displays or extracting information from 3D scenes through techniques like perspective or orthographic projection.

3D-to-2D

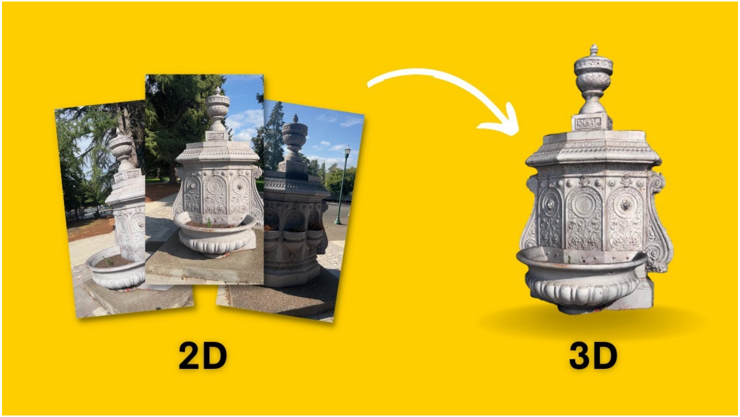
- Point to point mapping
- 3D to 2D projection



Slide credit: Kenton Anderson

- **2D-to-3D (3D reconstruction)**
  - Shape from Stereo
  - Structure from Motion
  - Single View Reconstruction

2D-to-3D reconstruction



Invariance and covariance

- Are detected corners invariant to photometric transformations and covariant to geometric transformations?
  - **Invariance:** image is transformed photometrically and corner locations do not change
  - **Covariance:** if we have two geometrically transformed versions of the same image, features should be detected in corresponding locations

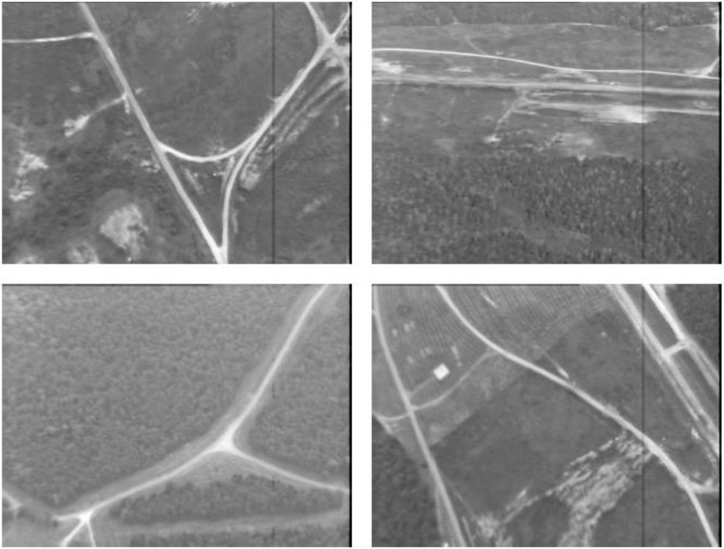
Image registration

- Process of transforming two images so that same features overlap

- Image registration is a fundamental process in computer vision and medical imaging that involves aligning and overlaying two or more images of the same scene or object taken at different times, from different viewpoints, or using different imaging modalities. The goal of image registration is to find the spatial transformation (such as translation, rotation, scaling, or deformation) that best aligns the features or content of the images, so they can be compared, combined, or analyzed together effectively.

Reference image

Image registration



Mission Images

Image registration

Registration = Computing transformation

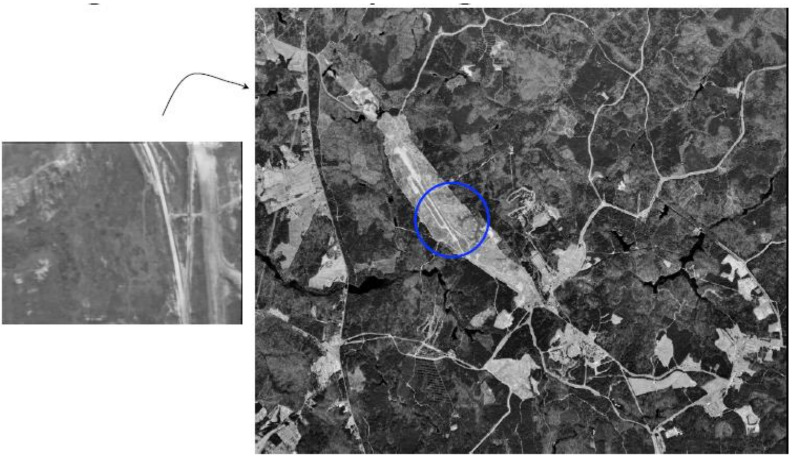


Image registration

Image registration

Panoramas

- Multiple images stitched together: Applications of 2D image registration

Panoramas

- Multiple images stitched together: Applications of 2D image registration

Panoramas


TOUR MAP Google
SHOW CONTROLS
FULLSCREEN
Applications of 2D Image Registration
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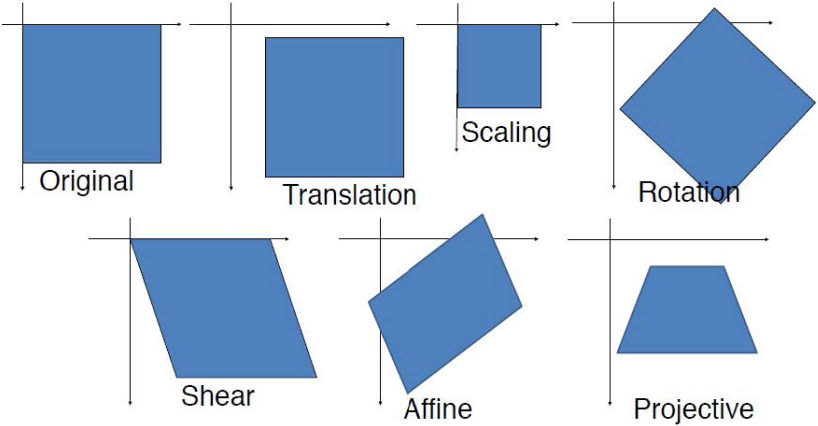
Spherical 360^o Imaging: Applications of 2D Image Registration

Lady Bug Camera[1] by Point Grey
60.8MP cameras image 75% of a full sphere

[1] https://www.flir.asia/iis/machine-vision/

Spherical 360^o Imaging: Applications of 2D Image Registration


Original
Translation
Scaling
Rotation
Shear
Affine
Projective

- Basic operations of all 2D transformations is matrix multiplication
  - Point to be transformed: $(x, y)^T$
  - Point after transformation: $(x', y')^T$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 x + a_2 y \\ a_3 x + a_4 y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

- Transformation Matrix $\rightarrow \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$
- Position before transformation $\rightarrow \begin{bmatrix} x \\ y \end{bmatrix}$
- Position after transformation $\rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix}$

2D Translation

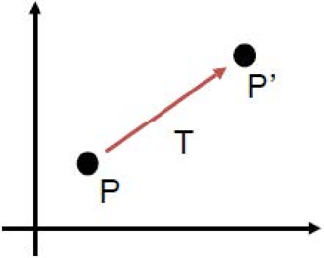
- Rigid body transformation (moves object without deforming it)
- Point to be transformed (x, y)
- Point after transformation (x', y')
- Translation distances t_x and t_y

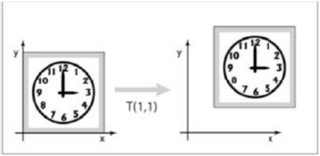
$$x' = x + t_x$$
$$y' = y + t_y$$

- To translate any shape, translate its vertices and redraw it

$$P = \begin{bmatrix} x \\ y \end{bmatrix}, P' = \begin{bmatrix} x' \\ y' \end{bmatrix}, T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$





2D Scaling

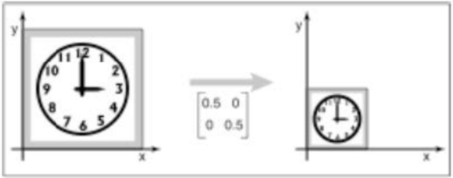
- Scaling (can change length and possibly direction)

$$x' = s_x x$$
$$y' = s_y y$$

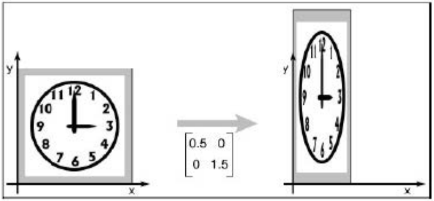
- In matrix form, it is represented as:

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

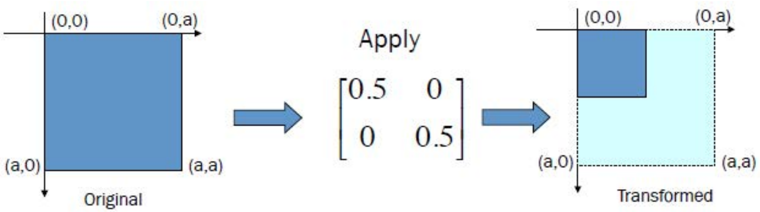


Uniform scaling



Non-uniform scaling

Example1



$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5a \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} 0.5a \\ 0.5a \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 0.5a \\ 0.5a \end{bmatrix} = \begin{bmatrix} 0.25a \\ 0.25a \end{bmatrix}$$

Example2

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix} = ?$$

In general, scaling (zoom / unzoom) transformation is given by

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

2D Reflection (Vertical and horizontal flipping)

- Reflection along x-axis

$$x' = x$$
$$y' = -y$$

- Reflection along y-axis

$$x' = -x$$
$$y' = y$$

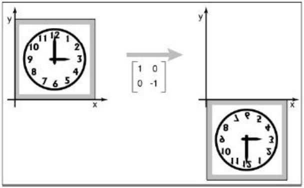
- In Matrix form

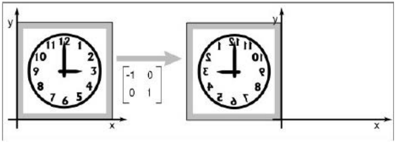
Reflection along x-axis

$$\rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Reflection along y-axis

$$\rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$





2D Reflection (Vertical and horizontal flipping

Do it!

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = ?$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = ?$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = ?$$

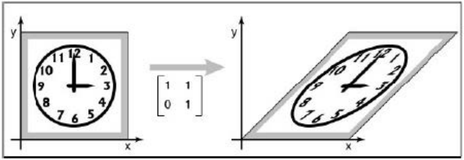
Horizontal Shearing

- Pushes things sideways
  - y-values remains unchanged
  - x-values changes

$$x' = x + sy$$
$$y' = y$$

- In Matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} x + sy \\ y \end{bmatrix}$$



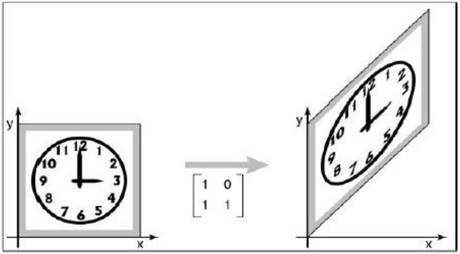
Vertical Shearing

- Pushes things sideways
  - x-values remains unchanged
  - y-values changes

$$x' = x$$
$$y' = y + sx$$

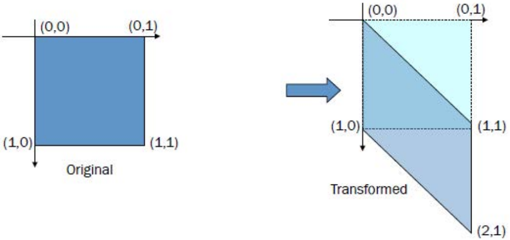
- In Matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} x \\ y + sx \end{bmatrix}$$



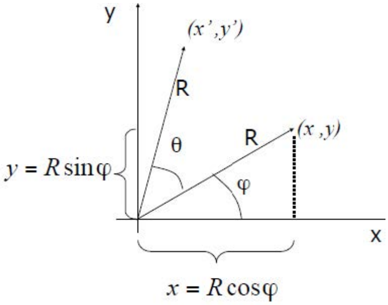
Shearing

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = ?$$



Rotation

$$x = R\cos\varphi$$
$$y = R\sin\varphi$$



$$x' = R\cos(\theta + \varphi)$$
$$y' = R\sin(\theta + \varphi)$$

$$x' = R\cos\theta\cos\varphi - R\sin\theta\sin\varphi$$
$$y' = R\sin\theta\cos\varphi + R\cos\theta\sin\varphi$$

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

R is rotation by θ **counterclockwise about origin**

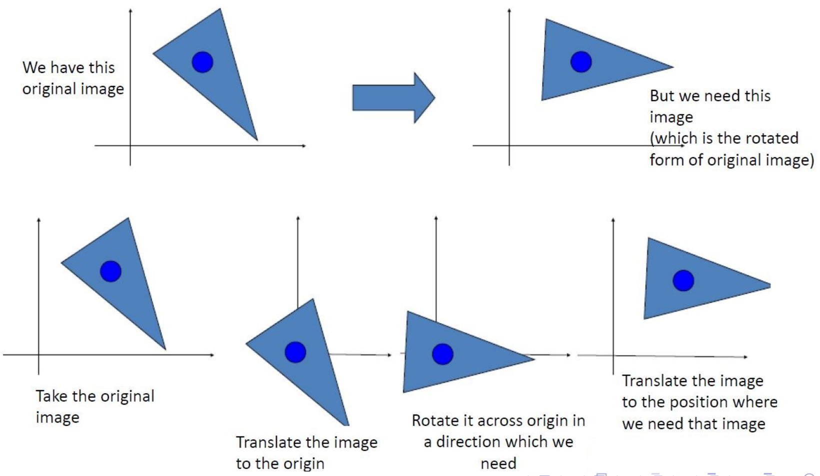
Rotation about an Arbitrary Point

- The rotation matrix that we have derived is for rotations about the origin
- We may want to rotate about some other point

Solution?

- Translate point of rotation to origin, rotate using normal rotation matrix, translate back

Rotation about an Arbitrary Point



Summary of the 2D transformation

Translation	$T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + T$
Scaling	$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$	$\begin{bmatrix} x' \\ y' \end{bmatrix} = S \begin{bmatrix} x \\ y \end{bmatrix}$
Horizontal shearing	$S_x = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} x' \\ y' \end{bmatrix} = S_x \begin{bmatrix} x \\ y \end{bmatrix}$
Vertical shearing	$S_y = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$	$\begin{bmatrix} x' \\ y' \end{bmatrix} = S_y \begin{bmatrix} x \\ y \end{bmatrix}$
Rotation	$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	$\begin{bmatrix} x' \\ y' \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix}$

Homogeneous coordinate system

- In general, a matrix multiplication allows us to linearly combine components of a vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

- This is sufficient for scaling, rotating, and skewing transformations
- But notice, we cannot add a constant offset, within the same format

Homogeneous coordinate system

- Solution is to use homogeneous coordinates for vectors

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

- Now we can rotate, scale and skew like before, AND translate (note how the multiplication works out, above)

Homogeneous Representation

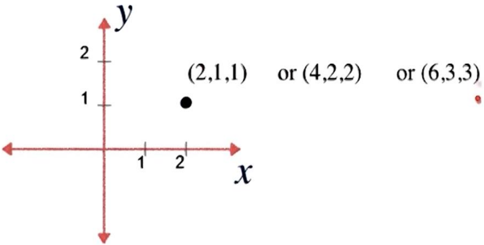
- Represent coordinates in 2D with a 3D

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1w \end{bmatrix}$$

- Add a 3rd coordinate to every 2D point

$$(x, y, w) \rightarrow \left(\frac{x}{w}, \frac{y}{w}\right)$$

- $(x, y, 0) \rightarrow$ infinity
- $(0, 0, 0)$ is not allowed



Translation

- in matrix form:

$$\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

- We could not have written T multiplicatively without using homogeneous coordinates

2D transformations

- Translation

$$T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = T \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Scaling

$$S = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = S \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Horizontal Shear

$$S_x = \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = S_x \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Vertical Shear

$$S_y = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = S_y \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Rotation

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = R \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Affine transformations

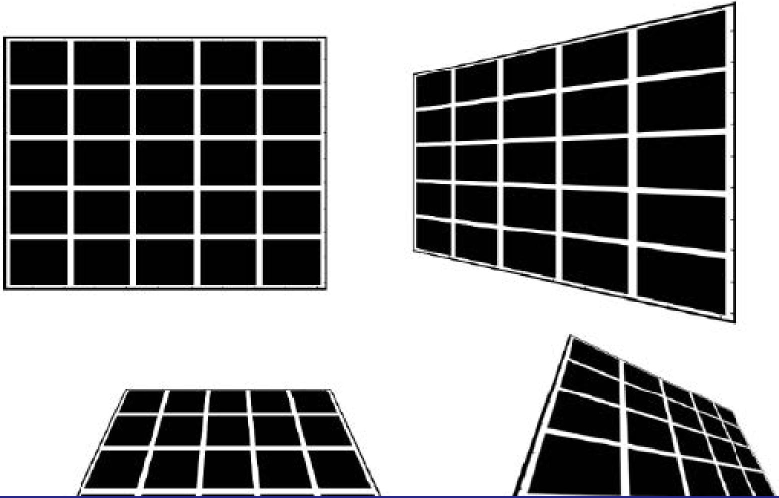
- Combine linear transformations and translation
- Properties:
  - Origin does not necessarily map to origin
  - Line map to line
  - Parallel lines remain parallel
- $$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$
- 6 parameters involved $\rightarrow$ 6 Degree of Freedom

Projective transformations

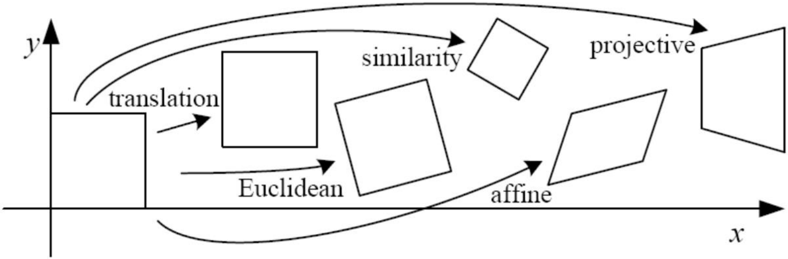
- Combination of affine transformations and projective warps
- Properties:
  - Origin does not necessarily map to origin
  - Line map to line
  - Parallel lines do not necessarily remain parallel
- $$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$
- 8 parameters involved $\rightarrow$ 8 Degree of Freedom

Projective transformations

Example



Classification of 2D Transformations



Order of transformations

- Suppose we first want to scale, then rotate

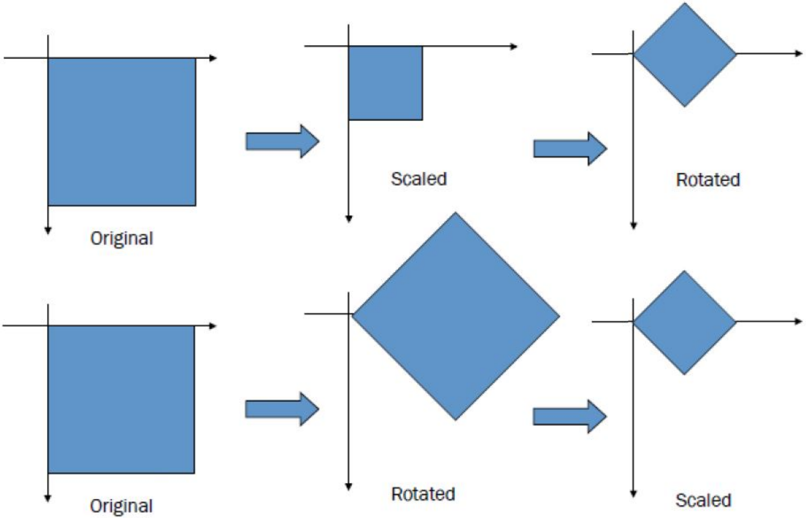
$$x' = Sx$$
$$x'' = Rx' = R(Sx) = (RS)x$$

- So two transformation can be represented by a single transformation matrix

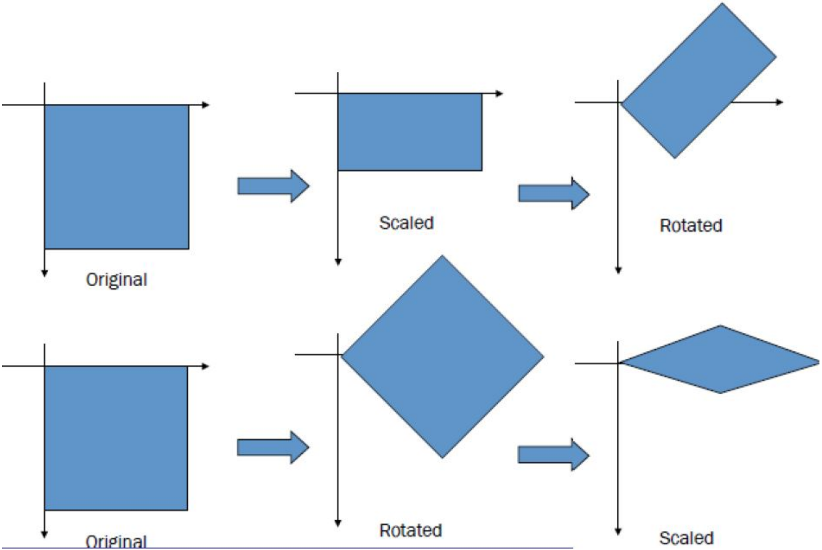
$$M = RS$$

- Important: read from right-side to get order of application of transformations

Order of transformations



Order of transformations



Order of transformations

- We can concatenate a large number of transformations into a single transformation
$$p_2 = T_{[dx\ \ dy]} S_{[S\ \ S]} R_\theta p_1$$
- Rule of matrix multiplication apply
- If we do not use homogeneous coordinates, what might be the problem here?

Order of transformations

$$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Then,

$$RS = \begin{bmatrix} s_x \cos\theta & -s_y \sin\theta \\ s_x \sin\theta & s_y \cos\theta \end{bmatrix}$$

$$SR = \begin{bmatrix} s_x \cos\theta & -s_x \sin\theta \\ s_y \sin\theta & s_y \cos\theta \end{bmatrix}$$

Order of transformations

- In general $AB \neq BA$
- However, in specific cases, this might hold true
- In the previous example, if $s_x = s_y$, then order of transformations does not matter

Order of transformations

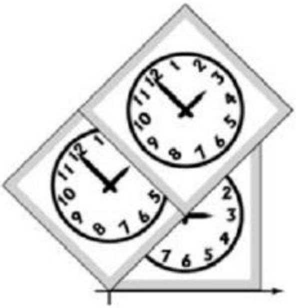
- Rotation/Scaling/Shear, followed by Translation

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & b_1 \\ a_3 & a_4 & b_2 \\ 0 & 0 & 1 \end{bmatrix}$$

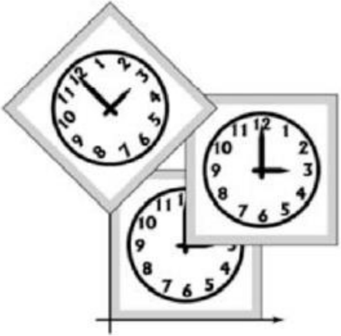
- Translation, followed by Rotation/Scaling/Shear

$$\begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_1 b_1 + a_2 b_2 \\ a_3 & a_4 & a_3 b_1 + a_4 b_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Order matters!



T(1,1)R(45)



R(45) T(1,1)

Inverse transformations

- Inverse transformations should undo the effect of original transformation
- Simply taking the matrix inverse will work $AA^{-1} = I$
- Inverse transformations

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- Remember that when inverting concatenation of transforms, their order reverses

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Inverse transformations

- Translation

$$T^{-1} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- Rotation

$$R^T = R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

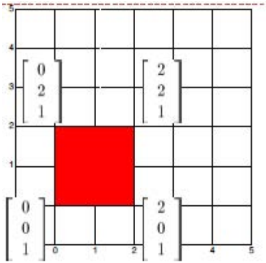
- Scaling

$$S^{-1} = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Examples

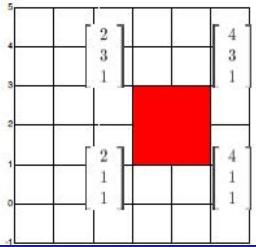
- Translate by $T(2,1)$
- Vertices to be transformed

$$\begin{bmatrix} x \\ y \\ h \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \\ h' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ h' \end{bmatrix} = \begin{bmatrix} 2 & 4 & 4 & 2 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



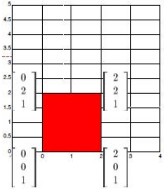
Examples

- Translate by $T(2,1)$, then rotate by $R(45)$
- Vertices to be transformed

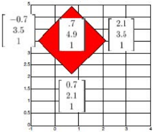
$$\begin{bmatrix} x \\ y \\ h \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ h' \end{bmatrix} = \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ h' \end{bmatrix} = \begin{bmatrix} 0.7 & 2.1 & 0.7 & -0.7 \\ 2.1 & 3.5 & 4.9 & 3.5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



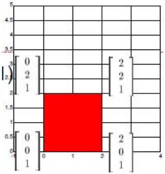
V(x,y,1)



Examples

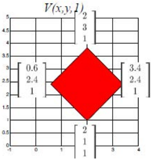
- Rotate by $R(45)$, then translate by $T(2,1)$
- Vertices to be transformed

$$\begin{bmatrix} x \\ y \\ h \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \\ h' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ h' \end{bmatrix} = \begin{bmatrix} 2 & 3.4 & 2 & 0.6 \\ 1 & 2.4 & 3.8 & 2.4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



Examples