VECTOR CALCULUS II

Divergence of a Vector

- Divergence of vector **A** at a given point *P* is the outward flux per unit volume as the volume shrinks about *P*

- Net outflow of the flux of a vector field **A** from a closed surface *S* is obtained from the integral:

$$\oint_S \mathbf{A} \cdot d\mathbf{S}$$

- The divergence of **A** as the net outward flow of flux per unit volume over a closed incremental surface:

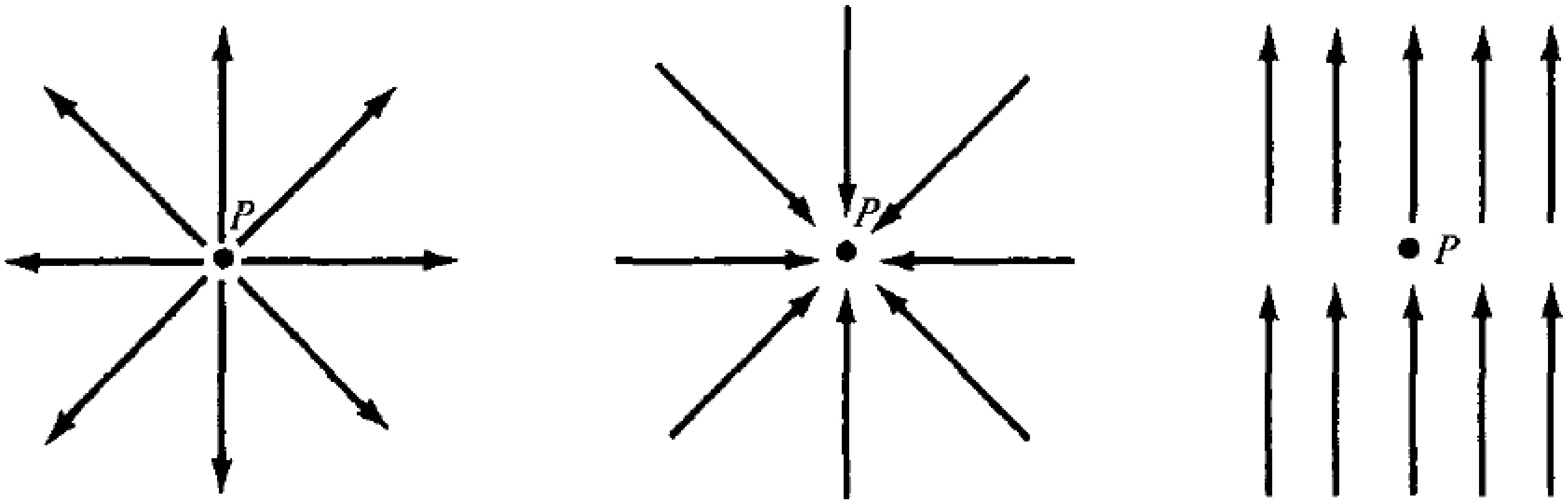
$$\operatorname{div} \mathbf{A} = \nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

- where Δv is the volume enclosed by the closed surface *S* in which *P* is located

Divergence of a Vector

- Physically, divergence of the vector field **A** at a given point is a measure of how much the field diverges or emanates from that point

- (a) Positive (b) Negative (c) Zero



(a) (b) (c)

Divergence of a Vector

- In Cartesian coordinate system:

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- Cylindrical coordinate system:

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

- Spherical coordinate system:

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Divergence Theorem

- From the definition of divergence of **A**, we have the divergence theorem as:

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \, dv$$

- The total outward flux of a vector field **A** through the closed surface **S** is the same as the volume integral of the divergence of **A**

- It will soon become apparent that volume integrals are easier to evaluate than surface integrals

- For this reason, to determine the flux of **A** through a closed surface, we simply find the right-hand side of the divergence theorem equation

Curl of a Vector

- We defined the circulation of a vector field **A** around a closed path *L* as the integral:

$$\oint_L \mathbf{A} \cdot d\mathbf{l}$$

- The curl of **A** is an axial (or rotational) vector whose magnitude is the maximum circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum

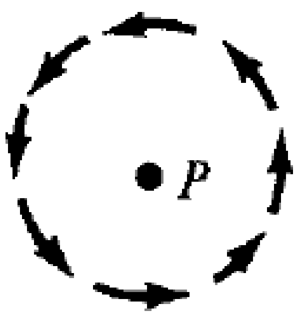
$$\text{curl}\,\mathbf{A} = \nabla \times \mathbf{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right)_{\max} \mathbf{a}_n$$

- where the area ΔS is bounded by the curve *L* and $\mathbf{a}_n$ is the unit vector normal to the surface ΔS and is determined using the right-hand rule

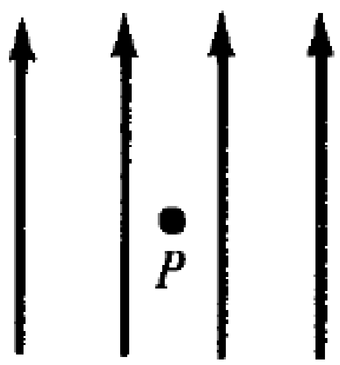
Curl of a Vector

- The curl provides the maximum value of the circulation of the field per unit area (or circulation density) and indicates the direction along which this maximum value occurs

- (a) curl at P points out of the page (b) curl at P is 0



(a)



(b)

Curl of a Vector

- Cartesian coordinates:

$$\nabla \times \mathbf{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\mathbf{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\mathbf{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\mathbf{a}_z$$

- Cylindrical coordinates:

$$\nabla \times \mathbf{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\mathbf{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\mathbf{a}_z$$

Curl of a Vector

- Spherical coordinates:

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi \sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\mathbf{a}_\phi$$

- Note the following properties of curl:

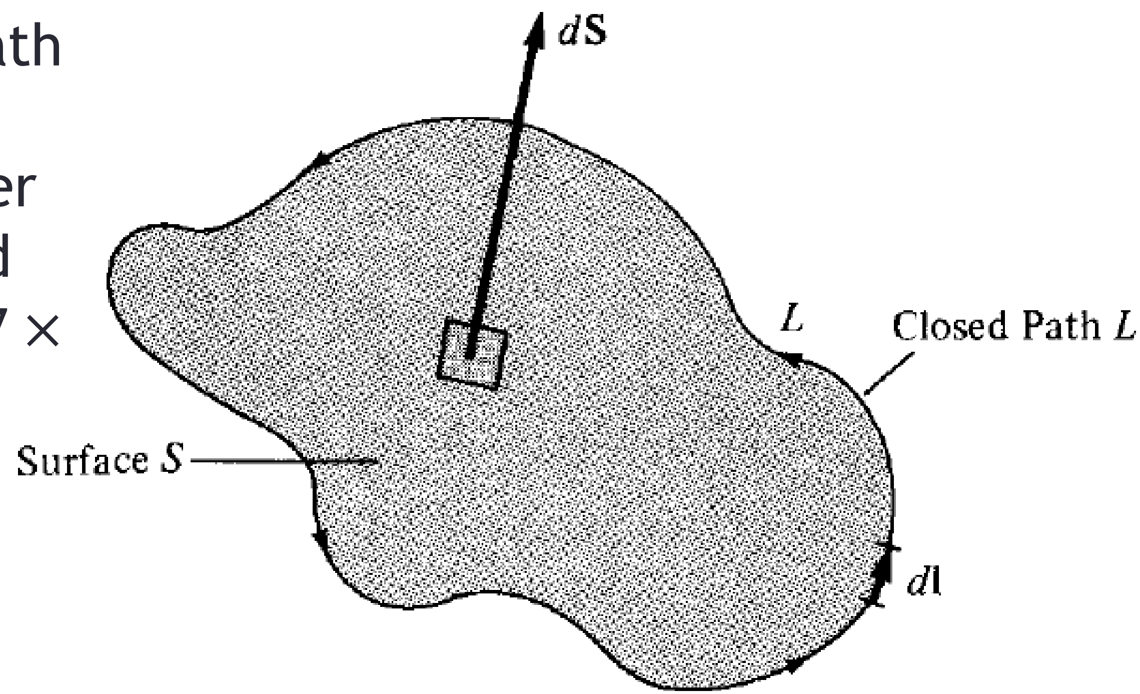
1. The curl of a vector field is another vector field
2. The curl of a scalar field V, $\nabla \times V$, makes no sense
3. The divergence of the curl of a vector field vanishes, that is, $\nabla \cdot (\nabla \times A) = 0$
4. The curl of the gradient of a scalar field vanishes, that is, $\nabla \times \nabla V = 0$

Stokes Theorem

- From the definition of the curl of **A**, we get:

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

- Stokes theorem states that the circulation of a vector field **A** around a (closed) path *L* is equal to the surface integral of the curl of **A** over the open surface *S* bounded by *L*, provided that **A** and **∇** × **A** are continuous on *S*



Stokes Theorem

- The direction of $d\mathbf{l}$ and $d\mathbf{S}$ in Stokes theorem equation must be chosen using the right-hand rule

- Using the right-hand rule, if we let the fingers point in the direction of $d\mathbf{l}$, the thumb will indicate the direction of $d\mathbf{S}$

- Note that the divergence theorem relates a surface integral to a volume integral

- Stokes's theorem relates a line integral (circulation) to a surface integral

Laplacian of a Scalar

- A useful operator which is the composite of gradient and divergence operators

- The Laplacian of a scalar field V, written as $\nabla^2 V$, is the divergence of the gradient of V

- In Cartesian coordinates, Laplacian is: $V = \nabla \cdot \nabla V = \nabla^2 V$

$$= \left[\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right] \cdot \left[\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right]$$

OR

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Laplacian of a Scalar

- In cylindrical coordinates:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

- In spherical coordinates:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Harmonic Scalar Field

- A scalar field *V* is said to be harmonic in a given region if its Laplacian vanishes in that region, that is:

$$\nabla^2 V = 0$$

- If the above equation is satisfied in a region, the solution for *V* in that region is harmonic

- Harmonic solution means it is of the form of sine or cosine

- Harmonic solution is familiar to the equation of harmonic motion

Problem-1

- Determine the flux of $\boldsymbol{D} = \rho^2 cos^2\varphi \boldsymbol{a_\rho} + zsin\varphi \boldsymbol{a_\varphi}$over the closed surface of the cylinder $0 \leq z \leq 1, \rho = 4$. Verify the divergence theorem for this case.