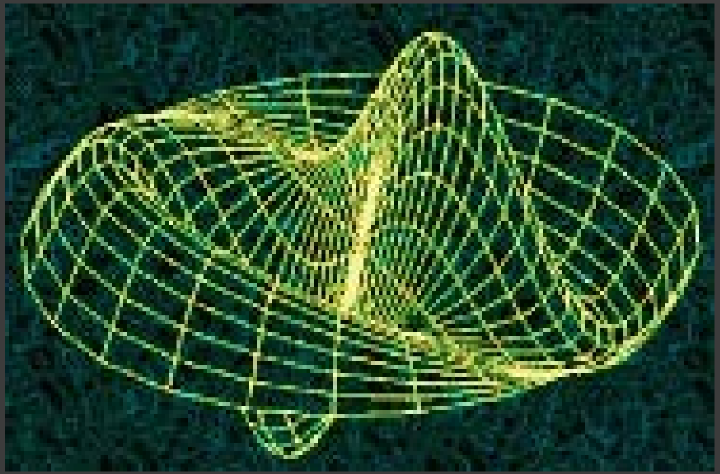
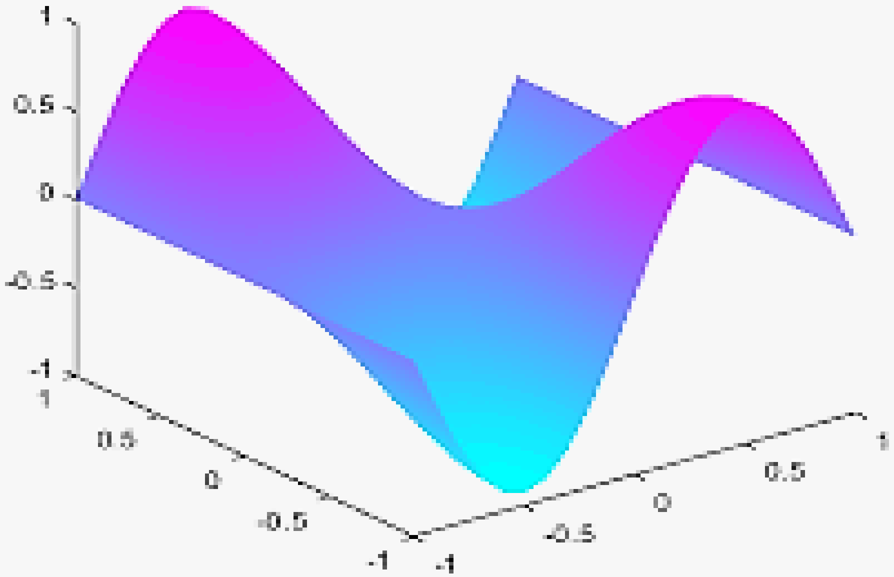
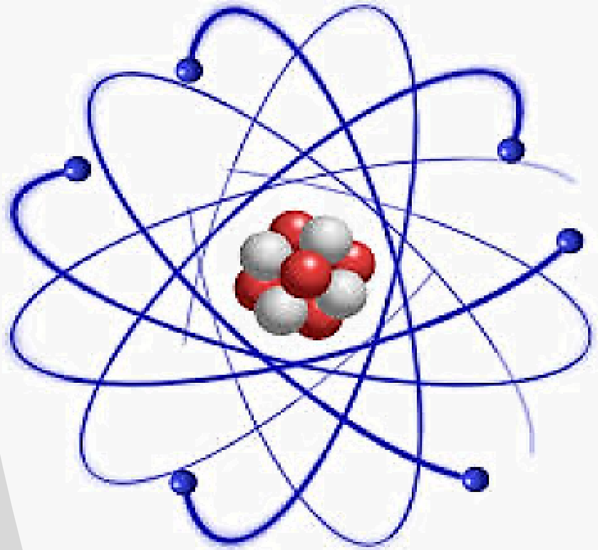
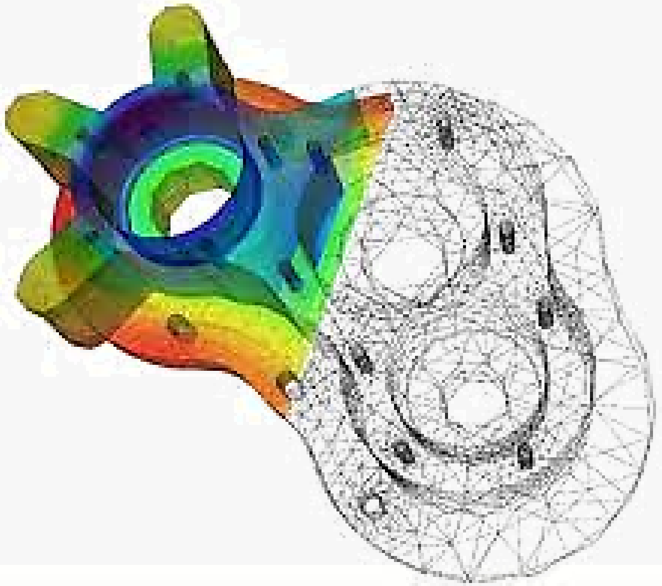


Partial Differential Equations







Vector Calculus(MATH-243)

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Partial Differential Equations

Book: Linear Partial Differential Equations for Scientists and Engineers (4th Edition) by Lokenath Debnath

- Chapter: 2
  - Sections: 2.2, 2.3, 2.7

Book: Advanced Engineering Mathematics (9th Edition) by Ervin Kreyszig

- Chapter: 12
  - Sections: 12.1

Different Integrals (Solutions) of a Partial Differential Equation

1. Complete Integral (solution):

Let

$$F\left(x, y, z, z_x, z_y\right) = 0, \qquad (1)$$

be the Partial Differential Equation. The **complete integral** of equation (1) is given by:

$$f(x, y, z, a, b) = 0. \qquad (2)$$

Thus, the system of surfaces (2) gives rise to a first-order partial differential equation. Role of complete integral is somewhat similar to that of a general solution for the case of an ordinary differential equation.

Different Integrals (Solutions) of a Partial Differential Equation

2. General Integral (solution):

Any relationship of the form:

$$f(\varphi, \psi) = 0 \qquad (3)$$

which involves an arbitrary function f of two known functions $\varphi = \varphi(x, y, z)$ and $\psi = \psi(x, y, z)$ and provides a solution of a first-order partial differential equation is called a **general solution** or **general integral** of this equation.

- Clearly, the general solution of a first-order partial differential equation depends on an arbitrary function. This is in striking contrast to the situation for ordinary differential equations where the general solution of a first order ordinary differential equation depends on one arbitrary constant.
- The general solution of a partial differential equation can be obtained from its complete integral.

Different Integrals (Solutions) of a Partial Differential Equation

2. General Integral (solution):

In equation (2), given as: $f(x, y, z, a, b) = 0$, assume an arbitrary relation of the form $b = b(a)$. Then (2) becomes:

$$f(x, y, z, a, b\,(a)) = 0 \quad \dots\dots\dots \ (4)$$

Differentiating (4) with respect to a, we get:

$$f_a\big(x, y, z, a, b\,(a)\big) + f_b\big(x, y, z, a, b\,(a)\big)b'(a) = 0. \qquad (5)$$

In principle, equation (3) can be solved for $a = a\,(x, y, z)$ as a function of x, y, and z. We substitute this result back in (4) to obtain

$$f\,\{x, y, z, a\,(x, y, z), b(a\,(x, y, z))\} = 0, \qquad (6)$$

where b is an arbitrary function. Indeed, the two equations (4) and (5) together define the general solution of first order PDE (1).

Different Integrals (Solutions) of a Partial Differential Equation

3. Particular Integral (solution):

- A solution obtained by giving particular values to the arbitrary constants in a $complete\ integral$ is called **particular solution**.
- When a definite $b(a)$ in (4) is prescribed, we obtain a particular solution from the $general\ solution$. Since the general solution depends on an arbitrary function, there are infinitely many solutions. In practice, only one solution satisfying prescribed conditions is required for a physical problem. Such a solution may be called a particular solution.

Different Integrals (Solutions) of a Partial Differential Equation

4. Singular Integral (solution):

The singular solution can easily be constructed from the complete solution representing a two-parameter family of surfaces. The envelope of this family is given by the system of three equations:

$$f(x, y, z, a, b) = 0, \qquad f_a = 0, \qquad f_b = 0. \qquad (7)$$

In general, it is possible to eliminate a and b from (7) to obtain the equation of the envelope which gives the singular solution. It may be pointed out that the singular solution cannot be obtained from the general solution. Its nature is similar to that of the singular solution of a first-order ordinary differential equation.

Techniques of Solving First Order PDEs: Separation of Variables

- Among various methods for solving partial differential equations, method of separation of variables is one of the most widely used technique to solve a PDE.
- The solution of the equation is separable, that is, the final solution can be represented as a product of several functions, for example, $u(x,y,z) = X(x)Y(y)Z(z) \neq 0,$ or sum of several functions, for example, $u(x,y,z) = X(x) + Y(y) + Z(z)$, each of which is only dependent upon a single independent variable.
- For instance, String displacement function $u(x,t) = X(x)T(t) \neq 0,$ is a product of two functions $X(x)$ & $T(t),$ where $X(x)$ is a function of only x, not t. On the other hand, $T(t)$ is a function of t, not x.
- By substituting the new product solution or sum solution form into the original PDE one can obtain a set of ordinary differential equations (hopefully), each of which involves only one independent variable. We solve these ODEs to determine solution of the given PDE.
- Usually, the first-order partial differential equation can be solved by separation of variables without the need for Fourier series.

Example:

Solve the initial-value problem:

$$u_x + 2u_y = 0; \quad u(0, y) = 4e^{-2y}. \qquad (1ab)$$

Solution:

We seek a separable solution $u(x, y) = X(x)Y(y) \neq 0$ and substitute into the given PDE $(1a)$ to obtain:

$$X'(x)Y(y) + 2X(x)Y'(y) = 0.$$

This can also be expressed in the form:

$$\frac{X'(x)}{2X\,(x)} = -\frac{Y'(x)}{Y\,(x)}. \qquad (2)$$

Since the left-hand side of this equation is a function of x only and the right-hand is a function of y only, it follows that (2) can be true if both sides are equal to the same constant value λ which is known as an arbitrary separation constant. Consequently, (2) gives two ordinary differential equations:

$$X'(x) - \ 2\lambda\, X(x) = 0, \qquad Y'(y) + \lambda\, Y(y) = 0. \qquad (3)$$

Solution:

By solving the set of ODEs (3), by the techniques used to solve ODEs, the solutions are respectively given as:

$$X(x) = Ae^{2\lambda x} \quad \text{and} \qquad Y(y) = Be^{-\lambda y},$$

where A and B are arbitrary integrating constants. Consequently, the general solution is given by:

$$u(x, y) = AB\exp(2\lambda x - \lambda y) = C\exp(2\lambda x - \lambda y), \qquad (4)$$

where $C = AB$ is an arbitrary constant. Using the given condition $(1b)$, we find:

$$4e^{-2y} = u(0, y) = Ce^{-\lambda y},$$

and hence, we deduce that $C = 4$ and $\lambda = 2$. Therefore, the (4) takes the form:

$$u(x, y) = 4\exp(4x - 2y).$$

Example:

Use $v(x,y) = \ln u(x,y)$ and $v(x,y) = f\ (x) + g\ (y)$ to solve the equation:

$$x^2 u_x^2 + y^2 u_y^2 = u^2. \qquad (1)$$

Solution:

Since $v = \ln u$ so $v_x = \frac{1}{u} u_x$ and $v_y = \frac{1}{u} u_y$. Thus, (1) takes the form:

$$x^2 v_x^2 + y^2 v_y^2 = 1. \qquad (2)$$

Using $v = f\ (x) + g\ (y)$ in (2) we get:

$$x^2\{f'(x)\}^2 + y^2\{g'(y)\}^2 = 1 \Longrightarrow x^2\{f'(x)\}^2 = 1 - y^2\{g'(y)\}^2 = \lambda^2,$$

where λ^2 is a separation constant. Thus, we obtain two ordinary differential equations:

$$f'(x) = \frac{\lambda}{x}, \qquad \text{and} \quad g'(y) = \frac{\sqrt{1-\lambda^2}}{y}. \qquad (3)$$

Solution:

Solving these two first order ODEs we get:

$$f(x) = \lambda \ln|x| + A \qquad \text{and} \qquad g(y) = \sqrt{1-\lambda^2}\ln|y| + B,$$

where A and B are arbitrary constants. Therefore, the general solution of (2) is given as:

$$v(x,y) = \lambda \ln|x| + \sqrt{1-\lambda^2}\ln|y| + \ln|C| = \ln\left(Cx^{\lambda}.\, y^{\sqrt{1-\lambda^2}}\right),$$

Where $\ln|C| = A + B$. Rewriting in terms of u, the final solution of the given PDE comes out to be:

$$u(x,y) = e^{v(x,y)} = Cx^{\lambda}.\, y^{\sqrt{1-\lambda^2}}.$$

Practice:

1. Solve the initial-value problem:

$$y^2u_x^2 + x^2u_y^2 = (xyu)^2; \quad u(x,0) = 3e^{(x^2/4)}.$$

2. Use the separation of variables $u(x,y) = f(x) + g(y)$ to solve the equation:

$$u_x^2 + u_y^2 = 1.$$

3. Use the separation of variables $u(x,y) = f(x) + g(y)$ to solve the equation:

$$u_x^2 + u_y + x^2 = 0.$$

Second Order Partial Differential Equations

- Partial differential equations arise frequently in formulating fundamental laws of nature and in the study of a wide variety of physical, chemical, and biological models.
- We will be interested in the study of a special type of second-order linear partial differential equation for the following reasons.
- First, second-order linear equations arise more frequently in a wide variety of applications.
- Second, their mathematical treatment is simpler and easier to understand than that of first-order equations in general.
- Usually, in almost all physical phenomena (or physical processes), the dependent variable $u = u\,(x, y, z, t)$ is a function of three space variables, x, y, z and time variable t.

Second Order Partial Differential Equations

The three basic types of second-order partial differential equations are:

(a) The wave equation

$$u_{tt} - c^2\,(u_{xx} + u_{yy} + u_{zz}) \;=\; 0.$$

(b) The heat equation

$$u_t - k\,(u_{xx} + u_{yy} + u_{zz}) \;=\; 0.$$

(c) The Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = \; 0.$$

Second Order Partial Differential Equations

We now list a few more common linear partial differential equations of importance in applied mathematics, mathematical physics, and engineering science.

(d) The Poisson equation

$$\nabla^2 u = f(x, y, z).$$

(e) The Helmholtz equation

$$\nabla^2 u + \lambda u = 0.$$

(f) The biharmonic equation

$$\nabla^4 u = \nabla^2 \left(\nabla^2 u\right) = 0.$$

(g) The biharmonic wave equation

$$u_{tt} + c^2 \nabla^4 u = 0.$$

(h) The telegraph equation

$$u_{tt} + a u_t + bu = c^2 u_{xx}.$$

(i) The Schrödinger equations in quantum physics

$$i\hbar\psi_t = \left[\left(-\frac{\hbar^2}{2m}\right)\nabla^2 + V(x, y, z)\right]\psi,$$

$$\nabla^2 \Psi + \frac{2m}{\hbar^2}\left[E - V(x, y, z)\right]\Psi = 0.$$

(j) The Klein–Gordon equation

$$\Box u + \lambda^2 u = 0,$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

is the Laplace operator in rectangular Cartesian coordinates (x, y, z),

$$\Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2},$$

is the d'Alembertian, and in all equations λ, a, b, c, m, E are constants and $h = 2\pi\hbar$ is the Planck constant.

Second Order PDEs: PDEs solvable as ODEs

For some PDEs which do not involve partial derivatives with respect to both independent variables x & y, we can solve them by solving their ODE analogues.

Example:

Determine solution of the PDE:

$$u_{xx} - u = 0,$$

where $u = u(x, y)$.

Solution:

Since no y −derivative occurs so we can solve this PDE like:

$$u'' - u = 0.$$

Solving above using the techniques of second order linear ODEs with constant coefficients

we get:

$$u(x, y) = A(y)e^{x} + B(y)e^{-x},$$

with arbitrary functions A and B.

Example:

Determine solution $u = u(x, y)$ of the PDE:

$$u_{xy} + u_x = 0,$$

Solution:

Let $u_x = p$ so that $u_{xy} = p_y$ and $p_y = -p$. Thus,

$$\frac{p_y}{p} = -1 \Longrightarrow \ln|p| = -y + \tilde{c}(x) \Longrightarrow p = c(x)e^{-y}.$$

Since $u_x = p$, so $u_x = f(x)e^{-y}$ and by integration with respect to x we get:

$$u(x, y) = f(x)e^{-y} + g(y),$$

where $f(x) = \int c(x)\, dx$ and $g(y)$ are arbitrary functions.

Practice Questions

Book: Linear Partial Differential Equations for Scientists and Engineers (4th Edition) by Lokenath Debnath

Chapter: 1

Exercise – 1.6: Q – 1 to 4, Q – 9 to 25.

Chapter: 2

Exercise – 2.8: Q – 1 to 2, Q – 25 to 29, Q – 31.

Book: Advanced Engineering Mathematics (9th Edition) by Ervin Kreyszig

Chapter: 12

Exercise – 12.1: Q – 2 to 13, Q – 16 to 23.