Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

Chapter: 11 (11.8, 11.9)

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

Chapter: 11 (11.8)

Calculus & Analytical Geometry MATH-101
Instructor: Dr. Naila Amir (SEECS, NUST)

$$Taylor\ Series: \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor & Maclaurin Series

$$Maclaurin\ Series: \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \qquad (1)$$

where:

- x is a variable.
- c_n are constants called the coefficients of the series.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots \qquad (2)$$

is called a power series centered at a.

Theorem

For a given power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n,$$

exactly one of the following three possibilities is true:

I. The series converges only when $x = a$.

II. The series converges for all x.

III. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Radius Of Convergence & Interval Of Convergence

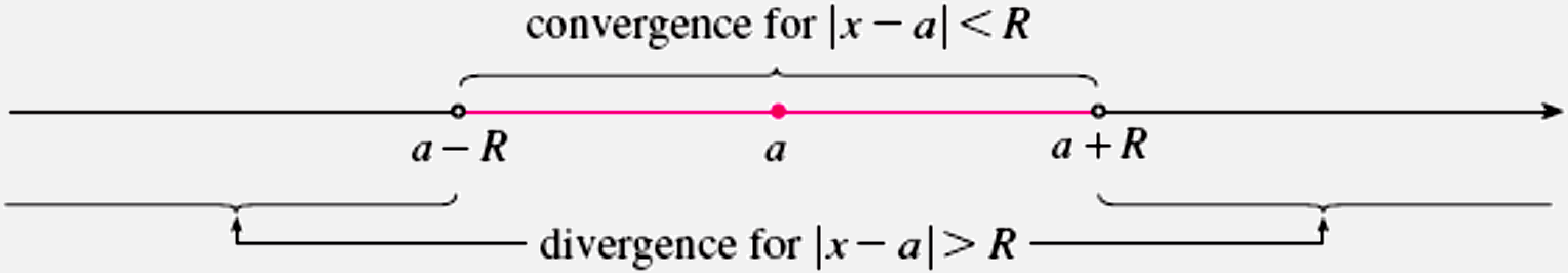
- The number R in (III) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case I and $R = \infty$ in case II.
- The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case I, the interval consists of just a single point a. In case II, the interval is $(-\infty, \infty)$. In case III, the interval of convergence is given by the inequality $|x - a| < R$ that can be rewritten as $a - R < x < a + R$.
- When x is an endpoint of the interval, that is, $x = a \pm R$, anything can happen: The series might converge at one or both endpoints. It might diverge at both endpoints.

Radius Of Convergence & Interval Of Convergence

Thus, in case III, there are four possibilities for the interval of convergence:

1. $(a - R, a + R)$
2. $(a - R, a + R]$
3. $[a - R, a + R)$
4. $[a - R, a + R]$

$|x - a| < R$



Example

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n n(x+3)^n}{4^n}.$$

Solution:

For the present case we have:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n(-1)(n+1)(x+3)^n(x+3)}{4^n(4)} \cdot \frac{4^n}{(-1)^n n(x+3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)(n+1)(x+3)}{4n}\right| = \frac{|x+3|}{4}.$$

Example

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+3|}{4} = L$$

By the Ratio Test, the given series is:

$\Rightarrow |x-a| < \underline{R}$

- Convergent, when $\frac{|x+3|}{4} < 1$ or $|x+3| < \boxed{4.}$
- Divergent, when $\frac{|x+3|}{4} > 1$ or $|x+3| > 4$,
- Thus, the **radius of convergence** is $R = 4$.

Note that: $|x+3| < 4 \Leftrightarrow -4 < x+3 < 4 \Leftrightarrow -7 < x < 1$. So, we test the series at the endpoints –7 and 1.

Example

$a_n = n \to \infty$

- When $x = -7$, the series becomes:

$$\sum_{n=0}^{\infty} \frac{(-1)^n n(x+3)^n}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n n(-4)^n}{4^n} = \sum_{n=0}^{\infty} (-1)^{2n} n = \sum_{n=0}^{\infty} n,$$

as $n \to \infty$

which is a divergent series by nth term test for divergence.

- If we put $x = 1$, the series is given as:

$$\sum_{n=0}^{\infty} \frac{(-1)^n n(x+3)^n}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n n(4)^n}{4^n} = \sum_{n=0}^{\infty} (-1)^n n.$$

This is a divergent alternating series since nth term does not approach 0 as $n \to \infty$. Thus, the given power series converges only when $-7 < x < 1$. Thus, the **interval of convergence** is: $(-7,1)$.

Example

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} n!\,(2x+1)^n.$$

$\sum c_n(x-a)^n$

$a = \frac{-1}{2}$

Solution:

For the present case we have:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(2x+1)^{n+1}}{n!\,(2x+1)^n}\right|$$

$$= \lim_{n\to\infty}|(n+1)(2x+1)| = \begin{cases} 0; & x = -1/2 \\ \infty; & x \neq -1/2 \end{cases}.$$

Example

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} 0; & x = -1/2 \\ \infty; & x \neq -1/2 \end{cases}$$

By the Ratio Test, the given series is:

- Convergent, only when $x = -1/2$.
- Divergent, when $x \neq -1/2$.
- Thus, the **radius of convergence** is: $R = 0$, because $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ when $x = -1/2$ and **interval of convergence** is: $\left\{-\frac{1}{2}\right\}$.

Taylor & Maclaurin Series

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$$Taylor\ Series: \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$Maclaurin\ Series: \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

Taylor and Maclaurin Series

Suppose that $f(x)$ is any function that can be represented by a power series:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \qquad \text{(I)}$$

with $|x - a| < \boldsymbol{R}$. Let's try to determine the coefficients c_n in terms of $f(x)$. To begin, notice that if we put $x = a$ in equation (I), then all terms after the first one are, 0 and we get:

$$f(a) = c_0.$$

We can differentiate the series in equation (I) term by term:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \qquad \text{(II)}$$

with $|x - a| < \boldsymbol{R}$, and substitution of $x = a$ in Equation 2 gives:

$$f'(a) = c_1.$$

Taylor and Maclaurin Series

Now we differentiate the series in equation (II) term by term and obtain:

$$f''(x) = 2c_2 + 2.3c_3(x-a) + 3.4c_4\,(x-a)^2 + \cdots \qquad \text{(III)}$$

with $|x - a| < \boldsymbol{R}$. Again, put $x = a$ in equation (III) gives:

$$f''(a) = 2c_2 = 2!\,c_2.$$

Let's apply the procedure one more time. Differentiation of the series in equation (III) gives:

$$f'''(x) = 2.3c_3 + 2.3.4c_4(x-a) + 3.4.5c_5\,(x-a)^2 + \cdots \qquad \text{(IV)}$$

with $|x - a| < \boldsymbol{R}$, and substitution of $x = a$ in equation (IV) gives:

$$f'''(a) = 2.3c_3 = 3!\,c_3.$$

Taylor and Maclaurin Series

By now we can see a pattern. If we continue to differentiate and substitute $x = a$, we obtain:

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot nc_n = n!\, c_n.$$

Solving this equation for the nth coefficient c_n, we get:

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

This formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)} = f$. Thus, we have proved the following theorem:

Theorem

- If $f(x)$ has a power series representation at a, i.e., if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n; \qquad |x-a| < R,$$

then its coefficients are given by the formula:

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Substituting this formula for c_n back into the series, we see that if $f(x)$ has a power series expansion at a, then it must be of the following form:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots. \quad (*)$$

Taylor Series

The series in equation $(*)$ is called the **Taylor series of the function $\boldsymbol{f(x)}$ at $\boldsymbol{a}$** (or **about $\boldsymbol{a}$** or **centered at $\boldsymbol{a}$**). Thus, we say that:

If $f(x)$ is a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series** generated by $f(x)$ at $x = a$ is:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

Taylor & Maclaurin Series

For the special case: $a = 0$, the Taylor series becomes:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots.$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

Example:

Determine the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Solution:

If $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all n. Therefore, the Taylor series for $f(x)$ at 0 (that is, the Maclaurin series) is given as:

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots.$$

To find the radius of convergence we let $a_n = x^n/n!$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} . \frac{n!}{x^n}\right| = |x| \lim_{n\to\infty} \left|\frac{n!}{(n+1).n!}\right| = |x| \lim_{n\to\infty} \left|\frac{1}{(n+1)}\right| = 0 < 1.$$

so, by the Ratio Test, the series converges for all x and the radius of convergence is $R = \infty$.

Taylor & Maclaurin Series

The conclusion we can draw from the theorem and the previous example is that if e^x has a power series expansion at 0, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So, a natural question that arises at this point is that how can we determine whether e^x does have a power series representation or in other words: under what circumstances is e^x equal to the sum of its power series representation?

Taylor & Maclaurin Series

- Let’s investigate the more general question: Under what circumstances is a function equal to the sum of its Taylor series?

- In other words, if $f(x)$ has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- As with any convergent series, this means that $f(x)$ is the limit of the sequence of partial sums.

Taylor & Maclaurin Series

In the case of the Taylor series, the partial sums are:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notice that $T_n(x)$ is a polynomial of degree n called the **$\boldsymbol{n}$th-degree Taylor polynomial** of $f(x)$ at a.

Taylor & Maclaurin Series

For instance, for the exponential function $f(x) = e^x$, the Taylor polynomials at 0 (or Maclaurin polynomials) with $n = 1, 2,$ and 3 are:

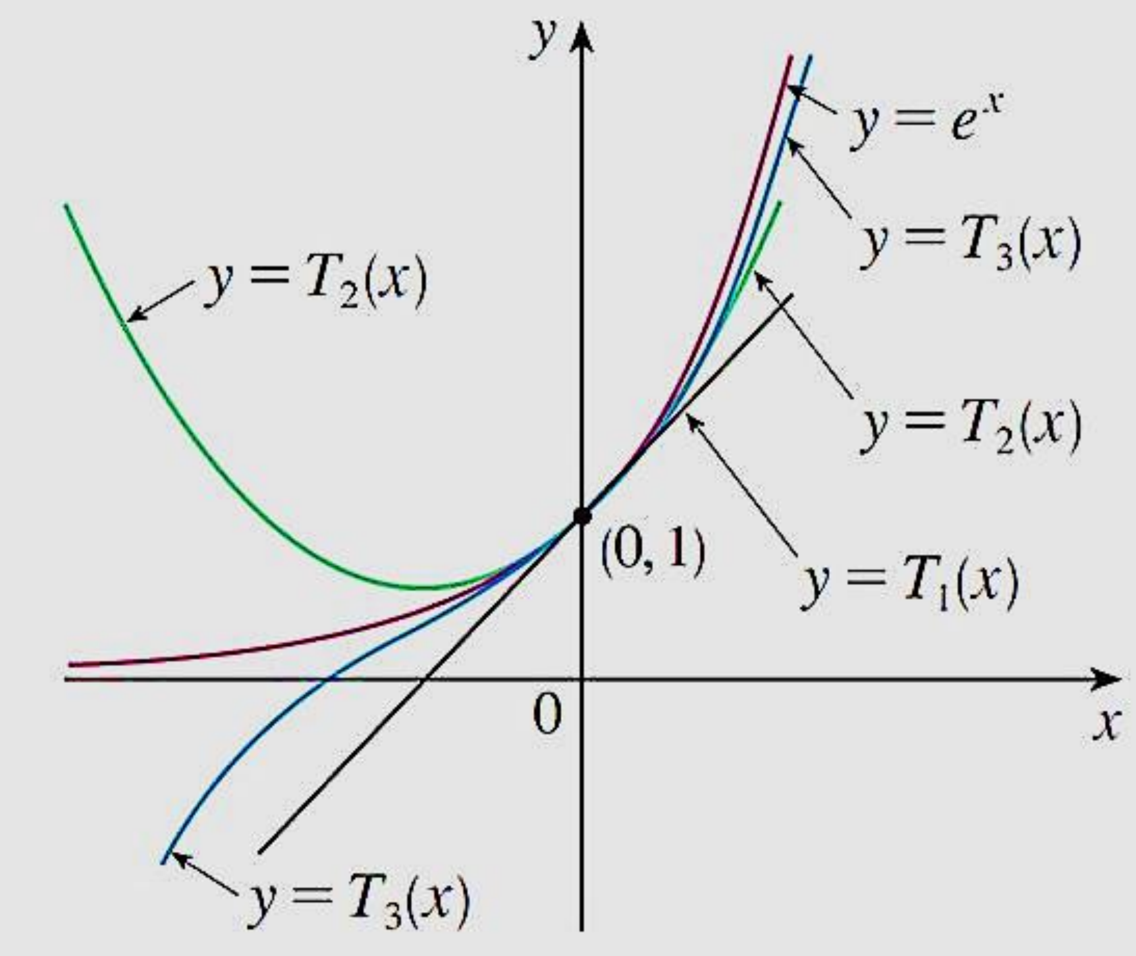
$$T_1(x) = 1 + x$$

$$T_2(x) = 1 + x + \frac{x^2}{2!}$$

$$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Taylor & Maclaurin Series

The graphs of the exponential function and the three Taylor polynomials with $n = 1, 2,$ and 3 are shown in the following figure:



$$T_1(x) = 1 + x$$

$$T_2(x) = 1 + x + \frac{x^2}{2!}$$

$$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Taylor & Maclaurin Series

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

$\{T_n(x)\}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

If we let

$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = \boxed{T_n(x)} + R_n(x)$$

then $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, for some c between x and a is called the **remainder** of the Taylor series. If we can show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that:

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}[f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) = f(x).$$

Taylor & Maclaurin Series

- We have therefore proved the following:

  If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the nth-degree Taylor polynomial of $f(x)$ at a and

  $$\lim_{n\to\infty} R_n(x) = 0,$$

  for $|x - a| < R$, then $f(x)$ is equal to the sum of its Taylor series on the interval $|x - a| < R$.

- In trying to show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function $f(x)$, we usually use the following theorems:

Theorems

Taylor's Inequality/The Remainder Estimation Theorem:
If $|f^{(n+1)}(c)| \le M$ for all c between x and a, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality:

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

If this condition holds for every n then, the series converges to $f(x)$.

Theorem:
If x is any real number, then

$$\lim_{n\to\infty} \frac{|x|^n}{n!} = 0.$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

0

Example

Prove that $f(x) = e^x$ is equal to the sum of its Maclaurin series.

Solution:

If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n. Note that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x); \qquad R_n(x) = \frac{e^c}{(n+1)!}x^{n+1} \text{ for some } c \text{ between } 0 \text{ and } x.$$

Since e^x is an increasing function of x, e^c lies between $e^0 = 1$ and e^x. When x is negative, so is c, and $e^c < 1$. When $x = 0$, $e^x = 1$ and $R_n(x) = 0$. When x is positive, so is c and $e^c < e^x$. Thus,

$$\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}; \quad \text{for } x \le 0 \quad \text{and} \quad \lim_{n\to\infty} |R_n(x)| < \lim_{n\to\infty} \frac{e^x|x|^{n+1}}{(n+1)!}; \text{for } x > 0$$

In both cases, $\lim_{n\to\infty} |R_n(x)| = 0 \Longrightarrow \lim_{n\to\infty} R_n(x) = 0$. Hence, e^x is equal to the sum of its Maclaurin series.

Example:

Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Solution:

We need to determine $f(2), f'(2), f''(2), \dots$. Differentiating the given function n times and finding values at $x = 2$, we get:

$$f(x) = x^{-1}; \qquad f(2) = (2)^{-1} = \frac{1}{2},$$

$$f'(x) = -x^{-2} = (-1)1!\,x^{-2}; \qquad f'(2) = -(2)^{-2} = -\frac{1}{2^2},$$

$$f''(x) = (-1)(-2)x^{-3} = (-1)^2 2!\,x^{-3}; \qquad \frac{f''(2)}{2!} = (-1)^2\frac{1}{2^3},$$

Solution:

- $$f'''(x) = (-1)(-2)(-3)x^{-4} = (-1)^3 3!\, x^{-4}; \qquad \frac{f'''(2)}{3!} = (-1)^3 \frac{1}{2^4},$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}; \qquad \frac{f^{(n)}(2)}{n!} = (-1)^n \frac{1}{2^{n+1}}.$$

Thus, the Taylor series generated by $f(x) = 1/x$ at $a = 2$ is:

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \frac{(x-2)^3}{2^4} + \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots.$$

This is a geometric power series with $a = 1/2$ and ratio $r = -(x-2)/2$. This series converges absolutely for $|x-2| < 2$ and its sum is given as:

$$\frac{1/2}{1 + (x-2)/2} = \frac{1}{x}.$$

Example:

Find the Maclaurin series generated by $f(x) = (1+x)^k$, where k is any real number.

Solution:

For the present case:

$$f(x) = (1+x)^k; \qquad f(0) = 1,$$

$$f'(x) = k(1+x)^{k-1}; \qquad f'(0) = k,$$

$$f''(x) = k(k-1)(1+x)^{k-2}; \qquad f''(0) = k(k-1),$$

$$f'''(x) = k(k-1)(k-2)(1+x)^{k-2}; \qquad f''(0) = k(k-1)(k-2),$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-3};$$

and $$f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1).$$

Example:

Thus, the Maclaurin series generated by $f\ (x) = (1+x)^k$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

This series is called the **binomial series**. Notice that if k is a nonnegative integer, then the terms are eventually 0 and so the series is finite. For other values of k none of the terms is 0 and so we can try the Ratio Test. For the present case:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)\,x^n}\right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{k-n}{n+1}\right| = |x|,$$

so, by the Ratio Test, the series converges when $|x| < 1$, and the radius of convergence is $R = 1$.

Some Important Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Multiplication & Division of Power Series

Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.

Solution:

Using the Maclaurin series for e^x and $\sin x$, we have:

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)$$

We multiply these expressions, collecting like terms just as for polynomials:

$$\begin{array}{rlllll}
 & 1 + x & + \frac{1}{2}x^2 & + \frac{1}{6}x^3 & + \cdots & \\
\times & \quad\; x & & - \frac{1}{6}x^3 & + \cdots & \\
\hline
 & \quad\; x + & x^2 & + \frac{1}{2}x^3 & + \frac{1}{6}x^4 & + \cdots \\
+ & & & - \frac{1}{6}x^3 & - \frac{1}{6}x^4 & - \cdots \\
\hline
 & \quad\; x + & x^2 & + \frac{1}{3}x^3 & + \cdots &
\end{array}$$

Multiplication & Division of Power Series

Thus, we have:

$$e^x \sin x = x + x^2 + \tfrac{1}{3}x^3 + \cdots$$

Solution: (b) $\tan x$.

Note that

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}$$

Multiplication & Division of Power Series

We use a procedure like long division:

$$\begin{array}{r@{}l}
& x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots \\ \cline{2-2}
1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \cdots \big) & x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \cdots \\
& x - \tfrac{1}{2}x^3 + \tfrac{1}{24}x^5 - \cdots \\ \cline{2-2}
& \tfrac{1}{3}x^3 - \tfrac{1}{30}x^5 + \cdots \\
& \tfrac{1}{3}x^3 - \tfrac{1}{6}x^5 + \cdots \\ \cline{2-2}
& \tfrac{2}{15}x^5 + \cdots
\end{array}$$

Thus, we get:

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$$

Practice Questions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Exercise: 11.8

  Q # 1 to Q # 28

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 11.8

  Q # 1 to Q # 6, Q # 9 to Q # 28