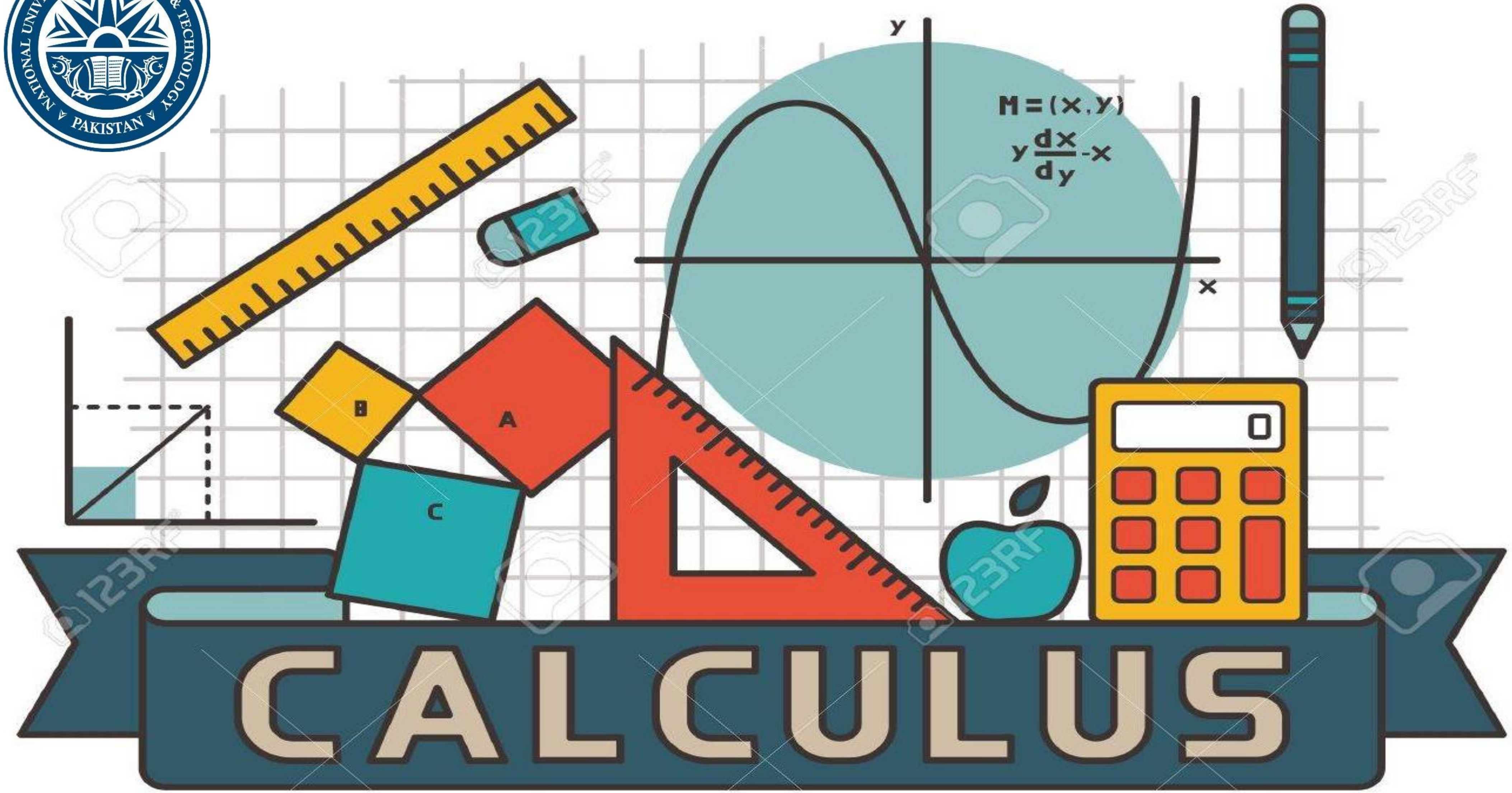


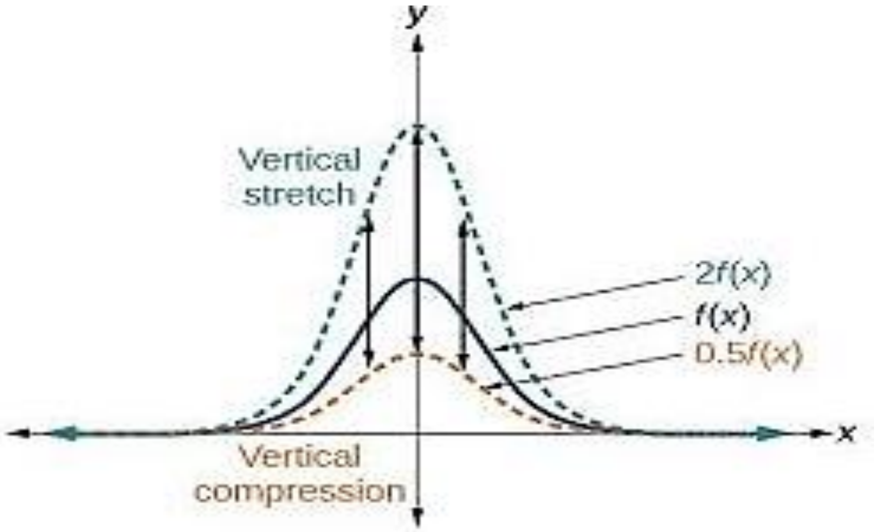
Calculus & Analytical Geometry
MATH- 101

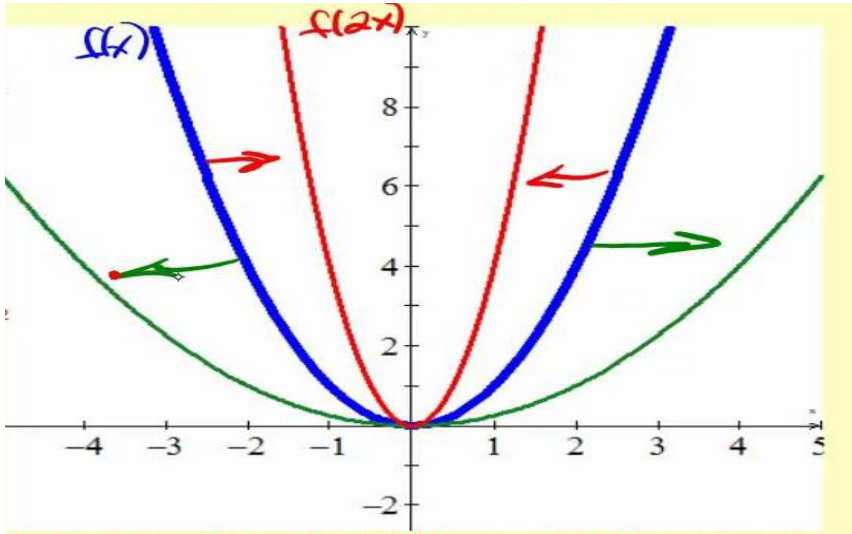
Instructor: Dr. Naila Amir
(SEECS, NUST)

Scaling Graphs

Scaling a graph of a function

- ❖ Scaling is a non-rigid translation in which the size and shape of the graph of a function is changed.
- ❖ To scale the graph of a function we stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function f, or the independent variable x, by an appropriate constant.
- ❖ A vertical scaling multiplies/divides every y −coordinate by a constant while leaving the x −coordinate unchanged.
- ❖ A horizontal scaling multiplies/divides every x −coordinate by a constant while leaving the y −coordinate unchanged.

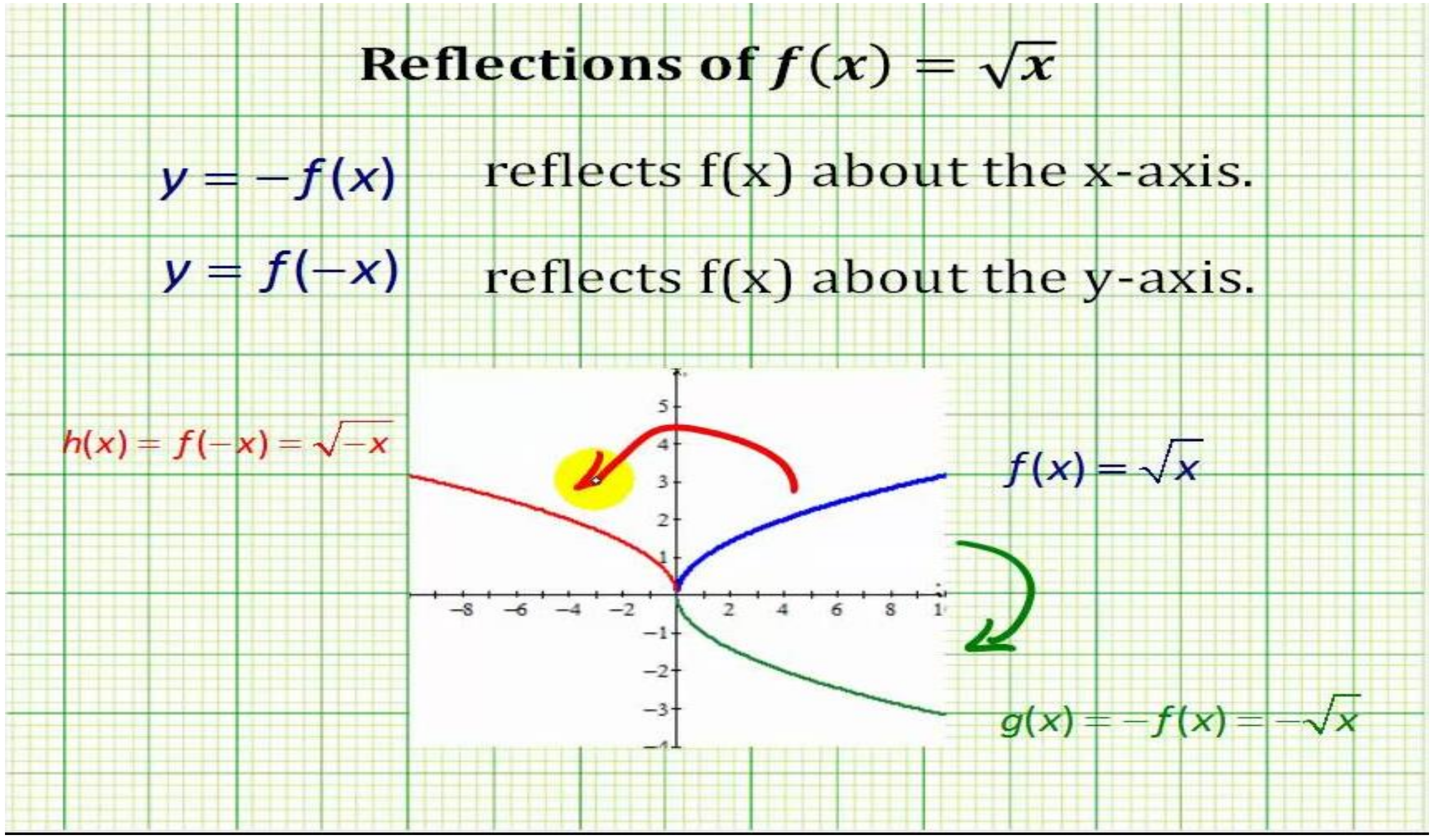




Horizontal Stretch and Horizontal Compression

Reflection a graph of a function

- A translation in which the graph of a function is mirrored about an axis.
- Reflections are just a special case of the scaling.
- To reflect about the y −axis, multiply every x by -1 to get $-x$.
- To reflect about the x-axis, multiply $f(x)$ by -1 to get $-f(x)$.



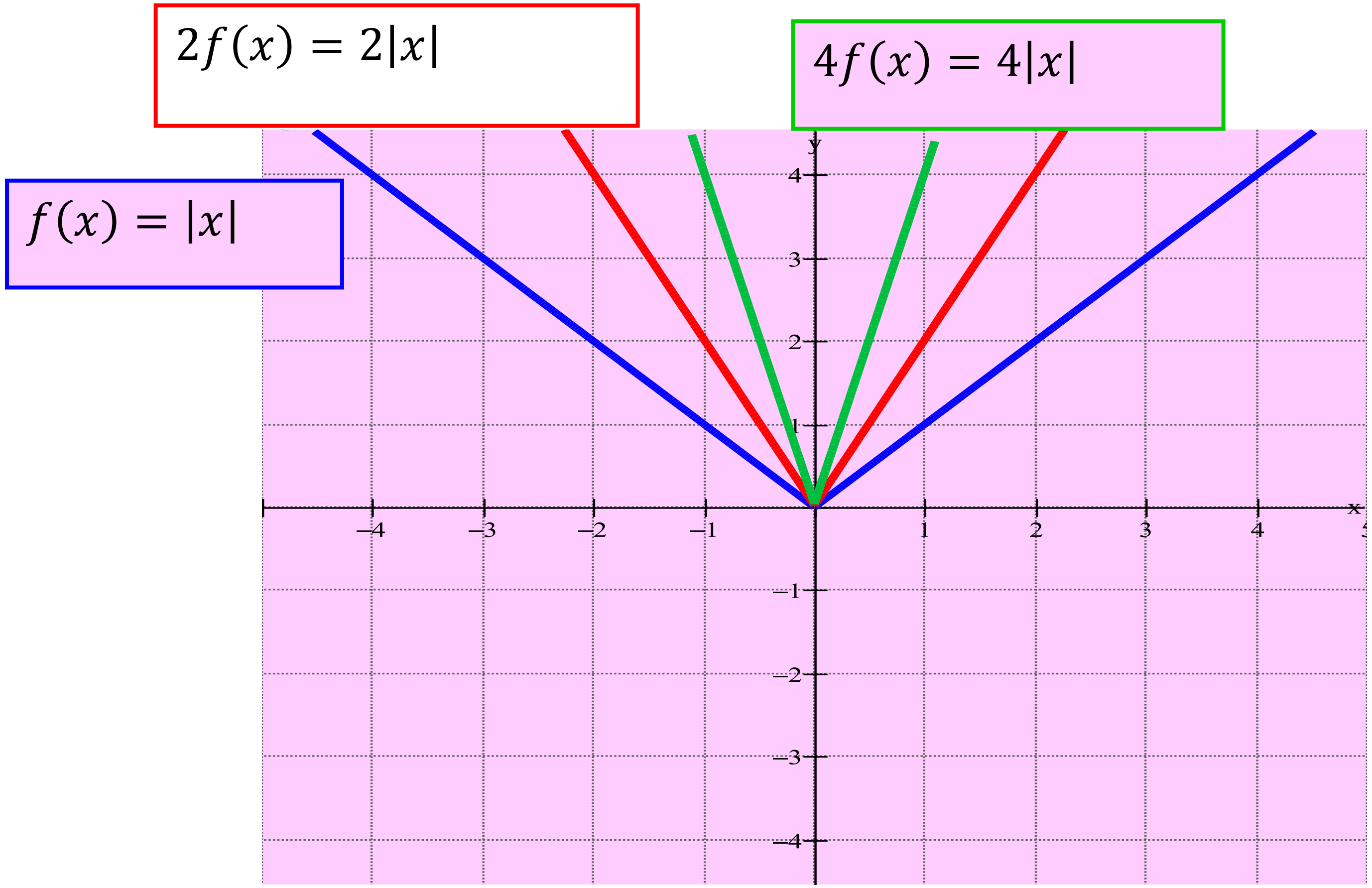
Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$,

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c}f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$,

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

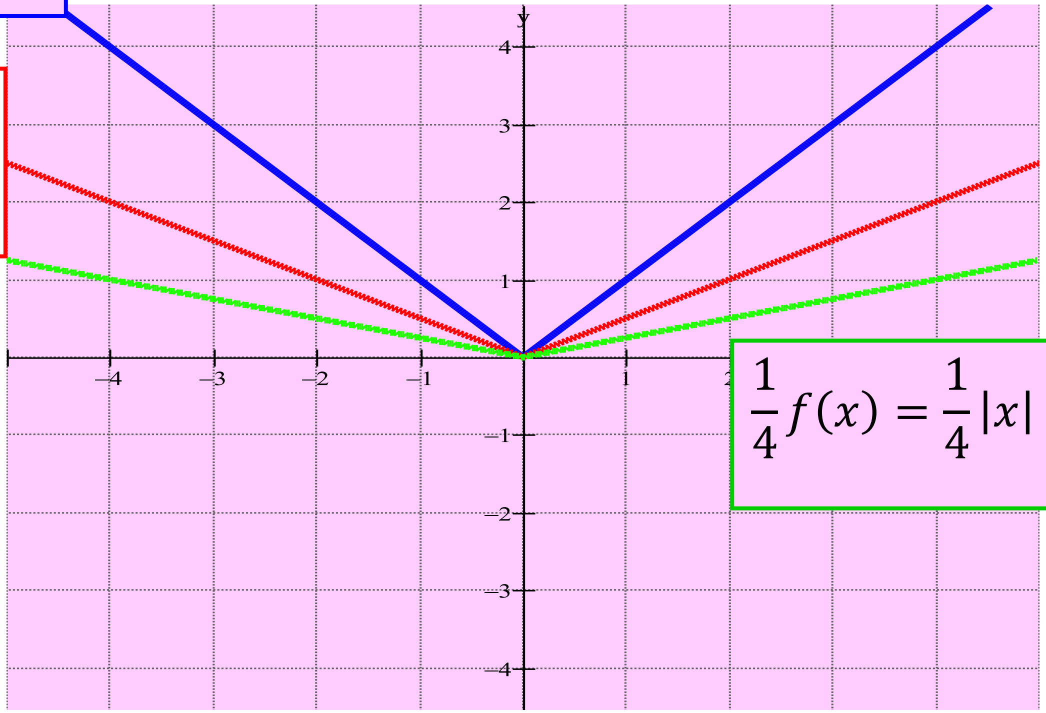


The graph $a\ f(x)$, where $a\ >\ 1$, is the graph of $f(x)$ but vertically stretched by a factor of "a".

What if the value of $\boldsymbol{a}$ was positive but less than 1?

$f(x) = |x|$

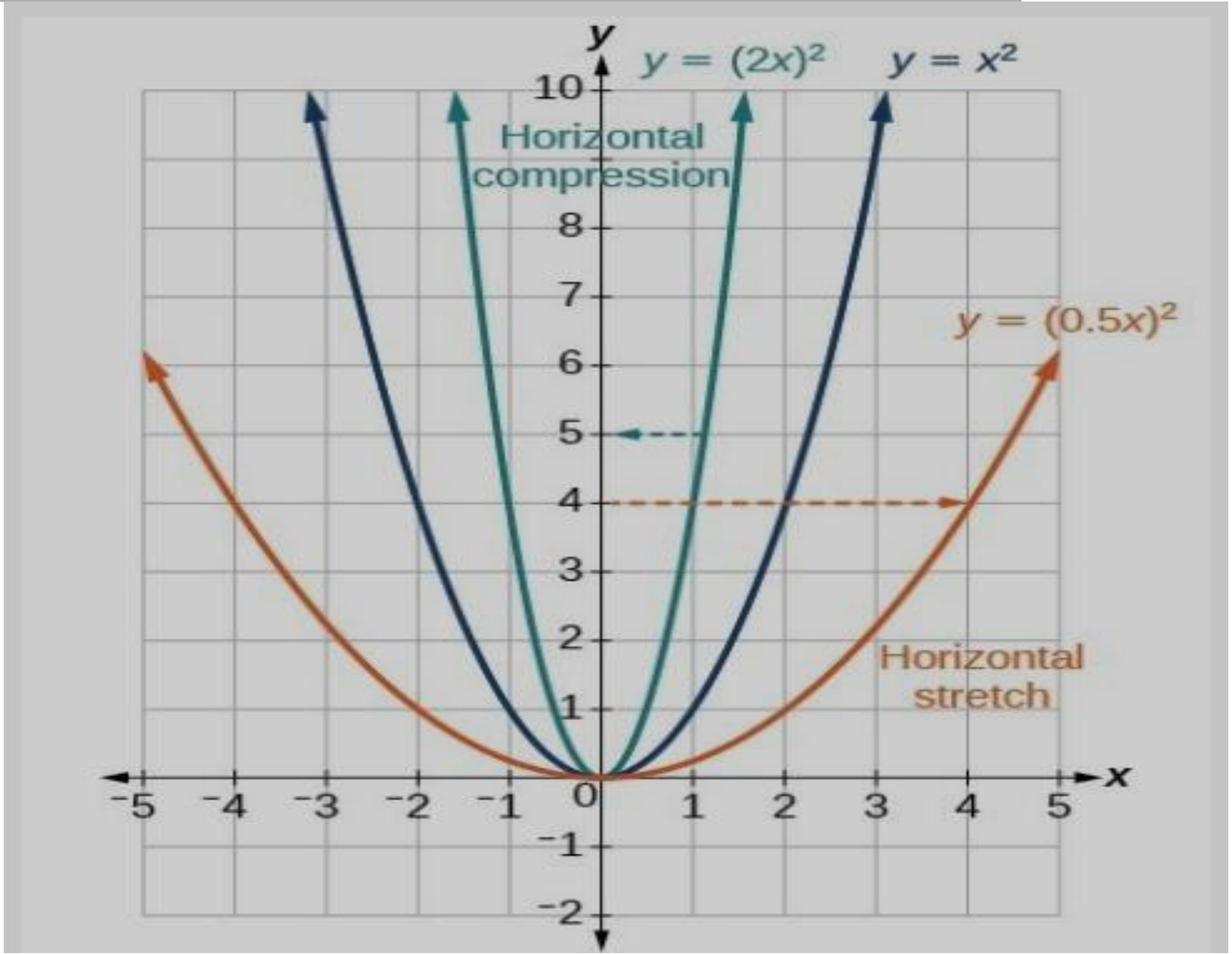
$\frac{1}{2} f(x) = \frac{1}{2}|x|$



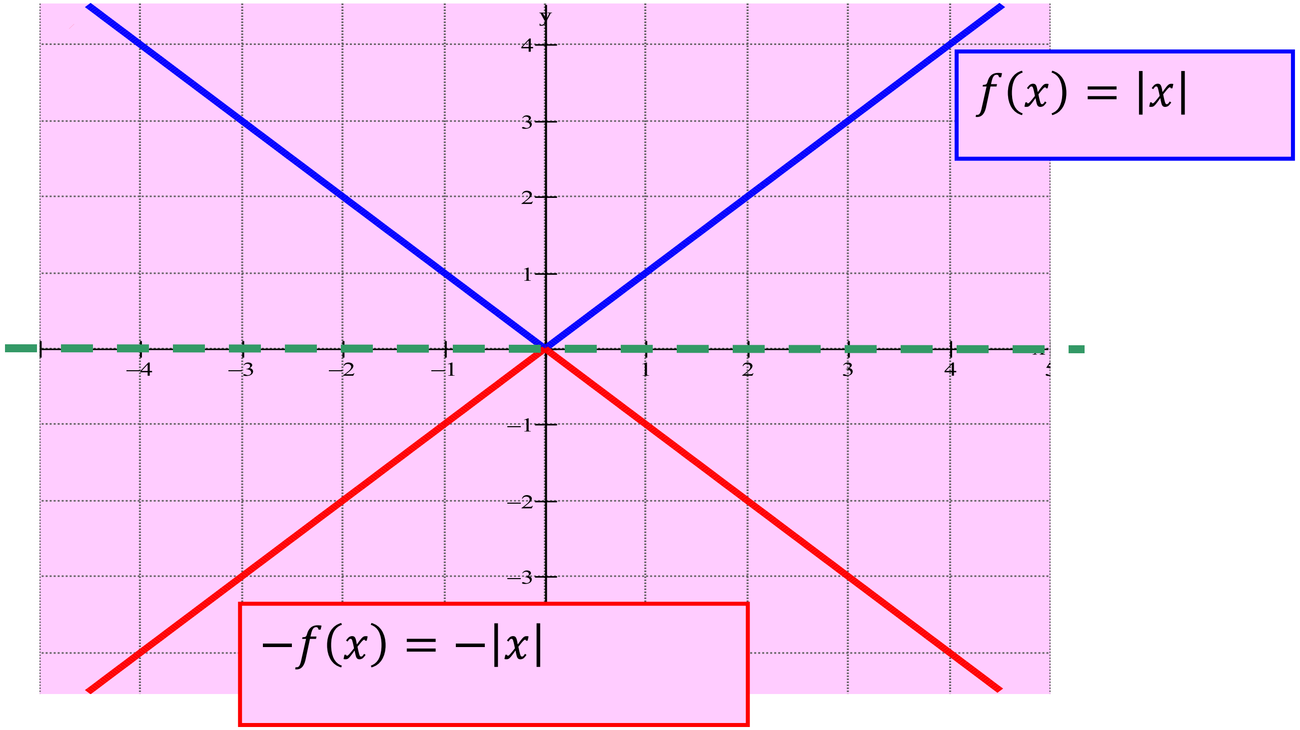
$\frac{1}{4} f(x) = \frac{1}{4}|x|$

The graph $af(x)$, where $0 < a < 1$, is the graph of $f(x)$ but vertically compressed by a factor of a.

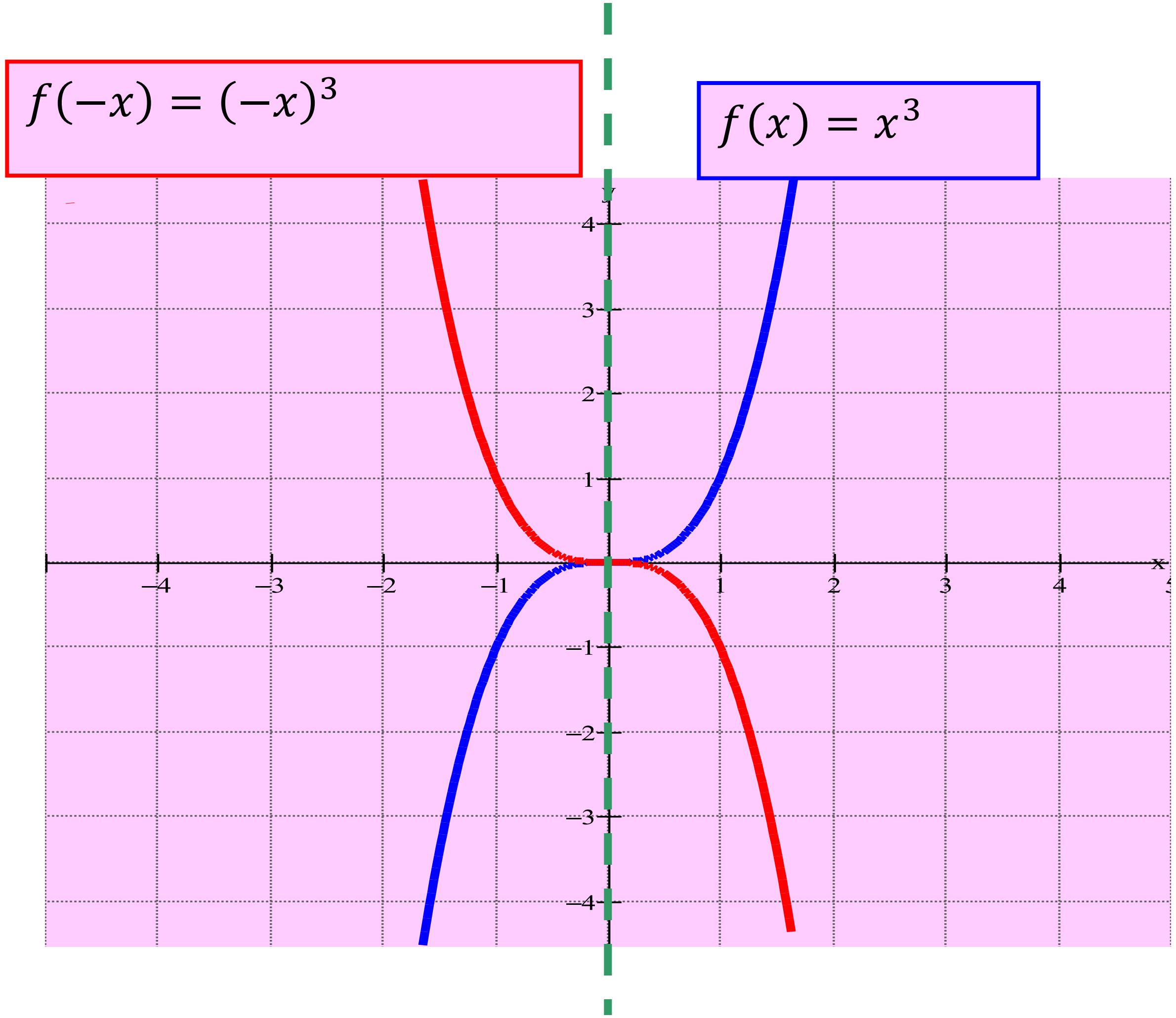
Horizontal Stretches and Compressions



What if the value of $\boldsymbol{a}$ was negative?



So the graph $-f(x)$ is a reflection about the x −axis of the graph of $f(x)$. The new graph is obtained by "flipping“ or reflecting the function over the x −axis.



The graph $f(-x)$ is a reflection about the y −axis of the graph of $f(x)$.

Summary

If $a > 1$, then vertical stretch by a factor of a.

If $0 < a < 1$, then vertical compression by a factor of a.

If $a < -1$, then reflection of graph.

$f(-x)$ reflection about y −axis

vertical translation by k

$$a\ f(x-h)+k$$

horizontal translation by h

Note: We always perform reflections before vertical and horizontal translations

Summary:

We can do all transformation **in one go** using this:

$$a\, f(\, b(x + c))+ d$$

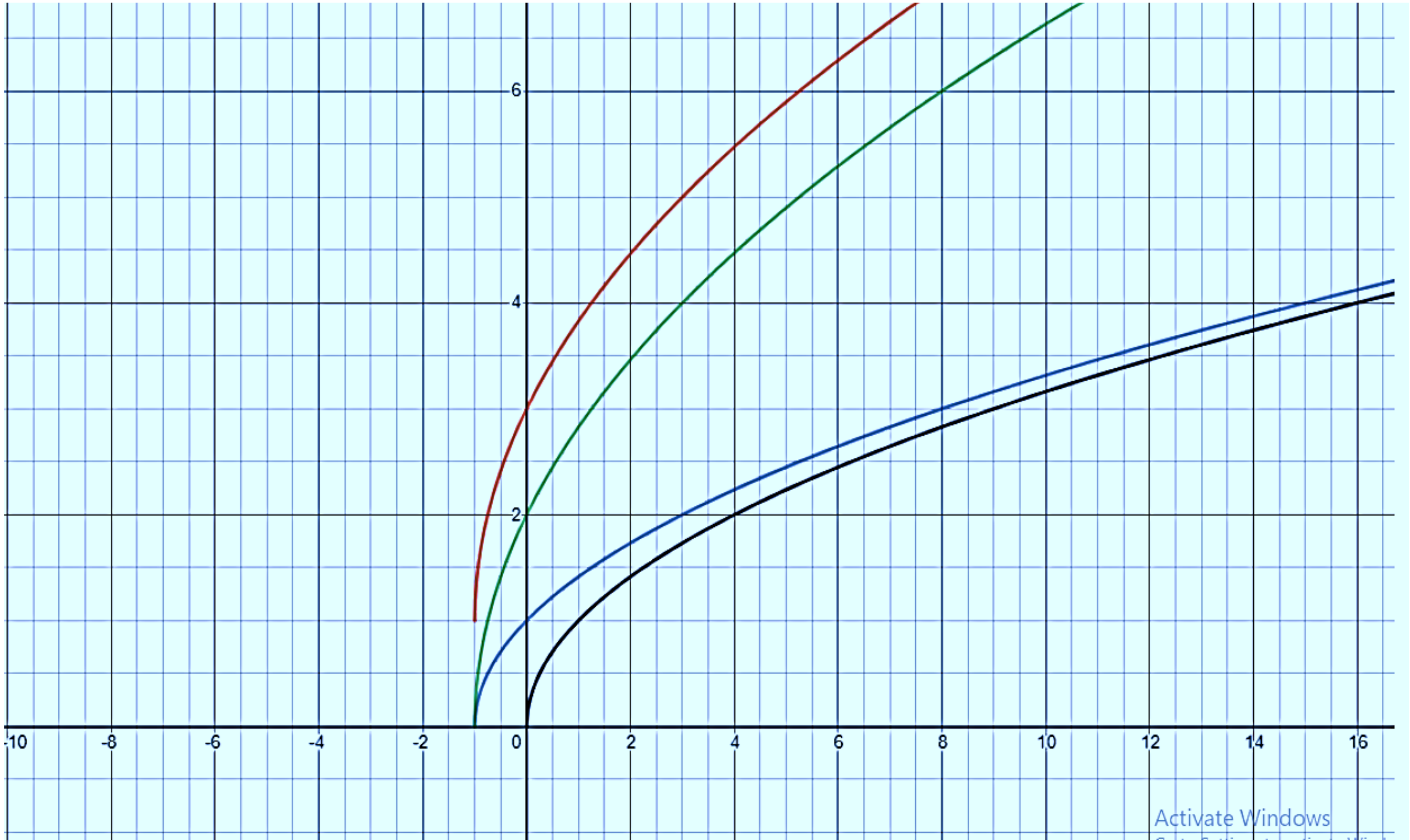
a is vertical stretch/compression	**b** is horizontal stretch/compression
• $|a| > 1$ stretches • $|a| < 1$ compresses • $a < 0$ flips the graph upside down	• $|b| > 1$ compresses • $|b| < 1$ stretches • $b < 0$ flips the graph left-right
c is horizontal shift	**d** is vertical shift
• $c < 0$ shifts to the right • $c > 0$ shifts to the left	• $d > 0$ shifts upward • $d < 0$ shifts downward

Example: $2\sqrt{(x+1)}+1$

a=2, c=1, d=1

Consider the square root function, and then

- Stretch it by 2 units in the y −direction.
- Shift it left 1 units, and
- Shift it up 1 units.



Combining Functions

Combining Functions

- There exist different ways to combine functions to make new functions:
  - Arithmetic Combinations of Functions that includes Sums, Differences, Products and Quotients.
  - Composition of Functions.

Arithmetic Combinations of Functions

Let $f(x) = 5x+2$ and $g(x) = x^2-1$. Evaluate each combination at the point $x=4$. Note that: $f(4)=5(4)+2=22$ and $g(4)=4^2-1=15$.

Expression	Combine, then evaluate			Evaluate, then combine	
$(f+g)(x)$	$(5x+2) + (x^2-1)$ $=x^2+5x+1$	$(f+g)(4)$	$4^2+5(4)+1$ $=16+20+1$ $=37$	$f(4)+g(4)$	$22+15$ $=37$
$(f-g)(x)$	$(5x+2) - (x^2-1)$ $=-x^2+5x+3$	$(f-g)(4)$	$-4^2+5(4)+3$ $=-16+20+3$ $=7$	$f(4)-g(4)$	$22-15$ $=7$
$(f\cdot g)(x)$	$(5x+2)*(x^2-1)$ $=5x^3+2x^2-5x-2$	$(f\cdot g)(4)$	$5(4^3)+2(4^2)-5(4)-2$ $=5(64)+2(16)-20-2$ $=330$	$f(4)\cdot g(4)$	$22(15)$ $=330$
$(f/g)(4)$	$(5x+2)/(x^2-1)$	$(f/g)(4)$	$[5(4)+2]/[4^2-1]$ $=22/15$	$f(4)/g(4)$	$22/15$

Arithmetic Combinations of Functions

- Two functions f and g can be combined to form new functions

$$f + g, \quad f - g, \quad fg, \quad f/g$$

in a manner similar to the way we add, subtract, multiply, and divide real numbers.

Sum of Functions

- We define the function $f + g$ by:

$$(f + g)(x) = f(x) + g(x)$$

  – The new function $f + g$ is called the sum of the functions f and g.

  – Its value at x is $f(x) + g(x)$.

Sum of Functions

- Of course, the sum on the right-hand side makes sense only if both $f(x)$ and $g(x)$ are defined, i.e., if x belongs to the domain of f and also to the domain of g.
  - So, if the domain of f is A and that of g is B, then the domain of $f + g$ is the intersection of these domains.
  - That is, $A \cap B$.
  - Note that the sum of two even functions is even, and the sum of two odd functions is odd.

Differences, Products, and Quotients

- Similarly, we can define:
  - the difference $(f - g)(x) = f(x) - g(x)$,
  - the product $(f.g)(x) = f(x).g(x)$,
  - the quotient $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, as long as $g(x) \neq 0$.

Note:

- The difference of two even functions is even, and the difference of two odd functions is odd.
- The product of two even functions is even, and the product of two odd functions is even. The product of an even function and an odd function is an odd function.
- The quotient of two even functions is even, and the quotient of two odd functions is even. The quotient of an even function and an odd function is an odd function.

Domain of Differences, Products, and Quotients

- The domain of each of these combinations is the intersection of the domain of f and the domain of g. In other words, both functions must be defined at a point for the combination to be defined. One additional requirement for the division of functions is that the denominator can't be zero.

- Their domains are $A \cap B$.

Algebra of Functions

- Let f and g be functions with domains A and B.
- Then, the functions $f + g, f - g, fg$, and f/g are defined as follows.

$$(f+g)(x) = f(x) + g(x) \qquad \text{Domain } A \cap B$$

$$(f-g)(x) = f(x) - g(x) \qquad \text{Domain } A \cap B$$

$$(fg)(x) = f(x)g(x) \qquad \text{Domain } A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{Domain } \{x \in A \cap B | g(x) \neq 0\}$$

Example: Combinations of Functions

Let $f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x}$

(a) Find the functions $f + g, f - g, fg$, and f/g and their domains.

(b) Find $(f + g)(4), (f - g)(4), (fg)(4)$, and $(f/g)(4)$.

Solution: (a)

- The domain of $f(x)$ is $\{x|x \neq 2\}$ and the domain of $g(x)$ is $\{x|x \geq 0\}$.

- The intersection of the domains of $f(x)$ and $g(x)$ is:

$$\{x \mid x \geq 0 \text{ and } x \neq 2\} = [0, 2) \cup (2, \infty).$$

- $(f+g)(x) = f(x) + g(x) = \frac{1}{x-2} + \sqrt{x}$ Domain$\{x|x \geq 0 \text{ and } x \neq 2\}$

- $(f-g)(x) = f(x) - g(x) = \frac{1}{x-2} - \sqrt{x}$ Domain$\{x|x \geq 0 \text{ and } x \neq 2\}$

Solution: (a)

- $(fg)(x) = f(x)g(x) = \frac{\sqrt{x}}{x-2}$ $\quad$ $\text{Domain}\{x \mid x \geq 0 \text{ and } x \neq 2\}$

- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{1}{(x-2)\sqrt{x}}$ $\quad$ $\text{Domain}\{x \mid x > 0 \text{ and } x \neq 2\}$

  – Note that, in the domain of f/g, we excluded 0 because $g(0) = 0$.

Solution: (b)

- Each of these values exist because $x = 4$ is in the domain of each function.
  - $(f+g)(4) = f(4) + g(4) = \frac{1}{4-2} + \sqrt{4} = \frac{5}{2}$.
  - $(f-g)(4) = f(4) - g(4) = \frac{1}{4-2} - \sqrt{4} = -\frac{3}{2}$.
  - $(fg)(4) = f(4)g(4) = \left(\frac{1}{4-2}\right)\sqrt{4} = 1$.
  - $\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)} = \frac{1}{(4-2)\sqrt{4}} = \frac{1}{4}$.

Arithmetic Combinations of Functions

Let $f(x) = 5x+2$ and $g(x) = x^2-1$. Evaluate each combination at the point $x=4$. Note that: $f(4)=5(4)+2=22$ and $g(4)=4^2-1=15$.

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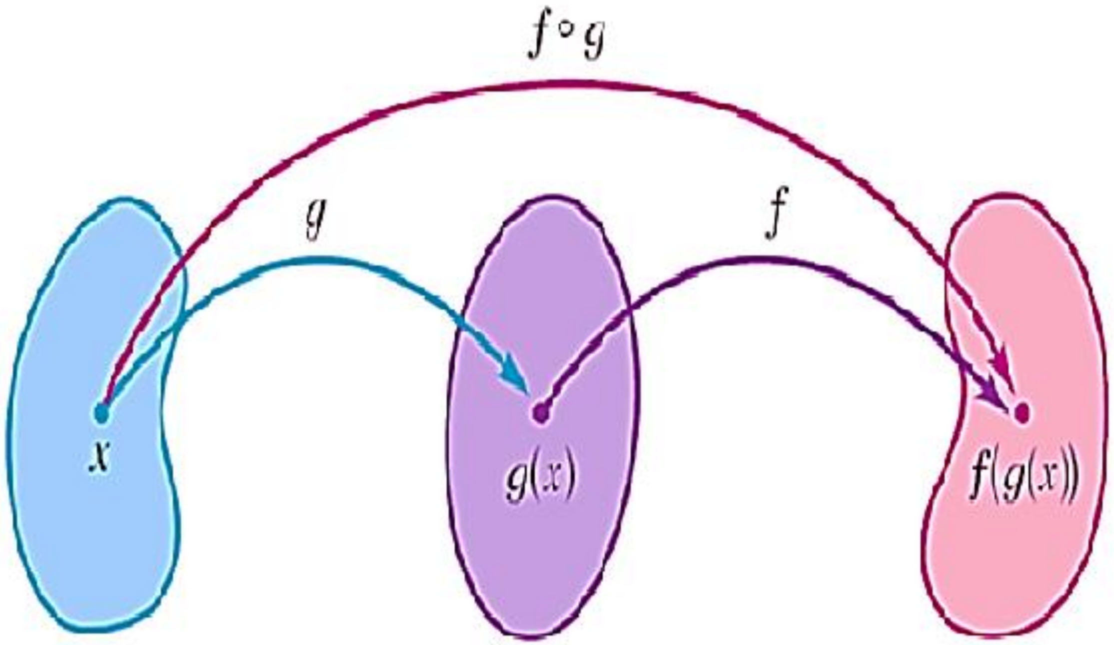
$(f/g)(4)$	$(5x+2)/(x^2-1)$	$(f/g)(4)$	$[5(4)+2]/[4^2-1]$ $=22/15$	$f(4)/g(4)$	$22/15$

Note:

- It doesn't matter if we combine and then evaluate or if we evaluate and then combine.
- In each of the above problems, the domain is all real numbers with the exception of the division. The domain in the division combination is all real numbers except for 1 and -1.

Composite Function



gof(x)=g(f(x))

fog(x)=f(g(x))

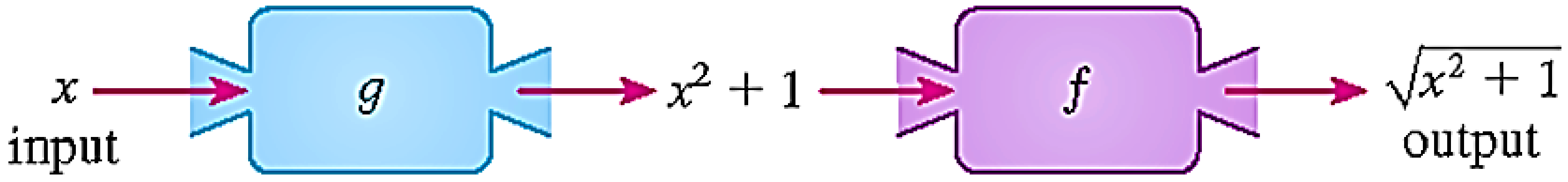
fog≠gof

Composition of Functions

- Now, let's consider a very important way of combining two functions to get a new function.
  - Suppose $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$.
  - We may define a function h as:

$$h(x) = (f \circ g)(x) = f\big(g(x)\big) = f(x^2 + 1) = \sqrt{x^2 + 1}.$$

  - The function h is made up of the functions f and g in an interesting way: Given a number x, we first apply to it the function g, then apply f to the result.



Composition of Functions

- In this case,
  - f is the rule "take the square root."
  - g is the rule "square, then add 1."
  - h is the rule "square, then add 1, then take the square root."
- In other words, we get the rule h by applying the rule g and then the rule f.

Composition of Functions

- In general, given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, we can then calculate the value of $f(g(x))$.
- The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the composition (or composite) of f and g and is denoted by $f \circ g$ ("f composed with g").
- The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

Example: Composition of Functions

Let $f(x) = x^2$ and $g(x) = x - 3$.

(a) Find the functions $f \circ g$ and $g \circ f$ and their domains.

(b) Find $(f \circ g)(5)$ and $(g \circ f)(7)$.

Solution:

(a) We have:

$$(f \circ g)(x) = f(g(x)) \; = f(x-3) \; = (x-3)^2$$

and

$$(g \circ f)(x) = g\big(f(x)\big) \; = g(x^2) \; = x^2 - 3\,.$$

The domains of both $f \circ g$ and $g \circ f$ are $\mathbb{R}$.

(b) We have:

$$(f \circ g)(5) = f(g(5)) = f(2) = 22 = 4$$

$$(g \circ f)(7) = g(f(7)) = g(49) = 49 - 3 = 46$$

Example: Composition of Functions

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find the following functions and their domains.

(a) $f \circ g$

(b) $g \circ f$

(c) $f \circ f$

(d) $g \circ g$

Solution:

(a) $(f \circ g)(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = \sqrt[4]{2-x}$

The domain of $f \circ g$ is:

$$\{x \mid 2 - x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2].$$

(b) $(g \circ f)(x) = g\big(f(x)\big) = g(\sqrt{x}) = \sqrt{2-\sqrt{x}}$

For $\sqrt{x}$ to be defined, we must have $x \geq 0$. For $\sqrt{2-\sqrt{x}}$ to be defined, we must have $2 - \sqrt{x} \geq 0$, that is $\sqrt{x} \leq 2$, or $x \leq 4$. Thus, we have: $0 \leq x \leq 4$. So, the domain of $g \circ f$ is the closed interval $[0, 4]$.

Practice: Compute part (c) and (d).