Polar Coordinates

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

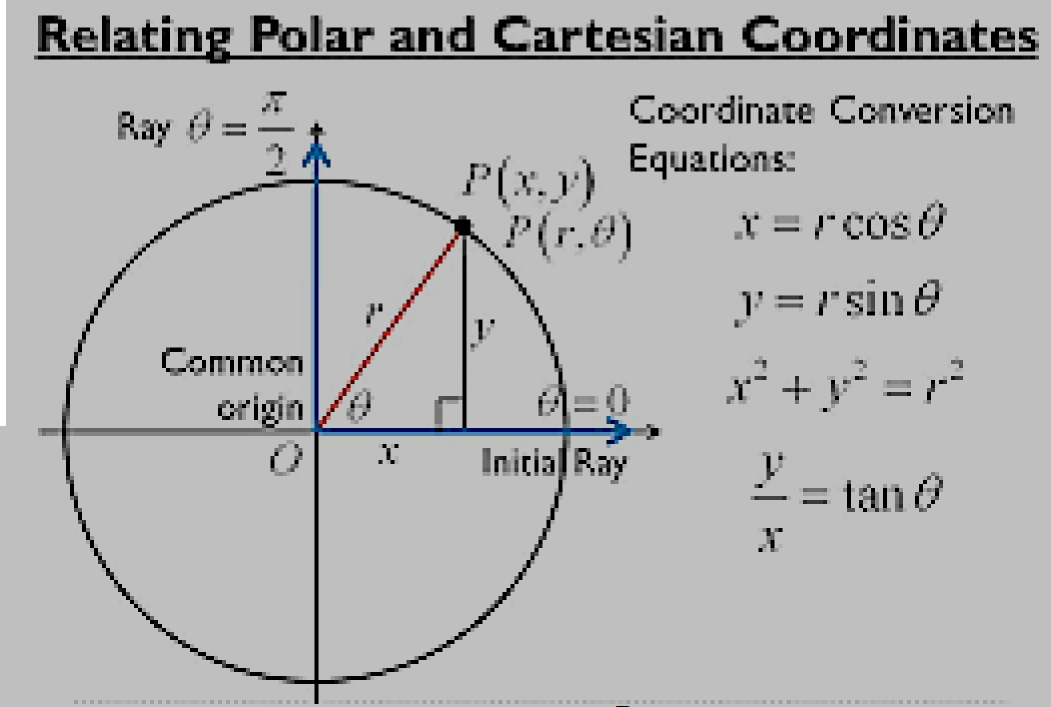
Chapter: 10 (10.5, 10.6, 10.7)

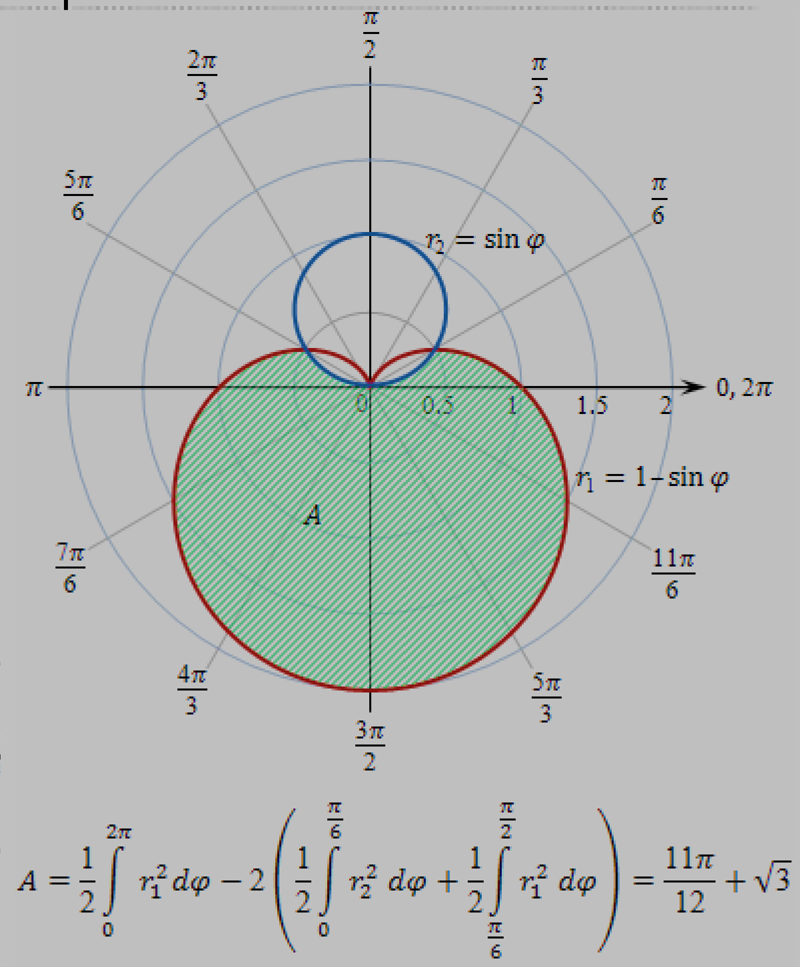
Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

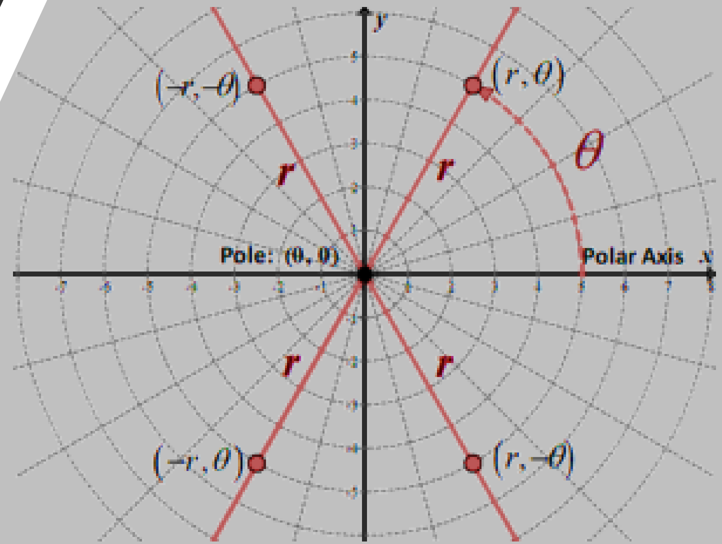
Chapter: 13 (13.3, 13.4)

Calculus & Analytical Geometry MATH-101

Instructor: Dr. Naila Amir (SEECS, NUST)

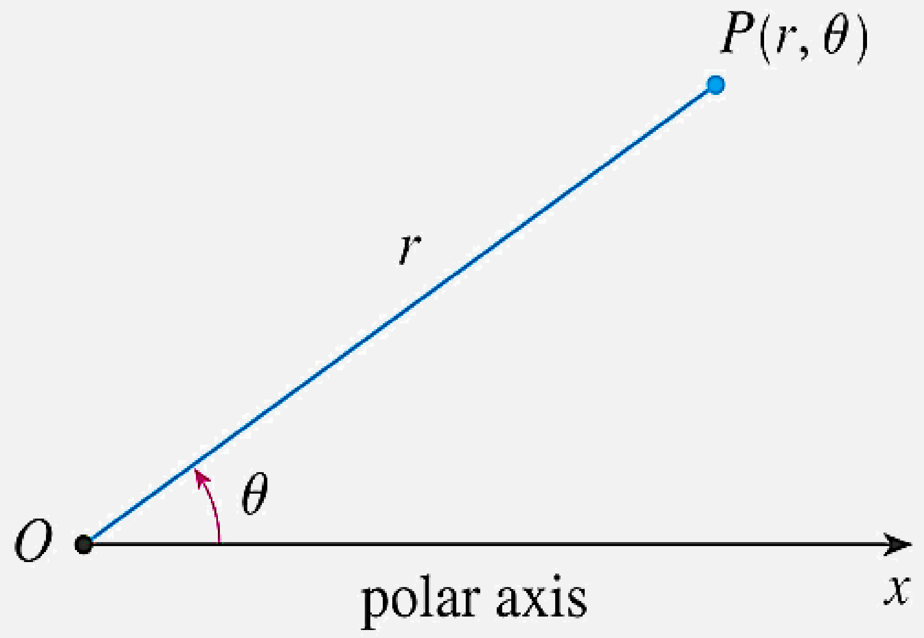






Polar Coordinates

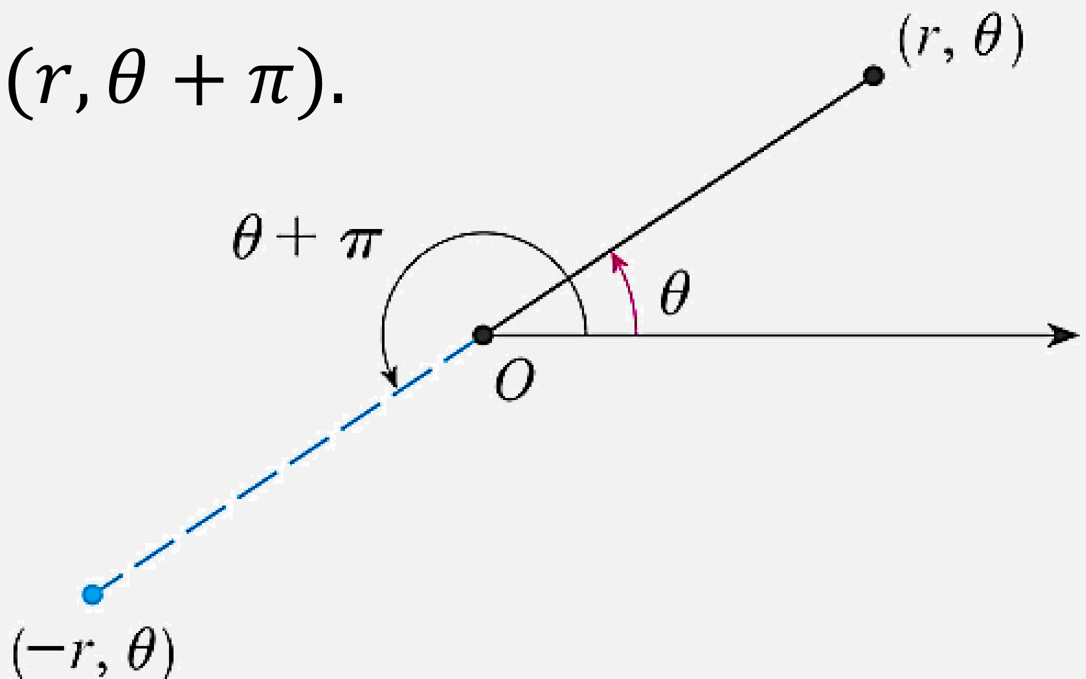
- If P is any point in the plane, let:
  - r be the distance from O to P.
  - θ be the angle (usually measured in radians) between the polar axis and the line OP.
- P is represented by the ordered pair (r, θ). r, θ are called polar coordinates of P.
- We use the convention that an angle is:
  - Positive—if measured in the counterclockwise direction from the polar axis.
  - Negative—if measured in the clockwise direction from the polar axis.



Polar Coordinates

- Note that the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O.
  - If $r > 0$, the point (r, θ) lies in the same quadrant as θ.
  - If $r < 0$, it lies in the quadrant on the opposite side of the pole.

- Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.



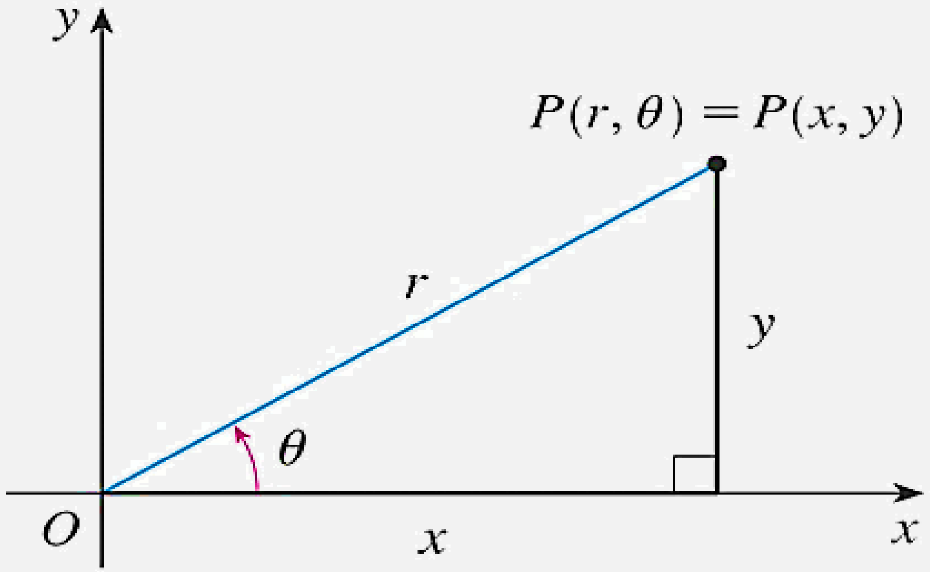
Cartesian & Polar Coordinates

- The connection between polar and Cartesian coordinates can be seen from the following figure. The pole corresponds to the origin and the polar axis coincides with the positive x −axis
- If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then:

$$x = r\cos\theta\,, \qquad y = r\sin\theta$$

- These equations allow us to find the Cartesian coordinates of a point when the polar coordinates are known.
- To find r and θ when x and y are known, we use the equations

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}.$$



Polar Equations and Graphs

- One way to graph a polar equation $r = f(\theta)$ is to make a table of (r, θ) $-$values plot the corresponding points, and connect them in order of increasing θ.
- This can work well if enough points have been plotted to reveal all the loops and dimples in the graph.
- Another method of graphing that is usually quicker and more reliable is to:
  - First graph $r = f(\theta)$ in the *Cartesian* $r\theta - \text{plane}$,
  - then use the Cartesian graph as a "table" and guide to sketch the *polar* coordinate graph.

Example

Graph the sets of points whose polar coordinates satisfy the following conditions:

a) $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$.

b) $-3 \leq r \leq 2$ and $\theta = \frac{\pi}{4}$.

c) $r \leq 0$ and $\theta = \frac{\pi}{4}$.

d) $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$ (no restriction on r).

$r = a \neq 0$

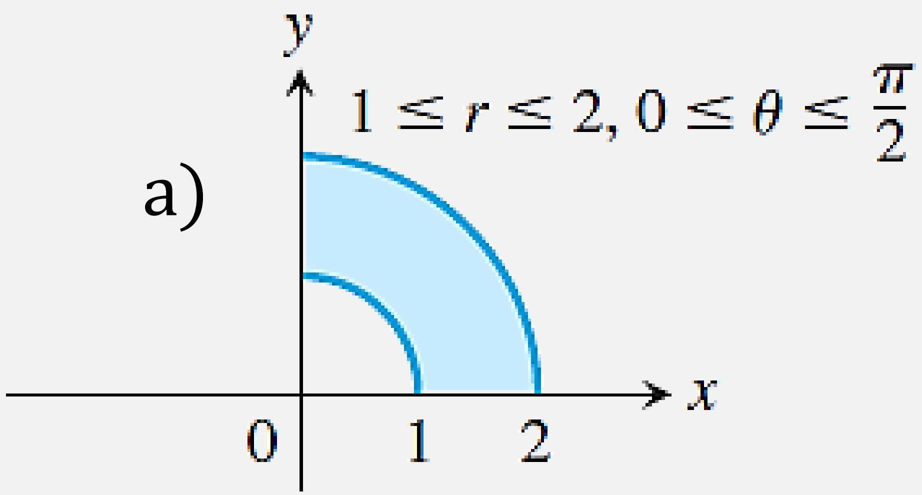
$0 \leq \theta < 2\pi$

$\theta = \theta_0$

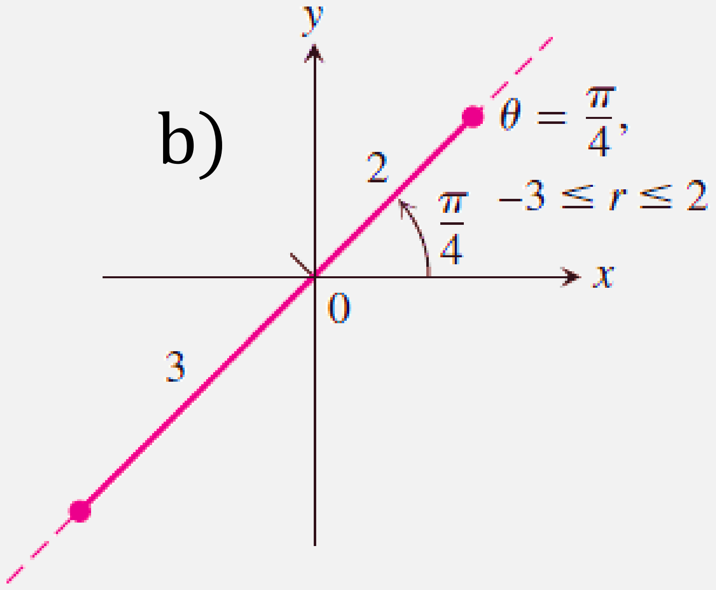
$-\infty < r < \infty$

Solution

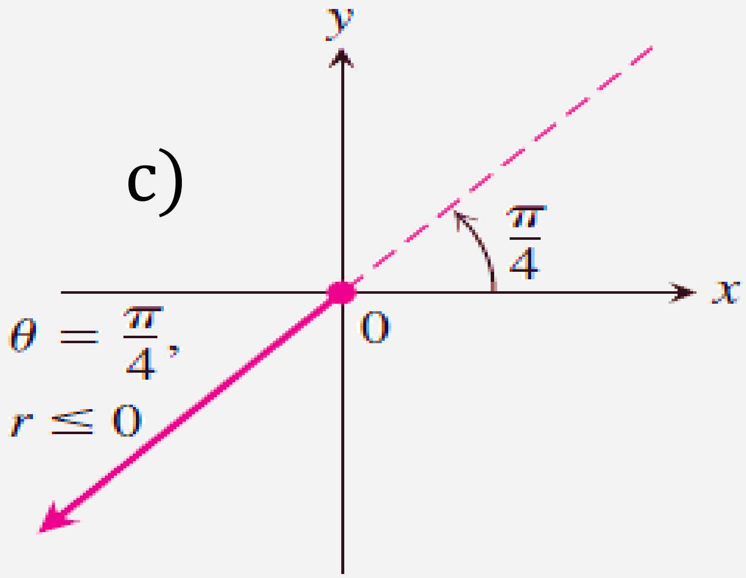
a)



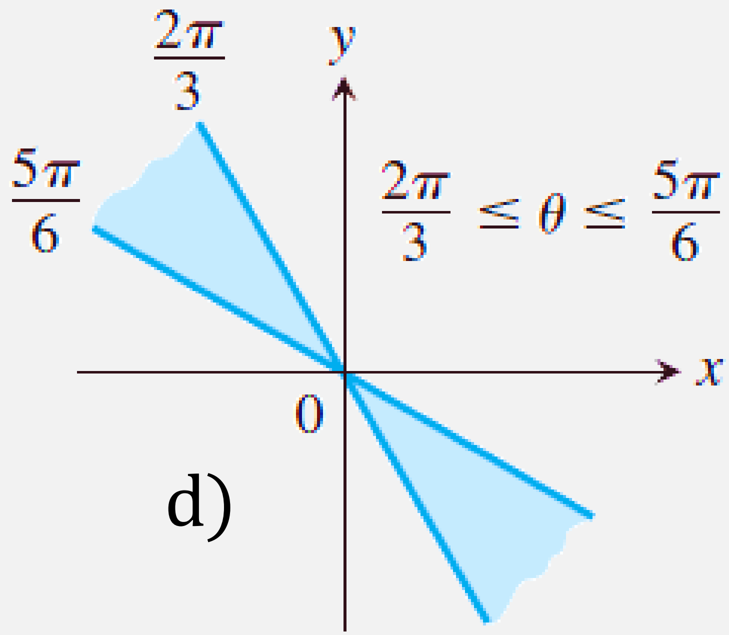
b)



c)



d)



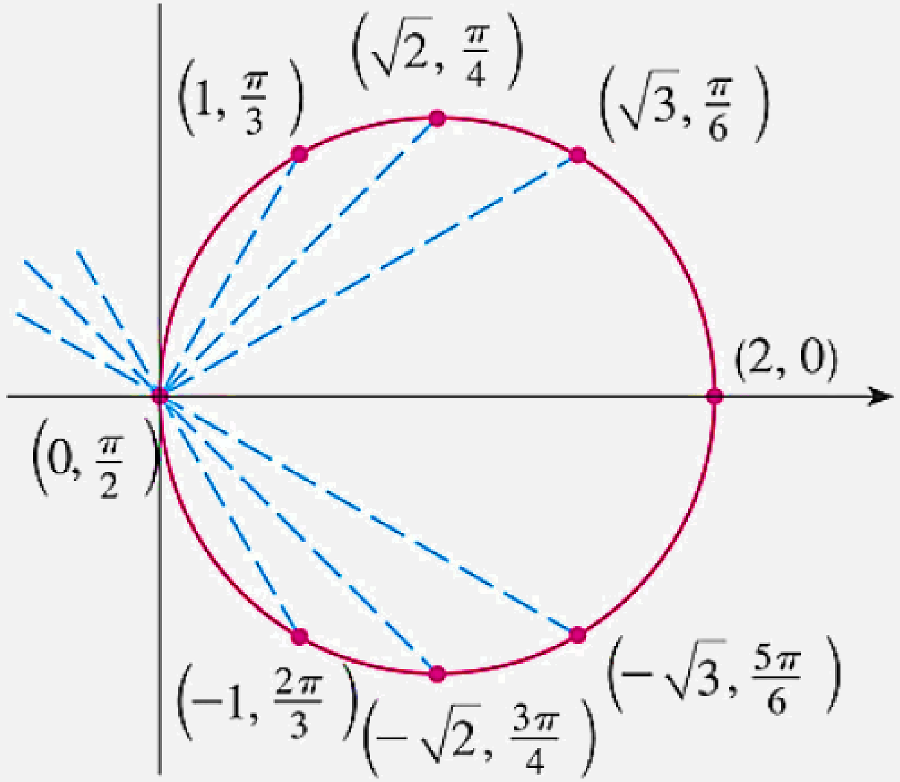
Example

Sketch the curve with polar equation $r = 2\cos\theta.$

Solution:

First, we find the values of r for some convenient values of $\theta.$ We plot the corresponding points (r, θ). Then, we join these points to sketch the curve.

The curve appears to be a **circle.** Note that we have used only values of θ between 0 and π. Since, if we let θ increase beyond π, we obtain the same points again.



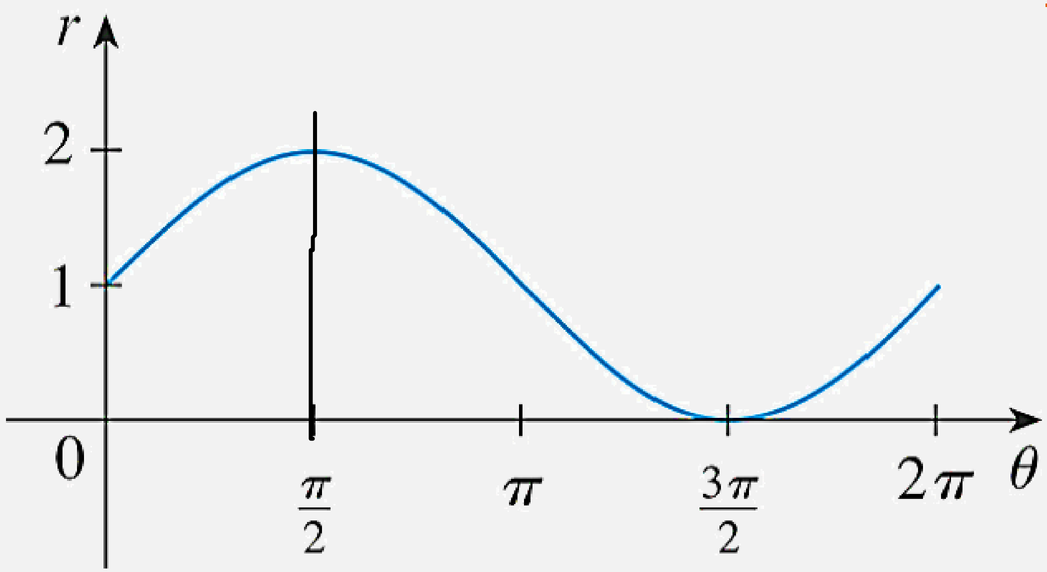
θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

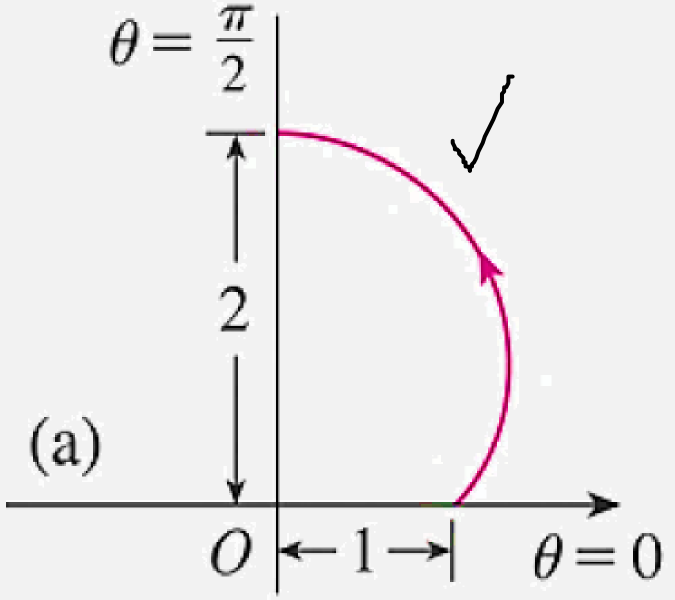
Example

Sketch the curve $r = 1 + \sin\theta$.

Solution:

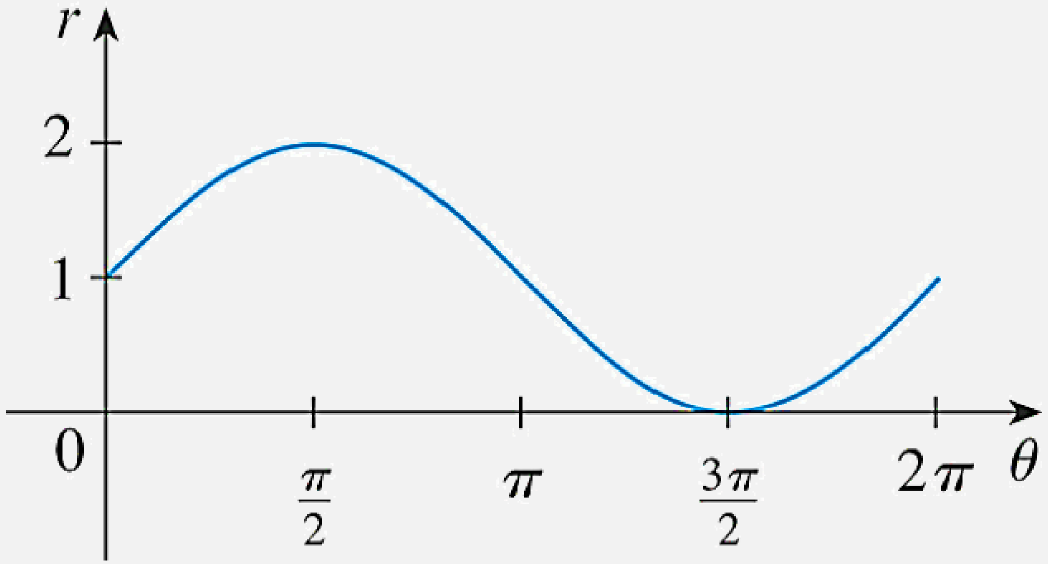
Here, we do not plot points as we did in previous example. Rather, we first sketch the graph of $r = 1 + \sin\theta$ in Cartesian coordinates by shifting the sine curve up one unit. This enables us to see immediately the values of r that correspond to increasing values of θ. For instance, we see that, as θ increases from 0 to $\pi/2$, r (the distance from O) increases from 1 to 2. So, we sketch the corresponding part of the polar curve.

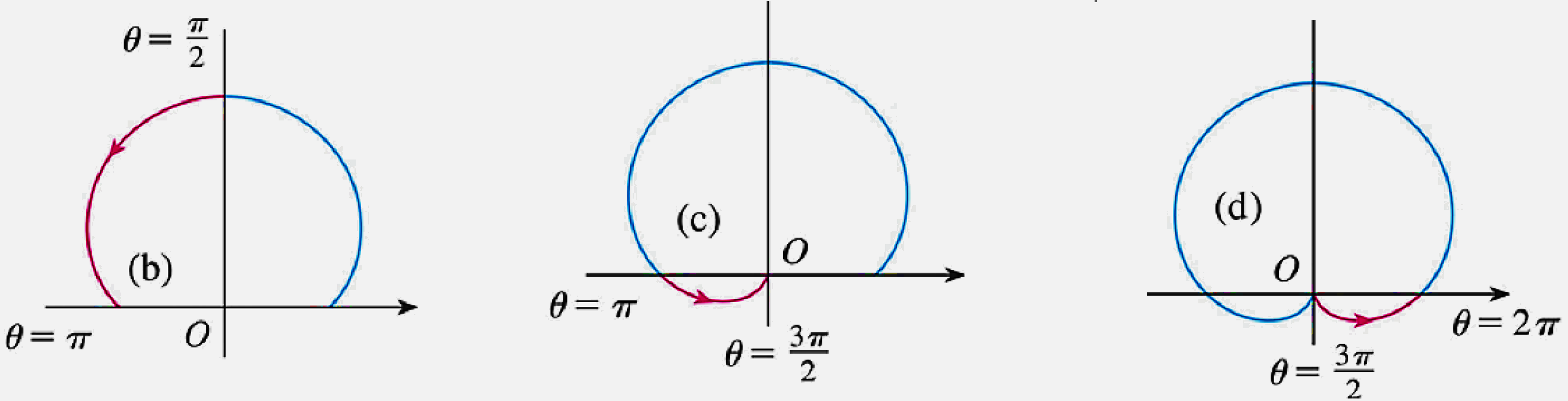




Solution

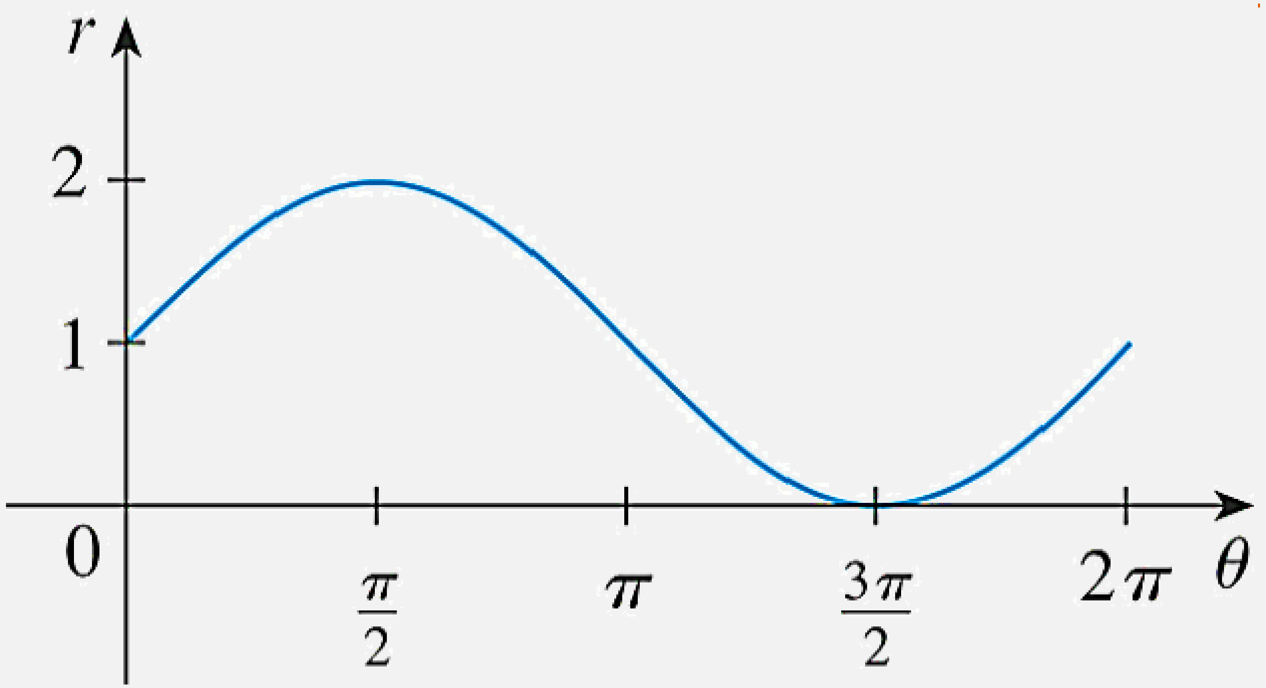
As θ increases from $\pi/2$ to π, the figure (b) shows that r decreases from 2 to 1. So, we sketch the next part of the curve. As θ increases from to π to $3\pi/2$, r decreases from 1 to 0, as shown in (c). Finally, as θ increases from $3\pi/2$ to 2π, r increases from 0 to 1, as shown in (d).

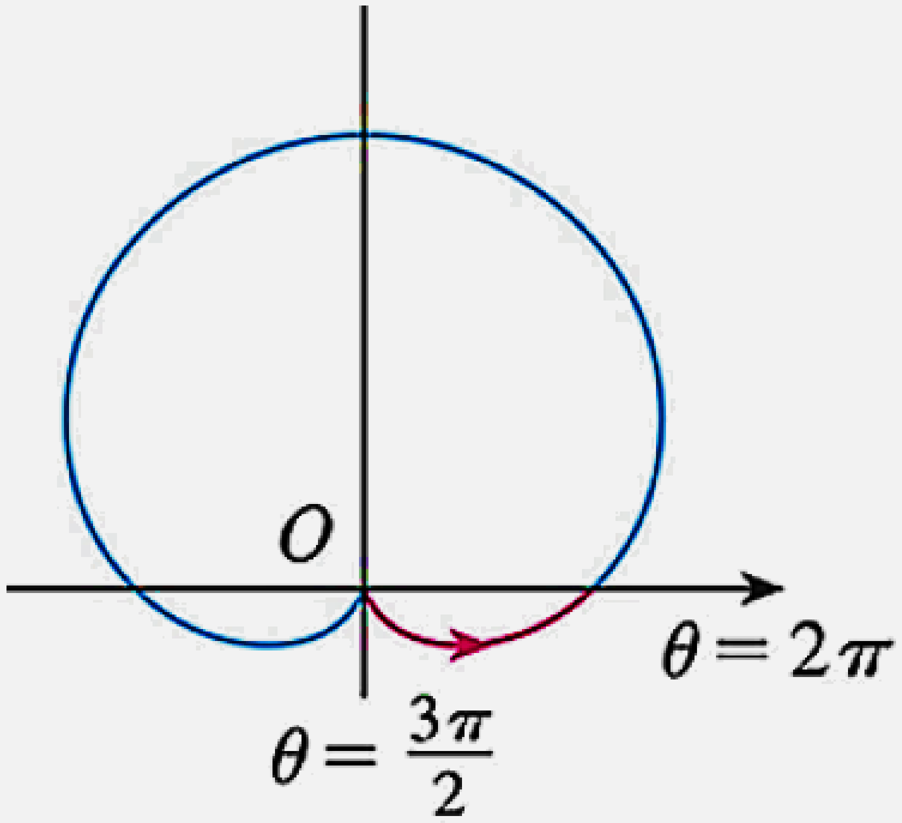




Solution

Note that, If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace our path. It is called a **cardioid**—because it's shaped like a heart.



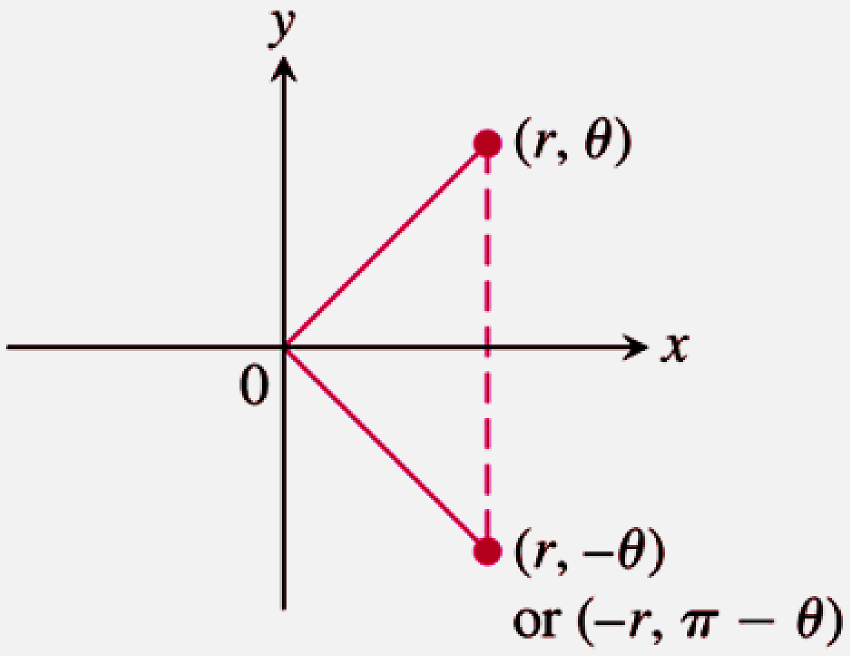


Symmetry

When we sketch polar curves, it is sometimes helpful to take advantage of symmetry.

RULE 1: Symmetry about the polar axis

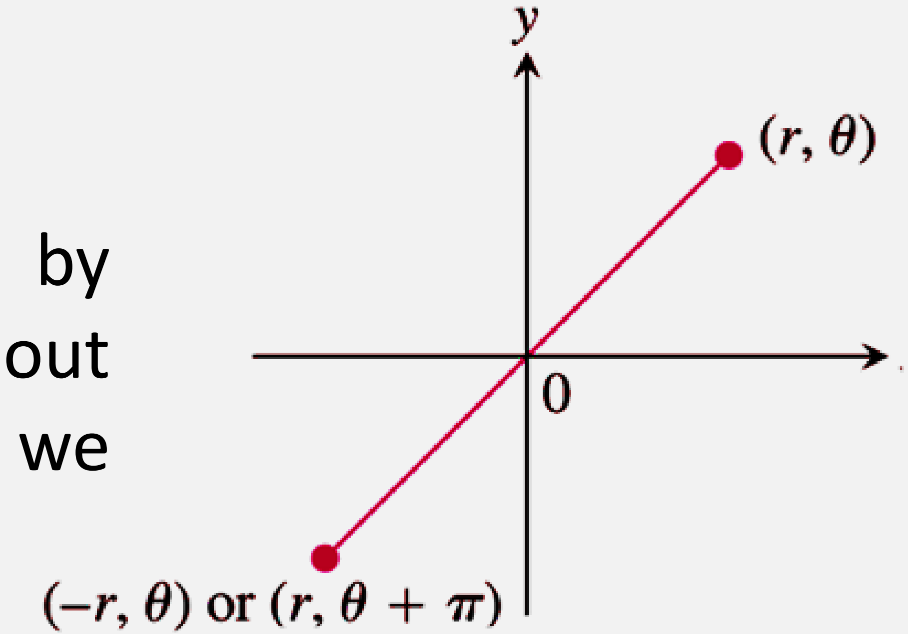
If a polar equation is unchanged when (r, θ) is replaced by either $(r, -\theta)$ or $(-r, \pi - \theta)$, the curve is symmetric about the polar axis.



Symmetry

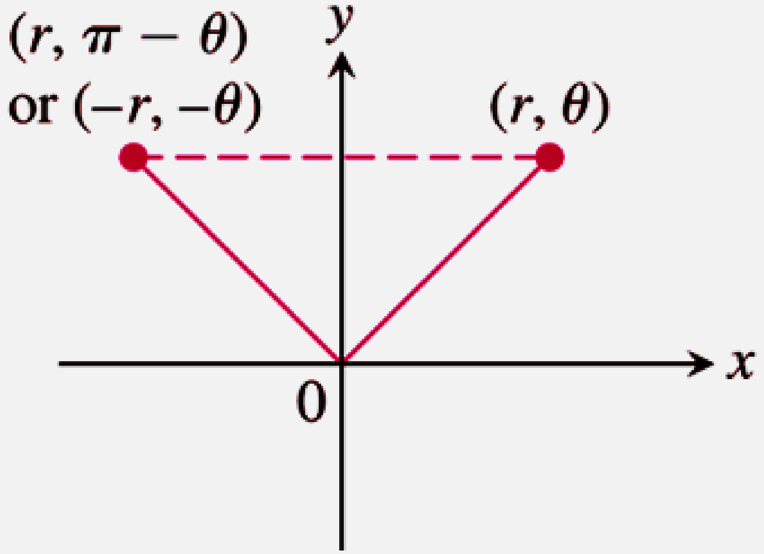
RULE 2: Symmetry about the pole

If the equation is unchanged when (r, θ) is replaced by $(-r, \theta)$, or by $(r, \theta + \pi)$, then the curve is symmetric about the pole. This means that the curve remains unchanged if we rotate it through 180° about the origin.



RULE 3: Symmetry about the vertical line

If the equation is unchanged when (r, θ) is replaced by $(r, \pi - \theta)$ or by $(-r, -\theta)$, the curve is symmetric about the vertical line $\theta = \pi/2$.



Example

Graph the curve $r^2 = 4\cos\theta$.

Solution:

The curve is symmetric about the polar axis because:

$$\begin{aligned}(r,\theta) \text{ on the graph } &\Longrightarrow r^2 = 4\cos\theta \\ &\Longrightarrow r^2 = 4\cos(-\theta) \qquad [\because \cos\theta = \cos(-\theta)] \\ &\Longrightarrow (r,-\theta) \text{ on the graph}\end{aligned}$$

The curve is also symmetric about the pole because:

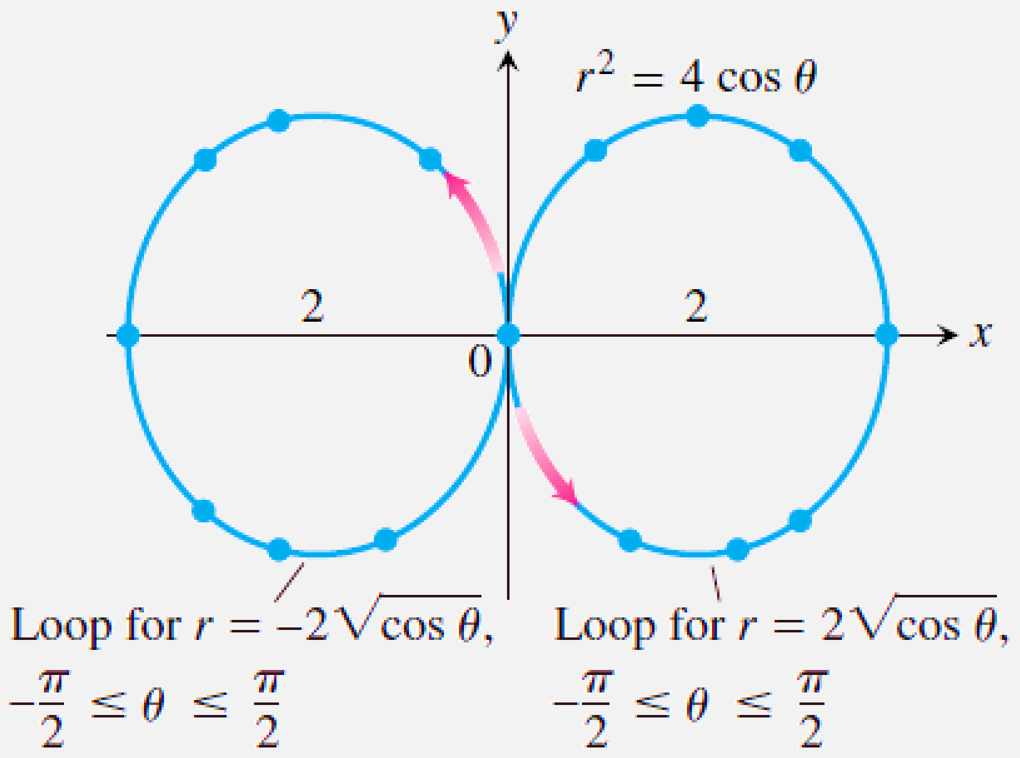
$$\begin{aligned}(r,\theta) \text{ on the graph } &\Longrightarrow r^2 = 4\cos\theta \\ &\Longrightarrow (-r)^2 = 4\cos\theta \\ &\Longrightarrow (-r,\theta) \text{ on the graph}\end{aligned}$$

Together, these two symmetries imply symmetry about the vertical line because (r,θ) on the graph $\Longrightarrow (-r,-\theta)$ on the graph.

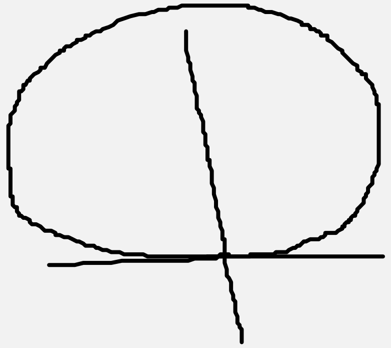
Solution

The curve passes through origin when $\theta = -\pi/2$ and $\theta = \pi/2$. Moreover, for each value of θ in the interval between $-\pi/2$ and $\pi/2$, the formula $r^2 = 4\cos\theta$ gives two values of r: $r = \pm 2\sqrt{\cos\theta}$. We make a small table of values, plot the corresponding points and use the information about symmetry to guide us in connecting the points with a smooth curve.

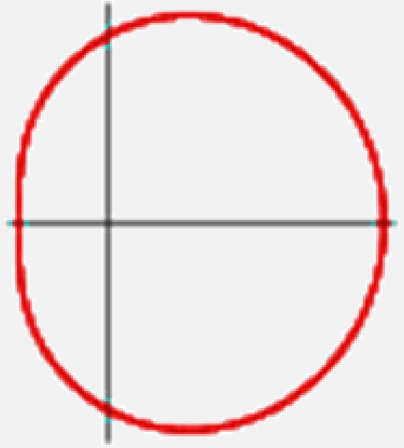
θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0



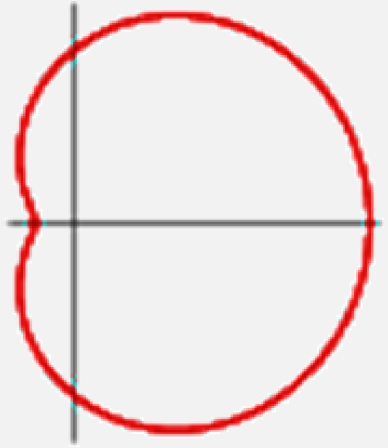
Special Polar Curves



$r = 2 + \cos\theta$



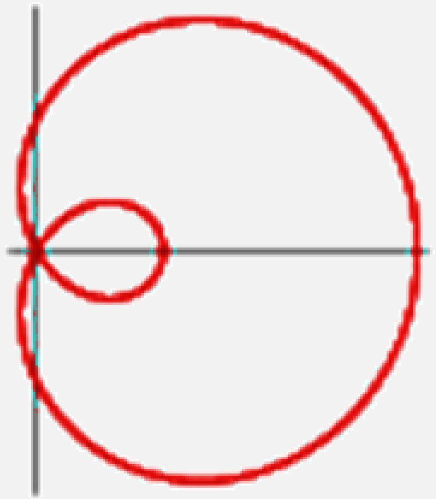
$r = 1.3 + \cos\theta$



$r = a + b\cos\theta\,;a > b$

Limacon with a dimple

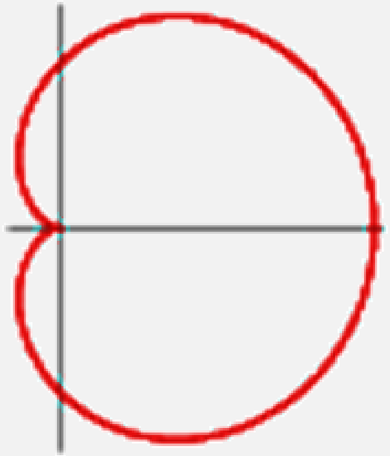
$r = 0.5 + \cos\theta$



$r = a + b\cos\theta\,;a < b$

Limacon with a loop

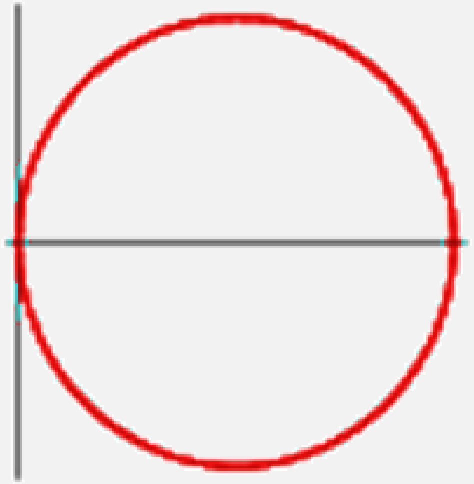
$r = 1 + \cos\theta$



$r = a + b\cos\theta\,;\; a = b$

Cardioid

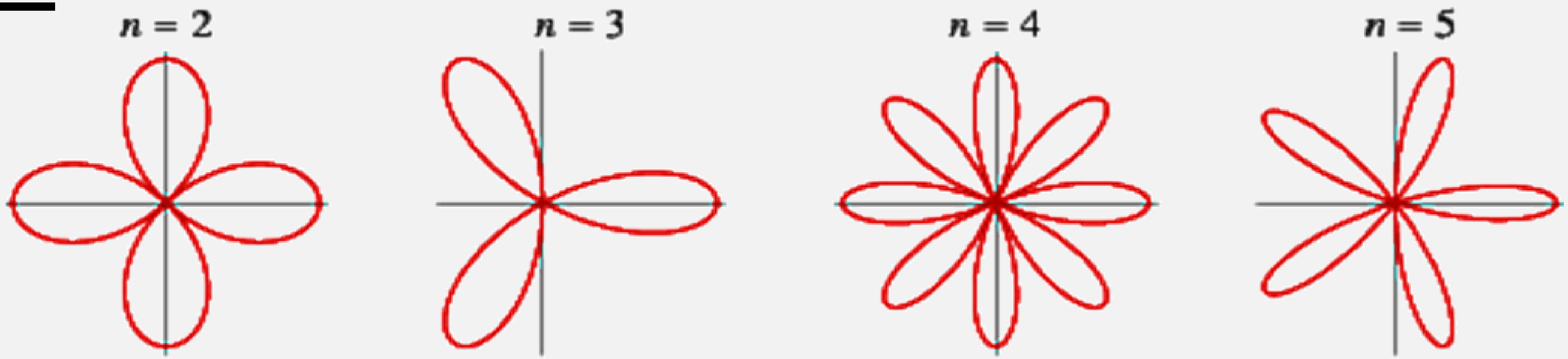
$r = \cos\theta$



$r = b\cos\theta$

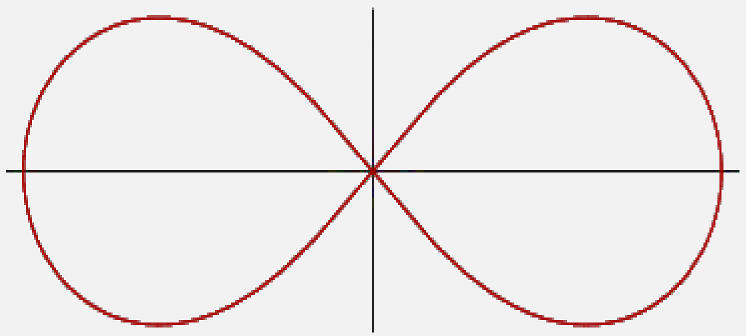
Circle

Special Polar Curves



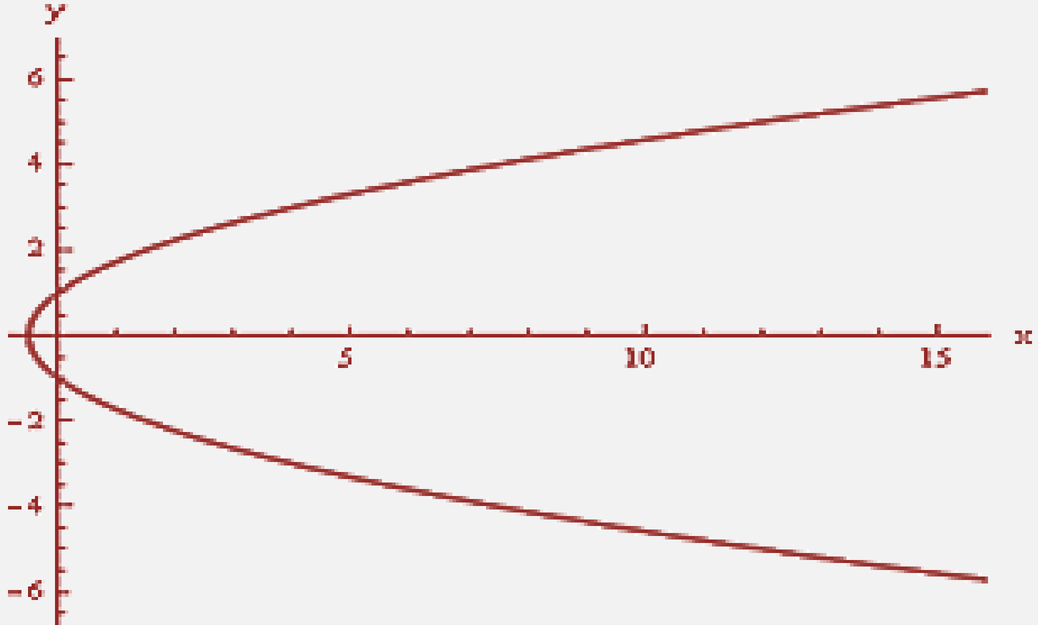
$r = a\cos(n\theta)$

Roses



$r^2 = a\cos(2\theta)$

Lemniscate



$$r = \frac{2}{a - a\cos\theta}$$

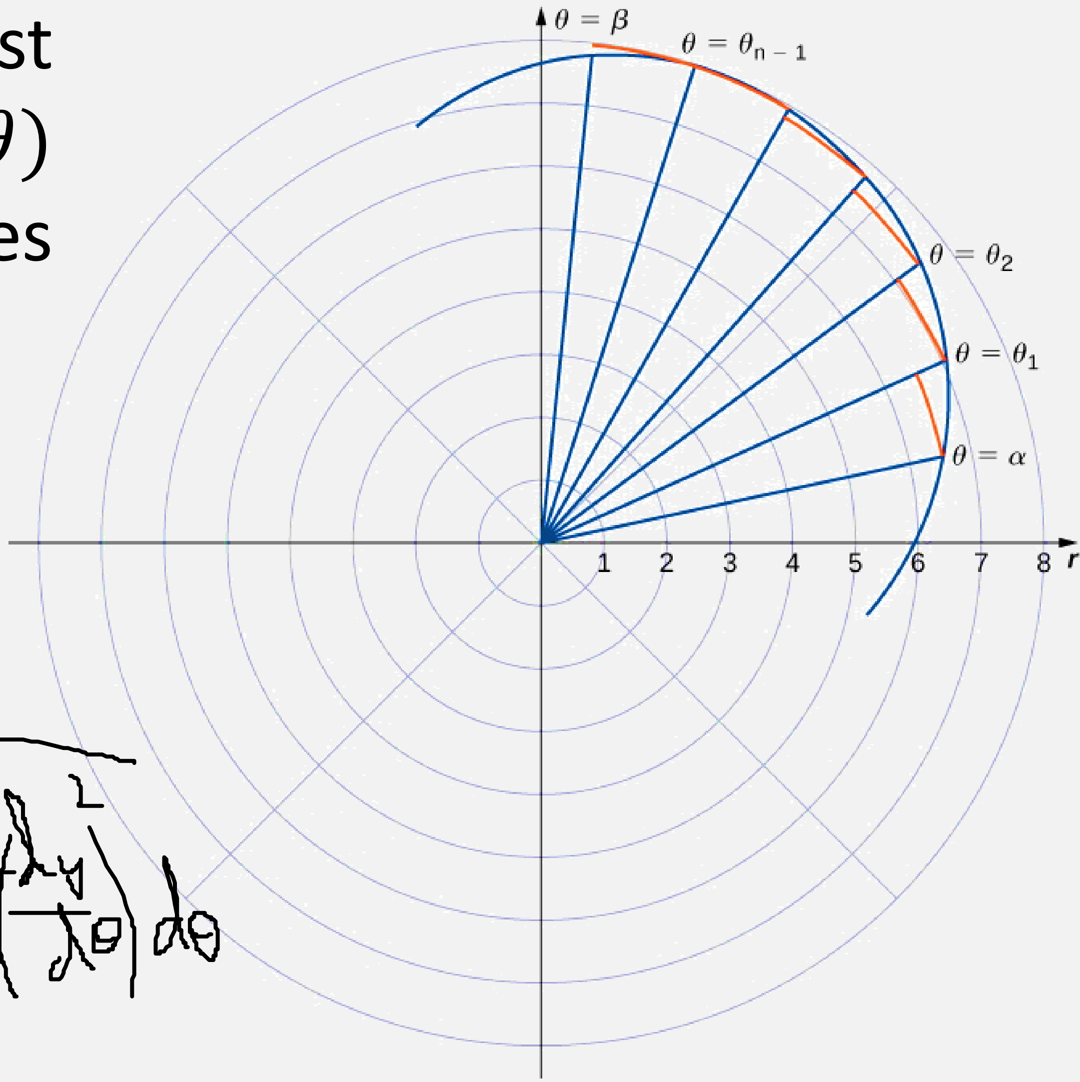
Parabola

Arclength Of a Polar Curve

If the curve $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ varies from α to β, then the length of the curve is:

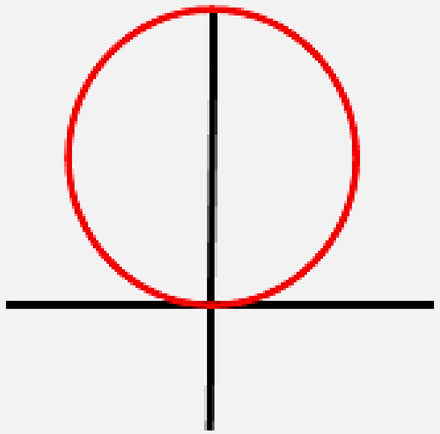
$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$



Example

Find the length of the curve $r = 5\sin\theta;\ 0 \leq \theta \leq \pi$.



Solution:

For the present case:

$$r = 5\,\sin\theta \Longrightarrow \frac{dr}{d\theta} = 5\cos\theta\,.$$

and

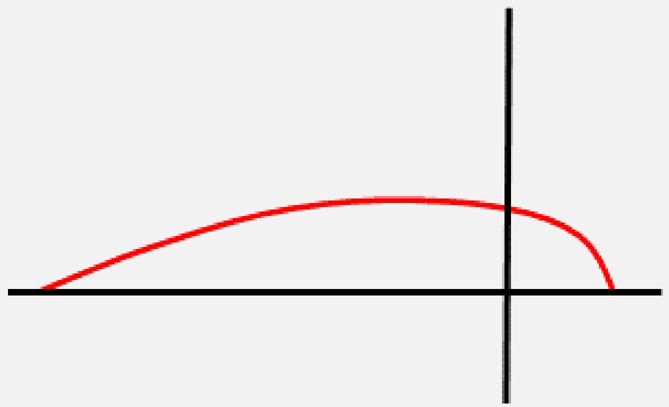
$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (5\sin\theta)^2 + (5\cos\theta)^2 = 25.$$

Thus,

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{\pi} \sqrt{25}\, d\theta = \int_{0}^{\pi} 5\, d\theta = 5\,\theta\Big|_0^{\pi} = 5\pi.$$

Example

Find the length of the curve $r = e^{\theta}; 0 \leq \theta \leq \pi$.



Solution:

For the present case:

$$r = e^{\theta} \Longrightarrow \frac{dr}{d\theta} = e^{\theta}.$$

and

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (e^{\theta})^2 + (e^{\theta})^2 = 2e^{2\theta}.$$

Thus,

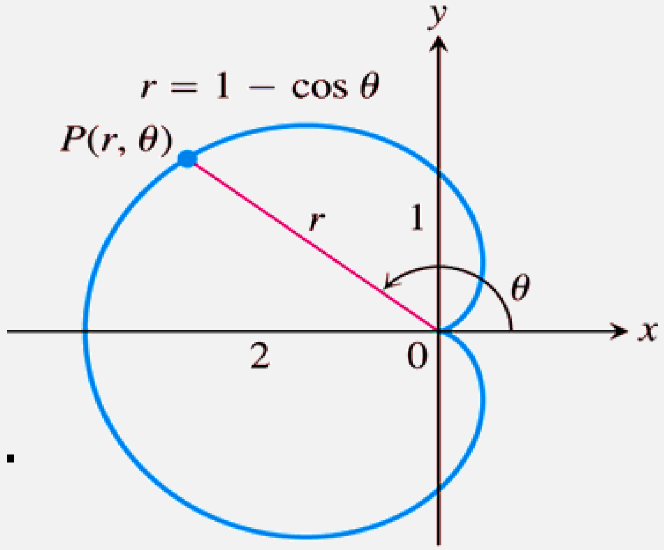
$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{\pi} \sqrt{2e^{2\theta}}\, d\theta = \sqrt{2} \int_{0}^{\pi} e^{\theta} d\theta = \sqrt{2}\, e^{\theta}\Big|_{0}^{\pi} = \sqrt{2}(e^{\pi} - 1).$$

Example

Find the length of the cardioid $r = 1 - \cos\theta$.

Solution:

Note that the point $P(r, \theta)$ traces the curve once, counterclockwise as runs from 0 to 2π, so these are the values we take for α and β. For the present case:



$$r = 1 - \cos\theta \Longrightarrow \frac{dr}{d\theta} = \sin\theta\,.$$

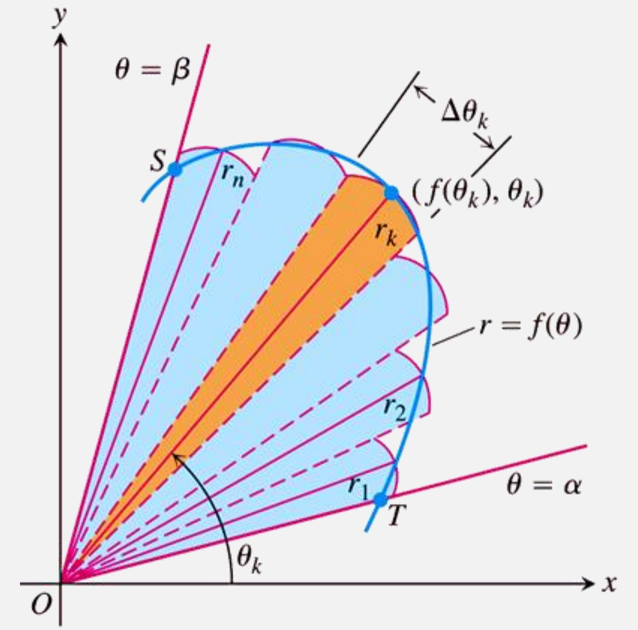
and

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - \cos\theta)^2 + (\sin\theta)^2 = 2(1 - \cos\theta) = 4\sin^2\left(\frac{\theta}{2}\right).$$

Thus,

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\, d\theta = 8.$$

Area Of a Polar Curve

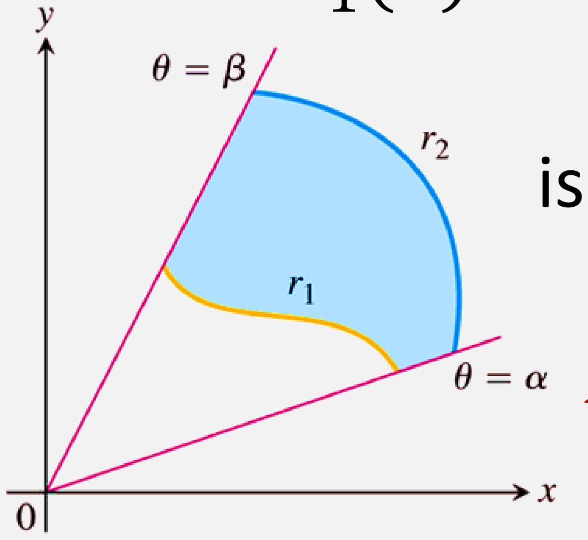


Area of the fan-shaped region between the pole and the curve $r = f(\theta)$; $\alpha \le \theta \le \beta$ is given as:

$$\text{Area of a polar curve} = A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta$$

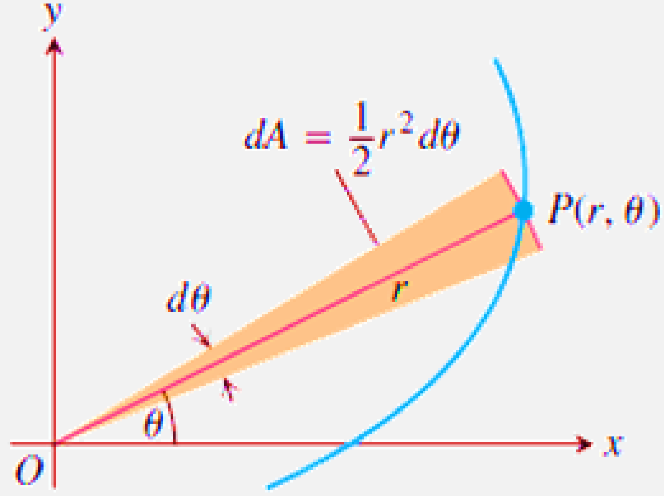
Area of the region bounded between two polar curves $r_1(\theta)$ and $r_1(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, i.e.,

$$0 \le r_1(\theta) \le r \le r_2(\theta); \quad \alpha \le \theta \le \beta$$



is given as:

$$\text{Area between polar curves} = A = \int_{\alpha}^{\beta} \frac{1}{2} ({r_2}^2 - {r_1}^2) d\theta \,.$$



Example

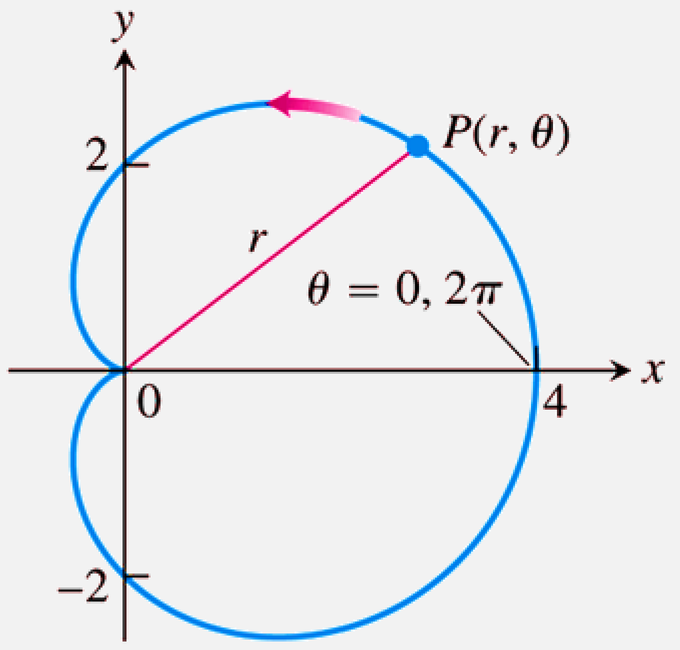
Find the area of the region enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution:

For the present case $r = 2(1 + \cos\theta)$ with $0 \leq \theta \leq 2\pi$. Thus,

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{0}^{2\pi} \frac{1}{2}(2 + 2\cos\theta)^2 d\theta$$

$$= 2\int_{0}^{2\pi} (1 + \cos\theta)^2 d\theta = 6\pi.$$



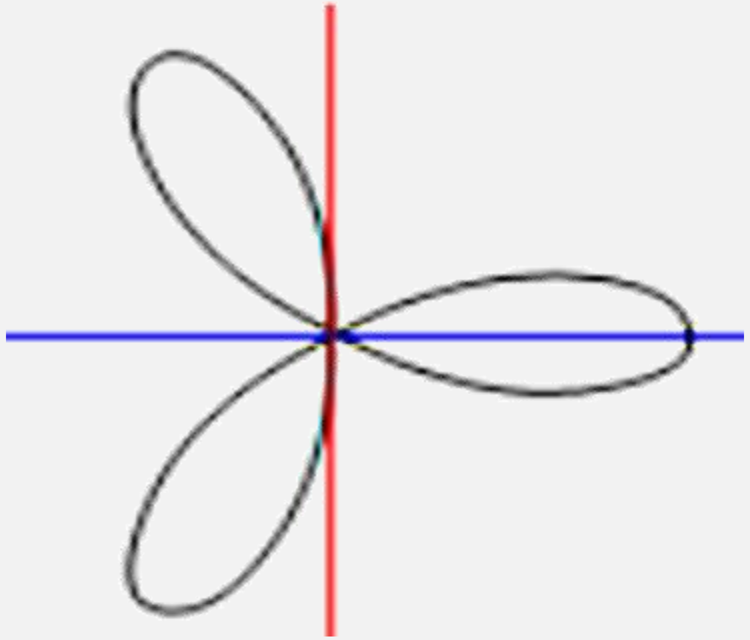
Example

Find the area of the region enclosed by the curve $r = 5\cos(3\theta)\,; 0 \le \theta \le \pi$.

Solution:

For the present case $r = 5\cos(3\theta)$ with $0 \le \theta \le \pi$. Thus, area of entire region is given as:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta \; = \int_{0}^{\pi} \frac{1}{2} (5\cos(3\theta))^2 d\theta = 19.635.$$



Area of one petal is given as:

$$A = \frac{19.635}{3} = 6.545.$$

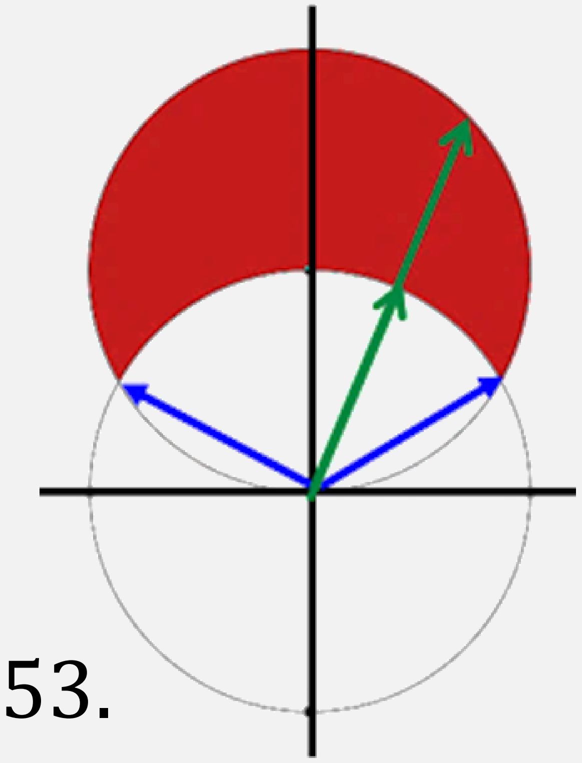
Example

Determine the Area of the region outside $r = 2$ and inside $r = 4\sin\theta$.

Solution:

For the present case, limits of integration can be determined by considering

$$2 = 4\sin\theta \Longrightarrow \sin\theta = \frac{1}{2} \Longrightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$



$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_2^2 - r_1^2)d\theta \; A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}[(4\sin\theta)^2 - 2^2]d\theta$$

$$= \frac{4}{2}\int_{\pi/6}^{5\pi/6} [4\sin^2\theta - 1]d\theta = 2\int_{\pi/6}^{5\pi/6} \left[4\left(\frac{1-\cos 2\theta}{2}\right) - 1\right]d\theta = 7.653.$$

Example

Determine the Area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

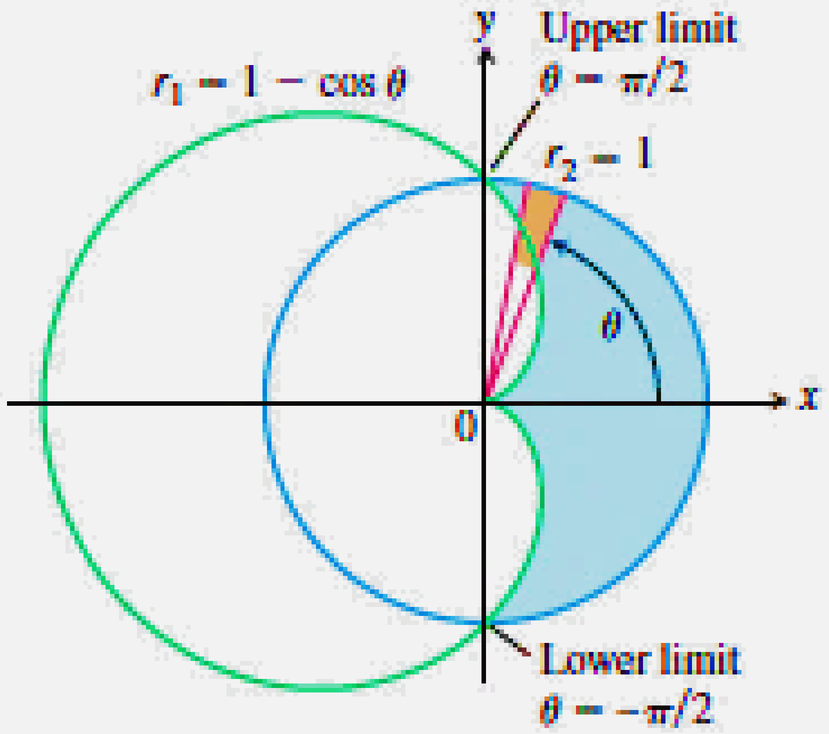
Solution:

For the present case, limits of integration can be determined by considering

$$1 - \cos\theta = 1 \Longrightarrow \cos\theta = 0 \Longrightarrow \theta = \frac{\pi}{2}, \frac{-\pi}{2}.$$

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_2^2 - r_1^2)d\theta \ A$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left((1)^2 - (1 - \cos\theta)^2\right)d\theta = 2 - \frac{\pi}{4}.$$



Example

Find the area of the region outside $r = 2 + 2sin\theta$, inside $r = 2 + 2cos\theta$, and in the first quadrant.

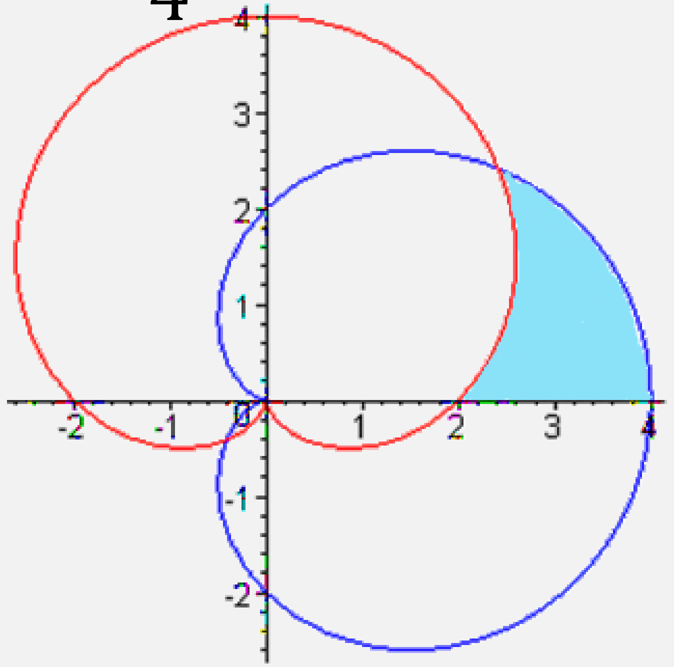
Solution:

For the present case, limits of integration can be determined by considering

$$2 + 2\sin\theta = 2 + 2\cos\theta \Longrightarrow 2\sin\theta = 2\cos\theta \Longrightarrow \tan\theta = 1 \Longrightarrow \theta = \frac{\pi}{4} = 0.785.$$

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_2^2 - r_1^2)d\theta \ A$$

$$= \int_{0}^{0.785} \frac{1}{2}\left((2 + 2\cos\theta)^2 - (2 + 2\sin\theta)^2\right)d\theta = 2.657.$$



Practice Questions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Exercise: 10.5

  Q # 1 - 6, Q # 23 – 62
- Exercise: 10.6

  Q # 1 - 16, Q # 21 – 24, Q # 31 – 42
- Exercise: 10.7

  Q # 1 - 16, Q # 19 - 27

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 13.3

  Q # 1 - 34, Q # 37 – 50
- Exercise: 13.4

  Q # 1 - 32

ESE: Total Marks: 100

- Q - 1: Blend of all CLOs (15+10+15 = 40 marks)
  MCQs + Fill in the blanks & True/False + Short question / Answers
  Basics of functions, limit, continuity, types of discontinuity, basics of derivatives, types of non-differentiabilty, applications of derivatives, applications of integrals, improper integral, CLO-3 complete excluding concept of sequence.

  Q - 2: CLO-1 (10 marks)
  Applications of derivatives: related rates, rate of change, extreme values, concavity, optimization problems.

  Q – 3: CLO-2 (25 marks) (There will be subparts of this question)
  Applications of integrals:
  Area, Arclength (cartesian & polar coordinates) and Volume (cartesian coordinates)

  Q – 4: CLO-3 (25 marks) (There will be subparts of this question)
  Infinite series, all tests, alternating series, power series, Taylor's & Maclaurin's series