PROPERTIES OF CTFT

Fourier Transform of Periodic Signals

Suppose
$$X(j\omega) = \delta(\omega - \omega_0)$$
$$\Downarrow$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

— periodic in t with frequency ω_o

That is

$$\boxed{e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)}$$

— All the energy is concentrated in one frequency — ω_o

More generally

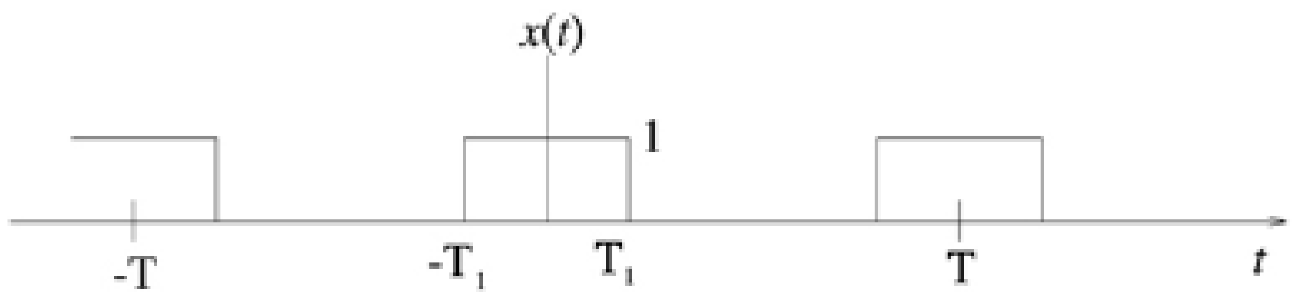
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

This is exactly the Fourier *series* representation of a periodic signal. Thus the Fourier transform of a periodic signal can be interpreted as a train of impulses occurring at the set of harmonically related frequencies.

FT of Periodic Signal – Square Wave

- Consider a periodic square wave of period $T = (2\pi / \omega_0)$ of the form:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



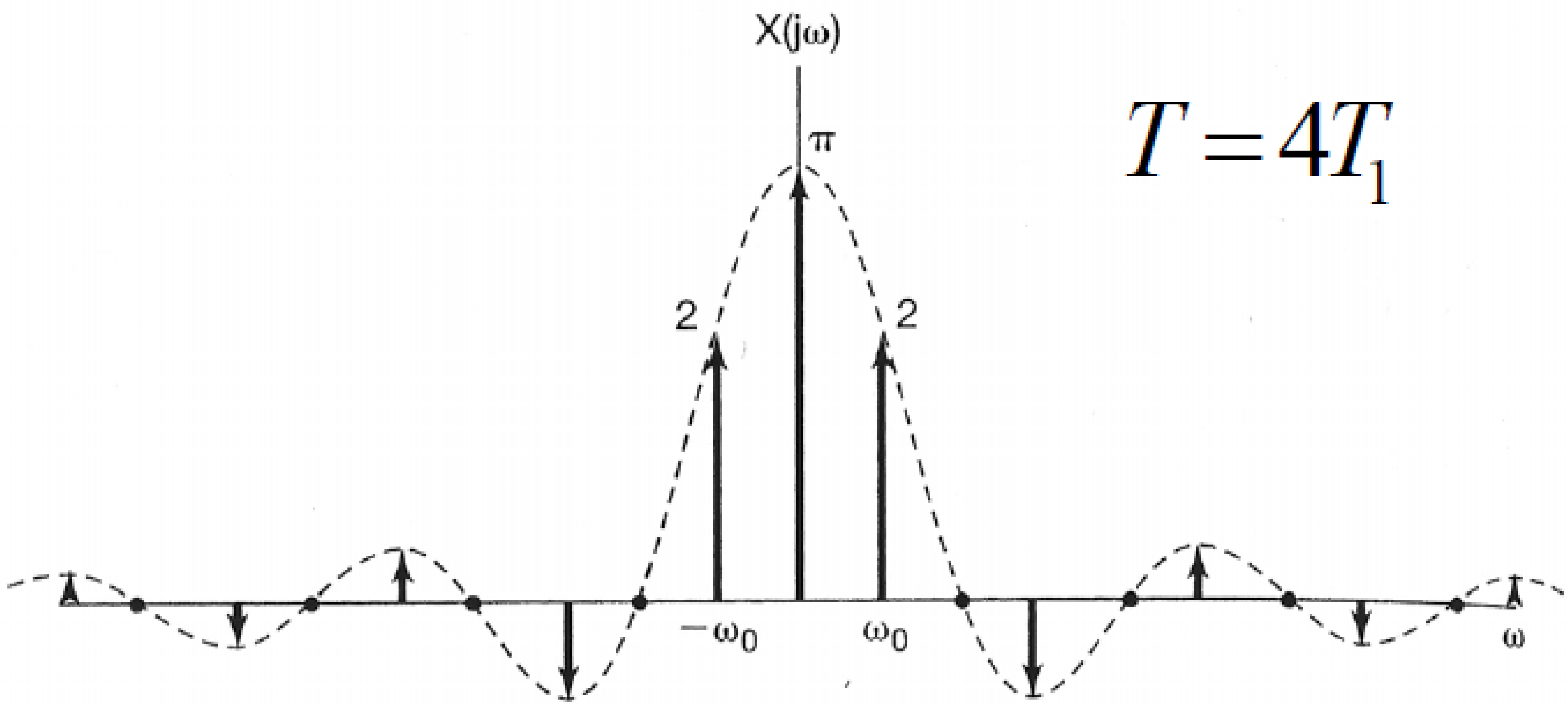
- Find the Fourier Transform using Fourier Series Coefficients

FT of Periodic Signal – Square Wave

- The Fourier Series coefficients and Fourier transform for this signal are:

$$a_k = \frac{\sin(k\omega_0 T_1)}{\pi k}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin(k\omega_0 T_1)}{k} \delta(\omega - k\omega_0)$$



$$T = 4T_1$$

FT of Periodic Signal

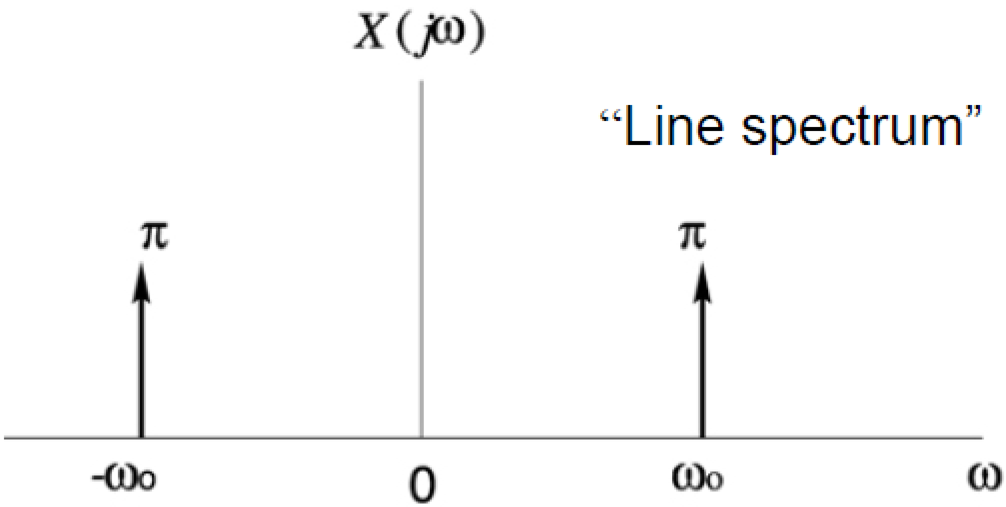
- Find the Fourier Transform using Fourier Series Coefficients

$x(t) = \cos \omega_0 t$ And $x(t) = \sin(\omega_0 t)$

FT of Periodic Signal

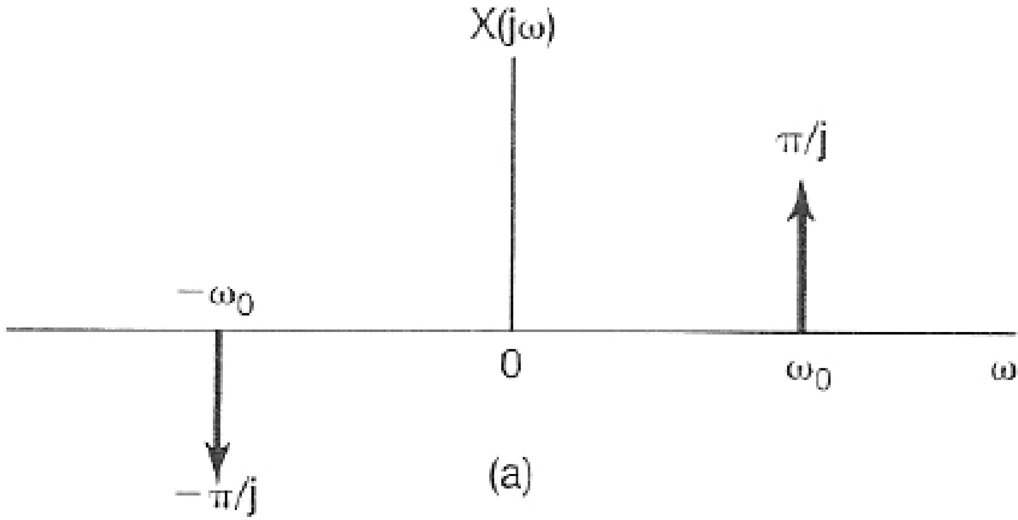
$$x(t) = \cos \omega_0 t$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



$$x(t) = \sin(\omega_0 t)$$

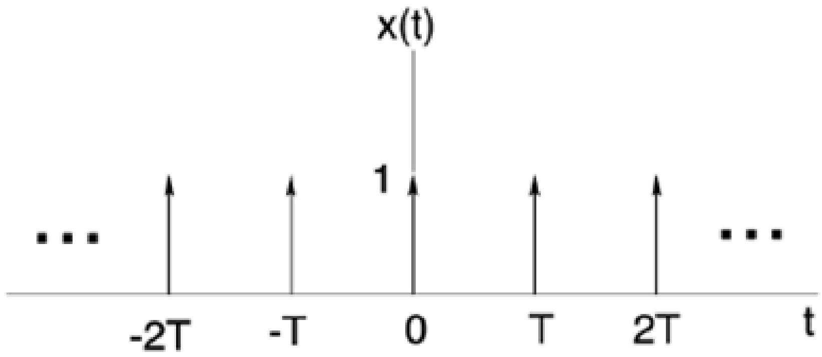
$$X(j\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$



FT of Periodic Signal – Sampling Function

— Sampling function

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



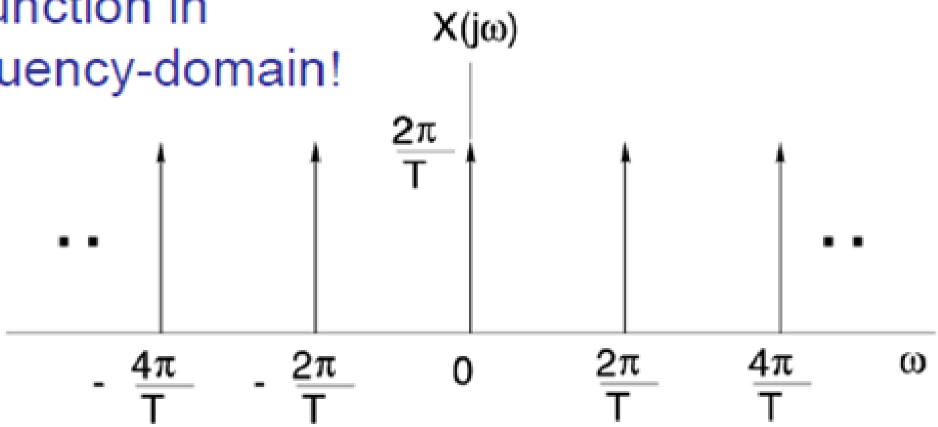
FT of Periodic Signal – Sampling Function

$$x(t) \leftrightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}$$

$$\Downarrow$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{2\pi}{T}}_{2\pi a_k} \delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_0})$$

Same function in
the frequency-domain!



Note: (period in t) T
$\Leftrightarrow$ (period in ω) $2\pi/T$
Inverse relationship again!

Notation

$$x(t) \xleftrightarrow{F} X(j\omega)$$

Some Examples

$$\frac{1}{a+j\omega} = F\left\{e^{-at}u(t)\right\}$$

$$e^{-at}u(t) = F^{-1}\left\{\frac{1}{a+j\omega}\right\}$$

$$e^{-at}u(t) \xleftrightarrow{F} \frac{1}{a+j\omega}$$

Properties of CTFT

1) Linearity: $ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega)$

2) Time Shifting: $x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)$

$$\text{Proof: } \int_{-\infty}^{\infty} x(\underbrace{t - t_0}_{t'})e^{-j\omega t} dt = e^{-j\omega t_0} \underbrace{\int_{-\infty}^{\infty} x(t')e^{-j\omega t'} dt'}_{X(j\omega)}$$

FT magnitude unchanged

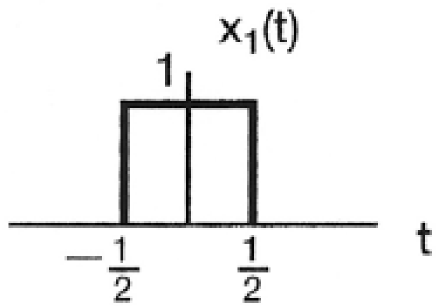
$$|e^{-j\omega t_0} X(j\omega)| = |X(j\omega)|$$

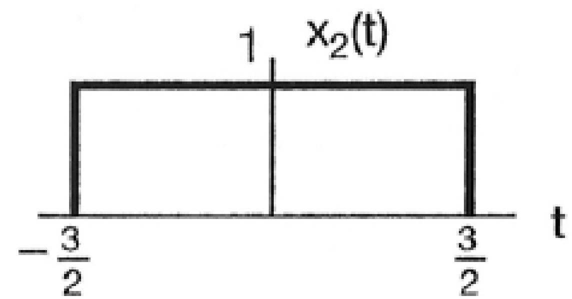
Linear change in *FT* phase

$$\angle(e^{-j\omega t_0} X(j\omega)) = \angle X(j\omega) - \omega t_0$$

Properties of CTFT

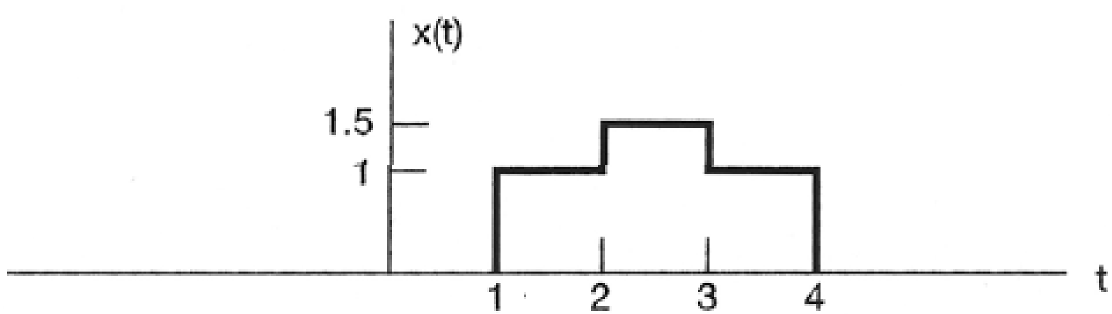
- Consider the signal $x_1(t)$ and $x_2(t)$ whose FTs are given.
- Determine the FT of x(t) using the Linearity and Time Shift property.





$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega}$$

$$X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}$$



Conjugation Property

- Conjugation property states that if:

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

- then

$$\boxed{x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)}$$

Conjugation Property

- Derivation:

$$X^*(j\omega) = \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right]^* = \int_{-\infty}^{\infty} x^*(t) e^{j\omega t} dt$$

- Replacing ω by $-\omega$ gives:

$$X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt \xleftrightarrow{\mathcal{F}} x^*(t)$$

- Case when $x(t)$ is real gives:

$$\boxed{X(-j\omega) = X^*(j\omega);\ x(t) \text{ real}}$$

Differentiation and Integration

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

- if we differentiate both sides of the FT synthesis eqn. we get:

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(j\omega) e^{j\omega t} d\omega$$

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)}$$

- if we integrate both sides of the FT synthesis eqn. we get:

$$\boxed{\int_{-\infty}^{t} x(\tau) d\tau \xleftrightarrow{FT} \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega)}$$

- The impulse term reflects the average value that results from integration

Time and Frequency Scaling

- If:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

- then:

$$\boxed{x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right); \quad a \text{ real constant}}$$

Time and Frequency Scaling

- Derivation:

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt$$

- Using the substitution $\tau = at$ gives:

$$\mathcal{F}\{x(at)\} = \begin{cases} \dfrac{1}{a}\displaystyle\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a > 0 \\ -\dfrac{1}{a}\displaystyle\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a < 0 \end{cases}$$

- Letting $a = -1$ gives:

$$\boxed{x(-t) \xleftrightarrow{\mathcal{F}} X(-j\omega)}$$

Time and Frequency Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

$$\Downarrow \; a = -1$$

$$x(-t) \longleftrightarrow X(-j\omega)$$

$$\Downarrow$$

E.g. $a > 1 \rightarrow at > t$
compressed in time $\Leftrightarrow$
stretched in frequency

a) $x(t)$ real and even

$$x(t) = x(-t)$$

$$\Rightarrow X(j\omega) = X(-j\omega) = X^*(j\omega) - \text{ Real \& even}$$

b) $x(t)$ real and odd

$$x(t) = -x(-t)$$

$$\Rightarrow X(j\omega) = -X(-j\omega) = -X^*(j\omega) - \text{ Purely imaginary \& odd}$$

END