INTEGRATION

Calculus & Analytical Geometry MATH-101

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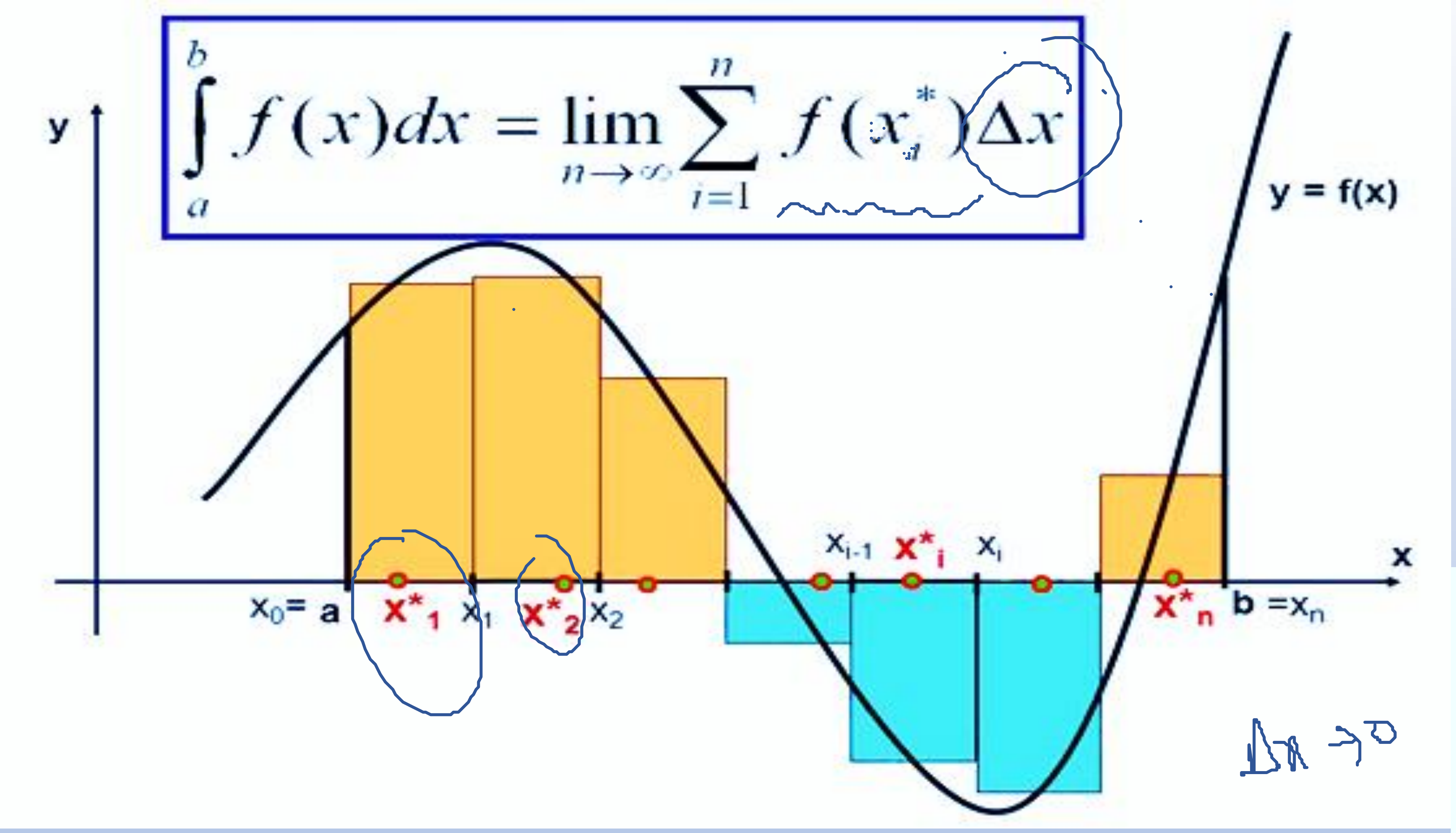
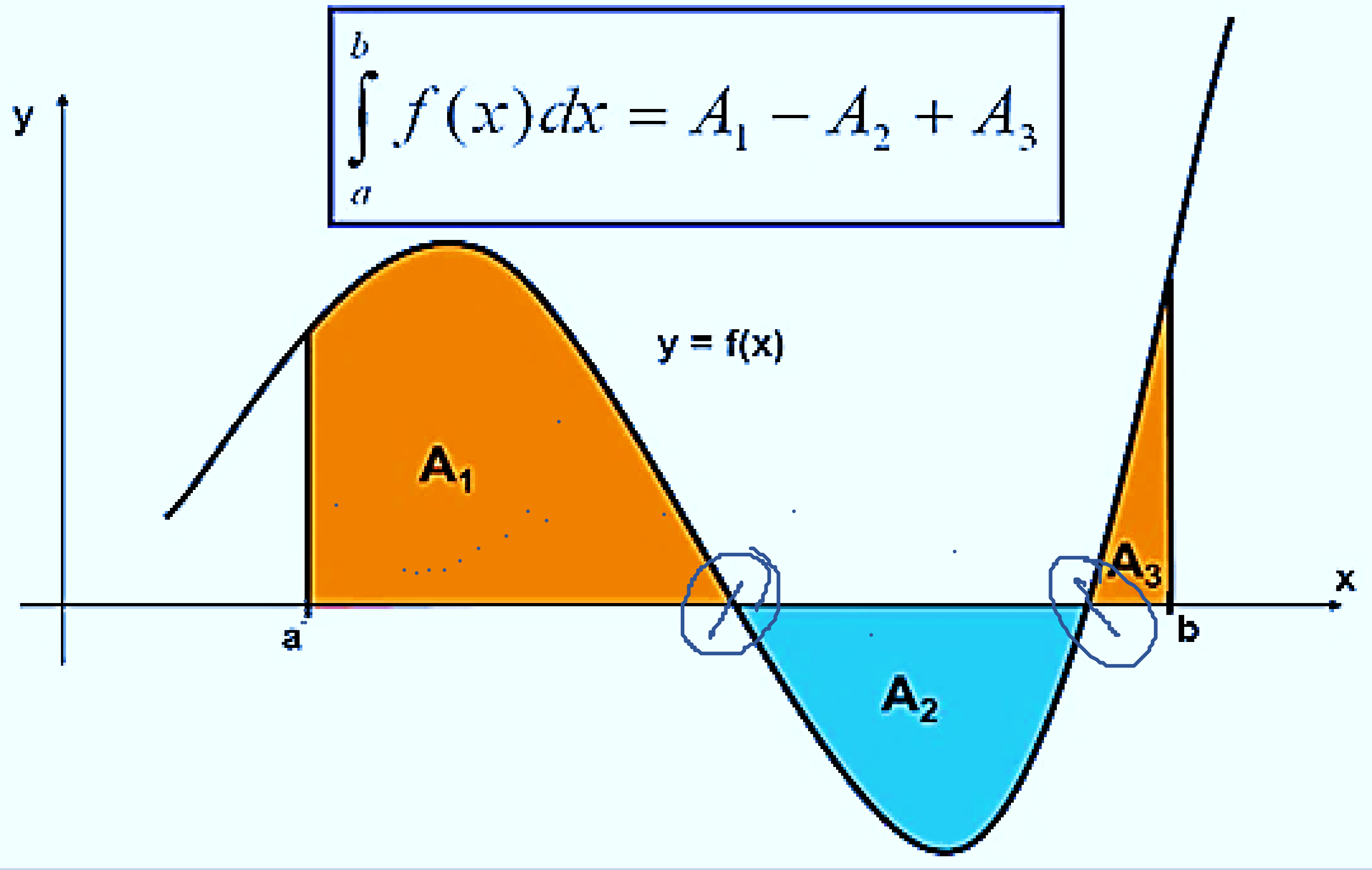


Definite Integrals
&
Area of a curve

$= A_1 + |-A_2| + A_3$



$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter:** 5
  - **Section:** 5.3, 5.4

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- **Chapter:** 5
  - **Section:** 5.4, 5.5, 5.6

Definite Integral

- The limit of Riemann sum is called the **Definite Integral** of $f(x)$ over $[a, b]$ and write

$$A = \lim_{n\to\infty} \sum_{i=1}^{i=n} f(x_i).\Delta x_i = \int_a^b f(x)dx \,.$$

- Definite integrals can be positive, negative, or zero.

- The definite integral is closely related to the area of certain region in a coordinate plane. We can easily calculate the area if the region is bounded by lines.

Rules of the Definite Integral

1. $$\int_a^b c\,dx = c(b-a)$$

$f(x)=x \to F(x)=\frac{x^2}{2}$

2. $$\int_a^b x\,dx = \frac{b^2}{2}-\frac{a^2}{2} = \frac{1}{2}\left[b^2-a^2\right]$$

3. $$\int_a^b x^2\,dx = \frac{b^3}{3}-\frac{a^3}{3} = \frac{1}{3}\left[b^3-a^3\right]$$

$$\boxed{\int_a^b x^n\,dx = \frac{1}{n+1}\left[b^{n+1}-a^{n+1}\right]}$$

$F(x)$ Antiderivative

$$\int_a^b f(x)\,dx = \boxed{F(b)-F(a)}$$

$f(x)=x^0$

$$\int_a^b c\,dx = \int_a^b c\cdot x^0\,dx = c\int_a^b x^0\,dx = c\,x\Big|_a^b = c(b-a)$$

Examples:

b

$c(b-a)$

c

1. $\int_{2}^{6} 4\, dx = 4(6-2) = 16$

a —

2. $\int_{4}^{8} x\, dx = \frac{8^2}{2} - \frac{4^2}{2} = 32 - 8 = 24$

3. $\int_{3}^{5} x^2\, dx = \frac{5^3}{3} - \frac{3^3}{3} = \frac{125}{3} - \frac{27}{3} = \frac{98}{3} = 32.67$

Examples:

$F(a)$

$F(b)$

$$4.\ \int_3^4 (x^2 + 3x - 2)\,dx = \frac{x^3}{3} + \frac{3x^2}{2} - 2x\Big|_3^4 = \frac{4^3}{3} + \frac{3(4)^2}{2} - 2(4) - \left(\frac{3^3}{3} + \frac{3(3)^2}{2} - 2(3)\right)$$

$4 \to b$, $3 \to a$

$$= 37.33 - 16.5 = 20.83.$$

$n \neq -1$

$F(32) - F(1)$

$$5.\ \int_1^{32} \frac{1}{x^{6/5}}\,dx = \int_1^{32} x^{-6/5}\,dx = \frac{x^{-1/5}}{-1/5}\Big|_1^{32} = -5\left(\frac{1}{(32)^{\frac{1}{5}}} - \frac{1}{(1)^{\frac{1}{5}}}\right) = -5\left(\frac{1}{2} - \frac{1}{1}\right) = -5\left(-\frac{1}{2}\right) = \frac{5}{2}.$$

$32 \to b$, $1 \to a$

$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

$$6.\ \int_{-1}^{1} |x|\,dx = \int_{-1}^{0} (-x)\,dx + \int_{0}^{1} (x)\,dx = -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{1} = -\left(0 - \frac{1}{2}\right) + \left(\frac{1}{2} - 0\right) = \frac{1}{2} + \frac{1}{2} = 1.$$

Example:

Evaluate

$$\int_1^3 (-x^2 + 4x - 3)\,dx,$$

By using the following values.

$$\int_1^3 x^2\,dx = \frac{26}{3}, \qquad \int_1^3 x\,dx = 4, \qquad \int_1^3 dx = 2$$

Solution:

$$\int_1^3 (-x^2 + 4x - 3)\,dx = \int_1^3 (-x^2)\,dx + \int_1^3 4x\,dx + \int_1^3 (-3)\,dx$$

$$= -\int_1^3 x^2\,dx + 4\int_1^3 x\,dx - 3\int_1^3 dx = -\left(\frac{26}{3}\right) + 4(4) - 3(2) = \frac{4}{3}$$

Definite integrals

A sufficient condition for a function $f(x)$ to be integrable on $[a, b]$ is that it is continuous on $[a, b]$.

Continuity Implies Integrability:

If a function $f(x)$ is continuous on the closed interval $[a, b]$, the $f(x)$ is integrable on $[a, b]$. That is:

$$\int_a^b f(x)\, dx,$$

exists.

Practice Questions

Evaluate the following:

1. $$\int_{-1}^{3} (x^3+1)^2 dx .$$

2. $$\int_{1}^{4} \left(5x - 2\sqrt{x} + \frac{32}{x^3}\right) dx.$$

3. $$\int_{2}^{10} \frac{3}{\sqrt{5x-1}} dx .$$

4. $$\int_{0}^{\pi/4} (1+\sin 2x)^3 \cos 2x \, dx.$$

5. $$\int_{-1}^{5} |x-2| dx .$$

6. $$\int_{0}^{6} f(x) dx, \quad \text{where } f(x) = \begin{cases} x^2, & x < 2 \\ 3x-2, & x \geq 2 \end{cases}$$

Practice Questions

Evaluate the following:

$$\int_1^4 \left(5x - 2\sqrt{x} + \frac{32}{x^3}\right) dx.$$

Sol:-

$$\int_1^4 \left(5x - 2\sqrt{x} + \frac{32}{x^3}\right) dx = \left. \frac{5x^2}{2} - \frac{2x^{3/2}}{3/2} + \frac{32x^{-2}}{(-2)} \right|_1^4$$

$$= \left. \frac{5}{2}x^2 - \frac{4}{3}x^{3/2} - \frac{16}{x^2} \right|_1^4$$

$$= \left[\frac{5}{2}(4)^2 - \frac{4}{3}(4)^{3/2} - \frac{16}{(4)^2}\right] - \left[\frac{5}{2} - \frac{4}{3} - 16\right]$$

$$\Rightarrow \int_1^4 \left(5x - 2\sqrt{x} + \frac{32}{x^3}\right)dx = \left(40 - \frac{32}{3} - 1\right) - \left(\frac{5}{2} - \frac{4}{3} - 16\right)$$

$$= 40 - \frac{32}{3} - 1 - \frac{5}{2} + \frac{4}{3} + 16$$

$$= \frac{240 - 64 - 6 - 15 + 8 + 96}{6}$$

$$= \frac{259}{6}.$$

Practice Questions

Evaluate the following:

$$\int_{0}^{\pi/4} (1+\sin 2x)^3 \cos 2x \, dx.$$

Sol :

Consider

$$\int (1+\sin 2x)^3 \cos 2x \, dx$$

$$u = 1 + \sin 2x$$

$$\Rightarrow du = 2 \cos 2x \, dx$$

$$\int (1+\sin 2x)^3 \cdot \cos 2x \, dx$$

$$= \int u^3 \cdot \frac{du}{2}$$

$$= \frac{1}{2} \int u^3 \, du$$

$$= \frac{u^4}{2 \times 4} = \frac{u^4}{8}$$

$$\int_0^{\pi/4} (1+\sin 2x)^3 \cos 2x \, dx = \left. \frac{(1+\sin 2x)^4}{8} \right|_0^{\pi/4}$$

$$= \frac{1}{8}\left[\left(1+\sin \cancel{2}\left(\frac{\pi}{4}\right)\right)^4 - (1+\sin 2(0))^4\right]$$

$$= \frac{1}{8}\left[\left(1+\sin \frac{\pi}{2}\right)^4 - (1+\sin 0)^4\right]$$

$$= \frac{1}{8}\left[(1+1)^4 - (1+0)^4\right]$$

$$= \frac{1}{8}\left[16 - 1\right] = \frac{15}{8} \checkmark$$

Practice Questions

Evaluate the following:

$$\int_0^6 f(x)dx, \quad \text{where } f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$$

Sol

$$\int_0^6 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx$$

$$= \int_0^2 x^2\,dx + \int_2^6 (3x-2)\,dx$$

$$= \frac{x^3}{3}\Big|_0^2 + \frac{3x^2}{2} - 2x\Big|_2^6$$

$$\Rightarrow \int_0^6 f(x)\,dx = \frac{1}{3}\left[(2)^3 - (0)^3\right] + \frac{3}{2}\left[(6)^2 - (2)^2\right] - 2\left[6 - 2\right]$$

$$= \frac{1}{3}\left[8 - 0\right] + \frac{3}{2}\left[36 - 4\right] - 2\left[4\right]$$

$$= \frac{128}{3}$$

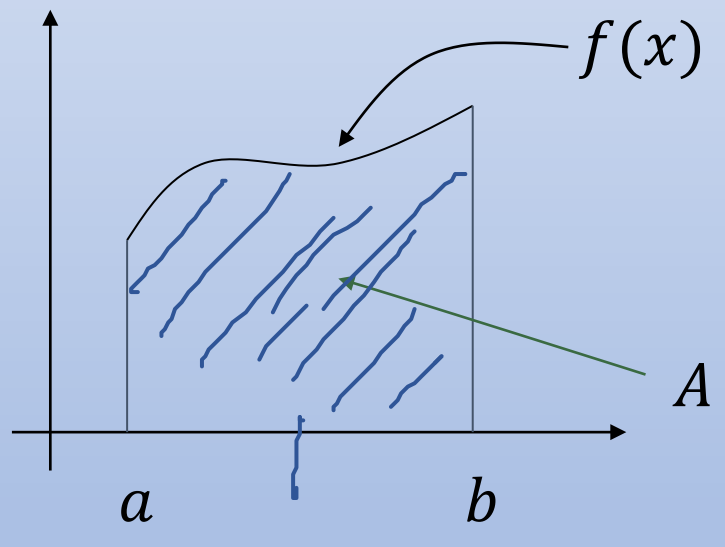
The area under a curve

For a definite integral to be Interpreted as an area, the function $f(x)$ must be continuous and non-negative on $[a, b]$, as stated in the following definition.

The Definite Integral as the Area of a Region:

If $f(x)$ is continuous and non-negative on the closed interval $[a, b]$, then the area of the region bounded by the graph of $f(x)$, the $x - axis$, and the vertical lines $x = a$ and $x = b$ is given as:

$$\text{Area} = \int_a^b f(x)\, dx.$$



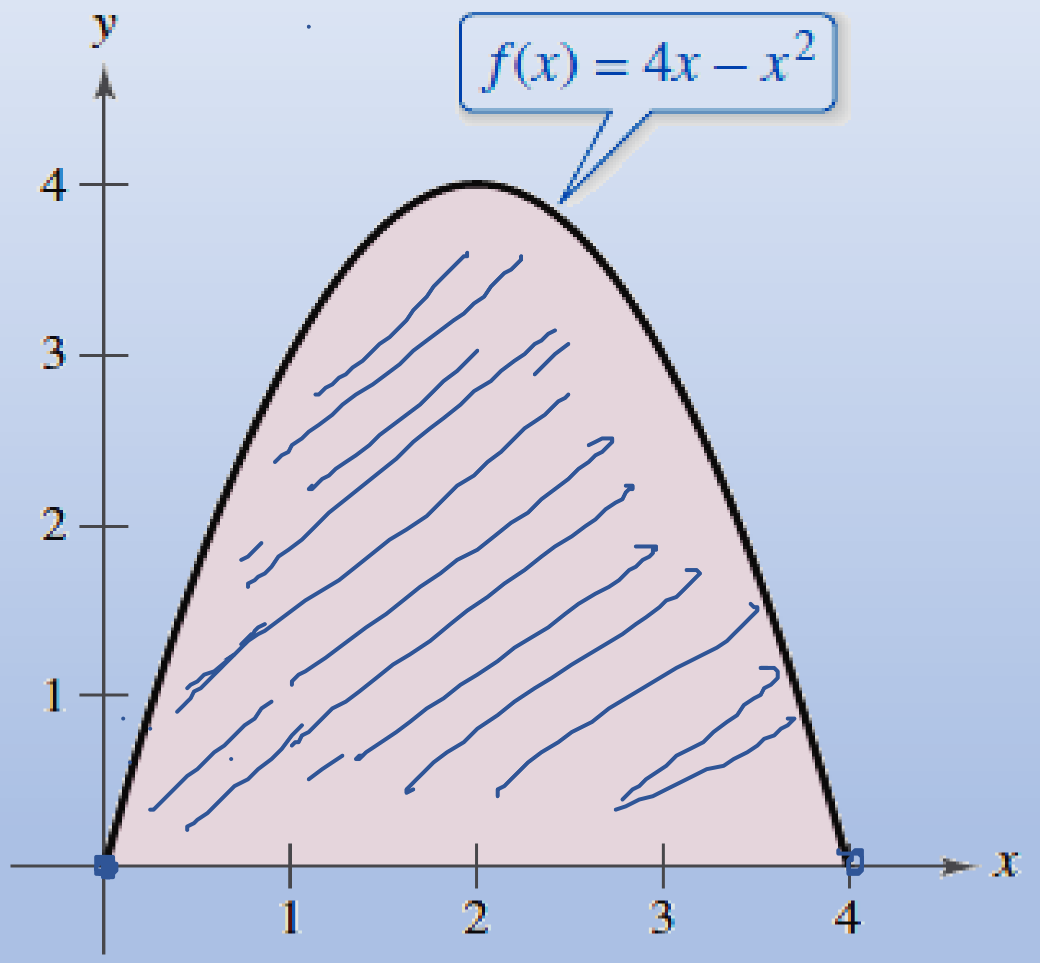
Example

Consider the region bounded by the graph of $f(x) = 4x - x^2$ and the x –axis, as shown in figure. Determine the area of the shaded region.

Solution:

Since $f(x)$ is continuous and non-negative on the closed interval $[0, 4]$, the area of the required region is given by:

$$\text{Area} = \int_0^4 (4x - x^2)\,dx.$$



$$\text{Area} = \int_0^4 (4x - x^2)\,dx$$

The area under a curve

In order to find area under the curve, we can evaluate a definite integral in two ways:

- either we can use the limit definition $A = \int_a^b f(x)\,dx$

  or

- we can check to see whether the definite integral represents the area of a common geometric region such as a rectangle, triangle, or a semicircle.

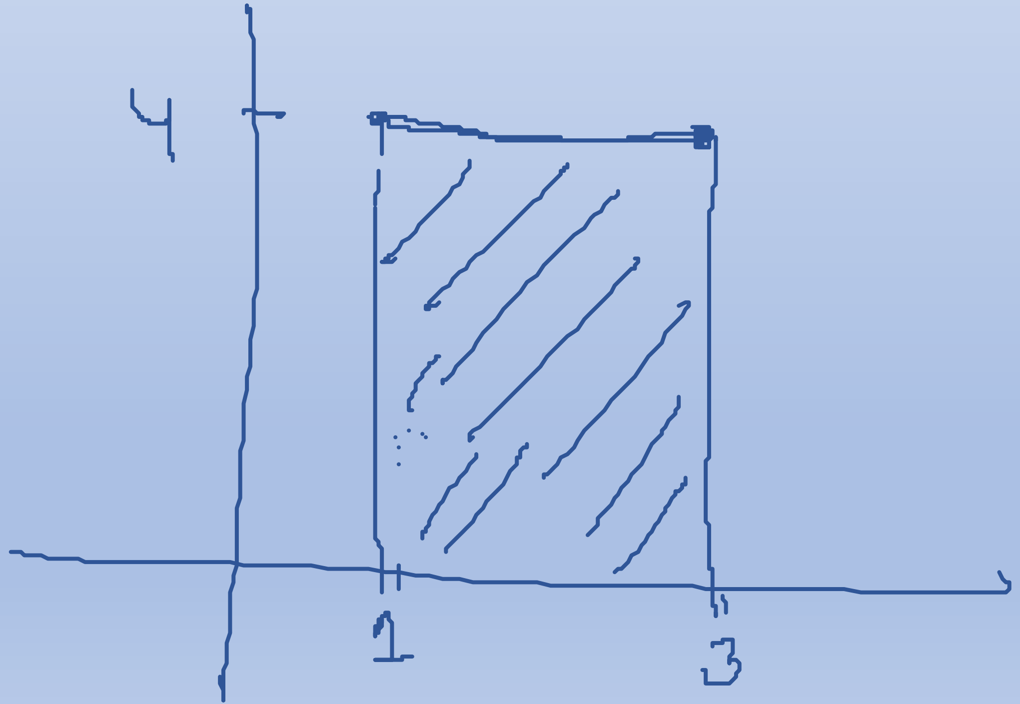
Examples– *Areas of common geometric figures*

Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

A. $\int_1^3 4\,dx$

$y = 4$

$[a, b] = [1, 3]$



B. $\int_0^3 (x + 2)\,dx$

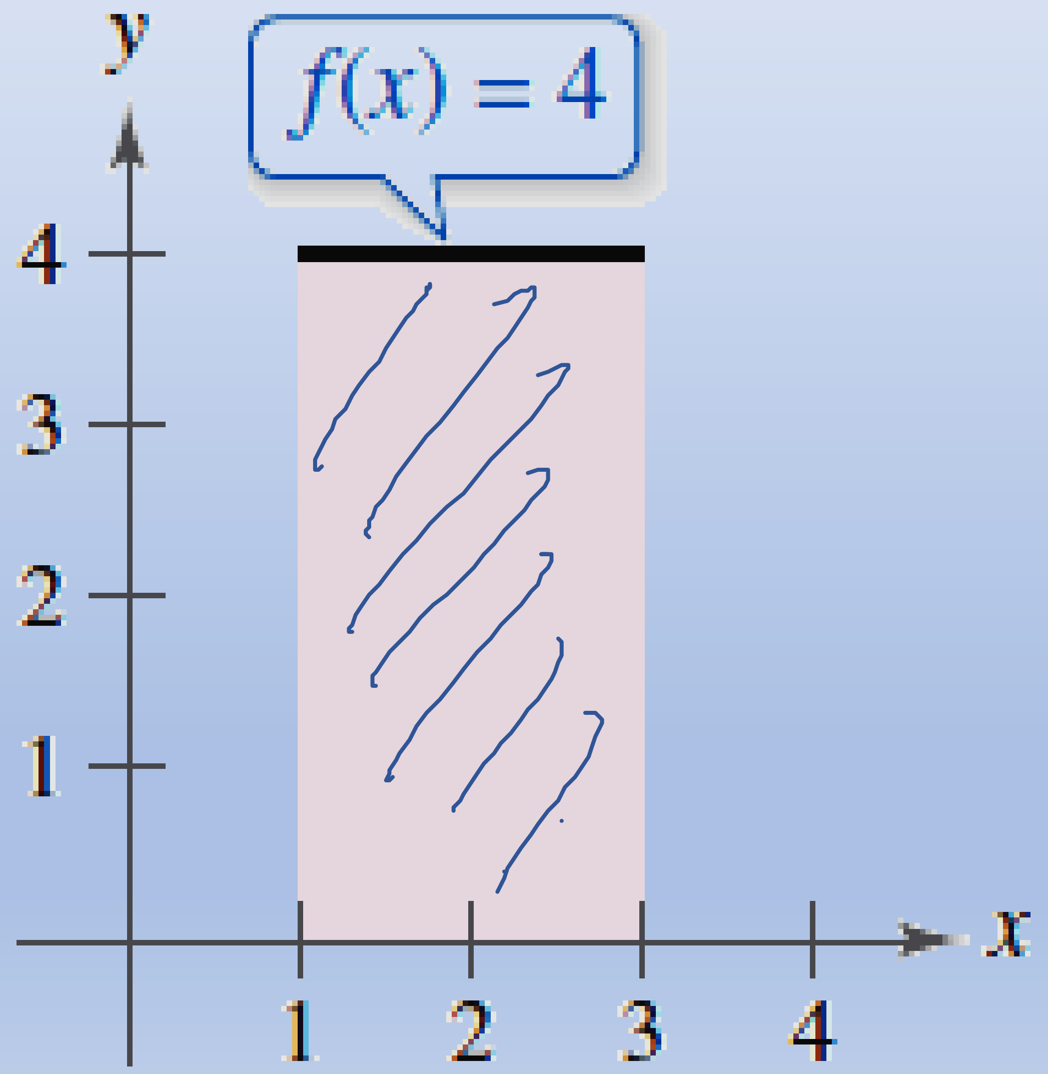
C. $\int_{-2}^{2} \sqrt{4 - x^2}\,dx$

Example (a) – *Solution*

$b - a = 3 - 1 = 2$

This region is a rectangle of height 4 and width 2.

$$\int_1^3 4\,dx = (\text{Area of rectangle}) = 4(2) = 8$$



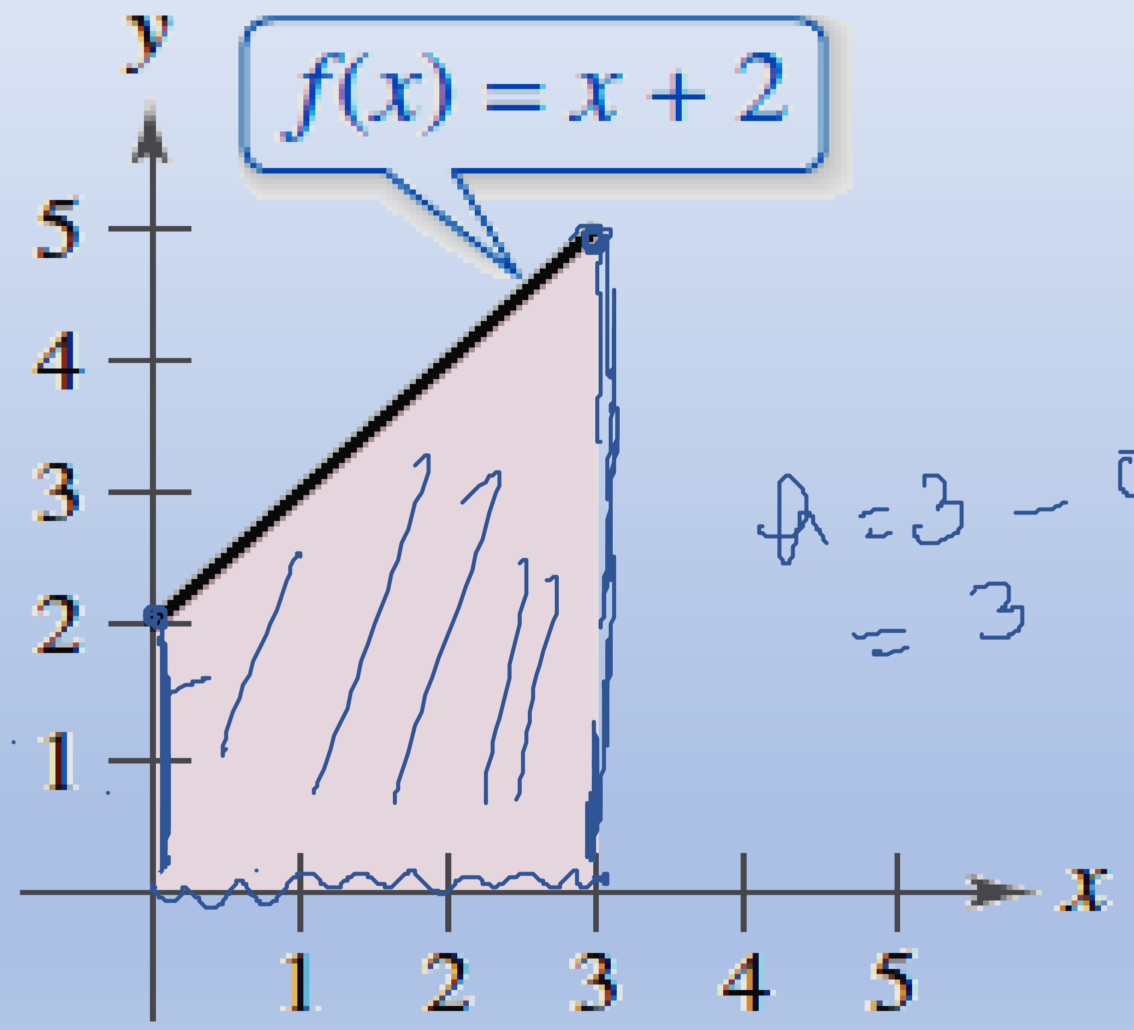
Example (b) – *Solution*

This region is a trapezoid with an altitude of 3 and parallel bases of lengths 2 and 5. The formula for the area of a trapezoid is:

$$\frac{1}{2}h(b_1 + b_2).$$

$$\int_0^3 (x+2)\,dx = \text{Area of trapezoid}$$

$$= \frac{1}{2}(3)(2+5) = \frac{21}{2}.$$



A = 3 − 0
= 3

Example (c) – *Solution*

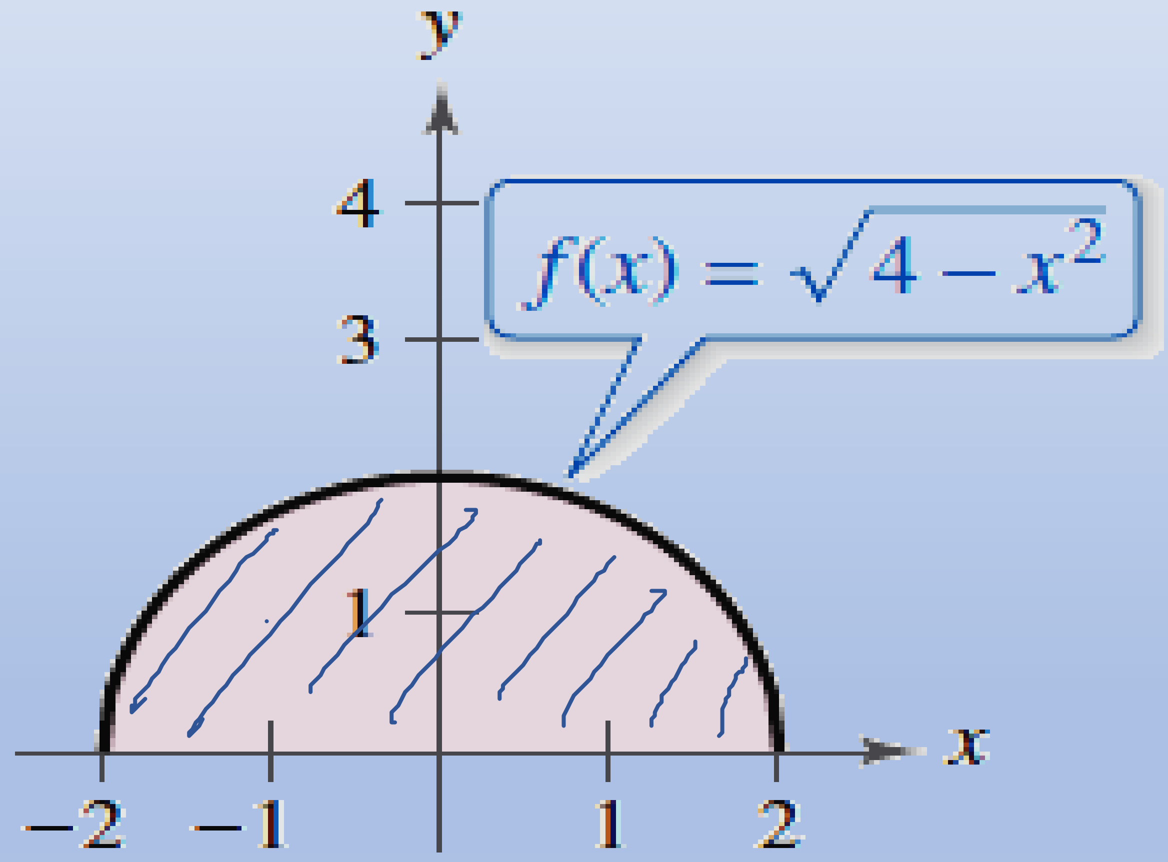
This region is a semicircle of radius 2. The formula for the area of a semicircle of radius r is:

$$\frac{1}{2}\pi r^2$$

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = \text{(Area of semicircle)}$$

$$= \frac{1}{2}\pi(2^2) = 2\pi$$

Practice



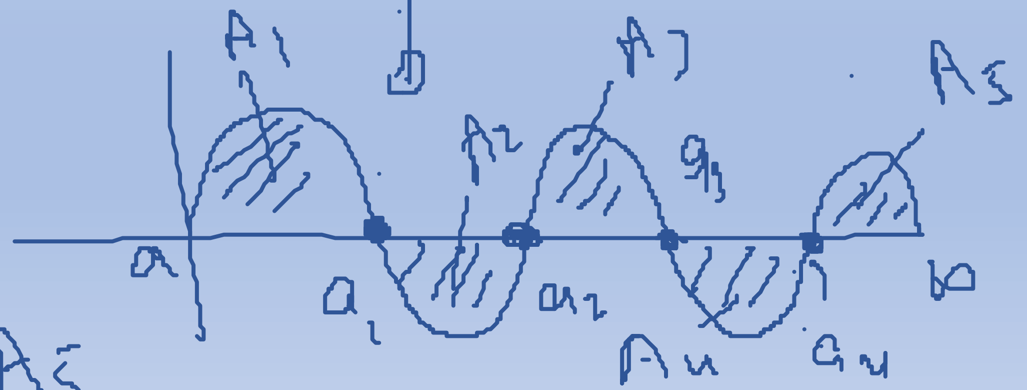
Total area of a curve

Total Area $= A =$
$= [A_1 + A_3 + A_5]$
$- [A_2 + A_4]$

- To compute the area of the region bounded by the graph of a function $y = f(x)$ and the x-axis requires more care when the function takes on both positive and negative values.
- We must be careful to break up the interval $[a, b]$ into subintervals on which the function doesn't change sign. Otherwise, we might get cancellation between positive and negative signed areas, leading to an incorrect total.

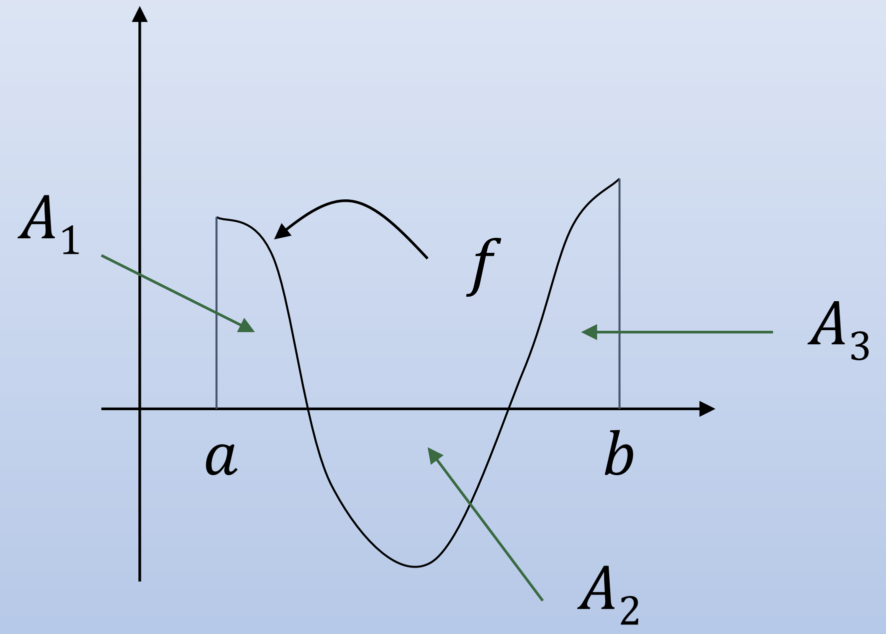
$f(x) = 0$ $[a, b]$

- The correct total area is obtained by adding the **Absolute value** of the definite integral over each subinterval where $f(x)$ does not change sign.
- The term "area" will be taken to mean *total area*.



$A = A_1 + |-A_2| + A_3 + |-A_4| + A_5$

Total area of a curve



A = area above – area below

$$\int_a^b f(x)dx = A_1 + A_3 - A_2$$

$$= A_1 + A_3 + |-A_2|.$$

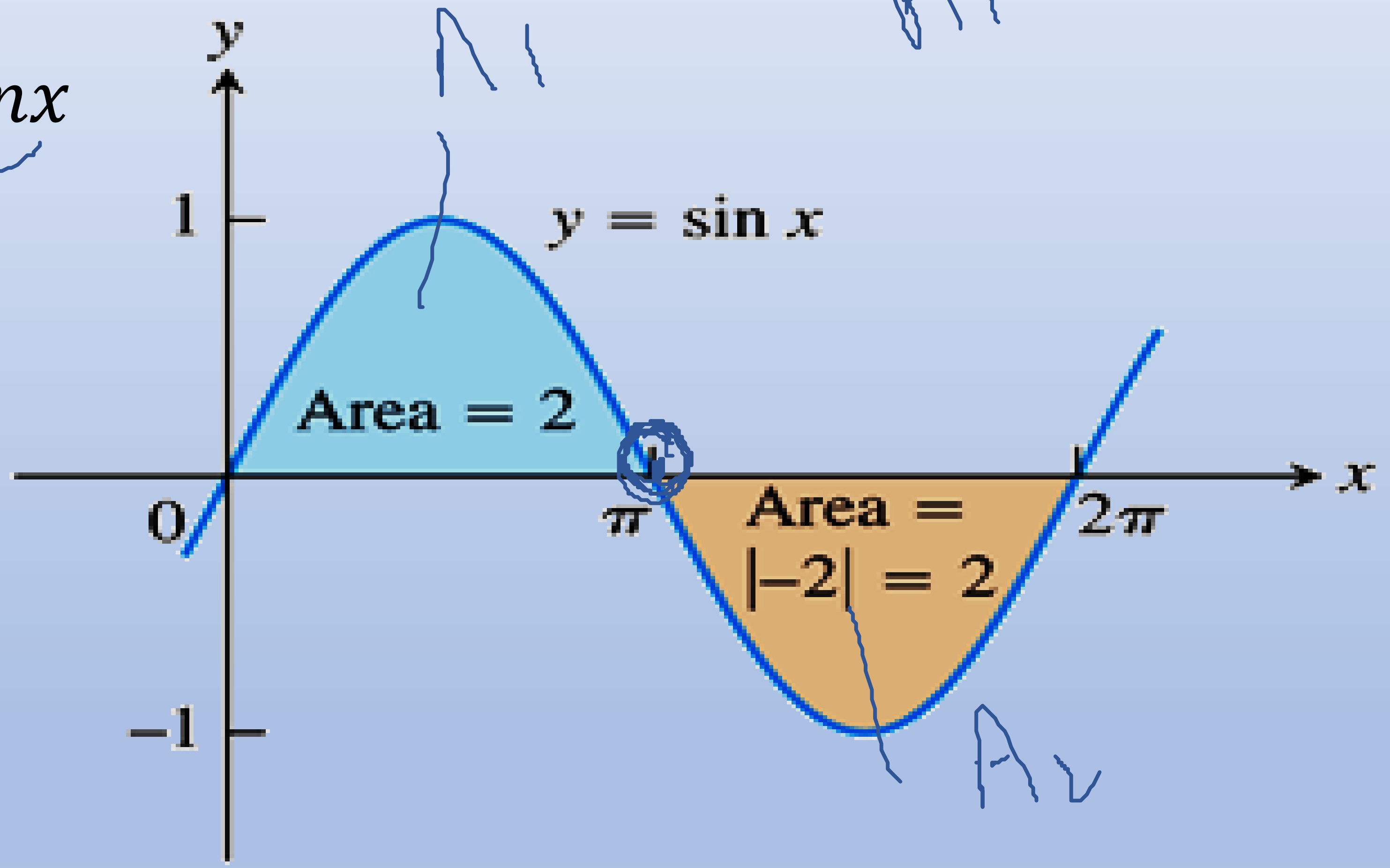
Difference between the Value of a Definite Integral and Total Area

$A = A_1 - A_2$

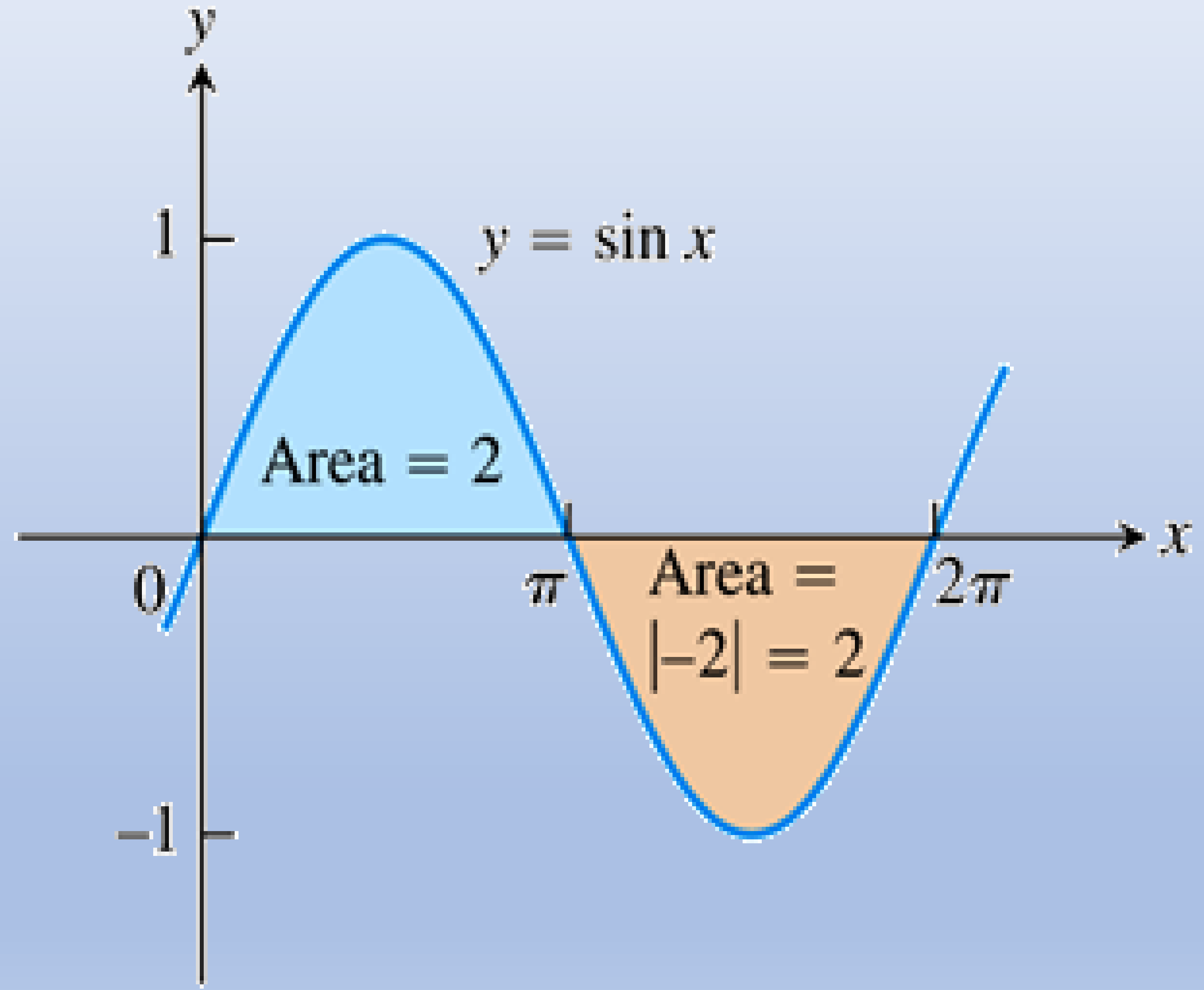
Figure shows the graph of the function $f(x) = sinx$ over the interval $[0, 2\pi]$. Compute

a) the definite integral of $f(x)$ over $[0, 2\pi]$.

b) the area between the graph of $f(x)$ and the x −axis over $[0, 2\pi]$.



Value of the Definite Integral



$$\int_{0}^{2\pi} sinx\, dx = -cosx\Big|_{0}^{2\pi}$$

$$= -cos(2\pi) - \big(-cos(0)\big)$$

$$= -(1) - (-1) = 0$$

Total Area

$$\int_0^{2\pi} \sin x\,dx = \underbrace{\int_0^{\pi} \sin x\,dx}_{A_1} - \underbrace{\int_{\pi}^{2\pi} \sin x\,dx}_{A_2} \qquad \because [A_1 - A_2]$$

$$= -\cos x \Big|_0^{\pi} - (-\cos x)\Big|_{\pi}^{2\pi}$$

$$= -\cos(\pi) - (-\cos(0)) - \big[\big(-\cos(2\pi) - (-\cos(\pi))\big)\big]$$

$$= -(-1) - (-1) - \big(-(1) - (1)\big)$$

$$= 2 - (-2)$$

$$= 4.$$

$$\int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx$$

Area between the graph of $y = f(x)$ and the x –axis over the interval $[a, b]$

To find the area between the graph of $y = f(x)$ and the x –axis over the interval $[a, b]$,

we do the following:

a) Subdivide $[a, b]$ at the zeros of $f(x)$. ✓

b) Integrate $f(x)$ over each subinterval. ✓

c) Add the absolute values of the integrals. ✓

Example:

$x = -1$ $x = 1$?? $[-1, 0]$ $[0, 1]$ $[1, 2]$

Determine the area of the region between the graph of $f(x) = x^3 - x^2 - 2x$, and the x −axis over the interval $[-1, 2]$.

Solution:

$[-1, 2]$

$[-1, 0]$ & $[0, 2]$

Zeros of $f(x)$:

$$f(x) = x^3 - x^2 - 2x = 0$$

$$\Longrightarrow x(x+1)(x-2) = 0$$

$$\Longrightarrow x = -1, 0, 2.$$

The zeros of $f(x)$partitions the given interval into two subintervals $[-1,0]$ and $[0,2]$. We integrate $f(x)$ on each subinterval and add the absolute values of the calculated values.

Solution:

Integral over $[-1, 0]$:

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0}$$

$$= 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

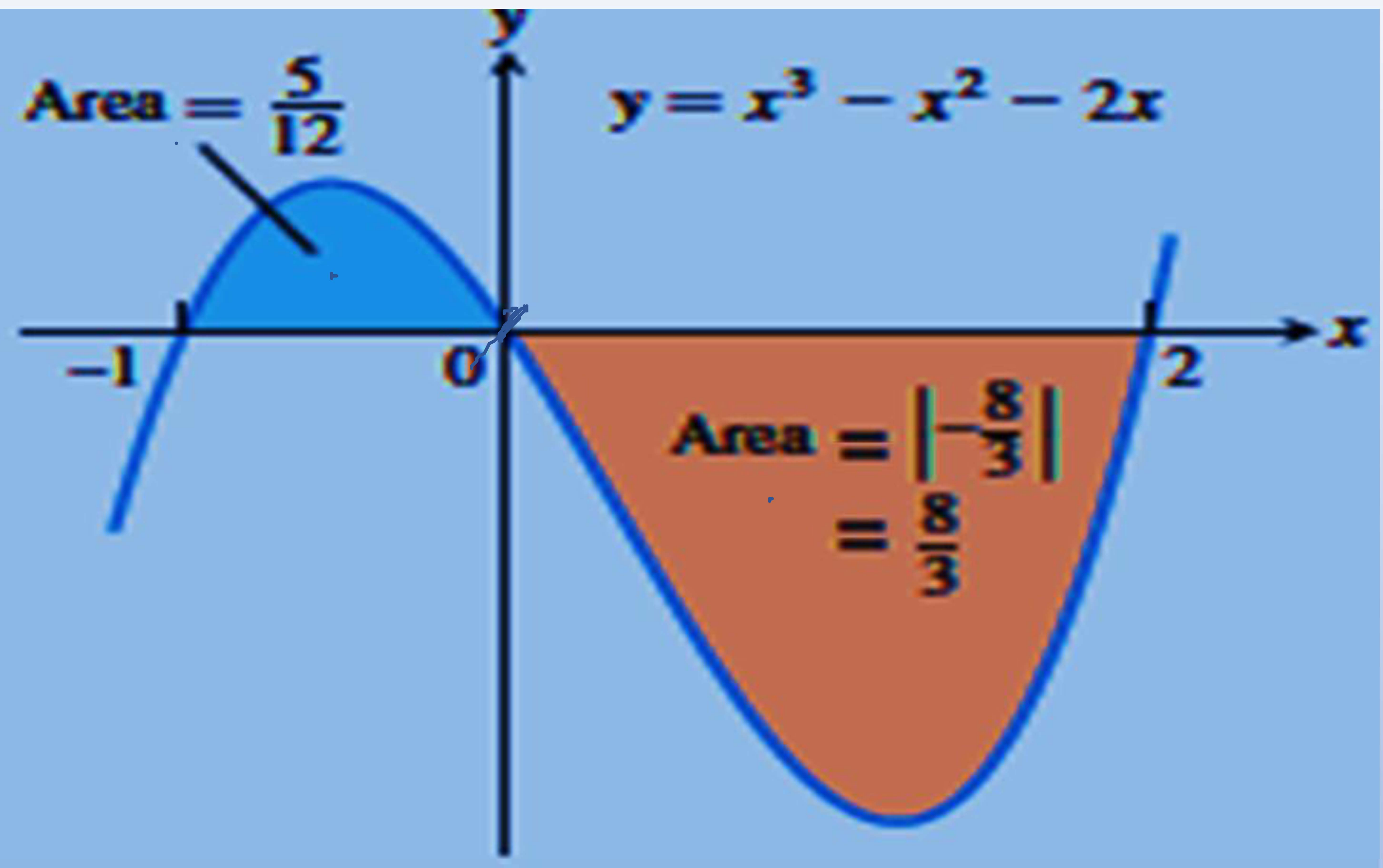
Integral over $[0, 2]$:

$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2}$$

$$= \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

Enclosed area:

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$



The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis

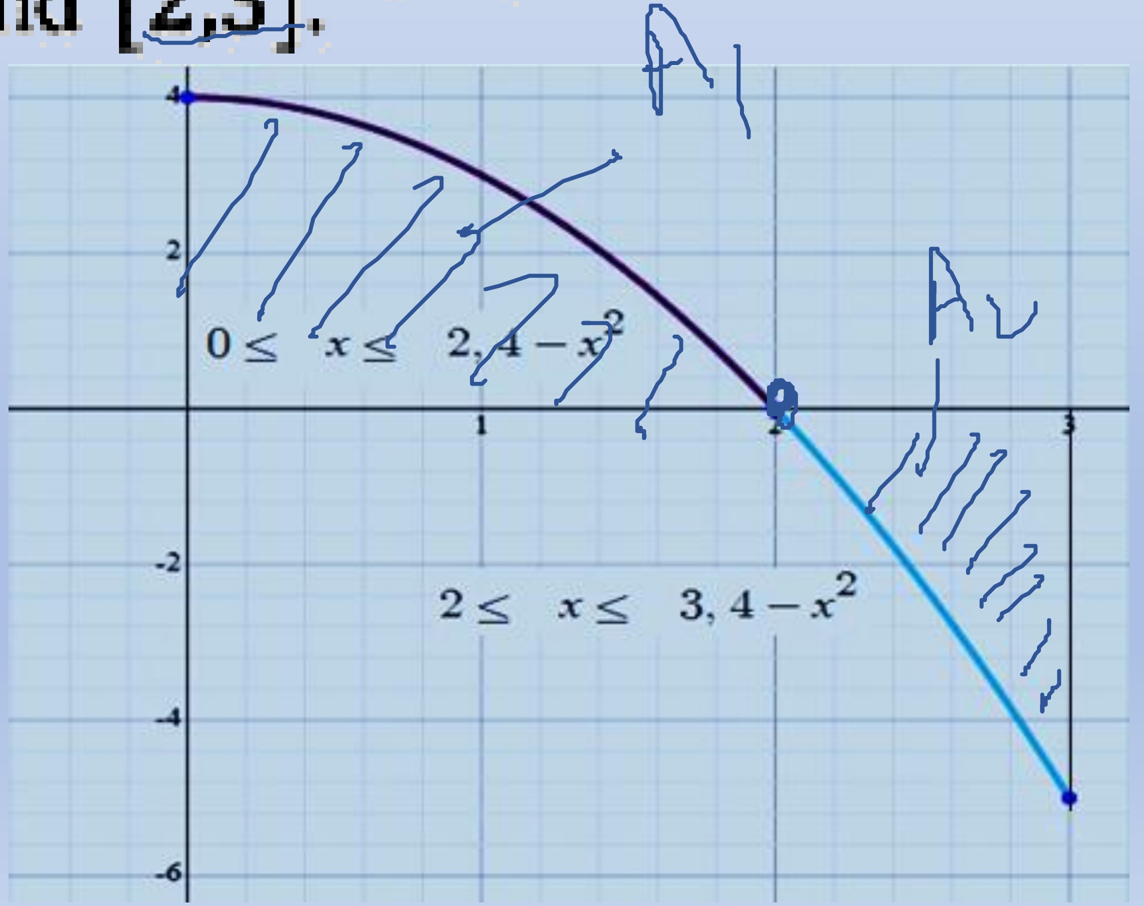
Example:

Find the area of the region between the curve $y = 4 - x^2, 0 \leq x \leq 3$, and the x-axis.

Solution:

The zeros of f are ± 2 of which 2 lies in the given interval [0,3]. It partitions the interval into two subintervals: [0,2] and [2,3].

$A = A_1 + |-A_2|$



Solution:

Integral over [0, 2]:

$$\int_0^2 (4 - x^2)\,dx = \int_0^2 4\,dx - \int_0^2 x^2\,dx$$

$$= 4(2-0) - \frac{(2)^3}{3}$$

$$= 8 - \frac{8}{3} = \frac{16}{3}$$

Integral over [2, 3]:

$$\int_2^3 (4 - x^2)\,dx = \int_2^3 4\,dx - \int_2^3 x^2\,dx$$

$$= 4(3-2) - \left(\frac{(3)^3}{3} - \frac{(2)^3}{3}\right)$$

$$= 4 - \frac{19}{3} = -\frac{7}{3}$$

The region's area: $\text{Area} = \frac{16}{3} + \left|-\frac{7}{3}\right| = \frac{23}{3}.$

Practice Questions

1. Calculate the areas of the segments contained between the curve $y = x(x - 1)(x - 2)$, the x −axis and the ordinates $x = 0$ and $x = 2$.

2. Find the area between x −axis, the curve $y = x(x - 3)$ and the ordinates $x = 0$ and $x = 5$.

3. Determine the area enclosed between the curve $y = x^3 - 4x^2 - 5x$, the x −axis and the ordinates $x = -1$ nd $x = 5$.

4. Calculate the area enclosed between the graph of the function $y = |x|$, the x −axis and the ordinates $x = -1$ and $x = 4$.

5. Calculate the area enclosed between the curve $y = 3 + \sqrt{4 - x^2}$, the x −axis and the ordinates $x = -2$ and $x = 2$.