Singularities, zeros & residues:

singular point

$f(z)$ is infinite at this point

There is a choice of value, and it is not possible to pick a particular one.

Here we shall be mainly concerned with singularities at which $f(z)$ has an infinite value.

A zero of $f(z)$ is a point in the z plane at which $f(z)=0$.

Singularities can be classified in terms of the Laurent series expansion of $f(z)$ about the point in question. If $f(z)$ has a Taylor series expansion, i.e, a Laurent series expansion with zero principal part, about the point $z=z_0$, then z_0 is a regular point of $f(z)$. If $f(z)$ has a Laurent series expansion with only a finite number of terms in the principal part, for example

$$f(z)=\frac{a_{-m}}{(z-z_0)^m}+\cdots\cdots+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\cdots+a_m(z-z_0)^m+\cdots\cdots$$

then $f(z)$ has a singularity at $z=z_0$ called a pole. If there are m terms in the principal part, then the pole is said to be of order m. If the principal part of the Laurent series for $f(z)$ at $z=z_0$ has infinitely many terms, then $z=z_0$ is called an essential singularity of $f(z)$.

EX: Find and classify singularities of $f(z)=\dfrac{e^z-\sin z-1}{z^2}$

[to read]

Sol: We know that $e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\cdots$

$$\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\cdots$$

hence, $$f(z)=\frac{\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\cdots\right)-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right)-1}{z^2}$$

$$=\frac{1}{2!}+\frac{2}{3!}z+\frac{1}{4}z^2+\frac{1}{6!}z^4+\frac{2}{7!}z^5+\cdots\cdots$$

The Laurent expansion of $f(z)$ has no negative powers of z. Therefore, $f(z)$ has a removable singularity at $z=0$.

Ex:- Let $f(z) = \frac{\pi z(1-z^2)}{\sin(\pi z)}$.

(a) Find all zero(s) of $f(z)$. (b) Find and classify all singularities of $f(z)$ (c) Identify the Laurent expansion of $f(z)$ around zero.

Sol:- We notice that

(a) $\lim\limits_{z\to 0} f(z) = \lim\limits_{z\to 0} \frac{\pi z(1-z^2)}{\sin(\pi z)} = 1 \neq 0$.

$\lim\limits_{z\to 1} f(z) = \lim\limits_{z\to 1} \frac{\pi z(1-z)(1+z)}{\sin(\pi(z-1)+\pi)}$

$= \lim\limits_{z\to 1} \frac{\pi z(1-z)(1+z)}{-\sin(\pi(z-1))} = 2\pi \neq 0$.

and $\lim\limits_{z\to -1} f(z) = \lim\limits_{z\to -1} \frac{\pi z(1-z)(1+z)}{\sin(\pi(1+z)-\pi)} = \lim\limits_{z\to -1} \frac{\pi z(1-z)(1+z)}{-\sin(\pi(z+1))} = -2\pi \neq 0$.

So, the function $f(z)$ has no zeros.

(b). $z = 0, 1,$ and -1 are removable singularities of $f(z)$

$z = k$, $k \neq 0, 1, -1$ are simple poles of the function $f(z)$.

(c) We know that $\frac{1}{\sin \pi z} = \operatorname{Cosec}(\pi z)$.

$= \frac{1}{\pi z}\left(1 + \frac{\pi^2 z^2}{3!} + \cdots\right)$

Thus the Laurent series expansion of f around $z = 0$ is

$f(z) = \pi z(1-z^2) \cdot \frac{1}{\pi z}\left(1 + \frac{\pi^2 z^2}{3!} + \cdots\right)$

$= 1 + \left(\frac{\pi^2}{6} - 1\right)z^2 + \cdots$

Ex:- Let $f(z) = \frac{1}{(z-1)(z-3)^2}$.

$f(z)$ has a pole of order one (simple pole) at $z = 1$

and a pole of order two at $z = 3$.

Residues: If a complex function $f(z)$ has a pole at the point $z = z_0$ then the coefficient a_{-1} of the term $1/(z-z_0)$ in the Laurent series expansion of $f(z)$ about $z = z_0$ is called the residue of $f(z)$ at the point $z = z_0$.

Let us consider the case when $f(z)$ has a simple pole at $z = z_0$. Then

$$f(z) = \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots\cdots$$

in an appropriate annulus $R_1 < |z-z_0| < R_2$. multiplying by $z-z_0$ gives

$$(z-z_0)f(z) = a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + \cdots$$

which is a Taylor series expansion of $(z-z_0)f(z)$. If we let z approach z_0, we then obtain the result

$$\text{Residue at a simple pole } z_0 = \lim_{z \to z_0} \left[(z-z_0)f(z)\right] = a_{-1}$$

Hence the limit gives a way of calculating the residue at a simple pole. Now suppose that we have a pole of order two at $z = z_0$. $f(z)$ has a Laurent series expansion of the form

$$f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots\cdots$$

$$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + \cdots\cdots$$

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$$\frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = a_{-1} + 2a_0(z-z_0) + \cdots\cdots$$

$$\lim_{z \to z_0}\left[\frac{d}{dz}\left((z-z_0)^2 f(z)\right)\right] = a_{-1}.$$

In general if $f(z)$ has a pole of order m at $z = z_0$,

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots\cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots\cdots$$

$$\text{residue at a pole of order } m \text{ at } z = z_0 = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)\right\}.$$

Ex:- Find the Laurent series for the functions below about the specified point and hence find the residue at the point.

(a). $f(z) = \dfrac{1}{z(z+1)}$, $z = 0$.

$f(z) = \frac{1}{z}(1+z)^{-1} = \frac{1}{z}(1 - z + z^2 - z^3 + \cdots\cdots)$ $\quad |z| < 1$

$= \frac{1}{z} - 1 + z - z^2 + z^3 + \cdots\cdots\cdots = \sum_{n=-1}^{\infty} (-1)^{n+1} z^n$.

The residue of a function $f(z)$ at $z = z_0$ is the Coefficient of $(z - z_0)^{-1}$ in its Laurent expansion around $z = z_0$.

Hence, $\operatorname{Res}[f(z), z = 0] = 1$.

(b). $f(z) = \dfrac{\sin z}{z^4}$, $z = 0$.

we know that $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$.

$\Rightarrow \dfrac{\sin z}{z^4} = \dfrac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots \right]$ [page 1.]

$= \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} z^{2n-3}$.

The residue of $f(z)$ at $z = 0$ is the Coefficient of $\frac{1}{z}$, $\text{Residue}\left[f(z), z = 0\right] = -\frac{1}{3!} = -\frac{1}{6}$.

Direct calculation Residue:

(a) $\operatorname{Res}\left[f(z) ; z = 0\right] = \lim_{z \to 0}\left[z f(z)\right] = \lim_{z \to 0} \frac{1}{z+1} = \frac{1}{0+1} = 1$.

(b) $\operatorname{Res}\left[f(z) ; z = 0\right] = \dfrac{1}{(4-1)!}\left[\dfrac{d^3}{dz^3}(\sin z)\right]_{z=0}$

$= \frac{1}{3!}\left[-\cos z\right]_{z=0} = -\frac{1}{3!} = -\frac{1}{6}$.

$f(z) = \sin z$
$f'(z) = \cos z$
$f''(z) = -\sin z$
$f'''(z) = -\cos z$

Ex:- Find the residues of $f(z)=\frac{1}{(3z+2)(2-z)}$ at $z=-\frac{2}{3}$ and $z=2$.

(i) Residue at $z=-2/3$: Let $w=z+\frac{2}{3} \Rightarrow z=-\frac{2}{3}+w$.

$$\frac{1}{(3z+2)(2-z)}=\frac{1}{[3(-\frac{2}{3}+w)+2][2-(-\frac{2}{3}+w)]}=\frac{1}{3w(\frac{8}{3}-w)}$$

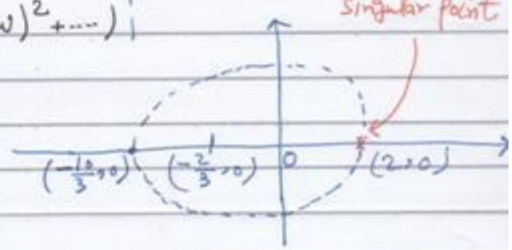
$|\frac{3}{8}w|<1 \Rightarrow |w|<\frac{8}{3}$

$|z+\frac{2}{3}|<\frac{8}{3}$

$$\frac{1}{(3z+2)(2-z)}=\frac{-1}{w(3w-8)}$$

$$=\frac{1}{8w}\frac{1}{(1-\frac{3}{8}w)}=\frac{1}{8w}\left(1+\frac{3}{8}w+(\frac{3}{8}w)^2+\cdots\right)$$

$$=\frac{1}{8w}+\frac{3}{64}+\frac{9}{512}w+\cdots\cdots$$



So, the residue at $w=0$ $(z=-\frac{2}{3})$ is $\frac{1}{8}$.

(ii) Residue at $z=2$: Let $z=2+w$

$$\frac{1}{(3z+2)(2-z)}=\frac{1}{(3(2+w)+2)(2-(2+w))}=\frac{1}{(-w)(3w+8)}$$

$|\frac{3}{8}w|<1$

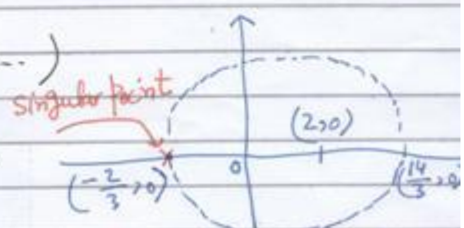
$|w|<\frac{8}{3}$

$|z-2|<\frac{8}{3}$

$$=\frac{-1}{8w}\frac{1}{(1+\frac{3}{8}w)}$$

$$=\frac{-1}{8w}\left(1-\frac{3}{8}w+(\frac{3}{8}w)^2-\cdots\cdots\right)$$

$$=-\frac{1}{8w}+\frac{3}{64}-\frac{9}{512}w+\cdots\cdots$$



So, the residue at $w=0$ $(z=2)$ is $\frac{-1}{8}$.

Direct Calculation of Residue:

$$\text{Residue}[f(z); z=-\tfrac{2}{3}]=\lim_{z\to-\frac{2}{3}}\left[[z-(-\tfrac{2}{3})]\,\frac{1}{3}\cdot\frac{1}{(z+\frac{2}{3})(2-z)}\right]$$

$$=\frac{1}{3}\lim_{z\to-2/3}\frac{1}{2-z}=\frac{1}{3}\cdot\frac{1}{2-(-2/3)}=\frac{1}{3}\cdot\frac{1}{8/3}=1/8.$$

$$\text{Residue}[f(z); z=2]=\lim_{z\to2}[(z-2)f(z)]=\lim_{z\to2}\left[(z-2)\frac{1}{(3z+2)(2-z)}\right]$$

$$=(-1)\lim_{z\to2}\frac{1}{3z+2}=-\frac{1}{8}.$$

EX:- Find the order of the pole of $f(z) = \dfrac{\sin z}{(1-\cos z)^2}$.

We know that $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\cdots$

$1-\cos z = 1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\cdots\right) = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\cdots$

$(1-\cos z)^2 = z^4\left(\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} + \cdots\right)^2$

$$\begin{array}{r|l} 2 & 576,720 \\ 2 & 288,360 \\ 2 & 144,180 \\ 2 & 72,90 \\ 9 & 36,45 \\ & 4,5 \end{array}$$

Therefore, $\dfrac{\sin z}{(1-\cos z)^2} = \dfrac{1}{z^3} \cdot \dfrac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\cdots}{\left(\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots\cdots\right)^2}$

$= \dfrac{1}{z^3} \dfrac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\cdots}{\frac{1}{4} - \frac{z^2}{24} + \frac{9}{2880}z^4 + \cdots\cdots}$

$\frac{4c_0}{z^3} = \frac{1}{z^3}$ $\Rightarrow 4c_0 = 1$, $c_0 = \frac{1}{4}$.

$\frac{1}{576} + \frac{1}{720} = \frac{5+4}{2880} = \frac{9}{2880}$.

$f(z)$ has a pole of order 3 at $z=0$.

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$$\frac{\frac{1}{z^3} - \frac{1}{6}\frac{1}{z} + \frac{1}{120}z}{\frac{1}{4} - \frac{z^2}{24} + \frac{9}{2880}z^4 + \cdots\cdots} = c_0\frac{1}{z^3} + c_1\frac{1}{z^2} + c_2\frac{1}{z} + c_3 + c_4 z + \cdots\cdots$$

$$c_0\frac{1}{4z^3} - \frac{c_0}{24}\frac{1}{z} + \frac{9}{2880}c_0 z + \cdots\cdots$$
$$+ \frac{c_1}{4}\frac{1}{z^2} - \frac{c_1}{24} + \frac{9c_1}{2880}z^2 + \cdots\cdots$$
$$+ \frac{c_2}{4}\frac{1}{z} - \frac{c_2}{24}z + \frac{9c_2}{2880}z^3 + \cdots\cdots$$
$$+ \frac{c_3}{4} - \frac{c_3}{24}z^2 + \frac{9c_3}{2880}z^4 + \cdots\cdots$$
$$+ \frac{c_4}{4}z - \frac{c_4}{24}z^3 + \frac{9c_4}{2880}z^5 + \cdots\cdots$$

Comparing coefficients, we get

$\frac{1}{z^3}: 1 = \frac{c_0}{4}$

$\frac{1}{z^2}: 0 = c_1/4$

$\frac{1}{z}: -\frac{1}{6} = -\frac{c_0}{24} + \frac{c_2}{4}$

$z^0: 0 = c_3/4$

$z^1: \frac{1}{120} = \frac{9c_0}{2880} - \frac{c_2}{24} + \frac{c_4}{4}$.

After solving for c_0, c_1, c_2, c_3 & c_4, we get

$c_0 = 4$, $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = -1/240$.

Hence, $f(z) = \frac{4}{z^3} + 0 + 0 + 0 - \frac{1}{240}z + \cdots\cdots$