LAPLACE TRANSFORM

Motivation for Laplace Transform

- CT Fourier transform enables us to do a lot of things, e.g.,
  - Analyze frequency response of LTI systems
  - Sampling
  - Modulation
- Why do we need yet another transform?

- One view of Laplace Transform is as an *extension* of the Fourier transform to allow analysis of broader class of signals and systems
- In particular, Fourier transform *cannot* handle large (and important) classes of signals and *unstable* systems, i.e., when

$$\int_{-\infty}^{\infty} |x(t)|dt = \infty$$

Motivation for Laplace Transform

- In many applications, we do need to deal with *unstable* systems, e.g.,
  - stabilizing an inverted pendulum
  - stabilizing an airplane or space shuttle
  - instability is *desired* in some applications, e.g., oscillators and lasers

- How do we analyze such signals/systems?
  Recall the eigenfunction property of LTI systems:

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt \qquad \text{(assuming this converges)}$$

  - e^{st} is an eigenfunction of *any* LTI system
  - $s = \sigma + j\omega$ can be complex in general

The Laplace Transform

- The response of an LTI system with impulse response $h(t)$ to a complex exponential input of the form e^{st} is:

$$y(t) = H(s)e^{st}$$

- where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

- for s imaginary (i.e., $s = j\omega$) the transform corresponds to the Fourier transform of $h(t)$.

- For general values of the complex variable, s, it is referred to as the *Laplace Transform* of the impulse response $h(t)$

The Laplace Transform

- The Laplace transform of a general signal, $x(t)$, is defined as:

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt}$$

- Note that the Laplace transform is a function of the independent variable s corresponding to the complex variable in the exponent of e^{-st}.
- The complex variable s can be written as $s = \sigma + j\omega$ with σ and ω being the real and imaginary parts of s
- We will denote the Laplace transform in operator form as $\mathcal{L}\{x(t)\}$ and denote the transform as

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

Relation Between Laplace and Fourier Transform

- When $s = j\omega \ (\sigma = 0)$ we get:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

- which corresponds to the Fourier transform of $x(t)$

$$X(s)\big|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

Relation Between Laplace and Fourier Transform

- Even when s is not purely imaginary the Laplace transform bears a straightforward relationship to the Fourier transform of the form:

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[x(t) e^{-\sigma t} \right] e^{-j\omega t}\, dt$$

- i.e., the FT of the sequence $x(t)e^{-\sigma t}$

Laplace Transform – Region of Convergence

$$x(t) \longleftrightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt = \mathcal{L}\{x(t)\}$$

$s = \sigma + j\omega$ is a *complex* variable – Now we explore the full range of s

Basic ideas:

absolute integrability needed

(1) $$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

Laplace Transform – Region of Convergence

(2) A critical issue in dealing with Laplace transform is convergence:
— $X(s)$ generally exists only for *some* values of s, located in what is called the *region of convergence* (ROC)

$$\text{ROC} = \{s = \sigma + j\omega \text{ so that } \int_{-\infty}^{\infty} \underbrace{\left|x(t)e^{-\sigma t}\right|}_{\substack{\text{Depends} \\ \text{only on } \sigma \\ \text{not on } \omega}} dt < \infty\}$$

absolute integrability condition

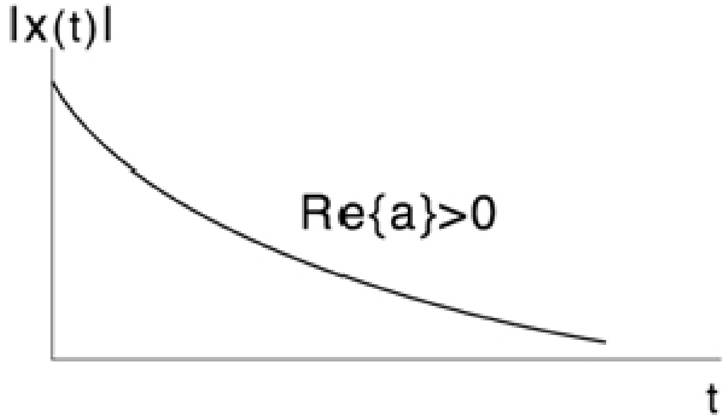
(3) If $s = j\omega$ is in the ROC (i.e., $\sigma = 0$), then

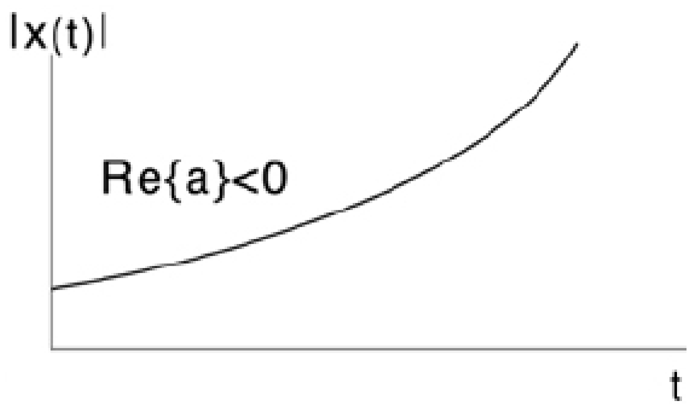
$$X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

Laplace Transform – Example 1

Find the Fourier and Laplace Transform of the signal:

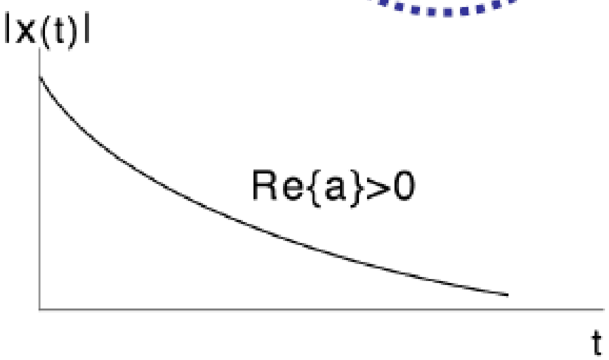
$$x(t) = e^{-at}u(t); (a > 0)$$

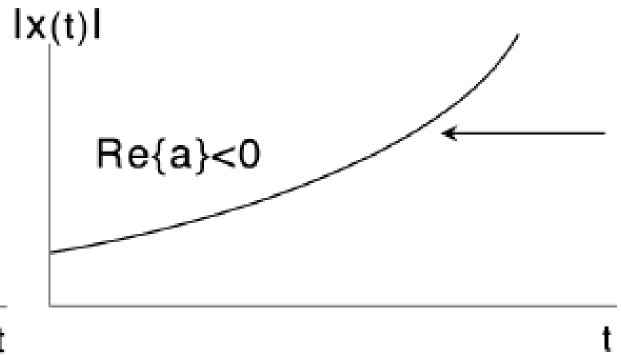




Laplace Transform – Example 1

$$x_1(t) = e^{-at}u(t) \qquad (a - \text{an arbitrary real or complex number})$$





Unstable:

- no *Fourier Transform*
- but *Laplace Transform* exists

$\boxed{\text{The Laplace transform can converge for some values of } \sigma = \mathcal{R}e\{s\} \text{ and not others.}}$

$$\Downarrow$$

$$X_1(s) = \frac{1}{s+a}, \quad \underbrace{\Re e\{s\} > -\Re e\{a\}}_{\text{ROC}}$$

Laplace Transform – Example 2

- Consider the signal

$$x(t) = -e^{-at}u(-t)$$

Laplace Transform – Example 2

$$X(s) = -\int_{-\infty}^{\infty} e^{-at} e^{-st} u(-t)\, dt = -\int_{-\infty}^{0} e^{-(s+a)t} dt$$

$$X(s) = \frac{1}{s+a}$$

- For convergence we require that $\mathrm{Re}\{s+a\} < 0$

or, equivalently, $\mathrm{Re}\{s\} < -a$

- Giving the Laplace transform pair:

$$-e^{-at} u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} < -a$$

Laplace Transform – ROC

- Examples 1 and 2 have the same algebraic expression for the Laplace transform
- However, the set of values of s for which the expression is valid is very different for the two examples.
- This illustrates the fact that in specifying the Laplace transform of a signal, both the algebraic expression and the range of values of s for which this expression is valid are required.
- The range of values of x for which the integral converges is called the *Region of Convergence (ROC)* of the Laplace transform

Laplace Transform – ROC

Example #1

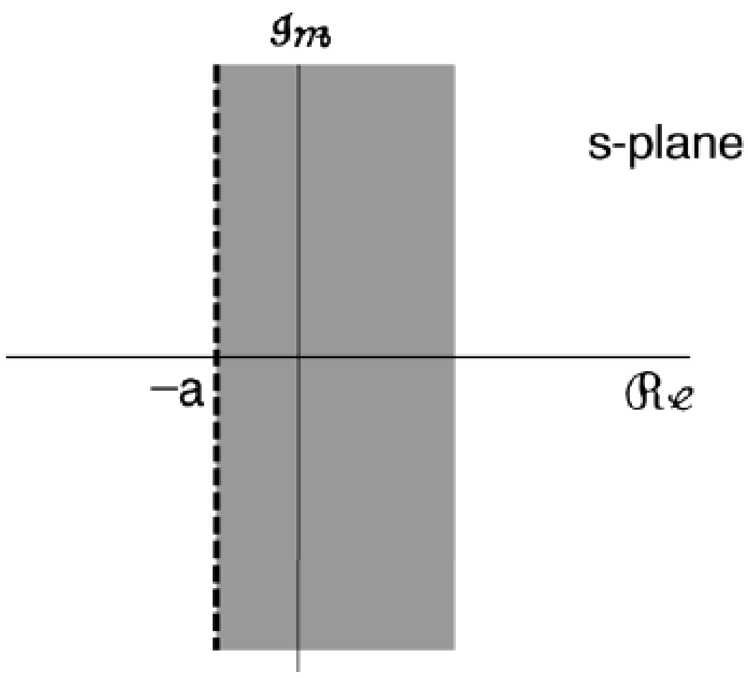
$$X_1(s) = \frac{1}{s+a}, \quad \Re e\{s\} > -\Re e\{a\}$$

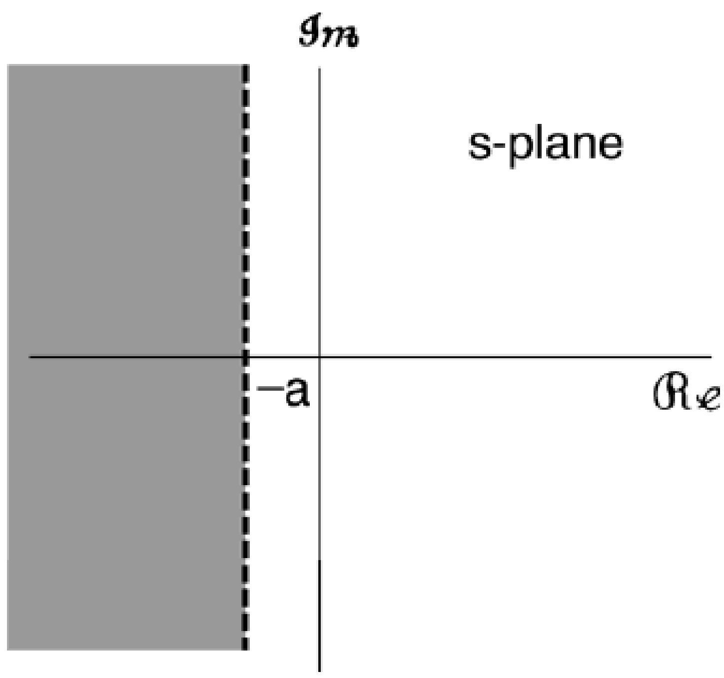
$x_1(t) = e^{-at}u(t)$ - right-sided signal

Example #2

$$X_2(s) = \frac{1}{s+a}, \quad \Re e\{s\} < -\Re e\{a\}$$

$x_2(t) = -e^{-at}u(-t)$ - left-sided signal





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