The Fourier transform

Very important tool in Science & Engineering. Fundamentally used for signal analysis in the Frequency domain.

Let $f(t)$ be an absolute integrable function in $-\infty < t < \infty$ (Area under graph of f $f(t)$ is finite) or $f(t) \longrightarrow 0$ as $|t| \longrightarrow \infty$.

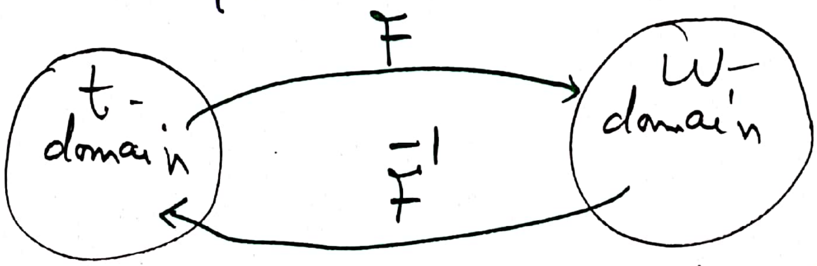
The Fourier transform of $f(t)$; denoted by $F(j\omega)$ is

$$\mathcal{F}[f(t)] = F(j\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt \quad \text{—— (i)}$$

Also,

$$\mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\, e^{j\omega t}\, d\omega \quad \text{—— (ii)}$$

(i) & (ii) give the F.T. pair, we sometime denote as: $f(t) \longleftrightarrow F(j\omega)$.



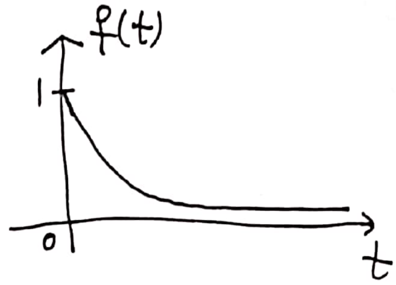
The Fourier transform pair (i) & (ii) enables us to switch between the time and frequency domain.

Tricky: $\mathcal{F}[1] = \int_{-\infty}^{\infty} (1)\, e^{-j\omega t}\, dt = \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-\infty}^{+\infty}$

$\lim_{t \to \infty} e^{-j\omega t} = \lim_{t \to \infty} (\cos\omega t - j\sin\omega t)$ DNE.

F.T. 1

Ex-1 Find the F.T. of the one-sided exponential function $f(t) = H(t)\, e^{-at}$ $(a>0)$.



where $H(t) = U(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

is the unit-step function.

Sol:
$$F[f(t)] = F(\omega) = \int_{-\infty}^{+\infty} H(t)\, e^{-at}\, e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t}\, dt = \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_0^{\infty} = 0 - \frac{1}{-(a+j\omega)} = \frac{1}{a+j\omega}.$$

Ex-2: Find the F.T. of $f(t) = t\, e^{-at} H(t)$, $a>0$.

Sol:
$$F[f(t)] = F(j\omega) = \int_{-\infty}^{+\infty} t\, e^{-at} H(t)\, e^{-j\omega t}\, dt.$$

$$= \int_0^{\infty} \underset{\text{I}}{t}\ \underset{\text{II}}{e^{-(a+j\omega)t}}\, dt.$$

$$= \left. t\, \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_0^{\infty} - \int_0^{\infty} (1)\, \frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\, dt.$$

$$= [0-0] + \frac{1}{a+j\omega} \int_0^{\infty} e^{-(a+j\omega)t}\, dt.$$

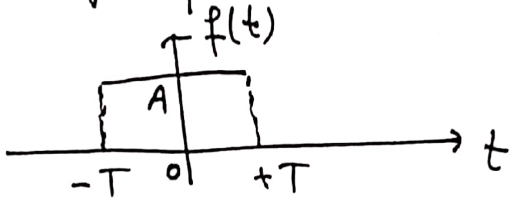
$$= \frac{1}{a+j\omega} \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty} = \frac{1}{a+j\omega} \left[0 - \frac{1}{-(a+j\omega)} \right].$$

$$= \frac{1}{(a+j\omega)} \left[\frac{1}{a+j\omega} \right] = \frac{1}{(a+j\omega)^2}.$$

F.T. 2

EX-3:- Calculate the F.T. of the rectangular pulse

$$f(t) = \begin{cases} A, & |t| \leq T \\ 0, & |t| > T. \end{cases}$$



Sol:-

$$\mathcal{F}[f(t)] = F(jw) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt$$

$$= \int_{-T}^{T} A\, e^{-jwt}\, dt = A \left. \frac{e^{-jwt}}{-jw} \right|_{-T}^{T}, \quad w \neq 0$$

$$= \frac{A}{-jw}\left[e^{-jwT} - e^{-jw(-T)} \right] = \frac{A}{w}\left[\frac{e^{jwT} - e^{-jwT}}{j} \right] = \frac{2A}{w}\left[\frac{e^{jwT} - e^{-jwT}}{2j} \right]$$

$$= \frac{2A}{w} \sin wT, \quad w \neq 0$$

for $w = 0$, $F(w) = \int_{-T}^{T} A\, dt = 2AT$.

Def: Sinc function:: $\operatorname{sinc} x = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$

In general, $A \operatorname{sinc}(ax) = \begin{cases} A \dfrac{\sin ax}{ax}, & x \neq 0 \\ A, & x = 0. \end{cases}$

In view of this definition, we summarize example-3 as:

$$F(jw) = \frac{2A}{w} \sin wT = 2AT \frac{\sin wT}{wT}$$

$$= 2AT \operatorname{sinc}(wT).$$ F.T. 3

The Continuous Frequency spectra:

$f(t) \longleftrightarrow F(jw)$, i.e, $\mathcal{F}[f(t)] = F(jw)$.

Writing $F(jw)$ in the complex exponential form

$$F(jw) = |F(jw)| \, e^{j \arg F(w)} .$$

plots of $|F(jw)|$ & $\arg F(jw)$ are called amplitude and phase spectra respectively of signal $f(t)$.

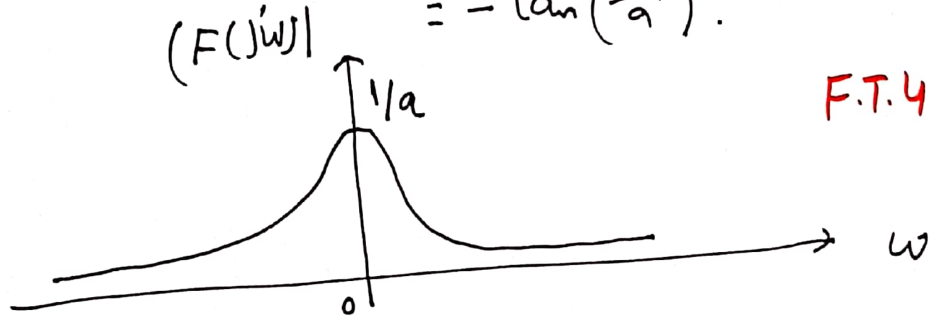
Ex-4 Determine the amplitude & phase spectra of the signal $f(t) = e^{-at} H(t)$, $a > 0$ and plot their graphs.

Sol: In example 1, we showed that $F(w) = \dfrac{1}{a+jw}$

$$|F(jw)| = \frac{1}{\sqrt{a^2+w^2}}, \quad \arg F(jw) = \tan^{-1}(1) - \tan^{-1}\left(\frac{w}{a}\right)$$

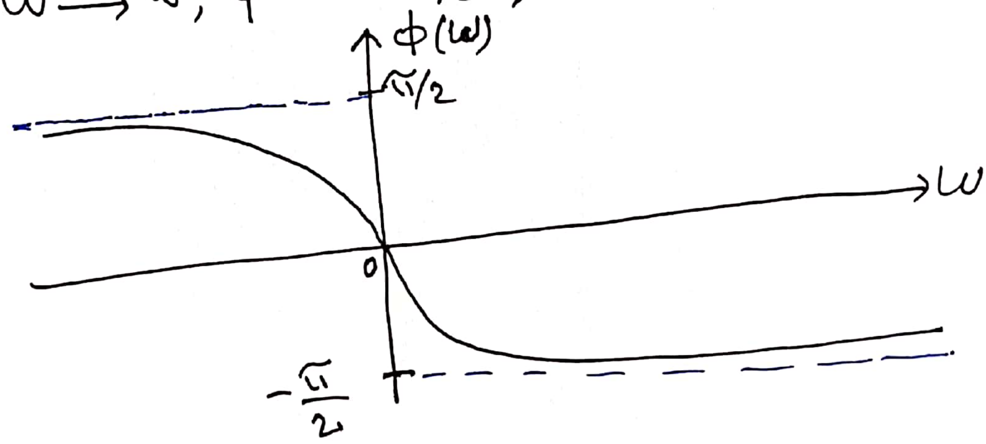
$$= -\tan^{-1}\left(\frac{w}{a}\right).$$

F.T.4



$\phi = \arg F(jw) = -\tan^{-1}(w/a)$

when $w \to \infty$; $\phi \to -\pi/2$, & $w \to -\infty$, $\phi \to -(-\pi/2) = \frac{\pi}{2}$.

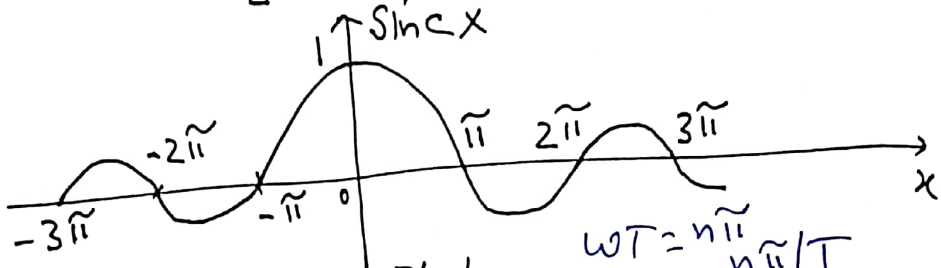


EX-5: Find & sketch the spectrum of the rectangular pulse in example - 3.

Sol: we know that $A\,rect\left(\frac{t}{2T}\right) \longleftrightarrow 2AT\,\mathrm{Sinc}\,\omega T$.

where $rect\left(\frac{t}{2T}\right) = \begin{cases} 1, & |t| \le T \\ 0, & \text{otherwise} \end{cases}$.

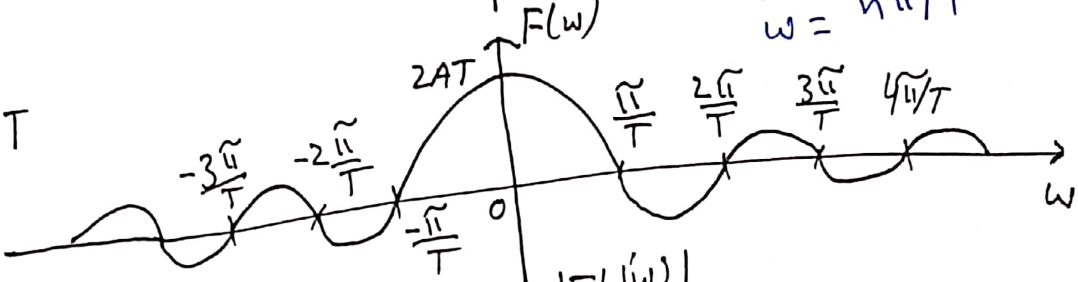
$\mathrm{Sinc}\,x = \frac{\sin x}{x}$



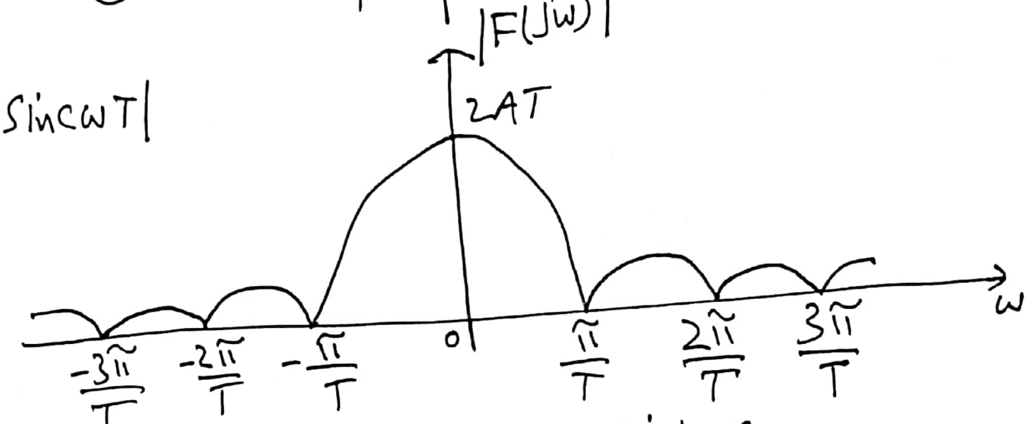
$\omega T = n\pi$

$\omega = n\pi / T$

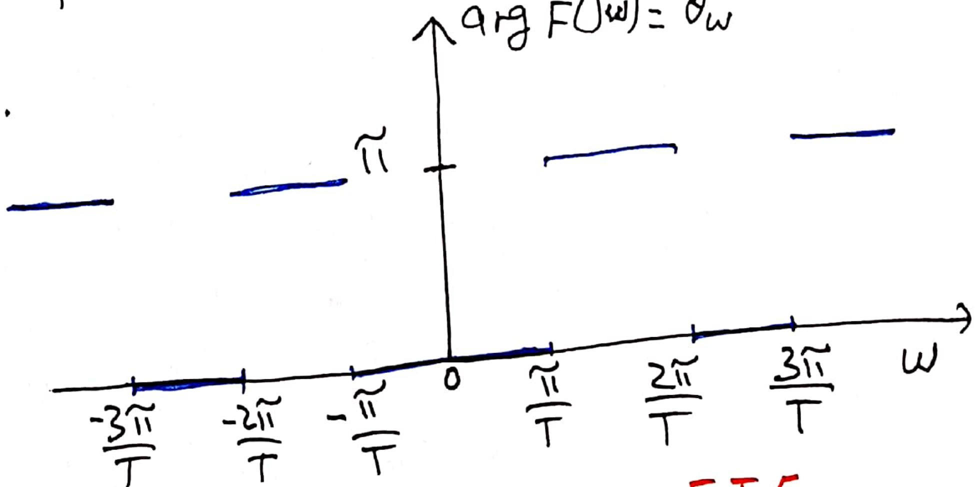
$F(\omega) = 2AT\,\mathrm{Sinc}\,\omega T$



$|F(\omega)| = 2AT\,|\mathrm{Sinc}\,\omega T|$



$\theta_\omega = \begin{cases} 0, & \sin \omega T \ge 0 \\ \pi, & \sin \omega T < 0 . \end{cases}$



Properties of the Fourier transform:-

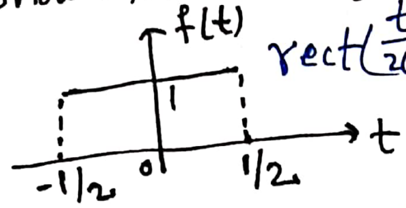
Time Shift: Let $f(t) \longleftrightarrow F(jw)$

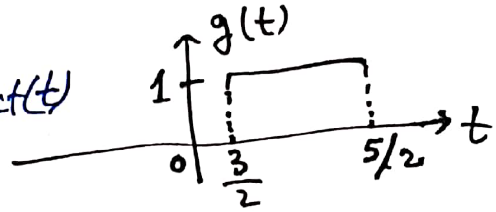
i.e, $\mathcal{F}[f(t)] = F(jw)$.

then $\mathcal{F}[f(t-a)] = e^{-jwa} F(jw)$.

Example: Calculate Fourier transform of $f(t)$ & $g(t)$ as shown in the figure:

$rect\left(\frac{t}{2(1/2)}\right) = rect(t)$





We know that $A\,rect\left(\frac{t}{2T}\right) \longleftrightarrow 2AT\,Sinc(WT)$

For $f(t)$, $A = 1$, $T = \frac{1}{2}$, $F(W) = (2)(1)(\frac{1}{2})\,Sinc(W.\frac{1}{2})$

$$\mathcal{F}[f(t)] = \mathcal{F}[rect(t)] = Sinc(W/2).$$

We observe that $g(t) = f(t-2)$.

Using time shifting property, we have

$$\mathcal{F}[g(t)] = G(jW) = e^{-2jW} F(jW) = e^{-2jW} Sinc(W/2).$$

Frequency spectrum under time shift:

Since, $|e^{-jwa}| = |Coswa - jSinwa| = \sqrt{Cos^2 wa + Sin^2 wa} = 1$

We have $|e^{-jwa} F(jw)| = |F(jw)|$

indicating that the amplitude spectrum of $f(t-a)$ is identical with that of $f(t)$. However,

$$\arg[e^{-jwa} F(jw)] = \arg F(jw) - \arg e^{jwa}$$
$$= \arg F(jw) - wa$$

indicating that each frequency is shifted by an amount proportional to its frequency w.

Frequency - shift property:

Let $f(t) \longleftrightarrow F(jw)$ and

$g(t) = e^{jw_0t} f(t)$ then

$$F[g(t)] = G(jw) = F(j(w-w_0)).$$

The property indicates that multiplication by e^{jw_0t} simply shifts the spectrum of $f(t)$ so that it is centred on the point $w = w_0$ in the frequency domain. This phenomena is the mathematical foundation for the process of modulation in communication theory.

The symmetry (Duality) property:

The symmetry/duality is very important property of the Fourier transform which provides a mechanism to explore new FT pairs.

If $\quad f(t) \longleftrightarrow F(w)$

then $\quad F(t) \longleftrightarrow 2\pi f(-w)$.

For example, $\quad rect(t) \longleftrightarrow \text{sinc}$

due to symmetry, $\quad \text{sinc} \longleftrightarrow rect(t)$.

observe if we directly try to calculate

$$F[\text{sinc}(at)] = \int_{-\infty}^{\infty} \text{sinc}(at)\, e^{-jwt}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{\sin at}{at} e^{-jwt}\, dt$$

(very difficult to evaluate this integral).

Fourier transform of Sinc function using symmetry

We know that $\text{rect}\left(\frac{t}{T}\right) \longleftrightarrow T\,\text{Sinc}(WT/2)$.

$$\text{rect}\left(\frac{t}{2T}\right) \longleftrightarrow 2T\,\text{Sinc}(WT)$$

$$A\,\text{rect}\left(\frac{t}{2T}\right) \longleftrightarrow 2AT\,\text{Sinc}(WT).$$

Using symmetry / Duality, we know that

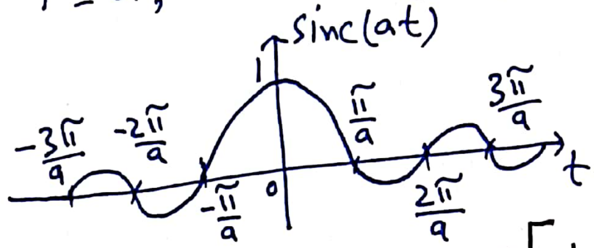
$$\text{rect}\left(\frac{t}{2T}\right) \longleftrightarrow 2T\,\text{Sinc}(WT)$$

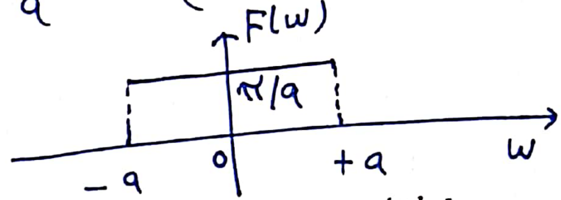
or, $$\frac{1}{2T}\text{rect}\left(\frac{t}{2T}\right) \longleftrightarrow \text{Sinc}(WT)$$

So, $$\text{Sinc}(tT) \longleftrightarrow 2\pi\left\{\frac{1}{2T}\text{rect}\left(-\frac{W}{2T}\right)\right] = \frac{\pi}{T}\text{rect}\left(\frac{W}{2T}\right)$$

rect(t) is an even function.

$T = a$, $$\text{Sinc}(at) \longleftrightarrow \frac{\pi}{a}\text{rect}\left(\frac{W}{2a}\right).$$



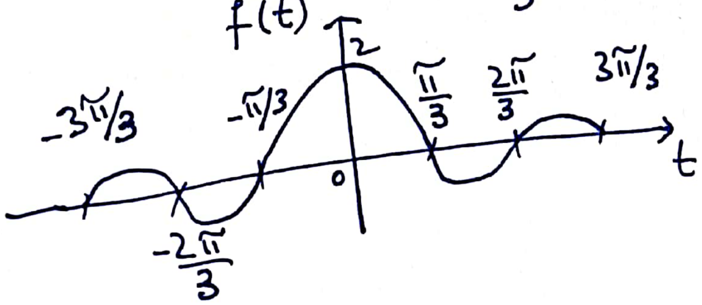


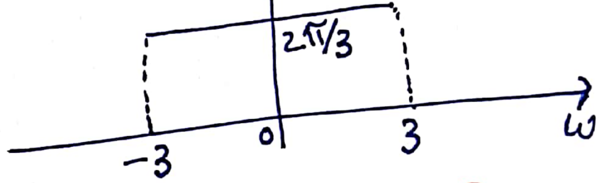
In general, $$F\left[K\,\text{sinc}(at)\right] = \frac{\pi K}{a}\text{rect}\left(\frac{W}{2a}\right).$$

EX: Calculate F.T. of $f(t) = 2\,\text{sinc}(3t)$ and sketch the spectrum.

Here, $K = 2$, $a = 3$, $$F\left[2\,\text{sinc}(3t)\right] = \frac{\pi(2)}{3}\text{rect}\left(\frac{W}{2(3)}\right)$$

$$F(jW) = \frac{2\pi}{3}\text{rect}\left(\frac{W}{6}\right).$$

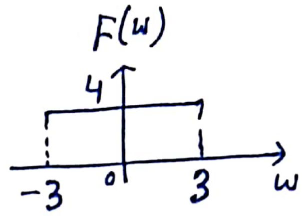




Ex:- (i). Let $f(t) = 4\,\frac{\sin 3t}{\pi t}$, calculate & sketch $F(j\omega)$.

$$f(t) = \frac{4}{\pi}(3)\,\frac{\sin 3t}{3t} = \frac{12}{\pi}\,\mathrm{sinc}(3t)$$

$$F(j\omega) = \frac{12}{\pi}\cdot\frac{\pi}{3}\,\mathrm{rect}\left(\frac{\omega}{2(3)}\right) = 4\,\mathrm{rect}\left(\frac{\omega}{6}\right)$$

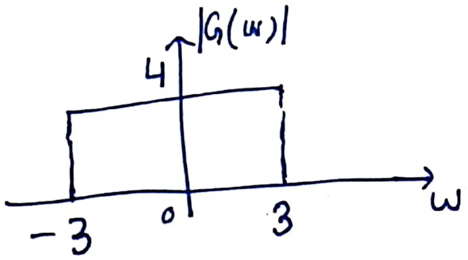


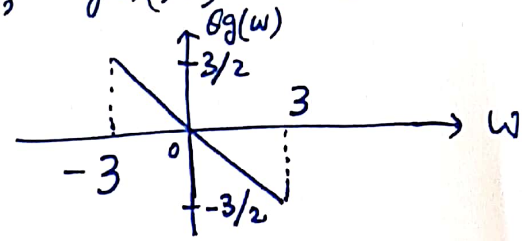
(ii). $g(t) = f(t-\frac{1}{2}) = \frac{4\sin 3(t-1/2)}{\pi(t-1/2)}$. Calculate $G(j\omega)$ & sketch the spectrum.

using time-shift property, we have

$$G(j\omega) = 4\,\mathrm{rect}\left(\frac{\omega}{6}\right)e^{-j\omega/2}.$$

Note that $|G(j\omega)| = 4\,\mathrm{rect}\left(\frac{\omega}{6}\right)$, $\arg G(j\omega) = -\frac{\omega}{2} = \theta_g(\omega)$





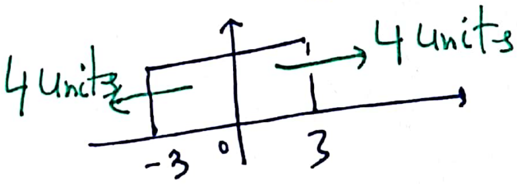
(iii). Let $h(t) = f(t)\cos 4t$

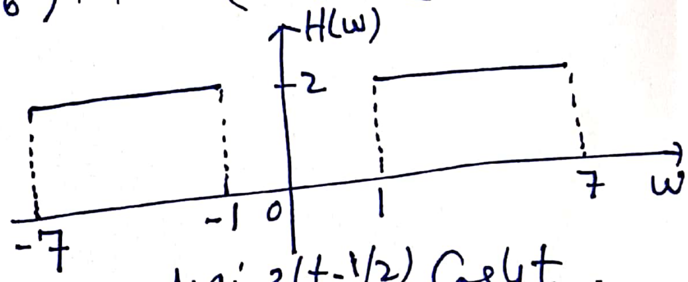
$$= f(t)\left[\frac{e^{j4t} + e^{-j4t}}{2}\right]$$

$$\mathcal{F}[h(t)] = \frac{1}{2}\left\{\mathcal{F}(f(t)e^{j4t}) + \mathcal{F}(f(t)e^{-j4t})\right]$$

$$= \frac{1}{2}\left\{F(j(\omega-4)) + F(j(\omega+4))\right].$$

$$= \frac{1}{2}\left[4\,\mathrm{rect}\left(\frac{\omega-4}{6}\right) + 4\,\mathrm{rect}\left(\frac{\omega+4}{6}\right)\right].$$





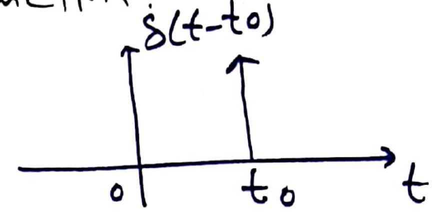
(iv). Try $f_1(t) = g(t)\cos 4t = \frac{4\sin 3(t-1/2)}{\pi(t-1/2)}\cos 4t$.

F.T.9

Transforms of the step & impulse function:

unit impluse / Dirac delta function:

$$\delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases}$$



total area: $$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

Sifting property: $$\int_{-\infty}^{+\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0).$$

F.T.10

Example: $$\int_{-\infty}^{\infty} t^2\,\delta(t-3)\,dt = (3)^2 = 9.$$

Fourier transform of Dirac delta function:

$$F[\delta(t)] = \int_{-\infty}^{+\infty} \delta(t)\,e^{-j\omega t}\,dt = e^{-j\omega(0)} = e^{0} = 1.$$

Thus, $\delta(t) \longleftrightarrow 1$. δ is an even function.

Using Symmetry, $1 \longleftrightarrow 2\pi\,\delta(-\omega) = 2\pi\,\delta(\omega)$

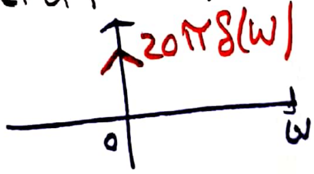
New pair: $F[1] = 2\pi\,\delta(\omega)$

In general, $F[k] = k\,F[1] = k(2\pi\,\delta(\omega))$, k is any constant

Before this, we were not able to find F.T. of such signals. As, $F[1] = \int_{-\infty}^{+\infty} (1)\,e^{-j\omega t}\,dt$, unable to solve this integral.

Ex:- $f(t) = 10, \; -\infty < t < \infty$, Calculate & sketch $F(j\omega)$.

$$F[10] = 10\,F[1] = 10(2\pi\,\delta(\omega)) = 20\pi\,\delta(\omega)$$



Also, $\mathcal{F}[\delta(t-a)] = \int_{-\infty}^{\infty} \delta(t-a)\, e^{-jwt}\, dt = e^{-jwa}$

Thus, $\delta(t-a) \longleftrightarrow e^{-jwa}$

$e^{-jat} \longleftrightarrow 2\pi\delta(-w-a) = 2\pi\delta(w+a)$.

New pair: $e^{-jat} \longleftrightarrow 2\pi\delta(w+a)$.

Also, $e^{jat} \longleftrightarrow 2\pi\delta(w-a)$.

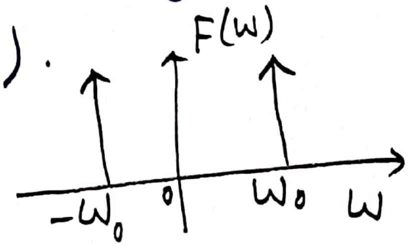
Our approach has therefore been successful, and we do indeed have a way of generating new pairs of transforms.

Ex:- Let $f(t) = \cos w_0 t$, $-\infty < t < \infty$. Calculate and sketch spectrum of $F(jw)$.

$$F(jw) = \mathcal{F}[\cos w_0 t] = \mathcal{F}\left[\frac{e^{jw_0 t} + e^{-jw_0 t}}{2}\right].$$

$$= \frac{1}{2}\left[2\pi\delta(w-w_0) + 2\pi\delta(w+w_0)\right].$$

$$= \pi\delta(w-w_0) + \pi\delta(w+w_0).$$



Likewise, we deduce the generalized F.T. pair.

$$\mathcal{F}[\sin w_0 t] = j\pi\left[\delta(w+w_0) - \delta(w-w_0)\right].$$