APPENDIX G

Mathematical Tables, Functions, and Transforms

Table G.1 Trigonometric Identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\cos A \cos B = (1/2)[\cos(A + B) + \cos(A - B)]$$

$$\sin A \sin B = (1/2)[\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = (1/2)[\sin(A + B) + \sin(A - B)]$$

$$\sin A + \sin B = 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A - B}{2}\right)\cos\left(\frac{A + B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \cos^2 A - \sin^2 A$$

$$\sin A/2 = \sqrt{(1 - \cos A)/2} \qquad \cos A/2 = \sqrt{(1 + \cos A)/2}$$

Table G.1 Trigonometric Identities (Continued)

$$\sin^2 A = (1 - \cos 2A)/2 \qquad \cos^2 A = (1 + \cos 2A)/2$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} \qquad \cos x = \frac{e^{jx} + e^{-jx}}{2} \qquad e^{jx} = \cos x + j\sin x$$

$$A\cos(\omega t + \phi_1) + B\cos(\omega t + \phi_2) = C\cos(\omega t + \phi_3)$$

where

$$C = \sqrt{A^2 + B^2 - 2AB\cos(\phi_2 - \phi_1)}$$

and

$$\phi_3 = \tan^{-1}\left[\frac{A\sin\phi_1 + B\sin\phi_2}{A\cos\phi_1 + B\cos\phi_2}\right]$$

$$\sin(\omega t + \phi) = \cos(\omega t + \phi - 90°)$$

$$\cos(\omega t + \phi) = \sin(\omega t + \phi + 90°)$$

Table G.2 Approximations

Taylor's series	$f(x) = f(a) + \dot{f}(a)\frac{(x-a)}{1!} + \ddot{f}(a)\frac{(x-a)^2}{2!} + \ ....$
Maclaurin's series	$f(0) = f(0) + \dot{f}(0)\frac{x}{1!} + \ddot{f}(0)\frac{x^2}{2!} + \ ....$
For small values of x (x<< 1)	$\frac{1}{1+x} \cong 1 - x$
	$(1+x)^n \cong 1 + nx \quad n \ge 1$
	$e^x \cong 1 + x$
	$\ln(1+x) \cong x$
	$\sin(x) \cong x$
	$\cos(x) \cong 1 - \frac{x^2}{2}$
	$\tan(x) \cong x$

Table G.3 Indefinite Integrals

$$\int \sin(ax)dx = -(1/a)\cos ax \qquad \int \cos(ax)dx = (1/a)\sin ax$$

$$\int \sin^2(ax)dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int x\sin(ax)dx = (1/a^2)(\sin ax - ax\cos ax)$$

$$\int x^2\sin(ax)dx = (1/a^3)(2ax\cos ax + 2\cos ax - a^2x^2\cos ax)$$

$$\int \sin(ax)\sin(bx) = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} \qquad (a^2 \neq b^2)$$

$$\int \sin(ax)\cos(bx) = -\left[\frac{\cos(a-b)x}{2(a-b)} + \frac{\cos(a+b)x}{2(a+b)}\right] \qquad (a^2 \neq b^2)$$

$$\int \cos(ax)\cos(bx) = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} \qquad (a^2 \neq b^2)$$

$$\int e^{ax}dx = \frac{e^{ax}}{a}$$

$$\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$\int x^2e^{ax}dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$

$$\int e^{ax}\sin(bx)dx = \frac{e^{ax}}{a^2+b^2}(a\sin(bx) - b\cos(bx))$$

$$\int e^{ax}\cos(bx)dx = \frac{e^{ax}}{a^2+b^2}(a\cos(bx) + b\sin(bx))$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int x\cos(ax)dx = (1/a^2)(\cos(ax) + ax\sin(ax))$$

$$\int x^2\cos(ax)dx = (1/a^3)(2ax\cos ax - 2\sin ax + a^2x^2\sin ax)$$

Table G.4 Definite Integrals

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\int_0^\infty e^{-r^2x^2} dx = \frac{\sqrt{\pi}}{2r}$$

$$\int_0^\infty x e^{-r^2x^2} dx = \frac{1}{2r^2}$$

$$\int_0^\infty x^2 e^{-r^2x^2} dx = \frac{\sqrt{\pi}}{4r^3}$$

$$\int_0^\infty x^n e^{-r^2x^2} dx = \frac{\Gamma[(n+1)/2]}{2r^{n+1}}$$

$$\Gamma(k) = (k-1)! \quad \text{for integers } k \geq 1$$

$$\int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2}, 0, -\frac{\pi}{2} \qquad \text{for } a > 0, a = 0, a < 0$$

$$\int_0^\infty \frac{\sin^2 x}{x} dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin^2 ax}{x^2} dx = |a|\frac{\pi}{2}$$

For m and n integers

$$\int_0^\pi \sin^2(mx) dx = \int_0^\pi \sin^2(x) dx = \int_0^\pi \cos^2(mx) dx = \int_0^\pi \cos^2(x) dx = \frac{\pi}{2}$$

$$\int_0^\pi \sin(mx)\cos(nx) dx = \begin{cases} \dfrac{(2m)}{(m^2 - n^2)} & \text{if } (m+n) \text{ odd} \\ 0 & \text{if } (m+n) \text{ even} \end{cases}$$

Table G.5 Functions

Rectangular	$rect\left(\frac{t}{T}\right) = \begin{cases} 1 & \lvert t\rvert \le T/2 \\ 0 & \lvert t\rvert > T/2 \end{cases}$
Triangular	$\Lambda\left(\frac{t}{T}\right) = \begin{cases} 1 - \frac{\lvert t\rvert}{T} & \lvert t\rvert \le T \\ 0 & \lvert t\rvert > T \end{cases}$
Sinc	$Sa(x) = \frac{\sin x}{x}$
Unit Step	$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$
Signum	$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$
Impulse	$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \ne 0 \end{cases}$
Bessel	$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin\theta - n\theta)}d\theta$
nth moment of a random variable X	$E[X^n] = \int_{-\infty}^{\infty} x^n p_X(x)dx$ where $n = 0,1,2,...$ and $p_X(x)$ is the *pdf* of X
nth central moment of X	$E[(X-\mu)^n] = \int_{-\infty}^{\infty} (x-\mu)^n p_X(x)dx$ where $\mu = E[X]$
Variance of X	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p_X(x)dx$ where $\mu = E[X]$

Table G.6 Probability Functions

Discrete distribution
Binomial
$$Pr(k) = \binom{n}{k} p^k q^{n-k} \qquad k = 0, 1, 2, \ldots\ldots\ldots n$$
$$= 0 \qquad \text{otherwise}$$
$0 < p < 1, \quad q = 1 - p$
$$p(x) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \delta(x-k)$$
$\bar{x} = np$
$\sigma_x^2 = npq$
Poisson
$$Pr(k) = \frac{\lambda^k e^{-\lambda}}{k!} \qquad k = 0, 1, 2, \ldots\ldots.$$
$$p(x) = \sum_{k=0}^{n} \frac{\lambda^k e^{-\lambda}}{k!} \delta(x-k)$$
$\bar{x} = \lambda$
$\sigma_x^2 = \lambda$
Continuous distribution
Exponential
$$p(x) = ae^{-ax} \qquad x > 0$$
$$= 0 \qquad \text{otherwise}$$
$\bar{x} = a^{-1}$
$\sigma_x^2 = a^{-2}$
Gaussian (normal)
$$p(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left[-\frac{(x-\bar{x})^2}{2\sigma_x^2}\right] \qquad -\infty \le x \le \infty$$
$E\{x\} = \bar{x}$
$E\{(x-\bar{x})^2\} = \sigma_x^2$

Table G.6 Probability Functions (Continued)

Bivariate Gaussian (normal)

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\bar{x}}{\sigma_x}\right)^2 + \left(\frac{y-\bar{y}}{\sigma_y}\right)^2 - \frac{2\rho}{\sigma_x\sigma_y}(x-\bar{x})(y-\bar{y})\right]\right\}$$

$$E\{x\} = \bar{x}$$

$$E\{y\} = \bar{y}$$

$$E\{(x-\bar{x})^2\} = \sigma_x^2$$

$$E\{(y-\bar{y})^2\} = \sigma_y^2$$

$\rho = \dfrac{E[xy] - \mu_x\mu_y}{\sigma_x\sigma_y}$ is the correlation coefficient

$$E\{(x-\bar{x})(y-\bar{y})\} = \sigma_x\sigma_y\rho$$

Rayleigh

The *pdf* of the envelope of Gaussian random noise having zero mean and variance σ_n^2

$$p(r) = \frac{r}{\sigma_n^2}\exp[-r^2/2\sigma_n^2] \qquad r \ge 0$$

$$E\{r\} = \bar{r} = \sigma_n\sqrt{\pi/2}$$

$$E\{(r-\bar{r})^2\} = \sigma_r^2 = \left(2 - \frac{\pi}{2}\right)\sigma_n^2$$

Ricean

The *pdf* of the envelope of a sinusoid with amplitude A plus zero mean Gaussian noise with variance σ^2

$$p(r) = \frac{r}{\sigma^2}\exp\left[-\frac{(r^2+A^2)}{2\sigma^2}\right] I_0\left(\frac{Ar}{\sigma^2}\right) \qquad \mathrm{r} \ge 0$$

For $A/\sigma \gg 1$, this is closely approximated by the following Gaussian PDF:

$$p(r) \cong \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(r-A)^2}{2\sigma^2}\right]$$

Table G.6 Probability Functions (Continued)

Uniform

$$p(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{elsewhere} \end{cases}$$

$$\bar{x} = \frac{a+b}{2}$$

$$\sigma_x^2 = \frac{(b-a)^2}{12}$$

Table G.7 Selected Fourier Transform Theorems

Operation	Function	Fourier Transform
Linearity	$a_1 w_1(t) + a_2 w_2(t)$	$a_1 W_1(f) + a_2 W_2(f)$
Time Delay	$w(t - T_d)$	$W(f)e^{-j\omega T_d}$
Scale Change	$w(at)$	$\frac{1}{\lvert a \rvert} W\left(\frac{f}{a}\right)$
Conjugation	$w^*(t)$	$W^*(-f)$
Duality	$W(t)$	$w(-f)$
Real Signal Frequency Translation [$w(t)$ is real]	$w(t)\cos(\omega_c t + \theta)$	$\frac{1}{2}[e^{j\theta}W(f-f_c) + e^{-j\theta}W(f+f_c)]$
Complex Signal Frequency Translation	$w(t)e^{j\omega_c t}$	$W(f-f_c)$
Bandpass Signal	$\mathrm{Re}\{g(t)e^{j\omega_c t}\}$	$\frac{1}{2}[G(f-f_c) + G^*(-f-f_c)]$
Differentiation	$\frac{d^n w(t)}{dt^n}$	$(j2\pi f)^n W(f)$
Integration	$\int_{-\infty}^{t} w(\lambda)d\lambda$	$(j2\pi f)^{-1}W(f) + \frac{1}{2}W(0)\delta(f)$
Convolution	$w_1(t) * w_2(t) = \int_{-\infty}^{\infty} w_1(\lambda) \cdot w_2(t-\lambda)d\lambda$	$W_1(f)W_2(f)$
Multiplication	$w_1(t)w_2(t)$	$W_1(f) * W_2(f) = \int_{-\infty}^{\infty} W_1(\lambda) \cdot W_2(f-\lambda)d\lambda$
Multiplication by t^n	$t^n w(t)$	$(-j2\pi)^{-1}\frac{d^n W(f)}{df^n}$

Table G.8 Selected Fourier Transform Pairs

Function	Time Waveform $w(t)$	Spectrum $W(f)$
Rectangular	$\mathrm{rect}\left(\frac{t}{T}\right)$	$T[\mathrm{Sa}(\pi f T)]$
Triangular	$\Lambda\left(\frac{t}{T}\right)$	$T[\mathrm{Sa}(\pi f T)]^2$
Unit Step	$u(t) \triangleq \begin{cases} +1, & t > 0 \\ -1, & t < 0 \end{cases}$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
Signum	$\mathrm{sgn}(t) \triangleq \begin{cases} +1, & t > 0 \\ -1, & t < 0 \end{cases}$	$\frac{1}{j\pi f}$
Constant	1	$\delta(f)$
Impulse at $t = t_0$	$\delta(t - t_0)$	$e^{-j2\pi f t_0}$
Sinc	$\mathrm{Sa}(2\pi W t)$	$\frac{1}{2W}\,\mathrm{rect}\left(\frac{f}{2W}\right)$
Phasor	$e^{j(\omega_0 t + \phi)}$	$e^{j\phi}\delta(f - f_0)$
Sinusoid	$\cos(\omega_c t + \phi)$	$\frac{1}{2}e^{j\phi}\delta(f - f_c) + \frac{1}{2}e^{-j\phi}\delta(f + f_c)$
Gaussian	$e^{-\pi(t/t_0)^2}$	$t_0 e^{-\pi(f t_0)^2}$
Exponential, One-sided	$\begin{cases} e^{-t/T}, & t \ge 0 \\ 0, & t < 0 \end{cases}$	$\frac{T}{1 + j2\pi f T}$
Exponential, Two-sided	$e^{-\lvert t\rvert/T}$	$\frac{2T}{1 + (2\pi f T)^2}$
Impulse Train	$\sum_{k=-\infty}^{k=\infty} \delta(t - kT)$	$f_0 \sum_{n=-\infty}^{n=\infty} \delta(f - n f_0)$, where $f_0 = 1/T$