NUST School of Electrical Engineering & Computer Science

MATH 232 - Complex Variables & transforms - Problem sheet - BEE9AB

Topic: Laurent Series, Classification of singularities, Poles & residues

Q-1 Discuss all the singularities of the following functions, including the type: pole-include order, essential, branch point etc. that each of these functions have in the finite z-plane. If the functions have a Laurent series around these singularities, write down the series.

$$\operatorname{sech}(z+2),\ \cot(1/z),\ \frac{z^{1/3}+1}{z+1},\ e^{\cot z},\ \operatorname{cosech} z,\ \frac{\log z}{z(z-2)},$$
$$\frac{z}{z^4+1},\ \cos\left(\frac{1}{z-1}\right),\ \frac{1}{z}\sinh(1/z).$$

Q-2. Let $f(z) = (z^2-1)^{1/2}$. Find the Laurent Series.

The branch points are $z = \pm 1$ and the branch cut connects them inside $|z|<1$. Thus, we need the expansion for a branch analytic outside the cut, i.e., Laurent expansion valid in $|z|>1$. $f(z) = z\left(1-\frac{1}{z^2}\right)^{1/2} = z\left(1-\frac{1}{2z^2}-\frac{1}{8z^4}+\cdots\right)$

Q-3 Classify the singularity of $f(z) = \dfrac{\sin z}{\cos(z^3)-1}$ at $z=0$ and calculate the residue. $\operatorname{Res}_{z=0} f(z) = \frac{-1}{60}$

Q-4 Find the Laurent Series of the function $f(z) = \dfrac{(z+4)}{z^2(z^2+3z+2)}$ in i) $0<|z|<1$, ii) $1<|z|<2$ iii) $|z|>2$ iv) $0<|z+1|<1$.

Q-5 Find the Laurent Series of $f(z) = \dfrac{1}{e^{z^2}-1}$ and show that the series converges in $0<|z|<\sqrt{2\pi}$.

$f(z) = \frac{1}{z^2} - \frac{1}{2} + \frac{z^2}{12} - \frac{z^6}{720} + \cdots$ $f(z)$ is analytic in $\{z: e^z-1\neq 0\} = \{z \neq 2n\pi i\}$. So it is analytic in $0<|z|<2\pi$. Therefore, $f(z^2)$ is analytic in $0<|z^2|<2\pi \Rightarrow 0<|z|<\sqrt{2\pi}$.

Q-6 Calculate the residue at each of singularity of $f(z) = \dfrac{1}{(\sin z)^2}$.

Q-7 Evaluate the integral $I = \frac{1}{2\pi i}\oint_c f(z)\,dz$, where c is the unit circle centered at the origin for following functions:

(a) $\dfrac{z+1}{z^3+a^3}$, $0<a<1$ (b) $\sin\left(\frac{1}{z}\right)$ (c) $\dfrac{\log(z+a)}{z+1/a}$, $a>1$, principal branch

(d) $z^2 e^{-1/z}$ (e) $\tan(2z)$.

(a) 0 (b) 1 (c) $\ln(a^2-1)/a$, (d) $-1/6$ (e) -1

Q-8 $f(z) = \dfrac{1}{e^{1/z}-1}$.

Q-8 Does $f(z) = \dfrac{1}{e^{1/z}-1}$ has an isolated singularity at $z=0$? Explain.

Q-31 (page 311) Consider $\sum_{k=1}^{\infty} \frac{(z-i)^k}{k\,2^k}$ about $(z-i)$, $u_n = \frac{1}{n\,2^n}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)\,2^{n+1}} \cdot n\,2^n\right| = \lim_{n\to\infty} \frac{1}{2}\left|\frac{n}{n+1}\right| = \frac{1}{2}.$$

The circle of convergence is $|z-i| = 2$. Thus, for any point z on the circle of convergence we have

$$\sum_{k=1}^{\infty} \left|\frac{(z-i)^k}{k\,2^k}\right| = \sum_{k=1}^{\infty} \frac{2^k}{k\,2^k} = \sum_{k=1}^{\infty} \frac{1}{k}.$$

This is the harmonic series that is divergent. Thus the power series is not absolutely convergent on the circle of convergence. Now consider the point $z=-2+i$, $|-2+i-i| = 2$, this point is on the circle of convergence. Furthermore, for $z=-2+i$, we have

$$\sum_{k=1}^{\infty} \frac{(z-i)^k}{k\,2^k} = \sum_{k=1}^{\infty} \frac{(-2)^k}{k\,2^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k\,2^k} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k}.$$

This is real alternating series which converges by AST. Therefore, $z=-2+i$ is a point on the circle of convergence at which the power series converges.

Ex:- Find the Laurent series expansion for $f(z) = \frac{1}{\sin z}$ near $z=0$. Find its radius of convergence.

$z=0$ is a simple zero of $g(z) = \sin z$, it is simple pole of $f(z)$.

Therefore we seek $\frac{1}{\sin z} = \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$

$$1 = \left(\frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

The coefficients $a_{-1}, a_0, a_1, \ldots$ are to be found by multiplying out the terms and equating the coefficients at z^n on both sides:

z^0: $1 = a_{-1}$

z^1: $0 = a_0$

z^2: $0 = -\frac{a_{-1}}{3!} + a_1 \Rightarrow a_1 = 1/3!$

z^3: $0 = -\frac{a_0}{3!} + a_2 \Rightarrow a_2 = 0$

z^4: $0 = \frac{a_{-1}}{5!} - \frac{a_1}{3!} + a_3$

$\Rightarrow a_3 = \left(\frac{1}{3!}\right)^2 - \frac{1}{5!} = \frac{7}{360}$

OR Long division:

$$\begin{array}{r|l}
 & 1/z + \frac{1}{6}z + \frac{7}{360}z^3 \\
\hline
z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots & 1 \\
 & 1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots \\
 & (-)\ (+)\ (-)\ (+) \\
\hline
 & \frac{z^2}{6} - \frac{z^4}{120} + \cdots \\
 & \frac{1}{6}z^2 - \frac{1}{36}z^4 + \cdots \\
 & (-)\ (+) \\
\hline
 & \frac{7}{360}z^4 + \cdots
\end{array}$$

Thus, $\frac{1}{\sin z} = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \cdots$

Since, $\sin(\pm\pi) = 0$, the series converges for $|z| < \pi$.

EX: Find the Laurent expansion of $f(z)=\dfrac{\sin z}{(1-\cos z)^2}$ and specify the order of the pole.

We know that $\sin z = z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dfrac{z^7}{7!}+\cdots\cdots$

$$1-\cos z = 1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)=\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\cdots\cdots$$

$$(1-\cos z)^2 = z^4\left(\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}+\cdots\right)^2$$

Therefore,
$$\frac{\sin z}{(1-\cos z)^2}=\frac{1}{z^3}\,\frac{1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots}{\left(\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}\cdots\right)^2}$$

2	576, 720
2	288, 360
2	144, 180
2	72, 90
9	36, 45
	4, 5

$$\frac{1}{576}+\frac{1}{720}$$
$$\frac{5+4}{2880}=\frac{9}{2880}$$

which confirms $f(z)$ has a pole of order 3 at $z=0$, for Laurent expansion above form needs simplification.

Now,
$$\frac{\frac{1}{z^3}-\frac{1}{6}\frac{1}{z}+\frac{1}{120}z+\cdots\cdots}{\frac14-\frac{1}{24}z^2+\frac{9}{2880}z^4+\cdots\cdots}=c_0\frac{1}{z^3}+c_1\frac{1}{z^2}+c_2\frac{1}{z}+c_3+c_4z+\cdots\cdots$$

$$\begin{aligned}\frac{1}{z^3}-\frac{1}{6}\frac{1}{z}+\frac{1}{120}z+\cdots = {} & c_0\frac{1}{4z^3}-\frac{c_0}{24}\frac{1}{z}+\frac{9}{2880}c_0z+\cdots\cdots\\ & +\frac{c_1}{4}\frac{1}{z^2}-\frac{c_1}{24}+\frac{9}{2880}c_2z^2+\cdots\cdots\\ & +\frac{c_2}{4}\frac{1}{z}-\frac{c_2}{24}z+\frac{9c_2}{2880}z^4+\cdots\cdots\\ & +\frac{c_3}{4}-\frac{c_3}{24}z^2+\frac{9c_3}{2880}z^4+\cdots\cdots\\ & +\frac{c_4}{4}z-\frac{c_4}{24}z^3+\frac{9c_4}{2880}z^5+\cdots\cdots\end{aligned}$$

Comparing coefficients and solving, we have $c_0=4$, $c_1=0$, $c_2=0$, $c_3=0$, $c_4=-\frac{1}{240}$.

Hence, Laurent Series is $f(z)=\dfrac{4}{z^3}-\dfrac{1}{240}z+\cdots\cdots\cdots$

EX: Expand $f(z)=e^z/\sin^2 z$ near its poles.

$f(z)$ has poles for $z\in[\ldots,-3\pi,-2\pi,-\pi,0,\pi,2\pi,\ldots]$
$z=n\pi,\ n\in\mathbb{Z}$.

Let $z-n\pi=w \Rightarrow z=w+n\pi$.

$$f=\frac{e^{w+n\pi}}{\sin^2(w+n\pi)}=e^{n\pi}\cdot\frac{e^w}{\sin^2 w}=\frac{e^{n\pi}\left(1+w+\frac{w^2}{2!}+\frac{w^3}{3!}+\frac{w^4}{4!}+\cdots\right)}{\left(w-\frac{w^3}{3!}+\frac{w^5}{5!}-\frac{w^7}{7!}+\cdots\right)^2}\longrightarrow\text{(i)}$$

$\sin(w+n\pi)$
$=\sin w\cos n\pi+\cos w\sin n\pi$
$=\pm\sin w$

Now,
$$\left(w-\frac{w^3}{6}+\frac{w^5}{120}-\frac{w^7}{7!}+\cdots\right)^2$$
$$=w^2+\frac{w^6}{36}+\frac{w^{10}}{120^2}-\frac{w^4}{3}+\frac{w^6}{60}-\frac{w^8}{360}+\cdots\cdots$$
$$=w^2-\frac{1}{3}w^4+\frac{2}{45}w^6+\cdots\cdots \quad\text{—(ii)}$$

$$\frac{1}{w^2}+\frac{1}{w}+5/6$$
$$w^2-\frac{w^4}{3}+\frac{2}{45}w^6+\cdots \;\Big)\; 1+w+\frac{w^2}{2}+\frac{w^3}{6}+\cdots$$
$$\underset{(-)}{1}\quad\underset{(+)}{-w^2/3}$$
$$w+\frac{5}{6}w^2+\frac{w^3}{6}+\cdots$$
$$\underset{(-)}{w}\quad\underset{(+)}{-w^3/3}+\cdots$$
$$\frac{5}{6}w^2+\frac{1}{2}w^3+\cdots$$
$$\frac{5}{6}w^2-\frac{5}{30}w^4$$

After long division, $f=e^{n\pi}\left(\dfrac{1}{w^2}+\dfrac{1}{w}+\dfrac{5}{6}+\cdots\right)$

$$f(z)=e^{n\pi}\left[\frac{1}{(z-n\pi)^2}+\frac{1}{z-n\pi}+\frac{5}{6}+\cdots\right]\quad\text{—(iii)}$$

$z=n\pi$ are poles of order 2.

OR: Using (ii) in (i), we have
$$\frac{e^{n\pi}\left(1+w+\frac{w^2}{2!}+\frac{w^3}{3!}+\frac{w^4}{4!}+\cdots\right)}{w^2-\frac{1}{3}w^4+\frac{2}{45}w^6+\cdots\cdots}=\frac{a_{-2}}{w^2}+\frac{a_{-1}}{w}+a_0+a_1w+a_2w^2+\cdots\cdots$$

EX: Find the Laurent expansion of $f(z) = \dfrac{\sin z}{(1-\cos z)^2}$ and specify the order of the pole.

We know that $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \cdots\cdots$

$$1-\cos z = 1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\cdots\right) = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\cdots$$

$$(1-\cos z)^2 = z^4\left(\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} + \cdots\cdots\right)^2$$

Therefore,
$$\frac{\sin z}{(1-\cos z)^2} = \frac{1}{z^3}\,\frac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}{\left(\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} \cdots\right)^2}$$

2	576,720
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$$\frac{1}{576} + \frac{1}{720}$$
$$\frac{5+4}{2880} = \frac{9}{2880}$$

which confirms $f(z)$ has a pole of order 3 at $z=0$, for Laurent expansion above form needs simplification.

Now,
$$\frac{\frac{1}{z^3} - \frac{1}{6}\frac{1}{z} + \frac{1}{120}z + \cdots\cdots}{\frac{1}{4} - \frac{1}{24}z^2 + \frac{9}{2880}z^4 + \cdots\cdots} = c_0\frac{1}{z^3} + c_1\frac{1}{z^2} + c_2\frac{1}{z} + c_3 + c_4 z + \cdots\cdots$$

$$\frac{1}{z^3} - \frac{1}{6}\frac{1}{z} + \frac{1}{120}z + \cdots = \begin{aligned}[t] & c_0\frac{1}{4z^3} - \frac{c_0}{24}\frac{1}{z} + \frac{9}{2880}c_0 z + \cdots\cdots \\ & + \frac{c_1}{4}\frac{1}{z^2} - \frac{c_1}{24} + \frac{9}{2880}c_1 z^2 + \cdots\cdots \\ & + \frac{c_2}{4}\frac{1}{z} - \frac{c_2}{24}z + \frac{9c_2}{2880}z^4 + \cdots\cdots \\ & + \frac{c_3}{4} - \frac{c_3}{24}z^2 + \frac{9c_3}{2880}z^4 + \cdots\cdots \\ & + \frac{c_4}{4}z - \frac{c_4}{24}z^3 + \frac{9c_4}{2880}z^5 + \cdots\cdots \end{aligned}$$

Comparing coefficients and solving, we have $c_0 = 4, c_1 = 0, c_2 = 0, c_3 = 0, c_4 = -\dfrac{1}{240}$.

Hence, Laurent Series is $f(z) = \dfrac{4}{z^3} - \dfrac{1}{240}z + \cdots\cdots$

EX: Expand $f(z) = e^z/\sin^2 z$ near its poles. $f(z)$ has poles for $z \in [\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, \ldots]$, $z = n\pi, n \in \mathbb{Z}$.

Let $z - n\pi = w \Rightarrow z = w + n\pi$.

$$f = \frac{e^{w+n\pi}}{\sin^2(w+n\pi)} = e^{n\pi}\cdot\frac{e^w}{\sin^2 w} = \frac{e^{n\pi}\left(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \frac{w^4}{4!} + \cdots\right)}{\left(w - \frac{w^3}{3!} + \frac{w^5}{5!} - \frac{w^7}{7!} + \cdots\right)^2} \longrightarrow \text{(i)}$$

$\sin(w+n\pi)$
$= \sin w \cos n\pi + \cos w \sin n\pi$
$= \pm \sin w$

Now, $\left(w - \frac{w^3}{6} + \frac{w^5}{120} - \frac{w^7}{7!} + \cdots\right)^2$

$$= w^2 + \frac{w^6}{36} + \frac{w^{10}}{120^2} - \frac{w^4}{3} + \frac{w^6}{60} - \frac{w^8}{360} + \cdots\cdots$$
$$= w^2 - \frac{1}{3}w^4 + \frac{2}{45}w^6 + \cdots\cdots \quad \text{—— (ii)}$$

$$\begin{array}{r|l} & \frac{1}{w^2} + \frac{1}{w} + 5/6 \\ \hline w^2 - \frac{w^4}{3} + \frac{2}{45}w^6 + \cdots & 1 + w + \frac{w^2}{2} + \frac{w^3}{6} + \cdots \\ & \underset{(-)}{1} \quad \underset{(+)}{-w^2/3} \\ \hline & w + \frac{5}{6}w^2 + \frac{w^3}{6} + \cdots \\ & \underset{(-)}{w} \quad \underset{(+)}{-w^3/3} + \cdots \\ \hline & \frac{5}{6}w^2 + \frac{1}{2}w^3 + \cdots \\ & \frac{5}{6}w^2 - \frac{5}{30}w^4 \\ \hline \end{array}$$

After long division, $f = e^{n\pi}\left(\dfrac{1}{w^2} + \dfrac{1}{w} + \dfrac{5}{6} + \cdots\right)$

$$f(z) = e^{n\pi}\left[\frac{1}{(z-n\pi)^2} + \frac{1}{z-n\pi} + \frac{5}{6} + \cdots\right] \quad \text{—— (iii)}$$

$z = n\pi$ are poles of order 2.

OR: Using (ii) in (i), we have

$$\frac{e^{n\pi}\left(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \frac{w^4}{4!} + \cdots\cdots\right)}{w^2 - \frac{1}{3}w^4 + \frac{2}{45}w^6 + \cdots\cdots} = \frac{a_{-2}}{w^2} + \frac{a_{-1}}{w} + a_0 + a_1 w + a_2 w^2 + \cdots\cdots$$

$$\Rightarrow e^{n\pi}\left(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \frac{w^4}{4!} + \cdots\right) = \left(\frac{a_{-2}}{w^2} + \frac{a_{-1}}{w} + a_0 + a_1 w + a_2 w^2 + \cdots\right)\left(w^2 - \frac{1}{3}w^4 + \cdots\right)$$

Comparing coefficients, we will reach at (iii).