LINEAR TIME-INVARIANT (LTI) SYSTEMS – DT CONVOLUTION

Linear Time-Invariant (LTI) Systems

- Systems that are linear and time-invriant
- Focus of most of this course
- LTI systems are of practical importance
- A basic fact: If we know the response of an LTI system to some inputs, we can find the response to many inputs
- LTI system can be characterized in terms of its response to a unit impulse (CT) or unit sample (DT)

System Properties – Linearity

A (CT) system is linear if it obeys the superposition property:

If $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$

then $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

$y[n] = x^2[n]$ Nonlinear, TI, Causal

$y(t) = x(2t)$ Linear, <u>not</u> TI, Noncausal

System Properties – Linearity

- Superposition is a combination of Additivity and Homogeneity

- Additivity: $x_1(t) + x_2(t) = y_1(t) + y_2(t)$

- Homogeneity: $ax(t) = ay(t)$
- Superposition

If $\quad x_k[n] \quad \longrightarrow \quad y_k[n]$

Then $\quad \sum_k a_k x_k[n] \quad \longrightarrow \quad \sum_k a_k y_k[n]$

- For linear systems, zero input $\rightarrow$ zero output

System Properties – Time Invariance

Informally, a system is **time-invariant** (TI) if its behavior does not depend on what time it is.

- Mathematically (in DT): A system $x[n] \rightarrow y[n]$ is TI if for <u>any</u> input $x[n]$ and <u>any</u> time shift n_0,

  If $\quad x[n] \rightarrow y[n]$

  then $\quad x[n-n_0] \rightarrow y[n-n_0]$.

- Similarly for a CT time-invariant system,

  If $\quad x(t) \rightarrow y(t)$

  then $\quad x(t-t_o) \rightarrow y(t-t_o)$.

LTI System Example

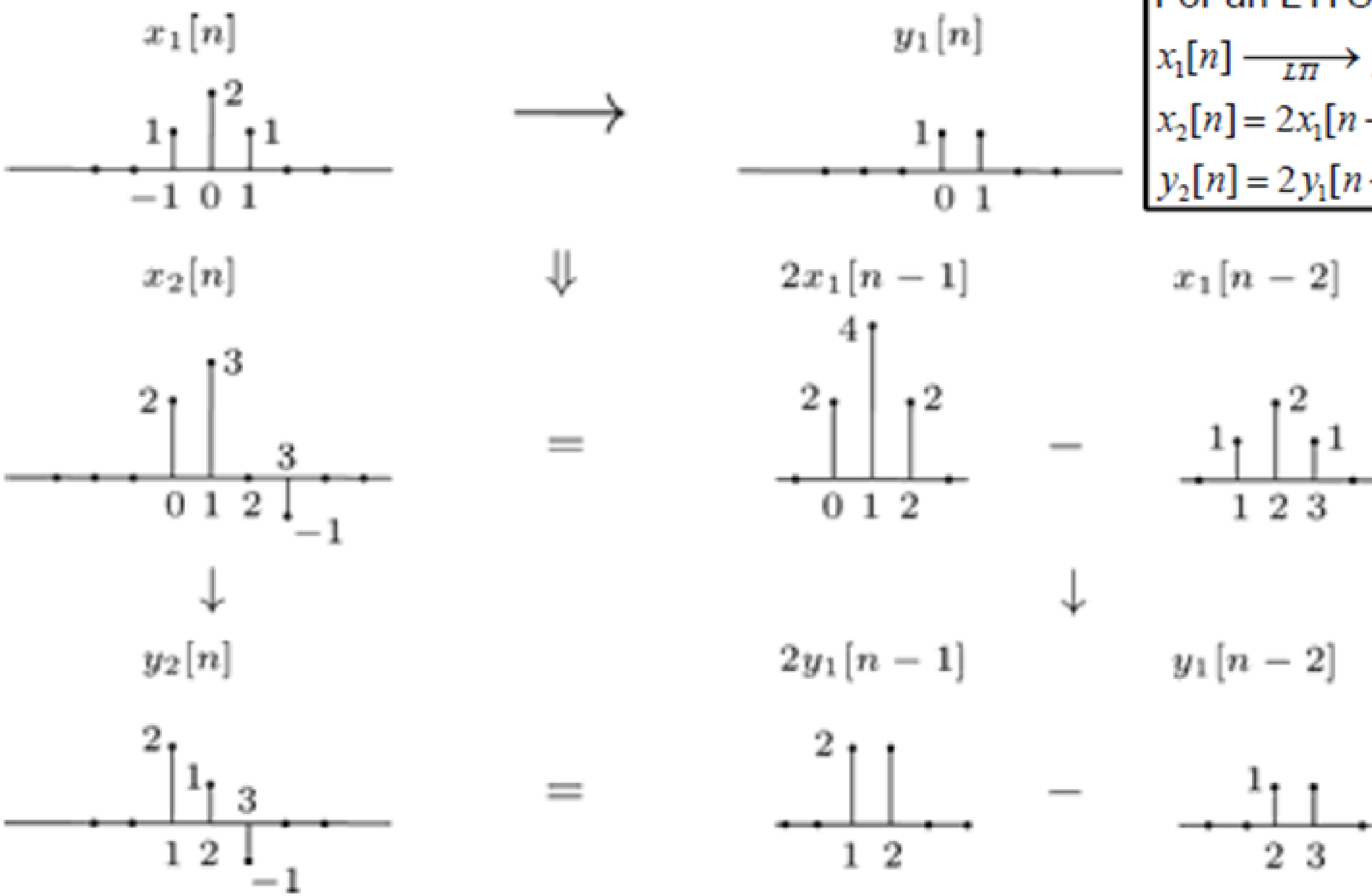
- The outputs for different inputs can be obtained as shown below:

For an LTI System:

$$x_1[n] \xrightarrow[LTI]{} y_1[n]$$

$$x_2[n] = 2x_1[n-1] - x_1[n-2]$$

$$y_2[n] = 2y_1[n-1] - y_1[n-2]$$



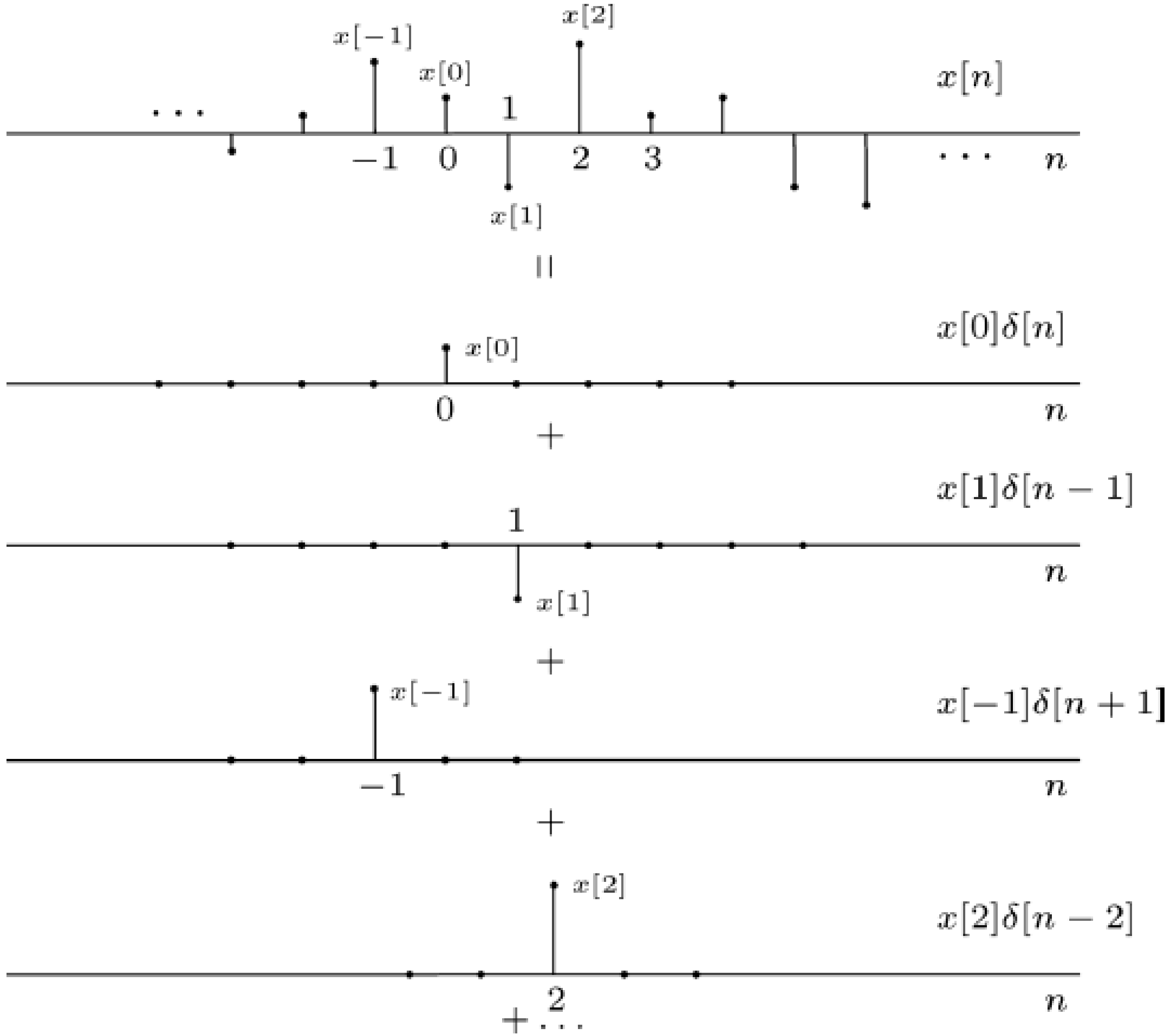
Representation of Signals

- Unit impulse (CT) or unit sample (DT) signals can be used as building blocks for LTI systems

DT	Shifted unit samples
CT	Shifted unit impulses

- For example, a DT LTI signal can be represented as the sum of scaled and shifted unit samples as shown in next slide

Representation of DT Signals



Representation of DT Signals

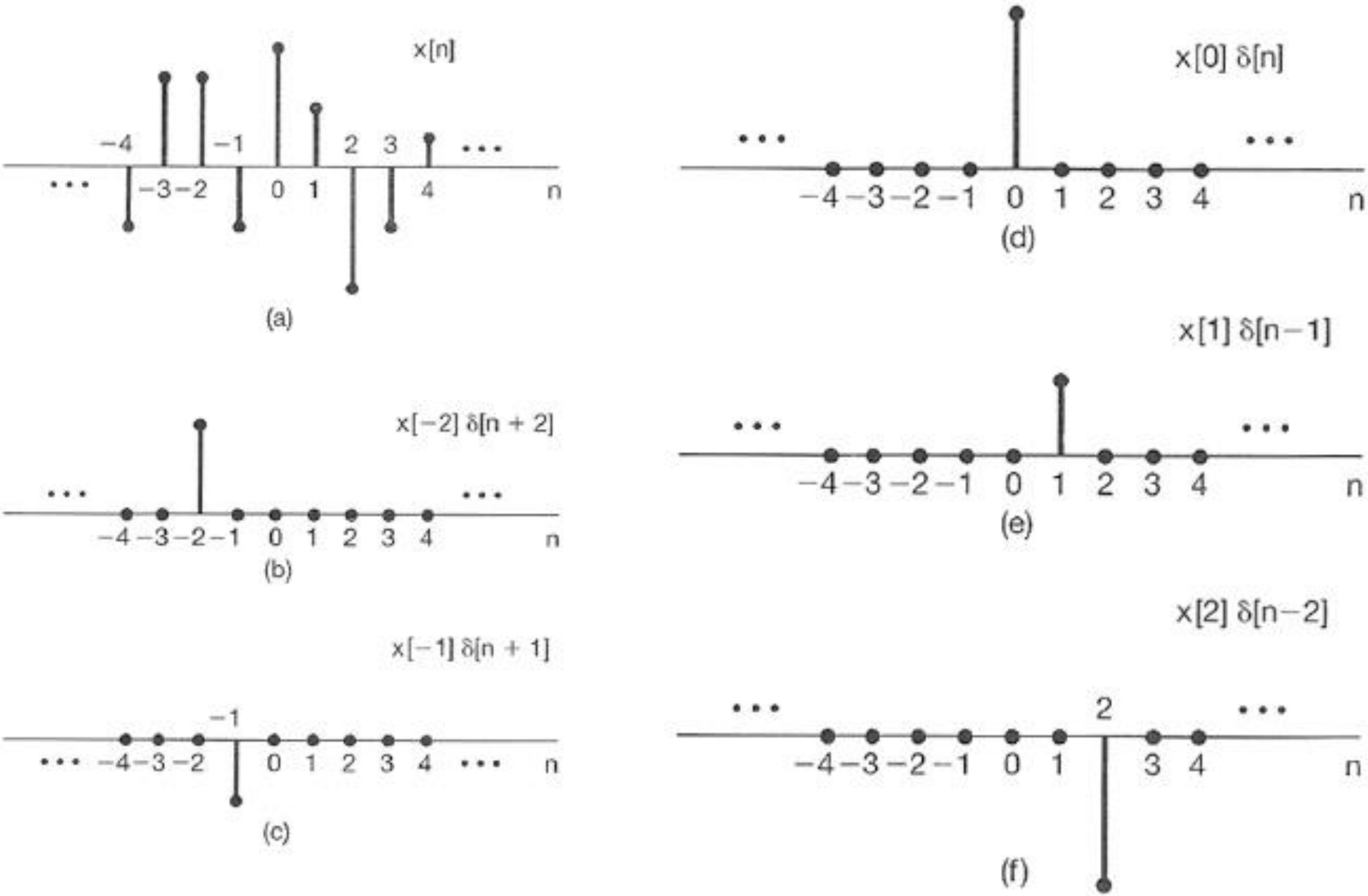
$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$

$$\Downarrow$$

$$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{Coefficients}} \underbrace{\delta[n-k]}_{\text{Basic Signals}}$$

The Sifting Property of the Unit Sample

Representation of DT Signals



Representation of DT Unit Step

- Consider the unit step, $x[n] = u[n]$ and its representation using shifted unit impulses:

$$u[k] = \begin{cases} 0, & k < 0 \\ 1, & k \geq 0 \end{cases}$$

thus we get the representation:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] -- \text{ Sifting Property of } \delta[n-k]$$

Summation "sifts" through the sequence of values, $x[k]$, and preserves only the value corresponding to $k = n$.

DT Unit Impulse Response

$$x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$

- Suppose the system is **linear**, and define $h_k[n]$ as the response to $\delta[n-k]$:

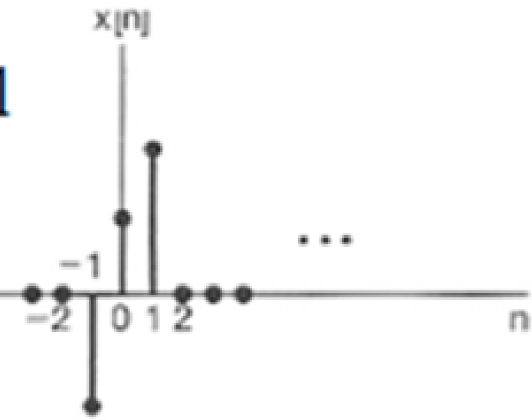
$$\delta[n-k] \rightarrow h_k[n]$$

From superposition: $\Downarrow$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

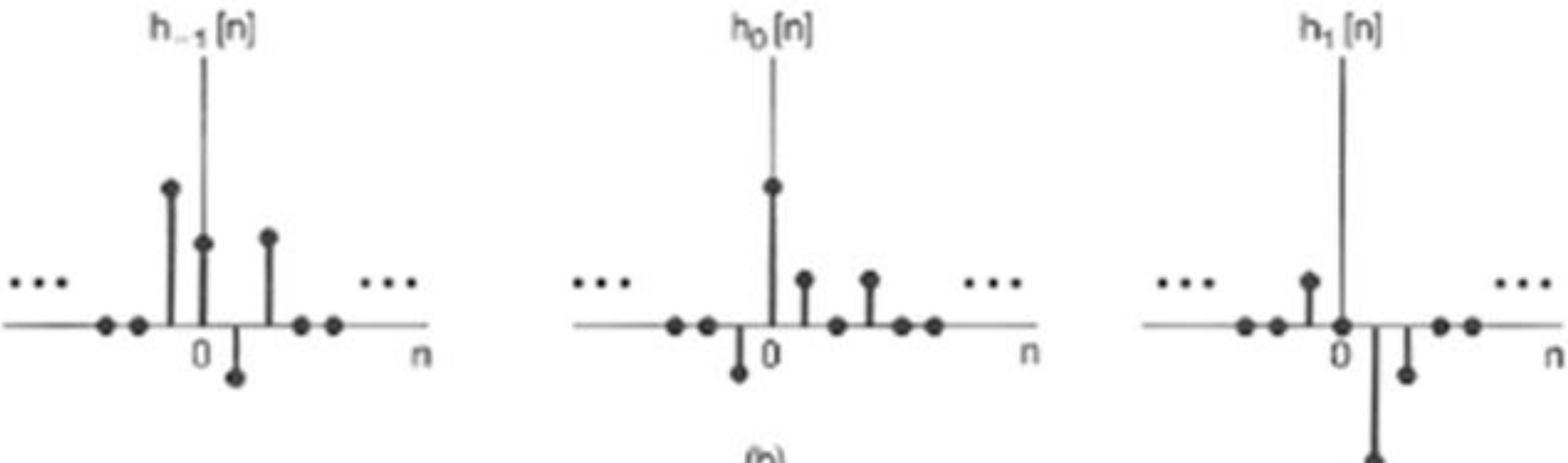
Superposition Sum

Input signal



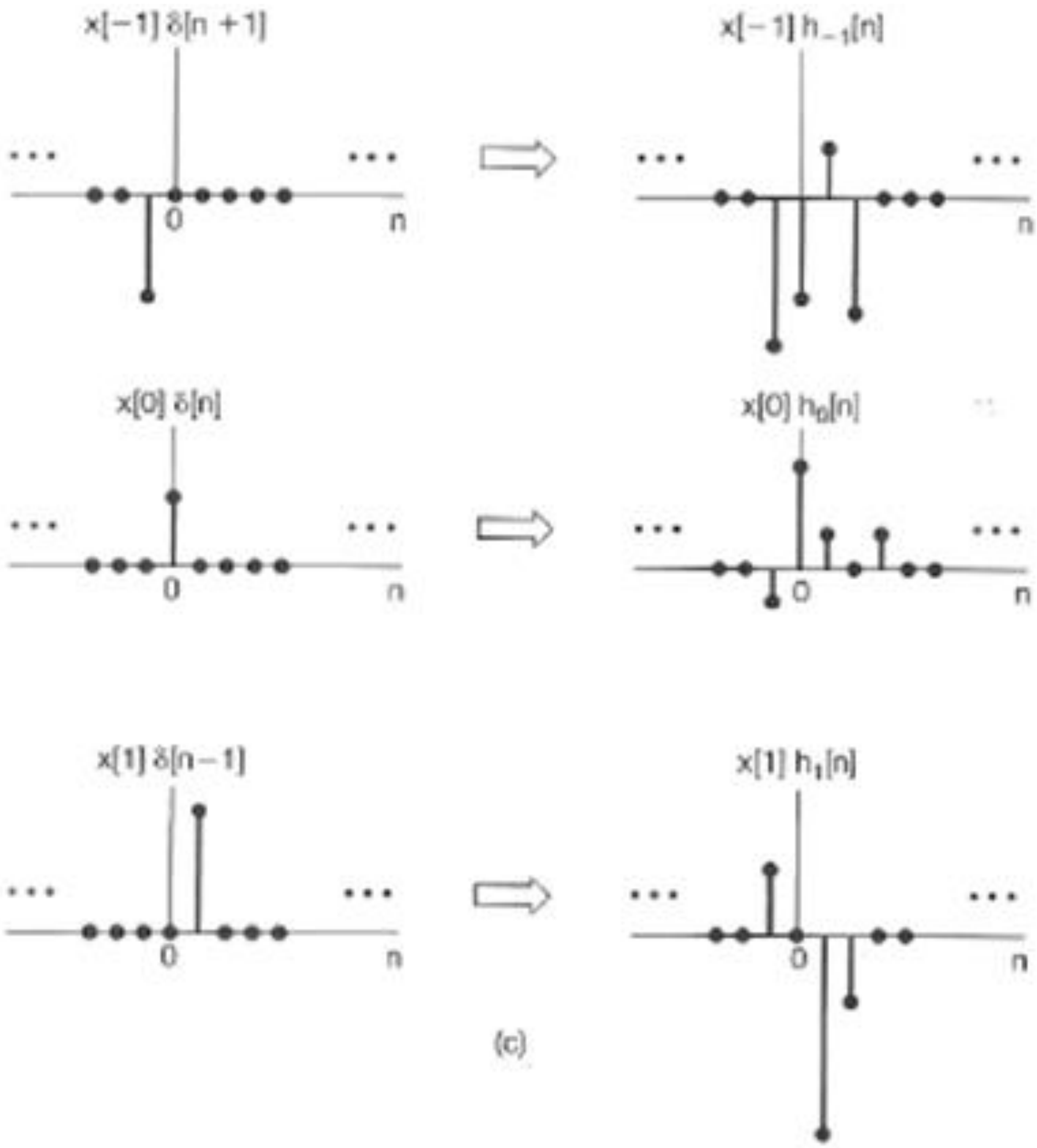
(a)

Impulse Responses of the system for $k = -1$, 0 and $+1$



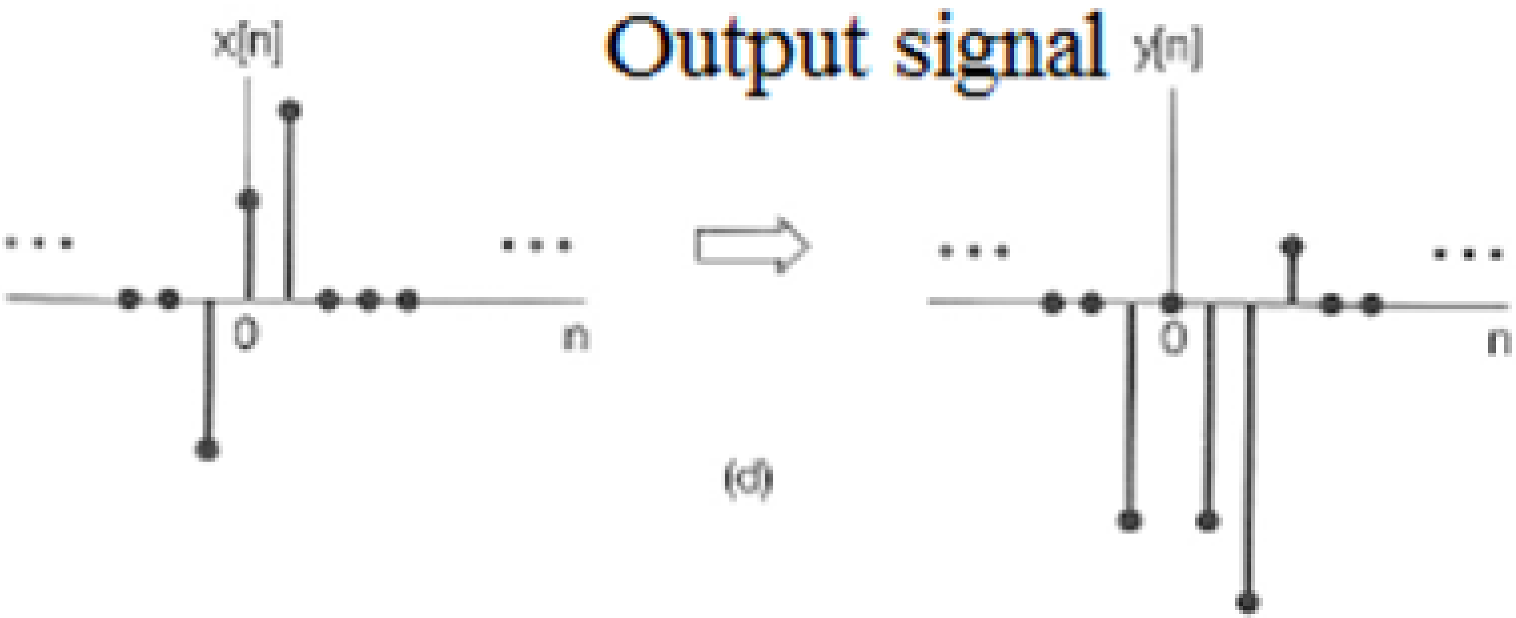
(b)

Superposition Sum



Superposition Sum

Since $x[n]$ is a sum of impulses, superposition says that the output is a superposition of responses to the sum of impulses



Convolution Sum

$$x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$

- Now suppose the system is **LTI**, and define the *unit sample response* $h[n]$*:*

$$\delta[n] \rightarrow h[n]$$

From TI:

$$\Downarrow$$

$$\delta[n-k] \rightarrow h[n-k]$$

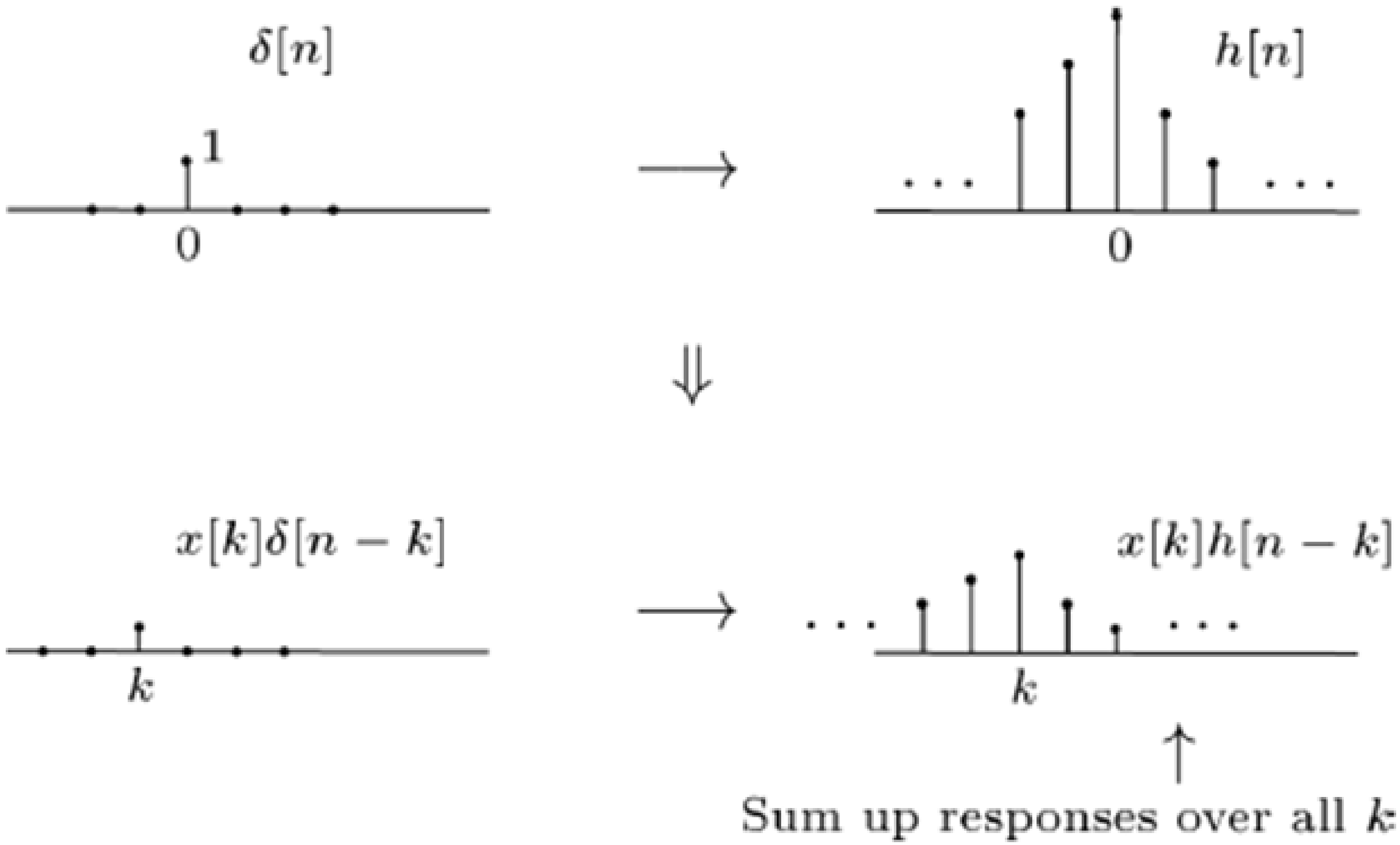
From LTI:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \underbrace{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}_{\text{Convolution Sum}}$$

Convolution Sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Interpretation

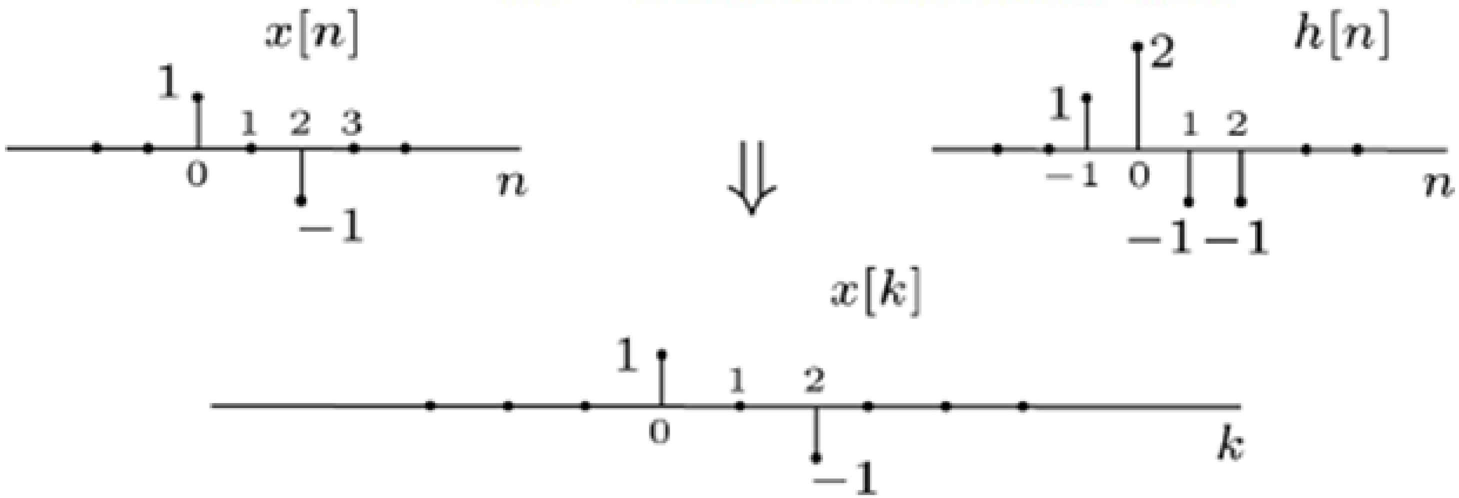


Convolution Sum

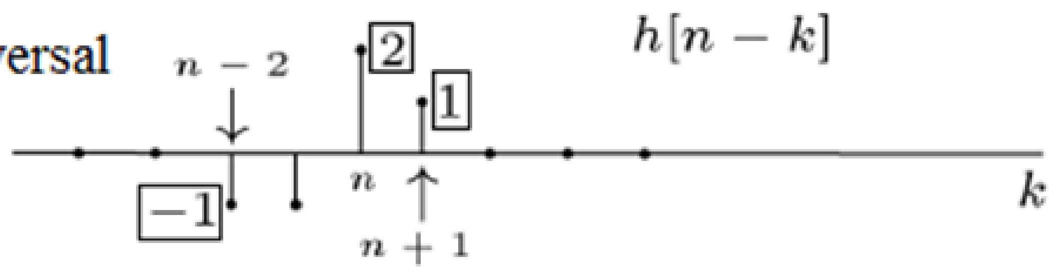
Visualizing the calculation of $y[n] = x[n] * h[n]$

$$y[n] = \sum_{k=\infty}^{\infty} x[k]h[n-k]$$

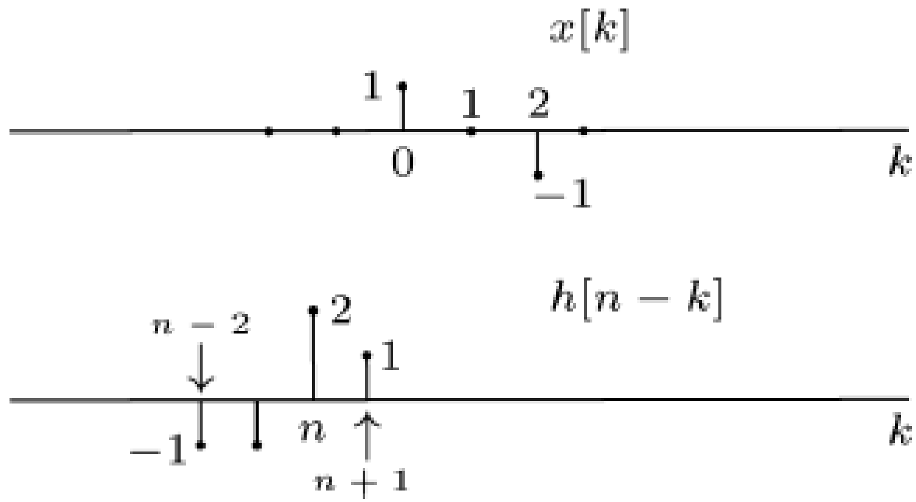
View as functions of k with n fixed



Time reversal



Convolution Sum



$$
\begin{aligned}
y[n] &= 0 \quad \text{for } n < -1 \\
y[-1] &= 1 \times 1 = 1 \\
y[0] &= 0 \times 1 + 1 \quad 2 = 2 \\
y[1] &= (-1) \times 1 + 0 \times 2 + 1 \times (-1) = -2 \\
y[2] &= (-1) \times 2 + 0 \times (-1) + 1 \times (-1) = -3 \\
y[3] &= (-1) \times (-1) + 0 \times (-1) = 1 \\
y[4] &= (-1) \times (-1) = 1 \\
y[n] &= 0 \quad \text{for } n > 4
\end{aligned}
$$

Consider only those values of n for which the values of k in both plots overlap

END