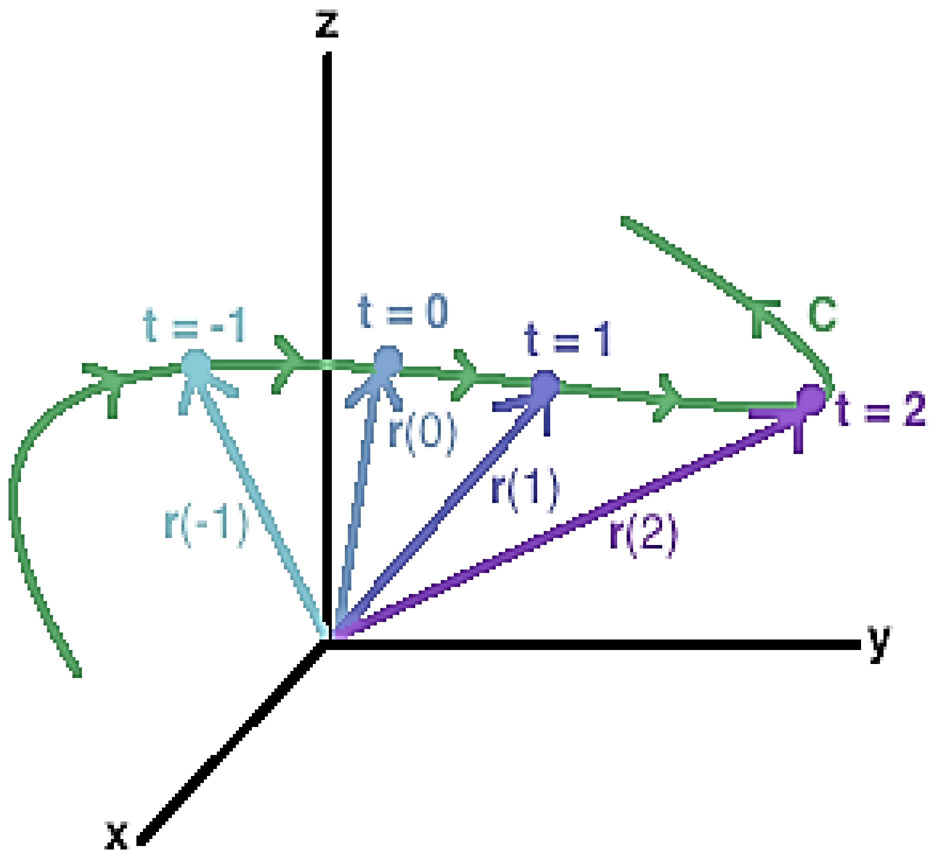
Vector Valued Functions & Space Curves



A curve C in three-dimensions represents by a vector-valued function r(t), where sample values t=-1, t=0, t=1, and t=2 are arbitrarily plotted.

Vector Calculus(MATH-243)
Instructor: Dr. Naila Amir

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Vectors And
The Geometry Of Space

Book: Thomas’ Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Chapter: 13 , Section: 13.1

Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.
Chapter: 13 , Section: 13.1

Vector Function

- In general, a function is a rule that assigns to each element in the domain an element in the range.

- A **vector-valued function**, or **vector function**, is simply a function whose:
  - Domain is a set of real numbers.
  - Range is a set of vectors.

- We are most interested in vector functions $\mathbf{r}$ whose values are three-dimensional (3-D) vectors.

- This means that, for every number t in the domain of $\mathbf{r}$, there is a unique vector in V_3 denoted by $\mathbf{r}(t)$.

Component Functions

If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f, g, and h which are real-valued functions, are called the component functions of $\mathbf{r}$. We can write:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \ = \ f(t)\mathbf{i} \ + \ g(t)\mathbf{j} \ + \ h(t)\mathbf{k}.$$

We usually use the letter t to denote the independent variable because it represents time in most applications of vector functions.

Example:

If

$$\mathbf{r}(t) = \langle \overset{f(t)}{t^3}, \overset{g(t)}{\ln(3 - t)}, \overset{h(t)}{\sqrt{t}} \rangle$$

then the component functions are:

$$f(t) = t^3, \qquad g(t) = \ln(3 - t), \qquad h(t) = \sqrt{t}.$$

By our usual convention, the domain of $\mathbf{r}$ consists of all values of t for which the expression for $\mathbf{r}(t)$ is defined. The expressions $t^3, \ln(3 - t)$, and $\sqrt{t}$ are all defined when $3 - t > 0$ and $t \geq 0$. Therefore, the domain of $\mathbf{r}$ is the interval $[0, 3)$.

Limit of a Vector Function

The limit of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions as follows:

Definition:

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

provided the limits of the component functions exist.

Note: Limits of vector functions obey the same rules as limits of real-valued functions.

Example:

Find $\lim_{t\to 0} \mathbf{r}(t)$, where

$$\mathbf{r}(t) = (\overset{f(t)}{1 + t^3})\mathbf{i} + \overset{g(t)}{te^{-t}}\mathbf{j} + \overset{h(t)}{\frac{\sin t}{t}}\mathbf{k}.$$

$\lim_{t\to 0}(1+t^3) = 1$

$\lim_{t\to 0}(te^{-t}) = 0$

$\lim_{t\to 0}\left(\frac{\sin t}{t}\right) = 1$

Solution:

We know that: $\lim_{t\to a} \mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle$. Thus,

$$\lim_{t\to 0} \mathbf{r}(t) = \left[\lim_{t\to 0}(1 + t^3)\right]\mathbf{i} + \left[\lim_{t\to 0}(te^{-t})\right]\mathbf{j} + \left[\lim_{t\to 0}\frac{\sin t}{t}\right]\mathbf{k}.$$

1   0   1

$$\Longrightarrow \lim_{t\to 0} \mathbf{r}(t) = \mathbf{i} + \mathbf{k}. = \langle 1, 0, 1\rangle$$

Continuity Criteria: A ftn $f: \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at a point "a", if:

(i) $f(a)$ is defined,

(ii) $\lim_{x\to a} f(x)$ exists,

(iii) $\lim_{x\to a} f(x) = f(a)$

$\vec{r}(t) = \langle 1+t^3, te^{-t}, \frac{\sin t}{t}\rangle$

$\vec{r}(0) = \langle f(0), g(0), h(0)\rangle$

$\rightarrow f(0)$ is defined

$\rightarrow g(0)$ is defined but

$\rightarrow h(0)$ is not defined

$\Rightarrow \vec{r}(0)$ is not defined

$\Rightarrow \vec{r}(t)$ is not continuous at $t=0$.

Continuous Vector Functions

A vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is **continuous** at a point a if:

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a).$$

In view of above definition, we see that $\mathbf{r}(t)$ is continuous at a if and only if its component functions $f(t)$, $g(t)$, and $h(t)$ are continuous at a, i.e.,

$$\lim_{t \to a} f(t) = f(a),$$

$$\lim_{t \to a} g(t) = g(a),$$

$$\lim_{t \to a} h(t) = h(a).$$

The function is **continuous** if it is continuous at every point in its domain.

Continuous Vector Functions & Space Curves

There is a close connection between continuous vector functions and space curves.

Suppose that f, g, and h are continuous real-valued functions on an interval I. Then, the set C of all points (x, y, z) in space, where:

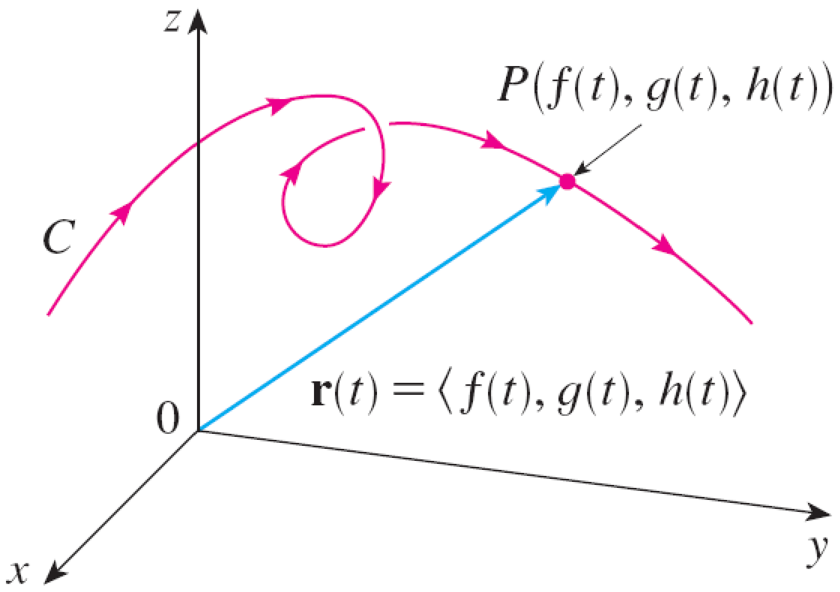
$$x = f(t); \quad y = g(t); \quad z = h(t) \qquad (*)$$

and t varies throughout the interval I is called a ***space curve***. Equations $(*)$ are called ***parametric equations*** of C, and "t" is called a ***parameter***. We can think of C as being traced out by a moving particle whose position at time t is:

$$(f(t), g(t), h(t)).$$

Space Curves

Let us consider the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C. Thus, any continuous vector function $\mathbf{r}$ defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$.



Example:

Describe the curve defined by the vector function:

$$\mathbf{r}(t) = \langle 1 + t, 2 + 5t, -1 + 6t \rangle.$$

Solution:

The corresponding parametric equations are:

$$x = 1 + t; \quad y = 2 + 5t; \quad z = -1 + 6t$$

We recognize these as parametric equations of a line passing through the point $(1, 2, -1)$ and parallel to the vector $\langle 1, 5, 6 \rangle$. Alternatively, we could observe that the function can be written as:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v},$$

where: $\mathbf{r}_0 = \langle 1, 2, -1 \rangle$ and $\mathbf{v} = \langle 1, 5, 6 \rangle$. This is the vector equation of a line.

Plane Curves

Plane curves can also be represented in vector notation. For instance, the curve given by the parametric equations:

$$x = t^2 - 2t \quad \text{and} \quad y = t + 1,$$

could also be described by the vector equation:

$$\mathbf{r}(t) = \langle t^2 - 2t, t + 1 \rangle = (t^2 - 2t)\mathbf{i} + (t + 1)\mathbf{j},$$

where $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

Example:

Sketch the curve whose vector equation is:

$$\mathbf{r}(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + t\, \mathbf{k}.$$

Solution:

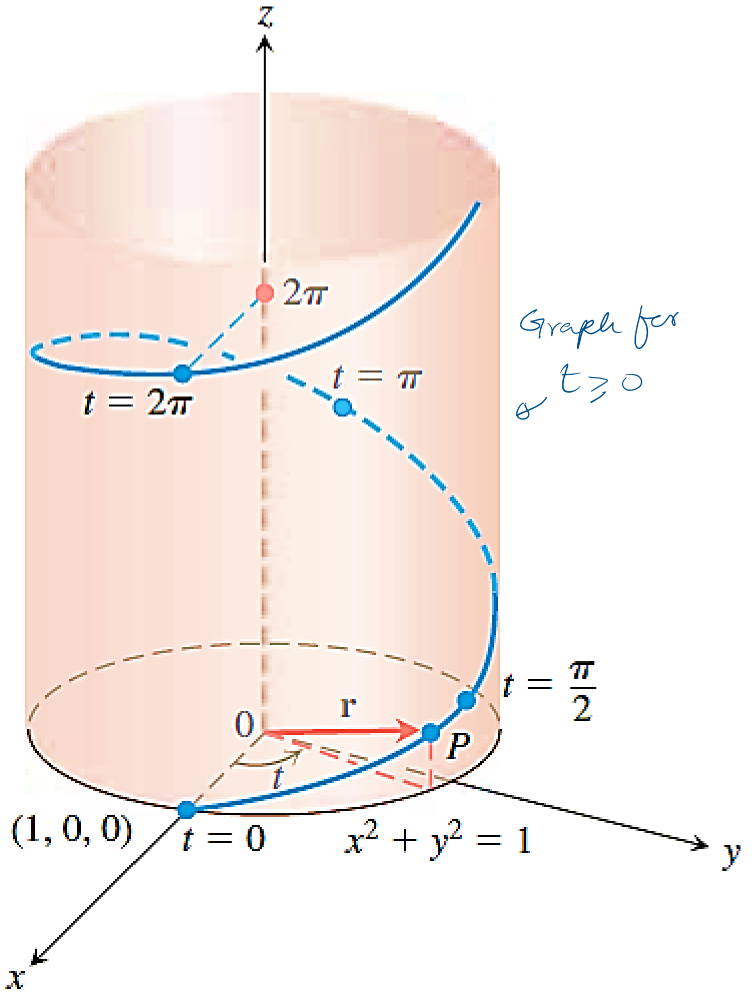
The parametric equations for this curve are:

$$x = \cos t\,; \qquad y = \sin t\,; \qquad z = t.$$

Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the curve traced by $\mathbf{r}$ must lie on the circular cylinder given by:

$$x^2 + y^2 = 1.$$

Since $z = t$, the curve spirals upward around the circular cylinder as t increases. Each time t increases by 2π, the curve completes one turn counterclockwise around the circular cylinder. The curve is called a **helix**. The domain is the largest set of points t for which all three equations are defined, i.e., $t \in (-\infty, \infty)$.



Graph for $t \geq 0$

Equation of a Line Segment

In general, we know that the vector equation of a line through the (tip of the) vector $\mathbf{r}_0$ in the direction of a vector $\mathbf{v}$ is:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

If the line also passes through (the tip of) $\mathbf{r}_1$, then we can take $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$ and the vector equation takes the form:

$$\mathbf{r} = \mathbf{r}_0 + \ t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - \ t)\mathbf{r}_0 + t\mathbf{r}_1.$$

The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the parameter interval $0 \leq t \leq 1$.

Example:

Find a vector equation and parametric equations for the line segment that joins the point $P(1,3,-2)$ to the point $Q(2,-1,3)$.

Solution:

We know that the vector equation for the line segment is given as:

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1; \qquad 0 \le t \le 1.$$

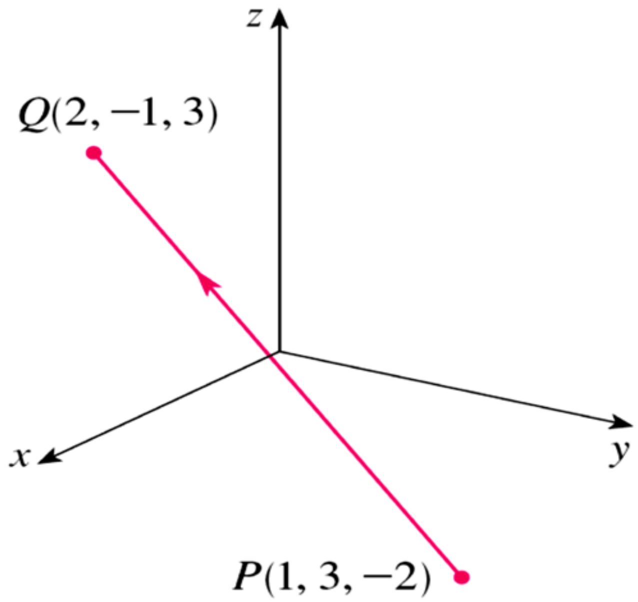
Using $\mathbf{r}_0 = \langle 1,3,-2\rangle$ and $\mathbf{r}_1 = \langle 2,-1,3\rangle$ in above equation we get:

$$\mathbf{r}(t) = (1-t)\langle 1,3,-2\rangle + t\langle 2,-1,3\rangle = \langle 1+t, 3-4t, -2+5t\rangle; \qquad 0 \le t \le 1.$$

The corresponding parametric equations are:

$$x = 1+t; \quad y = 3-4t; \quad z = -2+5t,$$

where $0 \le t \le 1$.

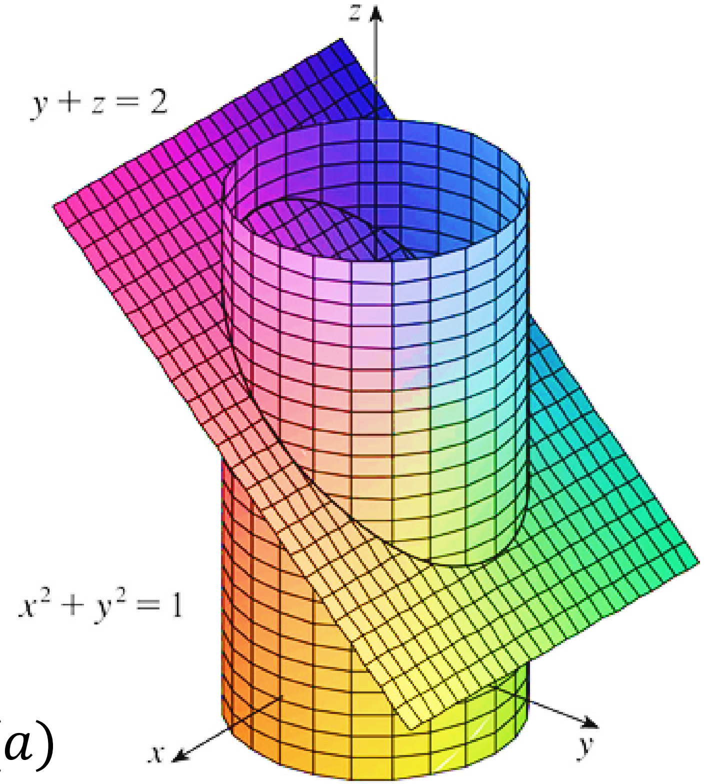


Example:

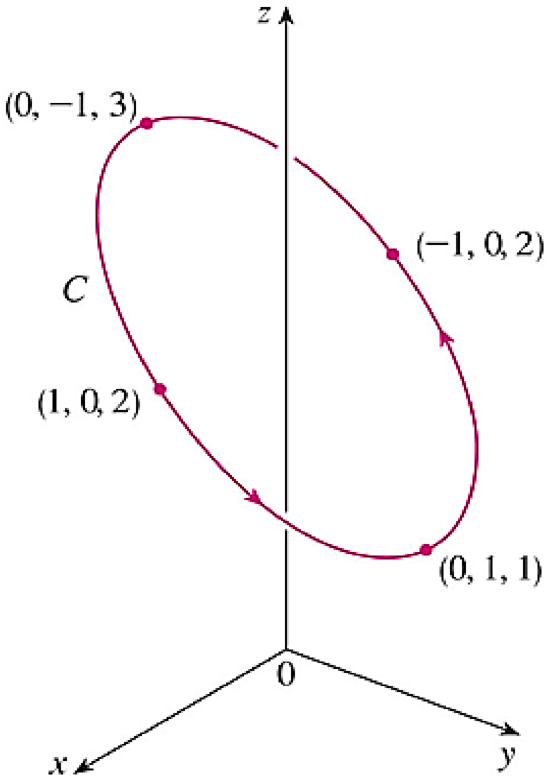
Find a vector function that represents the curve of intersection of the circular cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Solution:

Figure (a) shows how the plane and the cylinder intersect. Figure (b) shows the curve of intersection C, which is an ellipse.



(a)



(b)

Solution:

The projection of C onto the xy −plane is the circle:

$$x^2 + y^2 \; = \; 1; \quad z \; = \; 0.$$

So, we can write:

$$x = \cos t\,, \qquad y = \sin t\,,$$

where $0 \; \leq \; t \; \leq \; 2\pi$. From the equation of the plane, we have:

$$z \; = \; 2 - \; y \; = \; 2 - \sin t\,.$$

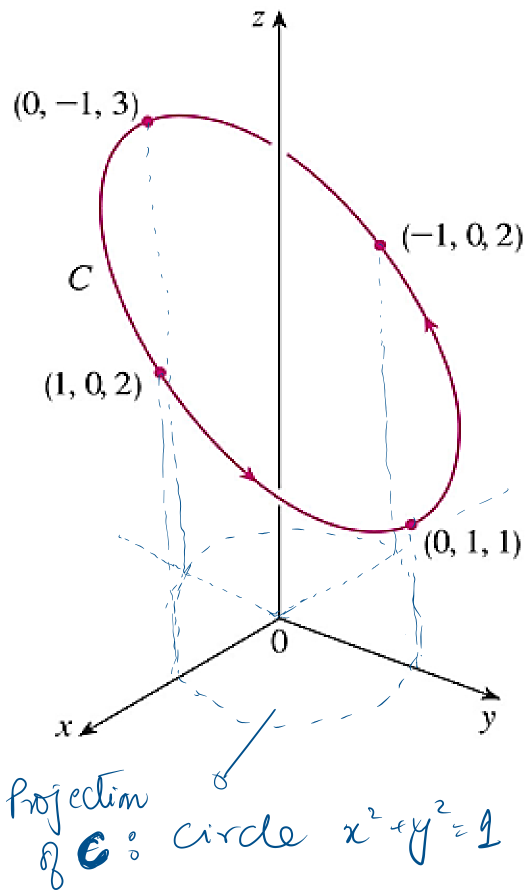
So, we can write parametric equations for C as:

$$x = \cos t\,; \quad y = \sin t\,; \quad z = 2 - \sin t\,,$$

where $0 \; \leq \; t \; \leq \; 2\pi$. The corresponding vector equation is:

$$\mathbf{r}(t) = (\cos t)\mathbf{i} \; + (\sin t)\mathbf{j} + (2 - \sin t)\mathbf{k},$$

where $0 \; \leq \; t \; \leq \; 2\pi$. This equation is called a parametrization of the curve C. The arrows indicate the direction in which C is traced as the parameter t increases.



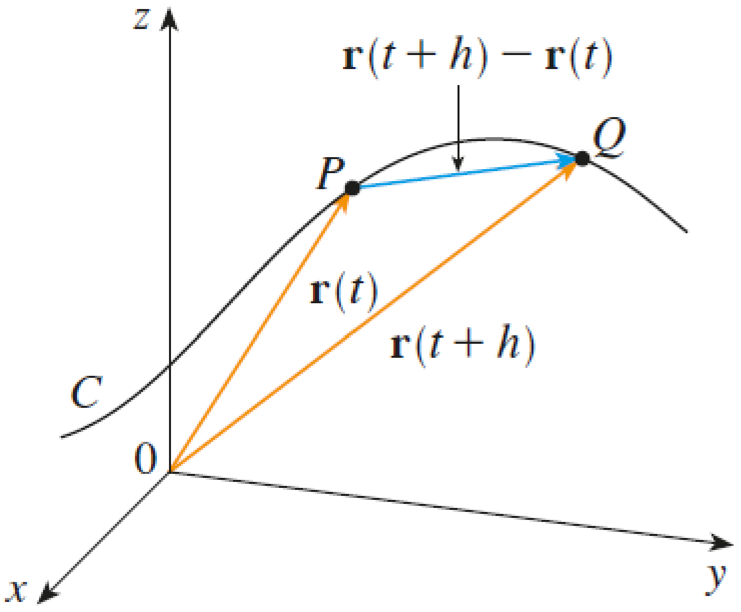
Practice Questions

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

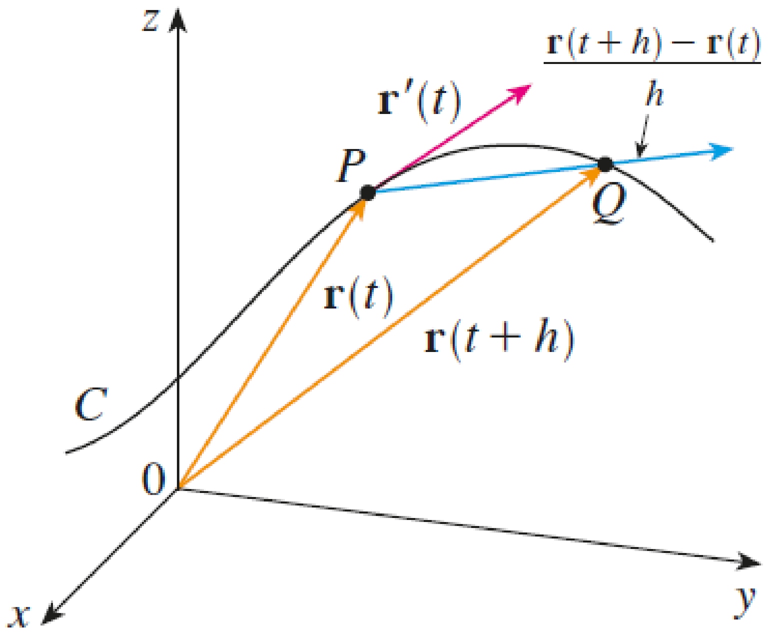
Chapter: 13

Exercise-13.1: Q – 1 to 28.

Derivatives of Vector Functions



(a) The secant vector



(b) The tangent vector

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Chapter: 13 , Section: 13.2

Derivatives

The derivative of a vector function $\mathbf{r}(t)$ is defined in much the same way as for real-valued functions.

Definition:

If $\mathbf{r}(t)$ is a vector function, then derivative $\mathbf{r}'(t)$ is given as:

$f: \mathbb{R} \to \mathbb{R}$

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

$f'(x) = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

provided this limit exists.

Derivative Geometric Significance

The geometric significance of this definition is shown as follows. If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t + h)$, then $\overrightarrow{PQ}$ represents the vector:

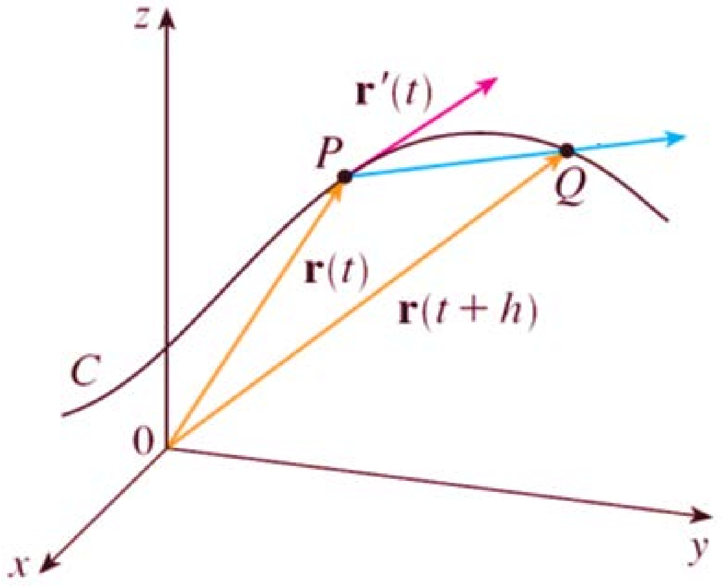
$$\mathbf{r}(t + h) - \mathbf{r}(t).$$

This can therefore be regarded as a secant vector. If $h > 0$, then the scalar multiple $(1/h)(\mathbf{r}(t + h) - \mathbf{r}(t))$ has the same direction as $\mathbf{r}(t + h) - \mathbf{r}(t)$. As $h \longrightarrow 0$, it appears that this vector approaches a vector that lies on the tangent line.

$\vec{r}(t) \longrightarrow$ position vector of the particle at time t.

$\vec{r}\,'(t) \longrightarrow$ velocity of the particle at t.

$$V(t) = \vec{r}\,'(t) = \lim_{h \to 0} \left[\frac{\vec{r}(t+h) - \vec{r}(t)}{h} \right]$$



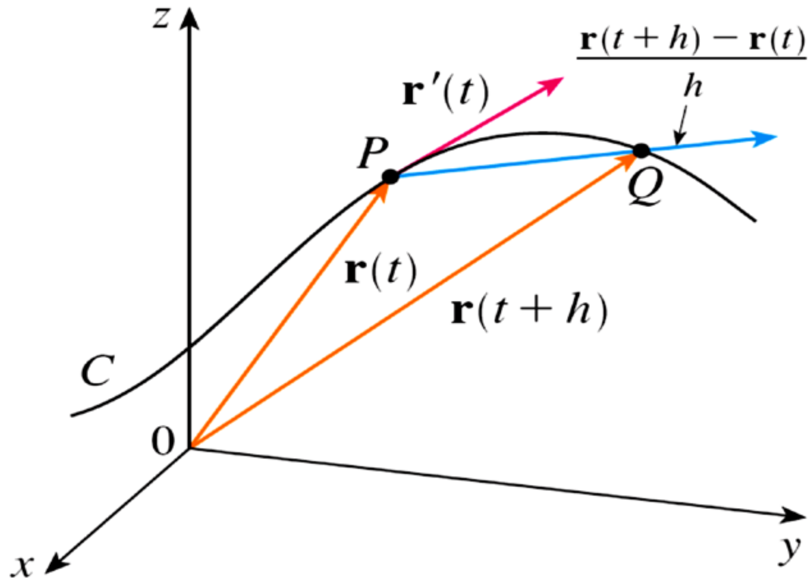
Derivative Geometric Significance

For this reason, the vector: $\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$ is called the **tangent vector** to the curve defined by $\mathbf{r}(t)$ at the point P, provided:

- **$\mathbf{r}'(t)$ exits**
- $\mathbf{r}'(t) \neq 0.$

The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The **unit tangent vector** is defined as:

$$T(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$



The tangent vector