Q1:- Determine the moments of inertia about the coordinate axes of a thin wire lying along the curve:

$$\vec{r}(t) = \langle t, \frac{2\sqrt{2}}{3} t^{3/2}, t^2/2 \rangle \; ; \; 0 \le t \le 2,$$

if the density function is $\rho(x,y,z) = \frac{1}{x+1}$.

<u>Solution</u>:- Given that:

$$\vec{r}(t) = \langle t, \frac{2\sqrt{2}}{3} t^{3/2}, t^2/2 \rangle \; ; \; 0 \le t \le 2.$$

Thus, the parametric equations are:

$$x = t \; , \; y = \frac{2\sqrt{2}}{3} t^{3/2} \; , \; z = \frac{t^2}{2}.$$

and

$$\vec{r}\,'(t) = \langle 1, \sqrt{2t}, t \rangle.$$

$$\Rightarrow |\vec{r}\,'(t)| = \sqrt{1 + 2t + t^2}$$

$$= \sqrt{(1+t)^2}$$

$$= 1+t.$$

Moreover,

$$\rho(x,y,z) = \frac{1}{x+1} \Rightarrow \rho(x,y,z) = \frac{1}{1+t} \quad [\because x = t]$$

We know that the moments of inertia about the coordinate axes are given as:

$$I_x = \int_C (y^2+z^2)\, \rho(x,y,z)\, ds,$$

$$I_y = \int_C (x^2+z^2)\, \rho(x,y,z)\, ds,$$

$$\& \quad I_z = \int_C (x^2+y^2)\, \rho(x,y,z)\, ds.$$

Now consider I_x first.

$$I_x = \int_C (y^2+z^2)\, \rho(x,y,z)\, ds$$

$$= \int_0^2 \left[\frac{8}{9}t^3 + \frac{t^4}{4}\right]\left(\frac{1}{1+t}\right)(1+t)\, dt \quad \left[\because \begin{array}{c} ds = |\vec{r}\,'(t)|\, dt \\ 0 \le t \le 2 \end{array}\right]$$

$$\Rightarrow I_x = \frac{2}{9}\left(\frac{t^4}{4}\right) + \frac{1}{4}\left(\frac{t^5}{5}\right)\Big|_0^2$$

$$= \frac{2}{9}\left[(2)^4 - 0\right] + \frac{1}{20}\left[(2)^5 - 0\right]$$

$$= \frac{32}{9} + \frac{32}{20} = 32\left[\frac{20+9}{120}\right] = \frac{232}{45}.$$

$$\Rightarrow \boxed{I_x = \frac{232}{45}}$$

Now $\quad I_y = \int_C (x^2 + z^2)\rho\, ds$

$$= \int_0^2 \left[t^2 + \frac{t^4}{4}\right]\left(\frac{1}{1+t}\right)(1+t)\, dt$$

$$= \frac{t^3}{3} + \frac{1}{4}\left(\frac{t^5}{5}\right)\Big|_0^2$$

$$= \frac{1}{3}\left[(2)^3 - 0\right] + \frac{1}{20}\left[(2)^5 - 0\right]$$

$$= \frac{8}{3} + \frac{32}{20} = \frac{64}{15}$$

$$\Rightarrow \boxed{I_y = \frac{64}{15}}$$

and $\quad I_z = \int_C (x^2 + y^2)\rho\, ds$

$$= \int_0^2 \left[t^2 + \frac{8}{9}t^3\right]\left[\frac{1}{1+t}\right](1+t)\, dt$$

$$= \frac{t^3}{3} + \frac{8}{9}\left(\frac{t^4}{4}\right)\Big|_0^2$$

$$= \frac{8}{3} + \frac{2}{9}(2^4 - 0)$$

$$= \frac{8}{3} + \frac{32}{9} = \frac{56}{9}$$

$$\Rightarrow \boxed{I_z = \frac{56}{9}}$$

Q2.- Calculate the surface area of the surface:

$$4x^2 + 4y^2 + z^2 - 6z + 5 = 0,$$

oriented inward.

Solution:- Consider the given surface:

$$4x^2 + 4y^2 + z^2 - 6z + 5 = 0$$

$$\Rightarrow 4x^2 + 4y^2 + z^2 - 2(z)(3) + 9 - 9 + 5 = 0$$

$$\Rightarrow 4x^2 + 4y^2 + (z-3)^2 - 4 = 0$$

$$\Rightarrow x^2 + y^2 + \frac{(z-3)^2}{4} - 1 = 0$$

$$\Rightarrow x^2 + y^2 + \frac{(z-3)^2}{4} = 1.$$

The given surface S is an ellipsoid centred at (0, 0, 3). In cylindrical coordinates, S consists of the points (r, θ, z) where, $0 \leq \theta \leq 2\pi$, $1 \leq z \leq 5$, and $r = \frac{1}{2}\sqrt{4-(z-3)^2}$.

working:

$$x^2 + y^2 + \frac{(z-3)^2}{4} = 1$$

$$\Rightarrow r^2 + \frac{(z-3)^2}{4} = 1 \quad \left[\because x = r\cos\theta,\ y = r\sin\theta,\ z = z \Rightarrow r^2 = x^2 + y^2\right]$$

$$\Rightarrow r^2 = 1 - \frac{(z-3)^2}{4}$$

$$\Rightarrow r = \pm\sqrt{\frac{4-(z-3)^2}{4}}$$

$$\Rightarrow \boxed{r = \frac{1}{2}\sqrt{4-(z-3)^2}}$$

[we ignored -ve root because r is radius which is never negative]

Also, at (0, 0, 3),

$$x = 0,\ y = 0,\ z = 3$$

$x = 0$ and $y = 0$ $\Rightarrow r = 0$

$$\Rightarrow \frac{1}{2}\sqrt{4-(z-3)^2} = 0$$

$$\Rightarrow 4-(z-3)^2 = 0$$

$$\Rightarrow (z-3)^2 = 4 \Rightarrow z-3 = \pm 2$$

$$\Rightarrow \boxed{z = 1, 5}$$

Therefore, we can parametrize the given surface S by using θ and z as variables and the vector valued function is given as:

$$\vec{r}(\theta, z) = \left\langle \frac{1}{2}\cos\theta\sqrt{4-(z-3)^2}, \frac{1}{2}\sin\theta\sqrt{4-(z-3)^2}, z \right\rangle.$$

$$\Rightarrow \vec{r}_\theta = \left\langle -\frac{1}{2}\sin\theta\sqrt{4-(z-3)^2}, \frac{1}{2}\cos\theta\sqrt{4-(z-3)^2}, 0 \right\rangle$$

and $$\vec{r}_z = \left\langle -\frac{1}{2}\frac{(z-3)\cos\theta}{\sqrt{4-(z-3)^2}}, -\frac{1}{2}\frac{(z-3)\sin\theta}{\sqrt{4-(z-3)^2}}, 1 \right\rangle$$

We want inward orientation, so we need a normal vector that is pointing downward at the upper tip of the ellipse. Thus, we consider

$$\vec{r}_z \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{1}{2}\frac{(z-3)\cos\theta}{\sqrt{4-(z-3)^2}} & -\frac{1}{2}\frac{(z-3)\sin\theta}{\sqrt{4-(z-3)^2}} & 1 \\ -\frac{1}{2}\sin\theta\sqrt{4-(z-3)^2} & \frac{1}{2}\cos\theta\sqrt{4-(z-3)^2} & 0 \end{vmatrix}$$

$$= \left\langle -\frac{1}{2}\cos\theta\sqrt{4-(z-3)^2}, -\frac{1}{2}\sin\theta\sqrt{4-(z-3)^2}, -\frac{1}{4}\cos^2\theta(z-3) - \frac{1}{4}\sin^2\theta(z-3) \right\rangle$$

$$= \left\langle -\frac{1}{2}\cos\theta\sqrt{4-(z-3)^2}, -\frac{1}{2}\sin\theta\sqrt{4-(z-3)^2}, -\frac{1}{4}(z-3) \right\rangle$$

$$\Rightarrow |\vec{r}_z \times \vec{r}_\theta| = \sqrt{\frac{1}{4}\cos^2\theta(4-(z-3)^2) + \frac{1}{4}\sin^2\theta(4-(z-3)^2) + \frac{1}{16}(z-3)^2}$$

$$= \frac{1}{2}\sqrt{4-(z-3)^2 + \frac{1}{4}(z-3)^2}$$

$$= \frac{1}{2}\sqrt{\frac{16-3(z-3)^2}{4}}$$

$$= \frac{1}{4}\sqrt{16-3(z-3)^2}.$$

We know that surface area of a surface can be calculated as:

$$\text{Surface area} = A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA.$$

$$\Rightarrow A(S) = \iint_D |\vec{r}_z \times \vec{r}_\theta| \, dA \quad [\text{where } \theta \text{ and } z \text{ are variables}]$$

$$= \int_1^5 \int_0^{2\pi} \frac{1}{4}\sqrt{16-3(z-3)^2} \, d\theta \, dz$$

$$= \frac{1}{4}\int_1^5 \sqrt{16-3(z-3)^2} \, |\theta|_0^{2\pi} \, dz$$

$$= \frac{2\pi}{4} \int_1^5 \sqrt{16-3(z-3)^2} \, dz \longrightarrow (1)$$

Let us solve $I = \int \sqrt{16-3(z-3)^2} \, dz$ separately.

Let $\quad a = \sqrt{3}(z-3)$

$\Rightarrow da = \sqrt{3} \, dz$

Thus,

$$I = \int \sqrt{16-3(z-3)^2} \, dz = \int \sqrt{16-a^2} \, \frac{da}{\sqrt{3}}$$

let $\quad a = 4\sin\theta \Rightarrow da = 4\cos\theta \, d\theta$

$$\Rightarrow I = \int \sqrt{16-16\sin^2\theta} \, \frac{(4\cos\theta)\,d\theta}{\sqrt{3}}$$

$$= \frac{16}{\sqrt{3}} \int (\cos^2\theta) \, d\theta = \frac{8}{\sqrt{3}} \int (1+\cos 2\theta) d\theta$$

$$= \frac{8}{\sqrt{3}} \left[\theta + \frac{\sin 2\theta}{2}\right]$$

$$= \frac{8}{\sqrt{3}} \left[\sin^{-1}\left(\frac{a}{4}\right) + \frac{1}{2}\frac{a\sqrt{16-a^2}}{8}\right] \left[\begin{array}{l} \because a = 4\sin\theta \\ \text{and } \sin 2\theta = 2\sin\theta\cos\theta \end{array}\right]$$

$$= \frac{8}{\sqrt{3}} \left[\sin^{-1}\left(\frac{\sqrt{3}(z-3)}{4}\right) + \frac{1}{16}[\sqrt{3}(z-3)]\sqrt{16-3(z-3)^2}\right] [\because a = \sqrt{3}(z-3)]$$

$$= \frac{1}{\sqrt{3}} \left[8\sin^{-1}\left(\frac{\sqrt{3}(z-3)}{4}\right) + \frac{\sqrt{3}}{2}(z-3)\sqrt{16-3(z-3)^2}\right] \rightarrow (2)$$

Using (2) in (1) we get:

$$A(S) = \frac{2\pi}{4\sqrt{3}} \left[8\sin^{-1}\left(\frac{\sqrt{3}(z-3)}{4}\right) + \frac{\sqrt{3}}{2}(z-3)\sqrt{16-3(z-3)^2}\right]$$

$$\Rightarrow \boxed{A(S) = 2\pi\left[1 + \frac{4\pi}{3\sqrt{3}}\right]}$$

Q3 :- Determine an equation for the plane tangent to the circular cylinder:

$$x^2 + (y-3)^2 = 9; \quad 0 \le z \le 5,$$

at the point $\left(\frac{3\sqrt{3}}{2}, \frac{9}{2}, 0\right)$.

Solution :- $x = r\cos\theta$

$y - 3 = r\sin\theta \Rightarrow y = 3 + r\sin\theta$

so that

$$x^2 + (y-3)^2 = r^2$$

For the present case $r = 3$, so

$$x = 3\cos\theta, \quad y = 3(\sin\theta + 1) \text{ and } z = z$$

provides us with the parametrization of the given surface. Thus,

$$\vec{r}(\theta, z) = \langle 3\cos\theta, 3(1+\sin\theta), z \rangle;$$

where $0 \le \theta \le 2\pi$ and $0 \le z \le 5$.

Now

$$\vec{r}_\theta = \langle -3\sin\theta, 3\cos\theta, 0 \rangle$$

and $\vec{r}_z = \langle 0, 0, 1 \rangle$

Thus,

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 3\cos\theta, 3\sin\theta, 0 \rangle$$

At the given point $\left(\frac{3\sqrt{3}}{2}, \frac{9}{2}, 0\right)$ the normal vector is given as:

$$\vec{r}_\theta \times \vec{r}_z \Big|_{\left(\frac{3\sqrt{3}}{2}, \frac{9}{2}, 0\right)} = \left\langle \frac{3\sqrt{3}}{2}, \frac{3}{2}, 0 \right\rangle$$

$\left[\because \text{At } \left(\frac{3\sqrt{3}}{2}, \frac{9}{2}, 0\right);\ \left.\begin{array}{l} x = 3\cos\theta \\ y = 3(1+\sin\theta) \end{array}\right\} \Rightarrow \theta = \frac{\pi}{6} \text{ and } z = 0 \right]$

Thus, the equation of tangent plane to the given surface at $\left(\frac{3\sqrt{3}}{2}, \frac{9}{2}, 0\right)$ is given as:

$$\frac{3\sqrt{3}}{2}\left(x - \frac{3\sqrt{3}}{2}\right) + \frac{3}{2}\left(y - \frac{9}{2}\right) + 0(z-0) = 0$$

$$\Rightarrow \frac{3}{2}\left[\sqrt{3}x - \frac{9}{2} + y - \frac{9}{2}\right] = 0$$

$$\Rightarrow \sqrt{3}x + y - 9 = 0$$

$$\Rightarrow \boxed{\sqrt{3}x + y = 9}$$

Q4:(1) For constants a, b, c and e consider the vector field:

$$\vec{F} = \langle ax + by + 5z,\ x + cz,\ 3y + ex \rangle.$$

(a) Suppose that the flux of $\vec{F}$ through any closed surface is 0. what does this tells us about the value of the constants a, b, c and e?

Solution:- If the flux of $\vec{F}$ through any closed surface is 0, then by the divergence theorem, the vector field must have zero divergence.

i.e.,

$$\text{div } \vec{F} = 0 \Rightarrow \vec{\nabla}\cdot\vec{F} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(ax + by + 5z) + \frac{\partial}{\partial y}(x + cz) + \frac{\partial}{\partial z}(3y + ex) = 0$$

$$\Rightarrow a + 0 + 0 = 0$$

$$\Rightarrow \boxed{a = 0}$$

From the given information we conclude that $a = 0$, however, we can not say anything about b, c or e.

(b) Suppose that the line integral of $\vec{F}$ around any closed curve is 0. What does this tells us about the values of the constants a, b, c and e?

Solution: If the line integral of $\vec{F}$ around any closed path is 0, this means that the vector field has curl equal to zero everywhere, i.e.,

$$\vec{\nabla} \times \vec{F} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ax+by+5z & x+cz & 3y+ex \end{vmatrix} = \vec{0}$$

$$\Rightarrow \langle 3-c, -e+5, 1-b \rangle = \langle 0,0,0 \rangle$$

$$\Rightarrow \left.\begin{aligned} 3-c&=0 \\ 5-e&=0 \\ 1-b&=0 \end{aligned}\right\} \quad \because \text{two vectors are equal if their corresponding elements are equal.}$$

$$\Rightarrow \boxed{c=3, \quad e=5} \text{ and } \boxed{b=1}$$

Thus, the given information provides us with the values of the constants b, c and e however, we can not extract any information about a from this information.

Q4:-

(a) Let S be the boundary surface of the solid given by $0 \le z \le \sqrt{4-y^2}$ and $0 \le x \le \pi/2$. Determine the unit outward normal vector on each of the four sides of S.

Solution: Four sides of the surface S are given by the equations:

$$z=0, \quad z=\sqrt{4-y^2}, \quad x=0 \text{ and } x=\frac{\pi}{2}.$$

→ On the surface $z=0$ (the bottom of S), the unit outward normal is given by: $\boxed{\hat{n}_1 = -\hat{k}}$.

→ On the side $x=0$ (one side of S), the unit outward normal is given by: $\boxed{\hat{n}_2 = -\hat{i}}$

→ On the side $x=\pi/2$ (other side of S), the unit outward normal is given by: $\boxed{\hat{n}_3 = \hat{i}}$

→ On the top surface $z=\sqrt{4-y^2}$ (the top of S), the outward unit vector can be determined by considering $z=f(x,y)=\sqrt{4-y^2}$, so that $\vec{r}(x,y) = \langle x, y, \sqrt{4-y^2}\rangle$ parametrize the top of S. For this case the outward unit normal is given as:

$$\hat{n}_4 = \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1+f_x^2+f_y^2}}$$

$$= \frac{\langle -y/\sqrt{4-y^2}, 1\rangle}{\sqrt{1+0+\frac{y^2}{4-y^2}}}$$

$$\Rightarrow \hat{n}_4 = \frac{\langle 0, y/\sqrt{4-y^2}, 1\rangle}{\sqrt{\frac{4-\cancel{y^2}+\cancel{y^2}}{4-y^2}}}$$

$$\Rightarrow \hat{n}_4 = \frac{\sqrt{4-y^2}}{2}\left\langle 0, \frac{y}{\sqrt{4-y^2}}, 1\right\rangle$$

$$\Rightarrow \boxed{\hat{n}_4 = \frac{1}{2}\langle 0, y, \sqrt{4-y^2}\rangle}$$

Q5:- Use the divergence theorem to calculate the outword flux of the field:

$$\vec{F}(x,y,z) = \langle z^2x , y^3/3 + \tan z , x^2z + y^2 \rangle,$$

through the surface S where S is the surface:

$$z = \sqrt{1-x^2-y^2} \; ; \; z>0,$$

oriented upward.

Solution: It is important to note that the surface S: $z=\sqrt{1-x^2-y^2}$; $z>0$ is not a closed surface and we can apply divergence theorem on closed surfaces only. So, in order to apply divergence theorem we introduce a disk $\{(x,y,0) : x^2+y^2 \le 1\}$ oriented downward so that the surface $S_2 = S \cup S_1$ is a closed surface bounding a region E, the semi ball $\{(x,y,z) : x^2+y^2+z^2 \le ; z \ge 0\}$, i.e., S_2 is boundary of the region E.

Thus, by divergence theorem we have:

$$\iint_{S_2} \vec{F}\cdot\vec{n}\, dS = \iiint_E \operatorname{div} \vec{F}\, dV.$$

Now

$$\operatorname{div} \vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(z^2x) + \frac{\partial}{\partial y}\left(\frac{y^3}{3} + \tan z\right) + \frac{\partial}{\partial z}(x^2z + y^2)$$

$$\Rightarrow \operatorname{div} \vec{F} = z^2 + y^2 + x^2 = \rho^2.$$

Hence,

$$\iint_{S_2} \vec{F}\cdot\vec{n}\, dS = \iiint_E \rho^2\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2 (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

$$\Rightarrow \iint_{S_2} \vec{F}\cdot\vec{n}\, dS = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^4 \sin\varphi \, d\rho\, d\varphi\, d\theta.$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{\rho^5}{5}\sin\varphi \Big|_0^1 \, d\varphi\, d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/2} \sin\varphi \, d\varphi\, d\theta$$

$$= \frac{1}{5}\int_0^{2\pi} \Big| -\cos\varphi \Big|_0^{\pi/2} d\theta$$

$$= \frac{1}{5}\int_0^{2\pi} [-0+1]\, d\theta$$

$$= \frac{1}{5}\,|\theta|_0^{2\pi} = \frac{2\pi}{5}.$$

$$\Rightarrow \iint_{S_2} \vec{F}\cdot\vec{n}\, dS = \frac{2\pi}{5} \longrightarrow (1)$$

Let us now calculate $\iint_{S_1} \vec{F}\cdot\vec{n}\, dS$, where S_1 is a disk $\{(x,y,0) : x^2+y^2 \le 1\}$. Note that for S_1 $\hat{n} = -\hat{k}$ is the outward unit normal vector and $z=0$. Thus, it follows that

$$\vec{F}\cdot\vec{n} = -y^2 = -r^2\sin^2\theta$$

Thus,

$$\iint_{S_1} \vec{F}\cdot\vec{n}\, dS = \int_0^{2\pi}\int_0^1 -(r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= -\int_0^{2\pi} \frac{r^4}{4}\sin^2\theta \Big|_0^1 \, d\theta$$

$$= \frac{-1}{4}\int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{-1}{8}\Big|\theta - \frac{\sin 2\theta}{2}\Big|_0^{2\pi} = -\frac{2\pi}{8} = -\frac{\pi}{4} \rightarrow (2)$$

Since $S_2 = S \cup S_1$, So

$$\iint_{S_2} \vec{F}\cdot\vec{n}\, ds = \iint_{S} \vec{F}\cdot\vec{n}\, ds + \iint_{S_1} \vec{F}\cdot\vec{n}\, ds$$

$$\Rightarrow \iint_{S} \vec{F}\cdot\vec{n}\, ds = \iint_{S_2} \vec{F}\cdot\vec{n}\, ds - \iint_{S_1} \vec{F}\cdot\vec{n}\, ds$$

$$\Rightarrow \iint_{S} \vec{F}\cdot\vec{n}\, ds = \frac{2\pi}{5} - \left(-\frac{\pi}{4}\right) \quad [\text{using } ① \;\&\; ②]$$

$$\Rightarrow \iint_{S} \vec{F}\cdot\vec{n}\, ds = \frac{2\pi}{5} + \frac{\pi}{4}$$

$$\Rightarrow \boxed{\iint_{S} \vec{F}\cdot\vec{n}\, ds = \frac{13\pi}{20}}.$$

Thus, the outward flux of the given field $\vec{F}$ through the surface S is $\frac{13\pi}{20}$.