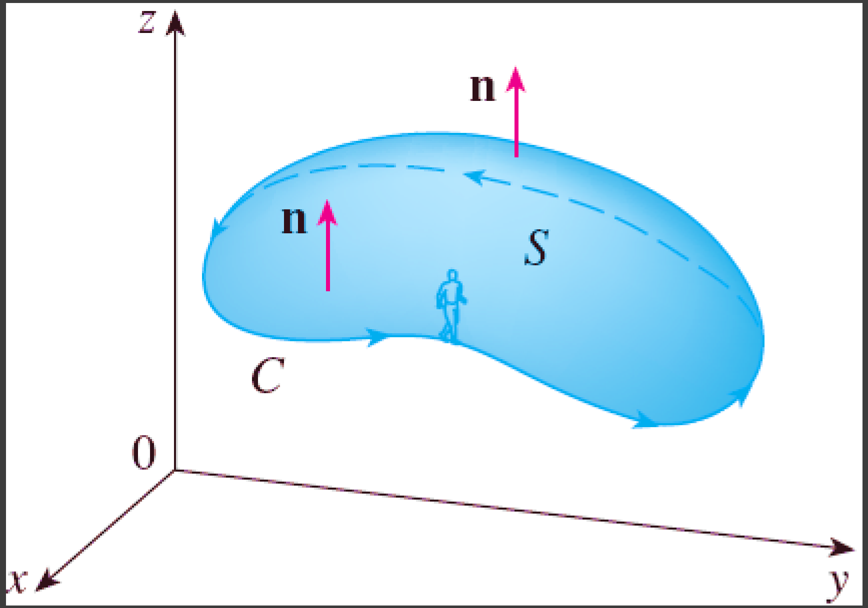
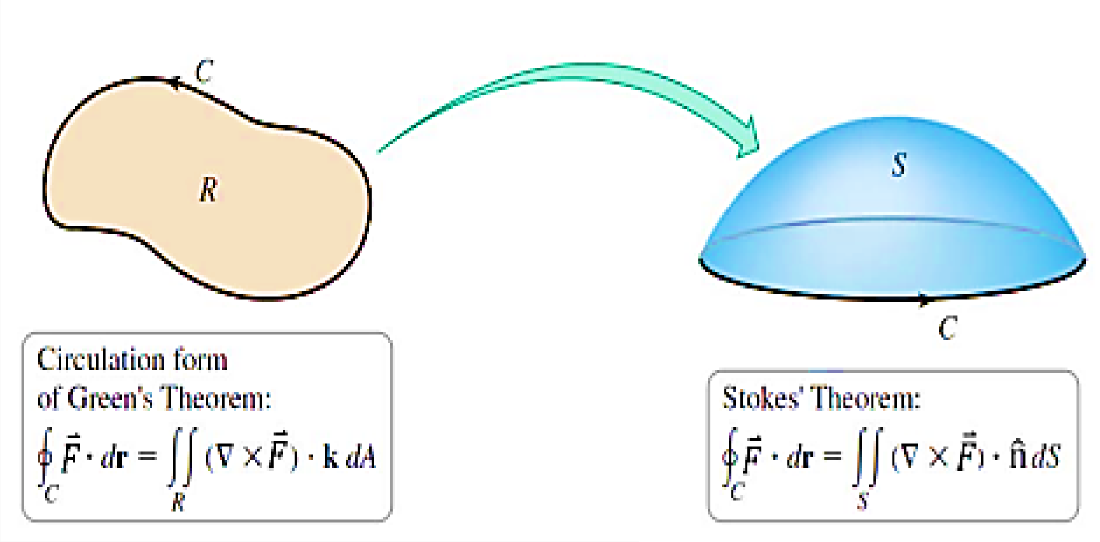
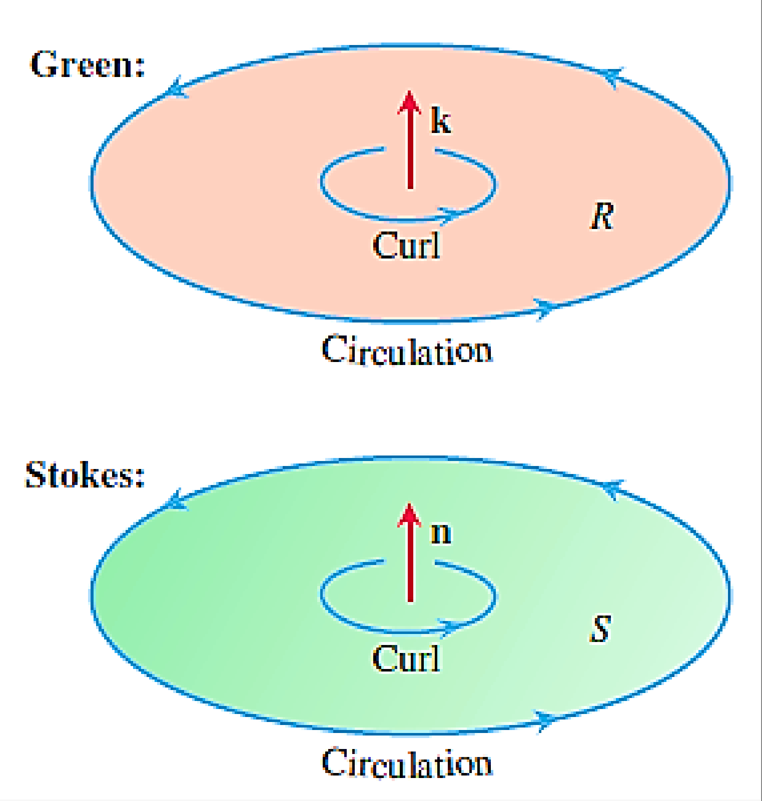



# Stokes' Theorem




Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

# 16

## Vector Calculus

**Book:** Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
    - **Section: 16.8**

**Book:** Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

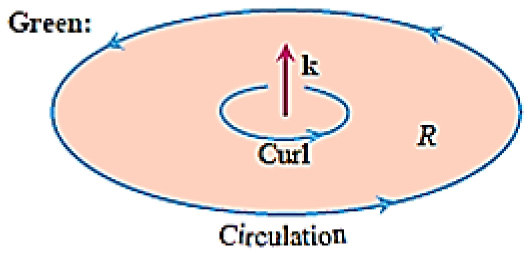
- **Chapter: 16**
    - **Section: 16.7**

# Stokes' Theorem & Green's Theorem

For a vector field $\mathbf{F}$ and a smooth surface $S$ the Green's theorem can be generalized to higher dimensions as follows:

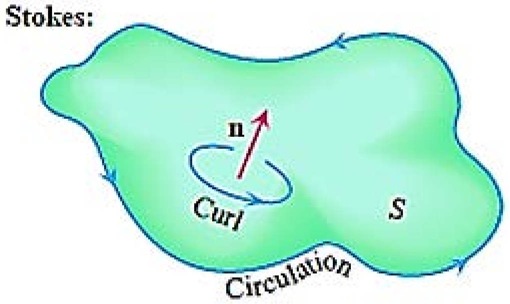
$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R (\text{curl}\, \mathbf{F}) \cdot \mathbf{k}\, dA$$

Circulation in $R$




$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\text{curl}\, \mathbf{F}) \cdot \mathbf{n}\, dS$$

Circulation in 3D



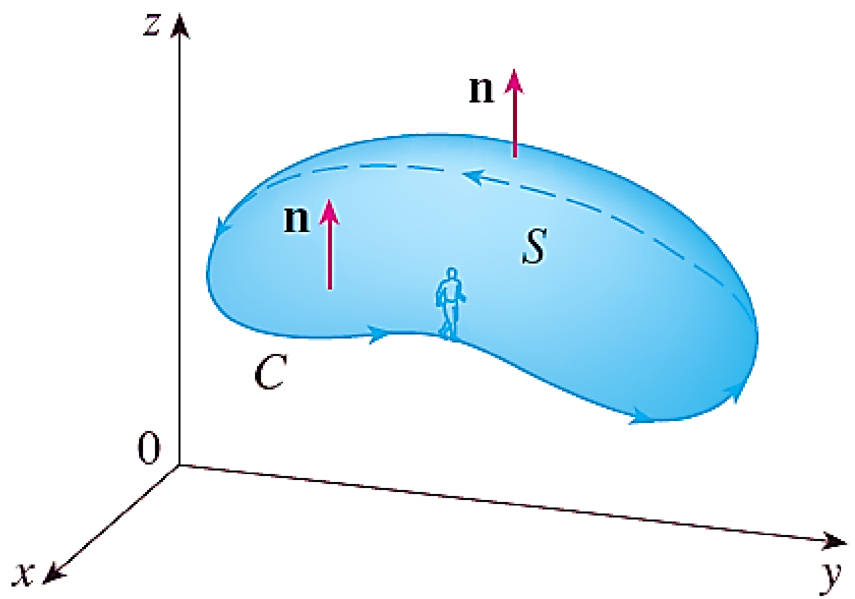

- Green's Theorem relates a double integral over a plane region $D$ to a line integral around its plane boundary curve.
- Stokes' Theorem relates a surface integral over a surface $S$ to a line integral around the boundary curve of $S$ (a space curve).

# Stokes' Theorem

Let $S$ be an oriented piecewise-smooth surface bounded by a simple, closed, piecewise-smooth boundary curve $C$ with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains $S$. Then,

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS \quad \text{or} \quad \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\operatorname{curl}\mathbf{F})\cdot d\mathbf{S},$$

where $\mathbf{n}$ is the unit normal vector at any point of $S$ drawn in the sense in which a right-handed screw would advance when rotated in the sense of the description of $C$.



## Question-17: Exercise (16.8)

A particle moves along line segments from the origin $O$ to the points $A(1,0,0)$, $B(1,2,1)$, $C(0,2,1)$, and back to the origin under the influence of the force field:

$$\mathbf{F}(x, y, z) = z^2\mathbf{i} + 2xy\,\mathbf{j} + 4y^2\mathbf{k}.$$

Find the work done.

**Solution:**

For the present case we are required to calculate:

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C \mathbf{F} \cdot d\mathbf{r},$$

where $C$ is a closed path which is composed of union of four paths $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}$, and $\overrightarrow{CO}$. To calculate the line integral directly, we need to parameterize each side of the path of the particle separately, calculate four separate line integrals, and add the result. This is not overly complicated, but it is time-consuming. By contrast, let's calculate the line integral using **Stokes' theorem** which connects this line integral with a surface integral of the normal component of the curl of $\mathbf{F}$.

## Solution:

In order to obtain equation of plane in which particle moves we will consider any two vectors that must lie in the plane. Suppose we consider $\overrightarrow{BA} = \langle 0, -2, -1 \rangle$ and $\overrightarrow{BC} = \langle -1,0,0 \rangle$. We know that the cross product of these two vectors will be orthogonal to both of these vectors and since both of these vectors lie in the plane so the cross product will also be orthogonal, or normal, to the plane. In other words, we can use the cross product of these two vectors as the normal vector to the plane. The cross product is: $\overrightarrow{BA} \times \overrightarrow{BC} = \langle 0,1,-2 \rangle$. Thus, the equation of plane is given as: $y - 2z = 0$ or $z = y/2$. Let $S$ be the planar region (sort of parallelogram) enclosed by the path of the particle, so $S$ is the portion of the plane:

$$z = f(x, y) = y/2;\ \text{for } 0 \le x \le 1, 0 \le y \le 2,$$

with upward orientation, for $(x, y)$ varying over the rectangular region with vertices $(0,0)$, $(1,0)$, $(1,2)$, and $(0,2)$ in the $xy$ −plane. Therefore, a parameterization of $S$ is given as:

$$\mathbf{r}(x, y) = \left\langle x, y, \frac{y}{2} \right\rangle; 0 \le x \le 1, 0 \le y \le 2$$

i.e., $x = x,\ y = y,\ z = y/2$. We then have:

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x, -f_y, 1 \rangle = \left\langle 0, \frac{-1}{2}, 1 \right\rangle.$$

## Solution:

Now,

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & 2xy & 4y^2 \end{vmatrix} = \langle 8y, 2z, 2y \rangle.$$

Therefore, by Stokes' theorem we have:

$$\oint_C \mathbf{F} \cdot d\mathbf{r}\, ds = \iint_S \text{curl}\,\mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \left[\text{curl}\,\mathbf{F} \cdot \left(\mathbf{r}_x \times \mathbf{r}_y\right)\right] dA\,,$$

where $D$ is the projected region on $xy$ −plane. Thus,

$$\oint_C \mathbf{F} \cdot d\mathbf{r}\, ds = \iint_D \left[\langle 8y, 2z, 2y \rangle \cdot \left\langle 0, \frac{-1}{2}, 1 \right\rangle\right] dA = \int_0^2 \int_0^1 \left(2y - \frac{y}{2}\right) dx dy = 3.$$

# Practice Questions

1. Calculate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \langle xy, x^2 + y^2 + z^2, yz \rangle$ and $C$ is the boundary of the parallelogram with vertices $(0,0,1)$, $(0,1,0)$, $(2,0,-1)$, and $(2,1,-2)$.
   Answer: 3.
   (*Solved in class*)
2. Use Stokes' theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle z^2, -3xy, x^3y^3 \rangle$ and $S$ is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume that $S$ is oriented upwards.
   Answer: 0.
3. Use Stokes' Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \langle -yz, 4y + 1, xy \rangle$ and $C$ is the circle of radius 3 at $y = 4$ and perpendicular to the $y$ −axis. $C$ has a clockwise rotation if we are looking down the $y$ −axis from the positive $y$ −axis to the negative $y$ −axis.
   Answer: $72\pi$.
4. Use Stokes' theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle z^2 - 1, z + xy^3, 6 \rangle$ and $S$ is the portion of $x = 6 - 4y^2 - 4z^2$ in front of $x = -2$ with orientation in the negative $x$ −axis direction.
   Answer: $2\pi$

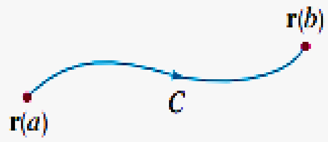
# Divergence Theorem

Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

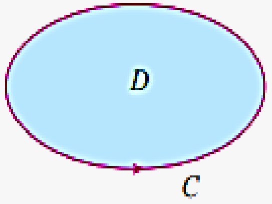

Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



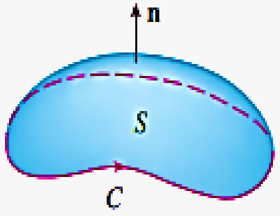

Green's Theorem
(Circulation form)

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$$



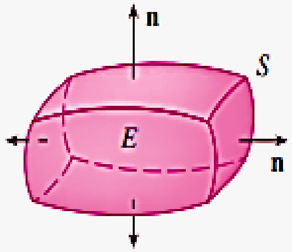

Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



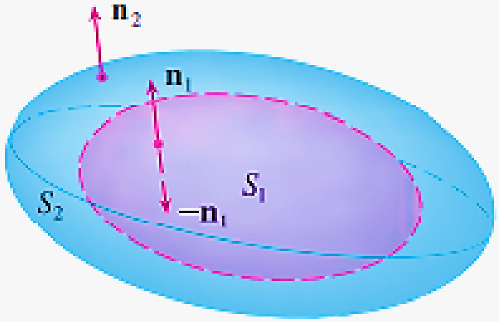

Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$




Divergence Theorem for regions that are finite unions of simple solid regions

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$$



# 16

# Vector Calculus

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- **Chapter: 16**
  - **Section: 16.9**

**Book:** Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

- **Chapter: 16**
  - **Section: 16.8**

# Divergence Theorem & Green's Theorem

For a vector field $\mathbf{F}$ and a smooth surface $S$ the Green's theorem can be generalized to higher dimensions as follows:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R (\operatorname{div} \mathbf{F})\, dA,$$

Total Flux across $R$

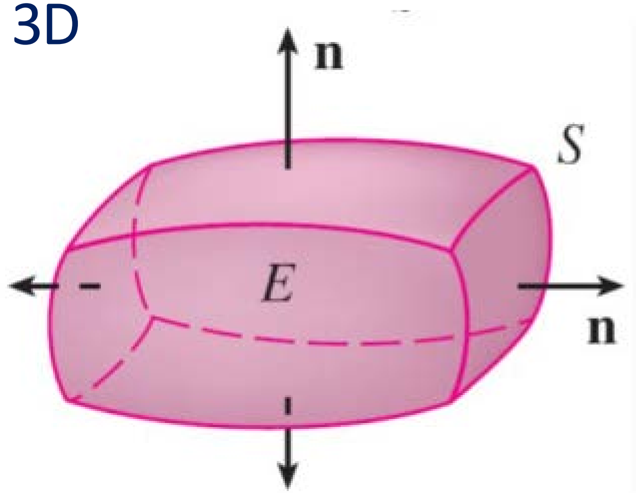
where $C$ is the positively oriented boundary curve of the plane region $R$.

↓

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E (\operatorname{div} \mathbf{F})\, dV,$$

Total Flux in 3D

where $S$ is the boundary surface of the solid region $E$.



# Gauss's Divergence Theorem

Let:

- $E$ be a simple solid region and let $S$ be the boundary surface of $E$, given with positive (outward) orientation.
- the unit normal vector $\mathbf{n}$ is directed outward from $E$.
- $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains $E$.

Then,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E (\operatorname{div} \mathbf{F})\, dV,$$

The Divergence Theorem states that:

Under the given conditions, the flux of $\mathbf{F}$ across the boundary surface $S$ of $E$ in the direction of the surface's outward unit normal $\mathbf{n}$ is equal to the triple integral of the divergence of $\mathbf{F}$ over the region $E$ enclosed by the surface $S$.

# Example: Finding Flux

Determine the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

**Solution:**

First, we compute the divergence of $\mathbf{F}$:

$$\text{div}\,\mathbf{F} = \nabla.\mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1.$$

The unit sphere $S$ is the boundary of the unit ball $E$ given by: $x^2 + y^2 + z^2 \leq 1$. So, the Divergence Theorem gives the flux as:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E (\text{div}\,\mathbf{F})\, dV = \iiint_E (1)\, dV = V(E) = \frac{4\pi}{3}(1)^3 = \frac{4\pi}{3}.$$