CONTINUITY EQUATION AND RELAXATION TIME

Continuity Equation

- **Principle of charge conservation:** the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume

- Thus, current I_{out} coming out of the closed surface is:

$$I_{\text{out}} = \oint \mathbf{J} \cdot d\mathbf{S} = \frac{-dQ_{\text{in}}}{dt}$$

- Where Q_{in} is the total charge enclosed by the closed surface

- The above equation is the integral form of continuity equation

- For differential form, we use divergence theorem, as:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{J}\, dv$$

Continuity Equation

- The rate of change of charge may also be written as:

$$\frac{-dQ_{\text{in}}}{dt} = -\frac{d}{dt}\int_v \rho_v \, dv = -\int_v \frac{\partial \rho_v}{\partial t} dv$$

- Rearranging the previously mentioned equations, we get:

$$\int_v \nabla \cdot \mathbf{J} \, dv = -\int_v \frac{\partial \rho_v}{\partial t} dv$$

- Or:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

- This is the *continuity of current equation* in point form

Continuity Equation

- From the physical interpretation of divergence, the continuity equation indicates that the current or charge per second, diverging from a small volume is equal to the time rate of decrease of charge per unit volume at every point

- The continuity equation is derived from the principle of conservation of charge which states that the outward flow of positive charge must be balanced by a decrease of positive charge (or perhaps an increase of negative charge) within the closed surface

- For steady currents, $\partial \rho_v / \partial t = 0$ and hence $\mathbf{\nabla}.\mathbf{J} = 0$, showing that the total charge leaving a volume is the same as the total charge entering it: *Kirchhoff's current law* follows from this

Relaxation Time

- We want to derive the equation for the decay of a volume charge density at some interior point of a material

- Therefore, consider the effect of introducing charge at some interior point of a given material (conductor or dielectric)

- We have the point form of Ohm's law as: $\mathbf{J} = \sigma \mathbf{E}$

- And the Gauss law as: $\nabla \cdot \mathbf{E} = \dfrac{\rho_v}{\varepsilon}$

- Substituting the above equations in the continuity equation, we get:

$$\nabla \cdot \sigma\mathbf{E} = \frac{\sigma\rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t}$$

Relaxation Time

Or: $$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0$$

- This is a homogenous linear ordinary differential equation
- By separating variables, we get: $$\frac{\partial \rho_v}{\rho_v} = -\frac{\sigma}{\varepsilon}\partial t$$
- Integrating both sides, we get: $$\ln \rho_v = -\frac{\sigma t}{\varepsilon} + \ln \rho_{vo}$$
- Here $\ln\rho_{vo}$ is a constant of integration
- Thus, we have the volume charge density as:

$$\rho_v = \rho_{vo} e^{-t/T_r} \qquad \text{where} \qquad T_r = \frac{\varepsilon}{\sigma}$$

Relaxation Time

- ρ_{vo} is the initial charge density (i.e., ρ_v at t = 0)

- The equation shows that as a result of introducing charge at some interior point of the material there is a decay of volume charge density ρ_v

- Associated with the decay is charge movement from the interior point at which it was introduced to the surface of the material

- The time constant T_r (in seconds) is known as the *relaxation time* or *rearrangement time*

Relaxation Time

- **Relaxation time** is the time it takes a charge placed in the interior of a material to drop to e^{-1}= 36.8 percent of its initial value

- It is short for good conductors and long for good dielectrics

- For example, for copper σ = 5.8 x 10^7 mhos/m, ε_r = 1, and so:

$$T_r = \frac{\varepsilon_r \varepsilon_o}{\sigma} = 1 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{1}{5.8 \times 10^7}$$
$$= 1.53 \times 10^{-19}\ \text{s}$$

- This shows a rapid decay of charge placed inside copper

Relaxation Time

- This implies that for good conductors, the relaxation time is so short that most of the charge will vanish from any interior point and appear at the surface (as surface charge)

- On the other hand, for fused quartz, for instance, σ = 10^{-17} mhos/m, ε_r = 5.0, we have:

$$T_r = 5 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{1}{10^{-17}}$$
$$= 51.2 \text{ days}$$

- showing a very large relaxation time

- Thus for good dielectrics, one may consider the introduced charge to remain wherever placed

Problem-1

- If $\mathbf{J} = 100/\rho^2 \mathbf{a}_\rho \mathrm{A/m^2}$, find:

a) The rate of increase in the volume charge density

b) The total current passing through surface defined by $\rho = 2, 0 < z < 1, 0 < \emptyset < 2\pi$.

Problem-2

- The current density in a certain region is approximated by $\mathbf{J} = \left(\frac{0.1}{r}\right)\exp(-10^6 t)\,\mathbf{a_r}$ A/m^2 in spherical coordinates:

a) At t = 1 μs, how much current is crossing the surface r = 5

b) Use the continuity equation to find $\rho_v(r, t)$, under the assumption that $\rho_v \rightarrow 0$ as $t \rightarrow \infty$

c) Find an expression for the velocity of the charge density

Problem-3

- Let: $\mathbf{J} = \frac{25}{\rho}\mathbf{a}_\rho - \frac{20}{\rho^2+0.01}\mathbf{a}_z$ A/m^2

a) Find the total current crossing the plane z = 0.2 in the $\mathbf{a}_z$ direction for $\rho \leq 0.4$

b) Calculate $\frac{\partial \rho_v}{\partial t}$