METHOD OF IMAGES

Boundary-value Problems

- We shall consider practical electrostatic problems where only electrostatic conditions (charge and potential) at some boundaries are known and it is desired to find **E** and *V* throughout the region

- Such problems are usually tackled using:

1. Poisson's equation
2. Or Laplace's equation
3. Or Method of Images

- These problems are usually referred to as boundary value problems

Method of Images

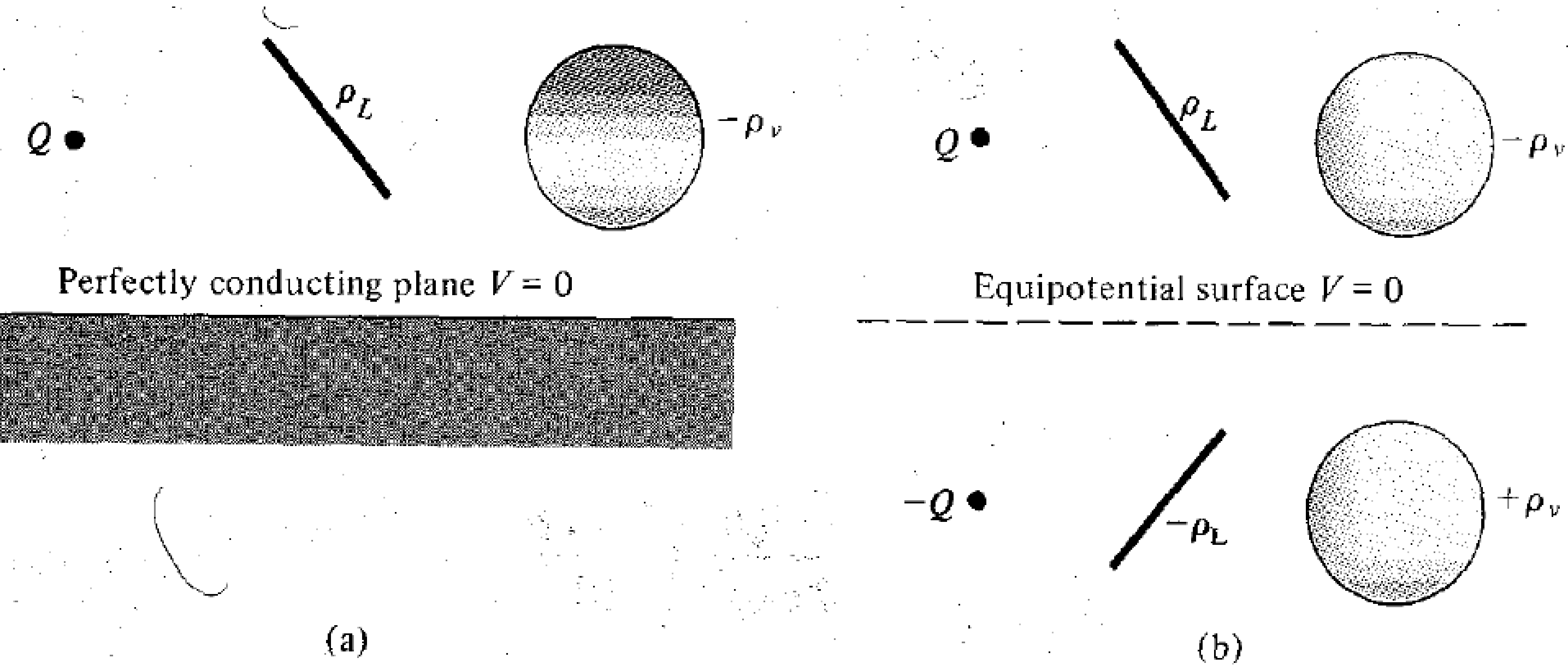
- The method of images, is commonly used to determine ***V***, **E**, **D**, and ρ_s due to charges in the presence of conductors

- By this method, we avoid solving Poisson's or Laplace's equation but rather utilize the fact that a conducting surface is equipotential

- The image theory states that *the field due to a charge above a perfectly conducting plane will remain the same if the conducting plane is removed and an opposite charge is placed at a symmetrical location below the plane*

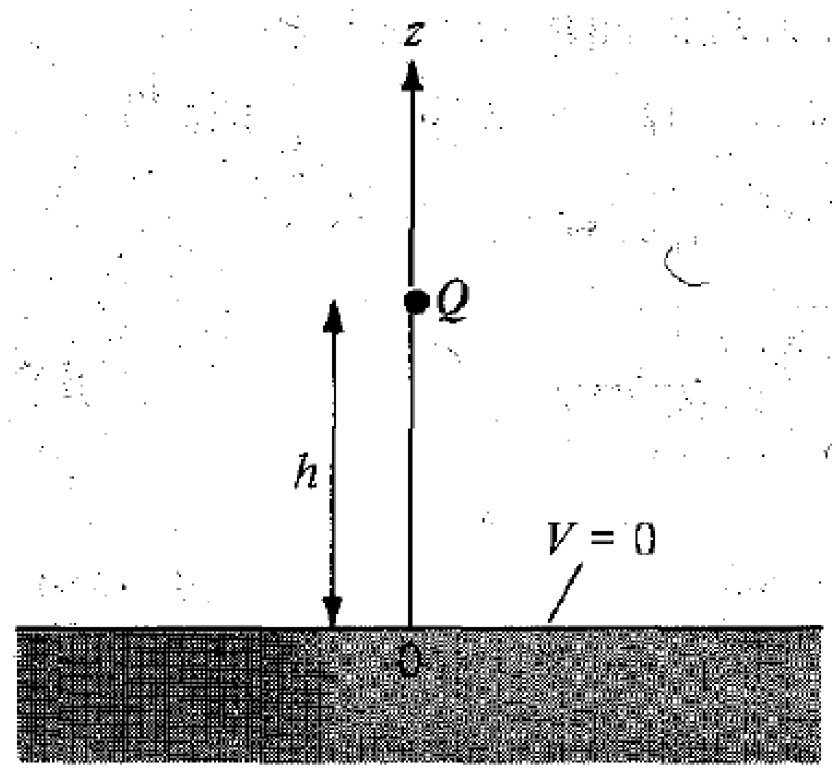
Method of Images

- Examples of point, line, and volume charge configurations are shown below



A Point Charge Above a Grounded Conducting Plane

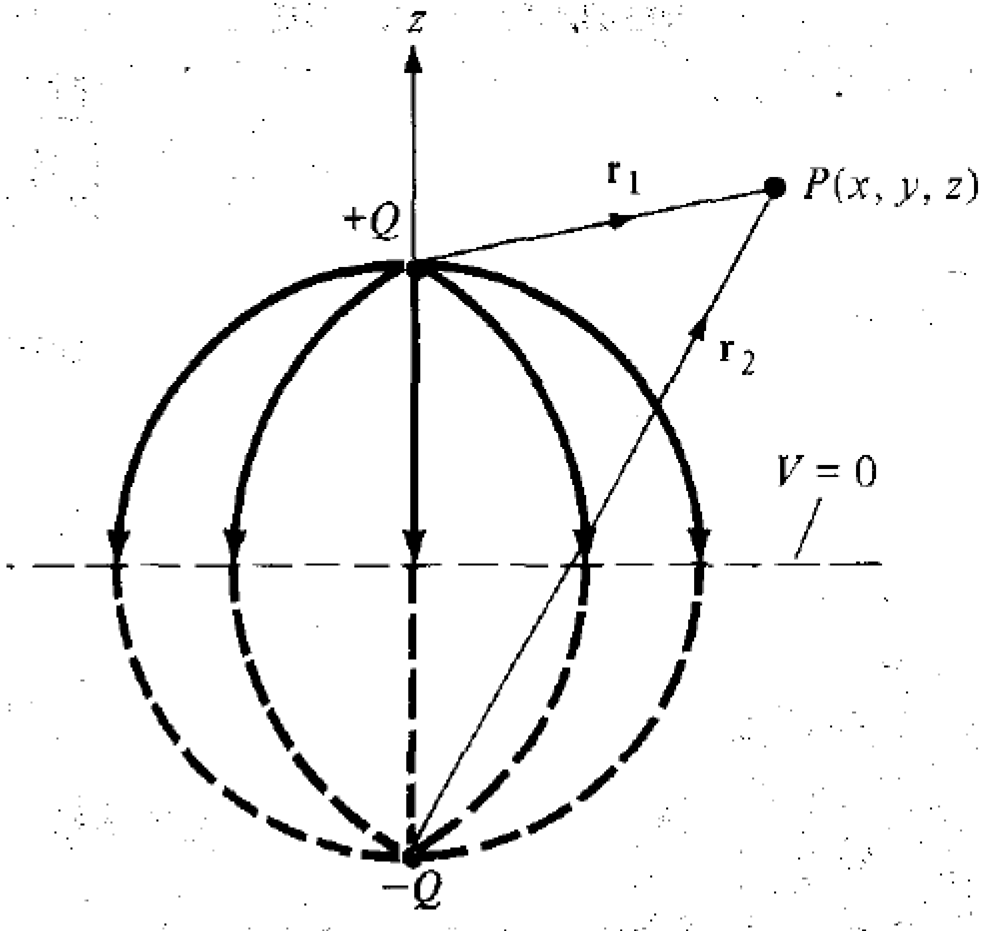
- Consider a point charge *Q* placed at a distance *h* from a perfect conducting plane of infinite extent as shown in Figure (a)



(a)

A Point Charge Above a Grounded Conducting Plane

- The image configuration is in Figure (b)



(b)

A Point Charge Above a Grounded Conducting Plane

- The electric field at point *P(x, y, z)* is given by:

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$$

$$= \frac{Q\,\mathbf{r}_1}{4\pi\varepsilon_o r_1^3} + \frac{-Q\,\mathbf{r}_2}{4\pi\varepsilon_o r_2^3}$$

- The distance vectors are given as:

$$\mathbf{r}_1 = (x, y, z) - (0, 0, h) = (x, y, z - h)$$

$$\mathbf{r}_2 = (x, y, z) - (0, 0, -h) = (x, y, z + h)$$

- Therefore:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o}\left[\frac{x\mathbf{a}_x + y\mathbf{a}_y + (z - h)\mathbf{a}_z}{[x^2 + y^2 + (z - h)^2]^{3/2}} - \frac{x\mathbf{a}_x + y\mathbf{a}_y + (z + h)\mathbf{a}_z}{[x^2 + y^2 + (z + h)^2]^{3/2}}\right]$$

A Point Charge Above a Grounded Conducting Plane

- It should be noted that when *z=0*, **E** has only the *z*-component, confirming that **E** is normal to the conducting surface

- The potential at *P* can be written as:

$$V = V_+ + V_-$$
$$= \frac{Q}{4\pi\varepsilon_o r_1} + \frac{-Q}{4\pi\varepsilon_o r_2}$$
$$V = \frac{Q}{4\pi\varepsilon_o}\left\{\frac{1}{[x^2 + y^2 + (z-h)^2]^{1/2}} - \frac{1}{[x^2 + y^2 + (z+h)^2]^{1/2}}\right\}$$

A Point Charge Above a Grounded Conducting Plane

- The surface charge density of the induced charge can be obtained as:

$$\rho_S = D_n = \varepsilon_o E_n \Big|_{z=0}$$

$$= \frac{-Qh}{2\pi[x^2 + y^2 + h^2]^{3/2}}$$

- So the total induced charge on the conducting plane is:

$$Q_i = \int \rho_S \, dS = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-Qh \, dx \, dy}{2\pi[x^2 + y^2 + h^2]^{3/2}}$$

- By changing variables, $\rho^2 = x^2 + y^2$, $dxdy = \rho d\rho d\emptyset$

A Point Charge Above a Grounded Conducting Plane

- Therefore:

$$Q_i = -\frac{Qh}{2\pi} \int_0^{2\pi} \int_0^{\infty} \frac{\rho \, d\rho \, d\phi}{[\rho^2 + h^2]^{3/2}}$$

- Or:

$$\begin{aligned} Q_i &= -\frac{Qh}{2\pi} 2\pi \int_0^{\infty} [\rho^2 + h^2]^{-3/2} \frac{1}{2} \, d(\rho^2) \\ &= \frac{Qh}{[\rho^2 + h^2]^{1/2}} \Bigg|_0^{\infty} \\ &= -Q \end{aligned}$$

- Therefore, all flux lines terminating on the conductor would have terminated on the image charge if the conductor were absent

A LINE Charge Above a Grounded Conducting Plane

- Consider an infinite charge with density ρ_L C/m located at a distance *h* from the grounded conducting plane *z* = *0*

- The same image system of point charge applies to the line charge as well except that *Q* is replaced by ρ_L

- The infinite line charge ρ_L may be assumed to be at $\boldsymbol{x = 0, z = h}$ and the image $-\rho_L$ at $\boldsymbol{x = 0, z = -h}$ so that the two are parallel to the y-axis

- The electric field at a point *P(x,y,z)* is given as: $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$

$$= \frac{\rho_L}{2\pi\varepsilon_o\rho_1}\mathbf{a}_{\rho 1} + \frac{-\rho_L}{2\pi\varepsilon_o\rho_2}\mathbf{a}_{\rho 2}$$

A LINE Charge Above a Grounded Conducting Plane

- The distance vectors are given as:

$$\boldsymbol{\rho}_1 = (x, y, z) - (0, y, h) = (x, 0, z - h)$$

$$\boldsymbol{\rho}_2 = (x, y, z) - (0, y, -h) = (x, 0, z + h)$$

- So we get:

$$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o}\left[\frac{x\mathbf{a}_x + (z - h)\mathbf{a}_z}{x^2 + (z - h)^2} - \frac{x\mathbf{a}_x + (z + h)\mathbf{a}_z}{x^2 + (z + h)^2}\right]$$

- Notice that when $z = 0$, **E** has only the z-component, confirming that **E** is normal to the conducting surface

A LINE Charge Above a Grounded Conducting Plane

- The potential at *P* is obtained from the line charges as:

$$V = V_+ + V_-$$
$$= -\frac{\rho_L}{2\pi\varepsilon_o}\ln\rho_1 - \frac{-\rho_L}{2\pi\varepsilon_o}\ln\rho_2$$
$$= -\frac{\rho_L}{2\pi\varepsilon_o}\ln\frac{\rho_1}{\rho_2}$$

- Substituting the magnitudes of the distance vectors, we get:

$$V = -\frac{\rho_L}{2\pi\varepsilon_o}\ln\left[\frac{x^2 + (z - h)^2}{x^2 + (z + h)^2}\right]^{1/2}$$

A LINE Charge Above a Grounded Conducting Plane

- The surface charge induced on the conducting plane is given by:

$$\rho_S = D_n = \varepsilon_o E_z \bigg|_{z=0} = \frac{-\rho_L h}{\pi(x^2 + h^2)}$$

- The induced charge per length on the conducting plane is:

$$\rho_i = \int \rho_S \, dx = -\frac{\rho_L h}{\pi} \int_{-\infty}^{\infty} \frac{dx}{x^2 + h^2}$$

- By letting $x = h \tan \propto$, the above equation becomes:

$$\rho_i = -\frac{\rho_L h}{\pi} \int_{-\pi/2}^{\pi/2} \frac{d\alpha}{h}$$
$$= -\rho_L$$

Problem-1

- A positive point charge Q is located at distance d_1 and d_2, respectively from two grounded (V = 0) perpendicular conducting half planes. Determine the force on charge Q caused by the charges induced on the planes.

Problem-2

- Let surface y=0 be a perfect conductor in free space. Two uniform infinite line charges of 30 nC/m each are located at x=0, y=1 and x=0, y=2. Let *V*=0 at the plane y=0, find **E** at *P*(1,2,0)