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Digital Signal Processing

Recap

Memoryless System : $y[n]$ depends on $x[n]$ at the same value of n

Ex. $y[n] = x[n+1]$
(a predictor sys.)

$$\int \rightarrow \Sigma$$

$$\Delta \rightarrow \text{Difference}$$

CT DT

» Compressor System

- $y_1(n) = x(Mn - n_0)$ Delaying input
- $y(n) = x(Mn)$

$y(n - n_0) = x(M(n - n_0))$ Delaying output

Not an LTI system

Causal System

output depends only on the current and previous samples.

$$y[n] = x[n] - x[n-1] \sim \text{Causal}$$

LPF : Smoothens an image

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Digital Signal Processing

LTI Systems

Input : $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

output : $y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$

$= x[n] * h[n]$

convolution

- Convolution is a property of LTI systems.
- If $y[n] = x[n] * h[n]$

$\Rightarrow$ length$(y[n])$ = length$(x(n))$ + length$(h(n))$ − 1

Stable and Causal LTI Systems

$$|y[n]| \leq B_x \sum_{k=-\infty}^{\infty} |h(k)|$$

bounded when absolute sum is finite

bounded input

If impulse response is absolute summable

- LTI System is causal if and only if :

$$h(k) = 0 \quad \text{for } k < 0$$

LCCD

$$\sum_{k=0}^{N} a_k\,y[n-k] = \sum_{k=0}^{M} b_k\,x[n-k]$$

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Digital Signal Processing

$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n]$$

$\Rightarrow$ • $y[n] = x[n] * h[n]$

Eigenfunctions

• $x[n] = e^{j\omega n}$ (To represent a generic case of any signal)

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega(n-k)}$$

$$= \left(\sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}\right) e^{j\omega n}$$

$$= \underbrace{H(e^{j\omega})}_{\text{eigenvalue}}\ \underbrace{e^{j\omega n}}_{\text{eigenfunction}}$$

• eigenvalue is called the frequency response of the system

$\rightarrow$ $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}$

More General Case:

$$x[n] = \sum_k a_k e^{j\omega_k n}$$

$$y[n] = \sum_k a_k H(e^{j\omega_k}) e^{j\omega_k n}$$

Discrete time frequency response is periodic with 2π.

$$H(e^{j(\omega + 2\pi r)}) = H(e^{j\omega})$$

where r is an int

DTFT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \{\text{inverse}\}$$

$$X(e^{j\omega}) = \sum_{n} x[n]\, e^{-j\omega n} \quad \{\text{forward}\}$$

Existance of DTFT

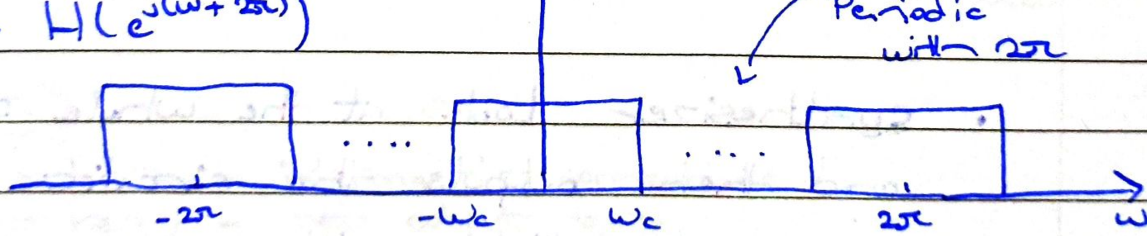
Absolute summability : $\sum_{n=-\infty}^{\infty} \Sigma |x[n]| < \infty$

- Sufficient condition

→ Example: Ideal LPF

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

$H(e^{j\omega}) = H(e^{j(\omega+2\pi)})$



$$h_{lp}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{\sin \omega_c n}{\pi n}$$

} not absolutely summable but it has a mean squared, in a sense, DTFT

date

Digital Signal Processing

$$x[n] = \frac{\sin(w_c n)}{\pi n}$$

$$X(e^{jw}) = \sum_n x[n] e^{-jwn}$$

$$= \frac{1}{\pi n} \sum_n \left[\frac{1}{2j} \left(e^{jw_c n} - e^{-jw_c n} \right) \right] e^{-jwn}$$

$$= \frac{1}{2j\pi n} \sum_n e^{j(w_c - w)n} - e^{-j(w_c + w)n}$$

$$= \frac{1}{2j\pi} \left[\sum_n \frac{e^{j(w_c - w)n}}{n} - \sum_n \frac{e^{-j(w_c + w)n}}{n} \right]$$

DTFT relation fails as it is not absolutely summable

$\rightarrow$ $x[n] = 1$ $\rightarrow$ impulse train

DTFT of $x[n]$ is also an impulse train

$$X(e^{jw}) = \sum_r 2\pi \delta(w + 2\pi r)$$

z-Transform

$z = re^{jw}$

$$X(z) = \sum_n x[n] z^{-n}$$

$z = e^{jw}$ will reduce zT to DTFT $(r = 1)$

re^{jw} $\rightarrow$ Circle : r : radius , $w = \phi$ phase

EXCELLENT

ate

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n] r^{-n}) e^{-j\omega n}$$

$$\sum_n |x[n] r^{-n}| < \infty$$

absolute summability may be made true by tuning r

Infinite & Finite Geometric Series

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

Example: $x[n] = \cos(\omega_0 n)$

$$X(z) = \sum_n x[n] z^{-n}$$

$$= \frac{1}{2}\left[\sum_n e^{j\omega_0 n} z^{-n} + \sum_n e^{-j\omega_0 n} z^{-n}\right]$$

$$= \frac{1}{2}\left[\sum_n (e^{j\omega_0} z^{-1})^n + \sum_n (e^{-j\omega_0} z^{-1})^n\right]$$

$$= \frac{1}{2}\left[\sum_{n=-\infty}^{-1} (e^{j\omega_0} z^{-1})^n + \sum_{n=0}^{\infty} (e^{j\omega_0} z^{-1})^n \ldots\right.$$

$$\left. + \sum_{n=-\infty}^{-1} (e^{-j\omega_0} z^{-1})^n + \sum_{n=0}^{\infty} (e^{-j\omega_0} z^{-1})^n\right]$$

$$= \frac{1}{2}\left[\sum_{0}^{\infty} (e^{-j\omega_0} z)^n - 1 + \sum_{n=0}^{\infty} (e^{j\omega_0} z^{-1})^n \ldots\right.$$

$$+ \sum_{n=0}^{\infty} (e^{j\omega_0} z)^n - 1 + \sum_{n=0}^{\infty} (e^{-j\omega_0} z^{-1})^n \Bigg]$$

$$= \frac{1}{2} \Bigg[\underbrace{\frac{1}{1-e^{-j\omega_0} z}}_{|z|<1} - 1 + \underbrace{\frac{1}{1-e^{j\omega_0} z^{-1}}}_{|z|>1} + \underbrace{\frac{1}{1-e^{j\omega_0} z}}_{|z|<1} - 1 \ldots$$

$$+ \underbrace{\frac{1}{1-e^{-j\omega_0} z^{-1}}}_{|z|>1} \Bigg]$$

→ RoC is none except $r = 1$.

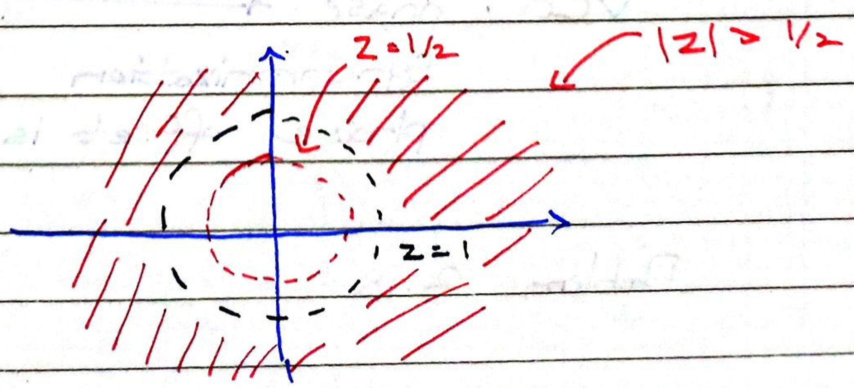
Q: $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n}$$

$$= \frac{1}{1-\frac{1}{2} z^{-1}} + \frac{1}{1-\frac{1}{3} z^{-1}}$$

$$= \frac{z}{z - 1/2} + \frac{z}{z - 1/3}$$

$|z| > 1/2 \cap |z| > 1/3$



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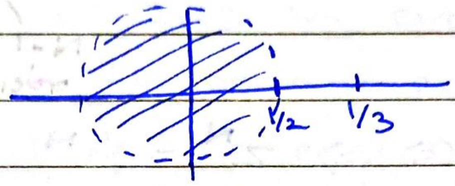
Digital Signal Processing

z - Transform

- $x[n] = \left(\frac{1}{2}\right)^n u[-n] + \left(\frac{1}{3}\right)^n u[-n]$

< innermost pole



- $x[n] = (-1/3)^n u[n] - (1/2)(1/2)^n u[-n-1]$

$$X[z] = \frac{z}{z+1/3} - \frac{1}{2}\sum_{-\infty}^{-1} (1/2)^n u[-n-1]$$

$$\boxed{|z| > |1/3|} \qquad - \frac{1}{2}\left[\sum_{0}^{\infty}\left[\left(\frac{1}{2}\right)^{-n} z^n\right] - 1\right]$$

$$- \frac{1}{2}\left[\frac{1}{1-(\frac{1}{2})^{-1}z} - 1\right]$$

$$- \frac{1}{2}\left[\frac{-2z}{1-2z}\right]$$

$$= \frac{z}{z+1/3} + \frac{z}{1-2z}$$

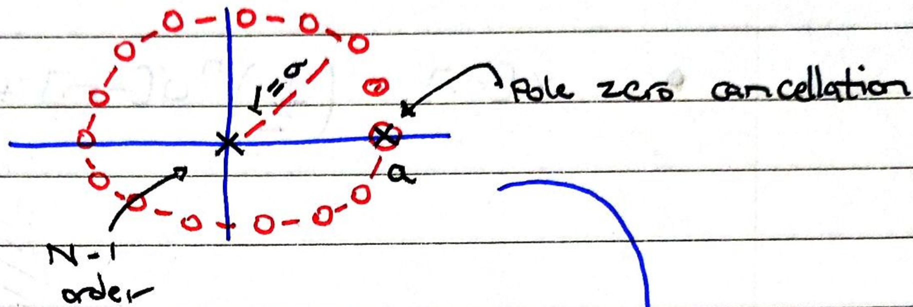
$|2z| < 1$
or $|z| < |1/2|$

$$|1/3| < |z| < |1/2|$$

- $x[n] = \begin{cases} a^n, & 0 \leq n \leq N-1 \\ 0 \end{cases}$ Finite

$$X(z) = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$



Zero(s) $\Rightarrow z^N = a^N$ $\underbrace{}_{}$ to match the form

$r^N e^{j\omega N} = a^N e^{j2\pi r}$

- $r = a$: $\omega N = 2\pi r$

$$\omega = \frac{2\pi r}{N}$$

RoC is all but origin

True for all Finite Sequences

DTFT exists if $|z|=1$ exists in RoC

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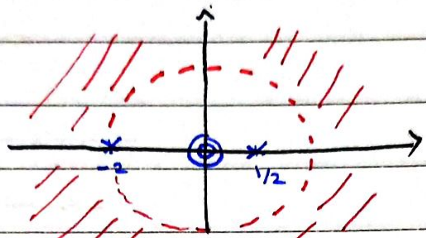
Digital Signal Processing

Stability, Causality and RoC

Causal

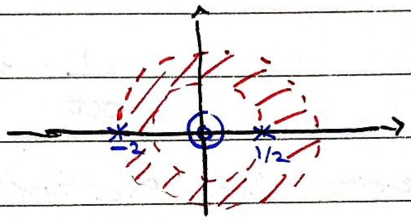
$u[n]$



$|z| > |2|$

Stable

$u[n]$ and $u[-n-1]$



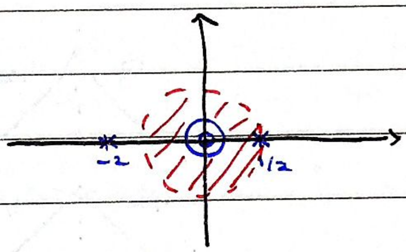
$|z| < |2|$

$|z| > |1/2|$

Non-causal

Non-stable

$u[-n-1]$



$|z| < |1/2|$

- For stability (DTFT exists), RoC must include unit circle $|z| = 1$

Inverse z-transform

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > 1/2$$

From inspection: $a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \ |z| > |a|$

$$x(n) = (1/2)^n u[n]$$

day/date

When there's no denominator in given expression:

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1+z^{-1})(1-z^{-1})$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

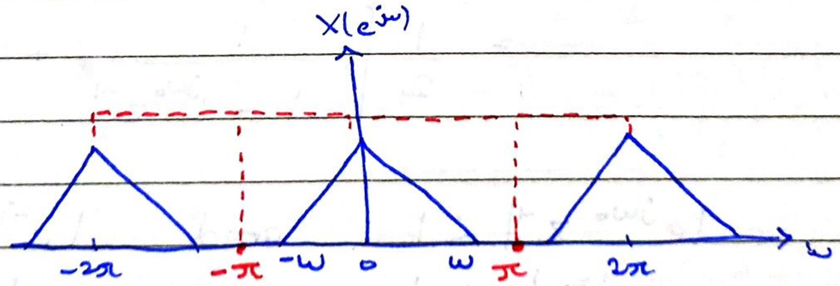
Property: $z^{-n_0} \longleftrightarrow \delta[n-n_0]$

Useful for finite length series

Problem

1. Show that π is the maximum radians frequency of any discrete signal.

$$x[n] \longleftrightarrow X(e^{j\omega}) = X(e^{j(\omega+2\pi r)})$$



To get an undistorted spectrum, maximum $\omega = \pi$.

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3. Find DTFT and z of:

a). $x[n] = \cos(w_0 n) u[n]$

- $X(e^{jw}) = \sum_{n=0}^{\infty} \cos(w_0 n) e^{-jw}$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} e^{jw_0 n} e^{-jw} + \sum_{n=0}^{\infty} e^{-jw_0 n} e^{-jw}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} e^{j(w_0-w)n} + \sum_{n=0}^{\infty} e^{-j(w_0+w)n}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j(w_0-w)}} + \frac{1}{1-e^{-j(w_0+w)}}\right]$$

$\underbrace{|e^{j(w_0-w)}| < 1}$

not true / DTFT does not exist

- $X(z) = \sum_{n=0}^{\infty} \cos(w_0 n) z^{-n}$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} (e^{jw_0} z^{-1})^n + \sum_{n=0}^{\infty} (e^{-jw_0} z^{-1})^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{jw_0} z^{-1}} + \frac{1}{1-e^{-jw_0} z^{-1}}\right]$$

$|e^{jw_0} z^{-1}| < 1$ and $|e^{-jw_0} z^{-1}| < 1$

$|z| > 1$ and $|z| > 1$

4. Find z transform of:

a). $x[n] = [2\ 0\ 0\ 5\ 6\ 7]$

with $x[-2] = 2$

$$x[n] = 2\delta[n+2] + 0(\delta[n+1] + \delta[n]) + \dots$$
$$5\delta[n-1] + 6\delta[n-2] + 7\delta[n-3]$$
$$= 2z^2 + 5z^{-1} + 6z^{-2} + 7z^{-3}$$

$$= \frac{2z^5 + 5z^2 + 6z + 7}{z^3}$$

Long division for poles & zeros

$$\rightarrow \quad X(z) = \frac{1}{1-0.5z^{-1}} + \frac{1}{1-1.5z^{-1}} \qquad 0.5 < |z| < 1.5$$

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{3}{2}\right)^n u[-n-1]$$

Given RoC ; find $x[n]$

$$\rightarrow \quad X(z) = \frac{1}{(z-0.5)(z-1.5)} = \frac{A}{z-0.5} + \frac{B}{z-1.5}$$

$$= \frac{-1}{z-0.5} + \frac{1}{z-1.5}$$

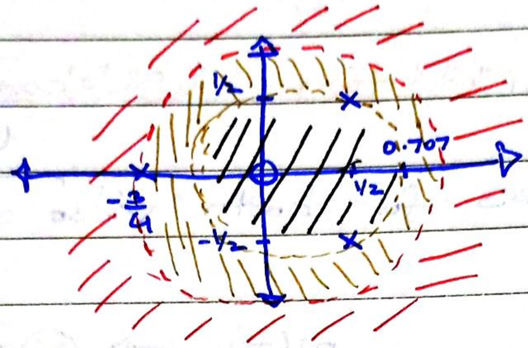
$$= -\frac{1}{2}\frac{z}{z-0.5} + \frac{1}{2}\frac{z}{z-1.5}$$

$$x[n] = -\left(\frac{1}{2}\right)^{n-1} u[n-1] - \left(\frac{3}{2}\right)^{n-1} u[-n]$$

$-a^n u[-n-1]$

Digital Signal Processing

Continued: Q7



- Outwards: $X(z) = \frac{z}{(z+3/4)(z-(0.5+0.5i))\dots(z-(0.5-0.5i))}$

  $|z| > 3/4$

- Middle: $\frac{\sqrt{2}}{2} < |z| < \frac{3}{4}$

- Inwards: $|z| < \frac{\sqrt{2}}{2}$

Expand using Partial Frac. then Inspect

Q8

a) $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$H(e^{jw}) = Y(e^{jw}) / X(e^{jw})$$

b) $y[n] - \frac{1}{2} y[n-1] = x[n] + 2x[n-1] + x[n-2]$

$$Y(e^{jw})\left[1 - \frac{e^{-jw}}{2}\right] = X(e^{jw})\left[1 + 2e^{-jw} + e^{-2jw}\right]$$

$$\boxed{H(e^{jw}) = \frac{1 + 2e^{-jw} + e^{-2jw}}{1 - e^{-jw}/2}}$$

Q9

$$x[n] \rightarrow h[n] \xrightarrow{v[n]} \bigotimes \rightarrow y[n]$$

with $u[n]$ into $\bigotimes$

where $h[n] = (1/4)^n u[n+10]$

→ a: $x_1[n] = \delta[n]$

$v_1[n] = (1/4)^n u[n+10]$

$\underline{y_1[n] = (1/4)^n u[n]}$

$x_2[n] = \delta[n-1]$

$v_2[n] = (1/4)^{n-1} u[n+9]$

$\underline{y_2[n] = (1/4)^{n-1} u[n]}$

$y_1[n] \neq y_2[n]$ } Not time invariant

→ Not an LTI system [check for $y_1[n-1]$ for time invariance]

→ b:

Non-Causal as $y_2[n] = 0$ for $n < 0$

$x_2[n] = 0$

→ c: Stable

Chapter 4 : Sampling of CT Signals

Aim is to preserve information

Periodic sampling

$x[n] = x_c(nT) \quad -\infty < n < \infty$

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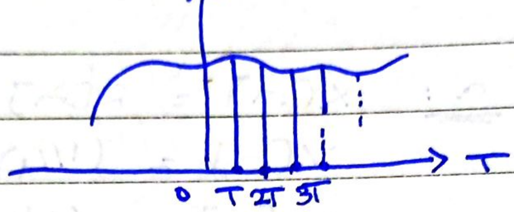
T = sampling period

$f_s = 1/T$

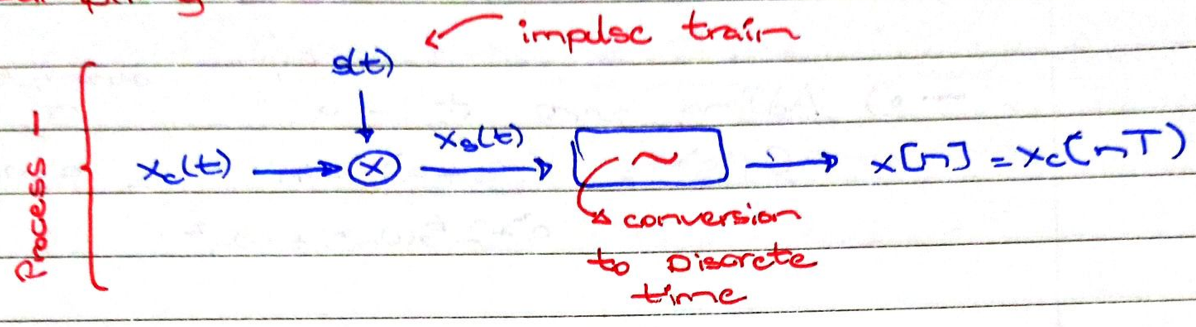
$\Omega_s = 2\pi f_s$ rad/sec (sampling frequency "caps - omega")



→ Loss of information in sampling is called aliasing

Digital Signal Processing

Sampling



CTFT

$$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t)\, e^{-j\Omega t}\, dt$$

$$x_c(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} X_c(j\Omega)\, e^{j\Omega t}\, d\Omega$$

Frequency Representation of Sampling

$$x_s(t) = x_c(t)\, s(t) = \sum_{n=-\infty}^{\infty} x_c(t)\, \delta(t - nT)$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

$$\rightarrow \quad S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

Working of Impulse Train

$a_k = 1/T$ of $s(t)$

$$e^{jk\omega_0 n} \longleftrightarrow 2\pi\, \delta(\omega - k\omega_0)$$

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 n} \quad \text{or} \quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$X(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

$$= \frac{1}{T}\left[X_c(j(\Omega - k\Omega_s))\right]$$

convolution property

Discrete Conversion

notation

$$\underbrace{X(e^{j\omega/T})}_{\text{notation}} = \frac{1}{T}\left[X_c(j(\omega/T - k\,2\pi/T))\right]$$

$\therefore\ \Omega = \omega/T \quad ; \quad \Omega_s = \omega_s/T = 2\pi/T$

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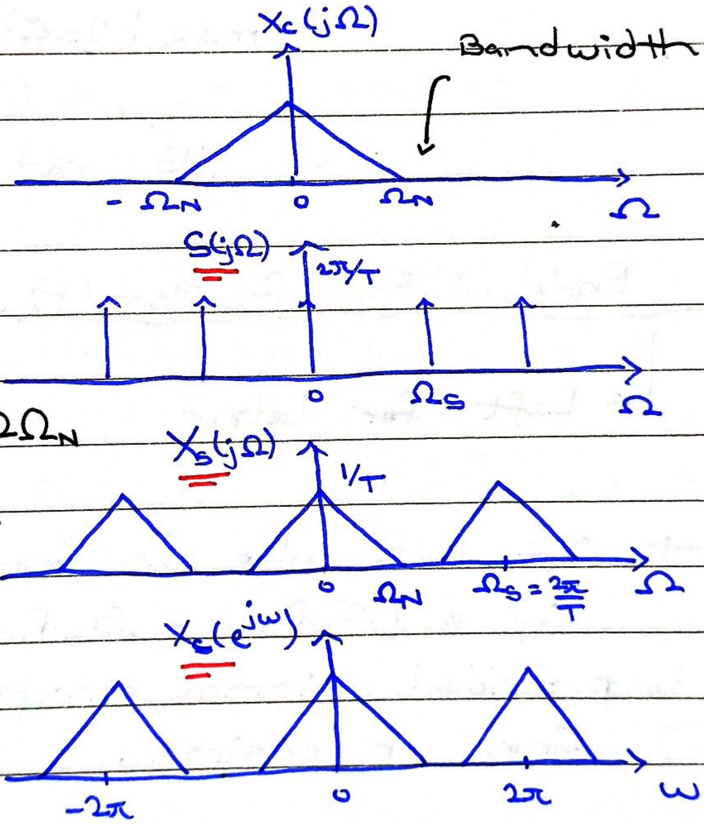
Digital Signal Processing

Sampling

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

Periodic with $2\pi/T$

Graphically,



Nyquist: $\Omega_s > 2\Omega_N$

rad/s

$\therefore \omega = \frac{\Omega}{T^{-1}}$

Normalized Frequency rad

Aliasing occurs when $\Omega_s < 2\Omega_N$

- Signals must be band limited for sampling to be possible
- ADC is a type of non-ideal sampler

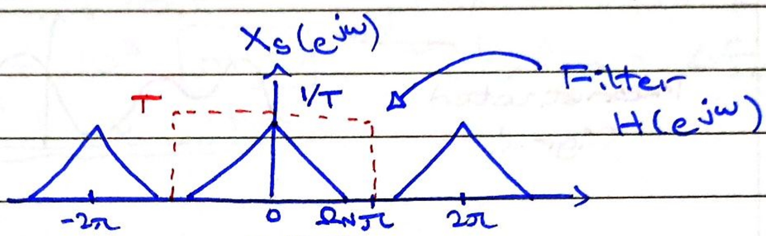
Alternative Representation:

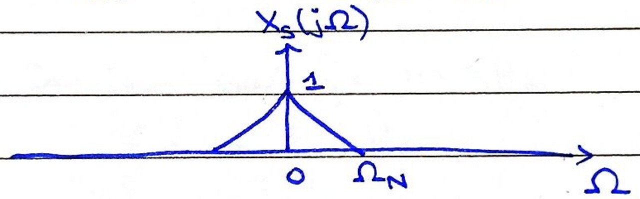
$$\Omega_s \geq 2\Omega_N$$

$$\frac{2\pi}{T} \geq 2\frac{\omega_N}{T}$$

$$\pi \geq \omega_N \quad \text{or} \quad \boxed{\omega_N \leq \pi}$$

Reconstruction:

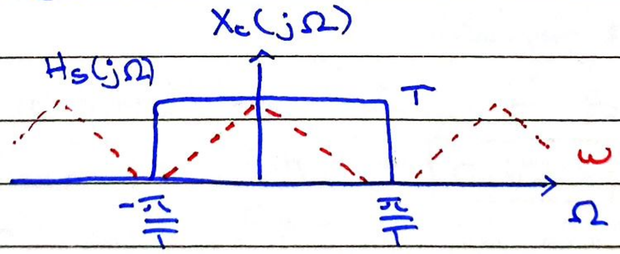




$\rightarrow$ $x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t-nT)$

$\hookrightarrow$ After LPF: $x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,h_r(t-nT)$

$\rightarrow$ $x_r(t) = x_c(t)$



$\rightarrow$ $h_r(t) = \frac{T\sin(\pi/T\, t)}{\pi t} = \frac{\sin(\pi/T \cdot t)}{\pi/T \cdot t}$

$\hookrightarrow$ $x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,\frac{\sin(\pi/T \cdot (t-nT))}{\pi/T \cdot (t-nT)}$

Conv. form
with t: actual & n: dummy

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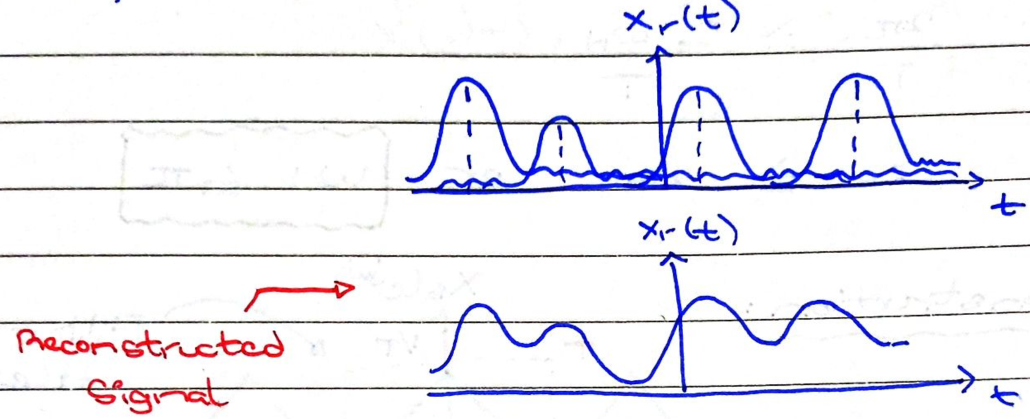
Equation is inferred as convolution of an impulse train and sinc () function.



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Digital Signal Processing

1 $\left[\begin{array}{l} x[n] = x_c(nT) \\ X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\omega/T - 2\pi k/T)) \end{array} \right.$

2 $\left[\begin{array}{l} y[n] = x[n] * h[n] \\ Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}) \\ \quad = \frac{1}{T} H(e^{j\omega}) \sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{T} - \frac{2\pi k}{T})) \end{array} \right.$

3

$$y_r(t) = y[n] * h_r(t)$$

$$= \sum_{n=-\infty}^{\infty} y[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}$$

$$Y_r(j\Omega) = Y(e^{j\Omega T}) H_r(j\Omega)$$

$$Y_r(j\Omega) = T\, Y(e^{j\Omega T}) \quad \{ |\Omega| \leq \frac{\pi}{T}$$

$$= H(e^{j\Omega T}) \cancel{\sum_{k=-\infty}^{\infty}} X_c(j(\Omega - k\Omega_s))$$

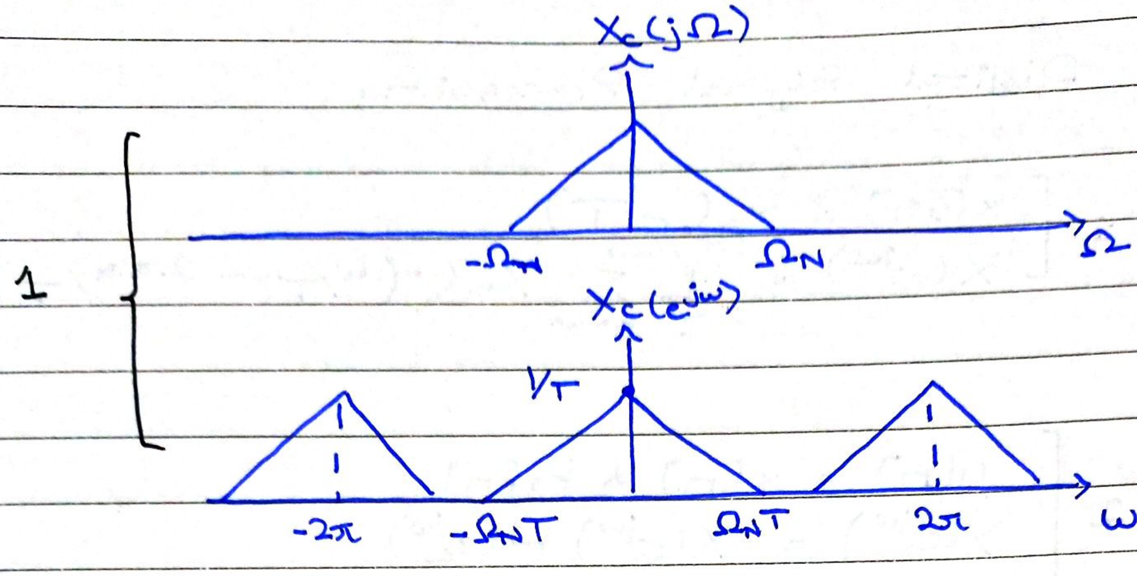
Summation goes bye-bye as $\{ |\Omega| \leq \frac{\pi}{T}$

$$= H_{eff}(e^{j\Omega T}) X_c(j\Omega)$$

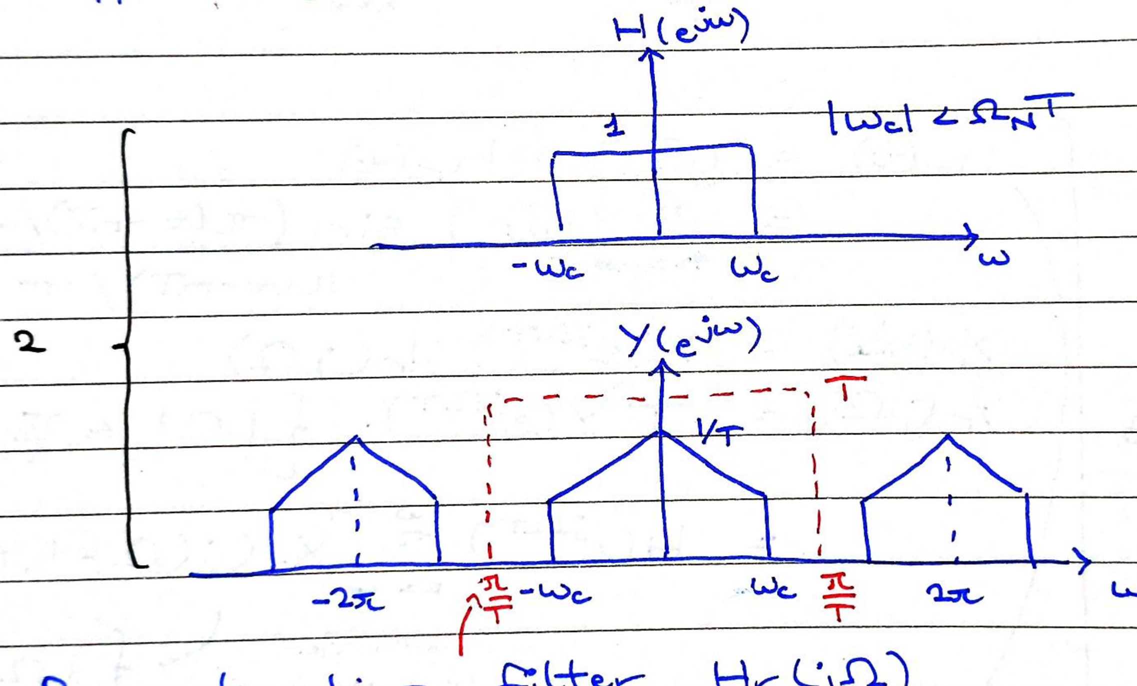
$$y_r(t) = h_{eff}[n] * x_c(t)$$

→ Spectral Domain Analysis

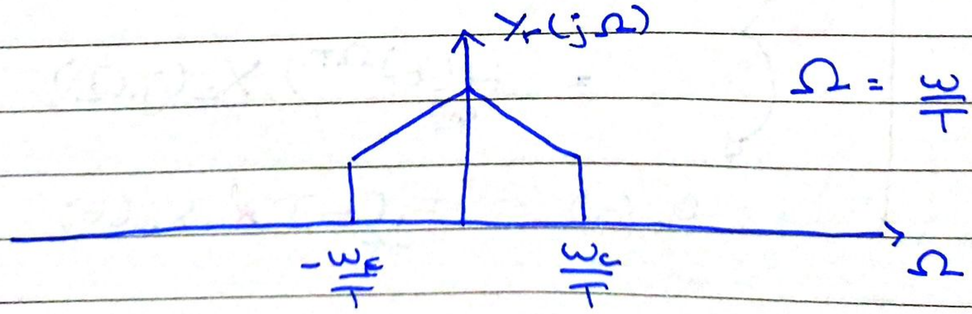
ate



Suppose system $H(e^{j\omega})$ is:



Reconstruction filter $H_r(j\Omega)$



14/03/23

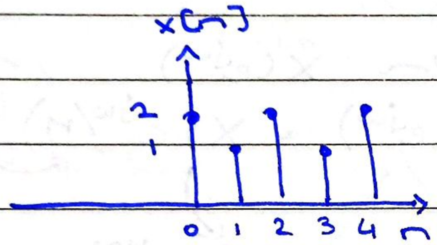
Digital Signal Processing

Changing the Sampling Rate

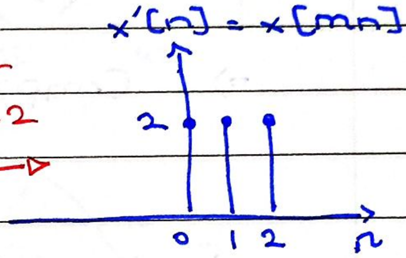
$x[n] = x_c(nT)$ &

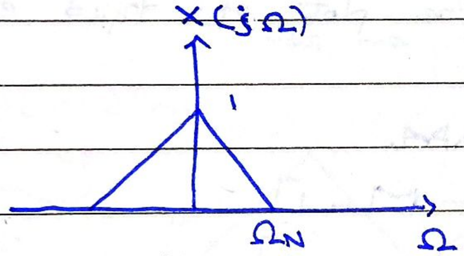
$x'[n] = x_c(nT') \Rightarrow T' = mT$

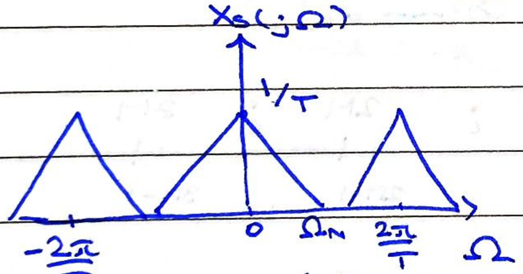
$x'[n] = x_c(mn)$

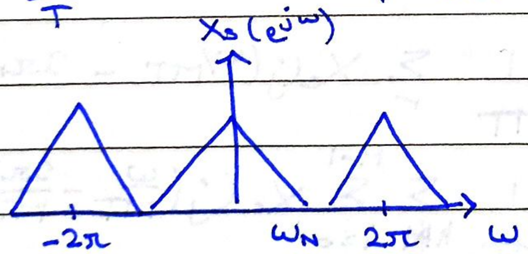


For $m=2$





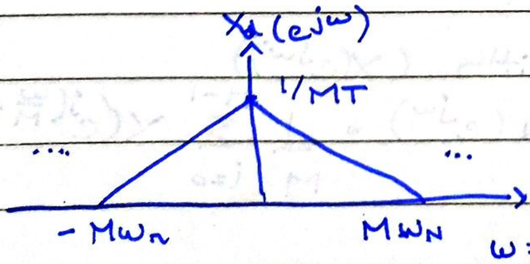




$\Omega = w/T$

$\Omega = w/T'$

$\Omega = w/MT$



$T' = MT$

compression in time domain is expansion in frequency domain

$w = \Omega MT$

2π will be shifted to $2\pi M$

i.e., for $M=2$; 4π

te

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k} X_c\left(j\left(\omega/T - 2\pi k/T\right)\right)$$

Now, $T' = TM$

$$\Rightarrow X_d(e^{j\omega}) = \frac{1}{T'} \sum_{k} X_c\left(j\left(\omega/T' - 2\pi k/T'\right)\right)$$

$$= \frac{1}{MT} \sum_{k} X_c\left(j\left(\omega/MT - 2\pi k/MT\right)\right)$$

$\Rightarrow$ Comparing with $X(e^{j\omega})$

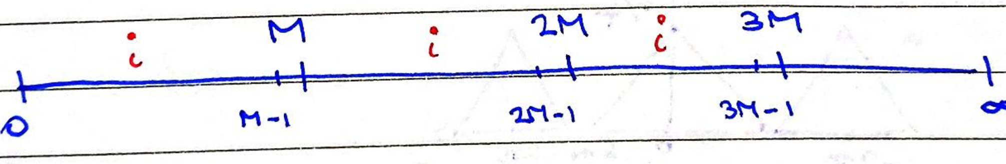
$$X_d(e^{j\omega}) = X(e^{j\omega/M})$$ notation

Last plot $\overset{?}{\sim}$ is the plot of ~~this~~ expression

$\Rightarrow$ $r = i + kM$

$i = [0 — M-1]$

$k = [-\infty — \infty]$



$$\Rightarrow X_d(e^{j\omega}) = \frac{1}{MT} \sum_{r} X_c\left(j\left(\omega/MT - 2\pi r/MT\right)\right)$$

$$= \frac{1}{MT} \sum_{k} \sum_{i=0}^{M-1} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi i}{MT} - \frac{2\pi kM}{MT}\right)\right)$$

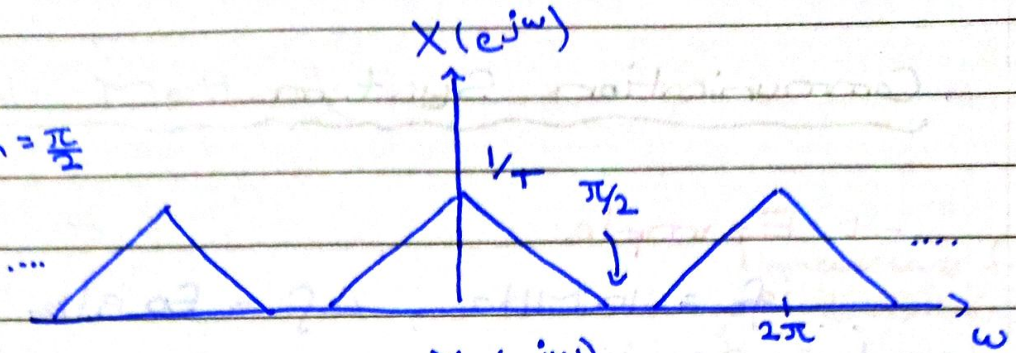
$\Rightarrow$ Comparing with $(X(e^{j\omega}))$

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)}\right)$$

For

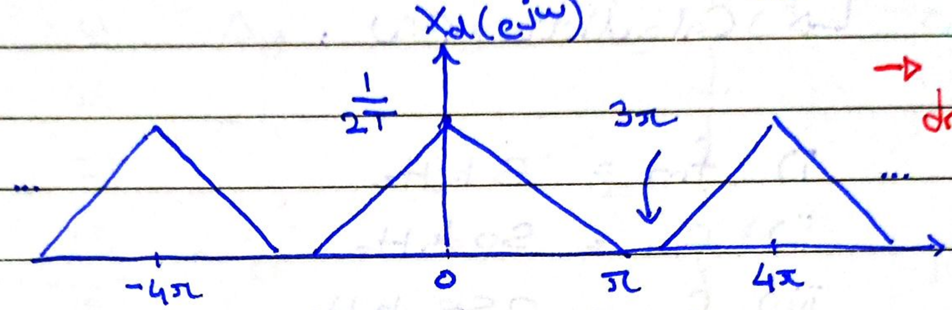
$M=2$ | $\omega_n = \frac{\pi}{2}$

$X_d(e^{j\omega})$



$i=0$

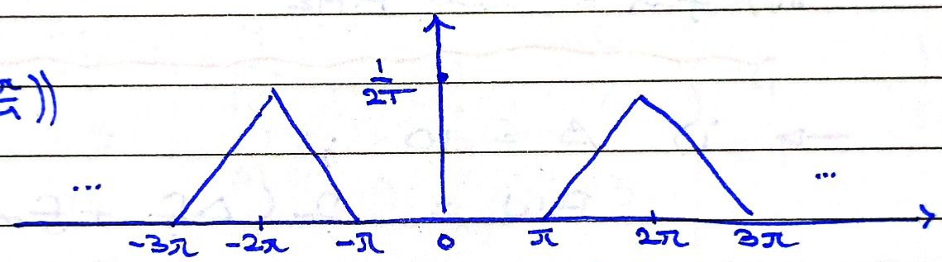
$\frac{1}{M} X(e^{j\omega/M})$



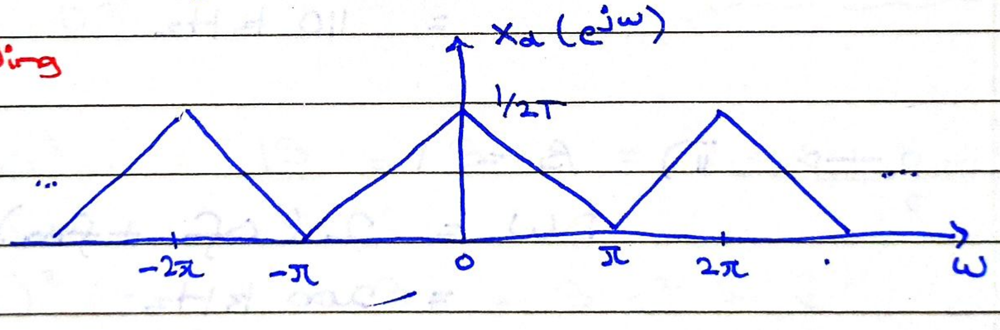
→ Not drawn to scale

$i=1$

$\frac{1}{M} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi}{M}\right)}\right)$

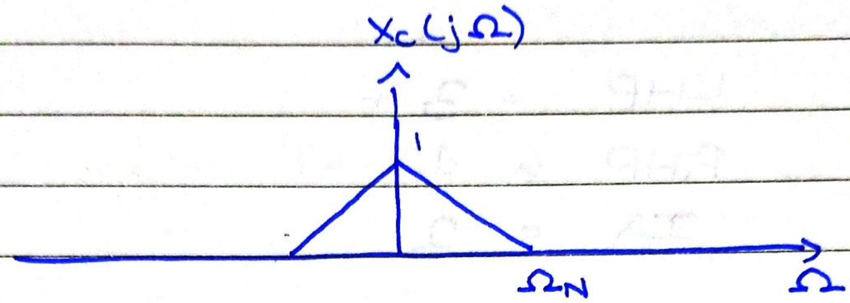


After adding it up



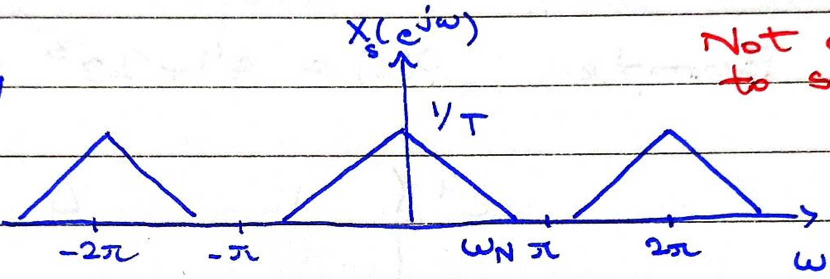
Digital Signal Processing

Recap :



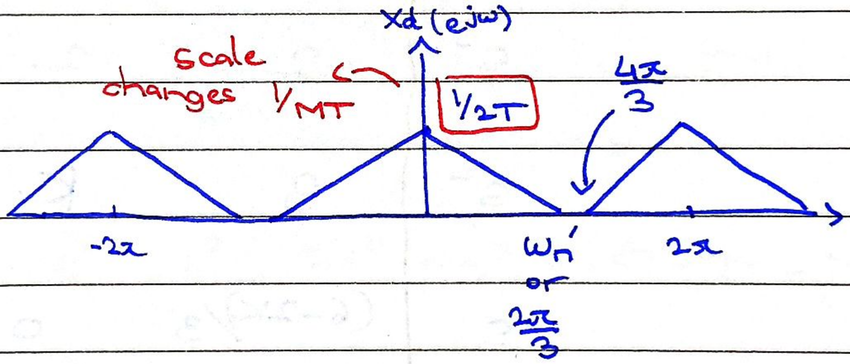
Assume $\left\{ \omega_N = \frac{\pi}{3},\ M = 2 \right\}$

Not drawn to scale



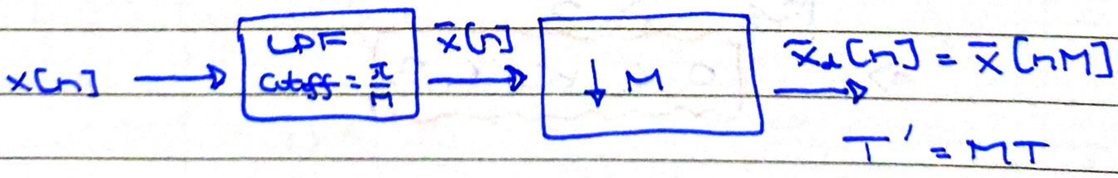
$\omega_n' = \frac{2\pi}{3}$

scale changes $1/MT$



$\hookrightarrow$ $\omega_n \leq \frac{\pi}{M}$ or $\Omega_N \leq \frac{\pi}{MT}$

otherwise aliasing occurs



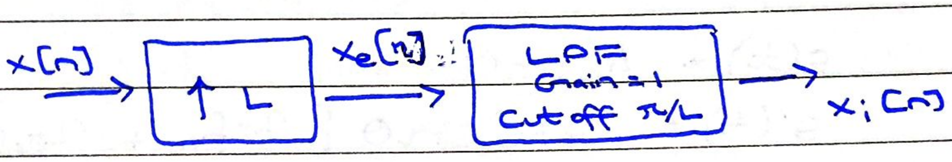
ate

21/03/23

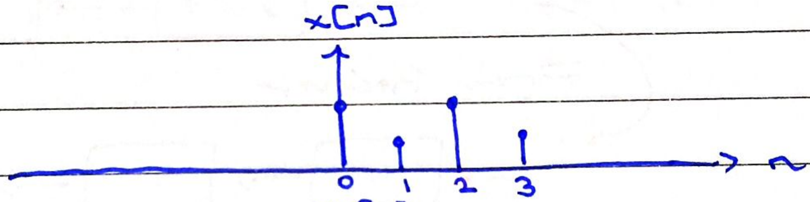
Digital Signal Processing

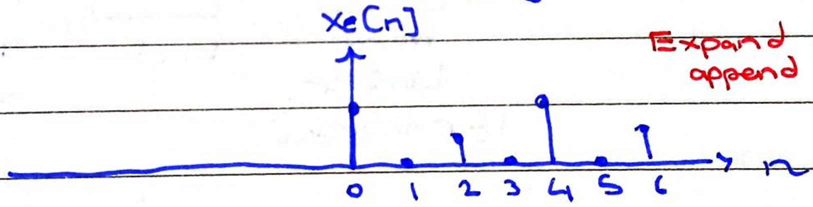
Upsampling

$$x_i[n] = x[n/L] = x_e[nT/L]$$



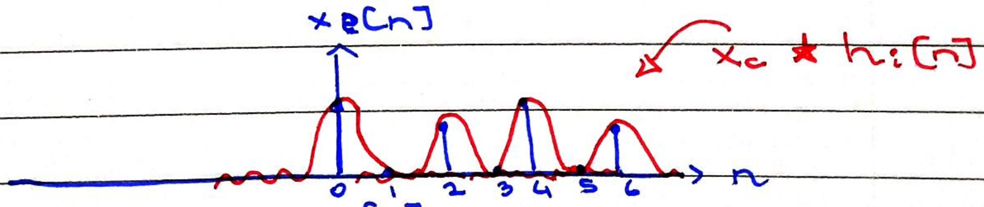
i) $x_e[n] = x[n/L]$

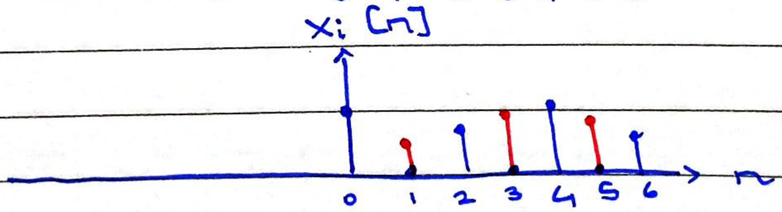




Expand & append zeros

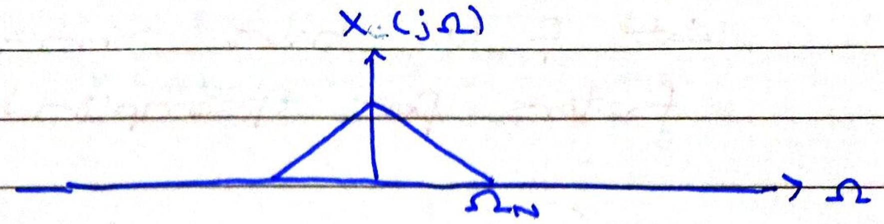
ii) Interpolation

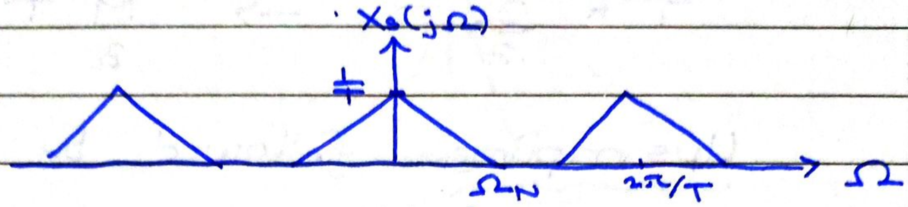


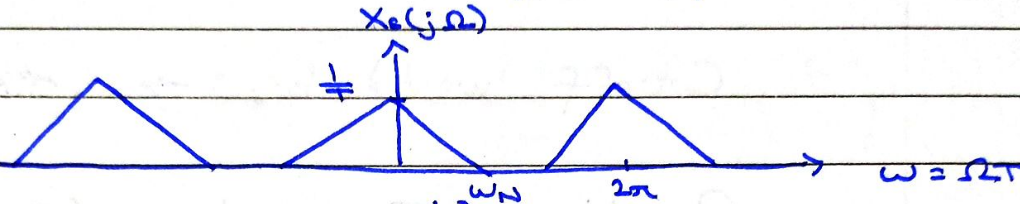


Frequency Domain Analysis

Next page

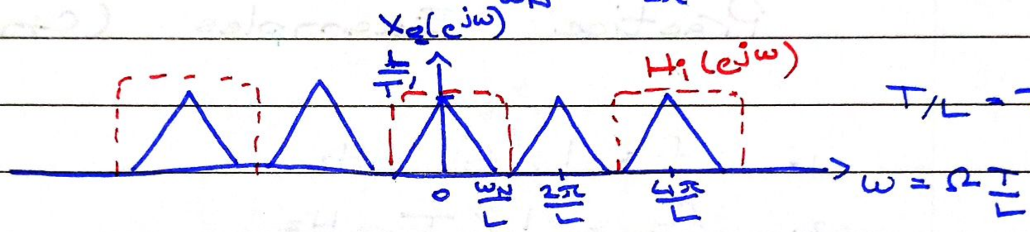






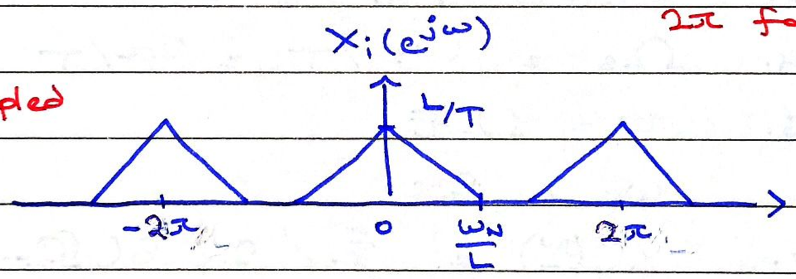
$H_i(e^{j\omega})$

Cutoff $= \frac{\pi}{L}$



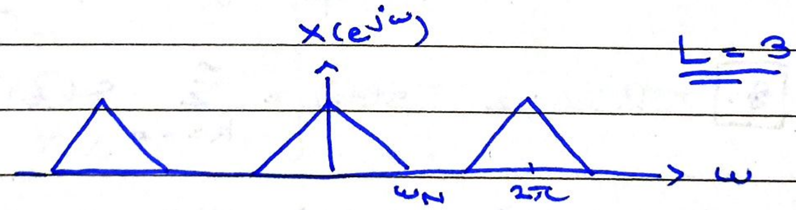
Periodicity by 2π fails to hold

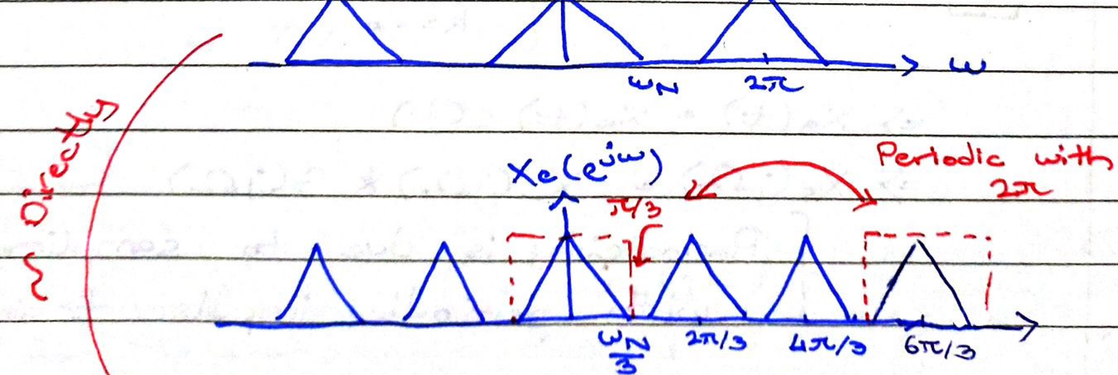
Assume its upsampled

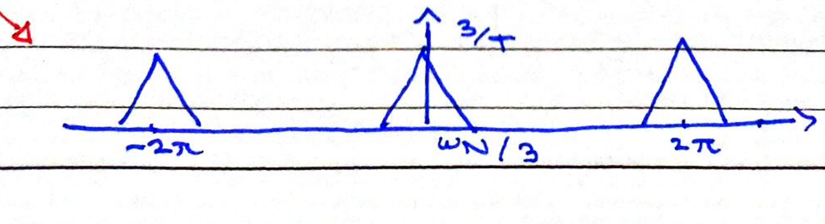


Example

$\underline{\underline{L=3}}$







Wat if we have non-integer factors for upsamplind / downsampling

$T' = \frac{2}{3}T \rightarrow \frac{M}{L}T$ $\left.\begin{matrix}\text{Upsample by } L \\ \text{Down sample by } M\end{matrix}\right\}$

Upsampling followed by {downsampling}

Cutoff would be → $\min(\pi/L, \pi/M)$

Practice Examples (Ch # 4)

1. $f = 1/t$ Hz
2. $f_s = 1/T$ Hz
3. $\Omega = \omega/t$ rad/s
4. $\Omega_s = \omega_s/T = 2\pi/T$
5. $\omega = \Omega t$
6. $\omega_s = 2\pi$
7. $X(j\Omega) = \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\Omega - k\Omega_s)\,(a_k)$

   where $a_k = \frac{1}{T}$ for impulse train

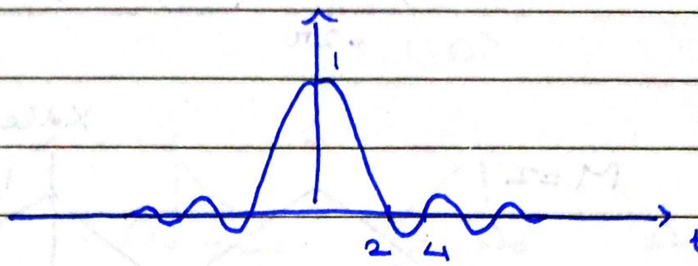
$\boxed{8.}$ Because $s(t) = \sum_{k=-\infty}^{\infty} \delta(t - nT)$

$\Rightarrow x_s(t) = x_c(t)\,s(t)$

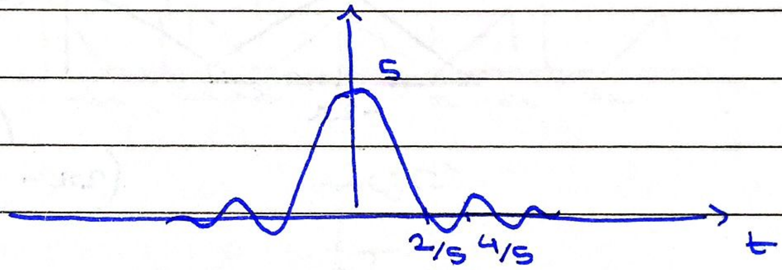
$\Rightarrow X_s(j\Omega) = X_c(j\Omega) * S(j\Omega)$

{ Periodicity is due to sampling with periodic impulse train }

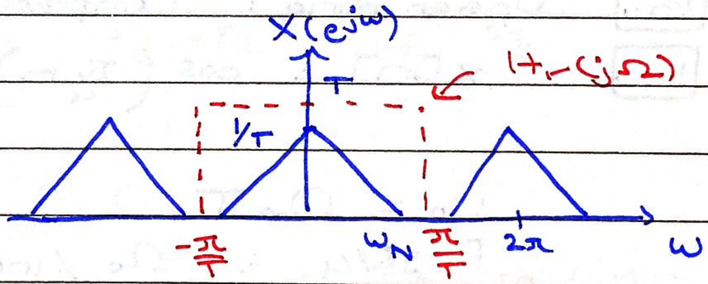
$\boxed{9.}$ $x(t) = \dfrac{\sin(\pi t/2)}{\pi t/2}$



→ <u>next one</u> : $\dfrac{\sin(5\pi t/2)}{\pi t/2} \left(\dfrac{5}{5}\right)$



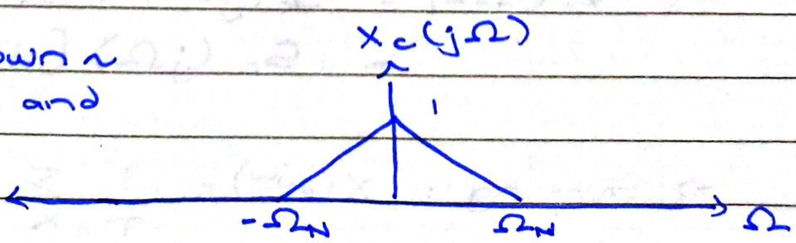
$\boxed{10.}$

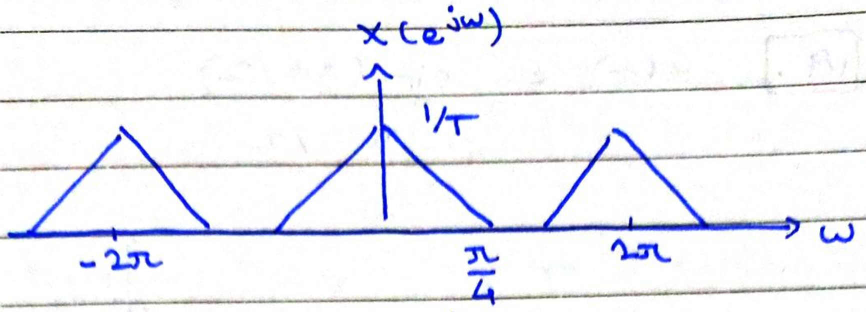


$$\begin{rcases} \Omega_s \geq 2\Omega_N \\ 2\pi/T \geq 2\Omega_N \\ \Omega_N \leq \pi/T \end{rcases} \text{because } \frac{\pi}{T} \text{ is the upper band}$$

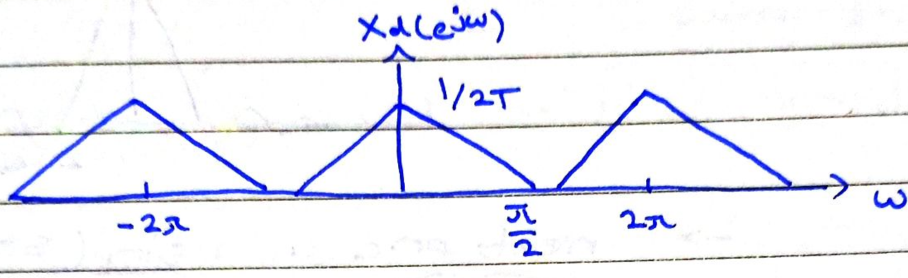
11.

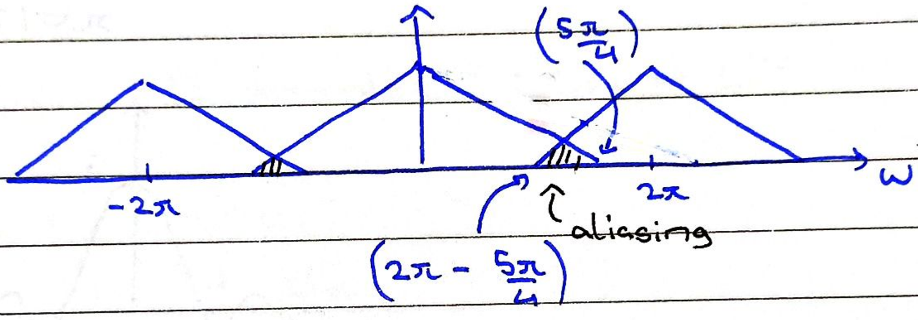
12. Perform down ~
for $M=2$ and
$w_N = \frac{\pi}{4}$





M = 2





$\boxed{13.}$ Upsampling : Compress $\underbrace{\text{axis}}_{}$; multiply magnitude
(axis: not the 2π)

$\boxed{14.}$ $x[n] = \cos\left(\frac{\pi}{4} n\right)$; $x_c(t) = \cos(\Omega_0 t)$

$w_0 = \Omega_0 T$

given sampling rate ~

- $\left[\begin{array}{l} \pi/4 = \Omega_0 / 1000 \\ \Omega_0 = 250\pi \text{ rad/s} \end{array}\right.$

- $\left[\begin{array}{l} (\pi/4 + 2\pi) = \Omega_0 / 1000 \\ \sim \text{Do the same} \Rightarrow \Omega_0 = 2250\pi \text{ rad/s} \end{array}\right.$

$\boxed{15.}$

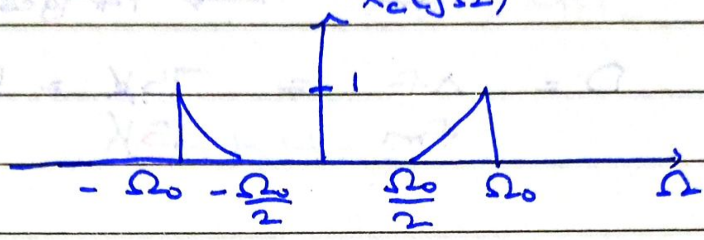
$$x_c(t) = s_c(t) + \alpha\, s_c(t - \tau_d)$$
$$X_c(j\Omega) = S_c(j\Omega) + \alpha e^{-j\tau_d \Omega} S_c(j\Omega)$$
$$= S_c(j\Omega)\left[1 + \alpha e^{-j\tau_d \Omega}\right]$$

$\Rightarrow$ Sampling : $X(e^{jw}) = \frac{1}{T} \sum_k X_c\left(j\left(\frac{w}{T} - \frac{2\pi k}{T}\right)\right)$

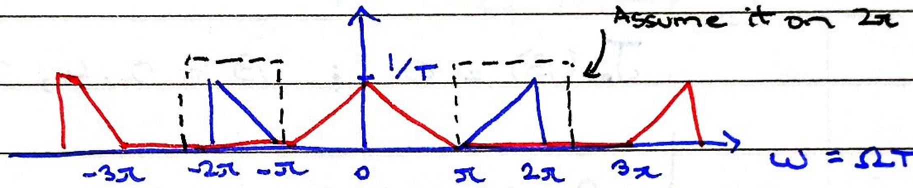
$$X(e^{j\omega}) = \frac{1}{T}\sum_{k} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)\left[1 + \alpha e^{-j\tau_d\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)}\right]$$

$\boxed{4.22.1}\ \boxed{16}$

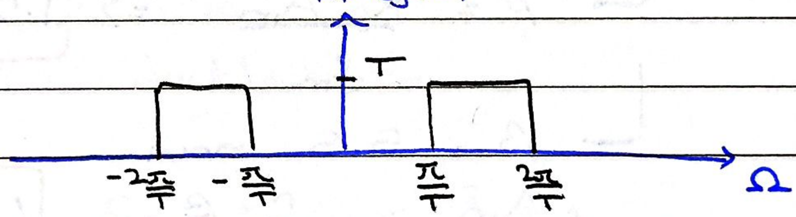
$T = \dfrac{2\pi}{\Omega_0}$



a)



b)



c) $\Omega_s = \dfrac{2\pi}{T} \geq 2\,\Omega_0$

$T \leq \pi/\Omega_0$ → however given T is > than this

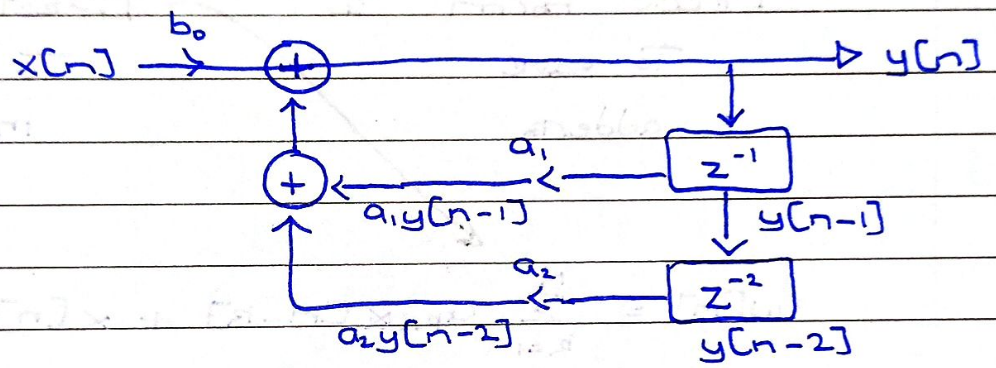
so ; $T \leq 2\pi/\Omega_0$

y/date 4/4/23

Digital Signal Processing

Structures for Discrete Time Systems

LCCD $\{ \; y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n]$



First values are garbage until we have $y[n-1]$, $y[n-2]$ ...

» $\left\{ \sum_{k=0}^{N} \hat{a}_k \, y[n-k] = \sum_{k=0}^{M} \hat{b}_k \, x[n-k] \right\}$

LCCD General Form

Direct Form 1

» $\frac{\hat{a}_0 y[n]}{\hat{a}_0} + \sum_{k=}^{N} \frac{\hat{a}_k}{\hat{a}_0} y[n-k] = \ldots$

» $a_0 y[n] - \sum_{k=1}^{N} a_k \, y[n-k]$ $\quad \therefore a_0 = 1$

$\therefore \underbrace{a_k = -\frac{\hat{a}_k}{\hat{a}_0}}_{\sim \text{Normalized}}$

$= \sum_{k=0}^{M} b_k \, x[n-k]$

structure on slides

$\therefore b_k = \frac{\hat{b}_k}{\hat{a}_0}$

$$Y(z) - \sum_{k=1}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$Y(z)\left[1 - \sum_{k=1}^{N} a_k z^{-k}\right] = X(z)\left[\sum_{k=0}^{M} b_k z^{-k}\right]$$

$$\frac{Y(z)}{X(z)} = H(z) = \boxed{\left(\sum_{k=0}^{M} b_k z^{-k}\right) \Big/ \left(1 - \sum_{k=1}^{N} a_k z^{-k}\right)}$$

Direct Form 1 — Direct Form 2

To save adders — To save memory

$$w[n] = \sum_{k=1}^{N} a_k w[n-k] + x[n]$$

$$y[n] = \sum_{k=0}^{M} b_k y[n-k] + w[n]$$

$$a_k = -\frac{\hat{a}_k}{\hat{a}_0}, \quad b_k = \frac{\hat{b}_k}{\hat{a}_0}$$

» $W(z) = \cancel{\left[X(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right]\right]} = X(z) + W(z) \sum_{k=1}^{N} a_k z^{-k}$

» $Y(z) = W(z) + W(z)\left[\sum_{k=0}^{M} b_k z^{-k}\right]$

» $Y(z) = \cancel{\left[X(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] + Y(z)\left[\sum_{k=0}^{M} b_k z^{-k}\right]\right]}$

» $$\frac{Y(z)}{X(z)} = H(z) = \boxed{\sum_{k=0}^{M} b_k z^{-k} \Big/ \left(1 - \sum_{k=1}^{N} a_k z^{-k}\right)}$$

Direct Form 1 ≡ Direct Form 2

day/date 6/4/23

Digital Signal Processing

Example

$$
\begin{cases}
w_1[n] = w_4[n] - x[n] \\
w_2[n] = \alpha w_1[n] \\
w_3[n] = w_2[n] + x[n] \\
w_4[n] = w_3[n-1] \\
y[n] = w_2[n] + w_4[n]
\end{cases}
$$

$$W_1(z) = W_4(z) - X(z)$$

$$W_2(z) = \alpha W_4(z) - \alpha(X(z))$$

$$W_3[z] = W_2(z) + X(z)$$

$$W_4(z) = [W_2(z) + X(z)] z^{-1}$$

$\gg$ $Y(z) = \alpha W_4(z) - \alpha X(z) + [W_2(z) z^{-1} + X(z) z^{-1}]$

$\gg$ $Y(z) = X(z)[z^{-1} - 1] + \alpha W_4(z) + W_2(z) z^{-1}$

↳ Find W_2 & W_4

$$W_2(z) = \alpha W_1(z)$$

$$W_2(z) = \alpha [W_4(z) - X(z)]$$

$$W_2(z) = \alpha [W_3(z) z^{-1} - X(z)]$$

$$W_2(z) = \alpha [W_2(z) z^{-1} + X(z) z^{-1} - X(z)]$$

$$W_2(z) = \alpha W_2(z) z^{-1} + \alpha X(z) z^{-1} - \alpha X(z)$$

$$W_2(z) - \alpha W_2(z) z^{-1} = X(z)[\alpha z^{-1} - \alpha]$$

$$W_2(z)[1 - \alpha z^{-1}] = \alpha X(z)[z^{-1} - 1]$$

$$\left\{ W_2(z) = \frac{\alpha X(z)[z^{-1} - 1]}{1 - \alpha z^{-1}} \right\}$$

day/date

Similarly, find $W_4(z)$ and substitute in $Y(z) = W_2(z) + W_4(z)$ & find $H(z)$. :)

Basic Structures for IIR Systems

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} \dots}{1 - a_1 z^{-1} - a_2 z^{-2} \dots}$$

write in form $\rightarrow \dfrac{(2^{nd} \text{ order})(2^{nd} \text{ order}) \dots}{(2^{nd} \text{ order})(2^{nd} \text{ order})}$

2^{nd} Order Rep.

$\rightarrow \dfrac{(b_0 + b_1 z^{-1} + b_2 z^{-2})}{1 - a_1 z^{-1} - a_2 z^{-2}}$

date

11/04/23

Digital Signal Processing

- To solve signal flow graphs, identify and mark adders before writing equations.

IIR Systems: Basic structure

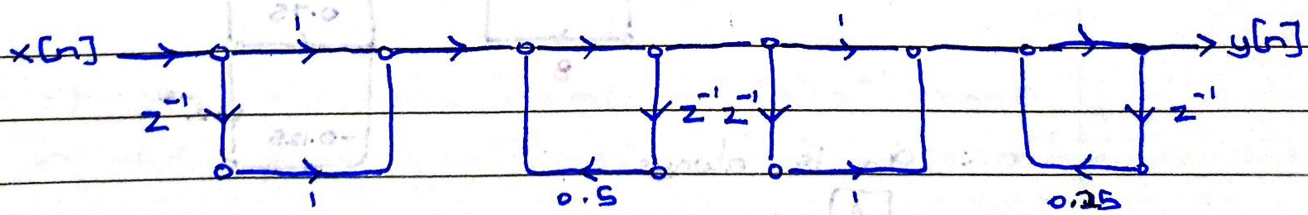
$$H(z) = \sum_{k=0}^{M} b_k z^{-k} \Big/ 1 - \sum_{k=1}^{N} a_k z^{-k}$$

$$\equiv \prod_{k=1}^{N_s} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$

Example

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75 z^{-1} + 0.125 z^{-2}}$$

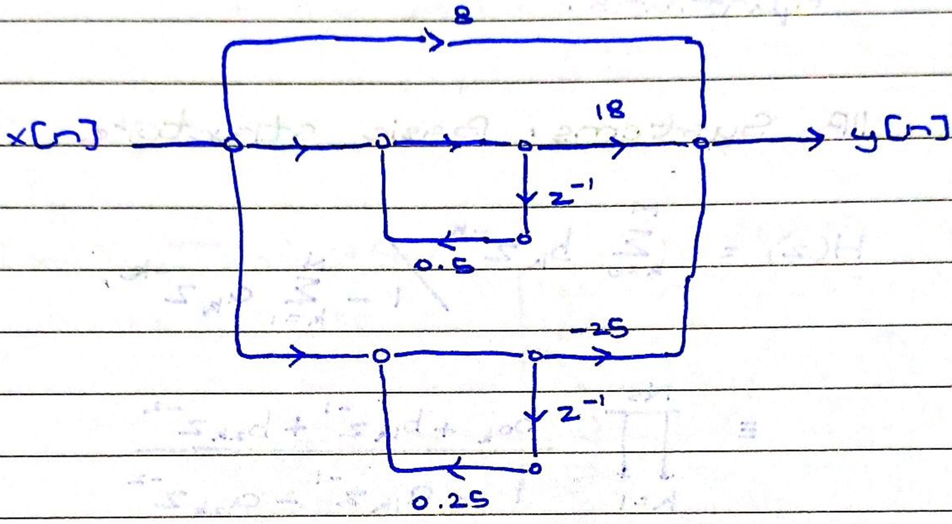
$$= \frac{(1 + z^{-1})(1 + z^{-1})}{(1 - 0.5 z^{-1})(1 - 0.25 z^{-1})}$$



DF I

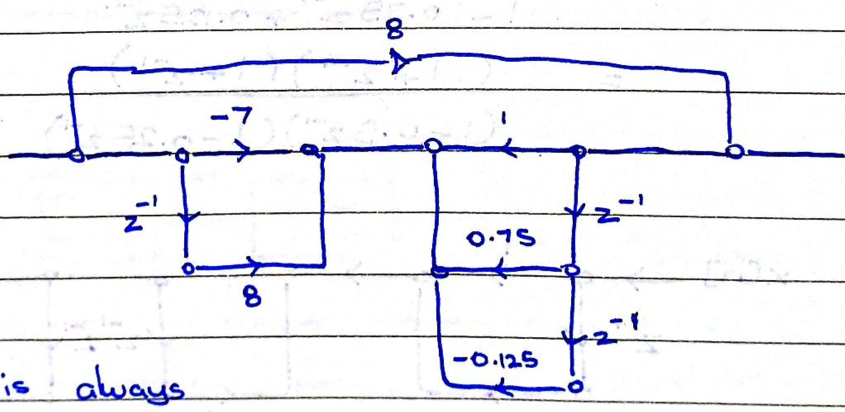
Example

$$H(z) = 8 + \frac{18}{1-0.5z^{-1}} - \frac{25}{1-0.25z^{-1}}$$



DF II

$$H(z) = 8 + \frac{-7 + 8z^{-1}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$



$\therefore$ a_0 is always $\boxed{1}$

DF I

ate

Transposed Forms

- Reverse directions of all branches
- Interchange I/O nodes (Flip)

DF II's transpose yields lower number of adders in conjunction with conserved memory

Practice Chapter 6

1. $y[n] = x[n] + 2r\cos\theta\, y[n-1] - r^2 y[n-2]$

$\xrightarrow{[ZT]} Y(z) = X(z) + 2r\cos\theta\, z^{-1} Y(z) - r^2 z^{-2} Y(z)$

$= Y(z)[1 - z^{-1} 2r\cos\theta + z^{-2} r^2] = X(z)$

$\Rightarrow H(z) = \dfrac{1}{1 - z^{-1} 2r\cos\theta + z^{-2} r^2}$

2. $w(n) = w[n-1]\, r\cos\theta + x[n] - r\sin\theta\, y[n-1]$

$y[n] = w[n-1]\, r\sin\theta + y[n-1]\, r\cos\theta$

$\Rightarrow Y(z) = W(z) z^{-1} r\sin\theta + Y(z) z^{-1} r\cos\theta$

$\Rightarrow W(z) = W(z) z^{-1} r\cos\theta + X(z) - Y(z) z^{-1} r\sin\theta$

$\rightarrow Y(z) = \dfrac{W(z)\, z^{-1} r\sin\theta}{1 - z^{-1} r\cos\theta}$

$\rightarrow W(z) = W(z) z^{-1} r\cos\theta + X(z) - \dfrac{W(z)(z^{-1} r\sin\theta)^2}{1 - z^{-1} r\cos\theta}$

EXCELLENT

day/date

$$\Rightarrow \quad W(z) = X(z)\left[1 - z^{-1}r\cos\theta + \frac{(z^{-1}r\sin\theta)^2}{1 - z^{-1}r\cos\theta}\right]^{-1}$$

$$\Rightarrow \quad Y(z) = X(z)\,\frac{z^{-1}r\sin\theta}{1-z^{-1}r\cos\theta}\;\frac{1}{1 - z^{-1}r\cos\theta + \frac{(z^{-1}r\sin\theta)^2}{1-z^{-1}r\cos\theta}}$$

$$= X(z)\,\frac{z^{-1}r\sin\theta}{1 - z^{-1}r\cos\theta - 1 + (z^{-1}r\cos\theta)^2 \ldots + (z^{-1}r\sin\theta)^2}$$

$$\Rightarrow \quad \boxed{H(z) = \frac{r\sin\theta\, z^{-1}}{1 - 2\cos\theta\, z^{-1} + r^2 z^{-2}}}$$

Something went wrong :(

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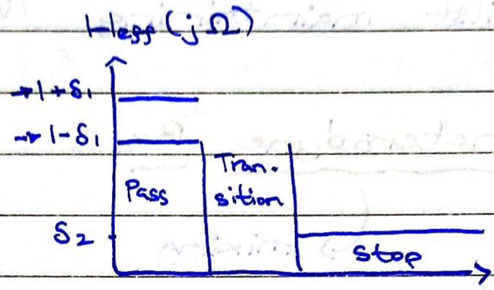
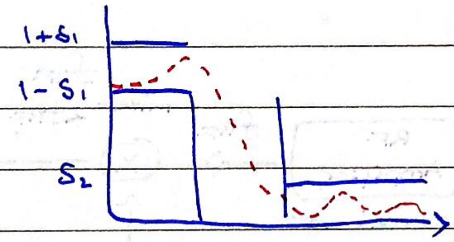
18/04/23

Digital Signal Processing

$$H(e^{j\omega}) = H_c(j\omega/T)$$

Filter Design Techniques





→ Buttersworth

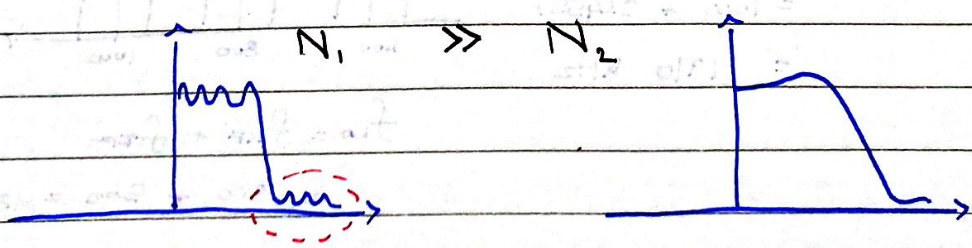
LPF

$$|H_c(j\Omega)|^2 = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}}$$

$$|H_a(s)|^2 = \frac{1}{1+(s/j\Omega_c)^{2N}}$$

Gain equations

Not Transfer functions



Higher N:

Sharper cutoff

More ripples (Gibbs phen.)

Vice Versa

day/date

$$S_k = 1 + \left(\frac{s}{j\Omega_c}\right)^{2N} = 0$$

$$\Rightarrow \left(\frac{s}{j\Omega_c}\right)^{2N} = -1$$

$$\Rightarrow \left(\frac{s}{j\Omega_c}\right) = (-1)^{1/2N}$$

$$\Rightarrow S_k = (-1)^{1/2N} j\Omega_c \quad , k = 1, 2 \dots N$$

$$\because j = e^{j(\pi/2 + k2\pi)} \qquad \because -1 = e^{j(\pi + k2\pi)}$$

$$\Rightarrow S_k = \Omega_c e^{(j\pi/2N)(2k + N - 1)} \quad , k = 1, 2 \dots N$$

Impulse Invariance

$$\because \underbrace{\omega}_{\text{rad}} = \overbrace{\Omega}^{\text{rad/s}} T_d$$

$$h[n] = h_c(t)\big|_{t = nT_d}$$

$$h[n] = T_d h_c(nT_d)$$

↳ To cancel T_d later

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k\right)$$

$\Rightarrow$ Band limited : $H_c(j\Omega) = 0 \quad ; |\Omega| \geq \pi/T_d$

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right) \quad ; |\omega| \leq \pi$$

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \xleftrightarrow{\text{Laplace T. Pair}} h_c(t) = \sum_{k=1}^{N} A_k e^{s_k t} \quad ; t \geq 0$$

$$h_c(t) = 0 \quad ; t < 0$$

EXCELLENT

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Substituting $h_c(t)$;

$$h[n] = h_c(nT_d)\,T_d = \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n]$$

$$= \sum_{k=1}^{N} T_d A_k \left(e^{s_k T_d}\right)^n u[n]$$

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

$\rightarrow$ $|e^{s_k T_d} z^{-1}| < 1$

$\rightarrow$ $|z| > |e^{s_k T_d}|$

$s = s_k$ transforms into pole at $e^{s_k T_d}$

EXCELLENT

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Digital Signal Processing

$$|H_c(s)|^2 = H_c(s)\,H_c(-s) = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}}$$

lowpass butterworth

filter order

→ Impulse Invariance is more or less just sampling

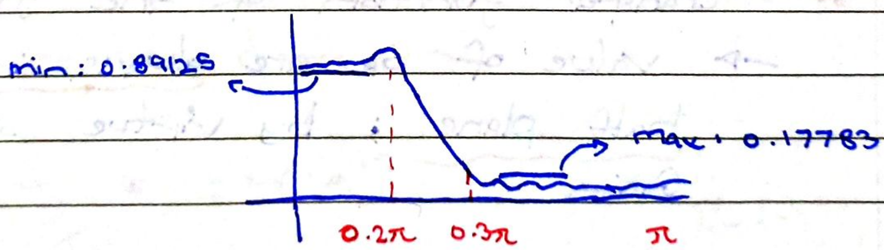
Example

pass band $\begin{cases} 0.89125 \le |H(e^{jw})| \le 1 \\ 0 \le w \le .2\pi \end{cases}$

stopband $\begin{cases} |H(e^{jw})| \le 0.17783 \\ 0.3\pi \le w \le \pi \end{cases}$

⇒ Convert specifications to analog/continuous domain ; With $T_d = 1$, $w = \Omega$

↳ Butterworth $1/1+(j\Omega/j\Omega_c)^{2N}$ cannot exceed 1



$|H_c(0.2\pi j)| \ge 0.89125$

$|H_c(0.3\pi j)| \le 0.17783$

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$$s = \sigma + j\omega \big|_{\sigma=0} = j\omega$$

→ Passband Equation:

$$(0.89125)^2 = \frac{1}{1 + (0.2\pi/\Omega_c)^{2N}}$$

→ Stopband Equation:

$$(0.17783)^2 = \frac{1}{1 + (0.3\pi/\Omega_c)^{2N}}$$

Solve simultaneously

⇒ $\text{ceil}(N) = \text{ceil}(5.8858) = 6$ } filter order

⇒ $\Omega_c = 0.70474$

put it in passband equation to get Ω_c

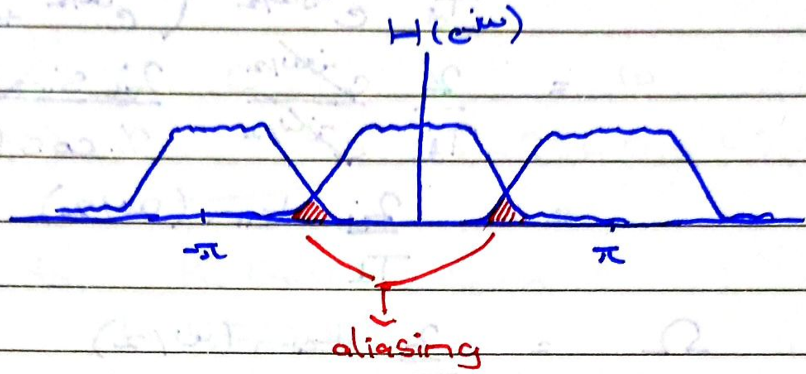
⇒ $s_k = \Omega_c \, e^{(j\pi/2N)(2k+N-1)}$ $k = 1, \ldots, N$

→ Substitute K's to construct the transfer function of the filter

→ value of s_k are always in the left half plane; by virtue of Butterworth Gain

$$H(s) = \prod_{k=1}^{N} \frac{s_k}{s_k - s}$$ } for Butterworth LPF

$H(z)$ would be aliased when $H(s)$ is found through impulse invariance.
$\hookrightarrow$ because we have lost control of T



Bilinear Transformation

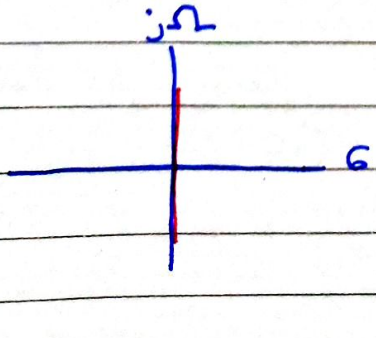
$$t \longrightarrow n$$
$$(\Omega)\ \text{CTFT} \longrightarrow \text{DTFT}\ (\omega)$$
$$(s)\ \text{LT} \longrightarrow \text{ZT}\ (z)$$

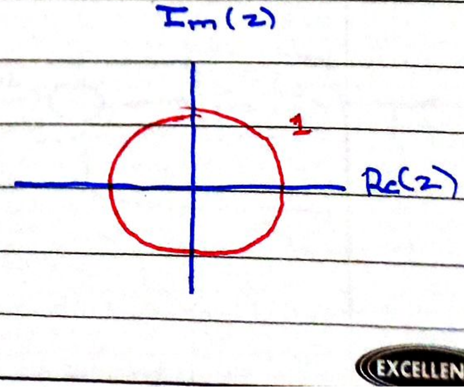
$$\Rightarrow\ s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \qquad \Rightarrow\ z = \frac{1+(T_d/2)s}{1-(T_d/2)s}$$

$$\Rightarrow\ z = \frac{1 + \sigma T_d/2 + j\Omega T_d/2}{1 - \sigma T_d/2 - j\Omega T_d/2} \quad \Bigg|\ s = \sigma + j\Omega$$

Stability in s-domain is carried over to z-domain

$$s = \overset{0}{\cancel{\sigma}} + j\omega = z = \overset{1}{\cancel{r}}\, e^{j\omega}$$





$$j\Omega = \frac{2}{T_d}\left(\frac{1 - e^{-jw}}{1 + e^{-jw}}\right)$$

$$= \frac{2}{T_d}\,\frac{e^{-jw/2}}{e^{-jw/2}}\left(\frac{e^{jw/2} - e^{-jw/2}}{e^{jw/2} + e^{-jw/2}}\right)$$

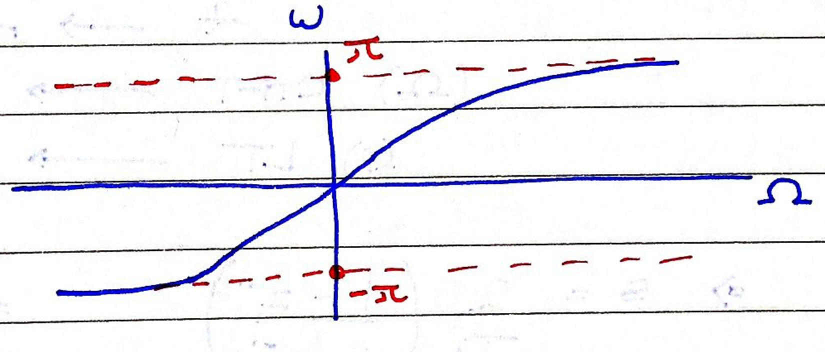
$$= \frac{\cancel{2}}{T_d}\,\frac{\cancel{e^{-jw/2}}}{\cancel{e^{-jw/2}}}\,\frac{2j\sin(w/2)}{\cancel{2}\cos(w/2)}$$

$$= \frac{2j}{T_d}\tan(w/2)$$

$$\Omega = \frac{2}{T_d}\tan(w/2)$$

$$w = 2\tan^{-1}\left(\frac{\Omega T_d}{2}\right)$$

Plot



Digital Signal Processing

IIR : Infinite Impulse Response

$h(t) = -\infty \longleftrightarrow \infty$ Same for $h[n]$

$H(s) = \dfrac{1}{(\ldots\ldots)}$ → Poles must be present for IIR; Zeros may or may not exist

FIR : Finite Impulse Response

$h(t) = T_1 \leq t \leq T_2$

$h[n] = N_1 \leq n \leq N_2$

$H(s) = (\ldots\ldots)$ → Only zeros exist
No poles

Filter Design by Windowing

$$H_d(e^{jw}) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-jwn}$$

$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{jw}) e^{jwn} \, dw$$

Ideal Case : We truncate

$$h[n] = \begin{cases} h_d[n] & 0 \leq n \leq M \\ 0 & \text{else} \end{cases}$$

OR $h[n] = h_d[n] \, w[n]$ where $w = 1$ for $0 \rightarrow M$ and 0 else

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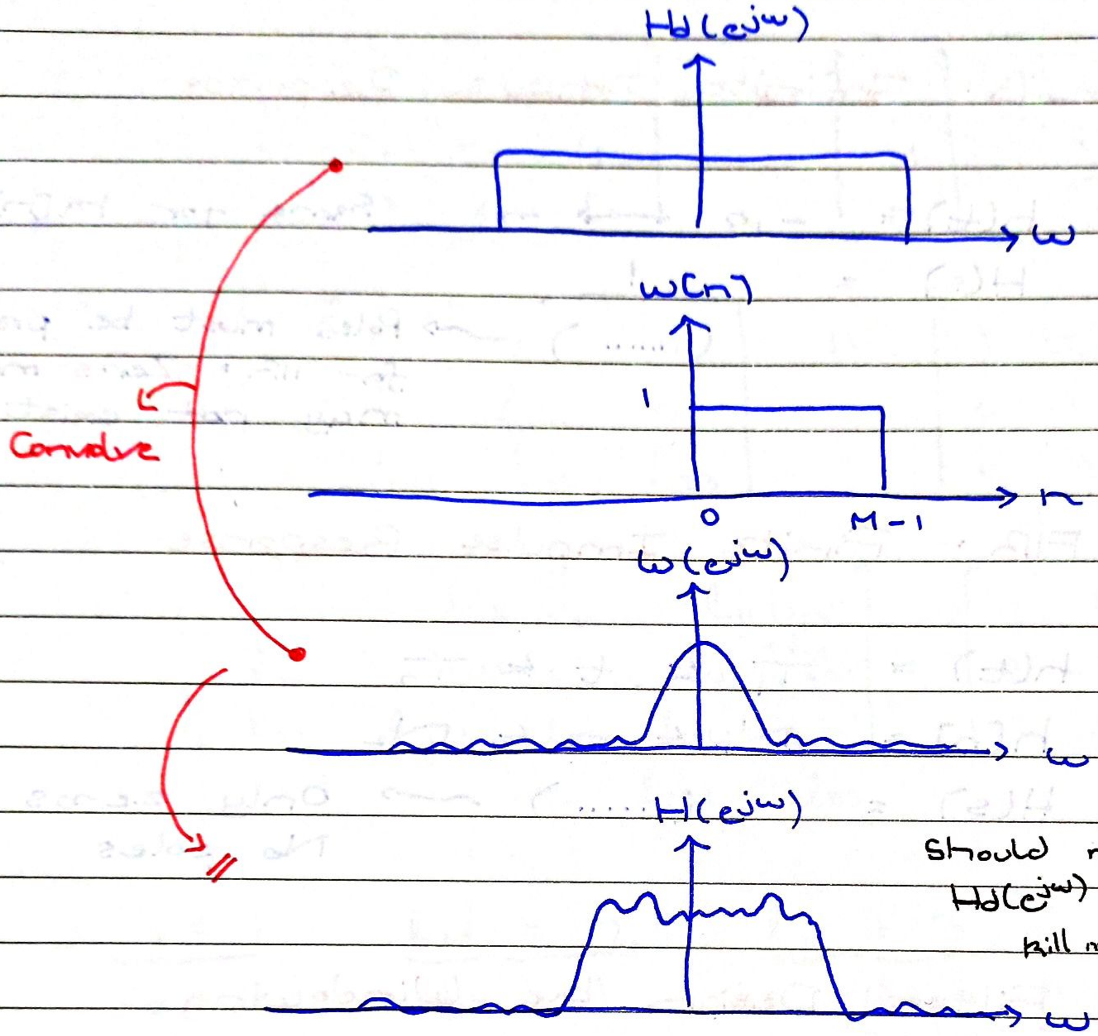
Assume $H_d(e^{jw})$ (Ideal Response) as:



Should match $H_d(e^{jw})$ OOPSIE! kill me :(

=> Sinc in its ideal form is an impulse.

=> When you chop in time domain sharply, $\underset{\downarrow \infty}{\Delta f} = \dfrac{1}{\Delta t \to 0}$, there occurs spectral leakage in frequency domain.

Rectangular Window

$w[n] = 1 \quad 0 \leq n \leq M-1$

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$$W(e^{j\omega}) = \sum_{n=-\infty}^{\infty} w[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{M-1} w[n] e^{-j\omega n} \quad \therefore w[n] = 1$$

$$= \sum_{n=0}^{M-1} e^{-j\omega n}$$

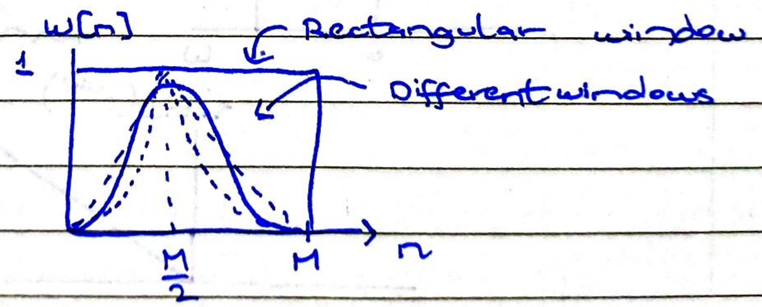
$$= \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\omega M/2} \left(e^{j\omega M/2} - e^{-j\omega M/2} \right)}{e^{-j\omega/2} \left(e^{j\omega/2} - e^{-j\omega/2} \right)}$$

$$= e^{-j\omega (M-1/2)} \frac{\sin[\omega M/2]}{\sin[\omega/2]}$$

Plot its magnitude: You'll get a Sinc, albeit an imperfect one

Rectangular window is not viable due to high spectral leakages



Hamming : Speech signals

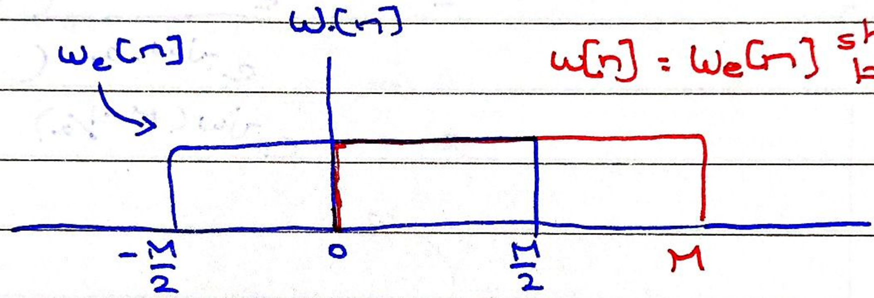
⋮ et cetera

Incorporation of Generalized Linear Phase

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & \end{cases}$$

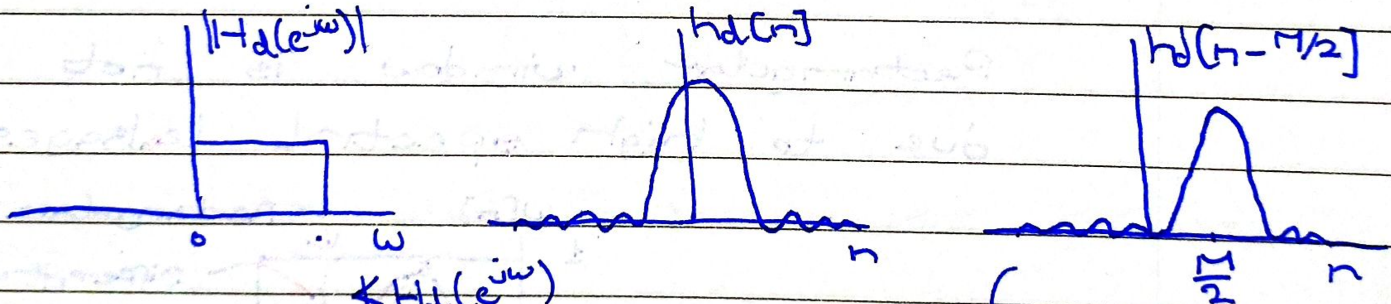
Symmetric about $M/2$

$$W(e^{j\omega}) = W_e(e^{j\omega})\, e^{-j\omega M/2}$$



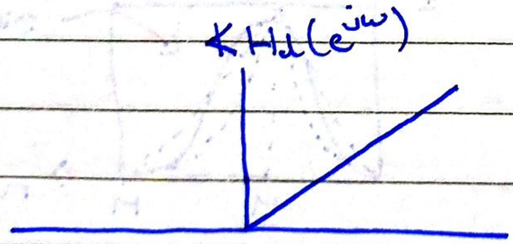
$w[n] = W_e[n]$ shifted by $M/2$

Shift $h_d[n]$ by $M/2$ as well



in f-domain

$$\boxed{H_d(e^{j\omega}) = e^{-j\omega\frac{M}{2}} H_e(e^{j\omega})}$$



[Phase Plot]

[Magnitude Plot remains same]

Subscript e implies even symmetric

Subscript d implies desired

Digital Signal Processing

Example

$$H_d(e^{jw}) = \begin{cases} 1 & -w_c \le w \le \pi \text{ or } w_c \\ 0 & \text{elsewhere} \end{cases}$$

if it was π, we wald get a perfect sinc

$$h_d(n) = \frac{\sin(w_c n)}{\pi n}$$

$$H_d(e^{jw}) = \begin{cases} e^{-jwM/2} & -w_c \le w \le w_c \\ 0 & \text{else} \end{cases}$$

$$h_d[n] = h_e(n - M/2]$$

$$h[n] = h_e[n]\, w[n] \quad \text{[wrong]}$$

$$h[n] = h_d[n]\, w[n] \quad \text{[Use shifted instead]}$$

$$h_d[n] = \frac{\sin(w_c(n - M/2))}{\pi(n - M/2)}$$

Kaiser Window Design Method

$$w[n] = \begin{cases} \dfrac{I_0\left[\beta\sqrt{1-((n-M/2)/M/2)^2}\right]}{I_0(\beta)} & 0 \le n \le M \\ 0 & \text{elsewhere} \end{cases}$$

where I_0 is zeroth order Bessel function of 1^{st} kind

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09/05/23

Digital Signal Processing

Kaiser Window Filter

$$w[n] = \begin{cases} \dfrac{I_0\left[\beta\sqrt{1-\left(\frac{n-M/2}{M/2}\right)^2}\right]}{I_0(\beta)}, & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

Example

$$\left\{ \omega_p = 0.4\pi \quad \omega_s = 0.6\pi \quad \begin{array}{l} \delta_1 = 0.01 \\ \delta_2 = 0.001 \end{array} \right\}$$

$\Delta\omega = \omega_s - \omega_p = 0.2\pi$

$\delta_1 = \delta_2 = 0.001$

If ω_c is not given, take mean of ω_s and ω_p » $\omega_c = 0.5\pi$

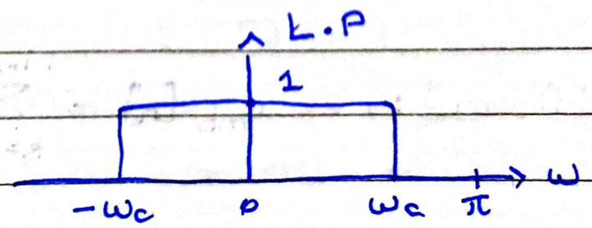
Using Kaiser window Parameters

$\{ M = 37 \; ; \; \beta = 5.653$

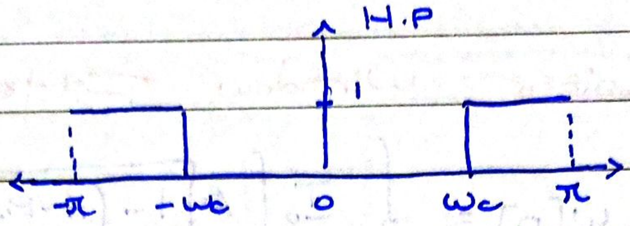
$$» \; h[n] = \begin{cases} \dfrac{\sin(0.5\pi(n-37/2))}{\pi(n-37/2)} \cdot \dfrac{I_0\left[5.653\sqrt{1-\left(\frac{n-37/2}{37/2}\right)^2}\right]}{I_0(5.653)} & 0 \le n \le 37 \\ 0 & \text{else} \end{cases}$$

General Frequency Selective Filter

Ideal Lowpass



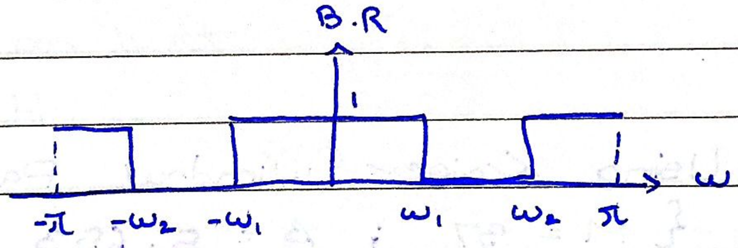
High Pass



- π is the maximum frequency (in order to satisfy Nyquist criteria)
- in Ω domain, we have control of T and hence aliasing

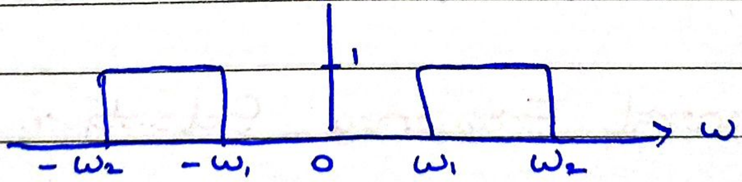
$$HP = 1 - LP$$

Band Reject



$$BR = (1 - LP_{\omega_2}) + LP_{\omega_1}$$

Band Pass



$$BP = 1 - BR_{(\omega_1, \omega_2)}$$

Multiband: $$h_{mb}[n] = \sum_{k=1}^{N_{mb}} (G_k - G_{k+1}) \frac{\sin(\omega_k(n - \frac{M}{2}))}{\pi(n - M/2)}$$

$$h_{mb}[n] = (G_1 - G_2)\frac{\sin(\omega_1(n-M/2))}{\pi(n-M/2)}$$

$$+ (G_2 - G_3)\frac{\sin(\omega_2(n-M/2))}{\pi(n-M/2)}$$

$$+ (G_3 - G_4)\frac{\sin(\omega_3(n-M/2))}{\pi(n-M/2)}$$

$$+ G_4 \frac{\sin(\pi(n-M/2))}{\pi(n-M/2)}$$

Expanding ~~the sum~~ Σ

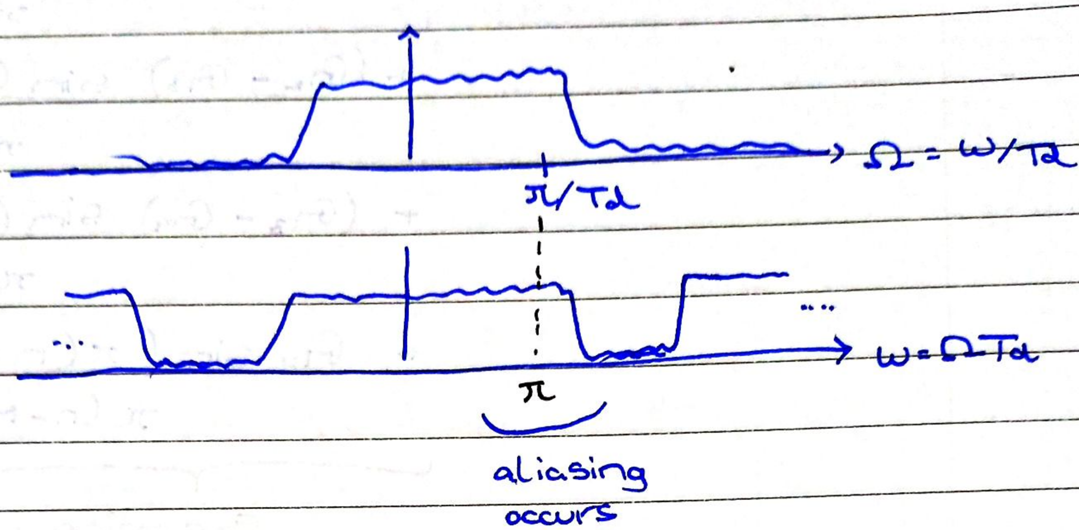
Practice

- $h[n] = h_d[n]\,\omega[n]$

1. Inverse relation : Higher the order ; Sharper the cutoff
2. 3. Done on the previous page
4. Analog : CTFT , Laplace
   Discrete : DTFT , Z - Transform
5. $|H(s)|^2 = 1/(1+(s/j\Omega_c)^{2N})$
   when $\sigma = 0$ in $s = \sigma + j\Omega$
   system is marginally stable
6. $\Omega = \omega / T_d$, $T_d = 1$
   When you simultaneously solve gain equations, T_d cancels out

ate

⑦ T_d is now out of our control and sampling frequency F_s



⑧ 1) $H(s) = \dfrac{1 + s^2 + 3s^3}{1 + s^2}$

2) $H(s) = \prod_{k=1}^{N} \dfrac{(\text{Same order})}{(\text{Same order})}$

3) $H(s) = \sum_{k=1}^{N} \dfrac{A_k}{(s - s_k)}$ — Partial Fraction Expansion (simplest ; by inspection)

⑨ $\omega = 2 \tan^{-1}\left(\dfrac{\Omega T_d}{2}\right)$ Yes ; non-linear sampling as a relation b/w ω and Ω is still being utilized

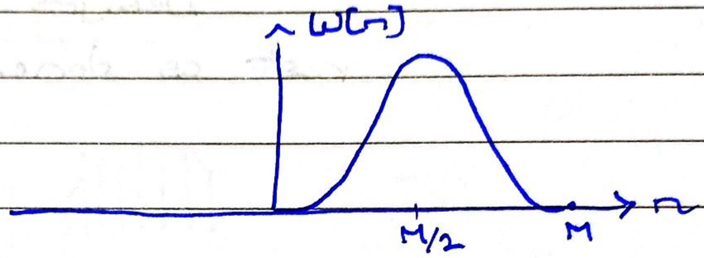
⑩ Formulas will be given

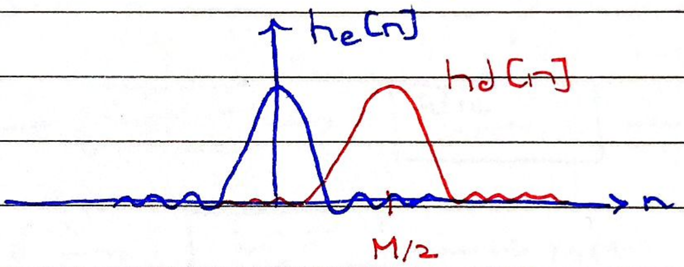
⑪ $h[n] = h_d[n]\, w[n]$
FIR IIR Window
Computationally efficient but loss of information

⑫ Aim of window is to truncate $h_d[n]$ and $[0 - M]$ ensures causality.

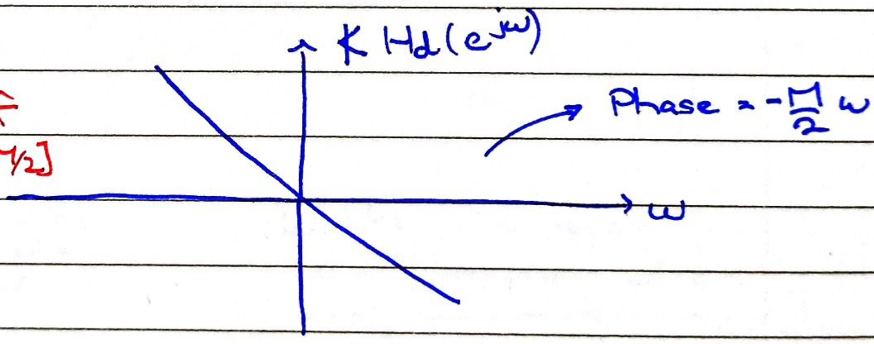
(13) If there wasn't even symmetry, IFFT would yield $h[n]$ along with e^{-jx} where x is the shift from the zero line. This exponential term is responsible for phase shift. Hence, we draw spectrums as even symmetric.

(14)





Magnitude Plot remains same of $h_d[n] = h_e[n - M/2]$



If a non-linear phase was present, we would not be able to predict and mitigate the error in time shifts.

Problems:

1. $\omega_p = 0.4\pi \qquad \omega_s = 0.6\pi$

$$\left.\begin{array}{l} 0.99 \le |H(e^{j\omega})| \le 1.01 \\ |H(e^{j\omega})| \le 0.001 \end{array}\right]$$

EXCELLENT

→ Divide by 1.01

Passband $\left\{ \frac{0.99}{1.01} \leq \left| \frac{H(e^{j\omega})}{1.01} \right| \leq 1 \right.$

Stopband $\left\{ \left| \frac{H(e^{j\omega})}{1.01} \right| \leq \frac{0.001}{1.01} \right.$

Design the rest as done in class

day/date

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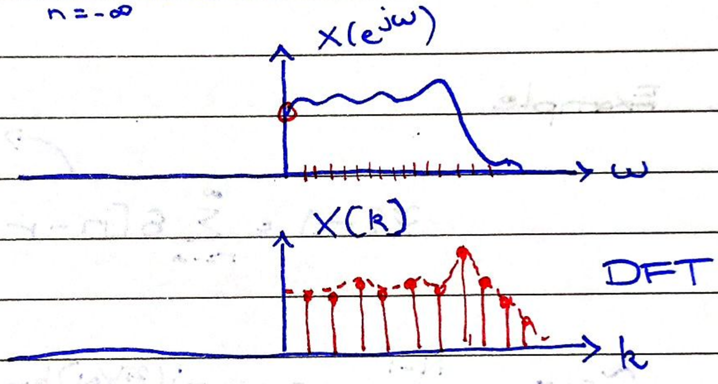
Digital Signal Processing

Discrete Fourier Transform

└ is a "shape" of DTFT
$\underbrace{\text{shape}}_{\text{sampled}}$

└ analogous to a discrete signal being a "shape" of continuous signal

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$



Discretizing ω-axis

$$\tilde{x}[n] = \tilde{x}[n + rN]$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k} \tilde{X}[k]\, e^{j(2\pi/N)kn}$$

For periodic signals

Due to periodicity of exponential; we only need N exponentials

$$\boxed{\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k]\, e^{j(2\pi/N)kn}}$$

$$\boxed{\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j(2\pi/N)kn}}$$

Discrete Fourier Series Pair

EXCELLENT

y/date

For convenience, we use :

$$W_N = e^{-j(2\pi/N)}$$

$$\Rightarrow \tilde{X}[k] = \sum_{k=0}^{N-1} \tilde{x}[n] \, W_N^{kn}$$

$$\Rightarrow \tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \, W_N^{-kn}$$

Synthesis & Analysis Eqns

→ Both $\tilde{x}$ and $\tilde{X}$ are periodic

Example

which is ~~or~~ just $\delta[n]$

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \begin{cases} 1 & n = rN \\ 0 & \text{else} \end{cases}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n] \, e^{-j(2\pi/N)kn} = e^{-j(2\pi/N)k(0)} = 1$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(2\pi/N)kn}$$

Example (Discrete Pulses)

$$N = 10 \Rightarrow \tilde{X}[k] = \sum_{k=0}^{4} e^{-j(2\pi/10)kn}$$

$$= \frac{1 - e^{-j(2\pi/10)k(5)}}{1 - e^{-j(2\pi/10)k}} = e^{-j\left(\frac{4\pi k}{10}\right)} \frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$

In its mag and phase plots, when mag = 0 ⇒ phase is indeterminate

16/05/23

Digital Signal Processing

Discrete Fourier Series

Some properties:

$$\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$$
$$\tilde{x}[n-m] \longleftrightarrow e^{-j2\pi/N\, km}\,\tilde{X}[k]$$
$$e^{j2\pi/N\, nm}\,\tilde{x}[n] \longleftrightarrow \tilde{X}[k-m]$$

Periodic Convolution

N superscript will/is representing period

$$\tilde{x}_1[n],\ \tilde{x}_2[n]$$
$$\tilde{x}_3[n]^{N} = \tilde{x}_1[n]^{N} * \tilde{x}_2[n]^{N} = \sum_{m=0}^{N-1} x_1[m]\,x_2[n-m]$$
$$\tilde{X}_3[k] = X_1[k]\,X_2[k]$$

Result will be periodic as well

$$\overset{M}{\tilde{x}_3[n]} = \overset{N}{\tilde{x}_1[n]} * \overset{M}{\tilde{x}_2[n]}$$

Period of $\tilde{x}_3[n]$ → max(N, M)

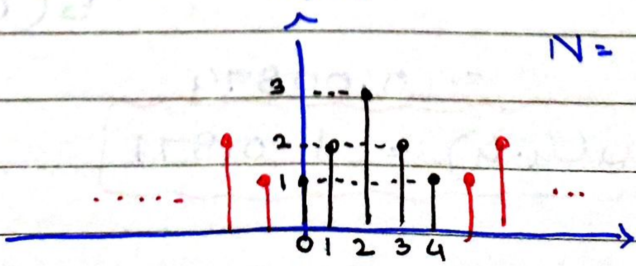
Proof of :

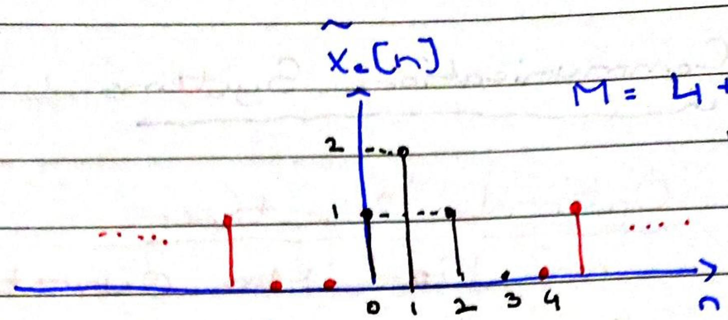
$$\tilde{x}_1[n] * \tilde{x}_2[n] = \tilde{X}_1[k]\,\tilde{X}_2[k]$$

in slides

Example

$\tilde{x}_1[n]$ $N = 5$

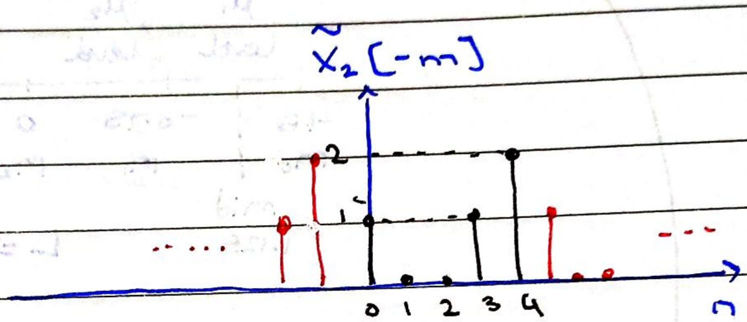




$M = 4 + \underset{\sim}{1}$

↳ pad zeros to make both periods same

swap $m = n$



Now, just compute sum of product of black colored sample and shift as $\tilde{x}_3[n] \to n$ goes on

$\tilde{x}_3[0] = 1 + 2 + 2 = 5$

$\tilde{x}_3[1] = 1 + 2 + 2 = 5$ $\rightsquigarrow \tilde{x}_2[-m+1]$

$\tilde{x}_3[2] = \sim = 8$

$\tilde{x}_3[3] = \sim = 10$

$\tilde{x}_3[4] = \sim = 8$

and then x_3 starts repeating :)

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Digital Signal Processing

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\Omega t k}\, dt$$

$$X(j\Omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\, \delta(\Omega - k\Omega_c)$$

For periodic signals (continuous)

$$\tilde{X}(e^{j\omega}) = \sum_{k=0}^{N-1} \frac{2\pi}{N} \underbrace{\tilde{X}[k]}_{\text{analogous to } a_k\text{'s}}\, \delta(\omega - 2\pi k/N)$$

Discrete Fourier Transform

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j 2\pi/N\, kn}$$

Now, assume $\tilde{x}[n]$ is not periodic;

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j 2\pi/N\, kn}$$ Implementation in computers

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$ Theoretical

Relation

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}$$

Sampling of ω $\boxed{\omega = \frac{2\pi k}{N}}$

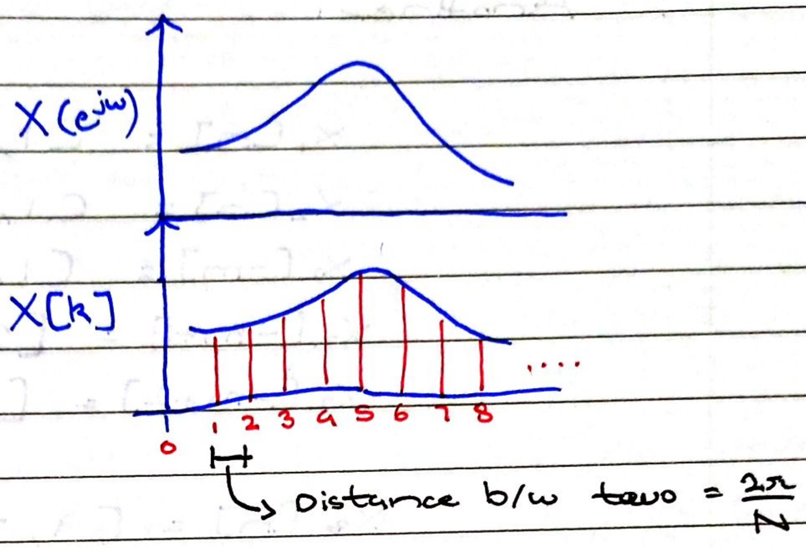
$\omega_0 = 0$

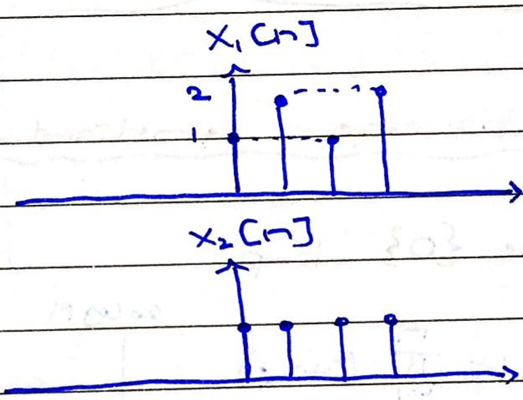
$\omega_1 = 2\pi/N$

$\omega_2 = 4\pi/N$

⋮ so on

date





$x_3[n] = x_1[n] * x_2[n]$

$X_3[k] = X_1[k]\, X_2[k]$

Circular Convolution ~ Just different terminology

$x_1[m] = [2, -3, 1, 2, 3]$

$x_2[m] = [1, 2, 3, 4, 0]$ ~ Pad zeros to make same length

$x_2[-m] = [1, 0, 4, 3, 2]$, $x_2[-m+1] = [2, 1, 0, 4, 3]$

circulating

$x_3[n] = [18, 18, 13, 3, -2]$

Another:

$x_1[n] = [1, 2, 1, 2]$

$x_2[n] = [1, 2, 3]$ — No Padding

$x_2[-m] = [1, 3, 2]$

$x_2[-m+1] = [2, 1, 3]$

$x_2[-m+2] = [3, 2, 1]$

$x_3[n] = [9, 7, 8]$ Due to absence of padding, information is lost

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Digital Signal Processing

$$X[k] = \sum_{n=0}^{4} x[n]\, e^{-j 2\pi k\, n/N}$$

$$= \sum_{n=0}^{4} e^{-j2\pi k \frac{n}{10}}$$

$$= \frac{1 - e^{-j\pi k}}{1 - e^{-j2\pi k/10}}$$

$$= \frac{e^{-j\pi k/2}}{e^{-j\pi k/10}} \left(\frac{\sin(\pi k/2)}{\sin(\pi k/10)} \right)$$

N represents resolution for DFT

Computers don't use nested loops
└ DFT > Multiplication > IDFT

DFT Multiplication's Inverse yields
CIRCULAR CONVOLUTION

DTFT Multiplication's Inverse yields
LINEAR CONVOLUTION

Convolution Lengths

$$\underbrace{x_3[n]}_{N+M-1} = \underbrace{x_1[n]}_{N} \underbrace{*}_{\text{linear}} \underbrace{x_2[n]}_{M}$$

$$\underbrace{x_3[n]}_{\max(N,M)} = \underbrace{x_1[n]}_{N} \underbrace{\circledast}_{\text{circular}} \underbrace{x_2[n]}_{M}$$

EXCELLENT

date

Assume $N > M$

$x_1[n]$ N

$x_2[n]$ M

Pad zeros to make:

$x_1[n]$ $N + M - 1$

$x_2[n]$ $M + N - 1$

Now perform circular convolution but the output will be equivalent to linear convolution

Using DFT

Mult

IDFT

If question says: → SIGNALS

given h[n] and x[n], compute

$y[n] = x[n] \circledast h[n]$

If question says: → SYSTEMS (LTI)

given h[n] and x[n], compute

$y[n] = x[n] \circledast h[n]$

↳ after padding zeros to make length equal to $N + M - 1$

Final result : $y[n] = x[n] * h[n]$