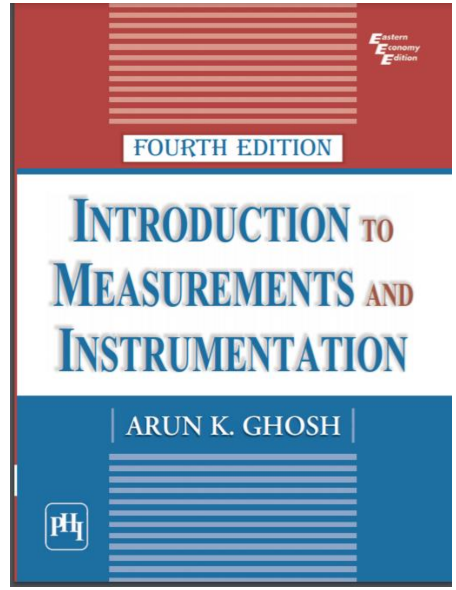
NUST
SCHOOL OF ELECTRICAL ENGINEERING
& COMPUTER SCIENCE

- Course: **EE383 Instrumentation and Measurements**
- Session: Fall 2022
- **Lectures: Week 6**
- Course Instructor: Dr. Shahzad Younis



Week 6

- **Chapter 3**

  Estimation of Static Errors & Reliability

- **Chapter 4**

  Dynamic Characteristics of Instruments

Week 6

- **Chapter 3**

  Estimation of Static Errors & Reliability

- **Chapter 4**

  Dynamic Characteristics of Instruments

Contents

- 3.1 → Definition of parameters
- 3.2 → Limiting Error
- 3.3 → Statistical Treatment
- 3.4 → Error Estimates from the normal(Gaussian) distribution
- 3.7 → Reliability Test

Reliability Principles

Three terms are defined in the context of reliability of measurement systems.

→Reliability

→Unreliability

→Mean Failure Rate

Reliability Principles

Reliability

Reliability *R (t)* of ***measurement system*** is defined as the probability that it will operate

- →To an agreed level of performance
- →For specified period of time
- →Under specified conditions
- →When used for the manner and purpose for which it is intended.

Example

Suppose, the agreed level of performance of a voltmeter is an accuracy of $\pm 2\%$ and the warranty period is 1 year. It should not be used above 40° and its maximum input should be 220 V. As long as the instrument is used in the specified conditions and it gives readings within this specified error limits, we consider it reliable. If it does not, although the system may be otherwise all right, it will be considered to have failed.

Because reliability is defined as a probability, its value always lies between 0 and 1. Quantitatively, reliability is defined as

$$R(t) = \Pr\{0 \text{ failures in } [0,t] | \text{ no failure at } t = 0\}$$

Reliability Principles

Reliability

n_0: total number of elements

n_s: number of elements working correctly at time t

n_f: number of elements that have failed after time t

$$R(t) = \frac{n_s}{n_0} = \frac{n_s}{n_s + n_f} \tag{3.19}$$

We have denoted $R(t)$ as a function of time because it is a common experience that an instrument that has just been checked and calibrated has $R(t) = 1$, but after a lapse of, say, six months it might have a reliability of 0.9. It is also a common experience that the reliability of a system always decreases with time.

Reliability of system always decreases with time

Reliability Principles

Unreliability

Is the complement of reliability

<reliability>

$$\boxed{R(t) = \frac{n_s}{n_0} = \frac{n_s}{n_s + n_f}}$$

<unreliability>

$$\boxed{F(t) = \frac{n_f}{n_0} = \frac{n_f}{n_s + n_f}}$$

Since an element or system can only have failed or not failed, the total probability, that is, the sum of reliability and unreliability, must be unity. Thus,

$$R(t) + F(t) = 1$$

Unreliability of system always increases with time

Reliability Principles

Mean Failure Rate

The mean failure rate λ is average of faults per device per unit time.

λ = Average Faults/device/unit time

Commonly expressed in failures per year or failure per million hour (FPMH).

Example

television has a failure rate of five per million hours, it means that one can watch one million one hour-long television shows and may experience a failure during only five shows.

Reliability Principles

Measurement systems are of two types:

- **Unrepairable Systems**
  - **Mean time to failure (MTTF)**
- **Repairable Systems**
  - **Availability**
  - **Unavailability**

Reliability Principles

Unrepairable systems are discarded once a fault occurs

→Artificial Satellites

→Missiles

→Microprocessor
→Hard disc drives, etc.

Reliability Principles

Unrepairable systems

Mean time to failure (MTTF)

Failure rates are calculated from MTTF data. Suppose, n_0 number of new devices comprise an unrepairable system. The devices and their conditions are identical and they are allowed to operate until they fail. The time taken for each device to fail is noted and once it fails, it is taken out of service. The average of these times, when all n_0 devices have failed, is the MTTF. Thus, if we say the total survival time or 'up time' for the i-th failure is T_i, then

$$\text{MTTF} = \frac{\text{Total up time}}{\text{Total number of failures}} = \frac{\sum_{i=1}^{n_0} T_i}{n_0} \tag{3.20}$$

Obviously then,

$$\lambda = \frac{\text{Total number of failures}}{\text{Total up time}} = \frac{n_0}{\sum_{i=1}^{n_0} T_i} = \frac{1}{\text{MTTF}} \tag{3.21}$$

If the units of λ are number of failures per hour, then those of MTTF are hours.

Reliability Principles

Unrepairable systems

Mean time to failure (MTTF)

We observe that at time $t = 0$, n_0 devices survive and at $t = T$, 0 devices survive. So, from Eq. (3.19) the reliability $R(i)$ is

$$R(i) = \frac{\text{Number of devices survived at } t = T_i}{\text{Total number of devices at } t = 0} = \frac{n_0 - i}{n_0} \tag{3.22}$$

From Eq. (3.22) we can construct a survival table as shown in Table 3.2. The $R(i)$ vs. t plot

Table 3.2 Survival table of devices

Time (t)	*Number of survivors*	*Reliability* $[R(i)]$
0	n_0	1
T_1	$n_0 - 1$	$\frac{n_0 - 1}{n_0}$
$\vdots$	$\vdots$	$\vdots$
T_i	$n_0 - i$	$\frac{n_0 - i}{n_0}$
$\vdots$	$\vdots$	$\vdots$
T	0	0

looks like Fig. 3.12 where rectangles have heights of $1/n_0$.

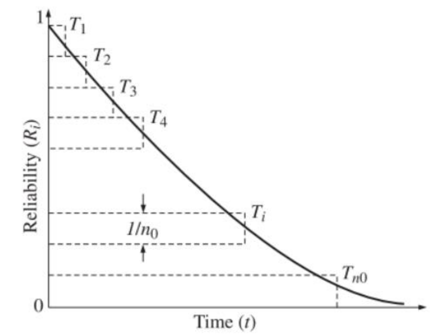


Fig. 3.12 Reliability vs. time plot for unrepairable systems.

The area under the curve is given by

$$\text{Area} = \left(T_1 \cdot \frac{1}{n_0}\right) + \left(T_2 \cdot \frac{1}{n_0}\right) + \left(T_3 \cdot \frac{1}{n_0}\right) + \cdots + \left(T_{n0} \cdot \frac{1}{n_0}\right)$$

$$= \frac{\sum_{i=1}^{n_0} T_i}{n_0} \tag{3.23}$$

Comparing Eqs. (3.20) and (3.23), we find that the area is the MTTF itself. By making $n_0 \to \infty$, we can write Eq. (3.23) as

$$\text{MTTF} = \int_0^\infty R(t)\, dt \tag{3.24}$$

Reliability $R(t)$ and failure rate λ are related. Let us consider a simple analysis here.

Reliability Principles

Repairable systems In this case, the failed components are repaired or replaced and the system is put back into service.

Useful metrics are

→Mean down time (MDT)

A particularly useful metric for repairable systems is the *mean time between failures* (MTBF). Let n_0 identical repairable systems or devices be tested over a period of time T. Once a fault occurs, it is recorded, the device repaired and put back into service. If T_i' be the down time of the i-th failure (Fig. 3.13), the total down time for n_f failures is $\sum_{i=1}^{n_f} T_i'$ and the *mean down time* (MDT) is

$$\mathrm{MDT} = \frac{\sum_{i=1}^{n_f} T_i'}{n_f}$$

Fig. 3.13 Failure pattern of repairable systems.

Reliability Principles

Repairable systems

Obviously, the total up time is then

$$\text{Total up time} = n_0 T - \sum_{i=1}^{n_f} T_i' = n_0 T - n_f(\text{MDT})$$

The MTBF is, therefore,

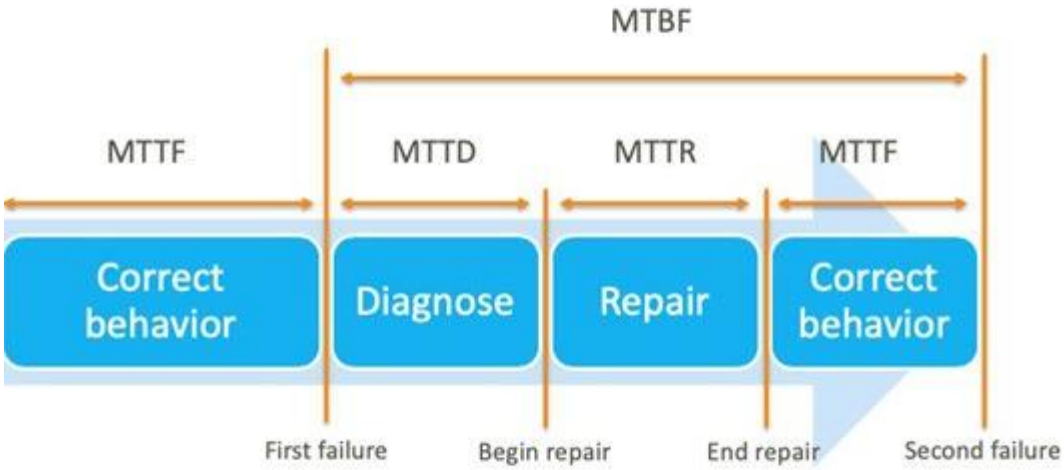
$$\text{MTBF} = \frac{\text{Total up time}}{\text{Total number of failures}} = \frac{n_0 T - n_f(\text{MDT})}{n_f} \tag{3.32}$$

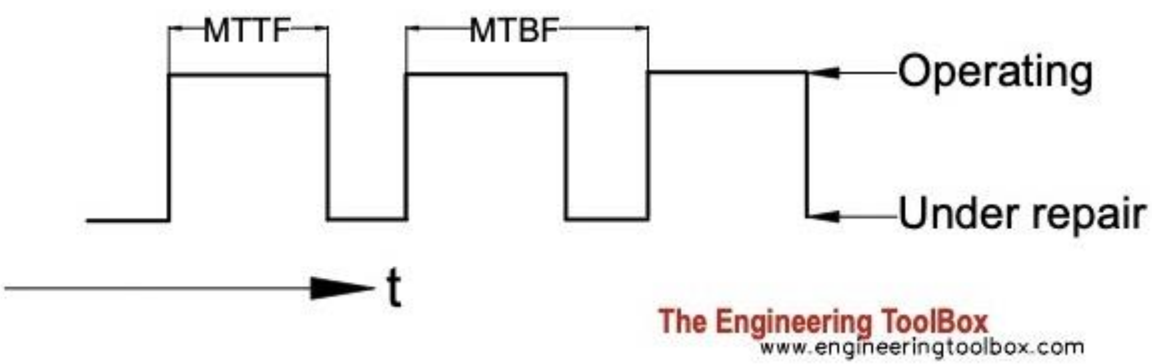
and the corresponding mean failure rate is

$$\lambda = \frac{1}{\text{MTBF}} \tag{3.33}$$

$$= \frac{n_f}{n_0 T - n_f(\text{MDT})} \tag{3.34}$$

MTTF and MTBF?





Example

$$\text{Total up time} = n_0 T - \sum_{i=1}^{n_f} T_i' = n_0 T - n_f(\text{MDT})$$

The MTBF is, therefore,

$$\text{MTBF} = \frac{\text{Total up time}}{\text{Total number of failures}} = \frac{n_0 T - n_f(\text{MDT})}{n_f} \tag{3.32}$$

Example 3.30

Calculate the MTBF and the mean failure rate if 100 faults were recorded for 300 transducers of a system during 1.5 years, the mean down time being 1 day.

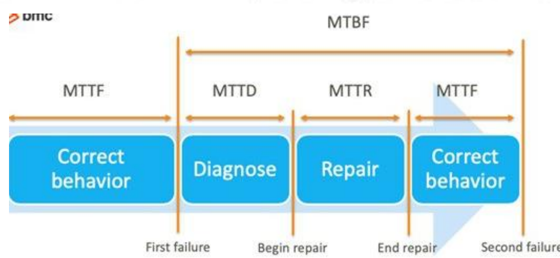
Solution

MDT = 1 day = $\frac{1}{365}$ yr. Therefore, from Eq. (3.32)

$$\text{MTBF} = \frac{(300)(1.5) - (100)(1/365)}{100} = 4.497 \text{ yrs}$$

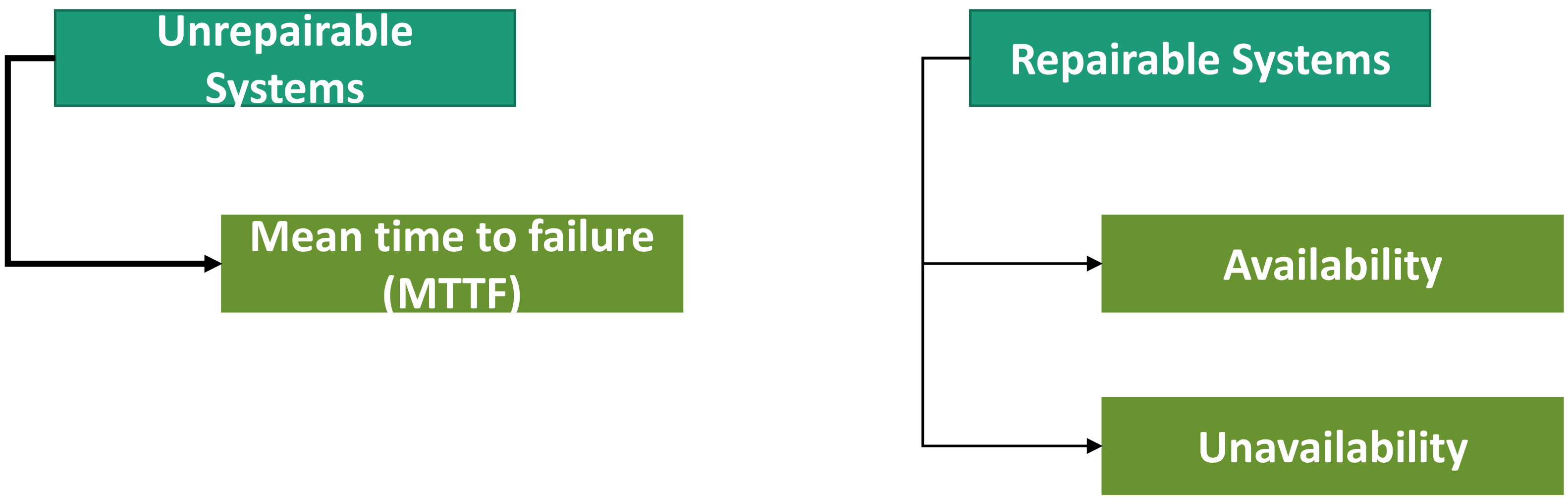
$$\text{Mean failure rate } \lambda = \frac{1}{4.497} = 0.222 \text{ per yr}$$

Note: Sometimes, the vendors mention a term *mean time to repair* (MTTR) which is lower than MDT because their MTTR is based on the assumption that a fully trained technician, complete with appropriate spares and test equipment, is ready for 24 h a day and that failed equipment is immediately available for repairs. So, the MTTR is rather an optimistic estimate while MDT, being the sum of MTTR and other delays, is more realistic.



Reliability Principles

Measurement systems are of two types:



Reliability Principles

Repairable systems

Availability

The availability A is the probability that a system will be functioning correctly when needed. In other words, it is the fraction of the total up time during a test interval. That is,

$$\begin{aligned} A &= \frac{\text{Total up time}}{\text{Test interval}} \\ &= \frac{\text{Total up time}}{\text{Total up time} + \text{Total down time}} \\ &= \frac{(n_f)(\text{MTBF})}{(n_f)(\text{MTBF}) + (n_f)(\text{MDT})} \\ &= \frac{\text{MTBF}}{\text{MTBF} + \text{MDT}} \qquad (3.35) \end{aligned}$$

Example

Example 3.30

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Solution

MDT = 1 day = $\dfrac{1}{365}$ yr. Therefore, from Eq. (3.32)

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Example 3.31

Calculate the availability of the system of Example 3.30.

Solution

In Example 3.30, we had MTBF = 4.497 yrs and MDT = (1/365) yr.
Therefore, from Eq. (3.35)

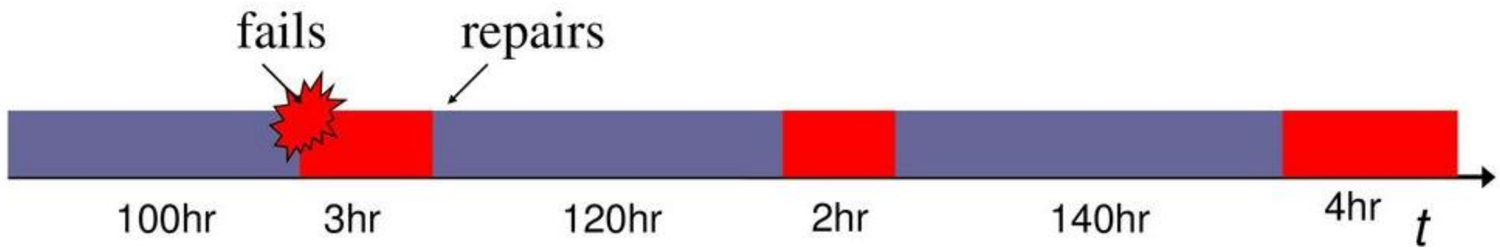
$$A = \frac{4.497}{4.497 + (1/365)} = 0.999$$

In other words, the availability is 99.9%.

Note: An availability of 99.9% is often called *three nines availability*. The one nine availability is not 9%, but 90%, two nines, 99% and five nines, 99.999%.

Concept Checks

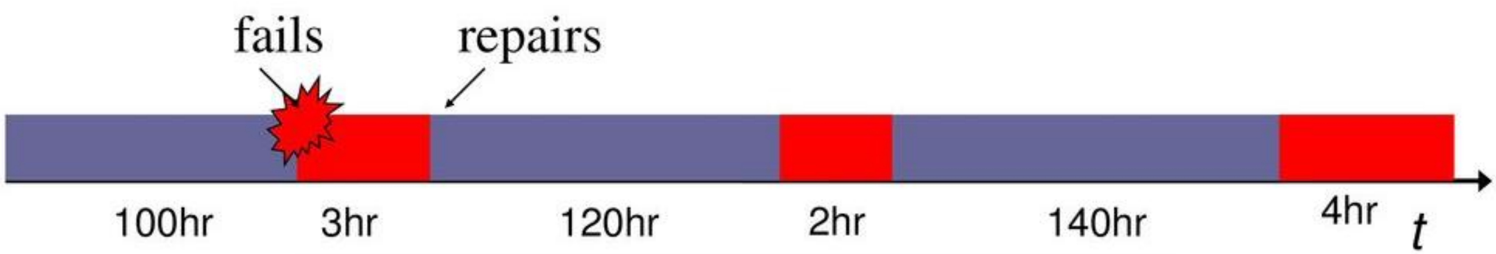
- What is Availability of this system ?



Concept Checks

Availability & MTTF & MTTR

- What is Availability of this system ?



$$A = MTTF / (MTTF+MTTR)$$

MTTF = (100 + 120 + 140) / 3 = 120 (hr/case)
MTTR = (3 + 2 + 4) / 3 = 3 (hr/case)

A = 120 / 120+3
= 120/123 = 0.975

Reliability Principles

Repairable systems

Unavailability

The unavailability U is the complement of availability. Since these are probabilities,

$$U = 1 - A$$
$$= \frac{\text{MDT}}{\text{MTBF} + \text{MDT}}$$

Substituting $1/\lambda$ for MTBF [see Eq. (3.33)], we get

the corresponding mean failure rate is

$$\lambda = \frac{1}{\text{MTBF}} \tag{3.33}$$

$$U = \frac{\lambda(\text{MDT})}{1 + \lambda(\text{MDT})} \approx \lambda(\text{MDT})$$

assuming $\lambda(\text{MDT}) \ll 1$.

Hazard rate

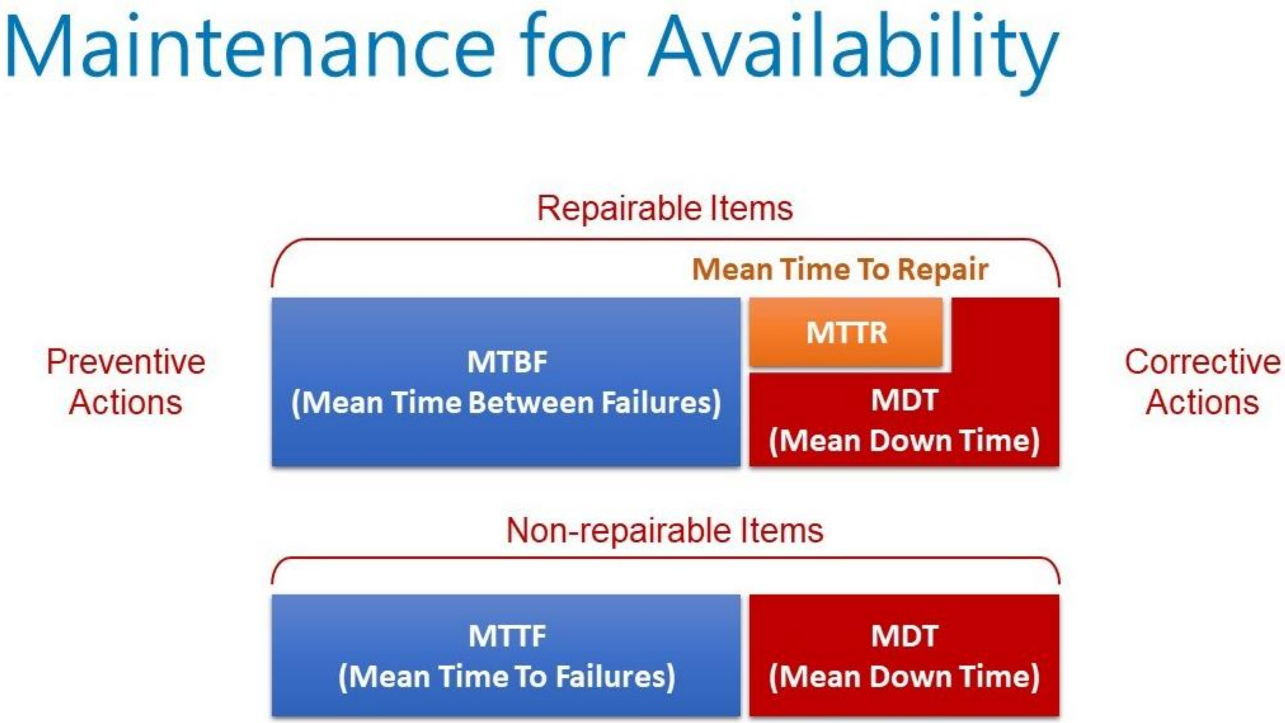
Often a *hazard rate* or *instantaneous failure rate* is defined if λ is not constant. Written as $\lambda(t)$, it is defined as

$$\lambda(t) = \frac{\text{Failure probability}}{\Delta t} = \frac{\Delta n_f}{n_0 \Delta t}$$

where Δt is a small span of time.

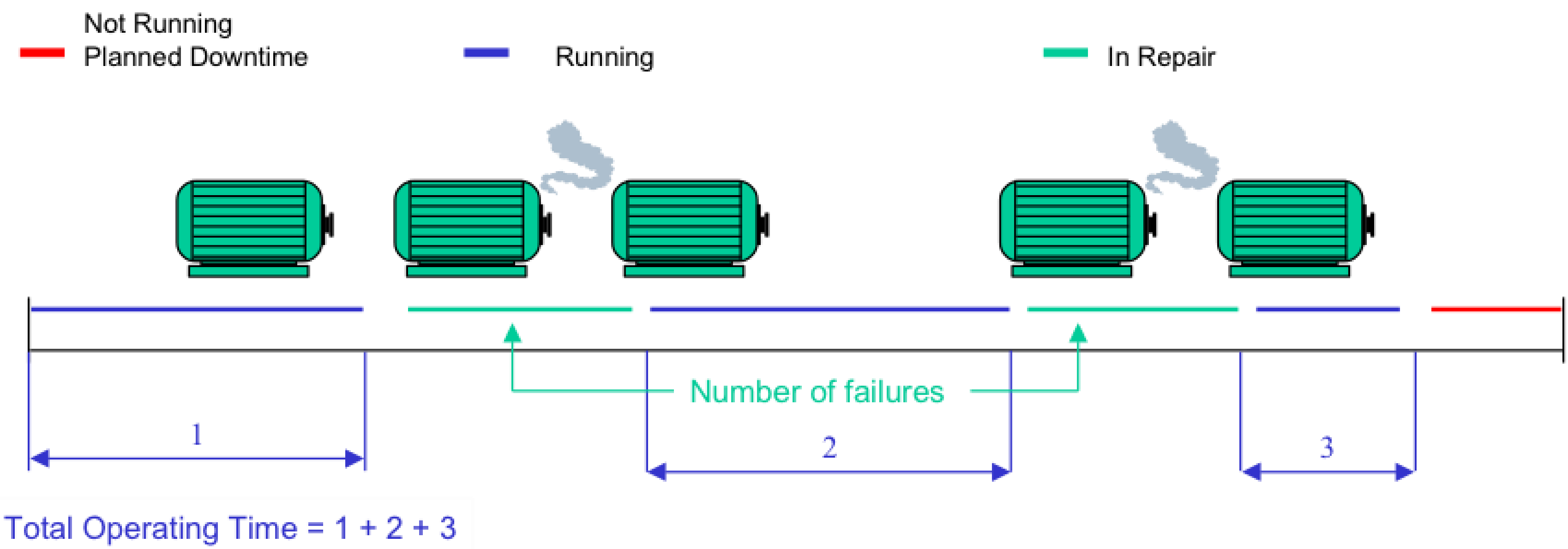
Concept Checks

MTTF and MTBF are for which Systems/Items…?



Concept Checks

MTBF & Reliability relationship?



$$\text{M.T.B.F} = \frac{\text{Total operating Time}}{\text{Number of failures}}$$

Comparing Eqs. (3.20) and (3.23), we find that the area is the MTTF itself. By making $n_0 \to \infty$, we can write Eq. (3.23) as

$$\text{MTTF} = \int_0^\infty R(t)\,dt \tag{3.24}$$

Reliability $R(t)$ and failure rate λ are related. Let us consider a simple analysis here.

As reliability increases
MTBF also increases

Week 6

- **Chapter 3**

  Estimation of Static Errors & Reliability

- **Chapter 4**

  Dynamic Characteristics of Instruments

4. DYNAMIC CHARACTERISTICS OF INSTRUMENTS **80–112**

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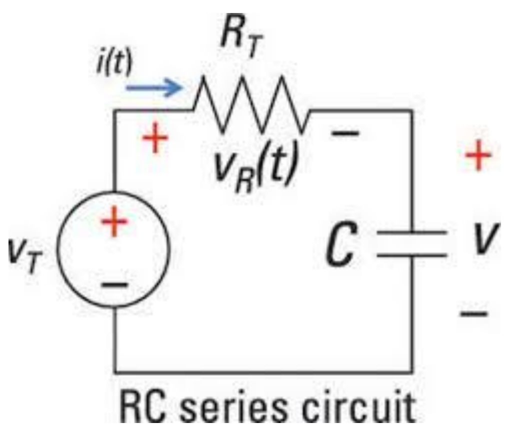
4.2 Standard Inputs to Study Time Domain Response *82*
4.3 Dynamic Characteristics *84*
4.4 Zero Order Instrument *85*
4.5 First Order Instrument *86*
4.6 Second Order Instrument *96*

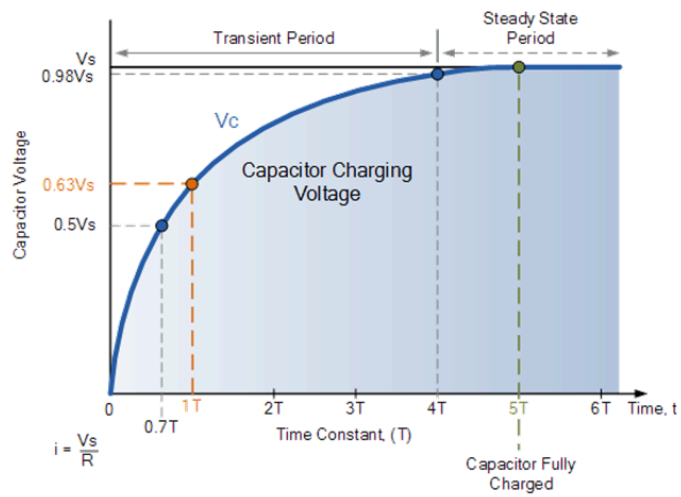
Dynamic Characteristics of Instruments

- **Static characteristics**
  - Characteristics concerning the steady-state reading that the instrument settles down to, such as accuracy of the reading.

Dynamic Characteristics of Instruments

- **Dynamic characteristics**
  - The behavior of a system with time when some input is given to the system.
  - ✓ The time between a measured quantity changes value and the time when the instrument output attains a steady value in response.





Dynamic Characteristics of Instruments

- **Dynamic characteristics**
  - The behavior of a system with time when some input is given to the system
  - ✓ The time between a measured quantity changes value and the time when the instrument output attains a steady value in response.

  - Like static characteristics, the specified dynamic characteristics are valid only under specified environmental conditions.
    - Outside these calibration conditions, some variation in the dynamic parameters can be expected.

Concept Checks

- Characteristics of instrument performance are usually subdivided into two classes on the basis of the frequency of the input signals.
- Static Characteristics
- Dynamic Characteristics

Concept Checks

- ***Static Characteristics*** describe the performance of instruments for dc or very low frequency inputs.
- The output for a wide range of constant inputs demonstrate the quality of the measurement, including nonlinear and statistical effects.

- ***Dynamic Characteristics*** require the use of differential and/or integral equations to describe the quality of the measurements.

Dynamic Characteristics of Instruments

- **Dynamic characteristics can also be divided into two basic categories**
  1. **Desirable**
  2. **Undesirable**

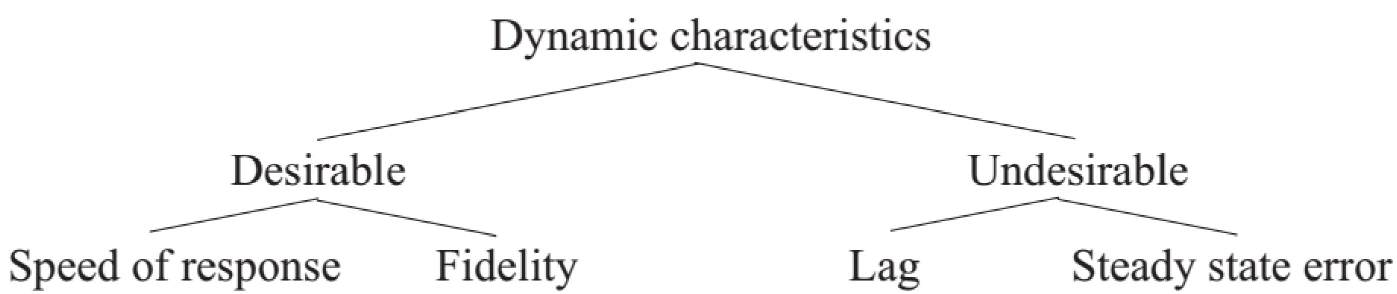


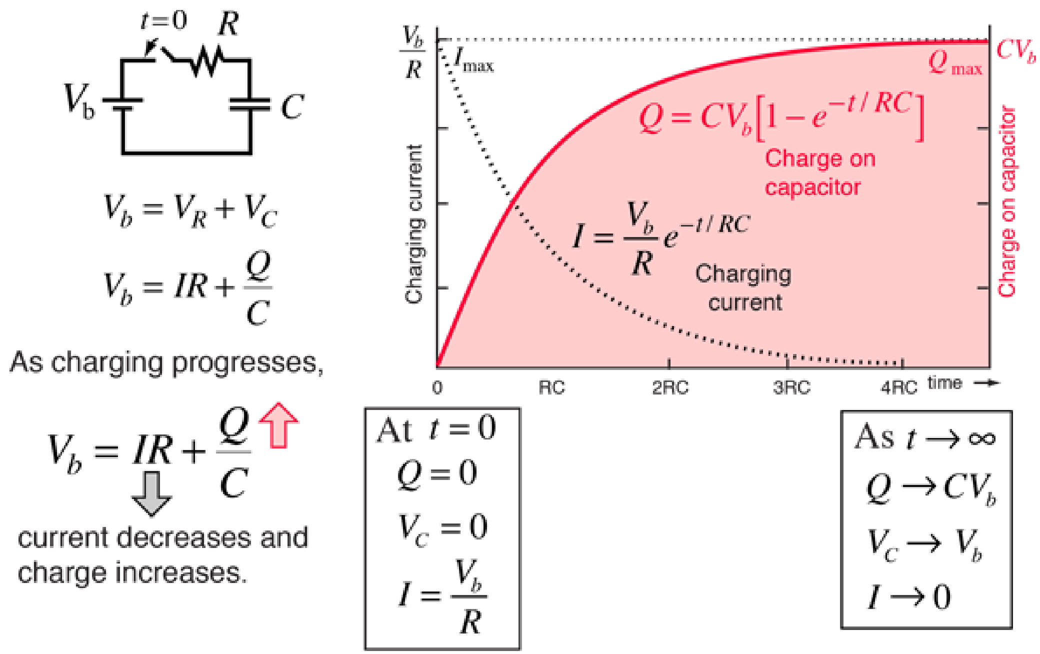
Fig. 4.5 Dynamic characteristics tree.

Dynamic Characteristics of Instruments: Desirable

- **Speed of response**
  - How quickly the system reacts to the input signal?

Dynamic Characteristics of Instruments: Desirable

- **Speed of response**
  - How quickly the system reacts to the input signal?



Dynamic Characteristics of Instruments: Desirable

- **Fidelity**
    - How faithfully the system outputs the input signal and what is the distortion, if any.

Dynamic Characteristics of Instruments: Desirable

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  - How faithfully the system outputs the input signal and what is the distortion, if any.

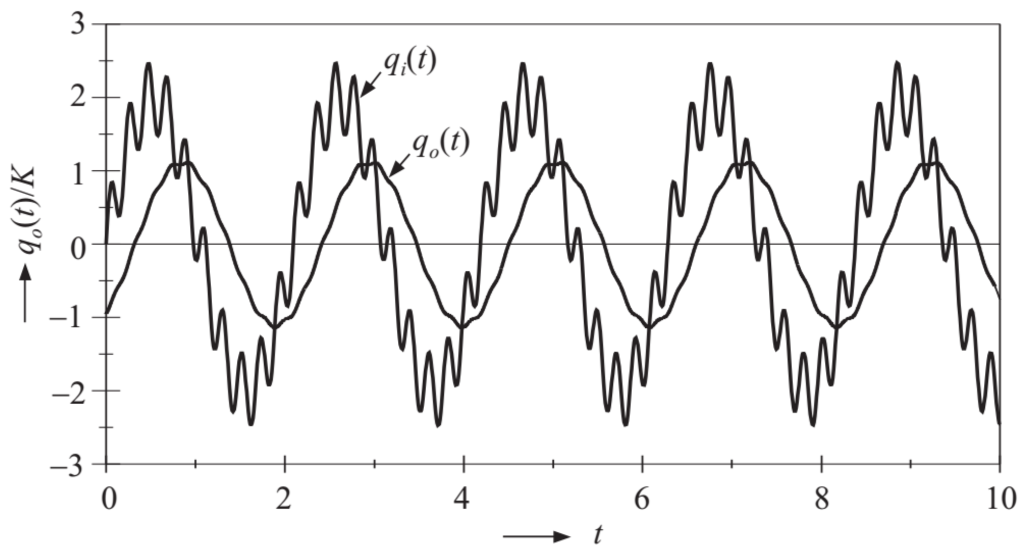

Fig. 4.11 Schematic presentation of frequency response (Example 4.4).

Dynamic Characteristics of Instruments: Undesirable

- **Lag**
  - The lag indicates what time the system takes to output the input signal

Dynamic Characteristics of Instruments: Undesirable

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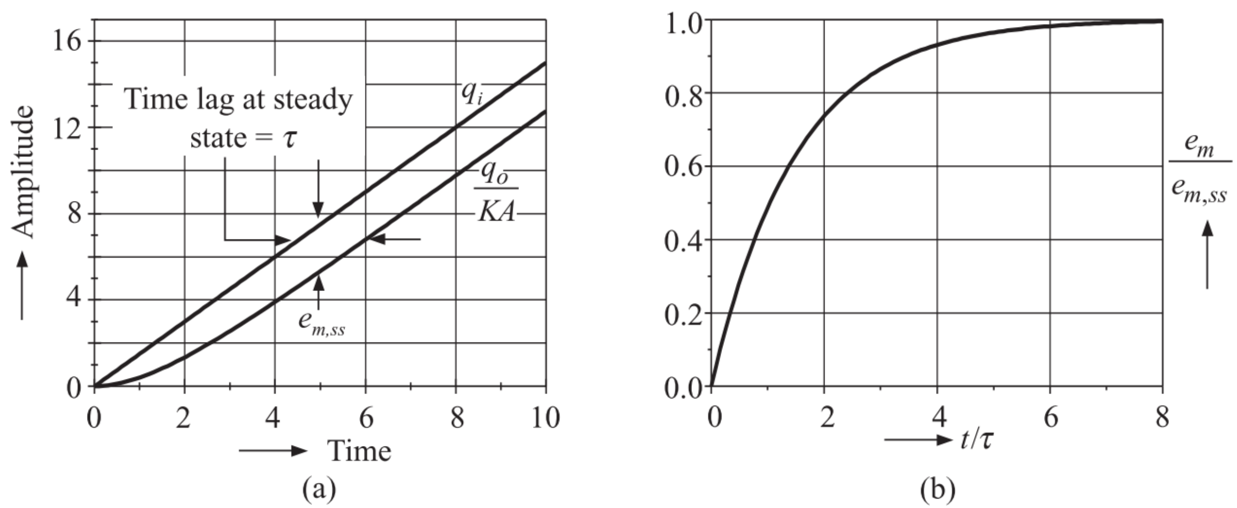

Fig. 4.8 Ramp response of the first order instrument: (a) actual response, and (b) error plot.

- The instrument's reading always lags behind the actual value
- The instrument shows a value what the input was τ seconds ago

Dynamic Characteristics of Instruments: Undesirable

- **Steady state error**

$$e_{ss} = \lim_{t \to \infty} e_m$$

$$e_m = q_i - \frac{q_o}{K}$$

q_i	is the input signal
q_i/K	is the normalised output
K	is the amplification factor

Dynamic Characteristics of Instruments: Undesirable

❑ Steady state error

$$e_{ss} = \lim_{t \to \infty} e_m$$

$$e_m = q_i - \frac{q_o}{K}$$

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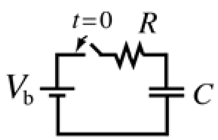


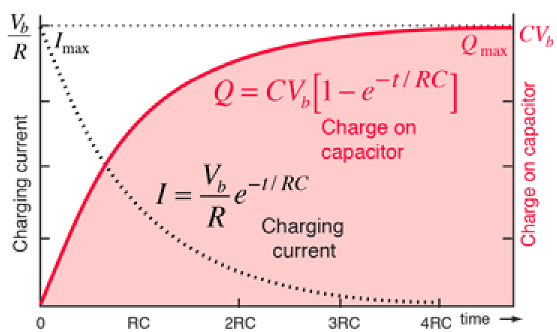
$V_b = V_R + V_C$

$V_b = IR + \dfrac{Q}{C}$

As charging progresses,

$V_b = IR + \dfrac{Q}{C}$

current decreases and charge increases.



At $t=0$
$Q = 0$
$V_C = 0$
$I = \dfrac{V_b}{R}$

As $t \to \infty$
$Q \to CV_b$
$V_C \to V_b$
$I \to 0$

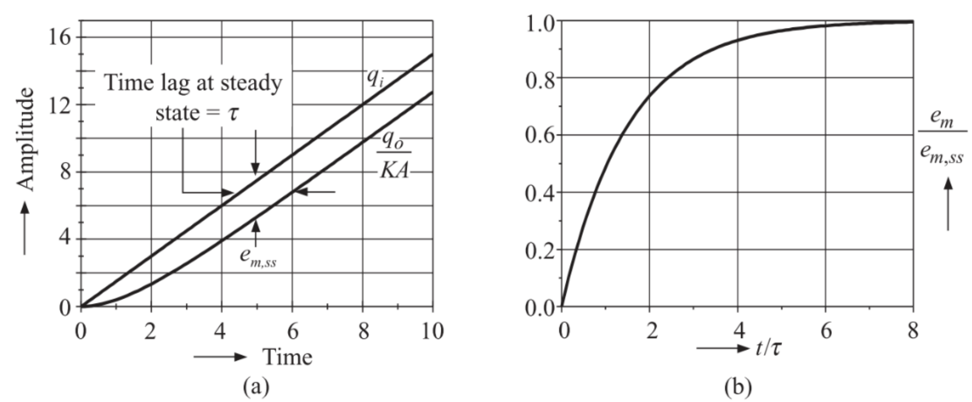


Fig. 4.8 Ramp response of the first order instrument: (a) actual response, and (b) error plot.

Few Concepts…
Before going towards Transfer Function Method

- *Role of Transfer Function?*
- *Laplace Transform and it's use?*
- *Steady-State Response*
- *Transient State Response*

Transfer Function

- The dynamic characteristics of an instrument include its transfer function, its frequency response, and its phase or time delay.
- Dynamic characteristics require the use of differential or integral equations to describe the quality of the measurements.
- Transfer functions are used to predict the stability of a system.

What is the transfer function of a circuit?

- The ratio of a circuit's output to its input in the s-domain:

$$H(s) = \frac{Y(s)}{X(s)}$$

- A single circuit may have many transfer functions, each corresponds to some specific choices of input and output.

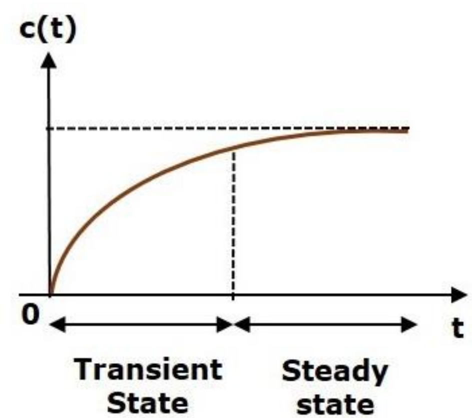
Few Concepts…

Transients and steady-state

- System response usually two parts
  - Transient that dies away over time
  - Steady-state response that continues, constant or not

- **Transient Reponse** = That part of the total response which decays to zero over time.

- **Steady State Response** = That part of the total response that does NOT decay to zero over time.



- Ex: Find transient and steady-state

$$\dot{x} + 2x = 1(t) \qquad x(0) = 0$$

$$x(t) = \underbrace{\frac{1}{2}}_{\text{steady-state}} \underbrace{- \frac{1}{2}e^{-2t}}_{\text{transient}} \quad \text{for } t \ge 0$$

Few Concepts…

The *transient response* (also called natural response) of a causal, stable LTI differential system is the homogeneous response, i.e., with the input set to zero.

The *steady-state response* (or forced response) is the particular solution corresponding to a constant or periodic input. We say that a stable system is in steady-state when the transient component of the output has practically disappeared. For example, consider the step response

$$s(t) = u(t) - e^{-5t}u(t) . \tag{8.35}$$

The transient part of this response is the term $e^{-5t}u(t)$, and the steady-state part is $u(t)$.

As another example, assume that a causal LTI differential system is subjected to the sinusoidal input signal $x(t) = \sin(\omega_0 t)u(t)$. Suppose that the resulting output is

$$y(t) = 2\sin(\omega_0 t - \phi)u(t) + e^{-2t}\cos(2t + \theta)u(t) . \tag{8.36}$$

Then the transient response of the system to the input is $e^{-2t}\cos(2t + \theta)u(t)$ while $2\sin(\omega_0 t - \phi)u(t)$ is the steady-state response.

It is a fact that the steady-state response of a causal, stable LTI system to a sinusoidal input of frequency ω_0 is also sinusoidal input of frequency ω_0, although in general with a different amplitude and phase.

http://www.cim.mcgill.ca/~boulet/304-303B/1999/L27.pdf

Transfer Function

- Transfer functions of systems are very useful to study their responses
  - The generalized relation between a particular input q_i and the corresponding output $q_{o,}$

$$a_n \frac{d^n q_o}{dt^n} + \cdots + a_1 \frac{dq_o}{dt} + a_0 q_o = b_m \frac{d^m q_i}{dt^m} + \cdots + b_1 \frac{dq_1}{dt} + b_0 q_i$$

    - ✓ where *a*’s and *b*’s are combination of system parameters assumed to be constant

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    - ✓ where *a*'s and *b*'s are combination of system parameters assumed to be constant

- Taking Laplace transform and assuming all initial conditions equal to zero,

$$(a_n s^n + \cdots + a_1 s + a_0) Q_o(s) = (b_m s^m + \cdots + b_1 s + b_0) Q_i(s)$$

About Laplace Transform

The Laplace transform, named after its inventor Pierre-Simon Laplace, is an integral transform that converts a function of a real variable (t) to a function of a complex variable (s).

The transform has many applications in science and engineering because it is a tool for solving differential equations

$$F(s) = \int_0^\infty f(t)e^{-st}t'$$

$$\text{where } s = \sigma + i\omega$$

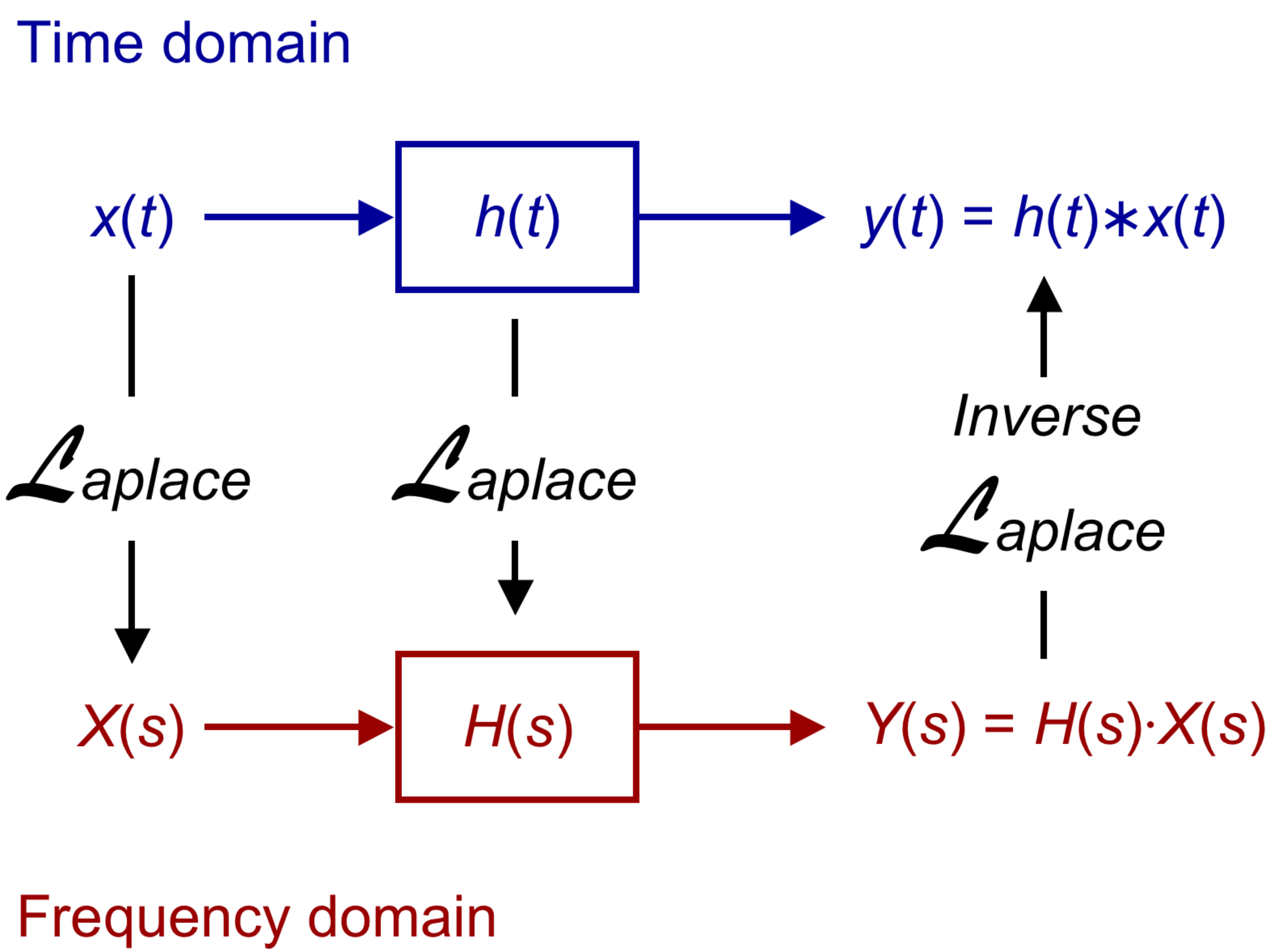
$F(s)$ = laplace transform

s = complex number

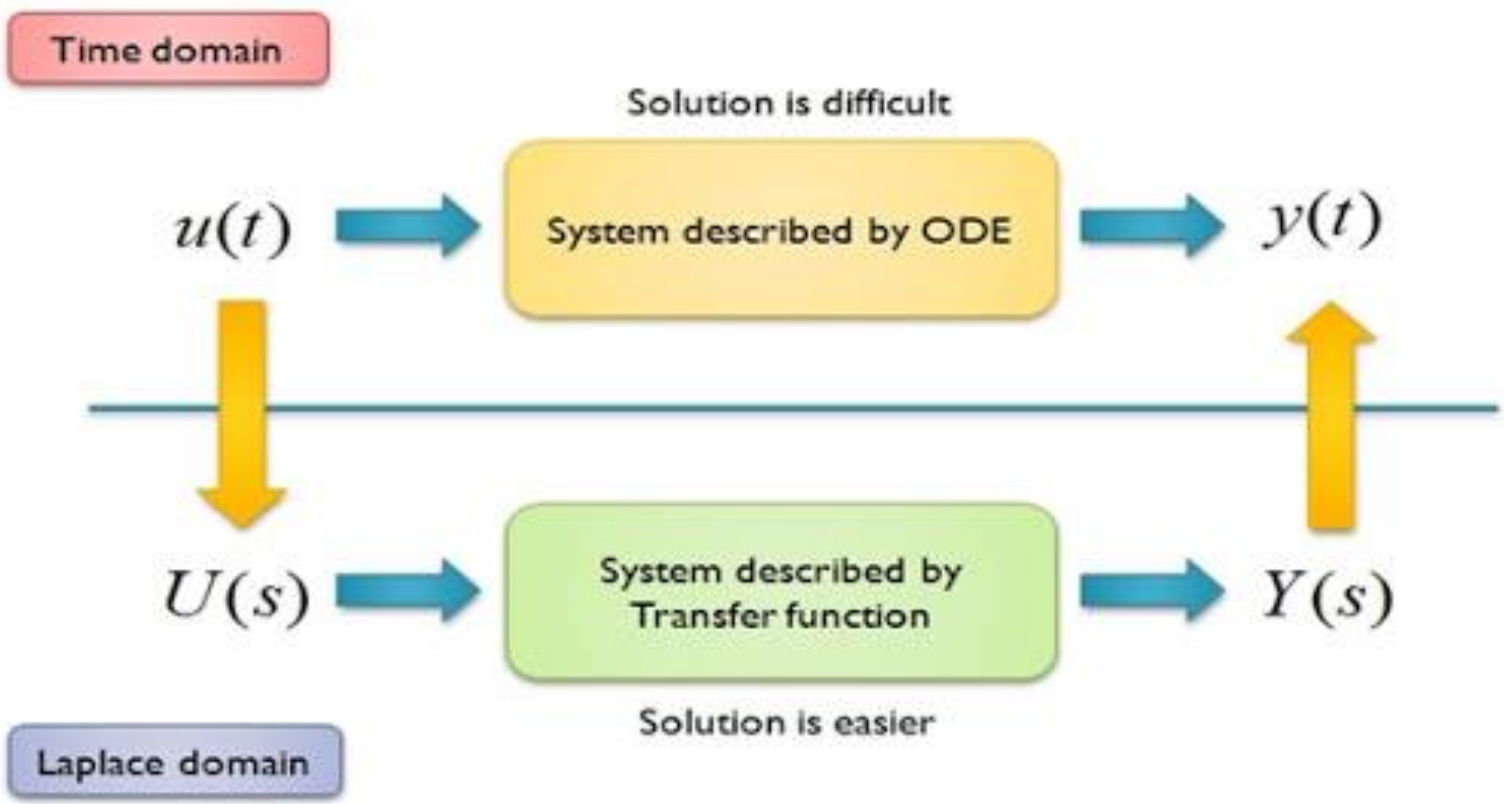
t = real number >= 0

t' = first derivative of the function f(t)

About Laplace Transform



About Laplace Transform



Method?

LAPLACE TRANSFORM FORMULA

$$F(s) = \int_{0}^{+\infty} f(t) \cdot e^{-s \cdot t} \cdot dt$$

There are certain steps which need to be followed in order to do a Laplace transform of a time function. In order to transform a given function of time f(t) into its corresponding Laplace transform, we have to follow the following steps:

- First multiply f(t) by e^{-st}, s being a complex number ($s = \sigma + j\,\omega$).
- Integrate this product w.r.t time with limits as zero and infinity. This integration results in Laplace transformation of f(t), which is denoted by F(s).

$$Laplace\ transform\ of\ f(t) = \pounds\,[f(t)] = F(s) = \int_0^\infty f(t) \cdot e^{-st} dt$$
$$when\ t \geq 0$$

The time function f(t) is obtained back from the Laplace transform by a process called inverse Laplace transformation and denoted by $\pounds^{-1}$

$$Inverse\ Laplace\ transform\ of\ F(s) = \pounds^{-1}\,[F(s)] = \pounds^{-1}\,[\pounds\, f(t)] = f(t)$$

https://www.electrical4u.com/laplace-transformation/

Examples

- First multiply f(t) by e^{-st}, s being a complex number ($s = \sigma + j\,\omega$).
- Integrate this product w.r.t time with limits as zero and infinity. This integration results in Laplace transformation of f(t), which is denoted by F(s).

$$Laplace\ transform\ of\ f(t) = \mathcal{L}\left[f(t)\right] = F(s) = \int_0^\infty f(t)\cdot e^{-st}dt$$

$when\ t \geq 0$

3) Let $f(t) = e^{at}$, $t \geq 0$

$$\mathcal{L}\{e^{at}\} = F(s) = \int_0^\infty e^{-st}e^{at}dt = \lim_{A\to\infty}\int_0^A e^{-(s-a)t}dt =$$

$$\lim_{A\to\infty}\frac{-e^{-(s-a)t}}{s-a}\Big|_0^A = \lim_{A\to\infty}\left[\frac{1}{s-a} - \frac{e^{-(s-a)A}}{s-a}\right] = \frac{1}{s-a},\ \text{for all } s > a.$$

4) Let $f(t) = \sin at$, $t \geq 0$

$$\mathcal{L}\{\sin at\} = F(s) = \int_0^\infty e^{-st}\sin at\ dt = \lim_{A\to\infty}\int_0^A e^{-st}\sin at\ dt =$$

$$\lim_{A\to\infty}\left[-\frac{e^{-st}\cos at}{a}\Big|_0^A - \frac{s}{a}\int_0^A e^{-st}\cos at\ dt\right] = \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos at\ dt =$$

$$\frac{1}{a} - \lim_{A\to\infty}\left[-\frac{e^{-st}\sin at}{a}\Big|_0^A + \frac{s^2}{a^2}\int_0^A e^{-st}\sin at\ dt\right] = \frac{1}{a} - \frac{s^2}{a^2}F(s)$$

then,

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2}F(s),\quad F(s)\left(\frac{a^2+s^2}{a^2}\right) = \frac{1}{a},\quad F(s) = \frac{a}{s^2+a^2},\ s \geq 0$$

https://byjus.com/maths/laplace-transform/

Transfer Function

- Transfer functions of systems are very useful to study their responses
  - The generalized relation between a particular input q_i and the corresponding output q_o,

$$a_n \frac{d^n q_o}{dt^n} + \cdots + a_1 \frac{dq_o}{dt} + a_0 q_o = b_m \frac{d^m q_i}{dt^m} + \cdots + b_1 \frac{dq_1}{dt} + b_0 q_i$$

    - ✓ **where *a*’s and *b*’s are combination of system parameters assumed to be constant**

- Taking Laplace transform and assuming all initial conditions equal to zero,

$$(a_n s^n + \cdots + a_1 s + a_0) Q_o(s) = (b_m s^m + \cdots + b_1 s + b_0) Q_i(s)$$

$$\boxed{G(s) \equiv \frac{Q_o(s)}{Q_i(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}}$$

❖ **The transfer function G(s)**

Properties of Transfer Function

- The transfer function is a general relation between the Laplace transforms of the output and input quantities; i.e., $Q_o\,(s)$ and $Q_i\,(s)$

- Example:

the relation between the current i and the emf e in an LCR circuit is given by

$$e(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)dt$$

Taking Laplace transform, we get

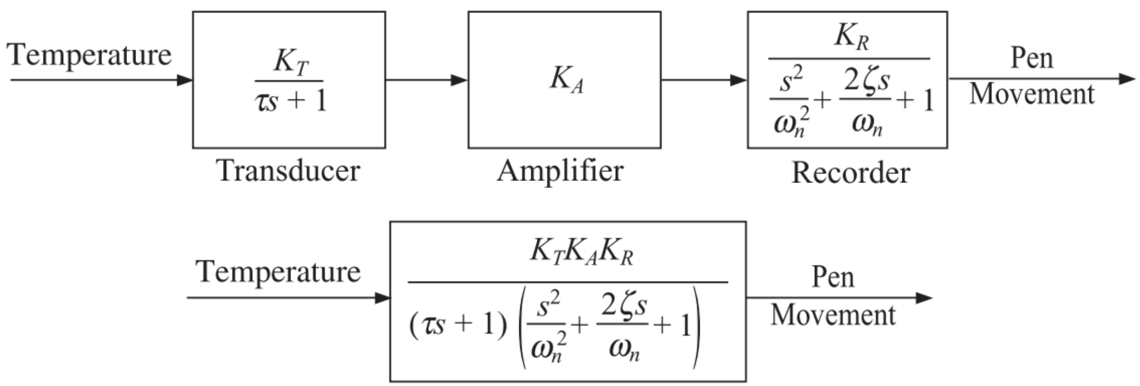
$$\mathscr{L}\{e(t)\} \equiv E(s) = sLI(s) + RI(s) + \frac{I(s)}{sC}$$

Therefore, the transfer function,

$$G(s) = \frac{I(s)}{E(s)} = \frac{1}{sL + R + (1/sC)}$$

Properties of Transfer Function

- The transfer function is a general relation between the Laplace transforms of the output and input quantities; i.e., $Q_o(s)$ and $Q_i(s)$
  - It is not the instantaneous ratio of the time varying quantities $q_o(t)$ and $q_i(t)$
- It offers a symbolic picture about the dynamic characteristics of the system
- If the transfer functions of individual components of the system are known, the overall characteristics of the system can be determined just by taking the product



Transfer Function

❑ Time domain analysis

So far while discussing dynamic characteristics, we have tacitly assumed that the inputs to the system are time-varying and we want to study the dynamic response of the system at different intervals of time. Such kind of a study is called a *time domain analysis*.

But time domain analysis is rather cumbersome and, of course, not necessary if the input varies periodically with time, such as $q_i = A_i \sin \omega t$. The output quantity q_o in such cases will also be a sine wave, once the transients die out. The only changes that are expected are in the amplitude and the phase of the output. Since the input and output frequency are the same, the output is completely specified by giving the amplitude ratio A_o/A_i, and the phase shift angle ϕ. Thus, the response of a system to a periodic input is completely studied if the amplitude ratio and phase shift are studied as a function of frequency (Fig. 4.2). This analysis is termed *frequency domain analysis*.

❑ Frequency domain analysis

- ❑ for periodic inputs (sinusoidal input), amplitude ratio and the phase between input and output matters

Transfer Function

- **Frequency domain analysis**

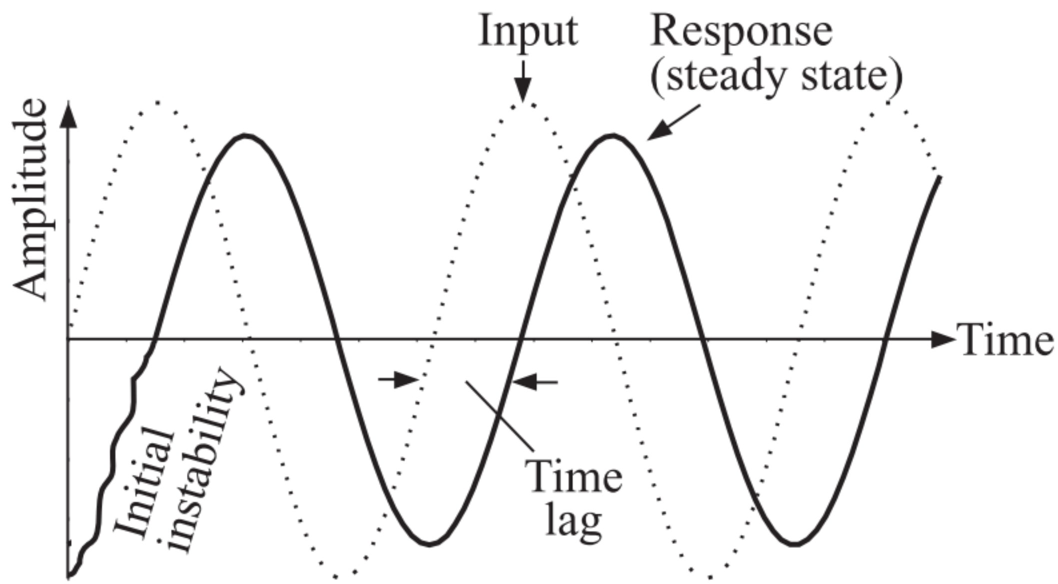


Fig. 4.2 A typical response of a system to a sinusoidal input

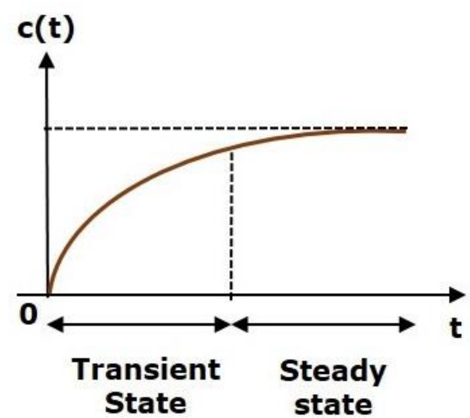
Recall: Few Concepts…

Transients and steady-state

- System response usually two parts
  - Transient that dies away over time
  - Steady-state response that continues, constant or not

- **Transient Reponse** = That part of the total response which decays to zero over time.

- **Steady State Response** = That part of the total response that does NOT decay to zero over time.



- Ex: Find transient and steady-state

$$\dot{x} + 2x = 1(t) \qquad x(0) = 0$$

$$x(t) = \underbrace{\frac{1}{2}}_{\text{steady-state}} \underbrace{- \frac{1}{2}e^{-2t}}_{\text{transient}} \quad \text{for } t \geq 0$$

Transfer Function

- **Frequency domain analysis**

$$\boxed{G(s) \equiv \frac{Q_o(s)}{Q_i(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}}$$

$$G(\jmath\omega) = \frac{Q_o(\jmath\omega)}{Q_i(\jmath\omega)} = \frac{b_m(\jmath\omega)^m + \cdots + b_1 \jmath\omega + b_0}{a_n(\jmath\omega)^n + \cdots + a_1 \jmath\omega + a_0}$$

Transfer Function

- **Frequency domain analysis**

$$G(j\omega) = \frac{Q_o(j\omega)}{Q_i(j\omega)} = \frac{b_m(j\omega)^m + \cdots + b_1 j\omega + b_0}{a_n(j\omega)^n + \cdots + a_1 j\omega + a_0}$$

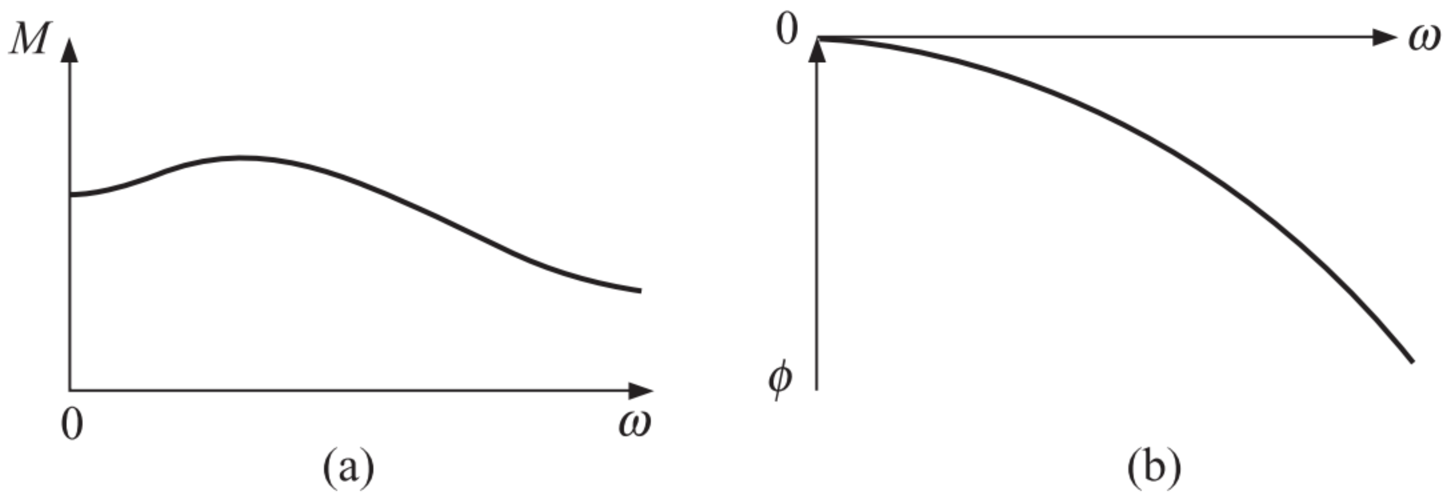
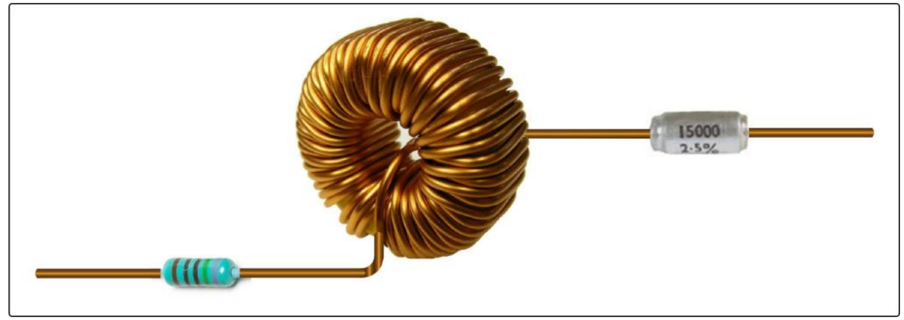


Fig. 4.3 Frequency response of system.

Transfer Function

- **Frequency domain analysis**



the sinusoidal transfer function for an LCR circuit is

$$G(\jmath\omega) = \frac{X(\jmath\omega)}{F(\jmath\omega)} = \frac{1}{\jmath\omega L + R + (1/\jmath\omega C)}$$

4. DYNAMIC CHARACTERISTICS OF INSTRUMENTS **80–112**

4.1 Transfer Function *80*
4.2 Standard Inputs to Study Time Domain Response *82*
4.3 Dynamic Characteristics *84*
4.4 Zero Order Instrument *85*
4.5 First Order Instrument *86*
4.6 Second Order Instrument *96*

Standard Inputs to Study Time Domain Response

Types of responses

- Free response
  - "Free" refers to lack of an input, i.e. how system behaves freely
  - Set right-hand side to zero
  - Usually non-zero initial conditions given

- Forced response
  - Input is specified
  - Assume zero initial conditions unless told otherwise
  - Also named according to the input
    - Step response $u(t) = 1(t)$
    - Impulse response $u(t) = \delta(t)$
    - Ramp response $u(t) = t \cdot 1(t)$

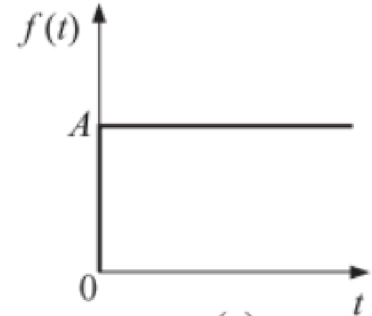
Step Input

The functional form [Fig. 4.4(a)] is given by

$$f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ A & \text{if } t > 0 \end{cases}$$

The Laplace transform of the step input is

$$\mathscr{L}\{f(t)\} = \frac{A}{s}$$



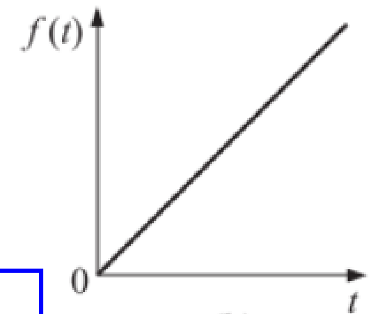
(a)

Ramp Input

The functional form [Fig. 4.4(b)] and the Laplace transform are

$$f(t) = At \qquad \text{and} \qquad F(s) = \frac{A}{s^2}$$

Ramp	$x = 0$	$t < 0$
	$x = at$	$t \geq 0$



(b)

Impulse Input

The impulse function, [Fig. 4.4(c)] related to the Dirac δ-function is defined as

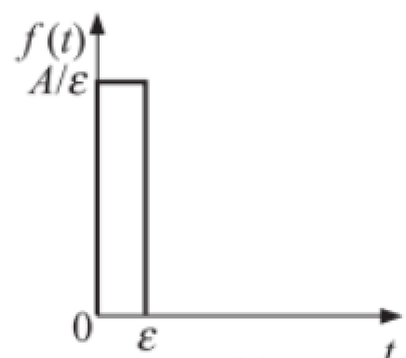
$$f(t) = A\delta(t)$$

where,

$$\delta(t) = \begin{cases} \lim\limits_{\varepsilon \to 0} \dfrac{1}{\varepsilon} & \text{for } 0 \le t \le \varepsilon \\ 0 & \text{for } t > \epsilon \end{cases}$$

By definition, $\int_0^\infty \delta(t)dt = 1$. Therefore, the corresponding Laplace transform[3] is $\mathscr{L}\{\delta(t)\} = 1$. Thus,

$$F(s) = A$$



(c)

Sinusoidal $\qquad x = X \sin(\omega t) \qquad t > 0$

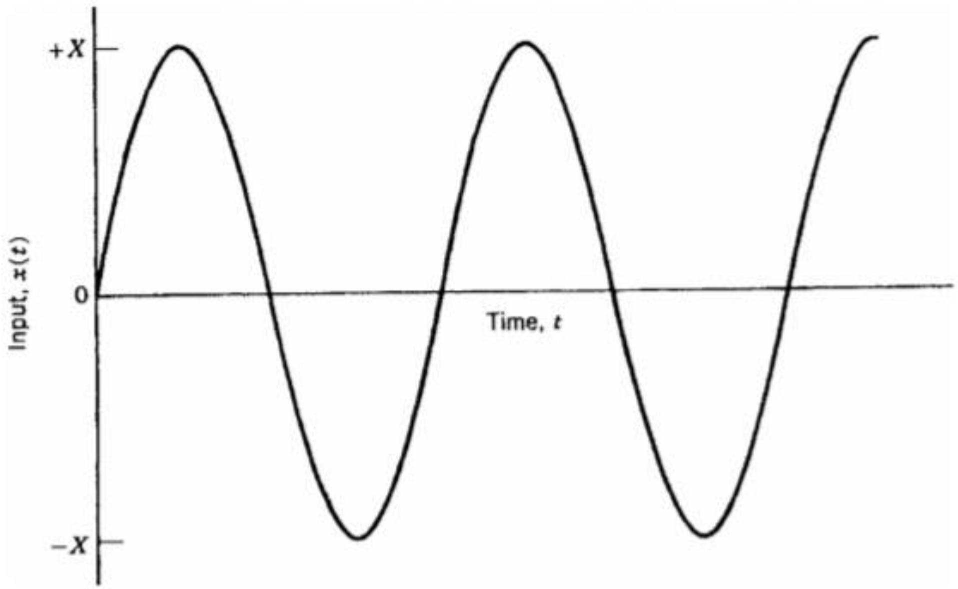


Figure F.3 Sinusoidally varying input to an instrument.

Order of an Instrument

Desirable and undesirable dynamic characteristic all depend on the speed of response, which in turn is related to time constant τ of the system.

Dynamic characteristics of instrument are largely dependent on the order of the system.

- Zero order systems
- First order systems
- Second order systems

Order of an Instrument?

F-1 GENERAL INSTRUMENT RESPONSE

The traditional way to investigate the dynamic response of an instrument is to consider the differential equation that describes the output. We assume that the instrument response can be modeled using a linear ordinary differential equation with constant coefficients [1]

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = bx(t) \qquad \text{(F.1)}$$

where y is the instrument output, x is the input, and n is the order of the instrument.

Instrument response to three different inputs will be discussed: (1) a step change, (2) a ramp input, and (3) a sinusoidal input. These are illustrated in Slide 65

Experimentation, Validation, and Uncertainty Analysis for Engineers, Third Edition H. W. Coleman and W. G. Steele **299**
 ISBN: 978-0-470-16888-2

Behavior of the Instrument?

Zero order

$$a_0 y = x(t)$$

First order

$$a_1 \frac{dy}{dt} + a_0 y = x(t)$$

Second order

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = x(t)$$

***n*th order**

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots\ldots\ldots\ldots a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = x(t)$$

Order of an Instrument?

- The order of a dynamic system is indicated by the highest power of the Laplace variable *s* in the rationalised denominator of the corresponding transfer function
  - ✓ The power of *s* is related to the degree of differential equation describing the instrumentation system
  - ✓ The degree of the differential equation determines the order of a dynamic system

Order of an Instrument?

- The order of a dynamic system is indicated by the highest power of the Laplace variable *s* in the rationalised denominator of the corresponding transfer function
  - ✓ The power of *s* is related to the degree of differential equation describing the instrumentation system
  - ✓ The degree of the differential equation determines the order of a dynamic system

$$G(s) = \frac{s+9}{s^2+2s+9}$$

❖ **The transfer function of a 2nd order system**

Order of an Instrument?

$$G(s) = \frac{s+9}{s^2+2s+9}$$

Here, the denominator is given by

$$s^2 + 2s + 9$$

which consists of a power of 2 for the Laplace variable s.

Now, the same transfer function can be written as

$$G(s) = \frac{1 + (9/s)}{s + 2 + (9/s)}$$

Does it indicate that it is a first order system? The answer is obviously 'no' because the denominator contains a fraction.

Zero Order Instrument

$$\boxed{a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = bx(t) \qquad \text{(F.1)}}$$

F-2 RESPONSE OF ZERO-ORDER INSTRUMENTS

Since $n = 0$ for a zero-order instrument, Eq. (F.1) reduces to an algebraic equation:

$$\boxed{y = Kx(t)} \tag{F.5}$$

where K $(= b/a_0)$ is called the *static gain*. Equation (F.5) shows that the output is always proportional to the input, so there is no error in the output due to the dynamic response. Of course, there will be static errors of the types we have discussed previously.

Zero Order Instrument

$$a_n \frac{d^n q_o}{dt^n} + \cdots + a_1 \frac{dq_o}{dt} + a_0 q_o = b_m \frac{d^m q_i}{dt^m} + \cdots + b_1 \frac{dq_1}{dt} + b_0 q_i$$

- Suppose all *a*'s and *b*'s except a_0 and b_0 are zero,

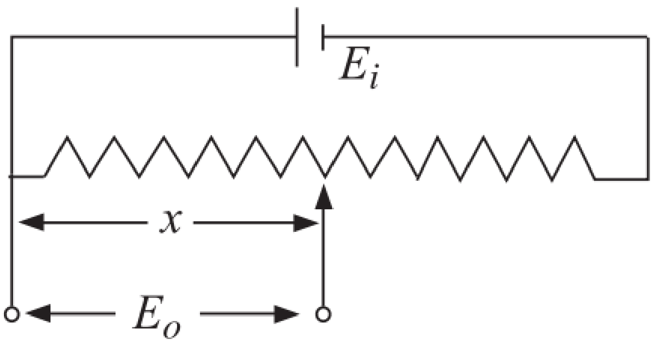
$$a_0 q_o(t) = b_0 q_i(t)$$

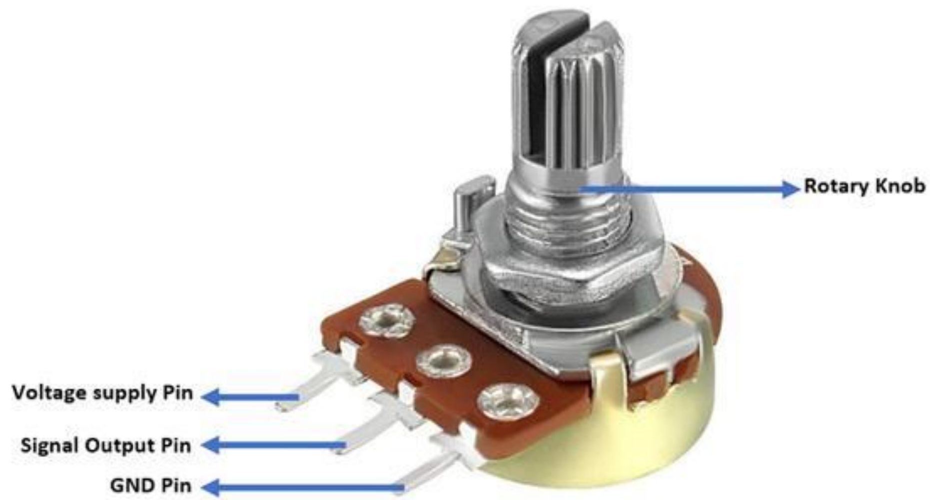
$$q_o(t) = \frac{b_0}{a_0} q_i(t) = K q_i(t)$$

- A zero order instrument closely obeys this equation over its range of operation.
- Output q_o faithfully follows the input q_i with no distortion or time lag of any sort.
- The zero order instrument, therefore, may be considered as ideal having a perfect dynamic response.

Example: Zero Order Instrument

- A potentiometer used for measuring displacements
  - A zero-order instrument





A potentiometer used for measuring displacements[4] may be shown to be a zero-order instrument. In such an arrangement, a wire-wound resistance, provided with a sliding contact, is excited with a voltage. Assuming that the resistance is distributed linearly along its length L, we have

$$E_o = \frac{x}{L} E_i \equiv Kx \tag{4.5}$$

where E_o and E_i are the output and input voltages, x is the displacement and K is a constant.

Zero Order Instrument

- **Step Response**
- Output q_o faithfully follows the input q_i with no distortion or time lag of any sort.
  - Following a step change in the measured quantity at time *t*, the instrument output moves immediately to a new value at the same time instant *t*

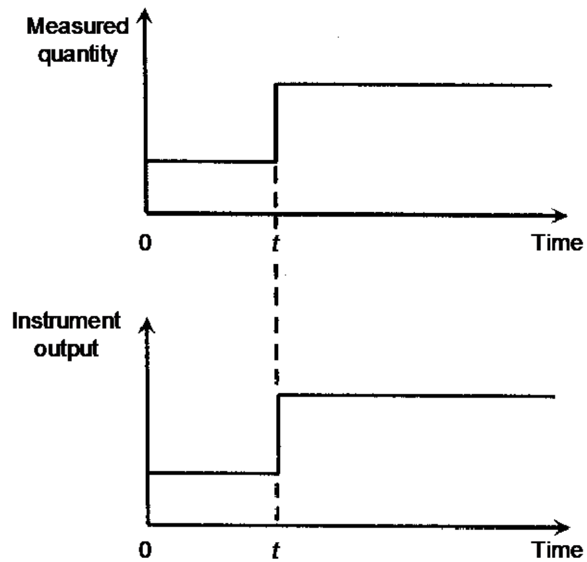


Figure 2.10
Zero-order instrument characteristic.

Dynamic Characteristics

Zero Order Systems

Speed of Response ✓

Fidelity ✓

Lag ✗

First Order Instrument

- **The dynamic relation between the input and output,**

$$a_1 \frac{dq_o(t)}{dt} + a_0 q_o(t) = b_0 q_i(t)$$

$$\Rightarrow \quad \tau \frac{dq_o(t)}{dt} + q_o(t) = K q_i(t)$$

- ***K* = *b*$_0$ /*a*$_0$ and τ = *a*$_1$ /*a*$_0$; τ is the time constant**

- In these instruments there is a time delay in their response to changes of input.
- The time constant *t* is a measure of the time delay.
- Thermometers for measuring temperature are first-order instruments.

First Order Instrument

- **The dynamic relation between the input and output,**

$$a_1 \frac{dq_o(t)}{dt} + a_0 q_o(t) = b_0 q_i(t)$$

$$\tau \frac{dq_o(t)}{dt} + q_o(t) = K q_i(t)$$

- **Taking Laplace transform**

$$\tau s Q_o(s) + Q_o(s) = K Q_i(s)$$

$$(1 + \tau s) Q_o(s) = K Q_i(s)$$

$$\boxed{G(s) \equiv \frac{Q_o(s)}{Q_i(s)} = \frac{K}{1 + \tau s}}$$

- **The transfer function G(s)**

First Order Instrument

- **The dynamic relation between the input and output,**

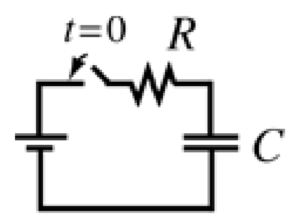
$$a_1 \frac{dq_o(t)}{dt} + a_0 q_o(t) = b_0 q_i(t)$$

$$\tau \frac{dq_o(t)}{dt} + q_o(t) = K q_i(t)$$

- **Example: *RC* circuit is a first order arrangement**

e_i (= voltage, input) and Q (= charge, output)

$$\boxed{\begin{aligned} R\frac{dQ}{dt} + \frac{Q}{C} &= e_i \\ \tau\frac{dQ}{dt} + Q &= Ke_i \end{aligned}}$$



where, $\tau = RC$ and $K = C$.

First Order Instrument

- **Mercury-in-glass thermometer**

Let V be the volume of the bulb
A_b be the area of the bulb conducting heat
γ_a be the coefficient of apparent expansion of mercury
θ_m be the temperature attained by mercury at any instant
x be the corresponding height of the mercury column (output)
A be the area of cross-section of the capillary tube
K_g be the thermal conductivity of glass
ρ be the density of mercury
c be the specific heat of mercury
θ_i be the temperature of the liquid (input)
θ be the wall thickness of the glass bulb,

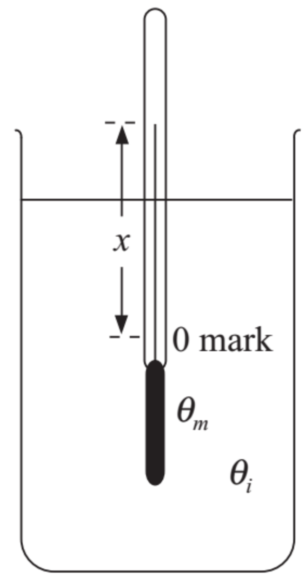


Fig. 4.6 Mercury-in-glass thermometer.

- **The dynamic relation between the input and output,**

$$\boxed{\tau \frac{dx}{dt} + x = K\theta_i}$$

$$\tau = \frac{\rho c V \theta}{K_g A_b}$$

$$K = \frac{\gamma_a V}{A}$$

Let V be the volume of the bulb
A_b be the area of the bulb conducting heat
γ_a be the coefficient of apparent expansion of mercury
θ_m be the temperature attained by mercury at any instant
x be the corresponding height of the mercury column (output)
A be the area of cross-section of the capillary tube
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c be the specific heat of mercury
θ_i be the temperature of the liquid (input)
θ be the wall thickness of the glass bulb,

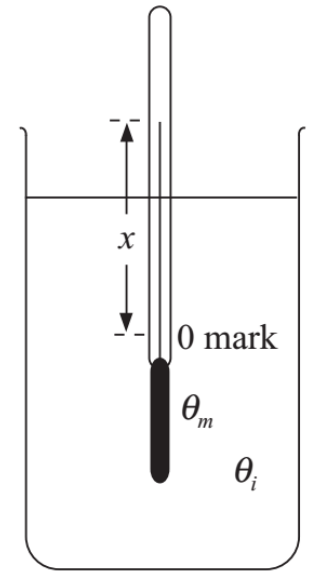


Fig. 4.6 Mercury-in-glass thermometer.

then $$x = \frac{\gamma_a V \theta_m}{A}$$

which gives $$\theta_m = \frac{Ax}{\gamma_a V}$$

Now, the heat conducted from the liquid to mercury through the bulb (i.e. heat lost by the liquid) during the interval dt equals

$$\frac{K_g A_b}{\theta}(\theta_i - \theta_m)dt$$

The corresponding heat gained by mercury in the bulb is

$$(\rho c V)\, d\theta_m$$

Equating heat lost to heat gained and substituting the value of θ_m, we get after some rearrangement

$$\tau \frac{dx}{dt} + x = K\theta_i$$

where, $$\tau = \frac{\rho c V \theta}{K_g A_b}$$

$$K = \frac{\gamma_a V}{A}$$

First Order Instrument

- **Thermocouple**
  - **A thermocouple which is dipped into a hot liquid, assuming**

(a) the heat transfer takes place only by conduction
(b) the emf vs temperature curve of the thermocouple is linear
(c) the other junction of the thermocouple is kept at room temperature

A is the heat transfer area of the thermocouple
K_t is the thermal conductivity of the thermocouple material
θ is the temperature attained by the thermocouple at any instant
θ_i is the temperature of the hot liquid
m is the mass of the thermocouple junction
c is the specific heat of the junction material
E is the developed emf in the thermocouple

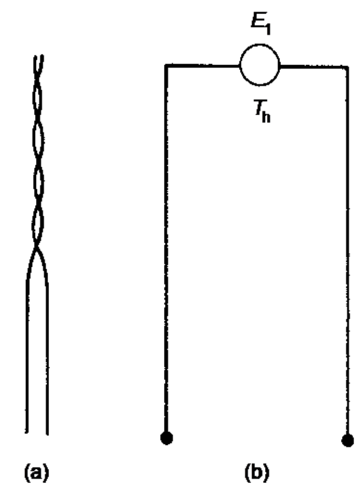


Figure 14.2
(a) Thermocouple and (b) equivalent circuit.

- **The dynamic relation between the input and output,**

$$\boxed{\tau \frac{dE}{dt} + E = K\theta_i}$$

$$\tau = \frac{mc}{K_t A}$$

Thermocouple

A thermocouple which is dipped into a hot liquid, assuming

we have

$$E = K\theta \tag{4.7}$$

where K is a constant. Heat conducted from the liquid to the thermocouple junction during a small time interval dt is

$$K_t A(\theta_i - \theta)dt$$

The corresponding heat gained by the thermocouple junction is

$$mc\, d\theta$$

Thus,

$$mc\, d\theta = K_t A(\theta_i - \theta)dt$$

which gives

$$\frac{mc}{K_t A}\frac{dE}{dt} + E = K\theta_i \quad \text{[applying Eq. (4.7)]} \tag{4.8}$$

or

$$\tau\frac{dE}{dt} + E = K\theta_i \tag{4.9}$$

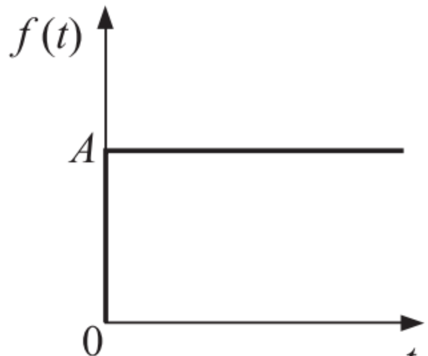
where

$$\tau = \frac{mc}{K_t A}$$

Equation (4.9) shows that the thermocouple is a first order system.

Dynamic Response of First Order Instruments

❑ Step Input



(a)

$$f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ A & \text{if } t > 0 \end{cases}$$

- **The Laplace transform of the step input is**

$$\mathscr{L}\{f(t)\} = \frac{A}{s}$$

Dynamic Response of First Order Instruments

- **Step Response**

$$\mathscr{L}\{f(t)\} = \frac{A}{s} \qquad G(s) \equiv \frac{Q_o(s)}{Q_i(s)} = \frac{K}{1+\tau s}$$

$$Q_o(s) = G(s)Q_i(s) = \frac{KA}{s(\tau s+1)} = KA\left(\frac{1}{s} - \frac{\tau}{\tau s+1}\right) = KA\left(\frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}\right)$$

- **Taking the inverse Laplace transform,**

$$\boxed{q_o(t) = KA[1-\exp(-t/\tau)]}$$

$$\boxed{\frac{q_o}{KA} = 1-\exp(-t/\tau)}$$

Example

2. <u>First Order System Model</u>

The first order system has only one pole as shown

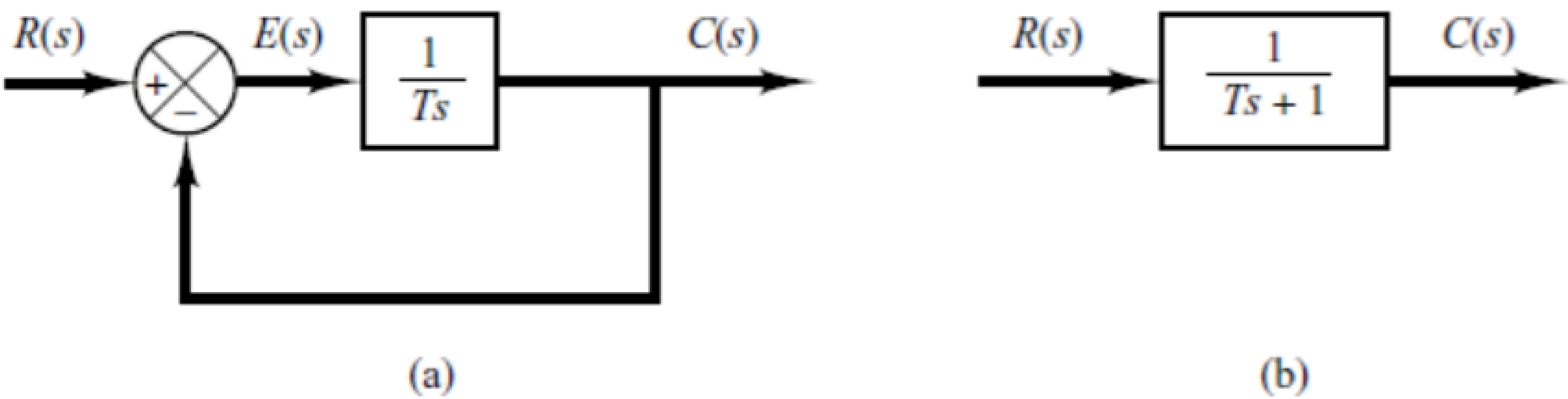


Figure 1: (a) Block Diagran of a first-order system; (b) Simplifed block Diagram

$$\frac{C(s)}{R(s)} = K\frac{1}{Ts+1} \qquad (1)$$

- Where ***K*** is the DC Gain and ***T*** is the time constant of the system.
- Time Constant is a measure of how quickly a 1^{st} order system response to a unit step input.
- DC gain of the system ration between the input signal and the steady state value of output.

2.3 Step Response of 1^{st} Order System

Consider the first order system in figure 1.

$$R(s) = \frac{1}{s}$$
$$C(s) = \frac{K}{Ts+1}\frac{1}{s}$$

In order to represent the response of the system in time domain we need to compute the inverse Laplace transform of the above equation, we have

$$c(t) = Ku(t) - e^{-\frac{t}{T}} \tag{4}$$

1) Where $u(t) = 1$

$$c(t) = K - e^{-\frac{t}{T}} \tag{5}$$

2) Where $t = T$

$$c(t) = K - e^{-1} = 0.632K \tag{6}$$

- For example, assume $\boldsymbol{K = 10, T = 1.5s}$

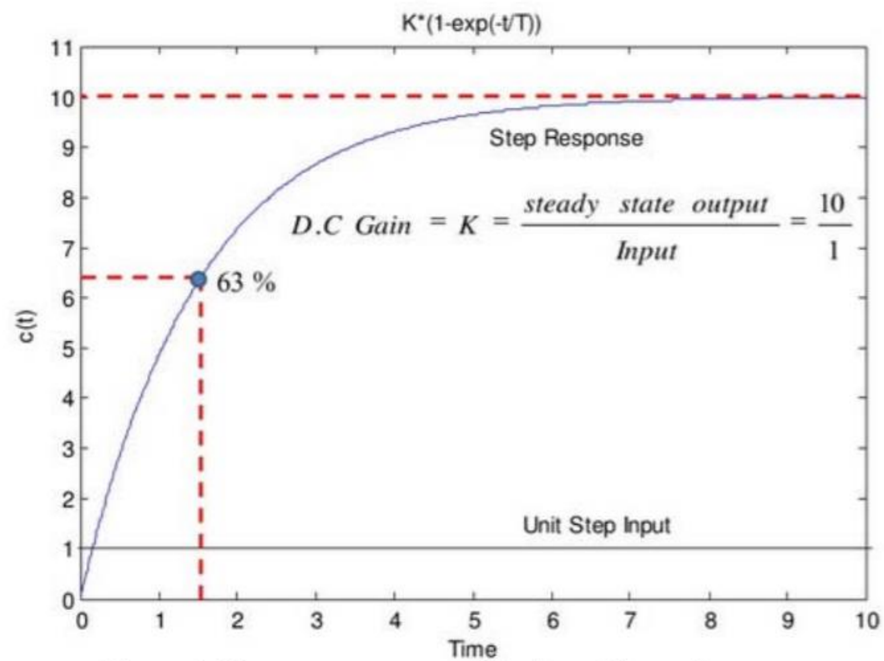


Figure 6: The step response specification of first order system

Dynamic Response of First Order Instruments

- **Step Response**

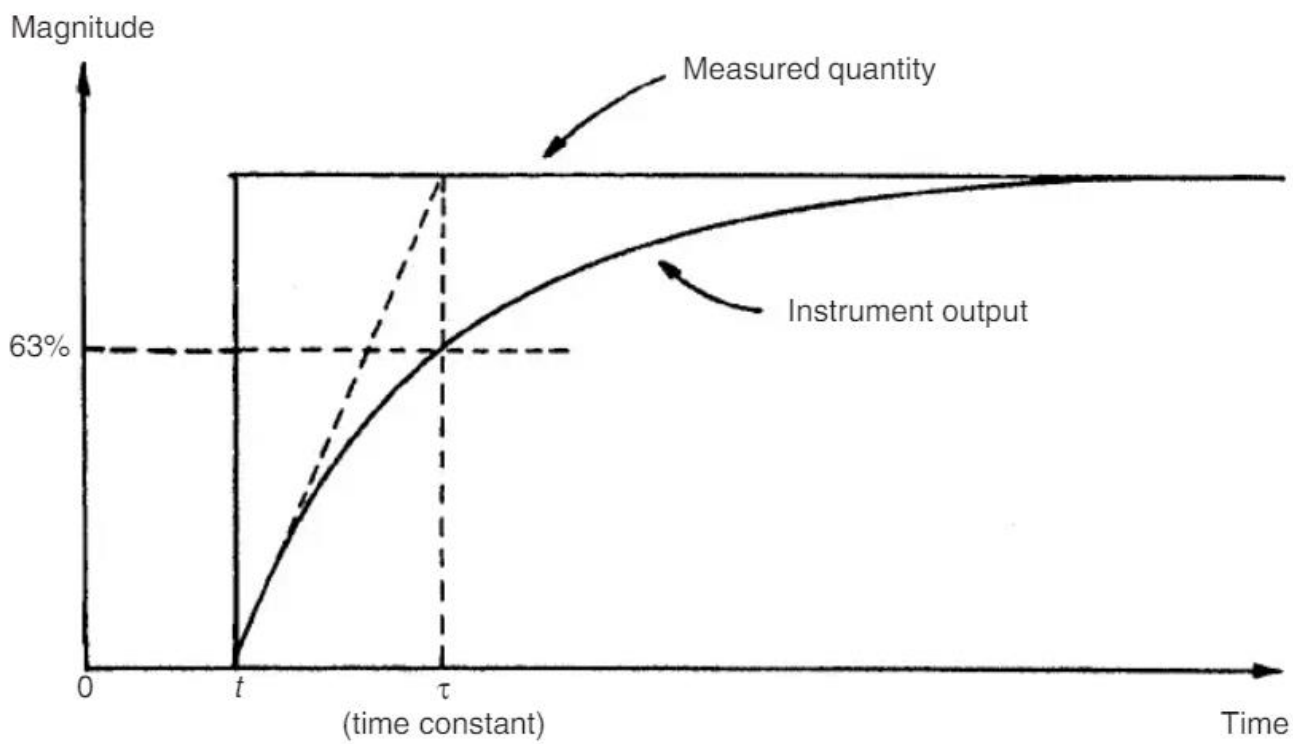


Fig. 2.11 First order instrument characteristic.

Dynamic Response of First Order Instruments

Step Response

$$\boxed{\frac{q_o}{KA} = 1 - \exp(-t/\tau)}$$

- **Measurement error**

$$e_m = q_i - \frac{q_o}{K} = A - A[1 - \exp(-t/\tau)]$$

$$\frac{e_m}{A} = \exp(-t/\tau)$$

- **Steady state error**

$$e_{ss} = \lim_{t \to \infty} e_m = 0$$

Dynamic Response of First Order Instruments

- **Step Response**

$$\boxed{\frac{q_o}{KA} = 1 - \exp(-t/\tau)}$$

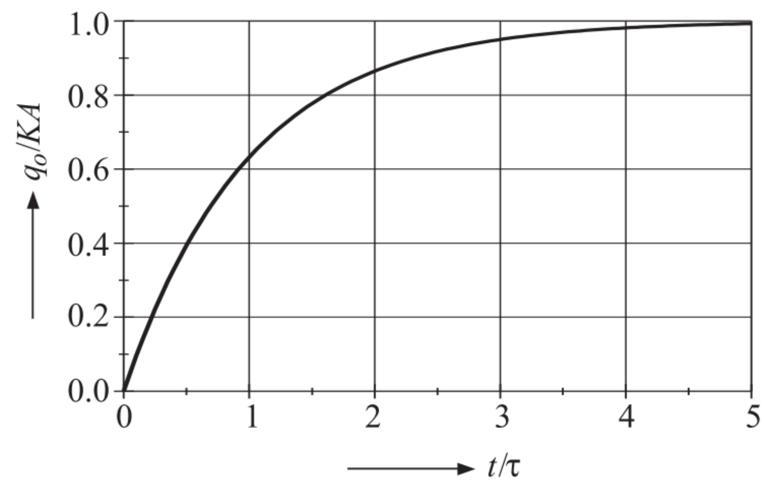


Fig. 4.7 Step response of the first order instrument.

Table 4.1 Values of non-dimensionalised parameters for step response of the first order instrument

t/τ	0	1	2	3	4	5	∞
q_o/KA	0.000	0.632	0.865	0.950	0.982	0.993	1.000

1. **The speed of response depends only on the value of τ**
2. **The response reaches within 5% of its final value at 3τ**
3. **The steady-state value can be assumed to have reached around 5τ**

First Order Instrument

❑ Step Response

Example 4.1

A thermometer, initially at 70°C, is suddenly dipped in a liquid at 300°C. After 3 s, the thermometer indicates 200°C. After what time is the thermometer expected to give a reliable reading, say well within 1% of the actual value?

Solution

This is obviously a case of a step input, the step being not from an initial zero value, but a finite value of $\theta_0 = 70°\text{C}$. So if we denote θ_3 as the thermometer reading after 3 s, our equation is

$$\theta_3 = (300 - 70)\left[1 - \exp\left(-\frac{3}{\tau}\right)\right] + 70 = 200\,°\text{C} \quad \text{(given)}$$

or

$$\exp\left(-\frac{3}{\tau}\right) = \frac{230 - 130}{230} = \frac{10}{23}$$

or

$$-\frac{3}{\tau} = \ln 10 - \ln 23 = -0.8329$$

or

$$\tau \cong 3.6 \text{ s}$$

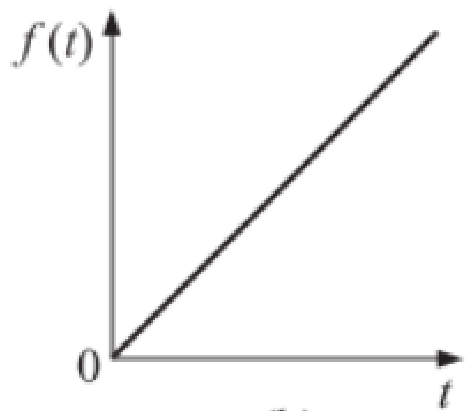
Since a reliable reading can be obtained at 5τ, the required time is 18 s.

First Order Instrument

Ramp Input

The functional form [Fig. 4.4(b)] and the Laplace transform are

$$f(t) = At \qquad \text{and} \qquad F(s) = \frac{A}{s^2}$$



(b)

First Order Instrument

- **Ramp Response**

Ramp response

Here $Q_i(s) = A/s^2$. Therefore,

$$Q_o(s) \equiv G(s)Q_i(s) = \frac{KA}{s^2(1+\tau s)} = KA\left(\frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{1+\tau s}\right)$$

On taking inverse Laplace transform, we get

$$q_o(t) = KA\Big[t - \tau\Big\{1 - \exp\Big(-\frac{t}{\tau}\Big)\Big\}\Big]$$

First Order Instrument

- **Ramp Response**

Measurement error. The measurement error is

$$e_m = At - A\left[t - \tau\left\{1 - \exp\left(-\frac{t}{\tau}\right)\right\}\right]$$

$$= A\tau\left\{1 - \exp\left(-\frac{t}{\tau}\right)\right\}$$

$$= -A\tau\exp\left(-\frac{t}{\tau}\right) + A\tau$$

transient error steady-state error

Steady-state error. The steady-state error is

$$e_{ss} = \lim_{t\to\infty} e_m = A\tau$$

First Order Instrument

- ❑ **Lag**
  - ❑ The lag indicates what time the system takes to output the input signal

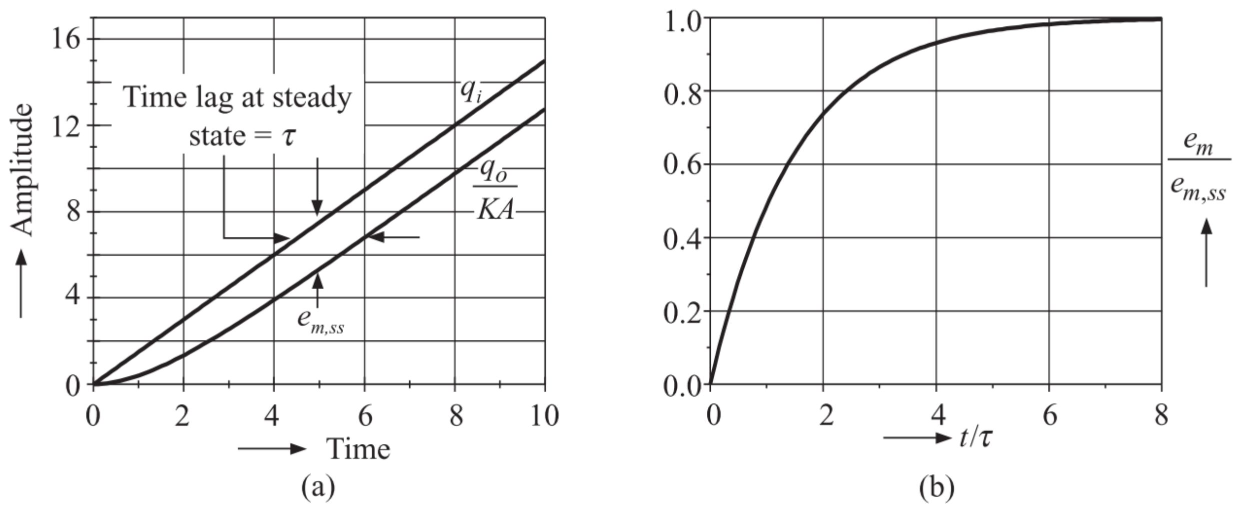
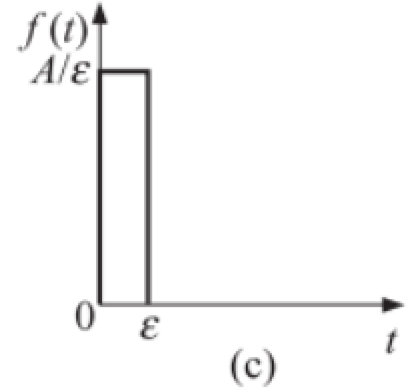


Fig. 4.8 Ramp response of the first order instrument: (a) actual response, and (b) error plot.

- ❑ The instrument's reading always lags behind the actual value
- ❑ The instrument shows a value what the input was τ seconds ago

First Order Instrument



Impulse Input

The impulse function, [Fig. 4.4(c)] related to the Dirac δ-function is defined as

$$f(t) = A\delta(t)$$

where,

$$\delta(t) = \begin{cases} \lim\limits_{\varepsilon \to 0} \dfrac{1}{\varepsilon} & \text{for } 0 \le t \le \varepsilon \\ 0 & \text{for } t > \epsilon \end{cases}$$

By definition, $\int_0^\infty \delta(t)dt = 1$. Therefore, the corresponding Laplace transform[3] is $\mathscr{L}\{\delta(t)\} = 1$. Thus,

$$F(s) = A$$

Armed with this background knowledge we will now study dynamic responses of different orders of instruments. But before that we will see what characteristics we want to watch. In other words, what are the dynamic characteristics of instruments.

Here, $Q_i(s) = A$. Therefore,

$$Q_o(s) = \frac{KA}{1+\tau s} = \frac{KA}{\tau}\frac{1}{s+(1/\tau)}$$

Taking inverse Laplace transform, we get

$$q_o(t) = \frac{KA}{\tau}\exp\left(-\frac{t}{\tau}\right)$$

It may be noted in this context that the realisation of an impulse input is rather impossible for a physical system. Because, for such an input, at $t = 0$, $q_i(t)$ has an infinite slope and it goes down to zero value at $t = \varepsilon$, where $\varepsilon \to 0$ (Fig. 4.9).

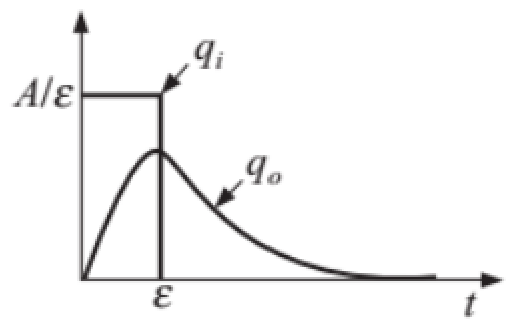


Fig. 4.9 Impulse response of first order instrument.

For any physical system to respond to such an input, it is necessary to transfer energy at an infinite rate which is not feasible. However, the derivative of a step function is an impulse function. Such a situation is observed at the output of a capacitor when a step input is applied to it.

Dynamic Characteristics

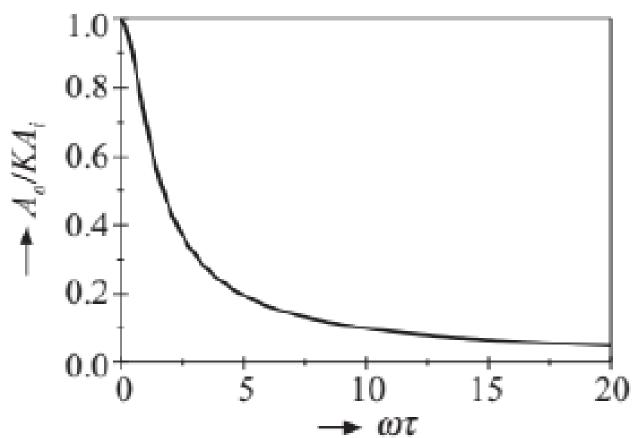
First Order Systems → Dynamic Response

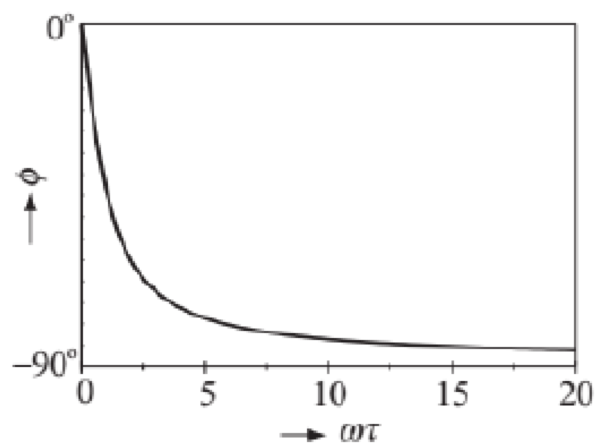
Frequency response:

$$G(s) \equiv \frac{Q_o(s)}{Q_i(s)} = \frac{K}{1+\tau s} \longrightarrow G(j\omega) = \frac{K}{1+j\omega\tau} = \frac{K}{\sqrt{1+\omega^2\tau^2}} \angle \tan^{-1}(-\omega\tau)$$

$$\frac{A_o}{A_i} = \frac{K}{\sqrt{1+\omega^2\tau^2}}$$

$$\phi = \tan^{-1}(-\omega\tau)$$





Second Order Instrument

- The dynamic relation between the input and output,

$$a_2 \frac{d^2 q_o}{dt^2} + a_1 \frac{dq_o}{dt} + a_0 q_o = b_0 q_i$$

$$\frac{1}{\omega_n^2} \frac{d^2 q_o}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dq_o}{dt} + q_o = K q_i$$

where,

$$K = b_0 / a_0 = \text{static sensitivity}$$

$$\omega_n = \sqrt{a_\circ / a_2} = \text{ natural frequency}$$

$$\zeta = a_1 / (2\sqrt{a_0 a_2}) = \text{damping ratio}$$

*The **damping ratio** is a dimensionless measure describing how oscillations in a system decay after a disturbance. The damping ratio is a measure describing how rapidly the oscillations decay from one bounce to the next.*

Damping Ratio?

Damping ratio describes how rapidly the Amplitude of a Vibrating system decays with respect to time.

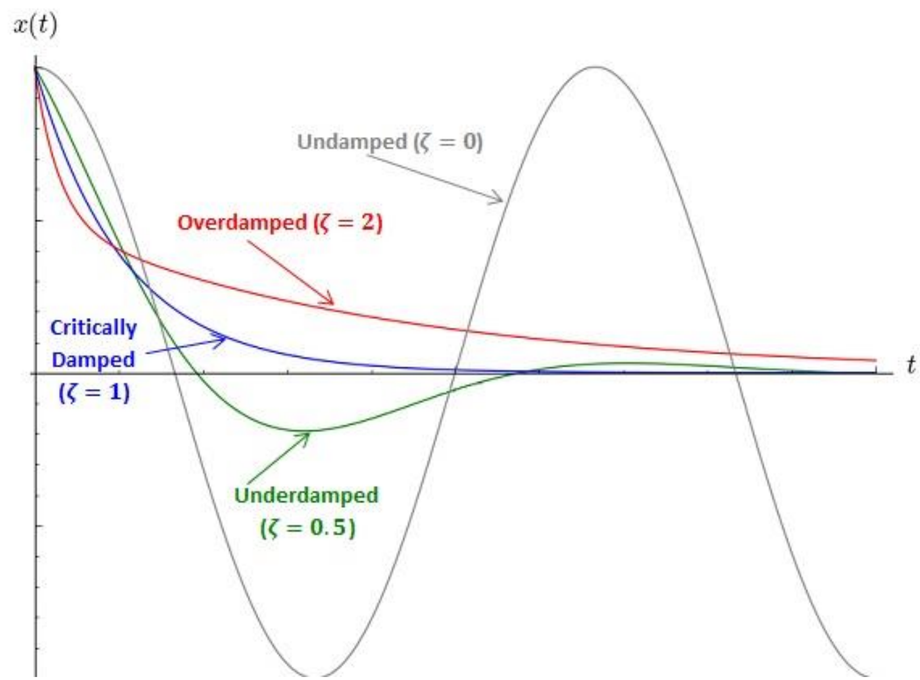
Depending on the value of ζ, three cases present themselves:

1. $\zeta > 1$, called the *overdamped* system
2. $\zeta = 1$, called the *critically-damped* system
3. $\zeta < 1$, called the *underdamped* system

Underdamped (structure oscillates to reach equilibrium)

Critically Damped (structure does not oscillate to reach equilibrium)

Overdamped (no oscillations and slower response to reach equilibrium)



Second Order Instrument

- The dynamic relation between the input and output,

$$a_2\frac{d^2q_o}{dt^2}+a_1\frac{dq_o}{dt}+a_0q_o=b_0q_i$$

$$\frac{1}{\omega_n^2}\frac{d^2q_o}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dq_o}{dt}+q_o=Kq_i$$

- **Taking Laplace transform**

$$\frac{s^2}{\omega_n^2}Q_o+\frac{2\zeta s}{\omega_n}Q_o+Q_o=KQ_i$$

- **Transfer function**

$$G(s)\equiv\frac{Q_o}{Q_i}=\frac{K}{(s^2/\omega_n^2)+(2\zeta s/\omega_n)+1}=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Second Order Instrument

- **Transfer function**

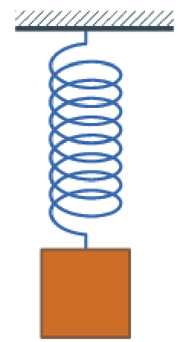
$$G(s) \equiv \frac{Q_o}{Q_i} = \frac{K}{(s^2/\omega_n^2) + (2\zeta s/\omega_n) + 1} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- **Denominator contains a quadratic expression which has two roots**

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Depending on the value of ζ, three cases present themselves:

1. $\zeta > 1$, called the *overdamped* system
2. $\zeta = 1$, called the *critically-damped* system
3. $\zeta < 1$, called the *underdamped* system



Second Order Instrument

❑ Mass-spring-damper system

When a force f is applied to the mass M downwards, it is acted upon by

1. An inertial force $f_M = M(d^2x/dt^2)$
2. A restoring force because of the spring stiffness $f_k = kx$
3. Another restoring force owing to viscous friction with the air $f_d = D(dx/dt)$

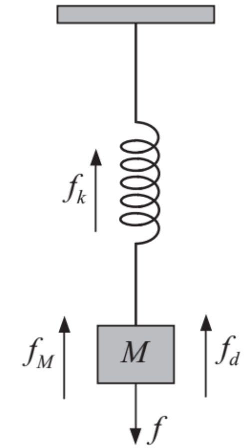


Fig. 4.12 Mass-spring-damper system.

$$\boxed{M\frac{d^2x}{dt^2} + D\frac{dx}{dt} + kx = f}$$

$$a_2\frac{d^2q_o}{dt^2} + a_1\frac{dq_o}{dt} + a_0q_o = b_0q_i$$

$$\frac{1}{\omega_n^2}\frac{d^2q_o}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dq_o}{dt} + q_o = Kq_i \qquad (4.18)$$

where, k is the stiffness constant of the spring and D is the damping constant. Comparing Eq. (4.19) with Eq. (4.18), we may write

$$\text{Natural frequency} \qquad \omega_n = \sqrt{\frac{k}{M}} \qquad (4.20)$$

$$\text{Damping ratio} \qquad \zeta = \frac{D}{2\sqrt{kM}} \qquad (4.21)$$

Second Order Instrument

- **Step response**

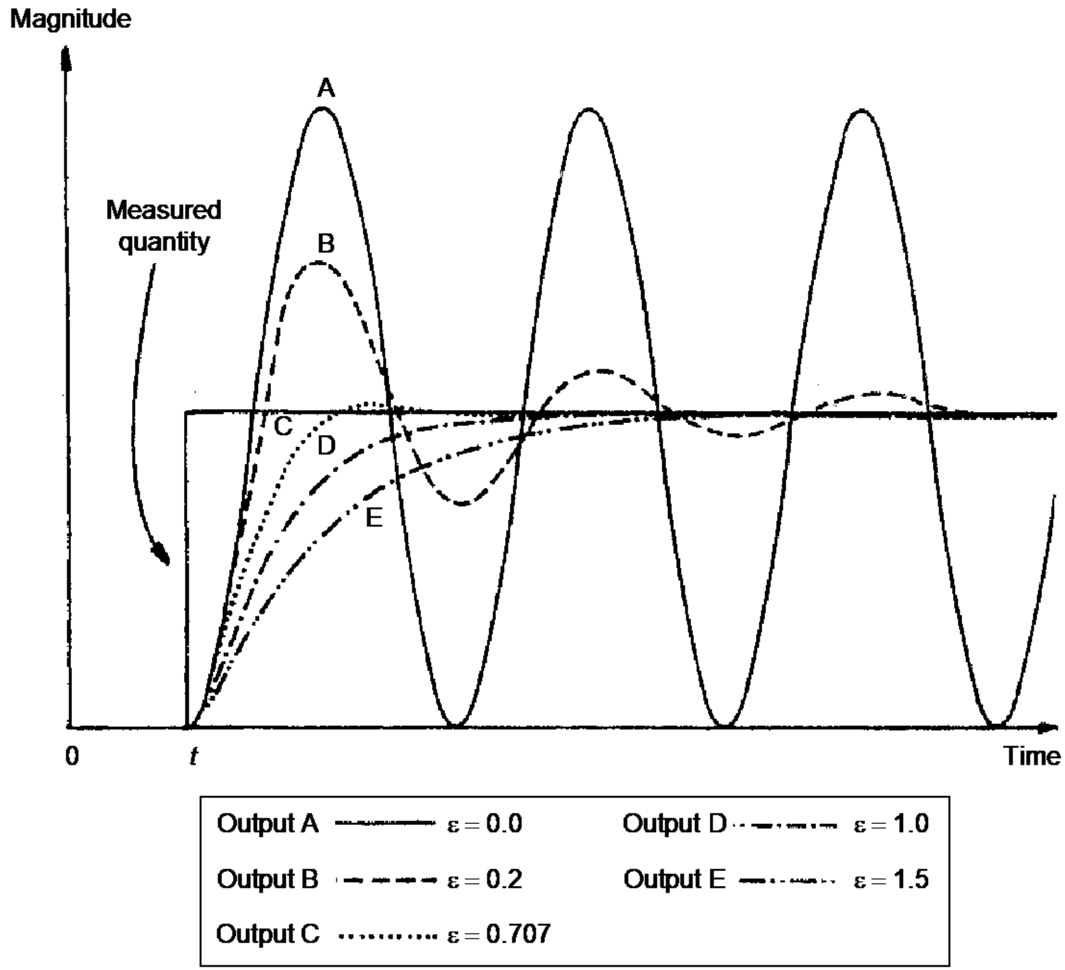


Figure 2.12
Response characteristics of second-order instruments.

Second Order Instrument

- **Step response**

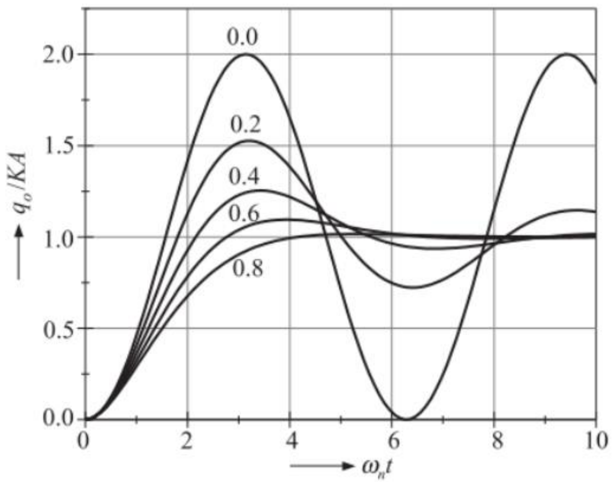


Fig. 4.13 Step response of a second order instrument for different damping ratios.

- **Ramp Response**

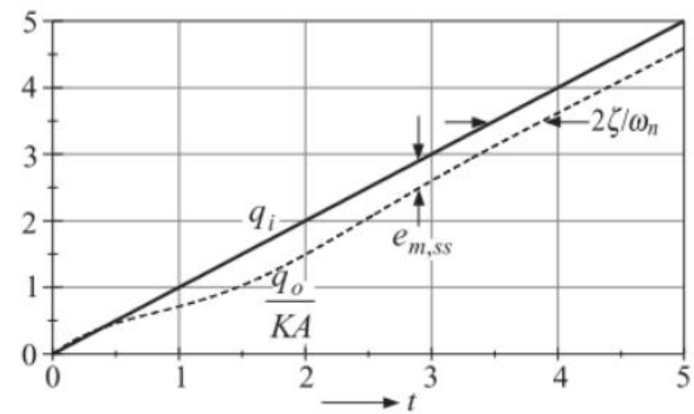


Fig. 4.16 Ramp response of second order instrument.

- **Impulse Response**

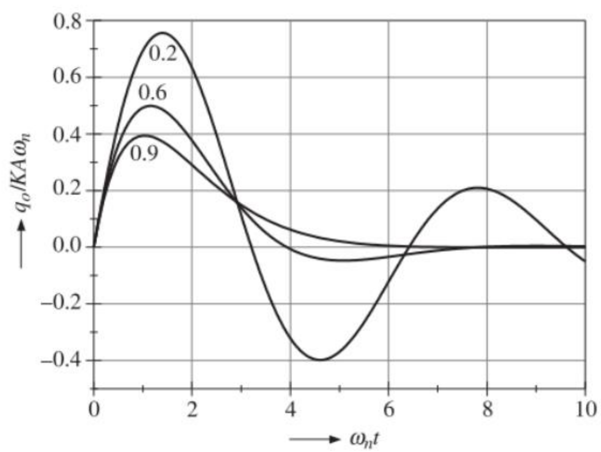


Fig. 4.17 Impulse response of a second order instrument for different values of ζ.

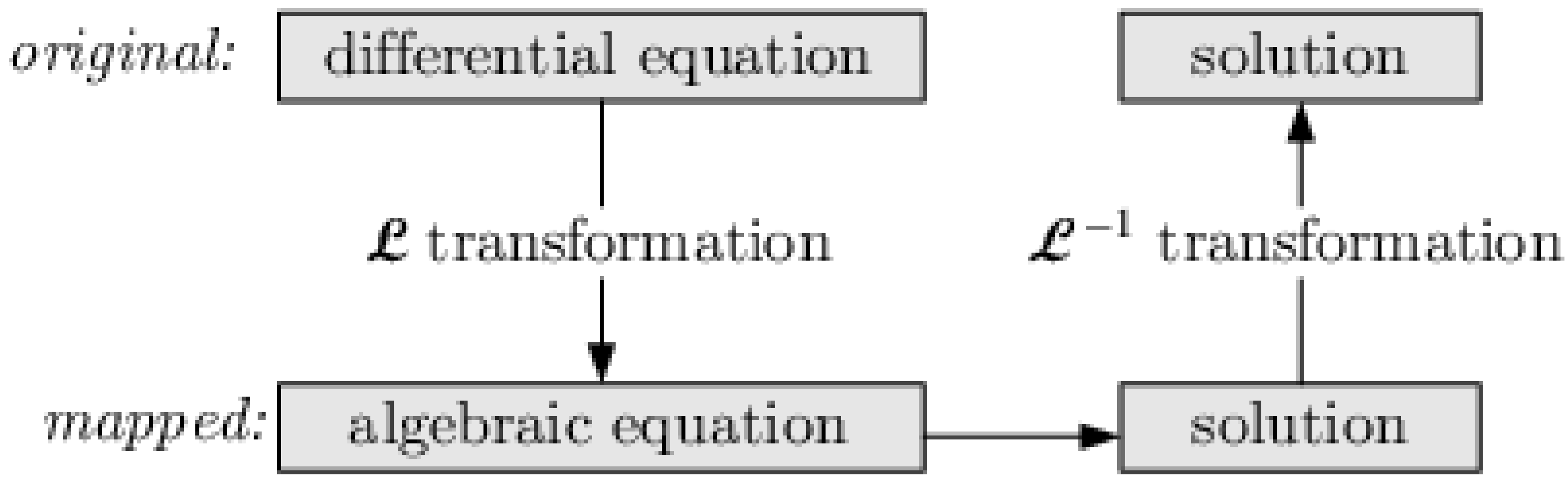
Summary

Laplace Transforms: Solution of ODE

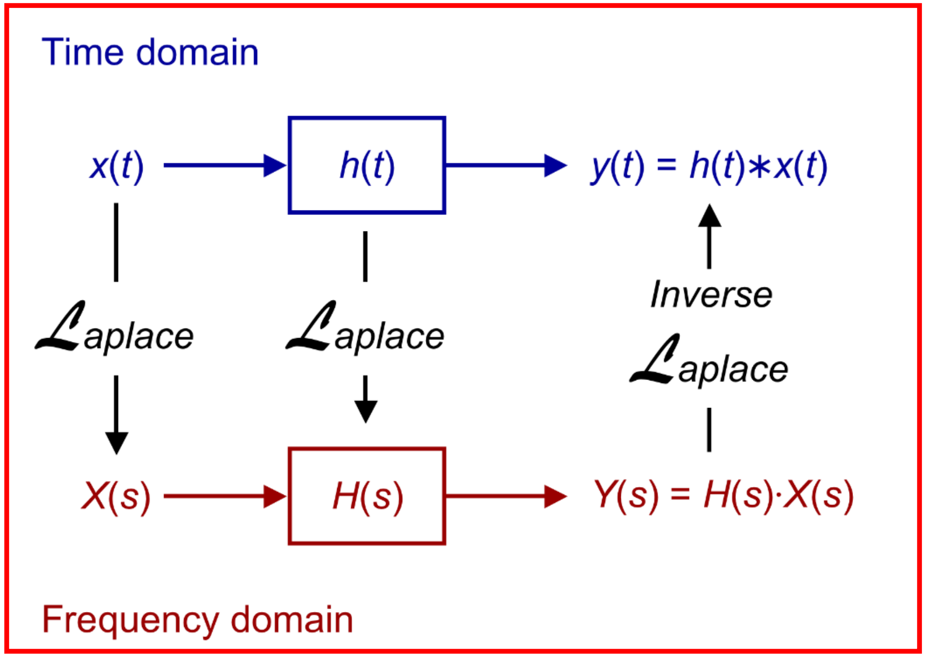
- The Laplace transform, is an elegant way for fast and schematic solving of linear differential equations with constant coefficients.
- Instead of solving the differential equation with the initial conditions directly in the original domain, the detour via a mapping into the frequency domain is taken, where only an algebraic equation has to be solved.
- Thus solving differential equations is performed in the following three steps (*schematic shown in next slide*):

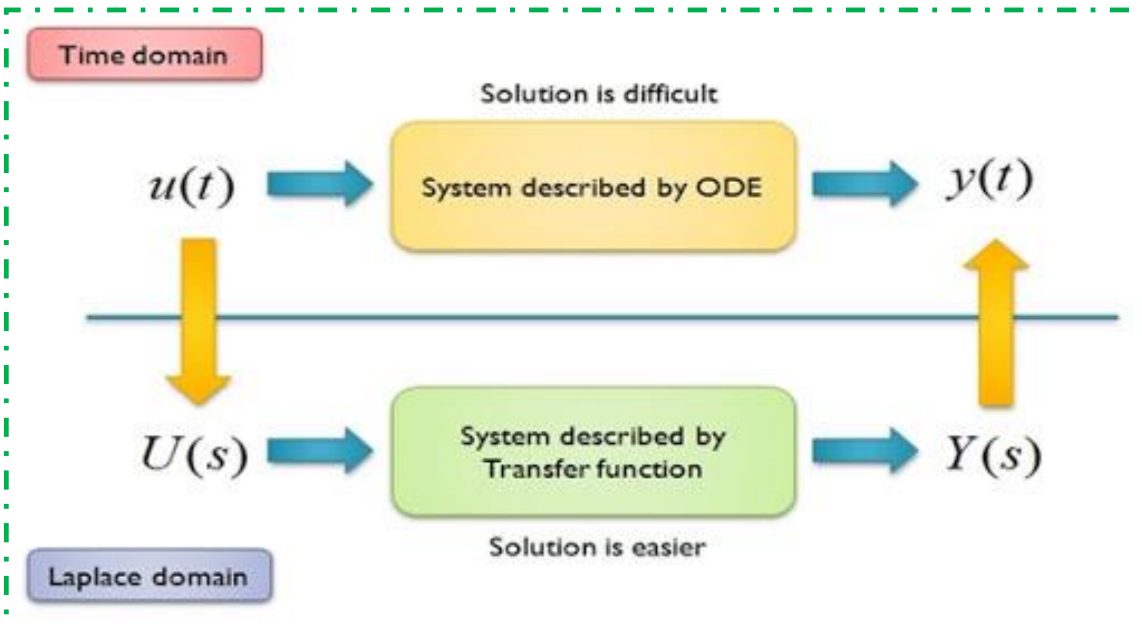
1) Transformation of the differential equation into the mapped space,
2) Solving the algebraic equation in the mapped space, and
3) Back transformation of the solution into the original space.

Schema for solving differential equations using the Laplace transformation



Dynamic Characteristics





Behavior of the Instrument?

Zero order

$$a_0 y = x(t)$$

First order

$$a_1 \frac{dy}{dt} + a_0 y = x(t)$$

Second order

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = x(t)$$

***n*th order**

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots\ldots\ldots\ldots a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = x(t)$$

Queries

Thanks!