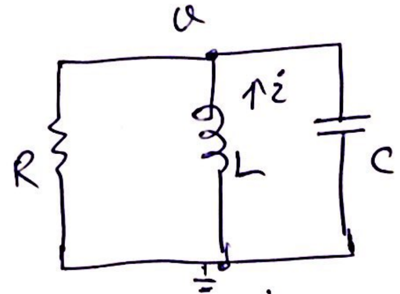
9.1 The Source-Free Parallel RLC Circuit

(pp 322 8th Ed HKD)

Consider:



— The direction of 'i' is arbitrary. And the initial conditions are

$$i(0^+) = I_0$$
$$\text{and } v(0^+) = V_0$$
HKD

{Sadiku $i(0) = I_0$ initial inductor current etc}

— The single nodal equation can be written as:-

$$\frac{v}{R} + \frac{1}{L}\int_{t_0}^{t} v\,dt' - i(t_0) + C\frac{dv}{dt} = 0$$

— This integro-differential equation can be differentiated to get a linear 2nd-order differential equation as:

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

— its solution $v(t)$ is the desired natural response.

— Note: For series RLC:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

Solution of the Differential Equation
(PP 322 8th Ed HKD)

The differential equation is

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

Let us assume the solution as

$$v = Ae^{st}$$

where A and s may be complex numbers, if necessary.

Then $$\frac{dv}{dt} = Ase^{st}$$

and $$\frac{d^2v}{dt^2} = As^2e^{st}$$

— Substituting the assumed solution in the second-order differential equation, we get

$$C(As^2e^{st}) + \frac{1}{R}(Ase^{st}) + \frac{1}{L}(Ae^{st}) = 0$$

So $$Cs^2 + \frac{1}{R}s + \frac{1}{L} = 0 \qquad \left\{ s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right\}$$

because $Ae^{st} \neq 0$

— This equation is usually called the auxilliary equation or the charateristic equation.

— Since this is a quadratic equation, there are two solutions

$$s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

———— contd

— Contd (324)

So we have the general form of the natural response as

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where A_1 and A_2 are two arbitrary constants which are to be selected to satisfy the two specified initial conditions.

Defintion of Frequency Terms
(PP 324 8th Ed HKD)

The general form of the natural response is

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Since the exponents $s_1 t$ and $s_2 t$ must be dimensionless, s_1 and s_2 must have the unit of some dimensionless quantity "per second".

— So from $s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$

we see that the units of $\frac{1}{2RC}$ and $\frac{1}{\sqrt{LC}}$ must also be s^{-1}.

— Units of this type are called frequencies.

— we define w_0 (Omega-zero)

$$w_0 = \frac{1}{\sqrt{LC}}$$

and reserve the term <u>resonant frequency</u> for it.

— We also call $\frac{1}{2RC}$ the <u>neper frequency</u> or the exponential damping coefficient α, so $\alpha = \frac{1}{2RC}$

— Note: α is called <u>exponential damping coefficient</u> as it is a measure of how rapidly the natural response decays to its steady final value (usually zero).

——— contd

— contd (324)

Finally s_1 and s_2 are called complex frequencies.

So the natural response of the parallel RLC circuit is

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC}$$ ($\alpha = \frac{R}{2L}$ for series)

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and A_1 and A_2 must be found by applying the given initial conditions.

Note: The ratio of α to ω_0 $\left(\frac{\alpha}{\omega_0}\right)$ is called the damping ratio by control system engineers and is designated by ξ (zeta).

Summary:

ω_0 = resonant frequency (undamped natural frequency)

ω_d = natural resonant frequency (damped natural frequency)

α = exponential damping coefficient

$\frac{\alpha}{\omega_0}$ = damping ratio