Energy & power: The total energy associated with the signal $f(t)$ is

$$E = \int_{-\infty}^{\infty} [f(t)]^2 \, dt \quad \text{——— (i)}$$

Parseval's theorem:

$$E = \int_{-\infty}^{\infty} [f(t)]^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(jw)|^2 \, dw \quad \text{——— (ii)}$$

equation (ii) relates the total energy of the signal $f(t)$ to the integral over all frequencies of $|F(jw)|^2$. For this reason, $|F(jw)|^2$ is called the energy spectral density.

Ex:- Determine the energy spectral density and total energy of the one-sided exponential function

$$f(t) = e^{-at} H(t) \quad , \quad a > 0 .$$

Sol:- we know that $F(jw) = \frac{1}{a+jw}$, $|F(w)|^2 = \frac{1}{a^2+w^2}$ is the energy-spectral density.

$$\text{Total energy} = E = \int_{-\infty}^{\infty} \left[e^{-at} H(t) \right]^2 = \int_{0}^{\infty} e^{-2at} \, dt$$

$$= \left[\frac{e^{-2at}}{-2a} \right]_0^{\infty} = \frac{-1}{2a} [0 - 1] = \frac{1}{2a} .$$

OR:

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(jw)|^2 \, dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a^2+w^2} \, dw$$

$$= \frac{1}{2\pi} \left[\frac{1}{a} \tan^{-1}\left(\frac{w}{a}\right) \right]_{-\infty}^{\infty} = \frac{1}{2\pi a} \left[\tan^{-1}(\infty) - \tan^{-1}(-\infty) \right]$$

$$= \frac{1}{2\pi a} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right] = \frac{1}{2\pi a} [\pi] = \frac{1}{2a} .$$

Ex:- Determine the energy of the signal:

$$f(t) = 2\,\frac{\sin 10(t-\pi/10)}{10(t-\pi/10)}.$$

Sol:- $E = \int_{-\infty}^{\infty} [f(t)]^2\,dt$; is difficult to evaluate.

So, we use Parseval's theorem, $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\,d\omega$.

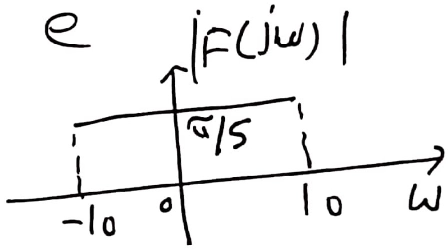
First, we calculate $F(j\omega) = \mathcal{F}[f(t)]$.

$$2\,\frac{\sin 10t}{10t} = 2\,\mathrm{sinc}(10t).$$

$$K\,\mathrm{sinc}(at) \longleftrightarrow \frac{\pi K}{a}\,\mathrm{rect}\left(\frac{\omega}{2a}\right).$$

$$2\,\mathrm{sinc}(10t) \longleftrightarrow \frac{\pi(2)}{10}\,\mathrm{rect}\left(\frac{\omega}{2(10)}\right) = \frac{\pi}{5}\,\mathrm{rect}\left(\frac{\omega}{20}\right).$$

$f(t)$ is delay, so $F(j\omega) = \frac{\pi}{5}\,\mathrm{rect}\left(\frac{\omega}{20}\right) e^{-j\frac{\omega}{10}\pi}$



$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\,d\omega$$

$$= \frac{1}{2\pi}\int_{-10}^{10} \left(\frac{\pi}{5}\right)^2 d\omega = \frac{1}{2\pi}\left(\frac{\pi^2}{25}\right)(20) = \frac{2\pi}{5}\ \mathrm{J}.$$

The idea of the "size" of a signal is crucial to many applications. It is nice to know, for example, how much electricity can be used in a defibrillator without ill effects or the amount of the signal driving a set of headphone. So, we use the concept of energy and power.

Defibrillation: Applying a controlled shock to allow restoration of the normal rythm in a serious cardic arrest.

Power of a signal:- There are important signals $f(t)$, defined in general for $-\infty < t < \infty$, for which the integral $\int_{-\infty}^{\infty} [f(t)]^2 dt$ is either unbounded or does not Converge to a finite limit. For such signals, instead of Considering energy, we consider the average power P, frequently referred to as the Power of the signal. This is defined by

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2 dt .$$

Ex:- Calculate the Power associated with the signal:

$$f(t) = \cos at , \quad -\infty < t < \infty .$$

Sol:- Note $E = \int_{-\infty}^{\infty} \cos^2 at$ is unbounded.

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \cos^2 at \, dt = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[\frac{1 + \cos 2at}{2} \right] dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \left[\frac{t}{2} + \frac{1}{2a} \sin 2at \right]_{-T/2}^{T/2} = \lim_{T \to \infty} \left[\frac{1}{T} \left(\frac{T}{2} - \left(-\frac{T}{2} \right) \right) - 0 \right]$$

$$P = \frac{1}{2} .$$

Def: The signals with finite associated energy are called energy signals. The signals with unbounded energy but finite power associated are called Power signals.

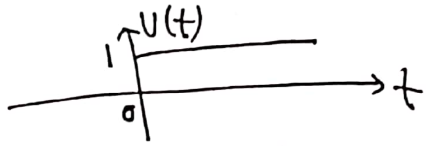
Note: The power (average) associated with energy signal is zero.

F.T. 14

Fourier transform of Heaviside unit-step function:-

Signum ftn: $\operatorname{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0. \end{cases}$

$$H(t) = U(t) = \frac{1}{2}\left[1 + \operatorname{sgn}(t)\right]$$



$\operatorname{sgn}(t)$ is not absolute integrable. However, if we approximate $e^{-\epsilon|t|}\operatorname{signum}(t)$, $\epsilon \to 0$,

$$F[\operatorname{sgn}(t)] = \lim_{\epsilon \to 0}\left[\int_{-\infty}^{0}(-1)\, e^{\epsilon t} \cdot e^{-j\omega t}\, dt + \int_{0}^{\infty}(1)\, e^{-\epsilon t} \cdot e^{-j\omega t}\, dt\right].$$

$$= \lim_{\epsilon \to 0}\left[-\frac{e^{(\epsilon - j\omega)t}}{\epsilon - j\omega}\Big|_{-\infty}^{0} + \frac{e^{-(\epsilon + j\omega)t}}{-(\epsilon + j\omega)}\Big|_{0}^{\infty}\right]$$

$$= \lim_{\epsilon \to 0}\left[-\left(\frac{1}{\epsilon - j\omega} - 0\right) + \left(0 - \frac{1}{-(\epsilon + j\omega)}\right)\right]$$

$$= \lim_{\epsilon \to 0}\left[-\frac{1}{\epsilon - j\omega} + \frac{1}{\epsilon + j\omega}\right] = \lim_{\epsilon \to 0}\left[\frac{\epsilon - j\omega - \epsilon - j\omega}{(\epsilon + j\omega)(\epsilon - j\omega)}\right]$$

$$= \lim_{\epsilon \to 0}\left[\frac{-2j\omega}{\epsilon^2 + \omega^2}\right] = -\frac{2j\omega}{\omega^2} = -\frac{2}{\omega}j = \frac{2}{j\omega}.$$

Hence, $F[U(t)] = F\left[\frac{1}{2}(1 + \operatorname{sgn}(t))\right]$

$$= \frac{1}{2}F(1) + \frac{1}{2}F[\operatorname{sgn}(t)].$$

$$= \frac{1}{2}(2\pi\delta(\omega)) + \frac{1}{2}\left(\frac{2}{j\omega}\right)$$

$$\Rightarrow F[U(t)] = \pi\delta(\omega) + \frac{1}{j\omega}.$$

Convolution: Let $f(t)$ & $g(t)$ be two functions defined for $-\infty < t < \infty$, the convolution of f & g is denoted by $f*g$, defined as

$$f*g = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)\,d\tau = g*f \quad \text{---}\ (i)$$

Convolution results involving Fourier transform:

Convolution in time: suppose, $f(t) \longleftrightarrow F(jw)$
$g(t) \longleftrightarrow G(jw)$

$$\mathcal{F}\left[f(t) * g(t)\right] = F(jw)\,G(jw) \quad \text{---}\ (ii)$$

(ii) indicates that a convolution in the time domain is transformed into a product in the frequency domain.

EX:- Let $f(x) = \begin{cases} 2xe^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$; $g(x) = \begin{cases} e^{-2x}, & x \geq 0 \\ 0, & x < 0 . \end{cases}$

Let $h(x) = (f*g)(x)$. Calculate $h(x)$, $H(jw)$ directly and using convolution theorem.

Sol:- $F(jw) = \int_{-\infty}^{\infty} 2xe^{-2x}e^{-jwx}\,dx = 2\int_0^{\infty} \underset{\mathrm{I}}{x}\ \underset{\mathrm{II}}{e^{-(2+jw)x}}\,dx.$

$$= 2\left[-x\frac{e^{-(2+jw)x}}{2+jw} - \frac{e^{-(2+jw)x}}{(2+jw)^2}\right]_0^{\infty} = 2\left[(0+0)-\left(0-\frac{1}{(2+jw)^2}\right)\right] = \frac{2}{(2+jw)^2}$$

Also, $G(jw) = \int_0^{\infty} e^{-2x}e^{-jwx}\,dx = \int_0^{\infty} e^{-(2+jw)x}\,dx = \left.\frac{e^{-(2+jw)x}}{-(2+jw)}\right|_0^{\infty} = \frac{1}{2+jw}$

using convolution theorem,

$$H(jw) = F(jw)\,G(jw) = \frac{2}{(2+jw)^2}\cdot\frac{1}{(2+jw)}$$

$$H(jw) = \frac{2}{(2+jw)^3} \quad \text{---}\ (iii)$$

F.T. 16

Now, we calculate $h(x)$ through convolution integral & then calculate its fourier transform.

$$f*g = \int_{-\infty}^{\infty} f(x-\tau)g(\tau)d\tau = \int_{0}^{\infty} f(x-\tau)g(\tau)d\tau .$$

$$f*g = \int_{0}^{\infty} f(x-\tau)e^{-2\tau}d\tau .$$

Now this integral vanishes if $x<0$, therefore $f*g=0$ for $x<0$.
for $x\geq 0$, the value of $f(x-\tau)$ is non-zero only for $0<\tau<x$.
where it equals $f(x-\tau) = 2(x-\tau)e^{-(x-\tau)2} = 2(x-\tau)e^{-2x+2\tau}$.
Therefore for $x\geq 0$, we have

$$f*g(x) = \int_{0}^{x} 2(x-\tau)e^{-2x+2\tau}d\tau \; (e^{-2\tau})$$

$$= 2\int_{0}^{x}(x-\tau)e^{-2x}d\tau = 2e^{-2x}\int_{0}^{x}(x-\tau)d\tau$$

$$= 2e^{-2x}\left(x\tau - \frac{\tau^2}{2}\right)\Big|_{\tau=0}^{\tau=x} = 2e^{-2x}\left(x^2-\frac{x^2}{2}\right) = 2e^{-2x}\left(\frac{x^2}{2}\right)$$

$$f*g(x) = \begin{cases} x^2e^{-2x}, & x\geq 0 \\ 0, & x<0 \end{cases} = h(x)$$

Now $H(\omega) = F[h(x)] = \int_0^{\infty} x^2e^{-2x}e^{-j\omega x}dx = \int_0^{\infty}\underset{\text{I}}{x^2}\,\underset{\text{II}}{e^{-(2+j\omega)x}}dx .$

$$= x^2\frac{e^{-(2+j\omega)x}}{-(2+j\omega)}\Big|_0^{\infty} - \int_0^{\infty}(2x)\frac{e^{-(2+j\omega)x}}{-(2+j\omega)}dx$$

$$= \frac{2}{2+j\omega}\int_0^{\infty}\underset{\text{I}}{x}\,\underset{\text{II}}{e^{-(2+j\omega)x}}dx = \frac{2}{2+j\omega}\left[x\frac{e^{-(2+j\omega)x}}{-(2+j\omega)}\Big|_0^{\infty} - \int_0^{\infty}(1)\frac{e^{-(2+j\omega)}}{-(2+j\omega)}dx\right]$$

$$= \frac{2}{(2+j\omega)}\frac{1}{(2+j\omega)}\left[\frac{e^{-(2+j\omega)x}}{-(2+j\omega)}\right]_0^{\infty} = \frac{2}{(2+j\omega)^3} \quad \text{———(iv)}$$

F.T.17

which is same as (iii).

EX:- Using Convolution theorem, Calculate

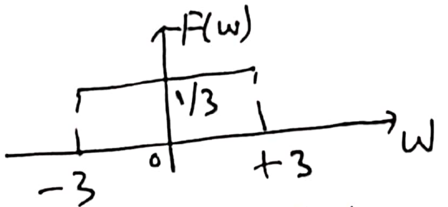
$$\mathcal{F}\left[\frac{1}{\pi}\frac{\sin 3t}{3t} * \frac{1}{\pi}\frac{\sin 5t}{5t}\right].$$

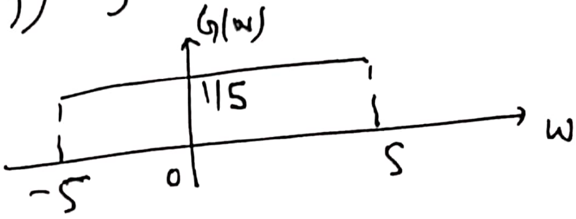
Solution:- $f(t) = \frac{1}{\pi}\frac{\sin 3t}{3t} = \frac{1}{\pi}\operatorname{sinc}(3t)$

$$F(jw) = \frac{1}{\pi}\frac{\pi}{3}\operatorname{rect}\left(\frac{w}{2(3)}\right) = \frac{1}{3}\operatorname{rect}\left(\frac{w}{6}\right).$$

$$g(t) = \frac{1}{\pi}\frac{\sin 5t}{5t} = \frac{1}{\pi}\operatorname{sinc}(5t)$$

$$G(jw) = \frac{1}{\pi}\frac{\pi}{5}\operatorname{rect}\left(\frac{w}{2(5)}\right) = \frac{1}{5}\operatorname{rect}\left(\frac{w}{10}\right)$$



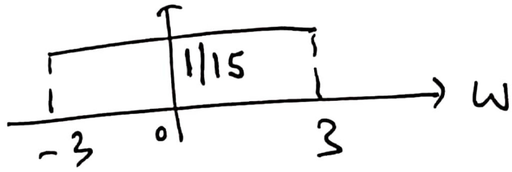


Using Convolution theorem, we have

$$\mathcal{F}[f(t) * g(t)] = F(jw)\,G(jw)$$

$$= \frac{1}{3}\operatorname{rect}\left(\frac{w}{6}\right)\frac{1}{5}\operatorname{rect}\left(\frac{w}{10}\right)$$

$$= \frac{1}{15}\operatorname{rect}\left(\frac{w}{6}\right)$$



What if we are interested in calculating

$\mathcal{F}[f(t)\,g(t)]$, transform of product

in the time, we need as :

Convolution in frequency: $f(t) \longleftrightarrow F(jw)$,

$g(t) \longleftrightarrow G(jw)$,

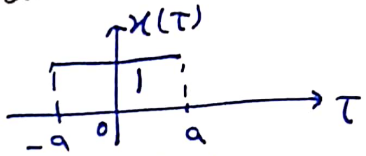
$$\mathcal{F}[f(t)\,g(t)] = F(jw) * G(jw).$$

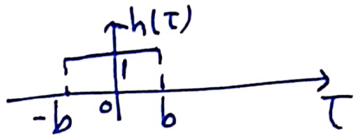
F.T. 18

To evaluate $\mathcal{F}\left[\frac{1}{\pi}\frac{\sin 3t}{3t}\cdot\frac{1}{\pi}\frac{\sin 5t}{5t}\right]$, we have to apply convolution in the frequency domain. i.e, we have to convolve two rectangular pulses.

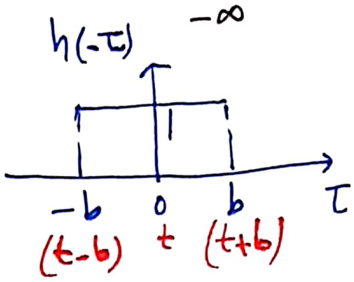
Convolution of two rectangular functions:-

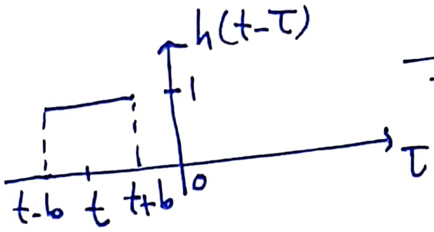
Ex: Let $x(t)=\text{rect}\left(\frac{t}{2a}\right)$, $h(t)=\text{rect}\left(\frac{t}{2b}\right)$, $a>b>0$. Calculate output $y(t)$ as $y(t)=x(t)*h(t)$.





Solution: $y(t)=\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$ —(i)

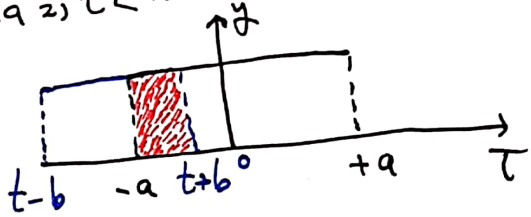




i) For $t+b<-a \Rightarrow t<-a-b$, no overlap between $x(\tau)$ and $h(t-\tau)$, $y(t)=0$, the output is zero.

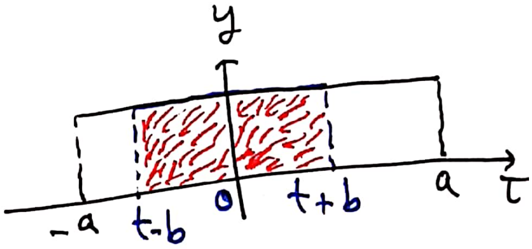
ii) when $t+b>-a \Rightarrow t>-a-b$, $t-b<-a \Rightarrow t<-a+b$,

$-a-b<t<-a+b$, $y(t)=\int_{-a}^{t+b}(1)(1)\,d\tau = t+b+a$



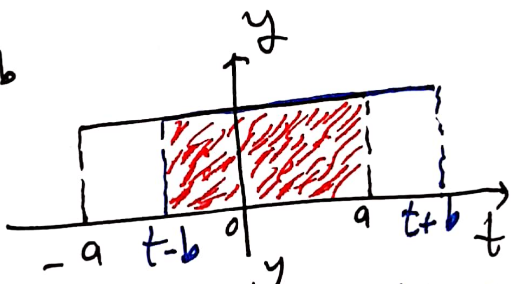
iii) when $t+b<a \Rightarrow t<a-b$
$t-b\geq -a \Rightarrow t\geq -a+b$ $\Big\}\Rightarrow -a+b\leq t<a-b$

$y(t)=\int_{t-b}^{t+b}(1)(1)\,d\tau = 2b$

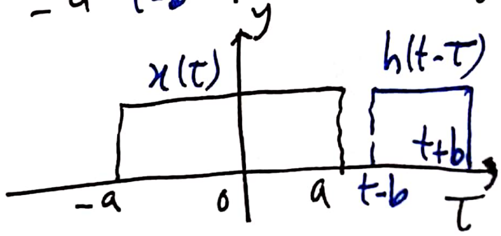


iv) when $t+b\geq a \Rightarrow t\geq a-b$
$t-b<a \Rightarrow t<a+b$ $\Big\}\Rightarrow a-b\leq t<a+b$

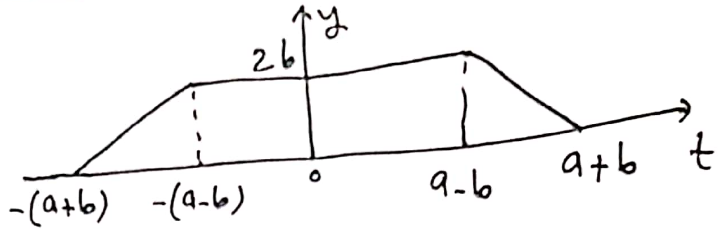
$y(t)=\int_{t-b}^{a}(1)(1)\,d\tau = -t+b+a$



v) when $t-b\geq a \Rightarrow t\geq a+b$,
no overlap.
$y(t)=0$.



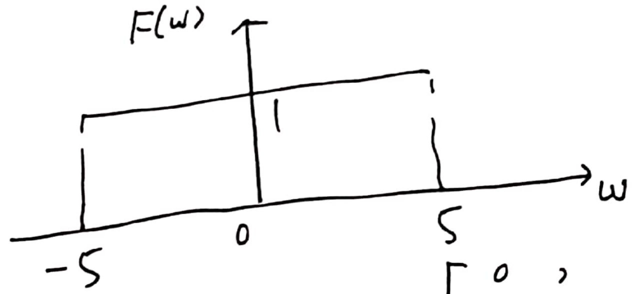
$$y(t) = \begin{cases} 0, & t < -a-b,\ t \geq a+b \\ t+a+b, & -a-b \leq t < -a+b \\ 2b, & -a+b \leq t < a-b \\ -t+a+b, & a-b \leq t < a+b. \end{cases}$$

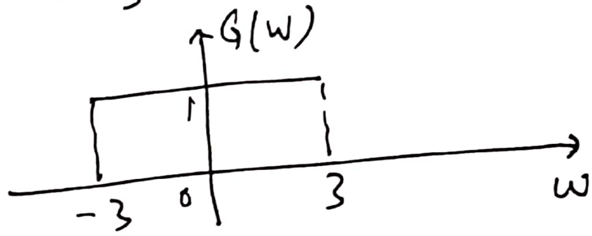


Ex:- Let $f(t) = \frac{1}{\pi}\frac{\sin 3t}{t}$, $g(t) = \frac{1}{\pi}\frac{\sin 5t}{t}$, $h(t) = f(t)\,g(t)$. Use convolution in the frequency domain to calculate $\mathcal{F}[h(t)] = H(w)$.

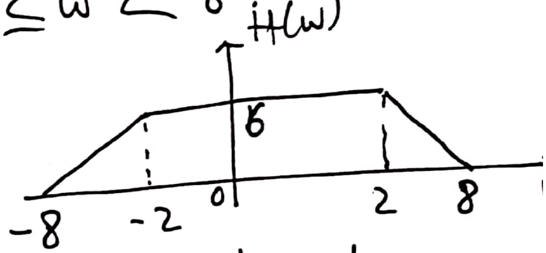
Sol:- $f(t) = \frac{1}{\pi} 3 \frac{\sin 3t}{3t} = \frac{3}{\pi}\operatorname{sinc}(3t)$, $F(w) = \frac{3}{\pi}\cdot\frac{\pi}{3}\operatorname{rect}\left(\frac{w}{6}\right)$

$g(t) = \frac{1}{\pi} 5 \frac{\sin 5t}{5t} = \frac{5}{\pi}\operatorname{sinc} 5t$, $G(w) = \frac{5}{\pi}\cdot\frac{\pi}{5}\operatorname{rect}\left(\frac{w}{10}\right)$





$$H(w) = F(w) * G(w) = \begin{cases} 0, & w < -8,\ w \geq 8 \\ w+8, & -8 \leq w < -2 \\ 6, & -2 \leq w < 2 \\ -w+8, & 2 \leq w < 8 \end{cases}$$



Exercise:

- Calculate $\operatorname{rect}(t) * \operatorname{rect}(t)$ & show that $\mathcal{F}[\operatorname{rect}(t) * \operatorname{rect}(t)] = \sin^2(w/2)$.

-- calculate & sketch Fourier transform of $f(t) = \left(\frac{\sin 2t}{t}\right)\left(\frac{\sin 6t}{6}\right)$.

F.T: 20