PROPERTIES OF FOURIER SERIES

Properties of CTFS

- We have studied the following properties of CTFS:

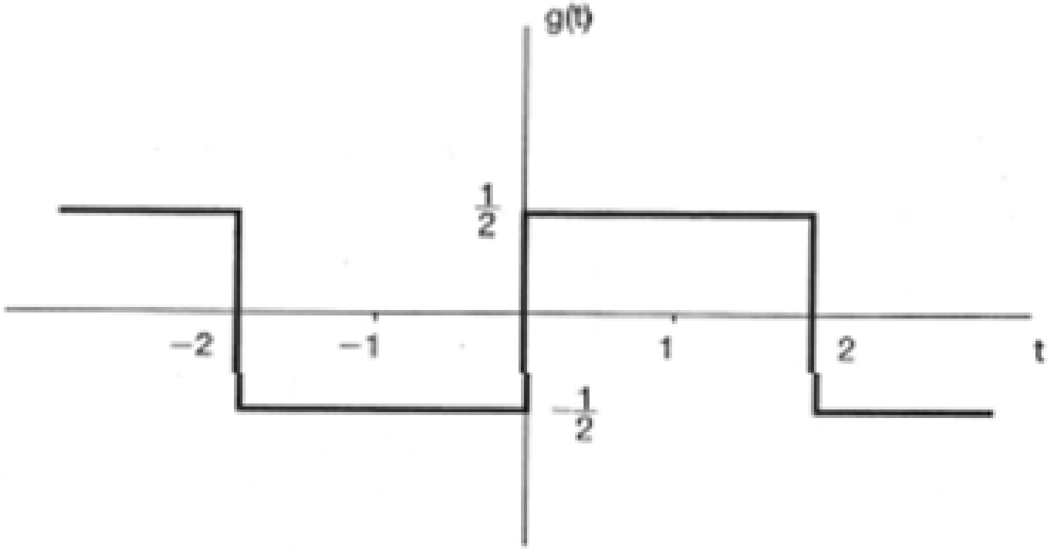
1. Linearity
2. Conjugate Symmetry
3. Time Shift
4. Time Reversal
5. Time Scaling

- Now we will study the remaining properties:

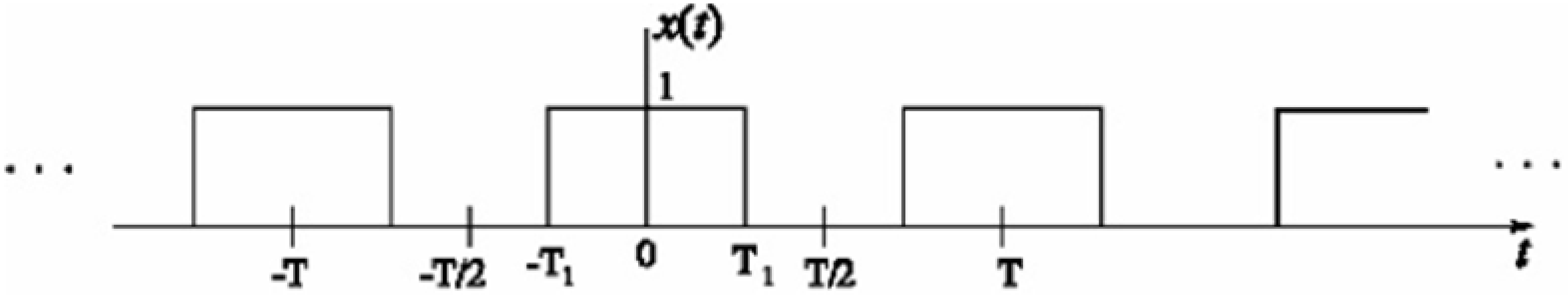
1. Multiplication
2. Parseval's Relation
3. Periodic Convolution

Properties of CTFS - Example

- Consider the periodic signal g(t) shown in figure below:



- Compute the FS coefficients d_k for g(t) using the Coefficients a_k for x(t) shown below:



Properties of CTFS - Example

$$d_k = \begin{cases} 0, & \text{for } k = 0 \\ \dfrac{\sin(\pi k / 2)}{\pi k} e^{-j\pi k/2} & \text{for } k \neq 0 \end{cases}$$

Properties of CTFS

- **Multiplication Property:**

$$\begin{array}{ccll} x(t) \leftrightarrow a_k, y(t) & \leftrightarrow & b_k & \text{(Both } x(t) \text{ and } y(t) \text{ are} \\ & \Downarrow & & \text{periodic with the same period } T) \end{array}$$

$$x(t) \cdot y(t) \quad \leftrightarrow \quad c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_k * b_k$$

Properties of CTFS

Proof:

$$\underbrace{\sum_l a_l e^{jl\omega_0 t}}_{x(t)} \cdot \underbrace{\sum_m b_m e^{jm\omega_0 t}}_{y(t)}$$

$$= \sum_{l,m} a_l b_m e^{j(l+m)\omega_0 t} \xrightarrow{l+m=k} \sum_k \underbrace{\left[\sum_l a_l b_{k-l}\right]}_{c_k} e^{jk\omega_0 t}$$

Properties of CTFS

- **Parseval's Relation**

$$\underbrace{\frac{1}{T}\int_T |x(t)|^2 dt}_{\text{Average signal power}} = \sum_{k=-\infty}^{\infty} \underbrace{|a_k|^2}_{\substack{\text{Power in the} \\ k_{th}\text{ harmonic}}}$$

Energy is the same whether measured in the time-domain or the frequency-domain

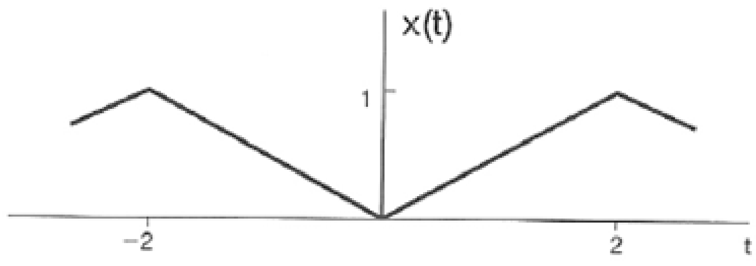
Properties – Differentiation and Integration

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Differentiation	$\dfrac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\dfrac{2\pi}{T}a_k$
Integration	$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\dfrac{1}{jk\omega_0}\right)a_k = \left(\dfrac{1}{jk(2\pi/T)}\right)a_k$

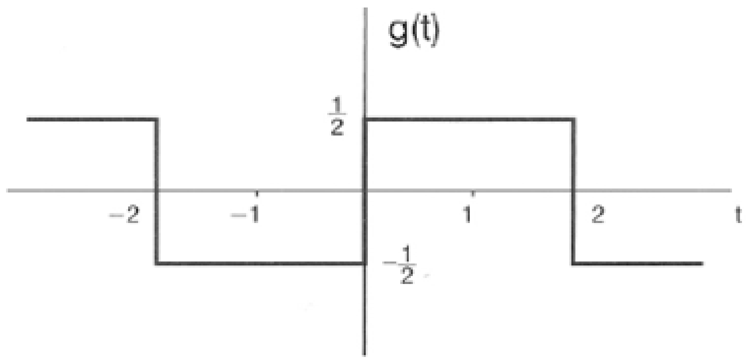
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \qquad \text{(Analysis equation)}$$

Differentiation - Example



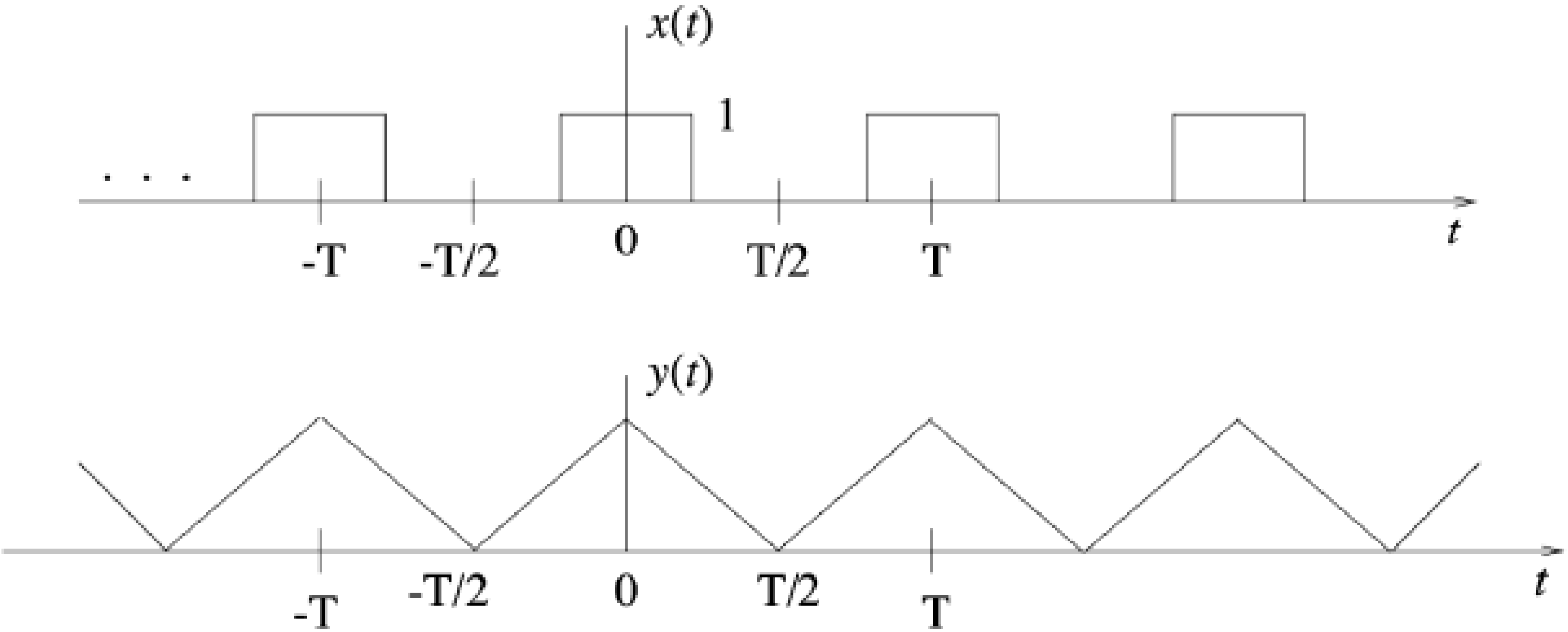
Triangular wave signal with period $T = 4$ and fundamental frequency $\omega_0 = \pi / 2$ with Fourier coefficients e_k



Derivative of triangular wave signal with Fourier coefficients d_k

Properties of CTFS – Periodic Convolution

$x(t)$, $y(t)$ **periodic with period** T



$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \quad - \text{ not very meaningful}$$

E.g. If both $x(t)$ and $y(t)$ are positive, then

$$x(t) * y(t) = \infty$$

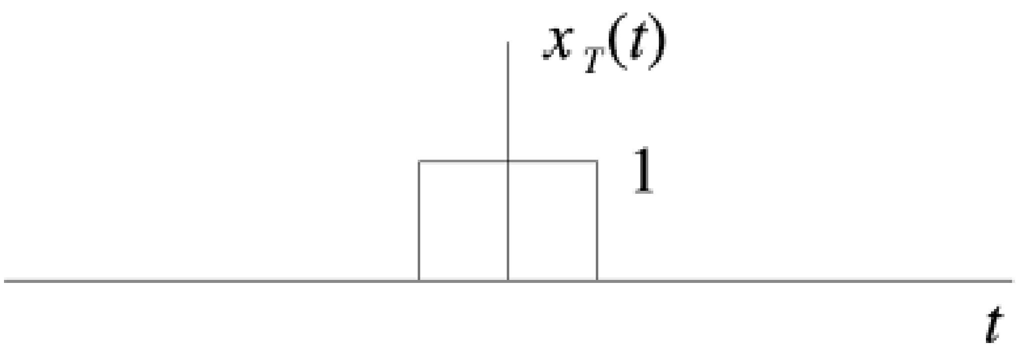
Properties of CTFS – Periodic Convolution

Periodic convolution: Integrate over *any* one period (e.g. $-T/2$ to $T/2$)

$$z(t) = \int_{-T/2}^{T/2} x(\tau)y(t-\tau)d\tau$$

where

$$x_T(t) = \begin{cases} x(t) & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$$



Periodic Convolution Facts

1) $z(t)$ is periodic with period T

- During convolution, one period of a signal slides out of the interval of integration and the next interval slides in

- Due to periodicity of the signals, the same convolution output will be achieved for the next interval

2) Doesn't matter what period over which we choose to integrate:

$$z(t) = \int_T x(\tau)y(t-\tau)d\tau = x(t) \otimes y(t)$$

Periodic Convolution Facts

3) **Periodic convolution in time**

$$x(t) \leftrightarrow a_k, y(t) \leftrightarrow b_k, z(t) \leftrightarrow c_k$$

$$\begin{aligned}
c_k &= \frac{1}{T}\int_T z(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T \left(\int_T x(\tau)y(t-\tau)d\tau\right) e^{-jk\omega_0 t}dt \\
&= \int_T \underbrace{\left(\frac{1}{T}\int_T y(t-\tau)e^{-jk\omega_0(t-\tau)}dt\right)}_{b_k} x(\tau)e^{-jk\omega_0\tau}d\tau \\
&= \int_T b_k x(\tau)e^{-jk\omega_0\tau}d\tau = Ta_kb_k
\end{aligned}$$

Multiplication in frequency!

END