Trigonometric Fourier Series:-

$\{1, \cos w_0t, \cos 2w_0t, \ldots, \cos nw_0t, \ldots, \sin w_0t, \sin 2w_0t, \ldots, \sin nw_0t\ldots\}$ $T_0 = 2\pi/w_0$.

$$\int_{T_0} \cos nw_0t \cos mw_0t \, dt = \begin{cases} 0, & m \neq n \\ T_0/2, & m = n \end{cases}, \quad \int_{T_0} \sin nw_0t \sin mw_0t \, dt = \begin{cases} 0, & m \neq n \\ T_0/2, & m = n. \end{cases}$$

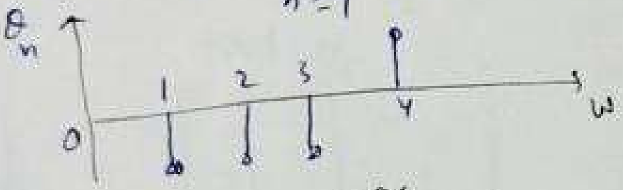
$g(t)$, $t_1 \leq t \leq t_1+T_0$, $g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nw_0t + \sum_{n=1}^{\infty} b_n \sin nw_0t$, $w_0 = 2\pi/T_0$.

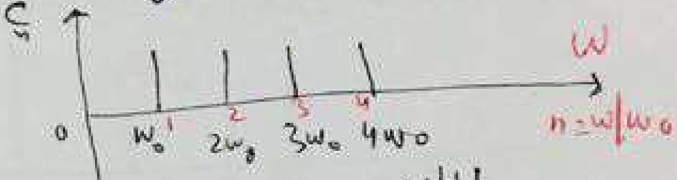
$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} g(t)\,dt, \quad a_n = \frac{2}{T_0}\int g(t)\cos(nw_0t)\,dt, \quad b_n = \frac{2}{T_0}\int g(t)\sin(nw_0t)\,dt.$$

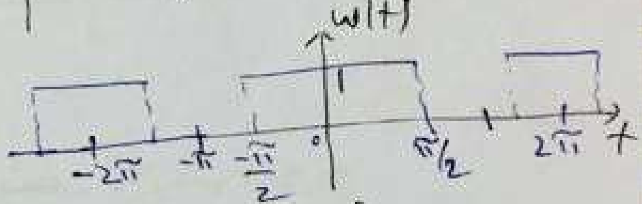
Compact TFS:- $a_n \cos nw_0t + b_n \sin nw_0t = C_n \cos(nw_0t + \theta_n)$

Where $C_n = \sqrt{a_n^2 + b_n^2}$, $\theta_n = \tan^{-1}(-b_n/a_n)$, $C_0 = a_0$.

CTFS: $g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(nw_0t + \theta_n)$, $t_1 \leq t \leq t_1 + T_0$,







EX:- $T_0 = 2\pi$, $w_0 = \frac{2\pi}{T_0} = 1$

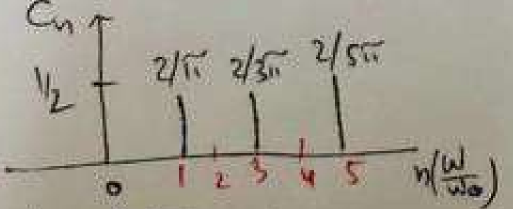
$a_0 = 1/2$, $a_n = \frac{2}{T}\int_{-T_0/4}^{T_0/4} \cos nwt\,dt = \frac{2}{n\pi}\sin\frac{n\pi}{2}$, $b_n = 0$

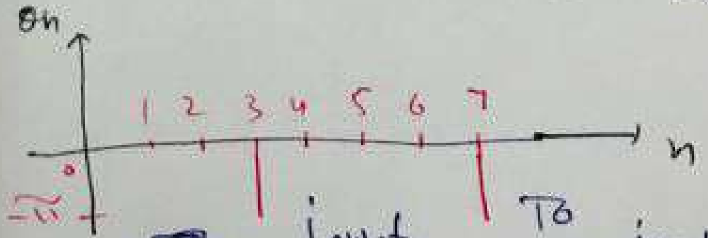
$a_n = 0$, for n even
$= \frac{2}{n\pi}$, for $n = 1, 5, 9, \ldots$
$= -\frac{2}{n\pi}$ for $n = 3, 7, 11, \ldots$

$w(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos w_0t - \frac{1}{3}\cos 3w_0t + \frac{1}{5}\cos 5w_0t - \frac{1}{7}\cos 7w_0t + \ldots\right)$

CFS: $w(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos w_0t + \frac{1}{3}\cos(3w_0t - \pi) + \frac{1}{5}\cos 5w_0t + \frac{1}{7}\cos(7w_0t - \pi) + \ldots\right.$

$\cos(x - \pi) = -\cos x$





Exponential Fourier Series:-

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jnw_0t}, \quad D_k = \frac{1}{T_0}\int_0^{T_0} g(t) e^{-jkw_0t}\,dt$$

$D_n = \frac{C_n}{2} e^{j\theta_n}$, $n = 1, 2, \ldots$, $D_0 = C_0$, $D_{-n} = D_n^* = \frac{C_n}{2} e^{-j\theta_n}$, $n = -1, -2, -3, \ldots$

Ex: $g(t) = 3\cos t + \sin(5t - \pi/6) - 2\cos(8t - \pi/3)$. (2)

(a) Sketch the amplitude and phase spectra of $g(t)$.

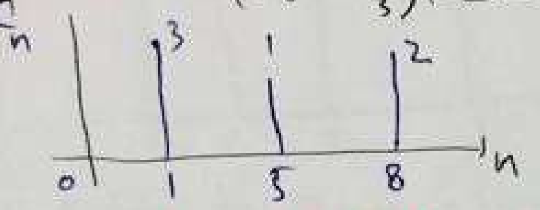
$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n), \text{ where } C_n \geq 0.$$

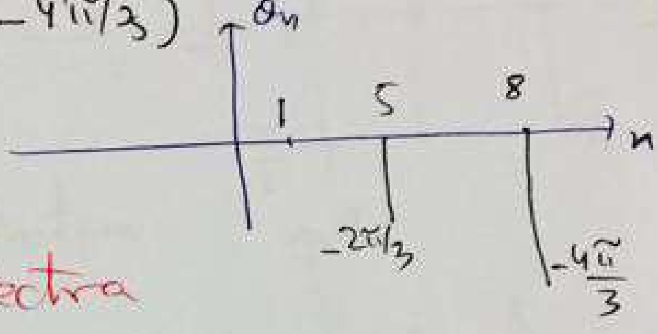
$\cos(\theta - \frac{\pi}{2}) = \sin\theta$

$\cos(\theta - \pi) = -\cos\theta$..

$$g(t) = 3\cos t + \cos(5t - \frac{\pi}{6} - \frac{\pi}{2}) + 2\cos(8t - \frac{\pi}{3} - \pi)$$

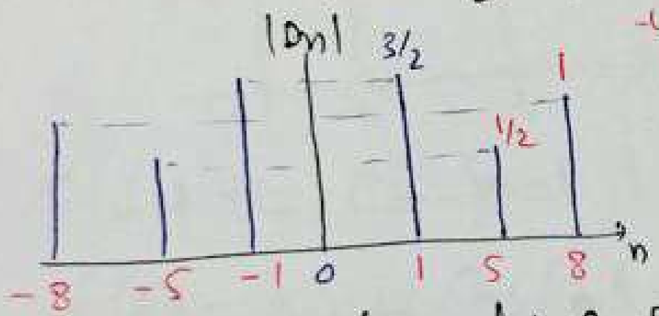
$$= 3\cos t + \cos(5t - 2\frac{\pi}{3}) + 2\cos(8t - 4\pi/3)$$

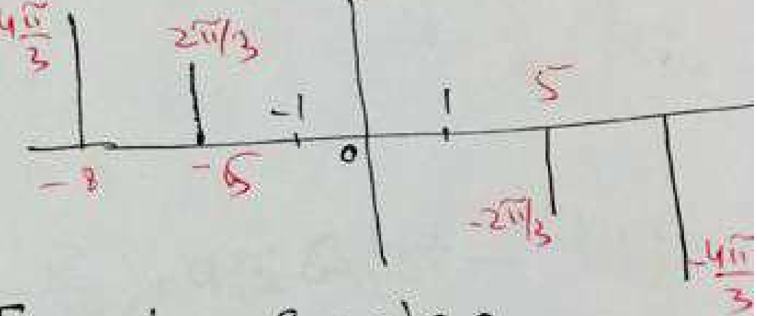




(b). Sketch exponential fourier spectra

$$D_0 = C_0, \ |D_n| = \frac{1}{2} C_n, \ \angle D_n = \theta_n, n > 0, \ \angle D_n = -\theta_n, n < 0.$$





(c). Write the exponential fourier series

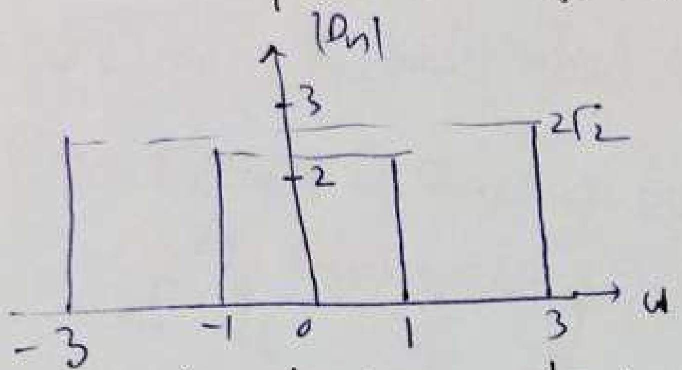
$$g(t) = D_{-8} e^{-j8t} + D_{-5} e^{-j5t} + D_{-1} e^{-jt} + D_1 e^{jt} + D_5 e^{j5t} + D_8 e^{j8t} \qquad D_1 = 3/2$$

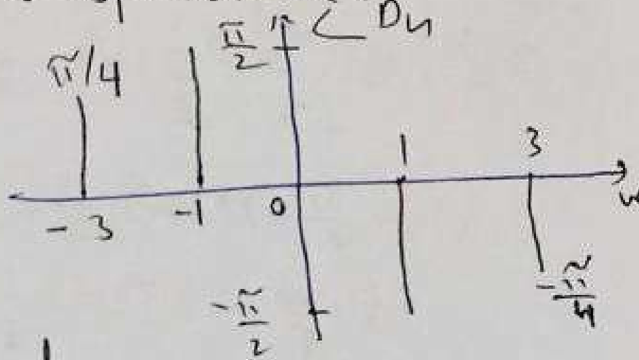
where, $D_{-8} = e^{j4\pi/3}$, $D_{-5} = \frac{1}{2} e^{j2\pi/3}$, $D_{-1} = 3/2$, $D_8 = e^{-j4\pi/3}$, $D_5 = \frac{1}{2} e^{-j2\pi/3}$.

Ex: The exponential F.S. of a periodic signal is given by

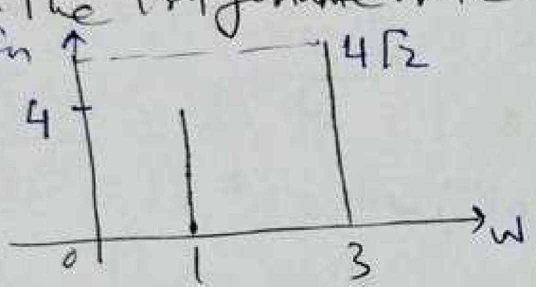
$$f(t) = (2+j2)e^{-j3t} + j2e^{-jt} + 3 - j2e^{jt} + (2-j2)e^{j3t}$$

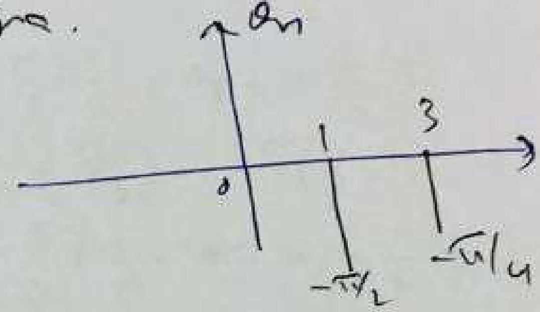
a) Sketch the exponential Fourier spectra.





b) Sketch the trigonometric spectra.





c). Find the Compact TFS.

$$f(t) = 3 + 4\cos\left(t - \frac{\pi}{2}\right) + 4\sqrt{2}\cos(3t - \pi/4).$$

Ex: write the following expression in the form of $C\cos(\omega_0 t+\theta)$

$f(t) = \cos 5t - \sin 5t$.

$C = \sqrt{1^2+1^2} = \sqrt{2}$, $\theta = \tan^{-1}(1) = \pi/4$

$f(t) = \sqrt{2}\cos(5t + \pi/4)$.

$f(t) = -\cos\omega_0 t - \sin\omega_0 t$, $C = \sqrt{2}$, $\theta = \tan^{-1}\left(-\frac{b}{a}\right) = \tan^{-1}\left(\frac{1}{-1}\right) = 3\pi/4$

$f(t) = \sqrt{2}\cos(\omega t + 3\pi/4)$.

$$f(t) = 3 + \sqrt{3}\cos 2t + \sin 2t + \sin 3t - \frac{1}{2}\cos(5t + \pi/3).$$

- $$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$C_n = \sqrt{a_n^2 + b_n^2},\ \theta_n = \tan^{-1}(-b_n/a_n),\ C_0 = a_0,\ D_n = \frac{1}{2}C_n e^{j\theta_n},\ D_{-n} = \frac{1}{2}C_n e^{-j\theta_n}$$

$$D_0 = C_0.$$

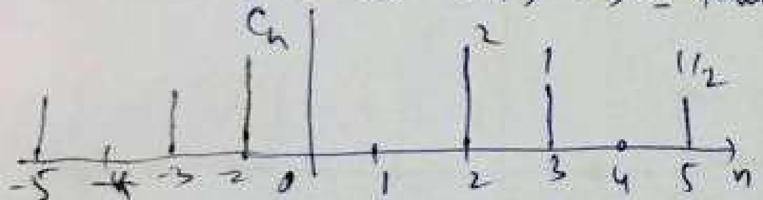
$$f(t) = 3 + \sqrt{3}\cos 2t + 1\cdot\sin 2t + 1\cdot\sin 3t + \frac{1}{2}\left(\cos\left(5t + \frac{\pi}{3} - \pi\right)\right).$$

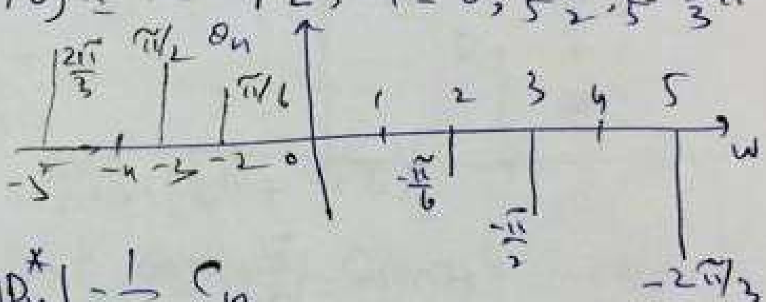
$$= \underbrace{3}_{a_0} + \underbrace{\sqrt{3}}_{a_2}\cos 2t + \underbrace{\sin 2t}_{b_2} + \underbrace{\sin 3t}_{b_3} + \underbrace{\frac{1}{2}}_{c_5}\cos(5t \underbrace{- \tfrac{2}{3}\pi}_{\theta_5}).$$

$$a_0 = 3,\ a_2 = \sqrt{3},\ b_2 = 1,\ a_3 = 0,\ b_3 = 1,\ C_5 = \frac{1}{2},\ \theta_5 = -\frac{2}{3}\pi.$$

$$C_0 = a_0 = 3,\ C_1 = 0,\ C_2 = \sqrt{a_2^2 + b_2^2} = 2,\ \theta_2 = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\pi/6.$$

$$C_3 = \sqrt{a_3^2 + b_3^2} = 1,\ \theta_3 = \tan^{-1}(-1/0) = -\pi/2,\ C_4 = 0,\ C_5 = \frac{1}{2},\ \theta_5 = -\frac{2}{3}\pi$$





Exponential spectra! $|D_n| = |D_{-n}| = |D_n^*| = \frac{1}{2}C_n$

$D_0 = C_0$, $D_0 = 3$, $|D_1| = |D_{-1}| = 0$, $|D_2| = |D_{-2}| = 1$, $\angle D_2 = -\angle D_{-2} = -\pi/6$

$|D_3| = |D_{-3}| = 1/2$, $\angle D_3 = -\angle D_{-3} = -\pi/2$,

$|D_4| = 0$, $|D_5| = |D_{-5}| = 1/4$, $\angle D_5 = -\angle D_{-5} = -2/3\,\pi$

$$f(t) = 3 + \left[e^{j(2t-\pi/6)} + e^{-j(2t-\pi/6)}\right] + \frac{1}{2}\left[e^{j(3t-\frac{\pi}{2})} + e^{-j(3t-\pi/2)}\right]$$

$$+ \frac{1}{4}\left[e^{j(5t-\frac{2}{3}\pi)} + e^{-j(5t-\frac{2}{3}\pi)}\right].$$

Problem IX [10 pts]
The trigonometric Fourier series of a periodic signal is given by

$$f(t) = 1 + 2\cos\left(t + \frac{\pi}{3}\right) - 4\sqrt{2}\sin\left(3t + \frac{\pi}{4}\right).$$

1. What is the fundamental period of this signal?
2. Sketch the trigonometric Fourier spectra.
3. Sketch the exponential Fourier spectra.

$$f(t) = 1 + 2\cos\left(t+\frac{\pi}{3}\right) + 4\sqrt{2}\sin\left(3t+\frac{5\pi}{4}\right)$$
$$= 1 + 2\cos\left(t+\pi/3\right) + 4\sqrt{2}\cos\left(3t+\frac{3\pi}{4}\right)$$

$1 \quad 3$

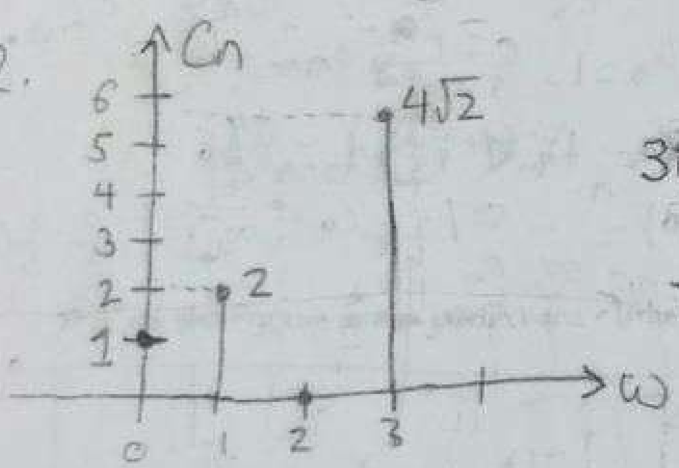
$\frac{2\pi}{1} = 2\pi, \ \frac{2\pi}{3}$

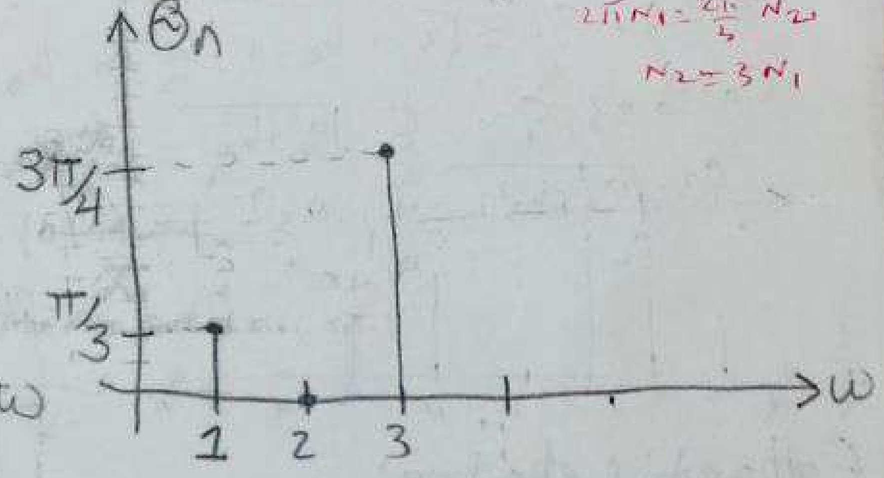
$2\pi N_1 = \frac{2\pi}{3} N_2$

$N_2 = 3N_1$

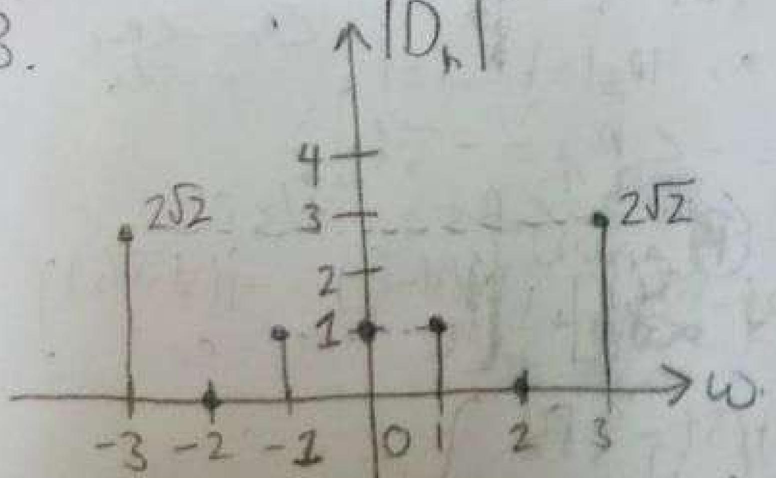
1. $\omega_0 = 1, \ T_0 = \frac{2\pi}{\omega_0} = 2\pi$

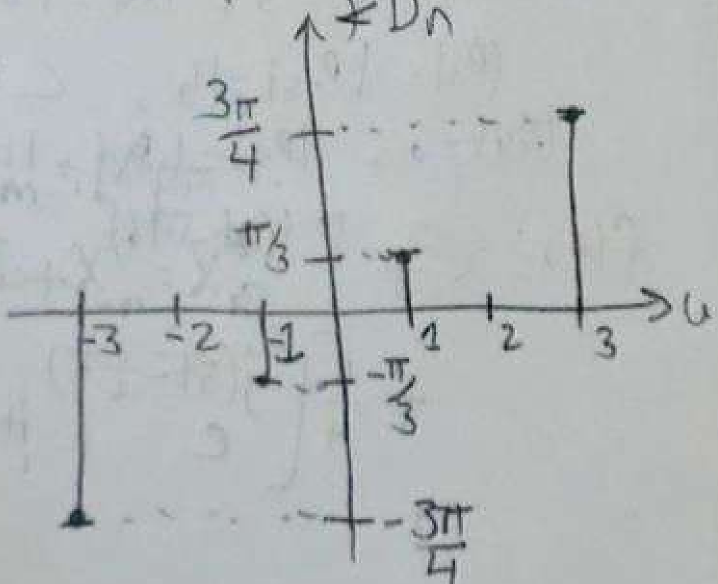
2.





3.





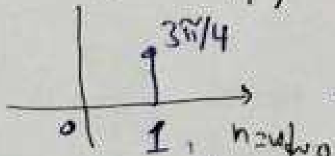
Ex:- A periodic signal is given by $g(t) = 5 - 2\cos 3t - 2\sin 3t$. (5)
Express the periodic signal in (a) TFS (b) Compact FS (c) exponential FS (d) Sketch the 1-sided spectra, and (e) sketch the 2-sided spectra.

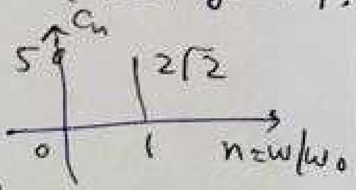
Sol:-
(a) Already in TFS, $\omega_0 = 3$ rad/sec, $a_0 = 5$, $a_1 = -2$, $b_1 = -2$

(b). $g(t) = 5 + \left((-2)^2 + (-2)^2\right)^{1/2} \cos\left(3t + \tan^{-1}\left(\frac{-(-2)}{-2}\right)\right)$ [we use 4-quadrant arctangent].

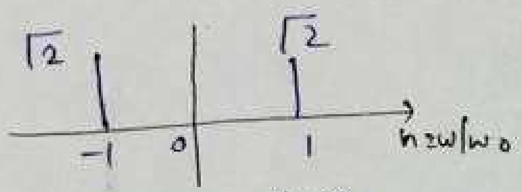
$= 5 + \sqrt{8}\cos\left(3t + \tan^{-1}\left(\frac{2}{-2}\right)\right) = 5 + 2\sqrt{2}\cos\left(3t + \frac{3\pi}{4}\right)$

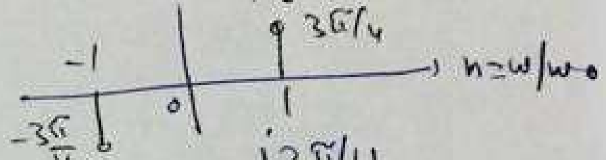
$C_0 = 5$, $C_1 = 2\sqrt{2}$, $\theta_1 = \frac{3\pi}{4}$ rad





(C).





$D_0 = 5$, $D_1 = \sqrt{2}\, e^{j3\pi/4}$, $D_{-1} = D_1^* = \sqrt{2}\, e^{-j3\pi/4}$

$g(t) = \left(\sqrt{2}\, e^{-j3\pi/4}\right) e^{-j3t} + 5 + \left(\sqrt{2}\, e^{j3\pi/4}\right) e^{j3t}$

Ex:- $g(t) = 3 - 2\sin\left(3t - \frac{\pi}{3}\right) - 2\cos\left(4t + \pi/3\right)$.

$\sin(3t - \pi/3) = \sin 3t \cos\frac{\pi}{3} - \cos 3t \sin\frac{\pi}{3}$, $\cos\left(4t + \frac{\pi}{3}\right) = \cos 4t \cos\frac{\pi}{3} - \sin 4t \sin\pi/3$

$g(t) = \underbrace{3}_{a_0} + \underbrace{\left(2\sin\frac{\pi}{3}\right)}_{a_3}\cos 3t + \underbrace{\left(-2\cos\frac{\pi}{3}\right)}_{a_4}\cos 4t + \underbrace{\left(-2\cos\frac{\pi}{3}\right)}_{b_3}\sin 3t + \underbrace{\left(2\sin\frac{\pi}{3}\right)}_{b_4}\sin 4t$

CFS:- $g(t) = 3 + 2\cos\left(3t - \frac{\pi}{3} - \frac{3\pi}{2}\right) + 2\cos\left(4t + \frac{\pi}{3} - \pi\right)$

$= \underset{C_0}{3} + \underset{C_3}{2}\cos\left(3t - \underset{\theta_3}{\frac{11\pi}{6}}\right) + \underset{C_4}{2}\cos\left(4t - \underbrace{2\pi/3}_{\theta_4}\right)$

EFS:- $g(t) = e^{j2\pi/3} e^{-j4t} + e^{j11\pi/6} e^{-j3t} + 3 + e^{-j11\pi/6} e^{j3t} + e^{-j\frac{2\pi}{3}} e^{j4t}$.