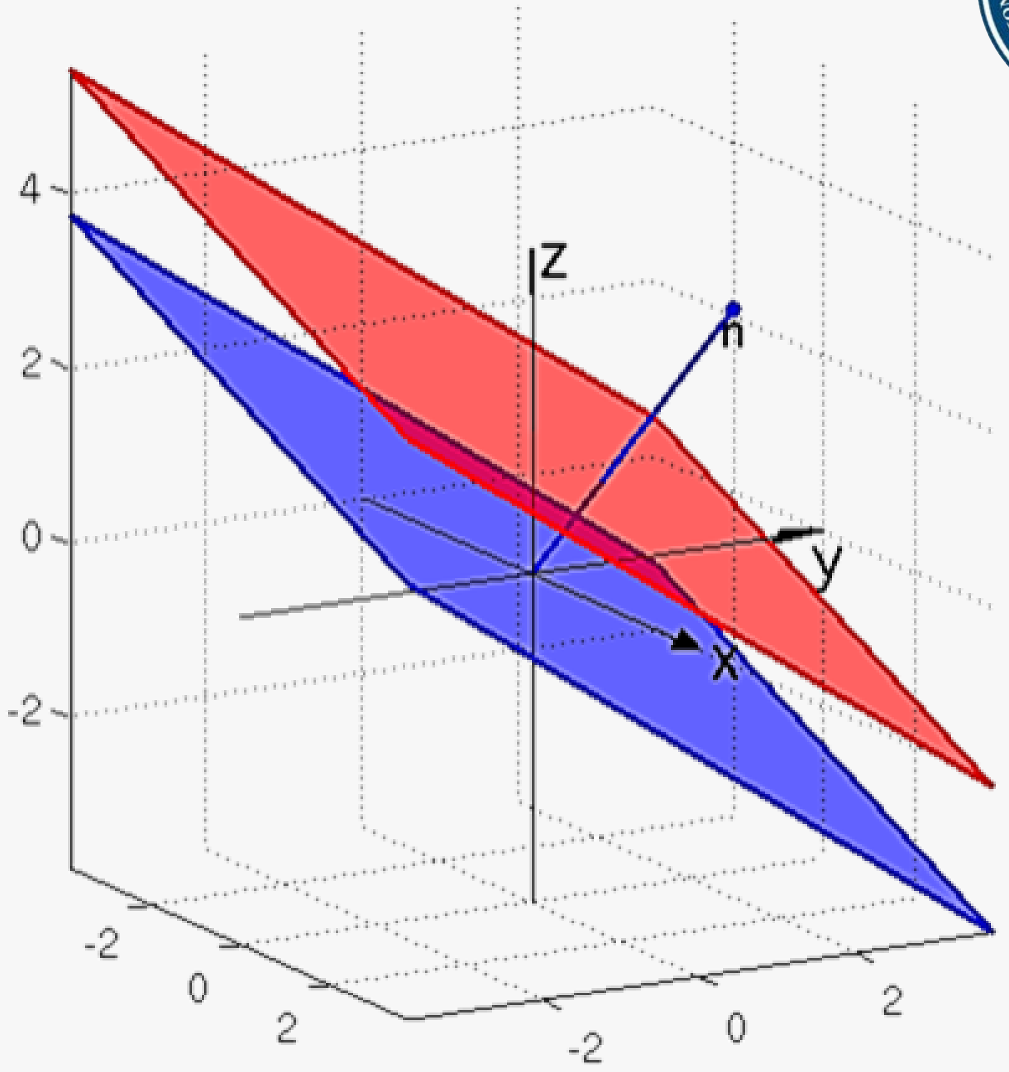
Equations of Lines and Planes



Vector Calculus(MATH-243)
Instructor: Dr. Naila Amir

12

Vectors And The Geometry Of Space

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Section: 12.5

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

Section: 12.5

Lines in 2-D & 3-D

- A line in the xy −plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written using the point-slope form. Otherwise, we can determine equation of a line in 2-D if information about two points on the line is known.

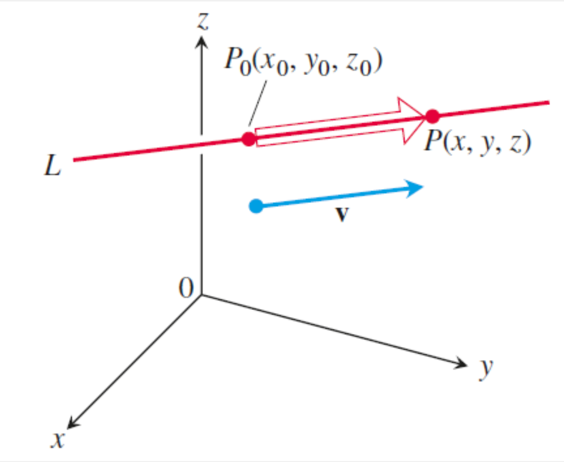
- A line L in 3-D space is determined when we have information about a point $P_0(x_0, y_0, z_0)$ on L and the direction of L, that can be determined with the help of a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, that is parallel to L.

Equation of a Line in 3-D

Let L be a line passing through the point $P_0(x_0, y_0, z_0)$ and is parallel to the vector: $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \langle a, b, c\rangle$. Let $P(x, y, z)$ be an arbitrary point on L. Then L is the set of points P for which $\mathbf{a} = \overrightarrow{P_0P}$ is parallel to $\mathbf{v}$. Since $\mathbf{a}$ is parallel to $\mathbf{v}$ so:

$\langle x-x_0, y-y_0, z-z_0\rangle = t\langle a, b, c\rangle = \langle ta, tb, tc\rangle$

$$\mathbf{a} = \overrightarrow{P_0P} = t\mathbf{v}; \quad t \in \mathbb{R}. \qquad (1)$$



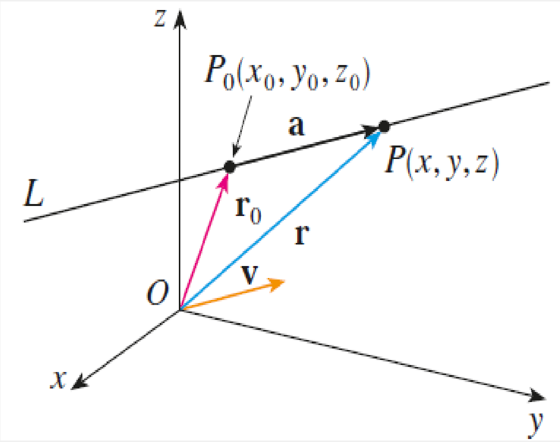
If $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on the line and $\mathbf{r}_0$ is the position vector of the point $P_0(x_0, y_0, z_0)$, then by using head to tail rule we get: $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$ and Equation (1) takes the form:

$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle$

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}; \qquad t \in \mathbb{R}, \qquad (2)$$

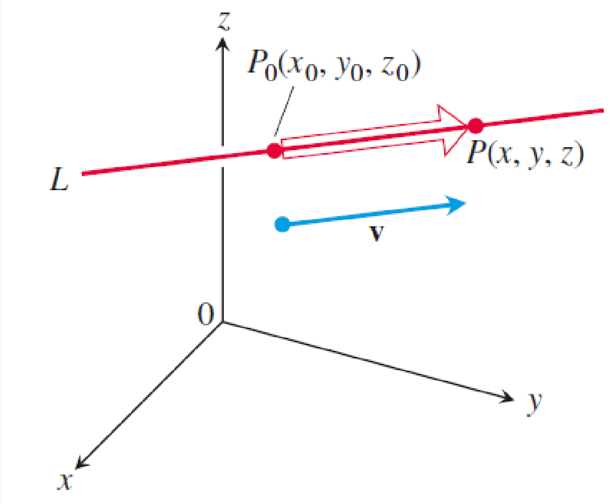
This is known as **vector equation** of a line in space

$\vec{v} = \underbrace{|\vec{v}|}_{\text{speed}} \left(\frac{\vec{v}}{|\vec{v}|}\right)$



Equation of a Line in 3-D

Now the equation of line L in space that is passing through the point $P_0(x_0, y_0, z_0)$ and is parallel to the vector: $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \langle a, b, c \rangle$ is given as:



$$\overrightarrow{P_0P} = t\mathbf{v}; \qquad t \in \mathbb{R}, \qquad (1)$$

Since $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$, so above equation can be written as:

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t\langle a, b, c \rangle,$$

From which we get three scalar equations:

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct; \qquad t \in \mathbb{R}. \qquad (3)$$

These equations are called **parametric equations** of the line L.

Equation of a Line in 3-D

Another way of describing a line is to eliminate the parameter from Equations (3). If none of a, b, or c is 0, we can solve each of these equations for the parameter t, equate the results, and obtain:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad \text{or} \quad \frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}. \qquad (4)$$

These equations are called **symmetric equations** of the line L. Notice that the numbers a, b, and c that appear in the denominators of Equations (4) are direction numbers of L, that is, components of a vector parallel to L. If one of a, b, or c is 0, we can still eliminate t. For instance, if $a = 0$, we could write the equations of L as:

$$x = x_0, \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}. \qquad (5)$$

This means that L lies in the vertical plane $x = x_0$.

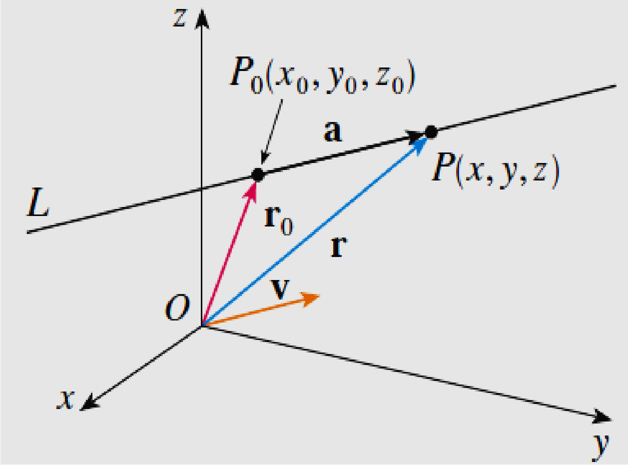
Equation of a Line in 3-D

The vector form of a line in space:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}; \qquad t \in \mathbb{R}, \qquad\qquad (2)$$

is more revealing if we think of a line as the path of a particle starting at position $P_0(x_0, y_0, z_0)$ and moving in the direction of vector **v**. Rewriting Equation (2), we have:

$$\mathbf{r} = \underbrace{\mathbf{r}_0}_{\text{Initial position}} + \underbrace{t}_{\text{Time}}\underbrace{|\mathbf{v}|}_{\text{Speed}}\underbrace{\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)}_{\text{Direction}}; \qquad t \in \mathbb{R}, \qquad\qquad (6)$$



In other words, the position of the particle at time t is its initial position plus its distance moved (speed × time) in the direction $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ of its straight-line motion.

Example:

A helicopter is to fly directly from a helipad at the origin in the direction of the point $(1, 1, 1)$ at a speed of $60\ \text{ft/sec}$. What is the position of the helicopter after $10\ \text{sec}$?

Solution:

We place the origin at the starting position (helipad) of the helicopter. Then the unit vector: $\mathbf{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$ gives the flight direction of the helicopter. From Equation (5), the position of the helicopter at any time t is given as:

$$\mathbf{r} = \mathbf{r}_0 + t(\text{speed})\mathbf{u} = \mathbf{0} + t(60)\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle = \left(20\sqrt{3}\,t\right)\langle 1,1,1\rangle.$$

At $t = 10\ \text{sec}$: $\mathbf{r} = 200\sqrt{3}\langle 1,1,1\rangle$. After $10\ \text{sec}$ of flight from the origin toward $(1, 1, 1)$, the helicopter is located at the point $\left(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3}\right)$ in space. It has traveled a distance of $(60\ \text{ft/sec})(10\ \text{sec}) = 600\ \text{ft}$, which is the length of the vector $\mathbf{r}$.

Example:

Find the parametric equations of the line passing through the points $(1, 2, -2)$ and $(3, -2, 5)$.

Solution:

First, we need to determine the direction vector and then by considering either of the points, simply use the parametric form to obtain the required line. Direction vector is obtained as:

$$\mathbf{v} = \langle 3 - 1, -2 - 2\ 5 - (-2) \rangle = \langle 2, -4, 7 \rangle.$$

Parametric equations of the line passing through the points $(1, 2, -2)$ and $(3, -2, 5)$ are given as:

$$x = 1 + 2t, \qquad y = 2 - 4t, \qquad z = -2 + 7t; \quad t \in \mathbb{R}.$$

Example:

a) Find the symmetric equations of the line that is passing through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.
b) Determine at what point this line will intersect the xy −plane.

Solution:

a) For the present case, the direction vector $\mathbf{v} = \langle a, b, c \rangle$, is obtained as:

$$\mathbf{v} = \langle 3 - 2, -1 - 4\ 1 - (-3) \rangle = \langle 1, -5, 4 \rangle.$$

Taking the point $(2, 4, -3)$ as the initial point, we see that the symmetric equations are given as:

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}.$$

Example:

a) Find the symmetric equations of the line that is passing through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.
b) Determine at what point this line will intersect the xy −plane.

Solution:

b) The line intersects the xy −plane when $z = 0$. So, we put $z = 0$ in the symmetric equations and obtain:

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{3}{4} \Longrightarrow x = \frac{11}{4} \quad \text{and} \quad y = \frac{1}{4}.$$

Thus, the line intersects the xy −plane at the point $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$.

Equation of Plane in Space

- Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe.

- A single vector parallel to a plane is not enough to convey the “direction” of the plane, but a vector perpendicular to the plane does completely specify its direction.

- Thus, a plane in space is determined by a point in the plane and a vector that is orthogonal to the plane. This orthogonal vector is called a **normal vector**.

Equation of Plane in Space

Let the plane M is passing through an arbitrary point $P_0(x_0, y_0, z_0)$, and is normal to the vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is perpendicular to $\mathbf{n}$. Means the normal vector $\mathbf{n}$ is orthogonal to every vector in the given plane, i.e.,

$\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0$

$\vec{r}_0 \to P_0$

$\vec{r} \to P$

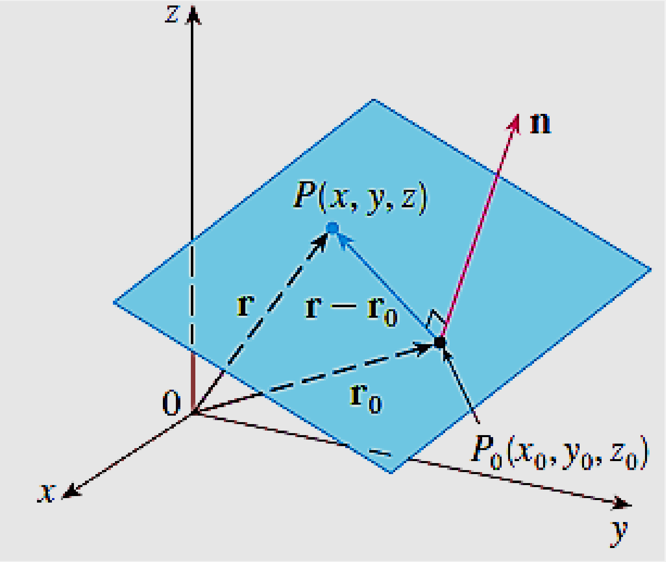
$$\mathbf{n}.\overrightarrow{P_0P} = 0. \qquad (1)$$

Equation (1) is the **vector equation** of plane. This equation can be written as:

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}).[(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$$

$$\Longrightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \qquad (2)$$

This is the **scalar equation** or **standard form** of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n}$.



Equation of Plane in Space

Note:

The equation (2)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \qquad (2)$$

can be simplified by using the distributive property and collecting like terms. This results in the **general form** given as:

$$ax + by + cz + d = 0. \qquad (3)$$

$d = -(ax_0 + by_0 + cz_0)$

Example:

Given the normal vector, $\langle 3, 1, -2 \rangle$ to the plane containing the point $(2, 3, -1)$, write the equation of the plane in both standard form and general form.

Solution:

Standard form of plane:

$\vec{n} = \langle a, b, c \rangle = \langle 3, 1, -2 \rangle$

$P_0 \ (x_0, y_0, z_0) = (2, 3, -1)$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

$$\Longrightarrow 3(x - 2) + 1(y - 3) - 2(z + 1) = 0.$$

General form of plane:

$$ax + by + cz + d = 0,$$

$$\Longrightarrow 3x + y - 2z - 11 = 0.$$

Example:

Given the points $\overset{P}{(1,2,-1)}$, $\overset{Q}{(4,0,3)}$ and $\overset{R}{(2,-1,5)}$ in a plane, find the equation of the plane in general form.

$(x_0, y_0, z_0) \leftarrow P$, Q, R $\quad \overrightarrow{PQ}$, $\overrightarrow{PR}$ $\quad \vec{n} = \langle a, b, c \rangle = \overrightarrow{PQ} \times \overrightarrow{PR}$

Solution:

Hint:

To write the equation of the plane we need a point (we have three) and a vector normal to the plane. So, we need to find a vector normal to the plane. First find two vectors in the plane, then recall that their cross product will be a vector normal to both those vectors and thus normal to the plane. Using all this information in the general equation of plane we will finally get:

Equation of the plane in general form:

$$2y + z - 3 = 0.$$

Some Properties of Lines and Planes

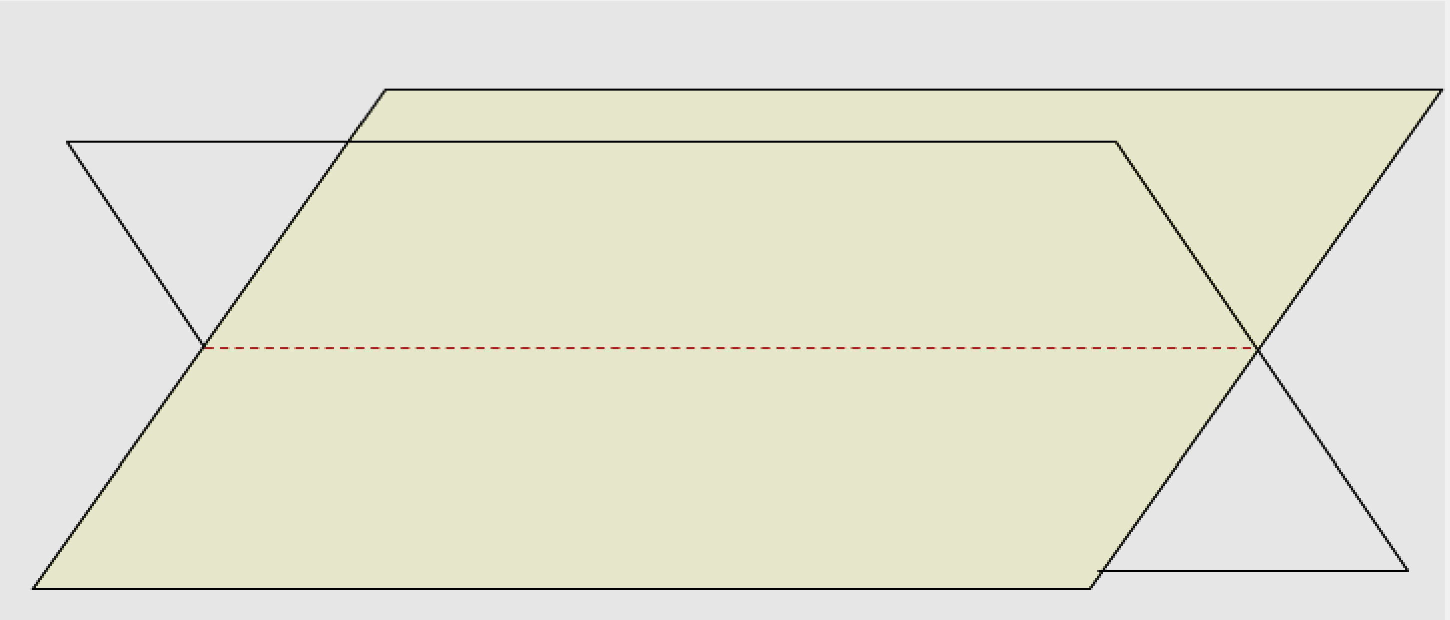
- Two lines are **parallel** if and only if they have the same direction.
- Two lines **intersect** at a point if they are lying in the same plane.
- The lines that do not intersect each other and are not parallel (and therefore do not lie in the same plane) are called **skew lines.**
- Intersection of a line and a plane is a point.
- Two planes are **parallel** if and only if their normals are parallel, i.e.,

$$\mathbf{n_1} = k\mathbf{n_2}; \quad \text{for some scalar } k.$$

- Two planes that are not parallel intersect in a line.

Intersecting Planes

Any two planes that are not parallel or identical will intersect in a line and to find the line, solve the equations simultaneously.



Example:

Find the line of intersection for the planes $x + 3y + 4z = 0$ and $x - 3y + 2z = 0$.

Solution:

To find the common intersection, solve the equations simultaneously. Multiply the first equation by –1 and add the two to eliminate x.

$$\left.\begin{array}{rl} -1 \cdot (x + 3y + 4z = 0) \Longrightarrow & -x - 3y - 4z = 0 \\ x - 3y + 2z = 0 \Longrightarrow & \underline{+x - 3y + 2z = 0} \end{array}\right\} \textbf{Add}$$

$$-6y - 2z = 0$$

$$\text{or} \qquad y = \frac{-1}{3} z$$

Back substitute y into one of the first equations and solve for x.

Example:

$$x + 3 \cdot \left(\frac{-1}{3} z\right) + 4z = 0$$

$$x - z + 4z = 0$$

$$x = -3z.$$

Finally, if we let $z \ = \ t$, the parametric equations for the line are:

$$x = -3t, \qquad y = \frac{-1}{3} t \ \text{ and } \ z = t.$$

Practice Questions

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 12

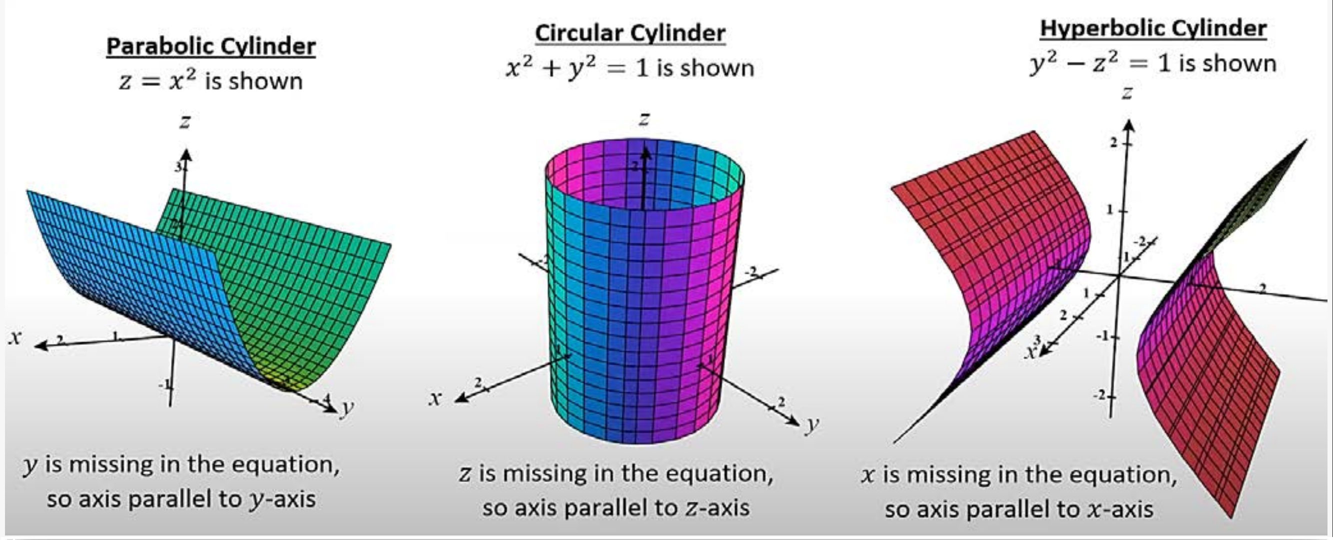
Exercise-12.5: Q – 1 to 12, Q – 21 to 26, Q – 57 to 64.

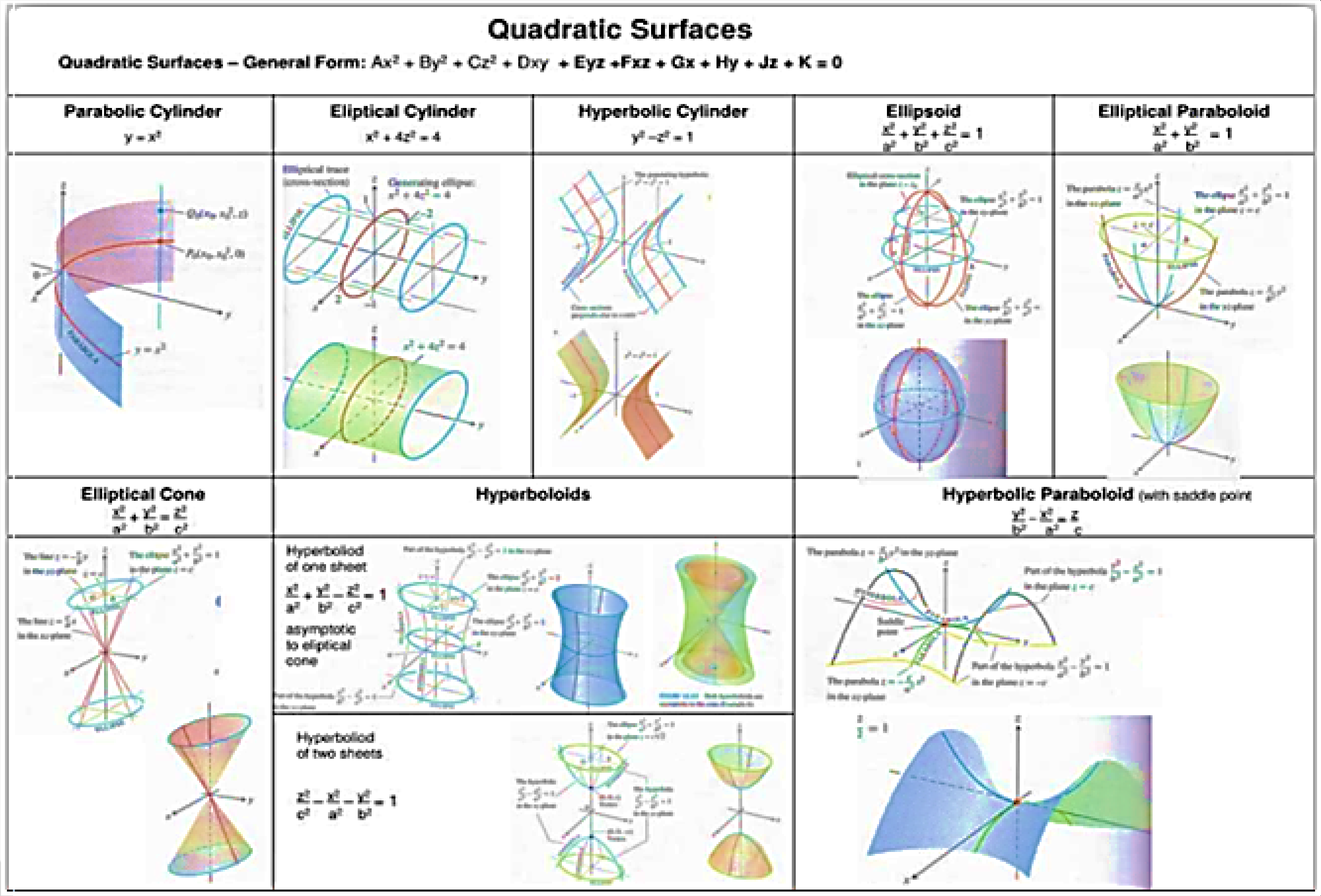
Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.

Chapter: 12

Exercise-12.5: Q – 1 to 12, Q – 19 to 36, Q – 43 to 45.

Cylinders & Quadric Surfaces





12

Vectors And
The Geometry Of Space

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Section: 12.6

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.
Section: 12.6