Applications of Derivatives

Calculus & Analytical Geometry MATH- 101

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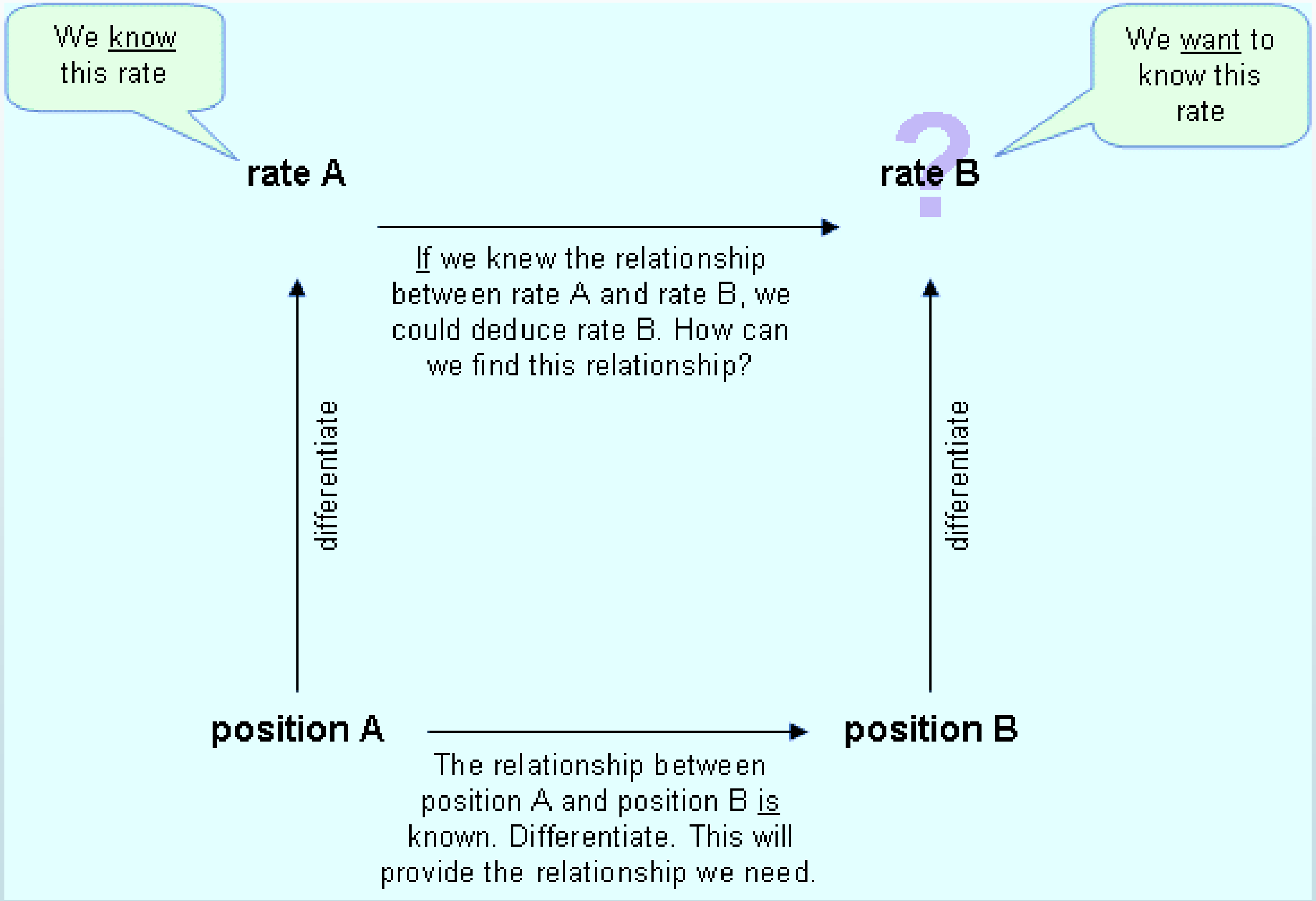
Related Rates



Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 3
  - Sections: 3.7

Steps for Related Rates Problems:

1. Draw a picture (sketch).
2. Write down known information.
3. Write down what you are looking for.
4. Write an equation to relate the variables.
5. Differentiate both sides with respect to t.
6. Evaluate.

EXAMPLE : A Highway Chase

A police cruiser, approaching a right angled intersection from the north is chasing a speeding car that has turned the corner and is now moving straight east. The cruiser is moving at $60\,mph$ and the police determine with radar that the distance between them is increasing at $20\,mph$. When the cruiser is $0.6\,mi$. north of the intersection and the car is $0.8\,mi$ to the east, what is the speed of the car?

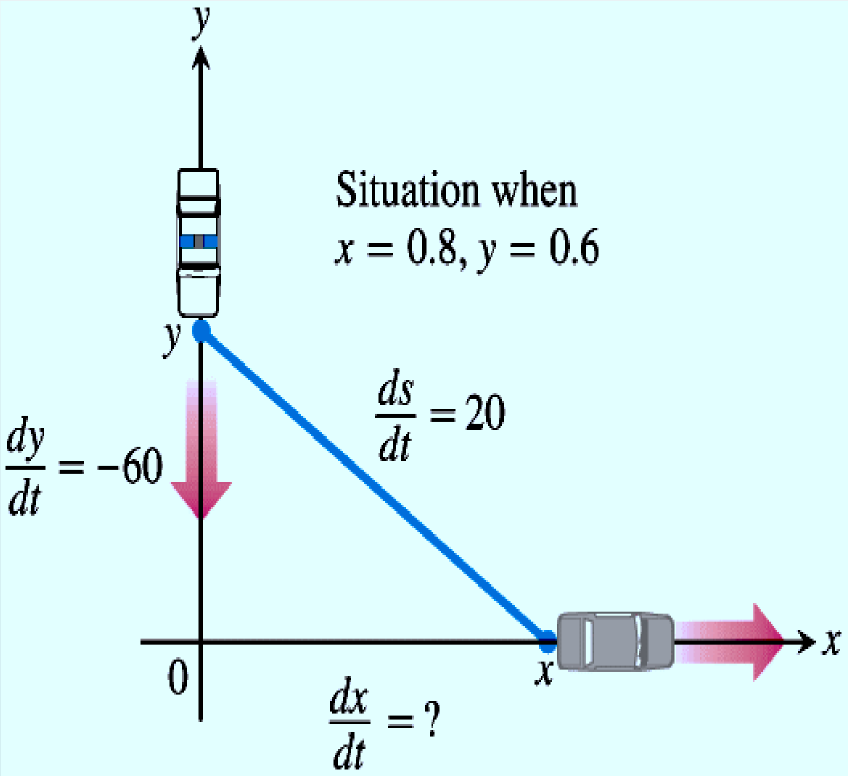
Solution:

We picture the car and cruiser in the coordinate plane, using the positive x –axis as the eastbound highway and the positive y –axis as the southbound highway. Let t represent time and set

x = position of car at time t

y = position of cruiser at time t

s = distance between car and cruiser at time t .



Solution:

Given: $\frac{ds}{dt} = 20mph,\ \ \frac{dy}{dt} = -60mph$

Find: $\frac{dx}{dt}$ when $x = .8, y = .6$

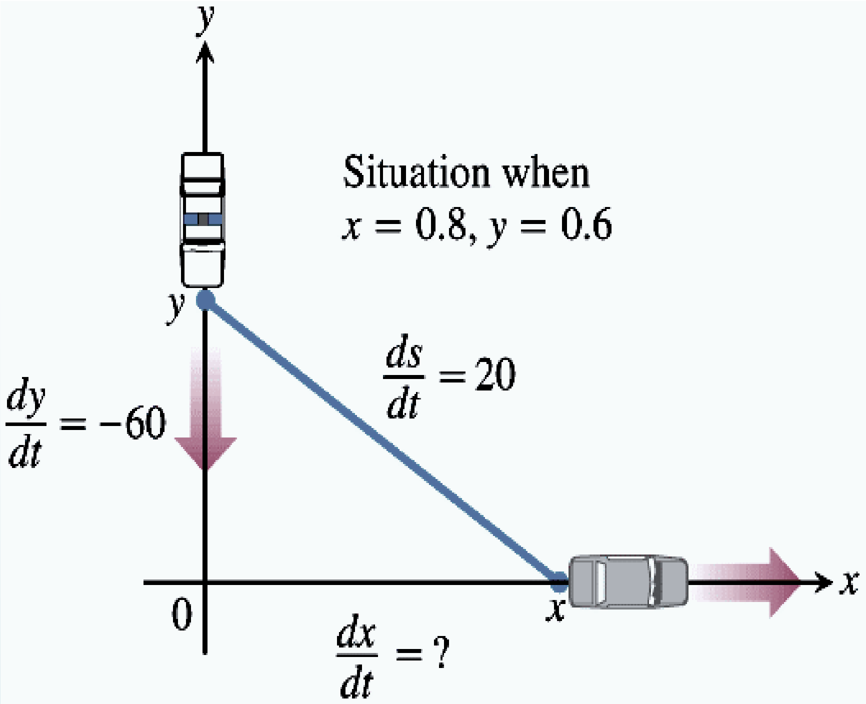
By Pythagoras theorem:

$$s^2 = x^2 + y^2$$

$$\Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

$$\Rightarrow 2(1)(20) = 2(.8)\frac{dx}{dt} + 2(.6)(-60).$$

$$\Rightarrow \frac{dx}{dt} = 70mph.$$



If $x = .8, y = .6$ then $s = 1$

Practice Questions

Book: Thomas Calculus (11th Edition) by Georg B.Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

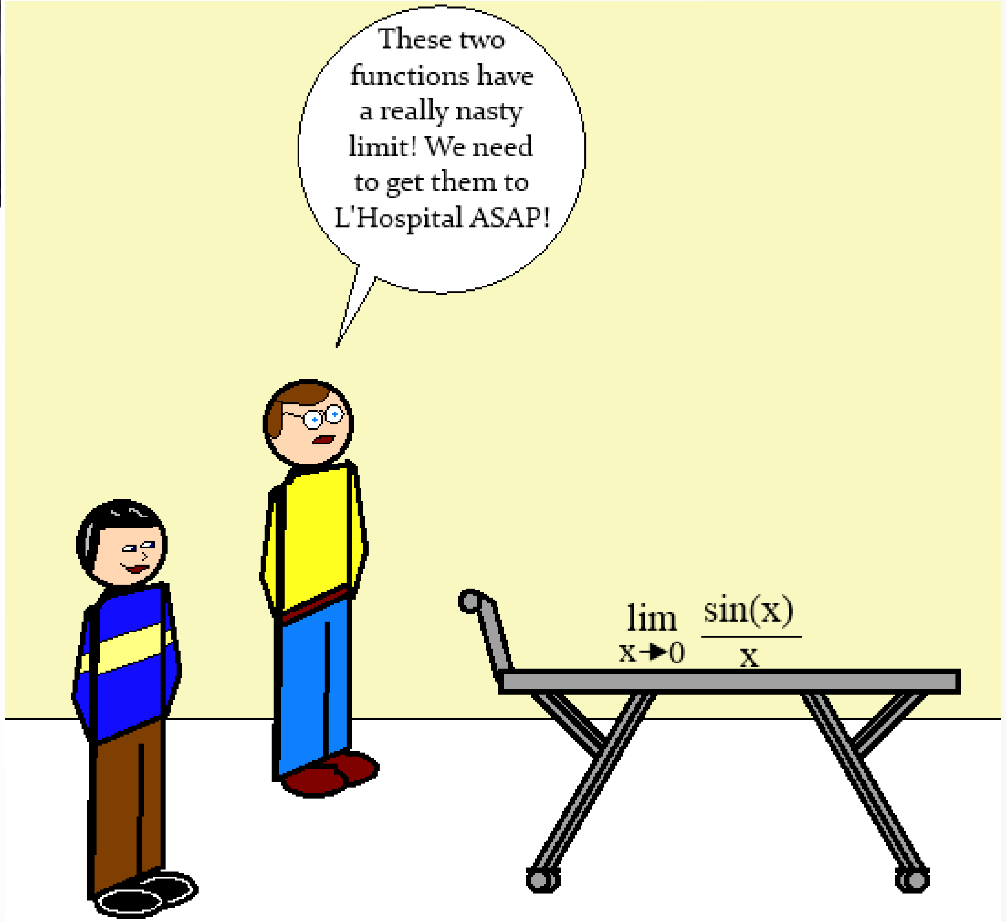
- **Chapter: 3**
  - **Exercise: 3.3**

    Q # 1 – 14, 19, 20, 23 – 30.
  - **Exercise: 3.7**

    Q # 1 – 38.

Indeterminate Forms and L' Hopital's Rule

Guillaume De l'Hôpital
1661 - 1704



Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 4
    - Sections: 4.6

Indeterminate Forms and L'Hopital's Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}.$$

Although $F(x)$ is not defined when $x = 1$, we need to know how $F(x)$ behaves near 1. In particular, we would like to know the value of the limit

$$\lim_{x \to 1} \frac{\ln x}{x - 1}.$$

In computing this limit we can't apply quotient rule of limits directly because limit of the denominator is "0".

Indeterminate Forms and L'Hopital's Rule

- In fact value of the limit is not obvious because both numerator and denominator approach "0" and $\frac{0}{0}$ is not defined.
- In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)},$$

where both $f(x)$ and $g(x)$ approach "0" as $x \to a$, then this limit may or may not exist and is called an indeterminate form of the type $\frac{0}{0}$.

Indeterminate Forms and L'Hopital's Rule

- For rational functions, we can cancel common factors:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x-1)}{(x+1)(x-1)}$$

$$= \lim_{x \to 1} \frac{x}{x+1}$$

$$= \frac{1}{2}$$

- We used a geometric argument to show that:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

- But these methods do not work for limits such as $\lim_{x \to 1} \frac{\ln x}{x-1}$.

Indeterminate Forms and L'Hopital's Rule

- Another situation in which a limit is not obvious, occurs when we look for limit of $F(x)=\frac{\ln x}{x-1}$ at infinity, i.e.,

$$\lim_{x\to\infty}\frac{\ln x}{x-1}.$$

- It is not obvious how to evaluate this limit because both numerator and denominator become large as $x\to\infty$.

- There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞; and if the denominator wins, then the answer will be 0. Or there may be some compromise, in which case the answer will be some finite positive number.

Indeterminate Forms and L'Hopital's Rule

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)},$$

where both $f(x)$ and $g(x)$ approach "∞" (or "$-\infty$") as $x \to a$, then the limit may or may not exist and is called an indeterminate form of the type $\frac{\infty}{\infty}$.

Indeterminate Forms and L'Hopital's Rule

- This type of limit can be evaluated for certain functions, including rational functions, by dividing numerator and denominator by the highest power of that occurs in the denominator. For instance,

$$\lim_{x\to\infty} \frac{x^2-1}{2x^2+1} = \lim_{x\to\infty} \frac{1-\dfrac{1}{x^2}}{2+\dfrac{1}{x^2}} = \frac{1-0}{2+0} = \frac{1}{2}$$

- However, this method does not work for limits such as $\lim_{x\to\infty} \frac{\ln x}{x-1}$.
- So we introduce a systematic method, known as ***L'Hopital's Rule***, for the evaluation of indeterminate forms.

Indeterminate Forms and L'Hopital's Rule

L'Hôpital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example

Determine $\lim_{x \to 1} \frac{\ln x}{x-1}$.

Solution:

Since $\lim_{x \to 1} \ln x = \ln 1 = 0$, and $\lim_{x \to 1} (x - 1) = 0$, therefore, we can apply L'Hopital's rule over here. Thus, we have

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x} = 1.$$

Indeterminate Forms and L'Hopital's Rule

- **Note1:**

  L'Hopital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of $f(x)$ and $g(x)$ before using L'Hospital's Rule.

- **Note2:**

  L'Hopital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+, x \to a^-, x \to \infty$, or $x \to -\infty$.

Examples

(a) $$\lim_{x\to 0^+} \frac{\sin x}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\cos x}{2x} = \infty \qquad \text{Positive for } x > 0.$$

(b) $$\lim_{x\to 0^-} \frac{\sin x}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0^-} \frac{\cos x}{2x} = -\infty \qquad \text{Negative for } x < 0.$$

Indeterminate Forms and L'Hopital's Rule

- **Note3:**

  To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by L'Hopital's Rule, continue to differentiate $f(x)$ and $g(x)$, so long as we still get the form $0/0$ or ∞/∞ at $x = a$. But as soon as one of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hopital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Examples

(a) $$\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$$ $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x}$$ Still $\frac{0}{0}$; differentiate again.

$$= \lim_{x\to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}$$ Not $\frac{0}{0}$; limit is found.

(b) $$\lim_{x\to 0} \frac{x - \sin x}{x^3}$$ $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{1 - \cos x}{3x^2}$$ Still $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{\sin x}{6x}$$ Still $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$ Not $\frac{0}{0}$; limit is found.

Indeterminate Product

Indeterminate Form of type $0.\infty$ or $\infty.0$

Indeterminate Product

- If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$) then it is not clear what will be value of
$$\lim_{x \to a} f(x)g(x).$$

- There is a struggle between $f(x)$ and$g(x)$. If f wins the limit will be 0; and if g wins, then the answer will be ∞ (or $-\infty$). Or there may be some compromise where the answer is a finite nonzero number.

- This kind of limit is known as an indeterminate form of type $0.\infty$.

Indeterminate Product

- We can deal with it by writing the product $f(x)g(x)$ as a quotient:

$$f(x)g(x) = \frac{f(x)}{1/g(x)} \quad \text{or} \quad f(x)g(x) = \frac{g(x)}{1/f(x)}.$$

- This converts the given limit into an indeterminate form of type $\frac{0}{0}$ $or \frac{\infty}{\infty}$, so that we can use the L'Hopital's Rule.

Example

Evaluate $\lim_{x \to 0} x \ln x$.

Solution:

The given limit is an indeterminate form of type $0.\infty$, because as $x \to 0$, the first factor x approaches 0 while the second factor $\ln x$ approaches ∞. Rewriting the given function as:

$$x \ln x = \frac{\ln x}{1/x},$$

so that as $x \to 0$ we get $\frac{\infty}{\infty}$ form. We can apply L'Hopital's rule now.

Example

Thus,

$$\lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{1/x} \qquad \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{x \to 0} \frac{1/x}{(-1/x^2)} \qquad \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{x \to 0} (-x)$$

$$= 0.$$

Indeterminate Difference

Indeterminate Form of type $\infty - \infty$.

Indeterminate Difference

- If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then the limit

$$\lim_{x\to a}[f(x) - g(x)],$$

  is known as an indeterminate form of type $\infty - \infty$.

- In order to evaluate such limit, we try to convert the difference into a quotient (for instance, by using a common denominator or rationalization or factoring out a common factor) so that we get an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example

Evaluate $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$.

Solution:

The given limit is an indeterminate form of type $\infty-\infty$, because

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\frac{1}{0}-\frac{1}{0}=\infty-\infty.$$

In order to evaluate this limit we first combine the fractions and find the common denominator, then apply L'Hopital's Rule to the result i.e.,

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) \qquad \frac{0}{0}\text{ form}$$

Solution

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) \qquad \frac{0}{0}\text{ form}$$

$$=\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x+x\cos x}\right) \qquad \frac{0}{0}\text{ form}$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{\cos x+\cos x-x\sin x}\right)$$

$$=\frac{0}{2}$$

$$=0.$$

Indeterminate Powers

Indeterminate Form of type $0^0, \infty^0, 1^\infty$.

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Indeterminate Powers

Each of these three cases can be treated either by taking the natural logarithm:

$$y = [f(x)]^{g(x)} \Longrightarrow \ln y = g(x) \ln f(x),$$

Or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}.$$

Both of the cases provides us the indeterminate product $g(x) \ln f(x)$, which is of type $0. \infty$ or $\infty. 0$.

Example

Calculate $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$.

Solution:

First notice that as $x \to 0^+$, $1+\sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is an indeterminate form of type 1^∞. Let

$$y=(1+\sin 4x)^{\cot x}$$

Then

$$\ln y=\ln[(1+\sin 4x)^{\cot x}]=\cot x\ln(1+\sin 4x).$$

Taking the limit of both sides we get:

$$\Longrightarrow \lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\cot x\ln(1+\sin 4x) \qquad \infty.0 \text{ form}$$

Solution

$$\Longrightarrow \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} \qquad \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x}$$

$$= 4.$$

So far we have computed the limit of $\ln y$, but we need limit of y. To find this we are going to use the fact that $y = e^{\ln y}$, and this is going to give us:

$$\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\left[\lim\limits_{x\to 0^+} \ln y\right]} = e^4.$$

Practice Questions

Book: Thomas Calculus (11th Edition) by Georg B.Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter: 4**
  - **Exercise: 4.6**

    Q # 1 – 26.