PROPERTIES OF CTFT

Integration Property - Example

- Determine the Fourier Transform of a unit step from the transform of an impulse by using the integration property:

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)}$$

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad \text{Synthesis equation}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad \text{Analysis equation}$$

Parseval's Relation

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Parseval's Relation - Derivation

- Derivation of Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t)\,dt = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega\right]dt$$

- Reversing order of integration gives

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]d\omega$$

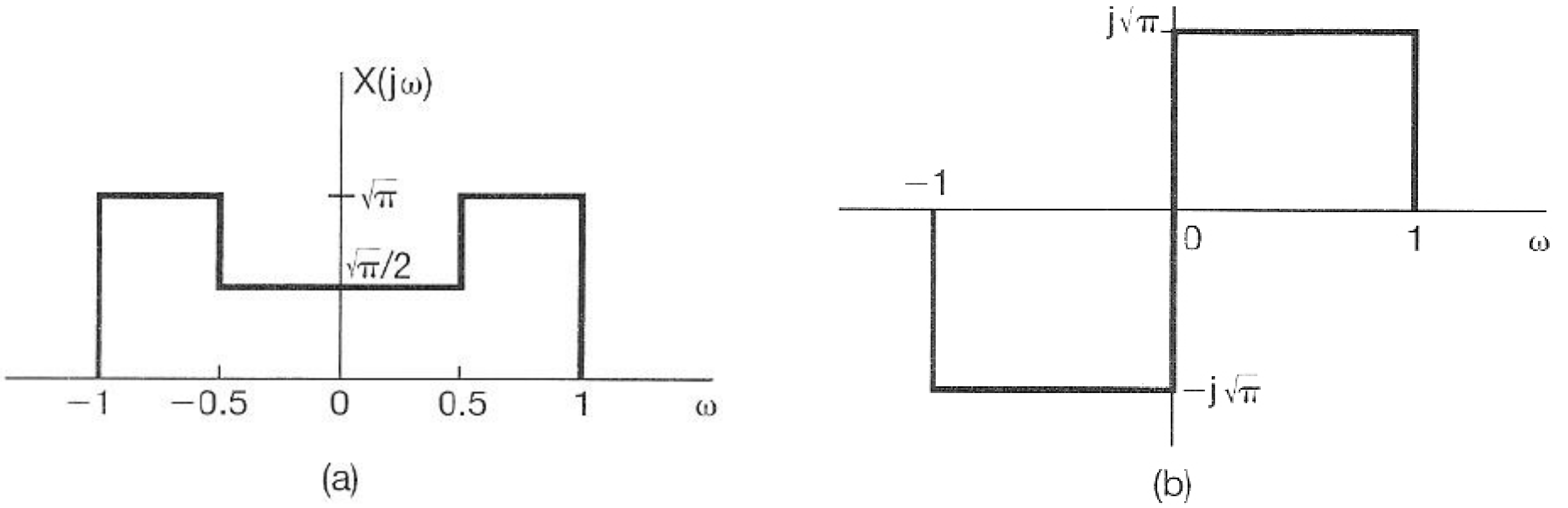
- The bracketed term is the Fourier transform of $x(t)$, giving

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\,d\omega$$

Parserval's Relation - Example

- For each of the Fourier transforms shown in the figure, evaluate the following time-domain expression

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$



Convolution Property

$$y(t) = h(t) * x(t) \longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

$$\text{where } h(t) \longleftrightarrow H(j\omega)$$

Convolution Property - Derivation

- Consider the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

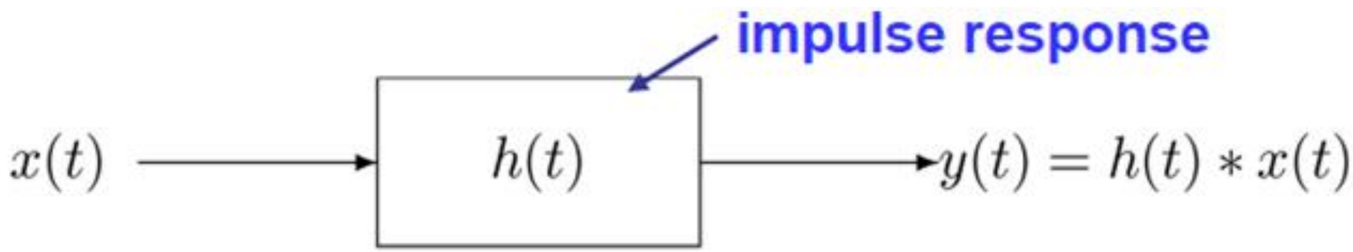
$$Y(j\omega) = FT\{y(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\underbrace{\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right]}_{e^{-j\omega\tau}H(j\omega)}d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau = H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau = H(j\omega)X(j\omega)$$

$$\boxed{y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(j\omega) = H(j\omega)X(j\omega)}$$

Frequency Response



$$Y(j\omega) = H(j\omega)X(j\omega)$$

frequency response

The frequency response of a CT LTI system is simply the Fourier transform of its impulse response

Frequency Response

Example:

$$x(t) = e^{j\omega_0 t} \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$$

Recall

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)$$

$$Y(j\omega) = H(j\omega)X(j\omega) = H(j\omega)2\pi\delta(\omega - \omega_0) = 2\pi H(j\omega_0)\delta(\omega - \omega_0)$$

$$\Downarrow \text{ inverse FT}$$

$$y(t) = H(j\omega_0)e^{j\omega_0 t}$$

Frequency Response - Differentiator

$$y(t) = \frac{dx(t)}{dt}$$ **- an LTI system**

Differentiation property: $Y(j\omega) = j\omega X(j\omega)$

$$\Downarrow$$

$$H(j\omega) = j\omega$$

1) Amplifies high frequencies (enhances sharp edges)

2) $+\pi/2$ phase shift $(j = e^{j\pi/2})$

Larger at high ω_o phase shift

$$\frac{d}{dt}\sin\omega_0 t = \omega_0 \cos\omega_0 t = \omega_0 \sin(\omega_0 t + \tfrac{\pi}{2})$$

$$\frac{d}{dt}\cos\omega_0 t = -\omega_0 \sin\omega_0 t = \omega_0 \cos(\omega_0 t + \tfrac{\pi}{2})$$

Convolution Property - Example

$$h(t) = e^{-t}u(t), \quad x(t) = e^{-2t}u(t)$$

$$y(t) = h(t) * x(t)$$

Multiplication Property

FT is highly symmetric,

$$x(t) \overset{\mathcal{F}^{-1}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega,\ X(j\omega) \overset{\mathcal{F}}{=} \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

We already know that: $x(t) * y(t) \longleftrightarrow X(j\omega)\cdot Y(j\omega)$

Then it isn't a surprise that: $x(t)\cdot y(t) \longleftrightarrow \frac{1}{2\pi}X(j\omega) * Y(j\omega)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)Y(j(\omega-\theta))d\theta$$

Convolution in ω

— A consequence of *Duality*

Multiplication Property - Example

- Determine the Fourier transform of the signal:

$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}$$

END