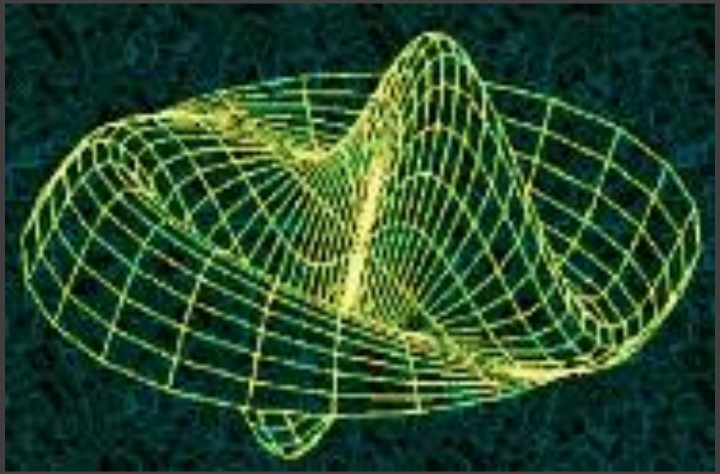
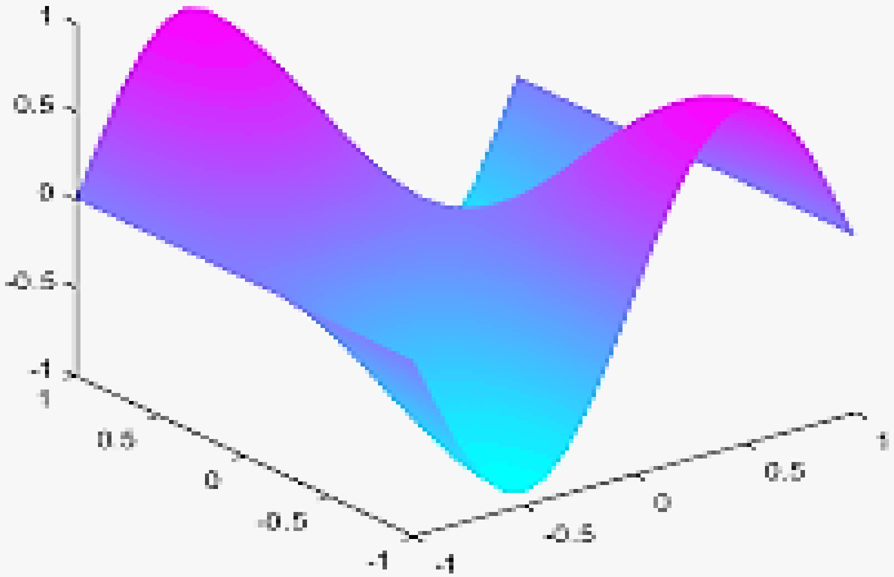
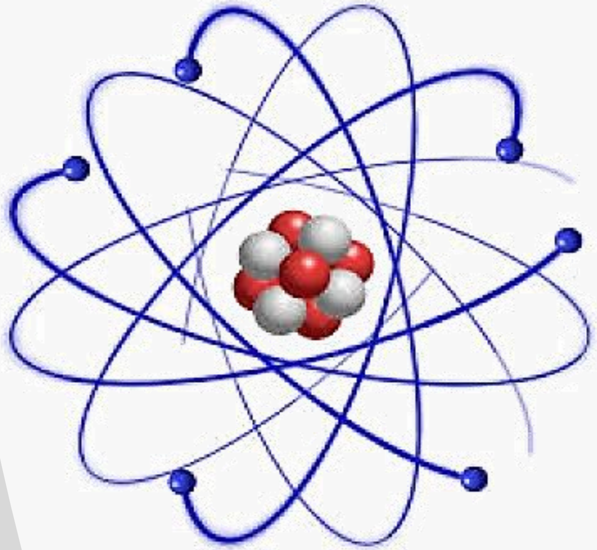


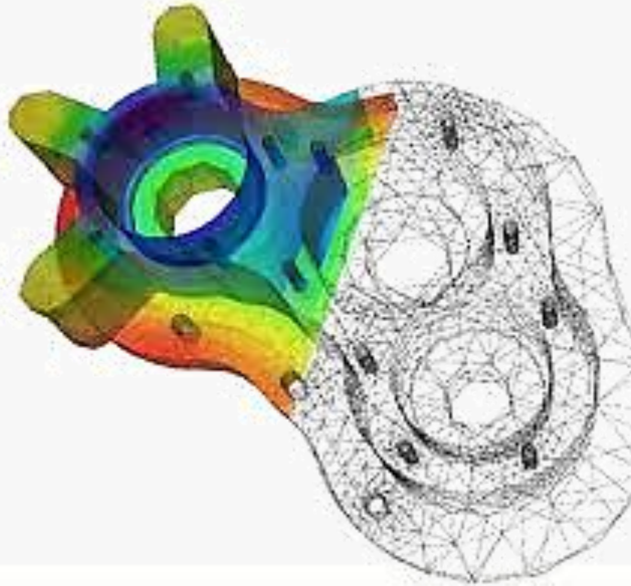
Partial Differential Equations

Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir







Partial Differential Equations

Book: Advanced Engineering Mathematics (9th Edition) by Ervin Kreyszig

- Chapter: 12
  - Sections: 12.5, 12.6

Book: Linear Partial Differential Equations for Scientists and Engineers (4th Edition) by Lokenath Debnath

- Chapter: 7
  - Sections: 7.1, 7.5

Book: Applied Partial Differential Equations With Fourier series and boundary value problems by Richard Haberman

- Chapter: 2
  - Sections: 2.1, 2.3

Method of Separation of Variables for Second Order PDEs.

- The method of separation of variables combined with the principle of superposition is widely used to solve initial boundary-value problems involving linear partial differential equations.
- The function $u(x, y)$ is expressed in the separable form $u\ (x, y)\ =\ X\ (x)\ Y\ (y)$, where X and Y are functions of x and y respectively.
- In many cases, the partial differential equation reduces to two ordinary differential equations for X and Y.
- A similar treatment can be applied to equations in three or more independent variables.
- This method of solution is also known as the Fourier method (or the method of eigenfunction expansion). Thus, the procedure outlined above leads to the important ideas of eigenvalues, eigenfunctions, and orthogonality, all of which are very general and powerful for dealing with linear problems.
- The method of separation of variables is used when the **partial differential equation and the boundary conditions are linear and homogeneous**.

Boundary Conditions

- We have just described the conditions on the separability of a given partial differential equation.
- Now, we shall take a look at the boundary conditions involved.
- There are several types of boundary conditions. The ones that appear most frequently in problems of applied mathematics and mathematical physics include:
  (i) **Dirichlet condition:** u is prescribed on a boundary
  (ii) **Neumann condition:** $(\partial u/\partial n)$ is prescribed on a boundary
  (iii) **Mixed condition:** $(\partial u/\partial n) + hu$ is prescribed on a boundary, where $(\partial u/\partial n)$ is the directional derivative of u along the outward normal to the boundary, and h is a given continuous function on the boundary.
- Besides these three boundary conditions, also known as, *the first, second, and third* boundary conditions, there are other conditions, such as the **Robin condition**; one condition is prescribed on one portion of a boundary, and another is given on the remainder of the boundary.
- We shall consider a variety of boundary conditions as we treat problems later.

Boundary Conditions

- To separate boundary conditions, such as the ones listed above, it is best to choose a coordinate system suitable to a boundary.

- For instance, we choose the Cartesian coordinate system (x, y) for a rectangular region such that the boundary is described by the coordinate lines $x =$constant and $y =$constant, and the polar coordinate system (r, θ) for a circular region so that the boundary is described by the lines r $=$ constant and θ $=$ constant.

- Another condition that must be imposed on the separability of boundary conditions is that boundary conditions, say at $x = x_0$, must contain the derivatives of u with respect to x only, and their coefficients must depend only on x. For example, the boundary condition:

$$[u + u_y]_{x=x_0} = 0$$

  cannot be separated. Needless to say, a mixed condition, such as $u_x + u_y$, cannot be prescribed on an axis.

Introduction to Fourier Series:

- The representation of a function in the form of a series is fairly common practice in mathematics. Probably the most familiar expansions are power series of the form:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

in which the resolved components or **base set** comprise the power functions $1, x, x^2, x^3, \dots, x^k, \dots$. Thus, a powerful way of studying a given function in terms of small constituents that share the same properties, is done by the use of a series. Power functions comprise only one example of a base set for the expansion of functions: a number of other base sets may be used.

- What about functions that exhibit repetitive behavior?

Such functions must be expressed in terms of fundamental functions that share repetitive behavior.

- We know that sinusoids (cosines and sines) are functions that repeat themselves (periodic), therefore, it would be interesting to study a series that contains both of them. Such a series is known as **Fourier series**, named after a French mathematician Joseph Fourier (1768-1830), who introduced it during his study of heat flow.

Background Knowledge:

- The concept of the Fourier series is based on representing a periodic signal as the sum of harmonically related sinusoidal functions. With the aid of Fourier series, complex periodic waveforms can be represented in terms of sinusoidal functions, whose properties are familiar to us. Thus, the response of a system to a complex waveform can be viewed as the response to a series of sinusoidal functions.

- A signal is periodic if, for some positive value of T,

$$f(x+T) = f(x) \quad \text{for all } x. \qquad (*)$$

  For example, $\sin x$ and $\cos x$ are periodic functions of period 2π.

- If $f(x)$ has period T, it also has the period $2T$ because $(*)$ implies $f(x+2T) = f(x+T) = f(x)$. This concept can be generalized. Thus, for any integer $k = 1,2,3, \dots,$ we have

$$f(x+kT) = f(x) \quad \text{for all } x. \qquad (**)$$

- For $k = 1$, we get the **fundamental period** of $f(x)$. It is the minimum positive, nonzero value of T for which $(*)$ is satisfied, and the value $\omega = 2\pi/T$ is referred to as the **fundamental frequency**.

- All other frequencies in the signal are integer multiples of the fundamental frequency. These various components are referred to as **harmonics,** with the **order** of a given harmonic indicated by the ratio of its frequency to the fundamental frequency.

Fourier Series

For a periodic function $f(x)$ of period T, defined on $x_0 \leq x \leq x_0 + T$, the **Fourier series** is defined as:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(k\omega x) + b_k \sin(k\omega x)], \qquad (1)$$

where a_0, a_k and b_k are the **Fourier coefficients**, given by:

$$a_0 = \frac{2}{T} \int_{x_0}^{x_0+T} f(x)\, dx, \qquad (2)$$

$$a_k = \frac{2}{T} \int_{x_0}^{x_0+T} f(x) \cos(k\omega x)\, dx \quad (k = 1,2,3, \dots), \qquad (3)$$

and

$$b_k = \frac{2}{T} \int_{x_0}^{x_0+T} f(x) \sin(k\omega x)\, dx \quad (k = 1,2,3, \dots) \qquad (4)$$

which are known as **Euler's formulae**.

Fourier Series: Important facts

- In electrical engineering it is common practice to refer to a_k and b_k respectively as the **in-phase** and **phase quadrature components** of the kth harmonic, this terminology arising from the use of the phasor notation:

$$e^{jk\omega t} = \cos(k\omega t) + j\sin(k\omega t). \qquad (5)$$

- The limits of integration in Euler's formulae may be specified over any period, so that the choice of x_0 is arbitrary and may be made in such a way as to help in the calculation of a_k and b_k. In practice, it is common to specify $f(x)$ over either the period: $\frac{-T}{2} \leq x \leq \frac{T}{2}$ or the period: $0 \leq x \leq T$, leading respectively to the limits of integration being $\frac{-T}{2}$ and $\frac{T}{2}$ (that is, $x_0 = \frac{-T}{2}$) or 0 and T (that is, $x_0 = 0$).
- It is also worth noting that an alternative approach may simplify the calculation of a_k and b_k. Using the formula (5), we have:

$$a_k + jb_k = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)e^{jk\omega x}dx\,. \qquad (6)$$

Evaluating this integral and equating real and imaginary parts on each side gives the values of a_k and b_k. This approach is particularly useful when only the amplitude $|a_k + jb_k|$ of the kth harmonic is required.

Fourier Series: Important facts

There are few formulae that we need as they help in simplifying Fourier coefficients: a_0, a_k and b_k.

$$\int_{x_0}^{x_0+T} \cos n\omega t \, \mathrm{d}t = \begin{cases} 0 & (n \neq 0) \\ T & (n = 0) \end{cases}$$

$$\int_{x_0}^{x_0+T} \sin n\omega t \, \mathrm{d}t = 0 \quad (\text{all } n)$$

$$\int_{x_0}^{x_0+T} \sin m\omega t \sin n\omega t \, \mathrm{d}t = \begin{cases} 0 & (m \neq n) \\ \frac{1}{2}T & (m = n \neq 0) \end{cases}$$

$$\int_{x_0}^{x_0+T} \cos m\omega t \cos n\omega t \, \mathrm{d}t = \begin{cases} 0 & (m \neq n) \\ \frac{1}{2}T & (m = n \neq 0) \end{cases}$$

$$\int_{x_0}^{x_0+T} \cos m\omega t \sin n\omega t \, \mathrm{d}t = 0 \quad (\text{all } m \text{ and } n)$$

The above results constitute the **orthogonality relations** for sine and cosine functions, and show that the set of functions

$$\{1, \cos \omega x \,, \cos 2\omega x \,, \dots, \cos k\omega x \,, \sin \omega x \,, \sin 2\omega x \,, \dots, \sin k\omega x\}$$

is an orthogonal set of functions on the interval $x_0 \leq x \leq x_0 + T$.

Alternative forms of Fourier Series

In general, there exist three forms of the **Fourier series,** given as:

- **Sine-Cosine Form/Trigonometric Fourier Series: (TFS)**

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(k\omega x) + b_k \sin(k\omega x)].$$

- **Amplitude-Phase Form: (APF)**

$$f(x) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega x + \phi_k)\,.$$

- **Complex Exponential Form/ Complex Fourier Series: (CFS)**

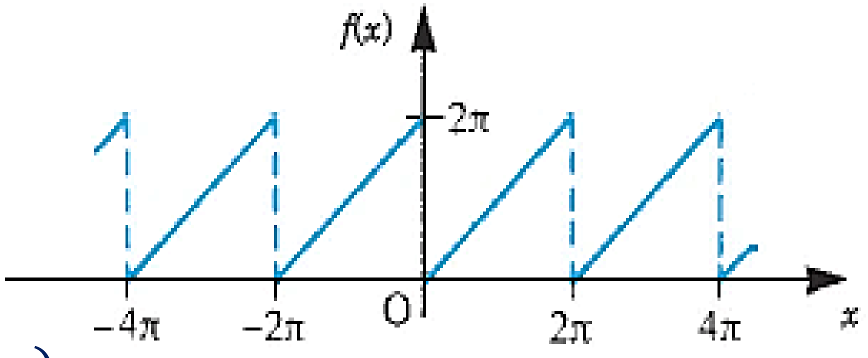
$$f(x) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega x}.$$

Example: Obtain the Fourier series expansion of the periodic function $f(x) = x;\ (0 \leq x \leq 2\pi)$ of period 2π i.e., $f(x) = f(x + 2\pi)$.

Solution: Given that $T = 2\pi$. Since, $T = \frac{2\pi}{\omega} \Rightarrow \omega = 1$. By using (2), (3) and (4) the Fourier coefficients are obtained as:

$$a_0 = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\,dx = \frac{2}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = 2\pi.$$

$$a_k = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\cos(k\omega x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos(kx)\,dx \qquad (k = 1,2,3,\dots).$$



Sawtooth wave

Using integration by parts we get $a_k = 0$.

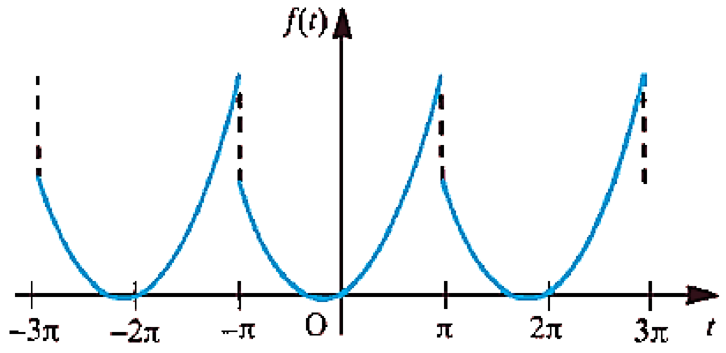
$$b_k = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\sin(k\omega x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin(kx)\,dx \qquad (k = 1,2,3,\dots),$$

which, on integration by parts, gives $b_k = -2/k$. Thus, the **Fourier series** (1) of $f(x)$ is given as:

$$f(x) = \pi - 2\sum_{k=1}^{\infty}\frac{\sin(kx)}{k} = \pi - 2\left[\sin x + \frac{\sin(2x)}{2} + \cdots + \frac{\sin(kx)}{k} + \cdots\right].$$

Example: A periodic function $f(t)$ with period 2π is defined by $f(t) = t^2 + t\ (-\pi \le t \le \pi)$. Sketch a graph of the function $f(t)$ for values of t from $t = -3\pi$ to $t = 3\pi$ and obtain a Fourier series expansion of the function.

Solution: Given that $T = 2\pi \Rightarrow \omega = 1$. By using (2), (3) and (4) the Fourier coefficients are obtained as:



$$a_0 = \frac{2}{T}\int_{x_0}^{x_0+T} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+t)\,dt = \frac{2}{3}\pi^2.$$

$$a_k = \frac{2}{T}\int_{x_0}^{x_0+T} f(t)\cos(k\omega t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+t)\cos(kt)\,dt = \frac{4}{k^2}(-1)^k, \qquad (k \ge 1).$$

$$b_k = \frac{2}{T}\int_{x_0}^{x_0+T} f(t)\sin(k\omega t)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+t)\sin(kt)\,dt = \frac{2}{k}(-1)^{k+1}, \qquad (k \ge 1).$$

Thus, by using (1) the **Fourier series** of $f(t)$ is given as:

$$f(t) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty}\left[\frac{(-1)^k 4\cos(kt)}{k^2} + \frac{(-1)^{k+1} 2\sin(kt)}{k}\right].$$

Alternative Method: By using equation (6):

$$a_k + jb_k = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)e^{jk\omega x}dx\,,$$

we have:

$$a_k + ib_k = \frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+t)\,e^{ikt}dt = (-1)^k\left[\frac{4}{k^2} - i\,\frac{2}{k}\right].$$

Equating real and imaginary parts, we get

$$a_k = \frac{4}{k^2}(-1)^k, \qquad \text{and} \qquad b_k = \frac{2}{k}(-1)^{k+1}, \qquad (k \geq 1).$$

Thus, by using (1) the **Fourier series** of $f(t)$ is given as:

$$f(t) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty}\left[\frac{(-1)^k 4\cos(kt)}{k^2} + \frac{(-1)^{k+1}2\sin(kt)}{k}\right].$$

Note: a_0 will be computed as before, i.e., by using (2).

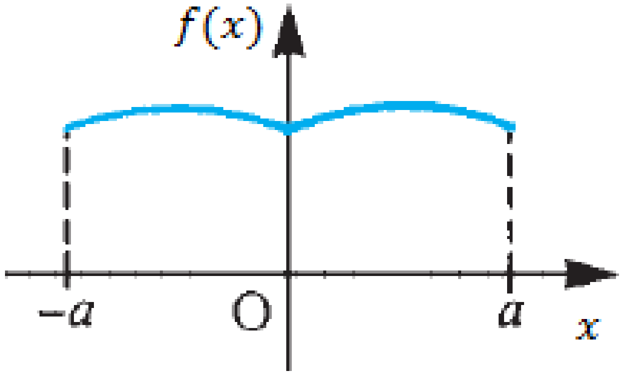
Even and odd functions

- A function $f(x)$ is called an even function if

$$f(x) = f(-x) \text{ for all } x.$$

The graph of an even function is symmetric about the vertical axis. From the definition of integration, it follows that if $f(x)$ is an even function then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

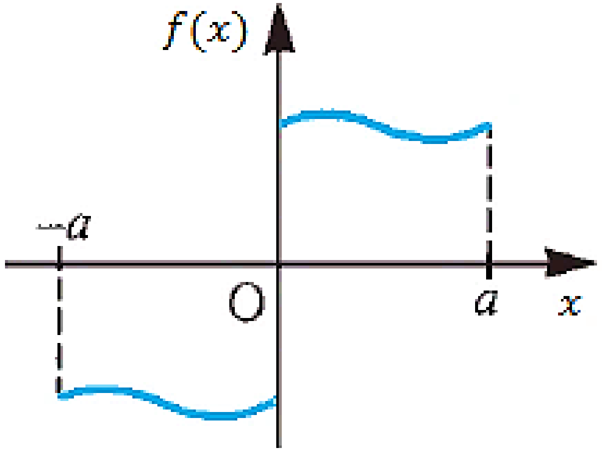


- A function $f(x)$ is said to be an odd function if

$$f(x) = -f(-x) \text{ for all } x.$$

The graph of the function is symmetric about the origin; that is, there is opposite-quadrant symmetry. From the definition of integration, it follows that if $f(x)$ is an odd function then

$$\int_{-a}^{a} f(x)dx = 0.$$



Examples: $\cos x$ is an even function while $\sin x$ is an odd function.

Properties of Even and odd functions

The following properties of even and odd functions are also useful for our purposes:

(a) the *sum* of two (or more) *odd* functions is an *odd* function;

(b) the *product* of two *even* functions is an *even* function;

(c) the *product* of two *odd* functions is an *even* function;

(d) the *product* of an *odd* and an *even* function is an *odd* function;

(e) the *derivative* of an *even* function is an *odd* function;

(f) the *derivative* of an *odd* function is an *even* function.

(g) the *composition* of an *odd* and an *even* function is an *even* function.

Fourier cosine series

If $f(x)$ is an ***even*** periodic function of period T then by using property (b), we have

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos(k\omega x)\,dx = \frac{4}{T}\int_{0}^{T/2} f(x)\cos(k\omega x)\,dx,$$

and by using property (d)

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin(k\omega x)\,dx = 0.$$

Thus, the Fourier series expansion of an even periodic function $f(x)$ with period T consists of cosine terms only and is given by:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega x)\,, \qquad (7)$$

with

$$a_0 = \frac{4}{T}\int_{0}^{T/2} f(x)\,dx \quad \text{and} \quad a_k = \frac{4}{T}\int_{0}^{T/2} f(x)\cos(k\omega x)\,dx, \qquad (k \geq 1). \qquad (8)$$

Fourier sine series

If $f(x)$ is an ***odd*** periodic function of period T then by using property (d), we have

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos(k\omega x)\,dx = 0, \qquad (k \geq 0)$$

and by using property (c)

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin(k\omega x)\,dx = \frac{4}{T}\int_{0}^{T/2} f(x)\sin(k\omega x)\,dx.$$

Thus, the Fourier series expansion of an odd periodic function $f(x)$ with period T consists of sine terms only and is given by:

$$f(x) = \sum_{k=1}^{\infty} b_k \sin(k\omega x)\,, \qquad (9)$$

with

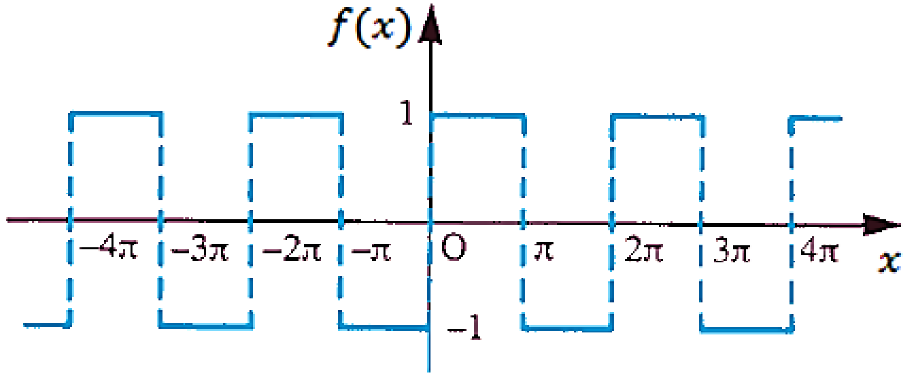
$$b_k = \frac{4}{T}\int_{0}^{T/2} f(x)\sin(k\omega x)\,dx. \qquad (k \geq 1) \qquad (10)$$

Example: Obtain the Fourier series expansion of the periodic function: $f(x) = \begin{cases} -1, & -\pi \le x \le 0 \\ 1, & 0 \le x \le \pi \end{cases}$

Solution: Note that $f(x) = -f(-x)$, thus, $f(x)$ is an odd function of x, so that its Fourier series expansion consists of sine terms only. Here $T = 2\pi$, that is $\omega = 1$, therefore by using (10) the Fourier coefficients are given by:

$$b_k = \frac{4}{T}\int_0^{T/2} f(x)\sin(k\omega x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx, \qquad (k \ge 1)$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(kx)\,dx = \frac{2[1-(-1)^k]}{k\pi} = \begin{cases} 0, & (\text{even } k) \\ \dfrac{4}{k\pi}, & (\text{odd } k) \end{cases}$$



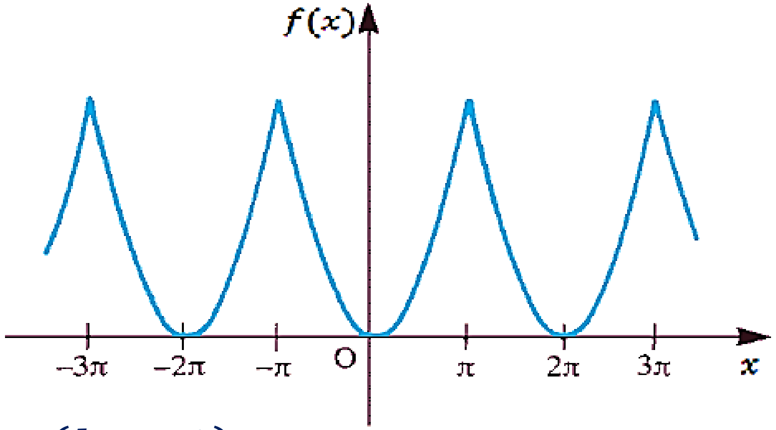
Square wave function

Thus, by using (9) the **Fourier sine series** of $f(x)$ is given as:

$$f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1} = \frac{4}{\pi}\left[\sin x + \frac{\sin(3x)}{3} + \cdots + \frac{\sin(2k-1)x}{2k-1} + \cdots\right].$$

Example: Obtain the Fourier series expansion of the periodic function $f(x) = x^2, -\pi \le x \le \pi$

Solution: Note that $f(x) = f(-x)$, thus $f(x)$ is an even function of x, so that its Fourier series expansion consists of cosine terms only. Here $T = 2\pi$, that is $\omega = 1$, therefore by using (8) the Fourier coefficients are given by:



$$a_0 = \frac{4}{T}\int_0^{T/2} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3}\pi^2,$$

$$a_k = \frac{4}{T}\int_0^{T/2} f(x)\cos(k\omega x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos(kx)\,dx = (-1)^k\frac{4}{k^2}, \qquad (k \ge 1).$$

Thus, by using (7) the **Fourier cosine series** of $f(x)$ is given as:

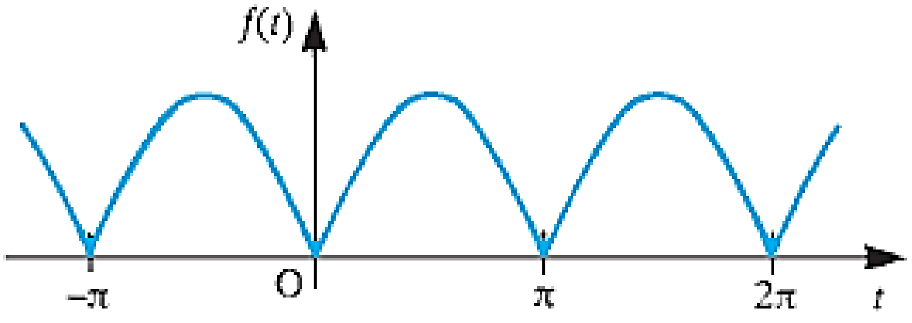
$$f(x) = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}\frac{(-1)^k\cos(kx)}{k^2} = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{\cos(2x)}{4} - \frac{\cos(3x)}{9} + \cdots + \frac{(-1)^k\cos(kx)}{k^2} + \cdots\right].$$

Example: Obtain the Fourier series expansion of the rectified sine wave $f(t) = |\sin t|$ and Sketch of the wave over the interval $-\pi \leq t \leq 2\pi$.

Solution: Note that $f(t) = f(-t)$, thus $f(t)$ is an even function of x, so that its Fourier series expansion consists of cosine terms only. Moreover, $f(t+\pi) = f(t)$, this means $T = \pi$, and $\omega = 2$. Thus, by using (8) the Fourier coefficients are given by:

$$a_0 = \frac{4}{\pi}\int_0^{\pi/2} f(t)\,dt = \frac{4}{\pi}\int_0^{\pi/2} \sin t\,dt = \frac{4}{\pi},$$

$$a_k = \frac{4}{\pi}\int_0^{\pi/2} \sin t\cos(2kt)\,dt = \frac{-4}{\pi}\left[\frac{1}{4k^2-1}\right], \qquad (k \geq 1).$$



Rectified wave $f(t) = |\sin t|$.

Thus, by using (7) the **Fourier cosine series** of $f(t)$ is given as:

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kt)}{4k^2-1} = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos(2t)}{3} + \frac{\cos(4t)}{15} + \cdots + \frac{\cos(2kt)}{4k^2-1} + \cdots\right].$$

Heat Flow from a Body in Space: Heat Equation

Physical Assumptions

1. The *specific heat* and the *density* of the material of the body are constant. No heat is produced or disappears in the body.
2. Experiments show that, in a body, heat flows in the direction of decreasing temperature, and the rate of flow is proportional to the gradient of the temperature; that is, the velocity $\mathbf{v}$ of the heat flow in the body is of the form:
$$\mathbf{v} = -K \operatorname{grad} u = -K\, \nabla u, \qquad (1)$$
where $u(x, y, z, t)$ is the temperature at a point (x, y, z) and time t.
3. The *thermal conductivity* K is constant, as is the case for homogeneous material and nonextreme temperatures.

Under these assumptions we can model heat flow as follows:

Let T be a region in the body bounded by a surface S with outer unit normal vector $\mathbf{n}$ such that the divergence theorem applies. Then $\mathbf{v} \cdot \mathbf{n}$ is the component of $\mathbf{v}$ in the direction of $\mathbf{n}$. Hence $|\mathbf{v} \cdot \mathbf{n}\, \Delta A|$ is the amount of heat *leaving* T (if $\mathbf{v} \cdot \mathbf{n} > 0$ at some point P) or *entering* T(if $\mathbf{v} \cdot \mathbf{n} < 0$ at P) per unit time at some point P of S through a small portion ΔS of S of area ΔA.

Heat Flow from a Body in Space: Heat Equation

Hence the total amount of heat that flows across S from T is given by the surface integral:

$$\iint_S \mathbf{v}\cdot\mathbf{n}\,dA\,.$$

Using Gauss's divergence theorem, we now convert our surface integral into a volume integral over the region T. Because of (1) this gives:

$$\iint_S \mathbf{v}\cdot\mathbf{n}\,dA = -K\iint_S \nabla u\cdot\mathbf{n}\,dA = -K\iiint_T \operatorname{div}(\nabla u)\,dV = -K\iiint_T \nabla^2 u\,dxdydz, \tag{2}$$

where $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$ is the **Laplacian** of u with respect to the Cartesian coordinates x, y, z. On the other hand, the total amount of heat in T is given as:

$$H = \iiint_T \sigma\,\rho\,u\,dxdydz,$$

where σ is the specific heat and ρ is the the *density* of the material of the body.

Heat Flow from a Body in Space: Heat Equation

Hence the time rate of decrease of H is given as:

$$-\frac{\partial H}{\partial t} = -\iiint_T \sigma\,\rho\,\frac{\partial u}{\partial t}\,dxdydz.$$

This must be equal to the amount of heat leaving T because no heat is produced or disappears in the body. From (2) we thus obtain:

$$-\iiint_T \sigma\,\rho\,\frac{\partial u}{\partial t}\,dxdydz = -K\iiint_T \nabla^2 u\,dxdydz$$

$$\Longrightarrow \iiint_T \left(\frac{\partial u}{\partial t} - c^2\nabla^2 u\right)dxdydz = 0,$$

where $c^2 = \frac{K}{\sigma\rho}$. The physical constant $\frac{K}{\sigma\rho}$ is denoted by c^2 (instead of c) to indicate that this constant is *positive,* a fact that will be essential to the form of the solutions.

Heat Flow from a Body in Space: Heat Equation

Since this holds for any region T in the body, the integrand (if continuous) must be zero everywhere. That is,

$$\frac{\partial u}{\partial t} - c^2\nabla^2 u = 0$$

$$\Longrightarrow \frac{\partial u}{\partial t} = c^2\nabla^2 u \qquad \text{or} \qquad \frac{\partial u}{\partial t} = c^2\left(u_{xx} + u_{yy} + u_{zz}\right).$$

This is the **heat equation** in three dimensions, the fundamental PDE modeling heat flow. It gives the temperature in a body of homogeneous material in space. The constant c^2 is the *thermal diffusivity.* K is the *thermal conductivity*, σ the *specific heat*, and ρ the *density* of the material of the body. $\nabla^2 u$ is the Laplacian of u with respect to the Cartesian coordinates x, y, z. The heat equation is also called the **diffusion equation** because it also models chemical diffusion processes of one substance or gas into another.