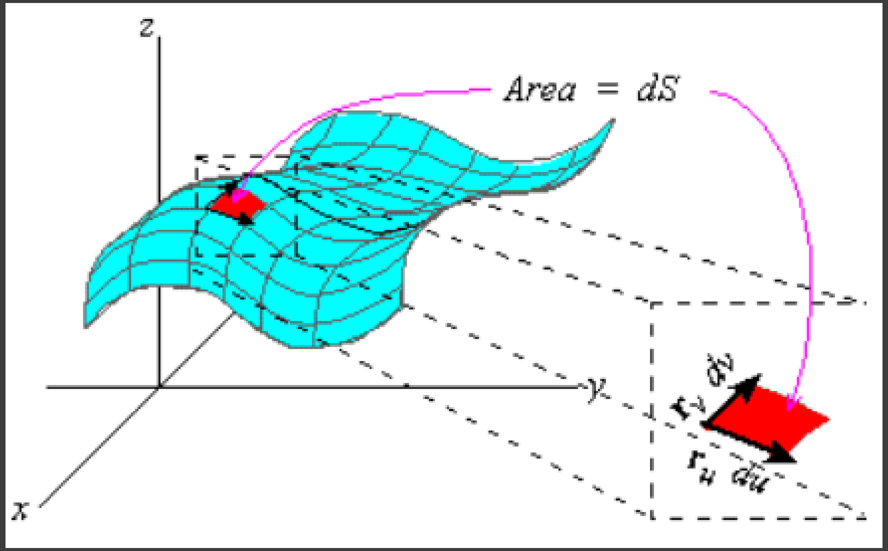
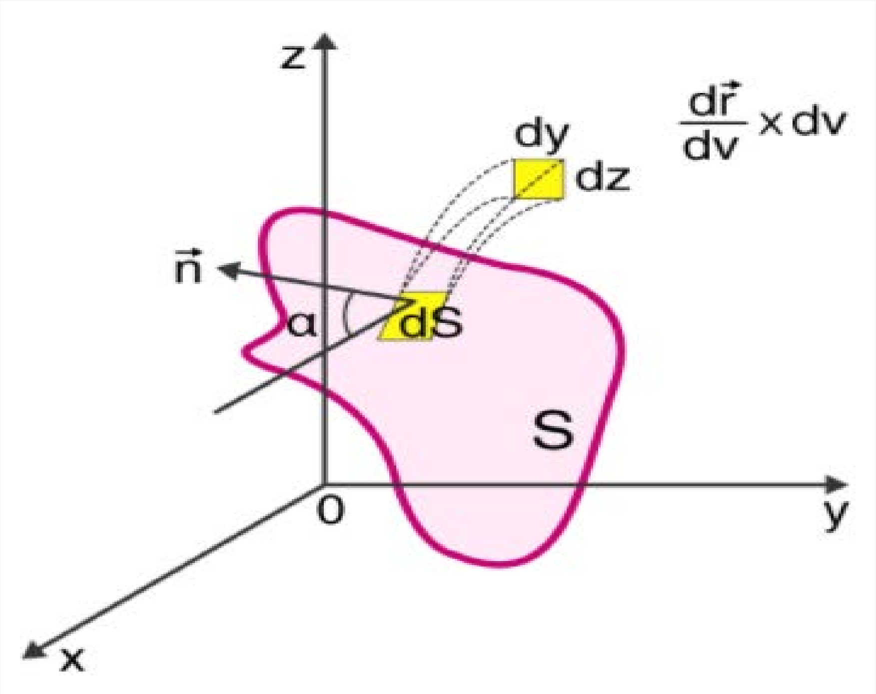


Surface Integrals

Vector Calculus(MATH-243)
Instructor: Dr. Naila Amir



Surface Integral of Vector Field

- If the surface "S" oriented is outward, then the surface integral of the vector field is given as:

$$\iint_S F(x,y,z).\,dS = \iint_S F(x,y,z).\,ndS = \iint_{D(u,v)} F[x(u,v),y(u,v),z(u,v))].\left[\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right] dudv.$$

- If the surface "S" oriented is inward, then the surface integral of the vector field is given as:

$$\iint_S F(x,y,z).\,dS = \iint_S F(x,y,z).\,ndS = \iint_{D(u,v)} F[x(u,v),y(u,v),z(u,v))].\left[\frac{\partial r}{\partial v} \times \frac{\partial r}{\partial u}\right] dudv$$

Where dS = ndS is known as the vector element of the surface.

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Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

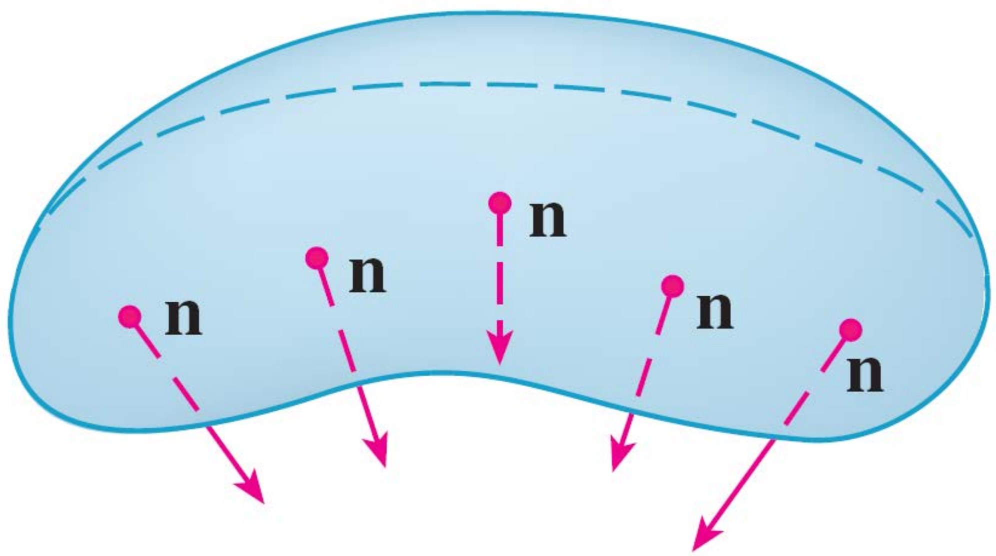
- **Chapter: 16**
  - **Section: 16.7**

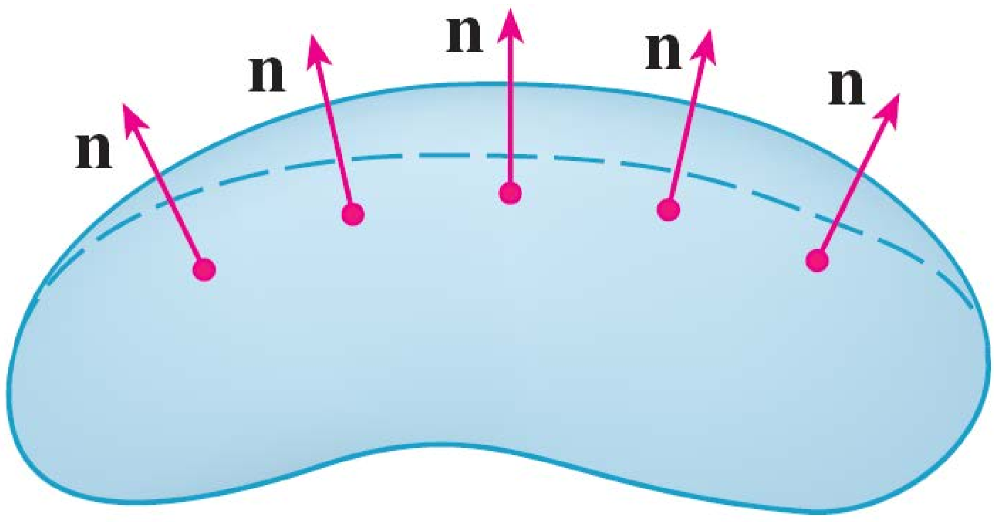
Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

- **Chapter: 16**
  - **Section: 16.6**

Oriented Surface & Orientation

When we can choose a continuous field of unit normal vectors $\mathbf{n}$ on a smooth surface S then we say that S is **orientable** (or two-sided). Spheres and other smooth surfaces that are the boundaries of regions in space are orientable, since we can choose an outward pointing unit vector $\mathbf{n}$ at each point to specify an orientation. An orientation is a way of consistently choosing one of the two sides of a surface. There are two possible orientations for any orientable (two-sided) surface.





Oriented Surface & Orientation

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector:

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

This gives the **upward orientation** of the surface. The opposite orientation is given by $-\mathbf{n}$. For a surface $z = g(x, y)$ given as the graph of g, we use:

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1 \rangle,$$

to associate with the surface a natural orientation given by the unit normal vector:

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} = \frac{\langle -g_x, -g_y, 1 \rangle}{\sqrt{1 + [g_x]^2 + [g_y]^2}}.$$

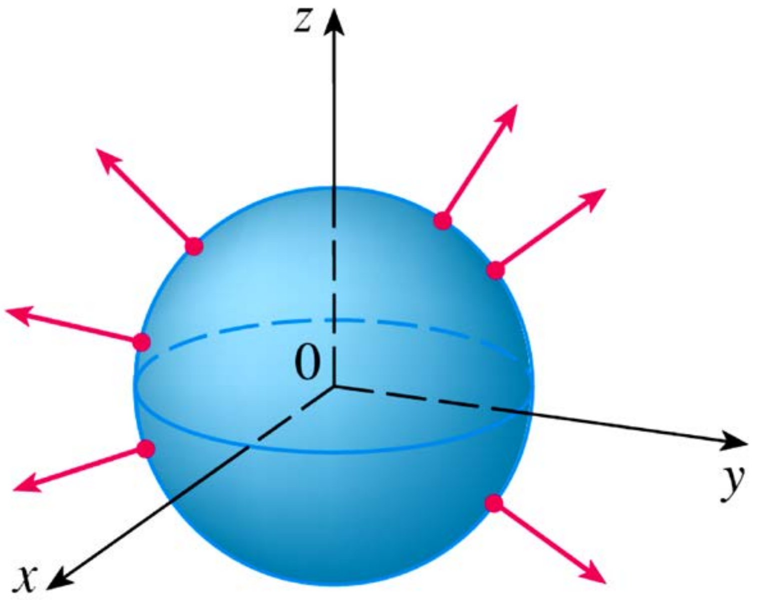
As the $\mathbf{k}$ $-$component is positive, this gives the **upward orientation** of the surface.

Oriented Surface & Orientation

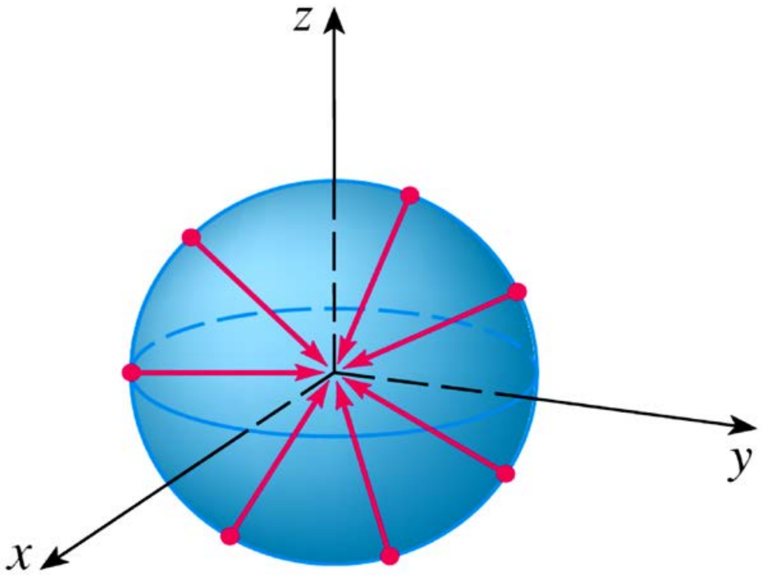
For a closed surface—a surface that is the boundary of a solid region E—the convention is that:

- The **positive orientation** is the one for which the normal vectors point outward from E. (Figure: a)

- Inward-pointing normal vectors give the **negative orientation**. (Figure: b)



(Figure: a)



(Figure: b)

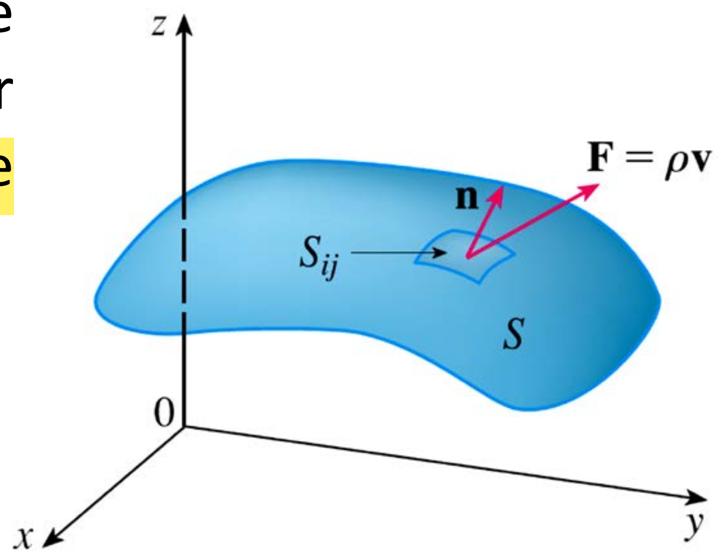
Surface Integrals of Vector Fields

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then, the the **surface integral** of $\mathbf{F}$ over S is given as:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \mathbf{F}\,d\boldsymbol{S} = \iint_D \left[\mathbf{F}\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}\right]|\mathbf{r}_u\times\mathbf{r}_v|\,dA = \iint_D [\mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)]\,dA,$$

where D is the parameter domain. This integral is also called the **flux integral** of $\mathbf{F}$ across S.

If $\mathbf{F}$ is the velocity field of a three-dimensional fluid flow, then the flux of $\mathbf{F}$ across S is the net rate at which fluid is crossing S per unit time in the chosen positive direction $\mathbf{n}$ defined by the orientation of S.



Example: Flux Integral

Find the flux of the vector field: $\mathbf{F}(x, y, z) = \langle z, y, x\rangle$ across the sphere: $x^2 + y^2 + z^2 = 1$.

Solution:

Using the parametric representation of sphere, we get:

$$\mathbf{r}(\varphi, \theta) = \langle \sin\varphi \cos\theta , \sin\varphi \sin\theta , \cos\varphi\rangle,$$

where, $0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi$, we have:

$$\mathbf{F}\big(\mathbf{r}(\varphi, \theta)\big) = \langle \cos\varphi , \sin\varphi \sin\theta , \sin\varphi \cos\theta\rangle.$$

Now:

$$\mathbf{r}_\varphi \times \mathbf{r}_\theta = \langle \sin^2\varphi \cos\theta , \sin^2\varphi \sin\theta , \sin\varphi \cos\varphi\rangle.$$

Therefore,

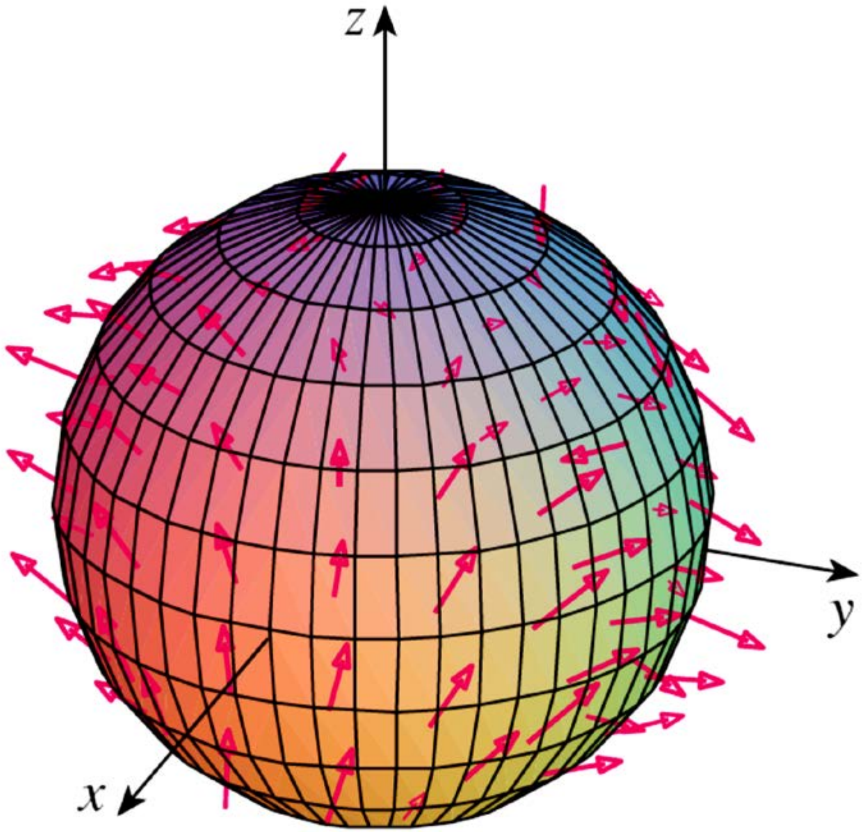
$$\mathbf{F}\big(\mathbf{r}(\varphi, \theta)\big) \cdot \big(\mathbf{r}_\varphi \times \mathbf{r}_\theta\big) = 2\cos\varphi \sin^2\varphi \cos\theta + \sin^3\varphi \sin^2\theta .$$

Thus, the flux of the vector field is given as:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D [\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)]\, dA = \int_0^{2\pi} \int_0^{\pi} [2\cos\varphi \sin^2\varphi \cos\theta + \sin^3\varphi \sin^2\theta] d\varphi d\theta = \frac{4\pi}{3}.$$

Example: Flux Integral

- The figure shows the vector field $\mathbf{F}$ at points on the unit sphere.
- If, for instance, the vector field $\mathbf{F}$ is a velocity field describing the flow of a fluid with density 1, then the answer, $4\pi/3$, represents: the rate of flow through the unit sphere in units of mass per unit time.



Surface Integrals of Vector Fields

In the case of a surface S given by a **graph of the function** $z = g(x, y)$, we can think of x and y as parameters and write:

$$\mathbf{F} \cdot \mathbf{n} = \frac{\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)}{|\mathbf{r}_x \times \mathbf{r}_y|} = \frac{\langle P, Q, R \rangle \cdot \langle -g_x, -g_y, 1 \rangle}{\sqrt{1 + [g_x]^2 + [g_y]^2}}$$

So that:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \left[-P g_x - Q g_y + R\right] dA.$$

Note:

- This formula assumes the upward orientation of S.
- For a downward orientation, we multiply by -1.
- Similar formulas can be worked out if S is given by $y = h(x, z)$ or $x = k(y, z)$.

Example:

Evaluate:

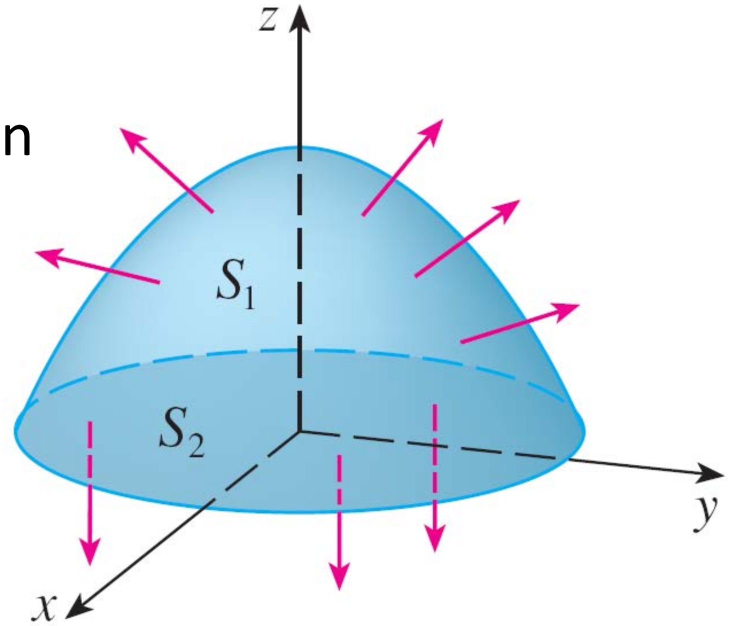
$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region *E* enclosed by the paraboloid $z = g(x, y) = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution:

S consists of a parabolic top surface S_1 and a circular bottom surface S_2. Since S is a closed surface, we use the convention of positive (outward) orientation.

This means that S_1 is oriented upward. Moreover, *D* is the projection of S_1 on the xy −plane, namely, the disk: $x^2 + y^2 \leq 1$.



Solution:

On S_1, $P(x,y,z) = y,\ Q(x,y,z) = x,\ R(x,y,z) = z = 1 - x^2 - y^2$. Also,

$$z = g(x,y) = 1 - x^2 - y^2,$$

so:

$$\frac{\partial g}{\partial x} = -2x, \qquad \frac{\partial g}{\partial y} = -2y.$$

So, we have:

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D [-Pg_x - Qg_y + R]\, dA = \iint_D [2xy + 2xy + 1 - x^2 - y^2]\, dA$$

$$= \iint_D [4xy + 1 - x^2 - y^2]\, dA = \int_0^{2\pi} \int_0^1 [1 - r^2 + 4r^2 \cos\theta \sin\theta]\, r dr d\theta = \frac{\pi}{2}.$$

Solution:

The disk S_2 is oriented downward. So, its unit normal vector is $\mathbf{n} = -\mathbf{k}.$ Moreover, $z = 0$ on S_2, thus we have:

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_2} \mathbf{F} \cdot (-\mathbf{k})\, dS = \iint_{D} [-z]\, dA = \iint_{D} [0]\, dA = 0.$$

Finally, we compute $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ as the sum of the surface integrals of $\mathbf{F}$ over the pieces S_1 and S_2, as:

$$\iint_{S} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS$$

$$= \frac{\pi}{2} + 0 = \frac{\pi}{2}.$$

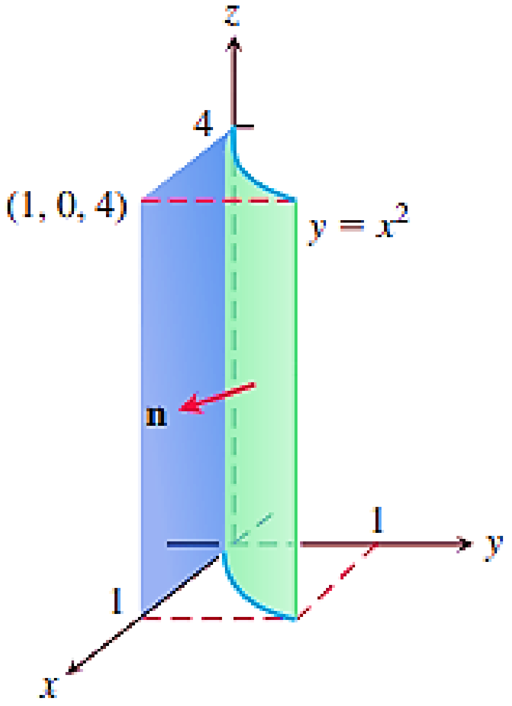
Example:

Find the flux of the vector field $\mathbf{F}(x,y,z) = yz\,\mathbf{i} + x\,\mathbf{j} - z^2\mathbf{k}$ through the parabolic cylinder $y = x^2; 0 \le x \le 1, 0 \le z \le 4$, in the direction $\mathbf{n}$ indicated in the accompanying figure.

Solution:

On the surface we have $x = x, y = x^2$, and $z = z$, Thus, for the present case we have:

$$\mathbf{r}(x,z) = \langle x, x^2, z\rangle; \quad 0 \le x \le 1, 0 \le z \le 4.$$



The cross product of tangent vectors is given as:

$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2x, -1, 0\rangle.$$

and

$$\mathbf{F}(x,y,z) = \langle x^2 z, x, -z^2\rangle.$$

Thus,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D [\mathbf{F}\cdot(\mathbf{r}_x \times \mathbf{r}_z)]\,dA = \int_0^4\int_0^1 [2x^3 z - x]\,dxdz = 2.$$

Applications

- Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations.
- For instance, if $\mathbf{E}$ is an electric field, the surface integral

$$\iint_S \mathbf{E} \cdot \mathbf{n}\, dS$$

is called the **electric flux** of **E** through the surface *S*.

- One of the important laws of electrostatics is **Gauss's Law**, which says that the net charge enclosed by a closed surface S is:

$$Q = \mathcal{E}_0 \iint_S \mathbf{E} \cdot \mathbf{n}\, dS$$

where $\mathcal{E}_0 \approx\ 8.8542 \times 10^{-12}\ \mathrm{C}^2/\mathrm{N} \cdot \mathrm{m}^2$ is a constant (called the permittivity of free space) that depends on the units used.

Applications

Another application occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then, the heat flow is defined as the vector field:

$$\mathbf{F} = -K\,\nabla u,$$

where K is an experimentally determined constant called the conductivity of the substance.

Then, the **rate of heat flow** across the surface S in the body is given by the surface integral:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -K \iint_S \nabla u \cdot \mathbf{n}\, dS.$$

Example:

The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

Solution:

Taking the center of the ball to be at the origin, we have:

$$u(x, y, z) = C(x^2 + y^2 + z^2),$$

where C is the proportionality constant.

Then, the heat flow is:

$$\mathbf{F}(x, y, z) = -K\,\nabla u = -KC(2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}),$$

where K is the conductivity of the metal.

Solution:

Instead of using the usual parametrization of the sphere, we observe that the outward unit normal to the sphere $x^2 + y^2 + z^2 = a^2$ at the point (x, y, z) is:

$$\mathbf{n} = \frac{1}{a}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

Thus,

$$\mathbf{F}\cdot\mathbf{n} = -\frac{2KC}{a}(x^2 + y^2 + z^2).$$

However, on S, we have: $x^2 + y^2 + z^2 = a^2$ and $\mathbf{F}\cdot\mathbf{n} = -2aKC.$ Thus, the rate of heat flow across S is:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = -2aKC\iint_S dS = -2aKCA(S) = -2aKC(4\pi a^2) = -8KC\pi a^3.$$

Practice Questions

Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.

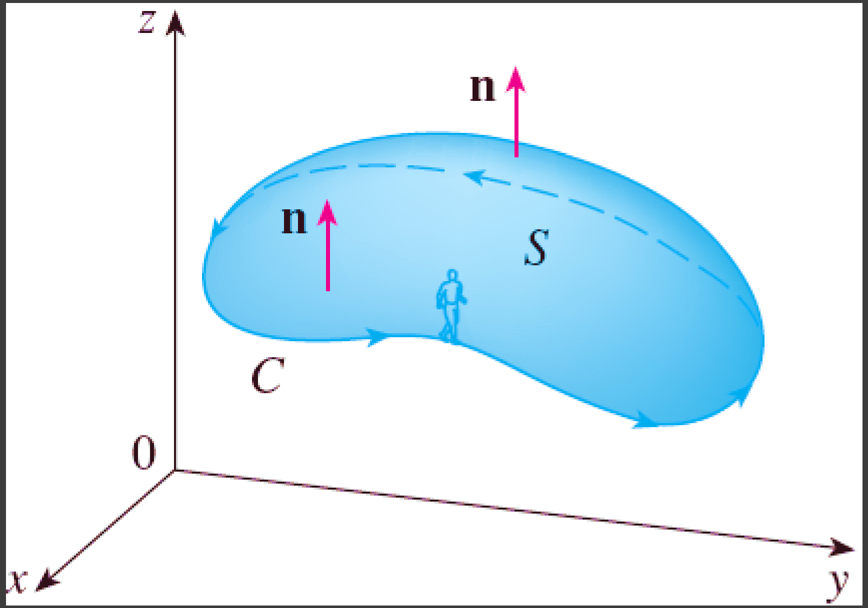
Chapter: 16

Exercise-16.7: Q – 5 to 30, Q – 33 to 47.

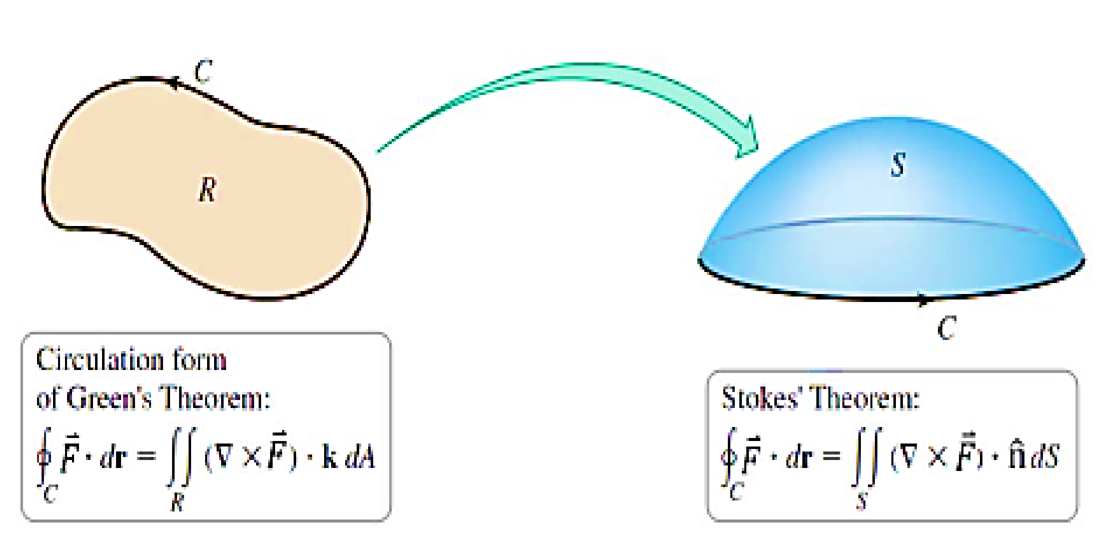
Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

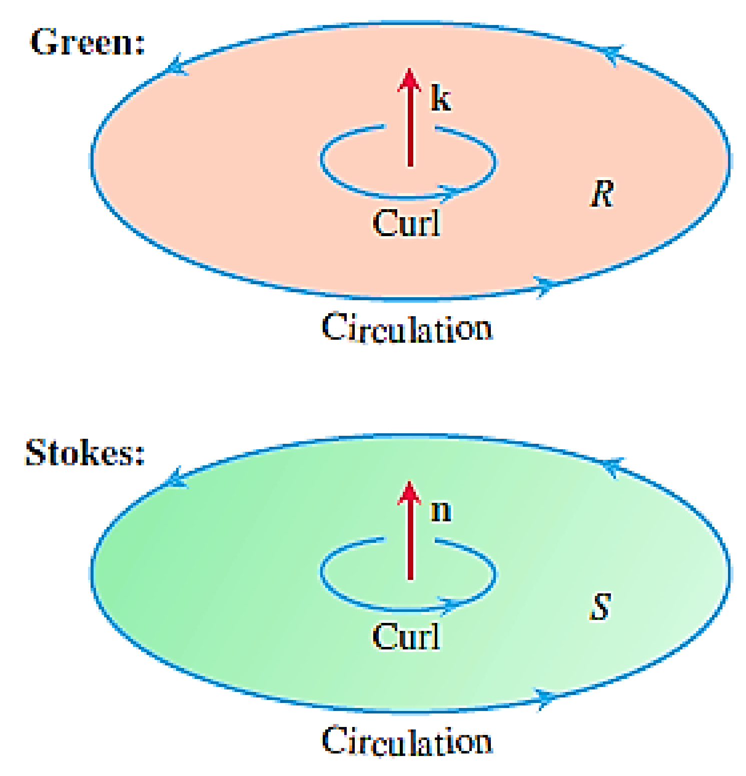
Chapter: 16

Exercise-16.6: Q – 1 to 46.



Stokes' Theorem





16

Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.8**

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

- **Chapter: 16**
  - **Section: 16.7**

Important Theorems we know

- Fundamental theorem of Calculus

$$\int_a^b f'(x)dx = f(b) - f(a)$$

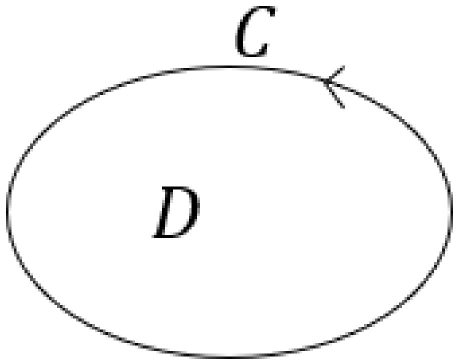
- Fundamental Theorem of Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- Green's Theorem

$$\iint_D (N_x - M_y)dxdy = \oint_C Mdx + Ndy$$

$$\iint_D (M_x + N_y)dxdy = \oint_C Mdy - Ndx$$



Every theorem relate an integral of a "derivative" to the original function on the boundary

Normal Form of Green's Theorem

In first form, Green's Theorem says that under suitable conditions the **outward flux** of a vector field across a simple closed curve in the plane equals the double integral of the divergence of the field over the region enclosed by the curve.

Green's Theorem (Flux-Divergence or Normal Form)

The outward flux of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ across a simple closed curve C equals the double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{Outward flux}} = \oint_C M\, dy - N\, dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy}_{\text{Divergence integral}}$$

$$= \iint_R \operatorname{div} \mathbf{F}\, dA.$$

Tangential Form of Green's Theorem

In second form, Green's Theorem says that the counterclockwise circulation of a vector field around a simple closed curve is the double integral of the $\mathbf{k}$ −component of the curl of the field over the region enclosed by the curve.

Green's Theorem (Circulation-Curl or Tangential Form)

The counterclockwise circulation of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ around a simple closed curve C in the plane equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M\, dx + N\, dy}_{\text{Counterclockwise circulation}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy}_{\text{Curl integral}}$$

$$= \iint_R (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dA.$$

Stokes' Theorem & Green's Theorem

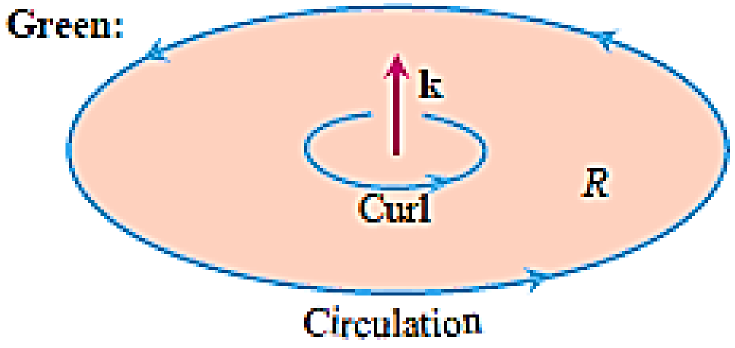
For a vector field $\mathbf{F}$ and a smooth surface S the Green's theorem can be generalized to higher dimensions as follows:

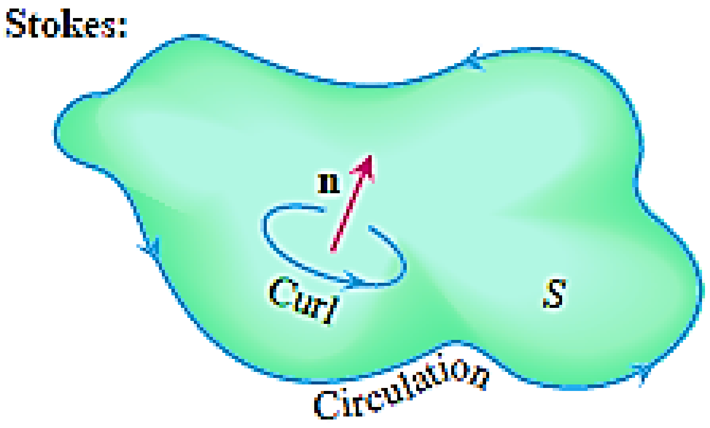
$$\oint_C \mathbf{F}.\mathbf{T}\, ds = \iint_R (\text{curl}\ \mathbf{F}) \cdot \mathbf{k}\, dA$$

Circulation in R

$$\oint_C \mathbf{F}.\mathbf{T}\, ds = \iint_S (\text{curl}\ \mathbf{F}) \cdot \mathbf{n}\, dS$$

Circulation in 3D





Stokes' Vs. Green's Theorem

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem.

- Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve.

- Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (a space curve).

Stokes' Theorem

Let S be an oriented piecewise-smooth surface bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then,

$$\oint_C \mathbf{F}.\mathbf{T}\, ds = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the unit normal vector at any point of S drawn in the sense in which a right-handed screw would advance when rotated in the sense of the description of C.