Complex (exponential) form of Fourier Series

$f(t)$, $d \le t \le d+T$, periodic of period T.

TFS: $$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t \quad \text{—— (i)}$$

we Know that

$$\sin n\omega t = \frac{e^{jn\omega t} - e^{-jn\omega t}}{2j}, \quad \cos n\omega t = \frac{e^{jn\omega t} + e^{-jn\omega t}}{2}$$

$$\text{(i)} \Rightarrow f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \frac{e^{jn\omega t} + e^{-jn\omega t}}{2} + b_n \frac{e^{jn\omega t} - e^{-jn\omega t}}{2j}$$

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[\frac{1}{2}(a_n - jb_n) e^{jn\omega t} + \frac{1}{2}(a_n + jb_n) e^{-jn\omega t} \right] \text{—— (ii)}$$

writing $c_0 = \frac{1}{2}a_0$, $c_n = \frac{1}{2}(a_n - jb_n)$, $c_{-n} = c_n^* = \frac{1}{2}(a_n + jb_n)$.

$$\text{(ii)} \Rightarrow f(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega t}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} + \sum_{n=-1}^{-\infty} c_n e^{jn\omega t}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}, \quad c_0 e^0 = c_0$$

(iii)

is the Complex/exponential form of F.S.

Where, $c_0 = \frac{1}{2} a_0 = \frac{1}{T} \int_{d}^{d+T} f(t)\, dt$ — (iv)

$$c_n = \frac{1}{2}(a_n - j b_n) = \frac{1}{T} \int_{d}^{d+T} f(t) \cos n\omega t \, dt - j \int_{d}^{d+T} f(t) \sin n\omega t \, dt$$

$$= \frac{1}{T} \int_{d}^{d+T} f(t) \left[\cos n\omega t - j \sin n\omega t\right] dt.$$

$$= \frac{1}{T} \int_{d}^{d+T} f(t)\, e^{-jn\omega t}\, dt \quad \text{— (v)}$$

(iii) – (v) gives complete calculations for Complex Fourier series.

Relationship with TFS:

$$c_n = \frac{1}{2}(a_n - j b_n), \quad c_{-n} = c_n^* = \frac{1}{2}(a_n + j b_n)$$

$$a_n - j b_n = 2 c_n, \quad a_n + j b_n = 2 c_n^*$$

giving, $a_n = c_n + c_n^*$, $b_n = j(c_n - c_n^*)$.

Note: If function/signal $f(t)$ is even,

$$c_n = c_n^*, \quad a_n = 2 c_n, \quad b_n = 0$$

The Complex Fourier Coefficients of an even signal are real (pure).

If $f(t)$ is odd, $c_n^* = -c_n$, $a_n = 0$, $b_n = 2j c_n$.

The Complex Fourier Coefficients of an odd function are pure imaginary.

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Ex-1: obtain the Complex Fourier Series expansion of the periodic function:

$$f(t) = t^2 \; (-\pi < t < \pi), \quad f(t+2\pi) = f(t), \text{ for all } t.$$

obtain the Corresponding trigmometric F.S.

Sol:- $T = 2\pi, \; \omega = 1, \quad f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnt}.$

$$\text{with } c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 \, dt = \frac{\pi^2}{3},$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underset{\text{I}}{t^2} \, \underset{\text{II}}{e^{-jnt}} \, dt = \frac{1}{2\pi}\left[\frac{-t^2}{jn} e^{-jnt} - \frac{2t}{(jn)^2} e^{-jnt} - \frac{2}{(jn)^3} e^{-jnt} \right]_{-\pi}^{\pi}$$

we simplify,

$$-\frac{2}{(jn)^3} e^{-jnt} = \frac{-2}{j^3 n^3} e^{-jnt} = \frac{-2}{n^3 (j)(j^2)} e^{-jnt} = \frac{-2}{n^3(-1)j} \times \frac{j}{j} e^{-jnt} = \frac{-2j}{n^3} e^{-jnt}.$$

$$\text{So, } c_n = \frac{1}{2\pi}\Bigg[\left(\frac{j\pi^2}{n} e^{-jn\pi} + \frac{2\pi}{n^2} e^{-jn\pi} - \frac{2j}{n^3} e^{-jn\pi} \right) - \frac{j\pi^2}{n} e^{jn\pi} - \frac{2\pi}{n^2} e^{jn\pi} - \frac{2j}{n^3} e^{jn\pi} \Big) \Bigg].$$

Since, $e^{jn\pi} = \cos n\pi + j\sin n\pi = \cos n\pi$, Also, $e^{-jn\pi} = \cos n\pi$,

$$c_n = \frac{1}{2\pi}\Big[\cancel{\frac{j\pi^2}{n}\cos n\pi} + \frac{2\pi}{n^2}\cos n\pi - \cancel{\frac{2j}{n^3}\cos n\pi} - \cancel{\frac{j\pi^2}{n}\cos n\pi} + \frac{2\pi}{n^2}\cos n\pi + \cancel{\frac{2j}{n^3}\cos n\pi} \Big].$$

$$c_n = \frac{1}{2\pi}\left[\frac{4\pi^2}{n^2}\cos n\pi \right] = \frac{2}{n^2}\cos n\pi = \frac{2}{n^2}(-1)^n, \; n \neq 0$$

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Hence, the Complex Form of Fourier Series of $f(t)$ is:

$$f(t) = \frac{\pi^2}{3} + \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{2}{n^2}(-1)^n e^{jnt}.$$

TFS: $a_0 = 2C_0 = \frac{2\pi^2}{3}$,

$$a_n - jb_n = \frac{4}{n^2}(-1)^n, \quad a_n + jb_n = \frac{4}{n^2}(-1)^n$$

giving, $b_n = 0$, $a_n = \frac{4}{n^2}(-1)^n$, the TFS

is $f(t) = \frac{1}{2}\left(\frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nt$.

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n \cos nt.$$

which is the same result as at page No. 9.

observe that $f(t)$ is an even function, C_n are real.

Homework: obtain Complex Series representation of the functions at page 5. Hence, obtain TFS.

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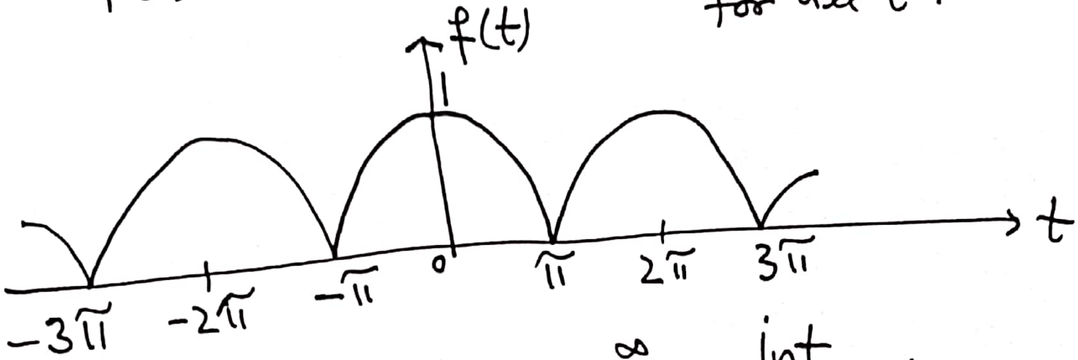
EX-2. Find the complex form of the F.S. expansion of the periodic function $f(t)$ defined by:

$$f(t) = \cos \tfrac{1}{2}t, \; -\pi < t < \pi. \quad f(t+2\pi) = f(t) \text{ for all } t.$$



Sol: Here, $T = 2\pi$, $\omega = 1$, $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnt}$, where

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\left(\tfrac{1}{2}t\right) e^{-jnt}\, dt.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{e^{jt/2} + e^{-jt/2}}{2}\right) e^{-jnt}\, dt.$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} \left[e^{-j(n-\frac{1}{2})t} + e^{-j(n+\frac{1}{2})t} \right] dt$$

$$= \frac{1}{4\pi}\left[\frac{-2e^{-j(2n-1)\frac{t}{2}}}{j(2n-1)} - \frac{2e^{-j(2n+1)t/2}}{j(2n+1)} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}(-2)\frac{1}{j}\left[\left(\frac{e^{-jn\pi}e^{j\pi/2}}{2n-1} + \frac{e^{-jn\pi}e^{-j\pi/2}}{2n+1} \right) - \left(\frac{e^{jn\pi}e^{-j\pi/2}}{2n-1} + \frac{e^{jn\pi}e^{j\pi/2}}{2n+1} \right) \right]$$

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We simplify as,

$$e^{j\pi/2} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = 0 + j(1) = j, \quad e^{-j\pi/2} = -j,$$

$$e^{jn\pi} = \cos n\pi + j\sin n\pi = \cos n\pi + j(0) = \cos n\pi = (-1)^n = e^{-jn\pi}, \text{ so that}$$

$$c_n = \frac{j}{2\pi}\left(\frac{j}{(2n-1)} - \frac{j}{2n+1} + \frac{j}{2n-1} - \frac{j}{2n+1}\right)\cos n\pi$$

$$= \frac{j}{2\pi}(j)(-1)^n\left(\frac{2}{2n-1} + \frac{2}{2n+1}\right)$$

$$= \frac{(-1)^n}{\pi}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) = \frac{(-2)(-1)^n}{\pi(4n^2-1)} = \frac{2(-1)^{n+1}}{\pi(4n^2-1)}.$$

Hence, the Complex F.S. of $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{2(-1)^{n+1}}{\pi(4n^2-1)} e^{jnt}.$$

TFS: $\quad a_0 = 2c_0, \; a_n = c_n + c_n^*, \; b_n = j(c_n - c_n^*).$

$$c_n = \frac{2(-1)^{n+1}}{\pi(4n^2-1)}, \quad c_0 = \frac{2(-1)^1}{\pi[4(0)^2-1]} = \frac{-2}{\pi(-1)} = \frac{2}{\pi}$$

$$a_0 = 2\left(\frac{2}{\pi}\right) = \frac{4}{\pi}, \quad a_n = 2c_n = 2\left(\frac{2(-1)^{n+1}}{\pi(4n^2-1)}\right)$$

Hence, TFS is

$$f(t) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1}\cos nt.$$

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