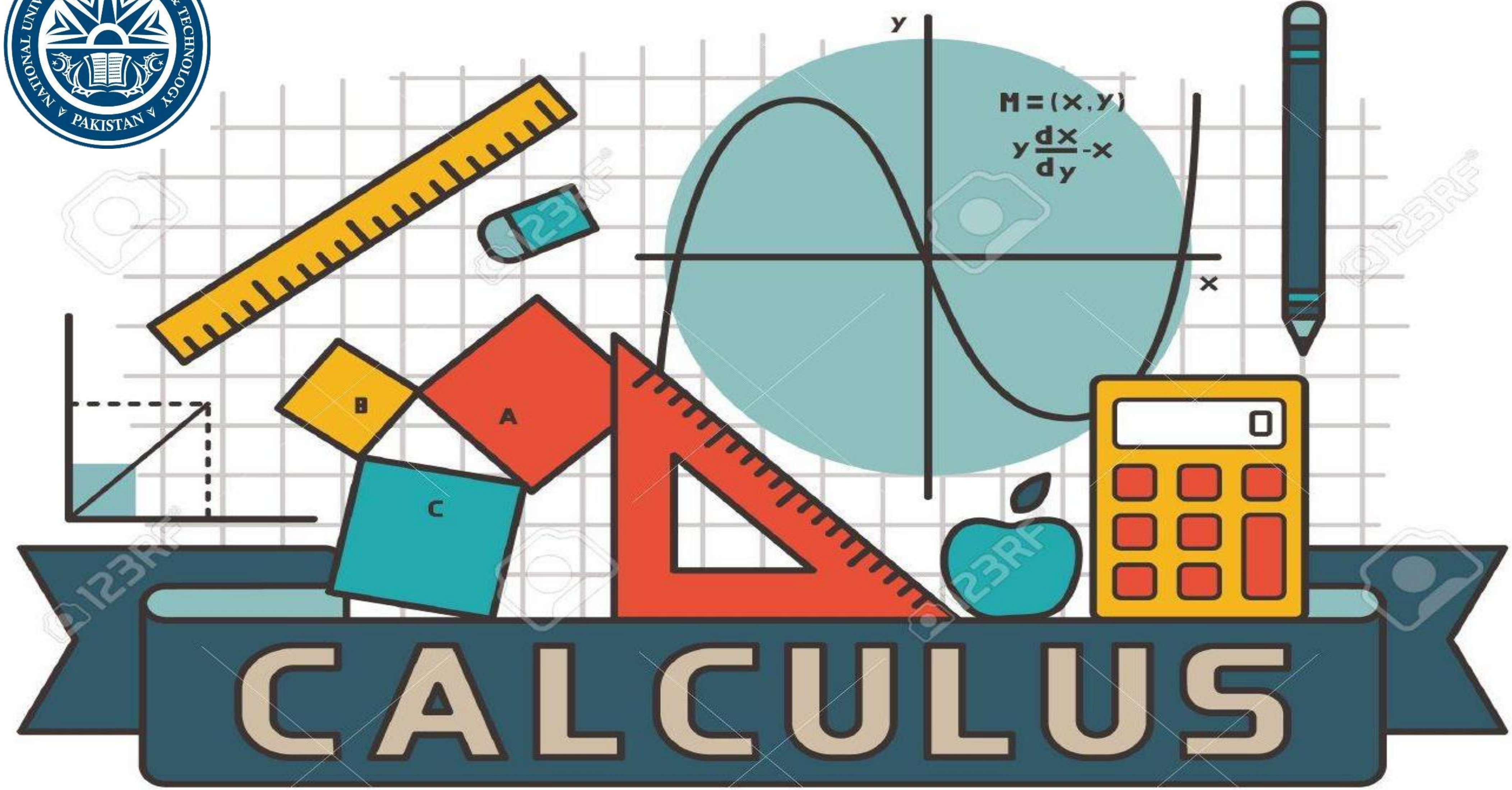

Calculus & Analytical Geometry
MATH- 101

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(SEECS, NUST)

Combining Functions

Combining Functions

- There exist different ways to combine functions to make new functions:
  - Arithmetic Combinations of Functions that includes Sums, Differences, Products and Quotients.
  - Composition of Functions.

Arithmetic Combinations of Functions

- Two functions f and g can be combined to form new functions

$$f + g,\ \ f - g,\ \ fg,\ \ f/g$$

in a manner similar to the way we add, subtract, multiply, and divide real numbers.

Algebra of Functions

- Let f and g be functions with domains A and B.
- Then, the functions $f + g, f - g, fg$, and f/g are defined as follows.

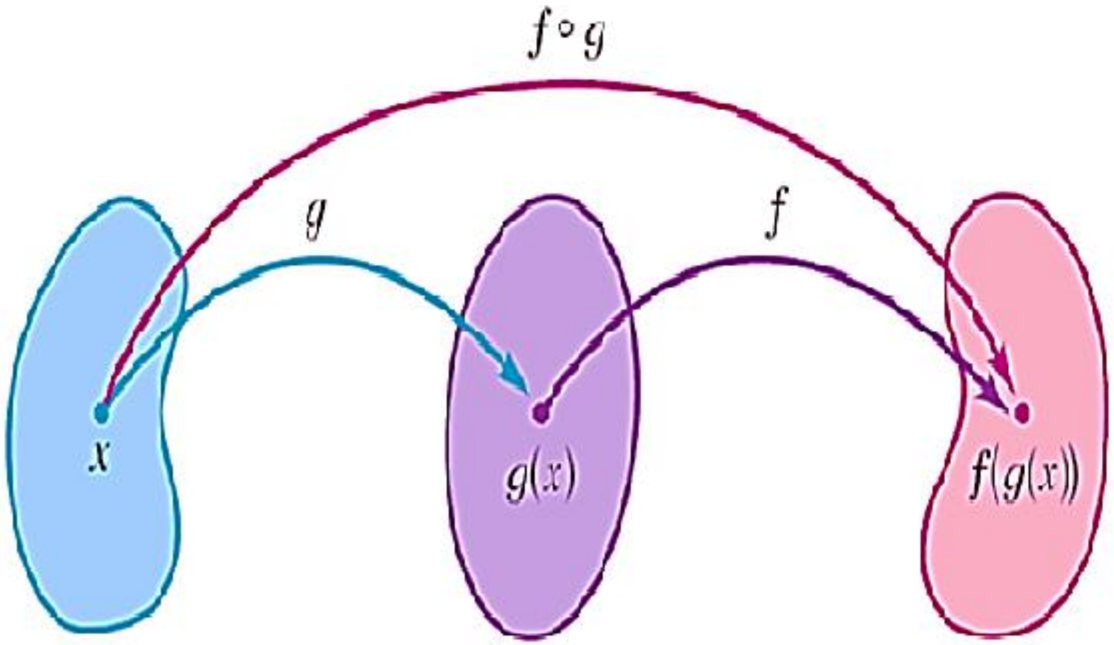
1. $(f+g)(x) = f(x) + g(x)$ Domain $A \cap B$

2. $(f-g)(x) = f(x) - g(x)$ Domain $A \cap B$

3. $(fg)(x) = f(x)g(x)$ **Note:** Determine domain on the basis of the final form of the function.

4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ **Note:** Determine domain on the basis of the final form of the function.

Composite Function



gof(x)=g(f(x))

fog(x)=f(g(x))

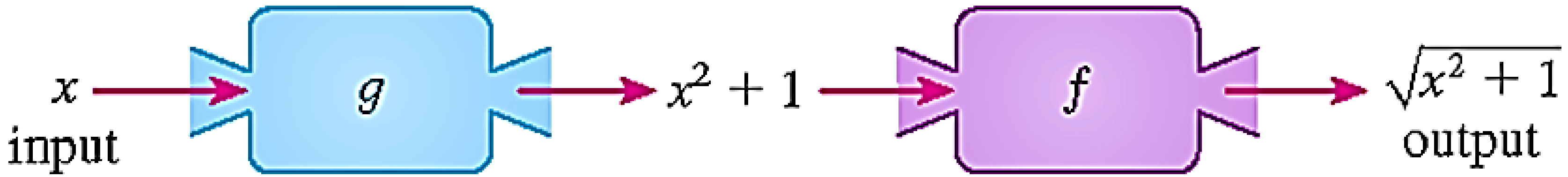
fog≠gof

Composition of Functions

- Now, let's consider a very important way of combining two functions to get a new function.
  - Suppose $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$.
  - We may define a function h as:

$$h(x) = (f \circ g)(x) = f\big(g(x)\big) = f(x^2 + 1) = \sqrt{x^2 + 1}.$$

  - The function h is made up of the functions f and g in an interesting way: Given a number x, we first apply to it the function g, then apply f to the result.



Composition of Functions

- In general, given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, we can then calculate the value of $f(g(x))$.
- The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the composition (or composite) of f and g and is denoted by $f \circ g$ ("f composed with g").
- The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

Example: Composition of Functions

Let $f(x) = x^2$ and $g(x) = x - 3$.

(a) Find the functions $f \circ g$ and $g \circ f$ and their domains.

(b) Find $(f \circ g)(5)$ and $(g \circ f)(7)$.

Solution:

(a) We have:

$$(f \circ g)(x) = f(g(x)) \;= f(x-3) \;= (x-3)^2$$

and

$$(g \circ f)(x) = g\big(f(x)\big) \;= g(x^2) \;= x^2 - 3\,.$$

The domains of both $f \circ g$ and $g \circ f$ are $\mathbb{R}$.

(b) We have:

$$(f \circ g)(5) = f(g(5)) = f(2) = 22 = 4$$

$$(g \circ f)(7) = g(f(7)) = g(49) = 49 - 3 = 46$$

Example: Composition of Functions

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find the following functions and their domains.

(a) $f \circ g$

(b) $g \circ f$

(c) $f \circ f$

(d) $g \circ g$

Solution:

(a) $(f \circ g)(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = \sqrt[4]{2-x}$

The domain of $f \circ g$ is:

$$\{x \mid 2 - x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2].$$

(b) $(g \circ f)(x) = g\big(f(x)\big) = g(\sqrt{x}) = \sqrt{2-\sqrt{x}}$

For $\sqrt{x}$ to be defined, we must have $x \geq 0$. For $\sqrt{2-\sqrt{x}}$ to be defined, we must have $2 - \sqrt{x} \geq 0$, that is $\sqrt{x} \leq 2$, or $x \leq 4$. Thus, we have: $0 \leq x \leq 4$. So, the domain of $g \circ f$ is the closed interval $[0, 4]$.

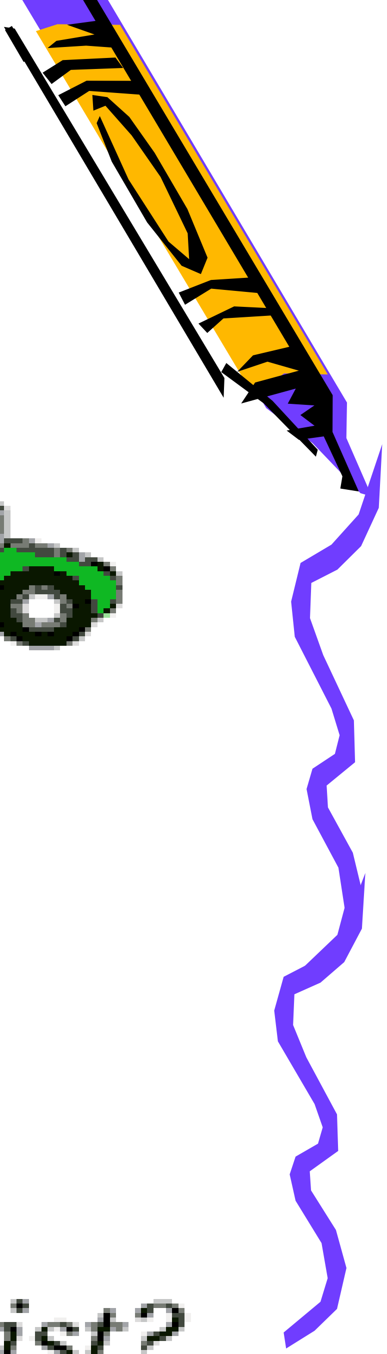
Practice: Compute part (c) and (d).

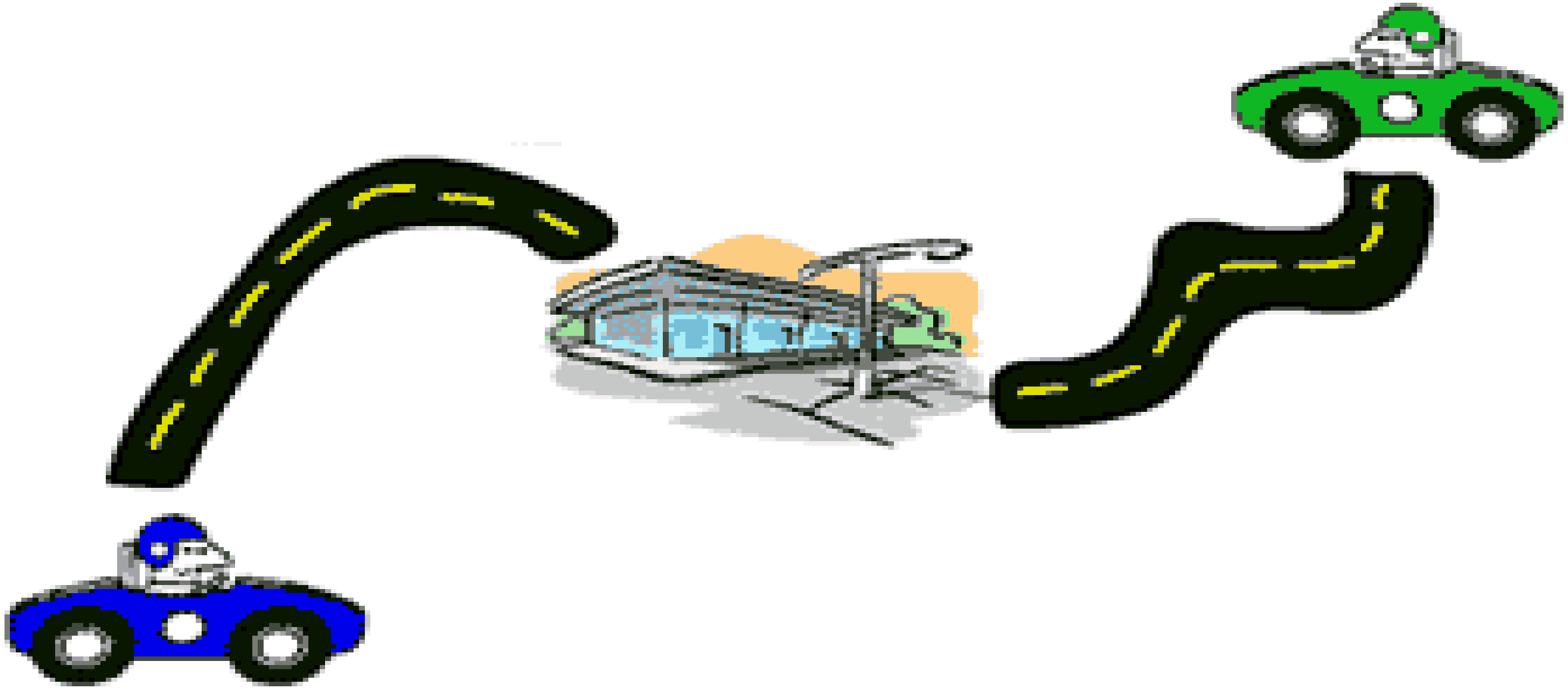
Properties of Composite functions

- The composition of two even functions is even, and the composition of two odd functions is odd.
- The composition of an even function and an odd function is even.
- The function composition of one-to-one function is always one to one.
- The function composition of two onto function is always onto.
- The inverse of the composition of two functions f and g is equal to the composition of the inverse of both the functions, such as:

$$(f \circ g)^{-1} = (g^{-1} \circ f^{-1}).$$

Limits





When does a limit exist?

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

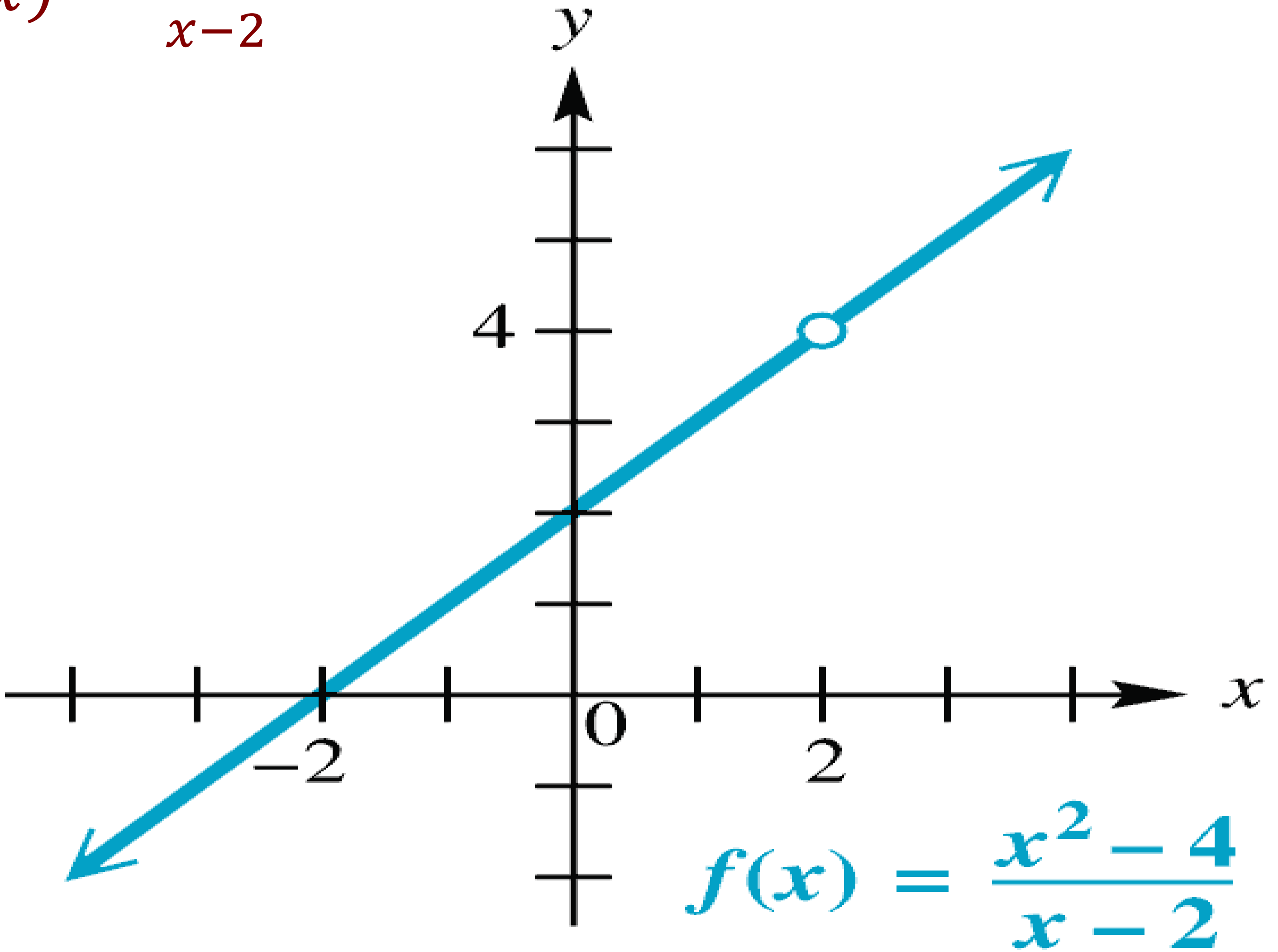
- Chapter: 2
  - Sections: 2.1

- **An Introduction To Limits**
- Laws for Calculating Limits
- One-Sided Limits
- Limits Involving Infinity

Approaching ...

Sometimes we can't work something out directly ... but we **can** see what it should be as we get closer and closer!

The function: $f(x) = \frac{x^2-4}{x-2}$



is not defined at $x = 2$, so its graph has a “hole” at $x = 2$.

Values of

$$f(x) = \frac{x^2 - 4}{x - 2}$$

may be computed near $x = 2$.

As x approaches 2

x	1.9	1.99	1.999→	←2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999→	←4.001	4.01	4.1

$f(x)$ approaches 4

The values of $f(x)$ get closer and closer to 4 as x gets closer and closer to 2.

We say that

"the limit of $\dfrac{x^2-4}{x-2}$ as x approaches 2 equals 4"

and write

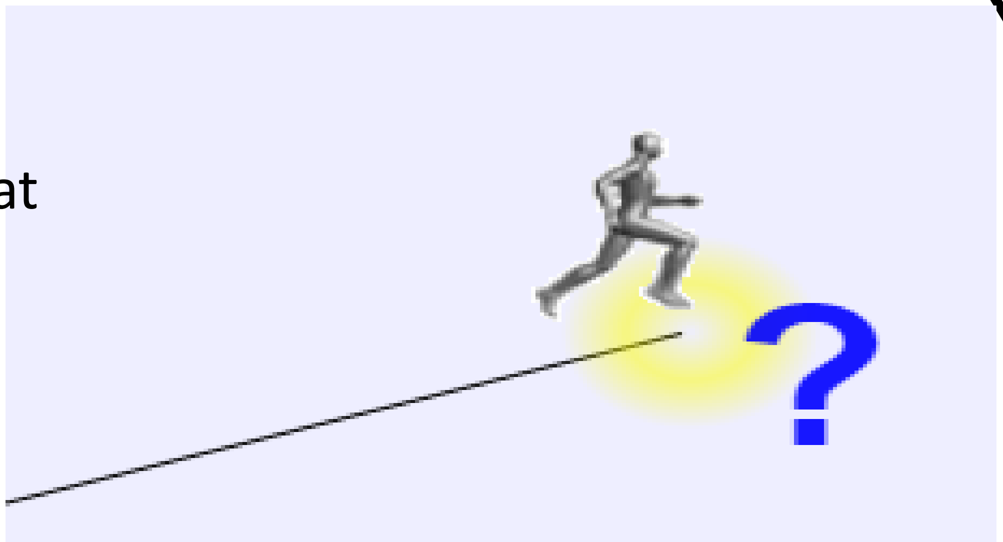
$$\boxed{\lim_{x\to 2}\frac{x^2-4}{x-2}=4.}$$

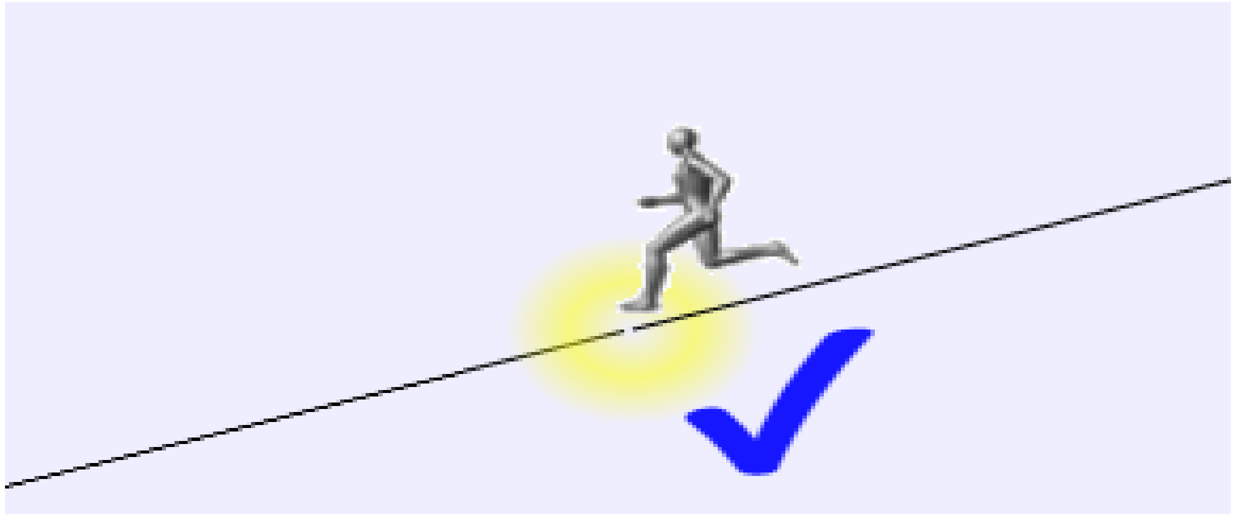
Test Both Sides!

It is like running up a hill and then finding the path **is magically "not there"...**

... but if we only check one side, who knows what happens?

So we need to test it **from both directions** to be sure where it "should be"!





Limits:

- **Limit of a function** is a fundamental concept in calculus concerning the behavior of that **function** near a particular input.
- It is fundamental to finding the tangent to a curve or the velocity of an object.
- We use limits to describe the way a function $f(x)$ varies. Some functions vary continuously; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump or vary erratically.
- The notion of limit gives a precise way to distinguish between these behaviors.
- The geometric application of using limits to define the tangent to a curve leads at once to the important concept of the derivative of a function which quantifies the way a function's values change.

Limit of a Function:

Let f be a function and let a and L be real numbers. $\boldsymbol{L}$ **is the limit of** $\boldsymbol{f(x)}$ **as** $\boldsymbol{x}$ **approaches** $\boldsymbol{a}$, written

$$\lim_{x \to a} f(x) = L,$$

if the following conditions are met.

1. As x assumes values closer and closer (but not equal) to a on both sides of a, the corresponding values of $f(x)$ get closer and closer (and are perhaps equal) to L.
2. The value of $f(x)$ can be made as close to L as desired by taking values of x arbitrarily close to a.

Finding the Limit of a Polynomial Function

Example: Find

$$\lim_{x \to 1} (x^2 - 3x + 4).$$

Solution: The behavior of

$$f(x) = x^2 - 3x + 4$$

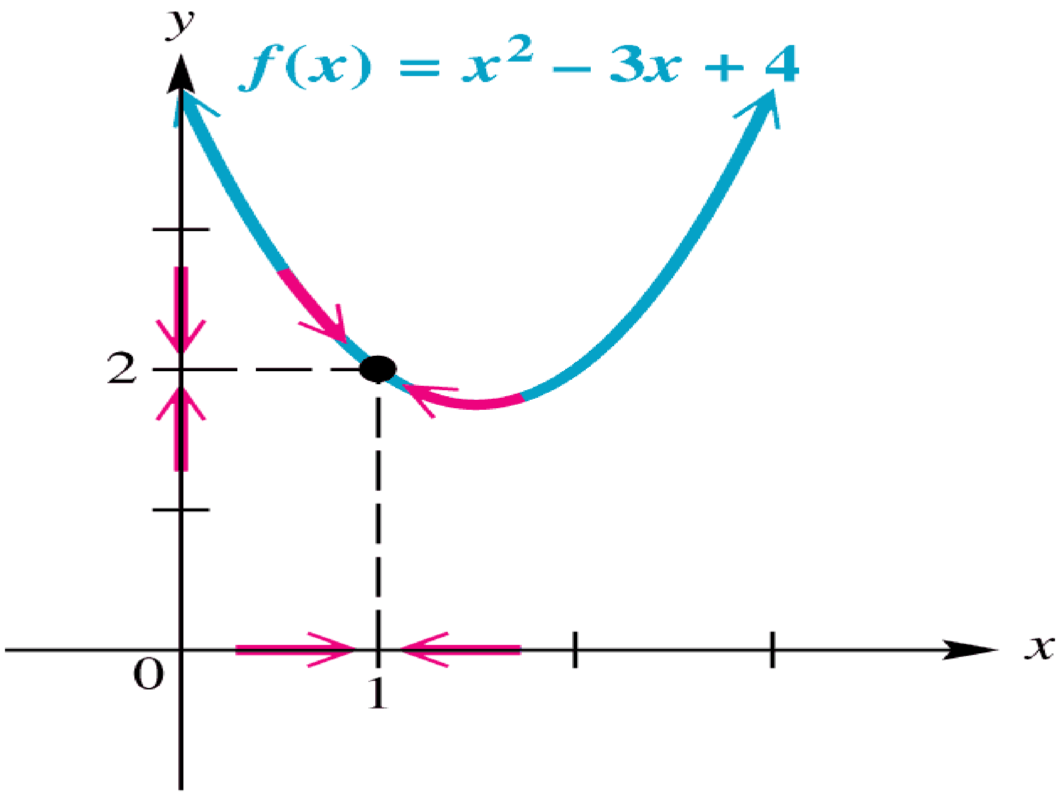
near $x = 1$ can be determined from a table of values,

As x approaches 1

x	.9	.99	.999→	←1.001	1.01	1.1
$f(x)$	2.11	2.0101	2.001→	←1.999	1.9901	1.91

$f(x)$ approaches 2

Alternatively: from a graph of $f(x)$.

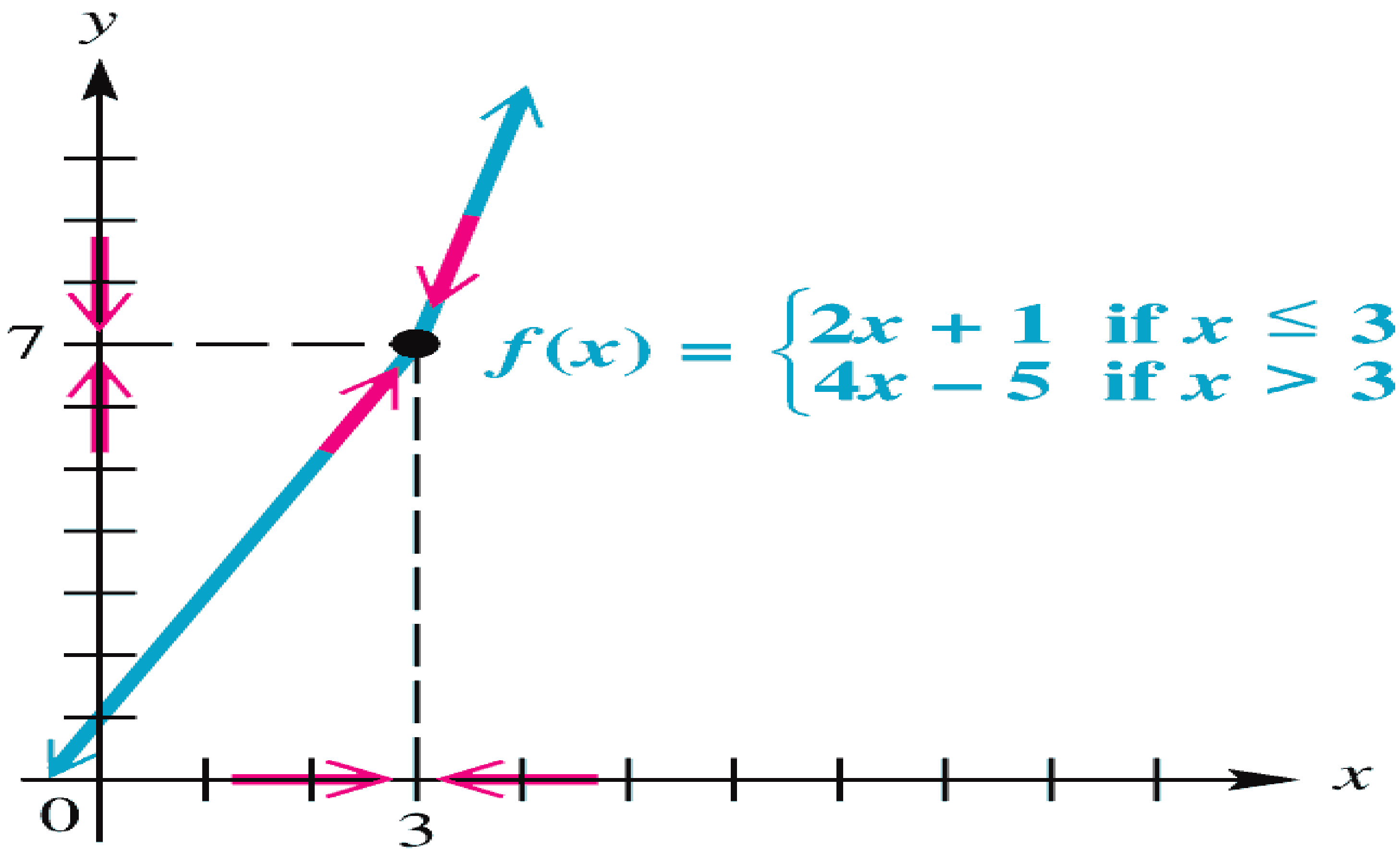


We see that

$$\lim_{x \to 1} (x^2 - 3x + 4) = 2.$$

Example: Find $\lim_{x \to 3} f(x)$ where

$$f(x) = \begin{cases} 2x+1 & \text{if} \quad x \le 3 \\ 4x-5 & \text{if} \quad x > 3 \end{cases}.$$



Solution:

As x approaches 3

x	2.9	2.99	2.999→	←3.001	3.01	3.1
$f(x)$	6.8	6.98	6.998→	←7.004	7.04	7.4

$f(x)$ approaches 7

Therefore,

$$\lim_{x \to 3} f(x) = 7.$$

Limits that do not exist

If there is no single value that is approached by $f(x)$ as x approaches a, we say that $f(x)$ does not have a limit as x approaches a,

or $\lim_{x \to a} f(x)$ does not exist.

Determining whether a limit exists

Example: Find $\lim_{x \to 2} f(x)$ where

$$f(x) = \begin{cases} 4x - 5 & \text{if} \quad x \le 2 \\ 3x - 5 & \text{if} \quad x > 2 \end{cases}.$$

Solution: Construct a table

x	1.9	1.99	1.999→	←2.001	2.01	2.1
$f(x)$	2.6	2.96	2.996→	←1.003	1.03	1.3

Solution:

- ❖ $f(x)$ approaches 3 as x gets closer to 2 from the left,

but

- ❖ $f(x)$ approaches 1 as x gets closer to 2 from the right.

Therefore, $\lim_{x \to 2} f(x)$ does not exist.

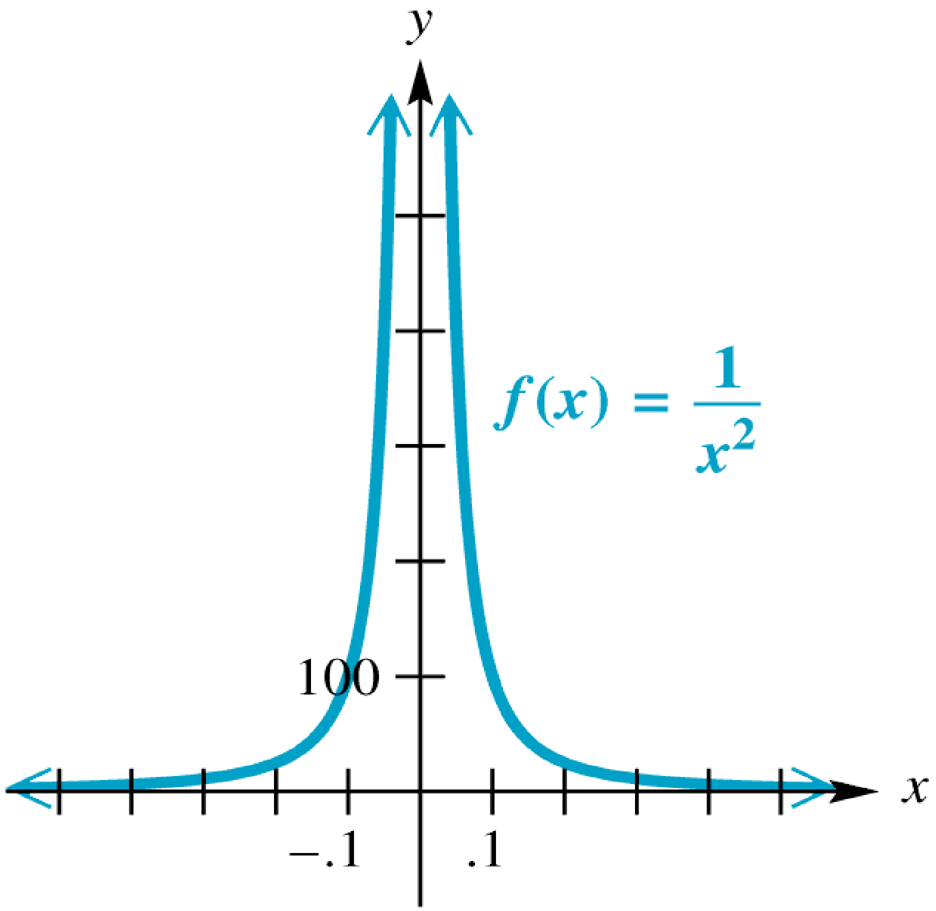
Example: Find $\lim_{x \to 0} f(x)$ where

$$f(x) = \frac{1}{x^2}.$$

Solution: Construct a table and graph

x	-.1	-.01	-.001→
$f(x)$	100	10,000	1,000,000→
x	←.001	.01	.1
$f(x)$	←1,000,000	10,000	100

Solution:



As x approaches 0, the corresponding values of $f(x)$ grow arbitrarily large. Therefore,

$$\lim_{x \to 0} \frac{1}{x^2} \text{ does not exist.}$$

Conditions under which $\lim\limits_{x \to a} f(x)$ fails to exist:

1. $f(x)$ approaches a number L as x approaches a from the left and $f(x)$ approaches a different number M as x approaches a from the right.

2. $f(x)$ becomes infinitely large in absolute value as x approaches a from either side.

3. $f(x)$ oscillates infinitely many times between two fixed values as x approaches a.

Example: A Function May Fail to Have a Limit at a Point in Its Domain

Discuss the behavior of the following functions as $x \to 0$.

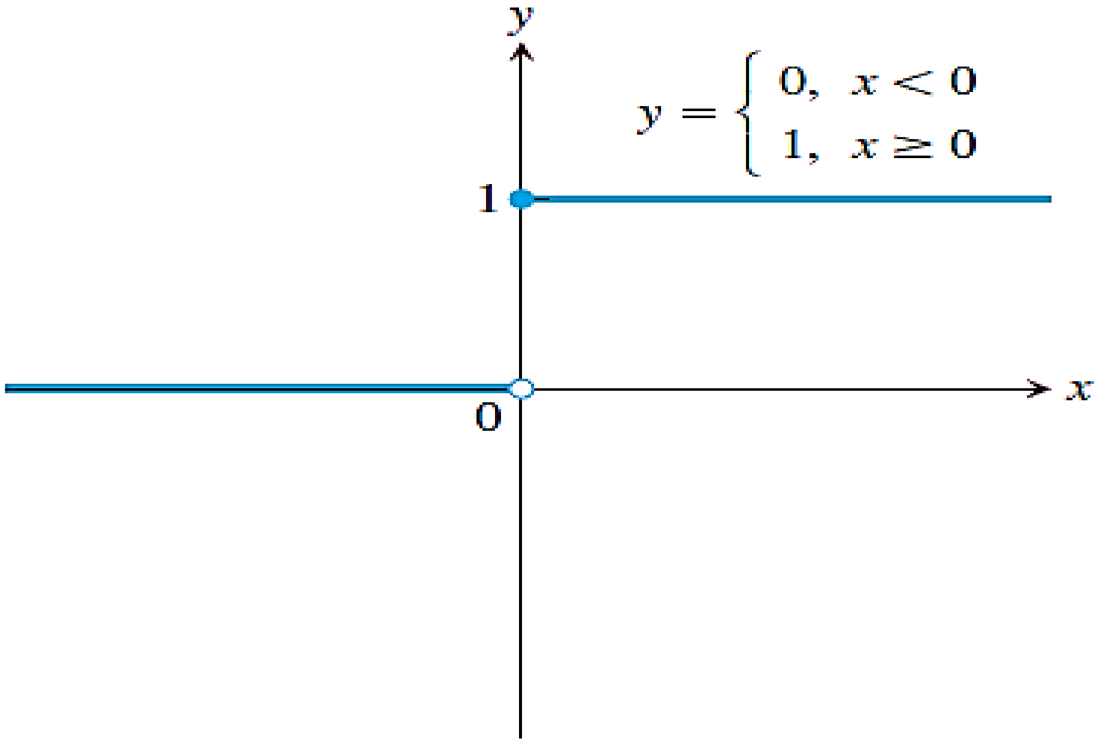
$$\textbf{(a)} \quad U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

$$\textbf{(b)} \quad g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\textbf{(c)} \quad f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \dfrac{1}{x}, & \mathrm{x} > 0 \end{cases}$$

Solution: A Function May Fail to Have a Limit at a Point in Its Domain

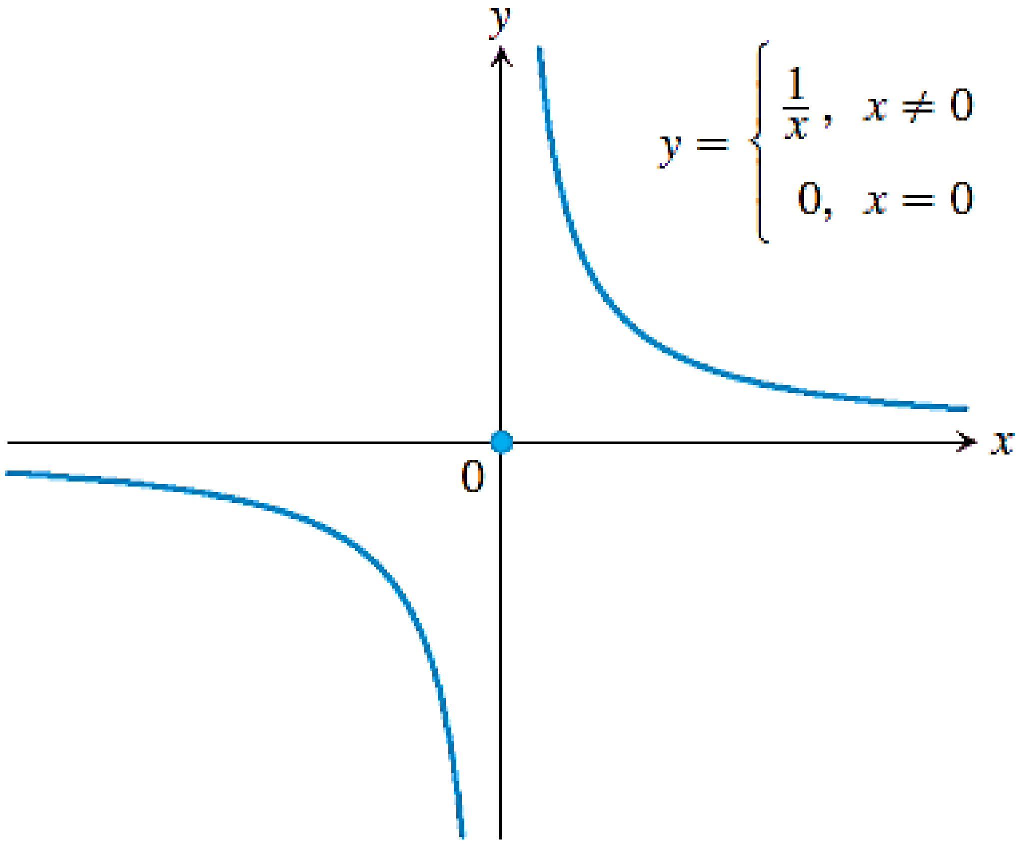
(a) It *jumps:* The **unit step function** $U(x) = y$ has no limit as because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \to 0$.



(a) Unit step function $U(x)$

Solution: A Function May Fail to Have a Limit at a Point in Its Domain

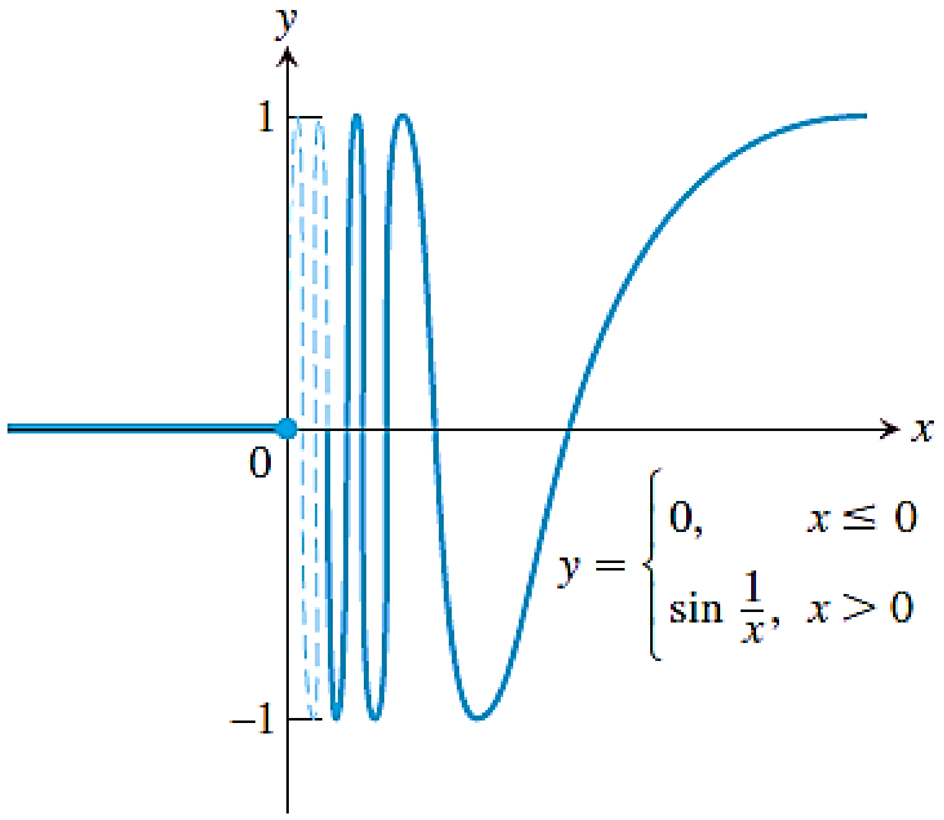
(b) It *grows too large to have a limit*: $g(x) = y$ has no limit $x \to 0$ as because the values of g grow arbitrarily large in absolute value as $x \to 0$ and do not stay close to *any* real number.



(b) $g(x)$

Solution: A Function May Fail to Have a Limit at a Point in Its Domain

(c) It *oscillates too much to have a limit*: $f(x) = y$ has no limit as $x \to 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$.



(c) $f(x)$