Power series:- A Series having the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\cdots \quad \text{—— (i)}$$

in which the coefficients a_n are real or complex and z_0 is a fixed point in the complex plane is called a power series about z_0 or a power series centred at z_0. When $z_0 = 0$, the series (i) reduces to power series centred at origin as:

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots \qquad (ii)$$

For example, the geometric series $\sum_{n=0}^{\infty} z^n$ has a sum to N terms

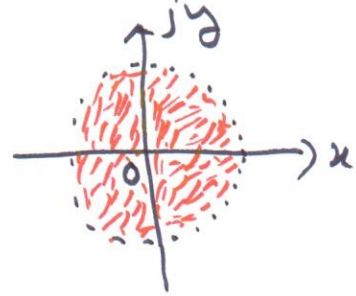
$$S_N = \sum_{n=0}^{N-1} z^n = \frac{1-z^N}{1-z}$$ and converges if $|z|<1$,

to the limit $\dfrac{1}{(1-z)}$ as $N \longrightarrow \infty$. If $|z| \geq 1$, the series diverges.

This requirement is identical as real variables series $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges for $|x|<1$.

However, in the complex case the geometrical interpertation is different. [Power Series 01]

Here the condition $|z|<1$ implies z lies inside the circle centred at the origin and radius 1 in the complex plane.



In general,

$$\sum_{n=0}^{\infty} a_n z^n \quad \begin{array}{l} \text{converges if } |z|<R \\ \text{diverges if } |z|>R. \end{array}$$

R is called the radius of convergence of the power series.

What happens when $|z|=R$ is normally investigated as a special case. Radius of convergence could be calculate using ratio test.

EX:- Find the radius of convergence for each power series:

(a) $$\sum_{n=0}^{\infty} n^2 (z-3)^n.$$

Consider $$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 (z-3)^{n+1}}{n^2 (z-3)^n}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)^2}{n^2}|z-3| = |z-3| \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = |z-3|$$

The series converges when $|z-3|<1$. Therefore, the radius of convergence is 1 and $|z-3|<1$ is the circle of convergence.

(b) $$\sum_{n=0}^{\infty} e^n (z+j)^n.$$

[Power Series 02]

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{e^{n+1}(z+j)^{n+1}}{e^n(z+j)^n}\right| = \lim_{n\to\infty} e|z+j|$$

$$= e|z+j|.$$

The series converges when $e|z+j|<1$ $\Rightarrow$ $|z+j|<\frac{1}{e}$. Therefore, $R = 1/e$.

(c).
$$\sum_{n=0}^{\infty} \frac{j^n n^2}{2^{2n}} z^n.$$

$$\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 z^{n+1}}{2^{2(n+1)}} \cdot \frac{2^{2n}}{n^2 z^n}\right|.$$

$$= \frac{|z|}{4}.$$

Thus we need $\frac{|z|}{4}<1$ for convergence. The series converges inside $|z|<4$. Therefore, the radius of convergence is $R=4$.

EX:- Find the power series, in the form indicated, representing the function $f(z)=\frac{1}{(z-3)}$ in the following regions:

(a) $|z|<3$; $\sum_{n=0}^{\infty} a_n z^n$ (b) $|z-2|<1$, $\sum_{n=0}^{\infty} a_n (z-2)^n$.

(c) $|z|>3$, $\sum_{n=0}^{\infty} \frac{a_n}{z^n}$.

(a) $|z| < 3 \Rightarrow \left|\frac{1}{3}z\right| < 1.$

$$f(z) = \frac{1}{z-3} = -\frac{1}{3}\left[\frac{1}{1-\frac{1}{3}z}\right] = -\frac{1}{3}\left(1-\frac{1}{3}z\right)^{-1}$$

$$= -\frac{1}{3}\left[1 + \frac{1}{3}z + \left(\frac{1}{3}z\right)^2 + \cdots + \left(\frac{1}{3}z\right)^n + \cdots\right]$$

giving

$$f(z) = \frac{1}{z-3} = -\frac{1}{3} - \frac{1}{9}z - \frac{1}{27}z^2 - \cdots \qquad (|z| < 3)$$

(b) $|z-2| < 1, \quad z-3 = (z-2)-1$

$$f(z) = \frac{1}{(z-3)} = \frac{1}{(z-2)-1} = \left[(z-2)-1\right]^{-1}$$

$$= -\left[1 + (z-2) + (z-2)^2 + \cdots\right] \qquad (|z-2| < 1)$$

[Power Series 04]

(c) $|z| > 3 \Rightarrow \left|\frac{3}{z}\right| < 1$

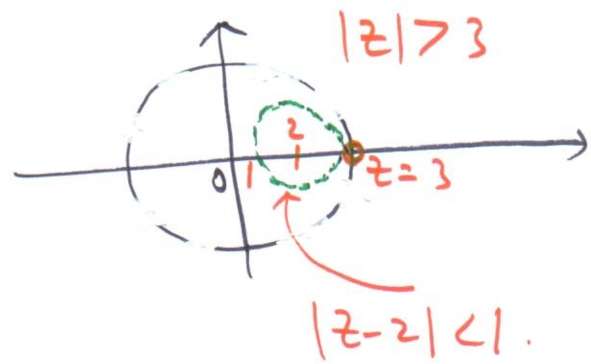
$$f(z) = \frac{1}{z-3} = \frac{1}{z}\left[\frac{1}{1-3/z}\right] = \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1}.$$

$$= \frac{1}{z}\left[1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \cdots\right]$$

giving the power ~~series~~

$$f(z) = \frac{1}{z-3} = \frac{1}{z} + \frac{3}{z^2} + \frac{9}{z^3} + \cdots \qquad (|z| > 3)$$

Note that none of the regions include the point $z = 3$.



Taylor Series:-

If $f(z)$ is complex function analytic inside and on a simple closed curve C (usually a circle) in the z-plane. If z_0 and z_0+h are two fixed points inside C then

$$f(z_0+h) = f(z_0) + h f'(z_0) + \frac{h^2}{2!} f''(z_0) + \cdots\cdots$$

where $f^{(k)}(z_0)$ is the kth derivative of $f(z)$ evaluated at z_0. Normally $z = z_0+h$ is introduced so that $h = z - z_0$, and the series expansion then becomes,

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots\cdots$$

$$= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0).$$

The region of convergence of this series is $|z-z_0| < R$, a disk centred at z_0 and of radius R.

When $z_0 = 0$, the series about origin is often called a Maclaurin Series expansion.

EX:- Determine the Taylor series expansion of the function $f(z) = \dfrac{1}{z(z-2j)}$ around the point $z = j$ directly and by using the binomial expansion.

$$f(z) = \frac{1}{z(z-2j)} = \frac{1}{2j}\left(\frac{1}{z-2j} - \frac{1}{z}\right) \quad \text{(by partial fraction)}$$

$$f(j) = 1, \quad f'(z) = \frac{1}{2j}\left[\frac{-1}{(z-2j)^2} + \frac{1}{z^2}\right], \quad f'(j) = 0.$$

$$f''(z) = \frac{1}{2j}\left[\frac{2}{(z-2j)^3} - \frac{2}{z^3}\right], \quad f^{(2)}(j) = -2.$$

$$f^{(3)}(z) = \frac{1}{2j}\left[\frac{-6}{(z-2j)^4} + \frac{6}{z^4}\right], \quad f^{(3)}(j) = 0.$$

$$f^{(4)}(z) = \frac{1}{2j}\left[\frac{24}{(z-2j)^5} - \frac{24}{z^5}\right], \quad f^{(iv)}(j) = 24.$$

[Power series of]

Hence,

$$f(z) = 1 - \frac{2}{2!}(z-j)^2 + \frac{24}{4!}(z-j)^4 + \cdots$$

$$= 1 - (z-j)^2 + (z-j)^4 + \cdots \quad \text{valid for } |z-j| < 1.$$

or

$$f(z) = \frac{1}{z(z-2j)} = \frac{1}{(z-j)^2 + 1} = \left[1 + (z-j)^2\right]^{-1}$$

$$= 1 - (z-j)^2 + (z-j)^4 - (z-j)^6 + \cdots$$

Note the singularities of $f(z)$ ($z = 0, z = 2j$) are precisely at distance 1 away from $z = j$.

Ex:- Find Taylor Series of $f(z) = \frac{1+z}{1-z}$ around $z = j$.
What is the radius of Convergence?

$f(z)$ is analytic except at $z=1$. The distance from $z=j$ to the singularity at $z=1$ is $\sqrt{2}$. So the series will converge in $|z-j| < \sqrt{2}$.

$$f(z) = \frac{(1+j)+(z-j)}{(1-j)-(z-j)} = \frac{(1+j)+(z-j)}{(1-j)}\left[1-\left(\frac{z-j}{1-j}\right)\right]^{-1}.$$

$$= \left[\frac{(1+j)}{1-j} + \frac{(z-j)}{1-j}\right]\left[1+\left(\frac{z-j}{1-j}\right)+\left(\frac{z-j}{1-j}\right)^2+\cdots\right]$$

$$= \left[j + \frac{(z-j)}{1-j}\right]\left[1+\left(\frac{z-j}{1-j}\right)+\frac{(z-j)^2}{(1-j)^2}+\cdots\right]$$

$$= j + (1+j)\left[\left(\frac{z-j}{1-j}\right)+\left(\frac{z-j}{1-j}\right)^2+\cdots\right]$$

$$= j + (1+j)\sum_{n=1}^{\infty}\left(\frac{z-j}{1-j}\right)^n.$$

for $\left|\frac{z-j}{1-j}\right| < 1 \Rightarrow |z-j| < |1-j|$

$$\Rightarrow |z-j| < \sqrt{2}.$$

[Power Series 07]

Ex: obtain Maclaurin series expansions of the functions:

(a). e^Z (b) $\sin Z$ (c). $\cos Z$ (d). $\frac{1}{1-Z}$ (e). $\frac{1}{1+Z}$.

(a). $f(Z) = e^Z$, $f^{(n)}(Z) = e^Z \Rightarrow f^{(n)}(0) = 1$ for all n. Hence

$$e^Z = 1 + Z + \frac{Z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{Z^n}{n!}, \quad |Z| < \infty.$$

(b). $f(Z) = \sin Z$, $f'(Z) = \cos Z$, $f''(Z) = -\cos Z$.

$$f^{(2n)}(Z) = (-1)^n \sin Z, \quad f^{(2n+1)}(Z) = (-1)^n \cos Z.$$

So $f^{(2n)}(0) = 0$, $f^{(2n+1)}(0) = (-1)^n$.

Hence

$$\sin Z = Z - \frac{Z^3}{3!} + \frac{Z^5}{5!} \cdots = \sum_{n=0} (-1)^n \frac{Z^{2n+1}}{(2n+1)!}, \quad |Z| < \infty$$

(c). Similarly,

$$\cos Z = 1 - \frac{Z^2}{2!} + \frac{Z^4}{4!} \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{Z^{2n}}{(2n)!}, \quad |Z| < \infty.$$

(d). $$\frac{1}{1-Z} = \sum_{n=0}^{\infty} Z^n, \quad |Z| < 1$$

(e). $$\frac{1}{1+Z} = \sum_{n=0}^{\infty} (-1)^n Z^n, \quad |Z| < 1.$$

The series in part (a), (b) & (c) are valid (convergent) for all values of Z.

[Power Series 08]

EX: write Series expansion for (a) $\sinh z$ (b) $\cosh z$, (c). $z^3 e^{2z}$.

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$= \frac{1}{2}\left[\left(1 + z + \frac{z^2}{2} + \cdots\right) - \left(1 - z + \frac{z^2}{2!} - \cdots\right)\right]$$

$$= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \ |z| < \infty$$

Similarly, using $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$,

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \ |z| < \infty$$

Also,
$$z^3 e^{2z} = z^3 \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n z^{n+3}}{n!}, \ |z| < \infty$$

[Power Series 09]

EX: Expand $f(z) = \dfrac{1}{1-z}$ in a Taylor series with center $z_0 = 2i$.

$$\frac{1}{1-z} = \frac{1}{1-z+2i-2i} = \frac{1}{(1-2i)-(z-2i)} = \frac{1}{1-2i} \cdot \frac{1}{1 - \frac{z-2i}{1-2i}}$$

we now write $\dfrac{1}{1 - \frac{z-2i}{1-2i}}$ as a power series

$$\frac{1}{1-z} = \frac{1}{1-2i}\left[1 + \frac{z-2i}{1-2i} + \left(\frac{z-2i}{1-2i}\right)^2 + \cdots\right]$$

$$= \frac{1}{1-2i} + \frac{1}{(1-2i)^2}(z-2i) + \frac{1}{(1-2i)^3}(z-2i)^2 + \cdots$$

The distance from the center $z_0 = 2i$ to the nearest singularity $z=1$ is $\sqrt{5}$. We conclude the region of convergence $|z-2i| < \sqrt{5}$

NUST School of Electrical Engineering & Computer Science
Complex variables & transforms - BEE3CD - Problem Sheet. No. 04

M T W T F S

Date 20/12/12

Q-1 Find the power series representation for the function $f(z) = \frac{1}{(z-j)}$ in the regions:

(a) $|z| < 1$ (b) $|z| > 1$ (c) $|z-1-j| < \sqrt{2}$.

Q-2. Find the power series representation of the function $f(z) = \frac{1}{1+z^2}$ in the disk $|z| < 1$.

Q-3 Find the first four non-zero terms of the Taylor series expansions about the points indicated, and determine the radius of convergence.

(a). $\frac{1}{z(z-4j)}$ $(z = 2j)$ (b). $\frac{1}{z^2}$ $(z = 1+j)$.

Q-4 Find the Maclaurin series of $f(z) = \frac{2-3z}{1+i+z}$.

Q-5. Find the Taylor series expansion of $\frac{z}{(z+2)(z-3)}$ center at $z_0 = i$. What is the radius of convergence.

Q-6 Find the Taylor series representation about $z_0 = j$ for the following functions:

(a). $f(z) = (z+j)\,e^{z}$, (b). $f(z) = \frac{1}{1-z}$.