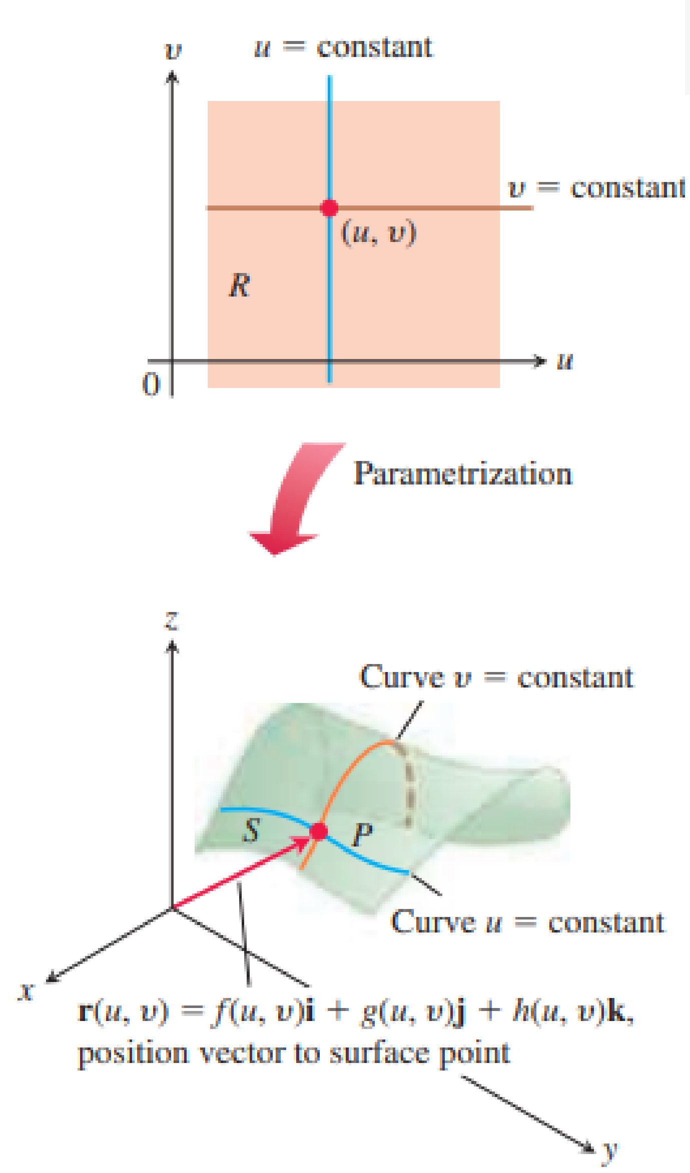
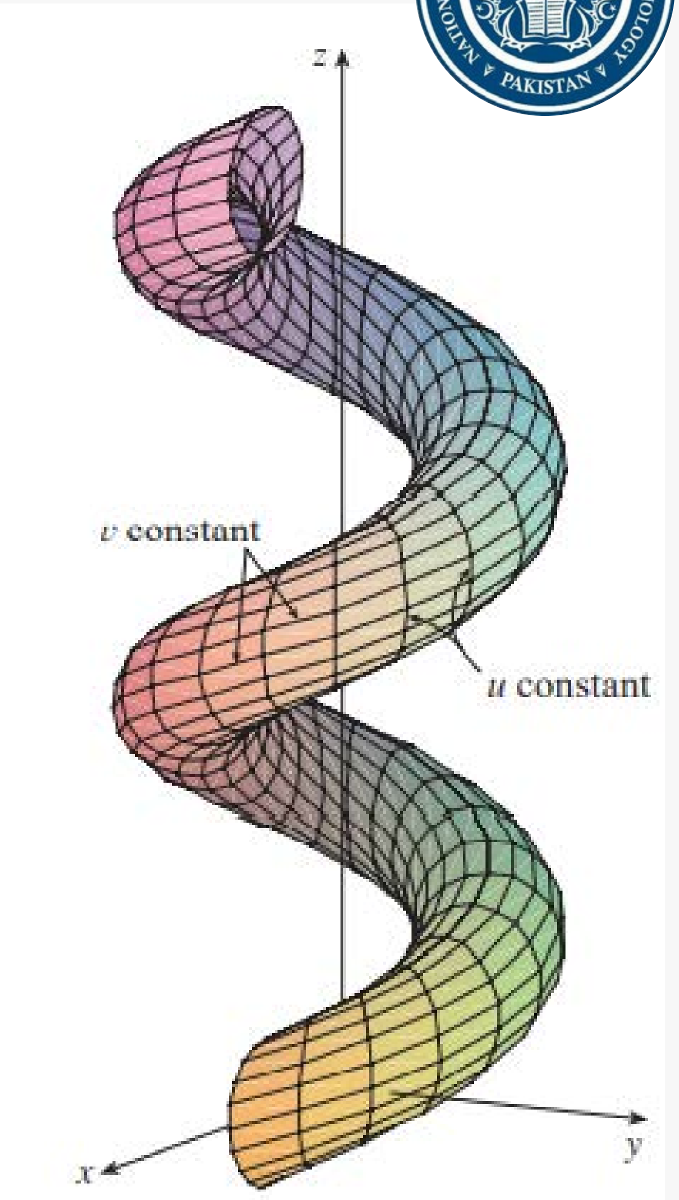
Parametrized Surfaces



A parametrized surface S expressed as a vector function of two variables defined on a region R.



Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

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Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.6**

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

- **Chapter: 16**
  - **Section: 16.5**

Parametric Surfaces

In much the same way that we describe a space curve by a vector function $\mathbf{r}(t)$ of a single parameter t, we can describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v. We suppose that:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \qquad (1)$$

is a vector-valued function defined on a region D in the uv −plane.

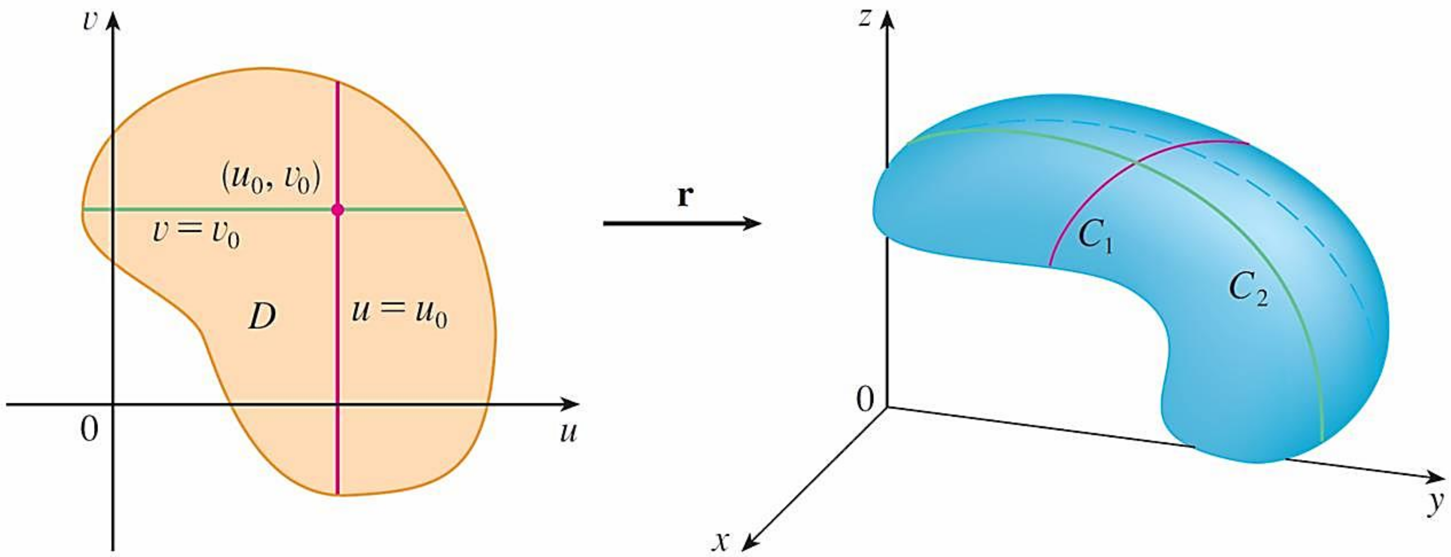
So, x, y, and z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain D. The set of all points (x, y, z) in $\mathbb{R}^3$ such that:

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \qquad (2)$$

and (u, v) varies throughout D, is called a **parametric surface** S and Equations (2) are called **parametric equations** of S.

Parametric Surfaces: Families of Curves

If a parametric surface S is given by a vector function $\mathbf{r}(u, v)$, then there are two useful families of curves that lie on S, one family with u constant and the other with v constant. These families correspond to **vertical** and **horizontal lines** in the uv −plane. If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a curve C_1 lying on S. Similarly, if we keep v constant by putting $v = v_0$, we get a curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S. We call these curves **grid curves**. For instance, in previous example, the grid curves obtained by letting u be constant are horizontal lines whereas the grid curves with v constant are circles.



Parametric Surfaces

Note:

- In general, a surface given as the **graph of a function** of x and y, that is, with an equation of the form $z = f(x, y)$, can always be regarded as a parametric surface by taking x and y as parameters and writing the parametric equations as:

$$x = x, \qquad y = y, \qquad z = f(x, y).$$

- Parametric representations (also called parametrizations) of surfaces are not unique.

  The next example shows two ways to parametrize a cone.

Tangent Planes

We now find the tangent plane to a parametric surface S traced out by a vector function:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k},$$

at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$.

If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_1 lying on S. The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v and is given as:

$$\mathbf{r}_v(u_0, v_0) = \frac{\partial x(u_0, v_0)}{\partial v}\mathbf{i} + \frac{\partial y(u_0, v_0)}{\partial v}\mathbf{j} + \frac{\partial z(u_0, v_0)}{\partial v}\mathbf{k}.$$

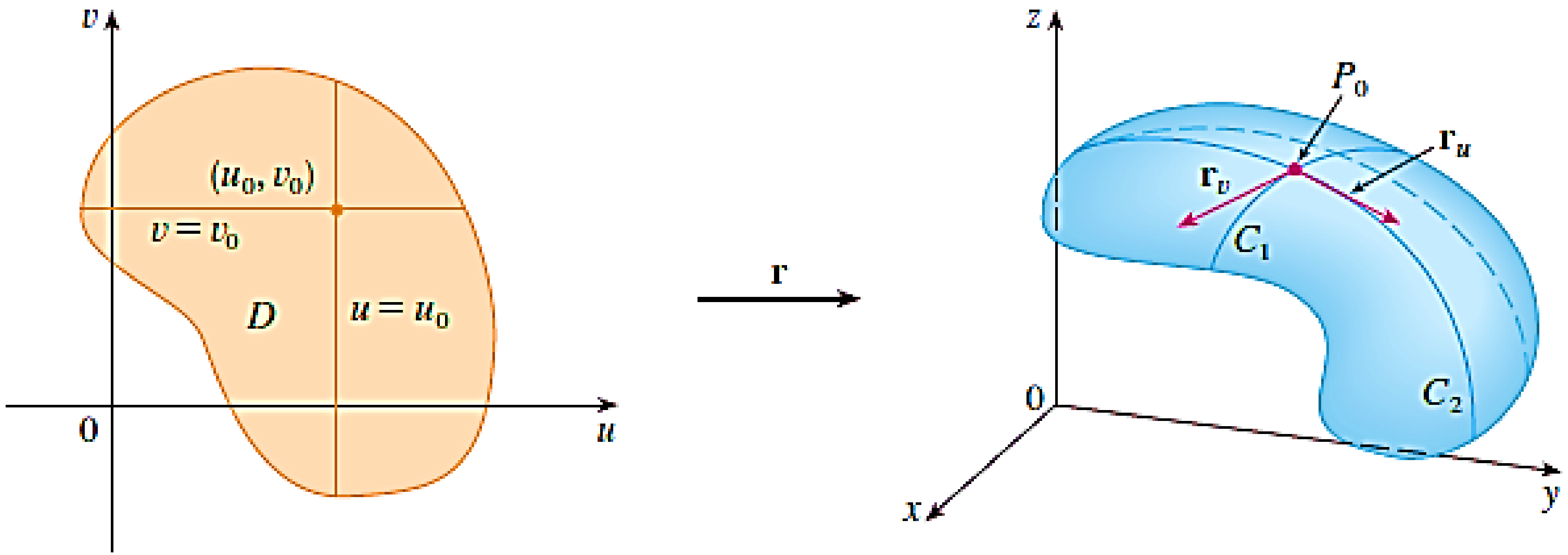
Similarly, if we keep v constant by putting $v = v_0$, we get a grid curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S, and its tangent vector at P_0 is:

$$\mathbf{r}_u(u_0, v_0) = \frac{\partial x(u_0, v_0)}{\partial u}\mathbf{i} + \frac{\partial y(u_0, v_0)}{\partial u}\mathbf{j} + \frac{\partial z(u_0, v_0)}{\partial u}\mathbf{k}.$$

Tangent Planes

- If $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$, then the surface *S* is called **smooth** (it has no "corners").

- For a smooth surface, the **tangent plane** is the plane that contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a **normal vector** to the tangent plane.



Unit Normal Vector

For a smooth surface S, the **unit normal vector** to the tangent plane is:

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

If the surface S is the graph of an equation $z = f(x, y)$ and if we let:

$$g(x, y, z) = z - f(x, y),$$

then S is also the graph of the equation $g(x, y, z) = 0$. Since the gradient of $g(x, y, z)$ is a normal vector to the graph of $g(x, y, z) = 0$ at the point (x, y, z), a **unit normal vector** can be obtained as follows:

$$\mathbf{n} = \frac{\nabla g(x, y, z)}{|\nabla g(x, y, z)|} = \frac{\langle g_x, g_y, g_z \rangle}{|\langle g_x, g_y, g_z \rangle|} = \frac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{1 + [f_x]^2 + [f_y]^2}}$$

Example:

Find the tangent plane to the surface with parametric equations:

$$x = u^2, \quad y = v^2, \quad z = u + 2v,$$

at the point $(1, 1, 3)$.

Solution:

We first compute the tangent vectors:

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} = 2u\mathbf{i} + \mathbf{k}.$$

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} = 2v\mathbf{j} + 2\mathbf{k}.$$

Thus, a normal vector to the tangent plane is given as:

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = \langle -2v, -4u, 4uv \rangle$$

Solution:

Notice that the point $(1, 1, 3)$ corresponds to the parameter values $u = 1$ and $v = 1$, so the normal vector at this point is given as:

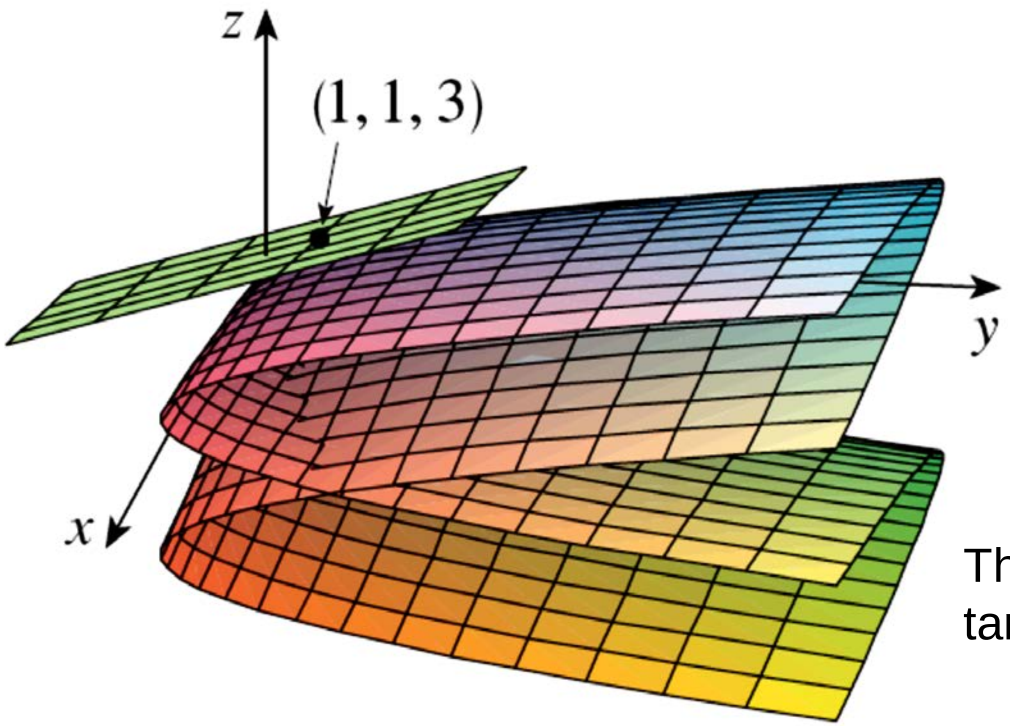
$$\mathbf{n} = \langle -2, -4, 4 \rangle.$$

Therefore, an equation of the tangent plane at $(1, 1, 3)$ is:

$$-2(x - 1) - 4(y - 1) + 4(z - 3) = 0$$

or

$$x + 2y - 2z + 3 = 0.$$



The figure shows the self-intersecting surface and its tangent plane at $(1, 1, 3)$.

Surface Area

Now we define the surface area of a general parametric surface. Suppose a smooth parametric surface S is given by equation:

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}; \qquad (u,v) \in D,$$

and S is covered just once as (u,v) ranges throughout the parameter domain D. Then, the surface area of S is given as:

$$\text{Surface Area} = A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where,

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \quad \text{and} \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$

Example:

Find the surface area of a sphere of radius a.

Solution:

The parametric representation of sphere of radius a is:

$$x = a\sin\varphi\cos\theta\,, \qquad y = a\sin\varphi\sin\theta\,, \qquad z = a\cos\varphi\,,$$

where the parameter domain is:

$$D = \{(\varphi,\theta)|0 \le \varphi \le \pi, 0 \le \theta \le 2\pi\}.$$

For the present case:

$$\mathbf{r}(\varphi,\theta) = \langle a\sin\varphi\cos\theta\,, a\sin\varphi\sin\theta\,, a\cos\varphi\rangle.$$

We first compute the tangent vectors:

$$\mathbf{r}_\varphi = \langle a\cos\varphi\cos\theta\,, a\cos\varphi\sin\theta\,, -a\sin\varphi\rangle.$$

and

$$\mathbf{r}_\theta = \langle -a\sin\varphi\sin\theta\,, a\sin\varphi\cos\theta\,, 0\rangle.$$

Solution:

We then compute the cross product of the tangent vectors $\mathbf{r}_\varphi$ and $\mathbf{r}_\theta$ as:

$$\mathbf{n} = \mathbf{r}_\varphi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \langle a^2\sin^2\varphi\cos\theta\,, a^2\sin^2\varphi\sin\theta\,, a^2\sin\varphi\cos\varphi \rangle.$$

Thus,

$$\left|\mathbf{r}_\varphi \times \mathbf{r}_\theta\right| = a^2\sin\varphi\,.$$

since $\sin\varphi \geq\ 0$ for $0\ \leq \varphi \leq \pi$. Hence, the area of the sphere is:

$$A(S) = \iint_D \left|\mathbf{r}_\varphi \times \mathbf{r}_\theta\right| dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin\varphi\, d\varphi d\theta = 4\pi a^2.$$

Surface Area of the Graph of a Function

Now, consider the special case of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives. Here, we take x and y as parameters. The parametric equations are:

$$x = x, \qquad y = y, \qquad z = f(x, y).$$

Thus,

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}, \qquad \mathbf{r}_x = \mathbf{i} + f_x\mathbf{k}, \qquad \mathbf{r}_v = \mathbf{j} + f_y\mathbf{k},$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle.$$

Thus, we have:

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + [f_x]^2 + [f_y]^2} = \sqrt{1 + [z_x]^2 + [z_y]^2}.$$

Then, the surface area formula can be rewritten as:

$$A(S) = \iint_D |\mathbf{r}_x \times \mathbf{r}_y| \, dA = \iint_D \sqrt{1 + [z_x]^2 + [z_y]^2} \, dA.$$

Surface Area of the Graph of a Function

Similarly, if we consider the surface S with equation $y = h(x,z)$, we have:

$$A(S) = \iint_D |\mathbf{r}_x \times \mathbf{r}_z|\, dA = \iint_D \sqrt{1 + [h_x]^2 + [h_z]^2}\, dA == \iint_D \sqrt{1 + [y_x]^2 + [y_z]^2}\, dA.$$

and if we consider the surface S with equation $x = k(y,z)$, we have:

$$A(S) = \iint_D |\mathbf{r}_y \times \mathbf{r}_z|\, dA = \iint_D \sqrt{1 + \left[k_y\right]^2 + [k_z]^2}\, dA == \iint_D \sqrt{1 + \left[x_y\right]^2 + [x_z]^2}\, dA.$$

Example:

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

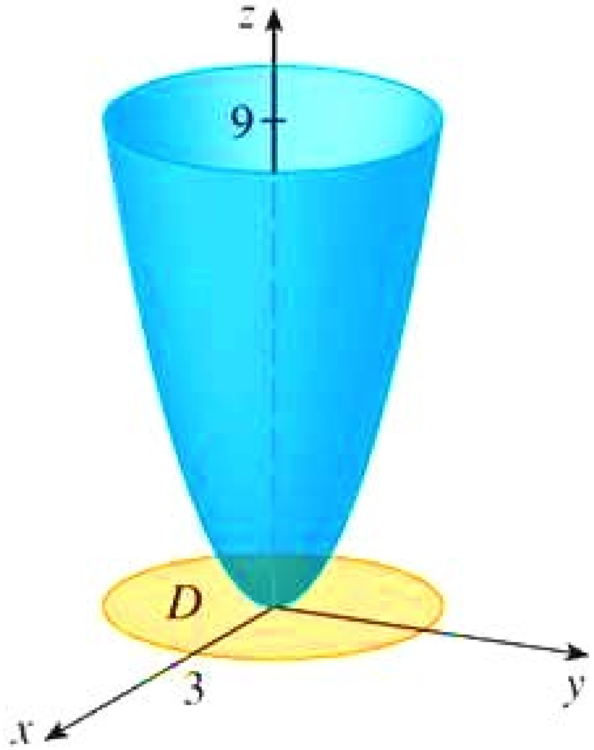
Solution:

The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Therefore, the given surface lies above the disk D with center the origin and radius 3. Hence, the surface area is:

$$A(S) = \iint_D \sqrt{1 + [z_x]^2 + [z_y]^2}\, dA$$

$$= \iint_D \sqrt{1 + [2x]^2 + [2y]^2}\, dA$$

$$= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA.$$



Using polar coordinates, we obtain:

$$A(S) = \int_0^{2\pi} \int_0^{3} \sqrt{1 + 4r^2}\, r\,dr\,d\theta = \frac{(37\sqrt{37} - 1)\pi}{6}.$$

Practice Questions

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

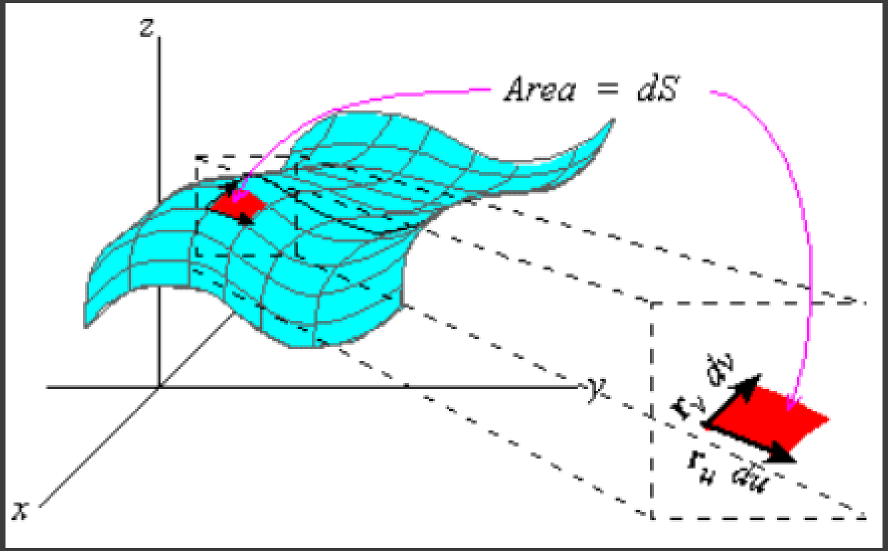
Chapter: 16

Exercise-16.6: Q – 1 to 26, Q – 33 to 47, Q – 56 to 57.

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 16

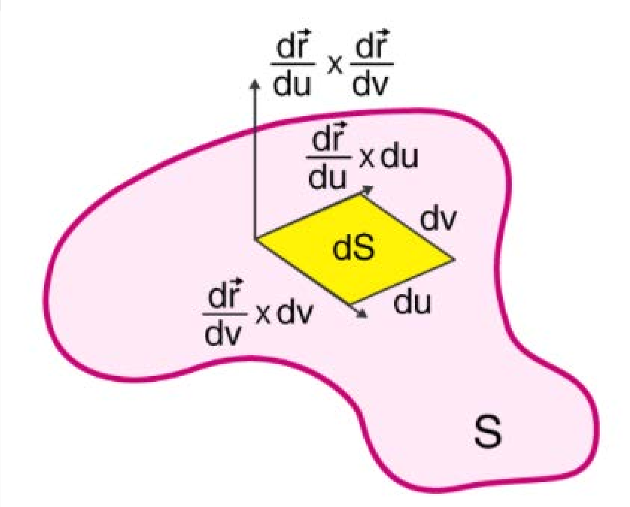
Exercise-16.5: Q – 1 to 30, Q – 33 to 56.



Surface Integrals

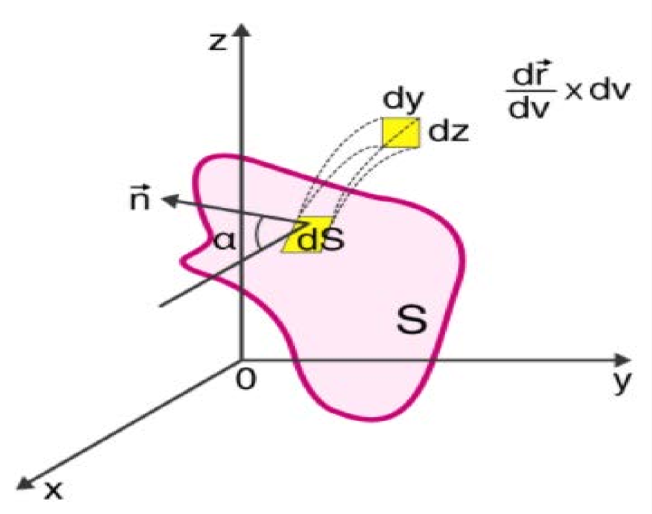
Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir



Surface Integral of Scalar Field

$$\iint_S f(x,y,z)dS = \iint_{D(u,v)} f[x(u,v),y(u,v),z(u,v)].\left|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right| dudv$$



Surface Integral of Vector Field

- If the surface "S" oriented is outward, then the surface integral of the vector field is given as:

$$\iint_S F(x,y,z).dS = \iint_S F(x,y,z).ndS = \iint_{D(u,v)} F[x(u,v),y(u,v),z(u,v))].\left[\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right] dudv.$$

- If the surface "S" oriented is inward, then the surface integral of the vector field is given as:

$$\iint_S F(x,y,z).dS = \iint_S F(x,y,z).ndS = \iint_{D(u,v)} F[x(u,v),y(u,v),z(u,v))].\left[\frac{\partial r}{\partial v} \times \frac{\partial r}{\partial u}\right] dudv$$

Where dS = ndS is known as the vector element of the surface.

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Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.7**

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

- **Chapter: 16**
  - **Section: 16.6**

Surface Integrals for Scalar Fields

- The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length.
- Suppose f is a function of three variables whose domain includes a surface S.
- We will define the surface integral of f over S such that the value of the surface integral is equal to the surface area of S in the case where $f(x, y, z) = 1$.
- If S is a smooth surface defined parametrically as:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}; \qquad (u, v) \in D,$$

and $f(x, y, z)$ is a continuous function defined on S, then the **integral of f over S** is:

$$\iint_S f(x, y, z)dS = \iint_D f(\mathbf{r}(u, v))|\mathbf{r}_u \times \mathbf{r}_v|dA.$$

- When using this formula, remember that $f(\mathbf{r}(u, v))$ is evaluated by writing $x = x(u, v)$, $y = y(u, v), z = z(u, v)$ in the formula for $f(x, y, z)$. Moreover, observe that:

$$\iint_S 1dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|dA = A(S).$$

Example:

Compute the surface integral

$$\iint_S x^2 dS,$$

where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution:

We use the parametric representation for the unit sphere:

$$x = \sin\varphi\cos\theta\,, \qquad y = \sin\varphi\sin\theta\,, \qquad z = \cos\varphi\,,$$

where, $D = \{(\varphi, \theta) | 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi\}$. That is,

$$\mathbf{r}(\varphi, \theta) = \langle \sin\varphi\cos\theta\,, \sin\varphi\sin\theta\,, \cos\varphi \rangle.$$

For the present case: $|\mathbf{r}_\varphi \times \mathbf{r}_\theta| = \sin\varphi$. Therefore, the surface integral can be calculated as:

$$\iint_S x^2 dS = \iint_D (\sin\varphi\cos\theta)^2 |\mathbf{r}_\varphi \times \mathbf{r}_\theta| dA = \int_0^{2\pi}\int_0^{\pi} (\sin\varphi\cos\theta)^2 \sin\varphi\, d\varphi d\theta = \frac{4\pi}{3}.$$

Graphs of a Function

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations:

$$x = x, \qquad y = y, \qquad z = g(x, y).$$

So, we have:

$$\iint_S f(x, y, z)\,dS = \iint_D f(x, y, g(x, y))\sqrt{1 + [z_x]^2 + [z_y]^2}\,dA.$$

Similar formulas apply when it is more convenient to project S onto the yz −plane or xy −plane.

Example:

Evaluate

$$\iint_S y dS,$$

where S is the surface $z = x + y^2, 0 \leq x \leq 1, 0 \leq y \leq 2$.

Solution:

For the present case we have: $x = x,\ y = y,\ z = x + y^2$. Thus,

$$\iint_S y dS = \iint_D y\sqrt{1 + [z_x]^2 + [z_y]^2} dA = \int_0^1 \int_0^2 y\sqrt{1 + [1]^2 + [2y]^2} dy dx = \frac{13\sqrt{2}}{3}.$$