National University of Sciences & Technology
School of Electrical Engineering and Computer Science
Department of Basic Sciences

MATH-243: Vector Calculus (3+0): BEE-2k20-ABC Fall 2021

Assignment # 3	
CLO-3: Develop analytical solutions of partial differential equations.	
Maximum Marks: 10	Instructor: Dr. Naila Amir
Announcement Date: 31st December 2021	Due Date: 31st December 2021

Instructions:

- Understanding the question is part of the assignment and copying is not allowed.
- Express your answer in the most simplified form. Direct calculations using calculator are not allowed, you need to show the detail of your work to get the maximum marks.
- This is a group assignment. Each group having **5 members** only. All group members are required to contribute equally. Each member of group will attempt one question and afterwards will discuss his/her attempt with other group members so that all group members have an idea of the solution of whole assignment. Name of student should be mentioned in the following table against the question attempted by him/her.
- Assignment must be neatly handwritten on A4 papers and properly bound. Do not use files or folders for submission of assignment.
- There are two pages in this assignment, including this cover page. These two pages should be part of every assignment.
- This is a class assignment and needs to be submitted at the end of class. Assignment is not acceptable after deadline.

Tasks: Attempt all questions.

Sr. No.	Students Name	CMS Id.	Question Attempted	Marks Obtained
1			Q # 1	
2			Q # 2	
3			Q # 3	
4			Q # 4	
5			Q # 5	

Total Marks	**Marks Obtained**	**Weight in 10**
50 Marks		

Q - 1: [10 marks]

Consider the heat boundary value problem:

$$u_t = u_{xx}; \quad 0 < x < 1, \qquad t > 0,$$
$$u_x(0,t) = -u(0,t), \quad u_x(1,t) = -u(1,t), \qquad t > 0,$$
$$u(x,0) = x; \qquad 0 < x < 1$$

This models a heat problem in a bar that is losing heat at its ends at the rate proportional to the temperature of the endpoints. Show that the temperature $u(x,t)$ in the bar is given as:

$$u(x,t) = A_0 e^{-x} e^{t} + \sum_{n=1}^{\infty} A_n[(n\pi)\cos(n\pi x) - \sin(n\pi x)]e^{n^2\pi^2 t},$$

where $A_0 = \frac{2e(e-2)}{e^2-1}$ and $A_n = \frac{2}{1+n^2\pi^2}\left[\frac{2(-1)^n-1}{n\pi}\right]; n = 1,2,\ldots$

[Hint: Using $u(x,0) = f(x)$ and the orthogonality relation $\int_0^1 X_i(x)X_j(x)\,dx = 0$, if $i \neq j$ we get: $A_0 = \frac{2e^2}{e^2-1}\int_0^1 f(x)e^{-x}dx$ and $A_n = \frac{2}{1+n^2\pi^2}\int_0^1 f(x)X_n(x)dx\,; n = 1,2,\ldots$]

Q - 2: [10 marks]

A quantum-mechanical particle on the line with an infinite potential outside the interval $(0, L)$ (particle in a box) is described by the Schrödinger equation:

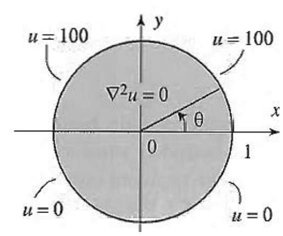
$$u_t = iu_{xx}; \quad 0 < x < L, \qquad t > 0,$$

with Dirichlet conditions $u(0,t) = 0$ and $u(L,t) = 0$ at the ends. Use separation of variables to find a representation formula for $u(x,t)$ as a series.

Q - 3: [10 marks]

The steady state temperature in a disk of radius 1 is described by a two-dimensional Laplace equation in polar coordinates as:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0; \quad 0 < r < 1; 0 < \theta < 2\pi.$$



Determine the solution when the upper half of the circumference is kept at 100° and the lower half is kept at 0°.

Q - 4: [10 marks]

(The hammer blow) Let $u(x,0) \equiv 0$ and $u_t(x,0) = g(x)$, where

$$g(x) = \begin{cases} 1, & |x| < 3 \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $u(x,t)$ provides solution to the one-dimensional wave equation: $u_{tt} = 4u_{xx}$. Using d' Alembert's solution, determine $u(x,t)$ and sketch the string profile at $t = 3/4$.

Q - 5: [10 marks]

A square membrane with $a = 1$, $b = 1$, and $c = 1/\pi$, is placed in the xy −plane. The edges of the membrane are held fixed, and the membrane is stretched into a shape modeled by the function $f(x, y) = xy(x-1)(y-1)$, $0 < x < 1$, $0 < y < 1$. Suppose that the membrane starts to vibrate from rest. Determine the position of each point on the membrane for $t > 0$. (Hint: $g(x,y) = 0 \Longrightarrow B^*_{mn} = 0$)