NUST School of Electrical Engineering & Computer Science

MATH 232 – Complex Variables & transforms – BEE 10D – Problem Sheet No. 1

Q-1 Given that 4 particles of masses 2, 1, 3, and 5 are located at the respective points $1+i$, $-3i$, $1-2i$, and -6. Find the center of mass of this system.

Q-2 Using the complex product $(1+i)(5-i)^4$ derive

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right).$$

Q-3 (a) Recall that the dot (scalar) product of two planar vectors $\bar{V_1} = (x_1, y_1)$ and $\bar{V_2} = (x_2, y_2)$ is given by $\bar{V_1}.\bar{V_2} = x_1x_2 + y_1y_2$. Show that the dot product of the vectors represented by the complex number z_1 & z_2 is given by

$z_1 . z_2 = Re(\bar{z}_1 z_2)$.

(b) Recall that in three dimensions the cross-product (vector product) of two vectors $\bar{V_1} = (x_1, y_1, 0)$, $\bar{V_2} = (x_2, y_2, 0)$ in the xy-plane is given by $\bar{V_1} \times \bar{V_2} = (0, 0, x_1y_2 - x_2y_1)$. Show that the third component of the cross-product of vectors in the xy-plane represented by the complex numbers z_1 & z_2 is given by $Im(\bar{z}_1 z_2)$.

(c) What can you deduce for vectors (2D) represented by the complex numbers z_1 and z_2 if

(i) $Re(\bar{z}_1 z_2) = 0$ (ii) $Im(\bar{z}_1 z_2) = 0$.

Q-4 Sketch the curves that are given by for $0 \le t \le 2\pi$

(a) $z(t) = e^{(1+i)t}$ (b) $z(t) = i + 2e^{it}$ (c) $z(t) = e^{-(1+i)t}$ (d) $z(t) = e^{(-1-i)t}$.

Q-5 Draw the regular polygons formed by the n nth roots of unity in the complex plane for $n = 3, 4, 5$, and 6

$w_n = (1)^{1/n}$.

Q-6 Write down three different multivalued $z^{1/3}$ with reference of angle. Use your functions f_1, f_2, and f_3 to calculate $(-i)^{1/3}$

Q-7. An analytic branch of $f(z) = \sqrt{z}$ is defined on the cut plane domain $D = \{z = re^{i\theta} : r > 0,\ 0 < \theta < 2\pi\}$ such that i is mapped to $\sqrt{i} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$. That is, an analytic function f is defined on the domain $D \subset \mathbb{C}$ such that $(f(z))^2 = z$ for every $z \in D$ and $f(i) = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$.

(a) Describe the function f by a formula.

(b) Compute the images $f(-1)$ and $f(-i)$ of -1 and $-i$, respectively.

(c) Describe the image of the domain D under f.

Q-8 An analytic branch of $z^{1/3}$ is defined on the cut plane domain $D = \{z = re^{i\theta},\ r > 0,\ \frac{\pi}{3} < \theta < \frac{7\pi}{3}\}$ such that i is mapped to $i^{1/3} = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$. That is, an analytic function f is defined on the domain D such that $(f(z))^3 = z$ for every $z \in D$ and $f(i) = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$. Compute the image $f(-i)$.

Q-9 An analytic branch of $\sqrt[6]{z}$ is defined on the cut plane domain $D = \mathbb{C} - \{z = ri : 0 < r \in \mathbb{R}\}$ (upper imaginary ray removed). i.e., $f(z)$ is analytic on D and $(f(z))^6 = z$ for all $z \in D$.
If $f(-1) = \frac{\sqrt{3}}{2} + i\frac{1}{2}$, what is $f(1)$?

Q-10 Let $w = (z^2+1)^{1/2}$. (a) If $w = 1$ when $z = 0$, and z describes the curve C_1 shown in Fig. 1, find the value of w when $z = 1$.
(b). If z describes the curve C_2 shown in Fig 2, is the value of w, when $z = 1$, the same as that obtained in (a)?

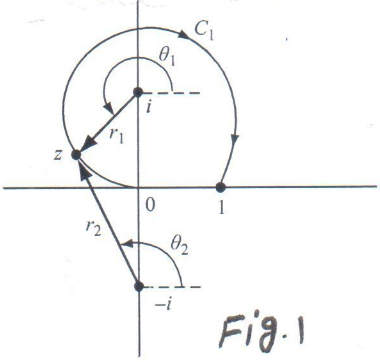

Fig. 1

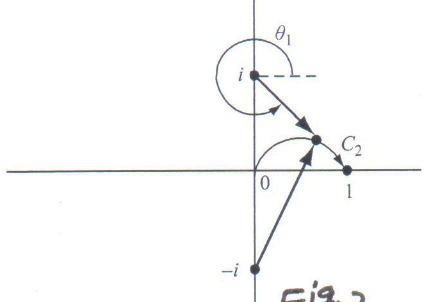

Fig. 2

Q-11 Consider the multiple-valued function $F(z) = (z-1+i)^{1/2}$.
(a) What is the branch point of F? Explain.
(b) Explicitly define two distinct branches f_1 & f_2 of F. In each case, state the branch cut [Ref: Section 2.6, Zill book].