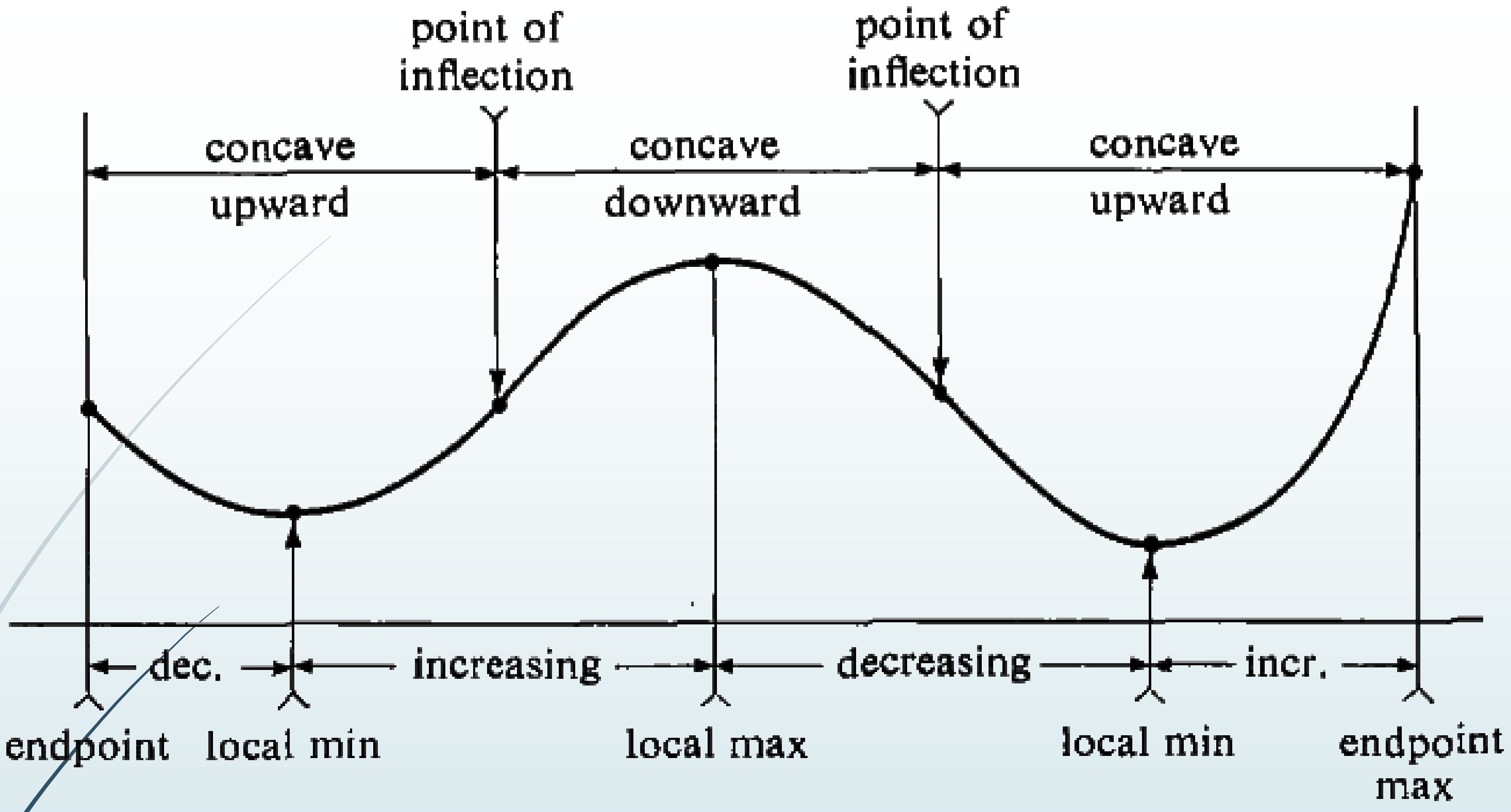
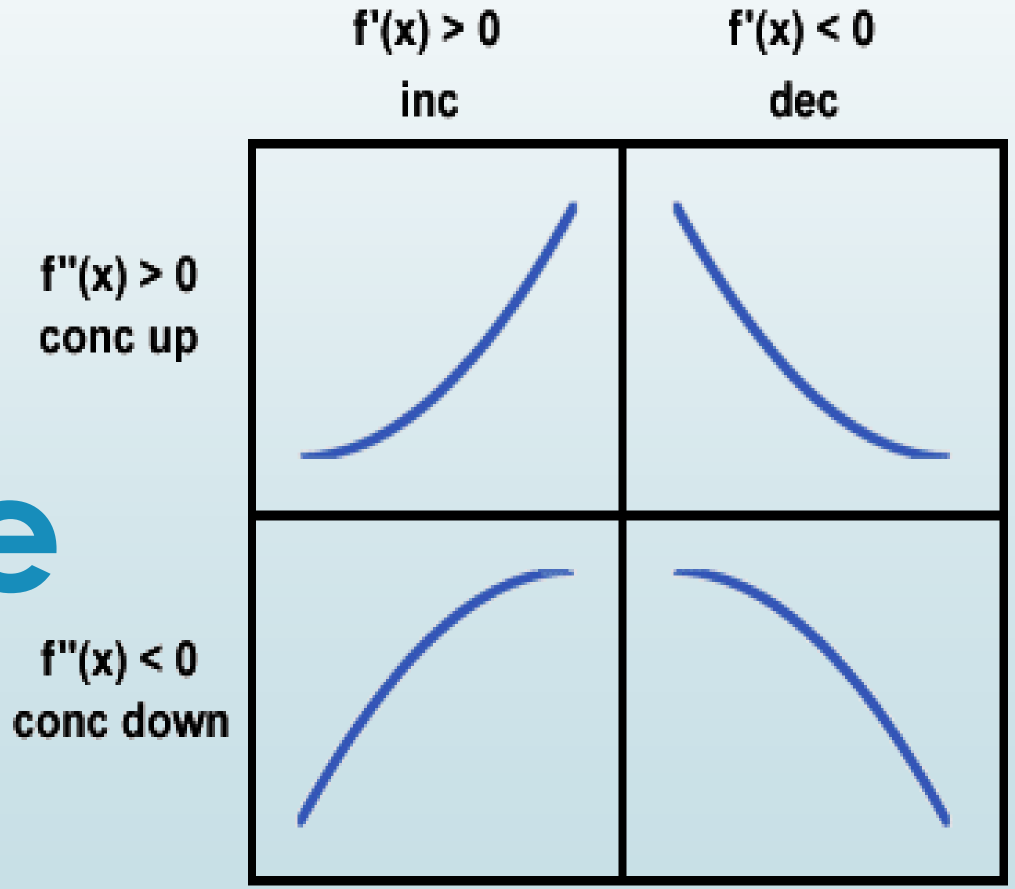
Applications of Derivatives

Calculus & Analytical Geometry MATH- 101
Instructor: Dr. Naila Amir (SEECS, NUST)



Concavity and Curve Sketching

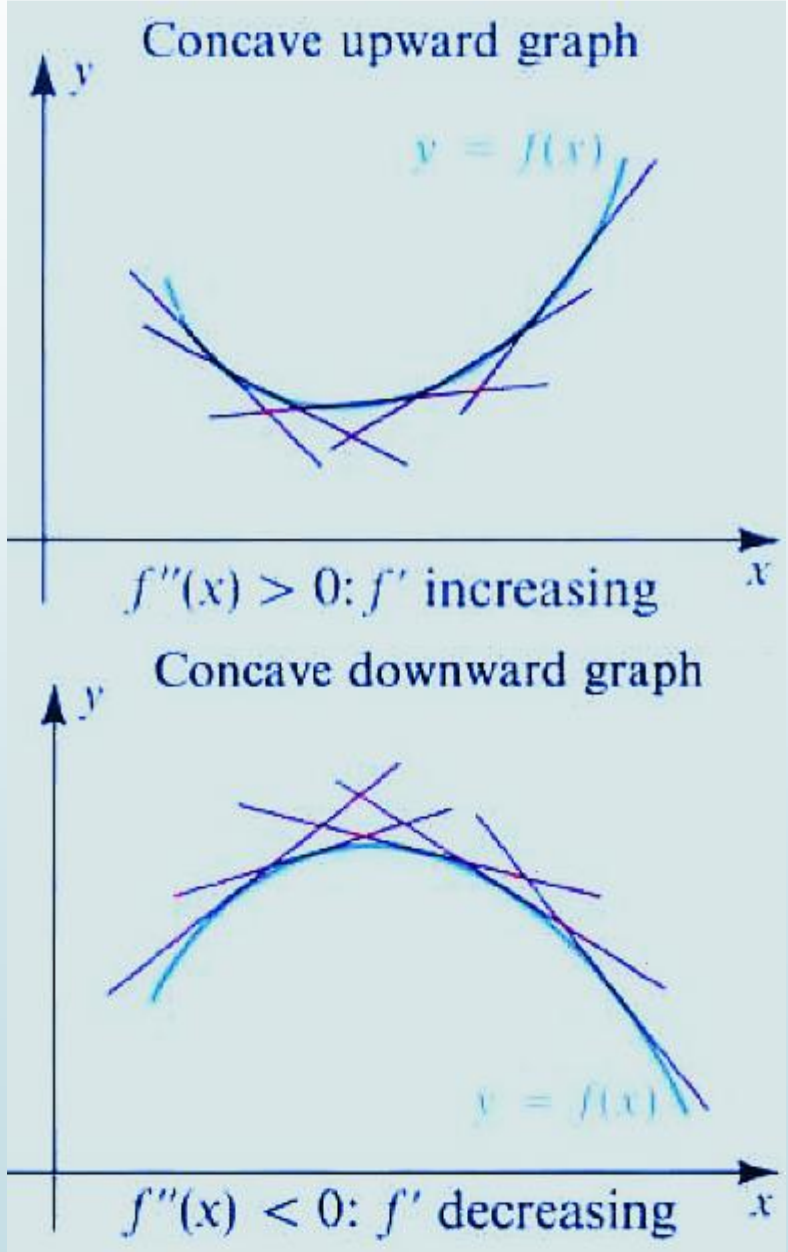


Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 4
  - Sections: 4.4

Concavity

- The **concavity** of the graph of a function is the notion of curving upward or downward.
- If the graph of a function lies above its tangents on some interval then the graph is called concave up on that interval.
- If the graph of a function lies below its tangents on some interval then the graph is called concave down on that interval.



Concavity and the Second Derivative Test

Let $y = f(x)$ be twice-differentiable on an interval I.

- If $f''(x) > 0$ on I, then $f'(x)$ is increasing on I. In this case, the slope of the tangent line of the graph of $f(x)$ is increasing as x is increasing and we say that the graph is <u>concave</u> <u>up</u> on I.

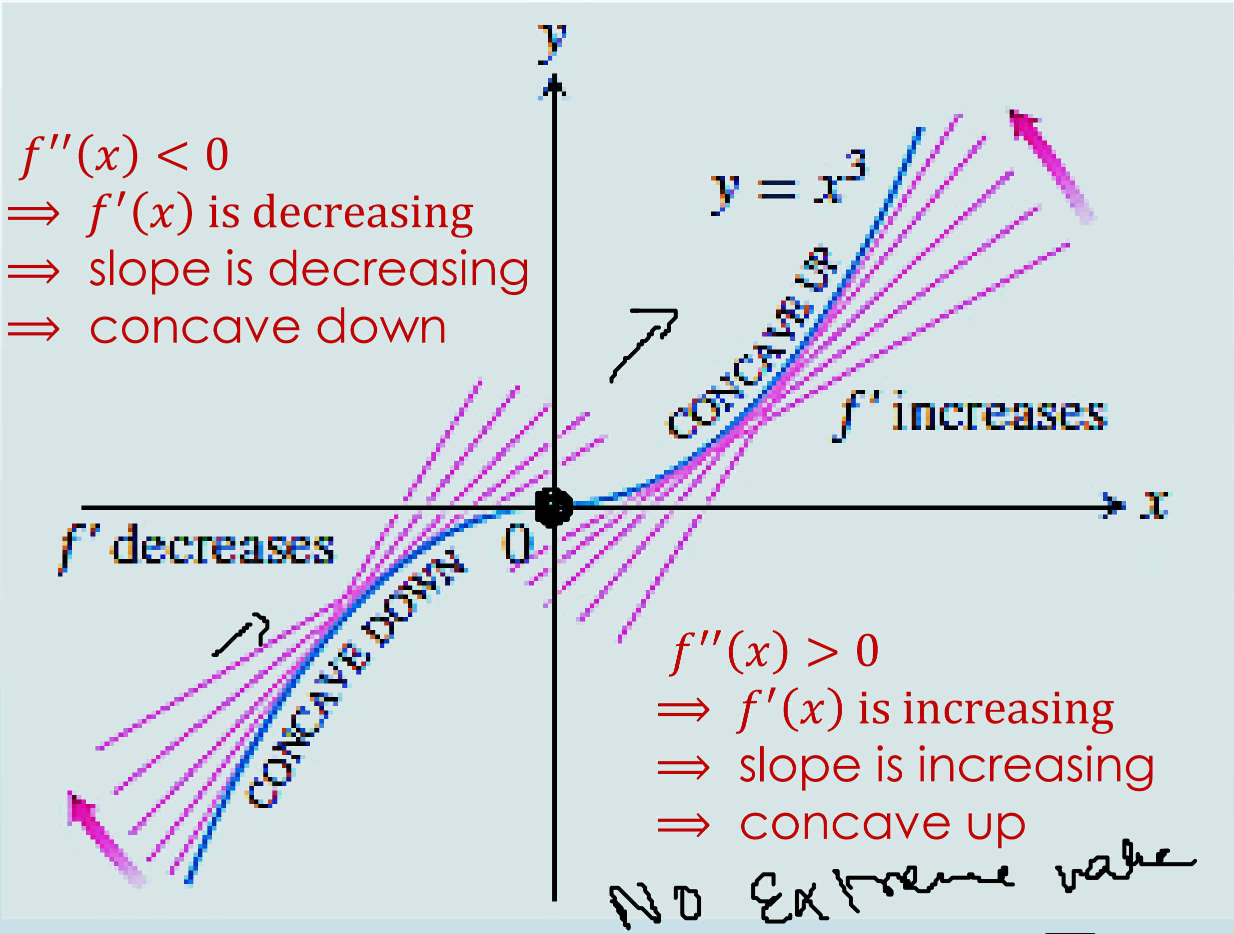
- If $f''(x) < 0$ on I, then $f'(x)$ is decreasing on I. In this case, the slope of the tangent line of the graph of $f(x)$ is decreasing as x is increasing and we say that the graph is <u>concave</u> <u>down</u> on I.

turning point

P.I

f' >0 <0

f'' <0 >0



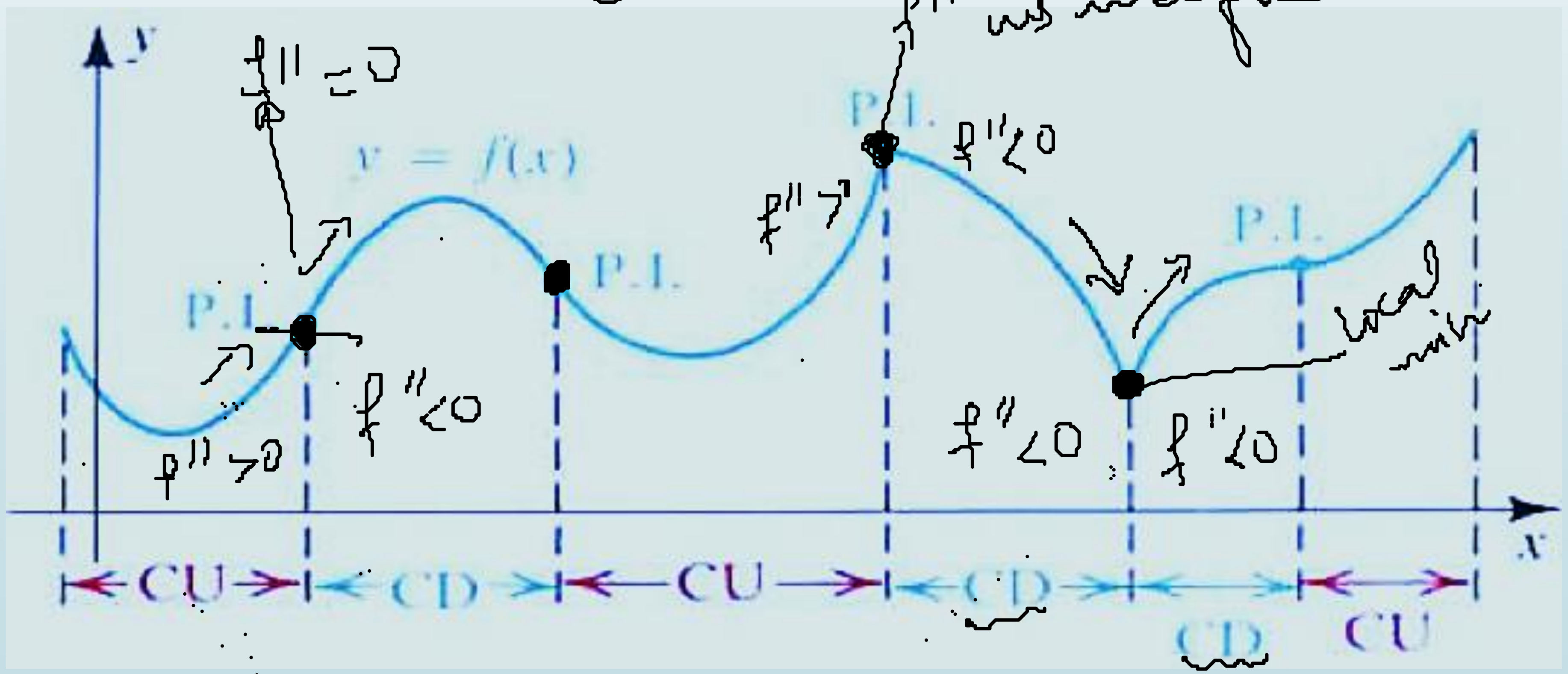
No Extreme value

$x_0 = 0$

CD CU

Inflection Points

- **Inflection points** are points where the graph changes concavity.
- In other words, a point on a curve where $f''(x)$ is positive on one side and negative on the other is a **point of inflection**.
- To determine the points of inflection we begin by finding the zeros of the second derivative (i.e., where $f''(x) = 0$) and the values where $f''(x)$ is undefined.



Note:
Observe that a corner or cusp mayor may not be a point of inflection.

Second derivative Test for Local Extrema

Instead of looking for sign changes in f' at critical points, we can sometimes use the following test to determine the presence and character of local extrema.

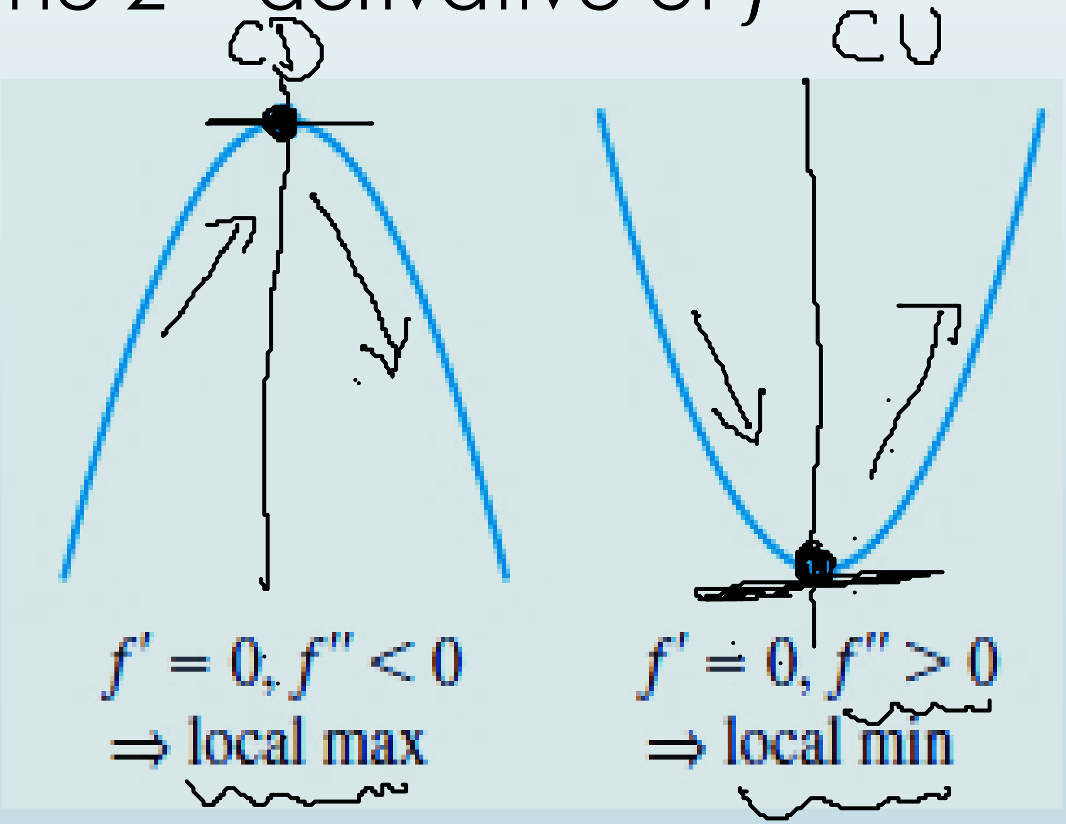
Theorem:

Let $f(x)$ be a function such that $f'(c) = 0$ and the 2nd derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.
3. If $f''(c) = 0$, then the test fails.

   Use the 1st Derivative Test.

$f'(c) = 0$
$f''(c) < 0$



Example :

If $f(x) = 12 + 2x^2 - x^4$, use the second derivative test to find the local extrema of $f(x)$. Discuss concavity, find the points of inflection, and sketch the graph of $f(x)$.

$f'(x) = 0$ CP

$f''(x) = 0$ P.I

Solution:

Differentiating $f(x)$ twice we get:

$$f'(x) = 4x - 4x^3 = 4x(1 - x^2)$$
$$f''(x) = 4 - 12x^2 = 4(1 - 3x^2)$$

The expression for $f'(x) = 0$ is used to find the critical numbers: $\mathbf{0\,,1, and -1}$.

Critical number c	$f''(c)$	Sign of $f''(c)$	Conclusion
-1	-8	$-$ CD	Local max: $f(-1) = 13$
0	4	$+$ CU	Local min: $f(0) = 12$
1	-8	$-$ CD	Local max: $f(1) = 13$

(-1, 13) L max

(0, 12) L min

(1, 13) L max

P.I are : $\frac{-\sqrt{3}}{3}$ and $\frac{\sqrt{3}}{3}$

To locate the possible points of inflection, we solve the equation $f''(x) = 0$ (that is, $4(1-3x^2)=0$), obtaining the solutions $\pm\sqrt{3}/3$. We next examine the sign of $f''(x)$ in each of the intervals:

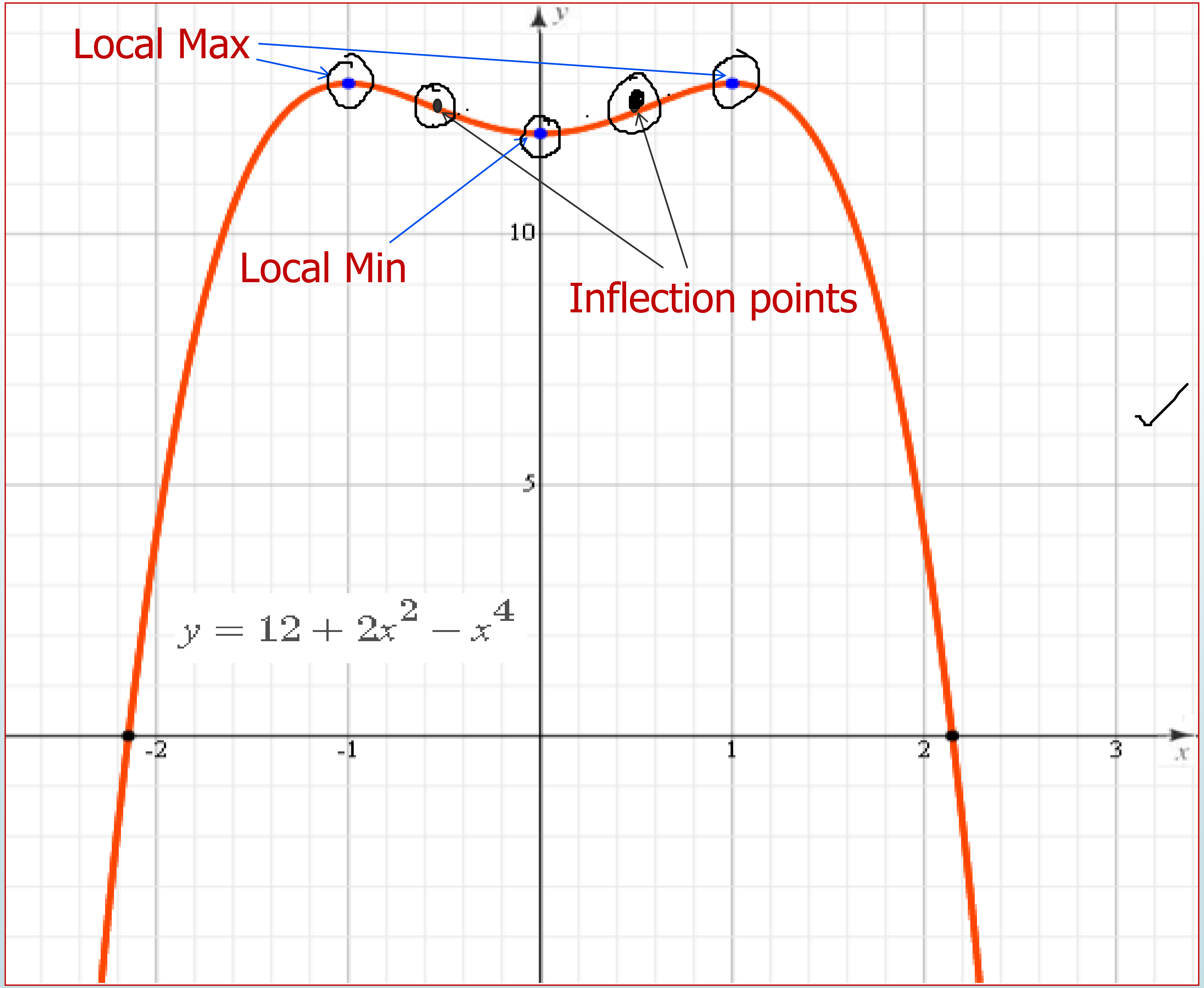
$(-\infty, -\sqrt{3}/3), (-\sqrt{3}/3, \sqrt{3}/3)$, and $(\sqrt{3}/3, \infty)$

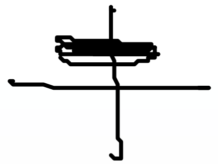
CD $-\sqrt{3}/3$ CU $\sqrt{3}/3$ CD

Interval	$(-\infty, -\sqrt{3}/3)$	$(-\sqrt{3}/3, \sqrt{3}/3)$	$(\sqrt{3}/3, \infty)$
k	-1	0	1
Test value $f''(k)$	$f''(-1) = -8$	$f''(0) = 4$	$f''(1) = -8$
Sign of $f''(x)$	$-$	$+$	$-$
Concavity	downward	upward	downward

CU $\longrightarrow$ $(-\sqrt{3}/3, \sqrt{3}/3)$

CD $\longrightarrow$ $(-\infty, -\frac{\sqrt{3}}{3}) \cup (\frac{\sqrt{3}}{3}, \infty)$

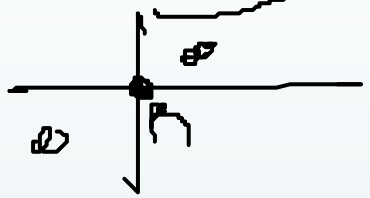

Local Max
Local Min
Inflection points
10
5
-2
-1
1
2
3
$y = 12 + 2x^2 - x^4$



$f(x) = f(-x) \to$ Even
Symmetric w.r.t y-axis

Strategy for Graphing $y = f(x)$

1. Identify the domain of $f(x)$ and any symmetries the curve may have.



$f(-x) = -f(-x) \to$ odd
Symmetric about origin

2. Determine $f'(x)$ ✓ and $f''(x)$ ✓.
3. Find the critical points of $f(x)$ and identify the function's behavior at each one of them.
4. Identify where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any, and determine the concavity of the curve.
6. Summarize the information from step 4 and step 5 and sketch a general shape.
7. Identify any asymptotes. Plot key points, such as the intercepts and the points found in Steps 3 – 5 and sketch the curve.

Example :

Graph the function:

$$y = f(x) = x^4 - 4x^3 + 10.$$

<u>Solution:</u>

Step 1. Domain: All real numbers.

Symmetry: None ($\because$ the given function is neither even nor odd)

Step 2. First and second derivative:

$$y' = f'(x) = 4x^3 - 12x^2$$

$$y'' = f''(x) = 12x^2 - 24x$$

$f'(x) = 0$

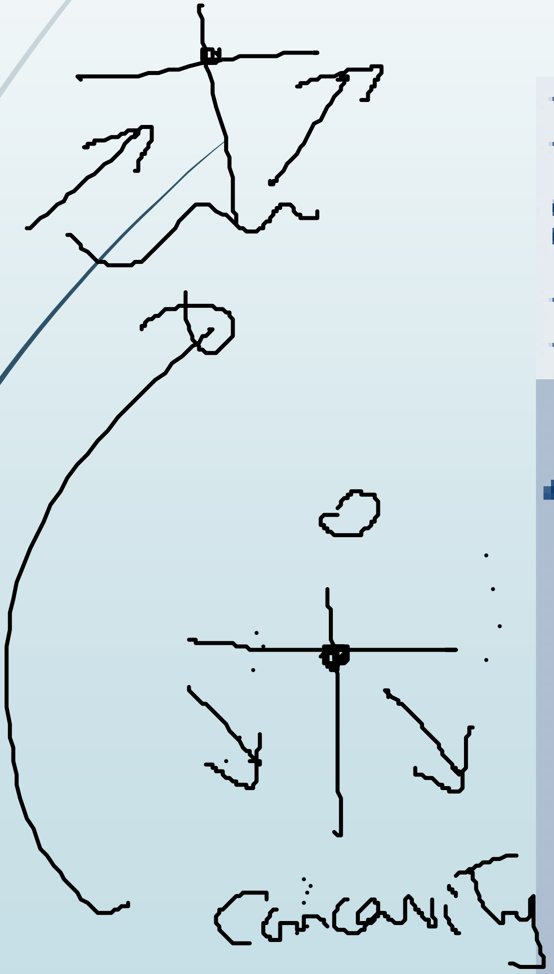
Step 3 & 4. Critical points, rise and fall:

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$
$$f'(x) = 0 \Rightarrow 4x^2(x-3) = 0$$

Critical points: $x = 0, x = 3$

$(-\infty, 0)$ $(3, \infty)$

Intervals	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing



$x = -1$

$f'(-1)$
$= 4(-1)^2(-1-3)$
$= 4(-4)$
$= -16 < 0$

Step 5. Concavity and points of inflection:

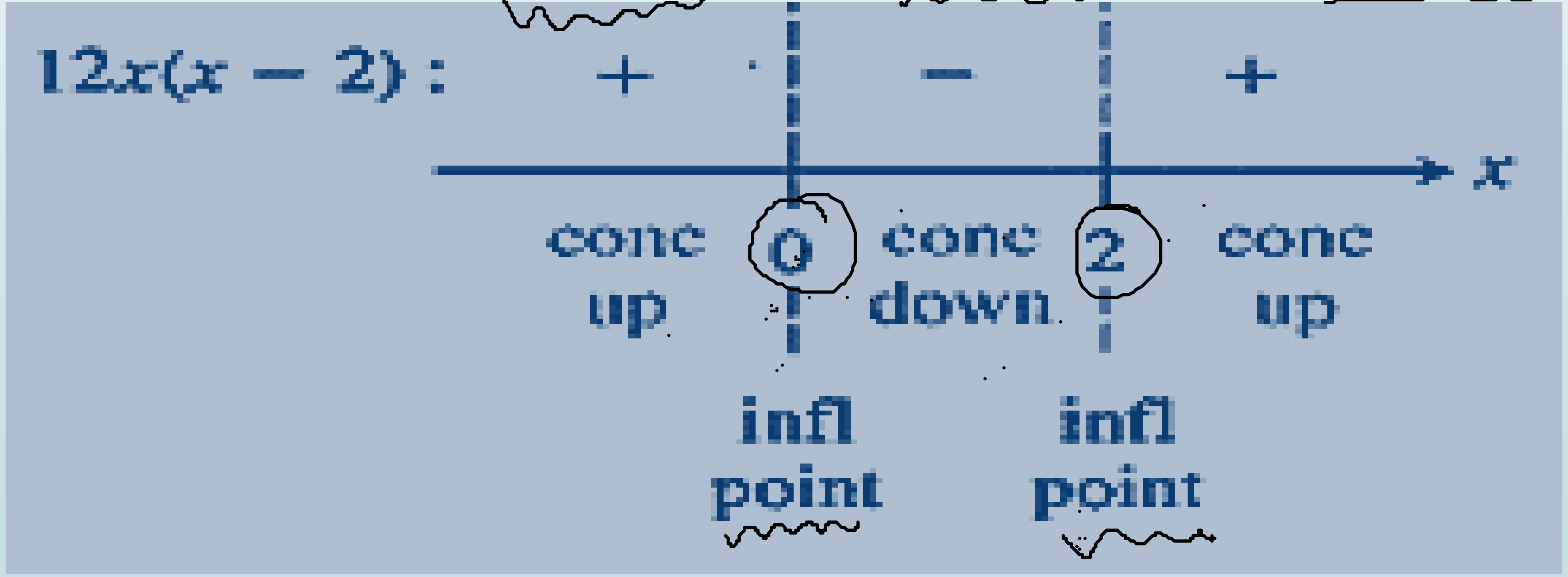
The second derivative $f''(x) = 12x^2 - 24x$ is zero when $x = 0$ and $x = 2$.

concave up $(-\infty, 0) \cup (2, \infty)$

C - d $(0, 2)$

Points of inflection: $x = 0$, $x = 2$.

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	+	−	+
Behavior of f	concave up	concave down	concave up



Step 6. Summarize the information from step 4 & 5 and sketch a general graph.

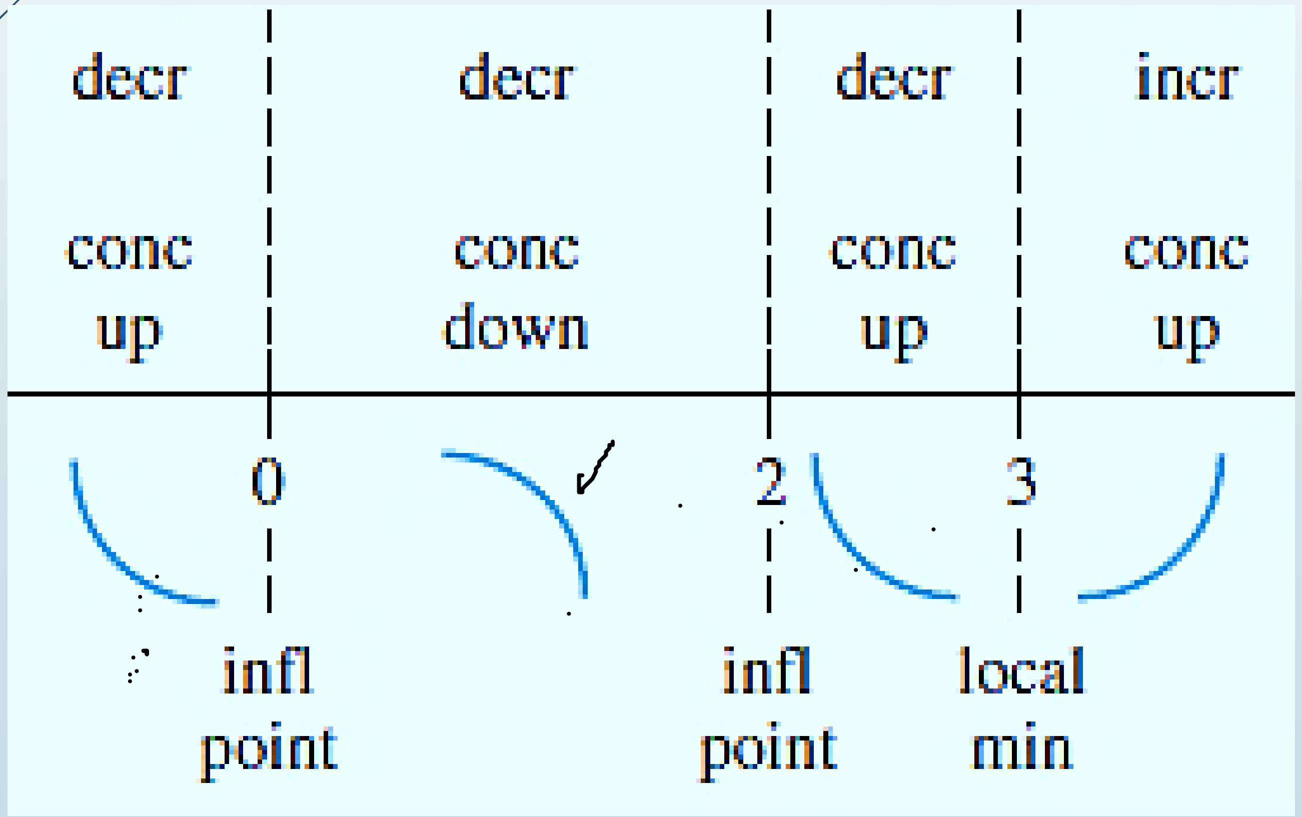
Intervals	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

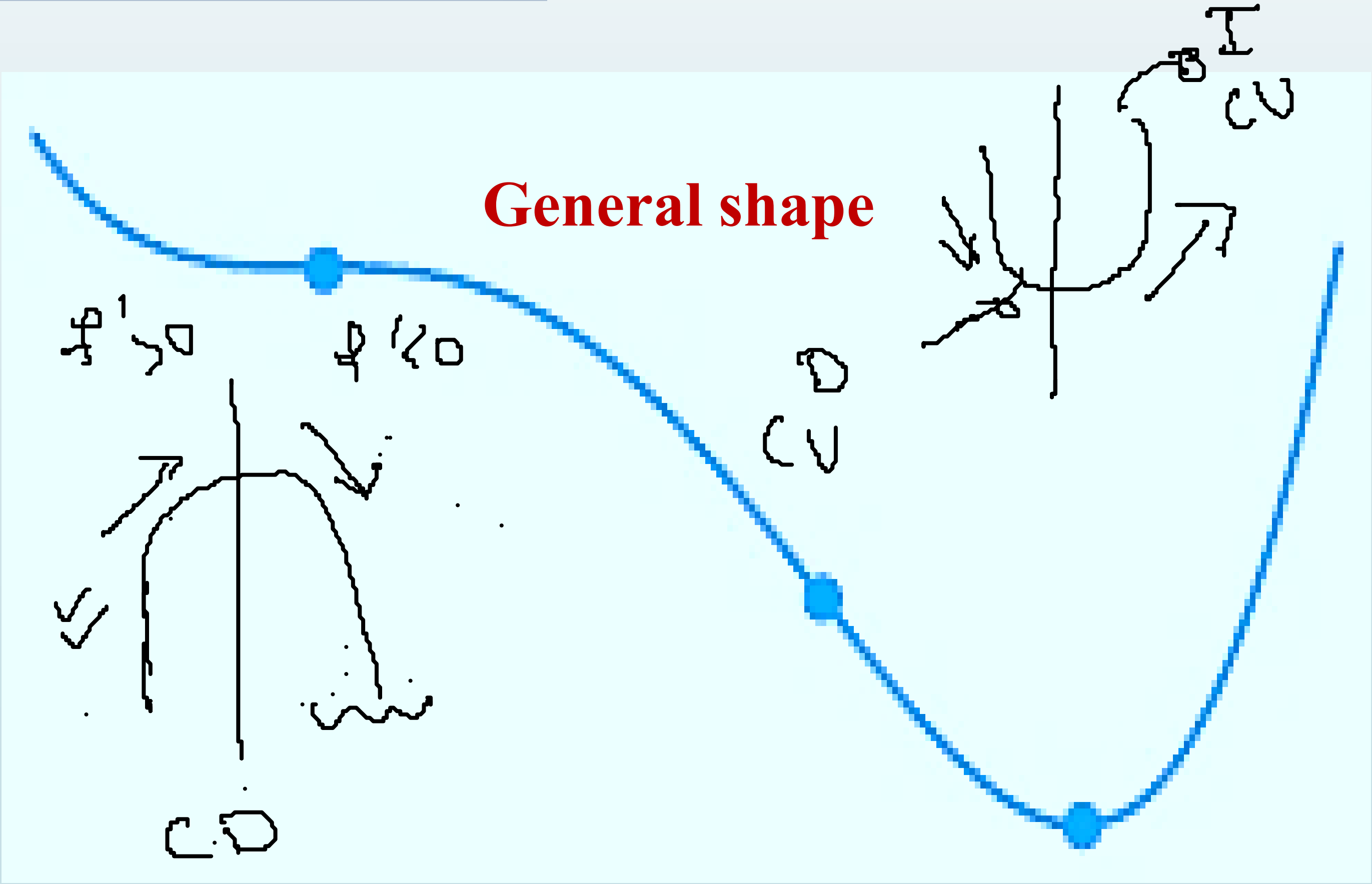
Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing concave up	decreasing concave down	decreasing concave up	increasing concave up

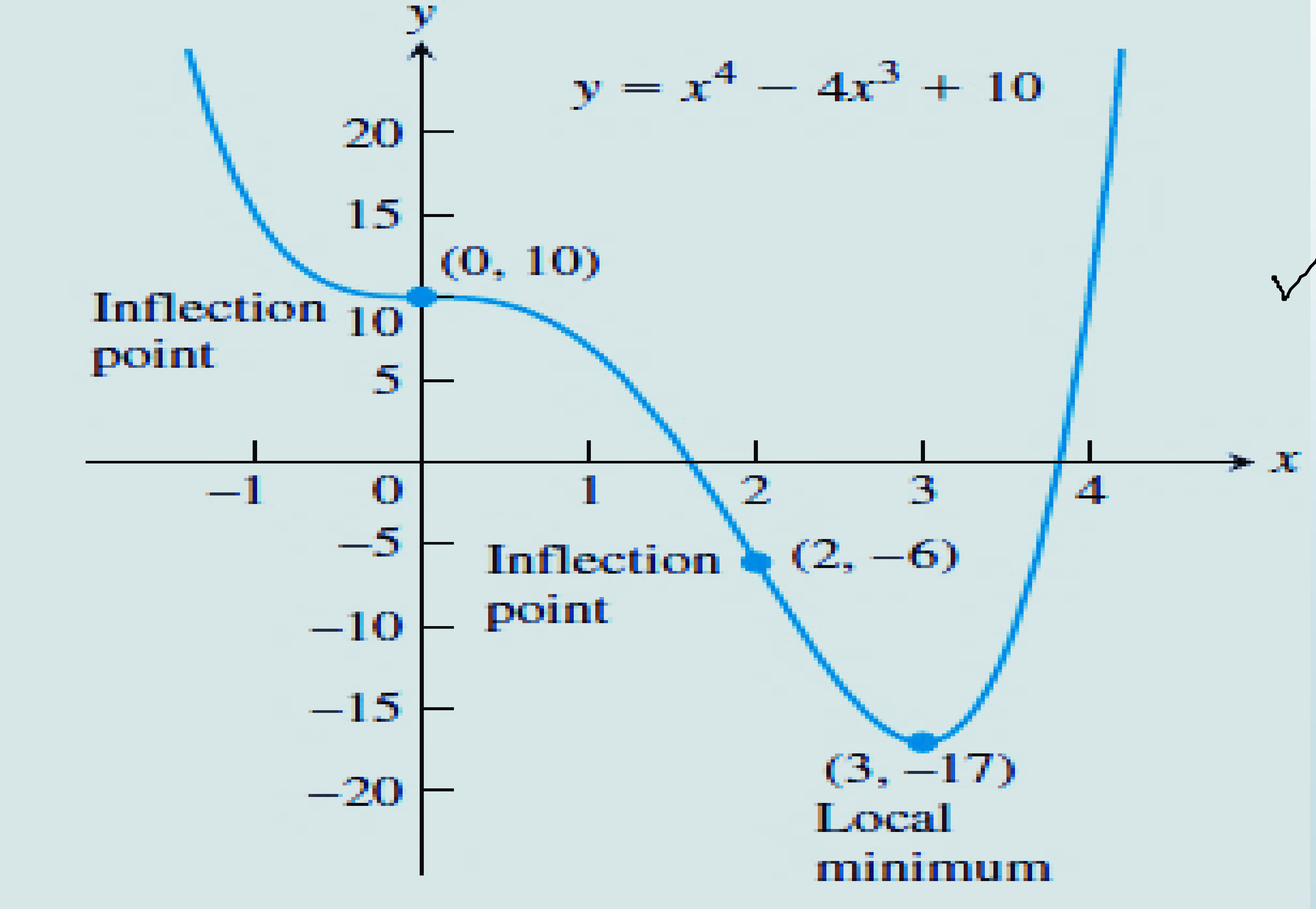
$0 < x < 3$

$0 < x < 2 \cup 2 < x < 3$





Step 7. Identify asymptotes (if any). Plot the curve's intercepts (if convenient) and the points where $f'(x)$ and $f''(x)$ are zero. Indicate any local extreme values and inflection points. Use the general shape in step 6 as a guide to sketch the curve. (Plot additional points as needed.)



Example :

Graph the function:

$$f(x) = \frac{2x^2}{9 - x^2}.$$

Solution:

Step 1. Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

Symmetry: $f(x) = f(-x) \Rightarrow$ Even $\Rightarrow$ Symmetric w.r.t y-axis

Step 2. First and second derivative:

$$f'(x) = 36x/(9-x^2)^2 \checkmark$$

$$f''(x) = \frac{108(x^2+3)}{(9-x^2)^4}$$

Step 3 & 4. Critical points, rise and fall:

$f'(x) = 36x / (9-x^2)^2$

$f'(x) = 0 \Rightarrow x = 0$

$f'(x)$ is undefined at $x = \pm 3 \notin$ Domain

So $x = \pm 3$ are not critical point

Critical points: $x = 0$

Intervals	$(-\infty, -3)$	$(-3, 0)$	$(0, 3)$	$(3, \infty)$
Sign of $f'(x)$	$-$	$-$	$+$	$+$
Behavior of $f(x)$	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

-3: No Extreme Value

0: local min

3: No Extreme value

$f(0) = 0$

$(0, 0)$ Local min

Step 5. Concavity and points of inflection:

$$f''(x) = \frac{108(x^2+3)}{(9-x^2)^3}$$

Numerator is always +ve

$f''(x) \neq 0$

$f''(x)$ is undefined at $x = \pm 3$

Points of inflection:

No points of inflection.

Intervals	$(-\infty, -3)$	$(-3, 3)$	$(3, \infty)$
Sign of $f''(x)$	$-$	$+$	$-$
Behavior of $f(x)$	CD	CU	CD

-3 $\quad$ 3

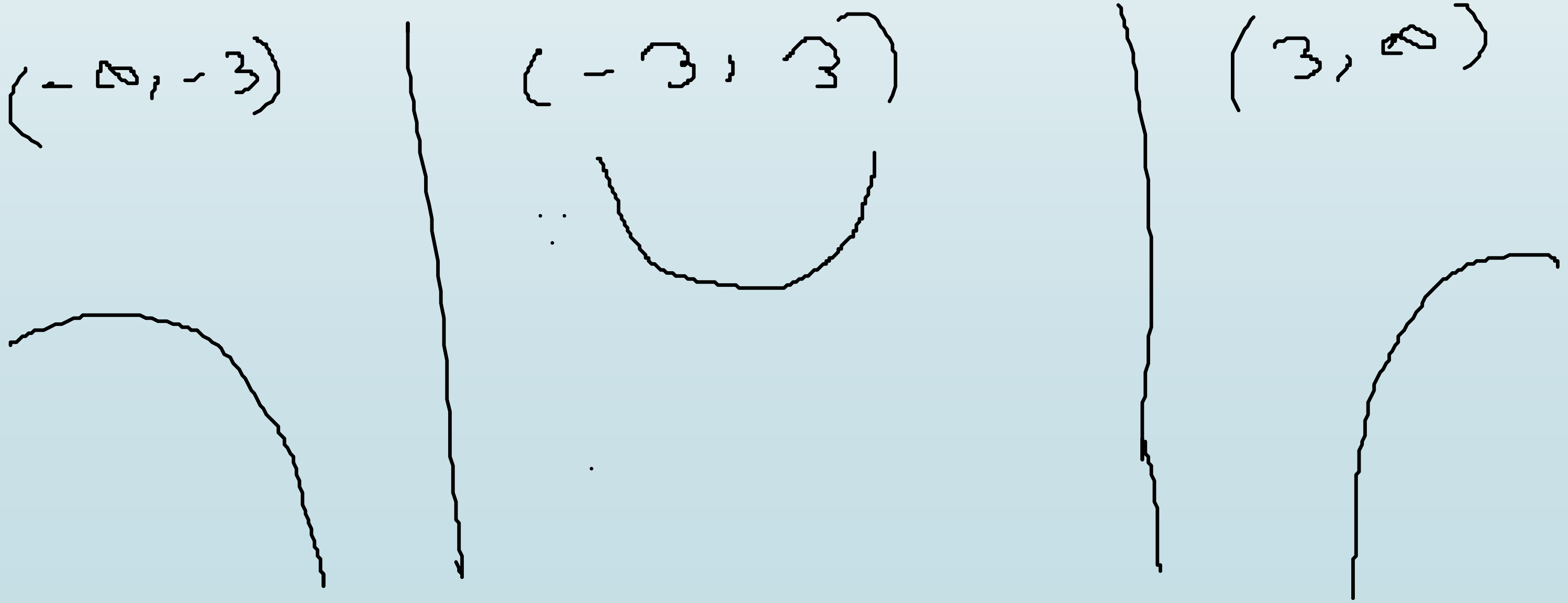
holes in graph at $x = \pm 3$

but fn is not cts at $x = \pm 3$ so $x = \pm 3$ are not P.I

Step 6. Summarize the information from step 4 and 5 and sketch a general graph.

Intervals	$(-\infty, -3)$	$(-3, 0)$	$(0, 3)$	$(3, \infty)$
Sign of $f'(x)$	$\searrow$ $-$	$-$ $\searrow$	$+$ $\nearrow$	$+$ $\nearrow$
Sign of $f''(x)$	$-$ CD	$+$ CU	$+$ CU	$-$ CD
Behavior of $f(x)$				

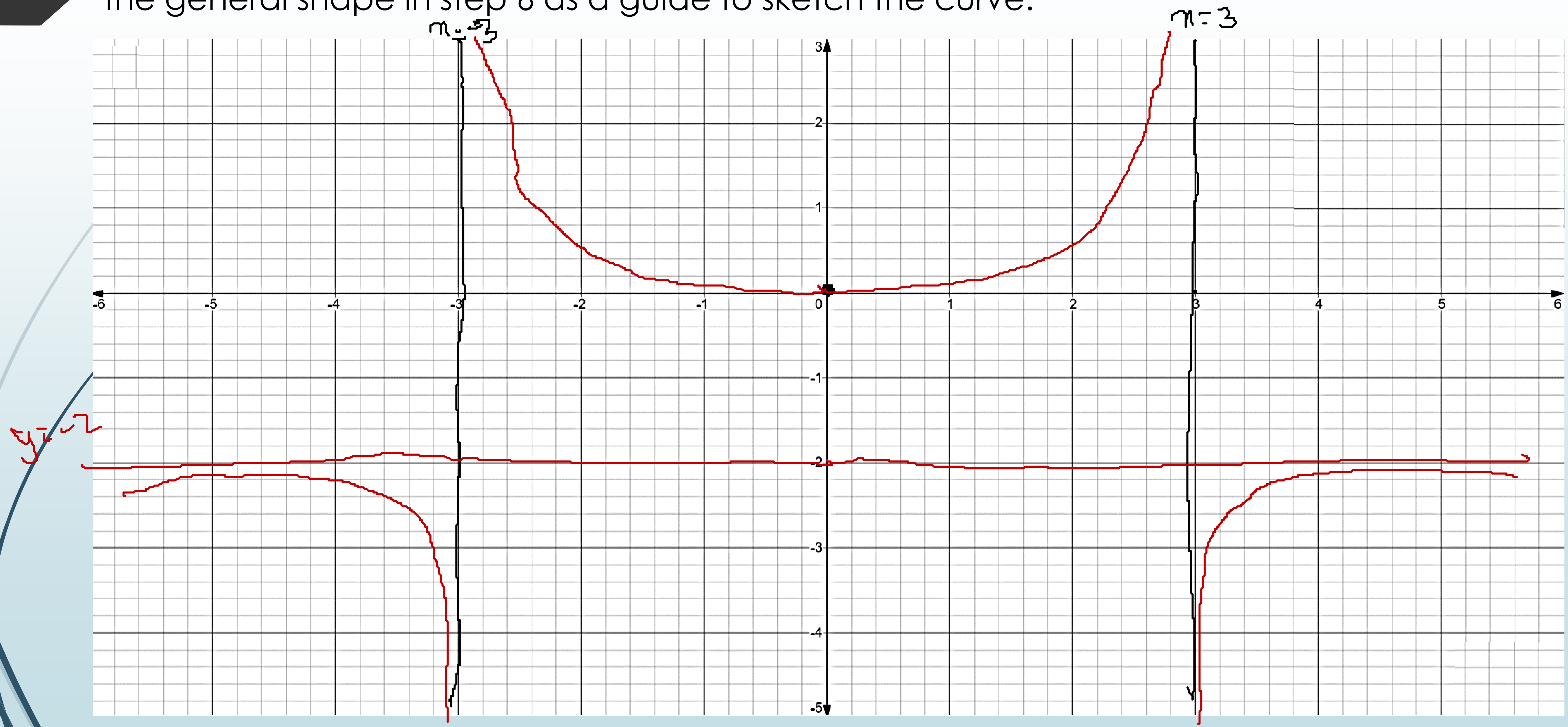
General shape



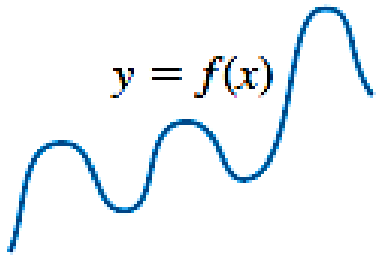
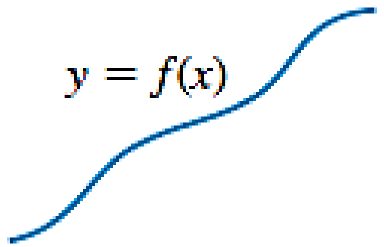
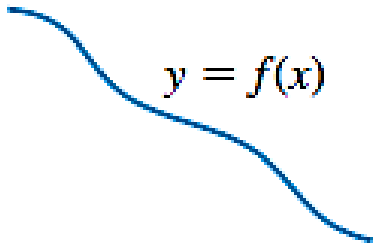
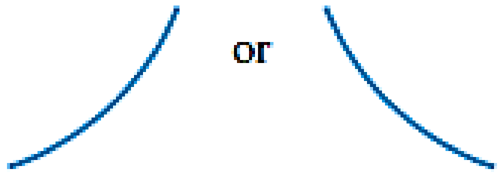
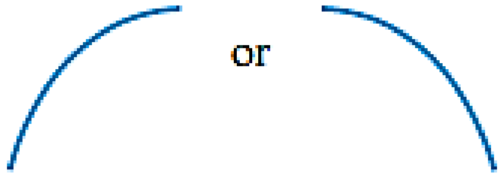
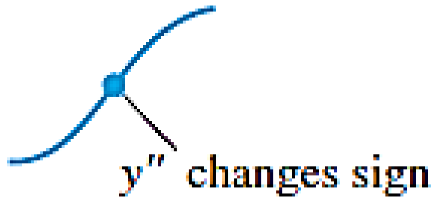
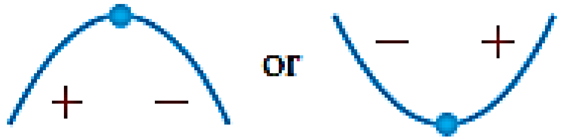
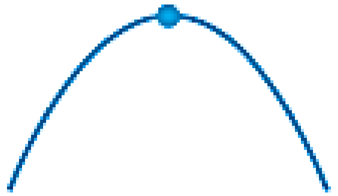
Step 7. Identify asymptotes (if any). Plot the curve's intercepts (if convenient) and the points where $f'(x)$ and $f''(x)$ are zero. Indicate any local extreme values and inflection points. Use the general shape in step 6 as a guide to sketch the curve.

Intercepts	Asymptotes
x-intercept $y = 0$ $\Rightarrow x = 0 \quad (0,0)$ y-intercept $x = 0$ $\Rightarrow y = 0 \quad (0,0)$	Vertical Asymptotes: $x = 3$ $x = -3$ Horizontal / Oblique $f(x) = \dfrac{2x^2}{9 - x^2}$ HA: $\boxed{y = -2}$ $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \left(\dfrac{2}{9/x^2 - 1} \right) =$ $-2 = \lim\limits_{x \to \infty} f(x)$

Step 7. Identify asymptotes (if any). Plot the curve's intercepts (if convenient) and the points where $f'(x)$ and $f''(x)$ are zero. Indicate any local extreme values and inflection points. Use the general shape in step 6 as a guide to sketch the curve.



Summary

 Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall	 $y' > 0 \Rightarrow$ rises from left to right; may be wavy	 $y' < 0 \Rightarrow$ falls from left to right; may be wavy
 $y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall	 $y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall	 Inflection point
 y' changes sign $\Rightarrow$ graph has local maximum or local minimum	 $y' = 0$ and $y'' < 0$ at a point; graph has local maximum	 $y' = 0$ and $y'' > 0$ at a point; graph has local minimum

Practice Questions

Book: Thomas Calculus (11th Edition) by Georg B.Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter: 4**
  - **Exercise: 4.4**

    Q # 1 – 70.