Applications of Eigenvalues and Eigenvectors

Assignment # 3

Section: BEE 12C

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Introduction

Eigenvalues and Eigenvectors

Definition: Eigenvalue

Eigenvalues are a special set of scalars associated with a matrix equation that are sometimes also known as characteristic roots, characteristic value, proper values, or latent roots. Often denoted by λ, it is a factor by which the eigenvector is scaled.

Mathematically, if,

$$\mathbf{A\,v = \lambda\,v}$$

for **v =/= 0**, we say that λ is the eigenvalue for v, and that v is an eigenvector for λ.

Definition: Eigenvector

An eigenvector or characteristic vector of a linear transformation is a nonzero vector that changes at most by a scalar factor when that linear transformation is applied to it.

"Eigenvectors are **by definition nonzero*****. Eigenvalues may be equal to zero."***

Applications

There are numerous applications of eigenvalues and eigenvectors in almost all scientific fields. Keeping the general concept and idea behind eigenvectors and the possible methods of solving the characteristic equation in our minds, we shall discuss the following five applications in this report.

I. Diagonalization
II. Solution of RLC Circuit
III. Markov Chain
IV. Principal Stress and Stress Tensor
V. Google's Page Rank Algorithm

I. Diagonalization

Definition :-

It is the process of transforming a non-diagonal matrix into an equivalent diagonal matrix by using eigen values and eigen vectors.

Uses :-

These are some application in which diagnolization plays an invaluable role.

i) Find higher powers of a given matrix.
ii) Solving first order differential equations.
iii) Solving second order differential equation.

Conditions for a matrix to be diagonalizable :-

① If the eigen values of a given matrix are **distinct**, then it is diagnolizable. But, In case they are **not distinct** then we need to check the following conditions:

i) The corresponding eigen vectors to an eigen value are Linearly independent.

ii) The algebric multiplicity of it's each eigen value is equal to the geometric multiplicity.

$$AM = GM$$

:- AM is the order of the root of the characteristic polynomial

:- GM of an eigen value is the nulity of $(A - I\lambda)$ matrix.

Also $\quad GM(\lambda) \leq AM(\lambda)$.

→ If any of these two conditions are false, matrix is non-diagonalizable.

Example :-

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = 4 - 4\lambda + \lambda^2 - 9 = 0$$

$$= \lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda+1)(\lambda-5) = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = 5$$

As the eigen values are distinct, this matrix is diagonalizable.

Process of Diagonalization:-

Let D be the diagonal matrix A, then D will be:

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

:- where λs are the eigen values of A.

And P be matrix containing the the eigen vectors associated with the eigen values. The are to be in the same order as the eigen values in D matrix.

$$P = \left[X^{(1)} \vdots X^{(2)} \vdots \cdots\cdots \vdots X^{(n)} \right]$$

The P^{-1} of it should also exist otherwise, it will be non-diagnolizable.

Then their relation is :-

$$P^{-1}AP = D \quad \text{—— eq (I)}$$

This equation is called similarity Transformation.

→ P is called <u>the modal matrix</u> of A.

<u>Example:-</u>

Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

As we found it's eigen values are $\lambda_1 = -1$ and $\lambda_2 = +5$.

Then the eigen vectors are :-

<u>:- for $\lambda_1 = -1$</u>

$$\begin{bmatrix} 2-(-1) & 3 \\ 3 & 2-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right] \quad R_2 - R_1$$

$$\left[\begin{array}{cc:c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

→ Let $\boxed{x_2 = 1}$ (it can be any number)

$3x_1 + 3x_2 = 0$

$x_1 = -x_2$

$\boxed{x_1 = -1}$

for $\lambda_2 = 5$

$$\begin{bmatrix} 2-5 & 3 \\ 3 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right] \quad R_2 - R_1$$

$$\left[\begin{array}{cc:c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $\boxed{x_2 = 1}$

$-3x_1 + 3x_2 = 0$

$x_1 = x_2$

$\boxed{x_1 = 1}$

The eigen vectors are :

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\Rightarrow \det(P) = -2$

$\Rightarrow \text{Adj}(P) = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

$$P^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

Now :-

$$D = P^{-1} A P$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} +1 & 5 \\ -1 & 5 \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} 2 & 0 \\ 0 & -10 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

This is the diagonal matrix with eigen values as it's entries. Just as expected.

Uses :-

① Matrix powers:-

Diagonalization can be used to find the power of a square matrix.

we know :-

$$D = P^{-1} A P$$

$$P D P^{-1} = P P^{-1} A P P^{-1}$$

$$A = P D P^{-1}$$

Now

$$A^2 = P D P^{-1} P D P^{-1}$$

$$A^2 = P D^2 P^{-1}$$

Similarly :-

$$\boxed{A^n = P D^n P^{-1}}$$

This provides us an easy way of finding high power of matrix A.

Example :-

Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$, we previously found. $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$

and $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \frac{1}{-2}$

Suppose we want to find A^{10}.

Then

we know

$$D^{10} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 5^{10} \end{bmatrix}$$

$$A^{10} = P D^{10} P^{-1}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9765625 \end{bmatrix} \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -9765625 & -9765625 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} -9765626 & -9765624 \\ -9765624 & -9765626 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 4882813 & 4882812 \\ 4882812 & 4882813 \end{bmatrix}$$

Thus by using diagonalization we found A^{10} in just a few steps instead of multiplying the A matrix 10 times.

② Solving First Order Differential Equations :-

If we have a pair of first order differential equation :-

$x' = -2x$ at $x(0) = 3$ i.e. $x = 3$ at $t=0$

$y' = -5y$ $\quad y(0) = 2$ i.e $y = 2$ at $t=0$

→ we know the solution of such equations is.

$x = K_1 e^{-s_1 t}$ were $K_1 = x(0) = 3$, $s_1 = -2$

$y = K_2 e^{-s_2 t}$ $\quad K_2 = y(0) = 2$, $s_2 = -5$

So

$$x = 3e^{-2t}$$
$$y = 2e^{-5t}.$$

But in case we have to solve such pair of equation in which both x and y are involved at the same time, then; we diagnolize it to get the solution as in the above equations.

Process :-

The first thing we need to do is write the equations in the form of matrices.

i.e $X' = AX$

where

X' is a $1 \times n$ matrix containing derivatives.

X is a $1 \times n$ matrix containing unknown functions.

A is a $n \times n$ matrix of the co-efficient.

Then we will find the modal matrix of A to do diagnolization. This will transform $X' = AX$ into $Y' = DY$. where D is the diagonal matrix. Y is an $1 \times n$ matrix that have the relation :- $X = PY$. P being modal matrix of A. Since P is a matrix of constants:-

$$X' = PY'$$

$$Y = \begin{bmatrix} r(t) \\ s(t) \end{bmatrix}$$

$$AX = PY'$$

$$APY = PY'$$

$$P^{-1}APY = P^{-1}PY'$$

$$DY = Y'$$

Implies $\rightarrow$

$$\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \cdots & 0 \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} r \\ s \\ \vdots \end{bmatrix} = \begin{bmatrix} r' \\ s' \\ \vdots \end{bmatrix}$$

Thus :-

$$r' = \lambda_1 r$$
$$s' = \lambda_2 s$$
$$\vdots$$

The solution of these will now be in the form of :- $r = Ce^{+\lambda_1 t}$, $s = Ke^{\lambda_2 t}$, ...

Example :-

Solve :-

$$x' = 4x + 2y$$
$$y' = -x + y$$

at $x(0) = 1$ & $y(0) = 0$

Here $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$; $A - \lambda I = \begin{bmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 0$

$(4-\lambda)(1-\lambda) + 2 = 0$

$\lambda^2 - 5\lambda + 4 + 2 = 0$

$(\lambda - 3)(\lambda - 2) =$

$\boxed{\lambda_1 = 3}, \boxed{\lambda_2 = 2}$

The corresponding vectors :-

$(A - \lambda I) X = 0$

For $\lambda_1 = 3$:-

$\left[\begin{array}{cc:c} 1 & 2 & 0 \\ -1 & -2 & 0 \end{array}\right]$ $R_2 + R_1$

$\left[\begin{array}{cc:c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Let $\boxed{x_2 = 1}$

$x_1 + 2x_2 = 0$

$\boxed{x_1 = -2}$

For $\lambda_2 = 2$

$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ -1 & -1 & 0 \end{array}\right]$ $2R_2 + R_1$

$\left[\begin{array}{cc:c} 2 & 2 & 0 \\ 0 & 0 & \end{array}\right]$

Let $\boxed{x_2 = 1}$

$2x_1 + 2x_2 = 0$

$\boxed{x_1 = -1}$

$P = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$ $\therefore$ $P^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$

$D = P^{-1}AP = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -6 & -2 \\ 3 & 2 \end{bmatrix}$

$D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$y' = DY$

$\begin{bmatrix} r' \\ s' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$

$r' = 3r$

$s' = 2s$

Then,

$s = Ke^{2t}$

$r = Ce^{3t}$

Now,

$$X = PY$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} Ce^{3t} \\ Ke^{2t} \end{bmatrix}$$

$$Y = \begin{bmatrix} r \\ s \end{bmatrix}$$

$$= \begin{bmatrix} -2Ce^{3t} - Ke^{2t} \\ Ce^{3t} + Ke^{2t} \end{bmatrix}$$

There fore :-

$$x = -2Ce^{3t} - Ke^{2t} \quad — ①$$
$$y = Ce^{3t} + Ke^{2t} \quad — ②$$

Using the initial Conditions :-

$x(0) = 1$
$y(0) = 0$

i.e.

$$1 = -2Ce^{3(0)} - Ke^{2(0)} \Rightarrow 1 = -2C - K \quad - ③$$

$$0 = Ce^{3(0)} + Ke^{2(0)} \Rightarrow 0 = C + K \quad - ④$$

Solving 3 & 4.

$C = -1$
$K = 1$

Putting in ① & ②

$$\boxed{x = 2e^{3t} - e^{2t}}$$
$$\boxed{y = -e^{3t} + e^{2t}}$$

Solution

③ Second Order Differential Equation:-

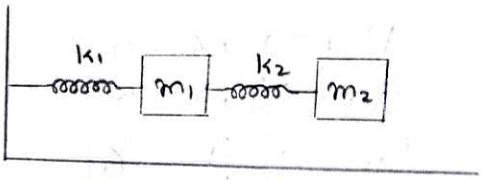
The method discussed above can also be used to solve a pair of second order differential equations. A second order D.E has the following form.

$$x'' = ax + by$$
$$y'' = cx + dy$$

where $X'' = AX$:- $X = \begin{bmatrix} x \\ y \end{bmatrix}$ & $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

These second order differential equations can represent any natural phenomenon.

Real Life Example



Q: Considering the system given, if the motion of the two masses m_1 and m_2 vibrating on the coupled springs, neglecting damping and spring masses is governed by :-

$$m_1 y_1'' = -k_1 y_1 + k_2 (y_2 - y_1)$$

$$m_2 y_2'' = -k_2 (y_2 - y_1)$$

Then give a function for the amplitude of the springs if $m_1 = m_2 = 1$, $k_1 = 3$, $k_2 = 2$, while the initial conditions are :-
$y_1(0) = 1$, $y_2(0) = 2$, $y_1'(0) = -2\sqrt{6}$, $y_2' = \sqrt{6}$.

Solution :-

If $m_1 = m_2 = 1$, $k_1 = 3$ & $k_2 = 2$:-

Then the equations become :-

$$y_1'' = -5y_1 + 2y_2$$

$$y_2'' = 2y_1 - 2y_2$$

Now,

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

→ Eigen values :-

The characteristic equation is :-

$$(A - \lambda I)X = 0.$$

For eigen values :-

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(-5-\lambda)(-2-\lambda) - 4 = 0$$

$$\lambda^2 + 7\lambda + 6 = 0$$

$$(\lambda + 6)(\lambda + 1) = 0$$

$$\boxed{\lambda_1 = -1} \quad \boxed{\lambda_2 = -6}$$

→ Associated Eigen Vectors:-

For $\lambda_1 = -1$

$$[A-(-1)I][X_1] = 0$$

$$\left[\begin{array}{cc|c} -5-(-1) & 2 & 0 \\ 2 & -2-(-1) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right] \quad 2R_2 + R_1$$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-4x_1 + 2x_2 = 0$$

Suppose $\boxed{x_2 = 2}$

$\boxed{x_1 = 1}$

$$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda_2 = -6$

$$[A-(-6)I][X_2] = 0$$

$$\begin{bmatrix} -5-(-6) & 2 \\ 2 & -2-(-6) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} +1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \quad R_2 - 2R_1$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 2x_2 = 0$$

Suppose $x_2 = 1$

$x_1 = -2$

$$X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

→ Now Diagonalization:-

$$P = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad \& \quad P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

Then

$$D = P^{-1}AP$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 12 \\ -2 & -6 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -5 & 0 \\ 0 & -30 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ as expected.}$$

As before :-

$$Y'' = DY$$

$$\begin{bmatrix} r'' \\ s'' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$$

$$r'' = -r \implies r'' = -\omega_1^2 r$$

$$s'' = -6s \implies s'' = -\omega_2^2 s$$

so $\boxed{\omega_1 = 1}$ & $\boxed{\omega_2 = \sqrt{6}}$

The general solution for a harmonic motion is :-

$$\underline{x = A\cos\omega_1 t + B\sin\omega_1 t}$$

Thus in the given case :-

$$r = K \cos t + L \sin t$$
$$s = M \cos(\sqrt{6}\,t) + N \sin(\sqrt{6}\,t)$$

Now :-

$$X = PY$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} K\cos t + L \sin t \\ M\cos(\sqrt{6}t) + N \sin(\sqrt{6}t) \end{bmatrix}$$

$$y_1 = K\cos t + L \sin t - 2M\cos(\sqrt{6}t) + 2N \sin(\sqrt{6}\,t) \quad — \text{A}$$

Using the initial condition $y_1(0) = 1$

$$1 = K - 2M \quad — 1$$

$$y_1' = -K\sin t + L\cos t + 2M\sqrt{6}\sin(\sqrt{6}\,t) - 2\sqrt{6}N\cos(\sqrt{6}\,t) \quad — \text{B}$$

Using the initial Condition $y_1'(0) = -2\sqrt{6}$

$$-2\sqrt{6} = L - 2\sqrt{6}N \quad — 2$$

$$y_2 = 2K\cos t + 2L\sin t + M\cos(\sqrt{6}\,t) + N\sin(\sqrt{6}\,t)$$

→ Using the initial condition $y_2(0) = 2$.

$$2 = 2K + M \quad — 3$$

$$y_2' = -2K\sin t + 2L\cos t - \sqrt{6}M\sin(\sqrt{6}t) + \sqrt{6}N\cos(\sqrt{6}t)$$

→ Using the initial Condition $y_2'(0) = \sqrt{6}$

$$\sqrt{6} = 2L + \sqrt{6}N \quad — 4$$

→ Solving 1 & 3

$$1 = k - 2\cancel{M}$$
$$4 = 4k + \cancel{2M}$$
$$5 = 5k$$

or $\boxed{k = 1}$

Putting $k = 1$ in eq 3 gives

$\boxed{M = 0}$

→ Solving 2 & 4

$$-2\sqrt{6} = L - \cancel{2\sqrt{6}N}$$
$$2\sqrt{6} = 4L + \cancel{2\sqrt{6}N}$$
$$0 = 5L$$

or $\boxed{L = 0}$

Putting value of $L = 0$ in eq 4 gives

$\boxed{N = 1}$

Putting the values of K; M, L & N in A & B.

$$y_1 = \cos t - 2\sin(\sqrt{6}\,t)$$
$$y_2 = 2\cos t + \sin(\sqrt{6}\,t)$$

Solution

These equations give the amplitude of the spring for any time.

In this way diagonalization and eigen vector / values can be used to solve differential equations, representing any real life phenomenon.

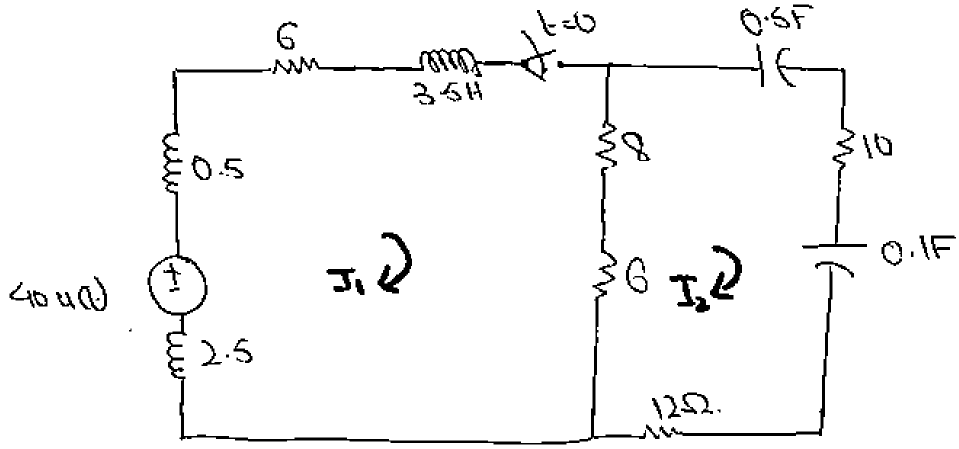
II. Solution of RLC Circuit

Introduction:

The RLC circuits are the circuits containing, an inductor, a capacitor and a resistor. The response of this circuit for any time $t>0$ can be predicted by the help of Eigenvalues and Eigenvectors.

Circuit:

Consider an RLC circuit as shown:



Q) Find the general response of the RLC circuit.

Thus we define two mesh currents I_1 & I_2

Note:

Note that there is no source present in the circuit prior to $t=0$ and the capacitors and inductors are initially uncharged.

Formulas:-

$$V_L = L\,di/dt$$
$$V_c = \int i\,dt$$

(i) Applying Mesh analysis at loop → (1)

$$\frac{5}{2}\frac{di_1}{dt} - 40 + \frac{1}{2}\frac{di_1}{dt} + 6i_1 + \frac{7}{2}\frac{di_1}{dt} + 8i_1 - 8i_2 + 6i_1 - 6i_2 = 0$$

By simplifying;

$$\frac{13}{2}\frac{di_1}{dt} + 20i_1 - 14i_2 = 40$$

Multiplying by 2:

$$13\frac{di_1}{dt} + 40i_1 - 26i_2 = 80$$

Rearranging

$$\frac{di_1}{dt} = 2i_2 - 3.076i_1 + 6.1538 \quad \rightarrow (1)$$

(ii) Applying Mesh analysis at loop: (2)

$$8i_2 - 8i_1 + 6i_2 - 6i_1 + 2\int i_2\,dt + 10i_2 + \int 10i_2\,dt + 12i_2 = 0$$

By simplifying we have:

$$36i_2 - 14i_1 + 12\int i_2\,dt = 0$$

By taking differential of whole equation:

$$36\frac{di_2}{dt} - 14\frac{di_1}{dt} + 12i_2 = 0 \quad \rightarrow (ii)$$

By using (i) in (ii)

$$36\frac{di_2}{dt} - 14(2i_2 - 3.076i_1 + 6.1538) + 12i_2 = 0$$

$$36\frac{di_2}{dt} = 16i_2 - 43.064i_1 + 86.1532.$$

$$\frac{di_2}{dt} = 0.444i_2 - 1.1962i_1 + 2.3931 \longrightarrow \text{(iii)}$$

The solution for the derivatives in equation (i) and (iii) in matrix system can be written as:

$$\frac{dK}{dt} = AK + V$$

Here;

$$A = \begin{bmatrix} -3.076 & 2 \\ -1.1962 & 0.444 \end{bmatrix}, \quad K = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$$V = \begin{bmatrix} 6.1538 \\ 2.3931 \end{bmatrix}$$

Now the characteristic equation for the matrix A becomes:

$$\det(I_2\lambda - A) = 0$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3.076 & 2 \\ -1.1962 & 0.444 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda + 3.076 & -2 \\ 1.1962 & \lambda - 0.444 \end{bmatrix} = 0$$

The determinant becomes:

$$(\lambda+3.076)(\lambda-0.444)+2(1.196)=0$$

$$\lambda^2+2.632\lambda-1.365+2.392=0$$
$$\lambda^2+2.632\lambda+1.027=0$$

The two roots of the characteristic equation become:

$$\lambda_1=-0.476$$
$$\lambda_2=-2.155$$

The eigenvector corresponding to the λ_1 becomes:

As:

$$A_1K+V_1=0$$
$$K=A_1^{-1}(-V)$$

Now:

$$A=\begin{bmatrix}-0.476+3.076 & -2 \\ 1.196 & -0.444\end{bmatrix}$$

Thus, the inverse of A becomes

$$A^{-1}=\begin{bmatrix}1.0504 & -2.2835 \\ 1.3655 & -4.0553\end{bmatrix}$$

Hence, putting it in the equation, "k" becomes:

$$K'=\begin{bmatrix}1.0504 & -2.28355 \\ 1.3655 & -4.0553\end{bmatrix}\begin{bmatrix}-6.1538 \\ -2.3931\end{bmatrix}$$

The first eigenvector corresponding to λ_1 becomes:

$$V_1=\begin{bmatrix}-0.9987 \\ 1.3020\end{bmatrix}$$

$$V_1 = 1.3020\begin{bmatrix}-0.7632\\1\end{bmatrix} = C_1\begin{bmatrix}-0.7632\\1\end{bmatrix}$$

Here; $C_1 = 1.3020$

Now the eigenvector V_2, corresponding to λ_2 will be:

$$A_2 K'' + V = 0$$
$$K' = A_2^{-1}(-V)$$

Now the matrix A_2 becomes:

$$A_2 = \begin{bmatrix}-2.155+3.076 & -2\\1.1962 & -2.155-0.444\end{bmatrix}$$

Hence, the inverse of the matrix A_2 becomes:

$$A_2^{-1} = \begin{bmatrix}2032.056 & -1563.721\\935.261 & -720.093\end{bmatrix}$$

Hence;

$$K_2' = A_2^{-1}(-V)$$

$$V_2 = \begin{bmatrix}2032.056 & -1563.721\\935.261 & -720.09\end{bmatrix}\begin{bmatrix}-6.1588\\-2.9931\end{bmatrix}$$

$$V_2 = \begin{bmatrix}-8762.7\\-4032.15\end{bmatrix}$$

$$V_2 = -4032.15\begin{bmatrix}2.1732\\1\end{bmatrix} = C_2\begin{bmatrix}2.1732\\1\end{bmatrix}$$

where $c_2 = -4032.15$

Hence, by applying the superposition principle we have:

$$K = K' + K''$$

Thus, the general solution for K in time domain beomes.

$$K = c_1 \begin{bmatrix} -0.7632 \\ 1 \end{bmatrix} e^{0.763t} + c_2 \begin{bmatrix} 2.1732 \\ 1 \end{bmatrix} e^{-2.1732t}.$$

Hence, by putting different values of t we can intrepret values of K for time t>0. "K" can be either voltage or current and in this case it is current.

III. Markov Chain

Introduction

The markov chain is a method used to forecast the value of a variable whose predicted value is influenced ONLY by its present situation. It is used in predicting traffic flows, communication networks and quesles and stuff.

The Problem

Suppose that at some initial point in time, 100,000 people live in a certain city and 25,000 live in its ~~suburbs~~ suburbs. The Regional Planning commision determines that each year 5% of city population moves to suburbs and 3% suburban people, move to the city. Over a long term, how will the population be distributed?

* Solution :-

First we represent the percentages in a matrix form:

$$\begin{array}{c|cc} & \text{State 1 City} & \text{State 2 Sub} \\ \hline \text{State 1 City} & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$}} \\ \text{State 2 Sub} & & \end{array}$$

Hence we have made a "transition matrix"

Transtion matrix:

A matrix for which all column vectors are probability vectors (sum is = 1).

Transition matrix

$$P = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$$

$$P\vec{X} = 1\vec{X}$$

$$P\vec{X} - \vec{X} = 0$$

$$\vec{X}\left[P - 1I\right] = 0$$

$$\vec{X}\left[\begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0$$

$$\vec{X}\begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix} = 0$$

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$\begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad R_2 = R_1 + R_2$$

$$\begin{bmatrix} -0.05 & 0.03 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Let $\underline{x_2 = t}$ so,

$$-0.05x_1 + 0.03x_2 = 0$$

Since $x_2 = t$,

$$-0.05x_1 + 0.03t = 0$$

$$x_1 = \frac{-0.03}{-0.05}t$$

$$\underbrace{x_1 = \frac{3}{5}t}$$

$$\vec{X} = \begin{bmatrix} \frac{3}{5}t \\ t \end{bmatrix}$$

Since it is a probability matrix, sum should equal to 1

$[x_1 + x_2 = 1]$

$$\frac{3}{5}t + t = 1$$

$$\underbrace{x_1 = \frac{3}{5}t}$$

$$* \quad \vec{X} = \begin{bmatrix} 3/5\, t \\ t \end{bmatrix}$$

Since it is a probability matrix, Sum should equal to 1

$$[x_1 + x_2 = 1]$$

$$\frac{3}{5}t + t = 1$$

$$\boxed{t = \frac{5}{8}}$$

$$\vec{X} = \begin{array}{c} \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix} \end{array} \begin{array}{l} \text{city} \\ \text{suburb} \end{array}$$

$x_1 \times$ Total population $= \frac{3}{8} \times 125{,}000 = \underline{46{,}875}$ city

$x_2 \times$ Total population $= \frac{5}{8} \times 125{,}000 = \underline{78{,}125}$ suburb

* Hence, we see that eigenvalue and eigenvectors were used in problems related to Markov chain.

IV. Principal Stress and Stress Tensor

- Cauchy Stress Tensor:

It is a second order tensor, consisting of nine components that completely define the state of stress inside a material in deformed configuration.

Mathematically;

The components of traction (vectorial) are given by Cauchy's formula:

$$T_i = \sigma_{ij} n_j \qquad - (1.1)$$

Since we are directing ourselves towards eigenvalue formulation, it is better to write the components in tensor form.

$$T \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Subtracting left side from the right side (through Matrix rules / properties yields:

$$\begin{bmatrix} \sigma_{11} - T & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - T & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - T \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

Which can be written as,

$$[\sigma - \underline{I}T][n] = 0$$

Where I is the identity matrix.

The above equation can only be solved if the determinant $|\sigma - IT|$ equals zero.

$$\begin{vmatrix} \sigma_{11} - T & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - T & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - T \end{vmatrix} = 0$$

- The values of T (eigenvalues) that solve this equation are principal values.
- The values of n (eigenvectors) are the principal directions.

Now we shall do some examples to put the knowledge into effect.

Example:

Stress at a point is given with respect to axis x_1, x_2, x_3 by:

$$[\sigma_{ij}] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}$$

Find a) principal values b) principal directions

a) Forming characteristic equation

$$\begin{vmatrix} 5-\sigma & 0 & 0 \\ 0 & -6-\sigma & -12 \\ 0 & -12 & 1-\sigma \end{vmatrix} = 0$$

$$(5-\sigma)[(-6-\sigma)(1-\sigma)-(144)] = 0$$

$$(5-\sigma)(\sigma-10)(15+\sigma) = 0$$

$\underline{\sigma_1 = 10}$, $\underline{\sigma_2 = 5}$, $\underline{\sigma_3 = -15}$

b) Using principal values :

$\sigma_1 : 10$ yield

$$\begin{bmatrix} 5-10 & 0 & 0 \\ 0 & -6-10 & -12 \\ 0 & -12 & 1-10 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$$\rightarrow -5n_1 = 0$$

$$-16n_2 - 12n_3 = 0$$

$$-12n_2 - 9n_3 = 0$$

Solving this with $n_1^2 + n_2^2 + n_3^2 = 1$ gives us ;

$$n_1 = -\left(\frac{3}{5}\right)e_2 + \left(\frac{4}{5}\right)e_3$$

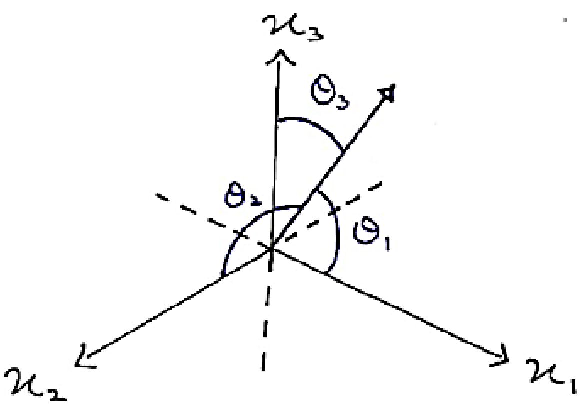
Note that n_1, n_2 and n_2 are direction cosines.

For σ_1, the angles with co-ordinate axis x_1, x_2 and x_3 are ;

$$\theta_1 = \cos^{-1}(0) = 90^\circ$$

$$\theta_2 = \cos^{-1}(-3/5) = 126.868^\circ$$

$$\theta_3 = \cos^{-1}(4/5) = 36.86^\circ$$



Now, we shall apply and extend aforementioned information to 2D-tensor.

Example:

Suppose the stress state at a point is given by:

$$\sigma_{ij} = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix} MPa$$ → Find principal values and directions

Forming the characteristic equation:

$$\begin{vmatrix} 10-T & 3 \\ 3 & 2-T \end{vmatrix} = 0$$

$$(10-T)(2-T) - (9) = 0$$

$$20 - 12T + T^2 - 9 = 0$$

$$T^2 - 12T + 11 = 0$$

$$\underline{T = 11\,MPa = \sigma_1}$$

$$\underline{T = 1\,MPa = \sigma_2}$$

Now for directions;

- Using $T = \sigma_1 = 11\,MPa$

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = 0$$

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 3 & -9 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 3 & -9 & 0 \end{array}\right] -R_1$$

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] R_2 - 3(R_1)$$

$$n_1 - 3n_2 = 0$$

$$\text{or,} \quad n_1 = 3n_2 \quad \Rightarrow \quad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = n_2 \underline{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}$$

Using $T = \sigma_2 = 1\,\text{MPa}$

$$\begin{bmatrix} 10-1 & 3 \\ 3 & 2-1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = 0$$

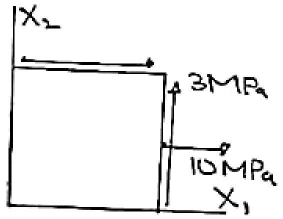
$$\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right] R_1/3$$

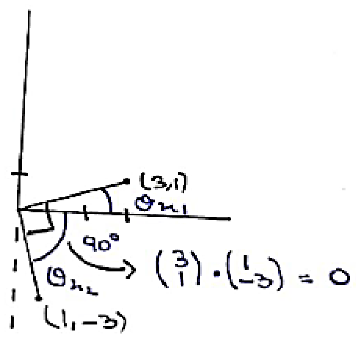
$$\left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] R_2 - R_1$$

$$3n_1 + n_2 = 0$$

$$n_2 = -3n_1 \quad \Rightarrow \quad n_1 \underline{\begin{pmatrix} 1 \\ -3 \end{pmatrix}}$$

Pictorially,





For σ_1

$$\theta_{n_1} = \tan^{-1}(n_2/n_1) = \tan^{-1}(1/3) = \boxed{18.5^\circ}$$

For σ_2

$$\theta_{n_1} = \tan^{-1}(n_2/n_1) = \tan^{-1}(-3) = \boxed{-71.5^\circ}$$

V. Google's Page Rank Algorithm

The use of eigenvalues & eigenvectors has led the google to an extraordinary level of success. With its help it can be easily sorted out the relevant searches and results.

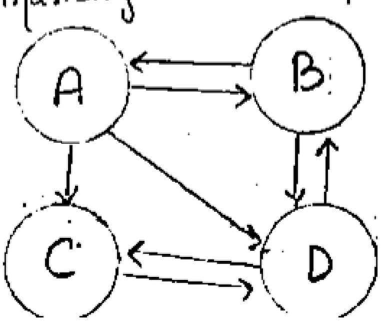
Statement:

Page-rank is a way of measuring the importance of website pages. It works by counting the number & quality of links to a page to determine the importance of the website.

PageRank is simply an algorithm used by Google web search engine to rank its pages & link to measure the importance of that website.

Assuming that a web page contains 4 pages. The author of page A links to all the other pages B, C and D. Meanwhile, the page B links to page D and A and page D links to the pages B and C. The page C only links to the page D.

Summarizing the concept:



Now, here comes the query of finding the most important page from the 4 pages related to the search made. Since A page has more links to the other pages it may be regarded as more important. But every time this is not the case. Other pages can be far more important due to the incoming links.

Formulation & Problem Solving:

Considering the 4 page web as illustrated before, we make a 'link matrix' representing the relative importance of links in & out of the page.

For instance, page A has 3 outgoing links (B, C and D) so for the first coloum we place 1/3 for the rows of B, C & D. In case of page B has 2 link going out to A & D so we place 1/2 in the respective row of the second coloumn. For the case of page C, the only out going link is to page D so we place 1 in the row of page D in coloumn C. Last, the page D is linked to 2 pages B & C so placing 1/2 in the respective rows. The matrix formed will be:

$$L = \begin{vmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{vmatrix}$$

Link Matrix

Now, we will find the eigenvalues:

Using the eq.:

$$p\lambda = \det(\lambda I_n - A) = 0$$

We have:

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1/2 & 0 & 0 \\ 1/3 & -\lambda & 0 & 1/2 \\ 1/3 & 0 & -\lambda & 1/2 \\ 1/3 & 1/2 & 1 & -\lambda \end{vmatrix} \quad C_1 + 2\lambda C_2$$

$$= \begin{vmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 - 2\lambda^2 & -\lambda & 0 & 1/2 \\ 1/3 & 0 & -\lambda & 1/2 \\ \lambda + 1/3 & 1/2 & 1 & -\lambda \end{vmatrix}$$

Expanding along row 1:

$$\Rightarrow (0)(-1)^{1+1} \begin{vmatrix} -\lambda & 0 & 1/2 \\ 0 & -\lambda & 1/2 \\ 1/2 & 1 & -\lambda \end{vmatrix} + (1/2)(-1)^{1+2} \begin{vmatrix} \frac{1}{3} - 2\lambda^2 & 0 & 1/2 \\ 1/3 & -\lambda & 1/2 \\ \lambda + 1/3 & 1 & \lambda \end{vmatrix}$$

$$+ (0)(-1)^{1+3} \begin{vmatrix} 1/3 - 2\lambda^2 & -\lambda & 1/2 \\ 1/3 & 0 & 1/2 \\ \lambda + 1/3 & 1/2 & -\lambda \end{vmatrix} + (0)(-1)^{1+4} \begin{vmatrix} \frac{1}{3} - 2\lambda^2 & \lambda & 0 \\ 1/3 & 0 & -\lambda \\ \lambda + 1/3 & 1/2 & 1 \end{vmatrix}$$

$$= -\frac{\begin{vmatrix} 1/3 - 2\lambda^2 & 0 & 1/2 \\ 1/3 & -\lambda & 1/2 \\ \lambda + 1/3 & 1 & -\lambda \end{vmatrix}}{2} \quad C_1 - (2/3 - 4\lambda^2)C_3$$

$$= \begin{vmatrix} 0 & 0 & 1/2 \\ 2\lambda^2 & -\lambda & 1/2 \\ -4\lambda^3 + \frac{2\lambda}{3} + 1/3 & 1 & -\lambda \end{vmatrix}$$

Expanding again along row 1:

Expanding again along row 1:

$$\Rightarrow (0)(-1)^{1+1}\begin{vmatrix} -\lambda & 1/2 \\ 1 & -\lambda \end{vmatrix} + (1/2)(-1)^{1+3}\begin{vmatrix} 2\lambda^2 & -\lambda \\ -4\lambda^3 + \frac{5\lambda}{3} + 1/3 & 1 \end{vmatrix} + 0(-1)^{1+4}\begin{vmatrix} 2\lambda^2 & 1/2 \\ -4\lambda^3 + \frac{5\lambda}{3} + 1/3 & -\lambda \end{vmatrix}$$

$$= \frac{\begin{vmatrix} 2\lambda^2 & -\lambda \\ -4\lambda^3 + \frac{5\lambda}{3} + 1/3 & 1 \end{vmatrix}}{2}$$

Taking the determinant of the 2×2 matrix & solving:

$$|L - \lambda I| = \lambda^4 - \frac{11\lambda^2}{12} - \frac{\lambda}{12}$$

Further solving to calculate the value of λ by putting $|L - \lambda I| = 0$

$\lambda = 0$

$\lambda = 1$

$\lambda = \frac{-3+\sqrt{6}}{6}$

$\lambda = \frac{-\sqrt{6}+3}{6}$

As we are supposed to use non-negative real values,

So $\lambda = 1$

Here, the eigenvalue becomes 1.

Using the $\lambda = 1$ (eigenvalue) to find the eigenvector:

$$A - I = \begin{bmatrix} -1 & 1/2 & 0 & 0 \\ 1/3 & -1 & 0 & 1/2 \\ 1/3 & 0 & -1 & 1/2 \\ 1/3 & 1/2 & 1 & -1 \end{bmatrix}$$

Changing to reduced echelon form:

$R_1 \times -1$

$$= \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 1/3 & -1 & 0 & 1/2 \\ 1/3 & 0 & -1 & 1/2 \\ 1/3 & 1/2 & 1 & -1 \end{bmatrix} \quad R_2 - \frac{R_1}{3}$$

$$= \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & -5/6 & 0 & 1/2 \\ 1/3 & 0 & -1 & 1/2 \\ 1/3 & 1/2 & 1 & -1 \end{bmatrix} \quad R_3 - \frac{R_1}{3}$$

$$= \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & -5/6 & 0 & 1/2 \\ 0 & 1/6 & -1 & 1/2 \\ 1/3 & 1/2 & 1 & -1 \end{bmatrix} \quad -\frac{6R_2}{5}$$

$$= \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & -3/5 \\ 0 & 1/6 & -1 & 1/2 \\ 0 & 2/3 & 1 & -1 \end{bmatrix} \quad R_1 + \frac{R_2}{2}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -3/10 \\ 0 & 1 & 0 & -3/5 \\ 0 & 1/6 & -1 & 1/2 \\ 0 & 2/3 & 1 & -1 \end{bmatrix} \quad R_3 - \frac{R_2}{6}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -3/10 \\ 0 & 1 & 0 & -3/5 \\ 0 & 0 & -1 & 3/5 \\ 0 & 2/3 & 1 & -1 \end{bmatrix} \quad R_4 - \frac{2R_2}{3}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -3/10 \\ 0 & 1 & 0 & -3/5 \\ 0 & 0 & -1 & 3/5 \\ 0 & 0 & 1 & -3/5 \end{bmatrix} \quad R_3 \times -1$$

$$= \begin{bmatrix} 1 & 0 & 0 & -3/10 \\ 0 & 1 & 0 & -3/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 1 & -3/5 \end{bmatrix} \quad R_4 - R_3$$

$$= \begin{bmatrix} 1 & 0 & 0 & -0.3 \\ 0 & 1 & 0 & -0.6 \\ 0 & 0 & 1 & -0.6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving the matrix, to find the null space:

$$\begin{bmatrix} 1 & 0 & 0 & -0.3 \\ 0 & 1 & 0 & -0.6 \\ 0 & 0 & 1 & -0.6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Taking $x_4 = t$, Then

$x_1 = \frac{3t}{10}$, $x_2 = \frac{3t}{5}$ & $x_3 = \frac{3t}{5}$

$$\vec{x} = \begin{bmatrix} 3t/10 \\ 3t/5 \\ 3t/5 \\ t \end{bmatrix} = \begin{bmatrix} 3/10 \\ 3/5 \\ 3/5 \\ 1 \end{bmatrix} t$$

The nullity of the matrix 1. Hence the page D has the highest pagerank.

This summarizes the use & effect of eigenvalues and eigenvectors in determining the page rank for a particular web search