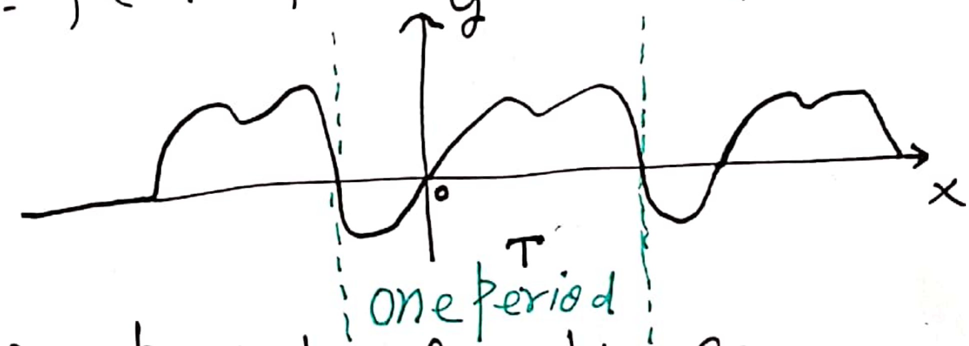
Fourier Series

Key idea: If $f(x)$ is piecewise continuous, periodic function, it could be expressed as a series of sines & cosines.

What is a periodic function?

- $f(x)$ is periodic of period T, if $f(x+T) = f(x)$, for all values of x.



The graph of a periodic function can be divided into 'Vertical strips' that are replicas of each other.

The interval between two successive replicas is called the 'period' of the function.

Mathematically,

$f(x+nT) = f(x)$, for any integer n.

To provide a measure of the number of repitions per unit of t, we define the frequency of a periodic function as $f = \frac{1}{\text{period}} = \frac{1}{T}$ ($\sec^{-1}$ or Hertz)

The term circular frequency is also used, and is defined by

$$\omega = 2\pi f = \frac{2\pi}{T} \text{ (rad/sec)}.$$

Angular frequency ω is a scalar measure of rotation rate. It refers to the angular displacement per unit time (in rotation).

One cycle is one radian, one revolution is equal to 2π radians.

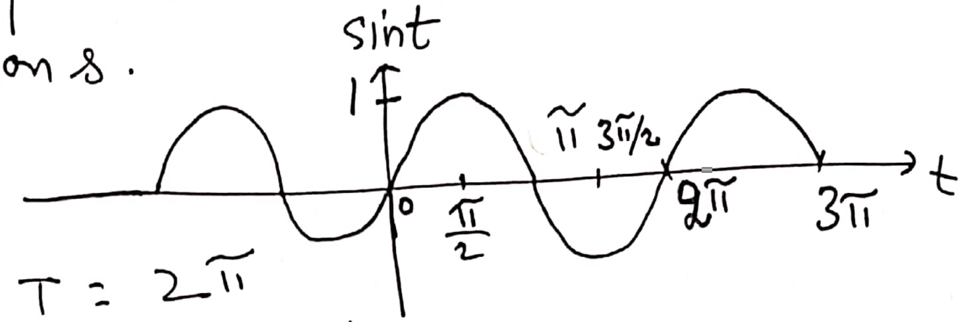
$$\omega = \frac{2\pi}{T} = 2\pi f,$$ T is measure in seconds.

F.S.2

Examples:-

Cosine & Sine are natural periodic functions.

i) $y = \sin t$



period $T = 2\pi$

$\omega = \frac{2\pi}{2\pi} = 1$ (rad/sec).

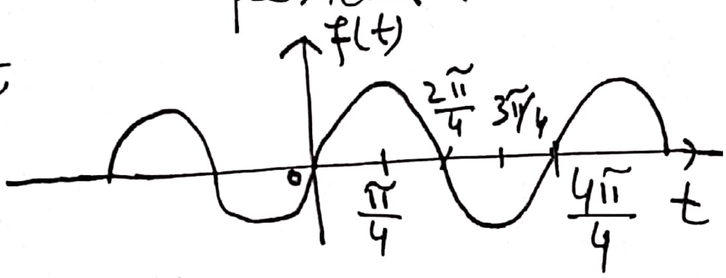
Mathematically, $f(t) = \sin t$

$f(t+T) = \sin(t+T) = f(t)$

$\Rightarrow \quad \sin t = \sin(t+T) = \sin(t+2n\pi)$

$T = 2n\pi$, $n=1$, $T = 2\pi$, fundamental period.

ii). $f(t) = \sin 2t$

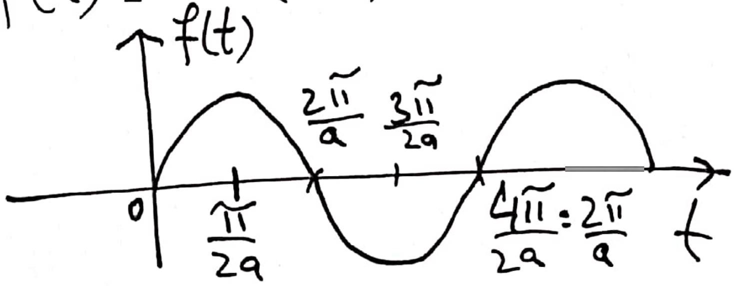


Period $T = \pi$ rad

$\omega = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$ (rad/sec)

In general, if $f(t) = \sin(at)$, then Period:

$T = \frac{2\pi}{a}$



If $f(t) = \sin(\frac{t}{2})$, Period: $T = \frac{2\pi}{1/2} = 4\pi$

All above is true for Cosat.

Conclusion

If $f(t) = \sin at$ 'or' $\cos at$

Period: $T = \frac{2\pi}{a}$, $w = \frac{2\pi}{T} = \frac{2\pi}{2\pi/a} = a$

EX: i) Find the period of the function

$$f(t) = \cos\left(\frac{t}{3}\right) + \cos\left(\frac{t}{4}\right).$$

$$\frac{2\pi}{1/3} = 6\pi, \quad \frac{2\pi}{1/4} = 8\pi$$

$$T = LCM(6\pi, 8\pi) = 24\pi.$$

OR: $6\pi N_1 = 8\pi N_2$

$N_1/N_2 = 8/6 = 4/3$, $N_1 = 4, N_2 = 3$

Giving, $T = 4(6\pi) = 8(3\pi) = 24\pi$.

(ii). $f(t) = \cos\left(\frac{3\pi}{5}t + \frac{\pi}{2}\right) + \sin\left(\frac{5\pi}{3}t + \frac{\pi}{3}\right) + \cos\left(\frac{\pi}{15}t\right)$.

$w_1 = \frac{3\pi}{5}$, $w_2 = \frac{5\pi}{3}$, $w_3 = \frac{\pi}{15}$ in rad/sec.

$T_1 = \frac{10}{3}$ Sec, $W_2 = \frac{6}{5}$ Sec, $T_3 = 30$ Sec (using $T = 2\pi/w$).

$\frac{T_1}{T_2} = \frac{25}{9}$, $T_1/T_3 = 1/9 \Rightarrow f$ is periodic.

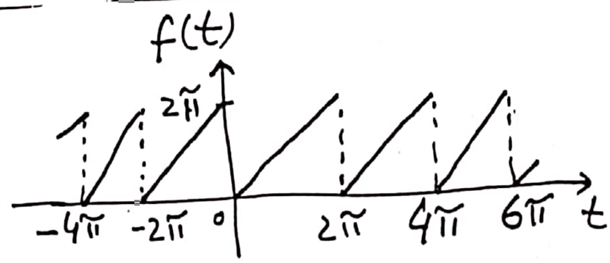
$T = LCM(10/3, \frac{6}{5}, 30) = 30$ Sec.

$w_0 = \frac{2\pi}{T} = \frac{2\pi}{30} = \frac{\pi}{15}$ rad/sec

is the fundamental frequency.

Examples of other periodic functions

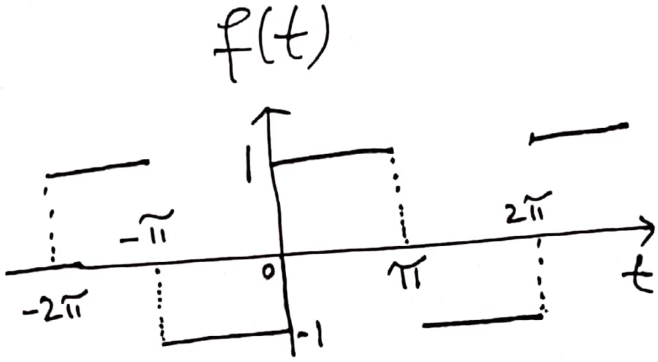
i). $f(t) = t, \; 0 < t < 2\pi$



$f(t + 2\pi) = f(t),$ for all t.

$T = 2\pi, \quad \omega = 1$

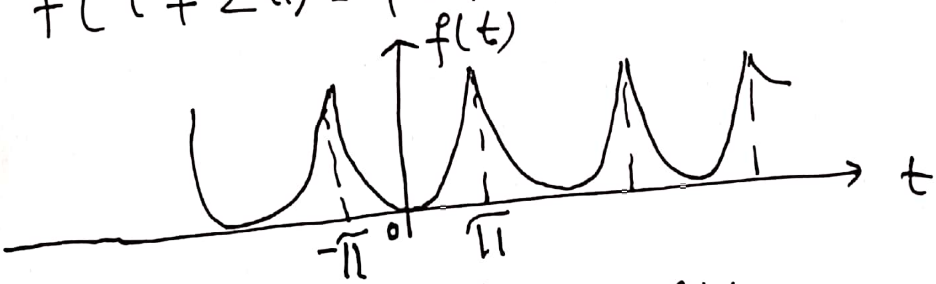
ii). $f(t) = \begin{cases} -1, & -\pi < t < 0 \\ +1, & 0 < t < \pi. \end{cases}$



$f(t + 2\pi) = f(t),$ for all t.

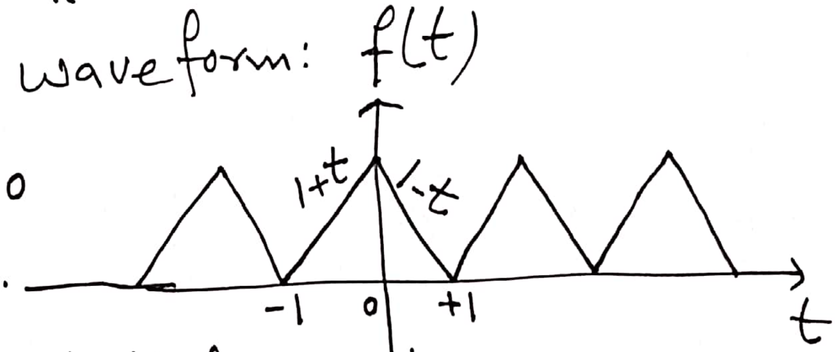
iii). $f(t) = t^2 \; (-\pi < t < \pi)$

$f(t + 2\pi) = f(t),$ for all values of t.



iv). Triangular waveform:

$f(t) = \begin{cases} 1 + t, & -1 < t < 0 \\ 1 - t, & 0 < t < 1. \end{cases}$



$f(t + 2) = f(t),$ for all t.

In this case, $T = 2, \; \omega = \frac{2\pi}{2} = \pi$.

Fourier Series (Trigonometric Form)

Let $f(t)$ be periodic of period T,

$d \le t \le d+T$, $\omega = 2\pi/T$

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right) \quad \text{— (i)}$$

where the Fourier Coefficients a_0, a_n, b_n are

$$a_0 = \frac{2}{T}\int_d^{d+T} f(t)\,dt \quad \text{— (ii)}$$

$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos(n\omega t)\,dt \quad \text{— (iii)}$$

$n = 1, 2, 3, \ldots$

$$b_n = \frac{2}{T}\int_d^{d+T} f(t)\sin(n\omega t)\,dt \quad \text{— (iv)}$$

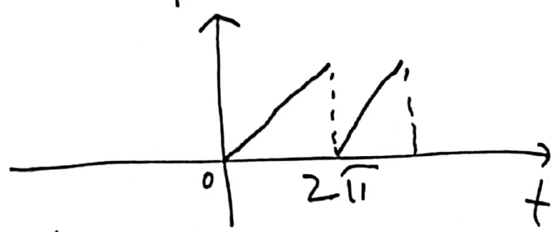
(i) gives the Trigonometric Form of Fourier Series (TFS), where a_0, a_n and b_n are calculated based on integrals (ii) — (iv)

Ex:- obtain the Fourier series expansion of the periodic function

$f(t) = t \ (0<t<2\pi)$,

$f(t+2\pi) = f(t)$

for all t.



Sol:- $T = 2\pi$, $w = \frac{2\pi}{2\pi} = 1$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

we calculate, $a_0 = \frac{2}{2\pi}\int_0^{2\pi} t\,dt = \frac{1}{\pi}\left[\frac{t^2}{2}\right]_0^{2\pi}$

$$a_0 = \frac{1}{2\pi}\left[(2\pi)^2 - (0)^2\right] = 2\pi .$$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} \underset{I}{t}\ \underset{II}{\cos nt}\, dt = \frac{1}{\pi}\left[t\,\frac{\sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left(\frac{2\pi}{n}\sin(2n\pi) + \frac{1}{n^2}\cos(2n\pi) - \frac{\cos 0}{n^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{2\pi}{n}(0) + \frac{1}{n^2}(1-1)\right) = 0 .$$

Also,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \underset{I}{t}\ \underset{II}{\sin(nt)}\, dt = \frac{1}{\pi}\left[-\frac{t}{n}\cos nt + \frac{\sin nt}{n^2}\right]_0^{2\pi}$$

$$b_n = \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos 2n\pi + 0\right] = \frac{-2}{n} .$$

Hence, the TFS expansion of $f(t)$ is

$$f(t) = \frac{1}{2}(2\pi) + \sum_{n=1}^{\infty} -\frac{2}{n}\sin(nt) .$$

EX:- Obtain the Fourier Series expansion of the periodic function:

$f(t) = t^2, \quad (-\pi < t < \pi),$

$f(t+2\pi) = f(t)$, for all t.

Sol: Here, $T = 2\pi$, $w = 1$,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).$$

where,

$$a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{1}{\pi}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{3\pi}\left[(\pi)^3 - (-\pi)^3\right] = \frac{1}{3\pi}\left[2\pi^3\right] = \frac{2}{3}\pi^2.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\mathrm{I}}{t^2} \underset{\mathrm{II}}{\cos nt}\,dt = \frac{1}{\pi}\left[\cancel{\left[\frac{t^2 \sin nt}{n}\right]_{-\pi}^{\pi}} \to 0 - \int_{-\pi}^{\pi}(2t)\frac{\sin nt}{n}\,dt\right].$$

$$a_n = -\frac{2}{n\pi}\int_{-\pi}^{\pi} \underset{\mathrm{I}}{t} \underset{\mathrm{II}}{\sin(nt)}\,dt$$

$$= -\frac{2}{n\pi}\left[t\frac{(-\cos nt)}{n}\Bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(1)\left(\frac{-\cos nt}{n}\right)dt\right].$$

$$= \frac{2}{n^2\pi}\left[\pi\cos n\pi - (-\pi\cos n\pi)\right] - 0$$

$$= \frac{2}{n^2\pi}\left[2\pi\cos n\pi\right] = \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n.$$

Now, we calculate b_n as

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} \underset{I}{t^2}\, \underset{II}{\sin(nt)}\,dt$$

$$= \frac{1}{\pi}\left[t^2\left(\frac{-\cos nt}{n}\right)\Bigg|_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}(2t)\left(-\frac{\cos nt}{n}\right)dt\right.$$

$$= -\frac{1}{n\pi}\left[\cancelto{0}{\pi^2\cos n\pi - \pi^2\cos(-n\pi)}\right] + \frac{2}{n\pi}\int_{-\pi}^{\pi} \underset{I}{t}\,\underset{II}{\cos nt}\,dt$$

$$= 0 + \frac{2}{n\pi}\left[\cancelto{0}{t\,\frac{\sin(tn)}{n}}\Bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(1)\,\frac{\sin(nt)}{n}\,dt\right]$$

$$= \left(\frac{2}{n\pi}\right)\left(-\frac{1}{n}\right)\int_{-\pi}^{\pi}\sin(nt)\,dt$$

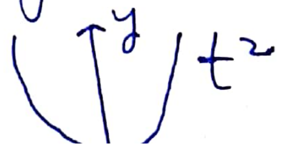
$$= -\frac{2}{n^2\pi}\left[-\frac{\cos nt}{n}\right]_{-\pi}^{\pi} = \frac{2}{n^3\pi}\left(\cancelto{0}{\cos n\pi - \cos(-n\pi)}\right)$$

giving $b_n = 0$. Therefore, TFS is

$$f(t) = \frac{1}{2}\left(\frac{2}{3}\pi^2\right) + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nt.$$

Note: Does b_n is accidently zero here?

Observation: $f(t) = t^2$, is



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$$= -\frac{1}{n\pi}\left[\cancel{\pi^2\cos n\pi - \pi^2\cos(-n\pi)}\right]^{\to 0} + \frac{2}{n\pi}\int_{-\pi}^{\pi}\underset{I}{t}\,\underset{II}{\cos nt}\,dt.$$

$$= 0 + \frac{2}{n\pi}\left[\cancel{t\,\frac{\sin(tn)}{n}}^{\to 0}\Bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(1)\frac{\sin(nt)}{n}dt\right].$$

$$= \left(\frac{2}{n\pi}\right)\left(-\frac{1}{n}\right)\int_{-\pi}^{\pi}\sin(nt)\,dt.$$

$$= -\frac{2}{n^2\pi}\left[-\frac{\cos nt}{n}\right]_{-\pi}^{\pi} = \frac{2}{n^3\pi}\left(\cancel{\cos n\pi - \cos(-n\pi)}\right)^{\to 0}$$

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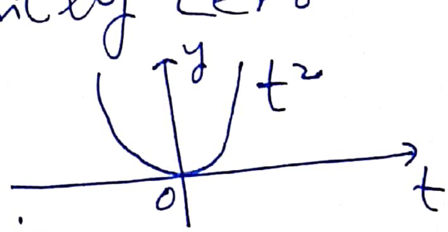
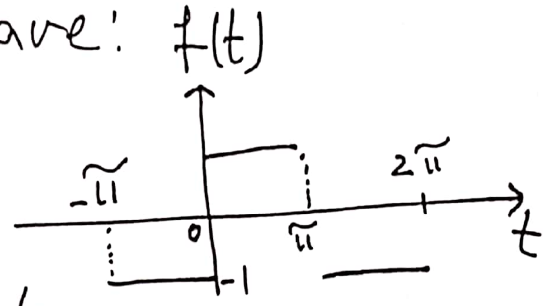


Observation: $f(t) = t^2$, is symmetric about y-axis, an even function, i.e., $f(-t) = f(t)$, $f(t)$ has only cosine terms in TFS representation.

F.S.9

Ex: Square periodic wave:



$$f(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases}$$

$f(t+2\pi) = f(t)$, for all t.

Obtain the F.S. expansion of $f(t)$.

Sol: Here, $T = 2\pi$, $\omega = 1$, giving

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt.$$

We calculate,

$$a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\,dt + \int_{0}^{\pi}(1)\,dt\right]$$

$$= \frac{1}{\pi}\left[-(t)\Big|_{-\pi}^{0} + (t)\Big|_{0}^{\pi}\right] = \frac{1}{\pi}\left[-(0+\pi)+(\pi-0)\right] = 0.$$

$$a_n = \frac{2}{2\pi}\left[\int_{-\pi}^{0}(-1)\cos nt\,dt + \int_{0}^{\pi}(1)\cos nt\,dt\right]$$

$$= \frac{1}{\pi}\left[-\frac{\sin nt}{n}\Big|_{-\pi}^{0} + \frac{\sin nt}{n}\Big|_{0}^{\pi}\right] = 0$$

$$b_n = \frac{2}{2\pi}\left[\int_{-\pi}^{0}(-1)\sin nt\,dt + \int_{0}^{\pi}(1)\sin nt\,dt\right].$$

$$= -\frac{1}{\pi}\left[-\frac{\cos nt}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{\cos nt}{n}\right]_{0}^{\pi}\Big\}.$$

$$= \frac{1}{n\pi}\left[1-\cos n\pi\right] - \frac{1}{n\pi}\left[\cos n\pi - 1\right]$$

$$= \frac{1}{n\pi}\left[1-\cos n\pi - \cos n\pi + 1\right] = \frac{2}{n\pi}\left[1-\cos n\pi\right].$$

$$b_n = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0, & n \text{ is even} \\ \frac{4}{n\pi}, & n \text{ is odd.} \end{cases}$$

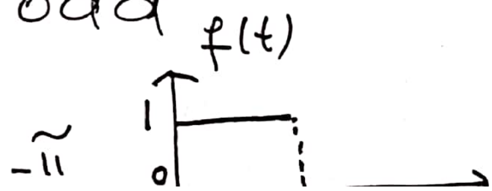
Thus the Fourier series expansion of $f(t)$ is

$$f(t) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\cdots\right)$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{2n-1}.$$

Observation: In this case, a_0 & a_n are zero, and the Fourier series of $f(t)$ has only sine terms.

In this case, $f(t)$ is an odd function, i.e., $f(-t) = -f(t)$



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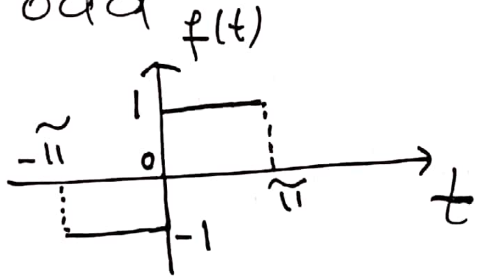
$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{2n-1}.$$

Observation: In this case, a_0 & a_n are Zero, and the Fourier Series of $f(t)$ has only Sine terms.

In this case, $f(t)$ is an odd function, i.e, $f(-t) = -f(t)$



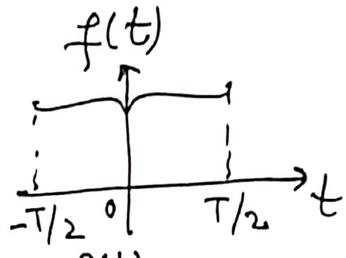
Conclusion:

If $f(t)$ is even, we have Fourier Cosine Series representation.

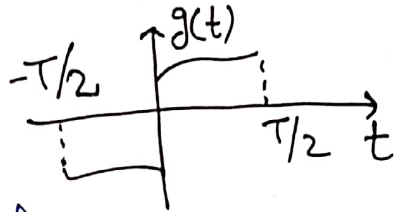
If $f(t)$ is an odd function, we have Fourier Sine Series representation.

Even & odd functions:

If $f(t)$ is even, $\int_{-T/2}^{T/2} f(t)dt = 2\int_{0}^{T/2} f(t)dt$



If $g(t)$ is odd, $\int_{-T/2}^{T/2} g(t)dt = 0$.



Result: Product of two odd/even functions is even.
Product of an even & an odd functions is odd.

Impact on Fourier Series:

If $f(t)$ is periodic of period T, $-\frac{T}{2} < t < \frac{T}{2}$,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t), \quad \text{——(i)}$$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt, \quad \text{——(ii)}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt \quad \text{——(iii)}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt \quad \text{——(iv)}$$

If $f(t)$ is even, $a_0 = \frac{4}{T}\int_{0}^{T/2} f(t)dt$

$$a_n = \frac{4}{T}\int_{0}^{T/2} f(t)\cos(n\omega t)dt$$

$b_n = 0$

If $f(t)$ is odd, $a_0 = 0$, $a_n = 0$,

$$b_n = \frac{4}{T}\int_{0}^{T/2} f(t)\sin(n\omega t)dt.$$

F.S.12

Example (Revisit)

i) $f(t) = t^2 \quad (-\pi < t < \pi)$

$f(t+2\pi) = f(t)$, for all t.

$f(-t) = (-t)^2 = t^2 = f(t)$, f is even

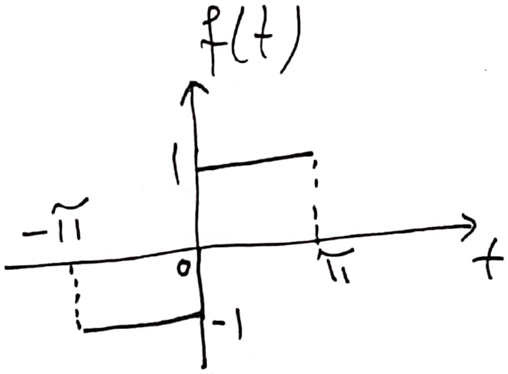
$$a_0 = \frac{4}{2\pi}\int_0^{\pi} t^2\,dt = \frac{2}{3}\pi^2$$

$$a_n = \frac{4}{2\pi}\int_0^{\pi} t^2 \cos nt\,dt = \frac{4}{n^2}(-1)^n$$

$$b_n = 0$$

ii) square wave (Periodic)

$$f(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases}$$



$f(t+2\pi) = f(t)$,

f is odd function.

$a_0 = 0$, $a_n = 0$ (Directly)

$$b_n = \frac{4}{2\pi}\int_0^{\pi} f(t)\sin(nt)\,dt$$

$$= \frac{2}{n\pi}\left[1 - \cos n\pi\right]$$

Confirm same as page 8 — 11

Functions defined on an arbitrary interval:

Ex: Expand $f(x) = x - x^2, \; -1 < x < 1, \; f(x+2) = f(x)$, as a fourier series.

Solution: Here, $T = 2$, $\omega = \frac{2\pi}{2} = \pi$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x),$$

$$a_0 = \frac{2}{2}\int_{-1}^{1}(x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1} = -\frac{2}{3}.$$

$$a_n = \frac{2}{2}\int_{-1}^{1}\underset{I}{(x - x^2)}\,\underset{II}{\cos(n\pi x)}\,dx = \left[(x - x^2)\frac{\sin(n\pi x)}{n\pi}\right]_{-1}^{1}$$

$$- \int_{-1}^{1}\underset{I}{(1 - 2x)}\,\underset{II}{\frac{\sin(n\pi x)}{n\pi}}\,dx.$$

$$a_n = 0 - \frac{1}{n\pi}\left[(1 - 2x)\,\frac{-\cos(n\pi x)}{n\pi}\right]_{-1}^{1} + \frac{1}{n\pi}\int_{-1}^{1}(0 - 2)\,\frac{-\cos(n\pi x)}{n\pi}\,dx.$$

$$a_n = \frac{1}{n^2\pi^2}\left[-\cos n\pi - 3\cos n\pi\right] + \frac{2}{n^2\pi^2}\left[\frac{\sin(n\pi x)}{n\pi}\right]_{-1}^{1}$$

$$a_n = \frac{1}{n^2\pi^2}\left[-4\cos n\pi\right] + 0 = \frac{-4(-1)^n}{n^2\pi^2} = \frac{4(-1)^{n+1}}{n^2\pi^2}.$$

$$b_n = \frac{2}{2}\int_{-1}^{1}\underset{I}{(x - x^2)}\,\underset{II}{\sin(n\pi x)}\,dx$$

$$= \left[(x - x^2)\,\frac{-\cos n\pi x}{n\pi}\right]_{-1}^{1} - \int_{-1}^{1}\underset{I}{(1 - 2x)}\,\underset{II}{\frac{-\cos(n\pi x)}{n\pi}}\,dx.$$

$$= -\frac{1}{n\pi}\left[0 + 2\cos(n\pi)\right] + \frac{1}{n\pi}\left[(1 - 2x)\,\frac{\sin(n\pi x)}{n\pi}\right]_{-1}^{1}$$

$$- \frac{1}{n\pi}\int_{-1}^{1}(0 - 2)\,\frac{\sin n\pi x}{n\pi}\,dx\Big].$$

$$b_n = \frac{-2(-1)^n}{n\pi} + 0 + \frac{2}{n^2\pi^2}\int_{-1}^{1} \sin(n\pi x)\,dx.$$

$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{2}{n^2\pi^2}\left[-\frac{\cos(n\pi x)}{n\pi}\right]_{-1}^{1}$$

$$= \frac{2(-1)^{n+1}}{n\pi} - \frac{2}{n^3\pi^3}\left[\cos(n\pi x)\right]_{-1}^{1}$$

$$b_n = \frac{2(-1)^{n+1}}{n\pi} - \frac{2}{n^3\pi^3}\left[\cos n\pi - \cos(-n\pi)\right]$$

$$b_n = \frac{2(-1)^{n+1}}{n\pi} - \frac{2}{n^3\pi^3}\left[(-1)^n - (-1)^n\right]$$

$$b_n = \frac{2(-1)^{n+1}}{n\pi} - 0 = \frac{2(-1)^{n+1}}{n\pi}.$$

Thus, F.S representation of $f(x)$ is

$$f(x) = -\frac{1}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^{n+1}}{n^2\pi^2}\cos(n\pi x) + \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x)\right]$$

$$= -\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x).$$