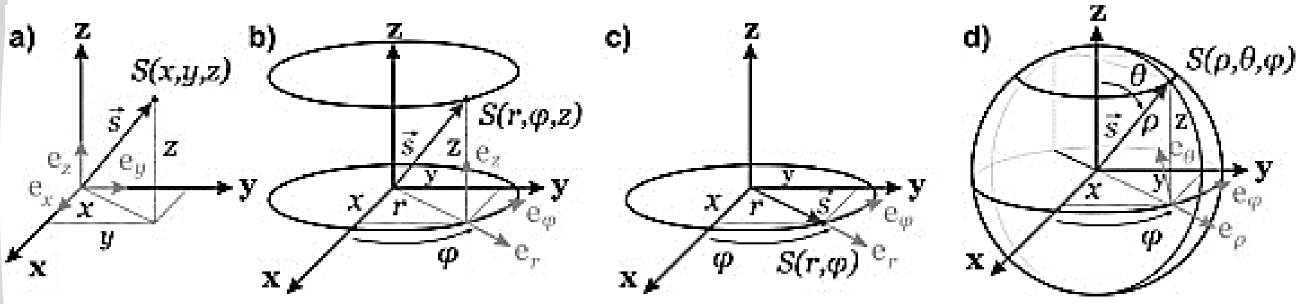
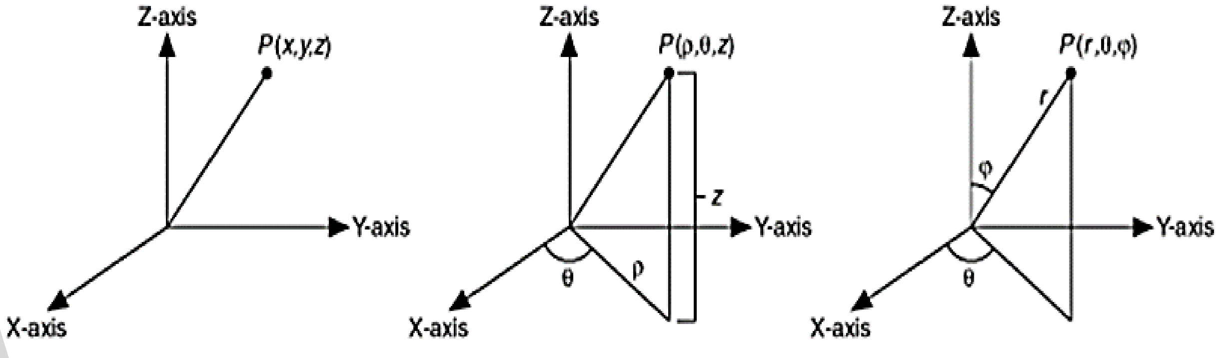
Coordinate Systems

- Rectangular or Cartesian
- Cylindrical
- Spherical





Cartesian Coordinate | Cylindrical Coordinate | Spherical Coordinate

Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

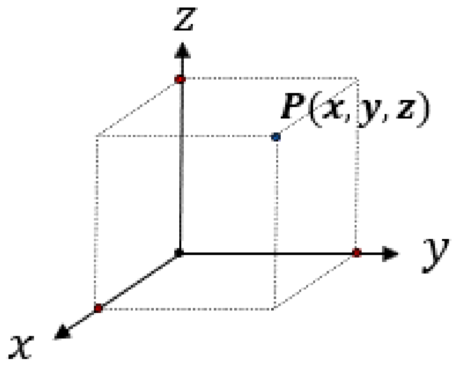
Orthogonal Coordinate Systems:

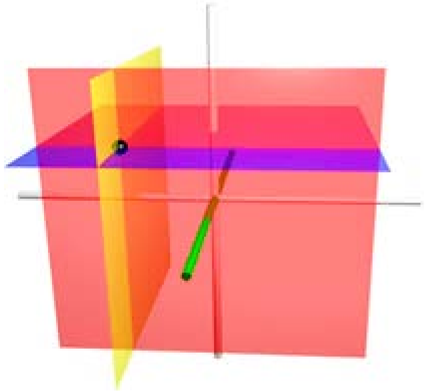
1. **Cartesian Coordinates**

Or

Rectangular Coordinates

$\boldsymbol{P}\ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$

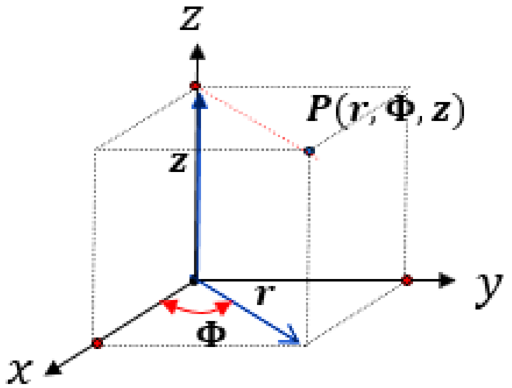


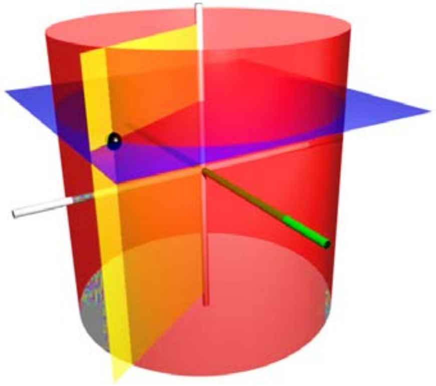


2. **Cylindrical Coordinates**

$\boldsymbol{P}\ (\boldsymbol{r}, \boldsymbol{\Phi}, \boldsymbol{z})$

$$\boxed{\begin{aligned} x &= r\cos\Phi, \\ y &= r\sin\Phi, \\ z &= z. \end{aligned}}$$

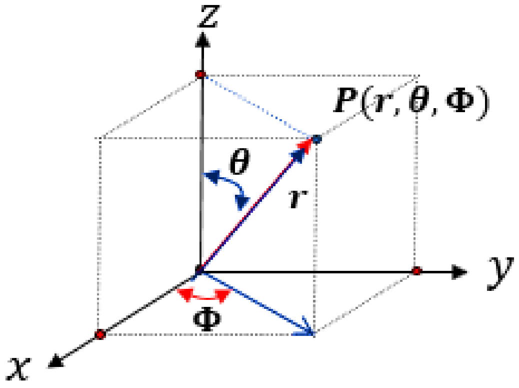


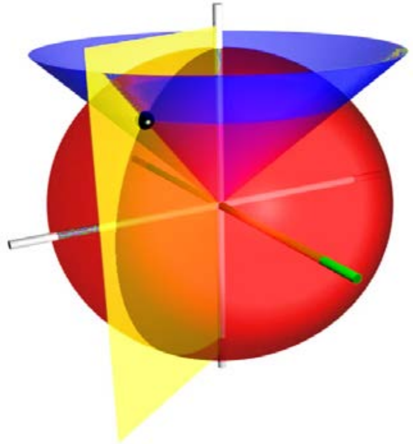


3. **Spherical Coordinates**

$\boldsymbol{P}\ (\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\Phi})$

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\Phi\,, \\ y &= r\sin\theta\sin\Phi, \\ z &= r\cos\theta\,. \end{aligned}}$$





12

VECTORS AND THE GEOMETRY OF SPACE

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

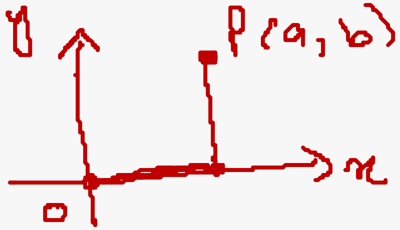
Section: 12.1

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

Section: 12.1

2-dimensional Coordinate Systems

To locate a point in a plane, two numbers are required.



- We know that any point in the plane can be represented as an ordered pair (a, b) of real numbers—where a is the x −coordinate and b is the y −coordinate.

- For this reason, a plane is called two-dimensional.

$\{(\underset{\hat{i}}{x}, \underset{\hat{j}}{y})\}$

$\{\langle 1, 0\rangle, \langle 0, 1\rangle\}$

$\mathbb{R}^2 = \underset{x}{\mathbb{R}} \times \underset{y}{\mathbb{R}}$

$(x, y) \neq (y, x)$

3-dimensional Coordinate Systems

To locate a point in space, three numbers are required.

- We represent any point in space by an ordered triple (a, b, c) of real numbers.

3-d Coordinate Systems

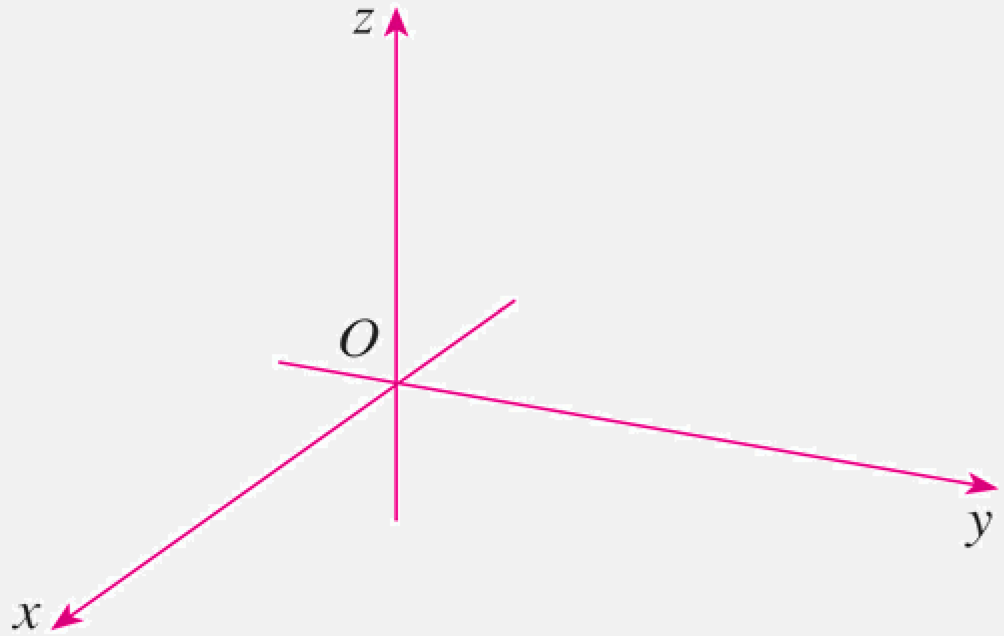
In order to represent points in space, we first choose:

- A fixed point O (the origin).
- Three directed lines through O that are perpendicular to each other.
- The three lines are called the ***coordinate axes***. They are labeled as:
  - x −axis
  - y −axis
  - z −axis

Note: Usually we think of the $x-$ and $y-$axes as being horizontal and the z-axis as being vertical.

Coordinate Axes

We draw the orientation of the axes as shown:

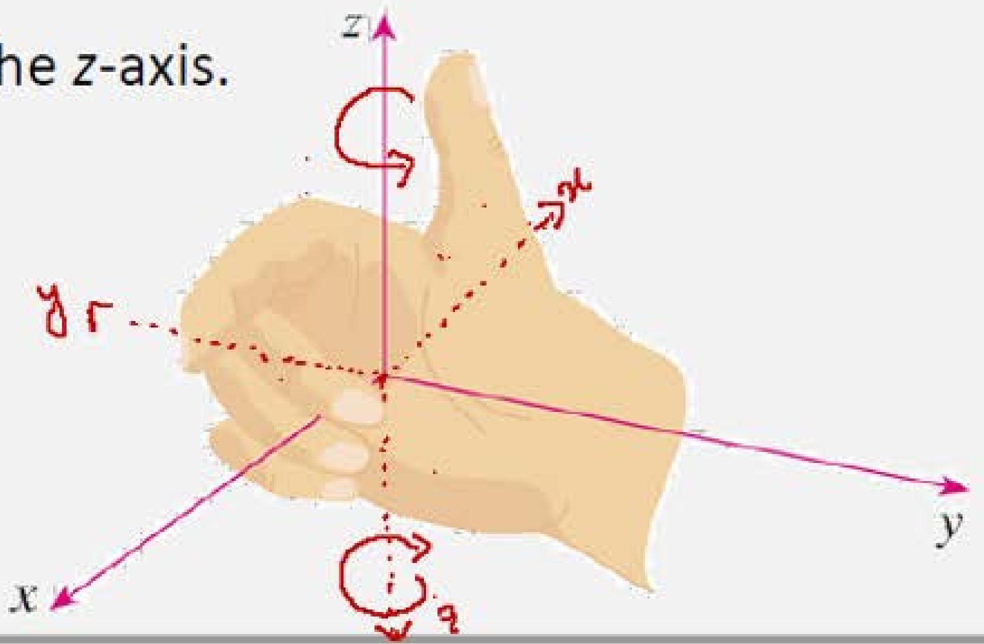


Coordinate Axes

The direction of the z −axis is determined by the right-hand rule, illustrated as follows:

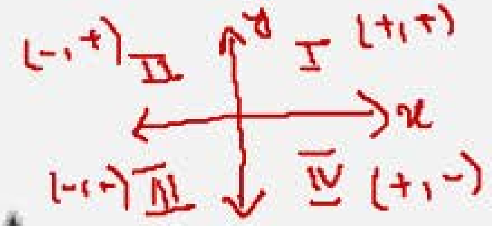
- Curl the fingers of right hand around the z −axis in the direction of a 90° counterclockwise rotation from the positive x −axis to the positive y −axis.

- Then, thumb points in the positive direction of the z-axis.

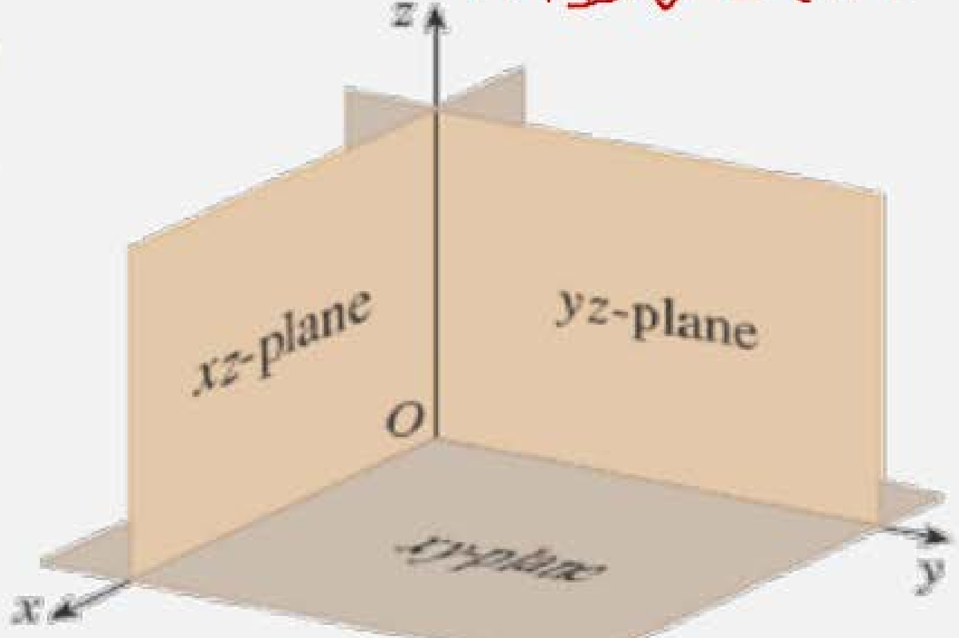


Coordinate Planes

The three coordinate axes determine the three coordinate planes.

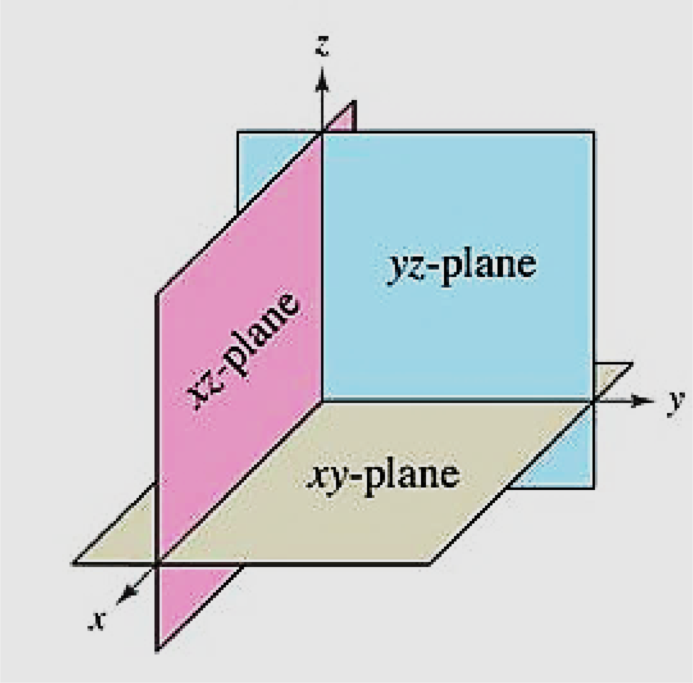


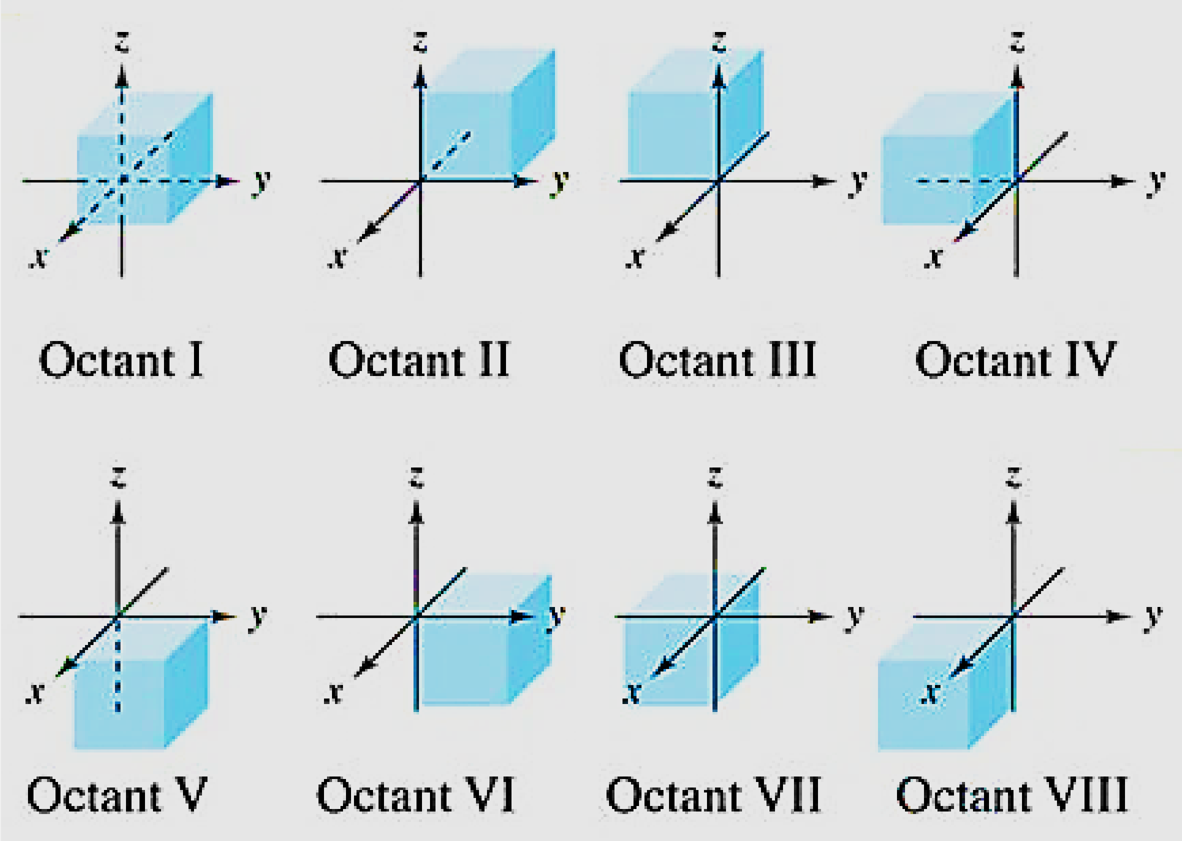
- The xy −plane contains the x − and y −axes.
- The yz −plane contains the y − and z −axes.
- The xz −plane contains the x − and z −axes.



Coordinate Planes & Octants

The coordinate planes divide space into eight parts, called octants. The first octant, in the foreground, is determine by the positive axes.





3-D Coordinate Systems

Now, let P be any point in space, and

- a is the (directed) distance from the yz $-$plane to P.
- b be the distance from the xz $-$plane to P.
- c be the distance from the xy $-$plane to P.

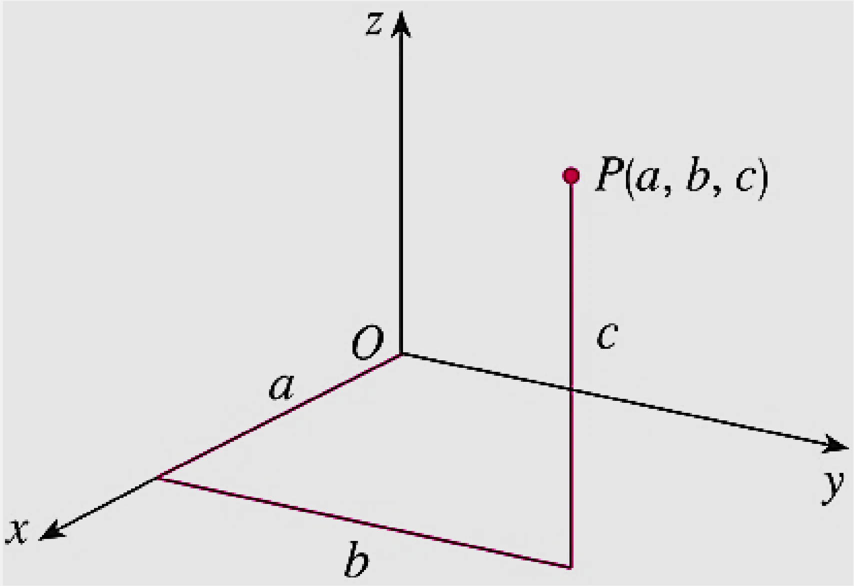
We represent the point P by the ordered triple of real numbers (a, b, c). We call a, b, and c the coordinates of P where:

- a is the x $-$coordinate.
- b is the y $-$coordinate.
- c is the z $-$coordinate.

3-D Coordinate Systems

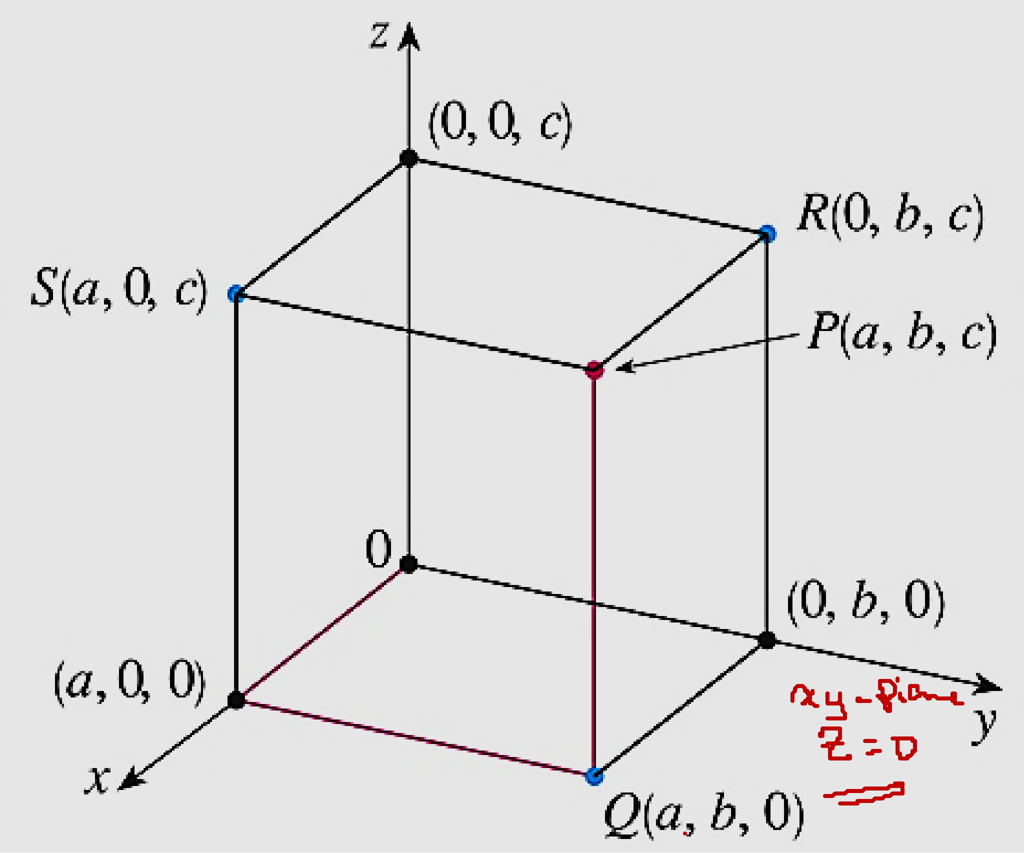
Thus, to locate the point $P(a, b, c)$, we can start from the origin O and proceed as follows:

- First, move a units along the x −axis.
- Then, move b units parallel to the y −axis.
- Finally, move c units parallel to the z −axis.



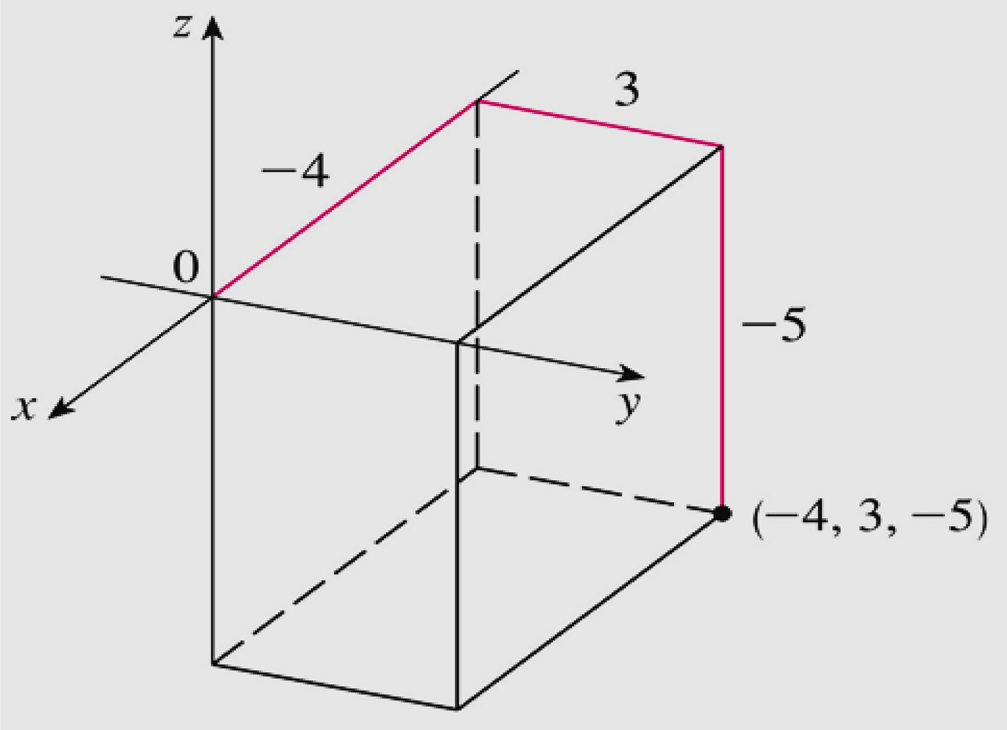
3-D Coordinate Systems & Projections

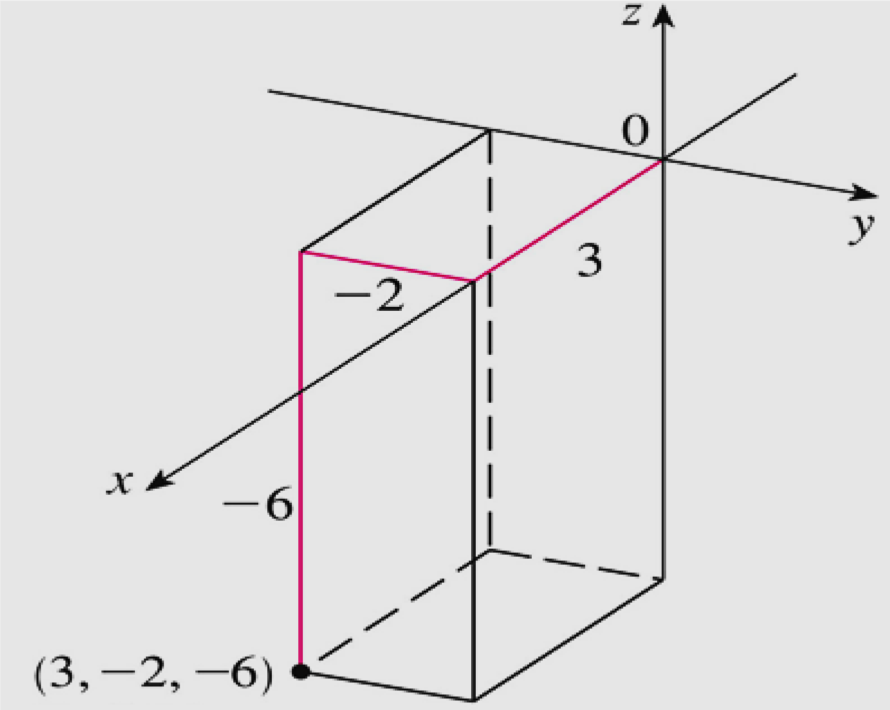
- The point $P(a,b,c)$ determines a rectangular box.
- If we drop a perpendicular from P to the xy −plane, we get a point Q with coordinates $(a,b,0)$. This is called the **projection** of P on the xy −plane.
- Similarly, $R(0,b,c)$ and $S(a,0,c)$ are the projections of P on the yz −plane and xz −plane, respectively.



3-D Coordinate Systems

As numerical illustrations, the points $(-4, 3, -5)$ and $(3, -2, -6)$ are plotted here.





3-D Coordinate Systems

In general, the Cartesian product:

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) | x, y, z \in \mathbb{R}\},$$

is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$.

$\hat{i} = \hat{e}_1 = \langle 1, 0, 0 \rangle$

$\hat{j} = \hat{e}_2 = \langle 0, 1, 0 \rangle$

$\hat{k} = \hat{e}_3 = \langle 0, 0, 1 \rangle$

3-D Rectangular Coordinate System

We have given a one-to-one correspondence between points P in space and ordered triples (a, b, c) in $\mathbb{R}^3$.

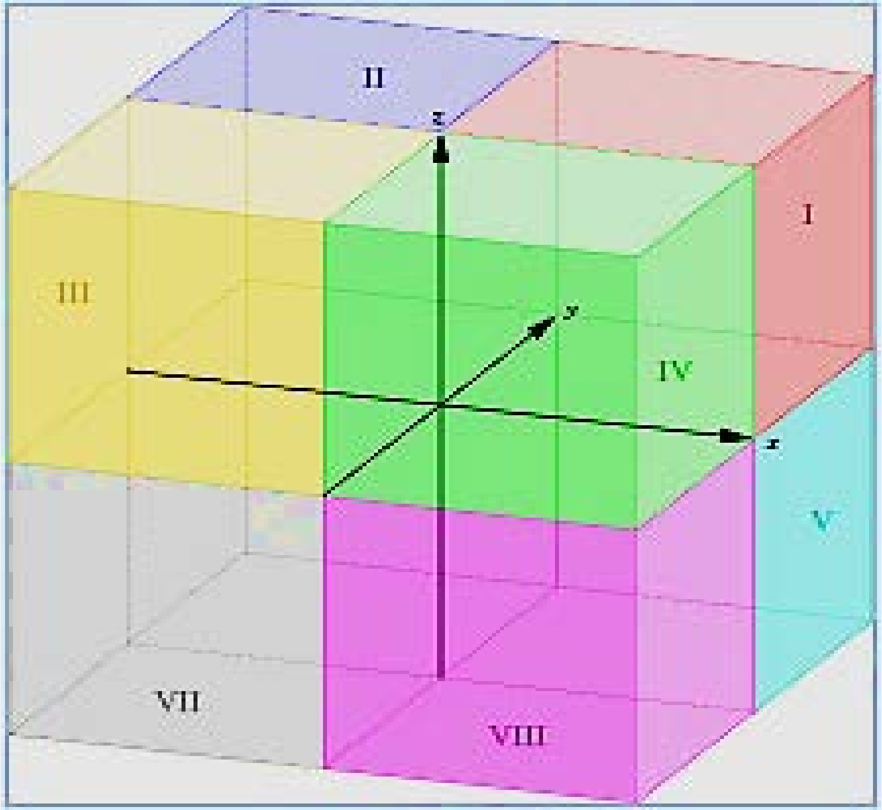
- It is called a 3-D rectangular coordinate system.

- Note that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

3-D Rectangular Coordinate System

Other octants can be described as below:

Number	Name	x	y	z
I	top-front-right	+	+	+
II	top-back-right	−	+	+
III	top-back-left	−	−	+
IV	top-front-left	+	−	+
V	bottom-front-right	+	+	−
VI	bottom-back-right	−	+	−
VII	bottom-back-left	−	−	−
VIII	bottom-front-left	+	−	−



2-D Vs. 3-D Analytic Geometry

- In 2-D analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$.

- In 3-D analytic geometry, an equation in x, y, and z represents a surface in $\mathbb{R}^3$.

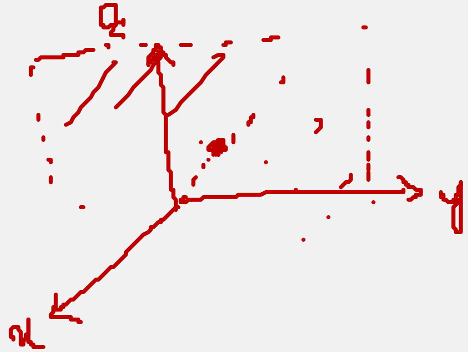
$x = -1$ in $\mathbb{R}^3$

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) = (-1,y,z)\}$

$x = -1$ in $\mathbb{R}^2$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

$= \{(x,y) = (-1,y)\}$



Example:

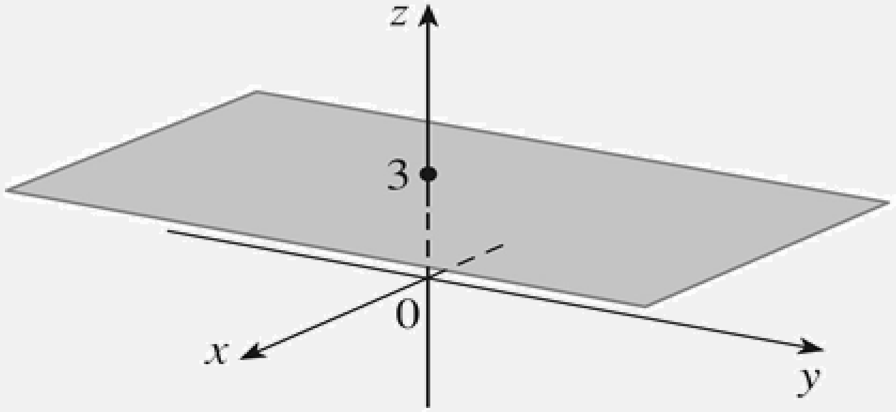
What surfaces in $\mathbb{R}^3$ are represented by the following equations?

a. $z = 3$
b. $y = 5$

Solution (a):

The equation $z = 3$ represents the set $\{(x, y, z) \mid z = 3\}$.

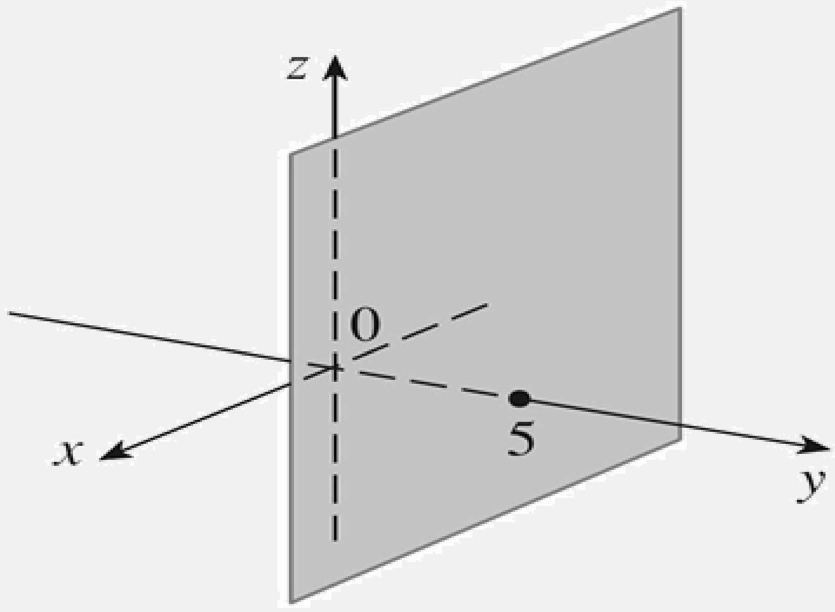
- This is the set of all points in $\mathbb{R}^3$ whose z −coordinate is 3.
- This is the horizontal plane that is parallel to the xy −plane and three units above it.



(a) $z = 3$, a plane in $\mathbb{R}^3$

Solution (b):

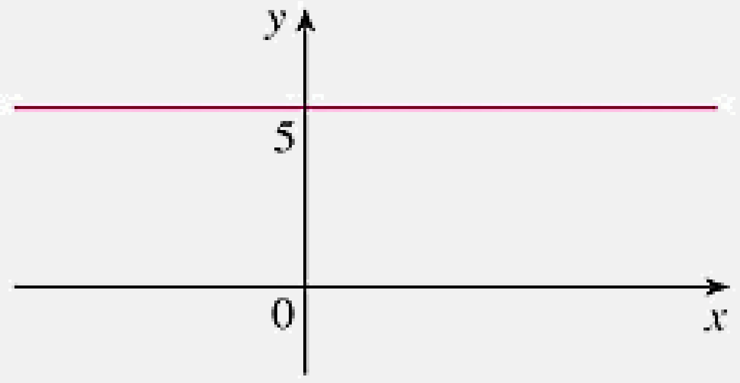
The equation $y = 5$ represents the set of all points in $\mathbb{R}^3$ whose y −coordinate is 5. This is the vertical plane that is parallel to the xz −plane and five units to the right of it.



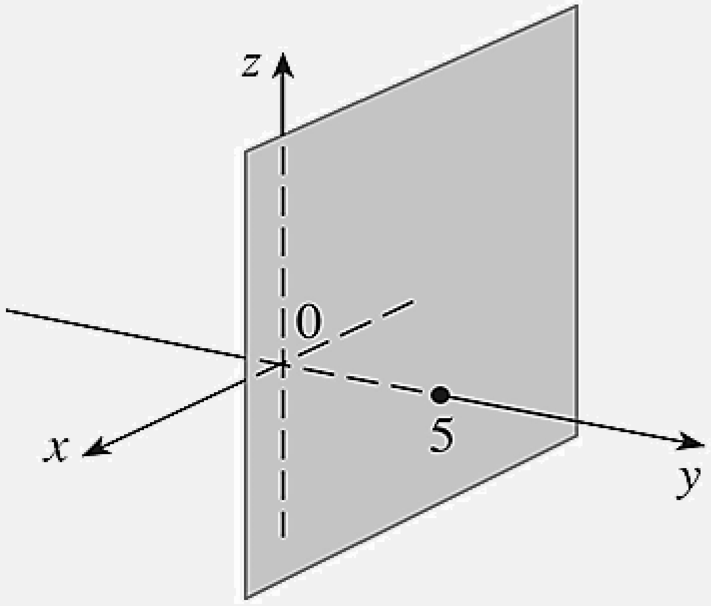
(b) $y = 5$, a plane in $\mathbb{R}^3$

Note:

When an equation is given, we must understand from the context whether it represents a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$. In Example, $y = 5$ represents a plane in $\mathbb{R}^3$. However, $y = 5$ can also represent a line in $\mathbb{R}^2$ if we are dealing with two-dimensional analytic geometry.



(c) $y = 5$, a line in $\mathbb{R}^2$



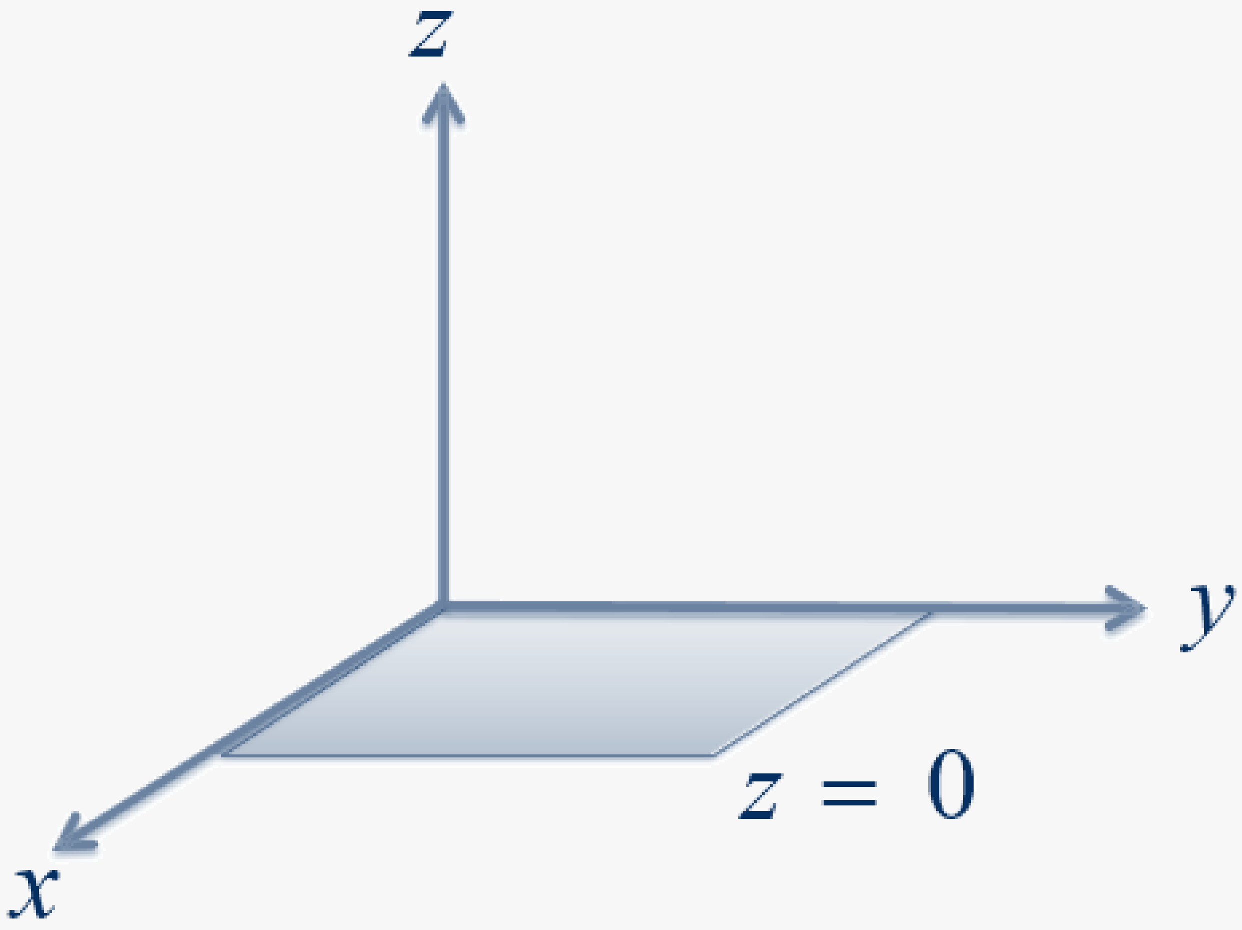
(b) $y = 5$, a plane in $\mathbb{R}^3$

Note:

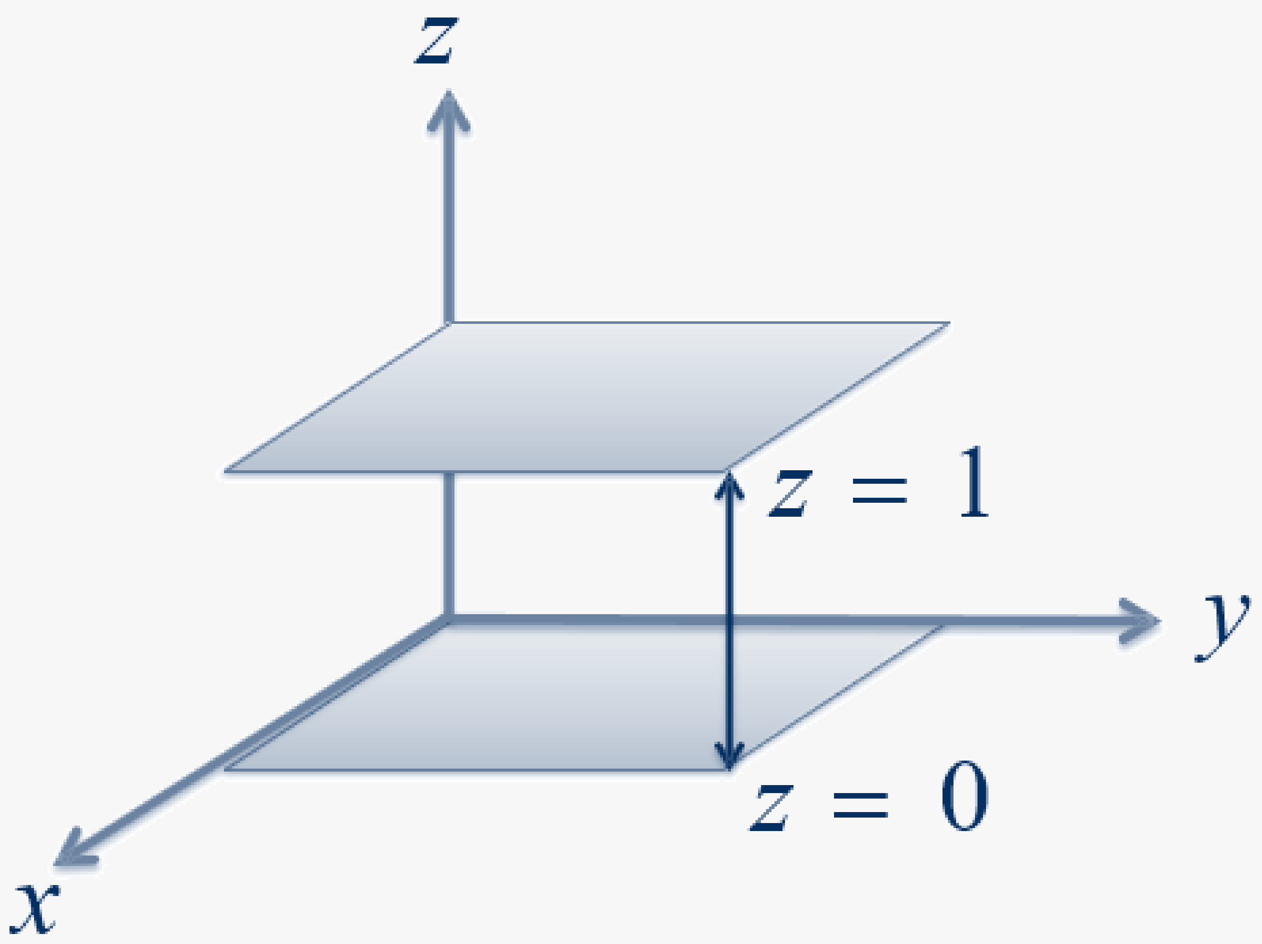
In general, if k is a constant, then

- $x = k$ represents a plane parallel to the yz $-$plane.
- $y = k$ is a plane parallel to the xz $-$plane.
- $z = k$ is a plane parallel to the xy $-$plane.

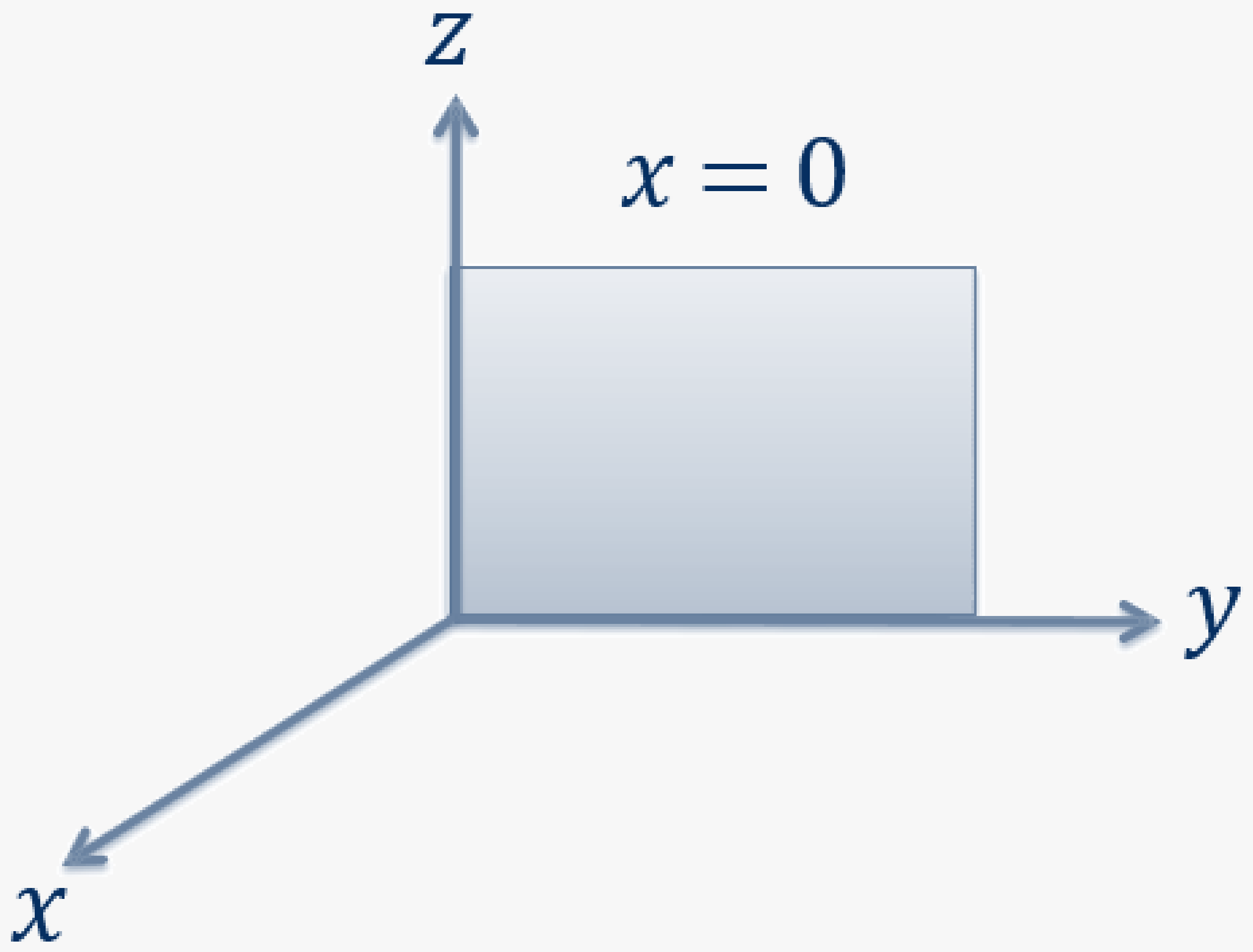
The equation of xy −plane is $z = 0$.



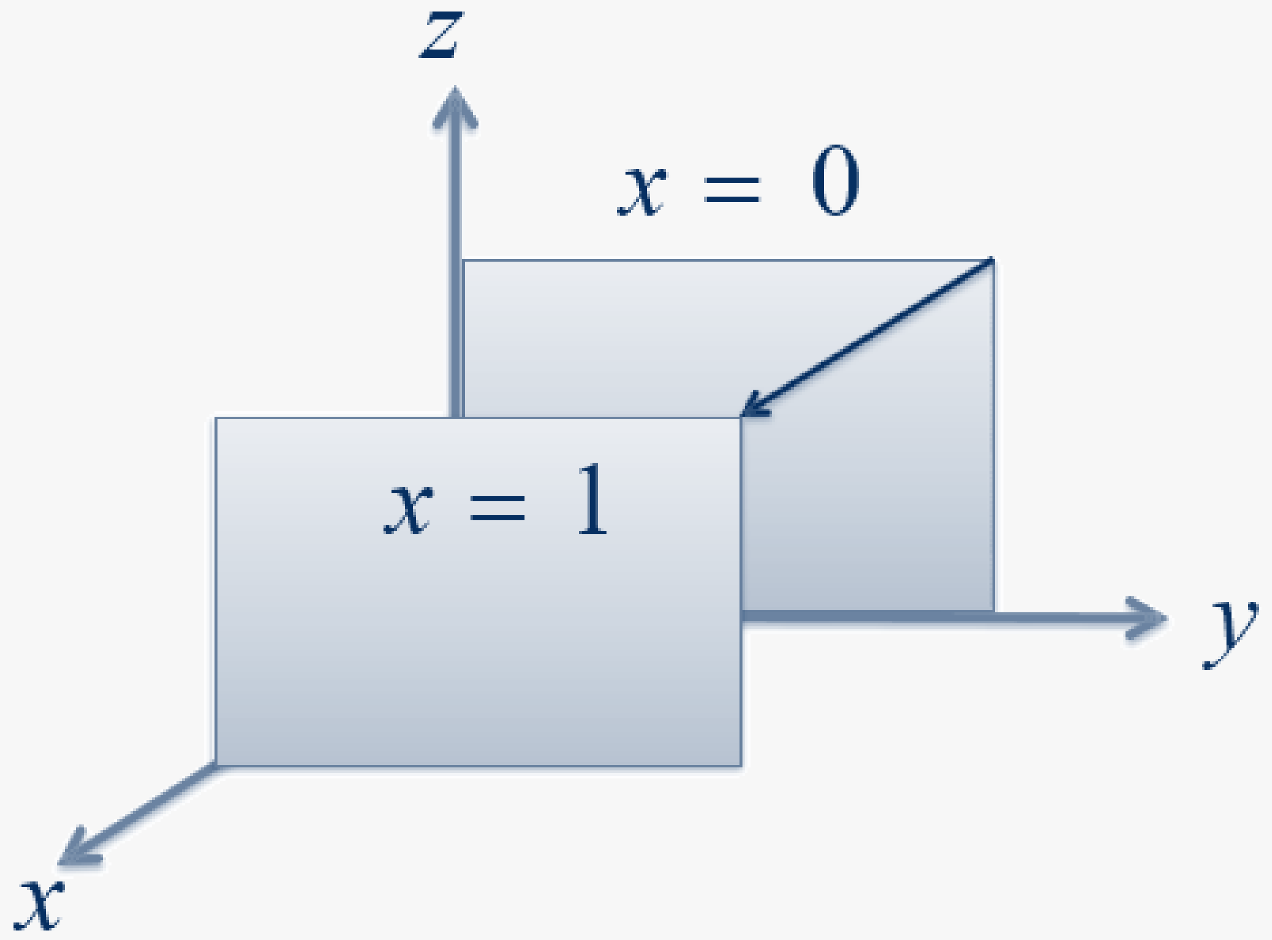
The equation of a plane parallel to $xy-$ plane and one unit above is $z = 1$.



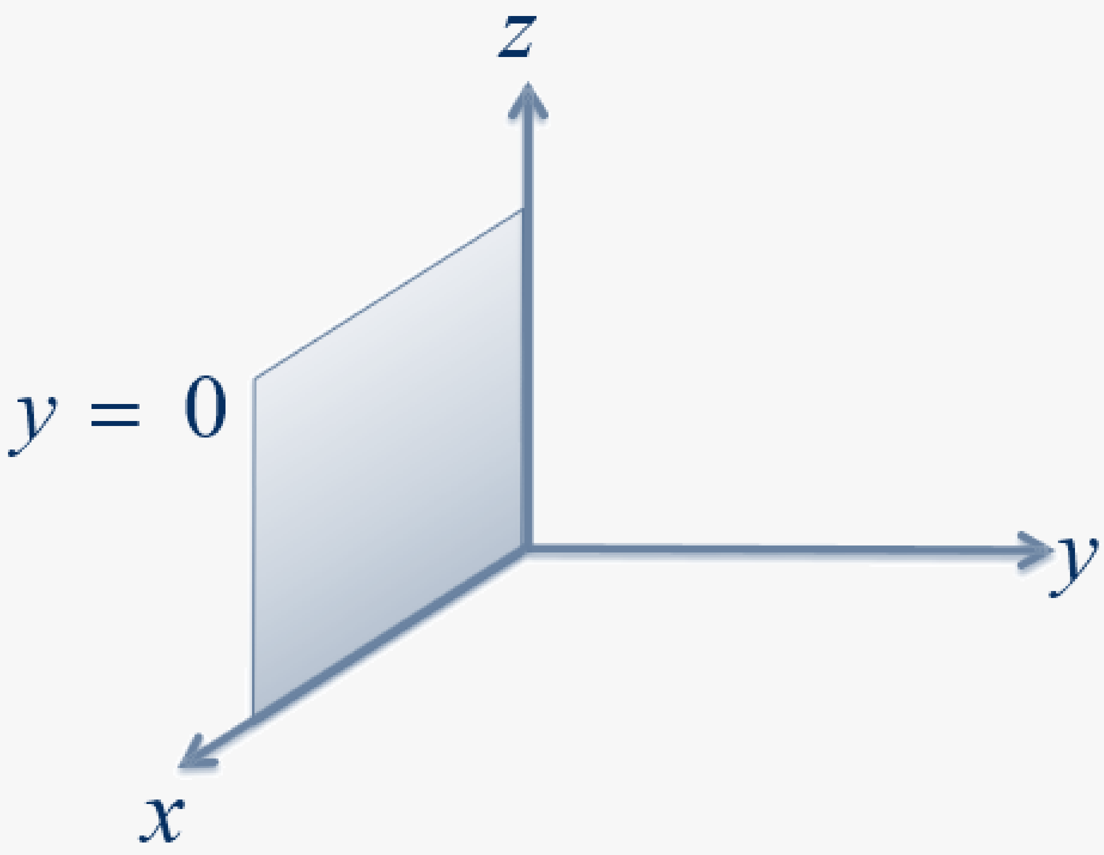
The equation of yz −plane is $x = 0$.



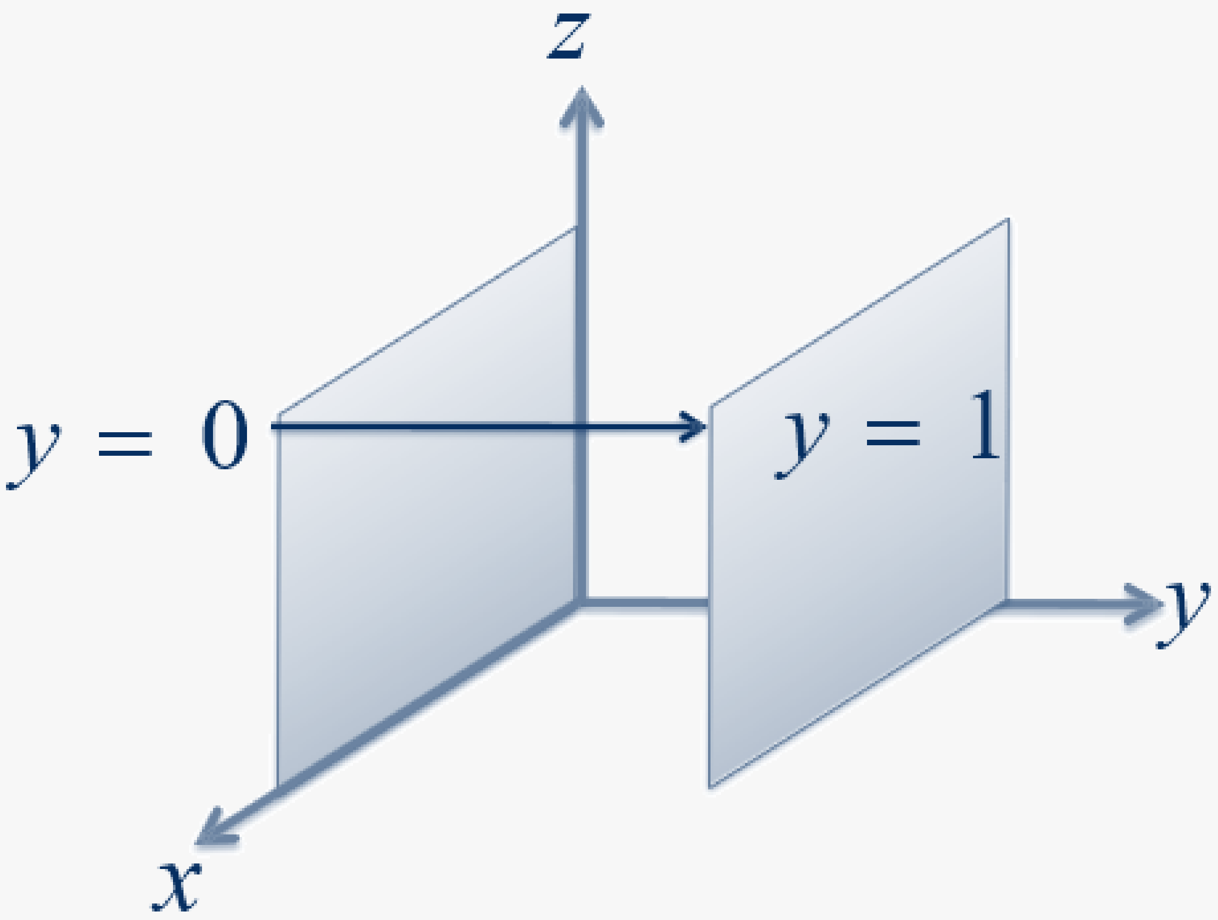
The equation of a plane parallel to $yz-$plane which is one unit upfront is $x = 1$.



The equation of xz −plane is $y = 0$.

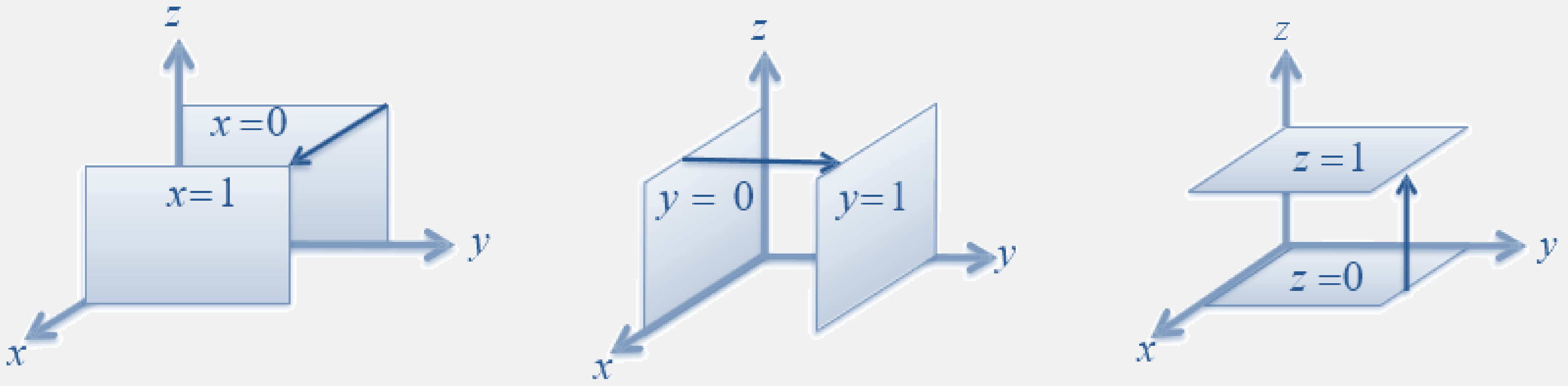


The equation of a plane parallel to $xz-$ plane and one unit to right is $y = 1$.



Traces

These planes are called ***traces*** and each three-dimensional surface can be thought as if it is made of curves in these planes such that the surface is obtained by gluing all such curves together.



Distance Formula In Three Dimensions

Radius = a

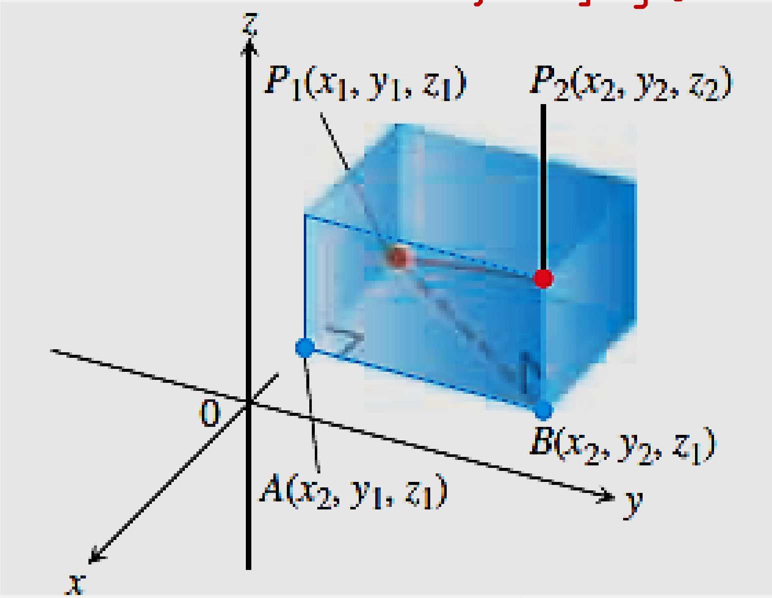
centre $P_0(x_0, y_0)$

$(x - x_0)^2 + (y - y_0)^2 = a^2$

The familiar formula for the distance between two points in a plane is easily extended to the following 3-D formula:

The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given as:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

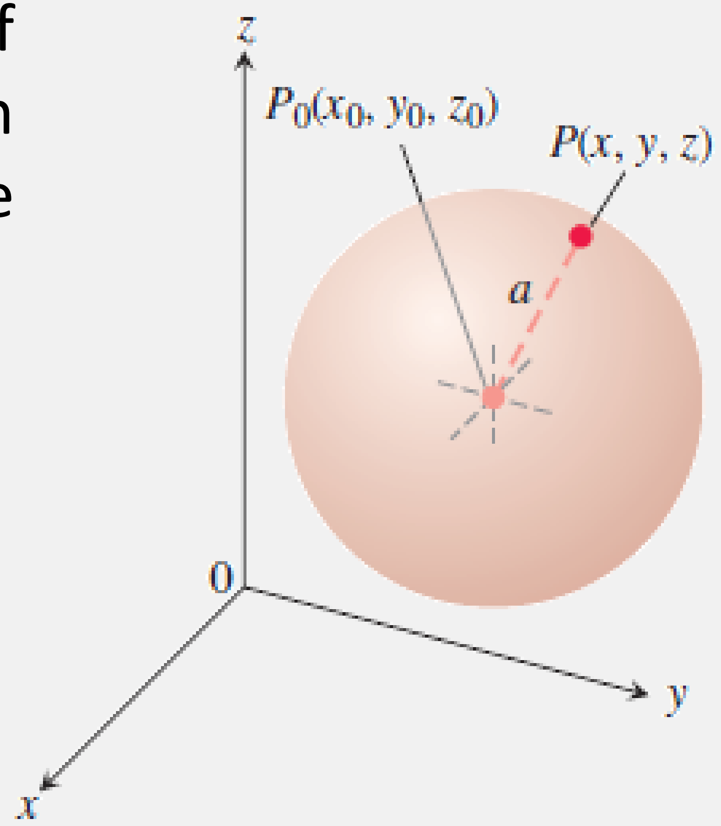


$d = |\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

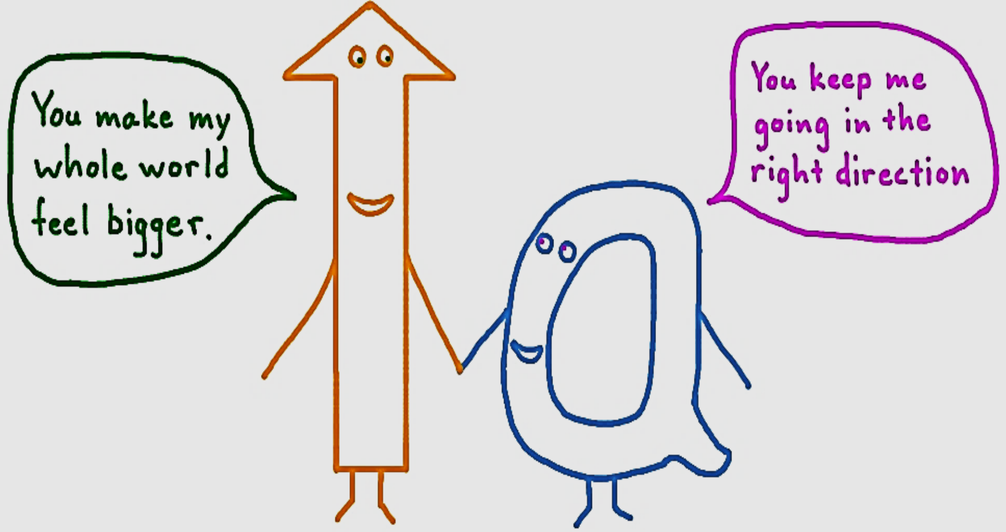
Sphere of Radius a and Center (x_0, y_0, z_0)

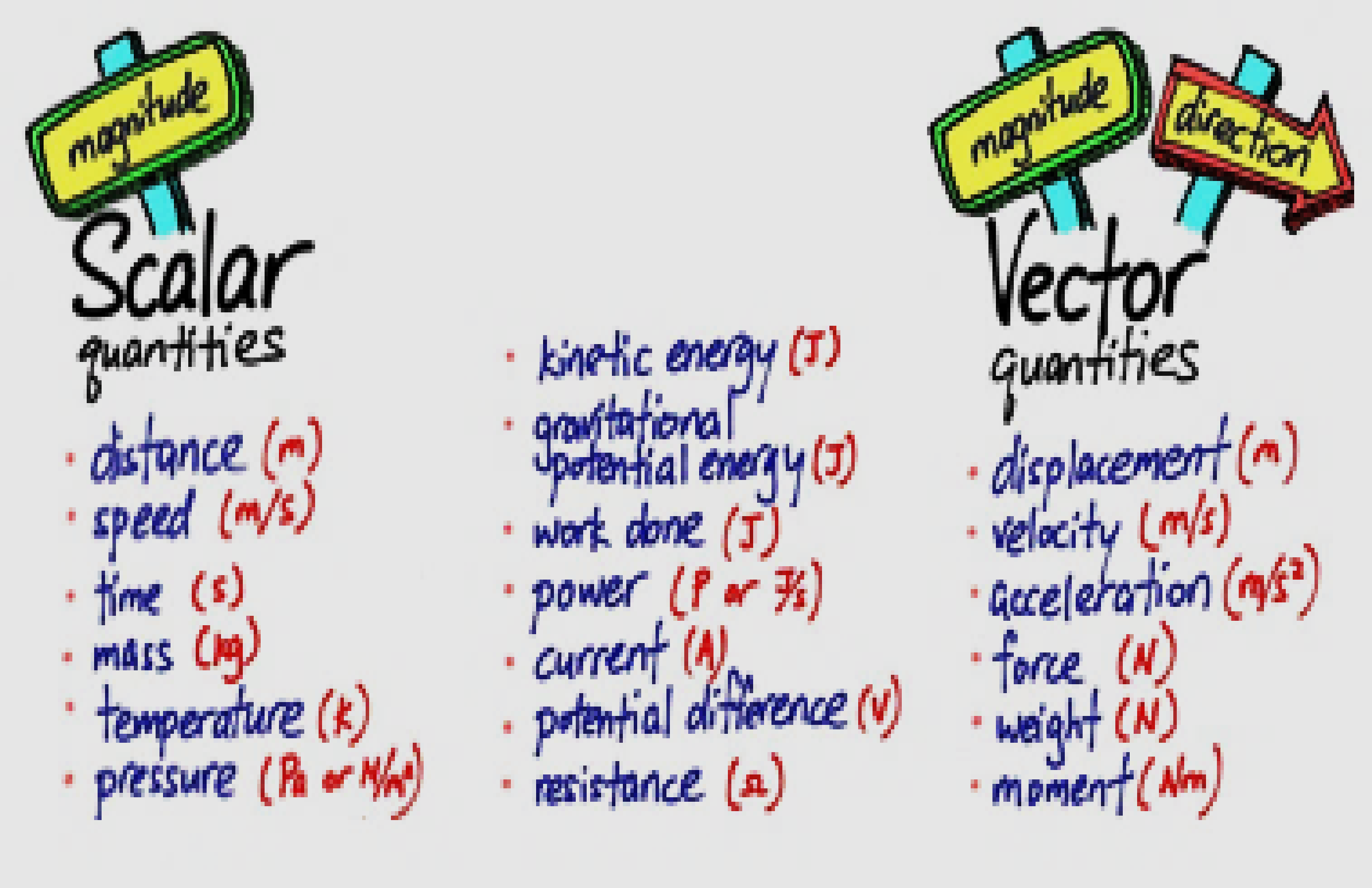
We can use the distance formula to write equation for sphere in space. A point $P(x, y, z)$ lies on the sphere of radius a centered at $P_0(x_1, y_0, z_0)$ precisely when $|P_0P| = a$. Thus, the standard equation of the sphere of radius a and center (x_0, y_0, z_0) is given as:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Vectors & Scalars





12

Vectors And
The Geometry Of Space

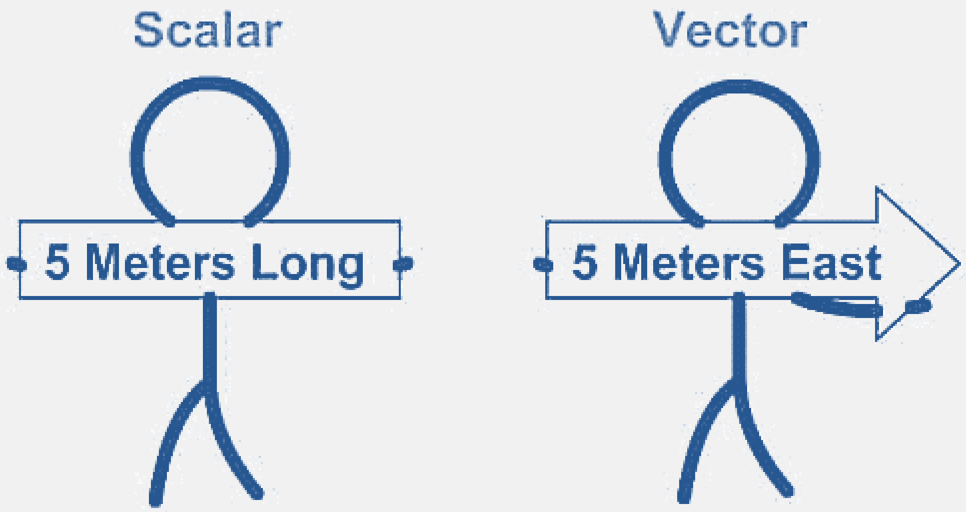
Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Section: 12.2

Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.
Section: 12.2

Vectors & Scalars

Scalar: A scalar is a quantity that has only one property- magnitude. Energy, speed, temperature, and mass are scalar quantities.

Vector: The term vector is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction.



Representing a Vector

A vector is often represented by an arrow or a directed line segment.

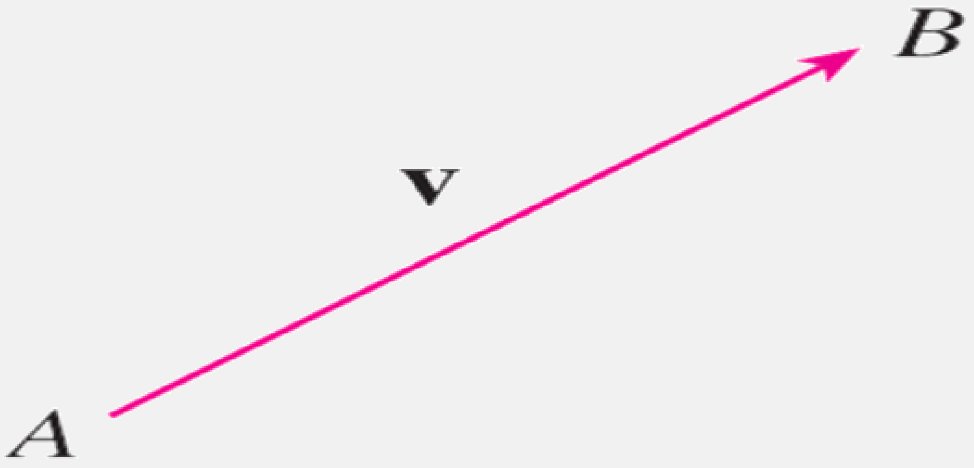
$\vec{v}$ or $\mathbf{v}$

or $\hat{v} \rightarrow$ of unit length

- The length of the arrow represents the magnitude of the vector.
- The arrow points in the direction of the vector.

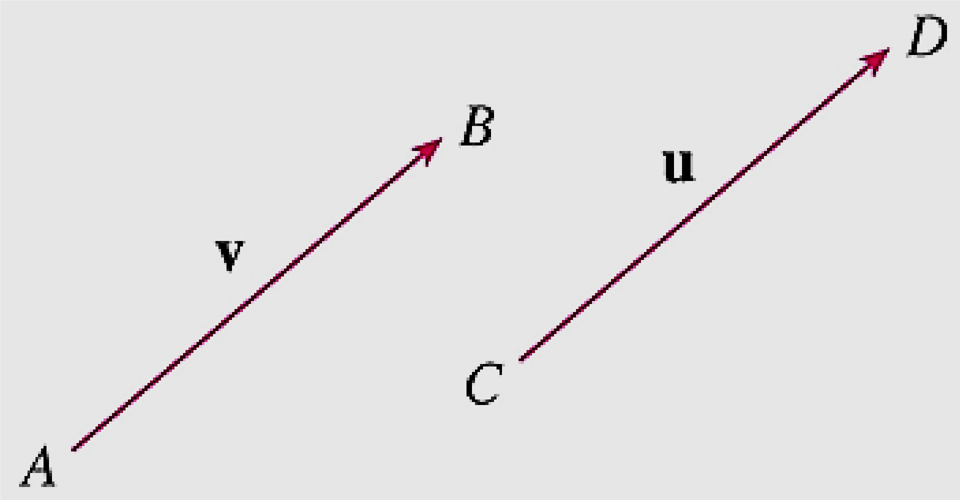
Vectors

For instance, suppose a particle moves along a line segment from point A to point B. The corresponding displacement vector $\mathbf{v}$ has initial point A (the tail) and terminal point B (the tip). We indicate this by writing $\mathbf{v} = \overrightarrow{AB}$.



Equivalent Vectors

Let us now consider two vectors $\mathbf{u}$ and $\mathbf{v}$, such that the vector $\mathbf{u}$ has initial point C (the tail) and terminal point D (the tip) i.e., $\mathbf{u} = \overrightarrow{CD}$ and the vector $\mathbf{v}$ has initial point A (the tail) and terminal point B (the tip) i.e., $\mathbf{v} = \overrightarrow{AB}$. Notice that the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as $\mathbf{v}$ even though it is at a different position. We say $\mathbf{u}$ and $\mathbf{v}$ are equivalent (or equal) and write $\mathbf{u} = \mathbf{v}$.



Zero Vectors

The zero vector, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

$\mathbf{0} = \vec{0} \in \mathbb{R}^3$

$= \langle 0, 0, 0 \rangle$

$\mathbf{0} \in M_{2\times 2}$

$\mathbf{0} = \vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Vectors in Coordinate Systems