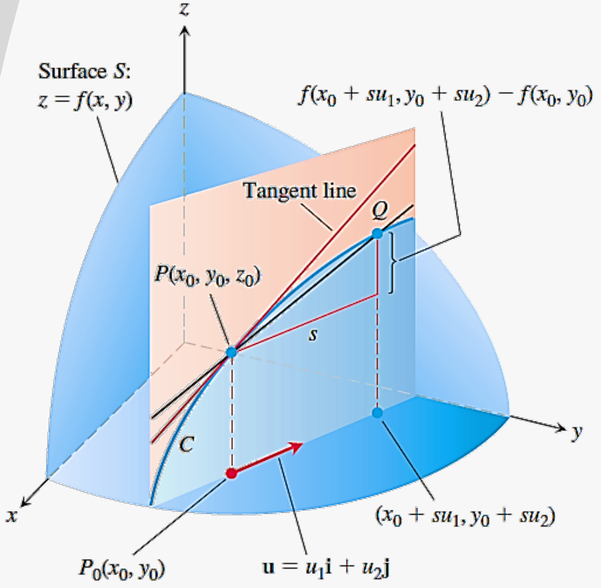
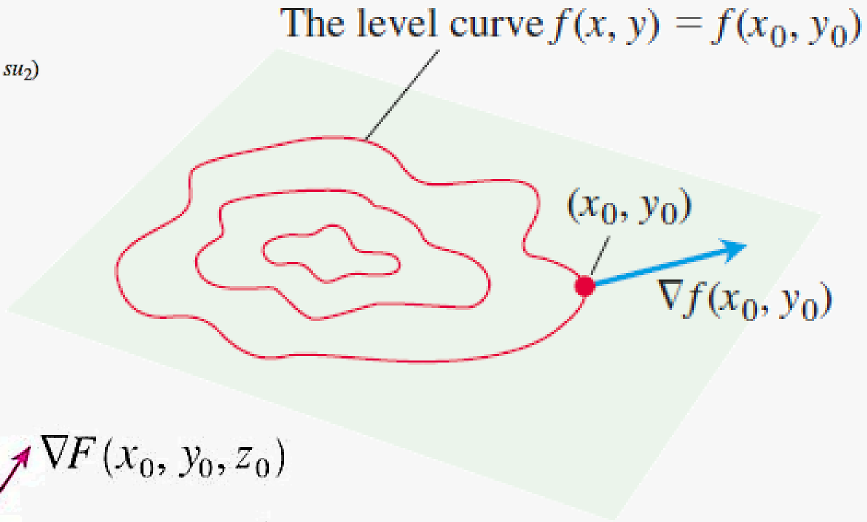
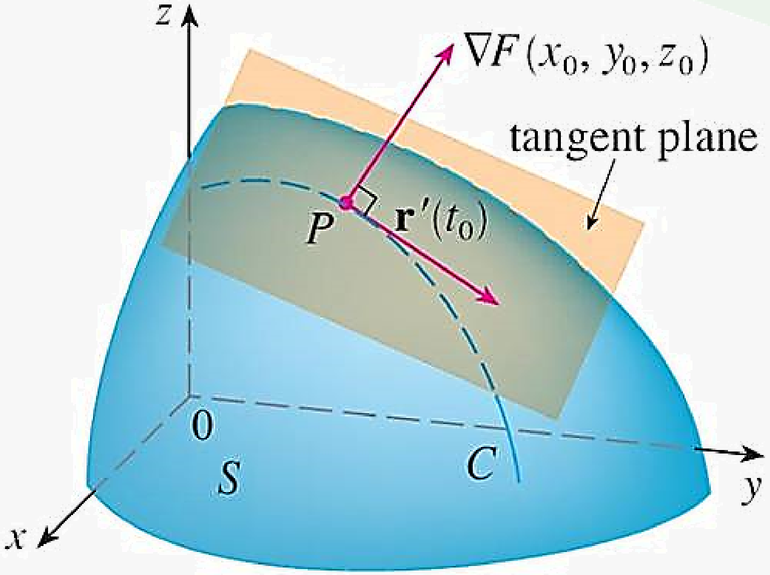
Directional Derivatives & Tangent Lines To Level Curves







Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

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Partial Derivatives

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Chapter: 14 , Section: 14.5, 14.6

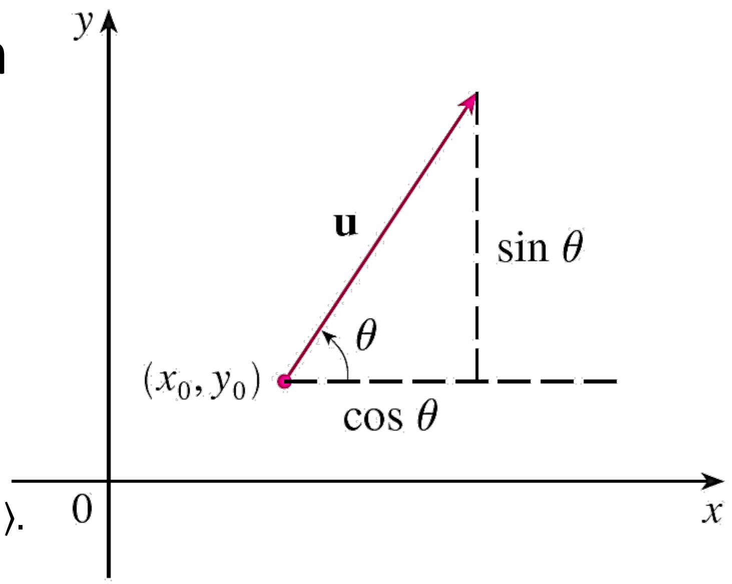
Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.
Chapter: 14 , Section: 14.1, 14.6

Directional Derivatives

Our objective is to introduce a type of derivative, called a *directional derivative*, that enables us to find the rate of change of a function of two or more variables in any direction. Recall that if $z = f\,(x, y)$, then the partial derivatives f_x and f_y are defined as:

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$
$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}, \qquad \text{(I)}$$

and represent the rates of change of z in the $x-$ and $y-$directions, that is, in the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$. Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$



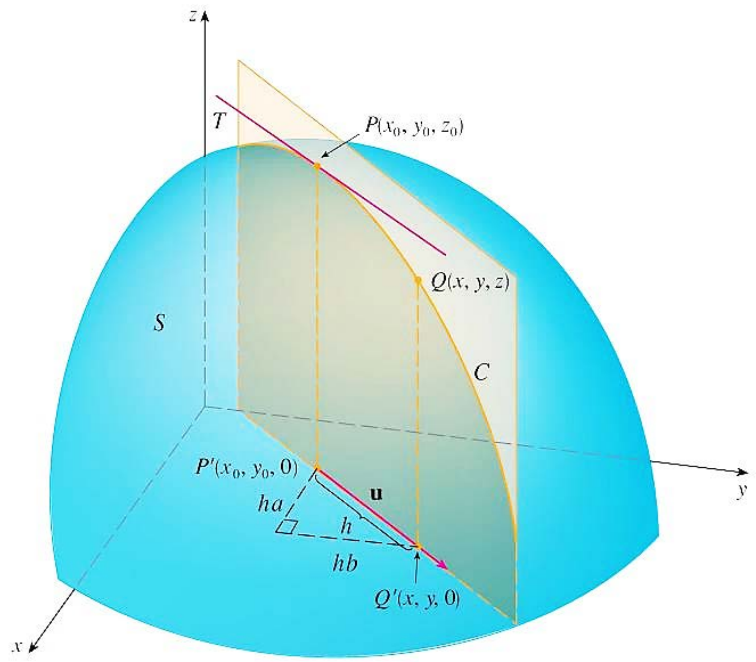
A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta\,, \sin\theta \rangle$.

Directional Derivatives

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u}$ is given as:

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h},$$

provided the limit exists. If $\mathbf{u} = \mathbf{i} = \langle 1,0 \rangle$, then $D_{\mathbf{i}} f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0,1 \rangle$, then $D_{\mathbf{j}} f = f_x$. In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.



Note:

The **slope** of the trace curve C of the surface $z = f(x, y)$ at (x_0, y_0) in the direction of a unit vector $\mathbf{u}$, is $\lim\limits_{Q \to P} \text{slope}\,(PQ)$; this is the **directional derivative:**

$$\left(\frac{df}{dh}\right)_{\mathbf{u},(x_0,y_0)} = D_{\mathbf{u}} f(x_0, y_0).$$

This represents the slope of the surface $z = f(x, y)$ at (x_0, y_0) in the direction of a unit vector $\mathbf{u}$

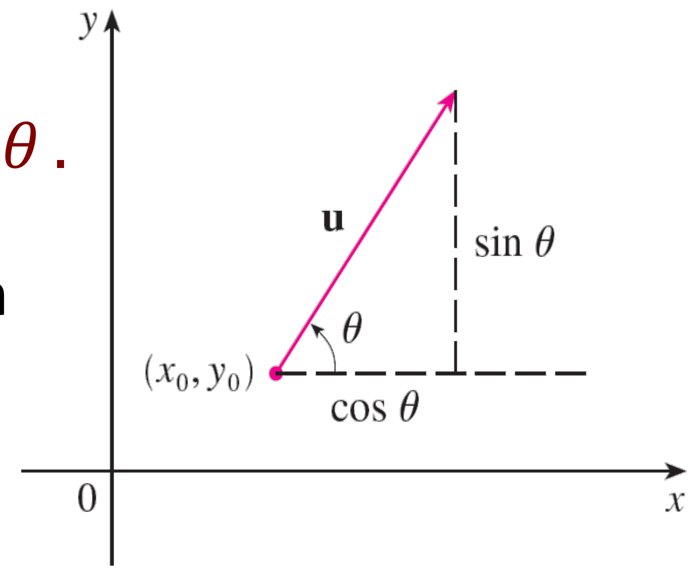
Directional Derivatives

We now develop an efficient formula to calculate the directional derivative for a differentiable function f. We begin with the line $x = x_0 + ha, y = y_0 + hb$, through $P(x_0, y_0)$ parametrized with the arc length parameter h increasing in the direction of the unit vector $\mathbf{u} = \langle a, b \rangle$. Then

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x −axis, then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta .$$



A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$.

Note that the directional derivative of a differentiable function can be written as the dot product of two vectors:

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)\, b$$

$$= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle = \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u}$$

The Gradient Vector

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

With the notation for the gradient vector, we can rewrite the directional derivative of a differentiable function f as:

$$D_{\mathbf{u}} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \cdot \mathbf{u} = \nabla f(x,y) \cdot \mathbf{u}.$$

This expresses the directional derivative in the direction of a unit vector $\mathbf{u}$ as the scalar projection of the gradient vector onto $\mathbf{u}$. Using properties of dot product, we have:

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta\,,$$

where θ is the angle between the vectors $\mathbf{u}$ and ∇f.

Properties of the Directional derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$, which means that $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is:

$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$

   ∇f points in the direction of **maximum rate of increase.**

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is:

$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|.$$

   $-\nabla f$ points in the direction of **maximum rate of decrease.**

3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and

$$D_{\mathbf{u}}f = |\nabla f| \cos\left(\frac{\pi}{2}\right) = 0.$$

Algebraic Rules for Gradient:

Let f and g be any functions of several variables and k is any constant then following rules are valid:

1. Constant multiple rule: $\nabla(kf) = k\nabla f$.
2. Sum rule: $\nabla(f+g) = \nabla f + \nabla g$.
3. Difference rule: $\nabla(f-g) = \nabla f - \nabla g$.
4. Product rule: $\nabla(fg) = f\nabla g + g\nabla f$.
5. Quotient rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Functions of Three Variables

For functions of three variables, we can define directional derivatives in a similar manner. Again, $D_{\mathbf{u}}f(x,y,z)$ can be interpreted as the rate of change of the function in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ and is given as:

$$D_{\mathbf{u}}f(x,y,z) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c$$

$$= \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle \cdot \mathbf{u}.$$

For a function f of three variables, the **gradient vector**, denoted by ∇f or $\mathbf{grad}\ f$, is:

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Then, just as with functions of two variables, the formula for the directional derivative can be rewritten as:

$$D_{\mathbf{u}}f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u} = |\nabla f| \cos\theta\,,$$

where θ is the angle between the vectors $\mathbf{u}$ and ∇f.

Example:

If $f(x,y,z) = x\sin(yz)$, find the gradient of f and the directional derivative of f at $(1,3,0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution:

The gradient of f is:

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle,$$

$$= \langle \sin(yz), xz\cos(yz), xy\cos(yz)\rangle.$$

At $(1,3,0)$ we have $\nabla f(1,3,0) = \langle 0,0,3\rangle = 3\mathbf{k}$. The unit vector in the direction of the vector: $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is given as:

$$\mathbf{u} = \frac{1}{\sqrt{6}}\langle 1,2,-1\rangle = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}.$$

Therefore,

$$D_{\mathbf{u}}f(1,3,0) = \nabla f(1,3,0)\cdot\mathbf{u} = 3\mathbf{k}\cdot\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\right) = -\frac{3}{\sqrt{6}} = -\sqrt{\frac{3}{2}}.$$

Maximizing the Directional Derivative

Suppose $f(\mathbf{x})$ is a differentiable function of two or three variables. $D_{\mathbf{u}}f(\mathbf{x})$ can be interpreted as the rate of change of the function in the direction of a unit vector $\mathbf{u}$ and is given as:

$$D_{\mathbf{u}}f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u} = |\nabla f| \cos\theta \,,$$

where θ is the angle between the vectors $\mathbf{u}$ and ∇f. The **maximum value** of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Note:

- $f(\mathbf{x}) = f(x,y)$, if it is representing a function of two variables.
- $f(\mathbf{x}) = f(x,y,z)$, if it is representing a function of three variables.

Example:

(a) If $f(x,y)=xe^y$, find the slope of f at the point $P(2,0)$ in the direction from P to $Q(1/2,2)$.

(b) In what direction does f has the maximum rate of change? Moreover, determine what is this maximum rate of change?

Solution:

(a) We first compute the gradient vector:

$$\nabla f(x,y)=\langle f_x(x,y),f_y(x,y)\rangle=\langle e^y,xe^y\rangle \Longrightarrow \nabla f(2,0)=\langle e^0,2e^0\rangle=\langle 1,2\rangle.$$

The unit vector in the direction of $\overrightarrow{PQ}=\langle -3/2,2\rangle$ is:

$$\mathbf{u}=\frac{\langle -3/2,2\rangle}{5/2}=\left\langle -\frac{3}{5},\frac{4}{5}\right\rangle,$$

so, the rate of change of f in the direction from P to Q is:

$$D_{\mathbf{u}}f(2,0)=\nabla f(2,0)\cdot\mathbf{u}=\langle 1,2\rangle\cdot\left\langle -\frac{3}{5},\frac{4}{5}\right\rangle=1.$$

Example:

(a) If $f(x, y) = xe^y$, find the rate of change of at the point $P(2,0)$ in the direction from P to $Q(1/2,2)$.

(b) In what direction does f has the maximum rate of change? Moreover, determine what is this maximum rate of change?

Solution:

(b) It is known that f increases fastest in the direction of the gradient vector $\nabla f(2, 0)$ at $(2, 0)$. Thus, the **direction of fastest change** is given as:

$$\frac{\nabla f(2, 0)}{|\nabla f(2, 0)|} = \frac{1}{\sqrt{5}} \langle 1,2 \rangle,$$

and the **maximum rate of change** is: $|\nabla f(2, 0)| = \sqrt{5}$.

Example:

Suppose that the temperature at a point (x, y, z) in space is given by:

$$T(x, y, z) = \frac{80}{1 + x^2 + 2y^2 + 3z^2},$$

where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1,1,-2)$? What is the maximum rate of increase?

Solution:

For the present case the temperature functions is given as:

$$T(x,y,z) = \frac{80}{1 + x^2 + 2y^2 + 3z^2}$$

The gradient of T is:

$$\nabla T(x,y,z) = \langle T_x(x,y,z), T_y(x,y,z), T_z(x,y,z) \rangle,$$

$$= \left\langle \frac{-160x}{(1 + x^2 + 2y^2 + 3z^2)^2}, \frac{-320y}{(1 + x^2 + 2y^2 + 3z^2)^2}, \frac{-480z}{(1 + x^2 + 2y^2 + 3z^2)^2} \right\rangle$$

$$= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2} \langle -x, -2y, -3z \rangle.$$

At $(1, 1, -2)$ the gradient vector is given as:

$$\nabla T(x,y,z) = \frac{160}{[1 + (1)^2 + 2(1)^2 + 3(-2)^2]^2} \langle -1, -2(1), -3(-2) \rangle = \frac{5}{8} \langle -1, -2, 6 \rangle.$$

Solution:

The temperature increases fastest in the direction of the gradient vector

$$\nabla T(x, y, z) = \frac{5}{8}\langle -1, -2, 6\rangle,$$

or, equivalently, in the direction of $\langle -1, -2, 6\rangle$. Thus, the direction of rapid increase in temperature T at $(1, 1, -2)$ is given by means of the unit vector:

$$\mathbf{u} = \frac{1}{\sqrt{41}}\langle -1, -2, 6\rangle = -\frac{1}{\sqrt{41}}\mathbf{i} - \frac{2}{\sqrt{41}}\mathbf{j} + \frac{6}{\sqrt{41}}\mathbf{k}.$$

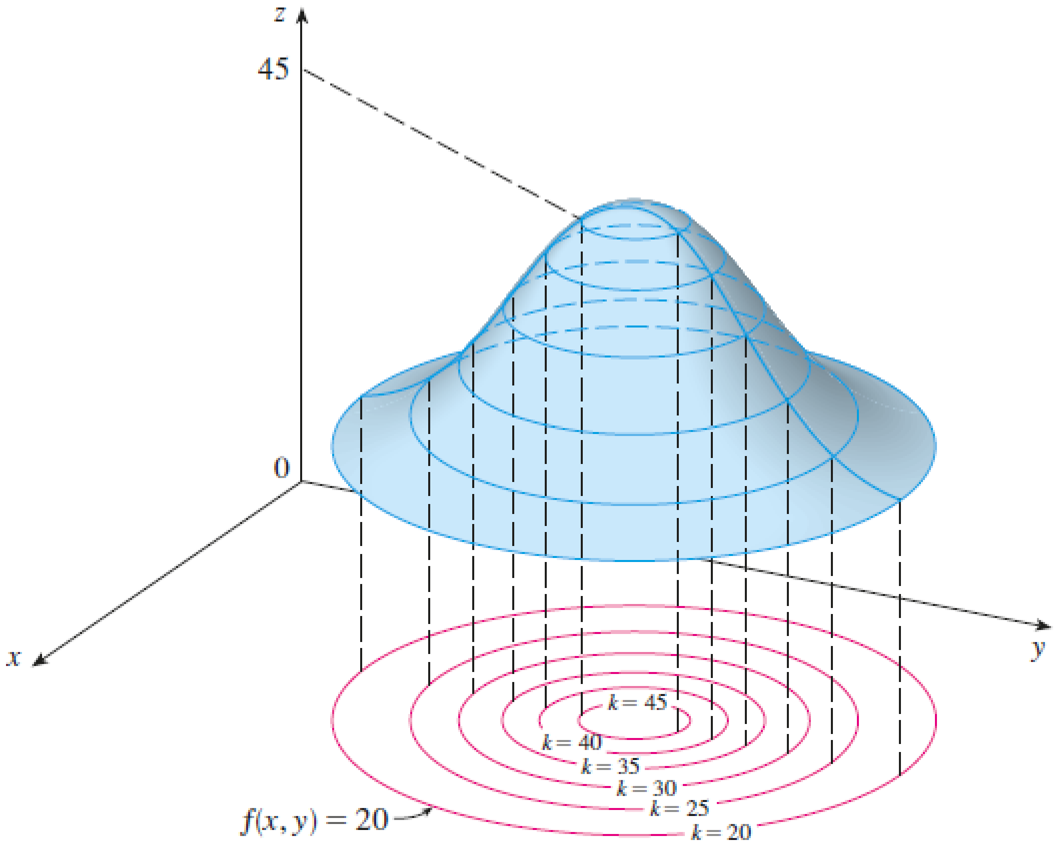
The maximum rate of increase is the length of the gradient vector at $(1, 1, -2)$:

$$|\nabla T(1, 1, -2)| = \frac{5}{8}\sqrt{41}.$$

Therefore, the maximum rate of increase of temperature is: $\frac{5}{8}\sqrt{41} \approx 4°\mathrm{C/m}.$

Level Curves

The **level curves** of a function f of two variables are the curves with equations $f(x,y) = k$, where k is a constant (in the range of f). A level curve $f(x,y) = k$ is the set of all points in the domain of f at which f takes on a given value k. In other words, it shows where the graph of f has height k. The level curves $f(x,y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy −plane.



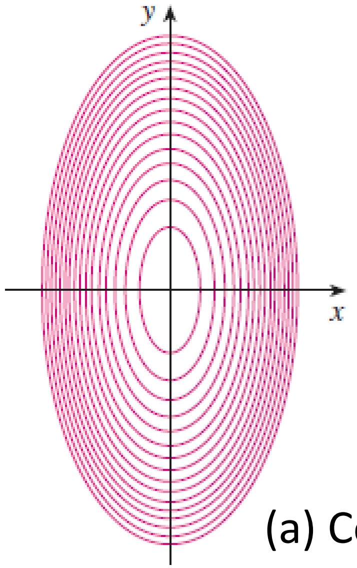
Example:

Sketch some level curves of the function $h(x, y) = 4x^2 + y^2$.

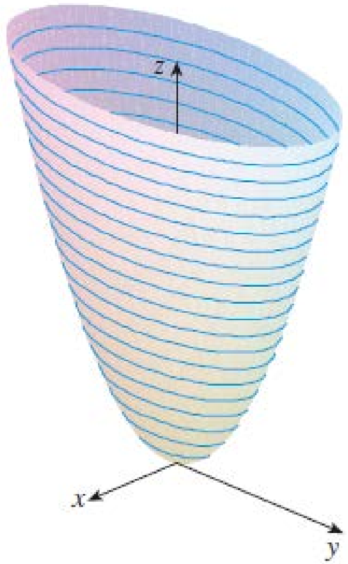
Solution: The level curves are given as:

$$4x^2 + y^2 = k \qquad \text{or} \qquad \frac{x^2}{k/4} + \frac{y^2}{k} = 1,$$

which, for $k > 0$, describes a family of ellipses with semiaxes $\sqrt{k}/2$ and $\sqrt{k}$. Figure (a) shows a contour map of h with level curves corresponding to $k = 0.25, 0.5, 0.75, \dots, 4$. Figure (b) shows these level curves lifted up to the graph of h (an elliptic paraboloid) where they become horizontal traces. We see from the figure how the graph of h is put together from the level curves.



(a) Contour map



(b) Horizontal traces are raised level curves.

Gradients to Level Curves

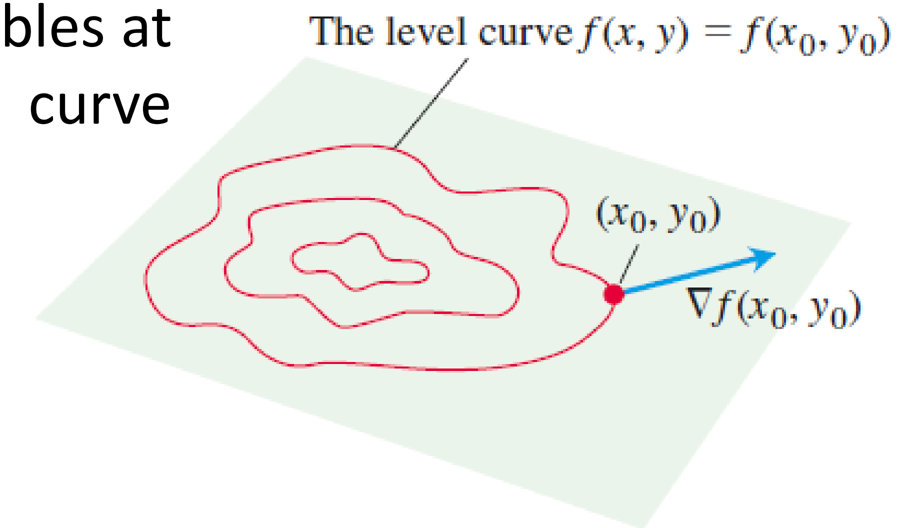
If $f(x, y)$ is a differentiable function of two variables and $f(x, y) = c$ along a smooth curve: $\mathbf{r}(t) = \langle g(t), h(t) \rangle$, making the curve part of a level curve of $f(x, y)$, then $f(g(t), h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the equations:

$$\frac{d}{dt} f\big(g(t), h(t)\big) = \frac{d}{dt}(c) \Longrightarrow \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0,$$

$$\Longrightarrow \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dg}{dt}, \frac{dh}{dt} \right\rangle = 0 \Longrightarrow \nabla f \cdot \frac{dr}{dt} = 0. \qquad (*)$$

Equation $(*)$ says that ∇f is normal to the tangent vector $\frac{dr}{dt}$, so it is normal to the curve.

Thus, the gradient of a differentiable function of two variables at a point (x_0, y_0) is always normal to the function's level curve through that point.



Tangent line to a Level Curve

- At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0).
- This observation enables us to find equations for tangent lines to level curves. They are the lines normal to the gradients.
- The line through a point (x_0, y_0) normal to a nonzero vector $\mathbf{N} = \langle A, B \rangle$ has the equation:

$$A(x - x_0) + B(y - y_0) = 0.$$

- If $\mathbf{N}$ is the gradient $\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$ and this gradient is a nonzero vector, then the equation for tangent line to level curve at (x_0, y_0) is given by:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

Example:

Find an equation for the tangent to the ellipse:

$$\frac{x^2}{4} + y^2 = 2,$$

at the point $(-2, 1)$.

Solution: The ellipse is a level curve of the function:

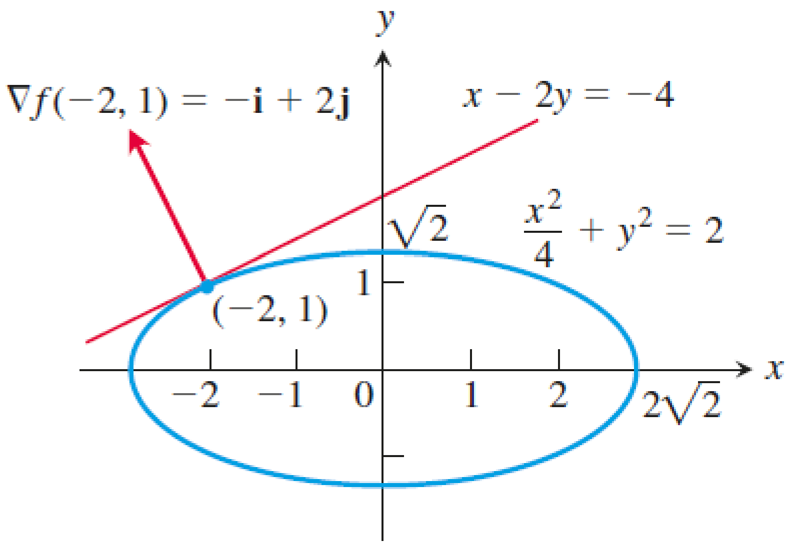
$$f(x, y) = \frac{x^2}{4} + y^2.$$

The gradient of f at $(2, -1)$ is:

$$\nabla f(-2,1) = \left\langle \frac{x}{2}, 2y \right\rangle \bigg|_{(-2,1)} = \langle -1,2 \rangle.$$

Because this gradient vector is nonzero, the tangent to the ellipse at $(-2,1)$ is the line:

$$(-1)(x + 2) + (2)(y - 1) = 0 \Longrightarrow x - 2y = -4.$$



We can find the tangent to the ellipse $\frac{x^2}{4} + y^2 = 2$ by treating the ellipse as a level curve of the function $f(x, y) = \frac{x^2}{4} + y^2$.