$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$$f_x(0,0) = \underset{h\to 0}{\text{limit}} \frac{f(0+h,0)-f(0,0)}{h} = \underset{h\to 0}{\text{limit}} \frac{f(h,0)-f(0,0)}{h}$$

$$= \underset{h\to 0}{\text{limit}} \frac{\frac{(h)(0)(h^2-0^2)}{h^2+0^2} - 0}{h} = \underset{h\to 0}{\text{limit}} \frac{0-0}{h} = \underset{h\to 0}{\text{limit}} (0) = 0 \quad \text{--- (i)}$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k} = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k}.$$

$$= \lim_{k\to 0} \frac{\frac{(0)(k)(0^2-k^2)}{0^2+k^2} - 0}{k} = \lim_{k\to 0} \frac{0-0}{k} = \lim_{k\to 0} (0) = 0 \quad \text{--- (ii)}$$

Now,

$$(f_{xy})\Big|_{(0,0)} = \left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]_{(0,0)} = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k}$$

$f_x(0,0) = 0$ by (i)

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} \longrightarrow \text{(iii)}$$

Now we calculate

$$f_x(0,k) = \lim_{h\to 0} \frac{f(0+h,k) - f(0,k)}{h} = \underset{h\to 0}{\text{limit}} \frac{f(h,k)-f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{hk(h^2-k^2)}{h^2+k^2} - \frac{(0)(k)(0^2-k^2)}{0^2+k^2}}{h} = \lim_{h\to 0} \frac{\cancel{h}k(h^2-k^2)}{\cancel{h}(h^2+k^2)}$$

$$\Rightarrow f_x(0,k) = \frac{-k^3}{k^2} = -k \quad \text{--- (iv)}$$

Using the value of $f_x(0,k)$ from (iv) in (iii), we get

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{-k-0}{k} = \lim_{k\to 0}(-1) = -1.$$

Hence, $f_{xy}(0,0) = -1$.

Similarly, it could be shown (DYS) that $f_{yx}(0,0) = 1$.

Hence, $f_{xy}(0,0) \neq f_{yx}(0,0)$.