Assignment #3

Q1:- Given that

$$(1) \leftarrow u_t = u_{xx} \; ; \quad 0 < x < 1 \; ; \; t > 0$$

$$(2) \leftarrow \begin{cases} u_x(0,t) = -u(0,t) \\ u_x(1,t) = -u(1,t) \end{cases} \; ; \quad t > 0$$

$$(3) \leftarrow u(x,0) = x \; ; \; 0 < x < 1$$

Using method of separation of variables we consider a function:

$$u(x,t) = X(x)\,T(t), \longrightarrow (4)$$

as a solution of (1). Using (4) in (1) we get

$$X(x)T'(t) = X''(x)T(t)$$

$$\Rightarrow \quad \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = K,$$

where K is a separation constant. From here we get two ODEs:

$$T'(t) = KT(t) \text{ or } T'(t) - KT(t) = 0 \rightarrow (5)$$

and $X''(x) = KX(x)$ or $X''(x) - KX(x) = 0 \rightarrow (6)$

Using (4) in (2) we get:

$$X'(0) = -X(0) \longrightarrow (7)$$

and $X'(1) = -X(1) \longrightarrow (8)$

Let us consider the BVP involving "x" which is given as:

$$X''(x) - KX(x) = 0 \rightarrow (6)$$

$$X'(0) = -X(0) \rightarrow (7)$$

$$X'(1) = -X(1) \rightarrow (8)$$

In order to solve (6) we will consider different values of K.

Case 1: $K=0$

For $K=0$, eq (6) reduces to:

$$X''(x) = 0$$

$$\Rightarrow X'(x) = c_1 \longrightarrow (9)$$

$$\Rightarrow X(x) = c_1 x + c_2 \longrightarrow$$

Using (7) and (8) in (9) and (10) we get

$$X'(0) = -X(0) \Rightarrow \boxed{c_1 = -c_2}$$

$$X'(1) = -X(1) \Rightarrow c_1 = -c_1 - c_2$$

$$\Rightarrow 2c_1 = -c_2$$

$$\Rightarrow 2c_1 = c_1 \quad [\because c_1 = -c_2]$$

$$\Rightarrow \boxed{c_1 = 0}$$

$$\Rightarrow \boxed{c_2 = 0} \quad [\because c_2 = -c_1]$$

Thus, (10) becomes:

$$X(x) = 0,$$

a trivial solution which is of no interest.

Case 2: $K > 0$

Let $K = p^2 > 0$. Using value of K in (6) we get:

$$X''(x) - p^2 X(x) = 0$$

$$\Rightarrow (D^2 - p^2) X(x) = 0$$

This is a second order linear homogenenous ODE. The characteristic equation is:

$$D^2 - p^2 = 0$$

$$\Rightarrow D^2 = p^2$$

$$\Rightarrow D = \pm p \quad \text{(distinct real roots)}$$

Thus, the general solution is given as:

$$X(x) = c_3 \cosh(px) + c_4 \sinh(px) \rightarrow (11)$$

$$X'(x) = pc_3 \sinh(px) + pc_4 \cosh(px) \rightarrow (12)$$

Using (7) and (8) in (11) and (12) we get:

$X'(0) = -X(0) \Rightarrow \boxed{pc_4 = -c_3}$ $[\because \cosh(0) = 1 \text{ \& } \sinh(0) = 0]$

$X'(1) = -X(1) \Rightarrow pc_3 \sinh(p) + pc_4 \cosh(p) = -c_3 \cosh(p) - c_4 \sinh(p)$

$\Rightarrow pc_3 \sinh(p) + \cancel{pc_4 \cosh(p)} = \cancel{pc_4 \cosh(p)} + \frac{c_3}{p} \sinh(p)$ $[\because -c_3 = pc_4]$

$\Rightarrow pc_3 \sinh(p) = \frac{c_3}{p} \sinh(p)$

$\Rightarrow p^2 c_3 \sinh(p) - c_3 \sinh(p) = 0$

$\Rightarrow (p^2 - 1) c_3 \sinh(p) = 0$

Since $p \neq 0$, $\sinh(p) = 0$ we consider $c_3 \neq 0$

Thus,

$p^2 - 1 = 0$

$\Rightarrow p^2 = 1 \Rightarrow k = 1$

$\Rightarrow \boxed{p = 1}$ [ignoring -ve root because $p^2 = k > 0$].

Using $p = 1$ in (11), we get

$X(x) = c_3 \cosh x + c_4 \sinh x$

$\Rightarrow X(x) = c_3 \cosh x - c_3 \sinh(x)$ $\left[\because pc_4 = -c_3 \text{ and } p = 1\right]$

$\Rightarrow X(x) = c_3 [\cosh x - \sinh x]$

$= c_3 \left[\left(\frac{e^x + e^{-x}}{2}\right) - \left(\frac{e^x - e^{-x}}{2}\right)\right]$

$= \frac{c_3}{2} [\cancel{e^x} + e^{-x} - \cancel{e^x} + e^{-x}]$

$\Rightarrow \boxed{X(x) = c_3 e^{-x}} \longrightarrow (13)$

Case 3 : $K < 0$

Let $K = -p^2 < 0$ so that (6) becomes

$$X''(x) + p^2 X(x) = 0$$

$$\Rightarrow (D^2 + p^2) X = 0$$

This is a second order linear homogeneous ODE. Characteristic equation is given as:

$$D^2 + p^2 = 0$$

$$\Rightarrow D = \pm ip \text{ (complex roots)}$$

Thus, general solution is given as:

$$\Rightarrow X(x) = c_5 \cos(px) + c_6 \sin(px) \longrightarrow (14)$$

$$\Rightarrow X'(x) = -c_5 p \sin(px) + c_6 p \cos(px) \rightarrow (15)$$

Using (7) and (8) in (14) and (15) we get

$X'(0) = -X(0) \Rightarrow \boxed{c_6 p = -c_5}$ $[\because \cos(0) = 1$ & $\sin(0) = 0]$

$X'(1) = -X(1) \Rightarrow -c_5 p \sin p + c_6 p \cos p = -c_5 \cos p - c_6 \sin p.$

$$\Rightarrow -c_5 p \sin p + c_6 \cancel{p} \cos p = c_5 p \cancel{\cos} p + \frac{c_5}{p} \sin p$$

$$\Rightarrow -c_5 p^2 \sin p - c_5 \sin p = 0$$

$$\Rightarrow (p^2+1) c_5 \sin p = 0$$

Since $p \neq 0$, we take $c_5 \neq 0$ (otherwise we will get a trivial solution). Thus, we get:

$$\sin p = 0$$

$$\Rightarrow \boxed{p = n\pi;} \quad n = 1, 2, 3, \ldots$$

Thus, (14) becomes

$$X_n(x) = -C_6(n\pi)\cos(n\pi x) + C_6 \sin(n\pi x) \quad [\because C_6 p = -C_5 \;\&\; p = n\pi]$$

$$\Rightarrow X_n(x) = -C_6\left[(n\pi)\cos(n\pi x) - \sin(n\pi x)\right]$$

$$\Rightarrow X_n(x) = C_7\left[(n\pi)\cos(n\pi x) - \sin(n\pi x)\right], \longrightarrow (16)$$

where $C_7 = -C_6$

Let us now consider equation (5) that is given as:

$$T'(t) - K\,T(t) = 0$$

$$\Rightarrow T'(t) = K\,T(t)$$

$$\Rightarrow \frac{dT(t)}{dt} = K\,T(t)$$

$$\Rightarrow \frac{dT(t)}{T(t)} = K\,dt$$

This is a first order separable ODE. Integrating both sides of above we get:

$$\ln|T(t)| = Kt + \ln|C_8|$$

$$\Rightarrow T(t) = C_8 e^{Kt}$$

$$\Rightarrow T_n(t) = \begin{cases} C_8 e^{t}; & \text{if } K = p^2 > 0 \;\&\; p = 1 \\ C_8 e^{-n^2\pi^2 t}; & \text{if } K = -p^2 < 0 \;\&\; p = n\pi \end{cases} \longrightarrow (17)$$

Since $u(x,t) = X(x)\,T(t) \Rightarrow u_n(x,t) = X_n(x)\,T_n(t)$

Using (13), (16) and (17) we get

$$u_n(x,t) = \begin{cases} A_0 e^{-x} e^{t} & ; \quad K = p^2 = 1 > 0 \\ A_n\left[n\pi\cos(n\pi x) - \sin(n\pi x)\right]; & K = -p^2 = -n^2\pi^2 < 0 \\ & n = 1, 2, \ldots \end{cases}$$

where $A_0 = C_3 C_8$ and $A_n = C_7 C_8$.

Superposition of these solutions provide us with

$$u(x,t) = A_0 e^{-x} e^{t} + \sum_{n=1}^{\infty} A_n \left[(n\pi)\cos(n\pi x) - \sin(n\pi x)\right] e^{-n^2\pi^2 t} \rightarrow (18)$$

Using the initial condition (3)

$$u(x,0) = x$$

we get

$$x = A_0 e^{-x} + \sum_{n=1}^{\infty} A_n \left[(n\pi)\cos(n\pi x) - \sin(n\pi x)\right] \rightarrow (19)$$

Using orthogonality relations

$$\int_0^1 X_m(x)\, X_n(x)\, dx = 0 \quad \text{if } m \neq n$$

we get

$$A_0 = \frac{2e^2}{e^2-1} \int_0^1 x e^{-x}\, dx \quad [\because f(x) = x]$$

$$= \frac{2e^2}{e^2-1} \left[-xe^{-x} \Big|_0^1 + \int_0^1 e^{-x}\, dx \right]$$

$$= \frac{2e^2}{e^2-1} \left[-e^{-1} + (-e^{-x}) \Big|_0^1 \right]$$

$$= \frac{2e^2}{e^2-1} \left[-e^{-1} - e^{-1} + 1 \right]$$

$$= \frac{2e^2}{e^2-1} \left[1 - 2e^{-1} \right]$$

$$A_0 = \frac{2e(e-2)}{e^2-1} \rightarrow (20)$$

And

$$A_n = \frac{2}{1+n^2\pi^2} \int_0^1 f(x)\, X_n(x)\, dx$$

$$\Rightarrow A_n = \frac{2}{1+n^2\pi^2} \int_0^1 x\left[(n\pi)\cos(n\pi x) - \sin(n\pi x)\right] dx$$

$$= \frac{2n\pi}{1+n^2\pi^2} \int_0^1 x \cos(n\pi x)\, dx - \frac{2}{1+n^2\pi^2} \int_0^1 x \sin(n\pi x)\, dx$$

$$= \frac{2n\pi}{1+n^2\pi^2}\left[x\, \frac{\sin(n\pi x)}{n\pi}\Big|_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\, dx\right]$$

$$- \frac{2}{1+n^2\pi^2}\left[x\, \frac{-\cos(n\pi x)}{n\pi}\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\, dx\right]$$

$$= \frac{2n\pi}{1+n^2\pi^2}\left[\frac{\cancel{\sin(n\pi)}^{0}}{n\pi} - 0 + \frac{1}{n^2\pi^2}\cos(n\pi x)\Big|_0^1\right]$$

$$- \frac{2}{1+n^2\pi^2}\left[-\frac{\cos(n\pi)}{n\pi} + 0 + \frac{1}{n^2\pi^2}\sin(n\pi x)\Big|_0^1\right]$$

$$= \frac{2n\pi}{1+n^2\pi^2}\left[\frac{1}{n^2\pi^2}\left(\cos(n\pi) - \cos(0)\right)\right]$$

$$- \frac{2}{1+n^2\pi^2}\left[\frac{-(-1)^n}{n\pi} + \frac{1}{n^2\pi^2}\left(\cancel{\sin(n\pi)}^{0} - \cancel{\sin(0)}^{0}\right)\right]$$

$$= \frac{2n\pi}{1+n^2\pi^2}\left[\frac{(-1)^n - 1}{n^2\pi^2}\right] - \frac{2}{1+n^2\pi^2}\left[\frac{-(-1)^n}{n\pi}\right]$$

$$= \frac{2}{1+n^2\pi^2}\left[\frac{(-1)^n - 1}{n\pi} + \frac{(-1)^n}{n\pi}\right]$$

$$\Rightarrow A_n = \frac{2}{1+n^2\pi^2}\left[\frac{2(-1)^n - 1}{n\pi}\right] \longrightarrow (21)$$

Using (20) and (21) in (19) we get

$$u(x,t) = \frac{2e(e-2)}{e^2-1} e^{-x} e^{t} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{2(-1)^n - 1}{n(1+n^2\pi^2)} \left[(n\pi)\cos(n\pi x) - \sin(n\pi x)\right] e^{-n^2\pi^2 t}.$$

Q21- Given that

$$u_t = i\, u_{xx} \;; \quad 0 < x < L \;; \quad t > 0 \longrightarrow (1)$$

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0 \longrightarrow (2)$$

Let $u(x,t) = X(x)T(t)$ be the solution of the given BVP. So, eq (1) & (2) becomes

$$X(x)\,T'(t) = i\, X''(x)\, T(t)$$

$$\Rightarrow \frac{T'(t)}{i\,T(t)} = \frac{X''(x)}{X(x)} = K,$$

where K is some constant. From here we get

$$T'(t) - iK\, T(t) = 0 \longrightarrow (3)$$

$$X''(x) - K\, X(x) = 0 \longrightarrow (4)$$

with

$$X(0) = 0 \quad \text{and} \quad X(L) = 0 \longrightarrow (5)$$

Let us consider BVP involving "x", which is given as:

$$X''(x) - K\, X(x) = 0 \longrightarrow (4)$$

$$X(0) = 0 \quad \text{and} \quad X(L) = 0 \longrightarrow (5)$$

Case 1 : $K = 0$

For $K = 0$ we get

$$X''(x) = 0$$

$$\Rightarrow X(x) = c_1 x + c_2$$

Using (5) we get

$$c_1 = 0 \quad \text{and} \quad c_2 = 0$$

$$\Rightarrow X(x) = 0,$$

a trivial solution which is of no interest.

Case 2: $k > 0$

Let $k = p^2 > 0$, so that (4) becomes:

$$X''(x) - p^2 X(x) = 0$$

$$\Rightarrow (D^2 - p^2) X(x) = 0$$

Characteristic equation is:

$$D^2 - p^2 = 0$$

$$\Rightarrow D = \pm p$$

Thus, general solution is given as:

$$X(x) = c_3 e^{-px} + c_4 e^{px} \rightarrow (6)$$

Using (5) in (6)

$$0 = c_3 + c_4 \quad [X(0) = 0]$$

and $$0 = c_3 e^{-pL} + c_4 e^{+pL} \quad [X(L) = 0]$$

Now $c_3 + c_4 = 0 \Rightarrow c_4 = -c_3$

So

$$c_3 e^{-pL} + c_4 e^{+pL} = 0$$

$$\Rightarrow -c_4 e^{-pL} + c_3 e^{pL} = 0$$

$$\Rightarrow c_4 \left[e^{pL} - e^{-pL}\right] = 0$$

$$\Rightarrow c_4 = 0 \quad \left[\because e^{pL} - e^{-pL} = 2 \sinh pL \neq 0\right]$$

Since $\boxed{c_4 = 0}$ and $c_3 = -c_4$ so $\boxed{c_3 = 0}$

Thus, (6) becomes

$$X(x) = 0,$$

again a trivial solution which is of no interest.

Case 3 : $K < 0$

Let $K = -p^2$, so that (4) becomes

$$X''(x) + p^2 X(x) = 0$$

$$\Rightarrow (D^2 + p^2) X = 0$$

Characteristic equation is:

$$D^2 + p^2 = 0 \Rightarrow D = \pm ip$$

Thus,

$$X(x) = C_5 \cos(px) + C_6 \sin(px) \rightarrow (7)$$

Using (5) in (7) we get

$$X(0) = 0 \Rightarrow \boxed{C_5 = 0}$$

and $X(L) = 0 \Rightarrow C_5 \cos(pL) + C_6 \sin(pL) = 0$

$$\Rightarrow C_6 \sin(pL) = 0 \quad [\because C_5 = 0]$$

For non-trivial solution $C_5 \neq 0$. Thus,

$$\sin(pL) = 0$$

$$\Rightarrow pL = n\pi$$

$$\Rightarrow p = \frac{n\pi}{L}; \quad n = 1, 2, 3, \ldots$$

Thus,

$$X_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right); \quad n = 1, 2, \ldots \rightarrow (8)$$

Let us now consider eq (3),

$$T'(t) - iKT(t) = 0$$

$$\Rightarrow T'(t) - i(-p^2)T(t) = 0 \quad [\because K = -p^2]$$

$$\Rightarrow T'(t) + ip^2 T(t) = 0$$

$$\Rightarrow T'(t) + i\left(\frac{n\pi}{L}\right)^2 T(t) = 0$$

$$\Rightarrow T'(t) = -i\left(\frac{n\pi}{L}\right)^2 T(t)$$

$$\Rightarrow \frac{dT(t)}{T(t)} = -i\left(\frac{n\pi}{L}\right)^2 dt$$

$$\Rightarrow \quad T_n(t) = B_n e^{-i(n\pi/L)^2 t}$$

$$\Rightarrow T_n(t) = B_n \cos\left(\frac{n\pi}{L}\right)t + B_n^* \sin\left(\frac{n\pi}{L}\right)t \quad \left[\begin{array}{l}\text{Using} \\ \text{Euler's formula}\end{array}\right] \to (9)$$

where $B_n^* = -iB_n$.

Using (8) and (9) we get

$$U_n(x,t) = X_n(x)\,T_n(t)$$

$$= \left[B_n \cos\left(\frac{n\pi}{L}\right)t + B_n^* \sin\left(\frac{n\pi}{L}\right)t\right] A_n \sin\left(\frac{n\pi}{L}x\right)$$

By super position principle we get

$$u(x,t) = \sum_{n=1}^{\infty}\left[E_n \cos\left(\frac{n\pi}{L}\right)t + E_n^* \sin\left(\frac{n\pi}{L}\right)t\right] \sin\left(\frac{n\pi}{L}\right)x,$$

where $E_n = B_n A_n$ and $E_n^* = B_n^* A_n$.

Q3:- The steady state (or time independent) temperature distribution in a circular disk of radius "1" with prescribed temperature at the boundary, satisfies the two-dimensional Laplace equation in polar coordinates and is given by:

$$(1) \leftarrow \nabla^2 u = \rho^2 u_{\rho\rho} + \rho u_\rho + u_{\theta\theta} = 0 \; ; \; 0<\rho<1, \; 0<\theta<2\pi$$

subject to the boundary conditions:

$$u(1,\theta) = f(\theta), \quad 0<\theta<2\pi \quad \to (2)$$

where $f(\theta) = \begin{cases} 100 & (0<\theta<\pi) \\ 0 & (\pi<\theta<2\pi) \end{cases}$

This is a problem of Laplace equation for a circular disk for which other boundary conditions are given as:

$$|u(0,\theta)| < \infty \longrightarrow (3)$$

and
$$\begin{aligned} u(r,0) &= u(r,2\pi) \\ u_\theta(r,0) &= u_\theta(r,2\pi) \end{aligned} \longrightarrow (4)$$

Following the method of separation of variables, we will look for product solutions of the form:

$$u(r,\theta) = R(r)\,\Theta(\theta) \longrightarrow (5)$$

Using (5) in (1), (3), and (4) we get

$$\Theta''(\theta) + K\,\Theta(\theta) = 0 \longrightarrow (6)$$
$$r^2 R''(r) + r R'(r) - K R(r) = 0 \longrightarrow (7)$$
$$\left.\begin{aligned}\Theta(0) &= \Theta(2\pi) \\ \Theta_\theta(0) &= \Theta(2\pi)\end{aligned}\right\} \longrightarrow (8)$$

and $|R(0)| < \infty \Rightarrow R(r) \longrightarrow \infty$ as $r \longrightarrow 0 \rightarrow (9)$

Let us solve the BVP involving 'θ' first.

$$\Theta''(\theta) + \lambda\,\Theta(\theta) = 0 \rightarrow (6)$$

and
$$\left.\begin{aligned}\Theta(0) &= \Theta(2\pi) \\ \Theta_\theta(0) &= \Theta_\theta(2\pi)\end{aligned}\right\} \rightarrow (8)$$

(6) is a second order linear homogeneous PDE which possesses a non-trivial solution for $K>0$. Thus, if $K=\lambda^2>0$, the general solution is given as:

$$\Theta(\theta) = c_1 \cos(\lambda\theta) + c_2 \sin(\lambda\theta) \longrightarrow (10)$$

Using (8) in (10) we find that the eigenvalues and corresponding eigenvectors ("circular harmonics") are given as:

$$K = \lambda^2 = n^2 \; ; \quad n = 0, 1, 2, \ldots$$

and

$$\Theta_n(\theta) = \begin{cases} a_0; & \text{for } n=0 \\ a_n \cos(n\theta) + b_n \sin(n\theta); & \text{for } n = 1, 2, \ldots \end{cases} \rightarrow (11)$$

respectively. For these values of K, equation (7) takes the form:

$$(12) \leftarrow r^2 R''(r) + r R'(r) - n^2 R(r) = 0 \; ; \quad n = 1, 2, \ldots$$

This is a 2nd order Cauchy-Euler differential equation. Let $r = e^t \Rightarrow t = \ln|r|$, so that
$rD = \Delta$ and $r^2 D^2 = \Delta(\Delta - 1)$

where $D = d/dr$ and $\Delta = d/dt$. Using above in (12) we get:

$$(\Delta^2 - n^2) R = 0$$
$$\Rightarrow R_n(t) = \begin{cases} c_3 + c_4 t \ ; & n = 0 \\ c_5 e^{nt} + c_6 e^{-nt} \ ; & n = 1, 2, \ldots \end{cases}$$

Since $t = e^t$ or $t = \ln|r|$ so we get

$$R_n(r) = \begin{cases} c_3 + c_4 \ln|r| \ ; & n = 0 \\ c_5 r^n + c_6 r^{-n} \ ; & n = 1, 2, \ldots \end{cases}$$

Using (9) we see that we must have c_4 and c_6 equal to zero. So we are left with

$$R_n(r) = \begin{cases} c_3 \ ; & n = 0 \\ c_5 r^n \ ; & n = 1, 2, \ldots \end{cases} \longrightarrow (13)$$

From (11) & (13) we get

$$u(r, \theta) = \begin{cases} A_0 \ ; & n = 0 \\ [A_n \cos(n\theta) + B_n \sin(n\theta)] r^n \ ; & n = 1, 2, \ldots \end{cases}$$

where $A_0 = a_0 c_3$, $A_n = a_n c_5$ and $B_n = b_n c_5$.

Superposing these solutions, we get

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(n\theta) + B_n \sin(n\theta)] r^n \rightarrow (14)$$

Using (2) in (14) we get

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

This is a trigonometric Fourier series of $f(\theta)$ with Fourier coefficients:

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta = \frac{1}{2\pi} \int_0^{\pi} (100)\, d\theta = \frac{50}{\pi} |\theta|_0^{\pi} = 50,$$

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta)\, d\theta = \frac{100}{\pi} \int_0^{\pi} \cos(n\theta)\, d\theta$$
$$= \frac{100}{\pi} \left| \frac{\sin(n\theta)}{n} \right|_0^{\pi} = \frac{100}{n\pi} [\cancel{\sin(n\pi)} - \cancel{\sin(0)}]$$
$$= 0$$

$$\Rightarrow \boxed{A_n = 0}$$

and

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta$$

$$= \frac{100}{\pi}\int_0^{\pi} \sin(n\theta)\,d\theta = -\frac{100}{\pi}\left|\frac{\cos(n\theta)}{n}\right|_0^{\pi}$$

$$= \frac{-100}{n\pi}\left[\cos(n\pi) - \cos(0)\right] = \frac{-100}{n\pi}\left[(-1)^n - 1\right]$$

$$= \frac{100\left[1 - (-1)^n\right]}{n\pi} = \begin{cases} \frac{100\,(0)}{n\pi} = 0 & ; \text{ if } n \text{ is even} \\ \frac{100(2)}{n\pi} = \frac{200}{n\pi} & ; \text{ if } n \text{ is odd} \end{cases}$$

Thus,

$$u(r,\theta) = 50 + \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{r^{(2n-1)}}{(2n-1)} \sin(2n-1)\theta.$$

Q4:- Given that

$$u_{tt} = 4u_{xx} \longrightarrow (1) \quad \left[\because u_{tt} = c^2 u_{xx}\right] \text{ wave equation}$$

Subject to the initial conditions

$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = g(x).$$

Using d' Alembert's formula, solution of (1) is given as

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

where $c = 2$, $f(x) = 0$ and $g(x) = \begin{cases} 1 & ; |x| < 3 \\ 0 & ; \text{otherwise.} \end{cases}$

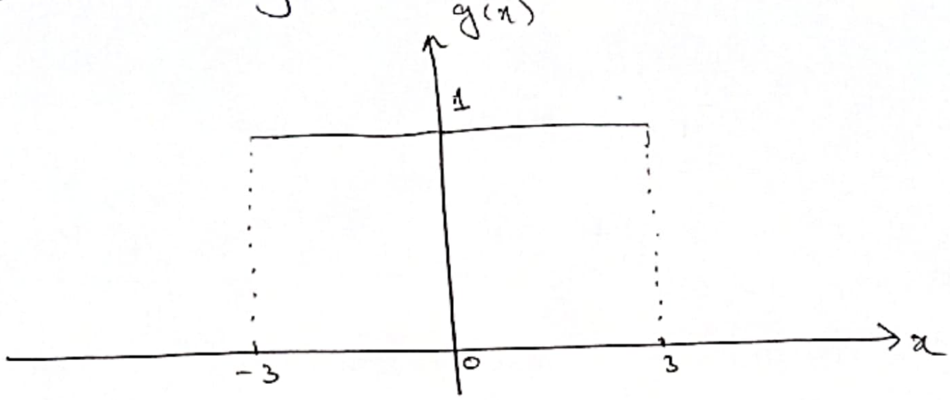
Thus, for the present case we have

$$u(x,t) = \frac{1}{2(2)}\int_{x-2t}^{x+2t} g(s)\,ds$$

$$\Rightarrow u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} g(s)\,ds,$$

where $g(x) = \begin{cases} 1 & ; |x| < 3 \\ 0 & ; \text{otherwise.} \end{cases}$

$g(x)$ is a rectangular pulse of width 6 and height 1.



For $t = 3/4$ sec, we have

$$u(x, 3/4) = \frac{1}{4}\int_{x-2(3/4)}^{x+2(3/4)} g(s)\, ds$$

$$= \frac{1}{4}\int_{x-3/2}^{x+3/2} g(s)\, ds,$$

where value of this integral depends on the location of the interval $(x-3/2, x+3/2)$ relative to $(-3, 3)$. In order to determine the location of the interval $(x-3/2, x+3/2)$ relative to the interval $(-3,3)$, we need to sketch string profile at $t = 3/4$ sec.

Case 1:- If $x + 3/2 \le -3$

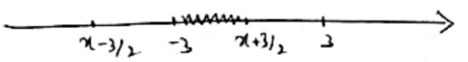
or $x \le -3 - \frac{3}{2}$

or $x \le -\frac{9}{2}$

Here $g(x) = 0$. Thus,

$$u(x, 3/4) = 0 \quad \text{if } x \le -9/2$$

Case 2:- If $x + \frac{3}{2} > -3$ and $x - \frac{3}{2} < -3$



or $x > -\frac{9}{2}$ and $x < -\frac{3}{2}$.

Here

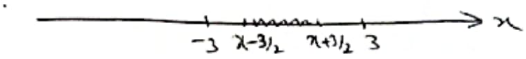
$$u(x, 3/4) = \frac{1}{4}\int_{-3}^{x+3/2} g(s)\, ds = \frac{1}{4}\int_{-3}^{x+3/2} (1)\, ds = \frac{1}{4}\left[x + \frac{3}{2} - (-3)\right]$$

$$\Rightarrow u(x,3/4) = \frac{1}{4}\left[x+\frac{3}{2}+3\right] = \frac{1}{4}\left[x+\frac{3+6}{2}\right] = \frac{1}{4}\left[x+\frac{9}{2}\right]$$

Thus,

$$u(x,3/4) = \frac{1}{4}\left(x+\frac{9}{2}\right) \quad \text{if} \quad -\frac{9}{2} < x < -\frac{3}{2}.$$

Case 3: If $x-\frac{3}{2} > -3$ and $x+\frac{3}{2} < 3$



or $x > -\frac{3}{2}$ and $x < \frac{3}{2}$

or $-\frac{3}{2} < x < \frac{3}{2}$

then,

$$u(x,3/4) = \frac{1}{4}\int_{x-3/2}^{x+3/2} g(s)\,ds = \frac{1}{4}\int_{x-3/2}^{x+3/2} (1)\,ds$$

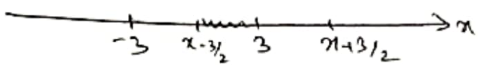
$$= \frac{1}{4}\Big|s\Big|_{x-3/2}^{x+3/2} = \frac{1}{4}\left[\left(x+\frac{3}{2}\right)-\left(x-\frac{3}{2}\right)\right]$$

$$= \frac{1}{4}\left[\cancel{x}+\frac{3}{2}-\cancel{x}+\frac{3}{2}\right] = \frac{1}{4}\left[\frac{6}{2}\right] = \frac{3}{4}$$

Thus,

$$u(x,3/4) = \frac{3}{4} \quad \text{if} \quad -\frac{3}{2} < x < \frac{3}{2}$$

Case 4: If $x-\frac{3}{2} < 3$ and $x+\frac{3}{2} > 3$



or $x < \frac{9}{2}$ and $x > \frac{3}{2}$

then

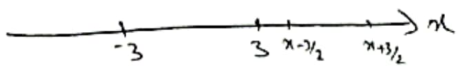
$$u(x,3/4) = \frac{1}{4}\int_{x-3/2}^{3} g(s)\,ds = \frac{1}{4}\int_{x-3/2}^{3} (1)\,ds = \frac{1}{4}\Big|s\Big|_{x-3/2}^{3}$$

$$= \frac{1}{4}\left[3-\left(x-\frac{3}{2}\right)\right] = \frac{1}{4}\left[3-x+\frac{3}{2}\right] = \frac{1}{4}\left[\frac{9}{2}-x\right]$$

Thus,

$$u(x,3/4) = \frac{1}{4}\left(\frac{9}{2}-x\right) \quad \text{if} \quad \frac{3}{2} < x < \frac{9}{2}$$

Case 5: If $x-3/2 \geq 3$



or $x \geq 9/2$

then $g(x) = 0$

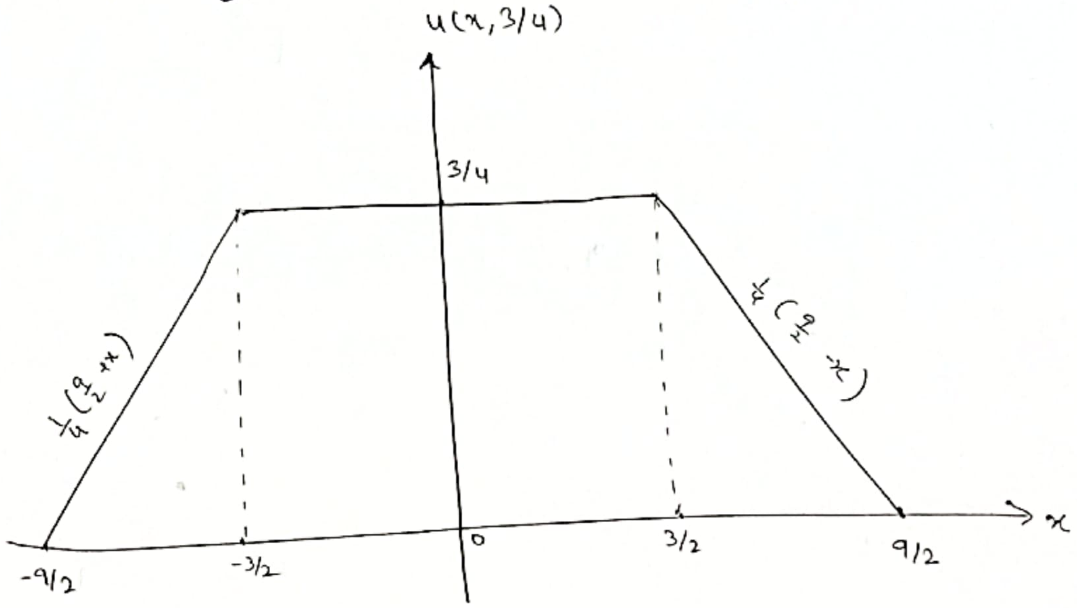
$$u(x,3/4) = 0$$

Thus, $u(x,3/4) = 0$ if $x \geq 9/2$

Let us combine all these results to write $u(x,t)$ for $t = 3/4$ sec.

$$u(x,3/4) = \frac{1}{4}\begin{cases} 0; & \text{if } x \le -9/2 \\ x+\frac{9}{2}; & \text{if } -\frac{9}{2} < x < -\frac{3}{2} \\ 3; & \text{if } -\frac{3}{2} < x < \frac{3}{2} \\ -x+\frac{9}{2}; & \text{if } \frac{3}{2} < x < 9/2 \\ 0; & \text{if } x \ge 9/2 \end{cases}$$

Graph of $u(x,3/4)$ is given as:



Q5.- Following BVP corresponds to the given physical situation:

① $\leftarrow$ $u_{tt} = c^2 [u_{xx} + u_{yy}]$; $0 < x < 1$; $0 < y < 1$, $t > 0$

② $\leftarrow$ $u(0,y,t) = 0 = u(1,y,t)$; $0 < y < 1$, $t \ge 0$

③ $\leftarrow$ $u(x,0,t) = 0 = u(x,1,t)$; $0 < x < 1$, $t \ge 0$

④ $\leftarrow$ $u(x,y,0) = f(x,y)$, $u_t(x,y,0) = 0$; $0 < x < 1$ & $0 < y < 1$
$\hookrightarrow$ ⑤

where $f(x,y) = xy(x-1)(y-1)$.

By method of separation of variable we choose:

$$u(x,y,t) = X(x)\, Y(y)\, Z(t) \rightarrow ⑥$$

Using (6) in (1) we get

$$X(x)\,Y(y)\,T''(t) = c^2\left[X''(x)\,Y(y)\,T(t) + X(x)\,Y''(y)\,T(t)\right]$$

$$\Rightarrow \quad \frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} \quad \left[\text{Dividing both sides by } X(x)\,Y(y)\,T(t)\right]$$

Since the left side is a function of t alone, and right side is a function of x & y only, the expressions on both sides must be equal to a constant. Expecting a periodic solution in t, we consider negative separation constants only. (We can also rule out the non-negative cases by arguing, as has been done in remaining questions that they only lead to trivial solutions). Thus,

$$\frac{T''}{c^2T} = -k^2 \quad \text{and} \quad \frac{X''}{X} + \frac{Y''}{Y} = -k^2 \quad (k>0)$$

The first equation yields:

$$T'' + k^2c^2T = 0 \rightarrow (7)$$

(with periodic solution s) and the second one yields:

$$\frac{X''}{X} = -\frac{Y''}{Y} - k^2.$$

Because in this last equation the right side depends only on y and the left side only on x, we infer that

$$\frac{X''}{X} = -\mu^2 \quad \text{and} \quad -\frac{Y''}{Y} - k^2 = -\mu^2, \quad \mu>0$$

or

$$X'' + \mu^2 X = 0 \rightarrow (8) \quad \text{and} \quad Y'' + \nu^2 Y = 0, \rightarrow (9)$$

where $\nu^2 = k^2 - \mu^2$. (Here again we have ruled out the non-negative values of the separation constant on the basis that they lead to trivial solutions).

Separating variables in the boundary conditions (2) & (3), we arrive at the equations:

$$X(0) = 0,\ X(1) = 0 \rightarrow (10)$$

$$Y(0) = 0,\ Y(1) = 0 \rightarrow (11)$$

Thus, we get following three separated equations:

$$X'' + \mu^2 X = 0 \ ;\ X(0) = 0,\ X(1) = 0$$

$$Y'' + \nu^2 Y = 0 \ ;\ Y(0) = 0,\ Y(1) = 0$$

and

$$T'' + c^2 K^2 T = 0 \ ;\quad K^2 = \mu^2 + \nu^2$$

Solution of the separated equations:
The general solutions of the last three differential equations are respectively given as:

$$X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

$$Y(y) = d_1 \cos(\nu y) + d_2 \sin(\nu y)$$

$$T(t) = e_1 \cos(ckt) + e_2 \sin(ckt) \quad [K^2 = \mu^2 + \nu^2]$$

From the boundary condition (10) and (11) for X and Y we get $c_1 = 0$ and $c_2 \sin(\mu) = 0$, $d_1 = 0$ and $d_2 \sin(\nu) = 0$. Thus,

$$\mu = \mu_m = m\pi \quad \text{and} \quad \nu = \nu_n = n\pi \ ;\ m, n = 1, 2, \ldots$$

and so

$$X_m(x) = \sin(m\pi x) \quad \text{and} \quad Y_n(y) = \sin(n\pi y).$$

Note that if $m = 0$ or $n = 0$, the solutions are identically zero, which are of no interest. Also negative choices of m and n would only change the signs of the solutions and hence would not contribute new solutions. For $m, n = 1, 2, \ldots$, we have:

$$K = K_{mn} = \sqrt{\mu^2_m + \nu^2_n} = \sqrt{m^2\pi^2 + n^2\pi^2}$$

and so,

$$T(t) = T_{mn}(t) = B_{mn} \cos(\lambda_{mn} t) + B^*_{mn} \sin(\lambda_{mn} t)$$

where $\lambda_{mn} = c\pi\sqrt{m^2+n^2}$. The λ_{mn}'s are called the characteristic frequencies of the membrane. In contrast to the one dimensional case of a vibrating string, the characteristic frequencies are not integers multiples of any basic frequency. We have thus derived the product solution satisfying the equations (1) & (2) & 3 and this solution is given as:

$$u_{mn}(x,y,t) = \sin(m\pi x)\sin(n\pi y)\left[B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right]$$

The functions u_{mn} are called the normal modes of the two dimensional wave equation.

In order to determine a solution that also satisfies the initial conditions (4) & (5), motivated by the superposition principle, we sum all the product solutions and try

$$(12) \leftarrow u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right]\sin(m\pi x)\sin(n\pi y).$$

From the initial condition $u(x,y,0) = f(x,y)$, we get

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn}\sin(m\pi x)\sin(n\pi y)$$

where $f(x,y) = x(x-1)y(y-1)$. The key to computing the coefficients B_{mn} is to observe that the functions $\sin(m\pi x)\sin(n\pi y)$ are "orthogonal" over the square $0\le x\le 1$, $0\le y\le 1$. This means, B_{mn} are Fourier coefficients and

$$B_{mn} = 4\int_0^1\int_0^1 f(x,y)\sin(m\pi x)\sin(n\pi y)\,dx\,dy$$

$$\Rightarrow B_{mn} = 4\int_0^1\int_0^1 x(x-1)\,y(y-1)\sin(m\pi x)\sin(n\pi y)\,dx\,dy$$

$$\Rightarrow B_{mn} = 4\int_0^1 \left[\int_0^1 x(x-1)\sin(m\pi x)\,dx\right] y(y-1)\sin(m\pi y)\,dy$$

$$= 4\int_0^1 I\, y(y-1)\sin(m\pi y)\,dy \longrightarrow (13)$$

where $\quad I = \int_0^1 x(x-1)\sin(m\pi x)\,dx$

$$\Rightarrow I = \int_0^1 (x^2-x)\sin(m\pi x)\,dx$$

$$= \int_0^1 x^2\sin(m\pi x)\,dx - \int_0^1 x\sin(m\pi x)\,dx$$

$$= -x^2\frac{\cos(m\pi x)}{m\pi}\Big|_0^1 + \frac{2}{m\pi}\int_0^1 x\cos(m\pi x)\,dx$$

$$-\left[-x\frac{\cos(m\pi x)}{m\pi}\Big|_0^1 + \frac{1}{m\pi}\int_0^1 \cos(m\pi x)\,dx\right]$$

$$= -\frac{\cos(m\pi)}{m\pi} + \frac{2}{m\pi}\left[x\,\cancel{\frac{\sin(m\pi x)}{m\pi}}^{\,0}\Big|_0^1 - \frac{1}{m\pi}\int_0^1 \sin(m\pi x)\,dx\right]$$

$$+\frac{\cos(m\pi)}{m\pi} - \frac{1}{m\pi}\left(\frac{\sin(m\pi x)}{m\pi}\right)\Big|_0^1$$

$$= -\cancel{\frac{(-1)^m}{m\pi}} - \frac{2}{m^2\pi^2}\left(-\frac{\cos(m\pi x)}{m\pi}\right)\Big|_0^1$$

$$+\cancel{\frac{(-1)^m}{m\pi}} - \frac{1}{m^2\pi^2}(0)$$

$$= \frac{2}{m^3\pi^3}\left[\cos(m\pi) - \cos(0)\right]$$

$$\Rightarrow I = \frac{2\left[(-1)^m - 1\right]}{m^3\pi^3} \longrightarrow (14)$$

Using (14) in (13) we get

$$B_{mn} = 4 \int_0^1 \left[\frac{2[(-1)^m - 1]}{m^3\pi^3} y(y-1) \sin(n\pi y) \right] dy$$

$$= \frac{8[(-1)^m - 1]}{m^3\pi^3} \int_0^1 y(y-1) \sin(n\pi y) dy$$

$$= \frac{8[(-1)^m - 1]}{m^3\pi^3} \left[\frac{2[(-1)^n - 1]}{n^3\pi^3} \right] \left[\text{using (14) for } x \to y \text{ and } m \to n \right]$$

$$= \frac{16[(-1)^m - 1][(-1)^n - 1]}{(m^3\pi^3)(n^3\pi^3)}.$$

Thus,

$$B_{mn} = \frac{16[(-1)^m - 1][(-1)^n - 1]}{\pi^6 m^3 n^3} \quad \forall\, m, n = 1, 2, \ldots$$

Note that if either of m or n is even we get $B_{mn} = 0$. If both m and n are odd

$$B_{mn} = \frac{64}{\pi^6 m^3 n^3}.$$

Hence,

$$B_{mn} = \begin{cases} \dfrac{64}{\pi^6 m^3 n^3} & ; \text{ if } m + n \text{ both are odd} \\ 0 & ; \text{ otherwise.} \end{cases}$$

Now arguing as before with the help of orthogonality, we obtain for the other condition (5)

$$B^*_{mn} = \frac{4}{\lambda_{mn}} \int_0^1 \int_0^1 g(x,y) \sin(m\pi x) \sin(n\pi y)\, dx\, dy.$$

Since for the present case $u_t = g(x,y) = 0$, so

$$B^*_{mn} = 0.$$

Thus, the solution (12) takes the form:

$$u(x,y,t) = \sum_{n\,odd} \sum_{m\,odd} \frac{64}{\pi^6 m^3 n^3} \sin(m\pi x)\sin(n\pi y)\cos\left(\sqrt{m^2+n^2}\,t\right)$$

$$= \sum_{l=1}^{\infty} \sum_{K=1}^{\infty} \left[\frac{64}{\pi^6 (2K-1)^3 (2l-1)^3} \sin((2K-1)\pi x)\sin((2l-1)\pi y) \times \cos\left(\sqrt{(2K-1)^2+(2l-1)^2}\,t\right)\right]$$

where $m = 2K-1$ and $n = 2l-1$, since only odd values will survive.