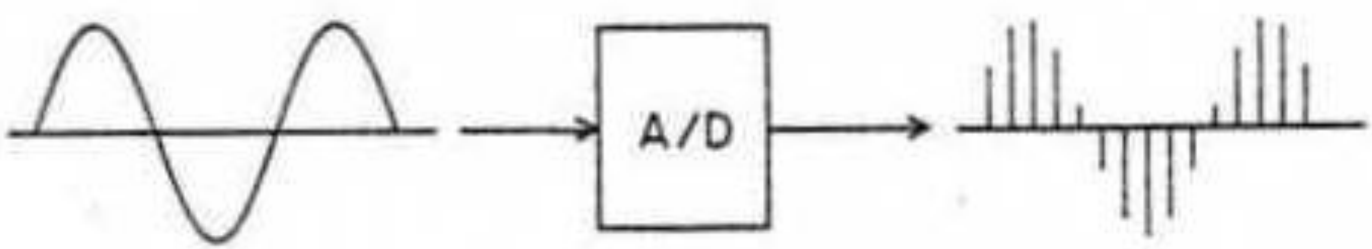
INTRODUCTION TO SAMPLING

Sampling

- In signal processing sampling is the reduction of a continuous time signal to a discrete time signal

- A sample refers to a value or set of values at a point in time and/or space



— sampling, taking snap shots of $x(t)$ every T seconds.

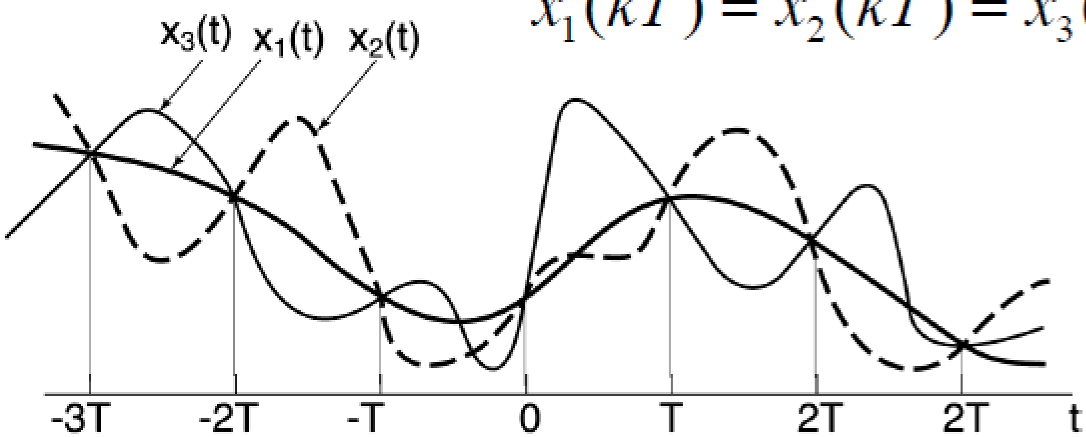
T – sampling period

$x[n] \equiv x(nT)$, $n = \ldots, -1, 0, 1, 2, \ldots$ — regularly spaced samples

Adequate set of samples

- Observation: *Lots* of signals have the same set of samples

$$x_1(kT) = x_2(kT) = x_3(kT)$$



Three continuous-time signals with identical values at integer multiples of *T*.

- By sampling we throw out lots of information
  - all values of $x(t)$ between sampling points are lost.

- **Key Question for Sampling**:

Under what conditions can we reconstruct the original CT signal $x(t)$ from its samples?

Sampling of Signals

- Key Question for Sampling:

Under what conditions can we reconstruct the original CT signal $x(t)$ from its samples?

If a signal is band limited, i.e., if its Fourier transform is zero outside a finite band of frequencies, and if the samples are taken sufficiently close together in relation to the highest frequency present in the signal, then the samples *uniquely* specify the signal, and we can reconstruct it perfectly!!

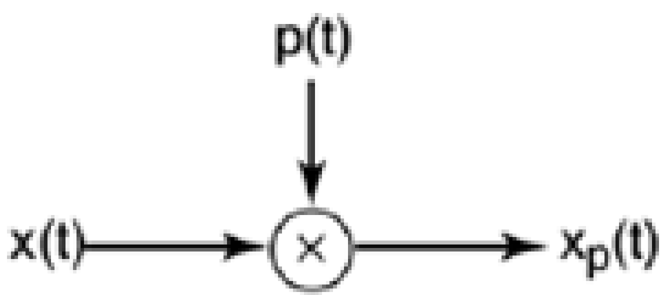
This result is known as the *sampling theorem* and is of profound importance for signal and system analysis.

Impulse Train Sampling

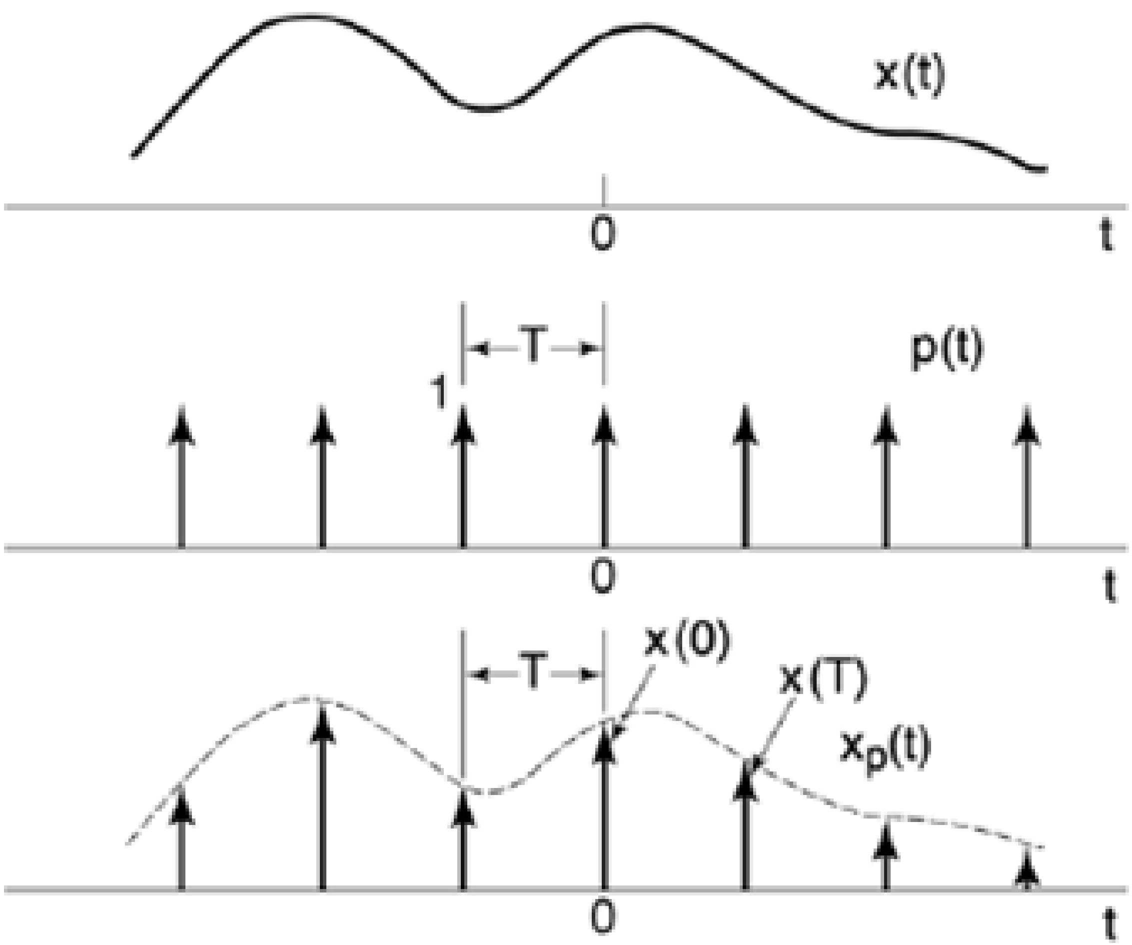
Impulse Train Sampling — Multiplying *x*(*t*) by a periodic train of impulses – called the sampling function

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$



Impulse Train Sampling



Analysis of Sampling in Frequency Domain

$$x_p(t) = x(t) \cdot p(t)$$

$$\text{Multiplication Property} \Rightarrow X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$\omega_s = \frac{2\pi}{T} = \text{Sampling Frequency}$$

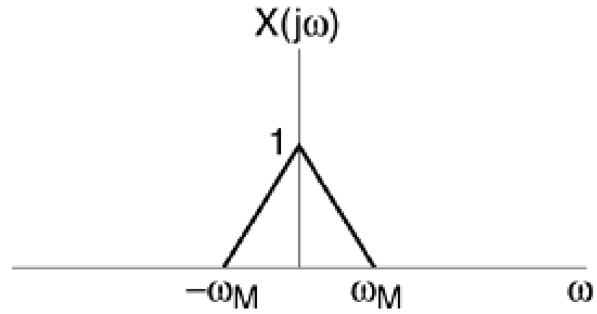
Important to note: $\omega_s \propto 1/T$

Analysis of Sampling in Frequency Domain

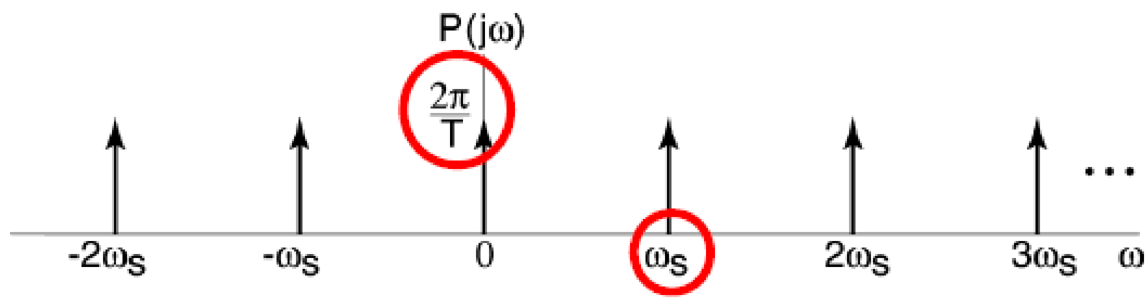
$$
\begin{aligned}
X_p(j\omega) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j\omega) * \delta(\omega - k\omega_s) \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))
\end{aligned}
$$

Analysis of Sampling in Frequency Domain

Illustration of sampling in the frequency-domain for a band-limited ($X(j\omega)=0$ for $|\omega|> \omega_M$) signal

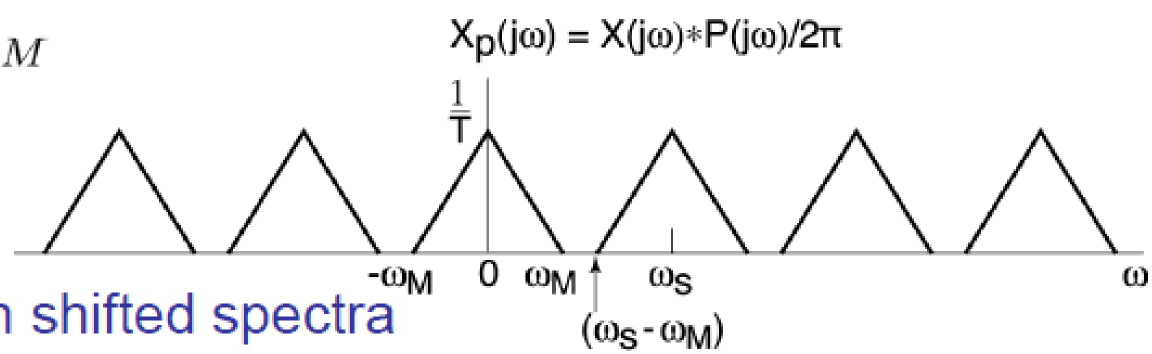


$X_p(j\omega)$ drawn assuming



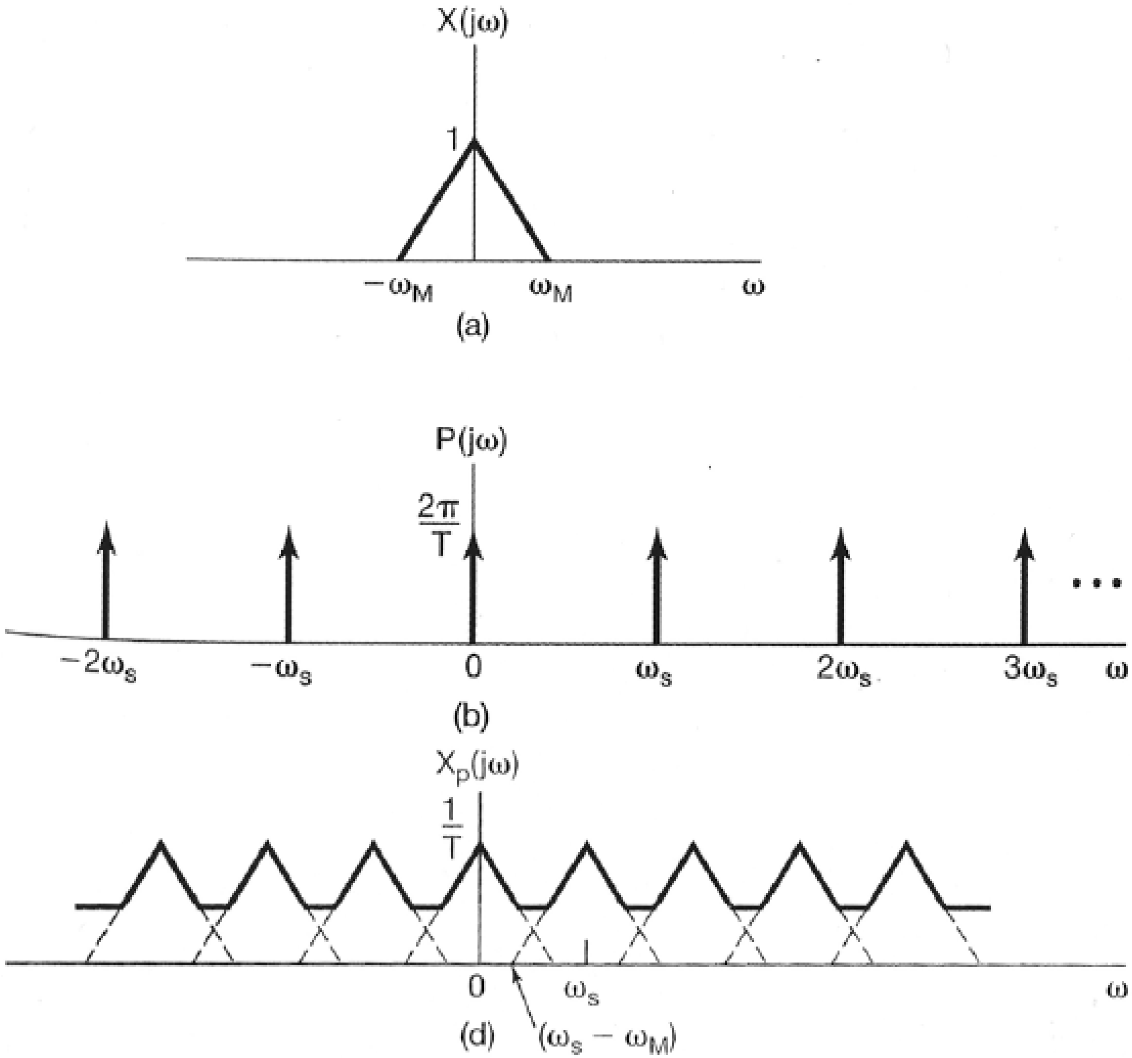
$$\omega_s - \omega_M > \omega_M$$

$$i.e. \qquad \omega_s > 2\omega_M$$



No overlap between shifted spectra

Under-sampling



(a) Spectrum of original signal;

(b) Spectrum of sampling function;

(c) Spectrum of sampled signal with $\omega_s > 2\omega_M$ (shown on previous slide)

(d) Spectrum of sampled signal with $\omega_s < 2\omega_M$

$X_p(j\omega)$ is a periodic function of ω consisting of a superposition of shifted replicas of $X(j\omega)$, scaled by $1/T$.

Sampling Theorem

Suppose $x(t)$ is bandlimited, so that

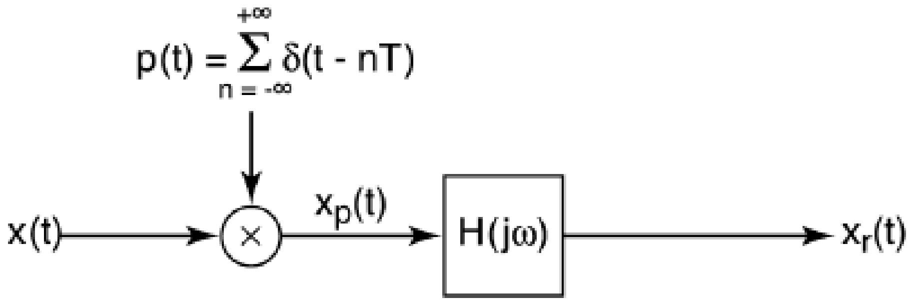
$$X(j\omega) = 0 \quad \text{for} \quad |\omega| > \omega_M$$

Then $x(t)$ is uniquely determined by its samples $\{x(nT)\}$ if

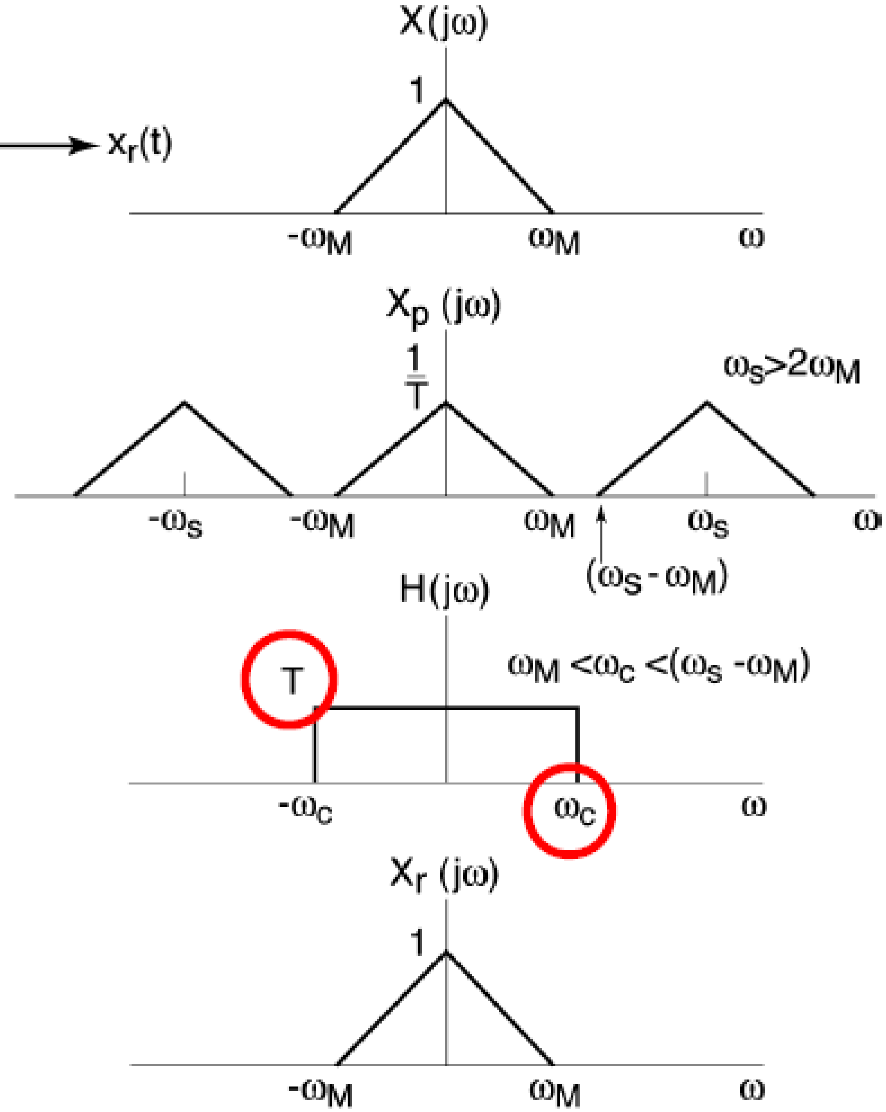
$$\boxed{\omega_s > 2\omega_M = \text{The Nyquist rate}}$$

$$\text{where } \omega_s = 2\pi/T$$

Signal Reconstruction



Reconstruction of $x(t)$ from sampled signals



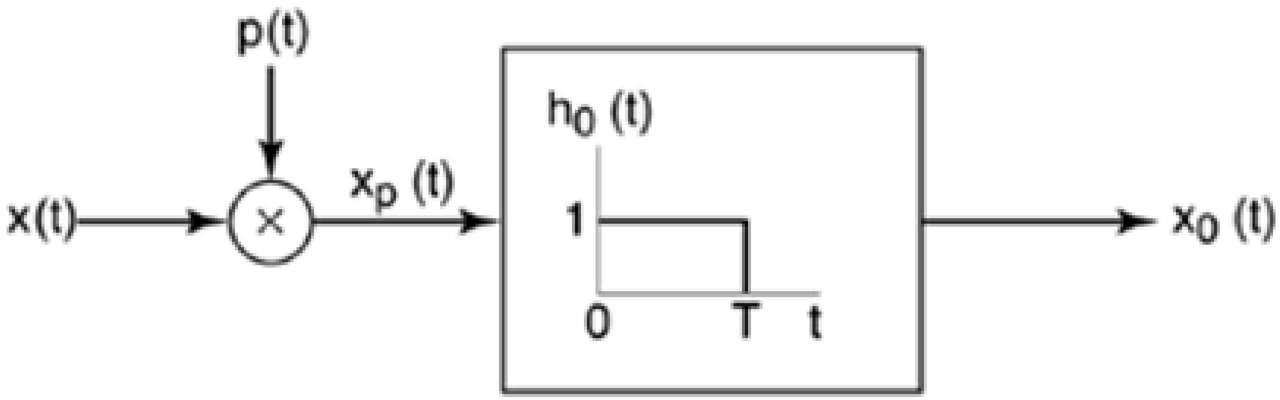
If there is no overlap between shifted spectra, a LPF can reproduce $x(t)$ from $x_p(t)$

Sampling with Zero-Order Hold

- Impulses (even narrow, large-amplitude pulses which approximate impulses) are hard to generate and transmit
- It is more convenient to generate the sampled signal as a *zero-order hold* as shown below.
- The zero-order hold samples $x(t)$ at a given instant, and holds that value until the next instant at which a sample is taken.
- Signal reconstruction can again be carried out by lowpass filtering.

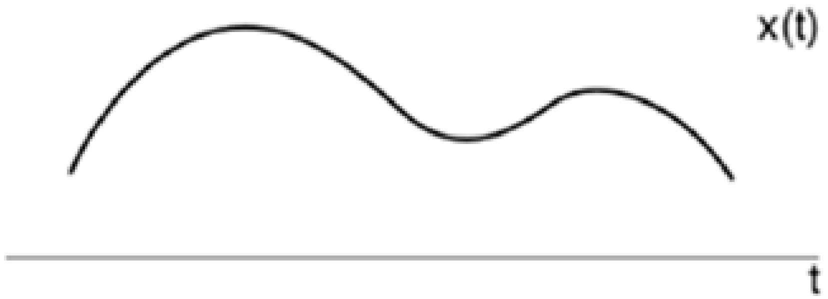
Zero-Order Hold Filtering

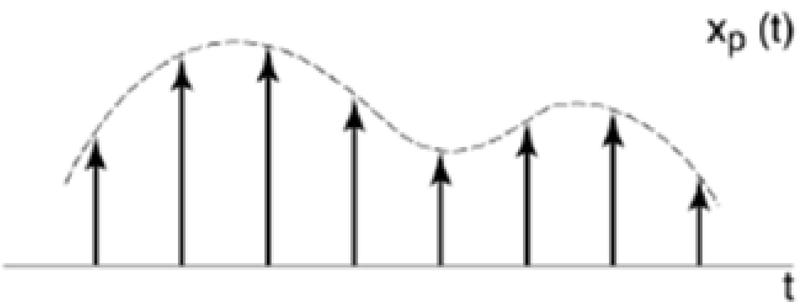
- The output from the zero-order hold, $x_o(t)$ can be generated by impulse-train sampling followed by an LTI system having a rectangular impulse response

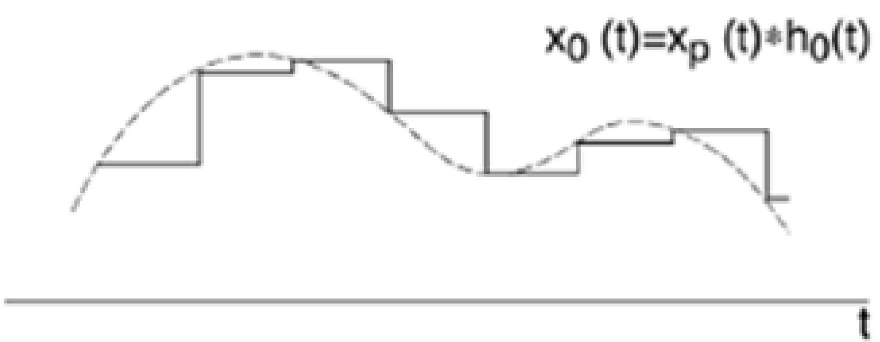


Zero-Order Hold Filtering

- Need to create a reconstruction filter that compensates for the zero-order hold frequency response and gives a flat combined response

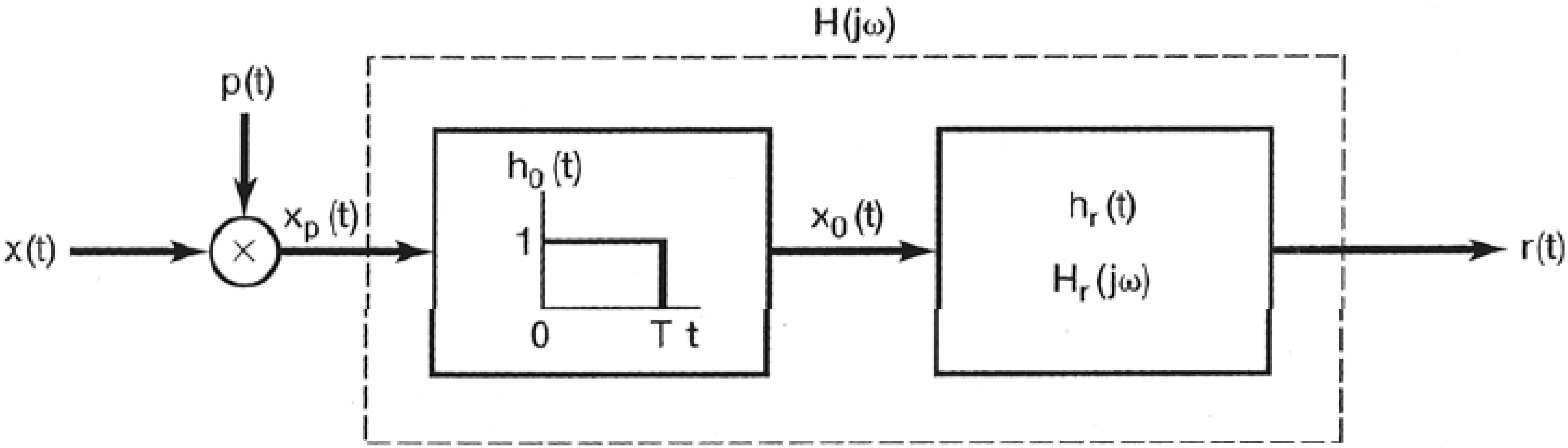






Signal Reconstruction from Zero-Order Hold

- To reconstruct $x(t)$ from $x_0(t)$, consider processing by LTI system with impulse response $h_r(t)$ and frequency response $H_r(j\omega)$



Signal Reconstruction from Zero-Order Hold

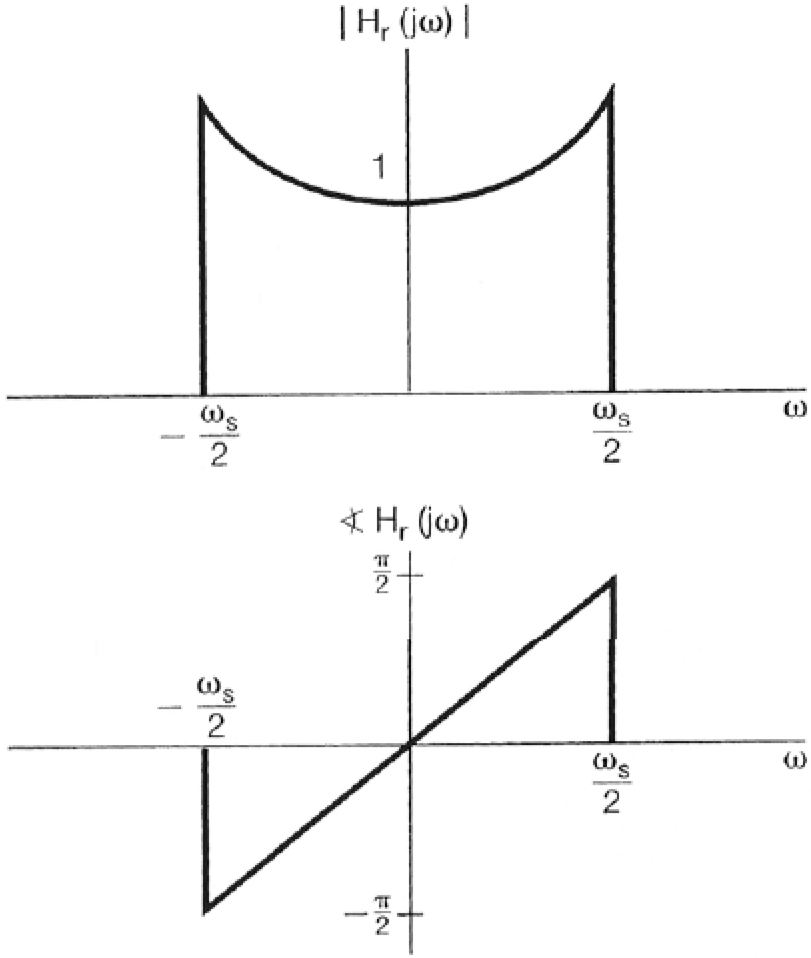
We want to choose

$H_r(j\omega)$ such that $r(t) = x(t) \Rightarrow H_r(j\omega) = H(j\omega)\left[H_0(j\omega)\right]^{-1}$ where $H(j\omega)$ is the ideal LPF used in the reconstruction process

$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right]$$

$$H_r(j\omega) = \frac{e^{j\omega T/2} H(j\omega)}{\dfrac{2\sin(\omega T/2)}{\omega}}$$

Reconstruction from Zero-Order Hold



Magnitude and phase for the reconstruction filter for a zero-order hold.

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