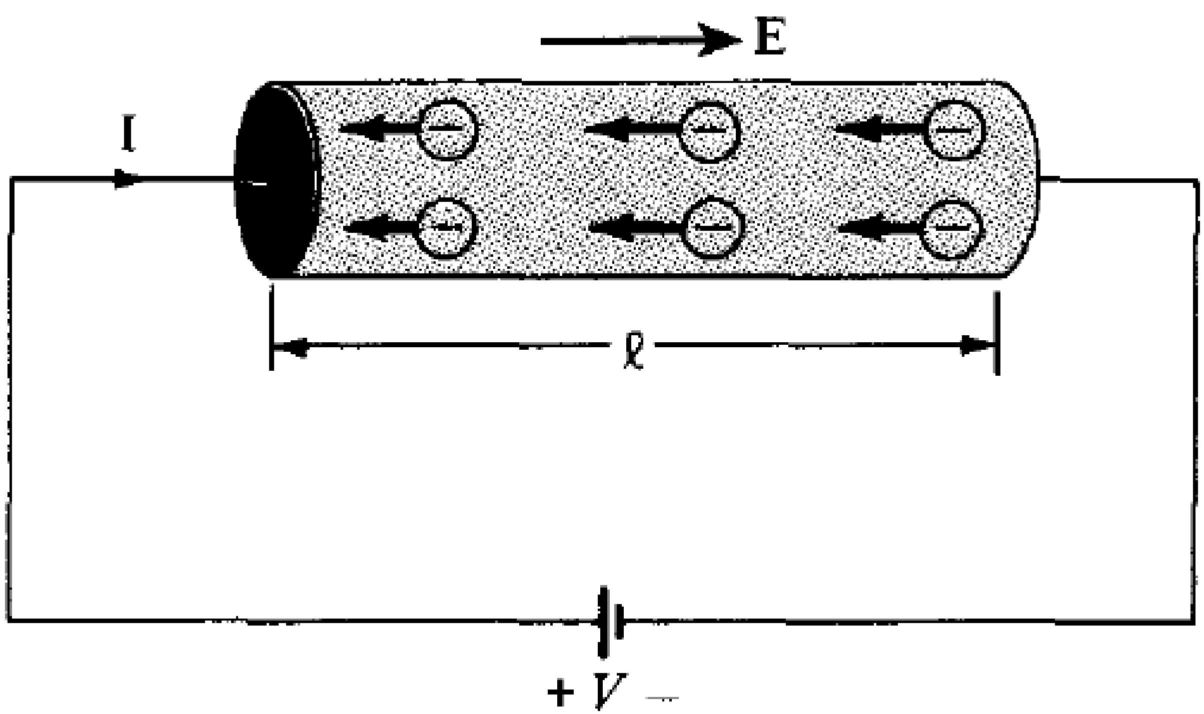
CONDUCTORS, DIELECTRICS AND POLARIZATION

Conductors

- Consider a conductor whose ends are maintained at a potential difference *V*, as shown in Figure below

- Note that in this case, $\mathbf{E} \neq 0$ inside the conductor (why ?)



Conductors

- There is no static equilibrium in this case since the conductor is not isolated but wired to a source of electromotive force

- The electromotive force compels the free charges to move and prevents the eventual establishment of electrostatic equilibrium

- Thus in this case, an electric field must exist inside the conductor to sustain the flow of current

- As the electrons move, they encounter some damping forces called resistance

- Based on Ohm's law ($J = \sigma E$) studied previously, we will derive the resistance of the conducting material

Conductivity

- From the previous lecture, we know that the *conduction current density* is given as:

$$\mathbf{J} = \rho_v \mathbf{u} = \frac{ne^2\tau}{m}\mathbf{E} = \sigma\mathbf{E}$$

- where $\boldsymbol{\sigma = ne^2\tau/m}$ is the conductivity of the conductor

- The above relationship is known as the point form of Ohm's law

- Therefore, we derived conductivity in terms of the parameters of a resistor

- Now we will relate this conductivity to the resistance

Resistivity of Conductors

- Suppose the conductor has a uniform cross section S and is of length l

- The direction of the electric field **E** produced is the same as the direction of the flow of positive charges or current I, (opposite to the direction of the flow of electrons)

- The electric field applied is uniform and its magnitude is given by:

$$E = \frac{V}{\ell}$$

- Since the conductor has a uniform cross-section, we have:

$$J = \frac{I}{S}$$

Resistivity of Conductors

- Substituting *J* and *E* in the previous equation, we have:

$$\frac{I}{S} = \sigma E = \frac{\sigma V}{\ell}$$

- Therefore:

$$R = \frac{V}{I} = \frac{\ell}{\sigma S} \qquad \text{OR} \qquad R = \frac{\rho_c \ell}{S}$$

- Here $\rho_c = 1/\sigma$ is the resistivity of the material

- The above equation is useful in determining the resistance of any conductor of uniform cross section

- The basic definition of resistance *R* as the ratio of the potential difference *V* between the two ends of the conductor to the current *I* through the conductor still applies

Joule's Law for Conductors

- The resistance of a conductor of non-uniform cross section is:

$$R = \frac{V}{I} = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\int \sigma \mathbf{E} \cdot d\mathbf{S}}$$

- Power P (in watts) is defined as the rate of change of energy *W* (in joules) or force times velocity, hence:

$$\int \rho_v \, dv \, \mathbf{E} \cdot \mathbf{u} = \int \mathbf{E} \cdot \rho_v \mathbf{u} \, dv \qquad \text{OR} \qquad P = \int \mathbf{E} \cdot \mathbf{J} \, dv$$

- This is known as ***Joule's law***

- For a conductor with uniform cross-section: $dv = dSdl$, so:

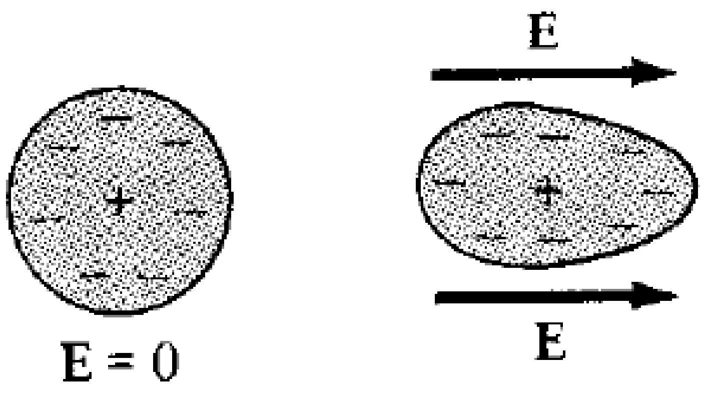
$$P = \int_L E \, dl \int_S J \, dS = VI \qquad \text{OR} \qquad P = I^2 R$$

Dielectrics

- The charges in a dielectric are not able to move about freely, they are bound by finite forces

- Therefore, we may certainly expect a displacement when an external force is applied

- Consider an atom of the dielectric as consisting of a negative charge - Q (electron cloud) and a positive charge +Q (nucleus)



Dielectrics-Dipole Moment

- When an electric field **E** is applied, the positive charge is displaced from its equilibrium position in the direction of **E** by the force $\mathbf{F}_+ = Q\mathbf{E}$ while the negative charge is displaced in the opposite direction by the force $\mathbf{F}\_ = Q\mathbf{E}$

- A dipole results from the displacement of the charges and the dielectric is said to be polarized

- In the polarized state, the electron cloud is distorted by the applied electric field **E**

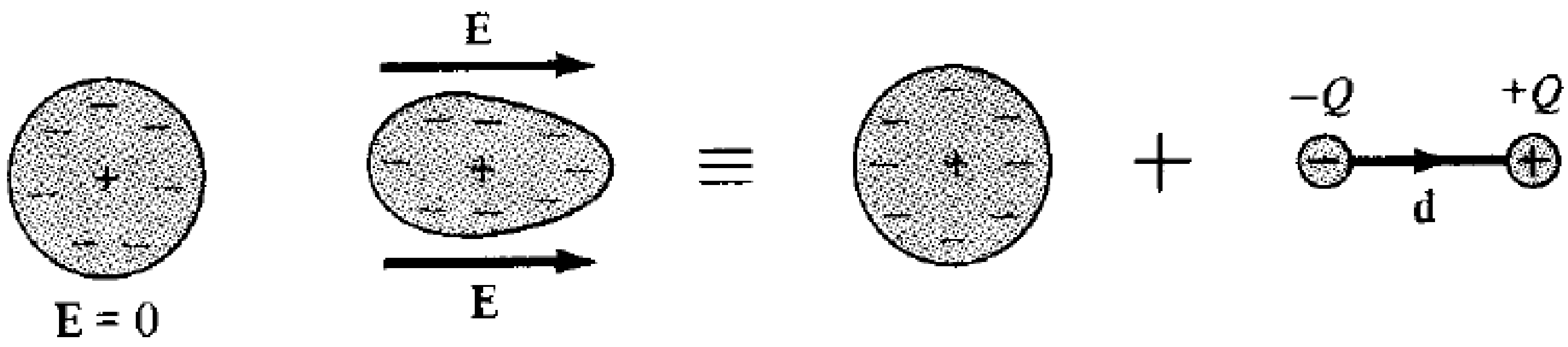
- This distorted charge distribution is equivalent, by the principle of superposition, to the original distribution plus a dipole whose moment is:

$$\mathbf{p} = Q\mathbf{d}$$

Polarization in Dielectrics

- Here **d** is the distance vector from −Q to +Q of the dipole as shown in Figure below
- If there are N dipoles in a volume Δv of the dielectric, the total dipole moment due to the electric field is:

$$Q_1\mathbf{d}_1 + Q_2\mathbf{d}_2 + \cdots + Q_N\mathbf{d}_N = \sum_{k=1}^{N} Q_k\mathbf{d}_k$$



Polarization in Dielectrics

- As a measure of intensity of the polarization, we define *polarization* **P** (in coulombs/meter square) as the dipole moment per unit volume of the dielectric; that is:

$$\mathbf{P} = \frac{\lim\limits_{\Delta v \to 0} \sum\limits_{k=1}^{N} Q_k \mathbf{d}_k}{\Delta v}$$

- Thus we conclude that the major effect of the electric field **E** on a dielectric is the creation of dipole moments that align themselves in the direction of **E**

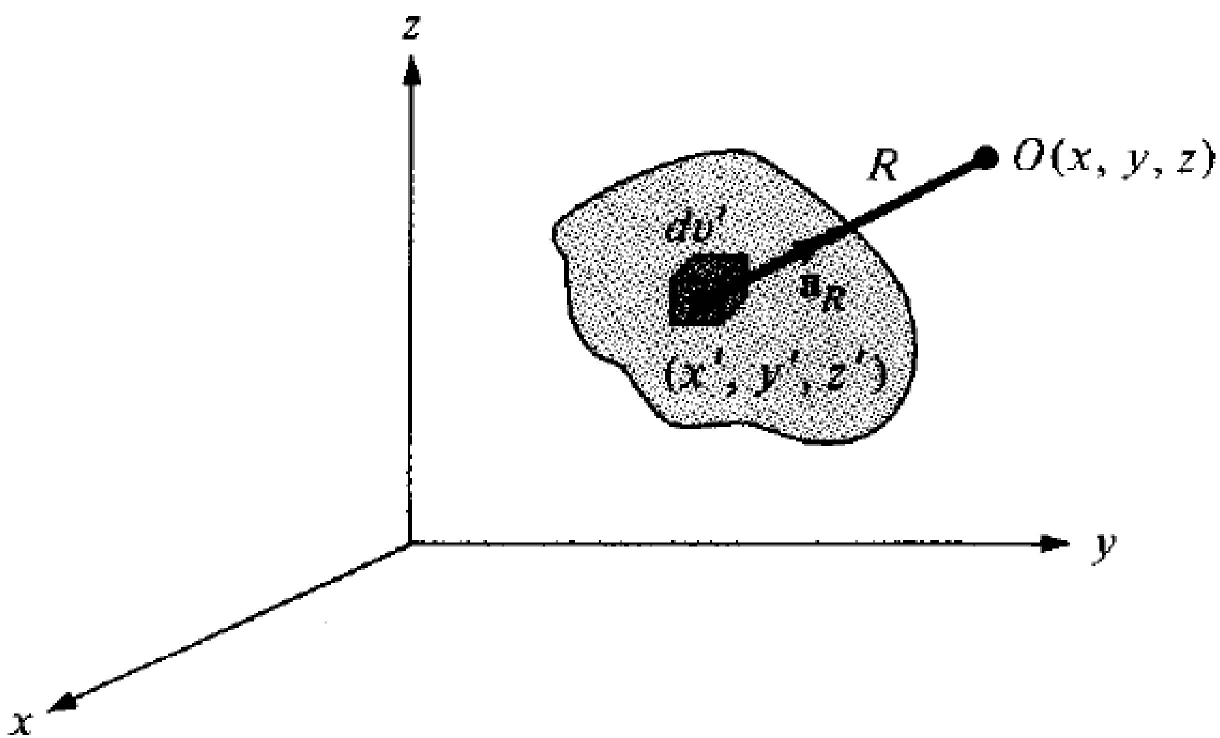
- This type of dielectric is said to be **nonpolar** (hydrogen, oxygen)

- Examples of **polar** are water, hydrochloric acid etc., have built-in permanent dipoles

Bound Charges

- We now calculate the field due to a polarized dielectric

- Consider the dielectric material shown in Figure below as consisting of dipoles with dipole moment **P** per unit volume



Bound Charges

- From the equation for dipole moment, the potential dV at an exterior point O due to the dipole moment $\mathbf{P}dv'$ is:

$$dV = \frac{\mathbf{p} \cdot \mathbf{a}_r}{4\pi\varepsilon_o r^2} = \frac{\mathbf{P} \cdot \mathbf{a}_R \, dv'}{4\pi\varepsilon_o R^2}$$

- where $R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$ and R is the distance between the volume element dv' at (x', y', z') and the field point O(x, y, z)
- We now transform the above equation into a form that facilitates physical interpretation

- It will be shown later that the gradient of $1/R$ with respect to the primed coordinates is:

$$\nabla'\left(\frac{1}{R}\right) = \frac{\mathbf{a}_R}{R^2}$$

Bound Charges

- So we have:

$$\frac{\mathbf{P}\cdot\mathbf{a}_R}{R^2} = \mathbf{P}\cdot\nabla'\left(\frac{1}{R}\right)$$

- Applying the vector identity: $\nabla'\cdot f\mathbf{A} = f\nabla'\cdot\mathbf{A} + \mathbf{A}\cdot\nabla' f$ to the equation above, we get:

$$\frac{\mathbf{P}\cdot\mathbf{a}_R}{R^2} = \nabla'\cdot\frac{\mathbf{P}}{R} - \frac{\nabla'\cdot\mathbf{P}}{R}$$

- Substituting the above equation into the equation for *dV* and integrating over the entire volume v' of the dielectric, we get:

$$V = \int_{v'} \frac{1}{4\pi\varepsilon_o}\left[\nabla'\cdot\frac{\mathbf{P}}{R} - \frac{1}{R}\nabla'\cdot\mathbf{P}\right]dv'$$

Bound Charges

- Applying divergence theorem to the first term leads finally to:

$$V = \int_{S'} \frac{\mathbf{P} \cdot \mathbf{a}'_n}{4\pi\varepsilon_o R} dS' + \int_{v'} \frac{-\nabla' \cdot \mathbf{P}}{4\pi\varepsilon_o R} dv'$$

- where $\mathbf{a}'_n$ is the outward unit normal to surface *dS'* of the dielectric

- Comparing the two terms on the right side of the above equation with the relation for potential shows that the two terms denote the potential due to surface and volume charge distributions with densities (upon dropping the primes):

$$\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n$$
$$\rho_{pv} = -\nabla \cdot \mathbf{P}$$

Bound Charges

- Equations above reveal that where polarization occurs, an equivalent bound volume charge density ρ_{pv} is formed throughout the dielectric

- While an equivalent bound surface charge density ρ_{ps} is formed over the surface of the dielectric

- Bound charges are those that are not free to move within the dielectric material; they are caused by the displacement that occurs on a molecular scale during polarization

- Whereas, free charges are those that are capable of moving over macroscopic distance as electrons in a conductor

Bound Charges

- The total positive bound charge on surface *S* bounding the dielectric is:

$$Q_b = \oint \mathbf{P} \cdot d\mathbf{S} = \int \rho_{ps}\, dS$$

- While the charge that is throughout the dielectric is:

$$-Q_b = \int_v \rho_{pv}\, dv = -\int_v \nabla \cdot \mathbf{P}\, dv$$

- Since the total charge of the dielectric material remains zero:

$$\text{Total charge} = \oint_S \rho_{ps}\, dS + \int_v \rho_{pv}\, dv = Q_b - Q_b = 0$$

- This is expected because the dielectric was electrically neutral before polarization

Problem-1

- A thin rod of cross section A extends along the x-axis from x = 0 to x = L. The polarization of the rod is along its length towards positive x-axis and is given by $P_x = ax^2 + b$. Calculate ρ_{ps} and ρ_{pv} at each end. Show explicitly that the total bound charge vanishes in this case.

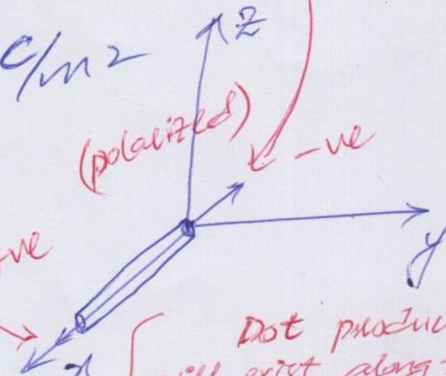
Problem-1

$$\rho_{ps} = \vec{P}\cdot\vec{a_x} = (ax^2+b)\vec{a_x}\cdot\vec{a_x}$$

$$\boxed{\rho_{ps} = ax^2+b}$$

$$\rho_{ps}\Big|_{x=0} = \vec{P}\cdot(-\vec{a_x})\Big|_{x=0} = -ax^2-b\Big|_{x=0} = \boxed{-b}\ C/m^2$$

$$\rho_{ps}\Big|_{x=L} = \vec{P}\cdot(\vec{a_x})\Big|_{x=L} = \boxed{aL^2+b}\ C/m^2$$



Dot product will exist along the ends.

$$\rho_{pv} = -\vec{\nabla}\cdot\vec{P} = -\frac{d}{dx}(ax^2+b)$$

$$\Rightarrow \boxed{\rho_{pv} = -2ax}$$

$$\boxed{\rho_{pv}\Big|_{x=0} = 0} \quad \text{and} \quad \boxed{\rho_{pv}\Big|_{x=L} = -2aL}$$

Problem-1

Proof:-

$$Q_s = \int \rho_{ps}\, dS = \int_{x=L} (aL^2+b)\, dA + \int_{x=0} (-b)\, dA$$

$$\Rightarrow Q_s = (aL^2+b)A - bA = \boxed{AaL^2}$$

$$Q_v = \int \rho_{pv}\, dV = \int_0^L -2ax\, A\, dx = -2aA\left[\frac{x^2}{2}\right]_0^L$$

fixed, given

$$\Rightarrow \boxed{Q_v = -aAL^2}$$

$$\Rightarrow Q_T = Q_s + Q_v = 0$$