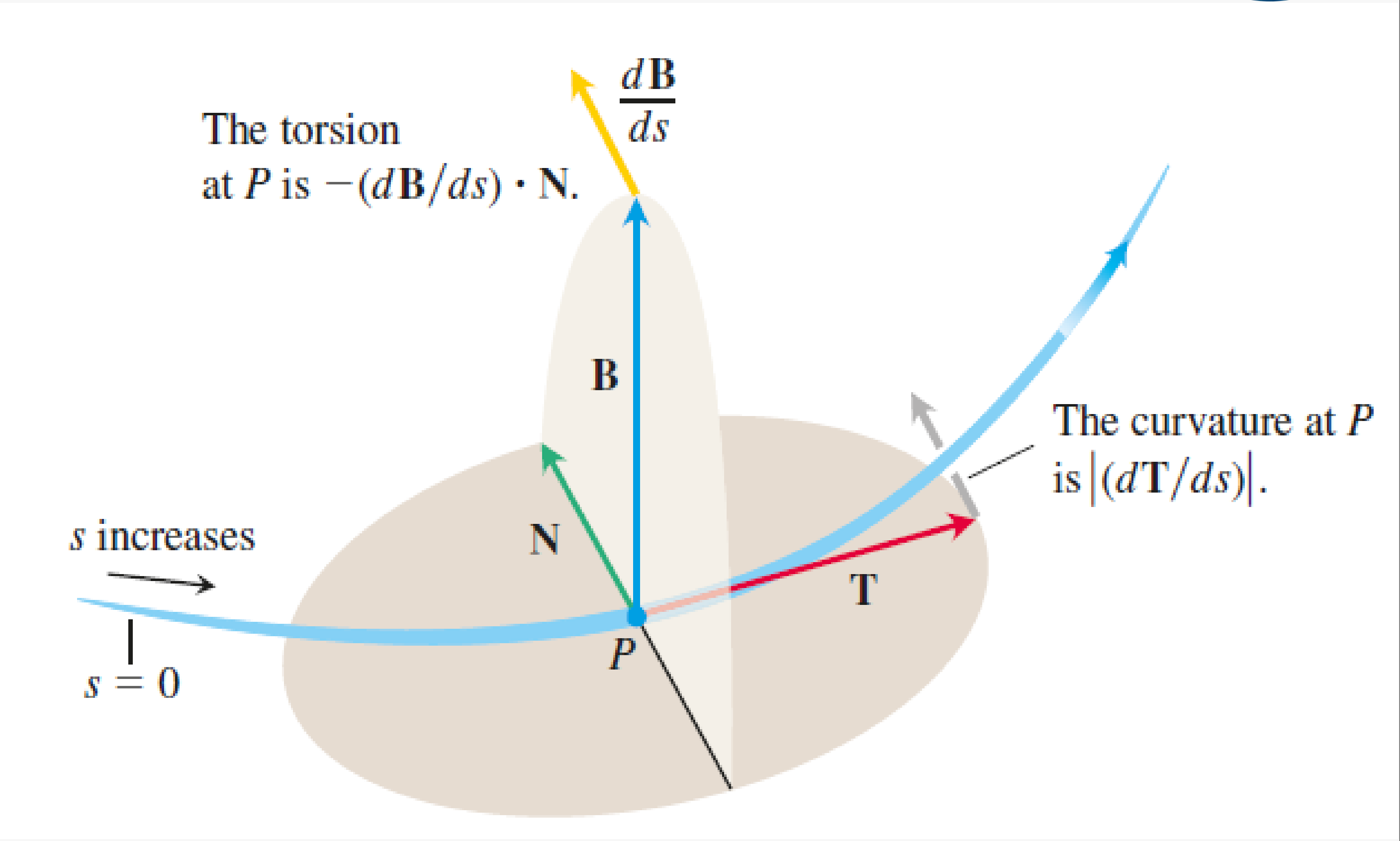
Torsion

Vector Calculus(MATH-243)
Instructor: Dr. Naila Amir



Every moving body travels with a **TNB** frame that characterizes the geometry of its path of motion.

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Vectors And
The Geometry Of Space

Book: Thomas’ Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Chapter: 13 , Section: 13.5

Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.
Chapter: 13 , Section: 13.3

Recap…

- **Arclength of a curve:**

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b |\mathbf{v}(t)|\, dt.$$

- **Arc length parameter for the curve (directed distance):**

If a vector-valued function $\mathbf{r}(t)$ represents the position of a particle in space as a function of time, then the arc-length function $s(t)$ measures how far that particle travels as a function of time. The formula for the arc-length function follows directly from the formula for arc length:

$$s(t) = \int_{t_0}^{t} |\mathbf{r}'(u)|\, du = \int_{t_0}^{t} \sqrt{\left[\frac{dx}{du}\right]^2 + \left[\frac{dy}{du}\right]^2 + \left[\frac{dz}{du}\right]^2}\, du.$$

If we use s as a variable, we get $\mathbf{r}(s)$, the *position in space in terms of distance along the curve*.

Recap…

- **Speed on a Smooth Curve:**

$$\frac{ds}{dt} = |\mathbf{r}\,'(t)| = |\mathbf{v}(t)|.$$

- **Curvature:**

Given a curve, if we are interested to measure, at various points, how sharply curved it is, then we can quantify this by means of *curvature*. Clearly this is related to how "fast" a tangent vector is changing direction. Curvature is defined as the rate of change of direction of a curve with respect to distance along the curve and is given as:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3}.$$

- **Unit tangent vector:**

If C is a smooth curve defined by $\mathbf{r}(t)$, then the unit tangent vector is given by:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}.$$

and indicates the *direction of the curve*.

Recap…

- **Unit normal vector:**

  If $\mathbf{r}(t)$ is a smooth curve, then at a point where $\kappa \neq 0$ we can define the **principal unit normal vector** (or simply **unit normal**) as:

$$\mathbf{N}(t) = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$

  The **normal vector** indicates the *direction in which the curve is turning* at each point.

- **Binormal vector:**

  The **binormal vector** of a curve in space is defined as:

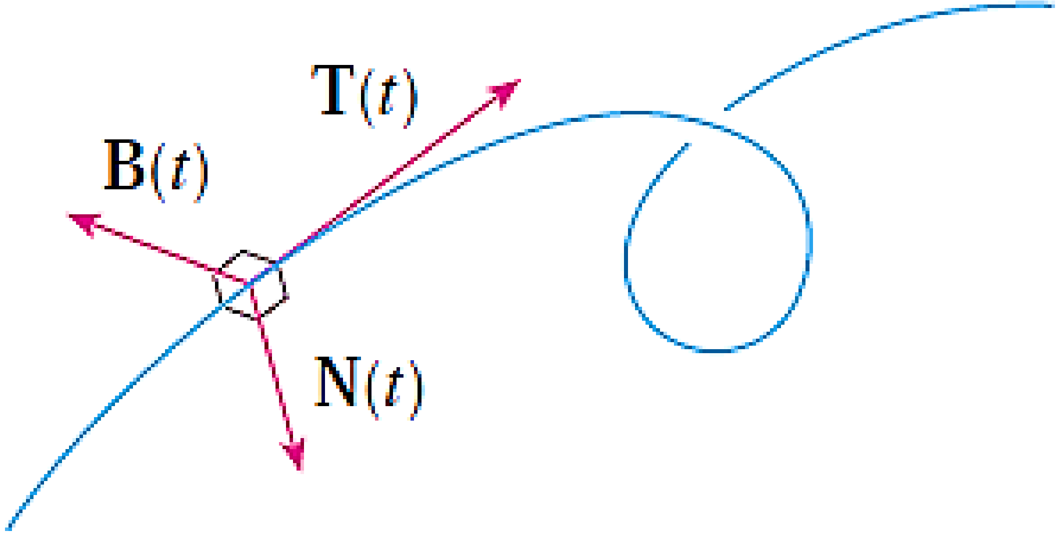
$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

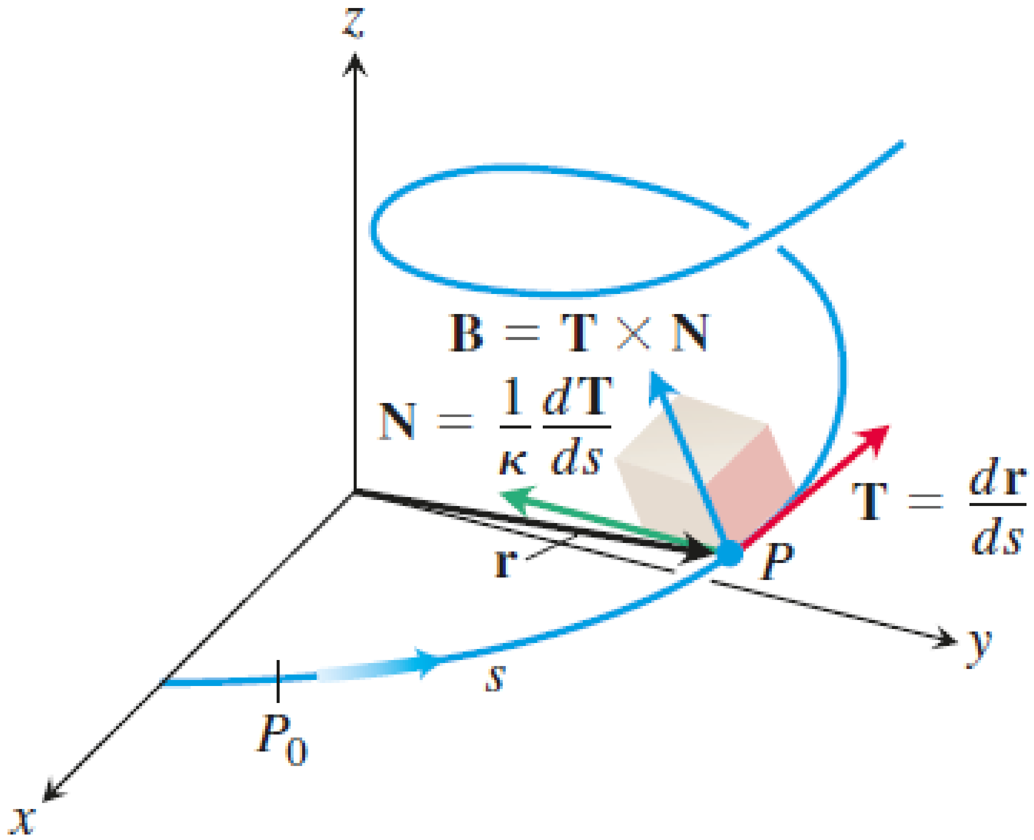
  It measures the *tendency of our motion to "twist"* out of the plane created by these vectors in the direction perpendicular to this plane.

Recap…

- **TNB Frame (Frenet Frame):**

  $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ define a moving righthanded vector frame that plays a significant role in calculating the paths of particles moving through space.

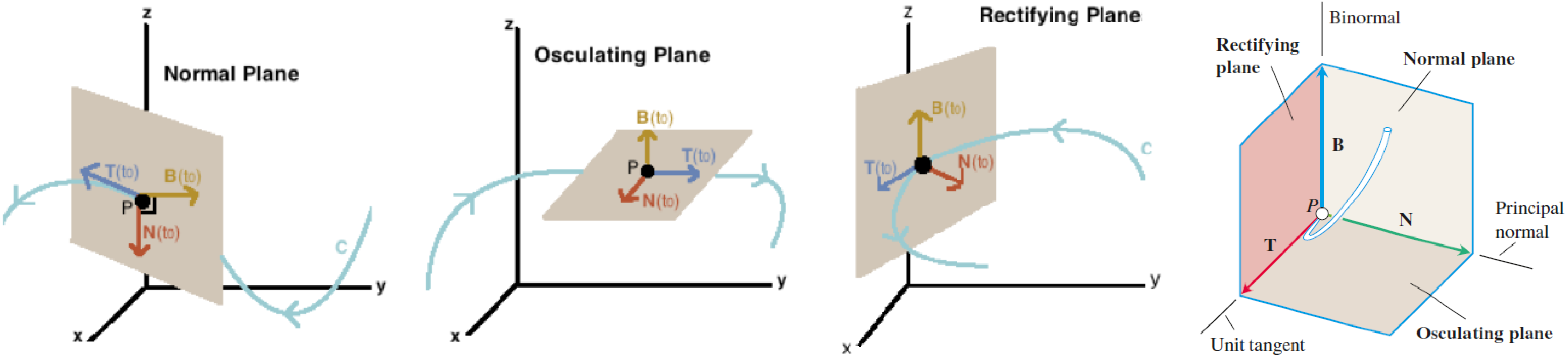




Normal, Osculating and Rectifying Planes

Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ be a vector function that represents the smooth curve C and let $P(x_0, y_0, z_0)$ be a point on C corresponding to $\mathbf{r}(t_0)$. Then:

- the **normal plane** of C at point P is the plane spanned by $\mathbf{N}(t_0)$ and $\mathbf{B}(t_0)$ with normal vector $\mathbf{T}(t_0)$. It consists of all lines that are orthogonal to the tangent vector.
- the **osculating plane** of C at point P is the plane spanned by $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$ with normal vector $\mathbf{B}(t_0)$.
- the **rectifying plane** of C at P is the plane spanned by $\mathbf{B}(t_0)$ and $\mathbf{T}(t_0)$ with normal vector $\mathbf{N}(t_0)$.



Torsion

The **torsion** function of a smooth curve is defined as:

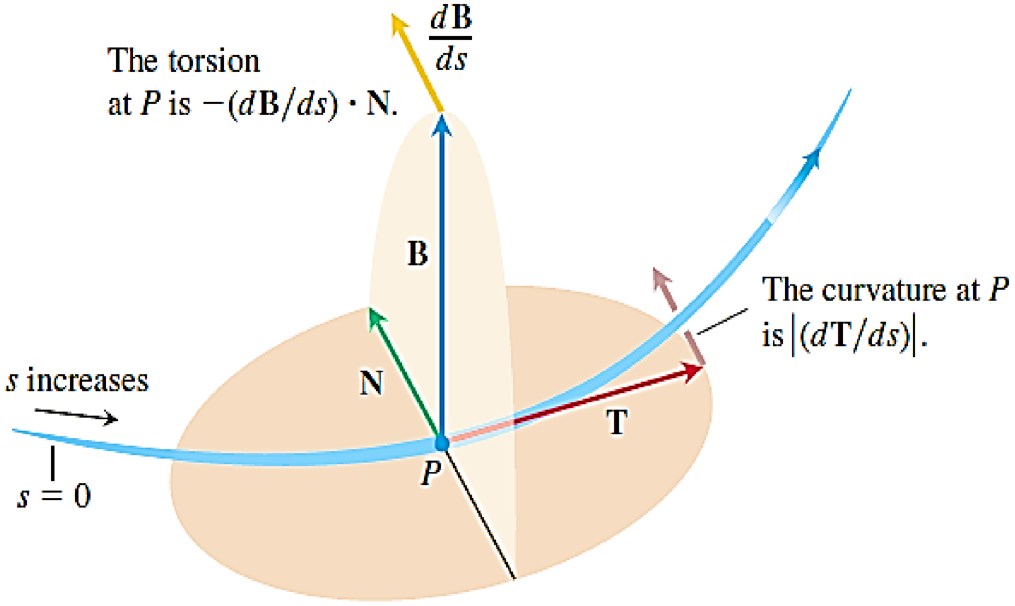
$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} = \frac{(\mathbf{r}'\times\mathbf{r}'').\,\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2},$$

where $\mathbf{N}$ is the unit normal vector and $\mathbf{B}$ represents the binormal vector.

Torsion measures *degree of twisting of a curve*.

Unlike the curvature κ, which is never negative,

the torsion τ may be positive, negative, or zero.



The curvature $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$ can be thought of as the

rate at which the *normal plane* turns as the

point P moves along its path. Similarly, the torsion $\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$ is the rate at which the *osculating plane* turns about $\mathbf{T}$ as P moves along the curve.

Torsion

The most widely used formula for torsion, derived in more advanced texts, is given in a determinant form.

$$\tau = \frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}; \quad \mathbf{v} \times \mathbf{a} \neq \mathbf{0}.$$

This formula calculates the torsion directly from the derivatives of the component functions: $x = f(t), y = g(t)$ and $z = h(t)$ that make up the vector function $\mathbf{r}(t)$. The determinant's first row comes from $\mathbf{v}$, the second row comes from $\mathbf{a}$, and the third row comes from $\mathbf{a}'(t) = \frac{d\mathbf{a}}{dt} = \mathbf{r}'''(t)$. This formula for torsion is traditionally written using Newton's dot notation for derivatives.

Example:

Determine the curvature and torsion of the helix:

$$\mathbf{r}(t) = \langle a\cos t\,, a\sin t\,, bt\rangle; \qquad a, b \geq 0, \qquad a^2 + b^2 \neq 0.$$

Solution:

For the present case:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -a\sin t\,, a\cos t\,, b\rangle,$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle -a\cos t\,, -a\sin t\,, 0\rangle,$$

$$\mathbf{a}'(t) = \mathbf{r}'''(t) = \langle a\sin t\,, -a\cos t\,, 0\rangle,$$

$$\mathbf{v}(t) \times \mathbf{a}(t) = \langle ab\sin t\,, -ab\cos t\,, a^2\rangle,$$

$$|\mathbf{v}(t)| = \sqrt{a^2 + b^2},$$

$$|\mathbf{v}(t) \times \mathbf{a}(t)| = a\sqrt{a^2 + b^2}.$$

Thus, the curvature of the helix is given as:

$$\kappa = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{a\sqrt{a^2 + b^2}}{(a^2 + b^2)^{3/2}} = \frac{a}{a^2 + b^2}.$$

Solution:

The torsion of the helix is given as:

$$\tau = \frac{\begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}}{\left(a\sqrt{a^2+b^2}\right)^2}$$

$$= \frac{b(a^2\cos^2 t + a^2\sin^2 t)}{a^2(a^2+b^2)}$$

$$= \frac{a^2 b}{a^2(a^2+b^2)}$$

$$= \frac{b}{a^2+b^2}.$$

Note that the torsion of a helix about a circular cylinder is constant. Constant curvature and constant torsion characterize the helix among all curves in space. Thus, *a space curve is a helix if and only if it has constant nonzero curvature and constant nonzero torsion.*

Example:

Consider the vector function:

$$\mathbf{r}(t) = \langle 2\sin(3t)\,, t, 2\cos(3t)\rangle; \qquad (0, \pi, -2).$$

Determine $\mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa$ and τ at the given point. Moreover, find equations of the normal plane, osculating plane and rectifying plane of the curve at the given point.

Solution:

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 6\cos(3t), 1, -6\sin(3t)\rangle$$

$$\vec{a}(t) = \vec{r}\,''(t) = \langle -18\sin(3t), 0, -18\cos(3t)\rangle$$

$$\vec{a}\,'(t) = \vec{r}\,'''(t) = \langle -54\cos(3t), 0, 54\sin(3t)\rangle$$

$$|\vec{v}(t)| = |\vec{r}\,'(t)| = \sqrt{36\cos^2(3t) + 1 + 36\sin^2(3t)} = \sqrt{37}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6\cos(3t) & 1 & -6\sin(3t) \\ -18\sin(3t) & 0 & -18\cos(3t) \end{vmatrix} = \hat{i}\left(-18\cos(3t)\right) - \hat{j}\left(-108\cos^2(3t) - 108\sin^2(3t)\right) + \hat{k}\left(18\sin(3t)\right)$$

$$= \langle -18\cos(3t), 108, 18\sin(3t)\rangle$$

Solution: $|\vec{v} \times \vec{a}| = \sqrt{(-18\cos(3t))^2 + (108)^2 + (18\sin(3t))^2} = \sqrt{324 + 11664} = 18\sqrt{37}.$

① $\vec{T}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} \Rightarrow \vec{T}(t) = \dfrac{1}{\sqrt{37}}\langle 6\cos(3t), 1, -6\sin 3t\rangle$

$P(0, \pi, -2) \longrightarrow \boxed{t = \pi}$

$$\boxed{\vec{T}(\pi) = \frac{1}{\sqrt{37}}\langle -6, 1, 0\rangle}$$

② $\vec{N}(t) = \dfrac{T'(t)}{|T'(t)|}$

$$\boxed{\vec{N}(\pi) = \langle 0, 0, 1\rangle}$$

$\vec{T}(t) = \dfrac{1}{\sqrt{37}}\langle 6\cos(3t), 1, -6\sin(3t)\rangle$

$\vec{T}'(t) = \dfrac{1}{\sqrt{37}}\langle -18\sin(3t), 0, -18\cos(3t)\rangle \Rightarrow |\vec{T}'(t)| = \dfrac{18}{\sqrt{37}}$

$\vec{N}(t) = \dfrac{1}{\sqrt{37}}\langle -18\sin(3t), 0, -18\cos(3t)\rangle \div \dfrac{18}{\sqrt{37}} = \langle -\sin(3t), 0, -\cos(3t)\rangle$

Solution: ③ $\vec{B} = \vec{T} \times \vec{N} \Rightarrow \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{6}{\sqrt{37}}\cos(3t) & \frac{1}{\sqrt{37}} & \frac{-6}{\sqrt{37}}\sin(3t) \\ -\sin(3t) & 0 & -\cos(3t) \end{vmatrix}$

$$= \hat{i}\left(\frac{-1}{\sqrt{37}}\cos(3t)\right) - \hat{j}\left(\frac{-6}{\sqrt{37}}\cos^2(3t) - \frac{6}{\sqrt{37}}\sin^2(3t)\right)$$

$$+ \hat{k}\left(\frac{1}{\sqrt{37}}\sin(3t)\right)$$

$$\vec{B}(t) = \frac{1}{\sqrt{37}}\langle -\cos(3t), 6, \sin(3t)\rangle$$

$$\Rightarrow \boxed{\vec{B}(\pi) = \frac{1}{\sqrt{37}}\langle 1, 6, 0\rangle}$$

④ $\kappa = \dfrac{|T'(t)|}{|\vec{v}(t)|}$ or $\kappa = \dfrac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \dfrac{18\sqrt{37}}{(\sqrt{37})^3} = \dfrac{18}{37}.$

Solution:

$$\kappa = \frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} \; ; \quad \vec{v} \times \vec{a} \neq \vec{0}$$

or

$$\kappa = \frac{(\vec{r}\,' \times \vec{r}\,'') \cdot \vec{r}\,'''}{|\vec{r}\,' \times \vec{r}\,''|^2} = \frac{(\vec{v} \times \vec{a}) \cdot \vec{a}\,'}{|\vec{v} \times \vec{a}|^2}$$

$$= \frac{\langle -18\cos(3t),\, 108,\, 18\sin(3t) \rangle \cdot \langle -54\cos(3t),\, 0,\, 54\sin(3t) \rangle}{(18\sqrt{37})^2}$$

$$= \frac{972\cos^2(3t) + 0 + 972\sin^2(3t)}{(18\sqrt{37})^2} = \frac{972}{(18\sqrt{37})^2} = \frac{3}{37}.$$

Solution: General Eq. of a plane is given as:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0,$$

where $P(x_0, y_0, z_0)$ is a point on the plane &

$\vec{n} = \langle a, b, c\rangle$ normal vector

(a) Normal Plane:

$\vec{N} \quad \vec{B}$

$\rightarrow \vec{T}(t_0) \longrightarrow$ normal vector

$$P(\underset{x_0}{0}, \underset{y_0}{\pi}, \underset{z_0}{-2}) \rightarrow t_0 = \pi$$

$$\vec{T}(\pi) = \frac{1}{\sqrt{37}}\langle \underset{a}{-6}, \underset{b}{1}, \underset{c}{0}\rangle$$

The required equation of the normal plane is given as:

$$\frac{-6}{\sqrt{37}}(x-0)+\frac{1}{\sqrt{37}}(y-\pi)+0(z+2)=0 \Rightarrow \boxed{y-6x=\pi}$$

$\rightarrow$ Eq. of normal plane.

Solution: (b) Osculating Plane $\left\langle \begin{array}{l}\vec{T}\\ \vec{N}\end{array}\right.$

↳ $\vec{B}(t_0)$ acts as normal vector plane.

$$(x_0, y_0, z_0) \longleftrightarrow (0, \pi, -2)$$

$$\vec{B}(\pi) = \frac{1}{\sqrt{37}} \langle 1, 6, 0 \rangle$$

$$\langle a, b, c \rangle \longleftrightarrow \frac{1}{\sqrt{37}} \langle 1, 6, 0 \rangle$$

The required eq. of the osculating plane is:

$$\frac{1}{\sqrt{37}}(x-0) + \frac{6}{\sqrt{37}}(y-\pi) + \frac{0}{\sqrt{37}}(z+2) = 0$$

$$\Rightarrow x + 6y - 6\pi = 0 \Rightarrow \boxed{x + 6y = 6\pi} \rightarrow \text{Eq. of Osculating Plane.}$$

Solution: (c) Rectifying Plane: $\langle \vec{T}, \vec{B}$

$\hookrightarrow$ $\vec{N}(t_0)$ acts as the normal vector

$$\vec{N}(\pi) = \langle 0, 0, 1 \rangle$$

Thus, $\langle a, b, c \rangle = \langle 0, 0, 1 \rangle$

and $(x_0, y_0, z_0) = (0, \pi, -2)$

Thus, the eq. of rectifying plane is given as:

$$0(x-0) + 0(y-\pi) + 1(z+2) = 0$$

$$\Rightarrow \boxed{z+2=0} \text{ or } \boxed{z=-2} \rightarrow \text{Eq. of Rectifying plane.}$$

Practice Questions

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

Chapter: 13

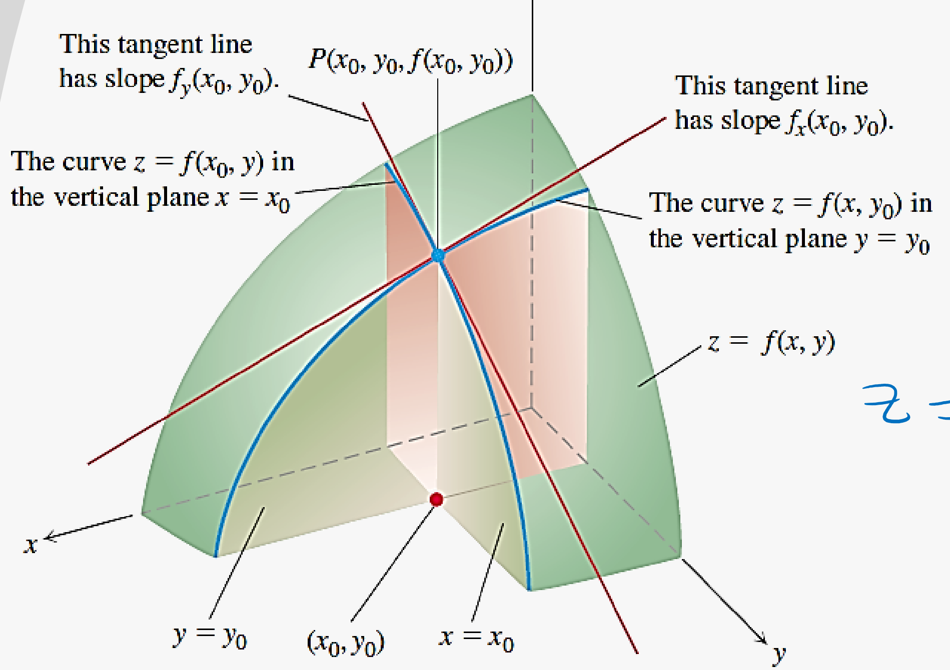
Exercise-13.3: Q – 56 to 58.

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 13

Exercise-13.5: Q – 9 to 16.

Partial Derivatives & Directional Derivatives



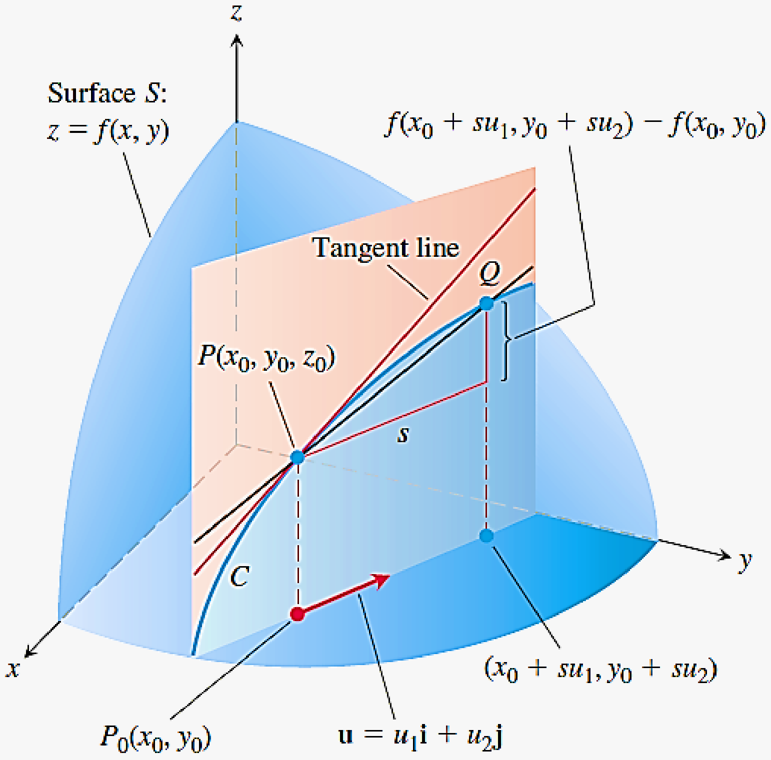
$z = f(x, y)$

$y = f(x)$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

$$\frac{dy}{dx} = f'$$



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Partial Derivatives

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Chapter: 14 , Section: 14.3, 14.5

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.
Chapter: 14 , Section: 14.3, 14.6

Partial Derivatives of a Function of Two Variables:

Suppose we have a multi-variable function of two variables $z = f(x, y)$, defined in domain D of xy −plane. Therefore, our function f depends on x and y, both. Now if we want to take derivative of f , then we have two options: either to take the derivative with respect to x or with respect to y. If f is a function of two variables, its partial derivatives are the functions $\frac{\partial f}{\partial x} = f_x$ and $\frac{\partial f}{\partial y} = f_y$ defined by:

$y = f(x)$

$f'(x) = \frac{dy}{dx}$

$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h},$$

provided the limit exists.

Notations for Partial Derivatives

There are many alternative notations for partial derivatives. If $z = f(x, y)$, we write:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$