# EE-381 Robotics-1

## UG ELECTIVE

**Lecture 9**

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# Time and Motion

- How pose changes over time?
- Linear and Angular velocity of rigid bodies/joints
- Motion of a Manipulator

# Basic-Linear Velocity

- Derivative of a vector

$$V_Q^B = \frac{d}{dt} Q^B = \lim_{\Delta t \to 0} \frac{Q^B(t + \Delta t) - Q^B(t)}{\Delta t}$$

- The velocity of a position vector can be thought of as the linear velocity of the point in space represented by the position vector.
- It is important to indicate the frame in which the vector is differentiated.
- Velocity vector when expressed in terms of frame {A}

$$\left(V_Q^B\right)^A = \left(\frac{d}{dt} Q^B\right)^A$$

# Basic-Linear Velocity

Frame wrt which position vector is differentiated

Frame wrt which resulting velocity vector is expressed

$$\left(V_Q^B\right)^A = \left(\frac{d}{dt} Q^B\right)^A$$

Leading superscript can be omitted when expressed in terms of itself

$$\left(V_Q^B\right)^B = V_Q^B$$

We can always remove the outer, leading superscript by explicitly including the rotation matrix that accomplishes the change in reference frame

$$\left(V_Q^B\right)^A = R_B^A \; V_Q^B$$

# Basic-Linear Velocity

- Generally, the velocity of the origin of a frame is considered relative to some understood universe reference frame

$$v_C = V_{CORG}^{U}$$

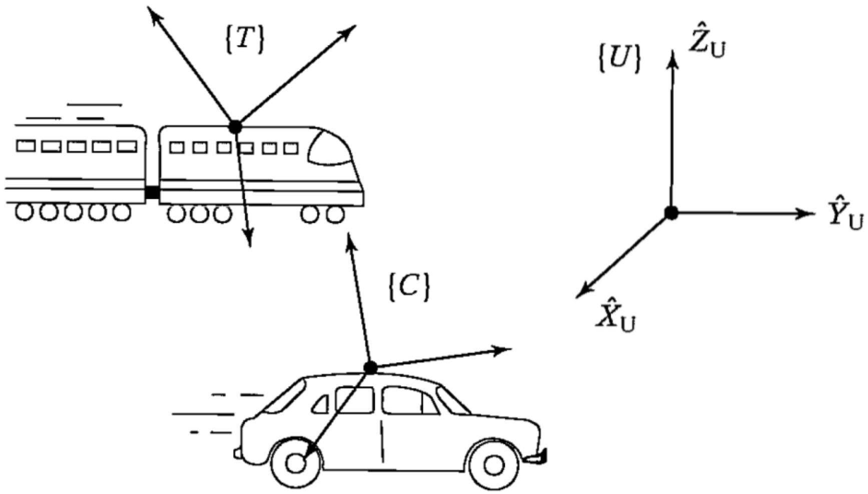
- We use the notation to refer to the velocity of the origin of frame {C}

- What is $v_C^A$?

Velocity of the origin of frame {C} expressed in terms of frame {A} (though differentiation was done relative to {U})

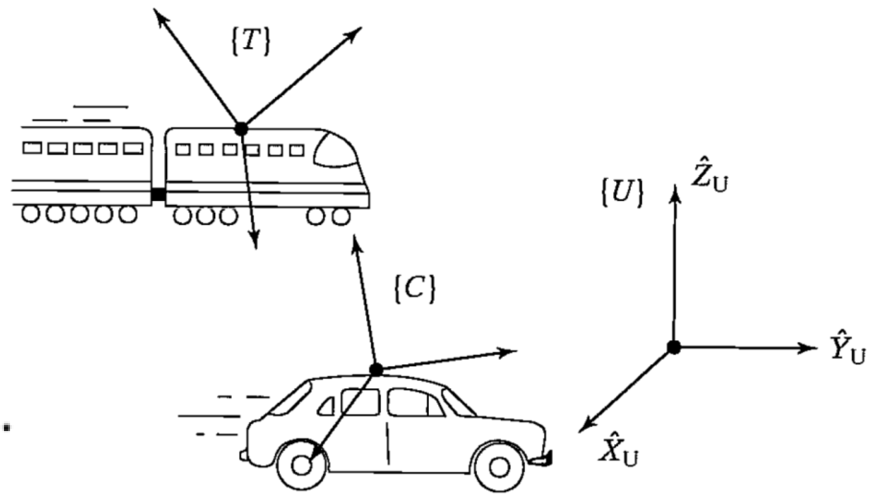
# Example

- Figure shows a fixed universe frame, {U}, a frame attached to a train travelling at 100mph, {T}, and a frame attached to a car travelling at 30mph, {C}. Both vehicles are heading in the $\hat{X}$ direction of {U}. The rotation matrices, $R_T^U$ and $R_C^U$, are known and constant



# Example

- What is $\left(\frac{d}{dt} P^U_{CORG}\right)^U$?




$$\frac{^U d}{dt} {}^U P_{CORG} = {}^U V_{CORG} = \upsilon_C = 30\hat{X}.$$

- What is ${}^C({}^U V_{TORG})$?

$${}^C({}^U V_{TORG}) = {}^C\upsilon_T = {}^C_U R \upsilon_T = {}^C_U R(100\hat{X}) = {}^U_C R^{-1}\, 100\hat{X}.$$

- What is ${}^C({}^T V_{CORG})$?

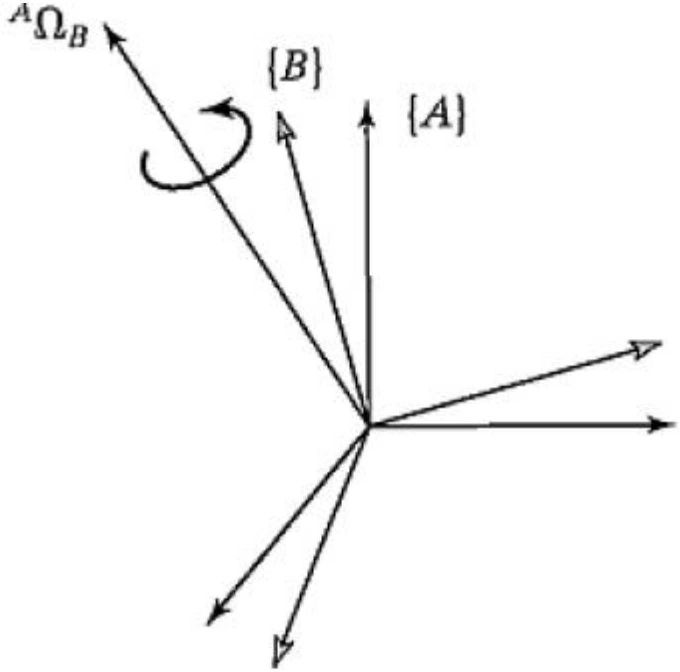
$${}^C({}^T V_{CORG}) = {}^C_T R\, {}^T V_{CORG} = -{}^U_C R^{-1}\, {}^U_T R\, 70\hat{X}.$$

# Basics-Angular Velocity

- Angular velocity vector –symbol $\Omega$
- **Linear velocity** describes an attribute of a **point**,
- **Angular velocity** describes an attribute of a **body**.
- We always attach a frame to the bodies, therefore angular velocity describes rotational motion of a frame

# Basics-Angular Velocity

- $\Omega_B^A$ describes the rotation of frame B relative to frame A




- Physically, at any instant, the direction of $\Omega_B^A$ indicates the instantaneous axis of rotation of {B} relative to {A}, and the magnitude of $\Omega_B^A$ indicates the speed of rotation.

# Basics-Angular Velocity

- An angular velocity vector may be expressed in any coordinate system, and so another leading superscript may be added; for example, $\left(\Omega_B^A\right)^C$ is the angular velocity of frame {B} relative to {A} expressed in terms of frame{C}
- **Simplified notation**: angular velocity of frame (C) relative to some understood reference frame, (U)

$$\omega_C = {}^U\Omega_C$$

- The angular velocity of frame (C) expressed in terms of (A) (though the angular velocity is with respect to (U)).

$${}^A\omega_C$$

# Linear and Angular Velocities of Rigid Bodies

- We focus on motion of rigid body with regards to velocity

- We therefore extend the notions of translations and orientations described earlier to the time-varying case

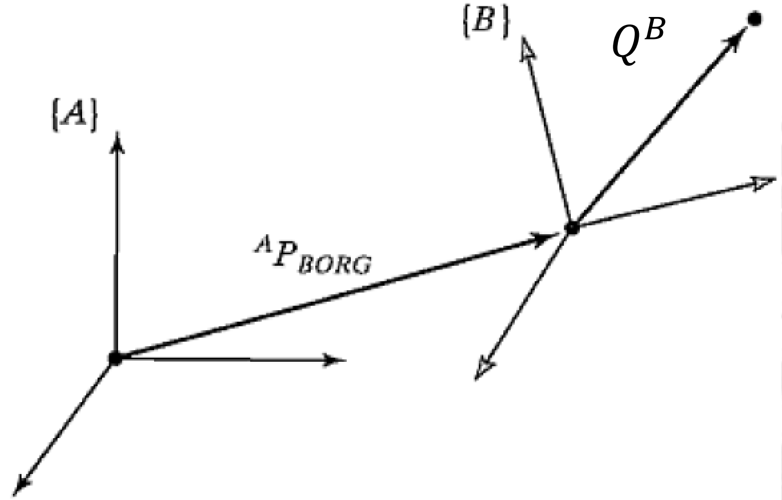
- We attach a coordinate system to any body that we wish to describe. Then, motion of rigid bodies can be equivalently studied as the motion of frames relative to one another.

# Linear and Angular Velocities of Rigid Bodies

## Linear Velocity

- Consider a frame {B} attached to a rigid body. We wish to describe the motion of {B} relative to frame {A}. For this time instant assume no change in orientation of B relative to A i.e. motion of Q is due to $P^{A}_{BORG}$ or $Q^{B}$ changing in time

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q}$$



# Skew-Symmetric Matrices

- A matrix is skew symmetric iff;

$$S + S^T = 0$$

- A result from linear algebra states that for any orthonormal matrix R, there exists a skew matrix S such that;

$$S = \dot{R}R^T$$

A skew-symmetric matrix of 3D is specified by three parameters $\mathrm{s} = [s_1, s_2, s_3]^{\mathrm{T}}$ as;

$$S = \begin{bmatrix} 0 & -s_1 & s_2 \\ s_1 & 0 & -s_3 \\ -s_2 & s_3 & 0 \end{bmatrix}$$

# Linear and Angular Velocities of Rigid Bodies

- We can derive an interesting relationship between the derivative of an orthonormal matrix and a certain skew-symmetric matrix as follows. For any n x n orthonormal matrix, R, we have

$$RR^T = I_n$$

n x n identity matrix

- Differentiating by product rule

$$\dot{R}R^T + R\dot{R}^T = 0_n$$

- Using the commutative property
- Let

$$\dot{R}R^T + (\dot{R}R^T)^T = 0_n.$$

$$S = \dot{R}R^T \qquad S + S^T = 0_n$$

# Linear and Angular Velocities of Rigid Bodies

## Velocity of a point due to rotating reference frame

- Consider a fixed vector $P^B$ unchanging with respect to frame (B). Its description in another frame {A} is given as

$$^{A}P = {}^{A}_{B}R\, {}^{B}P$$

If frame {B} is rotating (i.e., the derivative ${}^{A}_{B}\dot{R}$ is non zero), then $^{A}P$ will be changing even though $^{B}P$ is constant; that is,

$$^{A}\dot{P} = {}^{A}_{B}\dot{R}\, {}^{B}P \qquad \text{or}$$

$$^{A}V_P = {}^{A}_{B}\dot{R}\, {}^{B}P$$

Substituting for $^{B}P$

$$^{A}V_P = {}^{A}_{B}\dot{R}\, {}^{A}_{B}R^{-1}\, {}^{A}P.$$

$$^{A}V_P = {}^{A}_{B}S\, {}^{A}P.$$

The skew symmetric matrix we have introduced is called the **angular-velocity matrix**

# Linear and Angular Velocities of Rigid Bodies

## Skew Symmetric Matrices and Vector Cross-Product

- If we assign the elements in a skew-symmetric matrix S as

$$S = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$$

- and define the 3 x 1 column vector

$$\Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$$

Angular Velocity Vector

- then it is easily verified that

$$SP = \Omega \times P$$

- where P is any vector, and x is the vector cross-product, Hence

$${}^{A}V_{P} = {}^{A}_{B}S\,{}^{A}P \rightarrow {}^{A}V_{P} = {}^{A}\Omega_{B} \times {}^{A}P$$