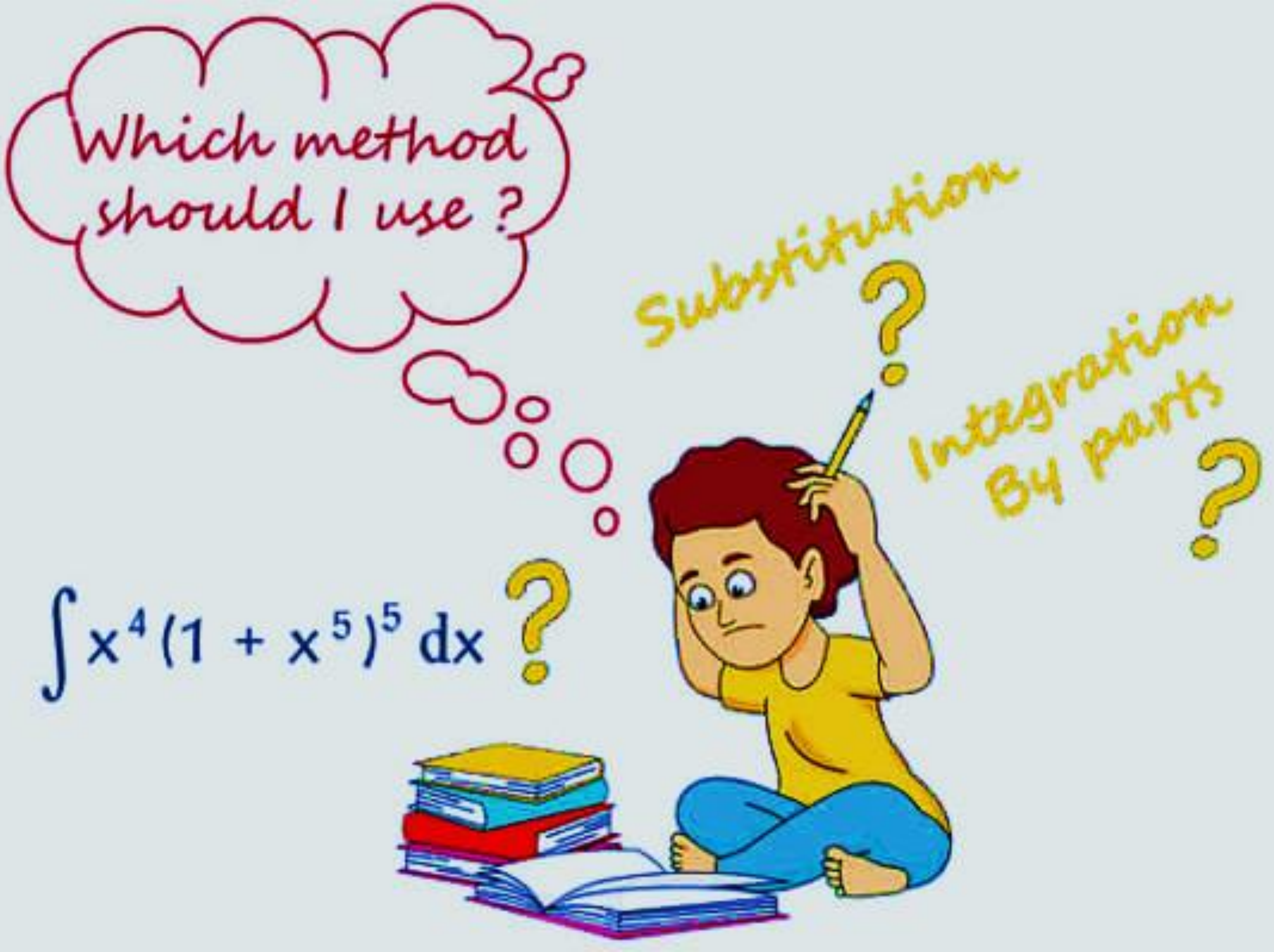
INTEGRATION

Calculus & Analytical Geometry MATH-101

Instructor: Dr. Naila Amir (SEECS, NUST)

TECHNIQUES OF INTEGRATION



Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter: 5**
  - **Section:** 5.5
- **Chapter: 8**
  - **Section:** 8.1, 8.2

Techniques of Integration

- Substitution Rule
- Integration by Parts
- Integration of Rational & Irrational Functions
- Trigonometric Integrals
- Trigonometric Substitution

Table of Integration Formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C \qquad \text{(any number } k\text{)}$
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \frac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad (a > 0)$
22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \qquad (u > a > 0)$

$$\int f(g(x))\, g'(x)\, dx$$

$$\int f(u)\, du$$

$$\frac{d}{dx}(x^2) \Rightarrow \frac{d(x^2)}{dx} = 2x$$

$$\Rightarrow d(x^2) = (2x)\,dx$$

$$\Rightarrow du = (2x)\,dx$$

Sections: 5.5 & 8.1
The Substitution Rule

$$\int \cos(x^2)\, 2x\, dx$$

$$\int \cos(u)\, du$$

The Substitution Rule

We now know how to solve the following integrals:

$$\int \sqrt[4]{x}\, dx \qquad \int \frac{1}{t^3} dt \qquad \int \cos w\, dw \qquad \int e^y\, dy$$

However, we can't solve the following integrals directly.

$$\int 18x^2 \sqrt[4]{6x^3+5}\, dx \qquad \int \frac{2t^3+1}{\left(t^4+2t\right)^3} dt$$

$$\int \left(1-\frac{1}{w}\right) \cos\left(w-\ln w\right) dw \qquad \int (8y-1)\, e^{4y^2-y}\, dy$$

The Substitution Rule

- In finding the antiderivative for some functions, many techniques fail.
- Substitution can sometimes remedy this problem.
- Substitution depends on the idea of a differential.
- If $u = f(x)$, then the differential of u, written du, is defined as $du = f'(x)dx$

Example:

If $u = 2x^3 + 1$, then $du = 6x^2\,dx$.

$u = f(x)$

$\frac{du}{dx} = f'(x)$

$\Rightarrow du = f'(x)dx$

Example:

Solve the following integral:

$$\int 18x^2 \sqrt[4]{6x^3+5}\, dx\,. \checkmark$$

Solution:

But using differentials and substitution we'll find the antiderivative:

$$\int 18x^2 \sqrt[4]{6x^3+5}\, dx = \int (\overbrace{6x^3+5}^{u})^{1/4}\, \overbrace{18x^2 dx}^{du}$$

$$= \int \mathrm{u}^{1/4} du$$

Now use the power rule

$$\int \mathrm{u}^{1/4} du = \frac{u^{5/4}}{5/4} + C$$

$$\int u^{n+1} du = \frac{u^{n+2}}{n+2}$$

Substitute $(6x^3 + 5)$ back in for u:

$$\int (6x^3 + 5)^{1/4} 18x^2 dx = \frac{4}{5}(6x^3 + 5)^{5/4} + C$$

Example:

Solve the following integral:

$$\int (2x^3 + 1)^4 6x^2 dx\,.$$

Solution:

But using differentials and substitution we'll find the antiderivative:

$$\int (2x^3 + 1)^4 6x^2 dx = \int \overbrace{(2x^3 + 1)}^{u}{}^4 \; \overbrace{6x^2\, dx}^{du}$$

$$= \int \mathrm{u}^4 du$$

Now use the power rule

$$\int \mathrm{u}^4 du = \frac{u^5}{5} + C.$$

Substitute $(2x^3 + 1)$ back in for u:

$$\int (2x^3 + 1)^4 6x^2 dx = \frac{(2x^3 + 1)^5}{5} + C.$$

SUBSTITUTION METHOD

Choosing u:

In general, for the types of problems we are concerned with, there are three cases.

We choose u to be one of the following:

1. the quantity under a root or raised to a power;
2. the quantity in the denominator;
3. the exponent one.

Remember that some integrands may need to be rearranged to fit one of these cases.

Example:

Evaluate:

$\frac{du}{3} = x^2 dx$

$$\int x^2\sqrt{x^3+1}\,dx.$$

u

$\int f(g(x))\cdot g'(x)\,dx$

$du \neq g'(x)\,dx$

$u = g(x)$

Solution:

Let $u = x^3 + 1$, then $du = 3x^2dx$. Note that there is an x^2 in the problem but no $3x^2$, so we need to multiply by 3. Multiplying by 3 changes the problem, so we need to counteract that 3 by also multiplying by $1/3$. Thus,

$$\int x^2\sqrt{x^3+1}\;dx = \frac{1}{3}\int 3x^2\sqrt{x^3+1}\;dx = \frac{1}{3}\int \sqrt{x^3+1}\;(3x^2dx)$$

$$= \frac{1}{3}\int \sqrt{u}\;du = \frac{1}{3}\int u^{\frac{1}{2}}du = \frac{1}{3}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{1}{3}\left(\frac{2}{3}\right)u^{\frac{3}{2}} + C = \frac{2}{9}u^{\frac{3}{2}} + C$$

Thus,

$$\int x^2\sqrt{x^3+1}\;dx = \frac{2}{9}(x^3+1)^{\frac{3}{2}} + C.$$

Example:

Evaluate:

$$\int \frac{(x+3)}{(x^2+6x)^2}\,dx\,.$$

Evaluate

$$\int \frac{3}{\underbrace{(x^2+6x)^2}_{??}}\,dx$$

Solution:

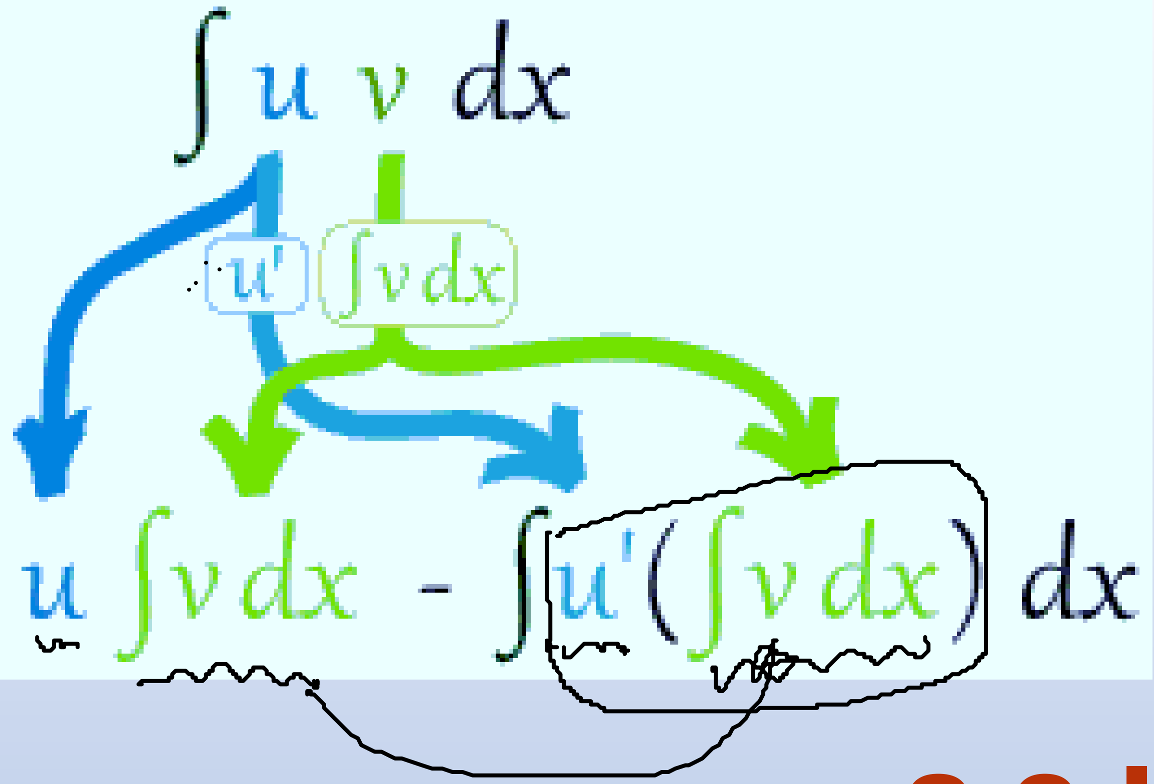
Let $u = x^2 + 6x$, then $du = (2x+6)dx = 2(x+3)dx.$

$$\int \frac{(x+3)}{(x^2+6x)^2}\,dx = \frac{1}{2}\int \frac{2(x+3)}{(x^2+6x)^2}\,dx$$

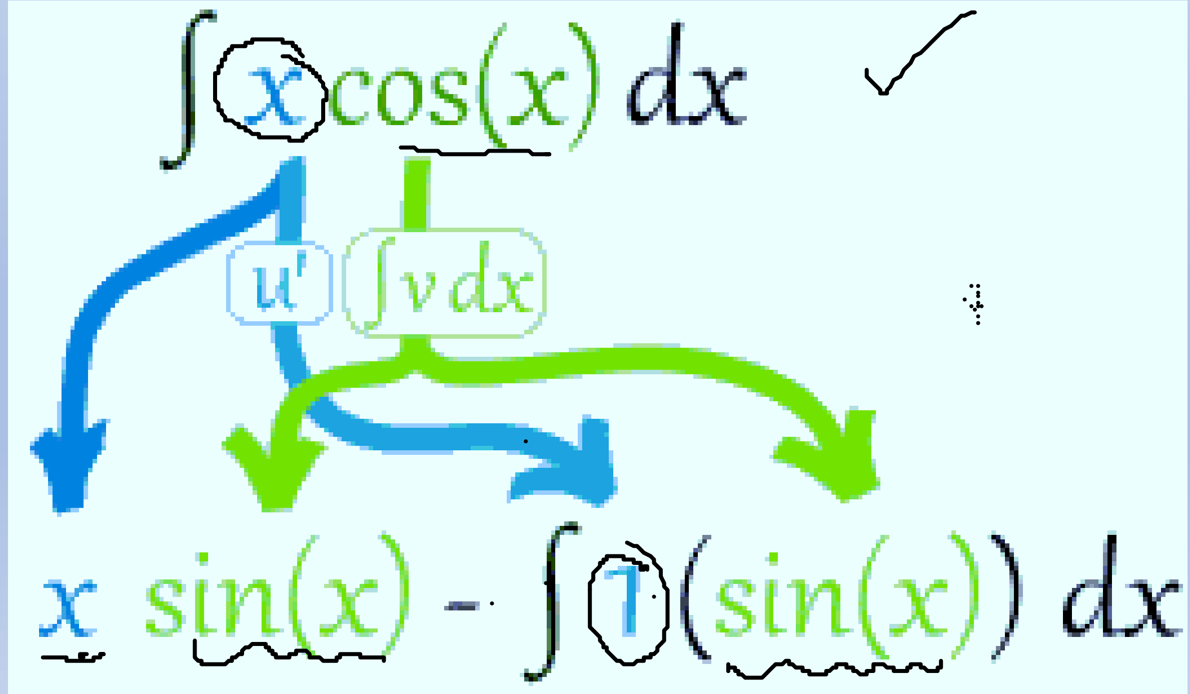
$$= \frac{1}{2}\int \frac{du}{u^2} = \frac{1}{2}\int u^{-2}\,du = \frac{1}{2}\left(\frac{u^{-1}}{-1}\right) + C = \frac{-1}{2u} + C. \checkmark$$

Thus,

$$\int \frac{(x+3)}{(x^2+6x)^2}\,dx = \frac{-1}{2(x^2+6x)} + C. \checkmark$$



8.2 Integration by Parts



Integration by parts

Every differentiation rule has a corresponding integration rule.

- For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation.
- The rule that corresponds to the Product Rule for differentiation is called the rule for integration by parts.

$\int uv\,dx$

Integration by parts

- The Product Rule states that, if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\Rightarrow \int d[f(x)\,g(x)] = \int [f(x)\,g'(x) + g(x)\,f'(x)]\,dx$$

- In the notation for indefinite integrals, this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\, dx = f(x)g(x)$$

or

$$\int f(x)g'(x)\, dx + \int g(x)f'(x)\, dx = f(x)g(x)$$

Integration by parts

$$\int f(x)\, g(x)\, dx = f(x) \cdot \int g(x)\, dx - \int \left[\int g(x)\, dx \cdot \frac{d}{dx}(f(x)) \right] dx$$

- We can rearrange this equation as:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx. \qquad \text{(I)}$$

This equation gives us the formula for **integration by parts.**

- It is perhaps easier to remember this formula in the following notation:
- Let $u = f(x)$ and $v = g(x)$. Then, the differentials are:

$$du = f'(x)dx \quad \text{and} \quad dv = g'(x)dx.$$

Thus, by the Substitution Rule, the formula for integration by parts becomes:

$$\int u dv = uv - \int v du. \qquad \text{(II)}$$

Example:

Evaluate:

$$(\mathrm{I}) \quad \text{---} \int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int g(x)\, f'(x)\, dx$$

$$\int x \sin x \, dx \, .$$

Solution:

Suppose we choose $f(x) = x$ and $g'(x) = \sin x$. Then, $f'(x) = 1$ and $g(x) = -\cos x$.

Thus, using (I) we have:

$$\int x \sin x \, dx = f(x)g(x) - \int g(x)f'(x)dx$$

$$= x(-\cos x) - \int (-\cos x)dx$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

Note:

- It is wise to check the answer by differentiating it.
- If we do so, we get:

$$
\begin{aligned}
\frac{d}{dx}(-x\cos x + \sin x + C) &= -\cos x + x\sin x + \cos x \\
&= x\sin x\,, \quad \checkmark
\end{aligned}
$$

as expected.

Alternative Method:

Let

$$u = x \qquad \text{and} \qquad dv = \sin x \, dx.$$

Then,

$$du = dx \qquad \text{and} \qquad v = -\cos x\,.$$

Using (II), we have:

$$\int x \sin x \, dx = \int \overbrace{x}^{u} \overbrace{\sin x \, dx}^{dv} = \overbrace{x}^{u} \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v} \overbrace{dx}^{du}$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C.$$

Note:

- Our aim in using integration by parts is to obtain a simpler integral than the one we started with.
- In previous example, we started with $\int x \sin x \, dx$ and expressed it in terms of the simpler integral $\int \cos x \, dx$.
- If we had chosen $u = \sin x$ and $dv = x \, dx$, then $du = \cos x \, dx$ and $v = x^2/2$.
- So, integration by parts gives:

$$\int x \sin x \, dx = (\sin x)\frac{x^2}{2} - \frac{1}{2}\int x^2 \cos x \, dx .$$

- Although this is also correct, but $\int x^2 \cos x \, dx$ is a more difficult integral than the one we started with.

Note:

Hence, when choosing u and dv, we usually try to keep $u = f(x)$ to be a function that becomes simpler when differentiated.

$$\int x^4 \ln x \, dx$$

u

- At least, it should not be more complicated.
- However, make sure that $dv = g'(x)\,dx$ can be readily integrated to give v. ✓

$\int \sin x \, dx = -\cos x$

$\int x \, dx = \frac{x^2}{2}$ ✓

$\frac{d}{dx}(\sin x) = \cos x$

$\int x \sin x \, dx$ ✓

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

Example:

Evaluate:

$$\int \ln x \, dx.$$

Solution:

Here, we don't have much choice for u and dv. Let

$$u = \ln x \quad \text{and} \quad dv = dx.$$

Then,

$$du = \frac{1}{x} dx \qquad \text{and} \qquad v = x.$$

$\int u\,dv = uv - \int v\,du \rightarrow \text{(II)}$

Integrating by parts, we get:

$u = \ln x$

$dv = dx$

$$\int \ln x \, dx = x \ln x - \int \cancel{x} \frac{dx}{\cancel{x}}$$

$du = \frac{1}{x}\,dx$

$v = x$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

$\int \ln x \, dx = x \ln x - \int \cancel{x} \cdot \frac{dx}{\cancel{x}}$

Example:

Evaluate:

$$\int t^2 e^t \, dt.$$

D

t^2

redu

I

Inc

Solution:

Notice that t^2 becomes simpler when differentiated. However, e^t remains unchanged when differentiated or integrated. So, we choose:

$$u = t^2 \; dv = e^t dt$$

Then,

$$du = 2tdtv = e^t$$

Integration by parts gives:

$$\int t^2 \, e^t dt = t^2 e^t - 2 \int \overset{u}{t} e^t \, dt$$

The integral that we obtained,

$$\int te^t\,dt,$$

is simpler than the original integral. However, it is still not obvious. So, in order to evaluate this integral, we need to use integration by parts again. This time, we choose

$$u = t \text{ and } dv = e^t dt$$

Then, $du = dt$ and $v = e^t$. So,

$$\int te^t\,dt = te^t - \int e^t dt = te^t - e^t + C.$$

Putting this in original equation, we get:

$$\int t^2 e^t dt = t^2 e^t - 2\int te^t\,dt = t^2 e^t - 2(te^t - e^t + C)$$
$$= t^2 e^t - 2te^t - 2e^t + C_1,$$

where $C_1 = -2C$.

Example:

Prove the reduction formula:

$$\int \sin^n x\, dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\, dx\,. \quad \checkmark$$

where $n \geq 2$ is an integer. This is called a reduction formula because the exponent n has been reduced to $n-1$ and $n-2$.

Solution:

$$\int \sin^n x\, dx = \int \sin^{n-1} x\,.\sin x\, dx\,.$$

Let

$$u = \sin^{n-1} x \quad \text{and} \quad dv = \sin x\, dx$$

Then,

$$du = (n-1)\sin^{n-2} x \cos x\, dx \quad \text{and} \quad v = -\cos x$$

So, integration by parts gives:

$$\int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx \,.$$

Since $\cos^2 x = 1 - \sin^2 x$, we have:

$$\int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, (1 - \sin^2 x) dx$$

$$\Rightarrow \int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

$$\Rightarrow \int \sin^n x \, dx + (n-1) \int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx$$

$$\Rightarrow n \int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx$$

$\int \sin^5 x \, dx$

$$\Rightarrow \int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{(n-1)}{n} \int \sin^{n-2} x \, dx \,.$$

- The reduction formula is useful.

- By using it repeatedly, we could express

$$\int \sin^n x\, dx\ 0, \qquad n \geq 2,$$

$\int \sin^5 x\, dx$

in terms of:

  - $\int \sin x\, dx = -\cos x + C$ (if n is odd) ✓
  - $\int (\sin x)^0 dx = \int dx = x + C$ (if n is even) ✓

Example:

Evaluate:

$$\int \cos^n x \, dx$$

$$\int \cos^n x \, dx = \int \cos^{n-1}(x) \cdot \cos x \, dx$$

$$u = \cos^{n-1}(x)$$

$$dv = \cos x \, dx$$

Practice

Practice Questions

Book: Thomas Calculus (11th Edition) by Georg B.Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Exercise: 5.5
  Q # 1 to Q # 48, Q # 53 to Q # 58. ✓
- Exercise: 8.1
  Q # 1 to Q # 46, Q # 53 to Q # 58. ✓
- Exercise: 8.2
  Q # 1 to Q # 30, Q # 39 to Q # 42. ✓