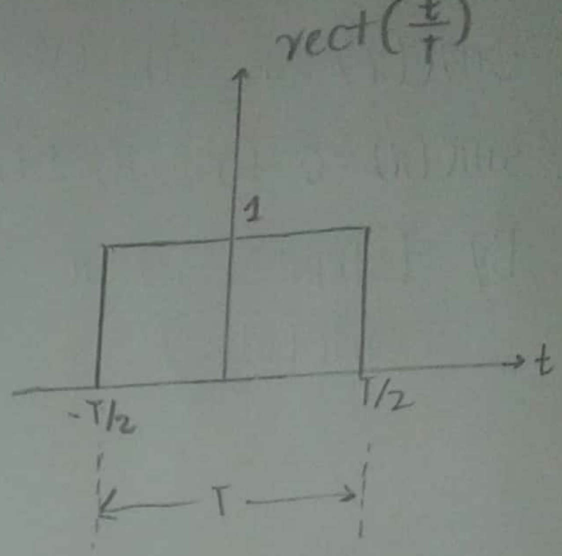
Some Useful Functions

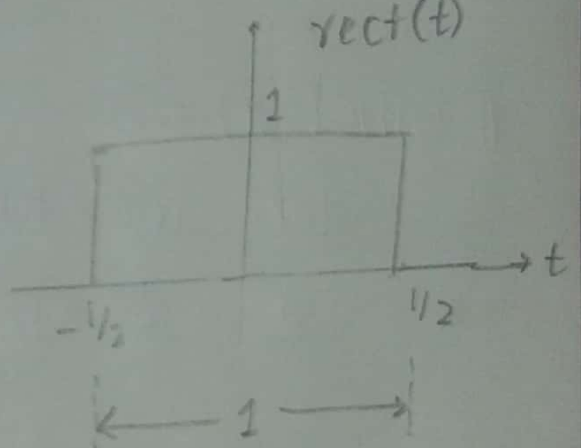
Gate Function

A gate function is a rectangular pulse.



$$rec\left(\frac{t}{T}\right) = \begin{cases} 1 & |t| < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

Note that here T is the pulse width.
So, $rect(t)$ is a pulse with width = 1.



Sampling Function or Interpolating Function or Sinc Function

The function $\frac{\sin x}{x}$ is "sine over argument" function, denoted by $\operatorname{sinc}(x)$.

This function plays an important role in signal processing. It is also known as the filtering or interpolating function.

$$\operatorname{sinc}(x) = \frac{\sin x}{x}$$

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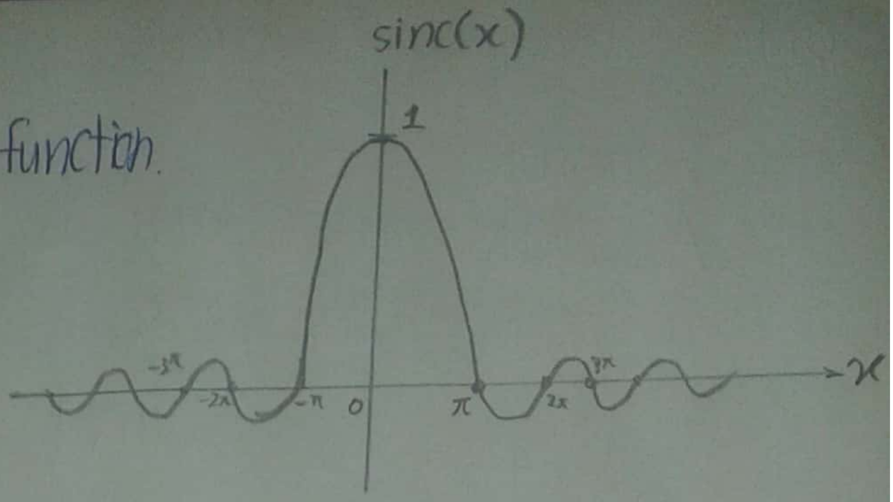
Few points:

1. $sinc(x)$ is an even function.
2. $sinc(x) = 0$ for $x = \pm n\pi$
3. By L'Hopital's rule:
   $sinc(0) = 1$

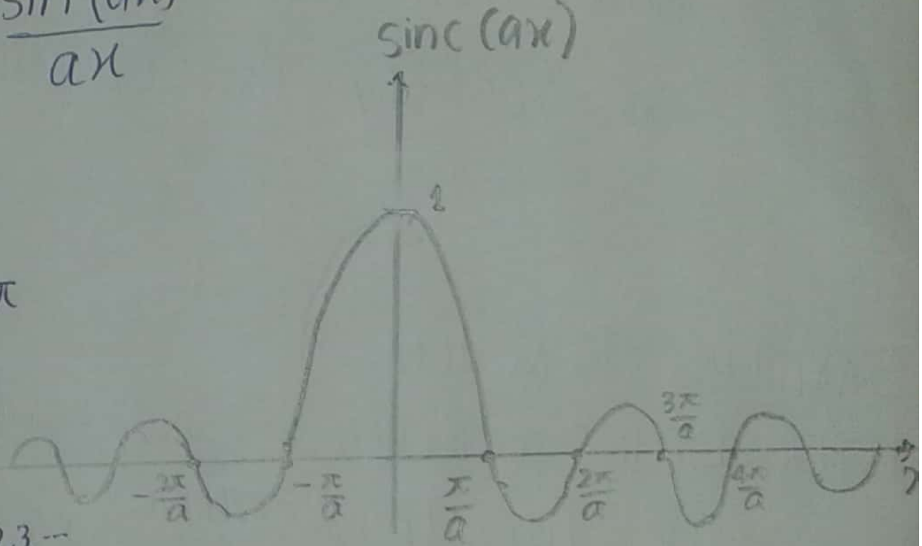


$$sinc(ax) = \frac{\sin(ax)}{ax}$$

$sinc(ax) = 0$
for $ax = \pm n\pi$

$\Rightarrow$ $x = \pm \frac{n\pi}{a}$

where $n = \cancel{0}, 1, 2, 3, \ldots$



FOURIER TRANSFORM OF THE GATE FUNCTION

$$f(t) = rect\left(\frac{t}{T}\right) = \begin{cases} 1 & |t| < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

By the definition of Fourier Transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t}\, dt$$

$$F(\omega) = \int_{-\infty}^{\infty} rect\left(\frac{t}{T}\right) e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{T/2} 1 \cdot e^{-j\omega t}\, dt$$

$$= \left[\frac{e^{-j\omega t}}{-j\omega} \right|_{-T/2}^{T/2}$$

By substitution: $e^{j\omega \frac{T}{2}} - e^{-j\omega \frac{T}{2}} = 2j \sin\left(\frac{\omega T}{2}\right)$

We have:

$$F(\omega) = \frac{1}{j\omega}\left(2j \sin \frac{\omega T}{2}\right) = \frac{2 \sin \frac{\omega T}{2}}{\omega}$$

$$= \frac{T \sin \frac{\omega T}{2}}{\frac{\omega T}{2}}$$

$$F(\omega) = T \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$

Thus, the Fourier Transform pair:

Half-pulse width

$$\mathrm{rect}\left(\frac{t}{T}\right) \longleftrightarrow T\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)$$

pulse width   pulse width

So,

$$\mathrm{rect}(t) \longleftrightarrow \mathrm{sinc}\left(\frac{\omega}{2}\right)$$

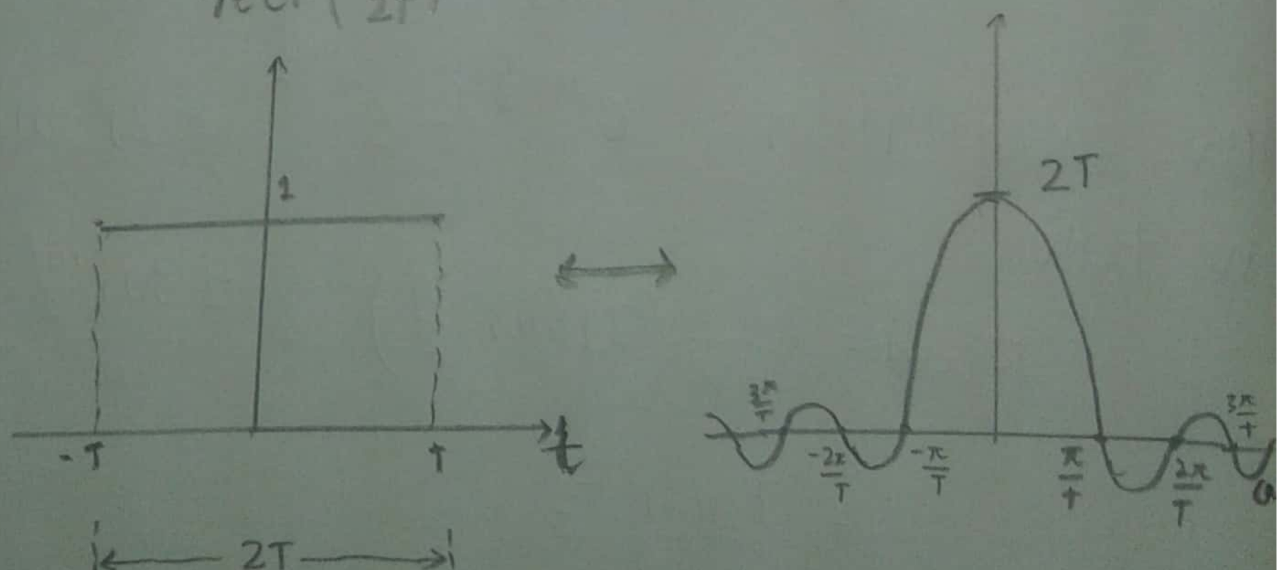
and

$$\mathrm{rect}\left(\frac{t}{2T}\right) \longleftrightarrow 2T\,\mathrm{sinc}(\omega T)$$

and

$$A\,\mathrm{rect}\left(\frac{t}{2T}\right) \longleftrightarrow 2AT\,\mathrm{sinc}(\omega T)$$

$$\mathrm{rect}\left(\frac{t}{2T}\right) \longleftrightarrow 2T\,\mathrm{sinc}(\omega T)$$



(time-domain)   (frequency-domain)

DUALITY or SYMMETRY PROPERTY

The duality property states that

if $\quad f(t) \longleftrightarrow F(\omega)$

then $\quad F(t) \longleftrightarrow 2\pi f(-\omega)$

Let's take a simple example:

EXAMPLE: Find the Fourier transform of e^{-jat}

SOLUTION:

We'll use the duality property:

So we write

③ $\delta(t-a) \longleftrightarrow$ ② $e^{-ja\omega}$

① $e^{-jat} \longleftrightarrow$ ④ $2\pi\,\delta((-\omega)-a)$

$= 2\pi\,\delta(\omega+a)$

Even Function $\delta(t)$

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EXAMPLE: Find $F\left[\frac{1}{1+jt}\right]$

SOLUTION: $e^{-t} \cdot u(t) \xleftrightarrow{3} \frac{1}{1+j\omega}$

$$1 \quad \frac{1}{1+jt} \xrightarrow{2} \frac{1}{1+j\omega}$$

$$e^{-t} \cdot u(t) \xrightarrow{4} e^{-(-\omega)} u(-\omega)$$

$$= e^{\omega} u(-\omega)$$

$$F\left[\frac{1}{1+jt}\right] = e^{\omega} u(-\omega)$$

FOURIER TRANSFORM OF sinc (at)

Given that:

$$\text{rect}\left(\frac{t}{2T}\right) \longleftrightarrow 2T\,\text{sinc}(\omega T)$$

Use duality to find $F[\text{sinc}(at)]$

SOLUTION:

We have:

$$\frac{1}{2T}\,\text{rect}\left(\frac{t}{2T}\right) \longleftrightarrow \text{sinc}(\omega T)$$

Duality

$$\text{sinc}(tT) \longleftrightarrow 2\pi\left[\frac{1}{2T}\,\text{rect}\left(\frac{-\omega}{2T}\right)\right]$$

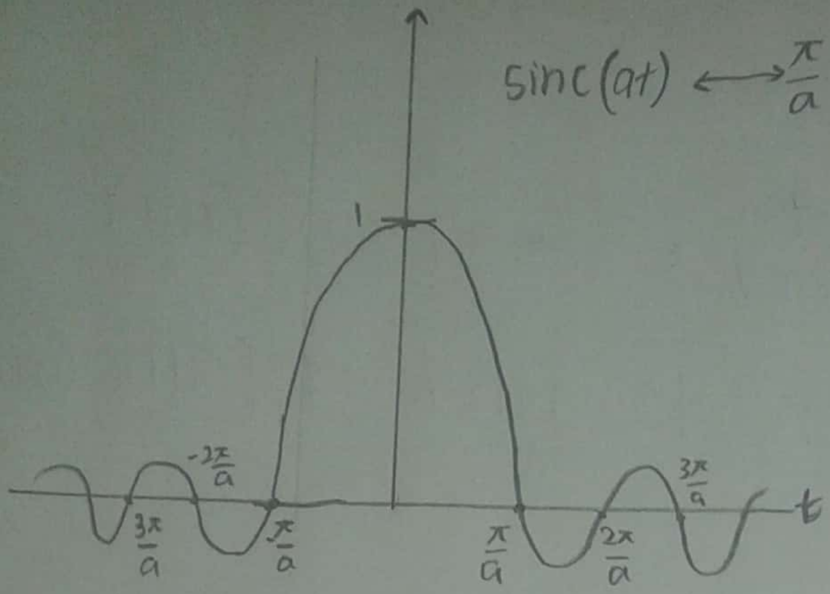
$$= \frac{\pi}{T}\,\text{rect}\left(\frac{\omega}{2T}\right)$$

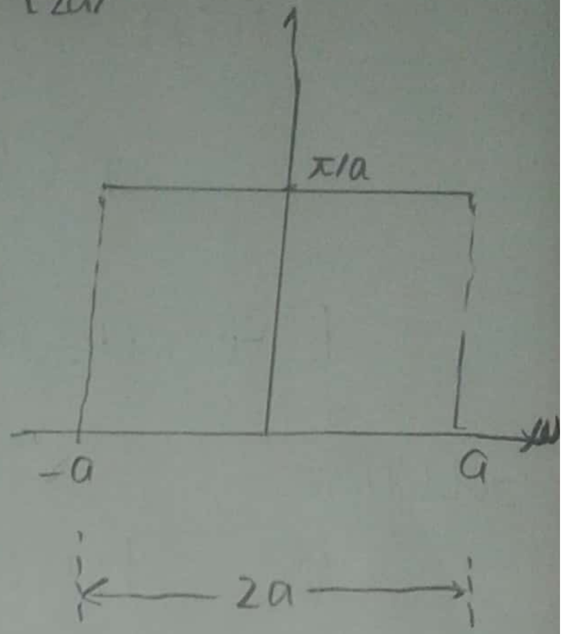
Replace T with a:

Even Function

$$\Rightarrow \boxed{\text{sinc}(at) \longleftrightarrow \frac{\pi}{a}\,\text{rect}\left(\frac{\omega}{2a}\right)}$$

$$\mathrm{sinc}(at) \longleftrightarrow \frac{\pi}{a}\,\mathrm{rect}\left(\frac{t}{2a}\right)$$





We summarize:

$$\mathrm{rect}\left(\frac{t}{T}\right) \longleftrightarrow T\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)$$

and

$$\mathrm{sinc}(at) \longleftrightarrow \frac{\pi}{a}\,\mathrm{rect}\left(\frac{\omega}{2a}\right)$$

Ex: Find $F[\mathrm{sinc}(t)]$.

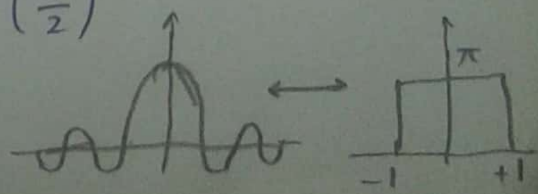
SOLUTION:

We know:

$$\mathrm{sinc}(at) \longleftrightarrow \frac{\pi}{a}\,\mathrm{rect}\left(\frac{\omega}{2a}\right)$$

For $a=1$:

$$\mathrm{sinc}(t) \longleftrightarrow \pi\,\mathrm{rect}\left(\frac{\omega}{2}\right)$$



A Comprehensive Example

(a) Using the Fourier Transform pair

$$\text{rect}\left(\frac{t}{T}\right) \longleftrightarrow T\,\text{sinc}\left(\frac{\omega T}{2}\right)$$

and duality, find $F[\text{sinc}(at)]$.

(b) Using the Fourier Transform pair

$$\text{sinc}(at) \longleftrightarrow \frac{\pi}{a}\,\text{rect}\left(\frac{\omega}{2a}\right)$$

and duality, find $F[\text{rect}\left(\frac{t}{T}\right)]$

SOLUTION:

(a) We have:

$$\frac{1}{T}\,\text{rect}\left(\frac{t}{T}\right) \longleftrightarrow \text{sinc}\left(\frac{\omega T}{2}\right)$$

compare

$$\text{sinc}(at) \longleftrightarrow ?$$

By comparing, we get:

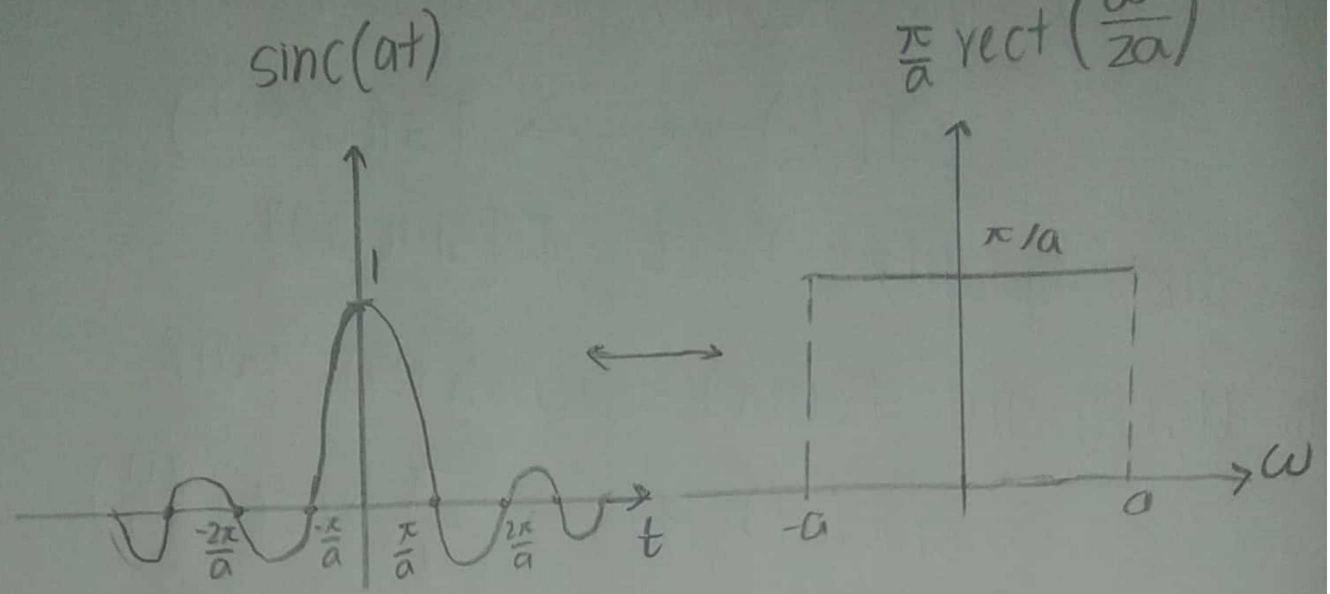
$$a = \frac{T}{2} \Rightarrow T = 2a$$

So that by duality:

$$\text{sinc}(at) \longleftrightarrow 2\pi\left[\frac{1}{2a}\,\text{rect}\left(\frac{\omega}{2a}\right)\right]$$

even function

$$\text{sinc}(at) \longleftrightarrow \frac{\pi}{a}\,\text{rect}\left(\frac{\omega}{2a}\right)$$



(b) We have:

$$\frac{a}{\pi}\,\text{sinc}(at) \longleftrightarrow \text{rect}\left(\frac{\omega}{2a}\right)$$

$$\text{rect}\left(\frac{t}{T}\right) \longleftrightarrow \;?$$

By comparing:

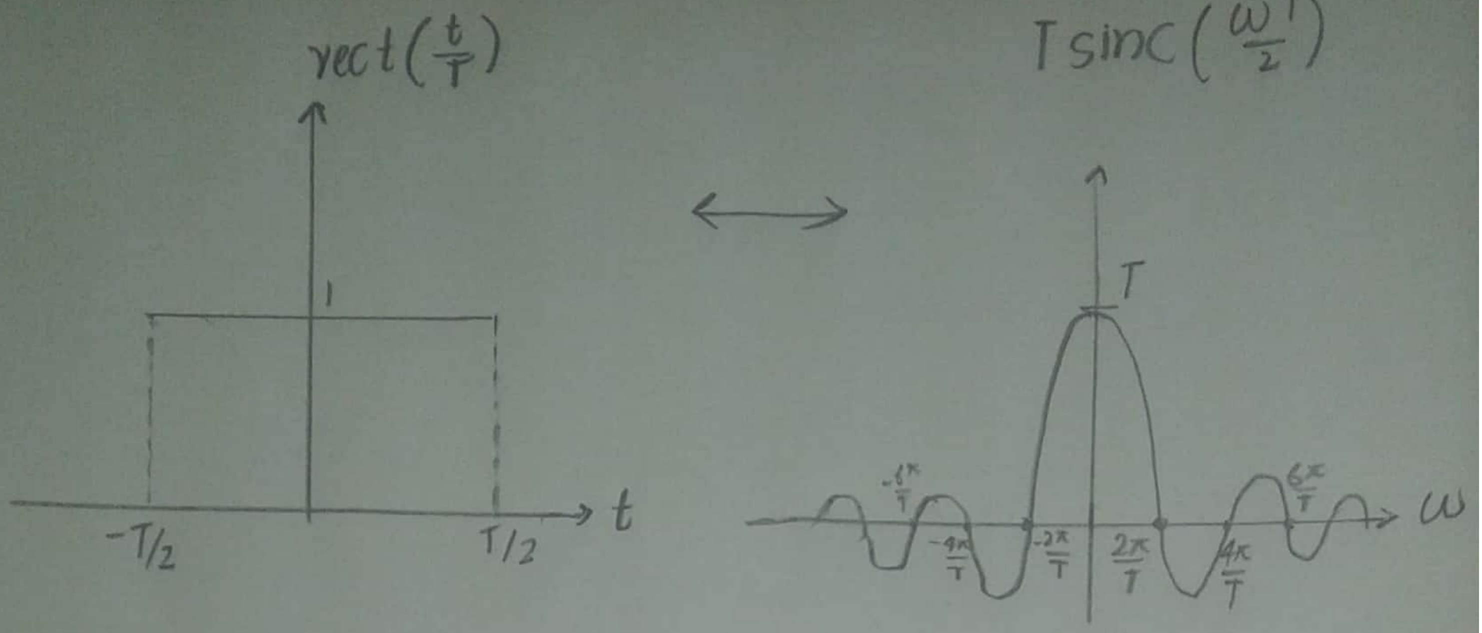
$$T = 2a \Rightarrow a = \frac{T}{2}$$

So that by duality:

even function

$$\text{rect}\left(\frac{t}{T}\right) \longleftrightarrow 2\pi\left[\frac{T/2}{\pi}\,\text{sinc}\left(\frac{T}{2}\omega\right)\right]$$

$$= T\,\text{sinc}\left(\frac{\omega T}{2}\right)$$


$rect(\frac{t}{T})$
1
$-T/2$
$T/2$
t
$T\,sinc(\frac{\omega T}{2})$
T
ω
$-\frac{6\pi}{T}$
$-\frac{4\pi}{T}$
$-\frac{2\pi}{T}$
$\frac{2\pi}{T}$
$\frac{4\pi}{T}$
$\frac{6\pi}{T}$