SPHERICAL COORDINATES

Spherical Coordinates (r,θ,Φ)

- Most appropriate when dealing with problems having a degree of spherical symmetry

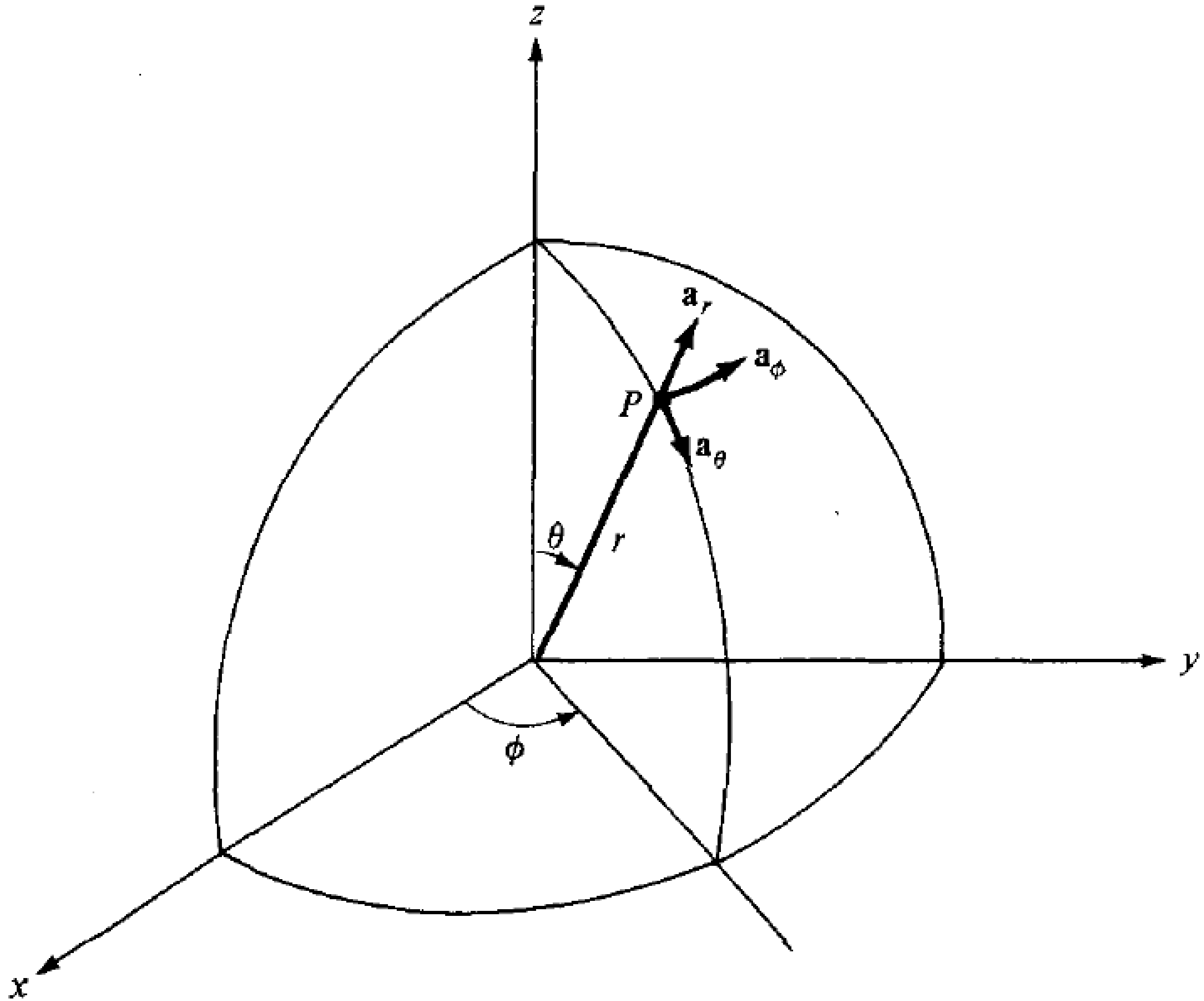
- A point P in spherical coordinates can be represented as (r,θ,Φ)

- r is defined as the distance from the origin to point P or the radius of a sphere centered at the origin and passing through P

- θ is the angle between the z-axis and the position vector of P

- Φ is measured from the x-axis and is the same angle as in cylindrical coordinates

Spherical Coordinates (r,θ,Φ)



Spherical Coordinates (r,θ,Φ)

- The ranges of the variables are:

$$0 \leq r < \infty$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi < 2\pi$$

- A vector in spherical coordinates may be written as:

$$(A_r, A_\theta, A_\phi) \qquad \text{or} \qquad A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\phi\mathbf{a}_\phi$$

- $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\Phi$ are unit vectors in the r, θ and Φ directions

- The magnitude of **A** is:

$$|\mathbf{A}| = (A_r^2 + A_\theta^2 + A_\theta^2)^{1/2}$$

Spherical Coordinates (r,θ,Φ)

- **$\mathbf{a}_r$** being directed along the radius – In the direction of increasing r
- **$\mathbf{a}_\theta$** in the direction of increasing θ
- **$\mathbf{a}_\Phi$** in the direction of increasing Φ. Therefore:

$$\mathbf{a}_r \cdot \mathbf{a}_r = \mathbf{a}_\theta \cdot \mathbf{a}_\theta = \mathbf{a}_\phi \cdot \mathbf{a}_\phi = 1$$

$$\mathbf{a}_r \cdot \mathbf{a}_\theta = \mathbf{a}_\theta \cdot \mathbf{a}_\phi = \mathbf{a}_\phi \cdot \mathbf{a}_r = 0$$

AND

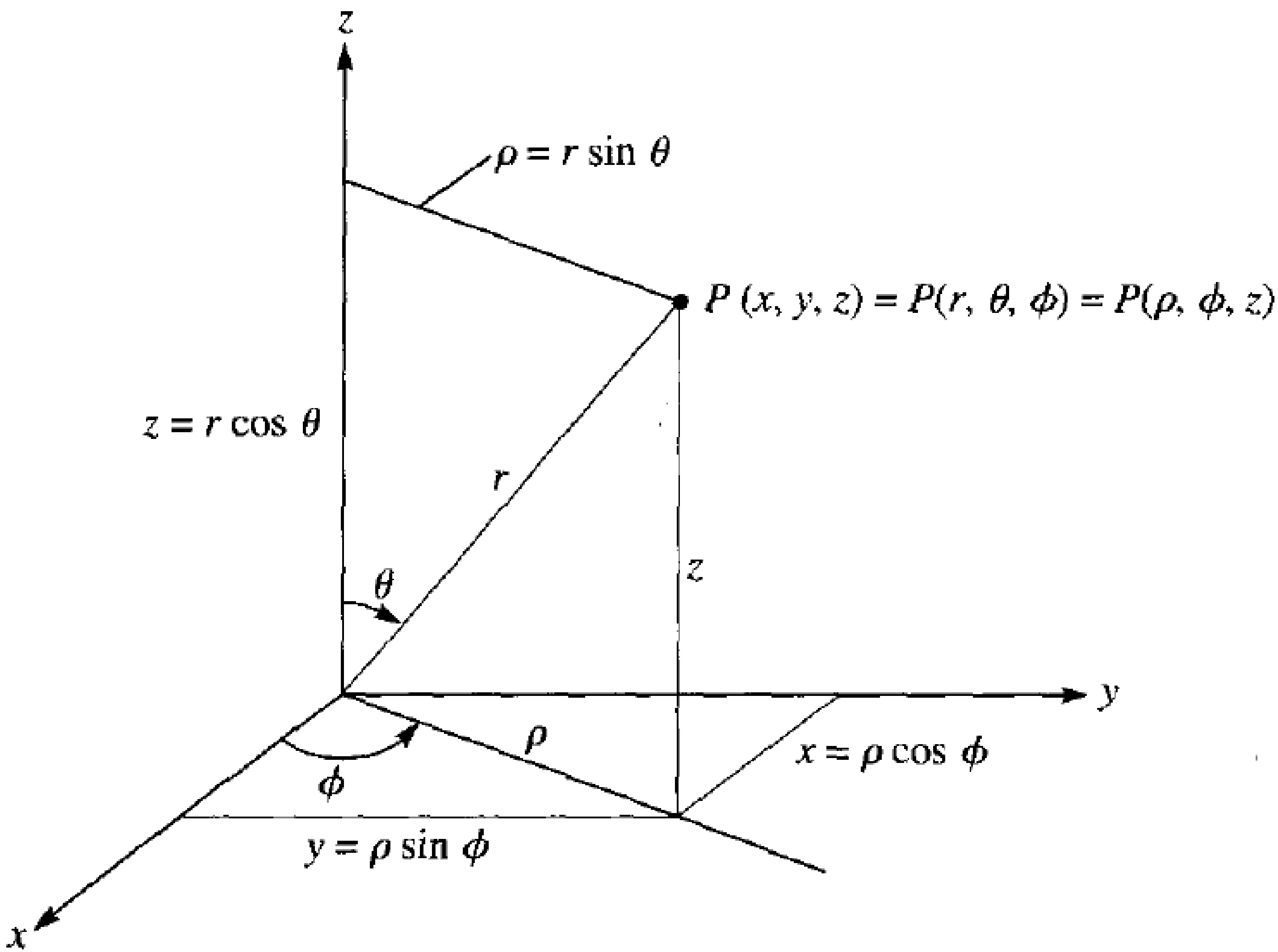
$$\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$$

$$\mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_r$$

$$\mathbf{a}_\phi \times \mathbf{a}_r = \mathbf{a}_\theta$$

Point Transformations

➢ Relation of space variables (x, y, z) in Cartesian coordinates with variables (r,θ,Φ) of a spherical coordinate system



Point Transformations

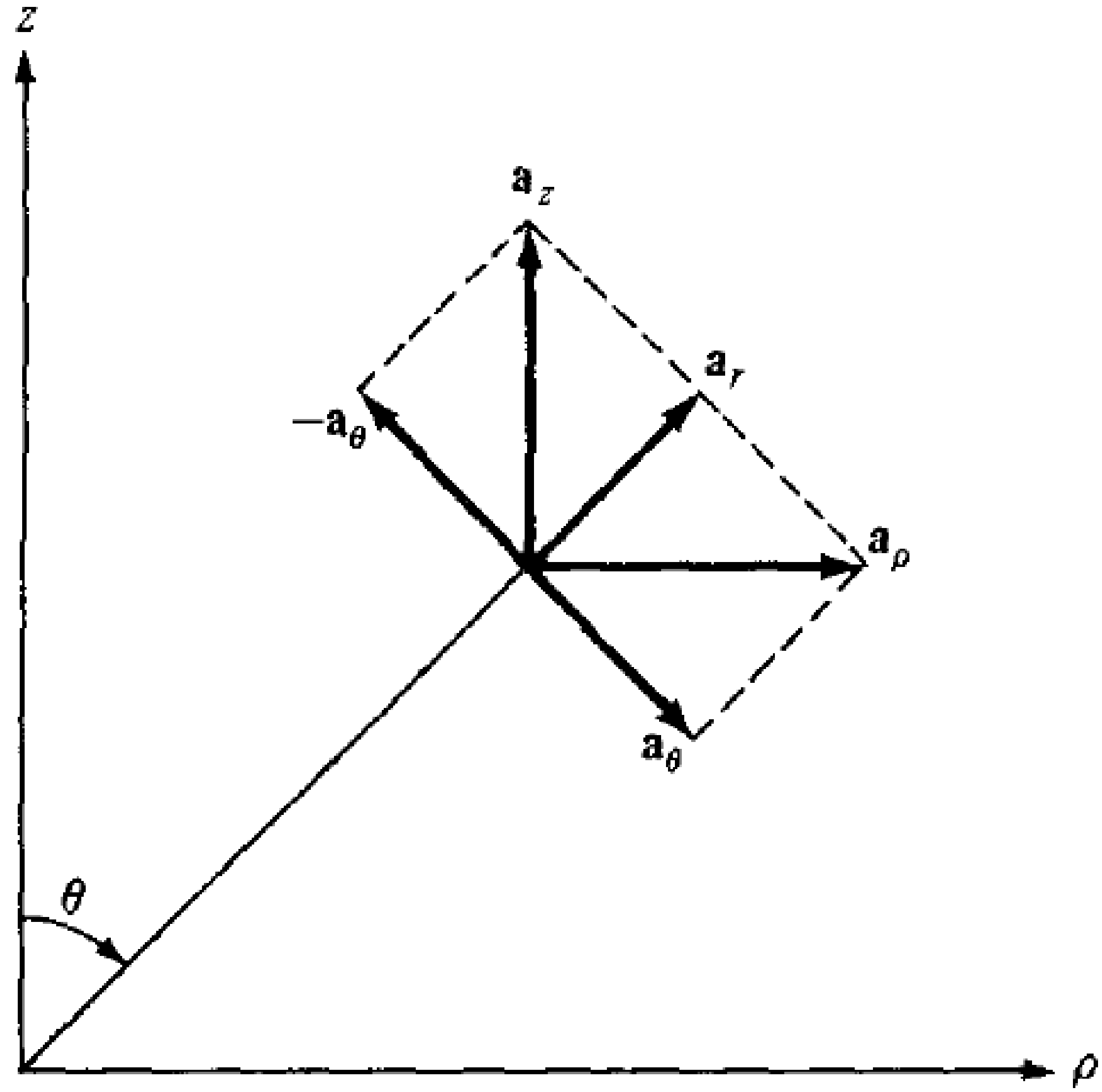
- For transforming a point from Cartesian (x, y, z) to spherical (r,θ,Φ) coordinates:

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

- For transforming a point from Spherical (r,θ,Φ) to Cartesian (x, y, z) coordinates:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

Unit Vector Transformations



Unit Vector Transformations

- The unit vectors ($\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$) and ($\mathbf{a}_r$, $\mathbf{a}_\theta$, $\mathbf{a}_\phi$) are related as follows:

$$\mathbf{a}_x = \sin\theta\cos\phi\,\mathbf{a}_r + \cos\theta\cos\phi\,\mathbf{a}_\theta - \sin\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_y = \sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_z = \cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta$$

OR

$$\mathbf{a}_r = \sin\theta\cos\phi\,\mathbf{a}_x + \sin\theta\sin\phi\,\mathbf{a}_y + \cos\theta\,\mathbf{a}_z$$

$$\mathbf{a}_\theta = \cos\theta\cos\phi\,\mathbf{a}_x + \cos\theta\sin\phi\,\mathbf{a}_y - \sin\theta\,\mathbf{a}_z$$

$$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$$

Vector Transformations

- Substitute the unit vector transformations into the equation below:

$$A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

- After collecting terms, we get:

$$\mathbf{A} = (A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta)\mathbf{a}_r + (A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta)\mathbf{a}_\theta + (-A_x \sin\phi + A_y \cos\phi)\mathbf{a}_\phi$$

OR

$$A_r = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta$$

$$A_\theta = A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

Vector Transformations

- The transformations may be written in matrix form as:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

AND

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Distance between two points

- In point or vector transformation the point or vector has not changed; it is only expressed differently

- For example, the magnitude of a vector will remain the same after the transformation and this may serve as a way of checking the result of the transformation

- The distance between two points is usually necessary in Electromagnetic theory

- The distance *d* between two points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is generally given by:

$$d = |\mathbf{r}_2 - \mathbf{r}_1|$$

Distance between two points

- Using point transformation, this distance may be expressed in Cartesian, cylindrical and spherical coordinates as below:

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \text{ (Cartesian)}$$

$$d^2 = \rho_2^2 + \rho_1^2 - 2\rho_1\rho_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2 \text{ (cylindrical)}$$

$$\begin{aligned} d^2 = r_2^2 + r_1^2 &- 2r_1r_2 \cos\theta_2 \cos\theta_1 \\ &- 2r_1r_2 \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1) \text{ (spherical)} \end{aligned}$$

Problem-1

a) Convert point P(0, - 4 , 3) from Cartesian to spherical coordinates

b) Evaluate **Q** at P in Cartesian and spherical coordinate systems

$$\mathbf{Q} = \frac{\sqrt{x^2 + y^2}\mathbf{a}_x}{\sqrt{x^2 + y^2 + z^2}} - \frac{yz\,\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}$$