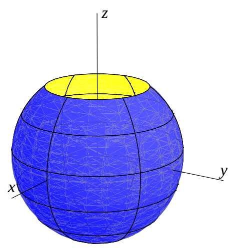
Stokes' Theorem

1. *Let $\vec{F}(x, y, z) = \langle -y, x, xyz \rangle$ and $\vec{G} = \operatorname{curl} \vec{F}$. Let $\mathcal{S}$ be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies below the plane $z = 4$, oriented so that the unit normal vector at $(0, 0, -5)$ is $\langle 0, 0, -1 \rangle$. Use Stokes' Theorem to find $\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S}$.*

   Solution. Here's a picture of the surface $\mathcal{S}$.

   To use Stokes' Theorem, we need to first find the boundary C of $\mathcal{S}$ and figure out how it should be oriented. The boundary is where $x^2 + y^2 + z^2 = 25$ and $z = 4$. Substituting $z = 4$ into the first equation, we can also describe the boundary as where $x^2 + y^2 = 9$ and $z = 4$.

   To figure out how C should be oriented, we first need to understand the orientation of $\mathcal{S}$. We are told that $\mathcal{S}$ is oriented so that the unit normal vector at $(0, 0, -5)$ (which is the lowest point of the sphere) is $\langle 0, 0, -1 \rangle$ (which points down). This tells us that the blue side must be the "positive" side.

   We want to orient the boundary so that, if a penguin walks near the boundary of $\mathcal{S}$ on the "positive" side (which we've already decided is the blue side), he keeps the surface on his left. If we imagine looking down on the surface from a really high point like $(0, 0, 100)$, then the penguin should walk clockwise (from our vantage point).

   So, using Stokes' Theorem, we have changed the original problem into a new one:

   > Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve described by $x^2 + y^2 = 9$ and $z = 4$, oriented clockwise when viewed from above.

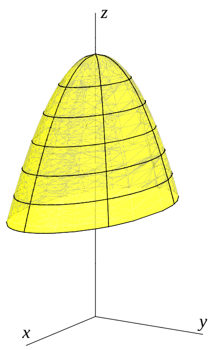
   Now, we just need to evaluate the line integral, using the definition of the line integral. (This is like #4(a) on the worksheet "Vector Fields and Line Integrals".) We start by parameterizing C. One possible parameterization is $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4 \rangle$, $0 \le t < 2\pi$.(1) If we look at this from above, it is oriented counterclockwise, which is the wrong orientation. Therefore,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= -\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt \\
&= -\int_0^{2\pi} \langle -3\sin t, 3\cos t, 36\cos t \sin t \rangle \cdot \langle -3\sin t, 3\cos t, 0 \rangle\ dt \\
&= -\int_0^{2\pi} 9\ dt \\
&= \boxed{-18\pi}
\end{aligned}$$

(1) To come up with this, remember that we can parameterize a circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ by $x = \cos t$, $y = \sin t$ (and, as t increases, this goes around the circle counterclockwise). Here, we're looking at $x^2 + y^2 = 9$; if we rewrite this as $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$, then we can write $\frac{x}{3} = \cos t$, $\frac{y}{3} = \sin t$.

2. *Let $\vec{F}(x, y, z) = \langle -y, x, z\rangle$. Let $\mathcal{S}$ be the part of the paraboloid $z = 7 - x^2 - 4y^2$ that lies above the plane $z = 3$, oriented with upward pointing normals. Use Stokes' Theorem to find $\iint_{\mathcal{S}} \operatorname{curl} \vec{F} \cdot d\vec{S}$.*

Solution. Here is a picture of the surface $\mathcal{S}$.



The strategy is exactly the same as in #1. The boundary is where $z = 7 - x^2 - 4y^2$ and $z = 3$, which is the same as $x^2 + 4y^2 = 4$ and $z = 3$.

Since $\mathcal{S}$ is oriented with normals pointing upward, the top side of the paraboloid (the yellow side in the picture) is the "positive" side. If we imagine looking down on the surface from above, then a penguin walking around on the "positive" (yellow, in this case) side keeps the surface on his left by walking counterclockwise.

Therefore, by Stokes' Theorem, the original problem can be rewritten as

> Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve described by $x^2 + 4y^2 = 4$ and $z = 3$, oriented counterclockwise when viewed from above.

A parameterization of this curve is $\vec{r}(t) = \langle 2\cos t, \sin t, 3\rangle$.[(2)] This goes counterclockwise when viewed from above (as we want), so

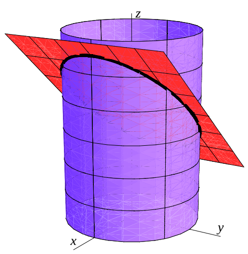
$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \\
&= \int_0^{2\pi} \langle -\sin t, 2\cos t, 3\rangle \cdot \langle -2\sin t, \cos t, 0\rangle\, dt \\
&= \int_0^{2\pi} (2\sin^2 t + 2\cos^2 t)\, dt \\
&= \int_0^{2\pi} 2\, dt \\
&= \boxed{4\pi}
\end{aligned}$$

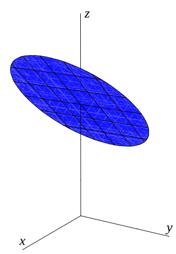
[(2)] To come up with this parameterization, rewrite $x^2 + 4y^2 = 4$ as $\left(\frac{x}{2}\right)^2 + y^2 = 1$ and then use $\frac{x}{2} = \cos t$, $y = \sin t$. It's easy to check that it's reasonable: if we plug in $x = 2\cos t$, $y = \sin t$, and $z = 3$, then the equations $x^2 + 4y^2 = 4$ and $z = 3$ are indeed satisfied.

3. *The plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$ intersect in a curve C. Suppose C is oriented counterclockwise when viewed from above. Let $\vec{F}(x, y, z) = \langle x^3 + 2y, \sin y + z, x + \sin z^2 \rangle$. Evaluate the line integral $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.*

   Solution. We'll use Stokes' Theorem. To do this, we need to think of an oriented surface $\mathcal{S}$ whose (oriented) boundary is C (that is, we need to think of a surface $\mathcal{S}$ and orient it so that the given orientation of C matches). Then, Stokes' Theorem says that $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{S}} \operatorname{curl} \vec{F} \cdot d\vec{S}$. Let's compute $\operatorname{curl} \vec{F}$ first. (It's worthwhile to do this first because, if we find out it's $\vec{0}$, then we know the integral will be 0 without any more work.) In this case, $\operatorname{curl} \vec{F} = \langle -1, -1, -2 \rangle$.

   Now, let's think of a surface whose boundary is the given curve C. We are told that C is the intersection of a plane and a cylinder (left picture), so one surface we could use is the part of the plane inside the cylinder (right picture):





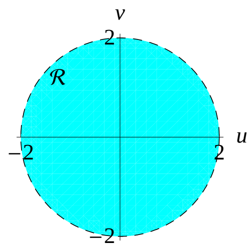
   Let's call this $\mathcal{S}$ and figure out how it should be oriented. We want to orient $\mathcal{S}$ so that, if a penguin walks along the given curve C (going counterclockwise when viewed from above) on the "positive" side of $\mathcal{S}$, he keeps the surface on his left. This means that we want the top side of $\mathcal{S}$ to be the "positive" side, so we should orient $\mathcal{S}$ with normals pointing upward.

   So, using Stokes' Theorem, we have changed the original problem into a new one:

Evaluate the flux integral $\displaystyle\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S}$, where $\mathcal{S}$ is the part of the plane $z = x + 4$ inside the cylinder $x^2 + y^2 = 4$, oriented with normals pointing upward, and $\vec{G}$ is the vector field $\vec{G}(x, y, z) = \langle -1, -1, -2 \rangle$.

   To do this new problem, let's follow the same three steps we used in #4(a) on the worksheet "Flux Integrals".

   First, we parameterize $\mathcal{S}$. Since the plane has equation $z = x + 4$, we can use x and y as our parameters. If we let $x = u$ and $y = v$, then $z = u + 4$. This gives the parameterization $\vec{r}(u, v) = \langle u, v, u + 4 \rangle$. Since we are only interested in the part of the plane inside the cylinder $x^2 + y^2 = 4$, we want $x^2 + y^2 < 4$. In terms of u and v, this says $u^2 + v^2 < 4$, so the region $\mathcal{R}$ in the uv-plane describing the possible parameter values is a disk:



Next, we need to see what orientation this parameterization describes. To do this, we compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned} \vec{r}_u &= \langle 1, 0, 1 \rangle \\ \vec{r}_v &= \langle 0, 1, 0 \rangle \\ \vec{r}_u \times \vec{r}_v &= \langle -1, 0, 1 \rangle \end{aligned}$$

This always points upward, which matches the orientation we want. So, the flux integral is

$$\begin{aligned} \iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= \iint_{\mathcal{R}} \vec{G}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA \\ &= \iint_{\mathcal{R}} \langle -1, -1, -2 \rangle \cdot \langle -1, 0, 1 \rangle\ dA \\ &= \iint_{\mathcal{R}} -1\ dA \end{aligned}$$

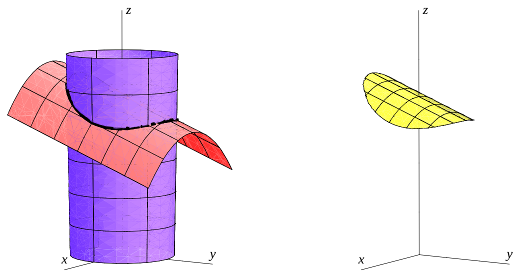
Although we could evaluate this double integral by converting it to an iterated integral, there is an easier way — remember that $\iint_{\mathcal{R}} 1\ dA$ gives the area of $\mathcal{R}$ (see #2(a) on the worksheet "Double Integrals"). Therefore,

$$\begin{aligned} \iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= -\iint_{\mathcal{R}} 1\ dA \\ &= -(\text{area of } \mathcal{R}) \\ &= \boxed{-4\pi} \end{aligned}$$

4. *Let C be the oriented curve parameterized by $\vec{r}(t) = \langle \cos t, \sin t, 8 - \cos^2 t - \sin t \rangle$, $0 \le t < 2\pi$, and let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = \left\langle z^2 - y^2, -2xy^2, e^{\sqrt{z}} \cos z \right\rangle$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.*

Solution. The line integral is very difficult to compute directly, so we'll use Stokes' Theorem. The curl of the given vector field $\vec{F}$ is $\operatorname{curl} \vec{F} = \langle 0, 2z, 2y - 2y^2 \rangle$.

To use Stokes' Theorem, we need to think of a surface whose boundary is the given curve C. First, let's try to understand C a little better. We are given a parameterization $\vec{r}(t)$ of C. In this parameterization, $x = \cos t$, $y = \sin t$, and $z = 8 - \cos^2 t - \sin t$. So, we can see that $x^2 + y^2 = 1$ and $z = 8 - x^2 - y$. In other words, C must be the intersection of the surface $x^2 + y^2 = 1$ (which is a cylinder) and the surface $z = 8 - x^2 - y$ (which we don't need to visualize particularly well, beyond noticing that it's the graph of a function $f(x, y) = 8 - x^2 - y$). So, one surface we could use is the part of the surface $z = 8 - x^2 - y$ inside the cylinder $x^2 + y^2 = 1$ (right picture).



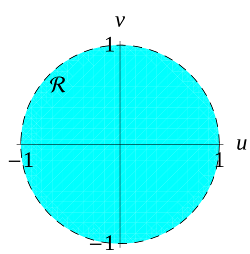
Let's call this surface $\mathcal{S}$ and figure out how it should be oriented. The original curve was parameterized using $x = \cos t$, $y = \sin t$, so when viewed from above, it was oriented counterclockwise. Therefore, we want to orient $\mathcal{S}$ so that is top is the "positive" side (a penguin walking on the top along the boundary, going counterclockwise when viewed from above, keeps the surface on his left). So, we should orient $\mathcal{S}$ with normals pointing upward.

So, using Stokes' Theorem, we have changed the original problem into a new one:

> Evaluate the flux integral $\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S}$, where $\mathcal{S}$ is the part of the surface $z = 8 - x^2 - y$ inside the cylinder $x^2 + y^2 = 1$, oriented with normals pointing upward, and $\vec{G}$ is the vector field $\vec{G}(x, y, z) = \langle 0, 2z, 2y - 2y^2 \rangle$.

To do this new problem, let's follow the same three steps we used in #4(a) on the worksheet "Flux Integrals".

First, we parameterize $\mathcal{S}$. Since the surface has equation $z = 8 - x^2 - y$, we can parameterize it as $\vec{r}(u, v) = \langle u, v, 8 - u^2 - v \rangle$. Since we are only interested in the part of the surface inside the cylinder $x^2 + y^2 = 1$, we want $x^2 + y^2 < 1$; in terms of u and v, this says $u^2 + v^2 < 1$, so the region $\mathcal{R}$ in the uv-plane describing the possible parameter values is a disk:



Next, we need to see what orientation this parameterization describes. To do this, we compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned} \vec{r}_u &= \langle 1, 0, -2u \rangle \\ \vec{r}_v &= \langle 0, 1, -1 \rangle \\ \vec{r}_u \times \vec{r}_v &= \langle 2u, 1, 1 \rangle \end{aligned}$$

This always points upward, which matches the orientation we want. So, the flux integral is

$$\begin{aligned} \iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= \iint_{\mathcal{R}} \vec{G}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA \\ &= \iint_{\mathcal{R}} \langle 0, 2(8 - u^2 - v), 2v - 2v^2 \rangle \cdot \langle 2u, 1, 1 \rangle\ dA \\ &= \iint_{\mathcal{R}} (16 - 2u^2 - 2v^2)\ dA \end{aligned}$$

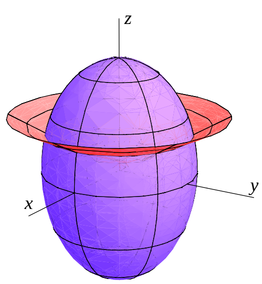
Since our region $\mathcal{R}$ is a disk, let's do this integral in polar coordinates. The disk $\mathcal{R}$ can be described as $0 \le r < 1$, $0 \le \theta < 2\pi$, so

$$\begin{aligned}
\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^1 (16 - 2r^2) \cdot r \; dr \; d\theta \\
&= \int_0^{2\pi} \int_0^1 (16r - 2r^3) \; dr \; d\theta \\
&= \int_0^{2\pi} \left(8r^2 - \frac{1}{2} r^4 \Big|_{r=0}^{r=1} \right) d\theta \\
&= \int_0^{2\pi} \frac{15}{2} \; d\theta \\
&= \boxed{15\pi}
\end{aligned}$$

5. *Let C be the curve of intersection of $2x^2 + 2y^2 + z^2 = 9$ with $z = \frac{1}{2}\sqrt{x^2 + y^2}$, oriented counterclockwise when viewed from above, and let $\vec{F}(x, y, z) = \langle 3y, 2yz, xz^3 + \sin z^2 \rangle$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.*

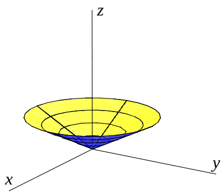
Solution. Again, we'll use Stokes' Theorem. The curl of the given vector field is curl $\vec{F} = \left\langle -2y, -z^3, -3 \right\rangle$.

We'll start by thinking of an oriented surface $\mathcal{S}$ whose (oriented) boundary is the given curve C. The curve is the intersection of an ellipsoid with a cone:

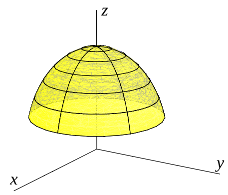


It appears from the picture that the curve lies in a plane parallel to the xy-plane. To verify this, let's look at the equations $2x^2 + 2y^2 + z^2 = 9$ and $z = \frac{1}{2}\sqrt{x^2 + y^2}$ defining the curve. The second equation can be rewritten as $x^2 + y^2 = 4z^2$. Plugging this into the first equation, $9z^2 = 9$, so $z = \pm 1$. Since the cone is only defined for $z \ge 0$, we know the intersection is where $z = 1$, in which case $x^2 + y^2 = 4$.

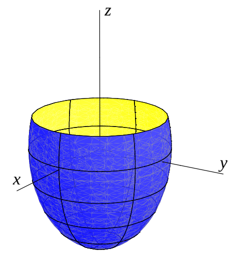
There are many surfaces whose boundary is the given curve C. From what we know so far, there are several surfaces that might come to mind:



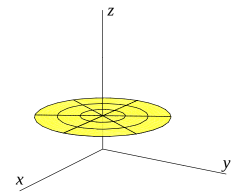
The part of the cone inside the ellipsoid.



The part of the ellipsoid lying above the curve.



The part of the ellipsoid lying below the curve.

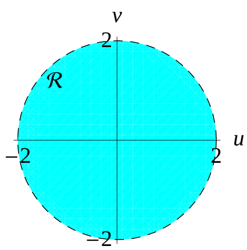


The part of the plane $z = 1$ lying inside $x^2 + y^2 = 4$.

Each of these should be oriented so that the yellow side is the "positive" side. The last is probably the simplest, so let's use that. To have the yellow side be the "positive" side, we want the normals to point upward. Thus, we have rewritten the original problem as:

> Evaluate the flux integral $\displaystyle\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S}$, where $\mathcal{S}$ is the part of the plane $z = 1$ lying inside $x^2 + y^2 = 4$, oriented with normals pointing upward, and $\vec{G}$ is the vector field $\vec{G}(x, y, z) = \left\langle -2y, -z^3, -3 \right\rangle$.

To do this, we first parameterize the surface. Since it is part of the plane $z = 1$, we can parameterize it by $\vec{r}(u, v) = \langle u, v, 1\rangle$. Since we want only the part inside $x^2 + y^2 = 4$, the region $\mathcal{R}$ in the uv-plane describing the possible parameter values is the disk $u^2 + v^2 < 4$:



Next, we see what orientation this parameterization describes. To do this, we compute $\vec{r}_u \times \vec{r}_v$:

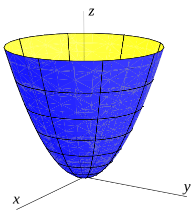
$$\begin{aligned}
\vec{r}_u &= \langle 1, 0, 0\rangle \\
\vec{r}_v &= \langle 0, 1, 0\rangle \\
\vec{r}_u \times \vec{r}_v &= \langle 0, 0, 1\rangle
\end{aligned}$$

This always points upward, so our parameterization matches the orientation we want. Therefore, the flux integral is

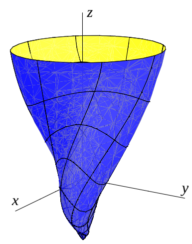
$$\begin{aligned}
\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= \iint_{\mathcal{R}} \vec{G}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA \\
&= \iint_{\mathcal{R}} \langle -2v, -1, -3\rangle \cdot \langle 0, 0, 1\rangle\ dA \\
&= \iint_{\mathcal{R}} -3\ dA \\
&= -3 \iint_{\mathcal{R}} 1\ dA \\
&= -3(\text{area of } \mathcal{R}) \text{ by the worksheet “Double Integrals”, \#2(a)} \\
&= \boxed{-12\pi}
\end{aligned}$$

6. *The two surfaces shown have the same boundary. Suppose they are both oriented so that the light side is the "positive" side. Is the following reasoning correct? "Since $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same (oriented) boundary, the flux integrals $\iint_{\mathcal{S}_1} \vec{G} \cdot d\vec{S}$ and $\iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S}$ must be equal for any vector field $\vec{G}$. Therefore, you can compute any flux integral using the simpler surface."*

$\mathcal{S}_1$



$\mathcal{S}_2$



Solution. $\boxed{\text{False.}}$ The statement *is* true if the vector field $\vec{G}$ is the curl of some other vector field, say $\vec{G} = \operatorname{curl} \vec{F}$. In that case, if C is the (properly oriented) boundary, Stokes' Theorem says that $\iint_{\mathcal{S}_1} \vec{G} \cdot d\vec{S}$ and $\iint_{\mathcal{S}_1} \vec{G} \cdot d\vec{S}$ are both equal to $\int_C \vec{F} \cdot d\vec{r}$. But there's no reason that $\vec{G}$ has to be the curl of some other vector field, so the statement is false in general.