(Optional)

9.4 The Underdamped Parallel RLC Circuit: $\alpha < \omega_0$

(pp 338 8th Ed H&D)

When we continue to increase R, we get an underdamped response.

— The damping coefficient $\alpha = \frac{1}{2RC}$ decreases while ω_0 remains constant.

— Thus α^2 becomes smaller than ω_0^2 and the radicand $\left(\sqrt{\alpha^2 - \omega_0^2}\right)$ becomes negative.

— Let us start with general response:-

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

Now $\sqrt{\alpha^2 - \omega_0^2} = \sqrt{-1}\sqrt{\omega_0^2 - \alpha^2}$

$$= j\sqrt{\omega_0^2 - \alpha^2}$$

where $j \equiv \sqrt{-1}$

— the new radical, real for the underdamped case is called ω_d, the <u>natural resonant frequency</u>:

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

also
ω_0 = undamped natural frequency
ω_d = damped natural frequency

So $s_{1,2} = -\alpha \pm j\omega_d$

— The response may be written as:

$$v(t) = e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

——— Contd

(Optional)

— contd (339)

— It can also be written as:-

$$v(t) = e^{-\alpha t}\left\{(A_1+A_2)\left[\frac{e^{j\omega_d t}+e^{-j\omega_d t}}{2}\right] + j(A_1-A_2)\left[\frac{e^{j\omega_d t}-e^{-j\omega_d t}}{j2}\right]\right.$$

so $$v(t) = e^{-\alpha t}\left[(A_1+A_2)\cos\omega_d t + j(A_1-A_2)\sin\omega_d t\right]$$

— The multiplying factors may be assigned new symbols:-

$$v(t) = e^{-\alpha t}\left[B_1\cos\omega_d t + B_2\sin\omega_d t\right]$$

— The two real constants B_1 and B_2 are selected to fit the given initial conditions.

(Note: Euler's identity $e^{j\theta} = \cos\theta + j\sin\theta$

and $e^{-j\theta} = \cos\theta - j\sin\theta$

— B_2 is real. Don't be misled by

$B_2 = j(A_1 - A_2)$ as

A_1 and A_2 are complex conjugates.

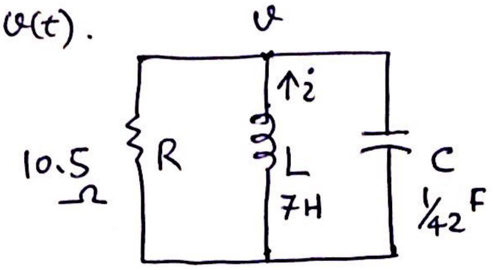
(PP 316 Nilsson & Riedel)

— Also $$\cos\omega_d t = \frac{e^{j\omega_d t} + e^{-j\omega_d t}}{}$$

and $$\sin\omega_d t = \frac{e^{j\omega_d t} + e^{-j\omega_d t}}{j2}$$

Example: The Underdamped Parallel RLC Circuit
(PP 339 8th Ed HKD)

Determine $v(t)$.



$v(0) = 0$ V

$i(0) = 10$ A

Note: R has been increased from 6 Ω to 8.57 Ω to 10.5 Ω.

Solution: As it is identified as a parallel RLC circuit,

so $\alpha = \dfrac{1}{2RC} = 2\ s^{-1}$ $\quad \left(\alpha = \dfrac{R}{2L} \text{ for series RLC}\right)$

and $\omega_0 = \dfrac{1}{\sqrt{LC}} = \sqrt{6}\ s^{-1} = 2.45\ s^{-1}$

because $\alpha < \omega_0$

So we determine

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{2} \text{ rad/s}$$

— The response will be of the form

$$v(t) = e^{-2t}\left(B_1 \cos\sqrt{2}\,t + B_2 \sin\sqrt{2}\,t\right)$$

— To determine initial B_1 and B_2.

At $v(0) = 0$

so

$$0 = B_1 \qquad (\text{because } \sin 0 = 0)$$

Hence

$$v(t) = e^{-2t}\left(B_2 \sin\sqrt{2}\,t\right) \text{ ——— (1)}$$

——— contd

— contd (340)

Now derivative of $v(t)$ is

$$\frac{dv}{dt} = \sqrt{2}\, B_2 e^{-2t} \cos\sqrt{2}t - 2B_2 e^{-2t} \sin\sqrt{2}t$$

At $t=0$

$$\left.\frac{dv}{dt}\right|_{t=0} = \frac{i_c(0)}{C} = 420 \quad \text{(calculated earlier)}$$

So $$420 = \sqrt{2}\, B_2 \qquad (\sin 0 = 0 \text{ and } \cos 0 = 1)$$

or $$B_2 = \frac{420}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = 210\sqrt{2}$$

— Putting the value in (1)

$$v(t) = e^{-2t}\, 210\sqrt{2} \sin\sqrt{2}t$$

or $$v(t) = 210\sqrt{2}\, e^{-2t} \sin\sqrt{2}t \quad V$$

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