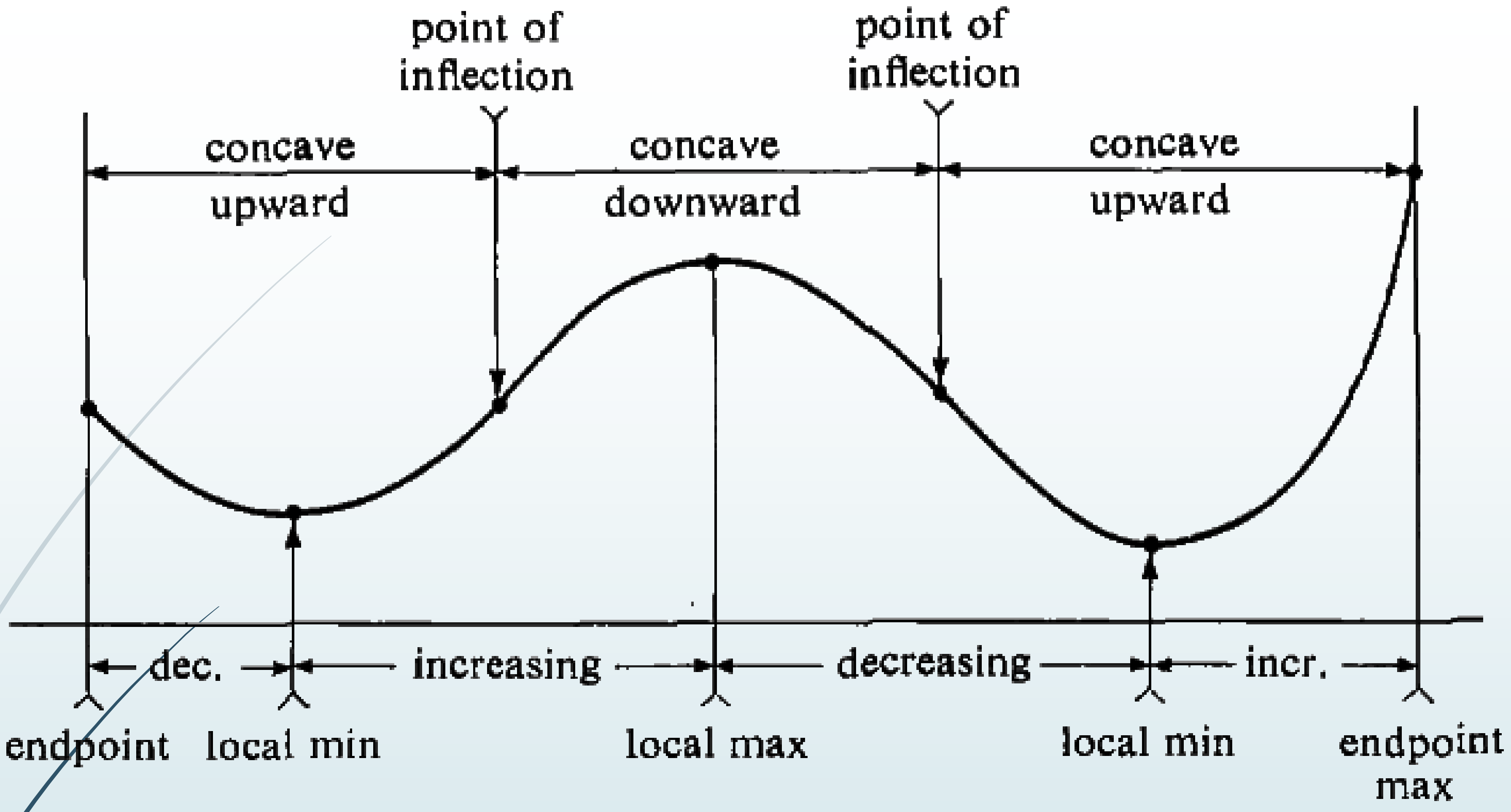
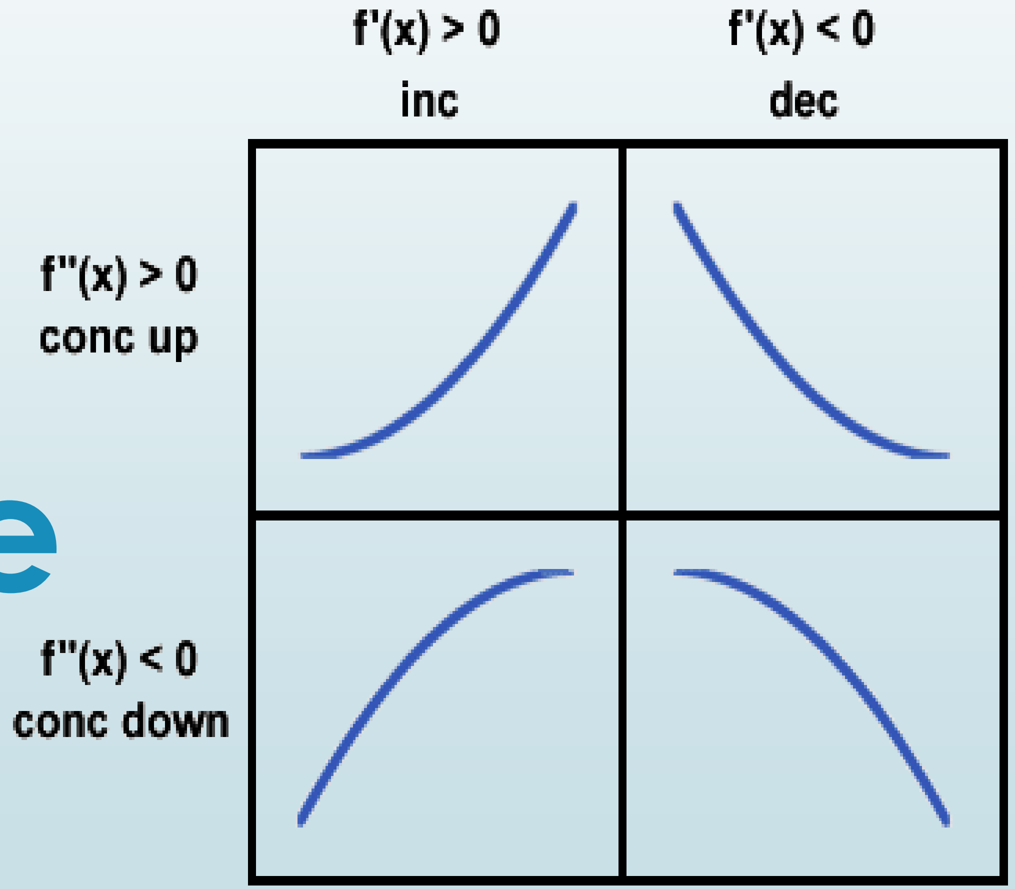
Applications of Derivatives

Calculus & Analytical Geometry MATH- 101
Instructor: Dr. Naila Amir (SEECS, NUST)



Concavity and Curve Sketching



Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 4
  - Sections: 4.4

Objectives

- So far, we have seen that how the first derivative tells us where a function is increasing and where it is decreasing. At a critical point of a differentiable function, the First Derivative Test tells us whether there is a local maximum or a local minimum, or whether the graph just continues to rise or fall there.

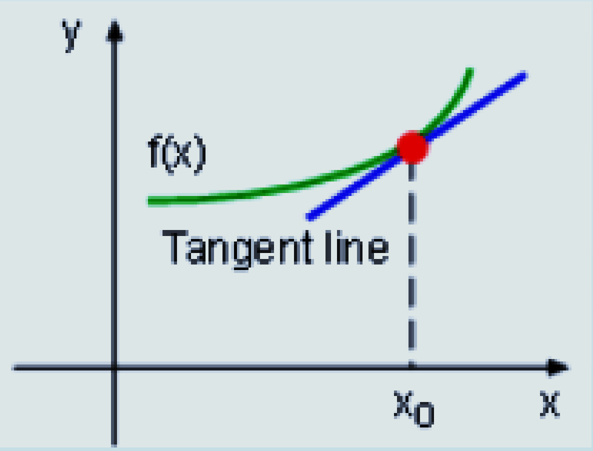
- We are now interested to see how the second derivative gives information about the way the graph of a differentiable function **bends or turns**. This additional information enables us to capture key aspects of the behavior of a function and its graph, and then present these features in a sketch of the graph.

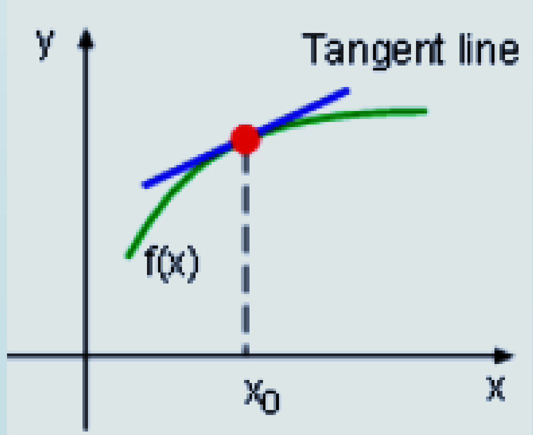
Objectives

- To determine the intervals on which the graph of a function is concave up or concave down.
- To find the inflection points of the graph of a function.
- Find extrema of a function using second derivative test.

Concavity

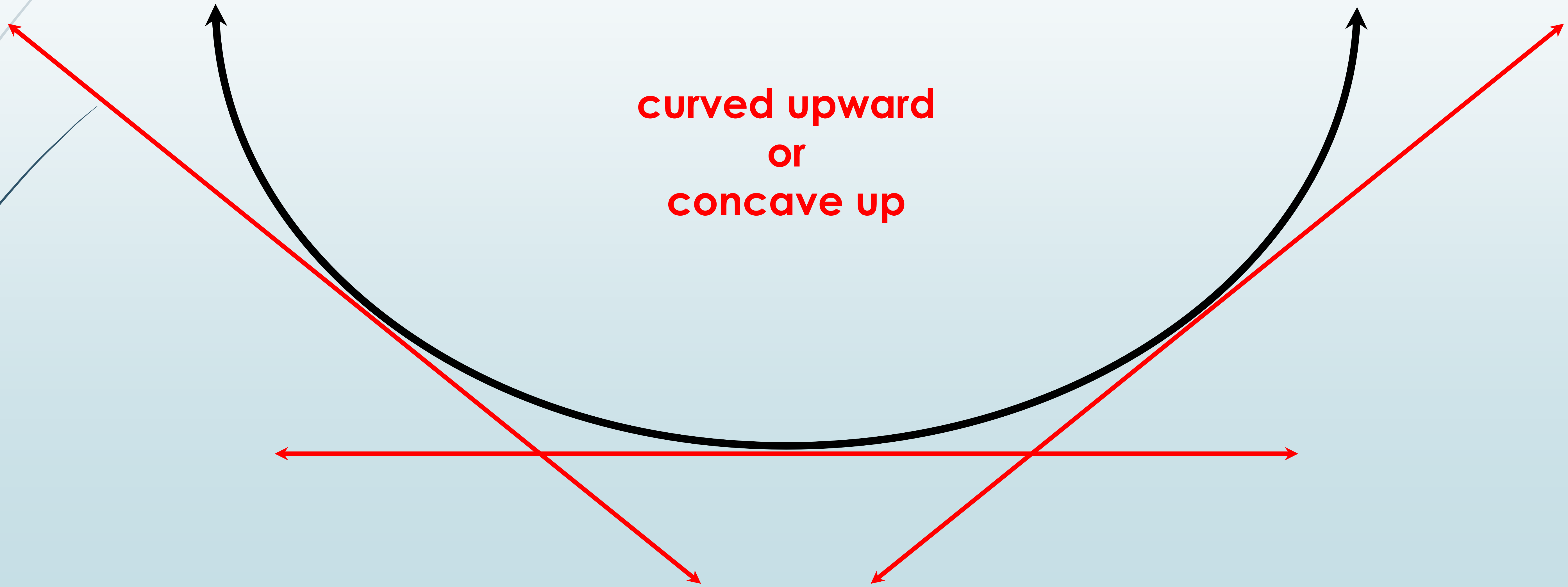
- The **concavity** of the graph of a function is the notion of curving <u>upward</u> or <u>downward</u>.
- If the graph of a function lies above its tangents on some interval then the graph is called <u>concave up</u> on that interval.
- If the graph of a function lies below its tangents on some interval then the graph is called <u>concave down</u> on that interval.





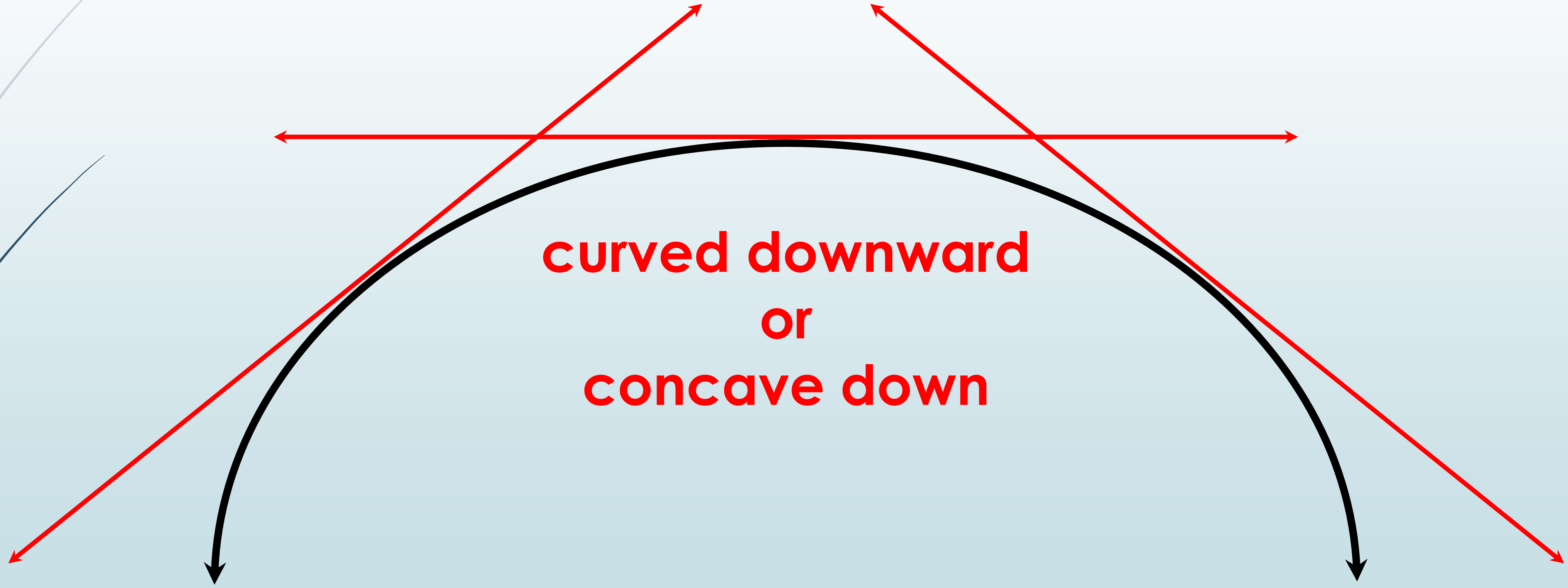
Concavity

Concave up: slope is increasing $\Rightarrow f'(x)$ is increasing



Concavity

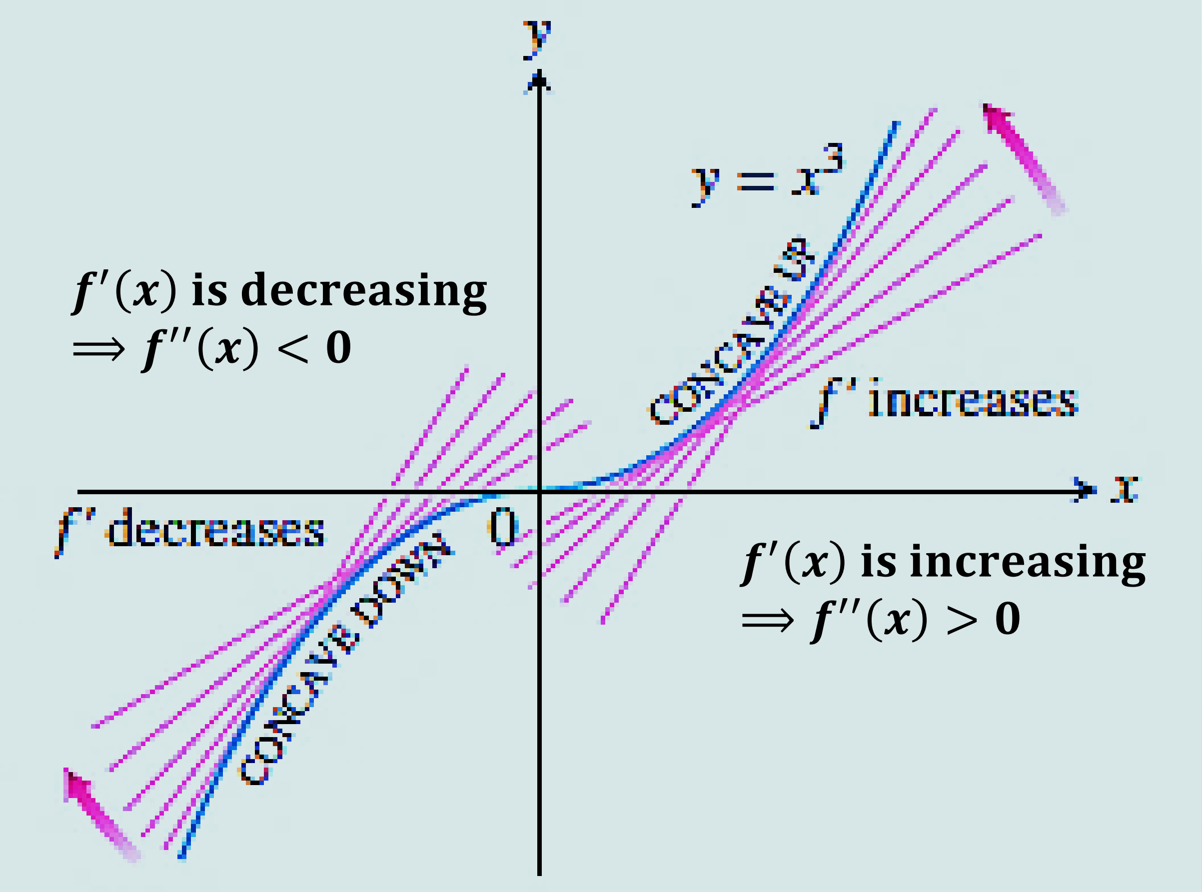
Concave down: slope is decreasing $\Rightarrow f'(x)$ is decreasing



DEFINITION: Concave Up, Concave Down

The graph of a differentiable function $y = f(x)$ is

- **concave up** on an open interval I if f' is increasing on I.
- **concave down** on an open interval I if f' is decreasing on I.



The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$

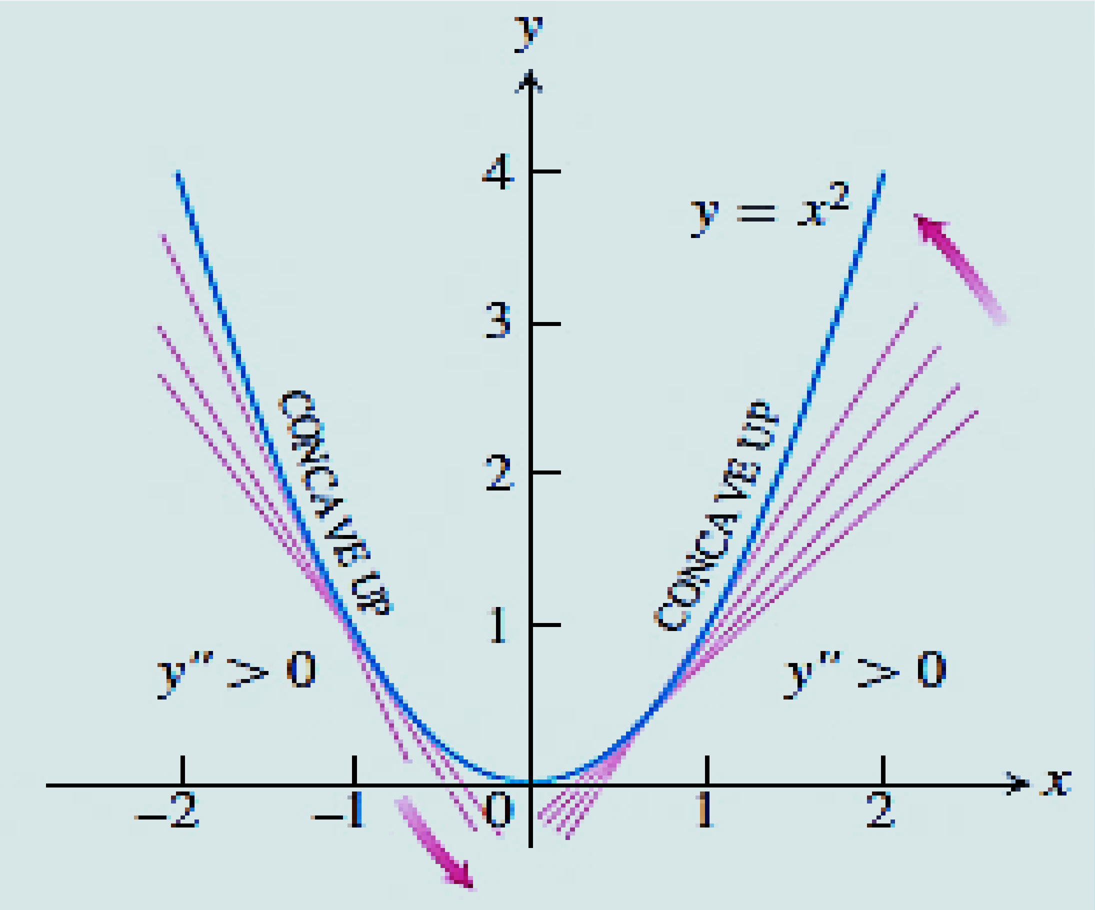
Concavity

- The concavity of a graph can be determined by using the second derivative.
- If the second derivative of a function is positive on a given interval, then the graph of the function is concave up on that interval.
- If the second derivative of a function is negative on a given interval, then the graph of the function is concave down on that interval.

The Second Derivative Test For Concavity

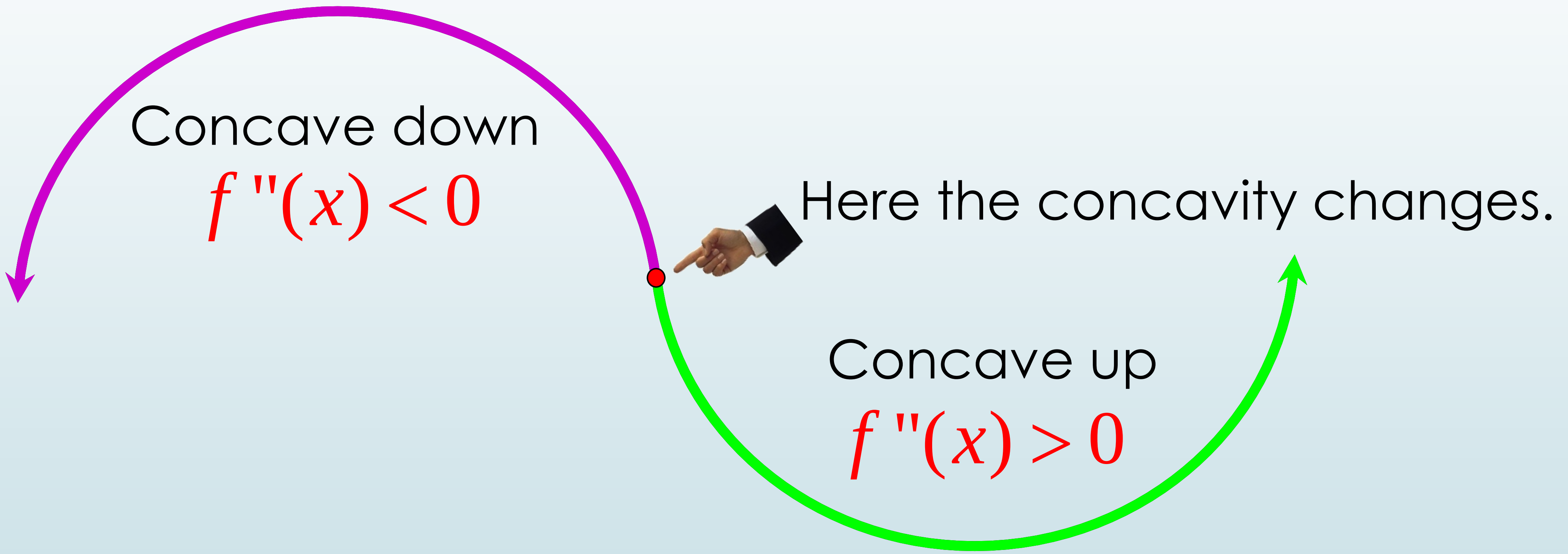
Let $y = f(x)$ be twice-differentiable on an interval I.

- If $f''(x) > 0$ on I, then the graph of $f(x)$ over I is concave up.
- If $f''(x) < 0$ on I, then the graph of $f(x)$ over I is concave down.



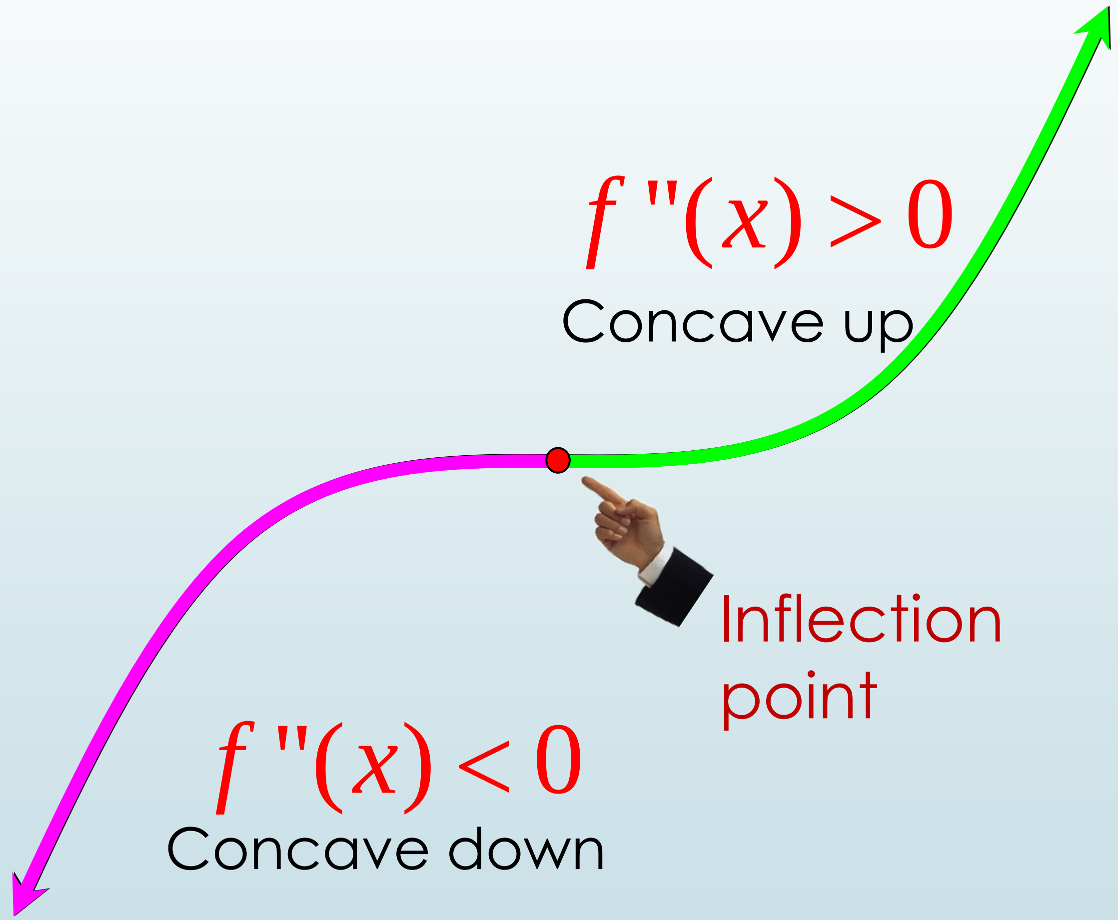
The graph of $f(x) = x^2$ is concave up on every interval

Concavity



This is called an **inflection point** (or point of inflection).

Concavity



Inflection Points

A point $(c, f(c))$ on the graph of f is a **point of inflection** if the following two conditions are satisfied:

(i) f is continuous at c.

(ii) There is an open interval (a, b) containing c such that the graph is concave upward on (a, c) and concave downward on (c, b), or vice versa.

Inflection Points

- **Inflection points** are points where the graph changes concavity.
- The second derivative will either equal zero or undefined at an inflection point.

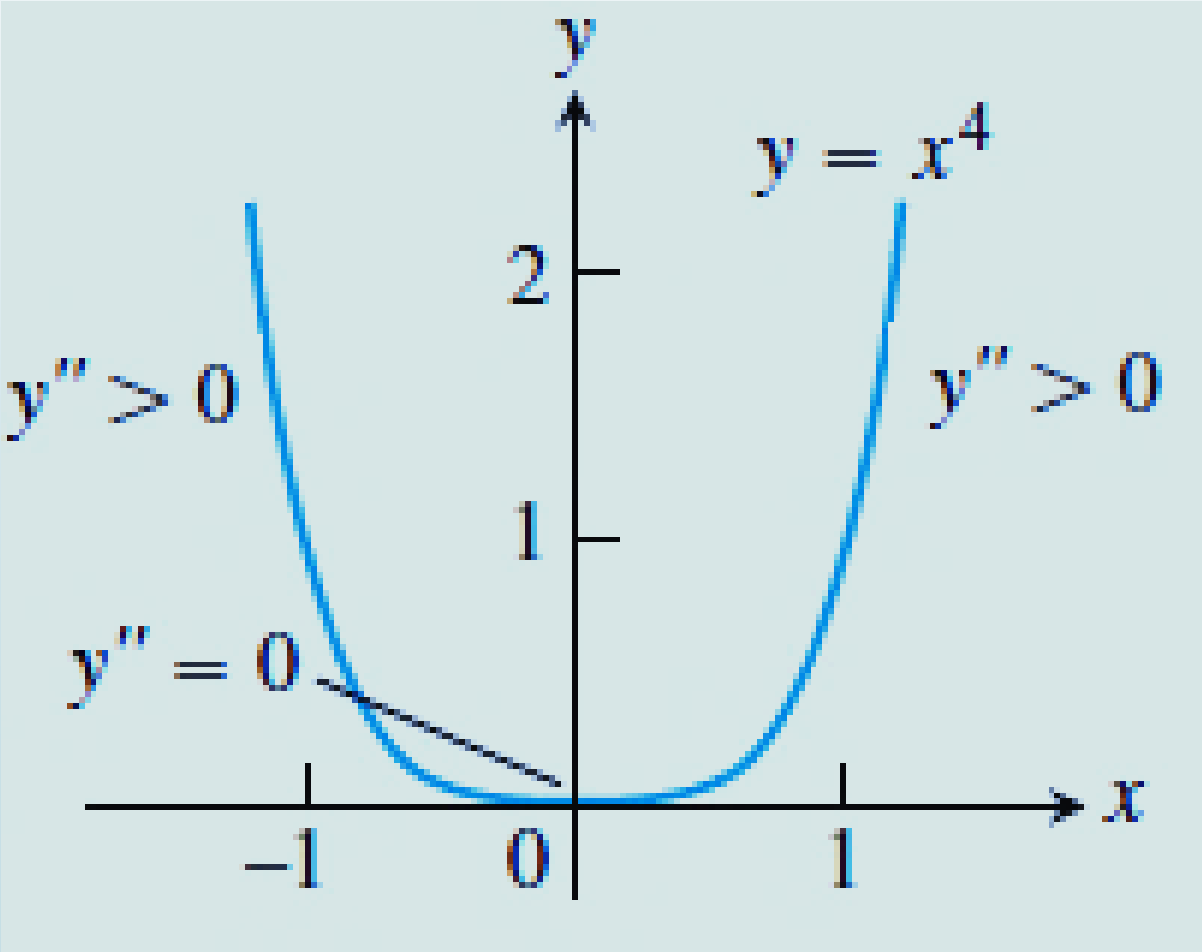
Inflection Points

- A point on a curve where $f''(x)$ is positive on one side and negative on the other is a **point of inflection**.

Note: If a local maximum or minimum occurs at a point then the first derivative is zero. It is not, however, true that when the derivative is zero, we necessarily have a local maximum or minimum. With a maximum we saw that the function changed from increasing to decreasing at that point. With a minimum it changed from decreasing to increasing. If the function has zero slope at a point, but is either increasing on either side of the point or decreasing on either side of the point we call that a ***point of inflection***.

Example: An inflection point may not exist where $f''(x) = 0$

The curve $y = f(x) = x^4$ has no inflection point at $x = 0$, even though $y'' = 12x^2$ is zero there, since it does not change sign.



Example: Studying Motion Along a Line

A particle is moving along a horizontal line with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

Find the velocity and acceleration and describe the motion of the particle.

Solution:

The velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t-1)(3t-11),$$

and the acceleration is

$$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t-7).$$

When the function $s(t)$ is increasing, the particle is moving to the right; when $s(t)$ is decreasing, the particle is moving to the left.

Note that the first derivative $v = s'$ is zero when $t = 1$ and $t = 11/3$.

Intervals	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	+	−	+
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

The particle is moving to the right in the time intervals $[0, 1)$ and $(11/3, \infty)$, and moving to the left in $(1,11/3)$. It is at rest at $t = 1$ and $t = 11/3$.

The acceleration $a(t) = s''(t) = 4(3t - 7)$ is zero when $t = 7/3$.

Intervals	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	−	+
Graph of s	concave down	concave up

The accelerating force is directed toward the left during the time interval $[0,7/3]$, is at rest at $t = 7/3$ and is directed toward the right thereafter.