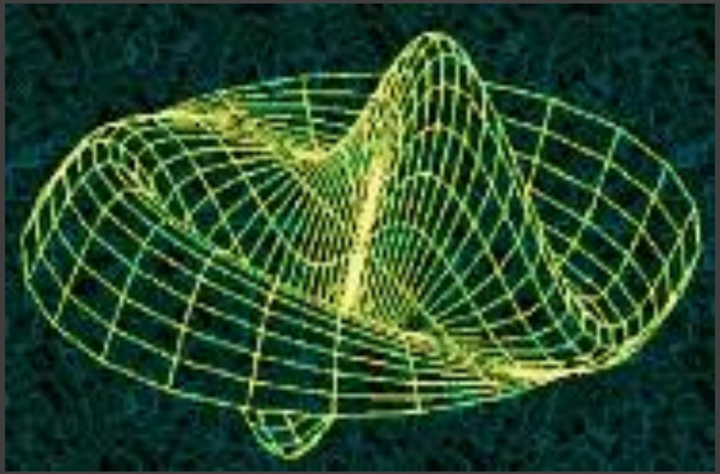
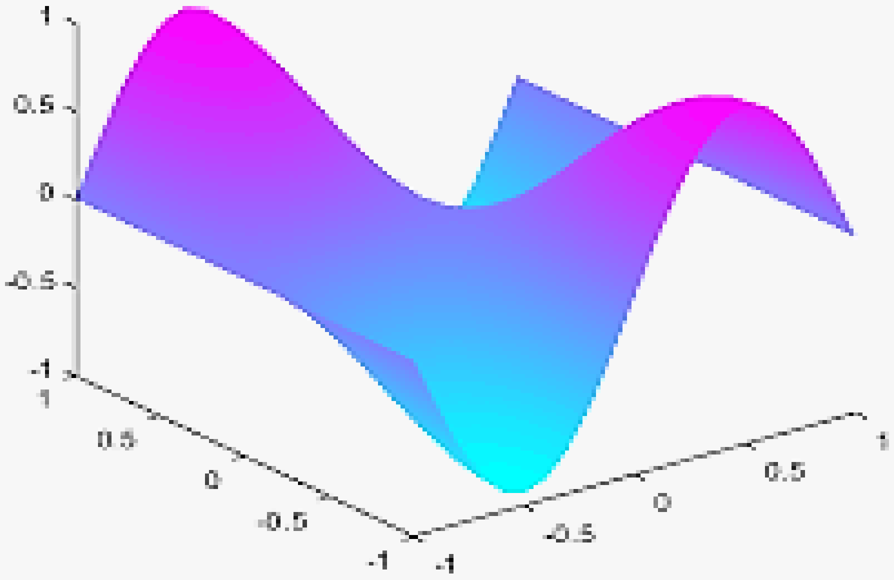
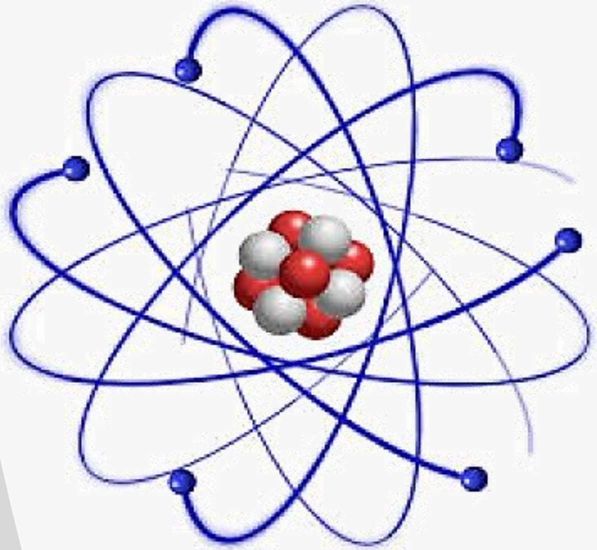
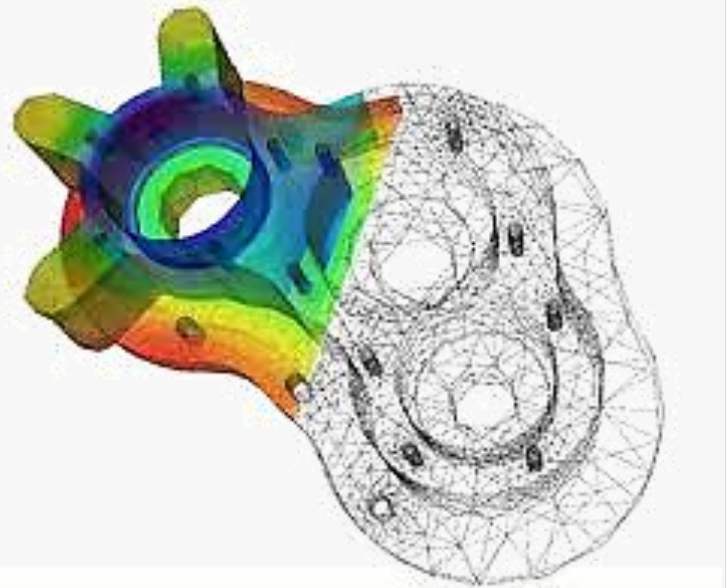


Partial Differential Equations







Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

Partial Differential Equations

Book: Advanced Engineering Mathematics

(9th Edition) by Ervin Kreyszig

- Chapter: 12
  - Sections: 12.3, 12.4, 12.9

The Vibrating String: Wave Equation

We solve the problem of the vibrating string (with its ends held fixed), with arbitrary initial position (displacement) and velocity by using the method of separation of variables. That is, we solve the boundary value problem for the wave equation that describes the vibrations of a string with fixed ends. The string is assumed to be stretched on the x −axis with ends fastened at $x = 0$ and $x = L$. The function $u(x,t)$, that represents the position at time t of the point x on the string, satisfies the **one-dimensional wave equation:**

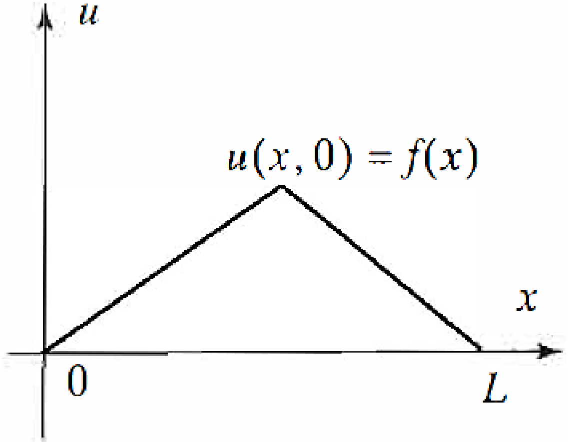
$$u_{tt} = c^2 u_{xx}; \qquad \text{where} \quad c^2 = \frac{T}{\rho}. \qquad (1)$$

To find $u(x,t)$, we solve this equation subject to the **boundary conditions**:

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t \geq 0, \qquad (2)$$

and the **initial conditions:**

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 \leq x \leq L \qquad (3)$$



Initial shape of a stretched string, $u(x,0)$.

The boundary conditions state that the ends of the string are held fixed for all time, while the initial conditions give the initial shape of the string $f(x)$ and its initial velocity $g(x)$.

The Vibrating String: Wave Equation

Two solutions of this problem are possible. One is based on a general method called the **method of separation of variables.** The second solution, due to **d'Alembert**, expresses $u(x,t)$ in closed form and leads to interesting geometric interpretations in terms of traveling waves. By using **method of separation of variables** solution to the BVP is given as:

$$u(x,t) = \sum_{n=1}^{\infty} [C_n \cos(\lambda_n t) + C_n^* \sin(\lambda_n t)] \sin\left(\frac{n\pi x}{L}\right), \qquad (4)$$

where the coefficients C_n and C_n^* are respectively given as:

$$C_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \qquad (5)$$

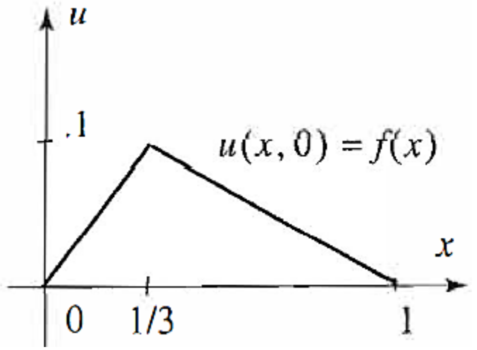
and

$$C_n^* = \frac{2}{cn\pi}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx. \qquad (6)$$

Example: Vibration of a stretched string with fixed ends

The ends of a stretched string of length $L = 1$ are fixed at $x = 0$ and $x = 1$. The string is set to vibrate from rest by releasing it from an initial triangular shape modeled by the function:

$$f(x) = \begin{cases} \dfrac{3x}{10}; & \text{if} \quad 0 \le x \le \dfrac{1}{3} \\ \dfrac{3(1-x)}{20}; & \text{if} \quad \dfrac{1}{3} \le x \le 1 \end{cases}$$



Determine the subsequent motion of the string, given that $c = 1/\pi$. Initial velocity $u_t(x,t) = 0$.

Solution:

For the present case $\lambda_n = \frac{cn\pi}{L} = n$. Moreover, $g(x) = 0$, so we have $C_n^* = 0$. Using (5) and integrating by parts, we get:

$$C_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx = \frac{3}{5}\int_0^{1/3} x\sin(n\pi x)\,dx + \frac{3}{10}\int_{1/3}^{1}(1-x)\sin(n\pi x)\,dx = \frac{9}{10n^2\pi^2}\sin\left(\frac{n\pi}{3}\right).$$

and the solution is given as:

$$u(x,t) = \frac{9}{10\pi^2}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{3}\right)}{n^2}\cos(nt)\sin(n\pi x).$$

D'Alembert solution of the wave equation

We have solved the wave equation by using Fourier series. But it is often more convenient to use the so-called ***d'Alembert solution to the wave equation.*** While this solution can be derived using Fourier series as well, It is easier and more instructive to derive this solution by making a correct change of variables to get an equation that can be solved by simple integration. Suppose we wish to solve the wave equation:

$$u_{tt} = c^2 u_{xx}; \qquad \text{where} \quad c^2 = \frac{T}{\rho}. \qquad (1)$$

subject to the side conditions

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t \geq 0, \qquad (2)$$

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 \leq x \leq L \qquad (3)$$

We will transform the equation into a simpler form where it can be solved by simple integration. We change variables to:

$$v = x + ct \quad \text{and} \quad w = x - ct. \qquad (4)$$

The chain rule says:

$$u_x = u_v v_x + u_w w_x = u_v + u_w.$$

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}.$$

D'Alembert solution of the wave equation

Transforming the other derivative in (1) by the same procedure, we find:

$$u_{tt} = c^2(u_{vv} - 2u_{vw} + u_{ww}).$$

Therefore, the wave equation (1) transforms into:

$$u_{vw} = 0.$$

It is easy to find the general solution to this equation by two successive integrations, first with respect to w and then with respect to v. This gives:

$$\frac{\partial u}{\partial v} = h(v) \qquad \text{and} \qquad u = \int h(v)\,dv + \psi(w).$$

Here $h(v)$ and $\psi(w)$ are arbitrary functions of v and w, respectively. Since the integral is a function of v, say, $\phi(v)$ the solution is of the form:

$$u = \phi(v) + \psi(w).$$

In terms of x and t, by (4), we thus have:

$$u(x,t) = \phi(x + ct) + \psi(x - ct). \quad (5)$$

This is known as **d'Alembert's solution** of the wave equation (1).

D'Alembert solution of the wave equation

Differentiating (5) w.r.t. "t" we get:

$$u_t(x, t) = c\phi'(x + ct) - c\psi'(x - ct)$$

Using (3) we have:

$$u(x, 0) = \phi(x) + \psi(x) = f(x), \qquad (6)$$

$$u_t(x, 0) = c\phi'(x) + c\psi'(x) = g(x). \qquad (7)$$

Dividing equation (7) by c and integrating with respect to x, we obtain:

$$\phi(x) - \psi(x) = k(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)\, ds, \qquad k(x_0) = \phi(x_0) - \psi(x_0). \qquad (8)$$

If we add this to (6), then $\psi(x)$ drops out and division by 2 gives

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\, ds + \frac{1}{2}\, k(x_0). \qquad (9)$$

D'Alembert solution of the wave equation

Similarly, subtraction of (8) from (6) and division by 2 gives:

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_{x_0}^{x} g(s)\, ds - \frac{1}{2} k(x_0). \tag{10}$$

In (9) we replace x by $x + ct$, we then get an integral from x_0 to $x + ct$. In (10) we replace x by $x - ct$ and get minus an integral from x_0 to $x - ct$ or plus an integral from $x - ct$ to x_0. Hence addition of $\phi(x + ct)$ and $\psi(x - ct)$ gives us:

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds.$$

If the initial velocity is zero, we see that this reduces to

$$u(x, t) = \tfrac{1}{2}[f(x + ct) + f(x - ct)].$$

This is called d' Alembert's solution of the vibrating string problem.

D'Alembert solution of the wave equation

Similarly, subtraction of (8) from (6) and division by 2 gives:

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\, ds - \frac{1}{2} k(x_0). \tag{10}$$

In (9) we replace x by $x + ct$, we then get an integral from x_0 to $x + ct$. In (10) we replace x by $x - ct$ and get minus an integral from x_0 to $x - ct$ or plus an integral from $x - ct$ to x_0. Hence addition of $\phi(x + ct)$ and $\psi(x - ct)$ gives us:

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds. \tag{11}$$

where f and g are odd periodic extensions. If the initial velocity is zero i.e., $g(x) = 0$, we see that (11) reduces to:

$$u(x, t) = \tfrac{1}{2}[f(x + ct) + f(x - ct)]. \tag{12}$$

This is called d' Alembert's solution of the vibrating string problem.

Geometric Interpretation of D' Alembert's Solution

When the initial velocity is zero, d'Alembert's solution takes on the simpler form:

$$u(x,t) = \tfrac{1}{2}[f(x+ct) + f(x-ct)]. \tag{12}$$

This has an interesting geometric interpretation. For fixed t, the graph of $f(x-ct)$ (as a function of x) is obtained by translating the graph of $f(x)$ by ct units to the right. As t increases, the graph represents a wave traveling to the right with velocity c. Similarly, the graph of $f(x+ct)$ is a wave traveling to the left with velocity c. We see from (12) that this solution of the wave equation is an average of two waves traveling in opposite directions with shapes determined from the initial shape of the string. The general form of d'Alembert's solution (11):

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds. \tag{11}$$

is harder to interpret geometrically. It does tell us, however, that the displacement at the point x at time $t > 0$ is determined entirely by the initial displacements at positions $x-ct$ and $x+ct$ and by the initial velocity on the interval between $x-ct$ and $x+ct$. To understand the contribution of the initial velocity to the motion, let G denote an antiderivative of g. Hence

Geometric Interpretation of D' Alembert's Solution

Hence,

$$G(x) = \int_a^x g(s)\,ds,$$

for some fixed number a. Note that:

$$G(x+2L) - G(x) = \int_x^{x+2L} g(s)\,ds = \int_{-L}^{L} g(s)\,ds = 0,$$

where the last equality follows because g is odd. Hence G is $2L$ periodic. The solution (11) may be rewritten in terms of f and G as follows:

$$\begin{aligned} u(x,t) = &\ \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}[G(x+ct) - G(x-ct)] \\ = &\ \frac{1}{2}\left[f(x-ct) - \frac{1}{c}G(x-ct)\right] + \frac{1}{2}\left[f(x+ct) + \frac{1}{c}G(x+ct)\right], \end{aligned}$$

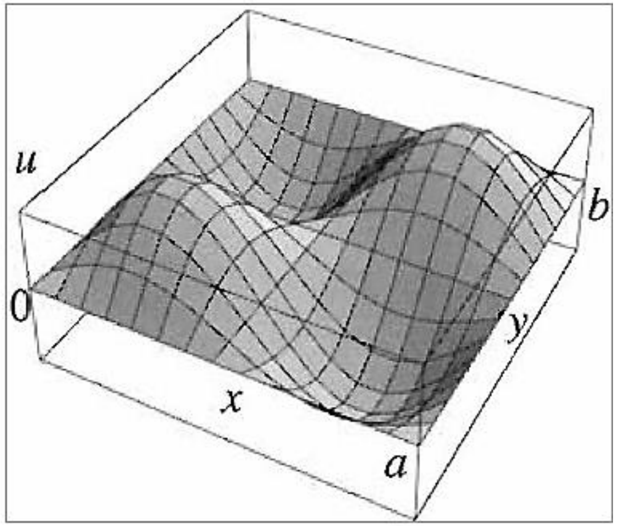
showing that, in general, the solution still consists of right- and left-moving waveforms. The main difference from the case represented by (12) is that here the two waveforms need no longer have the same shape.

Two-Dimensional Wave Equation

The vibrating string is a basic one-dimensional vibrational problem. The vertical displacement of a vibrating string satisfies the one-dimensional wave equation. Equally important is its two-dimensional analog, namely, the motion of an elastic membrane, such as a drumhead, that is stretched and then fixed along its edge. Suppose that a thin elastic membrane is stretched over a rectangular frame with dimensions a and b, and that the edges are held fixed. The membrane is then set to vibrate by displacing it vertically and then releasing it. The vibrations of the membrane are governed by the **two-dimensional wave equation:**

$$u_{tt} = c^2\left[u_{xx} + u_{yy}\right]; \quad 0 < x < a, \qquad 0 < y < b, \qquad t > 0 \qquad \text{where} \quad c^2 = \frac{T}{\rho}. \qquad (1)$$

Here $u = u(x, y, t)$ denotes the deflection at the point (x, y) at time t. The fact that the edges are held fixed is expressed by the condition $u(x, y, t) = 0$ on the boundary for all $t \geq 0$. More explicitly, we have the boundary conditions To find $u(x, t)$, we solve this equation subject to the **boundary conditions**:



$$\begin{aligned} u(0, y, t) &= 0 = u(a, y, t) = 0, \qquad 0 < y < b, \qquad t \geq 0, \\ u(x, 0, t) &= 0 = u(x, b, t) = 0, \qquad 0 < x < a, \qquad t \geq 0. \end{aligned} \qquad (2)$$

The Vibrating String: Wave Equation

The **initial conditions:**

$$u(x,y,0) = f(x,y), \qquad u_t(x,y,0) = g(x,y), \qquad 0 < x < a, \qquad 0 < y < b, \qquad (3)$$

represent, respectively, the shape and the velocity of the membrane at time $t = 0$. To determine the vibrations of the membrane, we must find the function $u(x,y,t)$ that satisfies $(1) - (3)$. By using method of separation of variables we get:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[B_{mn}\cos(\lambda_{mn}t) + B_{mn}^{*}\sin(\lambda_{mn}t)\right]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \qquad (4)$$

where the coefficients B_{mn} and B_{mn}^{*} are respectively given as:

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

and

$$B_{mn}^{*} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy,$$

where $\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$.

Practice Problems:

A. Solve for the motion of a string of length $L = \frac{\pi}{2}$ if $c = 1$ and the initial displacement and velocity are given by $f(x) = 0$ and $g(x) = x\cos x$, respectively.

B. Find $u(x,t)$ for the string of length $L = 1$ and $c^2 = 1$ when the initial velocity is zero and the initial deflection with small k (say, 0.01) is as follows:

1. $k\sin(3\pi x)$.

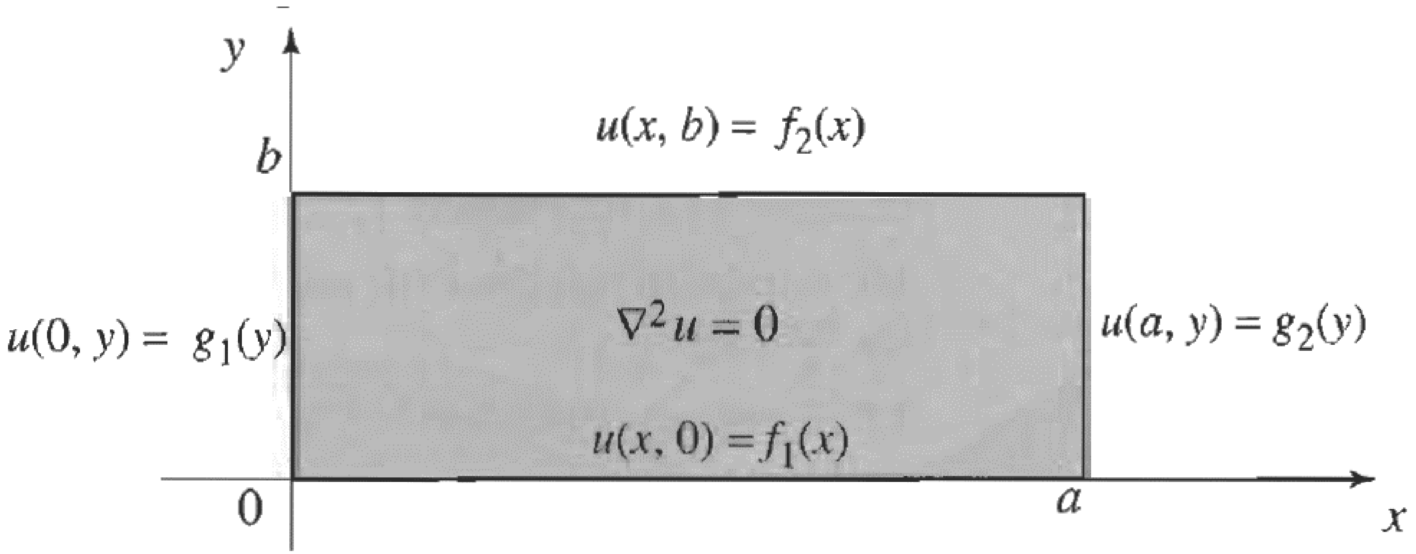
2. $k\left[\sin(\pi x) - \frac{1}{2}\sin(2\pi x)\right]$.

C. Determine $u(x,t)$ when $g(x) = 0$ and $c^2 = 1$, and

$$f(x) = \begin{cases} 4x; & \text{if} \quad 0 \le x \le \frac{1}{4} \\ -4\left(x - \frac{1}{2}\right); & \text{if} \quad \frac{1}{4} \le x \le \frac{3}{4} \\ 4(x-1); & \text{if} \quad \frac{3}{4} \le x \le 1. \end{cases}$$

Practice Problems:

D. Solve the Dirichlet problem in the following figure for the given data:



1. $a = 1,\ b = 2,\ f_2(x) = x,\quad f_1 = g_1 = g_2 = 0.$

2. $a = 1,\ \ b = 1,\ f_1(x){=}\ 0,\ f_2 = 100,\ g_1 = 0,\ g_2 = 100.$

3. $a = 2,\ \ b = 1,\ f_1(x){=}\ 100,\ f_2 = g_1 = 0,\ g_2(y) = 100(1 - y).$

4. $a = b = 1,\ f_1(x) = 1 - x,\ f_2(x) = x,\ g_1 = g_2 = 0.$

5. $a = b = 1,\ f_1(x) = \sin 7\pi x,\ f_2(x) = \sin \pi x,\ g_1(y) = \sin 3\pi y,\ g_2(y) = \sin 6\pi y.$

6. $a = b = \pi,\ f_1(x) = \cos x,\ f_2(x) = x \sin x,\ g_1(y) = \pi - y,\ g_2(y) = \begin{cases} 3 & \text{if } 0 < y < \pi/2 \\ 0 & \text{if } \frac{\pi}{2} < y < \pi. \end{cases}$

Practice Problems:

E. Solve the Laplace's equation:

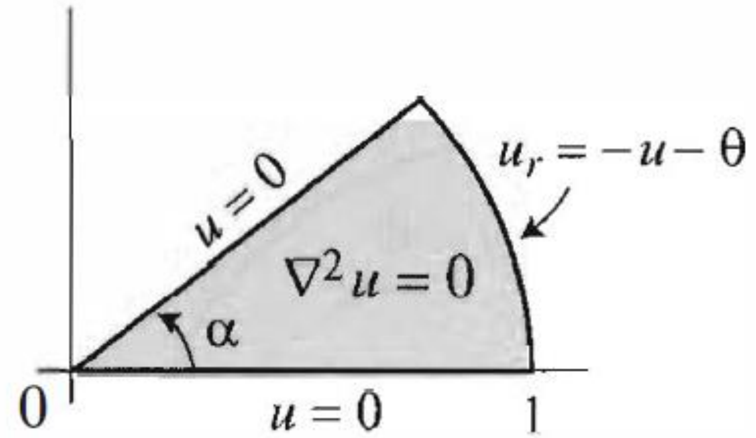
$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

over the wedge region, for $0 < r < 1$ and $0 < \theta < \alpha$, supplied with the boundary conditions:

$$u(r,0) = u(r,\alpha) = 0, \qquad u_r(1,\theta) = -u(1,\theta) - \theta.$$

The boundary conditions represent a wedge whose sides along the rays $\theta = 0$ and $\theta = \alpha$ are kept at 0 temperature, and the wedge is exchanging heat along its circular boundary at a rate given by the Robin condition $u_r(1,\theta) = -u(1,\theta) - \theta$. Presumably, the insulation of the wedge along the circular boundary is better for smaller values of θ, since the rate of heat loss is smaller for θ near 0.

The wedge-shaped region bounded by the rays $\theta = 0$ and $\theta = \alpha$, and the circle $r = 1$.



Practice Problems:

F. Solve the Dirichlet problem on the unit disk for the given boundary values:

1. $f(\theta) = \cos\theta$. **2.** $f(\theta) = \sin 2\theta$. **3.** $f(\theta) = \frac{1}{2}(\pi - \theta),\ 0 < \theta < 2\pi$.

4. $f(\theta) = \begin{cases} \pi - \theta & \text{if } 0 \le \theta \le \pi, \\ 0 & \text{if } \pi \le \theta < 2\pi. \end{cases}$ **5.** $f(\theta) = \begin{cases} 100 & \text{if } 0 \le \theta \le \pi/4, \\ 0 & \text{if } \pi/4 < \theta < 2\pi. \end{cases}$

G. Use d' Alembert's formula to solve the following boundary value problems for a string of unit length, subject to the given conditions.

1. $f(x) = \sin\pi x,\ g(x) = 0,\ \ c = \frac{1}{\pi}$.

2. $f(x) = \sin\pi x\cos\pi x,\ g(x) = 0,\ \ c = \frac{1}{\pi}$.

3. $f(x) = \sin\pi x + 3\sin 2\pi x,\ \ g(x) = \sin\pi x,\ c = 1$.

Practice Problems:

H. Solve the boundary value heat problem with the given data.

1. $L = \pi$, $c = 1$, $f(x) = 78$.

2. $L = \pi$, $c = 1$, $f(x) = 30 \sin x$.

3. $L = \pi$, $c = 1$,

$$f(x) = \begin{cases} 33x & \text{if } 0 < x \le \frac{\pi}{2}, \\ 33(\pi - x) & \text{if } \frac{\pi}{2} < x < \pi. \end{cases}$$

4. $L = \pi$, $c = 1$,

$$f(x) = \begin{cases} 100 & \text{if } 0 < x \le \frac{\pi}{2}, \\ 0 & \text{if } \frac{\pi}{2} < x < \pi. \end{cases}$$

5. $L = 1$, $c = 1$, $f(x) = x$.

6. $L = 1$, $c = 1$, $f(x) = e^{-x}$.

I. Solve the non-homogeneous boundary value problem for the given data.

1. $u(0,t) = 100$, $u(1,t) = 0$, $f(x) = 30\sin(\pi x)$, $L = 1$, $c = 1$.

2. $u(0,t) = 100$, $u(1,t) = 100$, $f(x) = 50\,x(1-x)$, $L = 1$, $c = 1$.

J. A square membrane with $a = 1$, $b = 1$, and $c = 1/\pi$, is placed in the xy −plane. The edges of the membrane are held fixed, and the membrane is stretched into a shape modeled by the function $f(x,y) = xy(x-1)(y-1), 0 < x < 1, 0 < y < 1$. Suppose that the membrane starts to vibrate from rest. Determine the position of each point on the membrane for $t > 0$. (hint: $g(x,y) = 0 \Longrightarrow B^*_{mn} = 0$)