16

Vector Calculus

Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.2**

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

- **Chapter: 16**
  - **Section: 16.2**

Line Integrals of Scalar Fields (Space curves)

We now suppose that C is a smooth space curve given by the parametric equations:

$$x = x(t), \qquad y = y(t), \qquad z = z(t); \quad a \le t \le b,$$

or by a vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Suppose f is a function of three variables that is continuous on some region containing C. Then, we define the line integral of f along C (w.r.t. arc length) as:

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

Observe that the integral can be written in the more compact vector notation:

$$\int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt.$$

Line integrals along C with respect to x, y, and z can also be defined. Thus, as with line integrals in the plane, we evaluate integrals of the form:

$$\int_C [P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz].$$

Line Integral Over a Space Curve

Physical interpretation of the line integral

$$\int_C f(x, y, z)\, ds,$$

Depends on the nature of its integrand $f(x, y, z)$. A function of three variables $f\ (x, y, z)$ can be interpreted as a scalar field that varies at each point (x, y, z).

Examples:

- Pressure: $P = P(x, y, z)$
- Temperature: $T = T(x, y, z)$
- Density: $\rho = \rho(x, y, z)$ density of an object occupying a region E in space.

Mass, Center of Mass and Moments

Let $\rho(x, y, z)$ represent the density function of a solid object that occupies the region E in units of mass per unit volume, at any given point (x, y, z), then mass of the wire is given by:

$$M = \int_C \rho(x, y, z)\, ds.$$

The center of mass of the wire with density function ρ is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \text{and} \qquad \bar{z} = \frac{M_{xy}}{M},$$

where:

$$M_{yz} = \int_C x\rho(x, y, z)\, ds, \qquad M_{xz} = \int_C y\rho(x, y, z)\, ds \quad \text{and} \quad M_{xy} = \int_C z\rho(x, y, z)\, ds,$$

represents first moments about the coordinate planes.

Moments of Inertia

the ***moments of inertia***, *or* ***second moments***, I_x, I_y and I_z about the coordinate axes are defined as:

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\, ds,$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\, ds,$$

and

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\, ds.$$

Moment of inertia about the origin is defined as:

$$2I_0 = 2\int_C (x^2 + y^2 + z^2)\rho(x, y, z)\, ds = I_x + I_y + I_z.$$

Total Electric Charge

The total electric charge on a solid object occupying a region E and having charge density $\rho(x, y, z)$ is given by:

$$Q = \int_C \rho(x, y, z)\, ds.$$

Centroids of Geometric Figures

When the density of a solid object is constant i.e., $\rho(x, y, z) = 1$ the center of mass is called the **centroid** of the object.

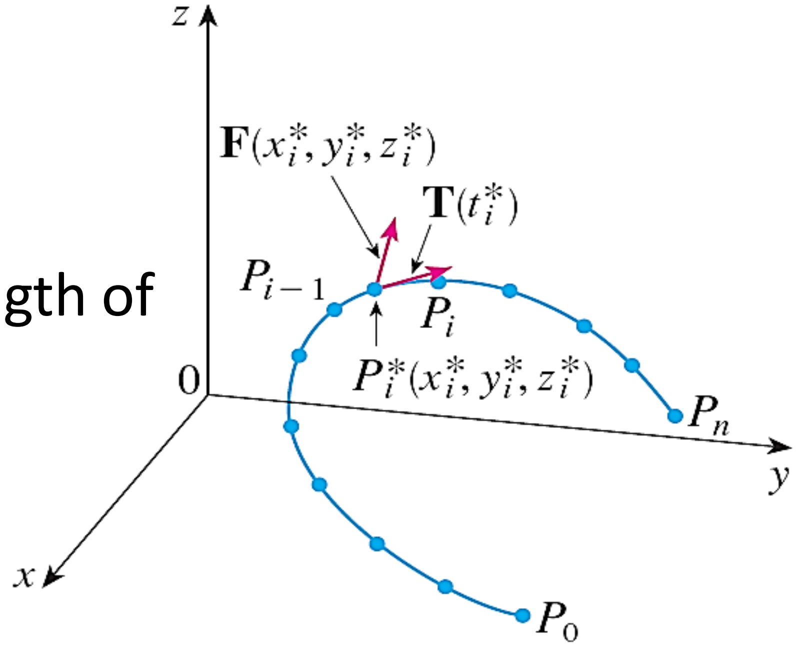
Line Integrals Of Vector Fields

Now, suppose that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$. We wish to compute the work done by this force in moving a particle along a smooth curve C. Thus, we define the work W done by the force field $\mathbf{F}$ as:

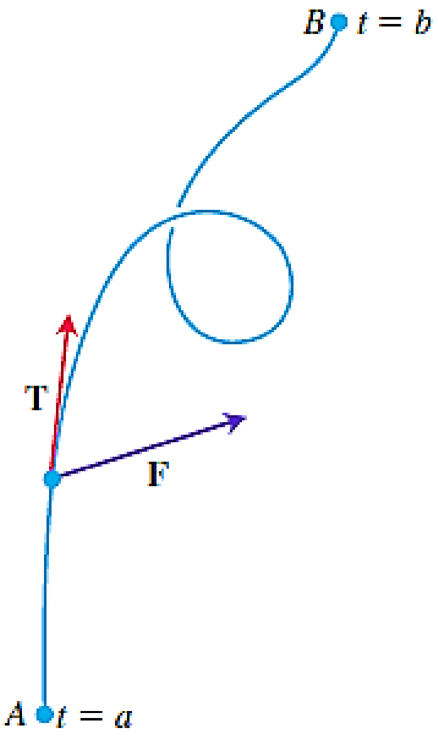
$$W = \int_C \mathbf{F}(x, y, z). \mathbf{T}(x, y, z)\, ds = \int_C \mathbf{F}. \mathbf{T}\, ds$$

where T is the unit tangent vector to the curve.

- This says that work is the line integral with respect to arc length of the tangential component of the force.



Line Integrals Of Vector Fields



If the curve C is given by the vector equation:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

Then

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Thus, we have

$$W = \int_C \mathbf{F}(x,y,z).\mathbf{T}(x,y,z)\,ds = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)).\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right]|\mathbf{r}'(t)|dt = \int_a^b \mathbf{F}(\mathbf{r}(t)).\mathbf{r}'(t)\,dt.$$

This integral is often abbreviated as: $\int_C \mathbf{F}.d\mathbf{r}$ and occurs in other areas of physics as well.

Vector fields

Thus, we make the following definition for the line integral of any continuous vector field. Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t), a \le t \le b$. Then, the line integral of $\mathbf{F}$ along C is:

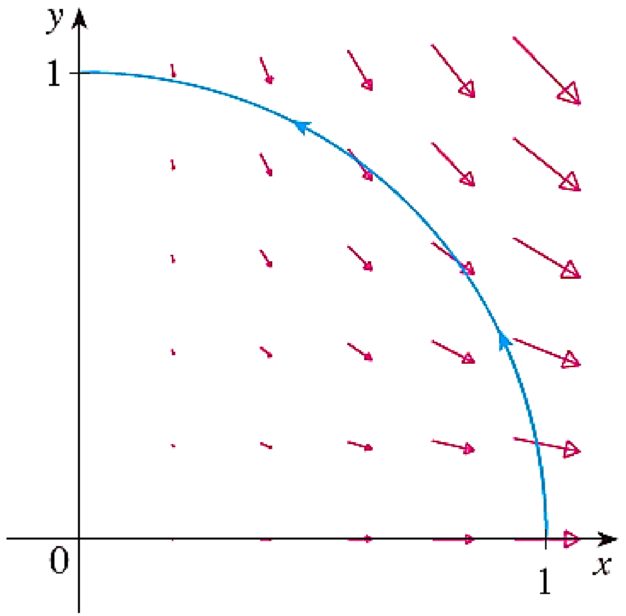
$$\int_C \mathbf{F}.\mathbf{T}\,ds = \int_a^b \mathbf{F}(\mathbf{r}(t)).\mathbf{r}'(t)\,dt = \int_C \mathbf{F}.d\mathbf{r}.$$

Note that $\mathbf{F}(\mathbf{r}(t))$ is just an abbreviation for $\mathbf{F}(x(t), y(t), z(t))$. So, we evaluate $\mathbf{F}(\mathbf{r}(t))$ simply by putting $x = x(t), y = y(t)$, and $z = z(t)$ in the expression for $\mathbf{F}(x, y, z)$.

Example

Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, 0 \ \leq \ t \ \leq \ \pi/2$$



Solution:

Since $x = \cos t$ and $y = \sin t$, we have:

$$\mathbf{F}\big(\mathbf{r}(t)\big) = \cos^2 t\,\mathbf{i} - \cos t \sin t\,\mathbf{j},$$

and

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}.$$

Therefore, the work done is:

$$W = \int_C \mathbf{F}\,.\,d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)).\,\mathbf{r}'(t)\,dt = \int_0^{\pi/2} (-2\cos^2 t \sin t)\,dt = 2\left[\frac{\cos^3 t}{3}\right]_0^{\pi/2} = -\frac{2}{3}.$$

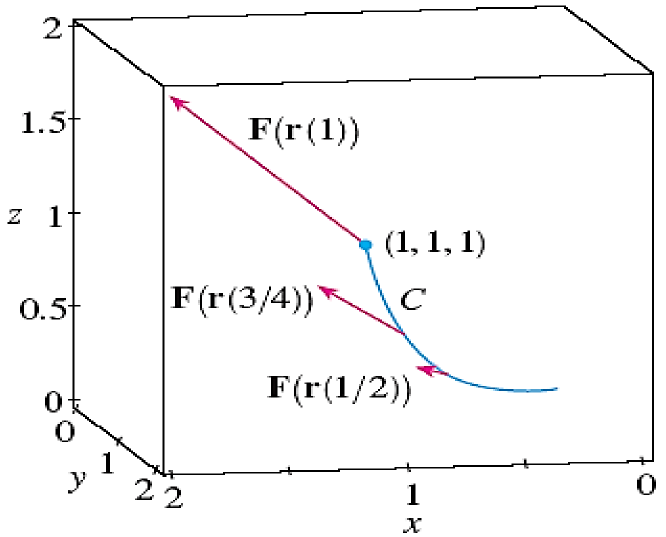
The figure shows the force field and the curve. The work done is negative because the field impedes movement along the curve.

Example

Evaluate $\int_C \mathbf{F} \, . \, d\mathbf{r}$ where: $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ and C is the twisted cubic given by

$$x = t, \qquad y = t^2, \qquad z = t^3; \quad 0 \le t \le 1.$$

Solution:



For the present case given that $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\ \mathbf{k}$, Thus, we have:

$$\mathbf{F}\big(\mathbf{r}(t)\big) = t^3\mathbf{i} + t^5\mathbf{j} + t^4\mathbf{i},$$

and

$$\mathbf{r}'(t) = \mathbf{i} + 2t\ \mathbf{j} + 3t^2\ \mathbf{k}.$$

Therefore:

$$\int_C \mathbf{F} \, . \, d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)). \mathbf{r}'(t)\, dt = \int_0^1 (t^3 + 5t^6)\, dt = \left[\frac{t^4}{4} + 5\,\frac{t^7}{7}\right]_0^1 = \frac{27}{28}.$$

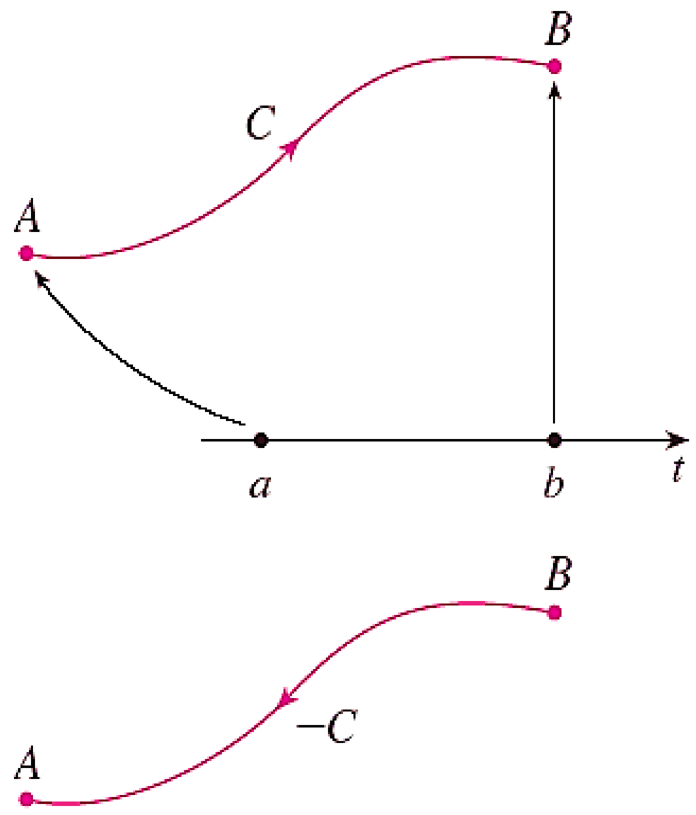
Curve Orientation

In general, a given parametrization:

$$x = x(t), \qquad y = y(t); \qquad a \le t \le b,$$

determines an orientation of a curve C, with the positive direction corresponding to increasing values of the parameter t. If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation, we have:

$$\int_{-C} \mathbf{F}\,.\,d\mathbf{r} = -\int_{C} \mathbf{F}\,.\,d\mathbf{r}$$



Line Integrals Of Vector & Scalar Fields

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field **F** on $\mathbb{R}^3$ is given in component form by:

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}.$$

We compute its line integral along C, as follows:

$$\int_C \mathbf{F} \,.\, d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)).\,\mathbf{r}'(t)\,dt$$

$$= \int_a^b [Px'(t) + Qy'(t) + Rz'(t)]\,dt$$

$$= \int_C [Pdx + Qdy + Rdz].$$

Practice Questions

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Chapter: 16

Exercise-16.2: Q – 19 to 22, Q – 39 to 45.

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Chapter: 16

Exercise-16.2: Q – 7 to 26.

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    - **Section: 16.3**

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    - **Section: 16.2, 16.3**

Evaluation of a Line Integral as a Definite Integral

Let f be continuous in a region containing a smooth curve C. If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \leq t \leq b$, then

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \leq t \leq b$, then

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt.$$

Definition of Line Integral of a Vector Field

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t)$, $a \leq t \leq b$. The **line integral** of $\mathbf{F}$ on C is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(x(t),\ y(t), z(t)) \cdot \mathbf{r}'(t)\, dt.$$

Note that for a force field **F**, the above integral is called the work done by **F** over the curve from a to b in the direction of the curve's unit tangent vector.

Fundamental Theorem of Line Integrals for Conservative Field

Let C be a piecewise smooth curve lying in an open region R and given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \leq t \leq b.$$

If $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative in R, and M and N are continuous in R, then

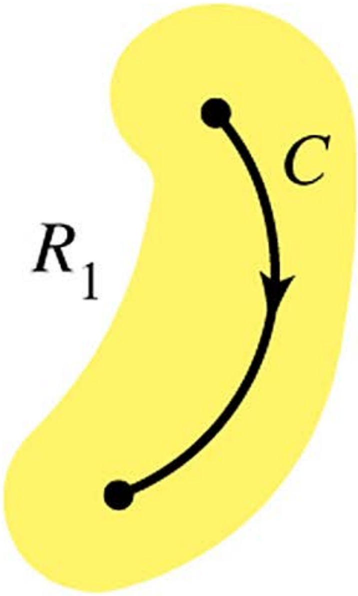
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a))$$

where f is a potential function of $\mathbf{F}$. That is, $\mathbf{F}(x, y) = \nabla f(x, y)$.

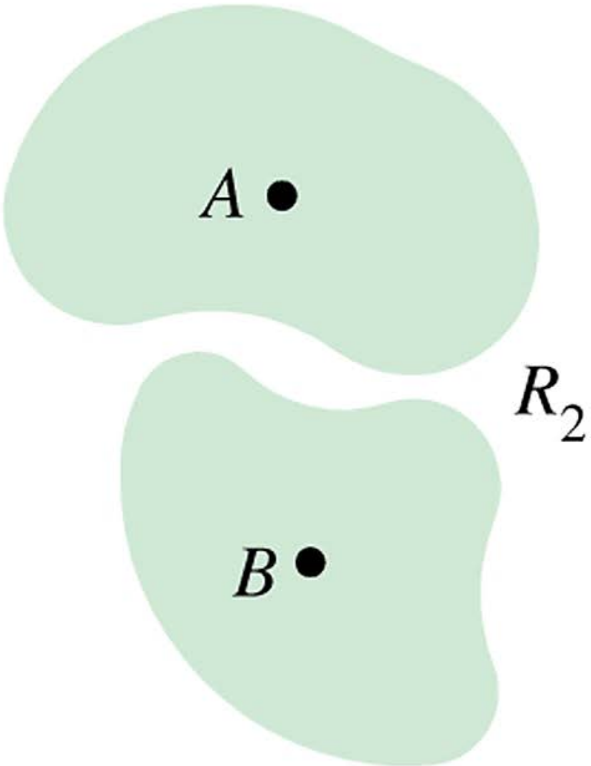
If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \leq t \leq b$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))$$

where f is a potential function of $\mathbf{F}$.



R_1 is connected.



R_2 is not connected.

Independence of Path and Conservative Vector Fields

Let $\mathbf{F}$ be a field defined on an open region D in space, and suppose that for any two points A and B in D the work $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ done in moving from A to B is the same over all paths from A to B. Then the integral $\int \mathbf{F} \cdot d\mathbf{r}$ is **path independent in D** and the field $\mathbf{F}$ is **conservative on D**.

It is important to note here that the line integral of a *conservative* vector field depends only on the initial point and terminal point of a curve. **Line integrals of conservative vector fields are independent of path.**

THEOREM

If $\mathbf{F}$ is continuous on an open connected region, then the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is independent of path if and only if $\mathbf{F}$ is conservative.

A curve is called **closed** if its terminal point coincides with its initial point, that is, $\mathbf{r}(b) = \mathbf{r}(a)$.

Equivalent Conditions

Let $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ have continuous first partial derivatives in an open connected region R, and let C be a piecewise smooth curve in R. The following conditions are equivalent.

1. $\mathbf{F}$ is conservative. That is, $\mathbf{F} = \nabla f$ for some function f.
2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.
3. $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every *closed* curve C in R.

Flow Integrals and Circulation for Velocity Fields

Instead of being a force field, suppose that **F** represents the velocity field of a fluid flowing through a region in space (a tidal basin or the turbine chamber of a hydroelectric generator, for example). Under these circumstances, the integral of **F**. **T** along a curve in the region gives the **fluid's flow along the curve.**

DEFINITIONS Flow Integral, Circulation

If $\mathbf{r}(t)$ is a smooth curve in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $t = a$ to $t = b$ is

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

The integral in this case is called a **flow integral**. If the curve is a closed loop, the flow is called the **circulation** around the curve.

Example: Finding Flow Along a Helix

A fluid's velocity field is $\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$. Find the flow along the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le \pi/2$.

Solution:

We evaluate $\mathbf{F}$ on the curve,

$$\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k} = (\cos t)\mathbf{i} + t\mathbf{j} + (\sin t)\mathbf{k}$$

and then find $d\mathbf{r}/dt$ and $\mathbf{F} \cdot (d\mathbf{r}/dt)$

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}.$$

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = (\cos t)(-\sin t) + (t)(\cos t) + (\sin t)(1)$$

$$= -\sin t \cos t + t \cos t + \sin t$$

so,

$$\text{Flow} = \int_{t=a}^{t=b} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{\pi/2} (-\sin t \cos t + t \cos t + \sin t)\, dt$$

$$= \left[\frac{\cos^2 t}{2} + t \sin t\right]_0^{\pi/2} = \left(0 + \frac{\pi}{2}\right) - \left(\frac{1}{2} + 0\right) = \frac{\pi}{2} - \frac{1}{2}.$$

Example: Finding Circulation Around a Circle

Find the circulation of the field $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$.

Solution:

On the circle, $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{j}$, and

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

Then

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + \sin^2 t + \cos^2 t$$

gives

$$\text{Circulation} = \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{2\pi} (1 - \sin t \cos t)\, dt$$

$$= \left[t - \frac{\sin^2 t}{2} \right]_0^{2\pi} = 2\pi.$$