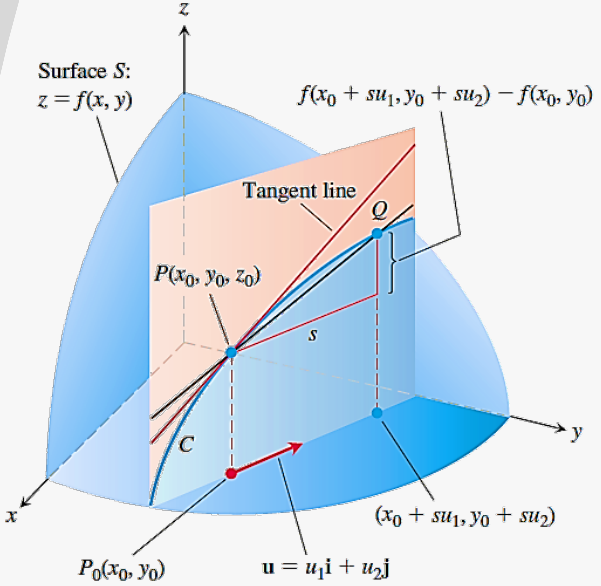
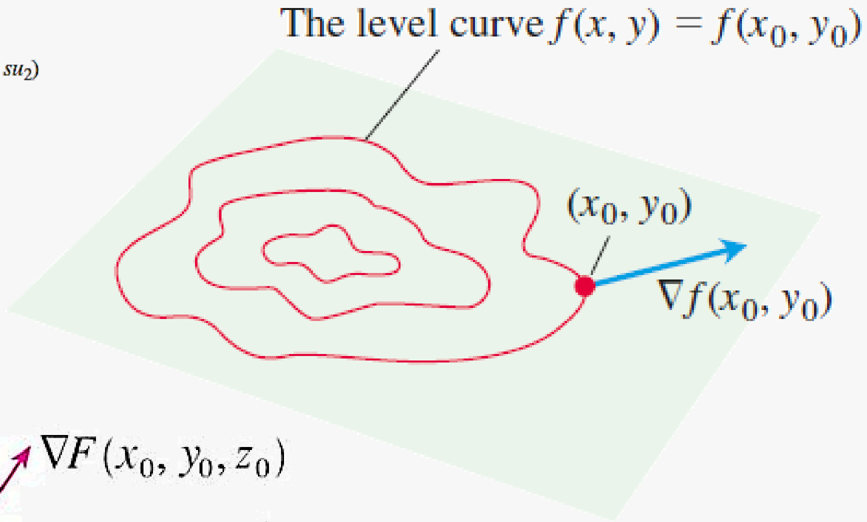
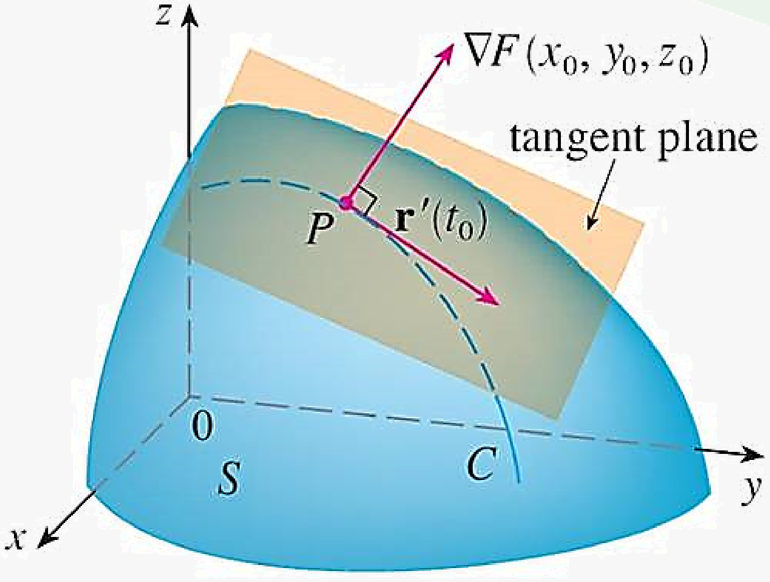
Tangent Planes & Normal Lines







Vector Calculus(MATH-243)
Instructor: Dr. Naila Amir

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Partial Derivatives

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Chapter: 14 , Section: 14.5, 14.6

Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.
Chapter: 14 , Section: 14.1, 14.6

Tangent Planes and Normal line

- Suppose that $f(x,y)$ has continuous partial derivatives. An equation of the **tangent plane** to the surface $z = f(x,y)$ at the point $P(x_0, y_0, z_0)$ is given as:

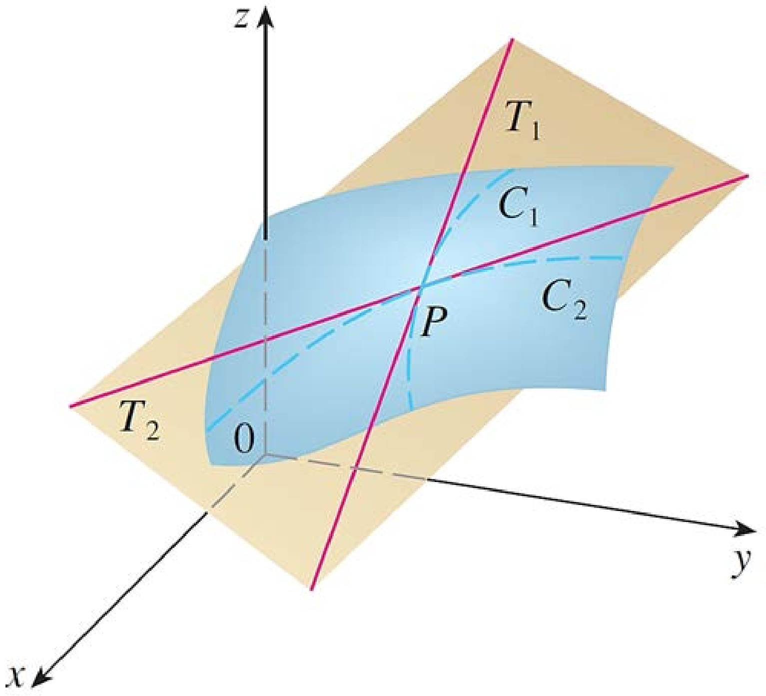
$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \qquad (2)$$

- Note the similarity between the equation of a tangent plane and the equation of a tangent line:

$$y - y_0 = f'(x_0)(x - x_0).$$

- The **normal line** to the surface $z = f(x,y)$ at P is the line passing through P and perpendicular to the tangent plane. Its direction is given by the gradient, and its symmetric equations are:

$$\frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = z - z_0. \qquad (3)$$



Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on the surface S and passes through the point P. Recall that the curve C is described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Let t_0 be the parameter value corresponding to P; that is, $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. Since C lies on S, any point $(x(t), y(t), z(t))$ must satisfy the equation of S, that is,

$$F\big(x(t), y(t), z(t)\big) = k. \qquad (4)$$

If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation (3) as follows:

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

$$\Longrightarrow \nabla F \cdot \mathbf{r}'(t) = 0. \qquad (5)$$

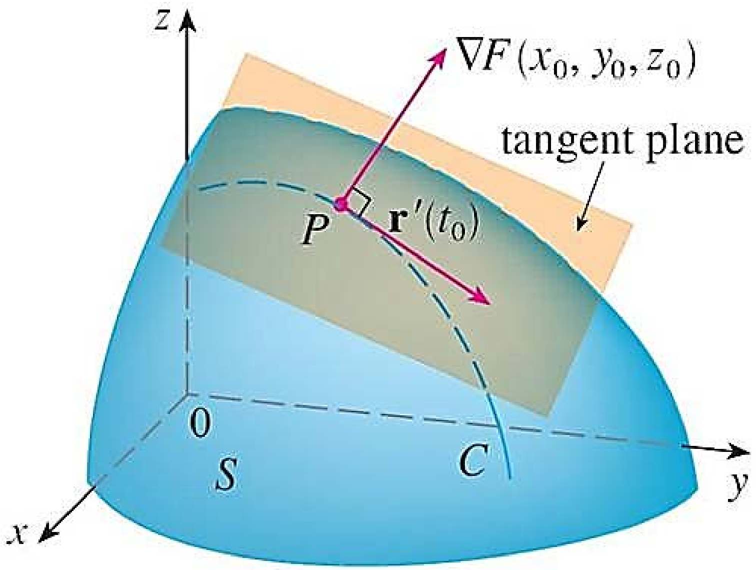
Tangent Planes to Level Surfaces

In particular, when $t = t_0$ we have:

$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0. \qquad (6)$$

Equation (6) says that *the gradient vector at* P, $\nabla F(x_0, y_0, z_0)$, *is perpendicular to the tangent vector* $\mathbf{r}'(t_0)$ *to any curve* C *on* S *that passes through* P. If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. Using the standard equation of a plane, we can write the equation of this tangent plane as:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0. \qquad (7)$$



Normal Line:

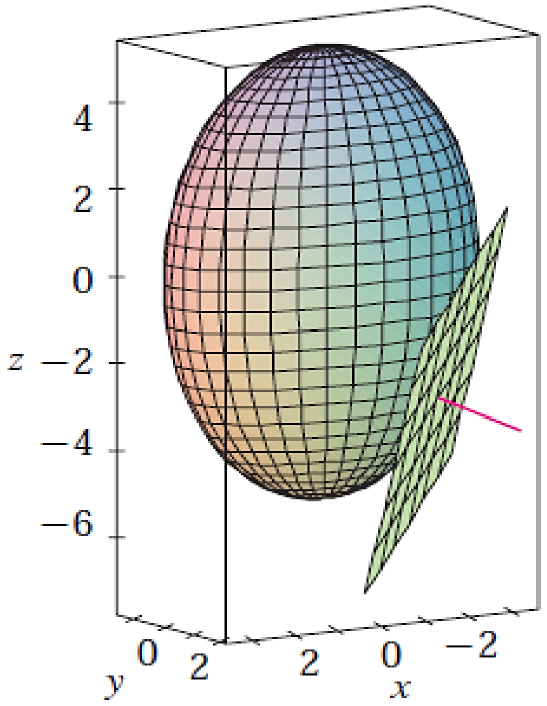
The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector: $\nabla F(x_0, y_0, z_0)$ and so, its symmetric equations are:

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}. \qquad (8)$$

Example:

Find the equations of the tangent plane and normal line at the point (–2, 1, –3) to the ellipsoid:

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$



Solution:

The ellipsoid is the level surface (with $k = 3$) of the function:

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}.$$

Therefore, we have:

$$F_x(x, y, z) = \frac{x}{2}; \quad F_y(x, y, z) = 2y; \quad F_z(x, y, z) = \frac{2z}{9},$$

$$\Longrightarrow F_x(-2, 1, -3) = -1; \quad F_y(-2, 1, -3) = 2; \quad F_z(-2, 1, -3) = -\frac{2}{3}.$$

Solution:

We know that the equation of the tangent plane at (x_0, y_0, z_0) is given as:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Thus, the equation of the tangent plane to the level surface at $(-2, 1, -3)$ is obtained as:

$$-1(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0,$$

$$\Longrightarrow 3x - 6y + 2z + 18 = 0.$$

By using (8), symmetric equations of the normal line are given as:

$$\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-2/3}.$$

Practice Questions

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

Chapter: 14

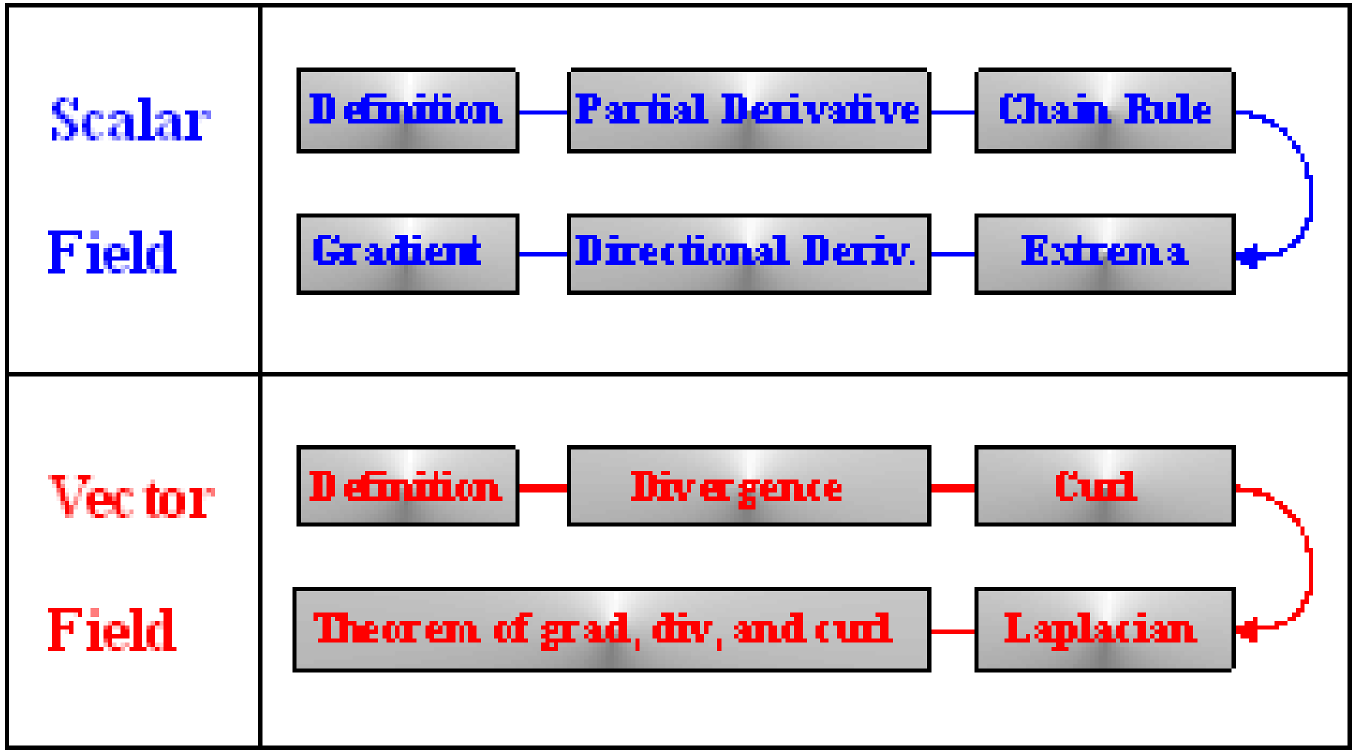
Exercise-14.3: Q – 15 to 42.
Exercise-14.6: Q – 4 to 26 , Q – 28 to 33, Q – 39 to 44.

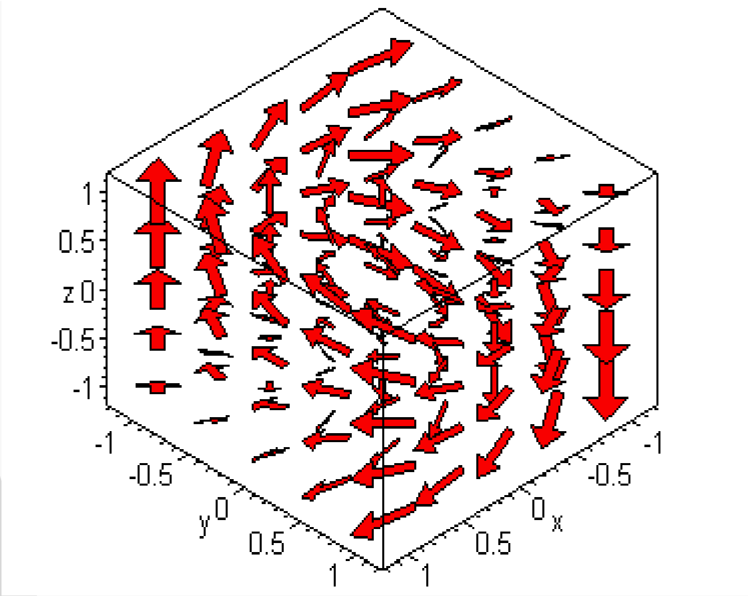
Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

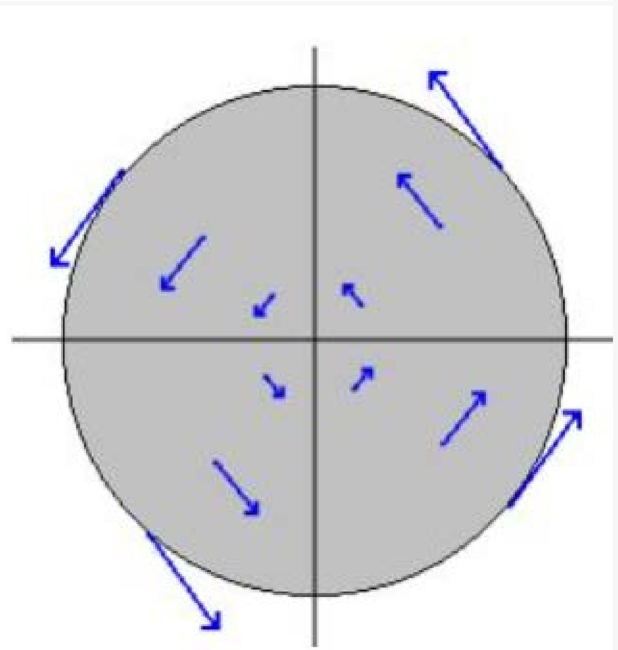
Chapter: 14

Exercise-14.3: Q – 1 to 22.
Exercise-14.5: Q – 1 to 28.
Exercise-14.6: Q – 1 to 14.

Scalar & Vector Fields







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Vector Calculus

Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.

Chapter: 16 , Section: 16.1

Field

- **Field**, is a region in which each point is affected by a force.

- Objects fall to the ground because they are affected by the force of earth’s gravitational field. A paper clip, placed in the magnetic field surrounding a magnet, is pulled toward the magnet, and two like magnetic poles repel each other when one is placed in the other’s magnetic field. An electric field surrounds an electric charge; when another charged particle is placed in that region, it experiences an electric force that either attracts or repels it.

- The strength of a field, or the forces in a particular region, can be represented by field lines; the closer the lines, the stronger the forces in that part of the field.

Field

- The concept of fields is quite common in Physics. For example, we encounter various physical fields like temperature, pressure or gravitational fields etc. Roughly speaking it represents a collection of numbers (scalars) or vectors.

- A field is a mathematical representation of the continuum arise from a physical process such that its value varies at each point. It depends on the process to require a scalar or a vector or a combination of both at each point to fully comprehend the dynamics involved.

- The former give rise **to scalar fields** and later are known as **vector fields.**

Two types of Fields

Scalar Fields:

(Magnitudes)

A scalar field can be regarded as a multi-variable function which gives numbers as an output which could be the values of:

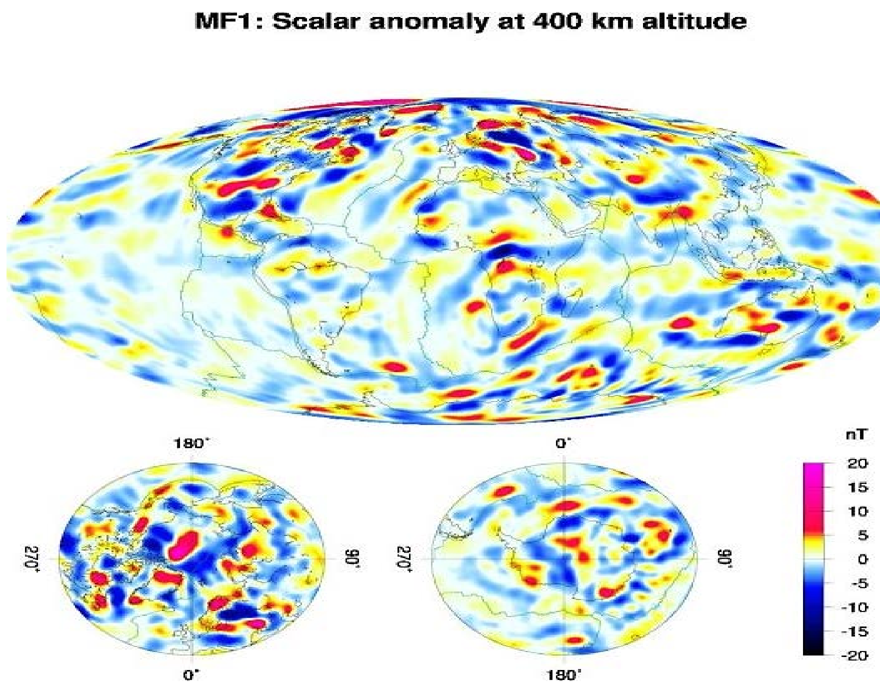
- Temperature
- Pressure
- Gravity anomaly
- Resistivity
- Elevation
- Maximum wind speed (without directional info)
- Energy
- Potential
- Density
- Time…

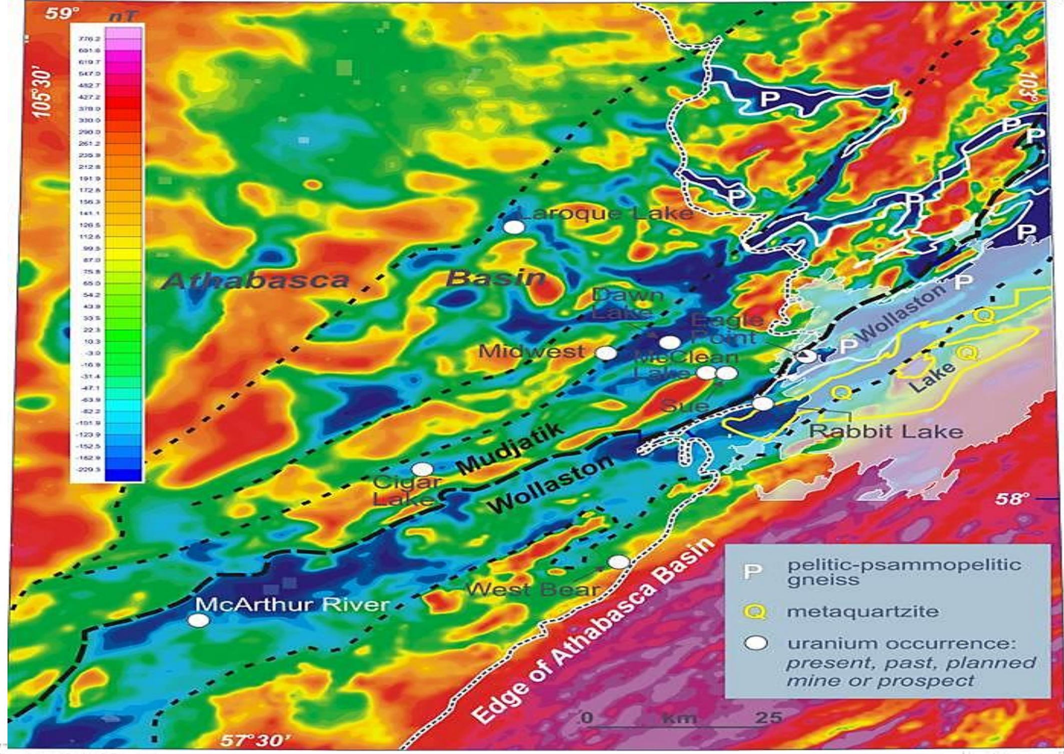
Vector Fields:

(Magnitude and direction)

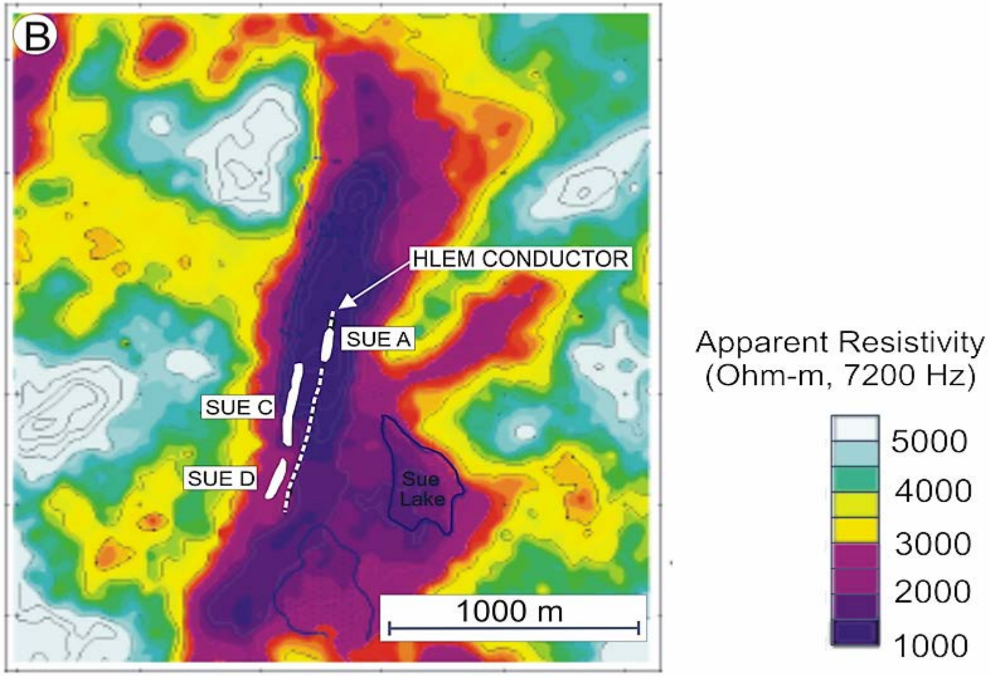
A vector field can be regarded as a multi-variable function which gives vectors in the output which could be the values of:

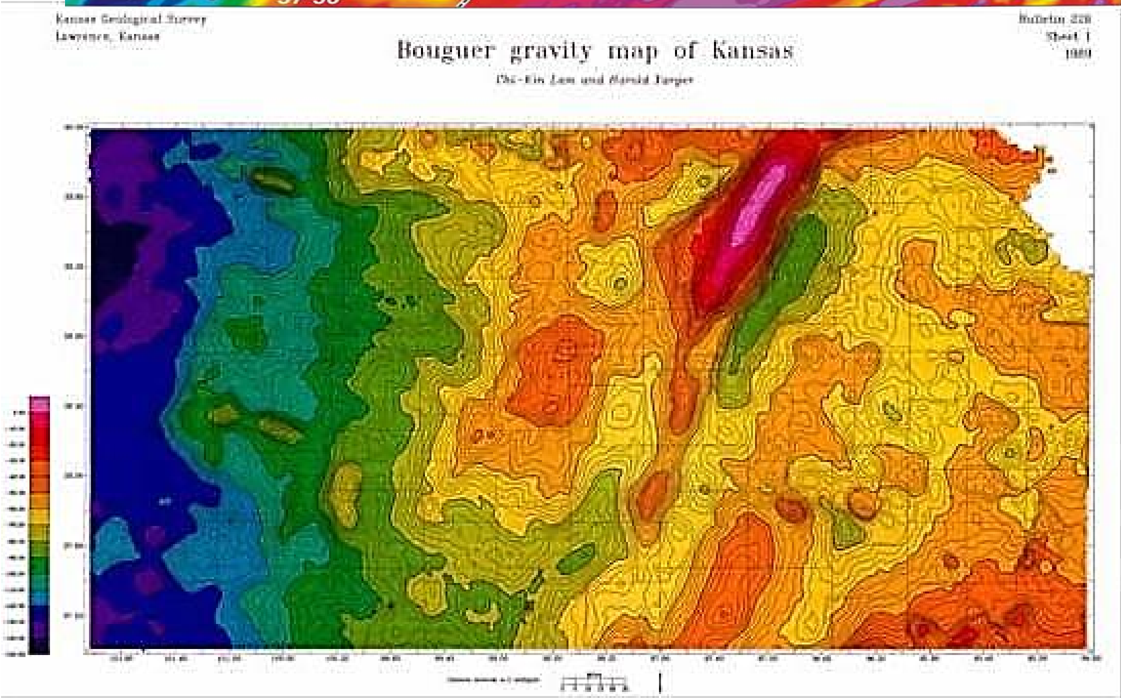
- Magnetic field (Scale Earth or mineral)
- Electric field
- Water velocity field
- Wind direction on a weather map
- Includes displacement, velocity, acceleration, force, momentum…



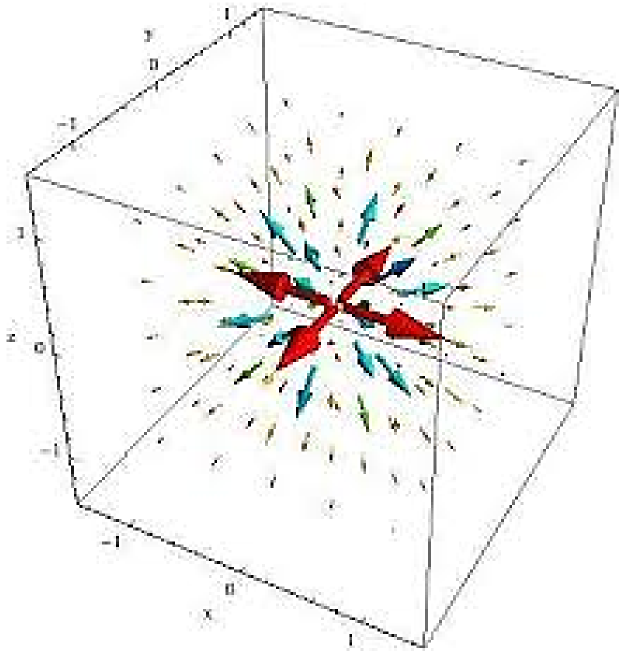


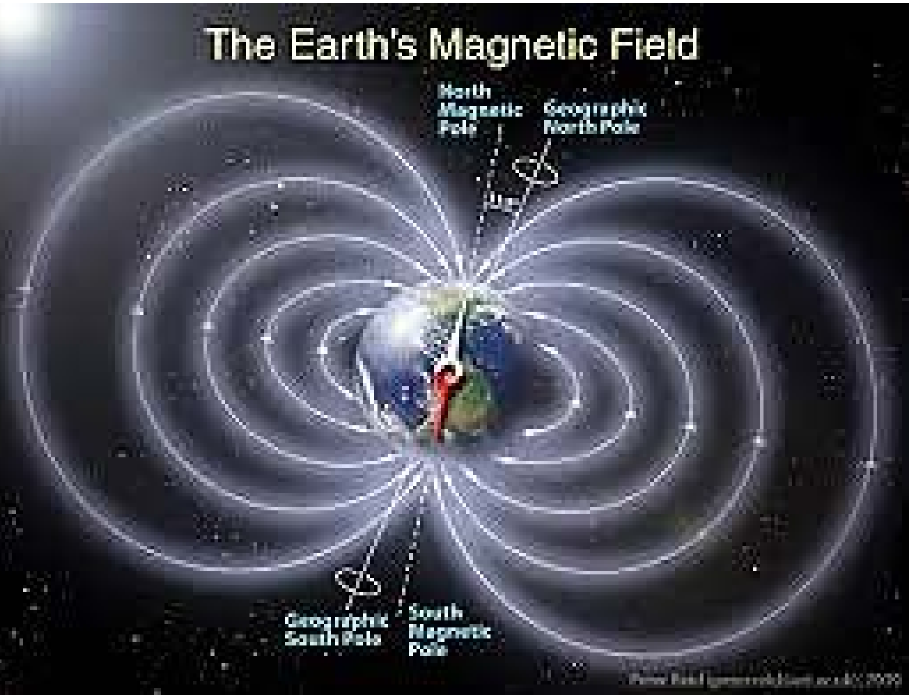
Scalar fields

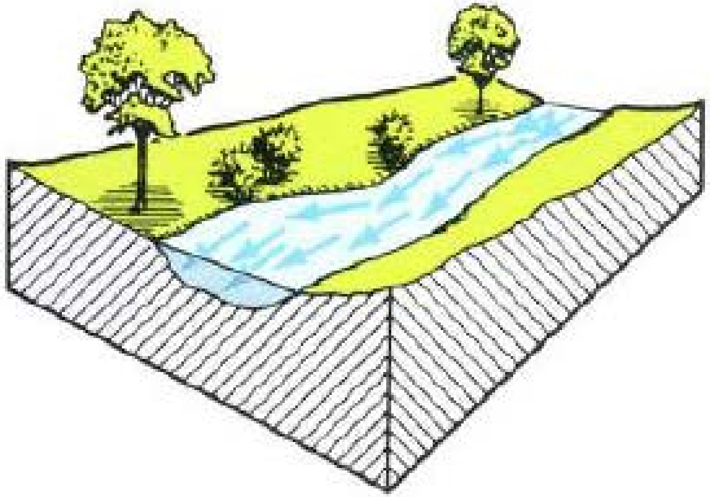


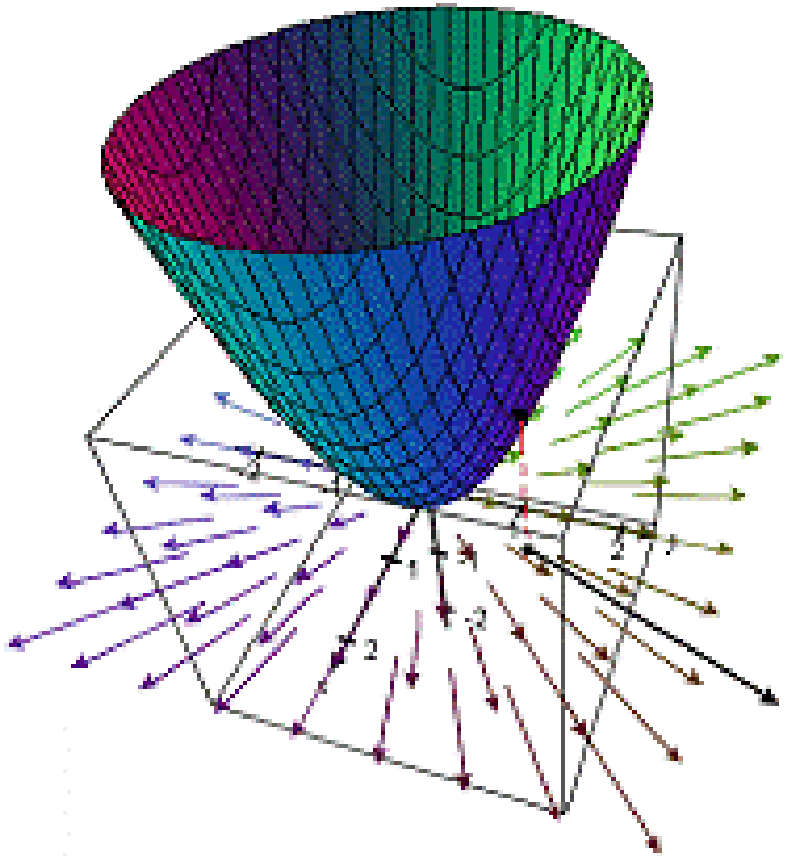


Vector fields





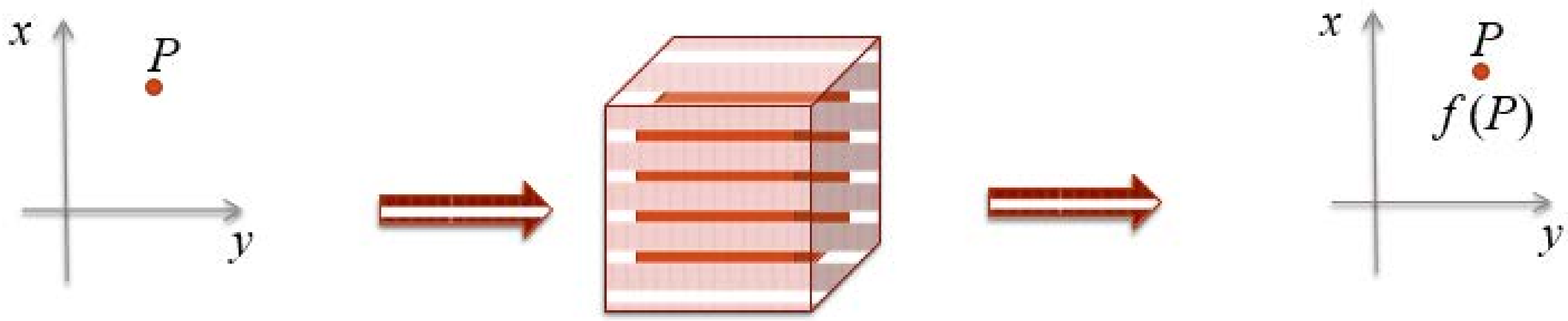




Scalar Fields (2D)

A scalar field in two dimensions can be represented by a multivariable function $f(x, y)$ such that its output is a number corresponding to a point (x, y) in a plane.

Example:



Mathematically,

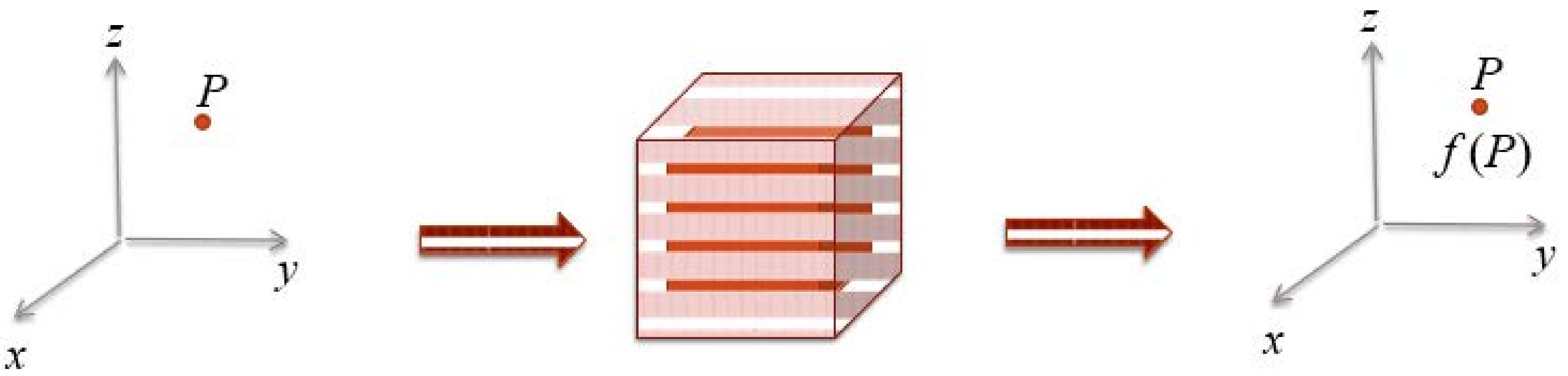
$$\boxed{f = f(P) = f(x, y)}$$

The collection of values of $f(x, y) = f(P)$, is called a ***scalar field.***

Scalar Fields (3D)

A scalar field in three dimensions can be represented by a multivariable function $f(x, y, z)$ such that its output is a number corresponding to a point (x, y, z) in space.

Example:



Mathematically,

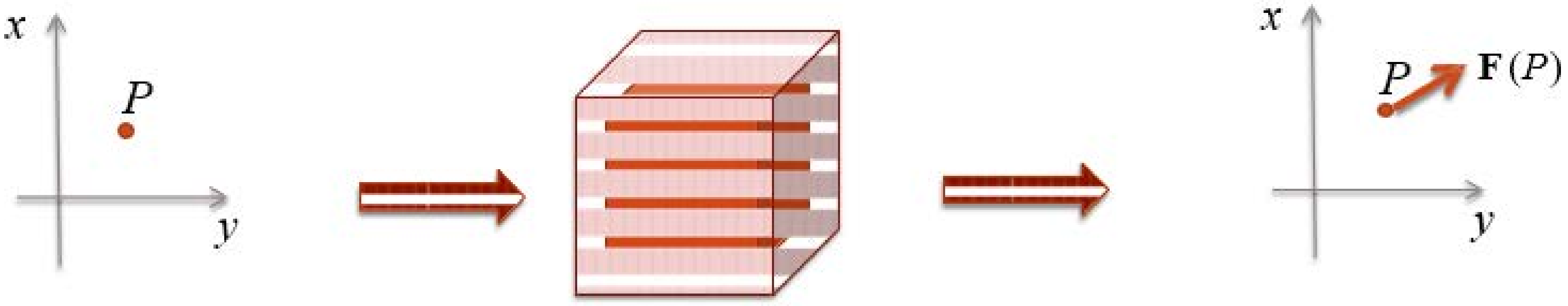
$$\boxed{f = f(P) = f(x, y, z)}$$

The collection of values of $f(x, y, z) = f(P)$, is called a ***scalar field.***

Vector Fields (2D)

A vector field can be regarded as a multi-variable function which gives vectors in the output.

Example:



Mathematically,

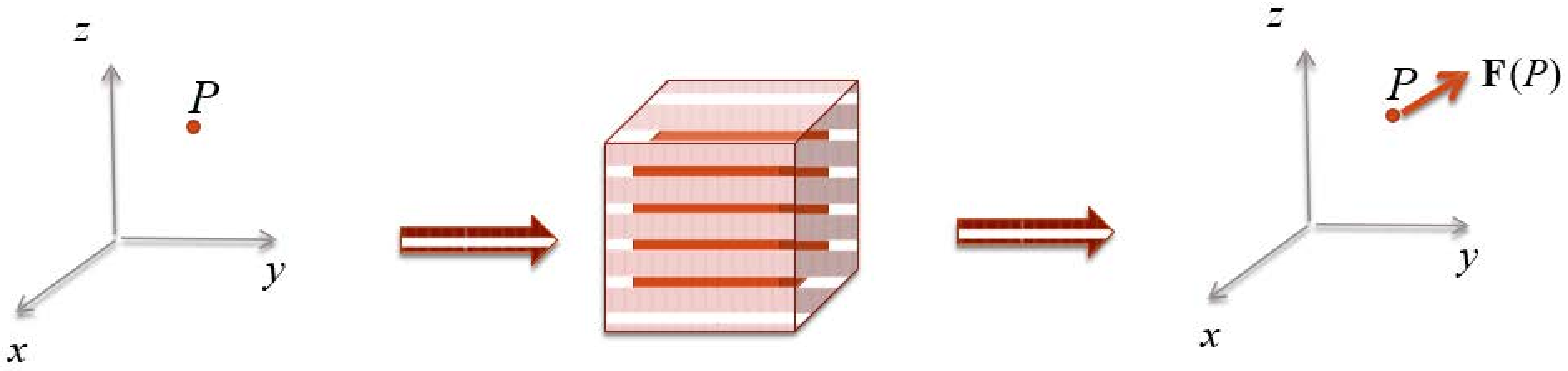
$$\mathbf{F} = \mathbf{F}(P) = \mathbf{F}(x, y) = \langle F_1, F_2 \rangle$$

The collection of values of $\mathbf{F}(P) = \mathbf{F}(x, y)$, is called a ***vector field.***

Vector Fields (3D)

A vector field can be regarded as a multi-variable function which gives vectors in the output.

Example:



Mathematically,

$$\mathbf{F} = \mathbf{F}(P) = \mathbf{F}(x, y, z) = \langle F_1, F_2, F_3 \rangle$$

The collection of values of $\mathbf{F}(P) = \mathbf{F}(x, y, z)$, is called a ***vector field.***

Vector Fields (2D & 3D)

A vector field on a domain in the plane or in space is a function that assigns a vector to each point in the domain. In general, a field of 2D vectors would look like this:

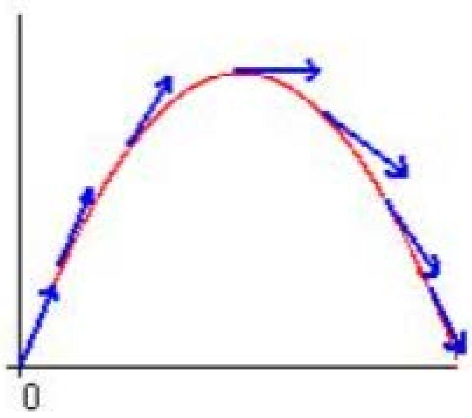
$$\mathbf{F} = \langle M(x, y), N(x, y) \rangle,$$

and a field of 3D vectors would look like this:

$$\mathbf{F} = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle.$$

If we can imagine attaching a projectile's velocity vector at each point on its trajectory in the plane of motion, then we have a 2D vector field defined on the trajectory as shown:

If we can imagine the gradient vector at each point on all level surfaces of a function of three variables, then we have a 3D vector field. Naturally, a gradient vector at each point of all level curves of a function of two variables creates a 2D vector field. Note that both of these would be extremely hard to draw.

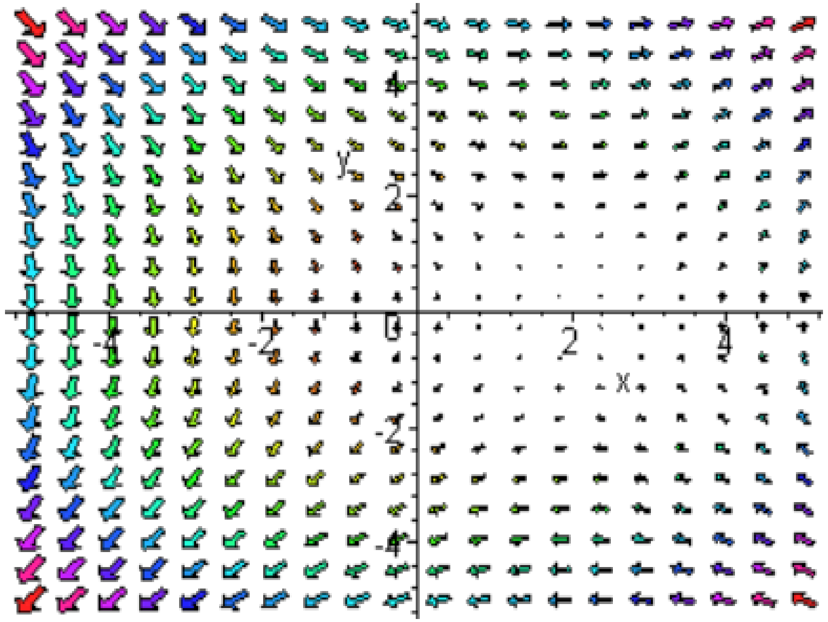


Gradient Vector Field

An important vector field that we have already encountered is the ***gradient vector field***. Let $f(x, y)$ be a differentiable function then the function that take a point (x_0, y_0) to $\nabla f(x_0, y_0)$ is a vector field since the gradient of a function at a point is a vector. For example, if $f(x, y) = 0.1xy - 0.2y$, then:

$$\nabla f = \langle 0.1y, 0.1x - 0.2 \rangle.$$

The sketch of the gradient is pictured below. The best way to sketch a vector field is to use a a computer, however, it is important to understand how they are sketched. We usually picture the gradient vector with its tail at (x, y), pointing in the direction of maximum increase. For example, we pick a point, say $(1,2)$ and plug it into the vector field: $\nabla f(1,2) = \langle 0.2, -0.1 \rangle$.Next, sketch the vector that begins at $(1,2)$ and ends at $(1 + 0.2, 2 - 0.1)$. Notice that when we sketch vector fields, we use the definition that involves two points rather than the definition that assumes all vectors emanate from the origin.



Inverse Square Field

In physics, many vector fields satisfy the ***inverse square law***. A vector field $\mathbf{F}$ satisfying the inverse square law has the property that if $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector for (x, y, z), and $\mathbf{u} = \mathbf{r}/|\mathbf{r}|$ denote the unit vector that has the same direction as $\mathbf{r}$, then a vector field $\mathbf{F}$ is an inverse square field if

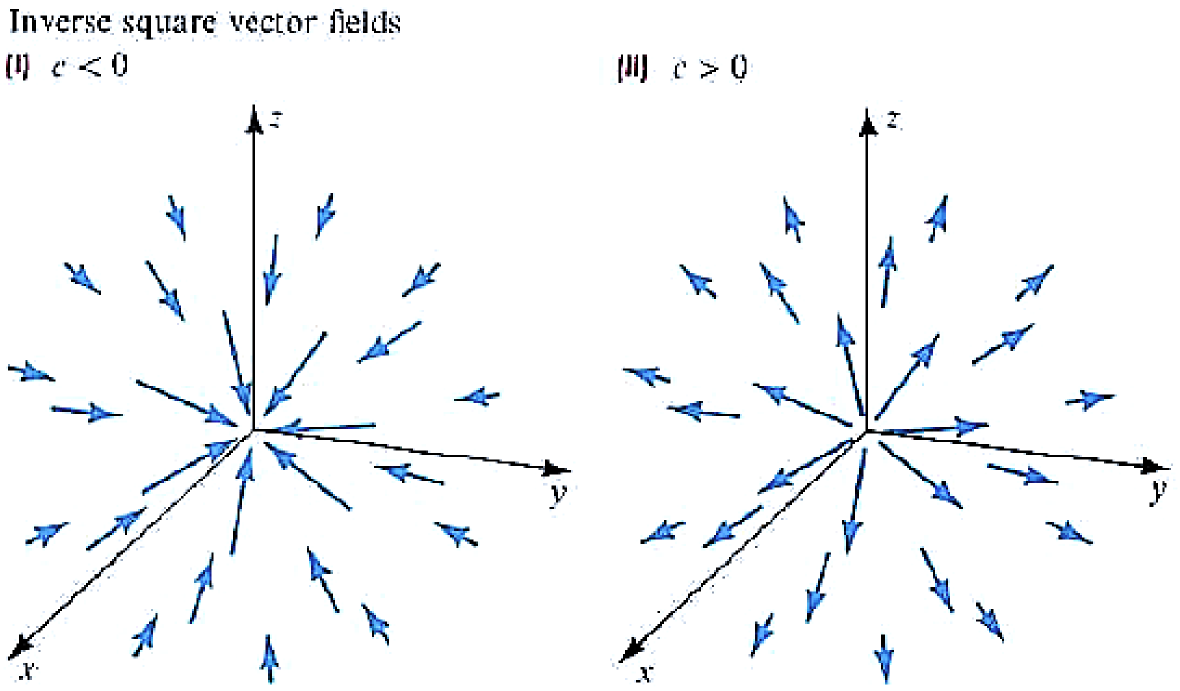
$$\mathbf{F}(x, y, z) = \frac{c}{|\mathbf{r}|^2}\mathbf{u} = \frac{c}{|\mathbf{r}|^3}\mathbf{r},$$

where c is a scalar.

Note:

In some books we have $|\mathbf{r}| = \|\mathbf{r}\|$.

Examples of force fields that satisfy the inverse square law are gravitational force fields and electric force fields. For gravity, this tells us that as we fly away from the earth, we experience less gravity, until it seems like weightlessness.



Example: 2D Vector Fields

Generate spectrum of the rotation vector field:

$$\mathbf{F}(x, y) = \langle -y, x \rangle.$$

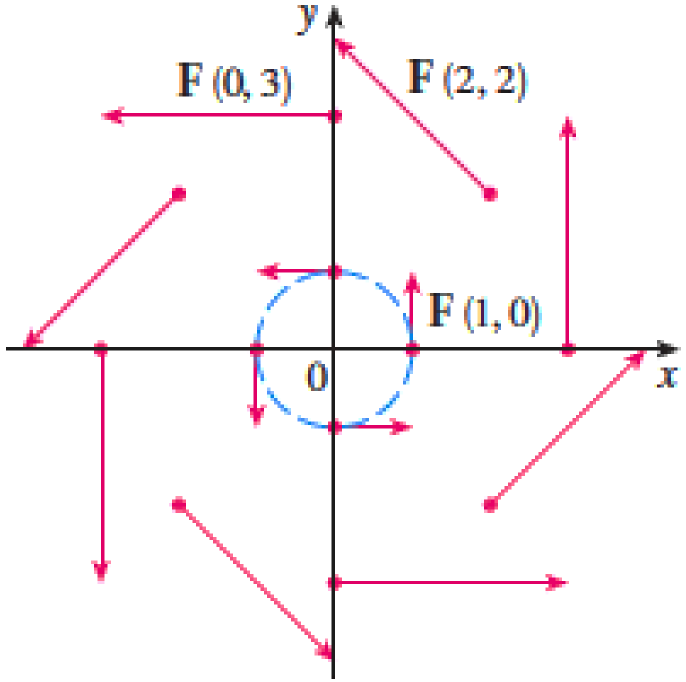
Solution:

At each of these points we find the images of the vector field which are given as:

$$\mathbf{F}(1,0) = \langle 0,1 \rangle; \quad \mathbf{F}(1,1) = \langle -1,1 \rangle; \quad \mathbf{F}(0,1) = \langle -1,0 \rangle.$$

Continuing in this way, we calculate several other representative values of $\mathbf{F}(x, y)$ in the table and draw the corresponding vectors to represent the vector field.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, 1 \rangle$	$(-1, 0)$	$\langle 0, -1 \rangle$
$(2, 2)$	$\langle -2, 2 \rangle$	$(-2, -2)$	$\langle 2, -2 \rangle$
$(3, 0)$	$\langle 0, 3 \rangle$	$(-3, 0)$	$\langle 0, -3 \rangle$
$(0, 1)$	$\langle -1, 0 \rangle$	$(0, -1)$	$\langle 1, 0 \rangle$
$(-2, 2)$	$\langle -2, -2 \rangle$	$(2, -2)$	$\langle 2, 2 \rangle$
$(0, 3)$	$\langle -3, 0 \rangle$	$(0, -3)$	$\langle 3, 0 \rangle$



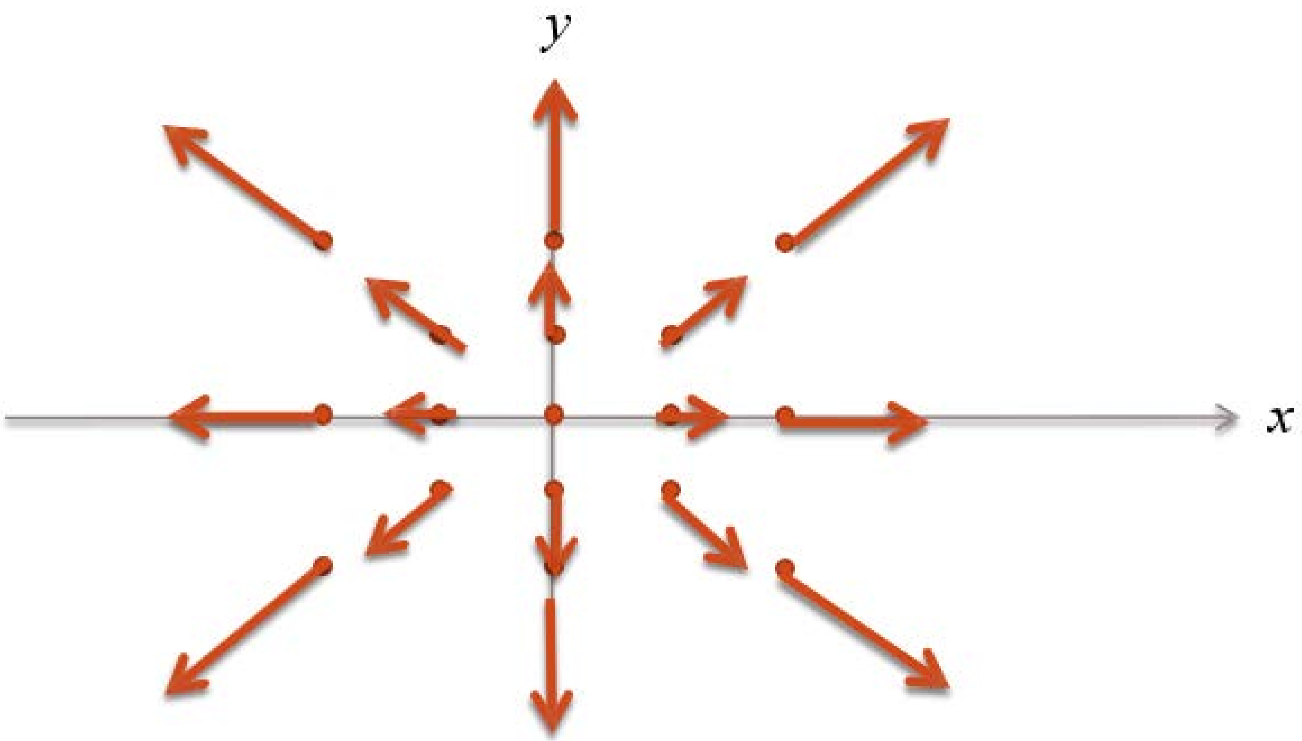
Example: 2D Vector Fields

Generate spectrum of the radial vector field:

$$\mathbf{F}(x, y) = \langle x, y \rangle.$$

Solution:

Following the same procedure, we can obtain a spectrum of vectors all emanating from the origin such that the field strength is increasing as we move away from origin.



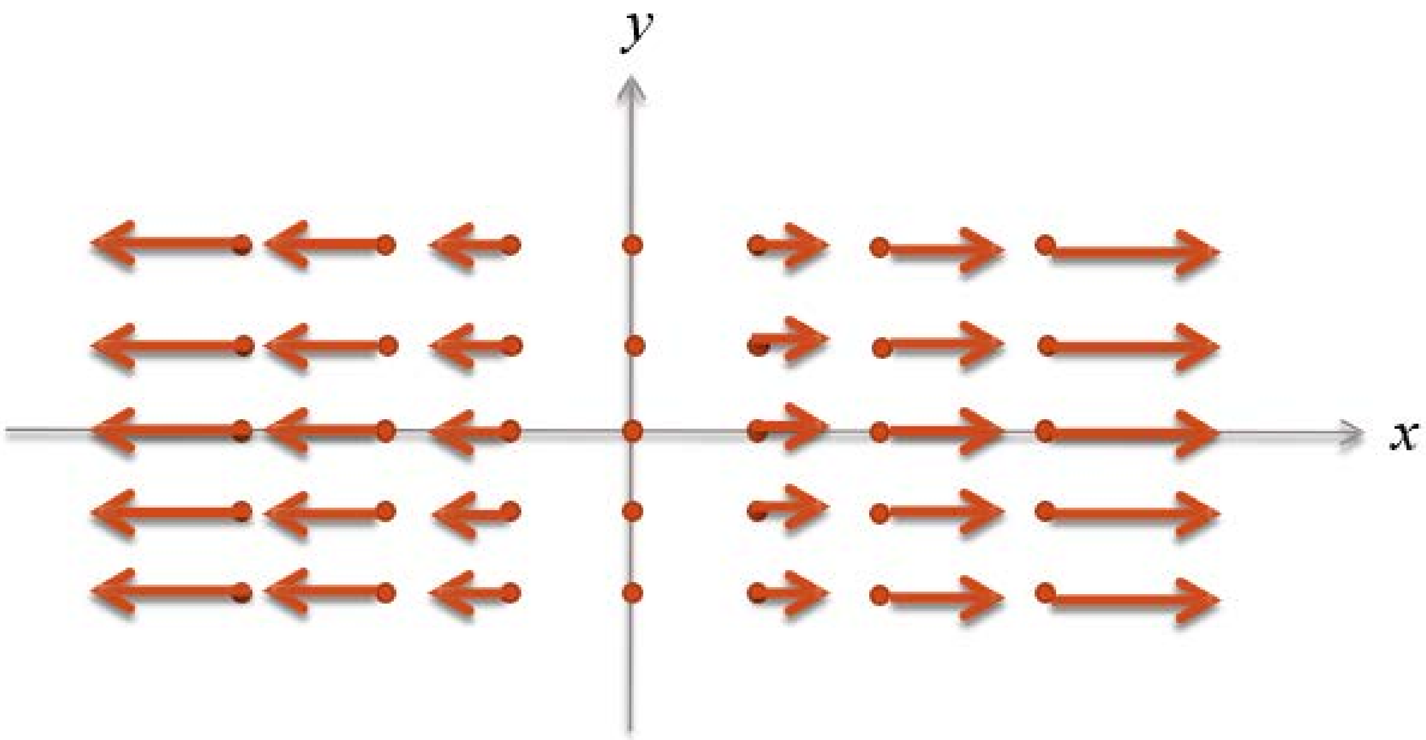
Example: 2D Vector Fields

Generate spectrum of the vector field:

$$\mathbf{F}(x, y) = \langle x, 0 \rangle.$$

Solution:

Following the same procedure, we can obtain a spectrum of vector field as:



Practice Questions: 2D Vector Fields

1. Generate the spectrum of the vector field: $\mathbf{F}(x, y) = \langle y, -x \rangle$.

2. Generate the spectrum of the vector field: $\mathbf{F}(x, y) = \langle x, \sin x \rangle$.

3. Generate the spectrum of the vector field: $\mathbf{F}(x, y) = \langle y, -1 \rangle$.

4. Generate the spectrum of the vector field: $\mathbf{F}(x, y) = \left\langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \right\rangle$.

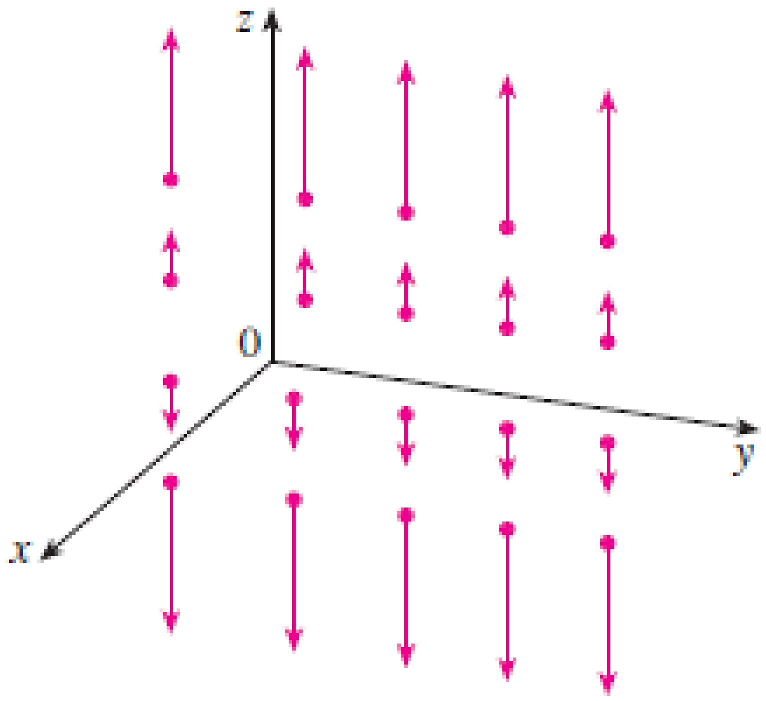
Example: 3D Vector Fields

Generate spectrum of the vector field:

$$\mathbf{F}(x, y, z) = \langle 0,0, z\rangle.$$

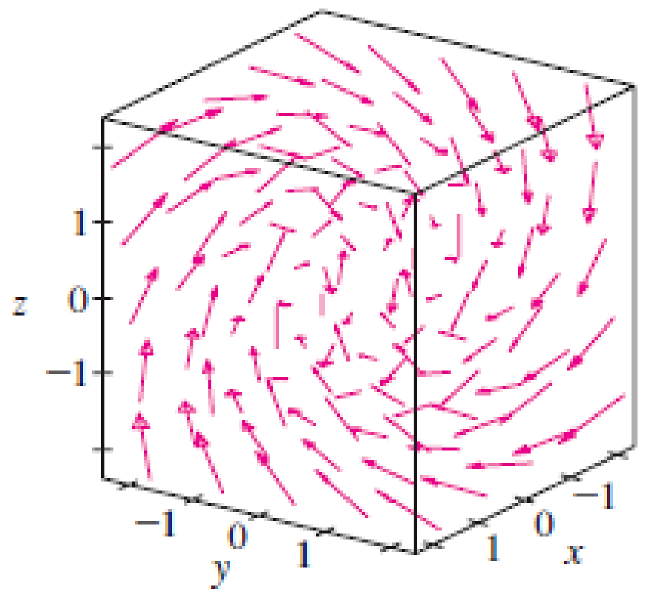
Solution:

The sketch is shown in the figure below. Notice that all vectors are vertical and point upward above the xy −plane or downward below it. The magnitude increases with the distance from the xy −plane.

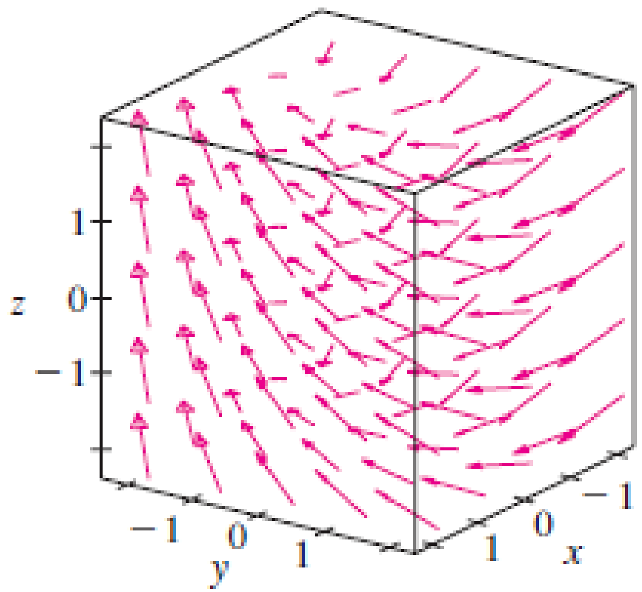


Example: 3D Vector Fields

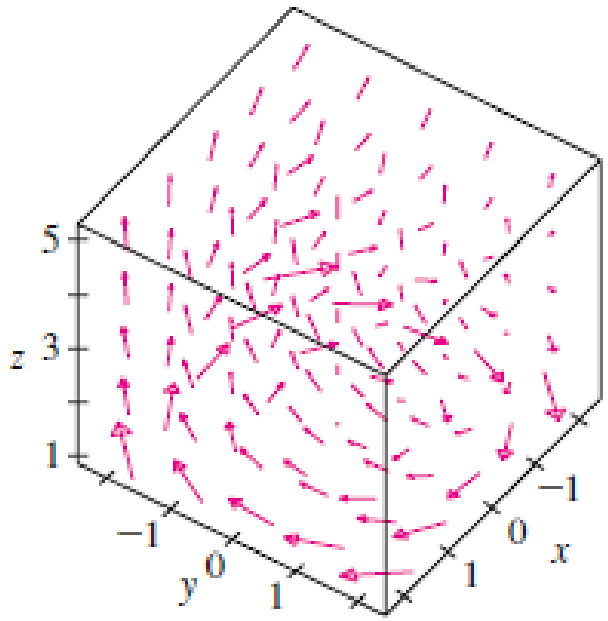
Note that we were able to draw the vector field by hand in previous example because of its particularly simple formula. Most three-dimensional vector fields, however, are virtually impossible to sketch by hand and so we need to resort to a computer algebra system.



$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$



$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2\,\mathbf{j} + x\,\mathbf{k}$



$\mathbf{F}(x, y, z) = \frac{y}{z}\,\mathbf{i} - \frac{x}{z}\,\mathbf{j} + \frac{z}{4}\,\mathbf{k}$