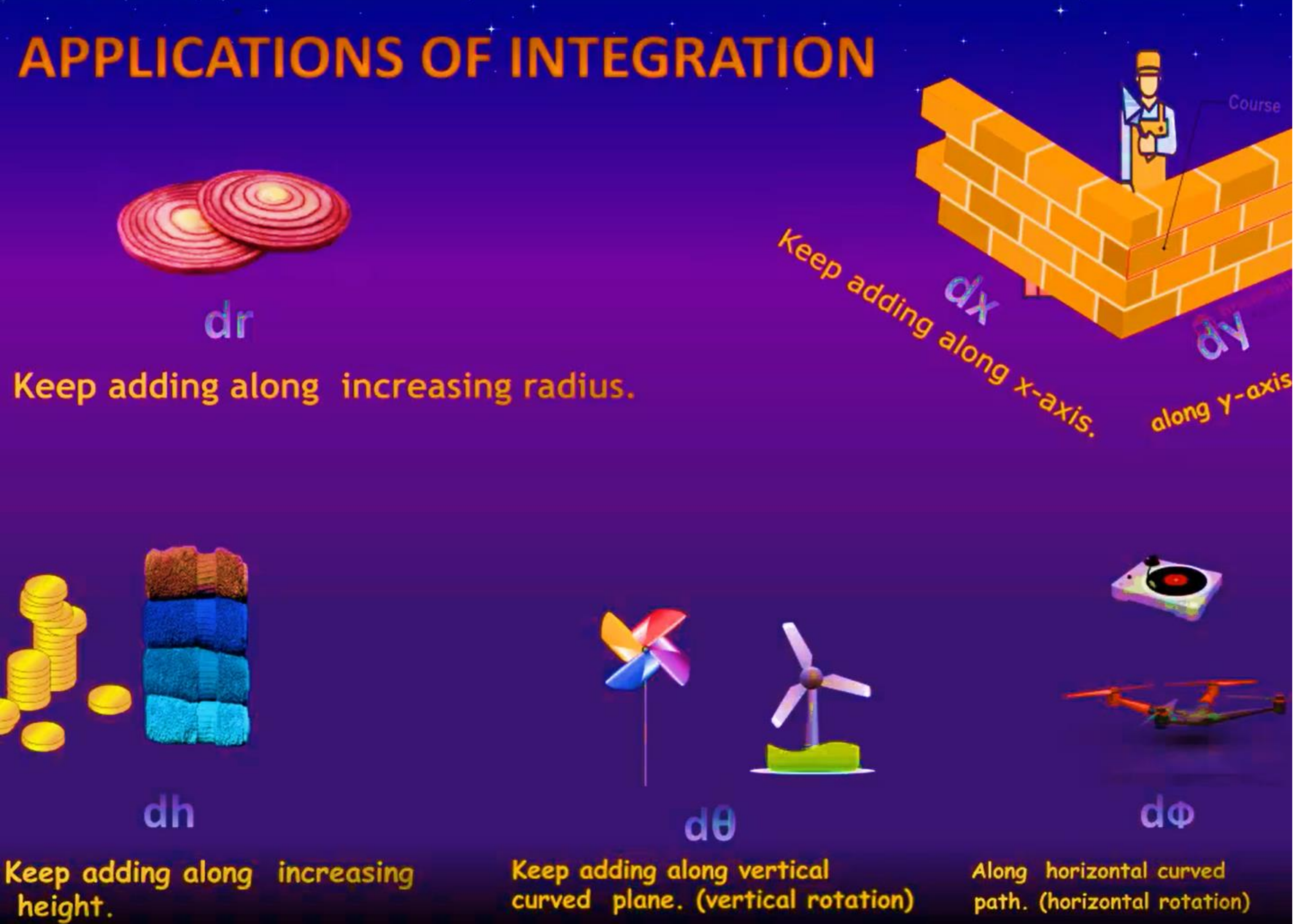


APPLICATIONS OF INTEGRATION

Calculus & Analytical Geometry MATH-101

Instructor: Dr. Naila Amir (SEECS, NUST)

APPLICATIONS OF INTEGRATION

- Our objective is to explore some of the applications of the definite integral by using it to compute areas between curves, volumes of solids, arclength of a curve and the work done by a varying force.

- The common theme is the following general method, which is similar to the one used to find areas under curves.

- We break up a quantity Q into a large number of small parts.

- Next, we approximate each small part by a quantity of the form $f(x_i{}^*)\Delta x$ and thus approximate Q by a Riemann sum.

- Then, we take the limit and express Q as an integral.

- Finally, we evaluate the integral.

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

6.1

Areas Between Curves

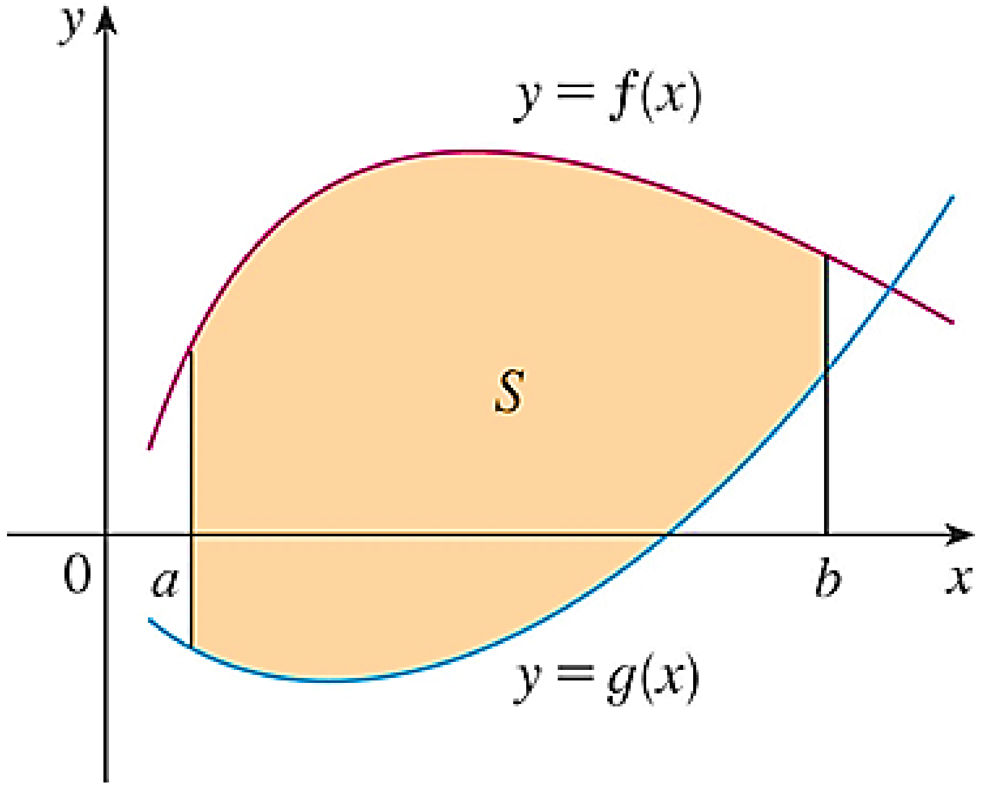
Our objective is to learn about:

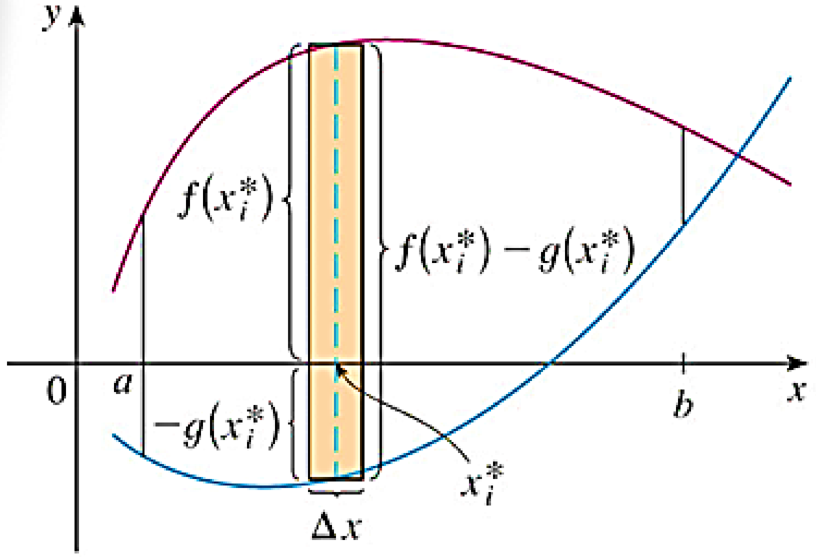
Using integrals to find areas of regions that lie

between the graphs of two functions.

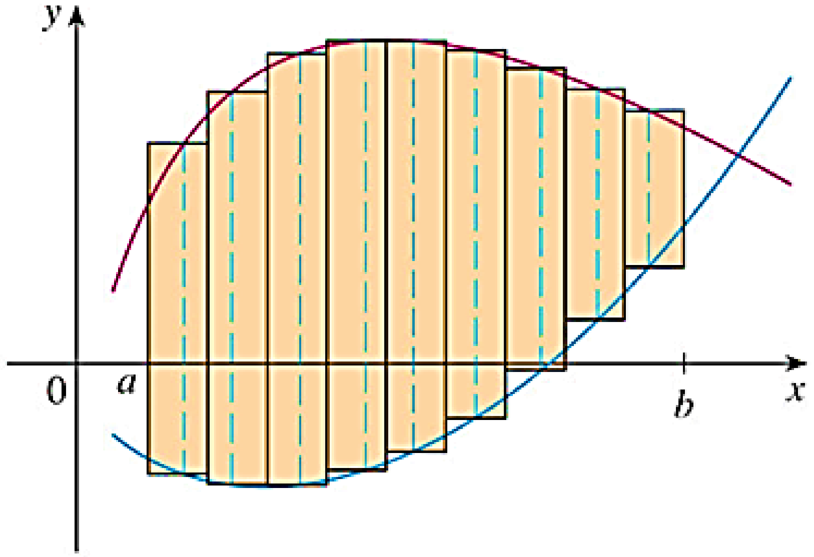
AREAS BETWEEN CURVES

- Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$. Here, f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$.
- As we did for areas under the curves, we divide S into n strips of equal width and approximate the ith strip by a rectangle with base Δx and height $f({x_i}^*) - g({x_i}^*)$.





(a) Typical rectangle



(b) Approximating rectangles

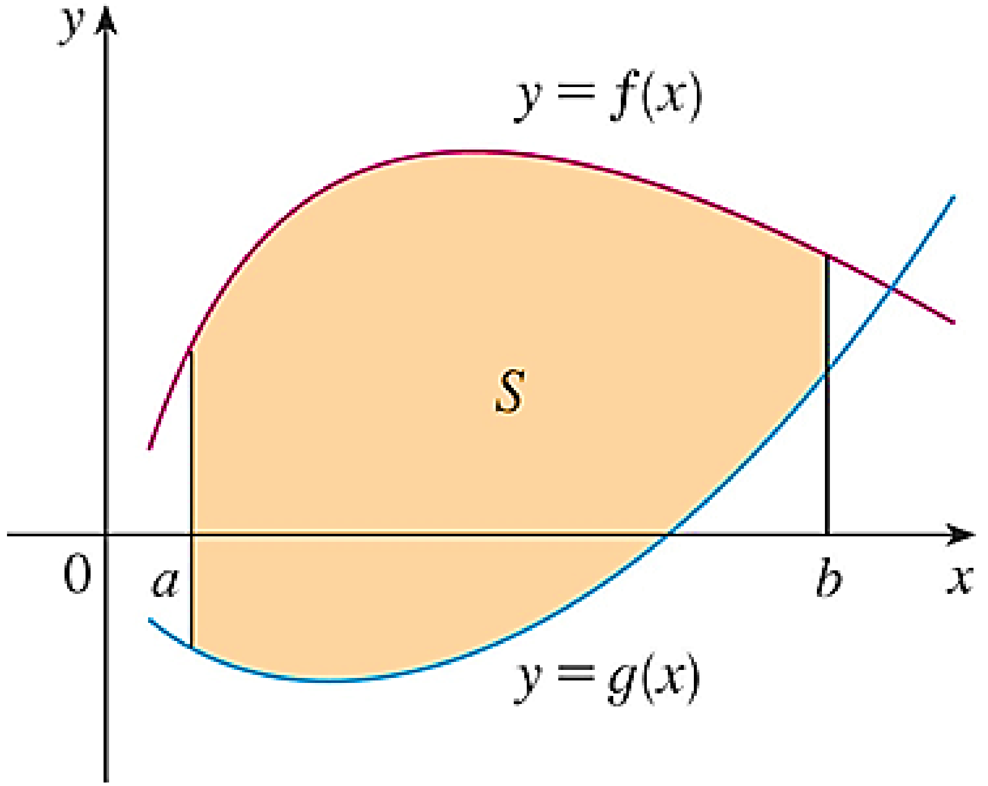
AREAS BETWEEN CURVES

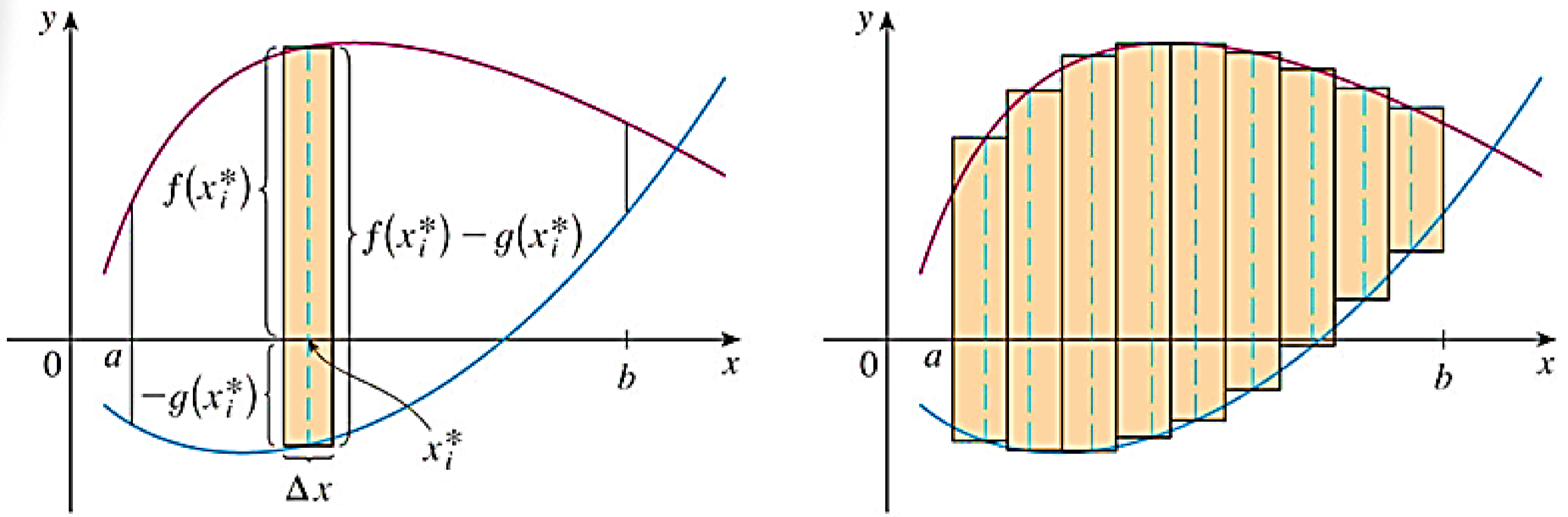
The Riemann sum

$$\sum_{i=1}^{n} [f(x_i{}^*) - g(x_i{}^*)]\Delta x\,,$$

is therefore an approximation to what we intuitively think of as the area of the region S. This approximation appears to become better and better as $n \to \infty$. We define the area A of the region S as:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i\,*) - g(x_i\,*)]\,\Delta x.$$





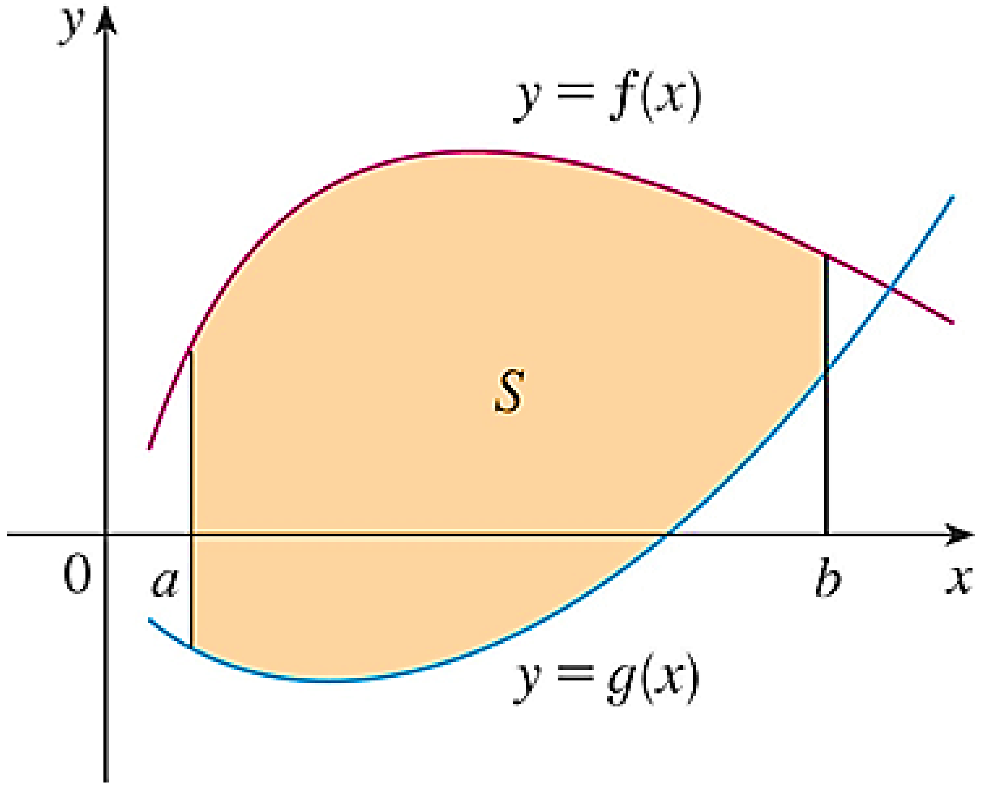
(a) Typical rectangle

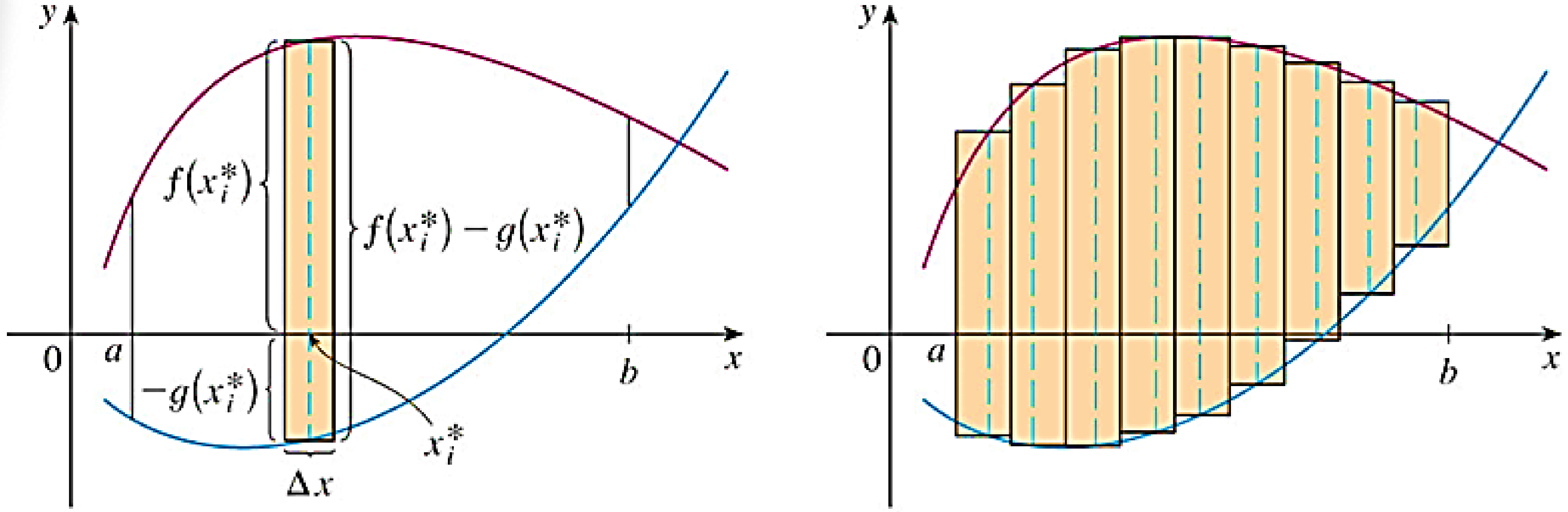
(b) Approximating rectangles

AREAS BETWEEN CURVES

Thus, the area A of the region bounded by the curves $y = f(x), y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is:

$$A = \int_a^b [f(x) - g(x)]\, dx$$





(a) Typical rectangle

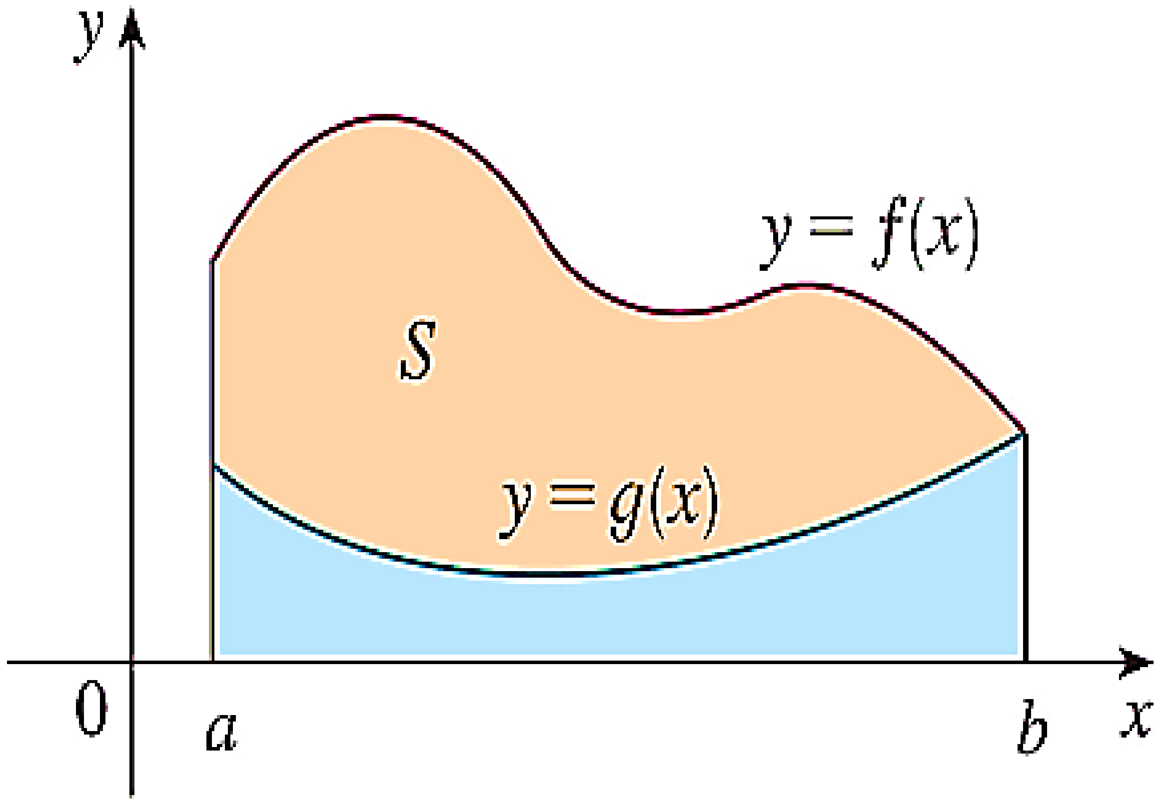
(b) Approximating rectangles

AREAS BETWEEN CURVES

$$A = [\text{area under } y = f(x)] - [\text{area under } y = g(x)]$$

$$= \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$= \int_a^b [f(x) - g(x)]\, dx.$$



Notice that, in the special case where $g(x) = 0$, S is the region under the graph of f and our general definition of area reduces to the definition of area under the curve.

Example: Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

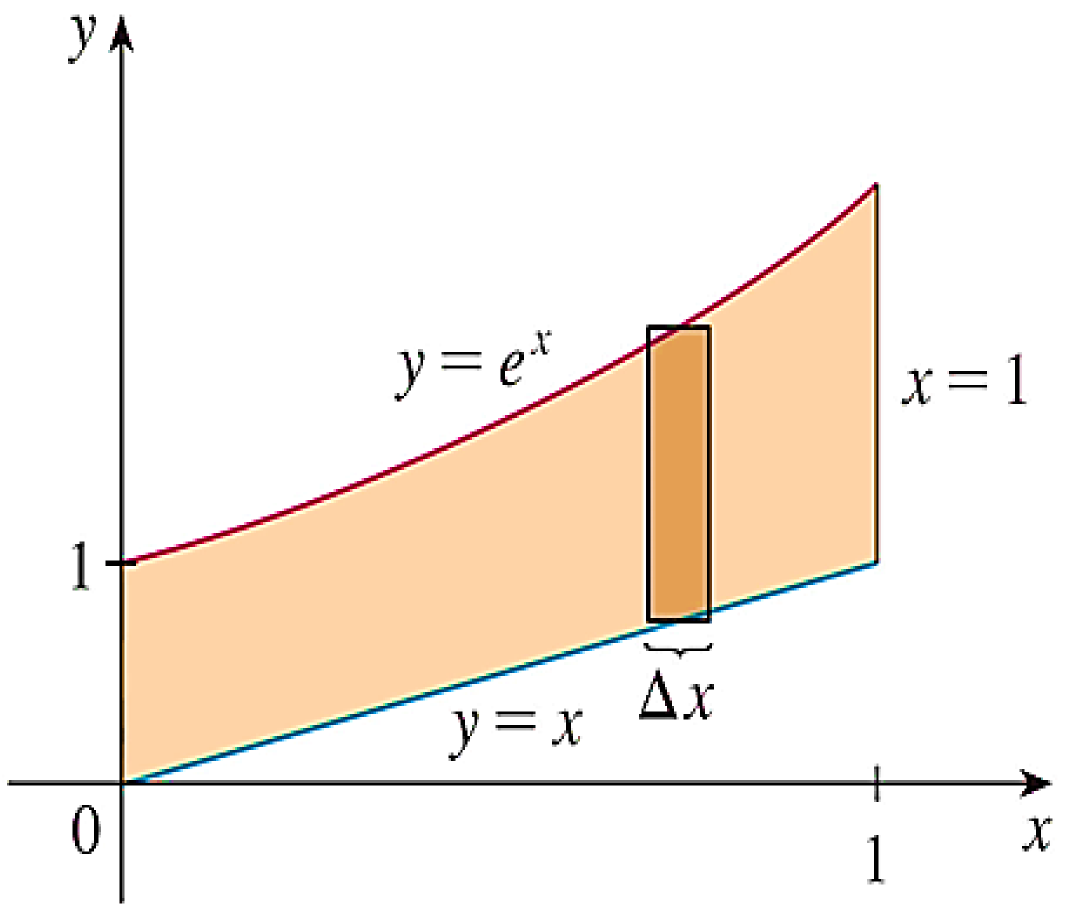
Solution:

For the present case, the upper boundary curve is $y = e^x$ and the lower boundary curve is $y = x$. So, we use the area formula with $f(x) = e^x$, $g(x) = x$, $a = 0$, and $b = 1$:

$$A = \int_0^1 (e^x - x)dx = e^x - \frac{1}{2}x^2\Big]_0^1$$

$$= e - \frac{1}{2} - 1$$

$$= e - 1.5.$$



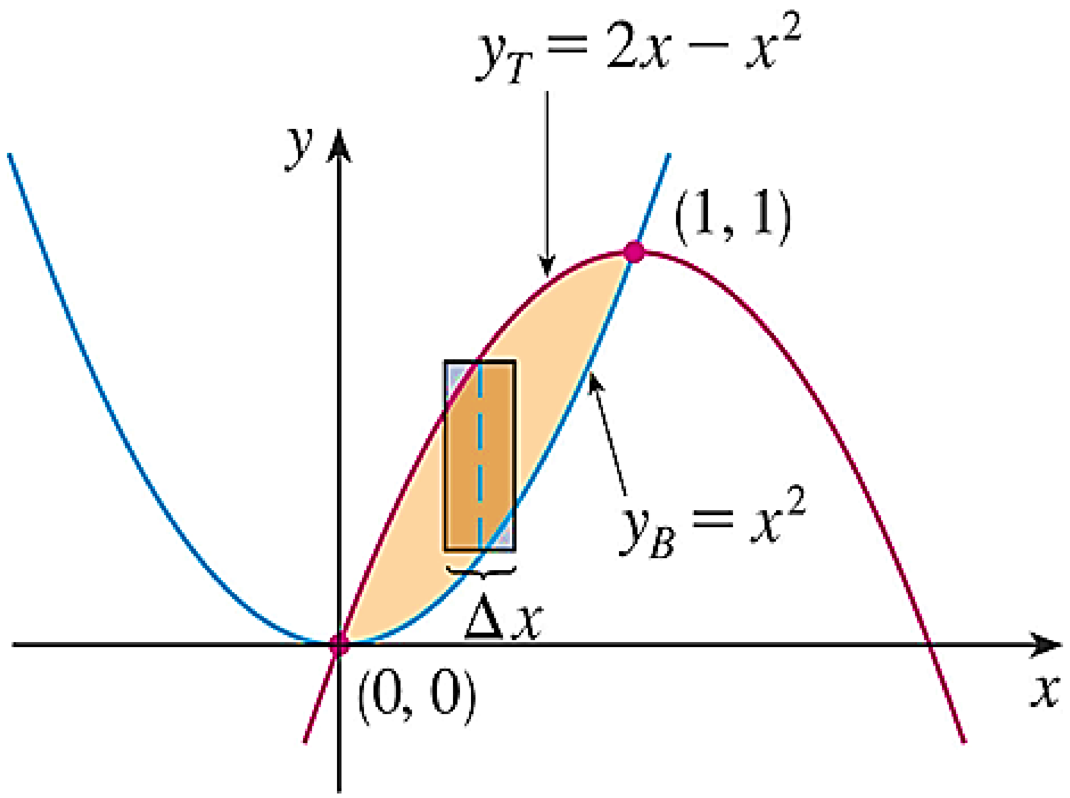
Example: Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Solution:

First, we find the points of intersection of the parabolas by solving their equations simultaneously. This gives:

$$x^2 = 2x - x^2, \quad \text{or} \quad 2x^2 - 2x = 0.$$

Thus, $2x(x - 1) = 0$, so $x = 0$ or 1. The points of intersection are: $(0, 0)$ and $(1, 1)$. For the present case, the top and bottom boundaries are: $y_T = 2x - x^2$ and $y_B = x^2$ respectively. So, the total area is:

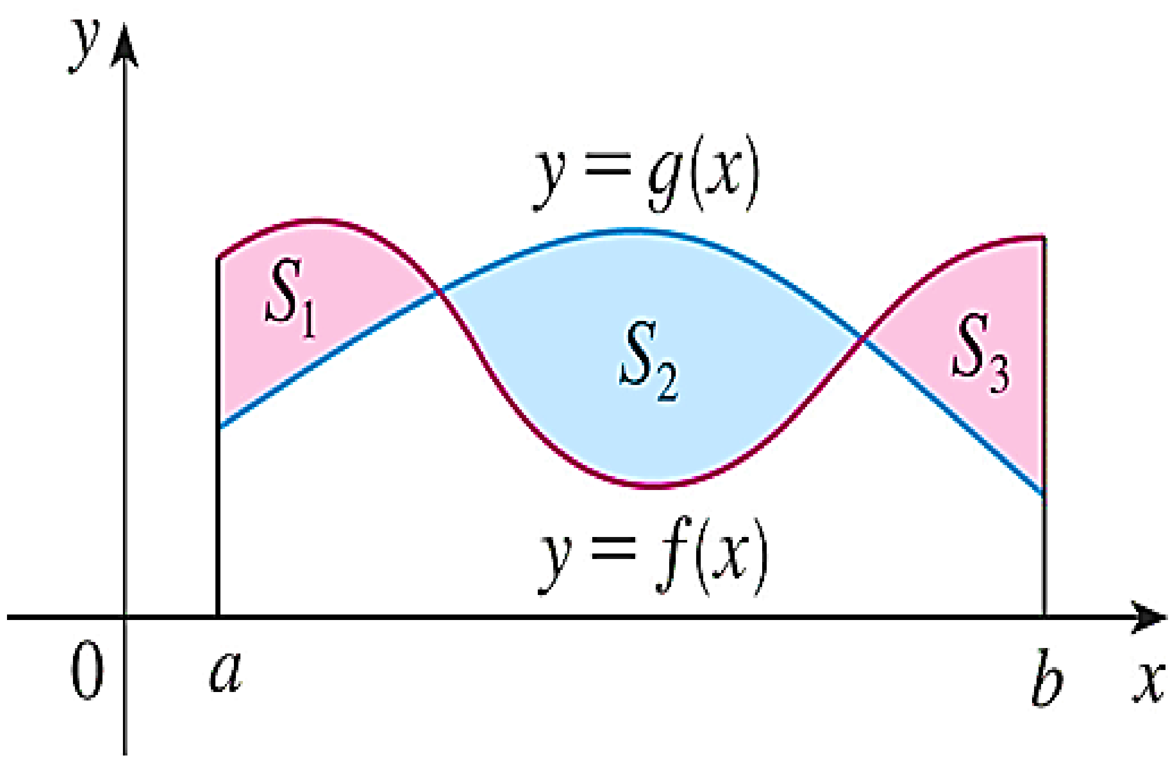


$$A = \int_0^1 [(2x - x^2) - x^2]\,dx = \int_0^1 (2x - 2x^2)\,dx$$

$$= 2\int_0^1 (x - x^2)\,dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.$$

AREAS BETWEEN CURVES

- To find the area between the curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values of x, split the given region S into several regions $S_1, S_2, \ldots$ with areas $A_1, A_2, \ldots$.



- Then, we define the area of the region S to be the sum of the areas of the smaller regions $S_1, S_2, \ldots$, that is:

$$A = A_1 + A_2 +, \ldots$$

Example: Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x, x = 0$, and $x = \pi/2$.

Solution:

The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi / 4$ (since $0 \leq x \leq \pi / 2$).

Observe that $\cos x \geq \sin x$ when $0 \leq x \leq \pi/4$ but $\sin x \geq \cos x$ when $\pi/4 \leq x \leq \pi/2$.

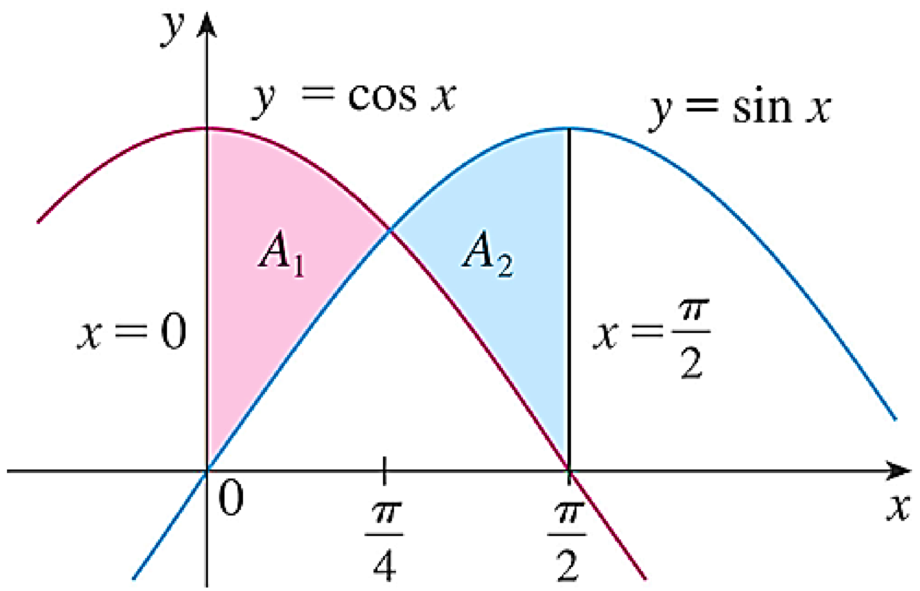
So, the required area is:

$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\, dx$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= 2\sqrt{2} - 2$$



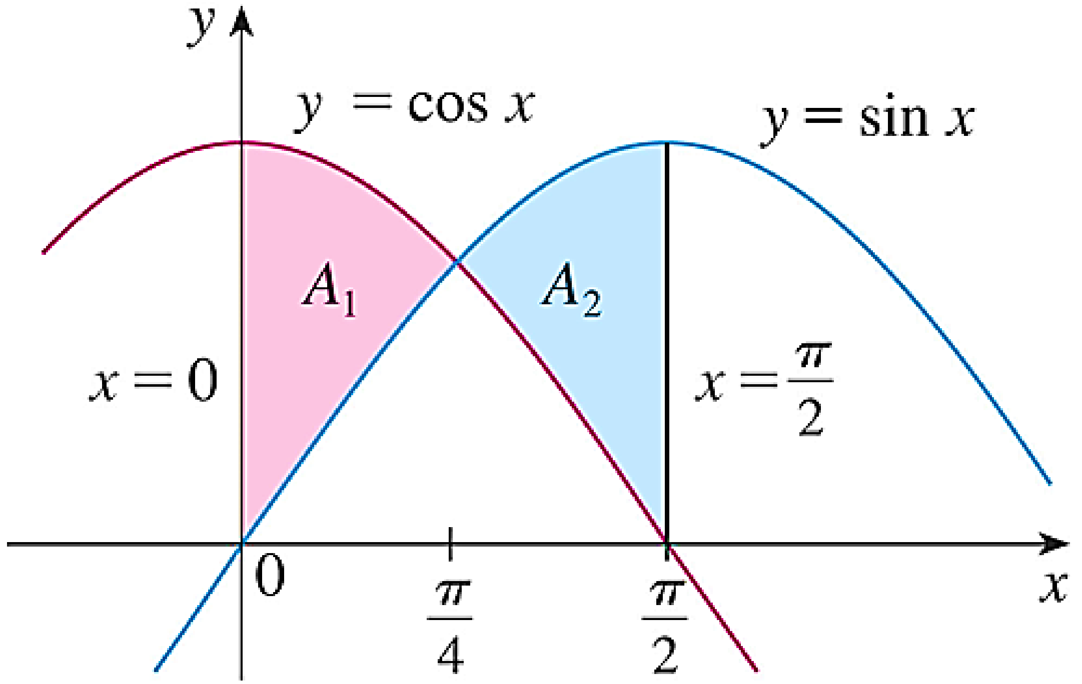
Example: Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x, x = 0$, and $x = \pi/2$.

Alternative Solution:

We could have saved some work by noticing that the region is symmetric about $x = \pi / 4$.

So, the area of the required region is given as:

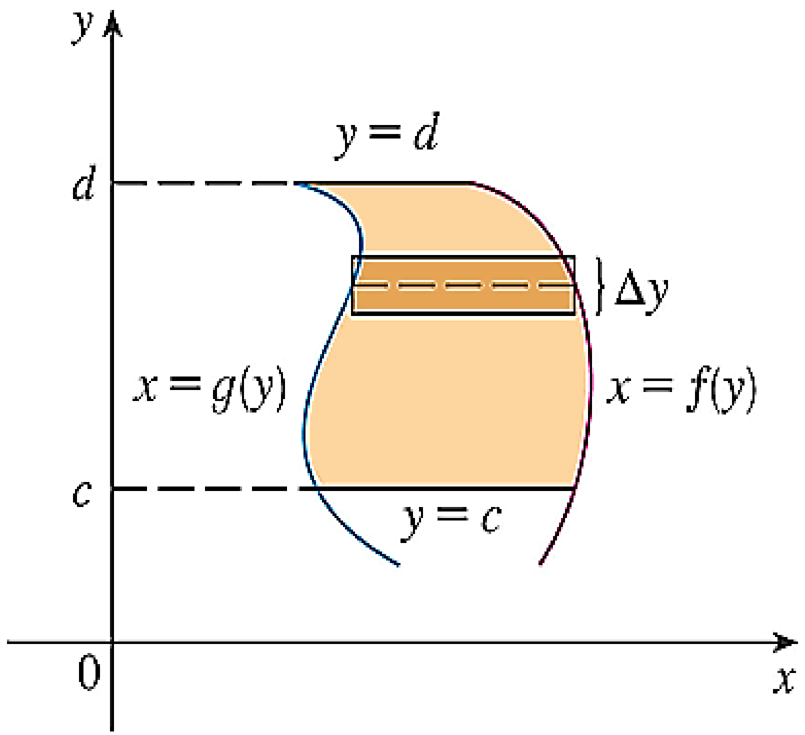
$$A = 2\int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx\,.$$



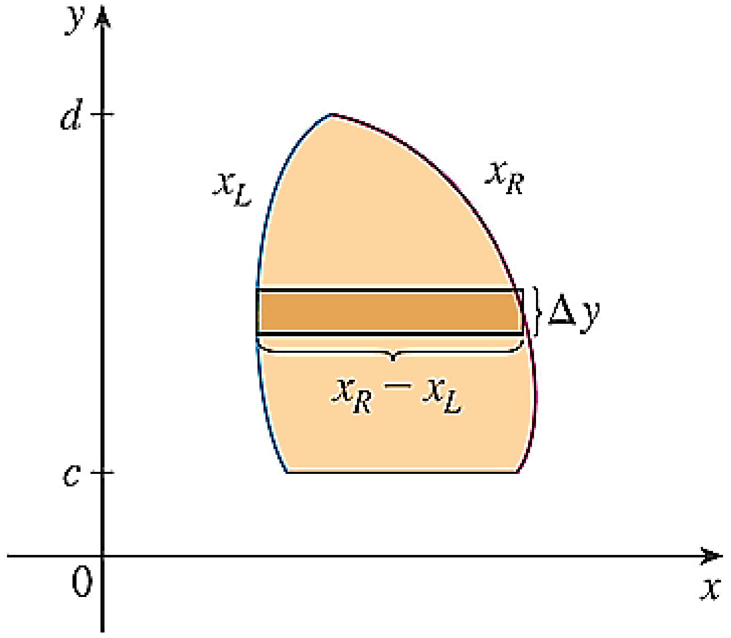
AREAS BETWEEN CURVES

Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations: $x = f(y), x = g(y), y = c$, and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$, then its area is:

$$A = \int_c^d [f(y) - g(y)]dy\,.$$



If we write x_R for the right boundary and x_L for the left boundary, we have:



$$A = \int_c^d (x_R - x_L)dy\,.$$

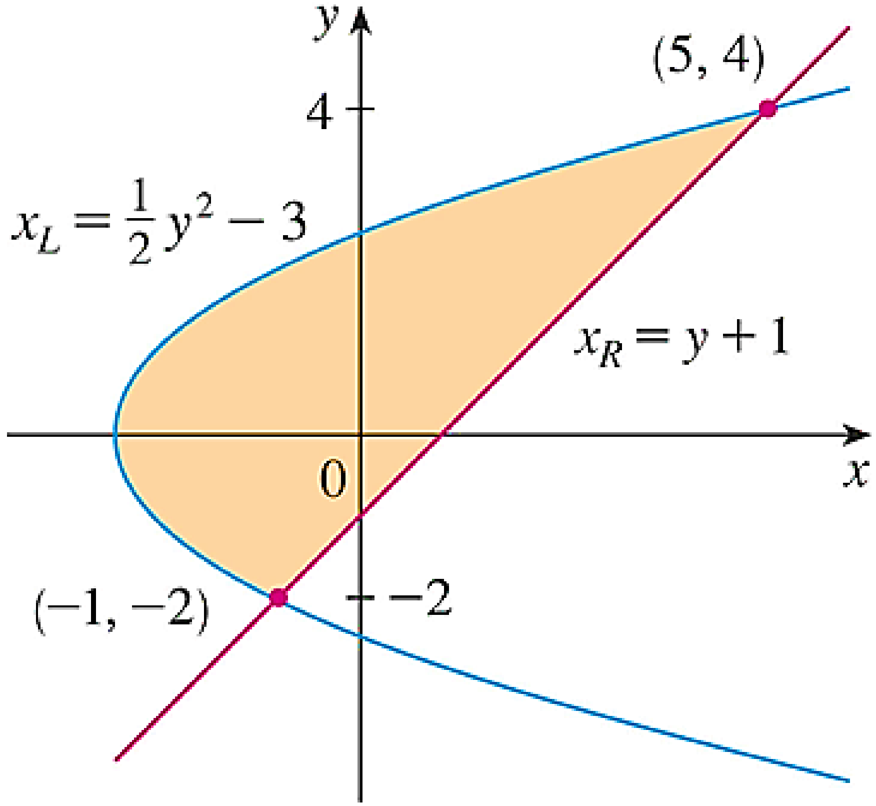
Example: Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution:

By solving the two equations, we find that the points of intersection are $(-1, -2)$ and $(5,4)$. We solve the equation of the parabola for x. Note that, the left and right boundary curves are:

$$x_L = \frac{1}{2}y^2 - 3 \quad \text{and} \quad x_R = y + 1,$$

respectively. We must integrate between the appropriate y −values, $y = -2$ and $y = 4$. Thus,



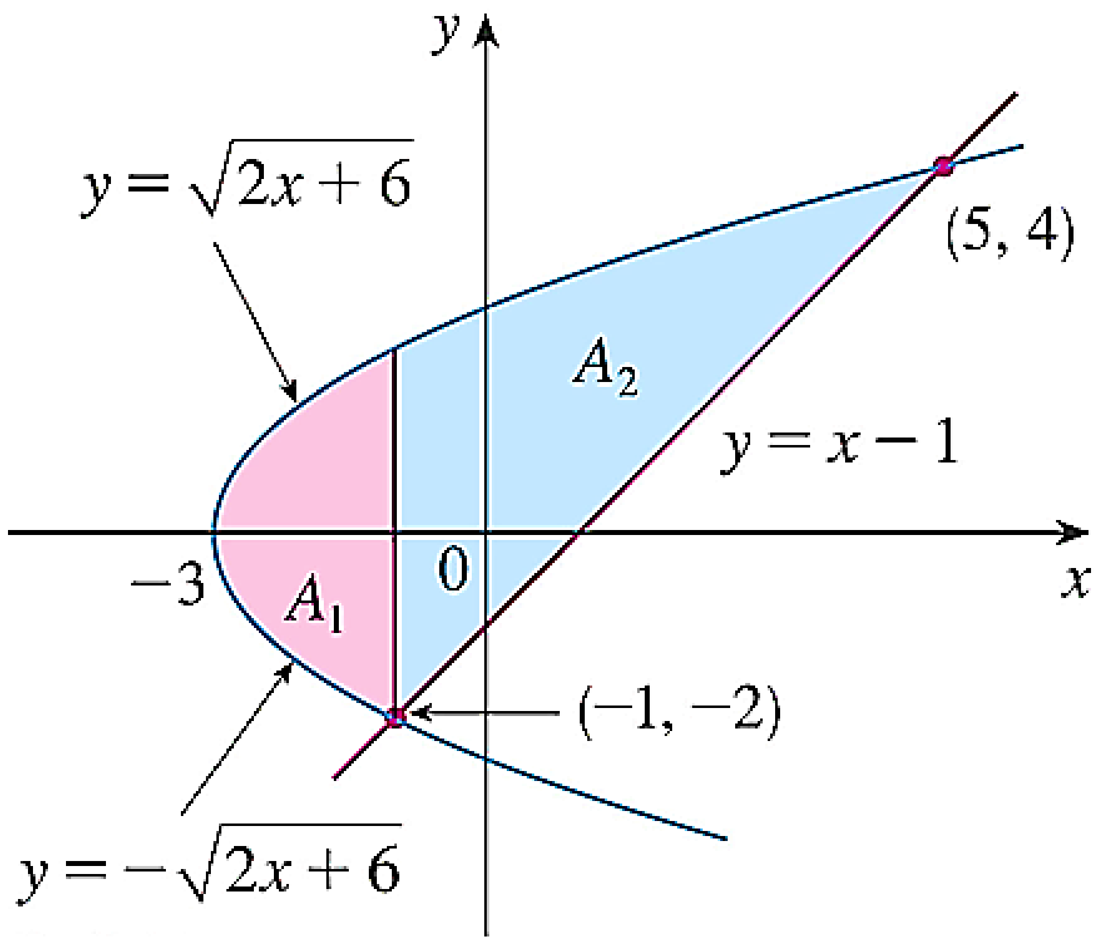
$$A = \int_{-2}^{4} \left[(y + 1) - \left(\frac{1}{2}y^2 - 3\right)\right] dy = \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4\right) dy$$

$$= -\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y\Big]_{-2}^{4} = -\frac{1}{6}(64) + 8 + 16 - \left(\frac{4}{3} + 2 - 8\right) = 18.$$

Example: Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Alternative Solution:

- We could have found the area by integrating with respect to x instead of y.
- However, the calculation is much more involved. It would have meant splitting the region in two and computing the areas labeled A_1 and A_2.
- The method used earlier was much easier.



The following guidelines may be helpful when working problems:

- Sketch the region, labeling the upper boundary $y = f(x)$ and the lower boundary $y = g(x)$. Find the smallest value $x = a$ and the largest value $x = b$ for points (x, y) in the region.
- Decide whether we will work in vertical or horizontal distances. Use the one that is easiest for the problem.
- Distance is always positive, remember to subtract the smaller value from the larger one, whether using x or y.

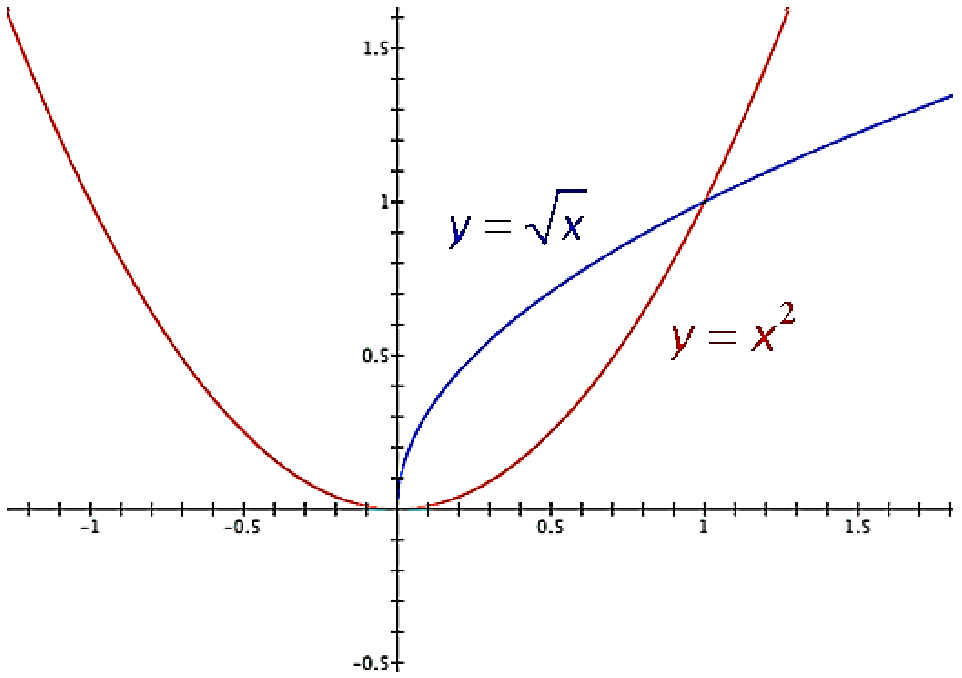
Example: Find the area of the region enclosed by the curves $y = x^2$ and $y = \sqrt{x}$.

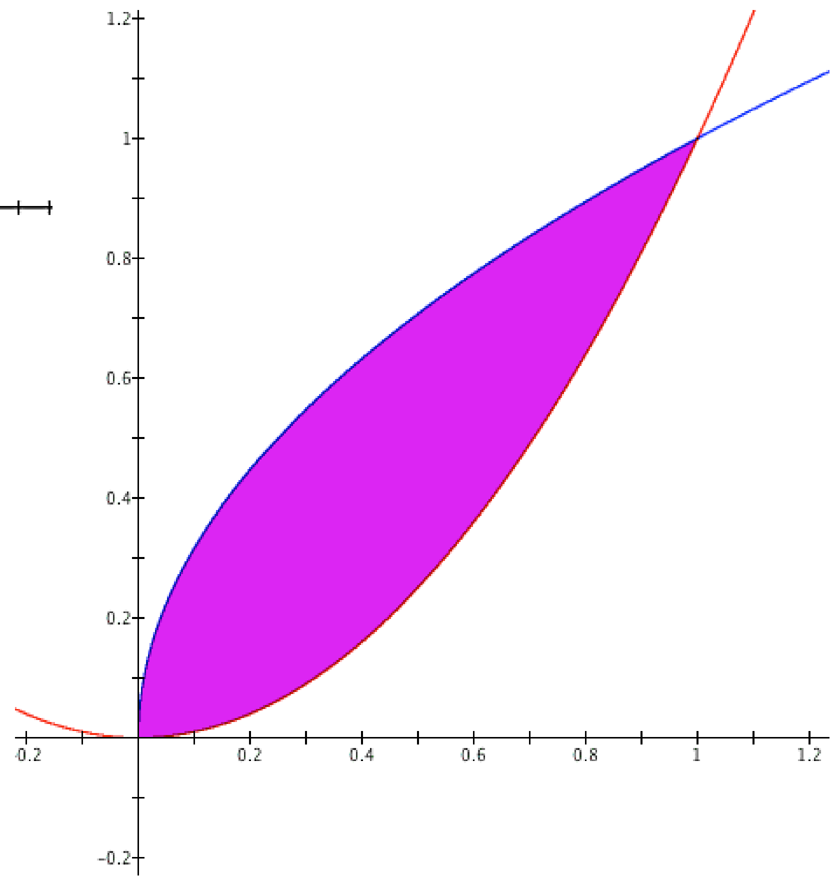
Solution:

First, we need to graph the given functions. We are interested in finding the area of the purple region. Note that the two curves intersect at $(0,0)$ and $(1,1)$. We can evaluate the area between the curves as:

$$\int_0^1 (\sqrt{x} - x^2)\, dx = \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right)\Bigg|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} - (0 - 0) = \frac{1}{3}.$$



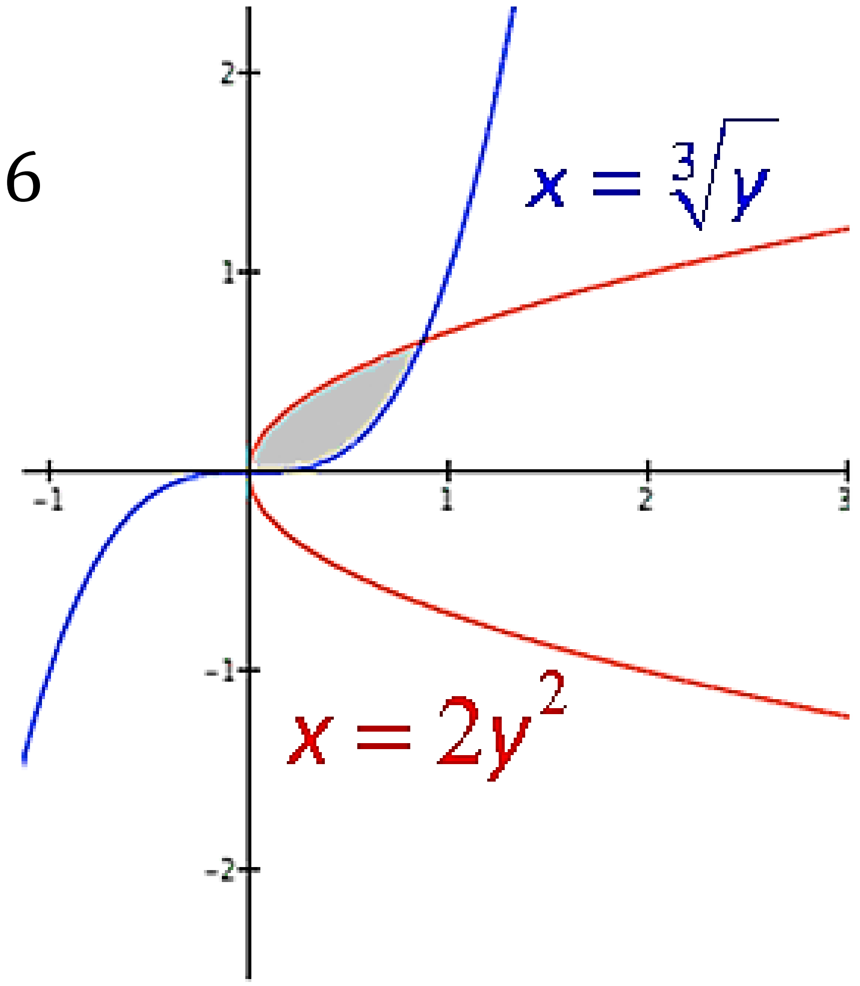


Example: Find the area of the region enclosed by the curves: $f(y) = x = \sqrt[3]{y}$ and $g(y) = x = 2y^2$.

Solution:

We see that the origin is one point of intersection. We need to find the other point of intersection. For that let us consider:

$$\sqrt[3]{y} = 2y^2 \Longrightarrow y = 8y^6 \Longrightarrow y^5 = \frac{1}{8} \Longrightarrow y = \left(\frac{1}{8}\right)^{1/5} \approx 0.66$$



Thus, the required area is given as:

$$A = \int_0^{(1/8)^{1/5}} (\sqrt[3]{y} - 2y^2)\, dy \approx 0.239.$$

Practice Questions

1. Find the area of the region bounded by the graphs of $y = x^2 + 1$ and $y = x^3$ and the vertical lines $x = -1$ and $x = 1$. $\left(Ans\text{: } \frac{8}{3}\right)$
2. Find the area of the region enclosed by the graphs of $2y^2 = x + 4$ and $y^2 = x$. $\left(Ans\text{: } \frac{32}{3}\right)$
3. Find the area of the region enclosed by the graphs of $y = x^3$ and $y = x$. $\left(Ans\text{: } \frac{1}{2}\right)$
4. Find the area of the region enclosed by the graphs of $y = 8 - x^2, y = 7x,$ and $y = 2x$ in the first quadrant. $\left(Ans\text{: } \frac{31}{6}\right)$

Practice Questions

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- **Exercise:** 6.1

    Q # 1 to Q # 36.

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

6.3

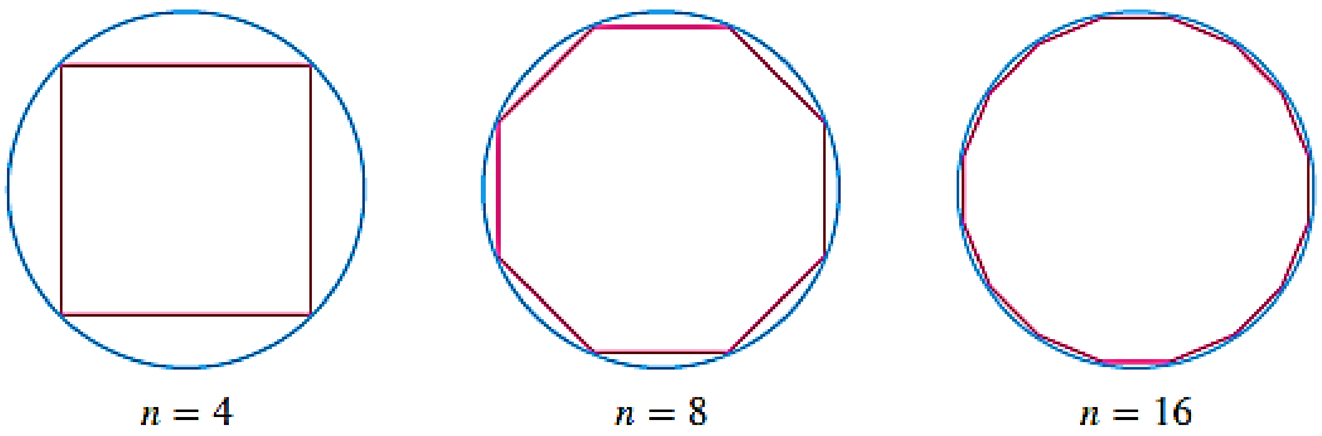
Arc Length of Plane curves

Our objective is to find:

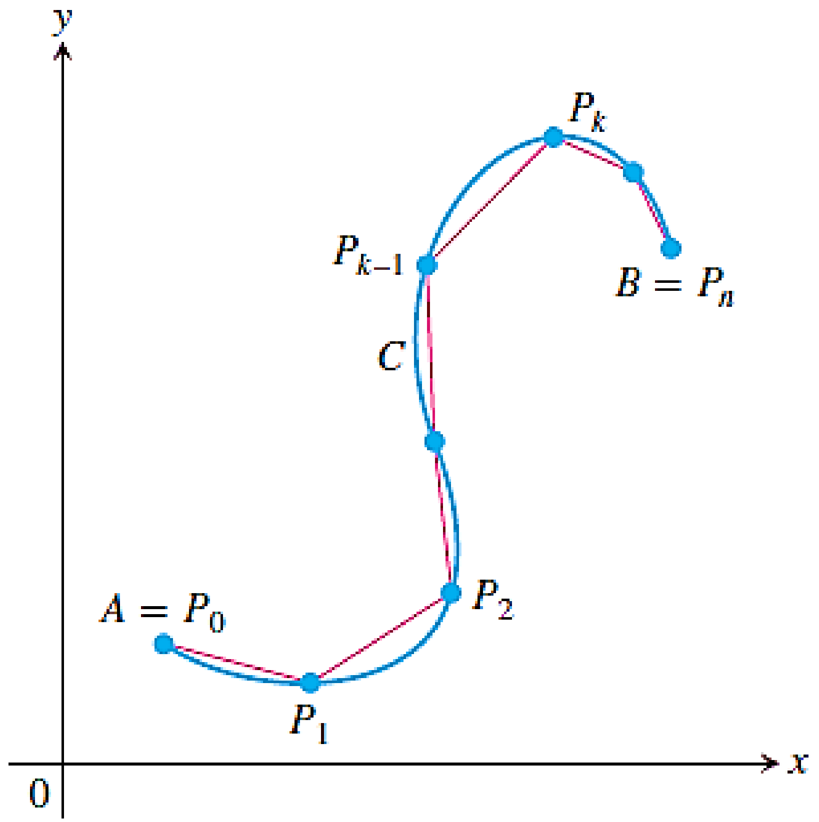
the arc length of a plane curve and

the arc length of a parametric curve.

Arc Length



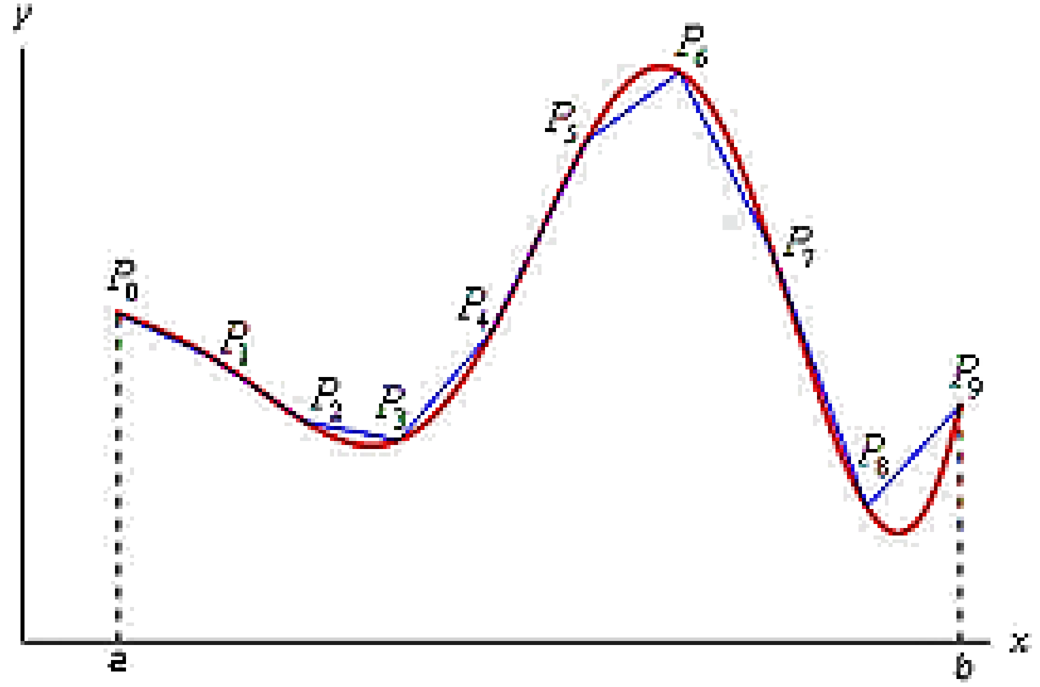
- We know what is meant by the length of a straight line segment, but without calculus, we have no precise notion of the length of a general winding curve.
- The idea of approximating the length of a curve running from point *A* to point *B* by subdividing the curve into many pieces and joining successive points of division by straight line segments dates back to the ancient Greeks.
- Archimedes used this method to approximate the circumference of a circle by inscribing a polygon of n sides and then using geometry to compute its perimeter.
- The extension of this idea to a more general curve is displayed in the accompanying figure.



Arc Length of a Plane Curve

If $f(x)$ is continuously differentiable on the closed interval $[a, b]$, the length of the curve $y = f(x)$ from $x = a$ to $x = b$ is given as:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$



Example: Determine the length of the curve: $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1; 0 \leq x \leq 1.$

Solution:

Given that: $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1 \Longrightarrow \frac{dy}{dx} = 2\sqrt{2}x^{1/2} \Longrightarrow \left(\frac{dy}{dx}\right)^2 = 8x.$

Thus, the length of the curve from $x = 0$ to $x = 1$ is given as:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx = \frac{2}{3} . \frac{1}{8}(1 + 8x)^{3/2}\Big|_0^1 = \frac{13}{6}.$$

Dealing with Discontinuities in dy/dx

At a point on a curve where dy/dx fails to exist, dx/dy may exist and we may be able to find the curve's length by expressing x as a function of y.

If $g(y)$ is continuously differentiable on the closed interval $[c, d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is given as:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Example: Determine the length of the curve: $y = \left(\frac{x}{2}\right)^{2/3}$; $0 \le x \le 2$.

Solution:

$$y = \left(\frac{x}{2}\right)^{2/3} \Rightarrow \frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

Note that $\frac{dy}{dx}$ is not defined at $x = 0$, so we cannot find the length of the curve by using $\frac{dy}{dx}$. We therefore rewrite the equation to express x in terms of y as:

$$y = \left(\frac{x}{2}\right)^{2/3} \Rightarrow x = 2y^{3/2}; \qquad 0 \le y \le 1.$$

Note that the derivative

$$\frac{dx}{dy} = 3\sqrt{y},$$

is continuous on $[0,1]$.

Example: Determine the length of the curve: $y = \left(\frac{x}{2}\right)^{2/3}$; $0 \leq x \leq 2.$

Solution:

Thus, the length of the curve from $y = 0$ to $y = 1$ is given as:

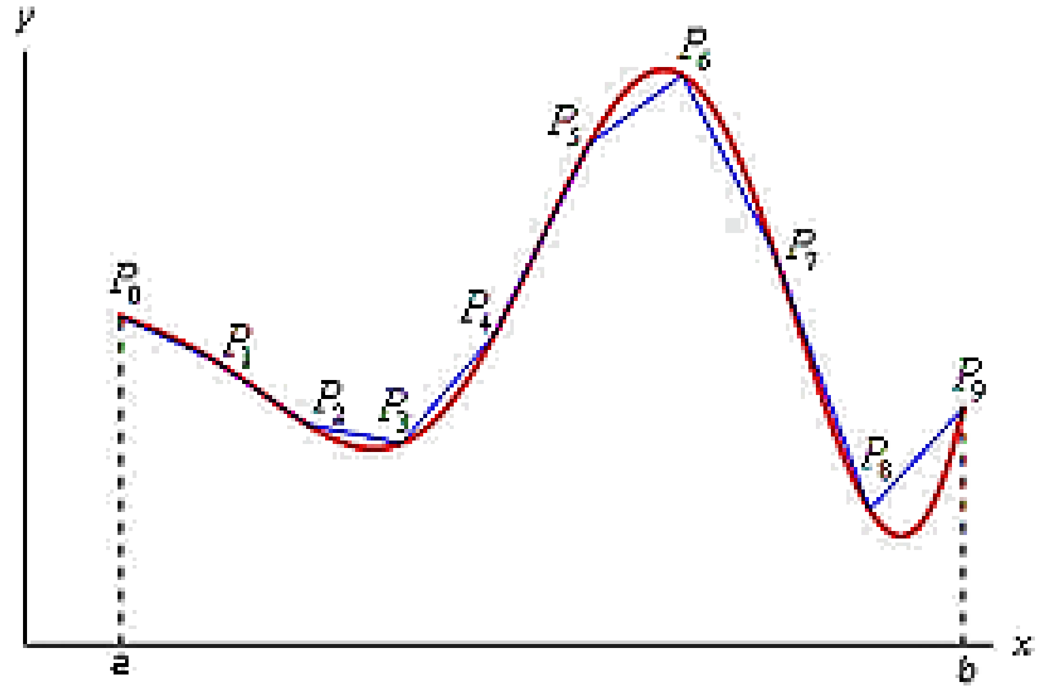
$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dx = \frac{1}{9} \cdot \frac{2}{3} (1 + 9y)^{3/2} \Big|_0^1$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \approx 2.27.$$

Arc Length of a Parametric Curve

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$; $a \leq t \leq b$, where $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then the length of the C is given as:

$$L = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$



Example: Find the length of the circle of radius *r* defined parametrically by:

$$x = r\cos t \quad \text{and} \quad y = r\sin t\,; \quad 0 \le t \le 2\pi.$$

Solution:

As t varies from 0 to 2π, the circle is traversed exactly once, so the circumference is given as:

$$L = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\,dt = \int_0^{2\pi} \sqrt{[-r\sin t]^2 + [r\cos t]^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{r^2[\sin^2 t + \cos^2 t]}\,dt = \int_0^{2\pi} r\,dt = r\,t\Big|_0^{2\pi} = 2\pi r.$$

Example: Find the length of the astroid defined parametrically by:

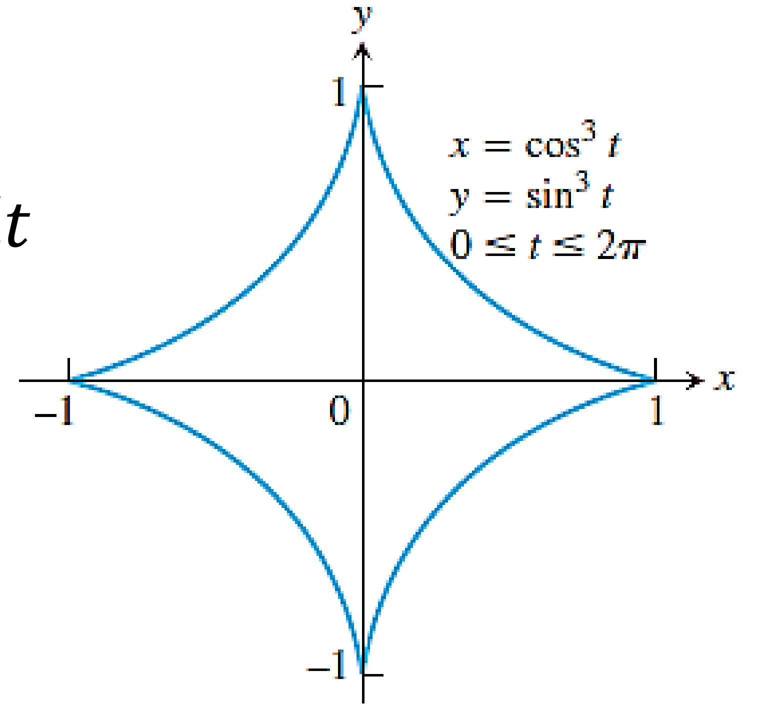
$$x = \cos^3 t \ \text{ and } \ y = \sin^3 t\,; \quad 0 \le t \le 2\pi.$$

Solution: Given that: $x = \cos^3 t$ and $y = \sin^3 t \Longrightarrow \frac{dx}{dt} = -3\cos^2 t \sin t$ and $\frac{dy}{dt} = 3\sin^2 t \cos t$. Thus, the length of the portion of astroid in first quadrant is given as:

$$L = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt = \int_0^{\pi/2} \sqrt{[-3\cos^2 t \sin t]^2 + [3\sin^2 t \cos t]^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{9\sin^2 t \cos^2 t\,[\sin^2 t + \cos^2 t]}\, dt = \int_0^{\pi/2} 3\sin t \cos t\, dt = \frac{3}{2}\int_0^{\pi/2} \sin(2t)\, dt$$

$$= -\frac{3}{4}\cos(2t)\Big|_0^{\pi/2} = \frac{3}{2}.$$



Total length of astroid is given as: $4L = 6$.

Practice Questions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Exercise:** 6.3

Q # 1 to Q # 16, Q # 29, Q # 30.