LAPLACE TRANSFORM - REGION OF CONVERGENCE

ROC – Example -1

- Consider signal that is the sum of two real exponentials:

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

ROC – Example -1

$$X(s)=\int_{-\infty}^{\infty}\left[3e^{-2t}u(t)-2e^{-t}u(t)\right]e^{-st}dt$$

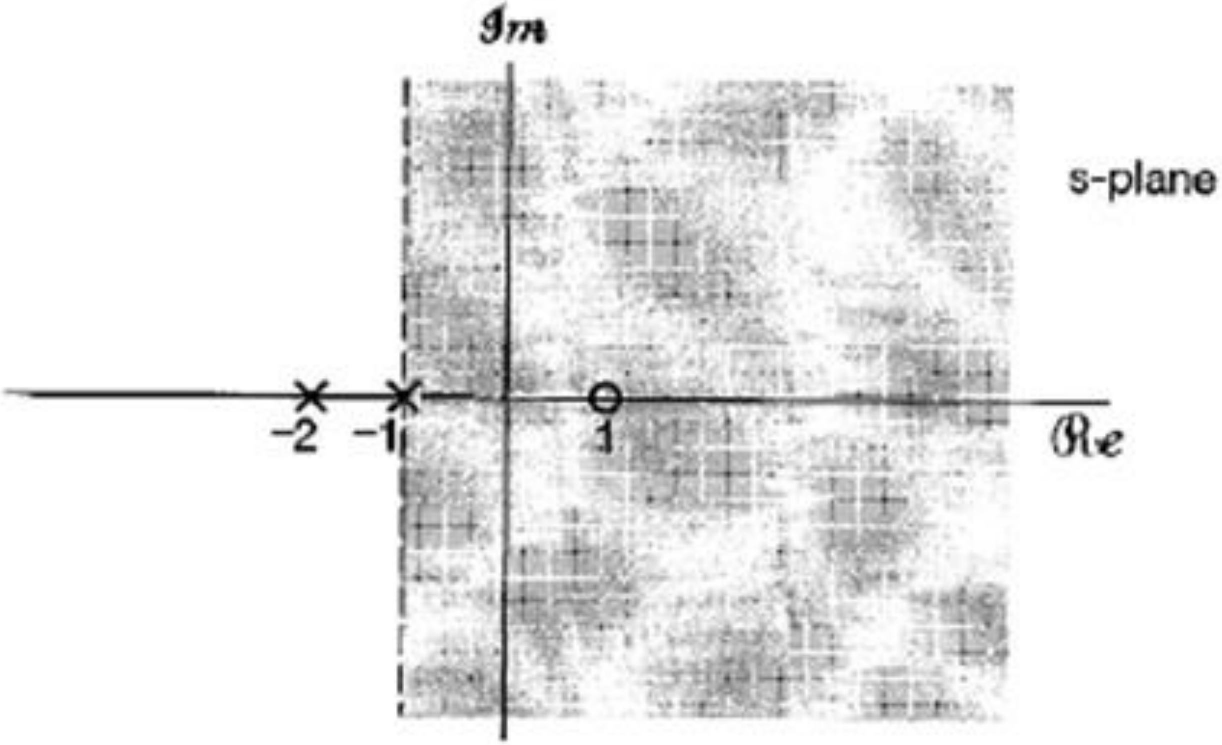
$$=3\int_{-\infty}^{\infty}e^{-2t}e^{-st}u(t)dt-2\int_{-\infty}^{\infty}e^{-t}e^{-st}u(t)dt$$

$$=\frac{3}{s+2}\left[\mathrm{Re}\{s\}>-2\right]-\frac{2}{s+1}\left[\mathrm{Re}\{s\}>-1\right]$$

$$=\frac{3}{s+2}-\frac{2}{s+1}=\frac{s-1}{s^2+3s+2};\quad \mathrm{Re}\{s\}>-1$$

- The common ROC is $\mathrm{Re}\{s\}>-1$

Laplace Transform – ROC



$$
\begin{aligned}
ROC &= ROC1 \cap ROC2 \\
&= \mathrm{Re}\{s\} > \max(-1, -2) \\
&= \mathrm{Re}\{s\} > -1
\end{aligned}
$$

ROC – Example -2

- Consider a signal that is a complex exponential:

$$x(t) = e^{-t}\left(\cos 3t\right)u(t)$$

ROC – Example -2

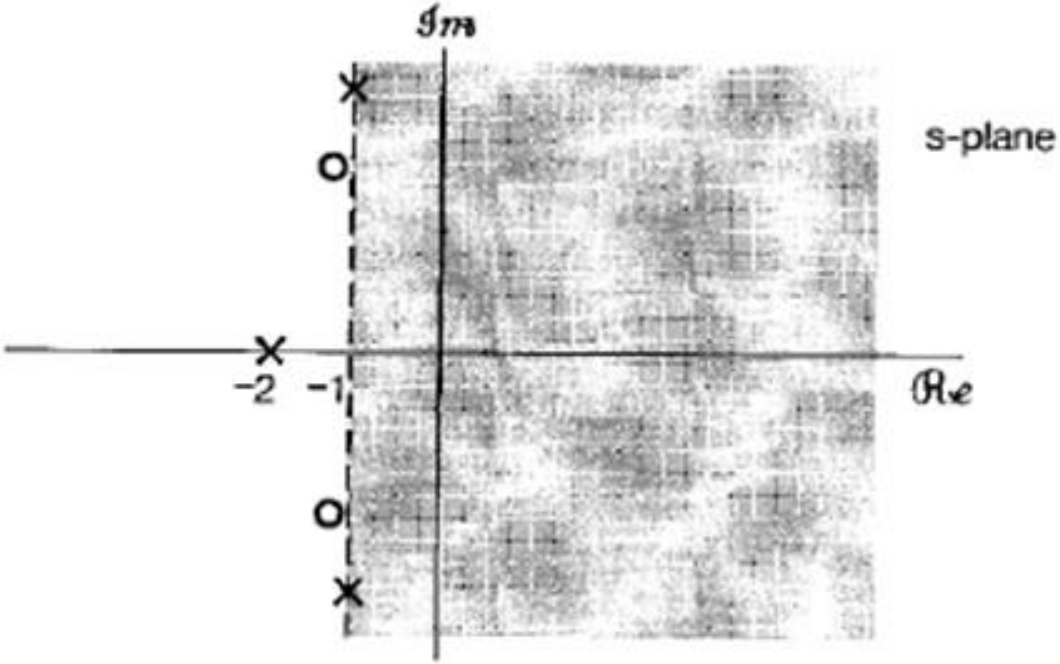
$$= u(t)\left[e^{-(1+3j)t} + e^{-(1-3j)t}\right]/2$$

$$X(s) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-(1+3j)t} e^{-st} u(t)dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-(1-3j)t} e^{-st} u(t)dt$$

$$= \frac{1}{2}\frac{1}{s+(1+3j)}\left[\mathrm{Re}\{s\} > -1\right] + \frac{1}{2}\frac{1}{s+(1-3j)}\left[\mathrm{Re}\{s\} > -1\right]$$

$$= \frac{s+1}{s^2+2s+10}; \quad \mathrm{Re}\{s\} > -1$$

Laplace Transform – ROC



$$ROC = ROC1 \cap ROC2$$

$$= \text{Re}\{s\} > \max(\text{Re}\{-1-3j, -1+3j\})$$

$$= \text{Re}\{s\} > -1$$

Rational Transforms

- Many (but by no means all) Laplace transforms of interest to us are rational functions of *s* (e.g., Examples 1 and 2), where:

$$X(s) = \frac{N(s)}{D(s)}, \quad N(s), D(s) - \text{polynomials in } s$$

- Roots of $N(s)$ = *zeros* of $X(s)$

- Roots of $D(s)$ = *poles* of $X(s)$

- Any $x(t)$ consisting of a linear combination of complex exponentials for $t > 0$ and for $t < 0$ (e.g., as in Example 1 and 2) has a rational Laplace transform.

Rational Transforms - Example

$$x(t) = 3e^{2t}u(t) - 2e^{-t}u(t)$$

Rational Transforms - Example

$$X(s) = \int_0^{\infty} [3e^{2t} - 2e^{-t}]e^{-st}dt$$

$$= 3\underbrace{\int_0^{\infty} e^{-(s-2)t}dt}_{\text{Requires } \Re e\{s\} > 2} - 2\underbrace{\int_0^{\infty} e^{-(s+1)t}dt}_{\text{Requires } \Re e\{s\} > -1}$$

BOTH required → ROC intersection

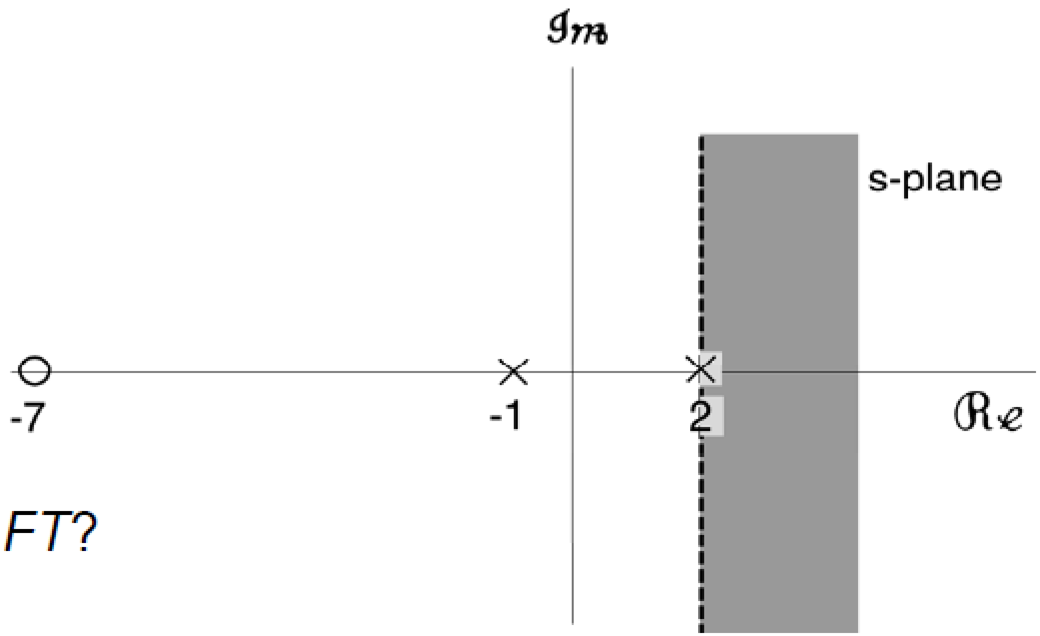
$\Downarrow$

$$X(s) = \frac{3}{s-2} - \frac{2}{s+1} = \frac{s+7}{(s-2)(s+1)} = \frac{s+7}{s^2-s-2} \quad \Re e\{s\} > 2$$

Notation:

× — *pole*

o — *zero*



Q: Does $x(t)$ have *FT*?

Rational Transforms - Example

- Consider the signal:

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$$

Rational Transforms - Example

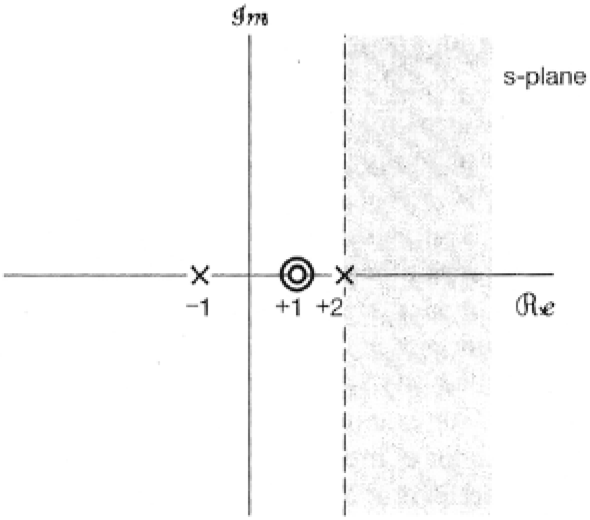
- The Laplace Transform of the impulse is:

$$L\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1 \quad [\text{valid for all } s]$$

- Hence the Laplace Transform for $x(t)$ is:

$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}; \quad \mathrm{Re}\{s\} > 2$$

$$= \frac{(s-1)^2}{(s+1)(s-2)}; \quad \mathrm{Re}\{s\} > 2$$



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