DISCRETE TIME FOURIER TRANSFORM (DTFT)

DTFT Pair

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$\boxed{\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ x[n] &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \end{aligned}}$$

– Analysis Equation
– DT FT

– Synthesis Equation
– DT Inverse FT
– Inverse DT FT

DTFT of Exponential

Recall CT result: $x(t) = e^{j\omega_0 t} \longleftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$

What about DT: $x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = ?$

a) We expect an impulse (of area 2π) at $\omega = \omega_o$

b) But $X(e^{j\omega})$ must be periodic with period 2π
In fact

$$X(e^{j\omega}) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)$$

DTFT of Exponential

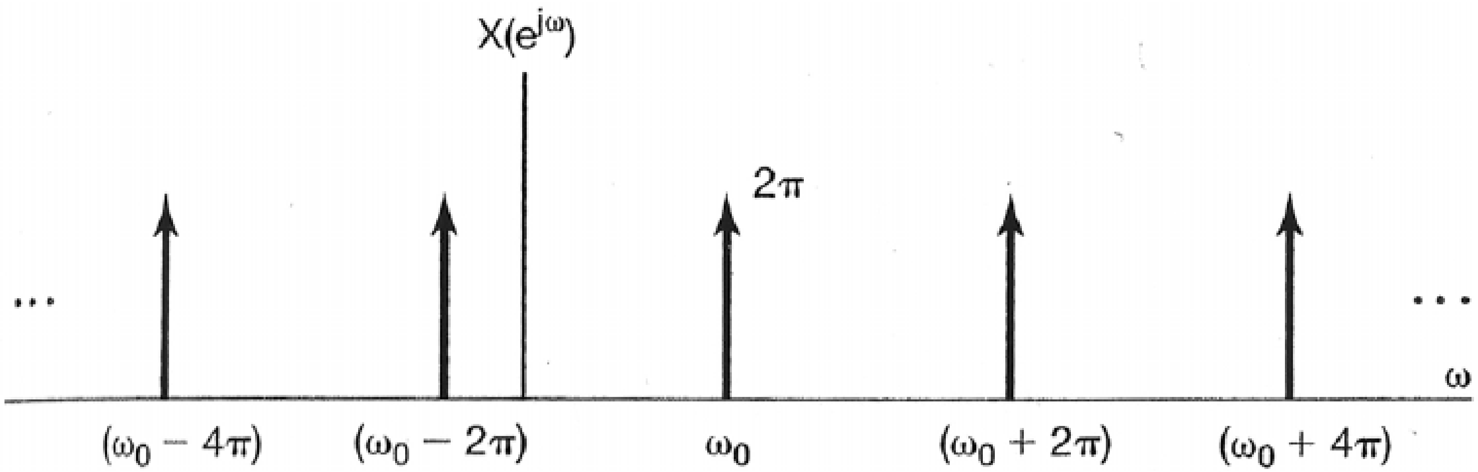
Note: The integration in the synthesis equation is over 2π period, only need $X(e^{j\omega})$ in *one* 2π period. Thus,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)}_{X(e^{j\omega})} e^{j\omega n} d\omega = e^{j\omega_0 n}$$

DTFT of Exponential

$$x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = ?$$

$$X(e^{j\omega}) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)$$



DTFT of Periodic Signals

$$x[n] = x[n+N]$$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N}$$

DTFS synthesis eq.

From the last page: $e^{jk\omega_0 n} \longleftrightarrow 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m)$

DTFT of Periodic Signals

$$X(e^{j\omega}) = \sum_{k=<N>} a_k \left[2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m) \right]$$

Linearity of DTFT

$$= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

The above shows that the Fourier transform of a periodic signal can be directly constructed from its Fourier coefficients.

DTFT of Periodic Signals

- Thus the Fourier transform of a periodic signal, $x[n]$, must be a linear combination of transforms of the form:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

- showing that the Fourier transform of a periodic signal can be directly constructed from its Fourier coefficients

DTFT of Periodic Signals

- If we choose the interval of summation as $k = 0, 1, \ldots, N-1,$ then

$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + a_2 e^{j2(2\pi/N)n} + \ldots + a_{N-1} e^{j(N-1)(2\pi/N)n}$$

- Thus $x[n]$ is a linear combination of signals at locations

$$\omega_0 = 0,\ 2\pi/N,\ 4\pi/N, \ldots, (N-1)2\pi/N$$

Next Slide: Interpretation of linear combination of signals

DTFT of Periodic Signals

- Consider the periodic signal

$$x[n] = \cos(\omega_0 n) = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}, \quad \text{with } \omega_0 = \frac{2\pi}{5}$$

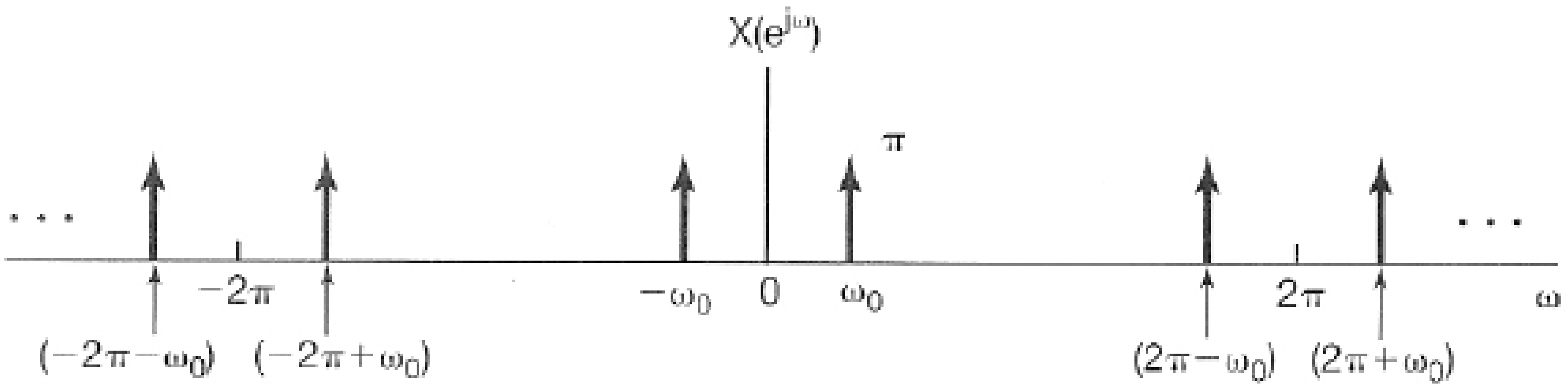
- The Fourier transform can be written as

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \pi\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{\infty} \pi\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right)$$

$$X(e^{j\omega}) = \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega + \frac{2\pi}{5}\right), \quad -\pi \le \omega < \pi$$

- and $X(e^{j\omega})$ repeats periodically with a period of 2π

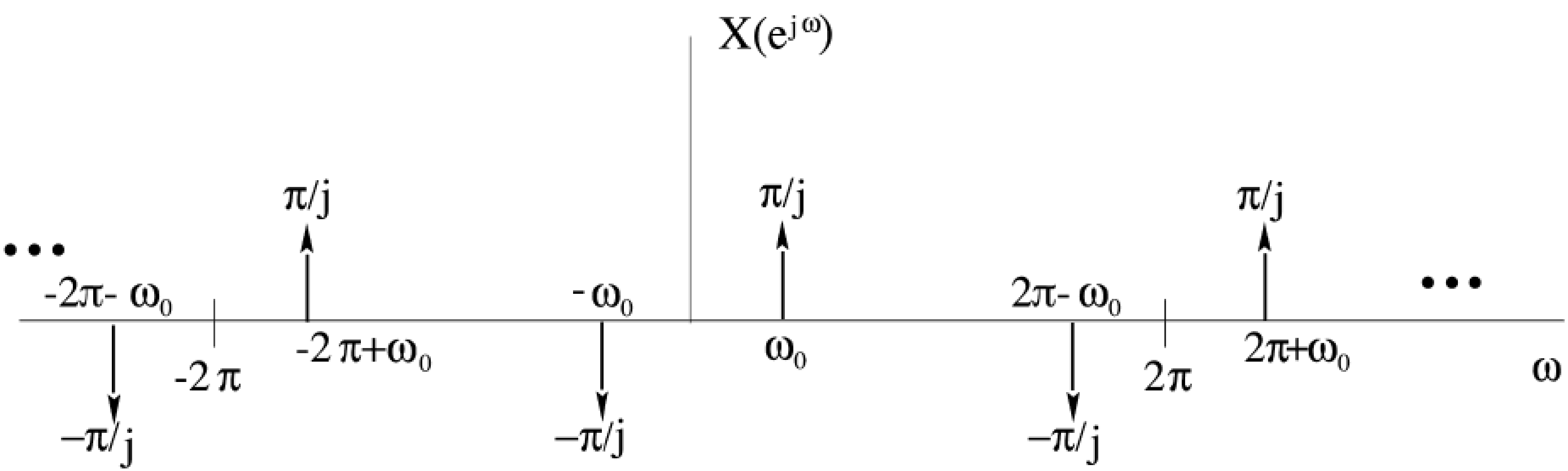
DTFT of Periodic Signals



DTFT of Periodic Signals

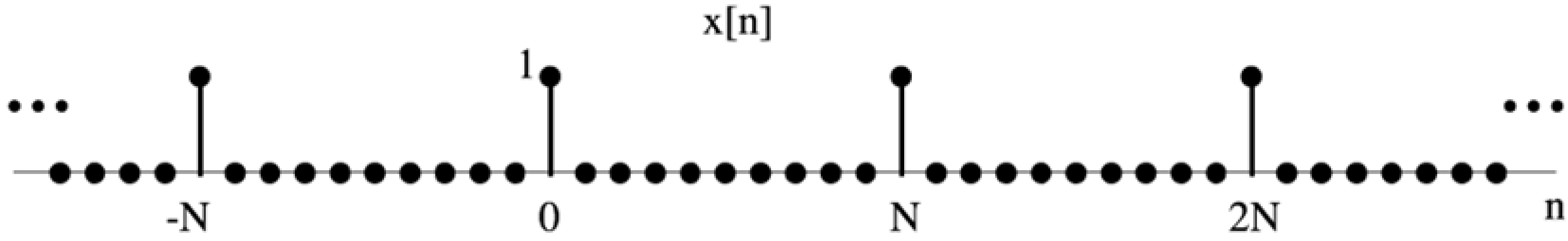
$$x[n] = \sin \omega_0 n = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m) - \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi m)$$



DTFT of Periodic Signals

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \qquad \omega_0 = 2\pi/N$$



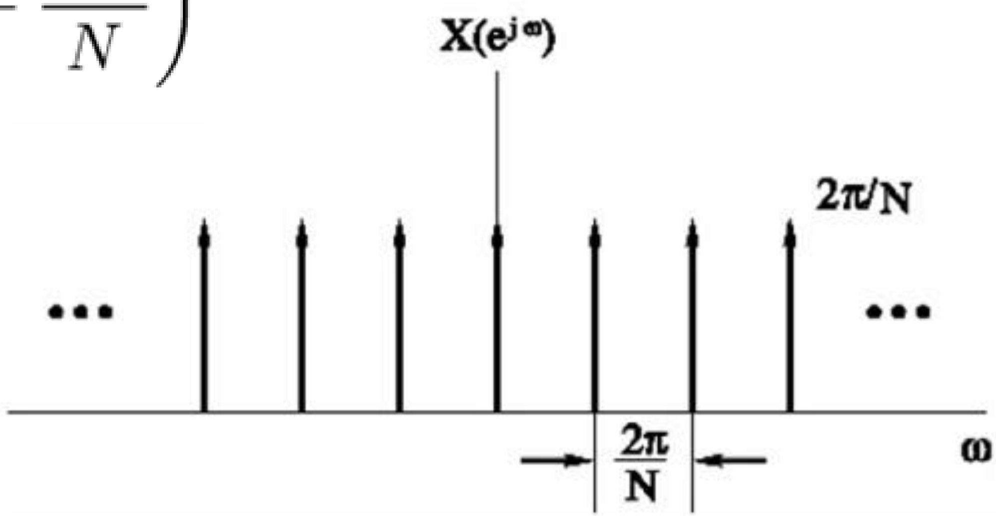
$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

DTFT of Periodic Signals

$$= \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{x[n]}_{=\delta[n]} e^{-jk\omega_0 n} = \frac{1}{N}$$

$$\Downarrow$$

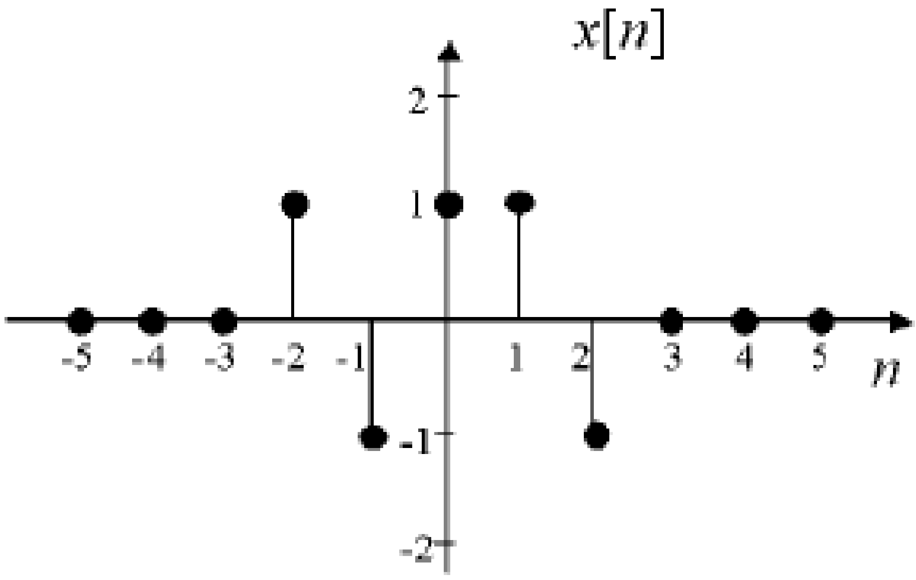
$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$



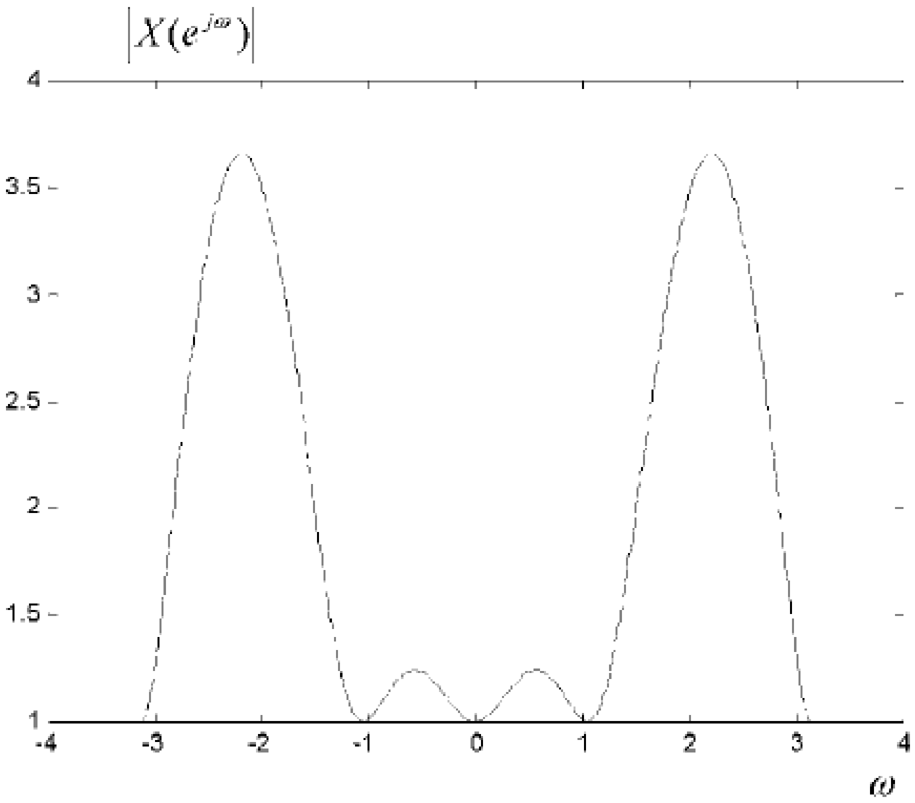
— Also periodic impulse train – in the frequency domain!

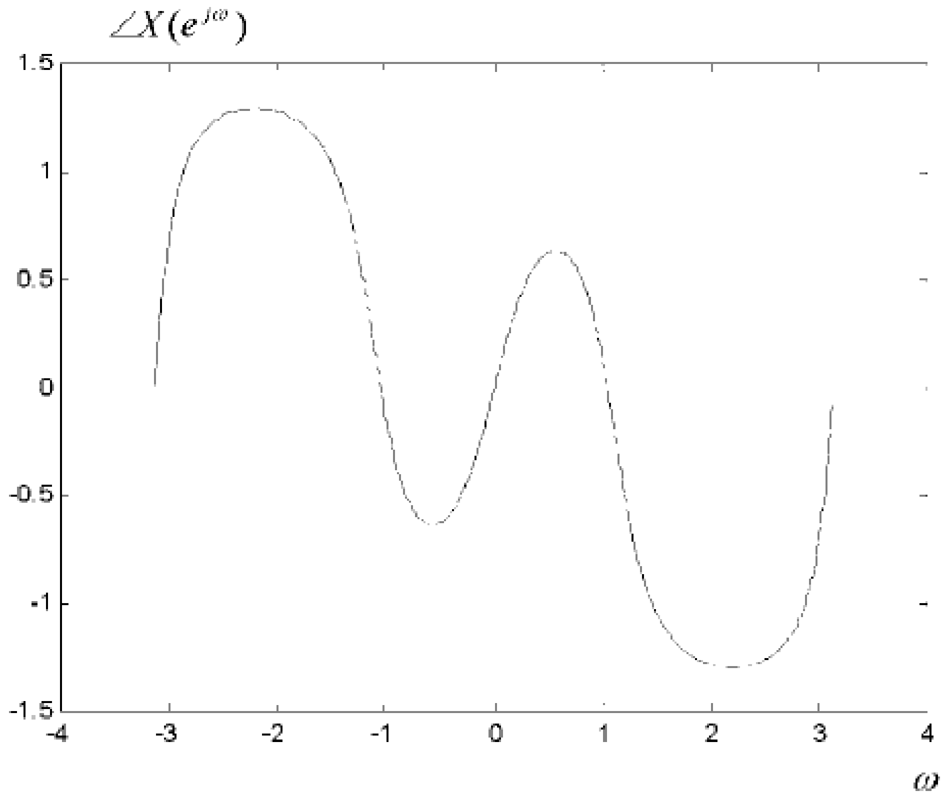
DTFT – Problem-1

- Compute the Fourier Transform of the signal shown below.



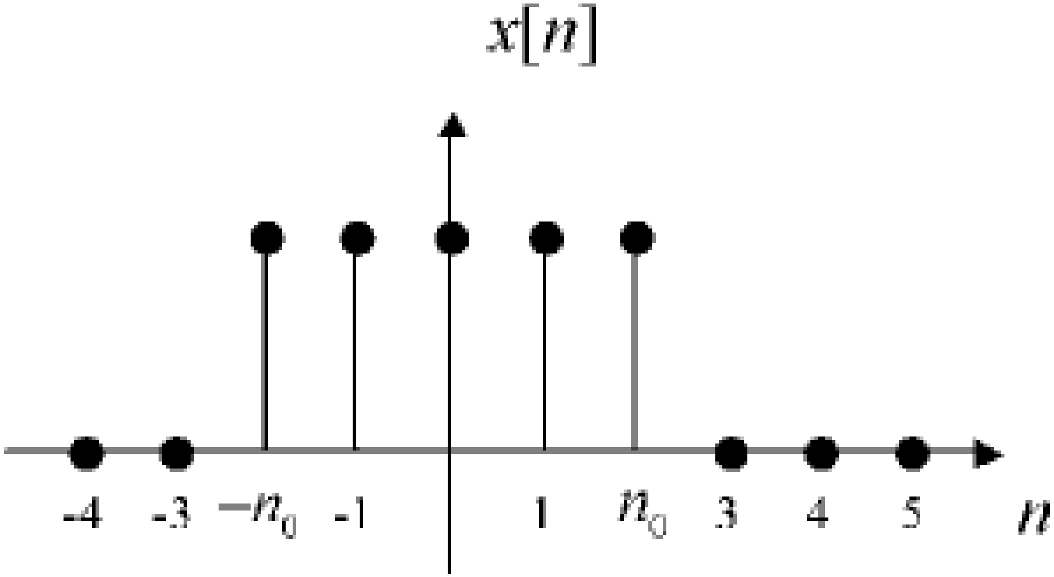
DTFT – Problem-1





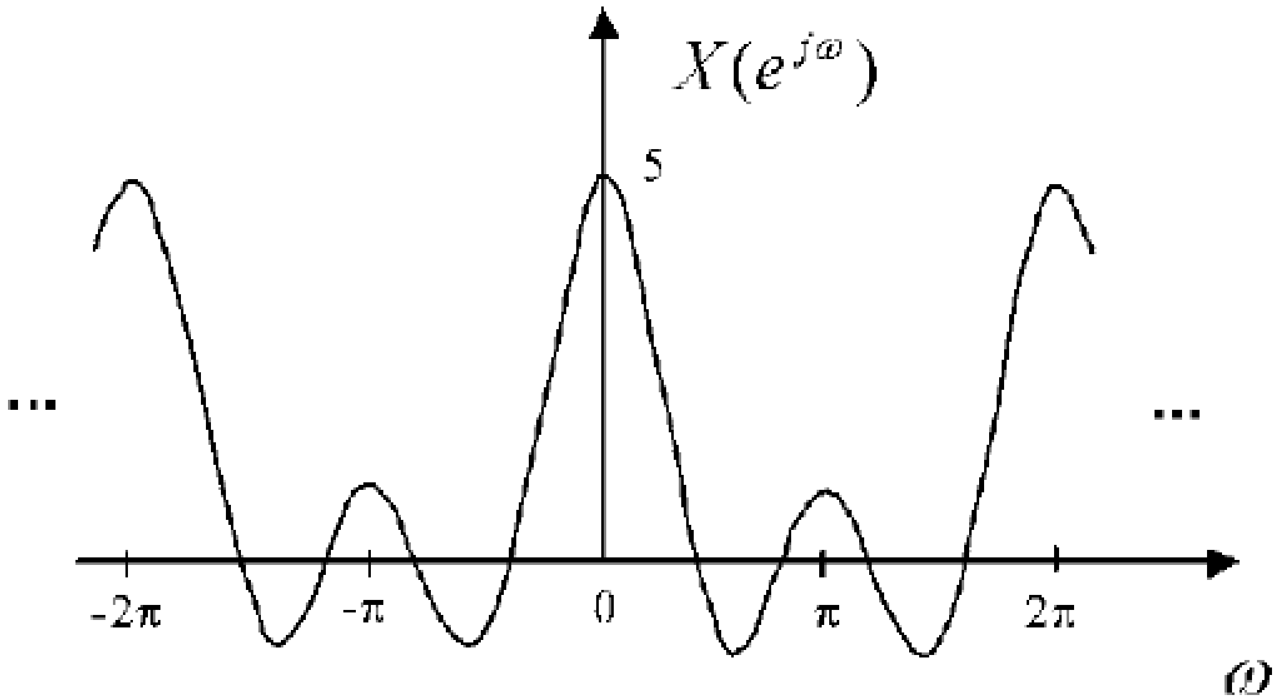
DTFT – Problem-2

- Compute the Fourier Transform of the signal shown below.



DTFT – Problem-2

- This function is the discrete-time counterpart of the sinc function that was the Fourier transform of the continuous-time rectangular pulse



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