Derivatives

Calculus & Analytical Geometry MATH- 101

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Parametric Equations

Conic	Parametric equations	Parameter	Range of parameter	Any point on the conic
Circle	$x = a\cos\theta$ $y = a\sin\theta$	θ	$0 \le \theta \le 2\pi$	'θ' or $(a\cos\theta, a\sin\theta)$
Parabola	$x = at^2$ $y = 2at$	t	$-\infty < t < \infty$	't' or $(at^2, 2at)$
Ellipse	$x = a\cos\theta$ $y = b\sin\theta$	θ	$0 \le \theta \le 2\pi$	'θ' or $(a\cos\theta, b\sin\theta)$
Hyperbola	$x = a\sec\theta$ $y = b\tan\theta$	θ	$-\pi \le \theta \le \pi$ except $\theta = \pm\frac{\pi}{2}$	'θ' or $(a\sec\theta, b\tan\theta)$

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 3
    - Sections: 3.5

Parametric Equations

If x and y are given as functions

$$x = f(t)\,, \qquad y = g(t).$$

Over an interval of t-values, then the set of points $(x, y) = \big(f(t), g(t)\big)$ defined by these equations is a **Parametric Curve.** The equations are **parametric equations** for the curve.

Derivative/Slope of Parametric Curves:

A parametric curve $x = f(t)\ \ and\ y = g(t)$ is differentiable at t if f and g are differentiable at t. Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{provided } \frac{dx}{dt} \neq 0.$$

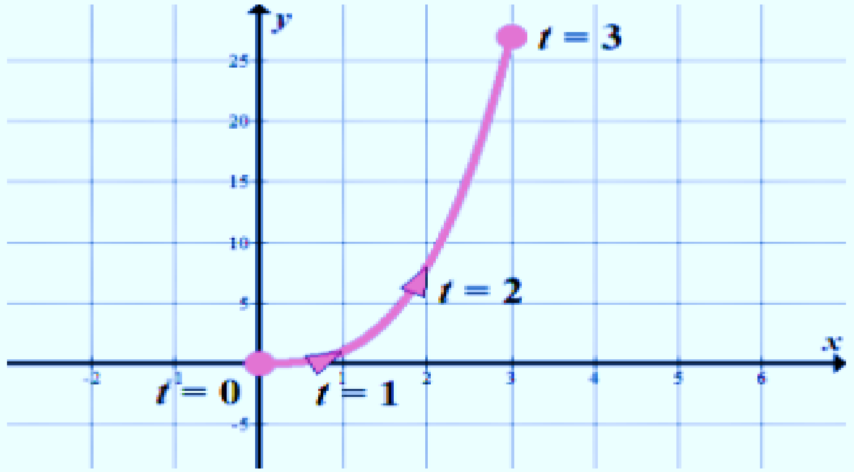
Parametric Equations

Parametric T-Chart	Parametric Graph

$x = t$, $y = t^3$ for t in $[0, 3]$

t	x	y
0	0	0
1	1	1
2	2	8
3	3	27

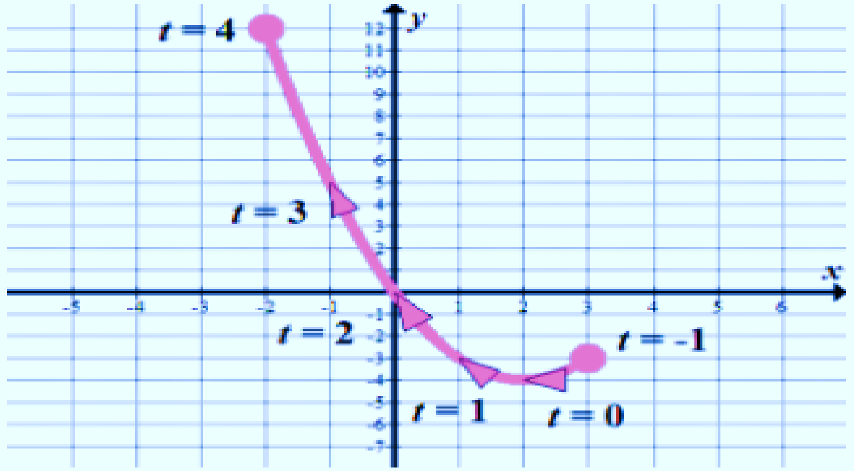
Domain: $[0, 3]$ **Range:** $[0, 27]$
End Points: $(0, 0)$ and $(3, 27)$



$x = -t + 2$, $y = t^2 - 4$ for t in $[-1, 4]$

t	x	y
–1	3	–3
0	2	–4
1	1	–3
2	0	0
3	–1	5
4	–2	12

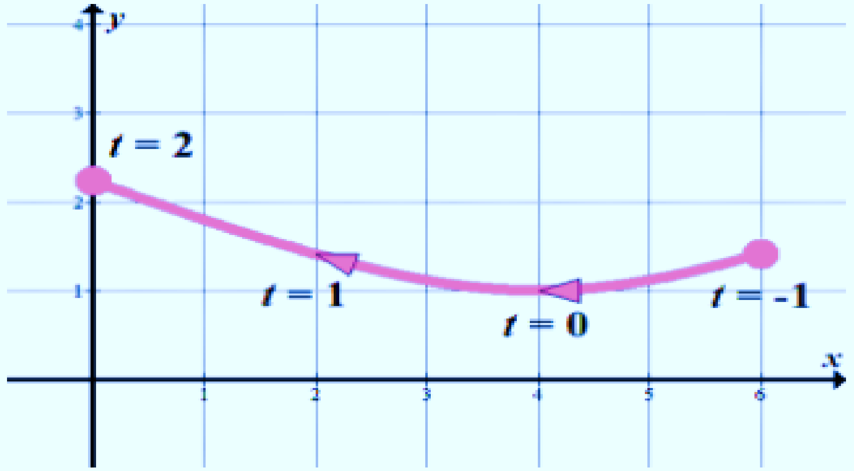
Domain: $[-2, 3]$ **Range:** $[-4, 12]$
End Points: $(3, -3)$ and $(-2, 12)$



$x = 4 - 2t$, $y = \sqrt{t^2 + 1}$ for t in $[-1, 2]$

t	x	y
–1	6	$\sqrt{2}$
0	4	1
1	2	$\sqrt{2}$
2	0	$\sqrt{5}$

Domain: $[0, 6]$ **Range:** $[1, \sqrt{5}]$
End Points: $(6, \sqrt{2})$ and $(0, \sqrt{5})$



Parametric Equations

Parametric Equations	Rectangular Equations
Eliminate the parameter and describe the resulting equation: $\begin{cases} x = 4t - 2 \\ y = 2 + 4t \end{cases}$	Solve for t in one of the equations and then substitute this in for the t in the other equation: $x = 4t - 2$ $x + 2 = 4t$ $t = \dfrac{x+2}{4}$ Plug this into the second equation: $y = 2 + 4t$ $y = 2 + 4\left(\dfrac{x+2}{4}\right)$ $y = x + 4 \ \text{(line)}$
Eliminate the parameter and describe the resulting equation: $\begin{cases} x = t - 3 \\ y = t^2 - 6t + 9 \end{cases}$	Solve for t in the **simplest equation** and then substitute this in for the t in the other equation: $x = t - 3$ $t = x + 3$ Plug this into the equation: $y = t^2 - 6t + 9$ $y = (x+3)^2 - 6(x+3) + 9$ $y = x^2 + 6x + 9 - 6x - 18 + 9$ $y = x^2 \ \text{(parabola)}$
Eliminate the parameter and describe the resulting equation: $\begin{cases} x = 5t \\ y = 3e^t \end{cases}$	Solve for t in the **simplest equation** and then substitute this in for the t in the other equation: $x = 5t$ $t = \dfrac{x}{5}$ Plug this into the second equation: $y = 3e^t$ $y = 3e^{\frac{x}{5}} \ \text{(exponential)}$
Eliminate the parameter and describe the resulting equation: $\begin{cases} x = \sqrt{t-3} \\ y = 2 + t \end{cases}$	Solve for t in the **simplest equation** and then substitute this in for the t in the other equation. Note that the second equation is simpler, since it doesn't have the square root. $y = 2 + t$ $t = y - 2$ Plug this into the second equation: $x = \sqrt{y-3}$ $x = \sqrt{(y-2)-3}$ $x = \sqrt{y-5}$ $x^2 = y - 5$ $y = x^2 + 5 \ \text{(parabola)}$
Eliminate the parameter and describe the resulting equation: $\begin{cases} x = 3t \\ y = -2t^2 - 1 \end{cases}$ $t \text{ on } [-1, 3]$	Solve for t in the **simplest equation** and then substitute this in for the t in the other equation. $t = \dfrac{x}{3}, \quad y = -2\left(\dfrac{x}{3}\right)^2 - 1$ $y = -\dfrac{2x^2}{9} - 1 \quad \text{(parabola)}$ Note that we are given an interval for t, so we are expected to find the domain and range for the rectangular equations. To find the domain and range, make a t-chart: $\begin{array}{c\|c\|c} t & x & y \\ \hline -1 & -3 & -3 \\ \hline 0 & 0 & -1 \\ \hline 1 & 3 & -3 \\ \hline 2 & 6 & -9 \\ \hline 3 & 9 & -19 \end{array}$ Domain is $[-3, 9]$ and Range is $[-19, -1]$.

Parametric Equations

Parametric Equations	Rectangular Equation
Eliminate the parameter and describe the resulting equation: $\begin{cases} x = 3\sin t + 2 \\ y = 3\cos t - 1 \end{cases}$ t on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	Solve for $\sin t$ in the first equation and $\cos t$ in the second (since it's too complicated to solve for t). Then use the Pythagorean identity $\sin^2 t + \cos^2 t = 1$, and substitute: $\begin{aligned} x &= 3\sin t + 2 \\ 3\sin t &= x - 2 \\ \sin t &= \frac{x-2}{3} \end{aligned}$ $\quad$ $\begin{aligned} y &= 3\cos t - 1 \\ 3\cos t &= y + 1 \\ \cos t &= \frac{y+1}{3} \end{aligned}$ $\quad$ $\begin{gathered} \sin^2 t + \cos^2 t = 1 \\ \left(\frac{x-2}{3}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1 \\ (x-2)^2 + (y+1)^2 = 9 \quad \text{(circle)} \end{gathered}$ This looks like a circle, but since the interval for t is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have a **semi-circle**.
Eliminate the parameter and describe the resulting equation: $\begin{cases} x = 4\sec t + 1 \\ y = 3\tan t \end{cases}$	Solve for $\sec t$ in the first equation $\tan t$ and in the second. Then use the Pythagorean identity $\sec^2 t - \tan^2 t = 1$, and substitute: $\begin{aligned} x &= 4\sec t + 1 \\ 4\sec t &= x - 1 \\ \sec t &= \frac{x-1}{4} \end{aligned}$ $\quad$ $\begin{aligned} y &= 3\tan t \\ \tan t &= \frac{y}{3} \end{aligned}$ $\quad$ $\begin{gathered} \sec^2 t - \tan^2 t = 1 \\ \left(\frac{x-1}{4}\right)^2 - \left(\frac{y}{3}\right)^2 = 1 \\ \frac{(x-1)^2}{16} - \frac{y^2}{9} = 1 \quad \text{(hyperbola)} \end{gathered}$
Eliminate the parameter and describe the resulting equation: $\begin{cases} x = 5\cot t - 4 \\ y = 4\csc t \end{cases}$	Solve for $\cot t$ in the first equation and $\csc t$ in the second. Then use the Pythagorean identity $\csc^2 t - \cot^2 t = 1$, and substitute: $\begin{aligned} x &= 5\cot t - 4 \\ 5\cot t &= x + 4 \\ \cot t &= \frac{x+4}{5} \end{aligned}$ $\quad$ $\begin{aligned} y &= 4\csc t \\ \csc t &= \frac{y}{4} \end{aligned}$ $\quad$ $\begin{gathered} \csc^2 t - \cot^2 t = 1 \\ \left(\frac{y}{4}\right)^2 - \left(\frac{x+4}{5}\right)^2 = 1 \\ \frac{y^2}{16} - \frac{(x+4)^2}{25} = 1 \quad \text{(hyperbola)} \end{gathered}$
Eliminate the parameter and describe the resulting equation: $\begin{cases} x = \sin t - 4 \\ y = 2\cos t + 2 \end{cases}$	Solve for $\sin t$ in the first equation and $\cos t$ in the second (since it's too complicated to solve for t). Then use the Pythagorean identity $\sin^2 t + \cos^2 t = 1$, and substitute: $\begin{aligned} x &= \sin t - 4 \\ \sin t &= x + 4 \end{aligned}$ $\quad$ $\begin{aligned} y &= 2\cos t + 2 \\ \cos t &= \frac{y-2}{2} \end{aligned}$ $\quad$ $\begin{gathered} \sin^2 t + \cos^2 t = 1 \\ (x+4)^2 + \left(\frac{y-2}{2}\right)^2 = 1 \\ \frac{(x+4)^2}{1} + \frac{(y-2)^2}{4} = 1 \quad \text{(ellipse)} \end{gathered}$
Eliminate the parameter and describe the resulting equation: $\begin{cases} x = 3t \\ y = \sin t \end{cases}$	Solve for t in the **simplest equation** and then substitute in the t in the other equation. Since we don't have a trig function in both equations, we can't use the Pythagorean identity like we did above. $\begin{aligned} x &= 3t \\ t &= \frac{x}{3} \end{aligned}$ $\quad$ Plug this into the second equation: $\quad$ $\begin{aligned} y &= \sin t \\ y &= \sin\left(\frac{x}{3}\right) \end{aligned}$ $\quad$ (sin graph)

Rate of Change

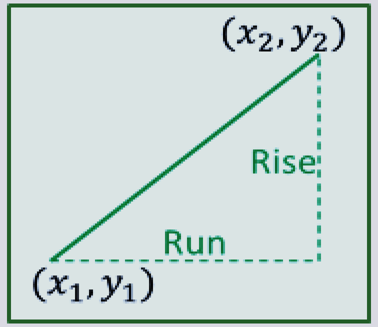
Rate of Change

A rate that describes how one quantity changes in relation to another quantity

It is represented by the Gradient of a line

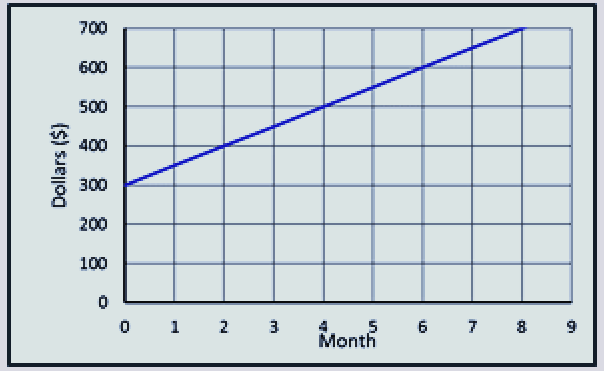
$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Gradient} = \frac{Rise}{Run}$$



Interpreting Rates of Change

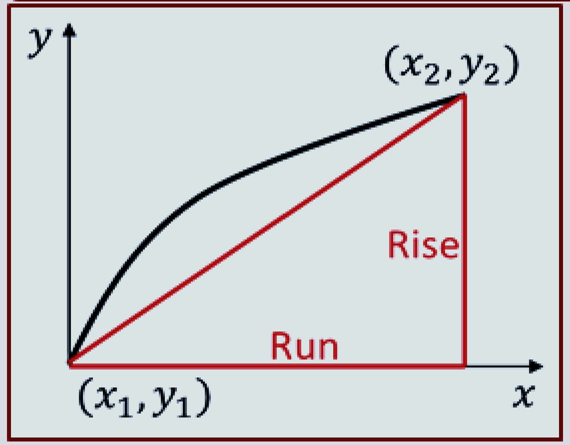
Gradient → Amount of (y) per Amount of (x)



Rate of change = $50 per month

Average rate of change

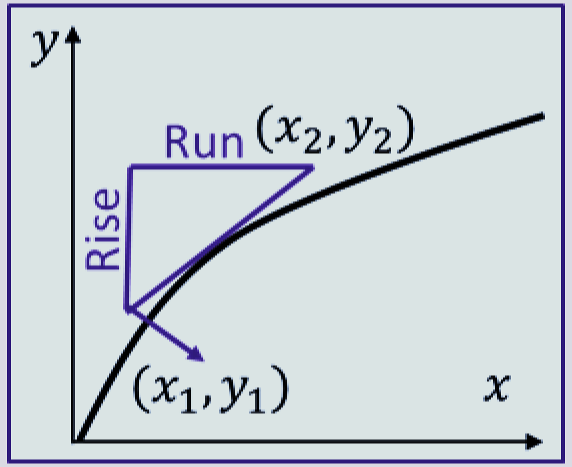
The rate of change **over a given interval**



- Create **chord** between two intervals
- Calculate gradient of chord
- Interpret gradient as a rate of change

Instantaneous rate of change

The rate of change at a **particular moment**



- Create **tangent** at specific point
- Calculate gradient of tangent
- Interpret gradient as a rate of change

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 3
  - Sections: 3.3

The Derivative as a Rate of Change

- When we initiated the study of derivatives we talked about average and instantaneous rates of change. We continue our investigations of applications in which derivatives are used to model the rates at which things change in the world around us.

- It is natural to think of change as change with respect to time, but other variables can be treated in the same way.

- For example, a physician may want to know how change in dosage affects the body's response to a drug. An economist may want to study how the cost of producing steel varies with the number of tons produced.

Instantaneous Rates of Change

- If we interpret the difference quotient $\frac{f(x+h)-f(x)}{h}$ as the average rate of change in $f(x)$ over the interval from x to $x + h$, we can interpret its limit as $h \to 0$ as the rate at which f is changing at the point x.

- The **instantaneous rate of change** of $f(x)$ with respect to x at x_0 is the derivative

$$f'(x) = \frac{f(x_0 + h) - f(x_0)}{h}$$

provided the limit exists.

- Thus, instantaneous rates are limits of average rates.

Example: How a Circle's Area Changes with Its Radius

The area A of a circle is related to its radius by the equation

$$A = \pi r^2$$

How fast does the area change with respect to the radius when the radius is 10 m?

Solution:

The rate of change of the area with respect to the diameter is

$$\frac{dA}{dr} = 2\pi r$$

When $r = 10m$ the area is changing at rate $2\pi(10) = 20\pi\, \frac{m^2}{m}$.

Motion Along a Line: Displacement, Velocity, Speed, Acceleration, and Jerk

- The **displacement** of the object over the time interval from t to $t + \Delta t$ is:

$$\Delta s = f(t + \Delta t) - f(t).$$

- **Velocity (instantaneous velocity)** $\boldsymbol{v(t)}$ is the derivative of position with respect to time, i.e.,

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}.$$

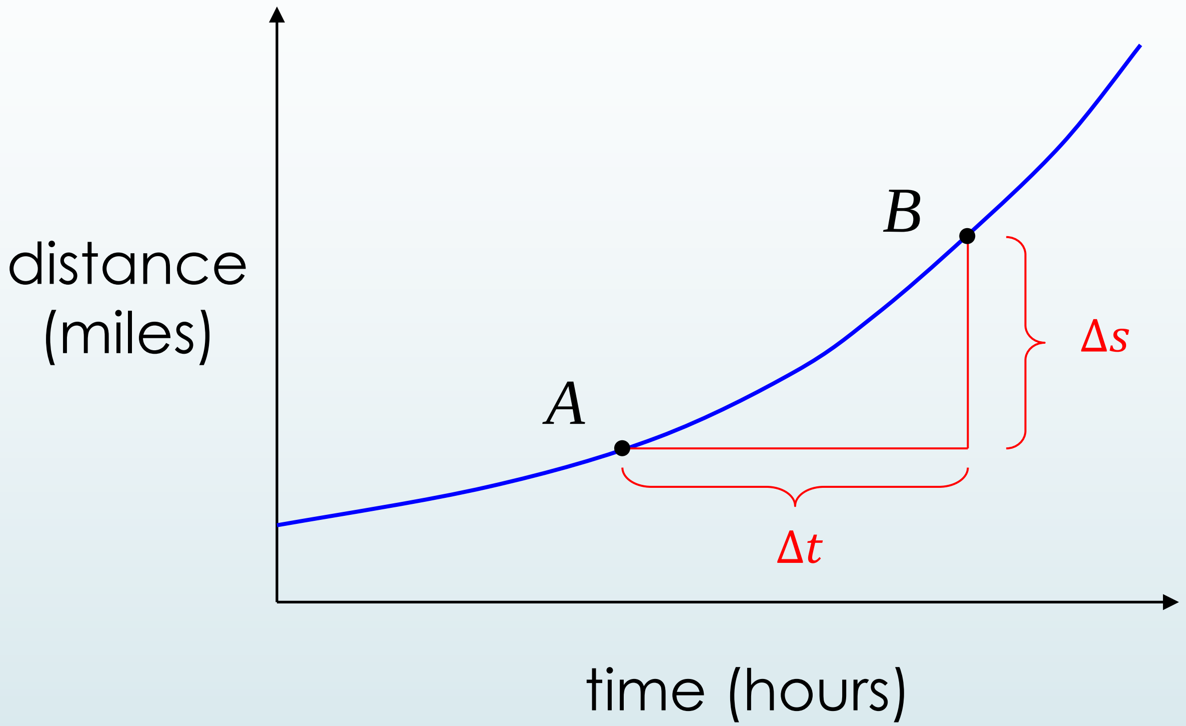
- **Speed** is the absolute value of velocity, i.e., $Speed \ = |v(t)|$.
- **Acceleration** is the derivative of velocity with respect to time, i.e.,

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

- **Jerk** is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Consider a graph of displacement (distance traveled) vs. time.



Average velocity can be found by taking:

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\Delta s}{\Delta t}$$

$$V_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

The speedometer in our cars does not measure average velocity, but instantaneous velocity.

(The velocity at one moment in time.)

$$V(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Example: Motion on a Spring

A body hanging from a spring is stretched 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5\cos t\,.$$

What are its velocity and acceleration at time t?

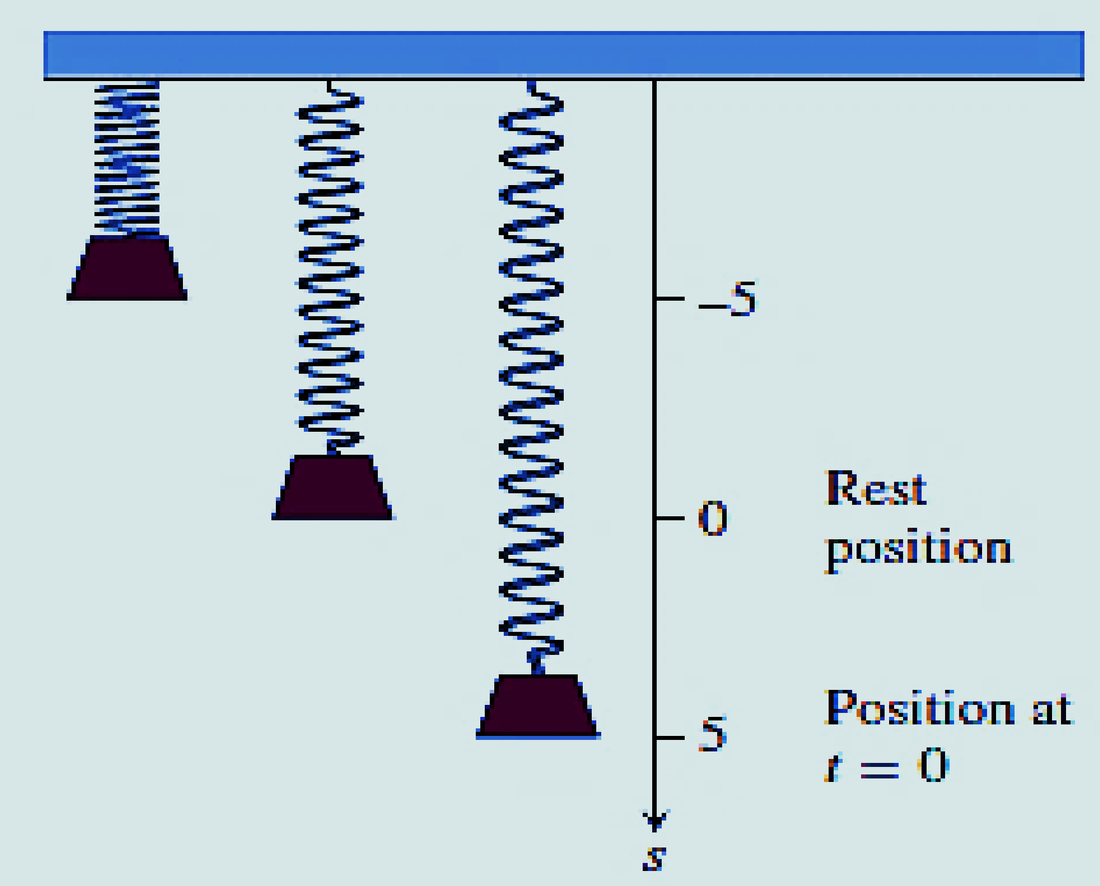


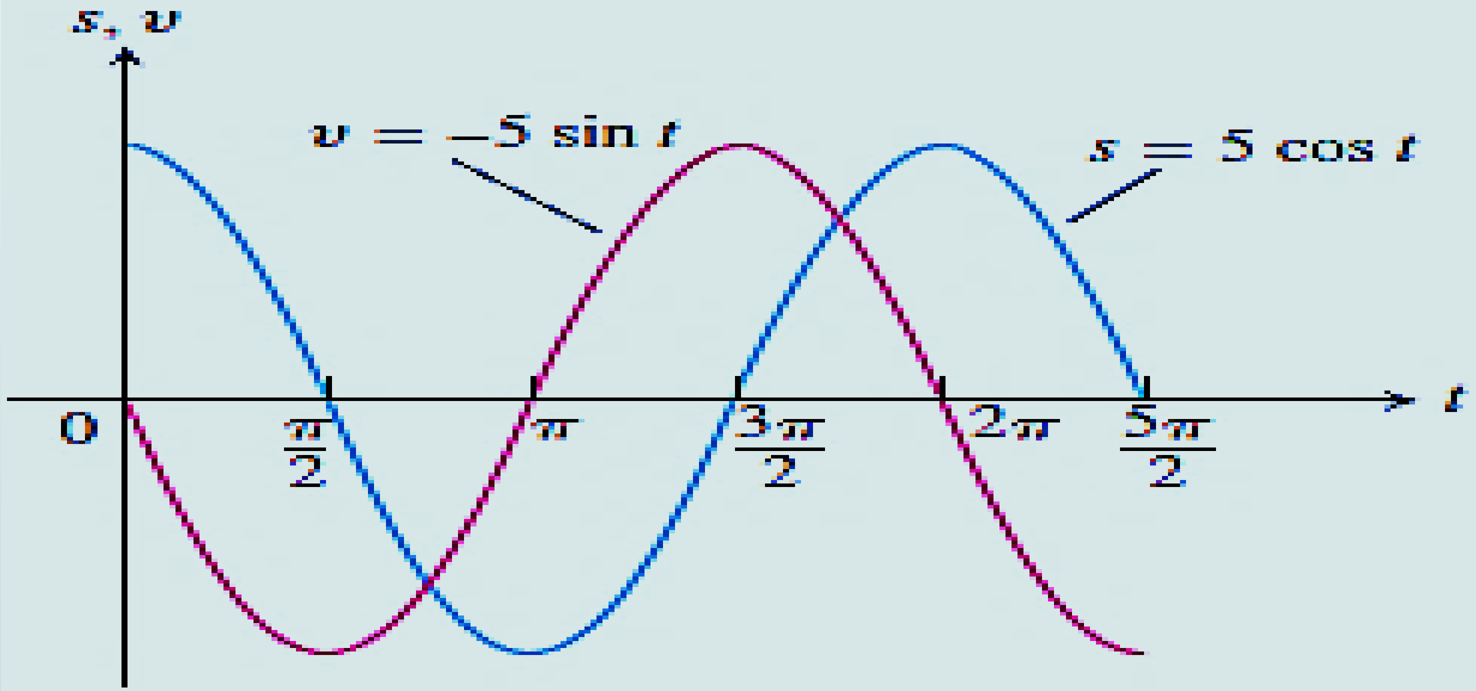
FIGURE A body hanging from a vertical spring and then displaced oscillates above and below its rest position. Its motion is described by trigonometric functions.

Solution We have

$$\text{Position:} \qquad s = 5\cos t$$

$$\text{Velocity:} \qquad v = \frac{ds}{dt} = \frac{d}{dt}(5\cos t) = -5\sin t$$

$$\text{Acceleration:} \qquad a = \frac{dv}{dt} = \frac{d}{dt}(-5\sin t) = -5\cos t.$$



The graphs of the position and velocity of the body.

Example: Jerk

The jerk of the simple harmonic motion considered in previous example is given by:

$$j = \frac{da}{dt} = \frac{d}{dt}(-5\cos t) = 5\sin t.$$

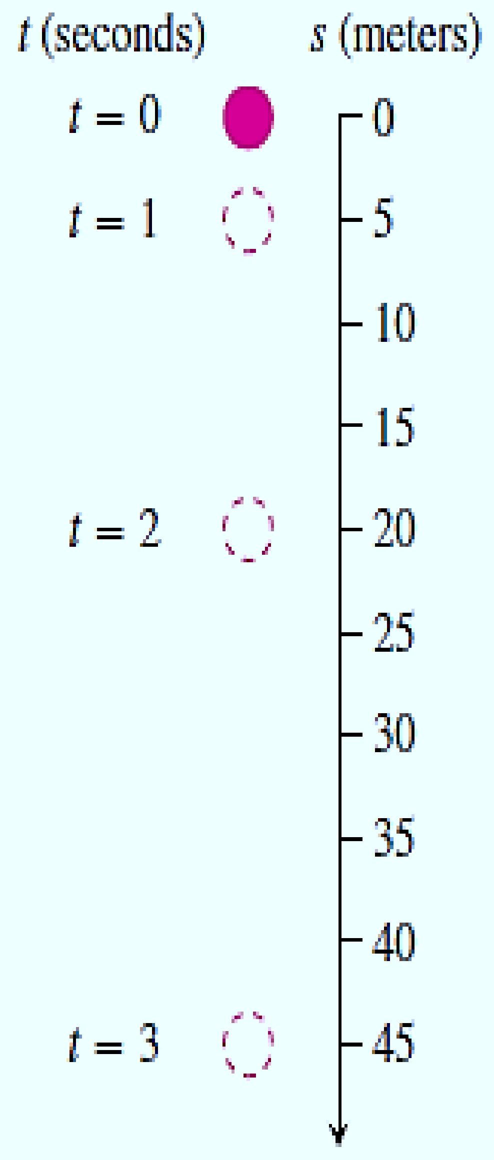
It has its greatest magnitude when $\sin t = \pm 1$, not at the extremes of the displacement but at the rest position, where the acceleration changes direction and sign.

EXAMPLE: Modeling Free Fall

Accompanying figure shows the free fall of a heavy ball bearing released from rest at time $t = 0\ sec$.

(a) How many meters does the ball fall in the first $2\ \text{sec}$?

(b) What is its velocity, speed, and acceleration then?



Solution:

(a) The free-fall equation is: $s = 4.9t^2$.

During the first $2\ \text{sec}$, the ball falls $s(2) = 4.9(2)^2 = 19.6m$.

(b) At any time t, velocity is: $v(t) = s'(t) = 9.8\,t$.

At $t = 2$, velocity is: $v(2) = 19.6\ m/sec$.

The speed at $t = 2$ is: Speed $= |v(2)| = 19.6\ m/sec$.

The acceleration at any time t is: $a(t) = v'(t) = s''(t) = 9.8$.

At $t = 2$, acceleration is: $a(2) = 9.8\ m/sec^2$.

Derivatives in Economics

- In a manufacturing operation, the *cost of production* $c(x)$ is a function of x, the number of units produced. The **marginal cost of production** is the rate of change of cost with respect to level of production, so it is $\frac{dc}{dx}$. Roughly speaking the marginal cost at some level of production x is the cost to produce the $x+1^{\text{st}}$ item.
- Suppose that $c(x)$ represents the dollars needed to produce x tons of steel in one week. It costs more to produce $x+h$ units per week, and the cost difference, divided by h, is the average cost of producing each additional ton:

$$\frac{c(x+h)-c(x)}{h} = \text{average cost of each of the additional } h \text{ tons of steel produced.}$$

- The limit of this ratio as $h \to 0$ is the *marginal cost* of producing more steel per week. When the current weekly production is x tons:

$$\frac{dc}{dx} = \lim_{h\to 0} \frac{c(x+h)-c(x)}{h} = \text{marginal cost of production.}$$

Example:

Suppose cost to produce x stoves is: $c(x) = x^3 - 6x^2 + 15x$.
The cost of producing one more stove a day is: $c'(x) = 3x^2 - 12x + 15$.
If we are currently producing 10 stoves, the 11th stove will cost approximately:

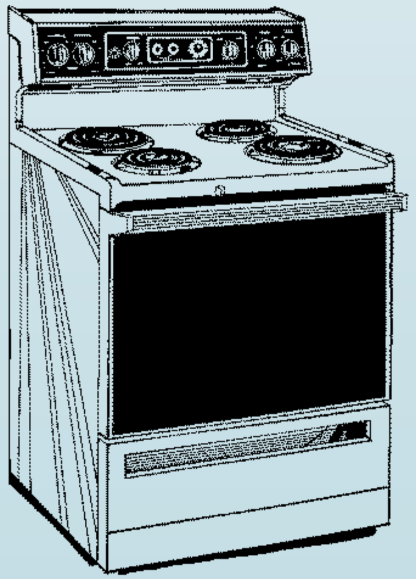
$$\begin{aligned} c'(10) &= 3 \cdot 10^2 - 12 \cdot 10 + 15 \\ &= 300 - 120 + 15 \\ &= \$195. \end{aligned}$$

marginal cost

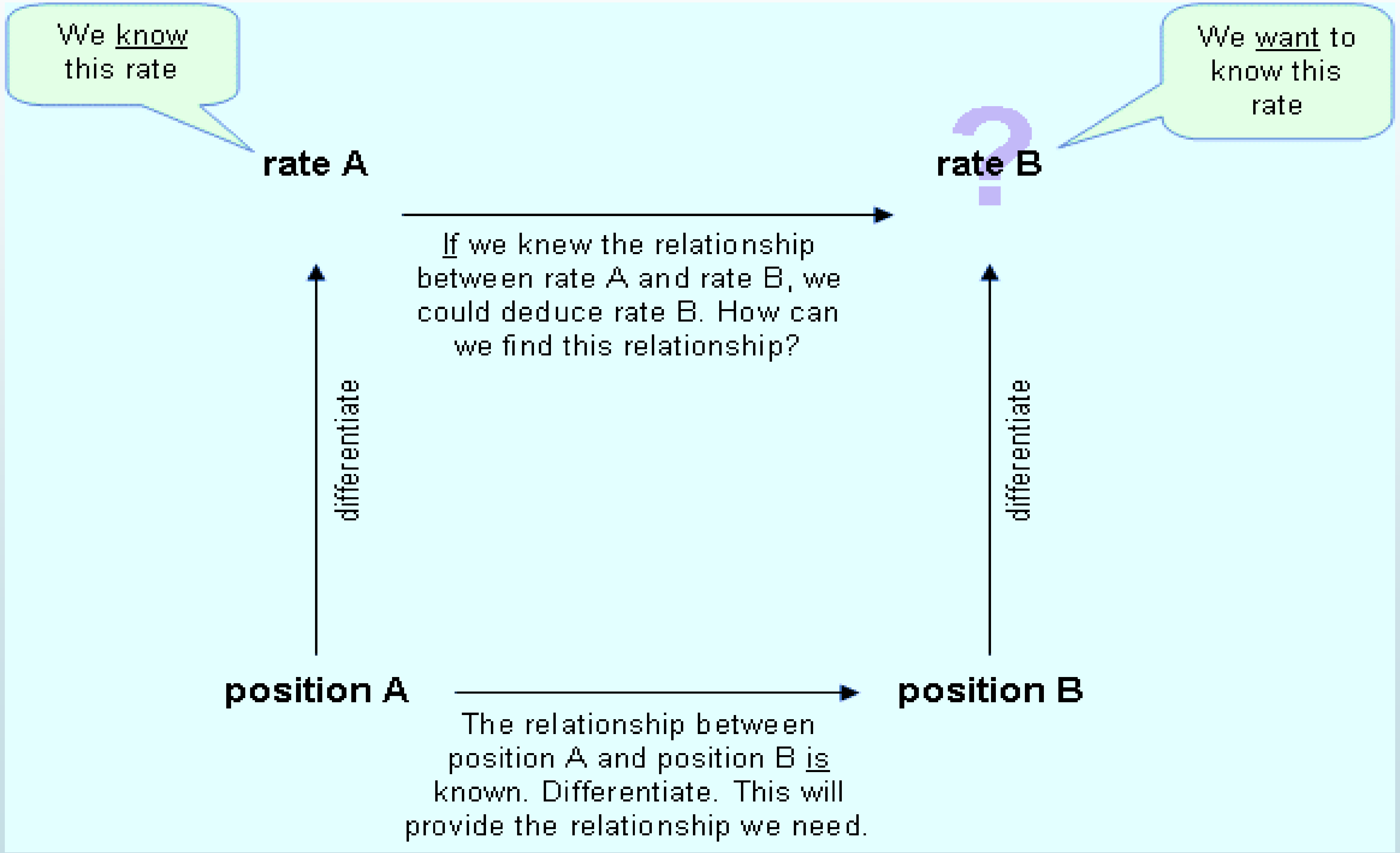
Suppose revenue of selling x stoves is: $r(x) = x^3 - 3x^2 + 12x$.
The revenue of producing one more stove a day is: $r'(x) = 3x^2 - 6x + 12$.
If we are currently producing 10 stoves, you can expect revenue to increase by about:

$$\begin{aligned} r'(10) &= 3x^2 - 6x + 12. \\ r'(10) &= 3(10)^2 - 6(10) + 12. \\ &= 300 - 60 + 12 = \$252. \end{aligned}$$

marginal revenue



Related Rates



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- Chapter: 3
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Related Rates

- We are now interested to look at problems that ask for the rate at which some variable changes.
- In each case the rate is a derivative that has to be computed from the rate at which some other variable (or perhaps several variables) is known to change.
- To find it, we write an equation that relates the variables involved and differentiate it to get an equation that relates the rate we seek to the rates we know.
- The problem of finding a rate you cannot measure easily from some other rates that you can is called a *related rates problem*.
- **Related rates** problems involve finding a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known. The rate of change is usually with respect to time.

Related Rates Equations

- Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then
$$V = \frac{4}{3}\pi r^3.$$
- Using the Chain Rule, we differentiate to find the related rates equation
$$\frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$
- So if we know the radius r of the balloon and the rate $\frac{dV}{dt}$ at which the volume is increasing at a given instant of time, then we can solve this last equation for $\frac{dr}{dt}$ to find how fast the radius is increasing at that instant.

Sphere problem:

Consider a sphere of radius $r = 10cm$. If the radius is changing at an instantaneous rate of $0.1\ cm/\sec$, how much does the volume change?

Solution:

If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3}\pi r^3.$$

Differentiating both sides of above w.r.t "t" we get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi(10\mathrm{cm})^2 \cdot \left(0.1\frac{\mathrm{cm}}{\mathrm{sec}}\right) = 40\pi\frac{\mathrm{cm}^3}{\mathrm{sec}}$$

Thus we conclude that the sphere is growing at a rate of $40\pi\ \mathrm{cm}^3/\mathrm{sec}$

Steps for Related Rates Problems:

1. Draw a picture (sketch).
2. Write down known information.
3. Write down what you are looking for.
4. Write an equation to relate the variables.
5. Differentiate both sides with respect to t.
6. Evaluate.

EXAMPLE : A Rising Balloon

A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of $0.14\, rad\ /\text{min}$. How fast is the balloon rising at that moment?

Solution:

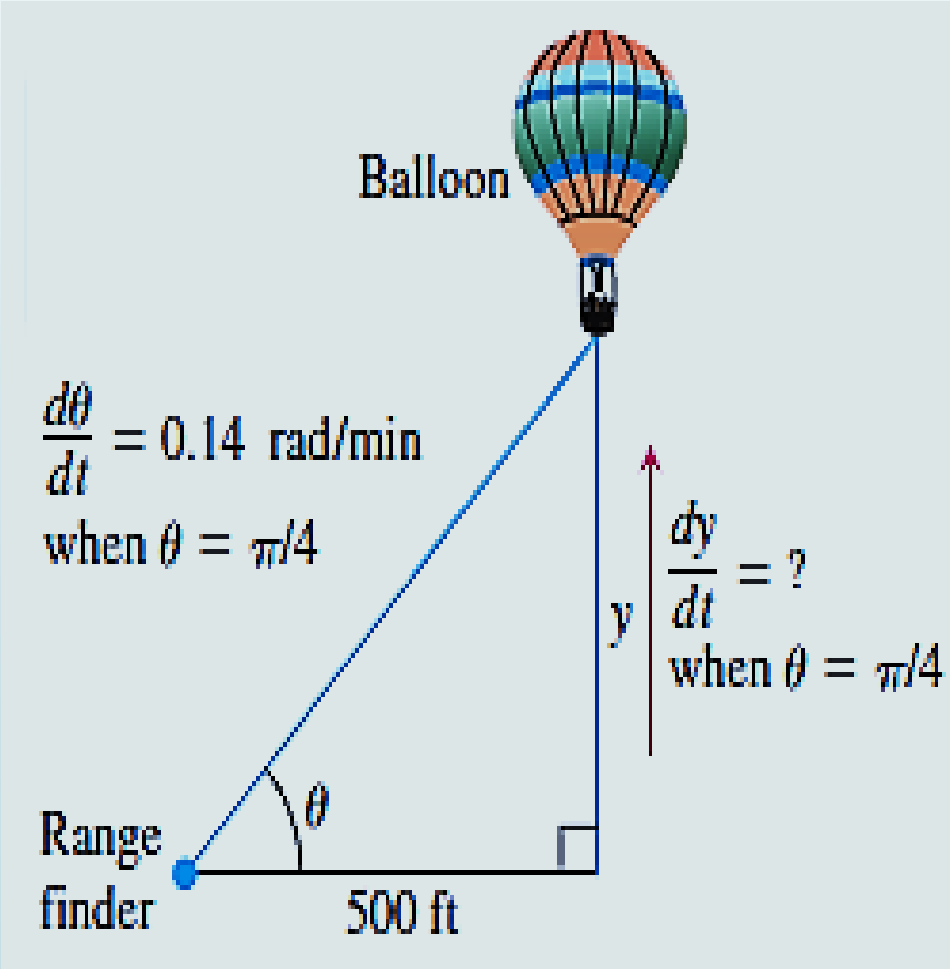
We let t represent time in minutes and assume that

$\theta =$ the angle in radians the range finder makes with the ground.

$y =$ the height in feet of the balloon.

such that θ and y are differentiable functions of t.

The one constant in the picture is the distance from the range finder to the liftoff point (500 ft). There is no need to give it a special symbol.



Given that: $\theta = \frac{\pi}{4} rad,\quad \frac{d\theta}{dt} = 0.14 \frac{rad}{min}$.

We are required to find that how fast is the balloon rising? For this we will find $\frac{dy}{dt}$.

Equation that is relating the given variables is given as:

$$\tan\theta = \frac{y}{500}$$

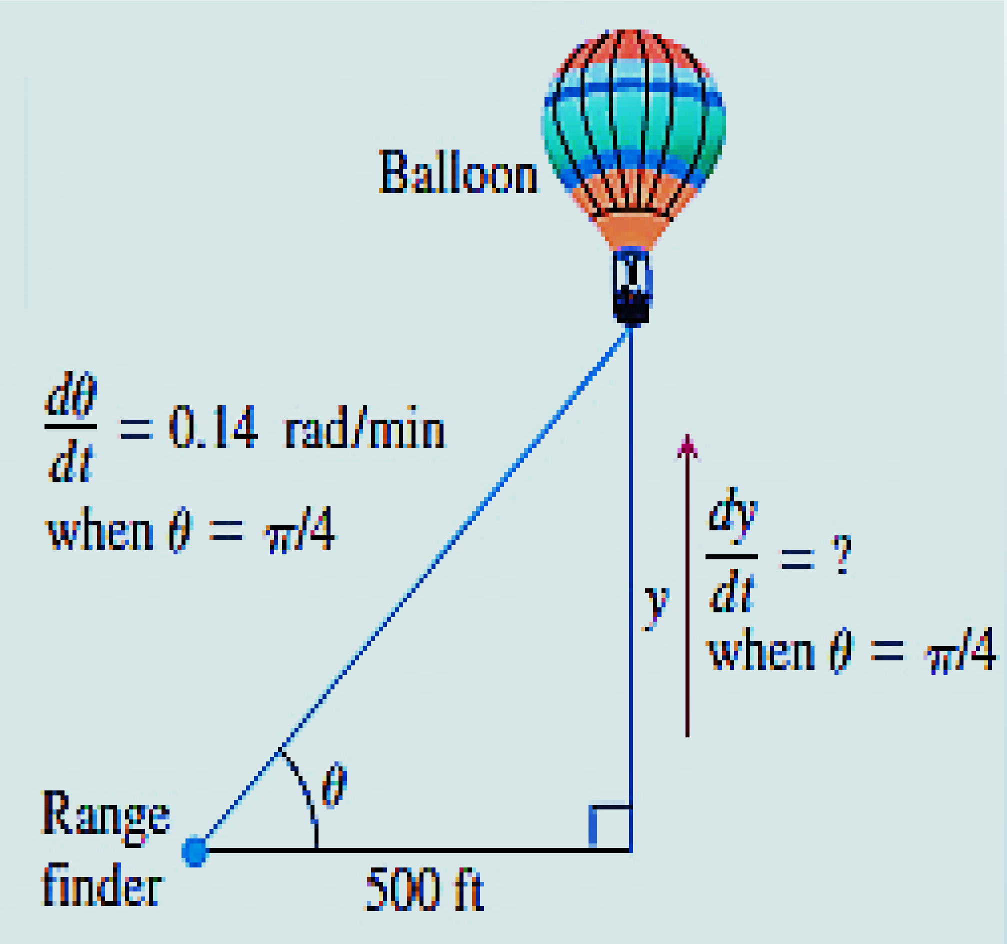
Differentiating both sides w.r.t "t" we get

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{500}\frac{dy}{dt}$$

$$\Rightarrow \left(\sec\frac{\pi}{4}\right)^2 (0.14) = \frac{1}{500}\frac{dy}{dt}$$

$$\Rightarrow \left(\sqrt{2}\right)^2 (0.14)\cdot 500 = \frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = 140\frac{ft}{min}.$$



EXAMPLE : A Highway Chase

A police cruiser, approaching a right angled intersection from the north is chasing a speeding car that has turned the corner and is now moving straight east. The cruiser is moving at $60\ mph$ and the police determine with radar that the distance between them is increasing at $20\ mph$. When the cruiser is $0.6\ mi$. north of the intersection and the car is $0.8\ mi$ to the east, what is the speed of the car?

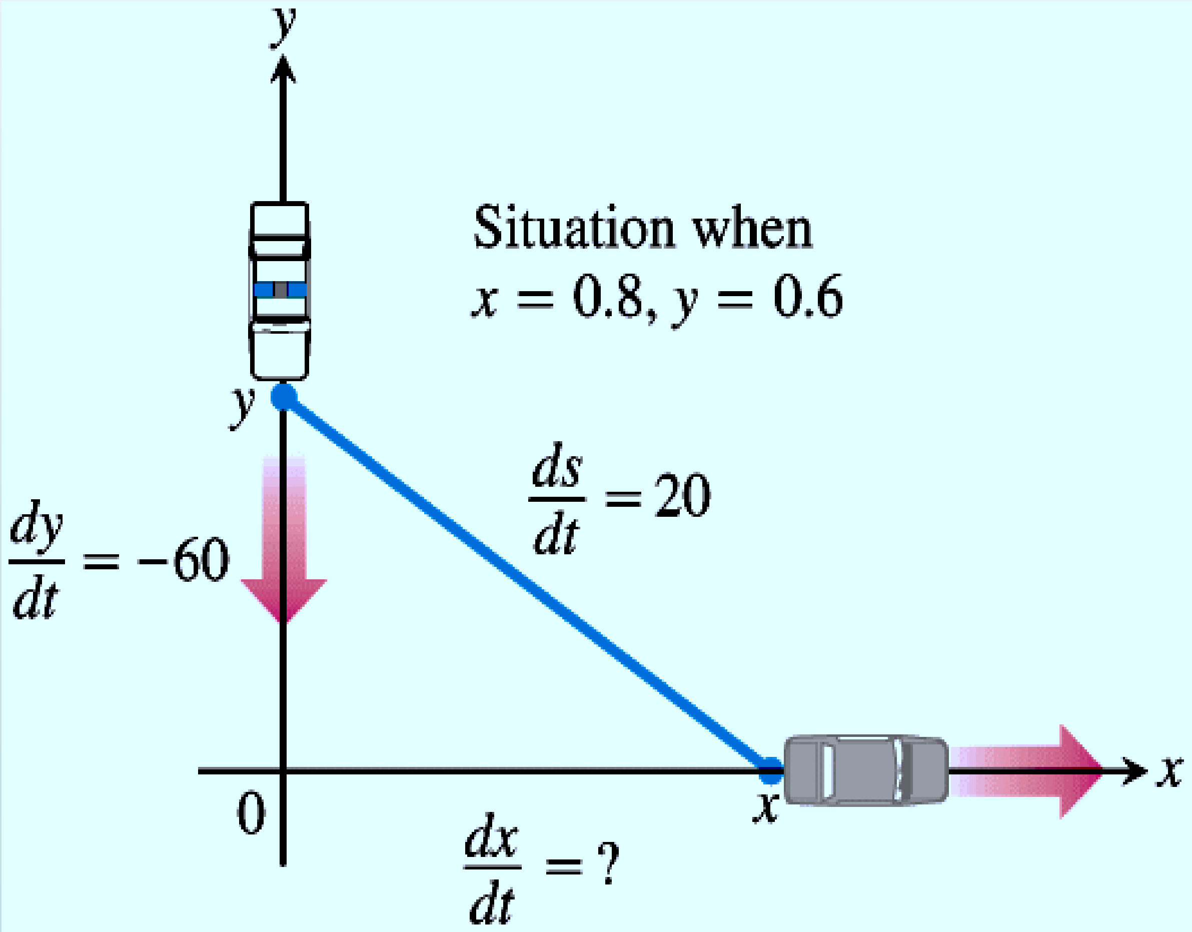
Solution:

We picture the car and cruiser in the coordinate plane, using the positive x −axis as the eastbound highway and the positive y −axis as the southbound highway. Let t represent time and set

x = position of car at time t

y = position of cruiser at time t

s = distance between car and cruiser at time t .



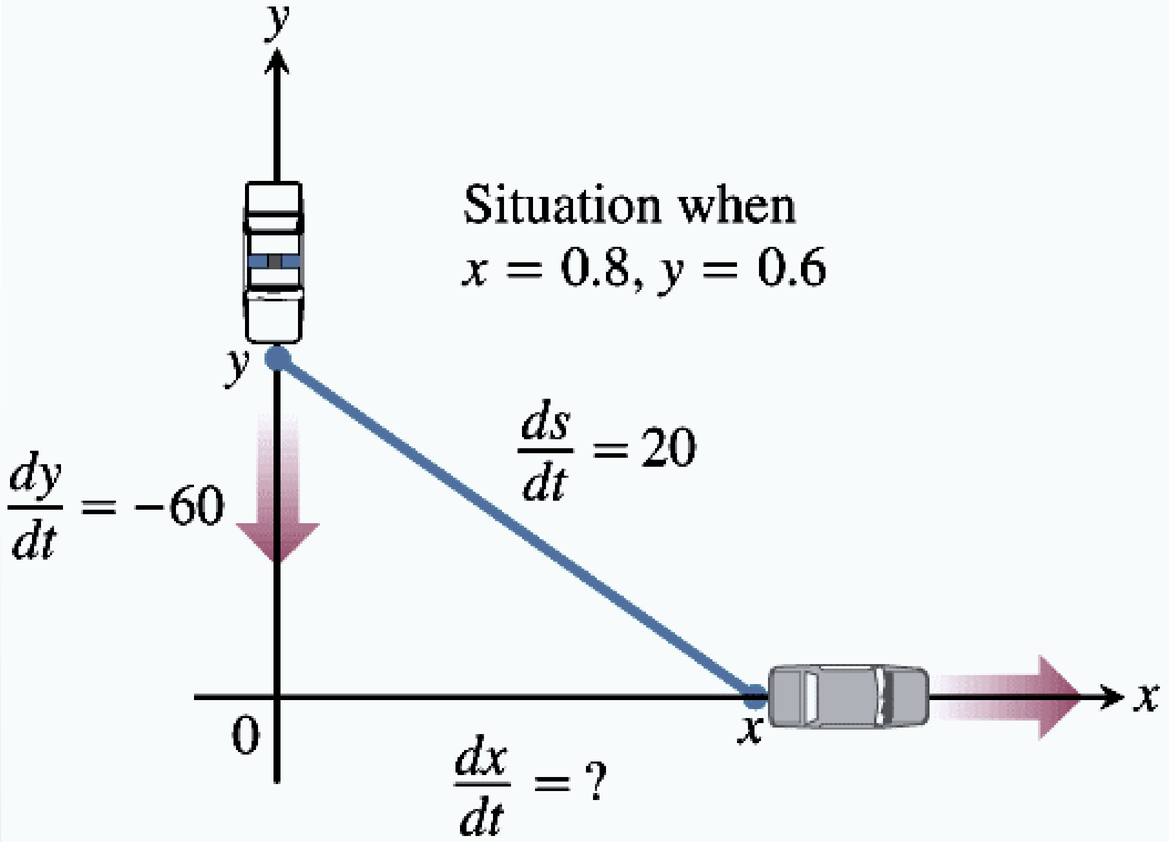
Solution:

Given: $\frac{ds}{dt} = 20mph$, $\frac{dy}{dt} = -60mph$

Find: $\frac{dx}{dt}$ when $x = .8, y = .6$

By Pythagoras theorem:

$$s^2 = x^2 + y^2$$
$$\Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$
$$\Rightarrow 2(1)(20) = 2(.8)\frac{dx}{dt} + 2(.6)(-60).$$
$$\Rightarrow \frac{dx}{dt} = 70mph.$$



If $x = .8, y = .6$ then $s = 1$

Practice Questions

Book: Thomas Calculus (11th Edition) by Georg B.Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter: 3**
  - **Exercise: 3.3**

    Q # 1 – 14, 19, 20, 23 – 30.
  - **Exercise: 3.7**

    Q # 1 – 38.