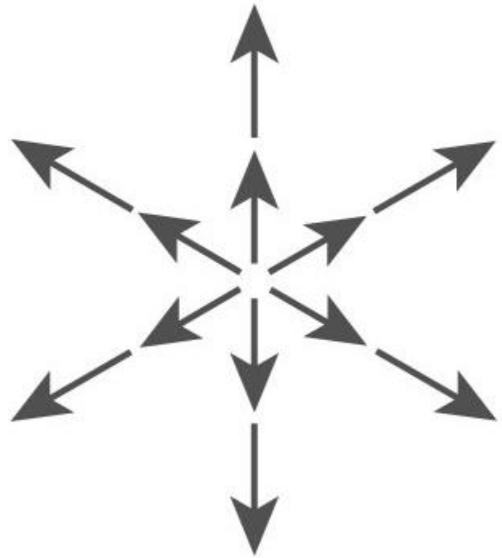
Divergence & Curl

Vector Calculus(MATH-243)

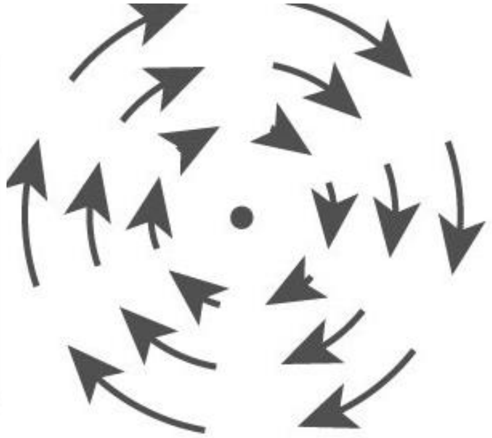
Instructor: Dr. Naila Amir

DIVERGENCE & CURL OF A VECTOR FIELD



$$\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\nabla \times \vec{V} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{pmatrix}$$



16

Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
    - **Section: 16.5**

Divergence

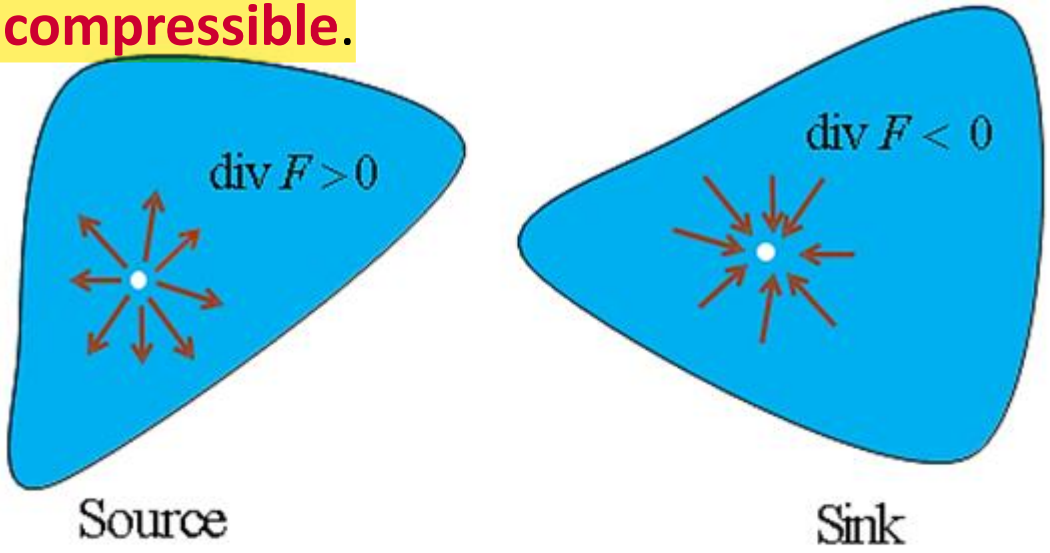
Let $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ be a differentiable vector field on a region in $\mathbb{R}^3$. Let $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ (the "del" or "nebla" operator), then the **divergence** of the vector field is defined as:

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

which is a scalar quantity.

Interpreting the Divergence

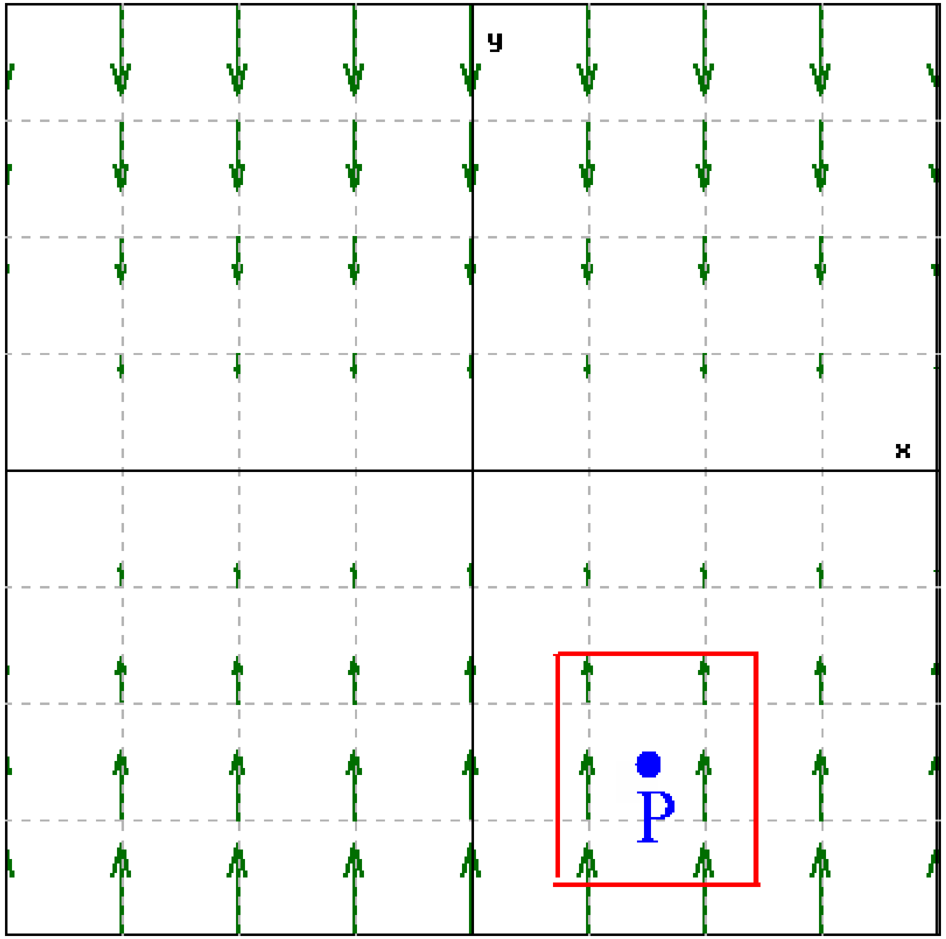
- Intuitively, the value of the divergence of a vector field at a particular point gives a measure of the net mass flow.
- Divergence (div) is "flux density"—the amount of flux entering or leaving a point. Think of it as the rate of flux expansion (positive divergence) or flux contraction (negative divergence).
- Alternatively, we can say that the divergence at a given point measures the **net flow** out of a small box around the point, that is, it measures what is produced (**source**) or consumed (**sink**) at a given point in space.
- In other words, $\operatorname{div}\mathbf{F}$ measures the tendency of the fluid to diverge from the point P. If $\operatorname{div}\mathbf{F} = 0$, then then the rate at which fluid is flowing into that point is equal to the rate at which fluid is flowing out. In this case, $\mathbf{F}$ is said to be **incompressible**.
- If we imagine $\mathbf{F}$ to be the velocity field of a gas (or a fluid), then $\operatorname{div}\mathbf{F}$ represents the rate of expansion per unit volume under the flow of the gas (or fluid).
- If $\operatorname{div}\mathbf{F} < 0$, the gas (or fluid) is compressing and $\operatorname{div}\mathbf{F} > 0$ represents that the gas (or fluid) is expanding.



Divergence

Example: Determine whether the divergence of each vector field at the indicated point P is positive, negative or zero

(a)



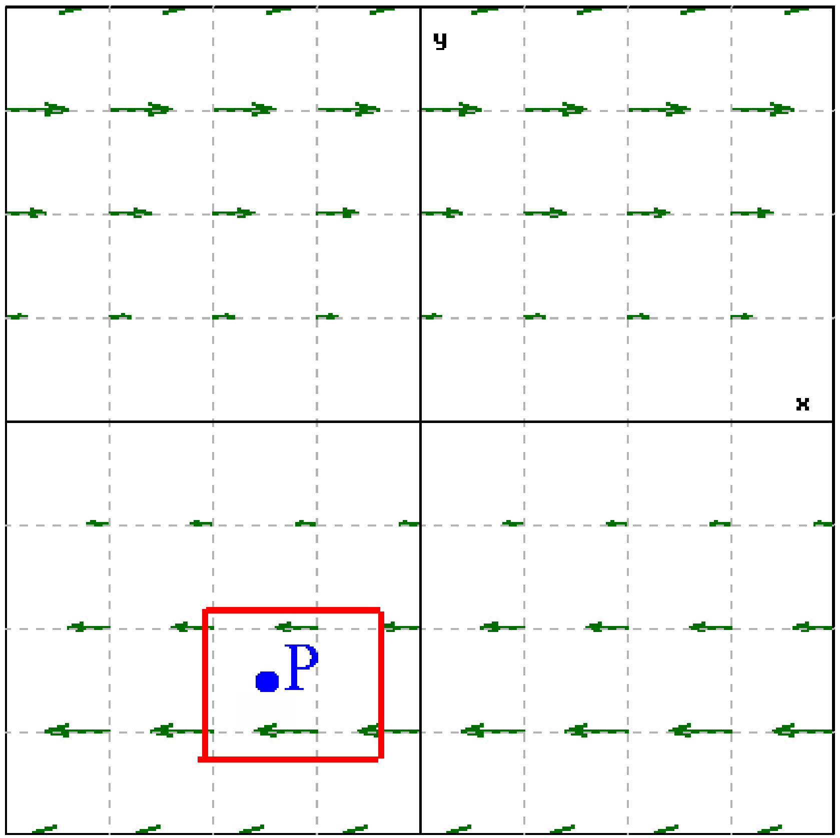
Draw a box around the point P.

We can clearly see from the magnitude of the vectors that the net flow out of the box is negative (more “stuff” coming in than out).

Thus, the divergence is **negative,** and we say the point P is a *sink*.

Divergence

(b)

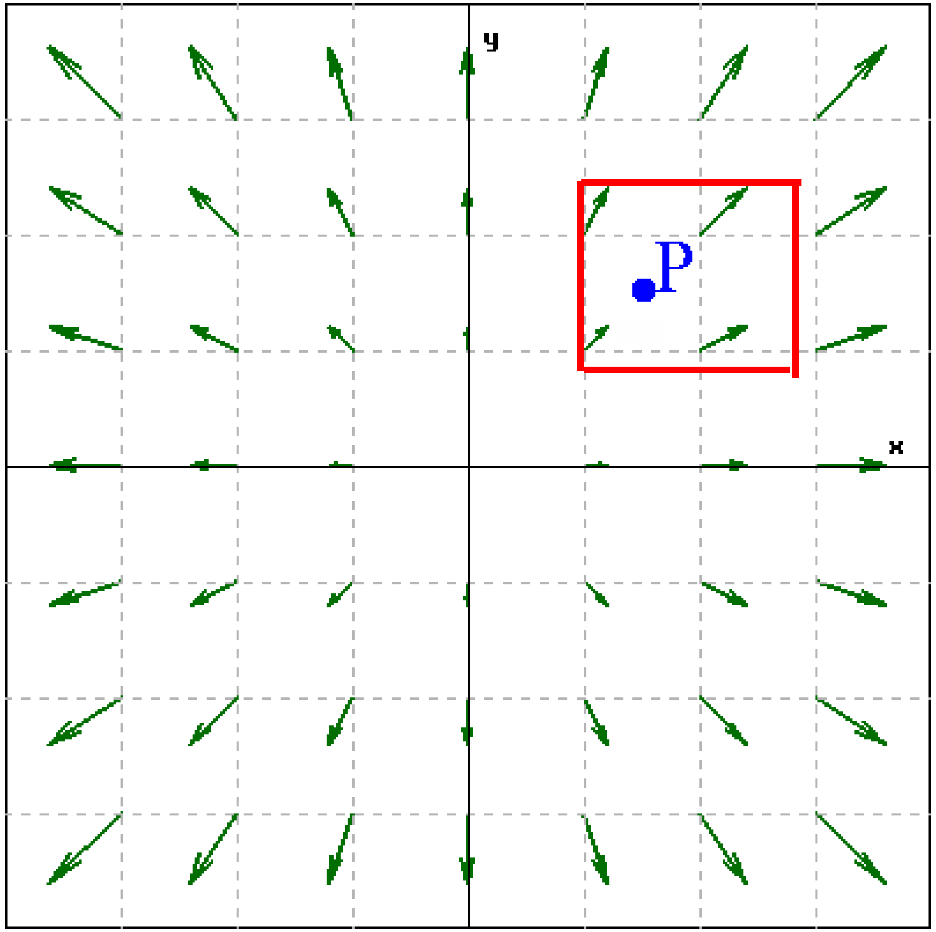


Draw a box around the point. We can see that the net flow out of the box is zero (same amount of "stuff" is coming in than coming out).

Thus, the divergence is **zero** and we say that the vector field is ***incompressible*** at the point.

Divergence

(c)



Draw a box around the point. The net flow is positive (less “stuff” coming in than out).

Thus, the divergence is **positive,** and we say that the point is a ***source*****.**

Curl

Let $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ be a vector field on $\mathbb{R}^3$ and assume that the partial derivatives of P, Q and R all exist. Let $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ (the "del" or "nebla" operator), then the **curl** of the vector field is defined as:

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

which is a vector quantity.

Examples:

Determine the curl of the following vector fields:

1. $\mathbf{F}(x, y, z) = \langle xyz, x^2 + 2yz, x^2 + y^2 + z^2 \rangle$.

   Solution: $\langle 0, xy - 2x, 2x - xz \rangle$.

2. $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$.

   Solution:$\langle 0,0,0 \rangle = \mathbf{0}$.

3. $\mathbf{F}(x, y, z) = \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle$.

   Solution:$\langle 0,0,0 \rangle = \mathbf{0}$.

Results

- A field with curl $\mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$ is called **curl free** or **irrotational** field.

- Recall that the gradient of a function of three variables is a vector field on $\mathbb{R}^3$ and so we can compute its curl. Thus, if $f(x, y, z)$ is a function of three variables that has continuous second order partial derivatives, then the **curl of a gradient vector field** is $\mathbf{0}$, i.e.,
$$\text{curl}(\nabla f) = \nabla \times (\nabla f) = \mathbf{0}.$$

- Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, above result can be rephrased as:

  If $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$.

  This provides a quick and easy way to verify that a 3D vector field is conservative.

- The 2D vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ can be written as a 3D vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y), 0 \rangle$. Then:
$$\nabla \times \mathbf{F} = \langle 0,0, Q_x - P_y \rangle.$$
  Therefore, in 2D, $\nabla \times \mathbf{F} = \mathbf{0}$ is equivalent to $Q_x = P_y$. Thus, a 2D vector field is conservative if $Q_x = P_y$.

Results

- The converse of previous statement that: if $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$. is not true in general, but the following theorem says the converse is true if $\mathbf{F}$ is defined everywhere. More generally, it is true if the domain of the vector field is simply connected– means **it has no hole** (in particular, the whole $\mathbb{R}^3$), then the converse of the previous statement is also true. In general, we say that:

"If $\mathbf{F}$ is a 3D vector field defined on all of $\mathbb{R}^3$ with continuous partial derivatives and curl $\mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative field."

Example:

Show that $\mathbf{F}(x, y, z) = \langle y, x + z, y + 6z \rangle$ is a conservative field. Moreover, find a function such that $\mathbf{F} = \nabla f$. In other words, determine the potential function f of the conservative field $\mathbf{F}$.

Solution:

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x+z & y+6z \end{vmatrix} = (1-1)\mathbf{i} - (0)\mathbf{j} + (1-1)\mathbf{k} = \mathbf{0}$$

F is irrotational, therefore there exists a potential function f such that $\mathbf{F} = \nabla f$

$$f_x = y \qquad f_y = x + z \qquad f_z = y + 6z$$

Integrate w.r.t x

$$f(x, y, z) = xy + g(y, z)$$

differentiate w.r.t y

$$f_y = x + g_y(y, z) = x + z$$

$$\Downarrow$$

$$g_y(y, z) = z \Longrightarrow g(y, z) = zy + h(z)$$

$$f(x, y, z) = xy + zy + h(z)$$

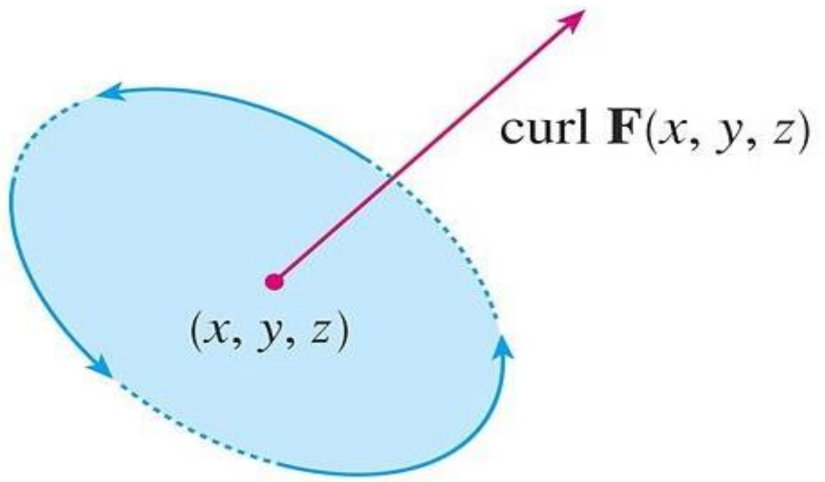
differentiate w.r.t z

$$f_z = y + h'(z) = y + 6z \longrightarrow h'(z) = 6z \longrightarrow h(z) = 3z^2 + K$$

$$\boxed{f(x, y, z) = xy + zy + 3z^2 + K}$$

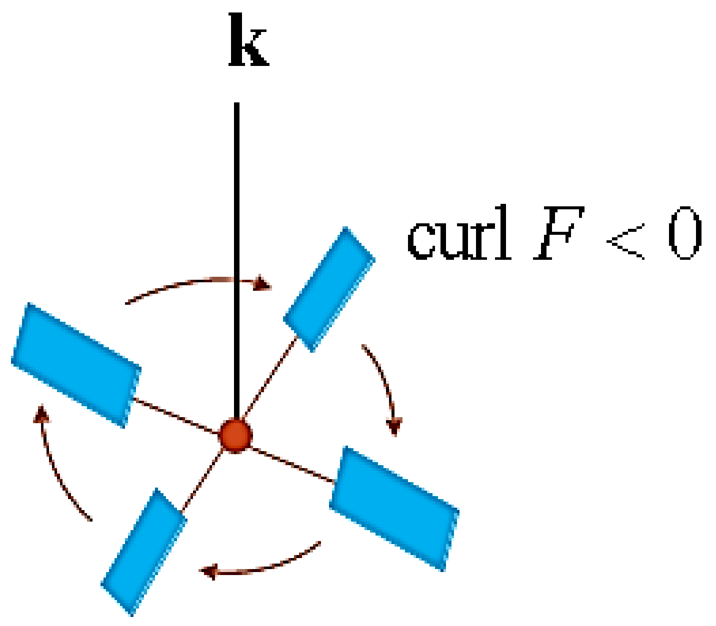
Curl

- The reason for the name *curl* is that the curl vector is associated with rotations.
- Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of curl $\mathbf{F}(x, y, z)$. The length of this curl vector is a measure of how quickly the particles move around the axis.
- If $\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$ at a point P, then the fluid is free from rotations at P and $\mathbf{F}$ is called irrotational at P. In other words, there is no whirlpool or eddy at P. If $\text{curl } \mathbf{F} = \mathbf{0}$, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis. If $\text{curl } \mathbf{F} \neq \mathbf{0}$, the paddle wheel rotates about its axis.

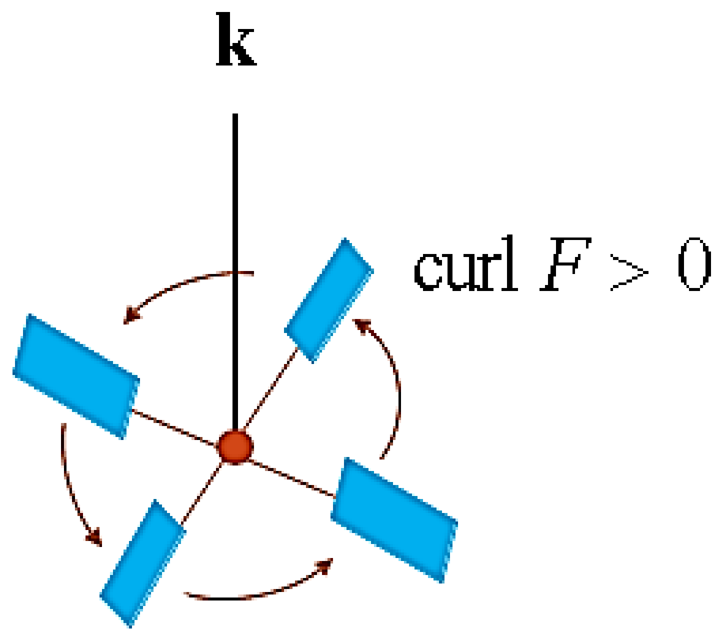


Interpreting the curl

- Curl is simply the circulation per unit area, circulation density, or rate of rotation (amount of twisting at a single point).
- Alternatively, if we think of the vector field as a velocity vector field of a fluid in motion, the curl measures rotation. Thus, at a given point, the curl is a vector parallel to the axis of rotation of flow lines near the point, with direction determined by the Right-Hand Rule.



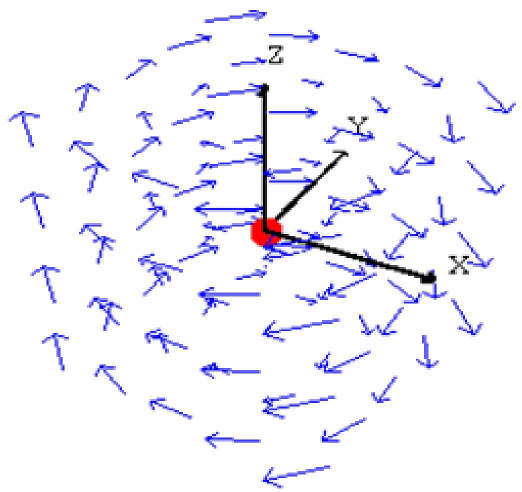
Clock wise rotation

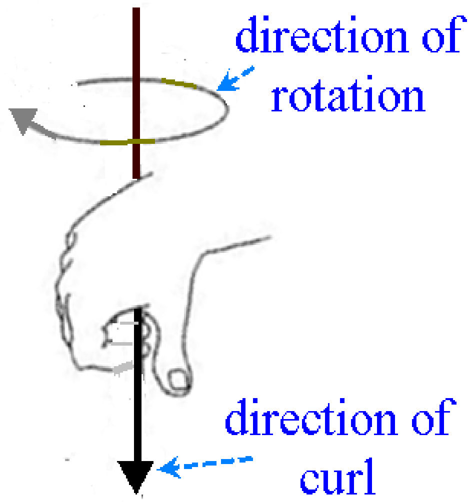


Counter-clock wise rotation

Curl

Example: Determine whether the curl of the vector field at the origin is the zero vector or points in the same direction as $\pm\,\mathbf{i}, \pm\mathbf{j}, \pm\,\mathbf{k}$.





Based on the direction of the rotation and the Right-Hand Rule, the curl will point in the $-\,\mathbf{k} = \langle 0,0,-1\rangle$ direction.

Practice:

For the vector fields:

1. $\mathbf{F}(x, y, z) = \langle x^2yz, xy^2z, xyz^2 \rangle$

2. $\mathbf{F}(x, y, z) = \langle e^x, e^{xy}, e^{xyz} \rangle$

determine the following:

a) The curl of $\mathbf{F}$.

b) The divergence $\mathbf{F}$.

c) Is the given field a conservative field? If yes, then determine the corresponding potential function.

Practice:

a) Show that

$$\mathbf{F}(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$$

is a conservative vector field.

b) Determine a function f such that $\mathbf{F} = \nabla f$.

Practice Questions

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

Chapter: 16

Exercise-16.5: Q – 1 to 18, Q – 23 to 32.