Taylor & Maclaurin Series

Book: Thomas Calculus (11th Edition) by
George B. Thomas, Maurice D. Weir,
Joel R. Hass, Frank R. Giordano

Chapter: 11 (11.8, 11.9)

Book: Calculus (5th Edition) by Swokowski,
Olinick and Pence

Chapter: 11 (11.8)

Calculus & Analytical Geometry MATH-101
Instructor: Dr. Naila Amir (SEECS, NUST)

$$Taylor\ Series: \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$Maclaurin\ Series: \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

Taylor & Maclaurin Series

If $f(x)$ is a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series** generated by $f(x)$ at $x = a$ is:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

For the special case: $a = 0$, the Taylor series becomes:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

Taylor & Maclaurin Series

- Let's investigate the more general question: Under what circumstances is a function equal to the sum of its Taylor series?

- In other words, if $f(x)$ has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- As with any convergent series, this means that $f(x)$ is the limit of the sequence of partial sums.

Taylor & Maclaurin Series

In the case of the Taylor series, the partial sums are:

$f(x) = T_n(x) + R_n(x)$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notice that $T_n(x)$ is a polynomial of degree n called the ***n*th-degree Taylor polynomial** of $f(x)$ at a.

Taylor & Maclaurin Series

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x)$$

then $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, for some c between x and a is called the **remainder** of the Taylor series. If we can show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that:

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}[f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) = f(x).$$

Taylor & Maclaurin Series

- We have therefore proved the following:

  If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the nth-degree Taylor polynomial of $f(x)$ at a and
  $$\lim_{n\to\infty} R_n(x) = 0,$$
  for $|x-a| < R$, then $f(x)$ is equal to the sum of its Taylor series on the interval $|x-a| < R$.

- In trying to show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function $f(x)$, we usually use the following theorems:

Theorems

Taylor's Inequality/The Remainder Estimation Theorem:
If $|f^{(n+1)}(c)| \leq M$ for all c between x and a, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

If this condition holds for every n then, the series converges to $f(x)$.

Theorem:
If x is any real number, then

$$\lim_{n\to\infty} \frac{|x|^n}{n!} = 0.$$

$\sum \frac{x^n}{n!}$

Example

Prove that $f(x) = e^x$ is equal to the sum of its Maclaurin series.

Solution:

If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n. Note that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x); \qquad R_n(x) = \frac{e^c}{(n+1)!}x^{n+1} \text{ for some } c \text{ between } 0 \text{ and } x.$$

Since e^x is an increasing function of x, e^c lies between $e^0 = 1$ and e^x. When x is negative, so is c, and $e^c < 1$. When $x = 0$, $e^x = 1$ and $R_n(x) = 0$. When x is positive, so is c and $e^c < e^x$. Thus,

$$\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}; \quad \text{for } x \le 0 \quad \text{and} \quad \lim_{n\to\infty} |R_n(x)| < \lim_{n\to\infty} \frac{e^x|x|^{n+1}}{(n+1)!}; \text{for } x > 0$$

In both cases, $\lim_{n\to\infty} |R_n(x)| = 0 \Longrightarrow \lim_{n\to\infty} R_n(x) = 0$. Hence, e^x is equal to the sum of its Maclaurin series.

Example:

Find the Maclaurin series generated by $f(x) = (1+x)^k$, where k is any real number.

Solution:

For the present case:

$$f(x) = (1+x)^k; \qquad f(0) = 1,$$

$$f'(x) = k(1+x)^{k-1}; \qquad f'(0) = k,$$

$$f''(x) = k(k-1)(1+x)^{k-2}; \qquad f''(0) = k(k-1),$$

$$f'''(x) = k(k-1)(k-2)(1+x)^{k-2}; \qquad f''(0) = k(k-1)(k-2),$$

$$\vdots \qquad \qquad \vdots$$

$$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-3};$$

and $$f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1).$$

Example:

Thus, the Maclaurin series generated by $f\ (x) = (1+x)^k$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

This series is called the **binomial series**. Notice that if k is a nonnegative integer, then the terms are eventually 0 and so the series is finite. For other values of k none of the terms is 0 and so we can try the Ratio Test. For the present case:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)\,x^n}\right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{k-n}{n+1}\right| = |x|,$$

$|x| < R$

so, by the Ratio Test, the series converges when $|x| < \boxed{1}$, and the radius of convergence is $R = 1$.

Some Important Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Multiplication & Division of Power Series

Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.

Solution:

Using the Maclaurin series for e^x and $\sin x$, we have:

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)$$

We multiply these expressions, collecting like terms just as for polynomials:

$$\begin{array}{rl}
 & 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \\
\times & x \qquad\qquad - \frac{1}{6}x^3 + \cdots \\
\hline
 & x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \cdots \\
+ & \qquad\qquad - \frac{1}{6}x^3 - \frac{1}{6}x^4 - \cdots \\
\hline
 & x + x^2 + \frac{1}{3}x^3 + \cdots
\end{array}$$

Multiplication & Division of Power Series

Thus, we have:

$$e^x \sin x = x + x^2 + \tfrac{1}{3}x^3 + \cdots$$

Solution: (b) $\tan x$.

Note that

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}$$

Multiplication & Division of Power Series

We use a procedure like long division:

$$
\begin{array}{r|l}
 & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\
\hline
1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \\
 & x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots \\
\cline{2-2}
 & \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots \\
 & \frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots \\
\cline{2-2}
 & \frac{2}{15}x^5 + \cdots
\end{array}
$$

Thus, we get:

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

Polar Coordinates

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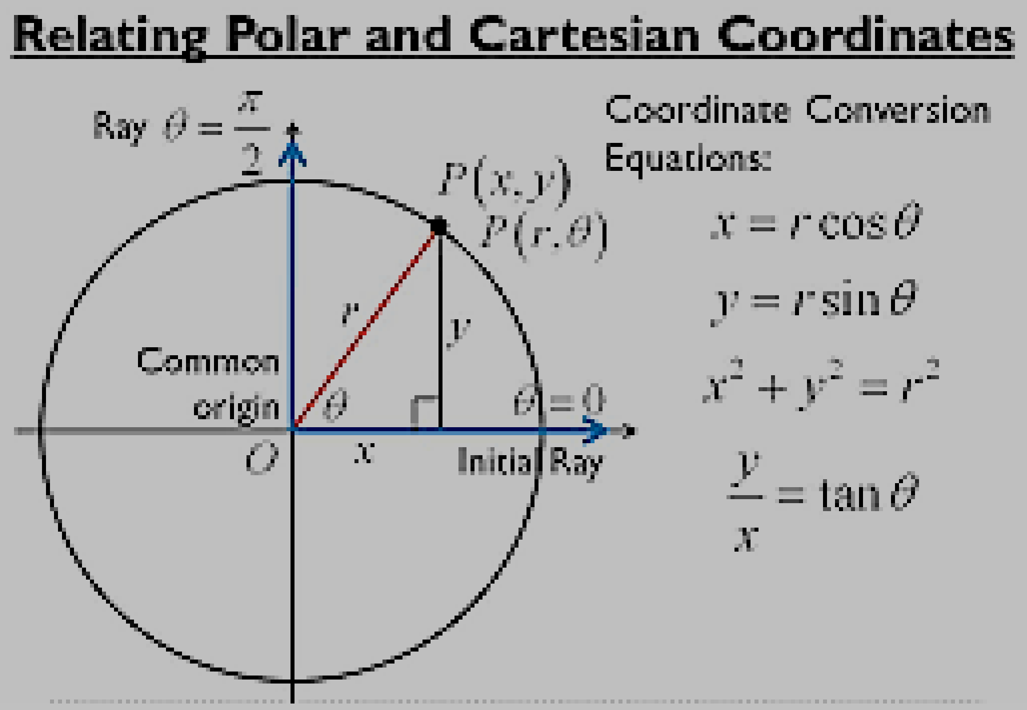
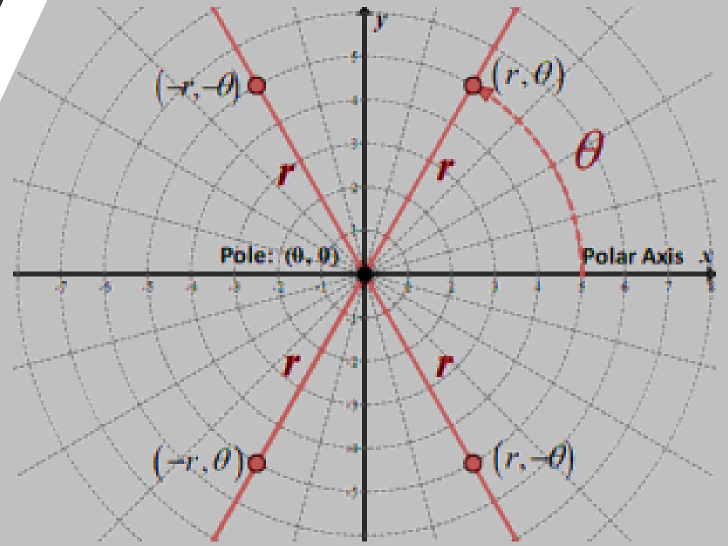
Chapter: 10 (10.5, 10.6)

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

Chapter: 13 (13.3)





Calculus & Analytical Geometry MATH-101

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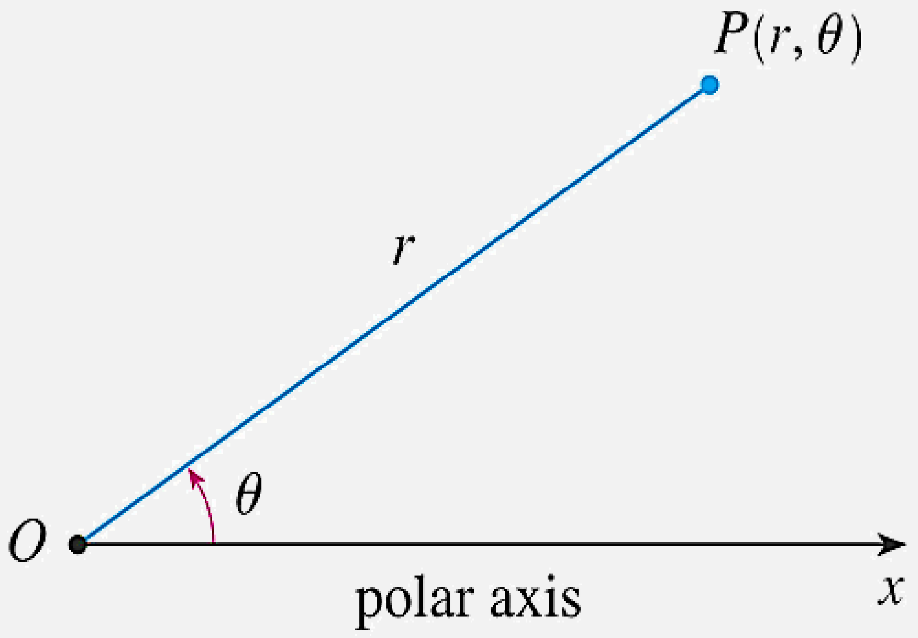
Polar Coordinates

- A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates.
- Usually, we use Cartesian coordinates, which are directed distances from two perpendicular axes.
- Here, we describe a coordinate system introduced by Newton, called the polar coordinate system.
- We choose a point in the plane that is called the **pole** (or origin) and is labeled O.
- Then, we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drawn corresponding to the positive x −axis in Cartesian coordinates.

Polar Coordinates

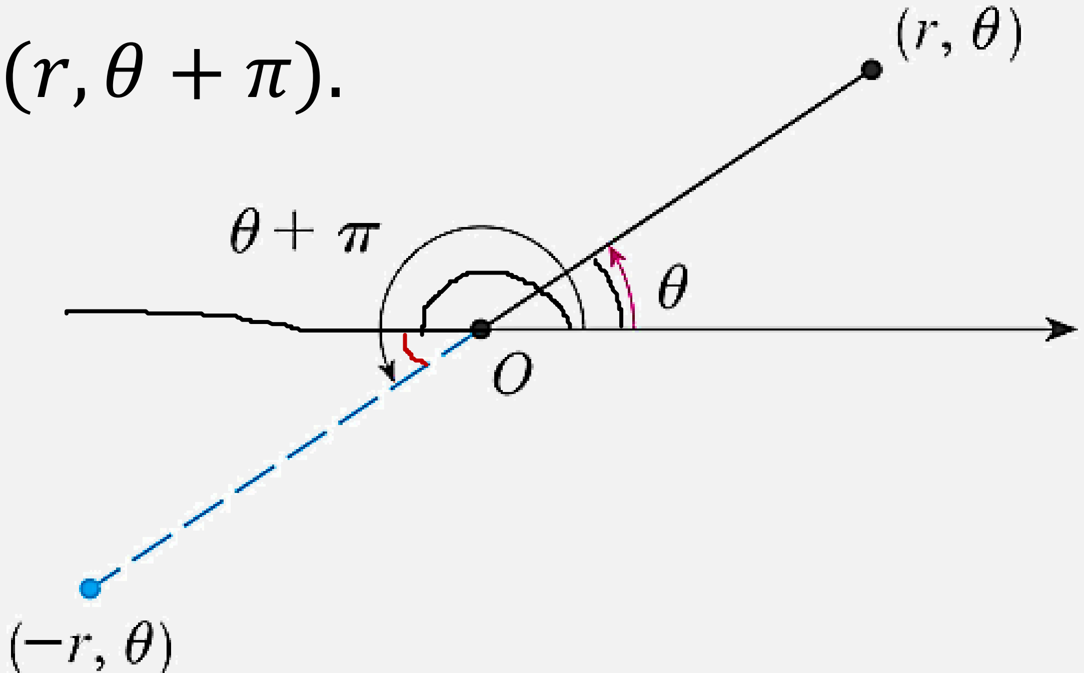
- If P is any point in the plane, let:
  - r be the distance from O to P.
  - θ be the angle (usually measured in radians) between the polar axis and the line OP.
- P is represented by the ordered pair (r, θ). r, θ are called polar coordinates of P.
- We use the convention that an angle is:
  - Positive—if measured in the counterclockwise direction from the polar axis.
  - Negative—if measured in the clockwise direction from the polar axis.



Polar Coordinates

- Note that the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O.
  - If $r > 0$, the point (r, θ) lies in the same quadrant as θ.
  - If $r < 0$, it lies in the quadrant on the opposite side of the pole.

- Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.



Example

Plot the points whose polar coordinates are given:

a. $(1, 5\pi/4)$

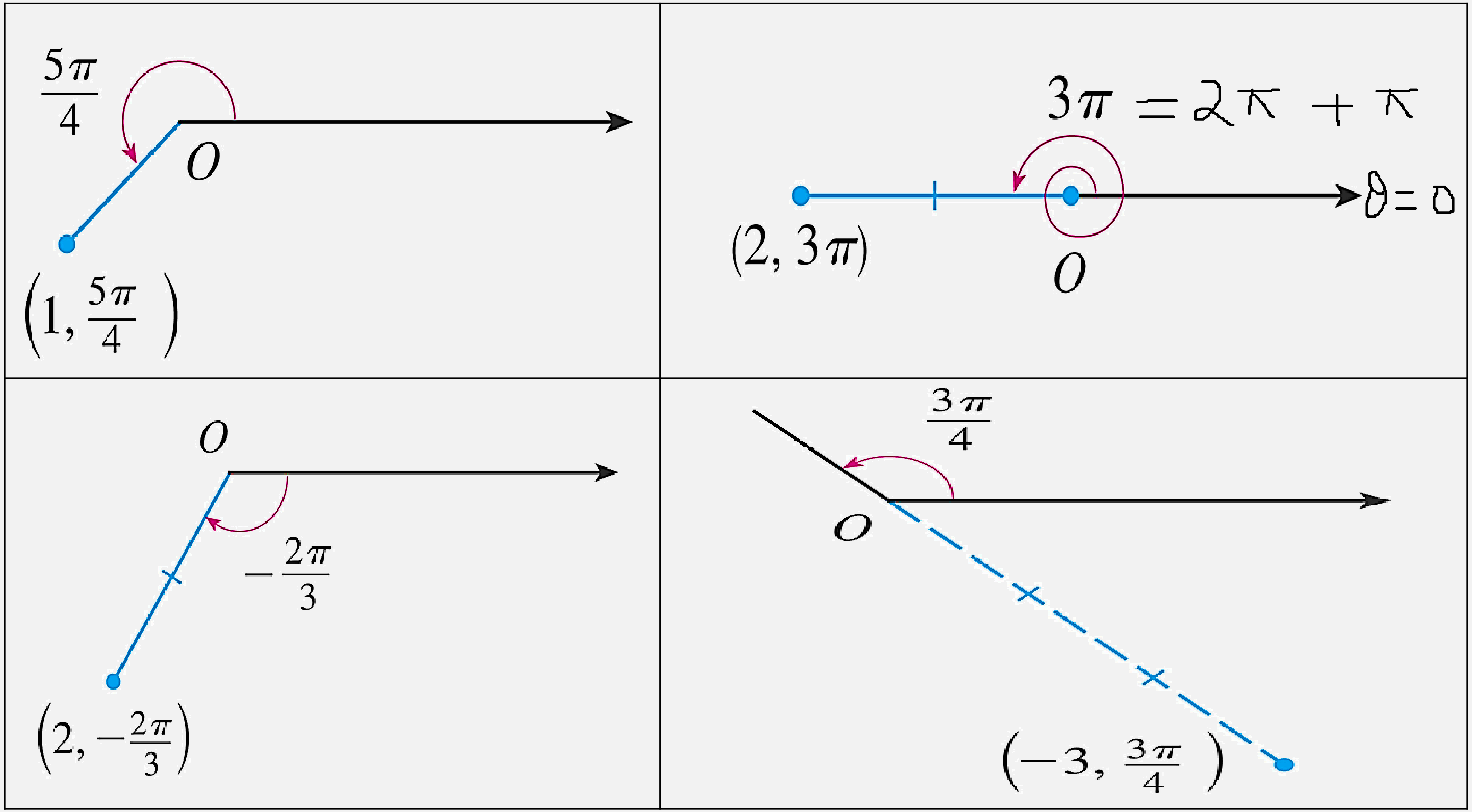
b. $(2, 3\pi)$

c. $(2, -2\pi/3)$

d. $(-3, 3\pi/4)$

Solution

—

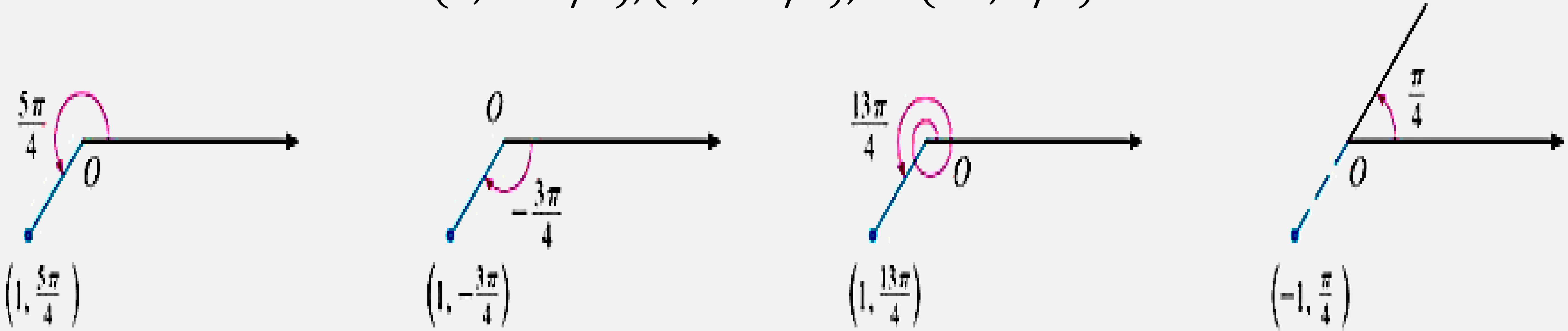


$\frac{5\pi}{4}$
O
$\left(1, \frac{5\pi}{4}\right)$
$3\pi = 2\pi + \pi$
$\theta = 0$
$(2, 3\pi)$
O
O
$-\frac{2\pi}{3}$
$\left(2, -\frac{2\pi}{3}\right)$
$\frac{3\pi}{4}$
O
$\left(-3, \frac{3\pi}{4}\right)$

Cartesian Vs. Polar Coordinates

- In the Cartesian coordinate system, every point has only one representation.
- However, in the polar coordinate system, each point has many representations.
- For instance, the point $(1, 5\pi/4)$ in previous example could be written as:

$$(1, -3\pi/4), (1, 13\pi/4), or\ (-1, \pi/4).$$



Example

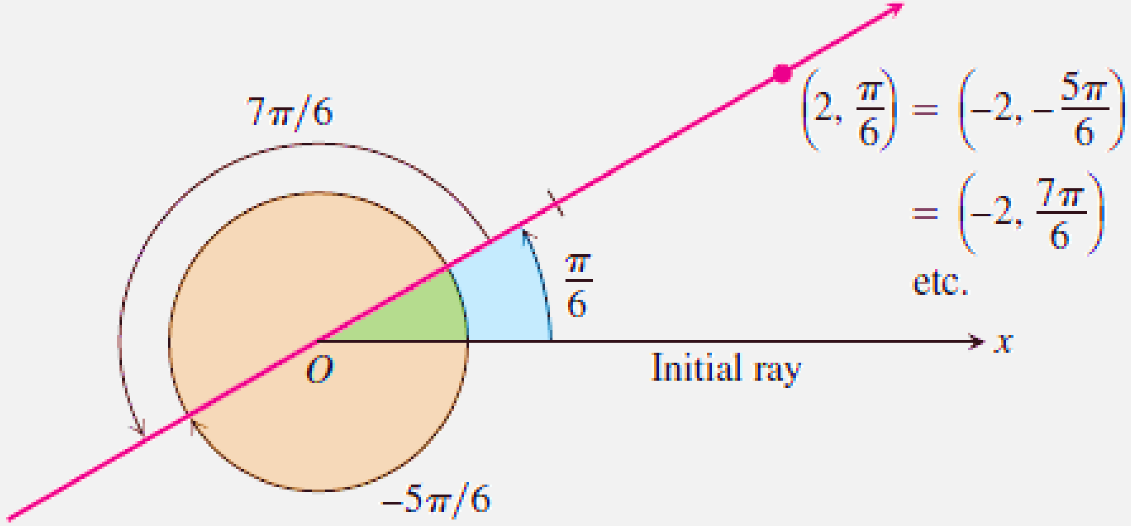
Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution:

We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$. We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

For $r = 2$, the complete list of angles is:

$$\frac{\pi}{6}, \frac{\pi}{6} \pm 2\pi, \frac{\pi}{6} \pm 4\pi, \frac{\pi}{6} \pm 6\pi, \ldots$$



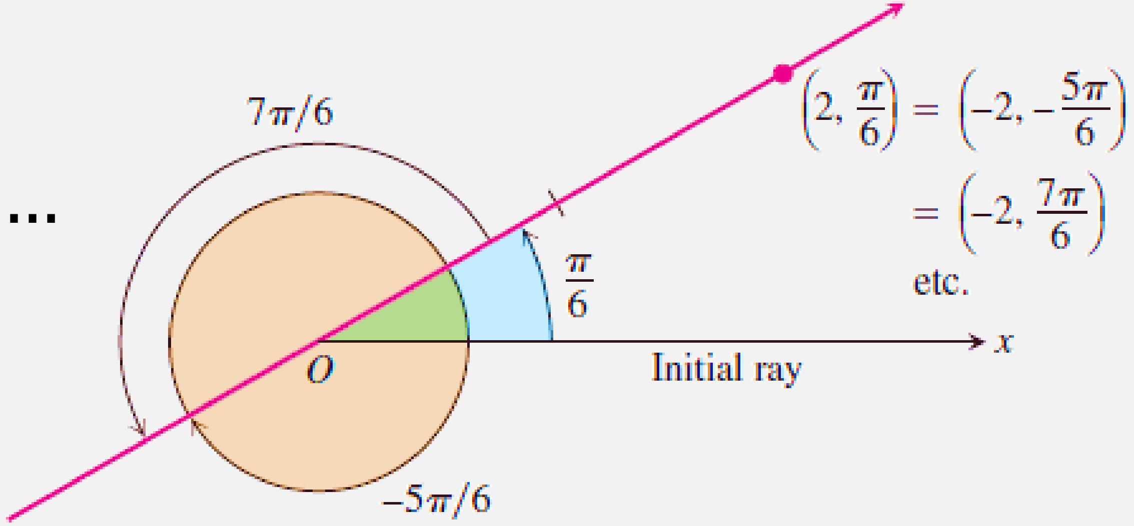
For $r = -2$, the angles are:

$$-\frac{5\pi}{6}, -\frac{5\pi}{6} \pm 2\pi, -\frac{5\pi}{6} \pm 4\pi, -\frac{5\pi}{6} \pm 6\pi, \ldots$$

Example

The corresponding coordinate pairs of P are:

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$



and

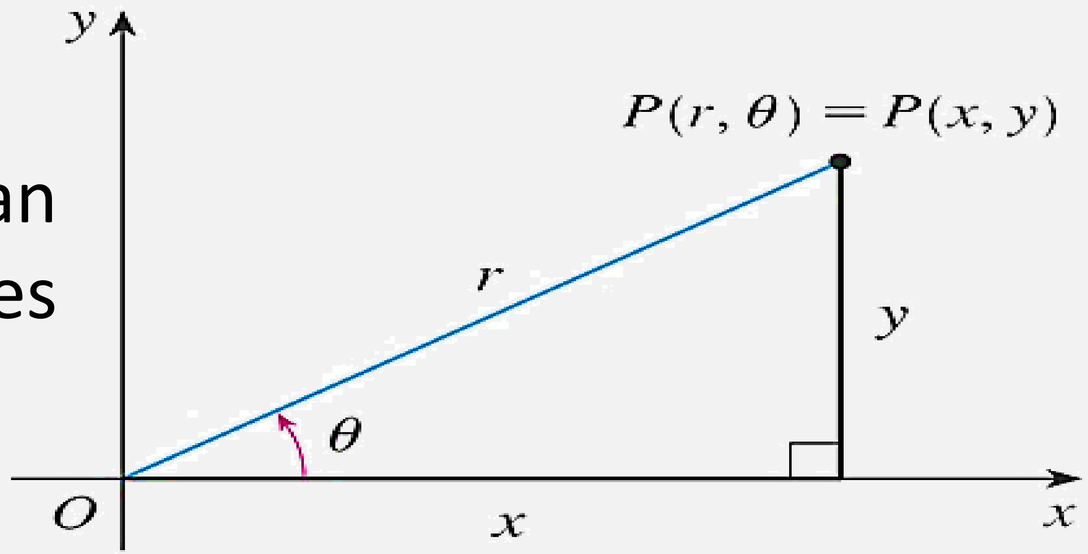
$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

Cartesian & Polar Coordinates

- The connection between polar and Cartesian coordinates can be seen from the figure.
  - The pole corresponds to the origin.
  - The polar axis coincides with the positive *x*-axis
- If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then from the figure, we have:

$$x = r\cos\theta\,, \qquad y = r\sin\theta$$

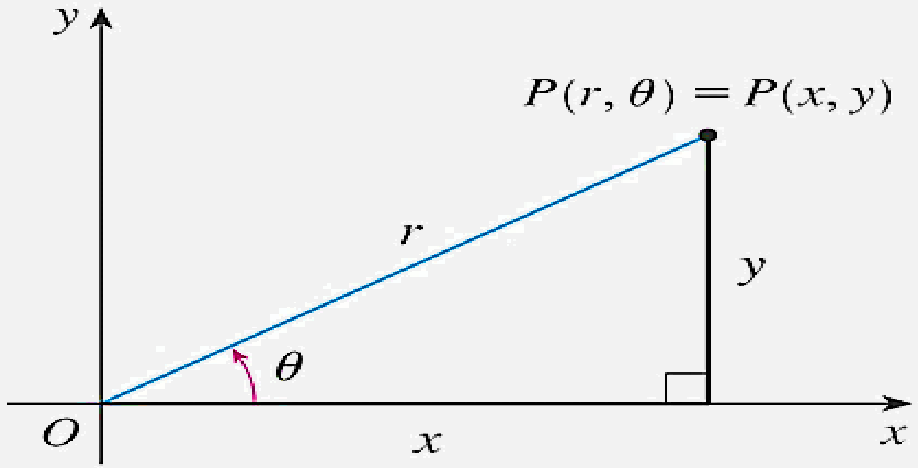
- These equations allow us to find the Cartesian coordinates of a point when the polar coordinates are known.



Cartesian & Polar Coordinates

To find r and θ when x and y are known, we use the equations

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}.$$



Example:
Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.
Solution:
For the present case $r = 2$ and $\theta = \pi/3$. Now

$$x = r\cos\theta = 2\cos(\pi/3) = 1,$$
$$y = r\sin\theta = 2\sin(\pi/3) = \sqrt{3}.$$

Thus, in Cartesian coordinates the point is: $(1, \sqrt{3})$.

Example

Convert the following equation in terms of polar coordinates:

$$x^2 + (y-3)^2 = 9. \quad (1)$$

Solution:

We know that: $x = r\cos\theta$ and $y = r\sin\theta$. Thus, eq. (1) takes the form:

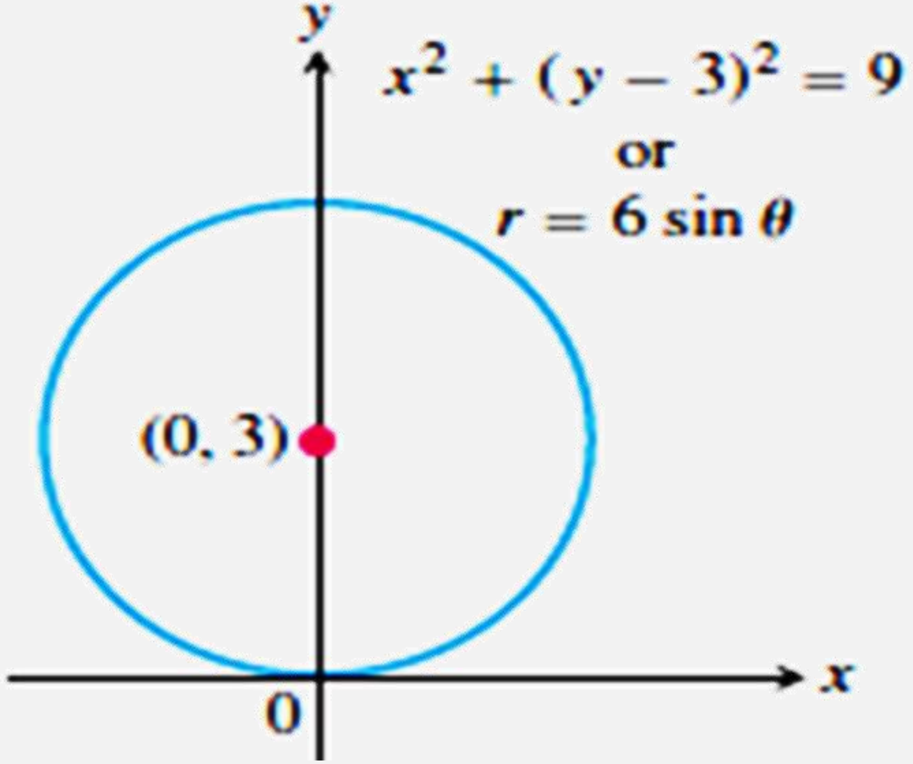
$$(r\cos\theta)^2 + (r\sin\theta - 3)^2 = 9$$

$$\Longrightarrow r^2\cos^2\theta + r^2\sin^2\theta - 6r\sin\theta + 9 = 9$$

$$\Longrightarrow r^2 - 6r\sin\theta = 0$$

$$\Longrightarrow r^2 = 6r\sin\theta$$

$$\Longrightarrow r = 6\sin\theta$$



Example

Determine the equivalent equations in terms of Cartesian coordinates for the following equations given in polar coordinates and identify their graphs.

a) $r^2 = 4r\cos\theta$

b) $r\cos\theta = -4.$

c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution

c) $r = \frac{4}{2\cos\theta - \sin\theta}$

$$\Longrightarrow r(2\cos\theta - \sin\theta) = 4$$

$$\Longrightarrow 2r\cos\theta - r\sin\theta = 4$$

$$\Longrightarrow 2x - y = 4$$

$$\Longrightarrow y = 2x - 4 = 2(x - 2).$$

This is equation of a line with slope $m = 2$, and y −intercept $c = -4$.

Polar Equations and Graphs

- One way to graph a polar equation $r = f(\theta)$ is to make a table of (r, θ) $-$values plot the corresponding points, and connect them in order of increasing θ.

- This can work well if enough points have been plotted to reveal all the loops and dimples in the graph.

- Another method of graphing that is usually quicker and more reliable is to:

  - First graph $r = f(\theta)$ in the *Cartesian* $r\theta - \text{plane}$,

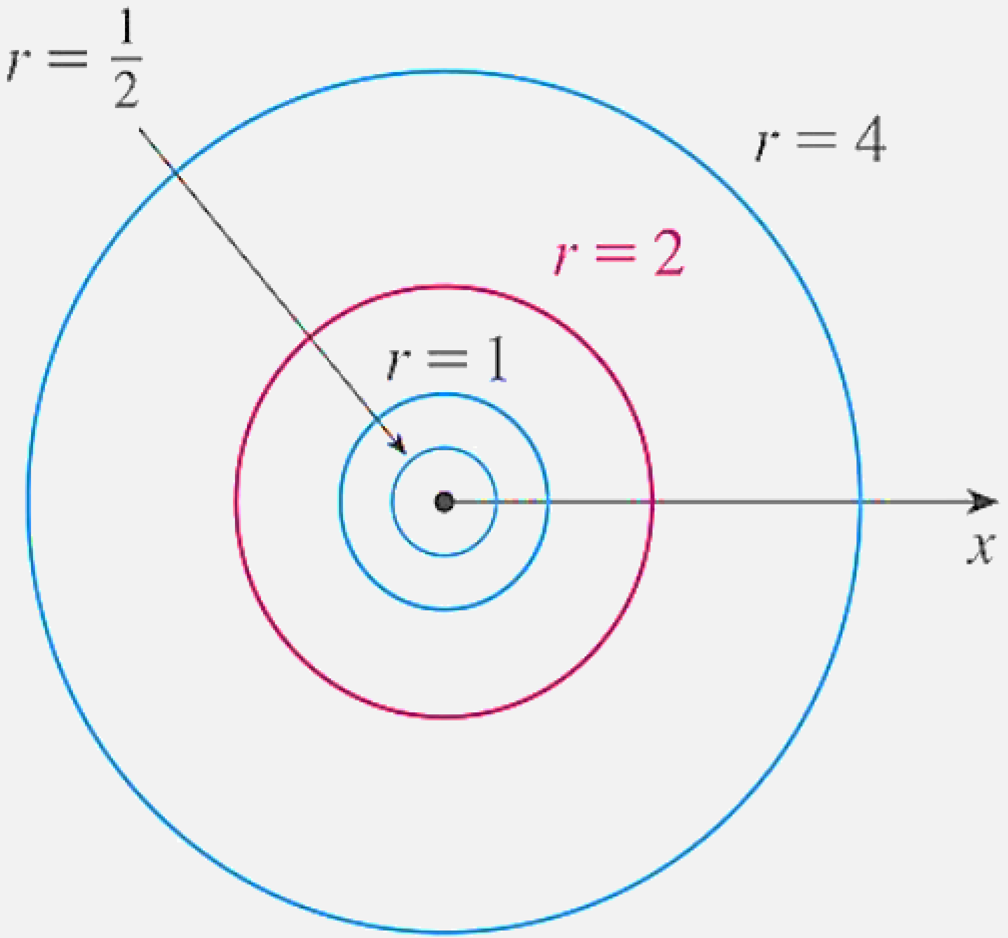
  - then use the Cartesian graph as a "table" and guide to sketch the *polar* coordinate graph.

Example

What curve is represented by the polar equation $r = 2$?

Solution:

The curve consists of all points (r, θ) with $r = 2$. Here r represents the distance from the point to the pole. Thus, the curve $r = 2$ represents the circle with center O and radius 2. In general, If we hold r fixed at a constant value $r = a \neq 0$, the point $P(r, \theta)$ will lie $|a|$ units from the origin O. As θ varies over any interval of length 2π, P then traces a circle of radius $|a|$ centered at O.

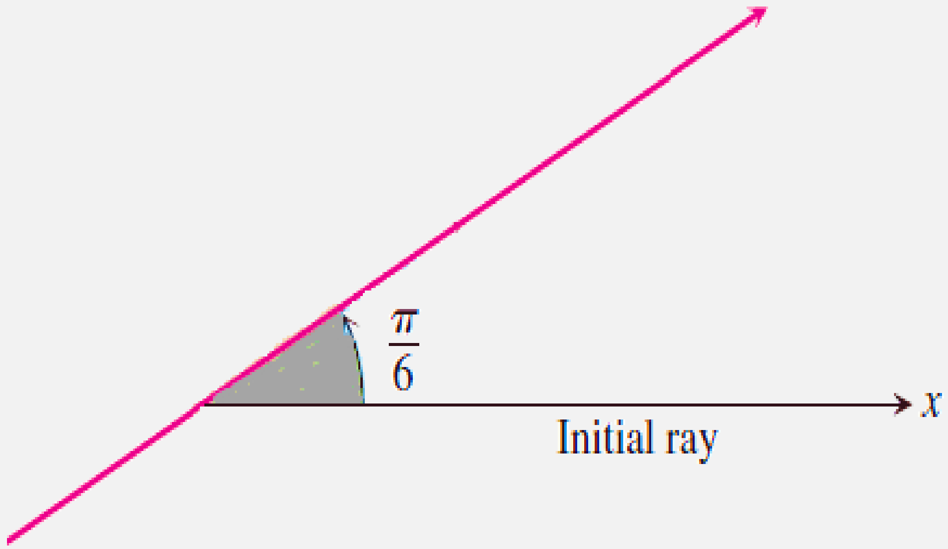


Example

What curve is represented by the polar equation $\theta = \pi/6$?

Solution:

In general, if we hold θ fixed at a constant value $\theta = \theta_0$, and let r vary between $-\infty$ and ∞, then the point $P(r, \theta)$ traces the line through O that makes an angle of measure θ_0 with the initial ray. Thus, $\theta = \pi/6$ is a line through O making an angle $\pi/6$ with the initial ray.



Example

Graph the sets of points whose polar coordinates satisfy the following conditions:

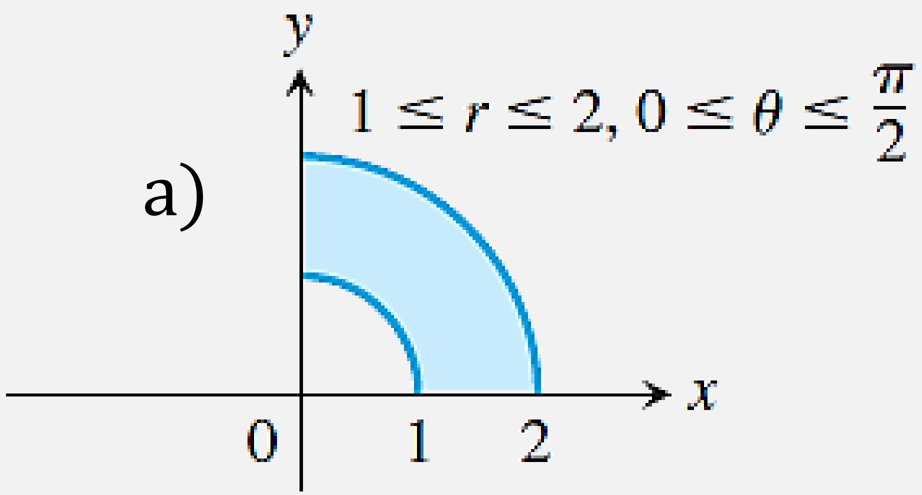
a) $1 \leq r \leq 2 \quad \text{and} \quad 0 \leq \theta \leq \frac{\pi}{2}$.

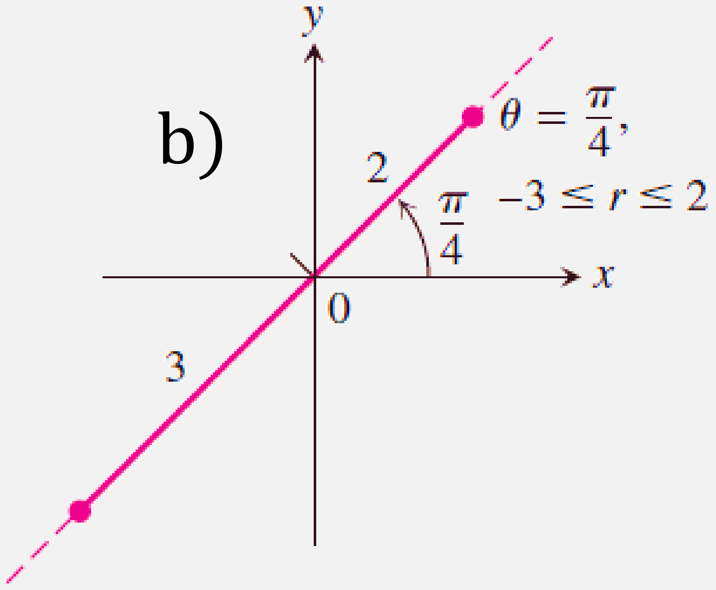
b) $-3 \leq r \leq 2 \quad \text{and} \quad \theta = \frac{\pi}{4}$.

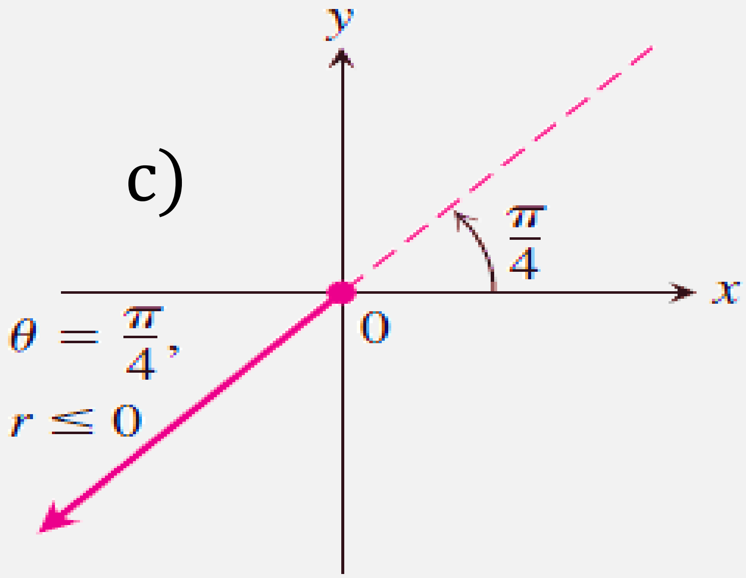
c) $r \leq 0 \quad \text{and} \quad \theta = \frac{\pi}{4}$.

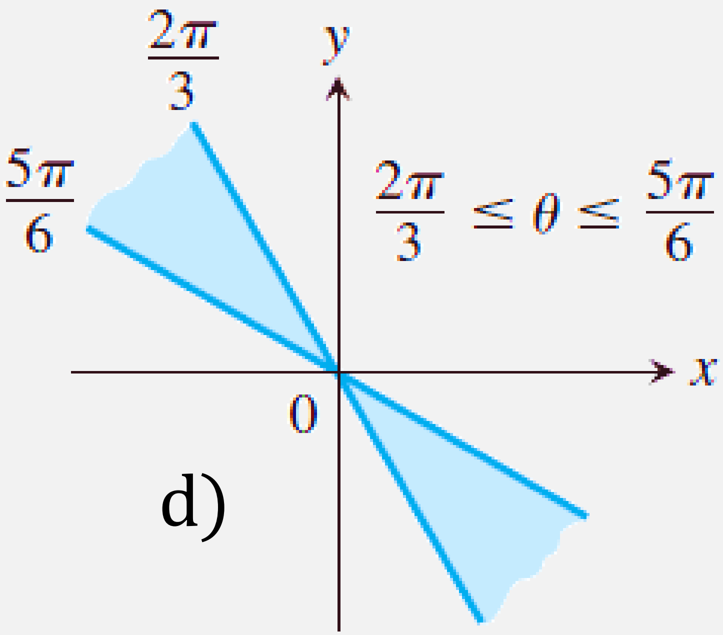
d) $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6} \qquad (\text{no restriction on } r)$.

Solution


a)
$1 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{2}$
y
x
0
1
2


b)
y
$\theta = \frac{\pi}{4}$,
$-3 \leq r \leq 2$
2
$\frac{\pi}{4}$
x
0
3


c)
y
$\frac{\pi}{4}$
x
$\theta = \frac{\pi}{4}$,
$r \leq 0$
0


$\frac{2\pi}{3}$
y
$\frac{5\pi}{6}$
$\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$
x
0
d)

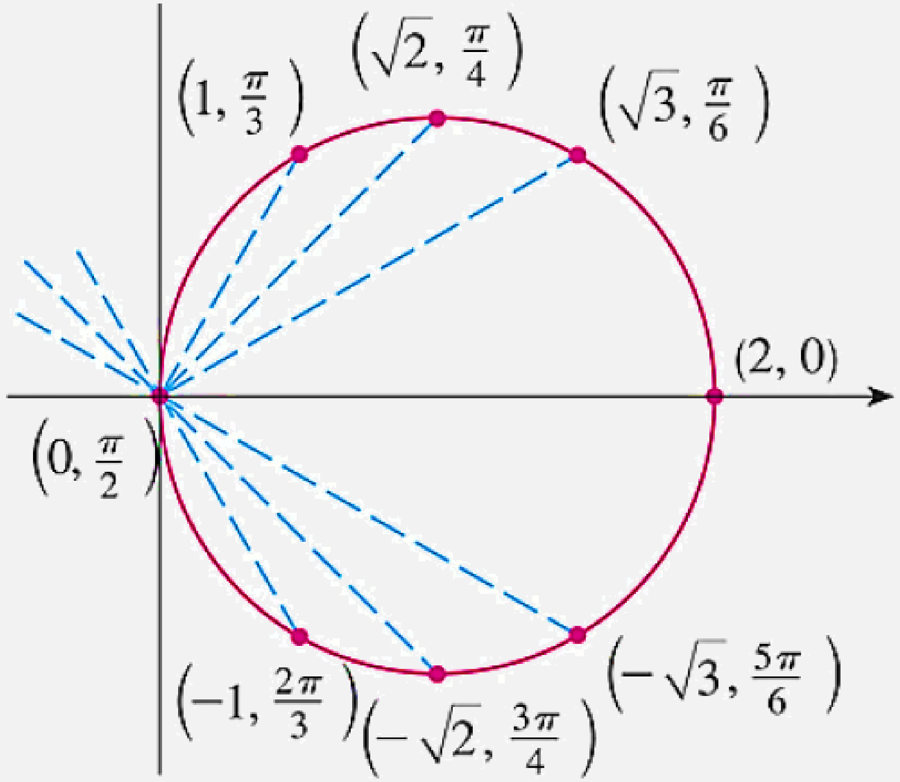
Example

Sketch the curve with polar equation $r = 2\cos\theta.$

Solution:

First, we find the values of r for some convenient values of $\theta.$ We plot the corresponding points (r, θ). Then, we join these points to sketch the curve.

The curve appears to be a **circle.** Note that we have used only values of θ between 0 and π. Since, if we let θ increase beyond π, we obtain the same points again.



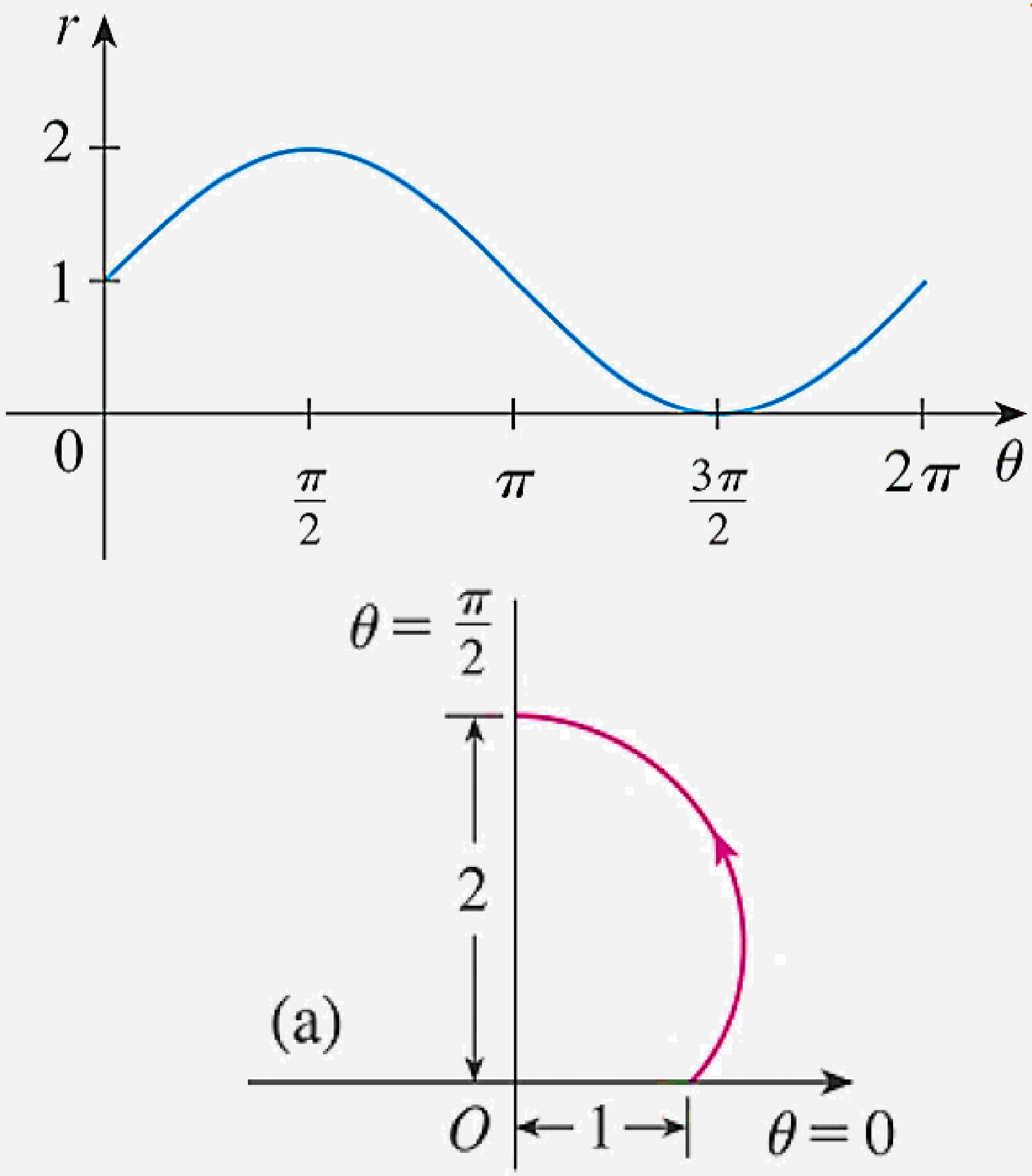
θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

Example

Sketch the curve $r = 1 + \sin\theta$.

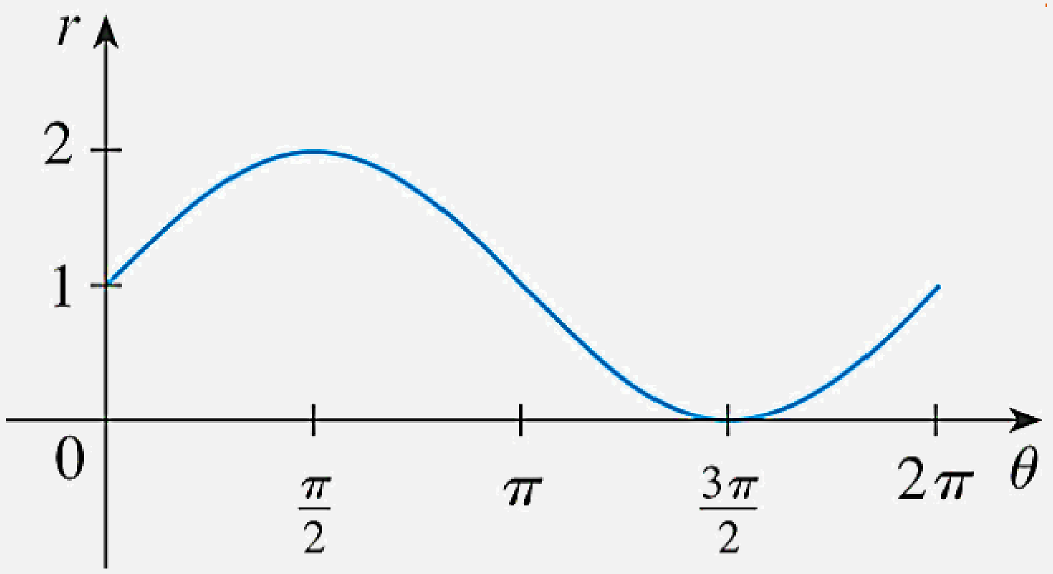
Solution:

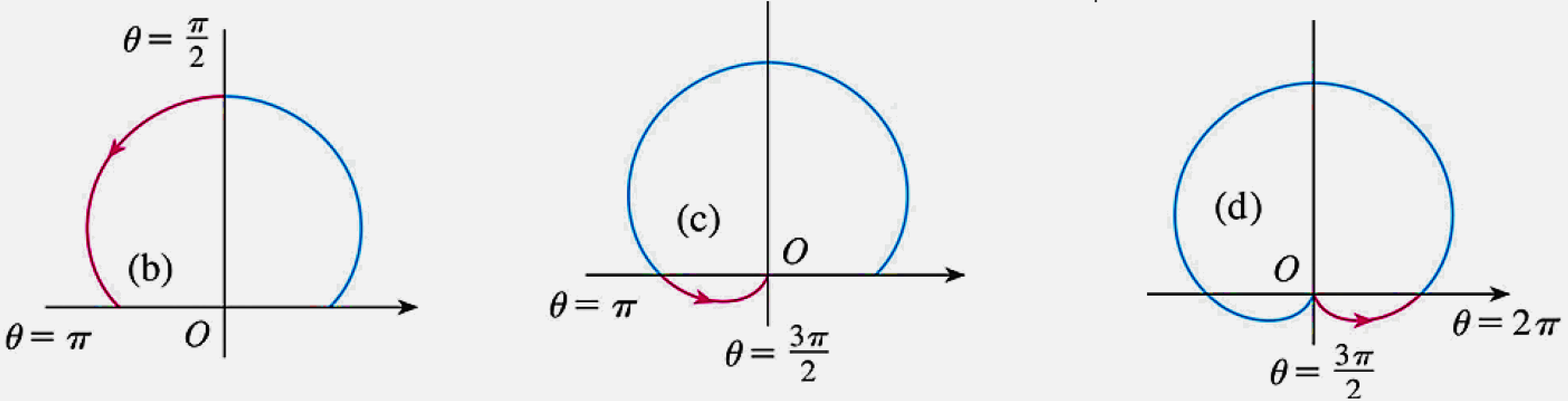
Here, we do not plot points as we did in previous example. Rather, we first sketch the graph of $r = 1 + \sin\theta$ in Cartesian coordinates by shifting the sine curve up one unit. This enables us to see immediately the values of r that correspond to increasing values of θ. For instance, we see that, as θ increases from 0 to $\pi/2$, r (the distance from O) increases from 1 to 2. So, we sketch the corresponding part of the polar curve.



Solution

As θ increases from $\pi/2$ to π, the figure (b) shows that r decreases from 2 to 1. So, we sketch the next part of the curve. As θ increases from to π to $3\pi/2$, r decreases from 1 to 0, as shown in (c). Finally, as θ increases from $3\pi/2$ to 2π, r increases from 0 to 1, as shown in (d).





Solution

Note that, If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace our path. It is called a **cardioid**—because it's shaped like a heart.