BOUNDARY CONDITIONS-I

Boundary Conditions

- So far, we have considered the existence of the electric field in a homogeneous medium
- If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called ***boundary conditions***
- These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known
- The conditions will be dictated by the types of material the media are made of

Boundary Conditions

- To determine the boundary conditions, we need to use the following two equations:

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{AND} \qquad \oint \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}}$$

- Also we need to decompose the electric field intensity **E** into two orthogonal components:

$$\mathbf{E} = \mathbf{E}_t + \mathbf{E}_n$$

- where $\mathbf{E}_t$ and $\mathbf{E}_n$ are, respectively, the tangential and normal components of **E** to the interface of interest

- Similar decomposition can be done for the electric flux density **D**

Boundary Conditions

- We shall consider the boundary conditions at an interface separating:

I. Dielectric (ε_{r1}) and dielectric (ε_{r2})

II. Conductor and dielectric

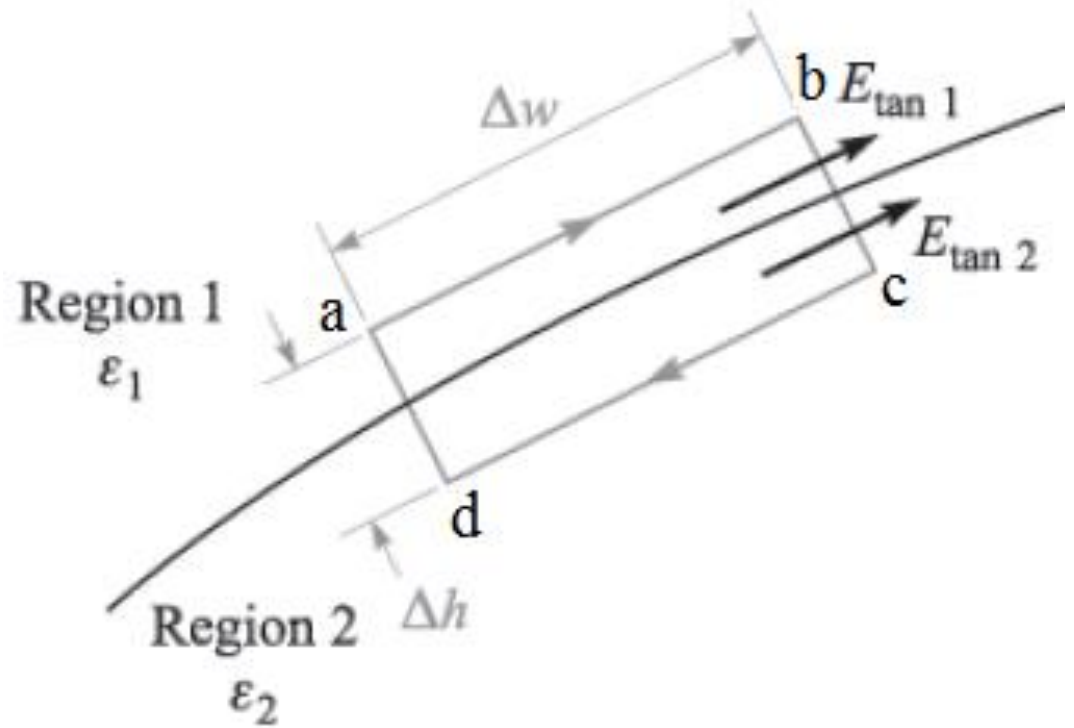
III. Conductor and free space

Dielectric-Dielectric

- Consider the **E** field existing in a region consisting of two different dielectrics characterized by $\varepsilon_1 = \varepsilon_o \varepsilon_{r1}$ and $\varepsilon_2 = \varepsilon_o \varepsilon_{r2}$

- **E**$_1$ and **E**$_2$ in media 1 and 2, respectively, can be decomposed as:

$$\mathbf{E}_1 = \mathbf{E}_{1t} + \mathbf{E}_{1n} \qquad \mathbf{E}_2 = \mathbf{E}_{2t} + \mathbf{E}_{2n}$$



Dielectric-Dielectric

- For the tangential components, we apply the line integral of **E** equation to the closed path abcda shown in Figure

- If the path is very small with respect to the variation of **E**, we obtain:

$$0 = E_{1t}\,\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\,\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}$$

- Here $E_t = |\mathbf{E_t}|$ and $E_n = |\mathbf{E_n}|$

- As $\Delta \boldsymbol{h} \rightarrow \mathbf{0}$, the above equation becomes: $E_{1t} = E_{2t}$

- Thus the tangential components of **E** are the same on the two sides of the boundary

Dielectric-Dielectric

- In other words, tangential component of **E** is said to be continuous across the boundary

- Since $\mathbf{D} = \varepsilon\mathbf{E} = \mathbf{D}_t + \mathbf{D}_n$, we have:

$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

Or:

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

- Therefore, tangential component of **D** is said to be discontinuous across the interface

Dielectric-Dielectric

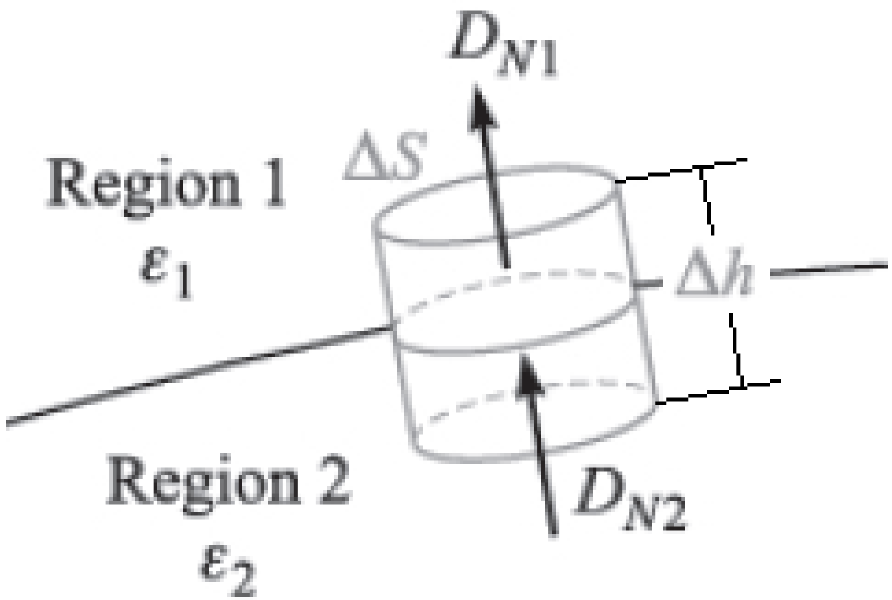
- For normal components, we apply Gauss's law equation to the Gaussian surface shown in the figure by making $\Delta h \to 0$

$$\Delta Q = \rho_S \Delta S = D_{1n} \Delta S - D_{2n} \Delta S$$

Or:

$$D_{1n} - D_{2n} = \rho_S$$

- where ρ_s is a free charge density placed at the boundary



Dielectric-Dielectric

- The equation above assumes that **D is directed from region 2 to region 1** and so the equation must be applied accordingly (negative sign)

- If no free charges exist at the interface, then $\rho_s = 0$ and the equation becomes:

$$D_{1n} = D_{2n}$$

- Thus the normal component of **D** is continuous across the interface; that is, D_n undergoes no change at the boundary

- Since $\mathbf{D} = \varepsilon\mathbf{E}$, the above equation can be written as:

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

- Hence, normal component of **E** is discontinuous at the boundary

Dielectric-Dielectric

- The equations derived are collectively referred to as boundary conditions

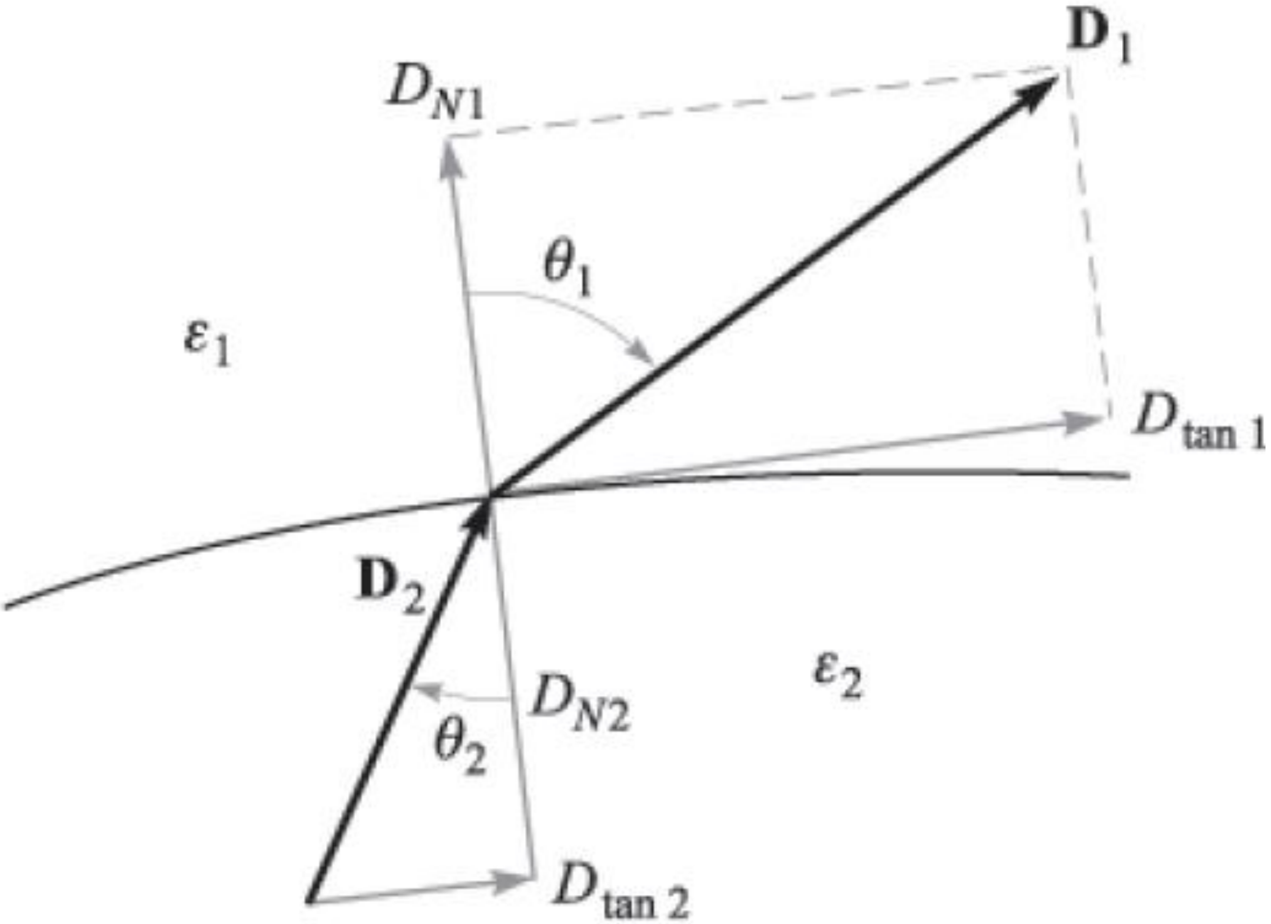
- They must be satisfied by an electric field at the boundary separating two different dielectrics

- The boundary conditions are applied in finding the electric field on one side of the boundary given the field on the other side

- Besides this, we can use the boundary conditions to determine the "***refraction***" of the electric field across the interface

Dielectric-Dielectric

- Consider $\mathbf{D}_1$ or $\mathbf{E}_1$ and $\mathbf{D}_2$ or $\mathbf{E}_2$ making angles θ_1 and θ_2 with the normal to the interface as illustrated in Figure below:



Dielectric-Dielectric

- For the tangential components, we have:

$$E_1 \sin\theta_1 = E_{1t} = E_{2t} = E_2 \sin\theta_2$$

Or:

$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$

- Similarly, for the normal components, we have:

$$\varepsilon_1 E_1 \cos\theta_1 = D_{1n} = D_{2n} = \varepsilon_2 E_2 \cos\theta_2$$

Or:

$$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2$$

- Dividing the above two equations, we get:

$$\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}$$

Dielectric-Dielectric

- Since $\varepsilon_1 = \varepsilon_o \varepsilon_{r1}$ and $\varepsilon_2 = \varepsilon_o \varepsilon_{r2}$, we have:

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

- This is the *law of refraction* of the electric field at a boundary free of charge (since $\rho_s = 0$ is assumed at the interface)

- Thus, in general, an interface between two dielectrics produces bending of the flux lines as a result of unequal polarization charges that accumulate on the sides of the interface

Problem-1

- Given that $\mathbf{E_1} = 10\mathbf{a_x} - 6\mathbf{a_y} + 12\mathbf{a_z}$ V/m in the Figure below, find: (a) $\mathbf{P_1}$ (b) $\mathbf{E_2}$ and the angle $\mathbf{E_2}$ makes with the y-axis.