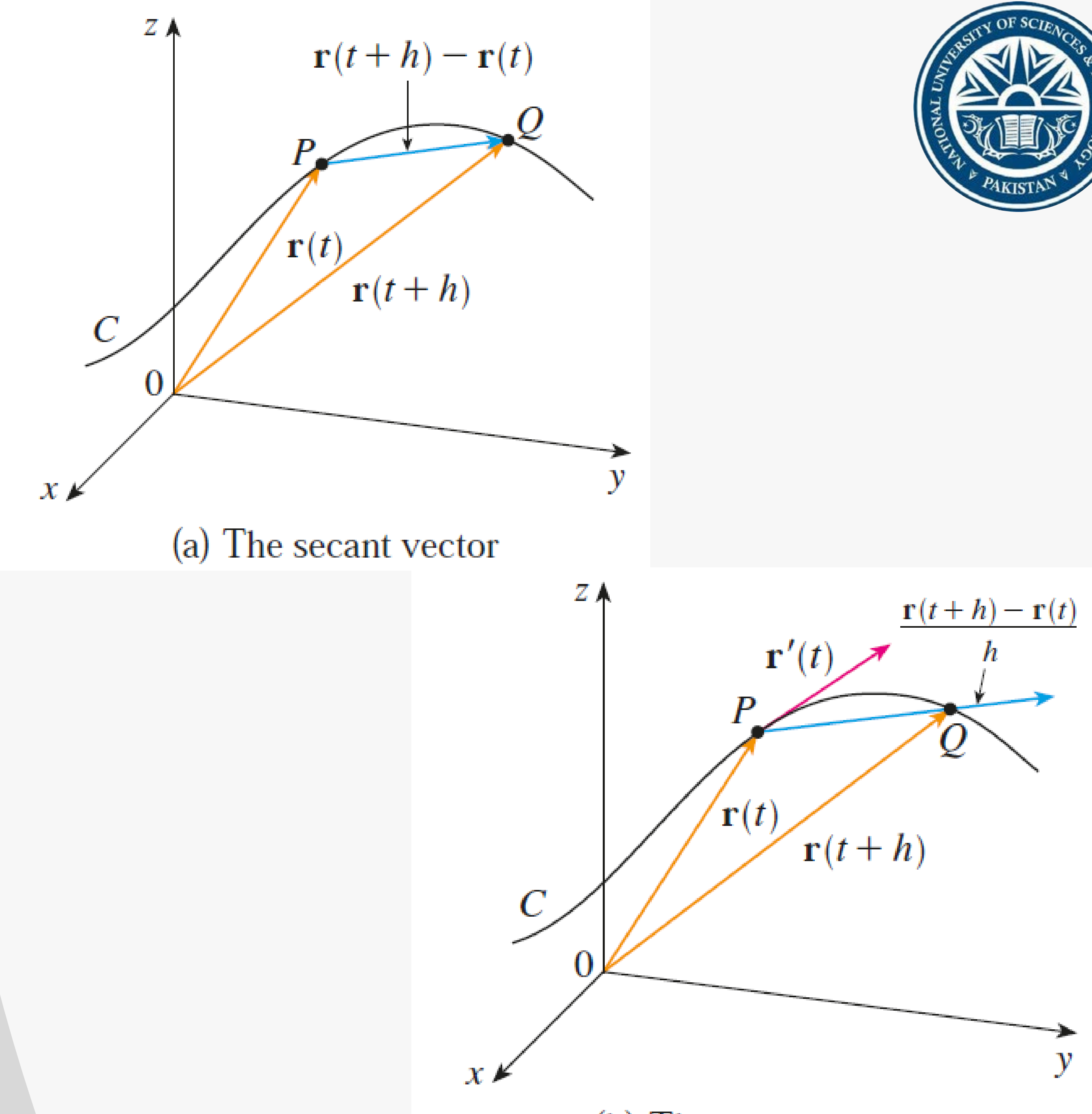
Derivatives of Vector Functions



(a) The secant vector

(b) The tangent vector

Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

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Vectors And
The Geometry Of Space

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Chapter: 13 , Section: 13.1

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.
Chapter: 13 , Section: 13.2

Derivatives

The derivative of a vector function $\mathbf{r}(t)$ is defined in much the same way as for real-valued functions.

Definition:

If $\mathbf{r}(t)$ is a vector function, then derivative $\mathbf{r}'(t)$ is given as:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

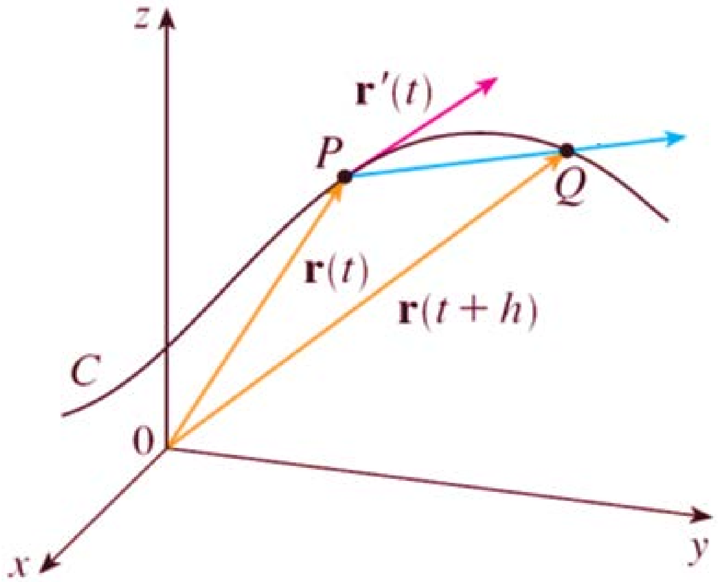
provided this limit exists.

Derivative Geometric Significance

The geometric significance of this definition is shown as follows. If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t + h)$, then $\overrightarrow{PQ}$ represents the vector:

$$\mathbf{r}(t + h) - \mathbf{r}(t).$$

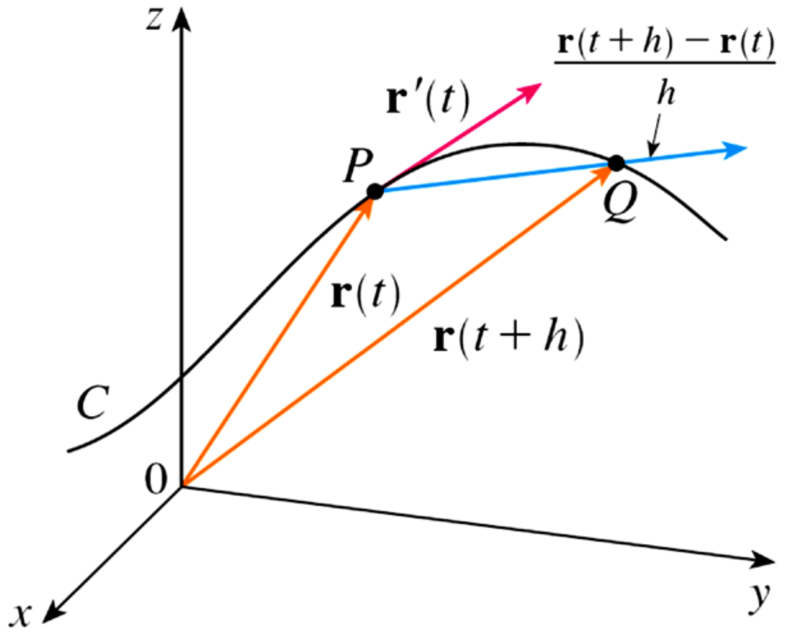
This can therefore be regarded as a secant vector. If $h > 0$, then the scalar multiple $(1/h)(\mathbf{r}(t + h) - \mathbf{r}(t))$ has the same direction as $\mathbf{r}(t + h) - \mathbf{r}(t)$. As $h \longrightarrow 0$, it appears that this vector approaches a vector that lies on the tangent line.



Derivative Geometric Significance

For this reason, the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided $\mathbf{r}'(t)$ exits and $\mathbf{r}'(t) \neq \mathbf{0}$. We require $\mathbf{r}'(t) \neq \mathbf{0}$ for a smooth curve to make sure the curve has a continuously turning tangent at each point. On a smooth curve, there are no sharp corners or cusps. The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The **unit tangent vector** is defined as:

$$T(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$



The tangent vector

Derivatives

The following theorem provides us with a convenient way for computing the derivative of a vector function $\mathbf{r}(t)$.

Theorem:

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions, then:

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Example:

a. Determine the derivative of the vector function:

$$\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \sin(2t)\,\mathbf{k}.$$

b. Moreover, find the unit tangent vector at the point where $t = 0$.

Solution:

(a) According to theorem: we differentiate each component of $\boldsymbol{r}(t)$ and get:

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + (1 - t)e^{-t}\mathbf{j} + 2\cos(2t)\,\mathbf{k}.$$

(b) As $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$, so the unit tangent vector at the point $(1, 0, 0)$ is given as:

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1 + 4}} = \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}.$$

Example:

For the curve: $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (2 - t)\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.

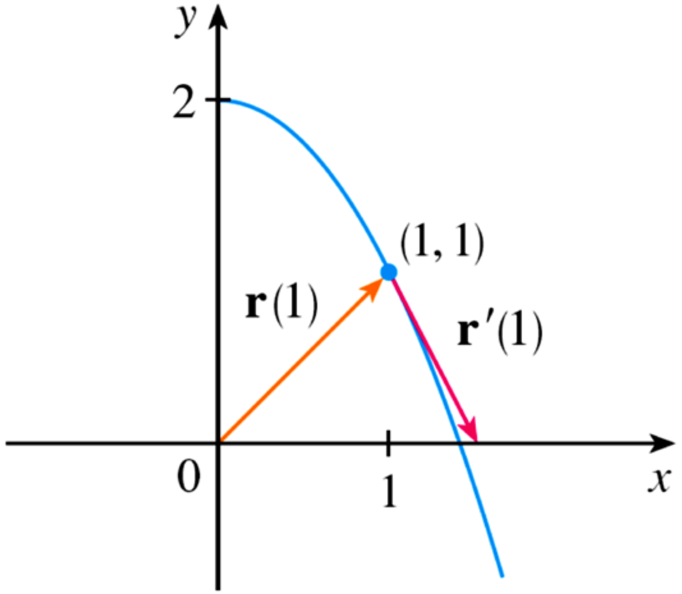
Solution:

We have: $\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} - \mathbf{j}$ and $\mathbf{r}'(1) = \frac{1}{2}\mathbf{i} - \mathbf{j}$. The given curve is a plane curve.

Elimination of the parameter from the equations $x = \sqrt{t}; y = 2 - t$ gives:

$$y = 2 - x^2, \qquad x \geq 0.$$

The position vector $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$ starts at the origin. The tangent vector $\mathbf{r}'(1)$ starts at the corresponding point $(1, 1)$.



Second Derivative

Just as for real-valued functions, the second derivative of a vector function $\mathbf{r}(\boldsymbol{t})$ is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$.

For instance, the second derivative of the function:

$$\mathbf{r}(\boldsymbol{t}) = \langle 1 + t^3, te^{-t}, \sin(2t) \rangle,$$

can be determined as:

$$\mathbf{r}'(\boldsymbol{t}) = \langle 3t^2, (1 - t)e^{-t}, 2\cos(2t) \rangle,$$

$$\mathbf{r}''(\boldsymbol{t}) = \langle 6t, (t - 2)e^{-t}, -4\sin(2t) \rangle.$$

Velocity and Acceleration Vectors

If $\mathbf{r}(t)$ is the position vector of a particle moving along a smooth curve in space, then $\mathbf{v}(t) = \mathbf{r}'(\boldsymbol{t})$ is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of $\mathbf{v}$ is the **direction of motion**, the magnitude of $\mathbf{v}$ is the particle's **speed**, and the derivative $\mathbf{a}(t) = \mathbf{v}'(\boldsymbol{t}) = \mathbf{r}''(\boldsymbol{t})$ when it exists, is the particle's **acceleration vector**. In conclusion,

1. Velocity is the derivative of position: $\mathbf{v}(t) = \mathbf{r}'(\boldsymbol{t})$.
2. Speed is the magnitude of velocity: Speed = $|\mathbf{v}(t)|$.
3. Acceleration is the derivative of velocity: $\mathbf{a}(t) = \mathbf{v}'(\boldsymbol{t}) = \mathbf{r}''(\boldsymbol{t})$.
4. The unit vector $\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$ is the direction of motion at time t.

Example:

A person on a hang glider is spiraling upward due to rapidly rising air on a path having position vector: $\mathbf{r}(\boldsymbol{t}) = \langle 3\cos t\,, 3\sin t\,, t^2\rangle$. Find

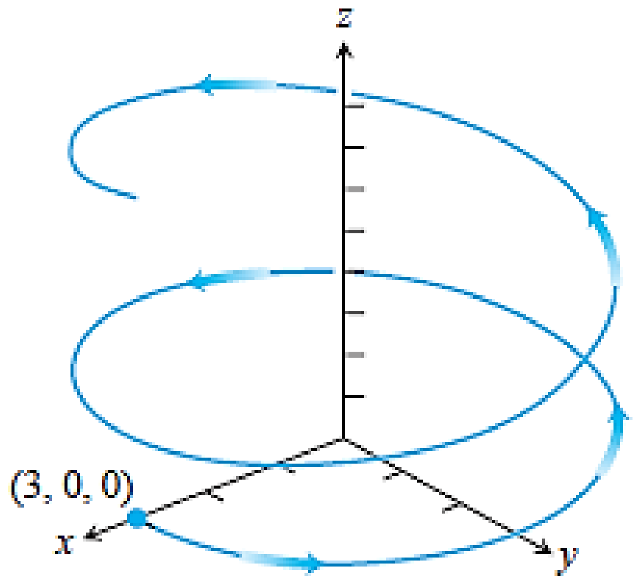
$$\left.\begin{array}{l} x = 3\cos t \\ y = 3\sin t \end{array}\right\} \longrightarrow \begin{array}{l} \text{circle in } xy\text{-plane} \\ x^2+y^2=(3)^2 \end{array}$$

$$z = t^2$$

(a) the velocity and acceleration vectors,

(b) the glider's speed at any time t,

(c) the times, if any, when the glider's acceleration is orthogonal to its velocity.



The path of a hang glider with position vector: $\mathbf{r}(\boldsymbol{t}) = \langle 3\cos t\,, 3\sin t\,, t^2\rangle$.

Solution:

(a) For the present case the velocity and acceleration vectors are respectively given as:

$$\mathbf{v}(t) = \mathbf{r}'(\boldsymbol{t}) = \langle -3\sin t\,, 3\cos t\,, 2t\rangle,$$

$$\mathbf{a}(t) = \mathbf{r}''(\boldsymbol{t}) = \langle -3\cos t\,, -3\sin t\,, 2\rangle.$$

Solution:

(b) The glider's speed at any time t is given as:

$$|\mathbf{v}(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2t)^2}$$
$$= \sqrt{9 + 4t^2}$$

The glider is moving faster and faster as it rises along its path.

(c) To find the times when $\mathbf{v}$ and $\mathbf{a}$ are orthogonal, we look for values of t for which

$$\mathbf{v}.\mathbf{a} \ = 0,$$
$$\Longrightarrow \langle -3\sin t \, , 3\cos t \, , 2t \rangle \cdot \langle -3\cos t \, , -3\sin t \, , 2 \, \rangle = 0,$$
$$\Longrightarrow 9\sin t\cos t - 9\sin t\cos t + 4t = 0 \Longrightarrow t = 0.$$

Thus, the only time the acceleration vector is orthogonal to $\mathbf{v}$ is when $t = 0$.

Differentiation Rules

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\dfrac{d}{dt}\mathbf{C} = \mathbf{0}$

2. *Scalar Multiple Rules:* $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

3. *Sum Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

4. *Difference Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$

5. *Dot Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

6. *Cross Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

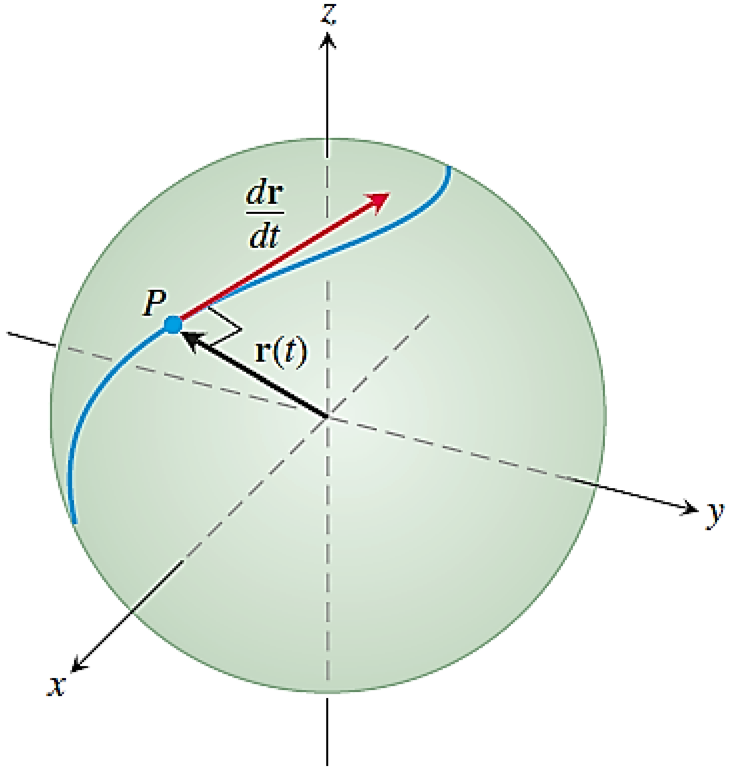
7. *Chain Rule:* $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Vector Functions of Constant Length

When we track a particle moving on a sphere centered at the origin, the position vector has a constant length equal to the radius of the sphere. The velocity vector $\mathbf{v}(t) = \mathbf{r}'(\boldsymbol{t})$, tangent to the path of motion, is tangent to the sphere and hence perpendicular to $\mathbf{r}(t)$. Thus, if $\mathbf{r}(t)$ is a differentiable vector function of t and the length of $\mathbf{r}(t)$ is constant, then:

$$\mathbf{r}(t) \cdot \mathbf{v}(t) = \mathbf{r}(t) \cdot \mathbf{r}'(\boldsymbol{t}) = 0.$$

The converse of above is also true.



Integrals of Vector Functions

INTEGRALS OF VECTOR-VALUED FUNCTIONS

$$\int_a^b \underline{r}(t)\,dt = \left[\underline{R}(t)\right]_a^b$$

$$= \underline{R}(b) - \underline{R}(a)$$

$$\int 4t^2\,\underline{i} + 3t\,\underline{j}\;dt$$

$$= \left(\frac{4}{3}t^3 + a\right)\underline{i} + \left(\frac{3t^2}{2} + b\right)\underline{j}$$

$$= \frac{4t^3}{3}\,\underline{i} + \frac{3t^2}{2}\,\underline{j} + \underline{c}$$

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Integrals

The definite integral of a continuous vector function:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

can be defined in much the same way as for real-valued functions—except that the integral is a vector. Thus,

$$\int_a^b \mathbf{r}(t)dt$$

$$= \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}.$$

This means that we can evaluate an integral of a vector function by integrating each component function.

Integrals

We can extend the Fundamental Theorem of Calculus to continuous vector functions:

$$\int_{\mathrm{a}}^{\mathrm{b}} \mathbf{r}(\mathrm{t})\, dt = \mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a),$$

Here, $\mathbf{R}$ is an antiderivative of $\mathbf{r}(t)$, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$.

For indefinite integrals, we have:

$$\int \mathbf{r}(t)\, dt = \mathbf{R}(t) + \mathbf{C}.$$

Example:

If $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}$, then

$$\int \mathbf{r}(t)dt = \left(\int 2\cos t\,dt\right)\mathbf{i} + \left(\int \sin t\,dt\right)\mathbf{j} + \left(\int 2t dt\right)\mathbf{k}$$

$$= 2\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t^2\mathbf{k} + \mathbf{C},$$

$\vec{C} = \langle c_1, c_2, c_3 \rangle$

where: $\mathbf{C}$ is a vector constant of integration.

If $0 \leq t \leq \frac{\pi}{2}$, then we can evaluate the corresponding definite integral as:

$$\int_0^{\frac{\pi}{2}} \mathbf{r}(t)\,dt = \left[2\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t^2\mathbf{k}\right]_0^{\frac{\pi}{2}} = 2\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\mathbf{k}.$$

Example: Revisiting the Flight of a Glider

Suppose that we did not know the path of the glider in a previous example, but only its acceleration vector is known to us:

$$\mathbf{a}(t) = \langle -3\cos t, -3\sin t, 2\rangle.$$

We also know that initially (at time $t = 0$), the glider departed from the point: $(3,0,0)$ with velocity $\mathbf{v}(0) = 3\mathbf{j}$. Find the glider's position as a function of t. Our goal is to find $\mathbf{r}(t)$ knowing the differential equation:

$$\mathbf{a}(t) = \frac{d^2\mathrm{r}}{dt^2} = -3\cos t\ \mathrm{i} - 3\sin t\ \mathrm{j} + 2\ \mathrm{k}. \quad \text{(I)}$$

The initial conditions: $\mathbf{v}(0) = 3\mathbf{j}$, $\mathbf{r}(0) = 3\mathbf{i}$.

Example: Revisiting the Flight of a Glider

Integrating eq. (I) with respect to t we get:

$$\mathbf{v}(t) = \frac{d\mathrm{r}}{dt} = -3\sin t\ \mathbf{i} + 3\cos t\ \mathbf{j} + 2t\,\mathbf{k} + \mathbf{C}_1.$$

Using the initial conditions: $\mathbf{v}(0) = 3\mathbf{j}$ in above we get:

$$\mathbf{v}(t) = \frac{d\mathrm{r}}{dt} = -3\sin t\ \mathbf{i} + 3\cos t\ \mathbf{j} + 2t\,\mathbf{k}. \quad (\mathrm{II})$$

Integrating eq. (II) with respect to t we get:

$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + t^2\mathbf{k} + \mathbf{C}_2. \qquad (\mathrm{III})$$

By using the initial condition: $\mathbf{r}(0) = 3\mathbf{i}$, in (III) we will get the glider's position as:

$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + t^2\mathbf{k}.$$

Practice Questions

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

Chapter: 13

Exercise-13.2: Q – 1, Q – 3 to 29, Q – 33 to 40.

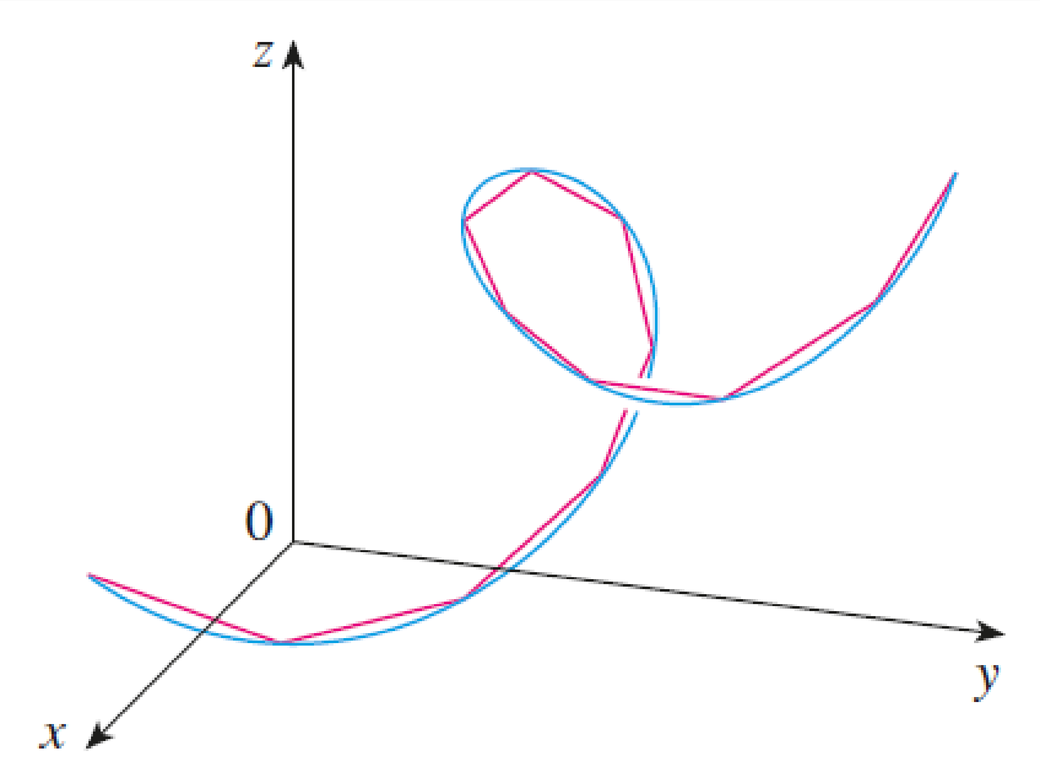
Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 13

Exercise-13.1: Q – 1 to 30.

Exercise-13.2: Q – 1 to 30.

Arc Length in Space



The length of a space curve is the limit of lengths of inscribed polygons.

$$L = \int_a^b |\mathbf{r}'(t)|\,dt$$
$$= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

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Arc Length

We defined the length of a **plane curve** with parametric equations:

$$x = f(t), \qquad y = g(t); \quad a \le t \le b,$$

as the limit of lengths of inscribed polygons and, for the case where $f'(t)$ and $g'(t)$ are continuous, we arrived at the formula:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt. \qquad (1)$$

The length of a space curve is defined in exactly the same way.

Arc Length

Suppose that the curve has the vector equation:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle, a \le t \le b,$$

or, equivalently, the parametric equations:

$$x = f(t), \qquad y = g(t), \qquad z = h(t); \quad a \le t \le b,$$

where $f'(t), g'(t)$, and $h'(t)$ are continuous. If the curve is traversed exactly once as increases from to , then it can be shown that its length is:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}\, dt. \quad (2)$$

Notice that both of the arc length formulas (1) and (2) can be put into the more compact form:

$$L = \int_a^b |\mathbf{r}'(t)|\, dt.$$