IMPORTANT SIGNALS - CONTINUOUS TIME

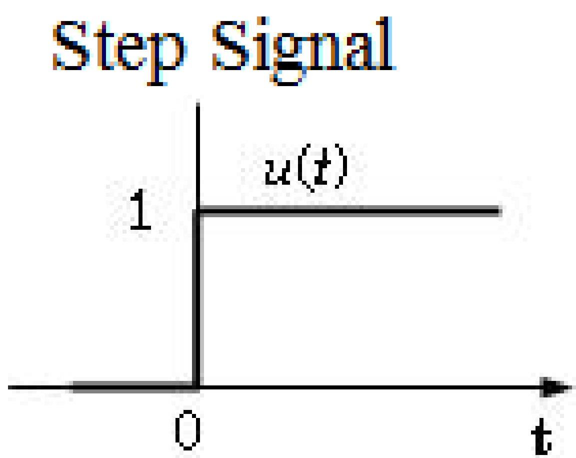
Motivation

- These signals occur frequently in nature
- They serve as basic building blocks from which we can construct many other signals
- Sinusoidal and periodic complex signals are used to describe the characteristics of many physical processes
- Constructing signals in this way will allow us to examine and understand more deeply the properties of both signals and systems

Continuous Time Unit Step Function

- A basic continuous-time signal unit step function:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

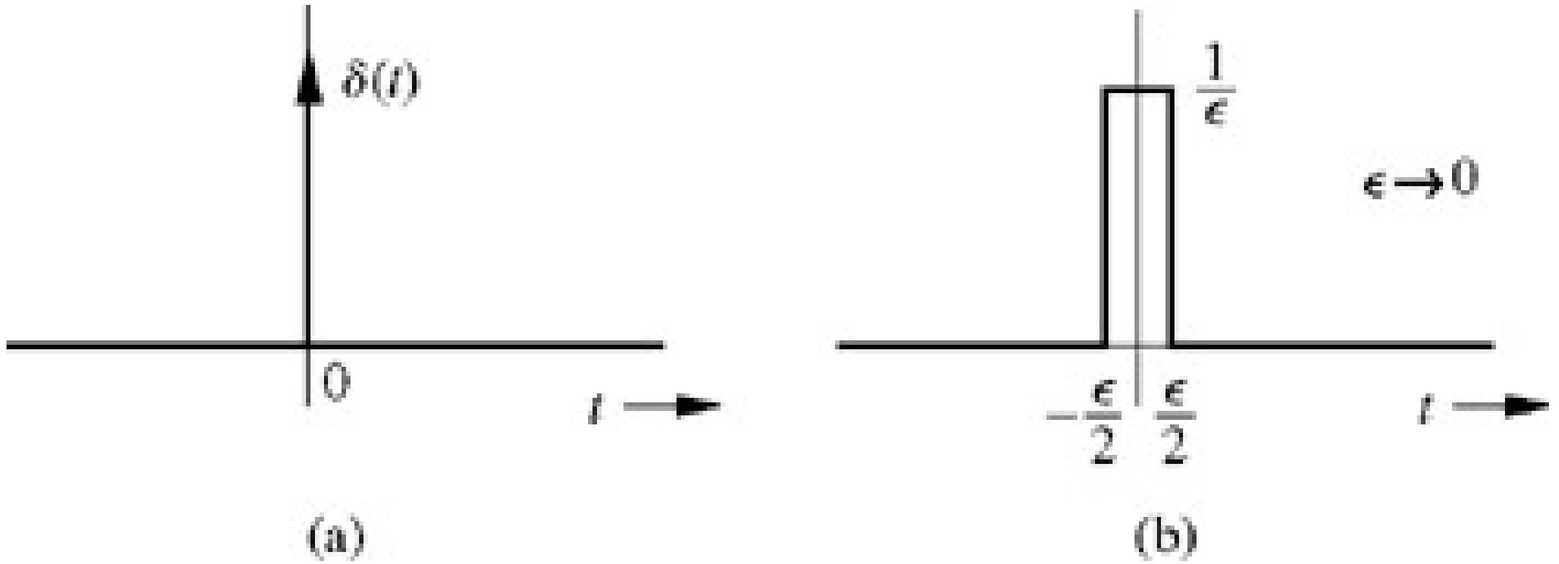


- Note that $u(t)$ is discontinuous at $t = 0$

- The unit step function will be very important in examination of the properties of the systems

Continuous Time Unit Impulse Function

- Figure shows the CT unit impulse function



- We can visualize an impulse as a tall, narrow, rectangular pulse of unit area

- The width of this rectangular pulse is a very small value $\varepsilon \rightarrow 0$, consequently, its height is a very large value $1/\varepsilon \rightarrow \infty$

Continuous Time Unit Impulse Function

- The unit impulse therefore can be regarded as a rectangular pulse with a width that has become infinitesimally small, a height that has become infinitely large, and an overall area that has been maintained at unity

- Thus $\delta(t) = 0$ everywhere except at $t = 0$, where it is undefined

- For this reason, a unit impulse is represented by the spear-like symbol

- Multiplication of a CT function $\varphi(t)$ with a unit impulse located at $t = 0$ results in an impulse, which is located at $t = 0$ and has strength $\emptyset(0)$ (the value of $\emptyset(t)$ at the location of the impulse)

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$

Unit Impulse Function

- The unit step function is the running integral of unit impulse function

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$= u(t)$$

- The CT unit impulse is the first derivative of the CT unit step as:

$$\delta(t) = \frac{du(t)}{dt}$$

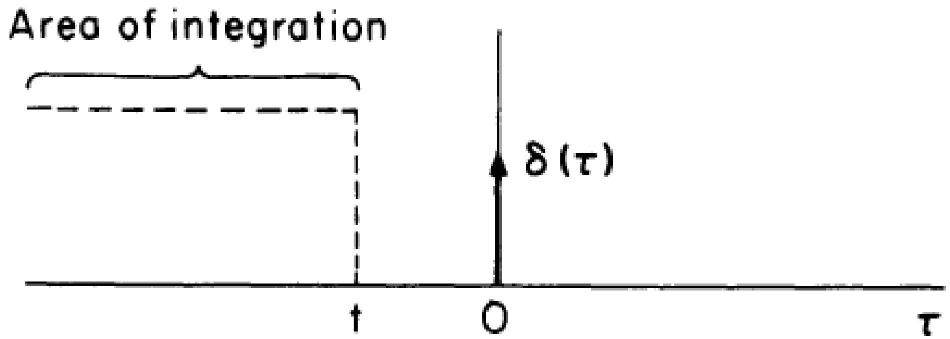
- Since u(t) is discontinuous at t = 0, it is not formally differentiable

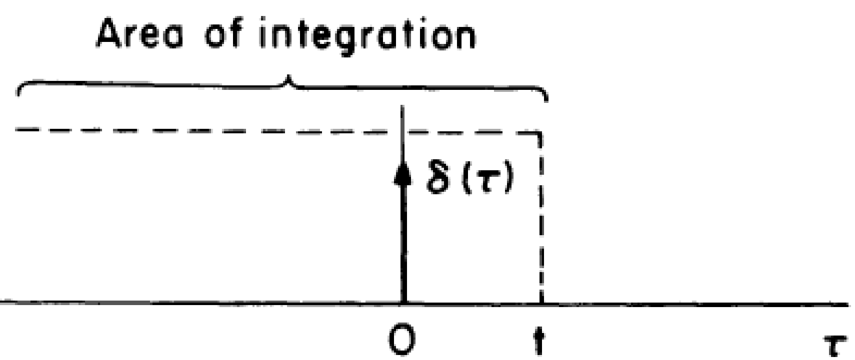
Continuous Time Unit Functions

$$\delta(t) = \frac{du(t)}{dt}$$

- The unit step expressed as the running integral of the unit impulse

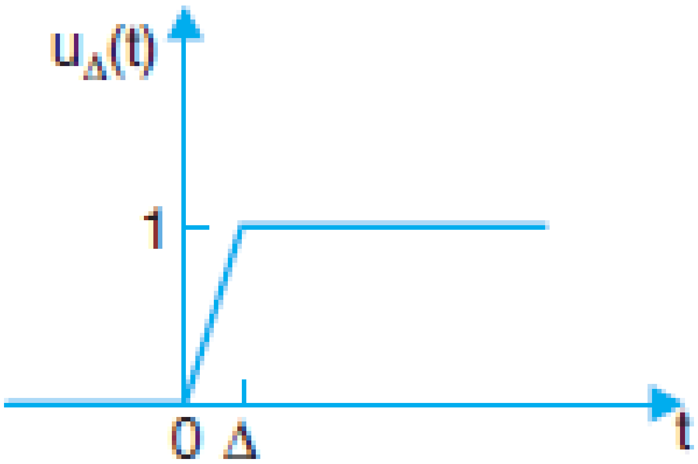
$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$



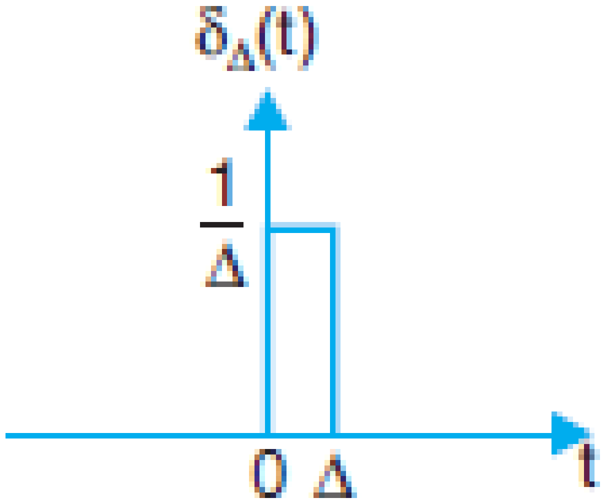


Practical Continuous Time Unit Functions

- In practice, the unit step $u_{\Delta}(t)$ rises from the value 0 to the value 1 in a short-time interval of length Δ



- The unit step can be thought of as an idealisation of $u_{\Delta}(t)$ for Limit $\Delta \rightarrow 0$



- As $\Delta \rightarrow 0$ and the derivative becomes the impulse in practical sense as shown - (Unity area)

Sinusoidal Signals

- General sinusoidal signal has the form:

$$x(t) = A\cos(\omega_0 t + \phi)$$

- Where t has the unit of seconds, ω_0 has the unit of radians per second, and Φ has the unit of radians. It is common to use the relation:

$$\omega_0 = 2\pi f_0$$

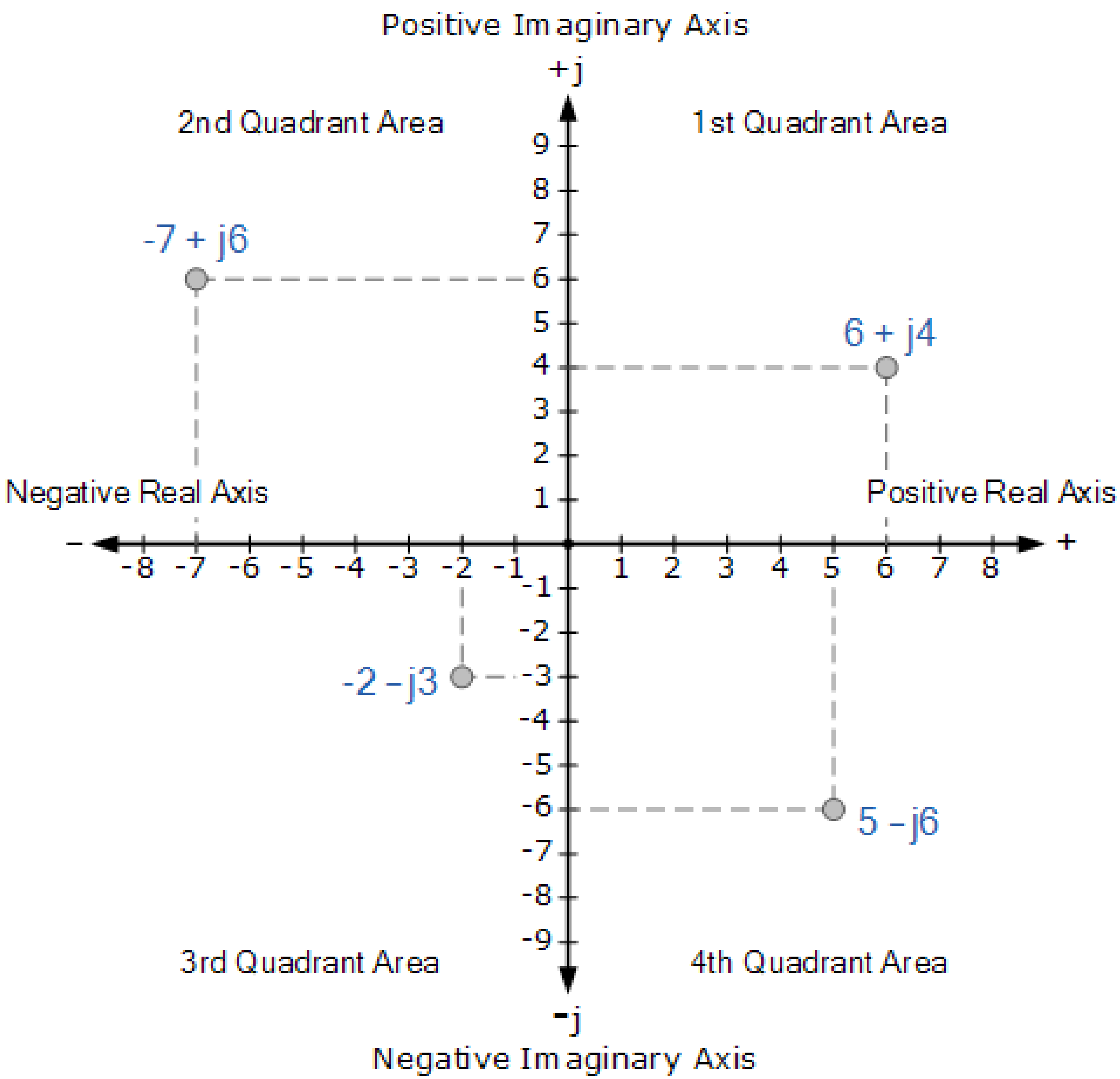
- Where f_0 has the units of cycles/second or Hertz

Complex Numbers

- Complex numbers are used when dealing with sinusoidal sources
- Complex Numbers represent points in a two dimensional complex plane or s-plane that are referenced to two distinct axes
- The horizontal axis is called the "real axis" while the vertical axis is called the "imaginary axis"
- The real and imaginary parts of a complex number, Z are abbreviated as Re(Z) and Im(Z), respectively

Complex Numbers

- Complex numbers shown on the complex plane



Complex Numbers

- We can use phasors to represent sinusoidal waveforms
- The amplitude and phase angle of phasors can be written in the form of a complex number
- A complex number can be represented in one of three ways:
  - $Z = x + jy$ » Rectangular Form
  - $Z = A \angle \Phi$ » Polar Form
  - $Z = A e^{j\Phi}$ » Exponential Form

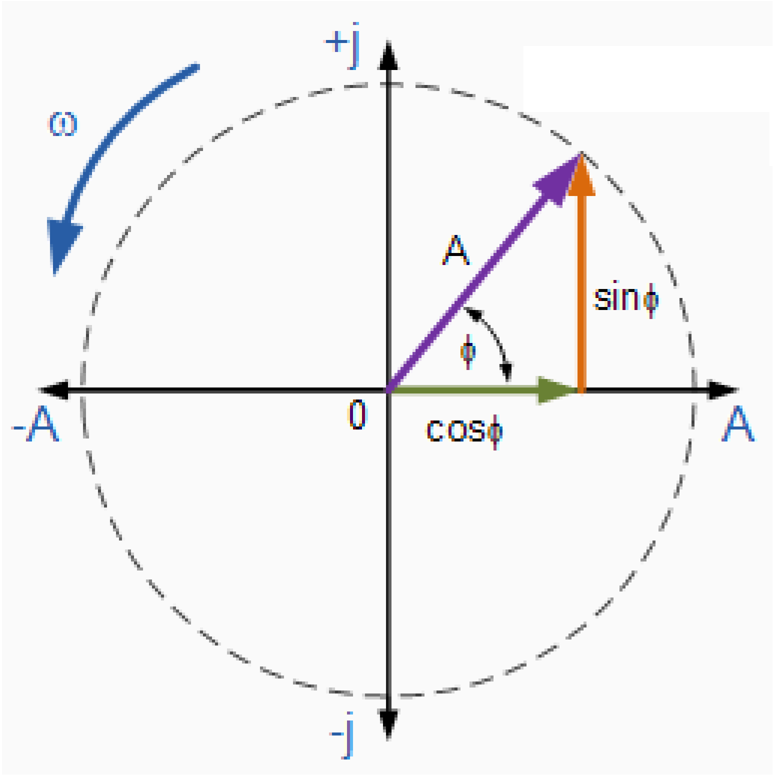
Complex Numbers

- Complex numbers in exponential form are represented as below:

$$Z = Ae^{j\phi}$$

$$Z = A\left(\cos\phi + j\sin\phi\right)$$

$$\text{Euler's identity: } e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$



- The phasor will rotate as the angle ϕ changes

Exponential Signals

Exponential signals

$$x(t) = Ae^{at}$$

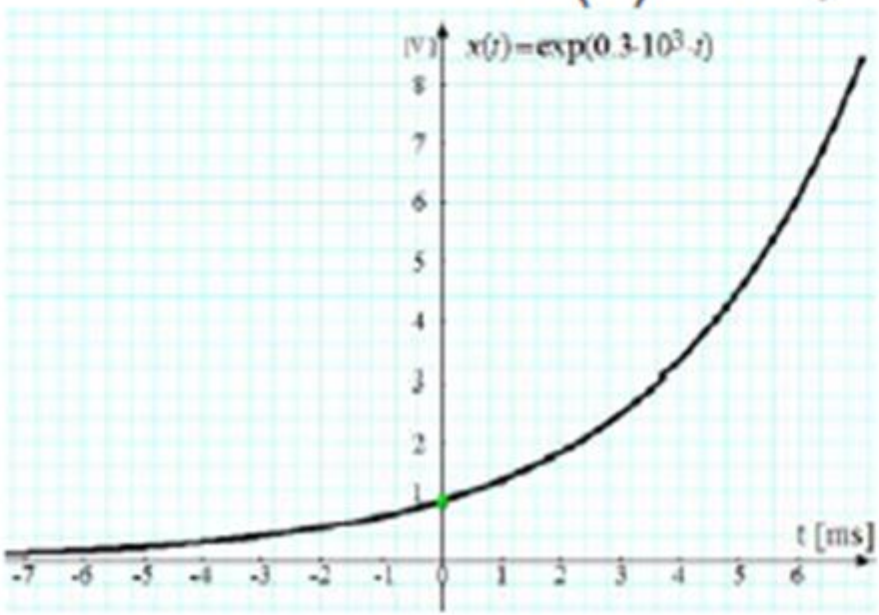
where A and a are complex numbers.

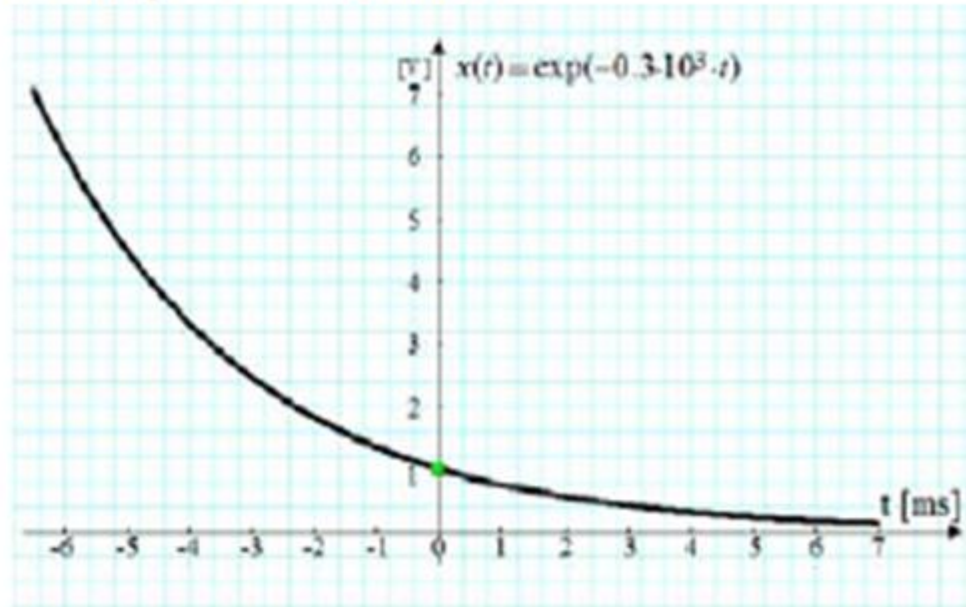
- Exponential and sinusoidal signals arise naturally in the analysis of linear systems
- Example: simple harmonic motion that you learned in physics
- There are several distinct types of exponential signals
  - A and a real
  - A and a imaginary
  - A and a complex (most general case)

Exponential Signals

- A and *a* are real

$$x(t) = e^{at}, \quad a \in \mathbb{R}, \quad e \sim 2.7182$$





$$a > 0 \ ; \quad \lim_{t \to -\infty} e^{at} = 0 \ ;$$

$$\lim_{t \to \infty} e^{at} = \infty \ ; \ e^{0} = 1$$

$$a < 0 \ ; \quad \lim_{t \to -\infty} e^{at} = \infty$$

$$\lim_{t \to \infty} e^{at} = 0 \ ; \ e^{0} = 1$$

Exponential Signals

- *a* is imaginary

$$x(t) = Ae^{at} = A(e^{a})^{t}$$

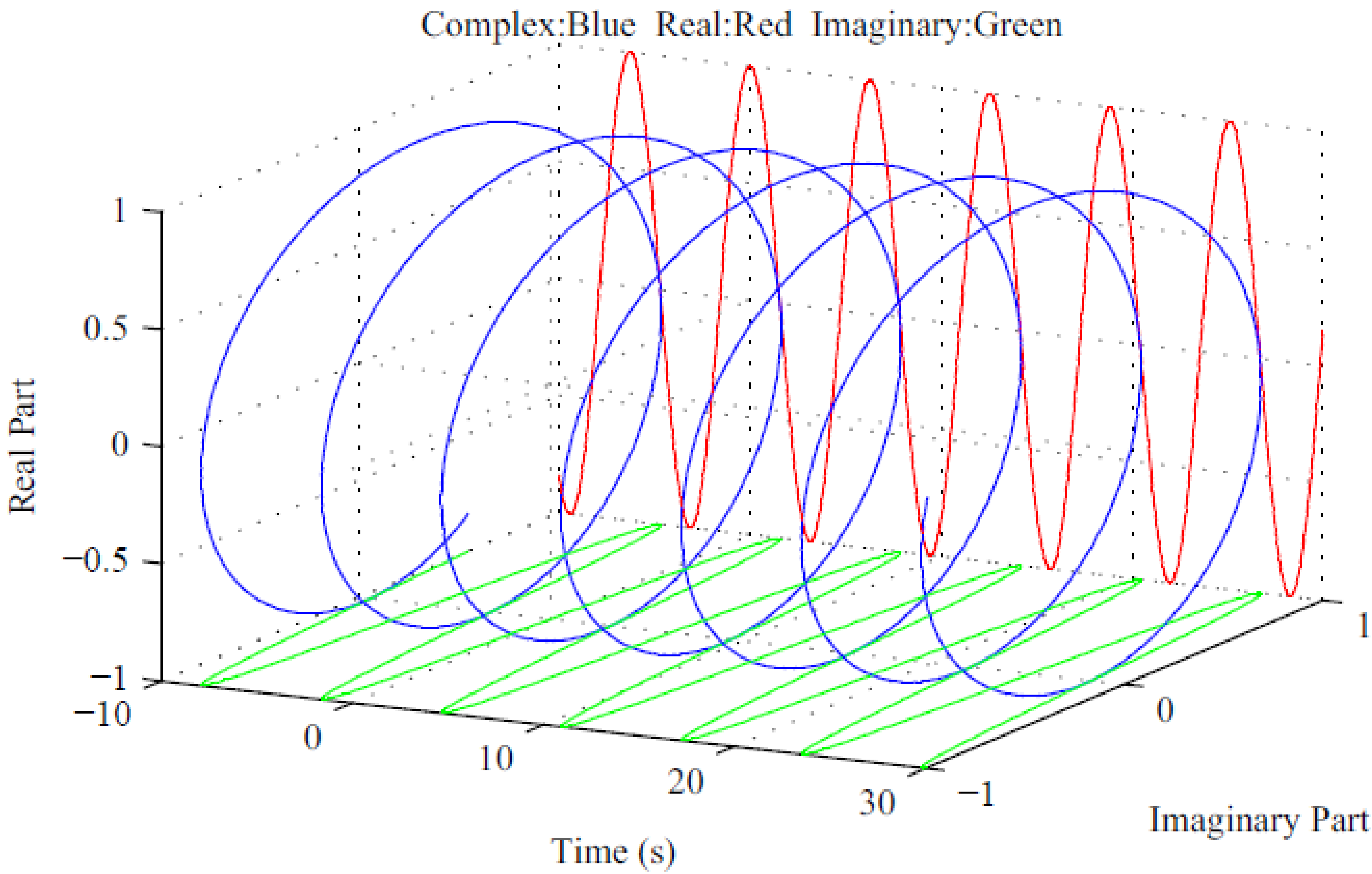
When a is imaginary, then Euler's equation applies:

$$\begin{aligned} e^{j\omega t} &= \cos(\omega t) + j\sin(\omega t) \\ e^{j\omega n} &= \cos(\omega n) + j\sin(\omega n) \end{aligned}$$

- Since $|e^{j\omega t}| = 1$, this looks like a coil in a plot of the complex plane versus time
- $e^{j\omega t}$ is Periodic with fundamental period $T = \frac{2\pi}{\omega}$
- Real part is sinusoidal: $\mathrm{Re}\{Ae^{j\omega t}\} = A\cos(\omega t)$
- Imaginary part is sinusoidal: $\mathrm{Im}\{Ae^{j\omega t}\} = A\sin(\omega t)$
- These signals have infinite energy, but finite (constant) average power

Exponential Signals

➢ *a* is imaginary: Ae^{at}, $A = 1$ **and** $a = j$



Sinusoidal Exponential Harmonics

- In order for $e^{j\omega t}$ to be periodic with period T, we require that

$$e^{j\omega t} = e^{j\omega(t+T)} = e^{j\omega t} e^{j\omega T} \text{ for all } t$$

- This implies $e^{j\omega T} = 1$ and therefore

$$\omega T = 2\pi k \text{ where } k = 0, \pm 1, \pm 2, \ldots$$

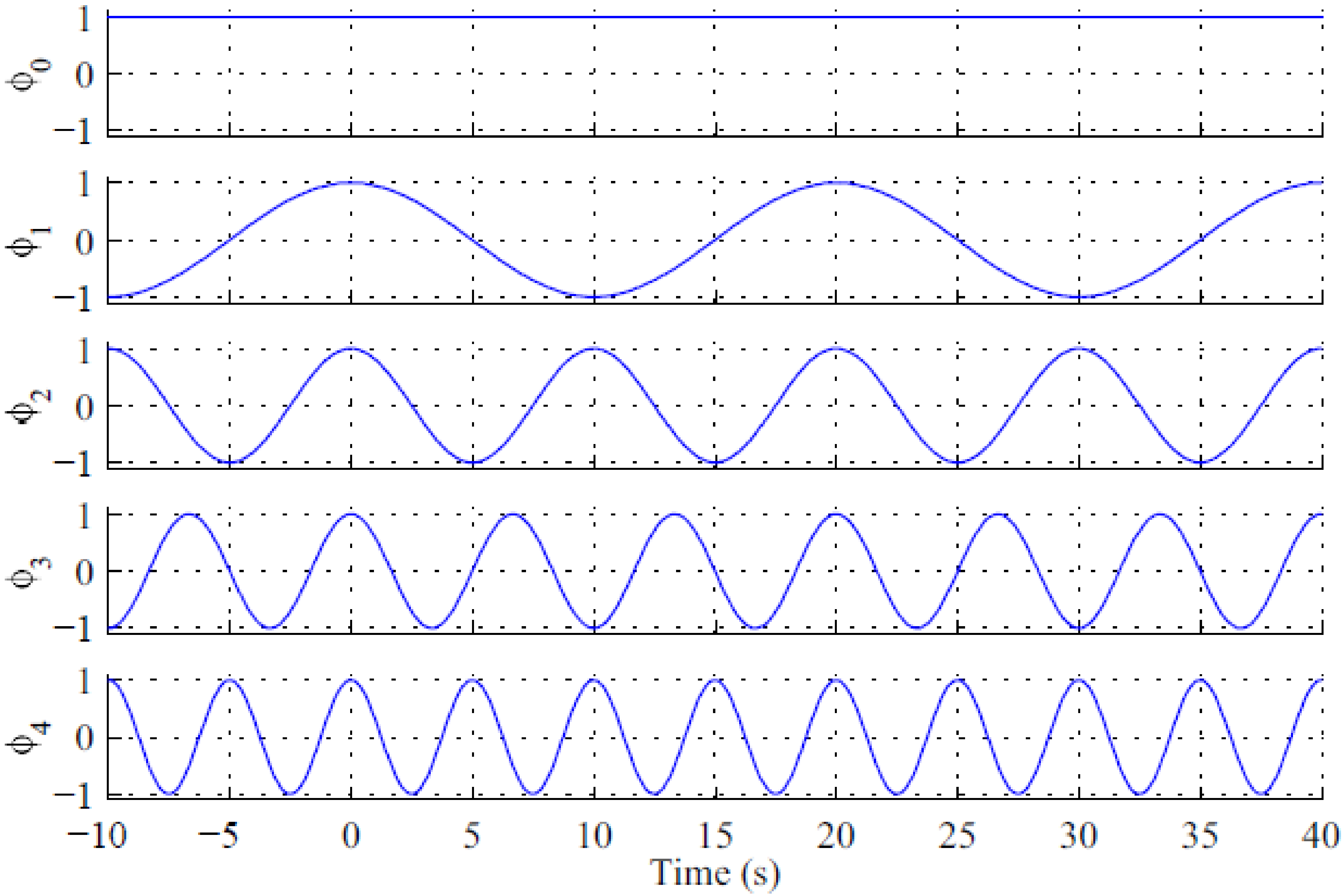
- There is more than one frequency ω that satisfies the constraint $x(t) = x(t+T)$ where $T = \frac{2\pi k}{\omega}$
- The **fundamental frequency** is given by $k = 1$:

$$\omega_0 = \frac{2\pi}{T_0}$$

- The other frequencies that satisfy this constraint are then integer multiples of ω_0

Sinusoidal Exponential Harmonics

- Example of continuous-time harmonics



Damped Complex Sinusoidal Exponentials

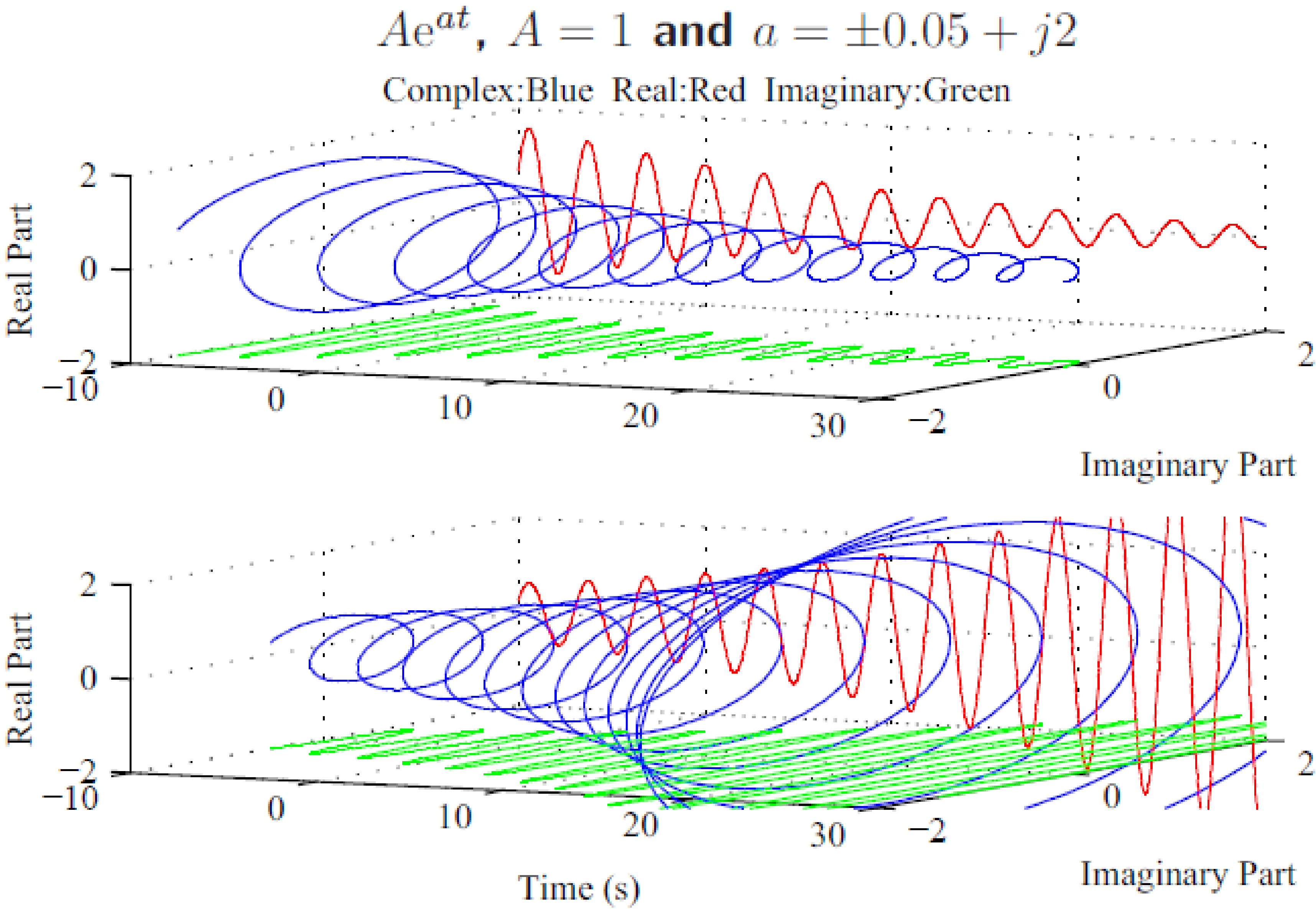
$$x(t) = Ae^{at}$$

- When a is complex, these become **damped** sinusoidal exponentials
- Let $a = \alpha + j\omega$. Then

$$x(t) = Ae^{at} = (Ae^{\alpha t}) \times e^{j\omega t}$$

- Thus, these are equivalent to multiplying an complex sinusoid by a real exponential

Damped Complex Sinusoidal Exponentials



Problem-1

- Express the sum of two complex exponentials as a product of a complex exponential and a single sinusoid:

$$x(t) = e^{j2t} + e^{j3t}$$

- Also, plot the magnitude of this signal.

END