9.3 Critical Damping $\alpha = \omega_0$

(PP 334 8th Ed HKD)

When $\alpha = \omega_0$, the roots are real and equal.

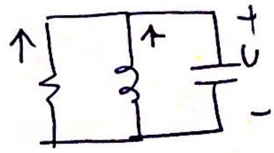
Consider $L = 7H$

$C = 1/42 F$

and $R = \frac{7\sqrt{6}}{2}$ (Earlier $R = 6\Omega$)

$v(0) = 0$

and $i(0) = 10A$



Now $\alpha = \frac{1}{2RC} = \sqrt{6}$

$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6}$

Hence $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

$s_{1,2} = -\sqrt{6}\ s^{-1}$

Returning to original differential

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

or $$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

(Now $\frac{1}{RC} = 2\alpha$)
and $\alpha^2 = \omega_0^2 = \frac{1}{LC}$

Thus $$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \alpha^2 v = 0$$

This $=^n$ has a standard solution:-

$$v = e^{-\alpha t}(A_1 t + A_2)$$

contd

— Contd (335)

So we get

$$v(t) = A_1 t e^{-\sqrt{6}t} + A_2 e^{-\sqrt{6}t}$$

Given $v(0) = 0$

$$0 = A_1 \times 0 \times 1 + A_2$$

So $A_2 = 0$

— Hence

$$v(t) = A_1 t e^{-\sqrt{6}t} + 0$$

or $v(t) = A_1 t e^{-\sqrt{6}t}$

— For 2nd constant

$$\frac{dv}{dt} = A_1 t e^{-\sqrt{6}t}(-\sqrt{6}) + A_1 e^{-\sqrt{6}t})$$

or $$\frac{dv}{dt} = A_1 e^{-\sqrt{6}t} - \sqrt{6} A_1 t e^{-\sqrt{6}t}$$

— but $$\left.\frac{dv}{dt}\right|_{t=0} = \frac{i_c(0)}{C} = \frac{\cancel{i_R(0)}}{\cancel{C}}^{=0} + \frac{i(0)}{C} = \frac{10}{1/42} = 420$$

Thus $A_1 = 420$

— Hence $$v(t) = 420 t e^{-2.45t}, \ V$$

(Supplementary)

* Critical Damping : Proof

Proceeding from: $\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \alpha^2 v = 0$

or $\frac{d}{dt}\left(\frac{dv}{dt} + \alpha v\right) + \alpha\left(\frac{dv}{dt} + \alpha v\right) = 0$

Let $\frac{dv}{dt} + \alpha v = f$ —— (A)

— Then $\frac{df}{dt} + \alpha f = 0$

— This is a 1st order eqn so

$f = A_1 e^{-\alpha t}$ is the solution

— So we get from (A)

$$\frac{dv}{dt} + \alpha v = A_1 e^{-\alpha t}$$

— or $e^{\alpha t}\frac{dv}{dt} + e^{\alpha t}(\alpha v) = A_1$

— This can be written as

$$\frac{d}{dt}\left(e^{\alpha t} v\right) = A_1$$

— Integrating both sides

$$e^{\alpha t} v = A_1 t + A_2$$

or $v = e^{-\alpha t}(A_1 t + A_2)$, V

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(Optional)

Graphical Representation of the Critically Damped Response

(PP 336 8th Ed H&D)

With the values of R, L and C such that $\alpha = \omega_0$, we have arrived at the end result :-

$$v(t) = 420t\, e^{-2.45t} \text{ V}$$

— This result confirms that the specified initial value is zero as $v(0) = 0$; $i_L(0) = 10$ A.

— It is not obvious that the response also approaches zero as t becomes infinitely large; which it should for the given source-free circuit.

— However, using L'Hospital's rule

$$\lim_{t\to\infty} v(t) = 420 \lim_{t\to\infty} \frac{t}{e^{2.45t}}$$

$$= 420 \lim_{t\to\infty} \frac{1}{2.45e^{2.45t}} = 0$$

— To determine the time t_m at which maximum value occurs, we differentiate wrt time and put the result equal to zero.

$$\frac{dv}{dt} = 420\left\{ t_m \times -2.45e^{-2.45t_m} + e^{-2.45t_m} \times 1 \right\} = 0$$

$$-2.45t_m = -1$$

$$t_m = 0.408 \text{ s} \qquad \text{(overdamped had } 0.358\text{)}$$

— Putting this value in $v(t)$, we get

$$v_m = 63.1 \text{ V} \qquad \text{(overdamped had } 49\text{V)}$$

——— contd

(Optional)

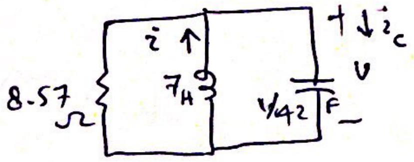
— Contd (334)

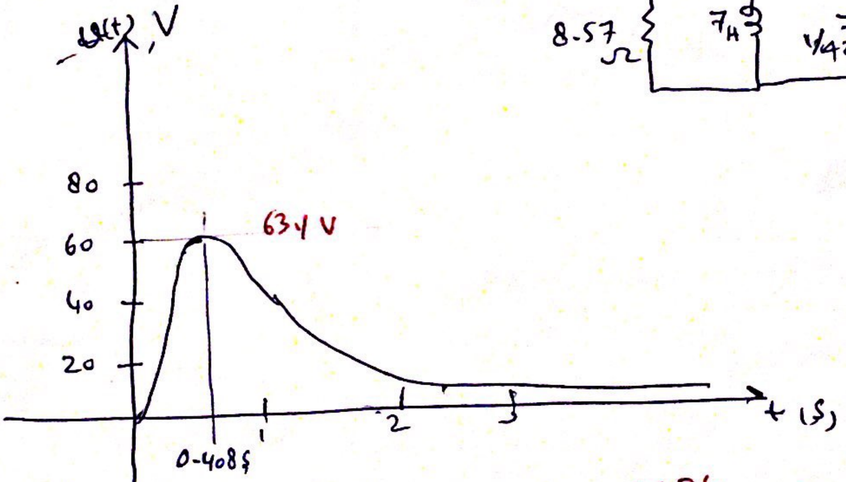
— The settling time at which 1% of max value is reached is $\frac{v_m}{100} = \frac{63.1}{100} = 420\, t_s\, e^{-2.45 t_s}$

— By trial and error;

$t_s = 3.12$ s (5.15 s overdamped)

Graphically:





(Plot of $v(t) = 420te^{-2.45t}$ V)