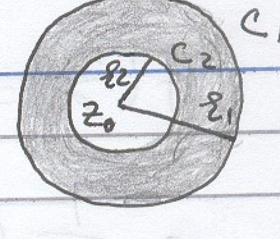
Laurent series: If $f(z)$ is analytic on a concentric circles C_1 & C_2 of radii r_1 and r_2 (with $r_2 < r_1$), centered at z_0 and also analytic throughout the region between the circles (i.e., an annular region), then for each point z within the Annulus $f(z)$ may be represented by the Laurent series



$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-z_0)^n .$$

$$= \cdots\cdots + \frac{c_{-r}}{(z-z_0)^r} + \frac{c_{-r+1}}{(z-z_0)^{r-1}} + \cdots + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots\cdots$$

where in general the coefficients c_r are complex. If $f(z)$ is analytic at z_0, then $c_n = 0$ for $n = -1, -2, \ldots\ldots$ and the Laurent series reduces to the Taylor series.

$$f(z) = \sum_{n=-\infty}^{-1} c_n (z-z_0)^n + \sum_{n=0}^{\infty} c_n (z-z_0)^n .$$

The first sum on the RHS, the 'Non-Taylor' part is called the principal part of the Laurent series.

EX: Determine the Laurent series expansions of

$$f(z) = \frac{1}{(z+1)(z+3)} \quad \text{valid for}$$

(a) $|z| < 1$.

Resolving into partial fractions, $f(z) = \frac{1}{2}\left[\frac{1}{1+z} - \frac{1}{3+z}\right]$

Now, $|z| < 1 \Rightarrow |z| < 3 \Rightarrow |z|/3 < 1$.

$$f(z) = \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}\left[1 - z + z^2 - z^3 + \cdots\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\cdots\right]$$

$$= \left(\frac{1}{2} - \frac{1}{6}\right) - \left(\frac{z}{2} - \frac{z}{18}\right) + \left(\frac{z^2}{2} - \frac{z^2}{54}\right) - \left(\frac{z^3}{2} - \frac{z^3}{162}\right) + \cdots$$

$f(z)$ admits Taylor series as $f(z)$ is analytic at all points.

(b) for $1 < |z| < 3$

since $|z| > 1$ and $|z| < 3$, we express $f(z)$ as $\quad |z|<3, \quad |z|>1$

$$f(z) = \frac{1}{2z}\frac{1}{\left(1+\frac{1}{z}\right)} - \frac{1}{6}\frac{1}{\left(1+\frac{1}{3}z\right)} \qquad \left|\frac{z}{3}\right|<1 \quad \left|\frac{1}{z}\right|<1$$

$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{1}{3}z\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{6}\left(1 - \frac{1}{3}z + \frac{1}{9}z^2 - \frac{1}{27}z^3 + \cdots\right)$$

$$= \cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{1}{18}z - \frac{1}{54}z^2 + \frac{1}{162}z^3 - \cdots$$

(c) for $|z| > 3$.

$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right)$$

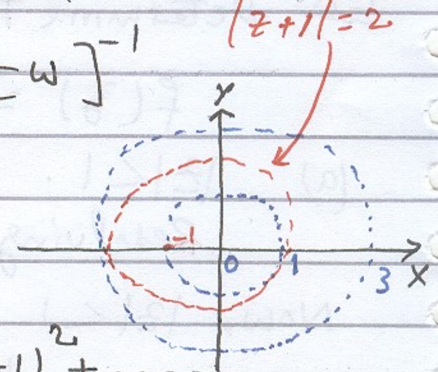
$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(d) for $0 < |z+1| < 2$

We can suppose $z+1 = w$, then $0 < |w| < 2$ and

$$f(w) = \frac{1}{w(w+2)} = \frac{1}{2w\left(1+\frac{w}{2}\right)} = \frac{1}{2w}\left[1 + \frac{1}{2}w\right]^{-1}$$

$$= \frac{1}{2w}\left(1 - \frac{1}{2}w + \frac{1}{4}w^2 - \frac{1}{8}w^3 + \cdots\right)$$



giving

$$f(z) = \frac{1}{(z+2)(z+1)} = \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots$$

Note in part (a), series representation do not involve negative powers of z. In (b) infinite terms, same happens in (c). In (d), $f(z)$ has finite terms in the Principal part

Singularities, zeros & residues:

singular point

$f(z)$ is infinite at this point

There is a choice of value, and it is not possible to pick a particular one.

Here we shall be mainly concerned with singularities at which $f(z)$ has an infinite value.

A zero of $f(z)$ is a point in the z plane at which $f(z) = 0$.

Singularities can be classified in terms of the Laurent series expansion of $f(z)$ about the point in question. If $f(z)$ has a Taylor series expansion, i.e., a Laurent series expansion with zero principal part, about the point $z = z_0$, then z_0 is a regular point of $f(z)$. If $f(z)$ has a Laurent series expansion with only a finite number of terms in the principal part, for example

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots\cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots\cdots + a_m(z-z_0)^m + \cdots\cdots$$

then $f(z)$ has a singularity at $z = z_0$ called a pole. If there are m terms in the principal part, then the pole is said to be of order m. If the principal part of the Laurent series for $f(z)$ at $z = z_0$ has infinitely many terms, then $z = z_0$ is called an essential singularity of $f(z)$.

EX: Find and classify singularities of $f(z) = \dfrac{e^z - \sin z - 1}{z^2}$.

Sol: We know that $e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots\cdots$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\cdots$$

hence,
$$f(z) = \frac{\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\cdots\right) - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) - 1}{z^2}$$

$$= \frac{1}{2!} + \frac{2}{3!} z + \frac{1}{4} z^2 + \frac{1}{6!} z^4 + \frac{2}{7!} z^5 + \cdots\cdots$$

The Laurent expansion of $f(z)$ has no negative powers of z. Therefore, $f(z)$ has a removable singularity at $z = 0$.

EX:- Let $f(z) = \dfrac{\pi z(1-z^2)}{\sin(\pi z)}$.

(a) Find all zero(s) of $f(z)$. (b) Find and classify all singularities of $f(z)$ (c) identify the Laurent expansion of $f(z)$ around zero.

Sol:- we notice that

(a) $$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\pi z(1-z^2)}{\sin(\pi z)} = 1 \neq 0.$$

$$\lim_{z\to 1} f(z) = \lim_{z\to 1} \frac{\pi z(1-z)(1+z)}{\sin(\pi(z-1)+\pi)}$$

$$= \lim_{z\to 1} \frac{\pi z(1-z)(1+z)}{-\sin(\pi(z-1))} = 2\pi \neq 0.$$

and $$\lim_{z\to -1} f(z) = \lim_{z\to -1} \frac{\pi z(1-z)(1+z)}{\sin(\pi(1+z)-\pi)} = \lim_{z\to -1} \frac{\pi z(1-z)(1+z)}{-\sin(\pi(z+1))} = -2\pi \neq 0.$$

So, the function $f(z)$ has no zeros.

(b). $z = 0, 1$, and -1 are removable singularities of $f(z)$.

$z = k$, $k \neq 0, 1, -1$ are simple poles of the function $f(z)$.

(c). we know that $\dfrac{1}{\sin \pi z} = \operatorname{Cosec}(\pi z)$.

$$= \frac{1}{\pi z}\left(1 + \frac{\pi^2 z^2}{3!} + \cdots\right)$$

Thus the Laurent series expansion of f around $z=0$ is

$$f(z) = \pi z(1-z^2) \cdot \frac{1}{\pi z}\left(1 + \frac{\pi^2 z^2}{3!} + \cdots\right)$$

$$= 1 + \left(\frac{\pi^2}{6} - 1\right) z^2 + \cdots$$

EX:- Let $f(z) = \dfrac{1}{(z-1)(z-3)^2}$

$f(z)$ has a pole of order one (simple pole) at $z=1$

and a pole of order two at $z=3$.

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Residues: If a complex function $f(z)$ has a pole at the point $z=z_0$ then the coefficient a_{-1} of the term $1/(z-z_0)$ in the Laurent series expansion of $f(z)$ about $z=z_0$ is called the residue of $f(z)$ at the point $z=z_0$.

Let us consider the case when $f(z)$ has a simple pole at $z=z_0$. Then

$$f(z) = \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots\cdots$$

in an appropriate annulus $R_1 < |z-z_0| < R_2$. multiplying by $z-z_0$ gives

$$(z-z_0)f(z) = a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + \cdots\cdots$$

which is a Taylor series expansion of $(z-z_0)f(z)$. If we let z approach z_0, we then obtain the result

$$\text{Residue at a simple pole } z_0 = \lim_{z \to z_0} \left[(z-z_0)f(z)\right] = a_{-1}$$

Hence the limit gives a way of calculating the residue at a simple pole.

Now suppose that we have a pole of order two at $z=z_0$. $f(z)$ has a Laurent series expansion of the form

$$f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots\cdots$$

$$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + \cdots\cdots$$

$$\frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = a_{-1} + 2a_0(z-z_0) + \cdots\cdots$$

$$\lim_{z \to z_0}\left[\frac{d}{dz}\left((z-z_0)^2 f(z)\right)\right] = a_{-1}.$$

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In general if $f(z)$ has a pole of order m at $z=z_0$,

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots\cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots\cdots$$

$$\text{residue at a pole of order } m \text{ at } z=z_0 = \frac{1}{(m-1)!} \lim_{z \to z_0}\left\{\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)\right\}.$$

EX:- Find the Laurent series for the functions below about the specified point and hence find the residue at the point.

(a). $f(z) = \dfrac{1}{z(z+1)}$, $z = 0$.

$$f(z) = \frac{1}{z}(1+z)^{-1} = \frac{1}{z}(1 - z + z^2 - z^3 + \cdots\cdots) \qquad |z| < 1$$

$$= \frac{1}{z} - 1 + z - z^2 + z^3 + \cdots\cdots\cdots = \sum_{n=-1}^{\infty} (-1)^{n+1} z^n .$$

The residue of a function $f(z)$ at $z = z_0$ is the Coefficient of $(z - z_0)^{-1}$ in its Laurent expansion around $z = z_0$.

Hence, $\text{Res}[f(z), z = 0] = 1$.

(b). $f(z) = \dfrac{\sin z}{z^4}$, $z = 0$.

we know that $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots\cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$.

$$\Rightarrow \frac{\sin z}{z^4} = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-3} .$$

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The residue of $f(z)$ at $z = 0$ is the Coefficient of $\frac{1}{z}$, $\text{Residue}\left[f(z), z = 0 \right] = -\dfrac{1}{3!} = -\dfrac{1}{6}$.

Direct calculation Residue:

(a) $\text{Res}\left[f(z) ; z = 0 \right] = \displaystyle\lim_{z \to 0} [z f(z)] = \lim_{z \to 0} \frac{1}{z+1} = \frac{1}{0+1} = 1$.

(b) $\text{Res}\left[f(z) ; z = 0 \right] = \dfrac{1}{(4-1)!} \left[\dfrac{d^3}{dz^3} (\sin z) \right]_{z=0}$

$= \dfrac{1}{3!} \left[-\cos z \right]_{z=0} = -\dfrac{1}{3!} = -\dfrac{1}{6}$.

$f(z) = \sin z$
$f'(z) = \cos z$
$f''(z) = -\sin z$
$f'''(z) = -\cos z$

Ex:- Find the residues of $f(z) = \dfrac{1}{(3z+2)(2-z)}$ at $z = -\dfrac{2}{3}$ and $z = 2$.

(i) Residue at $z = -2/3$: Let $w = z + \dfrac{2}{3} \Rightarrow z = -\dfrac{2}{3} + w$.

$w = z - (-2/3)$

$$\frac{1}{(3z+2)(2-z)} = \frac{1}{\left[3\left(-\frac{2}{3}+w\right)+2\right]\left[2-\left(-\frac{2}{3}+w\right)\right]} = \frac{1}{3w\left(\frac{8}{3}-w\right)}$$

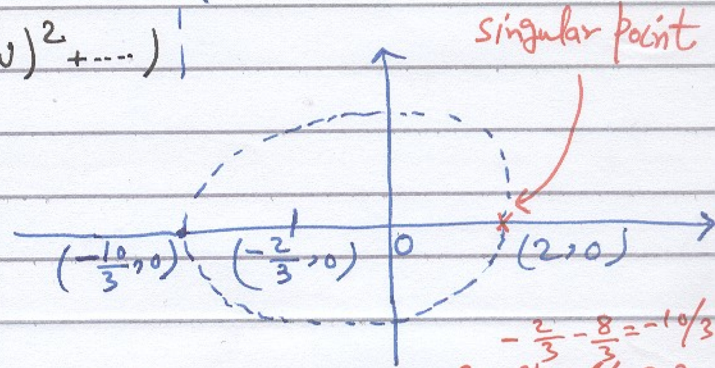
$$\frac{1}{(3z+2)(2-z)} = \frac{-1}{w(3w-8)}$$

$$= \frac{1}{8w}\,\frac{1}{\left(1-\frac{3}{8}w\right)} = \frac{1}{8w}\left(1+\frac{3}{8}w+\left(\frac{3}{8}w\right)^2+\cdots\right)$$

$$= \frac{1}{8w} + \frac{3}{64} + \frac{9}{512}w + \cdots$$

$\left|\frac{3}{8}w\right| < 1 \Rightarrow |w| < \frac{8}{3}$

$\left|z+\frac{2}{3}\right| < \frac{8}{3}$



So, the residue at $w = 0$ $\left(z = -\frac{2}{3}\right)$ is $\frac{1}{8}$.

$-\frac{2}{3} - \frac{8}{3} = -10/3$

$-\frac{2}{3} + \frac{8}{3} = 6/3 = 2$

(ii) Residue at $z = 2$: Let $z = 2 + w$

$w = z - 2$

$$\frac{1}{(3z+2)(2-z)} = \frac{1}{(3(2+w)+2)(2-(2+w))} = \frac{1}{(-w)(3w+8)}$$

$\left|\frac{3}{8}w\right| < 1$

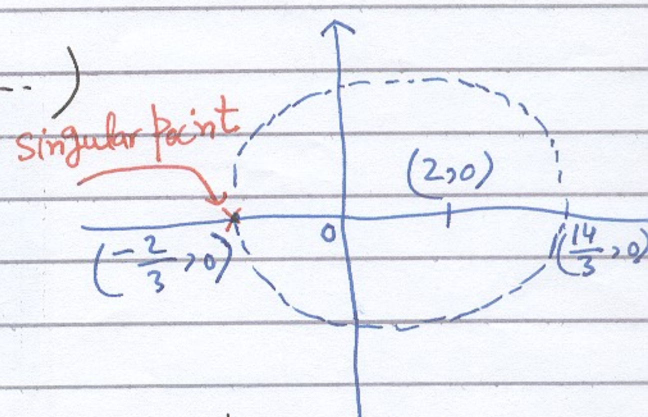
$|w| < \frac{8}{3}$

$|z-2| < \frac{8}{3}$

$$= -\frac{1}{8w}\,\frac{1}{\left(1+\frac{3}{8}w\right)}$$

$$= \frac{-1}{8w}\left(1-\frac{3}{8}w+\left(\frac{3}{8}w\right)^2 - \cdots\right)$$

$$= -\frac{1}{8w} + \frac{3}{64} - \frac{9}{512}w + \cdots$$



So, the residue at $w = 0$ $(z = 2)$ is $\frac{-1}{8}$.

Direct Calculation of Residue:

$$\text{Residue}\left[f(z); z = -\tfrac{2}{3}\right] = \lim_{z \to -\frac{2}{3}} \left[\left[z - \left(-\tfrac{2}{3}\right)\right] \frac{1}{3} \cdot \frac{1}{\left(z+\frac{2}{3}\right)(2-z)}\right]$$

$$= \frac{1}{3} \lim_{z \to -2/3} \frac{1}{2-z} = \frac{1}{3} \cdot \frac{1}{2-(-2/3)} = \frac{1}{3} \cdot \frac{1}{8/3} = 1/8.$$

$$\text{Residue}\left[f(z); z = 2\right] = \lim_{z \to 2} \left[(z-2) f(z)\right] = \lim_{z \to 2} \left[(z-2) \frac{1}{(3z+2)(2-z)}\right]$$

$$= (-1) \lim_{z \to 2} \frac{1}{3z+2} = -\frac{1}{8}.$$

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