Power Series (Tayler & Laurent Series)

Examples $f(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots$

$\sum_{n=0}^{\infty} x^n$ Converges for $|x|<1$ & diverges for $|x|>1$.

$\sum_{n=1}^{\infty} \frac{2^n x^n}{n}$ $\quad \left|\frac{u_{n+1}}{u_n}\right| = \frac{2^{n+1}|x|^{n+1}}{(n+1)} \frac{n}{2^n |x|^n} = \frac{2}{(1+\frac{1}{n})}|x|$

$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{2}{(1+\frac{1}{n})}|x| = 2|x| < 1$ $\quad$ Converges for $|x| < 1/2$.

$\sum_{n=1}^{\infty} n^2 (z+\frac{1}{2})^n$. $\quad \left|\frac{u_{n+1}}{u_n}\right| = \frac{(n+1)^2}{n^2} \cdot \frac{|(z+\frac{1}{2})^{n+1}|}{|(z+\frac{1}{2})^n|} = (1+\frac{1}{n})^2 |z+\frac{1}{2}|$

$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = |z+\frac{1}{2}|$, the series Converges when $|z+\frac{1}{2}|<1$.

$\sum_{n=0}^{\infty} \frac{(2+i)^n}{(z+i)^n (n+i)^2}$. $\quad \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{(|2+i|)^{n+1}}{|z+i|^n |n+i|^2} \times \frac{(z+i)^n (n+i)^2}{(2+i)^n}$

$= \lim_{n\to\infty} \frac{\sqrt{5}}{|z+i|} \left|\frac{n+i}{n+1+i}\right|^2 = \lim_{n\to\infty} \frac{\sqrt{5}}{|z+i|} \left|\frac{1+\frac{i}{n}}{1+\frac{1+i}{n}}\right|^2 = \frac{\sqrt{5}}{|z+i|}$

Series Converges absolutely for $\frac{\sqrt{5}}{|z+i|} < 1 \Rightarrow |z+i| > 5$

$1 + (z-1)^2 + (z-1)^4 + (z-1)^6 + \cdots$

$\frac{1}{1-z} = 1 + z + z^2 + \cdots$, $|z|<1$, $\frac{1}{1-(z-1)^2} = 1 + (z-1)^2 + (z-1)^4 + \cdots$

valid for $|z-1|<1$.

$1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$

$\frac{1}{1-\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$ $\quad \left|\frac{1}{z}\right| < 1$ or $|z|>1$

$\frac{z}{z-1} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \sum_{k=0}^{\infty} \frac{1}{z^k}$, $|z|>1$.

Expand $f(z) = \frac{1}{z}$, $z = i$.

$c_0 = f(i) = \frac{1}{i} = -i$, $\quad c_1 = \left.\frac{-1}{z^2}\right|_{z=i} = 1$,

$c_2 \Big|_{at\, z=i} = \left.\frac{2}{z^3}\right|_{z=i} = 2i$, $\quad c_3 \Big|_{z=i} = \left.-\frac{2(3)}{z^4}\right|_{z=i} = \frac{-6}{(i)^4} = -3!$

$$\frac{1}{z} = -i + (z-i) + \frac{1}{2!}(2i)(z-i)^2 + \frac{1}{3!}(-3i)(z-i)^3 \cdots$$

$$= -i + (z-i) + i(z-i)^2 - (z-i)^3 \cdots$$

(i) e^z, $z = 2+i$ (ii) $\operatorname{Log} z$, $z = e$ (iii) $\frac{1}{z^2}$, $z = 1+i$.

(iv) $\cosh z - \cos z$, $z = 0$.

$\log 1 = \ln 1 + i(0) = 0$.

(v) z^i, $z = 1$.

$z^i = e^{i \log z}$, $\quad c_0 = e^{i \log 1} = 1$.

For c_2: $\frac{d}{dz}\left(e^{i\log z} \cdot \frac{i}{z}\right)$
$= i\left[\frac{-1}{z^2} e^{i\log z} + \frac{1}{z} e^{i\log z}\left(\frac{i}{z}\right)\right]$
$= i(i-1)\frac{1}{z^2} e^{i\log z}$
$= i(i-1)\frac{1}{z^2} z^i = i(i-1) z^{i-2}$.

$c_1 \Big|_{z=1} = e^{i\log z} \cdot \left(\frac{i}{z}\right) = i z^{i-1} \Big|_{z=1} = i$

$c_2 = \frac{i(i-1) z^{i-2}}{2!} = \frac{i(i-1)}{2!}$

General term: $u_0 = 1$, $\quad u_n = \frac{(i)(i-1)\cdots[i-(n-1)]}{n!}(z-1)^n$, $n \geq 1$.

$e^{i\log z} = e^{i(\log r)} \cdot e^{-\theta}$ has branch point at $\theta = 0$

~~circle~~ radius of convergence $|z-1| = 1$

(vi) i^z, $z = 0$. $\quad i^z = e^{z \log i} = e^{i\frac{\pi}{2} z}$

$$i^z = 1 + i\frac{\pi}{2} z + \left(i\frac{\pi}{2} z\right)^2 \cdot \frac{1}{2!} + \left(i\frac{\pi}{2} z\right)^3 \cdot \frac{1}{3!} + \cdots$$

$u_n = \left(i\frac{\pi}{2} z\right)^n \cdot \frac{1}{n!}$, $n = 0, 1, 2, \ldots$ circle of convergence, center at 0, radius is ∞.

Find the sum of the following convergent series.

(i) $\sum_{n=0}^{\infty} \left(\frac{i}{3}\right)^n$ $\quad \sum_{n=0}^{\infty} \left(\frac{i}{3}\right)^n = \frac{1}{1-\frac{i}{3}} = \frac{9+3i}{10}$.

(ii) $\sum_{k=0}^{\infty} \frac{3}{(1+i)^k} = \frac{3}{1-\left(\frac{1}{1+i}\right)} = 3-3i$.

Using the ratio test, show that the following series converge.

(a) $\sum_{k=1}^{\infty} \frac{(3+i)^k}{k!}$ $\quad \lim_{k\to\infty} \left| \frac{(3+i)^{k+1}}{(k+1)!} \cdot \frac{k!}{(3+i)^k} \right| = \lim_{k\to\infty} \left|\frac{3+i}{k+1}\right| = 0$.

(b) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$, $\quad \lim_{k\to\infty} \left| \frac{(k+1)!}{(k+1)^{k+1}} \times \frac{k^k}{k!} \right| = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k = \lim_{k\to\infty} \frac{1}{(1+\frac{1}{k})^k} = \frac{1}{e} < 1$.

Using the ratio test, find a domain in which convergence holds for each of the following series of functions.

(a) $\sum_{k=0}^{\infty} \frac{(z-i)^k}{2^k}$ $\quad \lim_{k\to\infty} \left| \frac{(z-i)^{k+1}}{2^{k+1}} \cdot \frac{2^k}{(z-i)^k} \right| = \left|\frac{z-i}{2}\right|$. The series converges when $|z-i| < 2$.

(b) $\sum_{k=0}^{\infty} (z+5i)^{2k}(k+1)^2$. $\quad \lim_{k\to\infty} \left| \frac{(z+5i)^{2k+2}}{(z+5i)^{2k}} \cdot \frac{(k+2)^2}{(k+1)^2} \right| = |(z+5i)^2|$. The series converges when $|z+5i| < 1$.

Assume that for power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, we have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

Prove by the ratio test, that the radius of convergence of the power series is given by $R = 1/L$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n} \right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |z-z_0| = L|z-z_0|.$$

The ratio test implies that the series is convergent for $L|z-z_0| < 1$, yielding a radius of convergence of $1/L$.

Find the circle of convergence of $\sum_{k=1}^{\infty} \frac{(3-i)^k}{k^2}(z+2)^k$.

$$\lim_{k\to\infty} \left| \frac{(z+2)^{k+1}\cdot(3-i)^{k+1}}{(k+1)^2}\cdot\frac{k^2}{(3-i)^k(z+2)^k}\right|$$

$$= \lim_{k\to\infty}\left|(z+2)(3-i)\left(\frac{k}{k+1}\right)^2\right| = |(z+2)(3-i)|$$

$$|(z+2)\sqrt{1+9}| < 1 \Rightarrow |(z+2)| < \frac{1}{\sqrt{10}}.$$

Expand $f(z)=\frac{1}{1-z}$ in a Taylor series center at $z_0=2i$.

$$\frac{1}{1-z} = \frac{1}{1-z+2i-2i} = \frac{1}{(1-2i)-(z-2i)} = \frac{1}{1-2i}\,\frac{1}{1-\frac{z-2i}{1-2i}}$$

$$\frac{1}{1-z} = \frac{1}{(1-2i)}\left[1-\frac{z-2i}{1-2i}\right]^{-1}$$

$$= \frac{1}{(1-2i)}\left[1+\frac{z-2i}{1-2i}+\left(\frac{z-2i}{1-2i}\right)^2+\left(\frac{z-2i}{1-2i}\right)^3+\cdots\right]$$

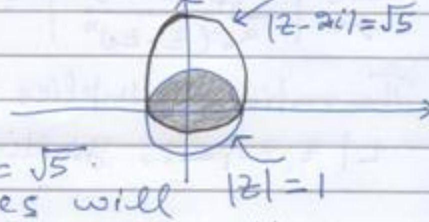
or $$\frac{1}{1-z} = \frac{1}{1-2i}+\frac{1}{(1-2i)^2}(z-2i)+\frac{1}{(1-2i)^3}(z-2i)^2+\cdots$$

$$= \sum_{n=0}^{\infty}\frac{1}{(1-2i)^{n+1}}(z-2i)^n.$$

Radius of Convergence: $\lim_{n\to\infty}\left|\frac{1}{(1-2i)^{n+2}}(z-2i)^{n+1}\cdot\frac{(1-2i)^{n+1}}{(z-2i)^n}\right|$

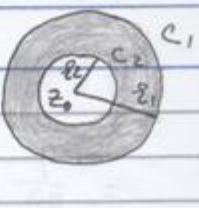
$$= \lim_{n\to\infty}\left|\frac{(z-2i)}{(1-2i)}\right| = \left|\frac{z-2i}{1-2i}\right| \Rightarrow |z-2i|<\sqrt{1+4} = 5.$$

Also,
$f(z)=\frac{1}{1-z}=1+z+z^2+z^3+\cdots$ converges for $|z|<1$.



Distance between $z=1$, $z=2i$ is $|1-2i|=\sqrt{5}$.
In the shaded region both the series will convergence. outside the colored region atleast one will diverge.

Laurent series If $f(z)$ is analytic on a concentric circles C_1 & C_2 of radii r_1 and r_2 (with $r_2 < r_1$), centered at z_0 and also analytic throughout the region between the circles (i.e. an annular region), then for each point z within the Annulus



$f(z)$ may be represented by the Laurent series

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n$$

$$= \cdots\cdots + \frac{c_{-r}}{(z-z_0)^r} + \frac{c_{-r+1}}{(z-z_0)^{r-1}} + \cdots + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots\cdots$$

where in general the coefficients c_r are complex. If $f(z)$ is analytic at z_0, then $c_n = 0$ for $n = -1, -2, \ldots$ and the Laurent series reduces to the Taylor series.

$$f(z) = \sum_{n=-\infty}^{-1} c_n (z-z_0)^n + \sum_{n=0}^{\infty} c_n (z-z_0)^n .$$

The first sum on the RHS, the 'Non-Taylor' part is called the principal part of the Laurent series.

EX: Determine the Laurent series expansions of

$$f(z) = \frac{1}{(z+1)(z+3)} \text{ valid for}$$

(a) $|z| < 1$.

Resolving into partial fractions, $f(z) = \frac{1}{2}\left[\frac{1}{1+z} - \frac{1}{3+z}\right]$

Now, $|z| < 1 \Rightarrow |z| < 3 \Rightarrow |z|/3 < 1$.

$$f(z) = \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}\left[1 - z + z^2 - z^3 + \cdots\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

$$= \left(\frac{1}{2} - \frac{1}{6}\right) - \left(\frac{z}{2} - \frac{z}{18}\right) + \left(\frac{z^2}{2} - \frac{z^2}{54}\right) - \left(\frac{z^3}{2} - \frac{z^3}{162}\right) + \cdots$$

$f(z)$ admits Taylor series as $f(z)$ is analytic at all points.

(b) for $1 < |z| < 3$

Since $|z| > 1$ and $|z| < 3$, we express $f(z)$ as

$$f(z) = \frac{1}{2z}\frac{1}{\left(1+\frac{1}{z}\right)} - \frac{1}{6}\frac{1}{\left(1+\frac{1}{3}z\right)} \qquad \begin{matrix} |z|<3, & |z|>1 \\ \left|\frac{z}{3}\right|<1 & \left|\frac{1}{z}\right|<1 \end{matrix}$$

$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{1}{3}z\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right) - \frac{1}{6}\left(1-\frac{1}{3}z+\frac{1}{9}z^2-\frac{1}{27}z^3+\cdots\right)$$

$$= \cdots - \frac{1}{2z^4}+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}-\frac{1}{6}+\frac{1}{18}z-\frac{1}{54}z^2+\frac{1}{162}z^3 - \cdots$$

(c) for $|z| > 3$.

$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right) - \frac{1}{2z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}+\cdots\right)$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(d) for $0 < |z+1| < 2$

We can suppose $z+1 = w$, then $0<|w|<2$ and

$$f(w) = \frac{1}{w(w+2)} = \frac{1}{2w\left(1+\frac{w}{2}\right)} = \frac{1}{2w}\left[1+\frac{1}{2}w\right]^{-1}$$

$$= \frac{1}{2w}\left(1-\frac{1}{2}w+\frac{1}{4}w^2-\frac{1}{8}w^3+\cdots\right)$$

giving

$$f(z) = \frac{1}{(z+2)(z+1)} = \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots$$

Note In part(a), series representation do not involve negative powers of z. In (b) infinite terms, same happens in (c). In (d), $f(z)$ has finite terms in the Principal Part.