Applications of Integration

Calculus & Analytical Geometry MATH-101

Instructor: Dr. Naila Amir (SEECS, NUST)

Recap

Applications of Integration

- **Area between the curves:**
  - $$A = \int_a^b [f(x) - g(x)]dx$$
  - $$A = \int_c^d [f(y) - g(y)]dy$$
- **Arclength of a curve:**
  - $$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
  - $$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$
  - $$L = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt$$

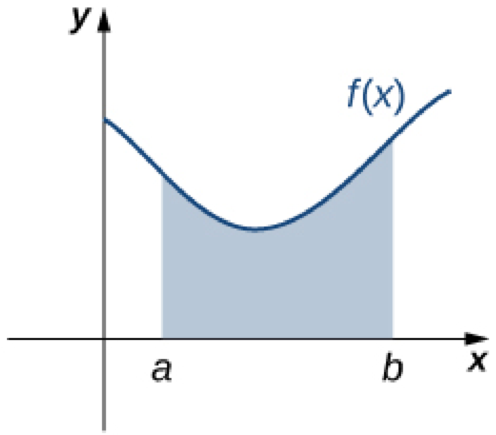
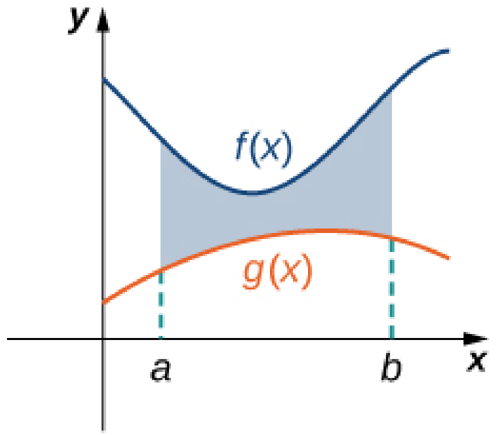
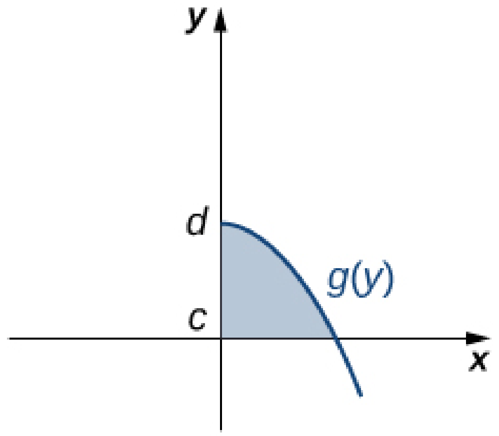
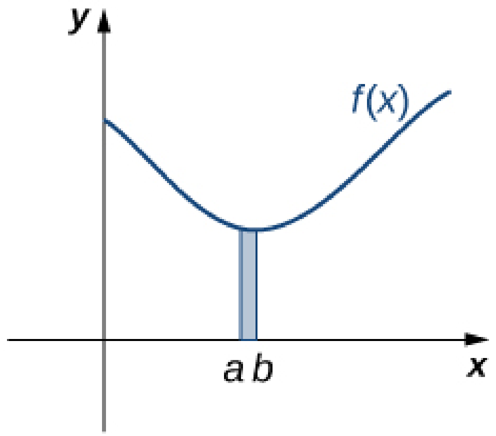
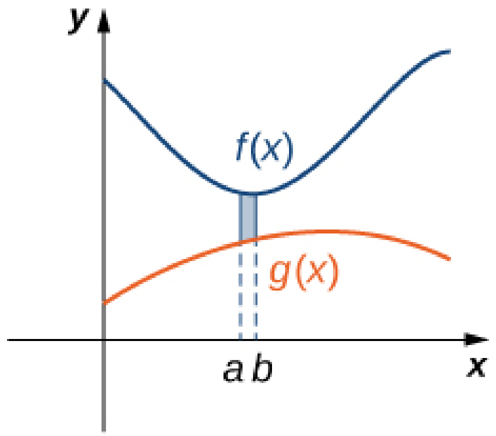
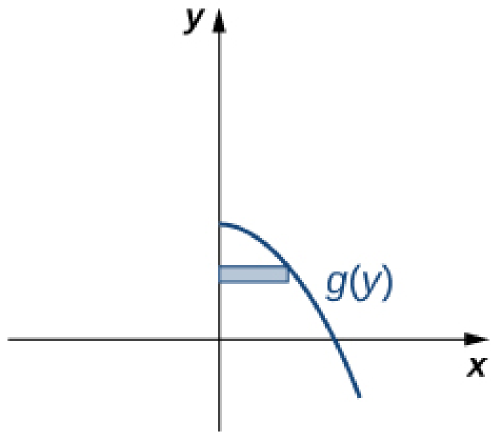
Volume of solids of revolution:

- **Book:** Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

  Chapter: 6 (Section: 6.1)

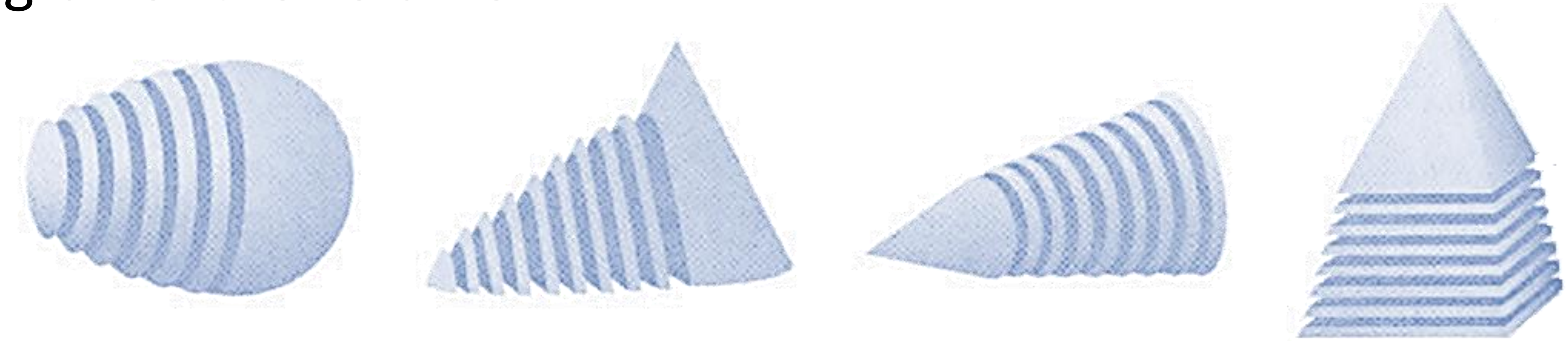
- **Book:** Calculus (5th Edition) by Swokowski, Olinick and Pence

  Chapter: 6 (Section: 6.2)

Compare	Disk Method	Washer Method	Shell Method
Volume formula	$V = \int_a^b \pi[f(x)]^2\,dx$	$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]\,dx$	$V = \int_c^d 2\pi y\, g(y)\,dy$
Solid	No cavity in the center	Cavity in the center	With or without a cavity in the center
Interval to partition	$[a, b]$ on x-axis	$[a, b]$ on x-axis	$[c, d]$ on y-axis
Rectangle	Vertical	Vertical	Horizontal
Typical region			
Typical element			

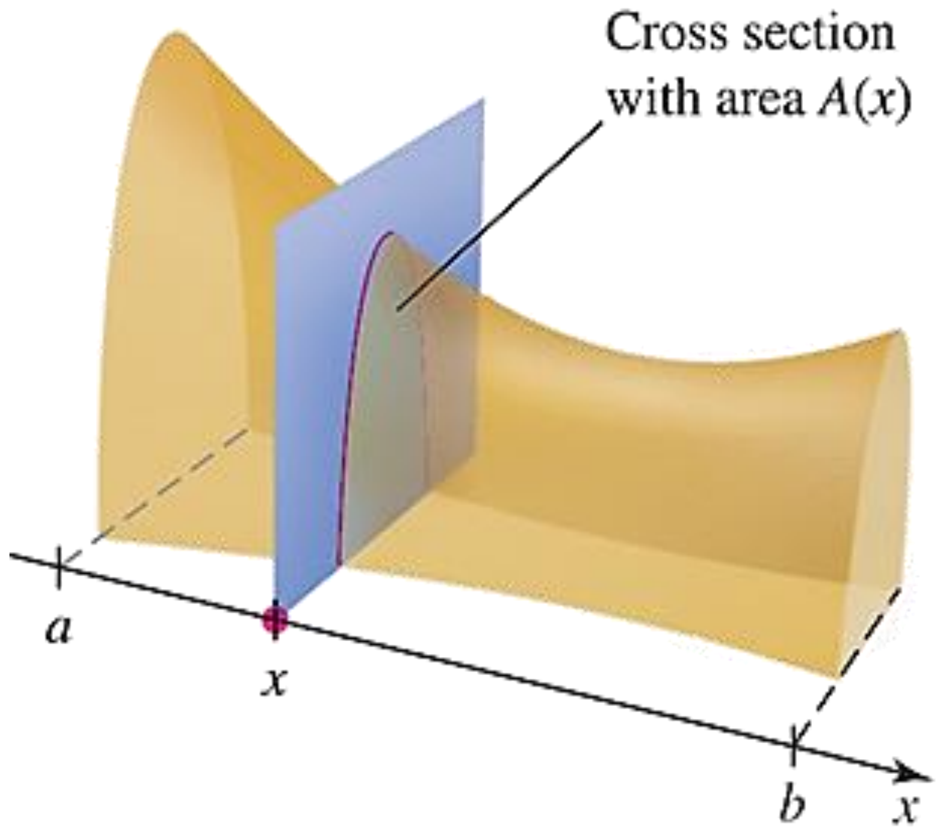
Volume

- Recall that the underlying principle for finding the area of a plane region is to divide the region into thin strips, approximate the area of each strip by the area of a rectangle, add the approximations to form a Riemann Sum, and take the limit of the Riemann Sums to produce an integral for the area.
- Under appropriate conditions, the same strategy can be used to find the volume of a solid. The idea is to divide the solid into thin slabs, approximate the volume of each slab, add the approximations to form a Riemann Sum, and take the limit of the Riemann Sums to produce an integral for the volume.



General Slicing Method

- Integrals are also used to find the volume of three-dimensional regions (or solids).
- Consider a solid object that extends in the x −direction from $x = a$ to $x = b$.
- Imagine cutting through the solid, perpendicular to the x −axis at a particular point x and suppose the area of the cross section created by the cut is given by a known integrable function A.
- To find the volume of this solid, we first divide $[a, b]$ into n subintervals of length $\Delta x = (b - a)/n$.
- The endpoints of the subintervals are the grid points $x_0 = a$, $x_1, x_2, \dots, x_n = b$.

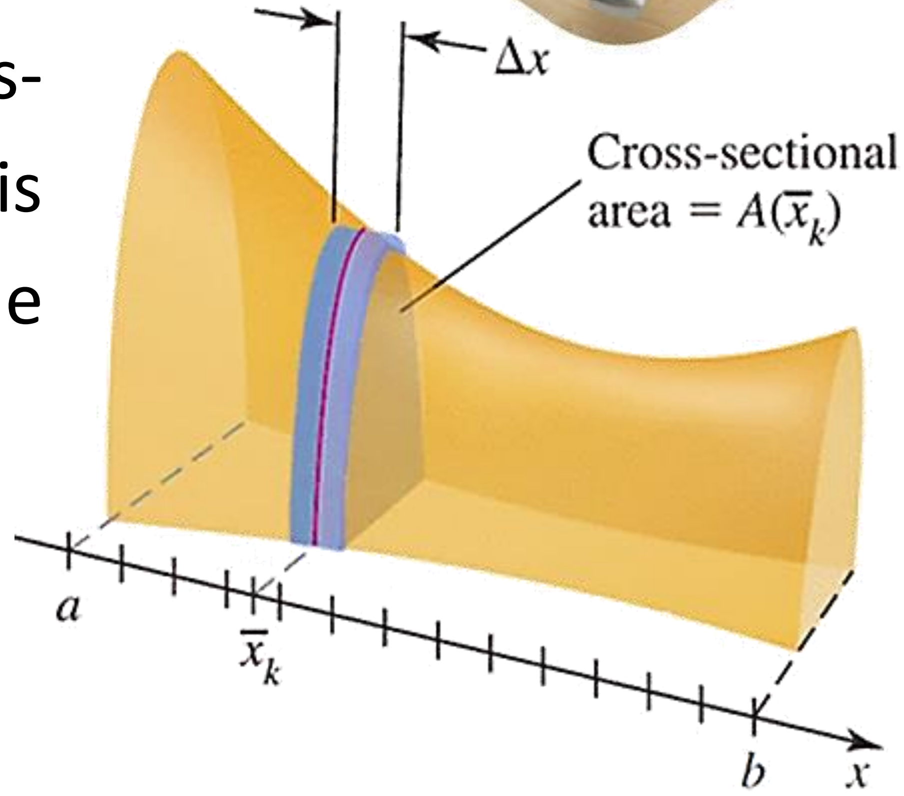


General Slicing Method

- We now make cuts through the solid perpendicular to the x −axis at each grid point, which produces n slices of thickness Δx. Imagine cutting a loaf of bread to create n slices of equal width.

- On each subinterval, an arbitrary point x_k^* is identified. The k^{th} slice through the solid has a Δx, and we take $A(x_k^*)$ as a representative cross-sectional area of the slice. Therefore, the volume of the k^{th} slice is approximately $A(x_k^*)\Delta x$. Summing the volumes of the slices, the approximate volume of the solid is:

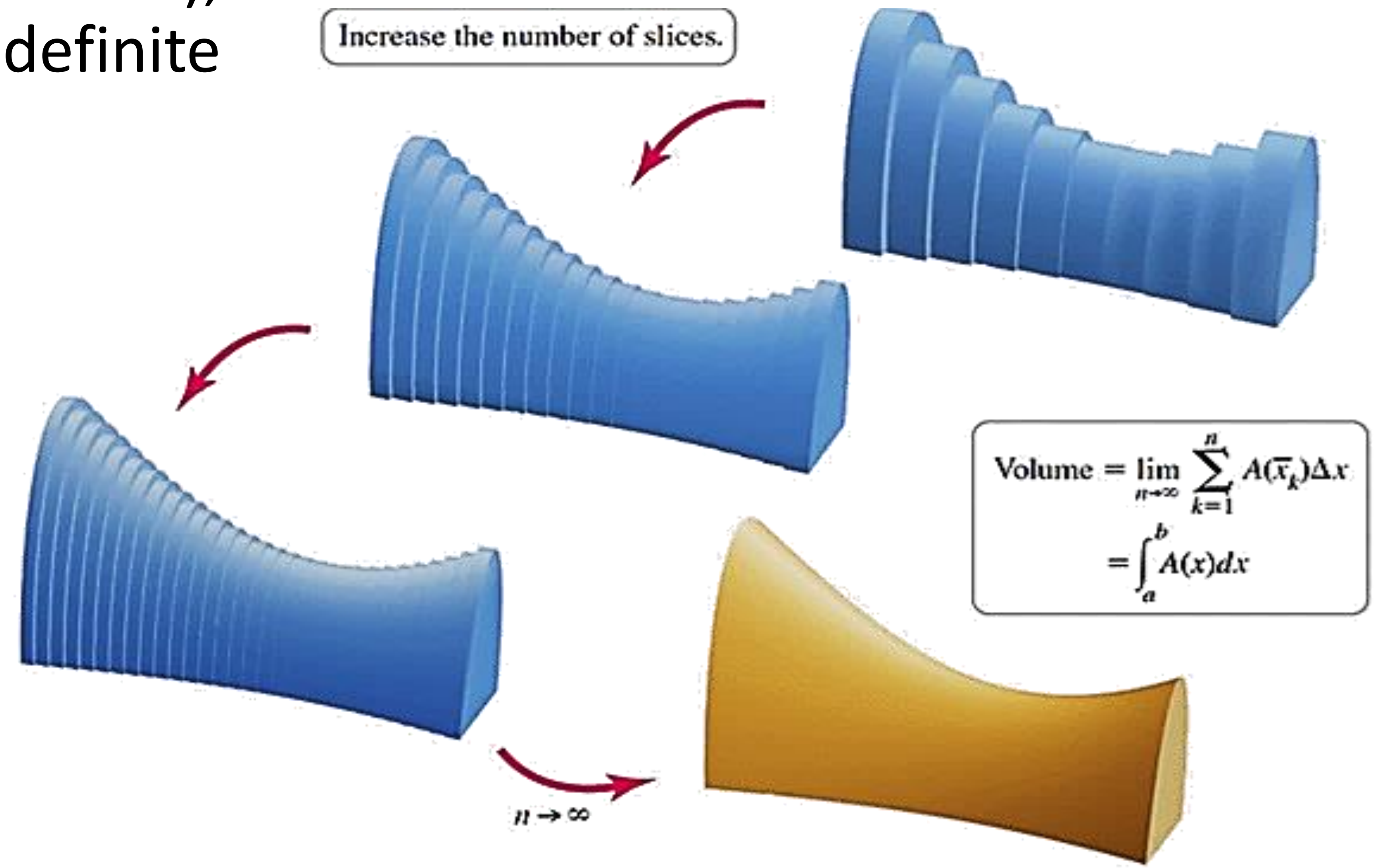
$$V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x.$$



General Slicing Method

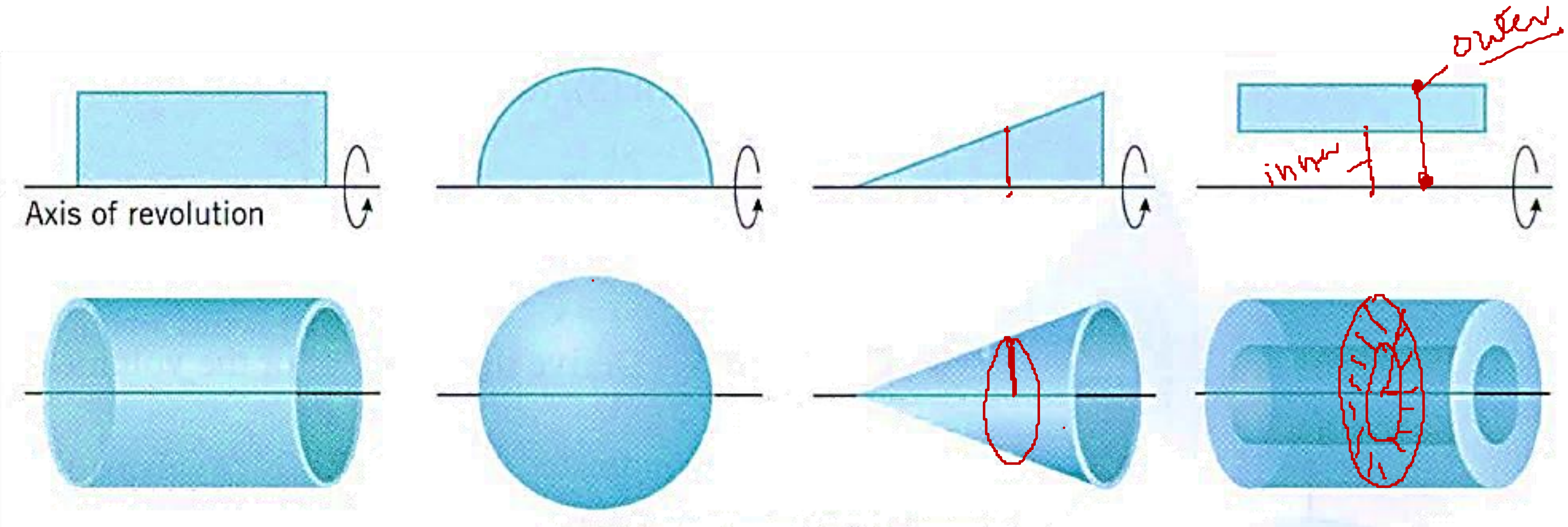
As the number of slices increases (i.e., $n \to \infty$) and the thickness of each slice goes to zero ($\Delta x \to 0$), the exact volume V is obtained in terms of a definite integral as:

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x = \int_a^b A(x)\,dx.$$



Solids of Revolution

A solid of revolution is a solid whose shape can be generated by revolving a plane region about a line that lies in the same plane as the region. The line is called the axis of revolution.



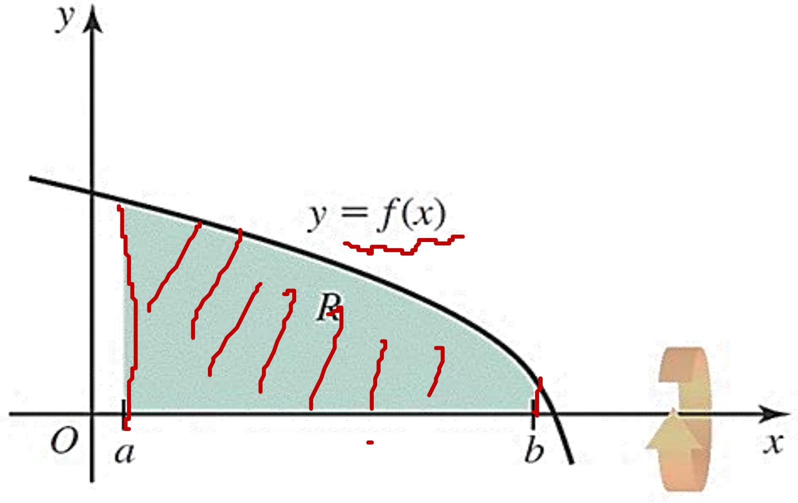
Some familiar solids of revolution

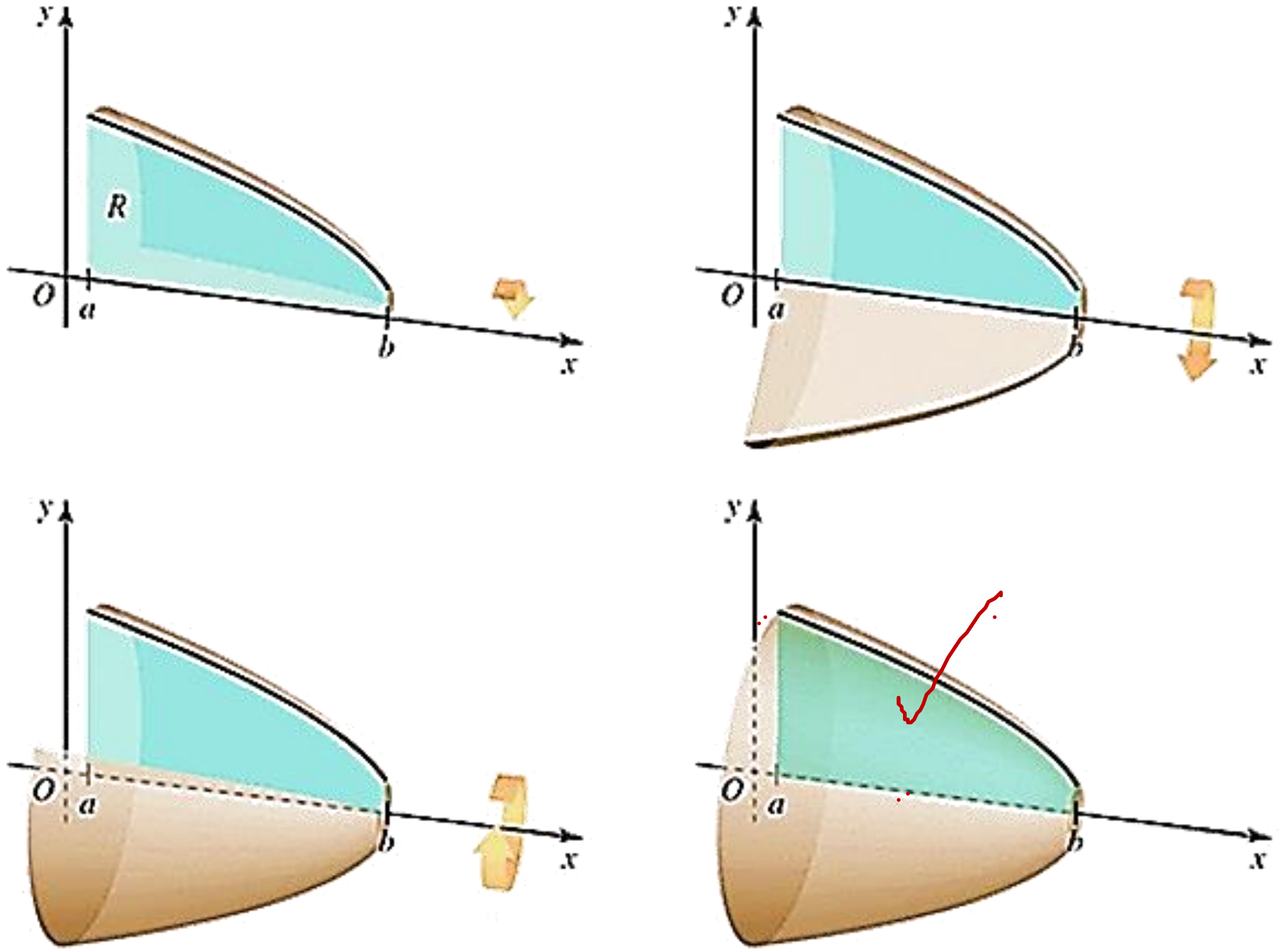
Radius

A(x)

Volume of Solid of Revolution

- Suppose $f(x)$ is a continuous function with $f(x) \geq 0$ on an interval $[a, b]$. Let R be the region bounded by the graph of $f(x)$, the x −axis, and the lines $x = a$ and $x = b$.





- Now revolve R around the x −axis. As R revolves once around the x −axis, it sweeps out a three-dimensional solid of revolution.
- The goal is to find the volume of the solid, and it may be done using the general slicing method.

Volume of Solid of Revolution

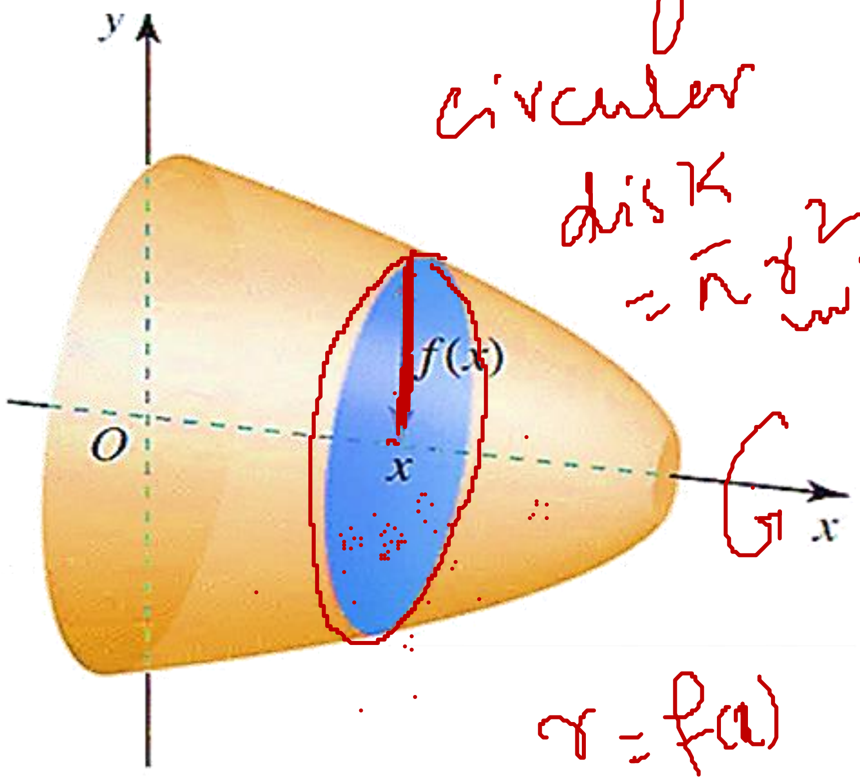
- With a solid of revolution, the cross-sectional area function has a special form since all cross sections perpendicular to the x –axis are *circular disks* with radius $f(x)$.
- Therefore, the cross section at the point x, where $a \leq x \leq b$, has area:

$$A(x) = \pi(radius)^2 = \pi[f(x)]^2.$$

- By the general slicing method, the volume of the solid is:

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[f(x)]^2\,dx.$$

Area of circular disk $= \pi r^2$



$r = f(x)$

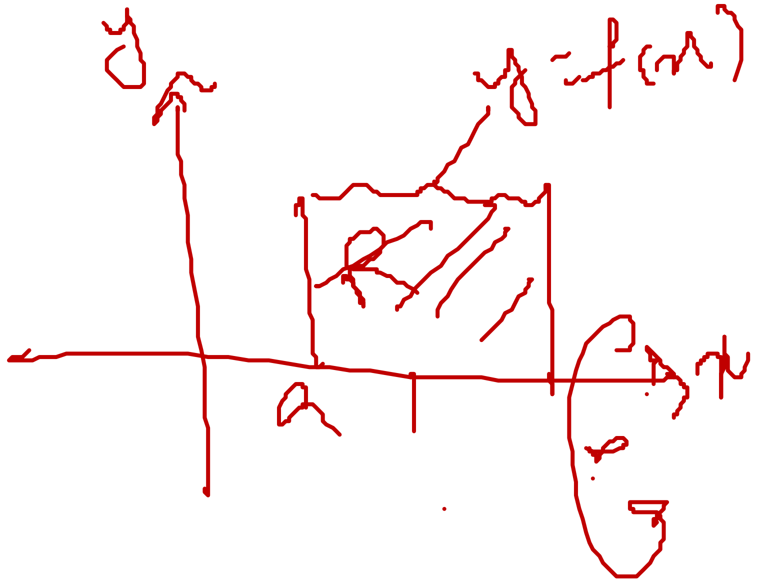
- Since each slice through the solid is a circular disk, the resulting method is called the ***disk method.***

The Disk Method

Disk Method About the x-axis:

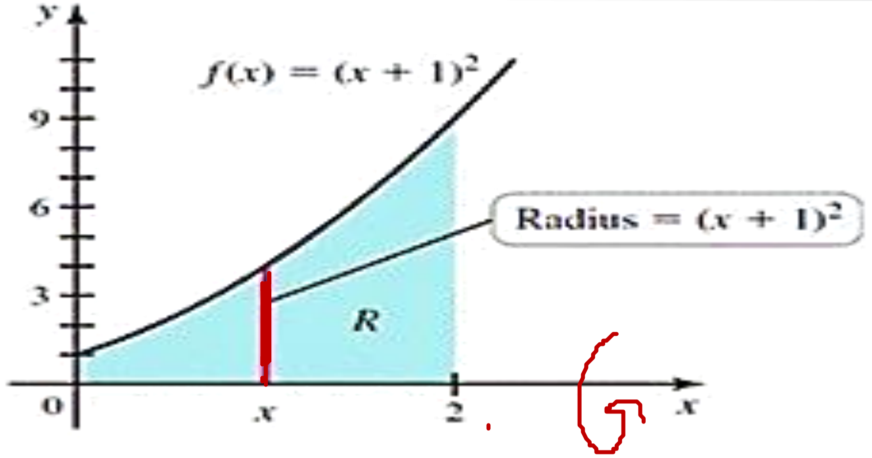
Let $f(x)$ be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of $f(x)$, the x −axis, and the lines $x = a$ and $x = b$ is revolved about the x −axis, the volume of the resulting solid of revolution is:

$$\boxed{V = \int_a^b \pi[f(x)]^2\, dx.}$$



Example:

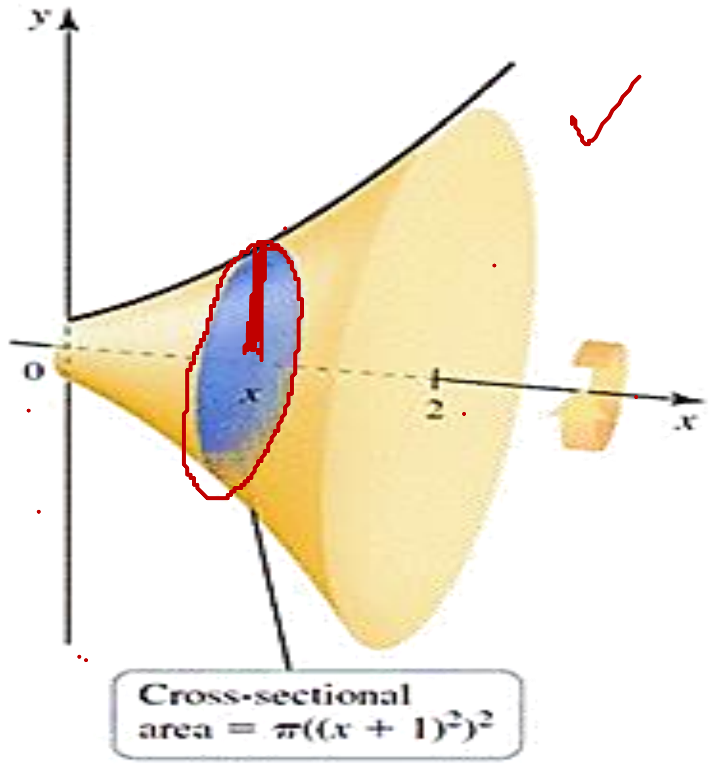
Let R be the region bounded by the curve $f(x) = (x + 1)^2$, the x −axis, and the lines $x = 0$ and $x = 2$. Find the volume of the solid of revolution obtained by revolving R about the x −axis.



Solution:

When the region R is revolved about the x −axis, it generates a solid of revolution. A cross section perpendicular to the x −axis at the point $0 \le x \le 2$ is a circular disk of radius $f(x)$. Therefore, a typical cross section has area:

$$A(x) = \pi[f(x)]^2 = \pi[(x + 1)^2]^2 = \pi(x + 1)^4.$$



Example:

Integrating this cross-sectional area between $x = 0$ and $x = 2$ gives the volume of the solid as:

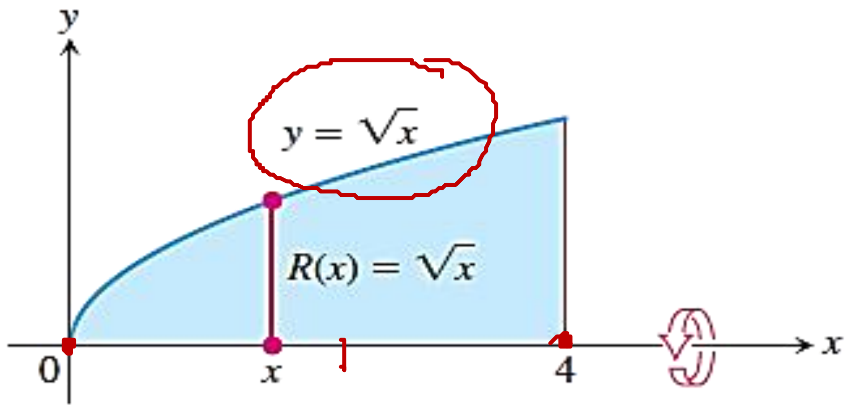
$$V = \int_a^b A(x)\,dx = \int_0^2 \pi(x+1)^4\,dx$$

$$= \left.\frac{\pi(x+1)^5}{5}\right|_0^2$$

$$= \frac{242}{5}\pi.$$

Example:

Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[0, 4]$ is revolved about the x −axis.



(a)

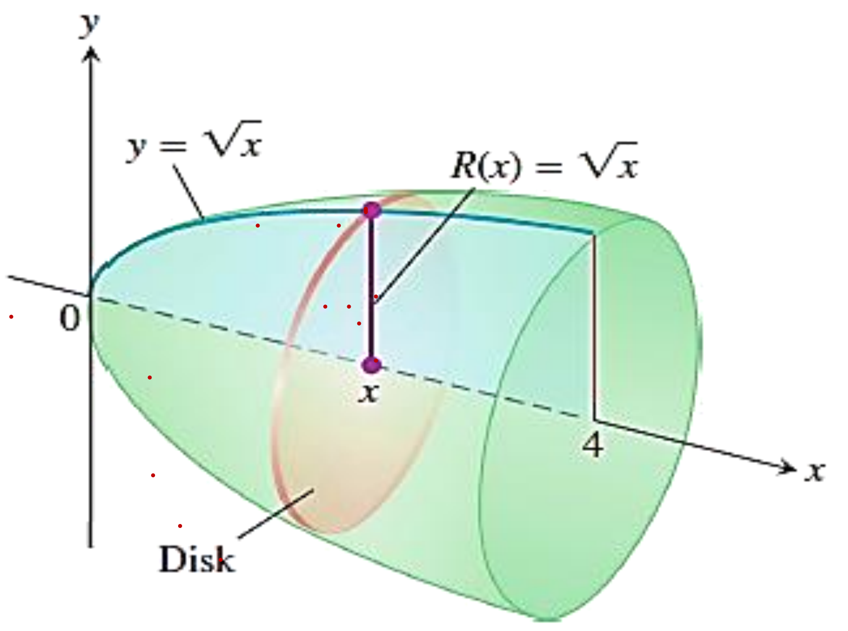
Solution:

For the present case, the cross-sectional area is given as:

$$A(x) = \pi[f(x)]^2 = \pi[\sqrt{x}]^2 = \pi x.$$

Thus, the required volume can be obtained as:

$$V = \int_a^b A(x)\,dx = \int_0^4 \pi x\,dx = \left.\frac{\pi x^2}{2}\right|_0^4 = 8\pi.$$



(b)

Example:

Derive the formula for the volume of a sphere of radius r.

Solution:

Equation of circle of radius r is:

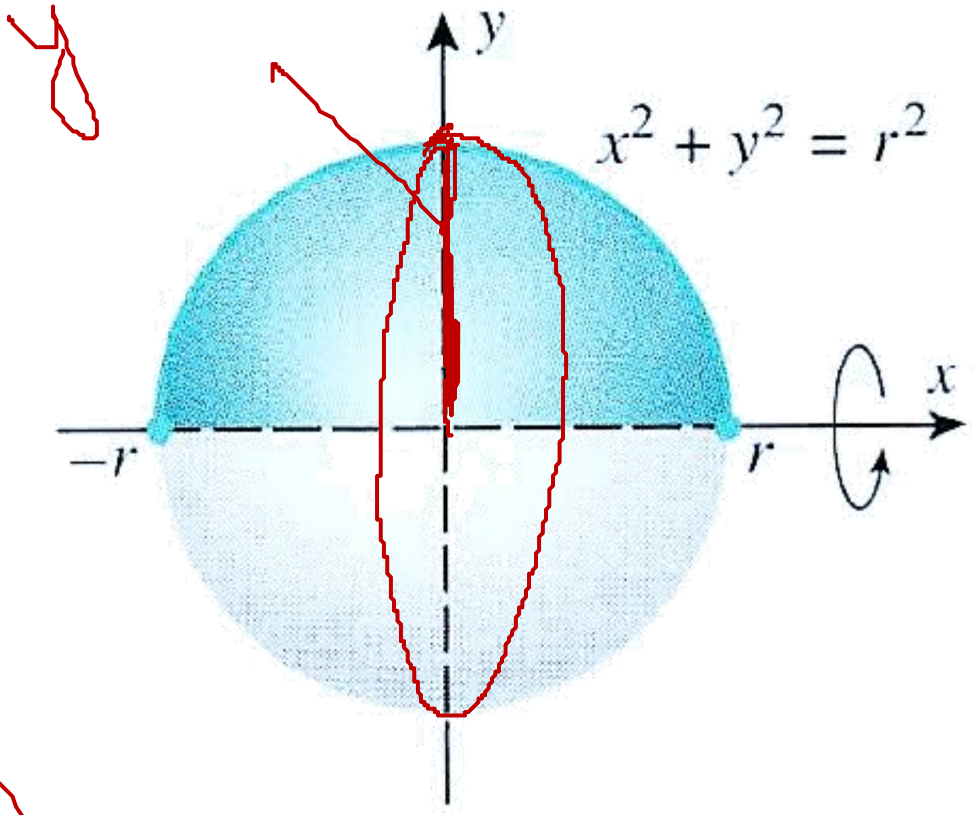
$$x^2 + y^2 = r^2 \implies \boxed{y^2 = r^2 - x^2}.$$

Thus, the cross-sectional area is given as:

$$A(x) = \pi[f(x)]^2 = \pi y^2 = \pi[r^2 - x^2].$$

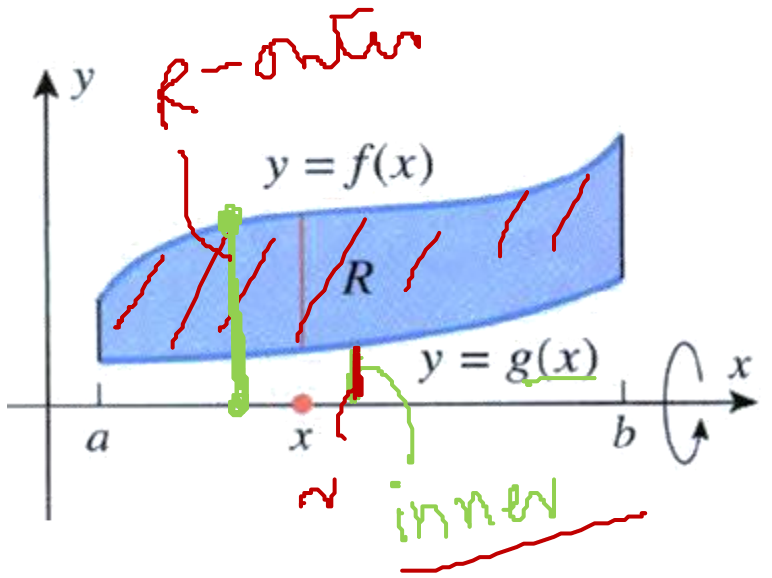
Hence, the required volume is:

$$V = \int_a^b A(x)\,dx = \int_{-r}^{r} \pi[r^2 - x^2]\,dx = \pi\left(r^2 x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r} = \frac{4}{3}\pi r^3.$$



Solid of Revolution: The Washer Method

Problem: Let $f(x)$ and $g(x)$ be continuous and nonnegative on $[a, b]$ and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$. Find the volume of the solid of revolution that is generated by revolving the region R about the x −axis.



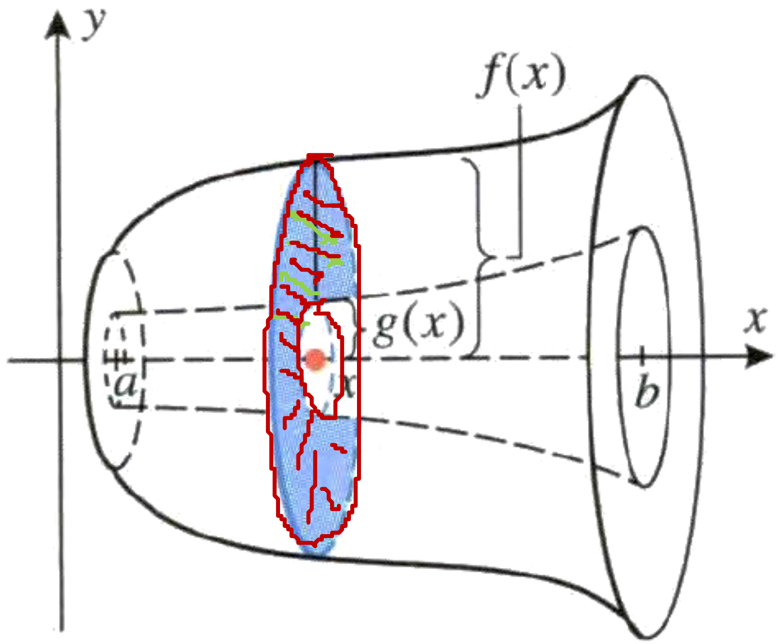
Solution: We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x −axis at the point x is the annular or "washer-shaped" region with inner radius $g(x)$ and outer radius $f(x)$. The cross-sectional area is given as:

$$A(x) = \pi[\underbrace{f(x)}_{R}]^2 - \pi[\underbrace{g(x)}_{r}]^2$$

$$V = \int_a^b A(x)\,dx$$

Thus, the volume is:

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2)dx$$



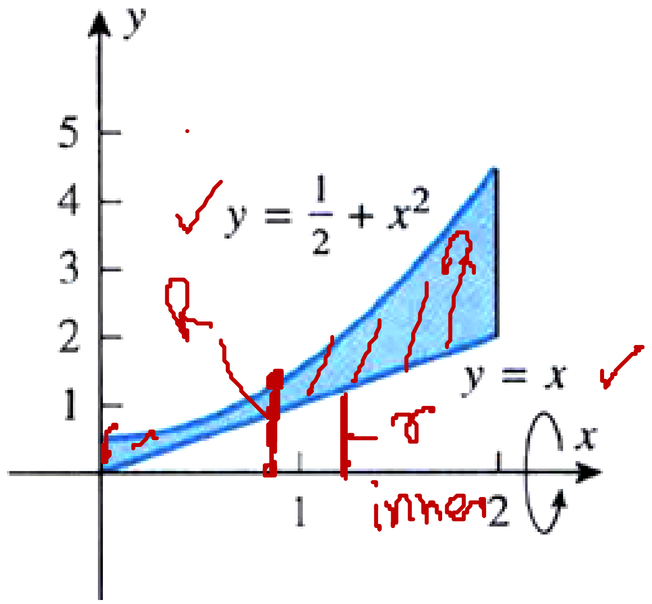
Example:

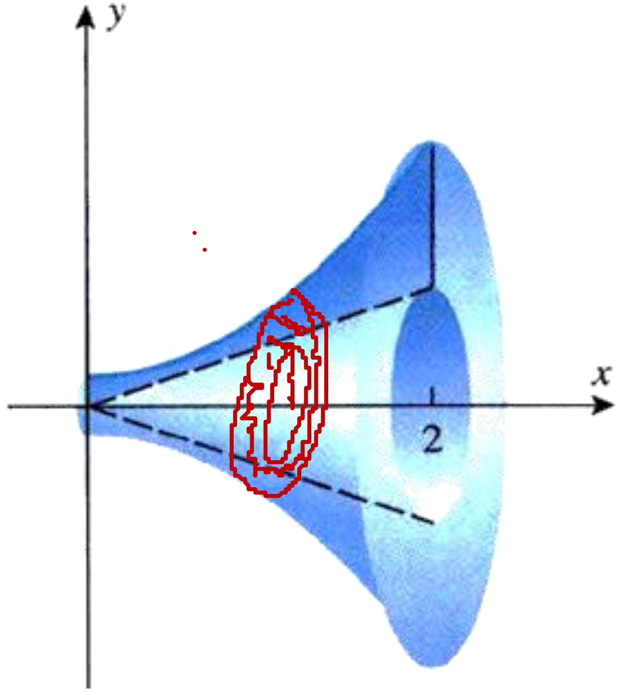
Find the volume of the solid generated when the region between the graphs of the equation $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x −axis.

Solution:

For the present case the outer radius is: $f(x) = \frac{1}{2} + x^2$ and inner radius is: $g(x) = x$. Thus, the required volume is given as:

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2)dx = \pi \int_0^2 \left(\left[\frac{1}{2} + x^2\right]^2 - x^2 \right) dx = \frac{69}{10}\pi.$$



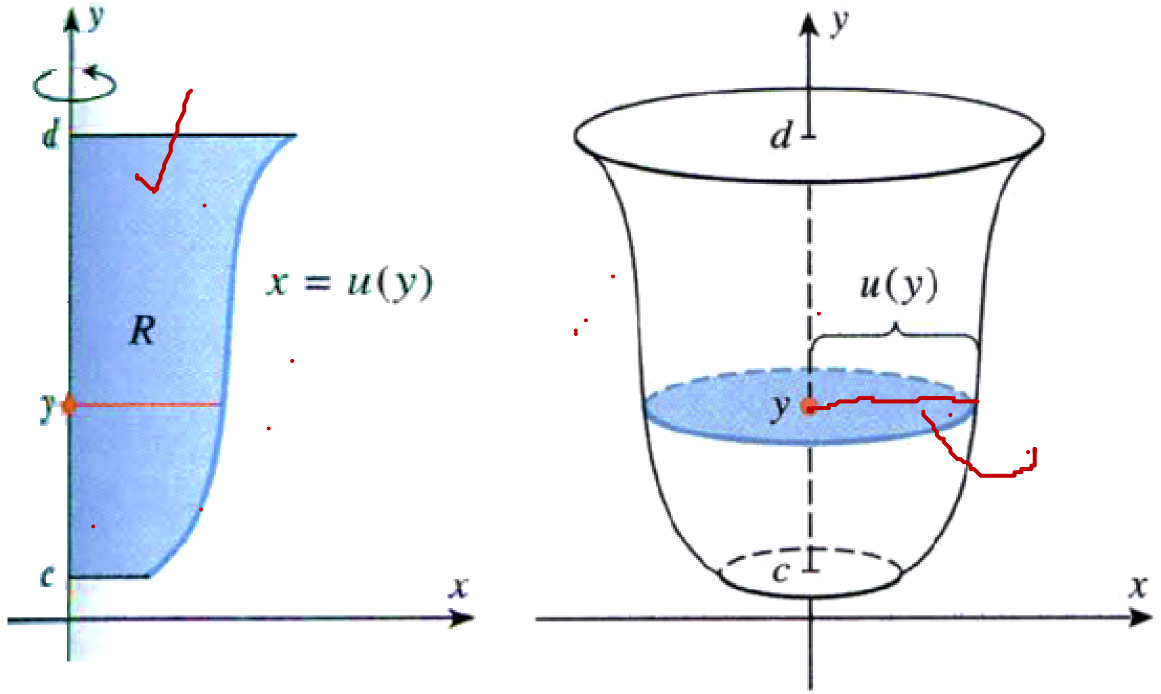


Disk and Washer Methods About the y −axis

The methods or disks and washers have analogs for regions that are revolved about the y −axis. Using the method of slicing, we can easily deduce the following formulas for the volumes of the solids.

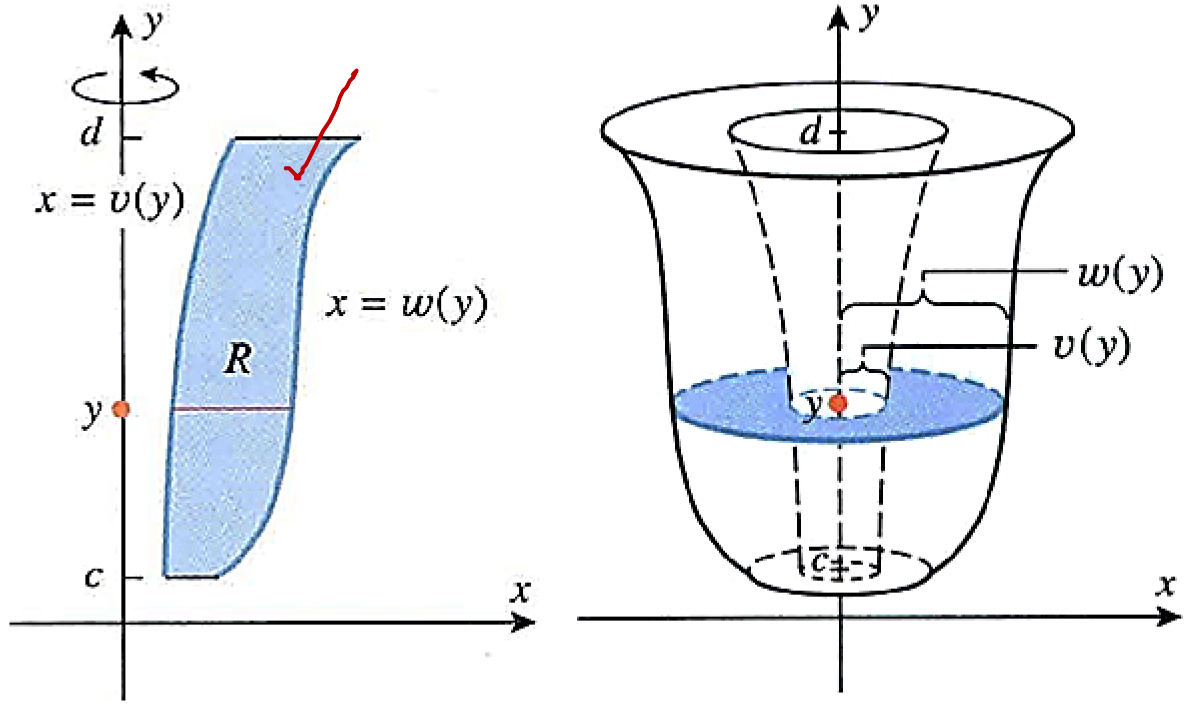
Disk Method:

$$V = \int_c^d \pi[u(y)]^2 dy .$$



Washer Method:

$$V = \pi \int_c^d ([w(y)]^2 - [v(y)]^2) dy .$$



Example:

Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}, y = 2$, and $x = 0$ is revolved about the y −axis.

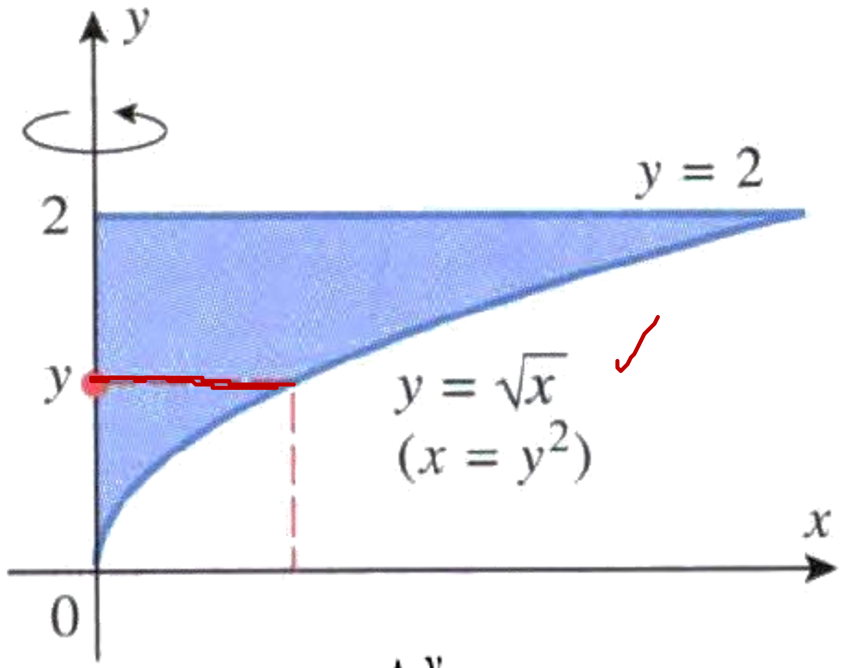
Solution:

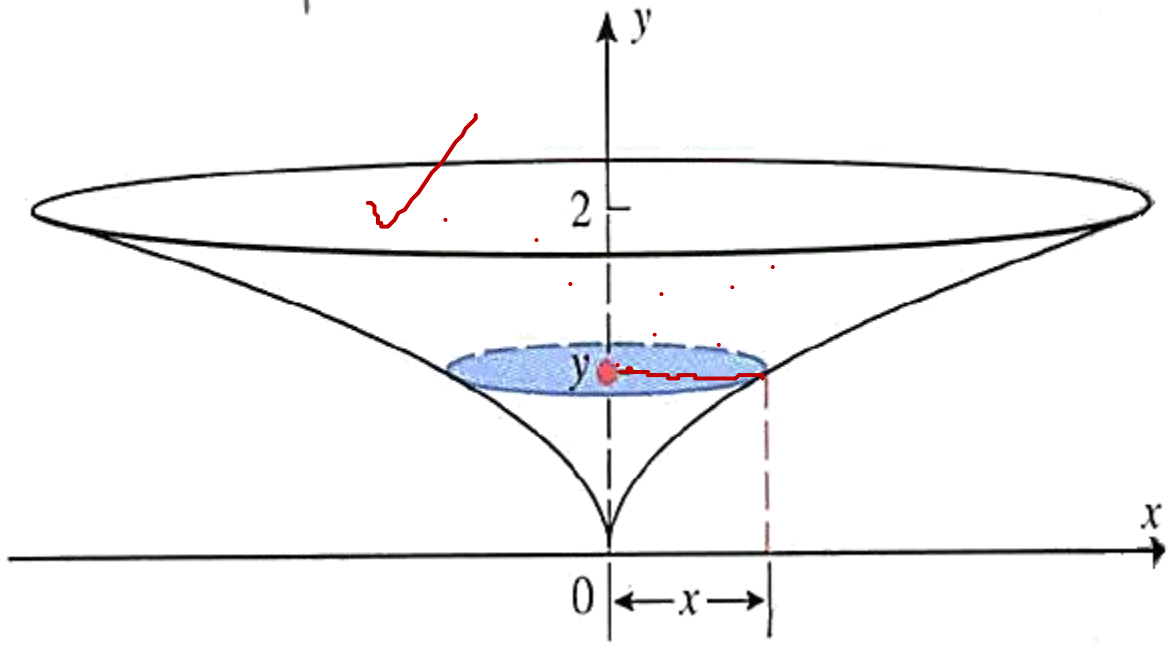
The required volume is given as:

$y = \sqrt{x}$

$\Rightarrow x = y^2$

$$V = \pi \int_0^2 (y^2)^2 dy = \frac{\pi y^5}{5}\Bigg]_0^2 = \frac{32}{5}\pi.$$





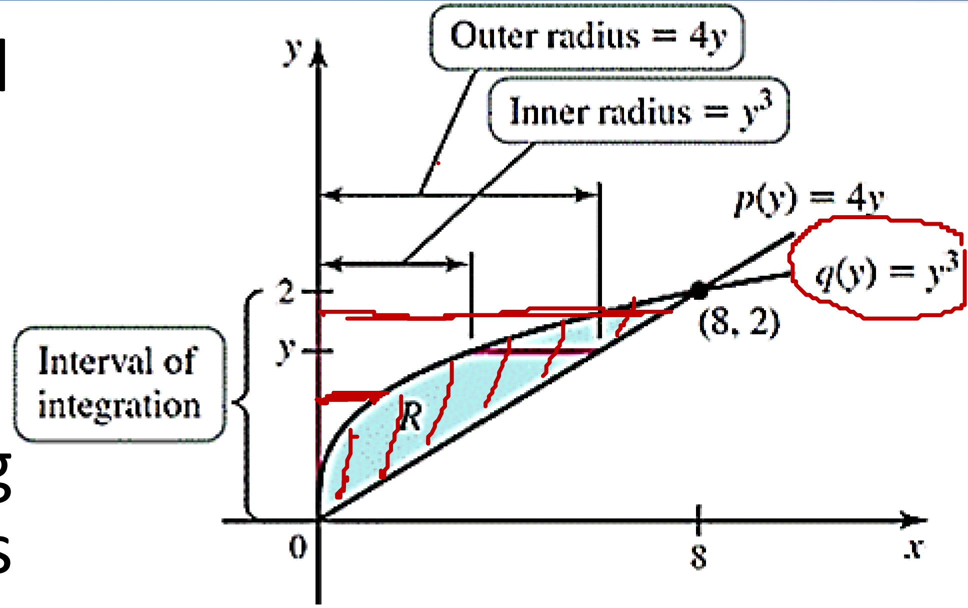
Example:

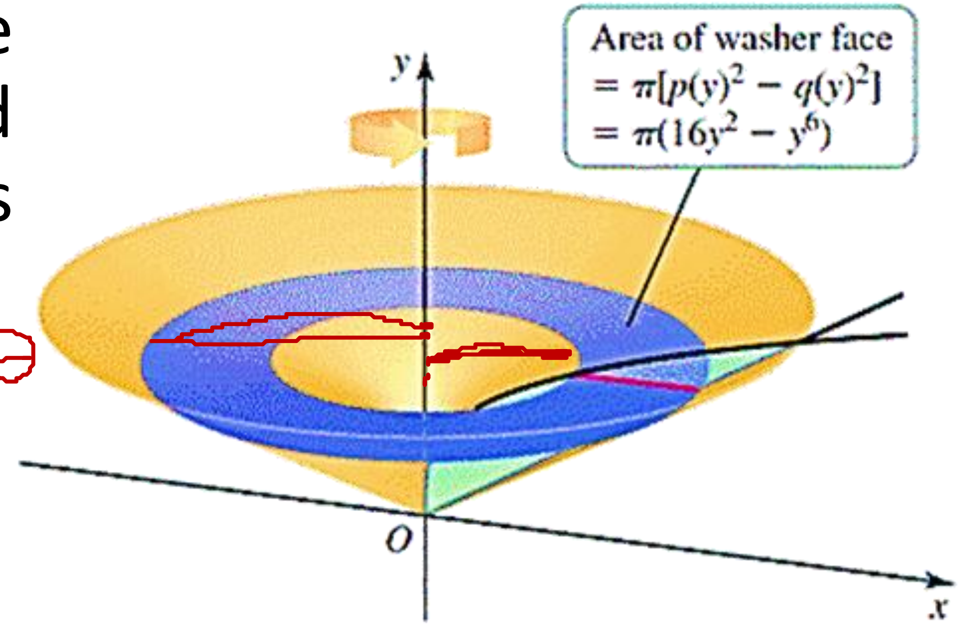
Let R be the region in the first quadrant bounded by the graphs of $x = y^3$ and $x = 4y$. Find the volume of the solid generated when R is revolved about the y −axis?

Solution:

Solving $y^3 = 4y$ or, equivalently, $y(y^2 - 4) = 0$ we find that the bounding curves of R intersect at the points $(0, 0)$ and $(8, 2)$. When the region R is revolved about the y −axis, it generates a funnel with a curved inner surface. The outer radius of the cross section at the point y is determined by the line $p(y) = 4y$. The inner radius of the cross section at the point y is determined by the curve $q(y) = y^3$. Applying the washer method, the volume of this solid is:

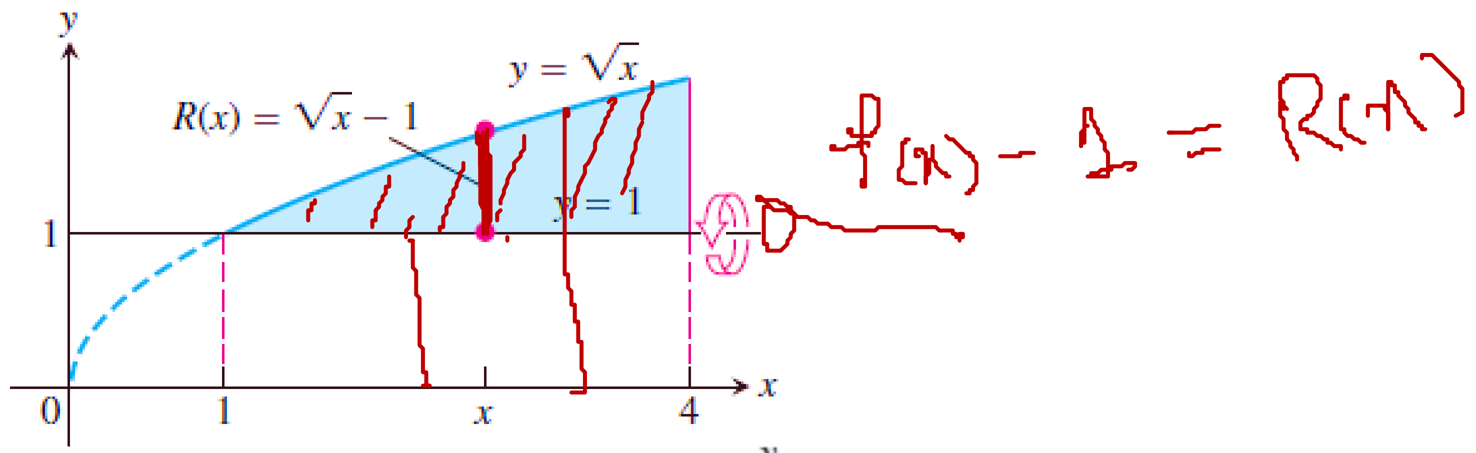
$$V = \pi \int_c^d ([p(y)]^2 - [q(y)]^2)dy = \pi \int_0^2 (16y^2 - y^6)dy = \frac{512}{21}\pi.$$

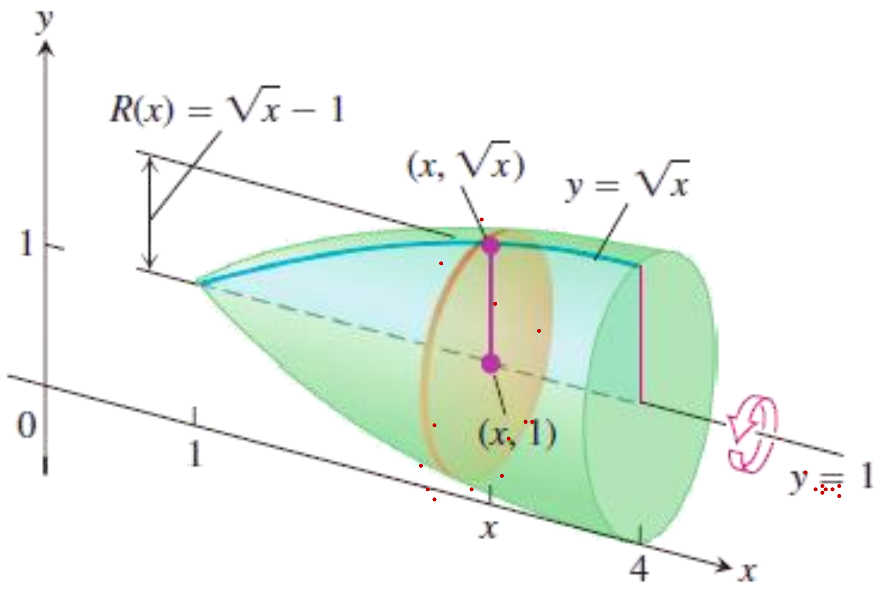


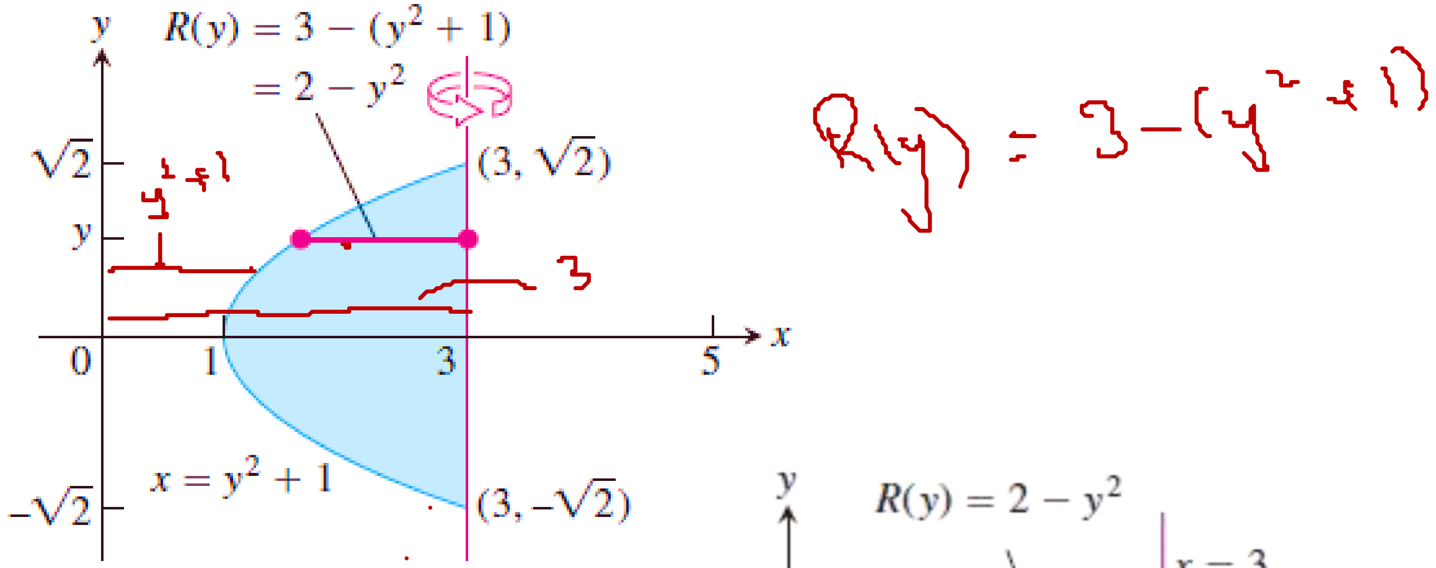


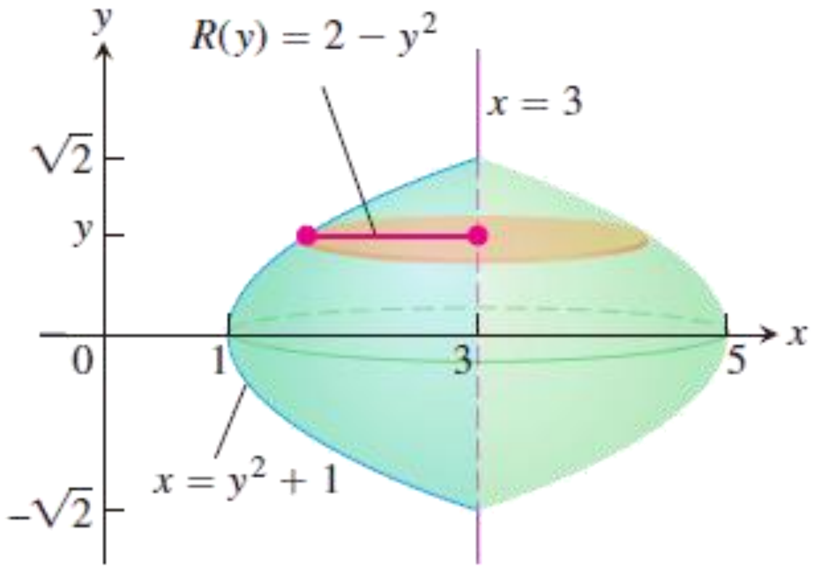
Rotation around a line other than x or y −axis

- Now, we will try to rotate a figure around a line other than the x or y −axis.
- We will use the idea of the outer and inner radius to find the correct formula.

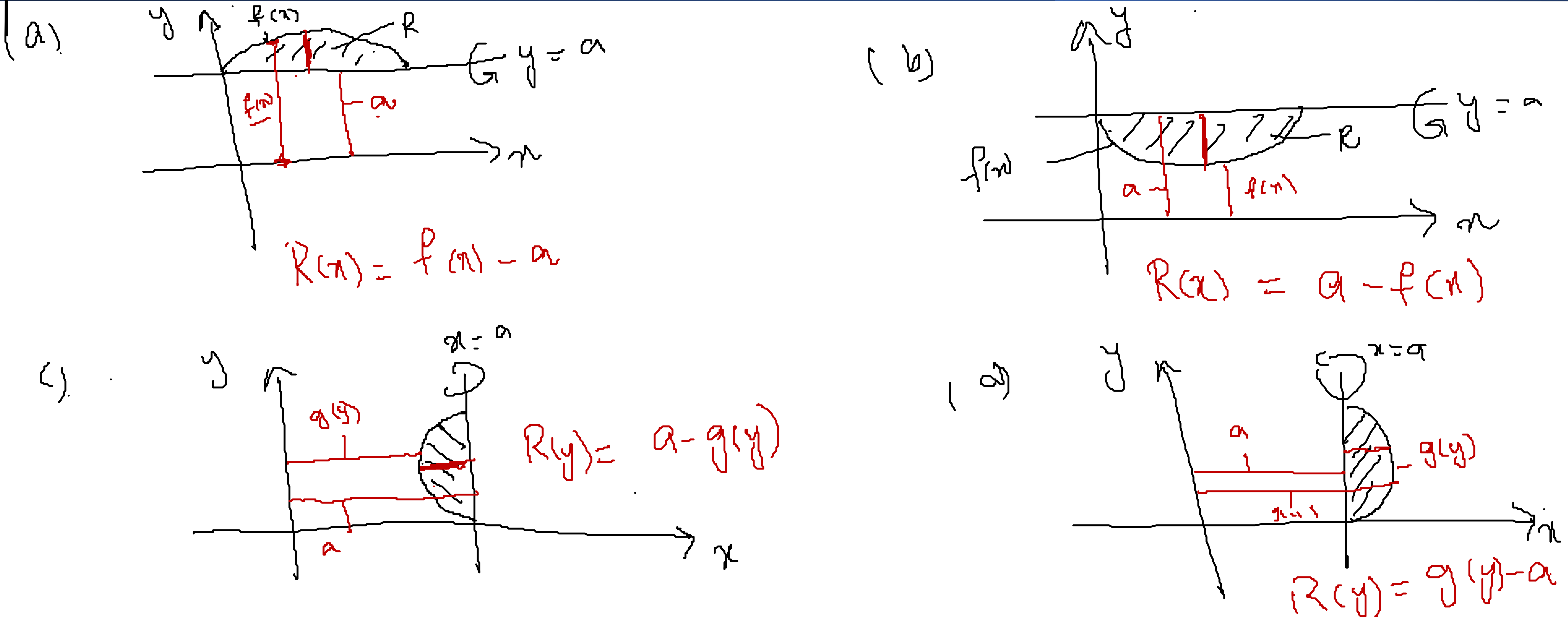








How to determine correct radius???



Example:

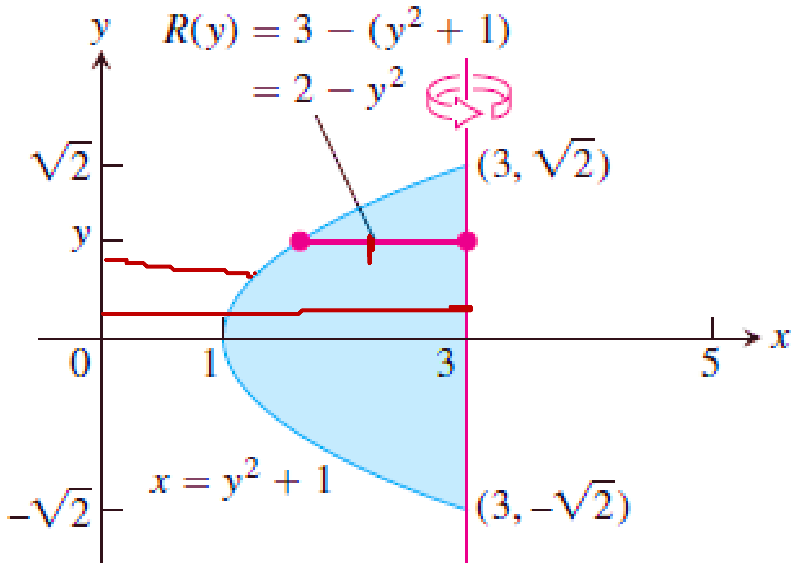
Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

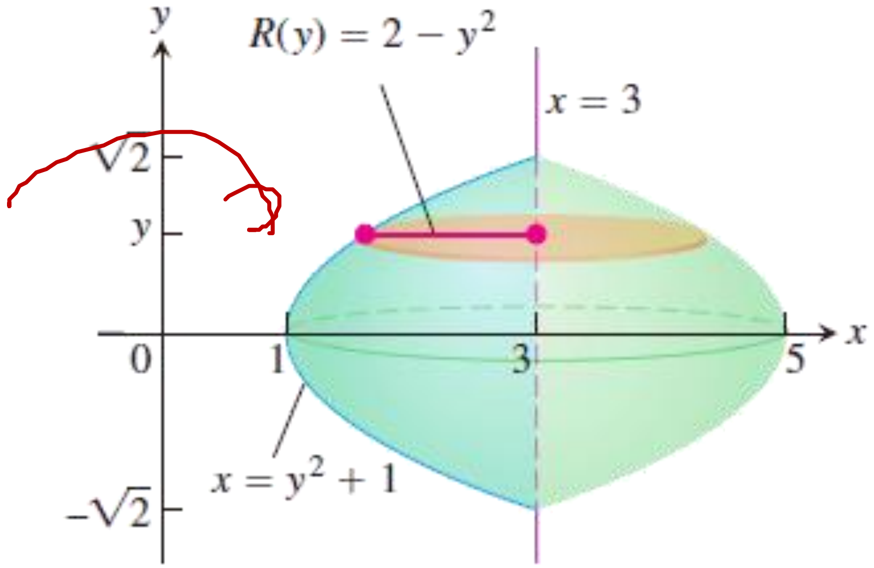
Solution:

Note that the cross-sections are perpendicular to the line $x = 3$. Thus, the volume of the generated solid is given as:

$$V = \pi \int_c^d [R(y)]^2 dy = \pi \int_{-\sqrt{2}}^{\sqrt{2}} [2 - y^2]^2 dy$$

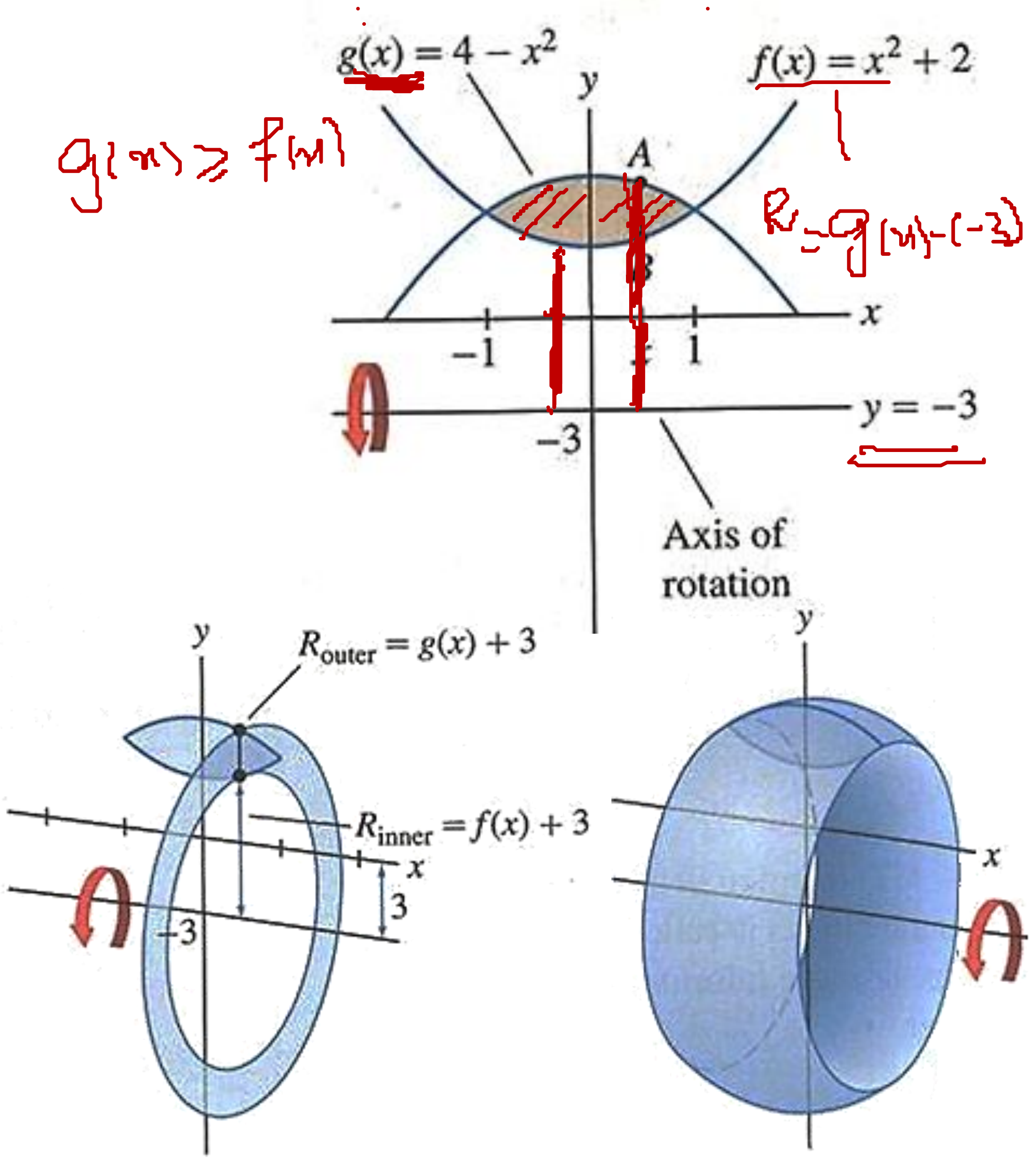
$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4] dy = \frac{64\sqrt{2}}{15}\pi.$$





Example:

Find the volume V of the solid obtained by rotating the region between the graphs of $f(x) = x^2 + 2$ and $g(x) = 4 - x^2$ about the horizontal line $y = -3$.



Solution:

$r = f(x) - (-3)$

Let us determine the points of intersections of the given curves first.

$$f(x) = g(x) \Longrightarrow x^2 + 2 = 4 - x^2 \Longrightarrow 2x^2 - 2 = 0 \Longrightarrow x = \pm 1.$$

Thus, the points of intersection are: $(-1,3)$ and $(1,3)$.

Moreover, $g(x) \geq f(x)$ for $-1 \leq x \leq 1$. Note that when we rotate about $y = -3$, the line segment AB generates a washer whose outer and inner radii are both 3 units larger:

Outer radius $= R(x) = g(x) - (-3) = (4 - x^2) + 3 = 7 - x^2$

Inner radius $= r(x) = f(x) - (-3) = (x^2 + 2) + 3 = x^2 + 5$

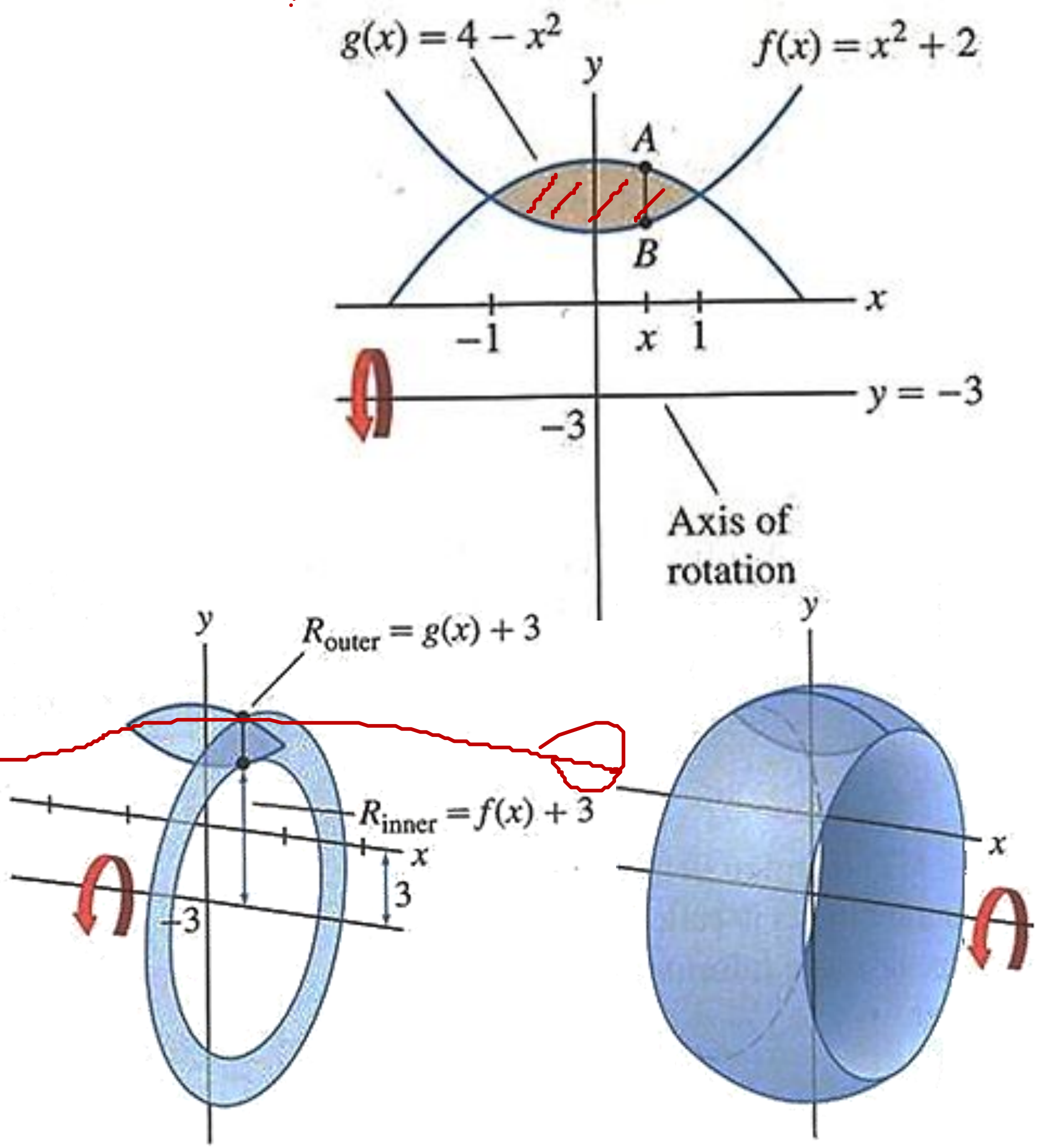
Solution:

The volume of revolution is equal to the integral of the area of this washer and is given as:

$$V = \pi \int_{a}^{b} (R^2 - r^2)dx = \pi \int_{-1}^{1} [(7 - x^2)^2 - (x^2 + 5)^2]dx$$

$$= \pi \int_{-1}^{1} [(49 - 14x^2 + x^4) - (x^4 + 10x^2 + 25)]\, dx$$

$$= \pi \int_{-1}^{1} (24 - 24x^2)dx = \pi(24x - 8x^3)\Big|_{-1}^{1} = 32\pi.$$



Example:

Setup the integrals to determine the volume of the solid obtained by rotating the graphs of $f(x) = 9 - x^2$ and $y = 12$ for $0 \leq x \leq 3$ about the line $y = 12$ and then, $y = 15$.

$R = 12 - f(x)$

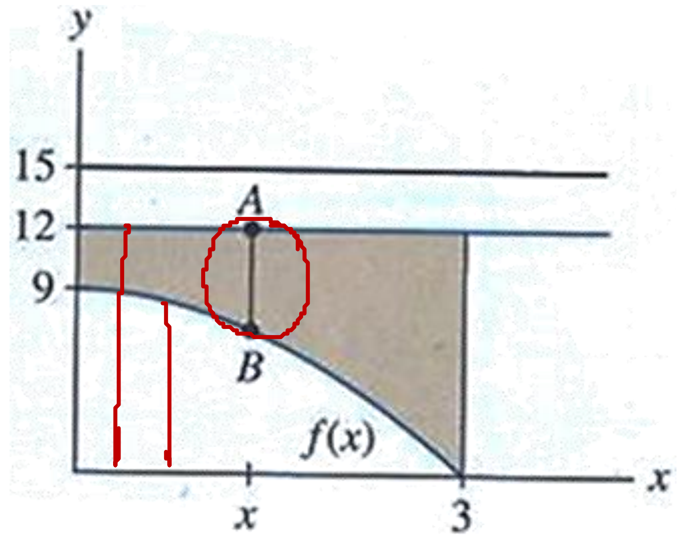
Solution:

Figure B shows that the line segment AB rotated about $y = 12$ generates a disk of radius:

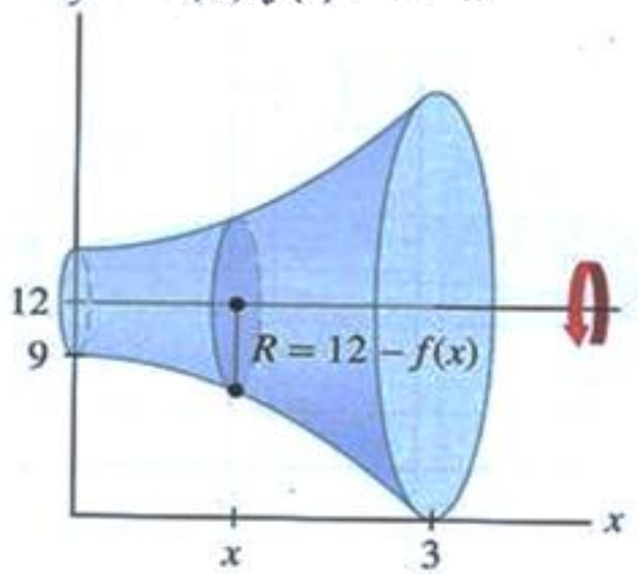
$$R = \text{length of line segment } AB = 12 - f(x) = 12 - (9 - x^2) = 3 + x^2.$$

Note that the length of line segment AB is $12 - f(x)$ rather than $f(x) - 12$ since the line $y = 12$ lies above the graph of $f(x)$. The volume when we rotate about the line $y = 12$ is:

$$V = \pi \int_a^b R^2 dx = \pi \int_0^3 (3 + x^2)^2 dx.$$



(A) $f(x) = 9 - x^2$



(B) Rotation about the line $y = 12$

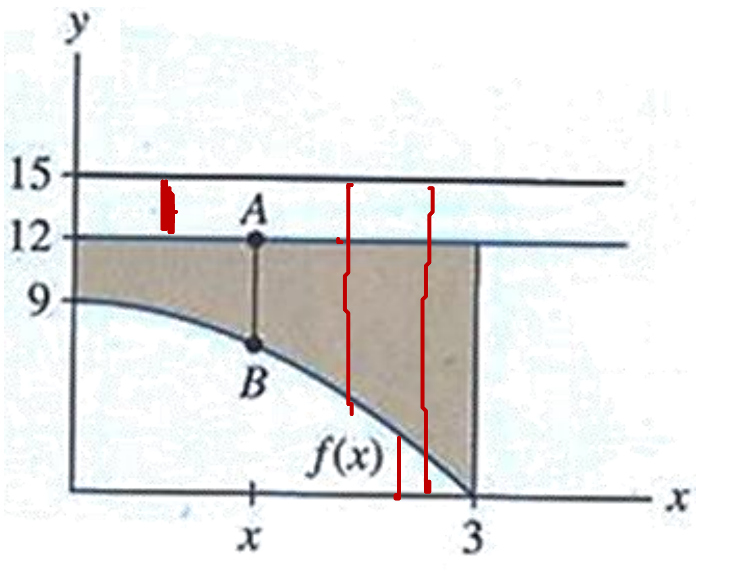
Solution:

Figure C shows that the line segment AB rotated about $y = 15$ generates a washer. The outer radius of this washer is the distance from B to the line $y = 15$:

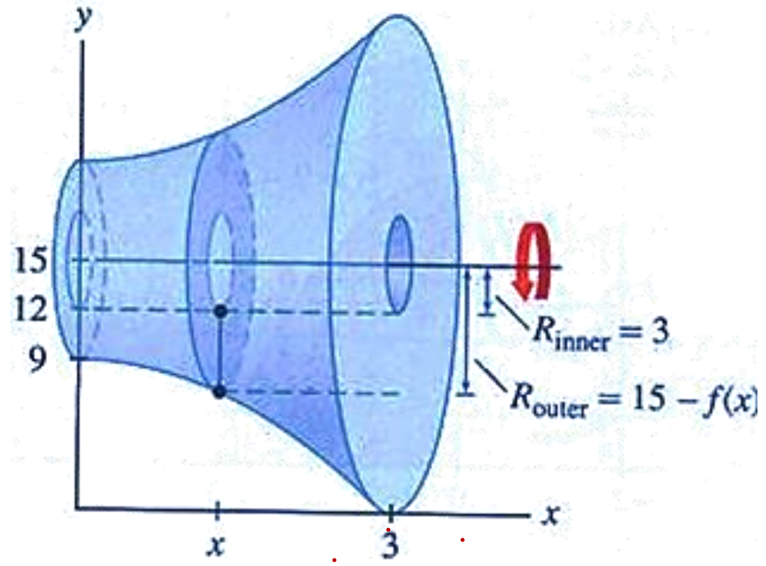
$$\text{Outer radius} = R(x) = 15 - f(x) = 15 - (9 - x^2) = 6 + x^2$$

The inner radius is $r(x) = 3$, so the volume of revolution about $y = 15$ is:

$$V = \pi \int_a^b (R^2 - r^2)dx = \pi \int_0^3 [(6 + x^2)^2 - (3)^2]dx.$$



(A) $f(x) = 9 - x^2$



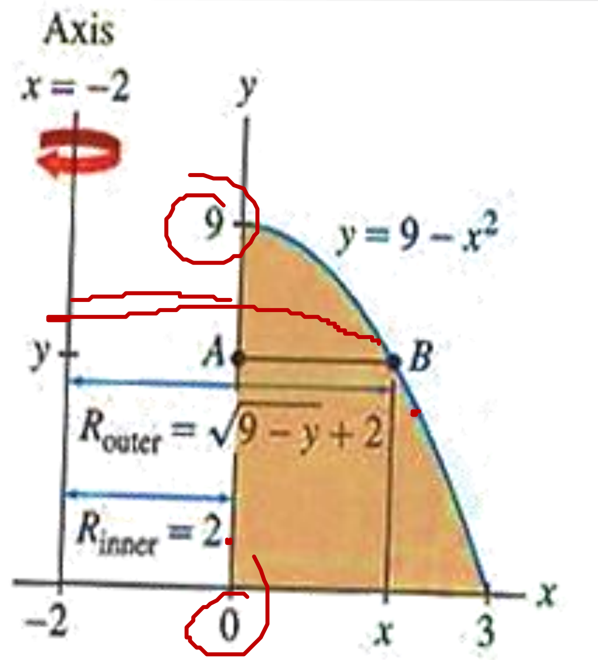
(C) Rotation about the line $y = 15$

Example:

Setup the integral to determine the volume of the solid obtained by rotating the graph of $f(x) = 9 - x^2$ for $0 \leq x \leq 3$ about the vertical line $x = -2$.

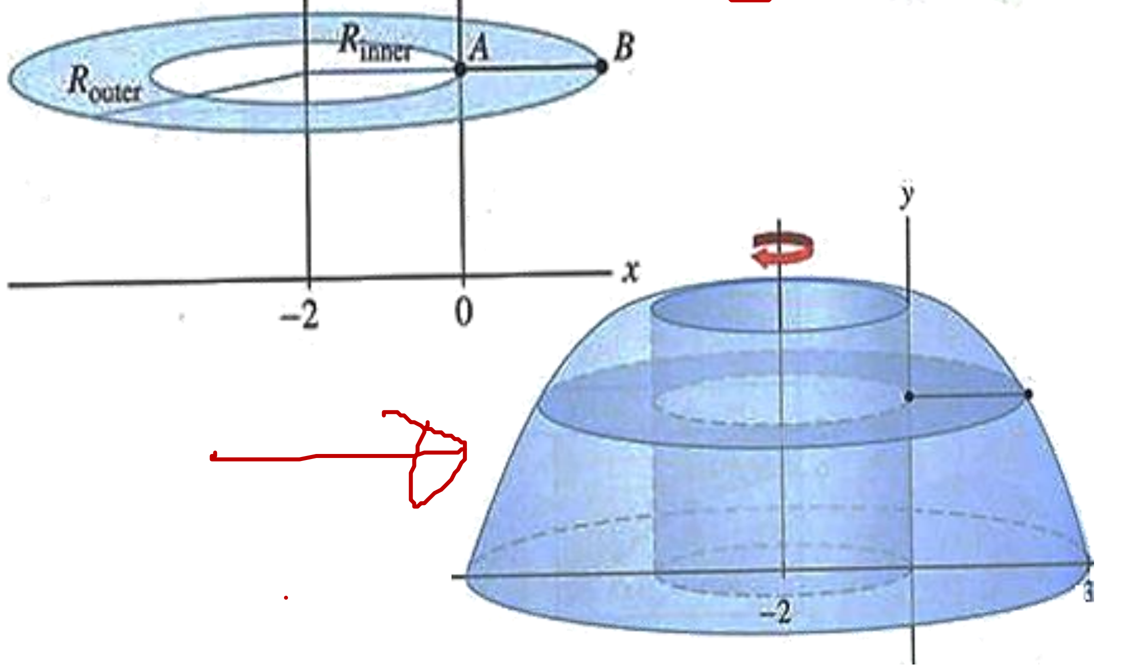
Solution:

The figure shows that the line segment AB sweeps out a horizontal washer when rotated about the vertical line $x = -2$. We are going to integrate with respect to y, so we need the inner and outer radii of this washer as functions of y. Solving for x in $y = 9 - x^2$, we obtain: $x^2 = 9 - y \Longrightarrow x = \sqrt{9-y} \because x \geq 0.$



Therefore:

$$R(y) = \sqrt{9-y} + 2 \quad \text{and} \quad r(y) = 2.$$



The region extends from $y = 0$ to $y = 9$ along the y −axis, so

$$V = \pi \int_c^d (R^2 - r^2)dy = \pi \int_0^9 \left[\left(\sqrt{9-y} + 2\right)^2 - (2)^2\right] dy.$$

Practice Questions

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Exercise: 6.1

  Q # 1 to Q # 48

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 6.2

  Q # 1 to Q # 34