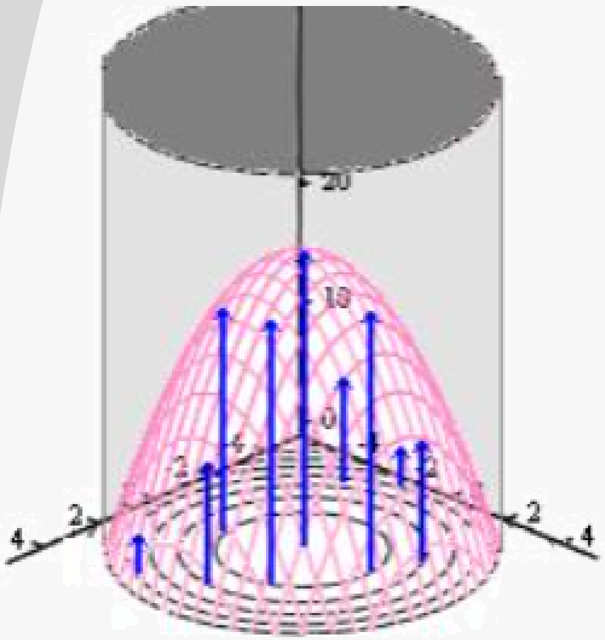
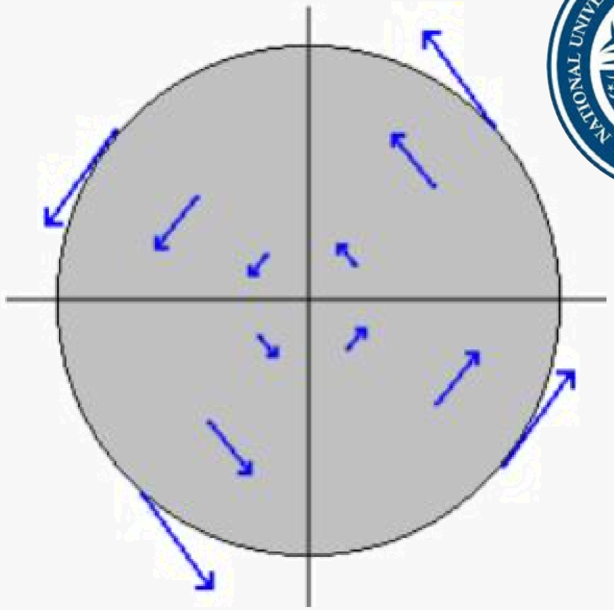
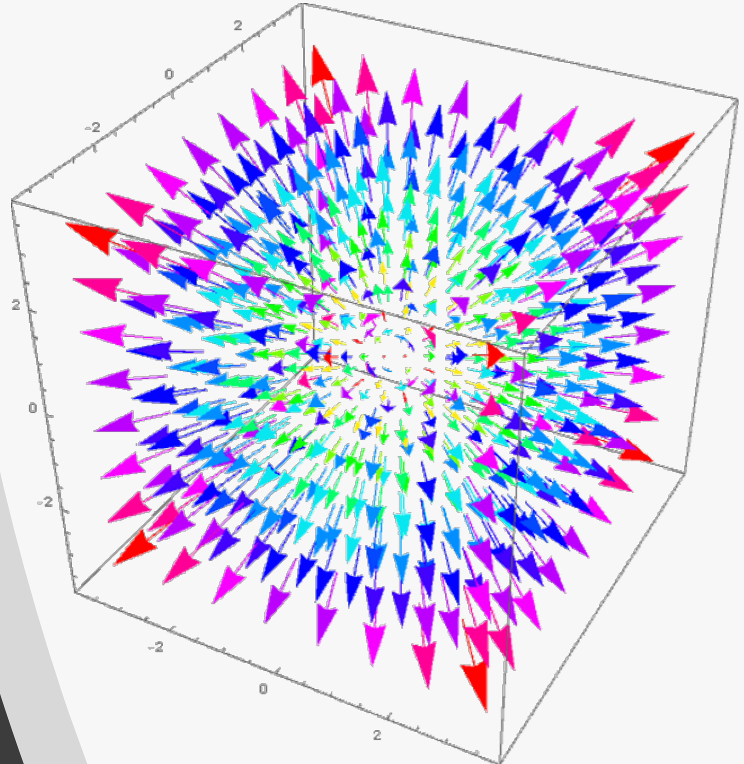
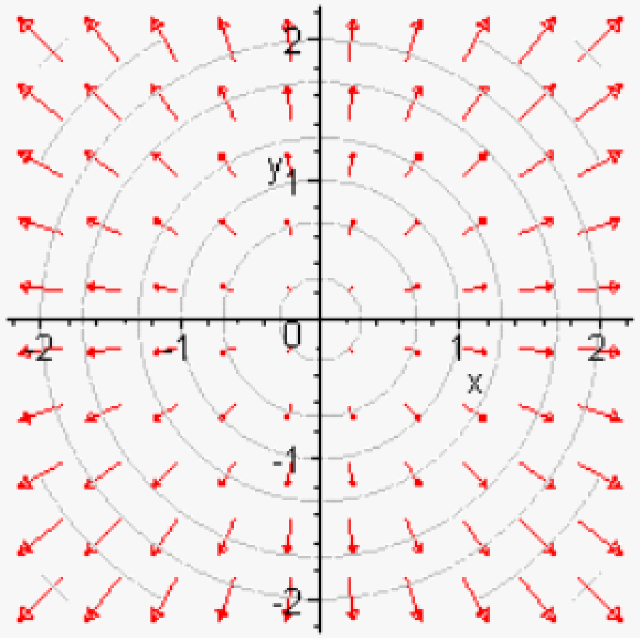
Scalar & Vector Fields









Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir

16

Vector Calculus

Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.1**

Field

- The concept of fields is quite common in Physics. For example, we encountered various physical fields like temperature, pressure or gravitational fields etc. Roughly speaking it represents a collection of numbers (scalars) or vectors.

- A field is a mathematical representation of the continuum arise from a physical process such that its value varies at each point. It depends on the process to require a scalar or a vector or a combination of both at each point to fully comprehend the dynamics involved.

- The former give rise **to scalar fields** and later are known as **vector fields.**

Two types of Fields

Scalar Fields:

(Magnitudes)

A scalar field can be regarded as a multi-variable function which gives numbers as an output which could be the values of:

- Temperature
- Pressure
- Gravity anomaly
- Resistivity
- Elevation
- Maximum wind speed (without directional info)
- Energy
- Potential
- Density
- Time…

Vector Fields:

(Magnitude and direction)

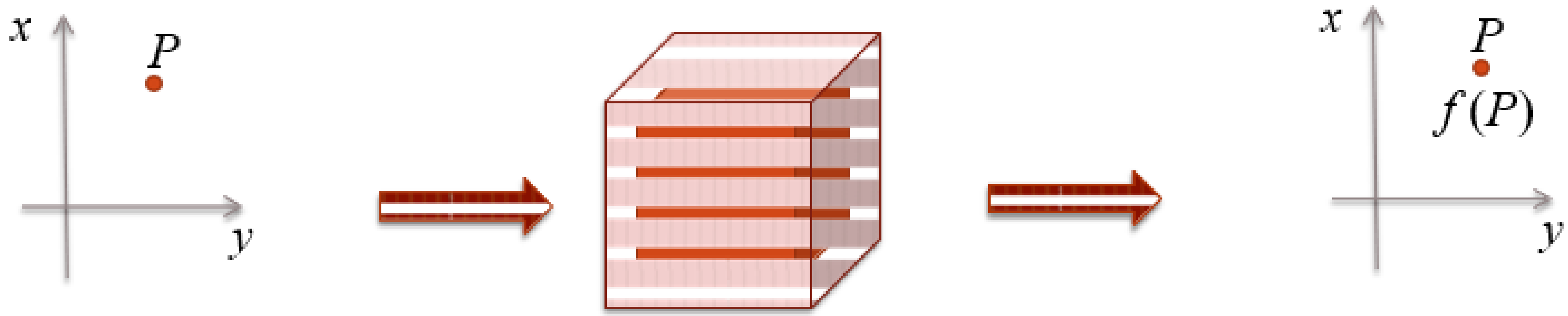
A vector field can be regarded as a multi-variable function which gives vectors in the output which could be the values of:

- Magnetic field (Scale Earth or mineral)
- Electric field
- Water velocity field
- Wind direction on a weather map
- Includes displacement, velocity, acceleration, force, momentum…

Scalar Fields (2D)

A scalar field in two dimensions can be represented by a multivariable function $f(x, y)$ such that its output is a number corresponding to a point (x, y) in a plane.

Example:



Mathematically,

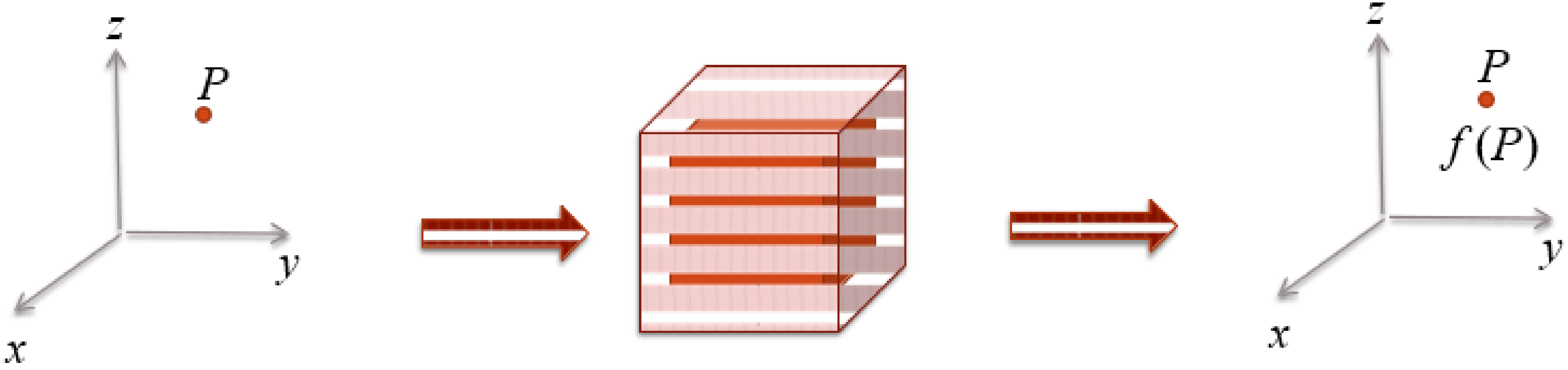
$$\boxed{f = f(P) = f(x, y)}$$

The collection of values of $f(x, y) = f(P)$, is called a ***scalar field.***

Scalar Fields (3D)

A scalar field in three dimensions can be represented by a multivariable function $f(x, y, z)$ such that its output is a number corresponding to a point (x, y, z) in space.

Example:



Mathematically,

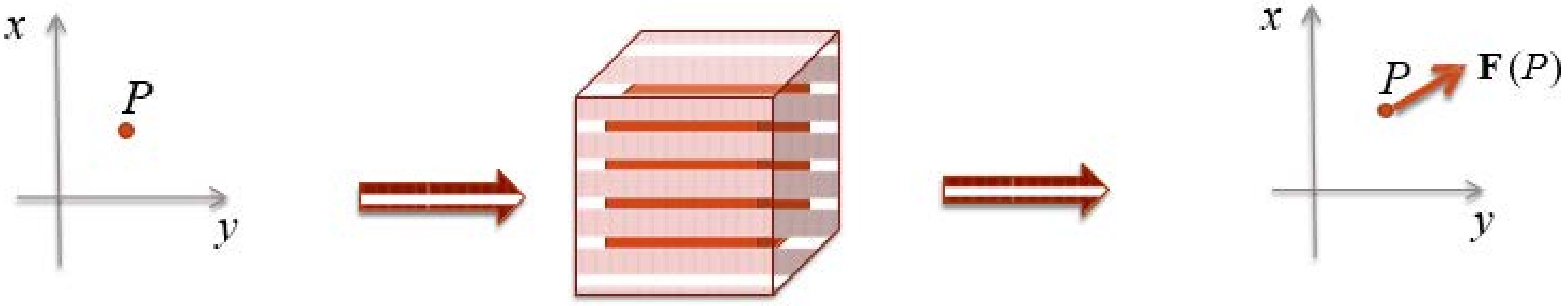
$$\boxed{f = f(P) = f(x, y, z)}$$

The collection of values of $f(x, y, z) = f(P)$, is called a ***scalar field.***

Vector Fields (2D)

A vector field can be regarded as a multi-variable function which gives vectors in the output.

Example:



Mathematically,

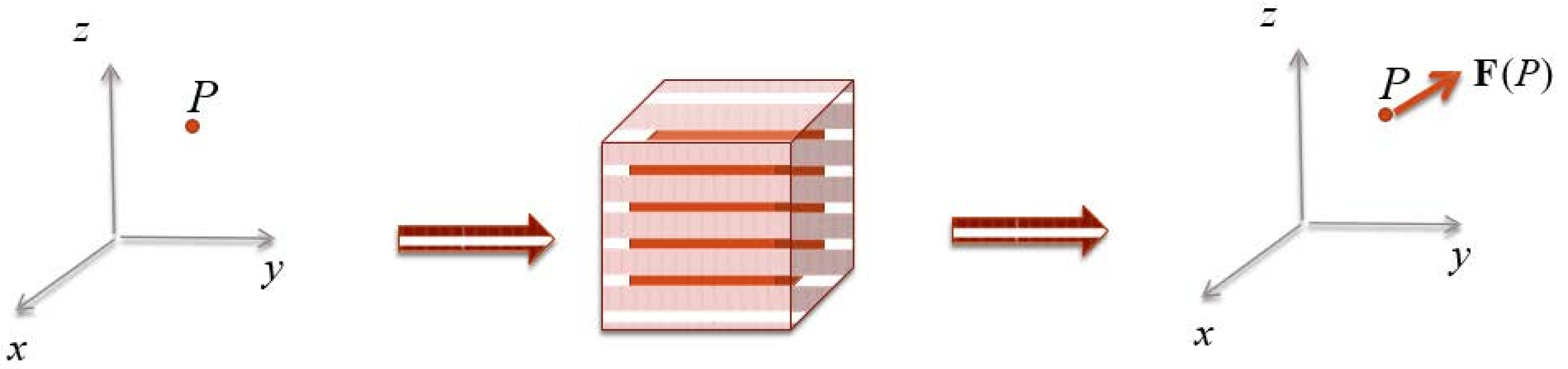
$$\mathbf{F} = \mathbf{F}(P) = \mathbf{F}(x, y) = \langle F_1, F_2 \rangle$$

The collection of values of $\mathbf{F}(P) = \mathbf{F}(x, y)$, is called a ***vector field.***

Vector Fields (3D)

A vector field can be regarded as a multi-variable function which gives vectors in the output.

Example:



Mathematically,

$$\mathbf{F} = \mathbf{F}(P) = \mathbf{F}(x, y, z) = \langle F_1, F_2, F_3 \rangle$$

The collection of values of $\mathbf{F}(P) = \mathbf{F}(x, y, z)$, is called a ***vector field.***

Vector Fields (2D & 3D)

A vector field on a domain in the plane or in space is a function that assigns a vector to each point in the domain. In general, a field of 2D vectors would look like this:

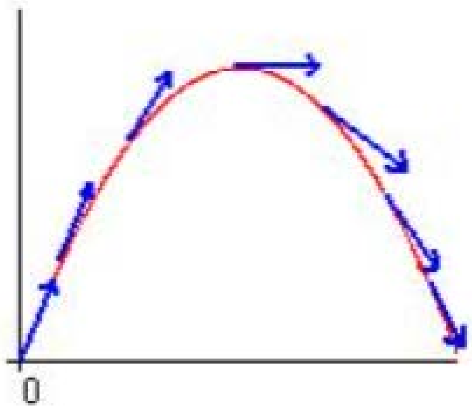
$$\mathbf{F} = \langle M(x, y), N(x, y) \rangle,$$

and a field of 3D vectors would look like this:

$$\mathbf{F} = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle.$$

If we can imagine attaching a projectile's velocity vector at each point on its trajectory in the plane of motion, then we have a 2D vector field defined on the trajectory as shown:

If we can imagine the gradient vector at each point on all level surfaces of a function of three variables, then we have a 3D vector field. Naturally, a gradient vector at each point of all level curves of a function of two variables creates a 2D vector field. Note that both of these would be extremely hard to draw.



Gradient Vector Field

If $f(x, y)$ is a scalar function of two variables, its gradient ∇f (or grad f) is defined by:

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle.$$

Therefore, ∇f is a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Likewise, if $f(x, y, z)$ is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by:

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle.$$

Example:

Determine the gradient vector field of the following functions:

1. $f(x, y) = x^2 \sin(5y)$.
2. $f(x, y, z) = ze^{-xy}$.

Solution:

1. $\nabla f(x, y) = \langle 2x \sin(5y), 5x^2 \cos(5y) \rangle$.
2. $\nabla f(x, y, z) = \langle -yze^{-xy}, -xze^{-xy}, e^{-xy} \rangle$.

Example:

Sketch the gradient vector field for $f(x, y) = x^2 + y^2$. Moreover, sketch several contours for the given function.

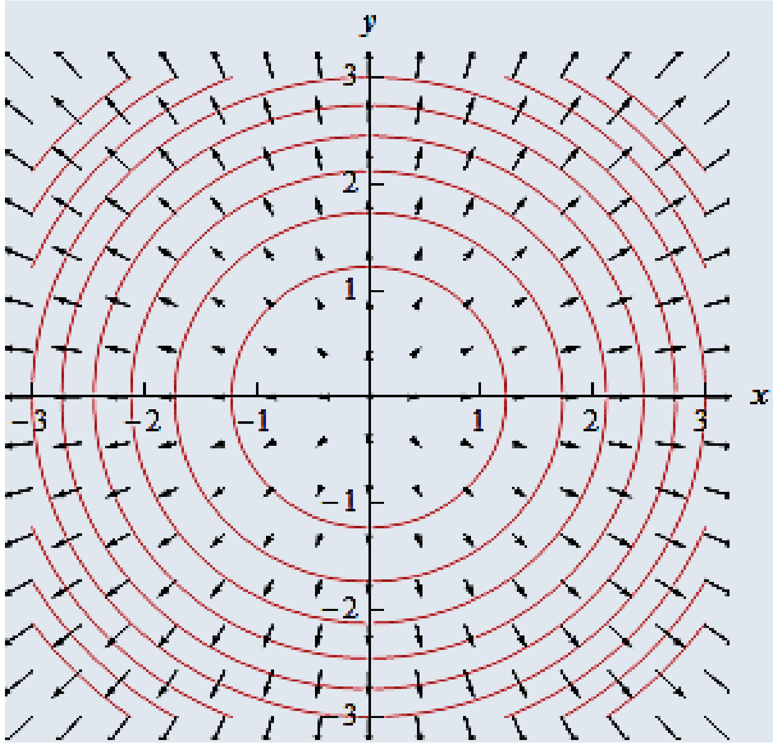
Solution: Recall that the contours for a function are nothing more than curves defined by, $f(x, y) = k$ for various values of k. So, for the present case the contours are defined by the equation:

$$x^2 + y^2 = k,$$

and they are circles centered at the origin with radius $\sqrt{k}$. The gradient vector field for this function is:

$$\nabla f(x, y) = \langle 2x, 2y \rangle.$$

Here is a sketch of several of the contours as well as the gradient vector field. Note that the vectors of the vector field are all perpendicular (or orthogonal) to the contours. This will always be the case when we are dealing with the contours of a function as well as its gradient vector field.



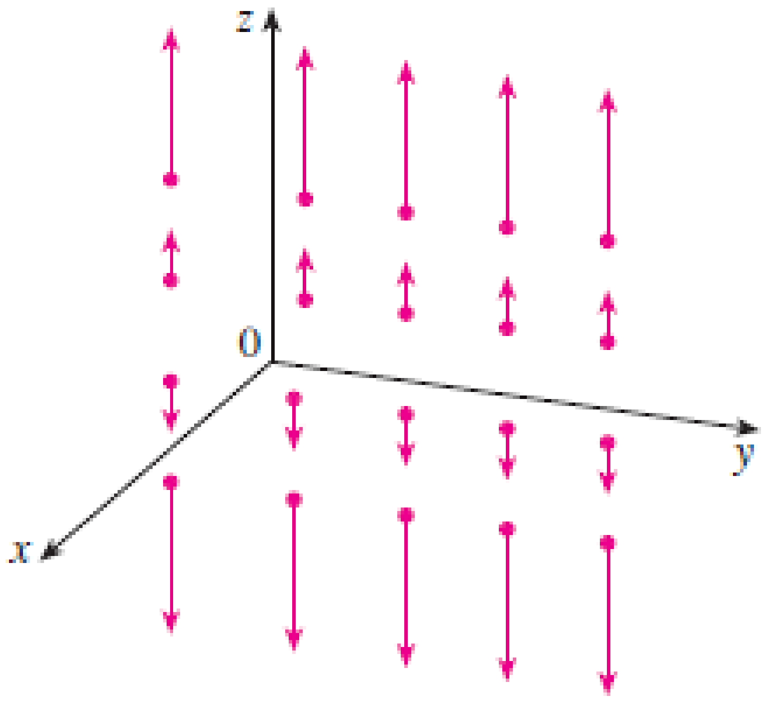
Example: 3D Vector Fields

Generate spectrum of the vector field:

$$\mathbf{F}(x, y, z) = \langle 0,0, z\rangle.$$

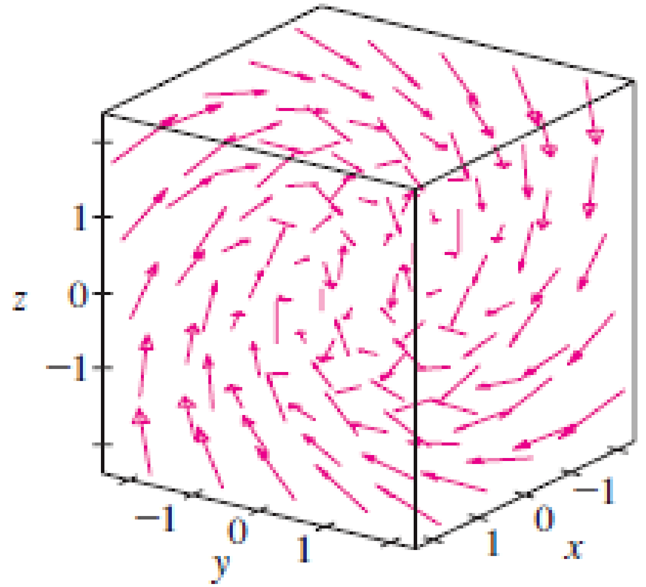
Solution:

The sketch is shown in the figure below. Notice that all vectors are vertical and point upward above the xy −plane or downward below it. The magnitude increases with the distance from the xy −plane.

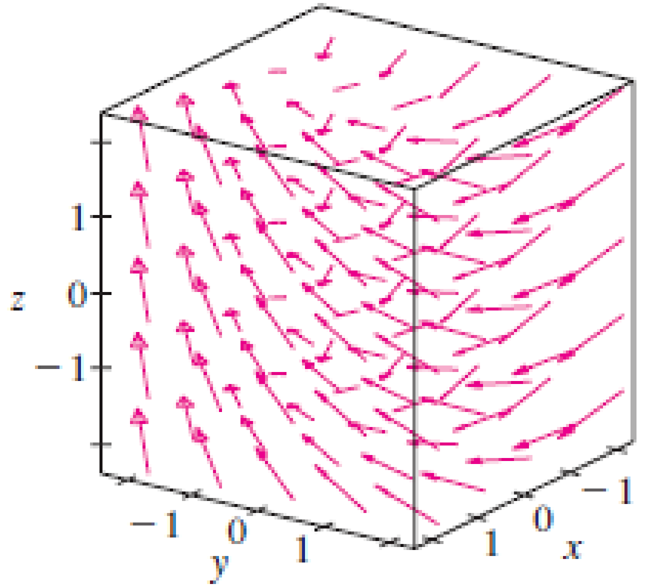


Example: 3D Vector Fields

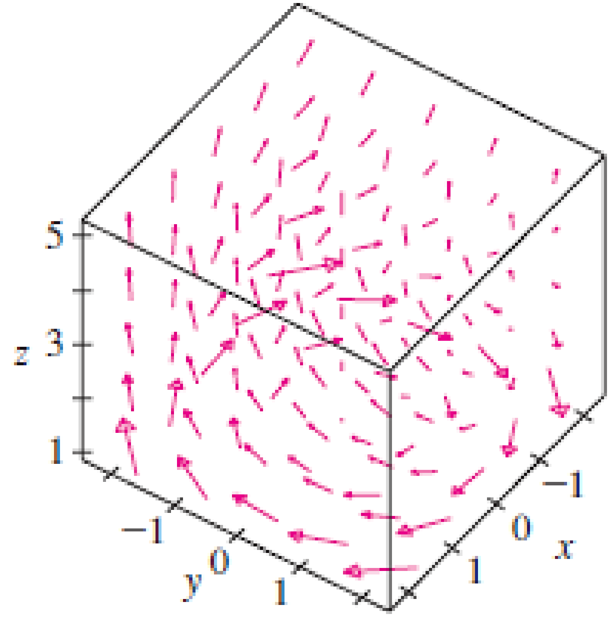
Note that we were able to draw the vector field by hand in previous example because of its particularly simple formula. Most three-dimensional vector fields, however, are virtually impossible to sketch by hand and so we need to resort to a computer algebra system.



$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$



$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2\,\mathbf{j} + x\,\mathbf{k}$



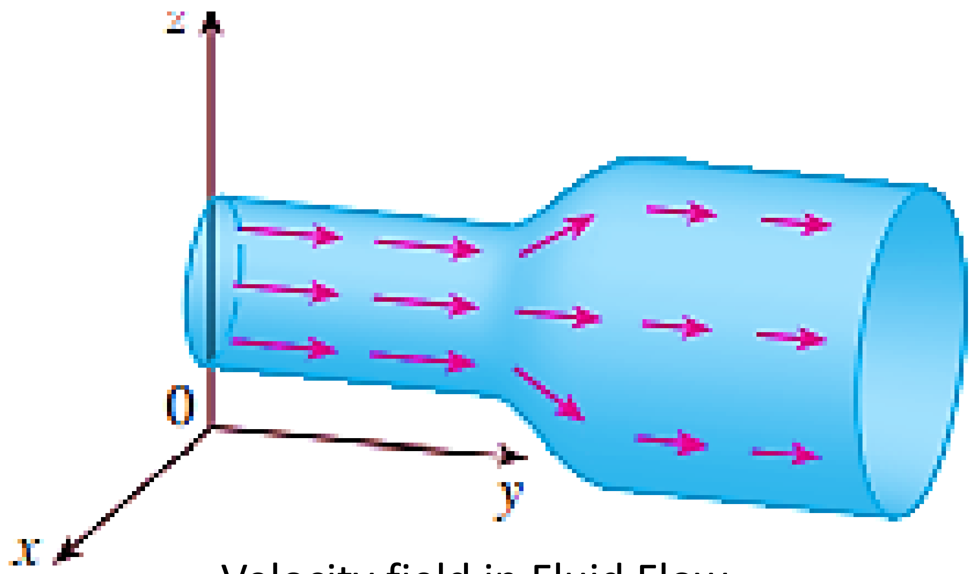
$\mathbf{F}(x, y, z) = \frac{y}{z}\,\mathbf{i} - \frac{x}{z}\,\mathbf{j} + \frac{z}{4}\,\mathbf{k}$

Example: Velocity Vector Fields

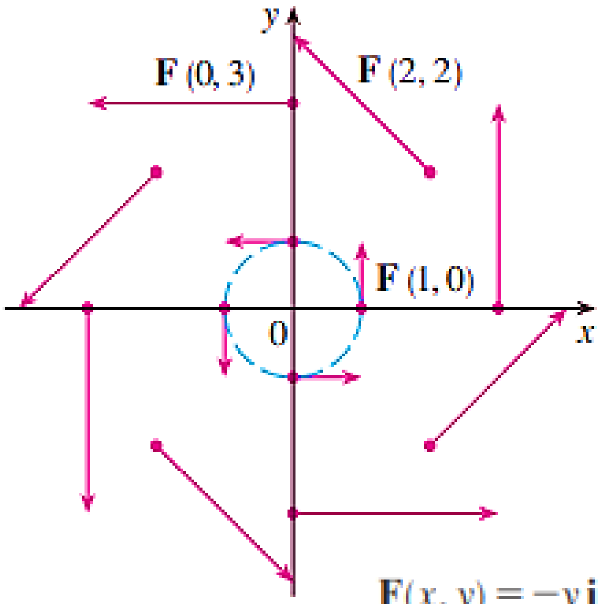
Imagine a fluid flowing steadily along a pipe and let be the velocity vector at a point . Then assigns a vector to each point in a certain domain (the interior of the pipe) and so is a vector field on called a **velocity field**. A possible velocity field is illustrated in the accompanying figure. The speed at any given point is indicated by the length of the arrow. Velocity fields also occur in other areas of physics. For instance, the vector field defined by:

$$\mathbf{F}(x, y) = \langle -y, x \rangle,$$

can be used as the velocity field describing the counterclockwise rotation of a wheel.



Velocity field in Fluid Flow



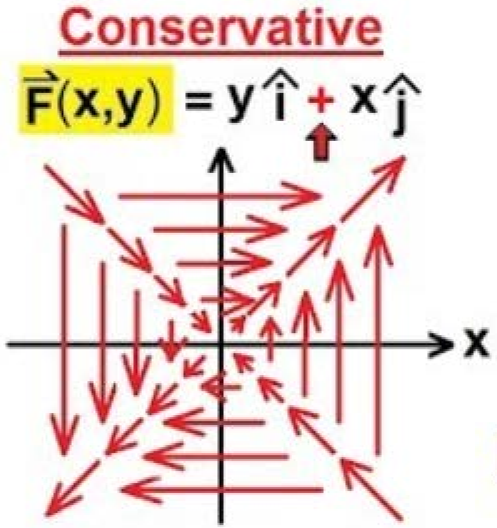
$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

Conservative Vector Field

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function such that $\mathbf{F} = \nabla f$. In this situation the scalar function f is called a **potential function** for the field $\mathbf{F}$. All this definition is saying is that a vector field is conservative if it is also a gradient vector field for some function.

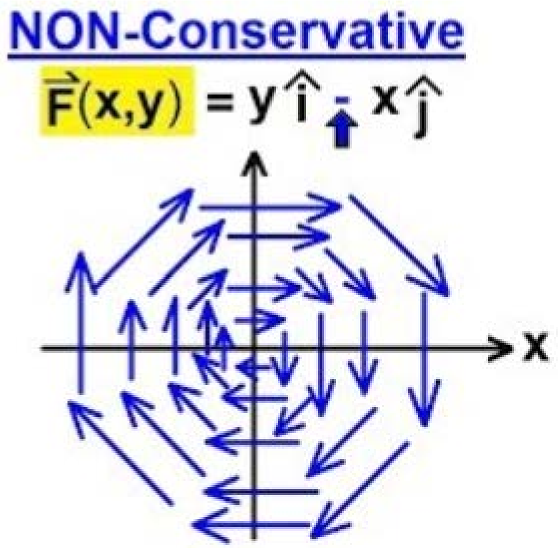
Note:

Every inverse square vector field is conservative.



Example:

The vector field $\mathbf{F} = \langle y, x \rangle$ is a conservative vector field with a potential function $f(x, y) = xy$ because $\mathbf{F} = \nabla f = \langle y, x \rangle$. On the other hand, the vector field $\mathbf{F} = \langle y, -x \rangle$ is not a conservative vector field since there is no function f such that $\mathbf{F} = \nabla f$.



Practice Questions

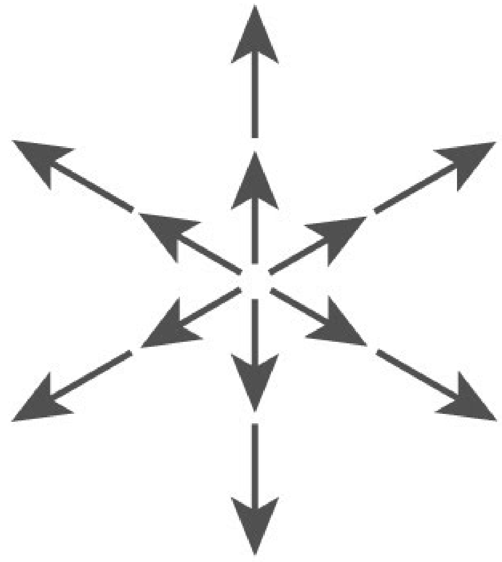
Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

Chapter: 16

Exercise-16.1: Q – 1 to 18, Q – 21 to 35.

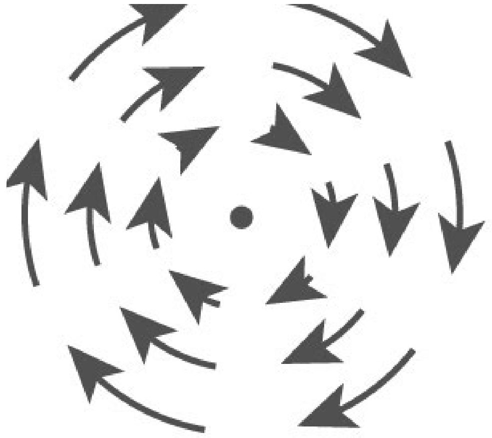
Divergence & Curl

DIVERGENCE & CURL OF A VECTOR FIELD



$$\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\nabla \times \vec{V} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{pmatrix}$$



16

Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.5**

Scalar Function : scalar function doesn't have any directions.

Vector Function : Vector function has directions i.e i,j,k

Vector Differential Operator(∇)

The symbol ∇ is read as "del or nabla"

i.e $\nabla = \bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

Gradient

let f(x,y,z) is scalar function, then Gradient of f is $\bar{\imath}\frac{\partial f}{\partial x} + \bar{\jmath}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$.

It is denoted by **grad f or ∇f**

i.e grad f = ∇f $= (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})f$

$$\nabla f = \bar{\imath}\frac{\partial f}{\partial x} + \bar{\jmath}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

Divergence

- Let $\bar{F} = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$ is vector function, then the divergence of $\bar{F}$ is $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$
- It is denoted by div $\bar{F}$ OR $\nabla . \bar{F}$
- i.e div $\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

curl

- Let $\bar{F} = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$ is vector function, then

$$\text{curl}\bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

- It is denoted by curl$\bar{F}$ OR $\nabla \times \bar{F}$

Field Operators: Grad, Div and Curl

Observe that:

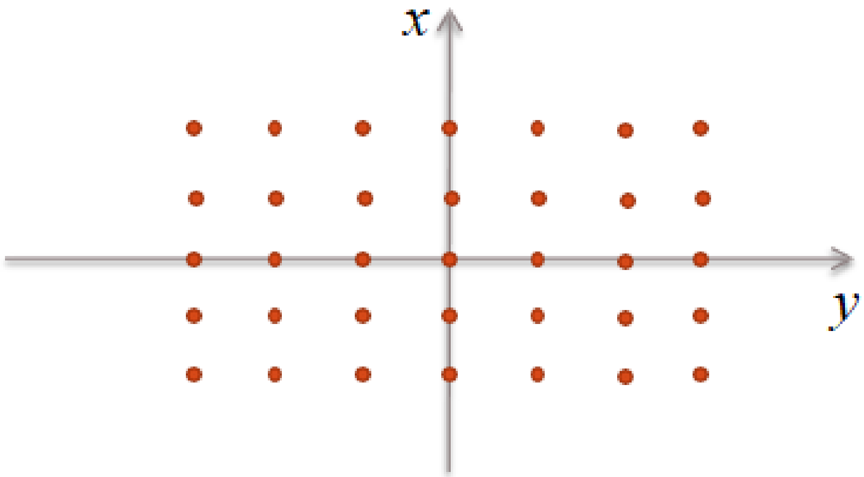
- $\text{grad}\, f$ is a vector field.

- $\text{curl}\, \mathbf{F}$ is a vector field.

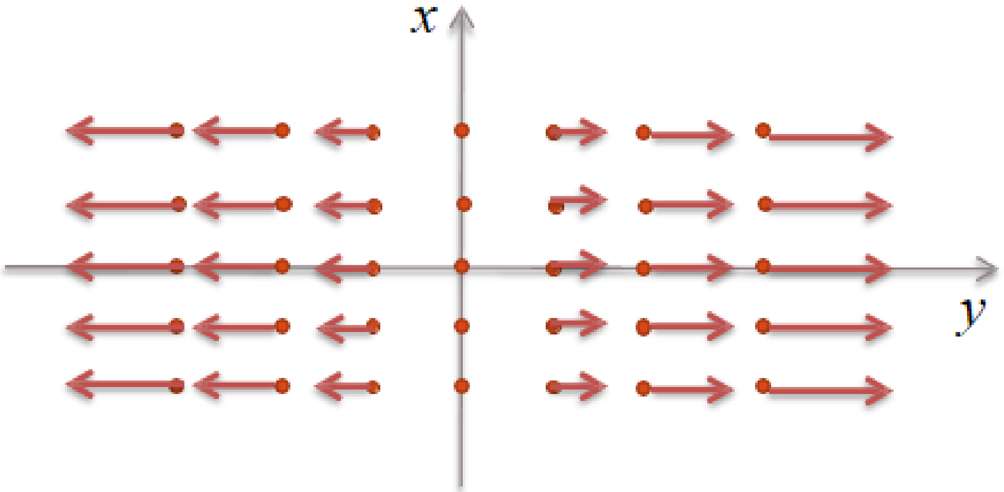
- $\text{div}\, \mathbf{F}$ is a scalar field.

Vector Calculus

As we have discussed that **scalar fields** represents the values of a dimensionless physical quantity at each point. For example, temperature on metallic plate is regarded as a scalar field which has a definite value at each point. Similarly, **vector fields** arise when we need to associate a direction besides strength at each point of a physical quantity. The gravitational field is a vector field for if it is determined at a point then not only it gives the strength but also gives the direction where gravitational field is pointing.



$$T(x, y) = x^2 + y^2$$

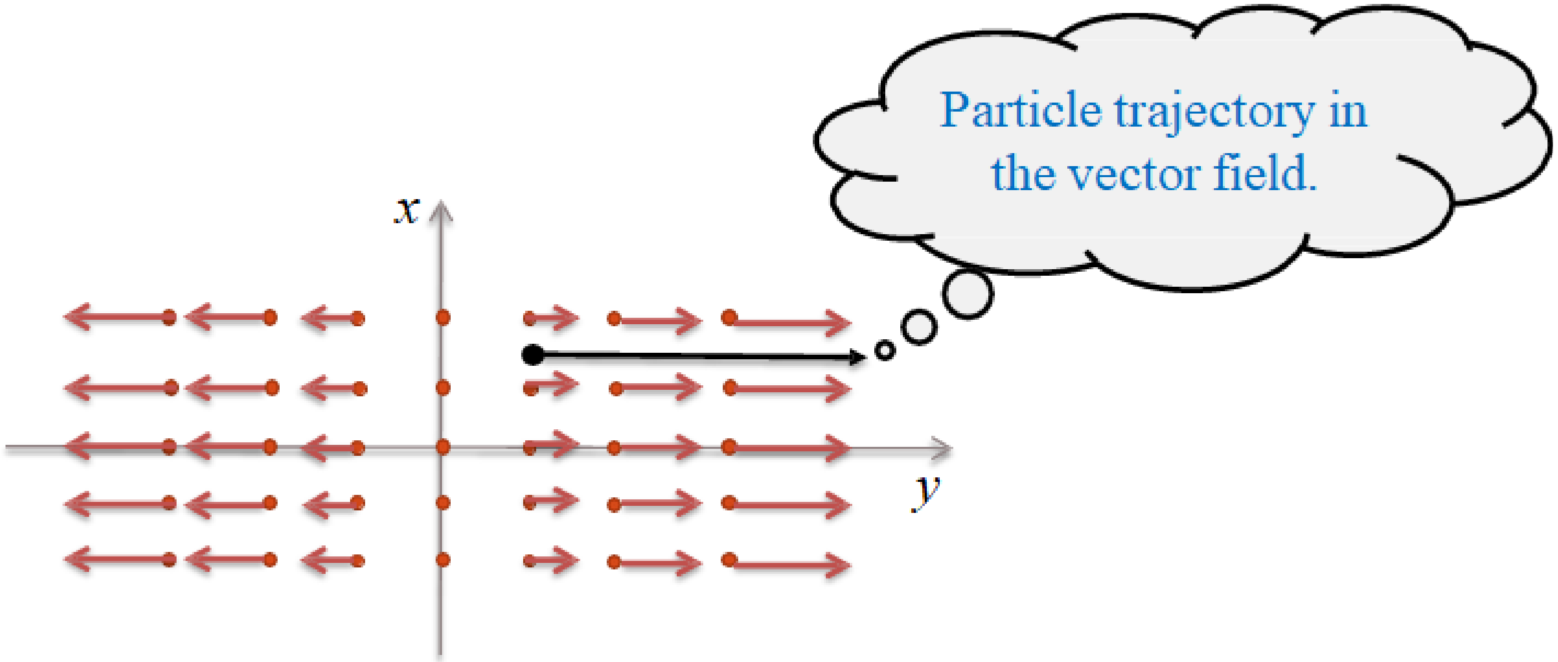


$$\mathbf{F}(x, y) = x\,\mathbf{i}$$

Vector Calculus

If a test particle (non-interacting) is placed in a vector field, then it is natural to see that particle will start moving. The motion is described by derivatives as we have studied. Therefore, we need to make sense of derivatives of a vector field.

$$\mathbf{F}(x, y) = x\,\mathbf{i}$$



Briefly, we have two ways to carry out derivatives:

$$\boxed{\frac{\partial}{\partial x}} \qquad \boxed{\frac{\partial}{\partial y}}$$

Vector Calculus

It does not make a proper sense to apply one of them on the vector field. In order to bring them on equal footing we introduce a vector operator known as del or nabla operator:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle.$$

We are now in a position to apply above vector operator on a vector field. We need to be careful as the multiplication of two vectors is not possible. However, we can take the natural dot and cross products as defined among vectors. We define two operations that can be performed on vector fields and that play a basic role in the applications of vector calculus to fluid flow and electricity and magnetism.

1. Flux Density (Divergence) $\boxed{\nabla \bullet \mathbf{F}}$

2. Circulation Density (Curl) $\boxed{\nabla \times \mathbf{F}}$

Each operation resembles differentiation. However, one produces a scalar field whereas the other produces a vector field.

Divergence

Let $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ be a differentiable vector field on a region in $\mathbb{R}^3$. Let $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ (the "del" or "nebla" operator), then the **divergence** of the vector field is defined as:

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

which is a scalar quantity.

Examples:

Determine the divergence of the following vector fields:

1. $\mathbf{F}(x, y, z) = \langle e^{xyz}, yz \sin x, x^2 + y^2 + z^2 \rangle$.

   Solution: $yze^{xyz} + z \sin x + 2z$.

2. $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$.

   Solution: 3.

3. $\mathbf{F}(x, y, z) = \langle xz, xyz, -y^2 \rangle$.

   Solution: $z + xz$.