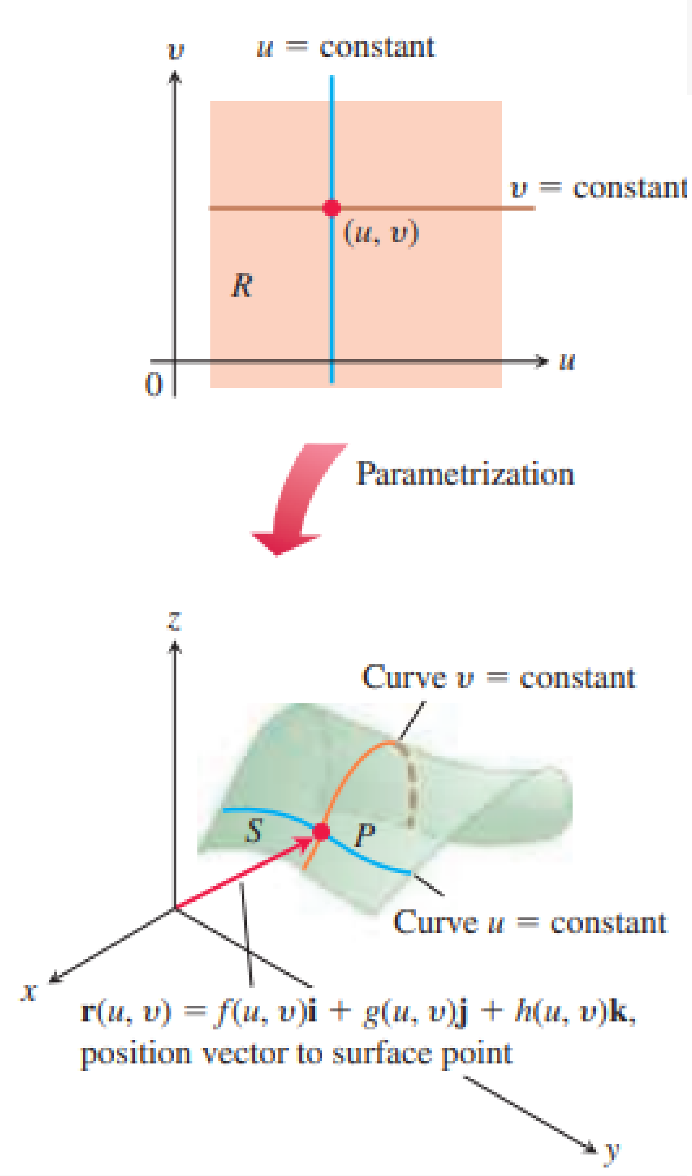
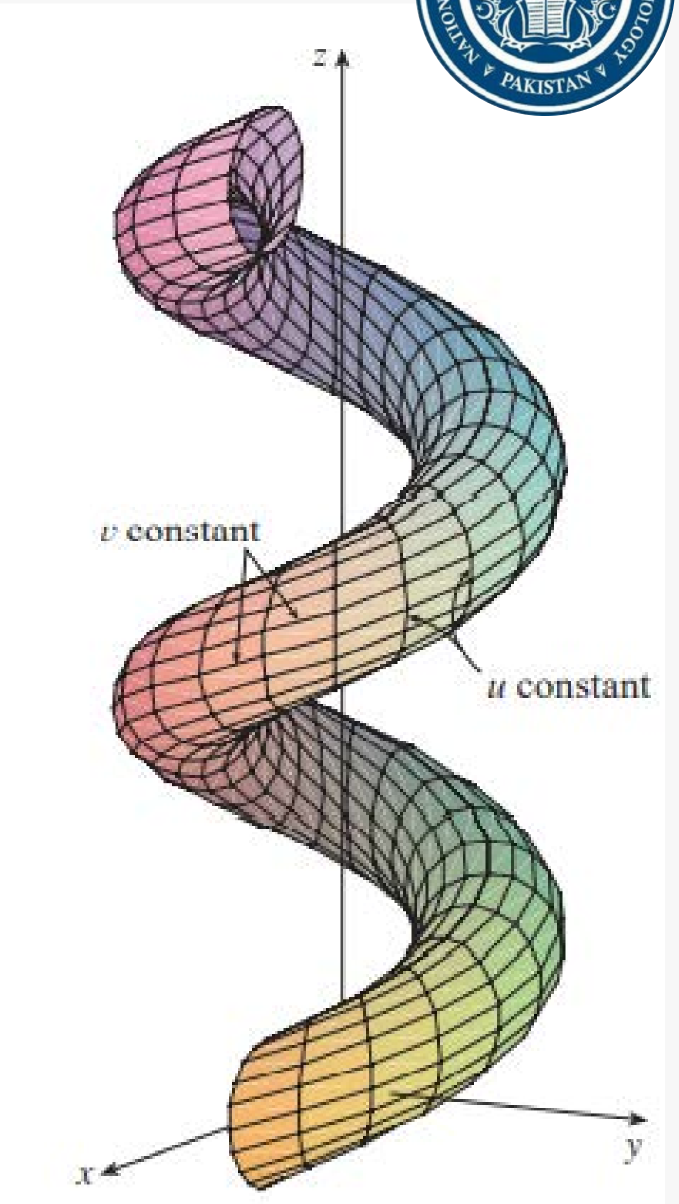
Parametrized Surfaces



A parametrized surface S expressed as a vector function of two variables defined on a region R.



Vector Calculus(MATH-243)

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Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.6**

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

- **Chapter: 16**
  - **Section: 16.5**

Parametric Surfaces

- So far, we have considered special types of surfaces:

  - Cylinders
  - Quadric surfaces
  - Graphs of functions of two variables
  - Level surfaces of functions of three variables.

- Now we aim to use vector functions to describe more general surfaces, called ***parametric surfaces***, and compute their areas.

Parametric Surfaces

In much the same way that we describe a space curve by a vector function $\mathbf{r}(t)$ of a single parameter t, we can describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v. We suppose that:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \qquad (1)$$

is a vector-valued function defined on a region D in the uv −plane.

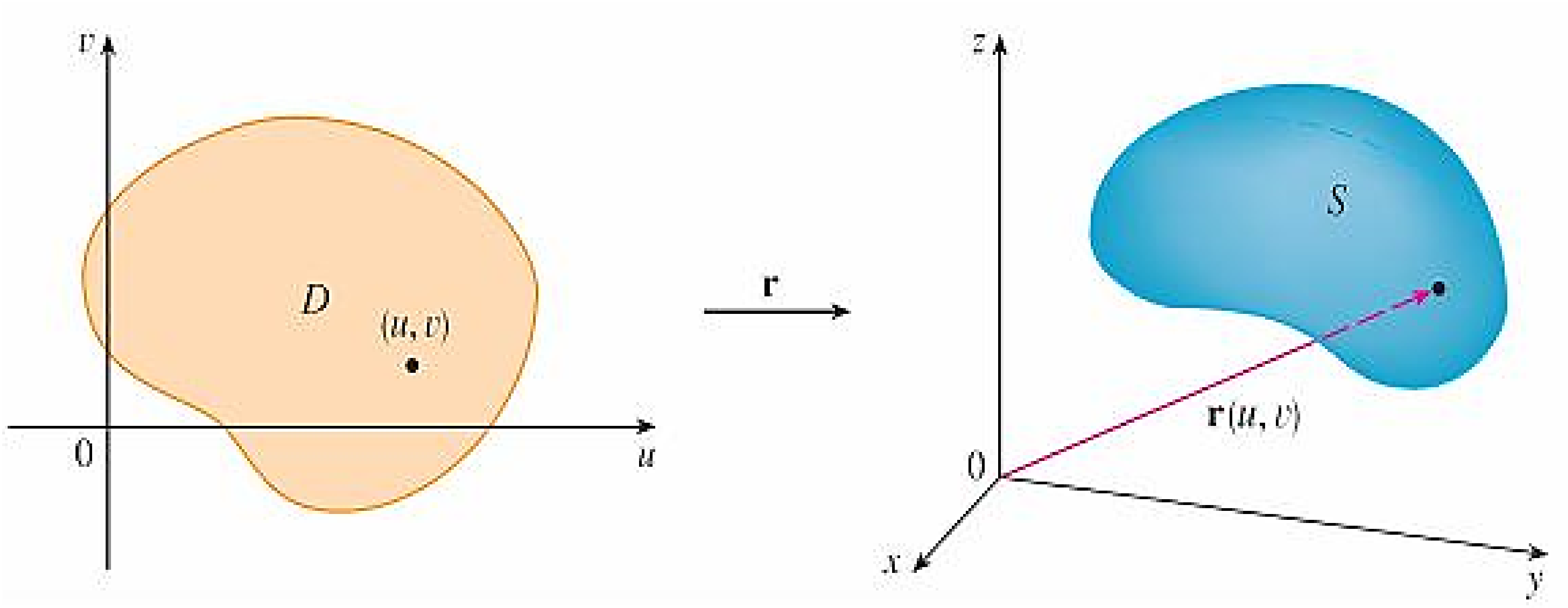
So, x, y, and z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain D. The set of all points (x, y, z) in $\mathbb{R}^3$ such that:

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \qquad (2)$$

and (u, v) varies throughout D, is called a **parametric surface** S and Equations (2) are called **parametric equations** of S.

Parametric Surfaces

Each choice of u and v gives a point on S; by making all choices, we get all of S. In other words, the surface is traced out by the tip of the position vector $\mathbf{r}(u, v)$ as (u, v) moves throughout the region D.



A parametric surface

Example:

Identify and sketch the surface with vector equation:

$$\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + v\,\mathbf{j} + 2\sin u\,\mathbf{k}.$$

Solution:

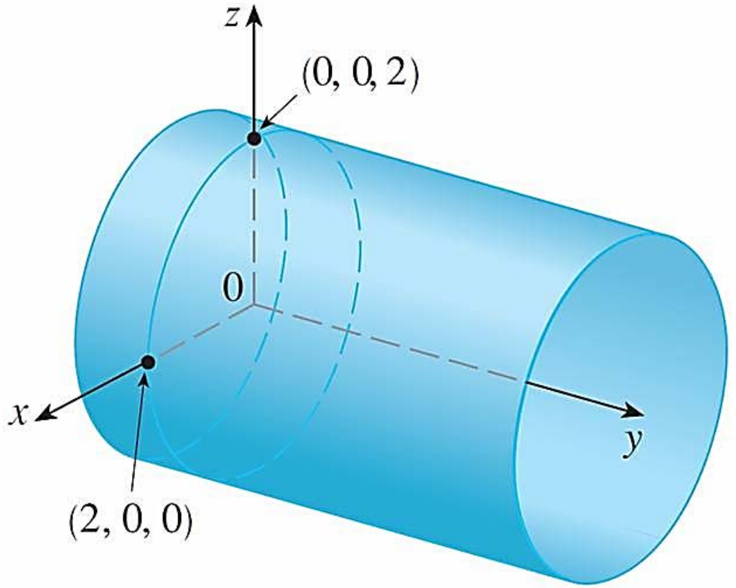
The parametric equations for this surface are:

$$x = 2\cos u\,, \qquad y = v, \qquad z = 2\sin u\,.$$

So, for any point (x, y, z) on the surface, we have:

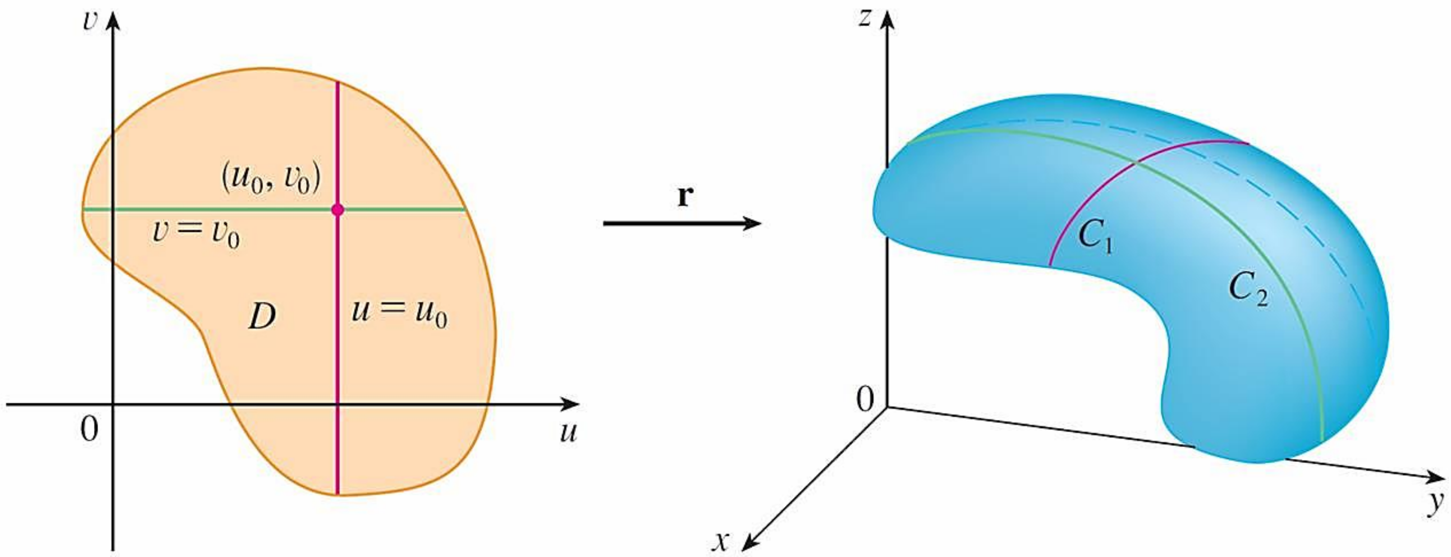
$$x^2 + z^2 = 4\cos^2 u + 4\sin^2 u = 4.$$

This means that vertical cross-sections parallel to the xz −plane (that is, with y constant) are all circles with radius 2. Since $y = v$ and no restriction is placed on v, the surface is a circular cylinder with radius 2 whose axis is the y −axis.



Parametric Surfaces: Families of Curves

If a parametric surface S is given by a vector function $\mathbf{r}(u, v)$, then there are two useful families of curves that lie on S, one family with u constant and the other with v constant. These families correspond to **vertical** and **horizontal lines** in the uv −plane. If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a curve C_1 lying on S. Similarly, if we keep v constant by putting $v = v_0$, we get a curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S. We call these curves **grid curves**. For instance, in previous example, the grid curves obtained by letting u be constant are horizontal lines whereas the grid curves with v constant are circles.



Example:

Identify the grid curves of the surface:

$$\boldsymbol{r}(u, v) = \langle (2 + \sin v)\cos u, (2 + \sin v)\sin u, u + \cos v \rangle$$

Which grid curves have u constant? Which have v constant?

Solution:

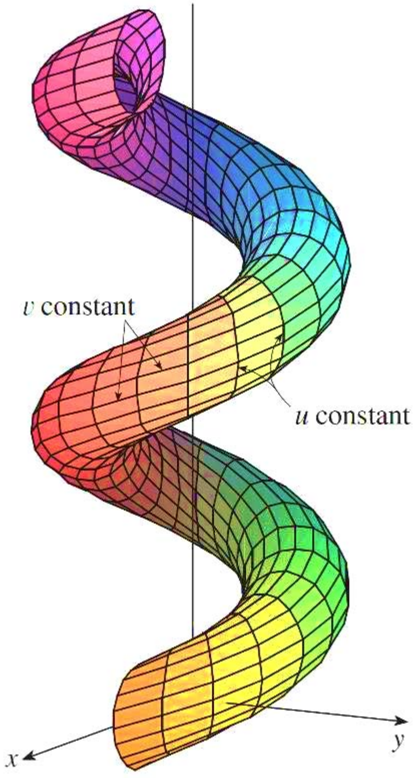
We graph the portion of the surface with parameter domain:

$$0 \le u \le 4\pi, \quad 0 \le v \le 2\pi.$$

It has the appearance of a spiral tube. To identify the grid curves, we write the corresponding parametric equations:

$$x = (2 + \sin v)\cos u,$$
$$y = (2 + \sin v)\sin u,$$
$$z = u + \cos v.$$

If v is constant, then $\sin v$ and $\cos v$ are constant, so the parametric equations resemble those of the helix. Thus, the grid curves with v constant are the spiral curves in the figure. Moreover, we deduce that the grid curves with u constant must be curves that look like circles in the figure.



Example:

Find a parametric representation of the sphere:

$$x^2 + y^2 + z^2 = a^2.$$

Solution:

The sphere has a simple representation $\rho = a$ in spherical coordinates, so let's choose the angles φ and θ in spherical coordinates as the parameters. Then, putting $\rho = a$ in the equations for conversion from spherical to rectangular coordinates, we obtain:

$$x = a \sin\varphi \cos\theta\,, \qquad y = a \sin\varphi \sin\theta\,, \qquad z = a \cos\varphi\,,$$

as the parametric equations of the sphere. The corresponding vector equation is:

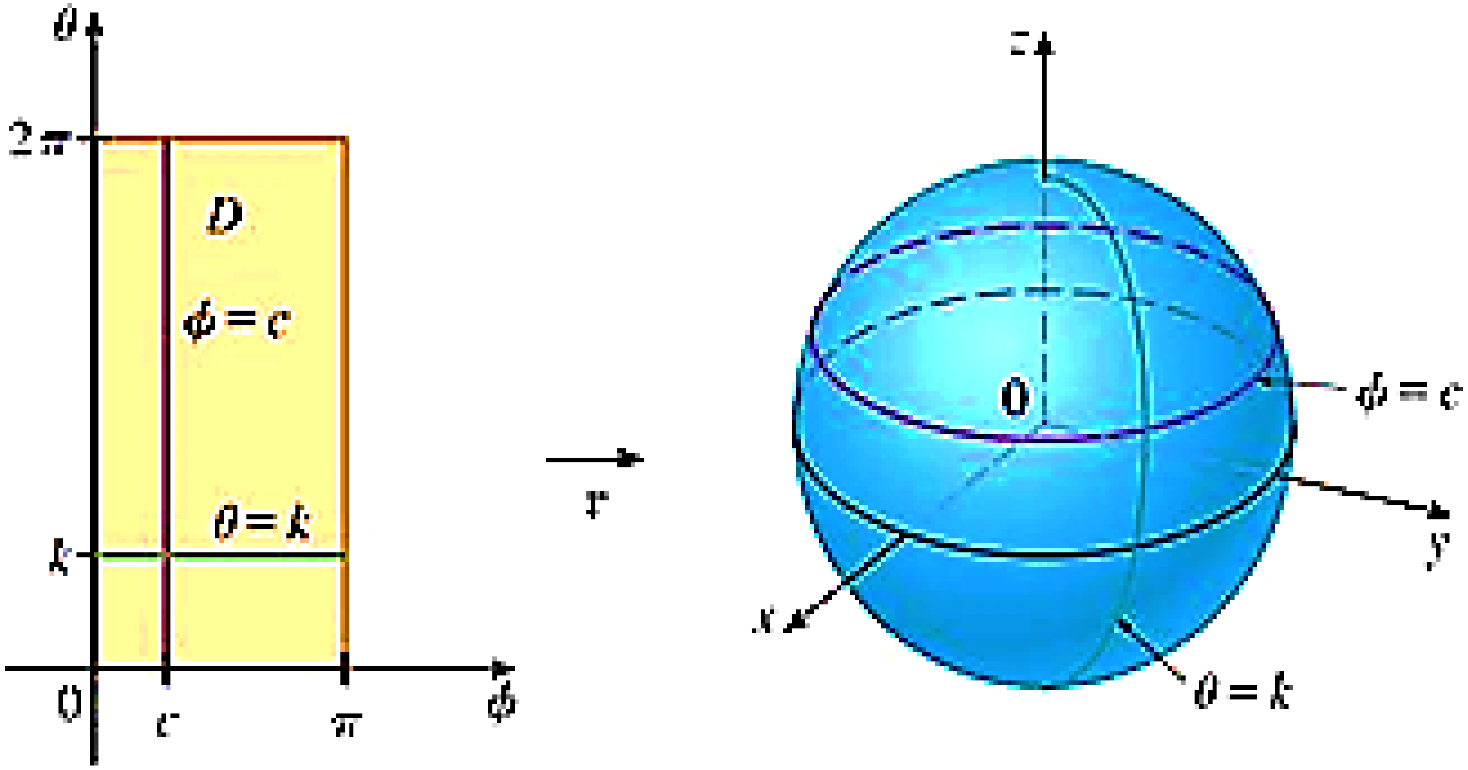
$$\mathbf{r}(\varphi, \theta) = a \sin\varphi \cos\theta\, \mathbf{i} + a \sin\varphi \sin\theta\, \mathbf{j} + a \cos\varphi\, \mathbf{k}.$$

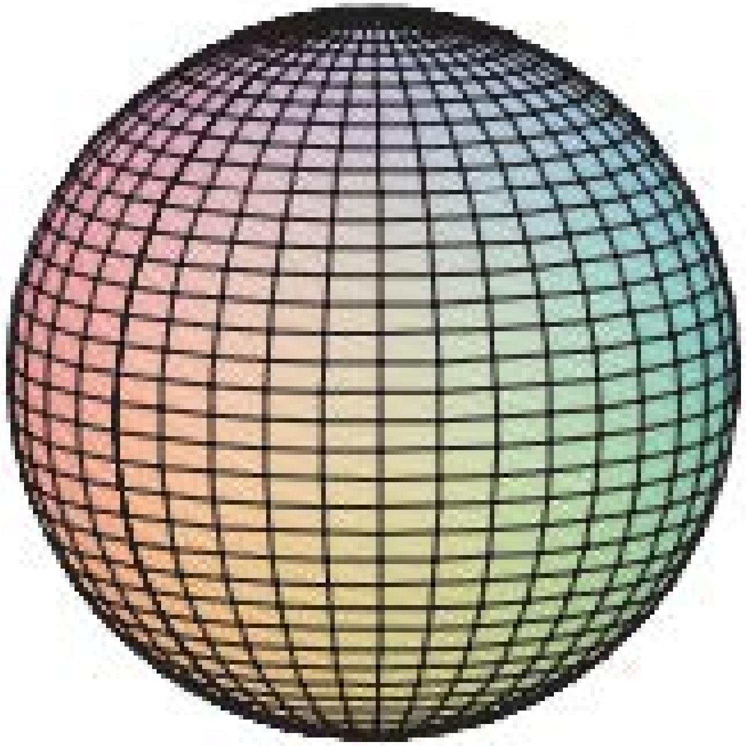
We have $0 \le \varphi \le \pi$ and $0 \le \theta \le 2\pi$, so the parameter domain is the rectangle:

$$D = [0, \pi] \times [0, 2\pi].$$

The grid curves with φ constant are the circles of constant latitude (including the equator). The grid curves with θ constant are the meridians (semi-circles), which connect the north and south poles.

Example:





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Parametric Surfaces

Note:

- In general, a surface given as the **graph of a function** of x and y, that is, with an equation of the form $z = f(x, y)$, can always be regarded as a parametric surface by taking x and y as parameters and writing the parametric equations as:

$$x = x, \qquad y = y, \qquad z = f(x, y).$$

- Parametric representations (also called parametrizations) of surfaces are not unique.

  The next example shows two ways to parametrize a cone.

Example:

Find a parametric representation for the surface:

$$z = 2(x^2 + y^2)^{1/2},$$

that is, the top half of the cone $z^2 = 4(x^2 + y^2)$.

Solution:

One possible representation is obtained by choosing x and y as parameters:

$$x = x, \qquad y = y, \qquad z = 2\sqrt{x^2 + y^2}.$$

So, the vector equation is:

$$\mathbf{r}(x, y) = \left\langle x, y, 2\sqrt{x^2 + y^2} \right\rangle.$$

Another representation results from choosing as parameters the polar coordinates r and θ.

A point (x, y, z) on the cone satisfies $x = r\cos\theta$, $y = r\sin\theta$, and $z = 2\sqrt{x^2 + y^2} = 2r$.
So, a vector equation for the cone is:

$$\mathbf{r}(r, \theta) = \langle r\cos\theta , r\sin\theta , 2r \rangle,$$

Where, $r \geq 0$ and $0 \leq \theta \leq 2\pi$.