Applications of Derivatives

Calculus & Analytical Geometry MATH-101

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Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter:** 4
  - **Section:** 4.5

Example:

The demand for x units of a product is related to a selling price of S dollars per unit by the equation:

$$2x + S^2 - 12{,}000 = 0.$$

a) Find the demand function, the marginal demand function, the revenue function, and the marginal revenue function.

b) Find the number of units and the price per unit that yield the maximum revenue.

c) Find the maximum revenue.

Solution:

a) Since $S^2 = 12{,}000 - 2x$ and S is positive, we see that the demand function $p(x)$ is given by:

$$S = p(x) = \sqrt{12000 - 2x}.$$

The domain of $p(x)$ consists of every x such that $12000 - 2x > 0$, or, equivalently, $2x < 12{,}000$. Thus, $0 \leq x < 6000$. The marginal demand function $p'(x)$ is given by:

$$p'(x) = \frac{-1}{\sqrt{12000 - 2x}}.$$

The negative sign indicates that a decrease in price is associated with an increase in demand.

Solution:

The revenue function $R(x)$ is given by:

$$R(x) = xp(x) = x\sqrt{12000 - 2x}.$$

Differentiating and simplifying gives us the marginal revenue function $R'(x)$ as:

$$R'(x) = \frac{12000 - 3x}{\sqrt{12000 - 2x}}.$$

Solution:

b) A critical number for the revenue function $R(x)$ is: $x = 12000/3 = 4000$. Observe that:

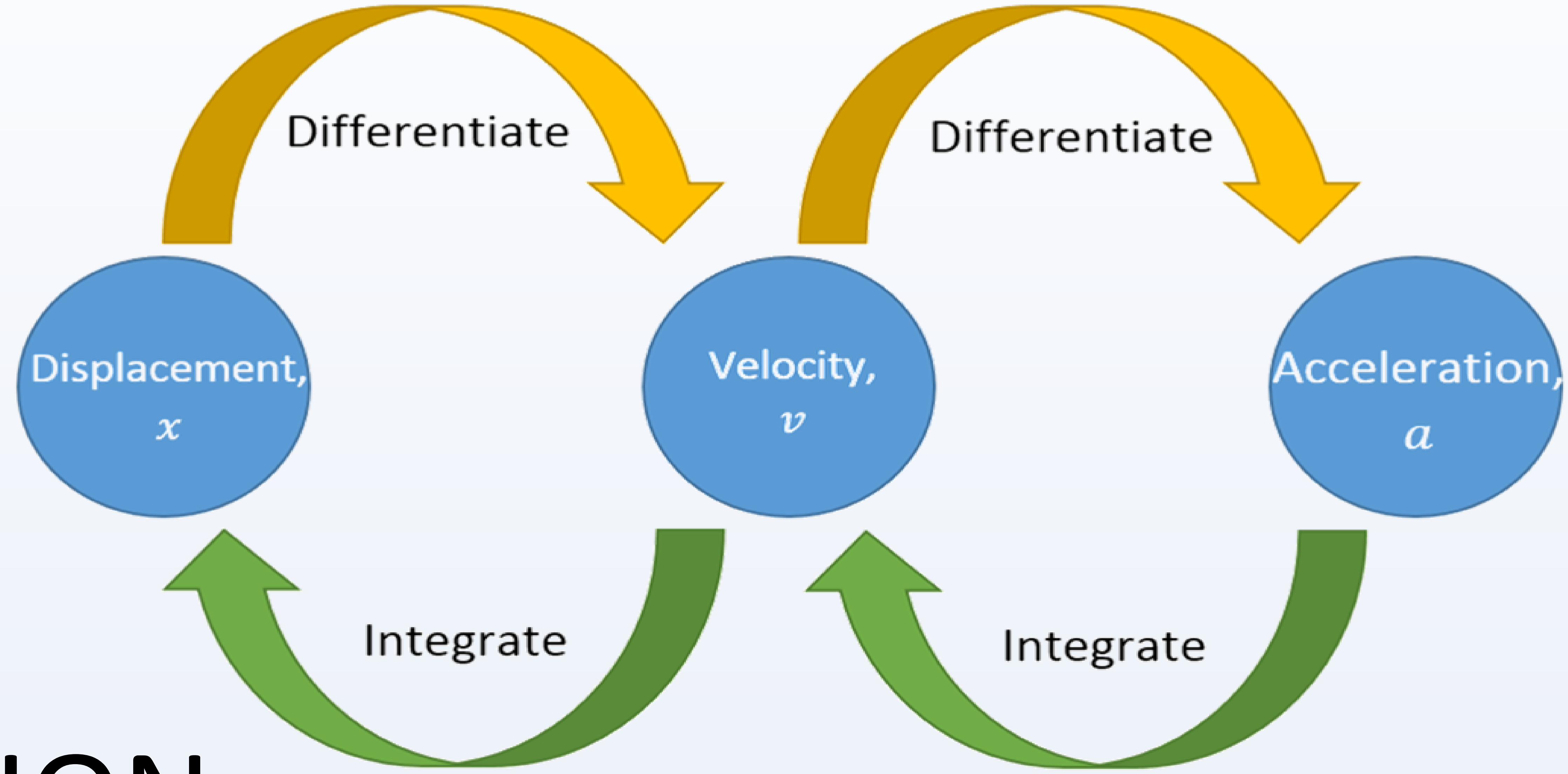
$$\begin{array}{ccc} + & & - \\ R'(x) > 0 & 4000 & R'(x) < 0 \end{array}$$

Since $R(x)$ is increasing for $0 \leq x < 4000$ and decreasing for $4000 < x < 6000$, the maximum revenue occurs when 4000 units a reproduced and sold. This corresponds to a selling price per unit of:

$$p(4000) = \sqrt{12000 - 2(4000)} \approx \$63.25\,.$$

c) The maximum revenue, obtained from selling 4000 units at \$63.25 per unit , is:

$$R(x) = xp(x) = 4000(63.25) = \$253{,}000.$$



INTEGRATION

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter:** 4
  - **Section:** 4.8

Antiderivatives

- So far, we have studied how to find the derivative of a function.
- However, many problems require that we recover a function from its known derivative (from its known rate of change).
- For instance, we may know the velocity function of an object falling from an initial height and need to know its height at any time over some period.
- More generally, we want to find a function F from its derivative f.
- If such a function F exists, it is called an ***antiderivative*** of f.

$dx \to$ differential $\frac{d}{dx}(10x) = 10$

Antiderivative

$d \to$ differential operator $\Rightarrow$ $d(10x) = (10)\,dx$

If $F'(x) = f(x)$, then $F(x)$ is an antiderivative of $f(x)$.

$\Rightarrow \int d(10x) = \int (10)\,dx$

$10x = \int (10)\,dx$

Examples:

- If $F(x) \;=\; 10x$, then $F'(x) =\; 10.$ $F(x)$ is the antiderivative of $f(x) =\; 10.$
- If $F(x) \;=\; x^2$, then $F'(x) =\; 2x$. $F(x)$ is the antiderivative of $f(x) \;=\; 2x$.

$\frac{d}{dx}(x^2) = 2x \Rightarrow d(x^2) = (2x)\,dx$

$\Rightarrow \int d(x^2) = \int (2x)\,dx$

$\Rightarrow x^2 = \int 2x\,dx$

Antiderivative

In the example we just did, we know that $F(x) = x^2$ is not the only function whose derivative is: $2x$

- $G(x) = x^2 + 2$ has $2x$ as the derivative.
- $H(x) = x^2 - 7$ has $2x$ as the derivative.

(Family of anti derivatives)

For any real number, C, the function:

$$F(x) = x^2 + C$$

general anti derivative of $f(x) = 2x$

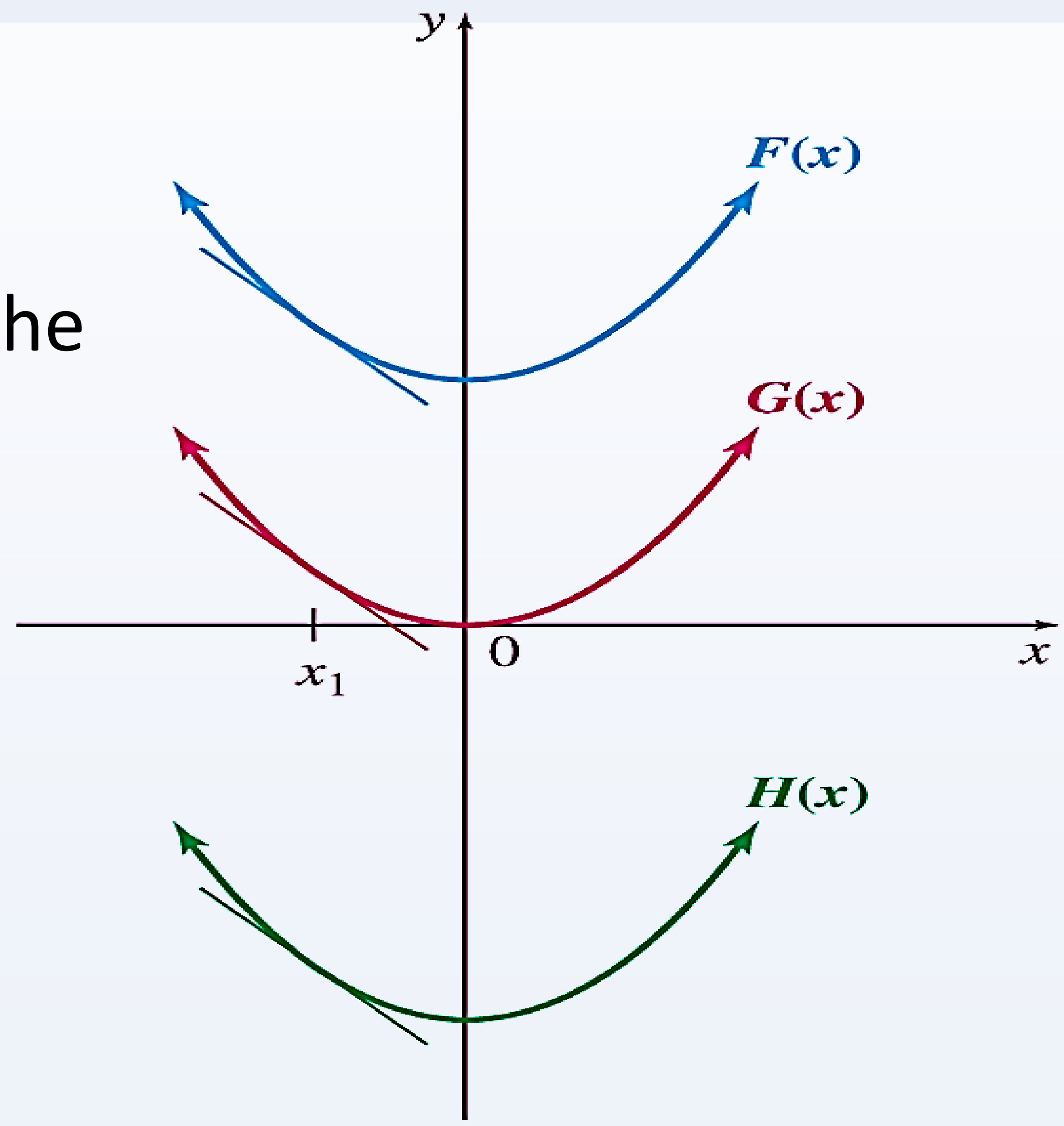
Constant of integration

has $f(x) = 2x$ as a derivative.

Antiderivative

- Since the functions $G(x) = x^2$, $F(x) = x^2 + 2$ and $H(x) = x^2 - 7$ differ only by a constant, the slope of the tangent line remains the same.

- There is a whole family of functions having $2x$ as a derivative and this family differs only by a constant.



Slopes of the tangent lines at $x = x_1$ are the same.

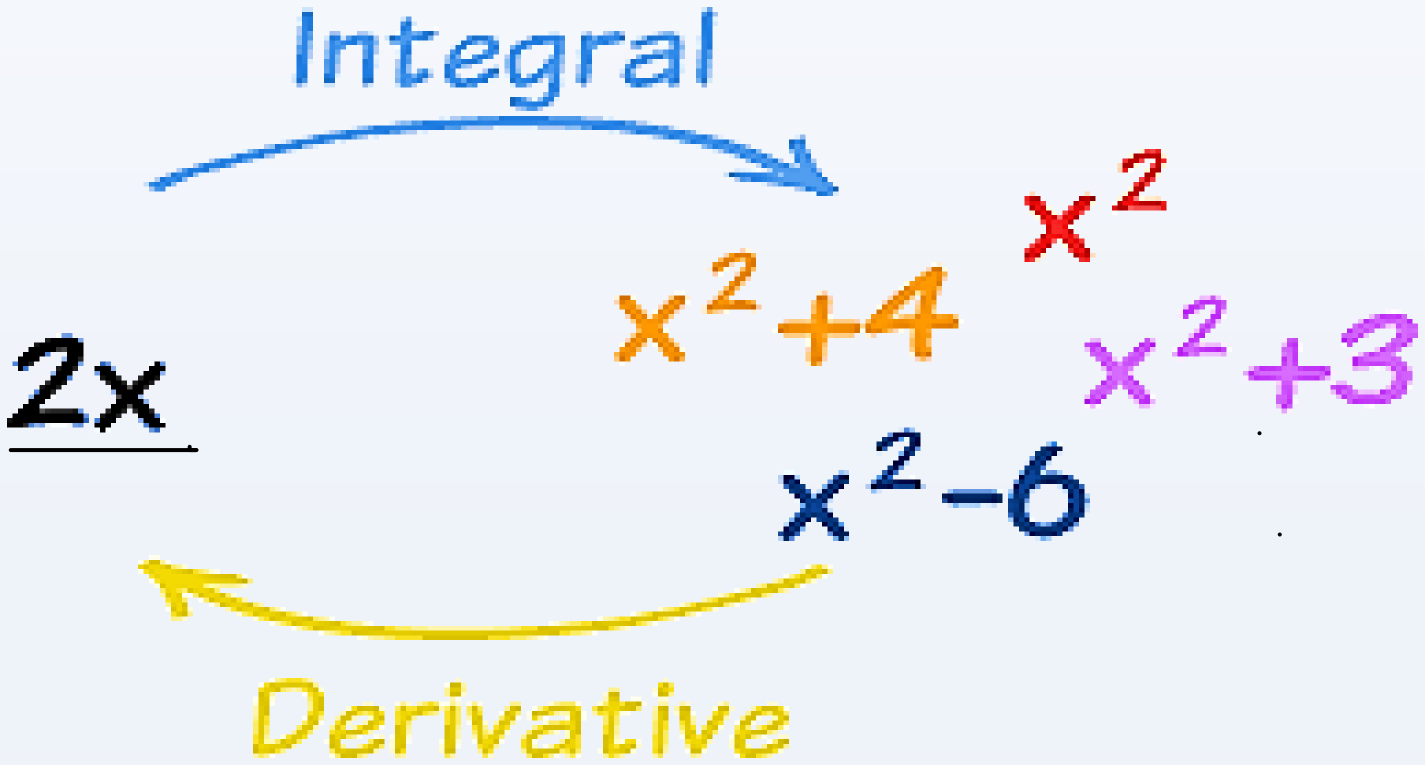
The family of antiderivatives can be represented by $\boldsymbol{F(x) + C}$

Antiderivative

If $F(x)$ and $G(x)$ are both antiderivatives of a function $f(x)$ on ,an interval, then there is a constant C such that:

$$F(x) - G(x) = C.$$

(Two antiderivatives of a function can differ only by a ,constant.) The arbitrary real number C is called an integration constant.



Antiderivative

Result:

If $F(x)$ is an antiderivative of $f(x)$ on an interval I, then the most general antiderivative of $f(x)$ on I is:

$$F(x) + C$$

where C is an arbitrary constant.

Thus, the most general antiderivative of $f(x)$ on I is a *family* of functions whose graphs are vertical translates of one another. We can select a particular antiderivative from this family by assigning a specific value to C.

Antiderivative

$\frac{d}{dx}(-\cos x) = \sin x$

Example: Finding a Particular Antiderivative

$\Rightarrow d(-\cos x) = (\sin x)\, dx$

Find an antiderivative of $f(x) = \sin x$ that satisfies $F(0) = 3$.

$\Rightarrow \int \sin x\, dx = \underbrace{-\cos x + C}_{F(x)}$

Solution:

Since the derivative of $-\cos x$ is $\sin x$, the general antiderivative

$$F(x) = -\cos x + C, \qquad (*)$$

gives all the antiderivatives of $f(x)$. The condition $F(0) = 3$ determines a specific value for C. Substituting $x = 0$ in $(*)$ we get:

$\Rightarrow C - 1 = 3$

$$F(0) = -\cos(0) + C = -1 + C. \qquad (**)$$

$\Rightarrow \boxed{C = 4}$

Given that $F(0) = 3$, thus, from $(**)$ we get: $C = 4$. Thus,

$$\boxed{F(x) = -\cos x + 4,}$$

is the antiderivative of $f(x)$ that satisfies $F(0) = 3$.

Initial Value Problems & Differential Equations

- Finding an antiderivative for a function $f(x)$ is the same problem as finding a function $y(x)$ that satisfies the equation:

$$\frac{dy}{dx} = f(x).$$

$$\frac{dy}{dx} = f(x)$$

$$d(y(x)) = f(x)\,dx$$

$$\Rightarrow y(x) = \int f(x)\,dx$$

- This is called a **differential equation**, since it is an equation involving an unknown function $y(x)$ that is being differentiated.

Family of curves { General antiderivative = GS — C — Solution — PS → particular solution — fix C

- To solve it, we need a function $y(x)$ that satisfies the equation. This function is found by taking the antiderivative of $f(x)$.

Initial Value Problems & Differential Equations

- We fix the arbitrary constant arising in the antidifferentiation process by specifying an initial condition:

at $x = x_0 \Rightarrow y = y_0$ $\qquad \frac{dy}{dx} = f(x)$

$$y(x_0) = y_0.$$

I.C

- This condition means the function $y(x)$ has the value y_0 when $x = x_0$. The combination of a differential equation and an initial condition is called an **initial value problem**.

- Such problems play important roles in all branches of science.

IVP

Initial Value Problems & Differential Equations

Example: Finding a Curve from Its Slope Function & a Point

Find the curve whose slope at the point (x, y) is $3x^2$ if the curve is required to pass through the point $(1, -1)$.

$x = 1,\ y = -1$ $\quad \boxed{y(1) = -1} \longleftrightarrow \boxed{\frac{dy}{dx} = 3x^2}$ $\quad$ IVP

Solution:

In mathematical language, we are asked to solve the initial value problem that consists of the following:

$y(x) = ?$

The differential equation: $\dfrac{dy}{dx} = 3x^2.$ ✓

$d[y(x)] = (3x^2)\,dx \Rightarrow y(x) = \int (3x^2)\,dx$

$\Rightarrow y(x) = x^3 + C$

The initial condition: $y(1) = -1.$

1. **Solve the differential equation:** The function $y(x)$ is an antiderivative of $f(x) = 3x^2$ so

$$y(x) = x^3 + C.$$ ✓✓

This result tells us that $y(x)$ equals $x^3 + C$ for some value of C. We find that value from the initial condition $y(1) = -1$. ✓

Initial Value Problems & Differential Equations

Solution:

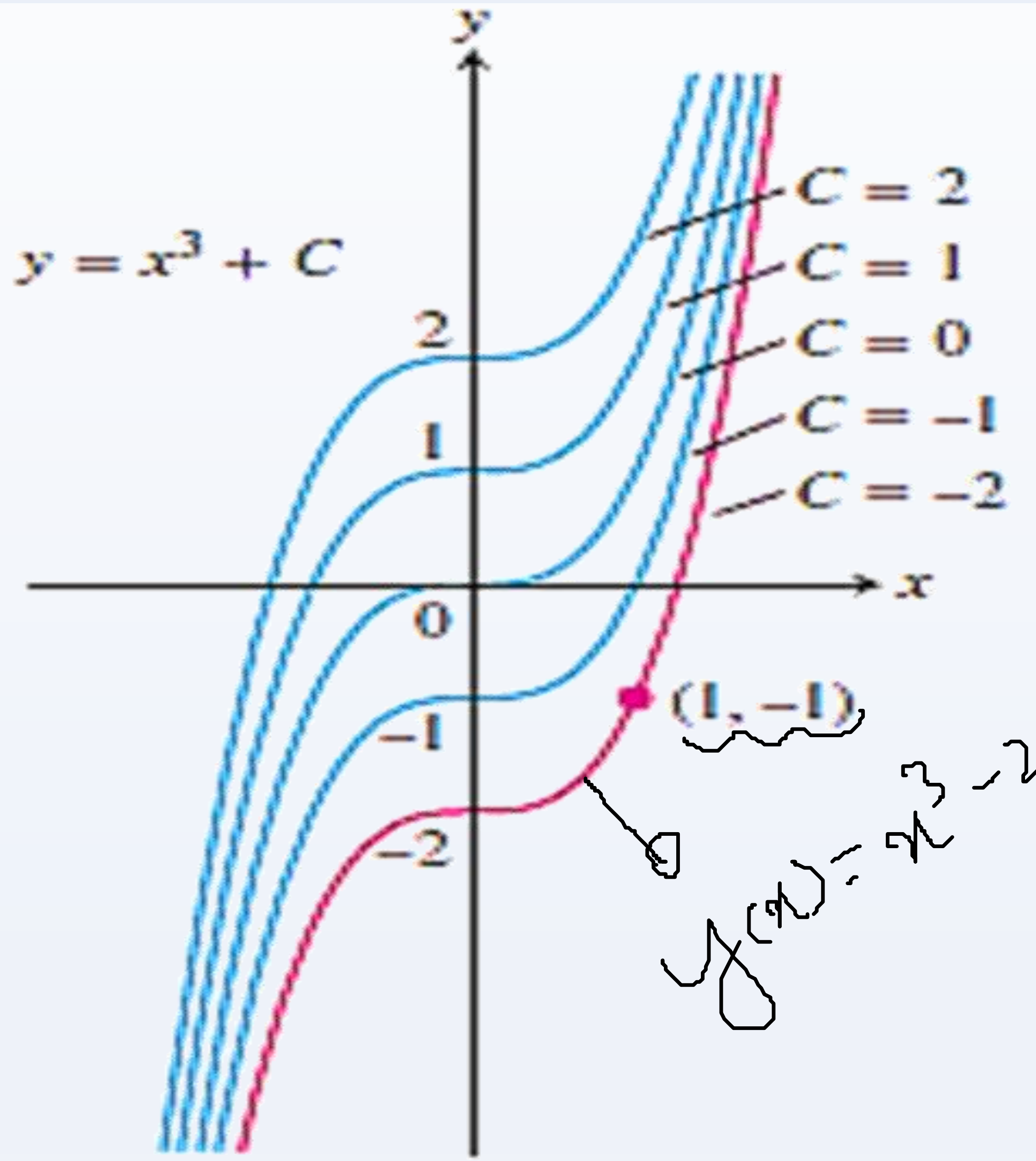
2. Evaluate C:

$$y(x) = x^3 + C.$$
$$-1 = (1)^3 + C$$
$$\Rightarrow C = -2. \checkmark$$

The required curve is: $y(x) = x^3 - 2.$

General solution: $y(x) = x^3 + C$

Particular solution $y(x) = x^3 - 2$



$y(x) = x^3 - 2 \checkmark$

Indefinite Integrals

- A special symbol is used to denote the collection of all antiderivatives of a function $f(x)$.
- The set of all antiderivatives of $f(x)$ is the **indefinite integral** of $f(x)$ with respect to x and is denoted by:

variable of integration

$$\int f(x)\,dx$$

This is called the indefinite integral and is the most general antiderivative of $f(x)$.

Integral sign

Integrand

$$\int f(x)\,dx = F(x) + C$$

$$F'(x) = f(x)$$

Indefinite Integrals

Definition:

If $F'(x) = f(x)$, then

$$\int f(x)\,dx = F(x) + C,$$

Constant of Integration

For any real number C.

Finding the Antiderivative

Finding the antiderivative is the reverse of finding the derivative. Therefore, the rules for derivatives leads to a rule for antiderivatives.

$$\frac{d}{dx}\left(x^{n+1}\right) = (n+1)x^{n}$$

Example:

$$\frac{d}{dx}x^5 = 5x^4$$

$$\Rightarrow d\left(x^{n+1}\right) = \left[(n+1)x^{n}\right]dx$$

So,

$$\Rightarrow \int d\left(x^{n+1}\right) = \int (n+1)\,x^{n}\,dx$$

$$\int 5x^4 dx = x^5 + C.$$

$$\Rightarrow x^{n+1} = (n+1)\int x^{n}dx$$

$$\Rightarrow \boxed{\int x^{n}dx = \frac{x^{n+1}}{n+1} + C}$$

Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\frac{x^{n+1}}{n+1} + C, \quad n \neq -1, n$ rational
2.	$\sin kx$	$-\frac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\frac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

Practice

Rules for Antiderivatives

Power Rule:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C\,; \qquad \text{for any real number } n \neq -1.$$

Examples:

1. $\displaystyle\int t^3 dt = \frac{t^{3+1}}{3+1} = \frac{t^4}{4} + C.$

2. $\displaystyle\int \frac{1}{t^2} dt = \int t^{-2} dt = \frac{t^{-1}}{-1} + C = \frac{-1}{t} + C.$

Power Rule:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C\,; \qquad \text{for any real number } n \neq -1.$$

Examples:

$n = 1/2$

3. $$\int \sqrt{u}\; du = \frac{2}{3}u^{\frac{3}{2}} + C.$$

$$\int x^0\, dx = \int 1\, dx$$

$n = ?$

4. $$\int dx = x + C.$$

Rules for Antiderivatives

Constant Multiple and Sum/Difference:

$\int$ is a linear Operator

linearity property

$$\int k \bullet f(x)dx = k\int f(x)dx\,; \quad \text{for any real number } k.$$

$$\int [f(x) \pm g(x)]dx = \int f(x)\,\mathrm{dx} \pm \int g(x)dx$$

Examples:

v – variable

1. $\int 2v^3 dv = 2\int v^3 dv = 2\left(\frac{v^4}{4}\right) + C = \frac{v^4}{2} + C.$

Rules for Antiderivatives

Examples:

2. $$\int \frac{12}{z^5}\,dz = \frac{-3}{z^4} + C.$$

3. $$\int (3z^2 - 4z + 5)\,dz = z^3 - 2z^2 + 5z + C.$$

4. $$\int \left[\frac{x^2+1}{\sqrt{x}}\right] dx = \int \left(\frac{x^2}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx = \int \left(\frac{x^2}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}}\right) dx$$

$$= \int \left(x^{\frac{3}{2}} + x^{\frac{-1}{2}}\right) dx = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + C.$$

Derivatives of Exponential Functions

We know that:

- If $f(x) = e^x$ then $f'(x) = e^x$. ✓
- If $f(x) = a^x$ then $f'(x) = (\ln a)a^x$. ✓
- If $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$. ✓
- If $f(x) = a^{kx}$ then $f'(x) = k(\ln a)a^{kx}$. ✓

- This leads to the following formulas:

Indefinite Integrals of Exponential Functions

$$\int e^x dx = e^x + C$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C; \quad k \neq 0$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int a^{kx} dx = \frac{a^{kx}}{k(\ln a)} + C; \quad k \neq 0$$

Practice

Examples:

1. $$\int 9e^{t} dt = 9\int e^{t}\, dt = 9e^{t} + C.$$

2. $$\int e^{9t} dt = \frac{e^{9t}}{9} + C.$$ ✓

3. $$\int 3e^{\frac{5}{4}u} du = 3\left(\frac{e^{\frac{5}{4}u}}{\frac{5}{4}}\right) + C = 3\left(\frac{4}{5}\right)e^{\frac{5}{4}u} + C = \frac{12}{5}e^{\frac{5}{4}u} + C.$$

Indefinite Integrals of x^{-1}

✓ $$\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + C.$$

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}$$

Note: if x takes on a negative value, then $\ln x$ will be undefined. The absolute value sign keeps that from happening.

Examples:

$$\Rightarrow d\left(\ln|x|\right) = \left(\frac{1}{x}\right)dx$$

1. $$\int \frac{4}{x}dx = 4\int \frac{1}{x}dx = 4\ln|x| + C.$$

$$\Rightarrow \ln|x| + C = \int \frac{1}{x}dx$$

2. $$\int \left(\frac{-5}{x} + e^{-2x}\right)dx = -5\ln|x| - \frac{1}{2}e^{-2x} + C.$$

Practice Questions

Book: Thomas Calculus (11th Edition) by Georg B.Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

Chapter: 4

Exercise: 4.8

Q # 1 to Q # 78, Q # 87, Q # 93 to Q # 100.