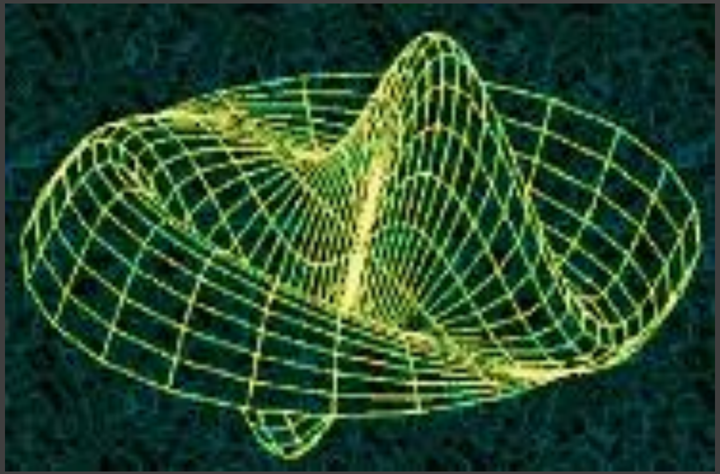
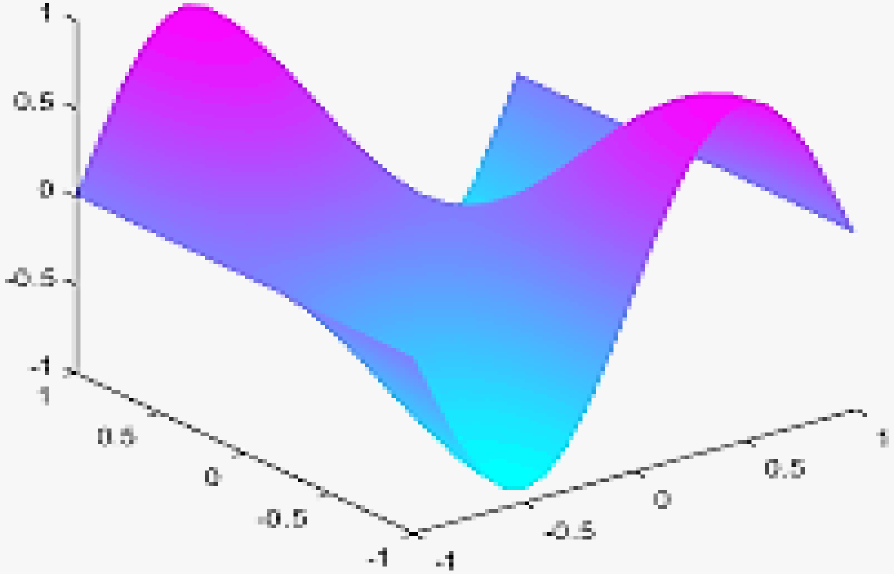
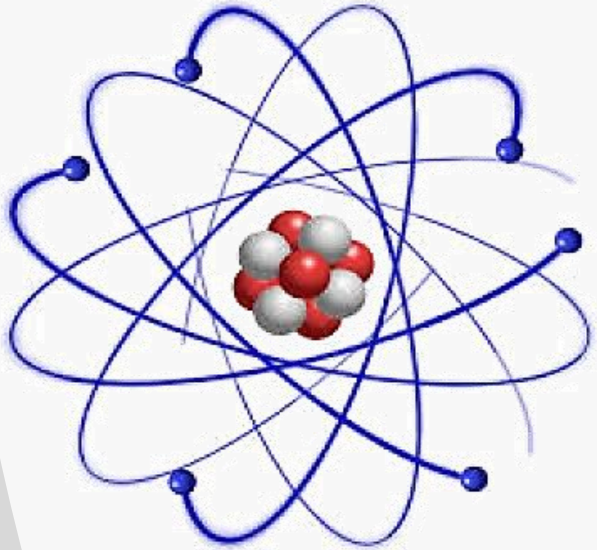


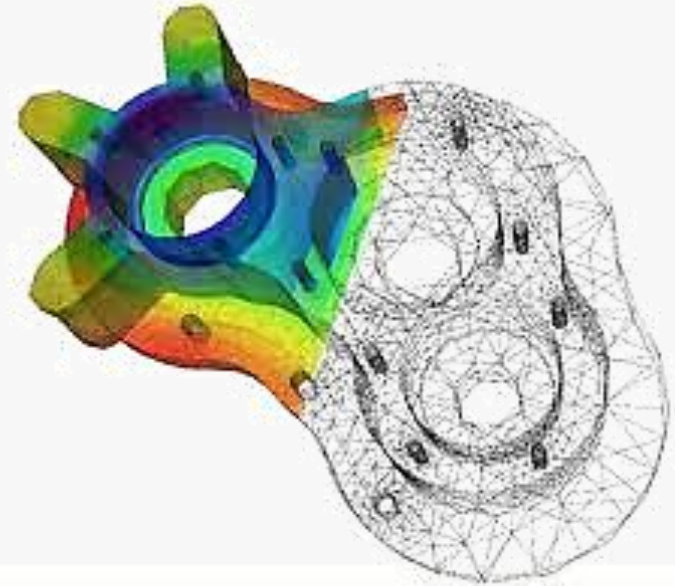
Partial Differential Equations

Vector Calculus(MATH-243)

Instructor: Dr. Naila Amir







Partial Differential Equations

Book: Advanced Engineering Mathematics (9th Edition) by Ervin Kreyszig

- Chapter: 12
  - Sections: 12.6

Book: Linear Partial Differential Equations for Scientists and Engineers (4th Edition) by Lokenath Debnath

- Chapter: 7
  - Sections: 7.1, 7.2

Book: Applied Partial Differential Equations With Fourier series and boundary value problems by Richard Haberman

- Chapter: 2
  - Sections: 2.3, 2.4

Conduction of Heat in a One-Dimensional Rod

- As an important application of the heat equation, let us first consider the temperature in a long thin metal bar of length L or wire of constant cross section and homogeneous material, which is oriented along the x −axis and is perfectly insulated laterally, so that heat flows in the x −direction only and no internal sources of heat, subject to certain boundary and initial conditions.

- Since the surface of the rod is insulated, and therefore, there is no heat loss through the boundary.

- Then besides time t, u depends only on x, so that the Laplacian reduces to $\nabla^2 u = u_{xx}$ and the heat equation becomes the **one-dimensional heat equation.**

- To describe the problem, let $u(x,t)$ $(0 < x < L, t > 0)$ represent the temperature of the point x of the bar at time t. Given that initial temperature distribution of the bar is $u(x,0) = f(x)$, and given that the ends of the bar are held at constant temperature 0, we are required to determine that what is $u(x,t)$ for $0 < x < L, t > O$?

Conduction of Heat in a One-Dimensional Rod

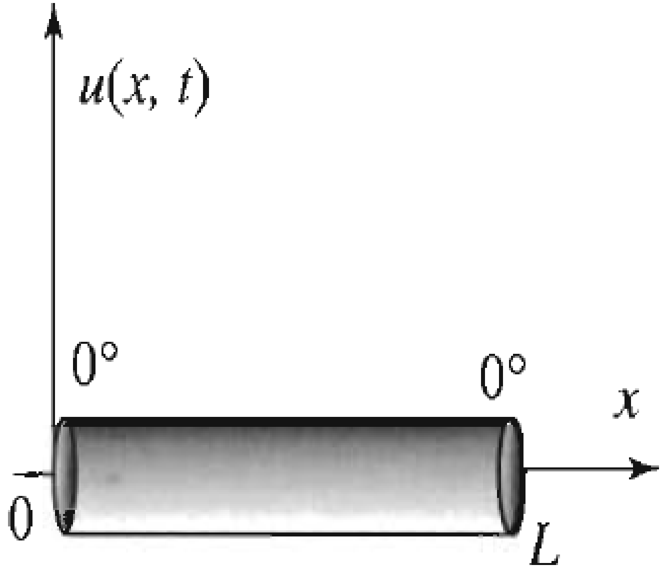
The temperature distribution of the rod is given by the solution of the initial boundary-value problem

$$(1) \qquad \frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, 0 < x < L, \qquad t > 0,$$

$$(2) \qquad u(0,t) = u(L,t) = 0, \quad t > 0, \quad \text{(Boundary conditions)}$$

$$(3) \qquad u(x,0) = f(x), \quad 0 < x < L. \quad \text{(Initial condition)}$$

We shall determine a solution of (1) satisfying (2) and (3)—one initial condition will be enough. Technically, our method will be separation of variables followed by the use of Fourier series.



Insulated bar with ends kept at 0°.

Conduction of Heat in a One-Dimensional Rod

Introducing solution of the form:

$$u(x,t) = X(x)T(t),$$

and substituting into the given I.V.P, we obtain:

$$X(x)T'(t) = c^2X''(x)T(t)\,, \qquad 0 < x < L, \qquad t > 0.$$

this leads to the following equation:

$$\frac{T'(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = \text{Constant} = \alpha.$$

Thus, we get a pair of ODEs given as:

$$T'(t) - c^2\alpha T(t) = 0, \qquad (4)$$

and

$$X''(x) - \alpha X(x) = 0. \qquad (5)$$

Conduction of Heat in a One-Dimensional Rod

(4) Implies hat we are looking for a non-trivial solution $X(x)$, satisfying:

$$X''(x) - \alpha X(x) = 0$$

$$X(0) = X(L) = 0 \quad \text{(Boundary conditions)}$$

We shall consider 3 cases:

$$\alpha = 0,\ \alpha > 0 \text{ and } \alpha < 0.$$

Case (I): $\alpha = 0.$

In this case we have:

$$X''(x) = 0 \Longrightarrow X'(x) = A \Longrightarrow X(x) = Ax + B$$

By using the boundary conditions, we get:

$$X(x) = 0, \text{ trivial solution.}$$

Conduction of Heat in a One-Dimensional Rod

Case (II): $\alpha > 0.$

Let $\alpha = p^2$, then the DE (4) gives:

$$X''(x) - p^2 X(x) = 0.$$

The fundamental solution set is: $\{e^{px}, e^{-px}\}$.

A general solution is given by:

$$X(x) = c_1 e^{px} + c_2 e^{-px}$$

Now

$$X(0) = 0 \Longrightarrow c_1 + c_2 = 0 \Longrightarrow c_1 = -c_2,$$

and

$$X(L) = 0 \Longrightarrow c_1 e^{pL} + c_2 e^{-pL} = 0 ,$$

hence

$$c_1(e^{pL} - e^{-pL}) = 0 \Longrightarrow c_1 = 0 \text{ and so is } c_2 = 0 .$$

Again, we have trivial solution $X(x) \equiv 0$.

Conduction of Heat in a One-Dimensional Rod

Case (III): $\alpha < 0$.

Let $\alpha = -p^2, p > 0$. The DE (4) takes the form:

$$X''(x) + p^2 X(x) = 0.$$

The auxiliary equation is:

$$D^2 + p^2 = 0, \qquad \text{or} \quad D = \pm p\, i\,.$$

The general solution:

$$X(x) = c_1 e^{ipx} + c_2 e^{-ipx}$$

or we prefer to write:

$$X(x) = c_1 \cos(px) + c_2 \sin(px)\,.$$

Now the boundary conditions $X(0) = X(L) = 0$ imply:

$c_1 = 0$ and $c_2 \sin(pL) = 0$. Thus, we have,

$$p = \frac{k\pi}{L} \quad \text{or} \quad \alpha = -\left(\frac{k\pi}{L}\right)^2, \qquad k = 1, 2, 3, \ldots$$

We set $X_k(x) = a_k \sin(k\pi x/L),\ k = 1, 2, 3, \ldots$

Conduction of Heat in a One-Dimensional Rod

Finally, we solve DE (5): $T'(t) - c^2\alpha T(t) = 0 \Longrightarrow T'(t) + \lambda_k^2 T(t) = 0$, where $\lambda_k^2 = \left(\frac{ck\pi}{L}\right)^2$.

This is a first order linear separable ODE. Solutions to above equation are given as:

$$T_k(t) = b_k e^{-\lambda_k^2 t}; \quad k = 1, 2, 3, \ldots$$

Thus, the function $u_k(x,t) = X_k(x)T_k(t)$ becomes:

$$u_k(x,t) = E_k \sin\left(\frac{k\pi x}{L}\right) e^{-\lambda_k^2 t}, \qquad \text{where } E_k = a_k b_k. \quad (6)$$

are solutions of the heat equation satisfying given boundary conditions. These are the **eigenfunctions** of the problem, corresponding to the **eigenvalues:** $\lambda_k = \frac{ck\pi}{L}$. So far, we have solutions (6) satisfying the boundary conditions (2). To obtain a solution that also satisfies the initial condition (3), we consider a series of these eigenfunctions,

$$\Rightarrow u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty} E_k \sin\left(\frac{k\pi x}{L}\right) e^{-\lambda_k^2 t}, \qquad (7)$$

where $\lambda_k^2 = \left(\frac{ck\pi}{L}\right)^2$.

Conduction of Heat in a One-Dimensional Rod

In order to satisfy the initial condition, we must have:

$$u(x,0) = \sum_{k=1}^{\infty} E_k \sin\left(\frac{k\pi x}{L}\right) = f(x).$$

This leads to the question: Is it possible to represent $f(x)$ by the Fourier sine series ?? And the answer is:

Yes!!!

Thus, we have:

$$f(x) = \sum_{k=1}^{\infty} E_k \sin\left(\frac{k\pi x}{L}\right),$$

where,

$$E_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx. \qquad (*)$$

Conduction of Heat in a One-Dimensional Rod

Thus,

$$u(x,t) = \sum_{k=1}^{\infty} E_k \sin\left(\frac{k\pi x}{L}\right) e^{-\lambda_k^2 t}, \qquad (**)$$

is solution of heat equation where the constants are given by equation:

$$E_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right) dx. \qquad (*)$$

Example: Temperature in a bar with ends held at 0° C

A thin bar of length π units is placed in boiling water (temperature 100°C). After reaching $100°$ C throughout, the bar is removed from the boiling water. With the lateral sides kept insulated, suddenly, at time $t = 0$, the ends are immersed in a medium with constant freezing temperature $0°\ C$. Taking $c^2 = 1$, find the temperature $u(x,t)$ for $t > 0$.

Solution: The boundary value problem that we need to solve is:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}\ , 0 < x < \pi, \qquad t > 0\ ,$$

$$u(0,t) = u(\pi,t) = 0\ ,\quad t > 0\ , \quad \text{(Boundary conditions)}$$

$$u(x,0) = 100, \quad 0 < x < \pi. \quad \text{(Initial condition)}$$

From previous slide, solution of a heat equation in one-dimension in general is given as:

$$u(x,t) = \sum_{k=1}^{\infty} E_k \sin\left(\frac{k\pi x}{L}\right) e^{-\lambda_k^2 t}, \qquad (**)$$

where $\lambda_k^2 = \left(\frac{ck\pi}{L}\right)^2$ and $E_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right) dx \quad (*)$.

Example: Temperature in a bar with ends held at 0° C

For the present case $L = \pi$ and $f(x) = 100$. Thus, we have:

$$u(x,t) = \sum_{k=1}^{\infty} E_k \sin(kx)\, e^{-k^2 t}, \qquad (***)$$

where,

$$E_k = \frac{2}{\pi}\int_0^{\pi} 100 \sin(kx)\, dx \Longrightarrow E_k = \frac{200}{k\pi}(1 - \cos k\pi).$$

Note that $\cos k\pi = (-1)^k$, so, $1 - \cos k\pi = 0$, if k is even and $1 - \cos k\pi = 2$, if k is odd. Thus, $(***)$ takes the form:

$$u(x,t) = \frac{400}{\pi}\sum_{k=1}^{\infty} \frac{e^{-(2k-1)^2 t}}{2k-1} \sin(2k-1)x\,.$$

Note that, if we plug a given value oft into the series solution, we obtain a function of x alone. This function gives the temperature distribution of the bar at the given time t. In particular, when $t = 0$, $u(x,0)$ yields the half-range sine series expansion of the initial temperature distribution $f(x)$.

Example:

Determine the temperature $u(x,t)$ in a laterally insulated copper bar 80 cm long if the initial temperature is $100\sin\left(\frac{\pi x}{80}\right)$ $^{\mathrm{o}}\mathrm{C}$ and the ends are kept at $0^{\mathrm{o}}\mathrm{C}$. How long will it take for the maximum temperature in the bar to drop to $50^{\mathrm{o}}\mathrm{C}$? *Physical data for copper:* density $8.92\,\frac{\mathrm{g}}{\mathrm{cm}^3}$, specific heat $0.092\,\frac{\mathrm{cal}}{\mathrm{g}\ ^{\mathrm{o}}\mathrm{C}}$, thermal conductivity $0.95\,\frac{\mathrm{cal}}{\mathrm{cm\ sec}\ ^{\mathrm{o}}\mathrm{C}}$.

Solution:

For the present case the initial condition is given as:

$$u(x,0) = \sum_{k=1}^{\infty} E_k \sin\left(\frac{k\pi x}{L}\right) = f(x) = 100\sin\left(\frac{\pi x}{80}\right).$$

Hence, by inspection we get:

$$E_1 = 100 \ \text{ and } \ E_k = 0 \ \forall\, k \geq 2.$$

Moreover,

$$\lambda_1^2 = \left(\frac{c\pi}{L}\right)^2,$$

where $c^2 = \frac{K}{\sigma\rho} = \frac{0.95}{(0.092)(8.92)} = 1.158\ \frac{\mathrm{cm}^2}{\mathrm{sec}}$.

Solution:

Hence, we obtain:

$$\lambda_1^2 = 0.001785 \text{ sec}^{-1}.$$

The solution is given as:

$$u(x,t) = 100 \sin\left(\frac{\pi x}{80}\right) e^{-0.001785\, t}.$$

Also, $100e^{-0.001785\, t} = 50 \Longrightarrow t = 388 \text{ sec} \approx 6.5 \text{ min.}$

Example:

Find the temperature in a laterally insulated bar of length π whose ends are kept at temperature $0^{\mathrm{o}}\mathrm{C}$, assuming that the initial temperature is:

$$f(x) = \begin{cases} x; & \text{if } \ 0 < x < \dfrac{\pi}{2} \\ \pi - x; & \text{if } \ \dfrac{\pi}{2} < x < \pi \end{cases}$$

Solution:

The boundary value problem that we need to solve is:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}\,, 0 < x < \pi, \qquad t > 0\,,$$

$$u(0,t) = u(\pi,t) = 0\,, \quad t > 0\,, \quad \text{(Boundary conditions)}$$

$$u(x,0) = f(x), \quad 0 < x < \pi. \quad \text{(Initial condition)}$$

where $f(x)$ is the function given in statement of the question.

Solution:

Solution of a heat equation in one-dimension in general is given as:

$$u(x,t)=\sum_{k=1}^{\infty}E_k\sin\left(\frac{k\pi x}{L}\right)e^{-\lambda_k^2 t},\qquad (**)$$

where $\lambda_k^2=\left(\frac{ck\pi}{L}\right)^2$ and $E_k=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx\quad (*)$.

For the present case $L=\pi$ and $f(x)=\begin{cases} x; & \text{if}\quad 0<x<\frac{\pi}{2} \\ \pi-x; & \text{if}\quad \frac{\pi}{2}<x<\pi \end{cases}$ thus we have:

$$E_k=\frac{2}{\pi}\left(\int_0^{\pi/2}x\sin(kx)\,dx+\int_{\pi/2}^{\pi}(\pi-x)\sin(kx)\,dx\right)\Rightarrow E_k=\begin{cases} 0, & (\text{even }\ k) \\ \frac{4}{k^2\pi^2}, & (k=1,5,9,\dots) \\ \frac{-4}{k^2\pi^2}, & (k=3,7,11,\dots) \end{cases}.$$

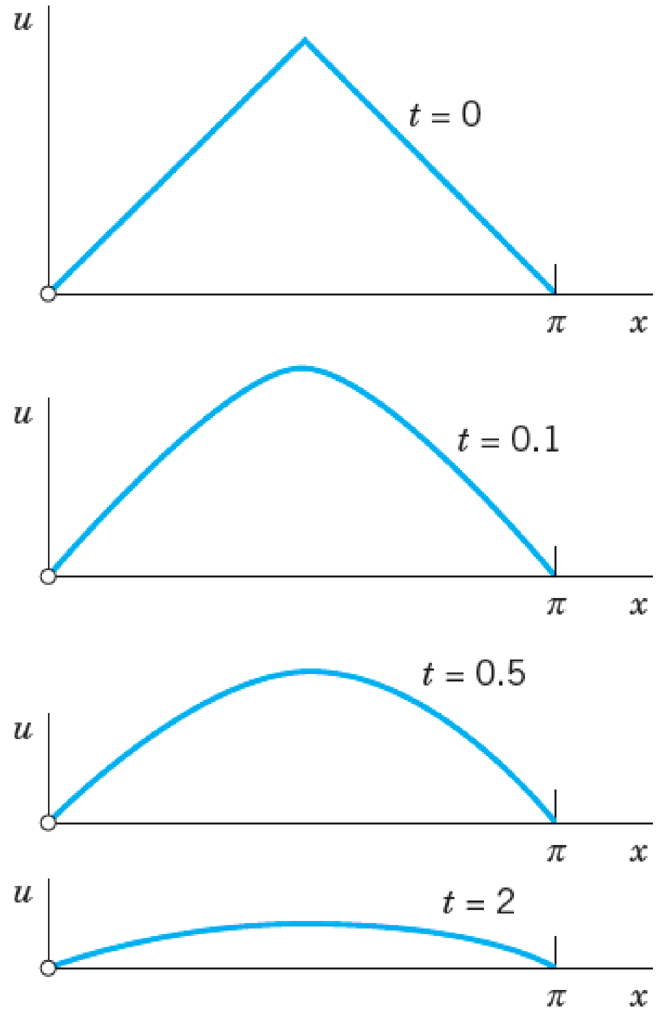
Hence the solution of the given B.V.P. is:

$$u(x,t)=\frac{4}{\pi^2}\left[\sin(x)\,e^{-c^2t}-\frac{\sin(3x)\,e^{-9c^2t}}{9}+\frac{\sin(5x)\,e^{-25c^2t}}{25}-\cdots\right].$$

Solution:

$$u(x,t) = \frac{4}{\pi^2}\left[\sin(x)\, e^{-c^2 t} - \frac{\sin(3x)\, e^{-9c^2 t}}{9} + \frac{\sin(5x)\, e^{-25c^2 t}}{25} - \cdots\right].$$

Note that the temperature is decreasing with increasing t, because of the heat loss due to the cooling of the ends.



Practice Problem: Heat Conduction in a Rod with Insulated Ends

Find the solution of the boundary value problem:

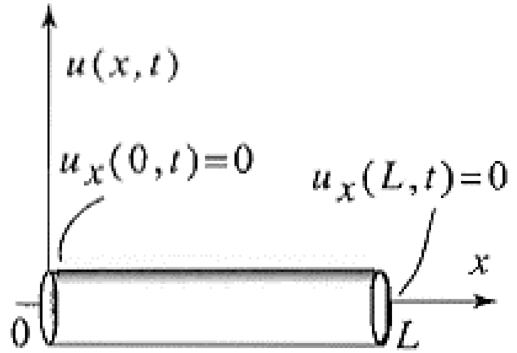
(1) $$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, 0 < x < L, \qquad t > 0,$$

(2) $u_x(0,t) = u_x(L,t) = 0$, $t > 0$, (Boundary conditions)

(3) $u(x,0) = f(x)$, $0 < x < L$. (Initial condition)

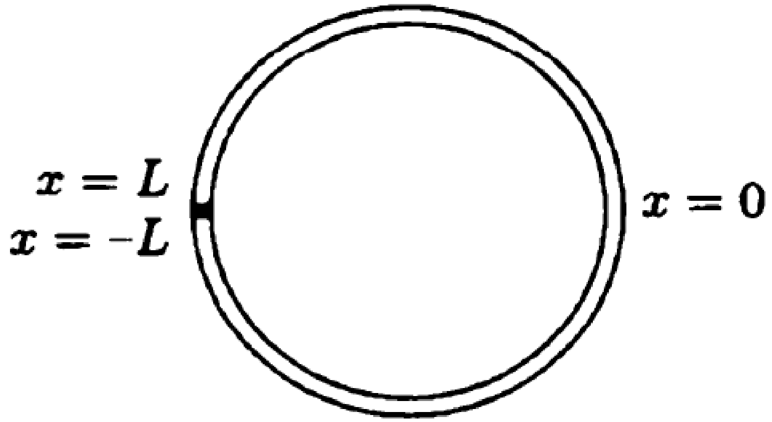
This is a heat conduction problem in a one-dimensional rod with constant thermal properties and no sources. This problem is quite similar to the problem treated earlier, the only difference being the boundary conditions. Here the ends are insulated, whereas in previous case the ends were fixed at 0°. Since, both the partial differential equation and the boundary conditions are linear and homogeneous, so, the method of separation of variables is applicable.

(Note: Solved in class)



Practice Problem: Heat Conduction in a Thin Circular Ring

Let us formulate the appropriate initial boundary value problem if a thin wire (with lateral sides insulated) is bent into the shape of a circle. Suppose that the wire have length $2L$. Since the circumference of a circle is $2\pi r$, the radius is $r = 2L/2\pi = L/\pi$. If the wire is thin enough, it is reasonable to assume that the temperature in the wire is constant along cross sections of the bent wire. In this situation the wire should satisfy a one-dimensional heat equation, where the distance is actually the arc length x along the wire. Let us assume that the wire has constant thermal properties and no sources. Moreover, it is assumed that the wire is very tightly connected to itself at the ends ($x = -L$ to $x = +L$). The conditions of perfect thermal contact should hold there. The temperature $u(x, t)$ is continuous there, Also, since the heat flux must be continuous there (and the thermal conductivity is constant everywhere), the derivative of the temperature is also continuous: These two conditions provide us with the boundary conditions for the partial differential equation. The initial condition is that the initial temperature is a given function of the position along the wire.



Practice Problem: Heat Conduction in a Thin Circular Ring

The mathematical problem consists of the linear homogeneous PDE subject to linear homogeneous BCs and is given as:

$$(1) \qquad \frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}\ , \ -L < x < L, \qquad t > 0\ ,$$

$$(2) \qquad \left.\begin{aligned} &u(-L,t) = u(L,t) \quad \text{and} \\ &u_x(-L,t) = u_x(L,t)\ , \quad t > 0\ , \end{aligned}\right\} \ \text{(Boundary conditions)}$$

$$(3) \qquad u(x,0) = f(x), \quad -L < x < L. \ \ \text{(Initial condition)}$$