FOURIER SERIES AND ITS PROPERTIES

How Do We Find the Fourier Coefficients

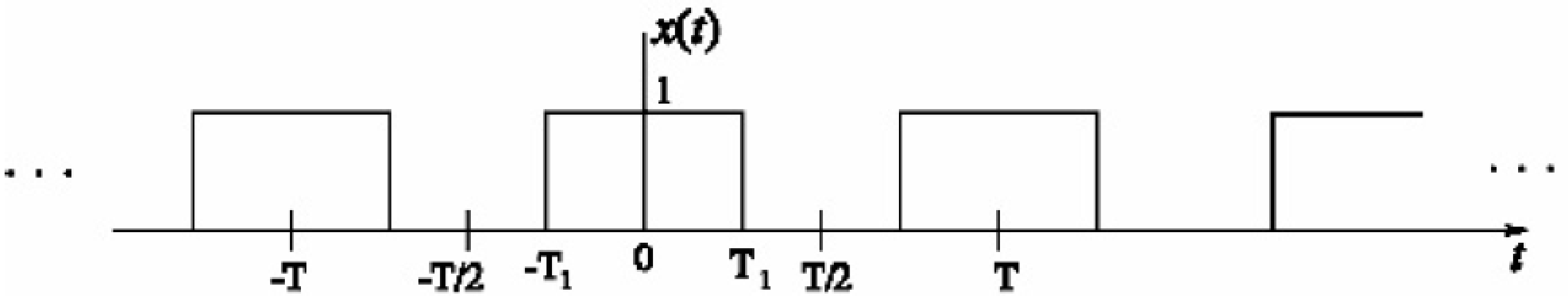
CT Fourier Series Pair $\left(\omega_0 = \frac{2\pi}{T}\right)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt \qquad \text{(Analysis equation)}$$

Fourier Series of Periodic Square Wave

- Determine the Fourier series of the signal below:

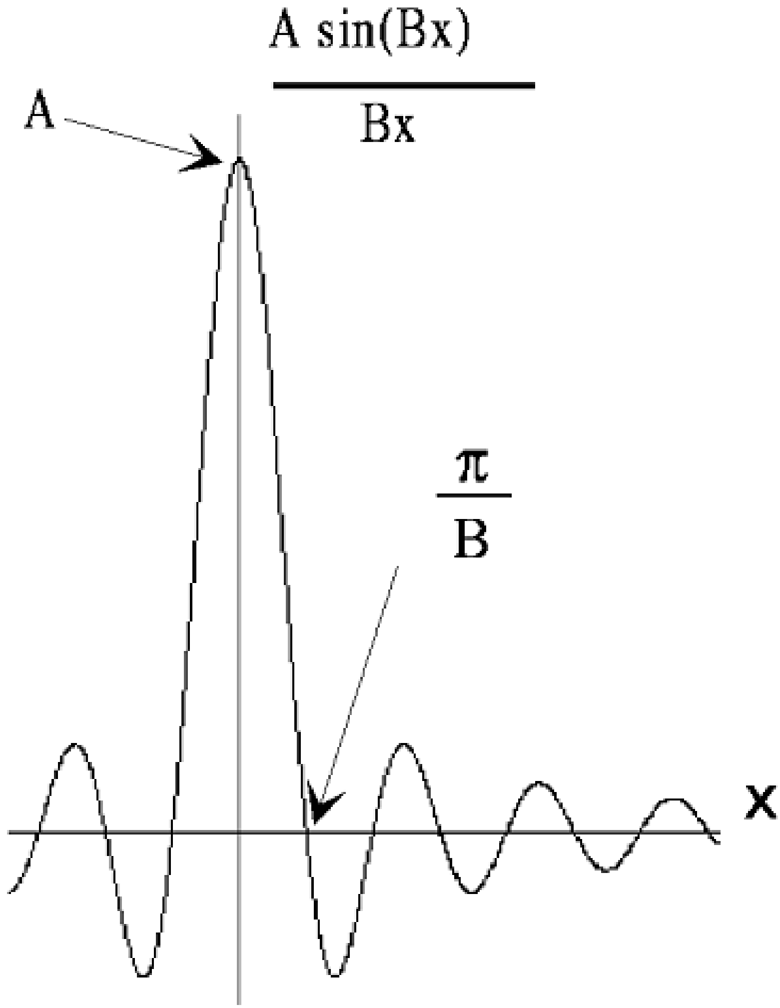


$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

- Signal is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi / T$
- Use interval from $-T/2 \le t < T/2$ as analysis interval (can use any interval of duration T)

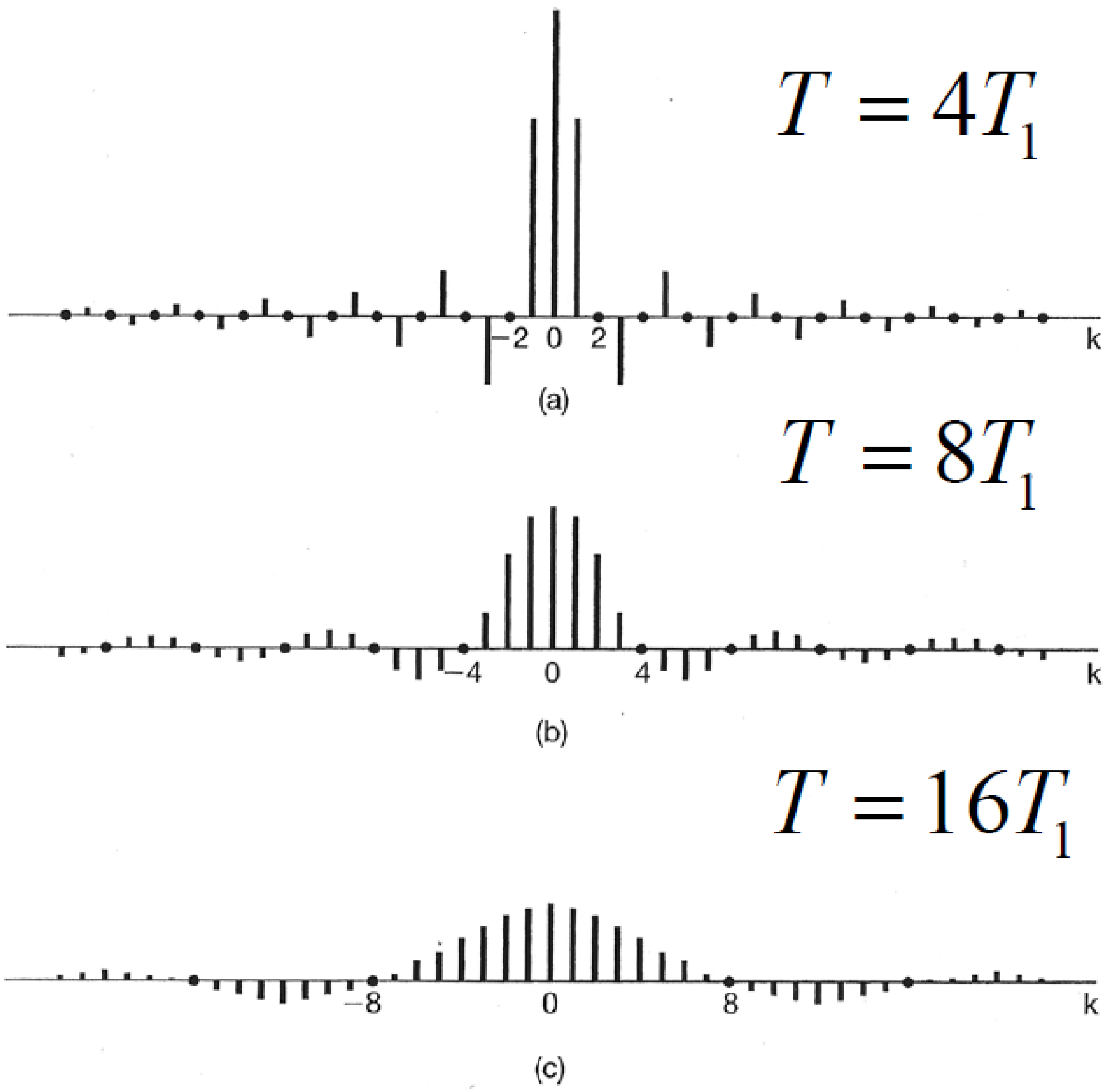
Fourier Series of Periodic Square Wave

- The Fourier series of a square wave is a sinc function as shown below:



Fourier Series of Periodic Square Wave

Magnitude of exponentials



Convergence of Fourier Series

- For a Fourier Series to be convergent, the following three conditions (called Dirichlet conditions) should be met

Condition 1. $x(t)$ is *absolutely integrable* over one period, i.e.,

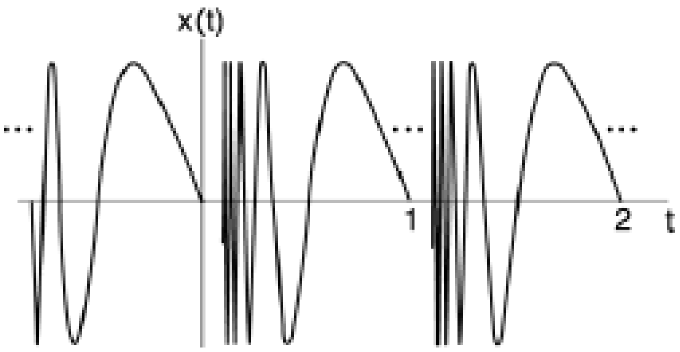
$$\int_T |x(t)|dt < \infty$$

And

Condition 2. In a finite time interval, $x(t)$ has a *finite* number of maxima & minima.

Ex. Example that violates Condition 2.

$$x(t) = \sin\left(\frac{2\pi}{t}\right) \quad 0 < t \leq 1$$



And

Condition 3. In a finite time interval, $x(t)$ has only a *finite* number of discontinuities.

Convergence of Fourier Series

- Dirichlet conditions are met for the signals we will encounter in the real world. Then

  - the Fourier series $= x(t)$ at points where $x(t)$ is continuous

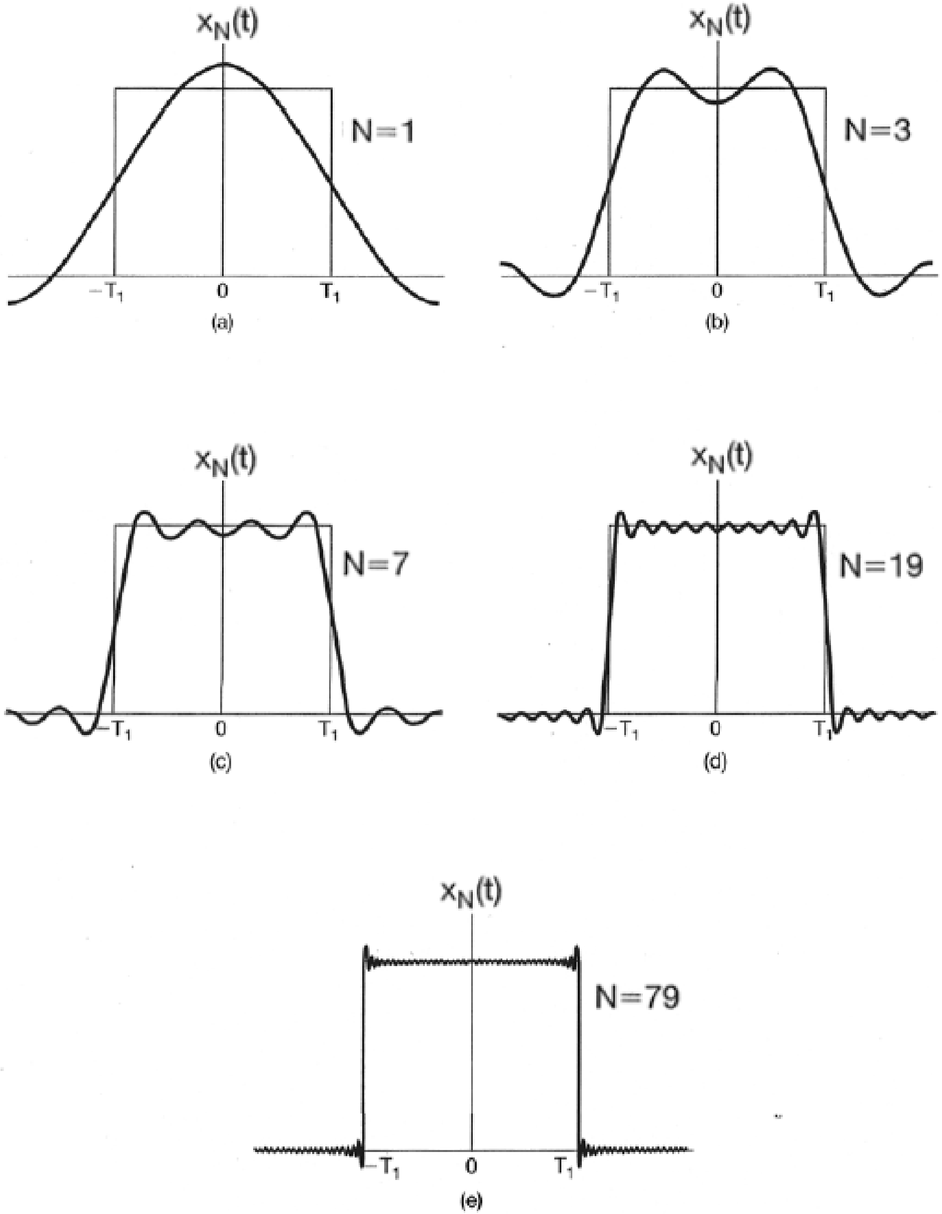
  - the Fourier series = "midpoint" at points of discontinuity

- Still, convergence has some interesting characteristics:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

  - As $N \to \infty$, $x_N(t)$ exhibits *Gibbs*' phenomenon at points of discontinuity

Fourier Series Gibbs Phenomenon



Convergence of the Fourier series representation of a square wave – an illustration of the Gibbs (overshoot) phenomenon.

Properties of CTFS - Linearity

- The properties can be proved by using the Fourier Series equations

- Let $x(t)$ and $y(t)$ be two periodic signals with period T, and with Fourier series coefficients denoted by a_k and b_k respectively.

$$x(t) \xleftrightarrow{FS} a_k$$

$$y(t) \xleftrightarrow{FS} b_k$$

- Any linear combination of the two periodic signals (with period T) must be periodic with period T.
- Furthermore, the Fourier series coefficients, c_k, of the linear combination are a linear combination of the Fourier series components; i.e.,

$$c(t) = Ax(t) + By(t) \xleftrightarrow{FS} c_k = Aa_k + Bb_k$$

Properties of CTFS

- ***Conjugate Symmetry***:

$$x(t) \text{ is real} \quad \Rightarrow \quad a_{-k} = a_k^*$$

$$\Downarrow$$

$$\begin{aligned} a_k &= Re\{a_k\} + jIm\{a_k\} \\ &= |a_k| e^{j\angle a_k} \end{aligned}$$

$$Re\{a_k\} \text{ is even, } Im\{a_k\} \text{ is odd}$$

- ***Time shift***:

$$\begin{aligned} x(t) &\leftrightarrow a_k \\ x(t - t_0) &\leftrightarrow a_k e^{-jk\omega_0 t_0} = a_k e^{-jk2\pi t_0/T} \end{aligned}$$

Introduces a linear phase shift $\propto t_o$

Properties of CTFS – Time Shift

- Time shift a periodic signal, $x(t)$, and the period, T, is preserved

- The Fourier series coefficients, b_k, of the resulting signal, $y(t) = x(t - t_0)$ can be expressed as:

$$b_k = \frac{1}{T}\int_T x(t - t_0)\, e^{-jk\omega_0 t}\, dt$$

- Let $\tau = t - t_0$ and using an interval of length T, gives:

$$\frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0(\tau + t_0)}\, d\tau = e^{-jk\omega_0 t_0}\,\frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau$$

$$= e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k$$

- where a_k is the k-th Fourier series coefficient of $x(t)$

Properties of CTFS – Time Shift

- Thus:

$$x(t) \xleftrightarrow{FS} a_k; \quad x(t-t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k$$

- When a periodic signal is shifted in time, the magnitudes of the Fourier series coefficients remain the same.

Properties of CTFS – Time Reversal

- let $x(t)$ be a periodic signal of period T; then if we have the signal $y(t) = x(-t)$, we can solve for the Fourier Series coefficients, b_k, of $y(t)$ in terms of the Fourier Series coefficients, a_k, of $x(t)$

- using the FS synthesis equation we get:

$$y(t) = x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}$$

Properties of CTFS – Time Reversal

- substituting $k=-m$ we get:

$$y(t)=x(-t)=\sum_{m=-\infty}^{\infty} a_{-m}e^{jm2\pi t/T}=\sum_{m=-\infty}^{\infty} b_m e^{jm2\pi t/T}$$

- we see that we have the relation between the FS coefficients:

  $b_m = a_{-m}$ (or equivalently) $b_k = a_{-k}$

- this implies that if $x(t)$ is even $(x(t)=x(-t))$, then $a_{-k}=a_k$

  $x(t)$ is odd $(x(t)=-x(-t))$, then $a_{-k}=-a_k$

Properties of CTFS – Time Scaling

- if $x(t)$ is periodic with period T (and fundamental frequency $\omega_0 = 2\pi / T$) then the time-scaled signal $y(t) = x(\alpha t)$ (where α is a positive real number) is periodic with period T / α and fundamental frequency $\alpha\omega_0$.

Properties of CTFS – Time Scaling

- it is easy to show that the FS coefficients for $y(t) = x(\alpha t)$ are exactly the same as those for $x(t)$; however since the fundamental frequency has changed, the representation is different

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$y(t) = x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\omega_0 \alpha)t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0' t}$$

END