INTEGRATION

Calculus & Analytical Geometry MATH-101

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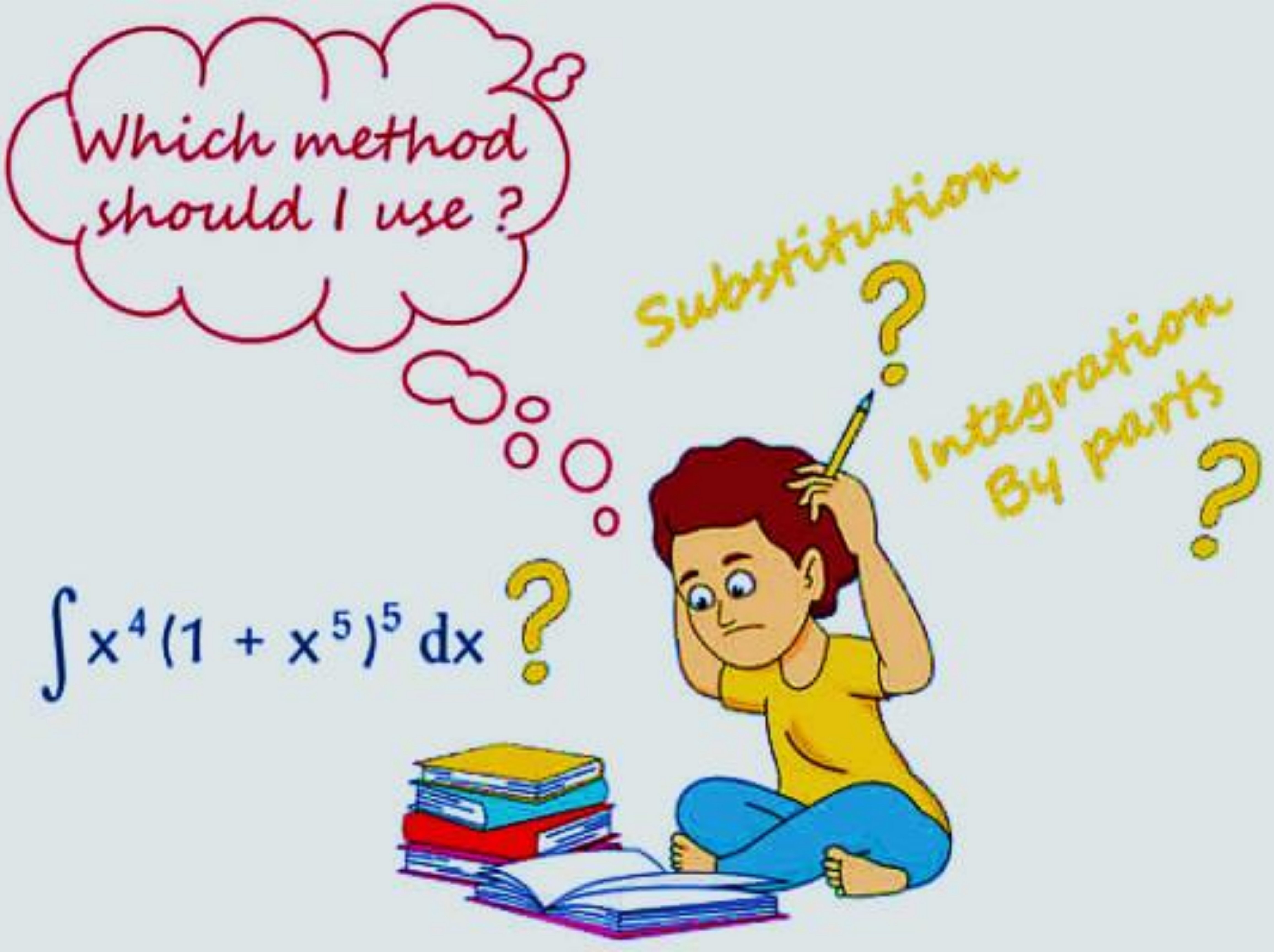
TECHNIQUES OF INTEGRATION



Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter:** 8
    - **Section:** 8.2, 8.3

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

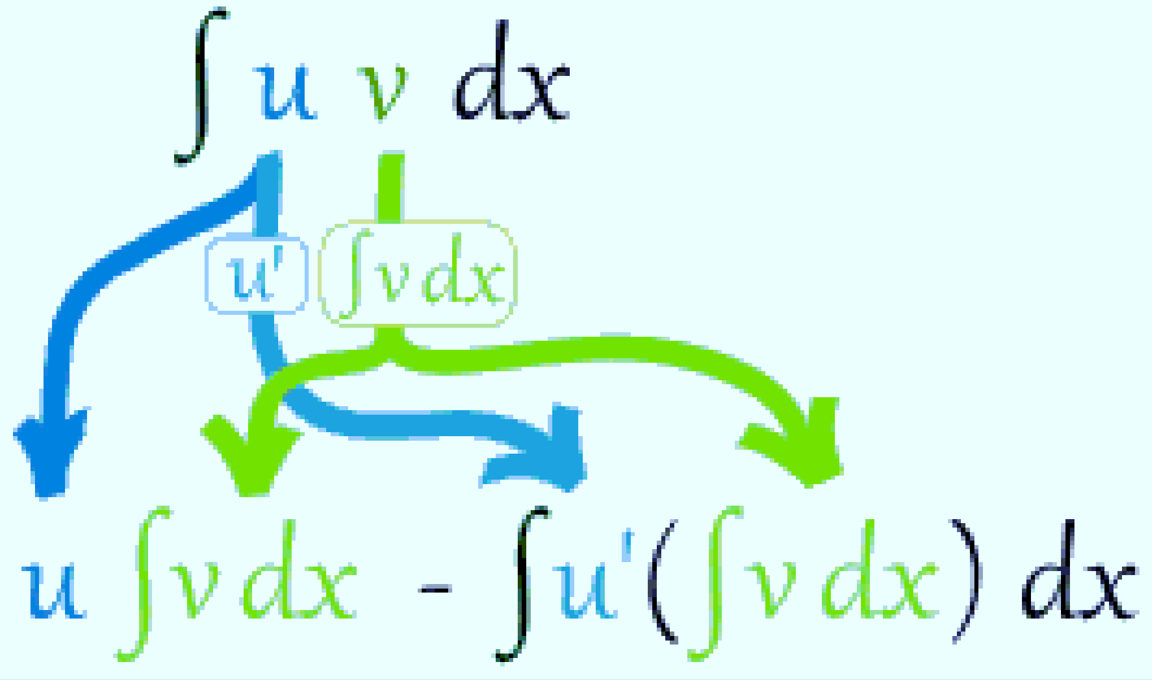
- **Chapter: 9**
    - **Section:** 9.4

Techniques of Integration

- Substitution Rule
- Integration by Parts
- Integration of Rational & Irrational Functions
- Trigonometric Integrals
- Trigonometric Substitution

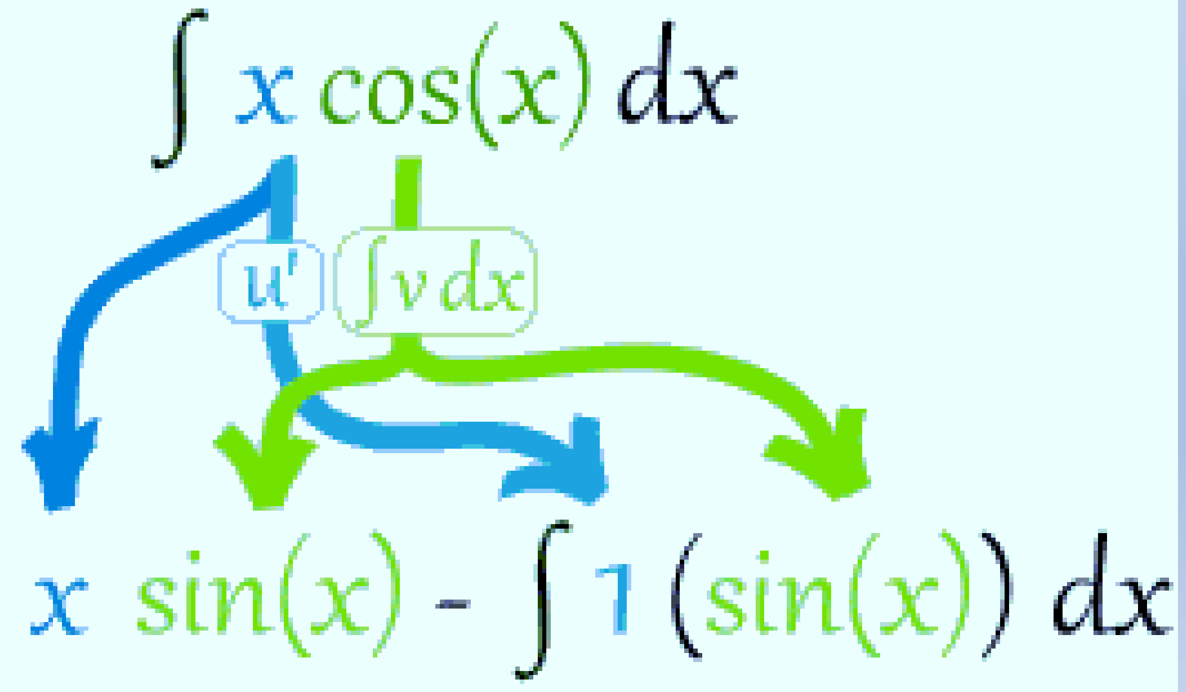
Table of Integration Formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C \qquad$ (any number k)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \dfrac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{u}{a}\right| + C$
21. $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (a > 0)$
22. $\int \dfrac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (u > a > 0)$



8.2 Integration by Parts

Reduction Formula



Integration by parts

$\int f(x)\, g(x)\, dx = f(x) \int g(x)\, dx - \int \left[\int g(x)\, dx \cdot f'(x) \right] dx$

- We can rearrange this equation as:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx. \qquad \text{(I)}$$

This equation gives us the formula for **integration by parts.**

- It is perhaps easier to remember this formula in the following notation:
- Let $u = f(x)$ and $v = g(x)$. Then, the differentials are:

$$du = f'(x)dx \quad \text{and} \quad dv = g'(x)dx.$$

Thus, by the Substitution Rule, the formula for integration by parts becomes:

$$\int udv = uv - \int vdu\,. \qquad \text{(II)}$$

Practice questions of Reduction Formula

Evaluate the following:

- $\int (\cos x)^n dx$. ✓
- $\int (\tan x)^n dx$. ✓

  Hint: $\int (\tan x)^n dx = \int (\tan x)^{n-2} (\tan x)^2 dx$.
- $\int (\sec x)^n dx$.

  Hint: $\int (\sec x)^n dx = \int (\sec x)^{n-2} (\sec x)^2 dx$.
- $\int x^n e^{ax} dx$.
- $\int x^m (\ln x)^n dx$, where m and n are positive integers.

Example:

Evaluate the following integral:

$$\int \tan^n x \, dx \, .$$

Solution:

$$\int \tan^n x \, dx = \int \tan^{n-2} x \cdot \tan^2 x \, dx$$

$$= \int \tan^{n-2} x \, (\sec^2 x - 1) \, dx$$

$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx \longrightarrow (1)$$

Now let $\tan x = u \Rightarrow \sec^2 x \, dx = du$

Thus,

$$\int \tan^{n-2} x \sec^2 x \, dx = \int u^{n-2} \, du$$

$$= \frac{u^{n-1}}{n-1} = \frac{\tan^{n-1} x}{n-1} \rightarrow (2)$$

Using (2) in (1)

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

Practice

Determine reduction formula

for $\int \tan^n x$

by using Integration by parts

Integration
of
Rational functions

- Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano
  - Chapter # 8 (Section: 8.3) ✓
- Calculus (5th Edition) by Swokowski, Olinick and Pence
  - Chapter # 9 (Section: 9.4) ✓

Rational Functions

Definition: A function of the type $P(x)/Q(x)$, where both $P(x)$ and $Q(x)$ are polynomials, and $Q(x) \neq 0$, is a **rational function**.

Example: $\dfrac{x^3+1}{x^2+x+1}$ is a rational function. ✓

The degree of the denominator of the above rational function is less than the degree of the numerator. First we need to rewrite the above rational function in a simpler form by performing polynomial division.

Rewriting

$$\frac{x^3+1}{x^2+x+1} = x - 1 + \frac{2}{x^2+x+1}.$$

For integration, it is always necessary to perform polynomial division first, if possible. To integrate the polynomial part is easy. Thus, polynomial division is the **first step** when integrating rational functions.

Partial Fraction Decomposition

The **second step** is to factor the denominator $Q(x)$ as far as possible.

For instance, if $Q(x) = x^4 - 16$ then

$$Q(x) = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$$

The **third step** is to express the proper rational function as a sum of *partial fractions* of the form:

$$\frac{A}{(ax+b)^k} \quad \text{or} \quad \frac{(Ax+B)}{(ax^2+bx+c)^k}$$

Example:

$$\frac{3x^2 + 3x + 2}{x^3 + x^2 + x + 1} = \frac{1}{1+x} + \frac{2x+1}{1+x^2}$$

$Q(x) = x^3 + x^2 + x + 1$

$= (1+x)(1+x^2)$

$\frac{3x^2+3x+2}{x^3+x^2+x+1} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$

The **fourth step** is to integrate the partial fractions.

Partial Fraction Decomposition

The partial fraction decomposition of a rational function $R(x) = P(x)/Q(x), Q(x) \neq 0$, with $\deg(P(x)) < \deg(Q(x))$ (*proper fraction*), depends on the factors of the denominator $Q(x)$. It may have following types of factors:

1. Simple, non-repeated linear factors $ax + b$. ✓
2. Repeated linear factors of the form $(ax + b)^k, k > 1$. ✓
3. Simple, non-repeated quadratic factors of the type $ax^2 + bx + c$. ✓
4. Repeated quadratic factors $(ax^2 + bx + c)^k$, $k > 1$. ✓

We will consider each of these four cases separately.

Simple Linear Factors

Case I:

Consider a rational function of the type:

$$\frac{\mathrm{P}(x)}{\mathrm{Q}(x)} = \frac{\mathrm{P}(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)}$$

where $a_j \neq 0$ for all $j, \dfrac{b_i}{a_i} \neq \dfrac{b_j}{a_j}$ for $i \neq j$, and $\deg(\mathrm{P}) < n, \deg(\mathrm{Q}) = n$.

Partial Fraction Decomposition: Case I

$$\frac{\mathrm{P}(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

for some uniquely defined numbers $A_k, k = 1, \ldots, n$.

Simple Linear Factors

$Ax^2 + Bx + C = ax^2 + bx + c$

only if equal

$A = a$
$B = b$
$C = c$

Example:

Consider a rational function:

$$\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)}.$$

By the result concerning Case I we can find numbers A and B such that

$$\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Compute these numbers in the following way:

$0x + 2 = 2$

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \Leftrightarrow \frac{2}{x^2-1} = \frac{A(x+1)}{(x-1)(x+1)} + \frac{B(x-1)}{(x+1)(x-1)}$$

$$\Leftrightarrow \frac{0\,x+2}{x^2-1} = \frac{(A+B)x+(A-B)}{x^2-1} \Leftrightarrow \begin{cases} A+B=0 \\ A-B=2 \end{cases} \Leftrightarrow \begin{cases} A=1 \\ B=-1 \end{cases}$$

To get the equations for A and B we use the fact that two polynomials are the same if and only if their coefficients are the same.

So the partial fraction decomposition is: $\dfrac{2}{x^2-1} = \dfrac{1}{x-1} - \dfrac{1}{x+1}.$

Repeated Linear Factors

Case II:

Consider a rational function of the type:

$$\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}; \quad \deg(\mathrm{P}) < \deg(\mathrm{Q}).$$

Assume that the denominator $\mathrm{Q}(x)$ has a repeated linear factor $(ax+b)^k, k > 1$.

Partial Fraction Decomposition: Case II

The repeated linear factor $(ax+b)^k$ of the denominator leads to terms of the type:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

in the partial fraction decomposition.

Repeated Linear Factors

Example:

The rational function:

$$\frac{4x^2+4x-4}{x^3+x^2-x-1} = \frac{4x^2+4x-4}{(x-1)(x+1)^2}$$

has a partial fraction decomposition of the type:

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}.$$

Thus,

$$\frac{4x^2+4x-4}{x^3+x^2-x-1} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

$$\Leftrightarrow \frac{4x^2+4x-4}{x^3+x^2-x-1} = \frac{A(x+1)(x-1)+B(x-1)+C(x+1)^2}{(x-1)(x+1)^2}$$

Repeated Linear Factors

Example:

$$\Leftrightarrow \frac{4x^2+4x-4}{x^3+x^2-x-1} = \frac{A(x+1)(x-1)+B(x-1)+C(x+1)^2}{(x-1)(x+1)^2}$$

$$\Leftrightarrow \frac{4x^2+4x-4}{x^3+x^2-x-1} = \frac{(A+C)x^2+(B+2C)x-A-B+C}{(x-1)(x+1)^2}$$

$$\Leftrightarrow \begin{cases} A+C=4 \checkmark \\ B+2C=4 \checkmark \\ -A-B+C=-4 \checkmark \end{cases} \Leftrightarrow \begin{cases} A=3 \checkmark \\ B=2 \checkmark \\ C=1 \checkmark \end{cases}$$

Equate the coefficients of the numerators.

We get

$$\frac{4x^2+4x-4}{x^3+x^2-x-1} = \frac{3}{x+1} + \frac{2}{(x+1)^2} + \frac{1}{x-1}.$$

Simple Quadratic Factors

Case III:

Consider a rational function of the type:

$$\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}; \quad \deg(\mathrm{P}) < \deg(\mathrm{Q}).$$

Assume that the denominator $\mathrm{Q}(x)$ has a quadratic factor: $ax^2 + bx + c$.

Partial Fraction Decomposition: Case III

The quadratic factor $ax^2 + bx + c$ of the denominator leads to a term of the type

$$\frac{Ax + B}{ax^2 + bx + c}$$

in the partial fraction decomposition.

Simple Quadratic Factors

Example:

The rational function:

$$\frac{3}{x^3 - 1} = \frac{3}{\underbrace{(x-1)}_{L}\underbrace{(x^2 + x + 1)}_{Q}}$$

has a term of the type $\frac{Ax+B}{x^2+x+1}$ in its partial fraction decomposition. Thus,

$$\frac{3}{x^3 - 1} = \frac{Ax + B}{x^2 + x + 1} + \frac{C}{x - 1}$$

$$\Leftrightarrow \frac{3}{x^3 - 1} = \frac{(Ax + B)(x - 1) + C(x^2 + x + 1)}{(x - 1)(x^2 + x + 1)}$$

$0x^2 + 0x + 3$

$$\Leftrightarrow \frac{3}{x^3 - 1} = \frac{(A + C)x^2 + (C + B - A)x + C - B}{x^3 - 1}$$

Simple Quadratic Factors

Example:

$$\Leftrightarrow \frac{3}{x^3-1} = \frac{(A+C)x^2 + (C+B-A)x + C - B}{x^3-1}$$

$$\Leftrightarrow \begin{cases} A + C = 0 \\ C + B - A = 0 \\ C - B = 3 \end{cases}$$

To get these equations use the fact that the coefficients of the two numerators must be the same.

$$\Leftrightarrow \begin{cases} A = -1 \\ B = -2 \\ C = 1 \end{cases}$$

Hence

$$\frac{3}{\mathrm{x}^3-1} = \frac{1}{x-1} - \frac{x+2}{x^2+x+1}$$

Repeated Quadratic Factors

Case IV:

Consider a rational function of the type:

$$\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}; \quad \deg(\mathrm{P}) < \deg(\mathrm{Q}).$$

Assume that the denominator $\mathrm{Q}(x)$ has a repeated quadratic factor: $(ax^2+bx+c)^k, k>1$.

Partial Fraction Decomposition: Case IV

The repeated quadratic factor $(ax^2+bx+c)^k$ of the denominator leads to terms of the type:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

in the partial fraction decomposition.

Repeated Quadratic Factors

Example:

The rational function:

$$\frac{2x^4+3x^2-x}{x^5-x^4+2x^3-2x^2+x-1}=\frac{2x^4+3x^2-x}{(x-1)(x^2+1)^2}$$

Practice

has a partial fraction decomposition of the type $\frac{A_1x+B_1}{x^2+1}+\frac{A_2x+B_2}{(x^2+1)^2}+\frac{C}{x-1}$. Thus,

$$\frac{A_1x+B_1}{x^2+1}+\frac{A_2x+B_2}{(x^2+1)^2}+\frac{C}{x-1}=\frac{(A_1x+B_1)(x^2+1)(x-1)+(A_2x+B_2)(x-1)+C(x^2+1)^2}{(x-1)(x^2+1)^2}$$

Computing in the same way as before we get: $A_1=B_1=A_2=C=1$, and $B_2=0$. Hence

$$\frac{2x^4+3x^2-x}{x^5-x^4+2x^3-2x^2+x-1}=\frac{x+1}{x^2+1}+\frac{x}{(x^2+1)^2}+\frac{1}{x-1}.$$

Integrating Partial Fraction Decompositions

After a general partial fraction decomposition one has to deal with integrals of the following types. There are four cases. Two first cases are easy.

1. $$\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C.$$ Here C is the constant of integration.

2. $$\int \frac{A}{(ax+b)^k}\,dx = \frac{A}{a}\left(\frac{(ax+b)^{1-k}}{1-k}\right) + C, k \neq 1.$$

In the remaining cases we have to compute integrals of the type:

3. $$\int \frac{Ax+B}{ax^2+bx+c}\,dx$$ and 4. $$\int \frac{Ax+B}{(ax^2+bx+c)^k}\,dx, k > 1.$$

Example: Compute $\int \frac{3}{x^3-1}dx$.

Observe that $x^3 - 1 = (x-1)(x^2+x+1)$. Hence

$$\frac{3}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

for some numbers A, B and C. To compute these numbers A, B and C we get

$$\frac{3}{x^3-1} = \frac{A(x^2+x+1)}{(x-1)(x^2+x+1)} + \frac{(Bx+C)(x-1)}{(x^2+x+1)(x-1)}$$

$$\Leftrightarrow \frac{3}{x^3-1} = \frac{(A+B)x^2+(A-B+C)x+A-C}{x^3-1}$$

$$\Leftrightarrow \begin{cases} A+B=0 \\ A-B+C=0 \\ A-C=3 \end{cases} \Leftrightarrow \begin{cases} A=1 \\ B=-1. \\ C=-2 \end{cases}$$

Hence

$$\frac{3}{x^3-1} = \frac{1}{x-1} - \frac{x+2}{x^2+x+1} \qquad (1)$$

Integrating both sides of (1) w.r.t x, we get

$$\int \frac{3}{x^3-1}dx = \int \frac{1}{x-1}dx - \int \frac{x+2}{x^2+x+1}dx$$

$$= \ln|x-1| - \frac{1}{2}\int \frac{2x+4}{x^2+x+1}dx$$

$$= \ln|x-1| - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{3}{2}\int \frac{1}{x^2+x+1}dx$$

$$= \ln|x-1| - \frac{1}{2}\ln(x^2+x+1) - \frac{3}{2}\int \frac{1}{(x+1/2)^2+3/4}dx$$

$$= \ln|x-1| - \frac{1}{2}\ln(x^2+x+1) - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C. \quad \left[\because \int \frac{dx}{1+x^2} = \arctan x\right]$$

$$\frac{d}{dx}(x^2+x+1) = 2x+1$$

$$2x+4 = (2x+1)+3$$

$$\frac{2x+4}{x^2+x+1} = \frac{2x+1}{x^2+x+1} + \frac{3}{x^2+x+1}$$

$$\int\left(\frac{a}{ax+b}\right)dx = \ln|ax+b|$$

Practice

$$-\frac{3}{2}\times\frac{4}{3}\int\frac{1}{1+\frac{4}{3}(x+\frac{1}{2})^2}dx$$

Example: Compute $\int \frac{x^3 - x + 2}{x^2 - 1} dx$.

We can simplify the function to be integrated by performing polynomial division first. This needs to be done whenever possible. We get:

$$\frac{x^3 - x + 2}{x^2 - 1} = x + \frac{2}{x^2 - 1}$$

$$\frac{2}{x^2-1} = \frac{2}{(x-1)\ (x+1)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1}$$

Partial fraction decomposition for the remaining rational expression leads to

$$\frac{x^3 - x + 2}{x^2 - 1} = x + \frac{2}{x^2 - 1} = x + \frac{1}{x - 1} - \frac{1}{x + 1}$$

$$\Rightarrow 2 = A(x+1) + B(x-1)$$

At $x = 1$

$$2 = 2A \Rightarrow A = 1$$

at $x = -1$

$$2 = -2B \Rightarrow B = -1$$

Now we can integrate

$$\ln|a| - \ln|b| = \ln\left(\frac{a}{b}\right)$$

$$\int \frac{x^3 - 1 + 2}{x^2 - 1} dx = \int \left(x + \frac{1}{x - 1} - \frac{1}{x + 1} \right) dx$$

$$= \frac{x^2}{2} + \ln|x - 1| - \ln|x + 1| + C = \frac{x^2}{2} + \ln\left|\frac{x - 1}{x + 1}\right| + C.$$

Practice Questions

Book: Thomas Calculus (11th Edition) by Georg B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Exercise: 8.3
  Q # 1 to Q # 34. ✓

Book: Calculus (5th Edition) by Swokowski, Olinick and Pence

- Exercise: 9.4
  Q # 1 to Q # 32. ✓