Complex Analysis Dennis G. Zill) Example

Chapter I:

Complex numbers and Complex Plane:

$$z = a + ib$$

$$i = \sqrt{-1}$$

$$\left.\begin{aligned} Re(z) &= a \\ Im(z) &= b \end{aligned}\right\}$$

Zer

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$$

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

$$\frac{z_1}{z_2} = \frac{(a_1 a_2 + b_1 b_2)}{a_2^2 + b_2^2} + i\,\frac{(a_2 b_1 - a_1 b_2)}{a_2^2 + b_2}$$

$$\left.\begin{aligned} z_1 + z_2 &= z_2 + z_1 \\ z_1 z_2 &= z_2 z_1 \end{aligned}\right\} \text{Commutative}$$

$$\left.\begin{aligned} z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 \\ z_1(z_2 z_3) &= (z_1 z_2) z_3 \end{aligned}\right\} \text{Associative}$$

$$\left.\begin{aligned} z_1(z_2 + z_3) &= z_1 z_2 + z_1 z_3 \\ z_1(z_2 + z_3) &= (z_2 + z_3) z_1 \end{aligned}\right\} \text{Distributive.}$$

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xample: $z_1 = 2+4i$
$z_2 = -3+8i$

$z_1+z_2 = (2-3) + i(4+8) = -1+12i$

$z_1 z_2 = [(2)(-3)-(4)(8)] + i[(2)(8)+(4)(-3)]$

$z_1 z_2 = [-6-32] + i[16-12]$

$z_1 z_2 = -38 + i(4) = -38+4i$

Zero and Unity:

$z+0 = 0+z = z$

$0 = 0+0i$

$z.1 = 1.z = z$

Conjugate:

$\bar{z} = a-ib$

$z = a+ib$

$\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$ $\quad z\bar{z} = |z|^2 = a^2+b^2$.

$\overline{z_1-z_2} = \bar{z}_1 - \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

ociative $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$; $\quad \bar{\bar{z}} = z$

butive. $z+\bar{z} = 2a$.

$z-\bar{z} = 2bi$

$z\bar{z} = a^2+b^2$.

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$$\left.\begin{aligned} z_1 &= 2-3i \\ z_2 &= 4+6i \end{aligned}\right\} \quad \frac{z_1}{z_2};$$

$$\frac{z_1}{z_2} = \frac{2-3i}{4+6i} = \frac{(2-3i)(4-6i)}{4^2+6^2}$$

$$\frac{z_1}{z_2} = \frac{[(2)(4)-(-3)(-6)]+i[(2)(-6)+(-3)(4)]}{16+36}$$

$$\frac{z_1}{z_2} = \frac{(8-18)+i(-12-12)}{52}$$

$$\frac{z_1}{z_2} = \frac{-10-24i}{52}$$

$$\frac{z_1}{z_2} = \frac{-10-24i}{2(26)} = \frac{-5-12i}{(26)}$$

$$\frac{z_1}{z_2} = -\frac{5}{26} - \frac{6}{13}i \Big\}$$

Inverse:

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{a^2+b^2} \Big\}.$$

$$z = 2-3i$$

$$\frac{1}{z} = \frac{2+3i}{(2)^2+(3)^2}$$

$\frac{1}{z}$

$|z| =$

$|z|$ is

abs

$d = |z_2$

$\sqrt{(x_2}$

$|z_1 z_2| =$

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$$\frac{1}{z} = \frac{2+3i}{4+9} = \frac{2+3i}{13} = \left(\frac{2}{13}\right) + i\left(\frac{3}{13}\right).$$

Exercise 1.1

1) (a) i^8

$$(i^2)^4 = (-1)^4 = 1.$$

(b) i^{11}

$$= i\,i^{10} = i(i^2)^5 = i(-1)^5 = -i$$

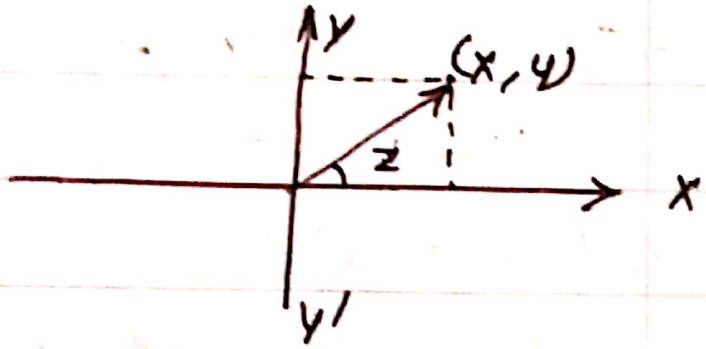
(c) $i^{42} = (i^2)^{21} = (-1)^{21} = -1.$

(d) $i^{105} = i\,i^{104} = i(i^2)^{52} = i(-1)^{52}$

$$i^{105} = i$$

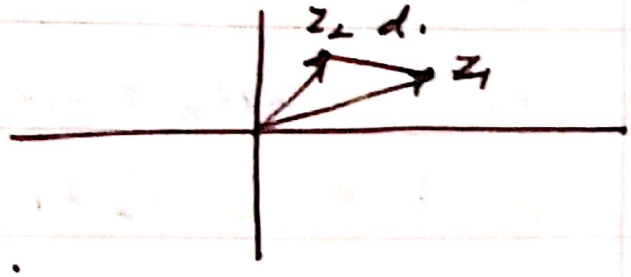
Complex Plane (Argan Diagram).

$$z = x + iy$$



$$|z| = \sqrt{x^2 + y^2}$$

$|z|$ is modulus/absolute of z.



$$d = |z_2 - z_1| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}.$$

$$|z_1 z_2| = |z_1||z_2|, \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad |z^2| = |z|^2$$

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Triangular Inequality:

$$\left.\begin{aligned} |z_1+z_2| &\le |z_1|+|z_2| \\ |z_1|-|z_2| &\le |z_1+z_2| \end{aligned}\right\}$$

$$\left|\frac{-1}{z^4+3z^2+2}\right| \quad \text{if } |z|=2;$$

$$\cancel{\frac{|-1|}{|z^4+3z^2+2|} \le M}$$

$$\cancel{\frac{1}{|z|^4+3|z|^2+2} \le M}$$

$$\cancel{\frac{1}{(2)^4+3(2)^2+2} \le M}$$

$$\cancel{M \ge \frac{1}{2+12+16}}$$

$$\cancel{M = \frac{1}{30}}$$

$$z^4+3z^2+2.$$

$$z^4+z^2+2z^2+2.$$

$$z^2(z^2+1)+2(z^2+1).$$

$$(z^2+1)(z^2+2).$$

$$\frac{1}{|(z^2+1)(z^2+2)|} \le M \longrightarrow |z^2+1|\,|z^2+2|.$$

$$|z_1+z_2| \ge |z_1|-|z_2|.$$

$$|z^2+1| \ge |z^2|-1$$

$$|z^2+2| \ge |z^2|-2.$$

$$|z^2| = |z|^2.$$

$$|z|^2-1 = 4-1 = 3$$

$$|z|^2-2 = 4-2 = 2$$

$$\left.\frac{1}{3\times 2} \le M = \frac{1}{6}\right\}.$$

Excercise 1.2

17) $\mathrm{Re}\left[(1+i)z-1\right]=0$

$$z = x+iy$$

$$(1+i)(x+iy) = (x-y)+i(x+y).$$

$$\mathrm{Re}\left[(x-y-1)+i(x+y)\right]=0.$$

$$x-y-1=0.$$

$$\boxed{y = x-1} \text{ line}$$

19) $|z-i| = |z-1|.$

$$z = x+iy$$

$$|z-i| = |x+i(y-1)| = \sqrt{x^2+(y-1)^2}.$$

$$|z-1| = \sqrt{(x-1)^2+y^2}$$

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$x^2 + (y-1)^2 = (x-1)^2 + (y-1)^2$

$x^2 + y^2 - 2y + 1 = x^2 + 1 - 2x + y^2.$

$\cancel{x^2} + \cancel{y^2} - 2y + \cancel{1} = \cancel{x^2} + \cancel{y^2} - 2x + \cancel{1},$

$-\cancel{2}y = -\cancel{2}x$

$\boxed{y = x}$ line.

21) $\operatorname{Im}|z^2| = 2.$

$z = x + iy$

$zz = z^2 = (x+iy)(x+iy).$

$\operatorname{Im}[(x^2 - y^2) + i(xy + xy)] = 2$

$xy + xy = 2.$

$\cancel{2}xy = \cancel{2}$

$\boxed{xy = 1}$ Hyperbola [Rectangular].

23). $|z-1| = 1.$

$z_1 = x + iy - 1$

$|z_1| = \sqrt{(x-1)^2 + y^2} = 1.$

$\cancel{x^2} + \cancel{1} - 2x + y^2 = \cancel{1}.$

$\boxed{x^2 + y^2 - 2x = 0.}$

$x^2 - 2x + y^2 = 0.$

$x(x-2) + y^2 = 0.$ (Circle).

25) $\operatorname{Re}(z) = |z-2|$

$x = \sqrt{(x-2)^2 + y^2}$

$x^2 = (x-2)^2 + y^2$

$\cancel{x^2} + 4 - 2x + y^2 = \cancel{x^2}$

$y^2 = 2x - 4.$

$y^2 = 4(\frac{1}{2})x - 4.$

$2(x-2).$

$y^2 = 4(1/2)[x-2].$ } Parabola.

$a = 1/2;\ c = -2.$

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$$\cancel{4x^4} + 9 - 6x^2 - 4x^2y^2 - 4 + 4xy + 4x^3y - 4ix^2 - 6ixy + 6i = \cancel{4x^4} - 4y^4 - \cancel{4xy^2} + 8ix^3y + \cancel{4x^2y^2} + 8ixy^3.$$

$$\cancel{4y^4} - \cancel{8ix^3y} - \cancel{8ixy^3} + \cancel{9} - \cancel{6x^2} - \cancel{4x^2y^2} - \cancel{4} + \cancel{4xy} + 4ix^3y - 4ix^2 - 6ixy + 6i = 0.$$

$$(4y^4 + \cancel{9} - 6x^2 - 4x^2y^2 - \cancel{4} + 4xy) + i(-\cancel{8ix^3y} - 8xy^3 + \cancel{4x^3y} - 4x^2 - 6xy + 6) = 0.$$

$$(4y^4 - 6x^2 - 4x^2y^2 + 4xy + 5) + i(-4x^3y - 8xy^3 - 4x^2 - 6xy + 6) = 0.$$

$$Re(z) = 0 \qquad Im(z) = 0.$$

$$\left.\begin{aligned} 4y^4 - 6x^2 - 4x^2y^2 + 4xy + 5 &= 0 \\ 4x^3y + 8xy^3 + 4x^2 + 6xy - 6 &= 0. \end{aligned}\right\}$$

$$\cancel{|z| - z = 2+i}$$

$$\cancel{|z|\left|1 - \frac{z}{|z|}\right| = 2+i}$$

$$\cancel{|z|\left(1 - \frac{1}{z}\right) = 2+i}$$

$$\left.\begin{aligned} 4y^4 - 4x^2y^2 + 4xy - 6x^2 + 5 &= 0 \\ 8xy^3 + 4x^3y + 6xy + 4x^2 - 6 &= 0 \end{aligned}\right]$$

$$|z| - z = 2 + i$$

$$z = x + iy$$

$$\sqrt{x^2+y^2} = (x+2) + i(y+1).$$

$$\sqrt{x^2+y^2} = x+2$$

$$(y+1) = 0.$$

$$\boxed{y = -1}$$

$$x^2 + y^2 = (x+2)^2$$

$$x^2 + y^2 = x^2 + 4 + 4x.$$

$$\cancel{x^2} + \cancel{1} = \cancel{x^2} + 4x + \cancel{4}.$$

$$4x + 3 = 0.$$

$$4x = -3$$

$$x = \frac{-3}{4}.$$

$$z = -\frac{3}{4} - i \}$$

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27) $|z| = 2$.

$8 \le |z+6+8i| \le 12$

$8 \le |z+6+8i|$.

$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|$

$8 = \big||z| - |6+8i|\big|$.

$8 = \big|2 - \sqrt{(6)^2+(8)^2}\big|$

$8 = 8$

$|z+6+8i| \le 12$

$|z| + |6+8i| = 12$.

$2 + \sqrt{6^2+8^2} = 12$

$2 + 10 = 12$ true.

29) $3z^2+2z+1$ $\quad |z| \le 1$.

Solution:-

$M \ge |3z^2+2z+1|$

~~$3z^2+2z+1$~~

~~$3z^2+3z-z+1$~~

$4-4(3)$.

$|3z^2+2z| + |1|$.

$|z|\,|3z+2| + |1|$

$1[3|z|+2| + |1|$.

$1[3+2] + 1$

$M \ge 6$

30) $|z| - z = 2+i$

$z = x+iy$;

$|z| = \sqrt{x^2+y^2}$

$\sqrt{x^2+y^2} - (x+iy) = 2+i$

$(x^2+y^2) + (x+iy)^2 - 2(x+iy)\sqrt{x^2+y^2} = (2)^2 + (i)^2 + 2i$

$x^2+\cancel{y^2} + x^2-\cancel{y^2} + 2xyi - 2(x+iy)\sqrt{x^2+y^2} = 4-1+2i$

$2x^2 + 2xyi - 2(x+iy)\sqrt{x^2+y^2} = 3+2i$

$2x^2-3 + 2xyi - 2i = 2(x+iy)\sqrt{x^2+y^2}$.

$(2x^2-3) + 2i(xy-1) = 2(x+iy)\sqrt{x^2+y^2}$.

$(2x^2-3)^2 + [2ixy-2i]^2 +$

$2i(2x^2-3)(xy-1) = 4(x^2-y^2+2ixy)(x^2+y^2)$.

$4x^4+9-6x^2+4i^2x^2y^2+4i^2-4i^2xy+4ix^3y-4ix^2-6ixy+6i$

$= 4x^4 - 4x^2y^2 + 8ix^3y + 4x^2y^2 - 4y^4 + 8ixy^3$.

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Polar Coordinates:

$(x,y) \to (r,\theta)$.

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \qquad \begin{aligned} r &= \sqrt{x^2+y^2}; \\ \theta &= \operatorname{atan}\left(\frac{y}{x}\right); \end{aligned}$$

$$z = x+iy = r\cos\theta + ri\sin\theta$$

$$z = r(\cos\theta + i\sin\theta);$$

Example: $-\sqrt{3}-i$;

$$r = \sqrt{(-\sqrt{3})^2+(-1)^2}$$

$$r = \sqrt{3+1}$$

$$\boxed{r=2}$$

$$\theta = \operatorname{atan}\left(\frac{-\sqrt{3}}{-\sqrt{3}}\right),$$

$$\theta = \operatorname{atan}\left(\frac{1}{\sqrt{3}}\right)$$

$$\theta = \frac{\pi}{6}; \quad +\pi$$

$$z = 2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]; \quad +\pi \quad +\pi$$

$$z = 2\left[\cos\left(\frac{7}{6}\pi\right) + i\sin\left(\frac{7}{6}\pi\right)\right].$$

$$-\pi < \operatorname{Arg}(z) \le \pi$$

$$\arg(z) = \operatorname{Arg}(z) + 2n\pi; \quad n \in \mathbb{Z}$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2).$$

$$\arg(z_1/z_2) = \arg(z_1) - \arg(z_2).$$

De Moivre's Theorem:

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$r = 1$$

$$z^n = \cos n\theta + i\sin n\theta;$$

Excercise 1.3:

i) 2

$$\theta = \operatorname{atan}\left(\frac{0}{2}\right) = 0$$

$$\arg(z) = 0 + 2n\pi;$$

$$\operatorname{Arg}(z) = 0.$$

$$z = 2(\cos 0 + i\sin 0)$$

$$z = 2(\cos 2n\pi + i\sin 2n\pi).$$

iii) $-3i$

$$\theta = \operatorname{atan}\left(\frac{-3}{0}\right) = \operatorname{atan}(-\infty) = -\frac{\pi}{2};$$

$$\operatorname{Arg}(z) = -\frac{\pi}{2}$$

$$\operatorname{Arg}(z) = -\frac{\pi}{2} + 2n\pi.$$

$$z = 3\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$$

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11) $-\sqrt{2}+\sqrt{7}i$

Solution:-

$r=\sqrt{(-\sqrt{2})^2+(\sqrt{7})^2}$

$r=\sqrt{2+7}$

$r=3$

$\theta=\underbrace{\operatorname{atan}\left(\frac{\sqrt{7}}{-\sqrt{2}}\right)}_{\approx 1\,rad\ =\ \frac{180}{\pi}}+\pi$

$Arg(z)\neq\theta=\theta+2\pi=8.344\,rads.$

$Arg(z)=\theta=2.0616\ rads$

$z=3\left[\cos(8.344)+i\sin(8.344)\right].$

$z=3\left[\cos(2.0616)+i\sin(2.0616)\right].$

35) $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^n=-1.$

Solution

$r=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}$

$r=\sqrt{\frac{3}{4}+\frac{1}{4}}=\sqrt{\frac{4}{4}}=1.$

$\phi=\tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=30=\frac{\pi}{6};$

$\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]^n=\left[\cos\left(\frac{n\pi}{6}\right)+i\sin\left(\frac{n\pi}{6}\right)\right]$

$\because$ By De Moivres Theorem.

$\cos\left(\frac{n\pi}{6}\right)+i\sin\left(\frac{n\pi}{6}\right)=-1+0i$

$\sin\left(\frac{n\pi}{6}\right)=0.$

$\cos\left(\frac{n\pi}{6}\right)=-1.$

$\frac{n\pi}{6}=\pi+2k\pi;\quad k\in\mathbb{Z}.$

$k=1.$

$\frac{n}{6}=1+2k.$

$k=0$

$n=6+12k=6(1+2k)$

$\boxed{n=6(1+2k)}$; $n=6$ -

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$$1+\cos\theta+\cos 2\theta+\cdots\cdots+\cos n\theta = \frac{1}{2}+\frac{\sin(n+1/2)\theta}{2\sin\theta/2}$$ } Langrange's Identity.

Proof:

$$1+z+z^2+z^3+\cdots+z^n = \frac{1-z^{n+1}}{1-z};$$

$$\sum_{i=0}^{n} z^i = \frac{1-z^{n+1}}{1-z};$$

$$\cancel{z = x+iy}$$

$$\cancel{z^2 = (x^2-y^2)+i(x^2y+y^2x)}$$

$$\cancel{z^2 = (x^2-y^2)+2ixy;}$$

$$z = re^{i\theta}; \quad r=1.$$

$$\cancel{1+re^{i\theta}+r^2e^{2i\theta}+r^3e^{3i\theta}+\cdots+r^ne^{ni\theta}}$$

$$\cancel{= \sum_{k=0}^{n} r^n e^{ni\theta}}$$

$$\cancel{\sum_{i=0}^{n} z^i + \sum_{i=-n}^{0} z^i = \sum_{-n}^{n} z^n}$$

$$\cancel{\frac{z^{-1/2}\,1-z_0z^n}{z^{1/2}(1-z)} = \frac{z^{-1/2}-z^{n+1-1/2}}{z^{-1/2}-z^{1/2}} \quad \frac{z^{-1/2}-z^{n+1/2}}{z^{-1/2}-z^{1/2}}}$$

$$\cancel{\sum_{i=-n}^{0} z^i = \frac{1-z^{0+1}}{1-z}}$$

$$\cancel{\frac{z^{1/2}-z^{-1/2}}{2} = \sinh(x/2)}$$

$$\cancel{z^{n+1/2}-z^{-n-1/2}}$$

$$\cancel{-\sum_{0}^{-n} z^i = -\frac{(1-z^{-n+1})}{(1-z)}}$$

$$\cancel{-2i\sin(x/2);}$$

$$\sum_{k=-n}^{n} z^k = \frac{(z^{-n}-z^{n+1})}{1-z};$$

$$\cancel{-\frac{(1-z\cdot\frac{1}{z^n})}{(1-z)}}$$

$$\frac{1}{z^n}+\cdots+\frac{1}{z^2}+\frac{1}{z}+$$

$$1+z+z^2+\cdots z^n = \frac{z^{-1/2}}{z^{-1/2}}\left[\frac{(z^{-n}-z^{n+1})}{1-z}\right] = \frac{z^{-n-1/2}-z^{n+1/2}}{z^{-1/2}-z^{1/2}} = \frac{z^{n+1/2}-z^{-n-1/2}}{z^{1/2}-z^{-1/2}}$$

$$z = re^{i\theta}; \quad = \frac{e^{i\theta(n+1/2)}-e^{-i\theta(n+1/2)}}{e^{i\theta/2}-e^{-i\theta/2}} = \frac{\sin(n+1/2)\theta}{\sin(\theta/2)};$$

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$$\sum_{-n}^{n} z^n = \sum_{-n}^{n} \cos n\theta + i\sum_{-n}^{n} \sin n\theta = \frac{\sin(n+1/2)\theta}{\sin(\theta/2)} + i0.$$

$r = 1.$

$$\sum_{-n}^{n} \cos n\theta = \frac{\sin(n+1/2)\theta}{\sin(\theta/2)}.$$

$$\cos(-n\theta) + \cos(-2\theta) + \cos(-\theta) + \cos 0 + \cos(\theta) + \cos 2\theta + \cdots + \cos n\theta.$$

$\because \cos(-x) = \cos(x)$

$$\cos 0 + 2\cos\theta + 2\cos 2\theta + 2\cos 3\theta + \cdots + 2\cos n\theta = \frac{\sin(n+1/2)\theta}{\sin(\theta/2)};$$

$$1 + 1 + 2\cos\theta + 2\cos 2\theta + \cdots + 2\cos n\theta = 1 + \frac{\sin(n+1/2)\theta}{\sin(\theta/2)};$$

$$2[1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos n\theta] = 1 + \frac{\sin(n+1/2)\theta}{\sin(\theta/2)};$$

$$\{1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos n\theta] = \frac{1}{2} + \frac{\sin(n+1/2)\theta}{2\sin(\theta/2)};$$

Hence, proved.

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Cube roots of a complex number:

$$\omega^3 = i$$

$$\omega = i^{1/3}.$$

$$i = e^{\frac{\pi}{2}i}$$

$$\omega^3 = e^{\frac{\pi}{2}i}$$

$$\omega = \left(e^{\frac{\pi}{6}i}\right).$$

$$\omega_0 = \left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right] = \frac{\sqrt{3}}{2} + \frac{i}{2};$$

$$\omega_1 = \left[\cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)\right] = -\frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$\omega_2 = \left[\cos\left(\frac{\pi}{6} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{4\pi}{3}\right)\right] = -i$$

$$\omega^n = re^{i\theta};$$

$$\omega^n = re^{(i\theta + 2k\pi i)}$$

$$\omega_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

Example: 2

Fourth roots of a complex number $1+i$;

$$\omega^4 = 1 + i = \sqrt{2}\, e^{\pi/4\, i}$$

$$\omega^4 = 2^{1/2} e^{\frac{\pi}{4}i}$$

$$\omega = \left(2^{1/2}\right)^{1/4} \left(e^{\pi/4\, i}\right)^{1/4}$$

$$\omega = 2^{1/8}\left(e^{\pi/16\, i}\right).$$

$$\omega_k = \sqrt[8]{2}\left[\cos\left(\frac{\frac{\pi}{4} + 2k\pi}{4}\right) + i\sin \cdots\right]$$

$$\omega_0 = \sqrt[8]{2}\left[0.981 + 0.195i\right].$$

$$\omega_1 = \sqrt[8]{2}\left[-0.195 + 0.981i\right]$$

$$\omega_2 = \sqrt[8]{2}\left[-0.981 - 0.195i\right]$$

$$\omega_3 = \sqrt[8]{2}\left[0.195 - 0.981i\right]$$

$\omega_0 = 1.0697 + 0.21265i$; $\omega_1 = -0.21265 + 1.0697i$; $\omega_2 = -1.0697 - 0.21265i$

$\omega_3 = 0.21265 - 1.0697i$;

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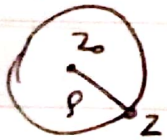
Sets of Points in the Complex Plane:



$\rho > 0$ $\quad \rho = |z - z_0|$.

$= \sqrt{(x-x_0)^2 + (y-y_0)^2}$;

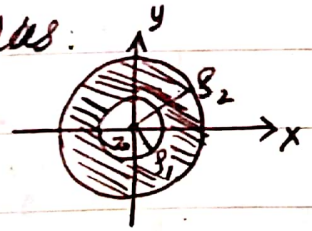
$|z - z_0| \le \rho$ Disk

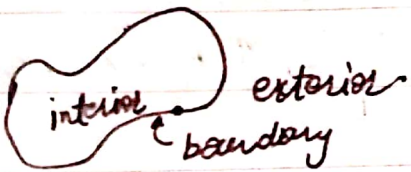
$|z - z_0| = \rho$ boundary points

$|z - z_0| < \rho$ interior points

$|z - z_0| > \rho$ exterior points

$\rho_1 < |z - z_0| \le \rho_2$ Annulus.





$|z - z_0| \le \rho$ Closed region, disk

$|z - z_0| < \rho$ open region disk

$0 < |z - z_0| \le \rho$ punctured disk

or $0 < |z - z_0| < \rho$

$0 < |z - z_0| < \rho \rightarrow$ deleted neighborhood of z_0,

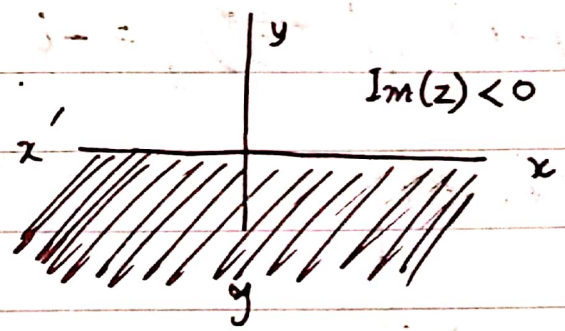
$|z - z_0| < \rho \rightarrow$ neighborhood of z_0,

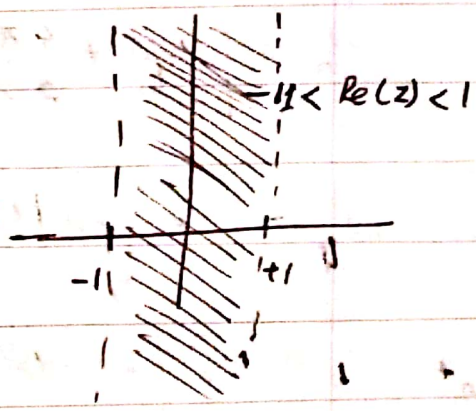
$$\underbrace{\rho_1 \le}_{\text{boundary points}} \overbrace{|z - z_0| < \rho_2}^{\text{interior points}} \Big\} \text{ Annular region.}$$

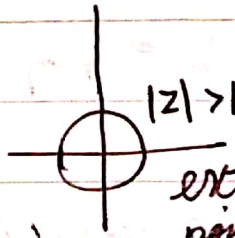
eg. $1 \le |z - 5| < 3$;

so it is neither open nor closed but it is an annular Region.

Open sets:

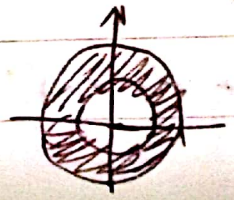






exteria points of unit circle.

$1 < |z| < 2$ Annular region



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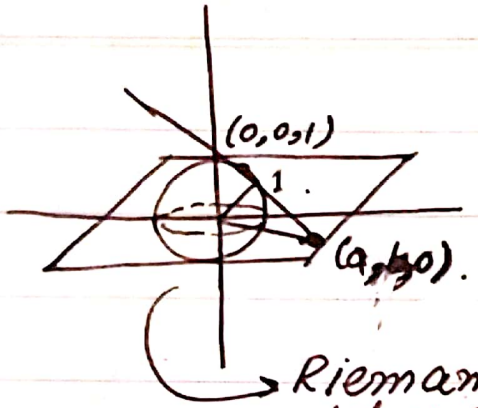
Riemann Sphere:

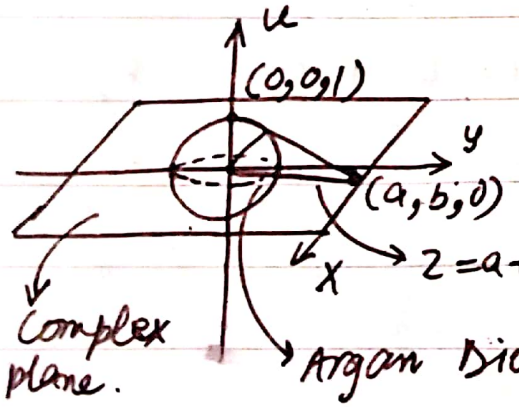
$$x^2 + y^2 + z^2 = 1.$$



Riemann sphere (a unit sphere to express complex number)

$z = a + ib$;

$$x^2 + y^2 + u^2 = 1.$$



Riemann sphere.

Quadratic Formula:

$$az^2 + bz + c = 0.$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

Example:

$$z^2 + (1-i)z - 3i = 0.$$

$$a = 1, \quad b = 1 - i, \quad c = -3i$$

$$z = \frac{-(1-i) \pm \sqrt{(1-i)^2 - 4(1)(-3i)}}{2(1)}.$$

$$(1-i)^2 = (1) + (i)^2 - 2(i)$$

$$= \cancel{1} - \cancel{1} - 2i = -2i$$

$$\frac{-1+i \pm \sqrt{\cancel{-2i} + \cancel{12i}}}{2} \quad = 10i$$

$$i = e^{\frac{\pi}{2}i}$$

$$\frac{-1+i \pm \sqrt{10}\, e^{\pi/4\, i}}{2}$$

$$e^{\pi/4\, i} = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right);$$

$$= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

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$$\frac{-1+i \pm \sqrt{10}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}{2}$$

$$\frac{-1+i \pm (\sqrt{5}+\sqrt{5}i)}{2}$$

$$z_1 = \frac{-1+i+\sqrt{5}+\sqrt{5}i}{2}$$

$$z_1 = \left(\frac{-1+\sqrt{5}}{2}\right) + i\left(\frac{1+\sqrt{5}}{2}\right).$$

$\therefore$ Golden Ratio $\varphi = \frac{1+\sqrt{5}}{2}$, $\varphi^* = \frac{-1+\sqrt{5}}{2}$;

$$\left.\begin{aligned} z_1 &= \varphi^* + i\varphi; \\ z_2 &= \frac{-1+i-\sqrt{5}-\sqrt{5}i}{2} = \frac{-(1+\sqrt{5})}{2} + i\,\frac{(1-\sqrt{5})}{2}. \end{aligned}\right\}$$

$$\left.z_2 = \varphi - i\varphi^*\right\}$$ Hence, solution set:

$$z = \{\varphi^* + i\varphi, \varphi - i\varphi^*\}$$

Differential equations:

$$y'' + 2y' + 2y = 0. \qquad y = e^{mx};$$

$$m^2 + 2m + 2 = 0.$$

$$m = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{-2}$$

$$m = \frac{-2 \pm 2i}{2} = -1 \pm i;$$

Case III: $m = \alpha \pm i\beta$.

$$y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

$$\alpha = -1, \beta = +1;$$

$$y(x) = e^{-x}(c_1 \cos x + c_2 \sin x).$$

Application of Complex number.

$$\begin{cases} e^z = e^x(\cos y + i \sin y). \\ z = x + iy \\ e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y). \end{cases}$$

Moving Charge in a Circuit:

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$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_0 \sin \gamma t;$$

$$i = \frac{dq}{dt}; \qquad L\frac{di}{dt} + Ri + \frac{q}{C} = E_0 \sin \gamma t;$$

$$q_p(t) = A\cos\gamma t + B\sin\gamma t;$$

$$q = \int i\,dt + q_0;$$

$$q'(t) = -A\gamma(\cos\gamma t)' + B\gamma(\sin\gamma t)'$$

$$q_p'(t) = -A\gamma\sin\gamma t + B\gamma\cos\gamma t$$

$$q_p''(t) = -A\gamma^2\cos\gamma t - B\gamma^2\sin\gamma t;$$

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt + \frac{q_0}{C} = E_0 \sin\gamma t.$$

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = E_0\cos\gamma t \Big\}.$$

$$L\left[-A\gamma^2\cos\gamma t - B\gamma^2\sin\gamma t\right] + R\left[-A\gamma\sin\gamma t + B\gamma\cos\gamma t\right] + \frac{A}{C}\cos\gamma + \frac{B}{C}\sin\gamma t = E_0\sin\gamma t;$$

$$\left[-LA\gamma^2 + RB\gamma + \frac{A}{C}\right]\cos\gamma t + \left[-LB\gamma^2 - RA\gamma + \frac{B}{C}\right]\sin\gamma t = E_0\sin\gamma t;$$

$$\left.\begin{aligned} E_0 &= -LB\gamma^2 - RA\gamma + \frac{B}{C} \\ 0 &= -LA\gamma^2 + RB\gamma + \frac{A}{C} \end{aligned}\right\}$$

$$A\left[-R\gamma\right] + B\left[\frac{1}{C} - L\gamma^2\right] = E_0$$

$$A\left[\frac{1}{C} - L\gamma^2\right] + B\left[R\gamma\right] = 0.$$

$$\begin{pmatrix} -R\gamma & 1/C - L\gamma^2 \\ \frac{1}{C} - L\gamma^2 & R\gamma \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} E_0 \\ 0 \end{pmatrix};$$

$$A = \frac{+E_0 R\gamma}{-R^2\gamma^2 - \left(\frac{1}{C} - L\gamma^2\right)^2}$$

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$$A = \frac{E_0 \gamma R}{-\left[R^2\gamma^2 + \left(\frac{1}{C} - L\gamma^2\right)^2\right]}, \quad B = \frac{\gamma E_0\left(\frac{1}{C} - L\gamma^2\right)}{\gamma\left[R^2\gamma^2 + \left(\frac{\gamma}{C\gamma} - L\gamma^2\right)^2\right]};$$

$$\because \left(\frac{1}{C\gamma} - L\gamma\right)^2 = X^2$$

$$X = L\gamma - \frac{1}{C\gamma} \qquad |Z| = \sqrt{R^2 + X^2} \qquad \left(L\gamma - \frac{1}{C\gamma}\right)^2 = X^2;$$

$$A = \frac{-E_0 \gamma R}{\left[R^2\gamma^2 + \left(\frac{\gamma}{C\gamma} - L\gamma^2\right)^2\right]} = \frac{-E_0 \gamma R}{R^2\gamma^2 + \gamma^2\left(\frac{1}{C\gamma} - L\gamma\right)^2}$$

Hence;

$$A = \frac{E_0 \gamma R}{\gamma^2\left[R^2 + X^2\right]} = \frac{-E_0 R}{\gamma Z^2}$$

$$q_p(t) = A\cos\gamma t + B\sin\gamma t.$$

$$A = \frac{-E_0 R}{\gamma |Z|^2}$$

$$B = \frac{E_0\gamma\left(\frac{1}{C\gamma} - L\gamma\right)}{\gamma^2\left[R^2 + \left(L\gamma - \frac{1}{C\gamma}\right)^2\right]} = \frac{-E_0 X}{\gamma |Z|^2}$$

$$|Z| = \sqrt{R^2 + X^2}$$

$$X = \left(L\gamma - \frac{1}{C\gamma}\right);$$

$$q_p(t) = \frac{-E_0}{\gamma |Z|^2}\left[R\cos\gamma t + X\sin\gamma t\right]; \qquad B = \frac{-E_0 X}{\gamma |Z|^2};$$

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$$i = \frac{dq}{dt}; \qquad L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = E_0 \cos \gamma t;$$

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E_0 \sin \gamma t;$$

$$i = \mathrm{Im}\{Ae^{j\gamma t}\} \qquad j = \sqrt{-1}$$

$$A = \frac{E_0}{R + jX}; \qquad |Z| = \sqrt{R^2 + X^2}$$

$$Z = |Z| e^{j\theta}$$

$$\theta = \mathrm{atan}\left[\frac{L\gamma - \frac{1}{C\gamma}}{R}\right];$$

$$i_p(t) = \mathrm{Im}\left\{\frac{E_0}{|Z| e^{j\theta}} e^{j\gamma t}\right\}$$

$$i_p(t) = \mathrm{Im}\left\{\frac{E_0}{\sqrt{R^2+X^2}} e^{j\gamma t - j\theta}\right\}.$$

$$i_p(t) = \frac{E_0}{\sqrt{R^2+X^2}} \sin[\gamma t - \theta]; \qquad \theta = \mathrm{atan}\left(\frac{L\gamma - 1/C\gamma}{R}\right)$$

$x^3 + px + q = 0$ By Cardan's Formula:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

Complex matrices:

$\bar{H}^t = H$ Hermitian

$\bar{H}^t = -H$ Skew-Hermitian

$\bar{A}^t = A^{-1}$. Unitary.

$$A = \begin{pmatrix} 3i & 10 & -10-2i \\ -10 & 0 & 4+i \\ 10-2i & -4+i & -5i \end{pmatrix}$$

$$\bar{A} = \begin{pmatrix} -3i & 10 & -10+2i \\ -10 & 0 & 4-i \\ 10+2i & -4-i & +5i \end{pmatrix}$$

$$\bar{A}^t = \begin{pmatrix} -3i & -10 & 10+2i \\ 10 & 0 & -4-i \\ -10+2i & 4-i & 5i \end{pmatrix}$$

$$\bar{A}^t = -\begin{pmatrix} 3i & 10 & -10-2i \\ -10 & 0 & +4+i \\ 10-2i & -4+i & -5i \end{pmatrix}$$

$\bar{A}^t = -A$ } Skew Hermitian.

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$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2+i}{\sqrt{10}} & \frac{-2+i}{\sqrt{10}} \\ 0 & \frac{2+i}{\sqrt{10}} & \frac{2-i}{\sqrt{10}} \end{pmatrix}$$

$$\bar{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2-i}{\sqrt{10}} & \frac{-2-i}{\sqrt{10}} \\ 0 & \frac{2-i}{\sqrt{10}} & \frac{2+i}{\sqrt{10}} \end{bmatrix}$$

$$\bar{B}^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2-i}{\sqrt{10}} & \frac{2-i}{\sqrt{10}} \\ 0 & \frac{-2-i}{\sqrt{10}} & \frac{2+i}{\sqrt{10}} \end{bmatrix}$$

$\bar{B}^t \neq B \neq -B$

Neither Hermitian nor Skew hermitian.

$$w = f(i) = (i)^2 - (2+i)(i)$$
$$= -1 - 2i - i^2$$
$$= \cancel{-1} - 2i \cancel{+1} = -2i$$

$$w = f(1+i) = (1+i)^2 - (1+i)(2+i).$$
$$= 2i - [2 - 1 + 3i]$$
$$= 2i - (1 + 3i) = -1 - i$$

Real and Imaginary Part of a complex function:

$$z = f(x,y) = x + iy$$
$$w = u(x,y) + i\,v(x,y).$$

eg.

$$f(x,y) = z^2 = (x+iy)^2$$
$$= (x^2 - y^2 + 2ixy);$$

Chapter II: Complex Functions and Mappings:

$w = f(z); \quad \forall z \in \mathbb{C}$

example: $w = z^2 - (2+i)z;$

$z = i, \quad z = i + 1;$

$$w = x^2 - y^2 + 2ixy$$
$$u(x,y) + i\,v(x,y).$$

$u(x,y) = x^2 - y^2$ (Hyperbola)

$v(x,y) = 2xy$ (Rectangular hyperbola).

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Hence:

$$f(z) = u(x,y) + i\,v(x,y).$$

Exponential function: $e^z = e^x(\cos y + i\sin y)$

$e^0 = 1$

$e^i = \cos 1 + i\sin 1,$

$e^{2+\pi i} = e^2(\cos\pi + i\sin\pi)$

$e^{2+\pi i} = -e^2;$

Polar Coordinates: (r,θ)

$$f(z) = z^2 = r^2(\cos\theta + i\sin\theta)^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

$$\left.\begin{aligned} u(r,\theta) &= r^2\cos 2\theta \\ v(r,\theta) &= r^2\sin 2\theta \end{aligned}\right\}$$

- Complex Function Mapping:

Complex functions are graphed in 4D Space with (x,y,z) as Cartesian space and one imaginary space 't'. Hence, Bernhard Riemann gave concept of Mapping.

-eg- image of line $x = 1$. under mapping: $w = z^2$

$z = x + iy$

$z = 1 + iy$

$w = z^2 = x^2 - y^2 + 2ixy$

$w = 1 - y^2 + 2iy$

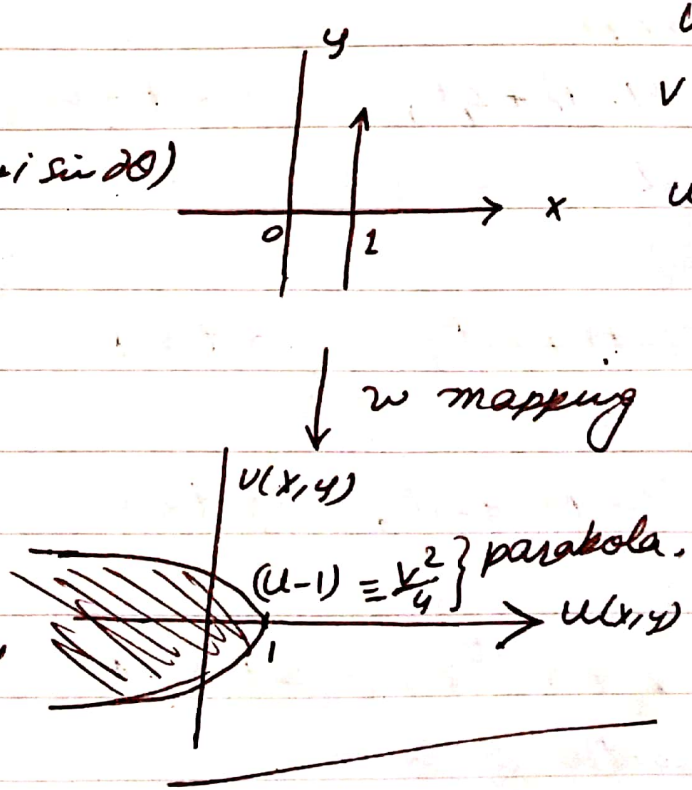
$u = 1 - y^2$

$v = 2y$

$u = 1 - \left(\frac{v}{2}\right)^2$

$u = 1 - \frac{v^2}{4}$

$(u-1) = -\frac{v^2}{4}$



$z = x + iy$

$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\} a \le t \le b$ is a parametric curve where 't' is a parameter

and $\left.\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}\right\}$ paramateric equations.

- Parametric Curves:

1) Line:

points (z_0, z_1)

$z(t) = z_0(1-t) + z_1 t; \quad -\infty < t + \infty.$

2) Line Segment:

points (z_0, z_1)

$z(t) = z_0(1-t) + z_1 t; \quad 0 \le t \le 1.$

3) Ray:

$z(t) = z_0(1-t) + z_1 t; \quad 0 \le t < \infty.$

4) Circle:

$z(t) = z_0 + re^{it}; \quad 0 \le t \le 2\pi.$

z_0 is center of circle

Image of a parametric curve,

$w = f(z)$

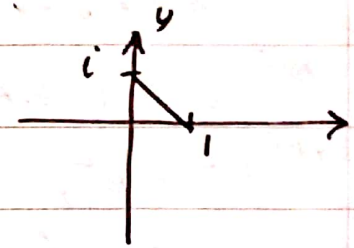
$\left.\begin{matrix} z(t) = x(t) + iy(t) \end{matrix}\right\} a \le t \le b.$

$w(t) = f(z(t)), \; a \le t \le b$

example:

Line: $(1, i)$

mapping $w = \overline{iz}$



$z(t) = 1(1-t) + it$

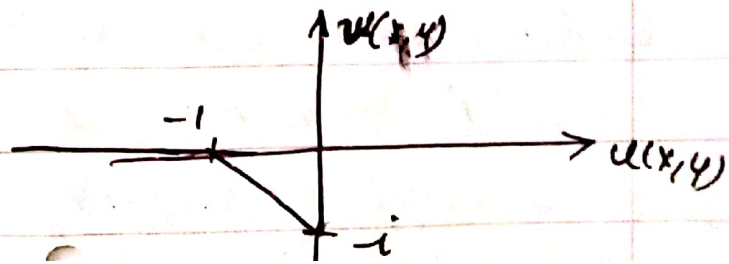
$z(t) = (1-t) + it$

$w = \overline{iz} = \overline{(i - it + i^2 t)}$

$= \overline{(i - it - t)} = \overline{(-t + i(1-t))}$

$w = \overline{iz} = -t - i(1-t).$

$w = -t - i(1-t); \quad 0 \le t \le 1$



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Example: Image of a Parametric Curve:

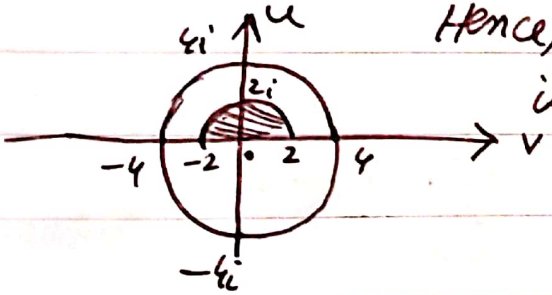
$w = z^2$ image of the semi circle.

$z(t) = z_0 + re^{it}$; $0 < t \le \pi$ } for semicircle.

$z(t) = 2e^{it}$; $0 \le t \le \pi$;

$w = z^2 = 4e^{2it}$; $2t = s$ new parameter

$w = 4e^{is}$; $0 \le s \le 2\pi$.



Hence, mapping of a semi-circle is a circle.

Exercise: 2.2

$w = z^2$

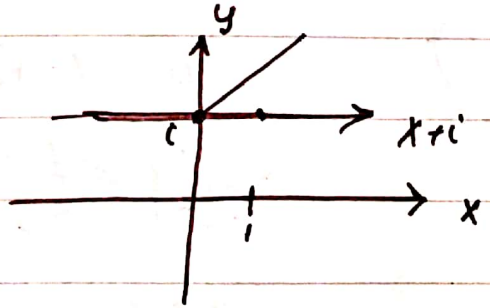
Q9: $y = 1$

$z = x + i$

$w = z^2 = x^2 - 1 + 2ix$

$u = x^2 - 1$

$v = 2x$;



$u = \frac{v^2}{4} - 1$

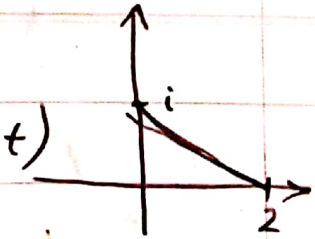
$(u+1) = \frac{v^2}{4}$

Parabola.

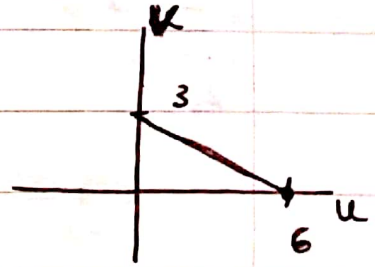
(5) $z(t) = 2(1-t) + it$, $0 \le t \le 1$

$f(z) = 3z$.

$3(2(1-t) + it)$



$f(z) = 6(1-t) + 3it$

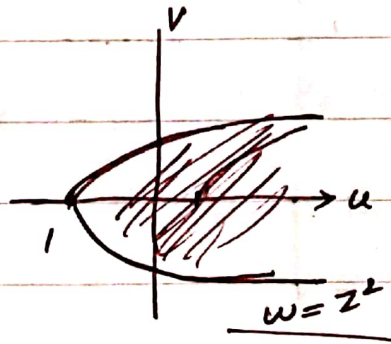


$V = 3t$

$u = 6(1-t)$

$u = 6\left[1 - \frac{V}{3}\right]$

$u = 6 - 2v$.



(7) $z(t) = 1 + 2e^{it}$

$z_0 = 1$

$r = 2$

$0 \le t \le 2\pi$;

$f(z) = z + 1 - i$;

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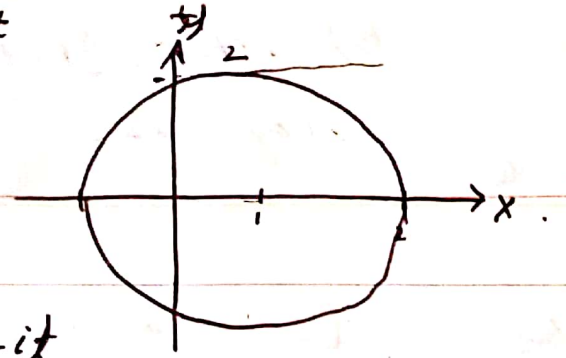
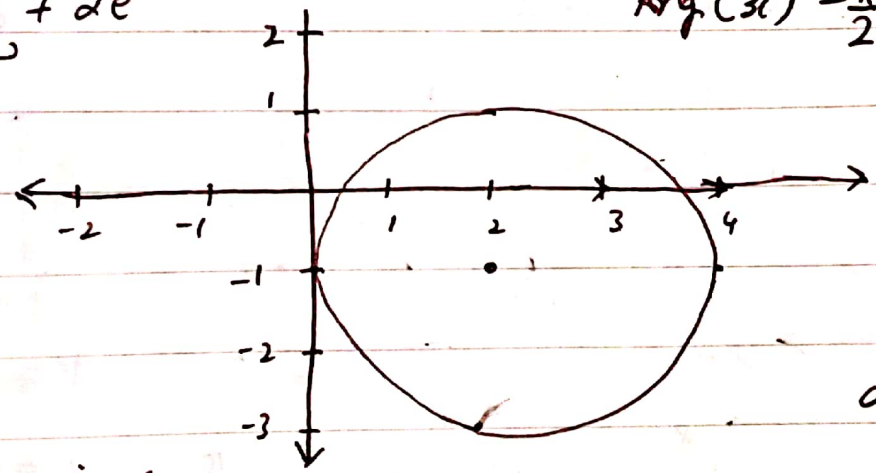
$z(t) = 1 + 2e^{it}$



$f(z) = (z + 1 - i)$

$= 1 + 2e^{it} + 1 - i$

$f(z) = \underbrace{2 - i}_{z_0} + 2e^{it}$



Linear Mappings:

$T(z) = az + b$; $a, b \neq 0$ (Translation)
$a = 1$

$T(z) = z \pm a$; Translation.

$T(z) = az$; $|a| = 1$ Rotation.

$a = e^{i\theta}$; θ is the rotation.

$T(z) = az$; $a > 0$ Magnification.

$T(z) = az$; $a < 0$ (Zoom out).

Exercise: 2-3

13) $f(z) = 3iz + 4$;

$b = 4$; Translation

$|3i| = 3$; Scaling

$\operatorname{Arg}(3i) = \frac{\pi}{2}$; Rotation

15) $f(z) = -\frac{1}{2}z + 1 - \sqrt{3}i$

$b = 1 - \sqrt{3}i$; Translation.

$a = -\frac{1}{2}$ } scaling (down)

(rotation) $\operatorname{Arg}(-1/2) = \pi$.

$z^{1/n} = \sqrt[n]{r}\, e^{\frac{i\theta + 2k\pi}{n}}$, $k = 0, \pm 1, \pm 2, \ldots$

$z^{1/2} = \sqrt{r}\, e^{i\theta/2 + k\pi}$.

$z^{1/n} = \sqrt[n]{r}\, e^{i\operatorname{Arg}(z)/n}$,

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Inverse of a function:

$w = f(z) = z + 3i$

$w = z + 3i$

$w - 3i = z.$

$f(z) = w$

$z = f^{-1}(w).$

$f^{-1}(w) = w - 3i$

$\boxed{f^{-1}(z) = z - 3i}$

- Vector Fields.

$F(x,y) = P(x,y)\hat{i} + i\,Q(x,y)\hat{j}$

$F(x,y) = (P(x,y), Q(x,y)).$

example:

$F(x,y) = (x+y)\hat{i} + (2xy)\hat{j}$

$F(1,2) = (1+2)\hat{i} + (2(1)(2))\hat{j}$

$F(1,2) = 3\hat{i} + 4\hat{j}$

$\nabla . F(x,y) = \frac{\partial}{\partial x}(x+y)\hat{i} + \frac{\partial}{\partial y}(2xy)\hat{j}$

$\nabla . F(x,y) = \hat{i} + 2x\hat{j} > 0$

$$\nabla \times F(x,y) = \cancel{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+y) & 2xy & 0 \end{vmatrix}}$$

$$= \hat{k}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ x+y & 2xy \end{vmatrix} + \frac{\partial}{\partial z}\begin{vmatrix} \hat{i} & \hat{j} \\ x+y & 2xy \end{vmatrix}$$

$\nabla \times F(x,y) = (2y - 1)\hat{k} > 0.$

Complex functions as Vector Fields.

$F(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$

$f(x,y) = u(x,y) + i\,v(x,y)$ } complex function.

Example: 3 Streamlines:

$f(z) = \bar{z}^2.$

$z = x + iy$

$\bar{z} = x - iy$

$\bar{z}^2 = (x-iy)(x-iy) = x^2 - y^2 - 2ixy$

$u(x,y) = x^2 - y^2$ }

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$f(x,y) = (x^2 - y^2) - 2xyi$;

$f(0,0) = 0^2 - 0^2 - 2(0)(0)i = 0 + 0i$

$f(0,1) = 0^2 - 1^2 - 2(0)(1)i = -1 + 0i$

$f(0,2) = 0^2 - 4 - 0i = -4 + 0i$

$f(0,3) = -9 + 0i$

$f(0,n) = -n^2 + 0i$

$f(1,0) = 1 + 0i$

$f(2,0) = 4 + 0i$

$f(n,0) = n^2 + 0i$

$f(1,1) = -2i$

$f(1,2) = -3 - 4i$

$f(2,1) = 3 - 4i$

$f(2,2) = -8i$

$f(2,3) = -5 - 12i$

$f(3,2) = 5 - 12i$

Excercise: 2.6

9) $f(z) = 1 - 2i$

11) $f(z) = iz$

$z = x + iy$

$iz = -y + ix$;

$f(z) = ix - y$;

$P(x,y) = -y$

$Q(x,y) = x$

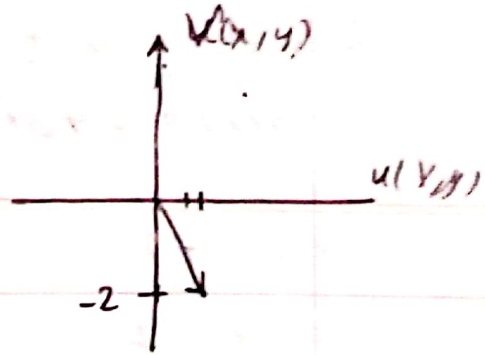
$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} = \frac{x}{-y}$

$\frac{dy}{dx} = -\frac{x}{y}$.

$\int y\,dy = \int -x\,dx$

$\frac{y^2}{2} = -\frac{x^2}{2} + c'$.

$y^2 = -x^2 + c$.



$\frac{dx}{dt} = P(x,y)$

$\frac{dy}{dt} = Q(x,y)$.

$\boxed{x^2 + y^2 = c}$

$c = a^2$.

$x^2 + y^2 = a^2$ equation of circle

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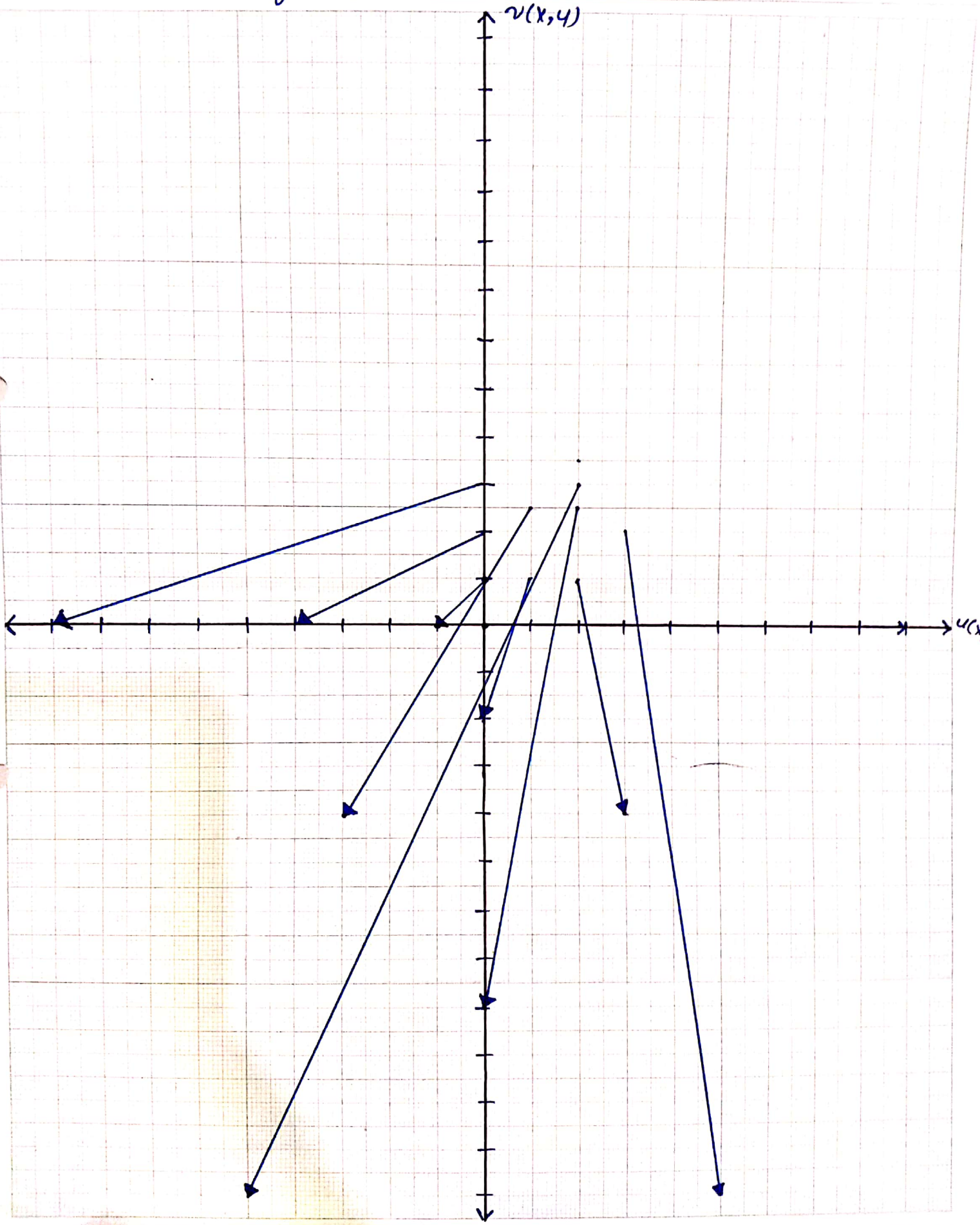
$f(x,y) = (x^2 - y^2)\hat{i} - 2xy\hat{j}$;

NAME Vector Field Diagram.

INDEX



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9) $f(z) = 1 - 2i$

$P(x,y) = 1$

$Q(x,y) = -2$

$\frac{dx}{dt} = P(x,y).$

$\frac{dy}{dt} = Q(x,y).$

$\frac{dy}{dx} = \frac{-2}{1}.$

$\frac{dy}{dx} = -2$

$\boxed{y(x) = -2x + C.}$

Chapter 3: Analytic Functions.

Limits:

$\lim_{z \to z_0} c f(z) = c \lim_{z \to z_0} f(z) = c \cdot L.$

$\lim_{z \to z_0} f(z) \pm g(z) = L \pm M$

$\lim_{z \to z_0} f(z) g(z) = L \cdot M$

$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L}{M};$

Continuity of function:

i) $\lim_{z \to z_0} f(z)$ exists; $\lim_{z \to z_0^-} f(z) = \lim_{z \to z_0^+} f(z).$

ii) $\lim_{z \to z_0} f(z)$ is defined.

iii) $\lim_{z \to z_0} f(z) = f(z_0).$

Continuity at $z_0 = 1 - i$

$f(z) = z^2 - iz + 2i$

$f(z_0) = (1-i)^2 - i(1-i) + 2$

$f(1-i) = \cancel{1 + i^2} - 2i - i + i^2 + 2 = 1 - 3i$

$\lim_{z_0 \to z_0} f(z) = 1 - 3i$ Hence, function is continuous.

Differentiation:

$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

$\left.\frac{dw}{dz}\right|_{z=z_0};$ a function $w = f(z)$ is analytic at point z_0 if it is differentiable at z_0 and at every

point in neighborhood of z_0.

$$\lim_{z \to 2+i} \frac{z^2-4z+5}{z^3-z-10i}$$

$$\frac{dc}{dz} = 0$$

$$\frac{(2z-4)}{(3z^2-1)}\Bigg|_{z=2+i}$$

$$\frac{dz^n}{dz} = nz^{n-1}$$

$$\frac{2(2+i)-4}{3(2+i)^2-1} = \frac{2i}{3(4-1+4i)-1}$$

$$\frac{d}{dz}(f(z) \pm g(z)) = f'(z) \pm g'(z)$$

$$\frac{d}{dz}(f(z) \cdot g(z)) = f'(z)g(z) + f(z)g'(z)$$

$$\frac{2i}{12-3+12i-1}$$

$$\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$$

$$\frac{2i}{8+12i}; \quad \frac{2i(8-12i)}{8^2+12^2} = \frac{24+16i}{64+144}.$$

$$\frac{d}{dz} f(g(z)) = f'(g(z))\, g'(z).$$

$$\begin{array}{r} 144 \\ 64 \\ \hline 208 \end{array} \qquad \frac{24+16i}{208}$$

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \Bigg\} \text{ L'Hopital's Rule.}$$

$$\frac{3}{26} - \frac{i}{13}; \quad \frac{3-2i}{26} \Bigg\}.$$

if $\left.\begin{array}{l} f(z_0) = 0 \\ g(z_0) = 0 \end{array}\right\}$ or any undeterminate form.

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Cauchy-Riemann Equation.

$f(z) = f(x,y) = u(x,y) + i\,v(x,y)$.
is differentiable at a point
$z = x + iy$;

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\}$$

$$\left.\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ f'(z) &= i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \end{aligned}\right\}$$

example:

$f(z) = z^2 + z$

$f(z) = x^2 - y^2 + 2ixy + x + iy$

$f(z) = (x^2 + x - y^2) + i(2xy + y)$;

$u(x,y) = x^2 + x - y^2$

$v(x,y) = 2xy + y$.

$\frac{\partial u}{\partial x} = 2x + 1$, $\frac{\partial v}{\partial y} = 2x + 1$

$\frac{\partial u}{\partial y} = -2y$, $\frac{\partial v}{\partial x} = 2y$.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$;

$\frac{\partial u}{\partial x} = 4x$

$\frac{\partial v}{\partial y} = 2y$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = 1$, $\frac{\partial v}{\partial x} = -1$.

Example:

$f(z) = z^2$

$z = x + iy$

$z^2 = x^2 - y^2 + 2ixy$.

$u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$.

$\frac{\partial u}{\partial x} = 2x$

$\frac{\partial v}{\partial y} = 2x$ $\quad \frac{\partial v}{\partial x} = 2y$.

$\frac{\partial u}{\partial y} = -2y$

$f(z) = 2x^2 + y + i(y^2 - x)$;
is not analytic.

$u(x,y) = 2x^2 + y$

$v(x,y) = y^2 - x$.

Cauchy-Riemann Equation.

$$\left\{\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta}; \\ \frac{\partial v}{\partial r} &= -\frac{1}{r}\frac{\partial u}{\partial \theta}; \end{aligned}\right\}$$

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Exercise 3.3:

1) $f(z) = z^3$.

$$z = x + iy$$

$$z^2 = (x^2 - y^2) + i(2xy)$$

$$z^3 = [(x^2 - y^2) + i(2xy)](x + iy)$$

$$z^3 = [x(x^2 - y^2) + i(2x^2y) + iy(x^2 - y^2) - 2xy^2].$$

$$[x^3 - xy^2 + i(2x^2y + x^2y - y^3) - 2xy^2]$$

$$f(z) = [x^3 - 3xy^2 + i(3x^2y - y^3)]$$

$$\left.\begin{aligned} u(x,y) &= x^3 - 3xy^2 \\ v(x,y) &= 3x^2y - y^3 \end{aligned}\right\}.$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2.$$

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$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$ Next;

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\frac{\partial u}{\partial y} = -3(2xy) = -6xy$$

$$\frac{\partial u}{\partial y} = -6xy; \qquad \frac{\partial v}{\partial x} = 6xy.$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ Hence, $f(z)$ is is differentiable.

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

$$f'(z) = 3x^2 - 3y^2 + i\,(6xy)$$

$$f'(z) = 3x^2 - 3y^2 + 6xyi$$ Solution.

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9) $f(z) = e^{-x}(\cos y - i\cancel{e^{-x}}\sin y)$.

$f(z) = e^{-x}\cos y - i\, e^{-x}\sin y$;

$$\left.\begin{aligned} u(x,y) &= e^{-x}\cos y \\ v(x,y) &= -e^{-x}\sin y \end{aligned}\right\}.$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= -e^{-x}\cos y \\ \frac{\partial v}{\partial y} &= -e^{-x}\cos y \end{aligned}\right\} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};$$

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= -e^{-x}\sin y \\ \frac{\partial v}{\partial x} &= e^{-x}\sin y \end{aligned}\right\} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x};$$

$$-e^{-x}\cos y + i(e^{-x}\sin y).$$

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$$f'(z) = -e^{-x}\cos y + ie^{-x}\sin y;$$

---

11) $f(z) = e^{x^2-y^2}\cos 2xy + ie^{x^2-y^2}\sin 2xy;$

Solution: $u(x,y) = e^{x^2-y^2}\cos 2xy$

$$v(x,y) = e^{x^2-y^2}\sin 2xy.$$

$$\frac{\partial u}{\partial x} = 2xe^{x^2-y^2}\cos 2xy + e^{x^2-y^2}-2y\sin 2xy$$

$$\frac{\partial u}{\partial x} = 2xe^{x^2-y^2}\cos 2xy - 2ye^{x^2-y^2}\sin 2xy;$$

$$\frac{\partial v}{\partial y} = -2ye^{x^2-y^2}\sin 2xy + 2xe^{x^2-y^2}\cos 2xy.$$

$$\left.\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right\}.$$

$$\frac{\partial u}{\partial y} = -2ye^{-x^2-y^2}\cos 2xy - 2xe^{x^2-y^2}\sin 2xy;$$

$$\frac{\partial v}{\partial x} = 2xe^{x^2-y^2}\sin 2xy + 2ye^{x^2-y^2}\cos 2xy$$

$$\left.\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right\}.$$

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$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \left(2xe^{x^2-y^2}\cos 2xy - 2ye^{x^2-y^2}\sin 2xy\right)$$

$$+i\left(2xe^{x^2-y^2}\sin 2xy + 2ye^{x^2-y^2}\cos 2xy\right)$$

15) $f(z) = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}$;

$u(r,\theta) = \frac{\cos\theta}{r}$; $v(r,\theta) = -\frac{\sin\theta}{r}$

$$\left.\begin{aligned}\frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta}\\ \frac{\partial v}{\partial r} &= -\frac{1}{r}\frac{\partial u}{\partial \theta}\end{aligned}\right\} \quad \begin{aligned}\frac{\partial u}{\partial r} &= -\frac{\cos\theta}{r^2};\\ \frac{\partial v}{\partial \theta} &= -\frac{\cos\theta}{r}\end{aligned}$$

$$\frac{\partial u}{\partial r} = -\frac{\cos\theta}{r^2} = \frac{1}{r}\frac{\partial v}{\partial \theta} = -\frac{\cos\theta}{r^2};$$

$$\frac{\partial v}{\partial r} = \frac{\sin\theta}{r^2}; \quad \frac{\partial u}{\partial \theta} = -\frac{\sin\theta}{r}$$

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$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\sin\theta}{r^2} = \frac{\partial v}{\partial r}$$

Hence, differentiable.

$$\therefore f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right);$$

$$e^{-i\theta}\left[-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right]$$

$$f'(z) = \frac{e^{-i\theta}}{r^2}\left[-\cos\theta + i\sin\theta\right]$$

Harmonic Function: A function is

$\phi(x,y)$ is Harmonic in D if:

$\nabla^2\phi = 0$

$\hookrightarrow$ Domain of $f(x,y)$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

$f(x,y) = u(x,y) + iv(x,y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\nabla^2 u = 0.$$

$$\nabla^2 v = 0$$

$$u(x,y) = x^3 - 3xy^2 - 5y$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial^2 u}{\partial x^2} = 6x;$$

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$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^3 - 3xy^2 - 5y\right).$$

$$\frac{\partial u}{\partial y} = -3x \cdot 2y - 5 \; ; \quad \frac{\partial^2 u}{\partial y^2} = -6x \; ;$$

$$= -6xy - 5$$

$$\frac{\partial^2 u}{\partial x^2} = 6x \, , \quad \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = \cancel{6x - 6x} = 0 .$$

$$\nabla^2 u = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$v(x,y)$ is the harmonic conjugate of $u(x,y)$.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 ;$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 .$$

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$$V(x,y) = \int 3x^2 - 3y^2 \, dy + g(x)$$

$$V(x,y) = 3x^2y - y^3 + g(x).$$

$$\frac{\partial V}{\partial x} = 6xy + g'(x) = -\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial y} = -6xy - 5; \qquad \frac{\partial u}{\partial y} = -\frac{\partial V}{\partial x};$$

$$6xy + g'(x) = -(-6xy - 5)$$

$$\cancel{6xy} + g'(x) = \cancel{6xy} + 5$$

$$g'(x) = 5$$

$$g(x) = 5x + C;$$

$$V(x,y) = 3x^2y - y^3 + 5x + C \Big\} \text{ solution.}$$

Excercise : 3.4

3) $u(x,y) = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial^2 u}{\partial y} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{2-2} = 0$$

$u(x,y)$ is Harmonic

$$\boxed{\nabla^2 u = 0}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};$$

$$\frac{\partial u}{\partial y} = 2x$$

$$v(x,y) = \int 2x\,dy + g(x).$$

$$v(x,y) = 2xy + g(x)$$

$$\frac{\partial v}{\partial x} = \cancel{2y} + g'(x) = -\frac{\partial u}{\partial y} = \cancel{2y}$$

$$g'(x) = 0$$

$$g(x) = c \Big\}.$$

$$v(x,y) = 2xy + c;$$

$$f(x,y) = x^2 - y^2 + i(2xy + c)$$

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5) $u(x,y) = \ln(x^2+y^2)$.

solution:-

$$\frac{\partial u}{\partial x} = \frac{2x}{(x^2+y^2)}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-(2x^2-2y^2)}{(x^2+y^2)^2};$$

$$\frac{\partial u}{\partial y} = \frac{2y}{(x^2+y^2)}; \quad \frac{\partial^2 u}{\partial y^2} = \frac{2(x^2+y^2) - 2y(2y)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\nabla^2 u = 0.$$

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$$\frac{\partial v}{\partial y} = \frac{2x}{(x^2+y^2)}$$

$$v(x,y) = \int \frac{2x\,dy}{x^2+y^2} + g(x).$$

$$v(x,y) = 2x\,\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) + g(x).$$

$$\frac{\partial v}{\partial x} = \frac{2}{y}\tan^{-1}\left(\frac{x}{y}\right) + \frac{2x}{y}\,\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y} + g'(x);$$

$$-\frac{\partial u}{\partial x} = \frac{-2y}{(x^2+y^2)};$$

$$= \frac{2}{y}\tan^{-1}(x/y) + \frac{2x}{\cancel{y^2}}\,\frac{\cancel{y^2}}{x^2+y^2} + g'(x)$$

$$-\frac{2}{y}\tan^{-1}(x/y) - \frac{2x-2y}{(x^2+y^2)} = g'(x).$$

$$g(x) = \int\left[\frac{2}{y}\tan^{-1}(x/y) + \frac{2x+2y}{x^2+y^2}\right]dx$$

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int\frac{x\,dx}{(x^2+1)}$$

$$x\tan^{-1}x - \frac{1}{2}\ln|x^2+1|$$

$$-g(x) = +\frac{2}{y}\left[x.\tan^{-1}\left(\frac{x}{y}\right) - \int\frac{y^2x\,dx}{(x^2+y^2)}\right.$$

$$\left. + \int\frac{2x\,dx}{(x^2+y^2)} + \int\frac{2y\,dx}{x^2+y^2}\right]$$

$$-g(x) = \frac{2}{y}\left[x\tan^{-1}\left(\frac{x}{y}\right) - \frac{y^2}{2}\ln|x^2+y^2|\right.$$

$$\left.\ln|x^2+y^2| + \cancel{2y}\,\frac{1}{\cancel{y}}\tan^{-1}\left(\frac{x}{y}\right)\right].$$

$$\frac{2}{y}\left[x\tan^{-1}\left(\frac{x}{y}\right) + 2\tan^{-1}\left(\frac{x}{y}\right) - \frac{y^2}{2}\ln|x^2+y^2|\right.$$

$$\left. + \ln|x^2+y^2|\right].$$

$$\frac{2(x+2)}{y}\tan^{-1}\left(\frac{x}{y}\right) - y\ln|x^2+y^2| + \frac{2}{y}\ln|x^2+y^2| + c.$$

$$v(x,y) = 2x.\int\frac{dy}{y^2+x^2} + g(x)$$

$$v(x,y) = 2x\left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right) + \frac{g(x)}{2x}\right].$$

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$$v(x,y) = 2\tan^{-1}\left(\frac{y}{x}\right) + g(x).$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{-2y}{x^2+y^2};$$

$$= 2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) + g'(x).$$

$$= \frac{2\cancel{x^2}}{x^2+y^2} \cdot \frac{-y}{\cancel{x^2}} + g'(x)$$

$$\cancel{\frac{-2y}{x^2+y^2}} + g'(x) = \cancel{-\frac{2y}{x^2+y^2}}$$

$$g'(x) = 0.$$

$$g(x) = C.$$

$$v(x,y) = 2\tan\left(\frac{y}{x}\right) + C;$$

$$f(x,y) = \ln(x^2+y^2) + i\left[2\tan^{-1}\left(\frac{y}{x}\right) + C\right]$$

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294897 Home Practice.

BEE11B

proof: $f'(z) = e^{-i\theta}(u_r + i v_r)$.

Solution: $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$$

$$u_r = \cos\theta\, u_x + \sin\theta\, u_y$$
$$u_\theta = -r\sin\theta\, u_x + r\cos\theta\, u_y$$
$$v_r = \cos\theta\, v_x + \sin\theta\, v_y$$
$$v_\theta = -r\sin\theta\, v_x + r\cos\theta\, v_y$$

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$$\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

$$R = AX$$

$$X = A^{-1}X$$

$$A^{-1} = \frac{\operatorname{adj} A}{|A|} = \frac{\begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix}}{(r\cos^2\theta + r\sin^2\theta)}$$

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_r \\ u_\theta \end{pmatrix}.$$

Similarly;

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_r \\ v_\theta \end{pmatrix}.$$

$$u_x = \cos\theta\, u_r - \frac{1}{r}\sin\theta\, u_\theta$$

$$v_x = \cos\theta\, v_r - \frac{1}{r}\sin\theta\, v_\theta$$

Using Cauchy Reimann Equation:

$$u_r = \frac{1}{r} v_\theta \quad ; \quad v_r = -\frac{1}{r} u_\theta$$

$$\left.\begin{aligned} u_x &= \cos\theta\, u_r + \sin\theta\, v_r \\ v_x &= \cos\theta\, v_r - \sin\theta\, u_r \end{aligned}\right\}$$

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Method: 2. Chain Rule

$$\left.\begin{aligned} r^2 &= x^2+y^2 \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}\right\} \text{polar coordinates}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} \\ \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} \end{aligned}\right\} \text{chain rule}$$

$$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2+y^2).$$

$$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x\frac{\partial x}{\cancel{\partial x}} \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{-y/x^2}{1+\left(\frac{y}{x}\right)^2}$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = -\frac{y}{r^2}.$$

So;

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta; \quad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}$$

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Using Cauchy-Reimann Equation.

$$u_r = \frac{1}{r} v_\theta \; ; \qquad v_r = -\frac{1}{r} u_\theta .$$

$$u_x = \cos\theta \, u_r - \frac{\sin\theta}{r} u_\theta$$

$$v_x = \cos\theta \, v_r - \frac{\sin\theta}{r} v_\theta$$

using CREs:

$$u_x = \cos\theta \, u_r + \sin\theta \, v_r$$

$$v_x = \cos\theta \, v_r - \sin\theta \, u_r$$

$$f'(z) = u_x + i v_x$$

$$f'(z) = \cos\theta \, u_r - i \sin\theta \, u_r + \sin\theta \, v_r + i \cos\theta \, v_r$$

$$f'(z) = (\cos\theta - i\sin\theta) u_r + i v_r (\cos\theta - i\sin\theta)$$

$$f'(z) = e^{-i\theta} (u_r + i v_r)$$

proved.

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15)

$u(r,\theta) = r^3 \cos 3\theta;$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{\partial u}{\partial r} = 3r^2 \cos 3\theta;$$

$$\frac{\partial^2 u}{\partial r^2} = 6r\cos 3\theta;$$

$$\frac{\partial u}{\partial \theta} = r^3(-3\sin 3\theta)$$

$$\frac{\partial^2 u}{\partial \theta^2} = -9r^3 \cos 3\theta;$$

$$\nabla^2 u = 6r\cos 3\theta + \frac{1}{\cancel{r}}(3r^{\cancel{2}}\cos 3\theta) + \frac{1}{\cancel{r^2}}(-9\cancel{r^3}\cos 3\theta).$$

$$\nabla^2 u = \cancel{6r\cos 3\theta} + \cancel{3r\cos 3\theta} - \cancel{9r\cos 3\theta}$$

$\nabla^2 u = 0$ Hence proved.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial r}{\partial \theta};$$

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$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta};$$

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} = r\left[3r^2\cos 3\theta\right].$$

$$v(r,\theta) = \int 3r^3\cos 3\theta\, d\theta + g(r).$$

$$v(r,\theta) = \cancel{3}r^3\frac{\sin 3\theta}{\cancel{3}} + g(r).$$

$$v(r,\theta) = r^3\sin 3\theta + g(r)$$

$$\frac{\partial v}{\partial r} = 3r^2\sin 3\theta + g'(r) = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

$$3r^2\sin 3\theta + g'(r) = -\frac{1}{\cancel{r}}\left[-3r^{\cancel{3}}\sin 3\theta\right].$$

$$\cancel{3r^2\sin 3\theta} + g'(r) = \cancel{3r^2\sin 3\theta}$$

$$g'(r) = 0$$

$$g(r) = c.$$

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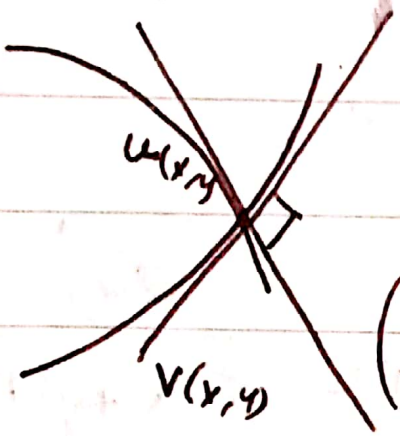
$v(r,\theta) = r^3 \sin 3\theta + c;$

$f(r,\theta) = r^3 \cos 3\theta + i(r^3 \sin 3\theta)$ Solution

$$\boxed{f(z) = z^3;}$$

- Orthogonal Families Curves.



$u(x,y) = c_1$

$v(x,y) = c_2$

$$\left(\frac{\partial u/\partial x}{\partial u/\partial y}\right)\left(-\frac{\partial v/\partial x}{\partial v/\partial y}\right) = -1$$

Gradient of Vector Field:

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$F(x,y) = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}$$

$$P(x,y) = \frac{\partial\phi}{\partial x}; \quad Q(x,y) = \frac{\partial\phi}{\partial y};$$

Divergence:

$$\nabla \cdot F = -\nabla\nabla\phi = -\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} = 0.$$

$F = -\nabla\phi$ } Gauss' Law.

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$$\phi = \frac{q q_0}{4\pi \varepsilon_0 r} \Bigg\} \frac{q q_0}{4\pi \varepsilon_0}\left[\frac{\partial}{\partial x}\cdot\frac{\hat{i}}{(x^2+y^2)^{1/2}} + \frac{\partial}{\partial y}\cdot\frac{\hat{j}}{(x^2+y^2)^{1/2}}\right]$$

$$F = -\nabla\phi,$$

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(\frac{1}{r}\right) &= (x^2+y^2)^{-1/2-1}\left(-\frac{1}{2}\right)\cdot 2x \\ \frac{\partial}{\partial x}\left(\frac{1}{r}\right) &= \left(\frac{-1}{2}\right)(x^2+y^2)^{-1/2-1}\, 2y \end{aligned}\right\}$$

$$F = -\frac{q q_0}{4\pi \varepsilon_0}\left[\frac{-\cancel{2}x\hat{i}}{\cancel{2}(x^2+y^2)^{+3/2}} - \frac{\cancel{2}y\hat{j}}{\cancel{2}(x^2+y^2)^{3/2}}\right]$$

$$F = \frac{q q_0}{4\pi \varepsilon_0}\left[\frac{x\hat{i}}{r^3} + \frac{y\hat{j}}{r^3}\right]$$

$$F = \frac{q q_0}{4\pi \varepsilon_0 r^3}\cdot(x\hat{i} + y\hat{j})$$

$$\vec{F} = \frac{q q_0}{4\pi \varepsilon_0 r^3}\vec{r} \Bigg\}$$ Coulomb's electrostatic Law of attraction.

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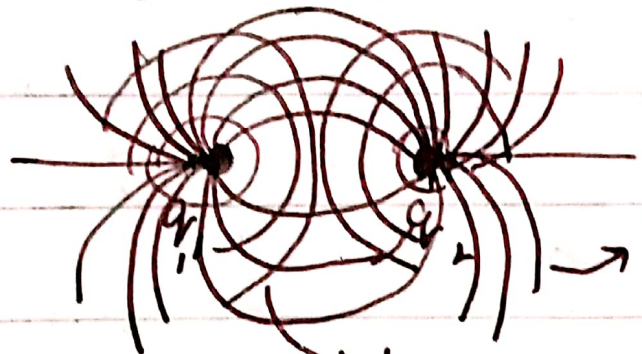
Complex Potentials:

$$\Omega(z) = \phi(x,y) + i\,\Psi(x,y).$$

Level curves:

$\phi(x,y) = c_1$ equipotential curves

$\Psi(x,y) = c_2$ lines of forces.



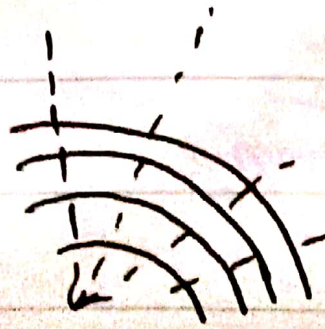
$\rightarrow \Psi(x,y) = c_2$

$\rightarrow \phi(x,y) = c_1$

streamlines $\Psi(x,y) = c_2$

Fluid Flow.

equipotential $\phi(x,y) = c_1$.



$\rightarrow \phi(x,y) = c_1$

isotherms

Flow lines of heat flux

$\Psi(x,y) = c_2$

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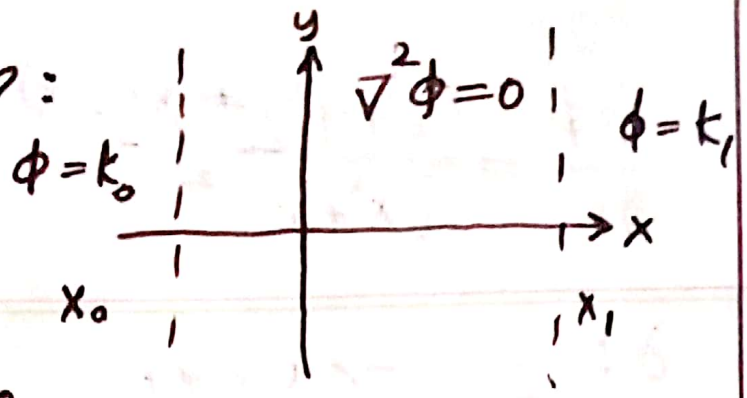
Dirichlet Problem:



$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

$$\left.\begin{aligned}\phi(x_0, y) &= k_0\\ \phi(x_1, y) &= k_1\end{aligned}\right\}.$$

$$x_0 < x < x_1$$
$$-\infty < y < +\infty$$

$$\because \frac{d^2\phi}{dx^2} = 0 \qquad \phi(x) = ax + b$$

$$\phi(x_0) = ax_0 + b = k_0$$
$$\phi(x_1) = ax_1 + b = k_1$$

$$ax_0 + b = k_0$$
$$ax_1 + b = k_1$$

$$\begin{bmatrix} x_0 & 1 \\ x_1 & 1 \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} k_0 \\ k_1 \end{pmatrix}.$$

$$\sim \left[\begin{array}{cc|c} x_0 & 1 & k_1 \\ x_1 - x_0 & 0 & k_1 - k_0 \end{array}\right]$$

$$a(x_1 - x_0) = (k_1 - k_0).$$

$$a = (k_1 - k_0)/(x_1 - x_0).$$

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$$a x_0 + b = k_0$$

$$b = k_0 - \frac{(k_1 - k_0)}{(x_1 - x_0)} x_0$$

$$b = \frac{k_0 x_1 - \cancel{k_0 x_0} - k_1 x_0 + \cancel{k_0 x_0}}{(x_1 - x_0)}$$

$$b = \frac{k_0 x_1 - k_1 x_0}{(x_1 - x_0)}$$

$$y = ax + b$$

$$\phi(x) = \left(\frac{k_1 - k_0}{x_1 - x_0}\right) x + \frac{k_0 x_1 - k_1 x_0}{(x_1 - x_0)}$$

$$\phi(x) = \left(\frac{k_1 - k_0}{x_1 - x_0}\right) x + k_0 - \left(\frac{k_1 - k_0}{x_1 - x_0}\right) x_0$$

$$\boxed{\phi(x) = \left(\frac{k_1 - k_0}{x_1 - x_0}\right)(x - x_0) + k_0}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial \psi}{\partial y} = \left(\frac{k_1 - k_0}{x_1 - x_0}\right), \quad \frac{\partial \psi}{\partial x} = 0.$$

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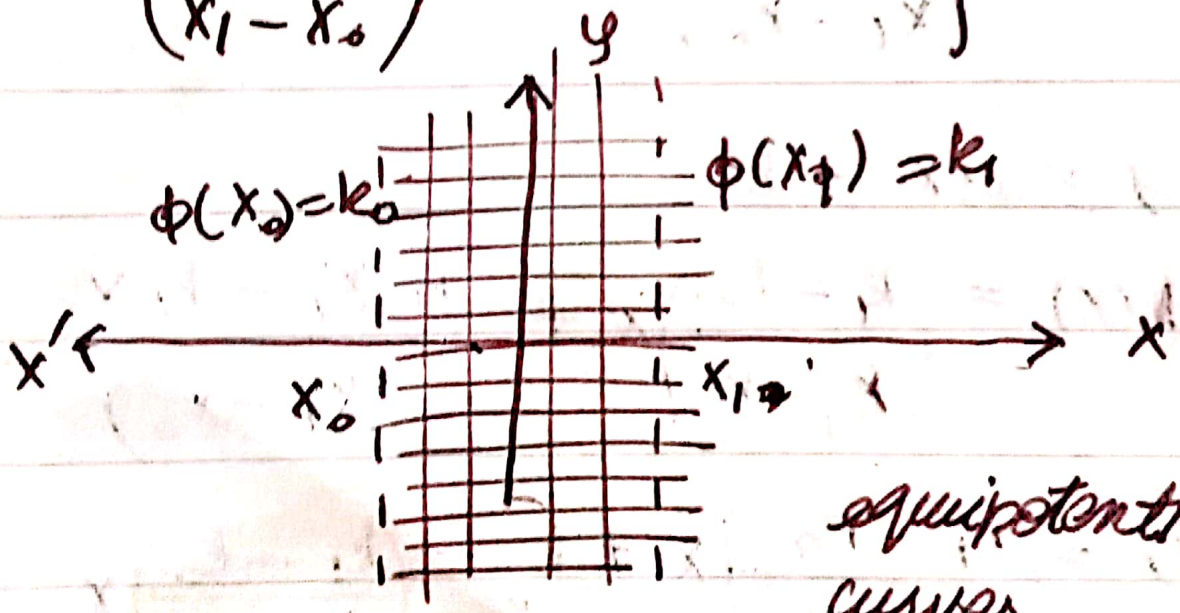
$$\psi(y) = \left(\frac{k_1 - k_0}{x_1 - x_0}\right) y ;$$

$$\Omega(z) = \phi(x) + i\,\psi(y).$$

$$\Omega(z) = \left(\frac{k_1 - k_0}{x_1 - x_0}\right)(x - x_0) + k_0 + i\left(\frac{k_1 - k_0}{x_1 - x_0}\right) y$$

$$\Omega(z) = \left(\frac{k_1 - k_0}{x_1 - x_0}\right)(z - x_0) + k_0 \Big\}.$$



equipotential curves.

Excercise: 3.5

9) $\phi(x, y) = \dfrac{x}{x^2 + y^2}$;

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \; ; \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

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$$\frac{\partial \phi}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2};$$

$$\frac{\partial \phi}{\partial y} = \frac{-x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}.$$

$$\frac{\partial \psi}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2}; \quad \frac{\partial \psi}{\partial x} = \frac{2xy}{(x^2+y^2)^2}.$$

$$\cancel{\psi(y) = \int \frac{-x^2+y^2}{(x^2+y^2)^2}\, dy + g(x).}$$

$$\psi = \int \frac{2xy}{(x^2+y^2)^2}\, dx + g(y).$$

$$\psi = y \int 2x(x^2+y^2)^{-2}\, dx + g(y).$$

$$\psi = y \cdot \frac{(x^2+y^2)^{-2+1}}{-2+1} + g(y).$$

$$\psi(y) = \frac{-y}{x^2+y^2} + g(y).$$

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$$\frac{\partial \psi(y)}{\partial y} = \frac{-1(x^2+y^2)+y\,2y}{(x^2+y^2)^2} + g'(y)$$

$$\frac{-x^2+y^2}{(x^2+y^2)} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} + g'(y)$$

$$g'(y) = 0.$$

$$g(y) = c$$

$$\Omega(z) = \phi(x,y) + i\psi(x,y).$$

$$= \frac{x}{(x^2+y^2)} + i\left[\frac{-y}{x^2+y^2} + c\right];$$

$$P(x,y)\hat{i} + Q(x,y)\hat{j} = F(x,y)$$

$$\frac{\partial \phi}{\partial x} = P(x,y) \; ; \quad \frac{\partial \phi}{\partial y} = Q(x,y).$$

$$\phi = \frac{x}{x^2+y^2}$$

$$\frac{\partial \phi}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2}, \quad \frac{\partial \phi}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

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(a) $F(x,y) = \dfrac{-x^2+y^2}{(x^2+y^2)^2}\hat{i} - \dfrac{2xy}{(x^2+y^2)^2}\hat{j}$ }

(b) $\Omega(z) = \dfrac{x - iy}{x^2+y^2}$;

$\Omega(z) = \dfrac{\bar{z}}{|z|^2}$

$\Omega(z) = 1/z$; }

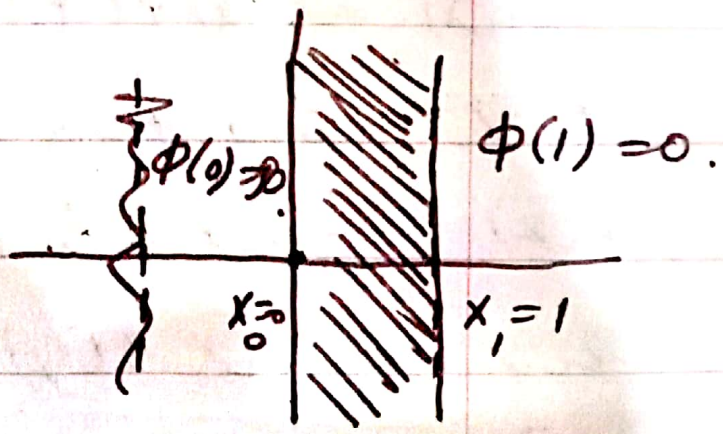
---

a) $\phi(x,y) = ?$ $\quad 0 < x < 1$

$\nabla^2\phi = 0$ $\quad -\infty < y < +\infty$

$\phi(0,y) = 50$

$\phi(1,y) = 0$



Solution:-

$\dfrac{d^2\phi}{dx^2} = 0$

$\phi(x) = ax+b$

$\phi(0) = 50 = a(0)+b$ $\quad b = 50$;

$\phi(1) = 0 = a+b = 0$ $\quad \boxed{a = -50}$

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$$\phi(x) = -50x + 50;$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}; \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x};$$

$$-50 = \frac{\partial \psi}{\partial y};$$

$$\psi(y) = -50y + g(x)$$

$$\frac{\partial \psi}{\partial x} = g'(x) = -\frac{\partial \phi}{\partial y} = 0.$$

$$g(x) = c$$

$$\psi(y) = -50y + c$$

$$\Omega(z) = \phi(x) + i\psi(y).$$

$$\Omega(z) = -50x + 50 + i(-50y + c).$$

$$\Omega(z) = -50x + 50 - 50iy.$$

$$\Omega(z) = -50(x + iy) + 50$$

$$\boxed{\Omega(z) = -50z + 50}$$

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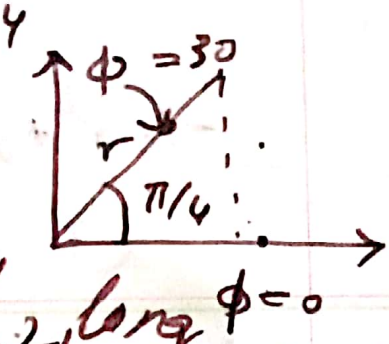
13). The potential $\phi(\theta)$ between the two infinitely long conducting plates.

$$\frac{d^2\phi}{d\theta^2} = 0.$$



$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\phi(\theta) = a\theta + b;$$

$$\phi(0) = 0 = b \qquad \boxed{b = 0}$$

$$\phi(\pi/4) = a\frac{\pi}{4} = 30.$$

$$a = \frac{120}{\pi};$$

$$\left.\phi(\theta) = \frac{120}{\pi}\theta\right\}$$

$$\left.\begin{aligned}\frac{\partial\phi}{\partial r} &= \frac{1}{r}\frac{\partial\psi}{\partial\theta}\\ \frac{\partial\phi}{\partial\theta} &= -\frac{1}{r}\frac{\partial\psi}{\partial r}\end{aligned}\right\}$$ Cauchy Riemann Equations.

$$\cancel{\frac{\partial\psi}{\partial\theta} = 0; \quad -\frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{120}{\pi};}$$

$$\cancel{\frac{\partial\psi}{\partial r} = -\frac{120}{\pi}r; \quad \psi(r) = -\frac{60}{\pi}r^2.}$$

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$$\phi(\theta) = \frac{120}{\pi}\theta.$$

$$\cancel{\psi(r) = -\frac{120}{\pi}\ln r}$$

$$\cancel{\frac{\partial\psi(r)}{\partial r} = -\frac{120}{\pi}\cdot\frac{1}{r}}$$

$$\frac{\partial\phi}{\partial\theta} = \frac{120}{\pi}; \quad \frac{\partial\phi}{\partial r} = 0.$$

$$\frac{\partial\phi}{\partial\theta} = +\frac{1}{r}\frac{\partial\psi}{\partial r}; \quad \frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta};$$

a)

$$\psi = -\frac{120}{\pi}\int\frac{dr}{r}$$

$$\cancel{-\frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{120}{\pi}; \quad \frac{\partial\psi}{\partial r} = -\frac{120}{\pi}r.}$$

$$\psi = -\frac{120}{\pi}\ln r.$$

$$\cancel{\psi(r) = -\frac{120}{\pi}\frac{r^2}{2} = -\frac{60}{\pi}r;}$$

(b)

$$\Omega(z) = \phi(\theta) + i\psi(r).$$

c

$$= -\frac{120}{\pi}\theta + \cancel{i\left(-\frac{60}{\pi}r\right)}\cdot i\left(-\frac{120}{\pi}\ln r\right)$$

$$\Omega(z) = -\frac{120}{\pi}\theta - \left.\begin{matrix}\frac{120}{\pi}\ln r \\ \cancel{\frac{60}{\pi}r}\end{matrix}\right\} - \frac{120}{\pi}i\left[\ln r + i\theta\right]$$

c

---

14). The steady state temperature $\phi(r)$ between the two concentric circular ~~coordinates~~ conducting cylinders satisfies a place equation.

$$\left\{ \nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}; \right\}$$

$$r^2 \frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} = 0.$$

a) Show that a solution of the equation subject to the boundary values

$\phi(a) = k_0$, $\phi(b) = k_1$, $\phi(r) = A\ln(r) + B$

$$A = \frac{k_0 - k_1}{\ln(a/b)}; \quad B = \frac{-k_0 \ln b + k_1 \ln a}{\ln(a/b)};$$

(b) Find $\Omega(z)$;

$$ax^2y'' + bxy' + cy = 0 \Big\}$$ Euler-Cauchy Differential equation.

$$\phi(r) = r^m$$

$$\phi'(r) = m r^{m-1}$$

$$\phi''(r) = m(m-1) r^{m-2}$$

$$m(m-1) r^m + r^m m = 0.$$

$$r^m(m^2 \cancel{- m + m}) = 0.$$

$m^2 = 0$. $\quad \phi(r) = c_1 r^0 + c_2 r^0 \ln r$

$m = 0$ $\quad \phi(r) = c_1 + c_2 \ln r$

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$$\phi(r) = c_1 + c_2 \ln r.$$

$$\phi(a) = c_1 + c_2 \ln a = k_0$$

$$\phi(b) = c_1 + c_2 \ln b = k_1$$

$$\sim \left(\begin{array}{cc|c} 1 & \ln a & k_0 \\ 1 & \ln b & k_1 \end{array}\right). \quad R_2 - R_1.$$

$$\sim \left(\begin{array}{cc|c} 1 & \ln a & k_0 \\ 0 & \ln b/a & k_1 - k_0 \end{array}\right) \quad c_2 \ln(b/a) = (k_1 - k_0)$$

$$c_2 = \frac{k_1 - k_0}{\ln(b/a)}$$

$$c_2 = \frac{k_0 - k_1}{\ln(a/b)};$$

$$c_1 + c_2 \ln a = k_0.$$

$$c_1 = k_0 - \ln a \left(\frac{k_0 - k_1}{\ln(a/b)} \right).$$

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$$c_1 = \frac{\cancel{k_0 \ln a} - k_0 \ln b - \cancel{k_0 \ln a} + k_1 \ln a}{\ln(a/b)}$$

$$c_1 = \frac{k_1 \ln a - k_0 \ln b}{\ln(a/b)}$$

$$\phi(r) = c_1 + c_2 \ln r.$$

$$c_2 = A, \quad c_1 = B.$$

$-k_0)$

$$\phi(r) = A \ln r + B.$$

$$A = \frac{k_0 - k_1}{\ln(a/b)}; \quad B = \frac{-k_0 \ln b + k_1 \ln a}{\ln(a/b)}.$$

Hence proved.

$$\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}; \quad \frac{\partial \phi}{\partial \theta} = -\frac{1}{r}\frac{\partial \psi}{\partial r}.$$

$$\frac{\partial \phi}{\partial r} = \frac{A}{r}; \quad \frac{\partial \phi}{\partial \theta} = 0.$$

$$\frac{A}{\cancel{r}} = \frac{1}{\cancel{r}}\frac{\partial \psi}{\partial \theta}; \quad \psi(\theta) = A\theta;$$

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$$\Omega(z) = \phi(r) + i\psi(\theta).$$

$$\Omega(z) = A\ln r + B + iA\theta.$$

$$A = \frac{k_0 - k_1}{\ln(a/b)}; \quad B = \frac{-k_0\ln b + k_1\ln a}{\ln(a/b)}$$

Chapter 4: Elementary Functions.

Exponential function:

$$e^{w} = e^{x}(\cos y + i\sin y).$$

$$\frac{d e^{z}}{dz} = e^{z}.$$

Example: $iz^4(z^2 - e^z)$.

$$\frac{d}{dz}\left(iz^4(z^2 - e^z)\right)$$

$$= 6iz^{6-1} - i\frac{d}{dz}(z^4 e^z).$$

$$= 6iz^5 - i(4z^3e^z + z^4e^z)$$

$$= 6iz^5 - 4iz^3e^z - iz^4e^z.$$

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$|e^z| = e^x$

$\arg(e^z) = y + 2n\pi, \quad n = 0, \pm 1, \pm 2, \pm 3 \ldots$

$\overline{e^z} = e^{\bar{z}}; \quad e^{z+2n\pi} = e^z$ } periodic $p = 2n\pi$;

Fundamental Region of e^z:

$-\infty < x < +\infty; \quad -\pi < y \leq +\pi.$

Complex Logarithm: $Ln(z) \rightarrow$ Principle value;

$Ln(z) = \ln r + i\theta;$

$Ln(z) = \ln\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)$ } polar and cartesian coordinate.

Example:

$e^w = 1+i;$

$w = \ln|1+i| = \ln(\sqrt{1^2+1^2}) + i\tan\left(\frac{1}{1}\right).$

$w = \frac{1}{2}\ln|2| + \frac{\pi}{4}i; \quad w = \frac{\ln 2}{2} + \frac{\pi}{4}i;$

$\ln(z_1 z_2) = \ln z_1 + \ln z_2$

$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2.$

$\ln(z^n) = n\ln z;$

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$\ln z = \ln|z| + i(\theta + 2n\pi)$.

$\operatorname{Ln} e^{1+3\pi/2 i}$

$\left|e^{1+3\pi/2}\right| = e$

$\operatorname{Arg}\left|e^{1+3\pi/2 i}\right| = \cancel{\tan^{-1}\left(\frac{3\pi}{}\right)}$

$z = e\left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right]$

$(0 - i)$. $\tan^{-1}\left(\frac{-1}{0}\right) = \tan^{-1}(-\infty)$
$= -\frac{\pi}{2}$;

$= \ln e + i\left(-\frac{\pi}{2}\right) = 1 - \frac{\pi}{2}i\}$. Ans.

Complex powers:

$z^{\alpha} = e^{\alpha \ln z}$; '$\alpha$' is a complex number.

a) i^{2i};

$z = i$; $\alpha = 2i$ $\ln i = \ln(\sqrt{0^2+1^2})$
$+ i\tan\left(\frac{1}{0}\right) + 2\pi n$

$e^{2i \ln i} = e^{2i\left(\frac{\pi}{2}\right)i}$

$e^{\cancel{2}i^2\pi/\cancel{2}} = e^{-\pi}$

$e^{-(4n+1)\pi}$

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$$z_0^{\alpha_1} z_0^{\alpha_2} = z_0^{\alpha_1+\alpha_2} ; \quad \frac{z^{\alpha_1}}{z^{\alpha_2}} = z^{\alpha_1-\alpha_2}$$

$$(z^{\alpha})^n = z^{n\alpha} . \quad n = 0, \pm 1, \pm 2, \pm 3 \ldots$$

Principal value of $z^{\alpha} = e^{\alpha \operatorname{Ln} z}$;

$$\frac{d \operatorname{Ln} z}{dz} = \frac{1}{z} ;$$

$$\frac{d z^{\alpha}}{dz} = \alpha z^{\alpha-1} ;$$

$$z^{2i} = i^{2i}$$

$$\frac{d}{dz} . z^{2i} = 2i z^{2i-1}$$

$$2i(i^{2i-1}) ,$$

Trigonometric and Hyperbolic Functions.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Example,

$$\tan z = \frac{\sin z}{\cos z} ; \quad \cos i = \frac{e^{i^2} + e^{-i^2}}{2}$$

Ex. $\cos i = \dfrac{e^{-1} + e}{2}$; $\sin(2+i) = ?$

$$\sin(2+i) = \frac{e^{i(2+i)} - e^{-i(2+i)}}{2i}$$

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$$= \frac{e^{2i}\, e^{i^2} - e^{-2i}\, e^{-i^2}}{2i} = \frac{e^{-1} e^{2i} - e^{-2i} e^{1}}{2i}$$

$$\sin(2+i) = \frac{e^{2i-1} - e^{-2i+1}}{2i},$$

$$= \frac{e^{-1}(\cos 2 + i \sin 2) - e(\cos 2 - i \sin 2)}{2i}$$

$$\left(\frac{e^{-1} - e}{2i}\right) \cos 2 + \cancel{i}\left(\frac{e^{-1} + e}{2\cancel{i}}\right) \sin 2.$$

$$\left(\frac{e^{-1} + e}{2}\right) \sin 2 - i\left(\frac{e^{-1} - e}{2}\right) \cos 2.$$

$$\left[\frac{e + e^{-1}}{2}\right] \sin 2 + i\left(\frac{e - e^{-1}}{2}\right) \cos 2 \Big\}.$$

$$\cos^2 z + \sin^2 z = 1$$

$$\sin(-z) = -\sin z, \quad \cos(-z) = +\cos z$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1.$$

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$\sin(2z) = 2\sin z \cos z;$

$\cos(2z) = \cos^2 z - \sin^2 z;$

$z = x + iy$

$$\sin z = \frac{e^{x}e^{iy} - e^{-x}e^{-iy}}{2i}$$

$$\frac{e^{x}(\cos y + i\sin y) - e^{-x}(\cos y - i\sin y)}{2i}$$

$$\frac{(e^{x} + e^{-x})}{2i}\sin y + \frac{i(e^{x} - e^{-x})}{2}\cos y$$

$\sin z = \cosh x \sin y + i \sinh x \cos y.$

$|\sin z| = \sqrt{\cosh^2 x \sin^2 y + \sinh^2 x \cos^2 y}.$

$\sinh^2 y$

$$\left.\begin{aligned} |\sin z| &= \sqrt{\sin^2 x + \sinh^2 y} \\ |\cos z| &= \sqrt{\cos^2 x + \sinh^2 y} \end{aligned}\right\}$$

Complex Hyperbolix functions.

z_2

$$\left.\begin{aligned} \sin z &= -i\sinh(iz) \\ \cos z &= \cosh(iz) \end{aligned}\right\}.$$

$\cosh^2 z - \sinh^2 z = 1.$

$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \sinh z_2 \cosh z_1$

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$\cosh(z_1+z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$

Inverse Trigonometric and Hyperbolic functions:

$\sin^{-1}(z) = -i\ln\left[iz+(1-z^2)^{1/2}\right]$;

$\cos^{-1}(z) = -i\ln\left[iz+i(1-z^2)^{1/2}\right]$.

$\cos^{-1}(z) = -i\ln\left[z+i(z^2-1)^{1/2}\right]$;

$\tan^{-1}(z) = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)$;

$\sin^{-1}(z) = -i\ln\left(iz+(1-z^2)^{1/2}\right)$;

$\cos^{-1}(z) = i\ln\left(z+i(1-z^2)^{1/2}\right)$;

$\tan^{-1}(z) = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)$;

$\sinh^{-1} z = \ln\left|z+\sqrt{z^2+1}\right|$

$\cosh^{-1} z = \ln\left|z+\sqrt{z^2-1}\right|$

$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$.

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$$\frac{d \sin z}{dz} = \cos z; \quad \frac{d \cos z}{dz} = -\sin z$$

$$\frac{d \tan z}{dz} = \sec^2 z; \quad \frac{d \cot z}{dz} = -\csc^2 z$$

$$\frac{d \sec z}{dz} = \sec z \tan z; \quad \frac{d}{dz} \csc z = -\csc z \cot z$$

$$\frac{d \sin^{-1} z}{dz} = \frac{1}{\sqrt{1-z^2}}; \quad \frac{d \cos^{-1} z}{dz} = \frac{-1}{\sqrt{1-z^2}}$$

$$\frac{d \tan^{-1} z}{dz} = \frac{1}{z^2+1};$$

$$\frac{d \sinh z}{dz} = \cosh z; \quad \frac{d \cosh z}{dz} = \sinh z$$

$$\frac{d \tanh}{dz} = \operatorname{sech}^2 z; \quad \frac{d \coth z}{dz} = -\operatorname{csch}^2 z$$

$$\frac{d \operatorname{sech} z}{dz} = -\operatorname{sech} z \tanh z; \quad \frac{d \operatorname{csch} z}{dz} = -\operatorname{csch} z \coth z.$$

$$\frac{d \sinh^{-1} z}{dz} = \frac{1}{\sqrt{z^2+1}}.$$

$$\frac{d \cosh^{-1} z}{dz} = \frac{1}{\sqrt{z^2-1}}$$

$$\frac{d \tanh^{-1} z}{dz} = \frac{1}{(1-z^2)};$$

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Harmonic Function and Analytic ~~Conf~~ Mapping

$$\phi(x,y) = \Phi(u(x,y), v(x,y))$$

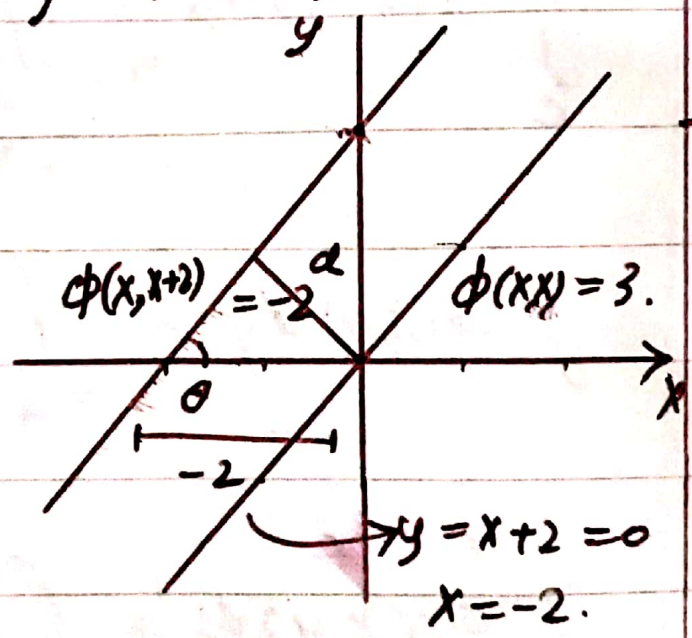
$$f(z) = u(x,y) + i\,v(x,y).$$

Example:

$$\left.\begin{aligned} y &= x \\ y &= x+2 \end{aligned}\right\} \quad \begin{aligned} \phi(x, x+2) &= -2 \\ \phi(x,x) &= 3 \end{aligned}$$

$f(z) = z = x + iy.$

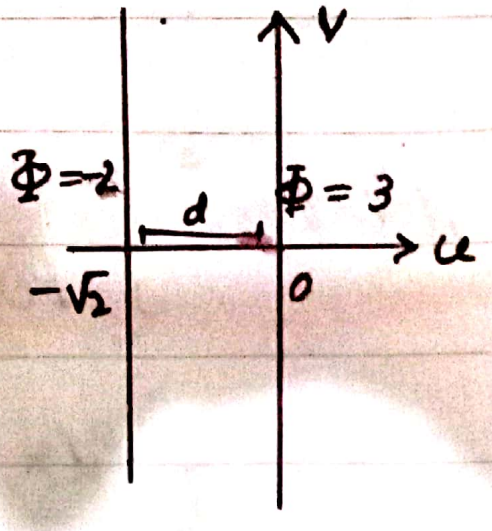
By rotating in $\frac{\pi}{4}$ anticlockwise:



$$f(z) = e^{i\pi/4} z.$$

$$\frac{d}{-2} = \sin\frac{\pi}{4}$$

$$d = -2\sin\frac{\pi}{4} = -\sqrt{2};$$



$$y - y_0 = m(x - x_0).$$

$$\Phi(u,v) - (-2) = \frac{3-(-2)}{0-(-\sqrt{2})}(u - (-\sqrt{2}))$$

$$\Phi(u,v) + 2 = \frac{5}{\sqrt{2}}(u + \sqrt{2})$$

$$\Phi(u,v) = \frac{5}{\sqrt{2}}u + 5 - 2$$

$$\boxed{\Phi(u,v) = \frac{5}{\sqrt{2}}x + 3}$$

$$f(z) = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)(x + iy)$$

3. $\rightarrow x$

$$\frac{\sqrt{2}}{2}(x-y) + \frac{\sqrt{2}}{2}(x+y)i$$

$$u(x,y) = \frac{\sqrt{2}}{2}(x-y) = \frac{(x-y)}{\sqrt{2}};$$

$$v(x,y) = \frac{(x+y)}{\sqrt{2}};$$

$$\phi(x,y) = \Phi(u(x,y), v(x,y)) = \frac{5}{\sqrt{2}}\left(\frac{(x-y)}{\sqrt{2}}\right) + 3$$

$$\phi(x,y) = \frac{5}{2}x - \frac{5}{2}y + 3$$

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$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}; \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x};$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = \frac{5}{2}; \quad \frac{\partial \phi}{\partial y} = -\frac{5}{2};$$

$$\psi(y) = \frac{5}{2}y + g(x)$$

$$\frac{\partial \psi}{\partial x} = g'(x) = -\frac{\partial \phi}{\partial y} = -\left(-\frac{5}{2}\right) = \frac{5}{2}.$$

$$g(x) = \frac{5}{2}x + c;$$

$$\psi(x,y) = \frac{5}{2}x + \frac{5}{2}y;$$

$$\boxed{\Omega(z) = \left(\frac{5}{2}x - \frac{5}{2}y + 3\right) + i\left(\frac{5}{2}x + \frac{5}{2}y\right).}$$

$$\cancel{\Omega(z) = \frac{5}{2}(x + iy)}$$

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Complex Variable and Transform
Assignment: 1

Q: 1

i) At $z_0 = i$. $f(z) = \begin{cases} \frac{z^2+1}{z-i}, & z \neq i \\ 3i, & z = i \end{cases}$

Solution:

For Continuity:

i) $\lim_{z \to z_0} f(z) =$ exists

ii) $\lim_{z \to z_0} f(z) = f(z_0)$

iii) $f(z)$ is defined at z_0

So,

$f(i) = 3i$

$$\lim_{z \to i} \frac{z^2+1}{z-i} = \frac{\cancel{(z-i)}(z+i)}{\cancel{(z-i)}}$$

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$$\lim_{z \to i} f(z) = 2i$$

Hence, the functions is not continuous because the value and limit doesnot coincide.

ii) At $z_0 = 0$, where

$$f(z) = \begin{cases} (Re(z))^2/|z|, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Solution:

$$f(0) = 0$$

$$= \lim_{z \to 0} \frac{(Re(z))^2}{|z|}$$

$$z = re^{i\theta}$$

$$|z| = r$$

$$Re(z) = r\cos\theta.$$

$$= \lim_{r \to 0} \frac{(r\cos\theta)^2}{r} = \lim_{r \to 0} r\cos^2\theta = 0\cos^2\theta = 0$$

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Hence, the given function is continuous.

iii) At $z_0 = 0$ where

$$f(z) = \begin{cases} (\bar{z})^3/z^2 & , z \neq 0 \\ 0 & , z = 0 \end{cases}$$

Solution:

Given that $f(0) = 0$.

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{(\bar{z})^3}{z^2}$$

$$\because \quad z = re^{i\theta} \rightarrow \bar{z} = re^{-i\theta}$$

$$= \lim_{r \to 0} \frac{(re^{-i\theta})^3}{(re^{i\theta})^2} = \lim_{r \to 0} \frac{re^{-3i\theta}}{e^{2i\theta}}$$

$$= \lim_{r \to 0} re^{-5i\theta} = 0.$$

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The $f(z)$ is continuous because it satisfies all conditions.

Q: 2 $\quad z(t) = z_0(1-t) + z_1(t)$;
$0 \leq t \leq 1$.

i) $T(z) = z + b$; $b \neq 0$

Solution: Given:

$$z(t) = z_0(1-t) + z_1 t$$

$$T(z) = z + b$$

$$T(z(t)) = z_0(1-t) + z_1 t + b; \quad b \neq 0$$

$\because b \Rightarrow b(1-t) + bt$ then;

$$T(z(t)) = (z_0 + b)(1-t) + (z_1 + b)t$$

$0 \leq t \leq 1$ and $b \neq 0$.

both z_0 and z_1 are shifted

$$z_0 \longrightarrow z_0 + b; \quad z_1 \longrightarrow z_1 + b$$

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$F(z) = z + b$

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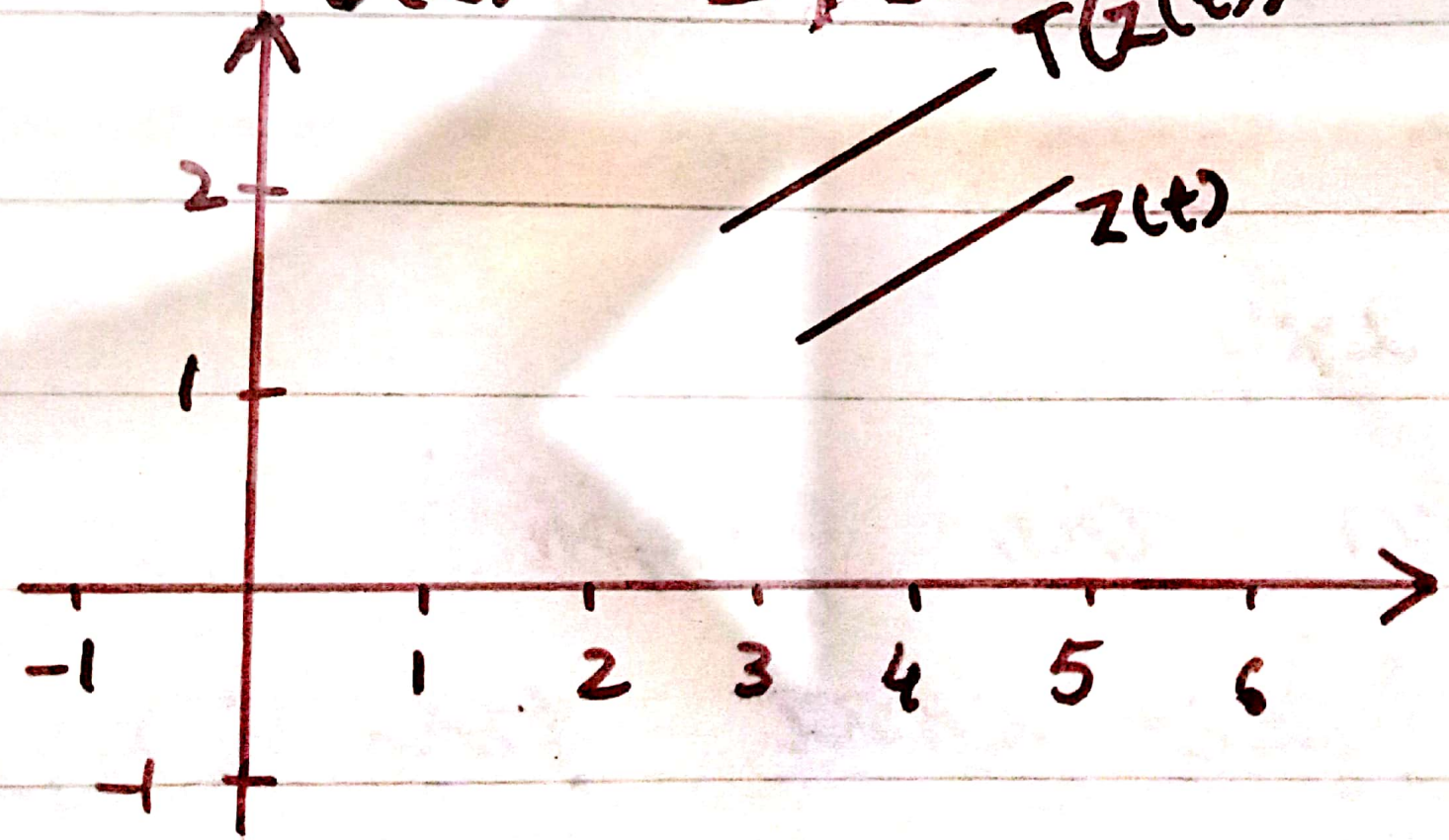

Figure.

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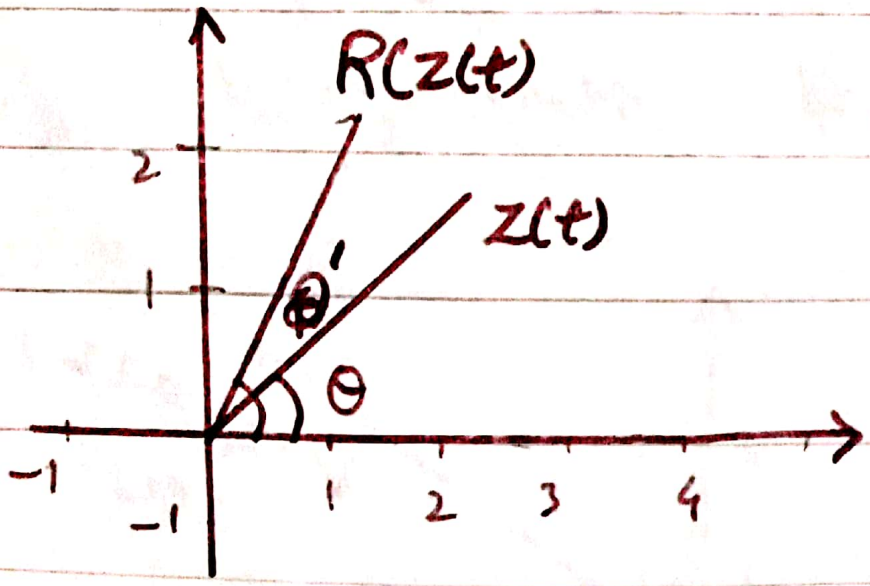
ii) $R(z) = az, \quad |a| = 1$

Solution:

$$R(z(t)) = a(z_0(1-t) + z_1 t)$$
$$= az_0(1-t) + az_1 t$$
$$0 \le t \le 1$$

The image is a line segment is rotated as shown in Figure:



$\theta' = (\theta + \pi)$

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iii) $M(z(t) = az, \; a > 0$

Solution.

$$z(t) = z_0(1-t) + z_1 t$$
$$0 \le t \le 1$$

$$M(z) = az_0(1-t) + az_1 t$$

(z_0, z_1) are scaled to az_0 and az_1

The image of line segment is magnified.

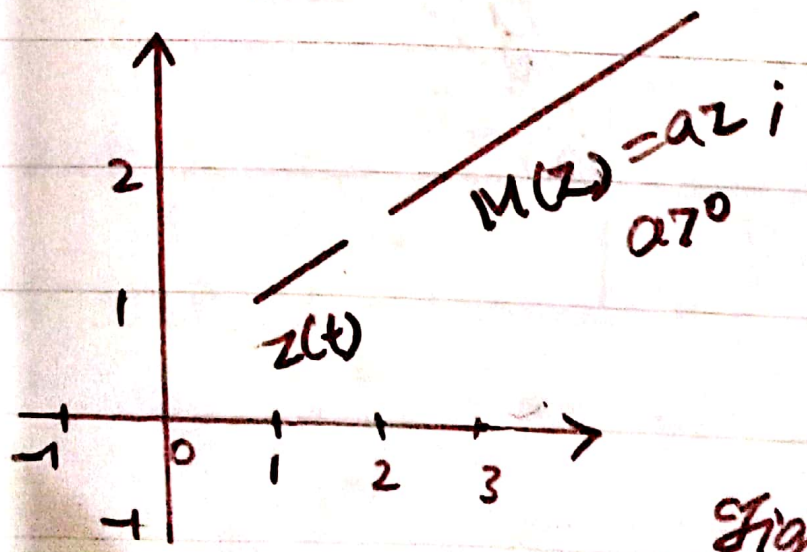

Figure.

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Q: 3

Solution:

$$w = f(z) = z^2 = x^2 - y^2 + 2ixy$$

$$u(x,y) = x^2 - y^2$$

$$v(x,y) = 2xy$$

$$A = \{z \in \mathbb{C} : z = x+iy,\ 0 < y^2 - x^2 < 3$$

Now;

$$0 < y^2 - x^2 < 3$$

$$0 < -u(x,y) < 3$$

$$0 > u(x,y) > -3$$

$$0 > u > -3$$

$$v = 2xy$$

As from the graph.

We can see that for $v(x,y) = 2xy$ we have set (x,y)

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and; the hyperbola and $y^2 = x^2$ never touch each other so we will have infinitely many pairs (x, y) and

$$-\infty < v(x,y) < +\infty$$

Hence, the region is between 0 and -3.

$$0 > u(x,y) > -3$$

or

$$-3 < u(x,y) < 0$$

$$-\infty < v(x,y) < +\infty$$

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Q: 4

Solution: Given,

$$A = \left\{ z \in \mathbb{C} : 0 < |z| \leq 3, \frac{\pi}{3} \leq \arg(z) \leq \frac{2\pi}{3} \right\}$$

$$z = re^{i\theta}; \quad w = \frac{1}{z} = \frac{1}{r}e^{-i\theta}$$

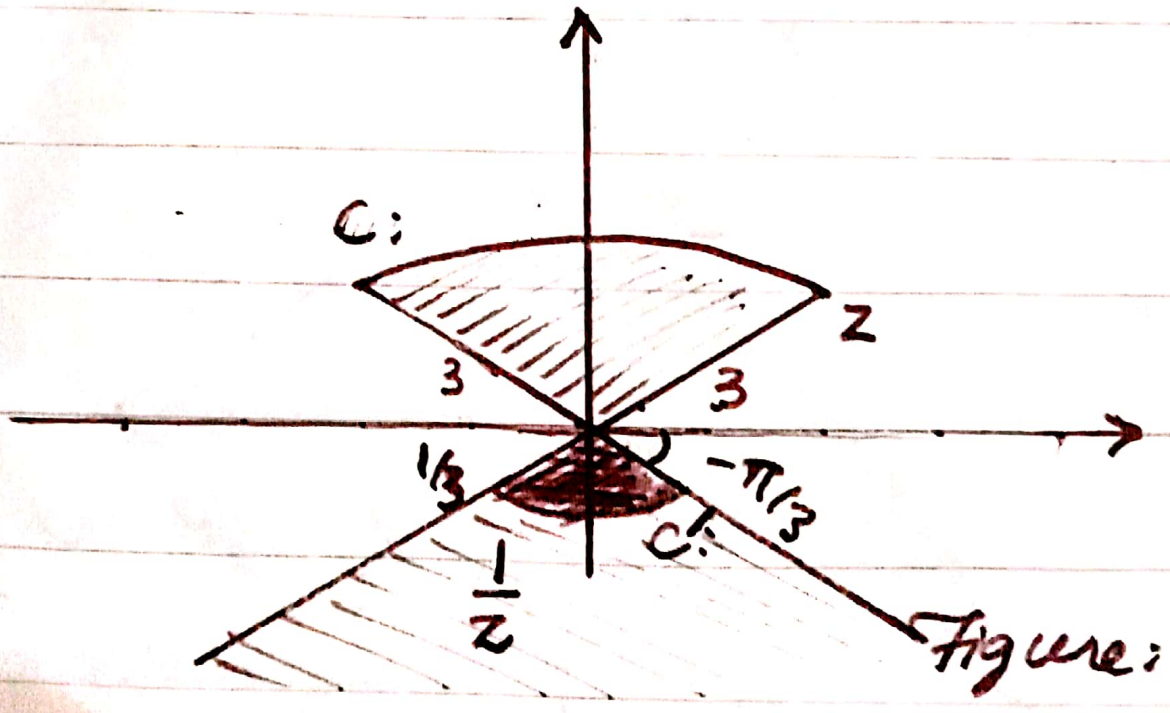

Figure:

$$\infty > |z| \geq \frac{1}{3} \quad \text{and}$$

$$-\frac{\pi}{3} \leq \arg(z) \leq -\frac{2\pi}{3}$$

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Q:5

i) Derivation of Laplace Equation in Polar Coordinates.

Solution:

$$\nabla^2 u = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

We know that, polar coordinates

$$x = r\cos\theta \; ; \quad y = r\sin\theta$$

$$r = \sqrt{x^2+y^2} \; ; \; \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

using these equations we get:

$$\frac{\partial r}{\partial x} = \cos\theta \quad ; \quad \frac{\partial r}{\partial y} = \sin\theta$$

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$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \quad ; \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

$$\frac{\partial u}{\partial x} = \cos\theta\, u_r - \frac{\sin\theta}{r} u_\theta$$

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial x^2} = \left[\cos\theta\, u_r - \frac{\sin\theta}{r} u_\theta\right]\left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right]$$

$$= \cos^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \cos\theta\sin\theta \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}(\cos\theta\, u_r).$$

$$= \cos^2\theta\, u_{rr} + \frac{\sin^2\theta}{r^2} u_{\theta\theta} - \cos\theta\sin\theta\left[\frac{u_{r\theta}}{r} - \frac{u_\theta}{r^2}\right]$$

$$+ \frac{\sin^2\theta}{r} u_r - \frac{\cos\theta\sin\theta}{r} u_{r\theta}$$

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$$u_{xx} = \cos^2\theta\, u_{rr} + \frac{\sin^2\theta}{r^2} u_{\theta\theta} -$$

$$2\frac{\cos\theta\sin\theta}{r} u_{r\theta} + \frac{\sin^2\theta}{r} u_r$$

$$+ \frac{\cos\theta\sin\theta}{r^2} u_r -$$

Now, further:

$$u_y = \left(\sin\theta\, u_r + \frac{\cos\theta}{r} u_\theta\right)$$

$$\frac{\partial}{\partial y} = \left[\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right].$$

$$u_{yy} = \sin^2\theta\, u_{rr} + \frac{\cos^2\theta}{r^2} u_{\theta\theta}$$

$$+ \cos\theta\sin\theta\left(\frac{\partial}{\partial r}(u_\theta/r)\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}(\sin\theta\, u_r)$$

u_{yy}

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$$u_{yy} = \sin^2\theta\, u_{rr} + \frac{\cos^2\theta}{r^2} u_{\theta\theta} + \frac{\cos^2\theta}{r} u_r$$

$$+ 2\frac{\cos\theta\sin\theta}{r} u_{r\theta} - \frac{\sin\theta\cos\theta}{r^2} u_\theta$$

Now adding; $u_{xx} + u_{yy} = 0$.

$$(\sin^2\theta + \cos^2\theta) u_{rr} + \frac{u_{\theta\theta}}{r^2}(\cos^2\theta + \sin^2\theta) + \frac{(\sin^2\theta + \cos^2\theta)}{r} u_r = 0.$$

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0.$$

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

is the Laplace equation in polar coordinates.

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(5)

ii)

Solution:

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \lim_{k \to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

Similarly:

$$\frac{\partial^2 f}{\partial y \partial x} = \lim_{h \to 0} \frac{f_y(x+h,y) - f_y(x,y)}{h}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \lim_{h \to 0} \frac{0-0}{h} = 0.$$

Hence, $f_{xy}(0,0) = f_{yx}(0,0) = 0$.

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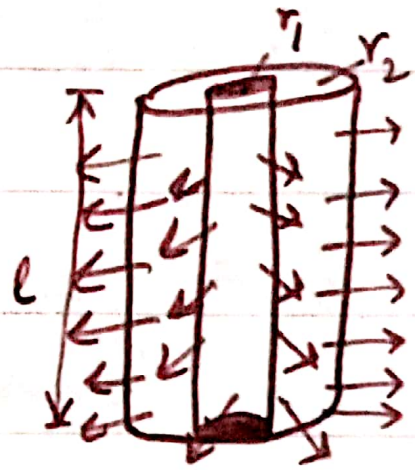
Assignment: 2

Question: 1

(a)



Fig#

Solution:

$r_2 > r_1$.

$\phi(r_2) = \phi_2$

$\phi(r_1) = \phi_1$

We know that:

$$\Omega(z) = \phi(r,\theta) + i\Psi(r,\theta)$$

Laplace equation (polar coordinates) is given as:

$$r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0.$$

Since, the field lines are radially outwards

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$$\frac{d^2\phi}{d\theta^2} = 0; \qquad r^2 \frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} = 0.$$

(Cauchy-Euler equation)

$$ax^n y^{(n)} + bx^{n-1} y^{(n-1)} + \cdots cy = 0.$$

the above equation has solution:

$$\phi(r) = A \ln r + B$$

$$\because \phi(r_2) = \phi_2 \quad \text{and} \quad \phi(r_1) = \phi_1$$

$$\phi_2 - \phi_1 = A \ln\left(\frac{r_2}{r_1}\right).$$

$$A = \frac{\phi_2 - \phi_1}{\ln(r_2/r_1)};$$

$$\phi_2 = \frac{\phi_2 - \phi_1}{\ln(r_2/r_1)} \ln r_2 + B.$$

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$$B = \frac{\cancel{\phi_2(\ln r_2 - \ln r_1)} - \cancel{\phi_2 \ln r_2} + \phi_1 \ln r_2}{\ln(r_2/r_1)}$$

(b)

$$B = \frac{\phi_1 \ln r_2 - \phi_2 \ln r_1}{\ln(r_2/r_1)};$$

$$\phi(r) = \left[(\phi_2 - \phi_1)\ln r + \phi_1 \ln r_2 - \phi_2 \ln r_1\right]\frac{1}{\ln(r_2/r_1)}$$

We know that, static electric field is:

$$E = \nabla\phi = \frac{\partial \phi}{\partial r}$$

$$E = \left[\frac{\phi_2 - \phi_1}{\ln(r_2/r_1)}\right]\cdot\frac{1}{z}; \quad \because E = \Omega'(z)$$

$$\Omega(z) = a\ln z + b$$

$$E = \left[\frac{\phi_2 - \phi_1}{\ln(r_2/r_1)}\right]\frac{1}{r}$$

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(b) Capacitance $= \dfrac{Q}{\Delta V}$

$\Delta V = \phi_2 - \phi_1$ (potential difference)

$\dfrac{Q}{\varepsilon} = \int_{enclosed} \vec{E}.d\vec{S}$; (Gauss' Law)

$\varepsilon = K\varepsilon_0$ $\qquad \varepsilon_0 = 8.85 \times 10^{-12}$ F/m.

$dS = r\,d\theta\,dz$

$\left.\begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le z \le l \end{array}\right\}$ cylinderical coordinates.

$$Q = K\varepsilon_0 \int_0^{l}\int_0^{2\pi} \left[\frac{\phi_2 - \phi_1}{\ln(r_2/r_1)}\right] \frac{1}{r}\, r\,d\theta\,dz$$

$$Q = 2\pi K\varepsilon_0 l \; \frac{\phi_2 - \phi_1}{\ln(r_2/r_1)}$$

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Question: 2

(a)

Solution: Let $z = 1e^{i\theta}$ (unit circle) and $z_{-a} = 2e^{i\pi/6}$

Translating circle 1 unit right:

$z + 1$; and applying inversion transformation.

$$\frac{1}{z+1}; \qquad w = \frac{1}{z+1}$$

$$z = x + iy; \quad w = \frac{1}{x+1+iy}$$

$$w = \frac{x+1-iy}{(x+1)^2+y^2} = \frac{x+1}{(x+1)^2+y^2} - i\frac{y}{(x+1)^2+y^2}$$

$$u(x,y) = \frac{x+1}{x^2+1+2x+y^2};$$

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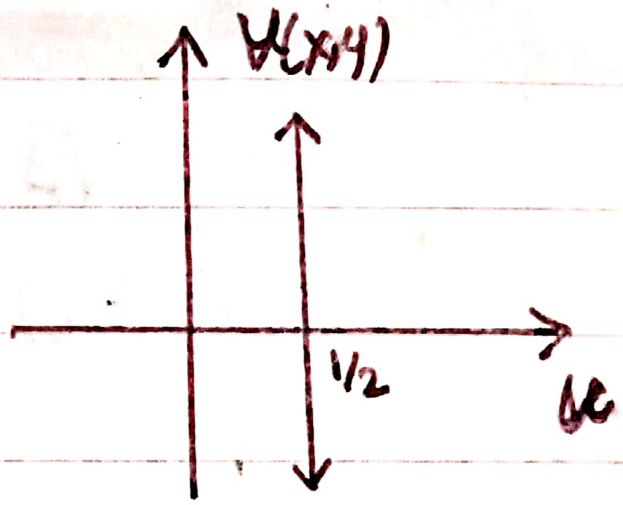
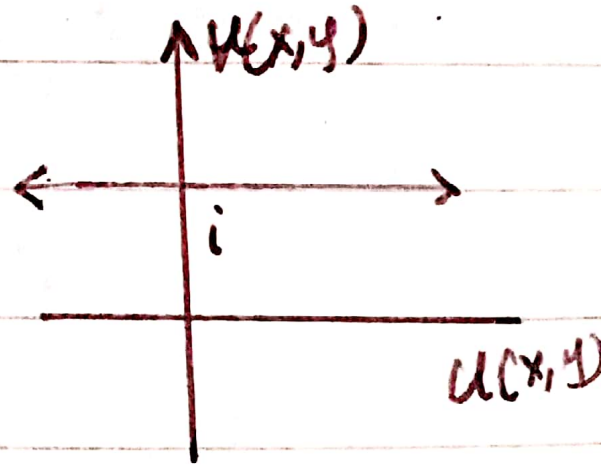
$$v(x,y) = \frac{-y}{x^2+1+2x+y^2};$$

$$u(1,0) = \frac{1}{2}; \quad v(1,0) = 0.$$

Magnifying by 2 and rotating by $\pi/2$ we have.



$$w = \frac{2i}{z+1}$$



furthermore, shifting down by 1 unit:

$$w = \frac{2i}{z+1} - i$$

$$w = \frac{i(\cancel{2}-z-\cancel{1})}{1+z} = \frac{i(1-z)}{(1+z)}$$

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Hence, we have:

$$w(z) = i\left(\frac{1-z}{1+z}\right)$$

$$z_{-q} = 2e^{i\pi/6} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$z_{-q} = \sqrt{3} + i$$

$$w(z_{-q}) = i\left[\frac{1-(\sqrt{3}+i)}{1+(\sqrt{3}+i)}\right]$$

$$w(z_{-q}) = 0.426\, e^{-i56.31^\circ}$$

Since, we have to ground unit circle that

$$w_{+q} = \overline{w_{-q}}$$

$$w_{+q} = 0.426\, e^{i56.31^\circ}$$

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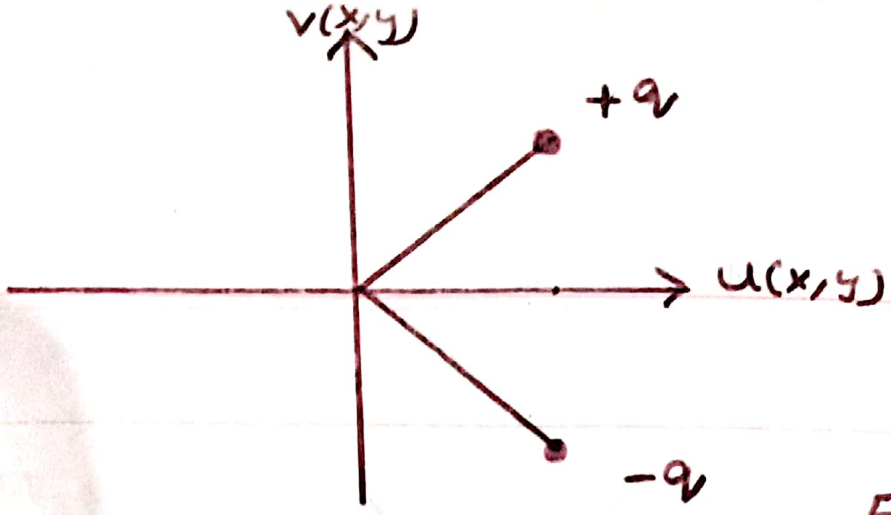

Figure

Now; the position of charge $+q$ in z-plane is given as:

$$w = i\left(\frac{1-z}{1+z}\right)$$

$$(1+z)w = i - iz$$

$$w + wz = i - iz$$

$$wz + iz = i - w$$

$$z(i+w) = i - w$$

$$z = \left(\frac{i-w}{i+w}\right) \Rightarrow z_{+q} = \frac{1}{2}e^{i\pi/6}$$

$$z_{-q} = 2e^{i\pi/6} \quad ; \quad z_{+q} = \frac{1}{2}e^{i\pi/6}$$

the positive charge will be

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(b)

Complex Potential:

$$\Omega(z) = 2q \ln\left(\frac{w - w_{-q}}{w - w_{+q}}\right)$$

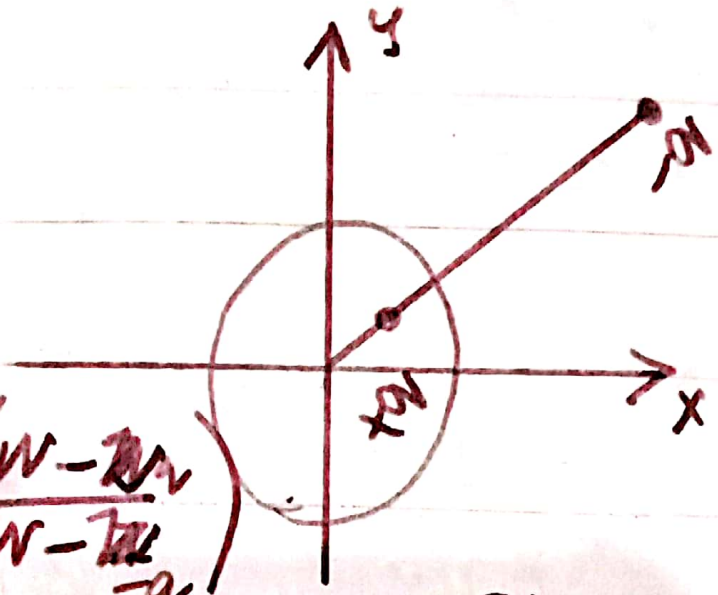

Figure.

$$z_{-q} = 2e^{i\pi/6} \quad ; \quad z_{+q} = \frac{1}{2}e^{i\pi/6}$$

$$w_{+q} = 0.426\, e^{+i56.31} \quad ; \quad w_{-q} = 0.462\, e^{-i56.31}$$

$$\Omega(z) = 2q \ln\left[\frac{w - 0.426\, e^{+i56.31}}{w - 0.426\, e^{-i56.31}}\right]$$

$$w = \frac{i(1-z)}{(1+z)}$$

Substituting in $\Omega(z)$ we get the complex potential:

$$\Omega(z) = 2q \ln \left[\frac{\frac{i(1-z)}{(1+z)} - 0.426 e^{i56.31}}{\frac{i(1-z)}{(1+z)} - 0.426 e^{-i56.31}} \right]$$

further:

$$\Omega(z) = 2q \ln \left[\frac{z(1.37 e^{-i100^\circ}) + 0.687 e^{i110^\circ}}{z(0.687 e^{-i110^\circ}) + 1.37 e^{i100}} \right].$$

is the required potential.

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Chapter 5: Integration in Complex Plane.

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Line Integral: $\int_C P(x,y) + Q(x,y)\,dx$

$$\int_C G(x,y)\,dx = \int_a^b G(x(t), y(t))\,x'(t)\,dt.$$
$$a \le t \le b$$

$$\int_C G(x,y)\,dy = \int_a^b G(x(t), y(t))\,y'(t)\,dt$$
$$a \le t \le b$$

$$\int_C G(x,y)\,ds = \int_a^b G(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt$$
$$a \le t \le b.$$

Notation:

$$\int_C P(x,y)\,dx + Q(x,y)\,dy.$$

$$\oint_C P\,dx + Q\,dy.$$

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Excercise 5.1

1) $\int_{-1}^{3} x(x-1)(x+2)\,dx.$

$x(x^2+2x-x-2)$

$x(x^2+x-2) = x^3+x^2-2x$

$\int_{-1}^{3}(x^3+x^2-2x)\,dx = \frac{x^4}{4}+\frac{x^3}{3}-x^2\Big|_{-1}^{3}$

$= \left[\frac{(3)^4}{4}+\frac{(3)^3}{3}-(3)^2\right]-\left[\frac{(-1)^4}{4}+\frac{(-1)^3}{3}-(-1)^2\right]$

$\left(\frac{81}{4}\right)+\cancel{3^2-3^2}-\left(\frac{1}{4}-\frac{1}{3}-1\right).$

$= \frac{81}{4}-\frac{1}{4}+\frac{1}{3}+1 = \frac{80}{4}+\frac{4}{3}$

$\begin{array}{r} 20 \\ 4\overline{)80} \\ -80 \\ \hline 0 \end{array}$

$= 20+4/3 = \frac{64}{3}$

3) $\int_{1/2}^{1} \sin 2\pi x\,dx = \frac{-\cos 2\pi x}{2\pi}\Big|_{1/2}^{1}$

$$-\frac{1}{2\pi}\left[\cos(2\pi) - \cos(\cancel{2\pi}\cdot \tfrac{1}{\cancel{2}})\right]$$

$$-\frac{1}{2\pi}\left[1-(-1)\right] = \frac{-1}{\cancel{2\pi}}(\cancel{2}) = -\frac{1}{\pi}:$$

$$\int_{1/2}^{1} \sin 2\pi x\,dx = -1/\pi.$$

3

5) $\displaystyle\int_0^4 \frac{dx}{2x+1}$; $\displaystyle\frac{1}{2}\int_0^4 \frac{2dx}{2x+1}$

1

1)

$$= \frac{1}{2}\ln|2x+1|\Big|_0^4.$$

$$= \frac{1}{2}\left[\ln|4+1| - \ln|0+1|\right].$$

$$\int_0^4 \frac{dx}{2x+1} = \frac{\ln 5}{2};$$

7) $\displaystyle\int_2^4 xe^{-x/2}\,dx.$

Solution:- $\displaystyle\frac{xe^{-x/2}}{(-1/2)} - \int \frac{e^{-x/2}}{(-1/2)}dx$

1/2

$$= -2xe^{-x/2} + 2\frac{e^{-x/2}}{(-1/2)} + c$$

$$= -2xe^{-x/2} - 4e^{-x/2}\Big|_2^4 =$$

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$$-2xe^{-x/2} - 4e^{-x/2}\Big|_{2}^{4}$$

$$=\left(-2(4)e^{-4/2} - 4e^{-4/2}\right) - \left(-2(2)e^{-2/2} - 4e^{-2/2}\right)$$

$$=\left(-8e^{-2} - 4e^{-2}\right) - \left(-4e^{-1} - 4e^{-1}\right).$$

$$= -12e^{-2} - \left(-8e^{-1}\right). = 8e^{-1} - 12e^{-2}.$$

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9) $\int_2^4 \frac{dx}{x^2-6x+5}$; $\quad x^2-6x+5=0$.

$$x = \frac{6 \pm \sqrt{(6)^2-4(5)(1)}}{2(1)}$$

$$x = \frac{6 \pm 4}{2} = \{5, 1\}.$$

$$\frac{1}{(x-5)(x-1)} = \frac{A}{(x-5)} + \frac{B}{(x-1)}$$

$$A = \frac{1}{(x-1)}\Big|_{x=5} = \frac{1}{4};$$

$$B = \frac{1}{(x-5)}\Big|_{x=1} = -\frac{1}{4};$$

$$\cancel{\left[\frac{1}{4}\ln\left|\frac{x-5}{x-1}\right|\right]_2^4 = \frac{1}{4}\ln\left|\frac{4-5}{4-1}\right|}$$

$$\frac{1}{4}\int_2^4\left[\frac{-dx}{(5-x)} - \frac{dx}{(x-1)}\right]$$

$$\frac{1}{4}\left[\ln|5-x| - \ln|x-1|\right]_2^4.$$

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$$\frac{1}{4}\left[\ln\left[\frac{5-x}{x-1}\right]\right]_2^4 = \frac{1}{4}\cdot\ln\left(\frac{5-4}{4-1}\right) - \ln\left(\frac{5-2}{2-1}\right)$$

$$= \frac{1}{4}\left[\ln\left(\frac{1}{3}\right) - \ln\left(\frac{3}{1}\right)\right].$$

$$\frac{1}{4}\left[-\ln(3) - \ln(3)\right] = \frac{-\cancel{2}\ln(3)}{\cancel{4}}$$

$$= -\frac{\ln 3}{2};$$

11) $G(x,y) = 2xy;$ $\quad x(t) = 5\cos t$

$y(t) = 5\sin t$

$0 \le t \le \pi/4.$

$$\int_C G(x,y)\,dx = \int_0^{\pi/4} G(x(t), y(t))\,x'(t)\,dt$$

$$\int_0^{\pi/4} 2(5\cos t)(5\sin t) \cdot +5\cos t\,dt.$$

$$2\cdot 5^3 \int_0^{\pi/4} \cos^2 t\,(-\sin t)\,dt.$$

$$\frac{d}{dt}\cos t = -\sin t.$$

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$$\cancel{\int_C G(x,y)\,dy = \int_a^b G(x(t),y(t))\,y'(t)\,dt}$$

$$\cancel{x = 5\cos t, \quad y = 5\sin t; \quad 0 \le t \le \pi/4}$$

$$\cancel{G(x,y) = 2xy}$$

$$\cancel{\int_0^{\pi/4} 2(5\cos t)(5\sin t)(5\cos t)\,dt}$$

$$\cancel{2(125)\int_0^{\pi/4} \cos^2 t \sin t\,dt}$$

---

ii) $G(x,y) = 2xy$

$$\left.\begin{aligned} x &= 5\cos t \\ y &= 5\sin t \end{aligned}\right\} 0 \le t \le \pi/4.$$

$$\int_C G(x,y)\,dx = \int_0^{\pi/4} 2xy\,x'(t)\,dt.$$

$$2\int_0^{\pi/4} (5\cos t)(5\sin t) - 5\sin t\,dt.$$

$$-2\,5^3 \int_0^{\pi/4} \sin^2 t \cos t\,dt.$$

$$-2\,5^3 \frac{\sin^3 t}{3}\Big|_0^{\pi/4} = -\frac{2\cdot 5^3}{3}\left[\left(\frac{\sqrt{2}}{2}\right)^3 - 0\right].$$

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$$-\frac{2(125)}{3}\cdot\frac{1}{2\sqrt{2}} = -\frac{125}{3\sqrt{2}};$$

$$\int_C G(x,y)\,dx = -\frac{125}{3\sqrt{2}};$$

$$\int_C G(x,y)\,dy = 2\,5^3\int_0^{\pi/4} \cos t \sin t \cos t\,dt.$$

$$= -2\cdot 5^3 \int_0^{\pi/4} \cos^2 t \frac{d}{dt}\cos t\,dt.$$

$$= -2(125)\int_0^{\pi/4} \frac{\cos^3 t}{3}\Big|_0^{\pi/4}.$$

$$-\frac{2(125)}{3}\left[\frac{1}{2\sqrt{2}} - 1\right]$$

$$\frac{-125}{3\sqrt{2}} + \frac{2(125)}{3} = \cancel{\frac{2(125)}{3\sqrt{2}}}$$

$$\frac{2(125)}{3} - \frac{125}{3\sqrt{2}} = \frac{125}{3}\left(2 - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{125(4-\sqrt{2})}{6}$$

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$$\int_C G(x,y)\,dy = \frac{125(4-\sqrt{2})}{6}.$$

$$\int_C G(x,y)\,ds = 2\int_0^{\pi/4} 5^2 \cos t \sin t \sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

$$\left.\begin{aligned} x'(t) &= -5\sin t \\ y'(t) &= 5\cos t \end{aligned}\right\} \quad \begin{aligned} x'(t)^2 + y'(t)^2 &= 5^2[\sin^2 t + \cos^2 t] \\ &= 5^2 \end{aligned}$$

$$= 5^3 \int_0^{\pi/4} 2\cos t \sin t\,dt.$$

$$= 125 \int_0^{\pi/4} \sin 2t\,dt.$$

$$= 125 \cdot \frac{-\cos 2t}{2}\Big|_0^{\pi/4} = \frac{-125}{2}\Big[\cos 2t\Big]_0^{\pi/4}.$$

$$\frac{-125}{2}\left[\cos\frac{\pi}{4} - \cos 0\right] = -\frac{125}{2}.$$

$$\int_C G(x,y)\,ds = -\frac{125}{2}.$$

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13) $G(x,y) = 3x^2+6y^2$; $y = 2x+1$,

$-1 \le x \le 0.$

Solution:-

$$\int_C G(x,y)\,dx = \int_a^b G(x, y(x))\,dx$$

$G(x,y) = 3x^2 + 6y^2$

$G(x, y(x)) = 3x^2 + 6(2x+1)^2 = 3x^2 + 6(4x^2+1+4x)$

$= 3x^2 + 24x^2 + 6 + 24x$

$= 27x^2 + 24x + 6$

$$\int_{-1}^{0} (27x^2 + 24x + 6)\,dx =$$

$$\cancel{27}\frac{x^3}{\cancel{3}} + \cancel{24}\frac{x^2}{\cancel{2}} + 6x \Big|_{-1}^{0}$$

$$9x^3 + 12x^2 + 6x \Big|_{-1}^{0}$$

$-(9(-1)^3 + 12(-1)^2 + 6(-1))$

$-(-9 + 12 - 6) = 3$ ~~9+6/2 = 9+4 = 13~~

$$\int_C G(x(y), y)\,dy.$$

$y = 2x + 1;\ -1 \le x \le 0$

$2x + 1 = y$

$y = 1 \quad x = 0.$

$2x = y - 1$

$y = -1 \quad x = -1.$

$x = \left(\frac{y-1}{2}\right);$

$$3x^2 + 6y^2 = 3\left(\frac{y-1}{2}\right)^2 + 6y^2.$$

$$\frac{3}{4}(y^2 + 1 - 2y) + 6y^2.$$

$$\frac{3}{4}y^2 + 6y^2 + \frac{3}{4} - \frac{3}{2}y.$$

$$G(x(y), y) = \frac{27}{4}y^2 - \frac{3}{2}y + \frac{3}{4};$$

$$\frac{\cancel{27}}{4} \cdot \frac{y^3}{\cancel{3}} - \frac{3}{2} \cdot \frac{y^2}{2} + \frac{3}{4}y \Big|_{-1}^{1}$$

$$\frac{9}{4}y^3 - \frac{3}{4}y^2 + \frac{3}{4}y \Big|_{-1}^{+1}$$

$$\left(\frac{9}{4}\right) - \left(-\frac{15}{4}\right) = \frac{9+15}{4} = \frac{\cancel{24}}{\cancel{4}} = 6.$$

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$$\int_C G(x,y)\,ds = \int_a^b G(x,y)\sqrt{1+y'^2(x)}\,dx.$$

$y = 2x+1$

$y' = 2$

$y^2 = 4.$

$\sqrt{1+4}$

$$\sqrt{5}\int_{-1}^{0}(7x^2+24x+6)\,dx.$$

$=3\sqrt{5}$ Ans.

$$\int_C (2x+y)\,dx + xy\,dy;$$

curve $\to (-1,2)$ to $(2,5)$

5) $y = x+3;$

$x(t) = x_0(1-t) + x_1(t).$ $\quad 0 \le t \le 1$

$x(t) = -1(1-t) + 2t$

$x(t) = -1+t+2t$

$x(t) = 3t-1;$ $\quad 0 \le t \le +1.$

$y(t) = 3t+2;$

$2x+y = 2(3t-1) + 3t+2$

$= 6t \cancel{-2} + 3t \cancel{+2}$

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$$\int_0^{+1} 9t\, d(3t-1) + (3t-1)(3t+2)\, d(3t+2).$$

$$\int_0^{+1} 9t\, 3dt + \int_0^{1} (9t^2 + \cancel{6t} - \cancel{3}t - 2)\, 3dt.$$

$$9 \cdot 3 \frac{t^2}{2}\Big|_0^{+1} + 3\int_0^{1} (9t^2 + 3t - 2)\, dt.$$

$$\frac{27}{2}(1^2 - 0^2) + 3\left[\frac{9}{3}t^3 + \frac{3}{2}t^2 - 2t\right]_0^1.$$

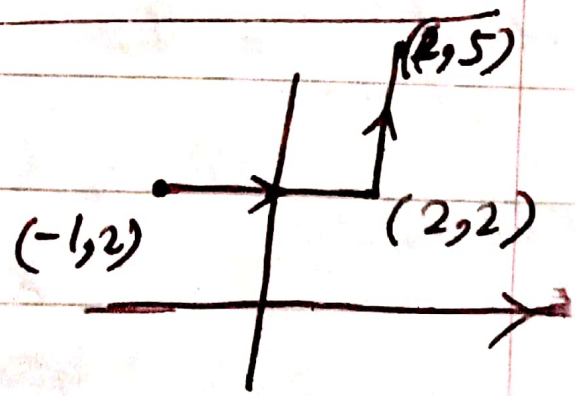
$$\frac{27}{2} + \left[9t^3 + \frac{9}{2}t^2 - 6t\right]_0^1.$$

$$\frac{27}{2} + 9 + \frac{9}{2} - 6 = 21$$

$$\int_C (2x+y)\,dx + xy\,dy = 21.$$

17)

$$\int_C P\,dx + Q\,dy =$$



$$\int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy.$$

C_1: $(-1,2) \longrightarrow (2,2)$ $\quad y = 2.$

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$(-1,2) \to (2,2)$.

$x(t) = -1(1-t) + 2t; \; 0 \le t \le 1$

$x(t) = -1 + t + 2t$

$x(t) = 3t - 1; \quad 0 \le t \le 1.$

$y = 2.$

$$\int_{C_1} (2x+y)\,dx + xy\,dy.$$

$$= \int_0^1 [2(3t-1)+2]\,3dt + (3t-1)(2)\,d(2).$$

$$= \int_0^1 (6t \cancel{-2+2})\,3dt$$

$$= \cancel{18}\cdot \frac{t^2}{\cancel{2}}\Big|_0^1 = 9; \qquad \int_{C_1} P\,dx + Q\,dy = 9.$$

$$\int_{C_2} P\,dx + Q\,dy; \qquad C_2 : (2,2) \to (2,5).$$

$x = 2.$

$(2x+y)\,dx + xy\,dy.$

$y(t) = y_0(1-t) + y_1(t); \; 0 \le t \le 1$

$y(t) = 2(1-t) + 5t.$

$y(t) = 2 - \cancel{2t} + 5t$

$\boxed{y(t) = 3t + 2;} \quad 0 \le t \le 1.$

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$$\int_0^1 (4+3t+2)\,dt + 2(3t+2)3\,dt$$

$$\int_0^1 6(3t+2)\,dt = \cancel{18}\,\frac{t^2}{\cancel{2}} + 12t\Big|_0^1.$$

$$\int_{C_2} \cdots = 9+12 \quad \int_C P\,dx + Q\,dy = 9+12+9.$$

$$18+12 = 30.$$

$$\int_C (2x-4y)\,dx + xy\,dy = 30.$$

---

(9) $\int_C y\,dx + x\,dy$; $\quad (0,0) \to (1,1)$

$y = x^2$.

$$x(t) = 0(1-t) + t;$$

$$0 \le t \le 1.$$

$$x(t) = t;$$

$$\int_0^1 t^2\,dt + t\,2x\,dx = \int_0^1 t^2\,dt + 2t^2\,dt.$$

$$\int_0^1 3t^2\,dt = t^3\Big|_0^1.$$

$$\int_C y\,dx + x\,dy = 1 \}.$$

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23) $\int_C (6x^2+2y^2)\,dx + 4xy\,dy.$

$x=\sqrt{t}\,;\quad y=t;\quad 4\le t\le 9.$

Solution:-

$$=\int_4^9 (6t+2t^2)\,d\sqrt{t} + 4\sqrt{t}\,t\,dt.$$

$$\frac{d\sqrt{t}}{dt} = \frac{1}{2\sqrt{t}}$$

$$=\int_4^9 \frac{(2t^2+6t)}{2\sqrt{t}}\,dt + 4t^{3/2}\,dt.$$

$$=\int_4^9 \left(t^{2-1/2} + 3\sqrt{t} + 4t^{3/2}\right)dt$$

$$\int_4^9 \left(t^{3/2} + 4t^{3/2} + 3t^{1/2}\right)dt$$

$$=\left[5\,\frac{t^{3/2+1}}{\left(\frac{3}{2}+1\right)} + 3\cdot\frac{t^{1/2+1}}{1+1/2}\right]_4^9.$$

$$\frac{\cancel{5}t^{5/2}}{\frac{\cancel{5}}{2}} + \frac{\cancel{3}t^{3/2}}{\cancel{3}/2}. = 2\left[t^{5/2}+t^{3/2}\right]$$

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$$= 2\left[(9)^{5/2} + (9)^{3/2} - (4)^{5/2} - (4)^{3/2}\right]$$

$$2\left\{(3^2)^{5/2} + (3^2)^{3/2} - (2^2)^{5/2} - (2^2)^{3/2}\right]$$

$$= 2\left[3^5 + 3^3 - 2^5 - 2^3\right]$$

$$= 460.$$

$$\int_C (6x^2 + 2y^2)\,dx + 4xy\,dy = 460.$$

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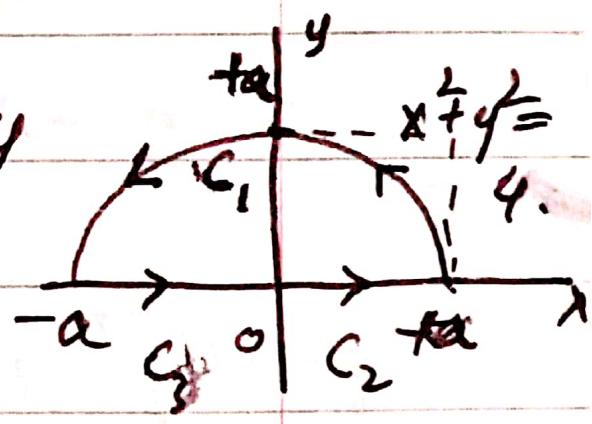
27) $\oint_C (x^2+y^2)\,dx - 2xy\,dy$



$\boxed{a = 2.}$

$x(t) = a\cos t, \quad y(t) = a\sin t.$

$x(t) = 2\cos t \quad ; \quad y(t) = 2\sin t$

$$\oint_C (x^2+y^2)\,dx - 2xy\,dy =$$

$$\oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy + \cdots$$

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$$\int_{C_1} (x^2+y^2)\,dx + 2xy\,dy -$$

$$\left.\begin{aligned} x(t) &= 2\cos t \\ y(t) &= 2\sin t \end{aligned}\right\} \quad 0 \le t \le \pi$$

$$\int_0^{\pi} -8\sin t\,dt - 2(4\cos t\sin t)\,2\cdot\cos t\,dt$$

$$= -8\int_0^{\pi} \sin t\,dt - 16\int_0^{\pi} \cos^2 t\sin t\,dt.$$

$$\frac{d}{dt}\cos t = -\sin t$$

$$= -8\,-\cos t\Big|_0^{\pi} + 16\int_0^{\pi} \cos^2 t\,\frac{d}{dt}\cos t\,dt$$

$$= 8\cos t\Big|_0^{\pi} + \frac{16}{3}\cos^3 t\Big|_0^{\pi}$$

$$= 8\left[-1-1\right] + \frac{16}{3}\left[(-1)^3 - (+1)^3\right].$$

$$= 8(-2) + \frac{16}{3}(-1-1) = -64/3$$

$$= -16 - \frac{32}{3} = \cancel{\frac{-24-32}{3} = -\frac{56}{3}}$$

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Complex Integrals:

$$z(t) = x(t) + i\,y(t); \quad a \le t \le b$$

$$\oint_c f(z)\,dz \;\&\; \oint_c f(z)\,dz$$ } Complex integrals in positive and negative directions

$$f(z) = u(x,y) + i\,v(x,y)$$

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}; \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

} Riemann-Cauchy equations.

$$\int_c f(z)\,dz = \int [u(x+y) + i\,v(x,y)][dx + i\,dy].$$

$$\int_c u\,dx - v\,dy + i\int_c v\,dx + u\,dy;$$

$$\left.\begin{array}{l} u(x(t), y(t)) \\ v(x(t), y(t)) \end{array}\right\} a \le t \le b$$

$$\int_c f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

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Example: $z(t) = 3t + it^2, \quad -1 \le t \le 4$

$$\int_C \bar{z}\, dz; \qquad dz = (3 + 2it)\,dt.$$

$$\bar{z}(t) = 3t - it^2$$

$$= \int_{-1}^{4} (3t - it^2)(3 + 2it)\, dt$$

$$= \int_{-1}^{4} (9t + \cancel{6it^2} - \cancel{3it^2} - 2i^2t^3)\, dt$$

$$= \int_{-1}^{4} (9t + 3it^2 + 2t^3)\, dt$$

$$= \left[\frac{\cancel{2}t^4}{\cancel{4}} + \frac{9t^2}{2} + \frac{\cancel{2}it^3}{\cancel{3}}\right]_{-1}^{4}$$

$$\left[\frac{t^4}{2} + \frac{9}{2}t^2 + \frac{3}{2}it^2\right]_{-1}^{4}$$

$$\left[\frac{(4)^4}{2} + \frac{9}{2}(4)^2 + \frac{3}{2}i(4)^2\right] - \left[\frac{(-1)^4}{2} + \left(\frac{9}{2}\right)(-1)^2 + \frac{3i}{2}(-1)^2\right]$$

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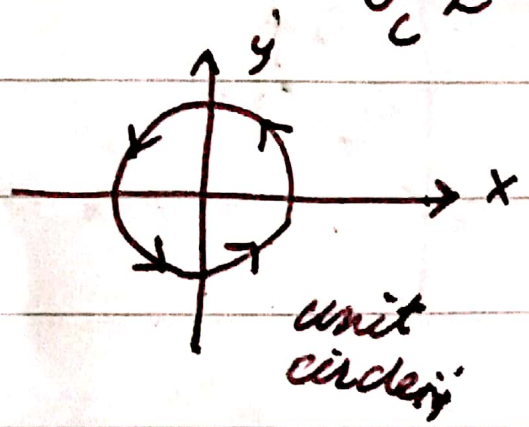
$(200 + 64i) - (5 + i)$.

$= 195 + 65i$ —

$$\int_c \bar{z}\,dz = 195 + 65i;$$

Example: $\oint_c \frac{1}{z}\,dz$; $\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} 0 \le t \le 2\pi$



$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2};$$

$$\oint_c (x - iy)(dx + i\,dy)$$

$$= \oint_c x\,dx + y\,dy - iy\,dx + ix\,dy$$

$x = \cos t \qquad dx = -\sin t\,dt.$

$y = \sin t \qquad dy = \cos t\,dt.$

$$\int_0^{2\pi} \cancel{\cos t(-\sin t)\,dt + \sin t \cos t\,dt} - i\left[y\,dx - x\,dy\right].$$

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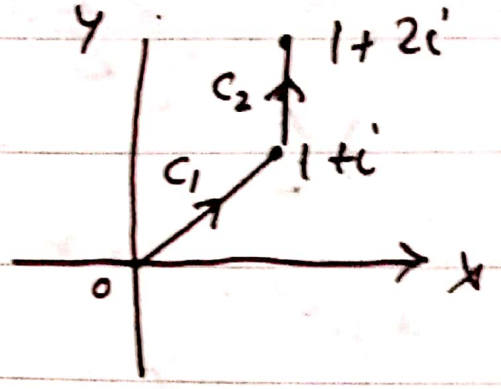
$$-i\int_0^{2\pi}\left[(\sin t)(-\sin t)\,dt-(\cos t)(\cos t\,dt)\right].$$

$$-i\int_0^{2\pi}(-\sin^2 t-\cos^2 t)\,dt = i\int_0^{2\pi}dt.$$

$$\oint_C \frac{dz}{z} = 2\pi i;\Big\}$$ Ans.

Example: 3

$$\int_C (x^2+iy^2)\,dz \;;$$



$C_1 : (0,0) \longrightarrow (1,1).$

$x(t) = 0(1-t)+t = t;\quad 0\le t\le 1.$

$y(t) = t\Big\}.$

$z(t) = x(t)[1+i].$

$dz = dt(1+i)$

$$\int_C (x^2+iy^2)\,dz$$

$$\int_0^1 (t^2+it^2)(1+i)\,dt.$$

$\cancel{(1+i2+2i)}$

$$\int_0^1 t^2(1+i)^2\,dt$$

$$= 2i\int_0^1 t^2\,dt$$

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$$\frac{2}{3}i\,t^3\Big|_0^1 = \frac{2}{3}i;$$

$$\int_{C_2}(x^2+iy^2)\,dz;$$

$C_2 : (1,1) \longrightarrow (1,2)$

$x(t) = 1(1-t) + t$

$y(t) = 1(1-t) + 2t.$

$$\left.\begin{aligned} x(t) &= 1 \\ y(t) &= t+1 \end{aligned}\right\} 0 \le t \le 1$$

$z = x + iy \quad = 1 + i(t+1).$

$dz = i\,dt.$

$$= \int_0^1 [1 + i(t+1)^2]\,i\,dt$$

$$i\int_0^1 [1 + i(t^2+2t+1)]\,dt$$

$$i\left[t + i\left(\frac{t^3}{3} + t^2 + t\right)\right]_0^1.$$

$$i\left[1 + i\left(\tfrac{1}{3} + 1 + 1\right)\right] \int_{C_2} \cdots = -\tfrac{7}{3} + i$$

$$i - \left(\tfrac{1}{3} + 2\right) \quad \int_C (x^2+iy^2)\,dz = -\frac{7}{3} + i + \frac{2}{3}i$$

$$= -\tfrac{7}{3} + \frac{5}{3}i$$

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Bounding Theorem:

$$\left|\int_C f(z)\,dz\right| \le ML;$$

$$z'(t) = x'(t) + i\,y'(t)$$

$$L = \int_a^b |z'(t)|\,dt;$$

$$|f(z)| \le M.$$

A bound for Contour Integral:

$$\oint_C \frac{e^z}{z+1}\,dz; \quad |z| = 4$$

Solution:

$$f(z) = \frac{e^z}{z+1}; \qquad |f(z)| = \left|\frac{e^z}{z+1}\right|;$$

$$M = \frac{e^{|z|}}{|z+1|} \le \frac{e^{|z|}}{|z|-1} = \frac{e^4}{4-1} = \frac{e^4}{3},$$

$$\cancel{L = \int_0^{2\pi} |z'(t)|\,dz;} \qquad \cancel{z(t) = z + iy}$$

$$z(t) = |z|e^{it}; \quad 0 \le t \le 2\pi$$

$$z'(t) = i|z|e^{it}.$$

$$\cancel{L = \int_0^{2\pi} |z|^2 i\, e^{it}\,dt}$$

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$$\cancel{|6|i\int_0^{2\pi} e^{it}\,dt = |6|\frac{e^{it}}{i}\Big|_0^{2\pi}}$$

$$|z'(t)| = \sqrt{(0)^2 + |z|^2 e^{2it}} = |z|\cdot e^{it}.$$

$$\cancel{\int_0^{2\pi} 4e^{it}\,dt = 4\cdot\int_0^{2\pi} e^{it}\,dt}$$

$$\cancel{\frac{4}{i}e^{it}\Big|_0^{2\pi} = -4i(\cos 2\pi - \cos 0)}$$

$$\cancel{= -4i(1-1) = 0}$$

$$z(t) = 1 + |z|e^{it}; \quad = 1 + 4e^{it}; \; 0 \le t \le 2\pi$$

$$z(t) = 1 + 4\cos t + 4i\sin t.$$

$$\cancel{|z(t)| = \sqrt{(1+4\cos t)^2 + (4\sin t)^2}}$$

$$z'(t) = -4\sin t + 4i\cos t.$$

$$|z'(t)| = \sqrt{(-4\sin t)^2 + (4\cos t)^2}.$$

$$L = \int_0^{2\pi} 4\sqrt{\cos^2 t + \sin^2 t}\,dt$$

$$L = 4(2\pi - 0) = 8\pi;$$

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$ML = \frac{8\pi e^4}{3}$; $\left|\oint_C \frac{e^z dz}{z+1}\right| \leq \frac{8\pi e^4}{3}$.

Excercise : 5.2

1) $\int_C (z+3)dz$; C is $x = 2t$; $y = 4t-1$;
$1 \leq t \leq 3$.

$z(t) = x(t) + iy(t)$.

$z(t) = 2t + i(4t-1)$

$dz = (2+4i)dt$. $1 \leq t \leq 3$

$= \int_1^3 ((2t+3) + i(4t-1))(2+4i)dt$.

$= \int_1^3 (2(2t+3) - 4(4t-1))dt$

$+ i\int_1^3 2(4t-1) + 4(2t+3)dt$.

$\cancel{= \int_1^3 (4t + 6 + 8t + 12)\, dt}$

$\int_1^3 (4t + 6 - 16t + 4)dt + i\int_1^3 (8t - 2 + 8t + 12)dt$

$= \int_1$

$\left|-\frac{18}{2}\right.$

$\cancel{83}$

$= -2$

3) $\int_C$

$= \int$

$-$

$\int$

-2

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$$= \int_1^3 (-12t + 10)\,dt + i\int_1^3 (16t + 10)\,dt.$$

$$\left| -\frac{12}{2}t^2 + 10t \right|_1^3 + i \left| \frac{16t^2}{2} + 10t \right|_1^3.$$

$$\cancel{83 + 102i - 3 - 18i = -36 + 84i}$$

$$= -28 + 84i \quad \underline{\text{Ans}}.$$

3) $\int_C z^2\,dz; \qquad z(t) = 3t + 2it; \quad -2 \le t \le 2$

$$dz = (3 + 2i)\,dt.$$

$$z^2 = (3t + 2it)^2 = 9t^2 + 4i^2t^2 + 12it^2$$

$$z^2 = 9t^2 - 4t^2 + 12it = 5t^2 + 12it;$$

$$= \int_{-2}^{+2} (5t^2 - 12it)(3 + 2i)\,dt$$

$$\cancel{\int_{-2}^{+2} (15t^2 + 24t + 10it^2 - 36it)\,dt.}$$

$$\cancel{\left[\frac{15}{3}t^3 + \frac{24}{2}t^2 + \frac{10}{3}it^3 - \frac{36}{2}it^2\right]_{-2}^{2}}$$

$$\cancel{\left(88 - \frac{136}{3}i\right) - \left(8 - \frac{296}{3}i\right) = 80 + 160i}$$

$$= \int_{-2}^{+2} t^2 (5+12i)(3+2i)\,dt \cancel{= (39-26i)\,\frac{t^3}{3}\Big|_{-2}^{+2}}$$

$$\cancel{= \left(13 - \frac{26i}{3}\right) t^3 \Big|_{-2}^{+2}}$$

$$\cancel{= \left(13 - \frac{26i}{3}\right) 16 = 208 - \frac{416i}{3}}$$

$$(3+2i)^3 \int_{-2}^{+2} t^2 = \frac{(3+2i)^3}{3}\, t^3 \Big|_{-2}^{+2}.$$

$$\frac{16}{3}(3+2i)^3 = -48 + \frac{736}{3} i \quad \text{Ans}$$

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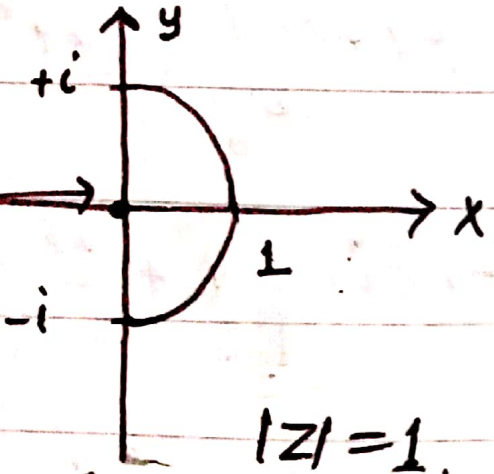
5) $\displaystyle\int_C \frac{z+1}{z}\,dz$



$z(t) = z_0 + e^{it}$

$|z| = 1.$

$0 \le t \le \cancel{2\pi}\ \pi$ (semicircle)

$z_0 = 0 + 0i,$ $\quad z(t) = e^{it}.$

$dz = ie^{it}\,dt.$

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$$\cancel{\int_0^{2\pi} \frac{(e^{it}+1)}{e^{it}}\, i e^{it}\, dt = i\int_0^{2\pi} (e^{it}+1)\, dt}$$

$$\cancel{= i\left[\int_0^{2\pi} e^{it} + \int_0^{2\pi} dt\right]}$$

$$\cancel{= i\left[\frac{e^{it}}{i}\Big|_0^{2\pi} + (2\pi - 0)\right]}$$

$$\cancel{= 2\pi i + e^{it}\Big|_0^{2\pi}}$$

$$\cancel{e^{it} = \cos t + i\sin t;}$$

$$\cancel{e^{2\pi t} = \cos 2\pi + i\sin 2\pi = -1}$$

$$\cancel{e^{0} = \cos 0 + i\sin 0 = 1}$$

$$\cancel{e^{2\pi t} - e^{0} = -1 - 1 = -2}$$

$$\cancel{(\pi i - 2) = (\pi i + 2i^2) = i(\pi + 2i)}$$

$$\cancel{\int_C \frac{z+1}{z}\, dz = \cancel{2\pi i};\ i(\pi + 2i)}$$

$$i\int_0^{\pi} \frac{(e^{it}+1)}{\cancel{e^{it}}}\, \cancel{e^{it}}\, dt = i\int_0^{\pi} (e^{it}+1)\, dt$$

$$i\left[-\frac{e^{it}}{i} + t\right]_0^{\pi} = (\pi + 2i)\, i$$

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7) $\oint_C Re(z)\,dz$, C is a circle. $|z| = 1$.

$z(t) = e^{it} = \cos t + i\sin t$; $0 \le t \le 2\pi$

$Re(z) = \cos t$; $dz = ie^{it}dt$.

$$\oint_C Re(z)\,dz = i\int_0^{2\pi} e^{it}\cos t\,dt.$$

$$\cos t = \frac{e^{it} + e^{-it}}{2} \Rightarrow \frac{e^{2it}}{2} + \frac{1}{2};$$

$$= i\int_0^{2\pi}\left(\frac{e^{2it}}{2} + \frac{1}{2}\right)dt.$$

$$= i\left[\frac{e^{2it}}{4i} + \frac{t}{2}\right]_0^{2\pi} = i\pi,$$

$$e^{2it} = \cos 2t + i\sin 2t$$

$$\oint_C Re(z)\,dz = \pi i.$$

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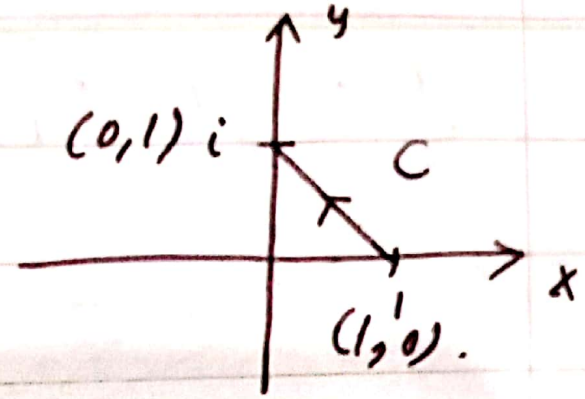
9. $\int_C (x^2+iy^3)\,dz$; C is a line from $z=1$ to $z=i$



$x(t) = 1(1-t) + 0t$

$x(t) = 1-t$;

$y(t) = t$; $0 \le t \le 1$

$z(t) = x(t) + iy(t)$.

$z'(t)dt = (-1+i)\,dt$.

$$\int_C (x^2+iy^3)\,dz = \int_0^1 [(1-t)^2 + it^3](-1+i)\,dt$$

$$= (-1+i)\int_0^1 (1+t^2-2t+it^3)\,dt.$$

$$= (-1+i)\left[t + \frac{t^3}{3} - t^2 + i\frac{t^4}{4}\right]_0^1.$$

$$= (-1+i)\left[\cancel{1} + \frac{1}{3} - \cancel{1} + \frac{i}{4}\right]$$

$$= (-1+i)\left(\frac{1}{3} + \frac{i}{4}\right) = -\frac{7}{12} + \frac{i}{12}, \text{ Ans}$$

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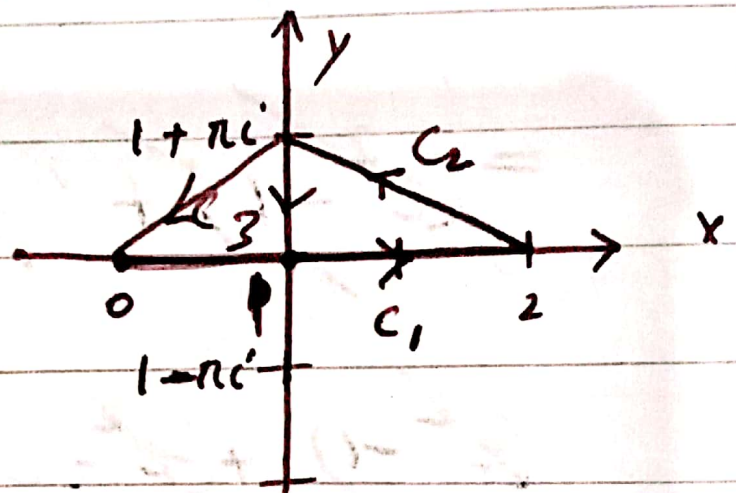
(1) $\int_C e^z dz$; C is a polygonal path consisting of the line segments from

$z=0 \to z=2$.

$z=2 \to z=1+i\pi$.

$$\oint_C e^z dz =$$

$$\int_{C_1} e^z dz + \int_{C_2} e^z dz + \int_{C_3} e^z dz.$$



$$\int_{C_1} e^z dz$$

$z(t) = x(t) + iy(t)$. $C_1: (0,0) \to (2,0)$

$x(t) = 0(1-t) + 2t$,

$x(t) = 2t; 0 \le t \le 1$

$y(t) = 0(1-t) + 0t = 0$.

$z = 2t$; $dz = 2dt$.

$$\int_0^1 e^{2t}\, 2\,dt = \cancel{2}\frac{e^{2t}}{\cancel{2}}\Big|_0^1 = (e^2 - e^0)$$

$$= (e^2 - 1)$$

$$\int_{C_2} e^z dz \qquad C_2 : (2,0) \to (1,\pi).$$

$$x(t) = 2(1-t) + t$$

$$x(t) = 2 - 2t + t$$

$$x(t) = 2 - t ; 0 \le t \le 1$$

$$y(t) = 0(1-t) + \pi t = \pi t ; \quad 0 \le t \le 1.$$

$$z(t) = x(t) + i y(t)$$

$$z(t) = 2 - t + \pi i t ;$$

$$z'(t)\,dt = (-1 + \pi i)\,dt$$

$$(\pi i - 1) \int_0^1 e^{2-t+\pi i t}\, dt$$

$$(\pi i - 1) e^2 \int_0^1 e^{(\pi i - 1)t}\, dt.$$

$$\cancel{(\pi i - 1)} e^2 \left. \frac{e^{(\pi i - 1)t}}{\cancel{(\pi i - 1)}} \right|_0^1 \qquad e^{\pi i} = -1$$

$$e^2 \left[e^{(\pi i - 1)} - e^0 \right]. \qquad e^2 \left[-e^{-1} - 1 \right].$$

$$e^2 \left[e^{\pi i} e^{-1} - 1 \right] = -e^2 - e^{+1}.$$

$C_3 : (1, \pi) \longrightarrow (0, 0)$

$$\left.\begin{aligned} x(t) &= 1 - t \\ y(t) &= \pi(1-t). \end{aligned}\right\} 0 \le t \le 1$$

$z(t) = (1-t) + i\pi(1-t)$

$dz(t) = (-1 - i\pi)\,dt.$

$$-(1+i\pi)\int_0^1 e^{(1-t)+i\pi(1-t)}\,dt$$

$$-(1+i\pi)\int_0^1 e\,e^{-t}\,e^{i\pi}\,e^{-i\pi t}\,dt$$

$e^{i\pi} = -1.$

$$(1+i\pi)e\int_0^1 e^{-(1+i\pi)t}\,dt.$$

$$\frac{\cancel{(1+i\pi)}e}{-\cancel{(1+i\pi)}}\left[e^{-(1+i\pi)t}\right]_0^1$$

$$-e\left[e^{-(1+i\pi)} - e^0\right].$$

$$-e\left[e^{-1}e^{-i\pi} - 1\right].$$

$\oint_C e^z\,dz = 0.$

for closed loop.

$\delta(z)$

$-e\left[-e^{-1}-1\right] = +e+1$

$$\int_C e^z dz = \int_{C_1} e^z dz + \int_{C_2} e^z dz + \underbrace{\int_{C_3} e^z dz}_{\text{excluded}}$$

$$= \cancel{(e^2-1)} + \cancel{(-e^2-e)} + \cancel{e+1}$$

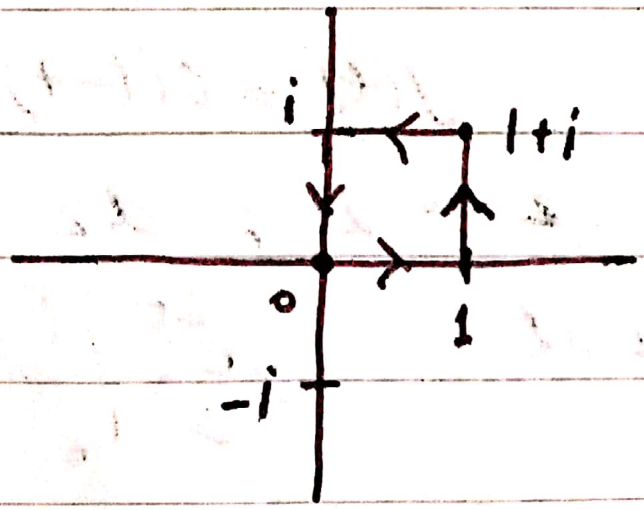
$$= 0$$

$$(e^2-1) + (-e^2-e)$$

$$(-e-1) = -(e+1)$$

15) $\oint_C ze^z dz$; C is Square; $z=0$, $z=1$, $z=1+i$ and $z=i$

Solution:-



$$\oint_C ze^z dz = \int_{C_1} ze^z dz + \int_{C_2} ze^z dz + \int_{C_3} ze^z dz.$$

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$C_1 : (0,1) \longrightarrow (1,1)$ C_3

$$\left.\begin{aligned} x(t) &= 0(1-t) + t = t \\ y(t) &= 1(1-t) + t = 1 \end{aligned}\right\} 0 \le t \le 1$$

$$z(t) = x(t) + i\,y(t) = t + i$$

$$z'(t)\,dt = dt.$$

$$= \int_0^1 (t+i)e^{t+i}\,dt = e^i \int_0^1 (t+i)e^t\,dt.$$

$$= e^i \left[\int_0^1 te^t\,dt + ie^t \Big|_0^1 \right].$$

$$= e^i \left[te^t - e^t \Big|_0^1 + i(e^1 - e^0) \right].$$

$$= e^i \left[\cancel{(e - e^1)} + (0 - e^0) + i(e-1) \right].$$

$$e^i(1 + ie - i)$$

$$= e^i + ie^{i+1} - ie^i \Big\} C_1$$

$$= e^{1+i}\left[(1+i)(e^{-1}-1) + (-2e^{-1}+1)\right] \Big\} C_3$$

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$C_3: (1,1) \longrightarrow (0,1)$.

$x(t) = 1(1-t) + 0t = (1-t)$

$y(t) = 1(1-t) + t = 1$.

$z(t) = 1 - t + i$

$z'(t) = -1$;

$$\int_0^1 (1-t+i)\, e^{1-t+i} (-1)\, dt.$$

$$-e^{1+i} \int_0^1 (1-t+i)\, e^{-t}\, dt.$$

$$-e^{1+i}(1+i)\frac{e^{-t}}{-1}\Big|_0^1 + e^{1+i}\int_0^1 t e^{-t}\, dt.$$

$$e^{1+i}(1+i)(e^{-1}-1) \qquad \frac{te^{-t}}{-1} + \int e^{-t} dt$$

$$+ e^{1+i}(-2e^{-1}+1). \qquad -te^{-t} - e^{-t};$$

$$(-e^{-1} - e^{-t}) - (0 - 1).$$

$$(-2e^{-1}+1); \Big| \quad e^{-t}\Big|_0^1 = e^{-1} - 1;$$

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C_2: $(1,0) \to (1,1)$.

$$\left.\begin{aligned} x(t) &= (1-t) + t = 1; \\ y(t) &= 0(1-t) + t = t; \end{aligned}\right\} 0 \le t \le 1$$

$$z(t) = 1 + it$$

$$z'(t) = i\frac{dt}{dt}.$$

$$\oint_{C_3} ze^z dz = \int_0^1 (1+it)e^{1+it}\, i\,dt$$

$$i\int_0^1 (1+it)\, e^1 e^{it}\, dt.$$

$$= ie\int_0^1 (e^{it} + ite^{it})\, dt.$$

$$= \cancel{i}e \cdot \frac{e^{it}}{\cancel{i}}\Big|_0^1 + i^2 e\left[\frac{te^{it}}{i} - \frac{1}{i^2}e^{it}\right]_0^1$$

$$= (e^i - e) - e(-ie^i + e^i + 0 - 1).$$

$$(e^i - e) - e(-ie^i + e^i - 1)$$

$$e^i - e + ie^{i+1} - e^{i+1} + e \;\}\; C_3.$$

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$C_4 : (0,1) \to (0,0)$

$x(t) = 0(1-t) + 0 = 0.$ $\quad 0 \le t \le 1$

$y(t) = 1(1-t) + 0t = 1-t,$

$z(t) = i(1-t).$

$z'(t) = -i,$

$$\int_{C_4} z e^z dz = \int_0^1 i(1-t) e^{i(1-t)} dt$$

$$= i e^i \int_0^1 (1-t) e^{(1-t)} dt.$$

$(1-t) = x$

$dx = -dt.$

$$ie^i \int_1^0 x e^x \, -dx = -ie^i \int_1^0 x e^x dx$$

$$+ie^i \; xe^x - e^x \Big|_0^1$$

$$(e - e^1) - (0-1) = 1.$$

$$= ie^i \underbrace{\qquad\qquad\qquad\qquad}_{C_2}$$

$$e^i + ie^{i+1} - ie^i + e^i - e + ie^{i+1} - e^{i+1} + e$$

$$+ e^{i+1}\left[(1+i)(e^{-1}-1) + (-2e^{-1}+1)\right] + ie^i$$

$$\cancel{e^i + ie^i + ie - ie^i + e^i - e + ie^i + ie - e^i - e}$$

$$e^i + ie^{i+1} - ie^i + e^i - e + ie^{i+1} - e^{i+1} + e + ie^i$$

$$e^{i+1}\left[e^{-1} - 1 + ie^{-1} - i - 2e^{-1} + 1\right]$$

$$\sim \cancel{e^i + ie^{i+1} + ie^i + e^i - 1 - e^{i+1} + ie^{i+1} - ie^i + e^i + e^{i+1} - ie^{i+1} - e^{i+1} + e}$$

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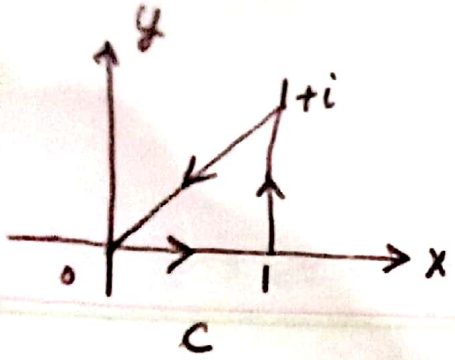

17) $\oint_C x\,dz$;

$C_1: (0,0) \to (1,0)$.

$x(t) = 0(1-t) + t = t$
$y(t) = 0$ $\Big\}\ 0 \le t \le 1$

$z(t) = t + i0 = t$

$z'(t) = 1$.

$$\int_{C_1} x\,dz = \int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2};$$

$C_2: (1,0) \to (1,1)$

$x(t) = 1(1-t) + t = 1 \cancel{-t+t} = 1$

$y(t) = 0(1-t) + t = t;$

$z(t) = 1 + it;$

$z'(t) = i;$

$$\int_{C_2} x\,dz = \int_0^1 i\,dt = i$$

$\int_{C_3} x\,dz = ?$ $\quad C_3: (1,1) \to (0,0)$.

$x(t) = 1(1-t) + 0t = 1-t;$
$y(t) = 1(1-t) + 0t = 1-t;$

$z(t)$

$(1$

$=$

$\oint_C$

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$z(t) = (1-t) + i(1-t); \quad 0 \le t \le 1$

$z'(t) = (-1 - i)$

$\int_0^1 (1-t)(-1-i)\,dt = (1+i)\int_0^1 (t-1)\,dt$

$(1+i)\left[\frac{t^2}{2} - t\right]_0^1 = \left(\frac{1}{2} - 1\right)(1+i).$

$= -\frac{1}{2}(1+i);$

$= -\frac{1}{2} + \frac{i}{2};$

$\oint_C x\,dz = \cancel{\frac{1}{2}} + i - \cancel{\frac{1}{2}} + \frac{i}{2} = \frac{i}{2};$

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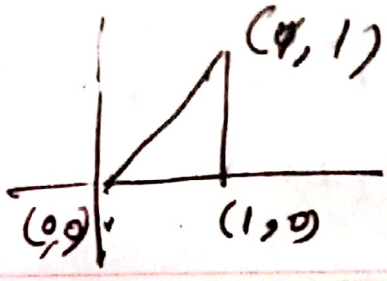
19) $\oint_C z^2 dz$;



C_1: $(0,0) \longrightarrow (1,0)$.

$z(t) = t; \quad 0 \le t \le 1$

$\int_0^1 t^2 dt = \quad \frac{t^3}{3} = \frac{1}{3}$;

C_2: $(1,0) \longrightarrow (1,1)$

$z(t) = 1 + it$

$z'(t) = i$

$i\int_0^1 (1+it)^2 dt = i\int_0^1 (1 - t^2 + 2it)$

$i\left[t - \frac{t^3}{3} + it^2\right]_0^1$

$i\left[1 - \frac{1}{3} + i\right] = \left[i - \frac{i}{3} - 1\right]$.

$\left[-1 + \frac{3-1}{3} i\right]$

$\left(-1 + \frac{2}{3} i\right)$

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$C_3: (1,1) \to (0,0).$

$z(t) = (1-t) + i(1-t); 0 \le t \le 1$

$z'(t) = -(1+i).$

$(1+i)\int_0^1 \cancel{(1-t)^2 + (1+t)^2} + 2i(1-t)^2 \, dt.$

$-(1+i)2i \int_0^1 (t^2+1-2t)\, dt.$

$-2i(1+i)\left| \frac{t^3}{3} - t^2 + t \right|_0^1$

$\cancel{\left| \frac{1}{3} - 1 + 1 \right|}$

$-\frac{2}{3}i(1+i) = -\frac{2}{3}i - \frac{2}{3}i^2.$

$= \left(\frac{2}{3} - \frac{2}{3}i\right).$

$\oint_C z^2 dz = \frac{1}{3} - \cancel{1 + \frac{2}{3}i} + \frac{2}{3} - \cancel{\frac{2}{3}i}.$

$\frac{1-3}{3} + \frac{2}{3} = \cancel{\frac{3i}{3}} = 0.$ Ans

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25)

$$\oint_c \frac{e^z}{z^2+1}\,dz \qquad |z|=5;$$

$$M = |f(z)| = \left|\frac{e^z}{z^2+1}\right|;$$

$$\left|\oint_c f(z)\,dz\right| \le ML.$$

$$\frac{e^{|z|}}{|z^2+1|} \le \frac{e^{|z|}}{|z|^2-1} = \frac{e^5}{5^2-1} = \frac{e^5}{24};$$

$$M = \frac{e^5}{24};$$

$$z(t) = 5e^{it}; \qquad 0 \le t \le 2\pi$$

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt;$$

$$z(t) = 5\cos t + 5i\sin t$$

$$\left.\begin{aligned} x'(t) &= -5\sin t \\ y'(t) &= 5\cos t \end{aligned}\right\}.$$

$$L = \int_0^{2\pi} 5\sqrt{\sin^2 t + \cos^2 t}\,dt.$$

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$L = 10\pi$

$$ML = \frac{\cancel{10}\pi e^5}{\cancel{24}} = \frac{5\pi e^5}{12};$$

$$\therefore \left| \oint_C \frac{e^z}{z^2+1}\, dz \right| \leq \frac{5\pi e^5}{12};$$

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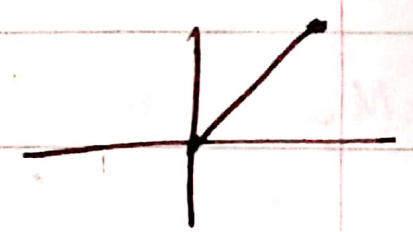
27) $\int_C (z^2+4)\,dz.$ $\quad z=0$ to $z=1+i$

Solution:-



$C = (0,0) \longrightarrow (1,1)$

$$\left.\begin{aligned} x(t) &= 0(1-t) + t = t \\ y(t) &= 0(1-t) + t = t \end{aligned}\right\} \quad 0 \leq t \leq 1$$

$z(t) = t(1+i)$

$z'(t) = (1+i);$

$|z| = \sqrt{(1)^2+(1)^2}$

$|z| = \sqrt{(2)}.$

$$|z^2+4| \leq |z|^2 + 4 = (\sqrt{2})^2 + 4 = 6.$$

$$\cancel{(1+i)\int_0^1 t^2(1+i)^2+4)\,dt}.$$

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$$(1+i^2+2i).$$

$$(1+i)\int_0^1 (2it^2+4)\,dt = 2i(1+i)\frac{t^3}{3}+4t\Big|_0^1$$

$$\frac{2}{3}i(1+i)+4(1+i) = \frac{2}{3}(-1+i)+4(1+i)$$

$$= -\frac{2}{3}+\frac{2}{3}i+4+4i = \frac{10}{3}+\frac{14}{3}i;$$

$$\int_C \le ML = 6\left(\frac{10}{3}+\frac{14}{3}i\right) = 20+28i.$$

$$L = \int_a^b \sqrt{x'(t)^2+y'(t)^2}\,dt.$$

$$z(t) = t+it;\ 0\le t\le 1$$

$$L = \int_0^1 \sqrt{1^2+1^2}\,dt = \sqrt{2}\,(1-0).$$

$$L=\sqrt{2};$$

$$ML = 6\sqrt{2};$$

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$$\left|\oint_C f(z)\,dz\right| \le ML.$$

$$\oint_C (z^2+4)\,dz \le 6\sqrt{2}$$ Ans.

Cauchy-Goursat Theorem:

$$\oint_C f(z)\,dz = 0.$$ iff $f'(z)$ is continuous in D.

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

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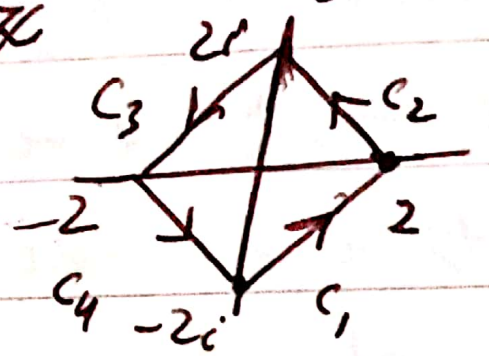
Exercise 5.3

$$\oint_C f(z)\,dz = 0. \qquad |z| = 1.$$

1) $\cancel{f(z) = z^3 - 1 + 3i}$

$\cancel{\oint_C z^3\,dz - (1+3i)\,dz}$

(9) $\oint_C \frac{1}{z}\,dz$



C_1 $(0,-2) \to (2,0)$.

$x(t) = 0(1-t) + 2t$

$y(t) = -2(1-t) + 0t$

$y(t) = 2t - 2, \quad x(t) = 2t; \quad 0 \le t \le 1$

$z(t) = x(t) + i\,y(t)$

$z(t) = 2t + 2it - 2i$

$z'(t) = (2 + 2i)$

$$\int_0^1 \frac{(2+2i)\,dt}{2t + 2it - 2i}$$

$$\int_0^1 \frac{(1+i)\,dt}{t(1+i) - i} = \int_0^1 \frac{dt}{\left[t - \left(\frac{i}{1+i}\right)\right]}$$

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$$\int_0^1 \frac{dt}{\left[t - \frac{1}{2}(1+i)\right]};$$

$$= \ln\left[t - \frac{1}{2}(1+i)\right]\Big|_0^1$$

$$= \ln\left[1 - \frac{1}{2} - \frac{i}{2}\right] - \ln\left[-\frac{1}{2} - \frac{1}{2}i\right]$$

$$= \mathrm{Ln}\left[\frac{1}{2} - \frac{i}{2}\right] - \ln\left[-\frac{1}{2} - \frac{1}{2}i\right].$$

$$\mathrm{Ln}\, z = \ln\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right).$$

$$\mathrm{Ln}\left[\frac{1}{2} - \frac{i}{2}\right] = \ln\left(\sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}\right) + i\tan\left(\frac{-1/2}{1/2}\right).$$

$$\frac{\sqrt{2}}{2}\angle -\pi/4; \quad = \cancel{\ln\left(\frac{\sqrt{2}}{2}\right)} - i\frac{\pi}{4};$$

$$\mathrm{Ln}\left[-\frac{1}{2} - \frac{1}{2}i\right] = \cancel{+\ln\left(\frac{\sqrt{2}}{2}\right)} - i\frac{3}{4}\pi.$$

$$\frac{\sqrt{2}}{2}\angle -3/4\,\pi$$

$$+$$

$$i\frac{\pi}{2};$$

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C_2: $(2,0) \to (0,2)$.

$x(t) = 2(1-t)$; $\quad 0 \le t \le 1$.

$y(t) = 2t$;

$z(t) = 2(1-t) + 2it$. $\qquad 2$

$z'(t) = (-2+2i)$.

$$\int_0^1 \frac{\cancel{-2}(1-i)}{\cancel{2}(1-t)+\cancel{2}it}\,dt.$$

$$\int_0^1 \frac{(-1+i)\,dt}{1-t+it} = \int_0^1 \frac{(-1+i)\,dt}{+t(-1+i)+1}$$

$$= \int_0^1 \frac{dt}{\left[t+\left(\frac{1}{-1+i}\right)\right]}$$

$$\int_0^1 \frac{dt}{\left[t-\frac{1}{2}-\frac{1}{2}i\right]} = \operatorname{Ln}\left[t-\frac{1}{2}-\frac{1}{2}i\right]\Big|_0^1$$

$$\operatorname{Ln}\left[\frac{1}{2}-\frac{1}{2}i\right] - \operatorname{Ln}\left[-\frac{1}{2}-\frac{1}{2}i\right] = i\frac{\pi}{2};$$

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$C_3 : (0,2) \longrightarrow (-2,0).$

$$\left.\begin{aligned} x(t) &= -2t; \\ y(t) &= 2(1-t) \end{aligned}\right\} \quad 0 \le t \le 1$$

$z(t) = x(t) + i\,y(t)$

$z(t) = -2t + 2i(1-t).$

$z'(t) = -2 - 2i$

$$= \int_0^1 \frac{(-2-2i)\,dt}{-2t+2i-2it}$$

$$= \int_0^1 \frac{(-2-2i)\,dt}{[t(-2-2i)+2i]} = \int_0^1 \frac{dt}{\left[t - \frac{1}{2} - \frac{1}{2}i\right]}$$

$$= \ln\left[t - \frac{1}{2} - \frac{i}{2}\right]\Big|_0^1$$

$= i\pi/2;$

$C_4 : (-2,0) \longrightarrow (0,-2).$

$x(t) = -2(1-t). \quad 0 \le t \le 1$

$y(t) = -2t;$

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$$\int_0^1 \frac{(2-2i)\,dt}{2(t-1)+2it} = \int_0^1 \frac{(1-i)\,dt}{(t-1-it)}.$$

$$\int_0^1 \frac{(1-i)\,dt}{t(1-i)-1} = \int_0^1 \frac{dt}{\left[t-\frac{1}{2}-\frac{1}{2}i\right]} = \frac{\pi i}{2};$$

$$\oint_C \frac{dz}{z} = 4\left(\frac{\pi i}{2}\right) = 2\pi i;$$

21) $\oint_C Ln|z+10)\,dz \qquad |z| = 2.$

Solution:- $\oint_C Ln(z+10)\,dz = 0.$

Because $Ln\,|z+10|$ is analytic inside and on C.

10)

$$\oint_C \frac{5}{z+1+i}\,dz \quad ; \quad C: x^4+y^4=16$$

$$z_0 = -1-i;$$

$$\oint \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n=1 \\ 0 & n \neq 1. \end{cases}$$

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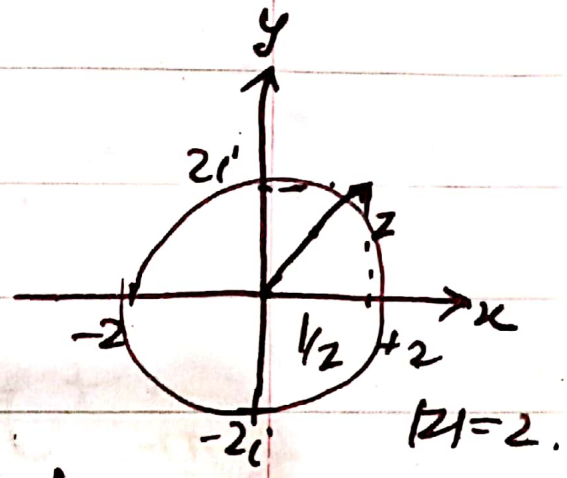
$5(2\pi i) = 10\pi i$;

11) $\oint_C \left(z + \frac{1}{z}\right) dz$; $|z| = 2$.

$= \oint_C z\,dz + \underbrace{\oint_C \frac{dz}{z}}$ $n=1$. $z_0 = 0$

$= 0 + 2\pi i$



12) $\oint_C \left(z + \frac{1}{z^2}\right) dz$, $|z| = 2$.

$\int \frac{dz}{(z-z_0)^n}$

Solution:- $\oint_C z\,dz + \int_C \frac{dz}{z^2}$

'z' is analytic function in and on C.

By Cauchy-Goursat Theorem:

$\oint_C f(z) = 0$ iff $f(z)$ is analytic on C.

$z_0 = 0$.

$0 + \int \frac{dz}{(z-z_0)^2} = 0$.

$\oint_C \left(z + \frac{1}{z^2}\right) dz = 0$.

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(3) $\oint_c \frac{z}{z^2-\pi^2}\,dz$; $|z| = 3$.

$$\frac{z}{(z-\pi)(z+\pi)} = \frac{A}{(z-\pi)} + \frac{B}{(z+\pi)}.$$

$A = 1$, $B = 1$.

$$= \oint_c \frac{dz}{(z-\pi)} + \oint_c \frac{dz}{(z+\pi)}$$

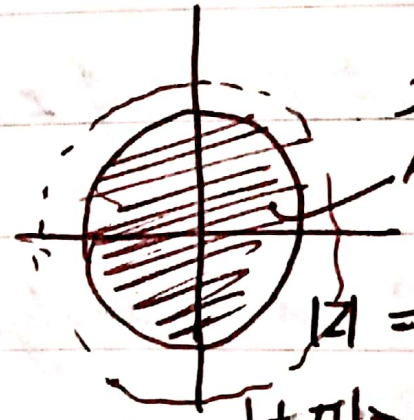
$z_0 = \pi$

$z_0 = -\pi$

$n = 1$.

$2\pi i + 2\pi i = 4\pi i$

$\lim_{|z|\to 3} f(z) = \frac{3}{3^2-\pi^2}$ exist.



$f(z)$.

Analytic.

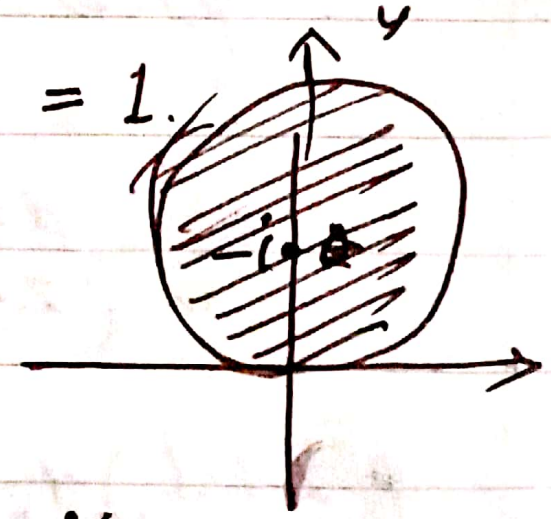
$|z| = 3$.

$|\pm\pi| > 3$

$$\oint_c \frac{z\,dz}{z^2-\pi^2} = 0$$

Cauchy-Goursat Theorem.

(4) $\oint_c \frac{10\,dz}{(z+i)^4}$; $|z+i| = 1$.



$$\oint_c \frac{10\,dz}{(z-(-i))^4} = 0.$$

$z_0 = -i$

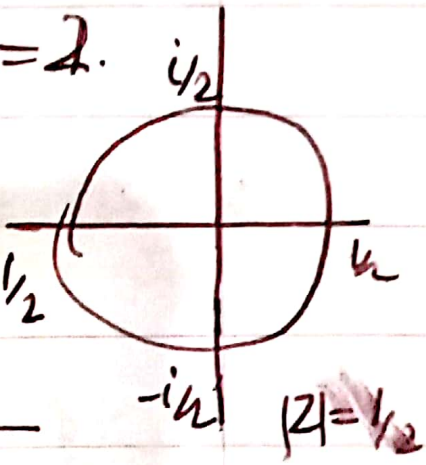
(5)

$$\therefore \oint \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n=1 \\ 0 & n\neq 1 \end{cases}.$$

$$n = 4.$$

15) $\oint_c \frac{2z+1}{z^2+z}\,dz$, (a): $|z| = 1/2$.

(b) $|z| = 2$.



$$\frac{2(z+1/2)}{z(z+1)} = \frac{A}{z} + \frac{B}{(z+1)}$$

$$A = \left.\frac{2z+1}{(z+1)}\right|_{z=0} = \frac{1}{1}$$

$$B = \left.\frac{2z+1}{z}\right|_{z=-1} = \frac{-2+1}{-1} = 1.$$

$$\oint_c \frac{dz}{z} + \int_c \frac{dz}{(z+1)}$$

$$\oint_c \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n=1 \\ 0 & n=0 \end{cases}$$

$z_0 = 0$.

$z_0 = -1$; $|z_0| = 1/2$.

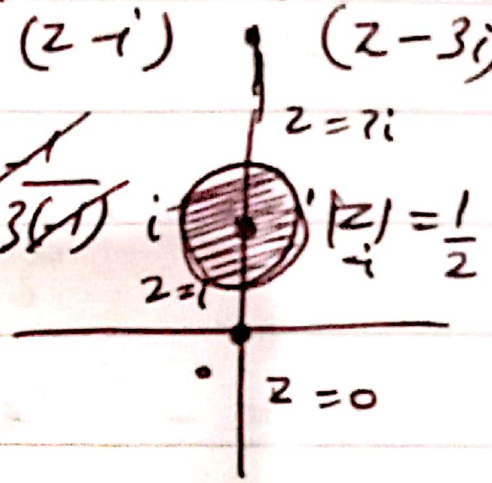
$$2\pi i + \int_c \frac{dz}{(z+1)} = 2\pi i$$

analytic at all points $\to 0$. (b) $= 4\pi i$

(a) $\oint_C \frac{(z-1)\,dz}{z(z-i)(z-3i)}$; $|z-i| = 1/2$.

$$\frac{(z-1)}{z(z-i)(z-3i)} = \frac{A}{z} + \frac{B}{(z-i)} + \frac{C}{(z-3i)}$$

$$A = \frac{(0-1)}{(0-i)(0-3i)} = \frac{-1}{3i^2} = \cancel{\frac{1}{3(-1)}}$$



$A = \frac{1}{3}$;

$$B = \frac{(i-1)}{i(i-3i)} = -\frac{1}{2} + \frac{1}{2}i \;;$$

$$C = \frac{(3i-1)}{3i(3i-i)} = \frac{1}{6} - \frac{1}{2}i$$

$$\cancel{; \oint_C f(z)\,dz = \frac{2\pi i}{3} + \left(-\frac{1}{2}+\frac{1}{2}i\right)2\pi i}$$

$\oint_{|z-i|=1/2} \frac{dz}{(z-0)}$ = 0 ; analytic;

$\oint_{|z-i|=1/2} \frac{dz}{(z-i)} = 2\pi i$

$$\cancel{\frac{2\pi i}{3} + (-1+i)\pi i = \frac{2\pi i}{3} - \pi i - \pi}$$

(a)

$\oint_{|z-i|=1/2} \frac{dz}{(z-3i)} = 0$. $f(z)$ is analytic.

(b)

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$$\cancel{-\pi + \frac{2\pi i - 3\pi i}{3} = -\pi \frac{\pi i}{3}}$$

$$\cancel{-\pi(1 - i/3)}$$

$$\cancel{2}\pi i\left(-\tfrac{1}{\cancel{2}} + \tfrac{1}{\cancel{2}} i\right)$$

$$\pi(-i + i^2) = +\pi(-i - 1) = -\pi(1 + i).$$

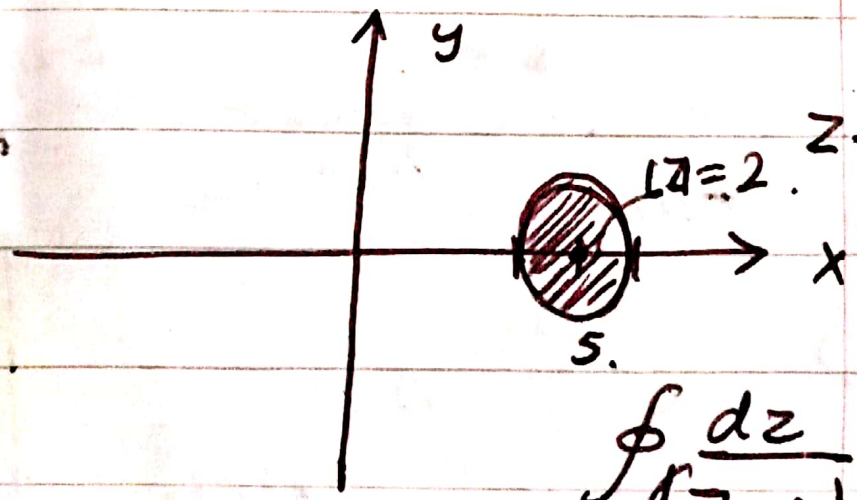
17) $$\oint_C \frac{(-3z+2)\,dz}{(z^2-8z+12)} \quad (a)\ |z| = 9 \quad (b)\ |z-5| = 2;$$

$$z^2 - 8z + 12 = 0.$$

$$z = \frac{8 \pm \sqrt{64 - 4(12)}}{2} = \frac{8 \pm 4}{2}; \{6, 2\}.$$

$$\oint_C \frac{(-3z+2)}{(z-2)(z-6)}\,dz = \oint \frac{(1)\,dz}{(z-2)} + \oint \frac{-4\,dz}{(z-6)}$$

πi



(a)

$-8\pi i$

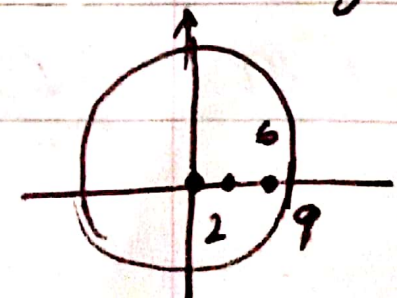
$$\oint_{|z-2|=2} \frac{dz}{(z-2)} = 0.$$

analytic

(b).

$$= 2\pi i - 8\pi i = -6\pi i;$$

πi



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Cauchy's Integral Formula

$$f(z_0) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)}\,dz;$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_c \frac{f(z)\,dz}{(z-z_0)^{n+1}}.$$

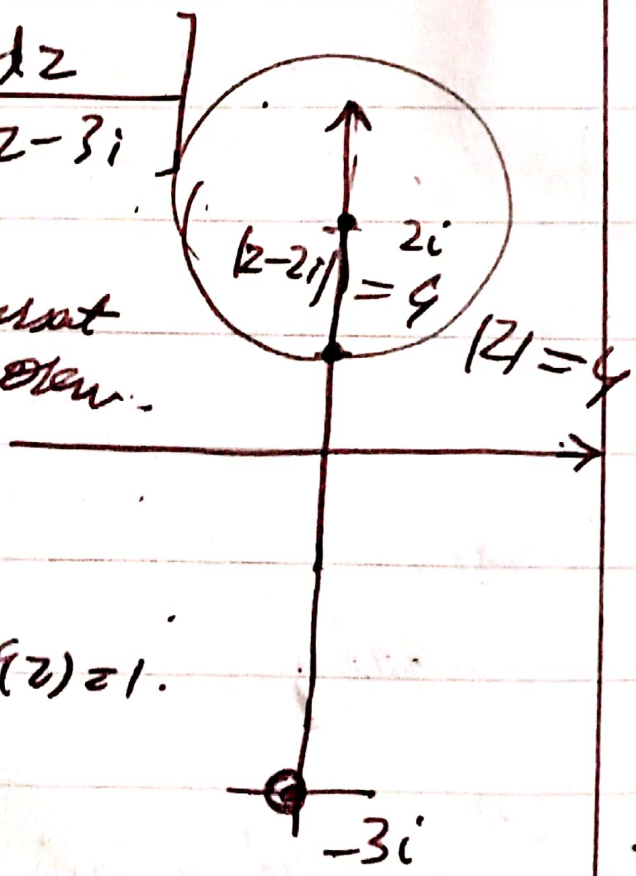
Example: $\oint_c \frac{z\,dz}{(z^2+9)}$ $\quad |z-2i| = 4;$

$(z+3i)(z-3i);$

$$= \frac{1}{2}\left[\oint \frac{dz}{(z+3i)} + \oint \frac{dz}{z-3i}\right]$$

$\to 0$ Goursat Theorem.

$$\frac{1}{2}\oint \frac{dz}{(z+3i)}.$$



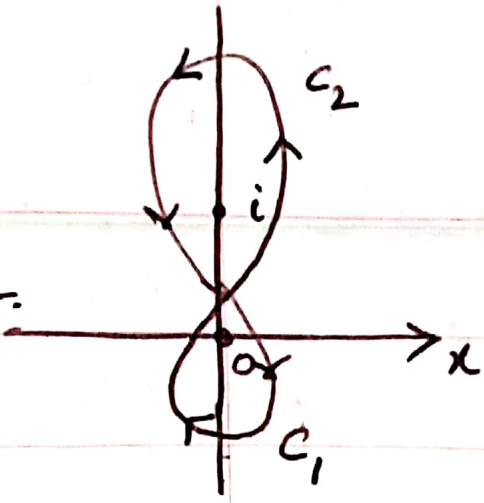
$$\pi i \cdot f(z_0) = \frac{1}{2}\oint \frac{dz}{(z-3i)}.$$

$f(z) = 1.$

$f(z_0) = 1$

$= \pi i$ Ans.

Example: $\int_c \frac{z^3+3}{z(z-i)^2}\,dz$;



$$\int_c \frac{(z^3+3)}{z(z-i)^2}\,dz = \int_{c_1} \frac{(z^3+3)}{z(z-i)^2}\,dz$$

$$+ \int_{c_2} \frac{(z^3+3)}{z(z-i)^2}\,dz.$$

$$= -\int_{c_1} \frac{(z^3+3)\,dz}{z(z-i)^2} + \int_{c_2} \frac{(z^3+3)}{z(z-i)^2}\,dz.$$

$$= -I_1 + I_2;$$

$$I_1 = \int_{c_1} \frac{\left(\frac{z^3+3}{(z-i)^2}\right)}{(z-0)}\,dz = 2\pi i\, f(z_0).$$

$$= 2\pi i \left[\frac{0^3+3}{(0-i)^2}\right] = \frac{2\pi i\, 3}{i^2}$$

$$\frac{6\pi}{i} = -6\pi i; \quad I_1 = -6\pi i;$$

$$I_2 = \int_{c_2} \frac{(z^3+3)/z}{(z-i)^2}\,dz.$$

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$$= 2\pi i\left(\frac{i^3+3}{i}\right) = 2\pi(i^2 i+3)$$

$$= 6\pi + 2\pi i^2 i = 6\pi - 2\pi i;$$

$$= -I_1 + I_2 = 4\pi i + 6\pi - 2\pi i$$

$$\oint_C \frac{(z^3+3)dz}{z(z-i)^2} = 4\pi i + 6\pi;$$

$$n+1 = 2 \quad ; \quad n = 1.$$

$$I_2 = 2\pi i f'(z_0) \qquad z_0 = i;$$

$$f(z) = \frac{z^3+3}{z} = z^2 + \frac{3}{z};$$

$$f'(z) = 2z - \frac{3}{z^2}; \quad f'(i) = 2i - \frac{3}{i^2}$$

$$= 2i - 3(-1) = 2i + 3$$

$$I_2 = 2\pi i(2i+3) = 4\pi i^2 + 6\pi i$$

$$I_2 - I_1 = -4\pi + 6\pi i + 6\pi i = 12\pi i - 4\pi$$

Ans

$$\oint_C \frac{(z^3+3)dz}{z(z-i)^2} = 12\pi i - 4\pi.$$

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Cauchy's Inequality

$r = |z - z_0|$

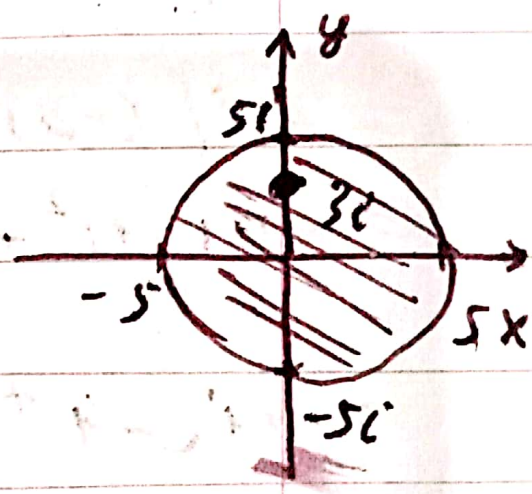
$$|f^{n}(z_0)| \le \frac{n!\,M}{r^n} \qquad M \ge |f(z)|$$

Excercise# 5.5

1) $\oint_C \frac{4\,dz}{(z-3i)}$; $|z| = 5$.

Solution:-

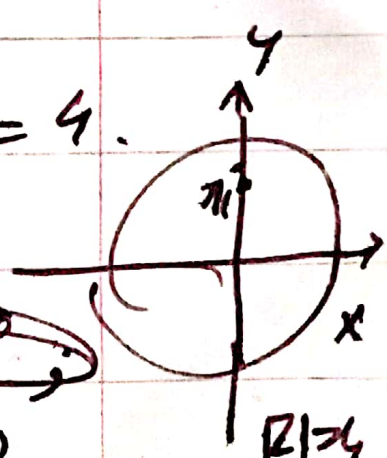
$$= 4\int_{|z|=5} \frac{dz}{(z-3i)^1}$$



$$\because \int_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i, & n = 1 \\ 0, & n \ne 1 \end{cases}$$

$$= 4(2\pi i) = 8\pi i;$$

3) $\oint_C \frac{e^z}{z-\pi i}\,dz$, $|z| = 4$.



Sol:- ~~$2\pi e^4$~~ $= 2\pi e^{\pi i}$;

$$\oint_C \frac{e^z}{(z-\pi i)}\,dz = \frac{1}{2\pi i}\int_{C\ |z|=4} \frac{f(z)}{(z-z_0)} = f(z_0).$$

$$2\pi i\, e^{\pi i} = -2\pi i$$

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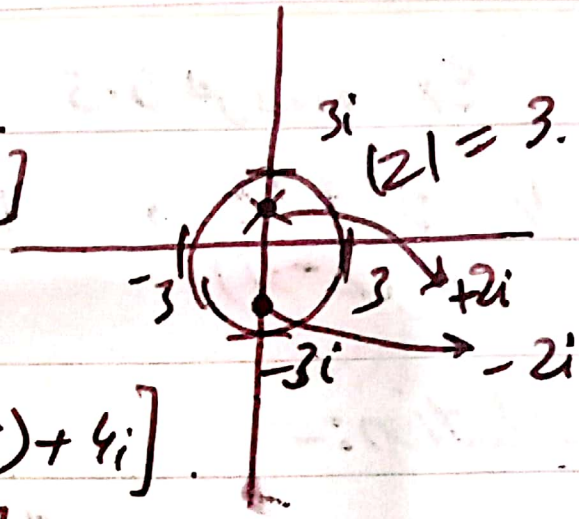
5) $\oint_C \frac{z^2 - 3z + 4i}{(z+2i)} dz, \quad |z| = 3.$

$$= \oint_{|z|=3} \frac{z^2 - 3z + 4i}{[z-(-2i)]}$$



$$= 2\pi i\, f(-2i)$$

$$= 2\pi i\left[(-2i)^2 - 3(-2i) + 4i\right].$$

$$= 2\pi i\left[4i^2 + 6i + 4i\right].$$

$$= 2\pi i(-4 + 10i) = 2\pi(-10 - 4i).$$

$$= -2\pi(+10 + 4i) = -4\pi(5 + 2i)$$

7) $\oint \frac{z^2\, dz}{z^2+4}$; (a) $|z - i| = 2$

(b) $|z + 2i| = 1.$

$$\cancel{\frac{z^2}{(z+2i)(z-2i)}} \qquad \frac{1}{(z+2i)} + \frac{1}{(z-2i)}$$

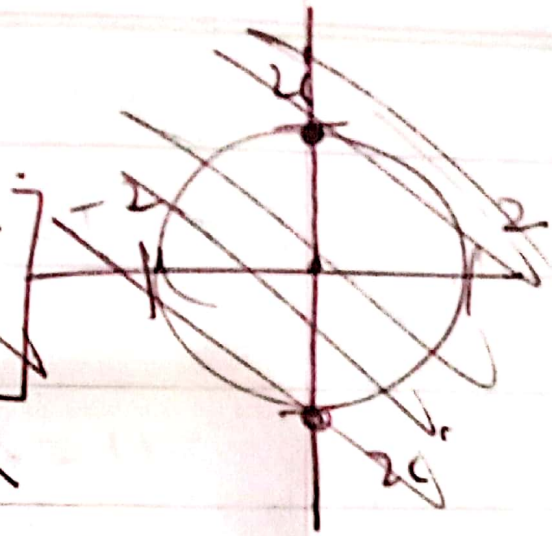
$$\cancel{\frac{z^2+4-4}{(z^2+4)} = 1 - \frac{4}{(z^2+4)}}.$$

$$\oint_{|z-i|=2} \left[1 - \frac{4}{(z+2i)} - \frac{4}{(z-2i)} \right] dz$$

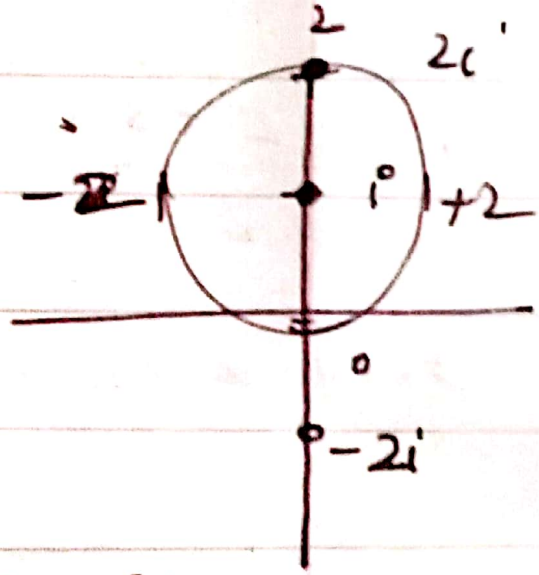
$$-4 \oint_{|z-i|=2} \left[\frac{dz}{(z+2i)} + \frac{dz}{(z-2i)} \right]$$

$$-4.(2\pi i + 2\pi i)$$

$$= -16\pi i.$$



$$\oint_{|z-i|=2} \frac{z^2\,dz}{(z+2i)(z-2i)}$$



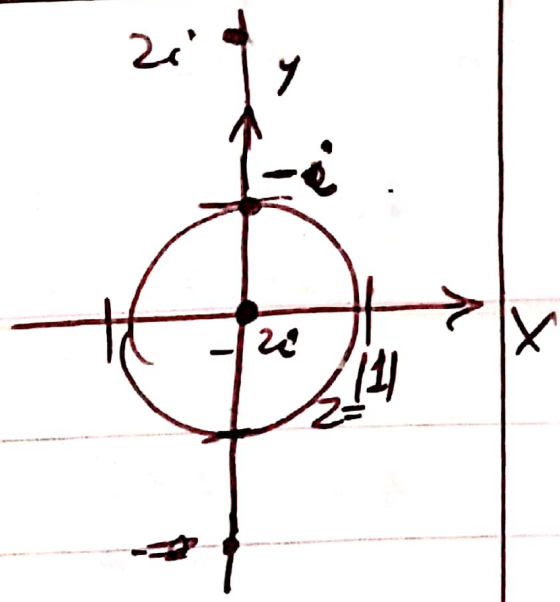
$$= \oint_{|z-i|=2} \frac{\left(\frac{z^2}{z+2i}\right)}{(z-2i)} dz$$

$z_0 = 2i$

$$= 2\pi i\, f(2i) = 2\pi i \cdot \frac{(2i)^2}{2i+4i} = \frac{(2i)^2}{4i}$$

$$= 2\pi i \frac{4i^2}{4i} = -2\pi i$$

(b) $|z+2i| = 1$.



$$\oint \frac{z^2\,dz}{z^2+4}$$

$$\oint \frac{z^2\,dz}{(z+2i)(z-2i)}$$

$|z+2i|=1$

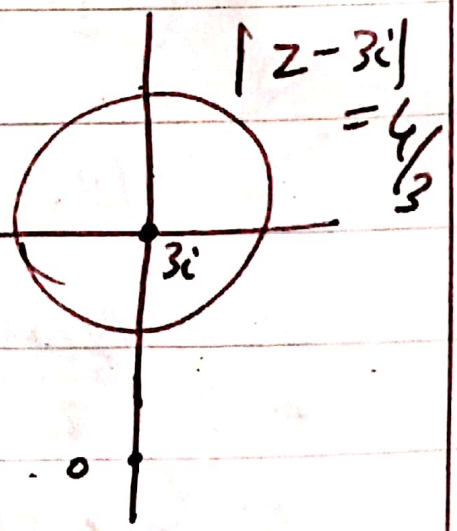
$$= 2\pi i\, f(-2i) = 2\pi i \frac{(-2i)^2}{(-2i-2i)}$$

$$= 2\pi i \frac{\cancel{4}i^2}{-\cancel{4}i} = -i^2\, 2\pi$$

$$= 2\pi \text{ Ans}$$

9) $\oint_C \frac{z^2+4\,dz}{(z^2-5iz-4)}$; $|z-3i| = \frac{4}{3}$; 11)

$$z = \frac{5i \pm \sqrt{(5i)^2-4(1)(-4)}}{2}$$



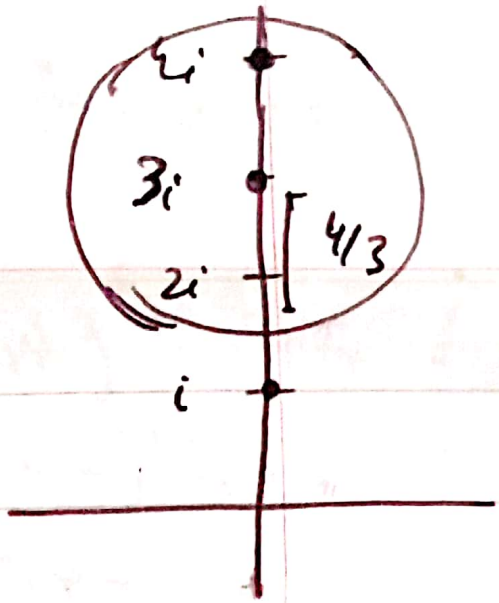
$$z = \frac{5i \pm \sqrt{-25+16}}{2}$$

$$z = \frac{5i \pm \sqrt{-9}}{2}$$

$$z = \{4i, i\}.$$

$$z = 5i \pm 3i/2$$

$$\oint_c \frac{(z^2+4)\,dz}{(z-4i)(z-i)}$$



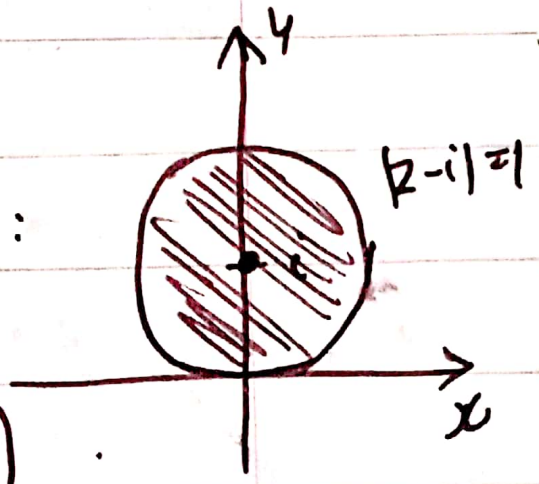
$$\int_{|z-3i|=4/3} \frac{\left(\frac{z^2+4}{z-i}\right)dz}{(z-4i)}$$

$$z_0 = 4i$$

$$2\pi i\left[\frac{(4i)^2+4}{4i-i}\right] = \frac{2\pi i(-16+4)}{(3i)}$$

$$\frac{2}{3}\pi(-12) = -8\pi$$

11) $\oint_c \frac{e^{z^2}\,dz}{(z-i)^3}$; $|z-i| = 1$



$$z_0 = i$$

$$n+1 = 3; \quad n = 2:$$

$$= 2\pi i\, f^{n}(z_0).$$

$$\frac{d^2}{dz^2}\left(e^{z^2}\right) = \frac{d}{dz}\left(e^{z^2}\cdot 2z\right)$$

$$= 2e^{z^2} + 2z\left(2ze^{z^2}\right) = 2e^{z^2} + 4z^2e^{z^2}$$

$$= e^{z^2}(4z^2+2)$$

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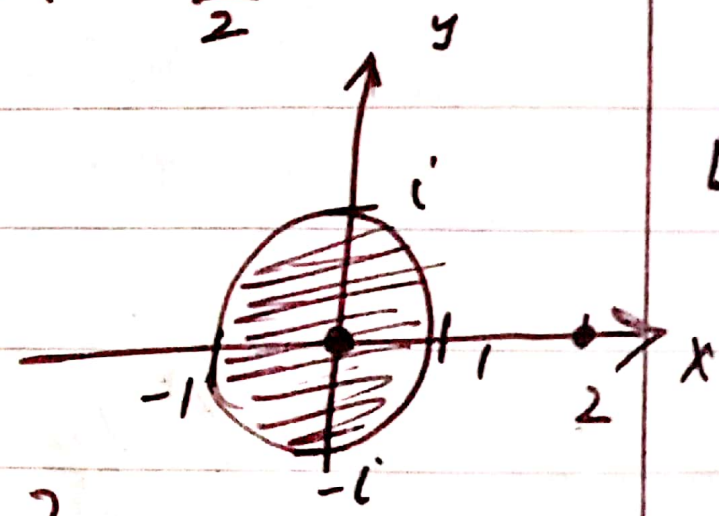
$$= 2\pi i \, e^{z^2}(4z^2+2)\Big|_{z=i}$$

$$= 2\pi i \, e^{i^2}(4i^2+2).$$

$$= 2\pi i e^{-1}(-4+2) = 2\pi e^{-1} i(-2).$$

$$= -4\pi e^{-1} i;$$

15) $\oint_C \frac{2z+5}{z^2-2z}$; (a) $|z| = \frac{1}{2}$.



$$= \oint_{|z|=1} \frac{2z+5 \, dz}{z(z-2)}$$

$$f(z) = \frac{2z+5}{(z-2)}$$

$z_0 = 0$;

$$= 2\pi i \left(\frac{2(0)+5}{0-2}\right) = \cancel{2}\pi i \left(\frac{5}{-\cancel{2}}\right)$$

$$= -5\pi i;$$

(b)

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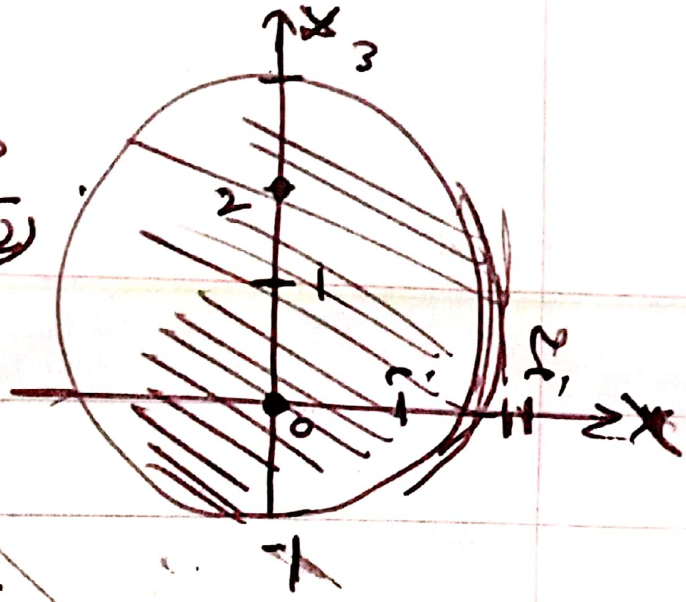
(b) $|z+1| = 2$.



$$\frac{2z+5}{z(z-2)} = \frac{A}{z} + \frac{B}{(z-2)}$$

$$A = \left.\frac{2z+5}{z-2}\right|_{z=0}$$

$$A = -\frac{5}{2}; \qquad B = \left.\frac{2z+5}{z}\right|_{z=2} = \frac{4+5}{2}$$

$$B = \frac{9}{2}; \qquad 2\pi i\left[\frac{9}{2}(2z+5)\right]_{z=2} + 2\pi i\left(-\frac{5}{2}\right)(2z+5)$$

$$= 2\pi i\left(\frac{9}{2}(4+5)\right) - 5\pi i(5)$$

$$= 81\pi i - 25\pi i = 56\pi i$$

$$-\frac{5}{2}\oint \frac{(2z+5)}{z}\,dz + \frac{9}{2}\oint \frac{(2z+5)}{(z-2)}\,dz$$

$$-\frac{5}{2}\,2\pi i(5) + \frac{9}{2}(2\pi i)(9) = 56\pi i$$

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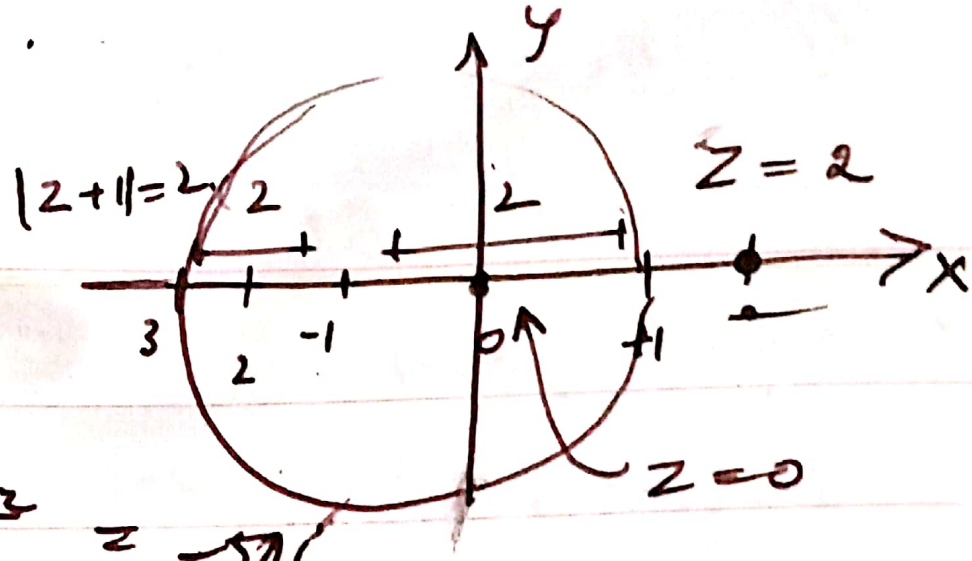
$|z+1| = 2$.



$$\int_{|z+1|=0} \frac{2z+5}{z(z-2)}\,dz = -5\pi i$$

13) $\oint \frac{\cos 2z}{z^5}\,dz$; $|z| = 1$.

$n+1 = 5$

$n = 4$;

$(\cos 2z)' = -2\sin 2z$

$(-2\sin 2z)' = (-4\cos 2z)$

$(-4\cos 2z)' = (8\sin 2z)$

$(8\sin 2z)' = 16\cos 2z$; $z_0 = 0$.

$$\oint_C dz = \frac{2\pi i \; 16\cos 2(0)}{4!} = 32\pi i \cos 0.$$

$$= \frac{32\pi i}{4!} = \frac{4}{3}\pi i$$

---

19) $\oint_C \left(\frac{e^{2iz}}{z^4} - \frac{z^4}{(z-i)^3}\right) dz$; $|z| = 6$

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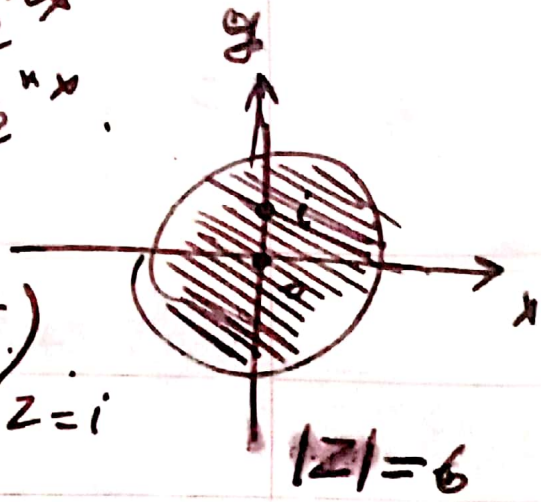
$$\left.\begin{array}{l} z=0 \\ z=i \end{array}\right\} \qquad \begin{array}{l} (e^{nx})' = n e^{nx} \\ (e^{nx})'' = n^2 e^{nx}. \end{array}$$

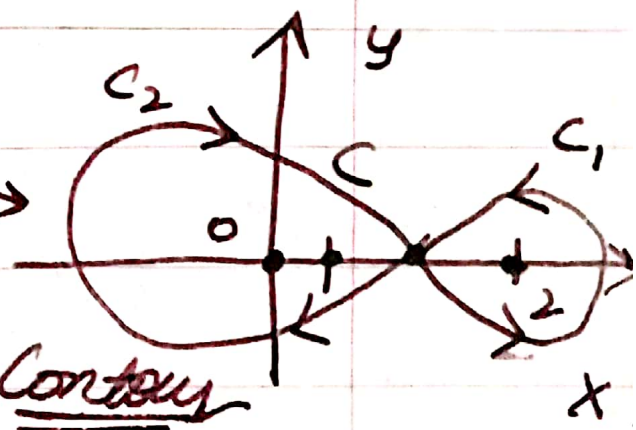


$$\frac{2\pi i\, e^{2i(0)} (2i)^3}{3!} - \frac{\cancel{2}\pi i}{\cancel{2}!}(12z^2)\Big|_{z=i}$$

$$= \frac{\cancel{2}\pi i \cdot 8 i i^2}{\cancel{6}_{3}} - 12\pi i (i)^2$$

$$= \frac{8\pi}{3} i^2 i^2 + 12\pi i = 12\pi i + \frac{8\pi}{3};$$

23) $$\oint_C \frac{(3z+1)dz}{z(z-2)^2}.$$



Contour

$$\oint_C \frac{(3z+1)}{z(z-2)^2}\, dz = \oint_{C_1} \frac{(3z+1)\,dz}{z(z-2)^2}$$

$$+ \int_{C_2} \frac{(3z+1)\,dz}{z(z-2)^2}.$$

$$+ \oint_{C_1} \frac{(3z+1)\,dz}{z(z-2)^2} - \oint_{-C_2} \frac{(3z+1)\,dz}{z(z-2)^2}.$$

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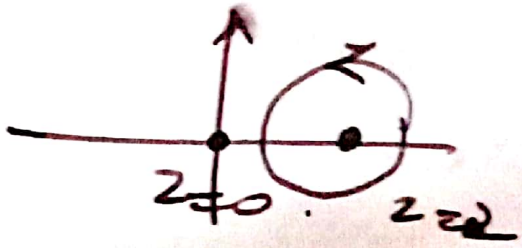
$$I_1 = \oint_{C_2} \frac{(3z+1)\,dz}{z(z-2)^2}$$



$$= \oint_{C_2} \frac{(3z+1)/z}{(z-2)^2}\,dz = \frac{2\pi i}{1!} f'(z_0)$$

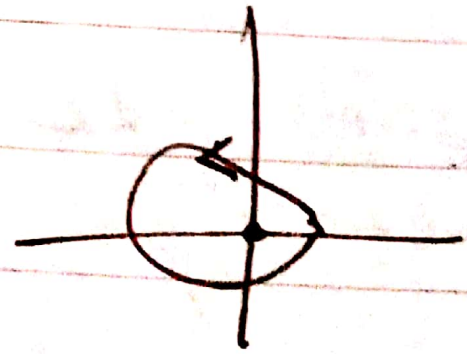
$$z_0 = 2.$$

$$f(z) = \frac{3z+1}{z} = 3 + \frac{1}{z};$$

$$f'(z) = -\frac{1}{z^2}; \qquad f'(2) = -1/4$$

$$= \cancel{2}\pi i\left(\frac{-1}{\cancel{4}}\right) = \frac{-\pi i}{2};$$

$$= \oint_{-C_2} \frac{(3z+1)/(z-2)^2}{z}\,dz$$



$$2\pi i\,\frac{(3(0)+1)}{(0-2)^2} = \frac{\cancel{2}\pi i}{\cancel{4}} = \frac{\pi i}{2};$$

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$$-\frac{\pi i}{2} - \frac{\pi i}{2} = -1(\pi i) = -\pi i$$

Chapter 6: Residues and Series.

- Residue Theorem.

$$\mathrm{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z);$$

$$\mathrm{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right]$$

Example: $f(z) = \dfrac{1}{(z-1)^2 (z-3)};$

$z = 3.$

$$\mathrm{Res}(f(z), 3) = \lim_{z \to 3} \frac{1}{(z-1)^2} = \frac{1}{(3-1)^2}$$

$= 1/4;$ $\quad z = 1.$

$$\mathrm{Res}(f(z), 1) = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d}{dz} \left(\frac{1}{(z-3)} \right)$$

$$\lim_{z \to 1} \left[\frac{-1}{(z-3)^2} \right] = \frac{-1}{(1-3)^2} = -\frac{1}{4};$$

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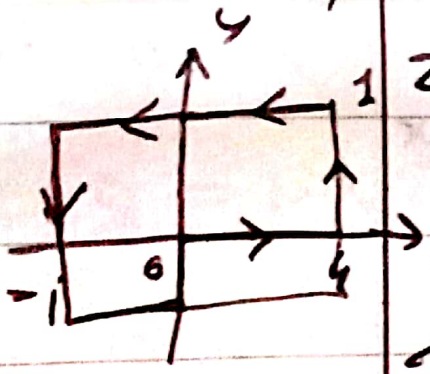
Cauchy's Residue Theorem:

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k).$$

Example: $\displaystyle\oint_C \frac{dz}{(z-1)^2(z-3)}$; (a)



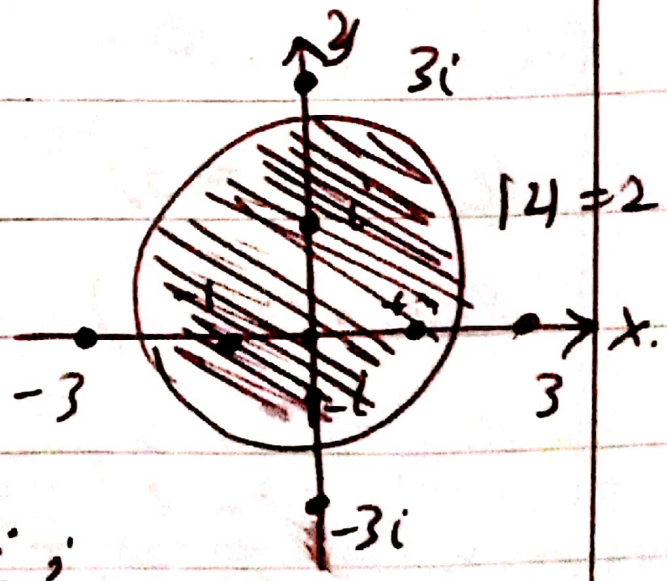
$$\operatorname{Res}(f(z), z=3) = \frac{1}{(3-1)^2} = \frac{1}{4};$$

$$\operatorname{Res}(f(z), z=1) = \frac{-1}{(z-3)^2} = -\frac{1}{(1-3)^2} = -\frac{1}{4};$$

$$= 2\pi i\left[\left(\frac{1}{4}\right) + \left(-\frac{1}{4}\right)\right]$$

(b) $|z| = 2$;

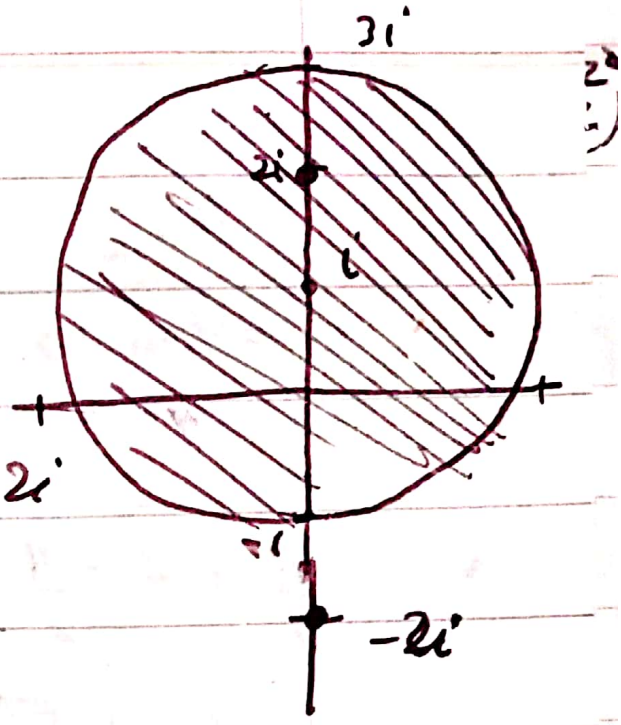
$z = 1$, ~~$z = 3$~~



$$= 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi}{2} i;$$

Example:

$$\oint_C \frac{2z+6}{z^2+4}\,dz \qquad C: |z-i| = 2.$$



$$z^2+4 = (z+2i)(z-2i)$$

$$\cancel{(z-2i)}\,\frac{2z+6}{(z+2i)\cancel{(z-2i)}}\Bigg|_{z=2i}$$

$$\text{Res}: = \frac{2(2i)+6}{2i+2i}$$

$$\frac{4i+6}{4i} = \frac{+4-6i}{4} = \underline{\qquad}$$

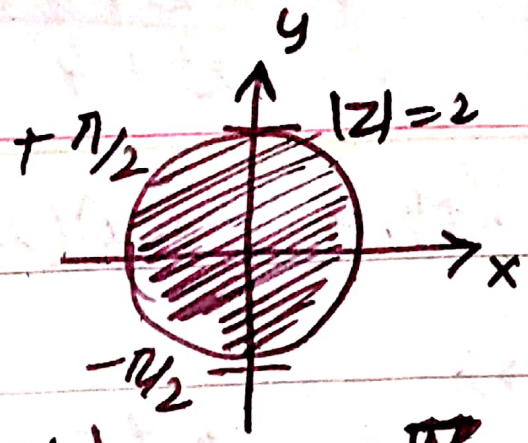
$$\text{Res}_1 = 1 - \frac{3}{2}i \Big\}. \qquad 2\pi i\left[1-\frac{3}{2}i\right].$$

$$2\pi i - 3\pi i^2 = 2\pi i + 3\pi$$

$$\oint_{|z-i|=2} \left(\frac{2z+6}{z^2+4}\right) dz = 2\pi i + 3\pi.$$

-1

Example: $\oint_C \tan z \, dz \quad C: |z| = 2.$



$$\tan z = \infty$$

$$z = \tan^{-1}\infty = \left(\frac{2n+1}{2}\right)\pi; \quad n \in \mathbb{Z}.$$

$$n = 0, \pm 1, \pm 2$$

$\frac{\pi}{2}$ and $-\frac{\pi}{2}$ are present in a circle ~~other~~ ;

$\cos(z) \to 0$

$$f(z) = \tan z = \frac{\sin z}{\cos z} = \frac{g(z)}{h(z)};$$

$$\left(f(z), \pi/2\right) = \left.\frac{\sin z}{-\sin z}\right|_{z=\pi/2} = \frac{1}{-1} = -1.$$

$$\operatorname{Res}\left(f(z), -\pi/2\right) = \frac{-1}{-(-1)} = -1;$$

$$2\pi i\left[-1 + (-1)\right] = -4\pi i;$$

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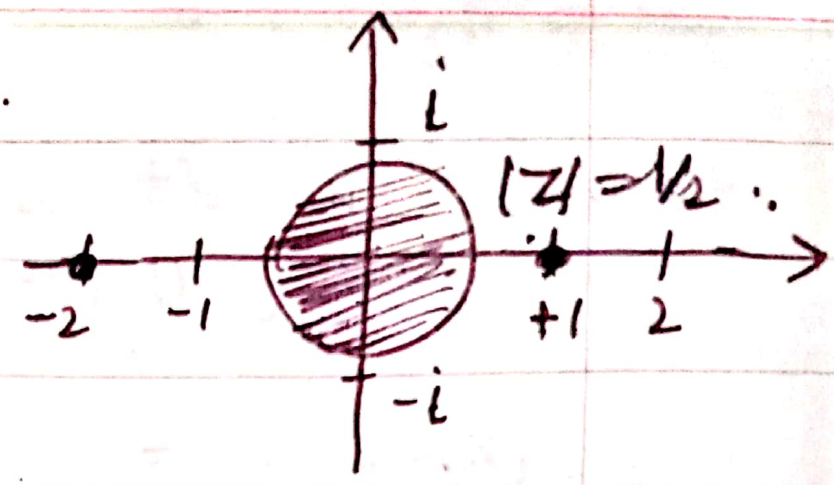
Excercise: 6-5

(7) $\oint_c \frac{dz}{(z-1)(z+2)^2}$;

(a) $|z| = 1/2$.



Ans = 0.

(b) $|z| = 3/2$.

$z = 1$.

$Res(f(z), 1) = \frac{1}{(1+2)^2} = \frac{1}{9}$;

$2\pi i\left(\frac{1}{9}\right) = \frac{2\pi}{9} i$

(c) $|z| = 3$;

$Res(f(z), -2) = \frac{1}{(2-1)!} \frac{d}{dz}\left[\frac{1}{(z+1)^2}\right]$

$\frac{-1}{(-2-1)^2} = \frac{-1}{9}$;

$\oint_c dz = \frac{2\pi i}{9} - \frac{2\pi i}{9} = 0$.

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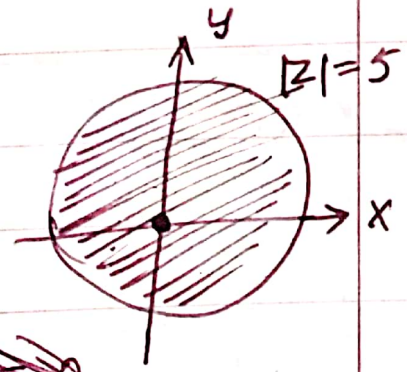
19) $\oint_C z^3 e^{-1/z^2}\,dz$; (a) $|z|=5$.

$$\oint_{|z|=5} \frac{z^3}{e^{1/z^2}}\,dz$$



$z=0$.

$\operatorname{Res}(f(z), z=0) = \lim_{z\to 0} z^3 = 0$.

$\infty e^{-0} = \infty$.

$\oint z^3 e^{-1/z^2}\,dz$

$\operatorname{Res}(z^3 e^{-1/z^2}, \infty)$.

$\operatorname{Res}(f(z), \infty) = -\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right)$.

$f(z) = z^3 e^{-1/z^2}$

$f\left(\frac{1}{z}\right) = \frac{e^{z^2}}{z^3}$;

$\left(\frac{e^{z^2}}{z^5}\right)\Big|_{z=0}$ $\quad \frac{2ze^{z^2}}{5z^4}$

$+\frac{8}{15}(2\pi i)$

$$\frac{2e^{z^2} + 4z^2e^{z^2}}{20z^3} = \frac{2(2ze^{z^2}) + 8ze^{z^2} + 4z^2(2ze^{z^2})}{60z^2}$$

$$= \frac{4ze^{z^2} + 8ze^{z^2} + 8z^3e^{z^2}}{60z^2}$$

$$= \frac{8z^3e^{z^2} + 12ze^{z^2}}{60z^2}$$

$$= \frac{8z^2e^{z^2} + 12e^{z^2}}{60z}$$

$$= \frac{8z^2(2ze^{z^2}) + 24e^{z^2} + 16ze^{z^2}}{60}$$

$$= \frac{16z^3e^{z^2} + 40ze^{z^2}}{60}$$

$$\frac{e^{-z^2}}{z^5} = \frac{e^{z^2} - 2z}{5z^4} = \frac{2e^{-z^2}}{5z^3} = \frac{-2(-2ze^{-z^2})}{15z^2} = \frac{4e^{-z^2}}{15z}$$

$$-\left(\frac{-8ze^{-z^2}}{15z}\right)\Big|_{z=\infty} + \frac{8}{15}$$

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$$\oint_{|z|=5} z^3 e^{-1/z^2}\, dz,$$

$$\frac{1}{z^2}\left(\frac{e^{-z^2}}{z^3}\right) = \frac{e^{-z^2}}{z^5} = \frac{-2z e^{-z^2}}{5z^4}$$

$$\frac{-2e^{-z^2}}{5z^3} = \frac{-2(-2z e^{-z^2})}{15 z^2} = \frac{4e^{-z^2}}{15z}$$

$$\frac{4(-2z e^{-z^2})}{15} = -\frac{8}{15}\frac{z}{e^{z^2}} = 0.$$

18) $\int_C z^3 e^{-1/z^2}\, dz \qquad C: |z| = 5$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{5!}$$

$$e^{-1/z^2} = 1 + \left(\frac{-1}{z^2}\right) + \frac{\left(\frac{-1}{z^2}\right)^2}{2!} + \frac{\left(\frac{-1}{z^2}\right)^3}{3!} + \cdots$$

$$= 1 - \frac{1}{z^2} + \frac{1}{2z^4} - \frac{1}{6z^6} + \frac{1}{24z^8} - \cdots$$

$$z^3 - z + \frac{1}{2z} - \frac{1}{6z^3} \cdots \approx z(z^2-1) + \frac{1}{2z}$$

$z = 0.$

$$\text{res}(f(z)=0) = z^4 - z^2 + \frac{1}{2}\Big|_{z=0}$$

$$2\pi i\left(\frac{1}{2}\right) = \pi i$$

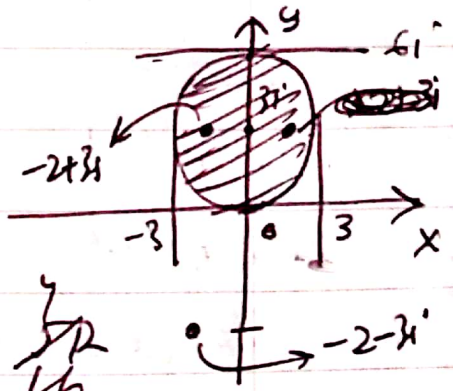
21) $\oint_C \frac{dz}{(z^2+4z+13)}; \qquad C: |z-3i| = 3.$

$$z = \frac{-4 \pm \sqrt{16-4(13)}}{2}$$

$$z = \frac{-4 \pm \sqrt{16-52}}{2}$$

$$z = \frac{-4 \pm 6i}{2}$$

$z = -2 \pm 3i;$



$$\frac{16}{36}$$

$z = -2+3i$

$z = -2-3i$ excluded

$$\oint \frac{dz}{(z+2-3i)(z+2+3i)}$$

$|z-3i| = 3.$

$$\text{Res}\left[f(z), -2+3i\right] = \frac{1}{(\cancel{-2}+3i+\cancel{2}+3i)} = \frac{1}{6i}$$

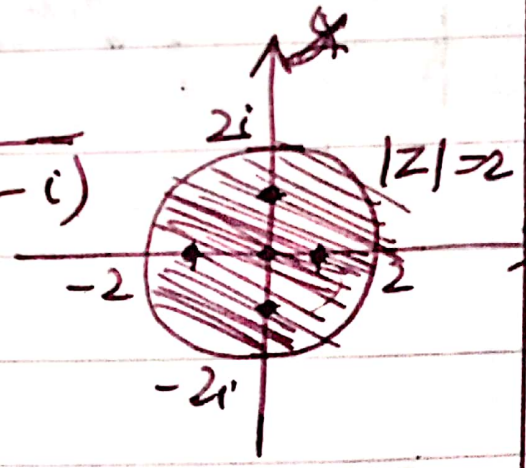
$$\cancel{\text{Res}\left[f(z), -2-3i\right] = \frac{1}{(\cancel{-2}-3i+\cancel{2}-3i)} = \frac{1}{-6i}}$$

$$2\pi\cancel{i}\left[\frac{1}{6\cancel{i}} \cancel{-\frac{1}{6i}}\right] = \pi/3 \Big\}.$$

23) $\oint_C \frac{z\,dz}{z^4-1}$; $C: |z-a| = 2$

$$(z^2-1)(z^2+1) = (z-1)(z+1)(z+i)(z-i)$$

$$\text{Res}\left[f(z), z=1\right] = \frac{1}{(2)(1+i)(1-i)}$$



$$= \frac{1}{4};$$

$$\text{Res}\left[f(z), z=-1\right] = \frac{1}{(-2)(-1+i)(-1-i)}.$$

$$= \frac{1}{4}$$

$$\oint_C f(z)\,dz = 2\pi$$

Res [

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$\oint_C$

$p(z) = ($

Res f

Res f

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$$\operatorname{Res}[f(z), i] = \frac{\cancel{i}}{(i-1)(i+1)(2\cancel{i})} = -\frac{1}{4}$$

$$\operatorname{Res}[f(z), -i] = \frac{-i}{(-i-1)(-i+1)(-i-i)} = -\frac{1}{4}$$

$$\oint_C f(z)\,dz = 2\pi i \sum \left(\cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} \right)$$

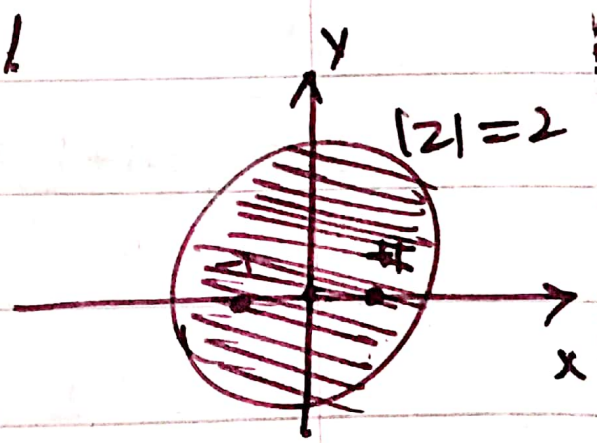
$$= 0.$$

---

25) $\displaystyle\oint_C \frac{z e^z}{z^2-1}\,dz; \quad |z| = 2.$

$$z = \pm 1$$

$$p(z) = (z-1)(z+1)$$



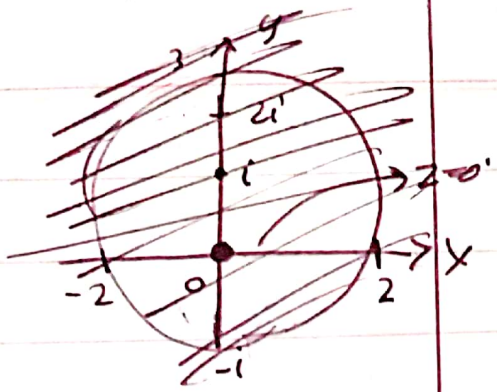
$$\operatorname{Res}[f(z), +1] = \frac{e}{2}$$

$$\operatorname{Res}[f(z), -1] = \frac{e^{-1}}{2}; = e^{-1}/2$$

$$\oint_C f(z)\,dz = 2\pi i \sum_{k}^{n} f(z_k) \; ; \quad 2\pi i \left[\frac{e + e^{-1}}{2} \right] = 2\pi i \cosh 1$$

$$= 2\pi i \cosh(1);$$

28) $\oint \frac{\tan z}{z}\,dz$; $C: |z-1| = 2$.



$f(z) = \frac{\tan z}{z}$

$f(z) = \frac{\sin z}{z\cos z}$;

$z = \left(\frac{2n+1}{2}\right)\pi$, $n = 0 \pm 1, \pm 2, \pm 3 \ldots$

$z = +\frac{\pi}{2}, -\frac{\pi}{2}$;

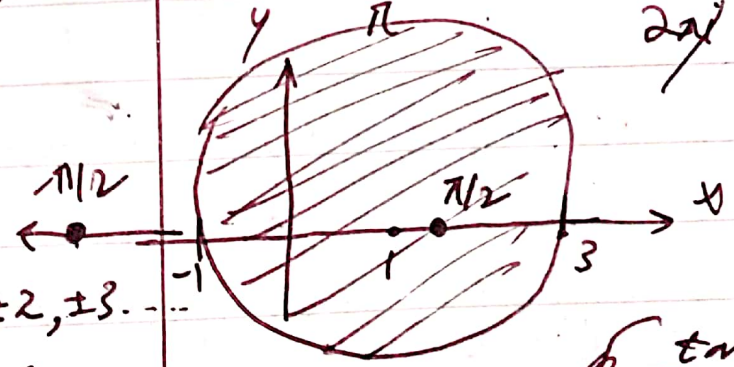
$$\operatorname{Res}\left[f(z), \frac{\pi}{2}\right] = \left.\frac{\sin z}{\cos z + z\sin z}\right|_{\pi/2}$$

$$= \frac{\sin(\pi/2)}{\cos(\pi/2) + \pi/2\,\sin(\pi/2)} = \frac{(1)}{-\pi/2(1)}$$

$$= -\frac{2}{\pi}; \quad \operatorname{Res}\left[f(z), -\frac{\pi}{2}\right]$$

$$= \frac{-1}{0 - \left(-\frac{\pi}{2}\right)(-1)} = \frac{1}{\pi/2}.$$

$$= \frac{2}{\pi}$$

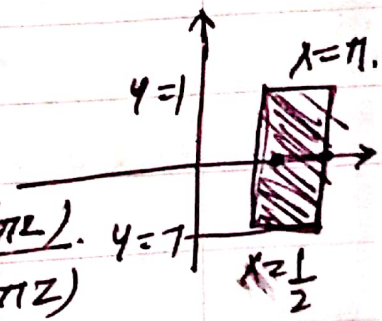


$$2\pi i\left(-\frac{2}{\pi}\right) = -4i$$

$$\oint_{|z-1|=2} \frac{\tan z\,dz}{z} = -4i$$

29) $\oint_C \cot(\pi z)\,dz$

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}.$$



$\sin(\pi z) = 0$; $\pi z = n\pi$; $0, \pm 1, \pm 2$

$n = 1.$

$\cancel{\pi z = \pi}$

$\boxed{z = 1}$

$\cancel{\text{Res}[f(z), 1] = \left.\dfrac{\cos \pi z}{\pi \cos \pi z}\right|_{z=1} = \dfrac{1}{\pi}}$

$\cancel{= 2\pi i \left(\dfrac{1}{\pi}\right) = \dfrac{2\pi i}{\pi} = 2i}$

$\cancel{\pi z = \pi n \qquad n = z; \qquad n = 0, \pm 1, \pm 2, \pm 3.}$

$\cancel{n = 1; \qquad z = 1}$

$\cancel{\dfrac{d}{dz}(\cot \pi z) = -\pi \csc^2 \pi z = \left.\dfrac{-\pi}{\sin^2 \pi z}\right|_{z=1}}$

$\cancel{\dfrac{-\pi}{1 + \cos^2 \pi z\big|_{z=1}} = \dfrac{-\pi}{1 + (-1)^2} = \left(-\dfrac{\pi}{2}\right)}$

$z = n = 0, \pm 1, \pm 2, \pm 3, \pm 4$

$x = \frac{1}{2}; \quad x = \pi.$

$\frac{1}{2} \le x \le \pi$

$n = 1, 2, 3$

$z = 1, 2, 3.$

$\text{Res}(f(z), 1) = \dfrac{1}{\pi}; \quad \text{Res}(f(z), 2) = \dfrac{\cancel{\cos 2\pi}}{\pi \cancel{\cos 2\pi}} = \dfrac{1}{\pi}.$

$\text{Res}(f(z), 3) = 1/\pi.$

$2\cancel{\pi i}\left(\dfrac{3}{\cancel{\pi}}\right) = 6i$

$\oint_C \cot \pi z = 6i$ Ans.

---

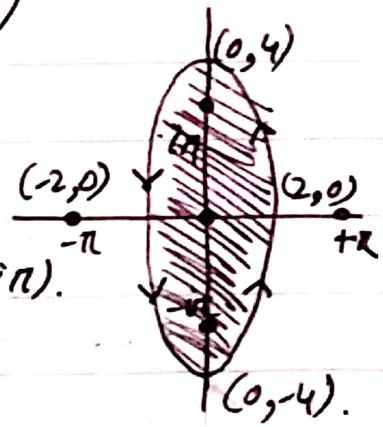
31) $\oint_C \left(\underbrace{z^2 e^{1/\pi z}}_{f_1(z)} + \underbrace{\dfrac{z e^{z}}{z^4 - \pi^4}}_{f_2(z)}\right) dz; \quad C: 4x^2 + y^2 = \pi$

$z = 0$

$(z^2 - \pi^2)(z^2 + \pi^2).$

$(z - \pi)(z + \pi)(z + i\pi)(z - i\pi).$

$z = \underbrace{\pi, -\pi}_{\text{excluded}}, +i\pi, -i\pi$



Figure

$\text{Res}(f_1(z), 0) = \dfrac{1}{6\pi}; \Big\}$ using Laurant Series.

$$f(z) = \frac{z e^{z}}{(z+\pi)(z-\pi)(z+i\pi)(z-i\pi)}.$$

$$\text{Res}\left[f(z), z = +i\pi\right] = \frac{\cancel{i\pi} e^{i\pi}}{(\pi+i\pi)(-\pi+i\pi)(\cancel{2i\pi})}.$$

$$= \frac{-1}{2(\pi+i\pi)(-\pi+i\pi)} = \frac{1}{2(\pi+i\pi)(\pi-i\pi)}.$$

$$\frac{1}{2\pi^2(1+i)(1-i)} = \frac{1}{4\pi^2};$$

$$\text{Res}\left[f(z), z = -i\pi\right] = \frac{\cancel{i\pi} e^{-i\pi}}{(-i\pi+\pi)(-i\pi-\pi)(\cancel{-2i\pi})}.$$

$$= \frac{-1}{2(\pi-i\pi)(-\pi-i\pi)} = \frac{1}{2\pi^2(1-i)(1+i)}$$

$$= \frac{1}{4\pi^2};$$

$$\cancel{2\pi i\left[\frac{1}{\pi} + \frac{1}{4\pi^2}\right] = 2\pi i\left[\frac{1}{\pi} + \frac{1}{2\pi^2}\right]}.$$

$$\cancel{2i + \frac{\cancel{2}\pi i}{\cancel{4}\pi^2} = 2i + \frac{\cancel{2}i}{\pi}}$$

$$i\left(2 + \frac{1}{\pi}\right);$$

$$z^2 e^{1/\pi z} \qquad e^{z} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$$

$$z^2\left(1 + \frac{1}{\pi z} + \frac{1}{2\pi^2 z^2} + \frac{1}{6\pi^3 z^3}\right)$$

$$z^2 + \frac{1}{\pi} + \frac{1}{2\pi^2} + \frac{1}{6\pi^3 z} -$$

$$\left(1/6\pi^3\right).$$

$$= 2\pi i\left(\frac{1}{6\pi^3} + \frac{1}{4\pi^2}\right)$$

$$= i\left(\frac{\cancel{2}\pi}{\cancel{6}\pi^3} + \frac{\cancel{2}\pi}{\cancel{4}\pi^2}\right) -$$

$$= i\left(\frac{1}{3\pi^2} + \frac{1}{\pi}\right) \quad \text{Ans}$$

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$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta. \Big\} \quad z = e^{i\theta}, 0 \le \theta \le 2\pi \qquad |z| = 1.$$

Example: $\displaystyle\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2};$$

$$z = e^{i\theta} \qquad dz = i e^{i\theta} d\theta.$$

$$\frac{dz}{iz} = d\theta;$$

$$= \oint_{|z|=1} \frac{\frac{dz}{iz}}{\left[2 + \frac{z+\frac{1}{z}}{2}\right]^2} = \frac{1}{i}\oint_{|z|=1} \frac{4\,dz}{z\left(4+\left(z+\frac{1}{z}\right)\right)^2}$$

$$-4i \oint_{|z|=1} \frac{dz}{z\left(\frac{4z+z^2+1}{\cancel{z}}\right)^{\cancel{2}}} = -4\oint \frac{z\,dz}{(z^2+4z+1)}$$

$$z = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm \sqrt{12}}{2}.$$

$$\frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3};$$

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$$\left.\begin{array}{l} z = -2+\sqrt{3} \\ z = -2-\sqrt{3} \end{array}\right\} \underline{\underline{\text{surds}}} \quad \begin{array}{l} (z+2-\sqrt{3}) = 0 \\ (z+2+\sqrt{3}) = 0. \end{array}$$

$\hookrightarrow$ outside the unit circle.

$$\text{Res}\left[f(z),\ z = -2+\sqrt{3}\right] = \cancel{\frac{-z+\sqrt{3}}{(-2+\sqrt{3}+2+\sqrt{3})}}$$

$$\cancel{\frac{-z}{(z+2+\sqrt{3})^3}}\Bigg|$$

$$\frac{d}{dz}\frac{z}{(z+2+\sqrt{3})^2}\Bigg|_{z=-2+\sqrt{3}}$$

$$\frac{(z+2+\sqrt{3})^2 - 2z(z+2+\sqrt{3})}{(z+2+\sqrt{3})^4}\Bigg|_{z=-2+\sqrt{3}}$$

$$-4i\left(\frac{\sqrt{3}}{18}\right) = \frac{2i\sqrt{3}}{9} = \frac{-2i}{3\sqrt{3}}.$$

$$2\pi i\left(\frac{-2i}{3\sqrt{3}}\right) = \left.\frac{4\pi}{3\sqrt{3}}\right\} \underline{\underline{\text{Ans}}}.$$

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$$\int_{-\infty}^{+\infty} f(x)\,dx. \qquad I_1 = \lim_{R\to\infty} \int_0^R f(x)\,dx.$$

$$I_2 = \lim_{R\to\infty} \int_{-R}^0 f(x)\,dx.$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = I_1 + I_2$$

$$= \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx.$$

Cauchy Principal value:

$$\int_{-R}^{0} f(x)\,dx = \int_0^R f(x).$$

$$\int_{-R}^{+R} f(x)\,dx = 2\int_0^R f(x)\,dx.$$

a = Principal value.

$$\int_0^{\infty} f(x)\,dx = \frac{a}{2}\int_{-\infty}^{+\infty} f(x)\,dx.$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = a\int_{-\infty}^{+\infty} f(x)\,dx.$$

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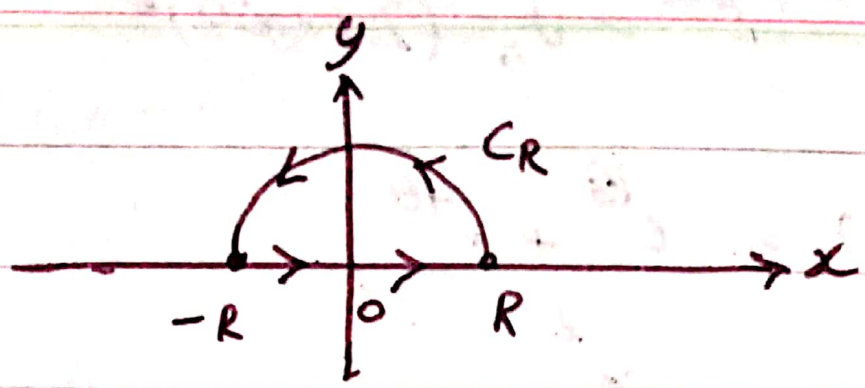
$$\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\{\cdots\}.$$



Example:

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)(x^2+9)};$$

$$\int_C \frac{dz}{(z^2+1)(z^2+9)} = \int_{C_R} \frac{dz}{(z^2+1)(z^2+9)} +$$

$$\int_{-R}^{R} \frac{dx}{(x^2+1)(x^2+9)} = 2\pi i \sum_{k=1}^{n} \mathrm{Res} f(z) -$$

$$(z+i)(z-i)(z+3i)(z-3i).$$

$$\mathrm{Res}(f(z), i) = \frac{1}{(2i)(3i+2i)(2i-3i)} = \frac{1}{16i}$$

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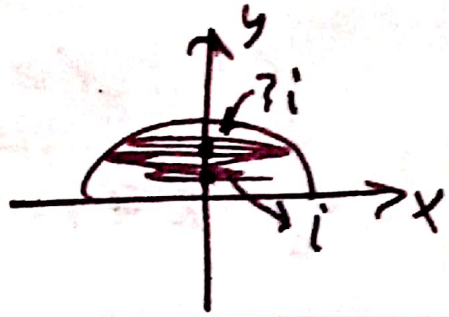
$$\operatorname{Res}(f(z), 3i) = -\frac{1}{48i};$$



$$= 2\pi i\left[\frac{1}{16i} - \frac{1}{48i}\right] = \frac{\pi}{12};$$

$$\int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}[f(z), z_k] - \int_{C_R} f(z)\,dz.$$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2+1)(x^2+9)} = \frac{\pi}{12} - \lim_{R\to\infty} \int_{C_R} f(z)\,dz.$$

$$\left.\begin{array}{l} z = Re^{i\theta}; \quad 0 \le \theta \le 2\pi \\ \text{As} \quad R \to \infty. \\ \displaystyle\int_{C_R} f(z)\,dz = 0. \end{array}\right\} \text{Theorem.}$$

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)(x^2+9)} = \frac{\pi}{12}.$$

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Example: $\int_{-\infty}^{+\infty} \frac{dx}{x^4+1}$;

$(x+1)(x-1)(x+i)(x-i)$.

$x=1,\ x=i\}$.

dz.

$Res[f(z), 1] = \frac{1}{4}$; $Res[f(z), i] = \frac{i}{4}$;

dz. $2\pi i\left(\frac{1}{4}+\frac{i}{4}\right) = \frac{\pi}{2}(i+i^2)$

$= \left|\frac{\pi}{2}(i-1)\right| = \frac{\pi\sqrt{2}}{2} = \frac{\pi}{\sqrt{2}}$;

Principal value; $\frac{\pi}{\sqrt{2}}$
as $R \to \infty$

$\int_{-\infty}^{+\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$;

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$$\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx.$$

$$\underbrace{\int_{-\infty}^{\infty} e^{\alpha i x} f(x)\,dx}_{I} = \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx + i\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$$

$$I_1 + i\,I_2.$$

$$\left.\begin{aligned} I_1 &= Re\{I\} \\ I_2 &= Im\{I\} \end{aligned}\right]$$

Example:

$$\int_0^{\infty} \frac{x\sin x}{x^2+9}\,dx.$$

$$2\int_0^{\infty} \frac{x\sin x}{x^2+9}\,dx = I_2.$$

$$\int_{-\infty}^{\infty} \frac{x\cos x}{x^2+9}\,dx + i\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+9}\,dx.$$

$$\int_{-\infty}^{+\infty} \frac{x e^{ix}}{x^2+9}\,dx.$$

$$= \lim_{R\to\infty} \int_{-R}^{+R} \frac{x e^{ix}\,dx}{x^2+9} = 2\pi i\,Res\left[f(z), z_k\right];$$

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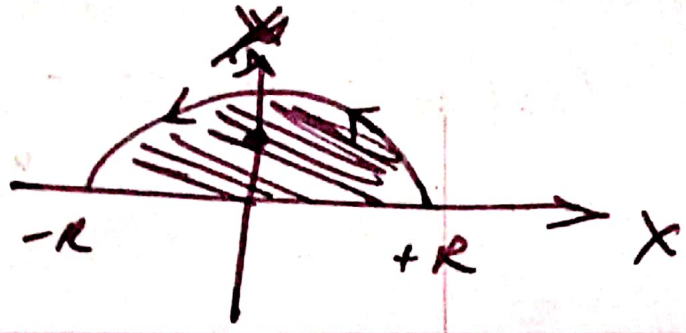
$$\oint_C \frac{ze^{iz}}{z^2+9}\,dz$$



$\cos x\,dx$

$\{3i, -3i\}$

$$\mathrm{Res}(f(z), 3i) = \frac{3ie^{3i(i)}}{(3i)^2+9i} = \frac{3ie^{-3}}{6i}$$

$$\frac{e^{-3}}{2};$$

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+9}\,dx = \frac{2\pi i}{2}e^{-3}.$$

$$= \frac{\pi}{e^3}i$$

$$\mathrm{Re}(I) = I_1 = 0$$

$$\mathrm{Im}(I) = I_2 = \frac{\pi}{e^3}.$$

$$\int_0^{\infty} \frac{x\sin x}{x^2+9}\,dx = \frac{\pi}{2e^3};$$

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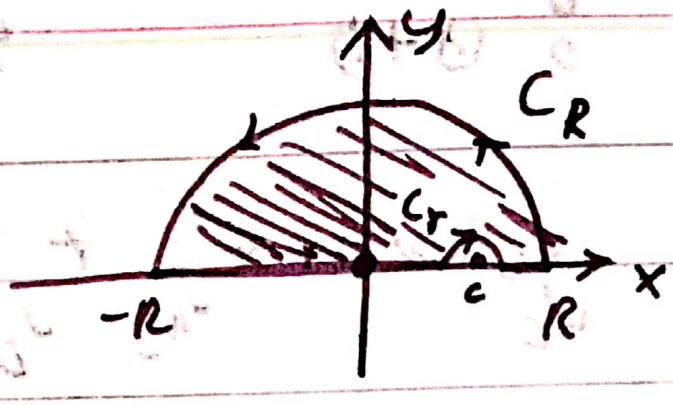
Indented Contours.

$$\lim_{r \to 0} \int_{C_r} f(z)\,dz = \pi i \operatorname{Res}(f(z), c)$$



$z = c + re^{i\theta}$

$0 \le \theta \le \pi$

Example :

$$\int_{-\infty}^{+\infty} \frac{\sin x \, dx}{x(x^2 - 2x + 2)}$$

$$\oint \frac{\sin z \, dz}{z(z^2 - 2z + 2)}$$

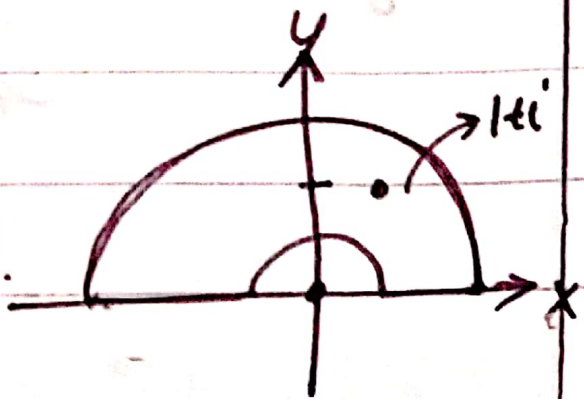
$$z^2 - 2z + 2 = 0.$$

$$z = \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$z = \frac{2 \pm 2i}{2} = 1 \pm i$$

$z = 0, 1 \pm i;$

$(z - 1 - i)(z - 1 + i).$



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$z=0; \quad z=1+i \Rightarrow \quad c=1.$

$$\oint_C = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_{r}^{R} = 2\pi i \, Res(f(z), c_1, c_2)$$

$$\int_{-C_r} = -\int_{C_r}^{(1+i)}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{e^{ix}\,dx}{x(x^2-2x+2)} = 2\pi i \sum_{k=1}^{n} (f(x), z_k) - \pi i \, Res(f(z), c).$$

$c=0.$

$$\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x(x^2-2x+2)} = 2\pi i \left[Res(f(z), z=1+i)\right] + \pi i \left[Res(f(z), 0)\right].$$

$$Res(f(z), 0) = \frac{e^0}{0-0+2} = \frac{1}{2};$$

$$Res(f(z), 1+i) = \frac{e^{(1+i)i}}{(1+i)(1+i-1-i)} = \frac{e^i e^{i^2}}{(1+i)2i}$$

$$\frac{e^{-1}e^{i}}{2(i+i^2)} = \frac{e^{-1+i}}{2(-1+i)};$$

$$\cancel{2}\pi i \frac{e^{-1+i}}{\cancel{2}(-1+i)} + \frac{\pi i}{2}; = \frac{\pi i}{2}\left[\frac{e^{-1}e^{i}}{(-1+i)} + 1\right]$$

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$$\frac{2\pi i}{3}\,\frac{e^{-1}(\cos 1+i\sin 1)}{(-1+i)}+\frac{\pi i}{2},$$

$$\frac{\pi}{2}e^{-1}i(-1-i)(\cos 1+i\sin 1)+\frac{\pi}{2}i$$

$$+\frac{\pi i}{2}\left[e^{-1}(-\cos 1-i\sin 1-i\cos 1+\sin 1)+1\right].$$

$$\frac{\pi}{2}\left[e^{-1}(-i\cos 1+\sin 1+\cos 1+i\sin 1)+i\right.$$

$$Im(I)=\int_{-\infty}^{+\infty}\frac{\sin x\,dx}{x(x^2-2x+2)}=\frac{\pi}{2}\left[e^{-1}(\sin 1-\cos 1)+1\right].$$

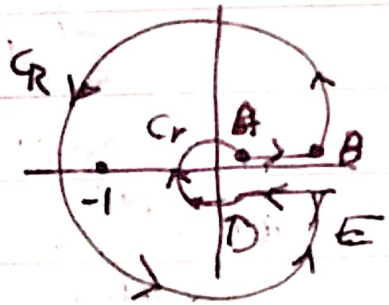
$$=\frac{\pi}{2}\left[1+e^{-1}(\sin(1)-\cos(1))\right]$$ Ans.

Integration using Branch Cut.

$$\int_{-\infty}^{\infty}f(x)\,dx$$

eg.: $\displaystyle\int_0^{\infty}\frac{x^{\alpha-1}\,dx}{(x+1)}$.

$x=0$.



$0<\alpha\le 1$

$$\int_0^{\infty}\frac{x^{1/2-1}\,dx}{(x+1)}=\int_0^{\infty}\frac{dx}{\sqrt{x}(x+1)}$$

$$\oint_C f(z)\,dz=\int_{AB}+\int_{DE}+\int_{C_r}+\int_{C_R}=Res.$$

$$2\pi i\,Res(f(z),-1))=2\int^{\infty}$$

$$\int_{AB}=\int_{DE}=\int_r^R\frac{x^{-1/2}}{x+1}\,dx$$

$$\lim_{\substack{r\to 0\\ R\to\infty}}2\int_r^R\frac{x^{-1/2}\,dx}{x+1}=2\pi i(Res\,f(z),-1)$$

$$\int_0^{\infty}\frac{dx}{\sqrt{x}(x+1)}=\pi i(-i)=\pi.$$

Ans:

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$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$$

substituting

$$\sin\theta = \frac{1}{2i}(z-z^{-1})$$

$$d\theta = \frac{dz}{iz}$$

$$\oint_c \frac{dz/iz}{\left[2+\frac{1}{2i}(z-z^{-1})\right]}$$

$$\oint_c \frac{2i\,dz/iz}{\left[4i+z-z^{-1}\right]}$$

$$\oint_c \frac{2\,dz}{(z^2+4i-1)}$$

~~Completing square~~:

$$z^2+4i-1=0$$

$$z_1 = -i(2+\sqrt{3})$$

$$z_2 = i(-2+\sqrt{3}).$$

We see that z_2 is inside the unit circle:

$$\oint_c \frac{2\,dz}{(z^2+4i-1)} = 2\pi i\left[\frac{2}{i(-2+\sqrt{3})+i(2+\sqrt{3})}\right]$$

$$\oint_c \frac{2\,dz}{(z^2+4i-1)} = \frac{2\pi}{\sqrt{3}} \Rightarrow \int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \frac{2\pi}{\sqrt{3}}.$$

Ans.

$$z_{1,2} = \frac{-4i \pm \sqrt{16i^2+4}}{2}$$

$$z_{1,2} = -2i \pm i\sqrt{3}$$

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Laurent's Series. $r < |z - z_0| < R$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0).$$

$\mathbb{C}$: curve inside R outs
z_0.

$$a_k = \frac{1}{2\pi i} \oint_{\mathbb{C}} \frac{f(s)}{(s - z_0)^{k+1}} \, ds$$

$$f(z) = \frac{z + 3}{(z-8)^2 (z-2)(z+1)^3}.$$

$$\frac{z+3}{(z-8)^2(z-2)(z+1)^3} = \frac{A}{(z-8)} + \frac{B}{(z-8)^2}$$

$$+ \frac{C}{(z-2)} + \frac{D}{(z+1)^2}$$

$$+ \frac{E}{(z+1)^2} + \frac{F}{(z+1)^3}.$$

$\cancel{z+3} =$

$$A = \frac{\partial}{\partial z}\left(\frac{z+3}{''}\right)\Bigg|_{z=8}.$$

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$$B = \left.\frac{z+3}{(z-2)(z+1)^3}\right|_{z=8} = \frac{8+3}{(8-2)(9)^3}$$

$$C = \left.\frac{z+3}{(z-8)^2(z+1)^3}\right|_{z=2} = \frac{2+3}{(2-8)^2(2+1)^3}$$

$$D = \left.\frac{\partial^2}{\partial z^2}\left(\frac{z+3}{(z-8)^2\cdots}\right)\right|_{z=1}$$

$$E = \left.\frac{\partial}{\partial z}\left(\frac{z+3}{(z-8)^2\cdots}\right)\right|_{z=-1}$$

$$F = \left.\frac{z+3}{(z-8)^2(z-2)}\right|_{z=-1} = \frac{-1+3}{(-1-8)^2(-1-2)}$$

$$B = \frac{11}{(6)(9)^3}; \quad C = \frac{5}{(36)(3)^3}$$

$$F = \frac{2}{9^2(-3)}; \quad B = \frac{11}{4374}, \quad C = \frac{5}{972}$$

$$F = 2/243$$

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$$\sum_{n=1}^{\infty} a_n z^n = z + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

example:

$$f(z) = \frac{1}{z(z-1)}$$

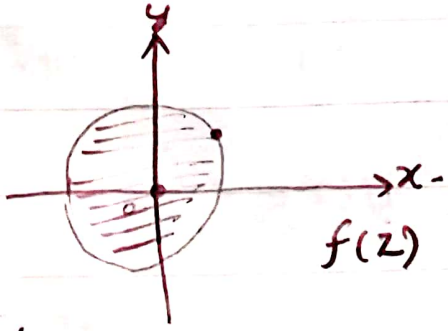
(a) $0 < |z| < 1$

(b) $1 < |z|$

(c) $0 < |z-1| < 1$

(d) $1 < |z-1|$.

$$f(z) = \frac{-1}{(z)} + \frac{1}{(z-1)}$$



$$f(z) = -\frac{1}{z} - \frac{1}{1-z}$$

Geometric Series:

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{k=0}^{\infty} z^k$$

$$1 + 2 + 2^2 + 2^3 + \cdots = \frac{1}{1-2} = -1.$$

$$\sum_{k=0}^{\infty} 2^k \stackrel{?}{=} -1.$$

$$f(z) = -\frac{1}{z} - \underbrace{\left(1 + z + z^2 + z^3 + \cdots\right)}_{\text{analytic part}}$$

$$f(z) = \underbrace{-\frac{1}{z}}_{\text{principal part}} - \overbrace{1 - z - z^2 - z^3 - \cdots}$$

$$f(z) = -\sum_{k=-1}^{\infty} z^k$$

$$f(z) = \frac{e^{z+1-1}}{z-1}$$

$$f(z) = \frac{e\, e^{z-1}}{(z-1)}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$f(z) = \frac{e}{(z-1)}\left[1 + \frac{(z-1)}{1} + \frac{(z-1)^2}{2} + \frac{(z-1)^3}{6} + \cdots\right]$$

$$f(z) = \frac{e}{(z-1)} + e + \frac{e(z-1)}{2} + \frac{e}{6}(z-1)^2 + \cdots$$

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(b) $1 < |z| \Rightarrow \left|\frac{1}{z}\right| < 1.$

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z^2} \cdot \frac{1}{\left(1-\frac{1}{z}\right)}.$$

$$\frac{1}{z^2}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k \Rightarrow f(z) = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)$$

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

(c) $0 < |z-1| < 1.$

$$f(z) = \frac{1}{z(z-1)}.$$

$$f(z) = \frac{1}{z} \cdot \frac{1}{(z-1)}.$$

$$f(z) = \frac{1}{z+1-1} \cdot \frac{1}{(z-1)}.$$

$$f(z) = \frac{1}{(z-1)} \cdot \frac{1}{1+(z-1)}$$ → expanding

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$$f(z) = \frac{1}{(z-1)} \cdot [$$

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = 1 + (-z) + (-z)^2 + (-z)^3 + \cdots$$

$$1 - z + z^2 - z^3 + z^4.$$

$$) = \frac{1}{(z-1)} \cdot \left[1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots\right]$$

$$f(z) = \frac{1}{(z-1)} - 1 + (z-1) - (z-1)^2 + \cdots$$

(d) $1 < |z-1| \Rightarrow \left|\frac{1}{z-1}\right| < 1.$

$$f(z) = \frac{1}{z(z-1)}$$

$$f(z) = \frac{1}{(z-1)} \cdot \frac{1}{1+(z-1)}$$

$$f(z) = \frac{1}{(z-1)^2} \cdot \frac{1}{1+\frac{1}{(z-1)}}.$$

$$f(z) = \frac{1}{(z-1)^2}\left[1 - \frac{1}{(z-1)} + \frac{1}{(z-1)^2} - \cdots\right].$$

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$$f(z) = \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} - \cdots$$

Example: 3

$$f(z) = \frac{1}{(z-1)^2(z-3)}$$

(a) $0<|z-1|<2$

(b) $0<|z-3|<2$

a) $0<|z-1|<2 \Rightarrow 0<\frac{|z-1|}{2}<1$

$$f(z) = \frac{1}{(z-3)} \cdot \frac{1}{2\frac{(z-1)^2}{2}}$$

$$f(z) = \frac{1}{2(z-3)} \cdot \frac{1}{-1+1+\frac{(z-1)^2}{2}}$$

$$f(z) = \frac{1}{2(z-3)} \cdot \left[\frac{1}{1-\left(1-\frac{(z-1)^2}{2}\right)}\right]$$

$$\frac{1}{2(z-3)}\left[1+\left(1-\frac{(z-1)^2}{2}\right)+\left(1-\frac{(z-1)^2}{2}\right)^2+\cdots\right]$$

$$\frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots = \frac{1}{1-\left(\frac{2-1}{2}\right)} = \frac{2}{2-1} = 2$$

$$f(z) = \frac{1}{(z-1)^2(z-3)} \; ; \; 0<\frac{|z-1|}{2}<1$$

$$f(z) = \frac{1}{(z-1)^2} \cdot \frac{1}{-2+(z-1)}$$

$$f(z) = \frac{1}{(z-1)^2} \cdot \frac{1}{-2} \cdot \frac{1}{\left(1-\frac{(z-1)}{2}\right)}$$

$$f(z) = \frac{1}{-2(z-1)^2}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}+\cdots\right]$$

$$f(z) = \frac{1}{-2(z-1)^2}+\frac{1}{-2^2(z-1)^{n-1}}+\frac{1}{-2^3}+\frac{(z-1)}{-2^4}\cdots$$

$$f(z) = -\sum_{k=-2}^{\infty} 2^{-k}(z-1)^{-k}$$

$$f(z) = -\frac{1}{4}\sum_{k=-2}^{\infty}\frac{1-p}{2^{k}(z-1)^{k}}$$

$$2^m(z-1)^n$$

$$m+1 = n-1$$

$$2+m = n$$

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(b) $f(z) = \dfrac{1}{(z-3)(z-1)^2}$

$$0 < |z-3| < 2$$

Solution:

$$0 < \frac{|z-3|}{2} < 1.$$

$$f(z) = \frac{1}{(z-3)} \cdot \frac{1}{(z-1-2+2)^2}$$

$$f(z) = \frac{1}{(z-3)} \cdot \frac{1}{2^2\left[1+\frac{(z-3)}{2}\right]^2}$$

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$$f(z) = \frac{1}{2^2(z-3)} \cdot \left[1 + \frac{(z-3)}{2}\right]^{-2}$$

Binomial expansion:

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$f(z) = \frac{1}{2^2(z-3)} \cdot \left[1 - 2\left(\frac{z-3}{2}\right) + \frac{(-2)(-2-1)}{2!}\left(\frac{z-3}{2}\right)^2 + \frac{(-2)(-2-1)(-2-2)}{3!}\left(\frac{z-3}{2}\right)^3 + \cdots\right]$$

$$f(z) = \frac{1}{2^2(z-3)}\left[1 - (z-3) + \frac{3}{4}(z-3)^2 - \frac{1}{2}(z-3)^3 + \frac{5}{16}(z-3)^4 - \cdots\right]$$

$$f(z) = \left[\frac{1}{4(z-3)} - \frac{1}{4} + \frac{3}{16}(z-3) - \frac{1}{8}(z-3)^2 + \frac{5}{64}(z-3)^3 - \cdots\right]$$

$$0 < |z-3| < 2$$

Example: 4 $f(z) = \dfrac{8z+1}{z(1-z)}$ $\quad 0 < |z| < 1.$

Solution

$$f(z) = \frac{1}{z}\left(\frac{8z+1}{1-z}\right)$$

$$f(z) = \frac{1}{z}\left[\frac{1}{1-z} + \frac{8z}{1-z}\right].$$

$$f(z) = \frac{1}{z}\left[\frac{1}{1-z} + 8\left(\frac{z+1-1}{1-z}\right)\right].$$

$$f(z) = \frac{1}{z}\left[\frac{1}{1-z} + 8\left(\frac{1}{1-z}\right) + 8\left(\frac{\overset{-1}{\cancel{z-1}}}{\cancel{1-z}}\right)\right].$$

$$= \frac{1}{z}\left[\frac{9}{1-z} + 8(-1)\right]$$

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$$f(z) = \frac{1}{z}\left[-8 + \frac{9}{1-z}\right].$$

$$= \frac{1}{z}\left[-8 + 9\left(1 + z + z^2 + z^3 + z^4 + \cdots\right)\right].$$

$$f(z) = \frac{1}{z}\left[1 + 9z + 9z^2 + 9z^3 + \cdots\right].$$

$$f(z) = \frac{1}{z} + 9 + 9z + 9z^2 + \cdots\cdots$$

Example: 5 $\quad f(z) = \dfrac{1}{z(z-1)}$

$$1 < |z-2| < 2$$

$$f(z) = \left[-\frac{1}{z} + \frac{1}{z-1}\right]$$

$$f_1(z) = -\frac{1}{z}\ ; \quad f_2(z) = \frac{1}{z-1}.$$

$$1 < |z-2| \ ; \qquad |z-2| < 2.$$

$$\left|\frac{1}{z-2}\right| < 1 \ ; \qquad \frac{|z-2|}{2} < 1.$$

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$$f_1(z) = \frac{-1}{z+2-2} = \frac{1}{2+(z+2)}.$$

$$f_1(z) = \frac{1}{2\left[1-\left(\frac{z+2}{2}\right)\right]}.$$

$$f_1(z) = \frac{1}{2}\left[1+\frac{(z+2)}{2}+\frac{(z+2)^2}{2^2}+\cdots\right].$$

$$f_2(z) = \frac{1}{z-1} = \frac{1}{z+2}$$

$$f_1(z) = \frac{-1}{(z-2)+2} = \frac{-1}{2+(z-2)}.$$

$$f_1(z) = \frac{-1}{2}\left[\frac{1}{1+\left(\frac{z-2}{2}\right)}\right]$$

$$f_1(z) = -\frac{1}{2}\left[1-\frac{(z-2)}{2}+\frac{(z-2)^2}{2^2}-\frac{(z-2)^3}{2^3}+-\right]$$

$$f_2(z) = \frac{1}{z-1} = \frac{1}{(z-2)+1} = \frac{1}{(z-2)}\cdot\frac{1}{\left(1+\frac{1}{(z-2)}\right)}$$

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$$f_2(z) = \frac{1}{(z-2)}\left[1 - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \cdots\right].$$

$$f(z) = f_1(z) + f_2(z).$$

$$\left(-\frac{1}{2} + \frac{(z-2)}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots\right)$$

$$+\left(\frac{1}{(z-2)} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} \cdots\right)$$

Excercise: 6.3

1) $f(z) = \dfrac{\cos z}{z}$

Solution:

$$\cancel{\cos z = 1 - \frac{z^2}{2!} + \frac{z^2}{2!} - z^3}$$

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

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$$\frac{\cos z}{z} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k-1}}{(2k)!}$$

$$f(z) = \left(\frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \cdots\cdots \right)$$

$$f(z) = \frac{1}{z(z-3)}$$

7) $0 < |z| < 3 \Rightarrow 0 < \frac{|z|}{3} < 1.$

$$f(z) = -\frac{1}{z} \cdot \frac{1}{(3-z)} = \frac{-1}{3z} \cdot \frac{1}{(1 - z/3)}.$$

$$f(z) = \frac{-1}{3z} \cdot \left(1 - \frac{z}{3}\right)^{-1}$$

$$= \left(\frac{-1}{3z}\right)\left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots\right).$$

$$f(z) = \left(\frac{-1}{3z} - \frac{1}{3^2} - \frac{z}{3^3} - \frac{z^2}{3^4} - \cdots\right)$$

9) $|z| > 3$

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$$f(z) = \frac{1}{z(z-3)} \qquad |z| \cancel{>} 3$$

$$\left|\frac{1}{z}\right| \cancel{<} 3$$

$$f(z) = \frac{1}{z^2(1-3/z)} \qquad \frac{3}{|z|} < 1.$$

$$f(z) = \frac{1}{z^2}\left(1-\frac{3}{z}\right)^{-1} = \frac{1}{z^2}\left(1+\frac{3}{z}+\frac{3^2}{z^2}+\frac{3^3}{z^3}+\cdots\right)$$

$$f(z) = \left(\frac{1}{z^2}+\frac{3}{z^3}+\frac{3^2}{z^4}+\frac{3^3}{z^5}+\cdots\right).$$

---

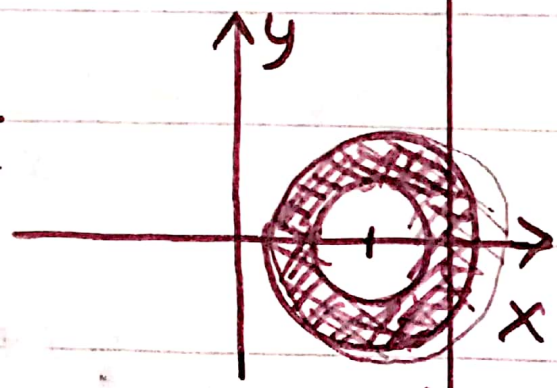
ii) $1 < |z-4| < 4$; $\qquad f(z) = \dfrac{1}{z(z-3)}$.

Annular Region

$$f(z) = \left[-\frac{1}{z}+\frac{1}{z-3}\right]\cdot\frac{1}{3}$$

$$f(z) = \frac{1}{3}\left[\frac{-1}{z}+\frac{1}{z-3}\right]$$

$$f_1(z) + f_2(z).$$



Annulus

$$|z-4| < 4. \Rightarrow \frac{|z-4|}{4} < 1.$$

$$f_1(z) = -\frac{1}{z} = \frac{-1}{z+4-4}$$

$$f_1(z) = \frac{-1}{4+(z-4)} = \frac{-1}{4}\left(1+\frac{(z-4)}{4}\right)^{-1}$$

$$f_1(z) = \frac{-1}{4}\left[1-\frac{(z-4)}{4}+\frac{(z-4)^2}{4^2}\cdots\right]$$

$$f_2(z) = \frac{1}{(z-3)} = \frac{1}{(z-4)+1}$$

$$|z-4|>1 \Rightarrow \frac{1}{|z-4|}<1$$

$$f_2(z) = \frac{1}{(z-4)}\cdot\frac{1}{1+\left(\frac{1}{z-4}\right)}$$

$$f_2(z) = \frac{1}{(z-4)}\left(1-\frac{1}{(z-4)}+\frac{1}{(z-4)^2}-\cdots\right)$$

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$$f(z) = \frac{1}{3}\left[f_1(z) + f_2(z)\right].$$

$$f(z) = \frac{1}{3}\left[-\frac{1}{4}\left(1 - \frac{(z-4)}{4} + \frac{(z-4)^2}{4^2} - \cdots\right) + \frac{1}{(z-4)}\left(1 - \frac{1}{(z-4)} + \frac{1}{(z-4)^2} - \cdots\right)\right].$$

$$f(z) = \frac{1}{3}\left[\cdots \frac{1}{(z-4)^3} - \frac{1}{(z-4)^2} + \frac{1}{(z-4)} - \frac{1}{4} + \frac{(z-4)}{4^2} - \frac{(z-4)^2}{4^3} + \cdots\right].$$

$$f(z) = \left(\cdots - \cdots \frac{1}{3(z-4)^3} - \frac{1}{3(z-4)^2} + \frac{1}{3(z-4)} - \frac{1}{12} + \frac{(z-4)}{3\cdot 4^2} - \frac{(z-4)^2}{3\cdot 4^3} + \cdots\right)$$

$$1 < |z-4| < 4$$

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$$f(z) = \frac{z}{(z+1)(z-2)}.$$ 19). $1 < |z| < 2$

$$f(z) = \frac{A}{(z+1)} + \frac{B}{(z-2)}.$$

Heaviside Formula:

$$A = \left.\frac{z}{(z-2)}\right|_{z=-1} = \frac{-1}{-1-2} = \frac{1}{3}.$$

$$B = \left.\frac{z}{z+1}\right|_{z=2} = \frac{2}{3};$$

$$f(z) = \frac{1}{3}\left[\frac{1}{z+1} + \frac{2}{(z-2)}\right].$$

$$f_1(z) = \frac{1}{z+1}$$ $1 < |z| \Rightarrow \frac{1}{|z|} < 1.$

$$f_1(z) = \frac{1}{z\left(1+\frac{1}{z}\right)} = \frac{1}{z}\left(1+\frac{1}{z}\right)$$

$$= \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} - \cdots\right)$$

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$$f_1(z) = \left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \frac{1}{z^5} - \cdots\right).$$

$$|z| < 2. \Rightarrow \frac{|z|}{2} < 1.$$

$$f_2(z) = \frac{2}{z+2} = \frac{2}{2+z}$$

$$f_2(z) = \frac{1}{1+\left(\frac{z}{2}\right)} = \left(1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \cdots\right)$$

$$\frac{1}{3}\left[f_1(z) + f_2(z)\right] = f(z).$$

$$f(z) = \frac{1}{3}\left[+\frac{1}{z^5} - \frac{1}{z^4} + \frac{1}{z^3} - \frac{1}{z^2} + \frac{1}{z} + 1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \cdots\right].$$

$$f(z) = \left[\cdots + \frac{1}{3z^5} - \frac{1}{3z^4} + \frac{1}{3z^3} - \frac{1}{3z^2} + \frac{1}{3z} + \frac{1}{3} - \frac{z}{3\cdot 2} + \frac{z^2}{3\cdot 2^2} - \frac{z^3}{3\cdot 2^3} + \frac{z^4}{3\cdot 2^4} - \cdots\right]$$

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MUHAMMAD SAUD ZAHIR 294897

BEE 11B

Assignment: 03 CVT

Question: 1 State the location of all the poles for $f(z) = \frac{1}{z^{1/2}\cos^2 z}$, give order of each pole. Use branch cut of the given function if there is any imbiguity.

Solution:

For singularities

$$z^{1/2}\cos^2 z = 0$$

$z^{1/2} = 0$; $\cos^2 z = 0$

$z = 0$; $\cos z = 0$.

$z = 0$; $z = (2n+1)\frac{\pi}{2}$; $n = 0, 1, 2 \ldots$

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There are no poles at $z = 0$, $n = -1, -2, -3, \ldots$ as they lie on principal branch cut of $z^{1/2}$

So, poles are at:

$$z_n = (2n+1)\frac{\pi}{2}; \quad n = 0, 1, 2, 3, \ldots$$

Now, for $n = 0$; $z_0 = \pi/2$

Let $w = z - \pi/2$,

$w + \pi/2 = z$.

$$= \frac{1}{\sqrt{w+\pi/2}} \cdot \frac{1}{\cos^2(w+\pi/2)};$$

$$\Rightarrow \frac{1}{\cos^2(w+\pi/2)} = \frac{1}{\sin^2 w};$$

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$$\sin^2 w = \frac{1-\cos 2w}{2}$$

Now, using taylor series:

$$\cos 2w = 1 - \frac{2^2}{2!}w^2 + \frac{2^4}{4!}w^4 - \frac{2^6}{6!}w^6 + \cdots$$

$$\frac{1-\cos(2w)}{2} = \frac{2}{2!}w^2 - \frac{2^3}{4!}w^4 + \frac{2^5}{6!}w^6 - \cdots$$

Similarly:

$$\left(w + \frac{\pi}{2}\right)^{-1/2} = \left(\frac{\pi}{2}\right)^{-1/2}\left(1 + \frac{2w}{\pi}\right)^{-1/2}$$

$$= \sqrt{\frac{2}{\pi}}\left(1 - \frac{w}{\pi} - \frac{3}{2\pi^2}w^2 - \frac{5}{2\pi^3}w^3 \cdots\right)$$

$$f(z) = \frac{1}{\sqrt{z}\cos^2 z} = \sqrt{\frac{2}{\pi}}\left(\frac{1 - \frac{w}{\pi} - \frac{3}{2\pi^2}w^2}{\frac{2}{2!}w^2 - \frac{2^3}{4!} + \cdots}\right)$$

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$$f(z) = \sqrt{\frac{2}{\pi}}\ \frac{\left(1 - \frac{z}{\pi} - \frac{3}{2\pi^2}z^2 - \frac{5}{2\pi^3}z^3 \ldots\right)}{z^2\left(1 - \frac{2^3}{4!}z^2 + \frac{2^5}{6!}z^4 - \ldots\right)}$$

By Long Division:

$$1 - z/\pi + \left(\frac{3}{2\pi^2} + 1/3\right)z^2 \ldots$$

$$1 - \frac{2^3}{4!}z^2 + \frac{2^5}{6!}z^4 - \ldots \;\Big)\; 1 - \frac{z}{\pi} - \frac{3}{2\pi^2}z^2 - \frac{5}{2\pi^3}z^3 - \ldots$$

$$\begin{array}{rl} & 1 \quad\quad - \frac{2^3}{4!}z^2 \ldots \\ + & \\ - & \quad\quad + \end{array}$$

$$-\frac{z}{\pi} + \left(\frac{3}{2\pi^2} + \frac{1}{3}\right)z^2 \ldots$$

$$-\frac{z}{\pi} \quad\quad\quad\quad -\frac{2^3}{4!}\frac{z^3}{\pi} -$$

$$\left(\frac{3}{2\pi^2} + \frac{1}{3}\right)z^2 + \ldots$$

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$$f(z) = \left(1 - \frac{z}{\pi} + \left(\frac{3}{2\pi^2} + \frac{1}{3}\right)z^2 - \cdots\right)\frac{1}{z^2}\sqrt{\frac{2}{\pi}}$$

rearranging:

$$f(z) = \sqrt{\frac{2}{\pi}}\left(\frac{1}{z^2} - \frac{1}{\pi z} + \left(\frac{3}{2\pi^2} + \frac{1}{3}\right) - \cdots\right)$$

So, degree of pole $z_1 = \frac{\pi}{2}$ is 2.

The same can be proven for

$z_n = (2n+1)\pi/2$; $n = 0, 1, 2, 3$

so;

poles are: $z_n = (2n+1)$, $n = 0, 1, 2,$

degree of poles $= 2$.

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Question:2

$$f(z) = \frac{\sin(z)}{\sinh^2(z)}$$

Solution:

(a) Let $\phi(z) = \dfrac{\sin z}{\sinh^2 z}$

$$\phi(z) = a_{-1} z^{-1} + a_0 + a_1 z^1 + a_2 z^2 +$$

We know that:

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

$$\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}$$

$$\phi(z)(\cosh 2z - 1) = 2\sin z$$

Using Half-angle Identity

$$\because \sinh^2 z = \frac{\cosh 2z - 1}{2}$$

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$$\left(\frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots\right)\left(\sum_{k=0}^{\infty} \frac{(2z)^{2k}}{(2k)!} - 1\right)$$

$$= 2 \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

$$\left(\frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots\right)\left(1 + \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} + \frac{(2z)^6}{6!} + \cdots - 1\right) = 2\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

We will only compare coefficients of $z^{-1}, z, z^3, z^5 \ldots$

z^{-1}: $\quad a_{-1}\frac{2^2}{2!} = \frac{2}{1!}$

$$a_{-1}\frac{2^4}{4!} + a_1\frac{2^2}{2!} = -\frac{2}{3!}$$

$$a_{-1}\frac{2^6}{6!} + a_1\frac{2^4}{4!} + a_3\frac{2^2}{2!} = \frac{2}{5!}$$

$$a_{-1}\frac{2^8}{8!} + a_1\frac{2^6}{6!} + a_3\frac{2^4}{4!} + a_5\frac{2^2}{2!} = -\frac{2}{7!}$$

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Solving we get:

$$a_{-1} = 1, \quad a_1 = -\frac{1}{2}, \quad a_3 = \frac{47}{360}$$

$$a_5 = -\frac{373}{15120}.$$

Hence, the Laurent series of $f(z)$ at $z = 0$. is

$$f(z) = \frac{1}{z} - \frac{z}{2} + \frac{47}{360} z^3 - \frac{373}{15120} z^5 \ldots$$

has pole at $z = 0$ and residue at that pole $a_{-1} = 1$.

(b) $$I = \oint_C \frac{\sin z}{\sinh^2 z} \, dz ; |z| = 3$$

Since $f(z)$ has only one pole of order 1 at $z = 0$. which is inside $|z| = 3$

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then using residue theorem:

$$\oint_{\mathbb{C}} f(z)\,dz = 2\pi i \, \mathrm{Res}(f(z), 0).$$

$$\mathrm{Res}(f(z), 0) = \left[\cancel{(z-0)} \cdot \frac{1}{\cancel{(z-0)}}\right] = 1$$

... which is a_{-1}.

$$\oint_{\mathbb{C}} \left(\frac{1}{z} - \frac{z}{2} + \frac{47}{360} z^3 - \cdots\right) dz = 2\pi i \, a_{-1}$$

Hence:

$$\oint_{\mathbb{C}} \frac{\sin z}{\sinh^2 z}\, dz = 2\pi i \; ; \; |z| = 3.$$

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$$\sum_{k=1}^{\infty} \frac{(-1)^{-k}}{(-2k+1)^3} = \frac{+1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3}$$

$$- \frac{1}{7^3} + \frac{1}{9^3} + \cdots\cdots$$

$$\sum_{-\infty}^{0} \frac{(-1)^{-k}}{(-2k+1)^3} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$$

$$= 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3}.$$

$$\because \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}}_{\beta(3).} = \sum_{k=1}^{\infty} \frac{(-1)^{-k}}{(-2k+1)}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)^3} = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$$

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$$\frac{\pi^3}{16} = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3};$$

$$\beta(3) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}$$

68)

Hence prove.

$$\sum_{k=0}^{\infty} \frac{1}{k^2+a^2} = \frac{1}{2a^2} + \frac{\pi}{2a}\coth(a\pi).$$

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+a^2} = -\operatorname{Res}\left[\frac{\pi\cot\pi z}{z^2+a^2}, z=+ai\right.$$

$$\left.\frac{\pi\cot\pi z}{z^2+a^2}, z=-ai\right];$$

$$\operatorname{Res}\left(\frac{\cot\pi z}{z+ai}, ai\right) = \frac{\cot(a\pi i)}{2ai}$$

$$\operatorname{Res}\left(\frac{\cot\pi z}{z-ai}, -ai\right) = \frac{\cot(-a\pi i)}{-2ai}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

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$$i\sin(-ix) = \frac{e^{-i^2x} - e^{i^2x}}{2} = \frac{e^{x} - e^{-x}}{2}$$

$$-i\sin ix = \sinh x. \qquad \tanh x = -i\tan(ix)$$

$$+\cos(ix) = \cosh x \qquad \coth x = i\cot ix.$$

$$-\pi\left[\frac{\cot a\pi i}{2ai} + \frac{\cot a\pi i}{2ai}\right]$$

$$-\pi\left(\frac{\coth a\pi}{ai^2}\right) = \frac{\pi\coth(a\pi)}{a}$$

$$\sum_{k=-\infty}^{\infty}\frac{1}{k^2+a^2} = \sum_{k=1}^{\infty}\frac{1}{k^2+a^2} + \sum_{k=-\infty}^{\infty}\frac{1}{k^2+a^2}$$

$$\frac{\pi\coth(a\pi)}{a} = 2\sum_{k=1}^{\infty}\frac{1}{k^2+a^2} + \frac{1}{a^2}$$

$$\frac{1}{a^2} + \frac{\pi\coth a\pi}{a} = 2\sum_{k=1}^{\infty}\frac{1}{k^2+a^2} + \frac{2}{a^2}$$

$$\frac{1}{a^2} + \frac{\pi\coth a\pi}{a} = 2\left[\sum_{k=0}^{\infty}\frac{1}{k^2+a^2}\right]$$

$$\sum_{k=0}^{\infty}\frac{1}{k^2+a^2} = \frac{1}{2a^2} + \frac{\pi\coth a\pi}{2};$$

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Excercise : 66

69) $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$;

$$\sum_{n=\infty}^{\infty} \frac{1}{(2k-1)^2} = -\sum \left(\frac{\pi \cot \pi z}{(2z-1)^2} \right)_{z=1/2}$$

$k = \frac{1}{2}$

$$-\frac{d}{dz}\left[\cancel{(z-1/2)^2} \; \frac{\pi \cot \pi z}{\cancel{(2z-1)^2}}\right]_{z=1/2}.$$

$$-\frac{\pi}{2^2} \frac{d}{dz} \cot \pi z \Big|_{z=1/2}$$

$$-\frac{\pi^2}{2^2} - \csc^2 \pi z \Big|_{z=1/2} = \frac{\pi^2}{4} \operatorname{cosec}^2 \pi z \Big|_{z=1/2}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{4};$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2k-1)^2} = \sum_{-\infty}^{-1} \frac{1}{(2k-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{(-2k+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2k-1)^2}.$$

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$$\sum_{k=-\infty}^{\infty} \frac{1}{(2k-1)^2} = 2\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

$$\frac{\pi^2}{4} = 2\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2};$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \pi^2/8$$

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(72) $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$;

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{k(k)} = -\sum_{j=1}^{r} \left[Res\left(\frac{\pi \csc \pi z}{k(z)}, z \right) \right]$$

$k = -1/2$;

$$-\frac{\pi}{2!} \frac{d^2}{dz^2} \left[(z+1/2)^3 \frac{\pi \csc(\pi z)}{(2z+1)^3} \right]_{z=-1/2}$$

$$-\frac{\pi}{16} \frac{d^2}{dz^2} \left[\csc(\pi z) \right]_{z=-1/2}$$

$$\frac{d^2}{dx^2} \operatorname{cosec}(ax) = +a^2 \left(\csc^3 ax + \csc^3 ax \cos^2 ax \right)$$

$$+\frac{\pi^3}{16} \cdot \csc\left(-\frac{\pi}{2}\right)^3 + \operatorname{cosec}\left(-\frac{\pi}{2}\right)^3 \cos\left(-\pi/2\right)^2$$

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{16};$$

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$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \sum_{k=-\infty}^{-1} \frac{(-1)^k}{(2k+1)^3} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$$

$$\beta(3) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}; \quad \text{Dirichlet Beta.}$$

$$\frac{\pi^3}{16} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(-2k+1)^3} + \beta(3).$$

$$\cancel{\sum_{k=1}^{\infty} \frac{(-1)^k}{(-2k+1)^3} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(-2k+1)^3} + \sum_{k=-\infty}^{0} \frac{(-1)^k}{(-2k+1)^3}}$$

ex). $$\cancel{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \beta(3).}$$

$$\cancel{\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(-2k+1)^3} = -\mathrm{Res}\left[\frac{\pi \csc \pi z}{(-2z+1)^3}, z=\frac{1}{2}\right]}$$

$$\cancel{-\frac{\pi}{(3-1)!} \frac{d^2}{dz^2}\left[\frac{\csc \pi z}{(-2z+1)^3}\left(z-\frac{1}{2}\right)^3\right]}$$

$$\cancel{\frac{+\pi}{2^4} \frac{d^2}{dz^2}(\csc \pi z)\Big|_{z=1/2}}.$$

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C: circle, line segment.

$$\oint_C f(z)\,dz = \int_a^b f(z(t)).z'(t)\,dt.$$

circle: $z(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$.

line segment: $z(t) = z_0(1-t) + z_1 t$,
$0 \le t \le 1$.

Cauchy-Goursat Theorem.

$\oint_C f(z)\,dz = 0$ If $f(z)$ is analytic inside C.

Corollary: $\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n=1 \\ 0 & n \neq 1 \end{cases}$

Cauchy's Integral Formula.

$$\oint_C \frac{f(z)}{(z-z_0)}\,dz = 2\pi i\, f(z_0).$$

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$$\oint_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i\, f^{(n-1)}(z_0)}{(n-1)!}$$

Power Series:

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

Taylor Series

$z = z_0$.

$z = 0$

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Maclaurin Series

$\downarrow$

Taylor Series $\quad a_k = \dfrac{f^k(z_0)}{k!}$, $\quad a_k = \dfrac{f^k(0)}{k!}$

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \; ; \quad |z|<1.$$

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \; ; \quad \sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$|z| < \infty.$$

Laurent Series.

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_0)^k$$

1) $|z| < a$.

$\dfrac{|z|}{a} < 1.$

2) $a < |z| < b$

$\dfrac{a}{|z|} < 1$; $\dfrac{|z|}{b} < 1.$ $\longleftarrow$ $a < |z|$; $|z| < b$

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$n=1$

$$\mathrm{Res}[f(z),(z_0)] = (z-z_0)f(z)\Big|_{z\to z_0}$$

n^{th} order pole:

$$\mathrm{Res}[f(z),(z_0)] = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z)$$

Cauchy's Residue Theorem.

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}[f(z), z_k].$$

$$\oint_C f(z)\,dz = 2\pi i a_{-1}$$

a_{-1} is coefficient of Laurent series.

$$\oint_C \left(\frac{z e^{\pi z}}{z^4-16} + z e^{\pi/z}\right) dz$$

$$C: 9x^2+y^2 = 9.$$

$z^4 = 16 \Rightarrow z^2 = \pm 4.$

$z^2 = +4$ $\qquad$ $z^2 = -4.$

$z = \pm 2$ $\qquad$ $z = \pm 2i$

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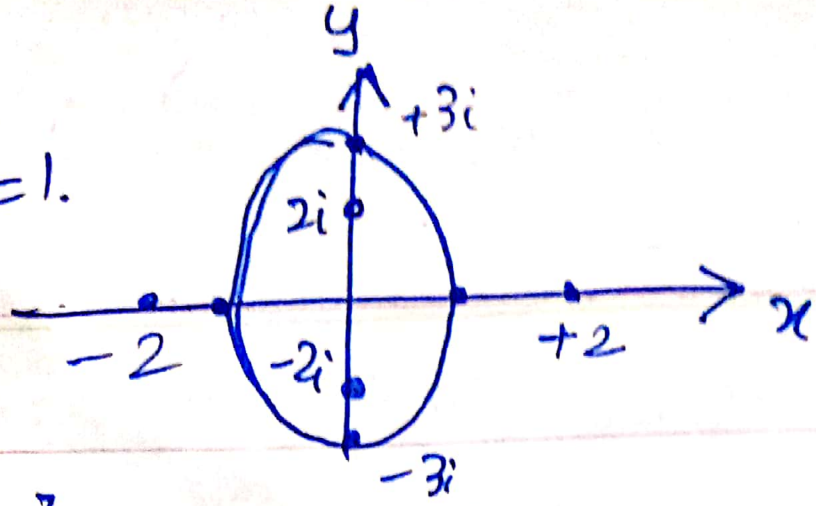
$$\frac{x^2}{1^2} + \frac{y^2}{3^2} = 1.$$



$$\mathrm{Res}[f(z), 2i].$$

$$= \frac{2i e^{2\pi i}}{(2i+2)(2i-2)(2i+2i)} = -\frac{1}{16}$$

$$\mathrm{Res}[f(z), -2i] = \frac{-2i e^{-2\pi i}}{(-2i+2)(-2i-2)(-2i-2i)} = -\frac{1}{16}.$$

$$z e^{\pi/z} = z\left[1 + \frac{\pi}{z} + \frac{\pi^2}{2!\, z^2} + \cdots\right].$$

$$= z + \pi + \frac{\pi^2}{2!}\frac{1}{z} + \cdots \qquad a_{-1} = \frac{\pi^2}{2!}$$

$$\oint_C \left(\frac{z e^{\pi z}}{z^4 - 16} + z e^{\pi/z}\right) = 2\pi i\left[-\frac{1}{8} + \frac{\pi^2}{2!}\right].$$

$$= 30.2208785$$

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$$f(x) = \frac{a_0}{2} + \sum_{K=0}^{\infty}\left(a_K \cos K\omega x + b_K \sin K\omega x\right).$$

$$x_0 \le x \le x_0 + T.$$

$$\omega = 2\pi/T$$

$$f(t) = A_0 + \sum_{K=0}^{\infty} A_K \cos(K\omega t + \phi_K)$$

$$A_K = \sqrt{a_K^2 + b_K^2} \; ; \; \phi_K = \tan^{-1}\left(-\frac{b_K}{a_K}\right)$$

$$f(t) = \sum_{K=-\infty}^{\infty} c_K e^{+jK\omega t}$$

$$c_K = \frac{1}{T}\int_{x_0}^{x_0+T} f(t)\, e^{-jK\omega t}\, dt$$

$$a_K = \frac{2}{T}\int_{x_0}^{x_0+T} f(t)\cos(K\omega t)\, dt$$

$$b_K = \frac{2}{T}\int_{x_0}^{x_0+T} f(t)\sin(K\omega t)\, dt.$$

$$a_0 = \frac{2}{T}\int_{x_0}^{x_0+T} f(t)\, dt.$$

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Even | Odd

$f(+t) = +f(-t)$ | $f(-t) = -f(t)$.

$a_k = \frac{4}{T}\int_0^{T/2} f(t)\cos k\omega t.$ | $a_k = 0$

$b_k = 0$ | $b_k = \frac{4}{T}\int_0^{T/2} f(t)\sin k\omega t.$

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta).$$

$$-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$$

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).$$

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).$$

$$c_0 = a_0/2 \; ; \quad c_k = \frac{a_k - j b_k}{2}$$

$$c_{-k} = c_k^* = \frac{a_k + j b_k}{2}.$$

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$$c_k = |c_k| e^{i\varphi_k}$$

$$|c_k| = \frac{1}{2}\sqrt{a_k^2 + b_k^2}.$$

Discrete Frequency Spectrum.

$$\omega_k = k\omega; \quad \omega = 2\pi/T.$$

Multiplication Theorem

and Parseval's Theorem.

$$\frac{1}{T}\int_{x_0}^{x_0+T} f(t)g(t)\,dt = \sum_{k=-\infty}^{\infty} c_k \bar{d}_k$$

$$= \frac{a_0\alpha_0}{2} + \sum_{k=1}^{\infty} \frac{a_k\alpha_k + b_k\beta_k}{2}$$

$$P = \frac{1}{T}\int_0^T x^2(t)\,dt = \sum_{k=-\infty}^{+\infty} |c_k|^2$$

$$P = \frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{+\infty}(a_k^2 + b_k^2)$$

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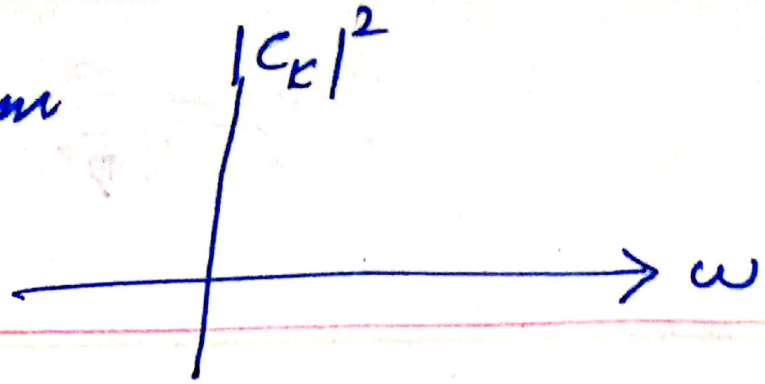
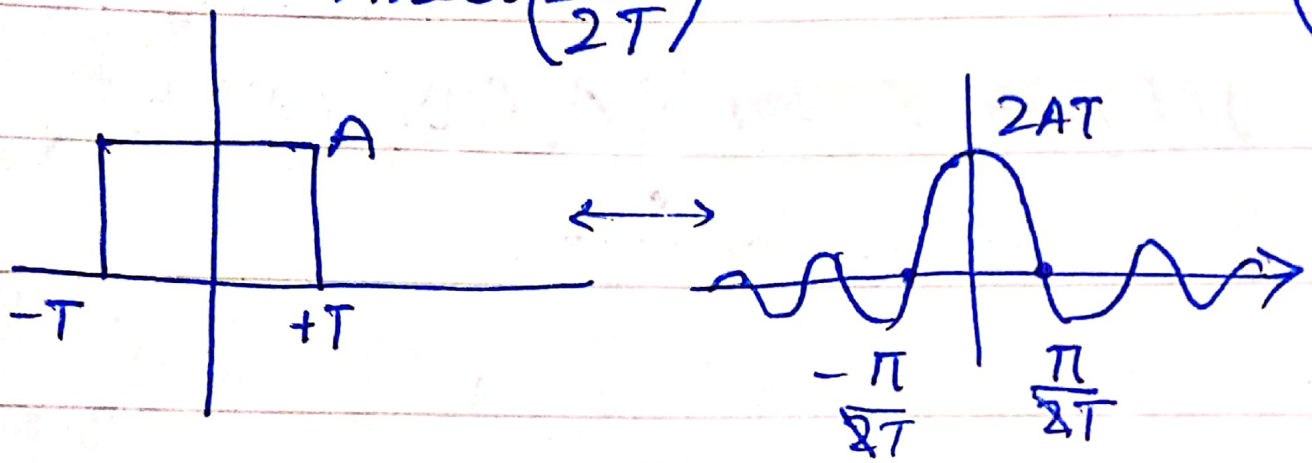
Power Spectrum



Fourier Transform.

$$A\,\mathrm{rect}\left(\frac{t}{2T}\right) \longleftrightarrow 2AT\,\mathrm{sinc}\left(\frac{\omega T}{1}\right)$$



Duality

$$f(t) \longleftrightarrow F(\omega)$$

$$F(t) \longleftrightarrow 2\pi f(-\omega)$$

$$F(j\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt.$$

$$E = \int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega$$

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BEE 11B

Assignment: 4 CVT

Question: 1 $v(t) = 5 - 3\cos\left(5t + \pi/2\right) - 2\sin\left(2t + \frac{3\pi}{2}\right)$

(a)

Solution: The fundamental period of $v(t)$ is 2π.
(as it is the L.C.M of given signal)

$T = 2\pi$; $f = \frac{1}{2\pi}$

$\omega_0 = 1$

The fourier series of given function is given as:

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$$v(t) = 5 - 3\cos(5t + \pi/2) - 2\sin(2t + 3\pi/2)$$

$\because$ $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\therefore$

$$\cos(5t + \pi/2) = -\sin 5t \sin\pi/2 = -\sin 5t$$

$$\sin(2t + 3\pi/2) = \cos 2t \sin 3\pi/2 = -\cos 2t$$

We get,

$$v(t) = 5 + 3\sin 5t + 2\cos 2t$$

$$= 5 + 2\cos 2t + 3\sin 5t$$

$$= 5 + \sum_{k=0}^{\infty} 2\delta(k-2)\cos kt + 3\delta(k-5)\sin kt$$

Hence; trigonometric fourier series is:

$$v(t) = 5 + \sum_{k=0}^{\infty} 2\delta(k-2)\cos kt + 3\delta(k-5)\sin kt$$

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Compact form of $v(t)$ is:

$$v(t) = 5 + \sum_{k=0}^{\infty} A_k \cos(kt + \phi_k)$$

$$A_k = \sqrt{3^2\delta^2(k-5) + 2^2\delta^2(k-2)}$$

$$\phi_k = \tan^{-1}\left(-\frac{3\delta(k-5)}{2\delta(k-2)}\right).$$

Complex form of $v(t)$ is:

$$v(t) = 5 + 2\cos 2t + 3\sin 5t.$$

$$v(t) = 5 + \cancel{2}\left(\frac{e^{+2jt} + e^{-2jt}}{\cancel{2}}\right) +$$

$$3\left(\frac{e^{+5jt} - e^{-5jt}}{2j}\right)$$

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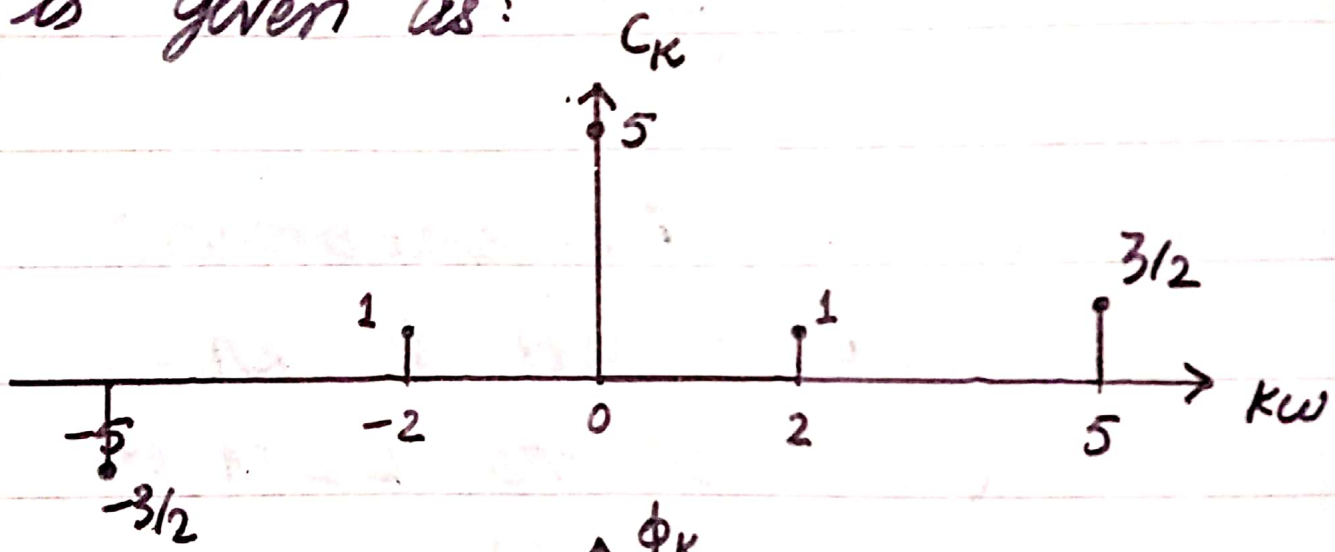
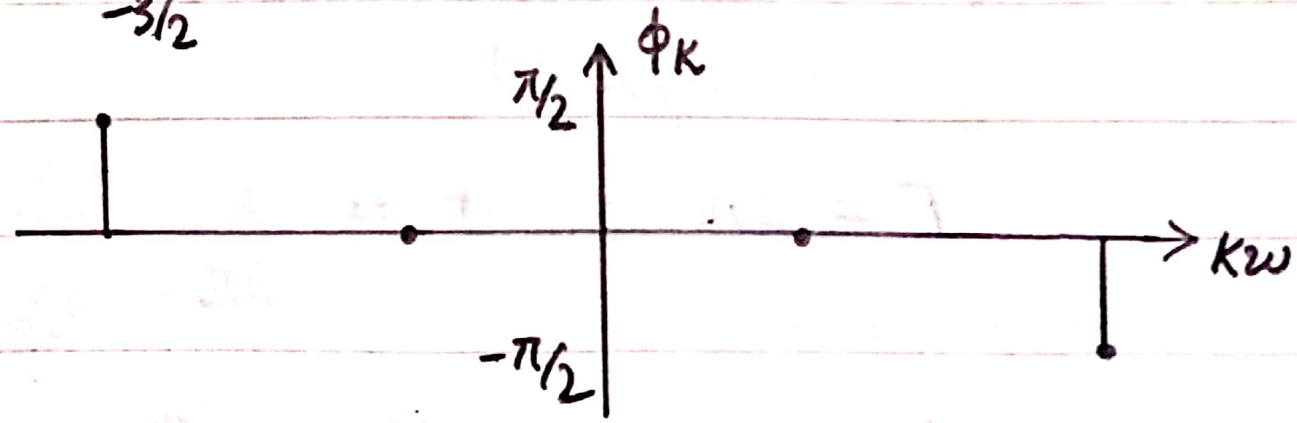
$$v(t) = \frac{3}{2} j e^{-5jt} + e^{-2jt} + 5 + e^{+2jt} - \frac{3}{2} j e^{+5jt}$$

is the complex form.

The 2-sided discrete fourier spectra is given as:





Also, the 1-sided fourier spectra is given as:

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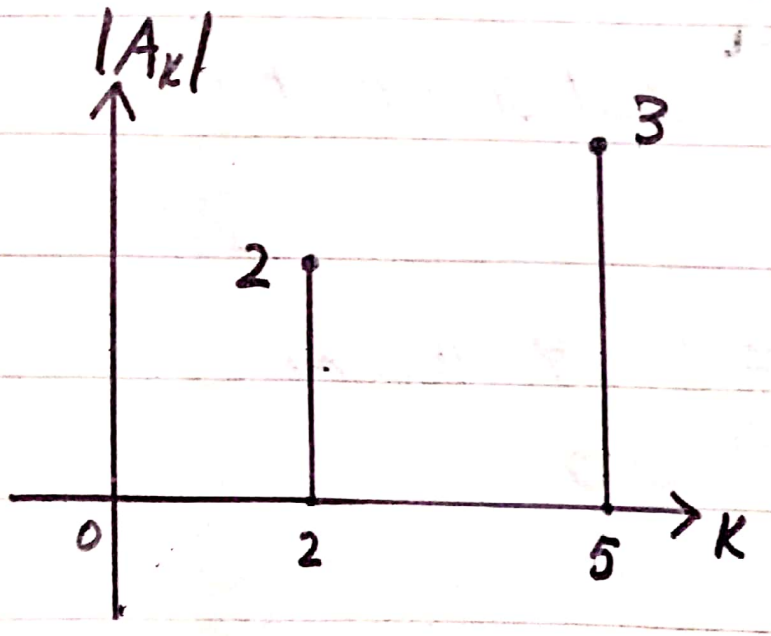
$$A_k = \sqrt{a_k^2 + b_k^2} \quad ; \quad \phi_k = \tan^{-1}\left(-\frac{b_k}{a_k}\right)$$

as: $v(t) = 5 + \sum_{k=0}^{\infty} 2\delta(k-2)\cos kt + 3\delta(k-5)\sin kt$

So;

$$A_k = \sqrt{2^2\delta^2(k-2) + 3^2\delta^2(k-5)}$$

$$\phi_k = \tan^{-1}\left(-\frac{3\delta(k-5)}{2\delta(k-2)}\right)$$



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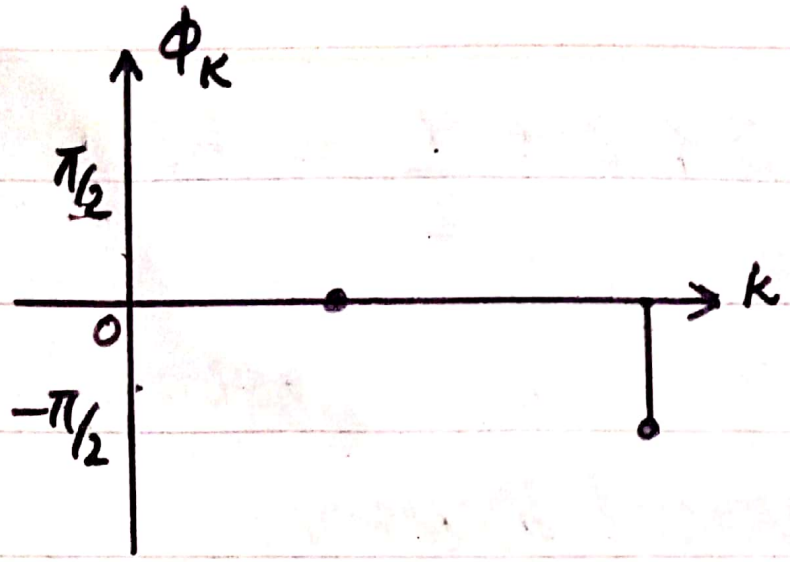

(b)

Solution:

Parseval's Theorem:

$$P = \frac{1}{T}\int_0^T f^2(t)\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Then average power delivered to the unit resistor is calculated as:

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2 = \left(\frac{3}{2}\right)^2 + (1)^2 + (5)^2 + (1)^2 + \left(\frac{3}{2}\right)^2$$

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$$P = 25 + 2 + \frac{9}{2} = \frac{63}{2} \text{ units.}$$

Now, using integral formula of Parseval's Theorem:

$$P = \frac{1}{2\pi} \int_0^{2\pi} (5 + 2\cos 2t + 3\sin 5t)^2 dt$$

$$\because (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.$$

$$P = \frac{1}{2\pi} \left[\int_0^{2\pi} (25 + 4\cos^2 2t + 9\sin^2 5t + 20\cos 2t + 6\cos 2t \sin 5t + 30\sin 5t) dt \right].$$

Using half angle identities

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$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} \; ; \; \sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}.$$

$$P = \frac{1}{2\pi}\left[25(2\pi) + 4\int_0^{2\pi}\left(\frac{1+\cancelto{0}{\cos 4t}}{2}\right)dt\right.$$

$$+ 9\int_0^{2\pi}\left(\frac{1-\cancelto{0}{\cos 10t}}{2}\right)dt +$$

$$20\int_0^{2\pi}\cancelto{0}{\cos 2t\,dt} + 30\int_0^{2\pi}\cancelto{0}{\sin 5t\,dt} +$$

$$\left.6\int_0^{2\pi}\cancelto{0}{\cos 2t \cdot \sin 5t\,dt}\right]$$

$\because \sin n\pi = 0 \quad \forall\, n \in \mathbb{Z}$

$$\cos n\pi \Big|_0^{2\pi} = 0$$

$$P = \frac{1}{\cancel{2\pi}}\left[25(\cancel{2\pi}) + 2(\cancel{2\pi}) + \frac{9}{2}(\cancel{2\pi}) + 0\right]$$

$P = 63/2$ units.

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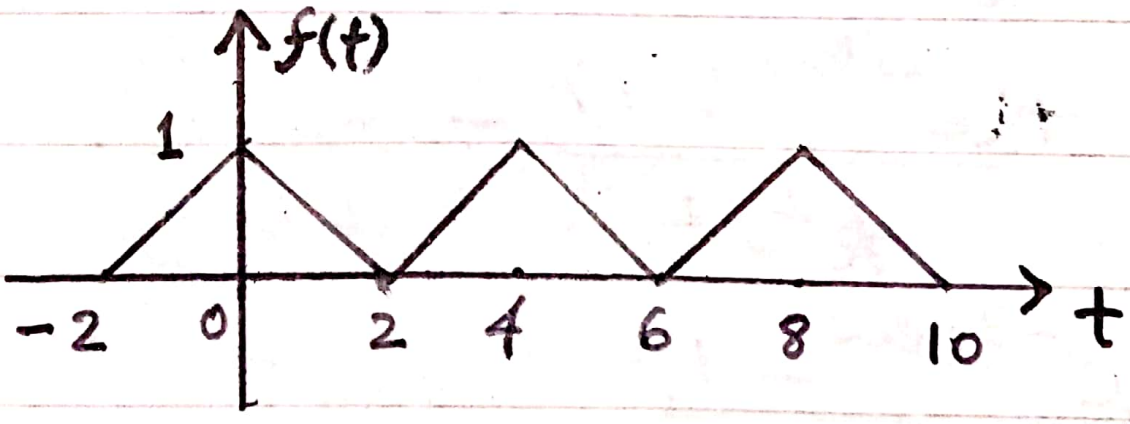
Question: 2

Solution:

We can extend the signal to negative time axis (as it is periodic)

$$f(t) = \begin{cases} 1 + \frac{t}{2} & -2 < t < 0 \\ 1 - \frac{t}{2} & 0 < t < 2 \end{cases}$$



We know that complex fourier series coefficient is

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{-ik\omega t}$$

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$$c_k = \frac{1}{T}\int_0^T f(t)\, e^{-ik\omega t}\, dt$$

$$c_k = \frac{1}{4}\int_0^4 f(t)\, e^{-ik\omega t}\, dt,$$

$$c_k = \frac{1}{4}\left[\int_{-2}^{0}\left(1+\frac{t}{2}\right)e^{-ik\omega t}\, dt + \int_0^{+2}\left(1-\frac{t}{2}\right)e^{-ik\omega t}\, dt\right]$$

$$c_k = \frac{1}{4}\left[\int_{-2}^{0} e^{-ik\omega t}\, dt + \int_0^{+2} e^{-ik\omega t}\, dt + \frac{1}{2}\int_{-2}^{0} t e^{-ik\omega t}\, dt - \frac{1}{2}\int_0^{+2} t e^{-ik\omega t}\, dt\right].$$

$$\because \int x e^{ax}\, dx = \frac{x e^{ax}}{a} - \frac{e^{ax}}{a^2} + C$$

$$c_k = \frac{1}{4}\left[\int_{-2}^{+2} e^{-ik\omega t}\, dt + \frac{1}{2}\int_{-2}^{0} t e^{-ik\omega t}\, dt - \frac{1}{2}\int_0^{+2} t e^{-ik\omega t}\, dt\right].$$

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$$c_k = \frac{1}{4}\left[\frac{e^{-2ik\omega} - e^{2ik\omega}}{-ik\omega} + \frac{1}{2}\int_{-2}^{0} te^{-ik\omega t}\,dt - \frac{1}{2}\int_{0}^{2} te^{-ik\omega t}\,dt\right]$$

$$c_k = \frac{1}{4}\left[\frac{2}{k\omega}\sin(2k\omega) + \frac{1}{2}\left(-\int_{0}^{-2} te^{-ik\omega t}\,dt - \int_{0}^{2} te^{-ik\omega t}\,dt\right)\right].$$

$$c_k = \frac{1}{4}\left[\frac{2}{k\omega}\sin(2k\omega) - \frac{1}{2}\left(\int_{0}^{2} -te^{ik\omega t}(-dt) + \int_{0}^{2} te^{-ik\omega t}\,dt\right)\right]$$

$$c_k = \frac{1}{4}\left[\frac{2}{k\omega}\sin(2k\omega) - \frac{1}{2}\left(\int_{0}^{2} te^{ik\omega t}\,dt + \int_{0}^{2} te^{-ik\omega t}\,dt\right)\right].$$

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$$c_k = \frac{1}{4}\left[\frac{2}{k\omega}\sin(k\omega) - \frac{1}{2}\left(\frac{te^{ik\omega t}}{ik\omega}\Big|_0^2 - \frac{e^{ik\omega t}}{i^2k^2\omega^2}\Big|_0^2 + \frac{te^{-ik\omega t}}{-ik\omega}\Big|_0^2 - \frac{e^{-ik\omega t}}{(+i^2)k^2\omega^2}\Big|_0^2\right)\right]$$

$$c_k = \frac{1}{4}\left[\frac{2}{k\omega}\sin(k\omega) - \frac{1}{2}\left(\frac{2e^{2ik\omega}}{ik\omega} + \frac{(e^{2ik\omega}-1)}{k^2\omega^2} - \frac{2e^{-2ik\omega}}{ik\omega} + \frac{(e^{-2ik\omega}-1)}{k^2\omega^2}\right)\right].$$

$$c_k = \frac{1}{4}\left[\frac{2}{k\omega}\sin(k\omega) - \frac{1}{2}\left(\frac{2(e^{2ik\omega}-e^{-2ik\omega})}{ik\omega} - \frac{2}{k^2\omega^2} + \left(\frac{e^{2ik\omega}+e^{-2ik\omega}}{k^2\omega^2}\right)\right)\right].$$

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$$c_k = \frac{1}{4}\left[\frac{2}{k\omega}\sin(2k\omega) - \frac{1}{2}\left(\frac{4}{k\omega}\sin(2k\omega)\right.\right.$$

$$\left.\left. - \frac{2}{k^2\omega^2} + \frac{2\cos(2k\omega)}{k^2\omega^2}\right)\right]$$

$$c_k = \frac{1}{4}\left[\frac{1-\cos 2k\omega}{k^2\omega^2}\right].$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}.$$

$$c_k = \frac{1}{4} \cdot \frac{(1-\cos k\pi)}{k^2 \frac{\pi^2}{4}}$$

Hence; $\because \cos(k\pi) = (-1)^k$

$$c_k = \frac{1}{k^2\pi^2}\left(1-(-1)^k\right); \quad k \neq 0$$

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$$c_0 = \frac{1}{4}\left[\int_{-2}^{0}\left(1+\frac{t}{2}\right)dt + \int_{0}^{2}\left(1-\frac{t}{2}\right)dt\right]$$

$$c_0 = \frac{1}{4}\left[t+\frac{t^2}{4}\Big|_{-2}^{0} + t-\frac{t^2}{4}\Big|_{0}^{2}\right]$$

$$c_0 = \frac{1}{4}\left[(0-(-2))\ \frac{0^2-(-2)^2}{4} + (2-0) - \frac{(2)^2-(0)^2}{4}\right].$$

$$\left[(2-1)+(2-1)\right]/4$$

$$c_0 = 1/2$$

Hence,

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{-ik\omega t}$$

$$c_k = \frac{1}{k^2\pi^2}(1-(-1)^k); \quad k \neq 0$$

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$$c_0 = 1/2 . \qquad \omega = \pi/2 .$$

$$f(t) = \frac{1}{2} + \sum_{k=1}^{+\infty} \frac{1-(-1)^k}{k^2\pi^2} e^{-ik\pi t/2}$$

Further, $f(t)$ is applied to a low pass filter with corner frequency $f_{cut\text{-}off} = 3.5 f_0$, $f_0 = \frac{1}{T}$. Then, the floor value would be considered. So output signal is given as,

$$y(t) = \frac{1}{2} + \frac{(1-(-1)^1)}{\pi^2} e^{-\frac{\pi i}{2}t} +$$

$$\frac{(1-(-1)^2)}{4\pi^2} e^{-\frac{2\pi i}{2}t} + \frac{(1-(-1)^3)}{9\pi^2} e^{-\frac{3\pi i}{2}t} .$$

$$y(t) = \frac{1}{2} + \frac{2e^{-\pi i/2 t}}{\pi^2} + \frac{2e^{-3\pi i/2 t}}{9\pi^2}$$

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$$y(t) = \frac{1}{2} + \frac{2}{\pi^2}\left[\cos\left(\frac{\pi}{2}t\right) - i\sin\left(\frac{\pi}{2}t\right)\right]$$

$$+ \frac{2}{9\pi^2}\left[\cos\left(\frac{3\pi}{2}t\right) - i\sin\left(\frac{3\pi}{2}t\right)\right]$$

is the required output.

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BEE 11B Assignment: 5

Complex Variable and Transforms

Q: 1

(i)

Solution: $c(t) = \cos(1000t)$

$m(t) = \cos(100t)$

$y(t) = c(t)m(t)$.

Then, the fourier transforms of given signals are:

$$C(j\omega) = \int_{-\infty}^{+\infty} c(t)e^{-j\omega t}dt \ldots$$

We know that (by duality theorem) the fourier transform of $\cos(at)$ is given as:

$$F(j\omega) = \int_{-\infty}^{+\infty} \cos(at)e^{-j\omega t}dt =$$

$$\pi\left[\delta(\omega+a) + \delta(\omega-a)\right].$$

Then:

$$C(j\omega) = \pi\left[\delta(\omega+1000) + \delta(\omega-1000)\right]$$

$$M(j\omega) = \pi\left[\delta(\omega+100) + \delta(\omega-100)\right]$$

Furthermore:

$$y(t) = c(t)m(t)$$

$$y(t) = \cos(1000t)\cos(100t)$$

$\because 2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

$$y(t) = \frac{\cos(1100t)}{2} + \frac{\cos(900t)}{2}$$

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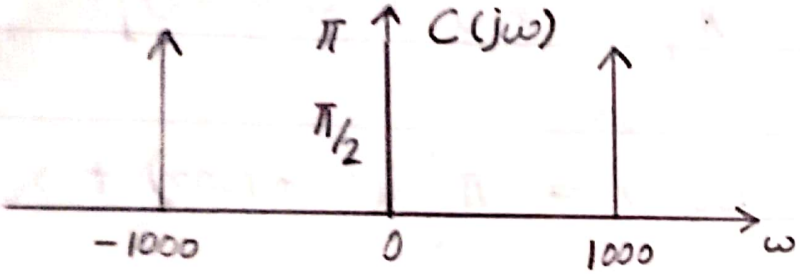
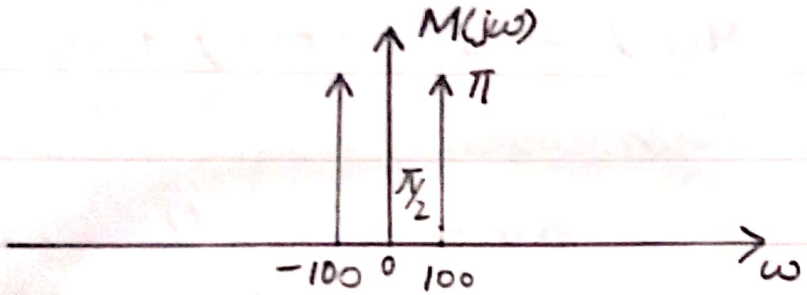
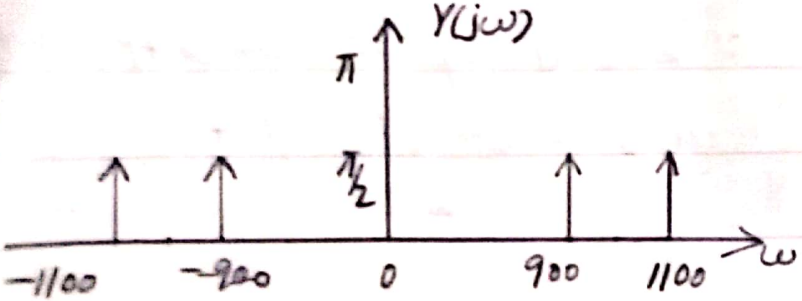
Then:

$$Y(j\omega) = \frac{\pi}{2}\Big[\delta(\omega+1100) + \delta(\omega-1100)$$

$$+ \delta(\omega+900) + \delta(\omega-900)\Big]$$







(ii)

Solution: $z(t) = y(t)\,c(t)$.

$$z(t) = \frac{1}{2}\Big[\cos(1100t) + \cos(900t)\Big]\cos(1000t)$$

$$\because 2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta).$$

$$z(t) = \frac{1}{2}\Big[\cos(1100t)\cos(1000t) + \cos(900t)\cos(1000t)\Big]$$

$$z(t) = \frac{1}{2}\left[\frac{\cos(2100t)}{2} + \frac{\cos(100t)}{2} + \frac{\cos(1900t)}{2} + \frac{\cos(100t)}{2}\right]$$

$$z(t) = \frac{1}{4}\Big[\cos 2100t + \cos 1900t\Big] + \frac{\cos 100t}{2}$$

$$Z(j\omega) = \frac{\pi}{4}\Big[\delta(\omega+2100) + \delta(\omega-2100) +$$

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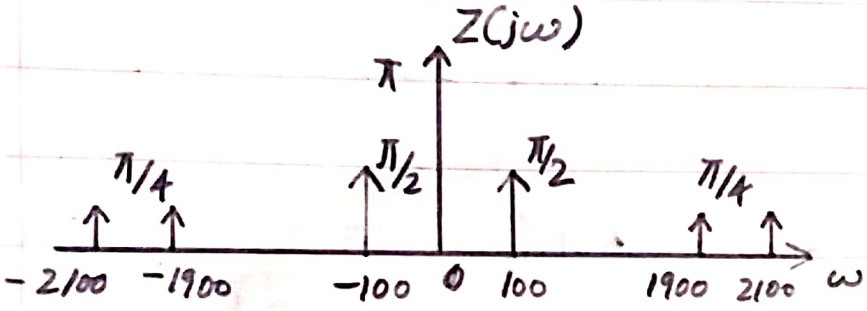
$$\delta(\omega+1900)+\delta(\omega-1900)]$$
$$+\frac{\pi}{2}\left[\delta(\omega+100)+\delta(\omega-100)\right]$$



(iii)

Solution: We have

$$z(t) = \frac{1}{4}\left[\cos 2100t + \cos 1900t\right]$$
$$+ \frac{\cos 100t}{2}.$$

Signal contains $\frac{m(t)}{2}$ and other high frequency signals.

To sift the original $m(t)$ from $z(t)$ we have to design a filter. An ideal low pass filter is $\text{rect}(t/T)$ function, $-T/2 \le t \le T/2$

Also we need amplitude of two; $(A=2)$ to cancel the $1/2$ scaling.

$$F(j\omega) = 2\,\text{rect}\left(\frac{\omega}{2\times 100}\right)$$

hence:

$F(j\omega) = 2\,\text{rect}\left(\frac{\omega}{200}\right)$; So when:

$Z(j\omega)F(j\omega) = M(j\omega)$, obviously we have:

$$m(t) = z(t) * f(t)$$

then:

$$f(t) \xleftrightarrow{\mathcal{F}} F(j\omega) = 2\,\text{Rect}\left(\frac{\omega}{200}\right)$$

$$f(t) \Rightarrow K\sin(at) = \pi K/a\ \text{rect}\left(\frac{\omega}{2a}\right)$$

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$$\frac{K\pi}{a} = 2 \Rightarrow K = \frac{2(100)}{\pi};$$

$$K = \frac{200}{\pi}. \quad a = 100$$

So, we have our signal:

$$f(t) = \frac{200}{\pi} \operatorname{sinc}(100t).$$

Q: 2

(i)

Solution:

$$x(t) = \operatorname{sinc}(6t)$$

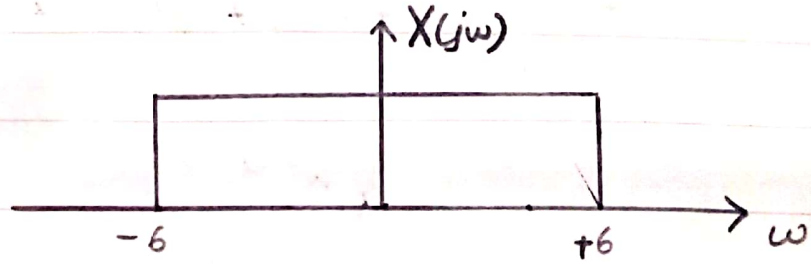
$$X(j\omega) = \mathcal{F}(x(t))$$

$$= \int_{-\infty}^{+\infty} \operatorname{sinc}(6t)\, e^{-j\omega t}\, dt.$$

$$\because \; K\operatorname{sinc}(at) \Leftrightarrow \frac{\pi K}{a} \operatorname{rect}\left(\frac{\omega}{2a}\right)$$

$$K = 1; \quad X(j\omega) = \frac{\pi}{6} \operatorname{rect}\left(\frac{\omega}{2\cdot 6}\right)$$

$$X(j\omega) = \frac{\pi}{6} \operatorname{rect}\left(\frac{\omega}{12}\right).$$



$$E = \int_{-\infty}^{+\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(j\omega)|^2\, d\omega$$

Using Parseval's Theorem:

$$E = \frac{1}{2\pi} \int_{-6}^{+6} \left|\frac{\pi}{6}\right|^2 d\omega$$

$$E = \frac{1}{2\pi} \cdot \frac{\pi^2}{36} \cdot (+6 - (-6))$$

$$E = \frac{\pi \cdot 2 \cdot 6}{2 \cdot 36} = \frac{\pi}{6} \qquad E = \frac{\pi}{6} \text{ units}$$

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