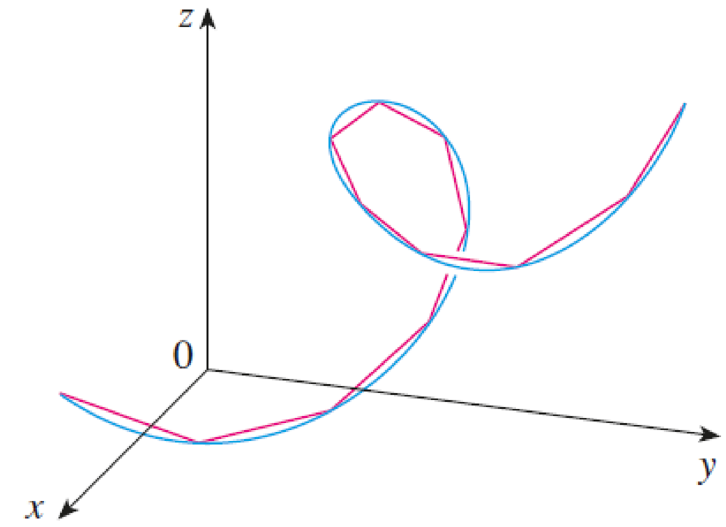
Arc Length, Curvature & TNB frame

Vector Calculus(MATH-243)
Instructor: Dr. Naila Amir

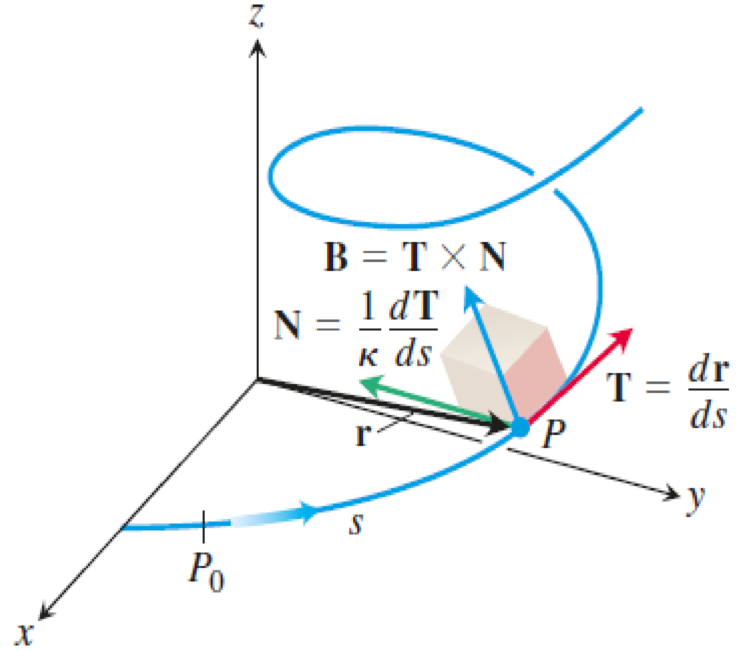


The length of a space curve is the limit of lengths of inscribed polygons.

$$L = \int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

The rate at which **T** turns per unit of length along the curve is called the *curvature*

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right|$$



13

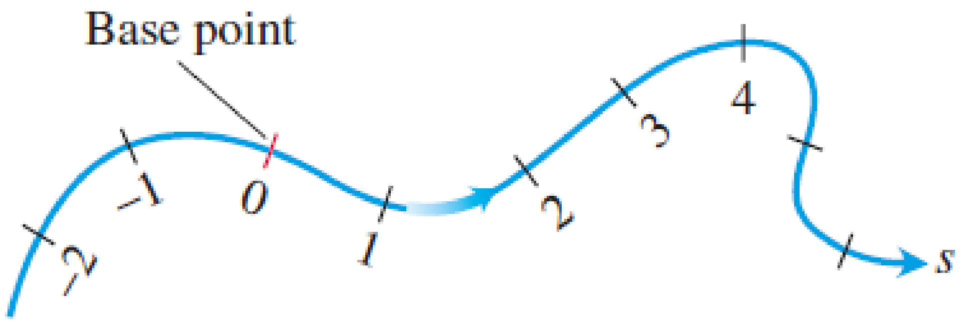
Vectors And
The Geometry Of Space

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.
Chapter: 13 , Section: 13.3, 13.4, 13.5

Book: Calculus Early Transcendentals (6^{th} Edition) By James Stewart.
Chapter: 13 , Section: 13.3

Arc Length

- One of the features of smooth space and plane curves is that they have a measurable length.
- This enables us to locate points along these curves by giving their directed distance s along the curve from some base point, the way we locate points on coordinate axes by giving their directed distance from the origin.



Smooth curves can be scaled like number lines, the coordinate of each point being its directed distance along the curve from a preselected base point.

Arc Length of a Plane Curve

We defined the length of a **plane curve** with parametric equations:

$$x = f(t), \qquad y = g(t); \qquad a \le t \le b,$$

as the limit of lengths of inscribed polygons and, for the case where $f'(t)$ and $g'(t)$ are continuous, we arrived at the formula:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt. \qquad (1)$$

The length of a space curve is defined in exactly the same way.

Arc Length of a Space Curve

Suppose that the curve has the vector equation:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle; \qquad a \le t \le b,$$

or, equivalently, the parametric equations:

$$x = f(t), \qquad y = g(t), \qquad z = h(t); \quad a \le t \le b,$$

where $f'(t), g'(t)$, and $h'(t)$ are continuous. If the curve is traversed exactly once as increases from a to b, then it can be shown that its length is:

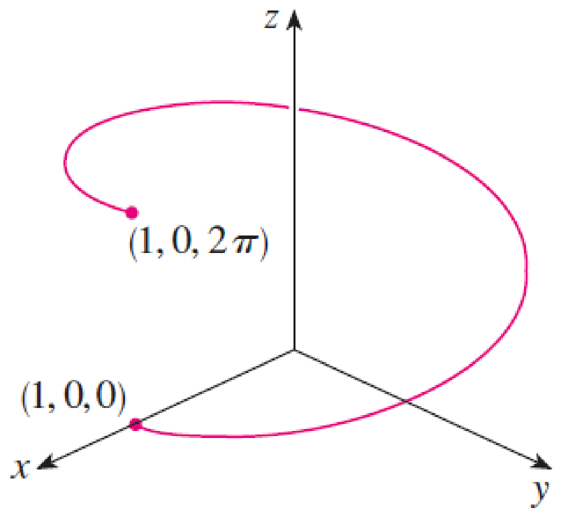
$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}\, dt. \quad (2)$$

Notice that both of the arc length formulas (1) and (2) can be put into the more compact form:

$$L = \int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b |\mathbf{v}(t)|\, dt. \qquad (3)$$

Example:

Find the length of the arc of the circular helix with vector equation:



$$\mathbf{r}(t) = \langle \cos t\,, \sin t\,, t \rangle,$$

from the point $(1{,}0{,}0)$ to the point $(1{,}0{,}2\pi)$.

Solution:

For the present case: $\mathbf{r}'(t) = \langle -\sin t\,, \cos t\,, 1 \rangle$. Thus,

$$|\mathbf{r}'(t)| = \sqrt{[-\sin t]^2 + [\cos t]^2 + [1]^2} = \sqrt{2}.$$

The arc from $(1{,}0{,}0)$ to $(1{,}0{,}2\pi)$ is described by the parameter interval $0 \le t \le 2\pi$ and the length of the arc of the circular helix is given as:

$$L = \int_a^b |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\pi.$$

Observation

A single curve can be represented by more than one vector function. For instance, the twisted cubic:

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle; \qquad 1 \le t \le 2, \qquad (4)$$

could also be represented by the function:

$$\mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle; \qquad 0 \le u \le \ln 2\,, \qquad (5)$$

where the connection between the parameters t and u is given by $t = e^u$. We say that Equations (4) and (5) are **parametrizations** of the curve C. If we were to use Equation (3) to compute the length of C using Equations (4) and (5), we would get the same answer. In general, it can be shown that when Equation (3) is used to compute arc length, the answer is **independent of the parametrization** that is used.

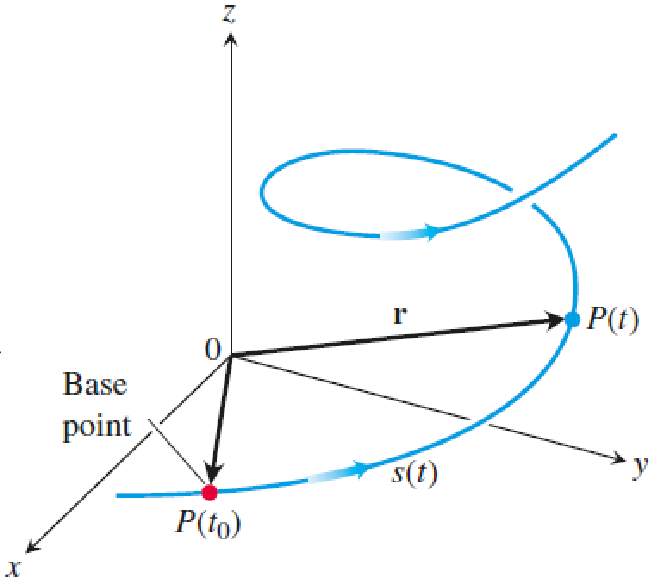
Arc Length Parameter with Base Point

If we choose a base point $P(t_0)$ on a smooth curve C parametrized by t, each value of t determines a point $P(t) = (x(t), y(t), z(t))$ on C and a "directed distance"

$$s(t) = \int_{t_0}^{t} |\mathbf{r}'(u)|\, du = \int_{t_0}^{t} \sqrt{\left[\frac{dx}{du}\right]^2 + \left[\frac{dy}{du}\right]^2 + \left[\frac{dz}{du}\right]^2}\, du. \qquad (6)$$

Each value of s determines a point on C, and this parametrizes C with respect to s. We call s an **arc length parameter** for the curve. The parameter's value increases in the direction of increasing t. This arc length parameter is particularly effective for investigating the turning and twisting nature of a space curve.

If a curve $\mathbf{r}(t)$ is already given in terms of some parameter t and $s(t)$ is the arc length function, then we may be able to solve for t as a function of s: $t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for t: $\mathbf{r} = \mathbf{r}(t(s))$. The new parametrization identifies a point on the curve with its directed distance along the curve from the base point.



Speed on a Smooth Curve:

Since $\mathbf{r}'(t)$ in Equation (6) is continuous (the curve is smooth), the Fundamental Theorem of Calculus tells us that s is a differentiable function of t with derivative:

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = |\mathbf{v}(t)|. \qquad (7)$$

Equation (7) says that the speed with which a particle moves along its path is the magnitude of $\mathbf{v}$, consistent with what we know. Although the base point $P(t_0)$ plays a role in defining s in (6), it plays no role in (7). The rate at which a moving particle covers distance along its path is independent of how far away it is from the base point.

Note that $\frac{ds}{dt} > 0$ since, by definition, $|\mathbf{v}(t)|$ is never zero for a smooth curve. Thus, we conclude that s is an increasing function of t.

Example:

Reparametrize the helix: $\mathbf{r}(t) = \langle \cos t\,, \sin t\,, t \rangle$, with respect to arc length measured from $(1,0,0)$ in the direction of increasing t.

Solution:

For the present case $P(t_0) = (1,0,0)$ which corresponds to the parameter value $t_0 = 0$. From our previous example we know that $|\mathbf{r}\,'(t)| = \sqrt{2}$. Thus, the arc length parameter along the helix from t_0 to t is given as:

$$s = s(t) = \int_{t_0}^{t} |\mathbf{r}'(u)|\, du = \int_{0}^{t} \sqrt{2}\, dt = \sqrt{2}t. \Longrightarrow t = \frac{s}{\sqrt{2}}.$$

Thus, by substituting $t = \frac{s(t)}{\sqrt{2}}$ into the position vector $\mathbf{r}(t)$, the required reparametrization for the helix is obtained as:

$$\mathbf{r}(t(s)) = \left\langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle.$$

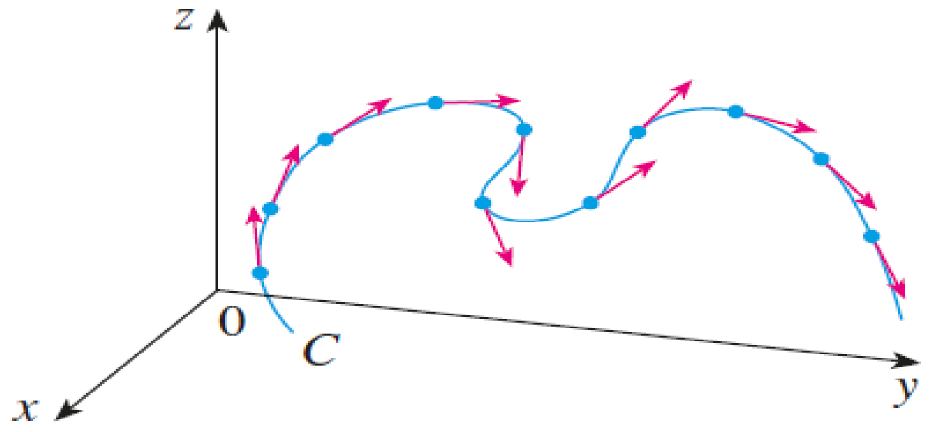
Curvature

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq 0$ on I. A curve is called **smooth** if it has a smooth parametrization. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously. If C is a smooth curve defined by the vector function $\mathbf{r}(t)$, recall that the unit tangent vector is given by:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}.$$

and indicates the direction of the curve. From the accompanying figure we can see that $\mathbf{T}(t)$ changes direction very slowly when C is fairly straight, but it changes direction more quickly when C bends or twists more sharply.

The **curvature** of C at a given point P is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the *magnitude of the rate of change of the unit tangent vector with respect to arc length.*

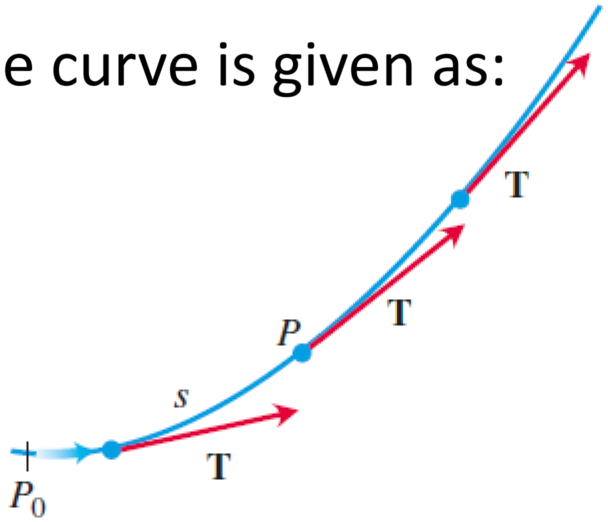


Unit tangent vectors at equally spaced points on C

Curvature

As a particle moves along a smooth curve, $\mathbf{T}$ turns as the curve bends. Since $\mathbf{T}$ is a unit vector, its length remains constant and only its direction changes as the particle moves along the curve. The rate at which $\mathbf{T}$ turns per unit of length along the curve is called the **curvature.** The traditional symbol for the curvature is the Greek letter κ ("kappa"). Thus, if $\mathbf{T}$ is the unit vector of a smooth curve, the **curvature** of the curve is given as:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|. \quad (8)$$



As P moves along the curve in the direction of increasing arc length, the unit tangent vector turns. The value of $|d\mathbf{T}/ds|$ at P is called the ***curvature*** of the curve at P.

The curvature is easier to compute if it is expressed in terms of the parameter t instead of s, so by using the chain rule we have: $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}$ and (8) takes the form

$$\kappa = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}. \quad (9)$$

$$\boxed{\because \frac{ds}{dt} = |\mathbf{r}\,'(t)|}$$

Example:

Show that the curvature of a circle of radius a is $1/a$.

Solution:

Consider the vector function: $\mathbf{r}(t) = \langle a\cos t\,, a\sin t\rangle$ that provides us with the parametrization the circle of radius a. For the present case: $\mathbf{r}'(t) = \langle -a\sin t\,, a\cos t\rangle$ and $|\mathbf{r}'(t)| = a$. Thus, the unit tangent vector is given as:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \langle -\sin t\,, \cos t\rangle.$$

This provides us with:

$$\mathbf{T}'(t) = \langle -\cos t\,, -\sin t\rangle \quad \text{and} \quad |\mathbf{T}'(t)| = 1.$$

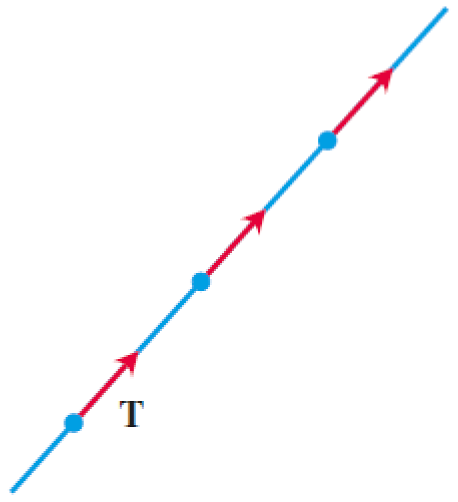
Thus, curvature of a circle of radius a is given as:

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{a},$$

as desired.

Curvature

- If $|d\mathbf{T}/ds|$ is large, $\mathbf{T}$ turns sharply as the particle passes through P, and the curvature at P is large.
- If $|d\mathbf{T}/ds|$ is close to zero, $\mathbf{T}$ turns more slowly and the curvature at P is smaller.
- The curvature is constant for straight lines and circles. The curvature of a straight line is always 0 because the tangent vector is constant. However, small circles have large curvature and large circles have small curvature.



Along a straight line, $\mathbf{T}$ always points in the same direction. The curvature, $|d\mathbf{T}/ds|$, is zero

Alternative Formula for Curvature

Although formula given in (9) can be used in all cases to compute the curvature, however, the formula given by the following theorem is often more convenient to apply.

Theorem:

The curvature of the curve given by the vector function $\mathbf{r}(t)$ is given as:

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Example:

Find the curvature of the twisted curve: $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at a general point and at the point $(0,0,0)$.

Solution:

For the present case we have:

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle, \qquad \mathbf{r}''(t) = \langle 0, 2, 6t \rangle, \qquad |\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4},$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t, 2 \rangle,$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}.$$

Thus, the curvature of the curve given by the vector function $\mathbf{r}(t)$ is given as:

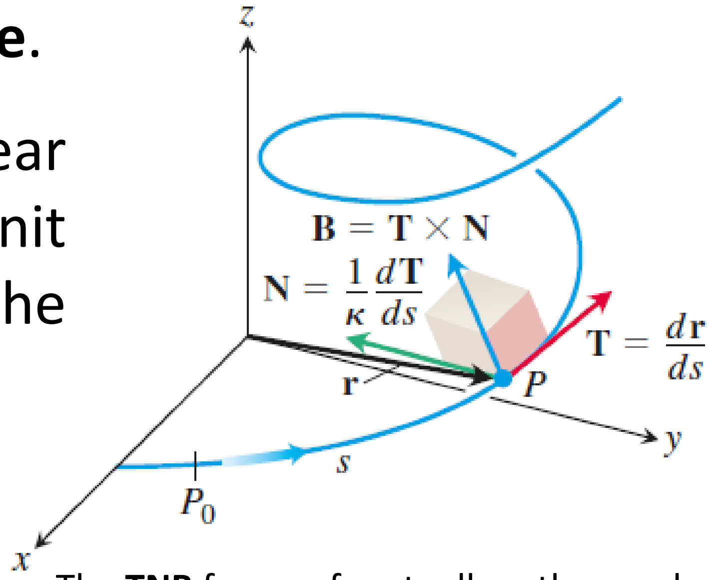
$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(1 + 4t^2 + 9t^4)^{3/2}}.$$

At the origin, where $t = 0$, the curvature is given as: $\kappa(0) = 2.$

The TNB Frame (Frenet Frame)

If we are traveling along a curve in space, the Cartesian $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ coordinate system for representing the vectors describing our motion may not be very relevant to us. Instead, the vectors that represent our forward direction (the *unit tangent vector* $\mathbf{T}$), the direction in which our path is turning (the *unit normal vector* $\mathbf{N}$), and the tendency of our motion to "twist" out of the plane created by these vectors in the direction perpendicular to this plane (defined by the *unit binormal vector* $\mathbf{B}$) are likely to be more important. Together $\mathbf{T}, \mathbf{N}$, and $\mathbf{B}$ define a moving righthanded vector frame that plays a significant role in calculating the paths of particles moving through space. It is called the **Frenet** ("fre-*nay*") **frame** (after Jean-Frédéric Frenet, 1816–1900), or the **TNB frame**.

Expressing the acceleration vector along the curve as a linear combination of this **TNB** frame of mutually orthogonal unit vectors traveling with the motion can reveal much about the nature of our path and our motion along it.



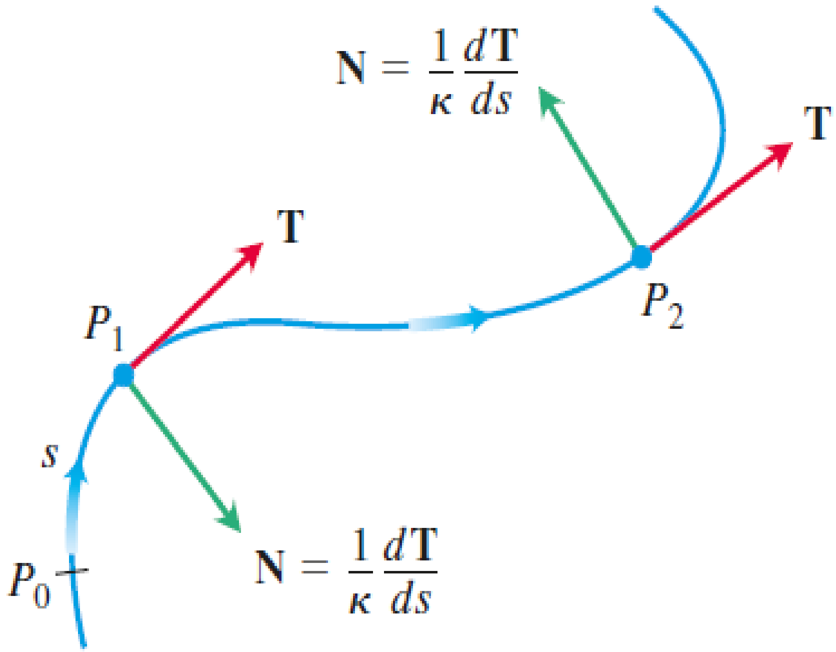
The **TNB** frame of mutually orthogonal unit vectors traveling along a curve in space.

The Normal Vector

At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$, among them there is one of particular significance because *it points in the direction in which the curve is turning*. since $|\mathbf{T}(t)| = 1$ for all t, we have $\mathbf{T}(t).\mathbf{T}'(t) = 0$ (by using fact that for a vector function of constant length $\mathbf{r}(t).\mathbf{r}'(t) = 0$ and converse is also true). So, $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$. Note that $\mathbf{T}'(t)$ itself is not a unit vector.

Thus, if $\mathbf{r}(t)$ is also smooth curve, then at a point where $\kappa \neq 0$ we can define the **principal unit normal vector** (or simply **unit normal**) as:

$$\mathbf{N}(t) = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$



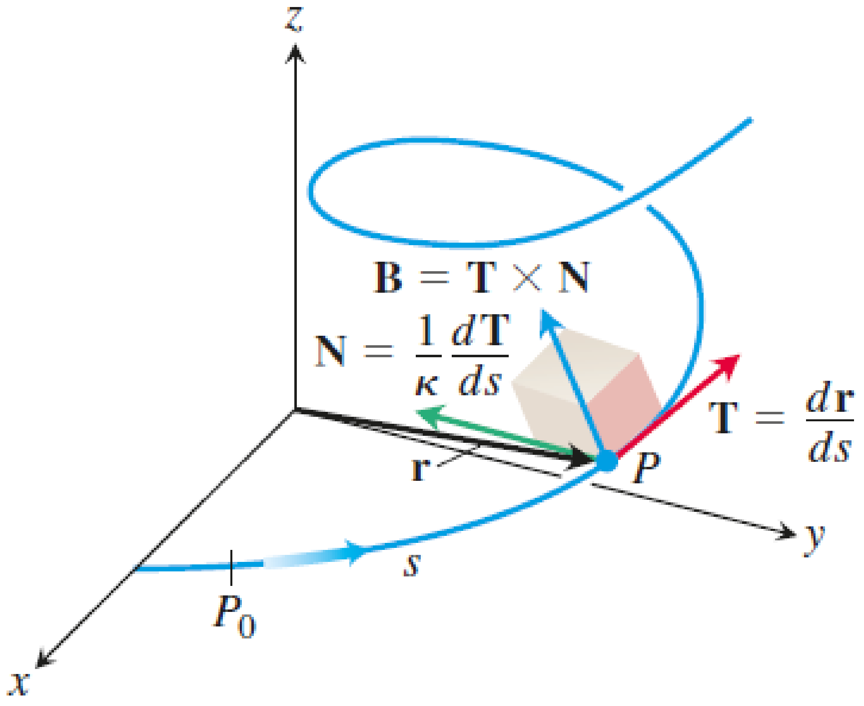
We can think of the **normal vector** as indicating the direction in which the curve is turning at each point.

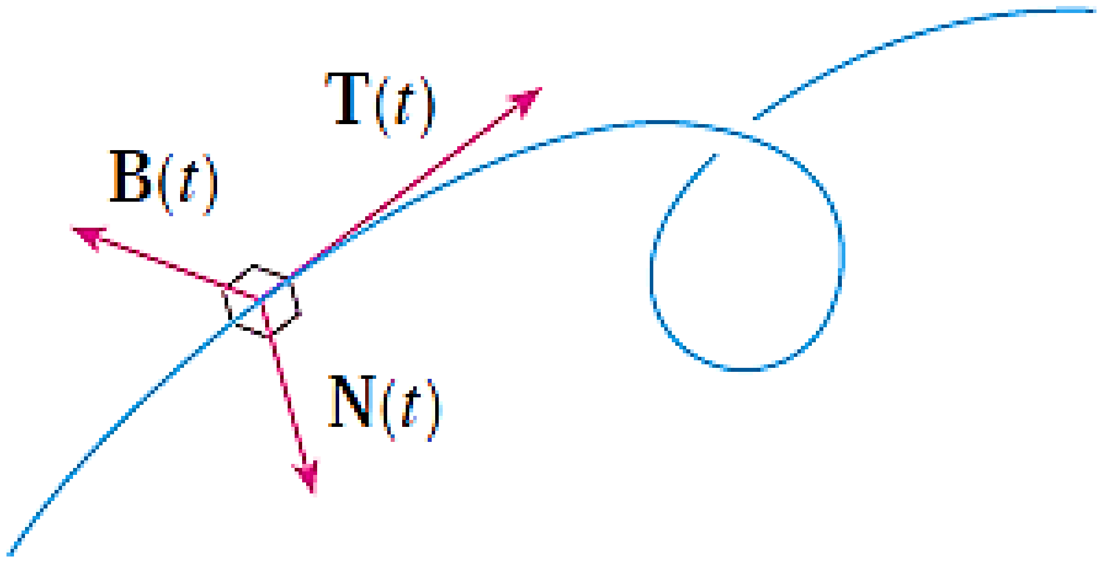
The Binormal Vector

The vector:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t),$$

is called the **binormal vector** of a curve in space. This is a unit vector that orthogonal to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ and is also a unit vector.





Example:

Find the unit normal and binormal vectors for the circular helix: $\mathbf{r}(t) = \langle \cos t\,, \sin t\,, t \rangle$.

Solution:

For the present case we have:

$$\mathbf{r}'(t) = \langle -\sin t\,, \cos t\,, 1 \rangle, \qquad \text{and} \qquad |\mathbf{r}'(t)| = \sqrt{2},$$

Thus,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}} \langle -\sin t\,, \cos t\,, 1 \rangle,$$

and

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}} \langle -\cos t\,, -\sin t\,, 0 \rangle, \qquad \text{and} \qquad |\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}.$$

Thus, the unit normal vector is given as:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\cos t\,, -\sin t\,, 0 \rangle.$$

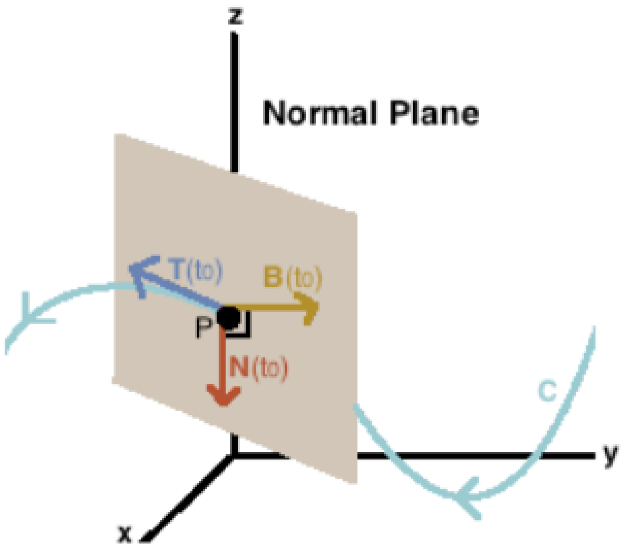
This shows that the normal vector at a point on the helix is horizontal and points toward the z −axis. The binormal vector is obtained as:

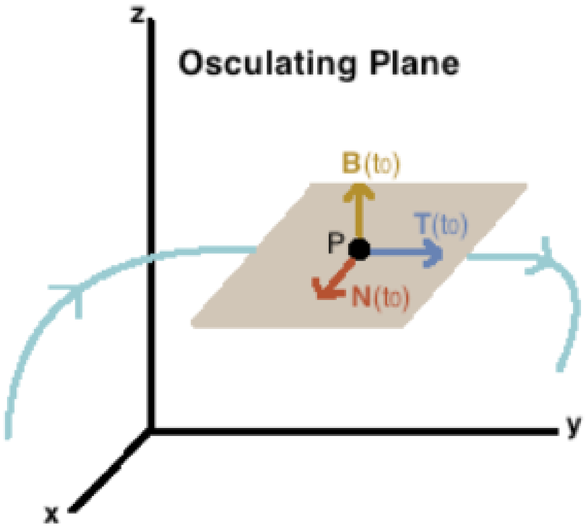
$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}} \langle \sin t\,, -\cos t\,, 1 \rangle.$$

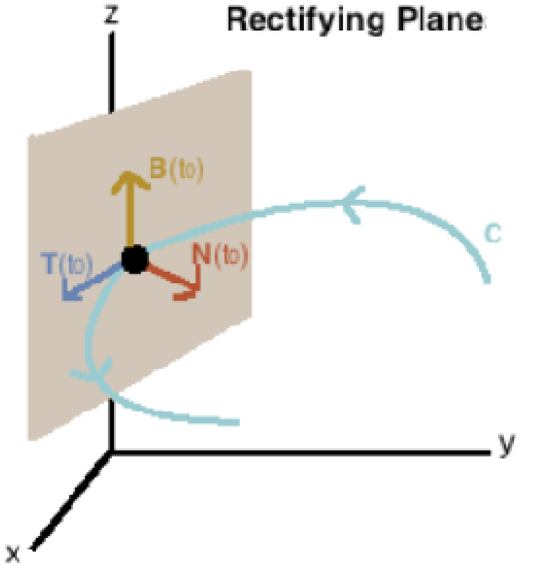
Normal, Osculating and Rectifying Planes

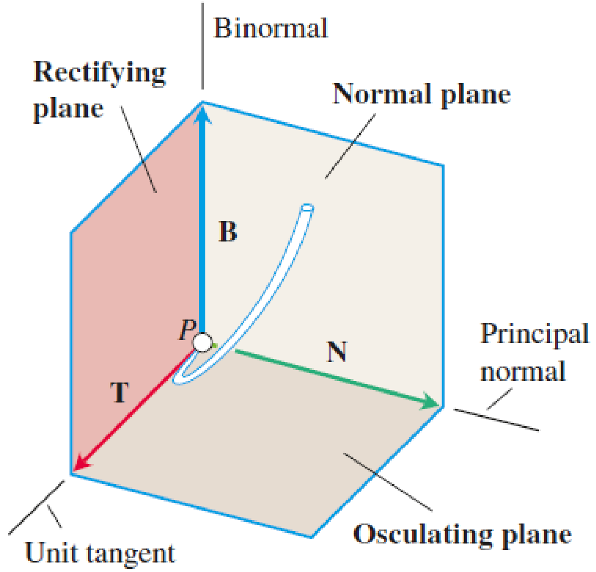
Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ be a vector function that represents the smooth curve C and let $P(x_0, y_0, z_0)$ be a point on C corresponding to $\mathbf{r}(t_0)$. Then:

- the **normal plane** of C at point P is the plane spanned by $\mathbf{N}(t_0)$ and $\mathbf{B}(t_0)$ with normal vector $\mathbf{T}(t_0)$. It consists of all lines that are orthogonal to the tangent vector.
- the **osculating plane** of C at point P is the plane spanned by $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$ with normal vector $\mathbf{B}(t_0)$.
- the **rectifying plane** of C at P is the plane spanned by $\mathbf{B}(t_0)$ and $\mathbf{T}(t_0)$ with normal vector $\mathbf{N}(t_0)$.









Example:

Find the equations of the normal plane and osculating plane of the circular helix:
$\mathbf{r}(t) = \langle \cos t\,, \sin t\,, t \rangle$ at the point $P(0,1,\pi/2)$.

Solution:

The normal plane at P has a normal vector:

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\langle -\sin t\,, \cos t\,, 1\rangle\bigg|_{t=\pi/2} = \frac{1}{\sqrt{2}}\langle -1,0,1\rangle.$$

So, equation of normal plane is given as:

$$\frac{-1}{\sqrt{2}}(x-0) + 0(y-1) + \frac{1}{\sqrt{2}}\left(z - \frac{\pi}{2}\right) = 0 \Longrightarrow z = x + \frac{\pi}{2}.$$

The normal vector of the osculating plane is the vector:

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\langle \sin t\,, -\cos t\,, 1\rangle\bigg|_{t=\pi/2} = \frac{1}{\sqrt{2}}\langle 1,0,1\rangle.$$

Thus, equation of the osculating plane is given as:

$$\frac{1}{\sqrt{2}}(x-0) + 0(y-1) + \frac{1}{\sqrt{2}}\left(z - \frac{\pi}{2}\right) = 0 \Longrightarrow z = -x + \frac{\pi}{2}.$$

Practice Questions

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

Chapter: 13

Exercise-13.3: Q – 1 to 14, Q – 17 to 26, Q – 43 to 46.

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 13

Exercise-13.3: Q – 1 to 8, Q – 11 to 14.
Exercise-13.4: Q – 1 to 4, Q – 9 to 16.
Exercise-13.5: Q – 7 to 8.