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Numerical Methods

Errors in numerical computations

- Inherent errors ; • Round off errors
- Truncation errors ; • Absolute errors
- Relative errors ; • Percentage errors

→ Absolute errors : $E_A = |x - \bar{x}|$ (x → true; $\bar{x}$ → approximate)

→ Relative errors : $E_R = \frac{E_A}{|x|} = \frac{|x-\bar{x}|}{|x|}$

→ Percentage errors : $E_p = E_R \times 100$

$$(\Delta u)_{max} = \left|\frac{\partial u}{\partial x}\Delta x\right| + \left|\frac{\partial u}{\partial y}\Delta y\right| + \left|\frac{\partial u}{\partial z}\Delta z\right|$$

$$\therefore \frac{\partial u}{\partial z} = -15\frac{xy^2}{z^4}$$

» $f(a)f(b) < 0$; at least one root exists (odd number)

» $f(a)f(b) > 0$; roots may or may not exist (even number of roots)

When one root ; it is bracket

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Numerical Methods

Two types of numerical methods:
- Closed or Bracketing
- Open or Iterative

Bisection Method

In interval (a, b) ; If $f(a)$ and $f(b)$ have opposite signs ; there exists at least one root in the interval

$\Rightarrow$ Assume root $x_0 = \frac{a+b}{2}$

$\therefore$ $f(a)$: +ive
$f(b)$: -ive

If $f(x_0)$ is +ive, root lies b/w x_0 and b.

And vice versa.

See example 1 from Lecture

- $x - \cos(x) = 0$
- $f(x) = x - \cos(x)$

$f(0) = -$ive $\quad f(0.5) = -$ive

$f(1) = +$ive $\quad f(0.75) = +$ive

$a = 0.5$; $b = 0.75$

$$x_0 = \frac{a+b}{2} = 0.625$$

- $f(x_0) = -0.1859$ (-ive)

$$x_0 = \frac{0.75 + 0.625}{2} = 0.6875$$

- $f(x_0) = -0.0853$ (-ive)

$$x_0 = \frac{0.75 + 0.6875}{2} = 0.71875$$

- $f(x_0) = -0.0338$ (-ive)

$$x_0 = \frac{0.75 + 0.71875}{2} = 0.734375$$

- $f(x_0) = -7.87 \times 10^{-3}$ (-ive)

$$x_0 = \frac{0.75 + 0.734375}{2} = 0.7421875$$

- $f(x_0) = 5.195 \times 10^{-3}$ (+ive)

$$x_0 = \frac{0.734375 + 0.7421875}{2} = 0.73828$$

⋮ continue until convergence

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Numerical Methods

Example $f(x) = x^3 - 9x + 1$; $(2, 3)$

$x_0 = 2.5$; $f(x_0) = -ive$

→ b/w 2.5 and 3 : $x_0 = \frac{2.5+3}{2} = 2.75$

$f(x_0) = -ive$

→ b/w 2.75 and 3 : $x_0 = \frac{2.75+3}{2} = 2.875$

$f(x_0) = -ive$

→ b/w 2.875 and 3 : $x_0 = \frac{2.875+3}{2} = 2.938$

$f(x_0) = -ive$

→ b/w 2.938 and 3 : $x_0 = \frac{2.938+3}{2} = 2.969$

$f(x_0) = +ive$

→ b/w 2.969 and 2.938 : $x_0 = \frac{2.938+2.969}{2}$

$f(x_0) = +ive$ $= 2.954$

→ b/w 2.954 and 2.938 : $x_0 = \frac{2.938+2.954}{2}$

$f(x_0) = +ive$ $= 2.946$

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→ b/w 2.946 and 2.938 : $x_0 = \frac{2.946 + 2.938}{2}$

$= 2.942$

→ $2.94 = x_0$ (three significant digits)

Example $\tan x + x = 0$; $(2, 2.1)$

$f(2) = -ive$; $f(2.1) = +ive$

$x_0 = 2.05$; $f(x_0) = +ive$
$x_0 = 2.03$; $f(x_0) = -ive$
$x_0 = \boxed{2.04}$; $f(x_0) = +ive$
$x_0 = \boxed{2.04}$; $f(x_0) = +ive$

Example $V = \pi h^2 \frac{[3R - h]}{3}$

$3V = 3\pi h^2 R - \pi h^3$
$90 = 9\pi h^2 - \pi h^3$
$f(h) = \pi h^3 - 9\pi h^2 + 90$]

Interval $(0, 3)$
$f(0) = +ive$; $f(3) = -ive$

$x_0 = 1.5$; $f(x_0) = +ive$
$x_0 = 2.25$; $f(x_0) = -ive$
$x_0 = 1.875$; $f(x_0) = +ive$
$x_0 = 2.0625$; $f(x_0) = -ive$
$x_0 = 1.968750$ (5 iters)

Regula - Falsi Method

Two point form of line :

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Chord joining $[a, f(a)], [b, f(b)]$

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

$y = 0$

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)} = x_0$$

Example $x^3 + x - 1 = 0$ $(0.5, 1)$

$f(0.5) = -\text{ive}$; $f(1) = +\text{ive}$

$= -0.375$; $= 1$

$x_0 = 0.636364$; $f(x_0) = -0.105935$

$x_0 = 0.671196$; $f(x_0) = -0.026428$

Example $x \log_{10} x - 1.2 = 0$; $(2, 3)$

$f(2) = -0.597940$; $f(3) = 0.231364$

$x_0 = 2.721014$; $f(x_0) = -0.017091$

$x_0 = 2.740205$; $f(x_0) = -0.000385$

$x_0 = 2.740636$; $f(x_0) = -0.000009$

$x_0 = 2.740646$

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Numerical Methods

Example: $f(x) = 4e^{-x}\sin(x) - 1$

$f(0) = -1$; $f(0.5) = 0.163145$

$a = 0$; $b = 0.5$

$x_0 = 0.429869$ (+ive) ; $f(x_0) = 0.084545$

; $b = 0.429869$: $f(b) = 0.084545$

$x_1 = 0.396359$: $f(x_1) = 0.038916$

$x_2 = 0.381512$: $f(x_2) = 0.016394$

$x_3 = 0.375358$: $f(x_3) = 0.007496$

$x_4 = 0.37248$

$x_5 = 0.37136$

$x_6 = 0.37089$

continue

Secant Method

Open, unlike Regula Falsi, iterative with $a = x_0$ and $b = x_1$; and so on

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Example: $x^3 - 4x - 9 = f(x)$

$x_0 = 2.5$; $x_1 = 3$ | $f(x_0) = -3.375$ $f(x_1) = 6$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 2.680$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \quad ; f(x_2) = -0.471$$

$$2.703 \quad ; f(x_3) = -0.058$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= 2.706$$

Root of $f(x) = \underline{2.706}$

Example $x = \log 2(x+1)$

$x_0 = 0.75$ $\qquad$ $x_1 = 1.25$

$f(x_0) = -0.05735$ $\qquad$ $f(x_1) = 0.08007$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 0.95867 \quad ; f(x_2) = -0.01121$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 0.994415 \quad ; f(x_3) = -0.0054$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = 1.00015 \quad ; f(x_4) = 0.00004$$

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} = 0.99999 \quad ; f(x_5) = 0.00000$$

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Numerical Methods

Example $x^3 - x - 1 = 0$

$x_0 = 1$; $x_1 = 1.5$
$f(x_0) = -1$; $f(x_1) = 0.87500$

Using : $x_{n+1} = \dfrac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$

$x_2 = 1.266667$; $f(x_2) = -0.234370$
$x_3 = 1.315962$; $f(x_3) = -0.037038$
$x_4 = 1.325214$; $f(x_4) = 0.002117$
$x_5 = 1.324714$; $f(x_5) = -0.000017$
✓ — $x_6 = \boxed{1.324718}$;

Example $x \log x = 1$

$x_0 = 2$; $x_1 = 3$
$f(x_0) = -0.39794$; $f(x_1) = 0.43136$

$x_2 = 2.479848$; $f(x_2) = -0.021886$
$x_3 = 2.504964$; $f(x_3) = -0.001016$
$x_4 = 2.506188$; $f(x_4) = 0.000003$
$x_5 = \boxed{2.506184}$

Example $\sin x + \cos x = 1$
$x_0 = 1$; $x_1 = 2$
$f(x_0) = 0.38177$; $f(x_1) = -0.50685$

$x_2 = 1.429624$; $f(x_2) = 0.130756$

$x_3 = 1.542793$; $f(x_3) = 0.023908$

$x_4 = 1.572766$; $f(x_4) = -0.001971$

$x_5 = 1.570772$; $f(x_5) = 0.000024$

$x_6 = \boxed{1.570796}$

Example $3x^2 + 5x - 40$

$x_0 = 2$; $x_1 = 3$

$f(x_0) = -18$; $f(x_1) = 2$

$x_2 = 2.900000$; $f(x_2) = -0.270000$

$x_3 = 2.911894$; $f(x_3) = -0.003144$

$x_4 = 2.912034$; $f(x_4) = -0.000005$

$x_5 = \boxed{2.912034}$

Newton Raphson

Equation of tangent to a curve $y = f(x)$ at point $P(x_0, y_0)$ is given by:

$$y - f(x_0) = (x - x_0) f'(x_0)$$

$$\Rightarrow x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad f'(x_k) \neq 0$$

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Numerical Methods

Newton Raphson Method

Example: $f(x) = x^3 - 5x + 1$ | $f(0) = 1$] Bracket
$f'(x) = 3x^2 - 5$ | $f(1) = -3$]

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$x_0 = 0.5$; $f(0.5) = -1.375$; $f(0.5) = -4.25$

$\Rightarrow$ ① $x_1 = x_0 - f(x_0)/f'(x_0) = 0.176471$; $f(x_1) = 0.123143$

$f'(x_1) = -4.906574$

$\Rightarrow$ ② $x_2 = x_1 - f(x_1)/f'(x_1) = 0.201569$

$f(x_2) = 0.000347$ $f'(x_2) = -4.878110$

$\Rightarrow$ ③ $x_3 = x_2 - f(x_2)/f'(x_2) = 0.201640$

$f(x_3) = 0.000000$ $f'(x_3) = -4.878024$

$\Rightarrow$ ④ $x_4 = x_3 - f(x_3)/f'(x_3) = \boxed{0.201640}$

Example: $f(x) = x\log_{10} x - 12.34$

$x_0 = 10$

$$f'(x) = x\left(\frac{1}{x\ln(10)}\right) + \log_{10} x\,(1) = 0.434294 + \log_{10} x$$

$$= 1/\ln(10) + \log_{10} x$$

$\Rightarrow$ ① $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 11.631464$

$f(x_1) = 0.054888$; $f'(x_1) = 1.499929$

$\Rightarrow$ ② $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 11.594870$

$f(x_2) = 0.000025$; $f'(x_2) = 1.498560$

$\Rightarrow$ ③ $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \boxed{11.594854}$

$f(x_3) = 0.000000$; $f'(x_3) = 1.498560$

Example $\left.\begin{array}{l} f(x) = x^3 - 6x + 4 = 0 \\ f'(x) = 3x^2 - 6 = 0 \end{array}\right]$ b/w 0 and 1

$x_0 = 0.5$; $f(x_0) = 1.125$; $f'(x) = -5.25$

$\Rightarrow$ ① $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.714286$

$f(x_1) = 0.078717$ $f'(x_1) = -4.4694$

$\Rightarrow$ ② $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.73190$

$f(x_2) = 0.00065$ $f'(x_2) = -4.39297$

$\Rightarrow$ ③ $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.73205$

~ $\rightarrow$ converges on next iter

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Example $e^x = 4x \Rightarrow f(x) = e^x - 4x$

$x_0 = 2$

$f'(x) = e^x - 4$

$x_1 = x_0 - f(x_0)/f'(x_0) = 2.1803$

1 $f(x_1) = 0.1276 \qquad f'(x_1) = 4.8487$

$x_2 = x_1 - f(x_1)/f'(x_1) = 2.1540$

2 $f(x_2) = 0.0033 \qquad f'(x_2) = 4.6193$

$x_3 = x_2 - f(x_2)/f'(x_2) = \boxed{2.1533}$

3 $f(x_3) = 0.0000 \qquad f'(x_3) = 4.6131$

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Numerical Methods

$x^3 - 5x + 1 = 0$

$\Rightarrow x = \frac{x^3+1}{5}$, $x = \sqrt[3]{5x-1}$, ... and more

Iteration Method

$x = \phi(x) \rightarrow \boxed{x_{k+1} = \phi(x_k)}$; $k = 0, 1, 2 \ldots$

$f(x) = 0$ written as $x_{k+1} = \phi(x_k)$ does not always converge.

To check for convergence : Take derivative of $\phi(x)$ and check for points near both a and b for interval (a, b).

Problem

$f(x) = x^3 - x - 10 = 0 \quad (2, 3)$

- $\phi(x) = x^3 - 10$

  $\phi'(x) = 3x^2$ (Does not converge)

- $\phi(x) = \sqrt[3]{x+10} = (x+10)^{1/3}$

  $\phi'(x) = \frac{1}{3(x+10)^{2/3}}$ (Converges)

$x_0 = 2.5$

$x_1 = \phi(x_0) = 2.309650$

$x_2 = \phi(x_1) = 2.308954$

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$x_3 = \phi(x_2) = 2.308908$

$x_4 = \phi(x_3) = 2.308907$

$x_5 = \phi(x_4) = 2.308907$

$x_6 = \phi(x_5) = \underline{2.308907}$

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Numerical Methods

$\Rightarrow f(x) = 0 \Rightarrow x = x + \alpha f(x) = \phi(x)$

For convergence ;

$\Rightarrow |\phi'(x_0)| = |1 + \alpha f'(x_0)| < 1$

Example 1.12 :

$f(x) = 3x^3 + 4x^2 + 4x + 1 = 0$; $(-1, 0)$

$x_0 = -0.5$

$\Rightarrow x = x + \alpha(3x^3 + 4x^2 + 4x + 1) = \phi(x)$

$\Rightarrow |\phi'(x)| = |1 + \alpha(9x^2 + 8x + 4)| < 1$

Put $x_0 \Rightarrow \left|1 + \frac{9\alpha}{4}\right| < 1$

$\Rightarrow -\frac{8}{9} < \alpha < 0$ (choose any α from this interval)

- Rate of Convergence

Relation b/w error of current and last iteration $\rightarrow \epsilon_{n+1} = A \epsilon_n^k$ ← order of convergence

Bisection Method

$\Rightarrow x_{n+1} = \frac{x_{n-1} + x_n}{2}$ for $n = 0, 1, 2 \dots$

$\Rightarrow \epsilon_{n+1} \approx \frac{\epsilon_n}{2}$ comparing

$k = 1$; $A = 1/2$

↳ denotes correctness upto significant figures

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Numerical Methods

Shifting Operator

Let $y = f(x)$; $x, x+h, x+2h \dots$ be the consecutive values of x.

$$\underline{Ef(x) = f(x+h)}$$

$\Rightarrow$ $E^n f(x) = f(x+nh)$

$E^{-n} f(x) = f(x-nh)$

Properties $E^m(E^n f(x)) = E^{m+n} f(x)$

$E^n(E^{-n} f(x)) = f(x)$

Forward Difference

Let $y = f(x)$; $y_1 - y_0$; $y_2 - y_1$ .... are the first differences of y.

$$\Delta y_{n-1} = y_n - y_{n-1}$$

$\Delta^2 y_1 = \Delta[\Delta y_1] = y_3 - 2y_2 + y_1$

second difference

Working : $\Delta[y_2 - y_1] = \Delta y_2 - \Delta y_1$

$= y_3 - y_2 - (+y_2 - y_1)$

$= y_3 - 2y_2 + y_1$

$\Delta^3 y_0 = \Delta^2[\Delta y_0] = \Delta^2[y_1 - y_0]$

$= \Delta^2 y_1 - \Delta^2 y_0$

$= \Delta[y_2 - y_1] - \Delta[y_1 - y_0]$

$= y_3 - y_2 - y_2 + y_1 - (y_2 - y_1 - y_1 + y_0]$

$= y_3 - 2y_2 + y_1 - y_2 + 2y_1 - y_0$

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$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

Example

$$\Delta e^{ax} = e^{a(x+h)} - e^{ax}$$
$$= e^{ax}(e^h - 1)$$

$$\Delta^2 e^x = \Delta(e^{x+h} - e^x)$$
$$= \Delta e^{x+h} - \Delta e^x$$
$$= e^{x+2h} - e^{x+h} - e^{x+h} + e^x$$
$$= e^x(e^{2h} - 2e^h + 1)$$

$$\Delta \sin x = \sin(x+h) - \sin(x)$$
$$= \sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)$$

$$\Delta \log x = \log(x+h) - \log(x)$$
$$= \log(x)\log(h) - \log(x)$$
$$= \log(x)(\log(h) - 1)$$

$$\Delta \tan^{-1}(x) = \tan^{-1}(x+h) - \tan^{-1}(x)$$
$= \sim$ From slides

Example

x	y	Δy	$\Delta^2 y$
0	0		
		0.174	
10	0.174		-0.001
		0.173	
20	0.347		-0.002
		0.171	
30	0.518		

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Numerical Methods

- Relation b/w E and Δ

$$\Delta f(x) = f(x+h) - f(x)$$
$$= Ef(x) - f(x)$$
$$= f(x)(E-1)$$

$$\Rightarrow \boxed{\Delta = E - 1}$$
$$\Rightarrow \boxed{E = 1 + \Delta}$$ Relation

$$\Rightarrow E\Delta f(x) = E(f(x+h) - f(x))$$
$$= Ef(x+h) - Ef(x)$$
$$= f(x+2h) - f(x+h)$$
$$= \Delta f(x+h)$$
$$= \Delta E f(x)$$

Commutative

Example 3.8

$$f(x+h) = Ef(x)$$
$$= (1+\Delta)f(x)$$
$$= f(x) + \Delta f(x)$$
$$\frac{f(x+h)}{f(x)} = 1 + \frac{\Delta f(x)}{f(x)}$$
$$\log\left(\frac{f(x+h)}{f(x)}\right) = \log\left(1 + \frac{\Delta f(x)}{f(x)}\right)$$
$$\log(f(x+h)) - \log(f(x)) = \log\left(1 + \frac{\Delta f(x)}{f(x)}\right)$$

$$\boxed{\Delta \log f(x) = \log\left(1 + \frac{\Delta f(x)}{f(x)}\right)}$$

Example 3.9

$$\left(\frac{\Delta^2}{E}\right)x^3$$

$$\Rightarrow \frac{(E^2-2E+1)}{E}x^3$$

$$\Rightarrow (E-2+E^{-1})x^3$$

$$\Rightarrow Ex^3 - 2x^3 + E^{-1}x^3$$

$$\Rightarrow (x+h)^3 - 2x^3 + (x-h)^3$$

$$\Rightarrow \cancel{x^3} + \cancel{h^3} + \cancel{3x^2h} + 3xh^2 - \cancel{2x^3} \ldots$$

$$\cancel{x^3} - \cancel{3x^2h} + 3xh^2 - \cancel{h^3}$$

$$\Rightarrow \boxed{6xh^2}$$

Example 3.10

$$e^x = \frac{\Delta^2}{E}e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$$

$$\Rightarrow (E-2+E^{-1})e^x \cdot \frac{e^{x+h}}{\Delta(e^{x+h}-e^x)}$$

$$\text{"} \quad \cdot \quad \frac{e^{x+h}}{e^{x+2h}-e^{x+h}-e^{x+h}+e^x}$$

$$\text{"} \quad \cdot \quad \frac{e^h}{(e^h-1)^2}$$

$$\Rightarrow (E-2+E^{-1})e^x \cdot \frac{e^h}{(e^h-1)^2}$$

$$\Rightarrow (e^{x+h}-2e^x+e^{x-h}) \cdot \frac{e^h}{(e^h-1)^2}$$

$$\Rightarrow e^x(e^{2h}-2e^h+1) \cdot \frac{1}{(e^h-1)^2}$$

$$\Rightarrow e^x\frac{\cancel{(e^h-1)^2}}{\cancel{(e^h-1)^2}} = \boxed{e^x}$$

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Numerical Methods

Backward Difference
(Itself minus previous)

$$\nabla y_n = y_n - y_{n-1}$$

$\rightarrow$ $\nabla f(x+h) = f(x+h) - f(x) = \Delta f(x)$

$\rightarrow$ Generalization: $\nabla f(x+nh) = \Delta f(x+(n-1)h)$

$\hookrightarrow$ 3rd $\nabla f(x+3h) = f(x+3h) - f(x+2h)$

$= \Delta f(x+2h)$

$\Rightarrow$ $\nabla^2 f(x+2h) = \nabla(f(x+2h) - f(x+h))$

$= [f(x+2h) - f(x+h)] - \ldots$

$[f(x+h) - f(x)]$

$= f(x+2h) - 2f(x+h) + f(x)$

$= \Delta^2 f(x)$

$\Rightarrow$ $\nabla^n f(x+nh) = \Delta^n f(x)$

Relation with E

$\nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1} f(x)$

$\nabla = 1 - E^{-1}$

Express polynomial in Factorial notation

$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_r x^r$

$\Delta f(x) = \cancel{a_0} + a_1(x+h) + a_2(x+h)^2 + \ldots\ldots$

$- [\cancel{a_0} + \cancel{a_1 x} + \cancel{a_2} x^2 + \ldots]$

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$= a_1 + 2a_2x + \cdots + ra_rx^{(r-1)}$

$\Delta^r f(x) = a_r r!$

Example 3.19

$f(x) = 3x^3 + x^2 + x + 1 \;;\; h = 1$

From slides

Example 3.20

$f(x) = 3x^3 - 4x^2 + 3x - 11 \;;\; h = 1$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	-11			
		2		
			10	
1	-9			18
		12		
			28	
2	3			
		40		
3	43			

$a_0 = -11$

$a_1 = 2$

$a_2 = 5$

$a_3 = 3$

$\Rightarrow (1-x)^2 = 1 + x^2 - 2x$

$\Rightarrow (1-x)^3 = 1 - x^3 - 3x + 3x^2$

$\Rightarrow (1-x)^4 = 1 - x^3 - 3x + 3x^2 \;\ldots$

$\quad - x + x^4 + 3x^2 - 3x^3$

$= x^4 - 4x^3 + 6x^2 - 4x + 1$

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Numerical Methods

Example 5.9

Unnecessary for cubic polynomial

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	25					
		-4				
1	21		1			
		-3		2		
2	18		3		4	
		0		6		-10
3	18		9		-6	
		9		0		10
4	27		9		4	
		18		4		-5
5	45		13		-1	
		31		3		
6	76		16			
		47				
7	123					

$$\frac{\text{Sum } \Delta^3 y}{\text{Total } n} = \frac{15}{5} = 3 = \frac{2+6+0+4+3}{5}$$

$$(1-x)^3 = 1 - 3x + 3x^2 \pm x^3$$

$$\left.\begin{aligned} &\Rightarrow \varepsilon(1) + 3 = 2 \\ &\Rightarrow \varepsilon(-3) + 3 = 6 \\ &\Rightarrow \varepsilon(3) + 3 = 0 \\ &\Rightarrow \varepsilon(-1) + 3 = 4 \end{aligned}\right\} \varepsilon = -1 \rightarrow y = 17$$

Newton's FD Interpolation

$x_i = x_0 + ih$

$\hookrightarrow a_n = \dfrac{\Delta^n y_0}{h^n n!}$

$\hookrightarrow x_0 + ph = x$ or $p = \dfrac{x - x_0}{h}$

ay/date

$$y_n(x) = y_0 + \frac{p\,\Delta y_0}{1!} + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots$$

$$\cdots \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}\Delta^n y_0$$

Example

$$y_n(x) = 24 + p(96) + \frac{p(p-1)}{2!}(120) + \cdots$$

$$\frac{p(p-1)(p-2)}{3!}(48)$$

$x = x_0 + ph \quad ; \; h = 2 \; ; \; x_0 = 1$

$p = \dfrac{x-1}{2}$

$$\Rightarrow \quad y_n(x) = 24 + \frac{(x-1)}{2}(96) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)(120)}{2!}$$

$$+ \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)(48)}{3!}$$

$$= x^3 + 6x^2 + 11x + 6$$

y/date

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Numerical Methods

Interpolation

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \dots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

→ Backward Difference

$$a_0 = y_n \; ; \; a_1 = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} \; ; \; a_n = \frac{\nabla^n y_n}{h^n n!}$$

→ $x = x_n + ph$

→ $y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n$

$\dots \; \nabla^n y_n \frac{p(p+1)(p+2)\dots(p+n-1)}{n!}$

Task $\quad p = (x-6)/1 = x - 6$

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$
3	13			
		-8		
4	21		2	
		-10		0
5	31		2	
		-12		
6	43			

Me confused

$\Delta y_0 = \nabla y_1$

Do it from home

$$y_n(x) = 43 + (x-6)(-12) + \frac{(x-6)(x-5)(2)}{2!} \dots + \cancel{0}$$

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$$= 43 + 72 - 12x + \frac{x^2 - 11x + 30}{2!} \cancel{(2)}$$

$$= x^2 - 11x - 12x + 145$$

$$= x^2 - 23x + 145$$

$$y(5.5) = \boxed{48.75}$$

Most likely wrong

Divided Difference Δ

$\Rightarrow \quad f(x_0, x_1) = \dfrac{f(x_0) - f(x_1)}{x_0 - x_1}$

$\Rightarrow \quad f(x_1, x_2) = \dfrac{f(x_1) - f(x_2)}{x_1 - x_2}$

Both of order one

$$\hookrightarrow \quad f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2}$$

$f(x) = \dfrac{1}{x}$; $f(a, b) = ?$

$f(a, b, c) = ?$

$$\Rightarrow \quad f(a, b) = \frac{f(a) - f(b)}{a - b} = \frac{\frac{1}{a} - \frac{1}{b}}{a - b}$$

$$= \frac{(b - a)/ab}{a - b}$$

$$= \frac{-1}{ab}$$

Similarly, $f(b, c) = -\dfrac{1}{bc}$

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$$f(a,b,c) = \frac{f(a,b) - f(b,c)}{a-c}$$

$$= \frac{-1/ab + 1/bc}{a-c}$$

$$= \frac{\frac{a-c}{abc}}{a-c} = \frac{1}{abc}$$

Example

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	0				
		9			
3	18		31/3		
		40		-1/3	
4	58		26/3		157/756
		66		43/28	
6	190		233/12		
		182.5			
10	920				

Newton's Divided Difference Formula

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) \dots + f(x_0, x_1, x_2 \dots x_n)(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

Example

x	$f(x)$
5	150
7	392
11	1452
13	2366
21	9702

~ From slides

EXCELLENT

te

$f(x_0) = 150$

$f(x_0, x_1) = 121$

$f(x_0, x_1, x_2) = 24$

$f(x_0, x_1, x_2, x_3) = 1$

$$f(x) = 150 + (x-5)(121) + (x-5)(x-7)(124)$$
$$+ (x-5)(x-7)(x-11)(1)$$
$$= 252$$

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Numerical Methods

Lagrange Interpolation Formula

$$P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

nth degree

where $L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \dfrac{x - x_j}{x_i - x_j}$

Suppose $P_3(x) = \sum_{i=0}^{3} \prod_{\substack{j=0 \\ j \neq i}}^{3} \dfrac{x - x_j}{x_i - x_j} f(x_i)$

Expansion

$$\left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)\left(\frac{x - x_3}{x_0 - x_3}\right) f(x_0) \ldots$$

$$+ \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right) f(x_1) \ldots$$

so on

Example 5.5

$(0, -12), (1, 0), (3, 6), (4, 12)$

$n = 2 \quad \{ x_0 = 0 \quad x_1 = 1 \quad x_2 = 3 \quad x_3 = 4 \}$

$$P_3(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)\left(\frac{x - x_3}{x_0 - x_3}\right) f(x_0) \ldots$$

$$+ \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right) f(x_1) \ldots$$

$$+ \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right) f(x_2) \ldots$$

$$+ \left(\frac{x - x_0}{x_3 - x_0}\right)\left(\frac{x - x_1}{x_3 - x_1}\right)\left(\frac{x - x_2}{x_3 - x_2}\right) f(x_3)$$

EXCELLENT

$$= \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}(-12)$$

$$+ \cancel{\frac{(x-0)(x-3)(x-4)}{}(0)}$$

$$+ \frac{(x-0)(x-1)(x-4)}{(3)(2)(-1)}(6)$$

$$+ \frac{(x-0)(x-1)(x-3)}{(4)(3)(1)}(12)$$

$$= (x-1)(x-3)(x-4) - (x)(x-1)(x-4)$$
$$+ (x)(x-1)(x-3)$$

$$= (x^2-4x+3)(x-4) - (x^2-x)(x-4)$$
$$+ (x^2-x)(x-3)$$

$$= x^3 - 4x^2 + 3x - 4x^2 - 16x - 12$$
$$- (x^3 - x^2 - 4x^2 + 4x) \ldots$$
$$+ (x^3 - x^2 - 3x^2 + 3x)$$

$$= x^3 - 7x^2 - 14x - 12$$

Example (Using same expansion)

$x_0 = 5 \quad x_1 = 6 \quad x_2 = 9 \quad x_3 = 11$

$$P_3(x) = \frac{(x-6)(x-9)(x-11)}{(-1)(-4)(-6)}(12) \ldots$$

$$+ \frac{(x-5)(x-9)(x-11)}{(1)(-3)(-5)}(13) \ldots$$

$$+ \frac{(x-5)(x-6)(x-11)}{(4)(3)(-2)}(14) \ldots$$

$$+ \frac{(x-5)(x-6)(x-9)}{(6)(5)(2)}(16)$$

$$= -\frac{(x^2-15x+54)(x-11)}{2} \cdots$$

$$+ \frac{(x^2-14x+45)(x-11)}{15}(13)$$

$$+ \frac{(x^2-11x+30)(x-11)}{12}(-7)$$

$$+ \frac{(x^2-11x+30)(x-9)}{15}(4)$$

$$= \left\{ -\frac{1}{2}(x^3-15x^2+54x-11x^2+165x-594) + \frac{13}{15} \quad \text{Just substitute above} \right\}$$

$$f(10) = \boxed{\frac{44}{3}}$$

Example

x	f	Cf	
30 - 40	a	a	< 40
40 - 50	b	a+b	< 50
50 - 60	c	a+b+c	< 60
60 - 70	d	a+b+c+d	< 70

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Spline Interpolation (Numerical Methods)

└ Piecewise Interpolation

$$\left[S_n(x) = a_n + b_n(x - x_n) \right]$$ Linear Family splines

n points : → (n − 1) splines

└ 2n unknowns : 2n equations

$$\left[S_n(x) = a_n + b_n(x - x_n) + c_n(x - x_n)^2 \right]$$

→ Quadratic Splines Family

└ 3n unknowns : 3n equations

2^n equations out of 3^n equations are found using linear step

---

Example (1, −8), (2, −1), (3, 18)

n = 2 } 2 splines

$S_0(x) = a_0 + b_0(x - 1) + c_0(x - 1)^2$

$S_1(x) = a_1 + b_1(x - 2) + c_1(x - 2)^2$

→ $S_0(1) = -8 = a_0$ — i

→ $S_1(2) = -1 = a_1$ — ii

→ $S_0(2) \Rightarrow -1 = -8 + b_0 + c_0$

$\Rightarrow b_0 + c_0 = 7$ — iii

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$S_1(3) \Rightarrow 18 = -1 + b_1 + c_1$

$\Rightarrow b_1 + c_1 = 19$ — iv

$$\frac{d}{dx} S_0(x)\Big|_{x=2} = \frac{d}{dx} S_1(x)\Big|_{x=2}$$

$\Rightarrow \quad b_0 + 2c_0(x-1)\Big|_{x=2} = b_1 + 2c_1(x-2)\Big|_{x=2}$

$\Rightarrow \quad b_0 + 2c_0 = b_1$

$\Rightarrow \quad b_0 - b_1 + 2c_0 = 0$ — v

Force $c_0 = 0$

$b_0 = b_1$; $\underbrace{b_0 = 7}_{\text{from iii}}$; $\underbrace{c_1 = 12}_{\text{from iv}}$

$$\left\{ S_n(x) = a_n + b_n(x - x_n) + c_n(x-x_n)^2 \dots + d_n(x-x_n)^3 \right\}$$

↳ cubic spline

└ $4n$ unknowns : $4n$ equations

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Numerical Methods

Numerical Differentiation

{Forward}

$$\frac{df}{dx} = \frac{1}{h}\left(\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 \dots\right)$$

$$\frac{d^2f}{dx^2} = \frac{1}{h^2}\left(\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \dots\right)$$

{Backward}

$$\frac{df}{dx} = \frac{1}{h}\left(\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n \dots\right)$$

$$\frac{d^2f}{dx^2} = \frac{1}{h^2}\left(\nabla^2 y_n + (p+1)\nabla^3 y_n + \dots\right)$$

Example

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$
1.5	3.375	3.265	3	0.75	-3	15
2.0	7.0	6.625	3.75	-2.25	12	
2.5	13.625	10.375	1.5	9.75		
3.0	24	11.875	11.25			
3.5	36.875	23.125				
4.0	59					

$$\frac{df}{dx} = 2\left(3.265 + \frac{2p-1}{2!}(3) + \frac{3p^2-6p+2}{3!}(0.75) \dots\right)$$

$$p = 0 \Rightarrow 2\left(3.265 - \frac{3}{2} + \frac{2}{3!}(0.75)\right)$$

$$\left.\frac{df}{dx}\right|_{x=1.5} = \boxed{4.75}$$

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Example

x	y	Δy_0	$\Delta^2 y_0$
0	0		
		32	
25	32		-6
		26	
50	58		-6
		20	
75	78		-6
		14	
100	92		-6
		8	
125	100		

a) velocity at $t = 25$

$$\left.\frac{df}{dx}\right|_{x=25} = \frac{1}{25}\left(32 + \frac{1}{2}(-6)\right)$$

$$= 1.16 \text{ km/s}$$

b) at $t = 50$

$$\left.\frac{df}{dx}\right|_{x=50} = \frac{1}{25}\left(32 + \frac{3}{2}(-6)\right)$$

$$= 0.92 \text{ km/s}$$

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Numerical Methods

$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots\right]$$

$$D^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \dots\right]$$

$$D^3 = \frac{1}{h^3}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \frac{7}{4}\Delta^5 - \dots\right]$$

Numerical Integration

$$\int_{x_0}^{x_n} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \dots\right]$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n\right]$$

Trapezoidal Rule

Simpson's Rule

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + \dots y_{n-1}) + \dots 2(y_2 + y_4 + \dots y_{n-2}) + y_n\right]$$

Problem

$$\int_{0}^{20} y\,dx = \frac{2}{3}\left[0 + 4(16 + 40 + 51 + 18 + 3) + 2(29 + 46 + 32 + 8) + 0\right]$$

$$= 494.667$$

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Numerical Methods

Simpson's 3 - 8 Rule

$$I = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 \cdots y_{n-1}) + 2(y_3 + y_6 + \cdots y_{n-3})\right]$$

Example 9.7

$$\int_0^1 \frac{1}{1+x^2}\,dx \qquad h = \frac{b-a}{n} = \frac{1}{6}$$

x	y
0	1
1/6	0.972
2/6	0.9
3/6	0.8
4/6	0.692
5/6	0.590
1	0.5

$$I = \frac{3(1/6)}{8}\left[1 + 0.5 + 3(0.972 + 0.9 + 0.692 + 0.590) + 2(0.8)\right]$$

$$= 0.785$$

Example

$$\int_0^{\pi/2} e^{\sin x}\,dx \qquad h = \pi/6$$

x	y
0	1
$\pi/6$	1.64872
$\pi/3$	2.37744
$\pi/2$	2.71828

$$I = \frac{3(\pi/6)}{8}\left[1 + 2.71828 + 3(1.64872 + 2.37744)\right]$$

$$= 3.10169$$

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Example $v = ds/dt \rightarrow s = \int_0^{20} v\,dt$

t	v
0	16
2	28.8
4	40
6	46.4
8	51.2
10	32
12	17.6
14	8
16	3.2
18	0
20	

$h = \frac{20}{10} = 2 \text{ min} \Rightarrow \frac{1}{30} \text{ hrs}$

$$I = \frac{1}{30.3}\left[0 + 0 + 4(16 + 40 + 51.2 + 17.6 + 3.2) + 2(28.8 + 46.4 + 32 + 8)\right]$$

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Numerical Methods

System of Linear Equations

- Consistent
  - Homogeneous
  - Non-homogeneous
- Non-consistent

» Consistent — Unique / Infinitely many } solution

» Inconsistent — No } solution

- Homogeneous

Rank(coefficient matrix A) : number of non-zero rows in echelon form

» If Rank(A) = n , system has a trivial solution and det(A) must not be = 0.

» If Rank(A) = n and det(A) = 0 , system have non trivial unique solution.

» If Rank(A) < n , system has infinitely many solutions.

- Non-homogeneous (C. Augmented matrix)

» If Rank(A) = Rank(C) = n ; consistent and unique solutions

» If Rank(A) = Rank(C) < n ; consistent and infinitely many solutions

» Inconsistent elsewise

Example

$$\underline{\text{At A}}: \quad 360 = x + t$$

$$\underline{\text{At B}}: \quad 250 + x = y$$

$$\underline{\text{At C}}: \quad t = 290 + z$$

$$\underline{\text{At D}}: \quad y = z + 250$$

→ $AX = B$

∴ $\boxed{A = LU}$ {Lower and Upper Δ}

→ $\boxed{LUX = B}$

→ $\boxed{LY = B}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

<u>Doolittle</u>

<u>Example</u>

$$A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

$u_{11} = 2$ $\quad u_{12} = -3$ $\quad u_{13} = 10$

$l_{21} u_{11} = -1$ $\quad l_{21} u_{12} + u_{22} = a_{22}$

$l_{21} = -1/2$ $\quad u_{22} = 5/2$

$l_{21} u_{12} + u_{23} = 2 \Rightarrow u_{23} = 7$

$l_{31} u_{11} = 5$ $\quad l_{32} = 19/5$

$l_{31} = 5/2$ $\quad u_{33} = -253/5$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 5/2 & 19/5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

$|L| = 1$ $\quad y_1 = \begin{vmatrix} 3 & 0 & 0 \\ 20 & 1 & 0 \\ -12 & 19/5 & 1 \end{vmatrix}$

$= 3$

$y_2 = 13/2$ $\quad y_3 = -506/5$

Do same and finally find [X]

y/date

Example 2

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

$l_{11} = 1$ $\quad$ $l_{21} = 2$ $\quad$ $l_{31} = 3$

$u_{12} = 1$ $\quad$ $u_{13} = 1$ $\quad$ ~ Rest from slides.

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Numerical Methods

Cholesky's Method

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ or } \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

$$A = LL^T$$

$$A = \begin{bmatrix} a^2 & ab & ad \\ ab & b^2+c^2 & bd+ce \\ ad & bd+ce & d^2+e^2+f^2 \end{bmatrix}$$

Example

$$\underset{A}{\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}}$$

$$A = LL^T$$

$a = 2$ $\quad$ $b = 5$ $\quad$ $d = 4$

$c^2 = 26 - 5^2 = 1 \Rightarrow c = 1$

$e = 6$ $\quad$ $f = 3$

$LL^TX = B$ $\quad$ $L^TX = Y$ $\quad$ $LY = B$

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$$\begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

$y_1 = 22$

$y_2 = 18$

$y_3 = 6$

$$\begin{bmatrix} 2 & 5 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 22 \\ 18 \\ 6 \end{bmatrix}$$

$$\boxed{\begin{aligned} x &= -8 \\ y &= 6 \\ z &= 2 \end{aligned}}$$

~ Solution

/date

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Numerical Methods

Iterative Methods (for $AX = B$)

→ Jacobi Method

Leading entries must be greather than other row entries

$$a_1 x + b_1 y + c_1 z = d_1 \quad \text{—i}$$
$$a_2 x + b_2 y + c_2 z = d_2 \quad \text{—ii}$$
$$a_3 x + b_3 y + c_3 z = d_3 \quad \text{—iii}$$

→ Condition : $|a_1| > |b_1| + |c_1|$ .... etc.

→ from i → $x = \dfrac{d_1 - b_1 y - c_1 z}{a_1}$

similarly → $y = \dfrac{d_2 - a_2 x - c_2 z}{b_2}$

$z = \dfrac{d_3 - a_3 x - b_3 y}{c_3}$

can be generalized

If no initial condition, use $(0,0,0)$

Example

$$8x - 3y + 2z = 20$$
$$6x + 3y + 12z = 35$$
$$4x + 11y - z = 33$$

swap 2nd and 3rd row

$$x_{r+1} = \frac{20 + 3y_r - 2z_r}{8}$$

$$y_{r+1} = \frac{33 + z_r - 4x_r}{11}$$

$$z_{r+1} = \frac{35 - 3y_r - 6x_r}{12}$$

$x_0 = 0 \quad y_0 = 0 \quad z_0 = 0$

$\Rightarrow x_1 = 20/8 \qquad y_1 = 33/11 \qquad z_1 = 35/12$

$\Rightarrow x_2 = 139/48 \qquad y_2 = 311/132 \qquad z_2 = 11/12$

$\Rightarrow x_3 = 3331/1056 \qquad y_3 = 67/33 \qquad z_3 = 929/1056$

$\Rightarrow x_4 = 12847/4224 \qquad y_4 = \sim \qquad z_4 = \sim$

Continue until
~ desired decimal
places

Vector Norms

$\vec{x} = [x_1, x_2, \dots, x_n]^t$ n col-vector

$$\|x\|_2 = \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{1/2}$$

$$\|x\|_\infty = \max_{1 \le i \le n} (x_i)$$

Gauss - Seidel Method

$$x_{r+1} = \frac{1}{a_1}(d_1 - b_1 y_r - c_1 z_r)$$

$$y_{r+1} = \frac{1}{b_1}(d_2 - a_2 x_{r+1} - c_2 z_r)$$

$$z_{r+1} = \frac{1}{c_1}(d_3 - a_3 x_{r+1} - b_3 y_{r+1})$$

Example Re $(x_0 = 0,\ y_0 = 0,\ z_0 = 0)$

$\Rightarrow$ $x_1 = 20/8$ $\quad y_1 = 2.09091 \quad z_1 = 1.14394$

$= 2.5$

$\Rightarrow$ $x_2 = 2.9908 \quad y_2 = 2.0137 \quad z_2 = 0.9142$

$\Rightarrow$ $x_3 = 3.02659 \quad y_3 = 1.98253 \quad z_3 = 0.90774$

$\Rightarrow$ $x_4 = 3.01651 \quad y_4 = 1.9852 \quad z_5 = 0.91211$

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Numerical Methods

Differential Equations

- ODE — when only one independent variable
  $dy/dx + y = 0$
- PDE — more than one i.v.
  $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$

Order and Degree

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 = 0$$ 1st degree, 2nd order

→ $y' + P(x)y = Q(x)$ — i

→ $y' + P(x)y = Q(x)y^n$ $n > 1$ — ii

— i : Linear, Non-Homogeneous, 1st order

— ii : Non-linear, Homogeneous, n th degree, 1st order

Taylor's Series Method

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \dots$$

Example from slide

→ Another form

ay/date

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$$

Euler's Method

$$y_{n+1} = y_n + h f(x_n, y_n) \qquad n = 0, 1, 2, \ldots N-1$$

first order IVP $\rightarrow \dfrac{dy}{dx} = f(x, y)$

Example from slides

Modified Euler Method

$$y_{n+1} = y_n + \frac{1}{2} h [f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]$$

where $y^*_{n+1} = y_n + h f(x_n, y_n)$

Runge Kutta

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' \ldots$$

RK1 RK2 RK3 ...

$\rightarrow$ RK2 : $y_n + \frac{1}{2}[k_1 + k_2]$; $n = 0, 1, 2, 3 \ldots$

$K_1 = h f(x_n, y_n)$

$K_2 = h f(x_n + h, y_n + k_1)$

Generally, RK3 : $y_{n+1} = y_n + 1/6 (k_1 + 4k_2 + k_3)$

$n = 0, 1, 2, 3 \ldots$

$$\begin{cases} K_1 = h f(x_n, y_n) \\ K_2 = h f(x_n + h/2, y_n + k_1/2) \\ K_3 = h f(x_n + h, y_n + h f(x_n + h, y_n + k_1)) \end{cases}$$

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Numerical Methods

Q: $\frac{dy}{dx} = \frac{2-y^2}{5x}$

$y(4)=1$, compute $y(4.4)$; $h=0.2$

$$y_{n+1} = y_n + \frac{1}{2}(k_1+k_2)$$

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n+h, y_n+k_1)$$

$$\Rightarrow k_1 = 0.2\left[\frac{2-1}{20}\right] = \frac{1}{100}$$

$$k_2 = 0.2\left[\frac{2-(1+1/100)^2}{5(4.2)}\right] = 0.00933$$

$$y(4.2) = 1 + \frac{1}{2}[1/100 + 0.00933]$$

$$= 1.00967$$

$$\Rightarrow k_1 = 0.2\left[\frac{2-(1.00967)^2}{5(4.2)}\right] = 0.00934$$

$$k_2 = 0.2\left[\frac{2-(1.00967+0.00934)^2}{5(4.4)}\right]$$

$$= 0.00874$$

$$\boxed{y(4.4) = 1.01871}$$

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Numerical Methods

$$x' = f(t, x, y)$$
$$y' = g(t, x, y)$$
$$x(t_0) = x_0 \qquad y(t_0) = y_0$$

Euler's Method: $y_{n+1} = y_n + h(g(t_n, x_n, y_n))$

$x_{n+1} = x_n + h(g(t_n, x_n, y_n))$

Example

Taylor Series:

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' \dots$$

$$x_{n+1} = x_n + h x_n' + \frac{h^2}{2!} x_n'' + \frac{h^3}{3!} x_n''' \dots$$

Q: $h = 0.1$ Use order 2 to approximate $x(0.2)$ $y(0.2)$

$$\begin{cases} x' = 6x + y + 6t \\ y' = 4x + 3y - 10t + 4 \\ x(0) = 0.5 \qquad y(0) = 0.2 \end{cases}$$

$\Rightarrow x'' = 6x' + y' + 6$

$\Rightarrow y'' = 4x' + 3y' - 10$

$x'(0) = 3.2 \qquad y'(0) = 6.6$

$x''(0) = 31.8 \qquad y''(0) = 22.6$

$$x_1 = x_0 + h x_0' + \frac{h^2}{2!} x_0'' = 0.979$$

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' = 0.973$$

⋮ Continue until $\{x_2, y_2\}$

IGNORE

$$\left[\begin{array}{l} \longrightarrow f(x+a) = f(a) + x f'(a) + \frac{x^2 f''(a)}{2!} \\ \qquad + \frac{x^3 f'''(a)}{3!} + \cdots \\ \longrightarrow f(x_0 + h) \sim \sim \end{array} \right\}$$

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Numerical Methods

RK4:

Example: $d^2y/dx^2 - x^2 dy/dx + 2xy = 1$

Let $\frac{dy}{dx} = z = f(x, y, z)$

$\frac{dz}{dx} = x^2 z - 2xy + 1 = g(x, y, z)$

$\Rightarrow \quad y_{n+1} = y_n + \sim$ Refer to slides

BVPs

$y(x+h) = y(x) + h y'(x) + \frac{h^2 y''(x)}{2!} + \dots$

$y(x-h) = y(x) - h y'(x) + \frac{h^2 y''(x)}{2!} + \dots$

$y(x+h) - y(x-h) = 2h y'(x)$

$$y'(x) = \frac{y(x+h) - y(x-h)}{2h}$$

Similarly, adding gives:

$y(x+h) + y(x-h) = 2y(x) + h^2 y''(x)$

$$y''(x) = \frac{y(x+h) + y(x-h) - 2y(x)}{h^2}$$

ate

Generalized FDM

Too fast ;-;

PDEs : Elliptic

Laplace   Poisson

Example

$$f_{xx} + f_{yy} = 0 \quad , \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

$$f(0,y) = 0 \quad , \quad f(x,0) = 0$$

$$f(1,y) = 100 \quad , \quad f(x,1) = 100 \qquad h = 1/3$$