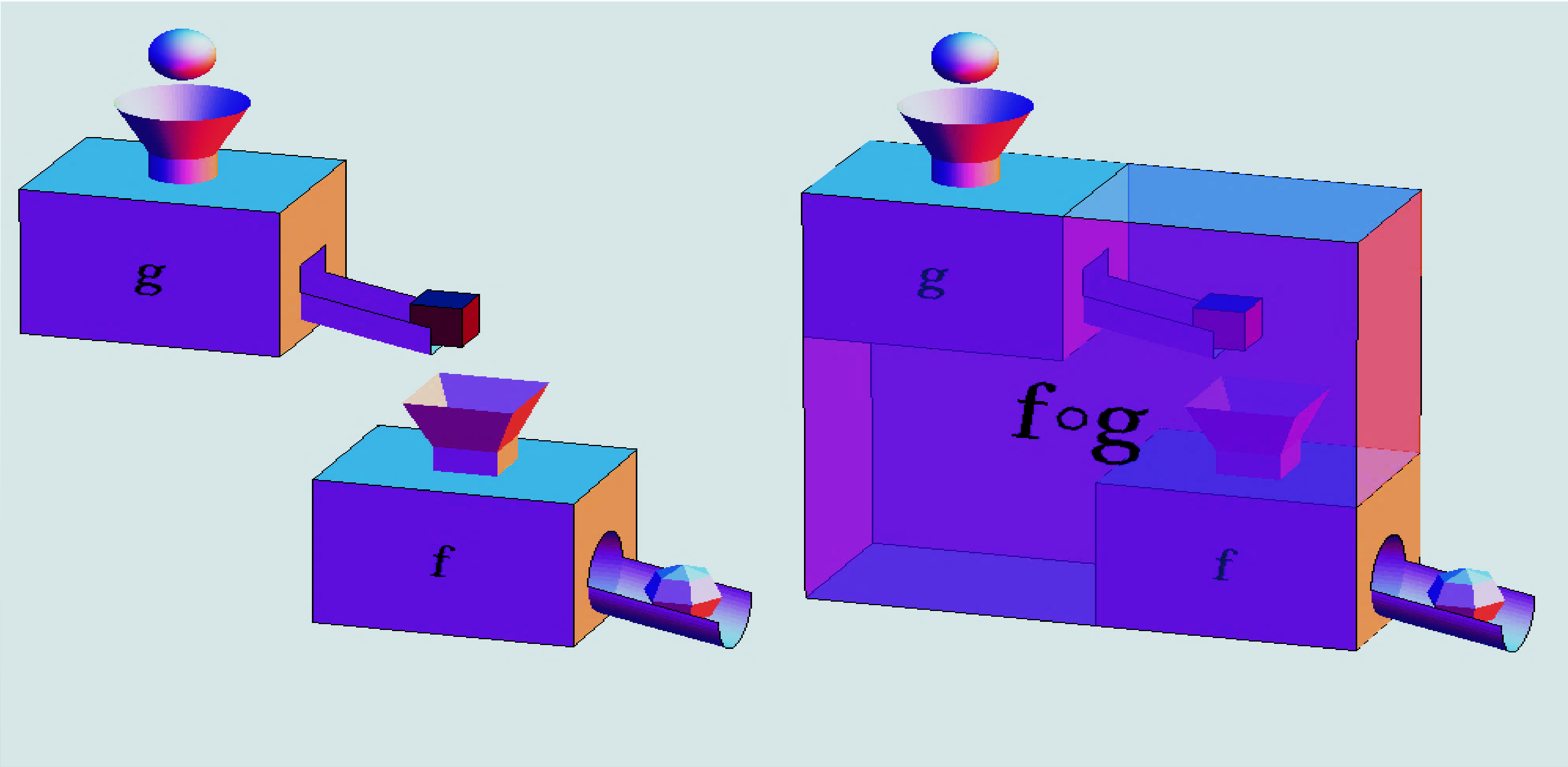
Derivatives

Calculus & Analytical Geometry MATH- 101

Instructor: Dr. Naila Amir (SEECS, NUST)

The Chain Rule



Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 3
  - Sections: 3.5

The Chain Rule

- Suppose we are asked to differentiate the function:

$$F(x) = \sqrt{x^2 + 1}$$

- Observe that $F(x)$ is a composite function. In fact, if we let $f(x) = \sqrt{x}$ and let $g(x) = x^2 + 1$, then we can write $F(x) = f(g(x))$, that is, $F(x) = (f \circ g)(x)$.
- We know how to differentiate both $f(x)$ and $g(x)$, but how do we differentiate a composite like $F(x)$?
- The differentiation formulas that we have studied so far do not tell us how to calculate $F'(x)$.
- So it would be useful to have a rule that tells us how to find the derivative of $F(x) = (f \circ g)(x)$ in terms of the derivatives of $f(x)$ and $g(x)$.
- The chain rule is one of the most important and widely used rules of differentiation.

The Chain Rule

- It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the ***chain rule.***
- It seems plausible if we interpret derivatives as rates of change. Regard du/dx as the rate of change of u with respect to x, dy/du as the rate of change of y with respect to u, and dy/dx as the rate of change of y with respect to x. If u changes twice as fast as x and y changes three times as fast as u, then it seems reasonable that y changes six times as fast as x, and so we expect that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

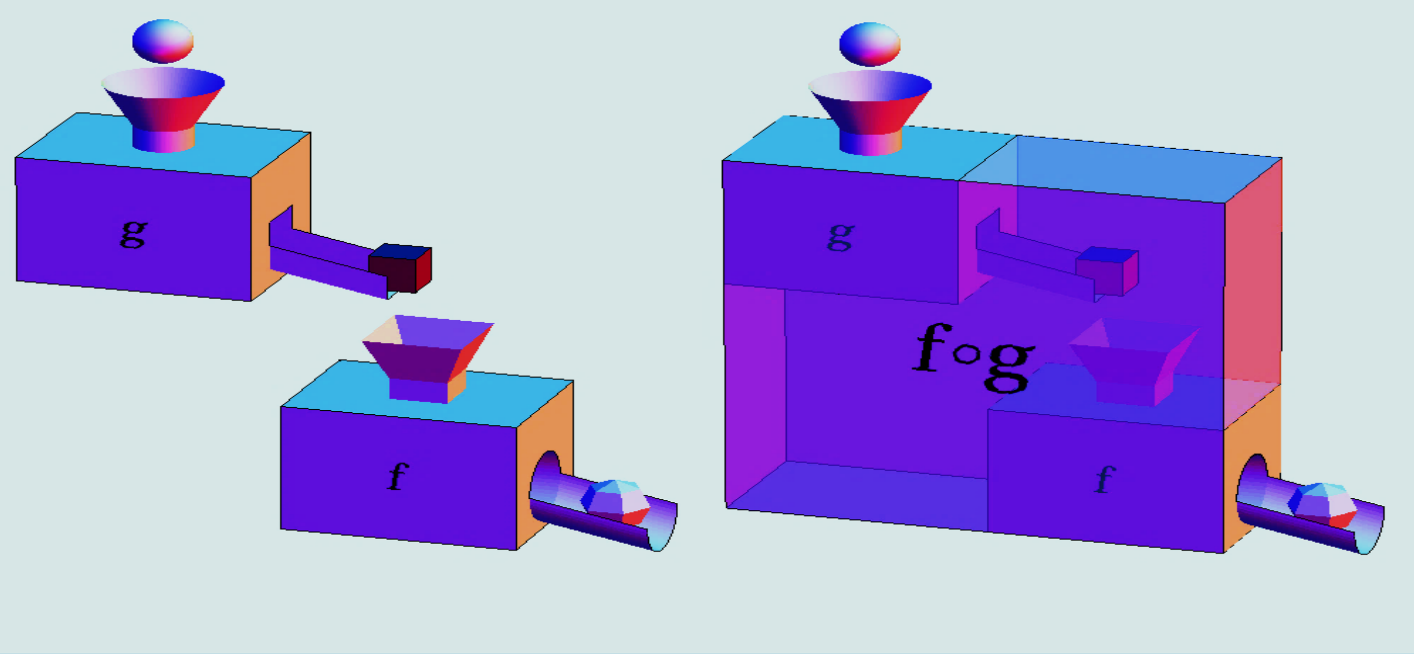
The Chain Rule

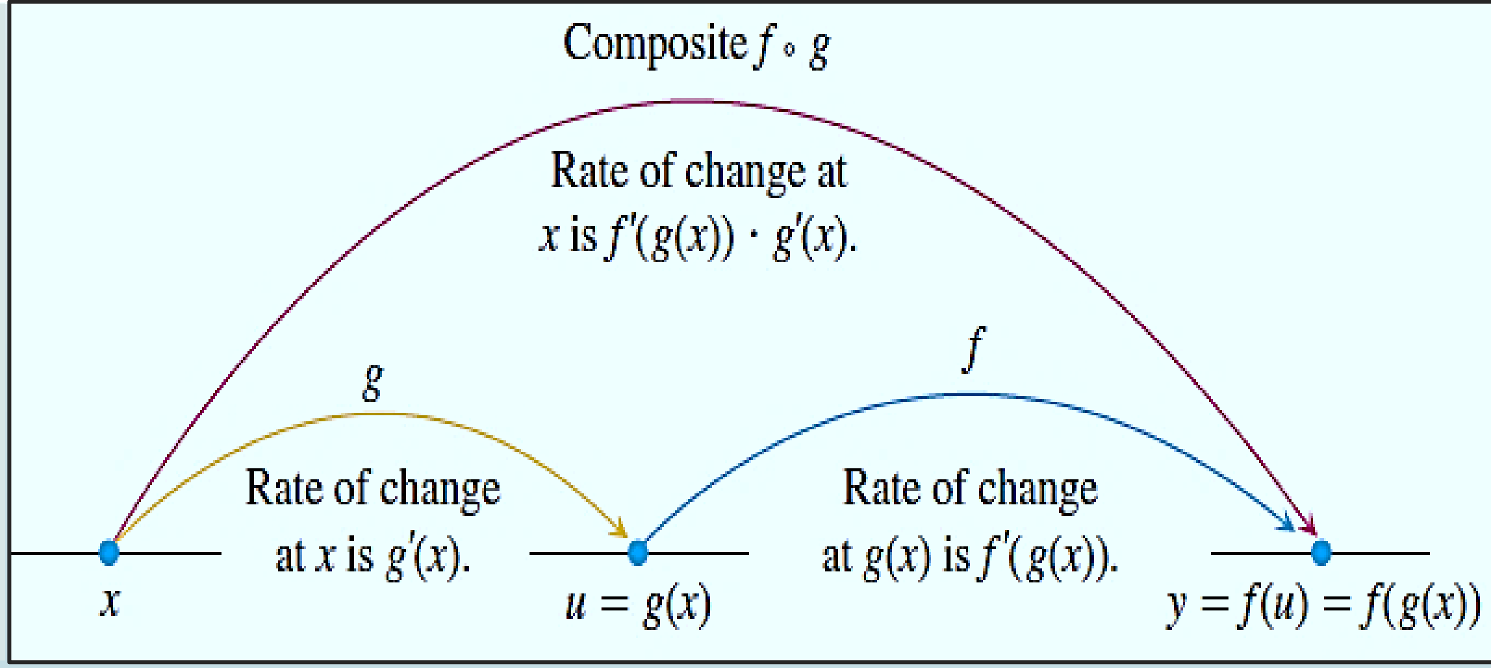
The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$





Example:

Determine $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

Solution:

We can express $F(x)$ as $F\,(x) = (f\ \circ g)(x) = f\,(g(x))$ where $f\,(u) = \sqrt{u}$ and $g\,(x)\ =\ x^2\ +\ 1$. Since

$$f'(u) = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad g'(x) =\ 2x,$$

we have

$$F'(x) = f'\big(g\,(x)\big)\cdot g'(x) = \frac{1}{2\sqrt{x^2\ +\ 1}}.\,2x = \frac{x}{\sqrt{x^2\ +\ 1}}.$$

Alternatively, if we let $u = g(x) = x^2 + 1$ and $y = f(u) = \sqrt{u}$, then

$$F'(x) = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x) = \frac{x}{\sqrt{x^2\ +\ 1}}.$$

The Chain Rule

- Let us consider the special case of the Chain Rule where the outer function $f(x)$ is a power function.

- If $y = [g(x)]^n$, where n is any real number, then we can write $y = u^n$, where $u = g(x)$. By using the Chain Rule and then the Power Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} = n[g(x)]^{n-1}g'(x).$$

- Alternatively, we can write it as:

$$\frac{d}{dx}([g(x)]^{n}) = n[g(x)]^{n-1}g'(x).$$

Example:

Differentiate $y = (x^3 - 1)^{100}$.

Solution:

Taking $u = g(x) = x^3 - 1$ and $n = 100$, we get

$$\frac{d}{dx}([g(x)]^n) = n[g(x)]^{n-1} g'(x)$$

$$= 100[x^3 - 1]^{100-1}(3x^2)$$

$$= 300x^2[x^3 - 1]^{99}.$$

Example

Differentiate $f(x) = (1 - \tan^2 x)^{3/2}$

Solution:

$$\begin{aligned}
\frac{df}{dx} &= \frac{d}{dx}(1 - \tan^2 x)^{3/2} \\
&= \frac{3}{2}(1 - \tan^2 x)^{\frac{3}{2}-1}\frac{d}{dx}(1 - \tan^2 x) \\
&= \frac{3}{2}(1 - \tan^2 x)^{\frac{1}{2}}\left(0 - 2\tan x \,.\, \frac{d}{dx}(\tan x)\right) \\
&= \frac{3}{2}(1 - \tan^2 x)^{\frac{1}{2}}(-2\tan x \ \sec^2 x) \\
&= -3(1 - \tan^2 x)^{\frac{1}{2}}\tan x \ \sec^2 x.
\end{aligned}$$

The Chain Rule

We can use the Chain Rule to differentiate an exponential function with any base $a > 0$. Recall that $a = e^{\ln a}$. So

$$a^x = (e^{\ln a})^x = e^{(\ln a)x}$$

and the Chain Rule gives

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{(\ln a)x}\right) = e^{(\ln a)x}\frac{d}{dx}[(\ln a)x] = e^{(\ln a)x}.(\ln a) = a^x \ln a\,.$$

Example:

In particular if $a = 2$, we get

$$2^x = (e^{\ln 2})^x = e^{(\ln 2)x}$$

and the Chain Rule gives

$$\frac{d}{dx}(2^x) = 2^x \ln 2\,.$$

The Chain Rule

- The reason for the name "Chain Rule" becomes clear when we make a longer chain by adding another link.
- Suppose that $y = f(u), u = g(x)$, and $x = h(t)$, where f, g, and h are differentiable functions.
- Then, to compute the derivative of y with respect to t, we use the Chain Rule twice:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt}.$$

Parametric Equations

If x and y are given as functions

$$x = f(t)\,, \qquad y = g(t).$$

Over an interval of t-values, then the set of points $(x, y) = \big(f(t), g(t)\big)$ defined by these equations is a **Parametric Curve.** The equations are **parametric equations** for the curve.

Derivative of Parametric Curves:

A parametric curve $x = f(t)$ and $y = g(t)$ is differentiable at t if f and g are differentiable at t. Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{provided } \frac{dx}{dt} \neq 0.$$

Example

The parametric equations for a curve are given by:

$$x = \mathrm{a}\cos t \quad \text{and} \quad y = b\sin t, \qquad 0 \le t \le 2\pi.$$

Find the line tangent to the curve at $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$.

Solution:

$$\text{slope of tangent} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{bcost}{-asint} = \frac{\frac{b}{a}x}{-\frac{a}{b}y} = -\frac{b^2x}{a^2y}$$

At $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$,

$$\left.\frac{dy}{dx}\right|_{\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)} = -\frac{b^2\frac{a}{\sqrt{2}}}{a^2\frac{b}{\sqrt{2}}} = -\frac{b}{a}.$$

Example

Equation of tangent line is given as:

$$y - \frac{b}{\sqrt{2}} = -\frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right).$$

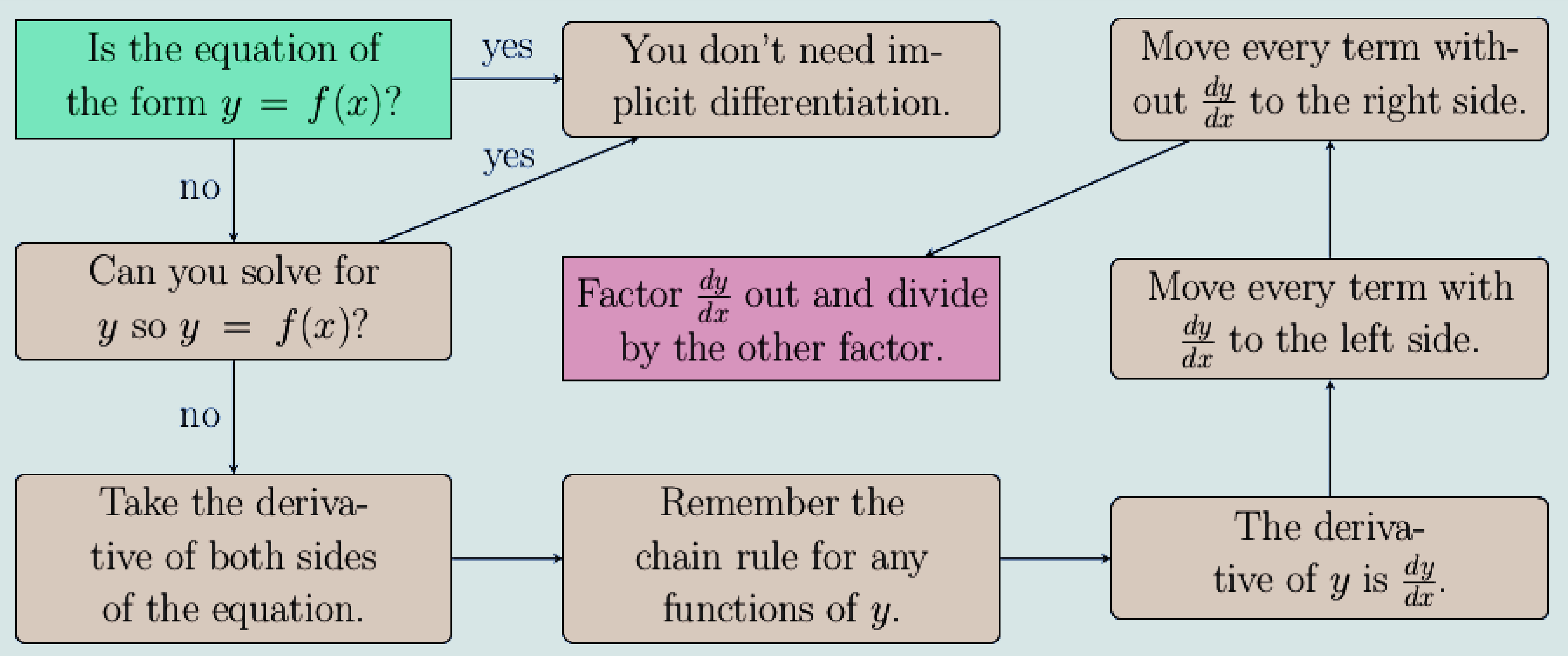
Practice Questions

Book: Thomas Calculus (11th Edition) by Georg B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- **Chapter: 3**
  - **Exercise: 3.5**

    Q # 1 – 66, 87 – 104

Implicit Differentiation



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- Chapter: 3
  - Sections: 3.6

Explicit Functions

If one variable (y) is described as function of other variable (x) then y is said to be described explicitly, i.e.,

$$y = f(x)$$

For example

- $y = \sqrt{x}$
- $y = \frac{1}{x}$
- $y = \frac{x+1}{x-1}$

Implicit Functions

If y is not expressed as function of other variable x then y is said to be described implicitly. In this case equation of a curve is represented by

$$F(x, y) = c$$

For example

$$x^2 + y^2 = c^2$$

Here y can be written explicitly as

$$y = \pm\sqrt{c^2 - x^2}$$

Implicit Differentiation

But in some cases, y can not be written explicitly as a function of x.

For example

$$\sin(xy) + x^2y = 1.$$

In order to differentiate such equations we use Implicit Differentiation.

Guidelines for Implicit Differentiation

In order to differentiate the functions defined implicitly by the equation $F(x, y) = c$, we follow the steps:

1. Differentiate both sides of the equation w.r.t. x (independent variable), treating y (dependent variable) as a differentiable function of x.

2. Collect the terms with $\frac{dy}{dx}$ on one side of the equation.

3. Solve the equation obtained in step 2 to find expression for $\frac{dy}{dx}$.

Example

Determine $\frac{dy}{dx}$ if $y^2 = x^2 + \sin(xy)$.

Solution:

Differentiating both sides w.r.t. x,

$$2y\frac{dy}{dx} = 2x + \cos(xy)\left(y + x\frac{dy}{dx}\right)$$

$$[2y - x\cos(xy)]\frac{dy}{dx} = 2x + y\cos(xy)$$

$$\frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}.$$

Example

Consider the curve:

$$x^3 + y^3 = 3xy.$$

a) Find $\frac{dy}{dx}$.

b) Find equations of tangent and normal lines to the curve at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Solution

a) Differentiate both sides w.r.t. x

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(3xy)$$

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

$$\Rightarrow (3y^2 - 3x)\frac{dy}{dx} = 3y - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}.$$

b) Slope of tangent at $\left(\frac{3}{2}, \frac{3}{2}\right)$ is

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2},\frac{3}{2}\right)} = \frac{\frac{3}{2} - \left(\frac{3}{2}\right)^2}{\left(\frac{3}{2}\right)^2 - \frac{3}{2}} = -1.$$

Solution

$$Slope\ of\ normal\ line = -\frac{1}{Slope\ of\ tangent\ line} = 1.$$

Thus the equation of tangent line is:

$$y - \frac{3}{2} = -1\left(x - \frac{3}{2}\right),$$

and the equation of normal line is given as:

$$y - \frac{3}{2} = 1\left(x - \frac{3}{2}\right).$$

Logarithmic Differentiation

We use Logarithm to differentiate the problems involving:

1. Complicated quotients and products
2. Variable powers of the functions.

Steps to follow:

1. Take natural logarithm on both sides.
2. Using logarithmic properties, write quotients, products and powers as differences, sums and scalar multiples of logarithmic functions.
3. Differentiate both sides using Implicit differentiation.
4. Solve for $\frac{dy}{dx}$.

Example

Find $\frac{dy}{dx}$ if $y = \frac{x\sqrt{x+1}}{\sqrt[3]{x+2}(x+3)^5}$.

Solution:

Taking natural logarithm on both sides of the given function:

$$\ln(y) = \ln\left(\frac{x\sqrt{x+1}}{\sqrt[3]{x+2}(x+3)^5}\right).$$

Using $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

$$\ln(y) = \ln\left(x\sqrt{x+1}\right) - \ln\left(\sqrt[3]{x+2}(x+3)^5\right).$$

Using $\ln(ab) = \ln a + \ln b$

$$\ln(y) = \ln(x) + \ln\left(\sqrt{x+1}\right) - \left[\ln\left(\sqrt[3]{x+2}\right) + \ln\left((x+3)^5\right)\right].$$

Using $\ln\left(a^b\right) = b\ln(a)$,

$$\ln(y) = \ln(x) + \frac{1}{2}\ln(x+1) - \frac{1}{3}\ln(x+2) - 5\ln(x+3).$$

Differentiating w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2(x+1)} - \frac{1}{3(x+2)} - \frac{5}{x+3}.$$

Isolating the terms involving $\frac{dy}{dx}$

$$\frac{dy}{dx} = y\left[\frac{1}{x} + \frac{1}{2(x+1)} - \frac{1}{3(x+2)} - \frac{5}{x+3}\right].$$

Since $y = \frac{x\sqrt{x+1}}{\sqrt[3]{x+2}(x+3)^5}$, so

$$\frac{dy}{dx} = \frac{x\sqrt{x+1}}{\sqrt[3]{x+2}(x+3)^5}\left[\frac{1}{x} + \frac{1}{2(x+1)} - \frac{1}{3(x+2)} - \frac{5}{x+3}\right].$$

Example

Find $\frac{dy}{dx}$ if $y = (\cos(2x))^{x^3}$.

Solution:

Taking natural logarithm on both sides,

$$\ln(y) = \ln\left((\cos\,(2x))^{x^3}\right)$$

Using $\ln(a^b) = b\ln(a)$,

$$\ln(y) = x^3 \ln(\cos(2x))$$

Differentiating w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = 3x^2 \ln(\cos(2x)) + x^3\left(\frac{-2\sin(2x)}{\cos(2x)}\right)$$

$$\Rightarrow \frac{dy}{dx} = (\cos(2x))^{x^3}\left[3x^2 \ln(\cos(2x)) + x^3\left(\frac{-2\sin(2x)}{\cos(2x)}\right)\right].$$

Practice Questions

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- **Chapter: 3**
  - **Exercise: 3.6**

    Q # 1 – 56