WAVE PROPAGATION IN LOSSY DIELECTRICS

Introduction

- Our goal is to derive EM wave motion in the following media:

1. Free space ($\sigma = 0, \varepsilon = \varepsilon_o, \mu = \mu_o$)
2. Lossless dielectrics ($\sigma = 0, \varepsilon = \varepsilon_r \varepsilon_o, \mu = \mu_r \mu_o$ or $\sigma \ll \omega\varepsilon$)
3. Lossy dielectrics ($\sigma \neq 0, \varepsilon = \varepsilon_r \varepsilon_o, \mu = \mu_r \mu_o$)
4. Good conductors ($\sigma \approx \infty, \varepsilon = \varepsilon_o, \mu = \mu_r \mu_o$ or $\sigma \gg \omega\varepsilon$)

- where ω is the angular frequency of the wave

- Case 3, for lossy dielectrics, is the most general case and will be considered first

- Remaining cases derived by changing the values of $\sigma, \varepsilon,$ and μ

Introduction

- Wave propagation in lossy dielectrics is a general case from which wave propagation in other types of media can be derived as special cases

- A lossy dielectric is a medium in which an EM wave loses power as it propagates due to partial conduction

- In other words, a lossy dielectric is a partially conducting medium (imperfect dielectric or imperfect conductor) with $\sigma \neq 0$

- We will consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge free ($\rho_v = 0$)

Maxwell’s Equations in Phasor Form

- The wave equation will be obtained from Maxwell’s equations for time-varying fields

- Maxwell’s equations in the phasor form will be used by removing the time-varying quantity

- Once we have the solution of the vector wave equation, the time varying quantity will be added back

- A sinusoidal current $I(t) = I_o \cos(wt + \theta)$, for example, equals the real part of $I_o e^{j\theta} e^{jwt}$

Maxwell's Equations in Phasor Form

- In performing mathematical operations, we must be consistent in our use of either the real part or the imaginary part of a quantity

- The current $I'(t) = I_o \sin(wt + \theta)$, which is the imaginary part of $I_o e^{j\theta} e^{jwt}$, can also be represented as the real part of $I_o e^{j\theta} e^{jwt} e^{-j90}$ because $\sin \propto = \cos(\propto - 90°)$

- The complex term $I_o e^{j\theta}$, which results from dropping the time factor e^{jwt} in $I(t)$ is called the phasor current, denoted by I_s:

$$I_s = I_o e^{j\theta} = I_0 \angle\theta$$

Maxwell's Equations in Phasor Form

- Thus $I(t) = I_o \cos(wt + \theta)$, the instantaneous form, can be expressed as:

$$I(t) = \mathrm{Re}\,(I_s e^{j\omega t})$$

- If a vector **A**(*x*, *y*, *z*, *t*) is a time-harmonic field, the phasor form of **A** is $\mathbf{A}_s(x, y, z)$; the two quantities are related as:

$$\mathbf{A} = \mathrm{Re}\,(\mathbf{A}_s e^{j\omega t})$$

- For example, if $\mathbf{A} = A_o \cos(wt - \beta x)$ $\mathbf{a}_y$, we can write **A** as:

$$\mathbf{A} = \mathrm{Re}\,(A_o e^{-j\beta x}\,\mathbf{a}_y e^{j\omega t})$$

- Therefore, the phasor form of **A** is:

$$\mathbf{A}_s = A_o e^{-j\beta x}\,\mathbf{a}_y$$

Maxwell's Equations in Phasor Form

- From the previous discussion:

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\partial}{\partial t}\operatorname{Re}\left(\mathbf{A}_s e^{j\omega t}\right) = \operatorname{Re}\left(j\omega \mathbf{A}_s e^{j\omega t}\right)$$

- Therefore:

$$\frac{\partial \mathbf{A}}{\partial t} \rightarrow j\omega \mathbf{A}_s$$

- Similarly:

$$\int \mathbf{A}\, \partial t \rightarrow \frac{\mathbf{A}_s}{j\omega}$$

- We shall now apply the phasor concept to time-varying EM fields

Maxwell's Equations in Phasor Form

- The fields quantities **E**(x, y, z, t), **D**(x, y, z, t), **H**(x, y, z, t), **B**(x, y, z, t), **J**(x, y, z, t), and $\rho_v(x, y, z, t)$ and their derivatives can be expressed in phasor form

- In phasor form, Maxwell's equations for time-harmonic EM fields in a linear, isotropic, and homogeneous medium can be written as:

$$\nabla \cdot \mathbf{E}_s = 0$$

$$\nabla \cdot \mathbf{H}_s = 0$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s$$

$$\nabla \times \mathbf{H}_s = (\sigma + j\omega\varepsilon)\mathbf{E}_s$$

Wave Propagation in Lossy Dielectrics

- Taking curl on both sides of the Maxwell's equation, we get:

$$\nabla \times \nabla \times \mathbf{E}_s = -j\omega\mu \, \nabla \times \mathbf{H}_s$$

- We have the vector identity:

$$\nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

- Applying the vector identity to the left side of the equation and by substituting Maxwell's remaining equations, we get:

$$\nabla (\cancelto{0}{\nabla \cdot \mathbf{E}_s}) - \nabla^2 \mathbf{E}_s = -j\omega\mu(\sigma + j\omega\varepsilon)\mathbf{E}_s$$

- Or:

$$\nabla^2 \mathbf{E}_s - \gamma^2 \mathbf{E}_s = 0$$

- Where:

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

Wave Propagation in Lossy Dielectrics

- The quantity γ is called the propagation constant (in per meter) of the medium

- By a similar procedure, it can be shown that for the **H** field:

$$\nabla^2 \mathbf{H}_s - \gamma^2 \mathbf{H}_s = 0$$

- These equations for **E** and **H** are known as homogeneous vector Helmholtz 's equations or simply vector wave equations

- In Cartesian coordinates, the wave equation for **E**, for example, is equivalent to three scalar wave equations, one for each component of **E** along $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$

Wave Propagation in Lossy Dielectrics

- Since γ is a complex quantity, we may write it as:

$$\gamma = \alpha + j\beta$$

- We obtain α and β from the previous equations as:

$$-\mathrm{Re}\,\gamma^2 = \beta^2 - \alpha^2 = \omega^2\mu\varepsilon$$

- And:

$$|\gamma^2| = \beta^2 + \alpha^2 = \omega\mu\sqrt{\sigma^2 + \omega^2\varepsilon^2}$$

- From the above equations we obtain:

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$$

Wave Propagation in Lossy Dielectrics

- For simplicity, we assume that the wave propagates along $+\mathbf{a}_z$ and that $\mathbf{E}_s$ has only an x-component, then:

$$\mathbf{E}_s = E_{xs}(z)\mathbf{a}_x$$

- Substituting the above into the wave equation, we get:

$$(\nabla^2 - \gamma^2)E_{xs}(z)$$

- Therefore:

$$\underbrace{\cancel{\frac{\partial^2 E_{xs}(z)}{\partial x^2}}}_{0} + \underbrace{\cancel{\frac{\partial^2 E_{xs}(z)}{\partial y^2}}}_{0} + \frac{\partial^2 E_{xs}(z)}{\partial z^2} - \gamma^2 E_{xs}(z) = 0$$

- Or:

$$\left[\frac{d^2}{dz^2} - \gamma^2\right]E_{xs}(z) = 0$$

Wave Propagation in Lossy Dielectrics

- This is a scalar wave equation, a linear homogeneous differential equation, with solution:

$$E_{xs}(z) = E_o e^{-\gamma z} + E'_o e^{\gamma z}$$

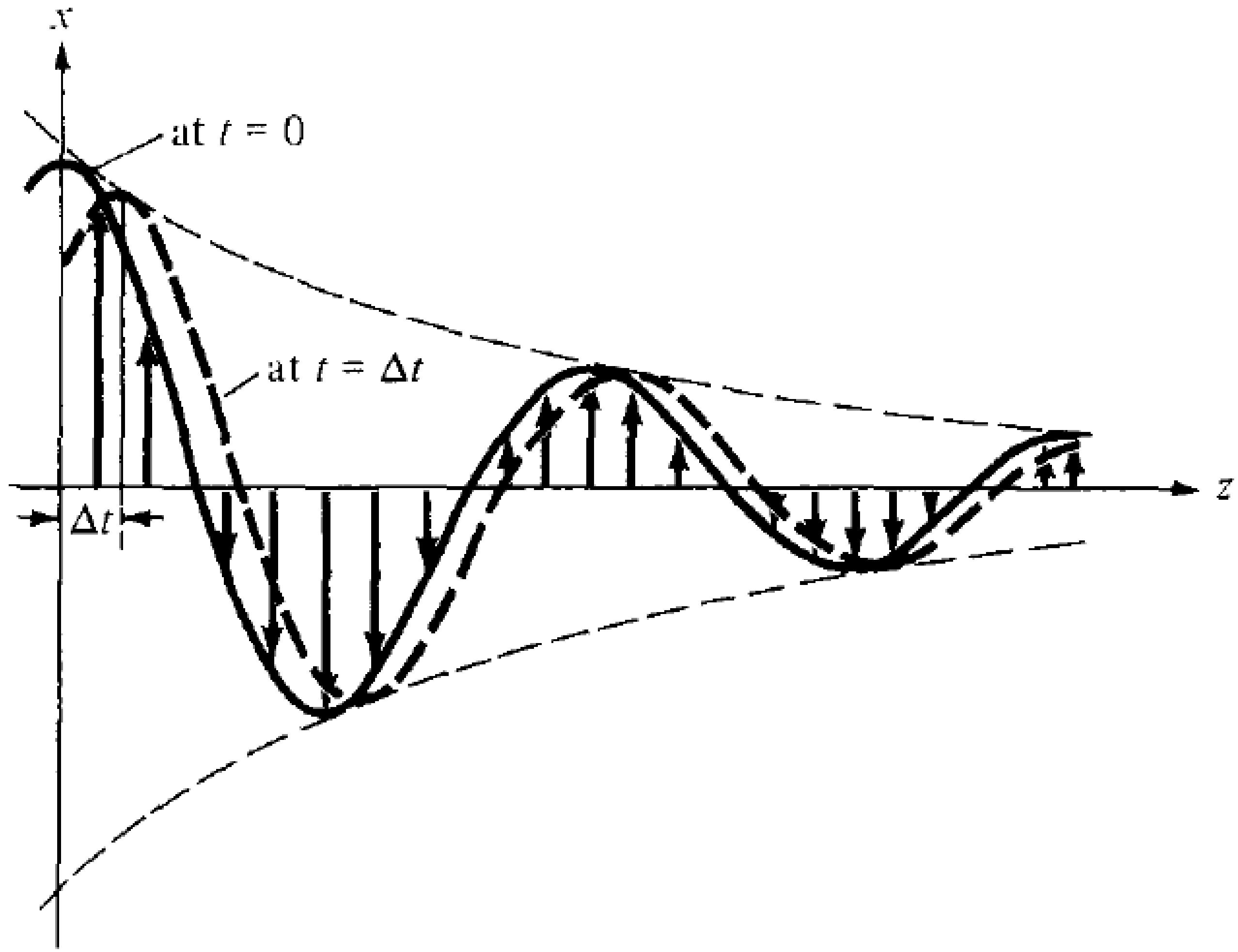
- where E_o and E'_o are constants
- The fact that the field must be finite at infinity requires that $E'_o = 0$
- Alternatively, because $e^{\gamma z}$ denotes a wave traveling along $-\mathbf{a}_z$ whereas we assume wave propagation along $\mathbf{a}_z$, => $E'_o = 0$
- Inserting the time factor e^{jwt} into the above equation and using value of γ, we obtain:

$$\mathbf{E}(z, t) = \text{Re}\,[E_{xs}(z)e^{j\omega t}\mathbf{a}_x] = \text{Re}\,(E_o e^{-\alpha z} e^{j(\omega t - \beta z)}\mathbf{a}_x)$$

Wave Propagation in Lossy Dielectrics

- Or: $$\mathbf{E}(z, t) = E_o e^{-\alpha z} \cos(\omega t - \beta z)\mathbf{a}_x$$

- A sketch of |E| at times t = 0 and t = Δt is shown , where it is evident that **E** has only an x-component and it is traveling along the +z-direction



Problem-1

- The equation of $\mathbf{E}(z,t)$ for an EM wave in a lossy dielectric medium is given below. Determine the equation for $\mathbf{H}(z,t)$ for the same EM wave.

$$\mathbf{E}(z, t) = \text{Re}\,[E_{xs}(z)e^{j\omega t}\mathbf{a}_x] = \text{Re}\,(E_o e^{-\alpha z} e^{j(\omega t - \beta z)}\mathbf{a}_x)$$

Problem-1

We will use the phasor form of Maxwell's equations:-

So $\vec{E} = Re\left[E_{xs}(z) e^{j\omega t} \vec{a_x} \right]$

$\Rightarrow \vec{E}_{xs} = E_0 e^{-(\alpha + j\beta) z}$ or $\vec{E_s} = E_{xs} \vec{a_x}$

From Maxwell's equation, we have

$$\vec{\nabla} \times \vec{E}_s = -j\omega\mu \vec{H_s}$$

or $\vec{H_s} = \dfrac{\vec{\nabla} \times \vec{E_s}}{-j\omega\mu}$

$$\nabla \times E_s = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{xs} & 0 & 0 \end{vmatrix} = 0\vec{a_x} - \vec{a_y}\left(-\frac{\partial E_{xs}}{\partial z}\right) + \cancel{\vec{a_z}\left(-\frac{\partial E_{xs}}{\partial y}\right)} \rightarrow 0$$

Problem-1

$$\vec{\nabla}\times\vec{E}_s = \frac{\partial E_{xs}}{\partial z}\vec{a_y} = -E_0(\alpha+j\beta)e^{-(\alpha+j\beta)z}\vec{a_y}$$

$$\vec{H}_s = \left(\frac{\alpha+j\beta}{j\omega\mu}\right)E_0 e^{-(\alpha+j\beta)z}\vec{a_y}$$

now ~~for~~ $\alpha+j\beta = \gamma = \sqrt{j\omega\mu(\sigma+j\omega\varepsilon)}$

$$\frac{\alpha+j\beta}{j\omega\mu} = \sqrt{\frac{\sigma+j\omega\varepsilon}{j\omega\mu}} = \frac{1}{\eta}$$

where η = intrinsic impedence in ohms of the medium.

$$\vec{H}_s = \frac{E_0}{\eta}e^{-(\alpha+j\beta)z}\vec{a_y}$$

OR

or $$\vec{H}(z,t) = Re\left\{\frac{E_0}{\eta}e^{-\alpha z}e^{j(\omega t-\beta z)}\vec{a_y}\right\}$$

Wave Propagation in Lossy Dielectrics

- Since η is a complex quantity, it may be written as:

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| \angle\theta_\eta = |\eta| e^{j\theta_\eta}$$

- With:

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}, \qquad \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}$$

- Where:

$$0 \leq \theta_\eta \leq 45^\circ$$

- Therefore, using the above quantities, **H** may be written as:

$$\mathbf{H} = \text{Re}\left[\frac{E_o}{|\eta| e^{j\theta_\eta}} e^{-\alpha z} e^{j(\omega t - \beta z)}\, \mathbf{a}_y\right]$$ OR $$\mathbf{H} = \frac{E_o}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\, \mathbf{a}_y$$