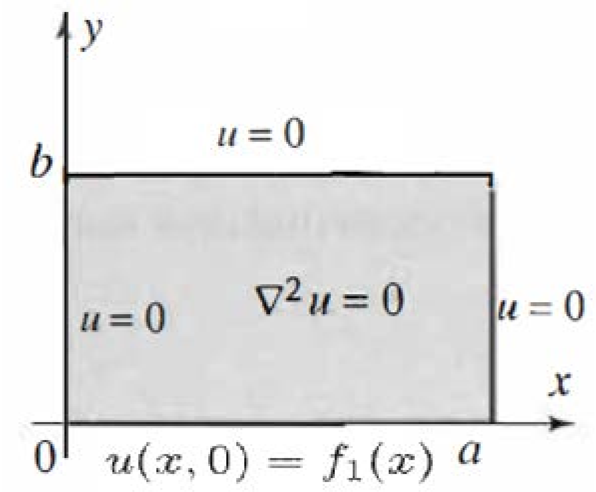
Dirichlet Problem in a Rectangle

Let us solve the boundary value problem described in the accompanying figure (subproblem -1) using the method of separation of variables.

We are required to determine solution of the BVP:



(1) $\quad \nabla^2 u = u_{xx} + u_{yy} = 0, \qquad 0 < x < a, \qquad 0 < y < b.$

(2) $\quad u(x,0) = f_1(x), \qquad u(x,b) = 0, \qquad 0 < x < a,$

(3) $\quad u(0,y) = 0, \qquad u(a,y) = 0, \qquad 0 < y < b.$

We begin by looking for product solution $u(x,y) = X(x)Y(y)$. Substituting into (1) and using the separation method, we arrive at the equations:

$$X''(x) + \alpha\, X(x) = 0, \qquad (4)$$

$$Y''(y) - \alpha\, Y(y) = 0, \qquad (5)$$

where α is the separation constant, with the boundary conditions:

$$X(0) = 0, \qquad X(a) = 0, \qquad \text{and} \qquad Y(b) = 0. \qquad (6)$$

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Let us consider the boundary value problem in X that is given as:

$$X''(x) + \alpha\, X(x) = 0,$$

$$X(0) = 0, \qquad X(a) = 0$$

We can check that the values $\alpha \leq 0$, lead to trivial solutions only. For $\alpha = \lambda^2 > 0$, we obtain the solutions:

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x\,.$$

Imposing the boundary conditions on X forces $c_1 = 0$, and

$$\lambda = \lambda_k = \frac{k\pi}{a} \quad or \quad \alpha = \left(\frac{k\pi}{a}\right)^2, \qquad k = 1,2,3,\ldots$$

and hence,

$$X_k(x) = c_k \sin\left(\frac{k\pi x}{a}\right), \qquad k = 1,2,3,\ldots \quad (7)$$

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Let us now consider (5), which on using $\alpha = \lambda_k^2 = \left(\frac{k\pi}{a}\right)^2$ takes the form:

$$Y''(y) - \left(\frac{k\pi}{a}\right)^2 Y(y) = 0. \quad (5')$$

The general solution of $(5')$ is a linear combination of exponentials or a linear combination of hyperbolic functions. For the present case it is convenient to express its general solution in terms of hyperbolic functions instead of exponential functions. Thus, the general solution of $(5')$ is given as:

$$Y(y) = Y_k(y) = d_k \cosh\left(\frac{k\pi y}{a}\right) + e_k \sinh\left(\frac{k\pi y}{a}\right). \quad (8)$$

Using the boundary condition $Y(b) = 0$ in (8) we get:

$$e_k = -d_k \coth\left(\frac{k\pi b}{a}\right).$$

Thus, (8) can be written as:

$$Y_k(y) = d_k \cosh\left(\frac{k\pi y}{a}\right) - d_k \coth\left(\frac{k\pi b}{a}\right) \sinh\left(\frac{k\pi y}{a}\right). \quad (8')$$

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Equation $(8')$ can be further simplified as:

$$Y_k(y) = \frac{d_k}{\sinh\left(\frac{k\pi b}{a}\right)}\left[\sinh\left(\frac{k\pi b}{a}\right)\cosh\left(\frac{k\pi y}{a}\right) - \cosh\left(\frac{k\pi b}{a}\right)\sinh\left(\frac{k\pi y}{a}\right)\right].$$

Since $\sinh(\alpha-\beta) = \sinh\alpha\cosh\beta - \cosh\alpha\sinh\beta$, so choosing $\alpha = \frac{k\pi b}{a}$ and $\beta = \frac{k\pi y}{a}$, above equation takes the form:

$$Y_k(y) = F_k \sinh\left(\frac{k\pi(b-y)}{a}\right); \quad \text{where} \quad F_k = \frac{d_k}{\sinh\left(\frac{k\pi b}{a}\right)}, \qquad k = 1,2,3,\dots \quad (9)$$

From (7) and (9) we obtain:

$$u_k(x,y) = X_k(x)Y_k(y) = A_k \sin\left(\frac{k\pi x}{a}\right)\sinh\left(\frac{k\pi(b-y)}{a}\right); \quad \text{where} \quad A_k = c_k F_k, \qquad (10)$$

as the **eigenfunctions** of our problem corresponding to eigenvalues $\lambda_k = \frac{k\pi}{a}$.

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Superposing these solutions (10), we get the general form of the solution as:

$$u(x,y)=\sum_{k=1}^{\infty} u_k(x,y)=\sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{a}\right)\sinh\left(\frac{k\pi y}{a}\right), \qquad (11)$$

where the constants E_k are to be chosen. Thus, by using $u(x,0)=f_1(x)$ we get:

$$f_1(x)=\sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{a}\right)\sinh\left(\frac{k\pi b}{a}\right). \qquad (12)$$

To meet this last requirement, we choose the coefficients $E_k \sinh\frac{k\pi b}{a}$ to be the Fourier sine coefficients of $f_2(x)$ on the interval $0<x<a$. Thus, it follows that:

$$A_k=\frac{2}{a\sinh\left(\frac{k\pi b}{a}\right)}\int_0^a f_2(x)\sin\left(\frac{k\pi x}{a}\right)dx, \qquad k=1,2,3,\ldots. \qquad (13)$$

The solution of the Dirichlet problem described in the figure is therefore given by (11) with coefficients determine by (13).