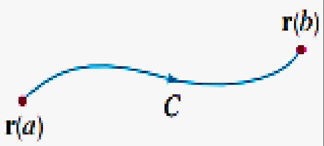
Divergence Theorem

Vector Calculus(MATH-243)

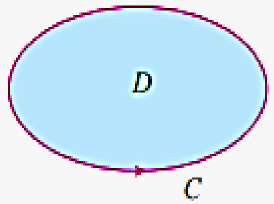
Instructor: Dr. Naila Amir

Fundamental Theorem of Calculus $\int_a^b F'(x)\,dx = F(b) - F(a)$

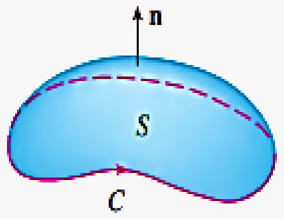
Fundamental Theorem for Line Integrals $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$



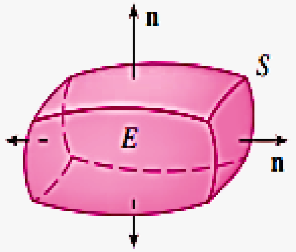
Green's Theorem
(Circulation form) $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$



Stokes' Theorem $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$

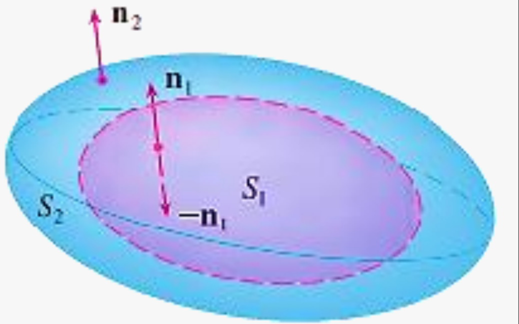


Divergence Theorem $\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$



Divergence Theorem for regions that are finite unions of simple solid regions

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$$



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Vector Calculus

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

- **Chapter: 16**
  - **Section: 16.9**

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

- **Chapter: 16**
  - **Section: 16.8**

Divergence Theorem & Green's Theorem

For a vector field $\mathbf{F}$ and a smooth surface S the Green's theorem can be generalized to higher dimensions as follows:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R (\operatorname{div} \mathbf{F})\, dA,$$

Total Flux across R

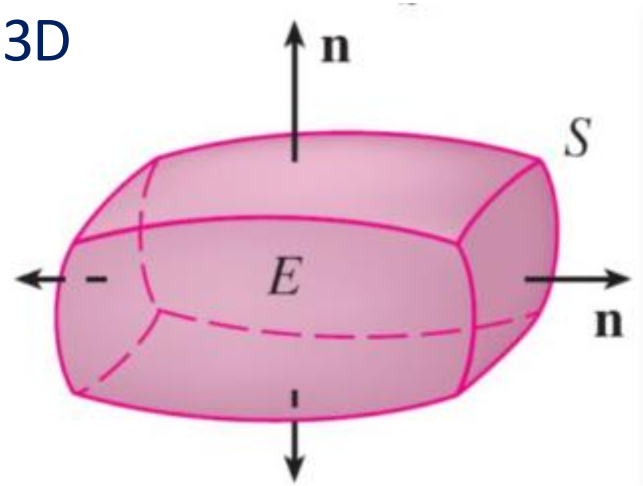
where C is the positively oriented boundary curve of the plane region R.

↓

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E (\operatorname{div} \mathbf{F})\, dV,$$

Total Flux in 3D

where S is the boundary surface of the solid region E.



Gauss’s Divergence Theorem

Let:

- E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation.
- the unit normal vector $\mathbf{n}$ is directed outward from E.
- $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E.

Then,

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E (\operatorname{div} \mathbf{F}) \, dV,$$

The Divergence Theorem states that:

Under the given conditions, the flux of $\mathbf{F}$ across the boundary surface S of E in the direction of the surface’s outward unit normal $\mathbf{n}$ is equal to the triple integral of the divergence of $\mathbf{F}$ over the region E enclosed by the surface S.

Coordinate Conversion Formulas in Triple Integrals

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

Example:

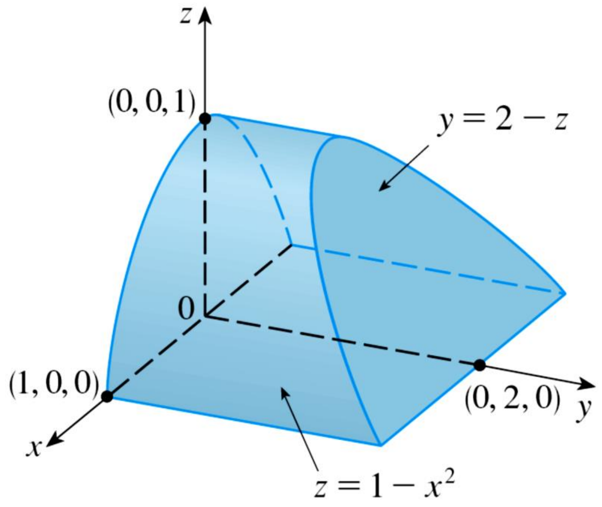
Evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

where, $\mathbf{F}(x, y, z) = \langle xy, y^2 + e^{xz^2}, \sin(xy)\rangle$ is the vector field and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0, y = 0$, and $y + z = 2$.

Solution:

Note that it would be extremely difficult to evaluate the given surface integral directly. We would have to evaluate four surface integrals corresponding to the four pieces of S. Thus, we apply divergence theorem to evaluate this integral. Note that, the divergence of $\mathbf{F}$ is much less complicated than $\mathbf{F}$ itself:



$$\mathrm{div}\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \frac{\partial}{\partial z}(\sin(xy))$$

$$= y + 2y = 3y.$$

Solution:

So, we use the Divergence Theorem to transform the given surface integral into a triple integral. The easiest way to evaluate the triple integral is to express E as:

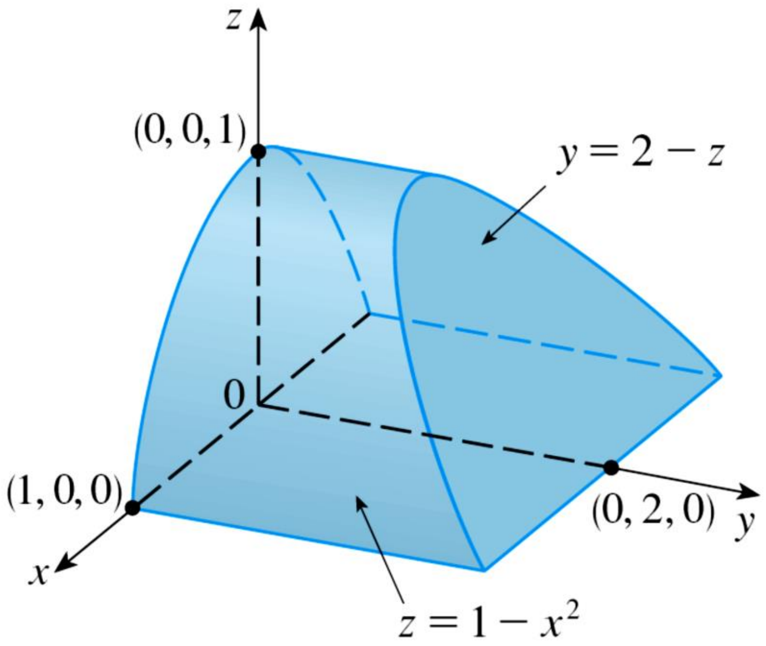
$$E = \{(x, y, z) | -1 \leq x \leq 1, 0 \leq z \leq 1 - x^2, 0 \leq y \leq 2 - z\}$$

Then, we have:

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_E \mathrm{div}\mathbf{F} dV$$

$$= \int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} (3y)\, dydzdx$$

$$= \frac{184}{35}.$$



Example:

Evaluate

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \mathbf{F}\cdot d\mathbf{S}$$

where, $\mathbf{F}(x,y,z) = \langle xy, -y^2/2, z\rangle$ and the surface consists of the surfaces, $z = 4 - 3x^2 - 3y^2$, $1 \le z \le 4$ on the top, $x^2 + y^2 = 1, 0 \le z \le 4$ on the sides and $z = 0$ on the bottom.

Solution:

The region E for the triple integral is the region enclosed by the given surfaces. Note that cylindrical coordinates would be a perfect coordinate system for this region. The limits for the ranges are given as:

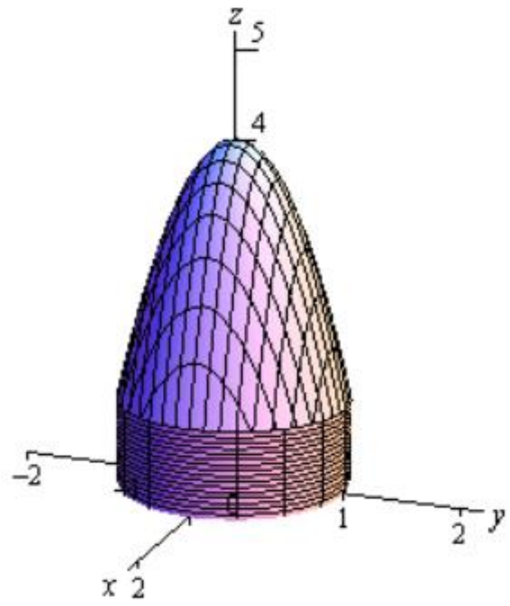
$$0 \le r \le 1, \qquad 0 \le \theta \le 2\pi, \qquad \text{and} \quad 0 \le z \le 4 - 3r^2,$$

and

$$\mathrm{div}\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(-\frac{y^2}{2}\right) + \frac{\partial}{\partial z}(z) = y - y + 1 = 1.$$

By divergence theorem we have:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \mathrm{div}\mathbf{F}\,dV\,.$$



Solution:

Thus,

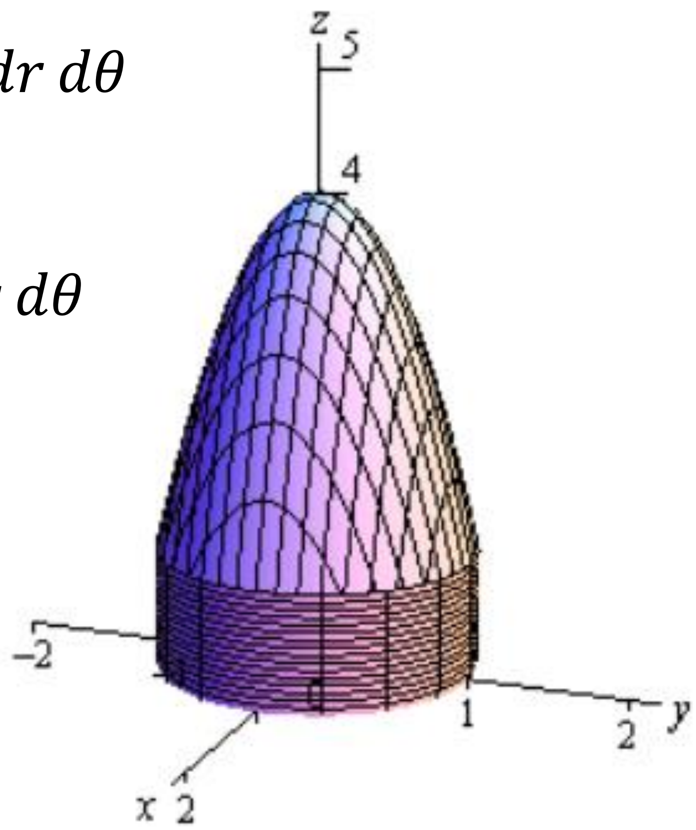
$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \mathrm{div}\mathbf{F}\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^{4-3r^2} (1)\,dz\,r dr\,d\theta = \int_0^{2\pi}\int_0^1 (z)\Big|_0^{4-3r^2}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (4-3r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (4r-3r^3)\;dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{4r^2}{2}-\frac{3r^4}{4}\right)\Big|_0^1 d\theta = \int_0^{2\pi}\left(\frac{5}{4}\right)d\theta$$

$$= \left(\frac{5}{4}\theta\right)\Big|_0^{2\pi} = \frac{5}{4}\pi.$$



Example:
Evaluate

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_S \mathbf{F}\cdot d\mathbf{S}$$

where, $\mathbf{F}(x,y,z) = \langle yx^2, xy^2 - 3z^4, x^3 + y^2\rangle$ and S is the surface of the sphere of radius 4 with $z \le 0$ and $y \le 0$. Note that all three surfaces of this solid are included in S.

Solution:

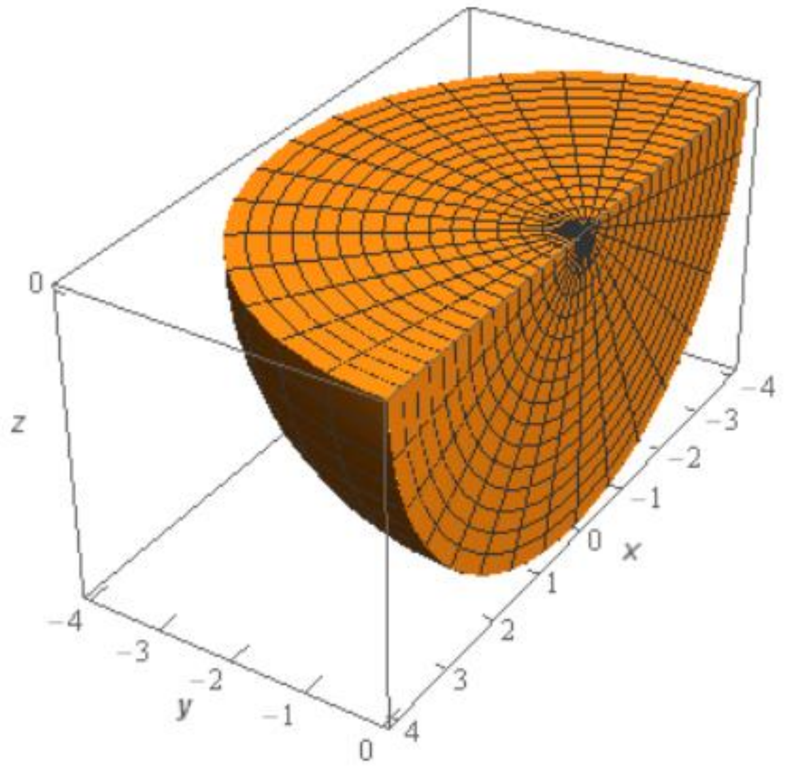
For the present case E is a portion of a sphere so its preferable to use spherical coordinates for the integration. The spherical limits we'll need to use for this region are given as:

$$0 \le \rho \le 4, \qquad \pi \le \theta \le 2\pi, \qquad \text{and} \quad \frac{\pi}{2} \le \varphi \le \pi,$$

and

$$\mathrm{div}\mathbf{F} = \frac{\partial}{\partial x}(yx^2) + \frac{\partial}{\partial y}(xy^2 - 3z^4) + \frac{\partial}{\partial z}(x^3 + y^2)$$

$$= 2xy + 2xy + 0 = 4xy.$$



Solution:

By divergence theorem we have:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F} dV = \iiint_E (4xy)\, dV$$

$$= \int_{\pi}^{2\pi}\int_{\pi/2}^{\pi}\int_0^4 [4(\rho\sin\varphi\cos\theta)(\rho\sin\varphi\sin\theta)]\,(\rho^2\sin\varphi)d\rho\, d\varphi\, d\theta$$

$$= \int_{\pi}^{2\pi}\int_{\pi/2}^{\pi}\int_0^4 [4\rho^4\sin^3\varphi\cos\theta\sin\theta]\, d\rho\, d\varphi\, d\theta$$

$$= \int_{\pi}^{2\pi}\int_{\pi/2}^{\pi}\left(\frac{4\rho^5}{5}\sin^3\varphi\cos\theta\sin\theta\right)\Big|_0^4\, d\varphi\, d\theta$$

$$= \frac{4096}{5}\int_{\pi}^{2\pi}\int_{\pi/2}^{\pi}[\sin\varphi\,(1-\cos^2\varphi)\cos\theta\sin\theta]\, d\varphi\, d\theta.$$

Solution:

$$\Rightarrow \iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \mathrm{div}\mathbf{F}dV \;= \frac{4096}{5}\int_{\pi}^{2\pi}\int_{\frac{\pi}{2}}^{\pi}[(\sin\varphi - \sin\varphi\cos^2\varphi)\cos\theta\sin\theta]\,d\varphi\,d\theta.$$

$$= \frac{4096}{5}\int_{\pi}^{2\pi}\left[\left(-\cos\varphi + \frac{1}{3}\cos^3\varphi\right)\cos\theta\sin\theta\right]\Bigg|_{\pi/2}^{\pi}\,d\theta$$

$$= \frac{8192}{15}\int_{\pi}^{2\pi}[\cos\theta\sin\theta]\,d\theta$$

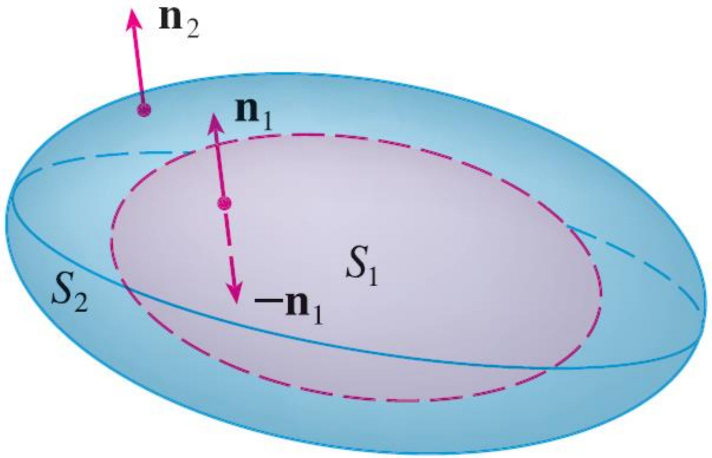
$$= \frac{8192}{15}\left(\frac{\sin^2\theta}{2}\right)\Bigg|_{\pi}^{2\pi}$$

$$= 0.$$

Divergence Theorem for Finite Unions Of Simple Solid Regions

The Divergence Theorem can also be proved for regions that are finite unions of simple solid regions. For example, let's consider the region E that lies between the closed surfaces S_1 and S_2, where S_1 lies inside S_2. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be outward normal vectors of S_1 and S_2 respectively. Then:

$$\begin{aligned}\iiint_E \operatorname{div}\mathbf{F}\,dV &= \iint_S \mathbf{F}\cdot\mathbf{n}\,dS \\ &= \iint_{S_1} \mathbf{F}\cdot(-\mathbf{n}_1)\,dS + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}_2\,dS \\ &= -\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}.\end{aligned}$$



$$\Longrightarrow \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iiint_E \operatorname{div}\mathbf{F}\,dV\,.$$

Example—Electric Field

Consider the electric field:

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\langle x, y, z\rangle|^3}\langle x, y, z\rangle,$$

where S_1 is a small sphere with radius a and center the origin. Determine the electric flux of $\mathbf{E}$ any closed surface S_2 that contains the origin and S_1.

Solution:

We can verify that div $\mathbf{E} = 0$. Thus, we have

$$\iint\limits_{S_2} \mathbf{E} \cdot \mathbf{n}\, dS = \iint\limits_{S_1} \mathbf{E} \cdot \mathbf{n}\, dS + \iiint\limits_{E} \text{div}\mathbf{E}\, dV = \iint\limits_{S_1} \mathbf{E} \cdot \mathbf{n} dS\,.$$

Now:

$$\begin{aligned}\mathbf{E} \cdot \mathbf{n} &= \frac{\varepsilon Q}{|\langle x, y, z\rangle|^3}\langle x, y, z\rangle \cdot \left(\frac{\langle x, y, z\rangle}{|\langle x, y, z\rangle|}\right), \\ &= \frac{\varepsilon Q}{|\langle x, y, z\rangle|^4}\langle x, y, z\rangle \cdot \langle x, y, z\rangle = \frac{\varepsilon Q}{|\langle x, y, z\rangle|^2} = \frac{\varepsilon Q}{a^2}\,.\end{aligned}$$

Solution:

Therefore:

$$\iint_{S_2} \mathbf{E}\cdot\mathbf{n}dS = \iint_{S_1} \mathbf{E}\cdot\mathbf{n}dS = \frac{\varepsilon Q}{a^2}\iint_{S_1} dS$$

$$= \frac{\varepsilon Q}{a^2}A(S_1) = \frac{\varepsilon Q}{a^2}4\pi a^2 = 4\pi\varepsilon Q.$$

This shows that the electric flux of $\mathbf{E}$ is $4\pi\varepsilon Q$ through any closed surface S_2 that contains the origin. This is a special case of Gauss's Law for a single charge. The relationship between ε and ε_0 is $\varepsilon = 1/4\pi\varepsilon_0$.

Summary:
Green’s Theorem and Its Generalization to Three Dimensions

Normal form of Green’s Theorem:

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \operatorname{div}\mathbf{F}(x,y)\,dA$$

Divergence Theorem:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{F}(x,y,z)\,dV$$

Tangential form of Green’s Theorem:

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R (\operatorname{curl}\mathbf{F}).\,\mathbf{k}\,dA$$

Stokes’Theorem:

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\operatorname{curl}\mathbf{F}).\,\mathbf{n}\,dS$$

Practice Questions

1. Evaluate

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \mathbf{F}\cdot d\mathbf{S}$$

where, $\mathbf{F}(x,y,z) = \langle \sin(\pi x), zy^3, z^2 + 4x\rangle$ is the vector field and S is the surface of the box with $-1 \le x \le 2$, $0 \le y \le 1$ and $1 \le z \le 4$. Note that all six sides of the box are included in S.

 Answer: $\frac{135}{2}$.

2. Verify the divergence theorem for vector field $\mathbf{F}(x,y,z) = \langle x-y, x+z, z-y\rangle$ and the surface S that consists of cone $x^2+y^2=z^2, 0 \le z \le 1$, and the circular top of the cone. Assume this surface is positively oriented.

   Answer: $\frac{2\pi}{3}$.

3. Determine the flux outward of the vector field: $\mathbf{F}(x,y,z) =< 0, yz,\ z^2 >$ through the surface S of the region E bounded by the cylinder $z^2+y^2=1,\ z \ge 0$ cut by the planes $x = 0$ and $x = 1$.

   Answer: 2.

Practice Questions

Book: Calculus Early Transcendentals (6th Edition) By James Stewart.

Chapter: 16

Exercise-16.9: Q – 1 to 14, Q – 17 to 24.

Book: Thomas' Calculus Early Transcendentals (14th Edition) By George B. Thomas, Jr., Joel Hass, Christopher Heil, Maurice D. Weir.

Chapter: 16

Exercise-16.8: Q – 9 to 20, Q – 27 to 28.