Chapter3: Gate-Level Minimization

Lecture1- Three and Four-Variables Function Simplification using Map Method

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Chapter Contents

Karnaugh Maps
SOP and POS Simplifications
Don’t Care Conditions
Quine McCluskey Minimization Algorithm
NAND and NOR Implementations
Parity Generation and Detection

Objectives

- Introduction to Map Method
- Plot and Labeling of minterms on Map
- Functions Simplification in Sum-of-Products (SOP) form using Three and Four-Variables Map

K-Map Method

- The Karnaugh Map (K-Map) method uses a simple procedure for minimizing Boolean functions.
  - The map is a diagram made up of squares with each square representing one minterm of the function.
  - The key is to learn to identify visual patterns.
  - The result is always an expression that is in one of the two standard forms, SOP or POS.
  - Much faster and more efficient than previous minimization techniques with Boolean algebra. It can be used to simplify functions of up to six variables.
  - It is possible to find two or more expressions that satisfy the minimization criteria.
  - Rules to consider
    - Every cell containing a 1 must be included at least once.
    - The largest possible “power of 2 rectangle” must be enclosed.
    - The 1’s must be enclosed in the smallest possible number of rectangles.

Two-Variable Map

- A two-variable map holds four minterms for two variables.
  - We mark the squares of the minterms that belong to a given function.
  - Combine adjacent squares to find minimal expression.

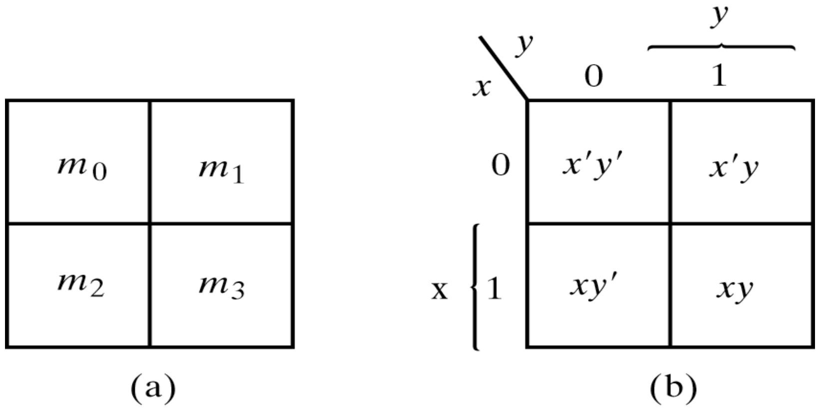


Fig. 3-1 Two-variable Map

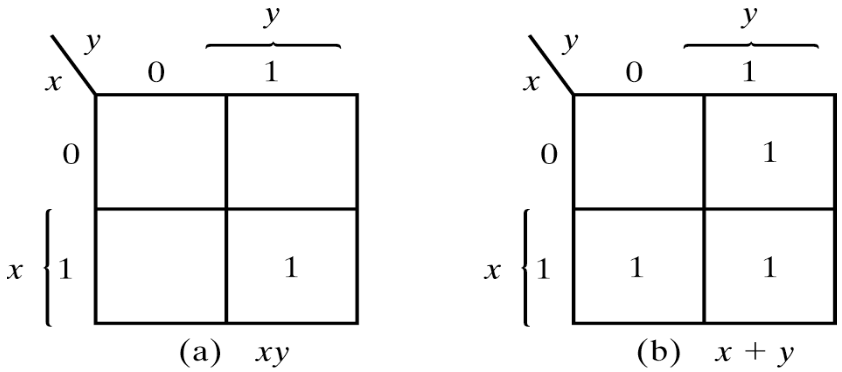


Fig. 3-2 Representation of Functions in the Map

Three-Variable Map

- A three-variable map holds eight minterms for three variables.
  - Again, we mark the squares of the minterms that belong to a given function.
  - Note that the sequence is arranged in Gray code to allow only one bit to change from column to column and row to row.
- Since any two adjacent cells in a 3-variable map represent a change in only a single bit, we use this to do minimization.
  - Consider the two cells for m_0 and m_1 where the difference is the negation of the bit z.
  - $F = m_0 + m_1 = x'y'z' + x'y'z = x'y'(z' + z) = x'y'$

m_0	m_1	m_3	m_2
m_4	m_5	m_7	m_6

(a)

$x \backslash yz$	00	01	11	10
0	$x'y'z'$	$x'y'z$	$x'yz$	$x'yz'$
1	$xy'z'$	$xy'z$	xyz	xyz'

(b)

Fig. 3-3 Three-variable Map

Minimization Example

- Each of the two adjacent pairs of entries can be simplified by eliminating the changing bit (z in both cases).
  - F (x,y,z) = x’y’ + xy

x \ yz	00	01	11	10
0	1	1		
1			1	1

$$F(x, y, z) = \sum(0,1,6,7)$$

x \ yz	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

Notes on Adjacency

- So far, we have assumed that adjacent cells in the map need to touch each other but this is not always the case.
  - ➢ m_0 and m_2 are considered adjacent
    - ○ $m_0 + m_2 = x'y'z' + x'yz' = x'z'(y' + y) = x'z'$
  - ➢ m_4 and m_6 are considered adjacent
    - ○ $m_4 + m_6 = xy'z' + xyz' = xz'(y' + y) = xz'$

x \ yz	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

x \ yz	00	01	11	10
0	$x'y'z'$	$x'y'z$	$x'yz$	$x'yz'$
1	$xy'z'$	$xy'z$	xyz	xyz'

3-Variable Map Patterns

- The number of adjacent squares that may be combined always represent a number that is a power of 2 such as 1, 2, 4, and 8.
  - One square represents one minterm with three literals.
  - Two adjacent squares represents a term of two literals.
  - Four adjacent squares represents a term of one literal.
  - Eight adjacent squares represents the entire map and produces a function that is always equal to 1.

Mapping Functions Example

- Given the function
  - ➢ F = x′z + xy′ + xy′z + yz

    F = ∑(1, 3, 4, 5, 7)

x \ yz	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

- Map the function
- Determine the sum of minterms equation
- Determine the minimum sum of products expression

x \ yz	00	01	11	10
0		1	1	
1	1	1	1	

- The minimum sum-of-Products (SOP) is

  F= = z + xy′

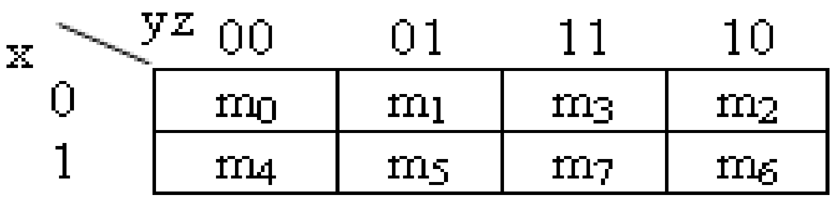
Example 3-1

Simplify the Boolean function F(x,y,z)= ∑(2,3,4,5) using map method

Solution:

- 1 is marked in each minterm that represents the function
- Find the possible adjacent squares and mark them with rectangles
- The upper right rectangle represents the area enclosed closed by x’y (eliminating the changing bit)
- Similarly lower left rectangle represents xy’
- The logical sum of these two terms gives:

F = x'y + xy'



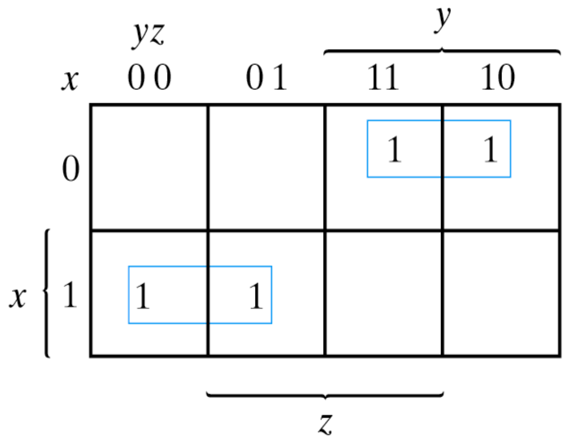


Fig. 3-4 Map for Example 3-1; $F(x, y, z) = \Sigma(2, 3, 4, 5) = x'y + xy'$

Example 3-2

Simply the function $F(x,y,z) = \sum(3,4,6,7)$

Solution:

- 1 is marked in each minterm that represents the function
- Find the possible adjacent squares and mark them with rectangles
- Two adjacent squares are combined in the third column to give a two-literal term yz
- The remaining two squares with 1's are enclosed in half rectangles. This gives two-literal term xz'
- The logical sum of these two terms gives: F = yz + xz'

x \ yz	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

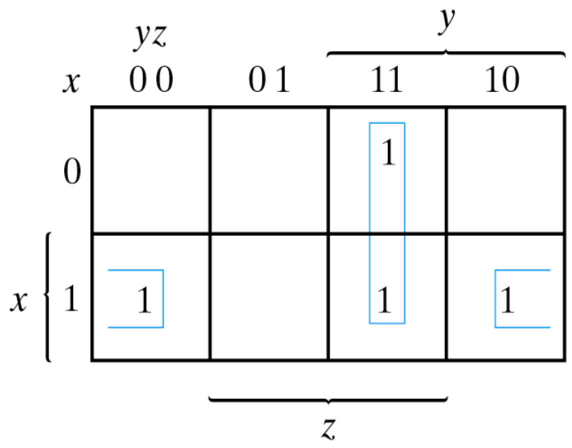


Fig. 3-5 Map for Example 3-2; $F(x, y, z) = \Sigma(3, 4, 6, 7) = yz + xz'$

Example 3-3

Simply the function F(x,y,z) =∑ S(0,2,4,5,6)

Solution:

- 1 is marked in each minterm that represents the function
- Find the possible adjacent squares and mark them with rectangles
- We combine four adjacent squares to get a single literal term z’ as $m_0+m_2+m_4+m_6$

  x'y'z'+x'yz'+xy'z'+xyz’= x'z'(y'+y) +xz'(y'+y)

  = x'z' + xz’ = z'
- The remaining two squares with 1’s are enclosed by a rectangle (with one square that is already used once). This gives two-literal term xy’
- The logical sum of these two terms gives:

  F = z’ + xy’

x \ yz	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

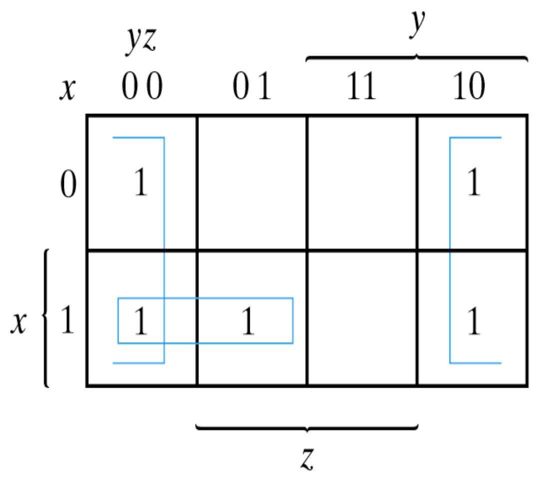


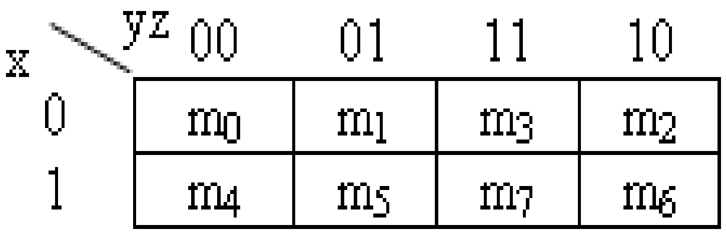
Fig. 3-6 Map for Example 3-3; $F(x, y, z) = \Sigma(0, 2, 4, 5, 6) = z' + xy'$

Example 3-4

Simplify F = A'C + A'B + AB'C + BC

Solution:

- The two squares corresponding to the first term A'C. (A' first row and C two middle columns)
- A'B has 1's in squares 011 and 010 in the same way. AB'C has 1 square 101 and BC has two 1's in squares 011 and 111
- The function has total of 5 minterms as shown in figure
- Find the possible adjacent squares and mark them with rectangles as shown in the map
- It can be simplified with only two terms giving: F = C + A'B



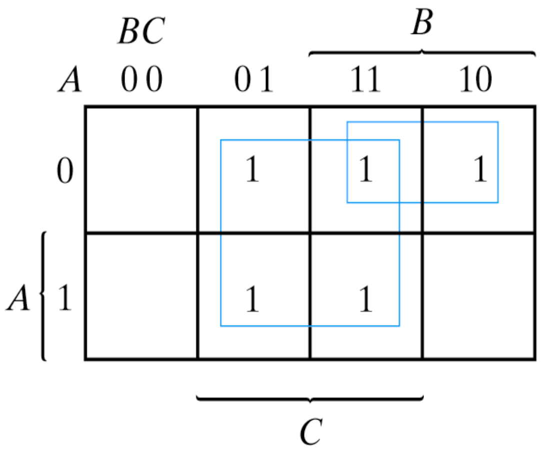


Fig. 3-7 Map for Example 3-4; $A'C + A'B + AB'C + BC = C + A'B$

Four-Variable Map

- A four-variable map holds 16 minterms for four variables.
  - ➢ Again, we mark the squares of the minterms that belong to a given function.
  - ➢ Note that the sequence is not arranged in a binary way.
  - ➢ The sequence used is a Gray code and allows only one bit to change from column to column and row to row.

m_0	m_1	m_3	m_2
m_4	m_5	m_7	m_6
m_{12}	m_{13}	m_{15}	m_{14}
m_8	m_9	m_{11}	m_{10}

(a)

$wx \backslash yz$	00	01	11	10
00	$w'x'y'z'$	$w'x'y'z$	$w'x'yz$	$w'x'yz'$
01	$w'xy'z'$	$w'xy'z$	$w'xyz$	$w'xyz'$
11	$wxy'z'$	$wxy'z$	$wxyz$	$wxyz'$
10	$wx'y'z'$	$wx'y'z$	$wx'yz$	$wx'yz'$

(b)

Fig. 3-8 Four-variable Map

4-Variable Map Patterns

- The number of adjacent squares that may be combined always represent a number that is a power of 2 such as 1, 2, 4, 8, and 16.
  - One square represents one minterm with four literals.
  - Two adjacent squares represents a term of three literals.
  - Four adjacent squares represents a term of two literals.
  - Eight adjacent squares represents a term of one literal.
  - Sixteen adjacent squares represents the entire map and produces a function that is always equal to 1.

Minimization Example

wx \ yz	00	01	11	10
00			1	1
01			1	1
11	1		1	1
10	1		1	1

$$F(w, x, y, z) = \sum(2, 3, 6, 7, 8, 10, 11, 12, 14, 15)$$

- The eight adjacent squares can be combined to form the one literal term y.
- Four adjacent squares can be combined by folding property to form the two literal term wz'.
- The simplified expression will be logical sum of two product terms producing the function

F = y+wz'

Another Example

wx \ yz	00	01	11	10
00	1		1	1
01	1			
11				
10	1		1	1

$F(w, x, y, z) = \sum(0, 2, 3, 4, 8, 10, 11)$

- Four adjacent corners can be combined to form the two literal term x′z′.
- Four adjacent squares can be combined to form the two literal term x′y.
- The remaining 1 is combined with a single adjacent 1 to obtain the three literal term w′y′z′.

$$F = x'z' + x'y + w'y'z'$$

Another Example

Simplify the function $F=A'BC'+A'CD'+ABC+AB'C'D'+ABC'+AB'C$ using map method

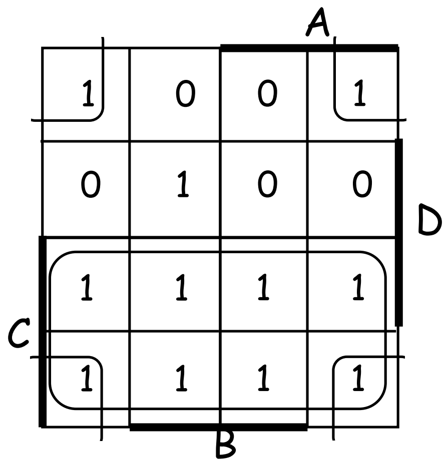
AB \ CD	00	01	11	10
00	0	0	0	1
01	1	1	0	1
11	1	1	1	1
10	1	0	1	1

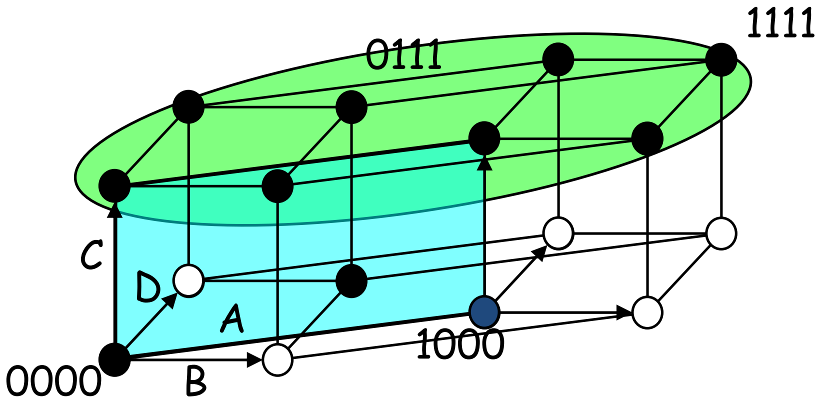
$F=BC'+CD'+AC+AD'$

Another Example

Simplify the function $F(A,B,C,D) = \sum m(0,3,5,8,9,10,11,12,13,14,15)$

$F = C + A'BD + B'D'$

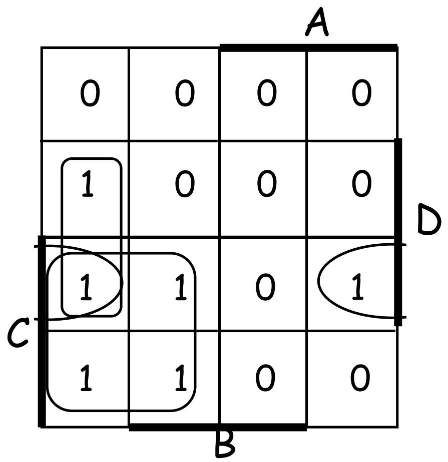




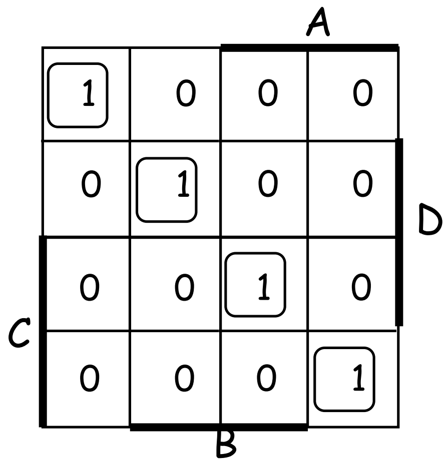
Solution set can be considered as a coordinate System!

Another Example

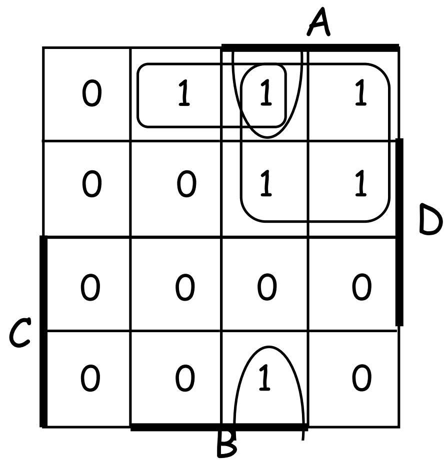
Magnitude Comparator



K-map for LT



K-map for EQ



K-map for GT

LT = A' B' D + A' C + B' C D

EQ = A'B'C'D' + A'BC'D + ABCD + AB'CD'

GT = B C' D' + A C' + A B D'

Example 3-5

$F(w,x,y,z) = \Sigma(0,1,2,4,5,6,8,9,12,13,14)$

Solution:

- 1 is marked in each minterm that represents the function
- Find the possible adjacent squares and mark them with rectangles
- We combine eight adjacent squares to get a single literal term y'
- The top two 1's on the right are combined with the top two 1's on the left to give the term w'z'
- We combine the single square left on right with three adjecent squares that are already used to give the term xz'
- The logical sum of these three terms gives:

  $F = y' + w'z' + xz'$

wx \ yz	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}

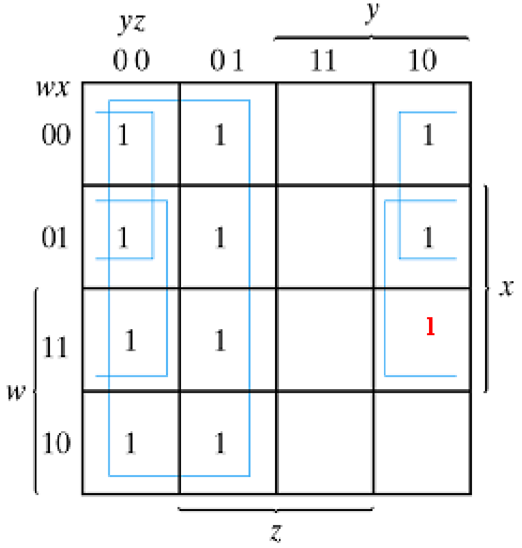


Fig. 3-9 Map for Example 3-5; $F(w, x, y, z) = \Sigma(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14) = y' + w'z' + xz'$

Example 3-6

F = A′B′C′+B′CD′+A′BCD′+AB′C′

Solution:

- Each of three literal term in map is represented by two squares and four literal term in map is represented by one square
- We combine the 1's in the four corners to give the term B′D′
- The two left hand 1's in the top row are combined with two 1's in the bottom row to give the term B′C′
- The remaining 1's may be combined in the two-square area to give the term A′CD′
- The logical sum of these three terms gives:

F = B′D′ + B′C′+ A′CD′

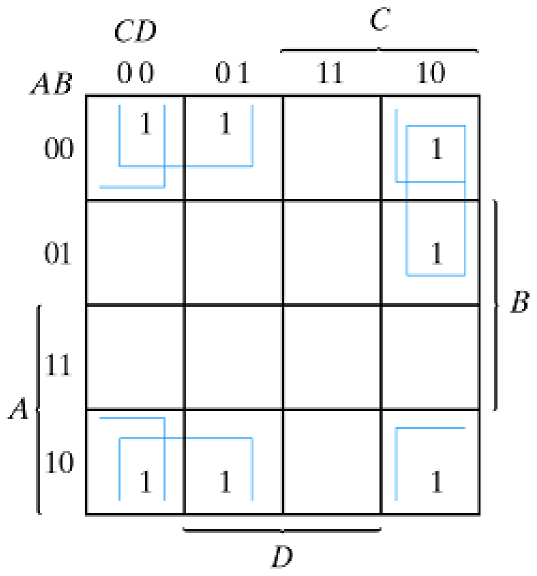
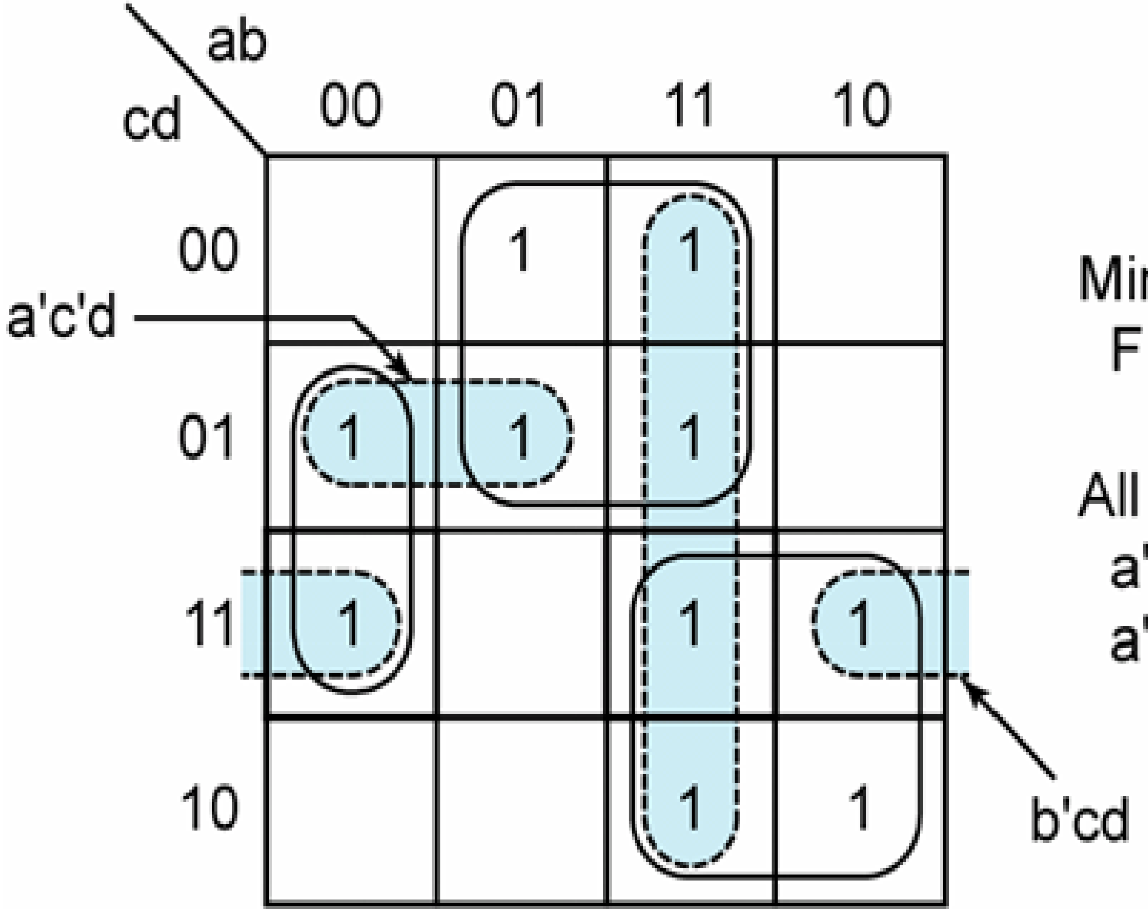


Fig.3-10 Map for Example 3-6; $A'B'C' + B'CD' + A'BCD' + AB'C' = B'D' + B'C' + A'CD'$

Prime Implicants Definitions

- **Implicant of a function F**
  a single 1 (= minterm) or any group of 1's which can be combined together in a K-map (i.e., 1's that are adjacent and which are grouped in a number that is always a power of 2). represents a product term which is called an implicant of a function F. An implicant represents a product term that can be used in a SOP expression for that function, that is, the function is 1 whenever the implicant is 1 (and maybe other times, as well …)
- **Prime Implicant**
  is an implicant that cannot be combined with another one to eliminate a literal. In other word each prime implicant corresponds to a product term in one of the minimum SOP expression for the function. A prime implicant is an implicant that is not fully contained in any other implicant.
- **Essential prime implicant**
  is a prime implicant that includes at least one 1 that is not included in any other prime implicant. In other words if a particular element of the on-set is covered by only one prime implicant, than that implicant is called an essential prime implicant.

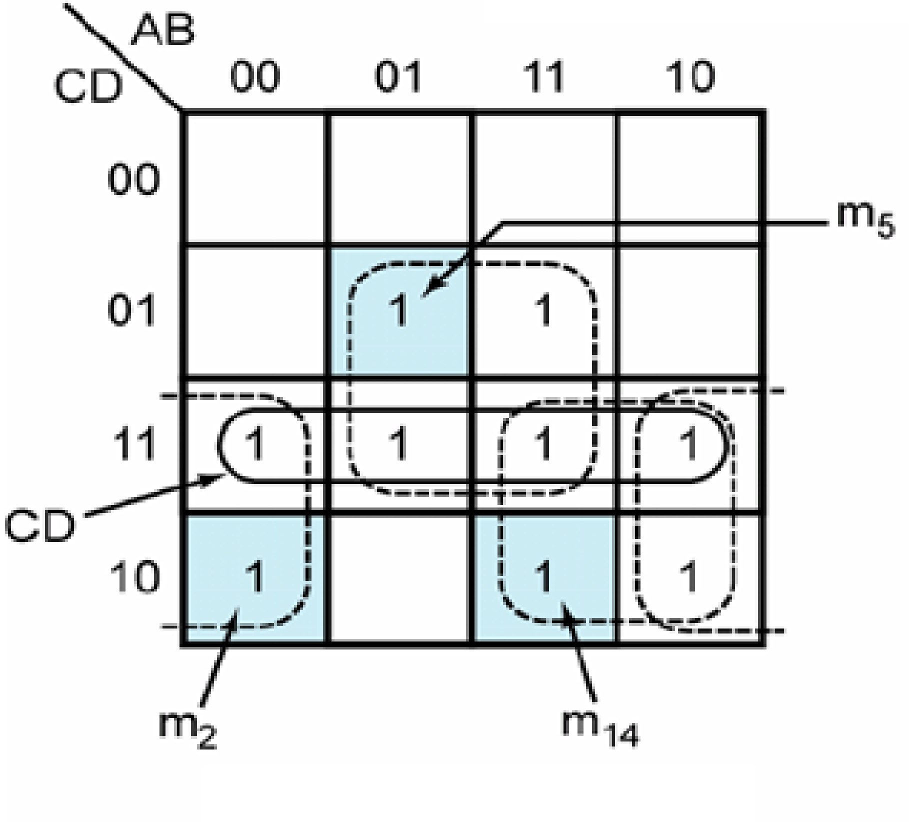
Example of Prime Implicants



Minimum solution:
F = a'b'd + bc' + ac

All prime implicants:
a'b'd, bc', ac,
a'c'd, ab, b'cd

Essential Prime Implicants



Essential Prime Implicants:

BD, AC, B′C

Distinguished 1-cells: m_2, m_5, m_{14}

Other Prime Implicants:

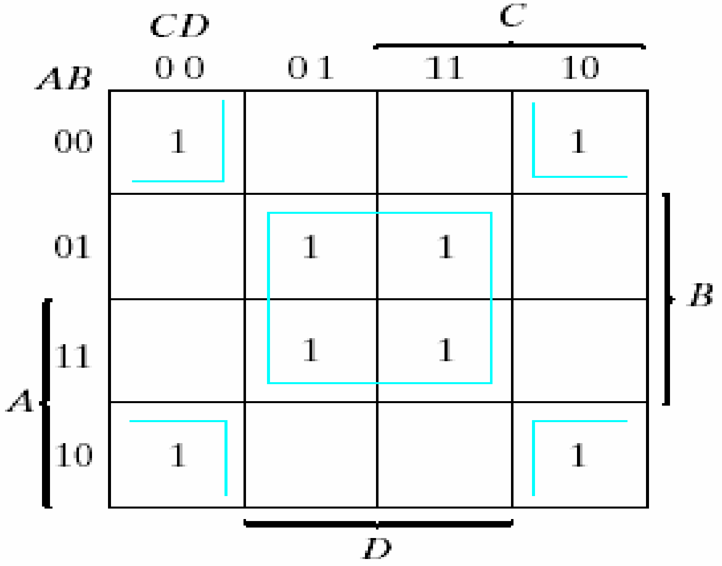
CD

F=BD+AC+B′C

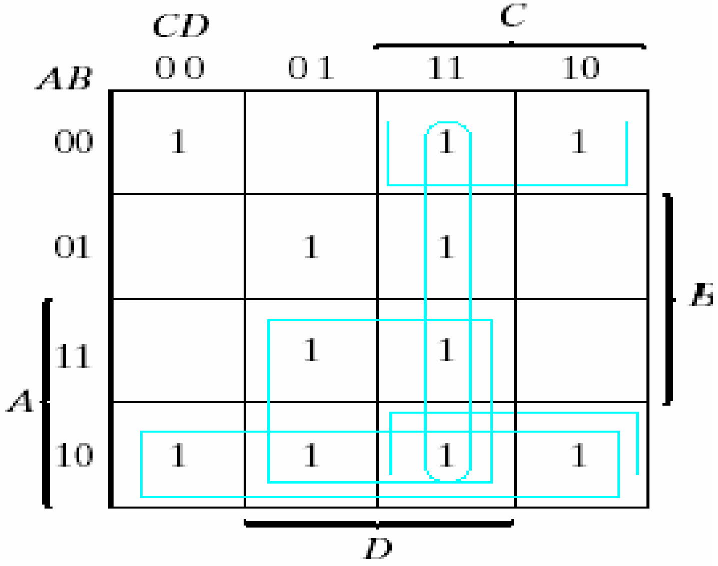
Functions with Multiple Solutions

Example: Find all the possible solutions of the following function using map method:-

$$F(A,B,C,D)=\sum(0,2,3,5,7,8,9,10,11,13,15)$$



(a) Essential prime implicants BD and B′D′



(b) Prime implicants CD, B′C AD, and AB′

Functions with Multiple Solutions Cont…

- All PIs: BD, B′D′,CD, B′C, AD, AB′
- EPIs: BD, B′D′
- All possible solutions are:-

F=BD+B′D′+CD+AD

F=BD+B′D′+CD+AB′

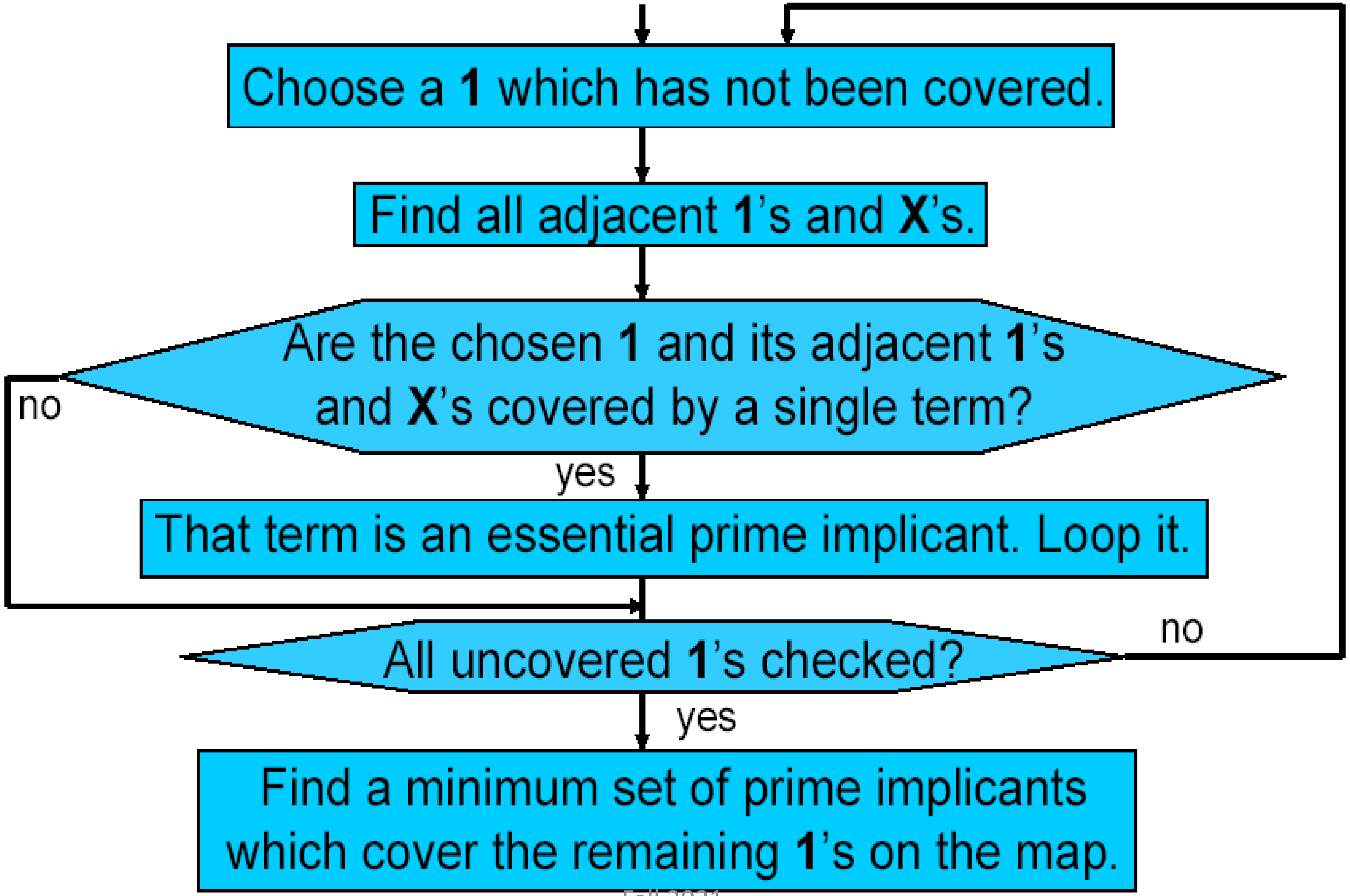
F=BD+B′D′+B′C+AD

F=BD+B′D′+B′C+AB′

OR

F= BD+B′D′+[(CD+AD) or (CD+AB′) or (B′C+AD) or (B′C+AB′)]

Algorithm for determining minimum SOP using a K-map



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