The Z-Transform

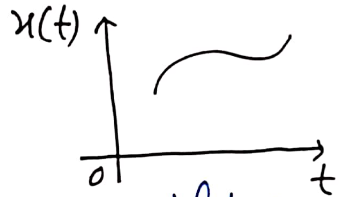
In general, time-domain functions/signals could be continuous & discrete as below:

Continuous

Time is continuous

Can be defined for all values of t, 1, 1.001, 1.002 etc

Independent variable is denoted by



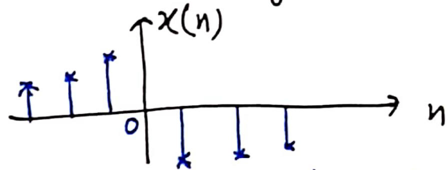
All amplitudes & all values

Discrete

samples are for discrete time

discrete sequence of time 1, 2, 3, 4, - -

Here independent variable is 'time' denoted by 'n'.



Amplitudes at discrete values of t.

In general a discrete time sequence is .... $x(-3), x(-2), x(-1), x(0), x(1), \ldots\ldots$

For a discrete time signal $x[n]$, the Z-transform; denoted by $X(Z)$ is defined as:

$$Z\left[x[n]\right] = X(Z) = \sum_{n=-\infty}^{\infty} x[n]\, Z^{-n} \quad \text{——(i)}$$

Z-transform is introduced to represent discrete-time signals (samples or sequences) in the Z-domain (Z-is a complex variable). $Z = re^{j\theta}$, in polar form where r is the magnitude of Z and θ is the argument of Z. $x[n]$ and $X(Z)$ are said to form a Z-transform pair

$$x[n] \longleftrightarrow X(Z).$$

[Z.transform 1]

EX-1: A finite sequence $x[n]$ is defined as $x[n] = [5, 3, -3, 0, 4, -2]$. Find $X(z)$.

Sol: First term is $x[0]$, fully right sided sequence

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{5} x[n] z^{-n}$$

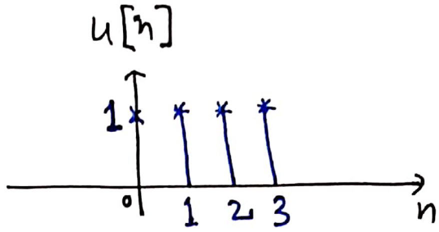
$a_0, a_1, a_2, a_3, \ldots\ldots$
$\uparrow$
$x[0]$

$$= (5) z^{0} + 3(z^{-1}) + (-3)(z^{-2}) + 4(z^{-4}) + (-2)(z^{-5})$$

$$= 5 + 3z^{-1} - 3z^{-2} + 4z^{-4} - 2z^{-5}.$$

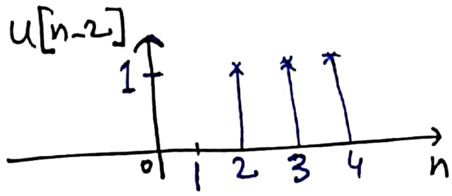
Unit-step sequence:-

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Shifted step sequence:

$$u[n-2] = \begin{cases} 1, & n \geq 2 \\ 0, & n < 2. \end{cases}$$



Z-transform of the unit-step sequence:-

$$X(z) = \sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=-\infty}^{-1} u[n] z^{-n} + \sum_{n=0}^{\infty} u[n] z^{-n}.$$

$$X(z) = \sum_{n=0}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^{n}$$

Important Summation: $\displaystyle\sum_{n=0}^{\infty} a^{n} = \frac{1}{1-a} \quad (|a| < 1).$

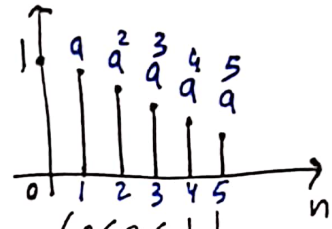
$$X(z) = \sum_{n=0}^{\infty} (z^{-1})^{n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}, \quad \left(\begin{array}{c} |z| > 1 \\ \cancel{|z| < 1} \end{array}\right)$$

$|z^{-1}| < 1$

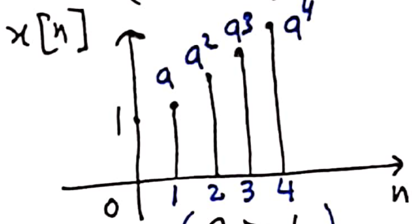
$|\frac{1}{z}| < 1$

[z.transform 2]

EX:- Find the Z-transform of the right sided exponential sequence: $x[n] = a^n u[n]$.



(0<a<1)



(a>1)

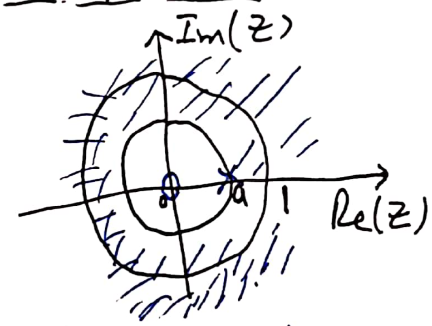
Sol:-
$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$
$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$
$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad \text{---(2)}$$

The region of Convergence (ROC) is $|az^{-1}| < 1 \Rightarrow |\frac{a}{z}| < 1$

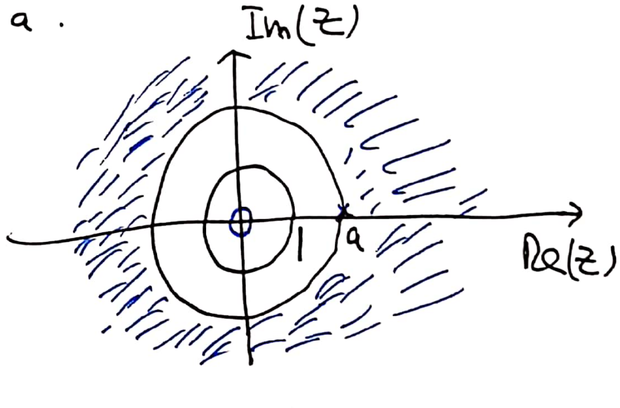
'or' equivalently, $|z| > |a|$. $\quad |\frac{z}{a}| > 1 \Rightarrow |z| > |a|$.

equation (2) suggests that $X(z)$ has a zero at $z=0$ and a simple pole at $z=a$.

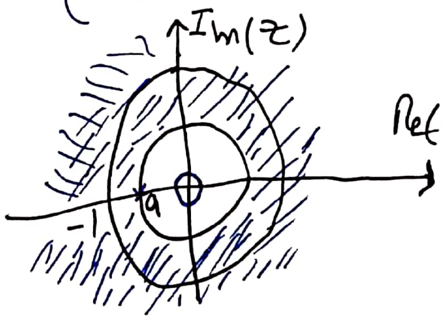
Plot of ROC:-



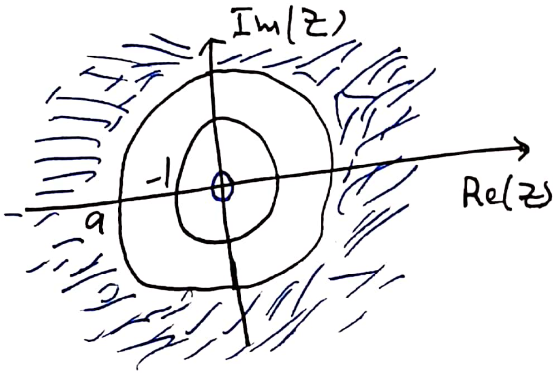
(0<a<1)



a>1



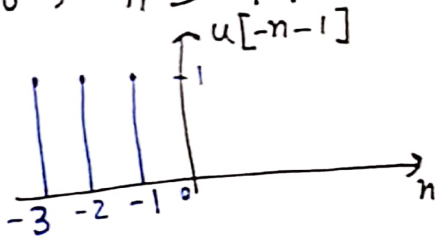
-1<a<0



a<-1

[Z transform 3]

Reverse step function:- $-u[-n-1] = \begin{cases} -1, & n \le -1 \\ 0, & n > -1 \end{cases}$



$$\mathcal{Z}\left\{-u[-n-1]\right\}$$

$$= \sum_{n=-\infty}^{\infty} -u[-n-1]\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (-1)\, z^{-n} = \sum_{n=1}^{\infty} (-1)\, z^{n} = 1 - \sum_{n=0}^{\infty} z^{n}$$

$$= 1 - \frac{1}{1-z} = \frac{1-z-1}{1-z} = \frac{-z}{1-z} = \frac{-z}{1-z} \quad (|z|<1).$$

OR: $X(z) = \sum_{n=1}^{\infty} (-1) z^{n} = \sum_{n=0}^{\infty} -(z^{k+1}) = -z \sum_{n=0}^{\infty} z^{n} = \frac{-z}{1-z}$ (same).

EX:- Find the z-transform of left-sided exponential sequence: $x[n] = -a^{n} u[-n-1]$.

Sol:- $X(z) = \sum_{n=-\infty}^{\infty} -a^{n} u[-n-1]\, z^{-n} = -\sum_{n=-\infty}^{-1} a^{n} z^{-n}$

$$= -\sum_{n=1}^{\infty} (a^{-1}z)^{n} = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^{n}.$$

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = -\frac{a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad (|z|<|a|)$$

(ii). $x[n] = a^{-n} u[-n-1]$.

$$X(z) = \sum_{n=-\infty}^{\infty} a^{-n} u[-n-1]\, z^{-n} = \sum_{n=-\infty}^{-1} (az)^{-n}$$

$$= \sum_{1}^{\infty} (az)^{n} = \sum_{n=0}^{\infty} (az)^{n} - 1 = \frac{1}{1-az} - 1$$

$$= \frac{1-1+az}{1-az} = \frac{az}{1-az} = \frac{z}{z-\frac{1}{a}}.$$

ROC: $|az| < 1 \Rightarrow |z| < 1/|a|$.

[z transform 4]

Ex:- Find the z-transform of the sequence:

$$x[n] = a^{n} u[-n-1].$$

Sol:-

$$X(z) = \sum_{n=-\infty}^{+\infty} a^{n} u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} (a z^{-1})^{n}$$

$$= \sum_{n=1}^{\infty} (a z^{-1})^{-n} = \sum_{n=0}^{\infty} (a^{-1} z)^{n} - 1$$

$$= \frac{1}{1-a^{-1}z} - 1 = \frac{1-1+a^{-1}z}{1-a^{-1}z} = \frac{a^{-1}z}{1-a^{-1}z} = \frac{z}{a-z}$$

ROC: $|a^{-1}z| < 1 \Rightarrow |\frac{z}{a}| < 1 \Rightarrow |z| < |a|.$

Conclusion:

$$a^{n} u[n] \longleftrightarrow \frac{z}{z-a}$$

$$-a^{n} u[-n-1] \longleftrightarrow \frac{z}{z-a}$$

$$a^{-n} u[n] \longleftrightarrow \frac{az}{az-1}$$

$$a^{-n} u[-n-1] \longleftrightarrow \frac{az}{az-1}.$$

Ex:-

$$2^{n} u[-n-1] \longleftrightarrow \frac{z}{2-z} = -\frac{z}{z-2}.$$

[z transform 5]

EX:- Find the Z-transform $X(Z)$ and sketch the pole-zero plot with the ROC of the sequence:

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{1}{3}\right)^n u[n]$$; sum of two exponential sequences.

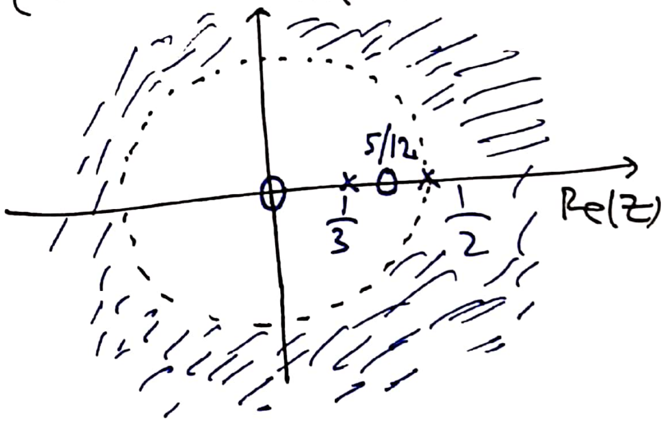
we know that $a^n u[n] \longleftrightarrow \dfrac{z}{z-a}$, $|z| > |a|$.

$$\left(\tfrac{1}{2}\right)^n u[n] \longleftrightarrow \frac{z}{z-\frac{1}{2}}, \quad |z| > \frac{1}{2}$$

Also, $$\left(\tfrac{1}{3}\right)^n u[n] \longleftrightarrow \frac{z}{z-\frac{1}{3}}, \quad |z| > \frac{1}{3}.$$

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{3}} \quad \left(|z| > \tfrac{1}{2}\right)$$

$$X(z) = \frac{2z\left(z-\frac{5}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}.$$



$X(z)$ has zeros at $z=0$ & $z = \frac{5}{12}$.

Simple poles at $z = \frac{1}{2}$ & $z = \frac{1}{3}$.

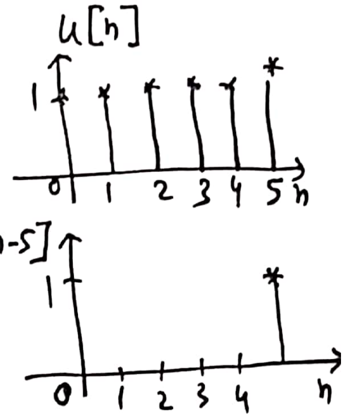
The ROC is $|z| > \frac{1}{2}$.

Practice: Find the Z-transform $X(z)$ and sketch the pole-zero plot with the ROC of the sequence:

$$x[n] = -\left(\tfrac{1}{2}\right)^n u[-n-1] - \left(\tfrac{1}{3}\right)^n u[-n-1].$$

EX:- Find the z-transform of the sequence:

$$x[n] = \left(\frac{1}{5}\right)^n \left[u(n) - u(n-5)\right].$$



Sol:-

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=0}^{4} \left(\frac{1}{5}\right)^n z^{-n}$$

$$= \sum_{n=0}^{4} \left(\frac{1}{5} z^{-1}\right)^n$$

$$= \frac{1 - \left(\frac{1}{5} z^{-1}\right)^{4+1}}{1 - \frac{1}{5} z^{-1}} = \frac{1 - (0.2)^5 z^{-5}}{1 - \frac{1}{5} z^{-1}}$$

$$X(z) = \frac{z^5 - (0.2)^5}{z^4 \left(z - \frac{1}{5}\right)}.$$

OR:

$$X(z) = Z\left\{\left(\frac{1}{5}\right)^n u(n)\right\} - Z\left\{\left(\frac{1}{5}\right)^n u(n-5)\right\}.$$

$$= \frac{z}{z - \frac{1}{5}} - \left(\frac{1}{5}\right)^5 Z\left\{\left(\frac{1}{5}\right)^{n-5} u(n-5)\right]$$

$$= \frac{z}{z - 1/5} - \left(\frac{1}{5}\right)^5 Z\left\{\left(\frac{1}{5}\right)^k u(k)\right\}, \quad k = n-5$$

$$= \frac{z}{z - 1/5} - (0.2)^5 z^{-5} \frac{z}{z - 1/5}.$$

$$= \frac{z - (0.2)^5 z^{-4}}{z - 1/5} = \frac{z^5 - (0.2)^5}{z^4 (z - 1/5)}.$$

Ex:- Find the Z-transform of $x[n] = \cos(n\omega)\, u(n)$.

Sol:-
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} \cos(n\omega)\, u(n)\, z^{-n} = \sum_{n=0}^{\infty} \cos(n\omega)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{e^{jn\omega} + e^{-jn\omega}}{2}\, z^{-n}$$

$$= \frac{1}{2}\left\{ \sum_{n=0}^{\infty} \left(e^{j\omega} z^{-1}\right)^{n} + \sum_{n=0}^{\infty} \left(e^{-j\omega} z^{-1}\right)^{n} \right\}$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j\omega} z^{-1}} + \frac{1}{1-e^{-j\omega} z^{-1}} \right] = \frac{1}{2}\left[\frac{z}{z-e^{j\omega}} + \frac{z}{z-e^{-j\omega}} \right]$$

$$X(z) = \frac{1}{2}\left[\frac{z(z-e^{-j\omega}) + z(z-e^{j\omega})}{z^2 - z e^{-j\omega} - z e^{j\omega} + 1} \right]$$

$$= \frac{1}{2}\left[\frac{z^2 - z e^{-j\omega} + z^2 - z e^{j\omega}}{z^2 + 1 - z(e^{j\omega} + e^{-j\omega})} \right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z(e^{j\omega} + e^{-j\omega})}{z^2 + 1 - z(e^{j\omega} + e^{-j\omega})} \right] = \frac{1}{2}\left[\frac{2z^2 - 2z\cos\omega}{z^2 + 1 - 2z\cos\omega} \right]$$

$$X(z) = \frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}.$$

Practice: show that

$$\mathcal{Z}\{\sin k\omega T\} = \frac{z \sin\omega T}{z^2 - 2z\cos\omega T + 1} \quad (|z|>1)$$

[z-transform 8]

The first shift property (delayed signal property):-

Let $\{x_k\}$, $\{y_k\}$ be two sequences such that

$$y_k = x_{k-k_0}$$

$$Z\{y_k\} = Z\{x_{k-k_0}\} = \frac{1}{z^{k_0}} Z\{x_k\}.$$

Ex:- The sequence $\{x_k\}$ is generated by

$$x_k = \left(\frac{1}{2}\right)^k, \ (k \geq 0).$$

Determine the z-transform of the shifted sequence x_{k-2}.

We know that $Z\{x_k\} = \dfrac{z}{z - 1/2}$ $\left(|z| > \frac{1}{2}\right)$

Thus, $Z\{x_{k-2}\} = \dfrac{1}{z^2} \dfrac{z}{z-\frac{1}{2}} = \dfrac{2}{z(2z-1)}$ $\left(|z| > \frac{1}{2}\right)$.

The second shift property (Advancing):

$$y_k = x_{k+1}, \ (k \geq 0).$$

$$Z\{y_k\} = \sum_{k=0}^{\infty} y_k z^{-k} = \sum_{k=0}^{\infty} x_{k+1} z^{-k} = z \sum_{k=0}^{\infty} x_{k+1} z^{-k-1}.$$

$$= z \sum_{n=1}^{\infty} x_n z^{-n} = z\left(\sum_{n=0}^{\infty} x_n z^{-n} - x_0\right)$$

$$= z X(z) - z x_0.$$

Similarly, $Z\{x_{k+2}\} = z^2 X(z) - z^2 x_0 - z x_1$

In general,

$$Z\{x_{k+k_0}\} = z^{k_0} X(z) - \sum_{n=0}^{k-1} x_n z^{k_0-n}.$$

Shift properties are used in solving difference equations using z-transform.

[Z-transform 9]

Ex:- Determine the Z-transform & sketch ROC of the signal:

$$x[n] = \begin{cases} \left(\frac{1}{3}\right)^n, & n \geq 0 \\ \left(\frac{1}{2}\right)^{-n}, & n < 0. \end{cases}$$

Sol:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{z}{2}\right)^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{3z}\right)^n.$$

$$= -1 + \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k + \sum_{n=0}^{\infty} \left(\frac{1}{3z}\right)^n$$

$$= \frac{1}{1-\frac{1}{3z}} + \frac{1}{1-(z/2)} - 1$$

$$X(z) = \frac{\frac{5}{3} z}{\left(z-\frac{1}{3}\right)(2-z)}, \quad \left(\frac{1}{3} < |z| < 2\right).$$

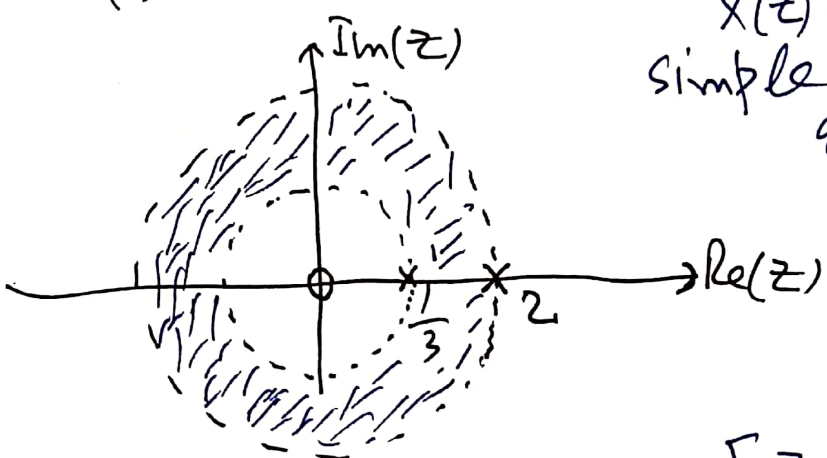
ROC:

$$\left|\frac{z}{2}\right| < 1 \Rightarrow |z| < 2$$

$$\left|\frac{1}{3z}\right| < 1 \Rightarrow |3z| > 1 \Rightarrow |z| > \frac{1}{3}$$

$$\Rightarrow \frac{1}{3} < |z| < 2).$$

$X(z)$ has a zero at $z = 0$, simple poles at $z = 2$ & $z = 1/3$.



[Z-transform 10]