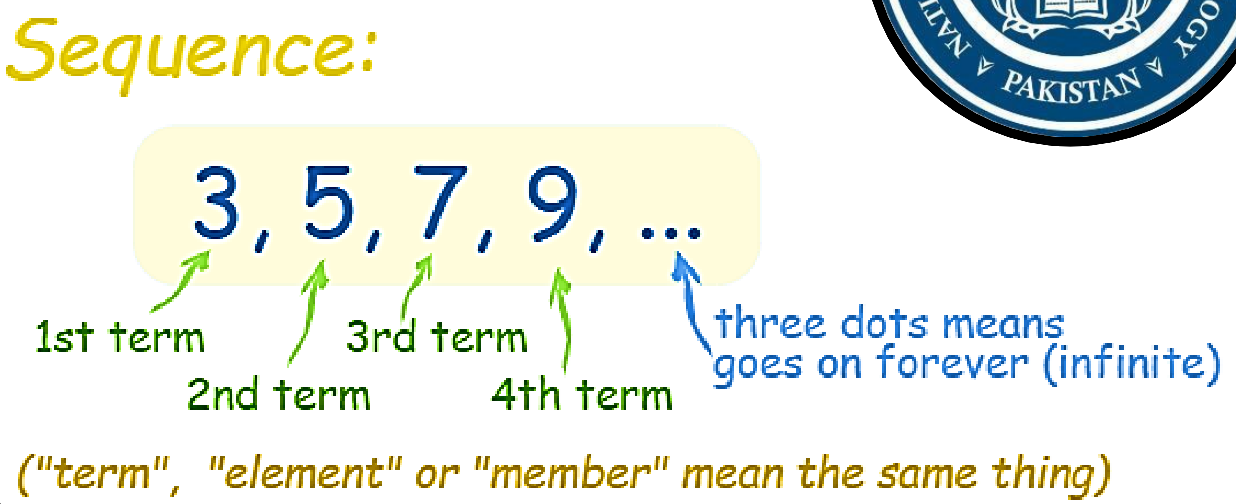
Infinite Sequences and Series

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

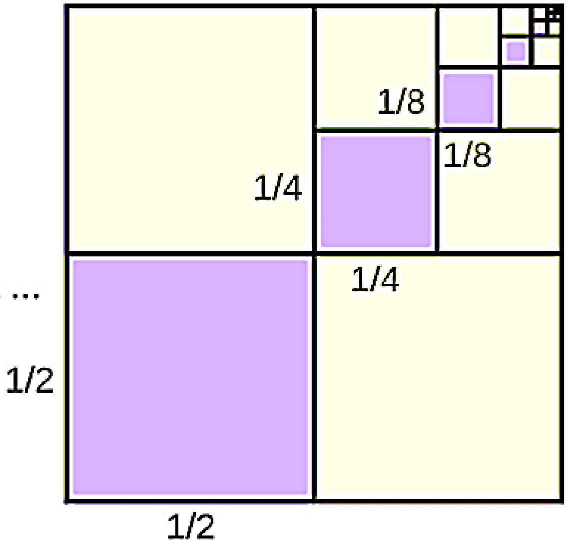
Chapter: 11

Section: 11.1, 11.2



Infinite Series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

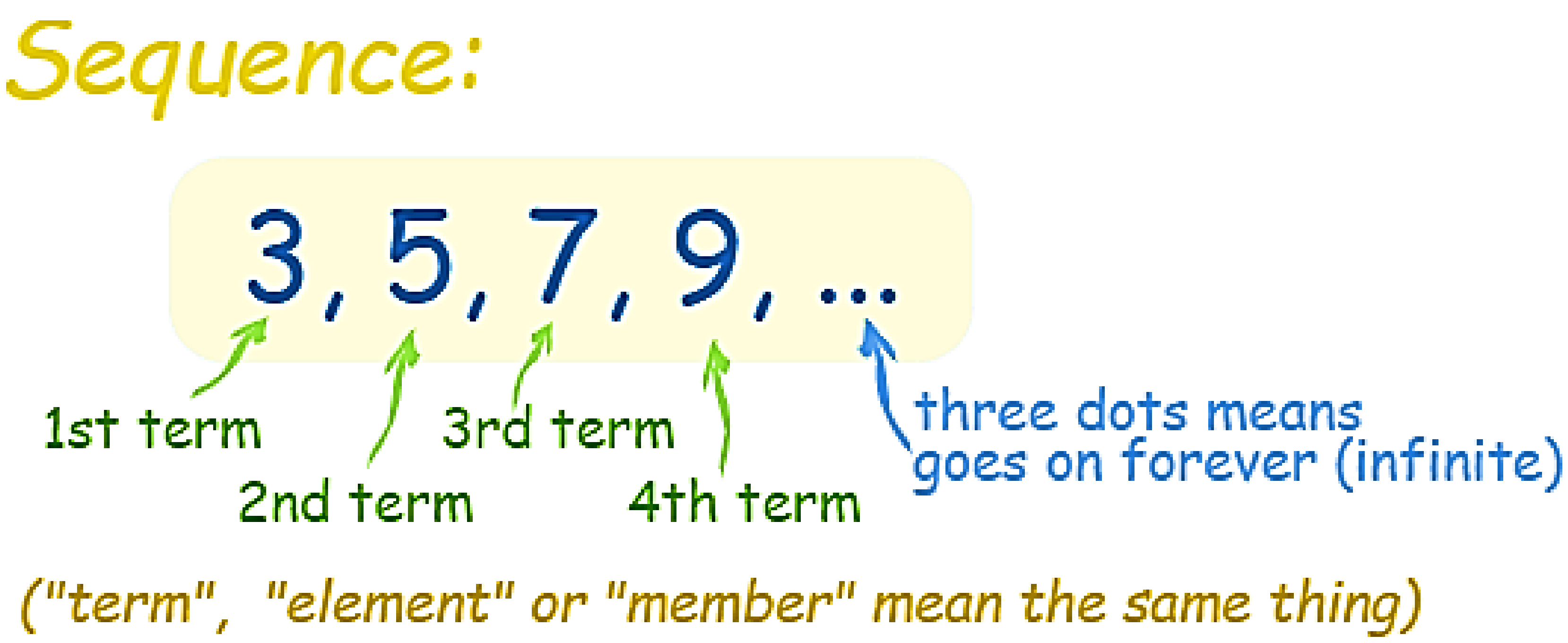


Calculus & Analytical Geometry MATH-101

Instructor: Dr. Naila Amir (SEECS, NUST)

Infinite Sequences

Section: 11.1



Infinite Sequence

- A Sequence is a list of things (usually numbers) that are in order.
- When the sequence goes on forever it is called an **infinite sequence**, otherwise, it is a **finite sequence**
- A Sequence is like a Set, except:
  - the terms are **in order** (with Sets the order does not matter)
  - the same value can appear many times (only once in Sets)
- An infinite sequence of numbers is a function whose domain is the set of positive integers (Natural number)

Infinite Sequence

- The values of the sequence $a: \mathbb{N} \to \mathbb{R}$, are usually written as:

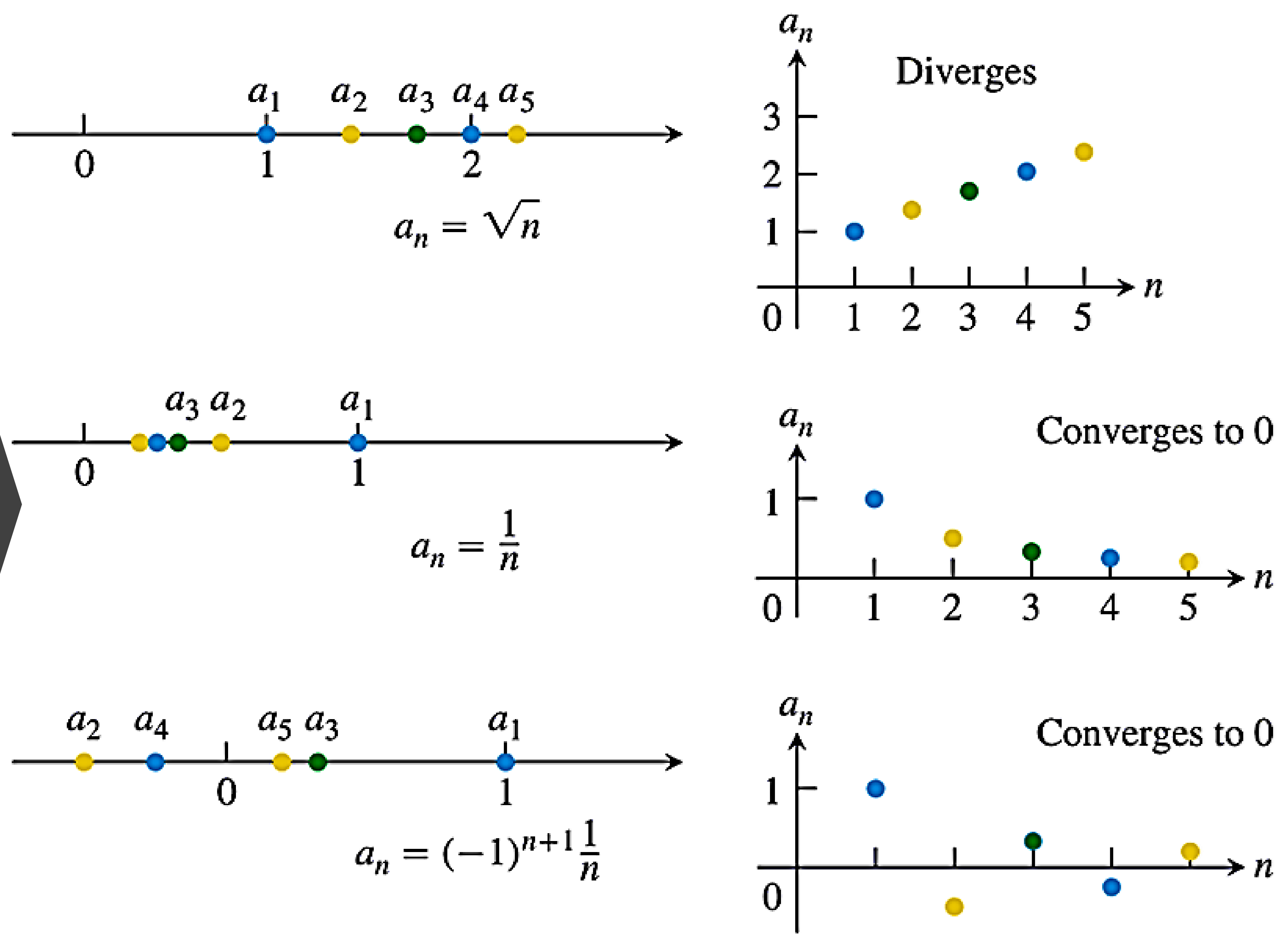
$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

instead of $a(1), a(2), \dots, a(n), \dots$ at the points $1,2, \dots, n, \dots$ of its domain $\mathbb{N}$.

- Each of the following are equivalent ways of denoting a sequence.

$$\{a_1, a_2, a_3, \dots, a_n, \dots\}, \qquad \{a_n\}, \qquad \{a_n\}_{n=1}^{\infty}$$

Infinite Sequence



Sequences can be represented as points on the real line or as points in the plane where the horizontal axis n is the index number of the term and the vertical axis a_n is its value.

Limit of a sequence (Convergence/Divergence)

- A sequence $\{a_n\}$ has the limit L if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon, \quad whenever\ n > N$$

- We write

$$\lim_{n\to\infty} a_n = L \quad or \quad a_n \to L\ as\ n \to \infty.$$

- If $\lim_{n\to\infty} a_n$ exists we say that the sequence **converges**. Note that for the sequence to converge, the limit must be finite.
- If the sequence does not converge, we will say that it **diverges.** Note that a sequence diverges if it approaches to infinity or if the sequence does not approach to anything

Limit Laws

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then

$$\lim_{n\to\infty} b_n = L$$

Monotonic Sequences

- A sequence $\{a_n\}$ is **nondecreasing** if

$$a_{n+1} \geq a_n; \quad \forall\, n,$$

and is **increasing** if $a_{n+1} > a_n;\ \forall\, n$.

- A sequence $\{a_n\}$ is **nonincreasing** if

$$a_{n+1} \leq a_n; \ \forall\, n,$$

and is **decreasing** if $a_{n+1} < a_n;\ \forall\, n$.

Bounded Sequences

- A sequence $\{a_n\}$ is **bounded above** if there exists a real number M such that

$$M \geq a_n; \quad \forall\, n,$$

  The number M is sometimes called an upper bound for the sequence.

- A sequence $\{a_n\}$ is **bounded below** if there exists a real number m such that

$$m \leq a_n; \quad \forall\, n,$$

  The number m is sometimes called a lower bound for the sequence.

- If the sequence is bounded below as well as bounded above, then we call the sequence **bounded**.

Theorem

- A bounded monotonic sequence is convergent.
- Note that we can make several variants of this theorem.
- If $\{a_n\}$ is an **increasing (or nondecreasing)** sequence which is **bounded above** i.e., there exists a real number M such that $M \geq a_n;\ \forall\, n$, then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = M.$$

- If $\{a_n\}$ is a **decreasing (or nonincreasing)** sequence which is **bounded below** i.e., there exists a real number m such that $m \leq a_n;\ \forall\, n$, then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = m.$$

Infinite Series

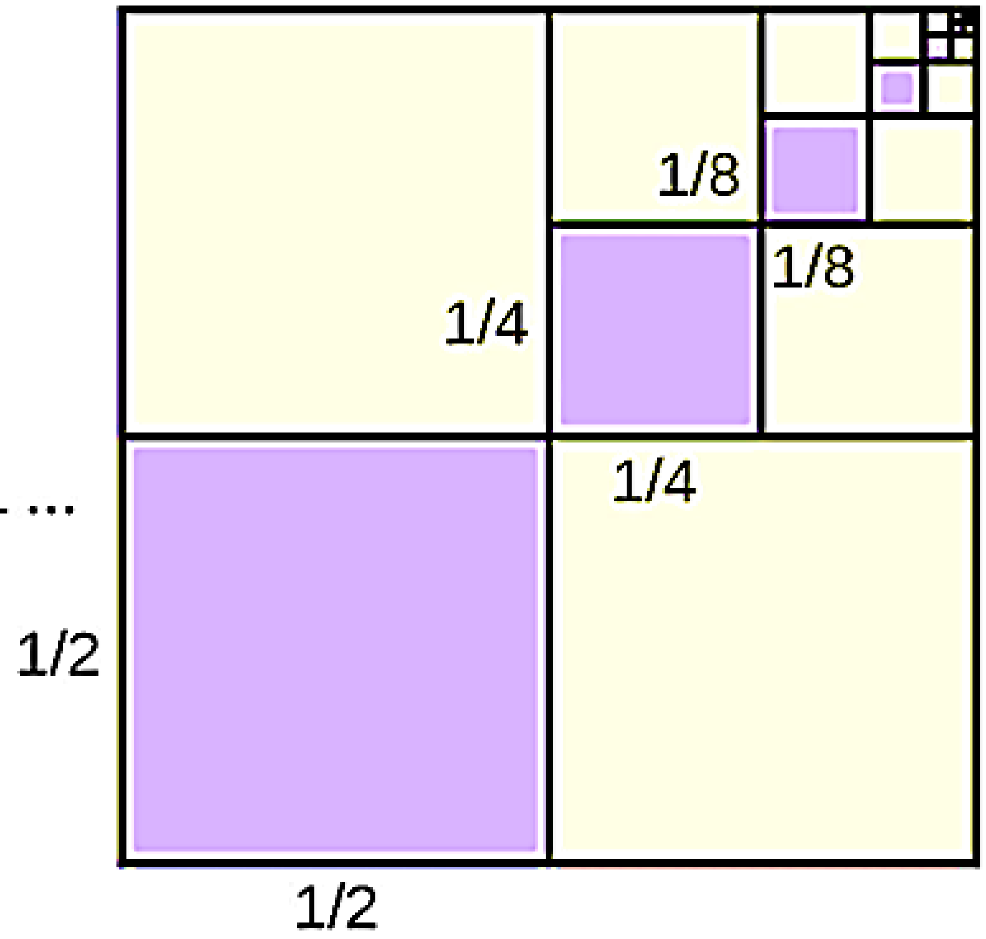
Section: 11.2

Infinite Sequence

$$\{a_n\} = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$$

Infinite Series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$



Infinite Series

- An infinite series is the **sum** of infinite terms that follow a rule.
- When we have an infinite sequence of values:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ \dots$$

which follow a rule (in this case each term is half the previous one), and we **add them all up**:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ \dots =$$

and we get an **infinite series**.

- "Series" sounds like it is the **list of numbers**, but it is actually when we add them together.

Example

- Given the sequence $\{a_n\} = \left\{\frac{1}{2^n}\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$, consider the following sums:

$$\begin{array}{ccccc} a_1 & = & 1/2 & = & 1/2 = 2-1/2 \\ a_1 + a_2 & = & 1/2 + 1/4 & = & 3/4 = 2^2-1/2^2 \\ a_1 + a_2 + a_3 & = & 1/2 + 1/4 + 1/8 & = & 7/8 = 2^3-1/2^3 \\ a_1 + a_2 + a_3 + a_4 & = & 1/2 + 1/4 + 1/8 + 1/16 & = & 15/16 = 2^4-1/2^4 \end{array}$$

- In general, we can show that:

$$a_1 + a_2 + a_3 + \cdots + a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

Example

- Let S_n be the sum of the first n terms of the sequence $\{1/2^n\}$. That is:

$$S_1 = a_1 = \frac{1}{2},$$

$$S_2 = a_1 + a_2 = \frac{3}{4},$$

$$\vdots$$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i = 1 - \frac{1}{2^n}.$$

- The S_n are called the **partial sums** and they form sequence, $\{S_n\}$.

Example

- For the present case, the limit of the sequence of partial sums $\{S_n\}$ is:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1.$$

- This limit can be interpreted as: the sum of all the terms of the sequence $\{1/2^n\}$ is 1.
- Moreover, note that:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i = 1.$$

- This example illustrates some interesting concepts that we are going to explore in this topic. We begin this exploration with some definitions.

Infinite Series

- Let $\{a_n\}$ be a sequence then the sum

$$\sum_{n=1}^{\infty} a_n ,$$

is known as an **infinite series** (or simply series).

- The sum of the first n terms $S_n = \sum_{i=1}^{n} a_i$, is called the $\boldsymbol{n^{th}}$ **partial sum** and the sequence $\{S_n\}$ is the sequence of partial sums.
- If the sequence $\{S_n\}$ converges to L, we say that the series $\sum_{n=1}^{\infty} a_n$ **converges** to L (or sum of the series is L) and we write

$$\sum_{n=1}^{\infty} a_n = L.$$

- If the sequence $\{S_n\}$ diverges, we say that the series $\sum_{n=1}^{\infty} a_n$ **diverges.**

Infinite Series

- Thus, an infinite series is an expression that can be written in the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

- It is the summation of all elements in a sequence $\{a_n\}$.
- Remember the difference: Sequence is a collection of numbers; a Series is its summation.
- Using our new terminology, we can state that the series $\sum_{n=1}^{\infty} 1/2^n$ converges and

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Example: Convergence of a Series

- It seems difficult to understand how it is possible that a sum of infinite numbers could be finite. For this let us consider an example.
- Note that:

$$\frac{1}{3} = 0.33333\ldots\ldots = 0.3 + 0.03 + 0.003 + 0.0003 + \ldots$$
$$= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots + \frac{3}{10^n} + \cdots$$
$$= \sum_{n=1}^{\infty} \frac{3}{10^n}$$

- Thus, we conclude that the series $\sum_{n=1}^{\infty} 3/10^n$ is convergent and sum of this series is $1/3$.

Geometric Series

Geometric series are series of the form:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots,$$

where a and r are fixed real numbers such that $a, r \neq 0$. r is called the common ratio.

Case 1: If $\boldsymbol{r = 1}$, then the n^{th} partial sum of the series is:

$$S_n = a + a(1) + a(1)^2 + a(1)^3 + \cdots + a(1)^{n-1} = an.$$

Observe that $\lim_{n\to\infty} S_n = \pm\infty$, depending upon the sign of a. This means that the series is divergent if $r = 1$.

Case 2: If $\boldsymbol{r = -1}$, then the n^{th} partial sum of the series is:

$$S_n = a + a(-1) + a(-1)^2 + a(-1)^3 + \cdots + a(-1)^{n-1}.$$

In this case the series diverges because the n^{th} partial sum alternate between a and 0.

Geometric Series

Case 3: If $|\boldsymbol{r}| \neq \mathbf{1}$, then

$$S_n = a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \cdots + a \cdot r^{n-1} \quad (1)$$

$$r \cdot S_n = a \cdot r + a \cdot r^2 + a \cdot r^3 + a \cdot r^4 + \cdots + a \cdot r^n \quad (2)$$

$$(1) - (2) \Longrightarrow S_n - r \cdot S_n = a - a \cdot r^n$$

$$\Longrightarrow S_n(1 - r) = a(1 - r^n)$$

$$\Longrightarrow S_n = \frac{a(1 - r^n)}{1 - r}, r \neq 1 \quad (3)$$

- If $|r| < 1$, then $S_n \longrightarrow \frac{a}{1-r}$ and the series converges.
- If $|r| > 1$, the terms of the series become larger and larger in magnitude, i.e., $|r^n| \longrightarrow \infty$ and the series diverges.

Examples

Determine sum of the following geometric series if it exist.

1. $$\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^{n-1}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1. \qquad \left(\because |r|=\frac{1}{2}<1\right)$$

2. $$\sum_{n=1}^{\infty}\frac{3}{5}\left(-\frac{4}{5}\right)^{n-1}=\frac{\frac{3}{5}}{1+\frac{4}{5}}=\frac{3}{9}=\frac{1}{3}. \qquad \left(\because |r|=\frac{4}{5}<1\right)$$

3. $\sum_{n=1}^{\infty}\frac{2}{3}(2)^{n-1}$. The series diverges. $(\because |r|=2>1)$

Example: Repeating decimals-Geometric Series

$$0.080\overline{808} = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \frac{8}{10^8} + \ldots$$

Here $a = \dfrac{8}{10^2}$ and $r = \dfrac{1}{10^2} < 1.$ Thus,

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} \frac{8}{10^2}\left(\frac{1}{10^2}\right)^{n-1}$$

Since $|r| < 1$, so the given series is convergent and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{\frac{8}{10^2}}{1-\frac{1}{10^2}} = \frac{8}{99}.$$

Thus, the repeating decimal is equivalent to 8/99.

Example

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

converges or diverges. If it converges, find the sum.

Solution:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \qquad \text{(Using partial fraction)}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \ldots$$

Example

This is a **Telescopic sum** which means that each term cancles part of the next term so that the sum reduces to only two terms. Thus,

$$S_n = \left(1 - \frac{1}{n+1}\right).$$

For the present case:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

Hence, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and its sum is 1.