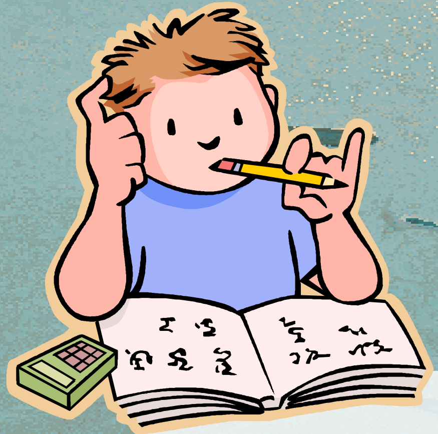
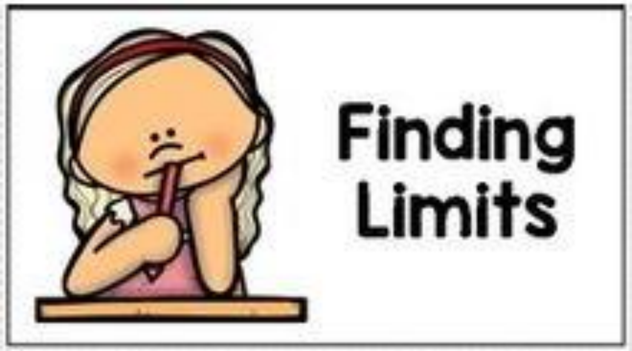


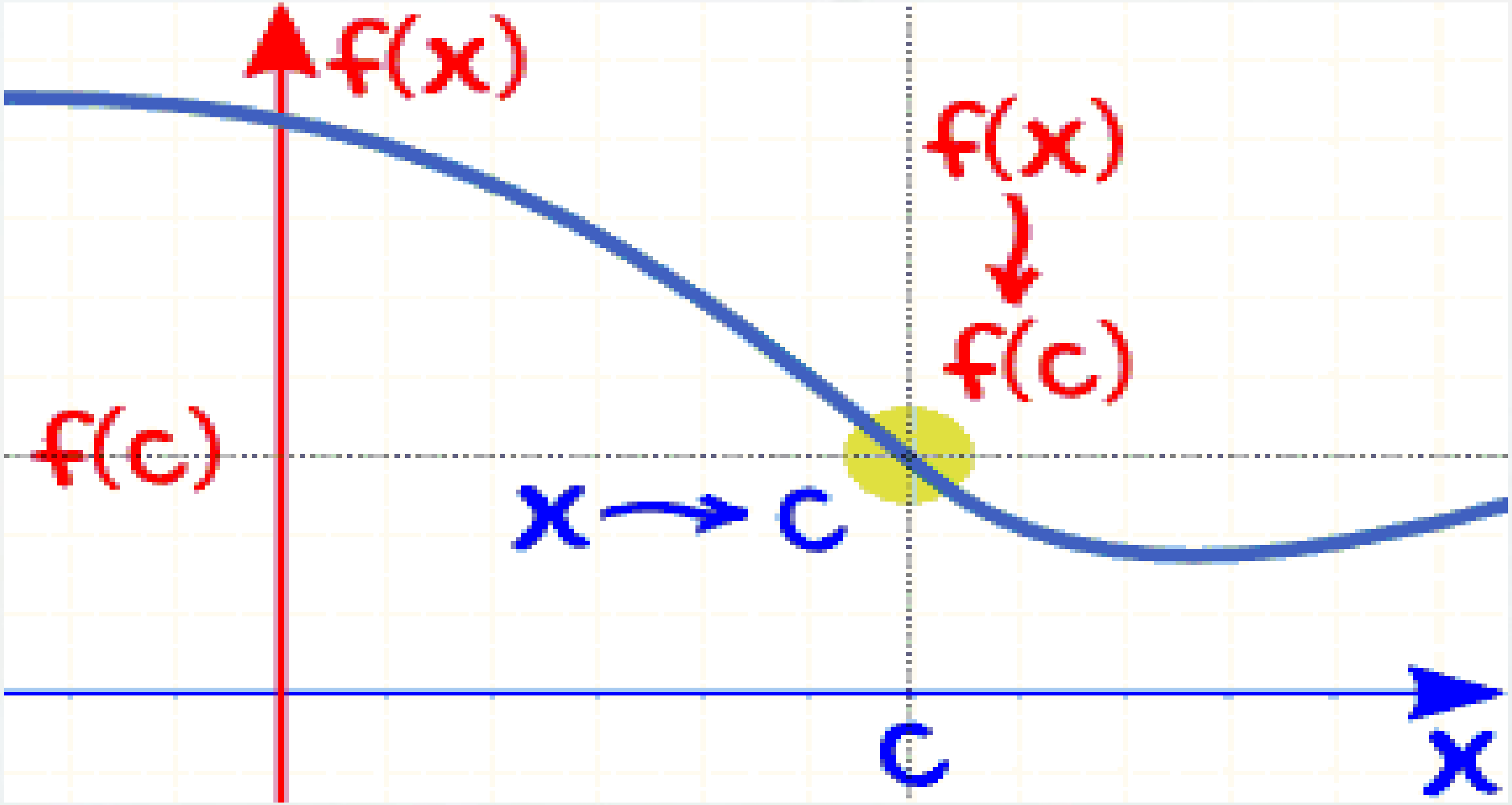
Continuity



Calculus & Analytical Geometry
MATH- 101

Instructor: Dr. Naila Amir
(SEECS, NUST)

Continuity



Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 2
    - Sections: 2.6

Objectives

- Determine continuity at a point and continuity on open and closed intervals.
- Use properties of continuity.
- Understand and use the Intermediate Value Theorem.

Continuity

- The idea of continuity is a direct consequence of the concept of limit.
- CONTINUOUS MOTION is motion that continues without a break.
- Calculus wants to describe that motion mathematically, both the distance traveled and the speed at any given time, particularly when the speed is not constant.
- In any real problem of continuous motion, the distance traveled will be represented by a "continuous function" of the time traveled because we always treat time as continuous.
- Therefore, we must investigate what we mean by a **continuous function.**

Continuity

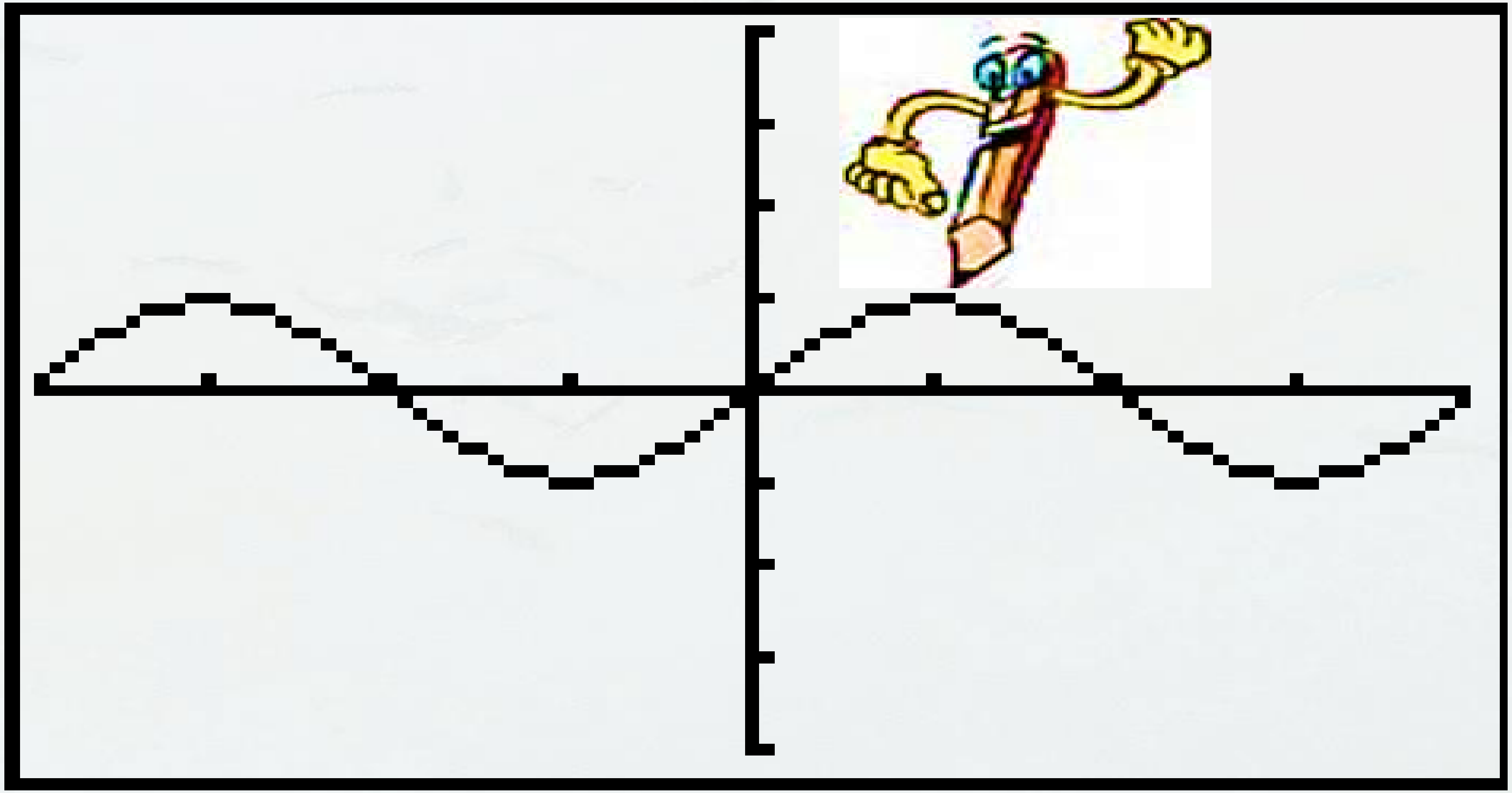
- In mathematics, the term *continuous* has much the same meaning as it has in everyday usage.

- Informally, to say that a function f is continuous at $x = c$ means that there is no interruption in the graph of $f(x)$ at c.

- That is, its graph is unbroken at c and there are no holes, jumps, or gaps.

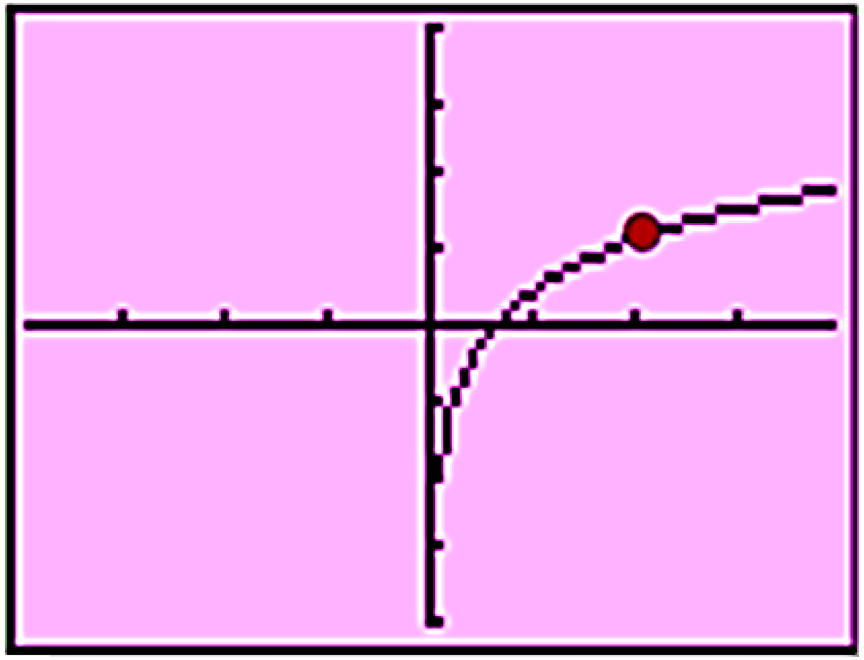
Most of the techniques of calculus require that functions are **continuous.**

A function is continuous if we can draw it in one motion without picking up pencil.



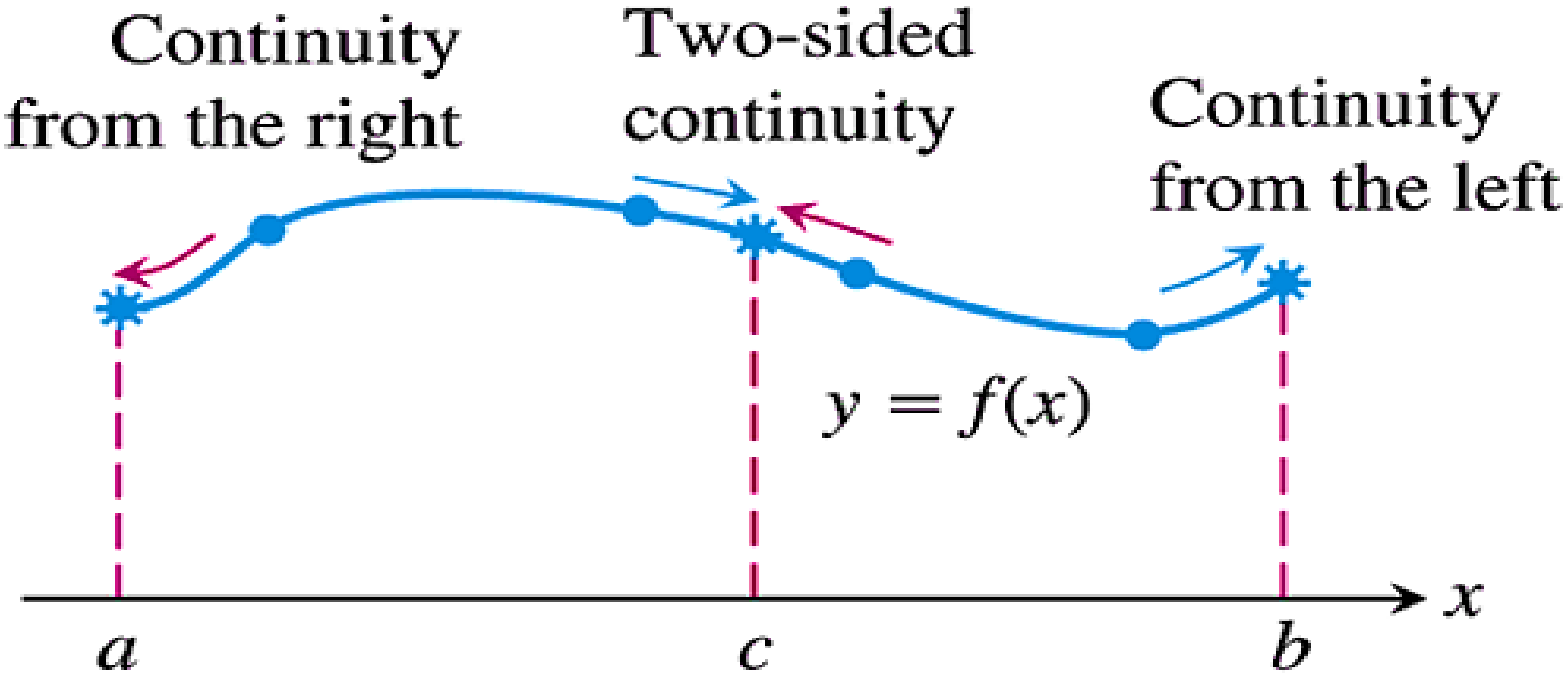
DEFINITION **Continuous at a Point**

Interior point: A function $y = f(x)$ is **continuous at an interior point** c of its domain if $\lim_{x\to c} f(x) = f(c)$.



Endpoint: A function $y = f(x)$ is **continuous at a left endpoint** a or is **continuous at a right endpoint** b of its domain if

$$\lim_{x\to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x\to b^-} f(x) = f(b), \quad \text{respectively.}$$



Continuity at points a, b, and c.

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value)

Example

Show that $g(x) = x^2 + 1$ is continuous at $x = 1$.

Solution:

1) $g(1) = 2$.

2) $\lim_{x \to 1} g(x) = 2$.

3) $\lim_{x \to 1} g(x) = g(1) = 2$.

Since all conditions are satisfied so we conclude that $g(x)$ is continuous at $x = 1$.

Example

Is the function $f(x) = \begin{cases} x+1; & x<2 \\ 2x-1; & x \geq 2 \end{cases}$ continuous at $x = 2$?

Solution:

1) $f(2) = 3.$

2) $\lim_{x \to 2^-} f(x) = 3,$

$\lim_{x \to 2^+} f(x) = 3,$

$\therefore \lim_{x \to 2} f(x)$ exists and $\lim_{x \to 2} f(x) = 3.$

3) $\lim_{x \to 2} f(x) = f(2) = 3.$

Since all conditions are satisfied so we conclude that $f(x)$ is continuous at $x = 2$.

Example

Is the function $f(x) = \begin{cases} x+1; & x<2 \\ 2x-1; & x>2 \end{cases}$ continuous at $x = 2$?

Solution:

Since $f(2)$ is not defined therefore the given function is not continuous at $x = 2$.

Example

Is the function $f(x) = \begin{cases} x+1 & x<2 \\ x^2 & x=2 \\ 2x-1 & x>2 \end{cases}$ continuous at $x = 2$?

Solution:

1) $f(2) = 4.$

2) $\lim_{x\to 2^-} f(x) = 3,\ \lim_{x\to 2^+} f(x) = 3,$

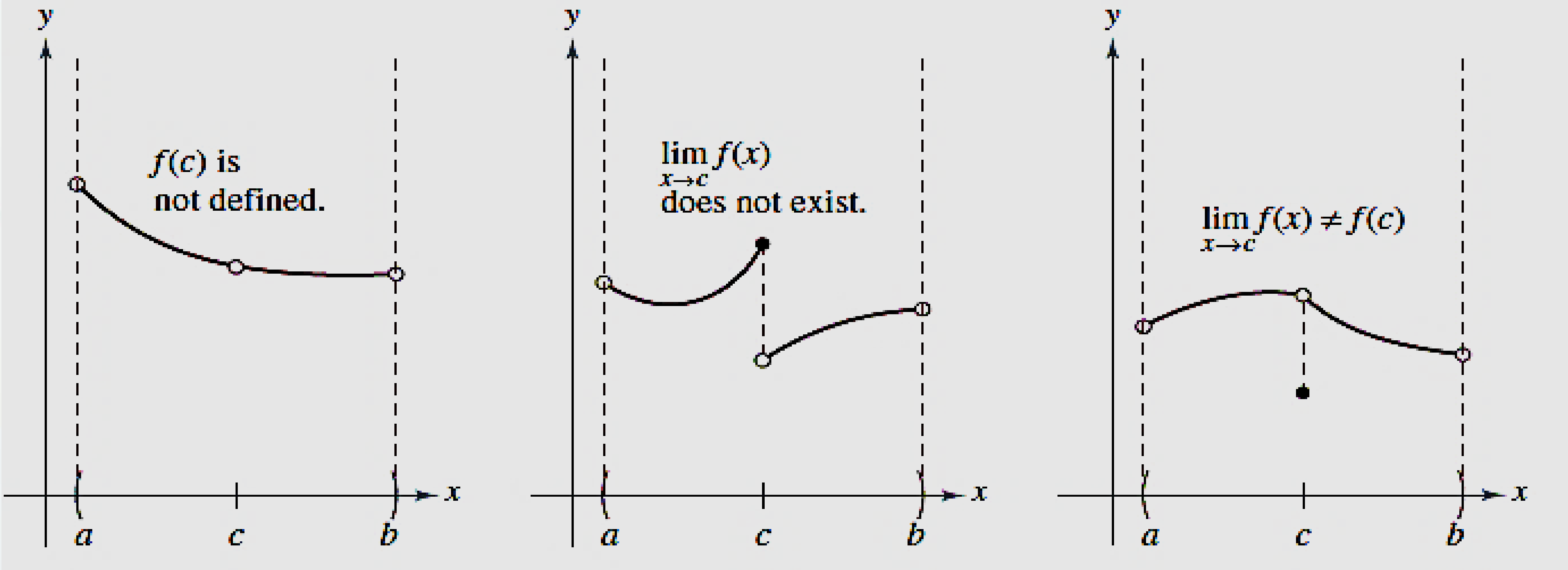
$\therefore \lim_{x\to 2} f(x)$ exists and $\lim_{x\to 2} f(x) = 3.$

3) $\lim_{x\to 2} f(x) \neq f(2).$

Since third condition fails to exist so we conclude that $f(x)$ is not continuous (or discontinuous) at $x = 2$.

Discontinuity

Following figure identifies three values of x at which the graph of $f(x)$ is *not* continuous. At all other points in the interval (a, b), the graph of $f(x)$ is uninterrupted and **continuous.**



Three conditions exist for which the graph of f is not continuous at $x = c$.

Discontinuity

In previous figure, it appears that continuity at $x = c$ can be destroyed by any one of the following conditions.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

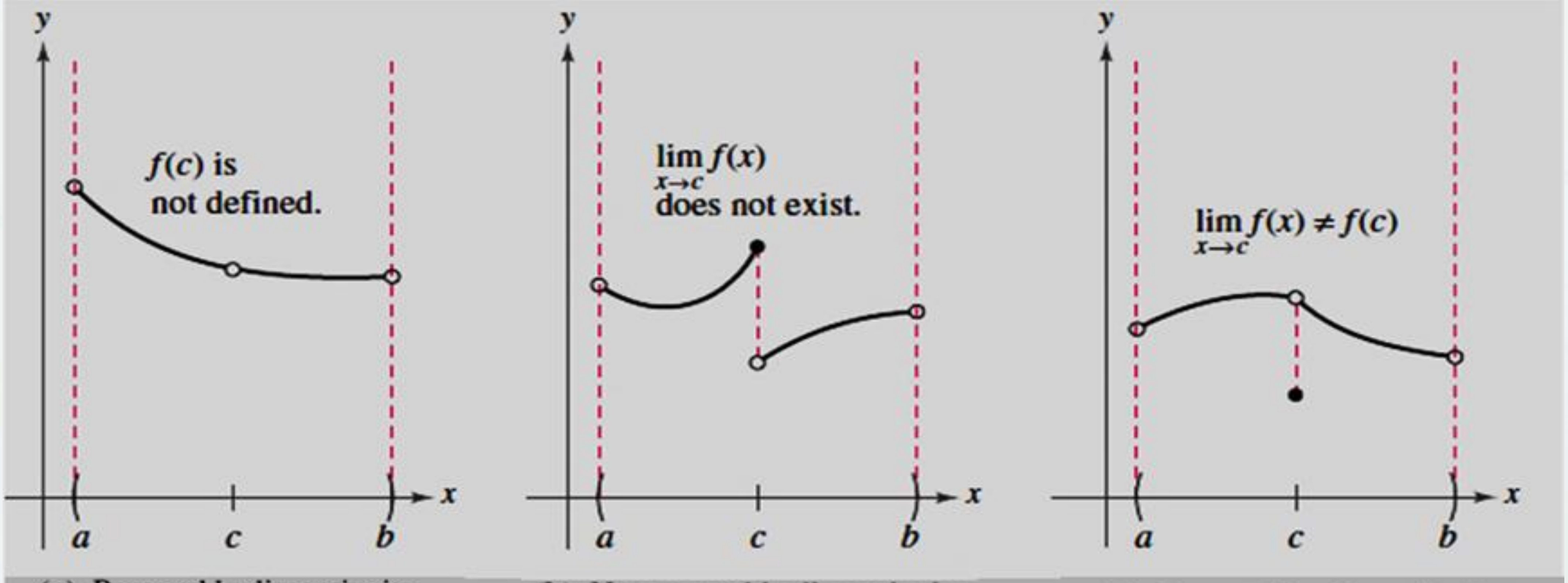
If *none* of the three conditions above is true, the function $f(x)$ is called **continuous at c**.

Types of Discontinuities

- Consider an open interval I that contains a real number c.
- If a function $f(x)$ is defined on I (except possibly at c), and $f(x)$ is not continuous at c, then f is said to have a **discontinuity** at c.
- Discontinuities fall into two categories: **removable** and **nonremovable.**
- A discontinuity at c is called removable if $f(x)$ can be made continuous by appropriately defining (or redefining $f(c)$).

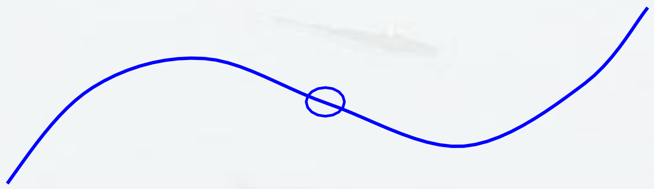
Types of Discontinuities

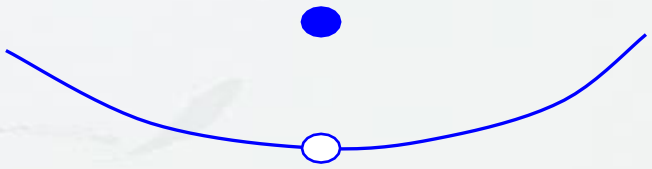
For instance, the functions shown in figures (a) and (c) have removable discontinuities at c and the function shown in (b) has a non-removable discontinuity at c.



(a) Removable discontinuity (b) Nonremovable discontinuity (c) Removable discontinuity

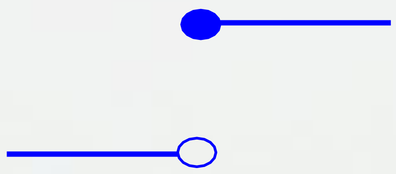
Removable Discontinuities:



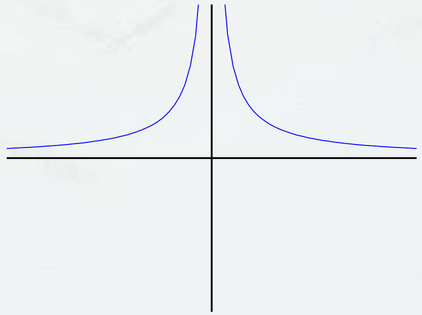


(We can fill the hole.)

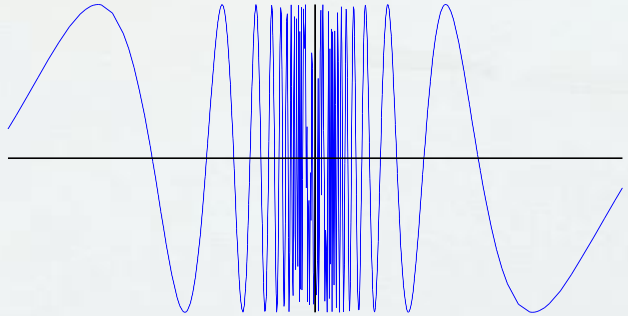
Nonremovable Discontinuities:



jump

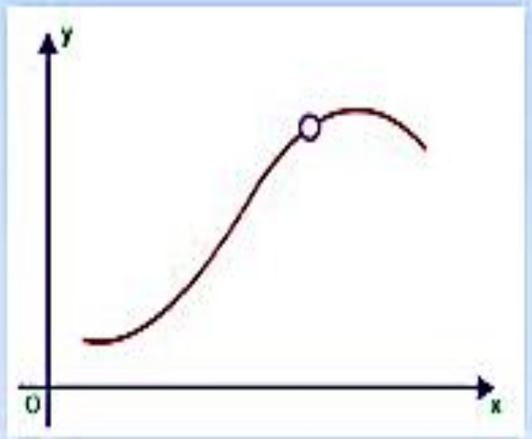
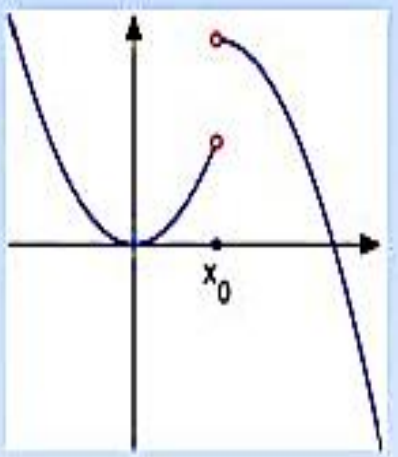
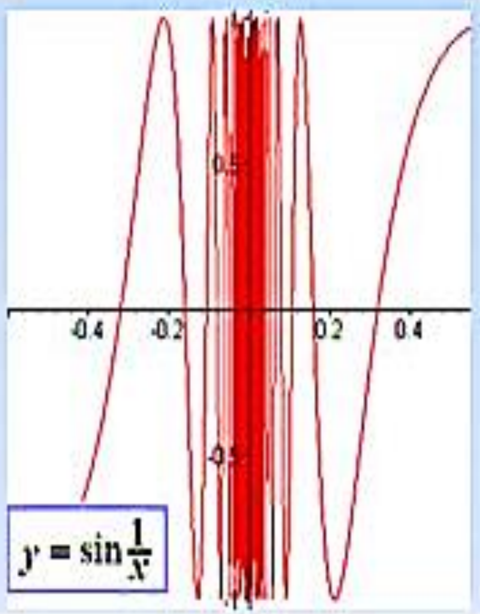
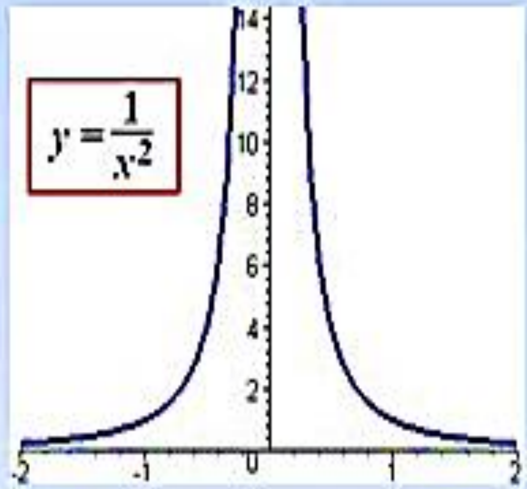


Infinite
(Essential)

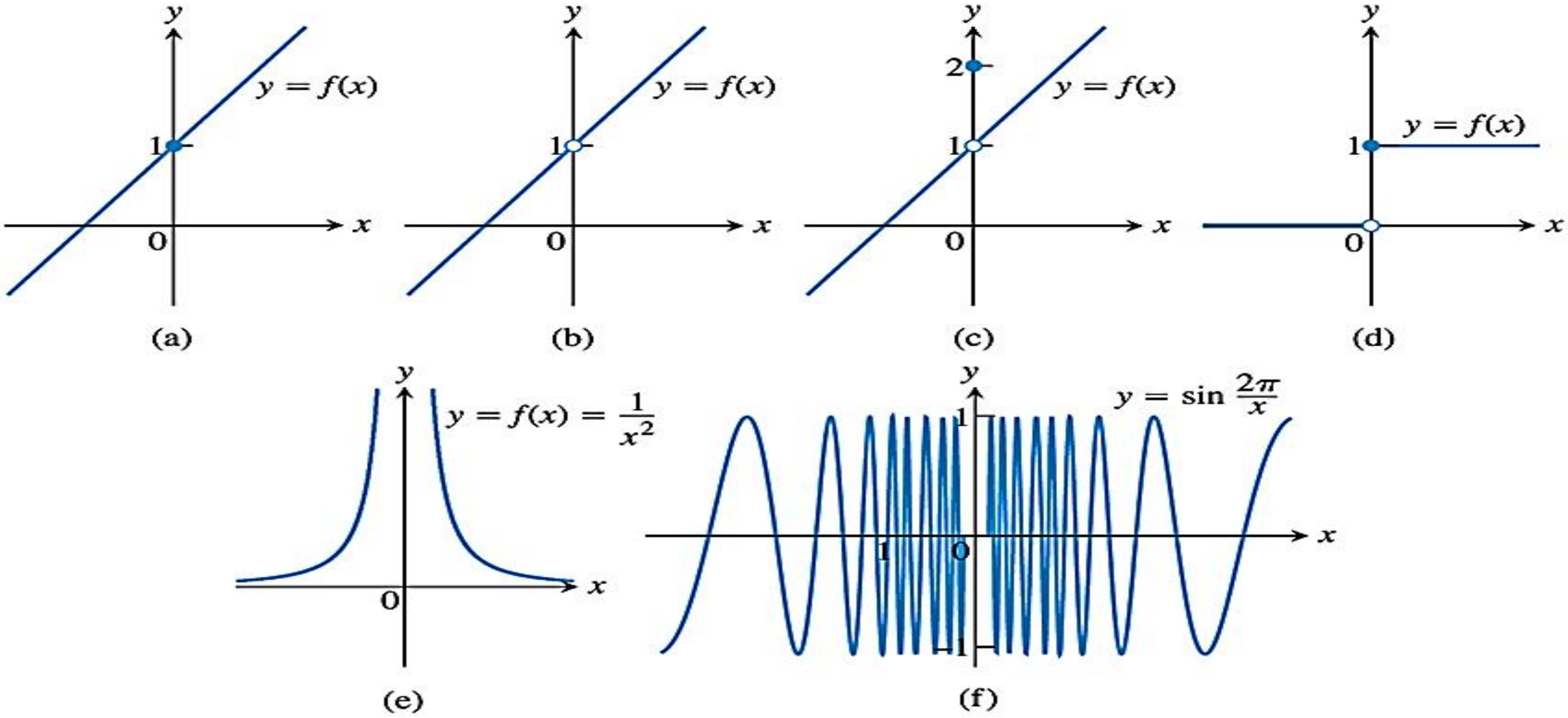


oscillating

Types of Discontinuities

Removable	Jump	Oscillating	Infinite
			

Examples



The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Places to test for continuity

- Rational Expression
  - Values that make denominator = 0
- Piecewise Functions
  - Changes in interval
- Absolute Value Functions
  - Use piecewise definition and test changes in interval
- Step Functions
  - Test jumps from 1 step to next.

Examples

Discuss the continuity of each function.

$$a. \quad f(x) = \frac{x^2 - 1}{x - 1}$$

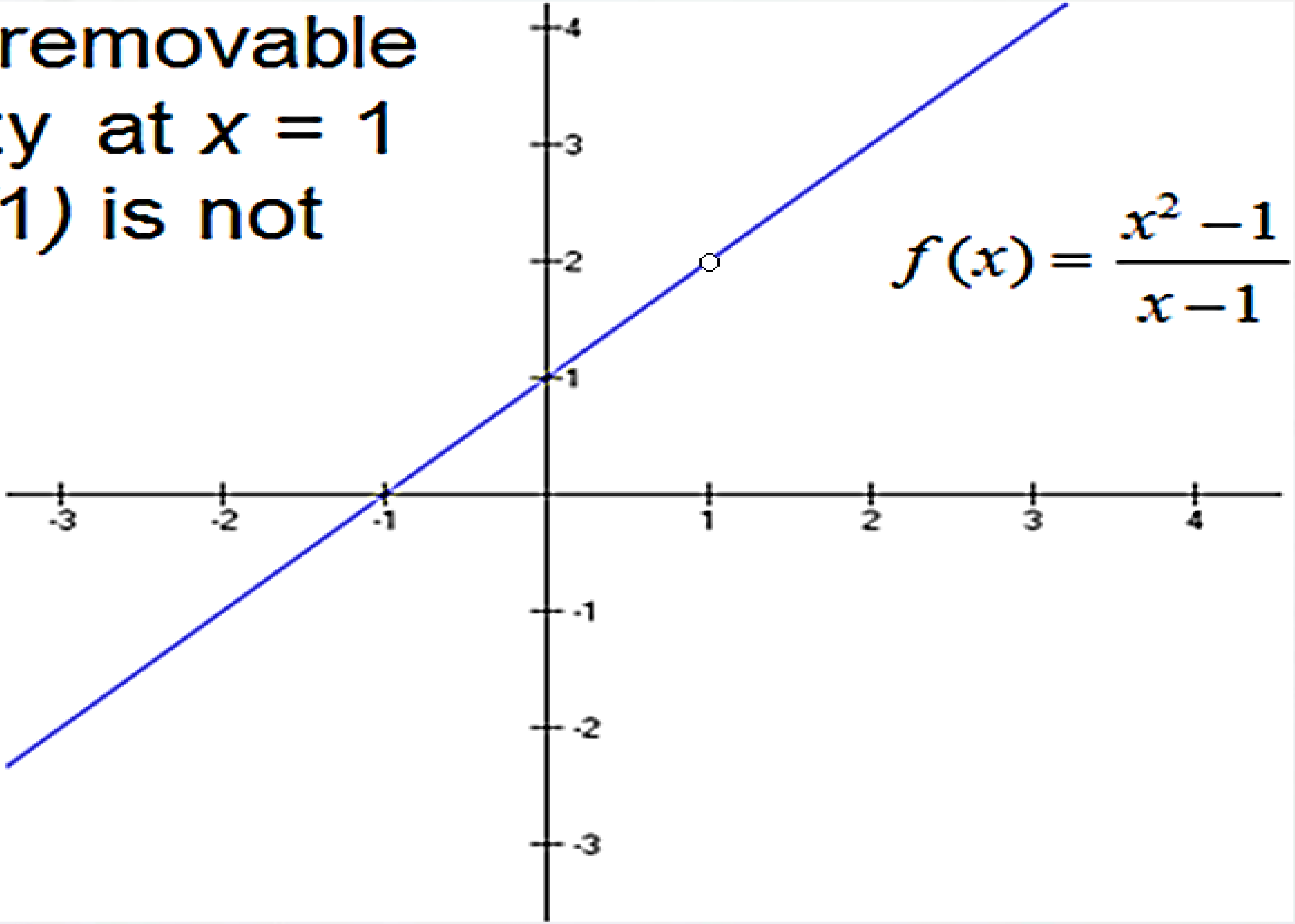
$$b. \quad g(x) = \frac{1}{x}$$

$$c. \quad h(x) = \begin{cases} x + 1, x \leq 2 \\ x^2 + 1, x > 2 \end{cases}$$

$$d. \quad i(x) = \begin{cases} x + 1, x \leq 0 \\ x^2 + 1, x > 0 \end{cases}$$

Solution (a)

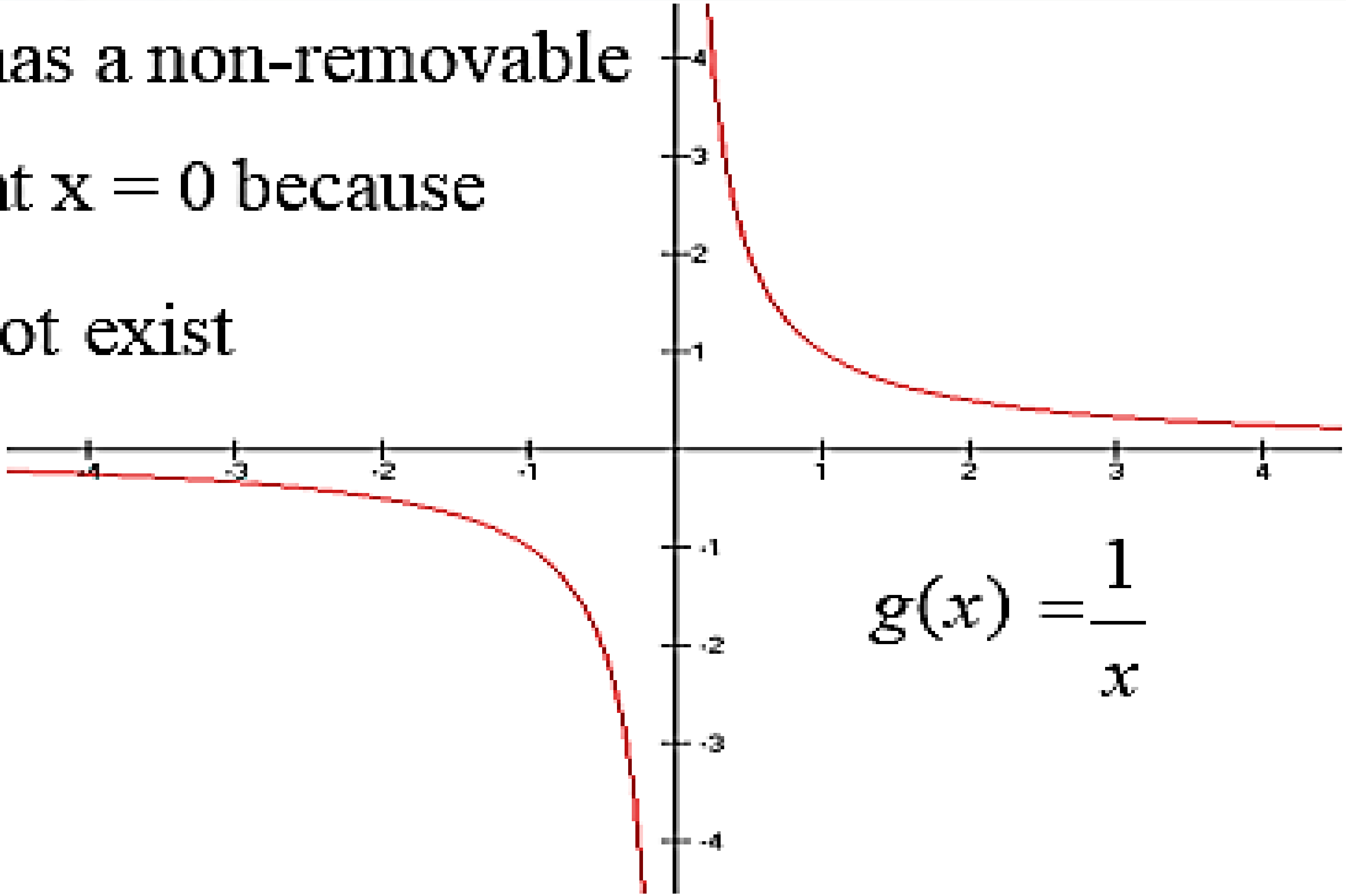
There is a removable discontinuity at $x = 1$ because $f(1)$ is not defined.



Solution (b)

The function has a non-removable discontinuity at x = 0 because

$\lim_{x \to 0} g(x)$ does not exist

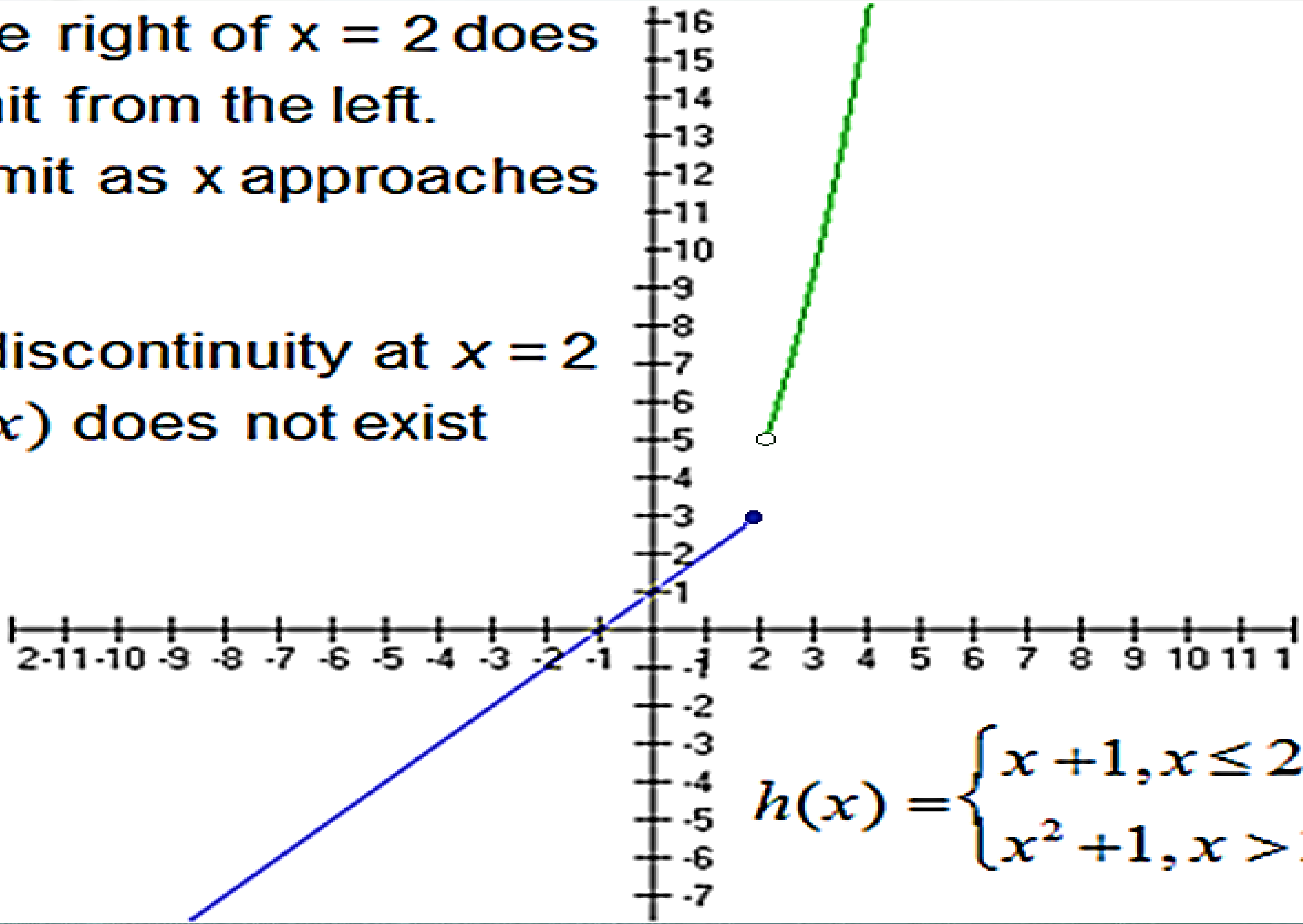


$$g(x) = \frac{1}{x}$$

Solution (c)

The limit from the right of x = 2 does not equal the limit from the left. Therefore, the limit as x approaches 2 does not exist.

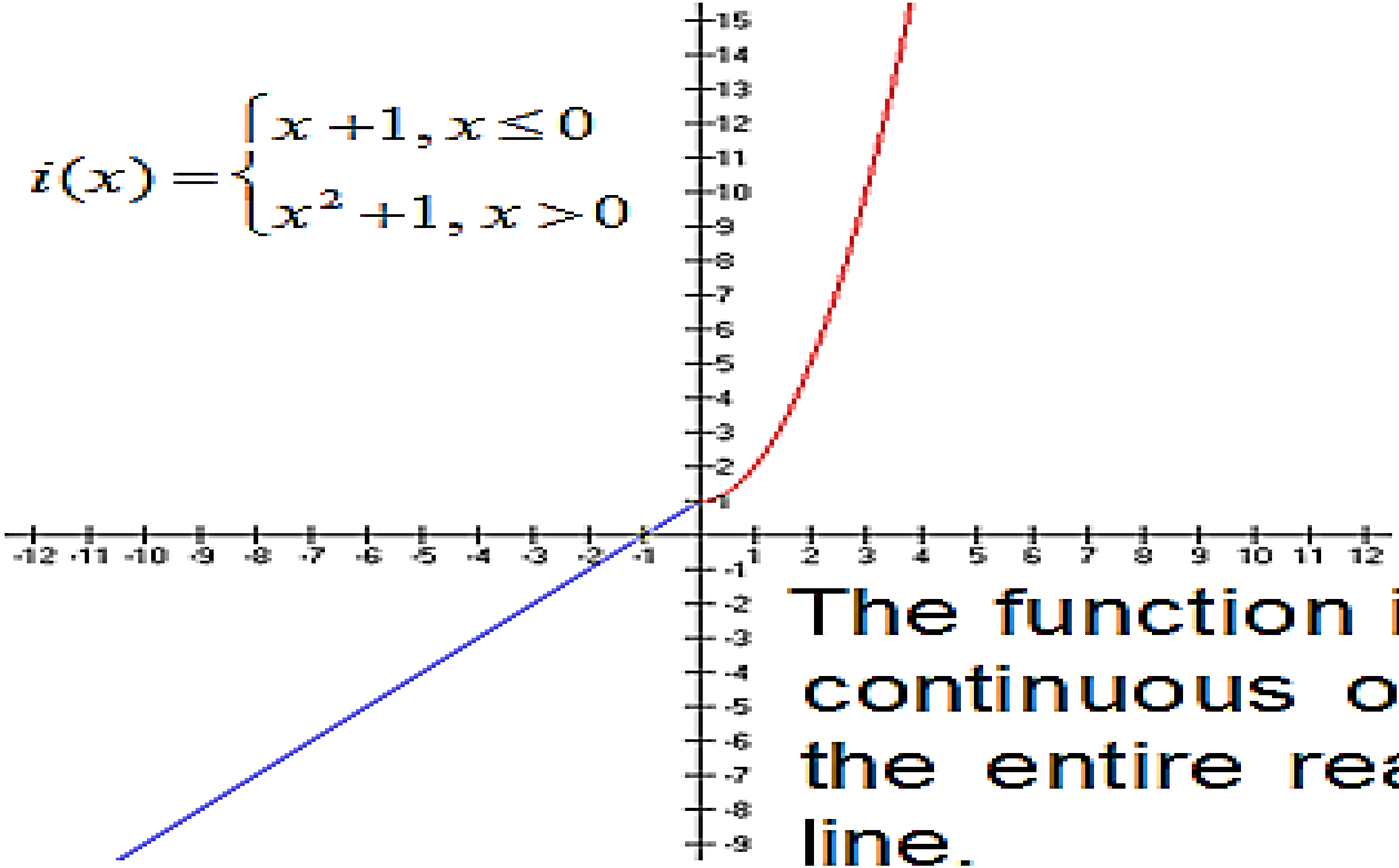
Function has a discontinuity at $x = 2$ because $\lim_{x \to 0} g(x)$ does not exist



$$h(x) = \begin{cases} x+1, x \le 2 \\ x^2+1, x > 2 \end{cases}$$

Solution (d)

$$i(x) = \begin{cases} x+1, & x \le 0 \\ x^2+1, & x > 0 \end{cases}$$



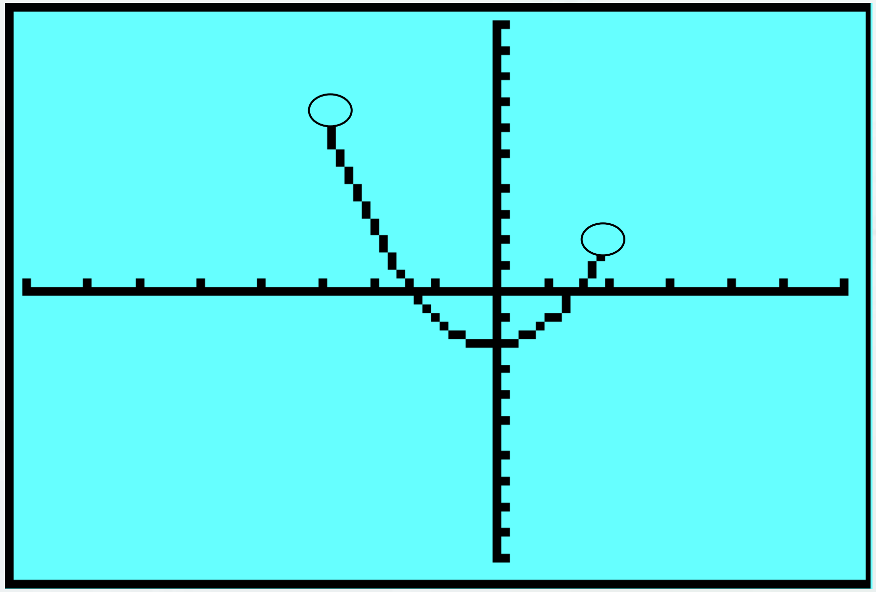
The function is continuous on the entire real line.

Continuity on an open interval

A function is continuous on **an open interval** (a, b) if it is continuous on each point in the interval. A function that is continuous on the entire real line **is every where continuous**.

Example:

$f(x)$ is continuous on $(-3,2)$.



Continuity on a closed interval

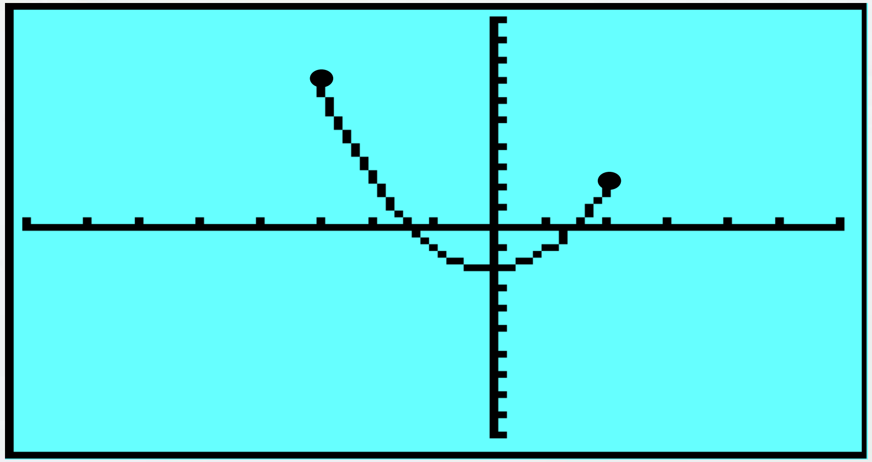
The concept of a one-sided limit allows us to extend the definition of continuity to closed intervals. A function $f(x)$ is continuous on the closed interval $[a, b]$ if it is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a) \text{ and } \lim_{x \to b^-} f(x) = f(b),$$

i.e., the function is continuous from the right at a and continuous from the left at b.

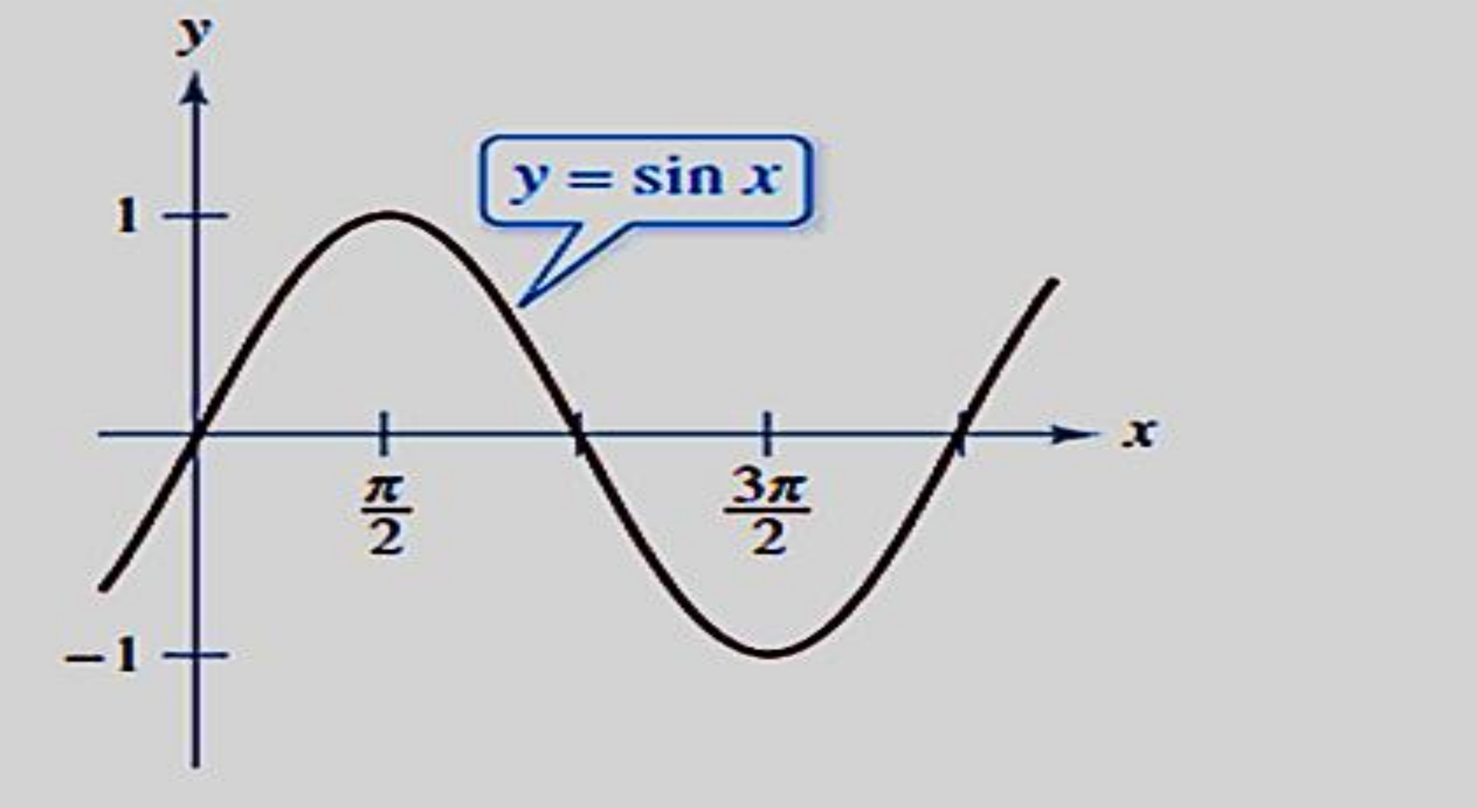
Example:

$f(x)$ is continuous on $[-3,2]$.



Example

The domain of the function $y = \sin x$ is the set of all real numbers. $f(x)$ is continuous on its entire domain, as shown in figure.



Continuous on entire real number line

Example

Discuss the continuity of $f(x) = \sqrt{1 - x^2}$.

Solution:

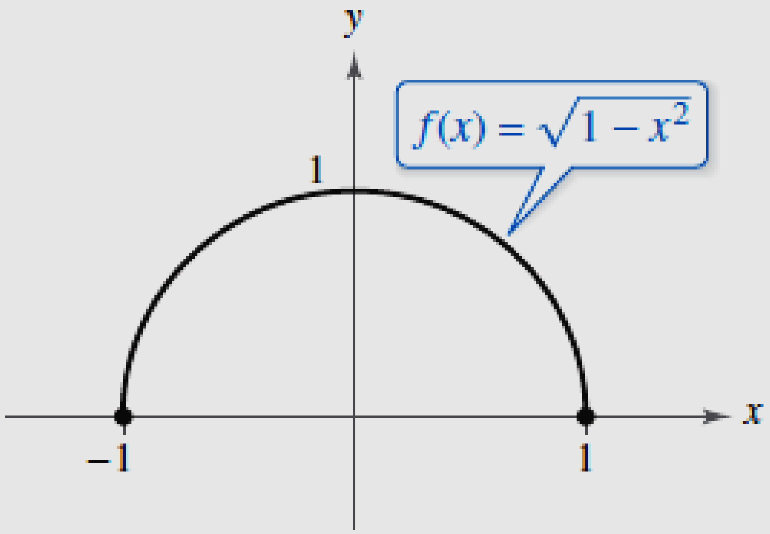
The domain of $f(x)$ is the closed interval $[-1, 1]$. At all points in the open interval $(-1, 1)$, the given function is continuous. Moreover,

$$\lim_{x \to -1^+} \sqrt{1 - x^2} = 0 = f(-1)$$ Continuous from the right

and

$$\lim_{x \to 1^-} \sqrt{1 - x^2} = 0 = f(1)$$ Continuous from the left

This implies that $f(x)$ is continuous on the closed interval $[-1, 1]$.



f is continuous on $[-1, 1]$.

Practice

Q#1: Discuss the continuity of the following functions:

1. $f(x) = \begin{cases} \dfrac{x-4}{\sqrt{x}-2}; & x \geq 0 \text{ and } x \neq 4 \\ 4; & x = 4 \end{cases}$ at $x = 4$.

2. $f(x) = \begin{cases} \dfrac{x^2-a^2}{x-a}; & 0 \leq x < a \\ a; & x = a \\ 2a; & x > a \end{cases}$ at $x = a$.

3. $f(x) = 2^{1/x}$ at $x = 0$.

4. $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1}; & x \neq 0 \\ 0; & x = 0 \end{cases}$ at $x = 0$.

Practice

Q#1: Discuss the continuity of the following functions:

5. $f(x) = \begin{cases} \dfrac{\sin 3x}{\sin 2x}; & x \neq 0 \\ 2/3; & x = 0 \end{cases}$ at $x = 0.$

6. $f(x) = x - |x|$ at $x = 1.$

7. $f(x) = \begin{cases} (1+x)^{1/x}; & x \neq 0 \\ 1; & x = 0 \end{cases}$ at $x = 0.$

Q#2: Find the constant "c", provided the function $f(x) = \begin{cases} \frac{1-\sqrt{x}}{x-1}; & 0 \leq x < 1 \\ c; & x = 1 \end{cases}$

is continuous for all $x \in [0,1]$.