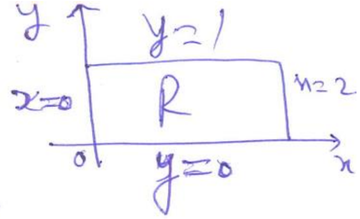
EX: Determine the region R' of the w-plane into which R is mapped under the transformations:



$$w = \sqrt{2}\, e^{\frac{\pi}{4} i} z + (1-2i).$$

$w = u+iv = (1+i)(x+iy) + (1-2i)$

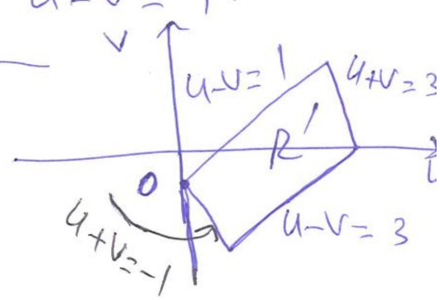
$u = x-y+1$, $v = x+y-2$

The lines $x=0$, $u = 1-y, v = 2-y \Rightarrow u+v = -1$

$y=0$, $u = x+1, v = x-2 \Rightarrow u-v = 3$

$x=2$, $u = 3-y, v = y \Rightarrow u+v = 3$

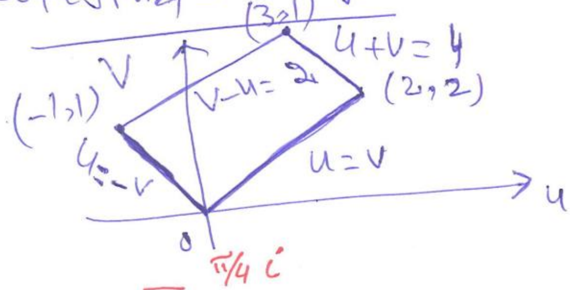
$y=1$ $u = x, v = x-1 \Rightarrow u-v = 1$.



$u = v$
$u+v = 4$
$2v = 4$
$v = 2$

OR stepwise:

Rotating scaling:



$-u+v = 2$
$2v = 2$
$v = 1$

$u+v = 4$
$-u+v = 2$
$2v = 6$
$v = 3$

$z_0 = 1-2i$

$A' \to -1+1 = 0$
$1-2 = -1$

$w = \sqrt{2}\, e^{\pi/4\, i}$

$wz = (1+i)(x+iy)$

$wz = (x-y) + i(x+y)$

$u = x-y$, $v = x+y$

Image of $x=2$: $u = 2-y$, $v = 2+y$
$u = 2 - y$
$v + u = 4$

Image of $y=1$: $u = x-1$
$v = x+1$
$u - v = -2$
'or' $v - u = 2$

Book section 2.3, 2.4 & 2.5

Using mappings to Solve a Dirichlet problem:

we know that

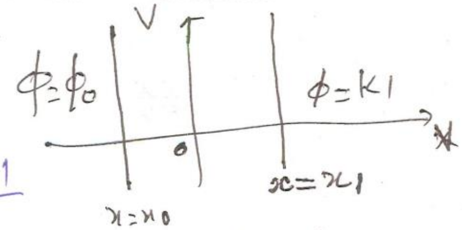


Fig 1

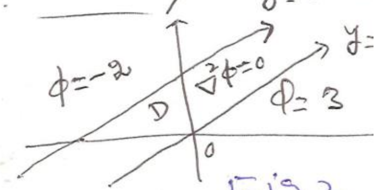
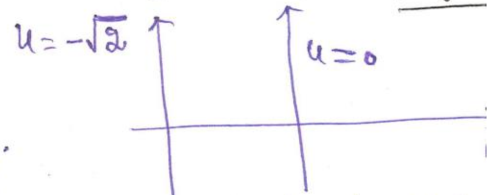


Fig 2

Solution is $\phi(x) = \frac{k_1 - k_0}{2} x + \frac{k_1 + k_0}{2}$ — (i)

We rotate the lines by an angle $\frac{\pi}{4}$ radians. to reach at vertical lines.

Fig 3



$R(z) = \sqrt{2}\, e^{i\pi/4}$

$w = e^{i\frac{\pi}{4}} z = \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right)(x + iy)$

$= \frac{1}{\sqrt{2}}(x - y) + \frac{1}{\sqrt{2}}(x + y)\, i$

giving transformation equations as $u = \frac{1}{\sqrt{2}}(x-y)$, $v = \frac{1}{\sqrt{2}}(x+y)$

maps $y = x$ on $u = 0$ & $y = x+2$ on $u = -\sqrt{2}$

To make map $u = -1$, we take $R(z) = \sqrt{2}\, e^{i\pi/4}$, we define translation by $z = -1$, $f(z) = \sqrt{2}\, e^{i\pi/4} + 1 = (1+i)z + 1$.

transformation of BC's: $w = f(z) = (1+i)(x+iy) = (x - y + 1) + i(x + y)$. (ii)

$y = x+2$: $w = u + iv = x - (x-2) + 1 + (x + (x+2))i = -1 + 2(x+1)i$

which is the line $u = -1$. Similarly,

$y = x$: $w = u + iv = x - x + 1 + (x + x)i = 1 + 2xi$.

Which is the line $u = 1$.

$\phi(x, x+2) = -2 = \Phi(-1, v)$, $\phi(x, x) = 3 = \Phi(1, v)$.

The solution due to equation (i) is

$\Phi(u, v) = \frac{3 - (-2)}{2} u + \frac{-2 + 3}{2} = \frac{5}{2} u + \frac{1}{2}$.

Due to (ii), we have $u(x, y) = x - y + 1$, and $v(x, y) = x + y$.

$\phi(x, y) = \Phi(u(x,y), v(x,y)) = \frac{5}{2}(x - y + 1) + \frac{1}{2} = \frac{5}{2}x - \frac{5}{2}y + 3$

See E.T., V. 9.4.5

Check: $\phi(x, x) = 3$, $\phi(x, x+2) = \frac{5}{2}x - \frac{5}{2}(x+2) + 3 = \frac{5}{2}x - \frac{5}{2}x - 5 + 3 = -2$