Applications of Derivatives

Calculus & Analytical Geometry MATH- 101

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Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 4
    - Sections: 4.5

Applied Optimization Problems

- To optimize something means to maximize or minimize some aspect of it.
- We will be mainly interested in solving problems such as:
  - Maximizing areas, volumes, and profits.
  - Minimizing distances, times, and costs.
- In solving such practical problems, the greatest challenge is often to b convert the word problem into a mathematical optimization problem—by setting up the function that is to be maximized or minimized.

Solving Applied Optimization Problems

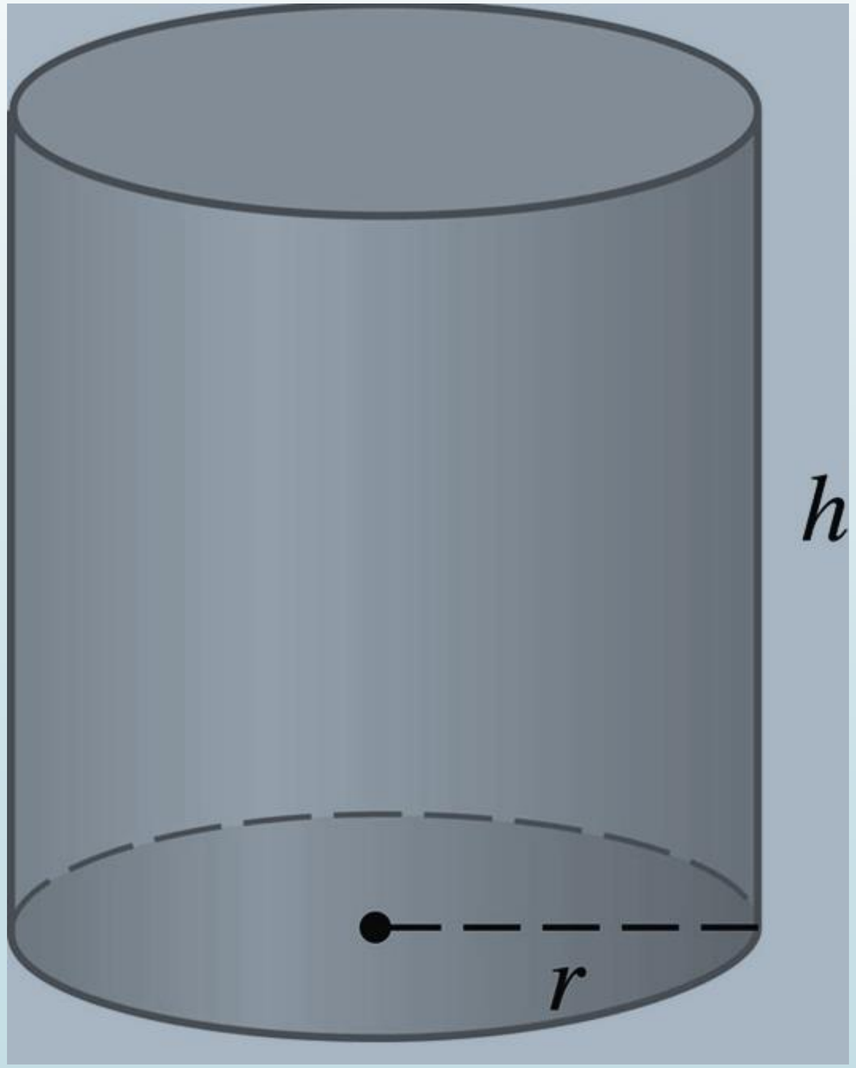
1. Assign symbols to all given quantities and quantities to be determined.
2. Write a primary equation for the quantity to be maximized or minimized.
3. Reduce the primary equation to one having a single independent variable. This may involve the use of secondary equation.
4. Determine the domain. Make sure it makes sense.
5. Determine the max or min by differentiation.

Example:

A cylindrical can is to be made to hold $1\,L$ of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Solution:

Let r be the radius and h be the height of the cylindrical can.



To minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides). We see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h. So, the surface area is:

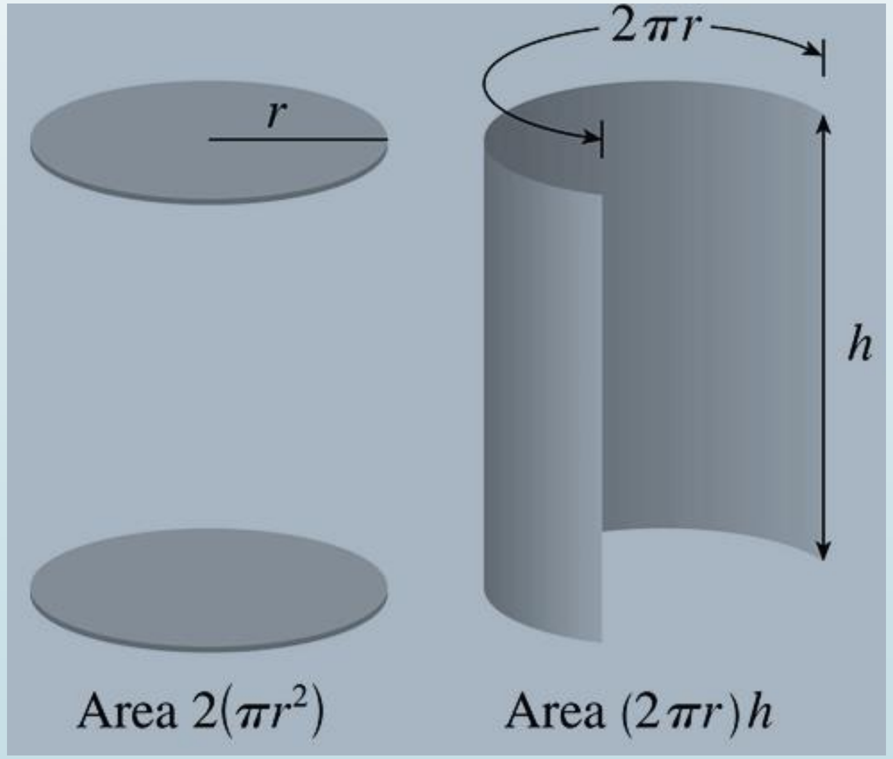
$$A = 2\pi r^2 + 2\pi rh$$

If r and h are measured in centimeters, then the volume of the can in cubic centimeters is given as:

$$\pi r^2 h = 1000 \qquad [\because 1\,L = 1000\,cm^3]$$

To eliminate h, we make use of the above equation and this gives: $h = \frac{1000}{\pi r^2}$. Substituting this in the expression for A gives:

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}.$$



So, the function that we want to minimize is:

$$A(r) = 2\pi r^2 + \frac{2000}{r}; \quad r > 0.$$

Since domain of A is $(0, \infty)$, so it is differentiable on an interval with no endpoints. Thus, A can have a minimum value only where its first derivative is zero. For the present case:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}.$$

Then,

$$A'(r) = 0 \implies \pi r^3 = 500.$$

So, the only critical number is: $r = \sqrt[3]{500/\pi}$. Observe that:

$-$ $\quad$ $+$

$A'(x) < 0$ $\quad$ $\sqrt[3]{500/\pi}$ $\quad$ $A'(x) > 0$

Thus, $r = \sqrt[3]{500/\pi}$ must give rise to a minimum value.

The value of *h* corresponding to $r = \sqrt[3]{500/\pi}$ is:

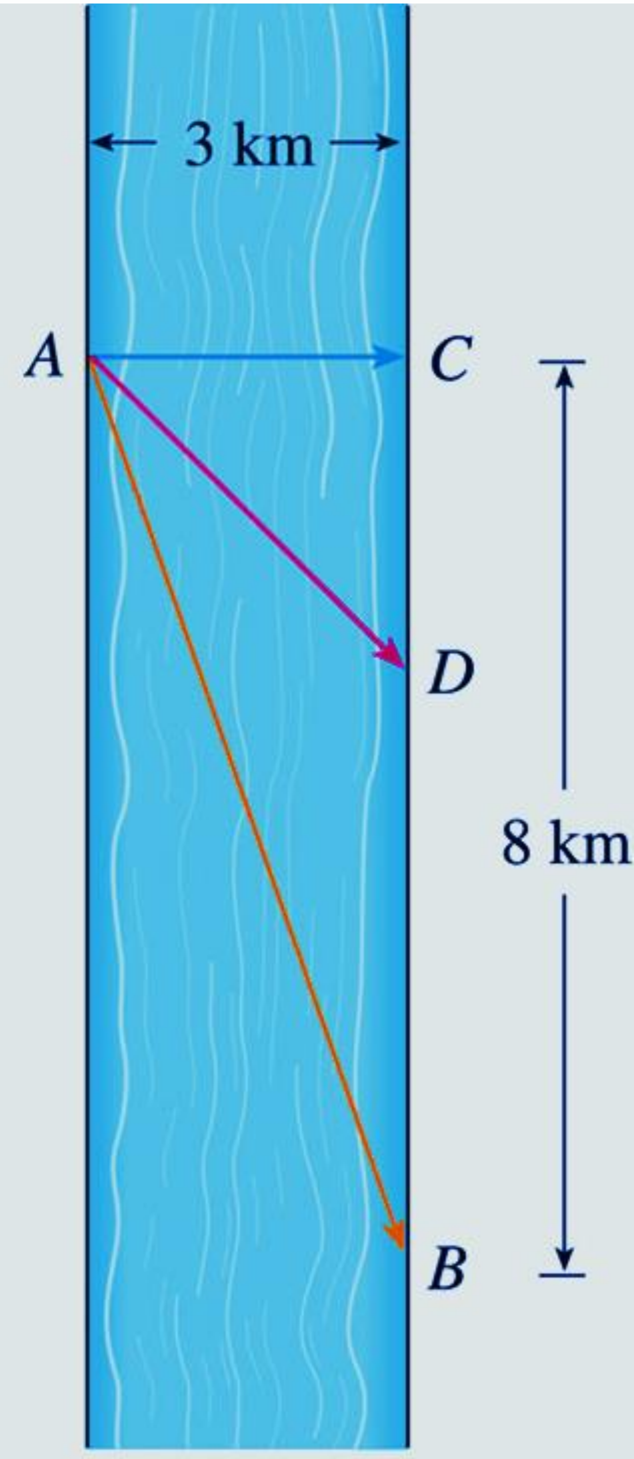
$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi (500/\pi)^{\frac{2}{3}}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

Thus, to minimize the cost of the can,

- The radius should be $r = \sqrt[3]{500/\pi}$ cm.
- The height should be equal to twice the radius—namely, the diameter

Example:

A man launches his boat from point A on a bank of a straight river, $3\,km$ wide, and wants to reach point B ($8\,km$ downstream on the opposite bank) as quickly as possible.



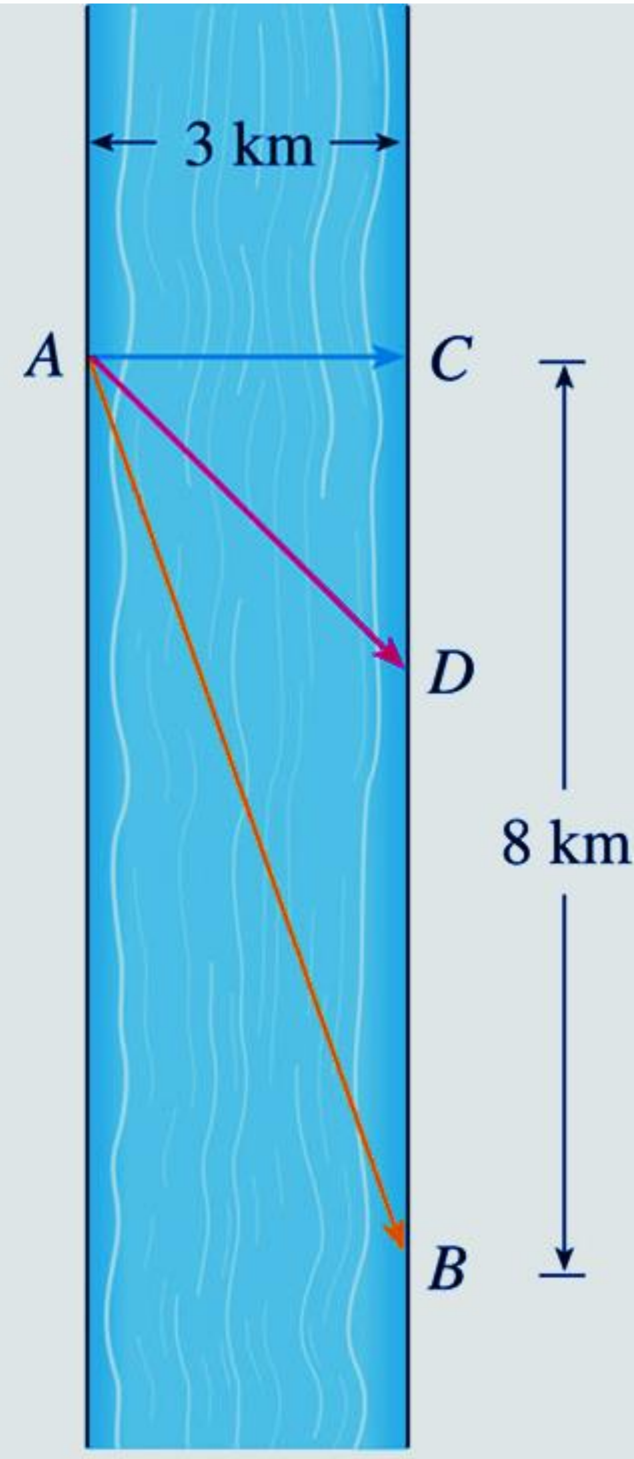
Solution:

He could proceed in any of three ways:

- Row his boat directly across the river to point C and then run to B.
- Row directly to B.
- Row to some point D between C and B and then run to B.

If he can row $6\ km/h$ and run $8\ km/h$, where should he land to reach B as soon as possible?

Note: we assume that the speed of the water is negligible compared with the speed at which he rows.



If we let x be the distance from C to D, then:

The running distance is: $|DB| = 8 - x$

The Pythagorean Theorem gives the rowing distance as:

$$|AD| = \sqrt{x^2 + 9}$$

Then, the rowing time is:

$$\frac{\sqrt{x^2+9}}{6}; \quad \text{because} \quad \text{time} = \frac{\text{distance}}{\text{rate}}.$$

The running time is: $\frac{8-x}{8}$

So, the total time T as a function of x is:

$$T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}.$$

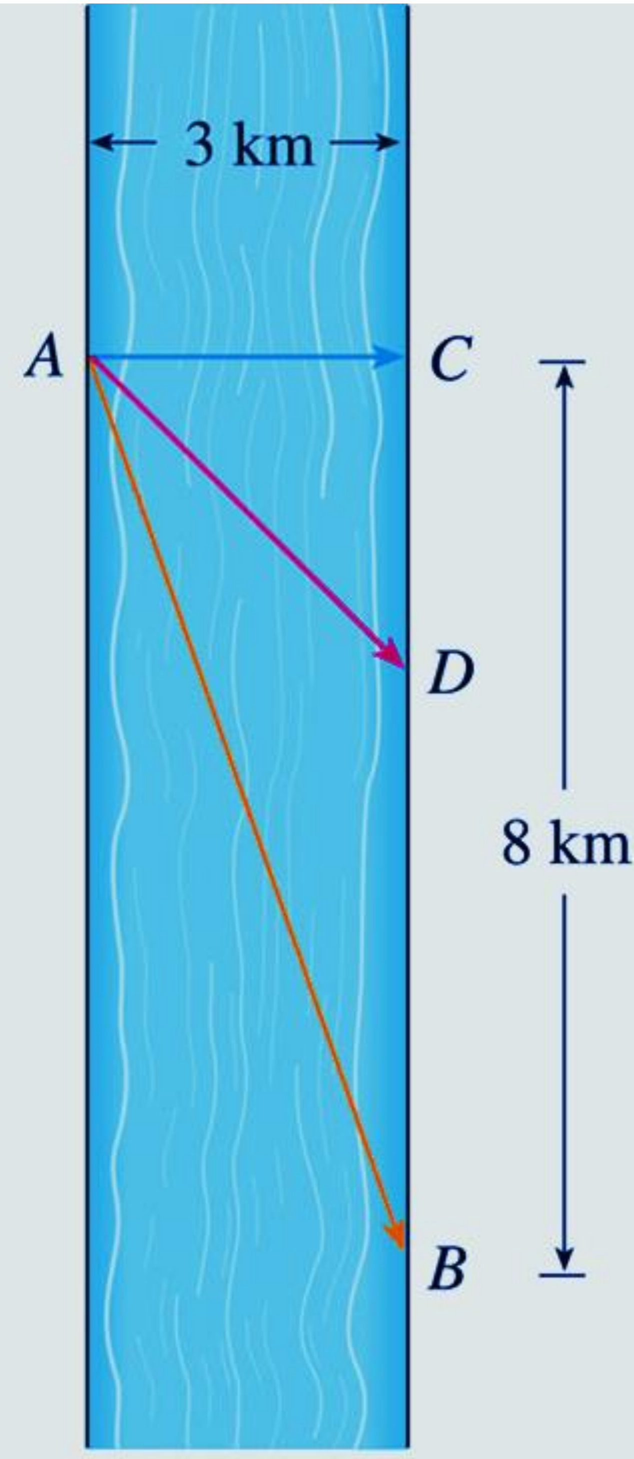
The domain of this function T is [0, 8]. Notice that if $x = 0$, he rows to C, and if $x = 8$, he rows directly to B. The derivative of T is given as:

$$T'(x) = \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8}.$$

Thus, using the fact that $x \geq 0$, we have:

$$T'(x) = 0 \Longrightarrow \frac{x}{6\sqrt{x^2+9}} = \frac{1}{8} \Longrightarrow x = \frac{9}{\sqrt{7}}.$$

The only critical number is: $x = \frac{9}{\sqrt{7}}$.



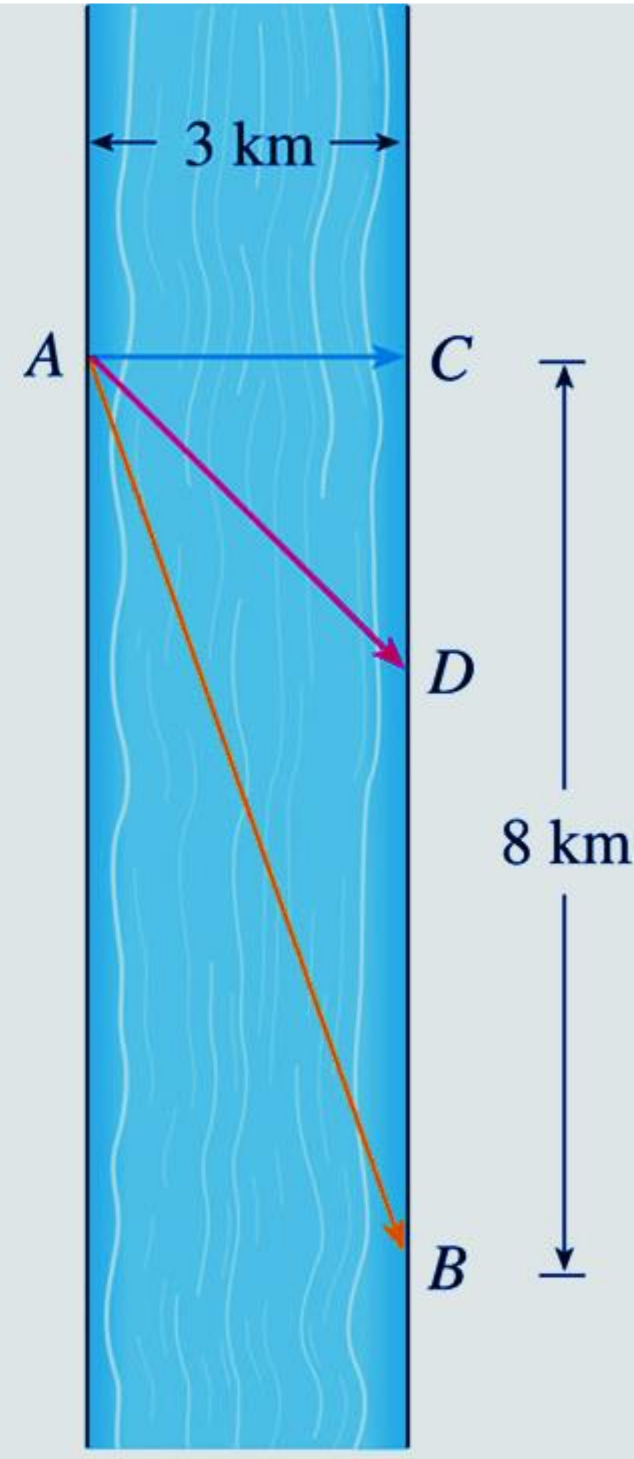
To see whether the minimum occurs at this critical number or at an endpoint of the domain $[0, 8]$, we evaluate T at all three points:

$$T(0) = 1.5$$

$$T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33$$

$$T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

Since the smallest of these values of T occurs when $x = \frac{9}{\sqrt{7}}$, the absolute minimum value of T must occur there. Thus, the man should land the boat at a point $x = \frac{9}{\sqrt{7}}$ $(\approx 3.4\ km)$ downstream from his starting point.

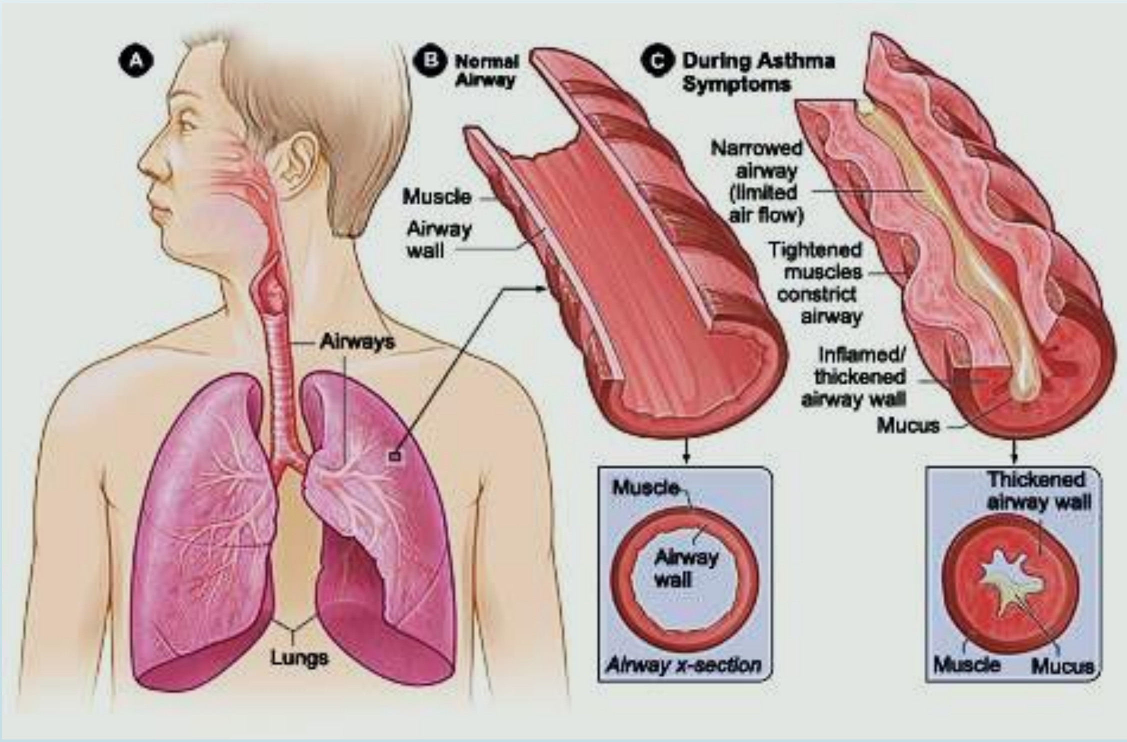


Example: A BIOLOGY PROBLEM

When a person coughs, the trachea (windpipe) contracts, allowing air to be expelled at a maximum velocity. It can be shown that during a cough the velocity v of airflow is given by the function:

$$v(r) = kr^2(R - r); \quad 0 \le r \le R,$$

where r is the trachea's radius (in centimeters) during cough, R is the trachea's normal radius (in centimeters), and k is a positive constant that depends on the length of the trachea. Find the radius r for which the velocity of airflow is greatest.



Solution:

Since a trachea has a maximum size, we can assume the closed interval $[0, R]$.

$v(r) = kr^2(R-r)$

$v'(r) = kr^2(-1) + 2kr(R-r)$ Product Rule

$v'(r) = -kr^2 + 2krR - 2kr^2$ Remove grouping symbols

$0 = 2krR - 3kr^2$ Combine like terms and set equal to 0

$0 = r(2kR - 3kr)$ Factor out r.

$r = 0, 2kR - 3kr = 0$ Set each factor equal to 0.

$r = 0, 2kR = 3kr$

$r = 0, \dfrac{2kR}{3k} = r$ Solving for r.

$r = 0, \dfrac{2}{3}R = r$ 2/3 R means that the radius is 2/3 the normal size of the trachea.

Determine $v(0), v(2R/3), v(R)$

$$v(0) = k(0)^2(R - 0) = 0.$$

minimum

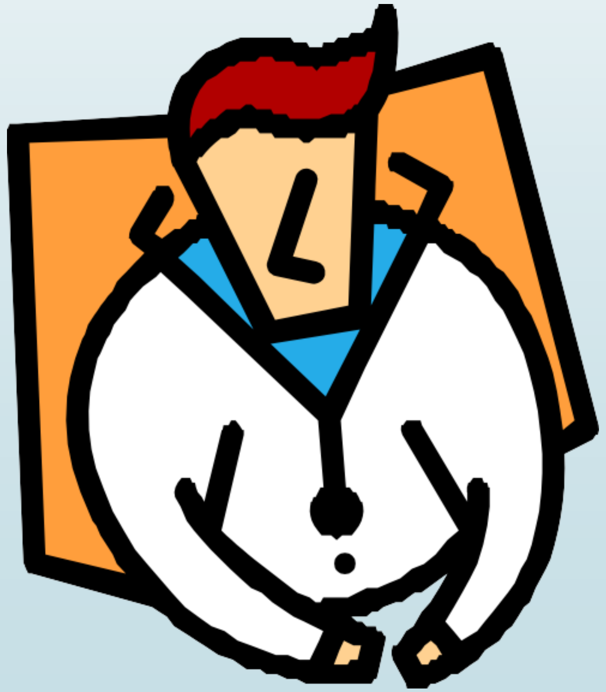
$$v\left(\frac{2}{3}R\right) = k\left(\frac{2}{3}R\right)^2\left(R - \frac{2}{3}R\right) = \frac{4kR^3}{27}.$$

maximum

$$v(R) = k(R)^2(R - R) = 0.$$

minimum

The airflow is greatest when the cough contracts the trachea to a radius of $2R/3$, or the velocity is greatest when the radius is $\frac{2R}{3}\,cm$.



Optimization Problems in Business and Economics.

MARGINAL COST FUNCTION

- Recall that if $C(x)$, the cost function, is the cost of producing x units of a certain product, then the marginal cost is the rate of change of C with respect to x.
- In other words, the marginal cost function is the derivative, $C'(x)$, of the cost function.

DEMAND FUNCTION

- Let $p(x)$ be the price per unit that the company can charge if it sells x units.
- Then, p is called the demand function (or price function), and we would expect it to be a decreasing function of x.

REVENUE FUNCTION

- If x units are sold and the price per unit is $p(x)$, then the total revenue is:

$$R(x) = xp(x)$$

  This is called the revenue function.

MARGINAL REVENUE FUNCTION

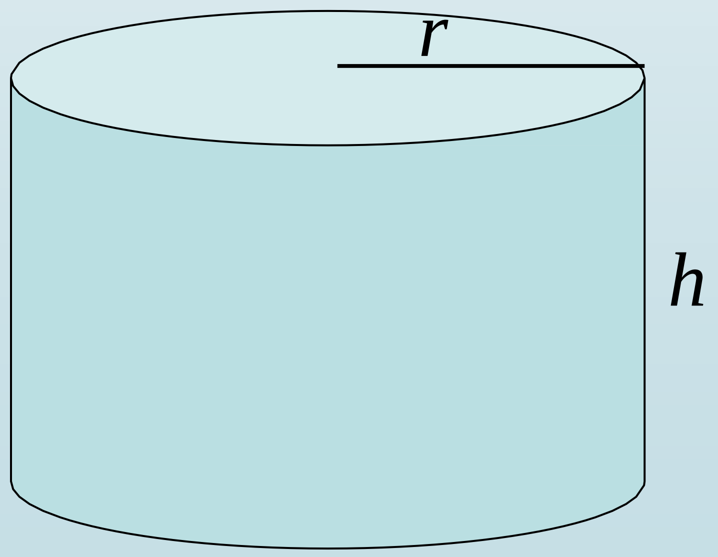
- The derivative $R'(x)$ of the revenue function is called the marginal revenue function.
- It is the rate of change of revenue with respect to the number of units sold.

MARGINAL PROFIT FUNCTION

- If x units are sold, then the total profit is: $P(x) = R(x) - C(x)$, and is called the profit function.
- The marginal profit function is $P'(x)$, the derivative of the profit function.

Example: MINIMIZING COST

A company needs to construct a cylindrical container that will hold $100cm^3$. The cost for the top and bottom of the can is 3 times the cost for the sides. What dimensions are necessary to minimize the cost.



$$V = \pi r^2 h$$

$$SA = 2\pi rh + 2\pi r^2$$

Solution:

$$V = \pi r^2 h$$

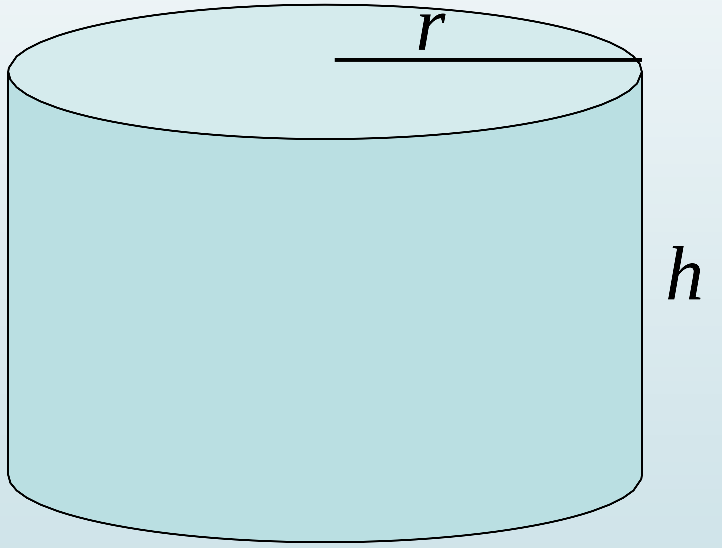
$$100 = \pi r^2 h$$

$$SA = 2\pi r h + 2\pi r^2$$

$$\frac{100}{\pi r^2} = h$$

$$SA = 2\pi r \frac{100}{\pi r^2} + 2\pi r^2$$

$$SA = \frac{200}{r} + 2\pi r^2$$



$$V = \pi r^2 h$$

$$SA = 2\pi r h + 2\pi r^2$$

Cost function is given as:

$$C(r) = \frac{200}{r} + 6\pi r^2$$

Domain: $r > 0$

$$C'(r) = \frac{-200}{r^2} + 12\pi r$$

Solution:

$$C'(r) = \frac{-200}{r^2} + 12\pi r$$

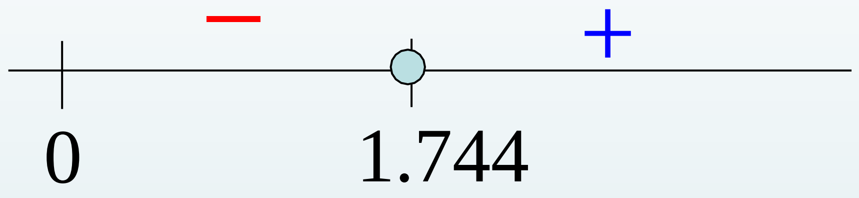
$$0 = \frac{-200}{r^2} + 12\pi r$$

$$\frac{200}{r^2} = 12\pi r$$

$$200 = 12\pi r^3$$

$$\frac{200}{12\pi} = r^3$$

$$r = \sqrt[3]{\frac{50}{3\pi}} \approx 1.744$$



C' changes from $-ve.$ to $+ve$ $\therefore$ Rel. min

$$\frac{100}{\pi r^2} = h$$

$$h = 10.464$$

The container will have a radius of 1.744 cm and a height of 10.464 cm

Example: MAXIMIZING PROFIT

The Sonic Company's total profit in dollars from manufacturing and selling x units of their loudspeaker system is given by:

$$P(x) = -0.02x^2 + 300x - 200{,}000; \quad (0 \le x \le 20{,}000)$$

How many units of the loudspeaker system must Sonic produce to maximize its profits.

Solution:

$$P(x) = -0.02x^2 + 300x - 200{,}000; \quad (0 \le x \le 20{,}000)$$
$$\Rightarrow P'(x) = -0.04x + 300.$$
$$\Rightarrow P'(x) = 0 \Rightarrow -0.04x + 300 = 0.$$
$$\Rightarrow 0.04x = 300$$
$$\Rightarrow x = 7500.$$

Determine $P(0), P(7500), P(20{,}000)$.

Note that

$P(0) = -0.02(0)^2 + 300(0) - 200{,}000 = -200{,}000$

$P(7500) = -0.02(7500)^2 + 300(7500) - 200{,}000 = 925{,}000$

$P(20000) = -0.02(20000)^2 + 300(20000) - 200{,}000 = -2{,}200{,}000$

Thus, the largest profit occurs at the critical point $x = 7500$ and is $\$925{,}000$.

Practice Questions

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- **Chapter: 4**
  - **Exercise: 4.5**

    Q # 1 to Q # 37, Q # 43 to Q # 52.