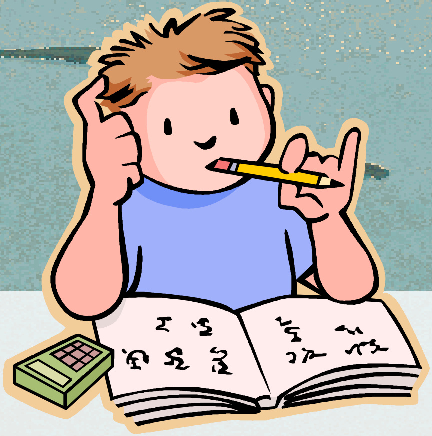
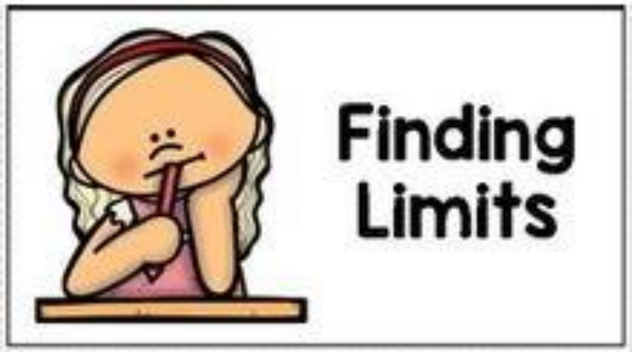


Limits



Calculus & Analytical Geometry
MATH- 101

Instructor: Dr. Naila Amir
(SEECS, NUST)

- An Introduction To Limits
- One-Sided Limits
- Laws for Calculating Limits
- **Limits Involving Infinity**
    - **Infinity as a Limit**
    - **Limit at infinity**

Book: Thomas Calculus (11th Edition) by George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano

- Chapter: 2
    - Sections: 2.2, 2.4, 2.5

Laws for Calculating Limits

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L \qquad \text{and} \qquad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $$\lim_{x \to c}(f(x) + g(x)) = L + M$$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $$\lim_{x \to c}(f(x) - g(x)) = L - M$$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $$\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$$

The limit of a product of two functions is the product of their limits.

Laws for Calculating Limits

4. *Constant Multiple Rule:* $$\lim_{x \to c} (k \cdot f(x)) = k \cdot L$$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:* $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule:* If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Laws for Calculating Limits

Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Some Useful Limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

LIMITS

Some Useful Limits

1. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e.$
2. $\lim_{x \to 0} (1 + x)^{1/x} = e.$
3. $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0.$
4. $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$

Example: Evaluate $\lim\limits_{x\to 0}\frac{a^x-1}{x}$.

Solution: Let $a^x - 1 = y$ so that $y \to 0$ as $x \to 0$

Now, $a^x = 1 + y$ or $x \ln a = \ln(1+y)$ or $x = \frac{\ln(1+y)}{\ln a}$

Thus, $$\frac{a^x-1}{x} = \frac{y\ln a}{\ln(1+y)} = \frac{\ln a}{\frac{1}{y}\ln(1+y)} = \frac{\ln a}{\ln(1+y)^{1/y}}$$

So that $$\lim_{x\to 0}\frac{a^x-1}{x} = \lim_{y\to 0}\left[\frac{\ln a}{\ln(1+y)^{1/y}}\right] = \ln a\,.\frac{1}{\lim\limits_{y\to 0}\ln(1+y)^{1/y}}$$

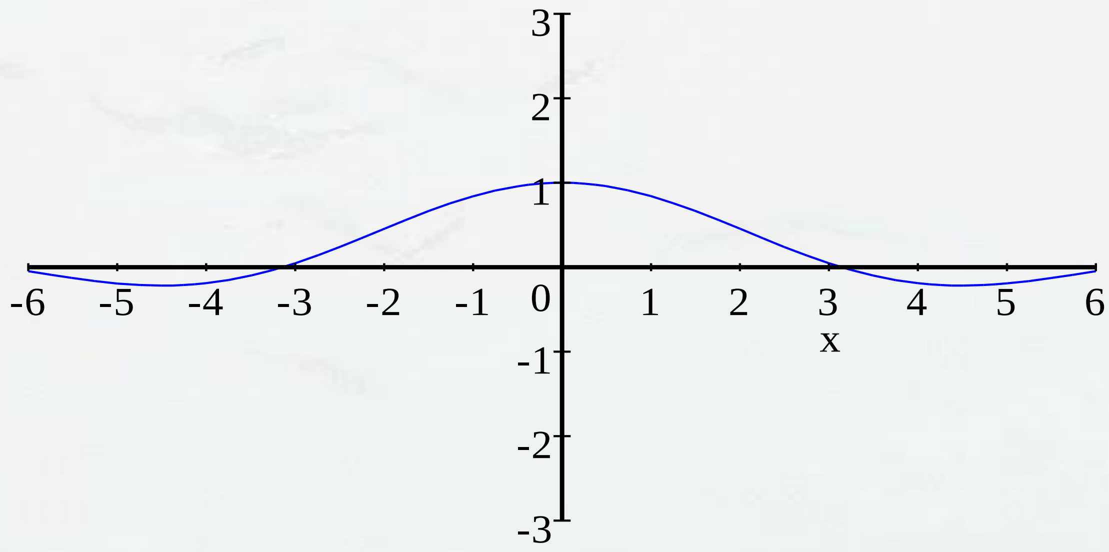
$$= \ln a\,.\frac{1}{\ln\left[\lim\limits_{y\to 0}(1+y)^{1/y}\right]} = \ln a\,.\frac{1}{\ln e} = \ln a$$

$\because \ln e = 1$

Example: Using the Sandwich theorem evaluate $\lim_{x \to 0} \dfrac{\sin x}{x}$

Solution:

If we graph $y = \dfrac{\sin x}{x}$, it appears that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

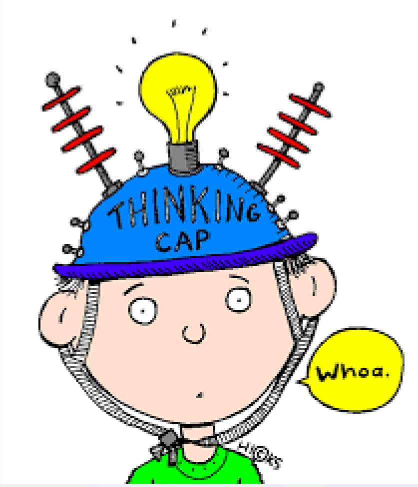


We might try to prove this using the sandwich theorem as follows:

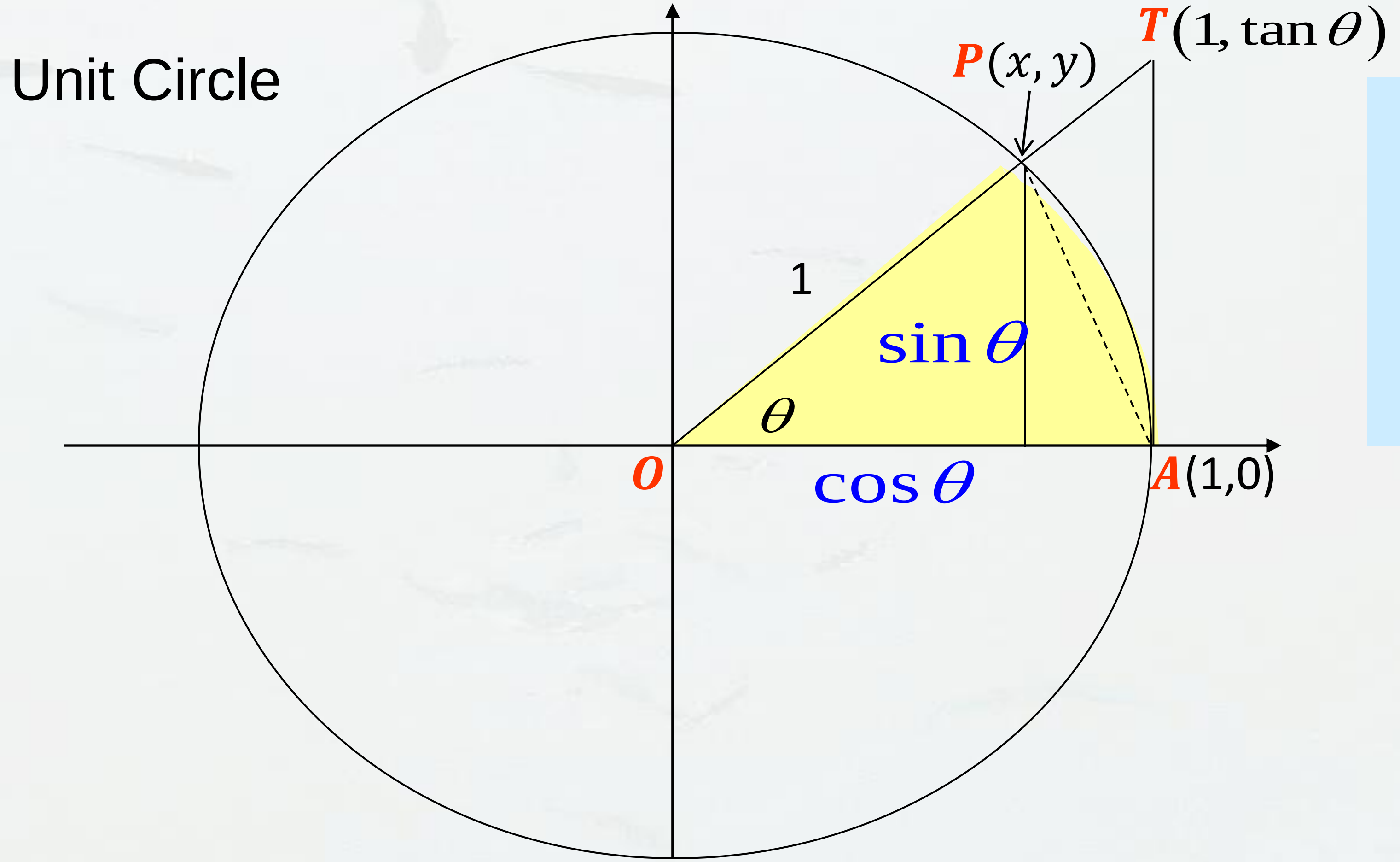
$$\text{Since} \quad -1 \le \sin x \le 1$$

$$\therefore \quad \lim_{x \to 0} \frac{-1}{x} \le \lim_{x \to 0} \frac{\sin x}{x} \le \lim_{x \to 0} \frac{1}{x}.$$

Unfortunately, neither of these new limits are defined, since the left and right hand limits do not match.



We will have to be more creative.



Area sector AOP

$$= \frac{\theta}{2\pi} \cdot \pi r^2$$

$$= \frac{\theta}{2}$$

$$\text{Area } \Delta\ AOP \le \text{Area of sector } AOP \le \text{Area } \Delta\ OAT$$

$$\frac{1}{2} \cdot 1 \cdot \sin\theta \le \frac{\theta}{2} \le \frac{1}{2} \cdot 1 \cdot \tan\theta$$

$$\frac{1}{2}\cdot 1\cdot \sin\theta \le \frac{\theta}{2} \le \frac{1}{2}\cdot 1\cdot \tan\theta$$

$$\sin\theta \le \theta \le \tan\theta$$ multiply by two

$$\sin\theta \le \theta \le \frac{\sin\theta}{\cos\theta}$$

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$ divide by $\sin\theta$

$$1 \ge \frac{\sin\theta}{\theta} \ge \cos\theta$$ Take the reciprocals, which reverses the inequalities.

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1$$ Switch ends.

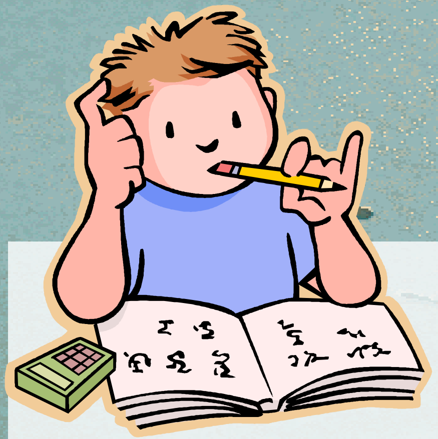
$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1$$

$$\lim_{\theta\to 0}\cos\theta \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le \lim_{\theta\to 0} 1$$

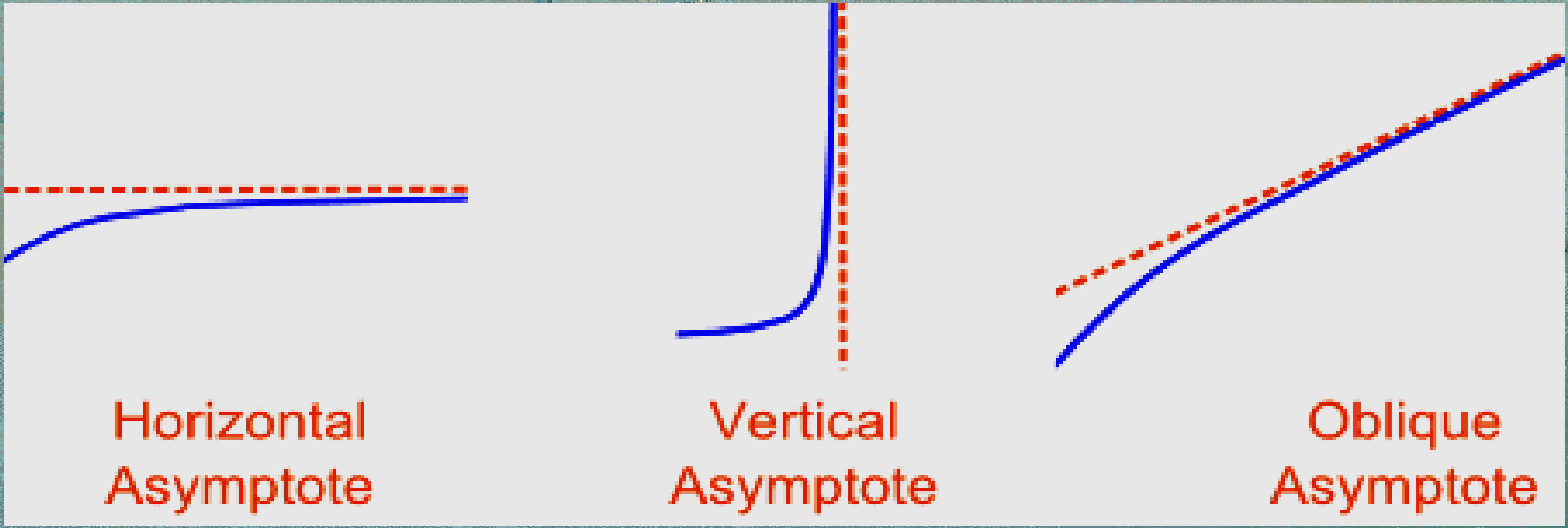
$$1 \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le 1$$

Thus, by the sandwich theorem:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$



Limits


Horizontal Asymptote
Vertical Asymptote
Oblique Asymptote

Recall…

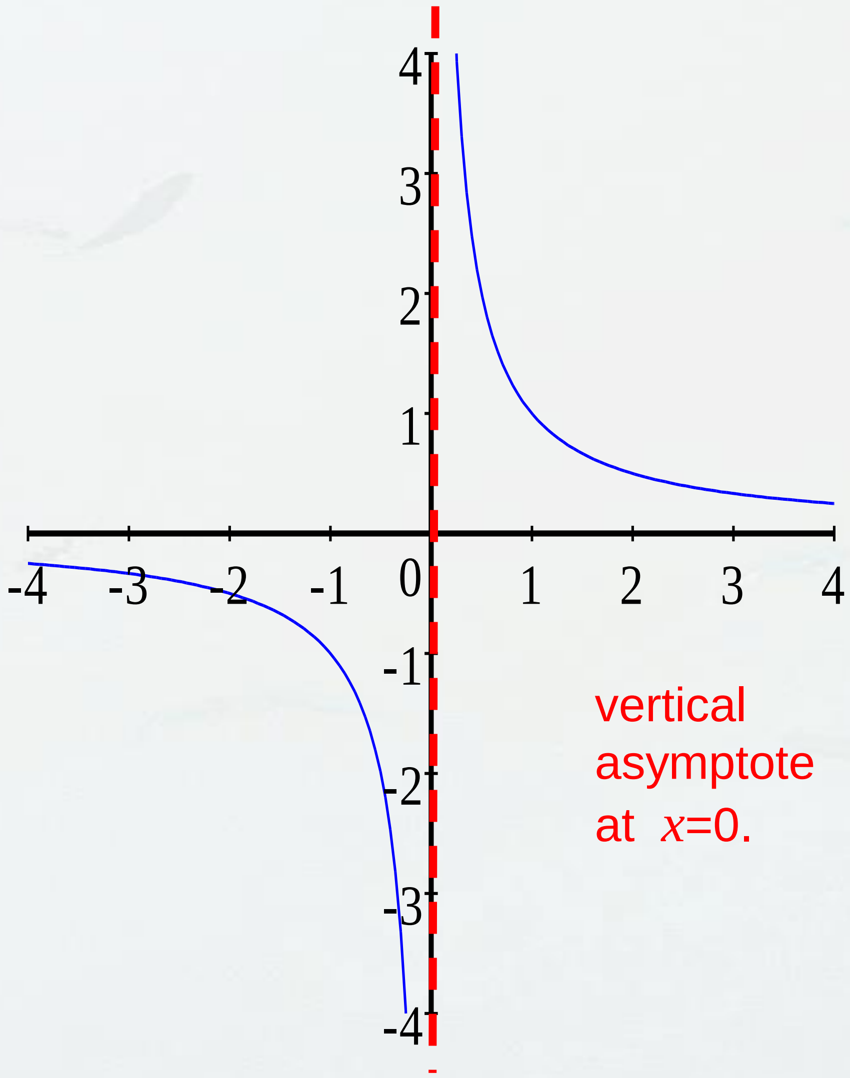
- The notation $\lim_{x \to c} f(x) = \infty$ tells us ***how*** the limit fails to exist by denoting the unbounded behavior of $f(x)$ as x approaches c.
- Infinity is **not** a number!

$$f(x) = \frac{1}{x}$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to 0} \frac{1}{x} \text{ does not exist.}$$



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Asymptotes

- An **asymptote** of a curve is a line or curve to which the curve converges. In other words, the curve and its asymptote get infinitely close, but they never meet at infinity.
- In most cases, the asymptote(s) of a curve can be found by taking the limit of a value where the function is not defined.
- Asymptotes are generally straight lines, unless mentioned otherwise.
- Asymptotes can be broadly classified into three categories: horizontal, vertical and oblique.

Vertical Asymptotes

- If $f(x)$ approaches infinity (or negative infinity) as x approaches a from the left or the right, then the line $x = a$ is a vertical asymptote of the graph of f.
- A function may have any number of vertical asymptotes.

The Existence of a Vertical Asymptote

If $h(x) = \frac{f(x)}{g(x)}$ is continuous around a and $g(x) \neq 0$ around a, then $x = a$ is a vertical asymptote of $h(x)$ if $f(a) \neq 0$ and $g(a) = 0$.

Big Idea: $x = a$ is a vertical asymptote if a ONLY makes the denominator zero.

Example

Determine all vertical asymptotes of $f(x) = \frac{x+1}{x^2-x-2}$.

When is the denominator zero:

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

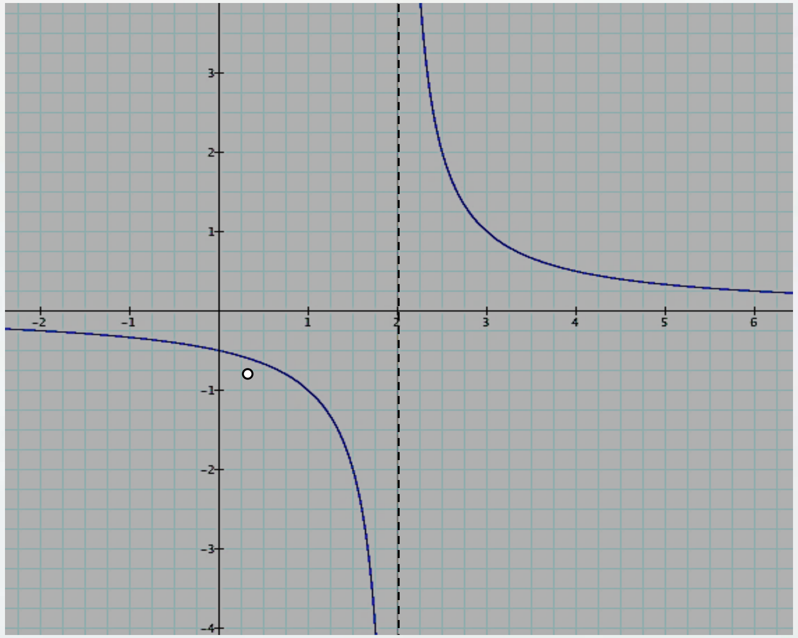
$$x = -1 \text{ or } x = 2$$

Do the x's make the numerator 0?

$-1+1=0$ Yes…

$2+1=3$ No!

$x = 2$ is a vertical asymptote



EXTRA: What about x = -1?

$$f(x) = \frac{x+1}{x^2-x-2} = \frac{\cancel{x+1}}{(\cancel{x+1})(x-2)} = \frac{1}{x-2}.$$

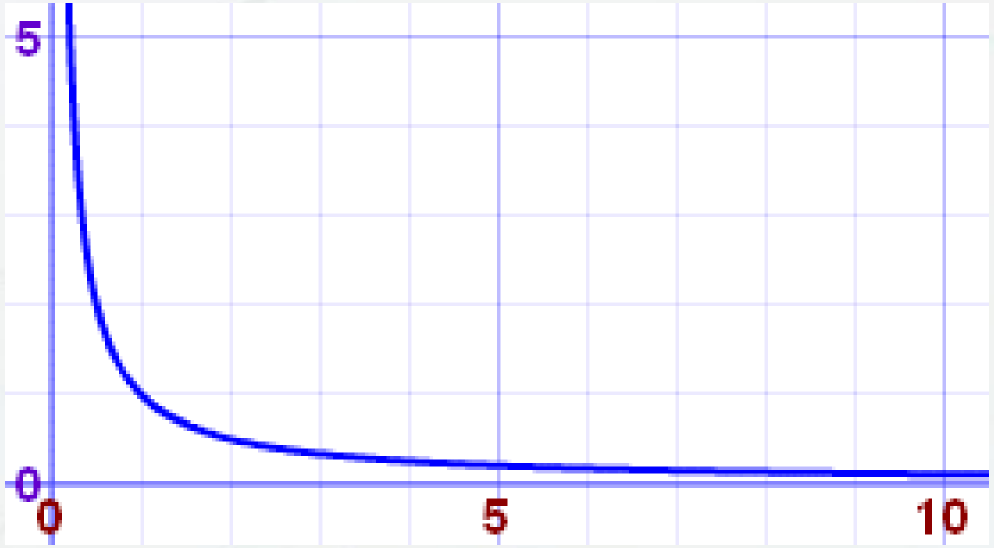
Therefore, $x = 1$ is a removable discontinuity

Digging deeper…

- Infinity is a very special idea. We know we can't reach it, but we can still try to work out the value of functions that have infinity in them.

But We Can Approach It!

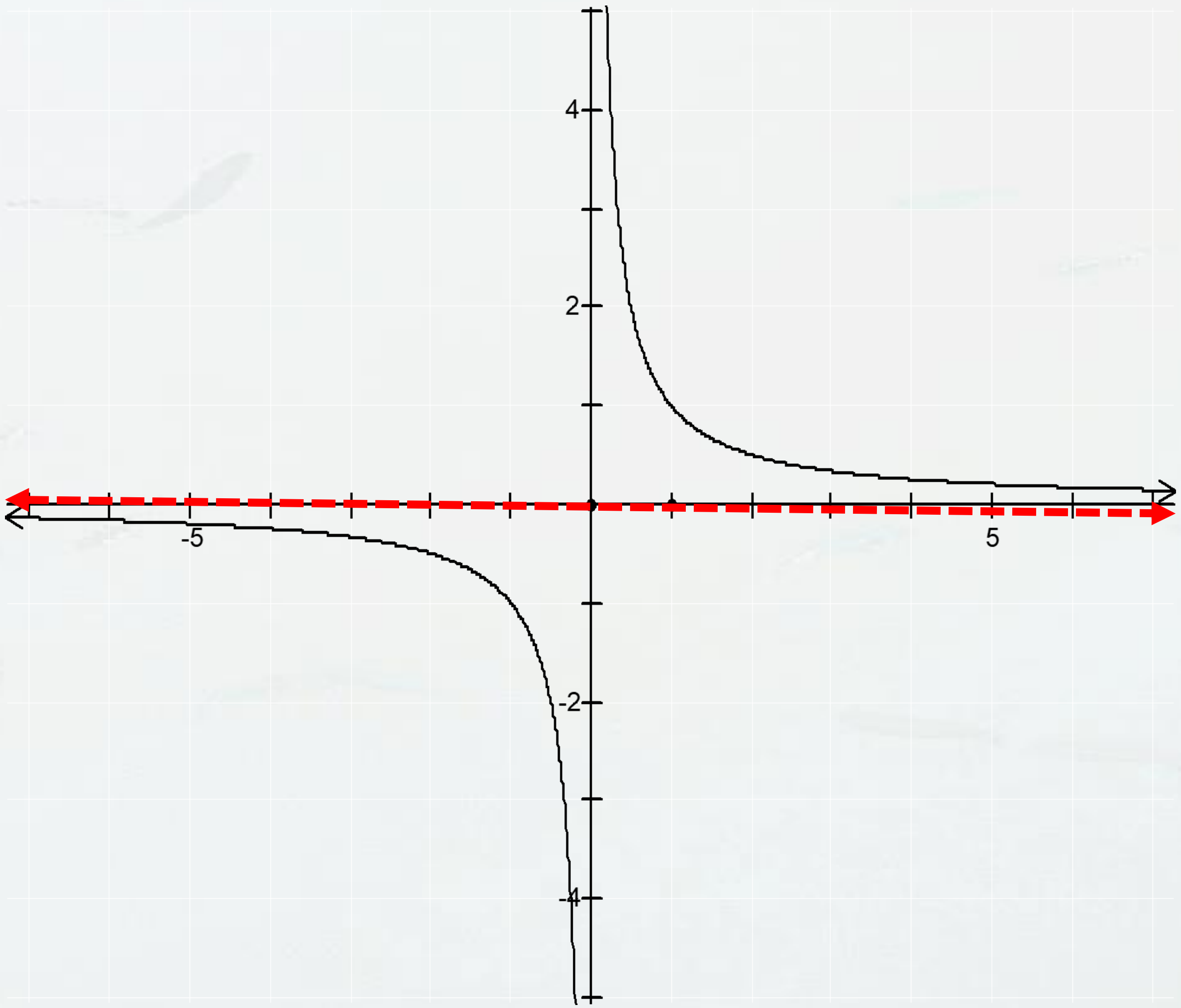
x	1/x
1	1.00000
2	0.50000
4	0.25000
10	0.10000
100	0.01000
1,000	0.00100
10,000	0.00010



Limits at infinity of $\frac{1}{x}$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

$$\lim_{x \to \infty} \frac{1}{x} = 0$$



Horizontal asymptote at $y = 0$

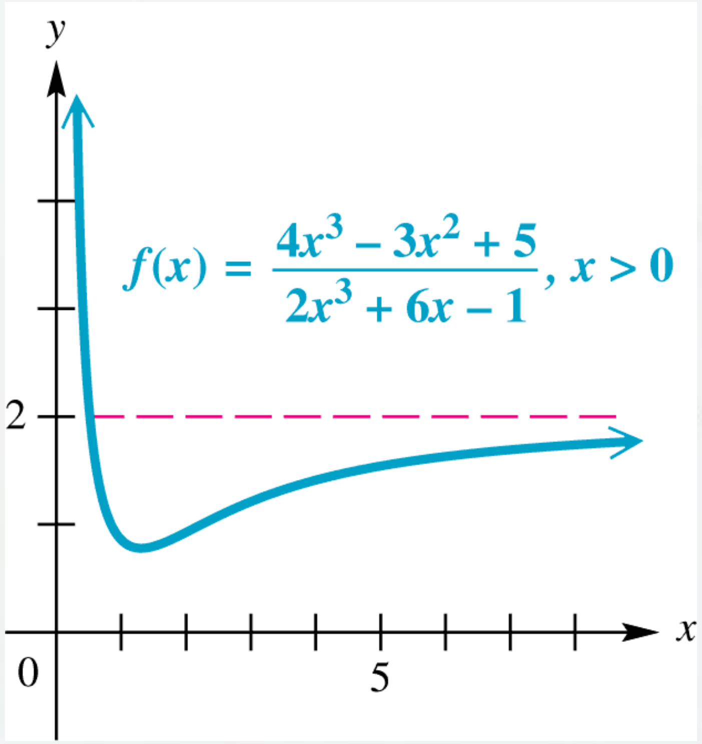
Limits at infinity of $\dfrac{1}{x^n}$

For any positive real number n,

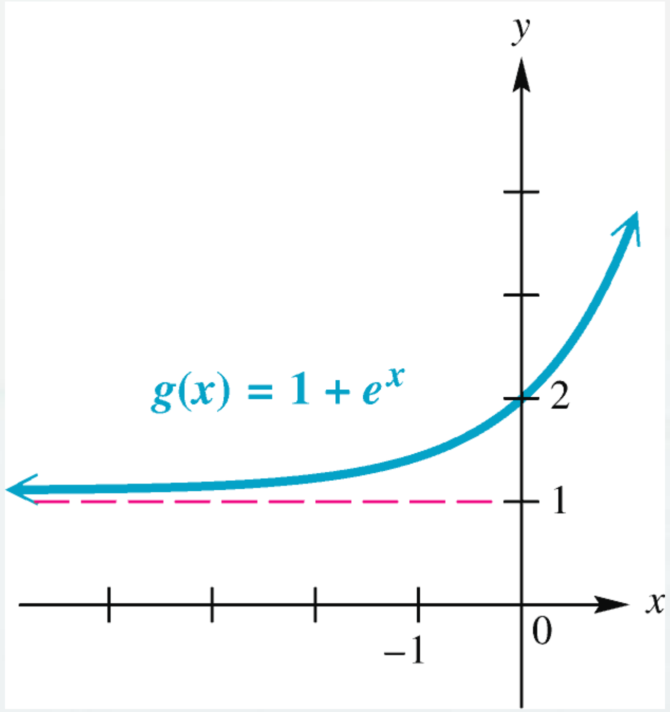
$$\lim_{x\to\infty}\frac{1}{x^n}=0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{1}{x^n}=0.$$

Limits as x approaches $\pm\infty$

A function may approach an asymptotic value as x moves in the positive or negative direction.



$$\lim_{x\to\infty} f(x) = 2$$



$$\lim_{x\to-\infty} g(x) = 1$$

The notation,

$$\boxed{\lim_{x \to \infty} f(x) = L}$$

is read:

“the limit of $f(x)$ as x approaches infinity is L.”

The values of $f(x)$ get closer and closer to L as x gets larger and larger.

The notation,

$$\boxed{\lim_{x \to -\infty} f(x) = L}$$

is read:

"the limit of $f(x)$ as x approaches negative infinity is L."

The values of $f(x)$ get closer and closer to L as x assumes negative values of larger and larger magnitude.

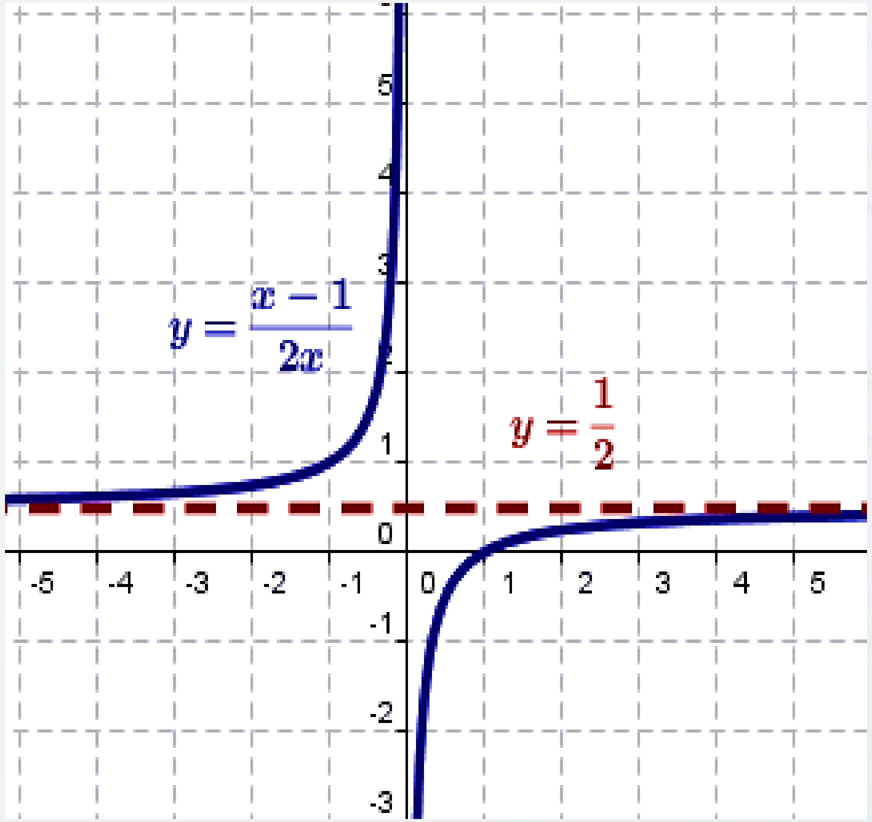
Horizontal Asymptotes

- The line $y = L$ is a horizontal asymptote of f if

$$\lim_{x\to\infty} f(x) = L$$

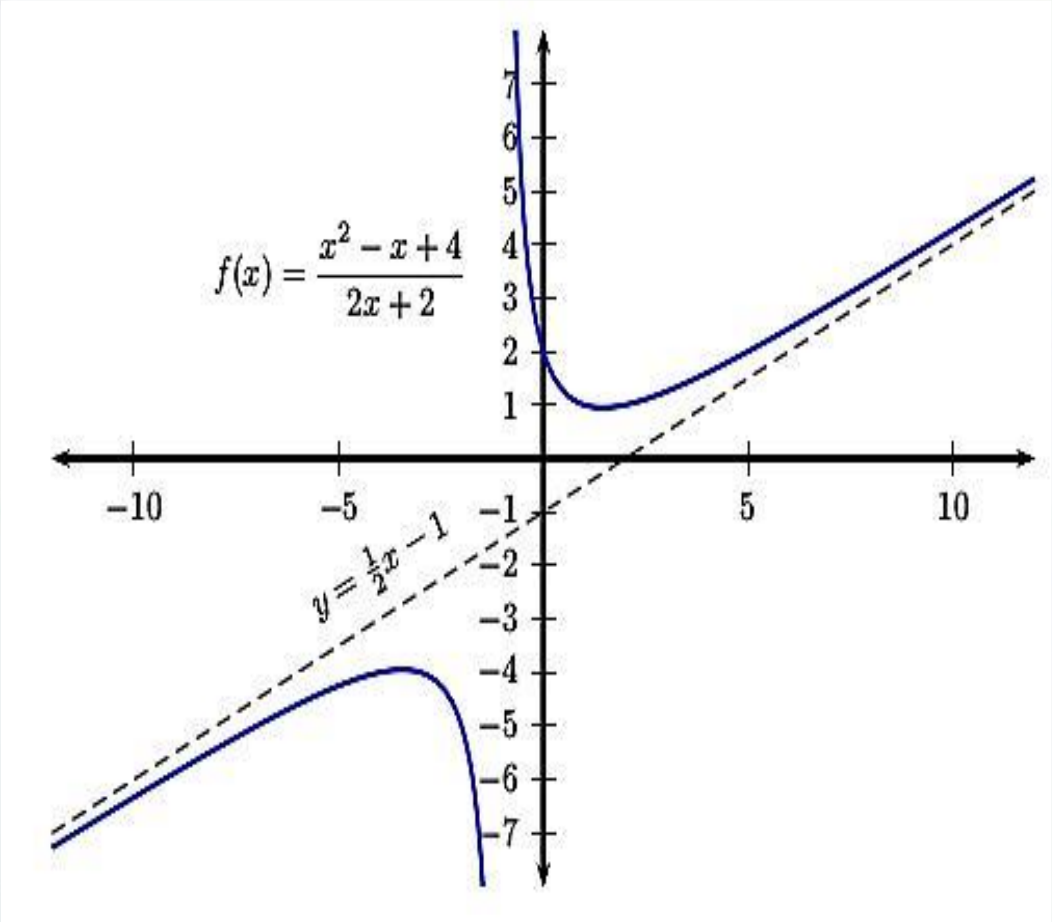
or
$$\lim_{x\to-\infty} f(x) = L$$

- Notice that a function can have at most two HORIZONTAL asymptotes (Why?)



Oblique Asymptotes

If the degree of the numerator of a rational function is one greater than the degree of the denominator, the graph has an **oblique (slanted) asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \infty$



THEOREM **Limit Laws as $x \to \pm\infty$**

If L, M, and k, are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \qquad \text{and} \qquad \lim_{x \to \pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to \pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to \pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x \to \pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

Example: Determine

$$\lim_{x\to\infty} \frac{5x^2 - 7x + 1}{2x^2 + x + 5}.$$

Solution: Divide numerator and denominator by the highest power of x involved, i.e., x^2.

$$\lim_{x\to\infty} \frac{5x^2 - 7x + 1}{2x^2 + x + 5} = \lim_{x\to\infty} \frac{5 - \frac{7}{x} + \frac{1}{x^2}}{2 + \frac{1}{x} + \frac{5}{x^2}}$$

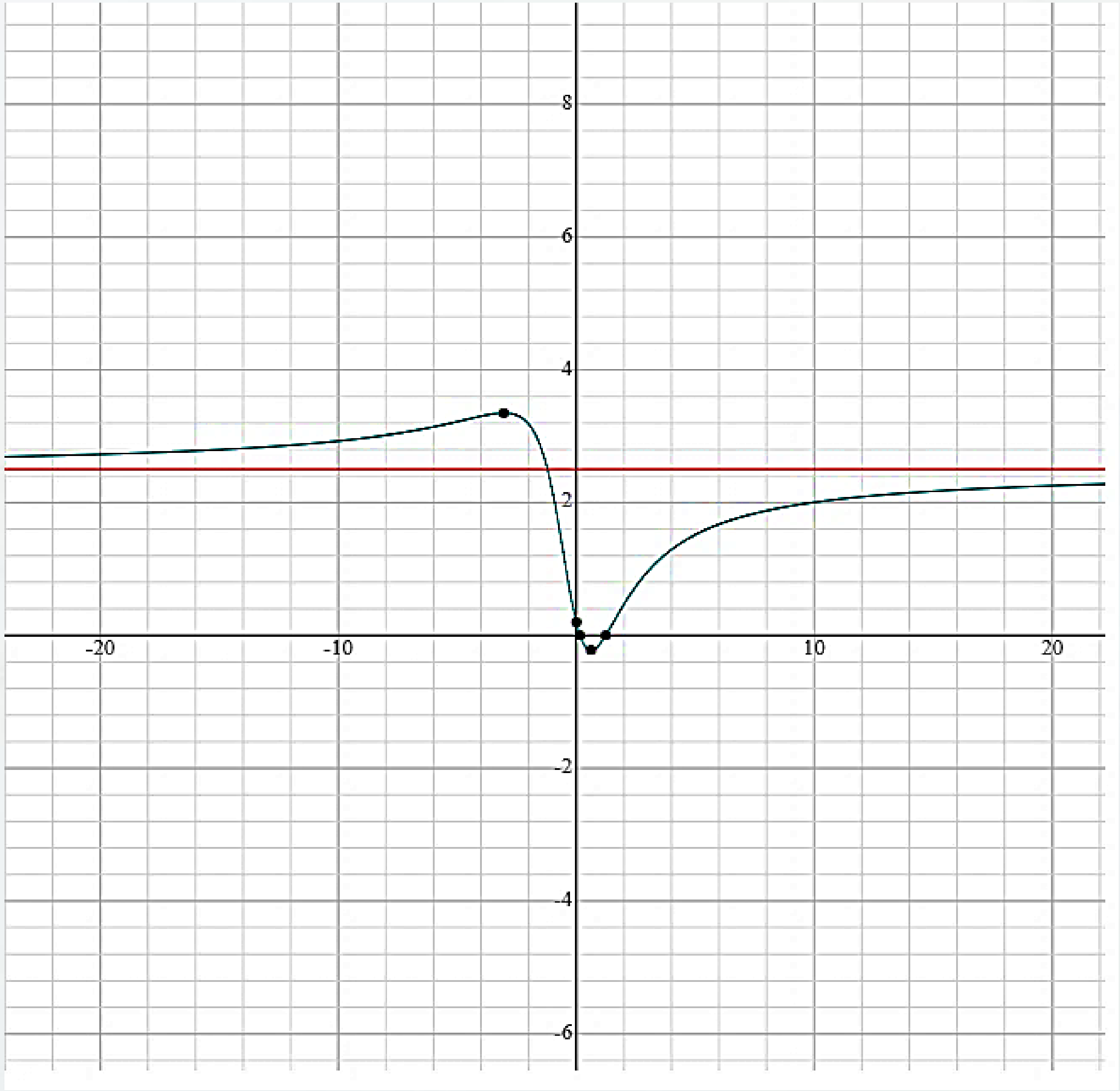
$$\lim_{x\to\infty}\frac{5x^2-7x+1}{2x^2+x+5}=\lim_{x\to\infty}\frac{5-\frac{7}{x}+\frac{1}{x^2}}{2+\frac{1}{x}+\frac{5}{x^2}}$$

$$=\frac{\lim\limits_{x\to\infty}5-\frac{7}{x}+\frac{1}{x^2}}{\lim\limits_{x\to\infty}2+\frac{1}{x}+\frac{5}{x^2}}$$

$$\lim_{x\to\infty}\frac{5x^2-7x+1}{2x^2+x+5}$$

$$=\frac{\lim\limits_{x\to\infty}5-7\cdot\lim\limits_{x\to\infty}\frac{1}{x}+\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty}2+\lim\limits_{x\to\infty}\frac{1}{x}+5\cdot\lim\limits_{x\to\infty}\frac{1}{x^2}}$$

$$=\frac{5-0+0}{2+0+0}=\frac{5}{2}$$

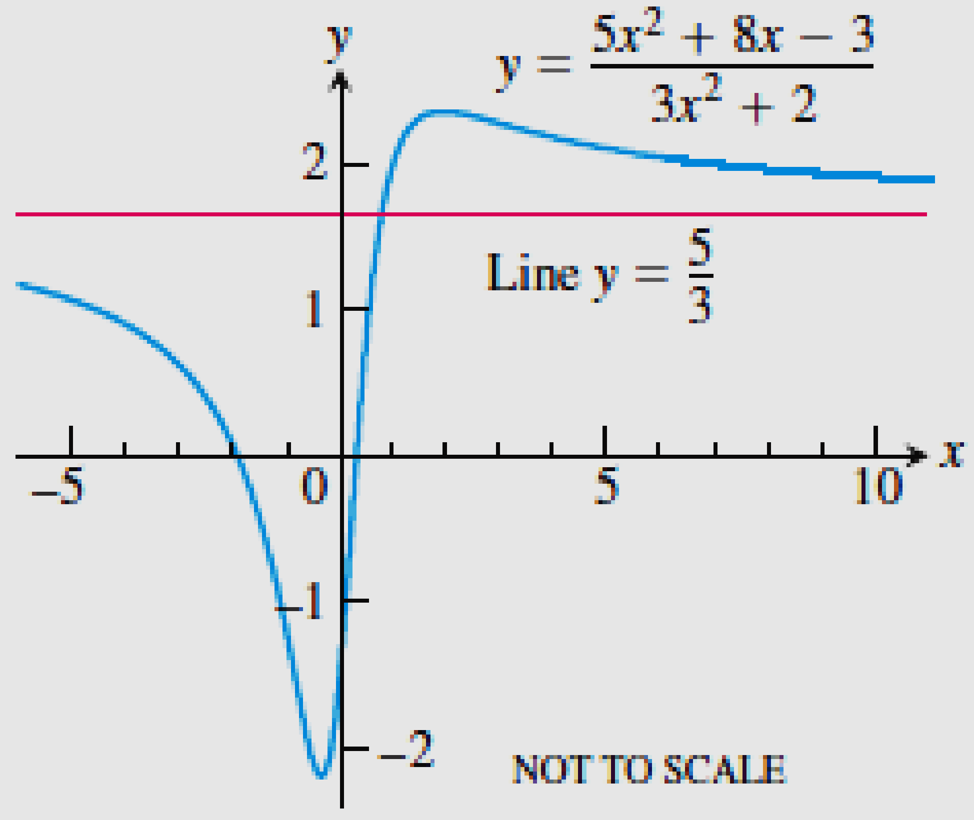


Example: Determine

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}.$$

Solution:

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}}=\frac{5}{3}.$$



The graph approaches the line $y = 5/3$ as $|x|$ increases.