Quantum Circuits and Quantum Machine Learning

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Linear Algebra

- **Hilbert Space** is a (finite-dimensional) complex vector space H equipped with an **inner product**:

$$\langle v|(|w\rangle) \equiv \langle v|w\rangle \equiv (|v\rangle, |w\rangle)$$

- A matrix U is **Unitary** if $UU^\dagger = U^\dagger U = I$
- A matrix X is **Self-adjoint** if $X^\dagger = X$
- A matrix Y is symmetric positive semi-definite if it is **Self-adjoint** and its **eigenvalues are positive.**
- Trace of an $n \times n$ matrix A: $Tr(A) = \sum_{i=1}^{n} a_{ii}$

Tensor Products

- Suppose V and W are Hilbert spaces of dimension m and n respectively. Then V $\otimes$ W has dimension m $\times$ n.
- Calculation rule of tensor product:

$$A \otimes B \equiv \left.\overbrace{\begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{bmatrix}}^{nq}\right\} mp \, .$$

Basic Quantum mechanics I

- To describe the *state space*(Hilbert Space) in the quantum physic system, we use the unit vector in the state space.
- The unit vector in the simplest state space $\mathbb{C}^2$ is called a **qubit**:

$$|\psi\rangle = a|0\rangle + b|1\rangle, \ \langle\psi|\psi\rangle = 1$$

  $|0\rangle$ and $|1\rangle$ are the orthonormal basis for $\mathbb{C}^2$, a, b are complex numbers. Hence, we could represent every state by this equation.

Basic Quantum mechanics II

The evolution of a closed quantum system is described by the Schrödinger Equation:

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

$\hbar$ is the *Planck's constant*, and H is a Hermitian (self-adjoint) operator.

Quantum Density& Measurement

- The **density** operator for the system is defined as:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

- $\mathrm{Tr}(\rho) = 1$ and ρ is a positive operator.
- Generalized measurement(**POVM**): **POVM** is a set of positive semi-definite matrices $\{T_i\}$, i is the outcome that may occur in the experiment, and it satisfies $\sum_{i=1}^{n} T_i = \mathbb{I}$.
- The probability of the measurement result i, $P(i)$ is:

$$P(i) = \langle\psi|T_i|\psi\rangle$$

Quantum Gates and Quantum Circuit I

- Quantum Logic Gates could operate on qubit.
- Quantum Gates are unitary matrices and reversible.
- A gate which acts on n qubits: $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$ is a $2^n \times 2^n$ matrix.
- Pauli Gates:

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \quad \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- $$\text{Control X}: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{SWAP}: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Quantum Gates and Quantum Circuit II

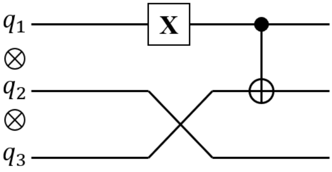
- The rotation operators are generated by the exponentiation of Pauli matrices:

$$\exp(\mathrm{i}P\theta) = \cos(\theta)I + \mathrm{i}\sin(\theta)P$$

- A rotation gate along z-axis by θ radians is:

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\theta} \end{bmatrix}$$

- An example quantum circuit composed by basic gates:



Representation Theory

- Definition and Tensor Product Representation
- Decomposition Theory
- Character
- $H \otimes H \otimes H$ and S_3

Definition and Tensor Product Representation

- **Definition:** Let V be a finite-dimensional complex vector space and G be a group. We say V is a representation of G, if $\rho : G \to GL(V)$ is homomorphism
- **Tensor Product Representation:**
- Let V, W be a representation of G
- $V \otimes W$ is a new representation of G

Decomposition Theory

- Let V be a representation of G
- V= $U_1 \oplus U_2 \oplus ... \oplus U_n$
- **Note:** Every U is a irreducible representation
- A representation is called irreducible if it has no non-trivial sub-representation
- Abelian group only has one dimensional representation

Character

- **Definition:** The character of a representation ρ is a function χ_ρ on G defined by $\chi_\rho(g) = Tr(\rho(g))$.
- **Another property:** The character is the same for elements related by conjugation. $\chi_\rho(g) = \chi_\rho(h^{-1}gh)$

	e	$(1\ 2)$	$(1\ 2\ 3)$
χ_{triv}	1	1	1
χ_{sign}	1	-1	1
χ_V	2	0	-1

TABLE 1. Character table of S_3.

$H \otimes H \otimes H$ and S_3

- **Flip:** $f : V \otimes W \to W \otimes V$
- $S_3 = \{e, (12), (13), (23), (123), (132)\}$
- $H \otimes H \otimes H$ is the irreducible representation of S_3
- suppose $h_1 \otimes h_2 \otimes h_3 \in H \otimes H \otimes H$, Then

$$\forall \sigma \in S_3 : \sigma(h_1 \otimes h_2 \otimes h_3) = h_{\sigma(1)} \otimes h_{\sigma(2)} \otimes h_{\sigma(3)}$$

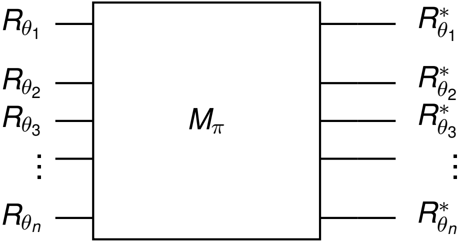
- **EX:** $\sigma_{(132)} \in S_3$

$$\sigma_{(132)}(h_1 \otimes h_2 \otimes h_3) = h_2 \otimes h_3 \otimes h_1$$

- It just a flip of $h_1 \otimes (h_2 \otimes h_3) \to (h_2 \otimes h_3) \otimes h_1$

Project goal - Permutation Matrix M_π

- Our goal is to find the permutation matrix M_π with $\pi \in S_n$ by a black-box quantum gate.



- Known inputs: $(R_{\theta_1}, R_{\theta_2}, ..., R_{\theta_n})$, each R_{θ_i} represents a rotation.
- Known outputs after measurement: $(R^*_{\theta_1}, R^*_{\theta_2}, ..., R^*_{\theta_n})$