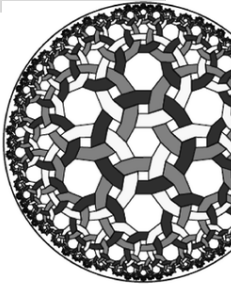
Understanding Holographic Entanglement

With gravitational path integral & tensor network

赖文昕 2019311369



Dunham:Escher on Escher'

Introduction

- **Question:** how can we *"prepare" / construct a state*, e.g.
  - vacuum: $\rho = |0\rangle\langle 0|$
  - thermal: $\rho \propto \sum_n e^{-\beta H} |n\rangle\langle n|$

  ... in a *holographic* system?
- **Answer:** via holography! (*duh...*)
  - Gravitational path integral
  - Tensor network
- Entanglement entropy captures some "structure" of the state

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Review: boundary entanglement as gravitational saddles

Recall: Path Integral in the Boundary & the Bulk

Lewkowycz:2013nqa: **Lewkowycz:2013nqa**

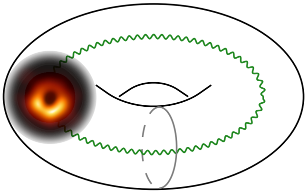

Figure: Thermal $\mathcal{Z}_\beta$

Thermal partition function
in the bulk & the boundary

Image made from **Benjamin:2020mfz**
and the EHT black hole photo

- In *any* field theory, a thermal state can be prepared by a path integral;
- In a *holographic* theory, $\mathcal{Z}_{\partial B} = \mathcal{Z}_{Bulk}$, a boundary state can be prepared by a bulk path integral.
  - e.g. the thermo-field double $\leftrightarrow$ the BTZ black hole
    Note that the Euclidean BTZ geometry is smooth:
    filling in the t_E cycle (*not* the ϕ cycle) of the torus
  - c.f. Chern-Simons/WZW: not quite the same

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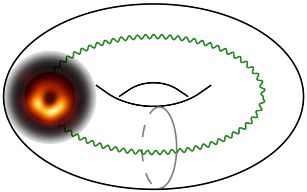

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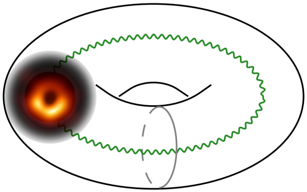

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$$\mathcal{Z}_{\partial B} = \mathcal{Z}_{Bulk} \tag{1}$$

- Replica trick: entanglement entropy for a region R:

$$S_R = \mathrm{Tr}\, \rho_R \log \rho_R \;\Leftarrow\; \mathrm{Tr}\, \rho_R^n \equiv \mathcal{Z}_n \tag{2}$$

i.e. reduced to the partition function of the n-replica.
It can be deployed in the boundary & the bulk!

  - Boundary: static geometry, but the field theory is usually strongly coupled — often difficult!
  - Bulk: dynamic geometry, weakly coupled gravity: gravity fills in the bulk smoothly, $\mathcal{Z} \sim \sum_i e^{-S_i[g_{\mu\nu}]}$: sum over classical saddles

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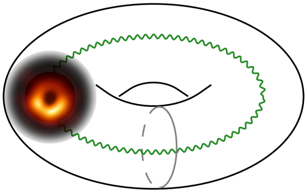

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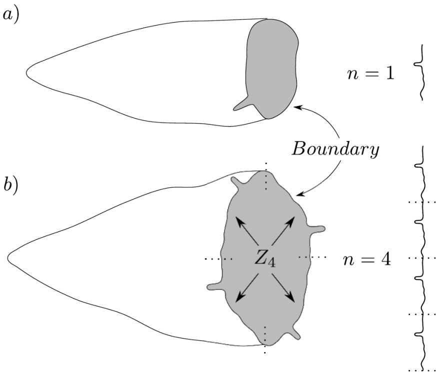

Figure: The bulk replica [**Lewkowycz:2013nqa**]

- $\mathcal{Z} \sim \sum_i e^{-S_i[g_{\mu\nu}]}$, saddle pt. approx.
  $\Rightarrow$ minimize $S[g_{\mu\nu}]$ on the n-replica $\widetilde{\mathcal{M}}_n$
- $\widetilde{\mathcal{M}}_n/\mathbb{Z}_n$: conical singularity at the $\mathbb{Z}_n$; $n \to 1$,
  $\Rightarrow$ minimize the area of the $\mathbb{Z}_n$ fixed point
  $\Rightarrow$ the extremal surface, the RT surface
- **Lesson:** use bulk path integral to:
  - prepare the states
  - compute the entanglement entropy

This requires holography,
but not necessarily $\mathrm{AdS}/\mathrm{CFT}$!

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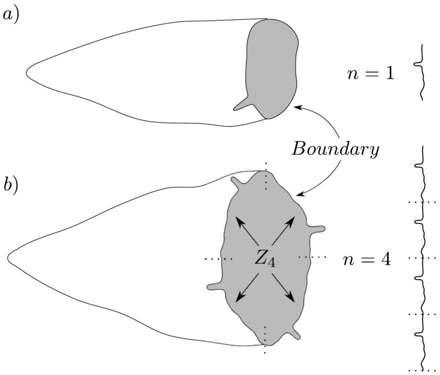

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Compare: bulk emergence from tensor networks

Prepare states via Tensor Networks

Vidal:2007hda, **Swingle:2009bg**. Reviewed by **Rangamani:2016dms**.

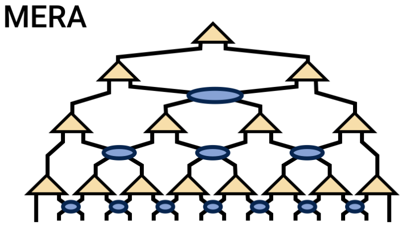

Figure: MERA

Multi-scale Entanglement Renormalization Ansatz

Image from `tensornetwork.org`

- Gravitational path integral: a spacetime perspective
  Tensor network: on a constant time slice
- States constructed with tensor networks:
  common in condensed matter (e.g. DMRG)
- To find the ground state of a system
  - Write down a tensor network
    as an **ansatz** for the ground state;
  - Vary the components of each tensor
    to achieve minimal energy — **optimization!**

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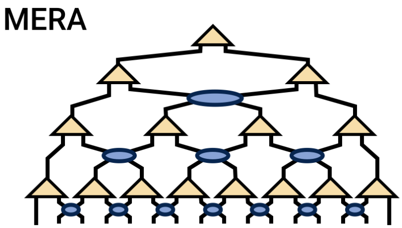

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AdS Bulk as a Tensor Network

Reviewed by **Harlow:2018fse**: **Harlow:2018fse**

`img/pentagonpush.pdf`

Figure: The HaPPY code [**Pastawski:2015qua**, **Harlow:2018fse**]

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- Dual geometry in the bulk is understood as the "continuous limit" of a *tensor network*
  - Node: tensor acting on the Hilbert space
  - Leg: index to be contracted
    - $1 \times$ free leg
      takes in bulk local operator insertions
    - $5 \times$ contracted leg
      propagates the bulk insertions to the boundary

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RT from the tensor network

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- "Entanglement" between R and $\bar{R}$:
  $\propto$ min $\#$ of links connecting the two regions
  Naturally, entropy $=$ bulk area!
- "Complexity": $\#$ of nodes
  Proposal: complexity $=$ bulk volume!
  See e.g. **Susskind:2014rva**
- Note: how to actually take the continuous limit?
  "Real time" path integral on a constant time slice
  - cMERA: **Nozaki:2012zj**
  - "Path-Integral Optimization": **Boruch:2021hqs**

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The Lesson

- The Ryu–Takayanagi proposal: $S \sim \frac{A}{4G_N}$
  - ... seems to be universal in holographic systems,
  - ... where boundary states can be constructed from some sort of bulk operations:
    - Gravitational path integral
    - Tensor network
- Applications: beyond standard $\mathrm{AdS}_3/\mathrm{CFT}_2$
  - Cutoff holography: **Lewkowycz:2019xse**
  - Flat holography: **Apolo:2020bld**, **Apolo:2020qjm**

Application: cutoff AdS_3 / $T\bar{T}$ deformed theory

Cutoff AdS_3 / $T\bar{T}$ deformed theory

McGough:2016lol

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- AdS_3 with finite cutoff:
  holographic renormalization of the boundary theory
  - This is clear in the tensor network picture "coarse-graining"
- Deform the boundary CFT_2 with some operator:
  $\mathrm{CFT}_2^{(\mathrm{UV})} \rightsquigarrow$ deformed theory $^{(\mathrm{IR})}$
- Surprisingly, we were able to find the deformed theory!
  $\delta S \propto \mu\,(T\bar{T})_\mu,\ T\bar{T} = \frac{1}{8}(T^{\alpha\beta}T_{\alpha\beta} - (T^\alpha_\alpha)^2)$
  - See e.g. **Smirnov:2016lqw**
  - **McGough:2016lol**: **McGough:2016lol**

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- If we *assume* that the tensor network intuition is valid, Then the RT proposal should still hold!
  - This is shown by **Lewkowycz:2019xse**
- Generalization of A in $S \sim \frac{A}{4G_N}$: A is actually the gravitational charge of the *replica symmetry*,
  - … *analytically continued* from $\mathbb{Z}_n$ to $U(1)$,
  - … corresponds to the Killing horizon generator / *modular flow generator* ξ.
  - This would in turn give us a hint of the modular flow in the $T\bar{T}$ deformed theory! (*ongoing work*)

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Further Reading & Outlook

- Single-trace $T\bar{T}$ duality and the flow towards to UV
  - **Apolo:2019zai**
- Quantum error correction:
  - **Jahn:2021uqr**
- Tensor network for flat spacetime?
  - **May:2016dgv**

References I