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Holographic duality beyond AdS/CFT

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Abstract: Holographic duality has been an ... This review aims to offer a bird's-eye review to recent developments on the holographic duality beyond $\mathrm{AdS}_3/\mathrm{CFT}_2$. The main content of this review is based on Wei Song's lectures at the Summer School on Strings and Stuff of Southeast University, 2021.

Contents

0 Introduction

Since the discovery of AdS/CFT duality [1], we have greatly furthered our understanding of quantum gravity in asymptotically AdS backgrounds. However, there are plenty of non-AdS geometries in our real worlds, including:

- The Kerr metric of a rotation black hole, which ... ;
- The asymptotically flat / Minkowski spacetime, which well approximates our current universe at a smaller length (and time) scale;
- The asymptotically de Sitter spacetime, which well approximates our current (and future) universe at a larger length scale, and also during the period of inflation;
- The FRW metric (or more precisely, the Friedmann-Lemaître-Robertson-Walker [2–5] metric), which describes the evolution of our homogeneous and isotropic universe from the big bang to its ... ;

- and more ...

Much less is known about quantum gravity in these backgrounds.

1 Brief review of AdS_3/CFT_2

2 Bottom-up approach: from asymptotic symmetry

3 Top-down approach: from string theory

String theory is a self-consistent theory of quantum gravity.

The first example of a microscopic counting of black hole entropy, discovered by Strominger-Vafa [6], comes from the D1-D5-P system in string theory.

The first incarnation of holographic principle was realized by Maldacena [1], by a stack of D3 branes in type IIB string theory.

3.1 The D1-D5-P system and its IR limit

Let us look at the D1-D5-P brane configuration in type IIB string theory. This is well-reviewed in [7]. This configuration allows for an open string description and a closed string description.

Geometry	$\mathbb{R}^{4,1}$		S^1	$\mathcal{M}_4 = T^4, \mathrm{K3}$			
Direction	0	1, 2, 3, 4	5	6	7	8	9
# D5 $= Q_5$	$\times$		$\times$	$\times$	$\times$	$\times$	$\times$
# D1 $= Q_1$	$\times$		$\times$				
P	$\times$		$\times$				

Table 1: Brane configuration of the D1-D5-P system. Here we are considering type IIB string theory on flat 6D spacetime, with a compactified $x^5 \in S^1$ direction, along with an internal $\mathcal{M}_4$ manifold. We use "$\times$" to mark the directions x^μ that an object occupies. Here $\mu = 0, 1, \cdots, 9$.

The D5 branes wrap the compact $\mathcal{M}_4$, while the D1 branes are localized on $\mathcal{M}_4$. Both the D1 and D5 branes extend along the fifth direction x^5, which is compactified to a circle S^1 with a large radius.

Open string excitations on the branes carry momentum and winding. Due to the large radius of S^1, we can focus on the momentum modes P along $x^5 \in S^1$ and neglect the winding modes. On the other hand, we will neglect momentum modes along the $\mathcal{M}_4$ directions, since $\mathcal{M}_4$ is assumed to be compact and small.

3.1.1 Closed string picture: gravity on AdS_3 background

In the IR limit, type IIB string theory is described by the low energy effective action of type IIB supergravity. The field content and the action of type IIB supergravity are well reviewed in the literature; see e.g. Appendix H of [8]. In particular, there is a pair of 2-form gauge potentials in type IIB supergravity. One of them is the NS-NS field B_2, and the other is the R-R field C_2. The D1 branes are electrically charged under C_2, while the D5 branes are magnetically charged under C_2.

The bosonic part of the string frame action is then given by [citations needed]:

$$\frac{1}{16\pi G}\int \mathrm{d}^{10}x\,\sqrt{-g}\,\Big(e^{-2\phi}\Big(R+4\,(\nabla\phi)^2-\frac{1}{12}H^2\Big)-\frac{1}{12}F^2\Big)\,, \tag{3.1}$$

$$H=\mathrm{d}B_2\,,\quad F=\mathrm{d}C_2 \tag{3.2}$$

where H and F are the 3-form field strengths, $H^2 = H_{\mu\nu\rho}H^{\mu\nu\rho} \propto H\wedge\star H$, and similar for F^2. After dimension reduction of the compact $\mathcal{M}_4$, the equations of motion admit a *black string* solution in 6D, where the metric is given by [citations needed]:

WL: convention for the coefficients?

$$\begin{aligned}\mathrm{d}s^2 &= (f_1 f_5)^{-1/2}\Big(-\mathrm{d}t^2+\mathrm{d}\phi^2+\frac{r_0^2}{r^2}\,(\cosh\sigma\,\mathrm{d}t+\sinh\sigma\,\mathrm{d}\phi)^2\Big)\\ &\quad+(f_1f_5)^{+1/2}\Big(\frac{\mathrm{d}r^2}{1-r_0^2/r^2}+r^2\,\mathrm{d}\Omega_3^2\Big)\,,\quad \phi\cong\phi+2\pi R,\end{aligned} \tag{3.3}$$

$$\text{where}\quad f_1=1+\frac{r_1^2}{r^2},\quad f_5=1+\frac{r_5^2}{r^2} \tag{3.4}$$

The parameters in this supergravity solution can be related to the brane construction as follows:

- $\phi\equiv x^5$ is the compactified S^1 direction along the D1 brane, normalized such that $\phi\cong\phi+2\pi R$, where R is the large radius of the S^1 circle.

  Upon dimension reduction of the ϕ direction, this 6D black string solution will become a 5D black hole solution. In fact the resulting 5D black hole solution is precisely the Strominger-Vafa black hole [6], which serves as the first example of a microscopic counting of the black hole entropy.

- r_0 marks the horizon of the black string, and it is related to the open string momentum P attached to the branes: $P\propto r_0^2\sinh 2\sigma$.

- r_1^2 and r_5^2 are related to the charges Q_1 and Q_5.

We further note that the above black string solution is asymptotically flat, consistent with our brane construction in string theory. On the other hand, if we zoom in to the near horizon region of this black string solution, we discover an $\mathrm{AdS}_3\times S^3$ geometry. This can be achieved by setting:

$$\ell^2 = r_1 r_5, \quad r \mapsto \lambda \ell r, \quad r_0 \mapsto \lambda \ell r_0, \quad t \mapsto t\ell/\lambda, \quad \phi \mapsto \phi\ell/\lambda, \tag{3.5}$$

where ℓ is the AdS radius, and now the ϕ coordinate is normalized such that $\phi \cong \phi + 2\pi$. More specially,

- For extremal black string with $r_0 = 0$ and thus $P = 0$, the near horizon limit leads to the zero mass BTZ geometry, with an additional S^3 factor:

$$\mathrm{d}s^2 = \ell^2 \left(r^2(-\,\mathrm{d}t^2 + \mathrm{d}\phi^2) + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\Omega_3^2 \right) \tag{3.6}$$

- For the near-extremal case with generic r_0, σ, the near horizon limit leads to the rotating BTZ geometry, again with an additional S^3 factor:

$$\mathrm{d}s^2 = \ell^2 \left(r^2(-\,\mathrm{d}t^2 + \mathrm{d}\phi^2) + \frac{\mathrm{d}r^2}{r^2 - r_0^2} + r_0^2 \left(\cosh\sigma \,\mathrm{d}t + \sinh\sigma \,\mathrm{d}\phi\right)^2 + \mathrm{d}\Omega_3^2 \right) \tag{3.7}$$

It is convenient to define the left and right-moving temperature:

$$T_L = \frac{1}{2\pi}\frac{r_0\, e^{\sigma}}{\ell^2}, \quad T_R = \frac{1}{2\pi}\frac{r_0\, e^{-\sigma}}{\ell^2} \tag{3.8}$$

On the other hand, the Hawking temperature T_H of this solution can be computed, and is given in terms of T_L, T_R as follows:

$$\frac{2}{T_H} = \frac{1}{T_L} + \frac{1}{T_R} \tag{3.9}$$

Within the framework of string theory, one can understand the IR black string geometry as the result of “integrating out” the dynamics of the branes, which includes the open string excitations. This process deforms the background geometry, and we end up with a closed string theory on the black string background. The near horizon limit brings us further to the IR fixed point, where the far region dynamics decouple and we are left with the AdS_3 geometry. WL: accurate?

In other words, strings on $\mathrm{AdS}_3 \times S^3 \times \mathcal{M}_4$ is understood as the *closed string description* of the IR fixed point of the D1-D5-P system.

3.1.2 Open string picture: worldvolume CFT_2 and the duality

On the other hand, we can consider the worldvolume theory of the brane construction. This provides the *open string description* of the D1-D5-P system.

After dimension reduction of the compact $\mathcal{M}_4$, we have a $(1+1)$ dimensional QFT living on the D1-D5 branes. This is a supersymmetric gauge theory with $\mathcal{N} = (4,4)$ supersymmetry. Similar to our previous discussions, we can consider the IR limit of this system.

It should flow to an IR fixed point, which is a $(1+1)$ dimensional superconformal field theory (SCFT_2).

The central charge of this SCFT_2 can be read off from the field contents of the world-volume theory: the number of bosonic fields is given by [citations needed]:

$$4Q_1Q_5 \tag{3.10}$$

and same for the fermions. The factor Q_1Q_5 comes from the open string excitations between the D1 and D5 branes, while the factor of 4 comes from the fact that the D1 branes can move inside the D5 branes along the compact $\mathcal{M}_4$ directions. In the end we have the central charge:

WL: is this language accurate?

$$c = 1 \times 4Q_1Q_5 + \frac{1}{2} \times 4Q_1Q_5 = 6Q_1Q_5 \tag{3.11}$$

We have thus discovered two equivalent descriptions of the D1-D5-P system, summarized as follows:

$$\begin{pmatrix} \text{Far-region dynamics} \\ + \\ \text{Strings on } \mathrm{AdS}_3 \times S^3 \times \mathcal{M}_4 \end{pmatrix} = \begin{pmatrix} \text{Far-region dynamics} \\ + \\ \mathrm{SCFT}_2 \text{ with target } \mathcal{M}_4 \end{pmatrix}$$

Closed string picture *Open string picture*

The far-region dynamics decouple on both sides, and we have the following proposal of an $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality [citations needed]:

$$\begin{aligned} &\text{Type IIB string theory on } \mathrm{AdS}_3 \times S^3 \times \mathcal{M}_4 \\ =\ &\text{Marginal deformations of some } \mathcal{N} = (4,4)\ \mathrm{SCFT}_2 \text{ with target } \mathcal{M}_4 \end{aligned} \tag{3.12}$$

Note that both sides of the duality includes a non-trivial moduli [citations needed]. In particular, the SCFT_2 on the right-hand side is proposed to be a *symmetric orbifold theory* [citations needed], together with its exactly marginal deformations along the conformal manifold.

WL: elaborate?

3.2 Type IIB string theory with NS-NS flux

Due to its non-trivial coupling to the R-R field C_2, the D1-D5 system is difficult to deal with on the worldsheet, within the framework of the RNS (or NSR) formalism. This difficulty can be avoided by considering its S-dual, the NS1-NS5 system.

First let us recall the $\mathrm{SL}(2,\mathbb{Z})$ duality of type IIB string theory, reviewed in §12 and §14 of [9]. In particular, the strong-weak S-duality is one of the generators of the $\mathrm{SL}(2,\mathbb{Z})$ action. When acted on the field strengths $H = \mathrm{d}B_2$ and $F = \mathrm{d}C_2$, it's given by:

$$S\colon \begin{pmatrix} H \\ F \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H \\ F \end{pmatrix} = \begin{pmatrix} F \\ -H \end{pmatrix}, \tag{3.13}$$
$$H = \mathrm{d}B_2\,, \quad F = \mathrm{d}C_2$$

We note that S-duality exchanges the field strengths H and F. On the other hand, the D1-D5 system couples to C_2 (electrically for D1, and magnetically for D5), while the NS1-NS5 system couples to B_2. Therefore S-duality corresponds to a map:

$$S\colon \quad \begin{array}{llcll} Q_5 & \text{D5 branes} & \longmapsto & k = Q_5 & \text{NS5 branes} \\ Q_1 & \text{D1 branes} & \longmapsto & p = Q_1 & \text{NS1 branes} \end{array} \tag{3.14}$$

We can then repeat the analysis in §3.1 and obtain a similar black string solution in 6D. By taking the near horizon limit, we further obtain an asymptotically AdS$_3$ geometry, namely the BTZ solutions as described in §3.1, but now with NS-NS flux instead of R-R flux. In particular, for $P = 0$ we have the massless BTZ solution [citations needed]:

WL: $k = \ell^2_{\mathtt{AdS}}/\ell_s^2$, here $\ell_s = 1$

$$\mathrm{d}s^2 = k\left(\mathrm{d}\rho^2 + e^{2\rho}\,\mathrm{d}\gamma\,\mathrm{d}\bar{\gamma}\right), \tag{3.15}$$

$$B_2 = \frac{1}{2}B_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\frac{k}{2}\,e^{2\rho}\,\mathrm{d}\gamma \wedge \mathrm{d}\bar{\gamma}\,, \quad e^{2\Phi} = \frac{k}{p}, \tag{3.16}$$

where we note that there is a non-trivial B-field and a dilaton profile Φ.

Why are we interested in this S-dual system? As we've mentioned before, unlike the original D1-D5 system, the NS1-NS5 system admits a worldsheet description as an RNS superstring theory. Therefore we have a well understood *worldsheet* CFT$_2$ for the bulk string theory. Furthermore, based on this worldsheet CFT$_2$, we are able to construct explicitly the *dual* CFT$_2$ on the asymptotic boundary of the spacetime. This is sometimes referred to as the *spacetime* CFT$_2$. Therefore the NS1-NS5 system serves as a realization of AdS$_3$/CFT$_2$ where both sides of the holographic duality are, in some sense, under control.

3.2.1 The worldsheet CFT$_2$

The bosonic action of the worldsheet theory is given by the familiar non-linear sigma model [citations needed]:

WL: conventions?

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2z\,\sqrt{-\eta}\left(\eta^{ab}G_{\mu\nu}(X) + \epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu \\ &= \frac{k}{2\pi}\int \mathrm{d}^2z\left(\partial\rho\,\bar{\partial}\rho + e^{2\rho}\,\partial\bar{\gamma}\,\bar{\partial}\gamma\right) \end{aligned} \tag{3.17}$$

where in the second line we've restricted to the AdS$_3$ part and plugged in the BTZ solution given in (3.15) and (3.16). We see that ρ is a Liouville field. We can further introduce an auxiliary field β, which leads us to the Wakimoto representation [citations needed]:

WL: conventions?

$$S[\rho,\gamma,\bar{\gamma},\beta,\bar{\beta}] = \frac{k}{2\pi}\int \mathrm{d}^2z\left(\partial\rho\,\bar{\partial}\rho + \beta\,\bar{\partial}\gamma + \bar{\beta}\,\partial\bar{\gamma} - \beta\bar{\beta}\,e^{-2\rho}\,\bar{\partial}\bar{\gamma}\,\partial\gamma\right) \tag{3.18}$$

It returns to the original action after we integrate out β. From this action it is clear that at large radius (large ρ), the last term in the action can be dropped, and the theory is well approximated by free fields, including a free scalar ρ, a pair of left-moving free fermions (β,γ), and a pair of right-moving ones $(\bar{\beta},\bar{\gamma})$.

An alternative representation of the worldsheet theory in the global AdS_3 background is given in terms of the $\widehat{\mathrm{SL}}(2,\mathbb{R})_k$ WZW model. This follows from a convenient accident, namely global AdS_3 is the universal cover of the group manifold $\mathrm{SL}(2,\mathbb{R})$. More specifically, consider the following metric of global AdS_3

$$\mathrm{d}s^2 = \ell_{\mathrm{AdS}}^2 \left(-\cosh^2\!\rho\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\!\rho\,\mathrm{d}\phi^2\right) \tag{3.19}$$

The (ρ, t, ϕ) coordinates can be packaged into the group element $g \in \mathrm{SL}(2,\mathbb{R})$, and a simple rewrite of (3.17) then gives us the action of $\widehat{\mathrm{SL}}(2,\mathbb{R})_k$ WZW model:

$$g = e^{iu\frac{\sigma_2}{2}} e^{\rho\sigma_3} e^{iv\frac{\sigma_2}{2}} \in \mathrm{SL}(2,\mathbb{R}), \tag{3.20}$$

$$\texttt{[TODO]} \quad S = \frac{k}{8\pi\alpha'} \int \mathrm{d}^2\sigma\, \sqrt{-\eta}\left(\eta^{ab} G_{\mu\nu}(X) + \epsilon^{ab} B_{\mu\nu}(X)\right) \partial_a X^\mu \partial_b X^\nu \tag{3.21}$$

where σ_i's are the Pauli matrices.

4 Holographic Dualities Beyond $\mathbf{AdS_3/CFT_2}$

In the previous sections, we have laid down some basic aspects of $\mathrm{AdS}_3/\mathrm{CFT}_2$ dualities in string theory. AdS/CFT duality is the most successful ever sofar to understand quantum gravity. However, this is only valid in Anti-de Sitter spacetime. One might wonder how far one can go away from the AdS space. This must resort to holographic dualities beyond AdS/CFT.

Starting from this section, we are going to introduce recent salient developments on holographic dualities beyond $\mathrm{AdS}_3/\mathrm{CFT}_2$. Our goal is to construct models of holographic duality by deformations of the AdS/CFT duality.

In particular, we will focus on $T\bar{T}$ deformation. This section is organised as follows:

4.1 Top-Down Approach: Deformations of $\mathbf{AdS_3/CFT_2}$ in String Theory

In this subsection, we are going to outline the basic idea of holographic dualities beyond $\mathrm{AdS}_3/\mathrm{CFT}_2$ by deformation. To making it concrete, we focus on the explicit type of $\mathrm{AdS}_3/\mathrm{CFT}_2$ dualities extensively discussed in section **??**. Recall that such a AdS_3 model can be obtained from Type IIB string on $AdS_3 \times S^3 \times M_4$ with NS 5-branes charge k and fundamental strings charge p. The main feature of such a model is that it admits weakly coupled worldsheet string description in terms of WZW model when $p \to \infty$, as been elucidiated in **??**. Furthermore, the 2d CFT dual to such a model in the long string sector is argued to be a symmetric product $\mathcal{M}_{6k}^p/S_p$.

zzzz **FX**: changing the picture!!!

With such a triangle relation, one can resort many CFT techniques to check the duality beyond $\mathrm{AdS}_3/\mathrm{CFT}_2$. The basic idea is to construct marginal deformation (if it is true

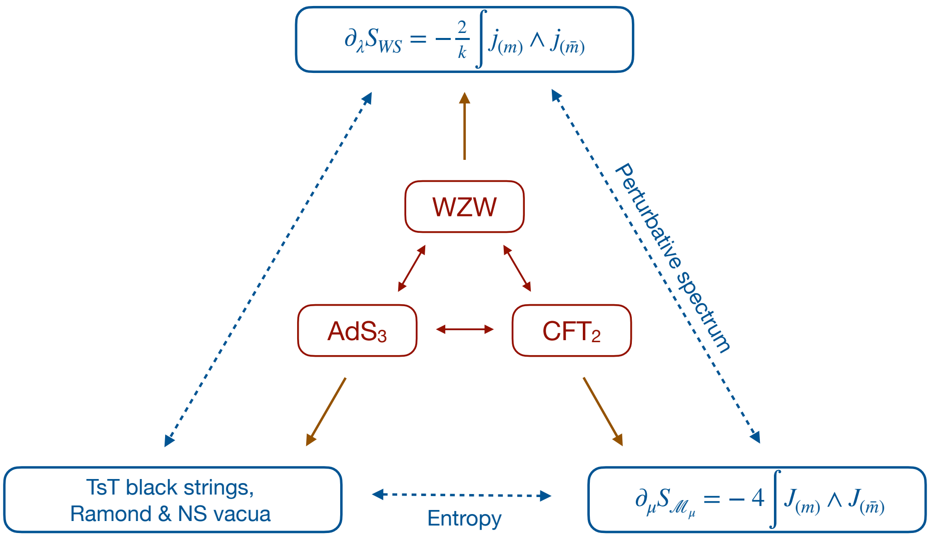

Figure 1: The holographic correspondence between string theory on $\mathrm{AdS}_3 \times M^7$ backgrounds and two-dimensional CFTs (inner triangle), and its relationship to the conjectured duality between TsT transformations of $\mathrm{AdS}_3 \times M^7$ spacetimes, bi-current deformations of the world-sheet action, and single-trace irrelevant deformations of two-dimensional CFTs (outer triangle). Adapted from [10]

marginal deformation) and that it would in general lead to another CFT. Since many spacetime operator can be constructed from the world-sheet operators, one would naturally construct the corresponding deformation, which perhaps in a good control way.

As indicated in **??**, we can construct the following marginal operator $D(x, \bar{x})$ in the world-sheet CFT [1] describing AdS_3 as

$$D(x,\bar{x}) = \int d^2z(\partial_x J\partial_x + 2\partial_x^2 J)(\partial_{\bar{x}}\bar{J}\partial_{\bar{x}} + 2\partial_{\bar{x}}^2\bar{J})\Phi_1 \tag{4.1}$$

where x denotes the set of spacetime coordinates whereas z denotes the set of world-sheet coordinates and

$$J(x,z) = 2xJ^3(z) - J^+(z) - x^2J^-(z). \tag{4.2}$$

Φ_1 here represents

$$\Phi_h(x,\bar{x};z,\bar{z}) = \frac{1}{\pi}\left(\frac{1}{(\gamma - x)(\bar{\gamma} - \bar{x})e^\rho + e^{-\rho}}\right)^{2h}, \tag{4.3}$$

with $h = 1$. To see that it is marginal, one expand it as

$$\begin{aligned}\int d^2xD(x,\bar{x}) &= \int d^2xd^2z\partial_x^2J\partial_{\bar{x}}^2\bar{J}\Phi_1(x,z) = 4\int d^2x\int d^2zJ^-(z)\bar{J}^-(\bar{z})\Phi_1(x,z)\\ &\sim \int d^2zJ^-(z)\bar{J}^-(\bar{z}),\end{aligned} \tag{4.4}$$

[1] we have combine the 2d spacetime coordinate (ϕ, t) into $x = \phi + t$ and $\bar{x} = \phi - t$.

where in the third equality we have used (4.2). Since $J^-(z)\bar{J}^-$ has dimension (2,2), so it is indeed a marginal operator on the world-sheet theory.

However, such a deformation operator $D(x,\bar{x})$, viewed from the 2d spacetime CFT, carries the conformal weight $(2,2)$ and it is irrelevant. One naturally would ask whether such a deformation operator plays the same role in the 2d spacetime CFT as $T(x)\bar{T}(\bar{x})$ with $T(x)$ the energy-momentum tensor, which carries the same conformal weight $(2,2)$ and is also irrelevant deformation.

It is then informative to look at their OPEs relations. Note that the energy-momentum tensor $T(x)$ in the dual symmetric product orbifold $\mathcal{M}^p/S_p$ CFT, due to the weakly coupling, can be written as

$$T(x) = \sum_{i=1}^{p} T_i(x), \tag{4.5}$$

where $T_i(x)$ denotes the energy-momentum tensor in the ith copy of the seed theory $\mathcal{M}$. And the deformation $T\bar{T}$ can be written as

$$T(x)\bar{T}(\bar{x}) = \sum_{i,j} T_i(x)\bar{T}_j(\bar{x}). \tag{4.6}$$

In terms of WZW world-sheet quantities, it further can be expressed as

$$T(x) = \frac{1}{2k}(\partial_x J \partial_x \Phi_1 + 2\partial_x^2 J \Phi_1)\bar{J}(\bar{x};\bar{z}) \tag{4.7}$$

With such identifications, one can use the world-sheet CFT relations to solve the OPEs between $T(x,\bar{x})D(y,\bar{y})$ and $T(x,\bar{x})T\bar{T}$, which are

$$T(x,\bar{x})D(y,\bar{y}) \sim \frac{c_{\mathcal{M}}/2}{(x-y)^4}\bar{T}(\bar{y}) + \cdots, \tag{4.8}$$

$$T(x,y)T\bar{T}(y,\bar{y}) \sim \frac{pc_{\mathcal{M}}/2}{(x-y)^4}\bar{T}(y) + \cdots, \tag{4.9}$$

respectively.

Therefore, one cannot identify $D(x,\bar{x})$ as $T(x)\bar{T}(\bar{x})$. Instead, inspired by the central charge difference, it turns out that

$$D(x,y) \sim \sum_{i}^{p} T_i(x)T_j(\bar{x}) \tag{4.10}$$

From this point, we call this deformation as single trace $T\bar{T}$ deformation. Whereas, the deformation associated $T\bar{T}$ are dubbed double trace $T\bar{T}$ deformation.

Similarly, we can also construct a marginal operator $A^a(x,\bar{x})$ in the world-sheet WZW model describing S^3 as

$$A^a(x,\bar{x}) = \int d^2z k^a(z)(\partial_{\bar{x}}\bar{J}\partial_{\bar{x}}\Phi_1 + 2\partial^2_{\bar{x}}\bar{J}\Phi_1) \tag{4.11}$$

where $k^a(z)$ describes the current algebra $\mathfrak{su}(2)$ in the S^3 direction. Note that this operator carries the conformal dimension $(1,2)$ in terms of the 2d spactime CFT algebra, which coincides with JT operator.

The question now is what do these deformation on the CFT of $AdS_3 \times S^3$ lead to? Does it lead to another CFT that corresponding to a string theory background? Since it has something to with the $T\bar{T}$ deformation, in order to fully answer this question, we are going to look close on the $T\bar{T}$ deformation in general.

4.2 A Short Review of $T\bar{T}$ Deformation

A detail review can be found in [11]. [12]

Following the line of [13], we consider a quantum field theory on Euclidean spacetime with an Energy-Momentum tensor $T^{\mu\nu}$. Assuming such a QFT is Lorentzian invariant, then the tensor $T^{\mu\nu}$ is conserved, i.e. $\partial_\mu T^{\mu\nu} = 0$. For brevity, we introduce the following short notations in the $(x,\bar{x})$ coordinates [2]

$$T := -2\pi T_{xx}, \quad \bar{T} := -2\pi T_{\bar{x}\bar{x}}, \quad , \Theta = 2\pi T_{x\bar{x}} \tag{4.13}$$

the conserved $T^{\mu\nu}$ then can be formulated as

$$\bar{\partial} T = \partial\Theta, \qquad \partial\bar{T} = \bar{\partial}\Theta, \tag{4.14}$$

One can then define a composite operator $\mathcal{O}_{T\bar{T}}$ from its determinant $\mathrm{Det}(T_{\mu\nu})$ as

$$\begin{aligned}\mathcal{O}_{T\bar{T}} &:= -\pi^2 \mathrm{Det}(T_{\mu\nu}) \\ &= -\pi^2(T_{xx}T_{\bar{x}\bar{x}} - T_{x\bar{x}}T_{\bar{x}x}) \\ &= -\pi^2(\mathrm{Lim}_{x\to y}(T_{xx}(x)T_{\bar{x}\bar{x}}(y) - T_{x\bar{x}}(x)T_{\bar{x}x}(y)) + \text{derivatives}.\end{aligned} \tag{4.15}$$

It turns out that this composite operator $\mathcal{O}_{T\bar{T}}$ is well-defined, since the non-derivative parts of $T_{xx}(x)T_{\bar{x}\bar{x}}(y)$ and $T_{x\bar{x}}(x)T_{\bar{x}x}(y)$ are cancelled by each other and hence it is finite up to a total derivative. When evaluated by the eigenstates of the Hamiltonian, such a total derivative leads to vanish value.

[2] In the (ϕ,t) coordinate, we have

$$T_{xx} = \frac{1}{4}(T_{\phi\phi} + T_{tt} - 2T_{\phi t}), \qquad \bar{T}_{\bar{x}\bar{x}} = \frac{1}{4}(T_{\phi\phi} + T_{tt} + 2T_{\phi t}), \qquad T_{x\bar{x}} = \frac{1}{4}(T_{\phi\phi} - T_{tt}) \tag{4.12}$$

Therefore, one can use the deformation coupling, denoting as μ, to define a one-parameter family of theories and hence a trajectory in the field theory space. We can denote the action at each point of this trajectory by S^{μ}, and obeys the following equation

$$\frac{\partial S_{\mathcal{M}_\mu}}{\partial \mu} = \int d^2x \mathcal{O}_{T\bar{T}} \tag{4.16}$$

Example:

Let us consider a free scalar model in flat 2d euclidean spacetime. The action is

$$S = \int d^2x \partial\phi\bar{\partial}\phi \tag{4.17}$$

It turns out that under the $T\bar{T}$ deformation, the deformed theory corresponds to a Nambu-Goto string action in a static gauge

$$S^{\mu} = \frac{1}{2\mu}(\sqrt{1 + 2\mu\partial\phi\bar{\partial}\phi + 1} - 1) \tag{4.18}$$

From this, one can see that the $T\bar{T}$ deformation has intriguing relation with string theory.

Let us make some remarks on the $T\bar{T}$ deformation before we move to the spectrum

• It is irrelevant, since the operator $\mathcal{O}_{T\bar{T}}$ has dimension $[\mathrm{Mass}]^4$, then the deformation coupling μ then is of dimension $[\mathrm{Length}]^2$.

• It turns out to be solvable [13] , which is in contrast to most of irrelevant deformation. This is the main reason why people are interested in studying them.

• It can also be reformulated as coupling to random geometry [14]. Furthermore, it was pointed out by [15] that $T\bar{T}$ deformation of a QFT can be viewed as being equivalent to coupling the QFT to a 2d Jackiw-Teitelboim-like topological gravity. The infinitesimal $T\bar{T}$ deformation of the partition function is equivalent to integrating over the variations of the underlying spacetime geometry, and the latter turns out to be a total derivative, which partly explains such irrelevant deformation is solvable. We will not touch these two aspect, but instead refer to the original references (or the recent review [11]) for more details.

• The partition function after the $T\bar{T}$ deformation turns out to be modular invariant [16], even though the deformed theory in general is not conformal. Similarly, the partition function of the JT deformation turns out to be modular covariance [17].

• The scattering amplitude can be shown as dressed by a phase factor of Castillejo-Dalitz-Dyson type.

4.2.1 Spectra of the Deformed Theories

In this subsection, we are going to derive the spectrum of the $T\bar{T}$ deformed theories. In order to obtain spectra, the standard way is to quantise it on a circle, for example, putting the theory on an infinite Cylinder, with $(x, \bar{x}) \sim (x + 2\pi R, \bar{x} + 2\pi R)$.

Denoting $|n,\mu>$ as the eigenstate of the deformed Hamilton, one can then evaluate the expectation value of the composite operator $\mathcal{O}_{T\bar{T}}$ as

$$\langle n,\mu|\mathcal{O}_{T\bar{T}}|n,\mu\rangle = \langle n,\mu|T\bar{T}|n,\mu\rangle - \langle n,\mu|\Theta\bar{\Theta}|n,\mu\rangle \tag{4.19}$$

The crucial point is that the above

Hence we have

$$\langle n,\mu|\mathcal{O}_{T\bar{T}}|n,\mu\rangle = \langle n,\mu|T|n,\mu\rangle\langle n,\mu|\bar{T}|n,\mu\rangle - \langle n,\mu|\Theta|n,\mu\rangle\langle n,\mu|\bar{\Theta}|n,\mu\rangle \tag{4.20}$$

From the definition of the energy-momentum tensor we have

$$\begin{aligned}\langle n,\mu|T^{\mu}_{tt}|n,\mu\rangle &= \frac{1}{R}\mathcal{E}_n,\\ \langle n,\mu|T^{\mu}_{\phi\phi}|n,\mu\rangle &= -\frac{d}{dR}\mathcal{E}_n\\ \langle n,\mu|T^{\mu}_{\phi\phi}|n,\mu\rangle &= -\frac{1}{R}J_n\end{aligned} \tag{4.21}$$

Furthermore, since the Hamiltonian density is same as the Euclidean Lagragian density, then the flow equation (**??**) tells us that

$$\begin{aligned}\partial_\mu E_n(R,\mu) &= -R\langle n,\mu|\mathrm{Det}(T_{\mu\nu})|n,\mu\rangle\\ &= -R(\langle n,\mu|T_{\phi\phi}|n,\mu\rangle\langle n,\mu|T_{tt}|n,\mu\rangle - (\langle n,\mu|T_{\phi t}|n,\mu\rangle)^2)\end{aligned} \tag{4.22}$$

Substituting (4.21) into (4.22), we have

$$\partial E_n(\mu,R) = E_n(\mu,R,)\partial E(0,R) + \frac{1}{R}J_n^2(\mu,R) \tag{4.23}$$

This is known as invisited Burger's equation. Since we start from a CFT ($\mu = 0$), the undeformed energy and momentum is known to be

$$E_n(0,R) = \frac{1}{R}(h_n + \bar{h}_n - \frac{c}{12}), \qquad J(0,R) = \frac{1}{R}(h_n + \bar{h}_n) \tag{4.24}$$

where $(h_n, \bar{h}_n)$ denote the conformal dimensions of the Virasoro generators $(L_0, \bar{L}_0)$ for the states $|n,\mu=0\rangle$. Note that the momentum is quantized since the CFT is on a circle, it is naturally expected that it is undeformed under the deformation. We can then solve the flow equation for the energy and momentum as

$$E(\mu,R) = -\frac{R}{2\mu}\left[1 - \sqrt{1 + \frac{4\mu}{R}E(0,R) + \frac{4\mu^2}{R^4}J(0,R)^2}\right], \qquad J(\mu,R) = J(0,R). \tag{4.25}$$

Note that there is a one-to-one correspondence between the state before and after the deformation, and there is no states generated or annihilated along the trajectory.

The ardent readers may find that the above spectrum has a different behaviour in terms of the sign of the deformation coupling μ.

• $\mu < 0$: it would lead to complex spectrum at very high energy. From the holographic dual perspective, this features leads people to proposal the deformed theory is dual to a cut-off in pure AdS_3 [16–19]. This is due to the fact that at high(but not too high) energy, the deformed entropy $S_{T\bar{T}}$can be viewed as a Cardy formular

$$S_{T\bar{T}}(E_n(\mu, R)) = S_C(E_n) \tag{4.26}$$

• $\mu > 0$: one can readily find that the ground state $E^{vac}(\mu)$ is given by

$$E^{vac}(\mu) = -\frac{R}{2\mu}(1 - \sqrt{1 - \frac{2k\mu}{R^2}}). \tag{4.27}$$

Note that it turns out to be complex if $\lambda := \frac{k\mu}{R^2} > \frac{1}{2}$. We denote the critical value as $\lambda_c = \frac{1}{2}$. Nevertheless, the high energy states always have real energies. And the entropy of the high energy state is given by

$$S = 2\pi[\sqrt{kRE_L(\mu)(1 + \frac{2\mu}{R}E_R(\mu))} + \sqrt{kRE_R(\mu)(1 + \frac{2\mu}{R}E_L(\mu))}] \tag{4.28}$$

Note that it has Hagedorn growth at very high energy $E(\mu) >> \frac{R}{\mu}$, $S_{T\bar{T}} \sim 2\pi\sqrt{2k\mu}E(\mu)$. From this, we know the temperatures $T_{L/R} := (\partial S_{T\bar{T}}/\partial E_{L/R})^{-1}$, have an upper bound $T_L T_R \leqslant \frac{1}{8\pi^2 k\mu}$.

Let's make some comments before we switch the topic. As we alluded before, there are two kinds of $T\bar{T}$ deformations in the context of symmetric product orbifold CFTs: so-called double trace $T\bar{T}$ deformation and single trace $T\bar{T}$ deformation. They share some properties, but differ in many ways from the holographic dual viewpoint. Here we list some main difference without present much details:

Holography for double trace $T\bar{T}$ deformations: 2d $T\bar{T}$ deformation corresponds to cutoff AdS_3 in Einstein gravity. And with Λ_2, it has patch of dS. Similiarly, $d > 2$, $T\bar{T}$ it also has cutoff AdS_{d+1} in Einstein gravity. note that the double trace $T\bar{T}$ deformation is universal, local geometry unchanged. changes the boundary condition.

Holography for single trace $T\bar{T}$ deformations. asymptotic geometry changed.

Embedded in string theory.

In the rest of this section, we will mainly focus on single-trace $T\bar{T}$ deformation. Let us first propose the following conjecture:

It was conjectured in [20] that performing the following TsT transformation on the string theory is holographically equivalent to deforming the seed of the dual symmetric orbifold,

$$\mathrm{TsT}_{(X^m, X^{\bar{m}}; \check{\mu})} \quad \Longleftrightarrow \quad \frac{\partial S_{\mathcal{M}_\mu}}{\partial \mu} = -4\int J_{(m)} \wedge J_{(\bar{m})}, \tag{4.29}$$

where $\mathrm{TsT}_{(X^m, X^{\bar{m}}; \check{\mu})}$ denotes T-duality on X^m, shift $X^{\bar{m}} \to X^{\bar{m}} - 2\check{\mu}X^m$, and T-duality on X^m, $S_{\mathcal{M}_\mu}$ is the action of the deformed seed, $\mu(\check{\mu})$ is the dimensionful(less) deformation parameter, and $J_{(m)}$, $J_{(\bar{m})}$ are the Noether currents generating translations along X^m, $X^{\bar{m}}$. It is interesting to note that from a purely worldsheet perspective, a TsT transformation is equivalent to a *marginal* bi-current deformation of the worldsheet action, the latter of which can be shown to be equivalent to a gauged $(SL(2,R)\times U(1))/U(1)$ WZW model [20].

4.3 TsT on the worldsheet

The string worldsheet action

$$S_{WS} = -\ell_s^{-2}\int d^2 z M_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu \tag{4.30}$$

where $M_{\mu\nu} := G_{\mu\nu} + B_{\mu\nu}$.

Assuming translational symmetry along $X^1, X^{\bar{2}}$, TsT transformations (T-duality along X^1, shift $X^{\bar{2}} = X^{\bar{2}} = 2\tilde{\lambda}X^1$, T-duality along X^1) lead to new string background.

$M = \tilde{M}(1+2\tilde{\lambda}\tilde{M})^{-1}$. $\Gamma_{\mu\nu} = \delta^1_\mu\delta^{\bar{2}}_\nu - \delta^{\bar{2}}_\mu\delta^1_\nu$, satisfies the differential equation:

$$\frac{\partial S_{WS}}{\partial\tilde{\lambda}} = -4\int j_1\wedge j_{\bar{2}} \tag{4.31}$$

where j_1 are world-sheet Noether 1-forms associated to ∂_{X^1}, and $\partial_{X^{\bar{2}}}$. Noether charges $p_{(m)} \sim \oint j_m$

Note that we have not specified the two $U(1)$ directions in $AdS_3\times S^3$. In fact, it turns out that these two $U(1)$ directions can be assigned to any directions with different physics. If these two $U(1)$ directions sit in the AdS_3 directions, then it corresponds to the $T\bar{T}$ deformation. If one of the U(1) direction is inside S^3, then it corresponds to $J\bar{T}$ deformation with

$$J_{(1)} = J_x dx + J_{\bar{x}} d\bar{x}, \quad J_{(2)} = T_{xx} dx + T_{x\bar{x}} d\bar{x} \tag{4.32}$$

.

Putting in a compact form, we propose the following conjecture:

4.4 Further evidence

From now on, we would like to provide some evidences supporting the above conjecture (**??**).

We will first elaborate that after the TsT transformation, the black hole charges under the new background and its themodynamics can be obtained from the $T\bar{T}$ deformed spacetime CFT.

Long string spectrum on the new world-sheet theory can be found to be matched with the spectrum of the $T\bar{T}$ deformed spacetime CFT.

Starting from IIB string theory on locally $AdS_3 \times \mathcal{N}$ with NS-NS background flux, The two $U(1)$s characterizing the TST transformation are both sitting in AdS_3 / one in AdS_3 and the other in $\mathcal{N}$ / both in $\mathcal{N}$ are dual to single trace $T\bar{T}$/ $J\bar{T}$ $(T\bar{J})/J\bar{J}$ deformations on the dual spacetime CFTs, respectively. This is inspired by the examples in AdS_5, which states that IIB string theory on $AdS_5 \times S^5$ with RR background flux, the TST transformation with two $U(1)$ directions sitting in both in AdS_5/ one in AdS_5 and the other in S^5/ both in S^5 are dual to the non-commutative/dipole/β deformations in the dual 4d $N = 4$ SCFTs. zzzz **FX**: boxing the above conjecture, Aesthetics need to be done!

4.5 TsT black strings and $T\bar{T}$ deformations

The generic solution corresponding to six-dimensional part of the $\mathrm{AdS}_3 \times S^3 \times M^4$ background such that the metric, B-field, and dilaton are given by

$$
\begin{aligned}
d\tilde{s}^2 &= \ell^2 \left\{ \frac{dr^2}{4\big(r^2 - 4T_u^2 T_v^2\big)} + r du dv + T_u^2 du^2 + T_v^2 dv^2 + d\Omega_3^2 \right\}, \\
\tilde{B} &= \frac{\ell^2}{4}\big(\cos\theta\, d\phi \wedge d\psi - 2r du \wedge dv\big), \\
e^{2\tilde{\Phi}} &= \frac{k}{p},
\end{aligned}
\tag{4.33}
$$

where $k = \ell^2/\ell_s^2$ with ℓ_s the string length and ℓ the radius of AdS, while $d\Omega_3^2$ is the metric of the 3-sphere that can be written as

$$
d\Omega_3^2 = \frac{1}{4}\Big[(d\psi + \cos\theta\, d\phi)^2 + d\theta^2 + \sin^2\theta\, d\phi^2 \Big]. \tag{4.34}
$$

The lightcone coordinates $u = \varphi + t/\ell$ and $v = \varphi - t/\ell$ of the AdS_3 factor satisfy

$$
(u, v) \sim (u + 2\pi, v + 2\pi), \tag{4.35}
$$

which implies that the dual CFT lives in a cylinder of size $2\pi\ell$. On the other hand, the coordinates of the 3-sphere satisfy

$$
\psi \sim \psi + 4\pi, \qquad \theta \sim \theta + \pi, \qquad \phi \sim \phi + 2\pi. \tag{4.36}
$$

Depending on the parameter T_u, T_v, the above solution corresponds to different object

- $T_u \geqslant 0, T_v \geqslant 0$, it corresponds to a BTZ balck hole.
- $T_u = T_v = 0$, it corresponds to a massless BTZ balck hole.
- $T_u < 0, T_v < 0$, it corresponds to a conical defect.
- $T_u^2 = T_v^2 = \frac{-1}{4}$, it corresponds to a global AdS_3.

zzzz **FX**: More details....

We now perform a TsT transformation along the $\tilde{X}^1 = u$ and $\tilde{X}^{\bar{2}} = v$ coordinates of the BTZ black hole by T-dualizing along u, shifting $v \to v - \frac{2\lambda}{k}u$, and T-dualizing along u once more. The deformed string-frame metric, B-field, and dilaton are then given by

$$\begin{aligned}
ds^2 &= \ell^2\left\{\frac{dr^2}{4\left(r^2 - 4T_u^2T_v^2\right)} + \frac{rdudv + T_u^2du^2 + T_v^2dv^2}{1 + 2\lambda r + 4\lambda^2T_u^2T_v^2} + d\Omega_3^2\right\},\\
B &= \frac{\ell^2}{4}\left[\cos\theta\, d\phi \wedge d\psi - \frac{2(r + 4\lambda T_u^2T_v^2)}{1 + 2\lambda r + 4\lambda^2T_u^2T_v^2}\, du \wedge dv\right],\\
e^{2\Phi} &= \frac{k}{p}\left(1 - 4\lambda^2T_u^2T_v^2\right)e^{2(\Phi_{\mathrm{TsT}} - \phi_0)} = \frac{k}{p}\left(\frac{1 - 4\lambda^2T_u^2T_v^2}{1 + 2\lambda r + 4\lambda^2T_u^2T_v^2}\right)e^{-2\phi_0},
\end{aligned} \tag{4.37}$$

where ϕ_0 is a constant that may depend on the temperatures and Φ_{TsT} is the value of the dilaton determined by Buscher's rule [21, 22]

$$e^{-2\Phi_{\mathrm{TsT}}} = e^{-2\tilde{\Phi}}\sqrt{\frac{\det\,\tilde{G}_{\mu\nu}}{\det\,G_{\mu\nu}}} = \frac{p}{k}\left(1 + 2\lambda r + 4\lambda^2T_u^2T_v^2\right), \tag{4.38}$$

where $G_{\mu\nu}$ and $\tilde{G}_{\mu\nu}$ respectively denote the deformed and undeformed string-frame metrics.

One can easily check that in the UV ($r \to \infty$), the above metric reduces to

$$ds^2 \sim dy^2 + dudv, \quad \partial_y\Phi \sim \lambda, \quad y \sim \ln r, \tag{4.39}$$

which is asymptotic to $R^{1,1} \times S^1 \times S^3$. Whereas, in the IR ($r \to 0$), it is asymptotic to locally AdS_3.

A quick look at the phase space, one can obtain that

• $T_u \geqslant 0, T_v \geqslant 0$, it corresponds to a TsT balck strings, with the horizon $r_+ = 2T_uT_v$, independent of λ.

• $T_u = T_v = 0, \phi_0 = 0$, it corresponds to a little string background of Giveon, Itzhaki and Kutasov [12].

• $T_u < 0, T_v < 0$, it corresponds to a conical defect.

• the upper bound $T_uT_v < \frac{1}{4\lambda^2}$ by requring real dilaton.

Furthermore, the ground state can be obtained by assuming $T_u^2 = T_v^2 = -\rho^2, \rho > 0$, and requiring the smoothness at the origin. More precisely, Using the following change of coordinates

$$r = 2\rho_o^2(2\rho^2 + 1), \tag{4.40}$$

the NS-NS vacuum can be written as

$$\begin{aligned}
ds^2 &= \ell^2\left\{\frac{d\rho^2}{\rho^2 + 1} + \frac{\rho^2d\varphi^2 - (\rho^2 + 1)dt^2}{1 + 2\lambda\rho^2} + d\Omega_3^2\right\},\\
B &= \frac{\ell^2}{4}\left[\cos\theta\, d\phi \wedge d\psi - \frac{2(\rho^2 + \rho_o)}{1 + 2\lambda\rho^2}\, du \wedge dv\right],\\
e^{2\Phi} &= \frac{k}{p}\frac{(1 - 2\lambda\rho_o^2)}{2\rho_o(1 + 2\lambda\rho^2)}e^{-2\phi_0}.
\end{aligned} \tag{4.41}$$

zzzz **FX**: More details...

4.6 The $T\bar{T}$ spectrum from the worldsheet

We now derive the spectrum of strings winding on the TsT-transformed background

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