Finite Temperature Field Theory №1

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Thermal propagator for harmonic oscilators

$$G(\tau) = \frac{1}{Z}\,\mathrm{Tr}\left(e^{-\beta H} x(\tau)\, x(0)\right) \tag{1}$$

The Hamiltonian is $H = \omega\big(a^\dagger a + \frac{1}{2}\big)$, while $x = \frac{1}{\sqrt{2m\omega}}\big(a + a^\dagger\big)$.

Operator formalism

Mode expansion of Lorentzian operator in the Heisenberg picture:

$$x(t) = e^{iHt}\, x(0)\, e^{-iHt} = \frac{1}{\sqrt{2m\omega}}\Big(a\, e^{-i\omega t} + a^\dagger e^{i\omega t}\Big) \tag{2}$$

This follows from the commutation relation between $a^{(\dagger)}$ and H, or from the quantum EOM in the Heisenberg picture. Wick-rotation to Euclidean signature: $\tau = it,\ t = -i\tau$, and we have:

$$x(\tau) = e^{H\tau}\, x(0)\, e^{-H\tau} = \frac{1}{\sqrt{2m\omega}}\Big(a\, e^{-\omega\tau} + a^\dagger e^{\omega\tau}\Big) \tag{3}$$

Insert $x(\tau)\, x(0)$ into the definition of $G(\tau)$, and note that only $aa^\dagger$ and $a^\dagger a$ contribute to non-zero amplitudes, then we have:

$$\begin{aligned} G(\tau) &= \frac{1}{Z}\sum_{n=0}^{\infty}\left\langle n\middle| e^{-\beta\omega(n+\frac{1}{2})}\,\frac{aa^\dagger e^{-\omega\tau} + a^\dagger a\, e^{\omega\tau}}{2m\omega}\middle| n\right\rangle \\ &= \frac{1}{Z}\sum_{n=0}^{\infty}\left\langle n\middle| e^{-\beta\omega(n+\frac{1}{2})}\,\frac{(n+1)\, e^{-\omega\tau} + n\, e^{\omega\tau}}{2m\omega}\middle| n\right\rangle \end{aligned} \tag{4}$$

Note that:

$$\sum_{n=0}^{\infty} n\, e^{-\beta\omega n} = -\frac{1}{\beta}\frac{\partial}{\partial\omega}\sum_{n=0}^{\infty} e^{-\beta\omega n} \tag{5}$$

The summations in $G(\tau)$ can be completed, and we get:

$$G(\tau) = \frac{1}{2m\omega}\frac{\cos\left(\left(\frac{\beta}{2} - \tau\right)\omega\right)}{\sin\left(\frac{\beta\omega}{2}\right)} \tag{6}$$

Path Integral

The mathematical trick in (5) is crucial in our path integral derivation of $G(\tau)$. We have:

$$G(\tau) = \big\langle x(\tau)\, x(0)\big\rangle = \frac{1}{Z}\int \mathscr{D}x\, e^{-S} x(\tau)\, x(0), \tag{7}$$

$$S = \int_0^\beta \mathrm{d}t\, \mathcal{L}_E, \quad \mathcal{L}_E = \frac{1}{2}\, m\dot{x}^2_{(\tau)} + \frac{1}{2}\, m\omega^2 x^2_{(\tau)} \tag{8}$$

Note that the path integral of a total derivative vanishes: $0 = \int \mathcal{D}x \, \frac{\delta}{\delta x}$ (see *Polchinski*), we have:

$$0 = \int \mathcal{D}x \, \frac{\delta}{\delta x(\tau)} \left\{ e^{-S} x(\tau') \right\} = \int \mathcal{D}x \, e^{-S} \left\{ -\frac{\delta S}{\delta x(\tau)} \, x(\tau') + \delta(\tau - \tau') \right\} \tag{9}$$

$\frac{\delta S}{\delta x(\tau)}$ yields precisely the quantum EOM:

$$\frac{\delta S}{\delta x(\tau)} = m \left(-\frac{\partial^2}{\partial \tau^2} + \omega^2 \right) x(\tau) \tag{10}$$

After some re-organization, we get the quantum equation for the Green's function (which, for a free theory, is identical to the classical one):

$$m \left(-\frac{\partial^2}{\partial \tau^2} + \omega^2 \right) \int \mathcal{D}x \, e^{-S} x(\tau) \, x(\tau') = \delta(\tau - \tau') \int \mathcal{D}x \, e^{-S}, \tag{11}$$

$$\left(-\frac{\partial^2}{\partial \tau^2} + \omega^2 \right) G(\tau, \tau') = \frac{1}{m} \, \delta(\tau - \tau'), \quad G(\tau) = G(\tau, 0) \tag{12}$$

Note that $\tau \in [0, \beta]$ and $x(\tau)$ satisfies the periodic boundary condition: $x(\beta) = x(0)$, hence also $G(\beta) = G(0)$; we can extend the domain so that $x(\tau)$ is a function with period β, defined for $\tau \in \mathbb{R}$. Furthermore, note that by definition $G(\tau, \tau') = G(\tau', \tau)$; this is the Feynman propagator! Integrating around $\tau = \tau'$ then setting $\tau' = 0$ gives:

$$G'(\beta^-) - G'(0^+) = \frac{1}{m} \tag{13}$$

While $G(\tau)$ away from $\tau = 0$ grows (or decays) exponentially: $G(\tau) \propto e^{\omega\tau} + e^{\omega(\beta - \tau)}$. This fixes $G(\tau)$ uniquely:

$$G(\tau) = \frac{1}{2m\omega} \frac{\cos\left(\left(\frac{\beta}{2} - \tau \right) \omega \right)}{\sin\left(\frac{\beta\omega}{2} \right)} \tag{14}$$