String Theory №1

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$\boxed{1}$ **Read *Polchinski* Sections 1.3 and 1.4**:

Read, *mostly* understood. $\qquad\square$

$\boxed{2}$ **Spinning Closed String in AdS Space**:

For a classical spinning string, we have Nambu–Goto action:

$$S_{NG} = -T \int \mathrm{d}\tau\,\mathrm{d}\sigma\,\sqrt{-\det\gamma_{ab}}, \quad \gamma_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \tag{1}$$

Here $G_{\mu\nu}$ is the spacetime metric. γ_{ab} can be treated as the induced metric on the worldsheet.

In AdS space we have:

$$ds^2 = R^2\big(-\cosh^2\!\rho\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\!\rho\,\mathrm{d}\Omega^2\big) \tag{2}$$

Where $\mathrm{d}\Omega^2$ is the metric of a unit $(d-2)$–sphere S^{d-2}. For convenience let's define unit S^{d-2} metric G^1_{ij}, and raise or lower the $i,j,\cdots$ indices using G^1_{ij} instead of G_{ij}, i.e.,

$$G^1_{ij} = G_{ij}\big/\big(R^2 \sinh^2\!\rho\big), \quad i,j = 2,\cdots,d-1 \tag{3}$$

Furthermore, we consider the special case that the closed string is *folded*, like a rubber band stretched along a line; in this case we can choose the worldsheet parameter $(\tau,\sigma) = (t,\rho)$ while $\Omega = \Omega(t,\rho) = \Omega(\tau,\sigma)$, which leads to the following decomposition:

$$\partial_a X^\mu = \delta_a^\mu + \delta_i^\mu\,\partial_a\Omega^i, \quad a = 0,1, \quad i = 2,\cdots,d-1, \tag{4}$$

$$\begin{aligned}
\gamma_{ab} &= G_{\mu\nu}\,\partial_a X^\mu \partial_b X^\nu \\
&= G_{ab} + G_{ij}\,\partial_a\Omega^i\,\partial_b\Omega^j \\
&= G_{ab} + R^2\sinh^2\!\rho\,G^1_{ij}\,\partial_a\Omega^i\,\partial_b\Omega^j \\
&= R^2\left\{ \begin{pmatrix} -\cosh^2\!\rho & \\ & 1 \end{pmatrix} + \sinh^2\!\rho \begin{pmatrix} (\partial_a\Omega)^2 & \partial_a\Omega\cdot\partial_b\Omega \\ \partial_b\Omega\cdot\partial_a\Omega & (\partial_b\Omega)^2 \end{pmatrix} \right\}
\end{aligned} \tag{5}$$

Here $\partial_a\Omega\cdot\partial_b\Omega \equiv \partial_a\Omega^i\,\partial_b\Omega_i \equiv G^1_{ij}\,\partial_a\Omega^i\,\partial_b\Omega^j$, and we have:

$$\begin{aligned}
\det\gamma_{ab} = (R^2)^2\Big\{ &\sinh^4\!\rho\,\det\big(\partial_a\Omega^i\partial_b\Omega_i\big) \\
&+ \sinh^2\!\rho\,\big((\partial_a\Omega)^2 - (\partial_b\Omega)^2\cosh^2\!\rho\big) \\
&- \cosh^2\!\rho \Big\},
\end{aligned} \tag{6}$$

$$\begin{aligned}
\sqrt{-\det\gamma_{ab}} = R^2\Big\{ &\cosh^2\!\rho - \sinh^2\!\rho\,\big((\partial_a\Omega)^2 - (\partial_b\Omega)^2\cosh^2\!\rho\big) \\
&- \sinh^4\!\rho\,\det\big(\partial_a\Omega^i\partial_b\Omega_i\big)\Big\}^{1/2}
\end{aligned}$$

Mark the end points of the string with $\rho = r(t)$, then the total length of such closed folded string is $\ell = 4r$. We then have:

$$S = -4TR^2\int \mathrm{d}t \int_0^r \mathrm{d}\rho\,\sqrt{\cosh^2\!\rho - \sinh^2\!\rho\,\big((\partial_a\Omega)^2 - (\partial_b\Omega)^2\cosh^2\!\rho\big) - \sinh^4\!\rho\,\det\big(\partial_a\Omega^i\partial_b\Omega_i\big)} \tag{7}$$

Further simplification comes from the fact that, due to rotational symmetry, the string's motion can be restricted in a plane where its position is characterized by some angle $\theta = \Omega^{i_0} \in \{\Omega^i\}_i$. In this case other angle parameters $\Omega^i|_{i \neq i_0} = 0$, and the action is further reduced to:

$$S = -4TR^2 \int \mathrm{d}t \int_0^r \mathrm{d}\rho \sqrt{\cosh^2\rho - \sinh^2\rho \left((\partial_a\theta)^2 - (\partial_b\theta)^2 \cosh^2\rho\right)} = \int \mathrm{d}t \int_0^r \mathrm{d}\rho\, \mathcal{L}, \qquad (8)$$

$$\mathcal{L} = -4TR^2 \sqrt{\cosh^2\rho - \omega^2 \sinh^2\rho}, \quad \omega = \partial_t\theta,\, \partial_\rho\theta = 0 \qquad (9)$$

We consider the special solution $\theta = \omega t$, while in general the endpoint $r = r(t)$ could be dynamical; variation of the action w.r.t. $r(t)$ gives[1]:

$$0 = \delta S = -4TR^2 \int \mathrm{d}t \int_r^{r+\delta r} \mathrm{d}\rho \sqrt{\cosh^2\rho - \omega^2 \sinh^2\rho} = -4TR^2 \int \mathrm{d}t \sqrt{\cosh^2 r - \omega^2 \sinh^2 r}\,\delta r\,, \quad (10)$$

$$\omega^2 = \frac{\cosh^2 r}{\sinh^2 r} = \coth^2 r \qquad (11)$$

Note that if ω is constant, then r must be fixed by (11). Taking θ as the only dynamical variable, it is then straight-forward to write the energy E and angular momentum J for such folded closed string:

$$\omega = \dot\theta, \quad \Pi = \frac{\partial \mathcal{L}}{\partial \omega} = 4TR^2 \frac{\omega \sinh^2\rho}{\sqrt{\cosh^2\rho - \omega^2 \sinh^2\rho}}, \qquad (12)$$

$$J = \int_0^r \mathrm{d}\rho\, \Pi = 4TR^2 \int_0^r \mathrm{d}\rho \frac{\omega \sinh^2\rho}{\sqrt{\cosh^2\rho - \omega^2 \sinh^2\rho}}, \qquad (13)$$

$$E = \int_0^r \mathrm{d}\rho\, (\Pi\omega - \mathcal{L}) = 4TR^2 \int_0^r \mathrm{d}\rho \frac{\cosh^2\rho}{\sqrt{\cosh^2\rho - \omega^2 \sinh^2\rho}}, \qquad (14)$$

In the large string limit, $r \to \infty$, $\omega = \coth r \to 1$. Expand in terms of $\epsilon = \omega - 1 > 0$, we find that $r = \frac{1}{2}\ln\left(1 + \frac{2}{\epsilon}\right) \sim \frac{1}{2}\ln\frac{2}{\epsilon}$, or alternatively, $e^{2r} \cdot \epsilon \sim 2$. With some help from Mathematica™, we get:

$$E - J = 4TR^2 \int_0^r \mathrm{d}\rho \frac{\cosh^2\rho - \omega \sinh^2\rho}{\sqrt{\cosh^2\rho - \omega^2 \sinh^2\rho}} = 4TR^2 \int_0^r \mathrm{d}\rho \left(1 + \frac{\epsilon^2}{8}\sinh^2(2\rho) + \mathcal{O}(\epsilon^3)\right)$$

$$= 4TR^2 \left(r\left(1 - \frac{\epsilon^2}{16} + \mathcal{O}(\epsilon^3)\right) + \mathcal{O}(1)\right) = \left(2TR^2 \ln\frac{2}{\epsilon}\right)\left(1 - \frac{\epsilon^2}{16} + \mathcal{O}(\epsilon^3)\right) \qquad (15)$$

$$\sim 2TR^2 \left(\ln\frac{2}{\epsilon}\right)$$

Similarly, $J \sim 4TR^2 \int_0^r \mathrm{d}\rho \sinh^2\rho \sim TR^2\left(\frac{2}{\epsilon}\right)$, this gives:

$$E - J \sim 2TR^2 \ln\frac{J}{TR^2} \qquad (16)$$

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[1] The above reasoning is confirmed in e.g. arXiv:hep-th/0204051.

3 | Special Conformal Transformations:

$$x^\mu \xrightarrow{\ K(a)\ } \tilde{x}^\mu = \frac{x^\mu + x^2 a^\mu}{1 + 2a \cdot x + a^2 x^2} \tag{17}$$

(a) Under special conformal transformation $K(a)$, metric $\delta_{\mu\nu} \mapsto g_{\mu\nu}$ while:

$$g_{\alpha\beta}\, d\tilde{x}^\alpha\, d\tilde{x}^\beta = \delta_{\mu\nu}\, dx^\mu\, dx^\nu\,, \quad g_{\alpha\beta} = \delta_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} \tag{18}$$

To calculate this we have to know the inverse transformation $x = K^{-1}(a)\,\tilde{x}$. First, notice the following decomposition[2] of $K(a)$:

$$\tilde{x}^\mu = \frac{\frac{x^\mu}{x^2} + a^\mu}{\frac{1}{x^2} + \frac{2a \cdot x}{x^2} + a^2} = \frac{\frac{x^\mu}{x^2} + a^\mu}{\left| \frac{x^\mu}{x^2} + a^\mu \right|^2}, \tag{19}$$

i.e. $\quad K(a): \quad x^\mu \xrightarrow{\ I\ } \frac{x^\mu}{x^2} \xrightarrow{\ T(a)\ } y^\mu = \frac{x^\mu}{x^2} + a^\mu \xrightarrow{\ I\ } \tilde{x}^\mu = \frac{y^\mu}{y^2}, \tag{20}$

i.e. $\quad \dfrac{\tilde{x}^\mu}{\tilde{x}^2} = \dfrac{y^\mu}{y^2} \Big/ \dfrac{1}{y^2} = y^\mu = \dfrac{x^\mu}{x^2} + a^\mu \tag{21}$

From (21), we see that the transformation parameter a^μ composes linearly: $K(b)\,K(a) = K(a+b)$, therefore $K^{-1}(a) = K(-a)$, and we have:

$$x^\mu = K(-a)\,\tilde{x}^\mu = \frac{\tilde{x}^\mu - \tilde{x}^2 a^\mu}{1 - 2a \cdot \tilde{x} + a^2 \tilde{x}^2} = \frac{\tilde{y}^\mu}{y^2}, \tag{22}$$

$$\frac{\partial x^\mu}{\partial \tilde{x}^\alpha} = \frac{\partial x^\mu}{\partial \tilde{y}^\sigma} \frac{\partial \tilde{y}^\sigma}{\partial \tilde{x}^\alpha} = \left(\frac{\partial}{\partial \tilde{y}^\sigma} \frac{\tilde{y}^\mu}{y^2} \right) \frac{\partial}{\partial \tilde{x}^\alpha} \left(\frac{\tilde{x}^\sigma}{\tilde{x}^2} - a^\sigma \right) = \left(\frac{\partial}{\partial \tilde{y}^\sigma} \frac{\tilde{y}^\mu}{y^2} \right) \left(\frac{\partial}{\partial \tilde{x}^\alpha} \frac{\tilde{x}^\sigma}{\tilde{x}^2} \right)$$
$$= \left(\tilde{y}^2 \delta^\mu_\sigma - 2\tilde{y}^\mu \tilde{y}_\sigma \right) \left(\tilde{x}^2 \delta^\sigma_\alpha - 2\tilde{x}^\sigma \tilde{x}_\alpha \right) \Big/ \left(\tilde{y}^4 \tilde{x}^4 \right), \tag{23}$$

$$g_{\alpha\beta} \xupero{(18)} \delta_{\mu\nu} \left(\tilde{y}^2 \delta^\mu_\sigma - 2\tilde{y}^\mu \tilde{y}_\sigma \right) \left(\tilde{x}^2 \delta^\sigma_\alpha - 2\tilde{x}^\sigma \tilde{x}_\alpha \right) \left(\tilde{y}^2 \delta^\nu_\rho - 2\tilde{y}^\nu \tilde{y}_\rho \right) \left(\tilde{x}^2 \delta^\rho_\beta - 2\tilde{x}^\rho \tilde{x}_\beta \right) \Big/ \left(\tilde{y}^8 \tilde{x}^8 \right)$$
$$\xupero{\sum_{\mu,\nu}} \tilde{y}^{-4} \delta_{\sigma\rho} \left(\tilde{x}^2 \delta^\sigma_\alpha - 2\tilde{x}^\sigma \tilde{x}_\alpha \right) \left(\tilde{x}^2 \delta^\rho_\beta - 2\tilde{x}^\rho \tilde{x}_\beta \right) \Big/ \tilde{x}^8 \tag{24}$$
$$\xupero{\sum_{\sigma,\rho}} \tilde{y}^{-4} \tilde{x}^{-4} \delta_{\alpha\beta}$$

We see that $g_{\alpha\beta} = f(x)\,\delta_{\alpha\beta}$, with coefficient:

$$f(x) = \tilde{y}^{-4} \tilde{x}^{-4} \xupero{(20)} \frac{x^4}{\tilde{x}^4} \xupero{(21)} \left(1 + 2a \cdot x + a^2 x^2 \right)^2 \tag{25}$$

$$\square_{(a)}$$

(b) In 2D with $z = x^1 + ix^2$, $x^\mu \sim (z, \bar{z})$, we see from (21) that:

$$\frac{x^\mu}{x^2} \sim \frac{z}{|z|^2} = \frac{1}{\bar{z}} \longmapsto \frac{1}{\bar{z}} + a, \quad \text{i.e.} \quad z \longmapsto w = \frac{1}{\frac{1}{z} + \bar{a}} = \frac{z}{1 + z\bar{a}} \tag{26}$$

Expand in the $\bar{a} \to 0$ limit, we find that $w = z\,(1 - z\bar{a} - \cdots) \sim z - z^2 \bar{a}$, i.e. it is generated by:

$$K_{\bar{z}} = -z^2 \partial_z = -z^2 \partial, \quad \partial \equiv \partial_z \tag{27}$$

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[2] See *Di Francesco et al*, and also github.com/davidsd/ph229.

Note that when considering non-holomorphic functions, we have to consider $(z, \bar{z})$ as *two* independent variables; hence the anti-holomorphic transformation $\bar{z} \mapsto \bar{w} = \frac{\bar{z}}{1+\bar{z}a} \sim \bar{z} - \bar{z}^2 a$ provides another degree of freedom, namely:

$$K_\mu \sim \left(K_{\bar{z}} = -z^2 \partial, \ K_z = -\bar{z}^2 \bar{\partial}\right), \tag{28}$$
$$\partial \equiv \partial_z, \ \bar{\partial} \equiv \partial_{\bar{z}}$$

Similarly, for translation $z \mapsto z + a$ and its conjugate, we have $P_\mu \sim \left(P_z = \partial, P_{\bar{z}} = \bar{\partial}\right)$. However, dilation and rotation are both encoded in a complex rescaling $z \mapsto \lambda z, \ \lambda = re^{i\theta} \in \mathbb{C}$; we have:

$$z \mapsto \lambda z, \quad \lambda = re^{i\theta} \in \mathbb{C}, \quad \begin{array}{l} \delta r \ \longleftrightarrow \ D = z\,\partial + \bar{z}\,\bar{\partial}, \\ \delta\theta \ \longleftrightarrow \ M = i\left(z\,\partial - \bar{z}\,\bar{\partial}\right), \end{array} \tag{29}$$

In summary, we have $\mathrm{span}_{\mathbb{R}}\left\{P_\mu, K_\mu, D, M\right\} = \mathfrak{so}(3,1)$ generating the "global" transformation subgroup of the 2D conformal group; here, the $\mathfrak{so}(3,1)$ boost is a linear combination[3] of P_μ and K_μ. More specifically, in 2D any holomorphic or anti-holomorphic function gives a conformal transformation, hence the (classical) 2D conformal group is generated by:

$$\ell_m = z^{m+1}\partial, \quad \bar{\ell}_m = \bar{z}^{m+1}\bar{\partial}, \quad m \in \mathbb{Z} \tag{30}$$

i.e. the *Witt algebra* (or Virasoro algbera $\mathbf{Vir}_c$ with $c = 0$). It is clear that a (complexified) $\mathfrak{so}(3,1)$ lives inside $\mathbf{Vir}_c$, i.e.,

$$\begin{aligned} \mathfrak{so}(3,1)^{\mathbb{C}} &= \mathrm{span}_{\mathbb{C}}\left\{P_\mu, K_\mu, D, M\right\} \\ &= \mathrm{span}_{\mathbb{C}}\left\{\ell_m, \bar{\ell}_m \,\middle|\, m = 0, \pm 1\right\} = \mathfrak{sl}(2,\mathbb{R})^{\mathbb{C}} \oplus_{\mathbb{C}} \mathfrak{sl}(2,\mathbb{R})^{\mathbb{C}} \subset \mathbf{Vir}_c \end{aligned} \tag{31}$$

$\blacksquare$

$\boxed{4}$ *bc* **CFT**:

$$S = \frac{1}{2\pi} \int \mathrm{d}^2 z \, b\,\bar{\partial}c \tag{32}$$

Stress tensor of a theory can be obtained via variation over the metric, or equivalently, over the fields ϕ^i with $\delta\phi$ induced by some *local* spacetime translation $x^\mu \mapsto x^\mu + \delta x^\mu$, $\delta x^\mu = \epsilon(x)\,a^\mu$. Here $\epsilon(x)$ is any compactly supported bump function, centered around some point x_0.

In 2D, we have $\mu = z, \bar{z}$; for $\phi(z, \bar{z})$ with conformal weight $(h, \bar{h})$, consider $z \mapsto z'$, $\bar{z} \mapsto \bar{z}'$. For convenience, let's first consider a generic variation $\delta z = \epsilon(z, \bar{z})$ before restricting to spacetime translation; we have:

$$\phi'(z', \bar{z}') = \left(\frac{\mathrm{d}z'}{\mathrm{d}z}\right)^{-h} \left(\frac{\mathrm{d}\bar{z}'}{\mathrm{d}\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}), \tag{33}$$

$$\tilde{\delta}\phi = \left(-h\,\partial\epsilon - \bar{h}\,\bar{\partial}\bar{\epsilon}\right)\phi, \tag{34}$$

$$\delta\phi = \tilde{\delta}\phi - \frac{\partial\phi}{\partial x^\mu}\delta x^\mu = \left(-h\,\partial\epsilon - \bar{h}\,\bar{\partial}\bar{\epsilon}\right)\phi - \epsilon\,\partial\phi - \bar{\epsilon}\,\bar{\partial}\phi, \tag{35}$$

Here we use $\tilde{\delta}\phi$ to denote the "internal" variation related to the conformal weights.

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[3] See e.g. github.com/davidsd/ph229.

Note that $\phi = b, c$ are anti-commuting Grassmann numbers, variation of the action gives:

$$\delta S\,[b, c, \delta b\,, \delta c] = \frac{1}{2\pi} \int \mathrm{d}^2 z \left(\delta b\, \bar\partial c + b\,\bar\partial\, \delta c\right)$$

$$= \frac{1}{2\pi} \int \mathrm{d}^2 z \left(-\bar\partial c\, \delta b - \bar\partial b\, \delta c\right) + \frac{1}{2\pi} \int \mathrm{d}^2 z\, \bar\partial(b\,\delta c) \tag{36}$$

For *unknown* b, c and arbitary $\delta b\,, \delta c$, the second term is reduced to a boundary term at infinity and can be dropped; imposing $\delta S = 0$ gives the equation of motion (EOM): $\bar\partial b = \bar\partial c = 0$.

On the other hand, for *on-shell* b, c and compactly supported $\phi = \delta b\,, \delta c$ given in (35), the first term in (36) vanishes while $\delta S_0 = 0$ still holds; this gives:

$$0 = \delta S_0 = \frac{1}{2\pi} \int \mathrm{d}^2 z\, \bar\partial(b\,\delta c) = \frac{1}{2\pi} \int \mathrm{d}^2 z\, \bar\partial \left(-(1 - \lambda)\, bc\, \partial\epsilon - b\, \partial c\, \epsilon\right)$$

$$= \frac{1}{2\pi} \int \mathrm{d}^2 z \left(-(1 - \lambda)\, bc\, \bar\partial\partial\epsilon - b\, \partial c\, \bar\partial\epsilon\right) \tag{37}$$

Here we've distributed the $\bar\partial$ operator and dropped all terms that vanish automatically by EOM. Next we shall collect the $\partial\epsilon, \bar\partial\epsilon$ terms; integrating by parts on the first integrand gives:

$$0 = \delta S_0 = \frac{1}{2\pi} \int \mathrm{d}^2 z \left((1 - \lambda)\, \partial(bc)\, - b\, \partial c\right) \bar\partial\epsilon$$

$$= \frac{1}{2\pi} \int \mathrm{d}^2 z \left((\partial b)\, c - \lambda\, \partial(bc)\right) \bar\partial\epsilon \tag{38}$$

$$= -\frac{1}{2\pi} \int \mathrm{d}^2 z\, \epsilon(z, \bar z)\, \partial_{\bar z}\big((\partial b)\, c - \lambda\, \partial(bc)\big)$$

Notice that we have obtained a conserved current using a generic $\delta z = \epsilon(z, \bar z), \delta\bar z = \bar\epsilon(z, \bar z)$; by setting $\epsilon = \epsilon(z)$, we get *a* energy momentum tensor[4]:

$$T(z) = {:}(\partial b)\, c{:} - \lambda\, \partial({:}bc{:}) \tag{39}$$

Normal ordering is added manually to remove singular terms.

To compute TT OPE, we need the OPE of $b(z)\, c(0)$; this is obtained by examining the following path integral, which is zero since the integrand is a total functional derivative:

$$0 = \int \mathcal{D}b\, \mathcal{D}c\, \frac{\delta}{\delta\phi} \left(e^{-S}\, \psi\right) \tag{40}$$

Taking $\phi, \psi = b, c$, this generates operator equations such as $\bar\partial\, b(z)c(0) = 2\pi\delta^2(z, \bar z)$. Note that $\bar\partial(\frac{1}{z}) = 2\pi\delta^2(z, \bar z)$, which gives:

$$b(z)\, c(0) \sim c(z)\, b(0) \sim \frac{1}{z}, \quad b(z)\, b(0) \sim 0 \sim c(z)\, c(0) \tag{41}$$

With the bc OPE in hand, the TT OPE is computed directly with brute force, by repeatedly applying Wick's theorem. This gives:

$$T(z)\, T(0) \sim \frac{-6\lambda^2 + 6\lambda - 1}{z^4} + \cdots \tag{42}$$

In general we have $-6\lambda^2 + 6\lambda - 1 = \frac{c}{2}$; for $\lambda = 2$ this gives $c = -26$. ∎

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[4] Note that the energy momentum tensor obtained in this way is generally *not* unique: it can be off by a boundary term or any term that vanish on-shell; see Luboš' comment at physics.stackexchange.com/a/96100, also arXiv:1601. 03616. However, it is possible to fix this redundancy by considering Tb OPE and match its conformal dimension. I would like to thank 林般 for pointing this out.

5 **Free Fermion CFT:**

$$S = \int \mathrm{d}^2 z \, \psi_i \, \bar{\partial} \psi^i, \quad \psi^i = \psi_i^*, \quad \psi_i = \psi_i(z) \tag{43}$$

(a) Mode expansion of such chiral fermion is given by:

$$\psi_i = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \frac{b_{ik}}{z^{k+\frac{1}{2}}}, \quad b_{ik} = \frac{1}{2\pi i} \oint \mathrm{d}z \, z^{k-\frac{1}{2}} \psi_i \tag{44}$$

Canonical quantization is achieved by simply imposing anti-commutation relations; this is justified by mapping the system onto a cylinder, then b_{ik}'s indeed map to modes on the spatial circle[5]. The only non-zero commutators are:

$$\{b_{ik}, b_q^{j\dagger}\} = \delta_{k+q,0} \, \delta_i^j \tag{45}$$

This gives the only non-zero 2-point functions:

$$
\begin{aligned}
\langle \psi_i(z) \, \psi^j(w) \rangle &= \sum_{k,q \in \mathbb{Z}+\frac{1}{2}} \frac{1}{z^{k+\frac{1}{2}}} \frac{1}{w^{q+\frac{1}{2}}} \langle b_{ik} b_q^{j\dagger} \rangle \\
&= \sum_{k,q \in \mathbb{Z}+\frac{1}{2}} \frac{1}{z^{k+\frac{1}{2}}} \frac{1}{w^{q+\frac{1}{2}}} \langle 0 | \{b_{ik}, b_q^{j\dagger}\} | 0 \rangle = \frac{\delta_i^j}{z-w}
\end{aligned} \tag{46}
$$

Note that $b_k^i |0\rangle = 0, \ \forall \ k \geq \frac{1}{2}$.

(b)(c) Combining two ψ expansions gives the mode expansion of $J_i{}^j = \, :\psi_i(z) \, \psi^j(z):$, namely:

$$J_i{}^j(z) = \sum_{k \in \mathbb{Z}} \frac{(J_i{}^j)_k}{z^{k+1}}, \quad (J_i{}^j)_k = \sum_{q \in \mathbb{Z}+\frac{1}{2}} :b_{iq} \, b_{k-q}^{j\dagger}: \tag{47}$$

It is in fact more convenient to obtain the JJ OPE first, and then use it to find the $[J_0, J_0]$ mode commutator[6]; note that $\psi_i(z) \, \psi^j(w)$ contraction gives $\frac{\delta_i^j}{z-w}$, we have:

$$J_i{}^j(z) \, J_k{}^l(0) \sim \frac{\delta_i^l \delta_k^j}{z^2} + \frac{\delta_k^j J_i{}^l(0) - \delta_i^l J_k{}^j(0)}{z}, \tag{48}$$

$$\left[(J_i{}^j)_0, (J_k{}^l)_0 \right] = \frac{1}{(2\pi i)^2} \oint_0 \mathrm{d}w \oint_w \mathrm{d}z \, J_i{}^j(z) \, J_k{}^l(w) = \delta_i^l \, (J_k{}^j)_0 - \delta_k^j \, (J_i{}^l)_0 \tag{49}$$

(d) Similar to bc CFT, we have:

$$T(z) = \frac{1}{2} \left(:\psi_i \, \partial \psi^i: - :\partial \psi_i \, \psi^i: \right), \quad T(z) \, T(w) \sim \frac{n/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{50}$$

With each (complex) field contributing $\frac{1}{2} \times 2$ central charge[7].

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[5] This can be proven rigorously by considering operator equations like in the bc CFT problem.

[6] I would like to thank 谷夏 for providing this hint.

[7] In fact a complex (Dirac) fermion can be "treated like" (*dual to*) a boson; this is *bosonization*.

(e) For real fermions, there is an additional reality condition:

$$\psi^i = \psi_i^{\star} = \psi_i \tag{51}$$

The canonical quantization still holds without the extra adjoint, same as the 2-point function:

$$\langle \psi_i(z)\,\psi_j(w) \rangle = \frac{\delta_{ij}}{z - w} \tag{52}$$

Similar holds for $J_{ij} = {:}\psi_i\psi_j{:}$ and its OPE, but we no longer need to distinguish upper/lower indices; we have:

$$J_{ij}(z)\,J_{kl}(0) \sim \frac{-\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{z^2} + \frac{-\delta_{ik}J_{jl}(0) + \delta_{il}J_{jk}(0) + \delta_{jk}J_{il}(0) - \delta_{jl}J_{ik}(0)}{z} \tag{53}$$

$$[(J_{ij})_0, (J_{kl})_0] = -\delta_{ik}(J_{jl})_0 + \delta_{il}(J_{jk})_0 + \delta_{jk}(J_{il})_0 - \delta_{jl}(J_{ik})_0 \tag{54}$$

This is precisely the $\mathfrak{o}(n)$ algebra. $\blacksquare$