

## 1 Spacelike Geodesic for Poincaré AdS

The Poincaré AdS<sub>d+1</sub> metric is given by:

$$ds^2 = G_{IJ} dX^I dX^J = \frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2} = \frac{dx^2 + dz^2}{z^2}, \quad (1.1)$$

$$dx^2 = -dt^2 + d\vec{x}^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad X^I \sim (t, \vec{x}, z) \sim (x^\mu, z), \quad (1.2)$$

Note that here we define  $dx^2$  using the flat metric  $\eta$ , while  $dX$  has an index that should be raised and lowered using the curved metric  $G_{IJ} = \frac{1}{z^2} \eta_{IJ}$ . Generally upper case tensors are handled with  $G_{IJ}$ , while lower case ones are handled with  $\eta$ . The full isometry of such spacetime is then given by  $SO(d, 2)$  with  $\frac{(d+2)(d+1)}{2}$  generators, including:

1. Among the  $x^\mu \sim (x^0, \vec{x}) \equiv (t, \vec{x})$  directions:
  - (a)  $\frac{d(d-1)}{2}$   $SO(d-1, 1)$  rotations  $x_\mu \partial_\nu - x_\nu \partial_\mu$ , boosts included;
  - (b)  $d$  translations  $\partial_\mu$ ;
2. Dilation with  $X^I \sim (x^\mu, z) \mapsto \lambda X^I = \lambda (x^\mu, z)$ , generated by  $\Delta = X^I \partial_I = z \partial_z + x^\mu \partial_\mu$ ;
3. Special conformal transformations; see [Appendix A](#) for an intuitive derivation. We have:

$$k_\mu = (z^2 + x^2) \partial_\mu - 2x_\mu \Delta \quad (1.3)$$

By Noether's theorem,  $Q_\Xi = V_I \Xi^I = G_{IJ} \Xi^I V^J$  is conserved along the geodesic; here  $V^I$  is the normalized tangent vector, while  $\Xi^I$  is some Killing vector of the spacetime, e.g. one from the list above. We can then write down the conserved charges along a geodesic  $\gamma$  in Poincaré AdS:

$$\begin{aligned} p_\mu &= G_{\mu I} V^I = V_\mu \\ m_{\mu\nu} &= x_\mu V_\nu - x_\nu V_\mu \\ \Delta &= X^I V_I = \frac{z V^z}{z^2} + x^\mu p_\mu \\ k_\mu &= (z^2 + x^2) p_\mu - 2x_\mu \Delta \end{aligned} \quad (1.4)$$

These are all integration constants along  $\gamma$ . Again note that  $X, V$  have indices that should be handled with  $G_{IJ}$ , in particular,

$$dX_\mu = G_{\mu I} dX^I = \frac{1}{z^2} dx_\mu, \quad V^\mu = \frac{dX^\mu}{d\lambda} = z^2 p^\mu \quad (1.5)$$

On the other hand,  $V^\mu$  should be properly normalized, therefore:

$$\|V\|^2 = g_{IJ} V^I V^J = \frac{(V^z)^2}{z^2} + z^2 p^2 = \delta = 0, \pm 1, \quad (1.6)$$

For  $p^2 = 0$ , we have  $\|V\|^2 = \frac{(V^z)^2}{z^2} \geq 0$ , thus  $\gamma$  can be either spacelike or null, but not timelike. In fact,  $\frac{1}{z} \frac{dz}{d\lambda} = 0, \pm 1$ , along with  $\frac{dx^\mu}{d\lambda} = z^2 p^\mu$ , we obtain:

$$\begin{aligned} p^2 = 0, \quad \text{spacelike: } & z(\lambda) = z(0) e^{\pm \lambda}, \quad x^\mu(\lambda) = x^\mu(0) \pm z(0)^2 p^\mu \frac{e^{\pm 2\lambda} - 1}{2}, \\ & \text{null: } z(\lambda) = z(0), \quad x^\mu(\lambda) = x^\mu(0) + z(0)^2 p^\mu \lambda, \end{aligned} \quad (1.7)$$

From now on we shall focus on the  $p^2 \neq 0$  situation. Note that we can complete the square on the right hand side of the  $k^\mu$  conservation such that:

$$k_\mu + p_\mu a^2 = p_\mu (z^2 + (x - a)^2), \quad \Delta = p_\mu a^\mu \quad (1.8)$$

For  $p^\mu \neq 0$ , we can always find some  $c_\mu$  such that  $c_\mu p^\mu \neq 0$ , therefore we have:

$$z^2 + (x - a)^2 = \frac{c_\mu k^\mu}{c_\mu p^\mu} + a^2 = \left(\frac{L}{2}\right)^2 = \text{const.}, \quad \Delta = p_\mu a^\mu \quad (1.9)$$

We see that  $\gamma$  lands on a “sphere” centered at  $z = 0, x = a$  in  $\mathbb{R}^{d+1}$ ; with the Lorentzian metric, it is actually a hyperboloid. Note that for now the radius  $\left(\frac{L}{2}\right)^2$  can actually be negative or zero; more specifically,

1.  $L^2 > 0$ : one-sheet hyperboloid;
2.  $L^2 < 0$ : two-sheet hyperboloid;
3.  $L^2 = 0$ : conic surface.

On the other hand, we can actually solve  $X^I$  completely by combining (1.5), (1.6); we have:

$$\frac{dx^\mu}{d\lambda} = z^2 p^\mu, \quad \frac{dz}{d\lambda} = \pm z \sqrt{\delta - z^2 p^2}, \quad \frac{dx^\mu}{dz} = \pm \frac{z p^\mu}{\sqrt{\delta - z^2 p^2}}, \quad \delta - z^2 p^2 = \left(\frac{V^z}{z}\right)^2 \geq 0 \quad (1.10)$$

$$\implies \gamma \subset x^\mu = a^\mu \pm \frac{p^\mu}{p^2} \sqrt{\delta - z^2 p^2}, \quad z^2 + (x - a)^2 = \frac{\delta}{p^2} = \left(\frac{L}{2}\right)^2 \quad (1.11)$$

This confirms our observation above, and further reveals that  $\gamma$  lies in a “plane” in the  $\mathbb{R}^d$  subspace consisting of the  $x^\mu$  coordinates. The behavior of  $\gamma$  is sensitive to the sign of  $\delta$  and  $p^2$ ; more specifically,

1.  $\delta = -1$ , i.e. for timelike  $\gamma$ , we must have  $p^2 < 0$ , and  $z \geq \frac{1}{\|p\|} = \frac{L}{2} > 0$ , namely  $z$  is bounded from below. This means that timelike geodesic can never reach the asymptotic boundary  $z \rightarrow 0$ . In this case,  $\gamma$  is a section of the one-sheet hyperboloid.
2.  $\delta = +1$ , i.e. for spacelike  $\gamma$ , we can have  $p^2 > 0$  or  $p^2 < 0$ .
  - (a) For  $p^2 > 0$ , again we have  $\frac{L}{2} = \frac{1}{\|p\|} > 0$ , and  $\gamma$  is again a cross section of the one-sheet hyperboloid. However, now we have  $z \leq \frac{L}{2}$ , namely  $z$  is bounded from above, and:

$$z \rightarrow 0, \quad x^\mu = a^\mu \pm \frac{p^\mu}{p^2} = a^\mu \pm \frac{L}{2} \hat{p}^\mu, \quad \hat{p}^\mu = \frac{p^\mu}{\|p\|} \quad (1.12)$$

We can also nicely parametrize  $X^I$  in terms of the proper length  $\lambda$ ; for convenience, set  $x^\mu(0) = a^\mu$ ,  $z(0) = \frac{L}{2}$ , then we have:

$$z(\lambda) = \frac{L}{2} \frac{1}{\cosh \lambda}, \quad x^\mu(\lambda) = a^\mu \pm \hat{p}^\mu \frac{L}{2} \tanh \lambda \quad (1.13)$$

- (b) For  $p^2 < 0$ , we have  $\left(\frac{L}{2}\right)^2 = \frac{1}{p^2} < 0$ , and  $\gamma$  is now a cross section of the two-sheet hyperboloid. Again as  $z \rightarrow 0$  it lands at  $x^\mu = a^\mu \pm \frac{L}{2} \hat{p}^\mu$ ; however,  $|x^\mu|$  grows with  $z$  and extends into the bulk instead of returning to the boundary, i.e.  $z, |x^\mu| \rightarrow \infty$ .

The differential equation in this case is almost the same as in (a), but now we have to choose a different initial condition, since  $\gamma$  won't even reach  $x = a$ . However, we can

actually set  $x(0), z(0) \rightarrow \infty$ ; we then have:

$$z(\lambda) = \left| \frac{L}{2} \right| \frac{1}{\sinh \lambda}, \quad x^\mu(\lambda) = a^\mu \pm \hat{p}^\mu \left| \frac{L}{2} \right| \coth \lambda \quad (1.14)$$

This is an evidence that  $z \rightarrow \infty$  might not be the “end of the world” after all; it’s likely that  $z \rightarrow \infty$  is only a horizon, since a spacelike geodesic  $\gamma$  can reach it within finite proper length  $\lambda$ .

In summary, there are 3 types of spacelike geodesics in Poincaré AdS, which closely resemble the 3 types of conic sections:

1.  $p^2 = 0$ : parabolic, given in (1.7), with one end going to  $z \rightarrow 0$  and the other end going to  $z \rightarrow \infty$ ; it takes infinite proper length  $\lambda$  for it to reach 0 or  $\infty$ .
2.  $p^2 > 0$ : hyperbolic, given in (1.11) and (1.14), also with one end going to  $z \rightarrow 0$  and the other end going to  $z \rightarrow \infty$ , but it reaches  $\infty$  within finite  $\lambda$ .
3.  $p^2 < 0$ : elliptic, given in (1.11) and (1.13), with both ends going to  $z \rightarrow 0$ , and it takes infinite proper length  $\lambda$  for it to reach either end.

Now consider two points  $x_1, x_2$  near the boundary  $z = \epsilon$  connected by a spacelike geodesic  $\gamma$ . This can only be the “elliptic type” discussed above. We have:

$$x^\mu = a^\mu \pm \hat{p}^\mu \sqrt{\left(\frac{L}{2}\right)^2 - z^2}, \quad z^2 + (x - a)^2 = \left(\frac{L}{2}\right)^2, \quad (1.15)$$

$$a = \frac{x_1 + x_2}{2}, \quad \hat{p} \propto x_2 - x_1 \quad (1.16)$$

It’s length is then given by:

$$\begin{aligned} A &= \int_{z=-\epsilon}^{z=\epsilon} \frac{dx^2 + dz^2}{z^2} \\ &= \int_{-\Lambda}^{\Lambda} \frac{(d \tanh \lambda)^2 + (d \operatorname{sech} \lambda)^2}{(\operatorname{sech} \lambda)^2}, \quad \Lambda = \cosh^{-1}\left(\frac{L}{2\epsilon}\right) \\ &= \int_{-\Lambda}^{\Lambda} d\lambda = 2\Lambda = 2 \cosh^{-1}\left(\frac{L}{2\epsilon}\right) \sim 2 \log \frac{L}{\epsilon}, \quad \epsilon \rightarrow 0 \end{aligned} \quad (1.17)$$

## 2 Einbein Action

The einbein action of a point particle is given by:

$$S[\eta, X] = \frac{1}{2} \int d\tau \left( \eta^{-1} \dot{X}_\mu \dot{X}^\mu - \eta m^2 \right) \quad (2.1)$$

Under worldline reparametrization:  $\tau \mapsto \tau' = f(\tau)$ , we have  $X'(\tau') = X(\tau)$ , i.e.  $X^\mu$  transforms like a **scalar** under worldline diffeomorphism;  $X(\tau) \mapsto X'(\tau) = X(f^{-1}(\tau))$ .

On the other hand,  $\eta$  should be treated like an einbein:  $\eta = \sqrt{-\gamma}$ , here  $\gamma = \gamma_{\tau\tau}$  is the worldline metric;  $\gamma < 0$  due to the Lorentzian signature. We have:

$$\eta = \sqrt{-\gamma} \mapsto \eta' = \eta \det \frac{\partial \tau}{\partial \tau'} = \eta \frac{\partial \tau}{\partial \tau'}, \quad (2.2)$$

$$\eta^{-1} = -\sqrt{-\gamma} \gamma^{-1} \mapsto (\eta')^{-1} = \eta^{-1} \left( \frac{\partial \tau}{\partial \tau'} \right) \left( \frac{\partial \tau'}{\partial \tau} \right) = \eta^{-1} \frac{\partial \tau'}{\partial \tau} \quad (2.3)$$

It is clear that the action is invariant under the transformation:

$$\begin{aligned} S' &= \frac{1}{2} \int d\tau' \left( (\eta')^{-1} \partial_{\tau'} X_\mu \partial_{\tau'} X^\mu - (\eta') m^2 \right) \\ &= \frac{1}{2} \int d\tau \frac{\partial \tau'}{\partial \tau} \left( \eta^{-1} \frac{\partial \tau'}{\partial \tau} \cdot \frac{\partial \tau}{\partial \tau'} \partial_\tau X_\mu \cdot \frac{\partial \tau}{\partial \tau'} \partial_\tau X^\mu - \eta \frac{\partial \tau}{\partial \tau'} m^2 \right) = S \end{aligned} \quad (2.4)$$

We can eliminate  $\eta$  classically by placing it on shell:

$$0 = \frac{\delta S}{\delta \eta} = -\eta^{-2} \dot{X}_\mu \dot{X}^\mu - m^2, \quad \eta[X] = \frac{1}{m} \sqrt{-\dot{X}_\mu \dot{X}^\mu} \quad (2.5)$$

Substitute this back to the action, and we have:

$$\begin{aligned} S[X] &= S[\eta = \eta[X], X] = \frac{1}{2} \int d\tau \left( m (-\dot{X}_\mu \dot{X}^\mu)^{-\frac{1}{2}} \dot{X}_\mu \dot{X}^\mu - m (-\dot{X}_\mu \dot{X}^\mu)^{+\frac{1}{2}} \right) \\ &= -m \int d\tau \sqrt{-\dot{X}_\mu \dot{X}^\mu} \end{aligned} \quad (2.6)$$

### 3 Ricci Tensor for Static Spherical Metric

Consider the metric:

$$ds^2 = -f(r) dt^2 + h(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) = \eta_{ab} e^a e^b \quad (3.1)$$

Here we've defined the following vierbein:

$$\begin{aligned} e^t &= \sqrt{f(r)} dt \\ e^r &= \sqrt{h(r)} dr \\ e^\theta &= r d\theta \\ e^\phi &= r \sin \theta d\phi \end{aligned} \quad (3.2)$$

## A Derivation of the special conformal transformations

Special conformal transformations can be understood as translations conjugated by *inversions*. Note that  $\frac{dz^2}{z^2}$  is invariant under  $z \mapsto \frac{1}{z}$ ; if we include the  $x^\mu$  directions, we can consider:

$$\mathcal{I}: \chi^I \mapsto \frac{\chi^I}{\chi^2}, \quad \chi^2 = -t^2 + \vec{x}^2 + z^2, \quad (A.1)$$

$$\mathcal{I}^2 = \mathbb{1}, \quad ds^2 \mapsto \left( \frac{\delta_J^I - 2 \frac{\chi^I \chi_J}{\chi^2}}{\chi^2} d\chi^J \right)^2 / \left( \frac{z}{\chi^2} \right)^2 = \frac{d\chi^2}{z^2} = ds^2 \quad (A.2)$$

We see that inversion  $\mathcal{I}$  is indeed a (discrete) symmetry of the metric. Here we've defined yet another lower case variable  $\chi^I \sim (x^\mu, z)$ , which as a contravariant vector has the same components as  $X^I$ , but with an index that should be lowered by the flat metric  $\eta_{IJ}$ , i.e.  $\chi_I = \eta_{IJ} \chi^J = \eta_{IJ} X^J$ . The  $d$

special conformal generators are then given by:

$$\begin{aligned}
k_\mu &= \frac{\partial}{\partial a^\mu} \left( \mathcal{I} \circ e^{a^\nu P_\nu} \circ \mathcal{I} \circ X^I \right)_{a=0} \frac{\partial}{\partial X^I} \\
&= \frac{\partial}{\partial a^\mu} \left( \frac{\frac{\chi^I}{\chi^2} + a^I}{\left| \frac{\chi^J}{\chi^2} + a^J \right|^2} \right)_{a=0} \frac{\partial}{\partial X^I} \\
&= \frac{\partial}{\partial a^\mu} \left( \frac{\chi^I + a^I \chi^2}{1 + 2a^I \chi_I + a^2 \chi^2} \right)_{a=0} \frac{\partial}{\partial X^I} \\
&= \chi^2 \partial_\mu - 2x_\mu X^I \partial_I \\
&= \chi^2 \partial_\mu - 2x_\mu \Delta
\end{aligned} \tag{A.3}$$