Finite Temperature Field Theory №2

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Pure QED Partition Function in Covariant Gauge

With the metric convention: $g \sim (-+++)$, we have:

$$\mathcal{Z} = \int \mathcal{D}A^\mu \; e^{-S} \, \delta\big[\partial_\mu A^\mu - f\big] \det\big[\partial^2 \delta^4(x-y)\big] \tag{1}$$

Here S is the Euclidean action:

$$(-S) = \int \mathrm{d}^4 x \, \mathcal{L}_{t=-i\tau}, \qquad \int \mathrm{d}^4 x = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x} \tag{2}$$

For pure QED, we have:

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} \tag{3}$$

Setting $t = -i\tau$ is equivalent to carrying out a Wick rotation: $x^0 \mapsto ix^0,\ A^0 \mapsto iA^0$, while:

$$g_{\mu\nu}A^\mu A^\nu = g'_{\mu\nu}A'^\mu A'^\nu \quad \Longrightarrow \quad g_{\mu\nu} \longmapsto g'_{\mu\nu} = \delta_{\mu\nu} \tag{4}$$

Under this convention, the Euclidean action is formally unchanged (due to its Lorentz invariance); same applies for the gauge-fixing and the ghost term:

$$\mathcal{L}_{t=-i\tau} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu}, \quad g_{\mu\nu} \longmapsto \delta_{\mu\nu}, \tag{5}$$

$$\delta\big[\partial_\mu A^\mu - f\big] \quad \Longrightarrow \quad \mathcal{L}_{gf} = -\frac{1}{2\rho}\,(\partial_\mu A^\mu)^2, \tag{6}$$

$$\det\big[\partial^2\delta^4(x-y)\big] \quad \Longrightarrow \quad \mathcal{L}_{gh} \sim (\partial^2\bar\eta)\,\eta \sim -\partial_\mu\bar\eta\,\partial^\mu\eta, \tag{7}$$

Here we've dropped some total derivative terms in the ghost Lagrangian. The partition function is then reduced to:

$$\mathcal{Z} = \int \mathcal{D}A^\mu \; \mathcal{D}\bar\eta \; \mathcal{D}\eta \; e^{-S'}, \quad (-S') = \int \mathrm{d}^4x \,\big(\mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh}\big)_{t=-i\tau} \tag{8}$$

The action can be further simplified by partial integration and dropping boundary terms:

$$\begin{aligned}
(-S') &= \int \mathrm{d}^4x \left(-\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\rho}\,(\partial_\mu A^\mu)^2 - \partial_\mu\bar\eta\,\partial^\mu\eta\right) \\
&= \int \mathrm{d}^4x \left(-\frac{1}{4}\,(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2\rho}\,(\partial_\mu A^\mu)^2 - \partial_\mu\bar\eta\,\partial^\mu\eta\right) \\
&= \int \mathrm{d}^4x \left(-\frac{1}{2}\,(\partial_\mu A_\nu\,\partial^\mu A^\nu - \partial_\nu A_\mu\,\partial^\mu A^\nu) - \frac{1}{2\rho}\,(\partial_\mu A^\mu\,\partial_\nu A^\nu) - \partial_\mu\bar\eta\,\partial^\mu\eta\right) \\
&\sim \int \mathrm{d}^4x \left(-\frac{1}{2}\,\big(-A_\nu\,\partial^2 A^\nu + A_\mu\,\partial^\mu\partial_\nu A^\nu\big) + \frac{1}{2\rho}\,(A^\mu\,\partial_\mu\partial_\nu A^\nu) + \bar\eta\,\partial^2\eta\right) \\
&= \int \mathrm{d}^4x \left(-\frac{1}{2}\,A^\mu\Big(-\delta_{\mu\nu}\partial^2 + \partial_\mu\partial_\nu - \tfrac{1}{\rho}\,\partial_\mu\partial_\nu\Big)A^\nu + \bar\eta\,\partial^2\eta\right) \\
&= -\frac{1}{2}\int \mathrm{d}^4x \left(A^\mu\Big(-\delta_{\mu\nu}\partial^2 + \big(1-\tfrac{1}{\rho}\big)\,\partial_\mu\partial_\nu\Big)A^\nu - 2\bar\eta\,\partial^2\eta\right)
\end{aligned} \tag{9}$$

With $\beta = \frac{1}{T}$, expand A^μ, η into dimensionless Fourier modes, and we have:

$$A^\mu = \frac{1}{\sqrt{TV}}\sum_k e^{ik_\nu x^\nu} A^\mu_k, \qquad \sum_k e^{ik_\nu x^\nu} = V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{n\in\mathbb{Z}} e^{i\omega_n\tau} \tag{10}$$

$$
\begin{aligned}
\sum_{p,k}\int \mathrm{d}^4x\, e^{i\,(p+k)\cdot x} &= \sum_{p,k}(2\pi)^4\,\delta^4(p+k) \\
&= V^2\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{k}}{(2\pi)^6}\sum_{m,n\in\mathbb{Z}}\beta\,\delta_{m,-n}\cdot(2\pi)^3\,\delta^3(\mathbf{p}+\mathbf{k}) \\
&= \beta V\cdot V\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\sum_{n\in\mathbb{Z}}\int \mathrm{d}^3\mathbf{p}\,\delta^3(\mathbf{p}+\mathbf{k})\sum_{n\in\mathbb{Z}}\delta_{m,-n} \\
&= \beta V\sum_{k}\int \mathrm{d}^3\mathbf{p}\,\delta^3(\mathbf{p}+\mathbf{k})\sum_{n\in\mathbb{Z}}\delta_{m,-n},
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
(-S') &= -\frac{1}{2TV}\sum_{p,k}\int \mathrm{d}^4x\,e^{i\,(p+k)\cdot x}\Big(A^\mu_p\Big(-\delta_{\mu\nu}(-k^2)+\big(1-\tfrac{1}{\rho}\big)\,(-k_\mu k_\nu)\Big)A^\nu_k-2\bar\eta_p\,(-k^2)\eta_k\Big) \\
&= -\frac{\beta V}{2TV}\sum_k\Big(A^\mu_{-k}\Big(k^2\delta_{\mu\nu}-\big(1-\tfrac{1}{\rho}\big)\,k_\mu k_\nu\Big)A^\nu_k+2\bar\eta_{-k}k^2\eta_k\Big) \\
&= -\frac{\beta^2}{2}\sum_k\Big(A^{\mu\dagger}_k D^{-1}_{\mu\nu}(k)A^\nu_k+2\bar\eta^\dagger_k k^2\eta_k\Big)
\end{aligned}
\tag{12}
$$

Here we've used the reality condition on A,η, namely $A^{\mu\dagger}_k = A^\mu_{-k}$, and defined the k–space inverse propagator $D^{-1}_{\mu\nu}(k)$. Similar result applies for η_k, except that we have to be careful about Grassmann variables. Carry out $\int\mathcal{D}A^\mu\,\mathcal{D}\bar\eta\,\mathcal{D}\eta$, and we have:

$$
\begin{aligned}
\mathcal{Z} &\sim \Big(\det_{\mu,\nu,k}\beta^2 D^{-1}_{\mu\nu}(k)\Big)^{-1/2}\Big(\det_k 2\beta^2k^2\Big)^{+1} \\
&= \prod_k\Big(\beta^{2\times4\times(-1/2)}\cdot\Big(\det_{\mu,\nu}D^{-1}_{\mu\nu}(k)\Big)^{-1/2}\cdot 2\beta^2k^2\Big) \\
&= \prod_k\Big(\beta^{-4}\Big(\det_{\mu,\nu}D^{-1}_{\mu\nu}(k)\Big)^{-1/2}\cdot 2\beta^2k^2\Big)
\end{aligned}
\tag{13}
$$

The determinant is evaluated as follows[1]:

$$
\begin{aligned}
\det_{\mu,\nu}D^{-1}_{\mu\nu} &= \det_{\mu,\nu}\Big(k^2\delta_{\mu\nu}-\big(1-\tfrac{1}{\rho}\big)\,k_\mu k_\nu\Big) \\
&= k^8\det_{\mu,\nu}\Big(\delta_{\mu\nu}-\big(1-\tfrac{1}{\rho}\big)\,\tfrac{k_\mu k_\nu}{k^2}\Big) \\
&= k^8\Big(1-\big(1-\tfrac{1}{\rho}\big)\,\tfrac{k^2}{k^2}\Big) \\
&= \tfrac{1}{\rho}k^8,
\end{aligned}
\tag{14}
$$

$$
\mathcal{Z}\sim\prod_k\tfrac{2}{\rho}\beta^{-4}k^{-4}\cdot\beta^2k^2\sim\prod_k\beta^{-2}k^{-2},
\tag{15}
$$

$$
\ln\mathcal{Z}\sim-\sum_k\ln\big(\beta^2k^2\big)
\tag{16}
$$

[1] Reference: `Wikipedia:` *Determinant # Sylvester's determinant theorem*.

We see that $\ln \mathcal{Z}$ is simply twice the result of a neutral scalar field, with mass $m \to 0$, i.e.

$$\ln \mathcal{Z} \sim -2 \times \frac{1}{2} \sum_k \ln\left(\beta^2 k^2\right) \sim -2 \sum_{\mathbf{k}} \left(\frac{1}{2} \beta E_k + \ln\left(1 - e^{-\beta E_k}\right) \right), \tag{17}$$

$$\mathcal{Z} \sim \prod_{\mathbf{k}} \left\{ \exp\left(-\frac{1}{2} \beta E_k - \ln\left(1 - e^{-\beta E_k}\right) \right) \right\}^2, \tag{18}$$

$$\Omega = -T \ln \mathcal{Z} = 2 \sum_{\mathbf{k}} \left(\frac{1}{2} E_k + T \ln\left(1 - e^{-E_k/T}\right) \right), \tag{19}$$

$$p = -\frac{\Omega}{V} = -2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left(\frac{1}{2} E_k + T \ln\left(1 - e^{-E_k/T}\right) \right), \tag{20}$$

Here $E_k = \|\mathbf{k}\|$. Ignore the vacuum contribution to p, and we have:

$$p = -2T \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \ln\left(1 - e^{-\|\mathbf{k}\|/T}\right) = 2\, \frac{\pi^2}{90}\, T^4 \tag{21}$$

Summary

1. We've completed the calculation of pure QED partition function $\mathcal{Z}$ and thermodynamic potential Ω (energy density p) by introducing a "soft" gauge fix $\mathcal{L}_{gf}$. Alternatively, we can simply impose a "hard" Lorenz gauge fix $\delta\left[\partial_\mu A^\mu\right]$; this can be achieved by taking $\rho \to \infty$ in the Gaussian packet $-\frac{1}{2\rho}\left(\partial_\mu A^\mu\right)^2$, or by integrating out A^0 directly — similar to (12), we have:

$$\begin{aligned} \mathcal{Z} &\sim \int \mathcal{D}A^\mu\, \mathcal{D}\bar{\eta}\, \mathcal{D}\eta\, \delta\left[\partial_\mu A^\mu\right] \exp\left(-\frac{\beta^2}{2} \sum_k \left(A_k^{\mu\dagger} \left(k^2 \delta_{\mu\nu} - k_\mu k_\nu\right) A_k^\nu + 2\bar{\eta}_k^\dagger k^2 \eta_k \right) \right) \\ &\sim \int \mathcal{D}A^i\, \mathcal{D}\bar{\eta}\, \mathcal{D}\eta\, \exp\left(-\frac{\beta^2}{2} \sum_k \left(A_k^{\mu\dagger} \left(k^2 \delta_{\mu\nu} - k_\mu k_\nu\right) A_k^\nu + 2\bar{\eta}_k^\dagger k^2 \eta_k \right) \right)_{A^0 = A^0[A^i]} \end{aligned} \tag{22}$$

$$A^0[A^i] = -\int \partial_i A^i\, \mathrm{d}\tau\,, \quad k_\mu A_k^\mu = 0, \tag{23}$$

Here we've omitted a non-dynamical Jacobian $\det\left[\theta(\tau - \tau')\right] = \det \int^\tau \mathrm{d}\tau''\, \delta(\tau'' - \tau')$. The ghost integral gives the same contribution, while the A^i integral yields:

$$\mathcal{Z}_A = \int \mathcal{D}A^i\, \exp\left(-\frac{\beta^2}{2} \sum_k \left(A_k^{\mu\dagger} \left(k^2 \delta_{\mu\nu}\right) A_k^\nu \right) \right), \tag{24}$$

$$\begin{aligned} A_k^{\mu\dagger} \left(k^2 \delta_{\mu\nu}\right) A_k^\nu &= A_k^{i\dagger} \left(k^2 \delta_{ij}\right) A_k^j + A_k^{0\dagger} \left(k^2\right) A_k^0 = A_k^{i\dagger} \left(k^2 \delta_{ij}\right) A_k^j + \frac{k^2}{\omega^2} A_k^{0\dagger} \left(\omega^2\right) A_k^0 \\ &\overset{(23)}{=} A_k^{i\dagger} \left(k^2 \delta_{ij}\right) A_k^j + \frac{k^2}{\omega^2} A_k^{i\dagger} (k_i k_j) A_k^0 = A_k^{i\dagger}\, k^2 \left(\delta_{ij} + \frac{k_i k_j}{\omega^2} \right) A_k^j \\ &= A_k^{i\dagger} D_{ij}^{-1}(k) A_k^j, \end{aligned} \tag{25}$$

$$\begin{aligned} \mathcal{Z}_A \sim \left(\det_{i,j,k} \beta^2 D_{ij}^{-1}(k) \right)^{-1/2} &= \prod_k \beta^{-3} k^{-3} \left(1 + \frac{\mathbf{k}^2}{\omega^2} \right)^{-1/2} = \prod_k \left(\beta^{-4} k^{-4} \cdot \beta\omega \right) \\ &= \prod_k \beta^{-4} k^{-4} \prod_{\mathbf{k}} \prod_n 2\pi n \sim \prod_k \beta^{-4} k^{-4} \end{aligned} \tag{26}$$

We see that the result from a "hard" Lorenz gauge fixing is the same as before, up to a non-dynamical overall coefficient.

2. In our previous calculations, we notice that after functional integration, ρ is just a non-dynamical overall coefficient in $\mathcal{Z}$, hence it can be safely dropped from the final expression; see eq. (15). Therefore, the result is independent of parameter ρ.