Qualifying Exam **Physics 2020 @ YMSC, Tsinghua**

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1 Gravity

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \tag{1.1}$$

$$f(r) = 1 - \frac{GM}{r} + \frac{Q^2}{r^2} = \frac{(r-r_+)(r-r_-)}{r^2} \tag{1.2}$$

1. Event horizon(s): $f(r) = 0$, we have:
   (a) $M > |Q|$, $r_\pm = M \pm \sqrt{M^2 - Q^2}$, 2 event horizons;
   (b) $M = |Q|$, $r_\pm = M$, 1 event horizon;
   (c) $M < |Q|$, no event horizon! "Naked" singularity.
2. New coordinate: $v = t + r^*$,

$$r^* = r + \frac{1}{2k_+}\ln\frac{|r-r_+|}{r_+} + \frac{1}{2k_-}\ln\frac{|r-r_-|}{r_-}, \quad k_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2} \tag{1.3}$$

We have:

$$\begin{aligned}
\mathrm{d}t = \mathrm{d}v - \mathrm{d}r^* &= \mathrm{d}v - \left(1 + \frac{1}{2k_+}\frac{1}{r-r_+} + \frac{1}{2k_-}\frac{1}{r-r_-}\right)\mathrm{d}r \qquad (1.4)\\
&= \mathrm{d}v - \left(1 + \frac{1}{r^2 f(r)}\frac{r_+^2(r-r_-) - r_-^2(r-r_+)}{r_+ - r_-}\right)\mathrm{d}r\\
&= \mathrm{d}v - \left(1 + \frac{1}{r^2 f(r)}\Big((r_+ + r_-)\,r - r_+ r_-\Big)\right)\mathrm{d}r\\
&= \mathrm{d}v - \frac{1}{f(r)}\,\mathrm{d}r \qquad (1.5)
\end{aligned}$$

Therefore,

(a) The new metric:

$$\begin{aligned}
\mathrm{d}s^2 &= -f(r)\left(\mathrm{d}v - \frac{1}{f(r)}\,\mathrm{d}r\right)^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \qquad (1.6)\\
&= -f(r)\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\,\mathrm{d}\Omega^2
\end{aligned}$$

It is only singular at $r = 0$.

Note: during the exam I panicked when I saw (1.3), and I made a very stupid mistake in step (1.4). However, I knew what this new coordinate is trying to achieve — it's aiming to eliminate the coordinate singularities in $\frac{1}{f}\,\mathrm{d}r^2$ by absorbing it into $\mathrm{d}v^2$, so I guessed the result (1.5) correctly and carried on. I hope they gave me some points for getting the right answer, despite with some wrong process (>_<).

(b) $\frac{\partial}{\partial v}$ is a Killing vector field, for the metric components are all v-independent. More precisely, since $\frac{\partial}{\partial v}$ itself is a coordinate basis, we have the Lie derivative:

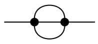
$$\mathcal{L}_{\frac{\partial}{\partial v}} g_{\mu\nu} = \partial_v g_{\mu\nu} = 0 \tag{1.7}$$

(c) $\left\|\frac{\partial}{\partial v}\right\|^2 = g_{\mu\nu}\delta^\mu_v\delta^\nu_v = g_{vv} = -f(r)$, therefore, for $M > |Q|$ we have:

- $\frac{\partial}{\partial v}$ timelike: $r > r_+$ and $r < r_-$
- spacelike: $r_- < r < r_+$
- null: $r = r_+$ and $r = r_-$

2 QFT

We shall restore the reasonable convention: $\eta_{\mu\nu} \sim (-,+,+,+)$.

1. 1PI: diagrammatic correction to the (1-particle) propagator that cannot be split into 2 disconnected parts by cutting one line; e.g. 
2. Consider the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2}Z(\partial\phi_r)^2 - \frac{1}{2}m^2 Z\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 - \frac{1}{2}\delta_Z(\partial\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4 \tag{2.1}$$

The convention here is rather bizarre; normally we write down the UV Lagrangian $\mathcal{L}_{\mathrm{UV}}$ and split it into 2 parts, one is the effective IR Lagrangian $\mathcal{L}_{\mathrm{IR}}$ and the other one is the counterterm:

$$\begin{aligned}\mathcal{L}_{\mathrm{UV}} &= -\frac{1}{2}Z(\partial\phi_r)^2 - \frac{1}{2}m^2 Z\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 \\ &= \left(-\frac{1}{2}(\partial\phi_r)^2 - \frac{1}{2}m_p^2\phi_r^2 - \frac{\lambda_p}{4!}\phi_r^4\right) - \left(-\frac{1}{2}\delta_Z(\partial\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4\right) \\ &= \mathcal{L}_{\mathrm{IR}} + \mathcal{L}_{\mathrm{ct}}\end{aligned} \tag{2.2}$$

Normally, we use $\mathcal{L}$ to denote the UV Lagrangian $\mathcal{L}_{\mathrm{UV}}$; this is the convention adopted by numerous standard textbooks, incl. *Peskin & Schroeder* [1], *Weinberg*, and also *Srednicki*. However, the Lagrangian in (2.1) seems to be $\mathcal{L}_{\mathrm{IR}}$ instead of $\mathcal{L}_{\mathrm{UV}}$. Anyway, we have:

$$Z + \delta_Z = 1, \quad m^2 Z + \delta_m = m_p^2, \quad \lambda + \delta_\lambda = \lambda_p \tag{2.3}$$

Where m_p, λ_p is the physical IR couplings, fixed by the renormalization scheme. The convention here is really confusing and somewhat inconsistent; e.g. if we choose to write the UV mass term as $-\frac{1}{2}m^2 Z\phi_r^2$, then the corresponding UV interaction term should look like $-\frac{\lambda}{4!}Z^2\phi_r^4$, but here we do not have the Z^2 factor. Also, we usually use m_0, λ_0 to denote bare couplings, but here it seems that they are denoted by m, λ.

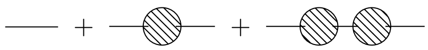
We can write down the renormalized Feynman rules nonetheless, despite some sign issues due to the conventions; to avoid further confusion, we will adopt the usual notation: m_0, λ_0 for bare couplings, and $m = m_p, \lambda = \lambda_p$ for physical couplings. We have:

- Renormalized propagator: $\frac{-i}{p^2+m^2-i\epsilon}$
- Renormalized vertex: $-i\lambda$
- Counterterm ϕ^2 vertex: $+i\big(\delta_Z(-p^2) + \delta_m\big)$, $-\!\!\otimes\!\!-$
- Counterterm ϕ^4 vertex: $+i\delta_\lambda$

3. The sum of all two point 1PI diagrams (no propagator on external legs) is given by:

$$-iM(p^2) = \quad \tag{2.4}$$

The full propagator is thus:



$$\begin{aligned} G(p^2) &= \quad \cdots \\ &= \tfrac{-i}{p^2+m^2} + \tfrac{-i}{p^2+m^2}(-iM)\tfrac{-i}{p^2+m^2} + \tfrac{-i}{p^2+m^2}(-iM)\tfrac{-i}{p^2+m^2}(-iM)\tfrac{-i}{p^2+m^2} + \cdots \end{aligned} \tag{2.5}$$

With $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$, we get:

$$G(p^2) = \frac{-i}{p^2+m^2} \cdot \frac{1}{1-(-iM)\frac{-i}{p^2+m^2}} = \frac{-i}{p^2+m^2+M(p^2)} \tag{2.6}$$

Here we've suppressed the $(-i\epsilon)$ prescription in the above expressions, but it's presence is always implied.

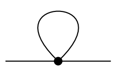
4. On-shell renormalization scheme — the full propagator:

$$G(p^2) = \frac{-i}{p^2+m^2+M(p^2)-i\epsilon} \xrightarrow{p^2\to -m^2} \frac{-i}{p^2+m^2-i\epsilon} \tag{2.7}$$

This means that $M(p^2=-m^2)=0$. Furthermore, $M(p^2) \sim \#\,(p^2+m^2)+\mathcal{O}\big(p^4\big)$, to ensure that the residue is 1 at the pole, we should have $\# \sim 0$, i.e.

$$M(p^2)\big|_{p^2=-m^2} = 0, \qquad \frac{\partial}{\partial(p^2)} M(p^2)\big|_{p^2=-m^2} = 0 \tag{2.8}$$

5. At 1-loop $\mathcal{O}(\lambda)$, if we do not include counterterm contributions, then there is only one diagram contributing to $M(p^2)$:



$$= (-i\lambda)\cdot\frac{1}{2}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{-i}{k^2+m^2-i\epsilon} \tag{2.9}$$

Here $\frac{1}{2}$ is the symmetry factor of the diagram; alternative, we can count the distinct ways of connecting the 4 legs of the ϕ^4 vertex and divide it by 4!, which is indeed $\frac{4\times 3}{4!}=\frac{1}{2}$.

The p^0 integral has poles at $p_0^2 = \mathbf{p}^2+m^2-i\epsilon$, i.e. $p^0 = \pm\sqrt{\mathbf{p}^2+m^2}\mp i\epsilon$, and it's regular everywhere else; we can thus compute the p^0 integral on the $\mathbb{C}$ plane using a right-tilted 8-shaped contour, which does not enclose the poles. Effectively, we've performed a Wick rotation $p^0 \mapsto ip^0$ so that the integral happens in Euclidean p space:

$$\frac{-i\lambda}{2}\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2+m^2} = \frac{-i\lambda}{2}\frac{A(S^d)}{(2\pi)^D}\int\frac{k^d\,\mathrm{d}k}{k^2+m^2} \tag{2.10}$$

Here $D=d+1$, d is the spatial dimension. There are many ways to regularize this integral; if we continue to work in general $D=d+1$ dimensions, then dimensional regularization is automatically implied. We have:

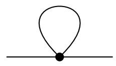
$$A(S^d) = \frac{2\pi^{D/2}}{\Gamma(D/2)}, \qquad \int\frac{k^d\,\mathrm{d}k}{k^2+m^2} = \frac{m^D}{m^2}\int\frac{t^d\,\mathrm{d}t}{1+t^2} \tag{2.11}$$

The t-integral is related to Beta functions; consider $t \mapsto \frac{t^2}{1+t^2}$, and we have:

$$\int_0^\infty \frac{t^d\,\mathrm{d}t}{1+t^2} = \frac{1}{2}\int_0^1 t^{\frac{D}{2}-1}(1-t)^{-\frac{D}{2}}\,\mathrm{d}t = \frac{\Gamma(\frac{D}{2})\,\Gamma(1-\frac{D}{2})}{2\Gamma(1)} = \frac{1}{2}\Gamma(\tfrac{D}{2})\,\Gamma(1-\tfrac{D}{2}) = \frac{\pi}{2\sin\frac{\pi D}{2}} \tag{2.12}$$

The last line is *Euler's reflection formula*, but here we actually don't need that since the $\Gamma(\frac{D}{2})$ factor is canceled by $A(S^d)$. In the end we have:

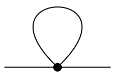
$$\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2+m^2} = \frac{\pi^{D/2}}{(2\pi)^D}\Gamma(1-\tfrac{D}{2})\,m^{D-2} = \frac{1}{(4\pi)^{D/2}}\Gamma(1-\tfrac{D}{2})\,m^{D-2}, \tag{2.13}$$



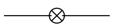
$$= \frac{-i\lambda}{2}\frac{1}{(4\pi)^{D/2}}\Gamma(1-\tfrac{D}{2})\,m^{D-2} \tag{2.14}$$

We then have to include counterterm contributions so that the renormalization condition (2.8) is satisfies; we have:

$$-iM(p^2) \sim$$



$$+$$



$$\begin{aligned} &= \frac{-i\lambda}{2}\frac{1}{(4\pi)^{D/2}}\Gamma(1-\tfrac{D}{2})\,m^{D-2} + i\big(\delta_Z(-p^2)+\delta_m\big) \\ &\sim 0 + 0\cdot(p^2+m^2) + \mathcal{O}\big(p^4\big) \end{aligned} \tag{2.15}$$

Therefore,

$$\delta_Z = 0, \quad \delta_m = \frac{\lambda}{2}\frac{1}{(4\pi)^{D/2}}\Gamma(1-\tfrac{D}{2})\,m^{D-2} \tag{2.16}$$

Alternatively, if we are working in $D = 4 = 3+1$ dimensions, it's easier to impose a naïve cutoff Λ, which gives:

$$\begin{aligned} \int^\Lambda \frac{k^d\,\mathrm{d}k}{k^2+m^2} &\sim \int^\Lambda k^{d-2}\,\mathrm{d}k + \int^\Lambda k^d\,\mathrm{d}k\left(\frac{1}{k^2+m^2}-\frac{1}{k^2}\right) \\ &= \int^\Lambda k^{d-2}\,\mathrm{d}k - m^2\int^\Lambda \frac{k^{d-2}\,\mathrm{d}k}{k^2+m^2}, \quad d = D-1 = 3 \\ &= \frac{\Lambda^2}{2} - \frac{m^2}{2}\ln\left(1+\frac{\Lambda^2}{m^2}\right), \end{aligned} \tag{2.17}$$

Similarly, with $A(S^3) = 2\pi^2$, we have:

$$\begin{aligned} \delta_Z = 0, \quad \delta_m &= \frac{\lambda}{2}\frac{2\pi^2}{(2\pi)^4}\left\{\frac{\Lambda^2}{2} - \frac{m^2}{2}\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right\} \\ &= \frac{\lambda}{32\pi^2}\left\{\Lambda^2 - m^2\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right\} \end{aligned} \tag{2.18}$$

References

[1] Michael E. Peskin & Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. Addison-Wesley, Reading, USA, **1995**. ISBN: 978-0-201-50397-5.