

## 1 Equivalence of categories is fully faithful:

$F: \mathcal{C} \rightarrow \mathcal{D}$  equivalence of categories, i.e.  $\exists G: \mathcal{D} \rightarrow \mathcal{C}$ , s.t.

$$G \circ F \simeq \mathbb{1}_{\mathcal{C}}, \quad F \circ G \simeq \mathbb{1}_{\mathcal{D}} \quad (1)$$

Here “ $\simeq$ ” means naturally isomorphic as functors, i.e.,

$$\exists \tau: G \circ F \Rightarrow \mathbb{1}_{\mathcal{C}}, \quad \sigma: F \circ G \Rightarrow \mathbb{1}_{\mathcal{D}} : \text{ natural isomorphisms} \quad (2)$$

By the definition of natural transformation, for  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ , we have:

$$\begin{array}{ccc} G \circ F(A) & \xrightarrow{G \circ F(f)} & G \circ F(B) \\ \tau_A \downarrow & & \downarrow \tau_B \\ A & \xrightarrow{f} & B \end{array}$$

$$\tau_B \circ (G \circ F)(f) \circ \tau_A^{-1} = f, \quad \forall f \in \text{Hom}_{\mathcal{C}}(A, B) \quad (3)$$

Here  $\tau_{A,B}$  are isomorphisms, which means that  $G \circ F$  must be a bijection between hom-sets, which further implies that  $F$  is injective and  $G$  is surjective. Switch the roles of  $F, G$ , we find that  $G$  is injective and  $F$  is surjective. Therefore,  $F, G$  are both fully faithful. ■

## 2 Forgetful functors to Set are often representable:

For  $F: \underline{\mathbf{Group}} \rightarrow \underline{\mathbf{Set}}$ , consider the free group generated by a single element  $\mathbb{Z}$ . We have:

$$\begin{aligned} \text{Hom}(\mathbb{Z}, -): \underline{\mathbf{Group}} &\longrightarrow \underline{\mathbf{Set}} \\ G &\longmapsto \text{Hom}(\mathbb{Z}, G) \end{aligned} \quad (4)$$

This is a covariant functor representable by  $\mathbb{Z}$ .

On the other hand,  $\text{Hom}(\mathbb{Z}, G)$  consists of group homomorphisms:

$$\text{Hom}(\mathbb{Z}, G) = \left\{ \begin{array}{c} \mathbb{Z} \rightarrow G \\ 1 \mapsto g \end{array} \middle| g \in G \right\} \quad (5)$$

More specifically, to fix any  $\mathbb{Z} \rightarrow G$ , we need only assign its generator<sup>1</sup>  $1 \mapsto g$ . Image of any other  $\mathbb{Z}$  element is generated automatically from the group law, without further specifications. This means that the hom-set is in one-to-one correspondence with  $G$  elements (as a set). Therefore,  $F \cong \text{Hom}_{\underline{\mathbf{Group}}}(\mathbb{Z}, -)$ , i.e. forgetful  $F: \underline{\mathbf{Group}} \rightarrow \underline{\mathbf{Set}}$  is representable by  $\mathbb{Z}$ . □

Similarly, for  $F: \underline{\mathbf{Ring}} \rightarrow \underline{\mathbf{Set}}$ , the free object generated by some generic element  $x$  is  $\mathbb{Z}[x]$ , the polynomial ring in one variable; we have:

$$F \cong \text{Hom}_{\underline{\mathbf{Ring}}}(\mathbb{Z}[x], -), \quad \text{Hom}_{\underline{\mathbf{Ring}}}(\mathbb{Z}[x], R) = \left\{ \begin{array}{c} \mathbb{Z}[x] \rightarrow R \\ x \mapsto r \end{array} \middle| r \in R \right\} \quad (6)$$

*Lesson:* Forgetful  $\underline{\mathbf{Cat}} \rightarrow \underline{\mathbf{Set}}$  are often representable by the free object in  $\underline{\mathbf{Cat}}$ . ■

<sup>1</sup>Note that  $0 \in \mathbb{Z}$  is the group identity of addition group  $\mathbb{Z}$ , not  $1 \in \mathbb{Z}$ .

### 3 Properties of contractible space:

(a)  $X$  contractible:  $\mathbb{1}_X \simeq f_0: X \rightarrow X$  some constant map,  $f_0(X) = \{x_0\}$ . We can restrict the codomain of  $f_0$  so that  $f_0: X \rightarrow \{x_0\}$ , in this way we have:

$$X \xrightarrow{f_0} \{x_0\} \hookrightarrow X \simeq \mathbb{1}_X, \quad (7.1)$$

$$\{x_0\} \hookrightarrow X \xrightarrow{f_0} \{x_0\} \simeq \mathbb{1}_{\{x_0\}}, \quad (7.2)$$

This means that  $f_0: X \rightarrow \{x_0\}$  isomorphic in  $\mathbf{hTop} = \mathbf{Top}/\simeq$ , which is precisely the definition of homotopic equivalence  $X \simeq \{x_0\}$ . ( $\Rightarrow$ )

On the other hand ( $\Leftarrow$ ), if  $X \simeq \{x_0\}$ , there exists some  $f_0: X \rightarrow \{x_0\}$  that fulfills (7). We can then extend the codomain s.t.  $f_0: X \rightarrow X$ , in this way (7.1) reads  $f_0 \simeq \mathbb{1}_X$ , i.e.  $X$  is contractible. Therefore,  $X$  contractible iff. homotopic equivalent to a single point.  $\blacksquare_{(a)}$

(b)  $\forall X$ : Topological space, we can define its *cone* as<sup>2</sup>:

$$CX = (X \times I)/(X \times \{0\}), \quad I = [0, 1] \quad (8)$$

i.e. gluing together one end of the cylinder  $X \times I$ . Naturally  $X \subset CX$  as a subspace; now we show that  $CX$  is contractible. Using (a), we need only show that  $\mathbb{1}_{CX} \simeq f_0$  some constant map.

In fact, any point in  $CX$  can be uniquely labeled by  $[x, h] \in X \times I$ , with the exception of the vertex  $v \sim [x, 0] \sim [x', 0]$ ,  $\forall x, x' \in X$ . We can then construct a homotopy  $F$  by shrinking the cone towards the vertex  $v$ :

$$\begin{aligned} F: CX \times I &\rightarrow CX, & F([x, h], t) &= [x, h \cdot t], \\ F|_{CX \times 0} &= v = \text{const}, & F|_{CX \times 1} &= \mathbb{1}_X \end{aligned} \quad (9)$$

This confirms that  $\mathbb{1}_{CX} \simeq v$ : constant map. By (a),  $CX$  is contractible.  $\blacksquare_{(b)}$

(c) For  $Y \simeq \{y_0\}$  contractible, given any  $g: X \rightarrow Y$ , we can deform the image  $g(X) \subset Y$  to a single point, hence  $g \simeq y_0$ : constant map. More precisely, we have:

$$\exists G: X \times I \rightarrow Y, \quad \text{s.t.} \quad G|_{X \times 0} = y_0 = \text{const}, \quad G|_{X \times 1} = g \quad (10)$$

Such  $G$  can be explicitly constructed using  $\mathbb{1}_Y \simeq y_0$ :

$$F: Y \times I \rightarrow Y, \quad F|_{Y \times 0} = y_0 = \text{const}, \quad F|_{Y \times 1} = \mathbb{1}_Y, \quad (11)$$

$$G(x, t) = F(g(x), t) \quad (12)$$

In summary, we have proven that  $g \simeq y_0$ ,  $\forall g \in \text{Hom}_{\mathbf{Top}}(X, Y)$ . By definition, this means that  $\text{Hom}_{\mathbf{hTop}}(X, Y) = \text{Hom}_{\mathbf{Top}}(X, Y)/\simeq = \{[y_0]\}$  a single point.  $\blacksquare_{(c)}$

(d) For  $X \simeq \{x_0\}$  contractible, similar to (11), we have homotopy  $F: X \times I \rightarrow X$ . Given any  $f: X \rightarrow Y$ , the composition  $f \circ F: X \times I \rightarrow Y$  yields  $f \simeq f(x_0)$ : constant map.

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<sup>2</sup>See Wikipedia: *Cone (topology)*.

Furthermore, for  $Y$ : path connected, there is a path  $\gamma: I \rightarrow Y$  connecting  $f(x_0)$  and some  $y_0 \in Y$ , therefore  $f(x_0) \simeq y_0: X \rightarrow Y$  constant maps. More precisely, we have:

$$\gamma: I \rightarrow Y, \quad \gamma(0) = y_0, \quad \gamma(1) = f(x_0), G: X \times I \rightarrow Y, \quad G(x, t) = \gamma(t) \quad (13)$$

Which gives  $f(x_0) \simeq y_0, \forall f$ , independent of the choice of  $f$ . This means that  $f \simeq f(x_0) \simeq y_0$ : constant map, therefore  $\text{Hom}_{\mathbf{hTop}}(X, Y) = \{[y_0]\}$  a single point.  $\blacksquare_{(d)}$

#### 4 Example of homotopic inequivalence<sup>3</sup>:

$$X = \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \right\}, \quad Y = \{0\} \cup \mathbb{Z}_+ \quad (14)$$

$X, Y \subset \mathbb{R}$ : subspace topology

Assume  $X \simeq Y$ , then similar to (7), we have  $Y \xrightarrow{g} X \xrightarrow{f} Y \simeq \mathbb{1}_Y$ . However, note that  $Y$  has discrete topology, in such case any map  $f \circ g$  homotopic to  $\mathbb{1}_Y$  must be  $\mathbb{1}_Y$  itself:  $f \circ g = \mathbb{1}_Y$ .

More specifically, consider:

$$F: Y \times I \rightarrow Y, \quad F|_{Y \times 0} = f \circ g, \quad F|_{Y \times 1} = \mathbb{1}_Y \quad (15)$$

Any point  $n \in Y$  is both open and closed, therefore its pre-image  $F^{-1}(n) \subset Y \times I$  is also both open and closed, and by  $F|_{Y \times 1} = \mathbb{1}_Y$  we know that  $F(y, 1) = y, (y, 1) \in F^{-1}(y)$ , therefore the only possibility is that  $F(\{y\} \times I) = y$ , i.e.  $f \circ g = \mathbb{1}_Y$ , which implies that  $g$  is injective and  $f$  is surjective.

However,  $f: X \rightarrow Y$  cannot be surjective due to the complication around  $0 \in X$ . Consider  $f^{-1}(f(0)) \ni 0$ , since  $f(0) \in Y$  both open and closed,  $f^{-1}(f(0)) \subset X$  must also be both open and closed. But any open set  $U \subset X$  is induced via subspace topology  $X \subset \mathbb{R}$ ; for  $0 \in U \subset X \subset \mathbb{R}$ ,  $U$  must contain  $\infty$ -many elements:

$$\left\{ \frac{1}{n} \mid n \geq N_0 \right\} \subset U \subset f^{-1}(f(0)), \quad \text{for some } N_0, \text{ for any } U \ni x \quad (16)$$

Hence  $f(X) = f(0) \cup f(\{\frac{1}{n} \mid n < N_0\})$ ,  $f(X) \subset Y$  a finite set, i.e.  $f: X \rightarrow Y$  is never surjective. Therefore,  $X \not\simeq Y$  by contradiction.  $\blacksquare$

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<sup>3</sup>This proof is produced thanks to helpful insights from 谷夏 and 於子雄.

**5 Fundamental group of topological group is abelian<sup>4</sup>:**

From a categorical point of view, the fundamental group  $\pi_1(G)$  of a topological group  $G$  can be seen as a functor:

$$G \in \underline{\mathbf{TopGroup}} \hookrightarrow \underline{\mathbf{Top}} \xrightarrow{\pi_1} \underline{\mathbf{Group}} \ni \pi_1(G) \quad (17)$$

$\underline{\mathbf{TopGroup}} \subset \underline{\mathbf{Top}}$  is a subcategory with additional group structure, i.e.  $(G, \cdot) \in \underline{\mathbf{TopGroup}}$  is a *group object*<sup>5</sup> in  $\underline{\mathbf{Top}}$ , with “ $\cdot$ ” denoting its product operation  $(\cdot): G \times G \rightarrow G$ . Correspondingly,  $\pi_1(\underline{\mathbf{TopGroup}})$  should be *group objects of  $\underline{\mathbf{Group}}$* , which have an *additional* group structure  $(\star) = \pi_1(\cdot)$ , along with the usual group product “ $*$ ” in  $\underline{\mathbf{Group}}$ .

In total, we have three different group structures (represented by their product operation):

$$(\cdot): G \times G \rightarrow G, \quad (18)$$

$$(*): \pi_1(G) \times \pi_1(G) \rightarrow \pi_1(G), \quad (19)$$

$$(\star) = \pi_1(\cdot): \pi_1(G) \times \pi_1(G) \rightarrow \pi_1(G), \quad (20)$$

Note that  $\pi_1(G) = \text{Aut}_{\Pi_1(G)} \mathbb{1}_G$ , i.e. loop classes  $[\gamma]$  in  $G$ ;  $(*)$  is defined as joining two loops, while  $(\star) = \pi_1(\cdot)$  is defined as the translation of loop classes by pointwise group product  $(\cdot)$ ,

$$[\gamma_1] \star [\gamma_2] = [\gamma_1 \cdot \gamma_2] \quad (21)$$

With the above definitions, we observe that:

$$([\gamma_1] \star [\gamma_2]) * ([\eta_1] \star [\eta_2]) = ([\gamma_1] * [\eta_1]) * ([\gamma_2] * [\eta_2]) \quad (22)$$

By definition, they are both equal to  $[(\gamma_1 \cdot \gamma_2) * (\eta_1 \cdot \eta_2)]$ . What’s surprising is that by using only the group axioms and “distributive law” (22), we can show that  $(\star)$  and  $(*)$  must always coincide:  $(\star) = (*)$ , and they have to be in fact, commutative. This is the *Eckmann–Hilton argument*<sup>6</sup>.

Proof of this argument is straight-forward; first, observe that the units of the two operations coincide:

$$1_\star = 1_\star \star 1_\star = (1_\star * 1_\star) \star (1_\star * 1_\star) \stackrel{(22)}{=} (1_\star \star 1_\star) * (1_\star \star 1_\star) = 1_\star * 1_\star = 1_\star \quad (23)$$

Further manipulation using (22) confirms that the two operations coincide and are commutative:

$$\begin{aligned} [\gamma] * [\eta] &= (1 \star [\gamma]) * ([\eta] \star 1) \stackrel{(22)}{=} (1 * [\eta]) \star ([\gamma] * 1) \\ &= [\eta] \star [\gamma] \\ &= ([\eta] * 1) \star (1 * [\gamma]) \stackrel{(22)}{=} ([\eta] \star 1) * (1 \star [\gamma]) \\ &= [\eta] * [\gamma] \end{aligned} \quad (24)$$

In summary, we find that the group objects in  $\underline{\mathbf{Group}}$  are indeed abelian groups, which means that  $\pi_1(G)$  for  $G \in \underline{\mathbf{TopGroup}}$  must be abelian. ■

<sup>4</sup>This proof is produced with the help of [math.stackexchange.com/q/727999](https://math.stackexchange.com/q/727999). Another (easier) proof lies in the fact that group translation induces  $\pi_1$  conjugation, therefore  $\gamma^{-1}\alpha\gamma = \alpha$ , hence abelian.

<sup>5</sup>See Wikipedia: *Group object*.

<sup>6</sup>See Wikipedia: *Eckmann–Hilton argument*.