

1 Gravity

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2 \quad (1.1)$$

$$f(r) = 1 - \frac{GM}{r} + \frac{Q^2}{r^2} = \frac{(r - r_+)(r - r_-)}{r^2} \quad (1.2)$$

1. Event horizon(s): $f(r) = 0$, we have:

- (a) $M > |Q|$, $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$, 2 event horizons;
- (b) $M = |Q|$, $r_{\pm} = M$, 1 event horizon;
- (c) $M < |Q|$, no event horizon! “Naked” singularity.

2. New coordinate: $v = t + r^*$,

$$r^* = r + \frac{1}{2k_+} \ln \frac{|r - r_+|}{r_+} + \frac{1}{2k_-} \ln \frac{|r - r_-|}{r_-}, \quad k_{\pm} = \frac{r_{\pm} - r_{\mp}}{2r_{\pm}^2} \quad (1.3)$$

We have:

$$dt = dv - dr^* = dv - \left(1 + \frac{1}{2k_+} \frac{1}{r - r_+} + \frac{1}{2k_-} \frac{1}{r - r_-} \right) dr \quad (1.4)$$

$$\begin{aligned} &= dv - \left(1 + \frac{1}{r^2 f(r)} \frac{r_+^2(r - r_-) - r_-^2(r - r_+)}{r_+ - r_-} \right) dr \\ &= dv - \left(1 + \frac{1}{r^2 f(r)} \left((r_+ + r_-)r - r_+ r_- \right) \right) dr \\ &= dv - \frac{1}{f(r)} dr \end{aligned} \quad (1.5)$$

Therefore,

(a) The new metric:

$$\begin{aligned} ds^2 &= -f(r) \left(dv - \frac{1}{f(r)} dr \right)^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2 \\ &= -f(r) dv^2 + 2 dv dr + r^2 d\Omega^2 \end{aligned} \quad (1.6)$$

It is only singular at $r = 0$.

Note: during the exam I panicked when I saw (1.3), and I made a very stupid mistake in step (1.4). However, I knew what this new coordinate is trying to achieve — it’s aiming to eliminate the coordinate singularities in $\frac{1}{f} dr^2$ by absorbing it into dv^2 , so I guessed the result (1.5) correctly and carried on. I hope they gave me some points for getting the right answer, despite with some wrong process ($>_<$).

(b) $\frac{\partial}{\partial v}$ is a Killing vector field, for the metric components are all v -independent. More precisely, since $\frac{\partial}{\partial v}$ itself is a coordinate basis, we have the Lie derivative:

$$\mathcal{L}_{\frac{\partial}{\partial v}} g_{\mu\nu} = \partial_v g_{\mu\nu} = 0 \quad (1.7)$$

(c) $\left\| \frac{\partial}{\partial v} \right\|^2 = g_{\mu\nu} \delta_v^\mu \delta_v^\nu = g_{vv} = -f(r)$, therefore, for $M > |Q|$ we have:

- $\frac{\partial}{\partial v}$ timelike: $r > r_+$ and $r < r_-$
- spacelike: $r_- < r < r_+$
- null: $r = r_+$ and $r = r_-$

2 QFT

We shall restore the reasonable convention: $\eta_{\mu\nu} \sim (-, +, +, +)$.

- 1PI: diagrammatic correction to the (1-particle) propagator that cannot be split into 2 disconnected parts by cutting one line; e.g. 
2. Consider the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2}Z(\partial\phi_r)^2 - \frac{1}{2}m^2Z\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 - \frac{1}{2}\delta_Z(\partial\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4 \quad (2.1)$$

The convention here is rather bizarre; normally we write down the UV Lagrangian \mathcal{L}_{UV} and split it into 2 parts, one is the effective IR Lagrangian \mathcal{L}_{IR} and the other one is the counterterm:

$$\begin{aligned} \mathcal{L}_{\text{UV}} &= -\frac{1}{2}Z(\partial\phi_r)^2 - \frac{1}{2}m^2Z\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 \\ &= \left(-\frac{1}{2}(\partial\phi_r)^2 - \frac{1}{2}m_p^2\phi_r^2 - \frac{\lambda_p}{4!}\phi_r^4 \right) - \left(-\frac{1}{2}\delta_Z(\partial\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4 \right) \\ &= \mathcal{L}_{\text{IR}} + \mathcal{L}_{\text{ct}} \end{aligned} \quad (2.2)$$

Normally, we use \mathcal{L} to denote the UV Lagrangian \mathcal{L}_{UV} ; this is the convention adopted by numerous standard textbooks, incl. *Peskin & Schroeder* [1], *Weinberg*, and also *Srednicki*. However, the Lagrangian in (2.1) seems to be \mathcal{L}_{IR} instead of \mathcal{L}_{UV} . Anyway, we have:

$$Z + \delta_Z = 1, \quad m^2Z + \delta_m = m_p^2, \quad \lambda + \delta_\lambda = \lambda_p \quad (2.3)$$

Where m_p, λ_p is the physical IR couplings, fixed by the renormalization scheme. The convention here is really confusing and somewhat inconsistent; e.g. if we choose to write the UV mass term as $-\frac{1}{2}m^2Z\phi_r^2$, then the corresponding UV interaction term should look like $-\frac{\lambda}{4!}Z^2\phi_r^4$, but here we do not have the Z^2 factor. Also, we usually use m_0, λ_0 to denote bare couplings, but here it seems that they are denoted by m, λ .

We can write down the renormalized Feynman rules nonetheless, despite some sign issues due to the conventions; to avoid further confusion, we will adopt the usual notation: m_0, λ_0 for bare couplings, and $m = m_p, \lambda = \lambda_p$ for physical couplings. We have:

- Renormalized propagator: $\frac{-i}{p^2 + m^2 - i\epsilon}$
- Renormalized vertex: $-i\lambda$
- Counterterm ϕ^2 vertex: $+i(\delta_Z(-p^2) + \delta_m)$, 
- Counterterm ϕ^4 vertex: $+i\delta_\lambda$

3. The sum of all two point 1PI diagrams (no propagator on external legs) is given by:

$$-iM(p^2) = \text{diagram with a shaded circle} \quad (2.4)$$

The full propagator is thus:

$$\begin{aligned} G(p^2) &= \text{diagram with a horizontal line} + \text{diagram with a shaded circle on a horizontal line} + \text{diagram with two shaded circles on a horizontal line} + \dots \\ &= \frac{-i}{p^2+m^2} + \frac{-i}{p^2+m^2}(-iM)\frac{-i}{p^2+m^2} + \frac{-i}{p^2+m^2}(-iM)\frac{-i}{p^2+m^2}(-iM)\frac{-i}{p^2+m^2} + \dots \end{aligned} \quad (2.5)$$

With $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$, we get:

$$G(p^2) = \frac{-i}{p^2+m^2} \cdot \frac{1}{1 - (-iM)\frac{-i}{p^2+m^2}} = \frac{-i}{p^2+m^2+M(p^2)} \quad (2.6)$$

Here we've suppressed the $(-i\epsilon)$ prescription in the above expressions, but it's presence is always implied.

4. On-shell renormalization scheme — the full propagator:

$$G(p^2) = \frac{-i}{p^2+m^2+M(p^2)-i\epsilon} \xrightarrow{p^2 \rightarrow -m^2} \frac{-i}{p^2+m^2-i\epsilon} \quad (2.7)$$

This means that $M(p^2 = -m^2) = 0$. Furthermore, $M(p^2) \sim \#(p^2+m^2) + \mathcal{O}(p^4)$, to ensure that the residue is 1 at the pole, we should have $\# \sim 0$, i.e.

$$M(p^2)|_{p^2=-m^2} = 0, \quad \frac{\partial}{\partial(p^2)} M(p^2)|_{p^2=-m^2} = 0 \quad (2.8)$$

5. At 1-loop $\mathcal{O}(\lambda)$, if we do not include counterterm contributions, then there is only one diagram contributing to $M(p^2)$:

$$\text{diagram with a loop on a horizontal line} = (-i\lambda) \cdot \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \frac{-i}{k^2+m^2-i\epsilon} \quad (2.9)$$

Here $\frac{1}{2}$ is the symmetry factor of the diagram; alternative, we can count the distinct ways of connecting the 4 legs of the ϕ^4 vertex and divide it by $4!$, which is indeed $\frac{4 \times 3}{4!} = \frac{1}{2}$.

The p^0 integral has poles at $p_0^2 = \mathbf{p}^2 + m^2 - i\epsilon$, i.e. $p^0 = \pm \sqrt{\mathbf{p}^2 + m^2} \mp i\epsilon$, and it's regular everywhere else; we can thus compute the p^0 integral on the \mathbb{C} plane using a right-tilted 8-shaped contour, which does not enclose the poles. Effectively, we've performed a Wick rotation $p^0 \mapsto ip^0$ so that the integral happens in Euclidean p space:

$$\frac{-i\lambda}{2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2+m^2} = \frac{-i\lambda}{2} \frac{A(S^d)}{(2\pi)^D} \int \frac{k^d dk}{k^2+m^2} \quad (2.10)$$

Here $D = d + 1$, d is the spatial dimension. There are many ways to regularize this integral; if we continue to work in general $D = d + 1$ dimensions, then dimensional regularization is automatically implied. We have:

$$A(S^d) = \frac{2\pi^{D/2}}{\Gamma(D/2)}, \quad \int \frac{k^d dk}{k^2+m^2} = \frac{m^D}{m^2} \int \frac{t^d dt}{1+t^2} \quad (2.11)$$

The t -integral is related to Beta functions; consider $t \mapsto \frac{t^2}{1+t^2}$, and we have:

$$\int_0^\infty \frac{t^d dt}{1+t^2} = \frac{1}{2} \int_0^1 t^{\frac{D}{2}-1} (1-t)^{-\frac{D}{2}} dt = \frac{\Gamma(\frac{D}{2}) \Gamma(1-\frac{D}{2})}{2\Gamma(1)} = \frac{1}{2} \Gamma(\frac{D}{2}) \Gamma(1-\frac{D}{2}) = \frac{\pi}{2 \sin \frac{\pi D}{2}} \quad (2.12)$$

The last line is *Euler's reflection formula*, but here we actually don't need that since the $\Gamma(\frac{D}{2})$ factor is canceled by $A(S^d)$. In the end we have:

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} = \frac{\pi^{D/2}}{(2\pi)^D} \Gamma(1 - \frac{D}{2}) m^{D-2} = \frac{1}{(4\pi)^{D/2}} \Gamma(1 - \frac{D}{2}) m^{D-2}, \quad (2.13)$$

$$\text{---} \bullet \text{---} = \frac{-i\lambda}{2} \frac{1}{(4\pi)^{D/2}} \Gamma(1 - \frac{D}{2}) m^{D-2} \quad (2.14)$$

We then have to include counterterm contributions so that the renormalization condition (2.8) is satisfied; we have:

$$\begin{aligned} -iM(p^2) &\sim \text{---} \bullet \text{---} + \text{---} \otimes \text{---} = \frac{-i\lambda}{2} \frac{1}{(4\pi)^{D/2}} \Gamma(1 - \frac{D}{2}) m^{D-2} + i(\delta_Z(-p^2) + \delta_m) \\ &\sim 0 + 0 \cdot (p^2 + m^2) + \mathcal{O}(p^4) \end{aligned} \quad (2.15)$$

Therefore,

$$\delta_Z = 0, \quad \delta_m = \frac{\lambda}{2} \frac{1}{(4\pi)^{D/2}} \Gamma(1 - \frac{D}{2}) m^{D-2} \quad (2.16)$$

Alternatively, if we are working in $D = 4 = 3 + 1$ dimensions, it's easier to impose a naïve cutoff Λ , which gives:

$$\begin{aligned} \int^\Lambda \frac{k^d dk}{k^2 + m^2} &\sim \int^\Lambda k^{d-2} dk + \int^\Lambda k^d dk \left(\frac{1}{k^2 + m^2} - \frac{1}{k^2} \right) \\ &= \int^\Lambda k^{d-2} dk - m^2 \int^\Lambda \frac{k^{d-2} dk}{k^2 + m^2}, \quad d = D - 1 = 3 \\ &= \frac{\Lambda^2}{2} - \frac{m^2}{2} \ln \left(1 + \frac{\Lambda^2}{m^2} \right), \end{aligned} \quad (2.17)$$

Similarly, with $A(S^3) = 2\pi^2$, we have:

$$\begin{aligned} \delta_Z = 0, \quad \delta_m &= \frac{\lambda}{2} \frac{2\pi^2}{(2\pi)^4} \left\{ \frac{\Lambda^2}{2} - \frac{m^2}{2} \ln \left(1 + \frac{\Lambda^2}{m^2} \right) \right\} \\ &= \frac{\lambda}{32\pi^2} \left\{ \Lambda^2 - m^2 \ln \left(1 + \frac{\Lambda^2}{m^2} \right) \right\} \end{aligned} \quad (2.18)$$

References

- [1] Michael E. Peskin & Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. Addison-Wesley, Reading, USA, **1995**. ISBN: 978-0-201-50397-5.