

## Thermal propagator for harmonic oscillators

$$G(\tau) = \frac{1}{Z} \text{Tr} (e^{-\beta H} x(\tau) x(0)) \quad (1)$$

The Hamiltonian is  $H = \omega (a^\dagger a + \frac{1}{2})$ , while  $x = \frac{1}{\sqrt{2m\omega}} (a + a^\dagger)$ .

### Operator formalism

Mode expansion of Lorentzian operator in the Heisenberg picture:

$$x(t) = e^{iHt} x(0) e^{-iHt} = \frac{1}{\sqrt{2m\omega}} (a e^{-i\omega t} + a^\dagger e^{i\omega t}) \quad (2)$$

This follows from the commutation relation between  $a^{(\dagger)}$  and  $H$ , or from the quantum EOM in the Heisenberg picture. Wick-rotation to Euclidean signature:  $\tau = it$ ,  $t = -i\tau$ , and we have:

$$x(\tau) = e^{H\tau} x(0) e^{-H\tau} = \frac{1}{\sqrt{2m\omega}} (a e^{-\omega\tau} + a^\dagger e^{\omega\tau}) \quad (3)$$

Insert  $x(\tau) x(0)$  into the definition of  $G(\tau)$ , and note that only  $aa^\dagger$  and  $a^\dagger a$  contribute to non-zero amplitudes, then we have:

$$\begin{aligned} G(\tau) &= \frac{1}{Z} \sum_{n=0}^{\infty} \left\langle n \left| e^{-\beta\omega(n+\frac{1}{2})} \frac{aa^\dagger e^{-\omega\tau} + a^\dagger a e^{\omega\tau}}{2m\omega} \right| n \right\rangle \\ &= \frac{1}{Z} \sum_{n=0}^{\infty} \left\langle n \left| e^{-\beta\omega(n+\frac{1}{2})} \frac{(n+1) e^{-\omega\tau} + n e^{\omega\tau}}{2m\omega} \right| n \right\rangle \end{aligned} \quad (4)$$

Note that:

$$\sum_{n=0}^{\infty} n e^{-\beta\omega n} = -\frac{1}{\beta} \frac{\partial}{\partial \omega} \sum_{n=0}^{\infty} e^{-\beta\omega n} \quad (5)$$

The summations in  $G(\tau)$  can be completed, and we get:

$$G(\tau) = \frac{1}{2m\omega} \frac{\cos((\frac{\beta}{2} - \tau)\omega)}{\sin(\frac{\beta\omega}{2})} \quad (6)$$

### Path Integral

The mathematical trick in (5) is crucial in our path integral derivation of  $G(\tau)$ . We have:

$$G(\tau) = \langle x(\tau) x(0) \rangle = \frac{1}{Z} \int \mathcal{D}x e^{-S} x(\tau) x(0), \quad (7)$$

$$S = \int_0^\beta dt \mathcal{L}_E, \quad \mathcal{L}_E = \frac{1}{2} m \dot{x}^2(\tau) + \frac{1}{2} m \omega^2 x^2(\tau) \quad (8)$$

Note that the path integral of a total derivative vanishes:  $0 = \int \mathcal{D}x \frac{\delta}{\delta x}$  (see *Polchinski*), we have:

$$0 = \int \mathcal{D}x \frac{\delta}{\delta x(\tau)} \left\{ e^{-S} x(\tau') \right\} = \int \mathcal{D}x e^{-S} \left\{ -\frac{\delta S}{\delta x(\tau)} x(\tau') + \delta(\tau - \tau') \right\} \quad (9)$$

$\frac{\delta S}{\delta x(\tau)}$  yields precisely the quantum EOM:

$$\frac{\delta S}{\delta x(\tau)} = m \left( -\frac{\partial^2}{\partial \tau^2} + \omega^2 \right) x(\tau) \quad (10)$$

After some re-organization, we get the quantum equation for the Green's function (which, for a free theory, is identical to the classical one):

$$m \left( -\frac{\partial^2}{\partial \tau^2} + \omega^2 \right) \int \mathcal{D}x e^{-S} x(\tau) x(\tau') = \delta(\tau - \tau') \int \mathcal{D}x e^{-S}, \quad (11)$$

$$\left( -\frac{\partial^2}{\partial \tau^2} + \omega^2 \right) G(\tau, \tau') = \frac{1}{m} \delta(\tau - \tau'), \quad G(\tau) = G(\tau, 0) \quad (12)$$

Note that  $\tau \in [0, \beta]$  and  $x(\tau)$  satisfies the periodic boundary condition:  $x(\beta) = x(0)$ , hence also  $G(\beta) = G(0)$ ; we can extend the domain so that  $x(\tau)$  is a function with period  $\beta$ , defined for  $\tau \in \mathbb{R}$ . Furthermore, note that by definition  $G(\tau, \tau') = G(\tau', \tau)$ ; this is the Feynman propagator! Integrating around  $\tau = \tau'$  then setting  $\tau' = 0$  gives:

$$G'(\beta^-) - G'(0^+) = \frac{1}{m} \quad (13)$$

While  $G(\tau)$  away from  $\tau = 0$  grows (or decays) exponentially:  $G(\tau) \propto e^{\omega\tau} + e^{\omega(\beta-\tau)}$ . This fixes  $G(\tau)$  uniquely:

$$G(\tau) = \frac{1}{2m\omega} \frac{\cos\left(\left(\frac{\beta}{2} - \tau\right)\omega\right)}{\sin\left(\frac{\beta\omega}{2}\right)} \quad (14)$$