String Theory №4

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1 Stringy Physics!

$$T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu: , \quad \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X^\mu \bar{\partial} X_\mu: , \tag{1}$$

$$V_k = :e^{ik\cdot X(z,\bar{z})}: , \quad G_{e,k} = e_{\mu\nu} :\partial X^\mu_z \, \bar{\partial} X^\nu_{\bar{z}} \, e^{ik\cdot X(z,\bar{z})}: , \tag{2}$$

We sometimes use subscripts like ∂X^μ_z to denote variable dependence to avoid clutter.

(a) The weight of a primary operator is given by its OPE with T and $\tilde{T}$. For exponential operators, there is a neat formula for cross contractions[1]:

$$\begin{aligned}
T(z)\, V_k(w,\bar{w}) &= \exp\left\{ \int \mathrm{d}^2 z' \int \mathrm{d}^2 w' \, \overbracket{X^\mu_{z'} X^\nu_{w'}} \frac{\delta}{\delta X^\mu_{z'}} \frac{\delta}{\delta X^\nu_{w'}} \right\} :T_z \, e^{ik\cdot X_w}: \\
&= \exp\left\{ \int \mathrm{d}^2 z' \, \overbracket{X^\mu_{z'} X^\nu_{w}} \frac{\delta}{\delta X^\mu_{z'}} \, ik_\nu \right\} :T_z \, e^{ik\cdot X_w}: \\
&= :\left\{ \exp\left(ik_\nu \int \mathrm{d}^2 z' \, \overbracket{X^\mu_{z'} X^\nu_{w}} \frac{\delta}{\delta X^\mu_{z'}} \right) T_z \right\} e^{ik\cdot X_w}: \\
&\sim -\frac{1}{\alpha'} :\left\{ 2\partial_z \left(ik_\sigma \overbracket{X^\mu_z X^\sigma_w} \right) \partial_z X_\mu + \partial_z \left(ik_\rho \overbracket{X^\mu_z X^\rho_w} \right) \partial_z \left(ik_\sigma \overbracket{X_{z,\mu} X^\sigma_w} \right) \right\} e^{ik\cdot X_w}: \\
&\sim -\frac{1}{\alpha'} :\left\{ 2\left(-\frac{\alpha'}{2} \frac{ik^\mu}{z-w} \right) \partial_z X_\mu + \left(-\frac{\alpha'}{2} \frac{ik^\mu}{z-w} \right) \left(-\frac{\alpha'}{2} \frac{ik_\mu}{z-w} \right) \right\} e^{ik\cdot X_w}: \\
&\sim \frac{\alpha' k^2}{4} \frac{V_k(w,\bar{w})}{(z-w)^2} + \frac{\partial V_k(w,\bar{w})}{z-w}
\end{aligned} \tag{3}$$

Here we've used the result that $ik_\sigma \overbracket{X^\mu_z X^\sigma_w} = ik^\mu (-\frac{\alpha'}{2}) \ln|z-w|^2$. We see that V_k is a primary of weight $(1,1)$ iff. $\frac{\alpha' k^2}{4} = 1$, or $m^2 = -k^2 = -\frac{4}{\alpha'}$. This is the mass shell condition for the closed string tachyon (at level 0). On the other hand,

$$G_{e,k} = e_{\mu\nu} G^{\mu\nu}_k , \tag{4}$$

$$\begin{aligned}
T(z)\, G^{\mu\nu}_k(0) \sim{}& :\overbracket{T_z \, \partial X^\mu_0} \, \bar{\partial} X^\nu_0 \, e^{ik\cdot X_0}: + :\cancel{\overbracket{T_z \, \partial X^\mu_0 \, \bar{\partial} X^\nu_0} \, e^{ik\cdot X_0}}: + :\overbracket{T_z \, \partial X^\mu_0 \, \bar{\partial} X^\nu_0 \, e^{ik\cdot X_0}}: \\
&+ :\cancel{\underbracket{T_z \, \partial X^\mu_0} \, \bar{\partial} X^\nu_0 \, e^{ik\cdot X_0}}: + :\underbracket{T_z \, \partial X^\mu_0} \, \bar{\partial} X^\nu_0 \, e^{ik\cdot X_0}: \\
\sim{}& \left(\frac{1}{z^2} G^{\mu\nu}_k(0) + \frac{1}{z} :\partial^2 X^\mu_0 \, \bar{\partial} X^\nu_0 \, e^{ik\cdot X_0}: \right) + \left(\frac{\alpha' k^2}{4} \frac{1}{z^2} G^{\mu\nu}_k(0) + \frac{1}{z} :\partial X^\mu_0 \, \bar{\partial} X^\nu_0 \, \partial \, e^{ik\cdot X_0}: \right) \\
&- \frac{2}{\alpha'} \left(-\frac{\alpha'}{2} \eta^{\sigma\mu} \frac{1}{z^2} \right) \left(-\frac{\alpha'}{2} \frac{ik_\sigma}{z} \right) :\bar{\partial} X^\nu_0 \, e^{ik\cdot X_0}: \\
\sim{}& ik^\mu :\bar{\partial} X^\nu_0 \, e^{ik\cdot X_0}: \left(-\frac{\alpha'}{2} \right) \frac{1}{z^3} + \left(1 + \frac{\alpha' k^2}{4} \right) \frac{G^{\mu\nu}_k(0)}{z^2} + \frac{\partial G^{\mu\nu}_k(0)}{z} ,
\end{aligned} \tag{5}$$

$$\tilde{T}(\bar{z})\, G^{\mu\nu}_k(0) \sim ik^\nu :\partial X^\mu_0 \, e^{ik\cdot X_0}: \left(-\frac{\alpha'}{2} \right) \frac{1}{\bar{z}^3} + \left(1 + \frac{\alpha' k^2}{4} \right) \frac{G^{\mu\nu}_k(0)}{\bar{z}^2} + \frac{\partial G^{\mu\nu}_k(0)}{\bar{z}} , \tag{6}$$

Therefore, $G_{e,k}$ is a primary of weight $(1,1)$ iff. $1 + \frac{\alpha' k^2}{4} = 1$ and $k^\mu e_{\mu\nu} = 0 = k^\nu e_{\mu\nu}$. The first equation gives the mass shell condition $m^2 = -k^2 = 0$ for a massless boson, while the second equation

[1] Reference: *Polchinski*, and physics.stackexchange.com/a/389193.

constrains the polarization to be transverse. These are the physical constraints for a massless gauge boson, which is the level 1 excitation for a bosonic closed string.

(b) The form of any primary 3-point function is completely fixed by $\mathrm{PSL}(2,\mathbb{C})$ invariance[2]. In fact, for any holomorphic $\phi_i(z_i)$ with weight h_i, by translational invariance, we have:

$$\langle \phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)\rangle = f(z_{12}, z_{23}, z_{31}), \quad z_{ij} = z_i - z_j, \tag{7}$$

Furthermore, scaling invariance requires that f is quasi-homogeneous:

$$\begin{aligned} z \mapsto z' = \lambda^{-1} z, \quad f \mapsto & \left\langle \lambda^{h_1}\phi_1(\lambda z_1)\,\lambda^{h_2}\phi_2(\lambda z_2)\,\lambda^{h_3}\phi_3(\lambda z_3)\right\rangle \\ &= \lambda^{h_1+h_2+h_3} f(\lambda z_{12}, \lambda z_{23}, \lambda z_{31}) \\ &= f(z_{12}, z_{23}, z_{31}), \end{aligned} \tag{8}$$

$$f = \sum_{a+b+c=\sum_i h_i} f_{abc} = \sum_{a+b+c=\sum_i h_i} \frac{C_{abc}}{z_{12}^a z_{23}^b z_{31}^c} \tag{9}$$

On the other hand, for special conformal transformation[3] $\frac{1}{z} \mapsto \frac{1}{z'} = \frac{1}{z} + a$, we have:

$$z \longmapsto z' = \frac{1}{\frac{1}{z} + \bar{a}} = \frac{z}{1 + z\bar{a}} = w(z), \quad \frac{\partial z}{\partial z'} = \frac{1}{(1 - z\bar{a})^2} = \frac{1}{\kappa^2}, \quad z_{ij} = \frac{z'_{ij}}{\kappa_i \kappa_j}, \tag{10}$$

$$f \longmapsto f\Big(w^{-1}(z_{12}), w^{-1}(z_{23}), w^{-1}(z_{31})\Big)\frac{1}{\kappa_1^{2h_1}\kappa_2^{2h_2}\kappa_3^{2h_3}} = f(z_{12}, z_{23}, z_{31}), \tag{11}$$

$$f_{abc}\Big(w^{-1}(z_{12}), w^{-1}(z_{23}), w^{-1}(z_{31})\Big) = f_{abc}(z_{12}, z_{23}, z_{31})\,\kappa_1^{c+a}\kappa_2^{a+b}\kappa_3^{b+c}, \tag{12}$$

We see that f is invariant under special conformal transformation iff. $f = f_{abc}$ where:

$$c + a = 2h_1, \quad a + b = 2h_2, \quad b + c = 2h_3, \tag{13}$$

$$\text{i.e.} \quad a = h_1 + h_2 - h_3, \quad b = h_2 + h_3 - h_1, \quad c = h_3 + h_1 - h_2, \tag{14}$$

In the above discussions we've restricted ϕ_i to be holomorphic; for *spin-less* $\phi_i = \phi_i(z, \bar{z})$, $h_i = \tilde{h}_i$, $\Delta_i = h_i + \tilde{h}_i$, the holomorphic and anti-holomorphic contributions can be nicely combined, and we have:

$$f = \frac{C}{|z_{12}|^{2a}|z_{23}|^{2b}|z_{31}|^{2c}}, \tag{15}$$

$$2a = \Delta_1 + \Delta_2 - \Delta_3, \quad 2b = \Delta_2 + \Delta_3 - \Delta_1, \quad 2c = \Delta_3 + \Delta_1 - \Delta_2, \tag{16}$$

$$\langle V_{k_1}(z_1, \bar{z}_1)\, V_{k_2}(z_2, \bar{z}_2)\, G_{e,k_3}(z_3, \bar{z}_3)\rangle = \frac{A(k_1, k_2, e)}{|z_{12}|^2|z_{23}|^2|z_{31}|^2} \tag{17}$$

(c) Following the recipe in (a), we have:

$$\begin{aligned} V_{k_1}(z_1, \bar{z}_1)\, V_{k_2}(z_2, \bar{z}_2) &= :\exp\left(ik_{1,\mu} ik_{2,\nu} \overbrace{X_1^\mu X_2^\nu}\right) e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: \\ &= \exp\left(\frac{\alpha'}{2}\, k_1 \cdot k_2 \ln|z_{12}|^2\right) :e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: \\ &= |z_{12}|^{\alpha' k_1\cdot k_2} :e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: \end{aligned} \tag{18}$$

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[2] Reference: Blumenhagen, *Introduction to CFT*, and also *Di Francesco et al.*

[3] See *Di Francesco et al*, and also github.com/davidsd/ph229.

Apply the on-shell conditions, and we find that:

$$\alpha' k_1 \cdot k_2 = \frac{\alpha'}{2}(k_1+k_2)^2 - \frac{\alpha'}{2}k_1^2 - \frac{\alpha'}{2}k_2^2 = \frac{\alpha'}{2}(-k_3)^2 - \frac{\alpha'}{2}k_1^2 - \frac{\alpha'}{2}k_2^2 = 0-2-2=-4 \tag{19}$$

We are interested in the $|z_{12}|^{-2}$ term in the OPE around z_2; it will contribute to the 3-point function discussed in (b). Note that:

$$\begin{aligned}
:e^{ik_1\cdot X_1}\, e^{ik_2\cdot X_2}: &= :\left(\cdots + \frac{1}{2}(ik_1\cdot X_1)^2 + \cdots\right) e^{ik_2\cdot X_2}: \\
&= :\left(\cdots - \frac{1}{2}\, k_1^\mu k_1^\nu \big(X_2 + z_{12}\,\partial X_2 + \bar z_{12}\,\bar\partial X_2 + \cdots\big)_\mu \big(\cdots\big)_\nu + \cdot\right) e^{ik_2\cdot X_2}: \\
&= :\left(\cdots - k_{1,\mu}k_{1,\nu}\left(z_{12}\bar z_{12}\,\partial X_2^\mu\,\bar\partial X_2^\nu\right) + \cdots\right) e^{ik_2\cdot X_2}: \\
&= \cdots - |z_{12}|^2\, k_{1,\mu}k_{1,\nu}\, G^{\mu\nu}_{k_2}(z_2,\bar z_2) + \cdots\,,
\end{aligned} \tag{20}$$

$$V_{k_1}(z_1,\bar z_1)\, V_{k_2}(z_2,\bar z_2) = \cdots + \frac{O_{k_1,k_2}(z_2,\bar z_2)}{|z_{12}|^2} + \cdots\,, \tag{21}$$

$$O_{k_1,k_2}(z_2,\bar z_2) = -k_{1,\rho}k_{1,\sigma}\, G^{\rho\sigma}_{k_2}(z_2,\bar z_2), \tag{22}$$

Consider the same limit: $z_1 \to z_2$ of the 3-point function, and we find that:

$$\begin{aligned}
z_1\to z_2,\quad \langle V_{k_1}(z_1,\bar z_1)\, V_{k_2}(z_2,\bar z_2)\, G_{e,k_3}(z_3,\bar z_3)\rangle &\to \frac{1}{|z_{12}|^2}\frac{A(k_1,k_2,e)}{|z_{23}|^4} \\
&\sim \frac{1}{|z_{12}|^2}\left\langle O_{k_1,k_2}(z_2,\bar z_2)\, G_{e,k_3}(z_3,\bar z_3)\right\rangle,
\end{aligned} \tag{23}$$

We see that in the $z_2 \to z_3$ limit, we should obtain:

$$O_{k_1,k_2}(z_2,\bar z_2)\, G_{e,k_3}(z_3,\bar z_3) = \cdots + \frac{A(k_1,k_2,e)}{|z_{23}|^4} + \cdots\,, \tag{24}$$

Note that $A(k_1,k_2,e) = A(k_1,k_2,e)\,\mathbb{1}$ is simply a number; therefore, when finding $A(k_1,k_2,e)$, it is safe to ignore all (non-identity) operator contributions, as they must cancel each other. Similar to (a), we have:

$$\begin{aligned}
G^{\rho\sigma}_{k_2}(z_2,\bar z_2)\, G^{\mu\nu}_{k_3}(z_3,\bar z_3) &= \cdots + :\partial X_2^\rho\,\bar\partial X_2^\sigma\, e^{ik_2\cdot X_2}\,\partial X_3^\mu\,\bar\partial X_3^\nu\, e^{ik_3\cdot X_3}: \\
&+ :\partial X_2^\rho\,\bar\partial X_2^\sigma\, e^{ik_2\cdot X_2}\,\partial X_3^\mu\,\bar\partial X_3^\nu\, e^{ik_3\cdot X_3}: \\
&+ :\partial X_2^\rho\,\bar\partial X_2^\sigma\, e^{ik_2\cdot X_2}\,\partial X_3^\mu\,\bar\partial X_3^\nu\, e^{ik_3\cdot X_3}: \\
&+ :\partial X_2^\rho\,\bar\partial X_2^\sigma\, e^{ik_2\cdot X_2}\,\partial X_3^\mu\,\bar\partial X_3^\nu\, e^{ik_3\cdot X_3}: + \cdots
\end{aligned} \tag{25}$$

$$\begin{aligned}
G^{\rho\sigma}_{k_2}(z_2,\bar z_2)\, G^{\mu\nu}_{k_3}(z_3,\bar z_3) \sim \cdots &+ \left(-\frac{\alpha'}{2}\eta^{\rho\mu}\frac{1}{z_{23}^2}\right)\left(-\frac{\alpha'}{2}\eta^{\sigma\nu}\frac{1}{\bar z_{23}^2}\right)\times 1 \\
&+ \left(-\frac{\alpha'}{2}\eta^{\rho\mu}\frac{1}{z_{23}^2}\right)\left(-\frac{\alpha'}{2}\frac{ik_3^\sigma}{\bar z_{23}}\right)\left(-\frac{\alpha'}{2}\frac{ik_2^\nu}{\bar z_{32}}\right) + \left(z\leftrightarrow\bar z,\ \rho\leftrightarrow\sigma,\ \mu\leftrightarrow\nu\right) \\
&+ \left(-\frac{\alpha'}{2}\frac{ik_2^\rho}{z_{23}}\right)\left(-\frac{\alpha'}{2}\frac{ik_2^\sigma}{\bar z_{23}}\right)\left(-\frac{\alpha'}{2}\frac{ik_3^\mu}{z_{32}}\right)\left(-\frac{\alpha'}{2}\frac{ik_3^\nu}{\bar z_{32}}\right) + \cdots
\end{aligned}$$

$$
\begin{aligned}
O_{k_1,k_2}(z_2,\bar{z}_2)\, G^{\mu\nu}_{k_3}(z_3,\bar{z}_3) \sim \cdots &- k_1^\mu k_1^\nu \left(\frac{\alpha'^2}{4}\right)\frac{1}{|z_{23}|^4} \\
&- i^2(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)\left(\frac{\alpha'}{2}(k_1\cdot k_3)\right)\left(\frac{\alpha'^2}{4}\right)\frac{1}{|z_{23}|^4} \\
&- i^4 k_3^\mu k_3^\nu \left(\frac{\alpha'}{2}(k_1\cdot k_2)\right)^2\left(\frac{\alpha'^2}{4}\right)\frac{1}{|z_{23}|^4} + \cdots
\end{aligned}
\tag{26}
$$

Again, apply the on-shell conditions, and we find that:

$$
\frac{\alpha'}{2}k_1\cdot k_2 = -2, \quad \frac{\alpha'}{2}k_1\cdot k_3 = -\frac{\alpha'}{2}k_1\cdot(k_1+k_2) = -\frac{\alpha'}{2}k_1^2 - \frac{\alpha'}{2}k_1\cdot k_2 = -2-(-2) = 0, \tag{27}
$$

$$
\begin{aligned}
A(k_1,k_2,e) &= -\frac{\alpha'^2}{4}\left(\cancel{4e_{\mu\nu}k_3^\mu k_3^\nu} + e_{\mu\nu}k_1^\mu k_1^\nu\right) = -\frac{\alpha'^2}{4}e_{\mu\nu}k_1^\mu k_1^\nu \\
&= -\frac{\alpha'^2}{4}e_{\mu\nu}(k_2+k_3)^\mu(k_2+k_3)^\nu = -\frac{\alpha'^2}{4}e_{\mu\nu}k_2^\mu k_2^\nu \\
&= -\frac{\alpha'^2}{8}e_{\mu\nu}\left(k_1^\mu k_1^\nu + k_2^\mu k_2^\nu\right) \\
&= -\frac{\alpha'^2}{8}e_{\mu\nu}\Big(k_{12}^\mu k_{12}^\nu + (k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)\Big),
\end{aligned}
\tag{28}
$$

On the other hand,

$$
0 = e_{\mu\nu}k_3^\mu k_3^\nu = e_{\mu\nu}(k_1+k_2)^\mu(k_1+k_2)^\nu = e_{\mu\nu}\Big(k_{12}^\mu k_{12}^\nu + 2(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)\Big) \tag{29}
$$

$$
A(k_1,k_2,e) = -\frac{\alpha'^2}{8}e_{\mu\nu}k_{12}^\mu k_{12}^\nu\left(1-\frac{1}{2}\right) = -\frac{\alpha'^2}{16}e_{\mu\nu}\,k_{12}^\mu k_{12}^\nu \tag{30}
$$

$\boxed{2}$ Strings Scattering Off a Heavy Particle:

A heavy particle can be modeled by some D0-brane with Neumann boundary condition in the X_0 direction[4]. The scattering of a closed string tachyon off the heavy particle can then be computed via a disc diagram with two insertions.

(a) The conformal Killing group (CKG) of the disc is $\mathrm{PSL}(2,\mathbb{R})$. It is a 3 dimensional $\mathbb{R}$ Lie group, generated by 3 conformal Killing vectors (CKV's); therefore, it is possible to partially fix the positions of the two insertions V_1, V_2. On the upper half plane, this can be implemented by putting z_1, z_2 on the imaginary axis, with z_2 fixed and z_1 integrated[5]:

$$
\mathcal{A} = g_c^2 e^{-\lambda}\int_0^{z_2}\mathrm{d}z_1\left\langle :c_1^x e^{ik_1\cdot X_1}: :c_2\tilde{c}_2\, e^{ik_2\cdot X_2}: \right\rangle, \quad z_2 = i, \quad z_1 = iy, \quad y\in[0,1] \tag{31}
$$

Here c^x comes from the CKV that brings $z_1 \to iy$. On the disc this can be taken to be a rotation around z_2; when mapped to the upper half plane and at around the imaginary axis, this is simply a translation along the $x = \frac{1}{2}(z+\bar{z})$ direction[6], i.e.

$$
\text{CKV: } \partial_x = \delta_x^a\,\partial_a \quad \Longrightarrow \quad \text{Ghost: } c^x, \tag{32}
$$

$$
c^x\partial_x + c^y\partial_y = c^z\partial_z + c^{\bar{z}}\partial_{\bar{z}}, \quad c^x = \frac{1}{2}\left(c^z + c^{\bar{z}}\right) = \frac{1}{2}\big(c(z)+\tilde{c}(\bar{z})\big), \tag{33}
$$

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4 Reference: arXiv:hep-th/9611214, arXiv:hep-th/9605168, and *Polchinski*.

5 Reference: arXiv:0812.4408. I would like to thank Lucy Smith for pointing this out.

6 Reference: *Polchinski*, Chapter 5 & 6.

The ghost contribution is then:

$$
\begin{aligned}
\left\langle c_1^x c_2 \tilde{c}_2 \right\rangle = \left\langle c^x(z_1)\, c(z_2)\, \tilde{c}(\bar{z}_2) \right\rangle &= \frac{1}{2}\Big(\left\langle c(z_1)\, c(z_2)\, \tilde{c}(\bar{z}_2) \right\rangle + \left\langle \tilde{c}(z_1)\, c(z_2)\, \tilde{c}(\bar{z}_2) \right\rangle \Big) \\
&= \frac{1}{2}\Big(\left\langle c(z_1)\, c(z_2)\, c(z_2') \right\rangle + \left\langle c(z_1')\, c(z_2)\, c(z_2') \right\rangle \Big), \quad z' = \bar{z}, \\
&= \frac{C^g_{D^2}}{2} \left(z_{12} z_{12'} z_{22'} + z_{1'2} z_{1'2'} z_{22'} \right), \quad z_1, z_2 \in i\mathbb{R}, \\
&= 2 C^g_{D^2} \left(z_1^2 - z_2^2 \right) z_2
\end{aligned}
\tag{34}
$$

On the other hand, the $e^{ik_j \cdot X_j}$ contributions is similar to what we've computed in [1], except that now we should be careful about the boundary conditions of X^μ, which affect the XX contraction in the formulae. For Neumann boundary condition: $\partial_y X^0 = 0$, the half-plane propagator from z' can be constructed with an image at $\bar{z}'$ with *the same charge*, i.e. we have:

$$
\overbracket{X_1^0 X_2^0} = -\frac{\alpha'}{2}\, \eta^{00} \ln |z_1 - z_2|^2 - \frac{\alpha'}{2}\, \eta^{00} \ln |z_1 - \bar{z}_2|^2 \tag{35}
$$

While for Dirichlet boundary $X^i = \text{const}$, we can always select the origin so that $X^i = 0$, and in this case the image should have *the opposite charge*, i.e.

$$
\overbracket{X_1^i X_2^j} = -\frac{\alpha'}{2}\, \delta^{ij} \ln |z_1 - z_2|^2 + \frac{\alpha'}{2}\, \delta^{ij} \ln |z_1 - \bar{z}_2|^2, \tag{36}
$$

$$
\begin{aligned}
\Longrightarrow \quad :e^{ik_1 \cdot X_1}::e^{ik_2 \cdot X_2}: &= \exp\left(ik_{1,\mu} ik_{2,\nu} \overbracket{X_1^\mu X_2^\nu} \right) :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}: \\
&= |z_{12}|^{\alpha' k_1 \cdot k_2} |z_{1\bar{2}}|^{\alpha' \left(-k_1^0 k_2^0 - \mathbf{k}_1 \cdot \mathbf{k}_2 \right)} :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}:
\end{aligned}
\tag{37}
$$

Before further calculations, we note that the normal ordering defined here on D^2 differs from that on the usual $\mathbb{C}^2$; in fact, there are also self-contractions with image charge[7]:

$$
\overbracket{X^\mu(z, \bar{z})\, X^\nu}(\bar{z}, z) = G_r^{\mu\nu}(z, \bar{z}) = \mp \frac{\alpha'}{2}\, \eta^{\mu\nu} \ln |z - \bar{z}|^2, \tag{38}
$$

$$
\Longrightarrow \quad \left\langle :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}: \right\rangle_{D^2} = \left\langle :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}: \right\rangle_{\mathbb{C}^2} \exp\left(\frac{1}{2} \sum_n ik_{n,\mu} ik_{n,\nu} \overbracket{X_n^\mu X_n^\nu} \right), \quad n = 1, 2 \tag{39}
$$

The "$\mp$" sign choice depends on the boundary condition.

Therefore,

$$
\begin{aligned}
\left\langle :e^{ik_1 \cdot X_1}::e^{ik_2 \cdot X_2}: \right\rangle_{D^2} &= \left\langle :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}: \right\rangle_{\mathbb{C}^2} \exp\left(ik_{1,\mu} ik_{2,\nu} \overbracket{X_1^\mu X_2^\nu} \right) \exp\left(\frac{1}{2} \sum_n ik_{n,\mu} ik_{n,\nu} \overbracket{X_n^\mu X_n^\nu} \right) \\
&= \left\langle :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}: \right\rangle_{\mathbb{C}^2} \exp\left(\frac{1}{2} \sum_{m,n} ik_{m,\mu} ik_{n,\nu} \overbracket{X_m^\mu X_n^\nu} \right) \\
&= \left\langle :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}: \right\rangle_{\mathbb{C}^2} |z_{12}|^{\alpha' k_1 \cdot k_2} |z_{1\bar{2}}|^{\alpha' \left(-k_1^0 k_2^0 - \mathbf{k}_1 \cdot \mathbf{k}_2 \right)} \prod_n |z_{n\bar{n}}|^{\frac{\alpha'}{2} \left(-(k_n^0)^2 - \mathbf{k}_n^2 \right)}
\end{aligned}
\tag{40}
$$

Note that X^i has no zero mode due to the Dirichlet boundary, hence $\int \mathscr{D} X$ gives a delta function in only the Neumann direction: $\delta\left(k_1^0 + k_2^0\right)$. Physically, this means that only the energy is conversed;

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[7] This is very much similar to the torus situation, where we also have to consider self-contractions with image charges. More rigorous discussion of G^r is given in *Polchinski*.

the momentum k^i is not conserved since the heavy D0-brane does not recoil. It is therefore convenient to define these on shell variables:

$$s=\omega^2=(k_1^0)^2=(k_2^0)^2,\quad t=-(\mathbf{k}_1+\mathbf{k}_2)^2=-\mathbf{k}_1^2-\mathbf{k}_2^2-2\mathbf{k}_1\cdot\mathbf{k}_2=2\left(-\omega^2-\mathbf{k}_1\cdot\mathbf{k}_2-\frac{4}{\alpha'}\right),\tag{41}$$

$$\mathbf{k_1}\cdot\mathbf{k_2}=-\frac{t}{2}-\omega^2-\frac{4}{\alpha'},\quad k_1\cdot k_2=-\omega(-\omega)+\mathbf{k_1}\cdot\mathbf{k_2}\tag{42}$$

Here we've used the on-shell condition: $m^2=-k^2=\omega^2-\mathbf{k}^2=-\frac{4}{\alpha'}$ for tachyons. The previous expressions can then be simplified to:

$$\begin{aligned}\left\langle :e^{ik_1\cdot X_1}::e^{ik_2\cdot X_2}:\right\rangle_{D^2}&=\left\langle :e^{ik_1\cdot X_1}\,e^{ik_2\cdot X_2}:\right\rangle_{\mathbb{C}^2}|z_{12}|^{-\frac{\alpha' t}{2}-4}|z_{1\bar{2}}|^{+\frac{\alpha' t}{2}+4+2\alpha'\omega^2}\prod_n|2z_n|^{-\alpha'\omega^2-2}\\&=iC_{D^2}^X\,2\pi\,\delta\big(k_1^0+k_2^0\big)\,|z_{12}|^{-\frac{\alpha' t}{2}-4}|z_{1\bar{2}}|^{+\frac{\alpha' t}{2}+4+2\alpha'\omega^2}\prod_n|2z_n|^{-\alpha'\omega^2-2}\\&=iC_{D^2}^X\,2\pi\,\delta\big(k_1^0+k_2^0\big)\,f\big(|z_{12}|,|z_{1\bar{2}}|,|z_1|,|z_2|\big),\end{aligned}\tag{43}$$

$$\begin{aligned}\mathcal{A}&=g_c^2\underline{e^{-\lambda}}\cdot i\underline{C_{D^2}^X}\,2\pi\,\delta\big(k_1^0+k_2^0\big)\cdot 2\underline{C_{D^2}^g}\int_0^{z_2}\mathrm{d}z_1\,\big(z_1^2-z_2^2\big)z_2\,f\big(|z_{12}|,|z_{1\bar{2}}|,|z_1|,|z_2|\big)\\&=g_c^2\underline{C_{D^2}}\,2\pi\,\delta\big(k_1^0+k_2^0\big)\cdot 2i\int_0^1 i\,\mathrm{d}y\,\big((iy)^2-i^2\big)\,i\cdot f\big(1-y,1+y,2y,2\big)\\&=-ig_c^2C_{D^2}\,2\pi\,\delta\big(k_1^0+k_2^0\big)\cdot 2\cdot 2^{-2\alpha'\omega^2-4}\int_0^1\mathrm{d}y\,(1-y^2)\,f\big(1-y,1+y,y,1\big),\end{aligned}\tag{44}$$

$$\begin{aligned}\int_0^1\mathrm{d}y\,(1-y^2)\,f\big(1-y,1+y,y,1\big)&=\int_0^1\mathrm{d}y\,(1-y)^{-\frac{\alpha' t}{2}-4+1}(1+y)^{+\frac{\alpha' t}{2}+4+2\alpha'\omega^2+1}y^{-\alpha'\omega^2-2}\\&=\int_0^1\mathrm{d}y\,y^{a-1}(1-y)^{2b-1}(1+y)^{-2a-2b+1},\quad y'=\frac{1-y}{1+y},\\&=-2^{1-2a}\int_0^1\mathrm{d}y'\,(-y')^{2b-1}(1-y'^2)^{a-1}\\&=2^{-2a}\int_0^1\mathrm{d}(y'^2)\,(y'^2)^{b-1}(1-y'^2)^{a-1}\\&=2^{-2a}B\left(a=-\alpha'\omega^2-1,\,b=-\frac{\alpha' t}{4}-1\right)\end{aligned}\tag{45}$$

Here $B(a,b)=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$ is the Euler Beta function.

Putting everything together, we obtain:

$$\begin{aligned}\mathcal{A}&=-ig_c^2C_{D^2}\,2\pi\,\delta\big(k_1^0+k_2^0\big)\cdot\frac{1}{2}\,B\left(-\alpha'\omega^2-1,\,-\frac{\alpha' t}{4}-1\right)\\&=-ig_c^2C_{D^2}\,\pi\,\delta\big(k_1^0+k_2^0\big)\,B\left(-\alpha'\omega^2-1,\,-\frac{\alpha' t}{4}-1\right)\end{aligned}\tag{46}$$

In fact $g_c^2C_{D^2}$ can be further computed by path integral or by comparing physical results. Here we settle for this generic coefficient since it's already enough for our following discussions[8].

[8] And I have run out of time and energy.

(b) The Regge limit is found by taking the high energy limit while keeping the momentum transfer fixed; in this case it is achieved by:

$$\text{Regge:}\quad s=\omega^2\to\infty,\quad t=-(\mathbf{k}_1+\mathbf{k}_2)^2\ \text{ fixed,} \tag{47}$$

$$\begin{aligned}\mathcal{A}\propto B\Big(a=-\alpha' s-1,\, b=-\frac{\alpha' t}{4}-1\Big)&=\frac{\Gamma(-\alpha' s-1)}{\Gamma\big(-\alpha' s-\frac{\alpha' t}{4}-2\big)}\,\Gamma\Big(-\frac{\alpha' t}{4}-1\Big)\\ &\sim\Big\{e\Big(\alpha' s+\frac{\alpha' t}{4}+3\Big)\Big\}^{\frac{\alpha' t}{4}+1}\,\Gamma\Big(-\frac{\alpha' t}{4}-1\Big)\\ &\sim\big(e\alpha'\omega^2\big)^{\frac{\alpha' t}{4}+1}\,\Gamma\Big(-\frac{\alpha' t}{4}-1\Big)\\ &\sim\big(\omega^2\big)^{\frac{\alpha' t}{4}+1}\,\Gamma\Big(-\frac{\alpha' t}{4}-1\Big)\end{aligned} \tag{48}$$

Here we've used the Stirling's approximation[9]: $\ln\Gamma(z+1)=\ln z!\sim z\ln z-z$. On the other hand, the hard scattering limit is found by keeping the scattering angle fixed, i.e.

$$\text{Hard scattering:}\quad s=\omega^2\to\infty,\quad (t/s)\equiv\lambda\ \text{ fixed,} \tag{49}$$

$$\mathcal{A}\propto B(a,b)=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\sim\exp\Big\{-\alpha'\Big(s\ln(\alpha' s)+\tfrac{t}{4}\ln\tfrac{\alpha' t}{4}+\tfrac{u}{4}\ln\tfrac{\alpha' u}{4}\Big)\Big\}, \tag{50}$$

$$s=\omega^2=(k_1^0)^2=(k_2^0)^2,\quad t=-(\mathbf{k}_1+\mathbf{k}_2)^2,\quad u=-(\mathbf{k}_1-\mathbf{k}_2)^2, \tag{51}$$

$$s+\frac{t}{4}+\frac{u}{4}=-\frac{4}{\alpha'}, \tag{52}$$

Here we've introduced an additional u variable, and we see that the result is symmetric under $t\leftrightarrow u$. We find that under the above limits, the amplitude exhibits similar behaviors as the Veneziano amplitude.

(c) Note that $\Gamma(z)$ has no zeros on $\mathbb{C}^2$, therefore the poles of $B(a,b)=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$ is given by the poles of $\Gamma(a)$ and $\Gamma(b)$:

$$a=-\alpha' s-1=0,-1,-2,\cdots,\qquad b=-\frac{\alpha' t}{4}-1=0,-1,-2,\cdots, \tag{53}$$

The first set of poles gives:

$$\omega^2=-\frac{1}{\alpha'},0,\frac{1}{\alpha'},\frac{2}{\alpha'},\cdots \tag{54}$$

This is precisely the open string spectrum in $D=26$. Going back to (43)$\sim$(45), we see that they come from the singularities as $y=|z_1|\to 0$ or $|z_{12}|\to 1$, i.e. the two insertions are far apart.

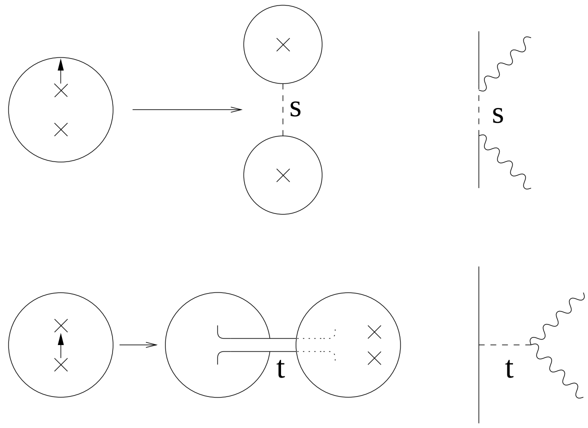
This is the s-channel contribution; physically, it means that the closed string is first absorbed by the D0-brane, then it propagates as an intermediate open string on the brane, and is re-emitted in the end.

[9] For the validity of Stirling's approximation when $z\in\mathbb{C}$, $\arg z=\pi-\epsilon$ and $|z|\to\infty$, see `Wikipedia:` *Stirling's formula for the gamma function*.

The other set of poles gives:

$$-t = (\mathbf{k}_1 + \mathbf{k}_2)^2 = \frac{4}{\alpha'}, 0, \cancel{-\frac{4}{\alpha'}, -\frac{8}{\alpha'}}, \cdots \tag{55}$$

Only two poles are realistic due to $\mathbf{k}^2 \geq 0$. This is the t-channel contribution, which corresponds to a closed string exchange between the D0-brane and the incoming particle. The above factorization channels are nicely illustrated in the diagram below, borrowed from arXiv:hep-th/9611214.



$\boxed{3}$ Compton Scattering Between $U(N)$ Gluons and Adjoint Tachyons:

Such scattering is captured by a disc diagram with $AATT$ insertions at the boundary; following the same recipe as before, we write down[10]:

$$\begin{aligned}\mathcal{A} = g_o'^2 g_o^2 e^{-\lambda} \int \mathrm{d}y \left\langle :e_\mu \partial_y X^\mu\, e^{ik\cdot X}(y): :c_1^y\, e_{1,\nu}\partial_y X_1^\nu\, e^{ik_1\cdot X_1}: :c_2^y e^{ik_2\cdot X_2}: :c_3^y e^{ik_2\cdot X_2}: \right\rangle \\ \times \operatorname{Tr}\left(\lambda^a \lambda^{a_1}\lambda^{a_2}\lambda^{a_3} : \text{in } (y, y_1, y_2, y_3) \text{ order}\right) \\ + (2 \leftrightarrow 3)\end{aligned} \tag{56}$$

Where $y_i = y_{1,2,3}$ are fixed while $y_0 = y$ is integrated. Note that for $\boxed{2}$ we use (x, y) to label the points on the upper half plane so that the boundary coordinate is $x = (x, 0)$; here in order to match with *Polchinski*, we label the boundary points with $y = (y, 0)$ instead.

We see that $\mathcal{A}$ is structurally similar to the 4-point Veneziano amplitude $\mathcal{A}_{TTTT}$ and also the 3-point amplitude $\mathcal{A}_{TTA}$; by analogy we can evaluate $\mathcal{A}$ in a similar fashion[11]:

$$\begin{aligned}\mathcal{A} = g_o'^2 g_o^2 e^{-\lambda} \prod_{i<j} |y_{ij}|^{2\alpha' k_i\cdot k_j + 1}\, e_\mu e_{1,\nu} \\ \times \int \mathrm{d}y \prod_j |y_{0j}|^{2\alpha' k\cdot k_j} \left\{ -2\alpha' \frac{\eta^{\mu\nu}}{y_{01}^2} + (-2\alpha')\left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}}\right)(-2\alpha')\left(\frac{ik^\nu}{y_{10}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}}\right)\right\} \\ \times \operatorname{Tr}(\cdots)\, iC_{D_2}^X C_{D_2}^g\, (2\pi)^{26}\delta^{26}\left(\textstyle\sum_i k_i\right) \\ + (2 \leftrightarrow 3)\end{aligned} \tag{57}$$

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[10] I would like to thank 谷夏 for help with this problem.

[11] For the ∂X contribution, see *Polchinski*, arXiv:1403.4553 and arXiv:1507.02172. Note that there are contractions between two ∂X fields.

Notice the doubling due to the boundary contraction $\overbracket{X_1^\mu X_2^\nu} = 2\times\left(-\frac{\alpha'}{2}\right)\ln y_{12}^2 = -2\alpha'\ln y_{12}$. Further simplifications can be achieved by using the on-shell and physical conditions for open string states:

$$k^2 = k_1^2 = 0, \quad e\cdot k = e_1\cdot k_1 = 0, \quad k_{2,3}^2 = \frac{1}{\alpha'} = -m^2, \tag{58}$$

$$\begin{aligned} s &= -\alpha'(k+k_1)^2 = -2\alpha' k\cdot k_1, \\ t &= -\alpha'(k+k_2)^2 = -2\alpha' k\cdot k_2 - 1, \\ u &= -\alpha'(k+k_3)^2 = -2\alpha' k\cdot k_3 - 1, \end{aligned} \tag{59}$$

$$s+t+u = -2, \tag{60}$$

$$2\alpha' k_1\cdot k_2 = \alpha'(k_1+k_2)^2 - \alpha' k_1^2 - \alpha' k_2^2 = \alpha'(k+k_3)^2 - 1 = -u-1 = 2\alpha' k\cdot k_3, \tag{61}$$

$$2\alpha' k_1\cdot k_3 = -t-1 = 2\alpha' k\cdot k_2, \tag{62}$$

$$2\alpha' k_2\cdot k_3 = \alpha'(k+k_1)^2 - 2 = -s-2, \tag{63}$$

$$\begin{aligned} \mathcal{A} &= i g_o'^2 g_o^2 C_{D^2}\, |y_{12}|^{-u}\, |y_{13}|^{-t}\, |y_{23}|^{-s-2}\, e_\mu e_{1,\nu} \\ &\quad \times \int \mathrm{d}y\, |y_{01}|^{-s} |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left\{ \frac{\eta^{\mu\nu}}{y_{01}^2} - 2\alpha' \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) \left(-\frac{ik^\nu}{y_{01}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\} \\ &\qquad \times \mathrm{Tr}\,(\cdots)\, (-2\alpha')(2\pi)^{26}\delta^{26}\left(\textstyle\sum_i k_i\right) \\ &\quad + \left(2 \leftrightarrow 3\right) \end{aligned} \tag{64}$$

Fix $y_1 \to \infty, y_2 = 0, y_3 = 1$, and we get:

$$\begin{aligned} \mathcal{A} &= i g_o'^2 g_o^2 C_{D^2}\, |y_1|^{-u-t}\, e_\mu e_{1,\nu} \\ &\quad \times \int \mathrm{d}y\, |y_{01}|^{-s} |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left\{ \frac{\eta^{\mu\nu}}{y_{01}^2} - 2\alpha' \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) \left(-\frac{ik^\nu}{y_{01}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\} \\ &\qquad \times \mathrm{Tr}\,(\cdots)\, \cdots + \cdots \end{aligned} \tag{65}$$

$$\begin{aligned} \mathcal{A} &= i g_o'^2 g_o^2 C_{D^2} \\ &\qquad \times \int \mathrm{d}y\, |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left| \frac{y_1^{s+2}}{y_{01}^s} \right| \left\{ \frac{e\cdot e_1}{y_{01}^2} - 2\alpha' e_\mu \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) e_{1,\nu} \left(-\frac{ik^\nu}{y_{01}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\} \\ &\qquad \times \mathrm{Tr}\,(\cdots)\, \cdots + \cdots \\ &= i g_o'^2 g_o^2 C_{D^2} \int \mathrm{d}y \left\{ (e\cdot e_1)\, f(t,u) - 2\alpha' g(t,u) \right\} \times \cdots + \cdots \end{aligned} \tag{66}$$

Note that the definition of $f(t,u)$ and $g(t,u)$ contains the $\mathrm{Tr}\,(\cdots)$ factor.

The limit $y_1 \to \infty$ should be treated with care. In fact, the integral along the boundary splits into three ranges[12]:

$$\int \mathrm{d}y = \left\{ \int_{y_1\to-\infty}^{y_3=0} + \int_{y_2=0}^{y_3=1} + \int_{y_3=1}^{y_1\to+\infty} \right\} \mathrm{d}y \tag{67}$$

[12] Reference: *Polchinski*'s discussion of $\mathcal{A}_{TTTT}$.

Notice the difference of $y_1 \to \pm\infty$; this is due to the S^1 topology of the boundary. Therefore,

$$\int \mathrm{d}y\, f(t,u) = \int \mathrm{d}y\, |y_{02}|^{-t-1}|y_{03}|^{-u-1}\left|\frac{y_1}{y_{01}}\right|^{s+2}\mathrm{Tr}\,(\cdots), \tag{68}$$

$$\mathrm{Tr}\left(\lambda^a\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\right) \equiv T_{0123}, \tag{69}$$

$$\begin{aligned}
\int_{y_1\to-\infty}^{y_2=0}\mathrm{d}y\, f(t,u) &= \int_{y_1\to-\infty}^{0}\mathrm{d}y\,(1-y)^{-t-1}(-y)^{-u-1}\left(\frac{-y_1}{y-y_1}\right)^{s+2}T_{1023} = T_{1023}B(-u,-s-1),\\
\int_{y_3=1}^{y_1\to+\infty}\mathrm{d}y\, f(t,u) &= \int_{1}^{y_1\to+\infty}\mathrm{d}y\,(y-1)^{-t-1}y^{-u-1}\left(\frac{y_1}{y_1-y}\right)^{s+2}T_{1230} = T_{1230}B(-t,-s-1),\\
\int_{y_3=0}^{y_2=1}\mathrm{d}y\, f(t,u) &= \int_0^1\mathrm{d}y\,(1-y)^{-t-1}y^{-u-1}t_{1203} = T_{1203}B(-t,-u),
\end{aligned} \tag{70}$$

$$\int \mathrm{d}y\, f(t,u) = T_{1203}B(-t,-u) + T_{1230}B(-t,-s-1) + T_{1023}B(-u,-s-1) \tag{71}$$

Under $2 \leftrightarrow 3$, we have $t \leftrightarrow u$; therefore,

$$\begin{aligned}
\int \mathrm{d}y\, f(t,u) + \big(2\leftrightarrow 3\big) &= \left(T_{1203}+T_{1302}\right)B(-t,-u)\\
&\quad + \left(T_{1230}+T_{1032}\right)B(-t,-s-1)\\
&\quad + \left(T_{1023}+T_{1320}\right)B(-u,-s-1)\\
&\equiv \alpha B(-t,-u) + \beta B(-t,-s-1) + \gamma B(-u,-s-1)
\end{aligned} \tag{72}$$

The $g(t,u)$ factor can be computed in a similar manner. In the end we have[13]:

$$\begin{aligned}
\mathcal{A} &= i g_o'^2 g_o^2 C_{D^2}\,(2\pi)^{26}\delta^{26}\big(\textstyle\sum_i k_i\big)\\
&\quad\times(-2\alpha')\Big\{(e_0\cdot e_1)\big(\alpha B(-t,-u)+\beta B(-t,-s-1)+\gamma B(-u,-s-1)\big)\\
&\qquad + (-2\alpha')(e_0\cdot k_2)(e_1\cdot k_3)\big(\alpha B(-t,-u)-\beta B(-t,-s-1)-\gamma B(-u,-s-1)\big) + \big(2\leftrightarrow 3\big)\Big\}
\end{aligned} \tag{73}$$

[13] Reference: arXiv:0801.3358.