

1 BRST Symmetry

The BRST transformation of c^a ghost is:

$$\delta c^a = \frac{1}{2} f_{bc}^a c^b c^c \Lambda, \quad \delta c = \delta c^a T_a = \frac{1}{2} [c, c\Lambda] \quad (1)$$

$D_\mu = \partial_\mu + A_\mu^a T_a$, T_a acts on c^a by adjoint representation: $(T_a)^c_b c^b = f_{ab}^c c^b$, i.e.

$$T_a \cdot c = (T_a)^c_b c^b T_c = f_{ab}^c T_c c^b = [T_a, T_b] c^b = [T_a, c], \quad (2)$$

$$D_\mu c = \partial_\mu c + [A_\mu, c] = [D_\mu, c], \quad (3)$$

$$D_\mu \delta c = \partial_\mu \delta c + [A_\mu, \delta c], \quad (4)$$

$$\begin{aligned} (D_\mu \delta c)^a &= \partial_\mu \delta c^a + [A_\mu, \delta c]^a \\ &= \partial_\mu \delta c^a + A_\mu^c [T_c, T_b]^a \delta c^b \\ &= \partial_\mu \delta c^a + A_\mu^c f_{cb}^a \delta c^b \\ &= \partial_\mu \delta c^a + A_\mu^c (T_c)^a_b \delta c^b \\ &= D_\mu (\delta c^a), \end{aligned} \quad (5)$$

$$\text{i.e. } (D_\mu \delta c)^a - \frac{1}{2} D_\mu (f_{bc}^a c^b c^c \Lambda) = (D_\mu \delta c)^a + \frac{1}{2} D_\mu (f_{bc}^a c^b \Lambda c^c) = 0 \quad (6)$$

2 Relativistic Particle

$$L_q = L + L_{gf} + L_{gh}, \quad (7)$$

$$L = \frac{1}{2e} \left(\frac{1}{c_0} \frac{dX}{dt} \right)^2 - \frac{e}{2} m^2 c_0^4, \quad (8)$$

$$L_{gh} = -e \dot{b} c \quad (9)$$

- For $t \mapsto t' = t - \xi(t)$, we have gauge transformation: $\delta X^\mu = \xi \dot{X}^\mu$, $\delta e = \frac{d}{dt}(e\xi)$, $\delta L = \frac{d}{dt}(\xi L)$, replace $\xi \mapsto c\Lambda$, and we have BRST transformation:

$$\delta X^\mu = c\Lambda \dot{X}^\mu = c\dot{X}^\mu \Lambda, \quad \delta e = \frac{d}{dt}(ec\Lambda) = \frac{d}{dt}(ec)\Lambda \quad (10)$$

- Assume nilpotency, and we have:

$$\begin{aligned} 0 &= \delta_\Lambda \delta_{\Lambda'} X^\mu = ((\delta_\Lambda c) \dot{X}^\mu + c \delta_\Lambda \dot{X}^\mu) \Lambda' \\ &= ((\delta_\Lambda c) \dot{X}^\mu + c (\dot{c} \dot{X}^\mu + c \ddot{X}^\mu) \Lambda) \Lambda' \\ &= (\delta_\Lambda c + c \dot{c} \Lambda) \dot{X}^\mu \Lambda', \end{aligned} \quad (11)$$

$$\boxed{\delta_\Lambda c = -c \dot{c} \Lambda} \quad (12)$$

$$\begin{aligned} \delta_\Lambda \delta_{\Lambda'} e &= \frac{d}{dt} ((\delta_\Lambda e) c + e \delta_\Lambda c) \Lambda' \\ &= \frac{d}{dt} \left(\frac{d}{dt} (ec) \Lambda c - e c \dot{c} \Lambda \right) \Lambda' \\ &= \frac{d}{dt} (e \dot{c} \Lambda c - e c \dot{c} \Lambda) \Lambda' = 0 \end{aligned} \quad (13)$$

- The BRST transformation for c is also nilpotent:

$$\begin{aligned}\delta_\Lambda \delta_{\Lambda'} c &= -((\delta_\Lambda c) \dot{c} + c \delta_\Lambda \dot{c}) \Lambda' \\ &= -(-c \dot{c} \Lambda \dot{c} - c c \dot{c} \Lambda) \Lambda' = 0\end{aligned}\quad (14)$$

- Gauge fixing $f = e(t) - 1 = 0$ can be imposed by:

$$\delta[f] \sim \int \mathcal{D}d(t) \exp\left(i \int dt d(t) f(t)\right), \quad (15)$$

$$L_{gf} = d(t)f(t) = d(t)(e(t) - 1) \quad (16)$$

The quantum action is $S_q = \int dt L_q$, $L_q = L[X, e] + L_{gf}[e, d] + L_{gh}[e, b, c]$. We want S_q to be BRST invariant, which will help determine transformation rules for b, d ; consider:

$$\begin{aligned}\delta(L_{gf} + L_{gh}) &= (e - 1) \delta d + d \delta e - \delta(e \dot{b} c) \\ &= (e - 1) \delta d + d(t) \frac{d}{dt}(ec) \Lambda + (ec) \delta \dot{b} \\ &= (e - 1) \delta d + \frac{d}{dt}(ec) (d(t) \Lambda - \delta b) + \frac{d}{dt}(ec \delta b)\end{aligned}\quad (17)$$

We find a natural choice of $\delta b, \delta d$:

$$\delta d = 0, \quad \delta b = d(t) \Lambda, \quad \delta(L_{gf} + L_{gh}) = \frac{d}{dt}(ec \delta b) \quad (18)$$

- The complete quantum action is BRST invariant, since:

$$\delta L = \frac{d}{dt}(\xi L)_{\xi \mapsto c\Lambda} = \frac{d}{dt}(cL) \Lambda, \quad \delta(L_{gf} + L_{gh}) = \frac{d}{dt}(ec \delta b), \quad (19)$$

$$\delta S_q = \int dt \delta L_q = \int dt (\delta L + \delta(L_{gf} + L_{gh})) = 0 \quad (20)$$

- Note that $\frac{\delta S_q}{\delta d} = 0 \Rightarrow f = e - 1 = 0$. Moreover,

$$\frac{\delta S_q}{\delta e} = 0 \implies -\frac{1}{2} \left(\frac{1}{e^2 c_0^2} \dot{X}^2 + m^2 c_0^4 \right) + d - \dot{b} c = 0, \quad (21)$$

$$d = d[X, b, c] = \frac{1}{2} \left(\frac{1}{c_0^2} \dot{X}^2 + m^2 c_0^4 \right) + \dot{b} c \quad (22)$$

Therefore, it is convenient to consider the reduced Lagrangian $L_q[X, b, c] = (L_q)_{e=1, d=d[X]}$, where e, d are integrated out¹. The symmetries are thus reduced to:

$$\delta X^\mu = c \dot{X}^\mu \Lambda, \quad \delta b = d[X, b, c] \Lambda, \quad \delta c = -c \dot{c} \Lambda, \quad (23)$$

$$\delta L_q = \frac{d}{dt} \left(c L \Lambda + ec \delta b \right)_{e=1} = \frac{d}{dt} \left(c L_{e=1} + \frac{c}{2} \left(\frac{1}{c_0^2} \dot{X}^2 + m^2 c_0^4 \right) \right) \Lambda = \frac{d}{dt} \left(c \left(\frac{1}{c_0} \dot{X} \right)^2 \right) \Lambda, \quad (24)$$

$$L_q = \frac{1}{2} \left(\frac{1}{c_0} \dot{X} \right)^2 - \frac{1}{2} m^2 c_0^4 - \dot{b} c \quad (25)$$

On the other hand, the *on-shell* variation is given by:

$$\delta_0 L_q = \frac{d}{dt} \left(\frac{\partial L_q}{\partial \dot{X}^\mu} \delta X^\mu + \frac{\partial L_q}{\partial \dot{b}} \delta b \right) = \frac{d}{dt} \left\{ c \left(\left(\frac{1}{c_0} \dot{X} \right)^2 + d[X, b, c] \right) \right\} \Lambda, \quad (26)$$

¹Reference: *Polchinski*.

The $\dot{b}c$ term in $d[X, b, c]$ is killed by the c multiplication: $c d[X, b, c] = c d[X]$. Therefore, the canonical BRST charge Q is given by:

$$0 = \delta_0 L_q - \delta L_q = \frac{dQ}{dt} \Lambda = \frac{d}{dt} (c d[X]) \Lambda, \quad (27)$$

$$Q = c d[X] = \frac{c}{2} \left(\frac{1}{c_0^2} \dot{X}^2 + m^2 c_0^4 \right) \quad (28)$$

- Note that $L_{gh} = -\dot{b}c = b\dot{c} - \frac{d}{dt}(bc) \sim b\dot{c}$; for future convenience, let's replace $(-\dot{b}c) \mapsto b\dot{c}$ in the Lagrangian, and we have:

$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{1}{c_0^2} \dot{X}_\mu, \quad p_c = \frac{\partial L}{\partial \dot{c}} \equiv (b\dot{c}) \frac{\overleftarrow{\partial}}{\partial \dot{c}} = b \quad (29)$$

Here we adopt the “right” derivative convention, in this case the Hamiltonian:

$$H = p_\mu \dot{X}^\mu + p_c \dot{c} - L_q = \frac{c_0^2}{2} p_\mu p^\mu + \frac{1}{2} m^2 c_0^4 = \frac{1}{2} (p^2 c_0^2 + m^2 c_0^4) \quad (30)$$

- We have:

$$Q = \frac{c}{2} \left(\frac{1}{c_0^2} \dot{X}^2 + m^2 c_0^4 \right) = \frac{c}{2} (p^2 c_0^2 + m^2 c_0^4) = cH \quad (31)$$

After canonical quantization, p_μ, p_c and H are promoted to Hermitian operators, and:

$$Q^2 = cH \cdot cH = 0 \quad (32)$$

- Note that:

$$[p_\mu, X^\nu] = -i\delta_\mu^\nu, \quad [p_\mu, \mathcal{F}(X)] = -i\partial_\mu \mathcal{F}(X), \quad (33)$$

i.e. p_μ acts on $\mathcal{F}(X)$ by X -derivative; from the path integral perspective, we have:

$$\langle p_\mu \mathcal{F}(X) \rangle = \int \mathcal{D}p \mathcal{D}X \mathcal{D}b \mathcal{D}c e^{iS[p, X, b, c]} p_\mu \mathcal{F}(X), \quad (34)$$

$$S[p, X, b, c] = \int dt \left(p_\mu \dot{X}^\mu + \dot{b}c - H[p] \right), \quad (35)$$

$$\int dt' \frac{\delta S}{\delta X^\mu(t')} = \int dt' \int dt p_\mu \partial_t \delta(t - t') \sim - \int dt' \dot{p}_\mu(t') = -p_\mu, \quad (36)$$

$$\langle p_\mu \mathcal{F}(X) \rangle = \int dt' \int \mathcal{D}p \mathcal{D}X \mathcal{D}b \mathcal{D}c \left(i \frac{\delta}{\delta X^\mu(t')} e^{iS[p, X, b, c]} \right) \mathcal{F}(X) \quad (37)$$

$$= \int \mathcal{D}p \mathcal{D}X \mathcal{D}b \mathcal{D}c e^{iS[p, X, b, c]} \int dt' \left(-i \frac{\delta}{\delta X^\mu(t')} \right) \mathcal{F}(X),$$

$$p_\mu \mathcal{F}(X) \sim \int dt' \left(-i \frac{\delta}{\delta X^\mu(t')} \right) \mathcal{F}(X) = -i \frac{\partial}{\partial X^\mu} \mathcal{F}(X) \quad (38)$$

For $e^{ik_\mu X^\mu} |0\rangle$, $\mathcal{F}(X) = e^{ik_\mu X^\mu}$, it is Q -closed iff.

$$0 = Q e^{ik_\mu X^\mu} |0\rangle = \frac{c}{2} (p^2 c_0^2 + m^2 c_0^4) e^{ik_\mu X^\mu} |0\rangle$$

$$= \frac{c}{2} (k^2 c_0^2 + m^2 c_0^4) (e^{ik_\mu X^\mu} |0\rangle), \quad (39)$$

$$k^2 c_0^2 = k_\mu k^\mu c_0^2 = -E^2 + \mathbf{k}^2 c_0^2 = -m^2 c_0^4, \quad (40)$$

Or $E^2 = \mathbf{k}^2 c_0^2 + m^2 c_0^4$. This is the dispersion relation of a relativistic particle.