# Gravity №1



*Bryan*

Compiled @ 2021/11/28

## 1 Spacelike Geodesic for Poincaré AdS

The Poincaré $\mathrm{AdS}_{d+1}$ metric is given by:

$$\mathrm{d}s^2 = G_{IJ}\,\mathrm{d}X^I\,\mathrm{d}X^J = \frac{-\,\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 + \mathrm{d}z^2}{z^2} = \frac{\mathrm{d}x^2 + \mathrm{d}z^2}{z^2}, \tag{1.1}$$

$$\mathrm{d}x^2 = -\,\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,, \quad X^I \sim (t, \vec{x}, z) \sim (x^\mu, z), \tag{1.2}$$

Note that here we define $\mathrm{d}x^2$ using the flat metric $\eta$, while $\mathrm{d}X$ has an index that should be raised and lowered using the curved metric $G_{IJ} = \frac{1}{z^2}\,\eta_{IJ}$. Generally upper case tensors are handled with $G_{IJ}$, while lower case ones are handled with $\eta$. The full isometry of such spacetime is then given by $\mathrm{SO}(d,2)$ with $\frac{(d+2)(d+1)}{2}$ generators, including:

1. Among the $x^\mu \sim (x^0, \vec{x}) \equiv (t, \vec{x})$ directions:
   (a) $\frac{d(d-1)}{2}$ $\mathrm{SO}(d-1,1)$ rotations $x_\mu\partial_\nu - x_\nu\partial_\mu$, boosts included;
   (b) $d$ translations $\partial_\mu$;
2. Dilation with $X^I \sim (x^\mu, z) \mapsto \lambda X^I = \lambda\,(x^\mu, z)$, generated by $\Delta = X^I\partial_I = z\partial_z + x^\mu\partial_\mu$;
3. Special conformal transformations; see Appendix A for an intuitive derivation. We have:

$$k_\mu = (z^2 + x^2)\,\partial_\mu - 2x_\mu\Delta \tag{1.3}$$

By Noether's theorem, $Q_\Xi = V_I\Xi^I = G_{IJ}\Xi^I V^J$ is conserved along the geodesic; here $V^I$ is the normalized tangent vector, while $\Xi^I$ is some Killing vector of the spacetime, e.g. one from the list above. We can then write down the conserved charges along a geodesic $\gamma$ in Poincaré AdS:

$$\begin{aligned} p_\mu &= G_{\mu I}V^I = V_\mu \\ m_{\mu\nu} &= x_\mu V_\nu - x_\nu V_\mu \\ \Delta &= X^I V_I = \frac{zV^z}{z^2} + x^\mu p_\mu \\ k_\mu &= (z^2 + x^2)\,p_\mu - 2x_\mu\Delta \end{aligned} \tag{1.4}$$

These are all integration constants along $\gamma$. Again note that $X, V$ have indices that should be handled with $G_{IJ}$, in particular,

$$\mathrm{d}X_\mu = G_{\mu I}\,\mathrm{d}X^I = \frac{1}{z^2}\,\mathrm{d}x_\mu\,, \quad V^\mu = \frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda} = z^2 p^\mu \tag{1.5}$$

On the other hand, $V^\mu$ should be properly normalized, therefore:

$$\|V\|^2 = g_{IJ}V^I V^J = \frac{(V^z)^2}{z^2} + z^2 p^2 = \delta = 0, \pm 1, \tag{1.6}$$

For $p^2 = 0$, we have $\|V\|^2 = \frac{(V^z)^2}{z^2} \geq 0$, thus $\gamma$ can be either spacelike or null, but not timelike. In fact, we have $\frac{V^z}{z} = \frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}\lambda} = 0, \pm 1$; along with $\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = z^2 p^\mu$, we obtain:

$$p^2 = 0, \qquad \begin{aligned} \text{spacelike:}\ & z(\lambda) = z(0)\,e^{\pm\lambda}, & x^\mu(\lambda) &= x^\mu(0) \pm z(0)^2\,p^\mu\,\tfrac{e^{\pm 2\lambda}-1}{2}, \\ \text{null:}\ & z(\lambda) = z(0), & x^\mu(\lambda) &= x^\mu(0) + z(0)^2\,p^\mu\lambda, \end{aligned} \tag{1.7}$$

From now on we shall focus on the $p^2 \neq 0$ situation. Note that we can complete the square on the right-hand side of the $k^\mu$ conservation such that:

$$k_\mu + p_\mu a^2 = p_\mu\big(z^2 + (x-a)^2\big), \quad \Delta = p_\mu a^\mu \tag{1.8}$$

For $p^\mu \neq 0$, we can always find some $c_\mu$ such that $c_\mu p^\mu \neq 0$, therefore we have:

$$z^2 + (x-a)^2 = \frac{c_\mu k^\mu}{c_\mu p^\mu} + a^2 = \left(\frac{L}{2}\right)^2 = \text{const.}, \quad \Delta = p_\mu a^\mu \tag{1.9}$$

We see that $\gamma$ lands on a "sphere" centered at $z = 0, x = a$ in $\mathbb{R}^{d+1}$; with the Lorentzian metric, it is actually a hyperboloid. Note that for now the radius $\left(\frac{L}{2}\right)^2$ can actually be negative or zero; more specifically,

1. $L^2 > 0$: one-sheet hyperboloid;
2. $L^2 < 0$: two-sheet hyperboloid;
3. $L^2 = 0$: conic surface.

On the other hand, we can actually solve $X^I$ completely by combining (1.5), (1.6); we have:

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = z^2 p^\mu, \quad \frac{\mathrm{d}z}{\mathrm{d}\lambda} = \pm z\sqrt{\delta - z^2p^2}, \quad \frac{\mathrm{d}x^\mu}{\mathrm{d}z} = \pm\frac{zp^\mu}{\sqrt{\delta - z^2p^2}}, \quad \delta - z^2p^2 = \left(\tfrac{V^z}{z}\right)^2 \geq 0 \tag{1.10}$$

$$\implies \quad \gamma \subset x^\mu = a^\mu \pm \frac{p^\mu}{p^2}\sqrt{\delta - z^2p^2}, \quad z^2 + (x-a)^2 = \frac{\delta}{p^2} = \left(\frac{L}{2}\right)^2 \tag{1.11}$$

This confirms our observation above, and further reveals that $\gamma$ lies in a "plane" in the $\mathbb{R}^d$ subspace consisting of the $x^\mu$ coordinates. The behavior of $\gamma$ is sensitive to the sign of $\delta$ and $p^2$; more specifically,

1. $\delta = -1$, i.e. for timelike $\gamma$, we must have $p^2 < 0$, and $z \geq \frac{1}{|\|p\||} = \frac{L}{2} > 0$, namely $z$ is bounded from below. This means that timelike geodesic can never reach the asymptotic boundary $z \to 0$. In this case, $\gamma$ is a section of the one-sheet hyperboloid.
2. $\delta = +1$, i.e. for spacelike $\gamma$, we can have $p^2 > 0$ or $p^2 < 0$.
   (a) For $p^2 > 0$, again we have $\frac{L}{2} = \frac{1}{\|p\|} > 0$, and $\gamma$ is again a cross-section of the one-sheet hyperboloid. However, now we have $z \leq \frac{L}{2}$, namely $z$ is bounded from above, and:

$$z \to 0, \quad x^\mu = a^\mu \pm \frac{p^\mu}{p^2} = a^\mu \pm \frac{L}{2}\,\widehat{p}^\mu, \quad \widehat{p}^\mu = \frac{p^\mu}{\|p\|} \tag{1.12}$$

   We can also nicely parametrize $X^I$ in terms of the proper length $\lambda$; for convenience, set $x^\mu(0) = a^\mu$, $z(0) = \frac{L}{2}$, then we have:

$$z(\lambda) = \frac{L}{2}\frac{1}{\cosh\lambda}, \quad x^\mu(\lambda) = a^\mu \pm \widehat{p}^\mu\frac{L}{2}\tanh\lambda \tag{1.13}$$

   (b) For $p^2 < 0$, we have $(\frac{L}{2})^2 = \frac{1}{p^2} < 0$, and $\gamma$ is now a cross-section of the two-sheet hyperboloid. Again as $z \to 0$ it lands at $x^\mu = a^\mu \pm \frac{L}{2}\,\widehat{p}^\mu$; however, $|x^\mu|$ grows with $z$ and extends into the bulk instead of returning to the boundary, i.e. $z, |x^\mu| \to \infty$.

   The differential equation in this case is almost the same as in (a), but now we have to choose a different initial condition, since $\gamma$ won't even reach $x = a$. However, we can actually set $x(0), z(0) \to \infty$; we then have:

$$z(\lambda) = \left|\frac{L}{2}\right| \frac{1}{\sinh \lambda}, \quad x^\mu(\lambda) = a^\mu \pm \widehat{p}^\mu \left|\frac{L}{2}\right| \coth \lambda \tag{1.14}$$

This is an evidence that $z \to \infty$ might not be the "end of the world" after all; it's likely that $z \to \infty$ is only a horizon, since a spacelike geodesic $\gamma$ can reach it within finite proper length $\lambda$.

In summary, there are 3 types of spacelike geodesics in Poincaré AdS, which closely resemble the 3 types of conic sections:

1. $p^2 = 0$: parabolic, given in (1.7), with one end going to $z \to 0$ and the other end going to $z \to \infty$; it takes infinite proper length $\lambda$ for it to reach 0 or $\infty$.
2. $p^2 > 0$: hyperbolic, given in (1.11) and (1.14), also with one end going to $z \to 0$ and the other end going to $z \to \infty$, but it reaches $\infty$ within finite $\lambda$.
3. $p^2 > 0$: elliptic, given in (1.11) and (1.13), with both ends going to $z \to 0$, and it takes infinite proper length $\lambda$ for it to reach either end.

Now consider two points $x_1, x_2$ near the boundary $z = \epsilon$ connected by a spacelike geodesic $\gamma$. This can only be the "elliptic type" discussed above. We have:

$$x^\mu = a^\mu \pm \widehat{p}^\mu \sqrt{\left(\frac{L}{2}\right)^2 - z^2}, \quad z^2 + (x-a)^2 = \left(\frac{L}{2}\right)^2, \tag{1.15}$$

$$a = \frac{x_1 + x_2}{2}, \quad \widehat{p} \propto x_2 - x_1 \tag{1.16}$$

It's length is then given by:

$$\begin{aligned} A &= \int_{z \geq \epsilon} \sqrt{\frac{\mathrm{d}x^2 + \mathrm{d}z^2}{z^2}} \\ &= \int_{|\lambda| \leq \Lambda} \sqrt{\frac{(\mathrm{d}\tanh \lambda)^2 + (\mathrm{d}\,\mathrm{sech}\,\lambda)^2}{(\mathrm{sech}\,\lambda)^2}}, \quad \Lambda = \cosh^{-1}(\tfrac{L}{2\epsilon}) \\ &= \int_{-\Lambda}^{\Lambda} \mathrm{d}\lambda = 2\Lambda = 2\cosh^{-1}\left(\frac{L}{2\epsilon}\right) \sim 2 \log \frac{L}{\epsilon}, \quad \epsilon \to 0 \end{aligned} \tag{1.17}$$

## 2 Einbein Action

The einbein action of a point particle is given by:

$$S[\eta, X] = \frac{1}{2} \int \mathrm{d}\tau \left(\eta^{-1} \dot{X}_\mu \dot{X}^\mu - \eta m^2\right) \tag{2.1}$$

Under worldline reparametrization: $\tau \mapsto \tau' = f(\tau)$, we have $X'(\tau') = X(\tau)$, i.e. $X^\mu$ transforms like a **scalar** under worldline diffeomorphism; $X(\tau) \mapsto X'(\tau) = X\big(f^{-1}(\tau)\big)$.

On the other hand, $\eta$ should be treated like an einbein: $\eta = \sqrt{-\gamma}$, here $\gamma = \gamma_{\tau\tau}$ is the worldline metric; $\gamma < 0$ due to the Lorentzian signature [1]. We have:

$$\eta = \sqrt{-\gamma} \longmapsto \eta' = \eta \det \frac{\partial \tau}{\partial \tau'} = \eta \frac{\partial \tau}{\partial \tau'}, \tag{2.2}$$

$$\eta^{-1} = -\sqrt{-\gamma}\, \gamma^{-1} \longmapsto (\eta')^{-1} = \eta^{-1} \left(\frac{\partial \tau}{\partial \tau'}\right) \left(\frac{\partial \tau'}{\partial \tau} \frac{\partial \tau'}{\partial \tau}\right) = \eta^{-1} \frac{\partial \tau'}{\partial \tau} \tag{2.3}$$

It is clear that the action is invariant under the transformation:

$$\begin{aligned} S' &= \frac{1}{2}\int \mathrm{d}\tau' \left((\eta')^{-1}\partial_{\tau'}X_\mu \partial_{\tau'}X^\mu - (\eta')m^2\right) \\ &= \frac{1}{2}\int \mathrm{d}\tau\,\frac{\partial\tau'}{\partial\tau}\left(\eta^{-1}\frac{\partial\tau'}{\partial\tau}\cdot\frac{\partial\tau}{\partial\tau'}\partial_\tau X_\mu\cdot\frac{\partial\tau}{\partial\tau'}\partial_\tau X^\mu - \eta\,\frac{\partial\tau}{\partial\tau'}\,m^2\right) = S \end{aligned} \tag{2.4}$$

We can eliminate $\eta$ classically by placing it on shell:

$$0 = \frac{\delta S}{\delta\eta} = -\eta^{-2}\dot{X}_\mu\dot{X}^\mu - m^2, \quad \eta[X] = \frac{1}{m}\sqrt{-\dot{X}_\mu\dot{X}^\mu} \tag{2.5}$$

Substitute this back to the action, and we have:

$$\begin{aligned} S[X] = S[\eta=\eta[X],X] &= \frac{1}{2}\int \mathrm{d}\tau\left(m\,(-\dot{X}_\mu\dot{X}^\mu)^{-\frac{1}{2}}\,\dot{X}_\mu\dot{X}^\mu - m\,(-\dot{X}_\mu\dot{X}^\mu)^{+\frac{1}{2}}\right) \\ &= -m\int \mathrm{d}\tau\,\sqrt{-\dot{X}_\mu\dot{X}^\mu} \end{aligned} \tag{2.6}$$

# 3 Ricci Tensor for Static Spherical Metric

Consider the metric:

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + h(r)\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) = \eta_{ab}\,e^a e^b \tag{3.1}$$

Here we've defined the following vierbein:

$$\begin{aligned} e^t &= \sqrt{f(r)}\,\mathrm{d}t \\ e^r &= \sqrt{h(r)}\,\mathrm{d}r \\ e^\theta &= r\,\mathrm{d}\theta \\ e^\phi &= r\sin\theta\,\mathrm{d}\phi \end{aligned} \tag{3.2}$$

To find the connection form, we first note the *Cartan's structure equation* [2]:

$$\begin{aligned} \mathrm{d}e^a = \mathrm{d}(e^a_\nu\,\mathrm{d}x^\nu) &= \partial_\mu e^a_\nu\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \\ &= \left(-\omega^a_{\mu b}e^b_\nu + \Gamma^\lambda_{\mu\nu}e^a_\lambda\right)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \\ &= -\omega^a{}_b\wedge e^b + T^a \end{aligned} \tag{3.3}$$

Where $T^a$ is the torsion tensor. Here we've used the relation between spin connection $\omega^a_{\mu b}$ and the affine connection $\Gamma^\lambda_{\mu\nu}$. For the torsion-free Levi-Civita connection, we have $\mathrm{d}e^a = -\omega^a{}_b\wedge e^b$.

Rigidity of the veilbein, namely $\nabla_\mu(g_{\mu\nu}e^\mu_a e^\nu_b) = \nabla_\mu\eta_{ab} = 0$, further implies that:

$$\omega_{\mu ab} = -\omega_{\mu ba} \tag{3.4}$$

With these constraints we can solve for the connection form, and we find that only the following components are non-vanishing[1]:

$$\omega^t{}_r = \frac{f'}{2\sqrt{fh}}\,\mathrm{d}t = -\omega^r{}_t \tag{3.5}$$

[1] Go to `https://github.com/bryango/Archive/blob/master/HW-Gravity/gravity1/nb/vielbein.wl` for a Mathematica script for this calculation.

$$\omega^r{}_\theta = \frac{-1}{\sqrt{h}}\,\mathrm{d}\theta = -\omega^\theta{}_r \tag{3.6}$$

$$\omega^r{}_\phi = \frac{-\sin\theta}{\sqrt{h}}\,\mathrm{d}\phi = -\omega^\phi{}_r \tag{3.7}$$

$$\omega^\theta{}_\phi = -\cos\theta\,\mathrm{d}\phi = -\omega^\phi{}_\theta \tag{3.8}$$

The curvature form is thus given by:

$$\Omega^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{3.9}$$

The exterior derivative is easily computed in the $\mathrm{d}x^\mu$ basis; however, to get to the Ricci tensor, it would be convenient to switch to the $e^a$ basis. Here we only write down the upper half of $\Omega^a{}_b$ since the lower half can be inferred from anti-symmetry:

$$\Omega^a{}_b \sim \begin{pmatrix} 0 & R^t{}_{rtr}\, e^t \wedge e^r & R^t{}_{\theta t\theta}\, e^t \wedge e^\theta & R^t{}_{\phi t\phi}\, e^t \wedge e^\phi \\ \cdots & 0 & R^r{}_{\theta r\theta}\, e^r \wedge e^\theta & R^r{}_{\phi r\phi}\, e^r \wedge e^\phi \\ \cdots & \cdots & 0 & R^\theta{}_{\phi\theta\phi}\, e^\theta \wedge e^\phi \\ \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{3.10}$$

$$\begin{aligned} R^t{}_{rtr} &= \frac{f'^2}{4f^2h} + \frac{f'h'}{4fh^2} - \frac{f''}{2fh}, \\ R^t{}_{\theta t\theta} &= \frac{-f'}{2rfh} = R^t{}_{\phi t\phi}, \\ R^r{}_{\theta r\theta} &= \frac{h'}{2rh^2} = R^r{}_{\phi r\phi}, \\ R^\theta{}_{\phi\theta\phi} &= \frac{h-1}{r^2h} \end{aligned} \tag{3.11}$$

The Ricci tensor in $e^a$ basis is given by $R_{ab} = R^c{}_{acb}$. Note that the components are non-zero iff. $a = b$, i.e. the Ricci tensor is diagonal. We have:

$$\begin{aligned} R_{ab} &\sim \mathrm{diag}\left(R^r{}_{trt} + R^\theta{}_{t\theta t} + R^\phi{}_{t\phi t}, \cdots, \cdots, \cdots\right) \\ &= \mathrm{diag}\left(R^t{}_{rtr} + R^t{}_{\theta t\theta} + R^t{}_{\phi t\phi}, \cdots, \cdots, \cdots\right) \\ &= \mathrm{diag}\Big( R^t{}_{rtr} + R^t{}_{\theta t\theta} + R^t{}_{\phi t\phi}, \\ &\qquad R^t{}_{rtr} + R^r{}_{\theta r\theta} + R^r{}_{\phi r\phi}, \\ &\qquad R^t{}_{\theta t\theta} + R^r{}_{\theta r\theta} + R^\theta{}_{\phi\theta\phi}, \\ &\qquad R^t{}_{\phi t\phi} + R^r{}_{\phi r\phi} + R^\theta{}_{\phi\theta\phi} \Big) \\ &= \mathrm{diag}\Big( R^t{}_{rtr} + 2R^t{}_{\theta t\theta}, \\ &\qquad R^t{}_{rtr} + 2R^r{}_{\theta r\theta}, \\ &\qquad R_{\theta\theta} = R^t{}_{\theta t\theta} + R^r{}_{\theta r\theta} + R^\theta{}_{\phi\theta\phi}, \\ &\qquad R_{\phi\phi} = R_{\theta\theta} \Big) \\ &= \mathrm{diag}\Big( \frac{f'^2}{4f^2h} + \frac{f'h'}{4fh^2} - \frac{f''}{2fh} - \frac{f'}{rfh}, \\ &\qquad \frac{f'^2}{4f^2h} + \frac{f'h'}{4fh^2} - \frac{f''}{2fh} + \frac{h'}{rh^2}, \\ &\qquad R_{\theta\theta} = -\frac{f'}{2rfh} + \frac{h'}{2rh^2} + \frac{h-1}{r^2h}, \\ &\qquad R_{\phi\phi} = R_{\theta\theta} \Big), \end{aligned} \tag{3.12}$$

To go back to the $\mathrm{d}x^\mu$ basis, we have $R_{\mu\nu} = R_{ab} e^a_\mu e^b_\nu$. In particular, here we have $R_{\mu\mu} = R_{aa}(e^a_\mu)^2$, therefore:

$$
\begin{aligned}
R_{\mu\nu} = \mathrm{diag}\Bigg( & \frac{f'^2}{4fh} + \frac{f'h'}{4h^2} - \frac{f''}{2h} - \frac{f'}{rh}, \\
& \frac{f'^2}{4f^2} + \frac{f'h'}{4fh} - \frac{f''}{2f} + \frac{h'}{rh}, \\
& R_{\theta\theta} = -\frac{rf'}{2fh} + \frac{rh'}{2h^2} + \frac{h-1}{h}, \\
& R_{\phi\phi} = R_{\theta\theta} \sin^2\theta \Bigg)
\end{aligned}
\tag{3.13}
$$

# 4 Jackiw–Teitelboim Gravity

The Jackiw–Teitelboim (JT) action is given by:

$$
S = \frac{1}{16\pi G} \int \mathrm{d}^2x \sqrt{-g}\, \Phi\, (R+2) \tag{4.1}
$$

$\frac{\delta S}{\delta \Phi} = 0$ gives us $R = -2$. Now consider $\frac{\delta S}{\delta g^{\mu\nu}}$, and we have:

$$
\begin{aligned}
\delta_g S &= \frac{1}{16\pi G} \int \mathrm{d}^2x\, \Phi\, \delta\Big(\sqrt{-g}\,(R+2)\Big), \quad R = g^{\mu\nu} R_{\mu\nu}, \\
&= \frac{1}{16\pi G} \int \mathrm{d}^2x \sqrt{-g}\, \Phi \left\{ \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R+2) \right) \delta g^{\mu\nu} + g^{\mu\nu}\, \delta R_{\mu\nu} \right\}
\end{aligned}
\tag{4.2}
$$

Note that the $g^{\mu\nu}\, \delta R_{\mu\nu}$ term is a total derivative[2]:

$$
\begin{aligned}
g^{\mu\nu}\, \delta R_{\mu\nu} &= \left(\nabla^\mu \nabla^\nu - g^{\mu\nu} \nabla^\lambda \nabla_\lambda\right) \delta g_{\mu\nu} \\
&= -\left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^\lambda \nabla_\lambda\right) \delta g^{\mu\nu} \\
&= \nabla_\lambda \Big( g^{\mu\nu}\, \delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\, \delta\Gamma^\nu_{\nu\mu} \Big)
\end{aligned}
\tag{4.3}
$$

In Einstein gravity this gets reduced to a boundary term. But here we have an additional factor of $\Phi$, so after integration by parts, we actually get the equation of motion (EoM) for $\Phi$, up to some boundary terms[3]:

$$
\delta_g S \sim \frac{1}{16\pi G} \int \mathrm{d}^2x \sqrt{-g} \left\{ \Phi \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R+2) \right) - \Big( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^\lambda \nabla_\lambda \Big) \Phi \right\} \delta g^{\mu\nu} \tag{4.4}
$$

In 2D, we have $R_{\mu\nu} \equiv \frac{1}{2} g_{\mu\nu} R$, so the EoM is simply:

$$
\left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^\lambda \nabla_\lambda + g_{\mu\nu}\right) \Phi = 0 \tag{4.5}
$$

Contraction with $g^{\mu\nu}$ further gives us $\big(\nabla^\lambda \nabla_\lambda - 2\big)\Phi = 0$, so in the end we have:

$$
\left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\right) \Phi = 0, \quad R = -2 \tag{4.6}
$$

[2] See the amazing lecture note by Matthias Blau at `http://www.blau.itp.unibe.ch/GRLecturenotes.html`.

[3] See e.g. Section 2 of [3].

# A Derivation of the special conformal transformations

Special conformal transformations can be understood as translations conjugated by *inversions*. Note that $\frac{\mathrm{d}z^2}{z^2}$ is invariant under $z \mapsto \frac{1}{z}$; if we include the $x^\mu$ directions, we can consider:

$$\mathcal{I}\colon\ \chi^I \mapsto \frac{\chi^I}{\chi^2}, \quad \chi^2 = -t^2 + \vec{x}^2 + z^2, \tag{A.1}$$

$$\mathcal{I}^2 = \mathbb{1}, \quad \mathrm{d}s^2 \mapsto \left(\frac{\delta^I_J - 2\,\frac{\chi^I \chi_J}{\chi^2}}{\chi^2}\,\mathrm{d}\chi^J\right)^2 \Big/ \left(\frac{z}{\chi^2}\right)^2 = \frac{\mathrm{d}\chi^2}{z^2} = \mathrm{d}s^2 \tag{A.2}$$

We see that inversion $\mathcal{I}$ is indeed a (discrete) symmetry of the metric. Here we've defined yet another lower case variable $\chi^I \sim (x^\mu, z)$, which as a contravariant vector has the same components as $X^I$, but with an index that should be lowered by the flat metric $\eta_{IJ}$, i.e. $\chi_I = \eta_{IJ}\chi^J = \eta_{IJ}X^J$. The $d$ special conformal generators are then given by:

$$\begin{aligned}
k_\mu &= \frac{\partial}{\partial a^\mu}\left(\mathcal{I} \circ e^{a^\nu P_\nu} \circ \mathcal{I} \circ X^I\right)_{a=0} \frac{\partial}{\partial X^I} \\
&= \frac{\partial}{\partial a^\mu}\left(\frac{\frac{\chi^I}{\chi^2} + a^I}{\left|\frac{\chi^J}{\chi^2} + a^J\right|^2}\right)_{a=0} \frac{\partial}{\partial X^I} \\
&= \frac{\partial}{\partial a^\mu}\left(\frac{\chi^I + a^I \chi^2}{1 + 2a^I \chi_I + a^2 \chi^2}\right)_{a=0} \frac{\partial}{\partial X^I} \\
&= \chi^2 \partial_\mu - 2x_\mu X^I \partial_I \\
&= \chi^2 \partial_\mu - 2x_\mu \Delta
\end{aligned} \tag{A.3}$$

# References


[1] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, **December 1998**. ISBN: 978-0-511-25227-3, 978-0-521-67227-6, 978-0-521-63303-1.

[2] Sean M. Carroll. *Lecture notes on general relativity*, **December 1997**. arXiv: `gr-qc/9712019`.

[3] Daniel Harlow & Daniel Jafferis. *The Factorization Problem in Jackiw-Teitelboim Gravity*. *JHEP*. **02**:177, **2020**. arXiv: `1804.01081 [hep-th]`.