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1 Equivalence of categories is fully faithful:

 $F: \mathcal{C} \to \mathcal{D}$ equivalence of categories, i.e. $\exists G: \mathcal{D} \to \mathcal{C}$, s.t.

$$G \circ F \simeq \mathbb{1}_{\mathcal{C}}, \quad F \circ G \simeq \mathbb{1}_{\mathcal{D}}$$
 (1)

Here "≃" means naturally isomorphic as functors, i.e.,

$$\exists \tau \colon G \circ F \Rightarrow \mathbb{1}_{\mathcal{C}}, \quad \sigma \colon F \circ G \Rightarrow \mathbb{1}_{\mathcal{D}} : \text{ natural isomorphisms}$$
 (2)

By the definition of natural transformation, for $f \in \text{Hom}_{\mathcal{C}}(A, B)$, we have:

$$G \circ F(A) \xrightarrow{G \circ F(f)} G \circ F(B)$$

$$\uparrow_{A} \downarrow \qquad \qquad \downarrow^{\tau_{B}} \downarrow$$

$$A \xrightarrow{f} B$$

$$\tau_{B} \circ (G \circ F)(f) \circ \tau_{A}^{-1} = f, \quad \forall \ f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$$

$$(3)$$

Here $\tau_{A,B}$ are isomorphisms, which means that $G \circ F$ must be a bijection between hom-sets, which further implies that F is injective and G is surjective. Switch the roles of F, G, we find that G is injective and F is surjective. Therefore, F, G are both fully faithful.

2 Forgetful functors to <u>Set</u> are often representable:

For $F: \mathbf{Group} \to \underline{\mathbf{Set}}$, consider the free group generated by a single element \mathbb{Z} . We have:

$$\operatorname{Hom}(\mathbb{Z}, -) \colon \operatorname{\underline{Group}} \longrightarrow \operatorname{\underline{Set}}$$

$$G \longmapsto \operatorname{Hom}(\mathbb{Z}, G)$$

$$(4)$$

This is a covariant functor representable by \mathbb{Z} .

On the other hand, $\operatorname{Hom}(\mathbb{Z},G)$ consists of group homomorphisms:

$$\operatorname{Hom}(\mathbb{Z}, G) = \left\{ \begin{array}{c} \mathbb{Z} \to G \\ 1 \mapsto g \end{array} \middle| g \in G \right\} \tag{5}$$

More specifically, to fix any $\mathbb{Z} \to G$, we need only assign its generator¹ $1 \mapsto g$. Image of any other \mathbb{Z} element is generated automatically from the group law, without further specifications. This means that the hom-set is in one-to-one correspondence with G elements (as a set). Therefore, $F \cong \operatorname{Hom}_{\mathbf{Group}}(\mathbb{Z}, -)$, i.e. forgetful $F \colon \mathbf{Group} \to \mathbf{\underline{Set}}$ is representable by \mathbb{Z} .

Similarly, for $F: \underline{\mathbf{Ring}} \to \underline{\mathbf{Set}}$, the free object generated by some generic element x is $\mathbb{Z}[x]$, the polynomial ring in one variable; we have:

$$F \cong \operatorname{Hom}_{\underline{\mathbf{Ring}}}(\mathbb{Z}[x], -), \quad \operatorname{Hom}_{\underline{\mathbf{Ring}}}(\mathbb{Z}[x], R) = \left\{ \begin{array}{c} \mathbb{Z}[x] \to R \\ x \mapsto r \end{array} \middle| r \in R \right\}$$
 (6)

Lesson: Forgetful $\underline{\mathbf{Cat}} \to \underline{\mathbf{Set}}$ are often representable by the free object in $\underline{\mathbf{Cat}}$.

¹Note that $0 \in \mathbb{Z}$ is the group identity of addiction group \mathbb{Z} , not $1 \in \mathbb{Z}$.

3 Properties of contractible space:

(a) X contractible: $\mathbb{1}_X \simeq f_0 \colon X \to X$ some constant map, $f_0(X) = \{x_0\}$. We can restrict the codomain of f_0 so that $f_0 \colon X \to \{x_0\}$, in this way we have:

$$X \xrightarrow{f_0} \{x_0\} \hookrightarrow X \simeq \mathbb{1}_X,\tag{7.1}$$

$$\{x_0\} \hookrightarrow X \xrightarrow{f_0} \{x_0\} \simeq \mathbb{1}_{\{x_0\}},$$
 (7.2)

This means that $f_0: X \to \{x_0\}$ isomorphic in $\underline{\mathbf{hTop}} = \underline{\mathbf{Top}}/\simeq$, which is precisely the definition of homotopic equivalence $X \simeq \{x_0\}$. (\Rightarrow)

On the other hand (\Leftarrow), if $X \simeq \{x_0\}$, there exists some $f_0 \colon X \to \{x_0\}$ that fulfills (7). We can then extend the codomain s.t. $f_0 \colon X \to X$, in this way (7.1) reads $f_0 \simeq \mathbb{1}_X$, i.e. X is contractible. Therefore, X contractible iff. homotopic equivalent to a single point.

(b) $\forall X$: Topological space, we can define its *cone* as²:

$$CX = (X \times I)/(X \times \{0\}), \quad I = [0, 1]$$
 (8)

i.e. gluing together one end of the cylinder $X \times I$. Naturally $X \subset CX$ as a subspace; now we show that CX is contractible. Using (a), we need only show that $\mathbb{1}_{CX} \simeq f_0$ some constant map.

In fact, any point in CX can be uniquely labeled by $[x,h] \in X \times I$, with the exception of the vertex $v \sim [x,0] \sim [x',0]$, $\forall x,x' \in X$. We can then construct a homotopy F by shrinking the cone towards the vertex v:

$$F: CX \times I \to CX, \quad F([x,h],t) = [x,h \cdot t],$$

$$F|_{CX \times 0} = v = \text{const}, \quad F|_{CX \times 1} = \mathbb{1}_X$$
(9)

This confirms that $\mathbb{1}_{CX} \simeq v$: constant map. By (a), CX is contractible.

(c) For $Y \simeq \{y_0\}$ contractible, given any $g: X \to Y$, we can deform the image $g(X) \subset Y$ to a single point, hence $g \simeq y_0$: constant map. More precisely, we have:

$$\exists G: X \times I \to Y, \quad \text{s.t.} \quad G|_{X \times 0} = y_0 = \text{const}, \quad G|_{X \times 1} = g \tag{10}$$

Such G can be explicitly constructed using $\mathbb{1}_Y \simeq y_0$:

$$F: Y \times I \to Y, \quad F|_{Y \times 0} = y_0 = \text{const}, \quad F|_{Y \times 1} = \mathbb{1}_Y,$$
 (11)

$$G(x,t) = F(g(x),t)$$
(12)

In summary, we have proven that $g \simeq y_0$, $\forall g \in \operatorname{Hom}_{\underline{\mathbf{Top}}}(X,Y)$. By definition, this means that $\operatorname{Hom}_{\underline{\mathbf{hTop}}}(X,Y) = \operatorname{Hom}_{\underline{\mathbf{Top}}}(X,Y)/_{\simeq} = \{[y_0]\}$ a single point.

(d) For $X \simeq \{x_0\}$ contractible, similar to (11), we have homotopy $F: X \times I \to X$. Given any $f: X \to Y$, the composition $f \circ F: X \times I \to Y$ yields $f \simeq f(x_0)$: constant map.

²See Wikipedia: Cone (topology).

Furthermore, for Y: path connected, there is a path $\gamma: I \to Y$ connecting $f(x_0)$ and some $y_0 \in Y$, therefore $f(x_0) \simeq y_0: X \to Y$ constant maps. More precisely, we have:

$$\gamma \colon I \to Y, \quad \gamma(0) = y_0, \quad \gamma(1) = f(x_0), G \colon X \times I \to Y, \quad G(x, t) = \gamma(t)$$
 (13)

Which gives $f(x_0) \simeq y_0$, $\forall f$, independent of the choice of f. This means that $f \simeq f(x_0) \simeq y_0$: constant map, therefore $\operatorname{Hom}_{\mathbf{hTop}}(X,Y) = \{[y_0]\}$ a single point.

4 Example of homotopic inequivalence³:

$$X = \{0\} \cup \left\{ \frac{1}{n} \middle| n \in \mathbb{Z}_+ \right\}, \quad Y = \{0\} \cup \mathbb{Z}_+$$

$$X, Y \subset \mathbb{R} \colon \text{subspace topology}$$

$$(14)$$

Assume $X \simeq Y$, then similar to (7), we have $Y \xrightarrow{g} X \xrightarrow{f} Y \simeq \mathbb{1}_Y$. However, note that Y has discrete topology, in such case any map $f \circ g$ homotopic to $\mathbb{1}_Y$ must be $\mathbb{1}_Y$ itself: $f \circ g = \mathbb{1}_Y$.

More specifically, consider:

$$F: Y \times I \to Y, \quad F|_{Y \times 0} = f \circ g, \quad F|_{Y \times 1} = \mathbb{1}_Y$$
 (15)

Any point $n \in Y$ is both open and closed, therefore its pre-image $F^{-1}(n) \subset Y \times I$ is also both open and closed, and by $F|_{Y \times 1} = \mathbb{1}_Y$ we know that F(y,1) = y, $(y,1) \in F^{-1}(y)$, therefore the only possibility is that $F(\{y\} \times I) = y$, i.e. $f \circ g = \mathbb{1}_Y$, which implies that g is injective and f is surjective.

However, $f \colon X \to Y$ cannot be surjective due to the complication around $0 \in X$. Consider $f^{-1}\big(f(0)\big) \ni 0$, since $f(0) \in Y$ both open and closed, $f^{-1}\big(f(0)\big) \subset X$ must also be both open and closed. But any open set $U \subset X$ is induced via subspace topology $X \subset \mathbb{R}$; for $0 \in U \subset X \subset \mathbb{R}$, U must contain ∞ -many elements:

$$\left\{ \frac{1}{n} \mid n \ge N_0 \right\} \subset U \subset f^{-1}(f(0)), \quad \text{for some } N_0, \text{ for any } U \ni x$$
 (16)

Hence $f(X) = f(0) \cup f(\left\{\frac{1}{n} \mid n < N_0\right\}), \ f(X) \subset Y$ a finite set, i.e. $f: X \to Y$ is never surjective. Therefore, $X \not\simeq Y$ by contradiction.

 $^{^3}$ This proof is produced thanks to helpful insights from 谷夏 and 於子雄.

5 Fundamental group of topological group is abelian⁴:

From a categorical point of view, the fundamental group $\pi_1(G)$ of a topological group G can be seen as a functor:

$$G \in \underline{\mathbf{TopGroup}} \hookrightarrow \underline{\mathbf{Top}} \xrightarrow{\pi_1} \underline{\mathbf{Group}} \ni \pi_1(G)$$
 (17)

<u>TopGroup</u> \subset <u>Top</u> is a subcategory with additional group structure, i.e. $(G, \cdot) \in \underline{\textbf{TopGroup}}$ is a $group \ object^5$ in $\underline{\textbf{Top}}$, with "·" denoting its product operation (·): $G \times G \to G$. Correspondingly, $\pi_1(\underline{\textbf{TopGroup}})$ should be $group \ objects \ of \underline{\textbf{Group}}$, which have an additional group structure $(\star) = \pi_1(\cdot)$, along with the usual group product "*" in $\underline{\textbf{Group}}$.

In total, we have three different group structures (represented by their product operation):

$$(\cdot)\colon G\times G\to G,\tag{18}$$

(*):
$$\pi_1(G) \times \pi_1(G) \to \pi_1(G)$$
, (19)

$$(\star) = \pi_1(\cdot) \colon \ \pi_1(G) \times \pi_1(G) \to \pi_1(G), \tag{20}$$

Note that $\pi_1(G) = \operatorname{Aut}_{\Pi_1(G)} \mathbb{1}_G$, i.e. loop classes $[\gamma]$ in G; (*) is defined as joining two loops, while $(\star) = \pi_1(\cdot)$ is defined as the translation of loop classes by pointwise group product (\cdot) ,

$$[\gamma_1] \star [\gamma_2] = [\gamma_1 \cdot \gamma_2] \tag{21}$$

With the above definitions, we observe that:

$$([\gamma_1] \star [\gamma_2]) * ([\eta_1] \star [\eta_2]) = ([\gamma_1] * [\eta_1]) \star ([\gamma_2] * [\eta_2])$$
(22)

By definition, they are both equal to $[(\gamma_1 \cdot \gamma_2) * (\eta_1 \cdot \eta_2)]$. What's surprising is that by using only the group axioms and "distributive law" (22), we can show that (\star) and (\star) must always coincide: $(\star) = (*)$, and they have to be in fact, commutative. This is the *Eckmann–Hilton argument*⁶.

Proof of this argument is straight-forward; first, observe that the units of the two operations coincide:

$$1_{\star} = 1_{\star} \star 1_{\star} = (1_{*} * 1_{\star}) \star (1_{\star} * 1_{*}) \xrightarrow{(22)} (1_{*} \star 1_{\star}) * (1_{\star} \star 1_{*}) = 1_{*} * 1_{*} = 1_{*}$$
 (23)

Further manipulation using (22) confirms that the two operations coincide and are commutative:

$$[\gamma] * [\eta] = (1 \star [\gamma]) * ([\eta] \star 1) \xrightarrow{\underline{(22)}} (1 * [\eta]) \star ([\gamma] * 1)$$

$$= [\eta] \star [\gamma]$$

$$= ([\eta] * 1) \star (1 * [\gamma]) \xrightarrow{\underline{(22)}} ([\eta] \star 1) * (1 \star [\gamma])$$

$$= [\eta] * [\gamma]$$

$$(24)$$

In summary, we find that the group objects in <u>Group</u> are indeed abelian groups, which means that $\pi_1(G)$ for $G \in \textbf{TopGroup}$ must be abelian.

⁴This proof is produced with the help of math.stackexchange.com/q/727999. Another (easier) proof lies in the fact that group translation induces π_1 conjugation, therefore $\gamma^{-1}\alpha\gamma = \alpha$, hence abelian.

⁵See Wikipedia: *Group object*.

 $^{^6\}mathrm{See}$ Wikipedia: Eckmann-Hilton argument.