

1 Example of limit in Vect:

$$I = \left\{ \begin{array}{c} \bullet \\ \star \end{array} \right\} \begin{array}{l} \xrightarrow{F} \left\{ \begin{array}{c} 0 \\ \mathbb{R} \end{array} \right\} \\ \xrightarrow{\Delta(V)} \left\{ \begin{array}{c} V \\ V \end{array} \right\} \end{array} \supset \underline{\mathbf{Vect}}: \mathbb{R} \text{ vector space ,} \quad (1)$$

(a) By definition, we have:

$$\begin{array}{ccc} & \Delta(\lim F) & \\ \exists! \Delta(g) \nearrow & \Downarrow \sigma & \\ \Delta(V) & \xrightarrow{\eta} & F \end{array} \quad (2)$$

Where $g: V \rightarrow \lim F$. More intuitively, for the above $F: I \rightarrow \underline{\mathbf{Vect}}$, this translates to the following diagram in Vect:

$$\begin{array}{ccc} & \eta_{\bullet} & \searrow \\ V & \xrightarrow{\exists! g} \lim F & \xrightarrow{\sigma_{\bullet}} 0 \\ & \eta_{\star} & \searrow \\ & & \mathbb{R} \end{array} \quad (3)$$

Now we verify that $\lim F = \mathbb{R}$, along with the following choice of σ :

$$\sigma_{\bullet} = 0, \quad \sigma_{\star} = \mathbb{1}_{\mathbb{R}} \quad (4)$$

In fact, for the above diagram (3) to be commutative, we must have $g = \eta_{\star}$. Note that such g is unique once σ is chosen; for our choice of σ , if $g \neq \eta_{\star}$, then the diagram *cannot* commute. Hence $\lim F = \mathbb{R}$, along with the above choice of $\sigma: \Delta(\mathbb{R}) \Rightarrow F$. In other words, we have:

$$\begin{array}{ccc} & \{ \bullet, \star \} & \\ \Delta(\mathbb{R}) \swarrow & \xRightarrow{\sigma} & \searrow F \\ \mathbb{R} & & \{ 0, \mathbb{R} \} \end{array} \quad (5)$$

□

(b)(c) From diagram (2) and discussions in (a), we know that:

$$\exists! \tau = \Delta(g): \Delta(V) \Longrightarrow \Delta(\mathbb{R}) \quad (6)$$

Here $g: V \rightarrow \mathbb{R}$ is fixed uniquely once σ is fixed. However, σ may vary up to isomorphism; therefore, a generic choice of σ is given by:

$$\sigma_{\bullet} = 0, \quad \sigma_{\star} = k \mathbb{1}_{\mathbb{R}}, \quad k \in \mathbb{R} \quad (7)$$

For such g , by the same arguments in (a), we have:

$$\exists! g = \frac{1}{k} \eta_*, \tau = \Delta(g) = \frac{1}{k} \Delta(\eta_*), \quad \text{s. t.} \quad (2), (3) \text{ commutes} \quad (8)$$

Note that $k \in \mathbb{R}$, for every k there is a different g and τ ; hence there are $\|\{k\}\| = \|\mathbb{R}\|$ many choices of τ to make the diagram commute. In particular, for $k = 1$ we recover $\tau = \Delta(\eta_*)$. ■

2 Limit and colimit of polynomial ring:

By definition,

$$\begin{array}{c}
 Q \\
 \downarrow \exists! f \\
 \lim F \\
 \begin{array}{ccccc}
 & \eta_{n+1} & & \eta_n & \\
 & \swarrow & \searrow & \swarrow & \searrow \\
 \dots & \xrightarrow{p_{n+1}} & \mathbb{Z}[x]/x^{n+1} & \xrightarrow{p_n} & \mathbb{Z}[x]/x^n & \xrightarrow{p_{n-1}} & \dots & \xrightarrow{p_1} & \mathbb{Z}[x]/x^1 = \mathbb{Z}
 \end{array} \\
 \begin{array}{ccccc}
 & \sigma_{n+1} & & \sigma_n & \\
 & \swarrow & \searrow & \swarrow & \searrow \\
 \dots & \xrightarrow{\tau_{n+1}} & \mathbb{Z}[x]/x^{n+1} & \xrightarrow{\tau_n} & \mathbb{Z}[x]/x^n & \xrightarrow{\tau_1} & \dots & \xrightarrow{\tau_1} & \mathbb{Z}[x]/x^1 = \mathbb{Z}
 \end{array} \\
 \text{colim } F \\
 \downarrow \exists! g \\
 R
 \end{array} \quad (9)$$

Here $p_n: \mathbb{Z}[x]/x^{n+1} \rightarrow \mathbb{Z}[x]/x^n$ is the natural projection.

Intuitively, if such $\lim F$ exists, it shall be the “smallest” object that “contains” $\mathbb{Z}[x]/x^n$ when $n \rightarrow \infty$. Note that $\mathbb{Z}[x]/x^n$ is naturally a \mathbb{Z}^n vector space:

$$\mathbb{Z}[x]/x^n \ni \sum_{m=0}^{n-1} a_m x^m \sim (a_0, a_1, \dots, a_{n-1}) \in \mathbb{Z}^n \quad (10)$$

While $n \rightarrow \infty$, this gives an ∞ -tuple which corresponds to the *formal power series*¹:

$$\mathbb{Z}[[x]] \ni \sum_{m=0}^{\infty} a_m x^m \quad (11)$$

The difference between $\mathbb{Z}[x]$ and $\mathbb{Z}[[x]]$ is that the latter may contain infinite series while the former may not. Now we confirm that, indeed, $\lim F = \mathbb{Z}[[x]]$, along with natural projections $\pi_n: \mathbb{Z}[[x]] \rightarrow \mathbb{Z}[x]/x^n$.

In fact, the $f: R \rightarrow \lim F$ in (9) can be explicitly written down as:

$$\begin{aligned}
 f &= \eta_1 + x \left(\frac{d}{dx} \eta_2 \right) + x^2 \left(\frac{1}{2} \frac{d}{dx} \eta_3 \right) + \dots = \sum_{m=0}^{\infty} x^m \left\{ \frac{1}{m!} \frac{d^m}{dx^m} \eta_{m+1} \right\} \\
 &\sim (\eta_{1,0}, \eta_{2,1}, \dots, \eta_{n,n-1}, \dots)
 \end{aligned} \quad (12)$$

¹ See Wikipedia: *Formal power series*. This is in fact the *adic completion* of $\mathbb{Z}[x]$. I would like to thank 刘逸华 & 谢贤进 for this hint.

Here $\frac{1}{m!} \frac{d^m}{dx^m}$ is used to extract the a_{n-1} coefficient in $\mathbb{Z}[x]/x^n$; this is the last component of η_n , denoted by $\eta_{n,n-1}$. Any other choice of f will break commutativity of (9), hence f is fixed uniquely by $\mathbb{Z}[[x]]$ and π_n 's. Therefore, $\lim F = \mathbb{Z}[[x]]$. \square

On the other hand, $\text{colim } F$ is the “largest” object that any map *out of* $\mathbb{Z}[x]/x^n$ must “pass through”. Also, this should hold for all $n \in \mathbb{Z}_+$. Naturally, projections $\sigma_n: \mathbb{Z}[x]/x^n \rightarrow \mathbb{Z}$ satisfy the above requirements; we have:

$$\sigma_n: x \mapsto 0, \quad \sum_{m=0}^{n-1} a_m x^m \mapsto a_0, \quad (13)$$

$$\sigma_{n+1} = p_1 \circ p_2 \circ \cdots \circ p_n \quad (14)$$

$g: \mathbb{Z} \rightarrow R$ in (9) is fixed uniquely for such choice of σ_n ; in fact, descend along the p_n tower in (9), and we have: $\tau_{n+1} = \tau_n \circ p_n = \tau_{n-1} \circ p_{n-1} \circ p_n = \cdots = \tau_1 \circ p_1 \circ p_2 \circ \cdots \circ p_n = \tau_1 \circ \sigma_{n+1}$, $\forall n \in \mathbb{Z}_+$, hence $\exists! g = \tau_1$. Therefore, $\text{colim } F = \mathbb{Z}$. \blacksquare

3 Example of push-out in Groupoid:

$$\begin{array}{ccc} \{0, 1\} & \xrightarrow{f_1} & \bullet \\ \downarrow f_2 & & \downarrow \tau_1 \\ \{0 \leftrightarrow 1\} & \xrightarrow{\tau_2} & P \end{array} \quad \begin{array}{c} \nearrow \eta_1 \\ \searrow \eta_2 \\ \downarrow \exists! g \end{array} \quad Q \quad (15)$$

Following the same observation as before, the push-out P is the “largest” object that any map out of \bullet and $\{0 \leftrightarrow 1\}$ must pass through. By such universal property, P can be no larger than the coproduct: $\{\bullet\} \coprod \{0 \leftrightarrow 1\}$. However, we should also consider the equivalence imposed by:

$$\bullet \xleftarrow{f_1} \{0, 1\} \xrightarrow{f_2} \{0 \leftrightarrow 1\} \quad (16)$$

Therefore, we simply have $P = \bullet$, with $\tau_{1,2}$ the natural projection. This can be verified with ease: we have $g = \eta_1$. It is unique since its image is a single point (with identity map to itself) $\star \in Q$, and the point \star is fixed by commutativity. \blacksquare

4 Product and coproduct in Ab:

For $G_\alpha \in \mathbf{Ab} \subset \mathbf{Group}$, note that we have:

$$\mathbf{Free} : \mathbf{Set} \rightleftharpoons \mathbf{Group} : \mathbf{Forget} \quad (17)$$

Therefore, for F : some diagram in Group, $\boxed{\lim (\mathbf{Forget} \circ F) = \mathbf{Forget} \circ \lim F}$ if $\lim F$ exists.

By definition, the product $\prod_\alpha G_\alpha \in \mathbf{Group}$ is a limit, hence it is identical (as in Set) to the *direct product*, with additional entry-wise group multiplication. Same applies for the full subcategory: abelian group $\mathbf{Ab} \subset \mathbf{Group}$.

On the other hand, the disjoint union of G_α 's as sets will not necessary be a group, the identities $1_\alpha \in G_\alpha$ must be glued together to produce a group structure. Furthermore, free-forgetful adjunction

(17) implies that for F' : some diagram in **Set**,

$$\text{colim}(\text{Free} \circ F') = \text{Free} \circ \text{colim} F', \quad (18)$$

Whenever $\text{colim} F'$ exists; in our case, $\text{colim} F'$ is the disjoint union of sets: $\coprod_{\alpha} \text{Forget}(G_{\alpha})$. Therefore, we should construct a free object in **Ab**.

Here we restrict our discussion to **Ab**, since the coproduct in **Ab** is *not* the same as in **Group** — the free product of abelian group is not necessary abelian. Hence, the coproduct in **Ab** shall be:

$$\coprod_{\alpha} G_{\alpha} = \bigoplus_{\alpha} G_{\alpha}, \quad i_{\alpha}: G_{\alpha} \hookrightarrow \bigoplus_{\alpha} G_{\alpha} \quad (19)$$

As a set, this is precisely the disjoint union with identities $0_{\alpha} \in G_{\alpha} \subset \text{Ab}$ glued together.

It is then straight-forward to verify its universal property: for $f_{\alpha}: G_{\alpha} \rightarrow H$,

$$\exists! f: \bigoplus_{\alpha} G_{\alpha} \longrightarrow H, \quad (g_{\alpha})_{\alpha} \mapsto \sum_{\alpha} f_{\alpha}(g_{\alpha}) \quad (20)$$

This is compatible with the abelian group multiplication. Note that for the summation to be well-defined, the coproduct must only contain finitely many components; otherwise it is identical to the product in **Ab**. ■

5 Composition of pull-backs:

(a) If A_1 is the pull-back of A_2 and A_2 is the pull-back of A_3 , then given Q with f_3, g_1 , we have $g_2 = \psi \circ g_1$, and f_2 is fixed uniquely by universal property of A_2 , while f_1 is fixed uniquely by universal property of A_1 . Hence, A_1 is the pull-back of A_3 .

(b) If A_1, A_2 are pull-backs of A_3 , then given Q with f_2, g_1 , we have $f_3 = \phi \circ f_2$, and f_2 is fixed uniquely by universal property of A_2 , and f_1 is fixed uniquely by universal property of A_1 . Hence, A_1 is the pull-back of A_3 .

☞ PAST WORK, AS TEMPLATE ☞

1 For $F_i \rightarrow E_i \xrightarrow{p_i} B$: coverings in $\text{Cov}_0(B)$ with E_i : connected and B : path connected and locally path connected, the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow p_1 & \swarrow p_2 \\ & B & \end{array} \quad \begin{array}{l} e_2 = f(e_1), \\ b = p_1(e_1) = p_2(e_2), \end{array}$$

To show that f is itself a covering, we need only verify that f is locally trivial with some discrete fiber F . In fact, given any $e_2 \in E_2$ and $b = p_2(e_2)$, there exists some neighborhood $U \subset B$ that the following diagram holds (by restriction):

$$\begin{array}{ccc} U \times F_1 & \xrightarrow{f} & U \times F_2 \\ & \searrow p_1 & \swarrow p_2 \\ & U & \end{array} \quad \begin{array}{l} e_1 = (b, k_1), \\ e_2 = (b, k_2(b, k_1)), \end{array} \quad k_i \in F_i$$

Generally, $k_2 = k_2(b, k_1)$ depends on the base point $b \in B$. However, since B is locally path connected, we can restrict U to be path connected, while $k_2 \in F_2$: discrete. Since continuous maps preserve path connectedness, k_2 is in fact independent of b , i.e. $k_2 = \varphi(k_1)$.

On the other hand, $\forall e_2 = (b, k_2) \in U \times \{k_2\} \subset E_2$, we have its preimage $f^{-1}(e_2) = \{b\} \times \varphi^{-1}(k_2)$. Note that E_2 is connected while $\varphi^{-1}(k_2) \in F_1$ is discrete; for the same reasoning as above, $\varphi^{-1}(k_2) = F$ is in fact independent of k_2 . This is the discrete fiber F we have been looking for. Hence f is also a covering map². ■

2 Cylinder with ends pinched — π_1 and universal cover:

$$Y = (X \times I) / (X \times \partial I), \quad I = [0, 1] \quad (22)$$

Note that Y is homeomorphic to two cones³ $CX_1 \amalg CX_2$ with “bases” $X_i \subset CX_i$ and “vertices” v_i respectively identified: $X_1 \sim X_2$, $v_1 \sim v_2 \equiv v$. X is path connected and so is Y , hence we are free to choose $\pi_1(Y) = \pi_1(Y, y_0)$.

First note that paths that do *not* pass through the vertex v are all homotopic, since they are contained in a cone and cones are contractible⁴. Therefore all contributions to $\pi_1(Y)$ are loop classes that *do* pass through the vertex v . In other words, morphisms in $\Pi_1 Y$ are in one-to-one correspondence with morphisms in:

$$\Pi_1([0, 1]_{/0 \sim 1}) = \Pi_1 S^1 \quad (23)$$

Therefore, $\pi_1(Y) \cong \pi_1(S^1) = \mathbb{Z}$. □

The universal cover \tilde{Y} of Y can be constructed by assigning an induced topology to the space of path classes, same as in the general proof of its existence. Since Y is “degenerate” at its vertex,

² Reference: math.stackexchange.com/a/109774.

³ See discussions from Problem Set №1.

⁴ $[\gamma_1] = [\gamma_2 \star \gamma_2^{-1} \star \gamma_1] = [\gamma_2]$.

this is equivalent to “cutting open” Y at its vertex v , and joining \mathbb{Z} copies them end-to-end. More explicitly, it can be written as:

$$\tilde{Y} = (X \times \mathbb{R}) / \sim, \quad (x, n) \sim (x', n), \quad \forall x \in X, n \in \mathbb{Z} \quad (24)$$

While the covering map: $\tilde{Y} \ni [x, t] \mapsto [x, t - [t]] \in Y$, here $[t]$ is the integer part of $t \in \mathbb{R}$. \blacksquare

3 π_1 of fiber in fibration:

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\quad} & E \\ \downarrow & \nearrow \exists \tilde{f} & \downarrow p \\ X \times I & \xrightarrow{f} & B \end{array}$$

For $F \rightarrow E \xrightarrow{p} B$: fibration, by homotopy lifting property (HLP), any homotopy in B can be uniquely lifted to path class in E , provided some “initial condition” $X \times \{0\}$. This leads to the following results:

(a) For B : simply-connected, take any loop class $[\tilde{\gamma}] \in \pi_1(E, e)$ as initial condition; its projection $[p \circ \tilde{\gamma}] \in \pi_1(B, b) = \{[\mathbb{1}_b]\}$ is trivial, i.e. $p \circ \tilde{\gamma} \simeq \mathbb{1}_b$. By HLP, such homotopy can be lifted into E , i.e.

$$p \circ \tilde{\gamma} \simeq \mathbb{1}_b \xrightarrow{\text{lift}} \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}' = \mathbb{1}_b \quad (25)$$

In other words, $\tilde{\gamma} \simeq \tilde{\gamma}' \subset p^{-1}(b)$, i.e. any loop in E is homotopic to some loop in $p^{-1}(b) \cong F$. This implies a surjective group homomorphism $\pi_1(p^{-1}(b), e) \rightarrow \pi_1(E, e)$, i.e. an epimorphism. \square

(b) For E : simply-connected, take any loop class $[\gamma] \in \pi_1(B, b)$ and consider its lifting $[\tilde{\gamma}]$. Note that in general $\tilde{\gamma}$ is *not* a loop; however, we have $p \circ \tilde{\gamma} = \gamma$, hence $\tilde{\gamma}(0), \tilde{\gamma}(1) \in p^{-1}(b)$. In general, we have:

$$\gamma \simeq \gamma' \xrightarrow{\text{lift}} \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}' = \gamma' \quad (26)$$

By continuity, $\tilde{\gamma}(0), \tilde{\gamma}'(0) \in F_0$: a path component of $p^{-1}(b)$; similarly, $\tilde{\gamma}(1), \tilde{\gamma}'(1) \in F_1$. In other words, the start and end points of $\tilde{\gamma}$ are confined in path components F_0 and F_1 , respectively. Hence a loop class in $\pi_1(B, b)$ maps to *transport* between path components:

$$\begin{aligned} T_{(\cdot)}(e): \pi_1(B, b) &\longrightarrow \pi_0(p^{-1}(b)) \\ [\gamma] &\longmapsto T_{[\gamma]}(e) \end{aligned} \quad (27)$$

As a matter of fact, $T_{(\cdot)}(e)$ is a bijection. For $T_{[\gamma]} = T_{[\gamma']}$, they are characterized by two lifted paths $\tilde{\gamma}, \tilde{\gamma}'$; since E is simply connected, they are always homotopic: $\tilde{\gamma} \simeq \tilde{\gamma}'$, hence $[\gamma] = [\gamma']$ by projection p . This means that T is injective. Surjectivity also follows from projection $\gamma = p \circ \tilde{\gamma}'$. Therefore, $T_{(\cdot)}(e)$ gives a bijection between $\pi_1(B, b)$ and $\pi_0(p^{-1}(b))$. \blacksquare

4 Pull-back of fibration is fibration:

$$\begin{array}{ccccc}
Y \times \{0\} & \longrightarrow & f^*(E) & \longrightarrow & E \\
\downarrow & \nearrow \exists \tilde{G} & \downarrow & \nearrow \exists \tilde{F} \text{ (HLP)} & \downarrow p \\
Y \times I & \xrightarrow{G} & X & \xrightarrow{f} & B
\end{array}$$

$$(x, e) \in f^*(E) \subset X \times E, \quad f(x) = p(e)$$

We need only verify that $f^*(E) \rightarrow X$ also has HLP, i.e. the existence of \tilde{F} in the above diagram⁵. By HLP of $E \xrightarrow{p} B$, $\exists \tilde{F}: Y \times I \rightarrow E$ as shown above. We can use \tilde{F} to construct \tilde{G} explicitly; in fact, first consider:

$$\begin{aligned}
\tilde{G}: Y \times I &\longrightarrow X \times E \\
(y, t) &\longmapsto (G(y, t), \tilde{F}(y, t))
\end{aligned} \tag{28}$$

Note that $f \circ G = p \circ \tilde{F}$; compared with the definition of $f^*(E)$, this implies that the image of \tilde{G} lies within $f^*(E) \subset X \times E$, hence after restriction of its codomain, \tilde{G} becomes a well-defined lifting of G into $f^*(E)$. Therefore, $f^*(E) \rightarrow X$ has HLP, i.e. it is also a fibration. ■

5 More properties of fibration:

(a) By HLP, given any initial condition $e \in p^{-1}(b_1)$, lifting of any path $b_1 \xrightarrow{\gamma} b_2$ exists. The lifted path with dependence of e can then be written as $F: p^{-1}(b_1) \times I \rightarrow E$. This is just a generalization of [3] for non-loop paths. □

(b) Similarly, transport $T_{[\gamma]}$ defined in [3] can be generalized for non-loop paths. $T_{[\gamma]}$ is well-defined for path class $[\gamma]$, since by HLP homotopic paths can be lifted to homotopy in E . Therefore, the transport is fixed up to homotopy, i.e.

$$\begin{aligned}
T: \text{Hom}_{\Pi_1 B}(b_0, b_1) &\longrightarrow \text{Hom}_{\mathbf{hTop}}(p^{-1}(b_0), p^{-1}(b_1)) \\
[\gamma] &\longmapsto T_{[\gamma]}
\end{aligned} \tag{29}$$

Note that T defined in this way is also independent of the choice of F , since F simply specifies the starting point of the lifted path; no matter which F we choose, the lifted paths will always be homotopic in E . Hence T is well-defined in the above sense. □

(c) T defined above is a functor: $\Pi_1 B \rightarrow \mathbf{hTop}$. To verify this, we need only check that it is compatible with composition and maps identity morphisms to identity morphisms. Indeed, $T_{[1_b]} = [1_{p^{-1}(b)}]$, and $T_{[\gamma'] \star [\gamma]} = T_{[\gamma' \star \gamma]} = T_{[\gamma']} \circ T_{[\gamma]}$ by joining two lifted paths (up to homotopy). □

(d) For B : path connected, there exists an isomorphism between any two objects in $\Pi_1 B$ (a path connecting any two points in B), which is mapped to isomorphisms between fibers $p^{-1}(b)$ in \mathbf{hTop} . Hence any two fibers of $E \xrightarrow{p} B$ have the same homotopy type. ■

⁵ Notice that $f^*(E)$ is the limit of the diagram, hence this is automatically true by the universal property of $f^*(E)$. I would like to thank 刘逸华 for pointing this out. For now, we will stick to a more traditional proof.