

## 1 BRST Quantization of Bosonic String:

$$S = S^X + S^{bc}, \quad (1)$$

$$S^X = \frac{1}{2\pi\alpha'} \int d^2z \partial X^\mu \bar{\partial} X_\mu, \quad S^{bc} = \frac{1}{2\pi} \int d^2z (b \bar{\partial} c + \tilde{b} \partial \tilde{c}) \quad (2)$$

This is the gauge fixed action. The corresponding BRST transformation is listed in *Polchinski*; for each of the subsystems, we have its energy-momentum:

$$T^X(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu :, \quad \tilde{T}^X(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X^\mu \bar{\partial} X_\mu :, \quad (3)$$

$$T^{bc}(z) = :(\partial b) c: - 2 \partial(:bc:), \quad \tilde{T}^{bc}(\bar{z}) = :(\bar{\partial} \tilde{b}) \tilde{c}: - 2 \bar{\partial}(:\tilde{b}\tilde{c}:), \quad (4)$$

(a) To get the energy-momentum of  $S$ , let's visit each of the subsystems respectively; first, BRST transformation of  $X$  is given by:

$$\delta X^\mu = i\epsilon (c\partial + \tilde{c}\bar{\partial}) X^\mu \quad (5)$$

Compared with the conformal transformation<sup>1</sup>:  $\delta X^\mu = -\epsilon (v\partial + \tilde{v}\bar{\partial}) X^\mu$ , we see that they are in fact identical under the equivalence  $-\epsilon v \sim i\epsilon c$ ,  $-\epsilon \tilde{v} \sim i\epsilon \tilde{c}$ , hence we can simply follow the derivation of conformal current and write down  $\delta S^X$ 's contribution to the conserved current:

$$j^X = c(z) T^X(z) \quad (6)$$

The transformation of  $b, c$  is less obvious; for holomorphic current, we need only focus on the holomorphic part of  $S^{bc}$ ; on-shell variation yields:

$$0 = \delta S^{bc} = \left( \frac{1}{2\pi} \int d^2z (-\bar{\partial} c \delta b - \bar{\partial} b \delta c) \right)_{=0} + \frac{1}{2\pi} \int d^2z \bar{\partial} (b \delta c) = \frac{1}{2\pi} \int d^2z \bar{\partial} \epsilon (-ibc \partial c) \quad (7)$$

Here we've plugged in  $\delta c = i\epsilon(z, \bar{z}) c \partial c$ , and we have moved  $\bar{\partial} \epsilon$  to the beginning of the expression, while respecting the anti-commuting nature of  $\epsilon$ . With a conventional  $i$  coefficient (which agrees with the convention of  $j^X$ ), we have  $bc$ 's contribution to the conserved current:

$$j^{bc} = i (-ibc \partial c) = bc \partial c \quad (8)$$

Note that  $j^{bc}$  is, in fact, related to the energy-momentum (at least classically):

$$\frac{1}{2} c T^{bc} = \frac{1}{2} c (\partial b) c - c \partial(bc) = -c \partial(bc) = -cb \partial c = bc \partial c = j^{bc} \quad (9)$$

Hence we have the classical BRST current:

$$j(z) = c(z) \left( T^X + \frac{1}{2} T^{bc} \right) \quad (10)$$

□

<sup>1</sup> We follow the convention of *Polchinski* unless otherwise stated.

For a quantum version, redefine  $j(z)$  with normal ordering<sup>2</sup>, and we have:

$$T(z) j(0) \sim T^X(z) T^X(0) c(0) + T^{bc}(z) c T^X(0) + T^{bc}(z) :bc \partial c:_{(0)}, \quad (11)$$

$$\text{where } T^X(z) T^X(0) c(0) \sim \left( \frac{D}{2z^4} + \frac{2}{z^2} T^X(0) + \frac{1}{z} \partial T^X(0) \right) c(0), \quad (12)$$

Here we've used the fact that  $X$  and  $b, c$  is de-coupled in the gauge-fixed action, hence their OPE is trivial. Also, we've expanded the first term using  $TT$  OPE of the free boson. Additionally, note that  $c(z)$  is primary with weight  $(-1, 0)$ , we have:

$$\begin{aligned} T^{bc}(z) c T^X(0) &\sim \{T^{bc}(z) c(0)\} T^X(0) \\ &\sim \left( \frac{-1}{z^2} c(0) + \frac{1}{z} \partial c(0) \right) T^X(0), \end{aligned} \quad (13)$$

The last term in (11) can be brute-forced as follows:

$$\begin{aligned} T^{bc}(z) :bc \partial c:_{(0)} &= (:(\partial b) c: - 2 \partial(:bc:))_{(z)} :bc \partial c:_{(0)}, \quad (14) \\ :(\partial b) c:_{(z)} :bc \partial c:_{(0)} &\sim :(\overbrace{(\partial b) c_{(z)} bc \partial c}_{(0)}) : + :(\overbrace{(\partial b) c_{(z)} bc \partial c}_{(0)}) : + :(\overbrace{(\partial b) c_{(z)} bc \partial c}_{(0)}) : \\ &\quad + :(\overbrace{(\partial b) c_{(z)} bc \partial c}_{(0)}) : + :(\overbrace{(\partial b) c_{(z)} bc \partial c}_{(0)}) : \\ &\sim \frac{-1}{z^2} (+1) :c_{(z)} b \partial c_{(0)} : + \frac{-2}{z^3} (-1) :c_{(z)} bc_{(0)} : + \frac{1}{z} (+1) : \partial b_{(z)} c \partial c_{(0)} : \\ &\quad + \frac{-1}{z^2} \cdot \frac{1}{z} (+1) \partial c(0) + \frac{-2}{z^3} \cdot \frac{1}{z} (-1) c(0) \\ &\sim \frac{-1}{z^2} (-j^{bc}(0) + \mathcal{O}(z^2)) + \frac{2}{z^3} \left( z j^{bc}(0) + \frac{z^2}{2} :bc \partial^2 c:_{(0)} + \mathcal{O}(z^3) \right) \\ &\quad + \frac{1}{z} (:(\partial b) c \partial c:_{(0)} + \mathcal{O}(z)) + \frac{-1}{z^3} \partial c(0) + \frac{2}{z^4} c(0) \\ &\sim \frac{4}{2z^4} c(0) + \frac{-1}{z^3} \partial c(0) + \frac{3}{z^2} j^{bc}(0) + \frac{1}{z} : (bc \partial^2 c + (\partial b) c \partial c) :_{(0)}, \\ &\sim \frac{4}{2z^4} c(0) + \frac{-1}{z^3} \partial c(0) + \frac{3}{z^2} j^{bc}(0) + \frac{1}{z} \partial j^{bc}(0), \end{aligned} \quad (15)$$

$$\begin{aligned} :bc:_{(z)} :bc \partial c:_{(0)} &\sim :(\overbrace{bc_{(z)} bc \partial c}_{(0)}) : + :(\overbrace{bc_{(z)} bc \partial c}_{(0)}) : + :(\overbrace{bc_{(z)} bc \partial c}_{(0)}) : \\ &\quad + :(\overbrace{bc_{(z)} bc \partial c}_{(0)}) : + :(\overbrace{bc_{(z)} bc \partial c}_{(0)}) : \\ &\sim \frac{1}{z} (+1) :c_{(z)} b \partial c_{(0)} : + \frac{1}{z^2} (-1) :c_{(z)} bc_{(0)} : + \frac{1}{z} (+1) :b_{(z)} c \partial c_{(0)} : \\ &\quad + \frac{1}{z} \cdot \frac{1}{z} (+1) \partial c(0) + \frac{1}{z^2} \cdot \frac{1}{z} (-1) c(0) \\ :bc:_{(z)} :bc \partial c:_{(0)} &\sim \frac{1}{z} (-j^{bc}(0)) + \frac{-1}{z^2} (z j^{bc}(0)) + \frac{1}{z} (j^{bc}(0)) + \frac{1}{z^2} \partial c(0) + \frac{-1}{z^3} c(0) \end{aligned}$$

<sup>2</sup> Normal ordering between  $\geq 3$  operators is in fact *not* associative; this directly leads to the ambiguity we are about to discover. See *Di Francesco et al* for more detailed discussions. Naïvely,  $:bc \partial c:_{(0)}$  is *defined* as  $b(0) c(z_1) \partial c(z_2)$  while  $z_1, z_2 \rightarrow 0$ , with singular terms subtracted; however, different ways of taking the limit might lead to different results. For example, we can first take  $z_1 \rightarrow 0$  then  $z_2 \rightarrow 0$ , or we can first take  $z_1 \rightarrow z_2$  then  $z_2 \rightarrow 0$ . This two procedures will differ by  $\frac{3}{2} \partial^2 c(z)$ , which is precisely the correction we are about to find out. *I suppose this is somehow related to topology, e.g. braid group?*

$$\sim \frac{-1}{z^3} c(0) + \frac{1}{z^2} \partial c(0) + \frac{-1}{z} j^{bc}(0), \quad (16)$$

$$\partial(:bc:)(z) : bc \partial c :_{(0)} \sim \frac{6}{2z^4} c(0) + \frac{-2}{z^3} \partial c(0) + \frac{1}{z^2} j^{bc}(0), \quad (17)$$

$$T^{bc}(z) : bc \partial c :_{(0)} \sim \frac{-8}{2z^4} c(0) + \frac{3}{z^3} \partial c(0) + \frac{1}{z^2} j^{bc}(0) + \frac{1}{z} \partial j^{bc}(0), \quad (18)$$

$$T(z) j(0) \sim ((12) + (13) + (18)) \sim \frac{D-8}{2z^4} c(0) + \frac{3}{z^3} \partial c(0) + \frac{1}{z^2} j(0) + \frac{1}{z} \partial j(0), \quad (19)$$

We see that  $j(z)$  defined with naïve normal ordering is *almost* but *not quite* a primary. It differs from primary OPE at  $\mathcal{O}(\frac{1}{z^4})$  and  $\mathcal{O}(\frac{1}{z^3})$ . However, it is possible to make it into a primary by adding extra terms that do not interfere with current conservation  $\bar{\partial} j = 0$ . To cancel the  $\frac{3}{z^3} \partial c(0)$  term, notice that  $b(z) \partial^2 c(0) \sim \frac{2}{z^3}$ , therefore it may be helpful to look at:

$$\begin{aligned} T(z) \partial^2 c(0) &\sim T^{bc}(z) \partial^2 c(0) \sim \partial_w^2 (T^{bc}(z) c(w))_{w \rightarrow 0} \\ &\sim \partial_w^2 \left( \frac{-1}{(z-w)^2} c(w) + \frac{1}{z-w} \partial c(w) \right)_{w \rightarrow 0} \\ &\sim \frac{-12}{2z^4} c(0) + \frac{-2}{z^3} \partial c(0) + \frac{1}{z^2} \partial^2 c(0) + \frac{1}{z} \partial^3 c(0), \end{aligned} \quad (20)$$

Again we've used  $Tc$  OPE of the primary  $c(w)$ . We see that indeed, the  $\frac{1}{z^3} \partial c(0)$  term can be canceled by shifting  $j(z)$ :

$$j(z) \mapsto j(z) + \frac{3}{2} \partial^2 c(z), \quad j(z) = cT^X + :bc \partial c: + \frac{3}{2} \partial^2 c, \quad (21)$$

$$T(z) j(0) \sim \frac{D-26}{2z^4} c(0) + \frac{1}{z^2} j(0) + \frac{1}{z} \partial j(0), \quad (22)$$

We see that  $j(z)$  defined in this way is a primary of weight  $(1, 0)$  in  $D = 26$ . This is the quantum BRST current.  $\square$

(b) For  $jj$  OPE, we have:

$$j = cT^X + j', \quad j' \equiv j^{bc} + \frac{3}{2} \partial^2 c, \quad j^{bc} = \frac{1}{2} : cT^{bc} : = : bc \partial c : , \quad (23)$$

$$\begin{aligned} j_z j_0 &\sim : \{ T_z^X T_0^X \} c_z c_0 : + : \{ c_z j'_0 \} T_z^X : + : \{ j'_z c_0 \} T_0^X : + j'_z j'_0 \\ &\sim : \{ T_z^X T_0^X \} c_z c_0 : + : \{ c_z j_0^{bc} \} T_z^X : + : \{ j_z^{bc} c_0 \} T_0^X : + j'_z j'_0, \end{aligned} \quad (24)$$

From now on, for convenience and clarity, we will use subscripts to denote variable dependence:  $c_z = c(z)$ . Let's compute this term by term. We have:

$$\begin{aligned} : \{ T_z^X T_0^X \} c_z c_0 : &\sim : \left( \frac{D}{2z^4} + \frac{2}{z^2} T_0^X + \frac{1}{z} \partial T_0^X \right) \left( z \partial c_0 + \frac{z^2}{2} \partial^2 c_0 + \frac{z^3}{6} \partial^3 c_0 \right) c_0 : \\ &\sim - \left( \frac{D}{2z^3} c \partial c_0 + \frac{D}{4z^2} c \partial^2 c_0 + \frac{D}{12z} c \partial^3 c_0 + \frac{2}{z} : T^X c \partial c_0 : \right), \end{aligned} \quad (25)$$

$$\begin{aligned} j_z^{bc} c_0 &\sim \frac{1}{2} : cT^{bc} :_z c_0 \sim \frac{1}{2} c_z \{ : T^{bc} :_z c_0 \} \sim \frac{1}{2} c_z \{ T_z c_0 \} \\ &\sim -\frac{1}{2} \left( \frac{-1}{z^2} c_0 + \frac{1}{z} \partial c_0 \right) (c_0 + z \partial c_0) \sim 0, \end{aligned} \quad (26)$$

$$j_0^{bc} c_z \sim 0, \quad (27)$$

$$\begin{aligned}
j'_z j'_0 &\sim j_z^{bc} j_0^{bc} + \frac{3}{2} j_z^{bc} \partial^2 c_0 + \frac{3}{2} \partial^2 c_z j_0^{bc} \\
&\sim \frac{1}{2} :cT^{bc}:_z j_0^{bc} + \frac{3}{2} (j_z^{bc} \partial^2 c_0 + \partial^2 c_z j_0^{bc}),
\end{aligned} \tag{28}$$

The task is now reduced to calculating terms in the above  $j'j'$  OPE, which can be laboriously computed following a similar procedure as before. Note that there will be a  $\frac{1}{z} :cT^{bc}: \partial c_0$  term which combines with the  $\frac{2}{z} :cT^X: \partial c_0$  term in (25). In total, we obtain the final  $jj$  OPE:

$$j_z j_0 \sim -\frac{D-18}{2z^3} c \partial c_0 - \frac{D-18}{4z^2} c \partial^2 c_0 - \frac{D-26}{12z} c \partial^3 c_0 \tag{29}$$

(c) Following the convention of *Polchinski*, expand  $X^\mu, b, c$  into modes  $\alpha_n^\mu, b_n, c_n$ , then a generic level 2 state of an open string can be created as<sup>3</sup>:

$$\begin{aligned}
|\psi\rangle &= (e_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu + \beta_\mu \alpha_{-1}^\mu b_{-1} + \gamma_\mu \alpha_{-1}^\mu c_{-1} \\
&\quad + \eta b_{-1} c_{-1} + e_\mu \alpha_{-2}^\mu + \beta b_{-2} + \gamma c_{-2}) |k; 0\rangle
\end{aligned} \tag{30}$$

Here  $e_{\mu\nu}$  is chosen to be symmetric since  $\alpha_{-1}^\mu \alpha_{-1}^\nu$  commutes. By acting on  $L_0$  (expanded in modes), we find that  $m^2 = -k^2 = \frac{1}{\alpha'} = l_s$ : massive.

The BRST charge  $Q = \frac{1}{2\pi i} \oint (dz j(z) - d\bar{z} \tilde{j}(z))$  can also be expanded in modes; note that:

$$Q^2 = \frac{1}{2} \{Q, Q\} \propto \oint \frac{dz}{2\pi i} \text{Res}_{z' \rightarrow z} j(z') j(z) + (\text{conjugate}) \tag{31}$$

Compared with the  $jj$  OPE, we see that  $Q$  is nilpotent iff.  $D = 26$ , i.e. the critical dimension of bosonic string theory. This condition is necessary for consistent BRST quantization.

The physical states are firstly,  $Q$ -closed; i.e.

$$Q_B |\psi\rangle = 0 \implies 4l_s k^\mu e_{\mu\nu} + l_s k_\nu \eta + e_\nu = 0, \quad 2\sqrt{2} l_s k^\mu + e_\nu^\nu e_\mu = 0, \quad \beta_\mu = \beta = 0, \tag{32}$$

This is also the negative-norm states.

On the other hand,  $Q$ -exact states generate gauge transformations; this gives:

$$\gamma_\nu \mapsto \gamma_\nu + \gamma'_\nu, \quad \gamma \mapsto \gamma + \gamma', \quad \eta \mapsto \eta + \eta', \quad e_{\mu\nu} \mapsto e_{\mu\nu} + l_s (\beta'_\mu k_\nu + \beta'_\nu k_\mu), \tag{33}$$

Here  $\beta'_\mu, \gamma'_\nu, \gamma', \eta'$  are arbitrary gauge parameters. For closed string the result can be obtained by the doubling trick, i.e. by introducing anti-holomorphic modes  $\tilde{\alpha}, \tilde{b}, \tilde{c}$  and imposing reality conditions. The result is similar. ■

## 2 Linear Dilaton CFT:

For  $z \mapsto z + \epsilon(z)$ , we have:

$$\delta X^\mu = -\epsilon \partial X^\mu - \bar{\epsilon} \bar{\partial} X^\mu - \frac{\alpha' V^\mu}{2} (\partial \epsilon + \bar{\partial} \bar{\epsilon}) \tag{34}$$

Note that the  $\alpha'$  term has no dependence on  $X$ .

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<sup>3</sup> Reference: Bram M. Wouters, *BRST quantization and string theory spectra*.

(a) For simplicity, assume for now  $X = X(z)$ : holomorphic. Note that the  $\alpha'$  term comes from the transformation of “*internal*” degrees of freedom, associated with the conformal properties of  $X$ . We have:

$$X'(z') - X(z) = -\frac{\alpha'V}{2} \partial\epsilon + \mathcal{O}(\epsilon^2) \quad (35)$$

This is a first order approximation of the finite transformation, where the transformation parameters are the modes  $\epsilon_n$  of  $\epsilon(z)$ ; namely, we have:

$$w(z) = z + \epsilon(z) + \mathcal{O}(\epsilon^2), \quad \epsilon(z) = \sum_n \epsilon_n z^n \quad (36)$$

$$F[w(z)] = X'(z') - X(z) \xrightarrow{w \rightarrow 0} -\frac{\alpha'V}{2} \partial\epsilon + \mathcal{O}(\epsilon^2) \quad (37)$$

What the above actually means is that:

$$\frac{\delta}{\delta\epsilon_n} F[w(z)]_{\epsilon \rightarrow 0} = -\frac{\alpha'V}{2} \frac{\delta}{\delta\epsilon_n} \partial_z (w(z) - z) = -\frac{\alpha'V}{2} n z^{n-1} \quad (38)$$

Where  $\epsilon \rightarrow 0$  corresponds to  $w \rightarrow z$ , i.e. the transformation goes to the identity. On the other hand,

$$\begin{aligned} \frac{\delta F}{\delta\epsilon_n} &= \frac{\partial F}{\partial w} \frac{\delta w}{\delta\epsilon_n} + \frac{\partial F}{\partial(\partial w)} \frac{\delta(\partial w)}{\delta\epsilon_n} + \frac{\partial F}{\partial(\partial^2 w)} \frac{\delta(\partial^2 w)}{\delta\epsilon_n} + \dots \\ &= \frac{\partial F}{\partial w} z^n + \frac{\partial F}{\partial(\partial w)} n z^{n-1} + \frac{\partial F}{\partial(\partial^2 w)} n(n-1) z^{n-2} + \dots \end{aligned} \quad (39)$$

By comparing the two above equations, and noting that  $\frac{\partial F}{\partial(\partial^\bullet w)}$  should have no dependence on  $n$ , we obtain the following constraints on the form of  $F[w(z)]$ :

$$F|_{w \rightarrow z} = 0, \quad \frac{\partial F}{\partial w} \Big|_{w \rightarrow z} = 0, \quad \frac{\partial F}{\partial(\partial w)} \Big|_{w \rightarrow z} = -\frac{\alpha'V}{2}, \quad \frac{\partial F}{\partial(\partial^k w)} \Big|_{w \rightarrow z} = 0, \quad k = 2, 3, \dots \quad (40)$$

We can think of this as the first order “Taylor” coefficients of  $F[w]$  in the functional space, around the point  $w(z) \rightarrow z$ . Note that  $\partial w|_{w \rightarrow z} = 1$ , while  $\partial^k w|_{w \rightarrow z} = 0$ , it is thus natural to consider the following ansatz:

$$F = F[\partial w], \quad F[1] = 0, \quad \frac{\partial F[x]}{\partial x} \Big|_{x \rightarrow 1} = -\frac{\alpha'V}{2} \quad (41)$$

In the end we shall obtain that<sup>4</sup>:

$$X'(z', \bar{z}') - X(z, \bar{z}) = -\frac{\alpha'V}{2} \ln \left( \frac{dz'}{dz} \frac{d\bar{z}'}{d\bar{z}} \right) \quad (42)$$

A better recipe to find finite transformations is to consider its properties under composition, which will lead to some constraints that can be solved to obtain the result<sup>5</sup>.

<sup>4</sup> I would like to thank Lucy Smith for helpful discussions.

<sup>5</sup> See [bryango.github.io/resources/archive/alpha/schwarzian.pdf](https://bryango.github.io/resources/archive/alpha/schwarzian.pdf) for some detailed discussions.

(b) Perform the usual Noether's procedure on the free boson action, and we have:

$$\delta\mathcal{L} \propto \frac{1}{\alpha'} (\partial \delta X^\mu \bar{\partial} X_\mu + \partial X^\mu \bar{\partial} \delta X_\mu) \sim \bar{\partial} \epsilon \left( V^\mu \partial^2 X^\mu - \frac{1}{\alpha'} \partial X^\mu \bar{\partial} X_\mu \right) \quad (43)$$

Here we've plugged in the holomorphic part of  $\delta X^\mu$ , used integration by parts to move  $\bar{\partial}$  before  $\epsilon$ , and collected the  $\bar{\partial}\epsilon$  coefficients. This gives:

$$T(z) = -\frac{1}{\alpha'} : \partial X^\mu \bar{\partial} X_\mu : + V^\mu \partial^2 X^\mu \quad (44)$$

With  $X_z^\mu X_0^\nu \sim -\frac{\alpha'}{2} \eta^{\mu\nu} \ln |z|^2$  unchanged, the  $TT$  OPE can be calculated following the usual procedure, as shown in great detail before. Here we can use the known result from free boson theory to speed up our calculation:

$$\begin{aligned} T_z T_0 &\sim (V_\mu \partial^2 X^\mu + T')_z (V_\mu \partial^2 X^\mu + T')_0 \\ &\sim V_\mu V_\nu \partial^2 X_z^\mu \partial^2 X_0^\nu + V_\mu \partial^2 X_z^\mu T'_0 + V_\mu T'_z \partial^2 X_0^\mu + T'_z T'_0 \end{aligned} \quad (45)$$

Here  $T'$  is the usual free boson stress tensor. Combining all terms yields:

$$T_z T_0 \sim \frac{D + 6\alpha' V^2}{2z^4} + \frac{2}{z^2} T_0 + \frac{1}{z} \partial T_0, \quad c = D + 6\alpha' V^2 \quad (46)$$

■

### 3 Bosonic Strings on $S^3$ :

For bosonic strings moving on  $S^3$  (radius  $R$ ) with background dilaton  $\Phi = \text{const.}$  and  $B$ -field:

$$B = R^2 \sin \theta (\psi - \sin \psi \cos \psi) d\theta \wedge d\phi \quad (47)$$

The corresponding  $\beta$ -functions and trace anomaly can be computed using the formulae given in *Polchinski*; here  $(\psi, \theta, \phi)$  is the usual spherical coordinates on  $S^3$ .

In fact, field strength:

$$H = dB = 2R^2 \sin \theta \sin \psi d\psi \wedge d\theta \wedge d\phi \quad (48)$$

While the spacetime curvature for a maximally symmetric sphere<sup>6</sup>:  $\mathcal{R}_{\mu\nu} = \frac{2}{R^2} g_{\mu\nu}$ ,  $\mathcal{R} = \frac{6}{R^2}$ . Plug in these results, and we have:

$$\beta^G = \beta^B = 0, \quad T_a^a \simeq -\frac{1}{2} \beta^\Phi \mathcal{R} = -\frac{D - 26 - \alpha' \mathcal{R}}{12} \mathcal{R} \quad (49)$$

(a) Compared with the trace anomaly formula of a CFT:  $T_a^a = -\frac{1}{12} c \mathcal{R}$ , where  $\mathcal{R}$  is the world-sheet Ricci scalar, we see that our theory is indeed conformally invariant with Weyl anomaly. Its central charge is given by:

$$c \simeq D - 26 - \alpha' \mathcal{R} = 3 - 26 - \frac{6\alpha'}{R^2} \quad (50)$$

This includes ghost contribution ( $-26$ ). If we do not gauge the conformal symmetry, then there will not be ghost contribution, and we will have  $c \simeq 3 - \frac{6\alpha'}{R^2}$ .

<sup>6</sup> I would like to thank 林般 for some very helpful hints.

(b) The background  $B$  field given above is not single-valued on the  $\psi$  circle. Note that we've encountered such difficulty in electromagnetism with a multi-valued  $A^\mu(x)$ . In fact, the resolution for this issue is very similar to Dirac's quantization of the magnetic monopole<sup>7</sup>: by allowing the action  $S$  to be invariant modulo  $2\pi$ , since  $e^{-(S+2\pi i)} = e^{-S}$ .

More specifically, for  $\psi \mapsto \psi + 2\pi$ , we have:

$$2\pi i n = \Delta S = \frac{i}{2\pi\alpha'} \Delta \int_{\Sigma} X^* B = \frac{i}{2\pi\alpha'} \Delta \int_{X(\Sigma)} B = \frac{i}{2\pi\alpha'} \Delta \int_M H \quad (51)$$

$B$  is a 2-form in  $S^3$ ,  $X^*B$  denotes its pullback to the worldsheet, and  $X(\Sigma) \subset S^3$  denotes the embedding of  $\Sigma$  into  $S^3$ . Note that  $H$  is proportional to the volume form in  $S^3$ , hence we have:

$$\Delta \int_M H = 2R^2 \Delta \text{Vol}(M) = 2R^2 \mathbb{Z} \text{Vol}(S^3) = 2R^2 2\pi^2 \mathbb{Z} = 4\pi^2 R^2 \mathbb{Z} \quad (52)$$

This leads to the following quantization:

$$\frac{R^2}{\alpha'} = n \in \mathbb{Z}, \quad R \geq \sqrt{\alpha'} \geq (\alpha'/\ell)^{1/3} \quad (53)$$

In particular, in string units:  $\alpha' = 1$ , we have  $R \geq 1$ . ■

#### 4 Anomalous Currents:

(a) For a conserved current in flat worldsheet to be anomalous in curved worldsheet, then its deviation from conservation must be proportional to the Ricci scalar:

$$\nabla_a j^a = QR, \quad Q = \text{const}. \quad (54)$$

The logic here is similar to the Weyl anomaly<sup>8</sup>:  $\nabla_a j^a$  is diff- and Poincaré-invariant with dimension 2, because we have preserved these symmetries, and it vanishes in the flat case; this leaves only one possibility —  $\nabla_a j^a \propto R$ : the Ricci scalar.

For conformal transformation  $z \mapsto z + \epsilon(z)$ ,  $\bar{z} \mapsto \bar{z} + \bar{\epsilon}(\bar{z})$ , we have:

$$\delta_\epsilon j(0) = -\text{Res}_{z \rightarrow 0} \epsilon(z) T(z) j(0) - \text{Res}_{\bar{z} \rightarrow 0} \bar{\epsilon}(\bar{z}) \tilde{T}(\bar{z}) j(0) \quad (55)$$

Hence the  $z^{-3}, \bar{z}^{-3}$  coefficients of the OPE reflect the  $\epsilon = z^2$ ,  $\bar{\epsilon} = \bar{z}^2$  transformation of  $j$ . By comparing the Weyl transformations<sup>9</sup>, this yields a total coefficient of  $4Q$ .

(b) For  $bc$  CFT with  $j = :cb:$ , the anomaly can be explicitly calculated using our results in (a), i.e. by calculating  $Tj$  OPE. Following the standard procedure<sup>10</sup>, we obtain that:

$$T_z j_0 \sim \frac{1-2\lambda}{z^3} + \mathcal{O}\left(\frac{1}{z^2}\right) \quad (56)$$

Note that the anti-holomorphic part is zero, therefore, we have:  $Q = \frac{1}{4}(1-2\lambda)$ . ■

<sup>7</sup> Reference: J. J. Sakurai, *Modern Quantum Mechanics*.

<sup>8</sup> See *Polchinski* for reference.

<sup>9</sup> Note that  $(\text{Conformal}) = (\text{Weyl}) + (\text{Translation})$ .

<sup>10</sup> For more detailed discussions, see Blumenhagen et al, *Basic Concepts of String Theory*.