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1 Morphism between coverings is covering:

For $F_i \to E_i \xrightarrow{p_i} B$: coverings in $Cov_0(B)$ with E_i : connected and B: path connected and locally path connected, the following diagram commutes:

$$E_1 \xrightarrow{f} E_2$$
 $e_2 = f(e_1),$ $b = p_1(e_1) = p_2(e_2),$

To show that f is itself a covering, we need only verify that f is locally trivial with some discrete fiber F. In fact, given any $e_2 \in E_2$ and $b = p_2(e_2)$, there exists some neighborhood $U \subset B$ that the following diagram holds (by restriction):

$$U \times F_1 \xrightarrow{f} U \times F_2 \qquad e_1 = (b, k_1), \\ e_2 = (b, k_2(b, k_1)), \qquad k_i \in F_i$$

Generally, $k_2 = k_2(b, k_1)$ depends on the base point $b \in B$. However, since B is locally path connected, we can restrict U to be path connected, while $k_2 \in F_2$: discrete. Since continuous maps preserve path connectedness, k_2 is in fact independence of b, i.e. $k_2 = \varphi(k_1)$.

On the other hand, $\forall e_2 = (b, k_2) \in U \times \{k_2\} \subset E_2$, we have its preimage $f^{-1}(e_2) = \{b\} \times \varphi^{-1}(k_2)$. Note that E_2 is connected while $\varphi^{-1}(k_2) \in F_1$ is discrete; for the same reasoning as above, $\varphi^{-1}(k_2) = F$ is in fact independent of k_2 . This is the discrete fiber F we have been looking for. Hence f is also a covering map¹.

2 Cylinder with ends pinched — π_1 and universal cover:

$$Y = (X \times I)/(X \times \partial I) , \quad I = [0, 1]$$
(1)

Note that Y is homeomorphic to two cones² $CX_1 \coprod CX_2$ with "bases" $X_i \subset CX_i$ and "vertices" v_i respectively identified: $X_1 \sim X_2$, $v_1 \sim v_2 \equiv v$. X is path connected and so is Y, hence we are free to choose $\pi_1(Y) = \pi_1(Y, y_0)$.

First note that paths that do *not* pass through the vertex v are all homotopic, since they are contained in a cone and cones are contractible³. Therefore all contributions to $\pi_1(Y)$ are loop classes that do pass through the vertex v. In other words, morphisms in $\Pi_1 Y$ are in one-to-one correspondence with morphisms in:

$$\Pi_1([0,1]/_{0\sim 1}) = \Pi_1 S^1 \tag{2}$$

Therefore,
$$\pi_1(Y) \cong \pi_1(S^1) = \mathbb{Z}$$
.

¹ Reference: math.stackexchange.com/a/109774.

 $^{^2~}$ See discussions from Problem Set $N\!\!\!_{2}1.$

 $^{^{3} [\}gamma_{1}] = [\gamma_{2} \star \gamma_{2}^{-1} \star \gamma_{1}] = [\gamma_{2}].$

The universal cover \tilde{Y} of Y can be constructed by assigning an induced topology to the space of path classes, same as in the general proof of its existence. Since Y is "degenerate" at its vertex, this is equivalent to "cutting open" Y at its vertex v, and joining \mathbb{Z} copies them end-to-end. More explicitly, it can be written as:

$$\tilde{Y} = (X \times \mathbb{R}) / \sim, \quad (x, n) \sim (x', n), \ \forall \ x \in X, \ n \in \mathbb{Z}$$
 (3)

While the covering map: $\tilde{Y} \ni [x, t] \mapsto [x, t - \lfloor t \rfloor] \in Y$, here $\lfloor t \rfloor$ is the integer part of $t \in \mathbb{R}$.

$\boxed{\bf 3}$ π_1 of fiber in fibration:



For $F \to E \xrightarrow{p} B$: fibration, by homotopy lifting property (HLP), any homotopy in B can be uniquely lifted to path class in E, provided some "initial condition" $X \times \{0\}$. This leads to the following results:

(a) For B: simply-connected, take any loop class $[\tilde{\gamma}] \in \pi_1(E, e)$ as initial condition; its projection $[p \circ \tilde{\gamma}] \in \pi_1(B, b) = \{[\mathbb{1}_b]\}$ is trivial, i.e. $p \circ \tilde{\gamma} \simeq \mathbb{1}_b$. By HLP, such homotopy can be lifted into E, i.e.

$$p \circ \tilde{\gamma} \simeq \mathbb{1}_b \quad \xrightarrow{\text{lift}} \quad \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}' = \mathbb{1}_b$$
 (4)

In other words, $\tilde{\gamma} \simeq \tilde{\gamma}' \subset p^{-1}(b)$, i.e. any loop in E is homotopic to some loop in $p^{-1}(b) \cong F$. This implies a surjective group homomorphism $\pi_1(p^{-1}(b), e) \to \pi_1(E, e)$, i.e. an epimorphism. \square

(b) For E: simply-connected, take any loop class $[\gamma] \in \pi_1(B, b)$ and consider its lifting $[\tilde{\gamma}]$. Note that in general $\tilde{\gamma}$ is *not* a loop; however, we have $p \circ \tilde{\gamma} = \gamma$, hence $\tilde{\gamma}(0), \tilde{\gamma}(1) \in p^{-1}(b)$. In general, we have:

$$\gamma \simeq \gamma' \xrightarrow{\text{lift}} \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}^{(\prime)} = \gamma^{(\prime)}$$
 (5)

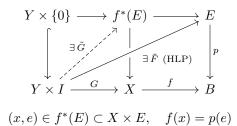
By continuity, $\tilde{\gamma}(0)$, $\tilde{\gamma}'(0) \in F_0$: a path component of $p^{-1}(b)$; similarly, $\tilde{\gamma}(1)$, $\tilde{\gamma}'(1) \in F_1$. In other words, the start and end points of $\tilde{\gamma}$ are confined in path components F_0 and F_1 , respectively. Hence a loop class in $\pi_1(B,b)$ maps to transport between path components:

$$T_{(\cdot)}(e) \colon \pi_1(B,b) \longrightarrow \pi_0(p^{-1}(b))$$

$$[\gamma] \longmapsto T_{[\gamma]}(e)$$
(6)

As a matter of fact, $T_{(\cdot)}(e)$ is a bijection. For $T_{[\gamma]} = T_{[\gamma']}$, they are characterized by two lifted paths $\tilde{\gamma}, \tilde{\gamma}'$; since E is simply connected, they are always homotopic: $\tilde{\gamma} \simeq \tilde{\gamma}'$, hence $[\gamma] = [\gamma']$ by projection p. This means that T is injective. Surjectivity also follows from projection $\gamma = p \circ \gamma'$. Therefore, $T_{(\cdot)}(e)$ gives a bijection between $\pi_1(B,b)$ and $\pi_0(p^{-1}(b))$.

4 Pull-back of fibration is fibration:



We need only verify that $f^*(E) \to X$ also has HLP, i.e. the existence of \tilde{F} in the above diagram⁴. By HLP of $E \xrightarrow{p} B$, $\exists \tilde{F} : Y \times I \to E$ as shown above. We can use \tilde{F} to construct \tilde{G} explicitly; in fact, first consider:

$$\tilde{G} \colon Y \times I \longrightarrow X \times E$$

$$(y,t) \longmapsto (G(y,t), \tilde{F}(y,t))$$
(7)

Note that $f \circ G = p \circ \tilde{F}$; compared with the definition of $f^*(E)$, this implies that the image of \tilde{G} lies within $f^*(E) \subset X \times E$, hence after restriction of its codomain, \tilde{G} becomes a well-defined lifting of G into $f^*(E)$. Therefore, $f^*(E) \to X$ has HLP, i.e. it is also a fibration.

5 More properties of fibration:

- (a) By HLP, given any initial condition $e \in p^{-1}(b_1)$, lifting of any path $b_1 \xrightarrow{\gamma} b_2$ exists. The lifted path with dependence of e can then be written as $F \colon p^{-1}(b_1) \times I \to E$. This is just a generalization of $\boxed{3}$ for non-loop paths.
- (b) Similarly, transport $T_{[\gamma]}$ defined in $\boxed{\mathbf{3}}$ can be generalized for non-loop paths. $T_{[\gamma]}$ is well-defined for path class $[\gamma]$, since by HLP homotopic paths can be lifted to homotopy in E. Therefore, the transport is fixed up to homotopy, i.e.

$$T: \operatorname{Hom}_{\Pi_{1}B}(b_{0}, b_{1}) \longrightarrow \operatorname{Hom}_{\underline{\operatorname{hTop}}} \left(p^{-1}(b_{0}), p^{-1}(b_{1}) \right)$$

$$[\gamma] \longmapsto T_{[\gamma]}$$
(8)

Note that T defined in this way is also independent of the choice of F, since F simply specifies the starting point of the lifted path; no matter which F we choose, the lifted paths will always be homotopic in E. Hence T is well-defined in the above sense.

- (c) T defined above is a functor: $\Pi_1 B \to \underline{\mathbf{hTop}}$. To verify this, we need only check that it is compatible with composition and maps identity morphisms to identity morphisms. Indeed, $T_{[\mathbb{I}_b]} = [\mathbb{1}_{p^{-1}(b)}]$, and $T_{[\gamma']\star[\gamma]} = T_{[\gamma'\star\gamma]} = T_{[\gamma']} \circ T_{[\gamma]}$ by joining two lifted paths (up to homotopy). \square
- (d) For B: path connected, there exists an isomorphism between any two objects in $\Pi_1 B$ (a path connecting any two points in B), which is mapped to isomorphisms between fibers $p^{-1}(b)$ in **hTop**. Hence any two fibers of $E \xrightarrow{p} B$ have the same homotopy type.

Notice that $f^*(E)$ is the limit of the diagram, hence this is automatically true by the universal property of $f^*(E)$. I would like to thank 刘逸华 for pointing this out. For now, we will stick to a more traditional proof.