

## 1 Example of limit in Vect:

$$I = \left\{ \begin{array}{c} \bullet \\ \star \end{array} \right\} \begin{array}{l} \xrightarrow{F} \left\{ \begin{array}{c} 0 \\ \mathbb{R} \end{array} \right\} \\ \xrightarrow{\Delta(V)} \left\{ \begin{array}{c} V \\ V \end{array} \right\} \end{array} \supset \underline{\mathbf{Vect}}: \mathbb{R} \text{ vector space ,} \quad (1)$$

(a) By definition, we have:

$$\begin{array}{ccc} & \Delta(\lim F) & \\ \exists! \Delta(g) \nearrow & \Downarrow \sigma & \\ \Delta(V) & \xrightarrow{\eta} & F \end{array} \quad (2)$$

Where  $g: V \rightarrow \lim F$ . More intuitively, for the above  $F: I \rightarrow \underline{\mathbf{Vect}}$ , this translates to the following diagram in Vect:

$$\begin{array}{ccc} & \eta_{\bullet} & \\ & \searrow & \\ V & \xrightarrow{\exists! g} \lim F & \xrightarrow{\sigma_{\bullet}} 0 \\ & \nearrow & \\ & \eta_{\star} & \end{array} \quad (3)$$

Now we verify that  $\lim F = \mathbb{R}$ , along with the following choice of  $\sigma$ :

$$\sigma_{\bullet} = 0, \quad \sigma_{\star} = \mathbb{1}_{\mathbb{R}} \quad (4)$$

In fact, for the above diagram (3) to be commutative, we must have  $g = \eta_{\star}$ . Note that such  $g$  is unique once  $\sigma$  is chosen; for our choice of  $\sigma$ , if  $g \neq \eta_{\star}$ , then the diagram *cannot* commute. Hence  $\lim F = \mathbb{R}$ , along with the above choice of  $\sigma: \Delta(\mathbb{R}) \Rightarrow F$ . In other words, we have:

$$\begin{array}{ccc} & \{ \bullet, \star \} & \\ \Delta(\mathbb{R}) \swarrow & \xRightarrow{\sigma} & \searrow F \\ \mathbb{R} & & \{ 0, \mathbb{R} \} \end{array} \quad (5)$$

□

(b)(c) From diagram (2) and discussions in (a), we know that:

$$\exists! \tau = \Delta(g): \Delta(V) \Longrightarrow \Delta(\mathbb{R}) \quad (6)$$

Here  $g: V \rightarrow \mathbb{R}$  is fixed uniquely once  $\sigma$  is fixed. However,  $\sigma$  may vary up to isomorphism; therefore, a generic choice of  $\sigma$  is given by:

$$\sigma_{\bullet} = 0, \quad \sigma_{\star} = k \mathbb{1}_{\mathbb{R}}, \quad k \in \mathbb{R} \quad (7)$$

For such  $g$ , by the same arguments in (a), we have:

$$\exists! g = \frac{1}{k} \eta_*, \tau = \Delta(g) = \frac{1}{k} \Delta(\eta_*), \quad \text{s. t.} \quad (2), (3) \text{ commutes} \quad (8)$$

Note that  $k \in \mathbb{R}$ , for every  $k$  there is a different  $g$  and  $\tau$ ; hence there are  $\|\{k\}\| = \|\mathbb{R}\|$  many choices of  $\tau$  to make the diagram commute. In particular, for  $k = 1$  we recover  $\tau = \Delta(\eta_*)$ . ■

## 2 Limit and colimit of polynomial ring:

By definition,

$$\begin{array}{c}
 Q \\
 \vdots \downarrow \exists! f \\
 \lim F \\
 \begin{array}{ccccc}
 & \eta_{n+1} & & \eta_n & \\
 & \swarrow & \pi_{n+1} & \searrow & \\
 \dots & \xleftarrow{p_{n+1}} & \mathbb{Z}[x]/x^{n+1} & \xrightarrow{p_n} & \mathbb{Z}[x]/x^n & \xrightarrow{p_{n-1}} & \dots & \xrightarrow{p_1} & \mathbb{Z}[x]/x^1 = \mathbb{Z}
 \end{array} \\
 \begin{array}{ccccc}
 & \sigma_{n+1} & & \sigma_n & \\
 & \swarrow & \tau_{n+1} & \searrow & \\
 & \searrow & \tau_n & \swarrow & \\
 & \text{colim } F & & & \\
 \vdots \downarrow \exists! g \\
 R
 \end{array}
 \end{array} \quad (9)$$

Here  $p_n: \mathbb{Z}[x]/x^{n+1} \rightarrow \mathbb{Z}[x]/x^n$  is the natural projection.

Intuitively, if such  $\lim F$  exists, it shall be the “smallest” object that “contains”  $\mathbb{Z}[x]/x^n$  when  $n \rightarrow \infty$ . Note that  $\mathbb{Z}[x]/x^n$  is naturally a  $\mathbb{Z}^n$  vector space:

$$\mathbb{Z}[x]/x^n \ni \sum_{m=0}^{n-1} a_m x^m \sim (a_0, a_1, \dots, a_{n-1}) \in \mathbb{Z}^n \quad (10)$$

While  $n \rightarrow \infty$ , this gives an  $\infty$ -tuple which corresponds to the *formal power series*<sup>1</sup>:

$$\mathbb{Z}[[x]] \ni \sum_{m=0}^{\infty} a_m x^m \quad (11)$$

The difference between  $\mathbb{Z}[x]$  and  $\mathbb{Z}[[x]]$  is that the latter may contain infinite series while the former may not. Now we confirm that, indeed,  $\lim F = \mathbb{Z}[[x]]$ , along with natural projections  $\pi_n: \mathbb{Z}[[x]] \rightarrow \mathbb{Z}[x]/x^n$ .

In fact, the  $f: R \rightarrow \lim F$  in (9) can be explicitly written down as:

$$\begin{aligned}
 f &= \eta_1 + x \left( \frac{d}{dx} \eta_2 \right) + x^2 \left( \frac{1}{2} \frac{d}{dx} \eta_3 \right) + \dots = \sum_{m=0}^{\infty} x^m \left\{ \frac{1}{m!} \frac{d^m}{dx^m} \eta_{m+1} \right\} \\
 &\sim (\eta_{1,0}, \eta_{2,1}, \dots, \eta_{n,n-1}, \dots)
 \end{aligned} \quad (12)$$

<sup>1</sup>See Wikipedia: *Formal power series*. This is in fact the *adic completion* of  $\mathbb{Z}[x]$ . I would like to thank 刘逸华 & 谢贤进 for this hint.

Here  $\frac{1}{m!} \frac{d^m}{dx^m}$  is used to extract the  $a_{n-1}$  coefficient in  $\mathbb{Z}[x]/x^n$ ; this is the last component of  $\eta_n$ , denoted by  $\eta_{n,n-1}$ . Any other choice of  $f$  will break commutativity of (9), hence  $f$  is fixed uniquely by  $\mathbb{Z}[[x]]$  and  $\pi_n$ 's. Therefore,  $\lim F = \mathbb{Z}[[x]]$ .  $\square$

On the other hand,  $\text{colim } F$  is the “largest” object that any map *out of*  $\mathbb{Z}[x]/x^n$  must “pass through”. Also, this should hold for all  $n \in \mathbb{Z}_+$ . Naturally, projections  $\sigma_n: \mathbb{Z}[x]/x^n \rightarrow \mathbb{Z}$  satisfy the above requirements; we have:

$$\sigma_n: x \mapsto 0, \quad \sum_{m=0}^{n-1} a_m x^m \mapsto a_0, \quad (13)$$

$$\sigma_{n+1} = p_1 \circ p_2 \circ \cdots \circ p_n \quad (14)$$

$g: \mathbb{Z} \rightarrow R$  in (9) is fixed uniquely for such choice of  $\sigma_n$ ; in fact, descend along the  $p_n$  tower in (9), and we have:  $\tau_{n+1} = \tau_n \circ p_n = \tau_{n-1} \circ p_{n-1} \circ p_n = \cdots = \tau_1 \circ p_1 \circ p_2 \circ \cdots \circ p_n = \tau_1 \circ \sigma_{n+1}$ ,  $\forall n \in \mathbb{Z}_+$ , hence  $\exists! g = \tau_1$ . Therefore,  $\text{colim } F = \mathbb{Z}$ .  $\blacksquare$

### 3 Example of push-out in Groupoid:

$$\begin{array}{ccc} \{0, 1\} & \xrightarrow{f_1} & \bullet \\ \downarrow f_2 & & \downarrow \tau_1 \\ \{0 \leftrightarrow 1\} & \xrightarrow{\tau_2} & P \end{array} \quad \begin{array}{c} \nearrow \eta_1 \\ \searrow \eta_2 \\ \downarrow \exists! g \end{array} \quad \begin{array}{c} \\ \\ Q \end{array} \quad (15)$$

Following the same observation as before, the push-out  $P$  is the “largest” object that any map out of  $\bullet$  and  $\{0 \leftrightarrow 1\}$  must pass through. By such universal property,  $P$  can be no larger than the coproduct:  $\{\bullet\} \coprod \{0 \leftrightarrow 1\}$ . However, we should also consider the equivalence imposed by:

$$\bullet \xleftarrow{f_1} \{0, 1\} \xrightarrow{f_2} \{0 \leftrightarrow 1\} \quad (16)$$

Therefore, we simply have  $P = \bullet$ , with  $\tau_{1,2}$  the natural projection. This can be verified with ease: we have  $g = \eta_1$ . It is unique since its image is a single point (with identity map to itself)  $\star \in Q$ , and the point  $\star$  is fixed by commutativity.  $\blacksquare$

### 4 Product and coproduct in Ab:

For  $G_\alpha \in \mathbf{Ab} \subset \mathbf{Group}$ , note that we have:

$$\mathbf{Free} : \mathbf{Set} \rightleftarrows \mathbf{Group} : \mathbf{Forget} \quad (17)$$

Therefore, for  $F$ : some diagram in Group,  $\boxed{\lim (\mathbf{Forget} \circ F) = \mathbf{Forget} \circ \lim F}$  if  $\lim F$  exists.

By definition, the product  $\prod_\alpha G_\alpha \in \mathbf{Group}$  is a limit, hence it is identical (as in Set) to the *direct product*, with additional entry-wise group multiplication. Same applies for the full subcategory: abelian group  $\mathbf{Ab} \subset \mathbf{Group}$ .

On the other hand, the disjoint union of  $G_\alpha$ 's as sets will not necessary be a group, the identities  $1_\alpha \in G_\alpha$  must be glued together to produce a group structure. Furthermore, free-forgetful adjunction

(17) implies that for  $F'$ : some diagram in **Set**,

$$\operatorname{colim} (\operatorname{Free} \circ F') = \operatorname{Free} \circ \operatorname{colim} F', \quad (18)$$

Whenever  $\operatorname{colim} F'$  exists; in our case,  $\operatorname{colim} F'$  is the disjoint union of sets:  $\coprod_{\alpha} \operatorname{Forget}(G_{\alpha})$ . Therefore, we should construct a free object in **Ab**.

Here we restrict our discussion to **Ab**, since the coproduct in **Ab** is *not* the same as in **Group** — the free product of abelian group is not necessary abelian. Hence, the coproduct in **Ab** shall be:

$$\coprod_{\alpha} G_{\alpha} = \bigoplus_{\alpha} G_{\alpha}, \quad i_{\alpha}: G_{\alpha} \hookrightarrow \bigoplus_{\alpha} G_{\alpha} \quad (19)$$

As a set, this is precisely the disjoint union with identities  $0_{\alpha} \in G_{\alpha} \subset \mathbf{Ab}$  glued together.

It is then straight-forward to verify its universal property: for  $f_{\alpha}: G_{\alpha} \rightarrow H$ ,

$$\exists! f: \bigoplus_{\alpha} G_{\alpha} \longrightarrow H, \quad (g_{\alpha})_{\alpha} \mapsto \sum_{\alpha} f_{\alpha}(g_{\alpha}) \quad (20)$$

This is compatible with the abelian group multiplication. Note that for the summation to be well-defined, the coproduct must only contain finitely many components; otherwise it is identical to the product in **Ab**. ■

### 5 Composition of pull-backs:

$$(21)$$

(a) If  $A_1$  is the pull-back of  $A_2$  and  $A_2$  is the pull-back of  $A_3$ , then given  $Q$  with  $f_3, g_1$ , we have  $g_2 = \psi \circ g_1$ , and  $f_2$  is fixed uniquely by universal property of  $A_2$ , while  $f_1$  is fixed uniquely by universal property of  $A_1$ . Hence,  $A_1$  is the pull-back of  $A_3$ .

(b) If  $A_1, A_2$  are pull-backs of  $A_3$ , then given  $Q$  with  $f_2, g_1$ , we have  $f_3 = \phi \circ f_2$ , and  $f_2$  is fixed uniquely by universal property of  $A_2$ , and  $f_1$  is fixed uniquely by universal property of  $A_1$ . Hence,  $A_1$  is the pull-back of  $A_3$ .



☞ PAST WORK, AS TEMPLATE ☞

**1** For  $F_i \rightarrow E_i \xrightarrow{p_i} B$ : coverings in  $\text{Cov}_0(B)$  with  $E_i$ : connected and  $B$ : path connected and locally path connected, the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow p_1 & \swarrow p_2 \\ & B & \end{array} \quad \begin{array}{l} e_2 = f(e_1), \\ b = p_1(e_1) = p_2(e_2), \end{array}$$

To show that  $f$  is itself a covering, we need only verify that  $f$  is locally trivial with some discrete fiber  $F$ . In fact, given any  $e_2 \in E_2$  and  $b = p_2(e_2)$ , there exists some neighborhood  $U \subset B$  that the following diagram holds (by restriction):

$$\begin{array}{ccc} U \times F_1 & \xrightarrow{f} & U \times F_2 \\ & \searrow p_1 & \swarrow p_2 \\ & U & \end{array} \quad \begin{array}{l} e_1 = (b, k_1), \\ e_2 = (b, k_2(b, k_1)), \end{array} \quad k_i \in F_i$$

Generally,  $k_2 = k_2(b, k_1)$  depends on the base point  $b \in B$ . However, since  $B$  is locally path connected, we can restrict  $U$  to be path connected, while  $k_2 \in F_2$ : discrete. Since continuous maps preserve path connectedness,  $k_2$  is in fact independence of  $b$ , i.e.  $k_2 = \varphi(k_1)$ .

On the other hand,  $\forall e_2 = (b, k_2) \in U \times \{k_2\} \subset E_2$ , we have its preimage  $f^{-1}(e_2) = \{b\} \times \varphi^{-1}(k_2)$ . Note that  $E_2$  is connected while  $\varphi^{-1}(k_2) \in F_1$  is discrete; for the same reasoning as above,  $\varphi^{-1}(k_2) = F$  is in fact independent of  $k_2$ . This is the discrete fiber  $F$  we have been looking for. Hence  $f$  is also a covering map<sup>2</sup>. ■

**2** Cylinder with ends pinched —  $\pi_1$  and universal cover:

$$Y = (X \times I) / (X \times \partial I), \quad I = [0, 1] \quad (22)$$

Note that  $Y$  is homeomorphic to two cones<sup>3</sup>  $CX_1 \amalg CX_2$  with “bases”  $X_i \subset CX_i$  and “vertices”  $v_i$  respectively identified:  $X_1 \sim X_2$ ,  $v_1 \sim v_2 \equiv v$ .  $X$  is path connected and so is  $Y$ , hence we are free to choose  $\pi_1(Y) = \pi_1(Y, y_0)$ .

First note that paths that do *not* pass through the vertex  $v$  are all homotopic, since they are contained in a cone and cones are contractible<sup>4</sup>. Therefore all contributions to  $\pi_1(Y)$  are loop classes that *do* pass through the vertex  $v$ . In other words, morphisms in  $\Pi_1 Y$  are in one-to-one correspondence with morphisms in:

$$\Pi_1([0, 1] / 0 \sim 1) = \Pi_1 S^1 \quad (23)$$

Therefore,  $\pi_1(Y) \cong \pi_1(S^1) = \mathbb{Z}$ . □

The universal cover  $\tilde{Y}$  of  $Y$  can be constructed by assigning an induced topology to the space of path classes, same as in the general proof of its existence. Since  $Y$  is “degenerate” at its vertex,

<sup>2</sup>Reference: [math.stackexchange.com/a/109774](https://math.stackexchange.com/a/109774).

<sup>3</sup>See discussions from Problem Set №1.

<sup>4</sup> $[\gamma_1] = [\gamma_2 \star \gamma_2^{-1} \star \gamma_1] = [\gamma_2]$ .

this is equivalent to “cutting open”  $Y$  at its vertex  $v$ , and joining  $\mathbb{Z}$  copies them end-to-end. More explicitly, it can be written as:

$$\tilde{Y} = (X \times \mathbb{R}) / \sim, \quad (x, n) \sim (x', n), \quad \forall x \in X, n \in \mathbb{Z} \quad (24)$$

While the covering map:  $\tilde{Y} \ni [x, t] \mapsto [x, t - [t]] \in Y$ , here  $[t]$  is the integer part of  $t \in \mathbb{R}$ .  $\blacksquare$

### **3 $\pi_1$ of fiber in fibration:**

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\quad} & E \\ \downarrow & \searrow \exists \tilde{f} & \downarrow p \\ X \times I & \xrightarrow{\quad f \quad} & B \end{array}$$

For  $F \rightarrow E \xrightarrow{p} B$ : fibration, by homotopy lifting property (HLP), any homotopy in  $B$  can be uniquely lifted to path class in  $E$ , provided some “initial condition”  $X \times \{0\}$ . This leads to the following results:

(a) For  $B$ : simply-connected, take any loop class  $[\tilde{\gamma}] \in \pi_1(E, e)$  as initial condition; its projection  $[p \circ \tilde{\gamma}] \in \pi_1(B, b) = \{[\mathbb{1}_b]\}$  is trivial, i.e.  $p \circ \tilde{\gamma} \simeq \mathbb{1}_b$ . By HLP, such homotopy can be lifted into  $E$ , i.e.

$$p \circ \tilde{\gamma} \simeq \mathbb{1}_b \xrightarrow{\text{lift}} \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}' = \mathbb{1}_b \quad (25)$$

In other words,  $\tilde{\gamma} \simeq \tilde{\gamma}' \subset p^{-1}(b)$ , i.e. any loop in  $E$  is homotopic to some loop in  $p^{-1}(b) \cong F$ . This implies a surjective group homomorphism  $\pi_1(p^{-1}(b), e) \rightarrow \pi_1(E, e)$ , i.e. an epimorphism.  $\square$

(b) For  $E$ : simply-connected, take any loop class  $[\gamma] \in \pi_1(B, b)$  and consider its lifting  $[\tilde{\gamma}]$ . Note that in general  $\tilde{\gamma}$  is *not* a loop; however, we have  $p \circ \tilde{\gamma} = \gamma$ , hence  $\tilde{\gamma}(0), \tilde{\gamma}(1) \in p^{-1}(b)$ . In general, we have:

$$\gamma \simeq \gamma' \xrightarrow{\text{lift}} \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}' = \gamma' \quad (26)$$

By continuity,  $\tilde{\gamma}(0), \tilde{\gamma}'(0) \in F_0$ : a path component of  $p^{-1}(b)$ ; similarly,  $\tilde{\gamma}(1), \tilde{\gamma}'(1) \in F_1$ . In other words, the start and end points of  $\tilde{\gamma}$  are confined in path components  $F_0$  and  $F_1$ , respectively. Hence a loop class in  $\pi_1(B, b)$  maps to *transport* between path components:

$$\begin{aligned} T_{(\cdot)}(e): \pi_1(B, b) &\longrightarrow \pi_0(p^{-1}(b)) \\ [\gamma] &\longmapsto T_{[\gamma]}(e) \end{aligned} \quad (27)$$

As a matter of fact,  $T_{(\cdot)}(e)$  is a bijection. For  $T_{[\gamma]} = T_{[\gamma']}$ , they are characterized by two lifted paths  $\tilde{\gamma}, \tilde{\gamma}'$ ; since  $E$  is simply connected, they are always homotopic:  $\tilde{\gamma} \simeq \tilde{\gamma}'$ , hence  $[\gamma] = [\gamma']$  by projection  $p$ . This means that  $T$  is injective. Surjectivity also follows from projection  $\gamma = p \circ \tilde{\gamma}'$ . Therefore,  $T_{(\cdot)}(e)$  gives a bijection between  $\pi_1(B, b)$  and  $\pi_0(p^{-1}(b))$ .  $\blacksquare$

### **4 Pull-back of fibration is fibration:**

$$\begin{array}{ccccc}
Y \times \{0\} & \longrightarrow & f^*(E) & \longrightarrow & E \\
\downarrow & \nearrow \exists \tilde{G} & \downarrow & \nearrow \exists \tilde{F} \text{ (HLP)} & \downarrow p \\
Y \times I & \xrightarrow{G} & X & \xrightarrow{f} & B
\end{array}$$

$$(x, e) \in f^*(E) \subset X \times E, \quad f(x) = p(e)$$

We need only verify that  $f^*(E) \rightarrow X$  also has HLP, i.e. the existence of  $\tilde{F}$  in the above diagram<sup>5</sup>. By HLP of  $E \xrightarrow{p} B$ ,  $\exists \tilde{F}: Y \times I \rightarrow E$  as shown above. We can use  $\tilde{F}$  to construct  $\tilde{G}$  explicitly; in fact, first consider:

$$\begin{aligned}
\tilde{G}: Y \times I &\longrightarrow X \times E \\
(y, t) &\longmapsto (G(y, t), \tilde{F}(y, t))
\end{aligned} \tag{28}$$

Note that  $f \circ G = p \circ \tilde{F}$ ; compared with the definition of  $f^*(E)$ , this implies that the image of  $\tilde{G}$  lies within  $f^*(E) \subset X \times E$ , hence after restriction of its codomain,  $\tilde{G}$  becomes a well-defined lifting of  $G$  into  $f^*(E)$ . Therefore,  $f^*(E) \rightarrow X$  has HLP, i.e. it is also a fibration. ■

#### 5 More properties of fibration:

(a) By HLP, given any initial condition  $e \in p^{-1}(b_1)$ , lifting of any path  $b_1 \xrightarrow{\gamma} b_2$  exists. The lifted path with dependence of  $e$  can then be written as  $F: p^{-1}(b_1) \times I \rightarrow E$ . This is just a generalization of [3] for non-loop paths. □

(b) Similarly, transport  $T_{[\gamma]}$  defined in [3] can be generalized for non-loop paths.  $T_{[\gamma]}$  is well-defined for path class  $[\gamma]$ , since by HLP homotopic paths can be lifted to homotopy in  $E$ . Therefore, the transport is fixed up to homotopy, i.e.

$$\begin{aligned}
T: \text{Hom}_{\Pi_1 B}(b_0, b_1) &\longrightarrow \text{Hom}_{\mathbf{hTop}}(p^{-1}(b_0), p^{-1}(b_1)) \\
[\gamma] &\longmapsto T_{[\gamma]}
\end{aligned} \tag{29}$$

Note that  $T$  defined in this way is also independent of the choice of  $F$ , since  $F$  simply specifies the starting point of the lifted path; no matter which  $F$  we choose, the lifted paths will always be homotopic in  $E$ . Hence  $T$  is well-defined in the above sense. □

(c)  $T$  defined above is a functor:  $\Pi_1 B \rightarrow \mathbf{hTop}$ . To verify this, we need only check that it is compatible with composition and maps identity morphisms to identity morphisms. Indeed,  $T_{[1_b]} = [1_{p^{-1}(b)}]$ , and  $T_{[\gamma'] \star [\gamma]} = T_{[\gamma' \star \gamma]} = T_{[\gamma']} \circ T_{[\gamma]}$  by joining two lifted paths (up to homotopy). □

(d) For  $B$ : path connected, there exists an isomorphism between any two objects in  $\Pi_1 B$  (a path connecting any two points in  $B$ ), which is mapped to isomorphisms between fibers  $p^{-1}(b)$  in  $\mathbf{hTop}$ . Hence any two fibers of  $E \xrightarrow{p} B$  have the same homotopy type. ■

<sup>5</sup>Notice that  $f^*(E)$  is the limit of the diagram, hence this is automatically true by the universal property of  $f^*(E)$ . I would like to thank 刘逸华 for pointing this out. For now, we will stick to a more traditional proof.