# String Theory №3

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## $\boxed{1}$ Strings on Curved Space:

$$S = \frac{1}{4\pi\alpha'}\int_M \mathrm{d}^2\sigma\,\sqrt{g}\,\Big(i\epsilon^{ab}B_{\mu\nu}(X)\,\partial_a X^\mu\partial_b X^\nu + \cdots\Big), \tag{1}$$

$$T^a{}_a = -\frac{1}{2\alpha'}\,\beta^G_{\mu\nu}\,g^{ab}\partial_a X^\mu\partial_b X^\nu + \cdots, \tag{2}$$

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} - \frac{1}{4}\,\alpha' H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega} + \cdots + \mathcal{O}\big(\alpha'^2\big) \tag{3}$$

We want to verify the coefficient of $\alpha' H^2$ term in $\beta^G_{\mu\nu}$; for convenience we've omitted non-related terms in the above expressions.

Note that at $\mathcal{O}(\alpha')$ such term does not depend on the metric $G_{\mu\nu}$, and it depends only on the field strength $H = \mathrm{d}B$, not the potential $B$, hence it's safe to assume:

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad B_{\mu\nu} = \frac{1}{3}H_{\mu\nu\rho}X^\rho, \quad H = \text{const}, \tag{4}$$

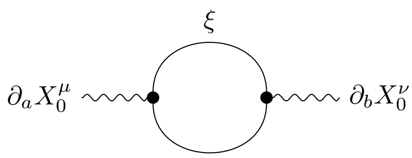
$$i\epsilon^{ab}B_{\mu\nu}(X)\,\partial_a X^\mu\partial_b X^\nu = \frac{i}{3}H_{\mu\nu\rho}\,X^\rho\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu, \tag{5}$$

We consider small perturbation away from the classical saddle: $X = X_0 + \xi$, then the 1-loop effective action is obtained by integrating over $\mathcal{O}\big(\xi^2\big)$ terms in the perturbed action[1]:

$$\Gamma^{(1)}[X_0] = -\ln\int \mathscr{D}\xi\; e^{-S^{(2)}[X_0,\xi]}, \tag{6}$$

$$\begin{aligned}
\mathcal{L}^{(2)} &= \frac{i}{3}H_{\mu\nu\rho}\,\epsilon^{ab}\Big(\xi^\rho\,\partial_a X_0^\mu\,\partial_b\xi^\nu + \xi^\rho\,\partial_a\xi^\mu\,\partial_b X_0^\nu + X_0^\rho\,\partial_a\xi^\mu\,\partial_b\xi^\nu\Big)\\
&\sim \frac{i}{3}H_{\mu\nu\rho}\,\epsilon^{ab}\Big(\xi^\rho\,\partial_a X_0^\mu\,\partial_b\xi^\nu - \xi^\rho\,\partial_a X_0^\nu\,\partial_b\xi^\mu - \xi^\mu\,\partial_a X_0^\rho\,\partial_b\xi^\nu\Big)\\
&= \frac{i}{3}H_{\mu\nu\rho}\,\epsilon^{ab}\cdot 3\xi^\rho\,\partial_a X_0^\mu\,\partial_b\xi^\nu\\
&= iH_{\mu\nu\rho}\,\epsilon^{ab}\,\partial_a X_0^\mu\,(\xi^\rho\partial_b\xi^\nu)
\end{aligned} \tag{7}$$

Here we've used the anti-symmetric properties of $H_{\mu\nu\rho}, \epsilon^{ab}$, and ignored any total derivative after integration by parts. This term introduces a cubic interaction vertex in the free background; therefore, $\Gamma^{(1)}$ can be expressed in the following diagram[2]:




$$\sim \frac{1}{2!}\left(\frac{1}{\alpha'}\right)^2\int \mathrm{d}^2p\,\Big(iH_{\mu\nu\rho}\,\epsilon^{ab}\,\partial_a X_0^\mu\,ip_b\Big)\frac{2}{p^4}\left(-\frac{\alpha'}{2}\right)^2\Big(iH_{\mu'}{}^{\nu\rho}\,\epsilon^{a'b'}\,\partial_{a'}X_0^{\mu'}\,ip_{b'}\Big) \tag{8}$$

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[1] Reference: Prof. Xi Yin's String Notes, see also `arXiv:0812.4408`.

[2] References:

- David Tong, *String Theory*;
- Callan & Thorlacius, *Sigma Models and String Theory*;
- Timo Weigand, *Introduction to String Theory*.

$$= \frac{2}{2!}\left(\frac{1}{\alpha'}\right)^2\left(-\frac{\alpha'}{2}\right)^2 H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega}\partial_a X_0^\mu\,\partial_b X_0^\nu \int \mathrm{d}^2p\,\frac{p^2g^{ab}-p^ap^b}{p^4} \tag{9}$$

$$= \frac{2}{2!}\left(-\frac{1}{2}\right)^2 H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega}\partial_a X_0^\mu\,\partial_b X_0^\nu \left(\frac{1}{2}\,g^{ab}\right)\int \mathrm{d}^2p\,\frac{1}{p^2} \tag{10}$$

$$= \frac{2}{2!}\left(-\frac{1}{2}\right)^2\left(\frac{1}{2}\right) H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega}\partial_a X_0^\mu\,\partial_b X_0^\nu\, g^{ab}\int \mathrm{d}^2p\,\frac{1}{p^2} \tag{11}$$

$$= \frac{1}{8}H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega}g^{ab}\partial_a X_0^\mu\,\partial_b X_0^\nu \int \mathrm{d}^2p\,\frac{1}{p^2} \tag{12}$$

Here the $\left(\frac{1}{\alpha'}\right)^2$ coefficient comes from the vertices, while $\left(-\frac{\alpha'}{2}\right)^2$ comes from the propagators. The $p^ap^b$ integral provides an additional $(\frac{1}{2})$ factor. The overall normalization is chosen to match the $\alpha' R_{\mu\nu}$ coefficient in $\beta^G_{\mu\nu} \subset T^a{}_a$, which is $\frac{1}{1!}\times(-\frac{1}{2})\times 1 = -\frac{1}{2}$. Therefore, we have:

$$T^a{}_a \supset \frac{1}{8}H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega}g^{ab}\partial_a X_0^\mu\,\partial_b X_0^\nu, \tag{13}$$

$$\beta^G_{\mu\nu} \supset -\frac{1}{4}\,\alpha' H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega} \tag{14}$$

■

$\boxed{2}$ **Classical Solutions of 11D SUGRA:** Following the convention of *Polchinski*, we have bosonic action:

$$S = \frac{1}{2\kappa^2}\int\left(\mathrm{d}^{11}x\,\sqrt{-g}\,\mathcal{R} - \frac{1}{2}\,G\wedge *G - \frac{1}{6}\,C\wedge G\wedge G\right), \tag{15}$$

Here $G = \mathrm{d}C$: a 4-form field. In components, the numerical coefficients would be $\frac{1}{2}\mapsto\frac{1}{2\times 4!}=\frac{1}{48}$, and $\frac{1}{6}\mapsto\frac{1}{6\times 3!\times 4!\times 4!}=\frac{1}{20736}$.

Variation of the action yields the EOMs of our theory[3]; Note that:

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\,\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu} \tag{16}$$

$\frac{\delta S}{\delta g^{\mu\nu}}$ is easier to compute in components; note that the $C\wedge G\wedge G$ term does not depend on $g^{\mu\nu}$, therefore it does not contribute to the EOM. We have the usual Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{17}$$

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{\kappa^2}\left(\frac{4}{48}G_{\mu\sigma_1\sigma_2\sigma_3}G_{\nu}{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{2}\,g_{\mu\nu}\cdot\frac{1}{48}\,G^{\sigma_1\sigma_2\sigma_3\sigma_4}G_{\sigma_1\sigma_2\sigma_3\sigma_4}\right) \\ &= \frac{1}{12\kappa^2}\left(G_{\mu\sigma_1\sigma_2\sigma_3}G_{\nu}{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{8}\,g_{\mu\nu}\,G^{\sigma_1\sigma_2\sigma_3\sigma_4}G_{\sigma_1\sigma_2\sigma_3\sigma_4}\right)\end{aligned} \tag{18}$$

[3] Reference: arXiv:hep-th/9912164. I would like to thank *Lucy Smith* for many helpful discussions.

On the other hand, $\frac{\delta S}{\delta C}$ is best carried out using differential forms:

$$
\begin{aligned}
0 = \delta_C S &= -\frac{1}{2\kappa^2}\int \left(\delta G \wedge * G + \frac{1}{6}\Big(\delta C \wedge G \wedge G - 2C \wedge \delta G \wedge G\Big)\right) \\
&= -\frac{1}{2\kappa^2}\int \left(\delta(\mathrm{d}C) \wedge * G + \frac{1}{6}\big(\delta C \wedge G \wedge G + 2\,\delta(\mathrm{d}C) \wedge C \wedge G\big)\right) \\
&= -\frac{1}{2\kappa^2}\int \left(-(-1)^3\,\delta C \wedge \mathrm{d} * G + \frac{1}{6}\big(\delta C \wedge G \wedge G - 2\,(-1)^3\,\delta C \wedge \mathrm{d}(C \wedge G)\big)\right) \\
&= -\frac{1}{2\kappa^2}\int \delta C \wedge \left(\mathrm{d} * G + \frac{1}{6}\Big(G \wedge G + 2\big(G \wedge G - C \wedge \cancel{\mathrm{d}^2 C}\big)\Big)\right) \\
&= -\frac{1}{2\kappa^2}\int \delta C \wedge \left(\mathrm{d} * G + \frac{1}{2}\,G \wedge G\right),
\end{aligned}
\tag{19}
$$

$$
\mathrm{d} * G + \frac{1}{2}\,G \wedge G = 0 \tag{20}
$$

(a) We hope to find a spacetime solution which is *maximally symmetric* in *some* directions; assume that these directions form a $d$-dimensional sub-manifold $\mathcal{M}_d$ with:

$$
\begin{aligned}
\text{Coordinates:} \quad & x^{\mu'}, \ \mu' \in \Delta \subset \{0, 1, \cdots, 11\}, \\
\text{Induced metric:} \quad & g' = g|_{\mathcal{M}_d}
\end{aligned}
\tag{21}
$$

The entire spacetime is then a direct product: $\mathcal{M}_d \times \widetilde{\mathcal{M}}_{11-d}$. For $\mathcal{M}_d$ to be maximally symmetric, we expect that $\kappa^2 T_{\mu'\nu'} = -\Lambda g'_{\mu'\nu'}$, i.e. the $G$-field serves as a cosmological constant $\Lambda$. By staring at (18) we find that this can be achieved with[4]:

$$
d = 4, \quad G_{\sigma_1\sigma_2\sigma_3\sigma_4} = \alpha\sqrt{|g'|}\,\epsilon_{\sigma_1\sigma_2\sigma_3\sigma_4}, \quad G^{\sigma_1\sigma_2\sigma_3\sigma_4} = \alpha\,\frac{\operatorname{sgn} g'}{\sqrt{|g'|}}\,\epsilon^{\sigma_1\sigma_2\sigma_3\sigma_4}, \quad \{\sigma_i\} \subset \Delta, \tag{22}
$$

$$
G_{\cdots\,\sigma\,\cdots} = 0, \quad \sigma \notin \Delta, \tag{23}
$$

$$
T_{\mu\nu} = (\operatorname{sgn} g')\,\frac{\alpha^2}{12\kappa^2}\left(3!\,g'_{\mu\nu} - \frac{4!}{8}\,g_{\mu\nu}\right) = (\operatorname{sgn} g')\,\frac{\alpha^2}{2\kappa^2}\left(g'_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\right), \tag{24}
$$

$$
\Lambda g_{\mu\nu} = \mp(\operatorname{sgn} g')\,\frac{\alpha^2}{4\kappa^2}\,g_{\mu\nu}, \qquad
\begin{cases}
-: \ \mu = \mu', \nu = \nu' \in \Delta, & \sim \mathcal{M}_4 \\
+: \ \mu, \nu \notin \Delta, & \sim \widetilde{\mathcal{M}}_7
\end{cases}
\tag{25}
$$

Matter EOM is trivially satisfied due to anti-symmetricity. We see that the other component $\widetilde{\mathcal{M}}_7$ is also maximally symmetric, but with an opposite sign in its cosmological constant.

The field equations in $\mathcal{M}_4$ and $\widetilde{\mathcal{M}}_7$ are both of the form $R_{\mu\nu} \propto g_{\mu\nu}$. For $\operatorname{sgn} g' = -1$ i.e. Lorentzian signature, the solution is flat, AdS or dS, depending on the sign of $\Lambda$; for $\operatorname{sgn} g' = -1$, the solution is flat, spherical or hyperbolic. Therefore, we have:

$$
\begin{aligned}
\operatorname{sgn} g' = -1, \quad \Lambda_{4,7} = \pm\frac{\alpha^2}{4\kappa^2}, \quad \mathcal{M}_4 = \mathrm{AdS}_{3,1}, \quad \widetilde{\mathcal{M}}_7 = S^7 \\
\operatorname{sgn} g' = +1, \quad \Lambda_{4,7} = \mp\frac{\alpha^2}{4\kappa^2}, \quad \mathcal{M}_4 = S^4, \quad \widetilde{\mathcal{M}}_7 = \mathrm{AdS}_{6,1}
\end{aligned}
\tag{26}
$$

[4] This is in fact the famous *Freund–Robin ansatz*; see `Wikipedia:` *Freund – Rubin compactification*, and also the original paper: Freund & Robin, *Dynamics of Dimensional Reduction*, 1980.

(b) Global supersymmetries of a theory with the above $\mathrm{AdS}_{4/7} \times S^{4/7}$ background are given by the solutions of:

$$0 = \delta_\eta \psi^\mu \equiv D^\mu \eta(x), \quad \eta\colon \text{spinor}, \tag{27}$$

$$\begin{aligned} D^\mu &= \nabla^\mu + \frac{1}{288}\, G_{\nu\rho\sigma\lambda}\big(\Gamma^{\mu\nu\rho\sigma\lambda} - 8g^{\mu\nu}\Gamma^{\rho\sigma\lambda}\big) \\ &= \nabla^\mu + \frac{1}{288}\, G_{\nu'\rho'\sigma'\lambda'}\big(\Gamma^{\mu\nu'\rho'\sigma'\lambda'} - 8g^{\mu\nu'}\Gamma^{\rho'\sigma'\lambda'}\big) \\ &= \nabla^\mu + \alpha \begin{cases} \dfrac{-8\times 3!}{288}\,(-\Gamma^\mu\gamma_5) = \dfrac{1}{6}\,\Gamma^\mu\gamma_5, \quad \mu = \mu' \in \Delta, & \sim \mathcal{M}_4 \\ \dfrac{4!}{288}\,(-\Gamma^\mu) = -\dfrac{1}{12}\,\Gamma^\mu, \quad \mu \notin \Delta, & \sim \widetilde{\mathcal{M}}_7 \end{cases} \end{aligned} \tag{28}$$

Note that we've replaced the $G$ indices with $\mathcal{M}_4$ indices, since $G$ vanish in $\widetilde{\mathcal{M}}_7$ directions; due to antisymmetricity, the $G$-term can be reduced to simple $\Gamma^\mu$ multiplications according to the $\mu$-direction[5]. Furthermore, the spin connection in $\nabla^\mu$ is also block diagonalized, same as $g_{\mu\nu}$; hence there is a natural separation of variable[6]:

$$\eta = \eta'(x')\,\eta''(x''), \quad D_{\mu'}\eta' = 0, \quad D_{\mu''}\eta'' = 0, \tag{29}$$

$$\mu', \eta', x' \sim \mathcal{M}_4, \quad \mu'', \eta'', x'' \sim \widetilde{\mathcal{M}}_7, \tag{30}$$

Due to the presence of an additional $\Gamma$, $D_{\mu'}\eta' = 0$ has only 4 linearly independent solutions labeled by $\mu'$, while $D_{\mu''}\eta'' = 0$ is Spin(8) (or Spin(7, 1), depending on the signature) invariant, and has $\frac{8\times 7}{2} = 28$ linearly independent solutions[7]. Hence the total number of SUSYs is $4 + 28 = 32$, for $\mathrm{AdS}_{4/7} \times S^{4/7}$ background.

## $\boxed{3}$ SUSY Sigma Models via Superspace:

$$\begin{aligned} D_{\bar\theta}\mathbf{X}^\nu &= \big(\partial_{\bar\theta} + \bar\theta\partial_{\bar z}\big)\big(X^\nu + i\theta\psi^\nu + i\bar\theta\tilde\psi^\nu + \theta\bar\theta F^\nu\big) \\ &= i\tilde\psi^\nu - \theta F^\nu + \bar\theta\,\bar\partial X^\nu - i\theta\bar\theta\,\bar\partial\psi^\nu, \\ D_\theta\mathbf{X}^\mu &= i\psi^\mu + \bar\theta F^\mu + \theta\,\partial X^\mu + i\theta\bar\theta\,\partial\tilde\psi^\mu, \end{aligned} \tag{31}$$

$$\begin{aligned} D_{\bar\theta}\mathbf{X}^\nu D_\theta\mathbf{X}^\mu &= \Big(i\tilde\psi^\nu - \theta F^\nu + \bar\theta\,\bar\partial X^\nu - i\theta\bar\theta\,\bar\partial\psi^\nu\Big)\Big(i\psi^\mu + \bar\theta F^\mu + \theta\,\partial X^\mu + i\theta\bar\theta\,\partial\tilde\psi^\mu\Big) \\ &= -\tilde\psi^\nu\psi^\mu - i\theta\Big(\tilde\psi^\nu\partial X^\mu + \psi^\mu F^\nu\Big) + i\bar\theta\Big(\psi^\mu\bar\partial X^\nu - \tilde\psi^\nu F^\mu\Big) \\ &\quad - \theta\bar\theta\Big(\bar\partial X^\nu\partial X^\mu + \tilde\psi^\nu\partial\tilde\psi^\mu - (\bar\partial\psi^\nu)\psi^\mu + F^\nu F^\mu\Big), \end{aligned} \tag{32}$$

$$\begin{aligned} G_{\mu\nu}(\mathbf{X}) &= G_{\mu\nu} + \Big(i\theta\psi^\lambda + i\bar\theta\tilde\psi^\lambda + \theta\bar\theta F^\lambda\Big)\,\partial_\lambda G_{\mu\nu} + \frac{1}{2}\Big\{i\theta\psi^\rho\partial_\rho,\, i\bar\theta\tilde\psi^\sigma\partial_\sigma\Big\}\,G_{\mu\nu} \\ &= G_{\mu\nu} + \Big(i\theta\psi^\lambda + i\bar\theta\tilde\psi^\lambda\Big)\,G_{\mu\nu,\lambda} + \theta\bar\theta\Big(F^\lambda G_{\mu\nu,\lambda} + \psi^\rho\tilde\psi^\sigma G_{\mu\nu,\rho\sigma}\Big), \end{aligned} \tag{33}$$

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[5] Reference for $\Gamma$-matrices and spinors: *Polchinski* Vol. II, Appendix B. I'm a bit confused about all the complicated conventions, therefore the coefficients might be off by some factors...

[6] See arXiv:hep-th/9912164 for more detailed discussions.

[7] Reference: Achilleas Passias, *Aspects of Supergravity in Eleven Dimensions.*

Note that $\int \mathrm{d}^2\theta = \partial_\theta \partial_{\bar\theta}$, hence we need only focus on the $\theta\bar\theta$ term in the Lagrangian:

$$
\begin{aligned}
4\pi S_G &= \int \mathrm{d}^2 z\, \mathrm{d}^2\theta\, G_{\mu\nu}(\mathbf{X})\, D_{\bar\theta}\mathbf{X}^\mu D_\theta \mathbf{X}^\nu = \int \mathrm{d}^2 z\, \mathrm{d}^2\theta\, (-\theta\bar\theta)\Big(G_{\mu\nu}\big(\partial X^\mu \bar\partial X^\nu + \cdots\big) + \cdots\Big) \\
&= \int \mathrm{d}^2 z\, \bigg( G_{\mu\nu}\Big(\partial X^\mu \bar\partial X^\nu + \tilde\psi^\nu \partial \tilde\psi^\mu - (\bar\partial \psi^\nu)\psi^\mu + F^\nu F^\mu\Big) \\
&\qquad + \tilde\psi^\nu \psi^\mu \Big(F^\lambda G_{\mu\nu,\lambda} + \psi^\rho \tilde\psi^\sigma G_{\mu\nu,\rho\sigma}\Big) \\
&\qquad - G_{\mu\nu,\lambda}\Big(\psi^\lambda\big(\psi^\mu \bar\partial X^\nu - \tilde\psi^\nu F^\mu\big) + \tilde\psi^\lambda\big(\tilde\psi^\nu \partial X^\mu + \psi^\mu F^\nu\big)\Big)\bigg)
\end{aligned}
\tag{34}
$$

Similar result holds for the $B$ contribution $S_B$. We see that there is no $\partial F$ term in the action, hence $F$ is not dynamical and can be integrated out; we have:

$$
0 = \delta_F S = \delta_F (S_G + S_B), \tag{35}
$$

$$
\begin{aligned}
4\pi\, \delta S_G &= \int \mathrm{d}^2 z\, \Big(2G_{\mu\nu} F^\mu\, \delta F^\nu + G_{\mu\nu,\lambda}\big(\tilde\psi^\nu \psi^\mu\, \delta F^\lambda - \tilde\psi^\nu \psi^\lambda\, \delta F^\mu - \tilde\psi^\lambda \psi^\mu\, \delta F^\nu\big)\Big) \\
&= \int \mathrm{d}^2 z\, \Big(2F_\lambda + \big(G_{\mu\nu,\lambda} - G_{\lambda\mu,\nu} - G_{\lambda\nu,\mu}\big)\, \tilde\psi^\nu \psi^\mu\Big)\, \delta F^\lambda \\
&= \int \mathrm{d}^2 z\, \Big(2F_\lambda - 2\Gamma_{\lambda\mu\nu} \tilde\psi^\nu \psi^\mu\Big)\, \delta F^\lambda\,,
\end{aligned}
\tag{36}
$$

$$
4\pi\, \delta S_B = \int \mathrm{d}^2 z\, \Big(0 + \big(B_{\mu\nu,\lambda} + B_{\lambda\mu,\nu} + B_{\nu\lambda,\mu}\big)\, \tilde\psi^\nu \psi^\mu\Big)\, \delta F^\lambda = \int \mathrm{d}^2 z\, H_{\lambda\mu\nu} \tilde\psi^\nu \psi^\mu\, \delta F^\lambda\,,
$$

$$
F_\lambda = \left(\Gamma_{\lambda\mu\nu} - \frac{1}{2} H_{\lambda\mu\nu}\right) \tilde\psi^\nu \psi^\mu, \tag{37}
$$

$$
F^\lambda = \left(\Gamma^\lambda_{\mu\nu} - \frac{1}{2} H^\lambda_{\mu\nu}\right) \tilde\psi^\nu \psi^\mu, \tag{38}
$$

Here we've used the (anti-)symmetry of $G_{\mu\nu}$ and $B_{\mu\nu}$, and we adopt the convention that the Levi-Civita connection $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} = G^{\lambda\lambda'}\Gamma_{\lambda'\mu\nu}$; similar holds for $B_{\mu\nu}$ and $H^\lambda_{\mu\nu}$.

Substitute $F_\lambda$ into $S$, collect the $\psi^0, \psi^2, \tilde\psi^2$ and $\psi^2\tilde\psi^2$ terms respectively, and we have:

$$
\begin{aligned}
4\pi S &= \int \mathrm{d}^2 z\, \bigg( \left(G_{\mu\nu} + B_{\mu\nu}\right) \partial X^\mu \bar\partial X^\nu \\
&\qquad + \left(G_{\mu\nu} + \cancel{B_{\mu\nu}}\right)\Big(\tilde\psi^\mu \partial \tilde\psi^\nu - (\bar\partial \psi^\mu)\psi^\nu\Big) \\
&\qquad - \left(G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda}\right)\Big(\psi^\lambda \psi^\mu \bar\partial X^\nu + \tilde\psi^\lambda \tilde\psi^\nu \partial X^\mu\Big) \\
&\qquad + G_{\mu\nu} F^\mu F^\nu - 2\left(\Gamma_{\lambda\mu\nu} - \frac{1}{2} H_{\lambda\mu\nu}\right) \tilde\psi^\nu \psi^\mu F^\lambda \\
&\qquad + \left(G_{\mu\nu,\rho\sigma} + B_{\mu\nu,\rho\sigma}\right) \tilde\psi^\nu \psi^\mu \psi^\rho \tilde\psi^\sigma \bigg) \\
&= \int \mathrm{d}^2 z\, \bigg( \left(G_{\mu\nu} + B_{\mu\nu}\right) \partial X^\mu \bar\partial X^\nu \\
&\qquad + G_{\mu\nu}\Big(\tilde\psi^\mu \partial \tilde\psi^\nu + \psi^\mu \bar\partial \psi^\nu\Big) - \left(G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda}\right)\Big(\psi^\lambda \psi^\mu \bar\partial X^\nu + \tilde\psi^\lambda \tilde\psi^\nu \partial X^\mu\Big) \\
&\qquad - F_\lambda F^\lambda + \left(G_{\mu\nu,\rho\sigma} + B_{\mu\nu,\rho\sigma}\right) \psi^\mu \psi^\rho \tilde\psi^\nu \tilde\psi^\sigma \bigg)
\end{aligned}
\tag{39}
$$

Here we've performed some integration by parts to clean up the result. Note that some terms involving $B_{\mu\nu}$ vanish conveniently (up to integration by parts) due to anti-symmetricity.

The $\psi^2, \tilde{\psi}^2$ terms in the integrand can be further simplified as follows:

$$
\begin{aligned}
\mathcal{L}_{\psi^2} &= G_{\mu\nu}\psi^\mu\bar{\partial}\psi^\nu - (G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda})\,\psi^\lambda\psi^\mu\bar{\partial}X^\nu \\
&= G_{\mu\nu}\psi^\mu\bar{\partial}\psi^\nu - \left(G_{\mu[\nu,\lambda]} + B_{\mu[\nu,\lambda]}\right)\psi^\lambda\psi^\mu\bar{\partial}X^\nu \\
&= G_{\mu\nu}\psi^\mu\bar{\partial}\psi^\nu - \left(-\Gamma_{\lambda\mu\nu} + \frac{1}{2}H_{\lambda\mu\nu}\right)\psi^\lambda\psi^\mu\bar{\partial}X^\nu \\
&= G_{\mu\nu}\psi^\mu\left(\bar{\partial}\psi^\nu + \left(\Gamma^\nu_{\rho\sigma} - \frac{1}{2}H^\nu_{\rho\sigma}\right)\psi^\rho\bar{\partial}X^\sigma\right) \\
&= G_{\mu\nu}\psi^\mu\left(\bar{\partial}\psi^\nu + \left(\Gamma^\nu_{\rho\sigma} + \frac{1}{2}H^\nu_{\rho\sigma}\right)\psi^\sigma\bar{\partial}X^\rho\right) = G_{\mu\nu}\psi^\mu\bar{\mathcal{D}}\psi^\nu, \\
\mathcal{L}_{\tilde{\psi}^2} &= G_{\mu\nu}\tilde{\psi}^\mu\partial\tilde{\psi}^\nu - (G_{\mu\nu,\lambda} + B_{\mu\nu,\lambda})\,\tilde{\psi}^\lambda\tilde{\psi}^\nu\partial X^\mu \\
&= G_{\mu\nu}\tilde{\psi}^\mu\left(\partial\tilde{\psi}^\nu + \left(\Gamma^\nu_{\rho\sigma} - \frac{1}{2}H^\nu_{\rho\sigma}\right)\tilde{\psi}^\sigma\partial X^\rho\right) = G_{\mu\nu}\tilde{\psi}^\mu\mathcal{D}\tilde{\psi}^\nu,
\end{aligned}
\tag{40}
$$

For the $\psi^2\tilde{\psi}^2$ term, recall that $R_{\mu\nu\rho\sigma} = e_\mu[\nabla_\rho\,, \nabla_\sigma\,]\,e_\nu, \nabla_\sigma e_\nu = e_\lambda\Gamma^\lambda_{\sigma\nu}$, and we have:

$$
\begin{aligned}
\mathcal{L}_{\psi^2\tilde{\psi}^2} &= \psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\left(G_{\mu\rho,\nu\sigma} + B_{\mu\rho,\nu\sigma} + \left(\Gamma_{\lambda\mu\rho} - \frac{1}{2}H_{\lambda\mu\rho}\right)\left(\Gamma^\lambda_{\nu\sigma} - \frac{1}{2}H^\lambda_{\nu\sigma}\right)\right) \\
&= \psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\left(G_{\mu\rho,\nu\sigma} + \Gamma_{\lambda\mu\rho}\Gamma^\lambda_{\nu\sigma} + B_{\mu\rho,\nu\sigma} - \frac{1}{2}\left(\Gamma^\lambda_{\mu\rho}H_{\lambda\nu\sigma} + \Gamma^\lambda_{\nu\sigma}H_{\lambda\mu\rho}\right) + \frac{1}{4}H^\lambda_{\mu\rho}H_{\lambda\nu\sigma}\right) \\
&= \mathcal{L}_G + \mathcal{L}_B + \frac{1}{4}H^\lambda_{\mu\rho}H_{\lambda\nu\sigma}\,\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma,
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
\mathcal{L}_G &= \psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\left(G_{\mu\rho,\nu\sigma} + \Gamma_{\lambda\mu\rho}\Gamma^\lambda_{\nu\sigma}\right) \\
&= \psi^{[\mu}\psi^{\nu]}\tilde{\psi}^{[\rho}\tilde{\psi}^{\sigma]}\left(G_{\mu\rho,\nu\sigma} + \Gamma_{\lambda\mu\rho}\Gamma^\lambda_{\nu\sigma}\right) \\
&= \frac{1}{2}\,\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\left\{\left(\frac{1}{2}\left(G_{\mu\rho,\nu\sigma} - G_{\mu\sigma,\nu\rho}\right) + \Gamma_{\lambda\mu\rho}\Gamma^\lambda_{\nu\sigma}\right) - \left(\cdots\right)_{\rho\leftrightarrow\sigma}\right\} \\
&= \frac{1}{2}R_{\mu\nu\rho\sigma}\,\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma, \\
\mathcal{L}_B &= \frac{1}{2}\nabla_\rho H_{\mu\nu\sigma}\,\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma,
\end{aligned}
\tag{42}
$$

Therefore, the total action is:

$$
\begin{aligned}
S = \frac{1}{4\pi}\int \mathrm{d}^2 z\,\Bigg( &(G_{\mu\nu} + B_{\mu\nu})\,\partial X^\mu\bar{\partial}X^\nu \\
&+ G_{\mu\nu}\left(\tilde{\psi}^\mu\mathcal{D}\tilde{\psi}^\nu + \psi^\mu\bar{\mathcal{D}}\psi^\nu\right) \\
&+ \left(\frac{1}{2}R_{\mu\nu\rho\sigma} + \frac{1}{2}\nabla_\rho H_{\mu\nu\sigma} + \frac{1}{4}H^\lambda_{\mu\rho}H_{\lambda\nu\sigma}\right)\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma\Bigg)
\end{aligned}
\tag{43}
$$

## $\boxed{4}$ Mixed Anomaly Between Diffeomorphism and Axial $U(1)$ Symmetry:

(a) Calculations of such anomaly is (schematically) similar to the usual axial anomaly; instead of the $A_\mu$ legs, we now have two $h_{\mu\nu}$ legs in the triangular diagram.

Again we chose the Pauli–Villars regularization with a regulator field $\psi'$ of mass $M \to \infty$. The $\partial^\mu J^A_\mu$ insertion is then reduced to:

$$
\partial^\mu J^A_\mu = \partial_\mu\left(i\bar{\psi}'\gamma^\mu\gamma^5\psi'\right) = i\bar{\psi}'(2M\gamma^5)\psi' \tag{44}
$$

The fermion–fermion–graviton vertex is given by $h_{\mu\nu}T^{\mu\nu}$, and (up to integration by parts) we have:

$$T^{\mu\nu} = \frac{i}{2}\bar{\psi}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\psi \sim \frac{i}{2}\bar{\psi}\gamma^{(\mu}\big(-2\partial^{\nu)}\big)\psi = -i\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi, \tag{45}$$

$$h_{\mu\nu}T^{\mu\nu} = \bar{\psi}\left(-ih_{\mu\nu}\gamma^{(\mu}\partial^{\nu)}\right)\psi, \tag{46}$$

This is very similar to the $A_\mu$ coupling, except that there is an extra derivative $\partial^\nu$. Denote the polarization of graviton as $\varepsilon_{\mu\nu}$, then in momentum space the interaction vertex $\sim \epsilon_{\mu\nu}\gamma^\mu(k_1^\nu + k_2^\nu)$, and we have:

$$\begin{aligned}
\langle\partial^\mu J^A_\mu\rangle_h &\sim \frac{1}{2!}\times 2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\left(2M\gamma_5\cdot\frac{\not{k}+M}{k^2+M^2}\cdot\cancel{\varepsilon_1(2k+p_1)}\cdot\frac{\not{k}+\not{p}_1+M}{(k+p_1)^2+M^2}\cdot\cancel{\varepsilon_2(2k+2p_1+p_2)}\cdot\frac{\not{k}+\not{p}_1+\not{p}_2+M}{(k+p_1+p_2)^2+M^2}\right)\\
&\sim \int\frac{\mathrm{d}^4k}{(2\pi)^4}\,2M^2(4\epsilon_{\mu\nu\rho\sigma})\,\varepsilon_1^{\mu\mu'}(2k+p_1)_{\mu'}\,p_1^\nu\,\varepsilon_2^{\rho\rho'}(2k+2p_1+p_2)_{\rho'}\,p_2^\sigma\left(\frac{1}{k^2+M^2}\cdots\right)\\
&\sim 8M^2\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(2k+p_1)_{\mu'}(2k+2p_1+p_2)_{\rho'}}{(k^2+M^2)\big((k+p_1)^2+M^2\big)\big((k+p_1+p_2)^2+M^2\big)}\\
&\sim 8M^2\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{4k_{\mu'}k_{\rho'}+p_{1,\mu'}p_{2,\rho'}}{(k^2+M^2)^3}
\end{aligned}\tag{47}$$

There are, in fact, 2 diagrams accounting for this amplitude with $1\leftrightarrow 2$ symmetry; here we simply take one contribution with an additional factor of 2, and imply $1\leftrightarrow 2$ symmetrization in the above expressions.

Note that due to the additional $k_{\mu'}k_{\rho'}$ the integral is no longer finite but logarithmic divergent: $\int^\Lambda \mathrm{d}^4k\,\frac{k^2}{k^6}\sim\ln\Lambda$. More specifically[8], we have:

$$\begin{aligned}
\langle\partial^\mu J^A_\mu\rangle_h &\sim 8M^2\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\,\frac{\mathrm{Vol}\,S^3}{(2\pi)^4}\int\left(\frac{4k_{\mu'}k_{\rho'}k^3\,\mathrm{d}k}{(k^2+M^2)^3}+p_{1,\mu'}p_{2,\rho'}\frac{k^3\,\mathrm{d}k}{(k^2+M^2)^3}\right)\\
&\sim 8M^2\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\,\frac{2\pi^2}{(2\pi)^4}\int\left(\delta_{\mu'\rho'}\frac{k^5\,\mathrm{d}k}{(k^2+M^2)^3}+p_{1,\mu'}p_{2,\rho'}\frac{k^3\,\mathrm{d}k}{(k^2+M^2)^3}\right)\\
&\sim 8M^2\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\,\frac{1}{8\pi^2}\left(\delta_{\mu'\rho'}\frac{1}{2}\ln\frac{\Lambda^2}{M^2}+p_{1,\mu'}p_{2,\rho'}\frac{1}{4M^2}\right)\\
&\sim \frac{1}{4\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\,\varepsilon_2^{\rho\rho'}\left(2\delta_{\mu'\rho'}M^2\ln\frac{\Lambda^2}{M^2}+p_{1,\mu'}p_{2,\rho'}\right)
\end{aligned}\tag{48}$$

The second term is very much similar to the axial anomaly result, while the first term diverges.

However, we believe that the divergent term must be canceled by other diagrams; otherwise, it will contribute a $p^\nu p^\sigma\,\delta_{\mu'\rho'}\varepsilon_1^{\mu\mu'}\varepsilon_2^{\rho\rho'} = p^\nu p^\sigma(\varepsilon_1)^\mu{}_\alpha(\varepsilon_2)^{\rho\alpha}\sim(\partial h)^2$ term in the final result, which is not diff-invariant. The second term, on the other hand, is diff-invariant:

$$R_{\mu\nu\alpha\beta} = p_\beta\,p_{[\nu}\,\varepsilon_{\mu]\alpha} - p_\alpha\,p_{[\nu}\,\varepsilon_{\mu]\beta}, \tag{49}$$

[8] References:

- David Tong, *Gauge Theory*;
- A. Zee, *QFT in a Nutshellz*;
- `arXiv:0802.0634`;
- `Wikipedia`: *Common integrals in quantum field theory*.

$$
\begin{aligned}
\langle \partial^\mu J^A_\mu \rangle_h &\sim \frac{1}{4\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,(\varepsilon^{\mu\mu'} p_{1,\mu'} p_1^\nu)(\varepsilon^{\rho\rho'} p_{2,\rho'} p_2^\sigma) \\
&\sim \frac{1}{4\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\frac{1}{4!\times 2\times 2}\times\frac{1}{2} R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} \\
&\sim \frac{1}{768\pi^2}\,\epsilon_{\mu\nu\rho\sigma} R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta}
\end{aligned}
\tag{50}
$$

(b) The next order contribution would come from the covariant derivative[9]:

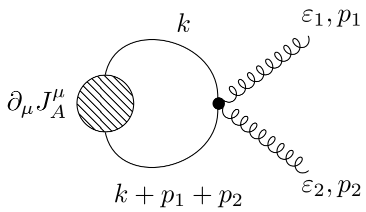
$$
\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{2}\,\omega_\mu{}^{ab}\sigma_{ab}\psi \tag{51}
$$

Where $\omega_\mu{}^{ab}$ is the spin connections, and $\sigma_{ab} = \frac{1}{4}[\gamma_a,\gamma_b]$; when linearized this contributes to the following interaction vertex:

$$
\mathcal{L}' = -\frac{i}{4}\,h_\lambda{}^\alpha \partial_\mu h_{\nu\alpha}\,\bar\psi\,\Gamma^{\mu\lambda\nu}\psi, \quad \Gamma^{\mu\lambda\nu} = \gamma^{[\mu}\gamma^\lambda\gamma^{\nu]}, \tag{52}
$$

$$
\text{Feynman rule:} \quad -\frac{i}{4}\,\Gamma^{\mu\lambda\nu}(p_1-p_2)_\mu\,(\varepsilon_1)_\lambda{}^\alpha(\varepsilon_2)_{\nu\alpha}, \tag{53}
$$

We see a $(\varepsilon_1)_\lambda{}^\alpha(\varepsilon_2)_{\nu\alpha}$ factor, much similar to the factor in the divergent term in (a). Note that this vertex already contains 3 $\gamma$-matrices; by joining it with the anomalous vertex $\partial_\mu j^\mu_A$, we obtain a simple 1-loop "seagull" diagram (with graviton wings) :




$$
\begin{aligned}
\langle \partial^\mu J^A_\mu \rangle'_h &\sim 2\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\,\mathrm{Tr}\left(2M\gamma_5\cdot\frac{\not k + M}{k^2+M^2}\cdot\left(-\frac{1}{4}\right)\cancel{\varepsilon_1\varepsilon_2(p_1-p_2)}\cdot\frac{\not k + \not p_1 + \not p_2 + M}{(k+p_1+p_2)^2+M^2}\right) \\
&\sim -\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\,M^2(4\epsilon_{\mu\nu\rho\sigma})\,\delta_{\mu'\rho'}\varepsilon_1^{\mu\mu'}\varepsilon_2^{\rho\rho'}(p_1-p_2)^\nu\,(p_1+p_2)^\sigma\left(\frac{1}{k^2+M^2}\cdots\right) \\
&\sim -4M^2\epsilon_{\mu\nu\rho\sigma}\,(2p_1^\nu p_2^\sigma)\,\varepsilon_1^{\mu\mu'}\varepsilon_2^{\rho\rho'}\int\frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{\delta_{\mu'\rho'}}{(k^2+M^2)^2} \\
&\sim -8M^2\epsilon_{\mu\nu\rho\sigma}\,p_1^\nu p_2^\sigma\,\varepsilon_1^{\mu\mu'}\varepsilon_2^{\rho\rho'}\,\frac{1}{8\pi^2}\left(\delta_{\mu'\rho'}\frac{1}{2}\ln\frac{\Lambda^2}{M^2}\right)
\end{aligned}
\tag{54}
$$

Compare with the result in (a), and we see that the divergences cancel each other out precisely.

(c) For an anomalous vertex with hypercharge $Y$, there will be an additional $Y$ factor in the front of $\langle\partial_\mu J^\mu_A\rangle$; summing over a family of matter gives the total anomaly[10]:

$$
\langle\partial_\mu J^\mu_A\rangle \propto \sum \mathrm{Tr}\,T_a T_b Y \propto \delta_{ab}\sum Y \tag{55}
$$

When the summation goes over all states in a complete generation, we have $\sum Y = 0$, i.e. the anomaly cancels.

---

[9] Reference: Alvarez-Gaume & Witten, *Gravitational Anomalies*.

[10] Reference: *Tong*, and `Wikipedia:` *Anomaly (physics) # Anomaly cancellation*.