Algebraic Topology №2

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$\boxed{1}$ Morphism between coverings is covering:

For $F_i \to E_i \xrightarrow{p_i} B$: coverings in $\mathrm{Cov}_0(B)$ with E_i: connected and B: path connected and locally path connected, the following diagram commutes:

$$
\begin{tikzcd}
E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\
& B &
\end{tikzcd}
\qquad
\begin{aligned}
e_2 &= f(e_1), \\
b = p_1(e_1) &= p_2(e_2),
\end{aligned}
$$

To show that f is itself a covering, we need only verify that f is locally trivial with some discrete fiber F. In fact, given any $e_2 \in E_2$ and $b = p_2(e_2)$, there exists some neighborhood $U \subset B$ that the following diagram holds (by restriction):

$$
\begin{tikzcd}
U \times F_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & U \times F_2 \arrow[dl, "p_2"] \\
& U &
\end{tikzcd}
\qquad
\begin{aligned}
e_1 &= \big(b, k_1\big), \\
e_2 &= \big(b, k_2(b, k_1)\big),
\end{aligned}
\quad k_i \in F_i
$$

Generally, $k_2 = k_2(b, k_1)$ depends on the base point $b \in B$. However, since B is locally path connected, we can restrict U to be path connected, while $k_2 \in F_2$: discrete. Since continuous maps preserve path connectedness, k_2 is in fact independence of b, i.e. $k_2 = \varphi(k_1)$.

On the other hand, $\forall\, e_2 = (b, k_2) \in U \times \{k_2\} \subset E_2$, we have its preimage $f^{-1}(e_2) = \{b\} \times \varphi^{-1}(k_2)$. Note that E_2 is connected while $\varphi^{-1}(k_2) \in F_1$ is discrete; for the same reasoning as above, $\varphi^{-1}(k_2) = F$ is in fact independent of k_2. This is the discrete fiber F we have been looking for. Hence f is also a covering map[1]. $\blacksquare$

$\boxed{2}$ Cylinder with ends pinched — π_1 and universal cover:

$$Y = (X \times I)/(X \times \partial I)\ , \quad I = [0, 1] \tag{1}$$

Note that Y is homeomorphic to two cones[2] $CX_1 \coprod CX_2$ with "bases" $X_i \subset CX_i$ and "vertices" v_i respectively identified: $X_1 \sim X_2$, $v_1 \sim v_2 \equiv v$. X is path connected and so is Y, hence we are free to choose $\pi_1(Y) = \pi_1(Y, y_0)$.

First note that paths that do *not* pass through the vertex v are all homotopic, since they are contained in a cone and cones are contractible[3]. Therefore all contributions to $\pi_1(Y)$ are loop classes that *do* pass through the vertex v. In other words, morphisms in $\Pi_1 Y$ are in one-to-one correspondence with morphisms in:

$$\Pi_1\big([0, 1]/_{0\sim 1}\big) = \Pi_1 S^1 \tag{2}$$

Therefore, $\pi_1(Y) \cong \pi_1(S^1) = \mathbb{Z}$. $\square$

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[1] Reference: `math.stackexchange.com/a/109774`.

[2] See discussions from Problem Set №1.

[3] $[\gamma_1] = [\gamma_2 \star \gamma_2^{-1} \star \gamma_1] = [\gamma_2]$.

The universal cover $\tilde{Y}$ of Y can be constructed by assigning an induced topology to the space of path classes, same as in the general proof of its existence. Since Y is "degenerate" at its vertex, this is equivalent to "cutting open" Y at its vertex v, and joining $\mathbb{Z}$ copies them end-to-end. More explicitly, it can be written as:

$$\tilde{Y} = (X \times \mathbb{R}) \big/ \sim, \quad (x, n) \sim (x', n), \ \forall\, x \in X,\, n \in \mathbb{Z} \tag{3}$$

While the covering map: $\tilde{Y} \ni [x, t] \mapsto [x, t - \lfloor t \rfloor] \in Y$, here $\lfloor t \rfloor$ is the integer part of $t \in \mathbb{R}$. ■

$\boxed{\textbf{3}}$ **π_1 of fiber in fibration:**

$$\begin{tikzcd}
X \times \{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \arrow[r, "f"'] \arrow[ur, dashed, "\exists\, \tilde{f}"] & B
\end{tikzcd}$$

For $F \to E \xrightarrow{p} B$: fibration, by homotopy lifting property (HLP), any homotopy in B can be uniquely lifted to path class in E, provided some *"initial condition"* $X \times \{0\}$. This leads to the following results:

(a) For B: simply-connected, take any loop class $[\tilde{\gamma}] \in \pi_1(E, e)$ as initial condition; its projection $[p \circ \tilde{\gamma}] \in \pi_1(B, b) = \big\{[\mathbb{1}_b]\big\}$ is trivial, i.e. $p \circ \tilde{\gamma} \simeq \mathbb{1}_b$. By HLP, such homotopy can be lifted into E, i.e.

$$p \circ \tilde{\gamma} \simeq \mathbb{1}_b \quad \xRightarrow{\text{lift}} \quad \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}' = \mathbb{1}_b \tag{4}$$

In other words, $\tilde{\gamma} \simeq \tilde{\gamma}' \subset p^{-1}(b)$, i.e. any loop in E is homotopic to some loop in $p^{-1}(b) \cong F$. This implies a surjective group homomorphism $\pi_1(p^{-1}(b), e) \to \pi_1(E, e)$, i.e. an epimorphism. □

(b) For E: simply-connected, take any loop class $[\gamma] \in \pi_1(B, b)$ and consider its lifting $[\tilde{\gamma}]$. Note that in general $\tilde{\gamma}$ is *not* a loop; however, we have $p \circ \tilde{\gamma} = \gamma$, hence $\tilde{\gamma}(0), \tilde{\gamma}(1) \in p^{-1}(b)$. In general, we have:

$$\gamma \simeq \gamma' \quad \xRightarrow{\text{lift}} \quad \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}^{(\prime)} = \gamma^{(\prime)} \tag{5}$$

By continuity, $\tilde{\gamma}(0), \tilde{\gamma}'(0) \in F_0$: a path component of $p^{-1}(b)$; similarly, $\tilde{\gamma}(1), \tilde{\gamma}'(1) \in F_1$. In other words, the start and end points of $\tilde{\gamma}$ are confined in path components F_0 and F_1, respectively. Hence a loop class in $\pi_1(B, b)$ maps to *transport* between path components:

$$\begin{aligned} T_{(\cdot)}(e)\colon\ \pi_1(B, b) \quad &\longrightarrow \quad \pi_0\big(p^{-1}(b)\big) \\ [\gamma] \quad &\longmapsto \quad T_{[\gamma]}(e) \end{aligned} \tag{6}$$

As a matter of fact, $T_{(\cdot)}(e)$ is a bijection. For $T_{[\gamma]} = T_{[\gamma']}$, they are characterized by two lifted paths $\tilde{\gamma}, \tilde{\gamma}'$; since E is simply connected, they are always homotopic: $\tilde{\gamma} \simeq \tilde{\gamma}'$, hence $[\gamma] = [\gamma']$ by projection p. This means that T is injective. Surjectivity also follows from projection $\gamma = p \circ \gamma'$. Therefore, $T_{(\cdot)}(e)$ gives a bijection between $\pi_1(B, b)$ and $\pi_0\big(p^{-1}(b)\big)$. ■

$\boxed{\textbf{4}}$ **Pull-back of fibration is fibration:**

$$\begin{tikzcd}
Y \times \{0\} \arrow[r] \arrow[d, hook] & f^*(E) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "G"] \arrow[ur, dashed, "\exists\, \tilde{G}"] \arrow[urr, "\exists\, \tilde{F}\ (\text{HLP})"'] & X \arrow[r, "f"] & B
\end{tikzcd}$$

$$(x, e) \in f^*(E) \subset X \times E, \quad f(x) = p(e)$$

We need only verify that $f^*(E) \to X$ also has HLP, i.e. the existence of $\tilde{F}$ in the above diagram[4]. By HLP of $E \xrightarrow{p} B$, $\exists\ \tilde{F} \colon Y \times I \to E$ as shown above. We can use $\tilde{F}$ to construct $\tilde{G}$ explicitly; in fact, first consider:

$$\begin{aligned} \tilde{G} \colon\ Y \times I \quad &\longrightarrow \quad X \times E \\ (y, t) \quad &\longmapsto \quad \big(G(y,t), \tilde{F}(y,t)\big) \end{aligned} \tag{7}$$

Note that $f \circ G = p \circ \tilde{F}$; compared with the definition of $f^*(E)$, this implies that the image of $\tilde{G}$ lies within $f^*(E) \subset X \times E$, hence after restriction of its codomain, $\tilde{G}$ becomes a well-defined lifting of G into $f^*(E)$. Therefore, $f^*(E) \to X$ has HLP, i.e. it is also a fibration. ■

5 More properties of fibration:

(a) By HLP, given any initial condition $e \in p^{-1}(b_1)$, lifting of any path $b_1 \xrightarrow{\gamma} b_2$ exists. The lifted path with dependence of e can then be written as $F \colon p^{-1}(b_1) \times I \to E$. This is just a generalization of **3** for non-loop paths. □

(b) Similarly, transport $T_{[\gamma]}$ defined in **3** can be generalized for non-loop paths. $T_{[\gamma]}$ is well-defined for path class $[\gamma]$, since by HLP homotopic paths can be lifted to homotopy in E. Therefore, the transport is fixed up to homotopy, i.e.

$$\begin{aligned} T \colon\ \ \mathrm{Hom}_{\,\Pi_1 B}(b_0, b_1) \quad &\longrightarrow \quad \mathrm{Hom}_{\,\underline{\mathbf{hTop}}}\big(p^{-1}(b_0), p^{-1}(b_1)\big) \\ [\gamma] \quad &\longmapsto \quad T_{[\gamma]} \end{aligned} \tag{8}$$

Note that T defined in this way is also independent of the choice of F, since F simply specifies the starting point of the lifted path; no matter which F we choose, the lifted paths will always be homotopic in E. Hence T is well-defined in the above sense. □

(c) T defined above is a functor: $\Pi_1 B \to \underline{\mathbf{hTop}}$. To verify this, we need only check that it is compatible with composition and maps identity morphisms to identity morphisms. Indeed, $T_{[\mathbb{1}_b]} = [\mathbb{1}_{p^{-1}(b)}]$, and $T_{[\gamma'] \star [\gamma]} = T_{[\gamma' \star \gamma]} = T_{[\gamma']} \circ T_{[\gamma]}$ by joining two lifted paths (up to homotopy). □

(d) For B: path connected, there exists an isomorphism between any two objects in $\Pi_1 B$ (a path connecting any two points in B), which is mapped to isomorphisms between fibers $p^{-1}(b)$ in $\underline{\mathbf{hTop}}$. Hence any two fibers of $E \xrightarrow{p} B$ have the same homotopy type. ■

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[4] Notice that $f^*(E)$ is the limit of the diagram, hence this is automatically true by the universal property of $f^*(E)$. I would like to thank 刘逸华 for pointing this out. For now, we will stick to a more traditional proof.