

1 Stringy Physics!

$$T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : , \quad \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X^\mu \bar{\partial} X_\mu : , \quad (1)$$

$$V_k = : e^{ik \cdot X(z, \bar{z})} : , \quad G_{e,k} = e_{\mu\nu} : \partial X_z^\mu \bar{\partial} X_{\bar{z}}^\nu e^{ik \cdot X(z, \bar{z})} : , \quad (2)$$

We sometimes use subscripts like ∂X_z^μ to denote variable dependence to avoid clutter.

(a) The weight of a primary operator is given by its OPE with T and \tilde{T} . For exponential operators, there is a neat formula for cross contractions¹:

$$\begin{aligned} T(z) V_k(w, \bar{w}) &= \exp \left\{ \int d^2 z' \int d^2 w' \overline{X_{z'}^\mu} X_{w'}^\nu \frac{\delta}{\delta X_{z'}^\mu} \frac{\delta}{\delta X_{w'}^\nu} \right\} : T_z e^{ik \cdot X_w} : \\ &= \exp \left\{ \int d^2 z' \overline{X_{z'}^\mu} X_w^\nu \frac{\delta}{\delta X_{z'}^\mu} ik_\nu \right\} : T_z e^{ik \cdot X_w} : \\ &= : \left\{ \exp \left(ik_\nu \int d^2 z' \overline{X_{z'}^\mu} X_w^\nu \frac{\delta}{\delta X_{z'}^\mu} \right) T_z \right\} e^{ik \cdot X_w} : \\ &\sim -\frac{1}{\alpha'} : \left\{ 2\partial_z (ik_\sigma \overline{X_z^\mu} X_w^\sigma) \partial_z X_\mu + \partial_z (ik_\rho \overline{X_z^\mu} X_w^\rho) \partial_z (ik_\sigma \overline{X_{z,w}^\mu} X_w^\sigma) \right\} e^{ik \cdot X_w} : \\ &\sim -\frac{1}{\alpha'} : \left\{ 2 \left(-\frac{\alpha'}{2} \frac{ik^\mu}{z-w} \right) \partial_z X_\mu + \left(-\frac{\alpha'}{2} \frac{ik^\mu}{z-w} \right) \left(-\frac{\alpha'}{2} \frac{ik_\mu}{z-w} \right) \right\} e^{ik \cdot X_w} : \\ &\sim \frac{\alpha' k^2}{4} \frac{V_k(w, \bar{w})}{(z-w)^2} + \frac{\partial V_k(w, \bar{w})}{z-w} \end{aligned} \quad (3)$$

Here we've used the result that $ik_\sigma \overline{X_z^\mu} X_w^\sigma = ik^\mu (-\frac{\alpha'}{2}) \ln |z-w|^2$. We see that V_k is a primary of weight $(1, 1)$ iff. $\frac{\alpha' k^2}{4} = 1$, or $m^2 = -k^2 = -\frac{4}{\alpha'}$. This is the mass shell condition for the closed string tachyon (at level 0). On the other hand,

$$G_{e,k} = e_{\mu\nu} G_k^{\mu\nu}, \quad (4)$$

$$\begin{aligned} T(z) G_k^{\mu\nu}(0) &\sim : T_z \overline{\partial X_0^\mu} \bar{\partial} X_0^\nu e^{ik \cdot X_0} : + : \cancel{T_z \overline{\partial X_0^\mu} \bar{\partial} X_0^\nu e^{ik \cdot X_0}} : + : T_z \overline{\partial X_0^\mu} \bar{\partial} X_0^\nu e^{ik \cdot X_0} : \\ &\quad + : \cancel{T_z \overline{\partial X_0^\mu} \bar{\partial} X_0^\nu e^{ik \cdot X_0}} : + : T_z \overline{\partial X_0^\mu} \bar{\partial} X_0^\nu e^{ik \cdot X_0} : \\ &\sim \left(\frac{1}{z^2} G_k^{\mu\nu}(0) + \frac{1}{z} : \partial^2 X_0^\mu \bar{\partial} X_0^\nu e^{ik \cdot X_0} : \right) + \left(\frac{\alpha' k^2}{4} \frac{1}{z^2} G_k^{\mu\nu}(0) + \frac{1}{z} : \partial X_0^\mu \bar{\partial} X_0^\nu \partial e^{ik \cdot X_0} : \right) \\ &\quad - \frac{2}{\alpha'} \left(-\frac{\alpha'}{2} \eta^{\sigma\mu} \frac{1}{z^2} \right) \left(-\frac{\alpha'}{2} \frac{ik_\sigma}{z} \right) : \bar{\partial} X_0^\nu e^{ik \cdot X_0} : \\ &\sim ik^\mu : \bar{\partial} X_0^\nu e^{ik \cdot X_0} : \left(-\frac{\alpha'}{2} \right) \frac{1}{z^3} + \left(1 + \frac{\alpha' k^2}{4} \right) \frac{G_k^{\mu\nu}(0)}{z^2} + \frac{\partial G_k^{\mu\nu}(0)}{z}, \end{aligned} \quad (5)$$

$$\tilde{T}(\bar{z}) G_k^{\mu\nu}(0) \sim ik^\nu : \partial X_0^\mu e^{ik \cdot X_0} : \left(-\frac{\alpha'}{2} \right) \frac{1}{\bar{z}^3} + \left(1 + \frac{\alpha' k^2}{4} \right) \frac{G_k^{\mu\nu}(0)}{\bar{z}^2} + \frac{\partial G_k^{\mu\nu}(0)}{\bar{z}}, \quad (6)$$

Therefore, $G_{e,k}$ is a primary of weight $(1, 1)$ iff. $1 + \frac{\alpha' k^2}{4} = 1$ and $k^\mu e_{\mu\nu} = 0 = k^\nu e_{\mu\nu}$. The first equation gives the mass shell condition $m^2 = -k^2 = 0$ for a massless boson, while the second equation

¹Reference: Polchinski, and physics.stackexchange.com/a/389193.

constrains the polarization to be transverse. These are the physical constraints for a massless gauge boson, which is the level 1 excitation for a bosonic closed string.

(b) The form of any primary 3-point function is completely fixed by $\text{PSL}(2, \mathbb{C})$ invariance². In fact, for any holomorphic $\phi_i(z_i)$ with weight h_i , by translational invariance, we have:

$$\langle \phi_1(z_1) \phi_2(z_2) \phi_3(z_3) \rangle = f(z_{12}, z_{23}, z_{31}), \quad z_{ij} = z_i - z_j, \quad (7)$$

Furthermore, scaling invariance requires that f is quasi-homogeneous:

$$\begin{aligned} z \mapsto z' = \lambda^{-1} z, \quad f &\mapsto \langle \lambda^{h_1} \phi_1(\lambda z_1) \lambda^{h_2} \phi_2(\lambda z_2) \lambda^{h_3} \phi_3(\lambda z_3) \rangle \\ &= \lambda^{h_1+h_2+h_3} f(\lambda z_{12}, \lambda z_{23}, \lambda z_{31}) \\ &= f(z_{12}, z_{23}, z_{31}), \end{aligned} \quad (8)$$

$$f = \sum_{a+b+c=\sum_i h_i} f_{abc} = \sum_{a+b+c=\sum_i h_i} \frac{C_{abc}}{z_{12}^a z_{23}^b z_{31}^c} \quad (9)$$

On the other hand, for special conformal transformation³ $\frac{1}{z} \mapsto \frac{1}{z'} = \frac{1}{z} + a$, we have:

$$z \mapsto z' = \frac{1}{\frac{1}{z} + \bar{a}} = \frac{z}{1 + z\bar{a}} = w(z), \quad \frac{\partial z}{\partial z'} = \frac{1}{(1 - z\bar{a})^2} = \frac{1}{\kappa^2}, \quad z_{ij} = \frac{z'_{ij}}{\kappa_i \kappa_j}, \quad (10)$$

$$f \mapsto f(w^{-1}(z_{12}), w^{-1}(z_{23}), w^{-1}(z_{31})) \frac{1}{\kappa_1^{2h_1} \kappa_2^{2h_2} \kappa_3^{2h_3}} = f(z_{12}, z_{23}, z_{31}), \quad (11)$$

$$f_{abc}(w^{-1}(z_{12}), w^{-1}(z_{23}), w^{-1}(z_{31})) = f_{abc}(z_{12}, z_{23}, z_{31}) \kappa_1^{c+a} \kappa_2^{a+b} \kappa_3^{b+c}, \quad (12)$$

We see that f is invariant under special conformal transformation iff. $f = f_{abc}$ where:

$$c + a = 2h_1, \quad a + b = 2h_2, \quad b + c = 2h_3, \quad (13)$$

$$\text{i.e. } a = h_1 + h_2 - h_3, \quad b = h_2 + h_3 - h_1, \quad c = h_3 + h_1 - h_2, \quad (14)$$

In the above discussions we've restricted ϕ_i to be holomorphic; for *spin-less* $\phi_i = \phi_i(z, \bar{z})$, $h_i = \tilde{h}_i$, $\Delta_i = h_i + \tilde{h}_i$, the holomorphic and anti-holomorphic contributions can be nicely combined, and we have:

$$f = \frac{C}{|z_{12}|^{2a} |z_{23}|^{2b} |z_{31}|^{2c}}, \quad (15)$$

$$2a = \Delta_1 + \Delta_2 - \Delta_3, \quad 2b = \Delta_2 + \Delta_3 - \Delta_1, \quad 2c = \Delta_3 + \Delta_1 - \Delta_2, \quad (16)$$

$$\langle V_{k_1}(z_1, \bar{z}_1) V_{k_2}(z_2, \bar{z}_2) G_{e,k_3}(z_3, \bar{z}_3) \rangle = \frac{A(k_1, k_2, e)}{|z_{12}|^2 |z_{23}|^2 |z_{31}|^2} \quad (17)$$

(c) Following the recipe in (a), we have:

$$\begin{aligned} V_{k_1}(z_1, \bar{z}_1) V_{k_2}(z_2, \bar{z}_2) &= : \exp \left(i k_{1,\mu} i k_{2,\nu} \overline{X_1^\mu} X_2^\nu \right) e^{i k_1 \cdot X_1} e^{i k_2 \cdot X_2} : \\ &= \exp \left(\frac{\alpha'}{2} k_1 \cdot k_2 \ln |z_{12}|^2 \right) : e^{i k_1 \cdot X_1} e^{i k_2 \cdot X_2} : \\ &= |z_{12}|^{\alpha' k_1 \cdot k_2} : e^{i k_1 \cdot X_1} e^{i k_2 \cdot X_2} : \end{aligned} \quad (18)$$

²Reference: Blumenhagen, *Introduction to CFT*, and also *Di Francesco et al.*

³See *Di Francesco et al.*, and also github.com/davidsd/ph229.

$$\begin{aligned}
O_{k_1, k_2}(z_2, \bar{z}_2) G_{k_3}^{\mu\nu}(z_3, \bar{z}_3) \sim \dots - k_1^\mu k_1^\nu \left(\frac{\alpha'^2}{4} \right) \frac{1}{|z_{23}|^4} \\
- i^2 (k_1^\mu k_2^\nu + k_1^\nu k_2^\mu) \left(\frac{\alpha'}{2} (k_1 \cdot k_3) \right) \left(\frac{\alpha'^2}{4} \right) \frac{1}{|z_{23}|^4} \\
- i^4 k_3^\mu k_3^\nu \left(\frac{\alpha'}{2} (k_1 \cdot k_2) \right)^2 \left(\frac{\alpha'^2}{4} \right) \frac{1}{|z_{23}|^4} + \dots
\end{aligned} \tag{26}$$

Again, apply the on-shell conditions, and we find that:

$$\frac{\alpha'}{2} k_1 \cdot k_2 = -2, \quad \frac{\alpha'}{2} k_1 \cdot k_3 = -\frac{\alpha'}{2} k_1 \cdot (k_1 + k_2) = -\frac{\alpha'}{2} k_1^2 - \frac{\alpha'}{2} k_1 \cdot k_2 = -2 - (-2) = 0, \tag{27}$$

$$\begin{aligned}
A(k_1, k_2, e) &= -\frac{\alpha'^2}{4} (4e_{\mu\nu} k_3^\mu k_3^\nu + e_{\mu\nu} k_1^\mu k_1^\nu) = -\frac{\alpha'^2}{4} e_{\mu\nu} k_1^\mu k_1^\nu \\
&= -\frac{\alpha'^2}{4} e_{\mu\nu} (k_2 + k_3)^\mu (k_2 + k_3)^\nu = -\frac{\alpha'^2}{4} e_{\mu\nu} k_2^\mu k_2^\nu \\
&= -\frac{\alpha'^2}{8} e_{\mu\nu} (k_1^\mu k_1^\nu + k_2^\mu k_2^\nu) \\
&= -\frac{\alpha'^2}{8} e_{\mu\nu} (k_{12}^\mu k_{12}^\nu + (k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)),
\end{aligned} \tag{28}$$

On the other hand,

$$0 = e_{\mu\nu} k_3^\mu k_3^\nu = e_{\mu\nu} (k_1 + k_2)^\mu (k_1 + k_2)^\nu = e_{\mu\nu} (k_{12}^\mu k_{12}^\nu + 2(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)) \tag{29}$$

$$A(k_1, k_2, e) = -\frac{\alpha'^2}{8} e_{\mu\nu} k_{12}^\mu k_{12}^\nu \left(1 - \frac{1}{2} \right) = -\frac{\alpha'^2}{16} e_{\mu\nu} k_{12}^\mu k_{12}^\nu \tag{30}$$

2 Strings Scattering Off a Heavy Particle:

A heavy particle can be modeled by some D0-brane with Neumann boundary condition in the X_0 direction⁴. The scattering of a closed string tachyon off the heavy particle can then be computed via a disc diagram with two insertions.

(a) The conformal Killing group (CKG) of the disc is $\text{PSL}(2, \mathbb{R})$. It is a 3 dimensional \mathbb{R} Lie group, generated by 3 conformal Killing vectors (CKV's); therefore, it is possible to partially fix the positions of the two insertions V_1, V_2 . On the upper half plane, this can be implemented by putting z_1, z_2 on the imaginary axis, with z_2 fixed and z_1 integrated⁵:

$$\mathcal{A} = g_c^2 e^{-\lambda} \int_0^{z_2} dz_1 \left\langle : c_1^x e^{ik_1 \cdot X_1} : : c_2 \tilde{c}_2 e^{ik_2 \cdot X_2} : \right\rangle, \quad z_2 = i, \quad z_1 = iy, \quad y \in [0, 1] \tag{31}$$

Here c^x comes from the CKV that brings $z_1 \rightarrow iy$. On the disc this can be taken to be a rotation around z_2 ; when mapped to the upper half plane and at around the imaginary axis, this is simply a translation along the $x = \frac{1}{2}(z + \bar{z})$ direction⁶, i.e.

$$\text{CKV: } \partial_x = \delta_x^a \partial_a \implies \text{Ghost: } c^x, \tag{32}$$

$$c^x \partial_x + c^y \partial_y = c^z \partial_z + c^{\bar{z}} \partial_{\bar{z}}, \quad c^x = \frac{1}{2}(c^z + c^{\bar{z}}) = \frac{1}{2}(c(z) + \tilde{c}(\bar{z})), \tag{33}$$

⁴Reference: [arXiv:hep-th/9611214](#), [arXiv:hep-th/9605168](#), and *Polchinski*.

⁵Reference: [arXiv:0812.4408](#). I would like to thank Lucy Smith for pointing this out.

⁶Reference: *Polchinski*, Chapter 5 & 6.

The ghost contribution is then:

$$\begin{aligned}
\langle c_1^x c_2 \tilde{c}_2 \rangle &= \langle c^x(z_1) c(z_2) \tilde{c}(\bar{z}_2) \rangle = \frac{1}{2} \left(\langle c(z_1) c(z_2) \tilde{c}(\bar{z}_2) \rangle + \langle \tilde{c}(z_1) c(z_2) \tilde{c}(\bar{z}_2) \rangle \right) \\
&= \frac{1}{2} \left(\langle c(z_1) c(z_2) c(z'_2) \rangle + \langle c(z'_1) c(z_2) c(z'_2) \rangle \right), \quad z' = \bar{z}, \\
&= \frac{C_{D^2}^g}{2} (z_{12} z_{12'} z_{22'} + z_{1'2} z_{1'2'} z_{22'}), \quad z_1, z_2 \in i\mathbb{R}, \\
&= 2C_{D^2}^g (z_1^2 - z_2^2) z_2
\end{aligned} \tag{34}$$

On the other hand, the $e^{ik_j \cdot X_j}$ contributions is similar to what we've computed in [1], except that now we should be careful about the boundary conditions of X^μ , which affect the XX contraction in the formulae. For Neumann boundary condition: $\partial_y X^0 = 0$, the half-plane propagator from z' can be constructed with an image at \bar{z}' with *the same charge*, i.e. we have:

$$\overline{X_1^0 X_2^0} = -\frac{\alpha'}{2} \eta^{00} \ln |z_1 - z_2|^2 - \frac{\alpha'}{2} \eta^{00} \ln |z_1 - \bar{z}_2|^2 \tag{35}$$

While for Dirichlet boundary $X^i = \text{const}$, we can always select the origin so that $X^i = 0$, and in this case the image should have *the opposite charge*, i.e.

$$\overline{X_1^i X_2^j} = -\frac{\alpha'}{2} \delta^{ij} \ln |z_1 - z_2|^2 + \frac{\alpha'}{2} \delta^{ij} \ln |z_1 - \bar{z}_2|^2, \tag{36}$$

$$\begin{aligned}
\Rightarrow :e^{ik_1 \cdot X_1} : :e^{ik_2 \cdot X_2} : &= \exp \left(ik_{1,\mu} ik_{2,\nu} \overline{X_1^\mu X_2^\nu} \right) :e^{ik_1 \cdot X_1} e^{ik_2 \cdot X_2} : \\
&= |z_{12}|^{\alpha' k_1 \cdot k_2} |z_{1\bar{2}}|^{\alpha' (-k_1^0 k_2^0 - \mathbf{k}_1 \cdot \mathbf{k}_2)} :e^{ik_1 \cdot X_1} e^{ik_2 \cdot X_2} :
\end{aligned} \tag{37}$$

Before further calculations, we note that the normal ordering defined here on D^2 differs from that on the usual \mathbb{C}^2 ; in fact, there are also self-contractions with image charge⁷:

$$\overline{X^\mu(z, \bar{z}) X^\nu(\bar{z}, z)} = G_r^{\mu\nu}(z, \bar{z}) = \mp \frac{\alpha'}{2} \eta^{\mu\nu} \ln |z - \bar{z}|^2, \tag{38}$$

$$\Rightarrow \left\langle :e^{ik_1 \cdot X_1} e^{ik_2 \cdot X_2} : \right\rangle_{D^2} = \left\langle :e^{ik_1 \cdot X_1} e^{ik_2 \cdot X_2} : \right\rangle_{\mathbb{C}^2} \exp \left(\frac{1}{2} \sum_n ik_{n,\mu} ik_{n,\nu} \overline{X_n^\mu X_n^\nu} \right), \quad n = 1, 2 \tag{39}$$

The “ \mp ” sign choice depends on the boundary condition.

Therefore,

$$\begin{aligned}
\left\langle :e^{ik_1 \cdot X_1} : :e^{ik_2 \cdot X_2} : \right\rangle_{D^2} &= \left\langle :e^{ik_1 \cdot X_1} e^{ik_2 \cdot X_2} : \right\rangle_{\mathbb{C}^2} \exp \left(ik_{1,\mu} ik_{2,\nu} \overline{X_1^\mu X_2^\nu} \right) \exp \left(\frac{1}{2} \sum_n ik_{n,\mu} ik_{n,\nu} \overline{X_n^\mu X_n^\nu} \right) \\
&= \left\langle :e^{ik_1 \cdot X_1} e^{ik_2 \cdot X_2} : \right\rangle_{\mathbb{C}^2} \exp \left(\frac{1}{2} \sum_{m,n} ik_{m,\mu} ik_{n,\nu} \overline{X_m^\mu X_n^\nu} \right) \\
&= \left\langle :e^{ik_1 \cdot X_1} e^{ik_2 \cdot X_2} : \right\rangle_{\mathbb{C}^2} |z_{12}|^{\alpha' k_1 \cdot k_2} |z_{1\bar{2}}|^{\alpha' (-k_1^0 k_2^0 - \mathbf{k}_1 \cdot \mathbf{k}_2)} \prod_n |z_{n\bar{n}}|^{\frac{\alpha'}{2} (-(k_n^0)^2 - \mathbf{k}_n^2)}
\end{aligned} \tag{40}$$

Note that X^i has no zero mode due to the Dirichlet boundary, hence $\int \mathcal{D}X$ gives a delta function in only the Neumann direction: $\delta(k_1^0 + k_2^0)$. Physically, this means that only the energy is conserved;

⁷This is very much similar to the torus situation, where we also have to consider self-contractions with image charges. More rigorous discussion of G^r is given in *Polchinski*.

the momentum k^i is not conserved since the heavy D0-brane does not recoil. It is therefore convenient to define these on shell variables:

$$s = \omega^2 = (k_1^0)^2 = (k_2^0)^2, \quad t = -(\mathbf{k}_1 + \mathbf{k}_2)^2 = -\mathbf{k}_1^2 - \mathbf{k}_2^2 - 2\mathbf{k}_1 \cdot \mathbf{k}_2 = 2\left(-\omega^2 - \mathbf{k}_1 \cdot \mathbf{k}_2 - \frac{4}{\alpha'}\right), \quad (41)$$

$$\mathbf{k}_1 \cdot \mathbf{k}_2 = -\frac{t}{2} - \omega^2 - \frac{4}{\alpha'}, \quad k_1 \cdot k_2 = -\omega(-\omega) + \mathbf{k}_1 \cdot \mathbf{k}_2 \quad (42)$$

Here we've used the on-shell condition: $m^2 = -k^2 = \omega^2 - \mathbf{k}^2 = -\frac{4}{\alpha'}$ for tachyons. The previous expressions can then be simplified to:

$$\begin{aligned} \left\langle :e^{ik_1 \cdot X_1} : :e^{ik_2 \cdot X_2} : \right\rangle_{D^2} &= \left\langle :e^{ik_1 \cdot X_1} e^{ik_2 \cdot X_2} : \right\rangle_{\mathbb{C}^2} |z_{12}|^{-\frac{\alpha' t}{2} - 4} |z_{1\bar{2}}|^{\frac{\alpha' t}{2} + 4 + 2\alpha' \omega^2} \prod_n |2z_n|^{-\alpha' \omega^2 - 2} \\ &= iC_{D^2}^X 2\pi \delta(k_1^0 + k_2^0) |z_{12}|^{-\frac{\alpha' t}{2} - 4} |z_{1\bar{2}}|^{\frac{\alpha' t}{2} + 4 + 2\alpha' \omega^2} \prod_n |2z_n|^{-\alpha' \omega^2 - 2} \quad (43) \\ &= iC_{D^2}^X 2\pi \delta(k_1^0 + k_2^0) f(|z_{12}|, |z_{1\bar{2}}|, |z_1|, |z_2|), \end{aligned}$$

$$\begin{aligned} \mathcal{A} &= g_c^2 e^{-\lambda} \cdot iC_{D^2}^X 2\pi \delta(k_1^0 + k_2^0) \cdot 2C_{D^2}^g \int_0^{z_2} dz_1 (z_1^2 - z_2^2) z_2 f(|z_{12}|, |z_{1\bar{2}}|, |z_1|, |z_2|) \\ &= g_c^2 C_{D^2} 2\pi \delta(k_1^0 + k_2^0) \cdot 2i \int_0^1 dy ((iy)^2 - i^2) i \cdot f(1-y, 1+y, 2y, 2) \quad (44) \\ &= -ig_c^2 C_{D^2} 2\pi \delta(k_1^0 + k_2^0) \cdot 2 \cdot 2^{-2\alpha' \omega^2 - 4} \int_0^1 dy (1-y^2) f(1-y, 1+y, y, 1), \end{aligned}$$

$$\begin{aligned} \int_0^1 dy (1-y^2) f(1-y, 1+y, y, 1) &= \int_0^1 dy (1-y)^{-\frac{\alpha' t}{2} - 4 + 1} (1+y)^{\frac{\alpha' t}{2} + 4 + 2\alpha' \omega^2 + 1} y^{-\alpha' \omega^2 - 2} \\ &= \int_0^1 dy y^{a-1} (1-y)^{2b-1} (1+y)^{-2a-2b+1}, \quad y' = \frac{1-y}{1+y}, \\ &= -2^{1-2a} \int_0^1 dy' (-y')^{2b-1} (1-y'^2)^{a-1} \quad (45) \\ &= 2^{-2a} \int_0^1 d(y'^2) (y'^2)^{b-1} (1-y'^2)^{a-1} \\ &= 2^{-2a} B\left(a = -\alpha' \omega^2 - 1, b = -\frac{\alpha' t}{4} - 1\right) \end{aligned}$$

Here $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Euler Beta function.

Putting everything together, we obtain:

$$\begin{aligned} \mathcal{A} &= -ig_c^2 C_{D^2} 2\pi \delta(k_1^0 + k_2^0) \cdot \frac{1}{2} B\left(-\alpha' \omega^2 - 1, -\frac{\alpha' t}{4} - 1\right) \\ &= -ig_c^2 C_{D^2} \pi \delta(k_1^0 + k_2^0) B\left(-\alpha' \omega^2 - 1, -\frac{\alpha' t}{4} - 1\right) \quad (46) \end{aligned}$$

In fact $g_c^2 C_{D^2}$ can be further computed by path integral or by comparing physical results. Here we settle for this generic coefficient since it's already enough for our following discussions⁸.

⁸And I have run out of time and energy.

(b) The Regge limit is found by taking the high energy limit while keeping the momentum transfer fixed; in this case it is achieved by:

$$\text{Regge: } s = \omega^2 \rightarrow \infty, \quad t = -(\mathbf{k}_1 + \mathbf{k}_2)^2 \text{ fixed}, \quad (47)$$

$$\begin{aligned} \mathcal{A} \propto B\left(a = -\alpha' s - 1, b = -\frac{\alpha' t}{4} - 1\right) &= \frac{\Gamma(-\alpha' s - 1)}{\Gamma(-\alpha' s - \frac{\alpha' t}{4} - 2)} \Gamma\left(-\frac{\alpha' t}{4} - 1\right) \\ &\sim \left\{ e\left(\alpha' s + \frac{\alpha' t}{4} + 3\right) \right\}^{\frac{\alpha' t}{4} + 1} \Gamma\left(-\frac{\alpha' t}{4} - 1\right) \\ &\sim (e\alpha' \omega^2)^{\frac{\alpha' t}{4} + 1} \Gamma\left(-\frac{\alpha' t}{4} - 1\right) \\ &\sim (\omega^2)^{\frac{\alpha' t}{4} + 1} \Gamma\left(-\frac{\alpha' t}{4} - 1\right) \end{aligned} \quad (48)$$

Here we've used the Stirling's approximation⁹: $\ln \Gamma(z + 1) = \ln z! \sim z \ln z - z$. On the other hand, the hard scattering limit is found by keeping the scattering angle fixed, i.e.

$$\text{Hard scattering: } s = \omega^2 \rightarrow \infty, \quad (t/s) \equiv \lambda \text{ fixed}, \quad (49)$$

$$\mathcal{A} \propto B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)} \sim \exp \left\{ -\alpha' \left(s \ln(\alpha' s) + \frac{t}{4} \ln \frac{\alpha' t}{4} + \frac{u}{4} \ln \frac{\alpha' u}{4} \right) \right\}, \quad (50)$$

$$s = \omega^2 = (k_1^0)^2 = (k_2^0)^2, \quad t = -(\mathbf{k}_1 + \mathbf{k}_2)^2, \quad u = -(\mathbf{k}_1 - \mathbf{k}_2)^2, \quad (51)$$

$$s + \frac{t}{4} + \frac{u}{4} = -\frac{4}{\alpha'}, \quad (52)$$

Here we've introduced an additional u variable, and we see that the result is symmetric under $t \leftrightarrow u$. We find that under the above limits, the amplitude exhibits similar behaviors as the Veneziano amplitude.

(c) Note that $\Gamma(z)$ has no zeros on \mathbb{C}^2 , therefore the poles of $B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$ is given by the poles of $\Gamma(a)$ and $\Gamma(b)$:

$$a = -\alpha' s - 1 = 0, -1, -2, \dots, \quad b = -\frac{\alpha' t}{4} - 1 = 0, -1, -2, \dots, \quad (53)$$

The first set of poles gives:

$$\omega^2 = -\frac{1}{\alpha'}, 0, \frac{1}{\alpha'}, \frac{2}{\alpha'}, \dots \quad (54)$$

This is precisely the open string spectrum in $D = 26$. Going back to (43)~(45), we see that they come from the singularities as $y = |z_1| \rightarrow 0$ or $|z_{12}| \rightarrow 1$, i.e. the two insertions are far apart.

This is the s -channel contribution; physically, it means that the closed string is first absorbed by the D0-brane, then it propagates as an intermediate open string on the brane, and is re-emitted in the end.

⁹For the validity of Stirling's approximation when $z \in \mathbb{C}$, $\arg z = \pi - \epsilon$ and $|z| \rightarrow \infty$, see [Wikipedia: Stirling's formula for the gamma function](#).

The other set of poles gives:

$$-t = (\mathbf{k}_1 + \mathbf{k}_2)^2 = \frac{4}{\alpha'}, 0, -\frac{4}{\alpha'}, -\frac{8}{\alpha'}, \dots \quad (55)$$

Only two poles are realistic due to $\mathbf{k}^2 \geq 0$. This is the t -channel contribution, which corresponds to a closed string exchange between the D0-brane and the incoming particle. The above factorization channels are nicely illustrated in the diagram below, borrowed from [arXiv:hep-th/9611214](#).



3 Compton Scattering Between $U(N)$ Gluons and Adjoint Tachyons:

Such scattering is captured by a disc diagram with $AATT$ insertions at the boundary; following the same recipe as before, we write down¹⁰:

$$\begin{aligned} \mathcal{A} = & g_o'^2 g_o^2 e^{-\lambda} \int dy \left\langle : e_\mu \partial_y X^\mu e^{ik \cdot X}(y) : : c_1^y e_{1,\nu} \partial_y X_1^\nu e^{ik_1 \cdot X_1} : : c_2^y e^{ik_2 \cdot X_2} : : c_3^y e^{ik_3 \cdot X_3} : \right\rangle \\ & \times \text{Tr} \left(\lambda^a \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} : \text{in } (y, y_1, y_2, y_3) \text{ order} \right) \\ & + (2 \leftrightarrow 3) \end{aligned} \quad (56)$$

Where $y_i = y_{1,2,3}$ are fixed while $y_0 = y$ is integrated. Note that for [2] we use (x, y) to label the points on the upper half plane so that the boundary coordinate is $x = (x, 0)$; here in order to match with *Polchinski*, we label the boundary points with $y = (y, 0)$ instead.

We see that \mathcal{A} is structurally similar to the 4-point Veneziano amplitude \mathcal{A}_{TTTT} and also the 3-point amplitude \mathcal{A}_{TTA} ; by analogy we can evaluate \mathcal{A} in a similar fashion¹¹:

$$\begin{aligned} \mathcal{A} = & g_o'^2 g_o^2 e^{-\lambda} \prod_{i < j} |y_{ij}|^{2\alpha' k_i \cdot k_j + 1} e_\mu e_{1,\nu} \\ & \times \int dy \prod_j |y_{0j}|^{2\alpha' k \cdot k_j} \left\{ -2\alpha' \frac{\eta^{\mu\nu}}{y_{01}^2} + (-2\alpha') \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) (-2\alpha') \left(\frac{ik^\nu}{y_{10}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\} \\ & \times \text{Tr}(\dots) iC_{D_2}^X C_{D_2}^g (2\pi)^{26} \delta^{26}(\sum_i k_i) \\ & + (2 \leftrightarrow 3) \end{aligned} \quad (57)$$

¹⁰I would like to thank 谷夏 for help with this problem.

¹¹For the ∂X contribution, see *Polchinski*, [arXiv:1403.4553](#) and [arXiv:1507.02172](#). Note that there are contractions between two ∂X fields.

Notice the doubling due to the boundary contraction $\overline{X_1^\mu X_2^\nu} = 2 \times (-\frac{\alpha'}{2}) \ln y_{12}^2 = -2\alpha' \ln y_{12}$. Further simplifications can be achieved by using the on-shell and physical conditions for open string states:

$$k^2 = k_1^2 = 0, \quad e \cdot k = e_1 \cdot k_1 = 0, \quad k_{2,3}^2 = \frac{1}{\alpha'} = -m^2, \quad (58)$$

$$\begin{aligned} s &= -\alpha'(k + k_1)^2 = -2\alpha'k \cdot k_1, \\ t &= -\alpha'(k + k_2)^2 = -2\alpha'k \cdot k_2 - 1, \\ u &= -\alpha'(k + k_3)^2 = -2\alpha'k \cdot k_3 - 1, \end{aligned} \quad (59)$$

$$s + t + u = -2, \quad (60)$$

$$2\alpha'k_1 \cdot k_2 = \alpha'(k_1 + k_2)^2 - \alpha'k_1^2 - \alpha'k_2^2 = \alpha'(k + k_3)^2 - 1 = -u - 1 = 2\alpha'k \cdot k_3, \quad (61)$$

$$2\alpha'k_1 \cdot k_3 = -t - 1 = 2\alpha'k \cdot k_2, \quad (62)$$

$$2\alpha'k_2 \cdot k_3 = \alpha'(k + k_1)^2 - 2 = -s - 2, \quad (63)$$

$$\begin{aligned} \mathcal{A} &= ig_o'^2 g_o^2 C_{D^2} |y_{12}|^{-u} |y_{13}|^{-t} |y_{23}|^{-s-2} e_\mu e_{1,\nu} \\ &\times \int dy |y_{01}|^{-s} |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left\{ \frac{\eta^{\mu\nu}}{y_{01}^2} - 2\alpha' \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) \left(-\frac{ik^\nu}{y_{01}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\} \\ &\times \text{Tr}(\dots) (-2\alpha')(2\pi)^{26} \delta^{26}(\sum_i k_i) \\ &+ (2 \leftrightarrow 3) \end{aligned} \quad (64)$$

Fix $y_1 \rightarrow \infty, y_2 = 0, y_3 = 1$, and we get:

$$\begin{aligned} \mathcal{A} &= ig_o'^2 g_o^2 C_{D^2} |y_1|^{-u-t} e_\mu e_{1,\nu} \\ &\times \int dy |y_{01}|^{-s} |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left\{ \frac{\eta^{\mu\nu}}{y_{01}^2} - 2\alpha' \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) \left(-\frac{ik^\nu}{y_{01}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\} \\ &\times \text{Tr}(\dots) \dots + \dots \end{aligned} \quad (65)$$

$$\begin{aligned} \mathcal{A} &= ig_o'^2 g_o^2 C_{D^2} \\ &\times \int dy |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left| \frac{y_1^{s+2}}{y_{01}^s} \right| \left\{ \frac{e \cdot e_1}{y_{01}^2} - 2\alpha' e_\mu \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) e_{1,\nu} \left(-\frac{ik^\nu}{y_{01}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\} \\ &\times \text{Tr}(\dots) \dots + \dots \\ &= ig_o'^2 g_o^2 C_{D^2} \int dy \left\{ (e \cdot e_1) f(t, u) - 2\alpha' g(t, u) \right\} \times \dots + \dots \end{aligned} \quad (66)$$

Note that the definition of $f(t, u)$ and $g(t, u)$ contains the $\text{Tr}(\dots)$ factor.

The limit $y_1 \rightarrow \infty$ should be treated with care. In fact, the integral along the boundary splits into three ranges¹²:

$$\int dy = \left\{ \int_{y_1 \rightarrow -\infty}^{y_3=0} + \int_{y_2=0}^{y_3=1} + \int_{y_3=1}^{y_1 \rightarrow +\infty} \right\} dy \quad (67)$$

¹²Reference: *Polchinski's* discussion of \mathcal{A}_{TTTT} .

Notice the difference of $y_1 \rightarrow \pm\infty$; this is due to the S^1 topology of the boundary. Therefore,

$$\int dy f(t, u) = \int dy |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left| \frac{y_1}{y_{01}} \right|^{s+2} \text{Tr}(\dots), \quad (68)$$

$$\text{Tr}(\lambda^a \lambda^{a_1} \lambda^{a_2} \lambda^{a_3}) \equiv T_{0123}, \quad (69)$$

$$\begin{aligned} \int_{y_1 \rightarrow -\infty}^{y_2=0} dy f(t, u) &= \int_{y_1 \rightarrow -\infty}^0 dy (1-y)^{-t-1} (-y)^{-u-1} \left(\frac{-y_1}{y-y_1} \right)^{s+2} T_{1023} = T_{1023} B(-u, -s-1), \\ \int_{y_3=1}^{y_1 \rightarrow +\infty} dy f(t, u) &= \int_1^{y_1 \rightarrow +\infty} dy (y-1)^{-t-1} y^{-u-1} \left(\frac{y_1}{y_1-y} \right)^{s+2} T_{1230} = T_{1230} B(-t, -s-1), \\ \int_{y_3=0}^{y_2=1} dy f(t, u) &= \int_0^1 dy (1-y)^{-t-1} y^{-u-1} T_{1203} = T_{1203} B(-t, -u), \end{aligned} \quad (70)$$

$$\int dy f(t, u) = T_{1203} B(-t, -u) + T_{1230} B(-t, -s-1) + T_{1023} B(-u, -s-1) \quad (71)$$

Under $2 \leftrightarrow 3$, we have $t \leftrightarrow u$; therefore,

$$\begin{aligned} \int dy f(t, u) + (2 \leftrightarrow 3) &= (T_{1203} + T_{1302}) B(-t, -u) \\ &\quad + (T_{1230} + T_{1032}) B(-t, -s-1) \\ &\quad + (T_{1023} + T_{1320}) B(-u, -s-1) \\ &\equiv \alpha B(-t, -u) + \beta B(-t, -s-1) + \gamma B(-u, -s-1) \end{aligned} \quad (72)$$

The $g(t, u)$ factor can be computed in a similar manner. In the end we have¹³:

$$\begin{aligned} \mathcal{A} &= i g_o'^2 g_o^2 C_{D^2} (2\pi)^{26} \delta^{26} (\sum_i k_i) \\ &\quad \times (-2\alpha') \left\{ (e_0 \cdot e_1) (\alpha B(-t, -u) + \beta B(-t, -s-1) + \gamma B(-u, -s-1)) \right. \\ &\quad \left. + (-2\alpha') (e_0 \cdot k_2) (e_1 \cdot k_3) (\alpha B(-t, -u) - \beta B(-t, -s-1) - \gamma B(-u, -s-1)) + (2 \leftrightarrow 3) \right\} \end{aligned} \quad (73)$$

¹³Reference: [arXiv:0801.3358](#).