Finite Temperature Field Theory №3

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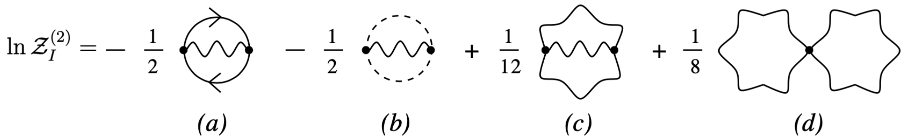
QCD Partition Function at $\mathcal{O}(g^2)$



$$\ln \mathcal{Z}_I^{(2)} = \ln \mathcal{Z}^{(a)} + \ln \mathcal{Z}^{(b)} + \ln \mathcal{Z}^{(c)} + \ln \mathcal{Z}^{(d)} \tag{1}$$

The contribution of (a) is given by:

$$\ln \mathcal{Z}^{(a)} = \frac{1}{2!}\,(-1)^1 \frac{T}{V}\sum_k \frac{T}{V}\sum_p \mathrm{Tr}\left(S(k)\,(g\gamma^\nu T^b)\,S(p)\,(g\gamma^\mu T^a)\right)\left(\frac{V}{T}\right)\delta_{ab}\,\Delta_{\mu\nu}(p-k) \tag{2}$$

Here the trace goes over spinor, color and flavor indices. $S(k)$ is the quark propagator, with suppressed spinor, color and flavor indices, while $\delta_{ab}\,\Delta_{\mu\nu}$ is the gluon propagator, where a, b are adjoint indices; each vertex contributes a $(g\gamma^\mu T^a)$ factor.

In our convention, $k = (\omega_n, \mathbf{k})$ stands for the Euclidean 4-momentum with ω_n: the discrete Matsubara frequency. Each $\sum_k$ comes with a factor $\frac{T}{V}$ while each momentum space delta function comes with an inverse factor: $\frac{V}{T}$; this is due to the fact that:

$$1 = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\,(2\pi)^4 \delta^4(k-k_0) \sim \frac{1}{\beta V}\sum_k \beta V \delta_{k,k_0} \tag{3}$$

Following the same recipe from QED, we can write down:

$$\begin{aligned}\ln \mathcal{Z}^{(a)} &= -\left(\mathrm{Tr}\left(T^a T^b\right)\delta_{ab}\right)\frac{1}{2}\,g^2 \frac{V}{T}\cdot\frac{T}{V}\sum_k \frac{T}{V}\sum_p \mathrm{Tr}\left(S(k)\,\gamma^\nu\,S(p)\,\gamma^\mu\right)\Delta_{\mu\nu}(p-k)\\ &= -\left(\frac{N_c^2-1}{2}\,N_f\right)\frac{g^2}{288}\frac{V}{T}\left(5T^4 + \frac{18}{\pi^2}\,T^2\mu^2 + \frac{9}{\pi^4}\,\mu^4\right)\end{aligned} \tag{4}$$

The (b) term is structurally similar to the (a) term; now the amplitude can be written down simply by replacing the propagator $S(k) \mapsto W(k)$ of the ghost, while the vertex is $(g\gamma^\mu T^a) \mapsto (-igk^\nu T^b)$ instead:

$$\ln \mathcal{Z}^{(b)} = \frac{1}{2!}\,(-1)^1 \frac{T}{V}\sum_k \frac{T}{V}\sum_p \mathrm{Tr}\left(W(k)\,(-igp^\nu T^b)\,W(p)\,(-igk^\mu T^a)\right)\left(\frac{V}{T}\right)\delta_{ab}\,\Delta_{\mu\nu}(p-k) \tag{5}$$

The trace now goes over suppressed *adjoint* indices of $W(k)_{ab} = -\delta_{ab}\Delta(k)$ and $(T_a)_{bc} = f_{abc}$, where f_{abc} is the structure constant of $\mathrm{SU}(N_c)$. Therefore[1],

$$\begin{aligned}\ln \mathcal{Z}^{(b)} &= -\left(\mathrm{Tr}\left(T^a T^b\right)\delta_{ab}\right)\frac{1}{2}\,g^2 \frac{V}{T}\cdot\frac{T}{V}\sum_k \frac{T}{V}\sum_p \Delta(k)\,\Delta(p)\,(-k^\mu p^\nu)\,\Delta_{\mu\nu}(p-k)\\ &= -\left(\frac{N_c^2-1}{2}\,2N_c\right)\frac{1}{2}\,g^2 \frac{V}{T}\cdot\frac{T}{V}\sum_k \frac{T}{V}\sum_p \Delta(k)\,\Delta(p)\,(-k^\mu p^\nu)\,(-g_{\mu\nu})\,\Delta(p-k)\end{aligned} \tag{6}$$

[1] $\mathrm{Tr}\left(T^a T^b\right)_{\mathrm{ad}}$ in the adjoint representation is precisely the *Killing form* of the $\mathfrak{su}(N_c)$ algebra, which is $2N_c$ times the $\mathrm{Tr}\left(T^a T^b\right)_0$ in the fundamental representation; see `Wikipedia:` *Killing form*.

Now we compute the remaining $\sum_{k,p}$. We have:

$$\Delta(k)\,\Delta(p)\,(k\cdot p)\,\Delta(p-k)=\frac{k\cdot p}{k^2p^2\,(p-k)^2} \tag{7}$$

The generic method to carry out such summation is by using the mixed representation of the propagator; for some propagator $D(k)$, we have:

$$\begin{aligned} D(k)=D(w_n,\mathbf{k}) &= \int_0^\beta \mathrm{d}\tau\, e^{-i\omega_n\tau}\, T\sum_m e^{i\omega_m\tau} D(w_m,\mathbf{k}) \\ &= \int_0^\beta \mathrm{d}\tau\, e^{-i\omega_n\tau}\, \tilde{D}(\tau,\mathbf{k}), \end{aligned} \tag{8}$$

$$\begin{aligned} \tilde{D}(\tau,\mathbf{k}) &= T\sum_m e^{i\omega_m\tau} D(w_m,\mathbf{k}) \\ &= T\sum_m e^{i\omega_m\tau}\int \frac{\mathrm{d}\omega}{2\pi}\,\frac{\rho(\omega,\mathbf{k})}{\omega+i\omega_m} \\ &= \int \frac{\mathrm{d}\omega}{2\pi}\,\rho(\omega,\mathbf{k})\, T\sum_m \frac{e^{i\omega_m\tau}}{\omega+i\omega_m} \\ &= \int \frac{\mathrm{d}\omega}{2\pi}\,\rho(\omega,\mathbf{k})\, e^{-\omega\tau}\big(1\pm n_\pm(\omega)\big), \end{aligned} \tag{9}$$

$$\rho(\omega,\mathbf{k})=\frac{1}{i}\big(D(\omega+i\epsilon)-D(\omega-i\epsilon)\big)=2\,\mathrm{Im}\,D(\omega+i\epsilon,\mathbf{k}),\quad n_\pm=\frac{1}{e^{\beta\omega}\mp 1}, \tag{10}$$

Then the Matsubara sum $\sum_{\omega_n}$ becomes a sum over exponentials like $e^{-i\omega_n\tau}$, which is easier to deal with. However, for this particular problem, there is a shortcut[2]; notice that the denominator of (7) is invariant under $p\mapsto k-p$, hence:

$$\begin{aligned} \sum_p \frac{k\cdot p}{k^2p^2\,(p-k)^2} &= \sum_{(k-p)} \frac{k\cdot(k-p)}{k^2\,(k-p)^2p^2} \\ &= \sum_p \frac{k\cdot(k-p)}{k^2p^2\,(p-k)^2} \\ &= \sum_p \frac{\frac{1}{2}\big(k\cdot p+k\cdot(k-p)\big)}{k^2p^2\,(p-k)^2} \\ &= \sum_p \frac{1}{2p^2\,(p-k)^2}, \end{aligned} \tag{11}$$

$$\begin{aligned} \frac{T}{V}\sum_k\frac{T}{V}\sum_p \Delta(k)\,\Delta(p)\,(k\cdot p)\,\Delta(p-k) &= \frac{1}{2}\,\frac{T}{V}\sum_p\frac{1}{p^2}\,\frac{T}{V}\sum_k\frac{1}{(p-k)^2} \\ &= \frac{1}{2}\left(\frac{T}{V}\sum_p\frac{1}{p^2}\right)^2 = \frac{1}{2}\left(\frac{T^2}{12}\right)^2, \end{aligned} \tag{12}$$

$$\boxed{\ln\mathcal{Z}^{(b)} = -\left(\frac{N_c^2-1}{2}\,2N_c\right)\frac{1}{2}\,g^2\frac{V}{T}\cdot\frac{1}{2}\left(\frac{T^2}{12}\right)^2 = -\frac{V}{T}\,N_c\big(N_c^2-1\big)\,\frac{1}{4}\,g^2\,\frac{T^4}{144}} \tag{13}$$

[2] Reference: Laine & Vuorinen, *Basics of Thermal Field Theory*.

The (c) term is structurally similar to the (b) term, but with a symmetrized 3-gluon vertex:

$$\left(\frac{1}{3!}\right) igf_{abc}\left(g_{\mu\nu}(k-p)_\rho + g_{\nu\rho}(p-q)_\mu + g_{\rho\mu}(q-k)_\nu\right) = \left(\frac{1}{3!}\right) igf_{abc}\, D_{\mu\nu\rho}(k,p,q) \tag{14}$$

To link the legs of two 3-gluon vertices as shown in (c), there are 3! possibilities. Therefore, we have:

$$\begin{aligned}
\ln \mathcal{Z}^{(c)} &= \frac{1}{2!}\cdot 3!\cdot\left(\frac{1}{3!}\right)^2 \frac{T}{V}\sum_k \frac{T}{V}\sum_p \Delta(k)\,\Delta(p)\left(\frac{V}{T}\right)\Delta(p-k) \\
&\quad\times\left(igf_{abc}D_{\mu\nu\rho}(k,-p,p-k)\right)\left(igf^{bac}D^{\nu\mu\rho}(p,-k,k-p)\right) \\
&= -\frac{1}{12}\frac{V}{T}\cdot\frac{T}{V}\sum_k \frac{T}{V}\sum_p \Delta(k)\,\Delta(p)\,\Delta(p-k) \\
&\quad\times\left(\frac{N_c^2-1}{2}\,2N_c\right) g^2 D_{\mu\nu\rho}(k,-p,p-k)\,D^{\mu\nu\rho}(p,-k,k-p) \\
&= -\left(\frac{N_c^2-1}{2}\,2N_c\right)\frac{1}{12}\,g^2\frac{V}{T}\cdot\frac{T}{V}\sum_k \frac{T}{V}\sum_p \Delta(k)\,\Delta(p)\,\Delta(p-k) \\
&\quad\times D_{\mu\nu\rho}(k,-p,p-k)\,D^{\mu\nu\rho}(p,-k,k-p),
\end{aligned} \tag{15}$$

$$\begin{aligned}
D_{\mu\nu\rho}(k,-p,p-k)\,D^{\nu\mu\rho}(p,-k,k-p) &= D_{\mu\nu\rho}(k,-p,p-k)\,D^{\mu\nu\rho}(-k,p,k-p) \\
&= -D_{\mu\nu\rho}(k,-p,p-k)\,D^{\mu\nu\rho}(k,-p,p-k) \\
&= -\boxed{g_{\mu\nu}g^{\mu\nu}}\left((k+p)^2+(k-2p)^2+(p-2k)^2\right) \\
&\quad - 2\,(k+p)\cdot(k-2p) \\
&\quad - 2\,(k-2p)\cdot(p-2k) \\
&\quad - 2\,(p-2k)\cdot(k+p) \\
&= -\boxed{d+1}\cdot 3\left(k^2+p^2+(k-p)^2\right) \\
&\quad - 2\,(-\tfrac{3}{2})\left(k^2+p^2+(k-p)^2\right) \\
&= -3d\left(k^2+p^2+(k-p)^2\right),
\end{aligned} \tag{16}$$

$$\begin{aligned}
\ln \mathcal{Z}^{(c)} &= -\left(\frac{N_c^2-1}{2}\,2N_c\right)\frac{1}{12}\,g^2\frac{V}{T}\cdot\frac{T}{V}\sum_k \frac{T}{V}\sum_p \frac{-3d\left(k^2+p^2+(k-p)^2\right)}{k^2p^2\,(p-k)^2} \\
&= \left(\frac{N_c^2-1}{2}\,2N_c\right)\frac{d}{4}\,g^2\frac{V}{T}\cdot\frac{T}{V}\sum_k \frac{T}{V}\sum_p \frac{k^2+p^2+(k-p)^2}{k^2p^2\,(p-k)^2},
\end{aligned} \tag{17}$$

$$\begin{aligned}
\frac{T}{V}\sum_k \frac{T}{V}\sum_p \frac{k^2+p^2+(k-p)^2}{k^2p^2\,(p-k)^2} &= \frac{T}{V}\sum_k \frac{T}{V}\sum_p \left(\frac{1}{p^2\,(p-k)^2}+\frac{1}{k^2\,(p-k)^2}+\frac{1}{k^2p^2}\right) \\
&= 3\left(\frac{T}{V}\sum_k \frac{1}{k^2}\right)^2 = 3\left(\frac{T^2}{12}\right)^2,
\end{aligned} \tag{18}$$

$$\boxed{\ln \mathcal{Z}^{(c)} = \left(\frac{N_c^2-1}{2}\,2N_c\right)\frac{d}{4}\,g^2\frac{V}{T}\cdot 3\left(\frac{T^2}{12}\right)^2 = +\frac{V}{T}\,N_c\left(N_c^2-1\right)\frac{3d}{4}\,g^2\frac{T^4}{144}} \tag{19}$$

Here we use d to denote spatial dimensions; for $d=3$ we have $\frac{3d}{4}=\frac{9}{4}$.

The (d) term is built around the symmetrized 4-gluon vertex:

$$\left(\frac{1}{3!}\right)\left(-\frac{g^2}{4}\right)\Big(f_{ad}{}^{\bullet} f_{bc\,\bullet}(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma}) + \boxed{(b,\nu) \leftrightarrow (d,\sigma)} + \boxed{(b,\nu) \leftrightarrow (c,\rho)}\Big) \tag{20}$$

Here we use "•" to denote a contracted adjoint index, and use $\boxed{(\cdots) \leftrightarrow (\cdots)}$ to mark a switch of indices relative to the *previous* term. By contracting 2 pairs of the indices, we obtain the desired diagram (d). There are 3 ways to do this, therefore:

$$\begin{aligned}\ln \mathcal{Z}^{(d)} &= \frac{1}{1!}\cdot 3\cdot\left(\frac{1}{3!}\right)\left(-\frac{g^2}{4}\right)\left(-\frac{N_c^2-1}{2}\,2N_c\right)\frac{T}{V}\sum_k \frac{T}{V}\sum_p \Delta(k)\,\Delta(p)\left(\frac{V}{T}\right)\delta_{+k-k+p-p} \\ &\quad\times\Big(-\big((d+1)^2-(d+1)\big)+0+\big((d+1)-(d+1)^2\big)\Big)\end{aligned} \tag{21}$$

One of the three terms in the vertex vanishes after contraction, due to the anti-symmetry of f_{abc}. Again, we've used the fact that $g_{\mu\nu}g^{\mu\nu} = d+1$, where d is the spatial dimension.

Note that momentum conservation is automatic at the vertex, which is indicated by the trivial delta function $\delta_{+k-k+p-p} = 1$. In the end, we have:

$$\boxed{\begin{aligned}\ln \mathcal{Z}^{(d)} &= \left(-\frac{g^2}{8}\right)\left(\frac{N_c^2-1}{2}\,2N_c\right)\frac{T}{V}\sum_k \frac{T}{V}\sum_p \Delta(k)\,\Delta(p)\left(\frac{V}{T}\right)2d\,(d+1) \\ &= \left(-\frac{g^2}{8}\right)\left(\frac{N_c^2-1}{2}\,2N_c\right)\left(\frac{V}{T}\right)2d\,(d+1)\left(\frac{T^2}{12}\right)^2 \\ &= -\frac{V}{T}\,N_c\big(N_c^2-1\big)\,\frac{d\,(d+1)}{4}\,g^2\,\frac{T^4}{144}\end{aligned}} \tag{22}$$

For $d=3$ we have $\frac{d\,(d+1)}{4} = 3$. Combining (a ~ d), we have the full $\mathcal{O}\big(g^2\big)$ partition function. ■