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## 1 T-duality of Heterotic Strings<sup>1</sup>

We use  $d \leq 10$  to denote the number of noncompact dimensions; the remaining  $m \geq d$  dimensions are compactified. For heterotic strings, the  $I \geq 10$  dimensions of the left-moving sector are already compactified on a lattice  $\Gamma_{16}$  or  $\Gamma_8 \times \Gamma_8$ . Here we use the label I to index the 16 internal dimensions.

(a) Generally, if we compactify an open string on the  $x^m$  direction:  $x^m \cong x^m + 2\pi R$  with constant backgrounds  $A_m$ , then its zero mode spectrum, with winding w = 0, can be obtained from canonical quantization of the effective point particle action, with an additional gauge action term in the form of a Wilson line<sup>2</sup>:

$$-W_q = -iq \int dx^m A_m \sim -iq \int d\tau A_m \dot{X}^m$$
 (1)

By imposing that the canonical momentum to be periodic along  $x^m$ , we find that:

$$k_m = \frac{n_m}{R} - qA_m \tag{2}$$

To obtain the winding states, we have to reproduce the above action from the world-sheet description. For heterotic strings with  $m < 10 \le I \le 26$ , this can be achieved by adding the following term to the usual world-sheet action<sup>3</sup>:

$$S_A \propto \int \mathrm{d}^2 \sigma \, \epsilon^{ab} A^I_\mu \, \partial_a X^\mu \, \partial_b X_I$$
 (3)

With proper normalization to match the result in (2).

Canonical quantization then produces<sup>4</sup>:

$$k_{m} = \frac{n_{m}}{R} \pm \frac{w_{m}R}{\alpha'} - q_{I}A_{m}^{I} - \frac{w_{n}R}{2}A_{I}^{n}A_{m}^{I}, \tag{4}$$

$$k_L^I = \sqrt{\frac{2}{\alpha'}} \left( q^I + w^m R A_m^I \right), \tag{5}$$

The " $\pm$ " signs in  $k_m$  correspond to the left and right-moving sector, respectively. Only the left-moving sector has an additional 16 dimensional internal torus, therefore  $k^I$  is labeled with an "L".

Note that the charge  $q^I$  now takes value on the  $\Gamma_{16}$  or  $\Gamma_8 \times \Gamma_8$  lattice, and:

$$l \circ l' = \frac{\alpha'}{2} \left( k_L^I k_{L,I}' + k_L^m k_{L,m}' - k_R^m k_{R,m}' \right) = q^I q_I' + 2nw \tag{6}$$

We can then see that the new "extended" lattice indeed satisfies the even and self-dual conditions, which follows from the even and self-dual properties of  $\Gamma_{16}$  or  $\Gamma_8 \times \Gamma_8$ .

(b) With m = 9 and  $G_{dd} = 1$ , we have:

$$W_q = \exp\left(-iq_I\theta^I\right), \quad A_9^I = -\frac{\theta^I}{2\pi R}$$
 (7)

Note that  $W_q$  captures the phase change of the paths that wind around  $x^9$ ; the extra phase from a non-trivial Wilson line might affect the boundary condition of some states while leaving others

<sup>&</sup>lt;sup>1</sup>I would like to thank Lucy Smith for help with this problem.

 $<sup>^2 {\</sup>it Reference: Polchinski},$  Chapter 8.

 $<sup>^3</sup>$ Reference: Blumenhagen et al., Basic Concepts of String Theory.

<sup>&</sup>lt;sup>4</sup>Reference: *Polchinski*, Chapter 11.

intact, thus breaking the original gauge symmetry. Our discussions here closely follow *Polchinski*, Chapter 11.

For the SO(32) theory with:

$$RA_9^I = \text{diag}\left(\left(\frac{1}{2}\right)^8, 0^8\right)$$
 (8)

Adjoint states are labeled by a pair of indices valued in  $1, \dots, 32$ ; those with one index from  $1 \le A \le 16$  and one from  $17 \le A \le 32$  are anti-periodic due to the additional phase  $e^{i\pi} = -1$  from the Wilson line, so the gauge symmetry is reduced to  $SO(16) \times SO(16)$ .

Similarly, for the  $E_8 \times E_8$  theory with:

$$R'A_9^I = \text{diag}(1, 0^7, 1, 0^7)$$
 (9)

Note that  $\Gamma_8$ , the root lattice of  $E_8$ , is basically the root lattice union an additional spinor weight lattice of SO(16). With the above Wilson line, the integer-charged states from the SO(16) root lattice in each  $E_8$  remain periodic, while the half-integer charged states from the SO(16) spinor lattices become anti-periodic, due to the additional phase  $e^{i\frac{1}{2}\cdot 2\pi} = -1$ . Again the gauge symmetry is broken down to SO(16)  $\times$  SO(16).

In summary, with the above Wilson line, the SO(32) and  $E_8 \times E_8$  theory shares an unbroken gauge of SO(16)  $\times$  SO(16). Consider the spectrum of the SO(16)  $\times$  SO(16) neutral states, i.e. those with internal momentum:

$$k_L^I = \sqrt{\frac{2}{\alpha'}} \left( q^I + wRA_9^I \right) = 0 \tag{10}$$

For the SO(32) theory, since  $q^I \in \Gamma_{16}$  while  $RA_9^I = \text{diag}\left((\frac{1}{2})^8, 0^8\right)$ , we must have w = 2m for this to hold. The same goes for the  $E_8 \times E_8$  theory; therefore, we have:

$$k_{L,R} = \frac{\tilde{n}}{R} \pm \frac{2mR}{\alpha'}, \quad k'_{L,R} = \frac{\tilde{n}'}{R'} \pm \frac{2m'R'}{\alpha'}, \tag{11}$$

$$\tilde{n} = n + 2m, \quad \tilde{n}' = n' + 2m'$$
 (12)

(c) If the two theories are related by T-duality, then we should expect:

$$(k_L, k_R) \longleftrightarrow (k'_L, -k'_R),$$
 (13)

Under suitable mapping of parameters. Indeed, it is straightforward to verify that  $(\tilde{n}, m) \leftrightarrow (m', \tilde{n}')$  realizes this, along with  $RR' = \alpha'/2$ . The above arguments can then be generalized to higher levels, by acting on fermionized left-moving fields  $\lambda^A$  and carefully organizing representations. We see that the two heterotic string theories are equivalent under T-duality.

## 2 String Junction<sup>5</sup>

For a string junction to be mechanically stable, the tension force exerted on the junction must cancel each other; this is a Newtonian mechanics problem, but with (p,q)-string tension given by the BPS bound:

$$\tau_{(p,q)} = \frac{\sqrt{p^2 + q^2/g^2}}{2\pi\alpha'} \tag{14}$$

Stability of the system implies that the BPS bound should be saturated.

<sup>&</sup>lt;sup>5</sup>Reference: arXiv:0812.4408.

From Newtonian mechanics, we know that three forces cannot cancel each other unless they are co-planar. Therefore, a 3-string junction must be co-planar in order to be stable. Suppose they lie in the  $(X^1, X^2)$  plane, then the tension exerted on the junction can expressed as:

$$\vec{T}_i = \tau_{(p_i, q_i)} \left(\cos \theta_i, \sin \theta_i\right), \quad i = 1, 2, 3, \tag{15}$$

 $\sum_{i} \vec{T}_{i} = 0$  gives two equations, and we have two independent unknowns (the angle between two pairs of strings); therefore if a solution exists, it should be unique up to rotations and reflections.

In fact, a solution can be found by simple observations:

$$\cos \theta_i = \frac{p_i}{\sqrt{p^2 + q^2/g^2}}, \quad \sin \theta_i = \frac{q_i/g}{\sqrt{p^2 + q^2/g^2}},$$
 (16)

It satisfies  $\sum_i \vec{T_i} = 0$  since that total (p,q) vanishes at each junction.

To find the remaining supersymmetries of this system, we start from the original supersymmetries of a (p,q) string (which saturates the BPS bound) extended along the  $\hat{X} = (\cos \theta, \sin \theta)$  direction:

$$\frac{1}{2L\tau_{(p,q)}} \left\{ \begin{bmatrix} Q_{\alpha} \\ \tilde{Q}_{\alpha} \end{bmatrix}, \begin{bmatrix} Q_{\beta}^{\dagger} \ \tilde{Q}_{\beta}^{\dagger} \end{bmatrix} \right\} = \delta_{\alpha\beta} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (\Gamma^{0}\Gamma^{\theta})_{\alpha\beta} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \tag{17}$$

$$\Gamma^{\theta} = \hat{X} \cdot \vec{\Gamma} = \Gamma^{1} \cos \theta + \Gamma^{2} \sin \theta \tag{18}$$

We see that the algebra depends on  $\theta$ , i.e. it is different for strings in different directions. However, if we can find a (maximal) subalgebra that is independent of  $\theta$ , then we would have found the remaining supersymmetries of the full system<sup>6</sup>.

We begin with diagonalizing the matrix on the RHS with:

$$U(\frac{\theta}{2}) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix},\tag{19}$$

$$\frac{1}{2L\tau_{(p,q)}} \left\{ U^{\mathrm{T}} \begin{bmatrix} Q_{\alpha} \\ \tilde{Q}_{\alpha} \end{bmatrix}, \left[ Q_{\beta}^{\dagger} \tilde{Q}_{\beta}^{\dagger} \right] U \right\} = \begin{bmatrix} (\mathbb{1} + \Gamma^{0} \Gamma^{\theta})_{\alpha\beta} & 0 \\ 0 & (\mathbb{1} - \Gamma^{0} \Gamma^{\theta})_{\alpha\beta} \end{bmatrix}, \tag{20}$$

Note that  $(\mathbb{1} + \Gamma^0 \Gamma^\theta)(\mathbb{1} - \Gamma^0 \Gamma^\theta) = 0$  and  $(\mathbb{1} + \Gamma^0 \Gamma^\theta) + (\mathbb{1} - \Gamma^0 \Gamma^\theta) = 2\mathbb{1}$ , i.e. they are orthogonal to each other; acting  $(\mathbb{1} \pm \Gamma^0 \Gamma^\theta)$  on both sides, we find the following combinations, which gives the 16 SUSYs of a (p,q) string:

$$(\mathbb{1} - \Gamma^0 \Gamma^\theta) \left( \cos \frac{\theta}{2} Q + \sin \frac{\theta}{2} \tilde{Q} \right)_{\alpha} = 0 = (\mathbb{1} + \Gamma^0 \Gamma^\theta) \left( -\sin \frac{\theta}{2} Q + \cos \frac{\theta}{2} \tilde{Q} \right)_{\beta}$$
(21)

<sup>&</sup>lt;sup>6</sup>The  $\tau_{(p,q)}$  factor can be absorbed by rescaling generators, hence does not matter in our discussions.

For further simplification, we can isolate the  $\theta$  dependence in  $\Gamma^{\theta}$  by working in a specific representation of the Clifford algebra, e.g. the Dirac representation given by *Polchinski*. Then  $\alpha$  is given by 10 D spinor components:  $\alpha = (s_0, s_1, s_2, s_3, s_4)$ ,  $s_i = \pm$ , with additional chirality constraints from both Q and  $\tilde{Q}$ :  $\prod_i s_i = +$ . In the end, we have 16 independent components as expected.

Details of the expansion are given in arXiv:0812.4408. When the dust settles, we find that the 16 SUSYs in (21) is given by:

$$\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}Q + \sin\frac{\theta}{2}\tilde{Q}\right)_{(++s)} + \cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}Q + \sin\frac{\theta}{2}\tilde{Q}\right)_{(--s)},\tag{22a}$$

$$\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}Q + \sin\frac{\theta}{2}\tilde{Q}\right)_{(+-s)} - \cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}Q + \sin\frac{\theta}{2}\tilde{Q}\right)_{(-+s)},\tag{22b}$$

$$\cos\frac{\theta}{2}\left(-\sin\frac{\theta}{2}Q + \cos\frac{\theta}{2}\tilde{Q}\right)_{(++s)} - \sin\frac{\theta}{2}\left(-\sin\frac{\theta}{2}Q + \cos\frac{\theta}{2}\tilde{Q}\right)_{(--s)},\tag{22c}$$

$$\cos\frac{\theta}{2}\left(-\sin\frac{\theta}{2}Q + \cos\frac{\theta}{2}\tilde{Q}\right)_{(+-s)} + \sin\frac{\theta}{2}\left(-\sin\frac{\theta}{2}Q + \cos\frac{\theta}{2}\tilde{Q}\right)_{(-+s)},\tag{22d}$$

$$s = (s_2 s_3 s_4), \quad \prod_i s_i = +$$
 (23)

By trial and error, we can find the 8 linear combinations that are independent of  $\theta$ ; they are:

$$(a) + (c) \implies \tilde{Q}_{(++s)} + Q_{(--s)}, \tag{24}$$

$$(b) + (d) \implies \tilde{Q}_{(+-s)} - Q_{(-+s)}, \tag{25}$$

Therefore, the string junction is  $\frac{8}{32} = \frac{1}{4}$  BPS.

## 3 Two and Three-Point Functions in AdS/CFT

Consider a scalar field  $\phi(x,z)$  in Poincaré AdS<sub>5</sub> (with radius R=1) satisfying:

$$(\nabla^2 - m^2) \phi(x, z) = 0, \quad \phi(x, z) \to \begin{cases} z^{\delta} \phi_0(x), & z \to 0, \\ \text{regular}, & z \to \infty, \end{cases} \quad \delta = 2 - \sqrt{m^2 + 4}$$
 (26)

It can be constructed via the boundary-to-bulk propagator  $K_{\Delta}$ :

$$\phi(x,z) = \int d^4x' K_{\Delta}(x,z;x') \phi(x'), \qquad (27)$$

$$K_{\Delta}(x,z;x') = \frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \left(\frac{z}{z^2 + \|x - x'\|^2}\right)^{\Delta}, \quad \Delta = 2 + \sqrt{m^2 + 4}$$
 (28)

(a) To verify this, we first check that the boundary conditions are indeed satisfied by  $K_{\Delta}$ ; note that  $\Delta \geq 2 > 0$ , and we have:

$$z \to 0, \quad l \neq 0, \quad \left(\frac{z}{z^2 + l^2}\right)^{\Delta} \to 0,$$
 (29)

i.e. the only contribution comes from the  $l \to 0$  case, where we have:

$$l = ||x - x'|| \to 0, \quad \int d^4 x' \, K_{\Delta}(x, z; x') = \frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \int_0^{\infty} 2\pi^2 l^3 \, dl \left(\frac{z}{z^2 + l^2}\right)^{\Delta}$$
$$= \frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \cdot \frac{2\pi^2}{2(\Delta - 1)(\Delta - 2)} z^{4 - \Delta}$$
$$= z^{\delta}$$
(30)

Therefore, we have:

$$z \to 0$$
,  $K_{\Delta}(x, z; x') \to z^{\delta} \delta^4(x - x')$ ,  $\phi(x, z) \to z^{\delta} \phi_0(x)$ , (31)

The other boundary condition is convenient to check; we have:

$$z \to \infty$$
,  $K_{\Delta}(x, z; x') \propto z^{-\Delta} \to 0$ ,  $\phi(x, z)$  regular. (32)

Now we need only check that  $K_{\Delta}$  satisfies the equation of motion; in Poincaré AdS<sub>5</sub> we have:

$$\nabla^2 = z^2 \left( \partial_z^2 - \frac{3}{z} \, \partial_z + \partial_x^2 \right) \tag{33}$$

With the help of Mathematica<sup>TM</sup>, it is straightforward to check that  $(\nabla^2 - m^2) \left(\frac{z}{z^2 + l^2}\right)^{\Delta} = 0$ , therefore  $(\nabla^2 - m^2) \phi(x, z) = 0$ .

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The AdS/CFT dictionary is given by:

$$\left\langle e^{\int d^4 x \, C_O \, \phi_0(x) \, O(x)} \right\rangle_{\text{CFT}} = e^{-S[\phi_0]} \tag{34}$$

Where  $S[\phi_0]$  the bulk effective action evaluated on the solution to the equation of motion:

$$S[\phi_0] = \int d^4x \, dz \, \sqrt{-G} \left\{ \frac{1}{2} \left( \partial_\mu \phi \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} g \phi^3 + \cdots \right\}$$
 (35)

(b) The CFT 2-point function  $\langle O(x) O(y) \rangle$  can be computed with the above dictionary, using the usual effective formalism, but with the bulk action instead of the boundary action:

$$\langle O(x) O(y) \rangle = \frac{1}{C_O^2} \left. \frac{\delta^2}{\delta \phi_0(x) \delta \phi_0(y)} e^{-S[\phi_0]} \right|_{\phi_0 = 0}$$
(36)

For 2-point function, we only need terms  $\sim \mathcal{O}(\phi^2)$ ; note that:

$$\frac{\delta\phi(x,z)}{\delta\phi_0(x')} = K_{\Delta}(x,z;x'),\tag{37}$$

$$\delta S \left[\phi_{0}\right] \sim \int d^{4}x \, dz \, \sqrt{-G} \left(-\nabla^{2} + m^{2}\right) \phi \, \delta\phi + \int_{z \to 0} d^{4}x \, \sqrt{-G} \, \partial^{z}\phi \, \delta\phi$$

$$= 0 - \int_{z \to 0} d^{4}x \, z^{-5}z^{2} \partial_{z}\phi \, \delta\phi = -\int_{z \to 0} d^{4}x \, z^{-3} \partial_{z}\phi \, \delta\phi \,,$$

$$\therefore \langle O(x) \, O(y) \rangle = + \frac{1}{C_{O}^{2}} \frac{\delta}{\delta\phi_{0}(x)} e^{-S[\phi_{0}]} \int_{z \to 0} d^{4}x' \, z^{-3} \partial_{z}\phi(x', z) \, K_{\Delta}(x', z; y) \Big|_{\phi_{0} = 0}$$

$$= \frac{1}{C_{O}^{2}} \int_{z \to 0} d^{4}x' \, z^{-3} \partial_{z} K_{\Delta}(x', z; x) \, K_{\Delta}(x', z; y)$$

$$= \frac{1}{C_{O}^{2}} \int_{z \to 0} d^{4}x' \, z^{-3} \partial_{z} K_{\Delta}(x', z; x) \, z^{\delta} \delta^{4}(x' - y)$$

$$= \frac{z^{\delta - 3}}{C_{O}^{2}} \partial_{z} K_{\Delta}(x, z; y)$$

$$\sim \frac{z^{\delta - 3}}{C_{O}^{2}} \frac{(\Delta - 1)(\Delta - 2)}{\pi^{2}} \frac{\Delta z^{\Delta - 1}}{\|x - y\|^{2\Delta}}$$

$$(38)$$

 $= \frac{1}{C_O^2} \frac{\Delta(\Delta - 1)(\Delta - 2)}{\pi^2} \frac{1}{\|x - y\|^{2\Delta}}$ 

Therefore, if we want  $\langle O(x) \, O(y) \rangle = \frac{1}{\|x-y\|^{2\Delta}}$ , then we have<sup>7</sup>:

$$C_O = \frac{1}{\pi} \sqrt{\Delta(\Delta - 1)(\Delta - 2)} \tag{40}$$

Here  $z \to 0$  is a cutoff parameter.

(c) Similarly, we can use the dictionary to compute 3-point functions; we have<sup>8</sup>:

$$\langle O(x_1) O(x_2) O(x_3) \rangle = \frac{1}{C_O^3} \left( -\frac{g}{3} \right) \int d^4 x \, dz \, \sqrt{-G} \, K_\Delta(x, z; x_1) \, K_\Delta(x, z; x_2) \, K_\Delta(x, z; x_3)$$
(41)

This is a difficult integral; as is suggested by arXiv:hep-th/9804058, we can use an important symmetry of AdS/CFT — the inversion  $\vec{x} \mapsto \frac{\vec{x}}{x^2}$ , to complete the integration.

By conformal symmetry, we know that the 3-point function is of the form:

$$\langle O(x_1) O(x_2) O(x_3) \rangle = A(x_1, x_2, x_3) = \frac{C_{OOO}}{|x_{12}|^{\Delta} |x_{23}|^{\Delta} |x_{31}|^{\Delta}}, \quad x_{ij} = x_i - x_j$$
 (42)

First set  $x_3 = 0$ , then perform inversion on all other points:

$$x_i = \frac{x_i'}{x_i'^2}, \quad (x, z) = \frac{(x', z')}{r'^2}, \quad r^2 = x^2 + z^2, \quad r^2 r'^2 = 1 = x_i^2 x_i'^2,$$
 (43)

$$\frac{d^4 x \, dz}{z^5} = \frac{d^4 x' \, dz'}{z'^5},\tag{44}$$

$$\frac{z}{z^{2} + \|x - x_{i}\|^{2}} = \frac{z}{r^{2} + x_{i}^{2} - 2x \cdot x_{i}} = \frac{z'/r'^{2}}{1/r'^{2} + 1/x_{i}'^{2} - 2x' \cdot x_{i}'/(r'^{2}x_{i}'^{2})}$$

$$= \frac{z'}{r'^{2} + x_{i}'^{2} - 2x' \cdot x_{i}'} x_{i}'^{2} = \frac{z'}{z'^{2} + \|x' - x_{i}'\|^{2}} x_{i}'^{2}, \tag{45}$$

$$K_{\Delta}(x, z; x_i) = K_{\Delta}(x', z'; x_i') |x_i'|^{2\Delta} = \frac{1}{|x_i|^{2\Delta}} K_{\Delta}(x', z'; x_i'), \tag{46}$$

With these in mind, we find that:

$$A(x_1, x_2, 0) = -\frac{g}{3C_O^3} \frac{1}{|x_1|^{2\Delta}} \frac{1}{|x_2|^{2\Delta}} \frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \int \frac{\mathrm{d}^4 x' \,\mathrm{d}z'}{z'^5} K_\Delta(x', z'; x_1') K_\Delta(x', z'; x_2') z'^\Delta \tag{47}$$

The integral can then be completed using Feynman parameters; in the end we obtain:

$$A(x_1, x_2, 0) \propto \frac{1}{|x_1|^{2\Delta}} \frac{1}{|x_2|^{2\Delta}} \frac{1}{|x_1' - x_2'|^{2\Delta}} = \frac{1}{|x_1|^{\Delta} |x_2|^{\Delta} |x_1 - x_2|^{\Delta}},$$
(48)

$$C_{OOO} = -\frac{g}{3C_O^3} \frac{1}{2\pi^4} \left( \frac{\Gamma(\frac{\Delta}{2})}{\Gamma(\Delta - 2)} \right)^3 \Gamma\left(\frac{3\Delta - 4}{2}\right)$$
 (49)

<sup>&</sup>lt;sup>7</sup>Reference: arXiv:hep-th/9804058. Again I would like to thank Lucy Smith for helpful hints.

<sup>&</sup>lt;sup>8</sup>Reference: arXiv:hep-th/9905111, and arXiv:hep-th/9804058.