A Modern Take on Renormalization

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Reason: The continuum limit $\Lambda \to \infty$ is **not** well-defined. Renormalization provides a way to *define* the theory when $\Lambda \to \infty$.

My belief: The only way to fully understand renormalization is through Wilson's arguments; all other "interpretations" of renormalization are only *heuristic*.

Keywords:

- Renormalization group
- Counterterms
- Regularization and cutoff

Contents

0 References **1**

1 Wilson's picture **2**

- 1.1 Free theory & CFT . . . 3
- 1.2 Massive theory & the space of theories . . . 4
- 1.3 Physical Scales vs the Cutoff . . . 5

2 Perturbative Renormalization **6**

- 2.1 Counterterms . . . 6
- 2.2 Perturbation . . . 7
- 2.3 Renormalization Schemes . . . 7

3 Renormalizability **8**

4 Effective Action **8**

0 References

- David Skinner's note:
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    * https://www.damtp.cam.ac.uk/user/dbs26/AQFT/Wilsonchap.pdf
    * https://www.damtp.cam.ac.uk/user/dbs26/AQFT/chap5.pdf
- Schwartz, Chapter 15
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1 Wilson's picture

Seed theory parameters (i.e. *bare parameters*): $(Z_\phi, g, \Lambda)_0$, where $g = (m, \lambda, \cdots)$ is the collection of all possible couplings. Z_ϕ is the coefficient of the kinetic term.

$$\begin{aligned} \phi_{\Lambda_0}(x) \sim \int^{\Lambda_0} \mathrm{d}p\, e^{ip\cdot x}\tilde{\phi}(p) &= \int^{\Lambda} \mathrm{d}p\, e^{ip\cdot x}\tilde{\phi}(p) + \int_{\Lambda}^{\Lambda_0} \mathrm{d}p\, e^{ip\cdot x}\tilde{\phi}(p) \\ &=: \phi_\Lambda(x) + \chi(x) \end{aligned} \tag{1.1}$$

$$\begin{aligned} \mathcal{D}\phi_{\Lambda_0}(x) \sim \prod_{\|p\|<\Lambda_0} \mathrm{d}\tilde{\phi}(p) &= \prod_{\|p\|<\Lambda} \mathrm{d}\tilde{\phi}(p) \prod_{\Lambda<\|p\|<\Lambda_0} \mathrm{d}\tilde{\phi}(p) \\ &\sim \mathcal{D}\phi_\Lambda(x)\; \mathcal{D}\chi(x) \end{aligned} \tag{1.2}$$

$$\begin{aligned} \mathcal{Z}(g_0, \Lambda_0) &= \int \mathcal{D}\phi_{\Lambda_0}\, e^{iS[\phi_{\Lambda_0}]} \\ &= \int \mathcal{D}\phi_\Lambda \int \mathcal{D}\chi\, e^{iS[\phi_\Lambda + \chi]} \\ &=: \int \mathcal{D}\phi_\Lambda\, e^{iS_{\text{eff}}[\phi_\Lambda]} =: \mathcal{Z}(g(\Lambda), \Lambda) \end{aligned} \tag{1.3}$$

Coarse-grained parameters: (Z_ϕ, g, Λ).

Subtlety: the notation above is only schematic; in practice we first Wick-rotate to Euclidean signature, so that the momentum cutoff is easily imposed: $\|p\| = \sqrt{p_0^2 + \mathbf{p}^2} < \Lambda$. In Lorentzian signature, it's hard to define a covariant cutoff since $p_\mu p^\mu = -p_0^2 + \mathbf{p}^2$. This process can be made rigorous; just think of the 8-shaped contour in loop integrals.

Effective action:

$$S_{\text{eff}}[\phi] = -i \ln \int \mathcal{D}\chi\, e^{iS[\phi+\chi]} \tag{1.4}$$

$\phi = \phi_\Lambda$ is treated as constant when doing $\int \mathcal{D}\chi$. Perturbatively, we integrate over loops with χ propagators as internal lines.

$$\begin{aligned} \mathcal{L}[\phi+\chi] &= -\frac{Z_\phi}{2}\, \partial_\mu(\phi+\chi)\, \partial^\mu(\phi+\chi) - \frac{1}{2}\, m^2(\phi+\chi)^2 - \frac{1}{4!}\, \lambda\, (\phi+\chi)^4 \\ &= \cdots \\ &= \mathcal{L}[\phi] + \Delta\mathcal{L}[\phi, \chi] \end{aligned} \tag{1.5}$$

$$S_{\text{eff}}[\phi_\Lambda] = S[\phi_\Lambda] - i \ln \int \mathcal{D}\chi\, e^{i\,\Delta S[\phi_\Lambda + \chi]} \tag{1.6}$$

If $\Lambda \lesssim \Lambda_0$, then S_{eff} is almost the same as the original S, with minor corrections from the $\int \mathcal{D}\chi$ term. Note that in such regularization scheme there will be no *quadratic* cross terms $\sim \phi\chi$, $\partial\phi\, \partial\chi$ in the effective action, since they have orthogonal Fourier modes. However, there will be non-vanishing *quartic* cross terms $\sim \phi^2\chi^2$, $\phi^3\chi$. After we integrate out χ, the $\phi^2\chi^2$ term will cause a shift in m^2, while the $\phi^3\chi$ will generate a new ϕ^6 vertex (by a $\chi\chi$ contraction).

Note that $\mathcal{Z}$ clearly does not depend on the intermediate scale Λ, and we have:

$$0 = \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda}\, \mathcal{Z}(g(\Lambda), \Lambda) = \left(\Lambda \frac{\partial}{\partial\Lambda} + \Lambda \frac{\partial g^{(i)}}{\partial\Lambda} \frac{\partial}{\partial g^{(i)}} \right) \mathcal{Z}(g(\Lambda), \Lambda) \tag{1.7}$$

This is an example of a **RG Equation**.

1.1 Free theory & CFT

We first observe that a massless free theory with $\mathcal{L}[\phi] \sim \int \mathrm{d}^d x\, (\partial\phi)^2$ is "not renormalized" when we integrate out high energy modes. Roughly speaking, we have:

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\phi_{\Lambda_0}\, e^{iS[\phi_0]} \\ &= \int \mathcal{D}\phi_\Lambda \int \mathcal{D}\chi\, e^{iS[\phi+\chi]} \\ &\sim \int \mathcal{D}\phi_\Lambda\, e^{iS[\phi]} \sqrt{Z_\phi^{\#}} \end{aligned} \tag{1.8}$$

Integrating over $\Lambda = s\Lambda_0 \leq |p| \leq \Lambda_0$ produces the Z_ϕ factor, where $\#$ is the total number of modes in this range. One can give an explicit expression of $\#$ with some IR cutoff L.

On the other hand, we can restore $\Lambda = s\Lambda_0$ back to Λ_0 by rescaling. This is easy to understand when we think of a lattice theory: to probe the IR behavor, we first "coarse-grain" by grouping points together, effectively lowering Λ; then we "zoom out" to see the bigger picture. Rescaling $\Lambda = s\Lambda_0$ back to Λ_0 is precisely the "zooming-out" process.

When taking $\Lambda \mapsto \Lambda_0 = \Lambda/s$, first note that the kinetic term $\int \mathrm{d}^d x\, (\partial\phi)^2$ is scale-invariant if we rescale ϕ accordingly:

$$\Lambda \mapsto \Lambda_0 = \Lambda/s, \quad x \mapsto x' = sx, \quad \phi(x) \mapsto \phi'(x') = s^{1-\frac{d}{2}}\phi(x) \tag{1.9}$$

The $s^{1-\frac{d}{2}}$ factor is consistent with the mass dimension of ϕ, and we have $S'[\phi'] = S[\phi]$ invariant under rescaling.

The path integral measure, however, is *not* scale-invariant; we have to add back some Fourier modes in between to return the path integral back to Λ_0. Luckily, the additional modes happen to cancels out the Z_ϕ factor in (1.8):

$$\sqrt{Z_\phi^{\#}}\, \mathcal{D}\phi_\Lambda \sim \sqrt{Z_\phi^{\#}} \prod_{\|p\|<\Lambda=s\Lambda_0} \mathrm{d}\tilde{\phi}(p) = \prod_{\|p'\|<\Lambda_0} \mathrm{d}\tilde{\phi}'(p') \sim \mathcal{D}\phi'_{\Lambda_0} \tag{1.10}$$

Therefore the complete path integral is invariant under RG flow & rescaling; Z_ϕ is also unchanged during this process. Therefore we can simply set $Z_\phi \equiv 1$ by an appropriate *normalization* of ϕ.

One can see how this process might fail in a curved background: in flat space the Fourier modes are equally spaced, and rescaling $\mathcal{D}\phi_\Lambda$ can be achived by simply adding back a total of $\#$ modes. However, in a curved background the Fourier modes are complicated (e.g. think of spherical harmonics) and we generally do not expect the Jacobian of $\mathcal{D}\phi_\Lambda \mapsto \mathcal{D}\phi'_{\Lambda_0}$ to cancel out the Z_ϕ factor precisely. This is a hint of *Weyl anomaly* in curved backgrounds. For a concrete discussion with path integral, see *Di Francesco et al*, 5.A.

Generally, Z_ϕ *will* change after rescaling:

$$Z_\phi \sim \Lambda^{-2\gamma_\phi}, \quad \gamma_\phi := -\frac{1}{2}\Lambda\frac{\partial \ln Z_\phi}{\partial \Lambda} \tag{1.11}$$

The scaling of Z_ϕ can be absorbed by *field strength renormalization* (or "wave function" renormalization); i.e. if we *demand* ϕ to retain the canonical normalization such that $Z_\phi \equiv 1$, then ϕ must scale with an *anomalous dimension* γ_ϕ:

$$\phi(x) \;\longmapsto\; \phi'(x') = s^{1-\frac{d}{2}-\gamma_\phi}\phi(x) \tag{1.12}$$

In practice we don't actually redefine the scaling of ϕ, just simplify keep track of it using the Z_ϕ factor; but we should know that due to quantum corrections, the mass dimension of ϕ is, effectively,

$$\Delta_\phi = \frac{d}{2} - 1 + \gamma_\phi \tag{1.13}$$

In summary, we've argued that the classical & quantum theory of a massless free boson on flat space is *scale-invariant*; $\gamma_\phi = 0$. In fact, this is a first example of free CFTs. CFTs are the *fixed points* of RG flow.

1.2 Massive theory & the space of theories

What about *massive* free theories? We do know that they don't receive loop corrections, therefore "invariant" under coarse-graining; however, mass *do* flow when we zoom out. This is very natural; as we zoom out the energy-momentum of all modes are enhanced by a factor of s^{-1}, and we have $m \mapsto m/s$. Therefore m is a *relevant* parameter as we flow towards IR; it grows and flows away from the massless free CFT. For an explicitly path integral calculation (along with ϕ^{2n} interactions), see *Hollowood*, 2.1.

Generally, for small g we can think of the theory (g, Λ) as a *deformation* away from the massless free CFT. A neat trick to stop worrying about rescaling is to re-define g as dimensionless couplings:

$$g(\Lambda) = \left(g^{(2)} = \frac{m^2_{(\Lambda)}}{\Lambda^2},\ g^{(4)} = \lambda,\ g^{(6)} = \cdots \right) \tag{1.14}$$

Here we use $m^2_{(\Lambda)}$ to denote the mass *after* coarse-graining but *before* rescaling. The mass term is then given by $\sim \int \mathrm{d}^d x\, m^2\phi^2 = \int \mathrm{d}^d x\, g^{(2)}\Lambda^2\phi^2$. g is thus invariant under rescaling, as the rescaling factor is absorbed by the Λ factor. We also don't have to worry about the anomalous path integral measure, as it's captured by the Z_ϕ factor.

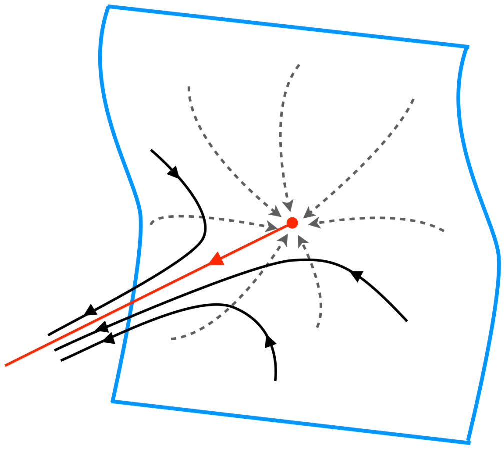
After rescaling $\Lambda \mapsto \Lambda_0$, the dimensionful mass is given by:

$$m'^2_{(\Lambda)} = g^{(2)}_{(\Lambda)}\Lambda_0^2 = m^2_{(\Lambda)}\left(\frac{\Lambda_0}{\Lambda}\right)^2 \tag{1.15}$$

In general, the rescaled, dimensional coupling is simply the dimensionless coupling times some factors of the *initial* cutoff Λ_0. Therefore its Λ dependence is identical to that of the dimensionless coupling. On the other hand, the coarse-grained, dimensionful couplings *before* rescaling is given by g_Λ times some power of Λ (instead of Λ_0); e.g.

$$m^2_{(\Lambda)} = g^{(2)}_{(\Lambda)}\Lambda^2 \xrightarrow{\text{free}} m_0^2 \tag{1.16}$$

i.e. it is constant for a free theory. Intuitively, this is because low & high energy modes are completely decoupled in a free theory.



1.3 Physical Scales vs the Cutoff

LSZ reduction (with the "mostly plus" metric convention, see *Srednicki*):

$$\int \mathrm{d}^d x\, e^{-ip\cdot x} \int \mathrm{d}^d y\, e^{+ik\cdot y} \big\langle \phi(x)\, \phi(y) \big\rangle \xmapsto{p,k \text{ on-shell}} \frac{-i\sqrt{Z_\phi}}{p^2+m^2-i\epsilon} \frac{-i\sqrt{Z_\phi}}{k^2+m^2-i\epsilon} \langle p|S|k\rangle \tag{1.17}$$

$$\big\langle \phi(x)\, \phi(y) \big\rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi_\Lambda\, e^{iS_\Lambda[\phi]}\, \phi(x)\, \phi(y) \tag{1.18}$$

$S \sim \mathbb{1} + i\mathcal{M}(\mu)$, physical amplitude $\mathcal{M}(\mu)$ scale with energy μ, but is cutoff-independent.

How to relate $\mathcal{M}(\mu)$ to $g(\Lambda)$? Note that the canonical Green function *does* scale with Λ! With the help of Poincaré invariance, we have:

$$Z_\Lambda \big\langle \phi(x)\, \phi(y) \big\rangle =: G_\Lambda\big(x, y; g(\Lambda)\big) = G_\Lambda\big(l; g(\Lambda)\big), \quad Z_\lambda := (Z_\phi)_\Lambda \tag{1.19}$$

Physical scale: $l = |x_1 - x_2|$, or center of mass momentum μ in amplitudes; $l \cdot \mu \sim 1$. Λ-dependence enters through Z_ϕ. For general n-pt function, RG flow:

$$\begin{aligned} \big\langle \phi^n(\cdots) \big\rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi_\Lambda\, e^{iS_\Lambda[\phi]}\, \phi^n(\cdots) &= Z_\Lambda^{-n/2} G_\Lambda\big(l; g(\Lambda)\big) \\ &= Z_{s\Lambda}^{-n/2} G_\Lambda\big(sl; g(s\Lambda)\big) \Big(s^{1-\frac{d}{2}}\Big)^{-n} \end{aligned} \tag{1.20}$$

This is a key result! We have:

$$G_\Lambda\big(l; g(\Lambda)\big) = G_\Lambda\big(sl; g(s\Lambda)\big) \left(s^{1-\frac{d}{2}} \sqrt{\frac{Z_\Lambda}{Z_{s\Lambda}}} \right)^n \simeq G_\Lambda\big(sl; g(s\Lambda)\big)\, s^{n\Delta_\phi} \tag{1.21}$$

Or in terms of energy scale μ,

$$\mathcal{M}\big(\mu; g(\Lambda)\big) = \mathcal{M}\big(\mu/s; g(s\Lambda)\big), \tag{1.22}$$

$$\frac{\partial}{\partial s} \;\Rightarrow\; \mu \frac{\partial \mathcal{M}}{\partial \mu} = \Lambda \frac{\partial g}{\partial \Lambda} \frac{\partial M}{\partial g} = \beta(g) \frac{\partial M}{\partial g} \tag{1.23}$$

2 Perturbative Renormalization

However, in naïve perturbation theory, we wish to complete the entire path integral $\mathcal{Z}(g_0, \Lambda_0)$. We can think of this as integrating out more and more high energy modes, until we reach the IR scale $\Lambda \to 0$.

When $\Lambda \ll \Lambda_0$, we have no reason to believe the renormalized couplings $g(\Lambda)$ are close to the original couplings g_0 at Λ_0. In fact, they may differ by a large (but finite) renormalization factor Z: $g_0 = Zg$.

2.1 Counterterms

In the above analysis, the theory flows from UV to IR. However, in reality, the IR results are known from experiments, and we are trying to *extrapolates* from IR to UV.

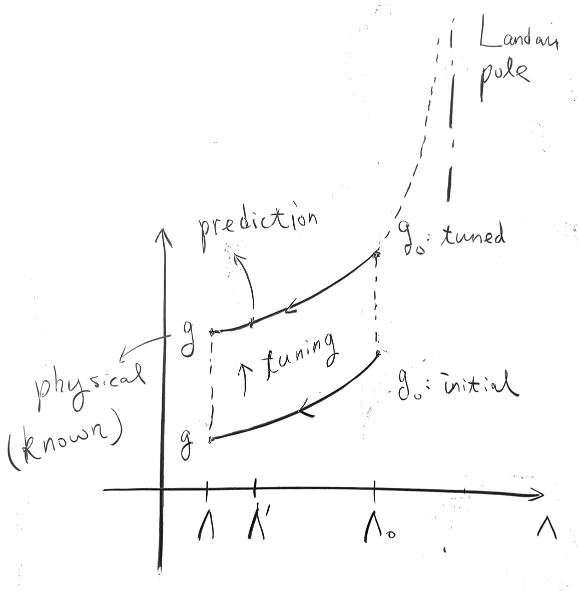
We achieve this by *tuning* the bare parameters (g_0, Λ_0) so that after RG flow, the IR results fit our experimental observations. If the IR couplings $g(\Lambda)$ are finite and small, then since $\Lambda \ll \Lambda_0$, we expect g_0 to be very large.

We often *split* g_0 into 2 parts for convenience:

$$g_0 = g + \delta g = g + (Z - 1)\, g \tag{2.1}$$

δg is the so-called *counterterm*; intuitively, it's the (large) correction that gets integrated out when we go from Λ_0 all the way to IR.

Basically, we have the following procedure:



0. Select some UV parameters (g_0, Λ_0)
1. Perform the RG flow: $(g_0, \Lambda_0) \to (g, \Lambda)$
2. Tune (redefine) g_0 so that (g, Λ) matches with experiments
3. Use the tuned data to predicts phenomena at a different scale (g', Λ')

Note that the tuning of UV parameters g_0 is *far from unique!* This is easy to understand: many UV theories might flow to the same IR theory. For this reason, some would say that RG is a *semi-group*.

However, for a *renormalizable* theory, we can restrict the tuning to a *finite* dimensional subspace formed by the *relevant* couplings, since most other parameters $g^{(i)}$ are *irrelevant* and get suppressed by Λ/Λ_0 in the IR. We shall see this in more details later. After such restriction to a relevant subspace, the RG flow is a group, and we can reverse the flow to *extrapolate* towards UV.

> **Subtlety:** the tuning process described above might encounter some serious obstruction: the tuned g_0 could blow up at some finite Λ_{UV}; this is the so-called *Landau pole*. This tells us that the theory only works under some Λ_{UV}, i.e. it is not *UV complete*; it's only an *effective* theory. One have to "embed" this Lagrangian into a bigger theory that works beyond Λ_{UV}; this is the non-trivial *UV completion* of an effective theory.

2.2 Perturbation

The above results are non-perturbative and should always hold. Perturbation theory is only a way to calculate the RG flow from UV to IR; it is reliable only if the IR coupling g is sufficiently small. In this case we can tune (g_0, Λ_0) with the following recursive / iterative algorithm:

1. Perturbative calculation of RG flow: $(g_0, \Lambda_0) \to (g, \Lambda)$ at $\mathcal{O}(g^n)$
2. Tune (redefine) (g_0, Λ_0) by adding counterterms, so that (g, Λ) matches with experiments
3. Increase order n and go to step 1

The (non-)renormalizability of a theory is evident in the perturbative expansion, by counting the **superficial degrees of divergence** D of the Feynman diagrams. Basically,

- Interaction vertices create loops, and loops create UV divergences. Higher order interaction vertices create more loops, which lead to more divergences.
- External lines suppress UV divergences by factors like $\frac{1}{\not p}$ or $\frac{1}{p^2}$.

For a renormalizable theory, there will be no divergence for diagrams with a sufficient number E of external legs; for a non-renormalizable theory, however, there will always be divergences, no matter how large E is.

2.3 Renormalization Schemes

There is a subtlety in the above procedure: how do we actually relate IR parameters (g, Λ) with actual physical quantities, e.g. amplitudes $\mathcal{M}(\mu)$?

In fact, we've assumed that $(g, \Lambda)_{\Lambda\to 0}$ gives the physical couplings that we are familiar with, e.g. mass, electric charge and so on. This is not quite true, since physical quantities are actually defined with scattering amplitudes. There are different choices of relating g with physical observables; this lead to various renormalization schemes:

- On-shell / pole-mass scheme
- Minimal subtraction (MS) & modified MS ($\overline{\text{MS}}$)

3 Renormalizability

As we've mentioned before, most parameters $g^{(i)} \in g$ are, in fact, *irrelevant* — such terms in the Lagrangian get suppressed by Λ/Λ_0 in IR. We shall see how this arise from dimensional anlaysis.

If the IR theory has only *relevant* couplings, then one should be able to recover their physical values by tuning a finite amount of relevant couplings in the UV, and usually the tuning is unique. This is the defining characteristic of a **renormalizable** theory. Basically, this means that we can naturally obtain a UV theory by extrapolation.

On the other hand, a theory is **non-renormalizable** if it contains irrelevant couplings in the IR. In this case the IR parameters g depend sensitively on small perturbations of the UV parameters g_0, and one has to tune infinitely many bare parameters to obtain the physical IR values. Such theory is hardly *fundamental*, since it depends on infinitely many parameters; but it's a good *effective theory* nonetheless.

4 Effective Action

After ϕ_Λ is completely integrated out, we have:

$$S_{\text{eff}}[\phi] \to W, \quad Z(g_0, \Lambda_0) = e^{iW} \tag{4.1}$$

Note that W no longer has any ϕ dependence, but it is a function of (g_0, Λ_0), which in turn is tuned by physical (g, Λ). W in fact contains all information about the seed theory, labeled by (g_0, Λ_0). To extract this information, we usually perturb the original action $S[\phi]$ with a source term; then we have:

$$S[\phi, J] = S[\phi] + \int \mathrm{d}x \; J(x)\, \phi(x), \quad W \to W[J] \tag{4.2}$$

Expand $W[J]$ in terms of J-modes, and its coefficient gives us physical coupling constants in the IR. More precisely, we can define the Legendre-transformed $\Gamma[\varphi]$.