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1 Spacelike Geodesic for Poincaré AdS

The Poincaré AdS_{d+1} metric is given by:

$$ds^{2} = G_{IJ} dX^{I} dX^{J} = \frac{-dt^{2} + d\vec{x}^{2} + dz^{2}}{z^{2}} = \frac{dx^{2} + dz^{2}}{z^{2}},$$
(1.1)

$$dx^{2} = -dt^{2} + d\vec{x}^{2} = \eta_{\mu\nu} dx^{\mu} dx^{\nu}, \quad X^{I} \sim (t, \vec{x}, z) \sim (x^{\mu}, z), \tag{1.2}$$

Note that here we define dx^2 using the flat metric η , while dX has an index that should be raised and lowered using the curved metric $G_{IJ} = \frac{1}{z^2} \eta_{IJ}$. Generally upper case tensors are handled with G_{IJ} , while lower case ones are handled with η . The full isometry of such spacetime is then given by SO(d,2) with $\frac{(d+2)(d+1)}{2}$ generators, including:

- 1. Among the $x^{\mu} \sim (x^0, \vec{x}) \equiv (t, \vec{x})$ directions:
 - (a) $\frac{d(d-1)}{2}$ SO(d-1,1) rotations $x_{\mu}\partial_{\nu} x_{\nu}\partial_{\mu}$, boosts included;
 - (b) d translations ∂_{μ}
- 2. Dilation with $X^I \sim (x^\mu, z) \mapsto \lambda X^I = \lambda(x^\mu, z)$, generated by $\Delta = X^I \partial_I = z \partial_z + x^\mu \partial_\mu$;
- 3. Special conformal transformations; see Appendix A for an intuitive derivation. We have:

$$k_{\mu} = (z^2 + x^2) \,\partial_{\mu} - 2x_{\mu} \Delta \tag{1.3}$$

By Noether's theorem, $Q_{\Xi} = V_I \Xi^I = G_{IJ} \Xi^I V^J$ is conserved along the geodesic; here V^I is the normalized tangent vector, while Ξ^I is some Killing vector of the spacetime, e.g. one from the list above. We can then write down the conserved charges along a geodesic γ in Poincaré AdS:

$$p_{\mu} = G_{\mu I} V^{I} = V_{\mu}$$

$$m_{\mu\nu} = x_{\mu} V_{\nu} - x_{\nu} V_{\mu}$$

$$\Delta = X^{I} V_{I} = \frac{z V^{z}}{z^{2}} + x^{\mu} p_{\mu}$$

$$k_{\mu} = (z^{2} + x^{2}) p_{\mu} - 2x_{\mu} \Delta$$
(1.4)

These are all integration constants along γ . Again note that X, V have indices that should be handled with G_{IJ} , in particular,

$$dX_{\mu} = G_{\mu I} dX^{I} = \frac{1}{z^{2}} dx_{\mu}, \quad V^{\mu} = \frac{dX^{\mu}}{d\lambda} = z^{2} p^{\mu}$$
(1.5)

On the other hand, V^{μ} should be properly normalized, therefore:

$$||V||^2 = g_{IJ}V^IV^J = \frac{(V^z)^2}{z^2} + z^2p^2 = \delta = 0, \pm 1,$$
 (1.6)

For $p^2=0$, we have $\|V\|^2=\frac{(V^z)^2}{z^2}\geq 0$, thus γ can be either spacelike or null, but not timelike. In fact, $\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}\lambda}=0,\pm 1$, along with $\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}=z^2p^\mu$, we obtain:

$$p^{2} = 0, \qquad \text{spacelike: } z(\lambda) = z(0) e^{\pm \lambda}, \quad x^{\mu}(\lambda) = x^{\mu}(0) \pm z(0)^{2} p^{\mu} \frac{e^{\pm 2\lambda} - 1}{2},$$

$$\text{null: } z(\lambda) = z(0), \qquad x^{\mu}(\lambda) = x^{\mu}(0) + z(0)^{2} p^{\mu} \lambda.$$

$$(1.7)$$

From now on we shall focus on the $p^2 \neq 0$ situation. Note that we can complete the square on the right hand side of the k^{μ} conservation such that:

$$k_{\mu} + p_{\mu}a^2 = p_{\mu}(z^2 + (x - a)^2), \quad \Delta = p_{\mu}a^{\mu}$$
 (1.8)

For $p^{\mu} \neq 0$, we can always find some c_{μ} such that $c_{\mu}p^{\mu} \neq 0$, therefore we have:

$$z^{2} + (x - a)^{2} = \frac{c_{\mu}k^{\mu}}{c_{\mu}p^{\mu}} + a^{2} = \left(\frac{L}{2}\right)^{2} = \text{const.}, \quad \Delta = p_{\mu}a^{\mu}$$
 (1.9)

We see that γ lands on a "sphere" centered at z=0, x=a in \mathbb{R}^{d+1} ; with the Lorentzian metric, it is actually a hyperboloid. Note that for now the radius $\left(\frac{L}{2}\right)^2$ can actually be negative or zero; more specifically,

- 1. $L^2 > 0$: one-sheet hyperboloid;
- 2. $L^2 < 0$: two-sheet hyperboloid;
- 3. $L^2 = 0$: conic surface.

On the other hand, we can actually solve X^{I} completely by combining (1.5), (1.6); we have:

$$\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda} = z^2 p^{\mu}, \quad \frac{\mathrm{d}z}{\mathrm{d}\lambda} = \pm z \sqrt{\delta - z^2 p^2}, \quad \frac{\mathrm{d}x^{\mu}}{\mathrm{d}z} = \pm \frac{zp^{\mu}}{\sqrt{\delta - z^2 p^2}}, \quad \delta - z^2 p^2 = \left(\frac{V^z}{z}\right)^2 \ge 0 \tag{1.10}$$

$$\implies \gamma \subset x^{\mu} = a^{\mu} \pm \frac{p^{\mu}}{p^2} \sqrt{\delta - z^2 p^2}, \quad z^2 + (x - a)^2 = \frac{\delta}{p^2} = \left(\frac{L}{2}\right)^2 \tag{1.11}$$

This confirms our observation above, and further reveals that γ lies in a "plane" in the \mathbb{R}^d subspace consisting of the x^{μ} coordinates. The behavior of γ is sensitive to the sign of δ and p^2 ; more specifically,

- 1. $\delta = -1$, i.e. for timelike γ , we must have $p^2 < 0$, and $z \ge \frac{1}{|||p|||} = \frac{L}{2} > 0$, namely z is bounded from below. This means that timelike geodesic can never reach the asymptotic boundary $z \to 0$. In this case, γ is a section of the one-sheet hyperboloid.
- 2. $\delta = +1$, i.e. for spacelike γ , we can have $p^2 > 0$ or $p^2 < 0$.
 - (a) For $p^2 > 0$, again we have $\frac{L}{2} = \frac{1}{\|p\|} > 0$, and γ is again a cross section of the one-sheet hyperboloid. However, now we have $z \leq \frac{L}{2}$, namely z is bounded from above, and:

$$z \to 0$$
, $x^{\mu} = a^{\mu} \pm \frac{p^{\mu}}{p^2} = a^{\mu} \pm \frac{L}{2} \, \hat{p}^{\mu}$, $\hat{p}^{\mu} = \frac{p^{\mu}}{\|p\|}$ (1.12)

We can also nicely parametrize X^I in terms of the proper length λ ; for convenience, set $x^{\mu}(0) = a^{\mu}$, $z(0) = \frac{L}{2}$, then we have:

$$z(\lambda) = \frac{L}{2} \frac{1}{\cosh \lambda}, \quad x^{\mu}(\lambda) = a^{\mu} \pm \hat{p}^{\mu} \frac{L}{2} \tanh \lambda$$
 (1.13)

(b) For $p^2 < 0$, we have $(\frac{L}{2})^2 = \frac{1}{p^2} < 0$, and γ is now a cross section of the two-sheet hyperboloid. Again as $z \to 0$ it lands at $x^\mu = a^\mu \pm \frac{L}{2}\,\widehat{p}^\mu$; however, $|x^\mu|$ grows with z and extends into the bulk instead of returning to the boundary, i.e. $z, |x^\mu| \to \infty$.

The differential equation in this case is almost the same as in (a), but now we have to choose a different initial condition, since γ won't even reach x = a. However, we can

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actually set $x(0), z(0) \to \infty$; we then have:

$$z(\lambda) = \left| \frac{L}{2} \right| \frac{1}{\sinh \lambda}, \quad x^{\mu}(\lambda) = a^{\mu} \pm \hat{p}^{\mu} \left| \frac{L}{2} \right| \coth \lambda$$
 (1.14)

This is an evidence that $z \to \infty$ might not be the "end of the world" after all; it's likely that $z \to \infty$ is only a horizon, since a spacelike geodesic γ can reach it within finite proper length λ .

In summary, there are 3 types of spacelike geodesics in Poincaré AdS, which closely resemble the 3 types of conic sections:

- 1. $p^2 = 0$: parabolic, given in (1.7), with one end going to $z \to 0$ and the other end going to $z \to \infty$; it takes infinite proper length λ for it to reach 0 or ∞ .
- 2. $p^2 > 0$: hyperbolic, given in (1.11) and (1.14), also with one end going to $z \to 0$ and the other end going to $z \to \infty$, but it reaches ∞ within finite λ .
- 3. $p^2 > 0$: elliptic, given in (1.11) and (1.13), with both ends going to $z \to 0$, and it takes infinite proper length λ for it to reach either end.

Now consider two points x_1, x_2 near the boundary $z = \epsilon$ connected by a spacelike geodesic γ . This can only be the "elliptic type" discussed above. We have:

$$x^{\mu} = a^{\mu} \pm \hat{p}^{\mu} \sqrt{\left(\frac{L}{2}\right)^2 - z^2}, \quad z^2 + (x - a)^2 = \left(\frac{L}{2}\right)^2,$$
 (1.15)

$$a = \frac{x_1 + x_2}{2}, \quad \widehat{p} \propto x_2 - x_1$$
 (1.16)

It's length is then given by:

$$A = \int_{z=-\epsilon}^{z=\epsilon} \frac{\mathrm{d}x^2 + \mathrm{d}z^2}{z^2}$$

$$= \int_{-\Lambda}^{\Lambda} \frac{(\mathrm{d}\tanh\lambda)^2 + (\mathrm{d}\operatorname{sech}\lambda)^2}{(\mathrm{sech}\lambda)^2}, \quad \Lambda = \cosh^{-1}(\frac{L}{2\epsilon})$$

$$= \int_{-\Lambda}^{\Lambda} \mathrm{d}\lambda = 2\Lambda = 2\cosh^{-1}\left(\frac{L}{2\epsilon}\right) \sim 2\log\frac{L}{\epsilon}, \quad \epsilon \to 0$$
(1.17)

2 Einbein Action

The einbein action of a point particle is given by:

$$S[\eta, X] = \frac{1}{2} \int d\tau \left(\eta^{-1} \dot{X}_{\mu} \dot{X}^{\mu} - \eta m^2 \right)$$
 (2.1)

Under worldline reparametrization: $\tau \mapsto \tau' = f(\tau)$, we have $X'(\tau') = X(\tau)$, i.e. X^{μ} transforms like a **scalar** under worldline diffeomorphism; $X(\tau) \mapsto X'(\tau) = X(f^{-1}(\tau))$.

On the other hand, η should be treated like an einbein: $\eta = \sqrt{-\gamma}$, here $\gamma = \gamma_{\tau\tau}$ is the worldline metric; $\gamma < 0$ due to the Lorentzian signature. We have:

$$\eta = \sqrt{-\gamma} \longmapsto \eta' = \eta \det \frac{\partial \tau}{\partial \tau'} = \eta \frac{\partial \tau}{\partial \tau'},$$
(2.2)

$$\eta^{-1} = -\sqrt{-\gamma} \, \gamma^{-1} \longmapsto (\eta')^{-1} = \eta^{-1} \left(\frac{\partial \tau}{\partial \tau'} \right) \left(\frac{\partial \tau'}{\partial \tau} \frac{\partial \tau'}{\partial \tau} \right) = \eta^{-1} \frac{\partial \tau'}{\partial \tau} \tag{2.3}$$

It is clear that the action is invariant under the transformation:

$$S' = \frac{1}{2} \int d\tau' \left((\eta')^{-1} \partial_{\tau'} X_{\mu} \partial_{\tau'} X^{\mu} - (\eta') m^{2} \right)$$

$$= \frac{1}{2} \int d\tau \frac{\partial \tau'}{\partial \tau} \left(\eta^{-1} \frac{\partial \tau'}{\partial \tau} \cdot \frac{\partial \tau}{\partial \tau'} \partial_{\tau} X_{\mu} \cdot \frac{\partial \tau}{\partial \tau'} \partial_{\tau} X^{\mu} - \eta \frac{\partial \tau}{\partial \tau'} m^{2} \right) = S$$
(2.4)

We can eliminate η classically by placing it on shell:

$$0 = \frac{\delta S}{\delta \eta} = -\eta^{-2} \dot{X}_{\mu} \dot{X}^{\mu} - m^2, \quad \eta[X] = \frac{1}{m} \sqrt{-\dot{X}_{\mu} \dot{X}^{\mu}}$$
 (2.5)

Substitute this back to the action, and we have:

$$S[X] = S[\eta = \eta[X], X] = \frac{1}{2} \int d\tau \left(m \left(-\dot{X}_{\mu} \dot{X}^{\mu} \right)^{-\frac{1}{2}} \dot{X}_{\mu} \dot{X}^{\mu} - m \left(-\dot{X}_{\mu} \dot{X}^{\mu} \right)^{+\frac{1}{2}} \right)$$

$$= -m \int d\tau \sqrt{-\dot{X}_{\mu} \dot{X}^{\mu}}$$
(2.6)

3 Ricci Tensor for Static Spherical Metric

Consider the metric:

$$ds^{2} = -f(r) dt^{2} + h(r) dr^{2} + r^{2} (d\theta^{2} + \sin^{2}\theta d\phi^{2}) = \eta_{ab} e^{a} e^{b}$$
(3.1)

Here we've defined the following vierbein:

$$e^{t} = \sqrt{f(r)} dt$$

$$e^{r} = \sqrt{h(r)} dr$$

$$e^{\theta} = r d\theta$$

$$e^{\phi} = r \sin \theta d\phi$$
(3.2)

A Derivation of the special conformal transformations

Special conformal transformations can be understood as translations conjugated by *inversions*. Note that $\frac{dz^2}{z^2}$ is invariant under $z \mapsto \frac{1}{z}$; if we include the x^{μ} directions, we can consider:

$$\mathcal{I}: \ \chi^I \mapsto \frac{\chi^I}{\chi^2}, \quad \chi^2 = -t^2 + \vec{x}^2 + z^2,$$
 (A.1)

$$\mathcal{I}^2 = 1, \quad ds^2 \mapsto \left(\frac{\delta_J^I - 2\frac{\chi^I \chi_J}{\chi^2}}{\chi^2} d\chi^J\right)^2 / \left(\frac{z}{\chi^2}\right)^2 = \frac{d\chi^2}{z^2} = ds^2 \tag{A.2}$$

We see that inversion \mathcal{I} is indeed a (discrete) symmetry of the metric. Here we've defined yet another lower case variable $\chi^I \sim (x^\mu, z)$, which as a contravariant vector has the same components as X^I , but with an index that should be lowered by the flat metric η_{IJ} , i.e. $\chi_I = \eta_{IJ}\chi^J = \eta_{IJ}X^J$. The d

special conformal generators are then given by:

$$k_{\mu} = \frac{\partial}{\partial a^{\mu}} \left(\mathcal{I} \circ e^{a^{\nu} P_{\nu}} \circ \mathcal{I} \circ X^{I} \right)_{a=0} \frac{\partial}{\partial X^{I}}$$

$$= \frac{\partial}{\partial a^{\mu}} \left(\frac{\chi^{I}}{\chi^{2}} + a^{I}}{\left| \frac{\chi^{J}}{\chi^{2}} + a^{J} \right|^{2}} \right)_{a=0} \frac{\partial}{\partial X^{I}}$$

$$= \frac{\partial}{\partial a^{\mu}} \left(\frac{\chi^{I} + a^{I} \chi^{2}}{1 + 2a^{I} \chi_{I} + a^{2} \chi^{2}} \right)_{a=0} \frac{\partial}{\partial X^{I}}$$

$$= \chi^{2} \partial_{\mu} - 2x_{\mu} X^{I} \partial_{I}$$

$$= \chi^{2} \partial_{\mu} - 2x_{\mu} \Delta$$
(A.3)