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## 1 Equivalence of categories is fully faithful:

 $F: \mathcal{C} \to \mathcal{D}$  equivalence of categories, i.e.  $\exists G: \mathcal{D} \to \mathcal{C}$ , s.t.

$$G \circ F \simeq \mathbb{1}_{\mathcal{C}}, \quad F \circ G \simeq \mathbb{1}_{\mathcal{D}}$$
 (1)

Here "≃" means naturally isomorphic as functors, i.e.,

$$\exists \tau \colon G \circ F \Rightarrow \mathbb{1}_{\mathcal{C}}, \quad \sigma \colon F \circ G \Rightarrow \mathbb{1}_{\mathcal{D}} : \text{ natural isomorphisms}$$
 (2)

By the definition of natural transformation, for  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ , we have:

$$G \circ F(A) \xrightarrow{G \circ F(f)} G \circ F(B)$$

$$\uparrow_{A} \downarrow \qquad \qquad \downarrow^{\tau_{B}} \downarrow$$

$$A \xrightarrow{f} B$$

$$\tau_{B} \circ (G \circ F)(f) \circ \tau_{A}^{-1} = f, \quad \forall \ f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$$

$$(3)$$

Here  $\tau_{A,B}$  are isomorphisms, which means that  $G \circ F$  must be a bijection between hom-sets, which further implies that F is injective and G is surjective. Switch the roles of F, G, we find that G is injective and F is surjective. Therefore, F, G are both fully faithful.

## **2** Forgetful functors to <u>Set</u> are often representable:

For  $F: \mathbf{Group} \to \underline{\mathbf{Set}}$ , consider the free group generated by a single element  $\mathbb{Z}$ . We have:

$$\operatorname{Hom}(\mathbb{Z}, -) \colon \operatorname{\underline{Group}} \longrightarrow \operatorname{\underline{Set}}$$

$$G \longmapsto \operatorname{Hom}(\mathbb{Z}, G)$$

$$(4)$$

This is a covariant functor representable by  $\mathbb{Z}$ .

On the other hand,  $\operatorname{Hom}(\mathbb{Z},G)$  consists of group homomorphisms:

$$\operatorname{Hom}(\mathbb{Z}, G) = \left\{ \begin{array}{c} \mathbb{Z} \to G \\ 1 \mapsto g \end{array} \middle| g \in G \right\} \tag{5}$$

More specifically, to fix any  $\mathbb{Z} \to G$ , we need only assign its generator<sup>1</sup>  $1 \mapsto g$ . Image of any other  $\mathbb{Z}$  element is generated automatically from the group law, without further specifications. This means that the hom-set is in one-to-one correspondence with G elements (as a set). Therefore,  $F \cong \operatorname{Hom}_{\mathbf{Group}}(\mathbb{Z}, -)$ , i.e. forgetful  $F \colon \mathbf{Group} \to \mathbf{\underline{Set}}$  is representable by  $\mathbb{Z}$ .

Similarly, for  $F: \underline{\mathbf{Ring}} \to \underline{\mathbf{Set}}$ , the free object generated by some generic element x is  $\mathbb{Z}[x]$ , the polynomial ring in one variable; we have:

$$F \cong \operatorname{Hom}_{\underline{\mathbf{Ring}}}(\mathbb{Z}[x], -), \quad \operatorname{Hom}_{\underline{\mathbf{Ring}}}(\mathbb{Z}[x], R) = \left\{ \begin{array}{c} \mathbb{Z}[x] \to R \\ x \mapsto r \end{array} \middle| r \in R \right\}$$
 (6)

*Lesson*: Forgetful  $\underline{\mathbf{Cat}} \to \underline{\mathbf{Set}}$  are often representable by the free object in  $\underline{\mathbf{Cat}}$ .

<sup>&</sup>lt;sup>1</sup>Note that  $0 \in \mathbb{Z}$  is the group identity of addiction group  $\mathbb{Z}$ , not  $1 \in \mathbb{Z}$ .

#### 3 Properties of contractible space:

(a) X contractible:  $\mathbb{1}_X \simeq f_0 \colon X \to X$  some constant map,  $f_0(X) = \{x_0\}$ . We can restrict the codomain of  $f_0$  so that  $f_0 \colon X \to \{x_0\}$ , in this way we have:

$$X \xrightarrow{f_0} \{x_0\} \hookrightarrow X \simeq \mathbb{1}_X,\tag{7.1}$$

$$\{x_0\} \hookrightarrow X \xrightarrow{f_0} \{x_0\} \simeq \mathbb{1}_{\{x_0\}},$$
 (7.2)

This means that  $f_0: X \to \{x_0\}$  isomorphic in  $\underline{\mathbf{hTop}} = \underline{\mathbf{Top}}/\simeq$ , which is precisely the definition of homotopic equivalence  $X \simeq \{x_0\}$ .  $(\Rightarrow)$ 

On the other hand ( $\Leftarrow$ ), if  $X \simeq \{x_0\}$ , there exists some  $f_0 \colon X \to \{x_0\}$  that fulfills (7). We can then extend the codomain s.t.  $f_0 \colon X \to X$ , in this way (7.1) reads  $f_0 \simeq \mathbb{1}_X$ , i.e. X is contractible. Therefore, X contractible iff. homotopic equivalent to a single point.

(b)  $\forall X$ : Topological space, we can define its *cone* as<sup>2</sup>:

$$CX = (X \times I)/(X \times \{0\}), \quad I = [0, 1]$$
 (8)

i.e. gluing together one end of the cylinder  $X \times I$ . Naturally  $X \subset CX$  as a subspace; now we show that CX is contractible. Using (a), we need only show that  $\mathbb{1}_{CX} \simeq f_0$  some constant map.

In fact, any point in CX can be uniquely labeled by  $[x,h] \in X \times I$ , with the exception of the vertex  $v \sim [x,0] \sim [x',0]$ ,  $\forall x,x' \in X$ . We can then construct a homotopy F by shrinking the cone towards the vertex v:

$$F: CX \times I \to CX, \quad F([x,h],t) = [x,h \cdot t],$$
  

$$F|_{CX \times 0} = v = \text{const}, \quad F|_{CX \times 1} = \mathbb{1}_X$$
(9)

This confirms that  $\mathbb{1}_{CX} \simeq v$ : constant map. By (a), CX is contractible.

(c) For  $Y \simeq \{y_0\}$  contractible, given any  $g: X \to Y$ , we can deform the image  $g(X) \subset Y$  to a single point, hence  $g \simeq y_0$ : constant map. More precisely, we have:

$$\exists G: X \times I \to Y, \quad \text{s.t.} \quad G|_{X \times 0} = y_0 = \text{const}, \quad G|_{X \times 1} = g \tag{10}$$

Such G can be explicitly constructed using  $\mathbb{1}_Y \simeq y_0$ :

$$F: Y \times I \to Y, \quad F|_{Y \times 0} = y_0 = \text{const}, \quad F|_{Y \times 1} = \mathbb{1}_Y,$$
 (11)

$$G(x,t) = F(g(x),t)$$
(12)

In summary, we have proven that  $g \simeq y_0$ ,  $\forall g \in \operatorname{Hom}_{\underline{\mathbf{Top}}}(X,Y)$ . By definition, this means that  $\operatorname{Hom}_{\underline{\mathbf{hTop}}}(X,Y) = \operatorname{Hom}_{\underline{\mathbf{Top}}}(X,Y)/_{\simeq} = \{[y_0]\}$  a single point.

(d) For  $X \simeq \{x_0\}$  contractible, similar to (11), we have homotopy  $F: X \times I \to X$ . Given any  $f: X \to Y$ , the composition  $f \circ F: X \times I \to Y$  yields  $f \simeq f(x_0)$ : constant map.

<sup>&</sup>lt;sup>2</sup>See Wikipedia: Cone (topology).

Furthermore, for Y: path connected, there is a path  $\gamma: I \to Y$  connecting  $f(x_0)$  and some  $y_0 \in Y$ , therefore  $f(x_0) \simeq y_0: X \to Y$  constant maps. More precisely, we have:

$$\gamma \colon I \to Y, \quad \gamma(0) = y_0, \quad \gamma(1) = f(x_0), G \colon X \times I \to Y, \quad G(x, t) = \gamma(t)$$
 (13)

Which gives  $f(x_0) \simeq y_0$ ,  $\forall f$ , independent of the choice of f. This means that  $f \simeq f(x_0) \simeq y_0$ : constant map, therefore  $\operatorname{Hom}_{\mathbf{hTop}}(X,Y) = \{[y_0]\}$  a single point.

## 4 Example of homotopic inequivalence<sup>3</sup>:

$$X = \{0\} \cup \left\{ \frac{1}{n} \middle| n \in \mathbb{Z}_+ \right\}, \quad Y = \{0\} \cup \mathbb{Z}_+$$

$$X, Y \subset \mathbb{R} \colon \text{subspace topology}$$

$$(14)$$

Assume  $X \simeq Y$ , then similar to (7), we have  $Y \xrightarrow{g} X \xrightarrow{f} Y \simeq \mathbb{1}_Y$ . However, note that Y has discrete topology, in such case any map  $f \circ g$  homotopic to  $\mathbb{1}_Y$  must be  $\mathbb{1}_Y$  itself:  $f \circ g = \mathbb{1}_Y$ .

More specifically, consider:

$$F: Y \times I \to Y, \quad F|_{Y \times 0} = f \circ g, \quad F|_{Y \times 1} = \mathbb{1}_Y$$
 (15)

Any point  $n \in Y$  is both open and closed, therefore its pre-image  $F^{-1}(n) \subset Y \times I$  is also both open and closed, and by  $F|_{Y \times 1} = \mathbb{1}_Y$  we know that F(y,1) = y,  $(y,1) \in F^{-1}(y)$ , therefore the only possibility is that  $F(\{y\} \times I) = y$ , i.e.  $f \circ g = \mathbb{1}_Y$ , which implies that g is injective and f is surjective.

However,  $f \colon X \to Y$  cannot be surjective due to the complication around  $0 \in X$ . Consider  $f^{-1}\big(f(0)\big) \ni 0$ , since  $f(0) \in Y$  both open and closed,  $f^{-1}\big(f(0)\big) \subset X$  must also be both open and closed. But any open set  $U \subset X$  is induced via subspace topology  $X \subset \mathbb{R}$ ; for  $0 \in U \subset X \subset \mathbb{R}$ , U must contain  $\infty$ -many elements:

$$\left\{ \frac{1}{n} \mid n \ge N_0 \right\} \subset U \subset f^{-1}(f(0)), \quad \text{for some } N_0, \text{ for any } U \ni x$$
 (16)

Hence  $f(X) = f(0) \cup f(\left\{\frac{1}{n} \mid n < N_0\right\}), \ f(X) \subset Y$  a finite set, i.e.  $f: X \to Y$  is never surjective. Therefore,  $X \not\simeq Y$  by contradiction.

 $<sup>^3</sup>$ This proof is produced thanks to helpful insights from 谷夏 and 於子雄.

# 5 Fundamental group of topological group is abelian<sup>4</sup>:

From a categorical point of view, the fundamental group  $\pi_1(G)$  of a topological group G can be seen as a functor:

$$G \in \underline{\mathbf{TopGroup}} \hookrightarrow \underline{\mathbf{Top}} \xrightarrow{\pi_1} \underline{\mathbf{Group}} \ni \pi_1(G)$$
 (17)

<u>TopGroup</u>  $\subset$  <u>Top</u> is a subcategory with additional group structure, i.e.  $(G, \cdot) \in \underline{\textbf{TopGroup}}$  is a  $group \ object^5$  in  $\underline{\textbf{Top}}$ , with "·" denoting its product operation (·):  $G \times G \to G$ . Correspondingly,  $\pi_1(\underline{\textbf{TopGroup}})$  should be  $group \ objects \ of \underline{\textbf{Group}}$ , which have an additional group structure  $(\star) = \pi_1(\cdot)$ , along with the usual group product "\*" in  $\underline{\textbf{Group}}$ .

In total, we have three different group structures (represented by their product operation):

$$(\cdot)\colon G\times G\to G,\tag{18}$$

(\*): 
$$\pi_1(G) \times \pi_1(G) \to \pi_1(G)$$
, (19)

$$(\star) = \pi_1(\cdot) \colon \ \pi_1(G) \times \pi_1(G) \to \pi_1(G), \tag{20}$$

Note that  $\pi_1(G) = \operatorname{Aut}_{\Pi_1(G)} \mathbb{1}_G$ , i.e. loop classes  $[\gamma]$  in G; (\*) is defined as joining two loops, while  $(\star) = \pi_1(\cdot)$  is defined as the translation of loop classes by pointwise group product  $(\cdot)$ ,

$$[\gamma_1] \star [\gamma_2] = [\gamma_1 \cdot \gamma_2] \tag{21}$$

With the above definitions, we observe that:

$$([\gamma_1] \star [\gamma_2]) * ([\eta_1] \star [\eta_2]) = ([\gamma_1] * [\eta_1]) \star ([\gamma_2] * [\eta_2])$$
(22)

By definition, they are both equal to  $[(\gamma_1 \cdot \gamma_2) * (\eta_1 \cdot \eta_2)]$ . What's surprising is that by using only the group axioms and "distributive law" (22), we can show that  $(\star)$  and  $(\star)$  must always coincide:  $(\star) = (*)$ , and they have to be in fact, commutative. This is the *Eckmann–Hilton argument*<sup>6</sup>.

Proof of this argument is straight-forward; first, observe that the units of the two operations coincide:

$$1_{\star} = 1_{\star} \star 1_{\star} = (1_{*} * 1_{\star}) \star (1_{\star} * 1_{*}) \xrightarrow{(22)} (1_{*} \star 1_{\star}) * (1_{\star} \star 1_{*}) = 1_{*} * 1_{*} = 1_{*}$$
 (23)

Further manipulation using (22) confirms that the two operations coincide and are commutative:

$$[\gamma] * [\eta] = (1 \star [\gamma]) * ([\eta] \star 1) \xrightarrow{\underline{(22)}} (1 * [\eta]) \star ([\gamma] * 1)$$

$$= [\eta] \star [\gamma]$$

$$= ([\eta] * 1) \star (1 * [\gamma]) \xrightarrow{\underline{(22)}} ([\eta] \star 1) * (1 \star [\gamma])$$

$$= [\eta] * [\gamma]$$

$$(24)$$

In summary, we find that the group objects in <u>Group</u> are indeed abelian groups, which means that  $\pi_1(G)$  for  $G \in \textbf{TopGroup}$  must be abelian.

<sup>&</sup>lt;sup>4</sup>This proof is produced with the help of math.stackexchange.com/q/727999. Another (easier) proof lies in the fact that group translation induces  $\pi_1$  conjugation, therefore  $\gamma^{-1}\alpha\gamma = \alpha$ , hence abelian.

<sup>&</sup>lt;sup>5</sup>See Wikipedia: *Group object*.

 $<sup>^6\</sup>mathrm{See}$  Wikipedia: Eckmann-Hilton argument.