

## 1 Ward Identity with Hard Thermal Loop

The hard thermal loop (HTL) approximation of the photon self-energy is given by:

$$\Pi_{\mu\nu}(Q) = m_E^2 \left( \frac{q^0}{q} \int \frac{d^2\Omega(\hat{\mathbf{k}})}{4\pi} \frac{\hat{K}_\mu \hat{K}_\nu}{\hat{\mathbf{k}} \cdot \hat{\mathbf{q}} + q^0/q} - g_{\mu 0} g_{\nu 0} \right) \quad (1)$$

Where  $m_E^2 = \frac{e^2 T^2}{3}$ . Here we've chosen the Lorentzian signature  $g \sim (-+++)$ , and  $Q^\mu \sim (q^0, \mathbf{q})$ , where  $q^0 = -i\omega$ ,  $q = \|\mathbf{q}\|$ , while the normalized  $\hat{K}_\mu \sim (1, \hat{\mathbf{k}})$ ,  $\hat{K}^\mu \sim (-1, \hat{\mathbf{k}})$ . We have:

$$Q^\mu \hat{K}_\mu = q^0 + \hat{\mathbf{k}} \cdot \mathbf{q} = q \left( \hat{\mathbf{k}} \cdot \hat{\mathbf{q}} + q^0/q \right), \quad Q^\mu g_{\mu 0} = q_0 = -q^0, \quad (2)$$

$$Q^\mu \Pi_{\mu\nu} = m_E^2 q^0 \left( \int \frac{d^2\Omega(\hat{\mathbf{k}})}{4\pi} \hat{K}_\nu + g_{\nu 0} \right) = 0 \quad (3)$$

More specifically, we have  $Q^\mu \Pi_{\mu 0} \propto \left( \int \frac{d^2\Omega(\hat{\mathbf{k}})}{4\pi} 1 \right) - 1 = 0$ , and also  $Q^\mu \Pi_{\mu 0} \propto \int \frac{d^2\Omega(\hat{\mathbf{k}})}{4\pi} \hat{K}_i = 0$ . Therefore, the self-energy is transverse.

## 2 Decomposition of $\Pi_{\mu\nu}$

As we know,  $Q^\mu \Pi_{\mu\nu} = 0$ , hence it can be nicely decomposed by various projections related to  $Q^\mu$ ; recall that:

$$E_{\mu\nu} = \frac{Q_\mu Q_\nu}{Q^2} = \hat{Q}_\mu \hat{Q}_\nu, \quad P_{\mu\nu} = g_{\mu\nu} - E_{\mu\nu}, \quad (4)$$

$$N^\mu = P^{\mu\nu}(0, \mathbf{q})_\nu = P^{\mu j} q_j = -P^{\mu 0} q_0 = P^{\mu 0} q^0, \quad Q_\mu N^\mu = 0, \quad (5)$$

$$P_L^{\mu\nu} = \hat{N}^\mu \hat{N}^\nu, \quad P_T^{\mu\nu} = P^{\mu\nu} - P_L^{\mu\nu}, \quad (6)$$

Note that  $P_T^{\mu\nu}$  annihilates both  $Q^\mu$  and  $N^\mu$ , by linearity it also annihilates both  $(1, \mathbf{0})$  and  $(0, \mathbf{q})$ , i.e.

$$P_T^{\mu 0} q_0 = 0, \quad P_T^{\mu 0} = 0, \quad (7)$$

$$P_T^{ij} = \delta^{ij} - \hat{Q}^i \hat{Q}^j - \hat{N}^i \hat{N}^j = \delta^{ij} - \hat{q}^i \hat{q}^j \quad (8)$$

The transverse self-energy can then be decomposed as:

$$\Pi^{\mu\nu} = \Pi_L P_L^{\mu\nu} + \Pi_T P_T^{\mu\nu}, \quad (9)$$

$$\Pi_\mu^\mu = \Pi_L + 2\Pi_T, \quad \Pi^{00} = \Pi_L P_L^{00}, \quad (10)$$

$$N^i = P^{ij} q_j = \left( 1 - \frac{q^2}{Q^2} \right) q^i = -\frac{q_0^2}{Q^2} q^i, \quad N^0 = \left( 0 - \frac{q^2}{Q^2} \right) q^0 = -\frac{q^2}{Q^2} q^0, \quad N^2 = -\frac{q_0^2 q^2}{Q^2} \quad (11)$$

$$P_L^{00} = \frac{N_0^2}{N^2} = -\frac{q^2}{Q^2} \quad (12)$$

We see that  $\Pi_{\mu\nu}$  is entirely determined by  $\Pi^{00}$  and  $\Pi^\mu{}_\mu$ . More explicitly, we have<sup>1</sup>:

$$\begin{aligned}\Pi_L &= -\frac{Q^2}{q^2} \Pi^{00} \\ \Pi_T &= \frac{1}{2} \left( \Pi^\mu{}_\mu + \frac{Q^2}{q^2} \Pi^{00} \right) = \frac{1}{2} \left( -\frac{q_0^2}{q^2} \Pi^{00} + \Pi^i{}_i \right)\end{aligned}\quad (13)$$

$\Pi^{00}, \Pi^i{}_i$  is computed explicitly as follows:

$$\Pi_{ij} = m_E^2 \frac{q^0}{q} \int \frac{d^2\Omega(\hat{\mathbf{k}})}{4\pi} \frac{\hat{k}_i \hat{k}_j}{\hat{\mathbf{k}} \cdot \hat{\mathbf{q}} + q^0/q}, \quad \Pi^{00} = m_E^2 \left( \frac{q^0}{q} \int \frac{d^2\Omega(\hat{\mathbf{k}})}{4\pi} \frac{1}{\hat{\mathbf{k}} \cdot \hat{\mathbf{q}} + q^0/q} - 1 \right) \quad (14)$$

$$\begin{aligned}\Pi^i{}_i &= m_E^2 \frac{q^0}{q} \int \frac{d^2\Omega(\hat{\mathbf{k}})}{4\pi} \frac{\hat{k}^i \hat{k}_i}{\hat{\mathbf{k}} \cdot \hat{\mathbf{q}} + q^0/q}, \quad \hat{k}^i \hat{k}_i = 1, \\ &= m_E^2 \frac{q^0}{q} \int_0^\pi \frac{2\pi \sin\theta d\theta}{4\pi} \frac{1}{\cos\theta + q^0/q} \\ &= m_E^2 \frac{q^0}{q} \int_{-1}^1 \frac{dz}{2} \frac{1}{z + q^0/q} \\ &= m_E^2 \frac{q^0}{q} H\left(\frac{q^0}{q}\right), \quad H(x) = \frac{1}{2} \ln \frac{x+1}{x-1},\end{aligned}\quad (15)$$

$$\Pi^{00} = \Pi^i{}_i - m_E^2, \quad (16)$$

$$\begin{aligned}\Pi_L &= -m_E^2 \frac{Q^2}{q^2} \left( \frac{q^0}{q} H\left(\frac{q^0}{q}\right) - 1 \right), \quad \frac{Q^2}{q^2} = 1 - \frac{q_0^2}{q^2}, \\ \Pi_T &= \frac{1}{2} \left( -\frac{q_0^2}{q^2} \Pi^{00} + \Pi^i{}_i \right) = \frac{1}{2} \left( \frac{Q^2}{q^2} \Pi^i{}_i + \frac{q_0^2}{q^2} m_E^2 \right) \\ &= \frac{1}{2} m_E^2 \frac{q^0}{q} \left( \frac{Q^2}{q^2} H\left(\frac{q^0}{q}\right) + \frac{q^0}{q} \right)\end{aligned}\quad (17)$$

### 3 HTL in QCD

The gluon self-energy at 1-loop is similar to the QED situation, except that now we should include the extra degrees of freedom in the summation.

**The quark 1-loop correction** is structurally identical to the QED fermion 1-loop correction, but enhanced  $N_f$  fold due to the flavor degrees of freedom. Also,  $\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$  gives an extra  $\frac{1}{2}$  factor. In the end, the quark loop contribution is given by the overall factor:

$$m_E^2 \mapsto m_{\text{quark}}^2 \delta^{ab} = \frac{1}{2} N_f \delta^{ab} \quad (18)$$

The rest is identical to the QED case.

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<sup>1</sup>Note that the metric convention here might result in signs that may differ from the textbook.



**The ghost loop** is also similar, except now we have a factor of  $N_c$  instead of  $\frac{1}{2}N_f$ , and with a frequency summation given by:

$$\begin{aligned} & \frac{T}{V} \sum_K \Delta(K) (-i(K-Q)^\nu) \Delta(K-Q) (-iK^\mu), \quad \Delta(K) = \frac{1}{K^2} \\ &= -\frac{T}{V} \sum_K \frac{(K-Q)^\nu K^\mu}{K^2(K-Q)^2} \simeq -\frac{T}{V} \sum_K \frac{K^\mu K^\nu}{K^2(K-Q)^2} \end{aligned} \quad (19)$$

We've computed a similar Matsubara summation before, while treating the QED fermionic loop; the method still applies here but with bosonic frequencies. This produces an additional factor of  $(-\frac{1}{8})$ , relative to the fermionic case<sup>2</sup>. In the end, we have:

$$m_E^2 \mapsto m_{\text{ghost}}^2 \delta^{ab} = -\frac{1}{8} N_c \delta^{ab} \quad (20)$$

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<sup>2</sup>Reference: Laine & Vuorinen, *Basics of Thermal Field Theory*.