String Theory №7

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1 T-duality of Heterotic Strings[1]

We use $d \leq 10$ to denote the number of noncompact dimensions; the remaining $m \geq d$ dimensions are compactified. For heterotic strings, the $I \geq 10$ dimensions of the left-moving sector are already compactified on a lattice Γ_{16} or $\Gamma_8 \times \Gamma_8$. Here we use the label I to index the 16 internal dimensions.

(a) Generally, if we compactify an open string on the x^m direction: $x^m \cong x^m + 2\pi R$ with constant backgrounds A_m, then its zero mode spectrum, *with winding* $w = 0$, can be obtained from canonical quantization of the effective point particle action, with an additional gauge action term in the form of a Wilson line[2]:

$$-W_q = -iq \int \mathrm{d}x^m \, A_m \sim -iq \int \mathrm{d}\tau \, A_m \dot{X}^m \tag{1}$$

By imposing that the canonical momentum to be periodic along x^m, we find that:

$$k_m = \frac{n_m}{R} - qA_m \tag{2}$$

To obtain the winding states, we have to reproduce the above action from the world-sheet description. For heterotic strings with $m < 10 \leq I \leq 26$, this can be achieved by adding the following term to the usual world-sheet action[3]:

$$S_A \propto \int \mathrm{d}^2\sigma \, \epsilon^{ab} A^I_\mu \, \partial_a X^\mu \, \partial_b X_I \tag{3}$$

With proper normalization to match the result in (2).

Canonical quantization then produces[4]:

$$k_m = \frac{n_m}{R} \pm \frac{w_m R}{\alpha'} - q_I A^I_m - \frac{w_n R}{2} A^n_I A^I_m, \tag{4}$$

$$k^I_L = \sqrt{\frac{2}{\alpha'}} \left(q^I + w^m R A^I_m \right), \tag{5}$$

The "$\pm$" signs in k_m correspond to the left and right-moving sector, respectively. Only the left-moving sector has an additional 16 dimensional internal torus, therefore k^I is labeled with an "L".

Note that the charge q^I now takes value on the Γ_{16} or $\Gamma_8 \times \Gamma_8$ lattice, and:

$$l \circ l' = \frac{\alpha'}{2} \left(k^I_L \, k'_{L,I} + k^m_L \, k'_{L,m} - k^m_R \, k'_{R,m} \right) = q^I q'_I + 2nw \tag{6}$$

We can then see that the new "extended" lattice indeed satisfies the even and self-dual conditions, which follows from the even and self-dual properties of Γ_{16} or $\Gamma_8 \times \Gamma_8$.

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[1] I would like to thank Lucy Smith for help with this problem.

[2] Reference: *Polchinski*, Chapter 8.

[3] Reference: *Blumenhagen et al.*, Basic Concepts of String Theory.

[4] Reference: *Polchinski*, Chapter 11.

(b) With $m = 9$ and $G_{dd} = 1$, we have:

$$W_q = \exp\left(-iq_I\theta^I\right), \quad A_9^I = -\frac{\theta^I}{2\pi R} \tag{7}$$

Note that W_q captures the phase change of the paths that wind around x^9; the extra phase from a non-trivial Wilson line might affect the boundary condition of some states while leaving others intact, thus breaking the original gauge symmetry. Our discussions here closely follow *Polchinski*, Chapter 11.

For the SO(32) theory with:

$$RA_9^I = \mathrm{diag}\left((\tfrac{1}{2})^8, 0^8\right) \tag{8}$$

Adjoint states are labeled by a pair of indices valued in $1, \cdots, 32$; those with one index from $1 \le A \le 16$ and one from $17 \le A \le 32$ are anti-periodic due to the additional phase $e^{i\pi} = -1$ from the Wilson line, so the gauge symmetry is reduced to SO(16) $\times$ SO(16).

Similarly, for the $E_8 \times E_8$ theory with:

$$R'A_9^I = \mathrm{diag}\left(1, 0^7, 1, 0^7\right) \tag{9}$$

Note that Γ_8, the root lattice of E_8, is basically the root lattice union an additional spinor weight lattice of SO(16). With the above Wilson line, the integer-charged states from the SO(16) root lattice in each E_8 remain periodic, while the half-integer charged states from the SO(16) spinor lattices become anti-periodic, due to the additional phase $e^{i\frac{1}{2}\cdot 2\pi} = -1$. Again the gauge symmetry is broken down to SO(16) $\times$ SO(16).

In summary, with the above Wilson line, the SO(32) and $E_8 \times E_8$ theory shares an unbroken gauge of SO(16) $\times$ SO(16). Consider the spectrum of the SO(16) $\times$ SO(16) *neutral states*, i.e. those with internal momentum:

$$k_L^I = \sqrt{\frac{2}{\alpha'}}\left(q^I + wRA_9^I\right) = 0 \tag{10}$$

For the SO(32) theory, since $q^I \in \Gamma_{16}$ while $RA_9^I = \mathrm{diag}\left((\tfrac{1}{2})^8, 0^8\right)$, we must have $w = 2m$ for this to hold. The same goes for the $E_8 \times E_8$ theory; therefore, we have:

$$k_{L,R} = \frac{\tilde{n}}{R} \pm \frac{2mR}{\alpha'}, \quad k'_{L,R} = \frac{\tilde{n}'}{R'} \pm \frac{2m'R'}{\alpha'}, \tag{11}$$

$$\tilde{n} = n + 2m, \quad \tilde{n}' = n' + 2m' \tag{12}$$

(c) If the two theories are related by T-duality, then we should expect:

$$(k_L, k_R) \longleftrightarrow (k'_L, -k'_R), \tag{13}$$

Under suitable mapping of parameters. Indeed, it is straightforward to verify that $(\tilde{n}, m) \leftrightarrow (m', \tilde{n}')$ realizes this, along with $RR' = \alpha'/2$. The above arguments can then be generalized to higher levels, by acting on fermionized left-moving fields λ^A and carefully organizing representations. We see that the two heterotic string theories are equivalent under T-duality.

$\boxed{2}$ String Junction[5]

For a string junction to be mechanically stable, the tension force exerted on the junction must cancel each other; this is a Newtonian mechanics problem, but with (p, q)-string tension given by the

[5] Reference: arXiv:0812.4408.

BPS bound:

$$\tau_{(p,q)} = \frac{\sqrt{p^2 + q^2/g^2}}{2\pi\alpha'} \tag{14}$$

Stability of the system implies that the BPS bound should be saturated.

From Newtonian mechanics, we know that three forces cannot cancel each other unless they are co-planar. Therefore, a 3-string junction must be co-planar in order to be stable. Suppose they lie in the (X^1, X^2) plane, then the tension exerted on the junction can expressed as:

$$\vec{T}_i = \tau_{(p_i,q_i)} \,(\cos\theta_i, \sin\theta_i), \quad i = 1, 2, 3, \tag{15}$$

$\sum_i \vec{T}_i = 0$ gives two equations, and we have two independent unknowns (the angle between two pairs of strings); therefore if a solution exists, it should be unique up to rotations and reflections.

In fact, a solution can be found by simple observations:

$$\cos\theta_i = \frac{p_i}{\sqrt{p^2 + q^2/g^2}}, \quad \sin\theta_i = \frac{q_i/g}{\sqrt{p^2 + q^2/g^2}}, \tag{16}$$

It satisfies $\sum_i \vec{T}_i = 0$ since that total (p,q) vanishes at each junction.

To find the remaining supersymmetries of this system, we start from the original supersymmetries of a (p,q) string (which saturates the BPS bound) extended along the $\hat{X} = (\cos\theta, \sin\theta)$ direction:

$$\frac{1}{2L\tau_{(p,q)}} \left\{ \begin{bmatrix} Q_\alpha \\ \tilde{Q}_\alpha \end{bmatrix}, \begin{bmatrix} Q^\dagger_\beta & \tilde{Q}^\dagger_\beta \end{bmatrix} \right\} = \delta_{\alpha\beta} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (\Gamma^0\Gamma^\theta)_{\alpha\beta} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \tag{17}$$

$$\Gamma^\theta = \hat{X} \cdot \vec{\Gamma} = \Gamma^1 \cos\theta + \Gamma^2 \sin\theta \tag{18}$$

We see that the algebra depends on θ, i.e. it is different for strings in different directions. However, if we can find a (maximal) subalgebra that is independent of θ, then we would have found the remaining supersymmetries of the full system[6].

We begin with diagonalizing the matrix on the RHS with:

$$U(\tfrac{\theta}{2}) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}, \tag{19}$$

$$\frac{1}{2L\tau_{(p,q)}} \left\{ U^{\mathrm{T}} \begin{bmatrix} Q_\alpha \\ \tilde{Q}_\alpha \end{bmatrix}, \begin{bmatrix} Q^\dagger_\beta & \tilde{Q}^\dagger_\beta \end{bmatrix} U \right\} = \begin{bmatrix} (\mathbb{1} + \Gamma^0\Gamma^\theta)_{\alpha\beta} & 0 \\ 0 & (\mathbb{1} - \Gamma^0\Gamma^\theta)_{\alpha\beta} \end{bmatrix}, \tag{20}$$

Note that $(\mathbb{1} + \Gamma^0\Gamma^\theta)(\mathbb{1} - \Gamma^0\Gamma^\theta) = 0$ and $(\mathbb{1} + \Gamma^0\Gamma^\theta) + (\mathbb{1} - \Gamma^0\Gamma^\theta) = 2\mathbb{1}$, i.e. they are orthogonal to each other; acting $(\mathbb{1} \pm \Gamma^0\Gamma^\theta)$ on both sides, we find the following combinations, which gives the 16 SUSYs of a (p,q) string:

$$(\mathbb{1} - \Gamma^0\Gamma^\theta)\Big(\cos\tfrac{\theta}{2}\, Q + \sin\tfrac{\theta}{2}\, \tilde{Q}\Big)_\alpha = 0 = (\mathbb{1} + \Gamma^0\Gamma^\theta)\Big(-\sin\tfrac{\theta}{2}\, Q + \cos\tfrac{\theta}{2}\, \tilde{Q}\Big)_\beta \tag{21}$$

[6] The $\tau_{(p,q)}$ factor can be absorbed by rescaling generators, hence does not matter in our discussions.

For further simplification, we can isolate the θ dependence in Γ^θ by working in a specific representation of the Clifford algebra, e.g. the Dirac representation given by *Polchinski*. Then α is given by 10 D spinor components: $\alpha = (s_0, s_1, s_2, s_3, s_4)$, $s_i = \pm$, with additional chirality constraints from both Q and $\tilde{Q}$: $\prod_i s_i = +$. In the end, we have 16 independent components as expected.

Details of the expansion are given in arXiv:0812.4408. When the dust settles, we find that the 16 SUSYs in (21) is given by:

$$\sin\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}\,Q + \sin\tfrac{\theta}{2}\,\tilde{Q}\right)_{(++s)} + \cos\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}\,Q + \sin\tfrac{\theta}{2}\,\tilde{Q}\right)_{(--s)}, \tag{22a}$$

$$\sin\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}\,Q + \sin\tfrac{\theta}{2}\,\tilde{Q}\right)_{(+-s)} - \cos\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}\,Q + \sin\tfrac{\theta}{2}\,\tilde{Q}\right)_{(-+s)}, \tag{22b}$$

$$\cos\tfrac{\theta}{2}\left(-\sin\tfrac{\theta}{2}\,Q + \cos\tfrac{\theta}{2}\,\tilde{Q}\right)_{(++s)} - \sin\tfrac{\theta}{2}\left(-\sin\tfrac{\theta}{2}\,Q + \cos\tfrac{\theta}{2}\,\tilde{Q}\right)_{(--s)}, \tag{22c}$$

$$\cos\tfrac{\theta}{2}\left(-\sin\tfrac{\theta}{2}\,Q + \cos\tfrac{\theta}{2}\,\tilde{Q}\right)_{(+-s)} + \sin\tfrac{\theta}{2}\left(-\sin\tfrac{\theta}{2}\,Q + \cos\tfrac{\theta}{2}\,\tilde{Q}\right)_{(-+s)}, \tag{22d}$$

$$s = (s_2 s_3 s_4), \qquad \prod_i s_i = + \tag{23}$$

By trial and error, we can find the 8 linear combinations that are independent of θ; they are:

$$(a) + (c) \quad\Longrightarrow\quad \tilde{Q}_{(++s)} + Q_{(--s)}, \tag{24}$$

$$(b) + (d) \quad\Longrightarrow\quad \tilde{Q}_{(+-s)} - Q_{(-+s)}, \tag{25}$$

Therefore, the string junction is $\frac{8}{32} = \frac{1}{4}$ BPS.

3 Two and Three-Point Functions in AdS/CFT

Consider a scalar field $\phi(x,z)$ in Poincaré AdS_5 (with radius $R=1$) satisfying:

$$\left(\nabla^2 - m^2\right)\phi(x,z) = 0, \quad \phi(x,z) \to \begin{cases} z^\delta \phi_0(x)\,, & z \to 0, \\ \text{regular}, & z \to \infty, \end{cases} \quad \delta = 2 - \sqrt{m^2+4} \tag{26}$$

It can be constructed via the boundary-to-bulk propagator K_Δ:

$$\phi(x,z) = \int \mathrm{d}^4x'\, K_\Delta(x,z;x')\,\phi(x'), \tag{27}$$

$$K_\Delta(x,z;x') = \frac{(\Delta-1)(\Delta-2)}{\pi^2}\left(\frac{z}{z^2 + \|x-x'\|^2}\right)^\Delta, \quad \Delta = 2 + \sqrt{m^2+4} \tag{28}$$

(a) To verify this, we first check that the boundary conditions are indeed satisfied by K_Δ; note that $\Delta \geq 2 > 0$, and we have:

$$z \to 0, \quad l \neq 0, \quad \left(\frac{z}{z^2+l^2}\right)^\Delta \to 0, \tag{29}$$

i.e. the only contribution comes from the $l \to 0$ case, where we have:

$$\begin{aligned} l = \|x-x'\| \to 0, \quad \int \mathrm{d}^4x'\, K_\Delta(x,z;x') &= \frac{(\Delta-1)(\Delta-2)}{\pi^2}\int_0^\infty 2\pi^2 l^3\,\mathrm{d}l\left(\frac{z}{z^2+l^2}\right)^\Delta \\ &= \frac{(\Delta-1)(\Delta-2)}{\pi^2}\cdot\frac{2\pi^2}{2\,(\Delta-1)(\Delta-2)}\,z^{4-\Delta} \\ &= z^\delta \end{aligned} \tag{30}$$

Therefore, we have:

$$z \to 0, \quad K_\Delta(x, z; x') \to z^\delta \delta^4(x - x'), \quad \phi(x, z) \to z^\delta \phi_0(x), \tag{31}$$

The other boundary condition is convenient to check; we have:

$$z \to \infty, \quad K_\Delta(x, z; x') \underset{\sim}{\propto} z^{-\Delta} \to 0, \quad \phi(x, z) \text{ regular.} \tag{32}$$

Now we need only check that K_Δ satisfies the equation of motion; in Poincaré AdS_5 we have:

$$\nabla^2 = z^2 \left(\partial_z^2 - \frac{3}{z}\, \partial_z + \partial_x^2 \right) \tag{33}$$

With the help of Mathematica™, it is straightforward to check that $\left(\nabla^2 - m^2\right) \left(\frac{z}{z^2 + l^2}\right)^\Delta = 0$, therefore $\left(\nabla^2 - m^2\right) \phi(x, z) = 0$.

The AdS/CFT dictionary is given by:

$$\left\langle e^{\int \mathrm{d}^4x\, C_O\, \phi_0(x)\, O(x)} \right\rangle_{\mathrm{CFT}} = e^{-S[\phi_0]} \tag{34}$$

Where $S[\phi_0]$ the bulk effective action evaluated on the solution to the equation of motion:

$$S[\phi_0] = \int \mathrm{d}^4x\, \mathrm{d}z\, \sqrt{-G} \left\{ \frac{1}{2} \left(\partial_\mu \phi\right)^2 + \frac{1}{2}\, m^2 \phi^2 + \frac{1}{3}\, g\phi^3 + \cdots \right\} \tag{35}$$

(b) The CFT 2-point function $\langle O(x)\, O(y) \rangle$ can be computed with the above dictionary, using the usual effective formalism, but with the bulk action instead of the boundary action:

$$\langle O(x)\, O(y) \rangle = \frac{1}{C_O^2}\, \frac{\delta^2}{\delta\phi_0(x)\, \delta\phi_0(y)}\, e^{-S[\phi_0]} \bigg|_{\phi_0 = 0} \tag{36}$$

For 2-point function, we only need terms $\sim \mathcal{O}\left(\phi^2\right)$; note that:

$$\frac{\delta\phi(x, z)}{\delta\phi_0(x')} = K_\Delta(x, z; x'), \tag{37}$$

$$\begin{aligned} \delta S\left[\phi_0\right] &\sim \int \mathrm{d}^4x\, \mathrm{d}z\, \sqrt{-G}\, \left(-\nabla^2 + m^2\right) \phi\, \delta\phi + \int_{z \to 0} \mathrm{d}^4x\, \sqrt{-G}\, \partial^z \phi\, \delta\phi \\ &= 0 - \int_{z \to 0} \mathrm{d}^4x\, z^{-5} z^2 \partial_z \phi\, \delta\phi = - \int_{z \to 0} \mathrm{d}^4x\, z^{-3} \partial_z \phi\, \delta\phi\,, \end{aligned} \tag{38}$$

$$\begin{aligned} \therefore \quad \langle O(x)\, O(y) \rangle &= + \frac{1}{C_O^2}\, \frac{\delta}{\delta\phi_0(x)}\, e^{-S[\phi_0]} \int_{z \to 0} \mathrm{d}^4x'\, z^{-3} \partial_z \phi(x', z)\, K_\Delta(x', z; y) \bigg|_{\phi_0 = 0} \\ &= \frac{1}{C_O^2} \int_{z \to 0} \mathrm{d}^4x'\, z^{-3} \partial_z K_\Delta(x', z; x)\, K_\Delta(x', z; y) \\ &= \frac{1}{C_O^2} \int_{z \to 0} \mathrm{d}^4x'\, z^{-3} \partial_z K_\Delta(x', z; x)\, z^\delta \delta^4(x' - y) \\ &= \frac{z^{\delta - 3}}{C_O^2}\, \partial_z K_\Delta(x, z; y) \\ &\sim \frac{z^{\delta - 3}}{C_O^2}\, \frac{(\Delta - 1)(\Delta - 2)}{\pi^2}\, \frac{\Delta z^{\Delta - 1}}{\|x - y\|^{2\Delta}} \\ &= \frac{1}{C_O^2}\, \frac{\Delta(\Delta - 1)(\Delta - 2)}{\pi^2}\, \frac{1}{\|x - y\|^{2\Delta}} \end{aligned} \tag{39}$$

Therefore, if we want $\langle O(x)\, O(y)\rangle = \frac{1}{\|x-y\|^{2\Delta}}$, then we have[7]:

$$C_O = \frac{1}{\pi}\sqrt{\Delta(\Delta-1)(\Delta-2)} \tag{40}$$

Here $z \to 0$ is a cutoff parameter.

(c) Similarly, we can use the dictionary to compute 3-point functions; we have[8]:

$$\langle O(x_1)\, O(x_2)\, O(x_3)\rangle = \frac{1}{C_O^3}\left(-\frac{g}{3}\right)\int \mathrm{d}^4x\,\mathrm{d}z\,\sqrt{-G}\,K_\Delta(x,z;x_1)\,K_\Delta(x,z;x_2)\,K_\Delta(x,z;x_3) \tag{41}$$

This is a difficult integral; as is suggested by arXiv:hep-th/9804058, we can use an important symmetry of AdS/CFT — the inversion $\vec{x} \mapsto \frac{\vec{x}}{x^2}$, to complete the integration.

By conformal symmetry, we know that the 3-point function is of the form:

$$\langle O(x_1)\, O(x_2)\, O(x_3)\rangle = A(x_1,x_2,x_3) = \frac{C_{OOO}}{|x_{12}|^\Delta |x_{23}|^\Delta |x_{31}|^\Delta}, \quad x_{ij} = x_i - x_j \tag{42}$$

First set $x_3 = 0$, then perform inversion on all other points:

$$x_i = \frac{x_i'}{x_i'^2}, \quad (x,z) = \frac{(x',z')}{r'^2}, \quad r^2 = x^2 + z^2, \quad r^2 r'^2 = 1 = x_i^2 x_i'^2, \tag{43}$$

$$\frac{\mathrm{d}^4x\,\mathrm{d}z}{z^5} = \frac{\mathrm{d}^4x'\,\mathrm{d}z'}{z'^5}, \tag{44}$$

$$\begin{aligned}\frac{z}{z^2 + \|x-x_i\|^2} &= \frac{z}{r^2 + x_i^2 - 2x\cdot x_i} = \frac{z'/r'^2}{1/r'^2 + 1/x_i'^2 - 2x'\cdot x_i'/(r'^2 x_i'^2)} \\ &= \frac{z'}{r'^2 + x_i'^2 - 2x'\cdot x_i'}\,x_i'^2 = \frac{z'}{z'^2 + \|x'-x_i'\|^2}\,x_i'^2,\end{aligned} \tag{45}$$

$$K_\Delta(x,z;x_i) = K_\Delta(x',z';x_i')\,|x_i'|^{2\Delta} = \frac{1}{|x_i|^{2\Delta}}\,K_\Delta(x',z';x_i'), \tag{46}$$

With these in mind, we find that:

$$A(x_1,x_2,0) = -\frac{g}{3C_O^3}\,\frac{1}{|x_1|^{2\Delta}}\,\frac{1}{|x_2|^{2\Delta}}\,\frac{(\Delta-1)(\Delta-2)}{\pi^2}\int \frac{\mathrm{d}^4x'\,\mathrm{d}z'}{z'^5} K_\Delta(x',z';x_1')\,K_\Delta(x',z';x_2')\,z'^\Delta \tag{47}$$

The integral can then be completed using Feynman parameters; in the end we obtain:

$$A(x_1,x_2,0) \propto \frac{1}{|x_1|^{2\Delta}}\,\frac{1}{|x_2|^{2\Delta}}\,\frac{1}{|x_1'-x_2'|^{2\Delta}} = \frac{1}{|x_1|^\Delta |x_2|^\Delta |x_1-x_2|^\Delta}, \tag{48}$$

$$C_{OOO} = -\frac{g}{3C_O^3}\,\frac{1}{2\pi^4}\left(\frac{\Gamma(\frac{\Delta}{2})}{\Gamma(\Delta-2)}\right)^3 \Gamma\left(\frac{3\Delta-4}{2}\right) \tag{49}$$

[7] Reference: arXiv:hep-th/9804058. Again I would like to thank Lucy Smith for helpful hints.

[8] Reference: arXiv:hep-th/9905111, and arXiv:hep-th/9804058.