

1 T-duality of Heterotic Strings¹

We use $d \leq 10$ to denote the number of noncompact dimensions; the remaining $m \geq d$ dimensions are compactified. For heterotic strings, the $I \geq 10$ dimensions of the left-moving sector are already compactified on a lattice Γ_{16} or $\Gamma_8 \times \Gamma_8$. Here we use the label I to index the 16 internal dimensions.

(a) Generally, if we compactify an open string on the x^m direction: $x^m \cong x^m + 2\pi R$ with constant backgrounds A_m , then its zero mode spectrum, *with winding* $w = 0$, can be obtained from canonical quantization of the effective point particle action, with an additional gauge action term in the form of a Wilson line²:

$$-W_q = -iq \int dx^m A_m \sim -iq \int d\tau A_m \dot{X}^m \quad (1)$$

By imposing that the canonical momentum to be periodic along x^m , we find that:

$$k_m = \frac{n_m}{R} - q A_m \quad (2)$$

To obtain the winding states, we have to reproduce the above action from the world-sheet description. For heterotic strings with $m < 10 \leq I \leq 26$, this can be achieved by adding the following term to the usual world-sheet action³:

$$S_A \propto \int d^2\sigma \epsilon^{ab} A_\mu^I \partial_a X^\mu \partial_b X_I \quad (3)$$

With proper normalization to match the result in (2).

Canonical quantization then produces⁴:

$$k_m = \frac{n_m}{R} \pm \frac{w_m R}{\alpha'} - q_I A_m^I - \frac{w_n R}{2} A_I^n A_m^I, \quad (4)$$

$$k_L^I = \sqrt{\frac{2}{\alpha'}} (q^I + w^m R A_m^I), \quad (5)$$

The “ \pm ” signs in k_m correspond to the left and right-moving sector, respectively. Only the left-moving sector has an additional 16 dimensional internal torus, therefore k^I is labeled with an “ L ”.

Note that the charge q^I now takes value on the Γ_{16} or $\Gamma_8 \times \Gamma_8$ lattice, and:

$$l \circ l' = \frac{\alpha'}{2} (k_L^I k'_{L,I} + k_L^m k'_{L,m} - k_R^m k'_{R,m}) = q^I q'_I + 2nw \quad (6)$$

We can then see that the new “extended” lattice indeed satisfies the even and self-dual conditions, which follows from the even and self-dual properties of Γ_{16} or $\Gamma_8 \times \Gamma_8$.

¹ I would like to thank Lucy Smith for help with this problem.

² Reference: *Polchinski*, Chapter 8.

³ Reference: *Blumenhagen et al.*, Basic Concepts of String Theory.

⁴ Reference: *Polchinski*, Chapter 11.

(b) With $m = 9$ and $G_{dd} = 1$, we have:

$$W_q = \exp(-iq_I \theta^I), \quad A_9^I = -\frac{\theta^I}{2\pi R} \quad (7)$$

Note that W_q captures the phase change of the paths that wind around x^9 ; the extra phase from a non-trivial Wilson line might affect the boundary condition of some states while leaving others intact, thus breaking the original gauge symmetry. Our discussions here closely follow *Polchinski*, Chapter 11.

For the $\text{SO}(32)$ theory with:

$$RA_9^I = \text{diag}\left(\left(\frac{1}{2}\right)^8, 0^8\right) \quad (8)$$

Adjoint states are labeled by a pair of indices valued in $1, \dots, 32$; those with one index from $1 \leq A \leq 16$ and one from $17 \leq A \leq 32$ are anti-periodic due to the additional phase $e^{i\pi} = -1$ from the Wilson line, so the gauge symmetry is reduced to $\text{SO}(16) \times \text{SO}(16)$.

Similarly, for the $E_8 \times E_8$ theory with:

$$R'A_9^I = \text{diag}(1, 0^7, 1, 0^7) \quad (9)$$

Note that Γ_8 , the root lattice of E_8 , is basically the root lattice union an additional spinor weight lattice of $\text{SO}(16)$. With the above Wilson line, the integer-charged states from the $\text{SO}(16)$ root lattice in each E_8 remain periodic, while the half-integer charged states from the $\text{SO}(16)$ spinor lattices become anti-periodic, due to the additional phase $e^{i\frac{1}{2} \cdot 2\pi} = -1$. Again the gauge symmetry is broken down to $\text{SO}(16) \times \text{SO}(16)$.

In summary, with the above Wilson line, the $\text{SO}(32)$ and $E_8 \times E_8$ theory shares an unbroken gauge of $\text{SO}(16) \times \text{SO}(16)$. Consider the spectrum of the $\text{SO}(16) \times \text{SO}(16)$ *neutral states*, i.e. those with internal momentum:

$$k_L^I = \sqrt{\frac{2}{\alpha'}} (q^I + wRA_9^I) = 0 \quad (10)$$

For the $\text{SO}(32)$ theory, since $q^I \in \Gamma_{16}$ while $RA_9^I = \text{diag}\left(\left(\frac{1}{2}\right)^8, 0^8\right)$, we must have $w = 2m$ for this to hold. The same goes for the $E_8 \times E_8$ theory; therefore, we have:

$$k_{L,R} = \frac{\tilde{n}}{R} \pm \frac{2mR}{\alpha'}, \quad k'_{L,R} = \frac{\tilde{n}'}{R'} \pm \frac{2m'R'}{\alpha'}, \quad (11)$$

$$\tilde{n} = n + 2m, \quad \tilde{n}' = n' + 2m' \quad (12)$$

(c) If the two theories are related by T-duality, then we should expect:

$$(k_L, k_R) \longleftrightarrow (k'_L, -k'_R), \quad (13)$$

Under suitable mapping of parameters. Indeed, it is straightforward to verify that $(\tilde{n}, m) \leftrightarrow (m', \tilde{n}')$ realizes this, along with $RR' = \alpha'/2$. The above arguments can then be generalized to higher levels, by acting on fermionized left-moving fields λ^A and carefully organizing representations. We see that the two heterotic string theories are equivalent under T-duality.

2 String Junction⁵

For a string junction to be mechanically stable, the tension force exerted on the junction must cancel each other; this is a Newtonian mechanics problem, but with (p, q) -string tension given by the

⁵ Reference: [arXiv:0812.4408](https://arxiv.org/abs/0812.4408).

BPS bound:

$$\tau_{(p,q)} = \frac{\sqrt{p^2 + q^2/g^2}}{2\pi\alpha'} \quad (14)$$

Stability of the system implies that the BPS bound should be saturated.

From Newtonian mechanics, we know that three forces cannot cancel each other unless they are co-planar. Therefore, a 3-string junction must be co-planar in order to be stable. Suppose they lie in the (X^1, X^2) plane, then the tension exerted on the junction can be expressed as:

$$\vec{T}_i = \tau_{(p_i, q_i)} (\cos \theta_i, \sin \theta_i), \quad i = 1, 2, 3, \quad (15)$$

$\sum_i \vec{T}_i = 0$ gives two equations, and we have two independent unknowns (the angle between two pairs of strings); therefore if a solution exists, it should be unique up to rotations and reflections.

In fact, a solution can be found by simple observations:

$$\cos \theta_i = \frac{p_i}{\sqrt{p^2 + q^2/g^2}}, \quad \sin \theta_i = \frac{q_i/g}{\sqrt{p^2 + q^2/g^2}}, \quad (16)$$

It satisfies $\sum_i \vec{T}_i = 0$ since that total (p, q) vanishes at each junction.

To find the remaining supersymmetries of this system, we start from the original supersymmetries of a (p, q) string (which saturates the BPS bound) extended along the $\hat{X} = (\cos \theta, \sin \theta)$ direction:

$$\frac{1}{2L\tau_{(p,q)}} \left\{ \begin{bmatrix} Q_\alpha \\ \tilde{Q}_\alpha \end{bmatrix}, \begin{bmatrix} Q_\beta^\dagger & \tilde{Q}_\beta^\dagger \end{bmatrix} \right\} = \delta_{\alpha\beta} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (\Gamma^0 \Gamma^\theta)_{\alpha\beta} \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}, \quad (17)$$

$$\Gamma^\theta = \hat{X} \cdot \vec{\Gamma} = \Gamma^1 \cos \theta + \Gamma^2 \sin \theta \quad (18)$$

We see that the algebra depends on θ , i.e. it is different for strings in different directions. However, if we can find a (maximal) subalgebra that is independent of θ , then we would have found the remaining supersymmetries of the full system⁶.

We begin with diagonalizing the matrix on the RHS with:

$$U(\frac{\theta}{2}) = \begin{bmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix}, \quad (19)$$

$$\frac{1}{2L\tau_{(p,q)}} \left\{ U^T \begin{bmatrix} Q_\alpha \\ \tilde{Q}_\alpha \end{bmatrix}, \begin{bmatrix} Q_\beta^\dagger & \tilde{Q}_\beta^\dagger \end{bmatrix} U \right\} = \begin{bmatrix} (\mathbb{1} + \Gamma^0 \Gamma^\theta)_{\alpha\beta} & 0 \\ 0 & (\mathbb{1} - \Gamma^0 \Gamma^\theta)_{\alpha\beta} \end{bmatrix}, \quad (20)$$

Note that $(\mathbb{1} + \Gamma^0 \Gamma^\theta)(\mathbb{1} - \Gamma^0 \Gamma^\theta) = 0$ and $(\mathbb{1} + \Gamma^0 \Gamma^\theta) + (\mathbb{1} - \Gamma^0 \Gamma^\theta) = 2\mathbb{1}$, i.e. they are orthogonal to each other; acting $(\mathbb{1} \pm \Gamma^0 \Gamma^\theta)$ on both sides, we find the following combinations, which gives the 16 SUSYs of a (p, q) string:

$$(\mathbb{1} - \Gamma^0 \Gamma^\theta) \left(\cos \frac{\theta}{2} Q + \sin \frac{\theta}{2} \tilde{Q} \right)_\alpha = 0 = (\mathbb{1} + \Gamma^0 \Gamma^\theta) \left(-\sin \frac{\theta}{2} Q + \cos \frac{\theta}{2} \tilde{Q} \right)_\beta \quad (21)$$

⁶ The $\tau_{(p,q)}$ factor can be absorbed by rescaling generators, hence does not matter in our discussions.

For further simplification, we can isolate the θ dependence in Γ^θ by working in a specific representation of the Clifford algebra, e.g. the Dirac representation given by *Polchinski*. Then α is given by 10D spinor components: $\alpha = (s_0, s_1, s_2, s_3, s_4)$, $s_i = \pm$, with additional chirality constraints from both Q and \tilde{Q} : $\prod_i s_i = +$. In the end, we have 16 independent components as expected.

Details of the expansion are given in [arXiv:0812.4408](#). When the dust settles, we find that the 16 SUSYs in (21) is given by:

$$\sin \frac{\theta}{2} \left(\cos \frac{\theta}{2} Q + \sin \frac{\theta}{2} \tilde{Q} \right)_{(++s)} + \cos \frac{\theta}{2} \left(\cos \frac{\theta}{2} Q + \sin \frac{\theta}{2} \tilde{Q} \right)_{(--s)}, \quad (22a)$$

$$\sin \frac{\theta}{2} \left(\cos \frac{\theta}{2} Q + \sin \frac{\theta}{2} \tilde{Q} \right)_{(+-s)} - \cos \frac{\theta}{2} \left(\cos \frac{\theta}{2} Q + \sin \frac{\theta}{2} \tilde{Q} \right)_{(-+s)}, \quad (22b)$$

$$\cos \frac{\theta}{2} \left(-\sin \frac{\theta}{2} Q + \cos \frac{\theta}{2} \tilde{Q} \right)_{(++s)} - \sin \frac{\theta}{2} \left(-\sin \frac{\theta}{2} Q + \cos \frac{\theta}{2} \tilde{Q} \right)_{(--s)}, \quad (22c)$$

$$\cos \frac{\theta}{2} \left(-\sin \frac{\theta}{2} Q + \cos \frac{\theta}{2} \tilde{Q} \right)_{(+-s)} + \sin \frac{\theta}{2} \left(-\sin \frac{\theta}{2} Q + \cos \frac{\theta}{2} \tilde{Q} \right)_{(-+s)}, \quad (22d)$$

$$s = (s_2 s_3 s_4), \quad \prod_i s_i = + \quad (23)$$

By trial and error, we can find the 8 linear combinations that are independent of θ ; they are:

$$(a) + (c) \implies \tilde{Q}_{(++s)} + Q_{(--s)}, \quad (24)$$

$$(b) + (d) \implies \tilde{Q}_{(+-s)} - Q_{(-+s)}, \quad (25)$$

Therefore, the string junction is $\frac{8}{32} = \frac{1}{4}$ BPS.

3 Two and Three-Point Functions in AdS/CFT

Consider a scalar field $\phi(x, z)$ in Poincaré AdS₅ (with radius $R = 1$) satisfying:

$$(\nabla^2 - m^2) \phi(x, z) = 0, \quad \phi(x, z) \rightarrow \begin{cases} z^\delta \phi_0(x), & z \rightarrow 0, \\ \text{regular}, & z \rightarrow \infty, \end{cases} \quad \delta = 2 - \sqrt{m^2 + 4} \quad (26)$$

It can be constructed via the boundary-to-bulk propagator K_Δ :

$$\phi(x, z) = \int d^4 x' K_\Delta(x, z; x') \phi(x'), \quad (27)$$

$$K_\Delta(x, z; x') = \frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \left(\frac{z}{z^2 + \|x - x'\|^2} \right)^\Delta, \quad \Delta = 2 + \sqrt{m^2 + 4} \quad (28)$$

(a) To verify this, we first check that the boundary conditions are indeed satisfied by K_Δ ; note that $\Delta \geq 2 > 0$, and we have:

$$z \rightarrow 0, \quad l \neq 0, \quad \left(\frac{z}{z^2 + l^2} \right)^\Delta \rightarrow 0, \quad (29)$$

i.e. the only contribution comes from the $l \rightarrow 0$ case, where we have:

$$\begin{aligned} l = \|x - x'\| \rightarrow 0, \quad \int d^4 x' K_\Delta(x, z; x') &= \frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \int_0^\infty 2\pi^2 l^3 dl \left(\frac{z}{z^2 + l^2} \right)^\Delta \\ &= \frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \cdot \frac{2\pi^2}{2(\Delta - 1)(\Delta - 2)} z^{4-\Delta} \\ &= z^\delta \end{aligned} \quad (30)$$

Therefore, we have:

$$z \rightarrow 0, \quad K_{\Delta}(x, z; x') \rightarrow z^{\delta} \delta^4(x - x'), \quad \phi(x, z) \rightarrow z^{\delta} \phi_0(x), \quad (31)$$

The other boundary condition is convenient to check; we have:

$$z \rightarrow \infty, \quad K_{\Delta}(x, z; x') \propto z^{-\Delta} \rightarrow 0, \quad \phi(x, z) \text{ regular}. \quad (32)$$

Now we need only check that K_{Δ} satisfies the equation of motion; in Poincaré AdS₅ we have:

$$\nabla^2 = z^2 \left(\partial_z^2 - \frac{3}{z} \partial_z + \partial_x^2 \right) \quad (33)$$

With the help of MathematicaTM, it is straightforward to check that $(\nabla^2 - m^2) \left(\frac{z}{z^2 + l^2} \right)^{\Delta} = 0$, therefore $(\nabla^2 - m^2) \phi(x, z) = 0$.



The AdS/CFT dictionary is given by:

$$\left\langle e^{\int d^4x C_O \phi_0(x) O(x)} \right\rangle_{\text{CFT}} = e^{-S[\phi_0]} \quad (34)$$

Where $S[\phi_0]$ the bulk effective action evaluated on the solution to the equation of motion:

$$S[\phi_0] = \int d^4x dz \sqrt{-G} \left\{ \frac{1}{2} (\partial_{\mu} \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} g \phi^3 + \dots \right\} \quad (35)$$

(b) The CFT 2-point function $\langle O(x) O(y) \rangle$ can be computed with the above dictionary, using the usual effective formalism, but with the bulk action instead of the boundary action:

$$\langle O(x) O(y) \rangle = \frac{1}{C_O^2} \frac{\delta^2}{\delta \phi_0(x) \delta \phi_0(y)} e^{-S[\phi_0]} \Big|_{\phi_0=0} \quad (36)$$

For 2-point function, we only need terms $\sim \mathcal{O}(\phi^2)$; note that:

$$\frac{\delta \phi(x, z)}{\delta \phi_0(x')} = K_{\Delta}(x, z; x'), \quad (37)$$

$$\begin{aligned} \delta S[\phi_0] &\sim \int d^4x dz \sqrt{-G} (-\nabla^2 + m^2) \phi \delta \phi + \int_{z \rightarrow 0} d^4x \sqrt{-G} \partial_z \phi \delta \phi \\ &= 0 - \int_{z \rightarrow 0} d^4x z^{-5} z^2 \partial_z \phi \delta \phi = - \int_{z \rightarrow 0} d^4x z^{-3} \partial_z \phi \delta \phi, \end{aligned} \quad (38)$$

$$\begin{aligned} \therefore \langle O(x) O(y) \rangle &= + \frac{1}{C_O^2} \frac{\delta}{\delta \phi_0(x)} e^{-S[\phi_0]} \int_{z \rightarrow 0} d^4x' z^{-3} \partial_z \phi(x', z) K_{\Delta}(x', z; y) \Big|_{\phi_0=0} \\ &= \frac{1}{C_O^2} \int_{z \rightarrow 0} d^4x' z^{-3} \partial_z K_{\Delta}(x', z; x) K_{\Delta}(x', z; y) \\ &= \frac{1}{C_O^2} \int_{z \rightarrow 0} d^4x' z^{-3} \partial_z K_{\Delta}(x', z; x) z^{\delta} \delta^4(x' - y) \\ &= \frac{z^{\delta-3}}{C_O^2} \partial_z K_{\Delta}(x, z; y) \\ &\sim \frac{z^{\delta-3}}{C_O^2} \frac{(\Delta-1)(\Delta-2)}{\pi^2} \frac{\Delta z^{\Delta-1}}{\|x-y\|^{2\Delta}} \\ &= \frac{1}{C_O^2} \frac{\Delta(\Delta-1)(\Delta-2)}{\pi^2} \frac{1}{\|x-y\|^{2\Delta}} \end{aligned} \quad (39)$$

Therefore, if we want $\langle O(x) O(y) \rangle = \frac{1}{\|x-y\|^{2\Delta}}$, then we have⁷:

$$C_O = \frac{1}{\pi} \sqrt{\Delta(\Delta-1)(\Delta-2)} \quad (40)$$

Here $z \rightarrow 0$ is a cutoff parameter.

(c) Similarly, we can use the dictionary to compute 3-point functions; we have⁸:

$$\langle O(x_1) O(x_2) O(x_3) \rangle = \frac{1}{C_O^3} \left(-\frac{g}{3}\right) \int d^4x dz \sqrt{-G} K_\Delta(x, z; x_1) K_\Delta(x, z; x_2) K_\Delta(x, z; x_3) \quad (41)$$

This is a difficult integral; as is suggested by [arXiv:hep-th/9804058](#), we can use an important symmetry of AdS/CFT — the inversion $\vec{x} \mapsto \frac{\vec{x}}{x^2}$, to complete the integration.

By conformal symmetry, we know that the 3-point function is of the form:

$$\langle O(x_1) O(x_2) O(x_3) \rangle = A(x_1, x_2, x_3) = \frac{C_{OOO}}{|x_{12}|^\Delta |x_{23}|^\Delta |x_{31}|^\Delta}, \quad x_{ij} = x_i - x_j \quad (42)$$

First set $x_3 = 0$, then perform inversion on all other points:

$$x_i = \frac{x'_i}{x_i'^2}, \quad (x, z) = \frac{(x', z')}{r'^2}, \quad r^2 = x^2 + z^2, \quad r^2 r'^2 = 1 = x_i^2 x_i'^2, \quad (43)$$

$$\frac{d^4x dz}{z^5} = \frac{d^4x' dz'}{z'^5}, \quad (44)$$

$$\begin{aligned} \frac{z}{z^2 + \|x - x_i\|^2} &= \frac{z}{r^2 + x_i^2 - 2x \cdot x_i} = \frac{z'/r'^2}{1/r'^2 + 1/x_i'^2 - 2x' \cdot x'_i/(r'^2 x_i'^2)} \\ &= \frac{z'}{r'^2 + x_i'^2 - 2x' \cdot x'_i} \frac{x_i'^2}{x_i'^2} = \frac{z'}{z'^2 + \|x' - x'_i\|^2} x_i'^2, \end{aligned} \quad (45)$$

$$K_\Delta(x, z; x_i) = K_\Delta(x', z'; x'_i) |x'_i|^{2\Delta} = \frac{1}{|x_i|^{2\Delta}} K_\Delta(x', z'; x'_i), \quad (46)$$

With these in mind, we find that:

$$A(x_1, x_2, 0) = -\frac{g}{3C_O^3} \frac{1}{|x_1|^{2\Delta}} \frac{1}{|x_2|^{2\Delta}} \frac{(\Delta-1)(\Delta-2)}{\pi^2} \int \frac{d^4x' dz'}{z'^5} K_\Delta(x', z'; x'_1) K_\Delta(x', z'; x'_2) z'^\Delta \quad (47)$$

The integral can then be completed using Feynman parameters; in the end we obtain:

$$A(x_1, x_2, 0) \propto \frac{1}{|x_1|^{2\Delta}} \frac{1}{|x_2|^{2\Delta}} \frac{1}{|x'_1 - x'_2|^{2\Delta}} = \frac{1}{|x_1|^\Delta |x_2|^\Delta |x_1 - x_2|^\Delta}, \quad (48)$$

$$C_{OOO} = -\frac{g}{3C_O^3} \frac{1}{2\pi^4} \left(\frac{\Gamma(\frac{\Delta}{2})}{\Gamma(\Delta-2)} \right)^3 \Gamma\left(\frac{3\Delta-4}{2}\right) \quad (49)$$

⁷ Reference: [arXiv:hep-th/9804058](#). Again I would like to thank Lucy Smith for helpful hints.

⁸ Reference: [arXiv:hep-th/9905111](#), and [arXiv:hep-th/9804058](#).