String Theory №6

Bryan Compiled @ 2021/12/04

1 Type 0 Superstrings

A closed superstring theory consists of sectors labeled by the boundary conditions $(-1)^\alpha$ of $(\psi, \tilde{\psi})$ along with suitable GSO projections $(-1)^F = \pm 1$. Here we follow the discussions of *Polchinski*, with R: $\alpha = 1$ and NS: $\alpha = 0$.

There are also some consistency conditions: by modular invariance, there must be at least one left-moving R sector and at least one right-moving R sector; on the other hand, the OPE must close, and since R $\times$ R = NS there must be some corresponding NS sector for each R sector.

If we include only the (NS, NS) and the (R, R) sectors, then both must exist due to the above conditions. In fact, closure of OPE implies that the (NS+, NS+) sector must exist. In addition, NS$-$ sector must be paired with another NS$-$ sector due to the level matching condition of the closed string, i.e. it is possible (but not required) to have a (NS$-$, NS$-$) sector.

All possibilities can then be generated by enumerating all possible (R, R) sectors (there are $2 \times 2 = 4$ of them), while applying an extra consistency check that all pairs of vertex operators O_1, O_2 are mutually local, i.e.

$$\exp i\pi\Big(F_1\alpha_2 - F_2\alpha_1 - \tilde{F}_1\tilde{\alpha}_2 + \tilde{F}_2\tilde{\alpha}_1\Big) = 1 \tag{1}$$

If $O_1 \in$ (NS+, NS+), then we have $\alpha_1 = \tilde{\alpha}_1 = 0 = F_1 = \tilde{F}_1$, hence the above factor is always trivial; for $O_1 \in$ (R, R), however, $\alpha_1 = \tilde{\alpha}_1 = 1$, which yields a non-trivial constraint for the second operator: $F_2 - \tilde{F}_2 = F_1\alpha_2 - \tilde{F}_1\tilde{\alpha}_2 = \alpha_2\big(F_1 - \tilde{F}_1\big) \pmod 2$, assuming $\alpha_2 = \tilde{\alpha}_2$. With $\alpha_2 = 0$ this gives $F_2 = \tilde{F}_2$, and with $\alpha_2 = 1$ this gives $F_2 - \tilde{F}_2 = F_1 - \tilde{F}_1$; this means that all (R, R) sectors have the same sign difference between F and $\tilde{F}$. The possible solutions can then be narrowed down to:

$$\text{0A}:\ (\text{NS}+, \text{NS}+),\ (\text{NS}-, \text{NS}-),\ (\text{R}+, \text{R}-),\ (\text{R}-, \text{R}+), \tag{2}$$

$$\text{0B}:\ (\text{NS}+, \text{NS}+),\ (\text{NS}-, \text{NS}-),\ (\text{R}+, \text{R}+),\ (\text{R}-, \text{R}-), \tag{3}$$

And additionally, (NS+, NS+) with any *single one* of the 4 possible (R, R) sectors. (4)

If there are two (R, R) sectors, then there must be an accompanying (NS$-$, NS$-$) sector due to the closure of OPE. It is straightforward to check that these possibilities are all valid under the above constraints: (0) level matching of closed strings, (1) mutual locality, (2) closure of OPE, and (3) (apparent) modular invariance (not sufficient yet, to be checked below).

(a) The torus partition function of the theory breaks up into a product of independent sums over the bosonic X and fermionic $(\psi, \tilde{\psi})$ oscillators. The bosonic part is identical to the bosonic string situation, therefore modular invariant; to check the total partition function for modular invariance, we will look at the fermionic contributions $Z = Z_{\psi,\tilde{\psi}}$ explicitly.

Similar to the Type II case, the building block of Z is given by:

$$Z^\alpha{}_\beta = \mathrm{Tr}_\alpha\left[(-1)^{\beta F} q^H\right], \quad q = e^{2\pi i\tau} \tag{5}$$

Where α, β labels the periodicity in the spatial and temporal directions (σ^1, σ^2); note that for fermionic fields, anti-periodicity in the time direction gives the simple trace, while the periodic path integral gives the trace weighted by $(-1)^F$, as is explained in *Polchinski*, Appendix A.

In 10 D, $\mu = 1, \cdots, 10$, in total there are $N = 10-2 = 4\times 2 = 8$ real, *transverse* spinor components in $(\psi^\mu, \tilde{\psi}^\mu)$; pairing them into complex chiral spinors like $\psi^1 \pm \psi^2$, each one of them contributes a factor of ${Z^\alpha}_0$ in the total partition function.

Note that for type II theories, the boundary conditions and GSO projections (α, F) for the left and right movers are "decoupled"; any possible (α, F) can be paired with any possible $(\tilde{\alpha}, \tilde{F})$, hence the left and right contributions can be calculated separately. For type 0 theories, however, the left and right (α, F)'s are coupled, hence we have to calculate their contributions together. With the above considerations, we have:

$$\begin{aligned}
\mathrm{Tr}_{(\mathrm{NS,NS})}\left[\frac{1+(-1)^{F-\tilde{F}}}{2}\,q^H\right] &= \frac{1}{2}\left\{\mathrm{Tr}_{(\mathrm{NS,NS})}\,q^H + \mathrm{Tr}_{(\mathrm{NS,NS})}\left[(-1)^{F-\tilde{F}}q^H\right]\right\} \\
&= \frac{1}{2}\left\{\left|({Z^0}_0)^{N/2}\right|^2 + \left|({Z^0}_1)^{N/2}\right|^2\right\} \\
&= \frac{1}{2}\left\{\left|{Z^0}_0\right|^N + \left|{Z^0}_1\right|^N\right\},
\end{aligned} \tag{6}$$

$$\begin{aligned}
\mathrm{Tr}_{(\mathrm{R,R})}\left[\frac{1\mp(-1)^{F-\tilde{F}}}{2}\,q^H\right] &= \frac{1}{2}\left\{\mathrm{Tr}_{(\mathrm{R,R})}\,q^H \mp \mathrm{Tr}_{(\mathrm{R,R})}\left[(-1)^{F-\tilde{F}}q^H\right]\right\} \\
&= \frac{1}{2}\left\{\left|{Z^1}_0\right|^N \mp \left|{Z^1}_1\right|^N\right\},
\end{aligned}$$

$$Z^{0\mathrm{A|B}} = \frac{1}{2}\left\{\left|{Z^0}_0\right|^N + \left|{Z^0}_1\right|^N + \left|{Z^1}_0\right|^N \mp \left|{Z^1}_1\right|^N\right\} \tag{7}$$

Similarly, for the situation in (4) with no (NS−, NS−) sector, depending on the GSO projections $(F, \tilde{F})$ in the single (R, R) sector, we have:

$$\begin{aligned}
Z' &= \left|\frac{1}{2}\left(({Z^0}_0)^{N/2} + ({Z^0}_1)^{N/2}\right)\right|^2 + \frac{1}{2}\left(({Z^1}_0)^{N/2} + (-1)^F({Z^1}_1)^{N/2}\right)\cdot\frac{1}{2}\left(({Z^1}_0)^{N/2} + (-1)^{\tilde{F}}({Z^1}_1)^{N/2}\right)^* \\
&= \frac{1}{2}\left\{Z^{0A|B} + \mathrm{Re}\,({Z^0}_0\overline{{Z^0}_1})^{N/2} + (-1)^{\tilde{F}}(\mathrm{Re}\,|\,i\,\mathrm{Im})\,({Z^1}_0\overline{{Z^1}_1})^{N/2}\right\}
\end{aligned} \tag{8}$$

To check for modular invariance, note that[1]:

$${Z^\alpha}_\beta(\tau) = {Z^\beta}_{-\alpha}(-\tfrac{1}{\tau}) = {Z^\alpha}_{\alpha+\beta-1}(\tau+1)\,\cdot\exp\left(-i\pi\,\frac{3\alpha^2-1}{12}\right) \tag{9}$$

We see that $Z^{0A|B}$ is indeed modular invariant, while $Z' = \frac{1}{2}Z^{0A|B} + (\cdots)$ is *not* modular invariant, due to the extra "mixing" terms in $(\cdots)$.

(b) Consider the ground states in the type 0 theories; the NS ground state is tachyonic:

$$m^2 = -k^2 = -\frac{1}{2\alpha'} \tag{10}$$

With $(-1)^F = -1$, while the level 1 states are massless and form a vector representation $\mathbf{8}_v$ of the massless little group SO(8). After GSO projections, the NS ground state becomes the (NS−) ground state, while the level 1 massless states become the (NS+) ground states.

[1] See *Polchinski*, Chapter 10. Note that the factor $\exp\left(-i\pi\,\frac{3\alpha^2-1}{12}\right)$ comes from a global gravitational anomaly, but does not matter in $Z^{0A|B}$ since we are taking absolute values.

On the other hand, the R ground state is massless. In general, 10 D Dirac spinors form a representation $\mathbf{32}_{\mathrm{Dirac}}$ of $\mathrm{SO}(9,1)$; however, in the massless case it can be further reduced into two Weyl spinors $\mathbf{16}+\mathbf{16}'$, labeled by chirality $\Gamma^{11}=(-1)^F$. They are spinor representations of $\mathrm{SO}(8)$. The on-shell condition (i.e. the Dirac equation) further reduces the representation into $\mathbf{8}$ and $\mathbf{8}'$, one for (R+) and one for (R−).

The closed string spectrum is then obtained by tensor product of the left and right moving part. For type 0 theories, we see that there is a tachyonic state: the (NS−, NS−) ground state is a $\mathbf{1}\times\mathbf{1}=\mathbf{1}$ scalar tachyon; with a momentum rescale $k\mapsto k/2$, the mass is now given by $m^2=-2/\alpha'$. The remaining massless states are:

$$(\mathrm{NS}+,\mathrm{NS}+):\qquad \mathbf{8}_v\times\mathbf{8}_v=[0]+[2]+(2) \tag{11}$$
$$(\mathrm{R}\pm,\mathrm{R}\pm):\qquad \mathbf{8}^{(\prime)}\times\mathbf{8}^{(\prime)}=[0]+[2]+[4]_\pm \tag{12}$$
$$(\mathrm{R}+,\mathrm{R}-):\qquad \mathbf{8}\times\mathbf{8}'=[1]+[3] \tag{13}$$

Where we've listed the irreducible decompositions of the various $\mathbf{8}\times\mathbf{8}$ tensor product, following the notations of *Polchinski*.

2 Kaluza–Klein Mechanism

The $D=d+1$ dimensional metric can be parameterized as follows:

$$\mathrm{d}s^2=G^D_{MN}\,\mathrm{d}x^M\,\mathrm{d}x^N=G_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu+e^{2\sigma}\left(\mathrm{d}x^d+A_\mu\,\mathrm{d}x^\mu\right)^2, \tag{14}$$
$$G^D_{\mu\nu}=G_{\mu\nu}+e^{2\sigma}A_\mu A_\nu,\qquad x^d\cong x^d+2\pi R \tag{15}$$

Where $\mu=0,1,\cdots,(d-1)$ labels the noncompact directions, and the x^d direction is compactified. $G_{\mu\nu},\sigma$ and A_μ should depend only on the noncompact coordinates x^μ, $A^\mu=G^{\mu\nu}A_\nu$.

G^D_{MN} can be inverted by solving $\delta^L_N=G_D^{LM}G^D_{MN}$, or in components:

$$0=G_D^{\mu\nu}\,e^{2\sigma}A_\nu+G_D^{\mu d}\,e^{2\sigma}\quad\Longrightarrow\quad G_D^{\mu d}=-G_D^{\mu\nu}A_\nu, \tag{16}$$
$$1=G_D^{\mu d}\,e^{2\sigma}A_\mu+G_D^{dd}\,e^{2\sigma}\quad\Longrightarrow\quad G_D^{dd}=e^{-2\sigma}-G_D^{\mu d}A_\mu=e^{-2\sigma}+G_D^{\mu\nu}A_\mu A_\nu, \tag{17}$$
$$\delta^\mu_\rho=G^{\mu\nu}G_{\nu\rho}=G_D^{\mu\nu}G^D_{\nu\rho}+G_D^{\mu d}\,e^{2\sigma}A_\rho, \tag{18}$$

Contract the last equation with A^ρ, and we can solve for $G_D^{\mu d}$ and then all other components. Alternatively, we can use the inversion formula for a block matrix[2]; either way, we obtain a nice and clean result:

$$G_D^{\mu d}=-A^\mu,\quad G_D^{dd}=e^{-2\sigma}+A^2,\quad G_D^{\mu\nu}=G^{\mu\nu}, \tag{19}$$

There is also a formula[3] for the determinant G_D; we have:

$$G_D^{-1}=G_d^{-1}\big(e^{-2\sigma}+A^2-A^2\big)=G_d^{-1}e^{-2\sigma},\quad G_D=G_d\,e^{2\sigma} \tag{20}$$

(a) The Christoffel symbols can hence be calculated explicitly, using the G^D_{MN} components; the Ricci scalar can then be computed with brute force[4]; in the end, we have:

$$R_D=R_d-2e^{-\sigma}\nabla^2e^\sigma-\frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}, \tag{21}$$

[2] See e.g. `Wikipedia`: *Block matrix # Block matrix inversion*.

[3] See e.g. `Wikipedia`: *Determinant # Block matrices*.

[4] Reference: `www.weylmann.com/kaluza.pdf`, and *Polchinski*, Chapter 8.

$$
\begin{aligned}
S &= \frac{1}{2\kappa_0^2}\int \mathrm{d}^D x\,\sqrt{-G_D}\,R_D \\
&= \frac{1}{2\kappa_0^2}\cdot 2\pi R\int \mathrm{d}^d x\,\sqrt{-G_d}\,e^{\sigma}R_D \\
&\sim \frac{\pi R}{\kappa_0^2}\int \mathrm{d}^d x\,\sqrt{-G_d}\,e^{\sigma}\Big(R_d - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}\Big)
\end{aligned}
\tag{22}
$$

Here we've dropped the $\nabla^2 e^{\sigma}$ term in the Einstein–Hilbert action, for it is a total derivative:

$$
\nabla^2 e^{\sigma} = \frac{1}{\sqrt{-G_d}}\,\partial_\mu\Big(\sqrt{-G_d}\,G_d^{\mu\nu}\partial_\nu e^{\sigma}\Big) \tag{23}
$$

However, if there is a D-dimensional dilaton Φ coupled to gravity: $\mathcal{L}_D \sim e^{-2\Phi}R_D$, then the $e^{-2\Phi}\nabla^2 e^{\sigma}$ term cannot be dropped, since it will contribute a Φ–σ coupling term. Here we are setting $\Phi \equiv 0$.

The e^{σ} factor before R_d can be absorbed by rescaling; first we eliminate the zero mode of σ by rescaling the coupling $\kappa_0 \to \kappa$:

$$
\sigma = \sigma_0 + \sigma', \quad \langle\sigma\rangle = \sigma_0, \quad \langle\sigma'\rangle = 0, \tag{24}
$$

$$
\frac{1}{\kappa_0^2}\,e^{\sigma} = \frac{1}{\kappa^2}\,e^{\sigma'}, \quad \kappa = \kappa_0\,e^{-\sigma_0/2}, \tag{25}
$$

Then we work on the remaining $\sigma' = \sigma - \sigma_0$. Note that:

$$
G'_{\mu\nu} = e^{2\omega(x)}G_{\mu\nu}, \quad G' = e^{2\omega\cdot d}G, \quad G'^{\mu\nu} = e^{-2\omega}G^{\mu\nu}, \tag{26}
$$

$$
R'_d = e^{-2\omega}\Big(R_d - 2\,(d-1)\nabla^2\omega - (d-2)(d-1)\,\partial_\mu\omega\,\partial^\mu\omega\Big), \tag{27}
$$

$$
\sqrt{-G}\,e^{\sigma'}R_d \sim \sqrt{-G'}\,R'_d \sim \sqrt{-G}\,e^{(d-2)\omega}R_d, \quad \omega = \frac{\sigma'}{d-2}, \tag{28}
$$

Before we proceed, let's first work out the Weyl transformation of the Laplacian:

$$
\begin{aligned}
\nabla'^2\sigma' &= \frac{1}{\sqrt{-G'}}\,\partial_\mu\Big(\sqrt{-G'}G'^{\mu\nu}\partial_\nu\sigma'\Big) \\
&= \frac{1}{\sqrt{-G}}\,e^{-\omega d}\,\partial_\mu\Big(\sqrt{-G}\,e^{+\omega d}e^{-2\omega}\,G^{\mu\nu}\partial_\nu\sigma'\Big) \\
&= e^{-\omega d}\,(\partial_\mu e^{\sigma'})\,G^{\mu\nu}\partial_\nu\sigma' + e^{-2\omega}\nabla^2\sigma' \\
&= G'^{\mu\nu}\partial_\mu\sigma'\partial_\nu\sigma' + e^{-2\omega}\nabla^2\sigma'
\end{aligned}
\tag{29}
$$

The transformed Ricci scalar can then be rewritten as:

$$
\begin{aligned}
R'_d &= e^{-2\omega}R_d - 2\,\frac{d-1}{d-2}\,e^{-2\omega}\nabla^2\sigma' - \frac{d-1}{d-2}\,\partial_\mu\sigma'\,\partial'^\mu\sigma' \\
&= e^{-2\omega}R_d - 2\,\frac{d-1}{d-2}\Big(\nabla'^2\sigma' - \partial_\mu\sigma'\,\partial'^\mu\sigma'\Big) - \frac{d-1}{d-2}\,\partial_\mu\sigma'\,\partial'^\mu\sigma' \\
&= e^{-2\omega}R_d - 2\,\frac{d-1}{d-2}\,\nabla'^2\sigma' + \frac{d-1}{d-2}\,\partial_\mu\sigma'\,\partial'^\mu\sigma'
\end{aligned}
\tag{30}
$$

Again, the $\nabla'^2\sigma'$ term is a total derivative and can be dropped in the action. In the end, we get:

$$
S \sim \frac{\pi R}{\kappa^2}\int \mathrm{d}^d x\,\sqrt{-G'_d}\Big(R'_d - \frac{d-1}{d-2}\,\partial_\mu\sigma'\,\partial'^\mu\sigma' - \frac{1}{4}e^{2(\sigma+\omega)}F_{\mu\nu}F'^{\mu\nu}\Big) \tag{31}
$$

This is the effective d-dimensional theory that we have been looking for, with a gauge field $F_{\mu\nu}$ and a massless dilaton σ'. Roughly speaking, the dilaton σ' can be treated as a Goldstone boson due to the breaking of scale invariance by compactification[5].

Following the convention of *Polchinski*, we define $A_\mu = R\tilde{A}_\mu,\ \rho = Re^\sigma,\ \rho_0 = \langle\rho\rangle = Re^{\sigma_0}$, then the gravitational and gauge couplings are given by:

$$\frac{1}{2\kappa_d^2} = \frac{\pi R}{\kappa^2}, \quad -\frac{1}{4g_d^2} = -\frac{1}{4}\, e^{2\langle\sigma+\omega\rangle} R^2 \cdot \frac{\pi R}{\kappa^2} = -\frac{1}{4}\, e^{2\sigma_0} R^2 \cdot \frac{1}{2\kappa_d^2}, \tag{32}$$

$$\therefore \quad \kappa_d^2 = \frac{\kappa^2}{2\pi R} = \frac{\kappa_0^2}{2\pi\rho_0}, \quad g_d^2 = \frac{2\kappa_d^2}{\rho_0^2} = \frac{\kappa_0^2}{\pi\rho_0^3}, \quad \rho_0 = Re^{\sigma_0} \tag{33}$$

(b) The above mechanism provides a natural theory of gravity and electromagnetism in $d = 4$. Note that the gravitational and gauge couplings are related with the radius of the compact dimension:

$$\frac{g_d^2}{\kappa_d^2} = \frac{2}{\rho_0^2} \tag{34}$$

In reality gravity is much weaker than electromagnetism, which means that $\rho_0 \to 0$, or $R \to 0$ if we gauge-fix $\sigma_0 \equiv 0$. In other words, the radius is constrained by the ratio of the couplings:

$$R \sim \sqrt{2}\,\frac{\kappa_d}{g_d} \tag{35}$$

3 Fiberwise T-Duality and the Dilaton

(a) For a bosonic string moving in a general background of massless fields in $D = d + 1 = 26$, its worldsheet action is given by:

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma\, \sqrt{g}\, \Big\{ \big(g^{ab} G_{MN}(X) + i\epsilon^{ab} B_{MN}(X) \big)\, \partial_a X^M \partial_b X^N + \alpha' \mathcal{R}\, \Phi(X) \Big\} \tag{36}$$

Where Φ is the worldsheet Ricci scalar. The $X^d \equiv X^{25}$ direction is to be compactified, and the background fields G_{MN}, B_{MN} and Φ depends only on X^μ, $\mu = 0, 1, \cdots, (d-1) = 24$.

G_{MN} can be further split into $G_{\mu\nu}, G_{\mu d}$ and G_{dd} with $d = 25$, in a way similar to (14), but here we are using a simpler convention, with $G^D_{\mu\nu} = G_{\mu\nu}$ instead of (15). Similar goes for B_{MN}, with $B_{\mu\nu}, B_{\mu d}$, and $B_{dd} = 0$, due to anti-symmetry.

(b) After replacing $\partial_a X^d \mapsto \partial_a X^d + A_a$ where A_a is an auxiliary abelian gauge field on the worldsheet, the X^d related parts in the Lagrangian become:

$$\mathcal{L}_d[X^d, A_a] = \frac{\sqrt{g}}{4\pi\alpha'} \Big\{ 2 \left(g^{ab} G_{\mu d} + i\epsilon^{ab} B_{\mu d} \right) (\partial_a X^d + A_a)\, \partial_b X^\mu + g^{ab} G_{dd}\, (\partial_a X^d + A_a)\, (\partial_b X^d + A_b) \Big\} \tag{37}$$

Consider a translation $X^d \mapsto X^d + \lambda$, where λ depends on $X^\mu = X^\mu(\sigma)$ and hence depends on the worldsheet coordinates σ; it is clear that:

$$\partial_a \big(X^d + \lambda \big) + (A_a - \partial_a \lambda) = \partial_a X^d + A_a, \quad \mathcal{L}_d\big[(X^d + \lambda), (A_a - \partial_a\lambda) \big] = \mathcal{L}_d\big[X^d, A_a \big] \tag{38}$$

i.e. X^d translation is equivalent to a local gauge transformation $A_a \mapsto A_a - \partial_a \lambda$.

[5] For a more careful discussion, see *Polchinski*. See also physics.stackexchange.com/q/138537.

In fact, we would like A_a to be "pure gauge", capturing only the X^d translational symmetry and nothing more; this can be achieved by adding yet another auxiliary field $\phi(\sigma)$ and an extra term:

$$\mathcal{L}_d \longmapsto \mathcal{L}_d + i\epsilon^{ab} F_{ab}\,\phi, \quad F_{ab} = \partial_a A_b - \partial_b A_a, \quad \int \mathscr{D}\phi \; e^{-i\epsilon^{ab}F_{ab}\,\phi} \sim \delta\big[\epsilon^{ab}F_{ab}\big] \tag{39}$$

Which forces $F_{12} \equiv 0$ in the remaining path integral. Note that the only non-zero independent component of F_{ab} in 2D is F_{12}, therefore $F_{12} \equiv 0$ implies that $F_{ab} \equiv 0$, or $F = \mathrm{d}A = 0$. On the plane, this implies that $A = \mathrm{d}\lambda$, i.e. it is indeed pure gauge[6].

We can then proceed to integrate out A_a. Since $A = \mathrm{d}\lambda$, we can gauge fix $A \equiv 0$, and the action reduces to the original one:

$$S'[X, \phi = 0, A_a = 0] = S[X] \tag{40}$$

Following the Faddeev–Popov procedure, we find that the path integral also reduces to the original one, up to some additional gauge volume determinant Δ_{FP}, which is independent of X. This implies that the theory for the fields (X, ϕ, A_a) is, indeed, equivalent to that of the original string theory which has only the X fields.

(c) Following our discussions in (b), we see that:

$$\partial_a X^d + A_a = 0 + A_a - \partial_a\lambda, \quad \lambda = -\partial_a X^d, \tag{41}$$

Before completing the path integral, we perform a gauge transformation $A_a \mapsto A_a - \partial_a\lambda$, with $\lambda = -X^d$. Assuming that there is no anomaly, we can ignore the functional Jacobian of the transformation, and the path integral shall be gauge invariant; in this case, the $\partial_a X^d$ term is canceled precisely by the gauge transformation, which is equivalent to setting $X^d = 0$ in the action:

$$S[X^\mu, \phi, A] = S'\big[X^\mu, X^d = 0, \phi, A\big] \tag{42}$$

Furthermore, the A_a related parts in the Lagrangian is now nice and quadratic:

$$\begin{aligned}
\mathcal{L}_A &= \frac{\sqrt{g}}{4\pi\alpha'}\Big\{2\left(g^{ab}G_{\mu d} + i\epsilon^{ab}B_{\mu d}\right)A_a\,\partial_b X^\mu + g^{ab}G_{dd}\,A_a A_b + i\epsilon^{ab}F_{ab}\,\phi\Big\} \\
&= \frac{\sqrt{g}}{4\pi\alpha'}\Big\{2\left(g^{ab}G_{\mu d} - i\epsilon^{ab}B_{\mu d}\right)\partial_a X^\mu A_b + g^{ab}G_{dd}\,A_a A_b + 2i\epsilon^{ab}\phi\,\partial_a A_b\Big\} \\
&\sim \frac{1}{4\pi\alpha'}\Big\{2\left(\delta^{ab}G_{\mu d} - i\epsilon^{ab}B_{\mu d}\right)\partial_a X^\mu A_b + \delta^{ab}G_{dd}\,A_a A_b - 2i\epsilon^{ab}\,\partial_a\phi\,A_b\Big\} \\
&= \frac{1}{4\pi\alpha'}\Big\{2\Big(\left(\delta^{ab}G_{\mu d} - i\epsilon^{ab}B_{\mu d}\right)\partial_a X^\mu - i\epsilon^{ab}\,\partial_a\phi\Big)A_b + \delta^{ab}G_{dd}\,A_a A_b\Big\}
\end{aligned} \tag{43}$$

Here we've fixed the conformal gauge $g_{ab} = \delta_{ab}$ and integrated by parts, so that $\phi\,\partial_a A_b \mapsto -\partial_a\phi\,A_b$.

It is convenient to define[7]:

$$J^b = \frac{1}{G_{dd}}\Big(\left(\delta^{ab}G_{\mu d} - i\epsilon^{ab}B_{\mu d}\right)\partial_a X^\mu - i\epsilon^{ab}\,\partial_a\phi\Big) \tag{44}$$

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6 However, if there are punctures on the worldsheet, then there is non-trivial cohomology, and A need not be $\mathrm{d}\lambda$. Instead, the gauge field can have non-trivial holonomy around the cycles of the worldsheet. One can show that these holonomies are gauge trivial if ϕ has periodicity 2π. In this case, the partition function is again equivalent to the original one. Reference: *Tong*, String Theory; see also `arXiv:0812.4408`.

7 Reference: *Blumenhagen et al*, Basic Concepts of String Theory, Chapter 14.

The path integral over A_a can then be completed as a Gaussian integral. Note that this time we integrate out A_a first, leaving ϕ in place; therefore we do not impose any gauge fixing. We have:

$$\mathcal{L}_A = \frac{G_{dd}}{4\pi\alpha'}\Big\{2J^b A_b + \delta^{ab} A_a A_b\Big\} = \frac{G_{dd}}{4\pi\alpha'}\Big\{(A+J)^2 - \delta^{ab} J_a J_b\Big\}, \tag{45}$$

$$S = \int \mathrm{d}^2\sigma\,\big(\mathcal{L}_A + \mathcal{L}_0\big), \quad \int \mathscr{D}A_a\; e^{-S} \sim \det\left[\frac{G_{dd}\big(X^\mu(\sigma)\big)}{2\pi\alpha'}\right]^{-\frac{1}{2}} e^{-\tilde{S}}, \tag{46}$$

$$\tilde{S} = \int \mathrm{d}^2\sigma\left(\mathcal{L}_0 - \frac{G_{dd}}{4\pi\alpha'}\,J^a J_a\right), \tag{47}$$

If we identify $X^d \cong X^d + 2\pi$, then the radius of the X^d circle is given by $R(X^\mu) = \sqrt{G_{dd}}$. When $R^2 \gg \alpha'$ or $R \gg \sqrt{\alpha'}$, the above path integral approaches the classical limit, and its main contribution comes from the classical saddle $A_a = -J_a$, which is included in the $e^{-\tilde{S}}$ factor. The functional determinant is sub-leading and can be ignored.

Expand the action $\tilde{S}$ in terms of (X^μ, ϕ), we find that:

$$\tilde{S} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\left\{\Big(\delta^{ab}\tilde{G}_{MN} + i\epsilon^{ab}\tilde{B}_{MN}\Big)\,\partial_a\tilde{X}^M\partial_b\tilde{X}^N\right\}, \quad \tilde{X} = (\tilde{X}^\mu, \tilde{X}^d) = (X^\mu, \phi), \tag{48}$$

$$\tilde{G}_{dd} = \frac{1}{G_{dd}}, \quad \tilde{G}_{\mu d} = \frac{1}{G_{dd}}B_{\mu d}, \quad \tilde{G}_{\mu\nu} = G_{\mu\nu} - \frac{1}{G_{dd}}\big(G_{\mu d}G_{\nu d} - B_{\mu d}B_{\nu d}\big), \tag{49}$$

$$\tilde{B}_{\mu d} = \frac{1}{G_{dd}}G_{\mu d}, \quad \tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{1}{G_{dd}}\big(G_{\mu d}B_{\nu d} - B_{\mu d}G_{\nu d}\big), \tag{50}$$

i.e. we've found the T-dual theory with $\tilde{R} \propto \frac{1}{R}$. Rescale $\phi \mapsto \phi/\sqrt{\alpha'}$ and $G_{dd} \mapsto \alpha' G_{dd}$, we recover the usual result: $\tilde{R} = \frac{\alpha'}{R}$.

(d) Now we return to the determinant; roughly speaking, we have:

$$\det\left[\frac{G_{dd}\big(X^\mu(\sigma)\big)}{2\pi\alpha'}\right]^{-\frac{1}{2}} = \exp\left(-\frac{1}{2}\ln\det\left[\cdots\right]\right) \sim \exp\left(-\frac{1}{2}\,\mathrm{tr}\,\frac{\ln G_{dd}}{\alpha'}\right) \tag{51}$$

Which appears to add a term $\sim -\frac{1}{2}\ln G_{dd}$ in the Lagrangian.

However, the "det" and "tr" in the above equation are divergent and ill-defined, and would only make sense after some careful regularization[8], which was introduced by *Buscher* [1] and nicely reviewed by *Alvarez et al* [2]. The regularized determinant along with the Jacobian adds the following contribution in the Lagrangian:

$$-\alpha'\mathcal{R}\cdot\frac{1}{2}\ln\frac{G_{dd}}{\alpha'} \tag{52}$$

Which is equivalent to a dilaton shift:

$$\tilde{\Phi} = \Phi - \frac{1}{2}\ln\frac{G_{dd}}{\alpha'} \tag{53}$$

In the limit of constant size $R = \sqrt{G_{dd}}$, note that the string coupling $g_s \sim e^{\Phi_0}$, and we recover the usual result: $\tilde{g}_s = g_s\sqrt{\alpha'}/R$.

[8] I would like to thank 谷夏 for hints about this problem.

$\boxed{4}$ Only One Coupling Constant in String Theory[9]

(a) Consider the open string one-loop diagram, which is topologically a cylinder. To represent such geometry on the place, we start from the "rectangular" torus:

$$w \cong w + 2\pi \cong w + 2\pi\tau, \quad \tau = it \tag{54}$$

And identify under an *involution*, i.e. a reflection through the imaginary axis:

$$w' = -\bar{w}, \quad \Longrightarrow \quad 0 \leq \mathrm{Re}\, w \leq \pi \tag{55}$$

The amplitude is similar to the torus amplitude; first, with a fixed $t = -i\tau$, we have:

$$\left\langle \prod_i :e^{ik_i \cdot X_i}: \right\rangle_t = iC\,(2\pi)^d\,\delta^d(\textstyle\sum_i k_i)\,\exp\left\{ -\sum_{i<j} k_i \cdot k_j\, G'(w_i, w_j) - \frac{1}{2}\sum_i k_i^2 G'(w_i, w_i) \right\} \tag{56}$$

But with a different propagator G', which can be obtained via the method of images; this leads to a doubling of the exponents compared to the torus amplitude:

$$G'(w, w') = G'_{T^2}(w, w') + G'_{T^2}(w, -\bar{w}'), \tag{57}$$

$$\left\langle \prod_i :e^{ik_i \cdot X_i}: \right\rangle_t = iC\,(2\pi)^d\,\delta^d(\textstyle\sum_i k_i)\prod_{i<j} |W_{ij}(t)|^{2\times\alpha' k_i \cdot k_j} \tag{58}$$

Here $W_{ij}(t)$ is the "corrected" distance on T^2; as $w \to w'$ we have $W_{ij} \sim w_{ij} = w_i - w_j$.

On the other hand, the vacuum amplitude is given by:

$$Z = iV_d \int_0^\infty \frac{\mathrm{d}t}{2t}\left(8\pi^2\alpha' t\right)^{-d/2} \eta(it)^{-(d-2)} \tag{59}$$

The bc ghost contributions is included in the $|\eta(it)|^2 = \big(\eta(it)\big)^2$ factor; note that for $t > 0$, $\eta(it) > 0$. Combining this and $\langle \prod_i :e^{ik_i \cdot X_i}:\rangle_t$, we obtain the final n-tachyon amplitude:

$$\mathcal{A} = ig_o^n\,(2\pi)^d\,\delta^d(\textstyle\sum_i k_i)\displaystyle\int_0^\infty \frac{\mathrm{d}t}{2t}\left(8\pi^2\alpha' t\right)^{-d/2} \eta(it)^{-(d-2)} \prod_k \int_{\partial M} \mathrm{d}w_k \prod_{i<j} |W_{ij}(t)|^{2\times\alpha' k_i \cdot k_j} \tag{60}$$

Here ∂M is the two ends of the cylinder.

As is suggested in arXiv:0812.4408, it is convenient to introduce the following parametrization for the operator insertions at each boundary:

$$w_i = \frac{1 - (-1)^\sigma}{2}\,\pi + 2\pi i t \cdot x_i, \quad 0 \leq x_i \leq 1 \tag{61}$$

$\sigma = 0, 1$ labels the left and right boundary.

We want 2 insertions at each boundary, labeled by $i = 1, 2$, $\sigma = 0$, and $i = 3, 4$, $\sigma = 1$; the amplitude can then be reduced to:

$$\begin{gathered} \mathcal{A} = ig_o^n\,(2\pi)^d\,\delta^d(\textstyle\sum_i k_i)\displaystyle\prod_k 2\int_0^1 \mathrm{d}x_k \int_0^\infty \frac{\mathrm{d}t}{2t}\left(8\pi^2\alpha' t\right)^{-d/2} \eta(it)^{-(d-2)} (2\pi t)^n \prod_{i<j} |W_{ij}(x_{ij}, t)|^{2\alpha' k_i \cdot k_j}, \\ W_{ij}(x_{ij}, t) = \eta(it)^{-3}\,\vartheta_{1,2}\big(ix_{ij}t \,\big|\, it\big)\,\exp\left(-\pi x_{ij}^2 t\right) \end{gathered} \tag{62}$$

There is an additional factor of 2 since $\int_{\partial M} \mathrm{d}\omega = 2\int \mathrm{d}x$, which includes the contribution after exchange of the two ends $12 \leftrightarrow 34$. Also, $\vartheta_{1,2} = \vartheta_1$ or ϑ_2, depending on whether the vertex operators

---

[9] Reference: arXiv:0812.4408.

i and j are on the same boundary or not; this is because:

$$\vartheta_1\big(ix_{ij}t - \tfrac{1}{2}\,\big|\,it\big) = -\vartheta_2\big(ix_{ij}t\,\big|\,it\big) \tag{63}$$

The $\eta(it)^{-3}$ factor in W_{ij} can be further extracted using the on-shell condition $\alpha' k_i^2 = 1$. More specifically, we have:

$$\sum_{i<j} 2\alpha' k_i \cdot k_j = \alpha' \left(\sum_i k_i\right)^2 - \alpha' \sum_i k_i^2 = 0 - n = -n \tag{64}$$

Therefore, we have:

$$\begin{gathered}\mathcal{A} = ig_o^n\,(2\pi)^d\,\delta^d\big(\textstyle\sum_i k_i\big) \displaystyle\prod_k \int_0^1 \mathrm{d}x_k \int_0^\infty \frac{\mathrm{d}t}{t}\,\big(8\pi^2\alpha' t\big)^{-d/2}(2\pi t)^n \eta(it)^{\boldsymbol{3n}-(d-2)} \prod_{i<j} \big|W'_{ij}(x_{ij},t)\big|^{2\alpha' k_i\cdot k_j}, \\ W'_{ij}(x_{ij},t) = \vartheta_{1,2}\big(ix_{ij}t\,\big|\,it\big)\,\exp\big(-\pi x_{ij}^2 t\big)\end{gathered} \tag{65}$$

To further simplify the expression, collect all the numerical coefficients:

$$ig_o^n\,(2\pi)^d\,\big(8\pi^2\alpha'\big)^{-d/2}(2\pi)^n = ig_o^n\,(2\pi)^d\,2^{-d/2}(2\pi)^{-d}\alpha'^{-d/2}(2\pi)^n = ig_o^n\,(2\pi)^n\,2^{-d/2}\alpha'^{-d/2} \tag{66}$$

In our case $n = 4$ and $d = 26$.

The t integral can be magically simplified using modular transformations of the ϑ functions[10]; with $t = \frac{1}{u}$, we have:

$$\begin{aligned} F(x) &= \int_0^\infty \mathrm{d}t\, t^{n-1-d/2} \eta(it)^{3n-(d-2)} \prod_{i<j} \big|W'_{ij}(x_{ij},t)\big|^{2\alpha' k_i\cdot k_j} \\ &= \int_0^\infty \mathrm{d}u\, \eta(iu)^{3n-(d-2)} \prod_{i<j} |\vartheta_{1,4}(x_{ij}|iu)|^{2\alpha' k_i\cdot k_j} \end{aligned} \tag{67}$$

The amplitude can then be neatly written as:

$$\mathcal{A} = ig_o^n\,(2\pi)^n\,2^{-d/2}\alpha'^{-d/2}\,\delta^d\big(\textstyle\sum_i k_i\big) \displaystyle\prod_k \int_0^1 \mathrm{d}x_k\, F(x) \tag{68}$$

(b) The "long cylinder" limit corresponds to the $t \to 0$ contributions in the above amplitude. Note that the full amplitude is an integral over the moduli $t = \frac{1}{u}$,

$$F(x) = \int_0^\infty \mathrm{d}u\, f(x,u), \quad u = \frac{1}{t}, \quad f(x,u) = \eta(iu)^{3n-(d-2)} \prod_{i<j} |\vartheta_{1,4}(x_{ij}|iu)|^{2\alpha' k_i\cdot k_j}, \tag{69}$$

We need only look at the integrand $f(x,u)$ as $u \to \infty$, or $q \equiv e^{-2\pi u} \to 0$. In this case $f(x,u)$ can be expanded as power series; ϑ_4 contributions turn out to be sub-leading, hence the product only needs to go over i,j on the same side, denoted as $(i,j)_\sigma$. We have:

$$f(x,u) = q^{\frac{3n-(d-2)}{24}} \prod_{(i<j)_\sigma} |2\sin\pi x_{ij}|^{2\alpha' k_i\cdot k_j} q^{\frac{2\alpha' k_i\cdot k_j}{8}} \big(1 + \mathcal{O}(q)\big) \tag{70}$$

[10] Reference: arXiv:0812.4408 and *Polchinski*'s summary of ϑ function properties.

Again we can simplify using on-shell conditions and Mandelstam variables; we have:

$$
\begin{aligned}
\sum_{(i<j)_\sigma} 2\alpha' k_i \cdot k_j &= \sum_{i<j} 2\alpha' k_i \cdot k_j - \sum_{i,\sigma=0} \sum_{j,\sigma=1} 2\alpha' k_i \cdot k_j \\
&= -n - 2\alpha' \sum_{i,\sigma=0} k_i \sum_{j,\sigma=1} k_j \\
&= -n - 2\alpha' s, \quad s = -\Big(\sum_{i,\sigma=0} k_i\Big)^2 = -\Big(\sum_{i,\sigma=1} k_i\Big)^2
\end{aligned} \tag{71}
$$

Where s is the mass squared of the intermediate state propagating along the long cylinder, from one end to another. With this we find that:

$$
\begin{aligned}
f(x,u) &= q^{\frac{3n-(d-2)}{24} + \frac{-n-2\alpha' s}{8}} \prod_{(i<j)_\sigma} |2\sin \pi x_{ij}|^{2\alpha' k_i \cdot k_j} \big(1 + \mathcal{O}(q)\big) \\
&= q^{-1-\frac{\alpha' s}{4}} \prod_{(i<j)_\sigma} |2\sin \pi x_{ij}|^{2\alpha' k_i \cdot k_j} \big(1 + \mathcal{O}(q)\big)
\end{aligned} \tag{72}
$$

Upon integrating over u, each q^k in the power series above produces a pole in s:

$$
\int_0^\infty \mathrm{d}u \, q^{-1-\frac{\alpha' s}{4}} q^k \propto \frac{1}{k - 1 - \frac{\alpha' s}{4}} \propto \frac{1}{s - \frac{4}{\alpha'}(k-1)} \tag{73}
$$

Apparently every integer power k appears in the expansion, hence we have the full closed string spectrum at $s = \frac{4}{\alpha'}(k-1)$, $k = 0, 1, 2, \cdots$.

Consider the tachyon pole, i.e. $k = 0$, $s = -\frac{4}{\alpha'}$; this contribution is represented as two disk diagrams linked by a closed string tachyon propagator; each disk has 3 insertions, two incoming (or outgoing) open string tachyons and one outgoing (or incoming) closed string tachyon.

By unitarity of the 4-point amplitude[11], sum of all such factorized diagrams should be equal to the original cylinder diagram; therefore, the strength of the tachyon pole, calculated from such two-disk diagram, should be equal to our previous calculations from the cylinder diagram.

On the other hand, by unitarity of the 3-point amplitude, the incoming and the outgoing disk diagrams have the same contributions. Therefore, we should expect that the closed string tachyon pole strength is equal to the *square* of the disk amplitude with 3 insertions.

(c) For the disk diagram, we have two open string insertions at the boundary, and one close string insertion in the disk; when mapped to the upper half plane, we can use the 3 CKVs to fix the closed string insertion at $(z, \bar{z})$ and one open string insertions at x_1, while integrating over the position x_2 of the remaining open string insertion:

$$
\begin{aligned}
\mathcal{A}_D &= g_c g_o^2 e^{-\lambda} \int \mathrm{d}x_2 \left\langle : c\tilde{c}\, e^{ik\cdot X} : : c_1^x \, e^{ik_1 \cdot X_1} : : e^{ik_2 \cdot X_2} : \right\rangle \\
&= g_c g_o^2 \, iC \, (2\pi)^d \, \delta^d(\textstyle\sum_i k_i) \, |z - x_1||\bar{z} - x_1||z - \bar{z}| \\
&\quad \times \int \mathrm{d}x_2 \, |z - \bar{z}|^{\alpha' k^2/2} |x_1 - x_2|^{2\alpha' k_1 \cdot k_2} \prod_i |z - x_i|^{2\alpha' k \cdot k_i}
\end{aligned} \tag{74}
$$

[11] Reference: *Polchinski*, Chapter 9.

Again note the doubling of exponents due to the image charges. With on-shell conditions, we get:

$$\begin{aligned}\mathcal{A}_D &= iCg_c g_o^2\,(2\pi)^d\,\delta^d(\textstyle\sum_i k_i)\,|z-x_1|^{-2}|z-\bar{z}|^3\displaystyle\int \mathrm{d}x_2\,|x_1-x_2|^2|z-x_2|^{-4}\\ &= iCg_c g_o^2\,(2\pi)^d\,\delta^d(\textstyle\sum_i k_i)\,\cdot 4\pi\end{aligned}\tag{75}$$

Furthermore, C can be fixed by comparing the 4-tachyon and 3-tachyon open string disk amplitudes[12]; we get:

$$C = \frac{1}{\alpha' g_o^2},\quad \mathcal{A}_D = \frac{4\pi i g_c}{\alpha'}\,(2\pi)^d\,\delta^d(\textstyle\sum_i k_i)\tag{76}$$

Combined with the closed string tachyon propagator, the factorized diagram described in (b) is then given by:

$$\mathcal{A}_0' = \left(\frac{4\pi i g_c}{\alpha'}\right)^2 \frac{i}{s-(-\frac{4}{\alpha'})}\,(2\pi)^d\,\delta^d(\textstyle\sum_i k_i)\tag{77}$$

On the other hand, the tachyon pole in (b) is given by:

$$\begin{aligned}\mathcal{A}_0 &= i g_o^4\,(2\pi)^4\,2^{-d/2}\alpha'^{-d/2}\,\delta^d(\textstyle\sum_i k_i)\displaystyle\prod_k\int_0^1 \mathrm{d}x_k\,F(x)\\ &\simeq i g_o^4\,(2\pi)^4\,2^{-d/2}\alpha'^{-d/2}\,\delta^d(\textstyle\sum_i k_i)\displaystyle\prod_k\int_0^1 \mathrm{d}x_k\left(-\sin^2(\pi x_{12})\,\sin^2(\pi x_{34})\,\frac{2^5}{\pi\alpha'}\frac{1}{s-(-\frac{4}{\alpha'})}\right)\\ &= -i g_o^4\,(2\pi)^3\,2^{-d/2+4}\alpha'^{-d/2-1}\,\delta^d(\textstyle\sum_i k_i)\displaystyle\frac{1}{s-(-\frac{4}{\alpha'})}\end{aligned}\tag{78}$$

Imposing $\mathcal{A}_0' = \mathcal{A}_0$ gives our desired relation, with $l_s = \sqrt{\alpha'}$:

$$g_o^2 = 2^{3(d-2)/4}\pi^{(d-1)/2}\alpha'^{(d-2)/4}g_c = 2^{18}\pi^{25/2}l_s^{12}g_c\tag{79}$$

References

[1] T.H. Buscher. *Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models*. *Phys. Lett. B.* **201**:466–472, **1988**.

[2] Enrique Alvarez, Luis Alvarez-Gaume & Yolanda Lozano. *An Introduction to T duality in string theory*. *Nucl. Phys. B Proc. Suppl.* **41**:1–20, **1995**. arXiv: `hep-th/9410237`.

[12] Reference: *Polchinski*, Chapter 6.