Gravity №1

Bryan

Compiled @ 2021/06/05

1 Spacelike Geodesic for Poincaré AdS

The Poincaré AdS_{d+1} metric is given by:

$$\mathrm{d}s^2 = G_{IJ}\,\mathrm{d}X^I\,\mathrm{d}X^J = \frac{-\,\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 + \mathrm{d}z^2}{z^2} = \frac{\mathrm{d}x^2 + \mathrm{d}z^2}{z^2}, \tag{1.1}$$

$$\mathrm{d}x^2 = -\,\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,, \quad X^I \sim (t, \vec{x}, z) \sim (x^\mu, z), \tag{1.2}$$

Note that here we define $\mathrm{d}x^2$ using the flat metric η, while $\mathrm{d}X$ has an index that should be raised and lowered using the curved metric $G_{IJ} = \frac{1}{z^2}\,\eta_{IJ}$. Generally upper case tensors are handled with G_{IJ}, while lower case ones are handled with η. The full isometry of such spacetime is then given by $\mathrm{SO}(d,2)$ with $\frac{(d+2)(d+1)}{2}$ generators, including:

1. Among the $x^\mu \sim (x^0, \vec{x}) \equiv (t, \vec{x})$ directions:
   (a) $\frac{d(d-1)}{2}$ $\mathrm{SO}(d-1,1)$ rotations $x_\mu\partial_\nu - x_\nu\partial_\mu$, boosts included;
   (b) d translations ∂_μ;
2. Dilation with $X^I \sim (x^\mu, z) \mapsto \lambda X^I = \lambda\,(x^\mu, z)$, generated by $\Delta = X^I\partial_I = z\partial_z + x^\mu\partial_\mu$;
3. Special conformal transformations; see Appendix A for an intuitive derivation. We have:

$$k_\mu = (z^2 + x^2)\,\partial_\mu - 2x_\mu\Delta \tag{1.3}$$

By Noether's theorem, $Q_\Xi = V_I\Xi^I = G_{IJ}\Xi^I V^J$ is conserved along the geodesic; here V^I is the normalized tangent vector, while Ξ^I is some Killing vector of the spacetime, e.g. one from the list above. We can then write down the conserved charges along a geodesic γ in Poincaré AdS:

$$\begin{aligned} p_\mu &= G_{\mu I}V^I = V_\mu \\ m_{\mu\nu} &= x_\mu V_\nu - x_\nu V_\mu \\ \Delta &= X^I V_I = \frac{zV^z}{z^2} + x^\mu p_\mu \\ k_\mu &= (z^2 + x^2)\,p_\mu - 2x_\mu\Delta \end{aligned} \tag{1.4}$$

These are all integration constants along γ. Again note that X, V have indices that should be handled with G_{IJ}, in particular,

$$\mathrm{d}X_\mu = G_{\mu I}\,\mathrm{d}X^I = \frac{1}{z^2}\,\mathrm{d}x_\mu\,, \quad V^\mu = \frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda} = z^2 p^\mu \tag{1.5}$$

On the other hand, V^μ should be properly normalized, therefore:

$$\|V\|^2 = g_{IJ}V^I V^J = \frac{(V^z)^2}{z^2} + z^2 p^2 = \delta = 0, \pm 1, \tag{1.6}$$

For $p^2 = 0$, we have $\|V\|^2 = \frac{(V^z)^2}{z^2} \geq 0$, thus γ can be either spacelike or null, but not timelike. In fact, $\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}\lambda} = 0, \pm 1$, along with $\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = z^2 p^\mu$, we obtain:

$$p^2 = 0, \qquad \begin{aligned} \text{spacelike:}\ & z(\lambda) = z(0)\,e^{\pm\lambda}, && x^\mu(\lambda) = x^\mu(0) \pm z(0)^2\,p^\mu\,\tfrac{e^{\pm 2\lambda}-1}{2}, \\ \text{null:}\ & z(\lambda) = z(0), && x^\mu(\lambda) = x^\mu(0) + z(0)^2\,p^\mu\lambda, \end{aligned} \tag{1.7}$$

From now on we shall focus on the $p^2 \neq 0$ situation. Note that we can complete the square on the right hand side of the k^μ conservation such that:

$$k_\mu + p_\mu a^2 = p_\mu\big(z^2 + (x-a)^2\big), \quad \Delta = p_\mu a^\mu \tag{1.8}$$

For $p^\mu \neq 0$, we can always find some c_μ such that $c_\mu p^\mu \neq 0$, therefore we have:

$$z^2 + (x-a)^2 = \frac{c_\mu k^\mu}{c_\mu p^\mu} + a^2 = \left(\frac{L}{2}\right)^2 = \text{const.}, \quad \Delta = p_\mu a^\mu \tag{1.9}$$

We see that γ lands on a "sphere" centered at $z = 0, x = a$ in $\mathbb{R}^{d+1}$; with the Lorentzian metric, it is actually a hyperboloid. Note that for now the radius $\left(\frac{L}{2}\right)^2$ can actually be negative or zero; more specifically,

1. $L^2 > 0$: one-sheet hyperboloid;
2. $L^2 < 0$: two-sheet hyperboloid;
3. $L^2 = 0$: conic surface.

On the other hand, we can actually solve X^I completely by combining (1.5), (1.6); we have:

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = z^2 p^\mu, \quad \frac{\mathrm{d}z}{\mathrm{d}\lambda} = \pm z\sqrt{\delta - z^2p^2}, \quad \frac{\mathrm{d}x^\mu}{\mathrm{d}z} = \pm\frac{zp^\mu}{\sqrt{\delta - z^2p^2}}, \quad \delta - z^2p^2 = \left(\tfrac{V^z}{z}\right)^2 \geq 0 \tag{1.10}$$

$$\implies \quad \gamma \subset x^\mu = a^\mu \pm \frac{p^\mu}{p^2}\sqrt{\delta - z^2p^2}, \quad z^2 + (x-a)^2 = \frac{\delta}{p^2} = \left(\frac{L}{2}\right)^2 \tag{1.11}$$

This confirms our observation above, and further reveals that γ lies in a "plane" in the $\mathbb{R}^d$ subspace consisting of the x^μ coordinates. The behavior of γ is sensitive to the sign of δ and p^2; more specifically,

1. $\delta = -1$, i.e. for timelike γ, we must have $p^2 < 0$, and $z \geq \frac{1}{|\|p\||} = \frac{L}{2} > 0$, namely z is bounded from below. This means that timelike geodesic can never reach the asymptotic boundary $z \to 0$. In this case, γ is a section of the one-sheet hyperboloid.
2. $\delta = +1$, i.e. for spacelike γ, we can have $p^2 > 0$ or $p^2 < 0$.
   (a) For $p^2 > 0$, again we have $\frac{L}{2} = \frac{1}{\|p\|} > 0$, and γ is again a cross section of the one-sheet hyperboloid. However, now we have $z \leq \frac{L}{2}$, namely z is bounded from above, and:

   $$z \to 0, \quad x^\mu = a^\mu \pm \frac{p^\mu}{p^2} = a^\mu \pm \frac{L}{2}\,\widehat{p}^\mu, \quad \widehat{p}^\mu = \frac{p^\mu}{\|p\|} \tag{1.12}$$

   We can also nicely parametrize X^I in terms of the proper length λ; for convenience, set $x^\mu(0) = a^\mu$, $z(0) = \frac{L}{2}$, then we have:

   $$z(\lambda) = \frac{L}{2}\frac{1}{\cosh\lambda}, \quad x^\mu(\lambda) = a^\mu \pm \widehat{p}^\mu\frac{L}{2}\tanh\lambda \tag{1.13}$$

   (b) For $p^2 < 0$, we have $(\frac{L}{2})^2 = \frac{1}{p^2} < 0$, and γ is now a cross section of the two-sheet hyperboloid. Again as $z \to 0$ it lands at $x^\mu = a^\mu \pm \frac{L}{2}\,\widehat{p}^\mu$; however, $|x^\mu|$ grows with z and extends into the bulk instead of returning to the boundary, i.e. $z, |x^\mu| \to \infty$.

   The differential equation in this case is almost the same as in (a), but now we have to choose a different initial condition, since γ won't even reach $x = a$. However, we can

actually set $x(0), z(0) \to \infty$; we then have:

$$z(\lambda) = \left|\frac{L}{2}\right| \frac{1}{\sinh\lambda}, \quad x^\mu(\lambda) = a^\mu \pm \widehat{p}^\mu \left|\frac{L}{2}\right| \coth\lambda \tag{1.14}$$

This is an evidence that $z \to \infty$ might not be the "end of the world" after all; it's likely that $z \to \infty$ is only a horizon, since a spacelike geodesic γ can reach it within finite proper length λ.

In summary, there are 3 types of spacelike geodesics in Poincaré AdS, which closely resemble the 3 types of conic sections:

1. $p^2 = 0$: parabolic, given in (1.7), with one end going to $z \to 0$ and the other end going to $z \to \infty$; it takes infinite proper length λ for it to reach 0 or ∞.
2. $p^2 > 0$: hyperbolic, given in (1.11) and (1.14), also with one end going to $z \to 0$ and the other end going to $z \to \infty$, but it reaches ∞ within finite λ.
3. $p^2 > 0$: elliptic, given in (1.11) and (1.13), with both ends going to $z \to 0$, and it takes infinite proper length λ for it to reach either end.

Now consider two points x_1, x_2 near the boundary $z = \epsilon$ connected by a spacelike geodesic γ. This can only be the "elliptic type" discussed above. We have:

$$x^\mu = a^\mu \pm \widehat{p}^\mu \sqrt{\left(\frac{L}{2}\right)^2 - z^2}, \quad z^2 + (x-a)^2 = \left(\frac{L}{2}\right)^2, \tag{1.15}$$

$$a = \frac{x_1 + x_2}{2}, \quad \widehat{p} \propto x_2 - x_1 \tag{1.16}$$

It's length is then given by:

$$\begin{aligned} A &= \int_{z=-\epsilon}^{z=\epsilon} \frac{\mathrm{d}x^2 + \mathrm{d}z^2}{z^2} \\ &= \int_{-\Lambda}^{\Lambda} \frac{(\mathrm{d}\tanh\lambda)^2 + (\mathrm{d}\,\mathrm{sech}\,\lambda)^2}{(\mathrm{sech}\,\lambda)^2}, \quad \Lambda = \cosh^{-1}(\tfrac{L}{2\epsilon}) \\ &= \int_{-\Lambda}^{\Lambda} \mathrm{d}\lambda = 2\Lambda = 2\cosh^{-1}\left(\frac{L}{2\epsilon}\right) \sim 2\log\frac{L}{\epsilon}, \quad \epsilon \to 0 \end{aligned} \tag{1.17}$$

2 Einbein Action

The einbein action of a point particle is given by:

$$S[\eta, X] = \frac{1}{2}\int \mathrm{d}\tau \left(\eta^{-1}\dot{X}_\mu \dot{X}^\mu - \eta m^2\right) \tag{2.1}$$

Under worldline reparametrization: $\tau \mapsto \tau' = f(\tau)$, we have $X'(\tau') = X(\tau)$, i.e. X^μ transforms like a **scalar** under worldline diffeomorphism; $X(\tau) \mapsto X'(\tau) = X\big(f^{-1}(\tau)\big)$.

On the other hand, η should be treated like an einbein: $\eta = \sqrt{-\gamma}$, here $\gamma = \gamma_{\tau\tau}$ is the worldline metric; $\gamma < 0$ due to the Lorentzian signature. We have:

$$\eta = \sqrt{-\gamma} \longmapsto \eta' = \eta \det\frac{\partial\tau}{\partial\tau'} = \eta\frac{\partial\tau}{\partial\tau'}, \tag{2.2}$$

$$\eta^{-1} = -\sqrt{-\gamma}\,\gamma^{-1} \longmapsto (\eta')^{-1} = \eta^{-1}\left(\frac{\partial\tau}{\partial\tau'}\right)\left(\frac{\partial\tau'}{\partial\tau}\frac{\partial\tau'}{\partial\tau}\right) = \eta^{-1}\frac{\partial\tau'}{\partial\tau} \tag{2.3}$$

It is clear that the action is invariant under the transformation:

$$\begin{aligned} S' &= \frac{1}{2}\int \mathrm{d}\tau' \left((\eta')^{-1}\partial_{\tau'}X_\mu \partial_{\tau'}X^\mu - (\eta')m^2\right) \\ &= \frac{1}{2}\int \mathrm{d}\tau \, \frac{\partial\tau'}{\partial\tau}\left(\eta^{-1}\frac{\partial\tau'}{\partial\tau}\cdot\frac{\partial\tau}{\partial\tau'}\partial_\tau X_\mu \cdot \frac{\partial\tau}{\partial\tau'}\partial_\tau X^\mu - \eta\,\frac{\partial\tau}{\partial\tau'}\,m^2\right) = S \end{aligned} \tag{2.4}$$

We can eliminate η classically by placing it on shell:

$$0 = \frac{\delta S}{\delta\eta} = -\eta^{-2}\dot{X}_\mu\dot{X}^\mu - m^2, \quad \eta[X] = \frac{1}{m}\sqrt{-\dot{X}_\mu\dot{X}^\mu} \tag{2.5}$$

Substitute this back to the action, and we have:

$$\begin{aligned} S[X] = S[\eta = \eta[X], X] &= \frac{1}{2}\int \mathrm{d}\tau \left(m\,(-\dot{X}_\mu\dot{X}^\mu)^{-\frac{1}{2}}\dot{X}_\mu\dot{X}^\mu - m\,(-\dot{X}_\mu\dot{X}^\mu)^{+\frac{1}{2}}\right) \\ &= -m\int \mathrm{d}\tau\,\sqrt{-\dot{X}_\mu\dot{X}^\mu} \end{aligned} \tag{2.6}$$

3 Ricci Tensor for Static Spherical Metric

Consider the metric:

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + h(r)\,\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big) = \eta_{ab}\,e^a e^b \tag{3.1}$$

Here we've defined the following vierbein:

$$\begin{aligned} e^t &= \sqrt{f(r)}\,\mathrm{d}t \\ e^r &= \sqrt{h(r)}\,\mathrm{d}r \\ e^\theta &= r\,\mathrm{d}\theta \\ e^\phi &= r\sin\theta\,\mathrm{d}\phi \end{aligned} \tag{3.2}$$

A Derivation of the special conformal transformations

Special conformal transformations can be understood as translations conjugated by *inversions*. Note that $\frac{\mathrm{d}z^2}{z^2}$ is invariant under $z \mapsto \frac{1}{z}$; if we include the x^μ directions, we can consider:

$$\mathcal{I}\colon\ \chi^I \mapsto \frac{\chi^I}{\chi^2}, \quad \chi^2 = -t^2 + \vec{x}^2 + z^2, \tag{A.1}$$

$$\mathcal{I}^2 = \mathbb{1}, \quad \mathrm{d}s^2 \mapsto \left(\frac{\delta^I_J - 2\,\frac{\chi^I\chi_J}{\chi^2}}{\chi^2}\,\mathrm{d}\chi^J\right)^2 \Big/ \left(\frac{z}{\chi^2}\right)^2 = \frac{\mathrm{d}\chi^2}{z^2} = \mathrm{d}s^2 \tag{A.2}$$

We see that inversion $\mathcal{I}$ is indeed a (discrete) symmetry of the metric. Here we've defined yet another lower case variable $\chi^I \sim (x^\mu, z)$, which as a contravariant vector has the same components as X^I, but with an index that should be lowered by the flat metric η_{IJ}, i.e. $\chi_I = \eta_{IJ}\chi^J = \eta_{IJ}X^J$. The d

special conformal generators are then given by:

$$
\begin{aligned}
k_\mu &= \frac{\partial}{\partial a^\mu}\Big(\mathcal{I}\circ e^{a^\nu P_\nu}\circ\mathcal{I}\circ X^I\Big)_{a=0}\,\frac{\partial}{\partial X^I} \\
&= \frac{\partial}{\partial a^\mu}\left(\frac{\frac{\chi^I}{\chi^2}+a^I}{\left|\frac{\chi^J}{\chi^2}+a^J\right|^2}\right)_{a=0}\frac{\partial}{\partial X^I} \\
&= \frac{\partial}{\partial a^\mu}\left(\frac{\chi^I+a^I\chi^2}{1+2a^I\chi_I+a^2\chi^2}\right)_{a=0}\frac{\partial}{\partial X^I} \\
&= \chi^2\partial_\mu - 2x_\mu X^I\partial_I \\
&= \chi^2\partial_\mu - 2x_\mu\Delta
\end{aligned}
\tag{A.3}
$$