

1 Read *Polchinski* Sections 1.3 and 1.4:

Read, *mostly* understood. □

2 Spinning Closed String in AdS Space:

For a classical spinning string, we have Nambu–Goto action:

$$S_{NG} = -T \int d\tau d\sigma \sqrt{-\det \gamma_{ab}}, \quad \gamma_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \quad (1)$$

Here $G_{\mu\nu}$ is the spacetime metric. γ_{ab} can be treated as the induced metric on the worldsheet.

In AdS space we have:

$$ds^2 = R^2 (-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega^2) \quad (2)$$

Where $d\Omega^2$ is the metric of a unit $(d-2)$ -sphere S^{d-2} . For convenience let's define unit S^{d-2} metric G_{ij}^1 , and raise or lower the i, j, \dots indices using G_{ij}^1 instead of G_{ij} , i.e.,

$$G_{ij}^1 = G_{ij} / (R^2 \sinh^2 \rho), \quad i, j = 2, \dots, d-1 \quad (3)$$

Furthermore, we consider the special case that the closed string is *folded*, like a rubber band stretched along a line; in this case we can choose the worldsheet parameter $(\tau, \sigma) = (t, \rho)$ while $\Omega = \Omega(t, \rho) = \Omega(\tau, \sigma)$, which leads to the following decomposition:

$$\partial_a X^\mu = \delta_a^\mu + \delta_i^\mu \partial_a \Omega^i, \quad a = 0, 1, \quad i = 2, \dots, d-1, \quad (4)$$

$$\begin{aligned} \gamma_{ab} &= G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \\ &= G_{ab} + G_{ij} \partial_a \Omega^i \partial_b \Omega^j \\ &= G_{ab} + R^2 \sinh^2 \rho G_{ij}^1 \partial_a \Omega^i \partial_b \Omega^j \\ &= R^2 \left\{ \begin{pmatrix} -\cosh^2 \rho & \\ & 1 \end{pmatrix} + \sinh^2 \rho \begin{pmatrix} (\partial_a \Omega)^2 & \partial_a \Omega \cdot \partial_b \Omega \\ \partial_b \Omega \cdot \partial_a \Omega & (\partial_b \Omega)^2 \end{pmatrix} \right\} \end{aligned} \quad (5)$$

Here $\partial_a \Omega \cdot \partial_b \Omega \equiv \partial_a \Omega^i \partial_b \Omega_i \equiv G_{ij}^1 \partial_a \Omega^i \partial_b \Omega^j$, and we have:

$$\begin{aligned} \det \gamma_{ab} &= (R^2)^2 \left\{ \sinh^4 \rho \det (\partial_a \Omega^i \partial_b \Omega_i) \right. \\ &\quad \left. + \sinh^2 \rho ((\partial_a \Omega)^2 - (\partial_b \Omega)^2 \cosh^2 \rho) \right. \\ &\quad \left. - \cosh^2 \rho \right\}, \quad (6) \\ \sqrt{-\det \gamma_{ab}} &= R^2 \left\{ \cosh^2 \rho - \sinh^2 \rho ((\partial_a \Omega)^2 - (\partial_b \Omega)^2 \cosh^2 \rho) \right. \\ &\quad \left. - \sinh^4 \rho \det (\partial_a \Omega^i \partial_b \Omega_i) \right\}^{1/2} \end{aligned}$$

Mark the end points of the string with $\rho = r(t)$, then the total length of such closed folded string is $\ell = 4r$. We then have:

$$S = -4TR^2 \int dt \int_0^r d\rho \sqrt{\cosh^2 \rho - \sinh^2 \rho ((\partial_a \Omega)^2 - (\partial_b \Omega)^2 \cosh^2 \rho) - \sinh^4 \rho \det (\partial_a \Omega^i \partial_b \Omega_i)} \quad (7)$$

Further simplification comes from the fact that, due to rotational symmetry, the string's motion can be restricted in a plane where its position is characterized by some angle $\theta = \Omega^{i_0} \in \{\Omega^i\}_i$. In this case other angle parameters $\Omega^i|_{i \neq i_0} = 0$, and the action is further reduced to:

$$S = -4TR^2 \int dt \int_0^r d\rho \sqrt{\cosh^2 \rho - \sinh^2 \rho ((\partial_a \theta)^2 - (\partial_b \theta)^2 \cosh^2 \rho)} = \int dt \int_0^r d\rho \mathcal{L}, \quad (8)$$

$$\mathcal{L} = -4TR^2 \sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho}, \quad \omega = \partial_t \theta, \partial_\rho \theta = 0 \quad (9)$$

We consider the special solution $\theta = \omega t$, while in general the endpoint $r = r(t)$ could be dynamical; variation of the action w.r.t. $r(t)$ gives¹:

$$0 = \delta S = -4TR^2 \int dt \int_r^{r+\delta r} d\rho \sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho} = -4TR^2 \int dt \sqrt{\cosh^2 r - \omega^2 \sinh^2 r} \delta r, \quad (10)$$

$$\omega^2 = \frac{\cosh^2 r}{\sinh^2 r} = \coth^2 r \quad (11)$$

Note that if ω is constant, then r must be fixed by (11). Taking θ as the only dynamical variable, it is then straight-forward to write the energy E and angular momentum J for such folded closed string:

$$\omega = \dot{\theta}, \quad \Pi = \frac{\partial \mathcal{L}}{\partial \omega} = 4TR^2 \frac{\omega \sinh^2 \rho}{\sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho}}, \quad (12)$$

$$J = \int_0^r d\rho \Pi = 4TR^2 \int_0^r d\rho \frac{\omega \sinh^2 \rho}{\sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho}}, \quad (13)$$

$$E = \int_0^r d\rho (\Pi \omega - \mathcal{L}) = 4TR^2 \int_0^r d\rho \frac{\cosh^2 \rho}{\sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho}}, \quad (14)$$

In the large string limit, $r \rightarrow \infty$, $\omega = \coth r \rightarrow 1$. Expand in terms of $\epsilon = \omega - 1 > 0$, we find that $r = \frac{1}{2} \ln \left(1 + \frac{2}{\epsilon}\right) \sim \frac{1}{2} \ln \frac{2}{\epsilon}$, or alternatively, $e^{2r} \cdot \epsilon \sim 2$. With some help from MathematicaTM, we get:

$$\begin{aligned} E - J &= 4TR^2 \int_0^r d\rho \frac{\cosh^2 \rho - \omega \sinh^2 \rho}{\sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho}} = 4TR^2 \int_0^r d\rho \left(1 + \frac{\epsilon^2}{8} \sinh^2(2\rho) + \mathcal{O}(\epsilon^3)\right) \\ &= 4TR^2 \left(r \left(1 - \frac{\epsilon^2}{16} + \mathcal{O}(\epsilon^3)\right) + \mathcal{O}(1)\right) = \left(2TR^2 \ln \frac{2}{\epsilon}\right) \left(1 - \frac{\epsilon^2}{16} + \mathcal{O}(\epsilon^3)\right) \\ &\sim 2TR^2 \left(\ln \frac{2}{\epsilon}\right) \end{aligned} \quad (15)$$

Similarly, $J \sim 4TR^2 \int_0^r d\rho \sinh^2 \rho \sim TR^2 \left(\frac{2}{\epsilon}\right)$, this gives:

$$E - J \sim 2TR^2 \ln \frac{J}{TR^2} \quad (16)$$

■

¹ The above reasoning is confirmed in e.g. [arXiv:hep-th/0204051](https://arxiv.org/abs/hep-th/0204051).

3 Special Conformal Transformations:

$$x^\mu \xrightarrow{K(a)} \tilde{x}^\mu = \frac{x^\mu + x^2 a^\mu}{1 + 2a \cdot x + a^2 x^2} \quad (17)$$

(a) Under special conformal transformation $K(a)$, metric $\delta_{\mu\nu} \mapsto g_{\mu\nu}$ while:

$$g_{\alpha\beta} d\tilde{x}^\alpha d\tilde{x}^\beta = \delta_{\mu\nu} dx^\mu dx^\nu, \quad g_{\alpha\beta} = \delta_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} \quad (18)$$

To calculate this we have to know the inverse transformation $x = K^{-1}(a) \tilde{x}$. First, notice the following decomposition² of $K(a)$:

$$\tilde{x}^\mu = \frac{\frac{x^\mu}{x^2} + a^\mu}{\frac{1}{x^2} + \frac{2a \cdot x}{x^2} + a^2} = \frac{\frac{x^\mu}{x^2} + a^\mu}{\left| \frac{x^\mu}{x^2} + a^\mu \right|^2}, \quad (19)$$

$$\text{i.e. } K(a): x^\mu \xrightarrow{I} \frac{x^\mu}{x^2} \xrightarrow{T(a)} y^\mu = \frac{x^\mu}{x^2} + a^\mu \xrightarrow{I} \tilde{x}^\mu = \frac{y^\mu}{y^2}, \quad (20)$$

$$\text{i.e. } \frac{\tilde{x}^\mu}{\tilde{x}^2} = \frac{y^\mu}{y^2} \Big/ \frac{1}{y^2} = y^\mu = \frac{x^\mu}{x^2} + a^\mu \quad (21)$$

From (21), we see that the transformation parameter a^μ composes linearly: $K(b)K(a) = K(a+b)$, therefore $K^{-1}(a) = K(-a)$, and we have:

$$x^\mu = K(-a) \tilde{x}^\mu = \frac{\tilde{x}^\mu - \tilde{x}^2 a^\mu}{1 - 2a \cdot \tilde{x} + a^2 \tilde{x}^2} = \frac{\tilde{y}^\mu}{y^2}, \quad (22)$$

$$\begin{aligned} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} &= \frac{\partial x^\mu}{\partial \tilde{y}^\sigma} \frac{\partial \tilde{y}^\sigma}{\partial \tilde{x}^\alpha} = \left(\frac{\partial}{\partial \tilde{y}^\sigma} \frac{\tilde{y}^\mu}{\tilde{y}^2} \right) \frac{\partial}{\partial \tilde{x}^\alpha} \left(\frac{\tilde{x}^\sigma}{\tilde{x}^2} - a^\sigma \right) = \left(\frac{\partial}{\partial \tilde{y}^\sigma} \frac{\tilde{y}^\mu}{\tilde{y}^2} \right) \left(\frac{\partial}{\partial \tilde{x}^\alpha} \frac{\tilde{x}^\sigma}{\tilde{x}^2} \right) \\ &= (\tilde{y}^2 \delta_\sigma^\mu - 2\tilde{y}^\mu \tilde{y}_\sigma) (\tilde{x}^2 \delta_\alpha^\sigma - 2\tilde{x}^\sigma \tilde{x}_\alpha) / (\tilde{y}^4 \tilde{x}^4), \end{aligned} \quad (23)$$

$$\begin{aligned} g_{\alpha\beta} &\stackrel{(18)}{=} \delta_{\mu\nu} (\tilde{y}^2 \delta_\sigma^\mu - 2\tilde{y}^\mu \tilde{y}_\sigma) (\tilde{x}^2 \delta_\alpha^\sigma - 2\tilde{x}^\sigma \tilde{x}_\alpha) (\tilde{y}^2 \delta_\rho^\nu - 2\tilde{y}^\nu \tilde{y}_\rho) (\tilde{x}^2 \delta_\beta^\rho - 2\tilde{x}^\rho \tilde{x}_\beta) / (\tilde{y}^8 \tilde{x}^8) \\ &\stackrel{\Sigma_{\mu,\nu}}{=} \tilde{y}^{-4} \delta_{\sigma\rho} (\tilde{x}^2 \delta_\alpha^\sigma - 2\tilde{x}^\sigma \tilde{x}_\alpha) (\tilde{x}^2 \delta_\beta^\rho - 2\tilde{x}^\rho \tilde{x}_\beta) / \tilde{x}^8 \\ &\stackrel{\Sigma_{\sigma,\rho}}{=} \tilde{y}^{-4} \tilde{x}^{-4} \delta_{\alpha\beta} \end{aligned} \quad (24)$$

We see that $g_{\alpha\beta} = f(x) \delta_{\alpha\beta}$, with coefficient:

$$f(x) = \tilde{y}^{-4} \tilde{x}^{-4} \stackrel{(20)}{=} \frac{x^4}{\tilde{x}^4} \stackrel{(21)}{=} (1 + 2a \cdot x + a^2 x^2)^2 \quad (25)$$

□_(a)

(b) In 2D with $z = x^1 + ix^2$, $x^\mu \sim (z, \bar{z})$, we see from (21) that:

$$\frac{x^\mu}{x^2} \sim \frac{z}{|z|^2} = \frac{1}{\bar{z}} \mapsto \frac{1}{\bar{z}} + a, \quad \text{i.e. } z \mapsto w = \frac{1}{\frac{1}{\bar{z}} + a} = \frac{z}{1 + z\bar{a}} \quad (26)$$

Expand in the $\bar{a} \rightarrow 0$ limit, we find that $w = z(1 - z\bar{a} - \dots) \sim z - z^2 \bar{a}$, i.e. it is generated by:

$$K_{\bar{z}} = -z^2 \partial_z = -z^2 \partial, \quad \partial \equiv \partial_z \quad (27)$$

² See *Di Francesco et al*, and also github.com/davidsd/ph229.

Note that when considering non-holomorphic functions, we have to consider (z, \bar{z}) as *two* independent variables; hence the anti-holomorphic transformation $\bar{z} \mapsto \bar{w} = \frac{\bar{z}}{1+\bar{z}a} \sim \bar{z} - \bar{z}^2 a$ provides another degree of freedom, namely:

$$K_\mu \sim (K_{\bar{z}} = -z^2 \partial, K_z = -\bar{z}^2 \bar{\partial}), \quad (28)$$

$$\partial \equiv \partial_z, \quad \bar{\partial} \equiv \partial_{\bar{z}}$$

Similarly, for translation $z \mapsto z + a$ and its conjugate, we have $P_\mu \sim (P_z = \partial, P_{\bar{z}} = \bar{\partial})$. However, dilation and rotation are both encoded in a complex rescaling $z \mapsto \lambda z$, $\lambda = re^{i\theta} \in \mathbb{C}$; we have:

$$z \mapsto \lambda z, \quad \lambda = re^{i\theta} \in \mathbb{C}, \quad \begin{aligned} \delta r &\longleftrightarrow D = z \partial + \bar{z} \bar{\partial}, \\ \delta \theta &\longleftrightarrow M = i(z \partial - \bar{z} \bar{\partial}), \end{aligned} \quad (29)$$

In summary, we have $\text{span}_{\mathbb{R}} \{P_\mu, K_\mu, D, M\} = \mathfrak{so}(3, 1)$ generating the “global” transformation subgroup of the 2D conformal group; here, the $\mathfrak{so}(3, 1)$ boost is a linear combination³ of P_μ and K_μ . More specifically, in 2D any holomorphic or anti-holomorphic function gives a conformal transformation, hence the (classical) 2D conformal group is generated by:

$$\ell_m = z^{m+1} \partial, \quad \bar{\ell}_m = \bar{z}^{m+1} \bar{\partial}, \quad m \in \mathbb{Z} \quad (30)$$

i.e. the *Witt algebra* (or Virasoro algebra \mathbf{Vir}_c with $c = 0$). It is clear that a (complexified) $\mathfrak{so}(3, 1)$ lives inside \mathbf{Vir}_c , i.e.,

$$\begin{aligned} \mathfrak{so}(3, 1)^\mathbb{C} &= \text{span}_{\mathbb{C}} \{P_\mu, K_\mu, D, M\} \\ &= \text{span}_{\mathbb{C}} \{\ell_m, \bar{\ell}_m \mid m = 0, \pm 1\} = \mathfrak{sl}(2, \mathbb{R})^\mathbb{C} \oplus_{\mathbb{C}} \mathfrak{sl}(2, \mathbb{R})^\mathbb{C} \subset \mathbf{Vir}_c \end{aligned} \quad (31)$$

■

4 *bc CFT*:

$$S = \frac{1}{2\pi} \int d^2 z b \bar{\partial} c \quad (32)$$

Stress tensor of a theory can be obtained via variation over the metric, or equivalently, over the fields ϕ^i with $\delta\phi$ induced by some *local* spacetime translation $x^\mu \mapsto x^\mu + \delta x^\mu$, $\delta x^\mu = \epsilon(x) a^\mu$. Here $\epsilon(x)$ is any compactly supported bump function, centered around some point x_0 .

In 2D, we have $\mu = z, \bar{z}$; for $\phi(z, \bar{z})$ with conformal weight (h, \bar{h}) , consider $z \mapsto z'$, $\bar{z} \mapsto \bar{z}'$. For convenience, let's first consider a generic variation $\delta z = \epsilon(z, \bar{z})$ before restricting to spacetime translation; we have:

$$\phi'(z', \bar{z}') = \left(\frac{dz'}{dz}\right)^{-h} \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}), \quad (33)$$

$$\tilde{\delta}\phi = (-h \partial \epsilon - \bar{h} \bar{\partial} \bar{\epsilon}) \phi, \quad (34)$$

$$\delta\phi = \tilde{\delta}\phi - \frac{\partial\phi}{\partial x^\mu} \delta x^\mu = (-h \partial \epsilon - \bar{h} \bar{\partial} \bar{\epsilon}) \phi - \epsilon \partial \phi - \bar{\epsilon} \bar{\partial} \phi, \quad (35)$$

Here we use $\tilde{\delta}\phi$ to denote the “internal” variation related to the conformal weights.

³ See e.g. github.com/davidsd/ph229.

Note that $\phi = b, c$ are anti-commuting Grassmann numbers, variation of the action gives:

$$\begin{aligned}\delta S[b, c, \delta b, \delta c] &= \frac{1}{2\pi} \int d^2 z (\delta b \bar{\partial} c + b \bar{\partial} \delta c) \\ &= \frac{1}{2\pi} \int d^2 z (-\bar{\partial} c \delta b - \bar{\partial} b \delta c) + \frac{1}{2\pi} \int d^2 z \bar{\partial}(b \delta c)\end{aligned}\quad (36)$$

For *unknown* b, c and arbitrary $\delta b, \delta c$, the second term is reduced to a boundary term at infinity and can be dropped; imposing $\delta S = 0$ gives the equation of motion (EOM): $\bar{\partial} b = \bar{\partial} c = 0$.

On the other hand, for *on-shell* b, c and compactly supported $\phi = \delta b, \delta c$ given in (35), the first term in (36) vanishes while $\delta S_0 = 0$ still holds; this gives:

$$\begin{aligned}0 = \delta S_0 &= \frac{1}{2\pi} \int d^2 z \bar{\partial}(b \delta c) = \frac{1}{2\pi} \int d^2 z \bar{\partial}(-(1-\lambda)bc\partial\epsilon - b\partial c\epsilon) \\ &= \frac{1}{2\pi} \int d^2 z (-(1-\lambda)bc\bar{\partial}\partial\epsilon - b\partial c\bar{\partial}\epsilon)\end{aligned}\quad (37)$$

Here we've distributed the $\bar{\partial}$ operator and dropped all terms that vanish automatically by EOM. Next we shall collect the $\partial\epsilon, \bar{\partial}\epsilon$ terms; integrating by parts on the first integrand gives:

$$\begin{aligned}0 = \delta S_0 &= \frac{1}{2\pi} \int d^2 z ((1-\lambda)\partial(bc) - b\partial c)\bar{\partial}\epsilon \\ &= \frac{1}{2\pi} \int d^2 z ((\partial b)c - \lambda\partial(bc))\bar{\partial}\epsilon \\ &= -\frac{1}{2\pi} \int d^2 z \epsilon(z, \bar{z}) \partial_{\bar{z}}((\partial b)c - \lambda\partial(bc))\end{aligned}\quad (38)$$

Notice that we have obtained a conserved current using a generic $\delta z = \epsilon(z, \bar{z}), \delta \bar{z} = \bar{\epsilon}(z, \bar{z})$; by setting $\epsilon = \epsilon(z)$, we get a energy momentum tensor⁴:

$$T(z) = :(\partial b)c: - \lambda\partial(bc) \quad (39)$$

Normal ordering is added manually to remove singular terms.

To compute TT OPE, we need the OPE of $b(z)c(0)$; this is obtained by examining the following path integral, which is zero since the integrand is a total functional derivative:

$$0 = \int \mathcal{D}b \mathcal{D}c \frac{\delta}{\delta\phi} (e^{-S} \psi) \quad (40)$$

Taking $\phi, \psi = b, c$, this generates operator equations such as $\bar{\partial} b(z)c(0) = 2\pi\delta^2(z, \bar{z})$. Note that $\bar{\partial}(\frac{1}{z}) = 2\pi\delta^2(z, \bar{z})$, which gives:

$$b(z)c(0) \sim c(z)b(0) \sim \frac{1}{z}, \quad b(z)b(0) \sim 0 \sim c(z)c(0) \quad (41)$$

With the bc OPE in hand, the TT OPE is computed directly with brute force, by repeatedly applying Wick's theorem. This gives:

$$T(z)T(0) \sim \frac{-6\lambda^2 + 6\lambda - 1}{z^4} + \dots \quad (42)$$

In general we have $-6\lambda^2 + 6\lambda - 1 = \frac{c}{2}$; for $\lambda = 2$ this gives $c = -26$. ■

⁴ Note that the energy momentum tensor obtained in this way is generally *not* unique: it can be off by a boundary term or any term that vanish on-shell; see Luboš' comment at physics.stackexchange.com/a/96100, also [arXiv:1601.03616](https://arxiv.org/abs/1601.03616). However, it is possible to fix this redundancy by considering Tb OPE and match its conformal dimension. I would like to thank 林般 for pointing this out.

5 Free Fermion CFT:

$$S = \int d^2z \psi_i \bar{\partial} \psi^i, \quad \psi^i = \psi_i^*, \quad \psi_i = \psi_i(z) \quad (43)$$

(a) Mode expansion of such chiral fermion is given by:

$$\psi_i = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{b_{ik}}{z^{k+\frac{1}{2}}}, \quad b_{ik} = \frac{1}{2\pi i} \oint dz z^{k-\frac{1}{2}} \psi_i \quad (44)$$

Canonical quantization is achieved by simply imposing anti-commutation relations; this is justified by mapping the system onto a cylinder, then b_{ik} 's indeed map to modes on the spatial circle⁵. The only non-zero commutators are:

$$\{b_{ik}, b_q^{j\dagger}\} = \delta_{k+q,0} \delta_i^j \quad (45)$$

This gives the only non-zero 2-point functions:

$$\begin{aligned} \langle \psi_i(z) \psi^j(w) \rangle &= \sum_{k,q \in \mathbb{Z} + \frac{1}{2}} \frac{1}{z^{k+\frac{1}{2}}} \frac{1}{w^{q+\frac{1}{2}}} \langle b_{ik} b_q^{j\dagger} \rangle \\ &= \sum_{k,q \in \mathbb{Z} + \frac{1}{2}} \frac{1}{z^{k+\frac{1}{2}}} \frac{1}{w^{q+\frac{1}{2}}} \langle 0 | \{b_{ik}, b_q^{j\dagger}\} | 0 \rangle = \frac{\delta_i^j}{z-w} \end{aligned} \quad (46)$$

Note that $b_k^i |0\rangle = 0, \forall k \geq \frac{1}{2}$.

(b)(c) Combining two ψ expansions gives the mode expansion of $J_i^j = :\psi_i(z) \psi^j(z):$, namely:

$$J_i^j(z) = \sum_{k \in \mathbb{Z}} \frac{(J_i^j)_k}{z^{k+1}}, \quad (J_i^j)_k = \sum_{q \in \mathbb{Z} + \frac{1}{2}} :b_{iq} b_{k-q}^{j\dagger}: \quad (47)$$

It is in fact more convenient to obtain the JJ OPE first, and then use it to find the $[J_0, J_0]$ mode commutator⁶; note that $\psi_i(z) \psi^j(w)$ contraction gives $\frac{\delta_i^j}{z-w}$, we have:

$$J_i^j(z) J_k^l(w) \sim \frac{\delta_i^l \delta_k^j}{z^2} + \frac{\delta_k^j J_i^l(w) - \delta_i^l J_k^j(w)}{z}, \quad (48)$$

$$[(J_i^j)_0, (J_k^l)_0] = \frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz J_i^j(z) J_k^l(w) = \delta_i^l (J_k^j)_0 - \delta_k^j (J_i^l)_0 \quad (49)$$

(d) Similar to bc CFT, we have:

$$T(z) = \frac{1}{2} (: \psi_i \partial \psi^i : - : \partial \psi_i \psi^i :), \quad T(z) T(w) \sim \frac{n/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \quad (50)$$

With each (complex) field contributing $\frac{1}{2} \times 2$ central charge⁷.

⁵ This can be proven rigorously by considering operator equations like in the bc CFT problem.

⁶ I would like to thank 谷夏 for providing this hint.

⁷ In fact a complex (Dirac) fermion can be “treated like” (*dual to*) a boson; this is *bosonization*.

(e) For real fermions, there is an additional reality condition:

$$\psi^i = \psi_i^\star = \psi_i \quad (51)$$

The canonical quantization still holds without the extra adjoint, same as the 2-point function:

$$\langle \psi_i(z) \psi_j(w) \rangle = \frac{\delta_{ij}}{z - w} \quad (52)$$

Similar holds for $J_{ij} = :\psi_i \psi_j:$ and its OPE, but we no longer need to distinguish upper/lower indices; we have:

$$J_{ij}(z) J_{kl}(0) \sim \frac{-\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}}{z^2} + \frac{-\delta_{ik} J_{jl}(0) + \delta_{il} J_{jk}(0) + \delta_{jk} J_{il}(0) - \delta_{jl} J_{ik}(0)}{z} \quad (53)$$

$$[(J_{ij})_0, (J_{kl})_0] = -\delta_{ik} (J_{jl})_0 + \delta_{il} (J_{jk})_0 + \delta_{jk} (J_{il})_0 - \delta_{jl} (J_{ik})_0 \quad (54)$$

This is precisely the $\mathfrak{o}(n)$ algebra. ■