

1 Equivalence of categories is fully faithful:

$F: \mathcal{C} \rightarrow \mathcal{D}$ equivalence of categories, i.e. $\exists G: \mathcal{D} \rightarrow \mathcal{C}$, s.t.

$$G \circ F \simeq \mathbb{1}_{\mathcal{C}}, \quad F \circ G \simeq \mathbb{1}_{\mathcal{D}} \quad (1)$$

Here “ \simeq ” means naturally isomorphic as functors, i.e.,

$$\exists \tau: G \circ F \Rightarrow \mathbb{1}_{\mathcal{C}}, \quad \sigma: F \circ G \Rightarrow \mathbb{1}_{\mathcal{D}} : \text{natural isomorphisms} \quad (2)$$

By the definition of natural transformation, for $f \in \text{Hom}_{\mathcal{C}}(A, B)$, we have:

$$\begin{array}{ccc} G \circ F(A) & \xrightarrow{G \circ F(f)} & G \circ F(B) \\ \tau_A \downarrow & & \downarrow \tau_B \\ A & \xrightarrow{f} & B \end{array}$$

$$\tau_B \circ (G \circ F)(f) \circ \tau_A^{-1} = f, \quad \forall f \in \text{Hom}_{\mathcal{C}}(A, B) \quad (3)$$

Here $\tau_{A,B}$ are isomorphisms, which means that $G \circ F$ must be a bijection between hom-sets, which further implies that F is injective and G is surjective. Switch the roles of F, G , we find that G is injective and F is surjective. Therefore, F, G are both fully faithful. ■

2 Forgetful functors to Set are often representable:

For $F: \underline{\mathbf{Group}} \rightarrow \underline{\mathbf{Set}}$, consider the free group generated by a single element \mathbb{Z} . We have:

$$\begin{aligned} \text{Hom}(\mathbb{Z}, -): \underline{\mathbf{Group}} &\longrightarrow \underline{\mathbf{Set}} \\ G &\longmapsto \text{Hom}(\mathbb{Z}, G) \end{aligned} \quad (4)$$

This is a covariant functor representable by \mathbb{Z} .

On the other hand, $\text{Hom}(\mathbb{Z}, G)$ consists of group homomorphisms:

$$\text{Hom}(\mathbb{Z}, G) = \left\{ \begin{array}{c} \mathbb{Z} \rightarrow G \\ 1 \mapsto g \end{array} \middle| g \in G \right\} \quad (5)$$

More specifically, to fix any $\mathbb{Z} \rightarrow G$, we need only assign its generator¹ $1 \mapsto g$. Image of any other \mathbb{Z} element is generated automatically from the group law, without further specifications. This means that the hom-set is in one-to-one correspondence with G elements (as a set). Therefore, $F \cong \text{Hom}_{\underline{\mathbf{Group}}}(\mathbb{Z}, -)$, i.e. forgetful $F: \underline{\mathbf{Group}} \rightarrow \underline{\mathbf{Set}}$ is representable by \mathbb{Z} . □

Similarly, for $F: \underline{\mathbf{Ring}} \rightarrow \underline{\mathbf{Set}}$, the free object generated by some generic element x is $\mathbb{Z}[x]$, the polynomial ring in one variable; we have:

$$F \cong \text{Hom}_{\underline{\mathbf{Ring}}}(\mathbb{Z}[x], -), \quad \text{Hom}_{\underline{\mathbf{Ring}}}(\mathbb{Z}[x], R) = \left\{ \begin{array}{c} \mathbb{Z}[x] \rightarrow R \\ x \mapsto r \end{array} \middle| r \in R \right\} \quad (6)$$

Lesson: Forgetful $\underline{\mathbf{Cat}} \rightarrow \underline{\mathbf{Set}}$ are often representable by the free object in $\underline{\mathbf{Cat}}$. ■

¹Note that $0 \in \mathbb{Z}$ is the group identity of addition group \mathbb{Z} , not $1 \in \mathbb{Z}$.

3 Properties of contractible space:

(a) X contractible: $\mathbb{1}_X \simeq f_0: X \rightarrow X$ some constant map, $f_0(X) = \{x_0\}$. We can restrict the codomain of f_0 so that $f_0: X \rightarrow \{x_0\}$, in this way we have:

$$X \xrightarrow{f_0} \{x_0\} \hookrightarrow X \simeq \mathbb{1}_X, \quad (7.1)$$

$$\{x_0\} \hookrightarrow X \xrightarrow{f_0} \{x_0\} \simeq \mathbb{1}_{\{x_0\}}, \quad (7.2)$$

This means that $f_0: X \rightarrow \{x_0\}$ isomorphic in $\mathbf{hTop} = \mathbf{Top}/\simeq$, which is precisely the definition of homotopic equivalence $X \simeq \{x_0\}$. (\Rightarrow)

On the other hand (\Leftarrow), if $X \simeq \{x_0\}$, there exists some $f_0: X \rightarrow \{x_0\}$ that fulfills (7). We can then extend the codomain s.t. $f_0: X \rightarrow X$, in this way (7.1) reads $f_0 \simeq \mathbb{1}_X$, i.e. X is contractible. Therefore, X contractible iff. homotopic equivalent to a single point. $\blacksquare_{(a)}$

(b) $\forall X$: Topological space, we can define its *cone* as²:

$$CX = (X \times I)/(X \times \{0\}), \quad I = [0, 1] \quad (8)$$

i.e. gluing together one end of the cylinder $X \times I$. Naturally $X \subset CX$ as a subspace; now we show that CX is contractible. Using (a), we need only show that $\mathbb{1}_{CX} \simeq f_0$ some constant map.

In fact, any point in CX can be uniquely labeled by $[x, h] \in X \times I$, with the exception of the vertex $v \sim [x, 0] \sim [x', 0]$, $\forall x, x' \in X$. We can then construct a homotopy F by shrinking the cone towards the vertex v :

$$\begin{aligned} F: CX \times I &\rightarrow CX, & F([x, h], t) &= [x, h \cdot t], \\ F|_{CX \times 0} &= v = \text{const}, & F|_{CX \times 1} &= \mathbb{1}_X \end{aligned} \quad (9)$$

This confirms that $\mathbb{1}_{CX} \simeq v$: constant map. By (a), CX is contractible. $\blacksquare_{(b)}$

(c) For $Y \simeq \{y_0\}$ contractible, given any $g: X \rightarrow Y$, we can deform the image $g(X) \subset Y$ to a single point, hence $g \simeq y_0$: constant map. More precisely, we have:

$$\exists G: X \times I \rightarrow Y, \quad \text{s.t.} \quad G|_{X \times 0} = y_0 = \text{const}, \quad G|_{X \times 1} = g \quad (10)$$

Such G can be explicitly constructed using $\mathbb{1}_Y \simeq y_0$:

$$F: Y \times I \rightarrow Y, \quad F|_{Y \times 0} = y_0 = \text{const}, \quad F|_{Y \times 1} = \mathbb{1}_Y, \quad (11)$$

$$G(x, t) = F(g(x), t) \quad (12)$$

In summary, we have proven that $g \simeq y_0$, $\forall g \in \text{Hom}_{\mathbf{Top}}(X, Y)$. By definition, this means that $\text{Hom}_{\mathbf{hTop}}(X, Y) = \text{Hom}_{\mathbf{Top}}(X, Y)/\simeq = \{[y_0]\}$ a single point. $\blacksquare_{(c)}$

(d) For $X \simeq \{x_0\}$ contractible, similar to (11), we have homotopy $F: X \times I \rightarrow X$. Given any $f: X \rightarrow Y$, the composition $f \circ F: X \times I \rightarrow Y$ yields $f \simeq f(x_0)$: constant map.

²See Wikipedia: *Cone (topology)*.

Furthermore, for Y : path connected, there is a path $\gamma: I \rightarrow Y$ connecting $f(x_0)$ and some $y_0 \in Y$, therefore $f(x_0) \simeq y_0: X \rightarrow Y$ constant maps. More precisely, we have:

$$\gamma: I \rightarrow Y, \quad \gamma(0) = y_0, \quad \gamma(1) = f(x_0), G: X \times I \rightarrow Y, \quad G(x, t) = \gamma(t) \quad (13)$$

Which gives $f(x_0) \simeq y_0, \forall f$, independent of the choice of f . This means that $f \simeq f(x_0) \simeq y_0$: constant map, therefore $\text{Hom}_{\mathbf{hTop}}(X, Y) = \{[y_0]\}$ a single point. $\blacksquare_{(d)}$

4 Example of homotopic inequivalence³:

$$X = \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \right\}, \quad Y = \{0\} \cup \mathbb{Z}_+ \quad (14)$$

$X, Y \subset \mathbb{R}$: subspace topology

Assume $X \simeq Y$, then similar to (7), we have $Y \xrightarrow{g} X \xrightarrow{f} Y \simeq \mathbb{1}_Y$. However, note that Y has discrete topology, in such case any map $f \circ g$ homotopic to $\mathbb{1}_Y$ must be $\mathbb{1}_Y$ itself: $f \circ g = \mathbb{1}_Y$.

More specifically, consider:

$$F: Y \times I \rightarrow Y, \quad F|_{Y \times 0} = f \circ g, \quad F|_{Y \times 1} = \mathbb{1}_Y \quad (15)$$

Any point $n \in Y$ is both open and closed, therefore its pre-image $F^{-1}(n) \subset Y \times I$ is also both open and closed, and by $F|_{Y \times 1} = \mathbb{1}_Y$ we know that $F(y, 1) = y, (y, 1) \in F^{-1}(y)$, therefore the only possibility is that $F(\{y\} \times I) = y$, i.e. $f \circ g = \mathbb{1}_Y$, which implies that g is injective and f is surjective.

However, $f: X \rightarrow Y$ cannot be surjective due to the complication around $0 \in X$. Consider $f^{-1}(f(0)) \ni 0$, since $f(0) \in Y$ both open and closed, $f^{-1}(f(0)) \subset X$ must also be both open and closed. But any open set $U \subset X$ is induced via subspace topology $X \subset \mathbb{R}$; for $0 \in U \subset X \subset \mathbb{R}$, U must contain ∞ -many elements:

$$\left\{ \frac{1}{n} \mid n \geq N_0 \right\} \subset U \subset f^{-1}(f(0)), \quad \text{for some } N_0, \text{ for any } U \ni x \quad (16)$$

Hence $f(X) = f(0) \cup f(\{\frac{1}{n} \mid n < N_0\})$, $f(X) \subset Y$ a finite set, i.e. $f: X \rightarrow Y$ is never surjective. Therefore, $X \not\simeq Y$ by contradiction. \blacksquare

³This proof is produced thanks to helpful insights from 谷夏 and 於子雄.

5 Fundamental group of topological group is abelian⁴:

From a categorical point of view, the fundamental group $\pi_1(G)$ of a topological group G can be seen as a functor:

$$G \in \underline{\mathbf{TopGroup}} \hookrightarrow \underline{\mathbf{Top}} \xrightarrow{\pi_1} \underline{\mathbf{Group}} \ni \pi_1(G) \quad (17)$$

$\underline{\mathbf{TopGroup}} \subset \underline{\mathbf{Top}}$ is a subcategory with additional group structure, i.e. $(G, \cdot) \in \underline{\mathbf{TopGroup}}$ is a *group object*⁵ in $\underline{\mathbf{Top}}$, with “ \cdot ” denoting its product operation $(\cdot): G \times G \rightarrow G$. Correspondingly, $\pi_1(\underline{\mathbf{TopGroup}})$ should be *group objects of $\underline{\mathbf{Group}}$* , which have an *additional* group structure $(\star) = \pi_1(\cdot)$, along with the usual group product “ $*$ ” in $\underline{\mathbf{Group}}$.

In total, we have three different group structures (represented by their product operation):

$$(\cdot): G \times G \rightarrow G, \quad (18)$$

$$(*): \pi_1(G) \times \pi_1(G) \rightarrow \pi_1(G), \quad (19)$$

$$(\star) = \pi_1(\cdot): \pi_1(G) \times \pi_1(G) \rightarrow \pi_1(G), \quad (20)$$

Note that $\pi_1(G) = \text{Aut}_{\Pi_1(G)} \mathbb{1}_G$, i.e. loop classes $[\gamma]$ in G ; $(*)$ is defined as joining two loops, while $(\star) = \pi_1(\cdot)$ is defined as the translation of loop classes by pointwise group product (\cdot) ,

$$[\gamma_1] \star [\gamma_2] = [\gamma_1 \cdot \gamma_2] \quad (21)$$

With the above definitions, we observe that:

$$([\gamma_1] \star [\gamma_2]) * ([\eta_1] \star [\eta_2]) = ([\gamma_1] * [\eta_1]) * ([\gamma_2] * [\eta_2]) \quad (22)$$

By definition, they are both equal to $[(\gamma_1 \cdot \gamma_2) * (\eta_1 \cdot \eta_2)]$. What’s surprising is that by using only the group axioms and “distributive law” (22), we can show that (\star) and $(*)$ must always coincide: $(\star) = (*)$, and they have to be in fact, commutative. This is the *Eckmann–Hilton argument*⁶.

Proof of this argument is straight-forward; first, observe that the units of the two operations coincide:

$$1_\star = 1_\star \star 1_\star = (1_\star * 1_\star) \star (1_\star * 1_\star) \stackrel{(22)}{=} (1_\star \star 1_\star) * (1_\star \star 1_\star) = 1_\star * 1_\star = 1_\star \quad (23)$$

Further manipulation using (22) confirms that the two operations coincide and are commutative:

$$\begin{aligned} [\gamma] * [\eta] &= (1 \star [\gamma]) * ([\eta] \star 1) \stackrel{(22)}{=} (1 * [\eta]) \star ([\gamma] * 1) \\ &= [\eta] \star [\gamma] \\ &= ([\eta] * 1) \star (1 * [\gamma]) \stackrel{(22)}{=} ([\eta] \star 1) * (1 \star [\gamma]) \\ &= [\eta] * [\gamma] \end{aligned} \quad (24)$$

In summary, we find that the group objects in $\underline{\mathbf{Group}}$ are indeed abelian groups, which means that $\pi_1(G)$ for $G \in \underline{\mathbf{TopGroup}}$ must be abelian. ■

⁴This proof is produced with the help of math.stackexchange.com/q/727999. Another (easier) proof lies in the fact that group translation induces π_1 conjugation, therefore $\gamma^{-1}\alpha\gamma = \alpha$, hence abelian.

⁵See Wikipedia: *Group object*.

⁶See Wikipedia: *Eckmann–Hilton argument*.