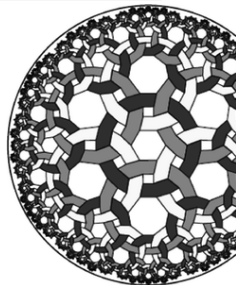


# Understanding Holographic Entanglement

With gravitational path integral & tensor network

*Bryan*



Dunham: Escher on Escher

# Introduction

■ **Question:** how can we “prepare” / construct a state, e.g.

■ vacuum:  $\rho = |0\rangle\langle 0|$

■ thermal:  $\rho \propto \sum_n e^{-\beta H} |n\rangle\langle n|$

... in a *holographic* system?

■ **Answer:** via holography! (*duh...*)

■ Gravitational path integral

■ Tensor network

■ Entanglement entropy captures some “structure” of the state

# Introduction

- **Question:** how can we “prepare” / construct a state, e.g.

- vacuum:  $\rho = |0\rangle\langle 0|$

- thermal:  $\rho \propto \sum_n e^{-\beta H} |n\rangle\langle n|$

... in a *holographic* system?

- **Answer:** via holography! (*duh...*)

- Gravitational path integral

- Tensor network

- Entanglement entropy captures some “structure” of the state

# Introduction

- **Question:** how can we “prepare” / construct a state, e.g.

- vacuum:  $\rho = |0\rangle\langle 0|$

- thermal:  $\rho \propto \sum_n e^{-\beta H} |n\rangle\langle n|$

... in a *holographic* system?

- **Answer:** via holography! (*duh...*)

- Gravitational path integral

- Tensor network

- Entanglement entropy captures some “structure” of the state

# Introduction

- **Question:** how can we “prepare” / construct a state, e.g.

- vacuum:  $\rho = |0\rangle\langle 0|$

- thermal:  $\rho \propto \sum_n e^{-\beta H} |n\rangle\langle n|$

... in a *holographic* system?

- **Answer:** via holography! (*duh...*)
  - Gravitational path integral
  - Tensor network
- Entanglement entropy captures some “structure” of the state

## Review: boundary entanglement as gravitational saddles

# Recall: Path Integral in the Boundary & the Bulk

Lewkowycz:2013nqa: Lewkowycz:2013nqa



Figure: Thermal  $\mathcal{Z}_\beta$

Thermal partition function  
in the bulk & the boundary

Image made from Benjamin:2020mfz  
and the EHT black hole photo

- In *any* field theory, a thermal state can be prepared by a path integral;
- In a *holographic* theory,  $\mathcal{Z}_{\partial B} = \mathcal{Z}_{Bulk}$ , a boundary state can be prepared by a bulk path integral.
  - e.g. the thermo-field double  $\leftrightarrow$  the BTZ black hole  
Note that the Euclidean BTZ geometry is smooth:  
filling in the  $t_E$  cycle (*not* the  $\phi$  cycle) of the torus
  - c.f. Chern-Simons/WZW: not quite the same

# Recall: Path Integral in the Boundary & the Bulk

Lewkowycz:2013nqa: Lewkowycz:2013nqa



Figure: Thermal  $\mathcal{Z}_\beta$

Thermal partition function  
in the bulk & the boundary

Image made from Benjamin:2020mfz  
and the EHT black hole photo

- In *any* field theory, a thermal state can be prepared by a path integral;
- In a *holographic* theory,  $\mathcal{Z}_{\partial B} = \mathcal{Z}_{Bulk}$ , a boundary state can be prepared by a bulk path integral.
  - e.g. the thermo-field double  $\leftrightarrow$  the BTZ black hole  
Note that the Euclidean BTZ geometry is smooth:  
filling in the  $t_E$  cycle (*not* the  $\phi$  cycle) of the torus
  - c.f. Chern-Simons/WZW: not quite the same



# Recall: Gravitational Path Integral

Lewkowycz:2013nqa: Lewkowycz:2013nqa

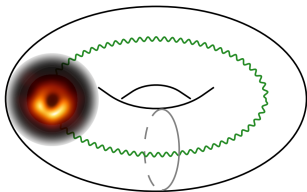


Figure: Thermal  $\mathcal{Z}_\beta$

Thermal partition function  
in the bulk & the boundary

Image made from Benjamin:2020mfz  
and the EHT black hole photo

$$\mathcal{Z}_{\partial B} = \mathcal{Z}_{Bulk} \quad (1)$$

- Replica trick: entanglement entropy for a region  $R$ :

$$S_R = -\text{Tr} \rho_R \log \rho_R \Leftrightarrow \text{Tr} \rho_R^n \equiv \mathcal{Z}_n \quad (2)$$

i.e. reduced to the partition function of the  $n$ -replica.  
It can be deployed in the boundary & the bulk!

- Boundary: static geometry, but the field theory is usually strongly coupled — often difficult!
- Bulk: dynamic geometry, weakly coupled gravity:  
gravity fills in the bulk smoothly,  
 $\mathcal{Z} \sim \sum_i e^{-S_i[g_{\mu\nu}]}$ : sum over classical saddles

# Recall: Gravitational Path Integral

Lewkowycz:2013nqa: Lewkowycz:2013nqa

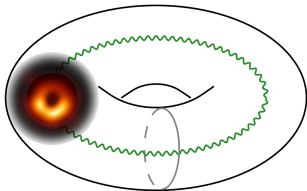


Figure: Thermal  $\mathcal{Z}_\beta$

Thermal partition function  
in the bulk & the boundary

Image made from Benjamin:2020mfz  
and the EHT black hole photo

$$\mathcal{Z}_{\partial B} = \mathcal{Z}_{Bulk} \quad (1)$$

- Replica trick: entanglement entropy for a region  $R$ :

$$S_R = -\text{Tr} \rho_R \log \rho_R \Leftrightarrow \text{Tr} \rho_R^n \equiv \mathcal{Z}_n \quad (2)$$

i.e. reduced to the partition function of the  $n$ -replica.  
It can be deployed in the boundary & the bulk!

- Boundary: static geometry, but the field theory is usually strongly coupled — often difficult!
- Bulk: dynamic geometry, weakly coupled gravity:  
gravity fills in the bulk smoothly,  
 $\mathcal{Z} \sim \sum_i e^{-S_i[g_{\mu\nu}]}$ : sum over classical saddles

# Recall: Replica Trick and the Ryu–Takayanagi Proposal

Lewkowycz:2013nqa: Lewkowycz:2013nqa



- $\mathcal{Z} \sim \sum_i e^{-S_i[g_{\mu\nu}]}$ , saddle pt. approx.  
 $\Rightarrow$  minimize  $S[g_{\mu\nu}]$  on the  $n$ -replica  $\widetilde{\mathcal{M}}_n$
- $\widetilde{\mathcal{M}}_n/\mathbb{Z}_n$ : conical singularity at the  $\mathbb{Z}_n$ ;  $n \rightarrow 1$ ,  
 $\Rightarrow$  minimize the area of the  $\mathbb{Z}_n$  fixed point  
 $\Rightarrow$  the extremal surface, the RT surface

■ **Lesson:** use bulk path integral to:

- prepare the states
- compute the entanglement entropy

This requires holography,  
 but not necessarily AdS/CFT!

Figure: The bulk replica  
 [Lewkowycz:2013nqa]

# Recall: Replica Trick and the Ryu–Takayanagi Proposal

Lewkowycz:2013nqa: Lewkowycz:2013nqa



- $\mathcal{Z} \sim \sum_i e^{-S_i[g_{\mu\nu}]}$ , saddle pt. approx.  
 $\Rightarrow$  minimize  $S[g_{\mu\nu}]$  on the  $n$ -replica  $\widetilde{\mathcal{M}}_n$
- $\widetilde{\mathcal{M}}_n/\mathbb{Z}_n$ : conical singularity at the  $\mathbb{Z}_n$ ;  $n \rightarrow 1$ ,  
 $\Rightarrow$  minimize the area of the  $\mathbb{Z}_n$  fixed point  
 $\Rightarrow$  the extremal surface, the RT surface
- **Lesson:** use bulk path integral to:
  - prepare the states
  - compute the entanglement entropy

This requires holography,  
but not necessarily AdS/CFT!

Figure: The bulk replica  
[Lewkowycz:2013nqa]

## Compare: bulk emergence from tensor networks

# Prepare states via Tensor Networks

Vidal:2007hda, Swingle:2009bg. Reviewed by Rangamani:2016dms.

## MERA



Figure: MERA

Multi-scale Entanglement  
Renormalization Ansatz

Image from [tensornetwork.org](http://tensornetwork.org)

- Gravitational path integral: a spacetime perspective  
Tensor network: on a constant time slice
- States constructed with tensor networks:  
common in condensed matter (e.g. DMRG)
- To find the ground state of a system
  - Write down a tensor network  
as an **ansatz** for the ground state;
  - Vary the components of each tensor  
to achieve minimal energy — **optimization!**

# Prepare states via Tensor Networks

Vidal:2007hda, Swingle:2009bg. Reviewed by Rangamani:2016dms.

## MERA



Figure: MERA

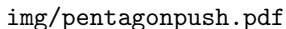
Multi-scale Entanglement  
Renormalization Ansatz

Image from [tensornetwork.org](http://tensornetwork.org)

- Gravitational path integral: a spacetime perspective  
Tensor network: on a constant time slice
- States constructed with tensor networks:  
common in condensed matter (e.g. DMRG)
- To find the ground state of a system
  - Write down a tensor network  
as an **ansatz** for the ground state;
  - Vary the components of each tensor  
to achieve minimal energy — **optimization!**

# AdS Bulk as a Tensor Network

Reviewed by **Harlow:2018fse**: **Harlow:2018fse**



img/pentagonpush.pdf

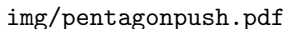
**Figure:** The HaPPY code  
[**Pastawski:2015qua**,  
**Harlow:2018fse**]

- Gravitational path integral: a spacetime perspective  
Tensor network: on a constant time slice
- Dual geometry in the bulk is understood as the “continuous limit” of a *tensor network*
  - Node: tensor acting on the Hilbert space
  - Leg: index to be contracted
    - $1 \times$  free leg  
takes in bulk local operator insertions
    - $5 \times$  contracted leg  
propagates the bulk insertions to the boundary



# AdS Bulk as a Tensor Network

Reviewed by **Harlow:2018fse**: **Harlow:2018fse**




img/pentagonpush.pdf

**Figure:** The HaPPY code  
[**Pastawski:2015qua**,  
**Harlow:2018fse**]

- Gravitational path integral: a spacetime perspective  
Tensor network: on a constant time slice
- Dual geometry in the bulk is understood as the “continuous limit” of a *tensor network*
  - Node: tensor acting on the Hilbert space
  - Leg: index to be contracted
    - $1 \times$  free leg  
takes in bulk local operator insertions
    - $5 \times$  contracted leg  
propagates the bulk insertions to the boundary

# RT from the tensor network

Harlow:2018fse: Harlow:2018fse



img/cutnetwork.pdf

Figure: The HaPPY code  
[Pastawski:2015qua,  
Harlow:2018fse]

- “Entanglement” between  $R$  and  $\bar{R}$ :  
 $\propto$  min # of links connecting the two regions  
 Naturally, entropy = bulk area!
- “Complexity”: # of nodes  
 Proposal: complexity = bulk volume!  
 See e.g. Susskind:2014rva
- Note: how to actually take the continuous limit?  
 “Real time” path integral on a constant time slice
  - cMERA: Nozaki:2012zj
  - “Path-Integral Optimization”:  
 Boruch:2021hqs

# RT from the tensor network

Harlow:2018fse: Harlow:2018fse



img/cutnetwork.pdf

Figure: The HaPPY code  
[Pastawski:2015qua,  
Harlow:2018fse]

- “Entanglement” between  $R$  and  $\bar{R}$ :  
 $\propto$  min # of links connecting the two regions  
 Naturally, entropy = bulk area!
- “Complexity”: # of nodes  
 Proposal: complexity = bulk volume!  
 See e.g. Susskind:2014rva
- Note: how to actually take the continuous limit?  
 “Real time” path integral on a constant time slice
  - cMERA: Nozaki:2012zj
  - “Path-Integral Optimization”:  
 Boruch:2021hqs

# RT from the tensor network

Harlow:2018fse: Harlow:2018fse



img/cutnetwork.pdf

Figure: The HaPPY code  
[Pastawski:2015qua,  
Harlow:2018fse]

- “Entanglement” between  $R$  and  $\bar{R}$ :  
 $\propto$  min # of links connecting the two regions  
 Naturally, entropy = bulk area!
- “Complexity”: # of nodes  
 Proposal: complexity = bulk volume!  
 See e.g. Susskind:2014rva
- Note: how to actually take the continuous limit?  
 “Real time” path integral on a constant time slice
  - cMERA: Nozaki:2012zj
  - “Path-Integral Optimization”:  
 Boruch:2021hqs

# The Lesson

- The Ryu–Takayanagi proposal:  $S \sim \frac{A}{4G_N}$ 
  - ... seems to be universal in holographic systems,
  - ... where boundary states can be constructed from some sort of bulk operations:
    - Gravitational path integral
    - Tensor network
- Applications: beyond standard  $\text{AdS}_3/\text{CFT}_2$ 
  - Cutoff holography: **Lewkowycz:2019xse**
  - Flat holography: **Apolo:2020bld**, **Apolo:2020qjm**

Application: cutoff  $\text{AdS}_3 / T\bar{T}$  deformed theory

# Cutoff $\text{AdS}_3$ / $T\bar{T}$ deformed theory

McGough:2016lol



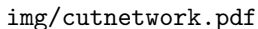
img/cutnetwork.pdf

Figure: The HaPPY code  
[Pastawski:2015qua,  
Harlow:2018fse]

- $\text{AdS}_3$  with finite cutoff:  
*holographic renormalization* of the boundary theory
  - This is clear in the tensor network picture  
“coarse-graining”
- Deform the boundary  $\text{CFT}_2$  with some operator:  
 $\text{CFT}_2^{(\text{UV})} \rightsquigarrow \text{deformed theory}^{(\text{IR})}$
- Surprisingly, we were able to find the deformed theory!  
 $\delta S \propto \mu (T\bar{T})_\mu, \quad T\bar{T} = \frac{1}{8} (T^{\alpha\beta} T_{\alpha\beta} - (T_\alpha^\alpha)^2)$ 
  - See e.g. Smirnov:2016lqw
  - McGough:2016lol:  
McGough:2016lol

# Cutoff $\text{AdS}_3$ / $T\bar{T}$ deformed theory

McGough:2016lol



img/cutnetwork.pdf

Figure: The HaPPY code  
[Pastawski:2015qua,  
Harlow:2018fse]

- $\text{AdS}_3$  with finite cutoff:  
*holographic renormalization* of the boundary theory
  - This is clear in the tensor network picture  
“coarse-graining”
- Deform the boundary  $\text{CFT}_2$  with some operator:  
 $\text{CFT}_2^{(\text{UV})} \rightsquigarrow \text{deformed theory}^{(\text{IR})}$
- Surprisingly, we were able to find the deformed theory!  
 $\delta S \propto \mu (T\bar{T})_\mu, \quad T\bar{T} = \frac{1}{8} (T^{\alpha\beta} T_{\alpha\beta} - (T_\alpha^\alpha)^2)$ 
  - See e.g. Smirnov:2016lqw
  - McGough:2016lol:  
McGough:2016lol



# Cutoff $\text{AdS}_3$ / $T\bar{T}$ deformed theory

McGough:2016lol



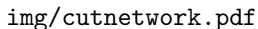
img/cutnetwork.pdf

Figure: The HaPPY code  
[Pastawski:2015qua,  
Harlow:2018fse]

- $\text{AdS}_3$  with finite cutoff:  
*holographic renormalization* of the boundary theory
  - This is clear in the tensor network picture  
“coarse-graining”
- Deform the boundary  $\text{CFT}_2$  with some operator:  
 $\text{CFT}_2^{(\text{UV})} \rightsquigarrow \text{deformed theory}^{(\text{IR})}$
- Surprisingly, we were able to find the deformed theory!  
 $\delta S \propto \mu (T\bar{T})_\mu, \quad T\bar{T} = \frac{1}{8} (T^{\alpha\beta} T_{\alpha\beta} - (T^\alpha_\alpha)^2)$ 
  - See e.g. Smirnov:2016lqw
  - McGough:2016lol:  
McGough:2016lol

# Holographic Entanglement in Cutoff $\text{AdS}_3$

Lewkowycz:2019xse



img/cutnetwork.pdf

Figure: The HaPPY code  
[Pastawski:2015qua,  
Harlow:2018fse]

- If we *assume* that the tensor network intuition is valid,  
Then the RT proposal should still hold!
  - This is shown by Lewkowycz:2019xse
- Generalization of  $A$  in  $S \sim \frac{A}{4G_N}$ :  $A$  is actually the  
gravitational charge of the *replica symmetry*,
  - ... *analytically continued* from  $\mathbb{Z}_n$  to  $U(1)$ ,
  - ... corresponds to the Killing horizon generator /  
*modular flow generator*  $\xi$ .
  - This would in turn give us a hint of the modular flow  
in the  $T\bar{T}$  deformed theory! (*ongoing work*)

# Holographic Entanglement in Cutoff $\text{AdS}_3$

Lewkowycz:2019xse



img/cutnetwork.pdf

Figure: The HaPPY code  
[Pastawski:2015qua,  
Harlow:2018fse]

- If we *assume* that the tensor network intuition is valid,  
Then the RT proposal should still hold!
  - This is shown by Lewkowycz:2019xse
- Generalization of  $A$  in  $S \sim \frac{A}{4G_N}$ :  $A$  is actually the  
gravitational charge of the *replica symmetry*,
  - ... *analytically continued* from  $\mathbb{Z}_n$  to  $U(1)$ ,
  - ... corresponds to the Killing horizon generator /  
*modular flow generator*  $\xi$ .
  - This would in turn give us a hint of the modular flow  
in the  $T\bar{T}$  deformed theory! (*ongoing work*)

## Further Reading & Outlook

- Single-trace  $T\bar{T}$  duality and the flow towards to UV
  - **Apolo:2019zai**
- Quantum error correction:
  - **Jahn:2021uqr**
- Tensor network for flat spacetime?
  - **May:2016dgv**

# References I