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1 Symmetry & Noether's Theorem

1.1 2D σ -Model

$$\mathcal{L} = -\frac{1}{2} \eta_{\alpha\beta} \eta_{\mu\nu} \partial^{\alpha} X^{\mu} \partial^{\beta} X^{\nu} = -\frac{1}{2} \partial^{\alpha} X_{\mu} \partial_{\alpha} X^{\mu}, \quad X^{\mu} \in \mathbb{R}^{1,D-1}$$
 (1)

• For $\delta X^{\mu} = a^{\mu} + \lambda^{\mu}_{\ \nu} X^{\nu}$, the Lagrangian (density) transforms as follows:

$$\begin{split} \delta \mathcal{L} &= -\partial^{\alpha} X_{\mu} \, \partial_{\alpha} \, \delta X^{\mu} \\ &= -\partial^{\alpha} X_{\mu} \, \partial_{\alpha} (a^{\mu} + \lambda^{\mu}_{\ \nu} X^{\nu}) \\ &= -\partial^{\alpha} X_{\mu} \, (\partial_{\alpha} a^{\mu} + X^{\nu} \, \partial_{\alpha} \lambda^{\mu}_{\ \nu} + \lambda^{\mu}_{\ \nu} \, \partial_{\alpha} X^{\nu}) \\ &= -\partial^{\alpha} X_{\mu} \, \partial_{\alpha} a^{\mu} - \partial^{\alpha} X^{\mu} \, \partial_{\alpha} X^{\nu} \, \lambda_{\mu\nu} - X^{\nu} \, \partial^{\alpha} X^{\mu} \, \partial_{\alpha} \lambda_{\mu\nu} \\ &= -\partial^{\alpha} X_{\mu} \, \partial_{\alpha} a^{\mu} - \partial^{\alpha} X^{\mu} \, \partial_{\alpha} X^{\nu} \, \lambda_{(\mu\nu)} - X^{\nu} \, \partial^{\alpha} X^{\mu} \, \partial_{\alpha} \lambda_{\mu\nu} \end{split}$$
(2)

Since a^{μ} and λ^{μ}_{ν} are independent, imposing $\delta L = 0$ yields $\partial_{\alpha} a^{\mu} = 0$, a = const. Furthermore, if $\delta L = 0$ is to hold for arbitrary X^{μ} fields, then $\partial_{\alpha} \lambda_{\mu\nu} = 0$, $\lambda_{(\mu\nu)} = 0$, i.e. $\lambda_{\mu\nu}$ is constant and anti-symmetric over its indices.

• Promote $\delta X \mapsto \epsilon(x) \, \delta X = \epsilon(x) \, (a^{\mu} + \lambda^{\mu}_{\ \nu} X^{\nu})$, with $\epsilon(x)$ some localized bump function; using (2) and considering *on-shell* variation, we have:

$$0 = \delta S = -\int d^2 x \left(\partial^{\alpha} X_{\mu} a^{\mu} \partial_{\alpha} \epsilon + X^{\nu} \partial^{\alpha} X^{\mu} \lambda_{\mu\nu} \partial_{\alpha} \epsilon \right)$$
$$= -\int d^2 x \left(\partial^{\alpha} X_{\mu} a^{\mu} + X_{[\nu} \partial^{\alpha} X_{\mu]} \lambda^{[\mu\nu]} \right) \partial_{\alpha} \epsilon$$
(3)

It is then evident (after partial integration) that the following currents are conserved; they are the Noether currents associated with a^{μ} and $\lambda^{[\mu\nu]}$:

$$j_{\mu}^{\alpha} = -\partial^{\alpha} X_{\mu}, \quad j_{\mu\nu}^{\alpha} = -X_{[\nu} \partial^{\alpha} X_{\mu]} = \frac{1}{2} \left(X_{\mu} \partial^{\alpha} X_{\nu} - X_{\nu} \partial^{\alpha} X_{\mu} \right) \tag{4}$$

Conserved charge $Q = \int d^2x \, j^0(x)$, we have:

$$P_{\mu} = -\int dx^{1} \,\partial^{0} X_{\mu} = \int dx^{1} \,\partial_{0} X_{\mu}, \quad M_{\mu\nu} = \frac{1}{2} \int dx^{1} \,(X_{\nu} \,\partial_{0} X_{\mu} - X_{\mu} \,\partial_{0} X_{\nu}) \tag{5}$$

They can be interpreted as spacetime momentum and spacetime angular momentum.

1.2 Real Scalar in (3+1) D

$$\mathcal{L} = -\frac{1}{2} \partial^{\mu} \phi \, \partial_{\mu} \phi - \frac{1}{2} m^2 \phi^2 \tag{6}$$

• For ϕ : scalar, under $x' = \lambda \circ x$, $\phi(x) \mapsto \phi'(x)$, while:

$$\phi'(x') = \phi(x) \implies \phi'(x) = \phi(\lambda^{-1} \circ x) \tag{7}$$

For $\lambda \sim \lambda^{\mu}_{\ \nu}$: Lorentz transformation, $\eta_{\mu\nu}\lambda^{\mu}_{\ \rho}\lambda^{\nu}_{\ \sigma}=\eta_{\rho\sigma}$, or equivalently, $(\lambda^{-1})^{\mu}_{\ \nu}=\lambda_{\nu}^{\ \mu}$. Therefore,

$$\phi'(x^{\mu}) = \phi(\lambda^{-1} \circ x^{\mu}) = \phi(x^{\nu} \lambda_{\nu}^{\mu}) \tag{8}$$

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• Under $x'^{\mu} = \lambda^{\mu}_{\ \nu} x^{\nu}$, we have:

$$\mathcal{L}'(x') = -\frac{1}{2} \, \partial'^{\mu} \phi'(x') \, \partial'_{\mu} \phi'(x') - \frac{1}{2} \, m^2 \phi'^2(x')
= -\frac{1}{2} \, \partial'^{\mu} \phi(x) \, \partial'_{\mu} \phi(x) - \frac{1}{2} \, m^2 \phi^2(x)
= -\frac{1}{2} \, \eta^{\mu\nu} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \, \partial_{\rho} \phi(x) \, \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \, \partial_{\sigma} \phi(x) - \frac{1}{2} \, m^2 \phi^2(x)
= -\frac{1}{2} \, \eta^{\rho\sigma} \partial_{\rho} \phi(x) \, \partial_{\sigma} \phi(x) - \frac{1}{2} \, m^2 \phi^2(x)
= \mathcal{L}(x)$$
(9)

Here we've used $\eta^{\mu\nu} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} = \eta^{\mu\nu} \lambda_{\mu}^{\rho} \lambda_{\nu}^{\sigma} = \eta^{\rho\sigma}$. Furthermore, $S' = \int d^4x \, \mathcal{L}'(x) = \int d^4x' \, \mathcal{L}'(x') = \int d^4x' \, \mathcal{L}(x) = \int d^4x' \, \mathcal{L}(x) = \int d^4x' \, \mathcal{L}(x) = \int d^4x' \, \mathcal{L}(x') = \int d^$

• Consider an infinitesimal Lorentz transformation: $\lambda \sim 1 + \omega$, then $\eta_{\mu\nu}\lambda^{\mu}_{\ \rho}\lambda^{\nu}_{\ \sigma} = \eta_{\rho\sigma}$ implies that $\omega_{\mu\nu}$ is anti-symmetric: $\omega_{\mu\nu} + \omega_{\nu\nu} = 0$. For $\delta x^{\mu} = \omega^{\mu}_{\ \nu} x^{\nu}$, we have:

$$\delta\phi = -\frac{\partial\phi}{\partial x^{\mu}} \,\delta x^{\mu} = -\omega^{\mu}_{\ \nu} x^{\nu} \,\partial_{\mu}\phi \tag{10}$$

To obtain the corresponding Noether charges, we can simply repeat the operations done in our previous problem; alternatively, we can try to derive a general recipe¹: for $\mathcal{L} = \mathcal{L}(\phi, \partial_{\mu}\phi)$ and $S = \int d^4x \, \mathcal{L}$, we have:

$$\delta S = \int d^4 x \, \delta \mathcal{L}$$

$$= \int d^4 x \left(\frac{\partial \mathcal{L}}{\partial \phi} \, \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \, \delta \partial_{\mu} \phi \right)$$

$$= \int d^4 x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \right) \delta \phi + \int d^4 x \, \partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \, \delta \phi \right)$$
(11)

If we vary S w.r.t. a symmetry of the system, we will have $\delta \mathcal{L} = \partial_{\mu} K^{\mu}$ some total derivative; when on-shell, such variation gives the conserved current with boundary term K^{μ} :

$$j^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \,\delta\phi - K^{\mu} \tag{12}$$

Back to our Lorentz transformation $\delta \phi = -\omega^{\mu}_{\ \nu} x^{\nu} \partial_{\mu} \phi$, we have symmetry variation:

$$\delta \mathcal{L} = -\omega^{\mu}_{, i} x^{\nu} \partial_{\mu} \mathcal{L} = -\partial_{\mu} (\omega^{\mu}_{, i} x^{\nu} \mathcal{L}) \tag{13}$$

We can write this down without explicit calculations, since we know \mathcal{L} itself is a Lorentz scalar, and that's how scalar transforms under Lorentz transformations.

This gives a boundary term $K^{\mu} = -\omega^{\mu}_{\ \nu} x^{\nu} \mathcal{L}$, and the Noether current and its corresponding conserved charge can be calculated as follows:

$$j^{\mu} = -\omega^{\sigma}_{\nu} x^{\nu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \, \partial_{\sigma} \phi - \delta^{\mu}_{\sigma} \mathcal{L} \right), \tag{14}$$

$$Q = \int d^3x \, j^0 = -\omega^{\sigma}_{\nu} \int d^3x \, x^{\nu} (\partial_0 \phi \, \partial_{\sigma} \phi - \delta^0_{\sigma} \mathcal{L}), \tag{15}$$

¹References: arXiv:1601.03616 and *Tong*: http://damtp.cam.ac.uk/user/tong/qft.html

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Note that $\omega^{\mu}_{\ \nu}$ is arbitrary, therefore Q can be decomposed into independent charges:

$$Q = \frac{1}{2} \omega_{\mu\nu} M^{\mu\nu}, \quad M^{\mu\nu} = -\int d^3x \, 2x^{[[\mu} \Big(\partial_0 \phi \, \partial^{\nu]} \phi - \eta^{\nu]0} \mathcal{L} \Big), \tag{16}$$

The indices of $M^{\mu\nu}$ are anti-symmetrized to match the degrees of freedom in $\omega_{\mu\nu}$. Note that the \mathcal{L} term only appears when one of the indices is 0.

Note that the canonical momentum:

$$\varpi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} = \partial_0 \phi \tag{17}$$

It is thus natural to re-organize M^{0i} in the following way:

$$M^{0i} = -M^{i0} = -\int d^3x \left(\varpi \left(x^0 \partial^i - x^i \partial^0 \right) \phi - x^i \mathcal{L} \right)$$

$$= -\int d^3x \left(x^0 \varpi \partial^i \phi + x^i \left(\varpi \dot{\phi} - \mathcal{L} \right) \right)$$

$$= -\int d^3x \left(x^0 \varpi \partial^i \phi + x^i \mathcal{H} \right)$$
(18)

We've obtained an interesting result: the expression for the boost generator M^{i0} contains the Hamiltonian density \mathcal{H} , weighted by the radial distance x^i . This is natural since a boost does indeed contains time evolution for excitations away from the origin. It's an important result utilized by the so-called *Rindler decomposition*; in fact, M^{i0} becomes the Hamiltonian for an accelerated observer in the Rindler patch².

For M^{ij} , we have:

$$M^{ij} = -\int d^3x \,\dot{\phi} \left(x^i \partial^j - x^j \partial^i \right) \phi \tag{19}$$

This is interpreted as the angular momentum of the field ϕ . Suppose ϕ is a wave packet localized around \mathbf{x} with momentum $\approx \mathbf{k}$, then we have the classical angular momentum up to some factor:

$$M^{ij} \sim \left(x^i k^j - x^j k^i\right) \int \mathrm{d}^3 x \, E\phi^2, \quad E = \sqrt{\mathbf{k}^2 + m^2}$$
 (20)

The $\int d^3x E\phi^2$ factor in the above expression is an $\mathcal{O}(1)$ normalization constant for a particle-like wave packet; to see this, note that $\phi \in \mathbb{R}$ has a phase factor $\phi \sim a e^{+ik \cdot x} + a^{\dagger} e^{-ik \cdot x} \sim \cos(k \cdot x)$,

$$E = \int d^3x \,\mathcal{H} = \int d^3x \left(\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + \cdots \right) \sim \left(E^2 + \mathbf{k}^2 + m^2 + \cdots \right) \int d^3x \, \frac{1}{2} \, \phi^2, \quad (21)$$

$$E \int d^3x \, \phi^2 \sim 1, \quad (22)$$

Indeed, we have: $M^{ij} \sim (x^i k^j - x^j k^i)$.

Canonical quantization:

$$[\phi(\mathbf{x}), \varpi(\mathbf{y})] = [\phi(\mathbf{x}), \dot{\phi}(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$$
(23)

Other equal-time commutators between ϕ, ϖ all just vanish. Operator products are then regularized by normal ordering: $M \mapsto :M:$, which can be explicitly implemented by normal

²See the lecture notes of Tom Hartman: hartmanhep.net/topics2015/gravity-lectures.pdf or Daniel Harlow arXiv:1409.1231.

ordering of the oscilator modes:

$$\phi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left(a_k e^{ik \cdot x} + a_k^{\dagger} e^{-ik \cdot x} \right), \quad [a_k, a_{k'}^{\dagger}] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')$$
 (24)

The k dependence in a_k, E_k is, in fact, only a **k** dependence; we've dropped the boldface in the subscripts simply for convenience.

For example, the first term in $M^{0i} = -M^{i0}$ can be expanded as:

$$-x^0 \int d^3x \,\varpi \,\partial^i \phi = x^0 \int \frac{d^3k}{(2\pi)^3} \,k^i a_k^{\dagger} a_k = x^0 P^i$$
 (25)

Here P^{μ} is the momentum operator on the Hilbert space, promoted from the classical $-i\partial^{\mu}$. M^{0i} is thus further reduced to:

$$M^{0i} = -M^{i0} = x^0 P^i - \int d^3 x \, x^i \mathcal{H}$$
 (26)

We note that this result is almost the classical $x^0P^i - x^iP^0$, but here x^iP^0 is replaced with the integral over energy density \mathcal{H} . The result can be nicely re-written with the stress tensor $T^{\mu\nu}$; just run the Noether's procedure with $\delta x^{\mu} = \epsilon^{\mu}$, then we shall obtain:

$$j'^{\mu} = \epsilon_{\nu} T^{\mu\nu}, \quad T^{\mu\nu} = \partial^{\mu} \phi \, \partial^{\nu} \phi + \eta^{\mu\nu} \mathcal{L},$$

$$Q' = \epsilon_{\mu} P^{\mu}, \quad P^{\mu} = \int d^{3}x \, T^{\mu 0} = \int d^{3}x \, \left(\partial^{0} \phi \, \partial^{\mu} \phi + \eta^{0\mu} \mathcal{L}\right)$$

$$M^{\mu\nu} = \int d^{3}x \, 2x^{[\mu} T^{\nu]0}$$
(27)

The quantization $M \mapsto :M:$ is thus reduced to the quantization of $T^{\nu 0}$, weighted by a x^{μ} factor.

First let's look at $T^{00} = \mathcal{H}$; note that:

$$\int d^3x \, x^i e^{i\mathbf{k} \cdot \mathbf{x}} = (2\pi)^3 \left(-i \, \frac{\partial}{\partial k_i} \right) \delta(\mathbf{k}) \tag{28}$$

One can then check explicitly with mode expansion that, up to normal ordering, we have³:

$$H = \int d^3x \,\mathcal{H} = \int \frac{d^3k}{(2\pi)^3} E_k \, a_k^{\dagger} a_k, \tag{29}$$

$$\mathcal{O}^{i} = \int d^{3}x \, x^{i} \mathcal{H} = \int \frac{d^{3}k}{(2\pi)^{3}} E_{k} \, a_{k}^{\dagger} \left(+i \, \frac{\partial}{\partial k_{i}} \right) a_{k}, \tag{30}$$

At first glance, derivative of $a_k = a_k$ with respect to k^i seems puzzling; however, note that $\left(+i\frac{\partial}{\partial k_i}\right)$ is precisely the x^i operator in "momentum-space", and one can make sense of it by

 $^{^3 \}text{See}$ a similar result in https://physics.stackexchange.com/q/27906. Moreover, the charge can be computed at arbitrary time slice t, but the $t\text{-dependence}~(\sim e^{\pm i E t})$ drops out in the final result, due to the on-shell condition $E_k^2 = \mathbf{k}^2 + m^2$ and symmetries, e.g. $\int \mathrm{d}^3 k \, k^i = 0.$ Note that $\frac{\partial E_k}{\partial k_i} = \frac{k^i}{E_k}.$

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considering a generic n-particle state:

$$|\Psi\rangle = \int d^3k_1 \cdots d^3k_n \, \Psi(\mathbf{k}_1, \cdots, \mathbf{k}_n) \, a_{k_1}^{\dagger} \cdots a_{k_n}^{\dagger} \, |0\rangle$$
 (31)

Where $\Psi(\mathbf{k}_1, \dots, \mathbf{k}_n)$ is the *n*-particle wave function; using $a_k a_{k'}^{\dagger} = a_{k'}^{\dagger} a_k + (2\pi)^3 \delta(\mathbf{k} - \mathbf{k'})$ recursively, we get in the end that:

$$\mathcal{O}_i |\Psi\rangle = \int d^3k_1 \cdots d^3k_n \left\{ \sum_{m=1}^n E_{k_m} \left(+i \frac{\partial}{\partial k_m^i} \right) \Psi(\mathbf{k}_1, \cdots, \mathbf{k}_n) \right\} a_{k_1}^{\dagger} \cdots a_{k_n}^{\dagger} |0\rangle$$
 (32)

We see that indeed \mathcal{O}_i acts as $E_k(+i\frac{\partial}{\partial k^i})$ on the momentum-space n-particle wave function, consistent with the result of ordinary quantum mechanics; ...

TODO: Detailed analysis! HINT: Ward identity!

Notice that $x^{[\mu}\partial^{\nu]} = \frac{1}{2}(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) = \frac{1}{2}D^{\mu\nu}$ is the Killing vector fields of $\mathbb{R}^{3,1}$, hence they naturally follow the commutation relations of $\mathfrak{so}(3,1)$ (up to a constant coefficient, or an isomorphism)⁴. We have:

$$[M^{\mu\nu}, M^{\rho\sigma}] = \int d^3x \int d^3y \left[\dot{\phi} D^{\mu\nu} \phi(x), \dot{\phi} D^{\rho\sigma} \phi(y) \right]$$
$$= \int d^3x \, \dot{\phi} \left[D^{\mu\nu}, D^{\rho\sigma} \right] \phi$$
(33)

Similar holds for M^{i0} . Therefore, $M^{\mu\nu}$'s indeed form the Lie algebra $\mathfrak{so}(3,1)$.

 $^{^4\}mathrm{I}$ would like to thank k for pointing this out.