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## Pure QED Partition Function in Covariant Gauge

With the metric convention:  $g \sim (-+++)$ , we have:

$$\mathcal{Z} = \int \mathcal{D}A^{\mu} e^{-S} \delta \left[ \partial_{\mu} A^{\mu} - f \right] \det \left[ \partial^{2} \delta^{4} (x - y) \right] \tag{1}$$

Here S is the Euclidean action:

$$(-S) = \int d^4x \, \mathcal{L}_{t=-i\tau}, \quad \int d^4x = \int_0^\beta d\tau \int d^3\mathbf{x}$$
 (2)

For pure QED, we have:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{3}$$

Setting  $t=-i\tau$  is equivalent to carrying out a Wick rotation:  $x^0\mapsto ix^0,\ A^0\mapsto iA^0,$  while:

$$g_{\mu\nu}A^{\mu}A^{\nu} = g'_{\mu\nu}A'^{\mu}A'^{\nu} \quad \Longrightarrow \quad g_{\mu\nu} \longmapsto g'_{\mu\nu} = \delta_{\mu\nu} \tag{4}$$

Under this convention, the Euclidean action is formally unchanged (due to its Lorentz invariance); same applies for the gauge-fixing and the ghost term:

$$\mathcal{L}_{t=-i\tau} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad g_{\mu\nu} \longmapsto \delta_{\mu\nu}, \tag{5}$$

$$\delta \left[ \partial_{\mu} A^{\mu} - f \right] \quad \Longrightarrow \quad \mathcal{L}_{gf} = -\frac{1}{2\rho} \left( \partial_{\mu} A^{\mu} \right)^{2}, \tag{6}$$

$$\det \left[ \partial^2 \delta^4(x - y) \right] \quad \Longrightarrow \quad \mathcal{L}_{gh} \sim \left( \partial^2 \bar{\eta} \right) \eta \sim -\partial_\mu \bar{\eta} \, \partial^\mu \eta, \tag{7}$$

Here we've dropped some total derivative terms in the ghost Lagrangian. The partition function is then reduced to:

$$\mathcal{Z} = \int \mathcal{D}A^{\mu} \mathcal{D}\bar{\eta} \mathcal{D}\eta \ e^{-S'}, \quad (-S') = \int d^{4}x \left(\mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh}\right)_{t=-i\tau}$$
 (8)

The action can be further simplified by partial integration and dropping boundary terms:

$$(-S') = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\rho} (\partial_{\mu} A^{\mu})^2 - \partial_{\mu} \bar{\eta} \partial^{\mu} \eta \right)$$

$$= \int d^4x \left( -\frac{1}{4} (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) (\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}) - \frac{1}{2\rho} (\partial_{\mu} A^{\mu})^2 - \partial_{\mu} \bar{\eta} \partial^{\mu} \eta \right)$$

$$= \int d^4x \left( -\frac{1}{2} (\partial_{\mu} A_{\nu} \partial^{\mu} A^{\nu} - \partial_{\nu} A_{\mu} \partial^{\mu} A^{\nu}) - \frac{1}{2\rho} (\partial_{\mu} A^{\mu} \partial_{\nu} A^{\nu}) - \partial_{\mu} \bar{\eta} \partial^{\mu} \eta \right)$$

$$\sim \int d^4x \left( -\frac{1}{2} (-A_{\nu} \partial^2 A^{\nu} + A_{\mu} \partial^{\mu} \partial_{\nu} A^{\nu}) + \frac{1}{2\rho} (A^{\mu} \partial_{\mu} \partial_{\nu} A^{\nu}) + \bar{\eta} \partial^2 \eta \right)$$

$$= \int d^4x \left( -\frac{1}{2} A^{\mu} \left( -\delta_{\mu\nu} \partial^2 + \partial_{\mu} \partial_{\nu} - \frac{1}{\rho} \partial_{\mu} \partial_{\nu} \right) A^{\nu} + \bar{\eta} \partial^2 \eta \right)$$

$$= -\frac{1}{2} \int d^4x \left( A^{\mu} \left( -\delta_{\mu\nu} \partial^2 + (1 - \frac{1}{\rho}) \partial_{\mu} \partial_{\nu} \right) A^{\nu} - 2\bar{\eta} \partial^2 \eta \right)$$

With  $\beta = \frac{1}{T}$ , expand  $A^{\mu}$ ,  $\eta$  into dimensionless Fourier modes, and we have:

$$A^{\mu} = \frac{1}{\sqrt{TV}} \sum_{k} e^{ik_{\nu}x^{\nu}} A_{k}^{\mu}, \qquad \sum_{k} e^{ik_{\nu}x^{\nu}} = V \int \frac{\mathrm{d}^{3}\mathbf{k}}{(2\pi)^{3}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{n\in\mathbb{Z}} e^{i\omega_{n}\tau}$$
(10)

$$\sum_{p,k} \int d^4 x \, e^{i \, (p+k) \cdot x} = \sum_{p,k} (2\pi)^4 \, \delta^4(p+k)$$

$$= V^2 \int \frac{d^3 \mathbf{p} \, d^3 \mathbf{k}}{(2\pi)^6} \sum_{m,n \in \mathbb{Z}} \beta \, \delta_{m,-n} \cdot (2\pi)^3 \, \delta^3(\mathbf{p} + \mathbf{k})$$

$$= \beta V \cdot V \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \sum_{n \in \mathbb{Z}} \int d^3 \mathbf{p} \, \delta^3(\mathbf{p} + \mathbf{k}) \sum_{n \in \mathbb{Z}} \delta_{m,-n}$$

$$= \beta V \sum_{k} \int d^3 \mathbf{p} \, \delta^3(\mathbf{p} + \mathbf{k}) \sum_{n \in \mathbb{Z}} \delta_{m,-n},$$
(11)

$$(-S') = -\frac{1}{2TV} \sum_{p,k} \int d^4x \, e^{i\,(p+k)\cdot x} \left( A_p^{\mu} \left( -\delta_{\mu\nu} (-k^2) + \left( 1 - \frac{1}{\rho} \right) (-k_{\mu}k_{\nu}) \right) A_k^{\nu} - 2\bar{\eta}_p \, (-k^2) \eta_k \right)$$

$$= -\frac{\beta V}{2TV} \sum_k \left( A_{-k}^{\mu} \left( k^2 \delta_{\mu\nu} - \left( 1 - \frac{1}{\rho} \right) k_{\mu}k_{\nu} \right) A_k^{\nu} + 2\bar{\eta}_{-k}k^2 \eta_k \right)$$

$$= -\frac{\beta^2}{2} \sum_k \left( A_k^{\mu\dagger} D_{\mu\nu}^{-1}(k) A_k^{\nu} + 2\bar{\eta}_k^{\dagger} k^2 \eta_k \right)$$

$$(12)$$

Here we've used the reality condition on  $A, \eta$ , namely  $A_k^{\mu\dagger} = A_{-k}^{\mu}$ , and defined the k-space inverse propagator  $D_{\mu\nu}^{-1}(k)$ . Similar result applies for  $\eta_k$ , except that we have to be careful about Grassmann variables. Carry out  $\int \mathcal{D}A^{\mu} \mathcal{D}\bar{\eta} \mathcal{D}\eta$ , and we have:

$$\mathcal{Z} \sim \left( \det_{\mu,\nu,k} \beta^2 D_{\mu\nu}^{-1}(k) \right)^{-1/2} \left( \det_k 2\beta^2 k^2 \right)^{+1} \\
= \prod_k \left( \beta^{2 \times 4 \times (-1/2)} \cdot \left( \det_{\mu,\nu} D_{\mu\nu}^{-1}(k) \right)^{-1/2} \cdot 2\beta^2 k^2 \right) \\
= \prod_k \left( \beta^{-4} \left( \det_{\mu,\nu} D_{\mu\nu}^{-1}(k) \right)^{-1/2} \cdot 2\beta^2 k^2 \right) \tag{13}$$

The determinant is evaluated as follows<sup>1</sup>:

$$\det_{\mu,\nu} D_{\mu\nu}^{-1} = \det_{\mu,\nu} \left( k^2 \delta_{\mu\nu} - \left( 1 - \frac{1}{\rho} \right) k_{\mu} k_{\nu} \right)$$

$$= k^8 \det_{\mu,\nu} \left( \delta_{\mu\nu} - \left( 1 - \frac{1}{\rho} \right) \frac{k_{\mu} k_{\nu}}{k^2} \right)$$

$$= k^8 \left( 1 - \left( 1 - \frac{1}{\rho} \right) \frac{k^2}{k^2} \right)$$

$$= \frac{1}{\rho} k^8, \tag{14}$$

$$\mathcal{Z} \sim \prod_{k} \frac{2}{\rho} \beta^{-4} k^{-4} \cdot \beta^2 k^2 \sim \prod_{k} \beta^{-2} k^{-2},$$
 (15)

$$\ln \mathcal{Z} \sim -\sum_{k} \ln \left(\beta^2 k^2\right) \tag{16}$$

<sup>&</sup>lt;sup>1</sup>Reference: Wikipedia: Determinant # Sylvester's determinant theorem.

We see that  $\ln \mathcal{Z}$  is simply twice the result of a neutral scalar field, with mass  $m \to 0$ , i.e.

$$\ln \mathcal{Z} \sim -2 \times \frac{1}{2} \sum_{k} \ln \left( \beta^2 k^2 \right) \sim -2 \sum_{k} \left( \frac{1}{2} \beta E_k + \ln \left( 1 - e^{-\beta E_k} \right) \right), \tag{17}$$

$$\mathcal{Z} \sim \prod_{\mathbf{k}} \left\{ \exp\left(-\frac{1}{2}\beta E_k - \ln\left(1 - e^{-\beta E_k}\right)\right) \right\}^2, \tag{18}$$

$$\Omega = -T \ln \mathcal{Z} = 2 \sum_{\mathbf{k}} \left( \frac{1}{2} E_k + T \ln \left( 1 - e^{-E_k/T} \right) \right), \tag{19}$$

$$p = -\frac{\Omega}{V} = -2 \int \frac{\mathrm{d}^3 \mathbf{k}}{(2\pi)^3} \left( \frac{1}{2} E_k + T \ln \left( 1 - e^{-E_k/T} \right) \right), \tag{20}$$

Here  $E_k = ||\mathbf{k}||$ . Ignore the vacuum contribution to p, and we have

$$p = -2T \int \frac{\mathrm{d}^3 \mathbf{k}}{(2\pi)^3} \ln\left(1 - e^{-\|\mathbf{k}\|/T}\right) = 2\frac{\pi^2}{90} T^4$$
 (21)

## Summary

1. We've completed the calculation of pure QED partition function  $\mathcal{Z}$  and thermodynamic potential  $\Omega$  (energy density p) by introducing a "soft" gauge fix  $\mathcal{L}_{gf}$ . Alternatively, we can simply impose a "hard" Lorenz gauge fix  $\delta[\partial_{\mu}A^{\mu}]$ ; this can be achieved by taking  $\rho \to \infty$  in the Gaussian packet  $-\frac{1}{2\rho}(\partial_{\mu}A^{\mu})^2$ , or by integrating out  $A^0$  directly — similar to (12), we have:

$$\mathcal{Z} \sim \int \mathcal{D}A^{\mu} \mathcal{D}\bar{\eta} \mathcal{D}\eta \,\delta \left[\partial_{\mu}A^{\mu}\right] \exp\left(-\frac{\beta^{2}}{2} \sum_{k} \left(A_{k}^{\mu\dagger} \left(k^{2} \delta_{\mu\nu} - k_{\mu} k_{\nu}\right) A_{k}^{\nu} + 2\bar{\eta}_{k}^{\dagger} k^{2} \eta_{k}\right)\right) \\
\sim \int \mathcal{D}A^{i} \mathcal{D}\bar{\eta} \mathcal{D}\eta \exp\left(-\frac{\beta^{2}}{2} \sum_{k} \left(A_{k}^{\mu\dagger} \left(k^{2} \delta_{\mu\nu} - k_{\mu} k_{\nu}\right) A_{k}^{\nu} + 2\bar{\eta}_{k}^{\dagger} k^{2} \eta_{k}\right)\right)_{A^{0} = A^{0}[A^{i}]} \\
A^{0}[A^{i}] = -\int \partial_{i} A^{i} \,d\tau \,, \quad k_{\mu} A_{k}^{\mu} = 0, \tag{23}$$

Here we've omitted a non-dynamical Jacobian det  $\left[\theta(\tau-\tau')\right] = \det \int^{\tau} d\tau'' \, \delta(\tau''-\tau')$ . The ghost integral gives the same contribution, while the  $A^i$  integral yields:

$$\mathcal{Z}_A = \int \mathcal{D}A^i \exp\left(-\frac{\beta^2}{2} \sum_k \left(A_k^{\mu\dagger} (k^2 \delta_{\mu\nu}) A_k^{\nu}\right)\right),\tag{24}$$

$$A_{k}^{\mu\dagger}(k^{2}\delta_{\mu\nu})A_{k}^{\nu} = A_{k}^{i\dagger}(k^{2}\delta_{ij})A_{k}^{j} + A_{k}^{0\dagger}(k^{2})A_{k}^{0} = A_{k}^{i\dagger}(k^{2}\delta_{ij})A_{k}^{j} + \frac{k^{2}}{\omega^{2}}A_{k}^{0\dagger}(\omega^{2})A_{k}^{0}$$

$$\stackrel{(23)}{=} A_{k}^{i\dagger}(k^{2}\delta_{ij})A_{k}^{j} + \frac{k^{2}}{\omega^{2}}A_{k}^{i\dagger}(k_{i}k_{j})A_{k}^{0} = A_{k}^{i\dagger}k^{2}\left(\delta_{ij} + \frac{k_{i}k_{j}}{\omega^{2}}\right)A_{k}^{j}$$

$$= A_{k}^{i\dagger}D_{ij}^{-1}(k)A_{k}^{j}, \qquad (25)$$

$$\mathcal{Z}_{A} \sim \left( \det_{i,j,k} \beta^{2} D_{ij}^{-1}(k) \right)^{-1/2} = \prod_{k} \beta^{-3} k^{-3} \left( 1 + \frac{\mathbf{k}^{2}}{\omega^{2}} \right)^{-1/2} = \prod_{k} \left( \beta^{-4} k^{-4} \cdot \beta \omega \right) \\
= \prod_{k} \beta^{-4} k^{-4} \prod_{k} \prod_{n} 2\pi n \sim \prod_{k} \beta^{-4} k^{-4} \tag{26}$$

We see that the result from a "hard" Lorenz gauge fixing is the same as before, up to a non-dynamical overall coefficient.

2. In our previous calculations, we notice that after functional integration,  $\rho$  is just a non-dynamical overall coefficient in  $\mathcal{Z}$ , hence it can be safely dropped from the final expression; see eq. (15). Therefore, the result is independent of parameter  $\rho$ .