Gravity №1

Bryan

Compiled @ 2021/11/27

1 Spacelike Geodesic for Poincaré AdS

The Poincaré AdS_{d+1} metric is given by:

$$\mathrm{d}s^2 = G_{IJ}\,\mathrm{d}X^I\,\mathrm{d}X^J = \frac{-\,\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 + \mathrm{d}z^2}{z^2} = \frac{\mathrm{d}x^2 + \mathrm{d}z^2}{z^2}, \tag{1.1}$$

$$\mathrm{d}x^2 = -\,\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,, \quad X^I \sim (t, \vec{x}, z) \sim (x^\mu, z), \tag{1.2}$$

Note that here we define $\mathrm{d}x^2$ using the flat metric η, while $\mathrm{d}X$ has an index that should be raised and lowered using the curved metric $G_{IJ} = \frac{1}{z^2}\,\eta_{IJ}$. Generally upper case tensors are handled with G_{IJ}, while lower case ones are handled with η. The full isometry of such spacetime is then given by $\mathrm{SO}(d,2)$ with $\frac{(d+2)(d+1)}{2}$ generators, including:

1. Among the $x^\mu \sim (x^0, \vec{x}) \equiv (t, \vec{x})$ directions:
   (a) $\frac{d(d-1)}{2}$ $\mathrm{SO}(d-1,1)$ rotations $x_\mu\partial_\nu - x_\nu\partial_\mu$, boosts included;
   (b) d translations ∂_μ;
2. Dilation with $X^I \sim (x^\mu, z) \mapsto \lambda X^I = \lambda\,(x^\mu, z)$, generated by $\Delta = X^I\partial_I = z\partial_z + x^\mu\partial_\mu$;
3. Special conformal transformations; see Appendix A for an intuitive derivation. We have:

$$k_\mu = (z^2 + x^2)\,\partial_\mu - 2x_\mu\Delta \tag{1.3}$$

By Noether's theorem, $Q_\Xi = V_I\Xi^I = G_{IJ}\Xi^I V^J$ is conserved along the geodesic; here V^I is the normalized tangent vector, while Ξ^I is some Killing vector of the spacetime, e.g. one from the list above. We can then write down the conserved charges along a geodesic γ in Poincaré AdS:

$$\begin{aligned} p_\mu &= G_{\mu I}V^I = V_\mu \\ m_{\mu\nu} &= x_\mu V_\nu - x_\nu V_\mu \\ \Delta &= X^I V_I = \frac{zV^z}{z^2} + x^\mu p_\mu \\ k_\mu &= (z^2 + x^2)\,p_\mu - 2x_\mu\Delta \end{aligned} \tag{1.4}$$

These are all integration constants along γ. Again note that X, V have indices that should be handled with G_{IJ}, in particular,

$$\mathrm{d}X_\mu = G_{\mu I}\,\mathrm{d}X^I = \frac{1}{z^2}\,\mathrm{d}x_\mu\,, \quad V^\mu = \frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda} = z^2 p^\mu \tag{1.5}$$

On the other hand, V^μ should be properly normalized, therefore:

$$\|V\|^2 = g_{IJ}V^I V^J = \frac{(V^z)^2}{z^2} + z^2 p^2 = \delta = 0, \pm 1, \tag{1.6}$$

For $p^2 = 0$, we have $\|V\|^2 = \frac{(V^z)^2}{z^2} \geq 0$, thus γ can be either spacelike or null, but not timelike. In fact, we have $\frac{V^z}{z} = \frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}\lambda} = 0, \pm 1$; along with $\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = z^2 p^\mu$, we obtain:

$$p^2 = 0, \qquad \begin{aligned} \text{spacelike:}\ \ & z(\lambda) = z(0)\,e^{\pm\lambda}, & x^\mu(\lambda) &= x^\mu(0) \pm z(0)^2\,p^\mu\,\tfrac{e^{\pm 2\lambda}-1}{2}, \\ \text{null:}\ \ & z(\lambda) = z(0), & x^\mu(\lambda) &= x^\mu(0) + z(0)^2\,p^\mu\lambda, \end{aligned} \tag{1.7}$$

From now on we shall focus on the $p^2 \neq 0$ situation. Note that we can complete the square on the right-hand side of the k^μ conservation such that:

$$k_\mu + p_\mu a^2 = p_\mu\big(z^2 + (x-a)^2\big), \quad \Delta = p_\mu a^\mu \tag{1.8}$$

For $p^\mu \neq 0$, we can always find some c_μ such that $c_\mu p^\mu \neq 0$, therefore we have:

$$z^2 + (x-a)^2 = \frac{c_\mu k^\mu}{c_\mu p^\mu} + a^2 = \left(\frac{L}{2}\right)^2 = \text{const.}, \quad \Delta = p_\mu a^\mu \tag{1.9}$$

We see that γ lands on a "sphere" centered at $z = 0, x = a$ in $\mathbb{R}^{d+1}$; with the Lorentzian metric, it is actually a hyperboloid. Note that for now the radius $\left(\frac{L}{2}\right)^2$ can actually be negative or zero; more specifically,

1. $L^2 > 0$: one-sheet hyperboloid;
2. $L^2 < 0$: two-sheet hyperboloid;
3. $L^2 = 0$: conic surface.

On the other hand, we can actually solve X^I completely by combining (1.5), (1.6); we have:

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = z^2 p^\mu, \quad \frac{\mathrm{d}z}{\mathrm{d}\lambda} = \pm z\sqrt{\delta - z^2p^2}, \quad \frac{\mathrm{d}x^\mu}{\mathrm{d}z} = \pm\frac{zp^\mu}{\sqrt{\delta - z^2p^2}}, \quad \delta - z^2p^2 = \left(\tfrac{V^z}{z}\right)^2 \geq 0 \tag{1.10}$$

$$\implies \quad \gamma \;\subset\; x^\mu = a^\mu \pm \frac{p^\mu}{p^2}\sqrt{\delta - z^2p^2}, \quad z^2 + (x-a)^2 = \frac{\delta}{p^2} = \left(\frac{L}{2}\right)^2 \tag{1.11}$$

This confirms our observation above, and further reveals that γ lies in a "plane" in the $\mathbb{R}^d$ subspace consisting of the x^μ coordinates. The behavior of γ is sensitive to the sign of δ and p^2; more specifically,

1. $\delta = -1$, i.e. for timelike γ, we must have $p^2 < 0$, and $z \geq \frac{1}{\|\|p\|\|} = \frac{L}{2} > 0$, namely z is bounded from below. This means that timelike geodesic can never reach the asymptotic boundary $z \to 0$. In this case, γ is a section of the one-sheet hyperboloid.
2. $\delta = +1$, i.e. for spacelike γ, we can have $p^2 > 0$ or $p^2 < 0$.
   (a) For $p^2 > 0$, again we have $\frac{L}{2} = \frac{1}{\|p\|} > 0$, and γ is again a cross-section of the one-sheet hyperboloid. However, now we have $z \leq \frac{L}{2}$, namely z is bounded from above, and:

$$z \to 0, \quad x^\mu = a^\mu \pm \frac{p^\mu}{p^2} = a^\mu \pm \frac{L}{2}\,\widehat{p}^\mu, \quad \widehat{p}^\mu = \frac{p^\mu}{\|p\|} \tag{1.12}$$

   We can also nicely parametrize X^I in terms of the proper length λ; for convenience, set $x^\mu(0) = a^\mu$, $z(0) = \frac{L}{2}$, then we have:

$$z(\lambda) = \frac{L}{2}\frac{1}{\cosh\lambda}, \quad x^\mu(\lambda) = a^\mu \pm \widehat{p}^\mu\frac{L}{2}\tanh\lambda \tag{1.13}$$

   (b) For $p^2 < 0$, we have $(\frac{L}{2})^2 = \frac{1}{p^2} < 0$, and γ is now a cross-section of the two-sheet hyperboloid. Again as $z \to 0$ it lands at $x^\mu = a^\mu \pm \frac{L}{2}\,\widehat{p}^\mu$; however, $|x^\mu|$ grows with z and extends into the bulk instead of returning to the boundary, i.e. $z, |x^\mu| \to \infty$.

   The differential equation in this case is almost the same as in (a), but now we have to choose a different initial condition, since γ won't even reach $x = a$. However, we can actually set $x(0), z(0) \to \infty$; we then have:

$$z(\lambda) = \left|\frac{L}{2}\right| \frac{1}{\sinh\lambda}, \quad x^\mu(\lambda) = a^\mu \pm \widehat{p}^\mu \left|\frac{L}{2}\right| \coth\lambda \tag{1.14}$$

This is an evidence that $z \to \infty$ might not be the "end of the world" after all; it's likely that $z \to \infty$ is only a horizon, since a spacelike geodesic γ can reach it within finite proper length λ.

In summary, there are 3 types of spacelike geodesics in Poincaré AdS, which closely resemble the 3 types of conic sections:

1. $p^2 = 0$: parabolic, given in (1.7), with one end going to $z \to 0$ and the other end going to $z \to \infty$; it takes infinite proper length λ for it to reach 0 or ∞.
2. $p^2 > 0$: hyperbolic, given in (1.11) and (1.14), also with one end going to $z \to 0$ and the other end going to $z \to \infty$, but it reaches ∞ within finite λ.
3. $p^2 > 0$: elliptic, given in (1.11) and (1.13), with both ends going to $z \to 0$, and it takes infinite proper length λ for it to reach either end.

Now consider two points x_1, x_2 near the boundary $z = \epsilon$ connected by a spacelike geodesic γ. This can only be the "elliptic type" discussed above. We have:

$$x^\mu = a^\mu \pm \widehat{p}^\mu \sqrt{\left(\frac{L}{2}\right)^2 - z^2}, \quad z^2 + (x-a)^2 = \left(\frac{L}{2}\right)^2, \tag{1.15}$$

$$a = \frac{x_1 + x_2}{2}, \quad \widehat{p} \propto x_2 - x_1 \tag{1.16}$$

It's length is then given by:

$$\begin{aligned} A &= \int_{z\geq\epsilon} \sqrt{\frac{\mathrm{d}x^2 + \mathrm{d}z^2}{z^2}} \\ &= \int_{|\lambda|\leq\Lambda} \sqrt{\frac{(\mathrm{d}\tanh\lambda)^2 + (\mathrm{d}\,\mathrm{sech}\,\lambda)^2}{(\mathrm{sech}\,\lambda)^2}}, \quad \Lambda = \cosh^{-1}(\tfrac{L}{2\epsilon}) \\ &= \int_{-\Lambda}^{\Lambda} \mathrm{d}\lambda = 2\Lambda = 2\cosh^{-1}\left(\frac{L}{2\epsilon}\right) \sim 2\log\frac{L}{\epsilon}, \quad \epsilon \to 0 \end{aligned} \tag{1.17}$$

2 Einbein Action

The einbein action of a point particle is given by:

$$S[\eta, X] = \frac{1}{2}\int \mathrm{d}\tau \left(\eta^{-1}\dot{X}_\mu \dot{X}^\mu - \eta m^2\right) \tag{2.1}$$

Under worldline reparametrization: $\tau \mapsto \tau' = f(\tau)$, we have $X'(\tau') = X(\tau)$, i.e. X^μ transforms like a **scalar** under worldline diffeomorphism; $X(\tau) \mapsto X'(\tau) = X\big(f^{-1}(\tau)\big)$.

On the other hand, η should be treated like an einbein: $\eta = \sqrt{-\gamma}$, here $\gamma = \gamma_{\tau\tau}$ is the worldline metric; $\gamma < 0$ due to the Lorentzian signature [1]. We have:

$$\eta = \sqrt{-\gamma} \longmapsto \eta' = \eta \det\frac{\partial\tau}{\partial\tau'} = \eta\frac{\partial\tau}{\partial\tau'}, \tag{2.2}$$

$$\eta^{-1} = -\sqrt{-\gamma}\,\gamma^{-1} \longmapsto (\eta')^{-1} = \eta^{-1}\left(\frac{\partial\tau}{\partial\tau'}\right)\left(\frac{\partial\tau'}{\partial\tau}\frac{\partial\tau'}{\partial\tau}\right) = \eta^{-1}\frac{\partial\tau'}{\partial\tau} \tag{2.3}$$

It is clear that the action is invariant under the transformation:

$$\begin{aligned}S' &= \frac{1}{2}\int \mathrm{d}\tau' \left((\eta')^{-1}\partial_{\tau'}X_\mu\partial_{\tau'}X^\mu - (\eta')m^2\right)\\ &= \frac{1}{2}\int \mathrm{d}\tau\, \frac{\partial\tau'}{\partial\tau}\left(\eta^{-1}\frac{\partial\tau'}{\partial\tau}\cdot\frac{\partial\tau}{\partial\tau'}\partial_\tau X_\mu\cdot\frac{\partial\tau}{\partial\tau'}\partial_\tau X^\mu - \eta\,\frac{\partial\tau}{\partial\tau'}\,m^2\right) = S\end{aligned} \tag{2.4}$$

We can eliminate η classically by placing it on shell:

$$0 = \frac{\delta S}{\delta\eta} = -\eta^{-2}\dot{X}_\mu\dot{X}^\mu - m^2, \quad \eta[X] = \frac{1}{m}\sqrt{-\dot{X}_\mu\dot{X}^\mu} \tag{2.5}$$

Substitute this back to the action, and we have:

$$\begin{aligned}S[X] = S[\eta=\eta[X],X] &= \frac{1}{2}\int \mathrm{d}\tau\left(m\,(-\dot{X}_\mu\dot{X}^\mu)^{-\frac{1}{2}}\dot{X}_\mu\dot{X}^\mu - m\,(-\dot{X}_\mu\dot{X}^\mu)^{+\frac{1}{2}}\right)\\ &= -m\int\mathrm{d}\tau\,\sqrt{-\dot{X}_\mu\dot{X}^\mu}\end{aligned} \tag{2.6}$$

3 Ricci Tensor for Static Spherical Metric

Consider the metric:

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + h(r)\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) = \eta_{ab}\,e^a e^b \tag{3.1}$$

Here we've defined the following vierbein:

$$\begin{aligned}e^t &= \sqrt{f(r)}\,\mathrm{d}t\\ e^r &= \sqrt{h(r)}\,\mathrm{d}r\\ e^\theta &= r\,\mathrm{d}\theta\\ e^\phi &= r\sin\theta\,\mathrm{d}\phi\end{aligned} \tag{3.2}$$

To find the connection form, we first note the *Cartan's structure equation* [2]:

$$\begin{aligned}\mathrm{d}e^a = \mathrm{d}(e^a_\nu\,\mathrm{d}x^\nu) &= \partial_\mu e^a_\nu\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\\ &= \left(-\omega^a_{\mu b}e^b_\nu + \Gamma^\lambda_{\mu\nu}e^a_\lambda\right)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\\ &= -\omega^a{}_b\wedge e^b + T^a\end{aligned} \tag{3.3}$$

Where T^a is the torsion tensor. Here we've used the relation between spin connection $\omega^a_{\mu b}$ and the affine connection $\Gamma^\lambda_{\mu\nu}$. For the torsion-free Levi-Civita connection, we have $\mathrm{d}e^a = -\omega^a{}_b\wedge e^b$.

Rigidity of the veilbein, namely $\nabla_\mu(g_{\mu\nu}e^\mu_a e^\nu_b) = \nabla_\mu\eta_{ab} = 0$, further implies that:

$$\omega_{\mu ab} = -\omega_{\mu ba} \tag{3.4}$$

With these constraints we can solve for the connection form, and we find that only the following components are non-vanishing[1]:

$$\omega^t{}_r = \frac{f'}{2\sqrt{fh}}\,\mathrm{d}t = -\omega^r{}_t \tag{3.5}$$

[1] Go to `https://github.com/bryango/Archive/blob/master/HW-Gravity/gravity1/nb/vielbein.wl` for a Mathematica script for this calculation.

$$\omega^r{}_\theta = \frac{-1}{\sqrt{h}}\,\mathrm{d}\theta = -\omega^\theta{}_r \tag{3.6}$$

$$\omega^r{}_\phi = \frac{-\sin\theta}{\sqrt{h}}\,\mathrm{d}\phi = -\omega^\phi{}_r \tag{3.7}$$

$$\omega^\theta{}_\phi = -\cos\theta\,\mathrm{d}\phi = -\omega^\phi{}_\theta \tag{3.8}$$

The curvature form is thus given by:

$$\Omega^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{3.9}$$

The exterior derivative is easily computed in the $\mathrm{d}x^\mu$ basis; however, to get to the Ricci tensor, it would be convenient to switch to the e^a basis. Here we only write down the upper half of $\Omega^a{}_b$ since the lower half can be inferred from anti-symmetry:

$$\Omega^a{}_b \sim \begin{pmatrix} 0 & R^t{}_{rtr}\,e^t\wedge e^r & R^t{}_{\theta t\theta}\,e^t\wedge e^\theta & R^t{}_{\phi t\phi}\,e^t\wedge e^\phi \\ \cdots & 0 & R^r{}_{\theta r\theta}\,e^r\wedge e^\theta & R^r{}_{\phi r\phi}\,e^r\wedge e^\phi \\ \cdots & \cdots & 0 & R^\theta{}_{\phi\theta\phi}\,e^\theta\wedge e^\phi \\ \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{3.10}$$

$$\begin{aligned} R^t{}_{rtr} &= \frac{f'^2}{4f^2h} + \frac{f'h'}{4fh^2} - \frac{f''}{2fh}, \\ R^t{}_{\theta t\theta} &= \frac{-f'}{2rfh} = R^t{}_{\phi t\phi}, \\ R^r{}_{\theta r\theta} &= \frac{h'}{2rh^2} = R^r{}_{\phi r\phi}, \\ R^\theta{}_{\phi\theta\phi} &= \frac{h-1}{r^2h} \end{aligned} \tag{3.11}$$

The Ricci tensor in e^a basis is given by $R_{ab} = R^c{}_{acb}$. Note that the components are non-zero iff. $a = b$, i.e. the Ricci tensor is diagonal. We have:

$$\begin{aligned} R_{ab} &\sim \mathrm{diag}\left(R^r{}_{trt} + R^\theta{}_{t\theta t} + R^\phi{}_{t\phi t}, \cdots, \cdots, \cdots\right) \\ &= \mathrm{diag}\left(R^t{}_{rtr} + R^t{}_{\theta t\theta} + R^t{}_{\phi t\phi}, \cdots, \cdots, \cdots\right) \\ &= \mathrm{diag}\Big(R^t{}_{rtr} + R^t{}_{\theta t\theta} + R^t{}_{\phi t\phi}, \\ &\qquad R^t{}_{rtr} + R^r{}_{\theta r\theta} + R^r{}_{\phi r\phi}, \\ &\qquad R^t{}_{\theta t\theta} + R^r{}_{\theta r\theta} + R^\theta{}_{\phi\theta\phi}, \\ &\qquad R^t{}_{\phi t\phi} + R^r{}_{\phi r\phi} + R^\theta{}_{\phi\theta\phi} \Big) \\ &= \mathrm{diag}\Big(R^t{}_{rtr} + 2R^t{}_{\theta t\theta}, \\ &\qquad R^t{}_{rtr} + 2R^r{}_{\theta r\theta}, \\ &\qquad R_{\theta\theta} = R^t{}_{\theta t\theta} + R^r{}_{\theta r\theta} + R^\theta{}_{\phi\theta\phi}, \\ &\qquad R_{\phi\phi} = R_{\theta\theta} \Big) \\ &= \mathrm{diag}\Big(\frac{f'^2}{4f^2h} + \frac{f'h'}{4fh^2} - \frac{f''}{2fh} - \frac{f'}{rfh}, \\ &\qquad \frac{f'^2}{4f^2h} + \frac{f'h'}{4fh^2} - \frac{f''}{2fh} + \frac{h'}{rh^2}, \\ &\qquad R_{\theta\theta} = -\frac{f'}{2rfh} + \frac{h'}{2rh^2} + \frac{h-1}{r^2h}, \\ &\qquad R_{\phi\phi} = R_{\theta\theta} \Big), \end{aligned} \tag{3.12}$$

To go back to the $\mathrm{d}x^\mu$ basis, we have $R_{\mu\nu} = R_{ab}e^a_\mu e^b_\nu$. In particular, here we have $R_{\mu\mu} = R_{aa}(e^a_\mu)^2$, therefore:

$$
\begin{aligned}
R_{\mu\nu} = \mathrm{diag}\Bigg(& \frac{f'^2}{4fh} + \frac{f'h'}{4h^2} - \frac{f''}{2h} - \frac{f'}{rh}, \\
& \frac{f'^2}{4f^2} + \frac{f'h'}{4fh} - \frac{f''}{2f} + \frac{h'}{rh}, \\
& R_{\theta\theta} = -\frac{rf'}{2fh} + \frac{rh'}{2h^2} + \frac{h-1}{h}, \\
& R_{\phi\phi} = R_{\theta\theta}\sin^2\theta \Bigg)
\end{aligned}
\tag{3.13}
$$

4 Jackiw–Teitelboim Gravity

The Jackiw–Teitelboim (JT) action is given by:

$$
S = \frac{1}{16\pi G}\int \mathrm{d}^2x\,\sqrt{-g}\,\Phi\,(R+2) \tag{4.1}
$$

$\frac{\delta S}{\delta \Phi} = 0$ gives us $R = -2$. Now consider $\frac{\delta S}{\delta g^{\mu\nu}}$, and we have:

$$
\begin{aligned}
\delta_g S &= \frac{1}{16\pi G}\int \mathrm{d}^2x\,\Phi\,\delta\Big(\sqrt{-g}\,(R+2)\Big), \quad R = g^{\mu\nu}R_{\mu\nu}, \\
&= \frac{1}{16\pi G}\int \mathrm{d}^2x\,\sqrt{-g}\,\Phi\left\{\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R+2)\right)\delta g^{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}\right\}
\end{aligned}
\tag{4.2}
$$

Note that the $g^{\mu\nu}\,\delta R_{\mu\nu}$ term is a total derivative[2]:

$$
\begin{aligned}
g^{\mu\nu}\,\delta R_{\mu\nu} &= \left(\nabla^\mu\nabla^\nu - g^{\mu\nu}\nabla^\lambda\nabla_\lambda\right)\delta g_{\mu\nu} \\
&= -\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^\lambda\nabla_\lambda\right)\delta g^{\mu\nu} \\
&= \nabla_\lambda\Big(g^{\mu\nu}\,\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\,\delta\Gamma^\nu_{\nu\mu}\Big)
\end{aligned}
\tag{4.3}
$$

In Einstein gravity this gets reduced to a boundary term. But here we have an additional factor of Φ, so after integration by parts, we actually get the equation of motion (EoM) for Φ, up to some boundary terms[3]:

$$
\delta_g S \sim \frac{1}{16\pi G}\int \mathrm{d}^2x\,\sqrt{-g}\left\{\Phi\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R+2)\right) - \Big(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^\lambda\nabla_\lambda\Big)\Phi\right\}\delta g^{\mu\nu} \tag{4.4}
$$

In 2D, we have $R_{\mu\nu} \equiv \frac{1}{2}g_{\mu\nu}R$, so the EoM is simply:

$$
\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^\lambda\nabla_\lambda + g_{\mu\nu}\right)\Phi = 0 \tag{4.5}
$$

Contraction with $g^{\mu\nu}$ further gives us $\big(\nabla^\lambda\nabla_\lambda - 2\big)\Phi = 0$, so in the end we have:

$$
\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\right)\Phi = 0, \quad R = -2 \tag{4.6}
$$

[2] See the amazing lecture note by Matthias Blau at http://www.blau.itp.unibe.ch/GRLecturenotes.html.

[3] See e.g. Section 2 of [3].

A Derivation of the special conformal transformations

Special conformal transformations can be understood as translations conjugated by *inversions*. Note that $\frac{\mathrm{d}z^2}{z^2}$ is invariant under $z \mapsto \frac{1}{z}$; if we include the x^μ directions, we can consider:

$$\mathcal{I}\colon\ \chi^I \mapsto \frac{\chi^I}{\chi^2}, \quad \chi^2 = -t^2 + \vec{x}^2 + z^2, \tag{A.1}$$

$$\mathcal{I}^2 = \mathbb{1}, \quad \mathrm{d}s^2 \mapsto \left(\frac{\delta^I_J - 2\,\frac{\chi^I \chi_J}{\chi^2}}{\chi^2}\,\mathrm{d}\chi^J \right)^2 \Big/ \left(\frac{z}{\chi^2} \right)^2 = \frac{\mathrm{d}\chi^2}{z^2} = \mathrm{d}s^2 \tag{A.2}$$

We see that inversion $\mathcal{I}$ is indeed a (discrete) symmetry of the metric. Here we've defined yet another lower case variable $\chi^I \sim (x^\mu, z)$, which as a contravariant vector has the same components as X^I, but with an index that should be lowered by the flat metric η_{IJ}, i.e. $\chi_I = \eta_{IJ}\chi^J = \eta_{IJ}X^J$. The d special conformal generators are then given by:

$$\begin{aligned}
k_\mu &= \frac{\partial}{\partial a^\mu} \left(\mathcal{I} \circ e^{a^\nu P_\nu} \circ \mathcal{I} \circ X^I \right)_{a=0} \frac{\partial}{\partial X^I} \\
&= \frac{\partial}{\partial a^\mu} \left(\frac{\frac{\chi^I}{\chi^2} + a^I}{\left| \frac{\chi^J}{\chi^2} + a^J \right|^2} \right)_{a=0} \frac{\partial}{\partial X^I} \\
&= \frac{\partial}{\partial a^\mu} \left(\frac{\chi^I + a^I \chi^2}{1 + 2a^I \chi_I + a^2 \chi^2} \right)_{a=0} \frac{\partial}{\partial X^I} \\
&= \chi^2 \partial_\mu - 2x_\mu X^I \partial_I \\
&= \chi^2 \partial_\mu - 2x_\mu \Delta
\end{aligned} \tag{A.3}$$

References

[1] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, **December 1998**. ISBN: 978-0-511-25227-3, 978-0-521-67227-6, 978-0-521-63303-1.

[2] Sean M. Carroll. *Lecture notes on general relativity*, **December 1997**. arXiv: gr-qc/9712019.

[3] Daniel Harlow & Daniel Jafferis. *The Factorization Problem in Jackiw-Teitelboim Gravity*. *JHEP*. **02**:177, **2020**. arXiv: 1804.01081 [hep-th].