

### 1 Partition Function for Compact Scalars

(a) Mode expansion of  $X$  CFT is<sup>1</sup>:

$$\partial X(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\alpha_m}{z^{m+1}}, \quad \bar{\partial} X(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}_m}{\bar{z}^{m+1}}, \quad (1)$$

$$X = x - i\sqrt{\frac{\alpha'}{2}} (\alpha_0 \ln z + \tilde{\alpha}_0 \ln \bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{1}{m} \left( \frac{\alpha_m}{z^m} + \frac{\tilde{\alpha}_m}{\bar{z}^m} \right), \quad (2)$$

Momentum  $p$  is the charge for *spacetime* translation; we have:

$$X \mapsto X + \text{const}, \quad j_a = \frac{i}{\alpha'} \partial_a X, \quad (3)$$

$$p = \frac{1}{2\pi i} \oint_C (dz j - d\bar{z} \tilde{j}) = \frac{1}{\alpha'} \sqrt{\frac{\alpha'}{2}} (\alpha_0 + \tilde{\alpha}_0) = \sqrt{\frac{1}{2\alpha'}} (\alpha_0 + \tilde{\alpha}_0) \quad (4)$$

Additionally, for compact free boson,  $X$  is only defined modulo  $2\pi R$ ; therefore, states after  $X + 2\pi R$  translation should be identical to the original states, i.e.

$$e^{ip(2\pi R)} = \mathbb{1}, \quad p = \frac{n}{R}, \quad n \in \mathbb{Z} \quad (5)$$

This, in fact, holds for any field theory<sup>2</sup> defined for  $X \in S^1$ , including the ordinary quantum mechanics (a classical field theory) on  $S^1$ .

On the other hand, there are additional constraints in string theory: for the state of a *single* closed string, there is a discrete translational symmetry on the *worldsheet*:

$$X(\sigma^1 + 2\pi) \cong X(\sigma^1), \quad X(\sigma^1 + 2\pi) = X(\sigma^1) + 2\pi R w, \quad w \in \mathbb{Z} \quad (6)$$

With some definite winding number  $w$ . In  $(z, \bar{z})$  coordinates, we have:

$$2\pi R w = X(z e^{2\pi i}, \bar{z} e^{-2\pi i}) - X(z, \bar{z}) = -i\sqrt{\frac{\alpha'}{2}} 2\pi i (\alpha_0 - \tilde{\alpha}_0) = 2\pi \sqrt{\frac{\alpha'}{2}} (\alpha_0 - \tilde{\alpha}_0), \quad (7)$$

$$p = \frac{p_L + p_R}{2}, \quad p_L = \sqrt{\frac{2}{\alpha'}} \alpha_0, \quad p_R = \sqrt{\frac{2}{\alpha'}} \tilde{\alpha}_0, \quad (8)$$

$$p_{L,R} = \frac{n}{R} \pm \frac{wR}{\alpha'}, \quad (9)$$

$$X = x - i\frac{\alpha'}{2} (p_L \ln z + p_R \ln \bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{1}{m} \left( \frac{\alpha_m}{z^m} + \frac{\tilde{\alpha}_m}{\bar{z}^m} \right), \quad (10)$$

For the oscillator expressions for  $L_0$ , recall that:

$$T(z) = -\frac{1}{\alpha'} : \partial X \partial X : = \sum_m \frac{L_m}{z^{m+2}}, \quad (11)$$

$$L_{m \neq 0} = \frac{1}{2} \sum_l \alpha_{m-l} \alpha_l, \quad L_0 = \frac{1}{2} : \sum_l \alpha_{-l} \alpha_l : \sim \frac{\alpha' p_L^2}{4} + \sum_{l>0} \alpha_{-l} \alpha_l, \quad (12)$$

<sup>1</sup>Again we follow the convention of *Polchinski*.

<sup>2</sup>Reference: discussions in *Polchinski*, Chapter 8.

The  $L_0$  expression may be off by some normal ordering constant; this ambiguity can be resolved by considering:

$$2L_0 |0, 0; n = w = 0\rangle = (L_1 L_{-1} - L_{-1} L_1) |0, 0; p_L = p_R = 0\rangle = 0 - 0 = 0 \quad (13)$$

Therefore the normal ordering constant is, in fact, trivial, and we have:

$$L_0 = \frac{\alpha' p_L^2}{4} + \sum_{l>0} \alpha_{-l} \alpha_l, \quad \tilde{L}_0 = \frac{\alpha' p_R^2}{4} + \sum_{l>0} \tilde{\alpha}_{-l} \tilde{\alpha}_l, \quad (14)$$

(b) The torus partition function is given by:

$$\langle \mathbb{1} \rangle_{T^2} \equiv Z(\tau = \tau_1 + i\tau_2) = \int \mathcal{D}X e^{-S} = \text{Tr} e^{-(2\pi\tau_2)H} e^{i(2\pi\tau_1)P} \quad (15)$$

Here  $P$  generates *worldsheet* translation along  $\sigma^1$ , not to be confused with  $p$  which generates *spacetime* translation; with  $z = e^{-iw}$ ,  $w = \sigma^1 + i\sigma^2$ ,

$$\begin{aligned} T_{-1}^0 &= \eta^{00} (\partial_0 \sigma^2) T_{21} = -iT_{12} = -i (T_{ww} (\partial_1 w)(\partial_2 w) + T_{\bar{w}\bar{w}} (\partial_1 \bar{w})(\partial_2 \bar{w})) \\ &= T_{ww} - T_{\bar{w}\bar{w}} \\ &= (T_{zz} (\partial_w z)^2 + \frac{c}{24}) - (T_{\bar{z}\bar{z}} (\partial_{\bar{w}} \bar{z})^2 + \frac{\tilde{c}}{24}) \\ &= T(z) (-iz)^2 - \tilde{T}(\bar{z}) (+i\bar{z})^2 + \frac{c - \tilde{c}}{24}, \end{aligned} \quad (16)$$

$$\begin{aligned} P &= \int \frac{d\sigma_1}{2\pi} (-T_{-1}^0) = - \int \frac{d\sigma_1}{2\pi} T(z) (-iz)^2 + \int \frac{d\sigma_1}{2\pi} \tilde{T}(\bar{z}) (+i\bar{z})^2 - \frac{c - \tilde{c}}{24} \\ &= + \oint \frac{dz}{2\pi(-iz)} T(z) (-iz)^2 + \oint \frac{d\bar{z}}{2\pi(+i\bar{z})} \tilde{T}(\bar{z}) (+i\bar{z})^2 - \frac{c - \tilde{c}}{24} \\ &= \oint \frac{dz}{2\pi i} z T(z) - \oint \frac{d\bar{z}}{2\pi i} \bar{z} \tilde{T}(\bar{z}) - \frac{c - \tilde{c}}{24} \\ &= L_0 - \tilde{L}_0 - \frac{c - \tilde{c}}{24} \\ &= (L_0 - \frac{c}{24}) - (\tilde{L}_0 - \frac{\tilde{c}}{24}), \end{aligned} \quad (17)$$

$$\begin{aligned} H &= \int \frac{d\sigma_1}{2\pi} T_0^0 = \int \frac{d\sigma_1}{2\pi} T_{22} \\ &= L_0 + \tilde{L}_0 - \frac{c + \tilde{c}}{24} \\ &= (L_0 - \frac{c}{24}) + (\tilde{L}_0 - \frac{\tilde{c}}{24}), \end{aligned}$$

Here we've used the fact that  $\oint \frac{d\bar{z}}{\bar{z}} = \oint \frac{d\bar{z}}{\bar{z}} = 2\pi i$ . Therefore,

$$Z(\tau) = \text{Tr} e^{-(2\pi\tau_2)H} e^{i(2\pi\tau_1)P} = \text{Tr} q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}, \quad q = e^{2\pi i \tau} \quad (18)$$

Note that here we are working in the grand canonical ensemble, where we have temperature  $\beta = 2\pi\tau_2$  and chemical potential  $2\pi\tau_1$ . At this stage  $P$  is *not* fixed, and we should sum over all states with various  $P$ . To go to the canonical ensemble, we do a Legendre transform and trade  $\tau_1$  for some  $P_0$ . This is implemented by a Laplace / Fourier transform  $\int d\tau_1 e^{-i(2\pi\tau_1)P_0}$  of the partition function.

In string theory, we should actually work with the canonical ensemble (and eventually, the *micro*-canonical ensemble), since we would like to impose the *level matching condition*, namely  $P_0 = 0$ .

However, in string theory, the Legendre transform is implicit as we integrate along  $\tau_1$  in the moduli space; roughly speaking, we have:

$$\int d\tau_1 e^{-i(2\pi\tau_1)(P-0)} \propto \delta(P-0) \quad (19)$$

As  $\tau_1$  gets integrated out, we are effectively computing  $\text{Tr}' e^{-\beta H}$  where the trace only goes over the  $P=0$  sector of the Hilbert space<sup>3</sup>.

On the other hand, since we are computing the grand partition function  $Z(\tau_1 + i\tau_2)$  for now, we should *not* impose the level matching condition, and the sum should go over all states with various  $P$ , although this is a much larger Hilbert space than the physical  $P=0$  subspace in string theory.

Using the expressions in (a), we find that  $L_0$  action on a state  $|\psi\rangle$  created by  $\alpha_{-l}, \tilde{\alpha}_{-l}$  yields the sum of occupation number  $N_l$  weighted by  $l$ :

$$L_0 |\psi\rangle = \left( \frac{\alpha' k_L^2}{4} + \sum_{l>0} l \cdot N_l \right) |\psi\rangle \quad (20)$$

With  $c = \tilde{c} = 1$ , we obtain:

$$\begin{aligned} Z(\tau) &= (q\bar{q})^{-\frac{1}{24}} \sum_{n,w} e^{-2\pi\tau_2 \alpha' \frac{k_L^2 + k_R^2}{4}} e^{2\pi i \tau_1 \alpha' \frac{k_L^2 - k_R^2}{4}} \sum_{(N_l), (\tilde{N}_l)} q^{\sum_{l>0} l \cdot N_l} \bar{q}^{\sum_{l>0} l \cdot \tilde{N}_l} \\ &= (q\bar{q})^{-\frac{1}{24}} \sum_{n,w} e^{-\pi\tau_2 \left( \frac{\alpha' n^2}{R^2} + \frac{w^2 R^2}{\alpha'} \right) + 2\pi i \tau_1 n w} \sum_{(N_l), (\tilde{N}_l)} \prod_{l>0} q^{l \cdot N_l} \bar{q}^{l \cdot \tilde{N}_l} \\ &= |\eta(\tau)|^{-2} \sum_{n,w} e^{-\pi\tau_2 \left( \frac{\alpha' n^2}{R^2} + \frac{w^2 R^2}{\alpha'} \right) + 2\pi i \tau_1 n w} \end{aligned} \quad (21)$$

We've simplified the contributions from the oscillator modes using  $\eta(\tau)$ , since they are identical to the oscillator contributions of the non-compact  $X \in \mathbb{R}^1$ :

$$\begin{aligned} (q\bar{q})^{-\frac{1}{24}} \sum_{(N_l), (\tilde{N}_l)} \prod_{l>0} q^{l \cdot N_l} \bar{q}^{l \cdot \tilde{N}_l} &= (q\bar{q})^{-\frac{1}{24}} \prod_{l>0} \sum_{N_l, \tilde{N}_l=0}^{\infty} q^{l \cdot N_l} \bar{q}^{l \cdot \tilde{N}_l} \\ &= (q\bar{q})^{-\frac{1}{24}} \prod_{l>0} \frac{1}{1-q^l} \frac{1}{1-\bar{q}^l} = |\eta(\tau)|^{-2} \end{aligned} \quad (22)$$

In the  $R \rightarrow \infty$  limit, only the  $w=0$  modes survive; all other modes are exponentially suppressed by the  $e^{-\pi\tau_2 w^2 R^2 / \alpha'}$  factor; i.e.

$$\begin{aligned} Z(\tau) &= |\eta(\tau)|^{-2} \sum_{n,w} \exp \left\{ -\pi\tau_2 \left( \frac{\alpha' n^2}{R^2} + \frac{w^2 R^2}{\alpha'} \right) + 2\pi i \tau_1 n w \right\} \\ &\rightarrow |\eta(\tau)|^{-2} \sum_n \exp \left\{ -\pi\tau_2 \frac{\alpha' n^2}{R^2} \right\}, \quad k = \frac{n}{R} \\ &\rightarrow |\eta(\tau)|^{-2} V \int \frac{dk}{2\pi} \exp \{ -\pi\tau_2 \alpha' k^2 \} \\ &= V |\eta(\tau)|^{-2} (4\pi^2 \alpha' \tau_2)^{-\frac{1}{2}} \\ &\equiv V \cdot Z_X(\tau) = 2\pi R Z_X(\tau) \end{aligned} \quad (23)$$

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<sup>3</sup>This is nicely explained in David Tong's *String Theory*. See also section 6.3 of Blumenhagen et al, *Basic Concepts of String Theory*. See also section 2.1, 2.2 of [arXiv:1912.07654](#) where the switch between ensembles is reviewed in the context of holography.

We recover the partition function  $V \cdot Z_X(\tau)$  for non-compact  $X$ , as expected.

(c) Using the Poisson resummation formula, we find that:

$$Z(\tau) = 2\pi R Z_X(\tau) \sum_{m,w} \exp\left(-\frac{\pi R^2 |m - w\tau|^2}{\alpha' \tau_2}\right) \quad (24)$$

$Z_X(\tau)$  is modular invariant by the properties of the Dedekind  $\eta(\tau)$  function, as is demonstrated for the non-compact  $X$  in *Polchinski*.

The sum, on the other hand, is naturally invariant under  $T: \tau \mapsto \tau + 1$ , by making a change of variables  $m \mapsto m + w$ . It is also invariant under  $S: \tau \mapsto -1/\tau$  with  $m \mapsto -w, w \mapsto m$ <sup>4</sup>. Therefore,  $Z(\tau)$  is modular invariant.

## **2 $\mathbb{Z}_2$ Orbifold**

The  $\mathbb{Z}_2$  orbifold is constructed by imposing an additional identification on  $X \in S^1$ :

$$X \cong -X \quad (25)$$

The target space is then reduced to  $S^1/\mathbb{Z}_2 \cong [0, \pi R]$ .

(a) The first contributions to the orbifold partition function comes from the states that are invariant under reflection  $r$ ; we have:

$$\text{Tr}_{S^1/\mathbb{Z}_2} = \text{Tr}_{S^1} \frac{1+r}{2} = \frac{1}{2} \text{Tr}_{S^1} + \frac{1}{2} \text{Tr}_{S^1} \circ r \quad (26)$$

Acting on  $q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}$ , the first term gives  $\frac{1}{2} Z_{S^1}(\tau)$  where  $Z_{S^1}$  is the  $S^1$  partition function we've obtained in [1](#).

For the second term, note that:

$$r: \left| (N_l), (\tilde{N}_l); n, w \right\rangle \mapsto (-1)^{\sum_l (N_l + \tilde{N}_l)} \left| (N_l), (\tilde{N}_l); -n, -w \right\rangle \quad (27)$$

In particular, it reverses  $n, w$ , hence  $r$  insertion gives vanishing amplitude unless  $n = w = 0$ ; the summation is very much similar to the  $Z_{S^1}$  case, i.e. we have:

$$\begin{aligned} \frac{1}{2} \text{Tr}_{S^1} \left( r q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \right) &= \frac{1}{2} (q\bar{q})^{-\frac{1}{24}} \prod_{l>0} \sum_{N_l, \tilde{N}_l=0}^{\infty} (-1)^{N_l + \tilde{N}_l} q^{l \cdot N_l} \bar{q}^{l \cdot \tilde{N}_l} \\ &= \frac{1}{2} (q\bar{q})^{-\frac{1}{24}} \prod_{l>0} \frac{1}{1 - (-q^l)} \frac{1}{1 - (-\bar{q}^l)} = \left| \frac{\eta(\tau)}{\theta_2(\tau)} \right| \end{aligned} \quad (28)$$

Where we've used the fact that<sup>5</sup>:  $q^{-\frac{1}{24}} \prod_{l>0} \frac{1}{1 - (-q^l)} = \sqrt{2} \sqrt{\frac{\eta(\tau)}{\theta_2(\tau)}}$ . Therefore, the total contributions from  $r$ -invariant states are:

$$\frac{1}{2} Z_{S^1}(\tau) + \left| \frac{\eta(\tau)}{\theta_2(\tau)} \right| \quad (29)$$

<sup>4</sup>Reference: *Polchinski*.

<sup>5</sup>Reference: Blumenhagen & Plauschinn, *Introduction to CFT*, and also *Polchinski*.

(b) With  $X \cong -X$ , new possibilities emerge as the boundary condition along  $\sigma^1$ :

$$X(\sigma^1 + 2\pi) \cong X(\sigma^1), \quad X(\sigma^1 + 2\pi) = \pm X(\sigma^1) + 2\pi R w, \quad w \in \mathbb{Z} \quad (30)$$

The “ $-$ ” sign corresponds to the *twisted states*. Due to the anti-periodicity,  $\partial X$  has a half-integer mode expansion:

$$\partial X(z e^{2\pi i}) = -\partial X(z), \quad (31)$$

$$\partial X(z) = -i \sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\alpha_{m-\frac{1}{2}}}{z^{m+\frac{1}{2}}}, \quad \bar{\partial} X(\bar{z}) = -i \sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}_{m-\frac{1}{2}}}{\bar{z}^{m+\frac{1}{2}}}, \quad (32)$$

$$X = x + i \sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{1}{m + \frac{1}{2}} \left( \frac{\alpha_{m+\frac{1}{2}}}{z^{m+\frac{1}{2}}} + \frac{\tilde{\alpha}_{m+\frac{1}{2}}}{\bar{z}^{m+\frac{1}{2}}} \right), \quad (33)$$

Apply the boundary condition on  $X$ , and we find that  $x = \pi R w'$ ; however, due to the identification  $X + 2\pi R \cong X \cong -X$ , there are only two inequivalent choices:  $x = 0$  and  $x = \pi R$ , which correspond to the string localized around either of the two fixed points of the  $\mathbb{Z}_2$  action.

Much similar to the case in [1], we have:

$$\left[ \alpha_{\frac{1}{2}+l}, \alpha_{-\frac{1}{2}-l} \right] = \frac{1}{2} + l, \quad (34)$$

$$L_{m \neq 0} = \frac{1}{2} \sum_l \alpha_{m-\frac{1}{2}-l} \alpha_{\frac{1}{2}+l}, \quad L_0 = \frac{1}{2} : \sum_l \alpha_{-\frac{1}{2}-l} \alpha_{\frac{1}{2}+l} : \sim \sum_{l \geq 0} \alpha_{-\frac{1}{2}-l} \alpha_{\frac{1}{2}+l} \quad (35)$$

We can use the same trick to fix the normal ordering constant in  $L_0$ ; this time it is non-trivial:

$$L_{-1} = \frac{1}{2} \alpha_{-\frac{1}{2}}^2 + \sum_{l \geq 0} \alpha_{-\frac{1}{2}-l} \alpha_{\frac{1}{2}+l}, \quad L_1 = \frac{1}{2} \alpha_{\frac{1}{2}}^2 + \sum_{l > 0} \alpha_{\frac{1}{2}-l} \alpha_{\frac{1}{2}+l}, \quad (36)$$

$$\begin{aligned} L_0 |0, 0; x\rangle &= \frac{1}{2} (L_1 L_{-1} - L_{-1} L_1) |0, 0; x\rangle \\ &= \frac{1}{2} \times \frac{1}{4} \alpha_{\frac{1}{2}}^2 \alpha_{-\frac{1}{2}}^2 |0, 0; x\rangle - 0 \\ &= \frac{1}{16} |0, 0; x\rangle, \end{aligned} \quad (37)$$

$$L_0 = \frac{1}{16} + \sum_{l \geq 0} \alpha_{-\frac{1}{2}-l} \alpha_{\frac{1}{2}+l} = \frac{1}{16} + \sum_{l \geq 0} \left( l + \frac{1}{2} \right) N_{l+\frac{1}{2}} = \frac{1}{16} + \sum_{l > 0} \left( l - \frac{1}{2} \right) N_{l-\frac{1}{2}}, \quad (38)$$

The trace can then be computed, following the same recipe as before:

$$\begin{aligned} \text{Tr}_{S^1} \left( \frac{1+r}{2} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\tilde{c}}{24}} \right) &= (q\bar{q})^{-\frac{1}{24} + \frac{1}{16}} \prod_{l+\frac{1}{2} \in \mathbb{Z}^+} \sum_{N_l, \tilde{N}_l=0}^{\infty} \frac{1 + (-1)^{N_l + \tilde{N}_l}}{2} q^{l \cdot N_l} \bar{q}^{l \cdot \tilde{N}_l} \times 2 \\ &= \frac{1}{2} (q\bar{q})^{+\frac{1}{48}} \left\{ \prod_{l>0} \left| \frac{1}{1 - q^{l-\frac{1}{2}}} \right|^2 + \prod_{l>0} \left| \frac{1}{1 + q^{l-\frac{1}{2}}} \right|^2 \right\} \times 2 \\ &= \left| \frac{\eta(\tau)}{\theta_4(\tau)} \right| + \left| \frac{\eta(\tau)}{\theta_3(\tau)} \right| \end{aligned} \quad (39)$$

There is an extra factor of 2 from the number of twisted sectors:  $x = 0$  and  $x = \pi R$ .

(c) The full partition function is therefore:

$$Z(\tau) = \frac{1}{2} Z_{S^1}(\tau) + \left| \frac{\eta(\tau)}{\theta_2(\tau)} \right| + \left| \frac{\eta(\tau)}{\theta_4(\tau)} \right| + \left| \frac{\eta(\tau)}{\theta_3(\tau)} \right| \quad (40)$$

The first term is modular invariant, as is proved in [1].

The remaining terms are also modular invariant, due to the transformational properties of  $\eta$  and  $\theta$  functions<sup>6</sup>:

$$T \circ \left| \frac{\eta(\tau)}{\theta_2(\tau)} \right| \xleftrightarrow{S} \left| \frac{\eta(\tau)}{\theta_4(\tau)} \right| \xleftrightarrow{T} \left| \frac{\eta(\tau)}{\theta_3(\tau)} \right| \circ S \quad (41)$$

Therefore, the full partition function is modular invariant.

### [3] Torus 4-point function in $bc$ CFT

$$\langle c(w_1) b(w_2) \tilde{c}(\bar{w}_3) \tilde{b}(\bar{w}_4) \rangle = \int \mathcal{D}b \mathcal{D}\tilde{b} \mathcal{D}c \mathcal{D}\tilde{c} c(w_1) b(w_2) \tilde{c}(\bar{w}_3) \tilde{b}(\bar{w}_4) e^{-S'} \equiv Z' \quad (42)$$

First we argue that only the zero modes of the insertions survive the path integral<sup>7</sup>. In fact, as anti-commuting replacements of the gauge degrees of freedom, ghost modes are *defined* to be the eigenvalues of  $P^\dagger P$ , where  $P$  is the conformal Killing differential<sup>8</sup>. More specifically, given a conformal Killing vector (CKV)  $\delta\sigma^a$ , the conformal Killing equation can be written as:

$$P \delta\sigma = 0 \quad (43)$$

While  $P^\dagger \delta'g = 0$  gives moduli variation  $\delta'g_{ab}$  of the metric. Roughly speaking,  $P$  captures the variation of gauge fixing under an arbitrary gauge transformation; naturally, CKV's are given by  $(\ker P)$ , while  $(\det P) \sim \Delta_{FP}$  is the Faddeev–Popov functional measure near the gauge slice.  $(\det P)$  can then be calculated with:

$$\delta\sigma^a \mapsto c^a, \quad \delta'g_{ab} \mapsto b_{ab}, \quad \Delta_{FP} \sim \det P \sim \int \mathcal{D}b \mathcal{D}\tilde{b} \mathcal{D}c \mathcal{D}\tilde{c} e^{-S'}, \quad (44)$$

$$S' = \frac{1}{2\pi} \int d^2\sigma g^{1/2} b_{ab} (P \cdot c)^{ab} = \frac{1}{2\pi} \int d^2w (b \bar{\partial}_w c + \tilde{b} \partial_w \tilde{c}) \quad (45)$$

In the end we have chosen conformal gauge, such that<sup>9</sup>  $P \sim (\bar{\partial}_w, \partial_w)$ ,  $P^\dagger P \sim -\bar{\partial}_w \partial_w = -\nabla^2$ . In the  $w = \sigma^1 + i\sigma^2$  coordinates, CKV's are simple translations:  $c^a = \text{const}$ ; with  $z = e^{-iw}$ , it gets mapped to  $c^z = c^w \partial_w z = c^w (-iz)$ , which agrees with the zero mode  $c_0$  in the  $c(z)$  expansion:

$$c(z) = \sum_{m=-\infty}^{\infty} \frac{c_m}{z^{m+1-\lambda}} = c_0 z + \sum_{m \neq 0} \frac{c_m}{z^{m-1}}, \quad \lambda = 2 \quad (46)$$

Now we are finally ready to prove our argument: for anti-commuting variables like  $c(z)$ ,

$$\int \mathcal{D}c \sim \prod_m \int dc_m \sim \prod_m \frac{\partial}{\partial c_m} \quad (47)$$

<sup>6</sup>Reference: *Blumenhagen & Plauschinn*.

<sup>7</sup>I would like to thank 谷夏 for some very helpful discussions about this problem.

<sup>8</sup>Reference: *Polchinski*, Chapter 3 & 5.

<sup>9</sup>References:

- Nakahara, *Geometry, Topology and Physics*;
- Blumenhagen et al, *Basic Concepts of String Theory*.

Since  $c_0$  corresponds to a CKV,  $P \cdot c_0 = 0$ , therefore it vanishes in  $S' = \int d^2\sigma (b \cdot P \cdot c)$ ; for the path integral to be non-zero, there has to be some additional  $c_0$  insertions, i.e.

$$Z' \sim \int \mathcal{D}b \mathcal{D}\tilde{b} \mathcal{D}c \mathcal{D}\tilde{c} c_0 b_0 \tilde{c}_0 \tilde{b}_0 e^{-S'} \sim \left(\frac{1}{\sqrt{\tau_2}}\right)^4 \int \mathcal{D}'b \mathcal{D}'\tilde{b} \mathcal{D}'c \mathcal{D}'\tilde{c} e^{-S'}, \quad \int \mathcal{D}'c \sim \prod_{m \neq 0} \int dc_m \quad (48)$$

Note the additional  $\left(\frac{1}{\sqrt{\tau_2}}\right)^4$  factor coming from the zero modes<sup>10</sup>; this has to do with the normalization of the zero modes, each contributing a factor of  $\frac{1}{\sqrt{A}}$ , where  $A \sim \tau_2$  is the volume (surface area) of the torus. On a different note, since it is very difficult, if not impossible, to keep track of various (often divergent) constant factors in the path integral, we have been and will be calculating  $Z'$  up to an overall constant coefficient.

Now we have to deal with the path integral over non-zero modes. Note that the holomorphic mode expansion (46) is incomplete for our purpose: it gives the *on-shell* mode expansion, while our path integral should go over all possible configurations, including the off-shell modes, which is *not* holomorphic. However, on  $T^2 = S^1 \times S^1$ , the full modes are simple<sup>11</sup>:

$$-\nabla^2 \psi_{n_1, n_2} = \lambda_{n_1, n_2} \psi_{n_1, n_2}, \quad (49)$$

$$\begin{aligned} \psi_{n_1, n_2} &= \exp\left(i(n_1 \tilde{\sigma}^1 + n_2 \tilde{\sigma}^2)\right), \quad \tilde{\sigma}^2 = \frac{\sigma^2}{\tau_2}, \quad \tilde{\sigma}^1 = \sigma^1 - \sigma^2 \frac{\tau_1}{\tau_2}, \\ &= \exp\left\{i\left(n_1 \sigma^1 + \frac{n_2 - n_1 \tau_1}{\tau_2} \sigma^2\right)\right\}, \end{aligned} \quad (50)$$

Here we first use the “rectangular” coordinates  $(\tilde{\sigma}^1, \tilde{\sigma}^2) \in [0, 2\pi]^2$  to write down the obvious eigenfunctions  $\psi_{n_1, n_2}$ , and then relate them back to the  $(\sigma^1, \sigma^2)$  coordinates. Therefore, we have:

$$\begin{aligned} \lambda_{n_1, n_2} &= \left\{n_1^2 + \left(\frac{n_2 - n_1 \tau_1}{\tau_2}\right)^2\right\} \\ &= \frac{1}{\tau_2^2} \left\{(n_1 \tau_2)^2 + (n_1 \tau_1 - n_2)^2\right\} \\ &= \frac{1}{\tau_2^2} |n_1 \tau - n_2|^2, \end{aligned} \quad (51)$$

$$\det' P \sim \left(\prod'_{n_1, n_2} \sqrt{\lambda_{n_1, n_2}}\right)^2 \sim \prod'_{n_1, n_2} \lambda_{n_1, n_2} \quad (52)$$

The determinant can be computed with  $\zeta$ -function regularization, as is performed in detail in *Di Francesco*; the result can be nicely summarized using the Eisenstein series, as shown in *Nakahara*:

$$E(\tau, s) = \sum_{n_1, n_2} \frac{\tau_2^s}{|n_1 \tau - n_2|^{2s}}, \quad (53)$$

$$\det' P \sim \prod'_{n_1, n_2} \frac{1}{\tau_2^2} |n_1 \tau - n_2|^2 \sim \tau_2 \exp\left\{-\partial_s E'(\tau, s)_{s=0}\right\} = \tau_2^2 |\eta(\tau)|^4 \quad (54)$$

Finally, we have:

$$Z' \sim \tau_2^{-2} \det' P \sim \tau_2^{-2} \tau_2^2 |\eta(\tau)|^4 \sim |\eta(\tau)|^4 \quad (55)$$

<sup>10</sup>Reference: *Di Francesco et al.*

<sup>11</sup>References: (1) *Nakahara*, (2) *Di Francesco et al.*, and (3) <http://theory.uchicago.edu/~sethi/Teaching/P483-W2018/p483-sol3.pdf>.

#### 4 Torus Propagator as a Trace

$$w' \rightarrow 0, \quad \langle \partial_w X(w) \partial_{w'} X(w') \rangle = \text{Tr} \left( \partial_w X(w) \partial_{w'} X(w') q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \right) \quad (56)$$

Here we've dropped the time ordering in the  $w' \rightarrow 0$  limit. Recall the mode expansion of  $\partial X$  in [1]; we see that only the “diagonal” components of  $\partial X(w) \partial X(w')$  survive in the trace, i.e.

$$\begin{aligned} \partial_w X(w) \partial_{w'} X(w') &= (\partial_w z)(\partial_{w'} z') \partial_z X(z) \partial_{z'} X(z'), \quad z = e^{-iw}, \quad 1 \leq |z| \leq e^{2\pi\tau_2} \\ &\sim -\frac{\alpha'}{2} \sum_{n=-\infty}^{\infty} \frac{\alpha_{-n} \alpha_n}{z^{-n+1} z'^{n+1}} (-iz)(-iz') \\ &= \frac{\alpha'}{2} \left( \alpha_0^2 + \sum_{n>0} \left( \left( \frac{z}{z'} \right)^n + \left( \frac{z'}{z} \right)^n \right) \alpha_{-n} \alpha_n + \sum_{n>0} n \left( \frac{z'}{z} \right)^n \right) \\ &= \frac{\alpha'}{2} \left( \alpha_0^2 + \sum_{n>0} \left( \left( \frac{z}{z'} \right)^n + \left( \frac{z'}{z} \right)^n \right) \alpha_{-n} \alpha_n + \frac{zz'}{(z-z')^2} \right) \end{aligned} \quad (57)$$

The last term is a normal ordering constant; here it is naturally regularized by  $\left(\frac{z'}{z}\right)^n$ .

The  $\alpha_0^2$  term can be substituted with spacetime momentum  $p$ ; we have:

$$p = \sqrt{\frac{1}{2\alpha'}} (\alpha_0 + \tilde{\alpha}_0) = \sqrt{\frac{1}{2\alpha'}} 2\alpha_0 = \sqrt{\frac{2}{\alpha'}} \alpha_0, \quad (58)$$

$$\partial_w X(w) \partial_{w'} X(w') \sim \frac{\alpha'}{2} \left( \frac{\alpha' p^2}{2} + \sum_{n>0} \left( \left( \frac{z}{z'} \right)^n + \left( \frac{z'}{z} \right)^n \right) n N_n \right) \quad (59)$$

On the other hand, the partition function is:

$$\begin{aligned} Z(\tau) &= \langle \mathbb{1} \rangle = (q\bar{q})^{-\frac{1}{24}} V \int \frac{dk}{2\pi} e^{-\pi\tau_2 \alpha' k^2} \sum_{(N_l), (\tilde{N}_l)} q^{\sum_{l>0} l \cdot N_l} \bar{q}^{\sum_{l>0} l \cdot \tilde{N}_l} \\ &= (q\bar{q})^{-\frac{1}{24}} V \int \frac{dk}{2\pi} e^{-\pi\tau_2 \alpha' k^2} \sum_{(N_l), (\tilde{N}_l)} \prod_{l>0} q^{l \cdot N_l} \bar{q}^{l \cdot \tilde{N}_l} \\ &= |\eta(\tau)|^{-2} V \int \frac{dk}{2\pi} e^{-\pi\tau_2 \alpha' k^2} \end{aligned} \quad (60)$$

We can work out  $Z^{-1} \langle \partial X \partial X \rangle$  by considering term by term insertion of the  $\partial X \partial X$  mode expansion into the above expression. For the  $\frac{\alpha' p^2}{2}$  term, we have a contribution of:

$$\frac{\int \frac{dk}{2\pi} \frac{\alpha' k^2}{2} e^{-\pi\tau_2 \alpha' k^2}}{\int \frac{dk}{2\pi} e^{-\pi\tau_2 \alpha' k^2}} = \frac{\alpha'}{2} \frac{1}{2 \cdot \pi \alpha' \tau_2} = \frac{1}{4\pi\tau_2} \quad (61)$$

For the  $n N_n$  insertion, we have a contribution of:

$$\begin{aligned} \frac{\sum_{(N_l)} n N_n q^{\sum_{l>0} l \cdot N_l}}{\sum_{(N_l)} q^{\sum_{l>0} l \cdot N_l}} &= \frac{\sum_{(N_l)} n N_n \prod_{l>0} q^{l \cdot N_l}}{\sum_{(N_l)} \prod_{l>0} q^{l \cdot N_l}} = \frac{\sum_{N_n=0}^{\infty} n N_n q^{n \cdot N_n}}{\sum_{N_n=0}^{\infty} q^{n \cdot N_n}} = \frac{n q^n \frac{\partial}{\partial(q^n)} \sum_{N_n=0}^{\infty} q^{n \cdot N_n}}{\sum_{N_n=0}^{\infty} q^{n \cdot N_n}} \\ &= \frac{n q^n \frac{\partial}{\partial(q^n)} \frac{1}{1-q^n}}{\frac{1}{1-q^n}} = \frac{n q^n}{1-q^n} \end{aligned} \quad (62)$$



Therefore, the complete result is given by:

$$\begin{aligned} \frac{1}{Z(\tau)} \langle \partial_w X(w) \partial_{w'} X(w') \rangle &= \frac{\alpha'}{2} \left( \frac{1}{4\pi\tau_2} + \sum_{n>0} \left( \left( \frac{z}{z'} \right)^n + \left( \frac{z'}{z} \right)^n \right) \frac{nq^n}{1-q^n} + \frac{zz'}{(z-z')^2} \right) \\ &\xrightarrow[z' \rightarrow 1]{w' \rightarrow 0} \frac{\alpha'}{2} \left( \frac{1}{4\pi\tau_2} + \sum_{n>0} (z^n + z^{-n}) \frac{nq^n}{1-q^n} + \frac{z}{(z-1)^2} \right) \end{aligned} \quad (63)$$

On the other hand, the torus propagator is given by:

$$G'(w, \bar{w}; w', \bar{w}') = -\frac{\alpha'}{2} \ln |f(w - w', \tau)|^2 + \frac{\alpha'}{4\pi\tau_2} (\text{Im}(w - w'))^2, \quad (64)$$

$$f(w, \tau) \equiv \theta_1 \left( \frac{w}{2\pi} \middle| \tau \right) = 2 e^{\frac{i\pi\tau}{4}} \sin \frac{w}{2} \prod_{m>0} (1 - q^m)(1 - z^{-1}q^m)(1 - zq^m), \quad z = e^{-iw} \quad (65)$$

We find that  $\partial_w \partial_{w'} G'$  contains the same zero mode contribution  $\frac{\alpha'}{8\pi\tau_2}$  and normal ordering contribution  $\frac{\alpha'}{2} \frac{z}{(z-1)^2}$  as in (63):

$$\partial_w \partial_{w'} G'(w, \bar{w}; w', \bar{w}')_{w'=0} = \frac{\alpha'}{8\pi\tau_2} + \frac{\alpha'}{2} \partial_w^2 \ln f(w, \tau), \quad (66)$$

$$\partial_w^2 \ln f(w, \tau) = \partial_w^2 \ln \sin \frac{w}{2} + \partial_w^2 \sum_{m>0} \left( \ln(1 - zq^m) + \ln(1 - z^{-1}q^m) \right), \quad (67)$$

$$\partial_w^2 \ln \sin \frac{w}{2} = \partial_w^2 \ln \sin \frac{w}{2} = -\frac{1}{4 \sin^2 \frac{w}{2}} = \frac{1}{2(\cos w - 1)} = \frac{1}{z + z^{-1} - 2} = \frac{z}{(z-1)^2}, \quad (68)$$

The remaining parts come from oscillator modes; they also match with (63), but the equivalence is less obvious: we have<sup>12</sup>:

$$\begin{aligned} \partial_w^2 \sum_{m>0} \ln(1 - zq^m) &= \partial_w^2 \sum_{m>0} \sum_{n>0} -\frac{1}{n} (zq^m)^n \\ &= \sum_{n>0} \partial_w^2 \left( -\frac{1}{n} z^n \right) \sum_{m>0} q^{mn}, \quad \partial_w = -iz \partial_z \\ &= \sum_{n>0} -\frac{(-in)^2}{n} z^n \cdot \frac{q^n}{1-q^n} \\ &= \sum_{n>0} z^n \frac{nq^n}{1-q^n}, \end{aligned} \quad (69)$$

$$\partial_w^2 \sum_{m>0} \ln(1 - z^{-1}q^m) = \sum_{n>0} z^{-n} \frac{nq^n}{1-q^n}, \quad (70)$$

This is precisely the contribution from oscillator modes in (63). Therefore, we have:

$$\frac{1}{Z(\tau)} \langle \partial_w X(w) \partial_{w'} X(w') \rangle_{w'=0} = \partial_w \partial_{w'} G'(w, \bar{w}; w', \bar{w}')_{w'=0} \quad (71)$$

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<sup>12</sup>Reference: <http://theory.uchicago.edu/~sethi/Teaching/P483-W2018/p483-sol13.pdf>. I would like to thank Lucy Smith for providing this hint.