String Theory №4

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1 Stringy Physics!

$$T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu :, \quad \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X^\mu \bar{\partial} X_\mu :, \tag{1}$$

$$V_k = :e^{ik\cdot X(z,\bar{z})}:, \quad G_{e,k} = e_{\mu\nu} :\partial X_z^\mu \, \bar{\partial} X_{\bar{z}}^\nu \, e^{ik\cdot X(z,\bar{z})}:, \tag{2}$$

We sometimes use subscripts like ∂X_z^μ to denote variable dependence to avoid clutter.

(a) The weight of a primary operator is given by its OPE with T and $\tilde{T}$. For exponential operators, there is a neat formula for cross contractions[1]:

$$
\begin{aligned}
T(z)\, V_k(w,\bar{w}) &= \exp\left\{ \int \mathrm{d}^2 z' \int \mathrm{d}^2 w' \, \overbrace{X_{z'}^\mu X_{w'}^\nu} \frac{\delta}{\delta X_{z'}^\mu} \frac{\delta}{\delta X_{w'}^\nu} \right\} :T_z\, e^{ik\cdot X_w} : \\
&= \exp\left\{ \int \mathrm{d}^2 z' \, \overbrace{X_{z'}^\mu X_w^\nu} \frac{\delta}{\delta X_{z'}^\mu} \, ik_\nu \right\} :T_z\, e^{ik\cdot X_w} : \\
&= :\left\{ \exp\left(ik_\nu \int \mathrm{d}^2 z' \, \overbrace{X_{z'}^\mu X_w^\nu} \frac{\delta}{\delta X_{z'}^\mu} \right) T_z \right\} e^{ik\cdot X_w} : \\
&\sim -\frac{1}{\alpha'} :\left\{ 2\partial_z\left(ik_\sigma \overbrace{X_z^\mu X_w^\sigma}\right) \partial_z X_\mu + \partial_z\left(ik_\rho \overbrace{X_z^\mu X_w^\rho}\right) \partial_z\left(ik_\sigma \overbrace{X_{z,\mu} X_w^\sigma}\right) \right\} e^{ik\cdot X_w} : \\
&\sim -\frac{1}{\alpha'} :\left\{ 2\left(-\frac{\alpha'}{2}\frac{ik^\mu}{z-w}\right) \partial_z X_\mu + \left(-\frac{\alpha'}{2}\frac{ik^\mu}{z-w}\right)\left(-\frac{\alpha'}{2}\frac{ik_\mu}{z-w}\right) \right\} e^{ik\cdot X_w} : \\
&\sim \frac{\alpha' k^2}{4}\frac{V_k(w,\bar{w})}{(z-w)^2} + \frac{\partial V_k(w,\bar{w})}{z-w}
\end{aligned}
\tag{3}
$$

Here we've used the result that $ik_\sigma \overbrace{X_z^\mu X_w^\sigma} = ik^\mu(-\frac{\alpha'}{2})\ln|z-w|^2$. We see that V_k is a primary of weight $(1,1)$ iff. $\frac{\alpha' k^2}{4} = 1$, or $m^2 = -k^2 = -\frac{4}{\alpha'}$. This is the mass shell condition for the closed string tachyon (at level 0). On the other hand,

$$G_{e,k} = e_{\mu\nu} G_k^{\mu\nu}, \tag{4}$$

$$
\begin{aligned}
T(z)\, G_k^{\mu\nu}(0) &\sim :\overbrace{T_z\, \partial X_0^\mu}\, \bar{\partial} X_0^\nu\, e^{ik\cdot X_0} : + :\overbrace{T_z\, \partial X_0^\mu\, \bar{\partial} X_0^\nu\, e^{ik\cdot X_0}} : + :\overbrace{T_z\, \partial X_0^\mu}\, \bar{\partial} X_0^\nu\, e^{ik\cdot X_0} : \\
&\quad + :\overbrace{T_z\, \partial X_0^\mu\, \bar{\partial} X_0^\nu}\, e^{ik\cdot X_0} : + :\overbrace{T_z\, \partial X_0^\mu}\, \bar{\partial} X_0^\nu\, e^{ik\cdot X_0} : \\
&\sim \left(\frac{1}{z^2} G_k^{\mu\nu}(0) + \frac{1}{z} :\partial^2 X_0^\mu\, \bar{\partial} X_0^\nu\, e^{ik\cdot X_0} : \right) + \left(\frac{\alpha' k^2}{4}\frac{1}{z^2} G_k^{\mu\nu}(0) + \frac{1}{z} :\partial X_0^\mu\, \bar{\partial} X_0^\nu\, \partial\, e^{ik\cdot X_0} : \right) \\
&\quad - \frac{2}{\alpha'}\left(-\frac{\alpha'}{2}\eta^{\sigma\mu}\frac{1}{z^2} \right)\left(-\frac{\alpha'}{2}\frac{ik_\sigma}{z} \right) :\bar{\partial} X_0^\nu\, e^{ik\cdot X_0} : \\
&\sim ik^\mu :\bar{\partial} X_0^\nu\, e^{ik\cdot X_0} : \left(-\frac{\alpha'}{2} \right)\frac{1}{z^3} + \left(1 + \frac{\alpha' k^2}{4} \right)\frac{G_k^{\mu\nu}(0)}{z^2} + \frac{\partial G_k^{\mu\nu}(0)}{z},
\end{aligned}
\tag{5}
$$

$$\tilde{T}(\bar{z})\, G_k^{\mu\nu}(0) \sim ik^\nu :\partial X_0^\mu\, e^{ik\cdot X_0} : \left(-\frac{\alpha'}{2} \right)\frac{1}{\bar{z}^3} + \left(1 + \frac{\alpha' k^2}{4} \right)\frac{G_k^{\mu\nu}(0)}{\bar{z}^2} + \frac{\partial G_k^{\mu\nu}(0)}{\bar{z}}, \tag{6}$$

Therefore, $G_{e,k}$ is a primary of weight $(1,1)$ iff. $1 + \frac{\alpha' k^2}{4} = 1$ and $k^\mu e_{\mu\nu} = 0 = k^\nu e_{\mu\nu}$. The first equation gives the mass shell condition $m^2 = -k^2 = 0$ for a massless boson, while the second equation

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[1]Reference: *Polchinski*, and physics.stackexchange.com/a/389193.

constrains the polarization to be transverse. These are the physical constraints for a massless gauge boson, which is the level 1 excitation for a bosonic closed string.

(b) The form of any primary 3-point function is completely fixed by $\mathrm{PSL}(2, \mathbb{C})$ invariance[2]. In fact, for any holomorphic $\phi_i(z_i)$ with weight h_i, by translational invariance, we have:

$$\langle \phi_1(z_1) \, \phi_2(z_2) \, \phi_3(z_3) \rangle = f(z_{12}, z_{23}, z_{31}), \quad z_{ij} = z_i - z_j, \tag{7}$$

Furthermore, scaling invariance requires that f is quasi-homogeneous:

$$z \mapsto z' = \lambda^{-1} z, \quad f \mapsto \left\langle \lambda^{h_1} \phi_1(\lambda z_1) \, \lambda^{h_2} \phi_2(\lambda z_2) \, \lambda^{h_3} \phi_3(\lambda z_3) \right\rangle$$

$$= \lambda^{h_1 + h_2 + h_3} f(\lambda z_{12}, \lambda z_{23}, \lambda z_{31}) \tag{8}$$

$$= f(z_{12}, z_{23}, z_{31}),$$

$$f = \sum_{a+b+c=\sum_i h_i} f_{abc} = \sum_{a+b+c=\sum_i h_i} \frac{C_{abc}}{z_{12}^a z_{23}^b z_{31}^c} \tag{9}$$

On the other hand, for special conformal transformation[3] $\frac{1}{\bar{z}} \mapsto \frac{1}{\bar{z}'} = \frac{1}{\bar{z}} + a$, we have:

$$z \longmapsto z' = \frac{1}{\frac{1}{z} + \bar{a}} = \frac{z}{1 + z\bar{a}} = w(z), \quad \frac{\partial z}{\partial z'} = \frac{1}{(1 - z\bar{a})^2} = \frac{1}{\kappa^2}, \quad z_{ij} = \frac{z_{ij}'}{\kappa_i \kappa_j}, \tag{10}$$

$$f \longmapsto f\left(w^{-1}(z_{12}), w^{-1}(z_{23}), w^{-1}(z_{31}) \right) \frac{1}{\kappa_1^{2h_1} \kappa_2^{2h_2} \kappa_3^{2h_3}} = f(z_{12}, z_{23}, z_{31}), \tag{11}$$

$$f_{abc}\left(w^{-1}(z_{12}), w^{-1}(z_{23}), w^{-1}(z_{31}) \right) = f_{abc}(z_{12}, z_{23}, z_{31}) \, \kappa_1^{c+a} \kappa_2^{a+b} \kappa_3^{b+c}, \tag{12}$$

We see that f is invariant under special conformal transformation iff. $f = f_{abc}$ where:

$$c + a = 2h_1, \quad a + b = 2h_2, \quad b + c = 2h_3, \tag{13}$$

$$\text{i.e.} \quad a = h_1 + h_2 - h_3, \quad b = h_2 + h_3 - h_1, \quad c = h_3 + h_1 - h_2, \tag{14}$$

In the above discussions we've restricted ϕ_i to be holomorphic; for *spin-less* $\phi_i = \phi_i(z, \bar{z})$, $h_i = \tilde{h}_i$, $\Delta_i = h_i + \tilde{h}_i$, the holomorphic and anti-holomorphic contributions can be nicely combined, and we have:

$$f = \frac{C}{|z_{12}|^{2a} |z_{23}|^{2b} |z_{31}|^{2c}}, \tag{15}$$

$$2a = \Delta_1 + \Delta_2 - \Delta_3, \quad 2b = \Delta_2 + \Delta_3 - \Delta_1, \quad 2c = \Delta_3 + \Delta_1 - \Delta_2, \tag{16}$$

$$\langle V_{k_1}(z_1, \bar{z}_1) \, V_{k_2}(z_2, \bar{z}_2) \, G_{e,k_3}(z_3, \bar{z}_3) \rangle = \frac{A(k_1, k_2, e)}{|z_{12}|^2 |z_{23}|^2 |z_{31}|^2} \tag{17}$$

(c) Following the recipe in (a), we have:

$$V_{k_1}(z_1, \bar{z}_1) \, V_{k_2}(z_2, \bar{z}_2) = \, :\exp\left(i k_{1,\mu} i k_{2,\nu} \overline{X_1^\mu X_2^\nu} \right) e^{i k_1 \cdot X_1} \, e^{i k_2 \cdot X_2} :$$

$$= \exp\left(\frac{\alpha'}{2} \, k_1 \cdot k_2 \ln |z_{12}|^2 \right) : e^{i k_1 \cdot X_1} \, e^{i k_2 \cdot X_2} : \tag{18}$$

$$= |z_{12}|^{\alpha' k_1 \cdot k_2} : e^{i k_1 \cdot X_1} \, e^{i k_2 \cdot X_2} :$$

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[2]Reference: Blumenhagen, *Introduction to CFT*, and also *Di Francesco et al.*

[3]See *Di Francesco et al*, and also github.com/davidsd/ph229.

Apply the on-shell conditions, and we find that:

$$\alpha' k_1 \cdot k_2 = \frac{\alpha'}{2}(k_1 + k_2)^2 - \frac{\alpha'}{2}k_1^2 - \frac{\alpha'}{2}k_2^2 = \frac{\alpha'}{2}(-k_3)^2 - \frac{\alpha'}{2}k_1^2 - \frac{\alpha'}{2}k_2^2 = 0 - 2 - 2 = -4 \qquad (19)$$

We are interested in the $|z_{12}|^{-2}$ term in the OPE around z_2; it will contribute to the 3-point function discussed in (b). Note that:

$$
\begin{aligned}
:e^{ik_1 \cdot X_1} e^{ik_2 \cdot X_2}: &= :\left(\cdots + \frac{1}{2}(ik_1 \cdot X_1)^2 + \cdots\right) e^{ik_2 \cdot X_2}: \\
&= :\left(\cdots - \frac{1}{2}k_1^\mu k_1^\nu \left(X_2 + z_{12}\,\partial X_2 + \bar{z}_{12}\,\bar{\partial}X_2 + \cdots\right)_\mu (\cdots)_\nu + \cdot\right) e^{ik_2 \cdot X_2}: \\
&= :\left(\cdots - k_{1,\mu}k_{1,\nu}\left(z_{12}\bar{z}_{12}\,\partial X_2^\mu\,\bar{\partial}X_2^\nu\right) + \cdots\right) e^{ik_2 \cdot X_2}: \\
&= \cdots - |z_{12}|^2 k_{1,\mu}k_{1,\nu}\,G_{k_2}^{\mu\nu}(z_2, \bar{z}_2) + \cdots,
\end{aligned}
\qquad (20)
$$

$$V_{k_1}(z_1, \bar{z}_1)\,V_{k_2}(z_2, \bar{z}_2) = \cdots + \frac{O_{k_1,k_2}(z_2, \bar{z}_2)}{|z_{12}|^2} + \cdots, \qquad (21)$$

$$O_{k_1,k_2}(z_2, \bar{z}_2) = -k_{1,\rho}k_{1,\sigma}\,G_{k_2}^{\rho\sigma}(z_2, \bar{z}_2), \qquad (22)$$

Consider the same limit: $z_1 \to z_2$ of the 3-point function, and we find that:

$$
\begin{aligned}
z_1 \to z_2, \quad \langle V_{k_1}(z_1, \bar{z}_1)\,V_{k_2}(z_2, \bar{z}_2)\,G_{e,k_3}(z_3, \bar{z}_3)\rangle &\to \frac{1}{|z_{12}|^2}\frac{A(k_1, k_2, e)}{|z_{23}|^4} \\
&\sim \frac{1}{|z_{12}|^2}\langle O_{k_1,k_2}(z_2, \bar{z}_2)\,G_{e,k_3}(z_3, \bar{z}_3)\rangle,
\end{aligned}
\qquad (23)
$$

We see that in the $z_2 \to z_3$ limit, we should obtain:

$$O_{k_1,k_2}(z_2, \bar{z}_2)\,G_{e,k_3}(z_3, \bar{z}_3) = \cdots + \frac{A(k_1, k_2, e)}{|z_{23}|^4} + \cdots, \qquad (24)$$

Note that $A(k_1, k_2, e) = A(k_1, k_2, e)\,\mathbb{1}$ is simply a number; therefore, when finding $A(k_1, k_2, e)$, it is safe to ignore all (non-identity) operator contributions, as they should cancel each other. Similar to (a), we have:

$$
\begin{aligned}
G_{k_2}^{\rho\sigma}(z_2, \bar{z}_2)\,G_{k_3}^{\mu\nu}(z_3, \bar{z}_3) = \cdots + &:\overset{\frown}{\partial X_2^\rho\,\bar{\partial}X_2^\sigma\,e^{ik_2 \cdot X_2}\,\partial X_3^\mu\,\bar{\partial}X_3^\nu\,e^{ik_3 \cdot X_3}}: \\
+ &:\overset{\frown}{\partial X_2^\rho\,\bar{\partial}X_2^\sigma\,e^{ik_2 \cdot X_2}\,\partial X_3^\mu\,\bar{\partial}X_3^\nu\,e^{ik_3 \cdot X_3}}: \\
+ &:\overset{\frown}{\partial X_2^\rho\,\bar{\partial}X_2^\sigma\,e^{ik_2 \cdot X_2}\,\partial X_3^\mu\,\bar{\partial}X_3^\nu\,e^{ik_3 \cdot X_3}}: \\
+ &:\overset{\frown}{\partial X_2^\rho\,\bar{\partial}X_2^\sigma\,e^{ik_2 \cdot X_2}\,\partial X_3^\mu\,\bar{\partial}X_3^\nu\,e^{ik_3 \cdot X_3}}: + \cdots
\end{aligned}
\qquad (25)
$$

$$
\begin{aligned}
G_{k_2}^{\rho\sigma}(z_2, \bar{z}_2)\,G_{k_3}^{\mu\nu}(z_3, \bar{z}_3) \sim \cdots + &\left(-\frac{\alpha'}{2}\eta^{\rho\mu}\frac{1}{z_{23}^2}\right)\left(-\frac{\alpha'}{2}\eta^{\sigma\nu}\frac{1}{\bar{z}_{23}^2}\right) \times 1 \\
+ &\left(-\frac{\alpha'}{2}\eta^{\rho\mu}\frac{1}{z_{23}^2}\right)\left(-\frac{\alpha'}{2}\frac{ik_3^\sigma}{\bar{z}_{23}}\right)\left(-\frac{\alpha'}{2}\frac{ik_2^\nu}{\bar{z}_{32}}\right) + \left(z \leftrightarrow \bar{z},\ \rho \leftrightarrow \sigma,\ \mu \leftrightarrow \nu\right) \\
+ &\left(-\frac{\alpha'}{2}\frac{ik_2^\rho}{z_{23}}\right)\left(-\frac{\alpha'}{2}\frac{ik_2^\sigma}{\bar{z}_{23}}\right)\left(-\frac{\alpha'}{2}\frac{ik_3^\mu}{z_{32}}\right)\left(-\frac{\alpha'}{2}\frac{ik_3^\nu}{\bar{z}_{32}}\right) + \cdots
\end{aligned}
$$

$$O_{k_1,k_2}(z_2, \bar{z}_2)\, G^{\mu\nu}_{k_3}(z_3, \bar{z}_3) \sim \cdots - k_1^\mu k_1^\nu \left(\frac{\alpha'^2}{4}\right) \frac{1}{|z_{23}|^4}$$

$$- i^2 (k_1^\mu k_2^\nu + k_1^\nu k_2^\mu) \left(\frac{\alpha'}{2}(k_1 \cdot k_3)\right)\left(\frac{\alpha'^2}{4}\right)\frac{1}{|z_{23}|^4} \tag{26}$$

$$- i^4 k_3^\mu k_3^\nu \left(\frac{\alpha'}{2}(k_1 \cdot k_2)\right)^2 \left(\frac{\alpha'^2}{4}\right)\frac{1}{|z_{23}|^4} + \cdots$$

Again, apply the on-shell conditions, and we find that:

$$\frac{\alpha'}{2}k_1 \cdot k_2 = -2, \quad \frac{\alpha'}{2}k_1 \cdot k_3 = -\frac{\alpha'}{2}k_1 \cdot (k_1 + k_2) = -\frac{\alpha'}{2}k_1^2 - \frac{\alpha'}{2}k_1 \cdot k_2 = -2 - (-2) = 0, \tag{27}$$

$$\begin{aligned}
A(k_1, k_2, e) &= -\frac{\alpha'^2}{4}\left(4\cancel{e_{\mu\nu}k_3^\mu k_3^\nu} + e_{\mu\nu}k_1^\mu k_1^\nu\right) = -\frac{\alpha'^2}{4}e_{\mu\nu}k_1^\mu k_1^\nu \\
&= -\frac{\alpha'^2}{4}e_{\mu\nu}(k_2+k_3)^\mu (k_2+k_3)^\nu = -\frac{\alpha'^2}{4}e_{\mu\nu}k_2^\mu k_2^\nu \\
&= -\frac{\alpha'^2}{8}e_{\mu\nu}\left(k_1^\mu k_1^\nu + k_2^\mu k_2^\nu\right) \\
&= -\frac{\alpha'^2}{8}e_{\mu\nu}\left(k_{12}^\mu k_{12}^\nu + (k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)\right),
\end{aligned} \tag{28}$$

On the other hand,

$$0 = e_{\mu\nu}k_3^\mu k_3^\nu = e_{\mu\nu}(k_1+k_2)^\mu (k_1+k_2)^\nu = e_{\mu\nu}\left(k_{12}^\mu k_{12}^\nu + 2(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)\right) \tag{29}$$

$$A(k_1, k_2, e) = -\frac{\alpha'^2}{8}e_{\mu\nu}k_{12}^\mu k_{12}^\nu \left(1 - \frac{1}{2}\right) = -\frac{\alpha'^2}{16}e_{\mu\nu}\,k_{12}^\mu k_{12}^\nu \tag{30}$$

$\boxed{2}$ Strings Scattering Off a Heavy Particle:

A heavy particle can be modeled by some D0-brane with Neumann boundary condition in the X_0 direction[4]. The scattering of a closed string tachyon off the heavy particle can then be computed via a disc diagram with two insertions.

(a) The conformal Killing group (CKG) of the disc is $\mathrm{PSL}(2,\mathbb{R})$. It is a 3 dimensional $\mathbb{R}$ Lie group, generated by 3 conformal Killing vectors (CKV's); therefore, it is possible to partially fix the positions of the two insertions V_1, V_2. On the upper half plane, this can be implemented by putting z_1, z_2 on the imaginary axis, with z_2 fixed and z_1 integrated[5]:

$$\mathcal{A} = g_c^2 e^{-\lambda} \int_0^{z_2} \mathrm{d}z_1 \left\langle :c_1^x e^{ik_1 \cdot X_1}::c_2\tilde{c}_2 e^{ik_2 \cdot X_2}: \right\rangle, \quad z_2 = i, \quad z_1 = iy, \quad y \in [0,1] \tag{31}$$

Here c^x comes from the CKV that brings $z_1 \to iy$. On the disc this can be taken to be a rotation around z_2; when mapped to the upper half plane and at around the imaginary axis, this is simply a translation along the $x = \frac{1}{2}(z + \bar{z})$ direction[6], i.e.

$$\text{CKV: } \partial_x = \delta_x^a\, \partial_a \quad \Longrightarrow \quad \text{Ghost: } c^x, \tag{32}$$

$$c^x \partial_x + c^y \partial_y = c^z \partial_z + c^{\bar{z}}\partial_{\bar{z}}, \quad c^x = \frac{1}{2}\left(c^z + c^{\bar{z}}\right) = \frac{1}{2}\left(c(z) + \tilde{c}(\bar{z})\right), \tag{33}$$

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[4]Reference: arXiv:hep-th/9611214, arXiv:hep-th/9605168, and *Polchinski*.

[5]Reference: arXiv:0812.4408. I would like to thank Lucy Smith for pointing this out.

[6]Reference: *Polchinski*, Chapter 5 & 6.

The ghost contribution is then:

$$
\begin{aligned}
\langle c_1^x c_2 \tilde{c}_2 \rangle = \langle c^x(z_1)\, c(z_2)\, \tilde{c}(\bar{z}_2) \rangle &= \frac{1}{2}\Big(\langle c(z_1)\, c(z_2)\, \tilde{c}(\bar{z}_2) \rangle + \langle \tilde{c}(z_1)\, c(z_2)\, \tilde{c}(\bar{z}_2) \rangle \Big) \\
&= \frac{1}{2}\Big(\langle c(z_1)\, c(z_2)\, c(z_2') \rangle + \langle c(z_1')\, c(z_2)\, c(z_2') \rangle \Big), \quad z' = \bar{z}, \\
&= \frac{C_{D^2}^g}{2}\big(z_{12} z_{12'} z_{22'} + z_{1'2} z_{1'2'} z_{22'} \big), \quad z_1, z_2 \in i\mathbb{R}, \\
&= 2 C_{D^2}^g \big(z_1^2 - z_2^2 \big) z_2
\end{aligned}
\tag{34}
$$

On the other hand, the $e^{ik_j \cdot X_j}$ contributions is similar to what we've computed in $\boxed{1}$, except that now we should be careful about the boundary conditions of X^μ, which affect the XX contraction in the formulae. For Neumann boundary condition: $\partial_y X^0 = 0$, the half-plane propagator from z' can be constructed with an image at $\bar{z}'$ with *the same charge*, i.e. we have:

$$
\overbracket{X_1^0 X_2^0} = -\frac{\alpha'}{2}\, \eta^{00} \ln |z_1 - z_2|^2 - \frac{\alpha'}{2}\, \eta^{00} \ln |z_1 - \bar{z}_2|^2
\tag{35}
$$

While for Dirichlet boundary $X^i = \text{const}$, we can always select the origin so that $X^i = 0$, and in this case the image should have *the opposite charge*, i.e.

$$
\overbracket{X_1^i X_2^j} = -\frac{\alpha'}{2}\, \delta^{ij} \ln |z_1 - z_2|^2 + \frac{\alpha'}{2}\, \delta^{ij} \ln |z_1 - \bar{z}_2|^2,
\tag{36}
$$

$$
\implies \quad :e^{ik_1 \cdot X_1} ::e^{ik_2 \cdot X_2}: = \exp\left(ik_{1,\mu} ik_{2,\nu} \overbracket{X_1^\mu X_2^\nu} \right) :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}:
\tag{37}
$$

$$
= |z_{12}|^{\alpha' k_1 \cdot k_2} |z_{1\bar{2}}|^{\alpha'\left(-k_1^0 k_2^0 - \mathbf{k}_1 \cdot \mathbf{k}_2 \right)} :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}:
$$

Before further calculations, we note that the normal ordering defined here on D^2 differs from that on the usual $\mathbb{C}^2$; in fact, there are also self-contractions with image charge[7]:

$$
\overbracket{X^\mu(z, \bar{z})\, X^\nu(\bar{z}, z)} = G_r^{\mu\nu}(z, \bar{z}) = \mp \frac{\alpha'}{2}\, \eta^{\mu\nu} \ln |z - \bar{z}|^2,
\tag{38}
$$

$$
\implies \quad \left\langle :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}: \right\rangle_{D^2} = \left\langle :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}: \right\rangle_{\mathbb{C}^2} \exp\left(\frac{1}{2} \sum_n ik_{n,\mu} ik_{n,\nu} \overbracket{X_n^\mu X_n^\nu} \right), \quad n = 1, 2
\tag{39}
$$

The "$\mp$" sign choice depends on the boundary condition.

Therefore,

$$
\begin{aligned}
\left\langle :e^{ik_1 \cdot X_1} ::e^{ik_2 \cdot X_2}: \right\rangle_{D^2} &= \left\langle :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}: \right\rangle_{\mathbb{C}^2} \exp\left(ik_{1,\mu} ik_{2,\nu} \overbracket{X_1^\mu X_2^\nu} \right) \exp\left(\frac{1}{2} \sum_n ik_{n,\mu} ik_{n,\nu} \overbracket{X_n^\mu X_n^\nu} \right) \\
&= \left\langle :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}: \right\rangle_{\mathbb{C}^2} \exp\left(\frac{1}{2} \sum_{m,n} ik_{m,\mu} ik_{n,\nu} \overbracket{X_m^\mu X_n^\nu} \right) \\
&= \left\langle :e^{ik_1 \cdot X_1}\, e^{ik_2 \cdot X_2}: \right\rangle_{\mathbb{C}^2} |z_{12}|^{\alpha' k_1 \cdot k_2} |z_{1\bar{2}}|^{\alpha'\left(-k_1^0 k_2^0 - \mathbf{k}_1 \cdot \mathbf{k}_2 \right)} \prod_n |z_{n\bar{n}}|^{\frac{\alpha'}{2}\left(-(k_n^0)^2 - \mathbf{k}_n^2 \right)}
\end{aligned}
\tag{40}
$$

Note that X^i has no zero mode due to the Dirichlet boundary, hence $\int \mathcal{D}X$ gives a delta function in only the Neumann direction: $\delta\big(k_1^0 + k_2^0 \big)$. Physically, this means that only the energy is conversed;

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[7] This is very much similar to the torus situation, where we also have to consider self-contractions with image charges. More rigorous discussion of G^r is given in *Polchinski*.

the momentum k^i is not conserved since the heavy D0-brane does not recoil. It is therefore convenient to define these on shell variables:

$$s = \omega^2 = (k_1^0)^2 = (k_2^0)^2, \quad t = -(\mathbf{k}_1 + \mathbf{k}_2)^2 = -\mathbf{k}_1^2 - \mathbf{k}_2^2 - 2\mathbf{k}_1 \cdot \mathbf{k}_2 = 2\left(-\omega^2 - \mathbf{k}_1 \cdot \mathbf{k}_2 - \frac{4}{\alpha'}\right), \quad (41)$$

$$\mathbf{k}_1 \cdot \mathbf{k}_2 = -\frac{t}{2} - \omega^2 - \frac{4}{\alpha'}, \quad k_1 \cdot k_2 = -\omega(-\omega) + \mathbf{k}_1 \cdot \mathbf{k}_2 \qquad (42)$$

Here we've used the on-shell condition: $m^2 = -k^2 = \omega^2 - \mathbf{k}^2 = -\frac{4}{\alpha'}$ for tachyons. The previous expressions can then be simplified to:

$$\left\langle :e^{ik_1 \cdot X_1} : :e^{ik_2 \cdot X_2} : \right\rangle_{D^2} = \left\langle :e^{ik_1 \cdot X_1} e^{ik_2 \cdot X_2} : \right\rangle_{\mathbb{C}^2} |z_{12}|^{-\frac{\alpha't}{2}-4} |z_{1\bar{2}}|^{+\frac{\alpha't}{2}+4+2\alpha'\omega^2} \prod_n |2z_n|^{-\alpha'\omega^2-2}$$

$$= iC_{D^2}^X 2\pi \delta(k_1^0 + k_2^0) |z_{12}|^{-\frac{\alpha't}{2}-4} |z_{1\bar{2}}|^{+\frac{\alpha't}{2}+4+2\alpha'\omega^2} \prod_n |2z_n|^{-\alpha'\omega^2-2} \qquad (43)$$

$$= iC_{D^2}^X 2\pi \delta(k_1^0 + k_2^0) f(|z_{12}|, |z_{1\bar{2}}|, |z_1|, |z_2|),$$

$$\mathcal{A} = g_c^2 \underline{e^{-\lambda}} \cdot i\underline{C_{D^2}^X} 2\pi \delta(k_1^0 + k_2^0) \cdot 2\underline{C_{D^2}^g} \int_0^{z_2} dz_1 (z_1^2 - z_2^2) z_2 f(|z_{12}|, |z_{1\bar{2}}|, |z_1|, |z_2|)$$

$$= g_c^2 \underline{C_{D^2}} 2\pi \delta(k_1^0 + k_2^0) \cdot 2i \int_0^1 i \, dy ((iy)^2 - i^2) i \cdot f(1-y, 1+y, 2y, 2) \qquad (44)$$

$$= -ig_c^2 C_{D^2} 2\pi \delta(k_1^0 + k_2^0) \cdot 2 \cdot 2^{-2\alpha'\omega^2-4} \int_0^1 dy (1-y^2) f(1-y, 1+y, y, 1),$$

$$\int_0^1 dy (1-y^2) f(1-y, 1+y, y, 1) = \int_0^1 dy (1-y)^{-\frac{\alpha't}{2}-4+1}(1+y)^{+\frac{\alpha't}{2}+4+2\alpha'\omega^2+1} y^{-\alpha'\omega^2-2}$$

$$= \int_0^1 dy \, y^{a-1}(1-y)^{2b-1}(1+y)^{-2a-2b+1}, \quad y' = \frac{1-y}{1+y},$$

$$= -2^{1-2a} \int_0^1 dy' (-y')^{2b-1}(1-y'^2)^{a-1} \qquad (45)$$

$$= 2^{-2a} \int_0^1 d(y'^2) (y'^2)^{b-1}(1-y'^2)^{a-1}$$

$$= 2^{-2a} B\left(a = -\alpha'\omega^2 - 1, b = -\frac{\alpha't}{4} - 1\right)$$

Here $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Euler Beta function.

Putting everything together, we obtain:

$$\mathcal{A} = -ig_c^2 C_{D^2} 2\pi \delta(k_1^0 + k_2^0) \cdot \frac{1}{2} B\left(-\alpha'\omega^2 - 1, -\frac{\alpha't}{4} - 1\right)$$

$$= -ig_c^2 C_{D^2} \pi \delta(k_1^0 + k_2^0) B\left(-\alpha'\omega^2 - 1, -\frac{\alpha't}{4} - 1\right) \qquad (46)$$

In fact $g_c^2 C_{D^2}$ can be further computed by path integral or by comparing physical results. Here we settle for this generic coefficient since it's already enough for our following discussions[8].

---

[8] And I have run out of time and energy.

(b) The Regge limit is found by taking the high energy limit while keeping the momentum transfer fixed; in this case it is achieved by:

$$\text{Regge:} \quad s = \omega^2 \to \infty, \quad t = -(\mathbf{k}_1 + \mathbf{k}_2)^2 \text{ fixed,} \tag{47}$$

$$\mathcal{A} \propto B\left(a = -\alpha's - 1, \, b = -\frac{\alpha't}{4} - 1\right) = \frac{\Gamma(-\alpha's - 1)}{\Gamma\left(-\alpha's - \frac{\alpha't}{4} - 2\right)} \Gamma\left(-\frac{\alpha't}{4} - 1\right)$$

$$\sim \left\{ e\left(\alpha's + \frac{\alpha't}{4} + 3\right)\right\}^{\frac{\alpha't}{4}+1} \Gamma\left(-\frac{\alpha't}{4} - 1\right)$$

$$\sim \left(e\alpha'\omega^2\right)^{\frac{\alpha't}{4}+1} \Gamma\left(-\frac{\alpha't}{4} - 1\right) \tag{48}$$

$$\sim \left(\omega^2\right)^{\frac{\alpha't}{4}+1} \Gamma\left(-\frac{\alpha't}{4} - 1\right)$$

Here we've used the Stirling's approximation[9]: $\ln\Gamma(z+1) = \ln z! \sim z\ln z - z$. On the other hand, the hard scattering limit is found by keeping the scattering angle fixed, i.e.

$$\text{Hard scattering:} \quad s = \omega^2 \to \infty, \quad (t/s) \equiv \lambda \text{ fixed,} \tag{49}$$

$$\mathcal{A} \propto B(a, b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \sim \exp\left\{-\alpha'\left(s\ln(\alpha's) + \frac{t}{4}\ln\frac{\alpha't}{4} + \frac{u}{4}\ln\frac{\alpha'u}{4}\right)\right\}, \tag{50}$$

$$s = \omega^2 = (k_1^0)^2 = (k_2^0)^2, \quad t = -(\mathbf{k}_1 + \mathbf{k}_2)^2, \quad u = -(\mathbf{k}_1 - \mathbf{k}_2)^2, \tag{51}$$

$$s + \frac{t}{4} + \frac{u}{4} = -\frac{4}{\alpha'}, \tag{52}$$

Here we've introduced an additional u variable, and we see that the result is symmetric under $t \leftrightarrow u$. We find that under the above limits, the amplitude exhibits similar behaviors as the Veneziano amplitude.

(c) Note that $\Gamma(z)$ has no zeros on $\mathbb{C}^2$, therefore the poles of $B(a, b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$ is given by the poles of $\Gamma(a)$ and $\Gamma(b)$:

$$a = -\alpha's - 1 = 0, -1, -2, \cdots, \quad b = -\frac{\alpha't}{4} - 1 = 0, -1, -2, \cdots, \tag{53}$$

The first set of poles gives:

$$\omega^2 = -\frac{1}{\alpha'}, 0, \frac{1}{\alpha'}, \frac{2}{\alpha'}, \cdots \tag{54}$$

This is precisely the open string spectrum in $D = 26$. Going back to $(43) \sim (45)$, we see that they come from the singularities as $y = |z_1| \to 0$ or $|z_{12}| \to 1$, i.e. the two insertions are far apart.
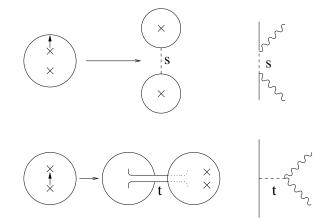
This is the s-channel contribution; physically, it means that the closed string is first absorbed by the D0-brane, then it propagates as an intermediate open string on the brane, and is re-emitted in the end.

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[9]For the validity of Stirling's approximation when $z \in \mathbb{C}$, $\arg z = \pi - \epsilon$ and $|z| \to \infty$, see Wikipedia: *Stirling's formula for the gamma function*.

The other set of poles gives:

$$-t = (\mathbf{k}_1 + \mathbf{k}_2)^2 = \frac{4}{\alpha'}, 0, -\frac{4}{\alpha'}, \frac{8}{\alpha'}, \cdots \tag{55}$$

Only two poles are realistic due to $\mathbf{k}^2 \geq 0$. This is the t-channel contribution, which corresponds to a closed string exchange between the D0-brane and the incoming particle. The above factorization channels are nicely illustrated in the diagram below, borrowed from `arXiv:hep-th/9611214`.

$\boxed{3}$ Compton Scattering Between $U(N)$ Gluons and Adjoint Tachyons:

Such scattering is captured by a disc diagram with $AATT$ insertions at the boundary; following the same recipe as before, we write down[10]:

$$
\begin{aligned}
\mathcal{A} = g_o'^2 g_o^2 e^{-\lambda} \int \mathrm{d}y \, &\Big\langle :e_\mu \partial_y X^\mu \, e^{ik \cdot X}(y) : :c_1^y \, e_{1,\nu} \partial_y X_1^\nu \, e^{ik_1 \cdot X_1} : :c_2^y e^{ik_2 \cdot X_2} : :c_3^y e^{ik_2 \cdot X_2} : \Big\rangle \\
&\times \mathrm{Tr}\left(\lambda^a \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} : \text{in } (y, y_1, y_2, y_3) \text{ order}\right) \\
&+ (2 \leftrightarrow 3)
\end{aligned}
\tag{56}
$$

Where $y_i = y_{1,2,3}$ are fixed while $y_0 = y$ is integrated. Note that for $\boxed{2}$ we use (x, y) to label the points on the upper half plane so that the boundary coordinate is $x = (x, 0)$; here in order to match with *Polchinski*, we label the boundary points with $y = (y, 0)$ instead.

We see that $\mathcal{A}$ is structurally similar to the 4-point Veneziano amplitude $\mathcal{A}_{TTTT}$ and also the 3-point amplitude $\mathcal{A}_{TTA}$; by analogy we can evaluate $\mathcal{A}$ in a similar fashion[11]:

$$
\begin{aligned}
\mathcal{A} = g_o'^2 g_o^2 e^{-\lambda} \prod_{i<j} &|y_{ij}|^{2\alpha' k_i \cdot k_j + 1} \, e_\mu e_{1,\nu} \\
&\times \int \mathrm{d}y \prod_j |y_{0j}|^{2\alpha' k \cdot k_j} \left\{ -2\alpha' \frac{\eta^{\mu\nu}}{y_{01}^2} + (-2\alpha') \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) (-2\alpha') \left(\frac{ik^\nu}{y_{10}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\} \\
&\times \mathrm{Tr}\,(\cdots)\, i C_{D_2}^X C_{D_2}^g \, (2\pi)^{26} \delta^{26}\big(\textstyle\sum_i k_i\big) \\
&+ (2 \leftrightarrow 3)
\end{aligned}
\tag{57}
$$

---

[10]I would like to thank 谷夏 for help with this problem.

[11]For the ∂X contribution, see *Polchinski*, `arXiv:1403.4553` and `arXiv:1507.02172`. Note that there are contractions between two ∂X fields.

Notice the doubling due to the boundary contraction $\overline{X_1^\mu X_2^\nu} = 2 \times \left(-\frac{\alpha'}{2}\right) \ln y_{12}^2 = -2\alpha' \ln y_{12}$. Further simplifications can be achieved by using the on-shell and physical conditions for open string states:

$$k^2 = k_1^2 = 0, \quad e \cdot k = e_1 \cdot k_1 = 0, \quad k_{2,3}^2 = \frac{1}{\alpha'} = -m^2, \tag{58}$$

$$s = -\alpha'(k + k_1)^2 = -2\alpha' k \cdot k_1,$$
$$t = -\alpha'(k + k_2)^2 = -2\alpha' k \cdot k_2 - 1, \tag{59}$$
$$u = -\alpha'(k + k_3)^2 = -2\alpha' k \cdot k_3 - 1,$$

$$s + t + u = -2, \tag{60}$$

$$2\alpha' k_1 \cdot k_2 = \alpha'(k_1 + k_2)^2 - \alpha' k_1^2 - \alpha' k_2^2 = \alpha'(k + k_3)^2 - 1 = -u - 1 = 2\alpha' k \cdot k_3, \tag{61}$$

$$2\alpha' k_1 \cdot k_3 = -t - 1 = 2\alpha' k \cdot k_2, \tag{62}$$

$$2\alpha' k_2 \cdot k_3 = \alpha'(k + k_1)^2 - 2 = -s - 2, \tag{63}$$

$$\mathcal{A} = ig_o'^2 g_o^2 C_{D^2} |y_{12}|^{-u} |y_{13}|^{-t} |y_{23}|^{-s-2} e_\mu e_{1,\nu}$$
$$\times \int dy \, |y_{01}|^{-s} |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left\{ \frac{\eta^{\mu\nu}}{y_{01}^2} - 2\alpha' \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) \left(-\frac{ik^\nu}{y_{01}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\}$$
$$\times \mathrm{Tr}\,(\cdots)\,(-2\alpha')(2\pi)^{26}\delta^{26}\left(\sum_i k_i\right)$$
$$+ (2 \leftrightarrow 3) \tag{64}$$

Fix $y_1 \to \infty, y_2 = 0, y_3 = 1$, and we get:

$$\mathcal{A} = ig_o'^2 g_o^2 C_{D^2} |y_1|^{-u-t} e_\mu e_{1,\nu}$$
$$\times \int dy \, |y_{01}|^{-s} |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left\{ \frac{\eta^{\mu\nu}}{y_{01}^2} - 2\alpha' \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) \left(-\frac{ik^\nu}{y_{01}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\}$$
$$\times \mathrm{Tr}\,(\cdots) \cdots + \cdots \tag{65}$$

$$\mathcal{A} = ig_o'^2 g_o^2 C_{D^2}$$
$$\times \int dy \, |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left| \frac{y_1^{s+2}}{y_{01}^s} \right| \left\{ \frac{e \cdot e_1}{y_{01}^2} - 2\alpha' e_\mu \left(\frac{ik_1^\mu}{y_{01}} + \frac{ik_2^\mu}{y_{02}} + \frac{ik_3^\mu}{y_{03}} \right) e_{1,\nu} \left(-\frac{ik^\nu}{y_{01}} + \frac{ik_2^\nu}{y_{12}} + \frac{ik_3^\nu}{y_{13}} \right) \right\}$$
$$\times \mathrm{Tr}\,(\cdots) \cdots + \cdots$$
$$= ig_o'^2 g_o^2 C_{D^2} \int dy \left\{ (e \cdot e_1) f(t,u) - 2\alpha' g(t,u) \right\} \times \cdots + \cdots \tag{66}$$

Note that the definition of $f(t,u)$ and $g(t,u)$ contains the $\mathrm{Tr}\,(\cdots)$ factor.

The limit $y_1 \to \infty$ should be treated with care. In fact, the integral along the boundary splits into three ranges[12]:

$$\int dy = \left\{ \int_{y_1 \to -\infty}^{y_3 = 0} + \int_{y_2 = 0}^{y_3 = 1} + \int_{y_3 = 1}^{y_1 \to +\infty} \right\} dy \tag{67}$$

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[12]Reference: *Polchinski*'s discussion of $\mathcal{A}_{TTTT}$.

Notice the difference of $y_1 \to \pm\infty$; this is due to the S^1 topology of the boundary. Therefore,

$$\int \mathrm{d}y\, f(t,u) = \int \mathrm{d}y\, |y_{02}|^{-t-1} |y_{03}|^{-u-1} \left| \frac{y_1}{y_{01}} \right|^{s+2} \mathrm{Tr}\,(\cdots), \tag{68}$$

$$\mathrm{Tr}\left(\lambda^a \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \right) \equiv T_{0123}, \tag{69}$$

$$\int_{y_1 \to -\infty}^{y_2=0} \mathrm{d}y\, f(t,u) = \int_{y_1 \to -\infty}^{0} \mathrm{d}y\, (1-y)^{-t-1} (-y)^{-u-1} \left(\frac{-y_1}{y-y_1} \right)^{s+2} T_{1023} = T_{1023} B(-u,-s-1),$$

$$\int_{y_3=1}^{y_1 \to +\infty} \mathrm{d}y\, f(t,u) = \int_{1}^{y_1 \to +\infty} \mathrm{d}y\, (y-1)^{-t-1} y^{-u-1} \left(\frac{y_1}{y_1-y} \right)^{s+2} T_{1230} = T_{1230} B(-t,-s-1),$$

$$\int_{y_3=0}^{y_2=1} \mathrm{d}y\, f(t,u) = \int_{0}^{1} \mathrm{d}y\, (1-y)^{-t-1} y^{-u-1} t_{1203} = T_{1203} B(-t,-u),$$

$$\tag{70}$$

$$\int \mathrm{d}y\, f(t,u) = T_{1203} B(-t,-u) + T_{1230} B(-t,-s-1) + T_{1023} B(-u,-s-1) \tag{71}$$

Under $2 \leftrightarrow 3$, we have $t \leftrightarrow u$; therefore,

$$\begin{aligned}
\int \mathrm{d}y\, f(t,u) + \left(2 \leftrightarrow 3\right) &= \left(T_{1203} + T_{1302}\right) B(-t,-u) \\
&\quad + \left(T_{1230} + T_{1032}\right) B(-t,-s-1) \\
&\quad + \left(T_{1023} + T_{1320}\right) B(-u,-s-1) \\
&\equiv \alpha B(-t,-u) + \beta B(-t,-s-1) + \gamma B(-u,-s-1)
\end{aligned} \tag{72}$$

The $g(t,u)$ factor can be computed in a similar manner. In the end we have[13]:

$$\begin{aligned}
\mathcal{A} = {}& i g_o'^2 g_o^2 C_{D^2} (2\pi)^{26} \delta^{26}\left(\textstyle\sum_i k_i\right) \\
& \times (-2\alpha') \Big\{ (e_0 \cdot e_1) \left(\alpha B(-t,-u) + \beta B(-t,-s-1) + \gamma B(-u,-s-1) \right) \\
& \qquad + (-2\alpha')(e_0 \cdot k_2)(e_1 \cdot k_3) \left(\alpha B(-t,-u) - \beta B(-t,-s-1) - \gamma B(-u,-s-1) \right) + \left(2 \leftrightarrow 3\right) \Big\}
\end{aligned} \tag{73}$$

---

[13]Reference: arXiv:0801.3358.