String Theory №5

Bryan Compiled @ 2023/03/04

1 Partition Function for Compact Scalars

(a) Mode expansion of X CFT is[1]:

$$\partial X(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{\alpha_m}{z^{m+1}}, \quad \bar\partial X(\bar z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{\tilde\alpha_m}{\bar z^{m+1}}, \tag{1}$$

$$X = x - i\sqrt{\frac{\alpha'}{2}}\left(\alpha_0\ln z + \tilde\alpha_0\ln\bar z\right) + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m}{z^m}+\frac{\tilde\alpha_m}{\bar z^m}\right), \tag{2}$$

Momentum p is the charge for *spacetime* translation; we have:

$$X\mapsto X+\text{const}, \quad j_a = \frac{i}{\alpha'}\,\partial_a X, \tag{3}$$

$$p = \frac{1}{2\pi i}\oint_C\left(\mathrm{d}z\, j - \mathrm{d}\bar z\,\tilde j\,\right) = \frac{1}{\alpha'}\sqrt{\frac{\alpha'}{2}}\left(\alpha_0+\tilde\alpha_0\right) = \sqrt{\frac{1}{2\alpha'}}\left(\alpha_0+\tilde\alpha_0\right) \tag{4}$$

Additionally, for compact free boson, X is only defined modulo $2\pi R$; therefore, states after $X+2\pi R$ translation should be identical to the original states, i.e.

$$e^{ip\,(2\pi R)} = \mathbb{1}, \quad p = \frac{n}{R}, \quad n\in\mathbb{Z} \tag{5}$$

This, in fact, holds for any field theory[2] defined for $X\in S^1$, including the ordinary quantum mechanics (a classical field theory) on S^1.

On the other hand, there are additional constraints in string theory: for the state of a *single* closed string, there is a discrete translational symmetry on the *worldsheet*:

$$X(\sigma^1+2\pi)\cong X(\sigma^1), \quad X(\sigma^1+2\pi) = X(\sigma^1)+2\pi Rw, \quad w\in\mathbb{Z} \tag{6}$$

With some definite winding number w. In $(z,\bar z)$ coordinates, we have:

$$2\pi Rw = X\big(z\,e^{2\pi i},\bar z\,e^{-2\pi i}\big) - X(z,\bar z) = -i\sqrt{\frac{\alpha'}{2}}\,2\pi i\left(\alpha_0-\tilde\alpha_0\right) = 2\pi\sqrt{\frac{\alpha'}{2}}\left(\alpha_0-\tilde\alpha_0\right), \tag{7}$$

$$p = \frac{p_L+p_R}{2}, \quad p_L = \sqrt{\frac{2}{\alpha'}}\,\alpha_0, \quad p_R = \sqrt{\frac{2}{\alpha'}}\,\tilde\alpha_0, \tag{8}$$

$$p_{L,R} = \frac{n}{R}\pm\frac{wR}{\alpha'}, \tag{9}$$

$$X = x - i\,\frac{\alpha'}{2}\left(p_L\ln z + p_R\ln\bar z\right) + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m}{z^m}+\frac{\tilde\alpha_m}{\bar z^m}\right), \tag{10}$$

For the oscillator expressions for L_0, recall that:

$$T(z) = -\frac{1}{\alpha'}:\partial X\,\partial X: = \sum_m\frac{L_m}{z^{m+2}}, \tag{11}$$

$$L_{m\neq 0} = \frac{1}{2}\sum_l \alpha_{m-l}\alpha_l, \quad L_0 = \frac{1}{2}:\sum_l\alpha_{-l}\alpha_l: \sim \frac{\alpha' p_L^2}{4} + \sum_{l>0}\alpha_{-l}\alpha_l, \tag{12}$$

---

[1] Again we follow the convention of *Polchinski*.

[2] Reference: discussions in *Polchinski*, §8.2.

The L_0 expression may be off by some normal ordering constant; this ambiguity can be resolved by considering:

$$2L_0\,|0,0;n=w=0\rangle = (L_1L_{-1}-L_{-1}L_1)\,|0,0;p_L=p_R=0\rangle = 0-0=0 \tag{13}$$

Therefore the normal ordering constant is, in fact, trivial, and we have:

$$L_0 = \frac{\alpha' p_L^2}{4} + \sum_{l>0}\alpha_{-l}\alpha_l, \quad \tilde{L}_0 = \frac{\alpha' p_R^2}{4} + \sum_{l>0}\tilde{\alpha}_{-l}\tilde{\alpha}_l, \tag{14}$$

(b) The torus partition function is given by:

$$\langle \mathbb{1}\rangle_{T^2} \equiv Z(\tau=\tau_1+i\tau_2) = \int \mathscr{D}X\, e^{-S} = \operatorname{Tr} e^{-(2\pi\tau_2)H} e^{i\,(2\pi\tau_1)P} \tag{15}$$

Here P generates *worldsheet* translation along σ^1, not to be confused with p which generates *spacetime* translation; with $z=e^{-iw},\ w=\sigma^1+i\sigma^2$,

$$\begin{aligned}
T^0{}_1 = \eta^{00}\,(\partial_0\sigma^2)\,T_{21} = -iT_{12} &= -i\left(T_{ww}\,(\partial_1 w)(\partial_2 w) + T_{\bar{w}\bar{w}}\,(\partial_1\bar{w})(\partial_2\bar{w})\right)\\
&= T_{ww} - T_{\bar{w}\bar{w}}\\
&= \left(T_{zz}\,(\partial_w z)^2 + \tfrac{c}{24}\right) - \left(T_{\bar{z}\bar{z}}\,(\partial_{\bar{w}}\bar{z})^2 + \tfrac{\tilde{c}}{24}\right)\\
&= T(z)\,(-iz)^2 - \tilde{T}(\bar{z})\,(+i\bar{z})^2 + \frac{c-\tilde{c}}{24},
\end{aligned} \tag{16}$$

$$\begin{aligned}
P = \int \frac{\mathrm{d}\sigma_1}{2\pi}\,(-T^0{}_1) &= -\int\frac{\mathrm{d}\sigma_1}{2\pi}\,T(z)\,(-iz)^2 + \int\frac{\mathrm{d}\sigma_1}{2\pi}\,\tilde{T}(\bar{z})\,(+i\bar{z})^2 - \frac{c-\tilde{c}}{24}\\
&= +\oint \frac{\mathrm{d}z}{2\pi\,(-iz)}\,T(z)\,(-iz)^2 + \oint\frac{\mathrm{d}\bar{z}}{2\pi\,(+i\bar{z})}\,\tilde{T}(\bar{z})\,(+i\bar{z})^2 - \frac{c-\tilde{c}}{24}\\
&= \oint\frac{\mathrm{d}z}{2\pi i}\,zT(z) - \oint\frac{\mathrm{d}\bar{z}}{2\pi i}\,\bar{z}\tilde{T}(\bar{z}) - \frac{c-\tilde{c}}{24}\\
&= L_0 - \tilde{L}_0 - \frac{c-\tilde{c}}{24}\\
&= \left(L_0 - \tfrac{c}{24}\right) - \left(\tilde{L}_0 - \tfrac{\tilde{c}}{24}\right),\\
H = \int\frac{\mathrm{d}\sigma_1}{2\pi}\,T^0{}_0 &= \int\frac{\mathrm{d}\sigma_1}{2\pi}\,T_{22}\\
&= L_0 + \tilde{L}_0 - \frac{c+\tilde{c}}{24}\\
&= \left(L_0 - \tfrac{c}{24}\right) + \left(\tilde{L}_0 - \tfrac{\tilde{c}}{24}\right),
\end{aligned} \tag{17}$$

Here we've used the fact that $\displaystyle\oint\frac{\mathrm{d}\bar{z}}{\bar{z}} = \oint\frac{\mathrm{d}\bar{z}}{\bar{z}} = 2\pi i$. Therefore,

$$Z(\tau) = \operatorname{Tr} e^{-(2\pi\tau_2)H} e^{i\,(2\pi\tau_1)P} = \operatorname{Tr} q^{L_0-\frac{c}{24}}\,\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}, \quad q=e^{2\pi i\tau} \tag{18}$$

Note that here we are working in the grand canonical ensemble, where we have temperature $\beta=2\pi\tau_2$ and chemical potential $2\pi\tau_1$. At this stage P is *not* fixed, and we should sum over all states with various P. To go to the canonical ensemble, we do a Legendre transform and trade τ_1 for some P'. This is implemented by a Laplace / Fourier transform of the partition function:

$$\int \mathrm{d}\tau_1\, e^{-i\,(2\pi\tau_1)P'} \tag{19}$$

In string theory, we should actually work with the canonical ensemble (and eventually, the *micro-canonical* ensemble), since we would like to impose the *level matching condition*, namely $P'=0$.

This is because we would like to gauge the worldsheet translation along σ^1 generated by P, which is understood as a redundancy.

However, in string theory, the Fourier transform is implicit as we integrate along τ_1 in the moduli space; roughly speaking, we have:

$$\int \mathrm{d}\tau_1 \, e^{-i\,(2\pi\tau_1)(P-0)} \sim \delta(P-0) \tag{20}$$

As τ_1 gets integrated out, we are effectively computing $\mathrm{Tr}'\, e^{-\beta H}$ where the trace only goes over the $P=0$ sector of the Hilbert space[3].

On the other hand, since we are computing the grand partition function $Z(\tau_1 + i\tau_2)$ for now, we should *not* impose the level matching condition, and the sum should go over all states with various P, although this is a much larger Hilbert space than the physical $P=0$ subspace in string theory.

Using the expressions in (a), we find that L_0 action on a state $|\psi\rangle$ created by $\alpha_{-l}, \tilde{\alpha}_{-l}$ yields the sum of occupation number N_l weighted by l:

$$L_0 \, |\psi\rangle = \left(\frac{\alpha' k_L^2}{4} + \sum_{l>0} l \cdot N_l \right) |\psi\rangle \tag{21}$$

With $c = \tilde{c} = 1$, we obtain:

$$\begin{aligned} Z(\tau) &= (q\bar{q})^{-\frac{1}{24}} \sum_{n,w} e^{-2\pi\tau_2\alpha' \frac{k_L^2 + k_R^2}{4}} e^{2\pi i \tau_1 \alpha' \frac{k_L^2 - k_R^2}{4}} \sum_{(N_l),(\tilde{N}_l)} q^{\sum\limits_{l>0} l \cdot N_l} \bar{q}^{\sum\limits_{l>0} l\cdot \tilde{N}_l} \\ &= (q\bar{q})^{-\frac{1}{24}} \sum_{n,w} e^{-\pi\tau_2 \left(\frac{\alpha' n^2}{R^2} + \frac{w^2 R^2}{\alpha'} \right) + 2\pi i \tau_1 n w} \sum_{(N_l),(\tilde{N}_l)} \prod_{l>0} q^{l\cdot N_l} \bar{q}^{l \cdot \tilde{N}_l} \\ &= |\eta(\tau)|^{-2} \sum_{n,w} e^{-\pi\tau_2 \left(\frac{\alpha' n^2}{R^2} + \frac{w^2 R^2}{\alpha'} \right) + 2\pi i \tau_1 n w} \end{aligned} \tag{22}$$

We've simplified the contributions from the oscillator modes using $\eta(\tau)$, since they are identical to the oscillator contributions of the non-compact $X \in \mathbb{R}^1$:

$$\begin{aligned} (q\bar{q})^{-\frac{1}{24}} \sum_{(N_l),(\tilde{N}_l)} \prod_{l>0} q^{l\cdot N_l} \bar{q}^{l\cdot\tilde{N}_l} &= (q\bar{q})^{-\frac{1}{24}} \prod_{l>0} \sum_{N_l, \tilde{N}_l = 0}^{\infty} q^{l\cdot N_l} \bar{q}^{l\cdot \tilde{N}_l} \\ &= (q\bar{q})^{-\frac{1}{24}} \prod_{l>0} \frac{1}{1-q^l} \frac{1}{1-\bar{q}^l} = |\eta(\tau)|^{-2} \end{aligned} \tag{23}$$

[3] This is nicely explained in David Tong's *String Theory*. See also §6.3 of Blumenhagen et al, *Basic Concepts of String Theory*. See also §2.1, 2.2 of `arXiv:1912.07654` where the switch between ensembles is reviewed in the context of holography.

In the $R \to \infty$ limit, only the $w = 0$ modes survive; all other modes are exponentially suppressed by the $e^{-\pi\tau_2 w^2 R^2/\alpha'}$ factor; i.e.

$$\begin{aligned}
Z(\tau) &= |\eta(\tau)|^{-2} \sum_{n,w} \exp\left\{ -\pi\tau_2 \left(\frac{\alpha' n^2}{R^2} + \frac{w^2 R^2}{\alpha'} \right) + 2\pi i \tau_1 n w \right\} \\
&\to |\eta(\tau)|^{-2} \sum_{n} \exp\left\{ -\pi\tau_2 \, \frac{\alpha' n^2}{R^2} \right\}, \quad k = \frac{n}{R} \\
&\to |\eta(\tau)|^{-2} \, V \!\int \frac{\mathrm{d}k}{2\pi} \exp\left\{ -\pi\tau_2 \, \alpha' k^2 \right\} \\
&= V \, |\eta(\tau)|^{-2} \left(4\pi^2 \alpha' \tau_2 \right)^{-\frac{1}{2}} \\
&\equiv V \cdot Z_X(\tau) = 2\pi R \, Z_X(\tau)
\end{aligned} \tag{24}$$

We recover the partition function $V \cdot Z_X(\tau)$ for non-compact X, as expected.

(c) Using the Poisson resummation formula, we find that:

$$Z(\tau) = 2\pi R \, Z_X(\tau) \sum_{m,w} \exp\left(-\frac{\pi R^2 |m - w\tau|^2}{\alpha' \tau_2} \right) \tag{25}$$

$Z_X(\tau)$ is modular invariant by the properties of the Dedekind $\eta(\tau)$ function, as is demonstrated for the non-compact X in *Polchinski*.

The sum, on the other hand, is naturally invariant under $T\colon \tau \mapsto \tau + 1$, by making a change of variables $m \mapsto m + w$. It is also invariant under $S\colon \tau \mapsto -1/\tau$ with $m \mapsto -w, w \mapsto m$ [4]. Therefore, $Z(\tau)$ is modular invariant.

$\boxed{2}$ $\mathbb{Z}_2$ Orbifold

The $\mathbb{Z}_2$ orbifold is constructed by imposing an additional identification on $X \in S^1$:

$$X \cong -X \tag{26}$$

The target space is then reduced to $S^1/\mathbb{Z}_2 \cong [0, \pi R]$.

(a) The first contributions to the orbifold partition function comes from the states that are invariant under reflection r; we have:

$$\mathrm{Tr}_{S^1/\mathbb{Z}_2} = \mathrm{Tr}_{S^1} \frac{1+r}{2} = \frac{1}{2} \mathrm{Tr}_{S^1} + \frac{1}{2} \mathrm{Tr}_{S^1} \circ r \tag{27}$$

Acting on $q^{L_0 - \frac{c}{24}} \, \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}$, the first term gives $\frac{1}{2} Z_{S^1}(\tau)$ where Z_{S^1} is the S^1 partition function we've obtained in $\boxed{1}$.

For the second term, note that the action of r on a state can be deduced from its action on the modes $\alpha_m, \tilde{\alpha}_m$, which is in turn induced from the action $r\colon X \mapsto -X$ through the mode expansion (10). In particular, we have $r : \alpha_{-m} \mapsto -\alpha_{-m}$, which means that[5]:

$$r \circ \alpha_{-m} \circ r^{-1} = -\alpha_{-m}, \quad r \, |N_m\rangle \sim r \circ \alpha_{-m}^{N_m} \, |0\rangle \sim (-1)^{N_m} \, |N_m\rangle \,, \tag{28}$$

[4] Reference: *Polchinski*.

[5] Reference: Blumenhagen et al, *Basic Concepts of String Theory*, §10.5.

$$r:\quad \Big|(N_l),(\tilde{N}_l);n,w\Big\rangle \longmapsto (-1)^{\sum_l (N_l+\tilde{N}_l)}\Big|(N_l),(\tilde{N}_l);-n,-w\Big\rangle \tag{29}$$

In particular, it reverses n, w, hence r insertion gives vanishing amplitude unless $n = w = 0$. The summation is very much similar to the Z_{S^1} case, i.e. we have:

$$\begin{aligned}\frac{1}{2}\,\mathrm{Tr}_{S^1}\left(r\,q^{L_0-\frac{c}{24}}\,\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}\right) &= \frac{1}{2}\,(q\bar{q})^{-\frac{1}{24}}\prod_{l>0}\sum_{N_l,\tilde{N}_l=0}^{\infty}(-1)^{N_l+\tilde{N}_l}q^{l\cdot N_l}\bar{q}^{l\cdot\tilde{N}_l}\\ &= \frac{1}{2}\,(q\bar{q})^{-\frac{1}{24}}\prod_{l>0}\frac{1}{1-(-q^l)}\frac{1}{1-(-\bar{q}^{\,l})} = \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right|\end{aligned} \tag{30}$$

Where we've used the fact that[6]: $q^{-\frac{1}{24}}\prod_{l>0}\frac{1}{1-(-q^l)} = \sqrt{2}\sqrt{\frac{\eta(\tau)}{\theta_2(\tau)}}$. Therefore, the total contributions from r–invariant states are:

$$\frac{1}{2}\,Z_{S^1}(\tau) + \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| \tag{31}$$

Physically, r invariant states are stationary waves:

$$\frac{1+r}{2}\,|(N_l),(\tilde{N}_l);n,w\rangle = \frac{1}{2}\Big(\,|(N_l),(\tilde{N}_l);n,w\rangle + (-1)^{\sum_l (N_l+\tilde{N}_l)}\,|(N_l),(\tilde{N}_l);-n,-w\rangle\Big), \tag{32}$$

which is a superposition of states with opposite momentum and winding, such that $\langle p_L\rangle = \langle p_R\rangle = 0$. Note that a general r invariant state is a linear combination of such states and may have non-vanishing momentum: $\langle p\rangle \neq 0$. In position space these states have unconstrained $x = x_L + x_R \in [0, \pi R]$. More precisely, if we define the position basis: $x\,|x'\rangle = \delta(x-x')\,|x'\rangle$, then we have:

$$\begin{aligned}\Big\langle (N_l),(\tilde{N}_l);x\Big|\frac{1+r}{2}\Big|(N_l),(\tilde{N}_l);n\Big\rangle &\propto \frac{1}{2}\Big(e^{i\frac{n}{R}x} + (-1)^{\sum_l (N_l+\tilde{N}_l)}e^{-i\frac{n}{R}x}\Big)\\ &= \begin{cases}\cos\frac{nx}{R}, & \sum_l (N_l+\tilde{N}_l)\ \text{even}\\ \sin\frac{nx}{R}, & \sum_l (N_l+\tilde{N}_l)\ \text{odd}\end{cases}\end{aligned} \tag{33}$$

The wave function ($\cos\frac{nx}{R}$) is what we expect: we get the same result if we turn off the oscillator modes $N, \tilde{N}$ altogether; in this case we have a worldline quantum mechanics with target $S^1/\mathbb{Z}_2$, which is equivalent to the spacetime QFT on $S^1/\mathbb{Z}_2$. Since we are only considering the worldline geometry S^1, we are restricting to the 1-particle sector of the spacetime QFT, which is again a quantum mechanical system. The wave functions compatible with $\mathbb{Z}_2$ action are precisely given by ($\cos\frac{nx}{R}$). On the other hand, the case with ($\sin\frac{nx}{R}$) doesn't seem to be $\mathbb{Z}_2$ invariant in the position space. This is due to the $\mathbb{Z}_2$ odd stringy excitations, which can be understood as some additional degrees of freedom, similar to the spin of a particle. This compensates for the $\mathbb{Z}_2$ odd ($\sin\frac{nx}{R}$), and the full wave function is $\mathbb{Z}_2$ invariant.

(b) With $X \cong -X$, new possibilities emerge as the boundary condition along σ^1:

$$X(\sigma^1+2\pi) \cong X(\sigma^1), \quad X(\sigma^1+2\pi) = \pm X(\sigma^1) + 2\pi R w, \quad w\in\mathbb{Z} \tag{34}$$

The "$-$" sign corresponds to the *twisted states.* Due to the anti-periodicity, ∂X has a half-integer mode expansion:

$$\partial X\big(z\,e^{2\pi i}\big) = -\partial X(z), \tag{35}$$

[6] Reference: Blumenhagen & Plauschinn, *Introduction to CFT*, and also *Polchinski.*

$$\partial X(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{\alpha_{m-\frac{1}{2}}}{z^{m+\frac{1}{2}}}, \quad \bar{\partial}X(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{\tilde{\alpha}_{m-\frac{1}{2}}}{\bar{z}^{m+\frac{1}{2}}}, \tag{36}$$

$$X = x + i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\frac{1}{m+\frac{1}{2}}\left(\frac{\alpha_{m+\frac{1}{2}}}{z^{m+\frac{1}{2}}} + \frac{\tilde{\alpha}_{m+\frac{1}{2}}}{\bar{z}^{m+\frac{1}{2}}}\right), \tag{37}$$

Apply the boundary condition on X, and we find that $x = \pi R w$; however, due to the identification $X + 2\pi R \cong X \cong -X$, there are only two inequivalent choices:

$$x = 0, \quad x = \pi R, \tag{38}$$

which correspond to the string localized around either of the two fixed points of the $\mathbb{Z}_2$ action. There is no momentum or winding, not even in the mode expansion.

Intuitively, the twisted boundary condition is realized only when the string is stretched around the fixed points of the orbifold action. For example, an open string that ends on X and $X+2\pi R \in \mathbb{R}^1$ becomes a closed winding string when we consider $\mathbb{R}^1/(2\pi R\,\mathbb{Z}) = S^1$. Similarly, an open string that ends on $\pm X$ becomes a closed string in the twisted sector, after we take $S^1/\mathbb{Z}_2$.

Much similar to the case in [1], we have:

$$\left[\alpha_{\frac{1}{2}+l}, \alpha_{-\frac{1}{2}-l}\right] = \frac{1}{2} + l, \tag{39}$$

$$L_{m\neq 0} = \frac{1}{2}\sum_l \alpha_{m-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l}, \quad L_0 = \frac{1}{2}:\sum_l \alpha_{-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l}: \sim \sum_{l\geq 0}\alpha_{-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l} \tag{40}$$

We can use the same trick to fix the normal ordering constant in L_0; this time it is non-trivial:

$$L_{-1} = \frac{1}{2}\alpha^2_{-\frac{1}{2}} + \sum_{l\geq 0}\alpha_{-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l}, \quad L_1 = \frac{1}{2}\alpha^2_{\frac{1}{2}} + \sum_{l>0}\alpha_{\frac{1}{2}-l}\alpha_{\frac{1}{2}+l}, \tag{41}$$

$$\begin{aligned} L_0\,|0,0;x\rangle &= \frac{1}{2}\left(L_1L_{-1} - L_{-1}L_1\right)|0,0;x\rangle \\ &= \frac{1}{2}\times\frac{1}{4}\alpha^2_{\frac{1}{2}}\alpha^2_{-\frac{1}{2}}\,|0,0;x\rangle - 0 \\ &= \frac{1}{16}\,|0,0;x\rangle, \end{aligned} \tag{42}$$

$$L_0 = \frac{1}{16} + \sum_{l\geq 0}\alpha_{-\frac{1}{2}-l}\alpha_{\frac{1}{2}+l} = \frac{1}{16} + \sum_{l\geq 0}\left(l+\frac{1}{2}\right)N_{l+\frac{1}{2}} = \frac{1}{16} + \sum_{l>0}\left(l-\frac{1}{2}\right)N_{l-\frac{1}{2}}, \tag{43}$$

The trace can then be computed, following the same recipe as before:

$$r:\ \left|(N_l),(\tilde{N}_l);x\right\rangle \longmapsto (-1)^{\sum_l(N_l+\tilde{N}_l)}\left|(N_l),(\tilde{N}_l);-x\right\rangle, \quad -x\equiv x, \quad x = 0,\pi R, \tag{44}$$

$$\begin{aligned} \mathrm{Tr}_{S^1}\left(\frac{1+r}{2}\,q^{L_0-\frac{c}{24}}\,\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}\right) &= (q\bar{q})^{-\frac{1}{24}+\frac{1}{16}}\prod_{l+\frac{1}{2}\in\mathbb{Z}^+}\sum_{N_l,\tilde{N}_l=0}^{\infty}\frac{1+(-1)^{N_l+\tilde{N}_l}}{2}\,q^{l\cdot N_l}\bar{q}^{l\cdot\tilde{N}_l}\times 2 \\ &= \frac{1}{2}(q\bar{q})^{+\frac{1}{48}}\left\{\prod_{l>0}\left|\frac{1}{1-q^{l-\frac{1}{2}}}\right|^2 + \prod_{l>0}\left|\frac{1}{1+q^{l-\frac{1}{2}}}\right|^2\right\}\times 2 \\ &= \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| \end{aligned} \tag{45}$$

There is an extra factor of 2 from the number of twisted sectors: $x = 0$ and $x = \pi R$.

(c) The full partition function is therefore:

$$Z(\tau) = \frac{1}{2}\, Z_{S^1}(\tau) + \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| \tag{46}$$

The first term is modular invariant, as is proven in $\boxed{1}$.

The remaining terms are also modular invariant, due to the transformational properties of η and θ functions[7]:

$$T \circlearrowright \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| \xleftrightarrow{S} \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right| \xleftrightarrow{T} \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| \circlearrowleft S \tag{47}$$

Therefore, the full partition function is modular invariant.

$\boxed{3}$ Torus 4-point function in bc CFT

$$\big\langle c(w_1)\, b(w_2)\, \tilde{c}(\bar{w}_3)\, \tilde{b}(\bar{w}_4)\big\rangle = \int \mathcal{D}b\,\mathcal{D}\tilde{b}\,\mathcal{D}c\,\mathcal{D}\tilde{c}\; c(w_1)\, b(w_2)\, \tilde{c}(\bar{w}_3)\, \tilde{b}(\bar{w}_4)\, e^{-S'} \equiv Z' \tag{48}$$

First we argue that only the zero modes of the insertions survive the path integral[8]. In fact, as anti-commuting replacements of the gauge degrees of freedom, ghost modes are *defined* to be the eigenvalues of $P^\dagger P$, where P is the conformal Killing differential[9]. More specifically, given a conformal Killing vector (CKV) $\delta\sigma^a$, the conformal Killing equation can be written as:

$$P\,\delta\sigma = 0 \tag{49}$$

While $P^\dagger \delta' g = 0$ gives moduli variation $\delta' g_{ab}$ of the metric. Roughly speaking, P captures the variation of gauge fixing under an arbitrary gauge transformation; naturally, CKV's are given by $(\ker P)$, while $(\det P) \sim \Delta_{FP}$ is the Faddeev–Popov functional measure near the gauge slice. $(\det P)$ can then be calculated with:

$$\delta\sigma^a \mapsto c^a, \quad \delta' g_{ab} \mapsto b_{ab}, \quad \Delta_{FP} \sim \det P \sim \int \mathcal{D}b\,\mathcal{D}\tilde{b}\,\mathcal{D}c\,\mathcal{D}\tilde{c}\, e^{-S'}, \tag{50}$$

$$S' = \frac{1}{2\pi}\int \mathrm{d}^2\sigma\, g^{1/2}\, b_{ab}\,(P\cdot c)^{ab} = \frac{1}{2\pi}\int \mathrm{d}^2 w\,\big(b\,\bar{\partial}_w c + \tilde{b}\,\partial_w \tilde{c}\big) \tag{51}$$

In the end we have chosen conformal gauge, such that[10] $P \sim (\bar{\partial}_w, \partial_w)$, $P^\dagger P \sim -\bar{\partial}_w \partial_w = -\nabla^2$. In the $w = \sigma^1 + i\sigma^2$ coordinates, CKV's are simple translations: $c^a = \text{const}$; with $z = e^{-iw}$, it gets mapped to $c^z = c^w\,\partial_w z = c^w\,(-iz)$, which agrees with the zero mode c_0 in the $c(z)$ expansion:

$$c(z) = \sum_{m=-\infty}^{\infty} \frac{c_m}{z^{m+1-\lambda}} = c_0\, z + \sum_{m\neq 0} \frac{c_m}{z^{m-1}}, \qquad \lambda = 2 \tag{52}$$

Now we are finally ready to prove our argument: for anti-commuting variables like $c(z)$,

$$\int \mathcal{D}c \sim \prod_m \int \mathrm{d}c_m \sim \prod_m \frac{\partial}{\partial c_m} \tag{53}$$

---

[7] Reference: *Blumenhagen & Plauschinn*.

[8] I would like to thank 谷夏 for some very helpful discussions about this problem.

[9] Reference: *Polchinski*, Chapter 3 & 5.

[10] References:

- Nakahara, *Geometry, Topology and Physics*;
- Blumenhagen et al, *Basic Concepts of String Theory*.

Since c_0 corresponds to a CKV, $P \cdot c_0 = 0$, therefore it vanishes in $S' = \int \mathrm{d}^2\sigma\,(b \cdot P \cdot c)$; for the path integral to be non-zero, there has to be some additional c_0 insertions, i.e.

$$Z' \sim \int \mathcal{D}b\mathcal{D}\tilde{b}\mathcal{D}c\mathcal{D}\tilde{c}\; c_0 b_0 \tilde{c}_0 \tilde{b}_0\, e^{-S'} \sim \left(\frac{1}{\sqrt{\tau_2}}\right)^4 \int \mathcal{D}'b\mathcal{D}'\tilde{b}\mathcal{D}'c\mathcal{D}'\tilde{c}\; e^{-S'}, \quad \int \mathcal{D}'c \sim \prod_{m\neq 0} \int \mathrm{d}c_m \tag{54}$$

Note the additional $\left(\frac{1}{\sqrt{\tau_2}}\right)^4$ factor coming from the zero modes[11]; this has to do with the normalization of the zero modes, each contributing a factor of $\frac{1}{\sqrt{A}}$, where $A \sim \tau_2$ is the volume (surface area) of the torus. On a different note, since it is very difficult, if not impossible, to keep track of various (often divergent) constant factors in the path integral, we have been and will be calculating Z' up to an overall constant coefficient.

Now we have to deal with the path integral over non-zero modes. Note that the holomorphic mode expansion (52) is incomplete for our purpose: it gives the *on-shell* mode expansion, while our path integral should go over all possible configurations, including the off-shell modes, which is *not* holomorphic. However, on $T^2 = S^1 \times S^1$, the full modes are simple[12]:

$$-\nabla^2 \psi_{n_1,n_2} = \lambda_{n_1,n_2} \psi_{n_1,n_2}, \tag{55}$$

$$\begin{aligned} \psi_{n_1,n_2} &= \exp\left(i\,(n_1\tilde{\sigma}^1 + n_2\tilde{\sigma}^2)\right), \quad \tilde{\sigma}^2 = \frac{\sigma^2}{\tau_2}, \quad \tilde{\sigma}^1 = \sigma^1 - \sigma^2\frac{\tau_1}{\tau_2}, \\ &= \exp\left\{ i\left(n_1\sigma^1 + \frac{n_2 - n_1\tau_1}{\tau_2}\,\sigma^2 \right) \right\}, \end{aligned} \tag{56}$$

Here we first use the "rectangular" coordinates $(\tilde{\sigma}^1, \tilde{\sigma}^2) \in [0, 2\pi]^2$ to write down the obvious eigenfunctions ψ_{n_1,n_2}, and then relate them back to the (σ^1, σ^2) coordinates. Therefore, we have:

$$\begin{aligned} \lambda_{n_1,n_2} &= \left\{ n_1^2 + \left(\frac{n_2 - n_1\tau_1}{\tau_2}\right)^2 \right\} \\ &= \frac{1}{\tau_2^2}\left\{ (n_1\tau_2)^2 + \left(n_1\tau_1 - n_2\right)^2 \right\} \\ &= \frac{1}{\tau_2^2}\left|n_1\tau - n_2\right|^2, \end{aligned} \tag{57}$$

$$\det{}' P \sim \left({\prod_{n_1,n_2}}' \sqrt{\lambda_{n_1,n_2}} \right)^2 \sim {\prod_{n_1,n_2}}' \lambda_{n_1,n_2} \tag{58}$$

The determinant can be computed with ζ-function regularization, as is performed in detail in *Di Francesco*; the result can be nicely summarized using the Eisenstein series, as shown in *Nakahara*:

$$E(\tau, s) = {\sum_{n_1,n_2}}' \frac{\tau_2^s}{\left|n_1\tau - n_2\right|^{2s}}, \tag{59}$$

$$\det{}' P \sim {\prod_{n_1,n_2}}' \frac{1}{\tau_2^2}\left|n_1\tau - n_2\right|^2 \sim \tau_2 \exp\left\{ -\partial_s E'(\tau, s)_{s=0} \right\} = \tau_2^2\left|\eta(\tau)\right|^4 \tag{60}$$

Finally, we have:

$$Z' \sim \tau_2^{-2} \det{}' P \sim \tau_2^{-2}\tau_2^2\left|\eta(\tau)\right|^4 \sim \left|\eta(\tau)\right|^4 \tag{61}$$

---

[11] Reference: *Di Francesco et al.*

[12] References: (1) *Nakahara*, (2) *Di Francesco et al.*, and (3) http://theory.uchicago.edu/~sethi/Teaching/P483-W2018/p483-sol3.pdf.

$\boxed{4}$ Torus Propagator as a Trace

$$w' \to 0, \quad \big\langle \partial_w X(w)\, \partial_{w'} X(w') \big\rangle = \mathrm{Tr}\left(\partial_w X(w)\, \partial_{w'} X(w')\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \right) \tag{62}$$

Here we've dropped the time ordering in the $w' \to 0$ limit. Recall the mode expansion of ∂X in $\boxed{1}$; we see that only the "diagonal" components of $\partial X(w)\, \partial X(w')$ survive in the trace, i.e.

$$\begin{aligned} \partial_w X(w)\, \partial_{w'} X(w') &= (\partial_w z)(\partial_{w'} z')\, \partial_z X(z)\, \partial_{z'} X(z'), \qquad z = e^{-iw}, \quad 1 \le |z| \le e^{2\pi\tau_2} \\ &\sim -\frac{\alpha'}{2} \sum_{n=-\infty}^{\infty} \frac{\alpha_{-n}\alpha_n}{z^{-n+1} z'^{n+1}} (-iz)(-iz') \\ &= \frac{\alpha'}{2} \left(\alpha_0^2 + \sum_{n>0} \left(\left(\frac{z}{z'}\right)^n + \left(\frac{z'}{z}\right)^n \right) \alpha_{-n}\alpha_n + \sum_{n>0} n \left(\frac{z'}{z}\right)^n \right) \\ &= \frac{\alpha'}{2} \left(\alpha_0^2 + \sum_{n>0} \left(\left(\frac{z}{z'}\right)^n + \left(\frac{z'}{z}\right)^n \right) \alpha_{-n}\alpha_n + \frac{zz'}{(z-z')^2} \right) \end{aligned} \tag{63}$$

The last term is a normal ordering constant; here it is naturally regularized by $\left(\frac{z'}{z}\right)^n$.

The α_0^2 term can be substituted with spacetime momentum p; we have:

$$p = \sqrt{\frac{1}{2\alpha'}}\, (\alpha_0 + \tilde{\alpha}_0) = \sqrt{\frac{1}{2\alpha'}}\, 2\alpha_0 = \sqrt{\frac{2}{\alpha'}}\, \alpha_0, \tag{64}$$

$$\partial_w X(w)\, \partial_{w'} X(w') \sim \frac{\alpha'}{2} \left(\frac{\alpha' p^2}{2} + \sum_{n>0} \left(\left(\frac{z}{z'}\right)^n + \left(\frac{z'}{z}\right)^n \right) n N_n \right) \tag{65}$$

On the other hand, the partition function is:

$$\begin{aligned} Z(\tau) = \langle \mathbb{1} \rangle &= (q\bar{q})^{-\frac{1}{24}}\, V \int \frac{\mathrm{d}k}{2\pi}\, e^{-\pi\tau_2\, \alpha' k^2} \sum_{(N_l),(\tilde{N}_l)} q^{\sum\limits_{l>0} l\cdot N_l}\, \bar{q}^{\sum\limits_{l>0} l\cdot \tilde{N}_l} \\ &= (q\bar{q})^{-\frac{1}{24}}\, V \int \frac{\mathrm{d}k}{2\pi}\, e^{-\pi\tau_2\, \alpha' k^2} \sum_{(N_l),(\tilde{N}_l)} \prod_{l>0} q^{l\cdot N_l} \bar{q}^{l\cdot \tilde{N}_l} \\ &= |\eta(\tau)|^{-2}\, V \int \frac{\mathrm{d}k}{2\pi}\, e^{-\pi\tau_2\, \alpha' k^2} \end{aligned} \tag{66}$$

We can work out $Z^{-1}\langle \partial X \partial X \rangle$ by considering term by term insertion of the $\partial X \partial X$ mode expansion into the above expression. For the $\frac{\alpha' p^2}{2}$ term, we have a contribution of:

$$\frac{\displaystyle\int \frac{\mathrm{d}k}{2\pi}\, \frac{\alpha' k^2}{2}\, e^{-\pi\tau_2\, \alpha' k^2}}{\displaystyle\int \frac{\mathrm{d}k}{2\pi}\, e^{-\pi\tau_2\, \alpha' k^2}} = \frac{\alpha'}{2}\, \frac{1}{2\cdot \pi\alpha'\tau_2} = \frac{1}{4\pi\tau_2} \tag{67}$$

For the nN_n insertion, we have a contribution of:

$$\begin{aligned} \frac{\sum\limits_{(N_l)} nN_n\, q^{\sum\limits_{l>0} l\cdot N_l}}{\sum\limits_{(N_l)} q^{\sum\limits_{l>0} l\cdot N_l}} &= \frac{\sum\limits_{(N_l)} nN_n \prod\limits_{l>0} q^{l\cdot N_l}}{\sum\limits_{(N_l)} \prod\limits_{l>0} q^{l\cdot N_l}} = \frac{\sum\limits_{N_n=0}^{\infty} nN_n\, q^{n\cdot N_n}}{\sum\limits_{N_n=0}^{\infty} q^{n\cdot N_n}} = \frac{nq^n \frac{\partial}{\partial(q^n)} \sum\limits_{N_n=0}^{\infty} q^{n\cdot N_n}}{\sum\limits_{N_n=0}^{\infty} q^{n\cdot N_n}} \\ &= \frac{nq^n \frac{\partial}{\partial(q^n)} \frac{1}{1-q^n}}{\frac{1}{1-q^n}} = \frac{nq^n}{1-q^n} \end{aligned} \tag{68}$$

Therefore, the complete result is given by:

$$\begin{aligned}\frac{1}{Z(\tau)}\langle\partial_w X(w)\,\partial_{w'}X(w')\rangle &= \frac{\alpha'}{2}\left(\frac{1}{4\pi\tau_2}+\sum_{n>0}\left(\left(\frac{z}{z'}\right)^n+\left(\frac{z'}{z}\right)^n\right)\frac{nq^n}{1-q^n}+\frac{zz'}{(z-z')^2}\right)\\ &\xrightarrow[z'\to 1]{w'\to 0}\frac{\alpha'}{2}\left(\frac{1}{4\pi\tau_2}+\sum_{n>0}\left(z^n+z^{-n}\right)\frac{nq^n}{1-q^n}+\frac{z}{(z-1)^2}\right)\end{aligned} \tag{69}$$

On the other hand, the torus propagator is given by:

$$G'(w,\bar{w};w',\bar{w}') = -\frac{\alpha'}{2}\,\ln|f(w-w',\tau)|^2+\frac{\alpha'}{4\pi\tau_2}\left(\mathrm{Im}\,(w-w')\right)^2, \tag{70}$$

$$f(w,\tau)\equiv\theta_1\left(\frac{w}{2\pi}\middle|\,\tau\right)=2\,e^{\frac{i\pi\tau}{4}}\sin\frac{w}{2}\prod_{m>0}^{\infty}(1-q^m)(1-z^{-1}q^m)(1-zq^m),\quad z=e^{-iw} \tag{71}$$

We find that $\partial_w\partial_{w'}G'$ contains the same zero mode contribution $\frac{\alpha'}{8\pi\tau_2}$ and normal ordering contribution $\frac{\alpha'}{2}\frac{z}{(z-1)^2}$ as in (69):

$$\partial_w\partial_{w'}G'(w,\bar{w};w',\bar{w}')_{w'=0}=\frac{\alpha'}{8\pi\tau_2}+\frac{\alpha'}{2}\partial_w^2\ln f(w,\tau), \tag{72}$$

$$\partial_w^2\ln f(w,\tau)=\partial_w^2\ln\sin\frac{w}{2}+\partial_w^2\sum_{m>0}\left(\ln\left(1-zq^m\right)+\ln\left(1-z^{-1}q^m\right)\right), \tag{73}$$

$$\partial_w^2\ln\sin\frac{w}{2}=\partial_w^2\ln\sin\frac{w}{2}=-\frac{1}{4\sin^2\frac{w}{2}}=\frac{1}{2\left(\cos w-1\right)}=\frac{1}{z+z^{-1}-2}=\frac{z}{(z-1)^2}, \tag{74}$$

The remaining parts come from oscillator modes; they also match with (69), but the equivalence is less obvious: we have[13]:

$$\begin{aligned}\partial_w^2\sum_{m>0}\ln\left(1-zq^m\right) &= \partial_w^2\sum_{m>0}\sum_{n>0}-\frac{1}{n}\left(zq^m\right)^n\\ &=\sum_{n>0}\partial_w^2\left(-\frac{1}{n}\,z^n\right)\sum_{m>0}q^{mn},\quad \partial_w=-iz\,\partial_z\\ &=\sum_{n>0}-\frac{(-in)^2}{n}\,z^n\cdot\frac{q^n}{1-q^n}\\ &=\sum_{n>0}z^n\frac{nq^n}{1-q^n},\end{aligned} \tag{75}$$

$$\partial_w^2\sum_{m>0}\ln\left(1-z^{-1}q^m\right)=\sum_{n>0}z^{-n}\frac{nq^n}{1-q^n}, \tag{76}$$

This is precisely the contribution from oscillator modes in (69). Therefore, we have:

$$\frac{1}{Z(\tau)}\langle\partial_w X(w)\,\partial_{w'}X(w')\rangle_{w'=0}=\partial_w\partial_{w'}G'(w,\bar{w};w',\bar{w}')_{w'=0} \tag{77}$$

[13] Reference: http://theory.uchicago.edu/~sethi/Teaching/P483-W2018/p483-sol3.pdf. I would like to thank Lucy Smith for providing this hint.