Algebraic Topology №3 [WIP]

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Compiled @ 2022/10/11

$\boxed{1}$ Example of limit in $\underline{\mathbf{Vect}}$:

$$\begin{tikzcd} & \left\{ \begin{array}{c} 0 \\ \mathbb{R} \end{array} \right\} \\ I = \left\{ \begin{array}{c} \bullet \\ \star \end{array} \right\} \arrow[ur, mapsto, "F"] \arrow[dr, mapsto, "\Delta(V)"'] & & \supset \ \underline{\mathbf{Vect}}\colon\ \mathbb{R} \text{ vector space }, \\ & \left\{ \begin{array}{c} V \\ V \end{array} \right\} \end{tikzcd} \tag{1}$$

(a) By definition, we have:

$$\begin{tikzcd} & \Delta(\lim F) \arrow[d, Rightarrow, "\sigma"] \\ \Delta(V) \arrow[ur, dashed, Rightarrow, "\exists!\,\Delta(g)"] \arrow[r, Rightarrow, "\eta"] & F \end{tikzcd} \tag{2}$$

Where $g\colon V \to \lim F$. More intuitively, for the above $F\colon I \to \underline{\mathbf{Vect}}$, this translates to the following diagram in $\underline{\mathbf{Vect}}$:

$$\begin{tikzcd} & & 0 \\ V \arrow[r, dashed, "\exists!\,g"] \arrow[urr, bend left, "\eta_\bullet"] \arrow[drr, bend right, "\eta_\star"'] & \lim F \arrow[ur, "\sigma_\bullet"] \arrow[dr, "\sigma_\star"'] & \\ & & \mathbb{R} \end{tikzcd} \tag{3}$$

Now we verify that $\lim F = \mathbb{R}$, along with the following choice of σ:

$$\sigma_\bullet = 0, \quad \sigma_\star = \mathbb{1}_{\mathbb{R}} \tag{4}$$

In fact, for the above diagram (3) to be commutative, we must have $g = \eta_\star$. Note that such g is unique once σ is chosen; for our choice of σ, if $g \neq \eta_\star$, then the diagram *cannot* commute. Hence $\lim F = \mathbb{R}$, along with the above choice of $\sigma\colon \Delta(\mathbb{R}) \Rightarrow F$. In other words, we have:

$$\begin{tikzcd} & \{\, \bullet,\ \star \,\} \arrow[dl, "\Delta(\mathbb{R})"'] \arrow[dr, "F"] & \\ \mathbb{R} \arrow[rr, phantom, "\overset{\sigma}{\Longrightarrow}" near start] & & \{\, 0,\ \mathbb{R} \,\} \end{tikzcd} \tag{5}$$

□

(b)(c) From diagram (2) and discussions in (a), we know that:

$$\exists !\ \tau = \Delta(g)\colon\ \Delta(V) \Longrightarrow \Delta(\mathbb{R}) \tag{6}$$

Here $g\colon V \to \mathbb{R}$ is fixed uniquely once σ is fixed. However, σ may vary up to isomorphism; therefore, a generic choice of σ is given by:

$$\sigma_\bullet = 0, \quad \sigma_\star = k\,\mathbb{1}_{\mathbb{R}}, \quad k \in \mathbb{R} \tag{7}$$

For such g, by the same arguments in (a), we have:

$$\exists!\ g = \frac{1}{k}\,\eta_\star\,,\ \ \tau = \Delta(g) = \frac{1}{k}\,\Delta(\eta_\star), \quad s.\,t. \quad (2),\,(3)\ \textit{commutes} \tag{8}$$

Note that $k \in \mathbb{R}$, for every k there is a different g and τ; hence there are $\|\{k\}\| = \|\mathbb{R}\|$ many choices of τ to make the diagram commute. In particular, for $k = 1$ we recover $\tau = \Delta(\eta_\star)$. ■

$\boxed{2}$ Limit and colimit of polynomial ring:

By definition,

$$\begin{tikzcd}
 & & Q \arrow[ddll] \arrow[ddl, "\eta_{n+1}"'] \arrow[d, dotted, "\exists! f"] \arrow[ddr, "\eta_n"] \arrow[ddrr] \arrow[ddrrr] & & & \\
 & & \lim F \arrow[dl, "\pi_{n+1}"] \arrow[dr, "\pi_n"'] & & & \\
\cdots \arrow[r, "p_{n+1}"] \arrow[ddrr] & \mathbb{Z}[x]/x^{n+1} \arrow[rr, "p_n"] \arrow[dr, "\sigma_{n+1}"] \arrow[ddr, "\tau_{n+1}"'] & & \mathbb{Z}[x]/x^n \arrow[r, "p_{n-1}"] \arrow[dl, "\sigma_n"'] \arrow[ddl, "\tau_n"] & \cdots \arrow[r, "p_1"] \arrow[ddll] & \mathbb{Z}[x]/x^1 = \mathbb{Z} \arrow[ddlll, "\tau_1"] \\
 & & \operatorname{colim} F \arrow[d, dotted, "\exists! g"] & & & \\
 & & R & & &
\end{tikzcd} \tag{9}$$

Here $p_n\colon \mathbb{Z}[x]/x^{n+1} \to \mathbb{Z}[x]/x^n$ is the natural projection.

Intuitively, if such $\lim F$ exists, it shall be the "smallest" object that "contains" $\mathbb{Z}[x]/x^n$ when $n \to \infty$. Note that $\mathbb{Z}[x]/x^n$ is naturally a $\mathbb{Z}^n$ vector space:

$$\mathbb{Z}[x]/x^n \ \ni\ \sum_{m=0}^{n-1} a_m x^m \ \sim\ (a_0, a_1, \cdots, a_{n-1}) \ \in\ \mathbb{Z}^n \tag{10}$$

While $n \to \infty$, this gives an ∞–tuple which corresponds to the *formal power series*[1]:

$$\mathbb{Z}[[x]] \ \ni\ \sum_{m=0}^{\infty} a_m x^m \tag{11}$$

The difference between $\mathbb{Z}[x]$ and $\mathbb{Z}[[x]]$ is that the latter may contain infinite series while the former may not. Now we confirm that, indeed, $\lim F = \mathbb{Z}[[x]]$, along with natural projections $\pi_n\colon \mathbb{Z}[[x]] \to \mathbb{Z}[x]/x^n$.

In fact, the $f\colon R \to \lim F$ in (9) can be explicitly written down as:

$$\begin{aligned} f &= \eta_1 + x\left(\frac{\mathrm{d}}{\mathrm{d}x}\,\eta_2\right) + x^2\left(\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}x}\,\eta_3\right) + \cdots = \sum_{m=0}^{\infty} x^m \left\{\frac{1}{m!}\,\frac{\mathrm{d}^m}{\mathrm{d}x^m}\,\eta_{m+1}\right\} \\ &\sim \big(\eta_{1,0},\, \eta_{2,1},\, \cdots,\, \eta_{n,n-1},\, \cdots\big) \end{aligned} \tag{12}$$

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[1] See `Wikipedia:` *Formal power series*. This is in fact the *adic completion* of $\mathbb{Z}[x]$. I would like to thank 刘逸华 & 谢贤进 for this hint.

Here $\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}$ is used to extract the a_{n-1} coefficient in $\mathbb{Z}[x]/x^n$; this is the last component of η_n, denoted by $\eta_{n,n-1}$. Any other choice of f will break communitativity of (9), hence f is fixed uniquely by $\mathbb{Z}[[x]]$ and π_n's. Therefore, $\lim F = \mathbb{Z}[[x]]$. $\square$

On the other hand, $\operatorname{colim} F$ is the "largest" object that any map *out of* $\mathbb{Z}[x]/x^n$ must "passes through". Also, this should hold for all $n \in \mathbb{Z}_+$. Naturally, projections $\sigma_n \colon \mathbb{Z}[x]/x^n \to \mathbb{Z}$ satisfy the above requirements; we have:

$$\sigma_n \colon \; x \longmapsto 0, \quad \sum_{m=0}^{n-1} a_m x^m \longmapsto a_0, \tag{13}$$

$$\sigma_{n+1} = p_1 \circ p_2 \circ \cdots p_n \tag{14}$$

$g \colon \mathbb{Z} \to R$ in (9) is fixed uniquely for such choice of σ_n; in fact, descend along the p_n tower in (9), and we have: $\tau_{n+1} = \tau_n \circ p_n = \tau_{n-1} \circ p_{n-1} \circ p_n = \cdots = \tau_1 \circ p_1 \circ p_2 \circ \cdots p_n = \tau_1 \circ \sigma_{n+1}, \; \forall \, n \in \mathbb{Z}_+$, hence $\exists! \; g = \tau_1$. Therefore, $\operatorname{colim} F = \mathbb{Z}$. $\blacksquare$

$\boxed{3}$ Example of push-out in $\underline{\textbf{Groupoid}}$:

$$\begin{tikzcd}
\{0,1\} \arrow[r, "f_1"] \arrow[d, "f_2"'] & \bullet \arrow[d, "\tau_1"] \arrow[ddr, bend left, "\eta_1"] & \\
\{0 \leftrightarrow 1\} \arrow[r, "\tau_2"] \arrow[drr, bend right, "\eta_2"'] & P \arrow[dr, dotted, "\exists!\, g"] & \\
 & & Q
\end{tikzcd} \tag{15}$$

Following the same observation as before, the push-out P is the "largest" object that any map out of $\bullet$ and $\{0 \leftrightarrow 1\}$ must pass through. By such universal property, P can be no larger than the coproduct: $\{\bullet\} \coprod \{0 \leftrightarrow 1\}$. However, we should also consider the equivalence imposed by:

$$\bullet \xleftarrow{f_1} \{0,1\} \xrightarrow{f_2} \{0 \leftrightarrow 1\} \tag{16}$$

Therefore, we simply have $P = \bullet$, with $\tau_{1,2}$ the natural projection. This can be verified with ease: we have $g = \eta_1$. It is unique since its image is a single point (with identity map to itself) $\star \in Q$, and the point $\star$ is fixed by commutativity. $\blacksquare$

$\boxed{4}$ Product and coproduct in $\underline{\textbf{Ab}}$:

For $G_\alpha \in \underline{\textbf{Ab}} \subset \underline{\textbf{Group}}$, note that we have:

$$\text{Free} : \underline{\textbf{Set}} \rightleftharpoons \underline{\textbf{Group}} : \text{Forget} \tag{17}$$

Therefore, for F: some diagram in $\underline{\textbf{Group}}$, $\boxed{\lim\left(\text{Forget} \circ F\right) = \text{Forget} \circ \lim F}$ if $\lim F$ exists.

By definition, the product $\prod_\alpha G_\alpha \in \underline{\textbf{Group}}$ is a limit, hence it is idential (as in $\underline{\textbf{Set}}$) to the *direct product*, with additional entry-wise group multiplication. Same applies for the full subcategory: abelian group $\underline{\textbf{Ab}} \subset \underline{\textbf{Group}}$.

On the other hand, the disjoint union of G_α's as sets will not necessary be a group, the identities $\mathbb{1}_\alpha \in G_\alpha$ must be glued together to produce a group structure. Furthermore, free-forgetful adjunction

(17) implies that for F': some diagram in **Set**,

$$\operatorname{colim}(\text{Free} \circ F') = \text{Free} \circ \operatorname{colim} F', \tag{18}$$

Whenever $\operatorname{colim} F'$ exists; in our case, $\operatorname{colim} F'$ is the disjoint union of sets: $\coprod_\alpha \text{Forget}(G_\alpha)$. Therefore, we should construct a free object in **Ab**.

Here we restrict our discussion to **Ab**, since the coproduct in **Ab** is *not* the same as in **Group** — the free product of abelian group is not necessary abelian. Hence, the coproduct in **Ab** shall be:

$$\coprod_\alpha G_\alpha = \bigoplus_\alpha G_\alpha, \quad i_\alpha \colon G_\alpha \hookrightarrow \bigoplus_\alpha G_\alpha \tag{19}$$

As a set, this is precisely the disjoint union with identities $0_\alpha \in G_\alpha \subset$ **Ab** glued together.

It is then straight-forward to verify its universal property: for $f_\alpha \colon G_\alpha \to H$,

$$\exists!\ f \colon \bigoplus_\alpha G_\alpha \longrightarrow H, \quad (g_\alpha)_\alpha \longmapsto \sum_\alpha f_\alpha(g_\alpha) \tag{20}$$

This is compatible with the abelian group multiplication. Note that for the summation to be well-defined, the coproduct must only contain finitely many components; otherwise it is identical to the product in **Ab**. ■

$\boxed{5}$ **Composition of pull-backs:**

$$\begin{tikzcd}
Q \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] \arrow[drrr, bend left, "f_3"] \arrow[ddr, bend right, "g_1"'] \arrow[ddrr, bend right, "g_2"] & & & \\
& A_1 \arrow[r, dashed] \arrow[d] \arrow[rr, bend left, "\phi'"] & A_2 \arrow[r, "\phi"] \arrow[d] & A_3 \arrow[d] \\
& B_1 \arrow[r, dashed, "\psi"'] \arrow[rr, bend right, "\psi'"'] & B_2 \arrow[r] & B_3
\end{tikzcd} \tag{21}$$

(a) If A_1 is the pull-back of A_2 and A_2 is the pull-back of A_3, then given Q with f_3, g_1, we have $g_2 = \psi \circ g_1$, and f_2 is fixed uniquely by universal property of A_2, while f_1 is fixed uniquely by universal property of A_1. Hence, A_1 is the pull-back of A_3.

(b) If A_1, A_2 are pull-backs of A_3, then given Q with f_2, g_1, we have $f_3 = \phi \circ f_2$, and f_2 is fixed uniquely by universal property of A_2, and f_1 is fixed uniquely by universal property of A_1. Hence, A_1 is the pull-back of A_3.

PAST WORK, AS TEMPLATE

$\boxed{1}$ For $F_i \to E_i \xrightarrow{p_i} B$: coverings in $\mathrm{Cov}_0(B)$ with E_i: connected and B: path connected and locally path connected, the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd} \qquad \begin{aligned} e_2 &= f(e_1), \\ b = p_1(e_1) &= p_2(e_2), \end{aligned}$$

To show that f is itself a covering, we need only verify that f is locally trivial with some discrete fiber F. In fact, given any $e_2 \in E_2$ and $b = p_2(e_2)$, there exists some neighborhood $U \subset B$ that the following diagram holds (by restriction):

$$\begin{tikzcd} U \times F_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & U \times F_2 \arrow[dl, "p_2"] \\ & U & \end{tikzcd} \qquad \begin{aligned} e_1 &= \big(b, k_1\big), \\ e_2 &= \big(b, k_2(b, k_1)\big), \end{aligned} \qquad k_i \in F_i$$

Generally, $k_2 = k_2(b, k_1)$ depends on the base point $b \in B$. However, since B is locally path connected, we can restrict U to be path connected, while $k_2 \in F_2$: discrete. Since continuous maps preserve path connectedness, k_2 is in fact independence of b, i.e. $k_2 = \varphi(k_1)$.

On the other hand, $\forall\, e_2 = (b, k_2) \in U \times \{k_2\} \subset E_2$, we have its preimage $f^{-1}(e_2) = \{b\} \times \varphi^{-1}(k_2)$. Note that E_2 is connected while $\varphi^{-1}(k_2) \in F_1$ is discrete; for the same reasoning as above, $\varphi^{-1}(k_2) = F$ is in fact independent of k_2. This is the discrete fiber F we have been looking for. Hence f is also a covering map[2]. ■

$\boxed{2}$ **Cylinder with ends pinched — π_1 and universal cover:**

$$Y = (X \times I)/(X \times \partial I)\,, \quad I = [0,1] \tag{22}$$

Note that Y is homeomorphic to two cones[3] $CX_1 \coprod CX_2$ with "bases" $X_i \subset CX_i$ and "vertices" v_i respectively identified: $X_1 \sim X_2$, $v_1 \sim v_2 \equiv v$. X is path connected and so is Y, hence we are free to choose $\pi_1(Y) = \pi_1(Y, y_0)$.

First note that paths that do *not* pass through the vertex v are all homotopic, since they are contained in a cone and cones are contractible[4]. Therefore all contributions to $\pi_1(Y)$ are loop classes that *do* pass through the vertex v. In other words, morphisms in $\Pi_1 Y$ are in one-to-one correspondence with morphisms in:

$$\Pi_1\big([0,1]/_{0\sim 1}\big) = \Pi_1 S^1 \tag{23}$$

Therefore, $\pi_1(Y) \cong \pi_1(S^1) = \mathbb{Z}$. □

The universal cover $\tilde{Y}$ of Y can be constructed by assigning an induced topology to the space of path classes, same as in the general proof of its existence. Since Y is "degenerate" at its vertex,

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[2] Reference: `math.stackexchange.com/a/109774`.

[3] See discussions from Problem Set №1.

[4] $[\gamma_1] = [\gamma_2 \star \gamma_2^{-1} \star \gamma_1] = [\gamma_2]$.

this is equivalent to "cutting open" Y at its vertex v, and joining $\mathbb{Z}$ copies them end-to-end. More explicitly, it can be written as:

$$\tilde{Y} = (X \times \mathbb{R})\Big/\sim, \quad (x, n) \sim (x', n),\ \forall\, x \in X,\, n \in \mathbb{Z} \tag{24}$$

While the covering map: $\tilde{Y} \ni [x, t] \mapsto [x, t - \lfloor t \rfloor] \in Y$, here $\lfloor t \rfloor$ is the integer part of $t \in \mathbb{R}$. ■

$\boxed{3}$ **π_1 of fiber in fibration:**

$$\begin{tikzcd}
X \times \{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \arrow[r, "f"] \arrow[ur, dashed, "\exists\, \tilde{f}"] & B
\end{tikzcd}$$

For $F \to E \xrightarrow{p} B$: fibration, by homotopy lifting property (HLP), any homotopy in B can be uniquely lifted to path class in E, provided some *"initial condition"* $X \times \{0\}$. This leads to the following results:

(a) For B: simply-connected, take any loop class $[\tilde{\gamma}] \in \pi_1(E, e)$ as initial condition; its projection $[p \circ \tilde{\gamma}] \in \pi_1(B, b) = \big\{[\mathbb{1}_b]\big\}$ is trivial, i.e. $p \circ \tilde{\gamma} \simeq \mathbb{1}_b$. By HLP, such homotopy can be lifted into E, i.e.

$$p \circ \tilde{\gamma} \simeq \mathbb{1}_b \quad \xRightarrow{\text{lift}} \quad \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}' = \mathbb{1}_b \tag{25}$$

In other words, $\tilde{\gamma} \simeq \tilde{\gamma}' \subset p^{-1}(b)$, i.e. any loop in E is homotopic to some loop in $p^{-1}(b) \cong F$. This implies a surjective group homomorphism $\pi_1(p^{-1}(b), e) \to \pi_1(E, e)$, i.e. an epimorphism. □

(b) For E: simply-connected, take any loop class $[\gamma] \in \pi_1(B, b)$ and consider its lifting $[\tilde{\gamma}]$. Note that in general $\tilde{\gamma}$ is *not* a loop; however, we have $p \circ \tilde{\gamma} = \gamma$, hence $\tilde{\gamma}(0), \tilde{\gamma}(1) \in p^{-1}(b)$. In general, we have:

$$\gamma \simeq \gamma' \quad \xRightarrow{\text{lift}} \quad \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}^{(\prime)} = \gamma^{(\prime)} \tag{26}$$

By continuity, $\tilde{\gamma}(0), \tilde{\gamma}'(0) \in F_0$: a path component of $p^{-1}(b)$; similarly, $\tilde{\gamma}(1), \tilde{\gamma}'(1) \in F_1$. In other words, the start and end points of $\tilde{\gamma}$ are confined in path components F_0 and F_1, respectively. Hence a loop class in $\pi_1(B, b)$ maps to *transport* between path components:

$$\begin{aligned} T_{(\cdot)}(e)\colon\ \pi_1(B, b) &\longrightarrow \pi_0\big(p^{-1}(b)\big) \\ [\gamma] &\longmapsto T_{[\gamma]}(e) \end{aligned} \tag{27}$$

As a matter of fact, $T_{(\cdot)}(e)$ is a bijection. For $T_{[\gamma]} = T_{[\gamma']}$, they are characterized by two lifted paths $\tilde{\gamma}, \tilde{\gamma}'$; since E is simply connected, they are always homotopic: $\tilde{\gamma} \simeq \tilde{\gamma}'$, hence $[\gamma] = [\gamma']$ by projection p. This means that T is injective. Surjectivity also follows from projection $\gamma = p \circ \gamma'$. Therefore, $T_{(\cdot)}(e)$ gives a bijection between $\pi_1(B, b)$ and $\pi_0\big(p^{-1}(b)\big)$. ■

$\boxed{4}$ **Pull-back of fibration is fibration:**

$$\begin{tikzcd}
Y \times \{0\} \arrow[r] \arrow[d, hook] & f^*(E) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "G"] \arrow[ur, dashed, "\exists\,\tilde{G}"] \arrow[urr, "\exists\,\tilde{F}\ \text{(HLP)}"] & X \arrow[r, "f"] & B
\end{tikzcd}$$

$$(x, e) \in f^*(E) \subset X \times E, \quad f(x) = p(e)$$

We need only verify that $f^*(E) \to X$ also has HLP, i.e. the existence of $\tilde{F}$ in the above diagram[5]. By HLP of $E \xrightarrow{p} B$, $\exists\ \tilde{F}\colon Y \times I \to E$ as shown above. We can use $\tilde{F}$ to construct $\tilde{G}$ explicitly; in fact, first consider:

$$\begin{aligned} \tilde{G}\colon\ Y \times I \quad &\longrightarrow \quad X \times E \\ (y, t) \quad &\longmapsto \quad \big(G(y, t), \tilde{F}(y, t)\big) \end{aligned} \tag{28}$$

Note that $f \circ G = p \circ \tilde{F}$; compared with the definition of $f^*(E)$, this implies that the image of $\tilde{G}$ lies within $f^*(E) \subset X \times E$, hence after restriction of its codomain, $\tilde{G}$ becomes a well-defined lifting of G into $f^*(E)$. Therefore, $f^*(E) \to X$ has HLP, i.e. it is also a fibration. ■

$\boxed{5}$ **More properties of fibration:**

(a) By HLP, given any initial condition $e \in p^{-1}(b_1)$, lifting of any path $b_1 \xrightarrow{\gamma} b_2$ exists. The lifted path with dependence of e can then be written as $F\colon p^{-1}(b_1) \times I \to E$. This is just a generalization of $\boxed{3}$ for non-loop paths. □

(b) Similarly, transport $T_{[\gamma]}$ defined in $\boxed{3}$ can be generalized for non-loop paths. $T_{[\gamma]}$ is well-defined for path class $[\gamma]$, since by HLP homotopic paths can be lifted to homotopy in E. Therefore, the transport is fixed up to homotopy, i.e.

$$\begin{aligned} T\colon\ \mathrm{Hom}_{\Pi_1 B}(b_0, b_1) \quad &\longrightarrow \quad \mathrm{Hom}_{\underline{\mathbf{hTop}}}\big(p^{-1}(b_0), p^{-1}(b_1)\big) \\ [\gamma] \quad &\longmapsto \quad T_{[\gamma]} \end{aligned} \tag{29}$$

Note that T defined in this way is also independent of the choice of F, since F simply specifies the starting point of the lifted path; no matter which F we choose, the lifted paths will always be homotopic in E. Hence T is well-defined in the above sense. □

(c) T defined above is a functor: $\Pi_1 B \to \underline{\mathbf{hTop}}$. To verify this, we need only check that it is compatible with composition and maps identity morphisms to identity morphisms. Indeed, $T_{[\mathbb{1}_b]} = [\mathbb{1}_{p^{-1}(b)}]$, and $T_{[\gamma'] \star [\gamma]} = T_{[\gamma' \star \gamma]} = T_{[\gamma']} \circ T_{[\gamma]}$ by joining two lifted paths (up to homotopy). □

(d) For B: path connected, there exists an isomorphism between any two objects in $\Pi_1 B$ (a path connecting any two points in B), which is mapped to isomorphisms between fibers $p^{-1}(b)$ in $\underline{\mathbf{hTop}}$. Hence any two fibers of $E \xrightarrow{p} B$ have the same homotopy type. ■

[5] Notice that $f^*(E)$ is the limit of the diagram, hence this is automatically true by the universal property of $f^*(E)$. I would like to thank 刘逸华 for pointing this out. For now, we will stick to a more traditional proof.