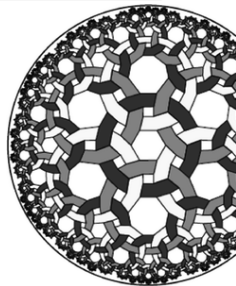


Understanding Holographic Entanglement

With gravitational path integral & tensor network

Bryan



Dunham: Escher on Escher

Introduction

- **Question:** how can we “prepare” / construct a state, e.g.

- vacuum: $\rho = |0\rangle\langle 0|$

- thermal: $\rho \propto \sum_n e^{-\beta H} |n\rangle\langle n|$

... in a *holographic* system?

- **Answer:** via holography! (*duh...*)

- Gravitational path integral

- Tensor network

- Entanglement entropy captures some “structure” of the state

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Review: boundary entanglement as gravitational saddles

Recall: Path Integral in the Boundary & the Bulk

Lewkowycz:2013nqa: Lewkowycz:2013nqa



Figure: Thermal \mathcal{Z}_β

Thermal partition function
in the bulk & the boundary

Image made from Benjamin:2020mfz
and the EHT black hole photo

- In *any* field theory, a thermal state can be prepared by a path integral;
- In a *holographic* theory, $\mathcal{Z}_{\partial B} = \mathcal{Z}_{Bulk}$, a boundary state can be prepared by a bulk path integral.
 - e.g. the thermo-field double \leftrightarrow the BTZ black hole
Note that the Euclidean BTZ geometry is smooth:
filling in the t_E cycle (*not* the ϕ cycle) of the torus
 - c.f. Chern-Simons/WZW: not quite the same

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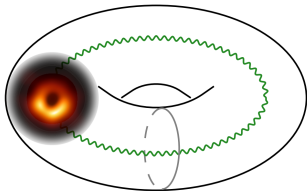


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$$\mathcal{Z}_{\partial B} = \mathcal{Z}_{Bulk} \quad (1)$$

- Replica trick: entanglement entropy for a region R :

$$S_R = -\text{Tr} \rho_R \log \rho_R \Leftrightarrow \text{Tr} \rho_R^n \equiv \mathcal{Z}_n \quad (2)$$

i.e. reduced to the partition function of the n -replica.
It can be deployed in the boundary & the bulk!

- Boundary: static geometry, but the field theory is usually strongly coupled — often difficult!
- Bulk: dynamic geometry, weakly coupled gravity:
gravity fills in the bulk smoothly,
 $\mathcal{Z} \sim \sum_i e^{-S_i[g_{\mu\nu}]}$: sum over classical saddles

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Recall: Replica Trick and the Ryu–Takayanagi Proposal

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- $\mathcal{Z} \sim \sum_i e^{-S_i[g_{\mu\nu}]}$, saddle pt. approx.
 \Rightarrow minimize $S[g_{\mu\nu}]$ on the n -replica $\widetilde{\mathcal{M}}_n$
- $\widetilde{\mathcal{M}}_n/\mathbb{Z}_n$: conical singularity at the \mathbb{Z}_n ; $n \rightarrow 1$,
 \Rightarrow minimize the area of the \mathbb{Z}_n fixed point
 \Rightarrow the extremal surface, the RT surface

■ **Lesson:** use bulk path integral to:

- prepare the states
- compute the entanglement entropy

This requires holography,
 but not necessarily AdS/CFT!

Figure: The bulk replica
 [Lewkowycz:2013nqa]

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Compare: bulk emergence from tensor networks

Prepare states via Tensor Networks

Vidal:2007hda, Swingle:2009bg. Reviewed by Rangamani:2016dms.

MERA



Figure: MERA

Multi-scale Entanglement
Renormalization Ansatz

Image from tensornetwork.org

- Gravitational path integral: a spacetime perspective
Tensor network: on a constant time slice
- States constructed with tensor networks:
common in condensed matter (e.g. DMRG)
- To find the ground state of a system
 - Write down a tensor network
as an **ansatz** for the ground state;
 - Vary the components of each tensor
to achieve minimal energy — **optimization!**

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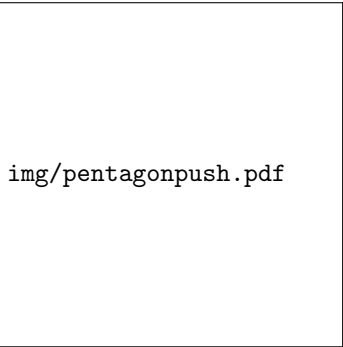
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AdS Bulk as a Tensor Network

Reviewed by **Harlow:2018fse**: **Harlow:2018fse**



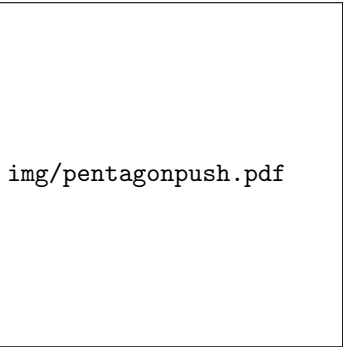
img/pentagonpush.pdf

Figure: The HaPPY code
[**Pastawski:2015qua**,
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- Dual geometry in the bulk is understood as the “continuous limit” of a *tensor network*
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 - Leg: index to be contracted
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takes in bulk local operator insertions
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propagates the bulk insertions to the boundary

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RT from the tensor network

Harlow:2018fse: Harlow:2018fse




img/cutnetwork.pdf

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- “Entanglement” between R and \bar{R} :
 \propto min # of links connecting the two regions
 Naturally, entropy = bulk area!
- “Complexity”: # of nodes
 Proposal: complexity = bulk volume!
 See e.g. Susskind:2014rva
- Note: how to actually take the continuous limit?
 “Real time” path integral on a constant time slice
 - cMERA: Nozaki:2012zj
 - “Path-Integral Optimization”:
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The Lesson

- The Ryu–Takayanagi proposal: $S \sim \frac{A}{4G_N}$
 - ... seems to be universal in holographic systems,
 - ... where boundary states can be constructed from some sort of bulk operations:
 - Gravitational path integral
 - Tensor network
- Applications: beyond standard $\text{AdS}_3/\text{CFT}_2$
 - Cutoff holography: **Lewkowycz:2019xse**
 - Flat holography: **Apolo:2020bld**, **Apolo:2020qjm**

Application: cutoff $\text{AdS}_3 / T\bar{T}$ deformed theory

Cutoff AdS_3 / $T\bar{T}$ deformed theory

McGough:2016lol



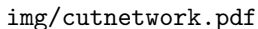
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- AdS_3 with finite cutoff:
holographic renormalization of the boundary theory
 - This is clear in the tensor network picture
"coarse-graining"
- Deform the boundary CFT_2 with some operator:
 $\text{CFT}_2^{(\text{UV})} \rightsquigarrow \text{deformed theory}^{(\text{IR})}$
- Surprisingly, we were able to find the deformed theory!
 $\delta S \propto \mu (T\bar{T})_\mu, \quad T\bar{T} = \frac{1}{8} (T^{\alpha\beta} T_{\alpha\beta} - (T_\alpha^\alpha)^2)$
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Holographic Entanglement in Cutoff AdS_3

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Then the RT proposal should still hold!
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- Generalization of A in $S \sim \frac{A}{4G_N}$: A is actually the
gravitational charge of the *replica symmetry*,
 - ... *analytically continued* from \mathbb{Z}_n to $U(1)$,
 - ... corresponds to the Killing horizon generator /
modular flow generator ξ .
 - This would in turn give us a hint of the modular flow
in the $T\bar{T}$ deformed theory! (*ongoing work*)

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Further Reading & Outlook

- Single-trace $T\bar{T}$ duality and the flow towards to UV
 - **Apolo:2019zai**
- Quantum error correction:
 - **Jahn:2021uqr**
- Tensor network for flat spacetime?
 - **May:2016dgv**

References I