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## 1 Spacelike Geodesic for Poincaré AdS

The Poincaré  $AdS_{d+1}$  metric is given by:

$$ds^{2} = G_{IJ} dX^{I} dX^{J} = \frac{-dt^{2} + d\vec{x}^{2} + dz^{2}}{z^{2}} = \frac{dx^{2} + dz^{2}}{z^{2}},$$
(1.1)

$$dx^{2} = -dt^{2} + d\vec{x}^{2} = \eta_{\mu\nu} dx^{\mu} dx^{\nu}, \quad X^{I} \sim (t, \vec{x}, z) \sim (x^{\mu}, z), \tag{1.2}$$

Note that here we define  $dx^2$  using the flat metric  $\eta$ , while dX has an index that should be raised and lowered using the curved metric  $G_{IJ} = \frac{1}{z^2} \eta_{IJ}$ . Generally upper case tensors are handled with  $G_{IJ}$ , while lower case ones are handled with  $\eta$ . The full isometry of such spacetime is then given by SO(d,2) with  $\frac{(d+2)(d+1)}{2}$  generators, including:

- 1. Among the  $x^{\mu} \sim (x^0, \vec{x}) \equiv (t, \vec{x})$  directions:
  - (a)  $\frac{d(d-1)}{2}$  SO(d-1,1) rotations  $x_{\mu}\partial_{\nu} x_{\nu}\partial_{\mu}$ , boosts included;
  - (b) d translations  $\partial_{\mu}$
- 2. Dilation with  $X^I \sim (x^\mu, z) \mapsto \lambda X^I = \lambda (x^\mu, z)$ , generated by  $\Delta = X^I \partial_I = z \partial_z + x^\mu \partial_\mu$ ;
- 3. Special conformal transformations; see Appendix A for an intuitive derivation. We have:

$$k_{\mu} = (z^2 + x^2) \,\partial_{\mu} - 2x_{\mu} \Delta \tag{1.3}$$

By Noether's theorem,  $Q_{\Xi} = V_I \Xi^I = G_{IJ} \Xi^I V^J$  is conserved along the geodesic; here  $V^I$  is the normalized tangent vector, while  $\Xi^I$  is some Killing vector of the spacetime, e.g. one from the list above. We can then write down the conserved charges along a geodesic  $\gamma$  in Poincaré AdS:

$$p_{\mu} = G_{\mu I} V^{I} = V_{\mu}$$

$$m_{\mu\nu} = x_{\mu} V_{\nu} - x_{\nu} V_{\mu}$$

$$\Delta = X^{I} V_{I} = \frac{z V^{z}}{z^{2}} + x^{\mu} p_{\mu}$$

$$k_{\mu} = (z^{2} + x^{2}) p_{\mu} - 2x_{\mu} \Delta$$
(1.4)

These are all integration constants along  $\gamma$ . Again note that X, V have indices that should be handled with  $G_{IJ}$ , in particular,

$$dX_{\mu} = G_{\mu I} dX^{I} = \frac{1}{z^{2}} dx_{\mu}, \quad V^{\mu} = \frac{dX^{\mu}}{d\lambda} = z^{2} p^{\mu}$$
(1.5)

On the other hand,  $V^{\mu}$  should be properly normalized, therefore:

$$||V||^2 = g_{IJ}V^IV^J = \frac{(V^z)^2}{z^2} + z^2p^2 = \delta = 0, \pm 1,$$
 (1.6)

For  $p^2=0$ , we have  $\|V\|^2=\frac{(V^z)^2}{z^2}\geq 0$ , thus  $\gamma$  can be either spacelike or null, but not timelike. In fact,  $\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}\lambda}=0,\pm 1$ , along with  $\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}=z^2p^\mu$ , we obtain:

$$p^{2} = 0, spacelike: z(\lambda) = z(0) e^{\pm \lambda}, x^{\mu}(\lambda) = x^{\mu}(0) \pm z(0)^{2} p^{\mu} \frac{e^{\pm 2\lambda} - 1}{2}, null: z(\lambda) = z(0), x^{\mu}(\lambda) = x^{\mu}(0) + z(0)^{2} p^{\mu} \lambda, (1.7)$$

From now on we shall focus on the  $p^2 \neq 0$  situation. Note that we can complete the square on the right hand side of the  $k^{\mu}$  conservation such that:

$$k_{\mu} + p_{\mu}a^2 = p_{\mu}(z^2 + (x - a)^2), \quad \Delta = p_{\mu}a^{\mu}$$
 (1.8)

For  $p^{\mu} \neq 0$ , we can always find some  $c_{\mu}$  such that  $c_{\mu}p^{\mu} \neq 0$ , therefore we have:

$$z^{2} + (x - a)^{2} = \frac{c_{\mu}k^{\mu}}{c_{\mu}p^{\mu}} + a^{2} = \left(\frac{L}{2}\right)^{2} = \text{const.}, \quad \Delta = p_{\mu}a^{\mu}$$
 (1.9)

We see that  $\gamma$  lands on a "sphere" centered at z=0, x=a in  $\mathbb{R}^{d+1}$ ; with the Lorentzian metric, it is actually a hyperboloid. Note that for now the radius  $\left(\frac{L}{2}\right)^2$  can actually be negative or zero; more specifically,

- 1.  $L^2 > 0$ : one-sheet hyperboloid;
- 2.  $L^2 < 0$ : two-sheet hyperboloid;
- 3.  $L^2 = 0$ : conic surface.

On the other hand, we can actually solve  $X^{I}$  completely by combining (1.5), (1.6); we have:

$$\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda} = z^2 p^{\mu}, \quad \frac{\mathrm{d}z}{\mathrm{d}\lambda} = \pm z \sqrt{\delta - z^2 p^2}, \quad \frac{\mathrm{d}x^{\mu}}{\mathrm{d}z} = \pm \frac{zp^{\mu}}{\sqrt{\delta - z^2 p^2}}, \quad \delta - z^2 p^2 = \left(\frac{V^z}{z}\right)^2 \ge 0 \tag{1.10}$$

$$\implies \gamma \subset x^{\mu} = a^{\mu} \pm \frac{p^{\mu}}{p^2} \sqrt{\delta - z^2 p^2}, \quad z^2 + (x - a)^2 = \frac{\delta}{p^2} = \left(\frac{L}{2}\right)^2 \tag{1.11}$$

This confirms our observation above, and further reveals that  $\gamma$  lies in a "plane" in the  $\mathbb{R}^d$  subspace consisting of the  $x^{\mu}$  coordinates. The behavior of  $\gamma$  is sensitive to the sign of  $\delta$  and  $p^2$ ; more specifically,

- 1.  $\delta = -1$ , i.e. for timelike  $\gamma$ , we must have  $p^2 < 0$ , and  $z \ge \frac{1}{|||p|||} = \frac{L}{2} > 0$ , namely z is bounded from below. This means that timelike geodesic can never reach the asymptotic boundary  $z \to 0$ . In this case,  $\gamma$  is a section of the one-sheet hyperboloid.
- 2.  $\delta = +1$ , i.e. for spacelike  $\gamma$ , we can have  $p^2 > 0$  or  $p^2 < 0$ .
  - (a) For  $p^2 > 0$ , again we have  $\frac{L}{2} = \frac{1}{\|p\|} > 0$ , and  $\gamma$  is again a cross section of the one-sheet hyperboloid. However, now we have  $z \leq \frac{L}{2}$ , namely z is bounded from above, and:

$$z \to 0$$
,  $x^{\mu} = a^{\mu} \pm \frac{p^{\mu}}{p^2} = a^{\mu} \pm \frac{L}{2} \, \hat{p}^{\mu}$ ,  $\hat{p}^{\mu} = \frac{p^{\mu}}{\|p\|}$  (1.12)

We can also nicely parametrize  $X^I$  in terms of the proper length  $\lambda$ ; for convenience, set  $x^{\mu}(0) = a^{\mu}$ ,  $z(0) = \frac{L}{2}$ , then we have:

$$z(\lambda) = \frac{L}{2} \frac{1}{\cosh \lambda}, \quad x^{\mu}(\lambda) = a^{\mu} \pm \hat{p}^{\mu} \frac{L}{2} \tanh \lambda$$
 (1.13)

(b) For  $p^2 < 0$ , we have  $(\frac{L}{2})^2 = \frac{1}{p^2} < 0$ , and  $\gamma$  is now a cross section of the two-sheet hyperboloid. Again as  $z \to 0$  it lands at  $x^\mu = a^\mu \pm \frac{L}{2}\,\widehat{p}^\mu$ ; however,  $|x^\mu|$  grows with z and extends into the bulk instead of returning to the boundary, i.e.  $z, |x^\mu| \to \infty$ .

The differential equation in this case is almost the same as in (a), but now we have to choose a different initial condition, since  $\gamma$  won't even reach x = a. However, we can

2 Einbein Action 3

actually set  $x(0), z(0) \to \infty$ ; we then have:

$$z(\lambda) = \left| \frac{L}{2} \right| \frac{1}{\sinh \lambda}, \quad x^{\mu}(\lambda) = a^{\mu} \pm \hat{p}^{\mu} \left| \frac{L}{2} \right| \coth \lambda$$
 (1.14)

This is an evidence that  $z \to \infty$  might not be the "end of the world" after all; it's likely that  $z \to \infty$  is only a horizon, since a spacelike geodesic  $\gamma$  can reach it within finite proper length  $\lambda$ .

In summary, there are 3 types of spacelike geodesics in Poincaré AdS, which closely resemble the 3 types of conic sections:

- 1.  $p^2 = 0$ : parabolic, given in (1.7), with one end going to  $z \to 0$  and the other end going to  $z \to \infty$ ; it takes infinite proper length  $\lambda$  for it to reach 0 or  $\infty$ .
- 2.  $p^2 > 0$ : hyperbolic, given in (1.11) and (1.14), also with one end going to  $z \to 0$  and the other end going to  $z \to \infty$ , but it reaches  $\infty$  within finite  $\lambda$ .
- 3.  $p^2 > 0$ : elliptic, given in (1.11) and (1.13), with both ends going to  $z \to 0$ , and it takes infinite proper length  $\lambda$  for it to reach either end.

Now consider two points  $x_1, x_2$  near the boundary  $z = \epsilon$  connected by a spacelike geodesic  $\gamma$ . This can only be the "elliptic type" discussed above. We have:

$$x^{\mu} = a^{\mu} \pm \hat{p}^{\mu} \sqrt{\left(\frac{L}{2}\right)^2 - z^2}, \quad z^2 + (x - a)^2 = \left(\frac{L}{2}\right)^2,$$
 (1.15)

$$a = \frac{x_1 + x_2}{2}, \quad \widehat{p} \propto x_2 - x_1$$
 (1.16)

It's length is then given by:

$$A = \int_{z=-\epsilon}^{z=\epsilon} \frac{\mathrm{d}x^2 + \mathrm{d}z^2}{z^2}$$

$$= \int_{-\Lambda}^{\Lambda} \frac{(\mathrm{d}\tanh\lambda)^2 + (\mathrm{d}\operatorname{sech}\lambda)^2}{(\mathrm{sech}\lambda)^2}, \quad \Lambda = \cosh^{-1}(\frac{L}{2\epsilon})$$

$$= \int_{-\Lambda}^{\Lambda} \mathrm{d}\lambda = 2\Lambda = 2\cosh^{-1}\left(\frac{L}{2\epsilon}\right) \sim 2\log\frac{L}{\epsilon}, \quad \epsilon \to 0$$
(1.17)

## 2 Einbein Action

The einbein action of a point particle is given by:

$$S[\eta, X] = \frac{1}{2} \int d\tau \left( \eta^{-1} \dot{X}_{\mu} \dot{X}^{\mu} - \eta m^2 \right)$$
 (2.1)

Under worldline reparametrization:  $\tau \mapsto \tau' = f(\tau)$ , we have  $X'(\tau') = X(\tau)$ , i.e.  $X^{\mu}$  transforms like a **scalar** under worldline diffeomorphism;  $X(\tau) \mapsto X'(\tau) = X(f^{-1}(\tau))$ .

On the other hand,  $\eta$  should be treated like an einbein:  $\eta = \sqrt{-\gamma}$ , here  $\gamma = \gamma_{\tau\tau}$  is the worldline metric;  $\gamma < 0$  due to the Lorentzian signature. We have:

$$\eta = \sqrt{-\gamma} \longmapsto \eta' = \eta \det \frac{\partial \tau}{\partial \tau'} = \eta \frac{\partial \tau}{\partial \tau'},$$
(2.2)

$$\eta^{-1} = -\sqrt{-\gamma} \, \gamma^{-1} \longmapsto (\eta')^{-1} = \eta^{-1} \left( \frac{\partial \tau}{\partial \tau'} \right) \left( \frac{\partial \tau'}{\partial \tau} \frac{\partial \tau'}{\partial \tau} \right) = \eta^{-1} \frac{\partial \tau'}{\partial \tau} \tag{2.3}$$

It is clear that the action is invariant under the transformation:

$$S' = \frac{1}{2} \int d\tau' \left( (\eta')^{-1} \partial_{\tau'} X_{\mu} \partial_{\tau'} X^{\mu} - (\eta') m^{2} \right)$$

$$= \frac{1}{2} \int d\tau \frac{\partial \tau'}{\partial \tau} \left( \eta^{-1} \frac{\partial \tau'}{\partial \tau} \cdot \frac{\partial \tau}{\partial \tau'} \partial_{\tau} X_{\mu} \cdot \frac{\partial \tau}{\partial \tau'} \partial_{\tau} X^{\mu} - \eta \frac{\partial \tau}{\partial \tau'} m^{2} \right) = S$$
(2.4)

We can eliminate  $\eta$  classically by placing it on shell:

$$0 = \frac{\delta S}{\delta \eta} = -\eta^{-2} \dot{X}_{\mu} \dot{X}^{\mu} - m^2, \quad \eta[X] = \frac{1}{m} \sqrt{-\dot{X}_{\mu} \dot{X}^{\mu}}$$
 (2.5)

Substitute this back to the action, and we have:

$$S[X] = S[\eta = \eta[X], X] = \frac{1}{2} \int d\tau \left( m \left( -\dot{X}_{\mu} \dot{X}^{\mu} \right)^{-\frac{1}{2}} \dot{X}_{\mu} \dot{X}^{\mu} - m \left( -\dot{X}_{\mu} \dot{X}^{\mu} \right)^{+\frac{1}{2}} \right)$$

$$= -m \int d\tau \sqrt{-\dot{X}_{\mu} \dot{X}^{\mu}}$$
(2.6)

## 3 Ricci Tensor for Static Spherical Metric

Consider the metric:

$$ds^{2} = -f(r) dt^{2} + h(r) dr^{2} + r^{2} (d\theta^{2} + \sin^{2}\theta d\phi^{2}) = \eta_{ab} e^{a} e^{b}$$
(3.1)

Here we've defined the following vierbein:

$$e^{t} = \sqrt{f(r)} dt$$

$$e^{r} = \sqrt{h(r)} dr$$

$$e^{\theta} = r d\theta$$

$$e^{\phi} = r \sin \theta d\phi$$
(3.2)

## A Derivation of the special conformal transformations

Special conformal transformations can be understood as translations conjugated by *inversions*. Note that  $\frac{dz^2}{z^2}$  is invariant under  $z \mapsto \frac{1}{z}$ ; if we include the  $x^{\mu}$  directions, we can consider:

$$\mathcal{I}: \ \chi^I \mapsto \frac{\chi^I}{\chi^2}, \quad \chi^2 = -t^2 + \vec{x}^2 + z^2,$$
 (A.1)

$$\mathcal{I}^2 = 1, \quad ds^2 \mapsto \left(\frac{\delta_J^I - 2\frac{\chi^I \chi_J}{\chi^2}}{\chi^2} d\chi^J\right)^2 / \left(\frac{z}{\chi^2}\right)^2 = \frac{d\chi^2}{z^2} = ds^2 \tag{A.2}$$

We see that inversion  $\mathcal{I}$  is indeed a (discrete) symmetry of the metric. Here we've defined yet another lower case variable  $\chi^I \sim (x^\mu, z)$ , which as a contravariant vector has the same components as  $X^I$ , but with an index that should be lowered by the flat metric  $\eta_{IJ}$ , i.e.  $\chi_I = \eta_{IJ}\chi^J = \eta_{IJ}X^J$ . The d

special conformal generators are then given by:

$$k_{\mu} = \frac{\partial}{\partial a^{\mu}} \left( \mathcal{I} \circ e^{a^{\nu} P_{\nu}} \circ \mathcal{I} \circ X^{I} \right)_{a=0} \frac{\partial}{\partial X^{I}}$$

$$= \frac{\partial}{\partial a^{\mu}} \left( \frac{\chi^{I}}{\chi^{2}} + a^{I}}{\left| \frac{\chi^{J}}{\chi^{2}} + a^{J} \right|^{2}} \right)_{a=0} \frac{\partial}{\partial X^{I}}$$

$$= \frac{\partial}{\partial a^{\mu}} \left( \frac{\chi^{I} + a^{I} \chi^{2}}{1 + 2a^{I} \chi_{I} + a^{2} \chi^{2}} \right)_{a=0} \frac{\partial}{\partial X^{I}}$$

$$= \chi^{2} \partial_{\mu} - 2x_{\mu} X^{I} \partial_{I}$$

$$= \chi^{2} \partial_{\mu} - 2x_{\mu} \Delta$$
(A.3)