On the transposition of computer programs

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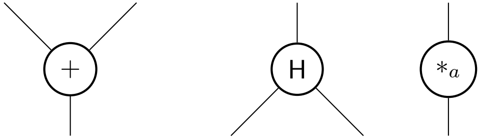
Plan

Arithmetic circuits

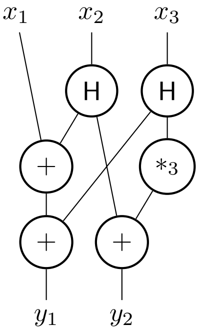
Algebraic complexity

Fix a ring R. We construct circuits that evaluate arithmetic functions.

Three gates: Addition, duplication, multiplication by a fixed element $a \in R$.



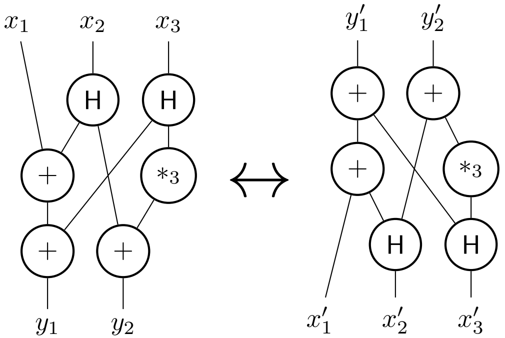
Arithmetic circuits



$$y_1 = x_1 + x_2 + x_3$$
$$y_2 = x_2 + 3x_3$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

Transposition of an arithmetic circuit



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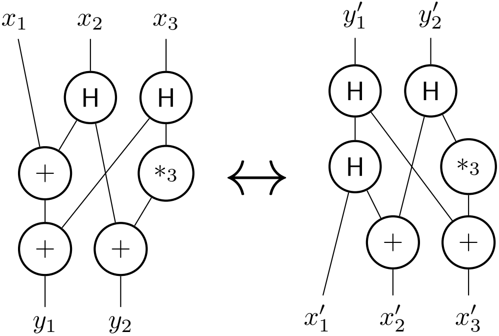
Transposition of an arithmetic circuit



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Arithmetic Circuits: uniform vs. non-uniform

Definition (Circuit family)

A *circuit family* is a family $(C_0, C_1, \ldots)$ of circuits indexed by $\mathbb{N}$ such that C_n has n inputs.

- Any Turing-undecidable problem has a trivial *polynomial-size* circuit family deciding it.

Definition (Uniform circuit family)

A circuit family $(C_0, C_1, \ldots)$ is said to be *uniform* if there is a $\log n$-space bounded touring machine which on input 1^n outputs a representation of C_n.

- We will extend the definition to allow families to be indexed by any (countable) set $\mathcal{P}$, called the *parameter space*.

From non-uniform to uniform circuits?

- The transposition theorem easily generalises to non-uniform circuits.
- It can even be directly applied to computer programs (under certain hypotheses).

```
for i = 0 to n-2 do
  a[i+1] = a[i] + a[i+1]
  a[i] = 0
end for
```

$$\begin{pmatrix} 0 & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots\cdots & 0 \\ 1 & \cdots\cdots & 1 \end{pmatrix}$$

From non-uniform to uniform circuits?

```
a[1] = a[0] + a[1]
a[0] = 0
a[2] = a[1] + a[2]
a[1] = 0
...
a[n-1] = a[n-2] + a[n-1]
a[n-2] = 0
```

```
a[n-2] = 0
a[n-2] = a[n-2] + a[n-1]
...
a[1] = 0
a[1] = a[1] + a[2]
a[0] = 0
a[0] = a[0] + a[1]
```

```
for i = n-2 to 0 do
  a[i] = 0
  a[i] = a[i] + a[i+1]
end for
```

$$\begin{pmatrix} 0 & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots\cdots & 0 \\ 1 & \cdots\cdots & 1 \end{pmatrix}$$

Plan

Minimal polynomials [Shoup '95]

Linear recurring sequences

An algebraic element $\sigma \in \mathbb{L} = \mathbb{K}[X]/f(X)$, the σ^i's satisfy a linear recurrence

$$\sigma^n = c_{n-1}\sigma^{n-1} + \cdots + c_1\sigma + c_0$$

Let $\ell \in \mathbb{L}^*$ a linear form, then

$$\ell(\sigma^n) = c_{n-1}\ell(\sigma^{n-1}) + \cdots + c_1\ell(\sigma) + c_0\ell(1)$$

Power projection

Given ℓ and σ, the power projection problem asks to compute

$$\ell(1), \ell(\sigma), \ell(\sigma^2), \ldots, \ell(\sigma^{n-1})$$

For fixed σ, it is a $\mathbb{K}$-linear map, its transpose is the map

$$g \mapsto g(\sigma) \bmod f$$

hence its transpose problem is *modular composition*.

Applications

Uses and generalisations of power projection

- Minimal polynomials in towers of extension fields [Shoup '95].
- Change of order in triangular sets.
- Change of order in Artin-Schreier towers [D.F., Schost '09], application to isogeny computation.

Other applications

- Generation of irreducible polynomials.
- Complexity bounds on evaluation/interpolation.
- Reverse mode in automatic differentiation.

Other motivations

"The author has not been able to find an example of a linear operator that is easy to apply but whose transpose is difficult to apply."

[Wiedemann '86]

"The transposition principle is very useful for proving the existence of algorithms, but actually coming up with an explicit, practical algorithm requires a bit more effort."

[Shoup '95]

"We offer no other proof of correctness other than the validity of this transformation technique (and the fact that it does indeed work in practice)."

[Shoup '95]

"Oulala ! Vous avez encore utilisé votre magie noire !"

François Morain, personal communication

Other motivations

```
void reduc_doit(GF2X& A0, GF2X& A1, const GF2X& A,
long init, long d, bool plusone){
  if (d <= 2){
    A0 = GF2X(0, coeff(A,init));
    A1 = GF2X(0, coeff(A,init+1));
    return;
  }

  long dp = d/2;
  GF2X A10, A11;

  reduc_doit(A0, A1, A, init, dp, plusone);
  reduc_doit(A10, A11, A, init+dp, dp, plusone);

  ShiftAdd(A0, A11, 1);
  if (plusone) A0 += A11;
  A1 += A10 + A11;

  long i = 1;
  bool even = true;
  while (2*i != d){
    ShiftAdd(A0, A10, i);
    ShiftAdd(A1, A11, i);
    i = 2*i;
    even = !even;
  }

  if (plusone && !even) {
    A0 += A10;
    A1 += A11;
  }
}
```

```
void treduc_doit(GF2X& A, const GF2X& A0, const GF2X& A1, long d,
bool plusone){
  if (d <= 2){
    SetCoeff(A, 0, coeff(A0, 0));
    SetCoeff(A, 1, coeff(A1, 0));
    return;
  }

  long dp = d/2;
  long hdp = dp/2;

  GF2X A00, A01, A10, A11;
  A00 = trunc(A0, hdp);
  A01 = trunc(A1, hdp);

  A10 = A01;
  if (plusone) A11 = A00;
  else A11 = 0;
  A11 += A01 + RightShift(trunc(A0, hdp+1), 1);
  long i = 1;
  bool even = true;
  while (2*i != d){
    A10 += RightShift(trunc(A0, hdp+i), i);
    A11 += RightShift(trunc(A1, hdp+i), i);
    i = 2*i;
    even = !even;
  }

  if (plusone && !even) {
    A10 += trunc(A0, hdp);
    A11 += trunc(A1, hdp);
  }

  GF2X B0, B1;
  treduc_doit(B0, A00, A01, dp, plusone);
  treduc_doit(B1, A10, A11, dp, plusone);
  A = B0 + LeftShift(B1,dp);
}
```

Other motivations

In developing transposed code for our ISSAC '09 paper, a very tricky mistake slowed down performances by more than a constant factor. The bug was so subtle that in the first place we didn't even think there was one; a machine wouldn't have made the mistake.

A striking similarity with reversible computation and quantum circuits.

A never (so I thought) enough studied relationship with automatic differentiation.

A long history of re-discoveries and many different formulations.

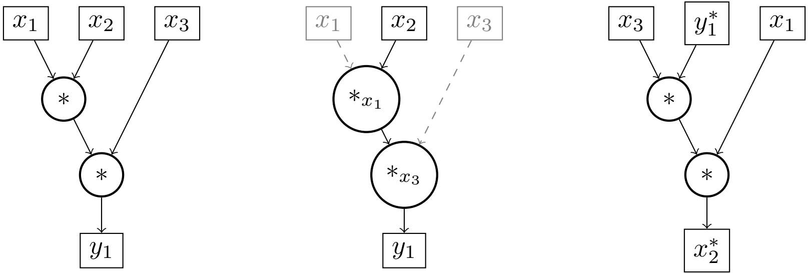
History

History

- Originally discovered in *electrical network theory* [Bordewijk '56] (only works for $\mathbb{C}$);
- [Bürgisser, Clausen, Shokrollahi] attribute the discovery to Tellegen, Bordewijk's director, but this is debated;
- Graph-theoretic approach by Fettweis (1971) for *digital filters*;
- [Fiduccia '73] and [Hopcroft, Musinski '73]: transposition of *bilinear chains*, the most complete formulation (non-commutative rings);
- Special case of *automatic differentiation* [Baur, Strassen '83];
- In *computer algebra*, popularized by Shoup, von zur Gathen, Kaltofen,...
- [Bostan, Lercerf, Schost '03] improve algorithms for polynomial evaluation and solve an open question on space complexity.

Plan

Multilinearity



- Almost anytime we want to transpose, we end-up *linearising* a circuit with multiplication nodes.
- Other constructs such as `if` statements and `for` loops need to be linearised too.

Circuit emulation

```
karatsuba [] y n = []

karatsuba x [] n = []

karatsuba x y n =
  if n <= 0
  then []
  else if n == 1
       then [(x!!0) * (y!!0)]
       else
         let h = n / 2 in
         let (a0, a1) = split x h in
         let (b0, b1) = split y h in
         let x0 = karatsuba a0 b0 h in
         let x2 = karatsuba a1 b1 (n-h) in
         let xx1 = karatsuba (a1 + a0) (b1 + b0) (n-h) in
         let x1 = xx1 + ((x0 + x2) * (- one)) in
         (shift x2 n) + (shift x1 h) + x0
```

Circuit emulation

```
karatsuba x n =
  if n <= 0
  then []
  else if n == 1
       then \y -> [(x!!0) * (y!!0)]
       else
         let h = n / 2 in
         let (a0, a1) = split x h in
         let x0 = karatsuba a0 h in
         let x2 = karatsuba a1 (n-h) in
         let xx1 = karatsuba (a1 + a0) (n-h) in
         let sp = \y -> split y h in
         let sh1 = \y -> shift y n in
         let sh2 = \y -> shift y h in

         \y -> ....
```

Circuit emulation

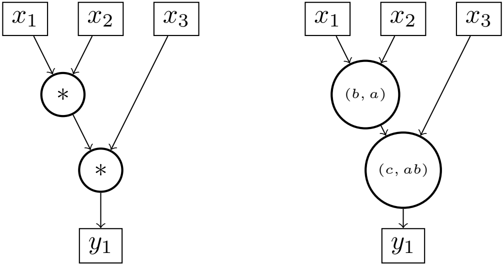
```
let h = n / 2 in
let (a0, a1) = split x h in
let x0 = karatsuba a0 h in
let x2 = karatsuba a1 (n-h) in
let xx1 = karatsuba (a1 + a0) (n-h) in
let sp = Circuit(\y -> split y h) in
let sh1 = Circuit(\y -> shift y n) in
let sh2 = Circuit(\y -> shift y h) in

proc y -> do
  (y0, y1) <- sp -< y
  s0 <- x0 -< y0
  s2 <- x2 -< y1
  ss1 <- (id <+> id).xx1 -< (y0, y1)
  s1 <- id <+> ((id <+> id) <*> (- one)) -< (ss1, (s0, s2))
  z <- sh1 <+> sh2 <+> id -<  (s2, (s1, s0))
  returnA -< z
```

Plan

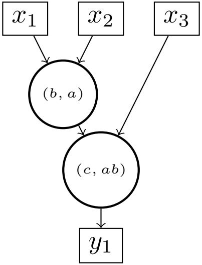
Automatic differentiation of circuits [Baur, Strassen '83]

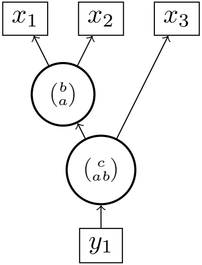
Transformation technique on circuits $R^n \to R^m$:
node $\mapsto$ its Jacobian at point x
(we are overly vulgarising, there's more than that in reality)



- By the chain rule, the result computes the Jacobian of the circuit.
- Evaluating on a vector gives a directional derivative
- n directional derivatives yield the whole Jacobian.

Reverse mode





Reverse mode

- After transposition, m directional derivatives suffice to compute the Jacobian,
- In particular, for a map $R^n \to R$, only one evaluation is needed to compute the gradient.

Transposing with AD tools

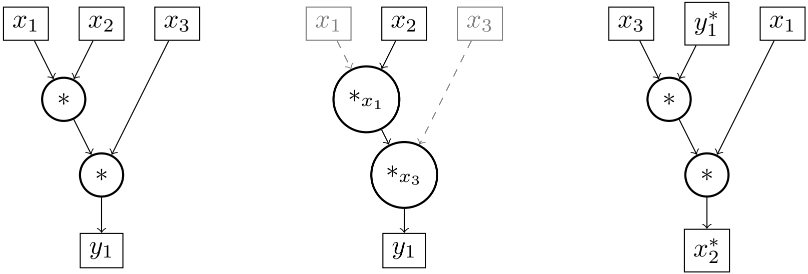
AD tools work on straight line programs, they implicitly implement transposition in reverse mode

But !

- They cost too much in space, because they have to precompute the circuit,
- Iterative statements can cause memory swell,
- They are only useful in the case $R^n \to R$,
- They can't handle multilinearity,
- They can't handle recursive calls (as far as I have seen).

Plan

Multilinearity



- Can we automatically deduce any possible linearisation of a program?
- Type inference systems can help us

Linearity inference

Suppose given a type `R` implementing a ring. We want to define types `L` (for *linear*) and `S` (for scalar) such that the following equations hold

```
plus :: L -> L -> L
plus :: S -> S -> S
times :: L -> S -> L
times :: S -> S -> S
zeroR :: L
zeroR :: S
oneR :: S
```

Linearity inference

Here's the solution

```
data L = L R
data S = S R

class Ring r where
  zero :: r
  (<+>) :: r -> r -> r
  neg :: r -> r
  (<*>) :: r -> S -> r

one = S oneR
(S a) == (S b) = a == b
```

Let's check it in a shell

Plan

The Transposable Algebraic Language

Algebraic types

- **Prototypes**: `Ring`, `Module`, (optionally `Algebra`, . . .)
- Declaring an algebraic type:

```
type Ring R
type Module(R) M
```

Declaring a function

```
fun (linear M A, const m)f(linear M Z, const M z, const n):
```

Other constructs

- Standard types (`int`, `bool`, . . .)
- `if`, `match`, recursion, `let` binding,
- Algebraic operators $+$, $\times$, projection/injection `a[n]`.

Automatic transposition: the general algorithm

```
fun (linear R res)scalar(linear M a, const n):
  if n = 0:
    res = 0
  else:
    res = a[n] + scalar(a, n-1)
```

The algortithm

- First run the algorithm in the normal direction to compute all the const values,
- then run the algorithm backwards transposing each instruction.

```
fun (linear M a)scalar^T(linear R res, const n):
  if n = 0:
    nop
  else:
    a[n] = res
    a += scalar^T(res, n-1)
```

Scalar prediction and tail recursion

- Permuting the order of the instructions may break tail/head recursion,
- this implies loss of efficiency,
- equivalently, in `for` loops we have to precompute all the `const` values of the loop,
- this seems to increase the space requirements of the algorithm, but does not affect the number of arithmetic operations.

Scalar prediction and tail recursion

```
fun (R a, R b)f(R c, R d):
  if d > 0:
    x, y = f(c, d - 1)
    a, b = x * y, y + 1
  else:
    a, b = c, d
```

```
fun (R c, R b)fT(R a, R d):
  # Forward sweep
  if (d > 0):
    _, y = f(a, d - 1)
    b = y + 1
  else:
    b = d

  # Reverse sweep
  if (d > 0):
    x = a * y
    c, y = fT(x, d - 1)
  else:
    c = a
```

Plan

A Python implementation of TransAL

`http://transalpyne.gforge.inria.fr/`

- Compiler/interpreter written in python,
- python-like syntax,
- automated constness inference,
- smart handling of array sizes,
- will compile to other languages (Haskell? OCaml?).

Karatsuba in `transalpyne`

```
def (M c)karatsuba(M a, M b, n):
  if n == 1:
    tmp = M.zero()
    tmp[0] += a[0]*b[0]
    c = tmp
  elif n > 1:
    a0, a1 = split(a, n/2, n)
    b0, b1 = split(b, n/2, n)
    x0 = karatsuba(a0, b0, n/2)
    x2 = karatsuba(a1, b1, n - n/2)
    x1 = karatsuba((a1 + a0), (b1 + b0), n - n/2) - x0 - x2
    c = shift(x2, n, n+1) + shift(x1, n/2, n+1) + x0
```

Karatsuba in transalpyne

```
(M b)karatsubaT(M a, M c, n)
  # Forward sweep
  if (n == 1):
    pass
  elif n > 1:
    a0, a1 = split(a, n / 2, n)
  # Reverse sweep
  if (n == 1):
    tmp = c
    _transAL_tmp_0[0] += a[0] * tmp[0]
    b = _transAL_tmp_0
  elif n > 1:
    x2 = trans shift(c, n, n + 1)
    x1 = trans shift(c, n / 2, n + 1)
    x0 = c
    b1 = trans karatsuba(x1, a1 + a0, n - n / 2)
    b0 = b1
    x0 += - x1
    x2 += - x1
    b1 += trans karatsuba(x2, a1, n - n / 2)
    b0 += trans karatsuba(x0, a0, n / 2)
    b = trans split(b0, b1, n / 2, n)
```

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