Dualities and Transposition Principle

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November 21, 2008

Tellegen's Principle

"From every *linear algorithm* computing a linear application we can deduce another *linear algorithm* computing the transpose application using *about* the same space and time resources."

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What's so special about transposition ?

Plan

1. Background
2. A new way of looking
3. Examples
4. Tellegen's principle into practice

History, motivations

History

- Originally discovered by Tellegen (1950), Bordewijk (1956) for *electrical network theory* and by Kalman (1960) for *control theory*;
- Graph-theoretic approach by Fettweis (1971) for *digital filters*;
- Fiduccia (1972): transposition of *bilinear algorithms*;
- Special case of reverse mode in *automatic differentiation*: Baur & Strassen (1983);
- In *computer algebra*, popularized by Shoup, von zur Gathen, Kaltofen,...
- [Bostan, Lecerf, Schost 2003] improve algorithms for polynomial evaluation.

Motivations

- Existence result in *complexity theory*;
- *Code transformation* technique;
- Improve $M^T \Leftrightarrow$ Improve M;
- **Divides by 2 the number of algorithms yet to be discovered.**

Classical proofs

Linear algebra

M computed as a sequence of *simple* linear applications U_i

$$M(v) = U_1 \circ U_2 \circ \cdots \circ U_n(v) \qquad \Leftrightarrow \qquad M^T(v) = U_n^T \circ \cdots \circ U_2^T \circ U_1^T$$

Graph-theoretic approach

- *Compile* the algorithm in a DAG;
- reverse the arrows of the DAG.

This works only for straight-line programs !

Graph-theoretic approach (cont'd)

Plan

Category theory, a gentle introduction...

- Category $\mathscr{C}$
- Objects $\mathrm{ob}(\mathscr{C})$

$A \quad B$

C

$D \quad E$

Category theory, a gentle introduction...

- Category $\mathscr{C}$
- Objects $\mathrm{ob}(\mathscr{C})$
- Arrows $\hom(\mathscr{C})$, $\mathrm{Hom}(A, B)$

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] & B \end{tikzcd}$$

$$\begin{tikzcd} & C \arrow[dr, "g"] & \\ D \arrow[ur, "f"] & & E \end{tikzcd}$$

Category theory, a gentle introduction...

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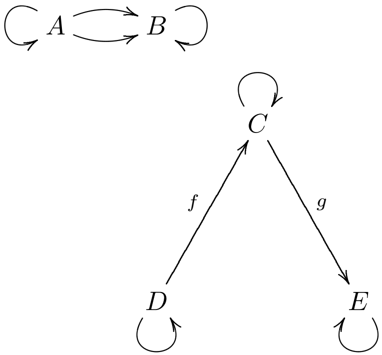
- Arrows $\hom(\mathscr{C})$, $\mathrm{Hom}(A, B)$
- Identities id_A



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$$
\begin{tikzcd}
A \arrow[loop left] \arrow[r, bend left] \arrow[r, bend right] & B \arrow[loop right]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & C \arrow[loop above] \arrow[dr, "g"] & \\
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\end{tikzcd}$$

Example : $\mathbf{FMod}_R$

- $\mathrm{ob}(\mathscr{C}) = R^n$ free R-modules,
- $\hom(\mathscr{C}) =$ linear applications.

Category theory, Functors

Covariant functor
$F : \mathscr{C} \to \mathscr{D}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[dr, "F(h)"'] & F(B) \arrow[d, "F(g)"] \\ & F(C) \end{tikzcd}$$

Contravariant functor
$F : \mathscr{C} \to \mathscr{D}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} F(A) & F(B) \arrow[l, "F(f)"'] \\ & F(C) \arrow[u, "F(g)"'] \arrow[ul, "F(h)"] \end{tikzcd}$$

Equivalence, duality

- Equivalence if $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ covariant
- Duality if $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ contravariant

and $\quad F \circ G \simeq \mathrm{Id}_{\mathscr{D}} \quad$ and $\quad G \circ F \simeq \mathrm{Id}_{\mathscr{C}}$.

Tellegen's principle

"From every *linear algorithm* computing a linear application we can deduce another *linear algorithm* computing the transpose application using *about* the same space and time resources."

An example

```
for i = 1 to n-2 do
  a[i+1] = a[i] + a[i+1]
  a[i] = 0
end for
```

$$\begin{pmatrix} 0 & \ldots\ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \ldots\ldots & 0 \\ 1 & \ldots\ldots & 1 \end{pmatrix}$$

Computations

Language, size

- Set of *instructions* $\mathscr{L} \subset \hom(\mathscr{C})$,
- Size function $\|.\| : \mathrm{ob}(\mathscr{C}) \to \mathbb{N}$.

Computation

Sequence $C_1 : b \to c$ of instructions.

$$\begin{tikzcd}
& {} \arrow[r] & {} \arrow[dr] & & {} \arrow[r] & {} \arrow[dr] & \\
a \arrow[ur] \arrow[rrr, dashed, "f_2"'] \arrow[rrr, phantom, shift left=6, "C_2"] & & & b \arrow[ur] \arrow[rrr, dashed, "f_1"'] \arrow[rrr, phantom, shift left=6, "C_1"] & & & c
\end{tikzcd}$$

Time and space cost

- $\mathrm{t}(C) =$ length of the computation,
- $\mathrm{s}(C) = \max_{o \in C} \|o\|$.

The case $\mathbf{FMod}_R$

[A. Bostan, G. Lecerf, E. Schost 2003] :

$$\begin{aligned} +_1 : R^2 &\to R^2 \\ (p,q) &\mapsto (p+q,q) \end{aligned} \qquad \begin{aligned} +_2 : R^2 &\to R^2 \\ (p,q) &\mapsto (p,p+q) \end{aligned} \qquad \begin{aligned} *_a : R &\to R \\ p &\mapsto ap \end{aligned}$$

$$\begin{aligned} \pi : R &\to 0 \\ p &\mapsto 0 \end{aligned} \qquad \begin{aligned} \iota : 0 &\to R \\ 0 &\mapsto 0 \end{aligned}$$

$$\mathscr{L} = \left\{ \mathrm{Id}_n \times \mathrm{op} \times \mathrm{Id}_m \middle| n,m \in \mathbb{N}, \mathrm{op} \in \{+_1, +_2, *_a, \pi, \iota \mid a \in R\} \right\}.$$

$$\|R^n\| = n$$

Our example

```
for i = 1 to n-2 do
  a[i+1] = a[i] + a[i+1]
  a[i] = 0
end for
```

$$\mathrm{Id}_i \times +_2 \times \mathrm{Id}_{n-2-i}$$
$$\mathrm{Id}_i \times *_0 \times \mathrm{Id}_{n-1-i}$$

$$R^n \xrightarrow{+_2 \times \mathrm{Id}_{n-2}} R^n \xrightarrow{{}^0 \times \mathrm{Id}_{n-1}} R^n \cdots\cdots\cdots\cdots R^n$$

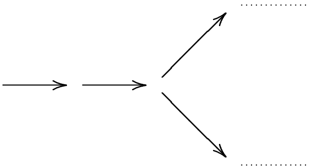
Our example

```
for i = 1 to n-2 do
  a[i+1] = a[i] + a[i+1]
  a[i] = 0
end for
```

$$\mathrm{Id}_i \times +_2 \times \mathrm{Id}_{n-2-i}$$
$$\mathrm{Id}_i \times *_0 \times \mathrm{Id}_{n-1-i}$$

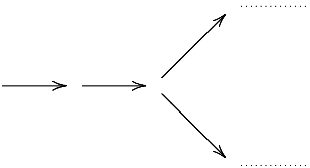
$$R^n \xrightarrow{+_2 \times \mathrm{Id}_{n-2}} R^n \xrightarrow{0 \times \mathrm{Id}_{n-1}} R^n \cdots\cdots\cdots\cdots R^n$$

Branchings



```
if a = (0,...,0) then
  ...
else
  ...
endif
```

Branchings



```
if n = 0 then
  ...
else
  ...
endif
```

Algorithm

Parameter space

Par a recursively enumerable

For example, $\quad \mathrm{Par} = \mathbb{N}$

Algorithm

A function $\quad A : \mathrm{Par} \to \mathscr{C}_{\to} \quad$ ($\mathscr{C}_{\to}$ = the computations)

Algorithm

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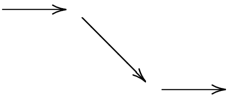
Algorithm

A function $\quad A : \mathrm{Par} \to \mathscr{C}_{\to} \quad (\mathscr{C}_{\to} =$ the computations$)$

1



Algorithm

Parameter space

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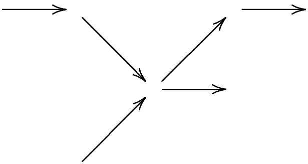
1

2



Algorithm

Parameter space

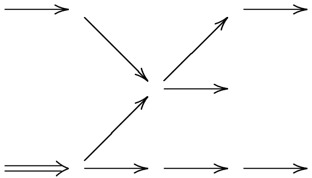
Par a recursively enumerable

For example, $\mathrm{Par} = \mathbb{N}$

Algorithm

A function $A : \mathrm{Par} \to \mathscr{C}_{\to}$ ($\mathscr{C}_{\to}$ = the computations)

1
2
3
$\vdots$



Complexity

Time complexity

$A : \mathrm{Par} \to \mathscr{C}_{\to}$ induces a function $\mathrm{t}_A : \mathrm{Par} \to \mathbb{N}$ given by

$$\mathrm{t}_A(x) = \mathrm{t}(A(x))$$

Space complexity

$A : \mathrm{Par} \to \mathscr{C}_{\to}$ induces a function $\mathrm{s}_A : \mathrm{Par} \to \mathbb{N}$ given by

$$\mathrm{s}_A(x) = \mathrm{s}(A(x))$$

Our example

```
for i = 1 to n-2 do
  a[i+1] = a[i] + a[i+1]
  a[i] = 0
end for
```

$$R^n \xrightarrow{+_2 \times \mathrm{Id}_{n-2}} R^n \xrightarrow{0 \times \mathrm{Id}_{n-1}} R^n \cdots\cdots\cdots\cdots R^n$$

Our example

```
a[1] = a[0] + a[1]
a[0] = 0
a[2] = a[1] + a[2]
a[1] = 0
...
a[n-1] = a[n-2] + a[n-1]
a[n-2] = 0
```

$$n \quad \mapsto \quad R^n \xrightarrow{+_2 \times \mathrm{Id}_{n-2}} R^n \xrightarrow{0 \times \mathrm{Id}_{n-1}} R^n \cdots\cdots\cdots\cdots R^n$$

Tellegen's theorem

Tellegen functor

A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be Tellegen if $F(\mathscr{L}_{\mathscr{C}}) \subset \mathscr{L}_{\mathscr{D}}$.

Tellegen's theorem

- $F : \mathscr{C} \to \mathscr{D}$ a Tellegen functor
- Par a parameter space
- $A : \mathrm{Par} \to \mathscr{C}_{\to}$ an algorithm

$F \circ A$, noted $F(A)$ is an algorithm $\mathrm{Par} \to \mathscr{D}_{\to}$ such that

- $\mathrm{t}_{F(A)} = \mathrm{t}_A$,
- $\mathrm{s}_{F(A)} \leq B(\mathrm{s}_A)$ if $B : \mathbb{N} \to \mathbb{N}$ is an upper bound for F.

Tellegen's theorem

Tellegen*ish* functor

A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be tellegenish if $F(\mathscr{L}_{\mathscr{C}}) \subset \mathscr{L}_{\mathscr{D}}$.

Tellegen's theorem

- $F : \mathscr{C} \to \mathscr{D}$ a tellegenish functor
- Par a parameter space
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The case $\mathbf{FMod}_R$

We know a (contravariant) functor $T : \mathbf{FMod}_R \to \mathbf{FMod}_R$ given by matrix transposition.

Tellegen's theorem for linear algebra

T is a tellegenish functor for the language $\mathscr{L}$ we gave before.

$$T(+_1) = +_2 \qquad T(*_a) = *_a \qquad T(\pi) = \iota$$

Plan

Our example

```
a[1] = a[0] + a[1]
a[0] = 0
a[2] = a[1] + a[2]
a[1] = 0
...
a[n-1] = a[n-2] + a[n-1]
a[n-2] = 0
```

```
a[n-2] = 0
a[n-2] = a[n-2] + a[n-1]
...
a[1] = 0
a[1] = a[1] + a[2]
a[0] = 0
a[0] = a[0] + a[1]
```

Our example

```
a[1] = a[0] + a[1]
a[0] = 0
a[2] = a[1] + a[2]
a[1] = 0
...
a[n-1] = a[n-2] + a[n-1]
a[n-2] = 0
```

```
a[n-2] = 0
a[n-2] = a[n-2] + a[n-1]
...
a[1] = 0
a[1] = a[1] + a[2]
a[0] = 0
a[0] = a[0] + a[1]
```

```
for i = n-2 to 0 do
  a[i] = 0
  a[i] = a[i] + a[i+1]
end for
```

$$\begin{pmatrix} 0 & \ldots\ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \ldots\ldots & 0 \\ 1 & \ldots\ldots & 1 \end{pmatrix}$$

Quantum Computing

The QC category $\mathscr{Q}$

$\mathrm{ob}(\mathscr{Q}) = \{(\mathbb{C}^2)^{\otimes n} \mid n \in \mathbb{N}\}$ $\qquad$ $\mathrm{hom}(\mathscr{Q}) = U$ unitaries ($U^* = U^{-1}$)

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i} \end{pmatrix}, \qquad CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Language, size

$\mathscr{L} = \left\{ \mathrm{Id}_2^{\otimes n} \otimes \mathrm{op} \otimes \mathrm{Id}_2^{\otimes m} \middle| n, m \in \mathbb{N}, \mathrm{op} \in \{H, R, R^*, CNOT\} \right\}$ $\qquad$ $\|(\mathbb{C}^2)^{\otimes n}\| = n$

The functor

$$* : U \mapsto U^*$$

Extension and restriction of scalars

A, B two rings, $f : A \to B$ a morphism,

- $E_f : M_A \mapsto M_A \otimes_A B$ maps A-modules to B-modules;
- $R_F : M_B \mapsto M_B$ maps B-modules to A-modules (by the law $am \equiv f(a)m$).

Extension

- Every algorithm written for $\mathbb{R}$-modules works for $\mathbb{C}$-modules;
- Every algorithm written for $\mathbb{Z}$-modules works for any module;
- Every algorithm written for $\mathbb{K}[X]$-modules works for $K[X]/P(X)$ modules.

Restriction

- Less important than extension;
- its adjoint, not its inverse (extension of scalars may loose information).

Caveat : automatic differentiation

Making AD tellegenish

- $\mathbb{R}$ the only object,
- arrows : all the analytic functions,
- **What functor ?**

One might be tempted to define the category whose arrows are the first derivatives of functions. But

$$f' \circ g' \neq (f \circ g)' \quad !!!$$

The composition is not even definable in this case !

- We can define an artificial category that lets us treat AD,
- but composition is not for free !

Plan

Automatic transposition of code

```
void reduc_doit(GF2X& A0, GF2X& A1, const GF2X& A,
long init, long d, bool plusone){
  if (d <= 2){
    A0 = GF2X(0, coeff(A,init));
    A1 = GF2X(0, coeff(A,init+1));
    return;
  }

  long dp = d/2;
  GF2X A10, A11;

  reduc_doit(A0, A1, A, init, dp, plusone);
  reduc_doit(A10, A11, A, init+dp, dp, plusone);

  ShiftAdd(A0, A11, 1);
  if (plusone) A0 += A11;
  A1 += A10 + A11;

  long i = 1;
  bool even = true;
  while (2*i != d){
    ShiftAdd(A0, A10, i);
    ShiftAdd(A1, A11, i);
    i = 2*i;
    even = !even;
  }

  if (plusone && !even) {
    A0 += A10;
    A1 += A11;
  }
}
```

```
void treduc_doit(GF2X& A, const GF2X& A0, const GF2X& A1, long d,
bool plusone){
  if (d <= 2){
    SetCoeff(A, 0, coeff(A0, 0));
    SetCoeff(A, 1, coeff(A1, 0));
    return;
  }

  long dp = d/2;
  long hdp = dp/2;

  GF2X A00, A01, A10, A11;
  A00 = trunc(A0, hdp);
  A01 = trunc(A1, hdp);

  A10 = A01;
  if (plusone) A11 = A00;
  else A11 = 0;
  A11 += A01 + RightShift(trunc(A0, hdp+1), 1);
  long i = 1;
  bool even = true;
  while (2*i != d){
    A10 += RightShift(trunc(A0, hdp+i), i);
    A11 += RightShift(trunc(A1, hdp+i), i);
    i = 2*i;
    even = !even;
  }

  if (plusone && !even) {
    A10 += trunc(A0, hdp);
    A11 += trunc(A1, hdp);
  }

  GF2X B0, B1;
  treduc_doit(B0, A00, A01, dp, plusone);
  treduc_doit(B1, A10, A11, dp, plusone);
  A = B0 + LeftShift(B1,dp);
}
```

Automatic transposition of code

I want a compiler that automatically transposes my code !

The problem

- Fix two computational categories and a tellegenish functor F,
- the languages have to be reasonable,
- composition has to be trivial,
- deciding the image of an instruction by F must be feasible.

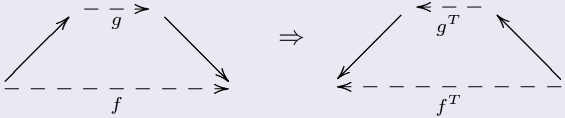
Straight line programs

- Easy.
- Read the program upside-down or bottom-up (depending if the F is covariant or contravariant),
- substitute each instruction with its dual.

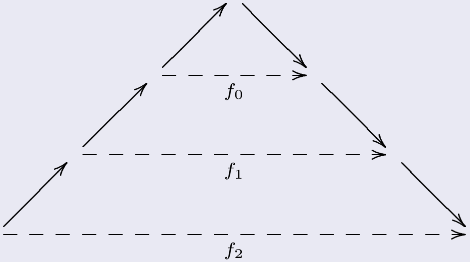
Subroutines

Subroutines

```
f(x, y, z) {
  ...
  a = g(x, y);
  ...
}
```



Recursion



Conditionals, loops

Conditionals

```
if n > 0 then
  ...
else
  ...
endif
```

- `n` must be in the parameter space,
- the conditional is left unchanged.

Loops

```
for i = 0 to n do
  ...
end for
```

- `n` must be in the parameter space,
- the loop is turned upside down (from `n` to 0).
- It also works for nested loops.

Are there any more complicated patterns ?

Who's in the parameter space?

The minimal parameter space

- Some variables **must** be in the parameter space.
- How do we find them ?

The maximal parameter space

- Any variable **can** be in the parameter space.
- Any decision problem can expressed in this category:

$$\mathrm{id}_Y \circlearrowleft Y \qquad\qquad N \circlearrowright \mathrm{id}_N$$

- Even when we fix $\mathbf{FMod}_{\mathbb{Z}}$,
- let our code compute a function $f : \{0,1\}^* \to \{0,1\}^*$,
- Consider $A : \{0,1\}^* \to \mathbf{FMod}_{\mathbb{Z}}$ such that $A(x) = *_{f(x)}$,
- all the variables are in the parameter space.

Who's in the parameter space?

The case of multiplication

```
Mult(x, y) {
  return x * y;
}
```

- Multiplication is not linear (it is bilinear),
- But *Transposed multiplication* (aka *Middle product*) is a very important operation :
- fix x, then $\mathtt{Mult}_{\mathtt{x}}$ is linear.

The solution

Put x in the parameter space !

How do we automatically find it ?

Conclusions

Towards theory

- Is this new point of view more enlightening?
- Are there any other interesting tellegenish functors?

Towards practice

- Is it always possible to find the minimal parameter space?
- How much does the code (conditionals and loops) change when we transpose?
- Can we dream of a working AT tool?