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## OPTIMAL CONTROL OF FAVORABLE GAMES WITH A TIME LIMIT

by

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We study how to control a stochastic process up to a given time when we have controls available that are favorable to us. To be more precise: The stochastic process  $X_t \in [a, b]$  for all t. There is a utility (objective) function u(x) and we wish to maximize  $E[u(X_T)]$  for some given time T. If u is increasing then there are controls available to us in order to make the process a submartingale and if u is decreasing then there are controls such that we can get a supermartingale.

First we solve the discrete time control problem of the random walk

$$X_{t} = X_{t-1} + \sigma(X_{t-1}, t-1) Y_{t-1}$$

where  $Y_{t-1}$  are iid with two point distributions and  $\sigma$  is the control variable. Then we study the asymptotic properties of its solution which turns out to be the same as those of the solution to the corresponding continuous time control problem

$$dX_t = \sigma(X_t, t) (\mu_t dt + dB_t)$$

where  $B_t$  is the standard Brownian motion,  $\mu_t$  is fixed, and  $\sigma$  is the control variable.

Lastly we will apply the results to obtain some inequalities for stochastic processes.

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Although he was fascinated by the probabilities of chance in any game he played, he was a bad player because it gave him no pleasure to win.

-Jorge Luis Borges,

The gospel according to Mark

#### 1 Introduction

How should we bet in a repeated betting game where the odds are favorable to use? What strategy to use will of course depend on the actual game, but more importantly it will depend on our objective. If we wish to maximize the expected fortune it is easy to see that we should bet our whole fortune every time. But this might not seem to thrilling to us when we note the fact that when the number of times we play increases, then the probability of ending up broke will approach one. To overcome this problem Kelly [1956] suggested maximizing the expected growth rate instead. He gave the optimal betting strategy for this objective which is to bet a fixed proportion of the fortune every time. The more favorable the game is the larger this proportion should be. This strategy of betting is now called the "Kelly criterion". Leo Breiman [1961] studied the same problem in more generality as well as its asymptotic properties. The continuous time version was solved by Heath, Orey, Pestien and Sudderth [1987].

Sometimes we might not be willing to wait an arbitrary amount of time to reach a fixed goal but want to reach it before some given time. For example, we may be concerned about the size of our pension account on the day of retirement; how much money there will be in the account on our 150th birthday is of less interest. This is the kind of problem that we study here. We will concern ourselves both with a 0/1 utility (objective) function, for which any fortune less than c is worth zero to us and any fortune greater than c is worth the same amount as at c, and with various continuous utility functions. In practice the Kelly criterion has been used for these kinds of

problems as well. The motivation for this has been based on intuition, a result of Breiman, and the fact that nothing better has been available. Leo Breiman [1961] discussed the 0/1 utility case for the discrete time game of red and black. He continued to show that U(f,t) - K(f,t) converges to zero uniformly as  $t\to\infty$ . Here t is the number of times left to play, f is the fortune, U is the probability to reach c if we play optimally and K is the probability to reach c if we use the Kelly criterion. We do not necessarily think that this is a very strong result. For  $\epsilon>0$  consider the following strategy: First throw away your whole fortune except  $\epsilon$  (if  $f\le\epsilon$  don't throw away anything) then play according to the Kelly criterion. Let  $K_\epsilon(f,t)$  be the probability to reach c if we use this strategy. This is clearly not a very good strategy if  $\epsilon$  is small but U(f,t) -  $K_\epsilon(f,t)$  converges to zero uniformly for any  $\epsilon>0$ . To see this note that  $K(f,t)=K_\epsilon(f,t)$  for  $f\le\epsilon$  and that  $K(\epsilon,t)\to 1$  as  $t\to\infty$ .

The intuition that we get from the more complete solution to the problem of red and black will enable us to get the asymptotic properties for the optimal strategy as well as for the optimal return function U(f,t). This will be dealt with in chapter two together with more general discrete time problems. In chapter three we will solve continuous time versions of the problem and the solutions to these turns out to be the same as those of the limits of the discrete time problem. In the fourth chapter we will give some inequalities for stochastic processes. These results follow from those of chapters two and three. There is no dependence between chapters two and three so they can be read independently.

The framework that we have used is that of Dubins and Savage [1965, 76].

### 2 Discrete Time Primitive Casinos

### 2.1 Description of problem

Suppose that we visit a casino with f dollars in our pocket. We will assume it to be a primitive casino. It is a casino with only one game available. In this game we will bet an amount s. With probability 1-w we will lose this stake and with probability w we will win the amount  $(\frac{1-r}{r})$ s. The numbers  $w \in (0,1)$  and  $r \in (0,1)$  are fixed by the casino. The stake can be any amount of our choice as long as it does not exceed our fortune at that moment. Negative stakes are not permitted.

There is a maximum number of times that we can gamble (we have to catch the train home). There is a utility function u(f) describing the value for us of having fortune f when we leave the casino. Our goal is to maximize the expected utility at the end of the game. One question we want to answer is what this expected utility is and another question is what stakes we should bet to obtain it. We shall let U(f,t) be the optimal expected utility at fortune f with f times left to play. Clearly f u(f,0) = f0.

One utility function that we will consider is u(f)=0 for  $f< c_1$  and  $u(c_1)=c_2$ . We will ignore fortunes larger than  $c_1$ . (That is equivalent to assuming  $u(f)=c_2$  for  $f>c_1$ ). We can assume that  $c_1=c_2=1$  without loss of generality. Aryeh Dvoretzky (see Dubins and Savage [1965, 76] pg 92) solved this problem for the case when  $w \le 1/2$  and r=1/2 (Red and Black). Dubins and Savage [1965, 76] lets us know the solution when w=1/2 and r>1/2 (The taxed coin). Both these cases are examples of unfavorable games and bold play is optimal for both. In the case of favorable games

Leo Breiman [1961] found U(f,t), together with one of the optimal strategies, when w > 1/2 and r = 1/2 (Red and Black). In section four we will give the solutions for other values of w and r but we will only concern ourself with the favorable case when w > r.

Another utility function that we will consider is  $u(f) = c_1 f$  for  $f < c_2$ . We can assume that  $c_1 = c_2 = 1$  without loss of generality. The case of unfavorable games, when w < r, was solved by an ancient moralizer: Never ever bet anything. In section three we will solve the favorable case for all possible values of w and r, i.e. for  $0 < r \le w < 1$ .

In section five we will consider some other utility functions for which the solution will follow from the first two problems.

To make things a little easier to read we will use the following notation:

$$F(k|t,p) = \begin{cases} \sum_{i=0}^{k} t \\ \sum_{i=0}^{k} (p^{i}(1-p)^{t-i}) & \text{if } k \in \{0,1,...t\} \\ 0 & \text{if } k < 0 \\ 1 & \text{if } k > t \end{cases}$$

This is of course  $P(X \le k)$  when  $X \sim Binomial(t,p)$ .

We will often write k instead of k(f,t). F,q and k are important to understand well.

### 2.2 Red and Black

We are first going to discuss the simplest case: red and black. We have u(f) = 0 for f < 1, u(1) = 1, r = 1/2 and w > 1/2. The intuition we will get from solving this problem will enable us to do the general case in the next sections as well as the limit problem. The proofs will only be sketched or hinted at in this section since these results follow from those in section 2.4. For an alternative approach to this problem see Breiman [1961]. He obtains U and gives an optimal strategy.

Definitions: A fortune f is binary at time t if f 2t is an integer.

A stake is binary at fortune f it we arrive at a binary fortune regardless of if we win or lose.

A strategy is binary if it only uses binary stakes.

### **Proposition**

 $U(f,t) = U(\frac{n}{2^t},t)$  where n is the integer such that  $\frac{n}{2^t} \le f < \frac{n+1}{2^t}$ .

Proof: By induction

This means that U(f,t) is a step function with jumps at the binary fortunes.

#### **Proposition**

If the initial fortune is binary then every optimal strategy is binary.

Proof: Use the previous proposition.

### <u>Proposition</u>

If we play t times there are 2<sup>t</sup> possible outcomes of the gamble. If the initial fortune is binary and the strategy is binary then exactly n of the 2<sup>t</sup> outcomes will result in reaching one and the remaining 2<sup>t</sup> -n outcomes will result in reaching zero.

<u>Proof</u>: We reach 0 or 1 for sure if we use a binary strategy and if w = 1/2 we get a martingale for the fortune so P (reaching one) =  $U(\frac{n}{2^t}, t) = n/2^t$  for integers n. Since all specific outcomes have the same probability  $1/2^t$  exactly n outcomes gets us to one. Hence the proposition is true for w = 1/2. Since the resulting fortune of an outcome does not depend on w it is true for all w.

For w > 1/2 we can get an upper bound on U by adding up the n outcomes with the highest probabilities.

### Corollary

Let  $k = k(\frac{n}{2^{t}}, t)$ . If n is an integer then

$$U(\frac{n}{2^t}\,,\,t) \leq F(k\text{-}1|t,\,1\text{-}w)\,+\,q(\frac{n}{2^t}\,,\,t)\,\,\tbinom{t}{k}\,\,(1\text{-}w)^k\,\,w^{t\text{-}k}$$

It turns out that this is not only an upper bound but the actual value of U. The way to show this is to find a strategy that conserves U.

With an initial fortune of  $n/2^t$  we need all outcomes with at most  $k(\frac{n}{2^t},t) - 1 = k - 1 \text{ losses to get us to one and there can be no outcome with more than k losses that brings us there.}$ 

Suppose now that we win our first gamble. There are

 $F(k-1|t-1,\frac{1}{2})$   $2^{t-1}$  outcomes which starts with a win and has at most k-1 losses. Since all these must lead us to our goal our new fortune in the case of a win must be at least F(k-1|t-1, 1/2). Likewise the new fortune must be at most F(k|t-1, 1/2). So

$$F(k-1|t-1, 1/2) - \frac{n}{2^t} \le s(\frac{n}{2^t}, t) \le F(k|t-1, 1/2) - \frac{n}{2^t}$$

If our first gamble is a loss instead there are  $F(k-2|t-1,\ 1/2)\ 2^{t+1}$  outcomes which starts with a loss and has a total of at most k-1 losses. This means that in the case of a loss we must end up with a fortune of at least  $F(k-2|t-1,\ 1/2)$ . Likewise our new fortune cannot exceed

$$F(k-1|\ t-1,\ 1-w). \ \ So \quad \ \frac{n}{2^t} - F(k-1|t-1,\ 1/2) \leq S(\frac{n}{2^t}\,,\ t) \leq \frac{n}{2^t} - F(k-2|\ t-1,\ 1/2) \ \ .$$

Putting these inequalities together we get:

### **Proposition**

If f is a binomial fortune with t times left to play than s(f, t) is an optimal stake if and only if

- i) s(f,t) is binomial
- ii)  $s(f,t) \le \min \left\{ \frac{n}{2^t} F(k-2|t-1, 1/2), F(k|t-1, 1/2) \frac{n}{2^t} \right\}$
- iii)  $s(f,t) \ge |F(k-1|t-1, 1/2) \frac{n}{2^t}|$

To verify that such a stake always exists use the formula:

$$F(k-1|t, 1/2) = \frac{1}{2}F(k-1|t-1, 1/2) + \frac{1}{2}F(k-2|t-1, 1/2)$$
 and the fact that

$$F(k-1|t, 1/2) \le f \le F(k|t, 1/2)$$
.

We also have:

Proposition (due to Breiman [1961])

 $U(f,t)=F(k-1|t,\ 1-w)+q(\frac{n}{2^t}\ ,\ t)\ \binom{t}{k}\ (1-w)^k\ w^{t-k}\ where\ n\ is\ the\ largest\ integer$  such that  $f\geq \frac{n}{2^t}$  .

### 2.3 u(f) = f

In this section we will consider the case in which u(f) = f. We first need a simple combinatorial lemma.

### Lemma 1

Let k = k (f, t).

If 
$$F(k-1|t-1, 1-r) \le f \le F(k|t, 1-r)$$
 (i.e.  $\frac{k}{t} \le q(f,t) < 1$ ) then

$$f = F(k-1|t, 1-r) + q(f,t) {t \choose k} (1-r)^k r^{t-k} =$$

$$= F(k-1|\ t-1,\ 1-r) + r\,\frac{t}{t-k}\,(q(f,t)-\frac{k}{t}\,)\,(\frac{t-1}{k})\,(1-r)^k\,r^{t-1-k}$$

If 
$$F(k-1|t, 1-r) \le f \le F(k-1|t-1, 1-r)$$
 (i.e.  $0 \le q(f,t) < \frac{k}{t}$ ) then

$$f = F(k-1|t, 1-r) + q(f,t) {t \choose k} (1-r)^k r^{t-k} =$$

$$= F(k-2|\ t-1,\ 1-r) + \left[r + \frac{t}{k} \left(1-r\right) \, q(f,t)\right] {t-1 \choose k-1} \, (1-r)^{k-1} \, r^{t-k}$$

$$\begin{split} & \underline{\text{Proof}} \colon F(k\text{-}1|\ t,1\text{-}r) + q(f,t) \ \binom{t}{k} \ (1\text{-}r)^k \ r^{t-k} = \\ & = r \ F(k\text{-}1|\ t\text{-}1,\ 1\text{-}r) + (1\text{-}r) \ F(k\text{-}2|t\text{-}1,\ 1\text{-}r) + q(f,t) \ \binom{t}{k} \ (1\text{-}r)^k \ r^{t-k} \\ & = F(k\text{-}2|\ t\text{-}1,\ 1\text{-}r) + r \ \binom{t\text{-}1}{k\text{-}1} \ (1\text{-}r)^{k\text{-}1} \ r^{t-k} + \frac{t}{k} \ (1\text{-}r) \ q(f,t) \ \binom{t\text{-}1}{k\text{-}1} \ (1\text{-}r)^{k\text{-}1} \ r^{t-k} \\ & = F(k\text{-}2|\ t\text{-}1,\ 1\text{-}r) + [r + \frac{t}{k} \ (1\text{-}r) \ q(f,t)] \ \binom{t\text{-}1}{k\text{-}1} \ (1\text{-}r)^{k\text{-}1} \ r^{t-k} \\ & = F(k\text{-}2|\ t\text{-}1,\ 1\text{-}r) + [r + \frac{t}{k} \ (1\text{-}r) \ q(f,t)] \ \binom{t\text{-}1}{k\text{-}1} \ (1\text{-}r)^{k\text{-}1} \ r^{t-k} \end{split}$$

$$= F(k-2|t-1, 1-r) + {t-1 \choose k-1} (1-r)^{k-1} r^{t-k} + \left[ \frac{t}{k} (1-r) q(f,t) + r-1 \right] {t-1 \choose k-1} (1-r)^{k-1} r^{t-k} =$$

$$= F(k-1|t-1, 1-r) + (1-r) \left[ \frac{t}{k} q(f,t) - 1 \right] r \frac{k}{t-k} {t-1 \choose k} (1-r)^{k-1} r^{t-1-k} =$$

$$= F(k-1|t-1, 1-r) + r \frac{t}{t-k} \left[ q(f,t) - \frac{k}{t} \right] {t-1 \choose k} (1-r)^{k-1} r^{t-1-k}$$

### Theorem 1

Let Q(f,t) = F(k-1| t, 1-w) + q(f,t)
$$\binom{t}{k}$$
(1-w)<sup>k</sup> w<sup>t-k</sup> and let f = F(k-1| t,1-r) + q(f,t) $\binom{t}{k}$ (1-r)<sup>k</sup> r<sup>t-k</sup>. If w > r then

- i) U(f,t) = Q(f,t)
- $\begin{aligned} &\text{ii)} & &\text{s}(f,t) \text{ is an optimal stake at time t with fortune f if and only if} \\ &\text{s}(f,t) \geq \text{max}\{f\text{-}F(k\text{-}1|t\text{-}1,\ 1\text{-}r),\ \frac{r}{1\text{-}r}\left(F(k\text{-}1|\ t\text{-}1,\ 1\text{-}r)\text{-}f\right)\} = \underline{s}(f,t) \text{ (minimum bet) and} \\ &\text{s}(f,t) \leq \text{min}\{f\text{-}F(k\text{-}2|\ t\text{-}1,\ 1\text{-}r),\ \frac{r}{1\text{-}r}\left(F(k|\ t\text{-}1,\ 1\text{-}r)\text{-}f\right)\} = \overline{s}(f,t) \text{ (maximum bet)}.} \end{aligned}$

Remark: As an alternate formula 
$$Q(f,t) = F(k-1,\,t,1-w) + \frac{f-F(k-1|t,\,1-r)}{\binom{t}{k}(1-r)^k r^{t-k}} \, \binom{t}{k} \, (1-w)^k w^{t-k} = \\ = F(k-1,\,t,\,1-w) + \left[f-F(k-1|t,\,1-r)\right] \, (\frac{1-w}{1-r})^k \, (\frac{w}{r})^{t-k}$$

Remark: Notice that the optimal stakes do no depend on w.

To graph U(f,t) just draw straight lines through the points  $(F(k|t,\,1-r),\,F(k|\,t,\,1-w)) \text{ when } k \text{ is increasing from 0 to t.}$ 

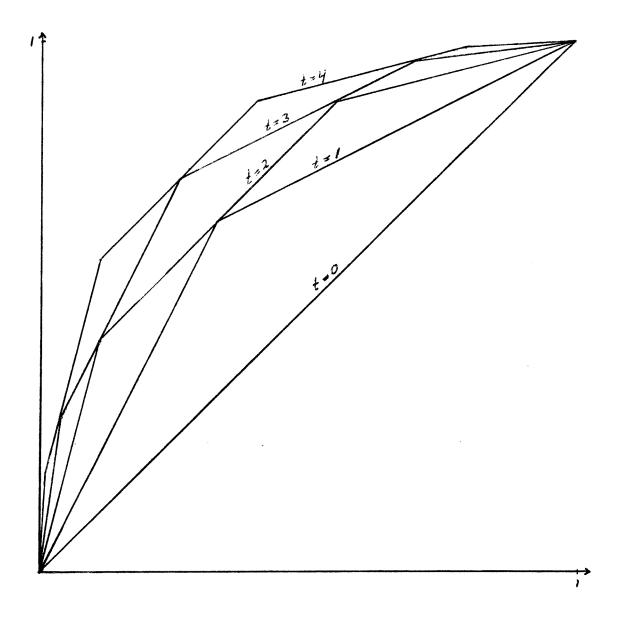


Figure 1: Example of U(f,t) with w=2/3, r=1/3 and t=0,1,2,3 and 4.

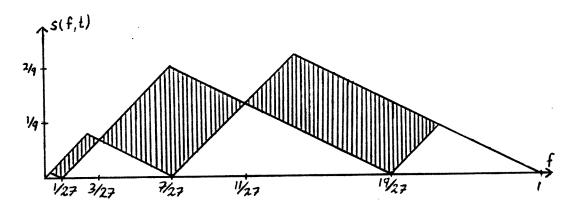


Figure 2: Optimal stakes, s(f,t), when r = 1/3 and t = 4.

Proof: By induction. It is trivially true for t=0.

First we are going to show that Q is obtainable  $(U \ge Q)$ :

Case 1: 
$$q(f,t) \ge \frac{k}{t}$$
 i.e.  $F(k-1|t-1, 1-r) \le f < F(k|t, 1-r)$ 

This case corresponds to the fortunes in figure 2 where the minimum stake is increasing. Let  $s(f,t) = f - F(k-1|t-1, 1-r) = \underline{s}(f,t)$ , the minimum stake. f can according for lemma one be written as:

$$f = F(k-1|t-1, 1-r) + r \frac{t}{t-k} (q(f,t) - \frac{k}{t}) {t-1 \choose k} (1-r)^k r^{t-1-k}$$

$$U(f,t) \geq (1-w) \ U(f-s,\,t-1) + w \ U(f+\frac{1-r}{r} \ s,\,t-1) = (1-w) \ Q \ (f-s,\,t-1) + w \ Q(f+\frac{1-r}{r} s,\,t-1) = (1-w) \ Q($$

$$= (1-w) Q (F(k-1|t-1, 1-r), t-1) +$$

+ wQ {F(k-1|t-1, 1-r) + 
$$\frac{t}{t-k}$$
 (q(f,t) -  $\frac{k}{t}$ )( $\frac{t-1}{k}$ )(1-r)k rt-1-k, t-1}

$$= (1-w) \; F(k-1|\; t-1,\; 1-w) \; + \; w \; \{F(k-1|t-1,\; 1-w) \; + \; \frac{t}{t-k} \; (q(f,t) \; - \; \frac{k}{t} \;) \binom{t-1}{k} \; (1-w)^k w^{t-1-k} \}$$

$$=F(k-1|t-1, 1-w) + w \frac{t}{t-k} (q(t,t) - \frac{k}{t}) {t-1 \choose k} (1-w)^k w^{t-1-k}$$

= 
$$F(k-1|t, 1-w) + q(f,t) {t \choose k} (1-w)^k w^{t-k}$$
 (lemma 1) =  $Q(f,t)$ 

Case 2: 
$$q(f,t) < \frac{k}{t}$$
 i.e.  $f < F(k-1|t-1, 1-r)$ 

Let  $s(f,t) = \frac{r}{1-r} (F(k-1|t-1, 1-r) - f) = \underline{s}(f,t)$ . f can according to lemma one be written as:  $f = F(k-2|t-1, 1-r) + (r + q(f,t)\frac{t}{k}(1-r))(\frac{t-1}{k-1})(1-r)^{k-1} r^{t-k}$ 

U (f,t) ≥ (1-w) Q(f-s, t-1) + w Q(f + 
$$\frac{1-r}{r}$$
s, t-1) =  
= (1-w)Q{F(k-2| t-1, 1-r) +  $\frac{t}{k}$ q(f,t)  $\binom{t-1}{k-1}$  (1-r)<sup>k-1</sup> r<sup>t-k</sup>, t-1} +

$$+ w Q(F(k-1|t-1, 1-r), t-1)$$

$$= (1-w) \{F(k-2|t-1, 1-w) + q(f,t) \binom{t}{k} (1-w)^{k-1} w^{t-k}\} + w F(k-1|t-1, 1-w)$$

$$= F(k-1|t, 1-w) + q(f,t) \binom{t}{k} (1-w)^k w^{t-k} = Q(f,t)$$

Hence Q is obtainable.

Now we will show that we cannot do any better than Q ( $U \le Q$ ). It might be a good idea to confer with figure one while reading the rest of this proof.

Let 
$$E_s(f,t) = (1-w) U(f-s, t-1) + w U(f + \frac{1-r}{r}s, t-1)$$
  
= constant +  $w \frac{1-r}{r} s (\frac{1-w}{1-r})^n (\frac{w}{r})^{t-1-n} - (1-w)s (\frac{1-w}{1-r})^m (\frac{w}{r})^{t-1-m}$ 

where m is such that  $F(m-1|t-1, 1-r) \le f-s < F(m|t-1, 1-r)$  and n is such that  $F(n-1|t-1, 1-r) \le f + \frac{1-r}{r}s < F(n|t-1, 1-r)$ .

Note that  $E_s$  (f,t) is almost everywhere differentiable with respect to s. We know that  $U(f,t) = \max_{S} E_s(f,t)$  (Dubins and Savage [1965, 76]).

To prove that s is an optimal bet if and only if  $\underline{s} \le s \le \overline{s}$  and that we cannot do any better than Q we will show that:

cannot do any better than Q we will show that 
$$\frac{d}{ds}E_s(f,t)$$
  $\begin{cases} > 0 & \text{when} & s < \underline{s} \\ = 0 & \text{when} & \underline{s} < s < \overline{s} \\ < 0 & \text{when} & s > \overline{s} \end{cases}$ 

for all s where Es is differentiable.

If  $s < \underline{s}$  then m = n so  $\frac{d}{ds} E_s(f,t) = w \, (\frac{1-w}{1-r})^n \, (\frac{w}{r})^{t-1-n} \, (\frac{1-r}{r} - \frac{1-w}{w}) > 0$  since w > r.

$$\text{If } \underline{s} < s < \overline{s} \text{ then } m = n-1 \text{ so } \frac{d}{ds} \, E_{S}(f,t) = \frac{(1-w)^{n} w^{t-n}}{(1-r)^{n-1} r^{t-n}} - \frac{(1-w)^{n} w^{t-n}}{(1-r)^{n-1} \ r^{t-n}} = 0 \ .$$

If  $s > \overline{s}$  then  $m \le n-2 = n-1-i$  for some positive integer i so

$$\frac{d}{ds} E_s (f,t) = \frac{(1-w)^{n-1} w^{t-n}}{(1-r)^{n-1} r^{t-n}} - \frac{(1-w)^{n-i} w^{t-n+i}}{(1-r)^{n-1-i} r^{t-n+i}} =$$

$$= \frac{(1-w)^{n-i} \ w^{t-n}}{(1-r)^{n-1-i} \ r^{t-n}} \left( \left( \frac{1-w}{1-r} \right)^{i} - \left( \frac{w}{r} \right)^{i} \right) < 0 \text{ since } w > r \text{ and } i \ge 1.$$

### 2.4 u(f) = 0 for f < 1 and u(1) = 1

In this section we will consider the case in which u(f) = 0 for f < 1 and u(f) = 1 for f = 1.

 $\underline{\text{Definition}} \colon \text{ A fortune f is binomial at time t if } q(f,t) \begin{pmatrix} t \\ k \end{pmatrix} \text{ is an integer}.$ 

Examples: i) The binomial fortunes when t = 2 are 0,  $(1-r)^2$ ,  $(1-r)^2 + r(1-r)$ ,  $(1-r)^2 + 2r(1-r) = 1-r^2$  and 1.

ii) When r = 1/2 then the binomial fortunes at times t are the integer multiples of  $(\frac{1}{2})^t$ .

<u>Definition</u>: A stake s(f,t) is binomial if we arrive at a binomial fortune regardless of whether we win or lose, i.e., if f - s(f,t) and  $f + \frac{1-r}{r} s(f,t)$  are binomial fortunes at time t-1.

### Theorem 2

If f is binomial then U(f,t) = Q(f,t) (as in theorem 1) and s(f,t) optimal, iff

i) 
$$s(f,t) \ge \max \{f-F(k-1|t-1, 1-r), \frac{r}{1-r}(F(k-1|t-1, 1-r) - f)\} = \underline{s}$$

ii) 
$$s(f,t) \le \min \{f-F(k-2|t-1, 1-r), \frac{r}{1-r}(F(k|t-1, 1-r) - f)\} = \overline{s}$$

iii) s(f,t) is a binomial stake

This means that for binomial f we can do just as well for this case as for the case of section 3 despite u(f) being smaller.

<u>Proof</u>: By induction. It is trivially true for t=0. The only thing we have to show is that there always exists a binomial stake fulfilling conditions one and two. The rest follows from theorem 1.

case 1:  $q(f,t) \ge \frac{k}{t}$ . We want to show that  $s(f,t) = f - F(k-1|t-1,1-r) (= \underline{s}(f,t))$  is a binomial stake.

- i) f-s = F(k-1|t-1, 1-r) which is binomial at time t-1.
- $\begin{aligned} &\text{ii)} & &f+\frac{1-r}{r}\,s=F(k-1|\ t-1,\ 1-r)+\frac{t}{t-k}\,(q(f,t)-\frac{k}{t})\,\binom{t-1}{k}(1-r)^k\,\,r^{t-1-k}\\ &=F(k-1|\ t-1,1-r)+q(f,t)\,\binom{t}{k}\,\,(1-r)^k\,\,r^{t-1-k}\,\,\text{which is binomial at time }t-1\\ &\text{since f being binomial at time t implies that }q(f,t)\,\binom{t}{k}\,\,\text{is an integer.} \end{aligned}$
- case 2:  $q(f,t) < \frac{k}{t}$ . Now we want to show that  $s(f,t) = \frac{r}{1-r} \left( F(k-1|t-1,\ 1-r) f \right) \ (= \underline{s}(f,t)) \text{ is a binomial stake}.$ 
  - i)  $f + \frac{1-r}{r}s = F(k-1|t-1, 1-r)$  which is binomial at time t-1.
  - ii)  $f s + F(k-2|t-1, 1-r) + \frac{t}{k} q(f,t) \binom{t-1}{k-1} (1-r)^{k-1} r^{t-k} = F(k-2|t-1, 1-r) + q(f,t) \binom{t}{k} (1-r)^{k-1} r^{t-k}$  which is binomial at time t-1 since  $q(f,t) \binom{t}{k}$  is integer valued.

For non binomial f we can use the fact that U is non decreasing to get a lower bound for U and we can use theorem 1 to get an upper bound. Let  $\widetilde{Q}(f,t) = F(k-1|t, 1-w) + [q(f,t)\binom{t}{k}](1-w)^k w^{t-k}$  where  $f = F(k-1|t, 1-r) + q(f,t)\binom{t}{k}(1-r)^k r^{t-k}$  ([x] is the integer part of x). This means that  $\widetilde{Q}$  takes jumps of magnitudes  $(1-w)^k w^{t-k}$  (k varies) at binomial fortunes and is constant otherwise. For binomial fortunes  $\widetilde{Q} = U$ .

### Corollary

 $\tilde{Q}(f,t) \le U(f,t) \le Q(f,t)$ 

Since  $|Q(f,t) - \widetilde{Q}(f,t)|$  converges to zero when  $t \to \infty$  the bounds will be pretty good for large t. For the case of  $r \le 1/2 \le w$  the result is stronger:

# Theorem 3 If $r \le 1/2 \le w$ then $U(f,t) = \tilde{Q}(f,t)$ .

It is not difficult to find a counter example to this theorem for general r and w. Just pick any r and w such that r < w < 1/2 or 1/2 < r < w and try it for sufficiently large t.

<u>Proof:</u> By induction. It is trivially true for t=0. Let  $f_1$ ,  $f_2$ , ... and  $g_1$ ,  $g_2$ , ... be the binomial fortunes at time t and t-1 respectively. Let  $g_x = f_j - \overline{s}$ ,  $g_m = f_j - \underline{s}$ ,  $\overline{g}_x = f_j + \frac{1-r}{r}\overline{s}$  and  $\overline{g}_m = f_j + \frac{1-r}{r}\underline{s}$ . We know that  $g_i - g_{i-1} = (1-r)^k r^{t-1-k}$  for some k and that  $U(g_i, t-1) - U(g_{i-1}, t-1) = (1-w)^k w^{t-1-k}$  for the same k.

Since U is non decreasing it is enough to show that  $U(f,t) = \widetilde{Q}(f,t)$  at fortunes  $f_j - \epsilon$  where  $\epsilon > 0$  is arbitrarily small. To be more precise we have to show that  $U(f_j - \epsilon, t) = U(f_{j-1}, t)$ . It is clear that an optimal stake can be found among those of the form  $f_j - \epsilon - g_j \ge 0$ . So  $U(f_j - \epsilon, t) =$ 

$$= \max_{i} \left\{ (1-w) \ U \ (g_i + \epsilon, \, t\text{-}1) \ + \ w \ U(f_j + \frac{1-r}{r} \left( f_j \text{-} \ \epsilon \text{-} \ g_i \right), \, t\text{-}1) \right\} \ \stackrel{\Delta}{=} \ max \ U_i \ .$$

Let us first try  $g_x \le g_i \le g_m$ :

$$U_i = (1-w) \ U(g_i + \varepsilon, t-1) + w \ U(\overline{g}_{x-(i-x)} - o(\varepsilon), t-1)$$
 so  $U_i = (1-w) \ U(g_i, t-1) + w \ U(\overline{q}_{x-(i-x)} - 1, t-1) =$ 

= 
$$(1-w) U(g_i, t-1) + w U(\overline{g}_{x-(i-x)}, t-1) - w (1-w)^k w^{t-1-k} =$$

$$= \, U(f_j,\,t) \, \cdot \, (1\!-\!w)^k \; w^{t\!-\!k} = U(f_{j\!-\!1},\,t).$$

Now let us try  $g_i < g_x$ :

$$\begin{array}{l} U_i = (1\text{-w}) \; U(g_i + \epsilon, \, t\text{--}1) + w \; U(g, \, t\text{--}1) \; \text{where} \; g < \overline{g}_{x \, + \, (x\text{-}i)} \; \text{since} \; r \leq 1/2 \\ \\ \text{so} \; U_i \leq (1\text{-w}) \; U(g_i, \, t\text{--}1) + w U(\overline{g}_{m\text{-}(i\text{-m})\text{--}1, t\text{--}1}) \leq \\ \\ \leq (1\text{-w}) \; \{U(g_x, \, t\text{--}1) - (x\text{--}i) \; (1\text{-w})^{k-1} \; w^{t-1\text{--}(k+1)}\} \; + \\ \\ w\{U(\overline{g}_x, \, t\text{--}1) + (x\text{--}i\text{--}1) \; (1\text{-w})^k \; w^{t\text{--}k} - (1\text{-w})^{k+1} \; w^{t\text{--}(k+1)}\} \; - \; (1\text{-w})^k \; w^{t\text{--}k} \leq \\ \\ \leq U(f_j, \, t) - (x\text{--}i\text{--}1) \; \{(1\text{-w})^k \; w^{t\text{--}k} \; (\text{since} \; w \geq 1/2) \; = \; U(f_{j\text{--}1}, \, t). \\ \\ \text{It remains to try} \; g_i > g_m: \\ \\ U_i = (1\text{-w}) \; U(g_i, \, t\text{--}1) + w U(g, \, t\text{--}1) \; \text{where} \; g < \overline{g}_{m\text{--}(i\text{-m})} \; \text{since} \; r \leq 1/2 \\ \\ \text{so} \; U_i \leq (1\text{-w}) \; U(g_i, \, t\text{--}1) + w \; U(\overline{g}_{m\text{--}(i\text{-m})\text{--}1}, \, t\text{--}1) = \\ \\ = (1\text{-w}) \; \{U(g_m, \, t\text{--}1) + (i\text{-m}) \; (1\text{-w})^k \; w^{t\text{--}1\text{--k}}\} \; + \\ \\ + \; w \; \{U(\overline{g}_m, \, t\text{--}1) - (i\text{-m}\text{--}1) \; (1\text{-w})^k \; w^{t\text{--}1\text{--k}}\} \; = \\ \\ = \; U(f_j, \, t) - (i\text{-m}) \; \{(1\text{-w})^k \; w^{t\text{--}1\text{--k}} \; (w\text{--}(1\text{-w}))\} - (1\text{-w})^k \; w^{t\text{--k}} \leq \\ \\ \leq U(f_i, \, t) - (1\text{-w})^k \; w^{t\text{--k}} \; (\text{since} \; w \geq 1/2) \; = \; U(f_{i\text{--}1}, t). \end{array}$$

To know which bets are optimal  $(r \le 1/2 \le w)$  we divide our fortune f into two parts:

- i)  $f \{q(f, t) \begin{pmatrix} t \\ k \end{pmatrix} [q(f, t) \begin{pmatrix} t \\ k \end{pmatrix}] \} (1-w)^k w^{t-k}$ , the valuable part
- ii)  $\{q(f,t) \begin{pmatrix} t \\ k \end{pmatrix} [q(f,t) \begin{pmatrix} t \\ k \end{pmatrix}]\} (1-w)^k w^{t-k}$ , the worthless part.

If we use the valuable part, and the money it generates during the course of the play, to bet according to theorem 2 we will achieve optimum. That means that we can use the worthless part, and the money it generates, for bets whenever we want, the strategy will not influence our chances to reach our goal. We could, of course, also use it, before we start playing, for some luck bringing endeavor such as throwing it in a fountain.

The strategies described here are not the only optimal ones except when the worthless part is zero (i.e. f is binomial).

### 2.5 Other utility functions.

The results of this section follow immediately from the previous theorems. Let  $S_1$  (f,t) and  $S_3$  (f,t) be the sets of optimal stakes in theorem 1 and 3 respectively.

### Corollary

If u(f) = Q(f,s) (or  $\tilde{Q}(f,s)$ ) then U(f,t) = Q(f,s+t) (respectively  $\tilde{Q}(f,s+t)$ ) and s(f,t) is optimal if it belongs to the set  $S_1(f,s+t)$  ( $S_3(f,s+t)$ ).

### 2.6 Limit results and approximation formulas.

All limits in this section will be the same for the cases of sections three and four so we will not make a distinction.  $\Phi$  denotes the cumulative distribution function of the standard normal distribution.

What happens to U(f,t) when  $t \to \infty$ ? It is easy to see that it converges to one pointwise. However, this is not the limit we are interested in. We want to know the limit of U(f,t) when w and r vary with t in such a way that w-r $\to$ 0 as t $\to\infty$ . This means that the bets will get less and less favorable but that will be compensated by the number of bets allowed, which will increase. We will do this in such a way that  $\sqrt{t}$  (w-r) equals a constant c for all t.

If 
$$\sqrt{t}$$
 (w-r) = c and w  $\rightarrow$  w<sub>0</sub>  $\in$  (0, 1) then  $\lim_{t\rightarrow\infty}$  U(f,t) =  $\Phi\left(\Phi^{-1}(f) + \frac{c}{\sqrt{w_0(1-w_0)}}\right)$ 

We will obtain this same function as the U for a similar continuous time problem in theorem 6.

Proof: 
$$f < F(k(f,t) | t, 1-r) = P(S_t \le k(f,t))$$
 where  $S_t \sim Bin(t, 1-r)$ 

$$= P\left(\frac{S_t - t(1-r)}{\sqrt{t \ r \ (1-r)}} \ \leq \ \frac{k(f,t) - t(1-r)}{\sqrt{t \ r \ (1-r)}}\right) = \Phi\left(\frac{k(f,t) - t(1-r)}{\sqrt{t \ r \ (1-r)}}\right) + \Delta_t$$

where  $\lim_{t\to\infty} \Delta_t = 0$  since  $\frac{k(f,t)-t(1-r)}{\sqrt{t\ r\ (1-r)}}$  converges.

so 
$$k(f,t) > \Phi^{-1} (f-\Delta_t) \sqrt{t r (1-r)} + t (1-r)$$

Likewise 
$$f \ge F(k(f,t)-1 \mid t, 1-r) = \Phi\left(\frac{k(f,t)-1-t(1-r)}{\sqrt{t \ r \ (1-r)}}\right) + \Delta't$$

so 
$$k(f,t) \le \Phi^{-1} (f-\Delta'_t) \sqrt{t r (1-r)} + t(1-r) + 1$$

Now, 
$$U(f,t) < F(k(f,t) | t, 1-w) = P(R_t \le k(f,t))$$
 where  $R_t \sim Bin(t, 1-w)$ 

$$=P\left(\frac{R_t-t(1-w)}{\sqrt{t\ w\ (1-w)}}\leq \frac{k(f,t)-t(1-w)}{\sqrt{t\ w\ (1-w)}}\right)=\Phi\left(\frac{k(f,t)-t(1-w)}{\sqrt{t\ w(1-w)}}\right)\\ +\epsilon_t\ where\ \lim_{t\to\infty}\ \epsilon_t=0$$

$$\leq \Phi\left(\frac{\Phi^{-1}(f-\Delta't)\sqrt{t\ r(1-r)}\ +\ t(1-r)+1\ -\ t(1-w)}{\sqrt{t\ w\ (1-w)}}\right) + \epsilon_t$$

$$= \Phi \left( \Phi^{-1}(f - \Delta'_t) \ \sqrt{\frac{r(1-r)}{w(1-w)}} \ + \ \frac{c}{\sqrt{w(1-w)^1}} - \ \frac{1}{\sqrt{t \ w(1-w)}} \ \right) + \ \epsilon_t$$

$$\text{Likewise U(f,t)} \geq F(k(f,t)-1 \mid t,1-w) + \Phi\left(\frac{k(f,t)-1-t(1-w)}{\sqrt{t \ w(1-w)}}\right) + \epsilon'_t$$

$$> \Phi \bigg( \Phi^{-1}(f - \Delta_t) \ \sqrt{\frac{r(1-r)}{w(1-w)}} \ + \ \frac{c}{\sqrt{w(1-w)}} \ - \ \frac{1}{\sqrt{t \ w \ (1-w)}} \bigg) + \ \epsilon'_t$$

so 
$$\lim_{\substack{t\to\infty\\ w\to w_0}} U(f,t) = \Phi\left(\Phi^{-1}(f) + \frac{c}{\sqrt{w_0(1-w_0)}}\right).$$

If we want to use this result to provide approximate values for U(f,t), we should perhaps use instead the following, which follows from the proof of the theorem:

### Corollary

If t, t w(1-w) and t r (1-r) are all large then

$$U(f,t) \simeq \Phi\left(\Phi^{-1}(f) \sqrt{\frac{r(1-r)}{w(1-w)}} + \frac{\sqrt{t(w-r)}}{\sqrt{w(1-w)}}\right)$$

We might also be interested in the limit of s(f,t) as  $t \to \infty$ . It is not difficult to see that it converges to zero uniformly (also when  $w-r \to 0$ ) but this doesn't tell us anything about how fast or in what way it converges. Neither does it give us an approximation formula for the bet sizes. Betting zero all the time will get us nowhere.

Instead we will consider the limit of  $\sqrt{t}$  s(f,t) as  $t\to\infty$ . Since s(f,t) is not unique we can choose different sequences of  $\sqrt{t}$  s(f,t), some of which may not converge. Which sequence should we choose? When f=F(k|t,1-r) the bet size is unique so the choice is obvious. For F(k-1|t,1-r) < f < F(k|t,1-r) we will choose a bet such that s(f,t) is between s(F(k-1|t,1-r),t) and s(F(k|t,1-r),t). In the case of section three there is always such an optimal bet. That is also true for the case of section four with the exception of when k=t/2.

### Theorem 5

 $\lim_{t\to\infty} \sqrt{t} \, s(f,t) = \sqrt{\frac{r}{1-r}} \, \phi \, \left(\Phi^{-1}(f)\right) \text{ where } \phi \text{ is the density function of the standard normal distribution.}$ 

Proof: It is enough to consider fortunes of the form F(k-1| t, 1-r).

Let 
$$f = F(k-1|t, 1-r)$$
 so  $s(f,t) = F(k-1|t, 1-r) - F(k-2|t-1, 1-r) =$ 

$$= (1-r) F(k-2|t-1, 1-r) + r F(k-1|t-1, 1-r) - F(k-2|t-1, 1-r) =$$

$$= F(k-1|t-1, 1-r) - F(k-2|t-1, 1-r)$$

and 
$$k(f,t) > \Phi^{-1}(f-\Delta_t) \sqrt{t r (1-r)} + t(1-r)$$

and 
$$k(f,t) \le \Phi^{-1}(f-\Delta'_t) \sqrt{t r (1-r)} + t (1-r) + 1$$

so 
$$k(f,t) = \Phi^{-1}(f-\widetilde{\Delta}_t) \sqrt{t \ r \ (1-r) + t(1-r)} + d \text{ where } d \in [0, 1].$$

$$\lim_{t\to\infty} \sqrt{t} \, s(f,t) \, = \lim_{t\to\infty} \sqrt{t} \, r \, \{ F(k-1|\ t-1,\ 1-r) \, - \, F(k-2|\ t-1,\ 1-r) \}$$

$$= r \lim_{t \to \infty} \sqrt{t} \; \mathsf{P}(\mathsf{S}_{t-1} = \mathsf{k}\text{-}1) = r \lim_{t \to \infty} \sqrt{t} \bigg( \frac{1}{\sqrt{t \; r(1-r)}} \, \varphi \left( \frac{\mathsf{k}(\mathsf{f},\mathsf{t})\text{-}1\text{-}\mathsf{t}(1-r)}{\sqrt{t \; r(1-r)}} \right) + \; \epsilon_t \bigg)$$

$$= \sqrt{\frac{r}{1-r}} \lim_{t \to \infty} \varphi \left( \Phi^{-1}(f + \widetilde{\Delta}_t) \right. \\ \left. + \frac{d-1}{\sqrt{t \ r \ (1-r)}} \right. \right) + r \lim_{t \to \infty} \sqrt{t} \ \epsilon_t = \sqrt{\frac{r}{1-r}} \ \varphi \left( \Phi^{-1}(f) \right)$$

$$\text{since} \mid r \lim_{t \to \infty} \sqrt{t} \, \epsilon_t \mid \leq r \lim_{t \to \infty} \sqrt{t} \, \left| \frac{1}{\sqrt{t} \, r \, \left(1 - r\right)} \left( \frac{A}{t} \, + \, \frac{B}{\sqrt{t}} \left( \frac{k - t \, \left(1 - r\right)}{\sqrt{t} \, r \, \left(1 - r\right)} \right)^2 \right) \right|$$

(A and B constants) = 
$$\sqrt{\frac{r}{1-r}} \lim_{t \to \infty} \left| \frac{A}{t} + \frac{B}{\sqrt{t}} \right| \left( \Phi^{-1}(f - \widetilde{\Delta}_t) + \frac{d}{\sqrt{tr}(1-r)} \right)^2 \right| = 0$$

For the bound on  $\varepsilon_t$  see Feller [1957] pg. 170.

### Corollary

If t and tr(1-r) are large then s(f,t)  $\simeq \frac{1}{\sqrt{t}} \sqrt{\frac{r}{1-r}} \, \varphi \; (\Phi^{-1}(f))$  .

For r=1/2 this is similar to the optimal stake for the continuous time problem with constant drift  $\mu_t$  (theorem 6).

The variance of a single bet is  $s(f,t)^2 \frac{w(1-w)}{r^2}$ . If we multiply the stake by  $\sqrt{\frac{r}{1-r}}$  we get variance  $\left(s(f,t)\sqrt{\frac{r}{1-r}}\right)^2 \frac{w(1-w)}{r^2} = s(f,t)^2 \frac{w(1-w)}{r(1-r)}$  which is equal to  $s(f,t)^2$  in the limit since w-r  $\rightarrow$ 0. This is the intuition behind the  $\sqrt{\frac{r}{1-r}}$  part of the formulas above.

### 3 Continuous time problem

Consider the stochastic process  $X_t$  such that  $dX_t = \sigma_t(\mu_t \, dt + dB_t)$  where  $B_t$  is the standard Brownian motion. In the gambling context  $X_t$  will represent the fortune at time t. The fortune space is of the form [a, b] and will without loss of generality be assumed to be [0, 1].  $\mu_t \leq M$  for some constant M and for all t, is fixed to us although it might vary with t. Zero and one are absorbing states so if  $X_t = 0$  or 1 then  $\sigma_S = 0$  for all  $s \geq t$ . Otherwise  $\sigma_t$ , which may only depend on what has happened up to time t, can be chosen to be any non negative values as long as

$$\int\limits_t^{T-\epsilon} \mid \sigma_s\mid^2 ds < \infty \text{ (which implies } \int\limits_t^{T-\epsilon} \mid \mu_s\sigma_s\mid ds < \infty) \ \ \forall \ \epsilon>0 \ \ .$$

T is a fixed stopping time and our goal is to maximize the expected value of a utility function u(f,T) which expresses the value for us of having fortune f at time T. In addition to this we want to know U(f,t); the expected utility at time t given that we have fortune f at time t and that we use the optimal strategy.

With  $\Phi$  we will denote the cumulative distribution function of the standard normal distribution and with  $\phi$  its density function.

Let 
$$S = \{(f,t): 0 \le f \le 1, 0 \le t \le T\}$$
 be the slate space. Let  $\mu_t^+ = \max\{0, \mu_t\}$ .

Definition: 
$$m_t = \sqrt{\int_t^T (\mu_s^+)^2 ds}$$
 a measure of the remaining amount of

"favorability". Note that when  $\mu_t = \mu \ \forall \ t$  then  $m_t = \sqrt{T-t} \ \mu$ .

Without loss of generality we assume that  $m_t > 0$  when t < T. This property can easily be obtained by a change of T since  $\sigma_t = 0$  is one of the optimal strategies when  $m_t = 0$ . We will study the problem with the utility function u(f,T) = f. The result for some other utility functions will follow as a corollary.

### Theorem 6

If 
$$u(f,T)=f$$
 then  $U(f,t)=\overset{\sim}{U}(f,t)\overset{\Delta}{=}\Phi\ (\Phi^{-1}(f)+m_t)$   $(f,t)\in S$  and  $\sigma^*\ (f,t)=\overset{\sim}{\sigma}^*\ (f,t)\overset{\Delta}{=}\phi\ (\Phi^{-1}(f))\ (\mu_t^+/m_t)$   $0\leq f\leq 1,\ 0\leq t< T$ 

Remark: For  $\mu_t = \mu \ \forall t$  and for r = T - t, the time left to play, we get the formulas

$$U(f,T-r) = \Phi (\Phi^{-1}(f) + \sqrt{r} \mu) \text{ and } \sigma^*(f,T-r) = \frac{1}{\sqrt{r}} \phi (\Phi^{-1}(f)).$$

Note that this result is the same as that for the limit of the discrete time problem of section 2.6. Note also that the bet size does not depend on  $\mu$  in this special case.

The main tool in proving the theorem is Ito's formula. Since the derivatives of U are not finite we will not to be able to use the formula directly but we will first have to look at a problem where the state space is reduced to  $S_{\epsilon} = \{(f,t): \ \epsilon \leq f \leq 1, \ 0 \leq t \leq T - \epsilon\}$  and where  $u(\epsilon,t) = \widetilde{U}(\epsilon,t), \ u(f,T-\epsilon) = \widetilde{U}(f,T-\epsilon)$  and u(f,t) = 0 otherwise.  $u(f,\tau)$  is the utility function at fortune f if we stop the process at time  $\tau$ . Let  $U_{\epsilon}$  and  $\sigma_{\epsilon}^*$  be the U and  $\sigma^*$  functions of this modified problem.

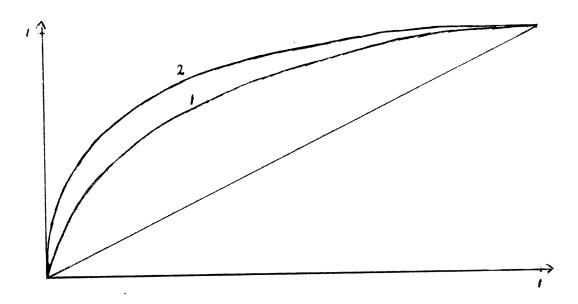


Figure 3: U(f, t) when  $\mu_t$  is constant equal to one and there is one and two time units left to play (m<sub>t</sub> = 1 and  $\sqrt{2}$ ).

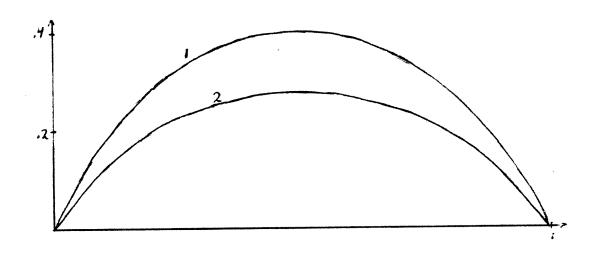


Figure 4:  $\sigma^*(f, t)$  when  $\mu_t$  is constant and there is one and two time units left to play  $(\mu_t^+/m_t=1$  and  $1/\sqrt{2})$ .

### Lemma 2

 $U_{\epsilon}\left(\mathbf{f,t}\right)=\overset{\sim}{U}\left(\mathbf{f,t}\right)\;\forall\left(\mathbf{f,t}\right)\in S_{\epsilon}\;\text{and}\;\sigma_{\epsilon}^{\;\star}\left(\mathbf{f,t}\right)=\overset{\sim}{\sigma}^{\star}\left(\mathbf{f,t}\right)\;\forall\;\left(\mathbf{f,t}\right)\in\;\text{interior of}\;S_{\epsilon}\;.$ 

Proof: Let  $U_{\epsilon}^{t}$  ( $U_{\epsilon}^{f}$ ) be the derivative of  $U_{\epsilon}$  with respect to time (fortune), let  $U_{\epsilon}^{ff}$  be the second derivative of  $U_{\epsilon}$  with respect to fortune and let  $I(\sigma) = U_{\epsilon}^{t}(X_{s},s) + U_{\epsilon}^{f}(X_{s},s) \sigma_{s}\mu_{s} + \frac{1}{2}U_{\epsilon}^{ff}(X_{s},s)\sigma_{s}^{2}$ . By definition  $U_{\epsilon}$  (f,t) =  $\tilde{U}$  (f,t) on

the boundaries of  $S_{\epsilon}$ . First we need to show that  $\widetilde{U}$   $(X_t, t)$  is a martingale for the strategy  $\sigma_{\epsilon}^*$  ( $\widetilde{U}$  is obtainable). Then we have to show that  $\widetilde{U}$   $(X_t, t)$  is a supermartingale for all strategies ( $\widetilde{U}$  is excessive). Let  $\tau$  be the stopping time when  $X_t$  hits the boundary of  $S_{\epsilon}$ . We note that  $U \in C^2$  on an open set containing  $S_{\epsilon}$  and that

$$\mathsf{E}\left(\int\limits_{t}^{T-\epsilon} |\mu_{s}\sigma_{s}^{\star}| \, ds\right) \leq \int\limits_{t}^{T-\epsilon} \mathsf{M} \, \sigma^{\star} \, \left(\frac{1}{2}, \, T\text{-}\epsilon\right) \, ds < \infty$$

and that 
$$E\left(\begin{array}{ccc} T-\epsilon \\ \int\limits_t^{} |\sigma_s{}^*|^2 \ ds \right) \leq \int\limits_t^{} |\sigma^*\left(\frac{1}{2},\,T-\epsilon\right)|^2 ds < \infty.$$

We can hence use Ito's formula:  $E[U_{\epsilon}(x_{\tau}, \tau) \mid X(t)] =$ 

since the integral with respect to Bs is zero

 $\begin{cases} = \ U_{\epsilon} \ (X_t, \, t) \ \text{if} \ I(\sigma) = 0 \quad \text{which makes} \ U_{\epsilon} \ (X_t, \, t) \ \text{a martingale} \\ \leq \ U_{\epsilon} \ (X_t, \, t) \ \text{if} \ I(\sigma) \leq 0 \quad \text{which makes} \ U_{\epsilon} \ (X_t, \, t) \ \text{a supermartingale} \end{cases}$  So it only remains to show that  $I(\sigma^*) = 0$  and that  $I(\sigma) \leq 0 \ \forall \sigma$ .

$$I(\sigma^{\star}) = - \phi(\Phi^{-1}(f) + m_S) \frac{(\mu_S^+)^2}{2m_S} + \frac{\phi(\Phi^{-1}(f) + m_S)}{\phi(\Phi^{-1}(f))} \phi(\Phi^{-1}(f)) \frac{\mu_S^+}{m_S} \mu_S^+ +$$

$$+\frac{1}{2}(-1)\frac{\phi(\Phi^{-1}(f))m_{S}}{\phi(\Phi^{-1}(f))^{2}}\phi(\Phi^{-1}(f))^{2}\frac{(\mu_{S}^{+})^{2}}{m_{S}^{2}}=\phi(\Phi^{-1}(f)+m_{S})\frac{(\mu_{S}^{+})^{2}}{m_{S}}(-\frac{1}{2}+1-\frac{1}{2})=0$$

$$0 = \frac{dI(\sigma)}{d\sigma} = U_{\varepsilon}^{f}(X_{S}, s) \mu_{s}^{+} + U_{\varepsilon}^{ff}(X_{S}, s)\sigma_{S} =$$

$$=\frac{\phi(\Phi^{-1}(f)+m_S)}{\phi(\Phi^{-1}(f))}\mu_S^+ + (-1)\frac{\phi(\Phi^{-1}(f)+m_S)m_S}{\phi(\Phi^{-1}(f))^2}\sigma_S =$$

$$=\frac{\varphi(\Phi^{-1}(f)+m_S)}{\varphi(\Phi^{-1}(f))}\bigg(\mu_S^+ \ -\ \frac{m_S\ \sigma_S}{\varphi(\Phi^{-1}(f))^2}\bigg) \text{ which implies that } \sigma_S=\varphi(\Phi^{-1}(f))\ (\mu_S^+/m_S)=\sigma_S^*$$

which maximizes 
$$I(\sigma)$$
 since  $\frac{d^2I(\sigma)}{d\sigma^2} = U_{\epsilon}^{ff}(X_s, s) = -\frac{\phi(\Phi^{-1}(f) + m_s)}{\phi(\Phi^{-1}(f))^2} m_s < 0.$ 

<u>Proof (of theorem 6)</u>: Since  $\widetilde{U}(f,T) = U(f,T) = u(f,T)$  and  $\widetilde{U}(o,t) = U(o,t) = u(o,t)$  and since  $\widetilde{U}$  and U are bounded continuous functions there exist a  $\delta_{\epsilon}$  > 0 for every  $\epsilon$  > 0 such that

i) 
$$\lim_{t\to\infty}\delta_{\varepsilon}=0$$

$$|U(f,t)-u(o,\,t)\mid\,<\delta_{\epsilon}\text{ and }|\;\widetilde{U}(f,\,t)-u(o,\,t)\mid\,<\delta_{\epsilon}\qquad\quad\forall\,\,f\leq\epsilon$$

$$|\; U(f,\,t) - u(f,\,o) \;| < \delta_\epsilon \; \text{and} \; |\; \widetilde{U}(f,\,t) - u(f,\,o)| < \delta_\epsilon \qquad \; \forall \; t \geq \; T - \epsilon.$$

Take any  $(f, t) \in S$ . We want to show that  $U(f, t) = \widetilde{U}(f, t)$ . On the boundaries of S it is true by definition. On the interior we have:

$$U(f, t) = \widetilde{U}(f,t) = U(f,t) - U_{\varepsilon}(f, t)$$
 for sufficiently small  $\varepsilon$  (lemma)

 $=\lim_{\epsilon\to 0}\,U(f,\,t)\,-\,U_\epsilon(f,\,t)\geq\lim_{\epsilon\to 0}\,E\!\int\! U(f_\tau,\,\tau)\;dv\;-\,E\!\int\! U_\epsilon(f_\tau,\,\tau)dv\;\text{where $\tau$ is as before}$ 

and v is the probability measure of (f\_{\tau}, t) under the strategy  $\sigma^{^{\star}}\epsilon$ 

$$= \lim_{\epsilon \to 0} \, \text{E}[\int \, U(f_\tau,\,\tau) \, - \, u_\epsilon(f_\tau,\,\tau) \, \, dv] \geq \lim_{\epsilon \to 0} \, \text{E}[\int \!\!\!\! -2\delta_\epsilon \, dv] = 0 \ .$$

Likewise U(f,t) - 
$$\widetilde{U}(f,t) \leq \lim_{\epsilon \to 0} EJU(f_{\tau},\,\tau) \,dw$$
 -  $EJU_{\epsilon}(f_{\tau},\tau)dw$ 

where w is the probability measure of  $(f_{\tau}, \tau)$  under the strategy  $\sigma^* = \lim_{\epsilon \to 0} \ E[\int U(f_{\tau,\tau}) - U_{\epsilon}(f_{\tau}, \tau) dw] \leq \lim_{\epsilon \to 0} \ E\int 2\delta_{\epsilon} \ dw = 0.$ 

Hence  $U(f,t)=\widetilde{U}(f,t)$  on the interior of S which implies that  $\sigma^*=\sigma^*_{\epsilon}$  on the interior of S.

### Corollary (to theorem 6)

If 
$$u(f, T) = \Phi(\Phi^{-1}(f) + m)$$
,  $m \ge 0$ , then 
$$U(f, t) = \Phi(\Phi^{-1}(f) + m + m_t) \text{ and } \sigma^*(f, t) = \phi(\Phi^{-1}(f)) \ \mu_t^+/(m + m_t) \ .$$

Remark: Note that the optimal strategy is no longer independent of  $\mu$  when  $\mu_t = \mu$  for all t. We think that the intuition behind this is that u(f, T) is no longer a convex function.

### 4 Inequalities for stochastic processes

In this chapter we will reformulate the problems into stochastic processes where  $\sigma$  (or s) that used to be a control variable is now arbitrary and possibly unknown. We wish to establish some inequalities of the probability of hitting an upper bound u before some given time T and before hitting a lower bound I.

Consider the random walk  $X_t = X_{t-1} + \sigma_{t-1} \ Y_{t-1}$  where  $Y_{t-1}$  has the two point distribution P(Y=-1)=1-w and  $P(Y=\frac{1-r}{r})=w$ , w>r and where  $\sigma_{t-1}$  is arbitrary. Let F, k and q to defined as in section 2.1 and let  $\tau_u=\min\{t\colon X_t\geq u\}$  and  $\tau_I=\min\{t\colon x_t\leq I\}$ .

### Inequality 1

If  $w \ge \frac{1}{2} \ge r$  and  $l < X_0 < u$  then  $P(\tau_u \le T, \tau_u < \tau_l \mid X_0) \le r$   $\le F(k-1|T, 1-w) + [q(f, T) \binom{T}{k}] (1-w)^k w^{T-k} < F(k|T, 1-w)$ where  $f = \frac{x_0 - l}{1 - l}$  and where [x] is the integer part of x.

<u>Proof</u>: Follows from theorem 3 where we found the strategy  $\sigma^*$  that maximizes the above probability.

### Inequality 2

For any 
$$w > r$$
 and if  $I < X_0 < u$  then  $P(\tau_u \le T, \tau_u < \tau_I \mid X_0) \le F(k-1|T, 1-w) + q(f,T) \binom{T}{k} (1-w)^k w^{T-k} < F(k|T, 1-w).$ 

<u>Proof</u>: Follows from the corollary to theorems 1 and 2.

Remark: Since  $P(X_T \ge u, X_s > I \forall s < T \mid X_0) \le P(\tau_u \le T, \tau_u < \tau_I \mid X_0)$  we get inequalities for this probability also.

We now turn to the continuous time process  $dX_t = \sigma_t \; (\mu_t dt + dB_t)$  where  $B_t$  is the standard Brownian motion and  $\sigma_t$  is arbitrary but nonnegative and fulfilling the conditions stated in the beginning of chapter three.

### Inequality 3

$$\begin{split} &\text{If } I < X_o < u \text{ then } P(X_T \geq u, \, X_S > I \, \forall \, \, s < T \mid X_o) \leq \, P(\tau_u \leq T, \, \tau_u \, < \tau_I \mid X_o) \\ &\leq \Phi \; (\Phi^{-1}(\frac{x_o \cdot I}{u \cdot I}) \, + \, m_t) \text{ where } m_t \text{ is defined as in chapter 3.} \end{split}$$

Proof: Follows from theorem 6.

<u>Remark</u>: The inequality will also hold for a process  $dX_t = \sigma_t$  ( $\widetilde{\mu}_t dt + dB_t$ ) where  $\sigma_t \widetilde{\mu}_t \le \sigma_t \mu_t$ .

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