Problem Set 1.2 :—

2. sketch these three lines and decide if the equations are solvable.

$x+2y=2$

$x-y=2$ 3 by 2 system

$y=1$

what happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

sol^n $x+2y=2$

$x-y=2$

$y=1$

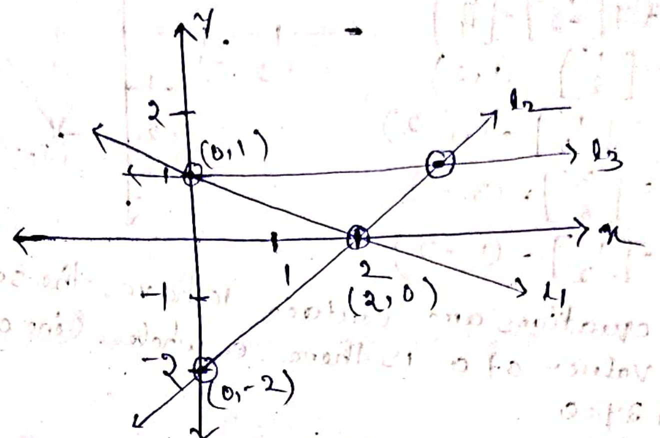
$l_1 : x+2y=2$

x	0	2
y	1	0

$l_2 : x-y=2$

x	0	2
y	-2	0

$l_3 : y=1$



It is solvable but it has singular solution.

(∵ as the right hand sides are 0 $|A|=0$)

$x+2y=2$

$x-y=-1$

$y=1$

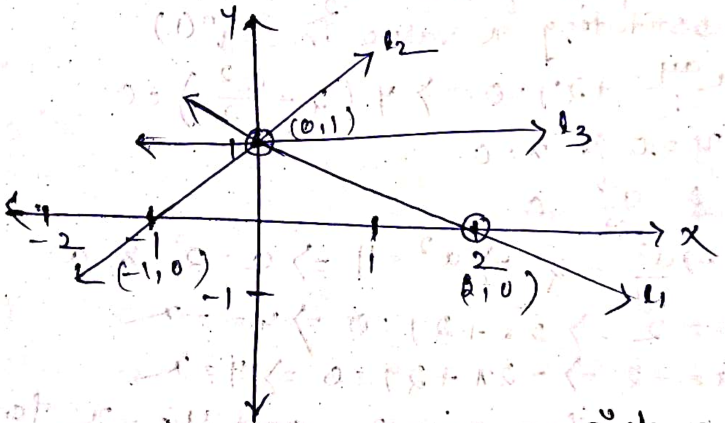
$l_1 : x+2y=2$

x	0	2
y	1	0

$l_2 : x-y=-1$

x	0	-1
y	1	0

$l_3 : y=1$



3. For the equations $x+y=4$; $2x-2y=4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector (4,4) on the right side)

sol^n Row pic

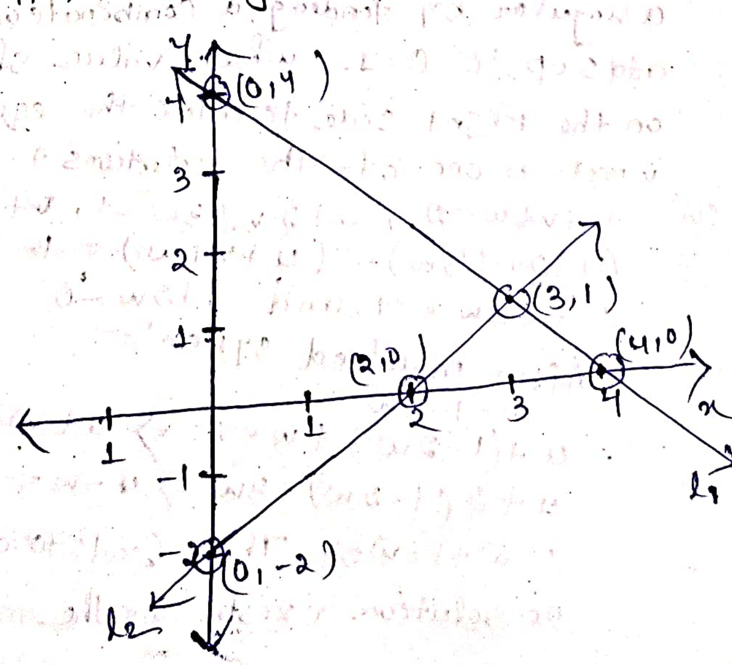
$l_1 : x+y=4$

x	0	4
y	4	0

l_2 $2x-2y=4 \Rightarrow 2(x-y)=4$

$\Rightarrow x-y=2$

x	2	0
y	0	-2



Column pic

$x+y=4$
$2x-2y=4$
$\Rightarrow 2(x-y)=4$
$x-y=2$

$x+y=4$
$x-y=2$
$2x=6$
$x=3$
$y=1$

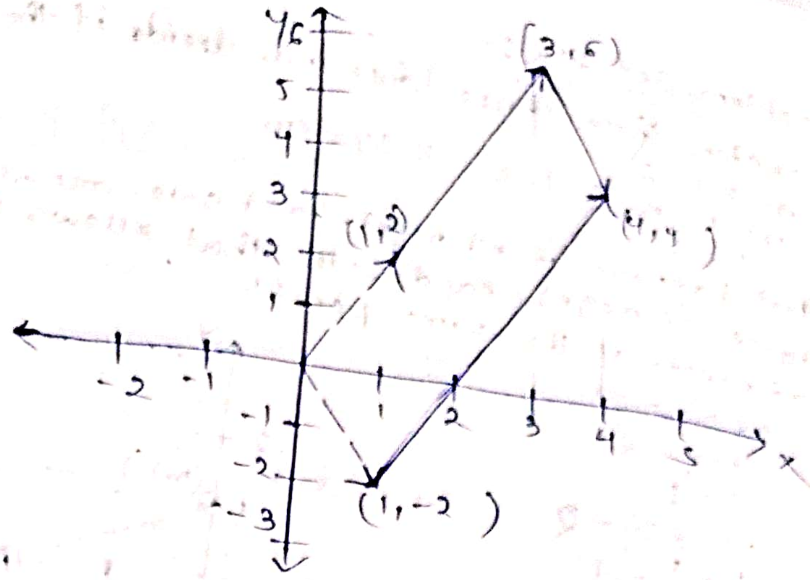
$x\begin{bmatrix}1\\2\end{bmatrix}+y\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix}$

Let $\vec{x_1}=\begin{bmatrix}1\\2\end{bmatrix}=(1,2)$

$\vec{x_2}=\begin{bmatrix}1\\-2\end{bmatrix}=(1,-2)$

$3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix}=(3,6)$

$1\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}1\\-2\end{bmatrix}=(1,-2)$



6) These equations are certain to have the solution $x=y=0$. For which values of a is there a whole line of solutions?

$ax+2y=0$
$2x+ay=0$

Solⁿ. $ax+2y=0$ —①
$2x+ay=0 \Rightarrow x=-\frac{ay}{2}$
substituting x value in eqⁿ(1)
$-a\frac{ay}{2}+2y=0 \Rightarrow y\left(2-\frac{a^2}{2}\right)=0$
$y=0$ & $x=0$
$2-\frac{a^2}{2}=0$
$\Rightarrow \frac{a^2}{2}=2 \Rightarrow a^2=4 \Rightarrow a=2$ & $a=-2$
$a=2 \Rightarrow 2x+2y=0 \Rightarrow y=-x$
$a=-2 \Rightarrow -2x+2y=0 \Rightarrow y=x$
$y=x$ for $a=-2$ and $y=-x$ for $a=2$

7) Explain why the system $u+v+w=2$, $u+2v+3w=1$, $v+2w=0$ is singular by finding a combination of the three equations that adds up to $0=1$. What value should replace the last zero on the right side to allow the equations to have solutions — and what is one of the solutions?

Solⁿ $u+v+w=2$, $u+2v+3w=1$, $v+2w=0$
$(u+2v+3w)-(u+v+w)=1-2 \Rightarrow v+2w=-1$
$\therefore v+2w=-1$ and $v+2w=0$
After modified system:—
$v=1-2w$
$u+(1-2w)+w=2 \Rightarrow u+w=1 \Rightarrow u=w+1$
$u+2(1-2w)+3w \Rightarrow u-w=-1 \Rightarrow u=w-1$
$\therefore w+1=w-1$ This leads to a contradiction, meaning no solution exists for the modified system.

8) (Recommended) Under what conditions on y_1, y_2, y_3 do the points $(0, y_1), (1, y_2), (2, y_3)$ lie on a straight line?

Soln The points $(0,y_1), (1,y_2), (2,y_3)$ lie on the straight line

$(0,y_1)$ in the form of (x_1, y_1)

$(1,y_2)$ in the form of (x_2, y_2)

$(2,y_3)$ in the form of (x_3, y_3)

So, $x_1 = 0$

$x_2 = 1$

$x_3 = 1$

$$\frac{y_3 - y_2}{x_3 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \frac{y_3 - y_1}{2-1} = \frac{y_2 - y_1}{1-0}$$

$$\Rightarrow y_3 - y_2 = y_2 - y_1$$

$$\Rightarrow y_3 - 2y_2 + y_1 = 0$$

$$\Rightarrow 2y_2 = y_1 + y_3$$

15) Draw the two pictures in two planes for the equations $x - 2y = 0$, $x + y = 6$

$x - 2y = 0$

$x + y = 6$

$\underline{(-)\ \ (-)\ \ \ (-)}$

$-3y = -6$

$y = 2$

$x + y = 6$

$\Rightarrow x + 2 = 6$

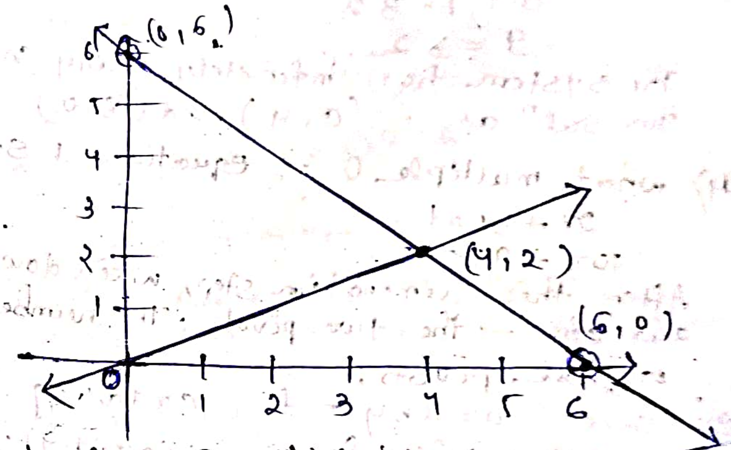
$\Rightarrow x = 4$

$l_1 : x - 2y = 0$

x	0	0
y	0	0

$l_2 : x + y = 6$.

x	0	6
y	6	0



18) In these equations, the third column (multiplying w) is the same as the right side b. The column form of the equations immediately gives what solution for (u, v, w)?

$6u + 7v + 8w = 8$ —— ①

$4u + 5v + 9w = 9$ —— ②

$2u - 2v + 7w = 7$ —— ③

Soln In form of vector equation:-

$$u\begin{bmatrix}6\\4\\2\end{bmatrix} + v\begin{bmatrix}7\\5\\2\end{bmatrix} + w\begin{bmatrix}8\\9\\7\end{bmatrix} = \begin{bmatrix}8\\9\\7\end{bmatrix}$$

multiplying w the third column same as the right side b.

$$w * \begin{bmatrix}8\\9\\7\end{bmatrix} = \begin{bmatrix}8\\9\\7\end{bmatrix}$$

$$\Rightarrow w = 1$$

Substituting $w = 1$ in eqn ①, ② & ③

$6u + 7v + 8(1) = 8 \Rightarrow 6u + 7v = 0$

$4u + 5v + 9(1) = 9 \Rightarrow 4u + 5v = 0$

$2u - 2v + 7(1) = 7 \Rightarrow 2u - 2v = 0 \Rightarrow 2u = 2v \Rightarrow u = v$

Putting $u = v$ in eqn ① $6u + 7u = 0$

$\Rightarrow 13u = 0 \Rightarrow u = 0 \Rightarrow v = 0$

So, $(u, v, w) = (0, 0, 1)$

Problem set 1.3   3, 4, 7, 8, 9, 10, 12, 14, 16, 32

3) Choose a coefficient b that makes this system singular. Then choose a right-hand side g that makes it solvable. Find two solutions in that singular case

$2x + by = 16$
$4x + 8y = g$

Sol^n multiplier used for elimination has to be 2.
8. So, b = 4

$$A = \begin{bmatrix} 2 & b \\ 4 & 8 \end{bmatrix}$$

let the augmented matrix be

$$A|B = \begin{bmatrix} 2 & b & 16 \\ 4 & 8 & g \end{bmatrix}$$

For singularity; $|A| = 0$.
so, $b = 4$
$2x + 4y = 16$
$0 = g - 32$
$g = 32$
The system has infinitely many solution
Two sol^n as : $(0, 4)$ and $(8, 0)$

4) what multiple ℓ of equation 1 should be subtracted from eq^n 2?
$2x + 3y = 1$
$10x + 9y = 11$
After this elimination step, write down the upper triangular system and circle the two pivots. The numbers 1 & 11 have no influence on those pivots.

Sol^n Given: $ax + by = f$, $cx + dy = g$
multiply eq^n(1) By c : $c(ax+by) = c(f) \Rightarrow acx + cby = cf$
multiply eq^n(2) by a : $a(cx+dy) = a(g) \Rightarrow acx + ady = ag$
∴ subtract eq^n(2) from the 1st,
$(acx + cby) - (acx + ady) = cf - ag \Rightarrow cby - ady = cf - ag$
$\Rightarrow y(cb - ad) = cf - ag$
Formula for pivot element :- $y = \dfrac{cf - ag}{cb - ad}$
The second element is missing when the denominator is zero i.e, when $cb - ad = 0 \Rightarrow cb = ad$

7) what test on b_1 & b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture
$3x - 2y = b_1$
$6x - 4y = b_2$

Sol^n The second equation is exactly twice the first equation.
$b_2 = 2b_1$
No. of column :- If $b_2 = 2b_1 \rightarrow$ Infinitely many columns.
If $b_2 \neq 2b_1 \rightarrow$ No solutions.

Column picture :-
• Coefficient matrix: $\begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}$. right-hand side - $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

OR

$$\begin{bmatrix} 3 & -2 & b_1 \\ 6 & -4 & b_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & b_1 \\ 6 & -4 & b_2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 3 & -2 & b_1 \\ 0 & 0 & b_2 - 2 \cdot b_1 \end{bmatrix}$$

$0 = b_2 - 2 \cdot b_1 \Rightarrow b_2 = 2b_1$

There is infinitely many sol^ns.

8). For which numbers a does elimination Break down (a) permanently, and (b) temporarily.

$ax + 3y = -3$
$4x + 6y = 6$

Sol^n :- Given eq^n :- $ax + 3y = -3$ $\quad 4x + 6y = 6$

multiplying by 2,

$2ax + 6y = -6$

Subtract $(2ax + 6y) - (4x + 6y) = -6 - 6 \Rightarrow (2a - 4)x = -12$

• Elimination breaks down permanently when $2a - 4 = 0$

• Elimination breaks down temporarily when $2a - 4 \neq 0$

The system when $a = 2$:-

$2x + 6y = 6 - 6$, $4x + 6y = 6$

exchange the row 1 to column

$4x + 6y = 6$
$2x + 6y = -6$

Subtract the second from first :-

$(4x + 6y) - (2x + 6y) = 6 - (-6) \Rightarrow 2x = 12 \Rightarrow x = 6$

Now, $2 \times 6 + 3y = -3 \Rightarrow y = -5$

The sol^n for $x = 6$ and $y = -5$ is obtained by row exchange

9) For which three numbers k does elimination break down? which is fixed by a row exchange? In each case is the number of solutions 0 or 1 or ∞?

$kx + 3y = 6$
$3x + ky = -6$

Sol^n multiply the 1st eq^n By $k = k(kx + 3y) = k6$

$\Rightarrow k^2x + 3ky = 6k$

multiply by 3 eq^n (2) :- $3(3x + ky) = 3(-6)$

$\Rightarrow 9x + 3ky = -18$

Subtract :- $(k^2x + 3ky) - (9x + 3ky) = 6k - (-18)$

$\Rightarrow (k^2 - 9)x = 6k + 18$

Elimination breaks down when coefficient of x is zero i.e, when

$\cancel{k} = k^2 - 9 = 0$

Case - 1 :- when $k = 3$:- 1) $3x + 3y = 6$ 2) $3x + 3y = -6$

These two eq^ns are ~~contra~~ ~~inconsistent~~ (No sol^n)

Case - 2 :- when $k = 3$:- 1) $-3x + 3y = 6$ 2) $3x - 3y = -6$

Adding :- $(-3x + 3y) + (3x - 3y) = 6 + (-6) \Rightarrow 0 = 0$ (Infinitely many sol^n)

Case - 3 :- $(k^2 - 9)x = 6x + 18 \Rightarrow x = \dfrac{6k + 18}{k^2 - 9}$ (one solution)

10) Which number b leads later to a row exchange? Which b leads to a missing pivot? In that singular case find a nonzero solutions x, y, z.

$x + by = 0$
$x - 2y - z = 0$
$y + z = 0$

Ans :- The corresponding matrix :

$$\left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{cccc} 1 & b & 0 & 0 \\ 0 & -2-b & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

~~If $-2-b=0 \Rightarrow b=-2$~~

To make system singular, we need to eliminate 3rd pivot get (zeroes insted of 1, 1 in 3rd row)

Let $b=-1$ (logical choice)

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For $b=-1$ System is singular, there is infinitely many soln's.

$x - y = 0 \Rightarrow x = y$

$y + z = 0 \Rightarrow y = -z$

$(z, z, -z)$, one soln is $(1, 1, -1)$.

12) Reduce this system to upper triangular form by two row operations.

$2x + 3y + z = 8$

$4x + 7y + 5z = 20$

$-2y + 2z = 0$

circles the pivots. solve by back-substitution for z, y, x.

Ans :- The augmented matrix :-
$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 4 & 7 & 5 & 20 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & -2 & 2 & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 8 & 8 \end{array}\right] \quad R_3 \to R_3 + 2R_2$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 8 \end{bmatrix}$$

$8z = 8 \Rightarrow z = 1$

$y + 3z = 4$

$\Rightarrow y + 3 = 4$

$\Rightarrow y = 1$

$\because 2x + 3y + z = 8$

$2x + 3 + 1 = 8$

$2x + 4 = 8$

$x = 2$

Pivots are :-
$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 8 & 8 \end{array}\right]$$

14) Which number q makes this system singular and which right-hand side t gives it infinitely many solutions ? Find the solution that has $z=1$

$x + 4y - 2z = 1$

$x + 7y - 6z = 6$

$3y + qz = t$

$$\begin{bmatrix} 1 & 4 & -2 & 1 \\ 1 & 7 & -6 & 6 \\ 0 & 3 & q & t \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 3 & q & t \end{bmatrix}$$

$$\Big\downarrow R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & q+4 & t-5 \end{bmatrix}$$

For $q = -4$, given system of eq^n is singular.

for $q = -4$ & $t = 5$ system has infinitely many solutions.

$q = -4, t = 5, z = 1$

$x + 4y - 2z = 1 \wedge 3y - 4z = 5 \xrightarrow{z=1} x + 4y - 2 = 1 \wedge 3y - 4 = 5$

$\Rightarrow x + 4y = 3 \wedge y = 3$

$\Rightarrow x + 12 = 3 \wedge y = 3$

$\Rightarrow x = -9 \wedge y = 3.$

$(x, y, z) = (-9, 3, 1)$, $q = -4$, $t = 5$

16) If rows 1 & 2 are the same, how far can you get with elimination

If columns 1 & 2 are the same, which pivot is missing?

$2x - y + z = 0$ $\qquad$ $2x + 2y + z = 0$

$2x - y + z = 0$ $\qquad$ $4x + 4y + z = 0$

$4x + y + z = 2$ $\qquad$ $6x + 6y + z = 2$

Ans

$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 4 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & -1 & 1 & 0 \\ 4 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ If row 1 & row 2 are same, there is no pivot in 3rd row

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 4 & 4 & 1 & 0 \\ 6 & 6 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 2 - R_1} \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 6 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 6 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 2 & 1 & 0 \\ 6 & 6 & 1 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 6 & 6 & 1 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{2} R_2} \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

There is no pivot in 2nd column.

32) use elimination to solve

$u + v + w = 6$ $\qquad$ $u + v + w = 7$

$u + 2v + 2w = 11$ $\quad$ & $\quad$ $u + 2v + 2w = 10$

$2u + 3v - 4w = 3$ $\qquad$ $2u + 3v - 4w = 3$

Ans

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 2 & 10 \\ 2 & 3 & -4 & 3 \end{array}\right]$$

$\Big\downarrow R_2 - R_1$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 1 & 3 \\ 2 & 3 & -4 & 3 \end{array}\right]$ $\xrightarrow{R_2 - R_1}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & -6 & -11 \end{array}\right]$ $\xrightarrow{R_3 - R_2}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 1 & 3 \\ 2 & 3 & -4 & -14 \end{array}\right]$

$-7w = -14 \Rightarrow w = 2$

$v + w = 3 \Rightarrow v + 2 = 3 \Rightarrow v = 1$

$u + v + w = 7 \Rightarrow u + v + 2 = 7 \Rightarrow u = 4$

$(u, v, w) = (4, 1, 2)$

Problem set 1.4 (4, 5, 11, 19, 21, 27, 28, 56)

4) Give 3 by 3 examples

a) a diagonal matrix : $a_{ij} = 0$ if $i \neq j$

B) a symmetric matrix : $a_{ij} = a_{ji}$ for all i & j

c) an upper triangular matrix : $a_{ij} = 0$ if $i > j$

d) a skew-symmetric : $a_{ij} = -a_{ji}$ for all i & j

Ans :- a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ c) $C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 4 \end{bmatrix}$ d) $D = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$

5) Compute the products

$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For the 3rd one, draw the column vectors (2,1) & (0,3). multiplying by (1,1) just adds the vectors.

Ans $\begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 17 \\ 4 \\ 17 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

11) True or False ? Give a specific counter example when false

(a) If column 1 & 3 of B are the same, so are columns 1 & 3 of AB

(b) If rows 1 & 3 of B are the same, so are rows 1 & 3 of AB.

(c) If rows 1 and 3 of A are the same, so are rows 1 & 3 of AB.

$(AB)^2 \cdot A^2B^2$

Ans a) False

A counter example :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 5 \\ 6 & 6 \end{bmatrix}$

$AB = \begin{bmatrix} 11 & 11 \\ 17 & 17 \end{bmatrix}$

columns 1 & 3 of B are same, but columns 1 & 2 of AB are same.

b) True
If rows 1 & 3 of B are same, the dot product of any row of A with rows 1 & 3 of B will be same, resulting the same rows 1 & 3 of AB.

c) False
Counter example :- $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$AB = \begin{bmatrix} 8 & 10 \\ 16 & 20 \end{bmatrix}$

Rows 1 & 3 of A are same, but rows 1 & 2 of AB are not same

d) False
Counter Example :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$ $A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$, $B^2 = \begin{bmatrix} 67 & 86 \\ 106 & 124 \end{bmatrix}$

$(AB)^2 = \begin{bmatrix} 861 & 986 \\ 1934 & 2222 \end{bmatrix}$ $A^2 B^2 = \begin{bmatrix} 1992 & 2186 \\ 3134 & 3942 \end{bmatrix}$

$(AB)^2 \neq A^2 B^2$

d) False.
Counter example :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 4 \\ 9 & 16 \end{bmatrix}$, $B^2 = \begin{bmatrix} 25 & 36 \\ 49 & 64 \end{bmatrix}$

$(AB)^2 = \begin{bmatrix} 361 & 484 \\ 1849 & 2500 \end{bmatrix}$ $A^2 B^2 = \begin{bmatrix} 221 & 292 \\ 666 & 1348 \end{bmatrix}$

$(AB)^2 \neq A^2 B^2$

19) Find the powers A^2, A^3 and B^2, B^3, C^2, C^3, what are A^k, B^k & C^k?

$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ & $C = AB = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

Ans. $A^2 = A \times A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$A^3 = A^2 \times A = A \times A = A^2 = A$

$B^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$B^3 = B \times B^2 = B \times I = B$

$C = AB \Rightarrow C^2 = C \times C = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$C^3 = C \times C^2 = C \times 0 = 0$

21) The matrix that rotates the x-y plane by an angle θ is
$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
Verify that $A(\theta_1) A(\theta_2) = A(\theta_1 + \theta_2)$ from the identities for $\cos(\theta_1 + \theta_2)$ & $\sin(\theta_1 + \theta_2)$. What is $A(\theta)$ times $A(-\theta)$?

Ans using $\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$
$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$

Ans $A(\theta_1)A(\theta_2) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$

$= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix}$

$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$

$= A(\theta_1+\theta_2)$

$\cos(0) = 1$ & $\sin(0) = 0$

$A(\theta)A(-\theta) = A(0)$

$= \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

27) which three matrices E_{21}, E_{31}, E_{32} put A into triangular form U.

$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$ and $E_{32}E_{31}E_{21}A = U$

multiply those E's to get one matrix M that does elimination $MA = U$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 4R_1} E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = M$

$\xrightarrow{R_3 \to 2R_1 + R_3} E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} =$

$\xrightarrow{R_3 \to R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$U = E_{32} * E_{31} * E_{21} * A$

$M = E_{32} * E_{31} * E_{21}$

$U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix}$ $M = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 6 & -1 & 1 \end{bmatrix}$ $MA = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix}$

28) This 4 by 4 matrix needs which elimination matrices E_{21} and E_{32} and E_{43}?

$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

$R_2 + \frac{R_1}{2} \to R_2$

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

Payal Priya - Rishi

$R_3 + \frac{2R_2}{3} \to R_3$

$$A \sim \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$R_4 \leftarrow R_4 + \frac{3R_3}{4}$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftarrow R_2 + \frac{R_1}{2}$, $E_{21} = R_2 + \frac{1}{2}R_1$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \leftarrow R_3 + \frac{2R_2}{3}$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 \leftarrow R_4 + \frac{3R_4}{4}$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix}$$

56) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that satisfy $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A$

$$A * \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} * A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} * \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$$

$a+b = a+c$
$a+b = b+d$
$c+d = a+c$
$c+d = b+d$

From eqn 1:
$a+b = a+c$
$\Rightarrow b = c$

From eqn 2:
$a+b = b+d$
$a = d$

From eqn 3:
$c+d = a+c$
$d = a$

From eqn 4:
$c+d = b+d$
$c = b$

$\therefore b = c$
$a = d$

~~$A = [[a, b], [b, a]]$~~ $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

~~$A * [1,1], [1,1]$~~

$$A * \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b, a+b, & b+a, b+a \\ a+b, a+b & a+b, a+b \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} * A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} * \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a+b, b+a & a+b, b+a \\ a+b, a+b & a+b, a+b \end{bmatrix}$$

$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ The solⁿ is correct as both resulting matrices are same.

1.5 (2, 7, 1, ~~9~~, 11, 19, 21, ~~25~~, 28, ~~32~~, 40, 41)

~~2) when is an upper triangular matrix nonsingular?~~

27) $A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$

$R_2 \to R_2 - 4R_1$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix}$

$R_3 \to R_3 + 2R_1$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$ $U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix} = M$

$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ $MA = U$

$E_{32}E_{31}E_{21}A = MA$

$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ $= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} = U$

56) $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$

$a+b = a+c$
$\Rightarrow b = c$ $\Rightarrow a = d$
$a+b = b+d$ $\Rightarrow d = a$
$c+d = a+c$
$c+d = b+d$ $\Rightarrow c = b$

1.5 (2, 7, 9, 11, ~~13~~ 19, 21, 25, 28, 32, 40, 41)

(2) when is an upper triangular matrix nonsingular?
we have find that when is an upper triangular matrix nonsingular & upper triangular matrix is matrix has all the elements below the main diagonal as zero. & also, the matrix which has elements above the main diagonal as zero is called lower triangular matrix.

~~Hence, ...~~

Hence, an upper triangular matrix is nonsingular & if all the diagonal entries of the matrix are nonzero.

7) Factor A into LU, & write down the upper triangular system $Ux = c$ which appears after elimination for

$Ax = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$

Preyanshi

<u>Ans</u> $A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}$

$R_3 \to R_3 - 3R_1$

$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} = U$

$R_3 \to R_3 + 3R_1$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ $LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}$

$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}$

$Ax = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$

$Ax = b$; $Ux = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

9) Apply elimination to produce the factors L and U for $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ & $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}$.

$A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 4R_1} \underbrace{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}}_{E_{21}} \underbrace{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}_{A} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$E_{21} A = U$

$U = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$. $E_{21} A = U \Rightarrow A = E_{21}^{-1} U$

using this, L has factor of $L = E_{21}^{-1}$.

$L = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}^{-1} = \frac{1}{1-0} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

factor of L & U for $A = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ & $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - \frac{1}{3}R_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \underbrace{\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 3 & \frac{1}{8} & 1 \\ 0 & \frac{8}{3} & \frac{2}{3} \\ 1 & 1 & 3 \end{bmatrix}}_{E_{21}A}$

$R_3 \to R_3 - \frac{1}{3}R_1$

$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix}}_{E_{31}} \underbrace{\begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{8}{3} & \frac{2}{3} \\ 1 & 1 & 3 \end{bmatrix}}_{E_{21}A} = \underbrace{\begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{8}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{8}{3} \end{bmatrix}}_{E_{31}E_{21}A}$

$R_3 \to R_3 - \cancel{R}\ \frac{1}{4}R_2$

$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{4} & 1 \end{bmatrix}}_{E_{32}} \underbrace{\begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 2/3 & 8/3 \end{bmatrix}}_{E_{31}E_{21}A} = \underbrace{\begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & 5/2 \end{bmatrix}}_{E_{32}E_{31}E_{21}A}$

$E_{32}E_{31}E_{21}A = U$ $\quad U = \begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{8}{3} & \frac{2}{3} \\ 0 & 0 & \frac{5}{2} \end{bmatrix}$

$E_{32}E_{31}E_{21}A = U$

$\Rightarrow A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$

$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{4} & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{4} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ \frac{1}{3} & & \\ \frac{1}{3} & \frac{1}{4} & 1 \end{bmatrix}$

factor of L & U for A is $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & 1 \end{bmatrix}$ & $\begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{8}{3} & \frac{2}{3} \\ 0 & 0 & \frac{5}{2} \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 1 & 4 & 8 \end{bmatrix}}_{E_{21}A}$

$R_3 \to R_3 - R_1$

$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{E_{31}} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 1 & 4 & 8 \end{bmatrix}}_{E_{21}A} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 7 \end{bmatrix}}_{E_{31}E_{21}A}$

$R_3 \to R_3 - R_2$ $\quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{E_{32}} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 7 \end{bmatrix}}_{E_{31}E_{21}A} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}}_{E_{32}E_{31}E_{21}A}$

$E_{32}E_{31}E_{21}A = U$ $\quad E_{32}E_{31}E_{21}A = U$

$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ $\quad \Rightarrow A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$

$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$ $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

factor of L & U for A is $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ & $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}$

11) Decide wheather the following systems are singular or nonsingular & wheather they have no solⁿ, one solⁿ or infinitely many solⁿ?

$v - w = 2$ $\quad v - w = 0$ $\quad v + w = 1$

$u - v = 2$ $\quad$ & $u - v = 0$ $\quad$ & $u + v = 1$

$u - w = 2$ $\quad u - w = 0$ $\quad u + w = 1$

$v - w = 2$

$u - v = 2$

$u - w = 2$

$\begin{bmatrix} 0 & 1 & -1 & 2 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_3 - R_1} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_3 - R_2}$

$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

$v - w = 0$

$u - v = 0$

$u - w = 0$

since there is no pivot in last, this is a singular case, no solⁿ.

Panigrahi

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

since there is no pivot in last row, it is singular case with infinitely many sol's.

$v + w = 1$
$u + v = 1$
$u + w = 1$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$u = \frac{1}{2}$ Non singular.
$v = \frac{1}{2}$
$w = \frac{1}{2}$

19) Find the $PA = LDU$ factorization for
$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ & $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

Ans $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{P_{21}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$\xrightarrow{DU} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$PA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

$LDU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_2 - 2R_1]{R_3 \leftrightarrow R_3 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_{23}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{DU} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

$LDU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

21) what three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular form $E_{32}E_{31}E_{21}A = U$? multiply by E_{32}^{-1}, E_{31}^{-1} & E_{21}^{-1} to factor A into LU where $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U.

$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$ $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix}$

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_3 \to R_3 - 3R_1$ $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}$, $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$ $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$; $E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$
$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

25) What two elimination matrices E_{21} & E_{32} put A into upper triangular form $E_{32}E_{21}A = U$? multiply by E_{32}^{-1} & E_{21}^{-1} to factor A into $LU = E_{21}^{-1}E_{32}^{-1}U$. $A = \begin{bmatrix}1&1&1\\2&4&5\\0&4&0\end{bmatrix}$

$$A = \begin{bmatrix}1&1&1\\0&2&3\\0&4&0\end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix}1&1&1\\0&2&3\\0&0&-6\end{bmatrix} = U$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + 2R_2$, $L = \begin{bmatrix}1&0&0\\2&1&0\\0&2&1\end{bmatrix}$

$$LU = \begin{bmatrix}1&0&0\\2&1&0\\0&2&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&2&3\\0&0&-6\end{bmatrix} = \begin{bmatrix}1&1&1\\2&4&5\\0&4&0\end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$E_{21} = \begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}$ $R_3 \to R_3 - 2R_2$ $E_{32} = \begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}$

$$E_{32}E_{21}A = \begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&1\\2&4&5\\0&4&0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix} \cdot \begin{bmatrix}1&1&1\\0&2&3\\0&0&-6\end{bmatrix}$$

$$U = \begin{bmatrix}1&1&1\\0&2&3\\0&0&-6\end{bmatrix}$$

$E_{32}^{-1} = \begin{bmatrix}1&0&0\\0&1&0\\0&2&1\end{bmatrix}$, $E_{21}^{-1} = \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}$ $E_{21}^{-1}E_{32}^{-1} = \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&2&1\end{bmatrix}$

$= \begin{bmatrix}1&0&0\\2&1&0\\0&2&1\end{bmatrix}$

$$E_{21}^{-1}E_{32}^{-1}U = \begin{bmatrix}1&0&0\\2&1&0\\0&2&1\end{bmatrix} = LU$$

$$= \begin{bmatrix}1&1&1\\2&4&5\\0&4&0\end{bmatrix} = A$$

$E_{21} = \begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}$ $E_{32} = \begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}$

28) Tridiagonal matrices have zero entries except on the main diagonal and the two adjacent diagonals. Factor these into $A = LU$ & $A = LDV$

$A = \begin{bmatrix}1&1&0\\1&2&1\\0&1&2\end{bmatrix}$ & $A = \begin{bmatrix}a&a&0\\a&a+b&b\\0&b&b+c\end{bmatrix}$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix}1&1&0\\0&1&1\\0&1&2\end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}^{-1} = L \Rightarrow L = \begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}$$

$$A = LU \quad \begin{bmatrix}1&1&0\\1&2&1\\0&1&2\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$$

$$A = LDV \quad \begin{bmatrix}1&1&0\\1&2&1\\0&1&2\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$$

$$A = \begin{bmatrix}a&a&0\\a&a+b&b\\0&b&b+c\end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix}a&a&0\\0&b&b\\0&b&b+c\end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix}a&a&0\\0&b&b\\0&0&c\end{bmatrix} = U$$

1 $\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}$ $A = LU$

$$\begin{bmatrix}a&a&0\\a&a+b&b\\0&b&b+c\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}a&a&0\\0&b&b\\0&0&c\end{bmatrix}$$

$A = LDV$

$$\begin{bmatrix}a&a&0\\a&a+b&b\\0&b&b+c\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix} = \begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$$

32) what are L & D for this matrix A? what is U in A = LU and what is the new U in A = LDU?

$$A = \begin{bmatrix}2&4&8\\0&3&9\\0&0&7\end{bmatrix}$$

Ans :- $A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&4&8\\0&3&9\\0&0&7\end{bmatrix}$ $A = LU$ $U = \begin{bmatrix}2&4&8\\0&3&9\\0&0&7\end{bmatrix}$.

A in the form of LDU

$$A = \begin{bmatrix}2&4&8\\0&3&9\\0&0&7\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&3&0\\0&0&7\end{bmatrix}\begin{bmatrix}1&2&4\\0&1&3\\0&0&1\end{bmatrix}$$

$L = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $D = \begin{bmatrix}2&0&0\\0&3&0\\0&0&7\end{bmatrix}$, $U = \begin{bmatrix}1&2&4\\0&1&3\\0&0&1\end{bmatrix}$

40) which permutation makes PA upper triangular? which permutation make P_1AP_2 lower triangular? multiplying A on the right by P_2 exchanges the _____ of A

$A = \begin{bmatrix}0&0&6\\1&2&3\\0&4&5\end{bmatrix}$ ~~Perform R_{12}, R_{23} on~~

Perform R_{12}, R_{23} on A !

$$A = \begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}$$

$P = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$; $PA = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&6\\1&2&3\\0&4&5\end{bmatrix} = \begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}$

A as $I_3AI_3 = A$

$$I_3AI_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}0&0&6\\1&2&3\\0&4&5\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}0&0&6\\1&2&3\\0&4&5\end{bmatrix} = A$$

Perform R_{23}, C_{13} on A

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}0&0&6\\1&2&3\\0&4&5\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} = \begin{bmatrix}6&0&0\\5&4&0\\3&2&1\end{bmatrix}$$

P_1AP_2 = Lower Triangular matrix

$P_1 = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$ & $P_2 = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$

P_2 gives column exchange C_{13} of A.

41) If P_1 & P_2 are permutations matrices so is P_1P_2. This still has the rows of I in some order. Give example with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$

Let $P_1 = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$, $P_2 = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$

$P_1P_2 = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ $P_2P_1 = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$ $P_1P_2 \neq P_2P_1$

$P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ & $P_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$P_3P_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $P_4P_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$P_3P_4 = P_4P_3$

1.6 2, 4, 5, 6, 10, 11, 12, 15, 17, 26, 41, 42, 44, 52, 54, 58

2) If the inverse of A^2 is B, show that the inverse of A is AB.

If m is the inverse of N

$NM^{-1} = M^{-1}N$

$NM^{-1} = I$.

$M^{-1}N = I$

A^2 is the inverse of matrix B

$A^2B = BA^2$

$A^2B = I$

$BA^2 = I$

using the eqn $A^2B = I$

$A^2B = I(AA)B = I$

(A^2 is AA) $A(AB) = I$

The product AB is the inverse of matrix A

Inverse of A is AB

4) Find the inverse of

$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$

$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$

echelon form of matrix :-

$[A_1\ e_1\ e_2\ e_3\ e_4] = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

$R_1 \leftrightarrow R_4 \left[\begin{array}{cccc|cccc} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$ $R_1 \to \frac{R_1}{4}$ $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$

$R_2 \leftrightarrow R_3 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$ $R_2 \to \frac{R_2}{3}$, $R_3 \to \frac{R_3}{2}$ $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$

$A_1^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}$

echlon form :-

$[A_2\ e_1\ e_2\ e_3\ e_4] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2/3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3/4 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 + \frac{1}{2}R_1$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3/4 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 + \frac{2}{3}R_2 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/3 & 2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_4 \to R_4 + \frac{3}{4}R_3$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/3 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1/4 & 1/2 & 3/4 & 1 \end{array}\right] \quad A_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/3 & 2/3 & 1 & 0 \\ 1/4 & 1/2 & 3/4 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

echelon form

$$\left[A_3 \; e_1 \; e_2 \; e_3 \; e_4\right] = \left[\begin{array}{cccc|cccc} a & b & 0 & 0 & 1 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to \frac{1}{a}R_1$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{b}{a} & 0 & 0 & 1/a & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 & 1 \end{array}\right] \quad R_2 \to R_2 + cR_1 \left[\begin{array}{cccc|cccc} 1 & b/a & 0 & 0 & 1/a & 0 & 0 & 0 \\ 0 & \frac{da-cb}{a} & 0 & 0 & -c/a & 1 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 \frac{a}{ad-bc} R_2$

$$\left[\begin{array}{cccc|cccc} 1 & b/a & 0 & 0 & 1/a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 & 1 \end{array}\right]$$

~~Add~~ $\cancel{(\frac{1}{ab})}$ ~~bc~~

$R_1 \to R_1 + \left(\frac{-1}{ab}\right) R_2$

$R_3 \to R_3 \frac{1}{a}$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & 1 & \frac{b}{a} & 0 & 0 & 1/a & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 & 1 \end{array}\right] \quad R_4 \to R_4 + cR_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & 1 & b/a & 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & \frac{ad-bc}{a} & 0 & 0 & -c/a & 1 \end{array}\right] \quad R_4 \to \frac{a}{ad-bc}R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & 1 & b/a & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$R_3 \to \left(-\frac{1}{ab}\right) R_4$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$A_3^{-1} = \begin{bmatrix} \frac{d}{ad-b} & -\frac{b}{ad-bc} & 0 & 0 \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

5) Show that $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ has no inverse by solving $Ax=0$, & by failing to solve

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$Ax = 0$

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x + y = 0$ A has no inverse

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a+c & b+d \\ 3a+3c & 3b+3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$a + c = 1$ $b + d = 0$

$3a + 3c = 0$ $3b + 3d = 1$

matrix A has no inverse

6) a) If A is invertible & $AB = AC$, Prove quickly that $B = C$

b) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, find an example with $AB = AC$ but $B \neq C$

a) $AB = AC$, multiply Both sides with A^{-1}

$A^{-1}(AB) = A^{-1}(AC)$

$A^{-1}AB = A^{-1}AC$

$(A^{-1}A)B = (A^{-1}A)C$ $\quad (\because A^{-1}A = I)$

$IB = IC \Rightarrow B = C$

b) Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $C = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1(a) + 0(c) + 1(b) + 0(d) \\ 0(a) + 0(c) + 0(b) + 0(d) \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$$

As $AB = AC$ $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$

$a = x$ & $b = y$

$AB = AC$ & $B \neq C$, $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $C = \begin{bmatrix} a & b \\ z & w \end{bmatrix}$ ~~$d \neq w$~~,

where $c \neq z$ & $d \neq w$

10) Use Gauss-Jordan method invert

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\det(A_1) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$\det(A_1) \neq 0$, ∴ A_1 is non singular

using Gauss Jordan method

$$A_2 = \begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$$

$$[A_1 I] = \begin{bmatrix}1&0&0&1&0&0\\1&1&1&0&1&0\\0&0&1&0&0&1\end{bmatrix} = [I\ A_1^{-1}] \qquad \det(A_2) = \begin{vmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{vmatrix} = 4$$

$$A_1^{-1} = \begin{bmatrix}1&0&0\\-1&1&-1\\0&0&1\end{bmatrix}$$

$\det(A_2) \neq 0$, A_2 is non singular

$$[A_2 I] = \begin{bmatrix}2&-1&0&1&0&0\\-1&2&-1&0&1&0\\0&-1&2&0&0&1\end{bmatrix}$$

$$\sim \begin{bmatrix}1&-1/2&0&1/2&0&0\\-1&2&-1&0&1&0\\0&-1&2&0&0&1\end{bmatrix} R_1 \to \tfrac{1}{2}R_1 \sim \begin{bmatrix}1&-1/2&0&1/2&0&0\\0&3/2&-1&1/2&1&0\\0&-1&2&0&0&1\end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix}1&-1/2&0&1/2&0&0\\0&1&-2/3&1/3&2/3&0\\0&-1&2&0&0&1\end{bmatrix} R_2 \to \tfrac{2}{3}R_2 \quad \begin{bmatrix}1&0&-1/3&2/3&1/3&0\\0&1&-2/3&1/3&2/3&0\\0&0&4/3&1/3&2/3&1\end{bmatrix}$$

$R_1 \to R_1 + \frac{1}{2}R_2$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix}1&0&-1/3&2/3&1/3&0\\0&1&-2/3&1/3&2/3&0\\0&0&1&1/4&1/2&3/4\end{bmatrix} R_3 \to \tfrac{3}{4}R_3 \begin{bmatrix}1&0&0&3/4&1/2&1/4\\0&1&0&1/2&1&1/2\\0&0&1&1/4&1/2&3/4\end{bmatrix}$$

$R_1 \to R_1 + \frac{1}{3}R_3$,

$R_2 \to R_2 + \frac{2}{3}R_3$

$$A_2^{-1} = \begin{bmatrix}3/4&1/2&1/4\\1/2&1&1/2\\1/4&1/2&3/4\end{bmatrix}$$

$$A_3 = \begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix} \qquad \det(A_3) = \begin{vmatrix}0&0&1\\0&1&1\\1&1&1\end{vmatrix} = -1$$

$\det(A_3) \neq 0$, A_3 is non-singular.

$$[A_3 I] = \begin{bmatrix}0&0&1&1&0&0\\0&1&1&0&1&0\\1&1&1&0&0&1\end{bmatrix} \sim \begin{bmatrix}1&1&1&0&0&1\\0&1&1&0&1&0\\0&0&1&1&0&0\end{bmatrix} R_3 \leftrightarrow R_1 \begin{bmatrix}1&0&0&0&-1&1\\0&1&1&0&1&0\\0&0&1&1&0&0\end{bmatrix}$$

$$R_1 \to R_1 - R_2 \begin{bmatrix}1&0&0&0&-1&1\\0&1&0&-1&1&0\\0&0&1&1&0&0\end{bmatrix} R_2 \to R_2 - R_3 = [I\ A_3^{-1}]$$

$$A_3^{-1} = \begin{bmatrix}0&-1&1\\-1&1&0\\1&0&0\end{bmatrix}$$

11) If B is square, show that $A = B + B^T$ is always symetric & $K = B - B^T$ is always skew symmetric, which means that $K^T = -K$. Find these matrices A & K when $B = \begin{bmatrix}1&3\\1&1\end{bmatrix}$, and write B as the sum of a symmetric matrix & a skew-symmetric matrix.

Let B be an $n \times n$ square matrix

$$B = \begin{bmatrix}b_{11}&b_{12}&\cdots&b_{1n}\\b_{21}&b_{22}&\cdots&b_{2n}\\b_{n2}&b_{n2}&&b_{nn}\end{bmatrix} \qquad B^T = \begin{bmatrix}b_{11}&b_{21}&\cdots&b_{n1}\\b_{12}&b_{22}&\cdots&b_{n2}\\\vdots&&&\\b_{1n}&b_{2n}&\cdots&b_{nn}\end{bmatrix}$$

$$A = B + B^T = \begin{bmatrix}2b_{11}&b_{12}+b_{21}&\cdots&b_{1n}+b_{n1}\\b_{21}+b_{12}&2b_{22}&\cdots&b_{n2}+b_{2n}\\b_{n1}+b_{1n}&b_{n2}+b_{2n}&\cdots&2b_{nn}\end{bmatrix}$$

$A = B + B^T$ $b_{ij} + b_{ji}$ $A = B + B^T$ is $b_{ji} + b_{ij}$

$b_{ij} + b_{ji} = b_{ji} + b_{ij}$

∴ $A = A^T$. $A = B + B^T$ is symmetric $K = B - B^T$

$$B - B^T = \begin{bmatrix} 0 & b_{12}-b_{21} & \cdots & b_{1n}-b_{n1} \\ b_{21}-b_{12} & 0 & \cdots & b_{n2}-b_{2n} \\ \vdots & & & \vdots \\ b_{n1}-b_{1n} & b_{n2}-b_{2n} & \cdots & 0 \end{bmatrix}$$

ijth term of $K = B - B^T$ is $b_{ij} - b_{ji}$

jith " " $K = B - B^T$ is $b_{ji} - b_{ij}$

$b_{ij} - b_{ji} = -(b_{ji} - b_{ij})$

ijth term of $K = -$ (jith term of K).

$K = -K^T$

$K = B - B^T$

$B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$. $B^T = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$ $A = B + B^T = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

$K = B - B^T = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ $\frac{1}{2}A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\frac{1}{2}K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \frac{1}{2}A + \frac{1}{2}K$

A is symmetric & K is skew symmetric.

$A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ symmetric matrix

$K = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ skew symmetric matrix

12) Compute the symm LDL^T factorization of

$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 36 \end{bmatrix}$ & $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 5R_1$

$R_3 \to R_3 - R_2$

$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ pivot positions are 1,2,3

$R_2 \to R_2 + 3R_1$
$R_3 \to R_3 + 5R_1$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$ $R_3 \to R_3 + R_2$ $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}$

$L^T = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ pivots are 1, 3 & 2

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

LDL factorization is

$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $R_2 \to R_2 - \frac{c}{a}R_1$ $A = \begin{bmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{bmatrix}$

$R_2 \to R_2 + \frac{b}{a}R_1$ $L = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$ $L^T = \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix}$

pivots are a & $d - \frac{bc}{a}$ $D = \begin{bmatrix} a & 0 \\ 0 & d - \frac{bc}{a} \end{bmatrix}$

LDT^{-1} factorization is

$A = \begin{bmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d - \frac{cb}{a} \end{bmatrix} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix}$

15) Under what cond^n on their entries are A & B invertible?

$A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$

$A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix}$ If $f = 0$, R_3 can never be pivot

$f \neq 0$

$\begin{bmatrix} f & 0 & 0 \\ d & e & 0 \\ a & b & c \end{bmatrix}$ $R_2 \to R_2 - \frac{d}{f} R_1$; $R_3 \to R_3 - \frac{a}{f} R_1$ $\begin{bmatrix} f & 0 & 0 \\ 0 & e & 0 \\ 0 & b & c \end{bmatrix}$

If $e = 0$, R_2 can never be pivot.

$e \neq 0$ $R_3 \to R_3 - \frac{b}{e} R_2$ $\begin{bmatrix} f & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & c \end{bmatrix}$

If $c \neq 0$ there is pivot in R_3, matrix A to be invertible if $c \neq 0$, $e \neq 0$ & $f \neq 0$

$B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$ If $e = 0$ R_3 never be pivot. B is invertible if $e \neq 0$

$R_2 \to R_2 - \frac{c}{a} R_1$ $\begin{bmatrix} a & b & 0 \\ 0 & d - \frac{c}{a} b & 0 \\ 0 & 0 & e \end{bmatrix}$

$d - \frac{c}{a} b \neq 0$

$ad - bc \neq 0$ If $c \neq 0$, $R_1 \leftrightarrow R_2$ $\begin{bmatrix} c & d & 0 \\ a & b & 0 \\ 0 & 0 & e \end{bmatrix}$

$R_2 \to R_2 - \frac{a}{c} R_1$ $\begin{bmatrix} c & d & 0 \\ 0 & b - \frac{a}{c} d & 0 \\ 0 & 0 & e \end{bmatrix}$

$b - \frac{a}{c} d \neq 0$

$bc - ad \neq 0$ $\therefore$ B is invertible either $a \neq 0$ & $ad - bc \neq 0$, or $c \neq 0$ & $bc - ad \neq 0$.

But $ad - bc \neq 0$ is equivalent to $bc - ad \neq 0$. & $e \neq 0$

17) Give examples of A & B such that
a) A+B is not invertible although A & B are invertible
b) A+B is invertible although A & B are not invertible
c) All of A, B & A+B are invertible
In the last case use $A^{-1}(A+B)B^{-1} = B^{-1} + A^{-1}$ to show that $C = B^{-1} + A^{-1}$ is also invertible and find a formula for C^{-1}.

a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\det A = 1 - 0 = 1$

$\det B = (-1)(-1) - 0 = 1$

$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\det(A+B) = 0$

b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(A+B) = 1 \neq 0$ $\det A = (1)(0) - (0)(0) = 0$

A & B are not invertible $\det B = (0)(1) - (0)(0) = 0$

c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\det A = 1 \neq 0$

$\det B = 1 \neq 0$

$A+B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\det(A+B) = 2 \neq 0$

A, B & A+B are invertible

d) $A^{-1}(A+B)B^{-1} = B^{-1} + A^{-1}$ $C = B^{-1} + A^{-1}$

$(B^{-1} + A^{-1})^{-1} = (A^{-1}(A+B)B^{-1})^{-1} = (B^{-1})^{-1}(A+B)^{-1}(A^{-1})$

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$, $(A^{-1})^{-1} = A$ $C^{-1} = B(A+B)^{-1}A$

$C = B^{-1} + A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$\det C = 2 \neq 0$ C is invertible

26) Find the numbers a & b that give the inverse of
eye(4) - ones(4,4):

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$$

what are a & b in the inverse of 6 * eye(5) - ones(5,5)?

using Gauss - Jordan method

$$\left[\begin{array}{cccc|cccc} 4 & -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_4 \rightarrow R_4 + R_3 + R_2 + R_1$

$$\left[\begin{array}{cccc|cccc} 4 & -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$R_1 \rightarrow R_4 + R_1$
$R_2 \rightarrow R_4 + R_2$
$R_3 \rightarrow R_4 + R_3$

$$\left[\begin{array}{cccc|cccc} 5 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 5 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 5 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$R_1 \rightarrow \frac{R_1}{5}$ $R_2 \rightarrow \frac{R_2}{5}$ $R_3 \rightarrow \frac{R_3}{5}$ $R_4 \rightarrow \frac{R_4}{5}$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 1/5 & 2/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 0 & 1/5 & 1/5 & 2/5 & 1/5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

Adding -ve of 1st 3 rows to last row

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 1/5 & 2/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 0 & 1/5 & 1/5 & 2/5 & 1/5 \\ 0 & 0 & 0 & 1 & 1/5 & 1/5 & 1/5 & 2/5 \end{array}\right]$$

$a = 2/5, \; b = 1/5$

6 = eye(5) - ones(5,5)

$a = 1/3, \; b = 1/6$

eye(k-1) - ones(k-1, k-1):
$a = 2/5, \; b = 1/5$

) True or false
a) A 4 by 4 matrix with a row of zeroes is not invertible
b) A matrix with 1s down the main diagonal is "
c) If A is invertible then A^{-1} is invertible
d) If A^T is invertible then A is invertible

a) 4 x 4 matrix with a row of zeros is not invertible.
True

b) matrix with 1s down the main diagonal is invertible.
False

c) $AA^{-1} = I$ & $A^{-1}A = I$
A^{-1} is invertible True

d) True if A^T is invertible then A also invertible

42) For which three nos. c is this matrix not invertible, & why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

For $c = 2$
$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 8 & 7 & 2 \end{bmatrix}$$
A is not invertible

For $c = 7$, $\begin{bmatrix} 2 & 7 & 7 \\ 7 & 7 & 7 \\ 8 & 7 & 7 \end{bmatrix}$

A is not invertible

For $c = 0$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 7 & 0 \end{bmatrix}$

A is not invertible

44) This matrix has a remarkable inverse, find A^{-1} by elimination on $[A\ I]$, extend to a 5 by 5 "alternating matrix" & guess its inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[A\ I] = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} R_2 \to R_2 + 2 \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_3 + R_2$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} R_3 \to R_4 + R_3 \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ (add row, column)

$$[B\ I] = \begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 + R_3$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_4 + R_3$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

52) Verify that $(AB)^T$ equals B^TA^T but those are diff. from A^TB^T $\quad A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ $AB = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

In case $AB = BA$, how do you prove that $B^TA^T = A^TB^T$

$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, B^T = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$A \cdot B = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$, $(A \cdot B)^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ $\quad B^TA^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = (A \cdot B)^T$

$A^T \cdot B^T = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} \neq (A \cdot B)^T$

$B^T \cdot A^T = (A \cdot B)^T$
$= (B \cdot A)^T$
$= A^T \cdot B^T \; (\because AB = BA)$

54) The row vector x^T times A times the column y produces what number?

$$x^T A y = [0\ 1]\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \_\_\_\_$$

b) This is the row $x^T A$ = $[4\ 5\ 6]$ times the column $y = (0,1,0)$

c) This is the row $x^T = [0\ 1]$ times the column Ay = ____

a) $x^T A y = [0\ 1]\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = [4\ 5\ 6]\begin{bmatrix}0\\1\\0\end{bmatrix} = 5$

b) $x^T A = [0\ 1]\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix} = [4\ 5\ 6]$

c) $x^T A y = x^T(Ay) = [0\ 1]\left(\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$

$= [0\ 1]\begin{bmatrix}2\\5\end{bmatrix}$

58) If $A = A^T$ & $B = B^T$, which of these matrices are certainly symmetric?

(a) $A^2 - B^2$ (b) $(A+B)(A-B)$ (c) ABA (d) $ABAB$

a) $A^2 - B^2$ is a symmetric matrix

b) $(A+B)(A-B)$ is not a symmetric matrix

c) ABA is a symmetric matrix

d) $ABAB$ is not a symmetric matrix.

13.03.25

2.1 2, 4, 5, 8, 10, 24, 26, 28

Q which of the following subsets of R^3 are actually subspace?

(a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$

(b) The plane of vectors b with $b_1 = 1$

(c) The vectors b with $b_2 b_3 = 0$ (this is the union of two subspaces, the plane $b_2 = 0$ & the plane $b_3 = 0$).

d) All combinations of two given vectors $(1,1,0)$ & $(2,0,1)$.

e) The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$

a) $A = \{(b_1, b_2, b_3) \mid b_1, b_2, b_3 \in R, b_1 = 0\}$

$(b_1, b_2, b_3) \in A$, $(c_1, c_2, c_3) \in A$ $b_1 = 0$ $c_1 = 0$

$(b_1, b_2, b_3) + (c_1, c_2, c_3) = (b_1 + c_1, b_2 + c_2, b_3 + c_3)$

$(0, b_2 + c_2, b_3 + c_3) \in A$ (since $b_1 = 0 \Rightarrow b_1 + c_1 = 0$)

A is under vector addition.

For scalar multiplication $c \in R$ & $(b_1, b_2, b_3) \in A$

$c(b_1, b_2, b_3) = (cb_1, cb_2, cb_3) = (0, cb_2, cb_3) \in A$

since $b_1 = 0 \Rightarrow cb_1 = 0$

subspaces of R^3.

b) $(1,2,3), (1,5,6) \in B$

$(1,2,3) + (1,5,6) = (2,7,9) \notin B$

$(2,7,9)$ the 1st component is not equal to 1

not subspace of R^3

c) $C = \{(b_1, b_2, b_3) \mid b_1, b_2, b_3 \in R, b_2 b_3 = 0\}$

$A = \{(b_1, b_2, b_3) \mid b_2 = 0\}$ $B = \{(b_1, b_2, b_3) \mid b_3 = 0\}$

$(1,0,2) \in C$ & $(1,5,0) \in C$

$(1,0,2)+(1,5,0)=(2,5,2) \notin C$

multiplying $b_1 2, b_1 3 = 5 \cdot 2 = 10 \neq 0$

not vector addition

not subspaces of R^3

d) $D=\{a(1,1,0)+b(2,0,1) \mid a,b \in R\}$

$(a_1(1,1,0)+b_1(2,0,1)) + (a_2(1,1,0)+b_2(2,0,1)) = (a_1+a_2)(1,1,0) + (b_1+b_2)(2,0,1) \in D$

satisfies vector addition for scalar multiplication.

$a_1(1,1,0)+b_1(2,0,1) \in D$

$c(a_1(1,1,0)+b_1(2,0,1)) = (ca_1(1,1,0)+cb_1(2,0,1)) \in D$

subspaces of R^3

e) $E=\{(b_1,b_2,b_3) \mid b_3-b_2+3b_1=0\}$

$(b_1,b_2,b_3) \in E, (c_1,c_2,c_3) \in E$

$(b_1,b_2,b_3)+(c_1,c_2,c_3)=(b_1+c_1, b_2+c_2, b_3+c_3) \in E$

$(b_3+c_3)-(b_2+c_2)+3(b_1+c_1)=(b_3-b_2+3b_1)+(c_3-c_2+3c_1)=0$

under vector addition.

$a \in R, (b_1,b_2,b_3) \in E$

$a(b_1,b_2,b_3)=(ab_1, ab_2, ab_3) \in E$

$ab_3-ab_2+3ab_1 = a(b_3-b_2+3b_1) = a \cdot 0 = 0$

E is under closed multiplication.

subspace of R^3.

4/a) $(1,0,1,0,\dots)+(0,1,0,1,\dots)=(1,1,1,1,\dots)$ not subspace of V

$(1,1,1,1,\dots) \notin V$ not subspace of R^∞.

b) $x+y=(x_1+y_1, x_2+y_2, \dots)$ $i=\max\{i,j\}$

$x_i=0$. $cx=(cx_1, cx_2, \dots)$ $cx_{\{i,j\}}=0$

subspace of R^∞

c) $(S_1+S_2)=(4,2,6,-2,-4\dots)$ decreasing

not subspace of R^∞

d) $\lim_{j\to\infty}(x_j+y_j)=\lim_{j\to\infty}x_j+\lim_{j\to\infty}y_j = P+Q$

(x_j+y_j) is convergent $(x_j+y_j) \in S$

$\lim_{j\to\infty} x_j = L$ $\lim_{j\to\infty}(cx_j)=c(\lim_{j\to\infty}x_j)=cL$

cx_j is convergent & $cx_j \in S$

subspace of R^∞

e) $x+y = x_1+y_1, x_2+y_2, \dots$

$x_2+y_2-x_1-y_1=d-p$ $x_3+y_3-x_2-y_2=d-p$

$\{cx_1, cx_2, cx_3 \dots + cx_{\{2\}} - cx_{\{1\}} = cx_{\{3\}} - cx_{\{2\}} = cd$

subspace of R^∞.

f) $(S_1+S_2)=(x_1+y_1, kx_1+ty_1+k^2x_1+t^2y_1, \dots)$

not vector addition.

not subspace of R^∞.

Column space of A = {b | b = Ax for x ∈ R²}

5) $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ Column space of A. $a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Let $x = \begin{bmatrix} u \\ v \end{bmatrix} \in R^2$ For nullspace $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$a\{1\} = -a\{2\}$

$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $u - v = 0$, $0 = 0$

$u - v = 0$ zerovector u & v same

N(A) is a line u = v in R^2. It construct null space $\begin{bmatrix} u \\ v \end{bmatrix}$ to be zero vector.

$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ $x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in R^3$.

$B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 3w \\ u+2v+3w \end{bmatrix}$

$C(B) = \{(3w, u+2v+3w) \mid u, v, w \in R\} = R^2$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $z = 0$, $x + 2y + 3z = 0$, $x = -2y$, $y = -\frac{1}{2}x$

$x \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}$ N(B) is line passing through $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Let $x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in R^3$

$C(C) = \{b \in R^2 \mid b = Cx\}$ for $x \in R^3$.

$Cx = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$N(C) = \{x \mid Cx = 0\}$ $Cx = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$N(C) = R^3$.

8) a) $x_1 + x_2 + x_3 = 0$
$x_1 + 2x_3 = 0$
$x\{1\} = -2x\{3\}$

substitute x_1 in 1st eqn
$-2x_3 + x_2 + x_3 = 0$
$x_2 = x_3$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} x_3$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ not form a plane

b) $x\{1\} = -2x\{3\}$ $-2x_3 + x_2 + x_3 = 0$, $x_2 = x_3$

Form a line

c) $(x_1, x_2, x_3) = x_3(-2, 1, 1)$

Ax = 0 not form a point

d) {x | Ax = 0} is null space of A

null space is subspace of R^3

Ax = 0 Form a null space

e) Forms a null space of A. Ax = 0 not a column space of A

f) Column space is not Correct,

it is not Correct.

10) $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 9 & 11 \end{bmatrix}$

$B = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\}$ $(a, b, c, d \in R$ & $ad - bc = 0)$

vector addn not closed A is not vector space, Let $\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} \in B$ $\qquad ad-bc = 3\cdot2 - 4\cdot1 = 2 \neq 0$

$\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ $\qquad$ B is not vector space

24) a) $\begin{bmatrix} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{bmatrix}$ $R_2 - 2R_1$ $\quad R_3 + R_1$ $\begin{bmatrix} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 \end{bmatrix}$

If $b_3 + b_1 = 0$ & $b_2 - 2b_1 = 0$

b) system is solvable if $b_3 + b_1 = 0$

26) $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\left\{ (b_1, b_2, b_3) \,\middle|\, \begin{bmatrix} b_1 \\ b_2 \\ b_2 \end{bmatrix} \right.$ is linear comb^n of $\left. \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\left\{ (b_1, b_2, b_3) \,\middle|\, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right.$ " " $\left. \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$

$C(A) = \{ (c_1 + 2c_2, 0, 0) \mid c_1, c_2 \in R \}$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$ $\qquad$ B is linear comb^n of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_2 \\ 0 \end{bmatrix}$

$C(B) = \{ (c_1, 2c_2, 0) \mid c_1, c_2 \in R \}$

Column space of B is x-y plane = all vectors $(x, y, 0)$

$C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$ $\qquad$ C is linear comb^n of $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_1 \\ 0 \end{bmatrix}$

$C(C) = \{ (c_1, 2c_1, 0) \mid c_1 \in R \} = \{ c(1, 2, 0) \mid c \in R \}$

28) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $\qquad$ $b_1 = x_1 + x_2 + x_3$

$b_2 = x_2 + x_3$

$b_3 = 0$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 - b_2 \\ b_2 - b_3 \\ b_3 \end{bmatrix}$ $\qquad$ has a sol^n if $(b_1, b_2, 0)$

(b_1, b_2, b_3) has a sol^n

2.2 (1, 2, 4, 5, 8, 12, 13, 34, 36, 44, 59, 56)

1) $u + v + 2w = 2$ — (1)

$2u + 3v - w = 5$ — (2)

$3u + 4v + w = c$ — (3)

$Ax = b$

$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ c \end{bmatrix}$

subtract ~~2~~ from both sides of eq^n (1) $\qquad u + v = 0$

Subtract v from both side $\qquad u = -v$

substituting $-v$ in eq^n (2) $\qquad 2(-v) + 3v - w = 5$

$\Rightarrow v - w = 5$

Substitute v in eq^n (3)

$(v-w)+(v+w)=5+c$

$\Rightarrow 2v=5+c \Rightarrow v=\frac{5+c}{2}$

Substitute v into $v-w=5$

$\frac{5+c}{2}-w=5$

$3(-v)+4v+w=c$

$\Rightarrow v+w=c$

$$\left[\begin{array}{ccc|c}1&1&2&2\\2&3&1&5\\3&4&1&c\end{array}\right] \xrightarrow{R_3\to R_3-R_2-R_1} \left[\begin{array}{ccc|c}1&1&2&2\\2&3&-1&5\\0&0&0&c-7\end{array}\right] \xrightarrow{R_2\to R_2-2R_1} \left[\begin{array}{ccc|c}1&1&2&2\\0&1&-5&1\\0&0&0&c-7\end{array}\right]$$

$c-7=0 \Rightarrow c=7$

$$A=LU=\begin{bmatrix}1&0&0\\2&1&0\\3&1&1\end{bmatrix}\begin{bmatrix}1&1&2\\0&1&-5\\0&0&0\end{bmatrix}$$

$Ux=c$ & $Lc=b$

$$\underbrace{\begin{bmatrix}1&0&0\\2&1&0\\3&1&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}2\\1\\0\end{bmatrix}}_{c}=\underbrace{\begin{bmatrix}2\\5\\7\end{bmatrix}}_{b}$$

$Ux=b$

$$\begin{bmatrix}1&1&2\\0&1&-5\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\1\\0\end{bmatrix} \Rightarrow \begin{cases}x_1=1-7x_3\\x_2=1+5x_3\\x_3\in R\end{cases}$$

Sol^n is $x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-7\\5\\1\end{bmatrix}$

2) $A=\begin{bmatrix}0&1&0&3\\0&2&0&6\end{bmatrix}$

$R_2\to R_2-2R_1$

$U=\begin{bmatrix}0&1&0&3\\0&0&0&0\end{bmatrix}$

echelon form of U is

$U=\begin{bmatrix}0&1&0&3\\0&0&0&0\end{bmatrix}$

let $x_1\ x_2\ x_3\ x_4$ are free variable

for $x_1=1, x_3, x_4=0$

for eq^n $Ux=0$;

$0*x_1+1*x_2+0*x_3+3*x_4=0$

$x=0$

1st special sol^n is $(1,0,0,0)$

for $x_3=1, x_1, x_4=0$

from $Ux=0$

$x_2+3*0=0 \quad x_2=0$

2nd special sol^n $=(0,0,1,0)$

for $x_4=1, x_1=x_3=0 \quad Ux=0$

$x_2+3*1=0 \Rightarrow x_2=-3$

3rd solⁿ: $(0,-3,0,1)$

$s_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $s_2 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$, $s_3 = \begin{bmatrix}0\\3\\0\\1\end{bmatrix}$

$Ax = b$

$$\begin{bmatrix}0 & 1 & 0 & 3\\ 0 & 2 & 0 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$$

so, $x_2 + 3x_4 = b_1$

$2x_2 + 6x_4 = b_2$

we have $b_2 = 2 * b_1$

we have $x_2 + 3x_4 = b_1 \Rightarrow x_2 = b_1 - 3x_4$

$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ where x_1, x_3, x_4 is free

$x_2 = b_1 - 3x_4$

$$x = \begin{bmatrix}x_1\\b_1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\-3\\0\\1\end{bmatrix} * x_4$$

which can be written

$$x = \begin{bmatrix}0\\b_1\\0\\0\end{bmatrix} + x_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + x_4\begin{bmatrix}0\\3\\0\\1\end{bmatrix}$$

4) (a) $\begin{bmatrix}1 & 2 & 2\\ 2 & 4 & 5\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2 & 2\\ 2 & 4 & 4\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$

$$-2\left[\begin{array}{ccc|c}1 & 2 & 2 & 1\\ 2 & 4 & 5 & 4\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 2 & 2 & 1\\ 0 & 0 & 1 & 2\end{array}\right] \xrightarrow{-2} \begin{bmatrix}u\\w\end{bmatrix} = \begin{bmatrix}-3\\2\end{bmatrix} - v\begin{bmatrix}2\\0\end{bmatrix}$$

i.e, $x_p = \begin{bmatrix}-3\\0\\2\end{bmatrix}$, $x_n = \begin{bmatrix}-2\\1\\0\end{bmatrix}$ (set $v=0$ for x_p, ignore rhs & set $v = 1$ for x_n)

b) $-2\left[\begin{array}{ccc|c}1 & 2 & 2 & 1\\ 2 & 4 & 2 & 4\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 2 & 2 & 1\\ 0 & 0 & 0 & 2\end{array}\right]$ inconsistent, no solution x_p

5) $A = \begin{bmatrix}1 & 2 & 0 & 1\\ 0 & 1 & 1 & 0\\ 1 & 2 & 0 & 1\end{bmatrix}$

$R_3 \to R_3 - R_1$ $\begin{bmatrix}1 & 2 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ echolen form

no. of nonzero rows, 2

Pivot Colⁿs are 1st & 2nd columns

3rd & 4th colⁿ are free variables

so, x_3, x_4

$Ax = 0$

Augmented matrix = $\left[\begin{array}{cccc|c}1 & 2 & 0 & 1 & 0\\ 1 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$

$x_1 + 2x_2 + x_4 = 0$

$\Rightarrow x_1 = -2x_2 - x_4$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

Solⁿ vector is: $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-2x_2 - x_4\\-x_3\\x_3\\x_4\end{bmatrix}$

~~write~~ $x_2 = -x_3$

$$\begin{bmatrix} -2(-x_3) - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix}$$

General sol^n $Ax=0$ is

$x = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ x_3 & x_4 are free parameter.

matrix $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Reduce B to echlon form

$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - 7R_1$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$ $R_2 \to R_2/3$, $R_3 \to R_3/-6$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ $R_3 \to R_3 - R_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

rank of $B = 2$, ∵ (2 non zero rows in echlon form).
3rd col^n is free variable (∵ does not have pivot)

$Bx = 0$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

from 1st row, $x_1 + 2x_3 + 3x_3 = 0 \Rightarrow x_1 = -2x_3 - 3x_3$

2nd row $x_2 + 2x_3 = 0 \Rightarrow x_2 = -2x_3$.

sol^n vector: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} -2x_2 - 3x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$

$x_2 = -2x_3$

$$\begin{bmatrix} -2(-2x_3) - 3x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_3 - 3x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$$

general sol^n to $Bx = 0$ x_3 is parameter.

$x = x_3 \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

8) a) $a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

Rank$(a) = 2$

$ax = 0$

echelon form $a = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

x_3 – pivot variable

x_2 – free "

Let $x_2 = 1$, $\Rightarrow x_3 = 0$

$x_1 + 2x_2 + 3x_3 = 0 \Rightarrow x_1 + 2(1) + 3(0) = 0 \Rightarrow x_1 = -2$

$x = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

b). $B = [A\ A]$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 2 & 4 & 6 & 2 & 4 & 6 \end{bmatrix}$$

Rank$(B) = 2$

$$x_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \quad x_4 = \begin{bmatrix} -3 \\ 0 \\ -1 \\ -3 \\ 0 \\ -1 \end{bmatrix}$$

c) $C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 2 & 4 & 6 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 2 & 4 & 6 & 0 & 0 & 0 \end{bmatrix}$$

Rank$(C) = 3$

$Cx = 0$

Echelon form of C is

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot col^{n}s are 1, 3, 4 & 6

Free col^{n}s are 2 & 5

$x_2 = 1, x_5 = 0$: $x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$x_2 = 0, x_5 = 1$: $x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

Special sol^n are:-

$$x_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

12) $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Augmented matrix $\begin{bmatrix} 1 & 2 & 0 & 3 & b_1 \\ 2 & 4 & 0 & 7 & b_2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - 2b_1 \end{bmatrix}$

$0x_1 + 0x_2 + 0x_3 + 1x_4 = b_2 - 2b_1$

$\Rightarrow x_4 = b_2 - 2b_1$

$AX = 0$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the echelon form: $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow x_4 = 0$

$x_1 + 2x_2 + 3x_4 = 0 \Rightarrow x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$

x_3 free variable

Let $x_2 = 1$ & $x_3 = 0 \Rightarrow x_1 = -2$

Let $x_2 = 0$ & $x_3 = 1$, $\Rightarrow x_1 = 0$

$$V_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$ Two vectors in null space

Complete solution:-

$Ax = b$

$x_4 = b_2 - 2b_1$

$x_1 + 2x_2 + 3x_4 = b_1$

$x_3 = t$ t is free variable

$x_2 = s$ s is free variable

substituting x_4 ~~...~~ into eq^n (1)

$x_1 + 2x_2 + 3(b_2 - 2b_1) = b_1$

$x_1 + 2x_2 = b_1 - 3(b_2 - 2b_1)$

$x_1 + 2x_2 = 7b_1 - 3b_2$

$x_1 = 7b_1 - 3b_2 - 2x_2$

Complete sol^n is $x = \begin{bmatrix} 7b_1 - 3b_2 - 2s \\ s \\ t \\ b_2 - 2b_1 \end{bmatrix}$

$$x = \begin{bmatrix} 7b_1 - 3b_2 \\ 0 \\ 0 \\ b_2 - 2b_1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

13) $Ux = 0$ & $Rx = 0$

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Ux = 0 \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 + 2x_4 = 0 \Rightarrow x_3 = -2x_4$

$x_1 + 2x_2 - 2x_4 = 0 \Rightarrow x_1 = -2x_2 + 2x_4$

Free variables :- x_2 & x_4

special sol^n 1 $x_2 = 1, x_4 = 0$:

$x_1 = -2, x_2 = 1, x_3 = 0, x_4 = 0$

special sol^n: $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

special sol^n $(x_2 = 0, x_4 = 1)$: $x_1 = 2, x_2 = 0, x_3 = -2, x_4 = 1$

Special sol^n:- $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

b) $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

$x_3 + 2x_4 = b$

$\Rightarrow x_3 = b - 2x_4$

$x_1 + 2x_2 + 3x_3 + 4x_4 = a$

substituting $x_3 = b - 2x_4$

$x_1 + 2x_2 + 3(b - 2x_4) + 4x_4 = a$

$x_1 + 2x_2 + 3b - 6x_4 + 4x_4 = a$

$x_1 + 2x_2 - 2x_4 = a - 3b$

$x_1 = a - 3b - 2x_2 + 2x_4$

Let $x_2 = s$ & $x_4 = t$, $x_1 = a - 3b - 2s + 2t$

$x_2 = s$, $x_3 = b - 2t$, $x_4 = t$

Complete sol^n: $x = \begin{bmatrix} a-3b-2s+2t \\ s \\ b-2t \\ t \end{bmatrix}$

$$x = \begin{bmatrix} a-3b \\ 0 \\ b \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

34) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$

$A_1 = \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{bmatrix}$ $\begin{matrix} R_2-2R_1 \\ R_3-2R_1 \\ R_4-3R_1 \end{matrix}$ $\begin{bmatrix} 1 & 2 & b_1 \\ 0 & 0 & b_2-2b_1 \\ 0 & 1 & b_3-2b_1 \\ 0 & 3 & b_4-3b_1 \end{bmatrix}$

$R_2 \leftrightarrow R_3$ $\begin{bmatrix} 1 & 2 & b_1 \\ 0 & 1 & b_3-2b_1 \\ 0 & 0 & b_2-2b_1 \\ 0 & 3 & b_4-3b_1 \end{bmatrix}$ $\begin{matrix} R_1-2R_2 \\ R_4-3R_2 \end{matrix}$ $\begin{bmatrix} 1 & 0 & 5b_1-2b_3 \\ 0 & 1 & b_3-2b_1 \\ 0 & 0 & b_2-2b_1 \\ 0 & 0 & b_4-3b_3+3b_1 \end{bmatrix}$

$b_2-2b_1=0$

$b_4-3b_3+3b_1=0$

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ $A_2 = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{bmatrix}$

$\begin{matrix} -2R_1 \\ -2R_1 \\ -3R_1 \end{matrix}$ $\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 1 & 1 & b_3-2b_1 \\ 0 & 3 & 3 & b_4-3b \end{bmatrix}$ R_3-2R_1 $\begin{matrix} R_1-2R_3 \\ R_4-3R_3 \end{matrix}$ $\begin{bmatrix} 1 & 0 & 1 & 5b_1-2b_3 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 1 & 1 & b_3-2b_1 \\ 0 & 0 & 0 & b_4-3b_3+3b_1 \end{bmatrix}$

R_{23} $\begin{bmatrix} 1 & 0 & 1 & (5b_1-2b_3) \\ 0 & 1 & 1 & (b_3-2b_1) \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_4-3b_3+3b_1 \end{bmatrix}$

2 pivot columns & 2 free columns

$b_2-2b_1=0$

$b_4-3b_3+3b_1=0$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\begin{bmatrix} 5b_1-2b_3 \\ b_3-2b_1 \end{bmatrix}$ is the general sol^n of 1st system.

$x_1+x_3 = 5b_1-2b_3$ $\qquad x_1 = -x_3+5b_1-2b_3$

$x_2+x_3 = b_3-2b_1$ $\qquad x_2 = -x_3+b_3-2b_1$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3+5b_1-2b_3 \\ -x_3+b_3-2b_1 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 5b_1-2b_3 \\ b_3-2b_1 \\ 0 \end{bmatrix}$

general sol^n of 2nd system

36) $x+3y+3z=1$

$2x+6y+9z=5$

$-x-3y+3z=5$

$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$

$[A|b] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{array}\right]$ $R_2 \to R_2-2R_1$ $\begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ -1 & -3 & 3 & 5 \end{bmatrix}$ $R_3 \to R_3+R_1$ $\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{array}\right]$

$R_3 \to R_3 - 2R_2$ $\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ 2nd row normalization $\to$ $\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$R_1 \to R_1 - 3R_2$ $\left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$Ax = 0 \iff Rx = 0$ $\quad$ y is free variable

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0 \quad \begin{cases} x = -3y \\ y \in R \\ z = 0 \end{cases}$$

special sol^n is $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ $\quad$ putting $y = 0$,
$Ax_p = b$
$\iff Rx_p = c$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x = -2 \\ z = 1 \end{cases}$$

Perticular sol^n is $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

Complete sol^n is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad R_2 \to R_2 - 2R_1 \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_2$ $\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ 2nd row normalization $\to$ $\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$R_1 \to R_1 - R_2$ $\left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x = 1/2 - 3y \\ y \in R \\ z = 1/2 - 2t \\ t \in R \end{cases}$

y, t are free variables

complete sol^n is

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1/2 - 3y \\ y \\ \frac{1}{2} - 2t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}}_{\text{Perticular sol}^n} + y \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{Special sol}^n} + t \underbrace{\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\text{special sol}^n}$$

44) $A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}$ $R_1 \to R_1 + 2R_2$ $\begin{bmatrix} 0 & 0 & 0 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}$ $R_3 \to R_3 + 3R_2$ $\begin{bmatrix} 0 & 0 & 0 \\ -3 & -2 & -1 \\ 0 & 0 & q-3 \end{bmatrix}$

its rank is 1 if it has two zero rows

$q = 3$

for rank $B = 1$

$\begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}$ $R_2 \to R_2 - 2R_1$ $\begin{bmatrix} 3 & 1 & 3 \\ q-6 & 0 & q-6 \end{bmatrix}$

it should have zero row, so $q = 6$

b) ~~rank~~ for rank A = 2

$A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & 1 \\ 6 & 0 & 9-3 \end{bmatrix}$ row must be ≠ 0 row

$\Rightarrow 9 \neq 3$

c) for rank B = 2

$\Rightarrow 9 \neq 6$

$r(A) = 3$ & $r(B) = 3$ are not possible because rank of matrix are linearly dependent

$r(A) = 3$ not possible

$r(B) = 3$ not possible

56). $Ux = C$ to form $Rx = d$

$$Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b$$

$$[Ab] = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & -3 & 3 \\ 2 & 0 & 4 & 9 & 10 \end{array}\right]$$

$$R_3 \to -2R_1 \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & -3 & 3 \\ 0 & 0 & 0 & 3 & 6 \end{array}\right] = [Uc]$$

2nd row normalization $\to$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 & 6 \end{array}\right] \xrightarrow[\text{normalization}]{\text{3rd row.}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] = [R\ d]$$

~~sp~~ special soln x_n

$Ax_n = 0$

$$\Leftrightarrow R x_n = 0 \Leftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = -2x_3 - 3x_4 \\ \quad = -2x_3 \\ x_2 = x_4 = 0 \\ x_3 \in \mathbb{R} \end{cases}$$

special soln is $x_n = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Perticular soln, x_p, $Ax_p = b$

$\Leftrightarrow R x_p = d$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{cases} x_1 = 2 - 2x_3 - 3x_4 \\ \quad = -4 \\ x_2 = 1 + x_4 \\ \quad = 3 \\ x_3 = 0 \\ x_4 = 2 \end{cases}$$

Perticular soln is $x_p = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$

59) $x - 3y - z = 0$

$\Rightarrow x = 3y + z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y + z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

y, z are free variables   x is pivot variable

$A = \begin{bmatrix} 1 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ special solns are (3,1,0) and (1,0,1)

Complete soln is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 + 3y + z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

2.3

Q - 3, 4, 7, 9, 13, 15, 16, 19, 23, 31, 40

3) a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ $R_2 - 3R_1 \to R_2$, $R_3 - 2R_1 \to R_3$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix}$

$R_2 \leftrightarrow R_3$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -5 & -7 \end{bmatrix}$ $R_3 - 5R_2 \to R_3$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 18 \end{bmatrix}$

linearly independent set

b) $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix}$ $R_2 + 3R_1 \to R_2$, $R_3 - 2R_1 \to R_3$ $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix}$ $R_3 + R_2 \to R_3$ $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$

Form linearly dependent set

4) Let $c_1v_1 + c_2v_2 + c_3v_3 = 0$

$\Rightarrow c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix} c_1+c_2+c_3 \\ c_2+c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$c_3 = 0$

$c_2 + c_3 = 0 \Rightarrow c_2 = 0$

$c_1 + c_2 + c_3 = 0 \Rightarrow c_1 = 0$

$c_1, c_2, c_3 = 0$ v_1, v_2, v_3 are linearly independent

let $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$

$\Rightarrow c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix} + c_4\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\begin{bmatrix} c_1+c_2+c_3+2c_4 \\ c_2+c_3+3c_4 \\ c_3+4c_4 \end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$c_3 + 4c_4 = 0 \Rightarrow c_3 = -4c_4$

$c_2 + c_3 + 3c_4 = 0$ $c_2 = -c_3 - 3c_4 = +4c_4 - 3c_4 = c_4$

$c_1 + c_2 + c_3 + 2c_4$

$c_1 = -c_2 - c_3 - 2c_4$

$= -c_4 + 4c_4 - 2c_4$

$c_1 = c_4$

If $c_4 = 1$ then $c_1 = 1, c_2 = 1, c_3 = -4$

$v_1 + v_2 - 4v_3 + v_4 = 0$

v_1, v_2, v_3, v_4 are linearly dependent.

7) $c_1 + c_2 + c_3 = 0$

$-c_1 = 0$

$-c_2 = 0$ v_1, v_2, v_3 are linearly independent

$-c_3 = 0$

$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$

$c_1\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} + c_4\begin{bmatrix}0\\1\\-1\\0\end{bmatrix} = 0$

$\begin{bmatrix} c_1+c_2+c_3 \\ -c_1+c_4 \\ -c_2-c_4 \\ -c_3 \end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

$c_1 + c_2 + c_3 = 0$ $c_4 = 0$

$-c_1 + c_4 = 0$ $c_2 = 0$ v_1, v_2, v_3, v_4 are linearly independent

$-c_2 - c_4 = 0$ $c_1 = 0$

$-c_4 = 0$ $c_3 = 0$

let $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 + c_5v_5 = 0$

$c_1\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} + c_4\begin{bmatrix}0\\1\\-1\\0\end{bmatrix} + c_5\begin{bmatrix}0\\1\\0\\-1\end{bmatrix} = 0$

$$\begin{bmatrix} c_1+c_2+c_3 \\ -c_1+c_4+c_5 \\ -c_2-c_4 \\ -c_3-c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1+c_2+c_3=0$ $\quad c_3=-c_5$
$-c_1+c_4+c_5=0$ $\quad c_2=-c_4$
$-c_2-c_4=0$ $\quad c_1=-c_2-c_3$
$-c_3-c_5=0$ $\quad =c_4+c_5$

$(c_4+c_5)v_1+(-c_4)v_2+(-c_5)v_3+c_4v_4+c_5v_5=0$

v_1,v_2,v_3,v_4,v_5,v_6 are linearly independent

This number How at spaced by V's.

9) $x+2y-3z-t=0$
$(1,0,0,1),(0,3,2,0)(1,1,1,0)$ – Three independent vectors.
$x+2y-3x-t=0$
Two independent vectors are $(1,0,0,1),(0,3,2,0)$

$$A=\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Nullspace of A is $x+2y-3z-t=0$

13) Let $c_1v_1+c_2v_2+c_3v_3+c_4v_4=0$

$$c_1\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}1\\0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}0\\0\\1\\1\end{bmatrix}+c_4\begin{bmatrix}0\\1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$c_1+c_2=0$ —— (1)
$c_1+c_4=0$ —— (2)
$c_2+c_3=0$ —— (3)
$c_3+c_4=0$ —— (4)

subtracting eq (2) from 1
$(c_1+c_2)-(c_1+c_4)=0$
$c_2-c_4=0$ $\quad c_2=c_4$
$-c_4-c_3=0$ from eq^n 4, $c_3=-c_4$
$c_1=-c_4$ i.e, $c_2=-c_1=c_4$
$c_1=-c_4, c_2=c_4, c_3=-c_4$
$-c_4v_1+c_4v_2-c_4v_3+c_4v_4=0$
If $c_4=1$ then $-v_1+v_2-v_3+v_4=0$
v_1,v_2,v_3,v_4 are dependent
They don't span R^4 because dimension of R^4 is 4.
$(0,0,0,1)$ $\quad c_1v_1+c_2v_2+c_3v_3+c_4v_4=(0,0,0,1)$
$c_1+c_2=0$ —— (5)
$c_1+c_4=0$ —— (6)
v_1,v_2,v_3,v_4 are linearly dependent & they don't span R^4 because dimension of R^4 is 4.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A has 2 pivot columns
domension of column space of A is 2
U has 2 pivot columns.
The domension of column space of U is 2.
Reduced row echelon form of A is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
The reduced row echelon form of A has 2 non-zero rows.
domension of rowspace of A is 2
U has 2 non-zero rows
domension of rowspace of U is 2.
The rowspace of A & row space of U are same because elementary row operations donot change the row space of a matrix

16) a) consider $\alpha \in \mathbb{R}^3$
$\alpha = a(1,1,-1) + b(-1,-1,1)$
$= a-b, a-b, -a+b$
Spanning space equal to $\{(a-b), a-b, -a+b / a, b \in \mathbb{R}\}$
this is a lone $\mathbb{R}^3$

b) the subspace of $\mathbb{R}^3$ spanned by the 3 vectors $(0,1,1), (1,1,0)$ & $(0,0,0)$ is a plane in $\mathbb{R}^3$.

c) The subspace of $\mathbb{R}^3$ formed by the given 3 by 4 echelon matrix with 2 pivots is a plane in $\mathbb{R}^3$.

d) the span of all vectors with +ve component will generate $\mathbb{R}^3$.

19) $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ many solns.
$x = 2y - 3z$
$y = s$ s & t are real no.s
$z = t$

From null theorem.
basis for null space $C_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$
C1 lies in X-Y plane & C2 lies in the line
$C_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ ← basis for intersection of the plane

Basis for null space
$BW = 0 \Leftrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $w = (w_1, w_2, w_3) \in \mathbb{R}^3$

$R_2 \to R_2 + \frac{1}{2}R_1 \to \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 \to R_1 - \frac{2}{3}R_2 \left[\begin{array}{ccc|c} 2 & 0 & -\frac{2}{3} & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$w_1 = \frac{1}{3}w_3$
$w_2 = -\frac{2}{3}w_3$
$w_3 \in \mathbb{R}$
basis are $w = w_3\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

2b) Column vectors of A are

$a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $a_2 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, $a_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$a_{\{1\}} = a_{\{2\}} = a_{\{3\}}$

By linear dependency $Ax = 0 \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$R_3 \to R_3 - R_1$

$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $R_1 \to R_1 - 3R_2$ $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} C_1 = C_3 \\ C_2 = -C_3 \\ C_3 \in \mathbb{R} \end{cases}$ $a_1 - a_2 + a_3 = 0$

$\Leftrightarrow a_1 = a_2 - a_3$ for $C_3 = 1$

bases for Column matrix A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$

For u is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Row matrix for A is $\{(1,0,-1), (0,1,1)\}$ &

For u is $\{(1,0,-1), (0,1,1)\}$

bases for null spaces of A & u :-

$x_1 - x_3 = 0$ & $x_2 + x_3 = 0$ x_3 is free variable.

Choose $x_3 = t$, t is a parameter

$x_1 = t$, $x_2 = -t$

$x = (x_1, x_2, x_3)$ $\quad x = (x_1, x_2, x_3)$

$= (t, -t, t)$

$= t.(1,-1,1)$

3!) $A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & c & d & 2 \end{bmatrix}$

$R_{\{3\}} : R_{\{3\}} + (-1) R_{\{2\}}$

$\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & c-c & d-2 & 0 \end{bmatrix}$ $\quad d - 2 = 0$

rank is 2

$d = 2$ rank of A is 2

$c = 2$ & $d = 2$, rank of matrices A & B will be 2 when $c = 0$ & $d = 2$

4o) a) $(1,2,0), (0,1,-1)$ is linearly independent as one vector is not scalar multiple of other

these vectors donot span R^3, which is not linear combination of $\{(1,2,0), (0,1,-1)\}$

The set is not basis for R3

b) The maxm no. is 3 & form a basis

$(1,1,-1), (2,3,4), (-4,1,-1), (0,1,-1)$ sets are not linearly independent. can not form a basis to R3.

c) assume: $a(1,2,2) + b(-1,2,1) + c(0,8,0) = (0,0,0)$

$a - b = 0$, $2a + 2b + 8c = 0$ $\quad$ linearly independent for

$2a + b = 0$ $\quad$ set $(1,2,2), (-1,2,1), (0,8,0)$

$\Rightarrow a = 0$ $\quad$ & the dimensions of R^3 is

$b = 0$, $c = 0$ $\quad$ equal to 3.

d) $a(1,2,2) + b(-1,2,1) + c(0,8,6) = 0$

$\Rightarrow a - b = 0$

$2b+8c=0$
$2a+b+6c=0$
$\Rightarrow a-b=0$
$a+b+4c=0$
$2a+b+6c=0$
$a=b$
$2b+4c=0 \quad a=b=-2c$

one of the vector is linearly dependent of other two,
they cannot form a basis to R^3.

2.4 (1, 2, 3, 6, 9, 13, 18, 29)

1) $A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$

$C(A)$ is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

Column space of A $C(A)=\left\{x\begin{bmatrix}1\\0\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\0\end{bmatrix} \mid x,y\in R\right\}$

null space $N(A)$, $AX=0$

$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $\quad x_2$ is 0, $x_3=0$

$x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}t\\0\\0\end{bmatrix}=t\begin{bmatrix}1\\0\\0\end{bmatrix}$

Basis of null space of A $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

null space is $N(A)=\left\{x\begin{bmatrix}1\\0\\0\end{bmatrix} \mid x\in R\right\}$

Transpose of A is $A^T=\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}$

Basis for column space is $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

column space $C(A^T)=\left\{x\begin{bmatrix}0\\1\\0\end{bmatrix}+y\begin{bmatrix}0\\0\\1\end{bmatrix} \mid x,y\in R\right\}$

null space of matrix $A^TX=0$

$\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $\quad x_1=0$, $x_2=0$ $\quad x$ is free variable. $x_3=t$

$x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\t\end{bmatrix}=t\begin{bmatrix}0\\0\\1\end{bmatrix}$

null space basis for $A^T=\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

null space is $N(A^T)=\left\{x\begin{bmatrix}0\\0\\1\end{bmatrix} \mid x\in R\right\}$.

2) dim row space for the matrix
$\dim C(A^T)=\dim C(U^T)=2$
Transpose the non zero rows $\left\{\begin{bmatrix}1\\2\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}$ forms the basis for the row space.
column space $\dim C(U)=2\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$

Payal
Priya
Panigrahi

substituting the values in dim N(A) = 2 $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$

null space is $Ux = 0$ $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$

$x_1 + 2x_2 + x_4 = 0$

$x_2 + x_3 = 0$

* substituting the values in matrix, $x = \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix}$

$= x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Left null space. $U^T x = 0$

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ x_1 is 0, x_2 is 0 $x_3 \in R$

soln is $x = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

left null space of A is null space A^T

$A^T x = 0$ $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ $x_1 + x_3$ is 0

x_2 is 0

$x_3 \in R$

$x = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

3) $A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ a. $A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$

$Ax = 0$

$Ux = 0$

$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $x_2 + 4x_3 = 0$

$x_2 = -4x_3$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ -4x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}$

null space of U = null space of A

$\left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_1, x_2, x_4 \in R \right\}$

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent vectors. dimension of null space U & dimension of null space A is 3

$A = \left\{ x \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid x \in R \right\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

$U = \left\{ x \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid x \in R \right\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$A^T = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 4 & 8 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 4 & 8 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_{12}, R_3 - 4R_2} \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Null space of A^T $A^T = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} : x_2 \in R \right\} = \left\{ x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} : x_2 \in R \right\}$

using row space of A is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, dimension is 1

$UT = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 4 & 0 \\ 0 & 0 \end{bmatrix}$ Column space of UT

$= \left\{ x \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} . \text{for } x \in R \right\}$

basis for Column space $= \left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$

row space is

$\left\{ x \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} : x \in R \right\}$

basis for row space of U is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$

6) $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

one linearly independent row of matrix A

$r = 1$ $A = UV^T$

$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix}$

Consider $A \in R^{2 \times 2}$ $A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}$ $C_1 = -(C_2)$

one linearily independent column of matrix A

$\Rightarrow r = 1$

$A = UV^T$

$\begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}$

9) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ Rank of matrix is 2

A has rank = number of rows.

matrix has full rowrank

A has right inverse C.

$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ Rank = 2

since matrix M has rank = no. of columns

matrix has full column rank.

M has left inverse B.

$B = (M^TM)^{-1} M^T$

$B = \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$B = \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$.

Consider $T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

when $a = b$ matrix T does not have inverse

when $a \neq b$ Rank of the matrix $= 2$ which is equal to both no. of rows & columns

It has both right & left inverse which are equal

left inverse B = right inverse C

$$\begin{bmatrix} \frac{1}{a} & -\frac{b}{a^2} \\ 0 & \frac{1}{a} \end{bmatrix}$$

13) $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$A = LU$

For row space: A & U have same basis as their rowspaces are same.

echelon form of U has 2 nonzero linearly independent rows.

Basis are:- $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

For column space:- pivot columns of U form basis for column space.

basis are:- 2 pivot columns i.e, 2nd & 4th column of U, 2nd & 4th columns of A form basis

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

For null space:- null spaces for A & U are same; they have same basis.

$Ux = 0$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x = 0 \iff \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$x_2 = -2x_3 + 2x_5$

$\iff x_4 = -2x_5 \quad x_1, x_3, x_5 \in \mathbb{R}$

$$\Rightarrow x = \begin{bmatrix} x_1 \\ -2x_3 + 2x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix} \Rightarrow x = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

where $x_1, x_3, x_5 \in \mathbb{R}$

Basis are,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

For left null space:- null space of A^T

$A^T x = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{bmatrix} x = 0 \iff \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Leftrightarrow x_1 + x_2 = 0$ $\qquad \Leftrightarrow \begin{matrix} x_1 = -x_2 \\ x_2 = -x_3 \end{matrix}$

$3x_1 + 4x_2 + x_3 = 0$

$x = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} \Rightarrow x = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ where, $x_3 \in \mathbb{R}$

basis are: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

18) Consider a eq^n: $x_1 + 2x_2 + 4x_3 = 0$

$A = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$ ← matrix form

we need to find a 1×3 matrix whose nullspace consist of all vectors in $\mathbb{R}^3$ that satisfy given eq^n.

$x \in N(A)$ satisfy.

$x_1 + 2x_2 + 4x_3 = 0$

$Ax = 0$

$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ $\qquad$ $x_1 + 2x_2 + 4x_3 = 0$

nullspace of A is:

$N(A) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 + 4x_3 = 0\}$

required 1×3 matrix is:

$B = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$

3×3 matrix is: $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix}$

29) $A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

Column space of matrix: Bases of column space are linearly independent columns of A.

dimension of column space is 2.

$\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

Row space of matrix: Bases of row space are linearly independent rows of A.

$\begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad$ dimension of column space is 2.

Null space of matrix: Bases of null space are linearly independent vectors in the null space of A. $Ax = 0$

Dimension of null space is 2.

bases are: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

Left null space of matrix: - Bases of left null space are linearly independent vectors in the nullspace of AT i.e, $A^T x = 0$ $\quad$ base is: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Dimension of left null space is 1.

Consider $A = \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 5 & 5 \end{bmatrix}$

Four subspaces are:

Column space of matrix: Bases of the column space are the linearly independent columns of A i.e, $\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

dimension of column space is 1.

Row space of matrix:-

Bases of the row space are linearly independent rows of A.

i.e, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ dimension of row space is 1.

Null space of matrix:- Bases of null space are linearly independent vectors in the null space of A.

i.e, in $Ax = 0$

bases is: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ Dimension of null space is 2.

- Left null space of matrix:-

Bases of the left null space are linearly independent vectors in the null space of A^T.

i.e, in $A^T x = 0$

base is: $\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$

Dimension of the left null space is 2.

13.03.25