2015-16 MATH1030 Linear Algebra Assignment 4 Solution

(Question No. referring to the 8^{th} edition of the textbook)

Section 3.1

3

To show that C is a vector space we must show that all eight axioms are satisfied.

A1

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i \\ &= (c+a)+(d+b)i \\ &= (c+di)+(a+bi)\end{aligned}$$

A2

$$\begin{aligned}(x+y)+z &= [(x_1+x_2i)+(y_1+y_2i)]+(z_1+z_2i) \\ &= (x_1+y_1+z_1)+(x_2+y_2+z_2)i \\ &= (x_1+x_2i)+[(y_1+y_2i)+(z_1+z_2i)] \\ &= \mathbf{x}+(\mathbf{y}+\mathbf{z})\end{aligned}$$

A3

$$(a+bi)+(0+0i)=(a+bi)$$

A4 If $z=a+bi$ then define $z=abi$. It follows that

$$\mathbf{z}+(\mathbf{z})=(a+bi)+(-a-bi)=0+0i=\mathbf{0}$$

A5

$$\begin{aligned}\alpha[(a+bi)+(c+di)] &= (\alpha a+\alpha c)+(\alpha b+\alpha d)i \\ &= \alpha(a+bi)+\alpha(c+di)\end{aligned}$$

A6

$$\begin{aligned}(\alpha+\beta)(a+bi) &= (\alpha+\beta)a+(\alpha+\beta)bi \\ &= \alpha(a+bi)+\beta(a+bi)\end{aligned}$$

A7

$$\begin{aligned}(\alpha\beta)(a+bi) &= (\alpha\beta)a+(\alpha\beta)bi \\ &= \alpha(\beta a+\beta bi)\end{aligned}$$

A8

$$1 \cdot (a + bi) = 1 \cdot a + 1 \cdot bi = a + bi$$

Since all 8 axioms are satisfied, C is a vector space with the given operators.

4

Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be arbitrary elements of $\mathbb{R}^{m \times n}$.

A1. Since $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ for each i and j it follows that $A + B = B + A$.

A2. Since

$$(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$$

for each i and j, it follows that

$$(A + B) + C = A + (B + C)$$

A3. Let O be the $m \times n$ matrix whose entries are all 0. If $M = A + O$ then

$$m_{ij} = a_{ij} + 0 = a_{ij}$$

Therefore, $A + O = A$.

A4. Define A to be the matrix whose ij^{th} entry is a_{ij} . Since

$$a_{ij} + (-a_{ij}) = 0$$

for each i and j, it follows that

$$A + (-A) = O$$

A5. Since

$$\alpha(a_{ij} + b_{ij}) = \alpha a_{ij} + \alpha b_{ij}$$

for each i and j, it follows that

$$\alpha(A + B) = \alpha A + \alpha B$$

A6. Sine

$$(\alpha + \beta)a_{ij} = \alpha a_{ij} +_{\beta} a_{ij}$$

for each i and j, it follows that

$$(\alpha + \beta)A = \alpha A + \beta A$$

A7. Since

$$(\alpha\beta)a_{ij} = \alpha(\beta a_{ij})$$

for each i and j, it follows that

$$(\alpha\beta)A = \alpha(\beta A)$$

A8. Since

$$1 \cdot a_{ij} = a_{ij}$$

for each i and j, it follows that

$$1A = A$$

6

Let f, g and h be arbitrary elements of P.

A1. For all $x \in \mathbb{R}$,

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x).$$

Therefore,

$$f + g = g + f$$

A2. For all $x \in \mathbb{R}$,

$$\begin{aligned}
[(f+g)+h](x) &= (f+g)(x) + h(x) \\
&= f(x) + g(x) + h(x) \\
&= f(x) + (g+h)(x) \\
&= [f + (g+h)](x)
\end{aligned}$$

Therefore,

$$[(f+g)+h] = [f + (g+h)]$$

A3. If z(x) is the zero polynomial, then for all $x \in \mathbb{R}$,

$$(f+z)(x) = f(x) + z(x) = f(x) + 0 = f(x)$$

Thus,

$$f + z = f$$

A4. Define $-f$ by

$$(-f)(x) = -f(x)$$

for all $x \in \mathbb{R}$. Since

$$(f + (-f))(x) = f(x) - f(x) = 0$$

for all $x \in \mathbb{R}$, it follows that

$$f + (-f) = z$$

A5. For each $x \in \mathbb{R}$,

$$\begin{aligned}
[\alpha(f+g)](x) &= \alpha f(x) + \alpha g(x) \\
&= (\alpha f)(x) + (\alpha g)(x)
\end{aligned}$$

Thus,

$$\alpha(f+g) = \alpha f + \alpha g$$

A6. For all $x \in \mathbb{R}$,

$$\begin{aligned}
[(\alpha+\beta)f](x) &= (\alpha+\beta)f(x) \\
&= \alpha f(x) + \beta f(x) \\
&= (\alpha f)(x) + (\beta f)(x)
\end{aligned}$$

Therefore,

$$(\alpha+\beta)f = \alpha f + \beta f$$

A7. For all $x \in \mathbb{R}$,

$$[(\alpha\beta)]f(x) = \alpha\beta f(x) = \alpha[\beta f(x)] = [\alpha(\beta f)](x)$$

Therefore,

$$(\alpha\beta)f = \alpha(\beta)f$$

A8. For any $x \in \mathbb{R}$.

$$1f(x) = f(x)$$

Therefore,

$$1f = f$$

7

Suppose there exists zeros elements $\mathbf{0}_1$ and $\mathbf{0}_2$ in the vector space V. By the definition of $\mathbf{0}_1$,

$$\mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_1$$

However, by the definition of $\mathbf{0}_2$,

$$\mathbf{0}_2 + \mathbf{0}_1 = \mathbf{0}_2$$

Since addition is commutative in V, so $\mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_2 + \mathbf{0}_1$. Therefore, it follows that $\mathbf{0}_1 = \mathbf{0}_2$. Hence, the zero element is unique.

8

For any $\mathbf{x} \in V$, there exists **-x** $\in V$ such that $(\textbf{-x}) + \mathbf{x} = \mathbf{0}$. Since

$$\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z}$$

it follows that

$$(-\mathbf{x}) + (\mathbf{x} + \mathbf{y}) = (-\mathbf{x}) + (\mathbf{x} + \mathbf{z})$$

Since addition is associative, so

$$\mathbf{y} = (-\mathbf{x} + \mathbf{x}) + \mathbf{y} = (-\mathbf{x}) + (\mathbf{x} + \mathbf{y}) = (-\mathbf{x}) + (\mathbf{x} + \mathbf{z}) = (-\mathbf{x} + \mathbf{x}) + \mathbf{z} = \mathbf{z}$$

9

(a)

If $\mathbf{y} = \beta\mathbf{0}$, then

$$\mathbf{y} + \mathbf{y} = \beta\mathbf{0} + \beta\mathbf{0} = \beta(\mathbf{0} + \mathbf{0}) = \beta\mathbf{0} = \mathbf{y}$$

and it follows that

$$\begin{aligned}
&(\mathbf{y} + \mathbf{y}) + (-\mathbf{y}) = y + (-\mathbf{y}) \\
&\mathbf{y} + [\mathbf{y} + (-\mathbf{y})] = \mathbf{0} \\
&\mathbf{y} + \mathbf{0} = \mathbf{0} \\
&\mathbf{y} = \mathbf{0}
\end{aligned}$$

(b)

If $\alpha\mathbf{x} = \mathbf{0}$ and $\alpha \neq 0$, then it follows from (a), A7 and A8 that

$$\mathbf{0} = \frac{1}{\alpha}\mathbf{0} = \frac{1}{\alpha}(\alpha\mathbf{x}) = (\frac{1}{\alpha}\alpha)\mathbf{x} = 1\mathbf{x} = \mathbf{x}$$

10

Axiom 6 fails to hold.

$$\begin{aligned}
(\alpha + \beta)\mathbf{x} &= ((\alpha + \beta)x_1, (\alpha + \beta)x_2) \\
\alpha\mathbf{x} + \beta\mathbf{x} &= ((\alpha + \beta)x_1, 0)
\end{aligned}$$

If $x_2 \neq 0$, then the above two equations do not agree.

11

Since addition is defined in the usual way, in order to check that V is a vector space with the given operators, we only need to show that V with the operators satisfies axioms A5-A8.
A5. For any $\alpha \in \mathbb{R}$,

$$\begin{aligned}\alpha \circ [(x_1, y_1) + (x_2, y_2)] &= \alpha \circ (x_1 + x_2, y_1 + y_2) = (\alpha(x_1 + x_2), y_1 + y_2) \\ &= (\alpha x_1 + \alpha x_2, y_1 + y_2) = (\alpha x_1, y_1) + (\alpha x_2, y_2) \\ &= \alpha \circ (x_1, y_1) + \alpha \circ (x_2, y_2)\end{aligned}$$

A6. For any $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}(\alpha + \beta) \circ (x_1, y_1) &= ((\alpha + \beta)x_1, y_1) = (\alpha x_1 + \beta x_1, y_1) \\ &= (\alpha x_1, y_1) + (\beta x_1, y_1) = \alpha \circ (x_1, y_1) + \beta \circ (x_1, y_1)\end{aligned}$$

A7. For any $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}(\alpha\beta) \circ (x_1, y_1) &= ((\alpha\beta)x_1, y_1) = (\alpha(\beta x_1), y_1) \\ &= \alpha \circ (\beta x_1, y_1) = \alpha \circ (\beta \circ (x_1, y_1))\end{aligned}$$

A8.

$$\mathbf{1} \circ (x_1, y_1) = (1 \cdot x_1, y_1) = (x_1, y_1)$$

Therefore, V is a vector space with the given operators.

12
A1.

$$x \oplus y = x \cdot y = y \cdot x = y \oplus x$$

A2.

$$(x \oplus y) \oplus z = x \cdot y \cdot z = x \oplus (y \oplus z)$$

A3. Since $x \oplus 1 = x \cdot 1 = x$ for all x, it follow that 1 is the zero vector.
A4. Let

$$-x = -1 \circ x = x^{-1} = \frac{1}{x}$$

It follows that

$$x \oplus (-x) = x \cdot \frac{1}{x} = 1$$

Therefore, $\frac{1}{x}$ is the additive inverse of x for the operation $\oplus$.
A5.

$$\begin{aligned}&\alpha \circ (x \oplus y) = (x \oplus y)^{\alpha} = (x \cdot y)^{\alpha} = x^{\alpha} \cdot y^{\alpha} \\ &\alpha \circ x \oplus \alpha \circ y = x^{\alpha} \oplus y^{\alpha} = x^{\alpha} \cdot y^{\alpha}\end{aligned}$$

A6.

$$\begin{aligned}&(\alpha + \beta) \circ x = x^{\alpha+\beta} = x^{\alpha} \cdot x^{\beta} \\ &\alpha \circ x \oplus \beta \circ x = x^{\alpha} \oplus x^{\beta} = x^{\alpha} \cdot x^{\beta}\end{aligned}$$

A7.

$$\begin{aligned}&(\alpha\beta) \circ x = x^{\alpha\beta} \\ &\alpha \circ (\beta \circ x) = \alpha \circ x^{\beta} = (x^{\beta})^{\alpha} = x^{\alpha\beta}\end{aligned}$$

A8.

$$1 \circ x = x^1 = x$$

Since all 8 axioms hold, $\mathbb{R}^+$ is a vector space under the operations $\circ$ and $\oplus$.

13
The system is not a vector space. Axioms A3, A4, A5, A6 fail to hold.

Section 3.2

2
(a)
For any $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$ with $x_1 + x_3 = y_1 + y_3 = 1$,

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

but $x_1 + y_1 + x_3 + y_3 = 2 \neq 1$. Therefore, it does not form a subspace of $\mathbb{R}^3$.
(b)
For any $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$ with $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$, we have

$$x_1 + y_1 = x_2 + y_2 = x_3 + y_3$$

Additionally, for any $\alpha \in \mathbb{R}$,

$$\alpha x_1 = \alpha x_2 = \alpha x_3$$

Therefore, it forms a subspace of $\mathbb{R}^3$.
(c)
For any $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$ with $x_3 = x_1 + x_2$ and $y_3 = y_1 + y_2$, we have

$$x_3 + y_3 = (x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2)$$

Additionally, for any $\alpha \in \mathbb{R}$,

$$\alpha x_3 = \alpha(x_1 + x_2) = (\alpha x_1) + (\alpha x_2)$$

Therefore, it forms a subspace of $\mathbb{R}^3$.
(d)
Let $(x_1, x_2, x_3) = (a, 0, a)$, then $x_1 = x_3$. And let $(y_1, y_2, y_3) = (0, b, b)$, then $y_2 = y_3$. Then

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (a, b, a + b)$$

But the equality $a = a + b$ and $b = a + b$ does not hold for all $a, b \in \mathbb{R}$. For example, take $a = 1$ and $b = 2$, then $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (1, 2, 3)$ does not lie in the given subset of $\mathbb{R}^3$. Therefore, it does not form a subspace of $\mathbb{R}^3$.

---

3 (Remark: Only sketches of the proofs are to be given. Students are expected to give more formal and rigorous proofs by considering $A = (a_{ij})$ for any $A \in \mathbb{R}^{2\times2}$.)
(a)
Since the addition of two diagonal matrices is still a diagonal matrix, and that the scalar multiplication of any diagonal matrix is still a diagonal matrix, so the set of 2×2 diagonal matrices form a subspace of $\mathbb{R}^{2\times2}$.
(b)

Since the addition of a lower triangular matrix with an upper triangular matrix does not give a triangular matrix in general, so the given set is not a subspace of $\mathbb{R}^{2\times 2}$.

(c)

Since the addition of two lower triangular matrices is still a lower triangular matrix, and that the scalar multiplication of any lower triangular matrix is still a lower triangular matrix, so the given set is a subspace of $\mathbb{R}^{2\times 2}$.

(d)

Note that the addition of two matrices from the given set has its 1-2-entry being $a_{12}+b_{12} = 1+1 = 2 \neq 1$, so the given set is not a subspace of $\mathbb{R}^{2\times 2}$.

(e)

Since $0+0 = 0$ and $\alpha 0 = 0$ for any $\alpha \in \mathbb{R}$, so the given set is closed under addition and scalar multiplication. Hence,the given set is a subspace of $\mathbb{R}^{2\times 2}$.

(f)

Since $a_{12}+b_{12} = a_{21}+b_{21}$ and $\alpha a_{12} = \alpha a_{21}$, so the given set is closed under addition and scalar multiplication. Hence,the given set is a subspace of $\mathbb{R}^{2\times 2}$.

(g)

In particular,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = I$$

is not singular, so the given set is not a subspace of $\mathbb{R}^{2\times 2}$.

5

(a)

For any polynomial $f(x) \in P_4$ of even degree, there exists $a, b, c \in \mathbb{R}$ such that $f(x) = ax^4 + bx^2 + c$. And for any two such polynomials,

$$f_1(x) + f_2(x) = (a_1 + a_2)x^4 + (b_1 + b_2)x^2 + (c_1 + c_2)$$

is still a polynomial in P_4 of even degree (though $a_1 + a_2$, $b_1 + b_2$ and $c_1 + c_2$ can be 0), where we take the zero polynomial to be of even degree also. Besides, for any $\alpha \in \mathbb{R}$,

$$\alpha f(x) = (\alpha a)x^4 + (\alpha b)x^2 + (\alpha c)$$

is still a polynomial in P_4 of even degree (though αa, αb and αc can be 0). Therefore, the given set is a subspace of P_4.

(b)

In particular,

$$f_1(x) + f_2(x) = (x^3 + x^2 + 1) + (-x^3 + x) = x^2 + x + 1$$

is not a polynomial of degree 3. Therefore, the given set is not a subspace of P_4.

(c)

For any polynomials in P_4 such that $p(0) = 0$, the constant term of $p(x)$ must be 0. Therefore,

$$\begin{aligned} p_1(x) + p_2(x) &= (a_1x^4 + b_1x^3 + c_1x^2 + d_1x) + (a_2x^4 + b_2x^3 + c_2x^2 + d_2x) \\ &= (a_1 + a_2)x^4 + (b_1 + b_2)x^3 + (c_1 + c_2)x^2 + (d_1 + d_2)x \end{aligned}$$

is still a polynomial in P_4 with its value being 0 when evaluating at $x = 0$. And clearly, the set is closed under scalar multiplication. Therefore, the given set is a subspace of P_4.

(d)

In particular,

$$p_1(x) + p_2(x) = (x^2 + 2x + 1) + (-x) = x^2 + x + 1$$

where $p_1(x)$ is a polynomial in P_4 having real double roots $x = 1$, and $p_2(x)$ is a polynomial in P_4 having real root $x = 0$. But $(p_1 + p_2)(x)$ is a polynomial in P_4 having no real root. Therefore, the given set is not a subspace of P_4.

8

If $B \in S_1$, then $AB = BAz$. It follows that

$$A(\alpha B) = \alpha AB = \alpha BA = (\alpha B)A$$

and hence $\alpha B \in S_1$. If B and C are in S_1, then

$$AB = BA \quad \text{and} \quad AC = CA$$

thus,

$$A(B + C) = AB + AC = BA + CA = (B + C)A$$

and hence $B + C \in S_1$. Therefore, S_1 is a subspace of $\mathbb{R}^{2\times2}$.

10

(a)

For any $B \in S_1$, for any $\alpha \in \mathbb{R}$,

$$(\alpha B)A = \alpha(BA) = \alpha\mathbf{0} = \mathbf{0}$$

Also, for any $C \in S_1$,

$$(B + C)A = BA + CA = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

Therefore, S_1 is a subspace of $\mathbb{R}^{2\times2}$.

(b)

Let $B \in S_1$, then we have $(-B) \in S_1$. Suppose S_1 is a subspace of $\mathbb{R}^{2\times2}$, then $B + (-B) \in S_1$. Hence, $\mathbf{0} \in S_1$. However, $\mathbf{0}A = A\mathbf{0}$, contradicting that $\mathbf{0} \in S_1$. Therefore, S_1 is not a subspace of $\mathbb{R}^{2\times2}$.

12

(c)

Note that

$$det(\begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & 2 & 0 \end{bmatrix}) = 0$$

Therefore, the three vectors are linearly dependent. Therefore, they do not form a spanning set for $\mathbb{R}^3$.

(d)

Note that

$$det(\begin{bmatrix} 2 & -2 & 4 \\ 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}) = 0$$

Therefore, the three vectors are linearly dependent. Therefore, they do not form a spanning set for $\mathbb{R}^3$.

(e)

Note that the given set consists of only two vectors in $\mathbb{R}^3$. However, $dim(\mathbb{R}^3) = 3 > 2$. Therefore, the given set must not be a spanning set of $\mathbb{R}^3$.

18

Let $S \neq \{0\}$ be a subspace of $\mathbb{R}^1$ and let a be an arbitrary element of $\mathbf{R}^1$. If s is a nonzero element of S, then we can define a scalar α to be the real number a/s. Since S is a subspace it follows that

$$\alpha s = \frac{a}{s}s = a$$

is an element of S. Therefore $S = \mathbb{R}^1$.

22
Let α be a scalar and let $\mathbf{x}$ and $\mathbf{y}$ be elements of $U \cap V$. The vectors $\mathbf{x}$ and $\mathbf{y}$ are elements of both U and V. Since U and V are subspaces it follows that

$$\begin{array}{lcl} \alpha\mathbf{x} \in U & \text{and} & \mathbf{x}+\mathbf{y} \in U \\ \alpha\mathbf{x} \in V & \text{and} & \mathbf{x}+\mathbf{y} \in V \end{array}$$

Therefore,

$$\alpha\mathbf{x} \in U \cap V \quad \text{and} \quad \mathbf{x}+\mathbf{y} \in U \cap V$$

Thus $U \cap V$ is a subspace of W.

Section 3.3

2
(b)
Since $dim(\mathbb{R}^3) = 3$ but we are given 4 vectors from $\mathbb{R}^3$, so the 4 vectors must be linearly dependent.
(d)
Note that

$$det(\begin{bmatrix} 2 & -2 & 4 \\ 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}) = 0$$

Therefore, the three vectors are linearly dependent.
(e)
Since we are given only 2 vectors from $\mathbb{R}^3$, so if the 2 vectors are linearly dependent, they must differ by only a scalar multiple. However, the first entry of the first vector is nonzero while that of the second vector is zero, plus the second vector is not the zero vector, we can conclude that the two vectors cannot be linearly dependent, that is, the two vectors are linearly independent.

4
(b)
Note that the vector

$$\begin{bmatrix} a & b \\ c & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

if and only if $a = b = c = 0$. Therefore, the 3 vectors are linearly independent.
(c)
Since

$$2\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + 3\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

Therefore, the 3 vectors are linearly dependent.

8

(c)

Note that

$$a(x+2)+b(x+1)+c(x^2-1)=cx^2+(a+b)x+(2a+b)=0$$

if and only if $c=0$, $a=-b$, and $a=-b/2$. But this is true if and only if $a=b=c=0$. Therefore, the 3 polynomials are linearly independent in P_3.

(d)

Note that

$$a(x+2)+b(x^2-1)=bx^2+ax+(2a-b)=0$$

if and only if $b=0$, $a=0$, and $2a=b$. But this is true if and only if $a=b=0$. Therefore, the 2 polynomials are linearly independent in P_3.

13

Let $v_1,\ldots,v_n$ be vectors in a vector space V . If one of the vectors, say v_1, is the zero vector then set

$$c_1=1, c_2=c_3=\cdots=c_n=\mathbf{0}$$

Since

$$c_1v_1+c_2v_2+\cdots+c_nv_n=\mathbf{0}$$

and $c_1\neq 0$, it follows that $v_1,\ldots,v_n$ are linearly dependent.

15

Let $v_1,v_2,\ldots,v_n$ be a linearly independent set of vectors and suppose there is a subset, say $v_1,\ldots,v_k$ of linearly dependent vectors. This would imply that there exist scalars $c_1,c_2,\ldots,c_k$, not all zero, such that

$$c_1v_1+c_2v_2+\cdots+c_kv_k=\mathbf{0}$$

but then

$$c_1v_1+\cdots+c_kv_k+0v_{k+1}+\cdots+0v_n=\mathbf{0}$$

This contradicts the original assumption that $v_1,v_2,\ldots,v_n$ are linearly independent.

19

Since $v_1,\ldots,v_n$ span V, we can write

$$v=c_1v_1+c_2v_2+\cdots+c_nv_n$$

Setting $c_1=-1$, we would have

$$c_1v_1+c_2v_2+\cdots+c_nv_n=\mathbf{0}$$

which would contradict the linear independence of $v_1,v_2,\ldots,v_n$.