Andalus Prime

Take any integer n greater than 3, and divide it by 6. That is, write $\boldsymbol{n = 6q + r}$ where q is a non-negative integer and the remainder r is one of 0, 1, 2, 3, 4, or 5.

If the remainder is 0, 2 or 4, then the number n is divisible by 2, and can not be prime. If the remainder is 3, then the number n is divisible by 3, and can not be prime.

So if n is prime(n>3), then the remainder r is either

1 (and $n = 6q + 1$ is one more than a multiple of six), or

5 (and $n = 6q + 5 = 6(q+1) - 1$ is one less than a multiple of six).

Remember that being one more or less than a multiple of six does not make a number prime.

5	7	11	13	17	19	23	25	29	31
35	37	41	43	47	49	53	55	59	61
65	67	71	73	77	79	83	85	89	91
95	97	101	103	107	109	113	115	119	121
125	127	131	133	137	139	143	145	149	151
155	157	161	163	167	169	173	175	179	181

all unmarked cells are guaranteed to be prime.

5	7	11	13	17	19	23	25	29	31
35	37	41	43	47	49	53	55	59	61
65	67	71	73	77	79	83	85	89	91
95	97	101	103	107	109	113	115	119	121
125	127	131	133	137	139	143	145	149	151
155	157	161	163	167	169	173	175	179	181

The index is starts from 1.(i.e 5 is found at index 1, 7 is found at index 2, ..)

To find the number x from index i
$x = 3i + 1$ if i is even
$x = 3i + 2$ if i is odd

If we multiply any two numbers that is expressed in the form of $6n \pm 1$ then the result is also expressed in the form of $6n \pm 1$.

For any prime p, all multiples of p larger than p are composite.
Mark all multiples of 5 greater than 5 that is expressed in $6n \pm 1$.

5	7	11	13	17	19	23	25	29	31
35	37	41	43	47	49	53	55	59	61
65	67	71	73	77	79	83	85	89	91
95	97	101	103	107	109	113	115	119	121
125	127	131	133	137	139	143	145	149	151
155	157	161	163	167	169	173	175	179	181

Mark all multiples of 7 greater than 7 that is expressed in $6n \pm 1$.

5	7	11	13	17	19	23	25	29	31
35	37	41	43	47	49	53	55	59	61
65	67	71	73	77	79	83	85	89	91
95	97	101	103	107	109	113	115	119	121
125	127	131	133	137	139	143	145	149	151
155	157	161	163	167	169	173	175	179	181

Mark all multiples of 11 greater than 11 that is expressed in $6n \pm 1$.

5	7	11	13	17	19	23	25	29	31
35	37	41	43	47	49	53	55	59	61
65	67	71	73	77	79	83	85	89	91
95	97	101	103	107	109	113	115	119	121
125	127	131	133	137	139	143	145	149	151
155	157	161	163	167	169	173	175	179	181

Mark all multiples of 13 greater than 13 that is expressed in $6n \pm 1$.

5	7	11	13	17	19	23	25	29	31
35	37	41	43	47	49	53	55	59	61
65	67	71	73	77	79	83	85	89	91
95	97	101	103	107	109	113	115	119	121
125	127	131	133	137	139	143	145	149	151
155	157	161	163	167	169	173	175	179	181

Now, all multiples of 5, 7, 11, 13 are marked. Since $\sqrt{N}$ ≈13.45, all unmarked cells are guaranteed to be prime.

5	7	11	13	17	19	23	25	29	31
35	37	41	43	47	49	53	55	59	61
65	67	71	73	77	79	83	85	89	91
95	97	101	103	107	109	113	115	119	121
125	127	131	133	137	139	143	145	149	151
155	157	161	163	167	169	173	175	179	181

Therefore, we can consider all unmarked numbers as primes.

5	7	11	13	17	19	23	25	29	31
35	37	41	43	47	49	53	55	59	61
65	67	71	73	77	79	83	85	89	91
95	97	101	103	107	109	113	115	119	121
125	127	131	133	137	139	143	145	149	151
155	157	161	163	167	169	173	175	179	181

Number	Index of their multiple in the table	Formula for the pattern n = 0,1,2,...	The difference between the starts
5	8 18 28 38 48 58 68 78 88 98 11 21 31 41 51 61 71 81 91	8 + 10n = 8 + 2 * 5n 11 + 10n = 11 + 2 * 5n	11 – 8 = 3
7	11 25 39 53 67 81 95 109 123 16 30 44 58 72 86 100 114 128	11 + 14n = 11 + 2 * 7n 16 + 14n = 16 + 2 * 7n	16 – 11 = 5
11	18 40 62 84 106 128 150 172 25 47 69 91 113 135 157 179	18 + 22n = 18 + 2 * 11n 25 + 22n = 25 + 2 * 11n	25 – 18 = 7
13	21 47 73 99 125 151 177 203 30 56 82 108 134 160 186 212	21 + 26n = 21 + 2 * 13n 30 + 26n = 30 + 2 * 13n	30 – 21 = 9
17	28 62 96 130 164 198 232 266 39 73 107 141 175 209 243 277	28 + 34n = 28 + 2 * 17n 39 + 34n = 39 + 2 * 17n	39 – 28 = 11
19	31 69 107 145 183 221 259 297 44 82 120 158 196 234 272 310	31 + 38n = 31 + 2 * 19n 44 + 38n = 44 + 2 * 19n	44 – 31 = 13
23	38 84 130 176 222 268 314 360 53 99 145 191 237 283 329 375	38 + 46n = 38 + 2 * 23n 53 + 46n = 53 + 2 * 23n	53 – 38 = 15
25	41 91 141 191 241 291 341 391 58 108 158 208 258 308 358 408	41 + 50n = 41 + 2 * 25n 58 + 50n = 58 + 2 * 25n	58 – 41 = 17
29	48 106 164 222 280 338 396 454 67 125 183 241 299 357 415 473	48 + 58n = 48 + 2 * 29n 67 + 58n = 67 + 2 * 29n	67 – 48 = 19
31	51 113 175 237 299 361 423 485 72 134 196 258 320 382 444 506	51 + 62n = 51 + 2 * 31n 72 + 62n = 72 + 2 * 31n	72 – 51 = 21

We can assign start=1, it is a_1 of the arithmetic sequence.
At each iteration it increase by 7 if start is odd or increase by 3 if start is even.
There are two different patterns for each number. Their only difference is the starting number. Let say the difference between the two a_1 is step. At the first time step = 1. For each iteration the step is increased by 2.

The two arithmetic sequences are
$$\text{start} + 2 * \text{number} * \text{n} \qquad \text{n} = 0, 1, 2, 3, \ldots$$
$$\text{start} + \text{step} + 2 * \text{number} * \text{n} \qquad \text{n} = 0, 1, 2, 3, \ldots$$

```
const int N = 1000000009;
bool prime[N / 3];
void andalus(){
   prime[0]=true;
   int p=5, i=0, step=1, start=1;
   int  last = N/3, s = sqrt(N);
     while (p<=s){
      i++;
      start += start%2? 7:3;
      step += 2;
      if(prime[i])continue;
      p = i%2? 3*i+2: 3*i+1;
      for(int k=start; k<last; k += 2*p)
         prime[k]=true;
      for(int k=start + step; k<last; k += 2*p)
         prime[k]=true;
   }
}
void display(){
   cout<<2 <<" "<<3 <<" ";
   int last = N/3;
   for(int i = 0; i<last; i++)
      if (!prime[[i])
         cout<<(i%2? 3*i+2: 3*i+1)<<" ";
}
```

Complexity Analysis

Space complexity of Andalus Prime is *N/3*

Time complexity of Andalus Prime is O(N)

$$\frac{1}{3}\sum_{x \in S}\left\lfloor \frac{N}{x} \right\rfloor$$

S is the set of prime numbers from 5 to

$M = \sqrt{N}$

The number of primes less than or equal to M approaches $n = \pi(M) = M/\log M$

The average gap of two primes less than or equal to M approaches $d = M \,/\, \pi(M) = \log M$

$$\frac{1}{3}\sum_{x \in S}\left\lfloor \frac{N}{x} \right\rfloor = \frac{N}{3}\sum_{x \in S}\frac{1}{x} = \frac{N}{3}\left(\frac{1}{5} + \frac{1}{5+d} + \frac{1}{5+2d} + \cdots\right) \;--$$

$$\Rightarrow \frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + \cdots + \frac{1}{a+(n-1)d} = \frac{1}{d}\ln\left(\frac{2a+(2n-1)d}{2a-d}\right) = S_n \qquad \boxed{d>0}$$

$a = 5$

$d = \log_e M = \ln M$

$n = M/\log_e M = M/\ln M$

$$\frac{N}{3} S_n = \frac{N}{3}\left(\frac{1}{\ln M}\ln\left(\frac{2\times 5 + (2M/\ln M - 1)\ln M}{2\times 5 - \ln M}\right)\right)$$

$$= \frac{N}{3}\left(\frac{1}{\ln M}\ln\left(\frac{10 + 2M - \ln M}{10 - \ln M}\right)\right) \simeq \frac{N}{3}\left(\frac{1}{\ln M}\ln\left(\frac{2M}{\ln M}\right)\right)$$

$$\simeq \frac{N}{3}\cdot\frac{1}{\ln M}\cdot\ln(M) \simeq \frac{N}{3} = O(N)$$