Problem Set - 1

Rates and Population properties of compact binary mergers

Mukesh Kumar Singh

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Problem 1: A simple 1 -dimensional example

There is an infinite number of audio speakers arranged along the x-axis. An audio detector is placed at some distance as in Fig. (). When one speaker at position x emits a sound, the detector records:

$$d = x + n \tag{1}$$

where x (in meters) is the true position of the speaker, and $n \sim \mathcal{N}(0, 1)$ is a random number drawn form a normal distribution with mean 0 and standard deviation 1 meter.

Let's assume that there exists a maximum value of x such that detectors placed at $x > x_{\max}$ would not produce a measurable sound. Hence the detection criteria:

$$d \leq x_{\max} \leftrightarrow \rho(d) > \rho_{\text{thr}} \tag{2}$$

Can you see that even a source with $x_{\text{true}} = x_{\max}$ will produce a detectable signal 50% of the time? Give reasoning!

Using the gaussian nature of noise, write the likelihood for the data d given x.

$$p(d|x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(d - x)^2\right] \tag{3}$$

Let us now focus on the inference of the property of the speakers, given a set of detections. We will assume that positions of the speakers that will play are scattered around some central value μ_λ^T and are drawn from a normal distribution:

$$\pi(x|\mu_\lambda, \sigma_\lambda) = \mathcal{N}(x - \mu_\lambda, \sigma_\lambda) \tag{4}$$

Someone has told us the value of σ_λ and our goal is find out the true value of μ_λ, μ_λ^T, given the data recorded by the detector.

Identify the number of hyper parameters ($\vec{\lambda}$) in the problem and what are they. First calculate selection effects $\alpha(\vec{\lambda})$. Assuming a flat prior on hyper parameter(s), write the down the posterior on hyper parameter(s) upto a normalization using eq. (108) in Savaltore et al.

$$p(\vec{\lambda}|D) = \frac{\pi(\vec{\lambda})}{p(D|\mathcal{H}_\Lambda)} \Gamma(N^{\text{tr}} - 1) \prod_{i=1}^{N^{\text{tr}}} \frac{\int d\vec{\theta} p(d_i|\vec{\theta}) \pi(\vec{\theta}|\vec{\lambda})}{\alpha(\vec{\lambda})} \tag{5}$$

Since $\vec{\lambda} = \mu_\lambda$ in our case, the overall posterior for μ_λ is:

$$p(\mu_\lambda|D) \propto \pi(\mu_\lambda) \prod_{i=1}^{N^{\text{tr}}} \frac{\exp\left[-\frac{(d_i - \mu_\lambda)^2}{2(1+\sigma_\lambda^2)}\right]}{\alpha(\mu_\lambda)} \tag{6}$$

Now given the analytical expression for the likelihood, a possible implementation of the algorithm to estimate the hyper parameter μ_λ would be:

- Choose the values of $\mu_\lambda^T, \sigma_\lambda, x_{\max}$.
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