CHAPTER 1

RELATIONS AND FUNCTIONS

IMPORTANT POINTS TO REMEMBER

- Relation R from a set A to a set B is subset of $A \times B$.
- $A \times B = \{(a, b) : a \in A, b \in B\}$.
- If $n(A) = r$, $n(B) = s$ from set A to set B then $n(A \times B) = rs$.

  and no. of relations $= 2^{rs}$
- ϕ is also a relation defined on set A, called the void (empty) relation.
- $R = A \times A$ is called universal relation.
- **Reflexive Relation :** Relation R defined on set A is said to be reflexive iff $(a, a) \in R \ \forall \ a \in A$
- **Symmetric Relation :** Relation R defined on set A is said to be symmetric iff $(a, b) \in R \Rightarrow (b, a) \in R \ \forall \ a, b, \in A$
- **Transitive Relation :** Relation R defined on set A is said to be transitive if $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R \ \forall \ a, b, c \in R$
- **Equivalence Relation :** A relation defined on set A is said to be equivalence relation iff it is reflexive, symmetric and transitive.
- **One-One Function :** $f : A \to B$ is said to be one-one if distinct elements in A has distinct images in B. *i.e.* $\forall \ x_1, x_2 \in A$ *s.t.* $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

  OR

  $\forall \ x_1, x_2 \in A$ s.t. $f(x_1) = f(x_2)$

  $\Rightarrow \ x_1 = x_2$

  One-one function is also called injective function.

- Onto function (surjective) : A function $f : A \rightarrow B$ is said to be onto iff $R_f = B$ *i.e.* $\forall \ b \in B$, there exist $a \in A$ s.t. $f(a) = b$
- A function which is not one-one is called many-one function.
- A function which is not onto is called into.
- **Bijective Function :** A function which is both injective and surjective is called bijective.
- **Composition of Two Function :** If $f : A \rightarrow B$, $g : B \rightarrow C$ are two functions, then composition of f and g denoted by gof is a function from A to C given by, $(gof)\ (x) = g\ (f\ (x)) \ \forall \ x \in A$

  Clearly gof is defined if Range of $f \subset$ domain of g. Similarly fog can be defined.
- **Invertible Function :** A function $f : X \rightarrow Y$ is invertible iff it is bijective.

  If $f : X \rightarrow Y$ is bijective function, then function $g : Y \rightarrow X$ is said to be inverse of f iff $fog = I_y$ and $gof = I_x$

  when I_x, I_y are identity functions.
- g is inverse of f and is denoted by f^{-1}.
- **Binary Operation :** A binary operation '*' defined on set A is a function from $A \times A \rightarrow A$. $*\ (a, b)$ is denoted by $a * b$.
- Binary operation * defined on set A is said to be commutative iff

$$a * b = b * a \ \forall \ a, b \in A.$$

- Binary operation * defined on set A is called associative iff $a * (b * c) = (a * b) * c \ \forall \ a, b, c \in A$
- If * is Binary operation on A, then an element $e \in A$ is said to be the identity element iff $a * e = e * a \ \forall \ a \in A$
- Identity element is unique.
- If * is Binary operation on set A, then an element b is said to be inverse of $a \in A$ iff $a * b = b * a = e$
- Inverse of an element, if it exists, is unique.

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. If A is the set of students of a school then write, which of following relations are. (Universal, Empty or neither of the two).

   R_1 = {(a, b) : a, b are ages of students and $|a - b| \geq 0$}

   R_2 = {(a, b) : a, b are weights of students, and $|a - b| < 0$}

   R_3 = {(a, b) : a, b are students studying in same class}

2. Is the relation R in the set A = {1, 2, 3, 4, 5} defined as R = {(a, b) : $b = a + 1$} reflexive?

3. If R, is a relation in set N given by

   $R = \{(a, b) : a = b - 3, b > 5\}$,

   then does elements $(5, 7) \in$ R?

4. If f : {1, 3} → {1, 2, 5} and g : {1, 2, 5} → {1, 2, 3, 4} be given by

   f = {(1, 2), (3, 5)}, g = {(1, 3), (2, 3), (5, 1)}

   Write down gof.

5. Let $g, f : R \to R$ be defined by

   $g(x) = \dfrac{x+2}{3}, f(x) = 3x - 2.$ Write fog.

6. If $f : R \to R$ defined by

   $f(x) = \dfrac{2x-1}{5}$

   be an invertible function, write $f^{-1}(x)$.

7. If $f(x) = \dfrac{x}{x+1} \forall x \neq -1,$ Write $fo\,f(x)$.

8. Let * is a Binary operation defined on R, then if

   (i) $a * b = a + b + ab$, write 3 * 2

(ii) $a * b = \frac{(a+b)^2}{3}$, Write $(2 * 3) * 4$.

9. If $n(A) = n(B) = 3$, Then how many bijective functions from A to B can be formed?

10. If $f(x) = x + 1$, $g(x) = x - 1$, Then (gof) (3) = ?

11. Is $f : N \to N$ given by $f(x) = x^2$ is one-one? Give reason.

12. If $f : R \to A$, given by

$f(x) = x^2 - 2x + 2$ is onto function, find set A.

13. If $f : A \to B$ is bijective function such that $n(A) = 10$, then $n(B) = ?$

14. If $n(A) = 5$, then write the number of one-one functions from A to A.

15. $R = \{(a, b) : a, b \in N, a \neq b$ and a divides $b\}$. Is R reflexive? Give reason?

16. Is $f : R \to R$, given by $f(x) = |x - 1|$ is one-one? Give reason?

17. $f : R \to B$ given by $f(x) = \sin x$ is onto function, then write set B.

18. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$.

19. If '*' is a binary operation on set Q of rational numbers given by $a * b = \frac{ab}{5}$ then write the identity element in Q.

20. If * is Binary operation on N defined by $a * b = a + ab \ \forall \ a, b \in N$. Write the identity element in N if it exists.

SHORT ANSWER TYPE QUESTIONS (4 Marks)

21. Check the following functions for one-one and onto.

(a) $f : R \to R, \quad f(x) = \frac{2x-3}{7}$

(b) $f : R \to R$, $f(x) = |x + 1|$

(c) $f : R - \{2\} \to R$, $f(x) = \frac{3x-1}{x-2}$

(d) $f : R \to [-1, 1]$, $f(x) = \sin^2 x$

22. Consider the binary operation * on the set {1, 2, 3, 4, 5} defined by $a * b$ = H.C.F. of a and b. Write the operation table for the operation *.

23. Let $f : R - \left\{\frac{-4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ be a function given by $f(x) = \frac{4x}{3x+4}$. Show that f is invertible with $f^{-1}(x) = \frac{4x}{4-3x}$.

24. Let R be the relation on set $A = \{x : x \in Z, 0 \le x \le 10\}$ given by $R = \{(a, b) : (a - b)$ is multiple of $4\}$, is an equivalence relation. Also, write all elements related to 4.

25. Show that function $f : A \to B$ defined as $f(x) = \frac{3x + 4}{5x - 7}$ where $A = R - \left\{\frac{7}{5}\right\}$, $B = R - \left\{\frac{3}{5}\right\}$ is invertible and hence find f^{-1}.

26. Let * be a binary operation on Q. Such that $a * b = a + b - ab$.

    (i) Prove that * is commutative and associative.

    (ii) Find identify element of * in Q (if it exists).

27. If * is a binary operation defined on $R - \{0\}$ defined by $a * b = \frac{2a}{b^2}$, then check * for commutativity and associativity.

28. If $A = N \times N$ and binary operation * is defined on A as

    $(a, b) * (c, d) = (ac, bd)$.

    (i) Check * for commutativity and associativity.

    (ii) Find the identity element for * in A (If it exists).

29. Show that the relation R defined by $(a, b) R (c, d) \Leftrightarrow a + d = b + c$ on the set $N \times N$ is an equivalence relation.

30. Let * be a binary operation on set Q defined by $a * b = \frac{ab}{4}$, *show that*

    (i) 4 is the identity element of * on Q.

(ii) Every non zero element of Q is invertible with

$$a^{-1} = \frac{16}{a}, \quad a \in Q - \{0\}.$$

31. Show that $f : R_+ \to R_+$ defined by $f(x) = \frac{1}{2x}$ is bijective where R_+ is the set of all non-zero positive real numbers.

32. Consider $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$ show that f is invertible with $f^{-1} = \frac{\sqrt{x+6}-1}{3}$.

33. If '*' is a binary operation on R defined by $a * b = a + b + ab$. Prove that * is commutative and associative. Find the identify element. Also show that every element of R is invertible except –1.

34. If $f, g : R \to R$ defined by $f(x) = x^2 - x$ and $g(x) = x + 1$ find (fog) (x) and (gof) (x). Are they equal?

35. $f : [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, find $f^{-1}(x)$.

36. $f : R \to R,\ g : R \to R$ given by $f(x) = [x]$, $g(x) = |x|$ then find

$$(fog)\left(\frac{-2}{3}\right) \text{ and } (gof)\left(\frac{-2}{3}\right).$$

ANSWERS

1. R_1 : is universal relation.

   R_2 : is empty relation.

   R_3 : is neither universal nor empty.

2. No, R is not reflexive.

3. $(5, 7) \notin R$

4. $gof = \{(1, 3), (3, 1)\}$

5. $(fog)(x) = x \ \forall \ x \in R$

6. $f^{-1}(x) = \dfrac{5x+1}{2}$

7. $(fof)(x) = \dfrac{x}{2x+1}, x \neq -\dfrac{1}{2}$

8. (i) 3 * 2 = 11

   (ii) $\dfrac{1369}{27}$

9. 6

10. 3

11. Yes, f is one-one $\because \forall\, x_1, x_2 \in N \Rightarrow x_1^2 = x_2^2$.

12. $A = [1, \infty)$ because $R_f = [1, \infty)$

13. $n(B) = 10$

14. 120.

15. No, R is not reflexive $\because (a, a) \notin R\ \forall\ a \in N$

16. f is not one-one functions

    $\because$ $f(3) = f(-1) = 2$

    $3 \neq -1$ *i.e.* distinct element has same images.

17. $B = [-1, 1]$

19. $e = 5$

20. Identity element does not exist.

21. (a) Bijective

    (b) Neither one-one nor onto.

    (c) One-one, but not onto.

    (d) Neither one-one nor onto.

22.

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

24. Elements related to 4 are 0, 4, 8.

25. $f^{-1}(x) = \dfrac{7x + 4}{5x - 3}$

26. 0 is the identity element.

27. Neither commutative nor associative.

28. (i) Commutative and associative.

(ii) (1, 1) is identity in $N \times N$

33. *0* is the identity element.

34. $(fog)\ (x) = x^2 + x$

$(gof)\ (x) = x^2 - x + 1$

Clearly, they are unequal.

35. $f^{-1}(x) = \dfrac{x + \sqrt{x^2 - 4}}{2}$

36. $(fog)\left(\dfrac{-2}{3}\right) = 0$

$(gof)\left(\dfrac{-2}{3}\right) = 1$