Harmonic Analysis

Definition:

The Process of finding the Fourier series for a function given by numerical values is known as harmonic analysis.

$$a_0 = 2\left[\text{Mean value of } y \text{ in } (0,2\pi)\right] = 2\left[\frac{\Sigma y}{n}\right]$$

$$a_n = 2\left[\text{Mean value of } y\cos nx \text{ in } (0,2\pi)\right] = 2\left[\frac{\Sigma y\cos nx}{n}\right]$$

$$b_n = 2\left[\text{Mean value of } y\sin nx \text{ in } (0,2\pi)\right] = 2\left[\frac{\Sigma y\sin nx}{n}\right]$$

Note: First harmonic

1. The terms $a_1\cos x + b_1\sin x$ is called the fundamental or first harmonic

2. The terms $(a_2\cos 2x + b_2\sin 2x)$ is called the second harmonic and so on.

Type 1

Given data's are in π - Form

Set your Calc in Radian mode

Problem

① Find the Fourier series expansion of period 2π for the function $y = f(x)$ in $(0,2\pi)$ defined by the table values given below. Find the series upto the third harmonic.

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

The last value of y is repetition of the first, so Consider only first six values.

WKT the fourier series upto the third harmonic is

$$y = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

x	y	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$\cos 3x$	$\sin 3x$	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$	$y\cos 3x$	$y\sin 3x$
0	1.0	1	0	1	0	1	0	1	0	1	0	1	0
$\pi/3$	1.4	0.5	0.866	-0.5	0.866	-1	0	0.7	1.212	-0.7	1.212	-1.4	0
$2\pi/3$	1.9	-0.5	0.866	-0.5	-0.866	1	0	-0.95	1.65	-0.95	-1.645	1.9	0
π	1.7	-1	0	1	0	-1	0	-1.7	0	1.7	0	-1.7	0
$4\pi/3$	1.5	-0.5	-0.866	-0.5	0.866	1	0	-0.75	-1.299	-0.75	1.299	1.5	0
$5\pi/3$	1.2	0.5	-0.866	-0.5	-0.866	-1	0	0.6	-1.039	-0.6	-1.039	-1.2	0
$\Sigma y : 8.7$								-1.1	0.524	-0.3	-0.173	0.1	0
								$\Sigma y\cos x$	$\Sigma y\sin x$	$\Sigma y\cos 2x$	$\Sigma y\sin 2x$	$\Sigma y\cos 3x$	$\Sigma y\sin 3x$

$a_0 = 2$ [Mean value of y in $(0, 2\pi)$]

$= 2 \frac{\Sigma y}{n}$

$= 2 \times \frac{8.7}{6} = 2.9$

$a_1 = 2$ [Mean values of $y\cos x$ in $(0, 2\pi)$]

$= 2 \frac{\Sigma y \cos x}{n}$

$= 2 \times \frac{(-1.1)}{6} = -0.37$

$a_2 = 2$ [Mean values of $y\cos 2x$ in $(0, 2\pi)$]

$= 2 \frac{\Sigma y \cos 2x}{n}$

$= 2 \times \frac{(-0.3)}{6} = -0.1$

$a_3 = 2$ [Mean values of $y\cos 3x$ in $(0, 2\pi)$]

$= 2 \frac{\Sigma y \cos 3x}{n}$

$= 2 \times \frac{0.1}{6} = 0.03$

$b_1 = 2$ [Mean value of $y\sin x$ in $(0, 2\pi)$]

$= 2 \frac{\Sigma y \sin x}{n}$

$= \frac{2}{6} \times 0.524 = 0.17$

$b_2 = 2$ [Mean value of $y\sin 2x$ in $(0, 2\pi)$]

$= 2 \frac{\Sigma y \sin 2x}{n}$

$= \frac{2}{6} \times (-0.173) = -0.06$

$b_3 = 2$ [Mean value of $y\sin 3x$ in $(0, 2\pi)$]

$= 2 \frac{\Sigma y \sin 3x}{n} = \frac{2}{6} \times (-0) = 0$

Substitute the values of a_0, a_1, a_2, a_3 and b_1, b_2, b_3 in ①

$$y = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x)$$

$$= 1.45 + (-0.37 \cos x + 0.17 \sin x) - (0.1 \cos 2x + 0.06 \sin 2x) + 0.03 \cos 3x.$$

② Obtain a Fourier series upto the second harmonic from the data

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	0.8	0.6	0.4	0.7	0.9	1.1	0.8

Soln:-

The First and last values are same, so consider first 6 values.

The Fourier series $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$ ①

x	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	0.8	1	1	0	0	0.8	0.8	0	0
$\pi/3$	0.6	0.5	-0.5	0.866	0.866	0.3	-0.3	0.5196	0.5196
$2\pi/3$	0.4	-0.5	-0.5	0.866	-0.866	-0.2	-0.2	0.3464	-0.3464
π	0.7	-1	1	0	0	-0.7	0.7	0	0
$4\pi/3$	0.9	-0.5	-0.5	-0.866	0.866	-0.45	-0.45	-0.7794	0.7794
$5\pi/3$	1.1	0.5	-0.5	-0.866	-0.866	0.55	-0.55	-0.9526	-0.9526
	4.5					0.3	0	-0.866	0
	Σy					$\Sigma y\cos x$	$\Sigma y\cos 2x$	$\Sigma y\sin x$	$\Sigma y\sin 2x$

$a_0 = 2$ [Mean value of y in $(0, 2\pi)$]

$= \dfrac{2\sum y}{n} = \dfrac{2}{6} \times 4.5 = 1.5$

$a_1 = 2$ [Mean value of $y\cos x$ in $(0, 2\pi)$]

$= \dfrac{2\sum y\cos x}{n}$

$= \dfrac{2}{6} \times 0.3 = 0.1$

$a_2 = 2$ [Mean value of $y\cos 2x$ in $(0, 2\pi)$]

$= \dfrac{2\sum y\cos 2x}{n}$

$= \dfrac{2}{6} \times 0 = 0$

$b_1 = 2$ [Mean value of $y\sin x$ in $(0, \pi)$]

$= \dfrac{2\sum y\sin x}{6}$

$= \dfrac{2}{6} \times (-0.866) = -0.2887$

$b_2 = 2$ [Mean value of $y\sin 2x$ in $(0, 2\pi)$]

$= 2\left[\dfrac{\sum y\sin 2x}{6}\right]$

$= \dfrac{2}{6} \times 0 = 0.$

Substituting these values of a_0, a_1, a_2 and b_1, b_2 in ①

$\therefore y = \dfrac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x)$

$= \dfrac{1.5}{2} + (0.1\cos x - 0.2887\sin x) + (0\cos 2x + 0\sin 2x)$

$= \dfrac{1.5}{2} + 0.1\cos x - 0.2887\sin x.$

③ Determine the 1st two harmonic of the F.S for the following values

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1.98	1.30	1.05	1.30	−0.88	−0.25	1.98

Ans: $a_0 = 1.5$, $a_1 = 0.373$, $a_2 = 0.023$, $b_1 = 1.005$, $b_2 = -0.109$

Type 2. Given data are in Degree Form

① Compute first two harmonic of the Fourier series of f(x) from the Table below:

x	0	$60°$	$120°$	$180°$	$240°$	$300°$	360
y	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Soln:

The First and Last values are same, so Let us Consider First 6 values.

The Fourier series $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$

x	x in θ	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
$0°$	0	1.98	1	1	0	0	1.98	1.98	0	0
$60°$	$\pi/3$	1.30	0.5	-0.5	0.866	0.866	0.65	-0.65	1.126	1.126
$120°$	$2\pi/3$	1.05	-0.5	-0.5	0.866	-0.866	-0.525	-0.525	0.909	-0.909
$180°$	π	1.30	-1	1	0	0	-1.30	-1.30	0	0
$240°$	$4\pi/3$	-0.88	-0.5	-0.5	-0.866	0.866	0.44	0.44	0.762	-0.762
$300°$	$5\pi/3$	-0.25	0.5	-0.5	-0.866	-0.866	-0.125	0.125	0.217	0.217
		4.5					1.12	0.07	3.014	-0.328
		Σy					$\Sigma y\cos x$	$\Sigma y\cos 2x$	$\Sigma y\sin x$	$\Sigma y\sin 2x$

$\therefore \quad a_0 = 2[\text{Mean value of } y]$

$$= \frac{2\sum y}{6} = \frac{2}{6} \times 4.5 = 1.5$$

$a_1 = 2[\text{Mean value of } y\cos x]$

$$= \frac{2\sum y\cos x}{n}$$

$$= \frac{2 \times 1.12}{6}$$

$$= 0.373$$

$a_2 = 2[\text{Mean value of } y\cos 2x]$

$$= \frac{2\sum y\cos 2x}{n}$$

$$= \frac{2}{6} \times 0.07 = 0.023$$

$b_1 = 2[\text{Mean value of } y\sin x]$

$$= 2\left[\frac{\sum y\sin x}{n}\right]$$

$$= 2\left[\frac{3.014}{6}\right] = 1.005$$

$b_2 = 2[\text{Mean value of } y\sin 2x]$

$$= \frac{2[\sum y\sin 2x]}{n}$$

$$= \frac{2[-0.328]}{6} = -0.109$$

$\therefore$ The Fourier Series

$$f(x) = \frac{a_0}{2} + \cancel{\sum_{n=1}} a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x$$

$$= \frac{1.5}{2} + 0.373\cos x + 0.023\cos 2x + 1.005\sin x - 0.109\sin 2x$$

② Find the First harmonic of the F.S of $f(x)$ from the following table

x	0	60	120	180	240	300	360
y	40.0	31.0	-13.7	20.0	3.7	-21.0	40.0

Ans $a_0 = 20$, $a_1 = 10$, $b_1 = 9.988$

Type 3 - Given Data are in T- Forms.

① The following table gives the variations of a periodic function over a period T

x	0	$T/6$	$T/3$	$T/2$	$\frac{2T}{3}$	$\frac{5T}{6}$	T
$f(x)$	1.98	1.3	1.05	1.3	-0.88	-0.25	1.98

Soln:-

Here the Last value repetition of first, so Consider first 6 values

$$f(x) = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta$$

Given $\theta = \frac{2\pi x}{T}$

when take x values $0, \frac{T}{6}, T/3, T/2, \frac{2T}{3}, 5T/6$

x	θ	y	$\cos\theta$	$\sin\theta$	$y\cos\theta$	$y\sin\theta$
0	0	1.98	1	0	1.98	0
$\frac{T}{6}$	$\pi/3$	1.30	0.5	0.866	0.65	1.125
$\frac{T}{3}$	$2\pi/3$	1.05	-0.5	0.866	-0.525	0.909
$\frac{T}{2}$	π	1.30	-1	0	-1.3	0
$\frac{2T}{3}$	$4\pi/3$	-0.88	-0.5	-0.866	0.44	0.762
$\frac{5T}{6}$	$5\pi/3$	-0.25	0.5	-0.866	-0.125	0.216
		4.5			1.12	3.01

$$a_0 = \frac{2[\Sigma y]}{6} = 2[\text{Mean value of } y]$$

$$= \frac{2}{6} \times 4.5$$

$$= 1.5$$

$$a_1 = 2[\text{Mean value of } y\cos x]$$

$$= \frac{2\,\Sigma y\cos x}{n}$$

$$= \frac{2 \times 1.12}{6}$$

$$= 0.37$$

$$b_1 = 2[\text{mean value of } y\sin x]$$

$$= \frac{2[\Sigma y\sin x]}{n}$$

$$= \frac{2}{6} \times 3.01 \qquad = 1.004$$

sub the values

$$f(x) = \frac{1.5}{2} + 0.37\cos x + 1.004\sin x$$

② obtain the constant term and the first two harmonic of the Fourier series of $f(x)$ from the following table

x	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
y	0	9.2	14.4	17.8	17.3	11.7	0

Ans

$a_0 = 23.47$, $a_1 = -7.733$, $a_2 = -2.833$, $b_1 = -1.559$

$b_2 = 0.115$.

Type A. Given data are in 'f' Form.

① Find the Fourier series upto second harmonic for the following data for y with period 6.

x	0	1	2	3	4	5
$f(x)$	9	18	24	28	26	20

Soln:

Here the length of the interval is 6 (not 2π)

(ie) $2l = 6$

$\boxed{l = 3}$

$\therefore$ The Fourier series $f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + a_2 \cos\frac{2\pi x}{3} + b_1 \sin\frac{\pi x}{3} + b_2 \sin\frac{2\pi x}{3}$

x	$\frac{\pi x}{3}$	$\frac{2\pi x}{3}$	y	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\sin\frac{2\pi x}{3}$
0	0	0	9	9	0	9	0
1	$\pi/3$	$2\pi/3$	18	9	15.7	9	15.7
2	$2\pi/3$	$4\pi/3$	24	-12	20.9	-12	-20.9
3	π	2π	28	-28	0	28	0
4	$4\pi/3$	$8\pi/3$	26	-13	-22.6	-13	-22.6
5	$5\pi/3$	$10\pi/3$	20	10	-17.4	-10	-17.4
			125	-25	-3.4	-7	-0.1
			Σy	$\Sigma y\cos\frac{\pi x}{3}$	$\Sigma y\sin\frac{\pi x}{3}$	$\Sigma y\cos\frac{2\pi x}{3}$	$\Sigma y\sin\frac{2\pi x}{3}$

$a_0 = 2\left[\text{Mean value of } y\right]$

$= \dfrac{2\,\Sigma y}{n} = 2 \times \dfrac{125}{6} = 41.66$

$a_1 = 2\left[\text{Mean value of } y \cos\dfrac{\pi x}{3}\right]$

$= \dfrac{2\,\Sigma y \cos\frac{\pi x}{3}}{n}$

$= \dfrac{2}{6} \times (-25) = -8.33$

$a_2 = 2\left[\text{Mean value of } y \cos\dfrac{2\pi x}{3}\right]$

$= 2\,\dfrac{\Sigma y \cos 2\pi x}{n}$

$= \dfrac{2 \times (-7)}{6} = -2.33$

$b_1 = 2\left[\text{Mean value of } y \sin\dfrac{\pi x}{3}\right]$

$= \dfrac{2\,\Sigma y \sin\frac{\pi x}{3}}{n} = \dfrac{2 \times (-3.4)}{6} = -1.13$

$b_2 = 2\left[\text{Mean value of } y \sin\dfrac{2\pi x}{3}\right]$

$= 2\,\dfrac{\Sigma y \sin\frac{2\pi x}{3}}{n}$

$= \dfrac{2}{6} \times (-0.1) = -0.03$

∴ The Fourier series

$f(x) = \dfrac{a_0}{2} + a_1 \cos\dfrac{\pi x}{3} + b_1 \sin\dfrac{\pi x}{3} + a_2 \cos\dfrac{2\pi x}{3} + b_2 \sin\dfrac{2\pi x}{3}$

$= \dfrac{41.66}{2} - 8.33 \cos\dfrac{\pi x}{3} - 1.13 \sin\dfrac{\pi x}{3} - 2.33 \cos\dfrac{2\pi x}{3} - 0.03 \sin\dfrac{2\pi x}{3}$

=x=

Compute the first harmonic of the Fourier series of f(x)

x	0	1	2	3	4	5
$f(x)$	4	8	15	7	6	2

Soln:-

Ans : $a_0 = 2\,\dfrac{\Sigma y}{6} = 2\left(\dfrac{42}{6}\right) = 14$

$a_1 = 2\,\dfrac{\Sigma y \cos\frac{\pi x}{3}}{6} = 2\left(\dfrac{-8.5}{6}\right) = -2.83$

$b_1 = 2\,\dfrac{\Sigma y \sin \pi x/3}{6} = 2\,\dfrac{(12.99)}{6} = 4.33$