Unit I

Fourier series.

1. Dirichlet's Conditions

2. General Fourier series
   i. Problems on $(0, 2\pi)$
   ii) Problems on $(0, 2l)$

3. odd and Even functions
   i). Problems on $(-\pi, \pi)$
   ii) Problems on $(-l, l)$
   iii) Problems on odd function in $(-\pi, \pi)$
   iv) Problems on Even function in $(-l, l)$

4. Half-Range sine series
   i) Fourier sine series in $(0, \pi)$
   ii) Fourier sine series in $(0, l)$

5 Half-Range Cosine series
   i). Fourier cosine series in $(0, \pi)$
   ii) Fourier Cosine series in $(0, l)$

parseval's identity

Harmonic Analysis.
   i). Type 1 :- π - Form
   ii) Type 2: T - Form
   iii) Type 3: Degree Form

Unit I Fourier Series

plan

Day 1 : Dirichlet's conditions, General Fourier series.

Day 2 : ODD and Even functions

Day 3 : Half - Range sine and cosine series.

Day 4 : parseval's identity

Day 5 : Harmonic Analysis.

Important formula

1. $\int dx = x + c$

2. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

3. $\int e^{ax} dx = \frac{e^{ax}}{a} + c$

4 $\int \sin ax \, dx = -\frac{\cos ax}{a} + c$

5 $\int \cos ax \, dx = \frac{\sin ax}{a} + c$

6 $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) + c$

7. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) + c$

8 $\int uv \, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

u' - differentiation

v_1 - Integration.

9. $\sin n\pi = 0$

$\sin 0 = 0$

$\sin \pi = 0$

$\sin 2\pi = 0$

$\sin 2n\pi = 0$.

10. $\cos n\pi = (-1)^n$

$\cos 0 = (-1)^0 = 1$

$\cos \pi = (-1)^1 = -1$

$\cos 2\pi = (-1)^2 = 1$

$\cos 3\pi = (-1)^3 = -1$

$\cos 2n\pi = (-1)^{2n} = 1$

$\cos(n+1)\pi = (-1)^{n+1}$

$\cos(n-1)\pi = (-1)^{n-1}$

11. $\sin(-\theta) = -\sin\theta$

12. $\cos(-\theta) = \cos\theta$

13. $\sin(2n+1)\frac{\pi}{2} = (-1)^n$

14. $\cos(2n+1)\frac{\pi}{2} = 0$

periodic functions:

A function $f(x)$ is said to be a period T if for all x,

$f(x+T) = f(x)$, where T is a positive constant. The least value of $T > 0$ is called the period of $f(x)$.

Example:

1. $f(x) = \sin x$

$f(x+2\pi) = \sin(x+2\pi) = \sin x = f(x)$

$\sin x$ is a periodic function with period 2π.

2. $f(x) = \cos x$

$f(x+2\pi) = \cos(x+2\pi) = f(x)$

$\cos x$ is a periodic function with period 2π.

3. $\tan x = \tan(\pi + x)$

$\tan x$ is a periodic function with period π.

4. The period of $\sin nx$ and $\cos nx$ is $\frac{2\pi}{n}$.

Dirichlet's Condition's.

A fourier series can be written for a function $f(x)$ if it will satisfy following properties/ Conditions.

1. $f(x)$ must be periodic, finite and single valued.
2. $f(x)$ has discontinuties at finite number of points.
3. $f(x)$ has at most a finite maxima and minima in any one period (interval)
4. $f(x)$ and $f'(x)$ are piecewise continuous.

Continuous function.

A function $f(x)$ is said to be Continuous at $x = a$ if

$$f(a-0) = f(a+0) = f(a)$$

Note: $f(x)$ is said to be Continuous in an interval (a, b) if it is Continuous at every point of the interval.

Dis Continuous function

A function $f(x)$ is said to be dis continuous at a point if it is not Continuous at the point.

Ex: $f(x) = \begin{cases} x, & x < 1 \\ x^2, & x > 1 \end{cases}$ here $x = 1$ is a point of dis Continuity.

piecewise Continuous:

A piecewise Continuous function is one that has atmost a finite number of finite dis continuities.

Fourier series:

The infinite trignometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$ is called the Fourier series of $f(x)$ which satisfies Dirichelet's Conditions in $c \leq x \leq c+2l$

Here a_0, a_n and b_n are Called Fourier Coefficients.

$$a_0 = \frac{1}{l} \int_0^{c+2l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_0^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_0^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx$$

The values of a_0, a_n and b_n are known as Euler's formula.

In $(0, 2\pi)$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

In $(-\pi, \pi)$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

In $(0, 2l)$

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l}\, dx$$

In $(-l, l)$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx$$

(17). Can $\tan x$ be expanded in Fourier series.

$\tan x$ Cannot be expanded as a Fourier series. Since it infinite number of infinite dis continuities.

Convergence of Fourier series.

Let $f(x)$ can be expanded as a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{in } (c, c+2l)$$

This series converges to

i) $f(x_0)$ if 'x_0' is a point of continuity

ii) $\frac{f(x_0+0)+f(x_0-0)}{2}$, if '$x_0$' is a point of dis continuity.

General Fourier series

Problems on $(0, 2\pi)$

① Expand in Fourier series of periodicity 2π of $f(x) = x^2$ for $0<x$

Soln:-

Step 1:

The Fourier series of $f(x)$ in $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—①}$$

Here $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx.$$

Step 2: To Find a_0.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^{2\pi} = \left[\frac{(2\pi)^3}{3} - 0\right] = \frac{8\pi^3}{3}$$

Step 3: To Find a_n.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx \, dx \quad \text{, using Bernoulli's formula}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

using Bernoulli's formula

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

u' - diff

v_1 - ing

$u = x^2$ $\quad V = \cos nx$

$u' = 2x$ $\quad V_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\quad V_2 = -\frac{\cos nx}{n^2}$

$u''' = 0$ $\quad V_3 = -\frac{\sin nx}{n^3}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} u\, v \, dx$$, Bernoulli's formula

$$= \frac{1}{\pi}\left[uv_1 - u'v_2 + u''V_3 + \dots\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2 \frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(4\pi^2 \frac{\cancel{\sin 2n\pi}}{n} + \frac{4\pi \cos 2n\pi}{n^2} - \frac{2\cancel{\sin 2n\pi}}{n^3}\right) - (0+0-0)\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right]$$

$\cos 2n\pi = 1$

$\sin n\pi = 0$

$$a_n = \frac{4}{n^2}$$

$\sin 2n\pi = 0$

Step 4: To Find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx \, dx$$

$\Rightarrow$ using Bernoulli's formula. $\int uv\,dx = uv_1 - u'v_2 + u''V_3 - \dots$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$u = x^2$ $\quad V = \sin nx$

$u' = 2x$ $\quad V_1 = -\frac{\cos nx}{n}$

$u'' = 2$ $\quad V_2 = -\frac{\sin nx}{n^2}$ $\quad V_3 = \frac{\cos nx}{n^3}$

$$= \frac{1}{\pi}\left[-x^2\frac{\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-4\pi^2\frac{\cos 2n\pi}{n} + 4\pi\frac{\sin 2n\pi}{n^2} + 2\frac{\cos 2n\pi}{n^3}\right) - \left(0+0+\frac{2\cos 0}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[-4\pi^2\frac{\cos 2n\pi}{n} + \cancel{4\pi\frac{\sin 2n\pi}{n^2}} + 2\frac{\cos 2n\pi}{n^3} - \frac{2}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \cancel{\frac{2}{n^3}} - \cancel{\frac{2}{n^3}}\right]$$

$\sin 2n\pi = 0$

$\cos 2n\pi = 1$

$$= -\frac{4\pi}{n}$$

Step 5: The required Fourier series

sub a_0, a_n and b_n in ①

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx + \sum_{n=1}^{\infty}\left(\frac{-4\pi}{n}\right)\sin nx$$

$$= \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nx}{n}.$$

=x=

2. Find the Fourier series of $f(x) = (\pi - x)^2$ in $(0, 2\pi)$ of periodicity 2π.

Soln:-

Step 1:

The Fourier series of the function $f(x)$ in the interval $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

here $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

Step 2 : To Find a_0

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} (\pi - x)^2\, dx$$

$$= \frac{1}{\pi} \left[\frac{(\pi - x)^3}{-3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{(\pi - (2\pi))^3}{-3} - \frac{(\pi - 0)^3}{-3} \right]$$

$$= \frac{1}{\pi} \left[\frac{(-\pi)^3}{-3} - \frac{\pi^3}{-3} \right]$$

$$= \frac{1}{\pi} \left[\frac{-\pi^3}{-3} + \frac{\pi^3}{3} \right] = \frac{1}{\pi} \left[\frac{\pi^3}{3} + \frac{\pi^3}{3} \right]$$

$$= \frac{1}{\pi} \left[\frac{2\pi^3}{3} \right] = \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

Step 3 : To Find a_n

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} (\pi - x)^2 \cos nx\, dx$$

using Bernoulli's formula

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

$u = (\pi - x)^2$ $\quad$ $v = \cos nx$

$u' = -2(\pi - x)$ $\quad$ $v_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\quad$ $v_2 = -\frac{\cos nx}{n^2}$

$u''' = 0$ $\quad$ $v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{1}{\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - 2(\pi-x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(\pi-x)^2\frac{\cancel{\sin nx}}{n} - 2(\pi-x)\frac{\cos nx}{n^2} - 2\frac{\cancel{\sin nx}}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-2(\pi-x)\frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$\sin 0 = 0$

$\sin 2n\pi = 0$

$$= \frac{1}{\pi}\left[-2(\pi-2\pi)\frac{\cos 2n\pi}{n^2} + 2(\pi-0)\frac{\cos 0}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-2(-\pi)\frac{1}{n^2} + \frac{2\pi}{n^2}\right]$$

$\cos 2n\pi = 1$

$\cos 0 = 1$

$$= \frac{1}{\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right]$$

$$\boxed{a_n = \frac{4}{n^2}}$$

step 4 : To Find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\, dx.$$

using Bernoulli's formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

$u = (\pi-x)^2$ $\quad$ $v = \sin nx$

$u' = 2(\pi-x)(-1)$ $\quad$ $v_1 = -\frac{\cos nx}{n}$

$u'' = 2$ $\quad$ $v_2 = -\frac{\sin nx}{n^2}$

$u''' = 0$ $\quad$ $v_3 = +\frac{\cos nx}{n^3}$

$$= \frac{1}{\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi-x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-(\pi-x)^2\frac{\cos nx}{n} - \cancel{2(\pi-x)\frac{\sin nx}{n^2}} + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$\sin 2n\pi = 0$

$\sin 0 = 0$

$$= \frac{1}{\pi}\left[-(\pi-x)^2\frac{\cos nx}{n} + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-(\pi-2\pi)^2\frac{\cos 2n\pi}{n} + \frac{2\cos 2n\pi}{n^3}\right) - \left(-(\pi-0)^2\frac{\cos 0}{n} + 2\frac{\cos 0}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\cancel{-\frac{\pi^2}{n}} + \cancel{\frac{2}{n^3}} + \cancel{\frac{\pi^2}{n}} - \cancel{\frac{2}{n^3}}\right]$$

$$= 0$$

$$\boxed{b_n = 0}$$

Step 5 : The required Fourier series

sub a_0, a_n and b_n in ①

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx + \sum_{n=1}^{\infty} 0 \sin nx$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

$$f(x) = \frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots\right]$$

$= x =$

(3) Find the Fourier series expansion of $f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi \\ 2\pi - x & \text{for } \pi \le x \le 2\pi \end{cases}$.

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{}$

Soln:

Step 1: The Fourier Series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Here $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx.$$

Step 2: To Find a_0

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

here $a_0 = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\, dx + \int_{\pi}^{2\pi} f(x)\, dx\right]$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \left(\left(4\pi^2 - \frac{4\pi^2}{2}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 2\pi^2 - \left(\frac{4\pi^2 - \pi^2}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 2\pi^2 - \frac{3\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{4\pi^2 - 3\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\pi^2\right]$$

$$\boxed{a_0 = \pi}$$

Step 3 : To Find a_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} f(x) \cos nx \, dx + \int_{\pi}^{2\pi} f(x) \cos nx \, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x \cos nx \, dx + \int_{\pi}^{2\pi} (2\pi - x) \cos nx \, dx\right\}$$

using Bernoulli's formula $\int uv \, dx = uv_1 - u'v_2 + u''v_3 - \ldots$

$u = x \qquad V = \cos nx \qquad\qquad u = (2\pi - x) \qquad V = \cos nx$

$u' = 1 \qquad V_1 = \frac{\sin nx}{n} \qquad\qquad u' = -1 \qquad V_1 = \frac{\sin nx}{n}$

$u'' = 0 \qquad V_2 = -\frac{\cos nx}{n^2} \qquad\qquad u'' = 0 \qquad V_2 = -\frac{\cos nx}{n^2}$

$$= \frac{1}{\pi}\left\{\left[x\left(\cancel{\frac{\sin nx}{n}}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\cancel{\frac{\sin nx}{n}}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] + \left[-\frac{\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2}\right]\right\}$$

$\sin n\pi = 0$

$\sin 0 = 0$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{\cos n\pi}{n^2}\right]$$

$$= \frac{2}{n^2\pi}\left[\cos n\pi - 1\right].$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$\therefore a_n = \begin{cases} 0, & \text{when 'n' is even} \\ -\frac{4}{n^2\pi}, & \text{when 'n' is odd} \end{cases}$$

Step 4 : To find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) . \sin nx \, dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} f(x) \sin nx \, dx + \int_{\pi}^{2\pi} f(x) \sin nx \, dx\right\}$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \sin nx \, dx + \int_{\pi}^{2\pi} (2\pi - x) \sin nx \, dx\right]$$

using Bernoulli's formula

$$\int u v \, dx = u v_1 - u' v_2 + u'' v_3 - \dots$$

$u = x$	$v = \sin nx$	$u = 2\pi - x$	$v = \sin nx$
$u' = 1$	$v_1 = -\frac{\cos nx}{n}$	$u' = -1$	$v_1 = -\frac{\cos nx}{n}$
$u'' = 0$	$v_2 = -\frac{\sin nx}{n^2}$	$u'' = 0$	$v_2 = -\frac{\sin nx}{n^2}$

$$= \frac{1}{\pi}\left\{\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\cancel{\frac{\sin nx}{n^2}}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\cancel{\frac{\sin nx}{n^2}}\right)\right]_{\pi}^{2\pi}\right\}$$

$\sin 0 = 0$

$\sin \pi = 0$

$\sin 2n\pi = 0$

$$= \frac{1}{\pi}\left\{\left(-x\frac{\cos nx}{n}\right)_0^{\pi} + \left[(2\pi - x)\left(-\frac{\cos nx}{n}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left(-\pi\frac{\cos n\pi}{n}\right) + \left(-\cancel{(2\pi - 2\pi)\frac{\cos 2n\pi}{n}} + (2\pi - \pi)\frac{\cos n\pi}{n}\right)\right\}$$

$$= \frac{1}{\pi}\left\{-\pi\cancel{\frac{(-1)^n}{n}} + \pi\cancel{\frac{(-1)^n}{n}}\right\}$$

$$= 0$$

$$\boxed{b_n = 0}$$

Step 5 : The required Fourier series

sub a_0, a_n and b_n in (1)

$$f(x) = \frac{a_0}{2} + \sum_{n=odd}\left(-\frac{4}{n^2\pi}\right)\cos nx$$

$$f(x) = \frac{a_0}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\dots}^{\infty}\frac{\cos nx}{n^2}$$

Here $a_n = \frac{1}{n^2}$

or

$b_n = 1/n$

Choice

$x = \pi, \pi/2, 0$

Step 6 : Deduction.

Let us choose $x = 0$

here $x = 0$ is a point of discontinuity

so the average value of $f(x)$.

Given $f(x) = \begin{cases} x & \text{in } 0 \le x \le \pi \\ 2\pi - x & \text{in } \pi \le x \le 2\pi \end{cases}$

$\therefore f(0) = 0$

$f(2\pi) = 2\pi - 2\pi = 0$

$$\frac{f(0)+f(2\pi)}{2} = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2}$$

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5\dots}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} = \frac{\pi}{2}$$

$$\sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} = \frac{\pi}{2} \times \frac{\pi}{4}$$

$$\frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

$=x=$

④ If $f(x) = \left(\frac{\pi - x}{2}\right)$ find the Fourier series of the period 2π in the interval $(0, 2\pi)$. Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$.

⑤ Express $f(x) = x\sin x$ as a Fourier series in $0 \le x \le 2\pi$.

$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

$2 \sin A \cos B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

Fourier series Problems on $(0, 2l)$

① Find the Fourier series expansion of period $2l$ for the function $f(x) = (l-x)^2$ in the range $(0, 2l)$. Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Soln:

Step 1:

The Fourier series of $f(x)$ in $(0, 2l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Step 2: To Find a_0.

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx$$

$$= \frac{1}{l} \int_0^{2l} (l-x)^2\, dx$$

$$= \frac{1}{l} \left[\frac{(l-x)^3}{-3} \right]_0^{2l}$$

$$= \frac{1}{l} \left[\frac{(l-2l)^3}{-3} - \frac{(l-0)^3}{-3} \right]$$

$$= \frac{1}{l} \left[\frac{l^3}{3} + \frac{l^3}{3} \right]$$

$$= \frac{2l^3}{3l} = \frac{2l^2}{3}$$

$$\boxed{a_0 = \frac{2l^2}{3}}$$

Step 3: To Find a_n

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{1}{l} \int_0^{2l} (l-x)^2 \cos \frac{n\pi x}{l}\, dx$$

using Bernoulli's formula $\int u v\, dx = u v_1 - u' v_2 + u'' v_3 - \dots$

$u = (l-x)^2$ $\qquad$ $v = \cos \frac{n\pi x}{l}$

$u' = 2(l-x)(-1)$ $\qquad$ $v_1 = \dfrac{\sin \frac{n\pi x}{l}}{n\pi / l}$

$u'' = 2$ $\qquad v_2 = \dfrac{-\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2}$

$u''' = 0$ $\qquad V_3 = \dfrac{-\sin\frac{n\pi x}{l}}{n^3\pi^3/l^3}$

$$= \frac{1}{l}\left[(l-x)^2\left(\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right) - 2(l-x)(-1)\left(\frac{-\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right) + 2\left(-\frac{\sin\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[(l-x)^2\left(\cancel{\frac{\sin\frac{n\pi x}{l}}{n\pi/l}}\right) - 2(l-x)\left(\frac{\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right) - \cancel{2\frac{\sin\frac{n\pi x}{l}}{n^3\pi^3/l^3}}\right]_0^{2l}$$

$$= \frac{1}{l}\left[-2(l-x)\frac{\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right]_0^{2l}$$

$\sin 0 = 0$

$\sin 2n\pi = 0$

$$= \frac{-2l}{n^2\pi^2}\left[(l-x)\cos\frac{n\pi x}{l}\right]_0^{2l}$$

$$= -\frac{2l}{n^2\pi^2}\left[(l-2l)\cos 2n\pi - (l-0)\cos 0\right]$$

$$= -\frac{2l}{n^2\pi^2}\left[-l-l\right]$$

$$= -\frac{2l}{n^2\pi^2}\left[-2l\right]$$

$$= \frac{4l^2}{n^2\pi^2}$$

$$\boxed{a_n = \frac{4l^2}{n^2\pi^2}}$$

Step 4 : To Find b_n

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^{l}(l-x)^2\sin\frac{n\pi x}{l}\,dx$$

using Bernoulli's formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \ldots$

$u = (l-x)^2$ $\qquad V = \sin\frac{n\pi x}{l}$

$u' = 2(l-x)(-1)$ $\qquad V_1 = \frac{-\cos\frac{n\pi x}{l}}{n\pi/l}$

$u'' = 2$ $\qquad V_2 = \frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}$

$u''' = 0$ $\qquad V_3 = \frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}$

$$= \frac{1}{l}\left[(l-x)^2\left(-\frac{\cos\frac{n\pi x}{l}}{n\pi/l}\right) - 2(l-x)(-1)\left\{-\frac{\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right\} + 2\left(\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[-(l-x)^2\frac{\cos\frac{n\pi x}{l}}{n\pi/l} - \cancel{2(l-x)\frac{\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}} + 2\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right]_0^{2l}$$

0

$\sin 0 = 0$

$\sin 2n\pi = 0$

$$= \frac{1}{l}\left[-(l-x)^2\frac{\cos\frac{n\pi x}{l}}{n\pi/l} + 2\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\left(-(l-2l)^2\frac{\cos 2n\pi}{n\pi/l} + 2\frac{\cos 2n\pi}{n^3\pi^3/l^3}\right) - \left(-(l-0)^2\frac{\cos 0}{n\pi/l} + \frac{2\cos 0}{n^3\pi^3/l^3}\right)\right]$$

$$= \frac{1}{l}\left[-l^2\cancel{\frac{1}{n\pi/l}} + \cancel{2\frac{1}{n^3\pi^3/l^3}} + l^2\cdot\cancel{\frac{1}{n\pi/l}} - \cancel{2\frac{1}{n^3\pi^3/l^3}}\right]$$

$= 0$

$$\boxed{b_n = 0}$$

Step 5 : The required Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

sub a_0, a_n and b_n.

$$f(x) = l^2/3 + \sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$= l^2/3 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}$$

Step 6: Deduction

Here $x = 0$ is a point of discontinuity

so the average value of $f(x)$ is

$$f(x) = (l-x)^2$$

$$f(0) = (l-0)^2 = l^2$$

$$f(2l) = (l-2l) = l^2$$

$\therefore$ The ends points are

$x=0, \quad x=2l$

$$f(x) = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}$$

$$\frac{f(0)+f(2l)}{2} = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{l^2+l^2}{2} = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$l^2 - \frac{l^2}{3} = \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{3l^2-l^2}{3} = \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{2\cancel{l^2}}{3} = \frac{4\cancel{l^2}}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{2\pi^2}{12} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2/6$$

$=x=$

(2) Find the Fourier series for $f(x) = 2x - x^2$ in the interval $0 < x < 2$

Soln

Step 1: The Fourier series of $f(x)$ in $(0, 2)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

here $2l = 2$

$\boxed{l = 1}$

Step 2: To Find a_0

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx$$

$$= \frac{1}{1} \int_0^{2} (2x - x^2)\, dx$$

$$= \int_0^{2} (2x - x^2)\, dx$$

$$= \left[\frac{2x^2}{2} - \frac{x^3}{3} \right]_0^2$$

$$= \left[\left(4 - \frac{8}{3}\right) - (0 - 0) \right]$$

$$= \frac{12 - 8}{3}$$

$$= \frac{4}{3}$$

$$\boxed{a_0 = \frac{4}{3}}$$

Step 3: To Find a_n

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

here $l = 1$

$f(x) = 2x - x^2$

$$= \frac{1}{1} \int_0^{2} (2x - x^2) \cos n\pi x\, dx$$

$$= \int_0^{2} (2x - x^2) \cos n\pi x\, dx$$

using Bernoulli's formule $\int uv\, dx = uv_1 - u'v_2 + u''v_3 - \ldots$

$u = (2x - x^2)$ $\quad$ $v = \cos n\pi x$

$u' = 2 - 2x$ $\quad$ $v_1 = \frac{\sin n\pi x}{n\pi}$

$u'' = -2$ $\quad$ $v_2 = -\frac{\cos n\pi x}{n^2\pi^2}$

$u''' = 0$

$v_3 = -\frac{\sin n\pi x}{n^3\pi^3}$

$$= \left[(2x-x^2)\left(\frac{\cancel{\sin n\pi x}}{n\pi} \right) - (2-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2} \right) + (-2)\left(-\frac{\cancel{\sin n\pi x}}{n^3\pi^3} \right) \right]_0^2$$

$$= \left[(2-2x)\frac{\cos n\pi x}{n^2\pi^2} \right]_0^2$$

$\sin 0 = 0$

$\sin 2n\pi = 0$

$$= \left[(2-4)\frac{\cos 2n\pi}{n^2\pi^2} - (2-0)\frac{\cos 0}{n^2\pi^2} \right]$$

$$= \left[-\frac{2}{n^2\pi^2} - \frac{2}{n^2\pi^2} \right]$$

$\cos 0 = 1$

$\cos 2n\pi = 1$

$$= \frac{-4}{n^2\pi^2}$$

$$\boxed{a_n = \frac{-4}{n^2\pi^2}}$$

Step 4: To Find b_n

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

here $l = 1$, $f(x) = 2x - x^2$

$$b_n = \int_0^2 (2x-x^2)\sin n\pi x\,dx$$

using Bernoulli's formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

$u = 2x - x^2$ $\quad$ $V = \sin n\pi x$

$u' = 2 - 2x$ $\quad$ $V_1 = -\frac{\cos n\pi x}{n\pi}$

$u'' = -2$ $\quad$ $V_2 = -\frac{\sin n\pi x}{n^2\pi^2}$

$u''' = 0$ $\quad$ $V_3 = \frac{\cos n\pi x}{n^3\pi^3}$

$$b_n = \left[(2x-x^2)\left(-\frac{\cos n\pi x}{n\pi} \right) - (2-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2} \right) - 2\left(\frac{\cos n\pi x}{n^3\pi^3} \right) \right]_0^2$$

omit sin terms $\quad$ $\sin 0 = 0$, $\sin 2n\pi = 0$

$$= \left[-(2x - x^2)\frac{\cos n\pi x}{n\pi} - 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left[\left(\cancel{-(4-4)\frac{\cos 4n\pi}{n\pi}} - 2\frac{\cos 2n\pi}{n^3\pi^3}\right) - \left(0 - 2\frac{\cos 0}{n^3\pi^3}\right)\right]$$

$$= \cancel{\frac{-2}{n^3\pi^3}} + \cancel{\frac{2}{n^3\pi^3}}$$

$\cos 2n\pi = 1$

$\cos 0 = 1$

$= 0$

Step 5 : The required Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

Sub the values of a_0, a_n and b_n

$$f(x) = \frac{4}{6} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi^2} \cos n\pi x$$

$$= \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi x .$$

=x=.

③ Find the Fourier series for $f(x) = \begin{cases} x, & \text{in } 0 \le x \le 3. \\ 6-x, & \text{in } 3 \le x \le 6. \end{cases}$

④ If $f(x) = \begin{cases} x, & 0 < x < l \\ 2l - x, & l < x < 2l, \end{cases}$ Expand $f(x)$ as a Fourier series.

Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$ in $0 < x < 3$.