UNIT IV

THE Z-TRANSFORM AND DIFFERENCE EQUATIONS

The Z-transform and Difference Equations.

- Z-Transform
- Elementary properties
- Inverse Z-transform (using partial fraction and residues)
- Convolution theorem
- Formation of difference Equations
- Solution of difference Equations using Z-Transform.

Z - TRANSFORMS

Definition of Z - Transform

Let $\{f(n)\}$ be a sequence defined for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$, then

Z - transform is defined as

$$Z[f(n)] = \sum_{n=-\infty}^{\infty} f(n) z^{-n} \quad , \quad Z \to \text{a complex number}$$

This is called two sided or bilateral Z - transform.

Definition: 2.

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$= F(Z)$$

This is called one sided Z - Transform.

Definition: Z - Transform for discrete values of 't'.

If $f(t)$ is defined for discrete values of 't', where $t = nT$,

$n = 0, 1, 2, 3, \ldots$

T being the sampling period, then

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$= F(z).$$

Z - Transform pair

F(z) and f(n) are called a Z - transform pair. It is denoted by

$$f(n) \xleftrightarrow{Z} F(z)$$

Define region of convergence?

The series $\sum_{k=0}^{\infty} f(k) z^{-k}$ will converge only for certain region in the Z - plane. This region is called the region of convergence of the Z - transform.

PROBLEMS ON SIMPLE FUNCTIONS

Find the z-transform of the following functions.

1) 1 2) n 3. $\frac{1}{n}$ 4. $\frac{1}{n+1}$ 5. $\frac{1}{n+2}$ 6. $\frac{1}{n-1}$ 7. $\frac{1}{n!}$ 8. $\frac{1}{(n+1)!}$

Example

1. Find $Z[1]$.

Apr 2005, 2008

Soln:-

WKT the z-transform

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}.$$

Given $f(n) = 1$

$$Z[1] = \sum_{n=0}^{\infty} 1 \, z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1} \quad \text{for } \left|\frac{1}{z}\right| < 1.$$

Binomial expansion $(1-x)^{-1} = 1 + x + x^2 + x^3.$

Here $x = 1/z$

$\therefore |x| = \left|\frac{1}{z}\right| < 1$

$\Rightarrow 1 < |z|$

$$= \frac{1}{1 - \frac{1}{z}}$$

$$= \frac{1}{\frac{z-1}{z}}$$

$$= \frac{z}{z-1}$$

$$\therefore Z[1] = \frac{z}{z-1}$$

2. Find $Z[n]$.

Dec 2010, Apr 2007, 2000

Soln:-

WKT the z-transform

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}.$$

Given $f(n) = n$

$$Z[n] = \sum_{n=0}^{\infty} n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{n}{z^n}$$

$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$

$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right]$

$= \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2}$

Formula

$\therefore (1-x)^{-2} = 1 + 2x + 3x^2 \cdots$ if $|x| < 1$.

$= \frac{1}{z} \frac{1}{(1-\frac{1}{z})^2}$

$= \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right)^2$

$= \frac{1}{z}\left(\frac{1}{\frac{z-1}{z}}\right)^2$

$= \frac{1}{z}\left(\frac{z}{z-1}\right)^2$

$= \frac{z}{(z-1)^2}$

$\therefore Z[n] = \frac{z}{(z-1)^2}$

3. Find $Z[\frac{1}{n}]$, $|z| > 1$, $n > 0$

Soln:-

The Z - transform is

$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

Given $f(n) = \frac{1}{n}$, Here $n > 0$.

$Z[\frac{1}{n}] = \sum_{n=1}^{\infty} \frac{1}{n} \cdot z^{-n}$

$= \sum_{n=1}^{\infty} \frac{1}{nz^n}$

$= \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{z^n}$

$= \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{z}\right)^n$

$= \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots$

Formula

$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ if $|x| < 1$

$= -\log\left(1 - \frac{1}{z}\right)$

$= -\log \frac{z-1}{z}$

$= \log\left(\frac{z-1}{z}\right)^{-1} = \log\left(\frac{z}{z-1}\right)$ //

4. Find $Z\left[\frac{1}{n+1}\right]$

Dec 99
Apr 2000
Dec 2005
Apr 2009
May 2010

Soln:

The Z-transform is

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

Given $f(n) = \frac{1}{n+1}$

$$Z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \frac{1}{4}\left(\frac{1}{z}\right)^3 + \dots$$

$$= 1 + \frac{(1/z)}{2} + \frac{(1/z)^2}{3} + \frac{(1/z)^3}{4} + \dots$$

Multiply and divided by z.

$$= z\left[\frac{1}{z} + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \dots\right]$$

$$= z\left[-\log\left(1 - \frac{1}{z}\right)\right]$$

Formula
$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

$$= -z \log\left(\frac{z-1}{z}\right)$$

$$= z \log\left(\frac{z-1}{z}\right)^{-1}$$

$$= z \log\left(\frac{z}{z-1}\right)$$

$$\therefore Z\left[\frac{1}{n+1}\right] = z \log\left(\frac{z}{z-1}\right).$$

5. Find $Z\left[\frac{1}{n+2}\right]$

Soln:-

The Z-transform is

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}.$$

Given $f(n) = \frac{1}{n+2}$

$$Z\left[\frac{1}{n+2}\right] = \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+2} \cdot \frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+2} \cdot \left(\frac{1}{z}\right)^n$$

$$= \frac{1}{2} + \frac{1}{3}\left(\frac{1}{z}\right) + \frac{1}{4}\left(\frac{1}{z}\right)^2 + \frac{1}{5}\left(\frac{1}{z}\right)^3 + \dots$$

Multiply and divided by z^2

$$= z^2 \left[\frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \frac{(1/z)^4}{4} + \dots \right]$$

$$= z^2 \left[-\frac{1}{z} + \frac{1}{z} + \underbrace{\frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \dots}\right]$$

$$= z^2 \left[-\frac{1}{z} - \log(1 - 1/z) \right]$$

$$= z^2 \left[-\frac{1}{z} - \log\left(1 - \frac{1}{z}\right) \right]$$

$$= z^2 \left[-\frac{1}{z} - \log\left(\frac{z-1}{z}\right) \right]$$

$$= -z - z^2 \log\left(\frac{z-1}{z}\right)$$

$$= -z + z^2 \log\left(\frac{z-1}{z}\right)^{-1}$$

$n \log m = \log m^n$

$$= -z + z^2 \log\left(\frac{z}{z-1}\right)$$

$$z\left[\frac{1}{n+2}\right] = z^2 \log\left(\frac{z}{z-1}\right) - z$$

6.. Find $z\left[\frac{1}{n-1}\right]$, $n > 1$.

Apr 2003
Apr 2006

Soln:

The z - transform is $z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

Given $f(n) = \frac{1}{n-1}$, Here $n > 1$

$$z\left[\frac{1}{n-1}\right] = \sum_{n=2}^{\infty} \frac{1}{n-1} z^{-n}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n-1} \left(\frac{1}{z}\right)^n$$

$$= \left(\frac{1}{z}\right)^2 + \frac{1}{2}\left(\frac{1}{z}\right)^3 + \frac{1}{3}\left(\frac{1}{z}\right)^4$$

$$= \frac{1}{z}\left[\frac{1}{z} + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \dots \right]$$

$$= \frac{1}{z}\left[-\log(1 - 1/z) \right]$$

$$= \frac{1}{z} \log\left(\frac{z}{z-1}\right)$$

$$\therefore z\left[\frac{1}{n-1}\right] = \frac{1}{z} \log\left(\frac{z}{z-1}\right)$$

7. Find $z\left[\frac{1}{n!}\right]$

Soln:-

The z - transform is $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

Given $f(n) = \frac{1}{n!}$

$$z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots$$

$$= 1 + \frac{(1/z)}{1} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots$$

$$= e^{1/z}$$

Formula

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$$z\left[\frac{1}{n!}\right] = e^{1/z}$$

8. Find $z\left[\frac{1}{(n+1)!}\right]$

Apr 2005
Dec 2007
Apr 2010

Soln:-

The z - transform is $z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

Given $f(n) = \frac{1}{(n+1)!}$

$$z\left[\frac{1}{(n+1)!}\right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{2!}\frac{1}{z} + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \frac{1}{4!}\left(\frac{1}{z}\right)^3 + \cdots$$

Multiply and divided by z

$$= z\left[\frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \frac{1}{4!}\left(\frac{1}{z}\right)^4 + \cdots\right]$$

Add and subtract 1

$$= z\left[1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots - 1\right]$$

$$= z\left[1 + \frac{(1/z)}{1} + \frac{(1/z)^2}{2!} + \cdots\right]$$

$$= z\left[e^{1/z} - 1\right]$$

Formula

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$z\left[\frac{1}{(n+1)!}\right] = z\left[e^{1/z} - 1\right]$$

9. Find $z[a^n]$.

Apr 99
June 2000
Apr 2009

soln:

The z-transform is $z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$.

Given $f(n) = a^n$

$$z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n \cdot \frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$

Formula
$(1-x)^{-1} = 1 + x + x^2 + \cdots$

$$= \frac{1}{1 - a/z}$$

$$= \frac{1}{\frac{z-a}{z}}$$

$$= \frac{z}{z-a}$$

$$z[a^n] = \frac{z}{z-a}$$

Examples.

1. $z[(1)^n] = \frac{z}{z-1}$

2. $z[(2)^n] = \frac{z}{z-2}$

3. $z[(-3)^n] = \frac{z}{z-(-3)} = \frac{z}{z+3}$.

PROBLEMS BASED ON BILATERAL Z-TRANSFORM

1. Find $Z[a^{|n|}]$.

Apr 2000
May 2002
Apr 2009
Dec 2010

Soln:

The bilateral z-transform is $Z[f(n)] = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$.

Given $f(n) = a^{|n|}$

$$Z[a^{|n|}] = \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n}.$$

$$= \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{n=0}^{\infty} a^{n} z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (az)^{-n} + \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n}.$$

$$= \sum_{1}^{\infty} (az)^{n} + \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n}$$

$$= \left[az + (az)^2 + (az)^3 + \cdots\right] + \left[1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots\right]$$

$$= az\left[1 + (az) + (az)^2 + \cdots\right] + \left[1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots\right]$$

$$= az\left[1 - az\right]^{-1} + \left[1 - \frac{a}{z}\right]^{-1}$$

Formula
$(1-x)^{-1} = 1 + x + x^2 + \ldots$

$$= az \cdot \frac{1}{1-az} + \frac{1}{1-\frac{a}{z}}$$

$$= \frac{az}{1-az} + \frac{z}{z-a}$$

$$= \frac{az(z-a) + z(1-az)}{(1-az)(z-a)}$$

$$= \frac{\cancel{az^2} - a^2 z + z - \cancel{az^2}}{(1-az)(z-a)}$$

$$= \frac{z - a^2 z}{(1-az)(z-a)}$$

2. Find $z[f(n)]$ if $f(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

Soln:-

The z - Transform is $z[f(n)] = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$

$$= \sum_{n=-\infty}^{-1} f(n) z^{-n} + \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$= 0 + \sum_{n=0}^{\infty} n . z^{-n}$$

$$= z[n]$$

$$= \frac{z}{(z-1)^2}$$ Already proved

3. Find $z[f(n)]$, $f(n) = \begin{cases} \frac{a^n}{n!}, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$

or

Find $z\left[\frac{a^n}{n!}\right]$

Soln:-

The z - transform is $z[f(n)] = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$

Here $n \geq 0$

$$z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

Given $f(n) = \frac{a^n}{n!}$

$$z\left[\frac{a^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n . \frac{1}{n!}$$

$$= 1 + \frac{1}{1!}\left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \cdots$$

$$= e^{a/z}$$ Formula $e^n = 1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots$

$$z\left[\frac{a^n}{n!}\right] = e^{a/z}$$

LINEAR PROPERTY

$$Z[af(n) \pm bg(n)] = aZ[f(n)] \pm bZ[g(n)]$$

Example

1. Find $Z(n-2)$

Soln:-

$$Z(n-2) = Z(n) - Z(2) \quad \text{using Linear property}$$

$$= \frac{z}{(z-1)^2} - 2Z(1)$$

$$= \frac{z}{(z-1)^2} - 2 \cdot \frac{z}{z-1}$$

$$= \frac{z - 2z(z-1)}{(z-1)^2} = \frac{z - 2z^2 + 2z}{(z-1)2}$$

$$= \frac{3z - 2z^2}{(z-1)^2}$$

2. Find $Z[\cos n\theta]$ and $Z[\sin n\theta]$

Apr 97
Dec 2002
Apr 2005
Apr 2008

Soln:-

WKT $Z[a^n] = \frac{z}{z-a}$

put $a = e^{i\theta}$

$$Z[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}}$$

$$Z[e^{in\theta}] = \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$= \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$Z[\cos n\theta + i\sin n\theta] = \frac{z}{(z-\cos\theta) - i\sin\theta} \times \frac{(z-\cos\theta) + i\sin\theta}{(z-\cos\theta) + i\sin\theta}$$

$$= \frac{z((z-\cos\theta) + i\sin\theta)}{(z-\cos\theta)^2 - i^2\sin^2\theta}$$

$$= \frac{z(z-\cos\theta) + i\sin\theta}{z^2 - 2z\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{z(z-\cos\theta) + i\sin\theta}{z^2 - 2z\cos\theta + 1}.$$

$$z[\cos n\theta] + i\,z[\sin n\theta] = \frac{z(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating real and imaginary part

$$z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$$

$$z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

3. Find $z[\cos n\pi/2]$.

Apr 2001
Apr 2003
Apr 2007
Dec 2008

Soln:..

WKT $z[\cos n\theta] = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$

put $\theta = \pi/2$

$$z[\cos n\theta] = \frac{z(z-\cos\pi/2)}{z^2 - 2z\cos\pi/2 + 1}$$

$$= \frac{z^2}{z^2+1}.$$

4. Find $z[\sin n\pi/2]$.

Soln:-

WKT $z[\sin n\theta] = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

put $\theta = \pi/2$

$$z[\sin n\pi/2] = \frac{z\sin\pi/2}{z^2 - 2z\cos\pi/2 + 1}$$

$$= \frac{z}{z^2+1}.$$

5. Find $z[t]$

Soln:-

WKT $z[f(t)] = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$

Given $f(t) = t$

$z[t] = \sum_{n=0}^{\infty} nT \cdot z^{-n}$

$f(t) = t$

Replace $t \to nT$

$$= T \sum_{n=0}^{\infty} n z^{-n}$$

$$= T\, Z[n]$$

$$= T \cdot \frac{z}{(z-1)^2}$$

$Z[n] = \frac{z}{(z-1)^2}$

$$Z[t] = \frac{zT}{(z-1)^2}.$$

6. Find $Z[e^{at}]$

Dec 2002
Apr 2005
Apr 2009

Soln:-

z - transform of $f(t)$ is

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}.$$

Given $f(t) = e^{at}$

$$Z[e^{at}] = \sum_{n=0}^{\infty} e^{anT} z^{-n}.$$

$$= \sum_{n=0}^{\infty} (e^{aT})^n z^{-n}$$

$$= Z[(e^{aT})^n]$$

$$= \frac{z}{z - e^{aT}}$$

Result $Z[a^n] = \frac{z}{z-a}$

Here $a = e^{aT}$

7. Find $Z[e^{at+b}]$

Soln:-

WKT $Z[e^{at}] = \frac{z}{z - e^{aT}}.$

$$Z[e^{at+b}] = Z[e^{at} \cdot e^{b}]$$

$$= e^{b}\, Z[e^{at}]$$

$$= e^{b} \cdot \frac{z}{z - e^{aT}}$$

Example

8. Find $Z[e^{3t+2}]$.

Soln:

$$Z[e^{3t+2}] = \frac{e^2 z}{z - e^{3T}}.$$

9. Find $Z\left[\frac{1}{(n+1)(n+2)}\right]$, $n>0$.

Dec 2003
Apr 2004
Dec 2008

Soln:

Consider $\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$

$$1 = A(n+2) + B(n+1)$$

put $n=-1$, $\Rightarrow A=1$

put $n=-2 \Rightarrow B=-1$

$$\therefore \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}.$$

Take Z - transform

$$Z\left[\frac{1}{(n+1)(n+2)}\right] = Z\left[\frac{1}{n+1}\right] - Z\left[\frac{1}{n+2}\right]$$

$$= z\log\left(\frac{z}{z-1}\right) - z^2\log\left(\frac{z}{z-1}\right) + z.$$

Result

$Z\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)$

$Z\left[\frac{1}{n+2}\right] = z^2\log\left(\frac{z}{z-1}\right) - z.$

10. Find $Z\left[\frac{2n+3}{(n+1)(n+2)}\right]$

Apr 2007, 08.
Dec 2002

Soln:

Consider $\frac{2n+3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$

$$2n+3 = A(n+2) + B(n+1)$$

put $n=-1 \Rightarrow -2+3 = A \Rightarrow A=1$

put $n=-2 \Rightarrow -4+3 = -B \Rightarrow B=1$

$$\frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}$$

$$Z\left[\frac{2n+3}{(n+1)(n+2)}\right] = Z\left[\frac{1}{n+1}\right] + Z\left[\frac{1}{n+2}\right].$$

$$= z\log\left(\frac{z}{z-1}\right) + z^2\log\left(\frac{z}{z-1}\right) - z.$$

Results

$Z\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)$

$Z\left[\frac{1}{n+2}\right] = z^2\log\left(\frac{z}{z-1}\right) - z$

FIRST SHIFTING THEOREM

If $Z[f(t)] = F(z)$, then $Z[e^{-at} f(t)] = F[z e^{aT}]$

PROOF:

WKT $Z[f(t)] = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$

$$Z[e^{-at} f(t)] = \sum_{n=0}^{\infty} e^{-anT} f(nT)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) \cdot (e^{aT})^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)\, (z e^{aT})^{-n}$$

$$= F(z e^{aT})$$

Note:

$$Z[e^{-at} f(t)] = [F(z)]_{z \to z e^{aT}}$$

$$Z[e^{at} f(t)] = [F(z)]_{z \to z e^{-aT}}$$

1. Find $Z[e^{-at} t]$

Soln:-

$$Z[e^{-at} f(t)] = [F(z)]_{z \to z e^{aT}} \qquad \therefore Z[f(t)] = F(z)$$

$$Z[e^{-at} \cdot t] = Z[t]_{z \to z e^{aT}}$$

$$= \left[\frac{Tz}{(z-1)^2}\right]_{z \to z e^{aT}}$$

Result

$Z[t] = \frac{Tz}{(z-1)^2}$

$$= \frac{T \cdot z e^{aT}}{(z e^{aT} - 1)^2}$$

DIFFERENTIATION IN Z-DOMAIN

$$Z[n\,f(n)] = -z \frac{d}{dz}[F(z)]$$

problems

1. Find $Z[n^2]$.

Soln:-

$$Z[n^2] = Z[n \times n]$$

$$= -z \frac{d}{dz}\{Z(n)\}$$

Result $Z(n) = \frac{z}{(z-1)^2}$

$$= -z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z \left[\frac{(z-1)^2 - z \cdot 2(z-1)}{(z-1)^4}\right]$$

$$= -z \left[\frac{(z-1)^2 - 2z(z-1)}{(z-1)^4}\right]$$

$$= -z \left[\frac{(z-1)(z-1-2z)}{(z-1)^4}\right]$$

$$= -z \left[\frac{z-1-2z}{(z-1)^3}\right]$$

$$= -z \left[\frac{-z-1}{(z-1)^3}\right]$$

$$= \frac{z^2+z}{(z-1)^3}$$

$$\therefore Z[n^2] = \frac{z^2+z}{(z-1)^3}.$$

2. Find $Z[n^3]$.

Soln: $Z[n^3] = Z[n \times n^2] = -z \frac{d}{dz}[Z(n^2)]$

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$$= \frac{z^3+4z^2+z}{(z-1)^4}.$$

3. Find $Z[(n+1)(n+2)]$

Soln:-

$$Z[(n+1)(n+2)] = Z[n^2+3n+2]$$

$$= Z[n^2] + 3Z[n] + 2Z[1]$$

$$= \frac{z^2+z}{(z-1)^3} + 3\frac{z}{(z-1)^2} + 2\frac{z}{z-1}$$

$$= \frac{z^2+z+3z(z-1)+2z(z-1)^2}{(z-1)^3}$$

$$= \frac{z^2+z+3z^2-3z+2z(z^2-2z+1)}{(z-1)^3}$$

$$= \frac{\cancel{4z^2}+2z^3-\cancel{4z^2}+\cancel{2z}-\cancel{2z}}{(z-1)^3}$$

$$= 2\left(\frac{z}{z-1}\right)^3$$

$$\therefore Z[(n+1)(n+2)] = 2\left(\frac{z}{z-1}\right)^3.$$

4. Find $Z[n(n-1)]$

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Soln:-

$$Z[n(n-1)] = Z[n^2-n] = Z[n^2] - Z[n]$$

$$= \frac{z^2+z}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$= \frac{z^2+z-z(z-1)}{(z-1)^3}$$

$$= \frac{z^2+z-z^2+z}{(z-1)^3}$$

$$= \frac{2z}{(z-1)^3}.$$

5. Find $Z[an^2+bn+c]$

Soln:

$$Z[an^2+bn+c] = aZ[n^2] + bZ(n) + cZ(1)$$

$$= a\frac{z^2+z}{(z-1)^3} + b.\frac{z}{(z-1)^2} + c.\frac{z}{z-1}$$

6. Find $Z[n(n-1)(n-2)]$

Soln:-

$$Z[n(n-1)(n-2)] = Z[n^3-3n^2+2n] = \frac{6z}{(z-1)^4}$$

DAMPING RULE (OR) SCALING IN Z-DOMAIN (OR) MULTIPLICATION BY a^n.

$Z[a^n f(n)] = F\left(\frac{z}{a}\right)$, then $F(z) = Z[f(n)]$.

Note: $Z[a^n f(n)] = F\left(\frac{z}{a}\right)$

$= F(z)_{z \to z/a}$.

problem.

1. Find $Z[na^n]$.

Soln:-

$Z[na^n] = Z[a^n \cdot n]$

$= Z[n]_{z \to z/a}$

$= \left[\frac{z}{(z-1)^2}\right]_{z \to z/a}$

$= \left[\frac{z/a}{(z/a - 1)^2}\right]$

$= \frac{z}{a} \cdot \frac{1}{\left(\frac{z-a}{a}\right)^2}$

$= \frac{z}{a} \cdot \frac{a^2}{(z-a)^2}$

$= \frac{za}{(z-a)^2}$ //

2. Find $Z\left[\frac{a^n}{n!}\right]$.

Soln:-

$Z\left[a^n \cdot \frac{1}{n!}\right] = F[z/a] = F(z)_{z \to z/a}$

$= Z\left[\frac{1}{n!}\right]_{z \to z/a}$

$= \left[e^{1/z}\right]_{z \to z/a}$

$= \left[e^{1/z/a}\right]$

$= e^{a/z}$ //

3. Find $z\left[\frac{a^n}{n}\right]$.

Soln:-

$$z\left[a^n \cdot \frac{1}{n}\right] = F\left[z/a\right]$$

$$= F[z]_{z \to z/a}$$

$$= z[f(n)]_{z \to z/a}$$

$$= z\left[\frac{1}{n}\right]_{z \to z/a}$$

$$= \left[\log\left(\frac{z}{z-1}\right)\right]_{z \to z/a}$$

$$= \log\left(\frac{z/a}{z/a - 1}\right)$$

$$= \log\left(\frac{z}{a} \cdot \frac{1}{\frac{z-a}{a}}\right)$$

$$= \log\left(\frac{z}{a} \cdot \frac{a}{z-a}\right)$$

$$= \log\left(\frac{z}{z-a}\right).$$

4. Find $z[a^n \cos n\theta] = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + 1}$

5. Find $z[a^n \sin n\theta] = \frac{az\sin\theta}{z^2 - 2az\cos\theta + 1}$.

6. Find $z[a^n \cos n\pi/2] = \frac{z^2}{z^2 + a^2}$

7. Find $z[a^n \sin n\pi/2] = \frac{az}{z^2 + a^2}$

8. Find $z[2^n \cos n\pi/2] = \frac{z^2}{z^2 + 4}$

9. Find $z[2^n \sin n\pi/2] = \frac{2z}{z^2 + 4}$.

SECOND SHIFTING THEOREM.

$$z[f(n+1)] = zF(z) - zF(0)$$

FINAL VALUE THEOREM.

If $z[f(t)] = F(z)$, then $\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1)F(z)$.

Z - TRANSFORM OF UNIT IMPULSE FUNCTION

definition :

A discrete unit impulse function is defined by

$$\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 . \end{cases}$$

Now $\delta(n-k)$ is defined by

$$\delta(n-k) = \begin{cases} 1, & n = k \\ 0, & n \neq k . \end{cases}$$

Example :

1. Find $z[\delta(k)]$

soln:-

$$\therefore z[\delta(k)] = \sum_{k=0}^{\infty} \delta(k) z^{-k}$$

$$= \delta(0) z^{-0} + \delta(1) z^{-1} + \dots$$

$$\therefore \delta(k) = 1, \quad k = 0$$
$$= 0, \quad \text{otherwise}$$

$$z[\delta(k)] = 1$$

UNIT STEP FUNCTIONS .

definition :

A discrete unit step function is defined as

$$u(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

problems

1. Find $z[u(k)]$

Soln:-

$$z[u(k)] = \sum_{k=0}^{\infty} u(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} z^{-k} \qquad \therefore u(k) = \begin{cases} 1, & k \geq 0 \\ 0 & k < 0 . \end{cases}$$

$$= \sum_{k=0}^{\infty} \frac{1}{z^k}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \dots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{1}{1 - \frac{1}{z}}$$

$$= \frac{1}{1-\frac{1}{z}}$$

$$= \frac{1}{\frac{z-1}{z}}$$

$$= \frac{z}{z-1}.$$

$$z[u(k)] = \frac{z}{z-1}$$

=x=.

Note: The z-transform pair for unit step function is

$$u(k) \overset{z}{\longleftrightarrow} \frac{z}{z-1}.$$

THE INVERSE Z - TRANSFORM.

If $z[f(n)] = F(z)$, then $z^{-1}[F(z)] = f(n)$ is called inverse z - transform of $F(z)$.

Example

1. $z[a^n] = \frac{z}{z-a}$ $\therefore z^{-1}\left[\frac{z}{z-a}\right] = a^n$

2. $z\left[\frac{1}{n!}\right] = e^{1/z}$ $\therefore z^{-1}\left[e^{1/z}\right] = \frac{1}{n!}$

Method of finding inverse z- transforms

1. Method of partial fraction
2. Method of residues
3. Long division method.

PARTIAL FRACTION METHOD

1. Find $z^{-1}\left[\frac{10z}{z^2-3z+2}\right]$

Soln:

$$F(z) = \frac{10z}{z^2-3z+2}$$

$$\frac{F(z)}{10z} = \frac{1}{(z-1)(z-2)}$$

Consider $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$$1 = A(z-2) + B(z-1)$$

put $z = 1 \Rightarrow A = -1$

put $z = 2 \Rightarrow B = 1$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\frac{F(z)}{10z} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$F(z) = \frac{-10z}{z-1} + \frac{10z}{z-2}$$

$$= -10\frac{z}{z-1} + 10\frac{z}{z-2}$$

$$z^{-1}\left[\frac{10z}{z^2-3z+2}\right] = -10\, z^{-1}\left[\frac{z}{z-1}\right] + 10\, z^{-1}\left[\frac{z}{z-2}\right]$$

$$= -10(1)^n + 10(2)^n, \quad n \geq 0$$

$$\therefore z^{-1}\left[\frac{10z}{z^2-3z+2}\right] = -10(1)^n + 10(2)^n. \quad n \geq 0.$$

HW

2. Find inverse z-transform of $\frac{z}{z^2-3z+2}$.

Soln:-

$$z^{-1}\left[\frac{z}{z^2-3z+2}\right] = 2^n - 1$$

3. Find the inverse z-transform of $\frac{z}{z^2+11z+24}$.

Soln:-

$$F(z) = \frac{z}{z^2+11z+24}$$

$$= \frac{z}{(z+8)(z+3)}$$

$$\frac{F(z)}{z} = \frac{1}{(z+8)(z+3)}$$

Consider $\frac{1}{(z+8)(z+3)} = \frac{A}{z+8} + \frac{B}{z+3}$

$$1 = A(z+3) + B(z+8)$$

put $z=-8 \Rightarrow A = -\frac{1}{5}$

put $z=-3 \Rightarrow B = \frac{1}{5}$

$$\therefore \frac{1}{(z+8)(z+3)} = \frac{1}{5}\left[\frac{1}{z+3} - \frac{1}{z+8}\right]$$

$$\frac{F(z)}{z} = \frac{1}{5}\left[\frac{1}{z+3} - \frac{1}{z+8}\right]$$

$$F(z) = \frac{1}{5}\left[\frac{z}{z+3} - \frac{z}{z+8}\right]$$

$$z^{-1}[F(z)] = \frac{1}{5}\left[z^{-1}\left[\frac{z}{z+3}\right] - z^{-1}\left[\frac{z}{z+8}\right]\right]$$

$$= \frac{1}{5}\left[(-3)^n - (-8)^n\right]$$

4. Find the inverse Z-transform of $\frac{z^2}{(z+2)(z^2+4)}$

Soln:-

$$F(z) = \frac{z^2}{(z+2)(z^2+4)}$$

$$\frac{F(z)}{z} = \frac{z}{(z+2)(z^2+4)}$$

Let $\frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+4}$

$\therefore z = A(z^2+4)+(Bz+C)(z+2)$

put $z=-2$, $\quad -2 = 8A$, $\quad A = -\frac{1}{4}$

put $z=0$, $\quad 0 = 4A+2C$ $\quad C = \frac{1}{2}$.

Equating the coefficients of z^2.

$0 = A+B$

$B = -A = \frac{1}{4}$

$\therefore \frac{z}{(z+2)(z^2+4)} = \frac{-1/4}{z+2} + \frac{\frac{1}{4}z+\frac{1}{2}}{z^2+4}$

$Z[a^n \cos n\pi/2] = \frac{z^2}{z^2+a^2}$

$Z[2^n \cos n\pi/2] = \frac{z^2}{z^2+4}$

$Z[2^n \sin n\pi/2] = \frac{2z}{z^2+4}$

$$\frac{F(z)}{z} = -\frac{1}{4}\frac{1}{z+2} + \frac{1}{4}\frac{z}{z^2+4} + \frac{1}{4}\frac{2}{z^2+4}$$

$$F(z) = -\frac{1}{4}\frac{z}{z+2} + \frac{1}{4}\frac{z^2}{z^2+4} + \frac{1}{4}\frac{2z}{z^2+4}.$$

$$z^{-1}[F(z)] = \frac{1}{4} z^{-1}\left[\frac{z}{z+2}\right] + \frac{1}{4} z^{-1}\left[\frac{z^2}{z^2+4}\right] + \frac{1}{4} z^{-1}\left[\frac{2z}{z^2+4}\right]$$

$$= \frac{1}{4}(-2)^n + \frac{1}{4} 2^n \cos\frac{n\pi}{2} + \frac{1}{4} 2^n \sin\frac{n\pi}{2}.$$

5. Find $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

Soln:-

$$F(z) = \frac{z^2}{(z-a)(z-b)}$$

$$\frac{F(z)}{z} = \frac{z}{(z-a)(z-b)}$$

Consider $\dfrac{z}{(z-a)(z-b)} = \dfrac{A}{z-a} + \dfrac{B}{z-b}$

$$z = A(z-b) + B(z-a)$$

put $z = a$, $\quad a = A(a-b) \Rightarrow A = \dfrac{a}{a-b}$

put $z = b$, $\quad b = A(b-a) \Rightarrow B = \dfrac{b}{b-a} = -\dfrac{b}{a-b}$

$$\therefore \frac{z}{(z-a)(z-b)} = \frac{a}{a-b} \cdot \frac{1}{z-a} - \frac{b}{a-b} \cdot \frac{1}{z-b}$$

$$\frac{F(z)}{z} = \frac{a}{a-b} \cdot \frac{1}{z-a} - \frac{b}{a-b} \frac{1}{z-b}$$

$$F(z) = \frac{a}{a-b} \frac{z}{z-a} - \frac{b}{a-b} \cdot \frac{z}{z-b}$$

$$z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{a}{a-b} z^{-1}\left[\frac{z}{z-a}\right] - \frac{b}{a-b} z^{-1}\left[\frac{z}{z-b}\right]$$

$$= \frac{a}{a-b} a^n - \frac{b}{a-b} b^n$$

METHOD OF RESIDUES

To Find inverse Z-transform using residue theorem

If $Z[f(n)] = F(Z)$, then $f(n)$ which gives the inverse Z-transform of $F(z)$ is obtained from the following result.

$$f(n) = \frac{1}{2\pi i} \int_C z^{n-1} F(z)\, dz.$$

where C is the closed contour which encloses all the poles of the integrand

By residue theorem

$$\int_C z^{n-1} F(z)\, dz = 2\pi i \, [\text{sum of residues}]$$

Find inverse Z-transform of the following using residue theorem

1. Find $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$

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Soln:

Let $\frac{z}{(z-1)(z-2)} = F(z)$, $\quad f(n) = Z^{-1}[F(z)]$

$$z^{n-1} F(z) = \frac{z^{n-1} \cdot z}{(z-1)(z-2)}$$

$$= \frac{z^n}{(z-1)(z-2)}$$

The poles are $z=1$, $z=2$ (simple poles)

$$\text{Res}\left[z^{n-1} F(z)\right]_{z=1} = \lim_{z \to 1} (z-1) \frac{z^n}{(z-1)(z-2)}$$

$$= -(1)^n$$

$$\text{Res}\left[z^{n-1} F(z)\right]_{z=2} = \lim_{z \to 2} (z-2) \cdot \frac{z^n}{(z-1)(z-2)}$$

$$= 2^n$$

$\therefore$ $f(n) =$ Sum of Residues $= 2^n - 1^n, \; n \geq 0$

$$Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right] = 2^n - (1)^n, \quad n \geq 0$$

2 Find $z^{-1}\left[\frac{z^2}{z^2+4}\right]$

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Soln.

$$z^{-1}\left[\frac{z^2}{z^2+4}\right] = 2^n \cdot \cos\frac{n\pi}{2}, \quad n \geq 0$$

CONVOLUTION METHOD

Convolution of two sequences.

The convolution of two sequence $\{f(n)\}$ and $\{g(n)\}$ is defined as

$$f(n) * g(n) = \sum_{r=0}^{n} f(r)\, g(n-r).$$

Convolution of two Functions.

The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \sum_{r=0}^{n} f(rT) \cdot g[(n-r)T]$$

where T is the sampling period

Convolution theorem

1. $Z[f(n) * g(n)] = F(z) \cdot G(z)$

2. $Z[f(t) * g(t)] = F(z) \cdot G(z).$

problem

1. Using convolution theorem, find $z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right]$

Soln:-

$$z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right] = z^{-1}\left[\frac{z}{z-4} \cdot \frac{z}{z-3}\right]$$

$$= z^{-1}\left[\frac{z}{z-4}\right] * z^{-1}\left[\frac{z}{z-3}\right]$$

$$= (4)^n * (3)^n.$$

By convolution

$$= \sum_{r=0}^{n} (4)^r (3)^{n-r}$$

$$= \sum_{r=0}^{n} (4)^r (3)^n (3)^{-r}$$

$$= 3^n \sum_{r=0}^{n} \left(\frac{4}{3}\right)^r$$

$$= 3^n \left[1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \cdots + \left(\frac{4}{3}\right)^n\right]$$

$$= 3^n \left[\frac{(4/3)^{n+1} - 1}{4/3 - 1} \right]$$

$$= 3^n \left[\frac{(4/3)^{n+1} - 1}{\frac{4-3}{3}} \right]$$

$$= 3^n \left[\frac{\frac{4^{n+1}}{3^{n+1}} - 1}{1/3} \right]$$

$$= 3^n \left[\frac{\frac{4^{n+1} - 3^{n+1}}{3^{n+1}}}{1/3} \right]$$

$$= 3^{n+1} \left[\frac{4^{n+1} - 3^{n+1}}{3^{n+1}} \right]$$

$$= 4^{n+1} - 3^{n+1}.$$

2. Find $z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$ using Convolution theorem.

Soln:-

$$z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = z^{-1}\left[\frac{z}{z-a} \cdot \frac{z}{z-a}\right]$$

$$= z^{-1}\left[\frac{z}{z-a}\right] * z^{-1}\left[\frac{z}{z-a}\right]$$

$$= (a)^n * (a)^n$$

By convolution $= \sum_{r=0}^{n} a^r \cdot a^{n-r}$

$$= a^n + a \cdot a^{n-1} + a^2 \cdot a^{n-2} + \dots + a^n$$

$$= a^n + a^n + \dots + a^n \quad (n+1) \text{ times}$$

$$z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = (n+1) \cdot a^n$$

3. using Convolution theorem, evaluate $z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$

Soln:-

$$z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = z^{-1}\left[\frac{z}{z-1} \cdot \frac{z}{z-3}\right]$$

$$= z^{-1}\left[\frac{z}{z-1}\right] * z^{-1}\left[\frac{z}{z-3}\right]$$

$$= (1)^n * (3)^n$$

$$= \sum_{r=0}^{n} 1^r \, 3^{n-r}$$

$$= \sum_{r=0}^{n} 3^{n-r}$$

$$= 3^n + 3^{n-1} + 3^{n-2} + \dots + 3^1 + 3^0$$

$$= 1 + 3 + 3^2 + \dots + 3^n$$

$$= \frac{3^{n+1} - 1}{3-1}$$

$$z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = \frac{3^{n+1}-1}{2}$$

4. Use convolution theorem, find the inverse $\frac{14z^2}{(7z-1)(2z-1)}$

Soln:-

$$z^{-1}\left[\frac{14z^2}{(7z-1)(2z-1)}\right] = z^{-1}\left[\frac{z^2}{(z-\frac{1}{7})(z-\frac{1}{2})}\right]$$

Dividing Nr and Dr in 14

$$= z^{-1}\left[\frac{z}{(z-\frac{1}{7})} \cdot \frac{z}{(z-\frac{1}{2})}\right]$$

$$= z^{-1}\left[\frac{z}{(z-\frac{1}{7})}\right] * z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right]$$

$$= \left(\frac{1}{7}\right)^n * \left(\frac{1}{2}\right)^n$$

By convolution

$$= \sum_{r=0}^{n} \left(\frac{1}{7}\right)^r \left(\frac{1}{2}\right)^{n-r}$$

$$= \sum_{r=0}^{n} \left(\frac{1}{7}\right)^r \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{-r}$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \left(\frac{2}{7}\right)^r$$

$$= \left(\frac{1}{2}\right)^n \left[1 + \frac{2}{7} + \left(\frac{2}{7}\right)^2 + \left(\frac{2}{7}\right)^3 + \dots + \left(\frac{2}{7}\right)^n\right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{(2/7)^{n+1} - 1}{\frac{2}{7} - 1}\right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{\frac{2^{n+1}}{7^{n+1}} - 1}{\frac{2-7}{7}}\right]$$

$$\left[=\left(\frac{1}{2}\right)^n\left[\frac{\frac{2^{n+1}-7^{n+1}}{2^{n+1}}}{\frac{2-7}{2}}\right]\right.$$

$$=\left(\frac{1}{2}\right)^n\left[\frac{\frac{2^{n+1}-7^{n+1}}{2^{n+1}}}{-5/2}\right]$$

$$=\left(\frac{1}{2}\right)^n\left(-\frac{2}{5}\right)\left[\frac{2^{n+1}-7^{n+1}}{2^{n+1}}\right]$$

$$=\left(\frac{1}{2}\right)^n\left[\frac{(2/7)^{n+1}-1}{-5/7}\right]$$

$$=\left(\frac{1}{2}\right)^n\left(-\frac{7}{5}\right)\left[\left(\frac{2}{7}\right)^{n+1}-1\right]$$

$$=\frac{7}{5}\left(\frac{1}{2}\right)^n-\left(\frac{1}{2}\right)^n\frac{7}{5}\left(\frac{2}{7}\right)^{n+1}\left[\left(\frac{2}{7}\right)^n\left(\frac{2}{7}\right)^1\right]$$

$$=\frac{7}{5}\left(\frac{1}{2}\right)^n-\frac{2}{5}\left(\frac{1}{7}\right)^n \qquad \to 2^n\,\frac{7}{5}\,\frac{2^{n+1}}{7^{n+1}}$$

$$2^n\left(\frac{7}{5}\right)\frac{2^n\,2}{7^n\,7} \qquad \left(\frac{2}{5}\right)\frac{1}{7^n}$$

⑤ Use convolution theorem, find $z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$

Soln:-

$$z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]=z^{-1}\left[\frac{z^2}{(z-\frac{1}{2})(z+\frac{1}{4})}\right]$$

$$=z^{-1}\left[\frac{z}{z-\frac{1}{2}}\cdot\frac{z}{z+\frac{1}{4}}\right]$$

$$=z^{-1}\left[\frac{z}{z-1/2}\right]*z^{-1}\left[\frac{z}{z+\frac{1}{4}}\right]$$

$$=\left(\frac{1}{2}\right)^n*\left(-\frac{1}{4}\right)^n$$

$$=\left(-\frac{1}{4}\right)^n*\left(\frac{1}{2}\right)^n$$

By convolution

$$=\sum_{r=0}^{n}\left(-\frac{1}{4}\right)^r\cdot\left(\frac{1}{2}\right)^{n-r}$$

$$=\sum_{r=0}^{n}\left(-\frac{1}{4}\right)^r\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{-r}$$

$$=\left(\frac{1}{2}\right)^n\sum_{r=0}^{n}\left(-\frac{2}{4}\right)^r$$

$$=\left(\frac{1}{2}\right)^n\sum_{r=0}^{n}\left(-\frac{1}{2}\right)^r$$

$$\frac{\left(-\frac{1}{4}\right)^r}{(1/2)^r}$$

$$\frac{1}{4}\times\frac{2}{1}$$

$$\frac{0}{0}=1$$

$$= \left(\frac{1}{2}\right)^n \left[1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^3 + \dots + \left(-\frac{1}{2}\right)^n \right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{\left(-\frac{1}{2}\right)^{n+1} - 1}{-\frac{1}{2} - 1} \right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{\left(-\frac{1}{2}\right)^{n+1} - 1}{-3/2} \right]$$

$$= \left(\frac{1}{2}\right)^n \left(-\frac{2}{3}\right) \left[\left(-\frac{1}{2}\right)^{n+1} - 1 \right]$$

$$= \left(\frac{1}{2}\right)^n \frac{2}{3} - \left(\frac{1}{2}\right)^n \left(\frac{2}{3}\right) \left(-\frac{1}{2}\right)^{n+1}$$

$$= \left(\frac{1}{2}\right)^n \frac{2}{3} - \left(\frac{1}{2}\right)^n \left(\frac{2}{3}\right) \left(-\frac{1}{2}\right)^n \left(-\frac{1}{2}\right)$$

$$= \left(\frac{1}{2}\right)^n \frac{2}{3} + \frac{1}{3} \left(-\frac{1}{2} \times \frac{1}{2}\right)^n$$

$$= \left(\frac{1}{2}\right)^n \frac{2}{3} + \frac{1}{3} \left(-\frac{1}{4}\right)^n$$

DIFFERENCE EQUATIONS

FORMATION OF DIFFERENCE EQUATIONS

1. Form the difference equation from $y_n = a + b3^n$.

Solution

Given $y_n = a + b\,3^n$ —①

$y_{n+1} = a + b \cdot 3^{n+1} = a + b \cdot 3\, 3^n = a + 3b3^n$ —②

$y_{n+2} = a + b\,3^{n+2} = a + b \cdot 3^2 \cdot 3^n = a + 9b3^n$ —③

Eliminating a and b from equation ①, ② & ③

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 3 \\ y_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$$y_n(9-3) - 1[9y_{n+1} - 3y_{n+2}] + 1[y_{n+1} - y_{n+2}] = 0$$

$$6y_n - 9y_{n+1} + 3y_{n+2} + y_{n+1} - y_{n+2} = 0$$

$$2y_{n+2} - 8y_{n+1} + 6y_n = 0$$

Divided by 2

$$y_{n+2} - 4y_{n+1} + 3y_n = 0,$$

2. Find the difference Equation from $y_n = (A + Bn)2^n$.

Soln:-

$y_n = (A+Bn)2^n = A2^n + Bn2^n$ —①

$y_{n+1} = (A + B(n+1))2^{n+1}$

$= A2^{n+1} + B(n+1)2^{n+1}$

$= 2A2^n + 2B(n+1)2^n$ —②

$y_{n+2} = (A + B(n+2))2^{n+2}$

$= A2^{n+2} + B(n+2) \cdot 2^{n+2}$

$= 4A2^n + 4B(n+2)2^n$ —③

Eliminating A and B.

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & 2 & 2(n+1) \\ y_{n+2} & 4 & 4(n+2) \end{vmatrix} = 0.$$

$$y_n\,[8(n+2) - 8(n+1)] - 1\,[4(n+2)y_{n+1} - 2y_{n+2}(n+1)] + n[4y_{n+1} - 2y_{n+2}] = 0.$$

$$8y_n\,[n+2-n-1] - 4(n+2)y_{n+1} + 2(n+1)y_{n+2} + 4n\,y_{n+1} - 2n\,y_{n+2} = 0$$

$$8y_n + y_{n+1}\,[4n - 4(n+2)] + y_{n+2}\,[2(n+1) - 2n] = 0$$

$$8y_n + y_{n+1}(-8) + 2y_{n+2} = 0$$

$\div 2$, $$y_{n+2} - 4y_{n+1} + 4y_n = 0$$

3. Form a difference equation by eliminating the arbitrary constant A from $y_n = A\cdot 3^n$.

Soln:

$$y_n = A\cdot 3^n$$

$$y_{n+1} = A\cdot 3^{n+1}$$

$$= 3A\cdot 3^n$$

$$y_{n+1} = 3y_n$$

$$y_{n+1} - 3y_n = 0.$$

HW

4. Form a difference equation by eliminating arbitrary constants from $y_n = A + B\,2^n$.

Soln: $$y_{n+2} - 3y_{n+1} + 2y_n = 0.$$

SOLVE THE DIFFERENCE EQUATIONS USING Z - TRANSFORMS.

Formula

$$Z[y(n+3)] = z^3\bar{y} - z^3 y(0) - z^2 y(1) - z y(2)$$

$$Z[y(n+2)] = z^2\bar{y} - z^2 y(0) - z y(1)$$

$$Z[y(n+1)] = z\bar{y} - z y(0)$$

$$Z[y(n)] = \bar{y}$$

Problems

1. Solve $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0$ and $y_1 = 1$ using Z-transform.

Soln:-

Given $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$

Taking z-transform

$$Z[y_{n+2}] + 4Z[y_{n+1}] + 3Z[y_n] = Z[2^n]$$

$$z^2\bar{y} - z^2 y_0 - z y_1 + 4[z\bar{y} - z y_0] + 3\bar{y} = \frac{z}{z-2}$$

$$z^2\bar{y} - z^2 y_0 - z y_1 + 4z\bar{y} - 4z y_0 + 3\bar{y} = \frac{z}{z-2}$$

Given $y_0 = 0$, $y_1 = 1$.

$$z^2\bar{y} - z^2(0) - z + 4z\bar{y} + 3\bar{y} = \frac{z}{z-2}$$

$$\bar{y}[z^2 + 4z + 3] - z = \frac{z}{z-2}$$

$$\bar{y}[z^2 + 4z + 3] = \frac{z}{z-2} + z$$

$$\bar{y} = \frac{z}{(z+1)(z+2)(z+3)} + \frac{z}{(z+1)(z+3)}$$

Now

$$\frac{z}{(z-2)(z+1)(z+3)} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{C}{z+3}$$

$$\frac{z}{(z-2)(z+1)(z+3)} = A(z+1)(z+3) + B(z-2)(z+3) + C(z-2)(z+1).$$

$$Z = A(Z+1)(Z+3) + B(Z-2)(Z+3) + C(Z-2)(Z+1)$$

put $Z = -1$

$-1 = -6B$

$B = \frac{1}{6}$

$Z = -3$

$-3 = 10C$

$C = -\frac{3}{10}$

put $Z = 2$

$2 = 15A$

$A = \frac{2}{15}$.

Now $\frac{Z}{(Z+1)(Z-2)(Z+3)} = \frac{2}{15}\frac{1}{Z-2} + \frac{1}{6}\frac{1}{Z+1} - \frac{3}{10}\frac{1}{Z+3}$.

Now consider $\frac{Z}{(Z+1)(Z+3)} = \frac{A}{Z+1} + \frac{B}{Z+3}$.

$$\cancel{Z} = A(Z+3) + B(Z+1)$$

put $Z = -3$

$-3 = -2B$

$B = 3/2$

$Z = -1$

$-1 = 2A$

$A = -\frac{1}{2}$

$\therefore \frac{Z}{(Z+1)(Z+3)} = -\frac{1}{2}\frac{1}{Z+1} + \frac{3}{2}\frac{1}{Z+3}$.

$$Z[y_n] = \frac{2}{15}\frac{1}{Z-2} + \frac{1}{6}\frac{1}{Z+1} - \frac{3}{10}\frac{1}{Z+3} - \frac{1}{2}\frac{1}{Z+1} + \frac{3}{2}\frac{1}{Z+3}.$$

$$y_n = \frac{2}{15}Z^{-1}\left[\frac{1}{Z-2}\right] + \frac{1}{6}Z^{-1}\left[\frac{1}{Z+1}\right] - \frac{3}{10}Z^{-1}\left[\frac{1}{Z+3}\right] - \frac{1}{2}Z^{-1}\left[\frac{1}{Z+1}\right] + \frac{3}{2}Z^{-1}\left[\frac{1}{Z+3}\right]$$

$$= \frac{2}{15}\left[2^{n-1}\right] + \frac{1}{6}(-1)^{n-1} - \frac{3}{10}(-3)^{n-1} - \frac{1}{2}(-1)^{n-1} + \frac{3}{2}(-3)^{n-1}$$

$$= \frac{2^n}{15} + \frac{(-1)^n}{6} - \frac{(-3)^n}{10} - \frac{(-1)^n}{2} + \frac{(-3)^n}{2}$$

$$= \frac{2^n}{15} + \left[\frac{1}{6} - \frac{1}{2}\right](-1)^n + \left[\frac{1}{2} - \frac{1}{10}\right](-3)^n$$

$$= \frac{2^n}{15} + \left[\frac{2-6}{12}\right](-1)^n + \frac{8}{20}(-3)^n$$

$$= \frac{2^n}{15} - \frac{4}{12}(-1)^n + \frac{8}{20}(-3)^n$$

$$= \frac{2^n}{15} - \frac{1}{3}(-1)^n + \frac{2}{5}(-3)^n.$$

2. Solve the difference equation using z- transform

$y_{n+2} + 6y_{n+1} + 9y_n = 2^n$, $\quad y_0 = y_1 = 0$

Soln:-

Given $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

Taking z transfrom

$$z[y_{n+2}] + 6z[y_{n+1}] + 9z[y_n] = z[2^n]$$

$$z^2\bar{y} - z^2y_0 - zy_1 + 6[z\bar{y} - zy_0] + 9\bar{y} = \frac{z}{z-2}$$

$$z^2\bar{y} - z^2y_0 - zy_1 + 6z\bar{y} - 6zy_0 + 9\bar{y} = \frac{z}{z-2}$$

Given $y_0 = y_1 = 0$

$$z^2\bar{y} + 6z\bar{y} + 9\bar{y} = \frac{z}{z-2}$$

$$\bar{y}(z^2+6z+9) = \frac{z}{z-2}$$

$$\bar{y}[(z+3)(z+3)] = \frac{z}{z-2}$$

$$\bar{y} = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{\bar{y}}{z} = \frac{1}{(z-2)(z+3)^2}$$

Consider $\frac{1}{(z-1)(z+3)^2} = \frac{A}{z-1} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$

$$1 = A(z+3)^2 + B(z+3)(z-2) + C(z-2)$$

put $z = 2$

$25A = 1$

$A = \frac{1}{25}$

put $z = -3$

$-5C = 1$

$C = -\frac{1}{5}$

Equating coefficients of z^2

$A + B = 0$

$B = -A$

$B = -\frac{1}{25}$

$$\frac{\bar{y}}{z} = \frac{1}{(z-2)(z+3)^2}$$

$$\frac{\bar{y}}{z} = \frac{1}{25}\frac{1}{z-2} - \frac{1}{25}\frac{1}{z+3} - \frac{1}{5}\frac{1}{(z+3)^2}$$

$$\bar{y} = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

$$y_n = \frac{1}{25} z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5} z^{-1}\left[\frac{z}{(z+3)^2}\right]$$

$$= \frac{1}{25} 2^n - \frac{1}{25}(-3)^n - \frac{1}{5} n(-3)^{n-1}$$

Results $z^{-1}\left[\frac{z}{(z+3)^2}\right] = n(-3)^{n-1}$

$$= \frac{1}{25} 2^n - \frac{1}{25}(-3)^n + \frac{1}{5} n(-3)^n, \quad n \geq 0$$

$z[a^n . n] = \frac{za}{(z-a)^2}$

3. Using z-transfrom solve $y(n+2) - 5y(n+1) + 6y(n) = 4^n$ given that $y(0) = 0, y(1) = 1$.

Soln:-

Given $y(n+2) - 5y(n+1) + 6y(n) = 4^n$

Taking z-transform

$$z[y(n+2)] - 5z[y(n+1)] + 6z[y(n)] = z[4^n]$$

$$z^2\bar{y} - z^2 y_0 - zy_1 - 5[z\bar{y} - zy_0] + 6\bar{y} = \frac{z}{z-4}$$

Applying $y_0 = 0, y_1 = 1$

$$z^2\bar{y} - z - 5[z\bar{y}] + 6\bar{y} = \frac{z}{z-4}$$

$$\bar{y}[z^2 - 5z + 6] - z = \frac{z}{z-4}$$

$$\bar{y}[z^2 - 5z + 6] = \frac{z}{z-4} + z$$

$$\bar{y}[(z-2)(z-3)] = \frac{z}{z-4} + z$$

$$\bar{y} = \frac{z}{(z-2)(z-3)(z-4)} + \frac{z}{(z-2)(z-3)}$$

Consider

$$\frac{z}{(z-2)(z-3)(z-4)} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{z-4}$$

$$z = A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)$$

put $z = 3$ (-1)(2)   put $z = 2$ put $z = 4$

$B = -3$ $A = 1$ $4 = 2C$

$C = 2$.

$$\therefore \frac{z}{(z-2)(z-3)(z-4)} = \frac{A}{z-2} + \frac{B}{z-3}$$

Consider $\frac{z}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$

$z = A(z-3) + B(z-2)$

put $z=3$ put $z=2$

$B=3$ $A=-2$

$$\frac{z}{(z-2)(z-3)} = -\frac{2}{z-2} + \frac{3}{z-3}.$$

$$\therefore \bar{y} = \frac{1}{z-2} - \cancel{\frac{3}{z-3}} + \frac{2}{z-4} - \frac{2}{z-2} + \cancel{\frac{3}{z-3}}$$

$$z[y_n] = \frac{2}{z-4} - \frac{1}{z-2}.$$

$$y_n = z^{-1}\left[\frac{2}{z-4}\right] - z^{-1}\left[\frac{1}{z-2}\right].$$

$$= 2(4)^{n-1} - (2)^{n-1}, \quad n \geq 1.$$

Result $z\left[\frac{1}{z-a}\right] = a^{n-1}$.

4. Solve the equation $y_{n+2} - 3y_{n+1} + 2y_n = 2^n$ given that $y_0 = y_1 = 0$

Soln:-

Given $y_{n+2} - 3y_{n+1} + 2y_n = 2^n$.

Taking z-transform

$$z[y_{n+2}] - 3z[y_{n+1}] + 2z[y_n] = z[2^n]$$

$$z^2\bar{y} - z^2 y_0 - zy_1 - 3[z\bar{y} - zy_0] + 2\bar{y} = \frac{z}{z-2}$$

Applying $y_0 = y_1 = 0$

$$z^2\bar{y} - 3z\bar{y} + 2\bar{y} = \frac{z}{z-2}$$

$$[z^2 - 3z + 2]\bar{y} = \frac{z}{z-2}$$

$$\bar{y}[(z-1)(z-2)] = \frac{z}{z-2}$$

$$\bar{y} = \frac{z}{(z-1)(z-2)^2}$$

$$y_n = z^{-1}\left[\frac{z}{(z-1)(z-2)^2}\right]$$

Consider $\dfrac{z}{(z-1)(z-2)^2}$

$$\frac{z}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2} \quad - (A)$$

$$z = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$$

put $z=2$ $C=2$

put $z=1$ $A=1$

~~put~~ Equating coefficients of z^2

$A+B=0$

$B=-A$

$B=-1$

sub $A=+1$, $B=-1$, $C=2$ in (A)

$$\frac{z}{(z-1)(z-2)^2} = \frac{1}{z-1} - \frac{1}{z-2} + \frac{2}{(z-2)^2}$$

$$z^{-1}\left[\frac{z}{(z-1)(z-2)^2}\right] = z^{-1}\left[\frac{1}{z-1}\right] - z^{-1}\left[\frac{1}{z-2}\right] + z^{-1}\left[\frac{2}{(z-2)^2}\right]$$

$$= (1)^{n-1} - 2^{n-1} + (n-1)2^{n-1}, \quad n \geq 1$$

$$= 1 - 2^{n-1} + n2^{n-1} - 2^{n-1}$$

$$= 1 + n2^{n-1} - 2 \cdot 2^{n-1}$$

Result

$$z^{-1}\left[\frac{2}{(z-2)^2}\right] = (n-1)2^{n-1}$$

(5) Solve, using z-transform $y_{n+2} - 4y_n = 3^n$ given $y_0 = 0$, $y_1 = 0$.

Soln:-

Given $y_{n+2} - 4y_n = 3^n$

Taking z-Transform on both sides

$$z[y_{n+2}] - 4z[y_n] = z[3^n]$$

$$z^2\bar{y} - z^2y_0 - zy_1 + 4[\bar{y}] = \frac{z}{z-3}$$

Applying $y_0 = y_1 = 0$

$$z^2\bar{y} - 4\bar{y} = \frac{z}{z-3}$$

$$\bar{y}[z^2-4] = \frac{z}{z-3}$$

$$\bar{y} = \frac{z}{(z-3)(z^2-4)}$$

$$= \frac{z}{(z-3)(z+2)(z-2)}$$

Consider $\dfrac{z}{(z-3)(z+2)(z-2)}$

$$\frac{z}{(z-3)(z+2)(z-2)} = \frac{A}{z-3} + \frac{B}{z+2} + \frac{C}{z-2} \quad \text{—①}$$

$$z = A(z+2)(z-2) + B(z-3)(z-2) + C(z-3)(z+2)$$

put $z=2$ | put $z=-2$ | put $z=3$
$2 = -4C$ | $-2 = 20B$ | $3 = 5A$
$C = -\frac{1}{2}$ | $B = -\frac{1}{10}$ | $A = \frac{3}{5}$

sub eqn ①

$$\bar{y} = \frac{3}{5}\frac{1}{z-3} - \frac{1}{10}\frac{1}{z+2} - \frac{1}{2}\frac{1}{z-2}$$

$$y_n = \frac{3}{5} z^{-1}\left[\frac{1}{z-3}\right] - \frac{1}{10} z^{-1}\left[\frac{1}{z+2}\right] - \frac{1}{2} z^{-1}\left[\frac{1}{z-2}\right]$$

$$= \frac{3}{5}(3)^{n-1} - \frac{1}{10}(-2)^{n-1} - \frac{1}{2}(2)^{n-1}.$$

6. Solve, using z-transform $y_{n+2} - 4y_{n+1} + 3y_n = 0$, given that $y_0 = 2$ and $y_1 = 4$.

Soln:-

Given $y_{n+2} - 4y_{n+1} - 3y_n = 0$

Taking z-transform on both sides

$$Z[y_{n+2}] - 4Z[y_{n+1}] - 3Z[y_n] = 0$$

$$z^2\bar{y} - z^2 y_0 - zy_1 - 4[\bar{y}z - zy_0] + 3\bar{y} = 0$$

Applying $y_0 = 2,\ y_1 = 4$.

$$z^2\bar{y} - 2z^2 - 4z - 4z\bar{y} + 8z + 3\bar{y} = 0$$

$$z^2\bar{y} - 4z\bar{y} + 3\bar{y} - 2z^2 - 4z + 8z = 0$$

$$\bar{y}[z^2 - 4z + 3] = 2z(z-2)$$

$$\bar{y}[(z-1)(z-3)] = 2z(z-2)$$

$$\bar{y} = \frac{2z(z-2)}{(z-1)(z-3)}$$

$$\frac{\bar{y}}{z} = \frac{2(z-2)}{(z-1)(z-3)}$$

Consider $\frac{2(z-2)}{(z-1)(z-3)}$

$$\frac{2(z-2)}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3} \quad \text{—①}$$

$$2(z-2) = A(z-3) + B(z-1)$$

put $z=1$ put $z=3$

$A = 1$ $2 = 2B$

$B = 1$

Sub in ①

$$\frac{\bar{y}}{z} = \frac{1}{z-1} + \frac{1}{z-3}$$

$$\bar{y} = \frac{z}{z-1} + \frac{z}{z-3}$$

$$z[y_n] = \frac{z}{z-1} + \frac{z}{z-3}$$

$$y_n = z^{-1}\left[\frac{z}{z-1}\right] + z^{-1}\left[\frac{z}{z-3}\right]$$

$$= 1^n + 3^n$$

$$y_n = 1 + 3^n$$