Matrix conic Assignment

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***Problem Statement* - The equation of the common tangent touching the circle $x^2+y^2=2$ and the parabola $y^2=8x$ above the x-axis is:**

Solution

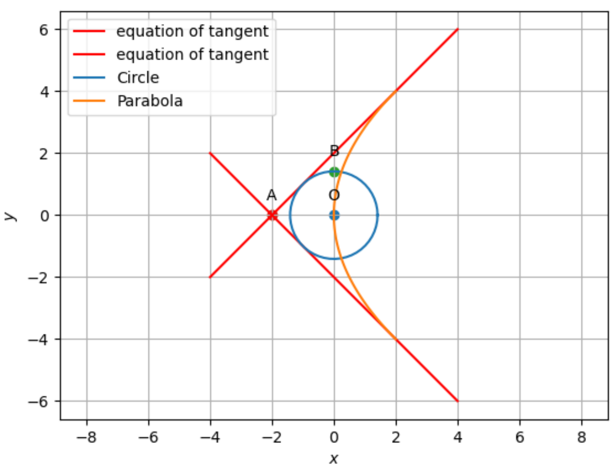

Figure 1: Two tangent is drawn to the circle and parabola

Solution

Part 1

Construction

The input parameters are equation of the curve and the point of contacts

Symbol	Value	Description
$\mathbf{O}$	$\begin{pmatrix}0\\0\end{pmatrix}$	Centre of circle
r	$\sqrt{2}$	Radius of circle
$\mathbf{F}$	$\begin{pmatrix}a\\0\end{pmatrix}$	Focus of parabola
a	2	Given value of a
$\mathbf{q}$	$\begin{pmatrix}x1\\y1\end{pmatrix}$	point of contact of parabola
q_1	$\begin{pmatrix}\frac{1.414}{\sqrt{16+y_1^2}}\\\frac{1.414}{\sqrt{4+y_1^2}}\end{pmatrix}$	point of contact of circle

Part 2

The standard equation of the parabola is given as :

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

The directrix of parabola is given as:

$$n_1^T x = c \tag{2}$$

where

$$\mathbf{X} = \begin{pmatrix}x\\y\end{pmatrix}, \tag{3}$$

$$\mathbf{n_1} = \begin{pmatrix}a\\0\end{pmatrix}, \tag{4}$$

$$f = 0 \tag{5}$$

$$c = -a \tag{6}$$

The equation of a parabola with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{7}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{8}$$

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{9}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{10}$$

$$e = 1 \tag{11}$$

$$\mathbf{V} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{12}$$

$$\mathbf{u} = \begin{pmatrix}-4\\0\end{pmatrix} \tag{13}$$

$$f = 0 \tag{14}$$

The equation of circle is given as :

$$\mathbf{x}^\top \mathbf{V_1} \mathbf{x} + 2\mathbf{u_1}^\top \mathbf{x} + f_1 = 0 \tag{15}$$

where

$$\mathbf{V_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{16}$$

$$\mathbf{u_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{17}$$

$$f_1 = -2 \tag{18}$$

Consider the equation of parabola : Given the point of contact $\mathbf{q}$, the equation of a tangent to $\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$ is

$$\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{19}$$

where

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{20}$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{21}$$

$$f = 0 \tag{22}$$

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{23}$$

$$\tag{24}$$

By substituting the above values in tangent equation we get :

$$\mathbf{-4x + yy_1 - 4x_1} = 0 \tag{25}$$

From the above equation the normal vector of tangent to the parabola is given by

$$\mathbf{n} = \begin{pmatrix} -4 \\ y1 \end{pmatrix} \tag{26}$$

Consider the equation of circle:

$$\mathbf{x}^\top \mathbf{V_1} \mathbf{x} + 2\mathbf{u_1}^\top \mathbf{x} + f_1 = 0 \tag{27}$$

If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to circle is given by

$$\begin{gathered} \mathbf{q_1} = \mathbf{V_1}^{-1}\left(\kappa_i \mathbf{n} - \mathbf{u_1}\right) \\ \text{where } \kappa = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V_1}^{-1}\mathbf{n}}} \end{gathered} \tag{28}$$

$$\kappa^2 \mathbf{n}^\top \mathbf{V_1}^{-1}\mathbf{n} - \mathbf{u_1}^\top \mathbf{V_1}^{-1}\mathbf{u_1} + f_1 = 0 \tag{29}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u_1}^\top \mathbf{V_1}^{-1}\mathbf{u_1} - f_1}{\mathbf{n}^\top \mathbf{V_1}^{-1}\mathbf{n}}} \tag{30}$$

By solving the above equation the point of contact of tangent to circle with normal vector $\mathbf{n}$ is given by :

$$\mathbf{q_1} = \begin{pmatrix} \frac{1.414}{\sqrt{16+y_1^2}} \\ \frac{1.414y1}{\sqrt{16+y_1^2}} \end{pmatrix} \tag{31}$$

The point of contact of tangent to parabola is :

$$q = \begin{pmatrix} \frac{y_1^2}{8} \\ y_1 \end{pmatrix} \tag{32}$$

The direction vector of tangent is given as :

$$\mathbf{q} - \mathbf{q_1} \tag{33}$$

In order to get the value of y1 :

$$\mathbf{n}^T(\mathbf{q} - \mathbf{q1}) = 0 \tag{34}$$

By simplifying the above equation we will be geting the value of y_1:

1. The value of y1 is :

$$y_1 = 4 \tag{35}$$

2. The point of contact of tangent to parabola is :

$$\mathbf{q} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{36}$$

3. The point of contact of tangent to circle is :

$$\mathbf{q_1} = \begin{pmatrix} 0.3516 \\ 0.3516 \end{pmatrix} \tag{37}$$

4. The normal vector of tangent is :

$$\mathbf{n} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} \tag{38}$$

Hence the equation of common tangents to the circle and parabola is :

$$\mathbf{n}^T(\mathbf{X} - \mathbf{q}) = 0 \tag{39}$$

1st tangent is :

$$\mathbf{x - y + 2} = 0 \tag{40}$$

similarly 2nd tangent is:

$$\mathbf{x + y + 2} = 0 \tag{41}$$