

MODELING AND USE OF FEM ON STATIC STRUCTURE

A SECOND YEAR PROJECT REPORT

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF B.Sc. IN COMPUTATIONAL MATHEMATICS

BY

1. Samrajya Raj Acharya (Exam Roll No.)
2. Bishesh Kafle (Exam Roll No.)
3. Priyanka Panta (Exam Roll No.)
4. Mukesh Tiwari (Exam Roll No.)



SCHOOL OF SCIENCE
KATHMANDU UNIVERSITY
DHULIKHEL, NEPAL

April, 2022

CERTIFICATION

This project entitled "Modeling and use of FEM on static structure" is carried out under my supervision for the specified entire period satisfactorily, and is hereby certified as a work done by following students

1. Samrajya Raj Acharya (Exam Roll No.)
2. Bishesh Kafle (Exam Roll No.)
3. Priyanka Panta (Exam Roll No.)
4. Mukesh Tiwari (Exam Roll No.)

in partial fulfillment of the requirements for the degree of B.Sc. in Computational Mathematics, Department of Mathematics, Kathmandu University, Dhulikhel, Nepal.

Dr. Gokul KC

Assistant Professor

Department of Mathematics,

School of Science, Kathmandu University,

Dhulikhel, Kavre, Nepal

Date: April 18, 2022

APPROVED BY:

I hereby declare that the candidate qualifies to submit this report of the Math Project (MATH-252) to the Department of Mathematics.

Head of the Department

Department of Natural Sciences

School of Science

Kathmandu University

Date: April 18, 2022

CONTENTS

CERTIFICATION	ii
LIST OF FIGURES	iv
LIST OF TABLES	v
LIST OF SYMBOLS	vi
1 MOTIVATION/INTRODUCTION	1
1.1 Context/Rationale/Background	1
1.2 History	1
1.3 Objectives	2
2 METHODOLOGY/MODEL EQUATION	3
2.1 Theoretical/Conceptual Framework	3
3 RESULTS AND DISCUSSIONS	9
3.1 Result1	9
4 CONCLUSIONS	10

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

Your parameters here.

CHAPTER 1

MOTIVATION/INTRODUCTION

1.1 Context/Rationale/Background

In this section, the author(s) shall discuss the historical background related to the work along with the introduction of the topic in brief.

Finite Element Method (FEM) is a numerical method for solving a differential or integral equation. It is a numerical method for finding approximate solutions to partial differential equations in two or three space variables. In the finite element method, a given domain is viewed as a collection of sub-domains, and over each subdomain the governing equation is approximated by any of the traditional variational methods.

1.2 History

The concept of FEM first originated with the need to solve complex elasticity and structural analysis problems in civil and aeronautical engineering. A. Hrennikoff[4], R. Courant[3], Ioannis Argyris[2] were pioneers from early 1940s. It was again rediscovered in China by Feng Kong in the later 1950s and early 1960s as Finite Difference Method based on variation principle. All of these works were based on mesh discretization of a continuous domain into a set of discrete sub-domains. Olgied Cecil Ziekiewicz[5] published his first paper in 1947 dealing with numerical approximation to the stress analysis of dams. He first recognised the general potential for using the finite element method to resolve problems in area outside of solid mechanics. Proper in-depth development of FEM began in the middle to late 1950s. By late 1950s, key concepts of stiffness matrix and element assembly existed essentially in the form used today. In 1965, NASA proposed for the development of FEM

software NASTRAN and sponsored the original version of it, and UC Berkeley made the finite element program SAP IV widely available. In 1976, Strang and Fix[6] published 'An Analysis Of The Finite Element Method' which provided a rigorous mathematical basis to FEM. FEM has since been generalized into a branch of applied mathematics for numerical modeling of physical systems in different disciplines like electromagnetism and fluid dynamics[7][1].

1.3 Objectives

1. To set up differential equations with variable boundary conditions.

Differential equations are the backbone for the mathematical modeling of physical systems. To translate physical problems into the mathematical ones, it is crucial to correctly specify the boundary conditions. So, the foremost objective is learning to correctly apply the differential equations and specify proper boundary conditions by working on various kinds of truss.

2. To learn about FEM, its variations, and its application to various mechanical problems.

It is important to have all the theoretical knowledge about the finite element method, its different types and application in different engineering disciplines such as thermal and fluid dynamics.

3. To learn to perform numericals manually and then implement those in a high level programming language

To understand the physical systems properly, we have to learn the solving techniques and processes manually first, through pen and paper, and then finally implement it into the computer so that the solutions can be verified.

4. Visualization and Project Writing.

Moreover, as the results created through FEM is a huge set of data. It is very difficult to gain information from a large data set and numbers. Visualization fills the gap, so that, we can explore solution through our eyes. Thus, we also aim to be able to visualize the raw data into information so that it is easy to grasp. Visualization is an important skill in today's world and we plan to learn problem specific visualization using python.

1.4 Significance/Scope

The use of numerical methods is prevalent in all fields of science and technology today. Our project focuses on one powerful numerical tool known as FEM which is theoretically sound and computationally efficient. The basic ideas of such tools which one is sure to encounter later will be very beneficial. This project also opens up the world of variational calculus and its applications which are generally not covered in undergraduate mathematics.

The computer implementation of a package that implements the algorithm for solving problem using FEM while handling inputs and visualizing the output is a stark contrast from the dummy math problem that are usually used in computer programming classes to teach concepts of general programming rather than mathematical programming. This project imparts the skill to convert mathematical knowledge into efficient and all around packages that solve problems that are based on real world applications and are similar to ones encountered later at work in industries or academia.

1.5 Limitations

The project only focuses on application of FEM to a single differential equation and thus despite being a great starting point for diving into the world of finite element method, it lacks behind in providing full demonstration of its potential. We have applied FEM to solve 2D truss structures. Thus, other physical structures might not be solved through this same approach. The computer implementation is also limited to this subset of problems and can work with only 2D trusses.

CHAPTER 2

METHODOLOGY/MODEL

EQUATION

2.1 Theoretical/Conceptual Framework

In this section, the author(s) should describe the theoretical/mathematical principles behind the whole work relative to the project. The information collected from literature review shall be relevant in this section.

Consider

$$-\frac{d}{dx} \left[a(x) \frac{du}{dx} \right] = f(x) \quad \text{for} \quad 0 < x < L. \quad (2.1)$$

$$(2.2)$$

for $u(x)$ subject to the boundary conditions

$$u(0) = u_0 \quad (2.3)$$

$$a(x) \frac{du}{dx} \Big|_{x=L} = Q_L \quad (2.4)$$

where $a(x)$ and $f(x)$ are known functions and u_0 and Q_L are known values. In case of bar (which is a axially loaded structure)

u = displacement

$a(x)$ = EA (stiffness)

f = distributed axial force

Q_L = axial load

Now, we want an approximation of $u(x)$ in the form

$$u(x) \approx u_N(x) = \sum_{j=1}^N c_j \phi_j(x) + \phi_0(x) \quad (2.5)$$

Substituting $U_N(x)$ in our Differential Equation,

$$-\frac{d}{dx} \left[a(x) \frac{dU_N}{dx} \right] = f(x) \quad \text{for} \quad 0 < x < L. \quad (2.6)$$

If this equally holds for all $x \in [0, L]$ the solution is exact. Since, we assume it only as an approximation. we define Residual Function $R(x, c_1, c_2, c_3, \dots, c_N)$ as

$$R = -\frac{d}{dx} \left[a(x) \frac{dU_N}{dx} \right] - f(x) \quad (2.7)$$

We have $R \neq 0 \forall x \in [0, L]$ as a U_N is only a approximation to solution.

Now we have various ways to minimize R in some senses over the domain to make this approximation close to the actual solution

1. Collocation Method

It forces that R is zero at selected N points of the domain.

i.e;

$$R(x, c_1, c_2, c_3, \dots, c_N) = 0 \quad \forall x = x_i, i = 1, 2, \dots, N \quad (2.8)$$

2. Least Square Method

$$\frac{\partial}{\partial c_j} \int_0^L R^2 dx = 0 \quad \forall i = 1, 2, \dots, N \quad (2.9)$$

3. Weighted Residual Method we desire that

$$\int_0^L w_i(x) R dx = 0 \quad \forall i = 1, 2, \dots, N \quad (2.10)$$

where $w_i(x)$ are N linearly independent functions called weight functions.

Now we convert our differential equation and boundary condition to weak form.

- Step 1 :

We write the weighted integral statement, i.e;

$$\int_0^L w \left[-\frac{d}{dx} \left(a(x) \frac{du}{dx} \right) - f \right] dx = 0 \quad (2.11)$$

It is equivalent to differential equations and does not include any boundary conditions and thus the variable u must be differentiable to as many order as is required by the differential equation.

- Step 2

Weakening Differentiability conditions

$$\int_0^L \left(w \left[-\frac{d}{dx} \left(a(x) \frac{du}{dx} \right) \right] - wf \right) dx = 0 \quad (2.12)$$

Integrating by parts, we get,

$$\int_0^L \left[a \frac{dw}{dx} \frac{du}{dx} - wf \right] dx - \left[wa \frac{du}{dx} \right]_0^L = 0 \quad (2.13)$$

we have used the fact that

$$\int_a^b \left(w \frac{dv}{dx} \right) dx = - \int_a^b v dw + [wv]_0^L = 0 \quad (2.14)$$

by considering,

$$v = -a \frac{du}{dx} \quad (2.15)$$

we can see that now we require that w be differentiable at least once but this also made that u needs to be differentiable only once even though the equation is of second order.// Now, we take care of the boundary conditions which are of two types

- Natural Boundary condition
- Essential Boundary Condition

Before defining these conditions we define primary and secondary variables.

Definition 1 *Primary Variable pass*

Definition 2 *Secondary Variable pass*

In our case , $a \frac{du}{dx}$ is secondary variable.

If secondary variable(SV) is specified in the boundary, then such conditions are called natural boundary conditions (NBC). If primary variable(PV) is specified in the boundary, then such conditions are called essential boundary conditions (NBC).

let us now define secondary variable as Q.

$$Q = a \frac{du}{dx} n_x \quad (2.16)$$

where n_x is ???

For 1D problems,

$$n_x = -1 \quad \text{at left} \quad (2.17)$$

$$n_x = 1 \quad \text{at right} \quad (2.18)$$

$$\int_0^L \left[a \frac{dw}{dx} \frac{du}{dx} - wf \right] dx - \left[wa \frac{du}{dx} \right]_0^L = 0 \quad (2.19)$$

\Rightarrow

$$\int_0^L \left[a \frac{dw}{dx} \frac{du}{dx} - wf \right] dx + wa \frac{du}{dx} \Big|_{x=L} - wa \frac{du}{dx} \Big|_{x=0} = 0 \quad (2.20)$$

since $n_x = -1$ at $x = 0$ and $n_x = 1$ at $x = L$

$$\int_0^L \left[a \frac{dw}{dx} \frac{du}{dx} - wf \right] dx - n_x wa \frac{du}{dx} \Big|_{x=L} - n_x wa \frac{du}{dx} \Big|_{x=0} = 0 \quad (2.21)$$

$$\int_0^L \left[a \frac{dw}{dx} \frac{du}{dx} - wf \right] dx - (wQ)_0 - (wQ)_L = 0 \quad (2.22)$$

Finally we require that weight functions vanishes at boundaries where essential boundary conditions are specified.

$$u(0) = u_0 \implies w(0) = 0 \quad (2.23)$$

Thus, we get

$$\int_0^L \left[a \frac{dw}{dx} \frac{du}{dx} - wf \right] dx - w(L)Q_L = 0 \quad (2.24)$$

$$B(w, u) = \int_0^L a \frac{dw}{dx} \frac{du}{dx} dx \quad (2.25)$$

$$l(w) = \int_0^L wf dx + w(L)Q_L \quad (2.26)$$

Hence, our weak form can be written as

$$0 = B(w, u) - l(w) \quad (2.27)$$

or ,

$$B(w, u) = l(w) \quad (2.28)$$

This is the variational form of the problem that is associated with our ODE and its boundary conditions.

When the differential equation is linear and of even order, the resulting weak form will have symmetric bi-linear form in u and w.

Let us now choose the approximate solutions that satisfy the two conditions for primary variables as other conditions are already included in the weak form. i.e,

$$u_h^e(x_a) = u_1^e \quad (2.29)$$

$$u_h^e(x_b) = u_2^e \quad (2.30)$$

let,

$$u_h^e(x) = c_1 + c_2x \quad (2.31)$$

Then by the conditions,

$$u_h^e(x_a) = c_1 + c_2 x_a = u_1^e \quad (2.32)$$

$$u_h^e(x_b) = c_1 + c_2 x_b = u_2^e \quad (2.33)$$

writing in matrix form, we obtain

$$\begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \begin{bmatrix} 1 & x_a \\ 1 & x_b \end{bmatrix} \begin{bmatrix} c_1^e \\ c_2^e \end{bmatrix} \quad (2.34)$$

We can rewrite the equations as

$$\begin{bmatrix} c_1^e \\ c_2^e \end{bmatrix} = \frac{1}{x_b - x_a} \begin{bmatrix} x_b & -x_a \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} \quad (2.35)$$

Hence we get,

$$c_1^e = \frac{1}{h_e} (x_b u_1^e - x_a u_2^e) \quad (2.36)$$

$$c_2^e = \frac{1}{h_e} (-u_1^e + u_2^e) \quad (2.37)$$

If we let,

$$\alpha_1^e = x_b$$

$$\alpha_2^e = -x_a$$

$$\beta_1^e = -1$$

$$\beta_2^e = 1$$

$$c_1^e = \frac{1}{h_e} (\alpha_1^e u_1^e + \alpha_2^e u_2^e) \quad (2.38)$$

$$c_2^e = \frac{1}{h_e} (\beta_1^e u_1^e + \beta_2^e u_2^e) \quad (2.39)$$

Hence,

$$U_h^e(x) = \frac{1}{h_e} [\alpha_1^e u_1^e + \alpha_2^e u_2^e + (\beta_1^e u_1^e + \beta_2^e u_2^e)x]$$

$$U_h^e(x) = \frac{1}{h_e} [\alpha_1^e + \beta_1^e x] u_1^e + \frac{1}{h_e} [\alpha_2^e + \beta_2^e x] u_2^e$$

$$U_h^e(x) = \sum_{j=1}^2 \frac{1}{h_e} [\alpha_j^e + \beta_j^e x] u_j^e \quad (2.40)$$

$$U_h^e(x) = \sum_{j=1}^2 \psi_j^e(x) u_j^e \quad (2.41)$$

where $\psi_j^e(x)$ are known as interpolation functions.

CHAPTER 3

RESULTS AND DISCUSSIONS

3.1 Result1

In this chapter, the author(s) shall present the results and simulations (in the form of numerical data or graphical form) of the theory described in CHAPTER-2 and finally discuss the results.

CHAPTER 4

CONCLUSIONS

In this section, the author(s) shall summarize the main points and the results of the work. Include key facts from the background research to help explain your results as needed.

Thus, we learned about the FEM, its variations and applications to various fields. We chose a truss structure and applied FEM to solve a 2D truss with straight bar elements that only undergo axial deformation. Initially we solved various examples manually and then implemented the same problems on a computer in python and verified results. We then verified results for the truss structure and were able to write a general code that would solve any other truss structures given essential parameters. Finally, we were able to visualize the initial problem along with the deformed structure.

REFERENCES

- [1] Ludmila Banakh, *Oscillations properties of the dynamic fractal structures*, Journal of Sound and Vibration **520** (2022), 116541.
- [2] Vinod Bandela and Saraswathi Kanaparthi, *Finite element analysis and its applications in dentistry*, Finite Element Methods and Their Applications, IntechOpen London, UK, 2020.
- [3] Richard Courant, *Variational methods for the solution of problems of equilibrium and vibrations*, Bulletin of the American mathematical Society **49** (1943), no. 1, 1–23.
- [4] Alexander Hrennikoff, *Solution of problems of elasticity by the framework method*, (1941).
- [5] Erwin Stein, *Olgierd c. zienkiewicz, a pioneer in the development of the finite element method in engineering science*, Steel Construction: Design and Research **2** (2009), no. 4, 264–272.
- [6] Gilbert Strang and George J Fix, *An analysis of the finite element method*(book- *an analysis of the finite element method.*), Englewood Cliffs, N. J., Prentice-Hall, Inc., 1973. 318 p (1973).
- [7] Olek C Zienkiewicz, Robert L Taylor, and Jian Z Zhu, *The finite element method: its basis and fundamentals*, Elsevier, 2005.