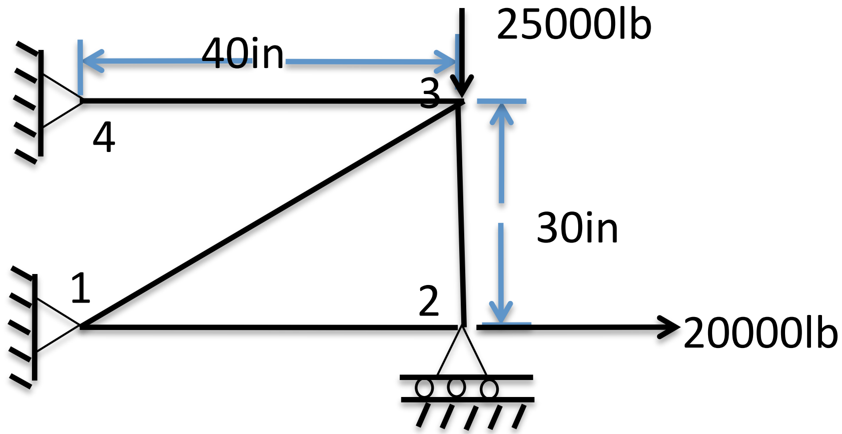
Lecture 12: FE Modeling – 2D and 3D Trusses

APL705 Finite Element Method

Example Problem

- Consider a planar 4-bar truss as shown below. Given that $E=29.5\times10^6$ psi, $A_e=1in^2$ for all elements.
- 1. Determine the element stiffness matrix.
  2. Assemble the structural stiffness matrix K
  3. Solve for the displacement
  4. Recover the stresses in each element
  5. Calculate the reaction forces



2D Truss Problem

- Nodal coordinates table

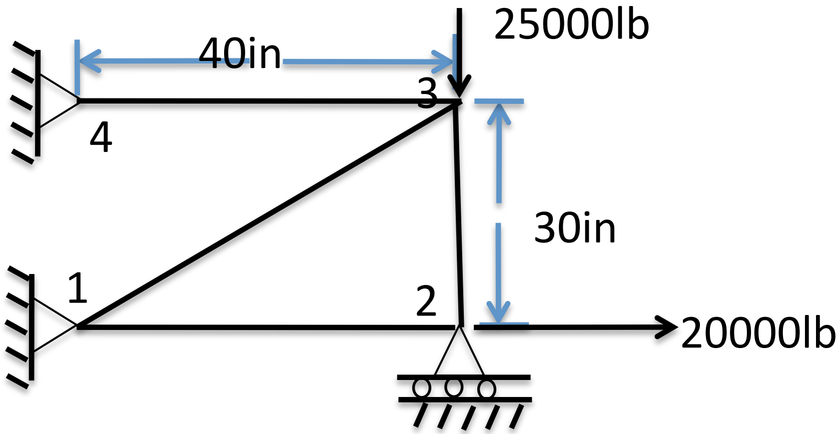
Node	x	y
1	0	0
2	40	0
3	40	30
4	0	30

- Element Connectivity

Element	1	2
1	1	2
2	3	2
3	1	3
4	4	3

- Direction cosines

Element	l_e	l	m
1	40	1	0
2	30	0	-1
3	50	0.8	0.6
4	40	1	0



2D Truss Problem

$$\mathbf{k}^1 = \frac{29.5 \times 10^6}{40} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \leftarrow \text{Global dof} \\ \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$\mathbf{k}^2 = \frac{29.5 \times 10^6}{30} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 3 \\ 4 \end{array}$$

$$\mathbf{k}^3 = \frac{29.5 \times 10^6}{50} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc} .64 & .48 & -.64 & -.48 \\ .48 & .36 & -.48 & -.36 \\ -.64 & -.48 & .64 & .48 \\ -.48 & -.36 & .48 & .36 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array}$$

$$\mathbf{k}^4 = \frac{29.5 \times 10^6}{40} \begin{array}{c} \begin{array}{cccc} 7 & 8 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 7 \\ 8 \\ 5 \\ 6 \end{array}$$

2D Truss Problem-Assembled K

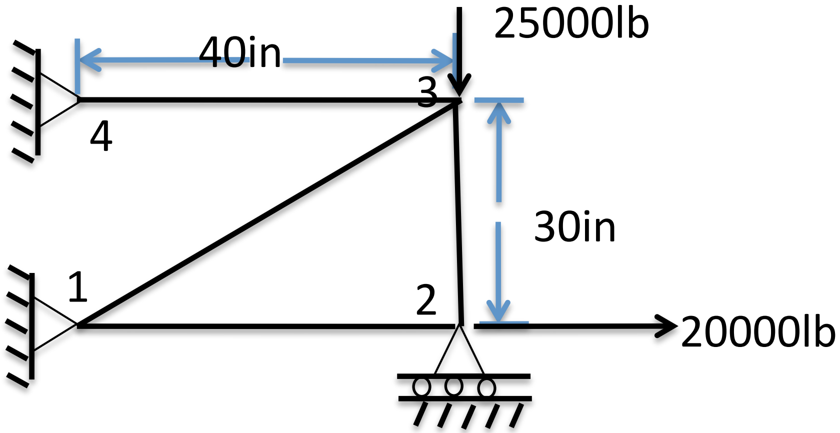
$$
\mathbf{K} = \frac{29.5 \times 10^6}{600}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\
5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\
-15.0 & 0 & 15.0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\
-7.68 & -5.76 & 0 & 0 & 22.68 & 5.76 & -15.0 & 0 \\
-5.76 & -4.32 & 0 & -20.0 & 5.76 & 24.32 & 0 & 0 \\
0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
$$

Truss Problem (contd.)

- We will now apply the BCs using the elimination approach.
- The rows and columns corresponding to dof 1,2,4,7 and 8 which belong to the supports as shown will be removed from the system equation

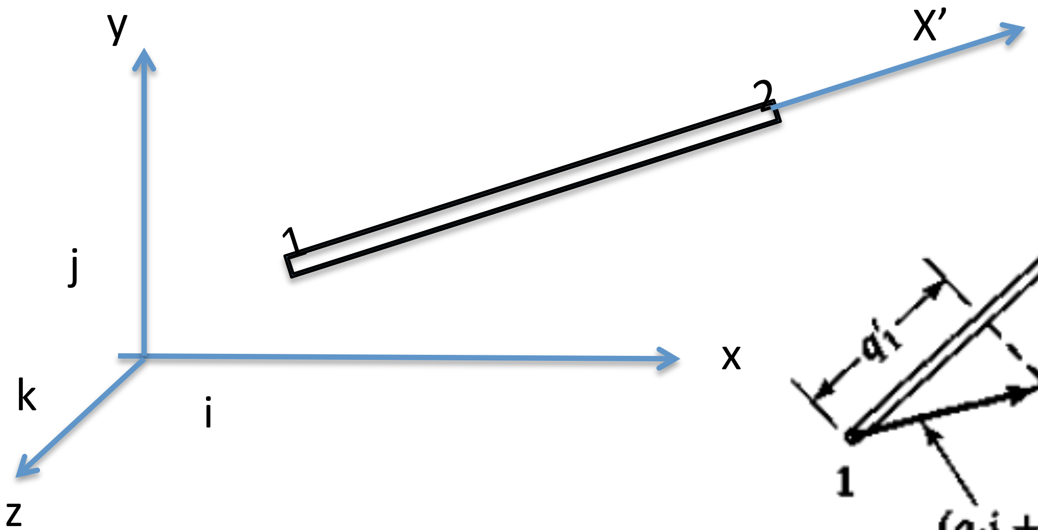
$$\frac{29.5 \times 10^6}{600}\begin{bmatrix} 15 & 0 & 0 \\ 0 & 22.68 & 5.76 \\ 0 & 5.76 & 24.32 \end{bmatrix}\begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 20\,000 \\ 0 \\ -25\,000 \end{Bmatrix}$$

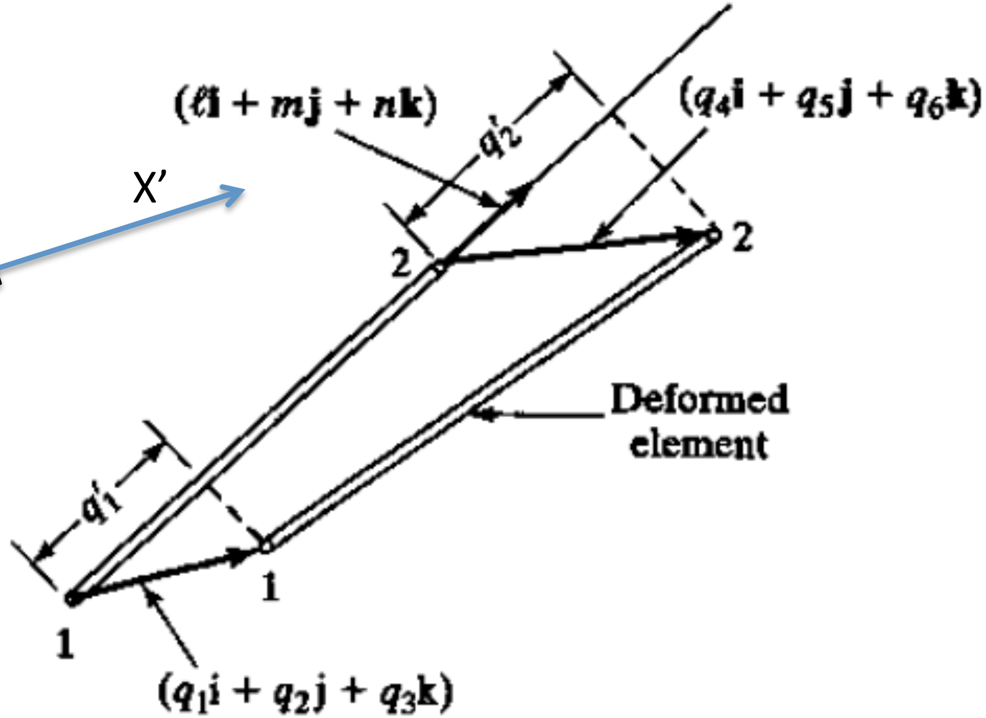
$$\begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 27.12 \times 10^{-3} \\ 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \end{Bmatrix} \text{in.}$$



3D Trusses

- Three dimensional truss is a general form of truss and 2D truss is a special case of it. For considering these orientations, we define **local** and **global** coordinates as in the case of 2D truss. A 3-D truss element is shown here in both local and global coordinates.





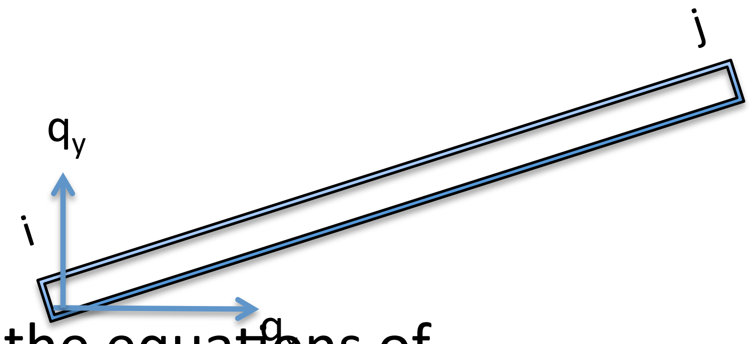
3D Truss Element

- Displacement in local coordinates

$$q' = [q_1^{'}, q_2^{'}]^T$$

- The element displacement vector in global coordinates is

$$q = [q_1, q_2, q_3, q_4, q_5, q_6]^T$$



- Using this definitions we express the equations of displacements in a matrix form as follows

$$q' = Lq \qquad \text{where } L = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix}$$

Truss Element Stiffness Matrix

- As we know the truss element is an one-dimensional element when viewed from local coordinate system. Therefore from our discussion of 1D elements, we write the element stiffness matrix as

$$k^e = \frac{E_e A_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- To derive the element stiffness matrix in global coordinates, we start from the element strain energy in local coordinates as $U_e = \frac{1}{2} q'^T k' q'$ substituting for q'=Lq, we get

$$U_e = \frac{1}{2} q'^T [L^T k' L] q'$$

Truss Element Stiffness Matrix

- The strain energy in global coordinates is

$$U_e = \frac{1}{2} q^T k q$$

- Comparing the two equations we have $k = L^T k' L$
- Substituting for L and k, we have

$$k = \frac{E_e A_e}{l_e} \begin{bmatrix} l^2 & lm & \ln & -l^2 & -lm & -\ln \\ lm & m^2 & mn & -lm & -m^2 & -mn \\ lm & mn & n^2 & -\ln & -mn & -n^2 \\ -l^2 & -lm & -\ln & l^2 & lm & \ln \\ -lm & -m^2 & -mn & lm & m^2 & mn \\ -\ln & -mn & -n^2 & \ln & mn & n^2 \end{bmatrix}$$

Calculation of Direction cosines

- The direction cosines can be calculated from the element geometry

$$l = \frac{x_2 - x_1}{l_e}$$

$$m = \frac{y_2 - y_1}{l_e}$$

$$n = \frac{z_2 - z_1}{l_e}$$

$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Calculation of Element Stresses

- Since truss element is a two-force member, we have in local coordinates

$$\sigma = E_e \varepsilon = E_e \frac{q'_2 - q'_1}{l_e}$$

$$= \frac{E_e}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q'_1 \\ q'_2 \end{bmatrix}$$

Substituting q'=Lq

$$\sigma = \frac{E_e}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix} Lq$$

Now substituting for L

$$\sigma = \frac{E_e}{l_e} \begin{bmatrix} -l & -m & -n & l & m & n \end{bmatrix} q$$