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$\boxed{\text{States} \rightarrow}$ $\delta\omega = \omega_{B/R} = \omega_{B/N} - \omega_{R/N}$, $\sigma_{BR} = \sigma_{BN} - \sigma_{RN}$

Goal :

$\boxed{\text{PART} \rightarrow \text{A}}$

$\delta\omega \longrightarrow 0$

$\sigma_{BR} \longrightarrow 0$.

Motions of equation : $I[\dot{\omega}] = -[\tilde{\omega}][I]\,\omega + u$

→ Lyapunov function

$$V = \frac{1}{2}\omega^T I \omega + 2K\ln(1+\sigma^T\sigma)$$

→ here ω is ω_{BR}

differentiate

→ It is body frame derivative

$$\dot{V} = \delta\omega^T[I]\dot{\delta\omega} + 2K\frac{1}{(1+\sigma^T\sigma)} \times 2\sigma\dot{\sigma}$$

$$\dot{V} = \delta\omega^T[I]\dot{\delta\omega} + \frac{4K\sigma\dot{\sigma}}{(1+\sigma^2)} \qquad \left[\sigma^T\sigma = \sigma^2\right]$$

$$\dot{\sigma} = \frac{1}{4}[B(\sigma)]\delta\omega$$

$$\dot{\sigma} = \frac{1}{4}\left[(1-\sigma^2)I + 2[\tilde{\sigma}] + 2\sigma\sigma^T\right]\delta\omega$$

~~Sets~~

Multiply by σ both sides we get

$$\sigma\dot{\sigma} = \frac{1}{4}\left[\sigma(1-\sigma^2)I + \cancel{2\sigma[\tilde{\sigma}]}^{0} + 2\sigma\sigma^T\sigma\right]\delta\omega$$

$$\sigma\sigma = \frac{1}{4}\left[\sigma(1-\sigma^2+\sigma^2)\right]\delta\omega^T$$

$$\sigma\dot{\sigma} = \frac{\sigma}{4}(1+\sigma^2)\,\delta\omega^T$$

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$$\dot{V} = \delta\omega^T [I] \delta\omega \quad + \frac{\cancel{4}K}{\cancel{1+\sigma^2}} \left(\frac{1}{\cancel{4}}\sigma \cancel{(1+\sigma^2)}\right) \delta\omega^T$$

$$\dot{V} = \delta\omega^T [I] \delta\dot{\omega} \quad + K\sigma_{B/R}\, \delta\omega^T .$$

3×1 3×1

$$\dot{V} = \delta\omega^T \left([I] \frac{{}^B d}{dt}(\delta\omega) \quad + K\sigma_{B/R} \right).$$

To guarantee stability we force $\dot{V}$ to be negative semi-definite by setting it equal to

$$\dot{V} = -\delta\omega^T [P] \delta\omega . \quad \left[P = P^T > 0 \right].$$

Now we have

$$\delta\omega^T \left[[I] \frac{{}^B d}{dt}(\delta\omega) \quad + K\sigma_{B/R} \right] = -\delta\omega^T [P] \delta\omega .$$

$$\Rightarrow \boxed{[I] \frac{{}^B d}{dt}(\delta\omega) \quad + K\sigma_{B/R} \quad + P\delta\omega = 0}$$

↳ closed loop dynamics.

now $\frac{{}^B d}{dt}(\delta\omega) = \dot{\omega}_{B/N} - \dot{\omega}_{R/N} + \omega_{B/N} \times \omega_{R/N}$.

$$[I] \left(\dot{\omega}_{B/N} - \dot{\omega}_{R/N} + \omega_{B/N} \times \omega_{R/N} \right) + [P] \left[\omega_{B/N} - \omega_{R/N} \right]$$

$$+ K\sigma = 0 .$$

substitute $\underline{\underline{EOM}}$.

$$[I] \dot{\omega} = -[\tilde{\omega}][I]\omega \quad + u .$$

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$$\boxed{u = -K\sigma_{B/R} - P(\omega_{B/N} - \omega_{R/N}) + [I](\dot{\omega}_{R/N} - \omega_{B/N} \times \omega_{R/N} + [\tilde{\omega}][I]\omega}$$

PART-B

we had $\dot{V} = -\delta\omega^T[P]\delta\omega$

negative ➡ global stabilizing $(\delta\omega \to \infty \quad \dot{V} \to \infty)$

It is a semi-definite as it does not have states and is definite w.r.t to $\delta\omega$

Note $\dot{V} = 0 \Rightarrow \Omega = \{\delta\omega = 0\}$.

2^{nd} derivative $\ddot{V} = -2\delta\omega^T[P]\delta\dot{\omega}$

$$\ddot{V}(\sigma, \delta\omega = 0) = 0$$

3rd derivative

$$\dddot{V} = -2\delta\dot{\omega}^T[P]\delta\dot{\omega}$$

- substituting $\delta\dot{\omega}$ from closed loop dynamics we get.

$$\dddot{V}(\sigma, \delta\omega = 0), = -K^2\sigma^T([I]^{-1})[P][I]\sigma$$

$\Rightarrow$ It is negative definite and control is asymptotic global stability

$\Rightarrow$ Mukherjee chr. theorem

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(PART-2)

PART-A

AIM → $\omega_{B/N} \to 0$.

Lyapunov function → $V = \frac{1}{2}\omega^T[I]\omega$.

EOM → $[I]\dot{\omega} = -[\tilde{\omega}][I][\omega] + u$

$$\dot{V} = \omega^T[I]\dot{\omega} = \omega^T\left[-[\tilde{\omega}][I][\omega] + u\right]$$

$$\dot{V} = \omega^T u.$$

To guarantee stability we force u to be $-[P]\omega$ with $[P] = P^T > 0$

$$\dot{V} = \underbrace{-\omega^T[P]\omega}_{\text{is a global asymptotically stable}}$$

(1) V is +ve definite
$V = 0$, $(\omega = 0)$
$V > 0 \quad \forall \omega \in R$

(2) $\dot{V}$ is negative definite
$\dot{V} = 0 \quad (\omega = 0)$
$\dot{V} < 0 \quad \forall \quad \omega \neq 0$.

$$\boxed{u = -[P]\omega}$$

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PART-B

Now since the control is limited to 0 and $\pm u_{max}$

we cannot use $u = -[P]\omega$.

For stable system we need $\dot{V} \le 0$.

$\dot{V} = \sum \omega_i u_i$ and aiming to minimize it

we can define $u_i = -u_{imax}\, sgn(\omega_i)$

so now we get a control.

$u = -u_{max}\, sgn(\omega)$

$\dot{V} = -u_{max}\, \omega\, sgn(\omega)$

Stability →

1. V is +ve definite.
2. V is continuously differentiate w.r.t
3. $\dot{V}$ is negative definite.

$\dot{V}(\omega = 0) = 0$

$\dot{V}(\omega < 0) = -u_{imax}\, \omega \cdot sign(\omega) = u_{imax}\omega < 0$

$\dot{V}(\omega > 0) = -u_{imax}\, \omega < 0$

control is asymptotically stabl

Global stability

$\omega \to \infty \quad V \to \infty$.

Hence it is gobally stable

$\boxed{\text{PART-C}}$ The control in part B creates chatter issues around 0 rate condition due to bang-bang control.

New control.

$$u = \begin{cases} -[P]\omega & \text{for } |[P]\omega| \le u_{max} \\ -u_{max}\,\mathrm{sgn}(\omega) & \text{for } |[P]\omega| > u_{max} \end{cases}$$

→ $\dot{V}$ is global asymptotic ~~s~~ stabilising