4OR 1: 225–241 (2003)

DOI: 10.1007/s10288-003-0017-3

Bottleneck shortest paths on a partially ordered scale

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Received: June 2002 / Revised version: March 2003

Abstract. In *bottleneck* combinatorial problems, admissible solutions are compared with respect to their maximal elements. In such problems, one may work with an ordinal evaluation scale instead of a numerical scale. We consider here a generalization of this problem in which one only has a *partially ordered scale* (instead of a completely ordered scale). After the introduction of a *mappimax* comparison operator between sets of evaluations (which boils down to the classical max operator when the order is complete), we establish computational complexity results for this variation of the shortest path problem. Finally, we formulate our problem as an *algebraic shortest path* problem and suggest adequate algorithms to solve it in the subsequent semiring.

Key words: Shortest path, partial order, algebraic methods, *bottleneck* problems

AMS classification: 05C50, 16Y60, 90B06

1 Introduction

In most combinatorial problems, the quality of a potential solution is evaluated through an objective function to be optimized. This objective function is often defined as the sum of the costs of elementary components (in the shortest path problem, the sum of the evaluations of the arcs along the path). We consider here situations in which preferences on solutions are not always representable by such a numerical function.

Indeed, the evaluation scale can be qualitative, i.e. there is no way to do cardinal comparisons between evaluations. The easiest and most common way to deal

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with ordinal scales is to define the evaluation of a solution as the largest evaluation of its components. However, such an evaluation does not always exist when the scale is only partially ordered. Thus, a difficulty arises to find a rule that extends a preference relation on a set of evaluations to a preference relation on its power set.

The use of ordinal evaluation scales in combinatorial problems has been investigated in several works. Firstly, the interested reader may consider the book by Zimmermann dealing with combinatorial optimization in linearly ordered algebraic structures (Zimmermann, 1981). In this work, an ordinal evaluation scale is explicitly used and the evaluation of a solution is done thanks to a concatenation operation on the evaluations of its elementary components. More recently, combinatorial problems involving a partial order on the elementary components have been studied, such as the problem of determining the set of minimal spanning trees of an ordered graph (Flament and Leclerc, 1983; Bossong and Schweigert, 1999; Schweigert, 1999), or the problem of determining the set of minimal paths from a vertex to another in an ordered digraph (Bossong and Schweigert, 1997). In these latter papers, no evaluation scale is assumed and the preference relation is extended without using a concatenation operation. However, the authors focus on an extension rule that is an ordinal counterpart of the addition operation. Consequently, the suggested algorithms are not suitable to deal with bottleneck problems. Finally, we can also mention some works dealing with refinements of *bottleneck* problems under a linearly ordered scale(Della Croce et al., 1999; Fortemps and Dubois, 2001).

In this work, we first justify the introduction of the idea of incomparability between evaluations (Sect. 3), then we propose a generalization of the min max criterion that allows to work with a *partially ordered scale* (Sect. 4). Next, we establish some computational complexity results concerning our problem (Sect. 5), with special emphasis on bottleneck problems with missing evaluations. Finally, the problem is embedded in a semiring structure (see, e.g., Gondran and Minoux, 1979, 2000; Rote, 1990) (Sect. 6), which enables the use of the *Jacobi* algorithm (Sect. 7).

2 Preliminary definitions

We first recall some elementary definitions on binary relations.

Definition 1. *For any binary relation $\lesssim$ on a set E, the asymmetric part and symmetric part of $\lesssim$ are the relations $<$ and $\sim$ defined as follows:*

$$(e < e') \iff (e \lesssim e') \text{ and not}(e' \lesssim e) \text{ for all } e, e' \in E,$$
$$(e \sim e') \iff (e \lesssim e') \text{ and } (e' \lesssim e) \text{ for all } e, e' \in E.$$

Definition 2. *A binary relation is said to be:*

- *a* partial preorder *if it is reflexive and transitive.*

- *a* complete preorder *if it is reflexive, transitive and complete.*

- *a* partial order *if it is reflexive, antisymmetric and transitive.*
- *a* complete order *if it is reflexive, antisymmetric, transitive and complete.*
- *an* equivalence relation *if it is reflexive, symmetric and transitive*[1].

Definition 3. *For any partial preorder $\lesssim$ defined on a finite set E, the sets of minimal and maximal elements of a subset A in E are respectively defined by:*

$$\min(A, \lesssim) = \{e \in A : not(e' < e) \text{ for all } e' \in A\},$$
$$\max(A, \lesssim) = \{e \in A : not(e < e') \text{ for all } e' \in A\}.$$

When there is no ambiguity, these sets are denoted $A_{\min}$ and $A_{\max}$.

Here, $\lesssim$ represents a weak preference relation and therefore $<$ is the corresponding strict preference relation. Finally, we introduce formally the notion of evaluation scale.

Definition 4. *We call* evaluation scale *a set of evaluations. Every element in that set is called an* evaluation.

3 A hazardous materials routing problem

Let us consider a transportation network (see Figure 3.1), where each arc represents a direct link between two vertices, and assume that there is an emergency service at each vertex. Consequently, if a breakdown arises during the journey from a vertex to another, it is the nearest breakdown service that is called in. Each link is evaluated on two non-commensurate criteria:

- the distance (1, 2, 3) between the two endpoints,
- the unsecurity level (a, b, c) on the link (the smaller the better).

We would like to minimize both the worst intervention time (i.e., the distance between the two endpoints) in case of a breakdown and the worst unsecurity level. The comparison of two links can be done thanks to the pointwise dominance relation $\leq_D$ between two vectors x and y: $x \leq_D y \iff x_i \leq y_i$ for all i. Clearly, when taking into account only one criterion, the choice of a "best" path from the source vertex to the destination vertex in the network amounts to search for a path minimizing the greatest evaluation of the arcs along it (*bottleneck* problem). Thus, when taking into account only one criterion, the extension of the preference relation on the set of evaluations to its power set reduces to a max operation with respect to the linearly ordered evaluation scale. When taking into account two criteria, we have to generalize the extension rule so that a partially ordered evaluation scale could be handled. We suggest to use the *mappimax* criterion, that we introduce below.

[1] If $\sim$ is an equivalence relation on a set X then the *quotient set* denoted by $X/\sim$ is the set consisting of the equivalence classes of $\sim$.

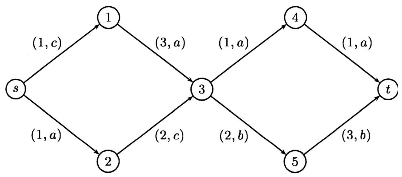

Fig. 3.1. An instance of the hazardous material routing problem

4 The *mappimax* criterion

4.1 Definition and first properties

We present here a rule to extend a partial order on a set to a partial preorder on its power set, that we call *mappimax*. This generalizes the extension of a complete order with respect to the max operator. Let $\precsim_{\max}$ denote this partial preorder.

Definition 5. *Let $(E, \leq_E)$ be a partially ordered set. For any two sets $A, B \subseteq E$, we define $\precsim_{\max}$ on $\mathcal{P}(E)$ as follows:*

$$A \precsim_{\max} B \iff \exists f : A \to B \text{ a mapping s.t. } \forall a \in A, a \leq_E f(a).$$

Remark. If $\leq_E$ is a complete order, it is easy to see that *mappimax* reduces to the usual bottleneck criterion.

Example. Consider the graph on Figure 3.1. The set of evaluations along the upper path $(s, 1, 3, 4, t)$ is $U = \{(1, c), (3, a), (1, a)\}$ whereas the set of evaluations along the lower path $(s, 2, 3, 5, t)$ is $L = \{(1, a), (2, c), (2, b), (3, b)\}$. Therefore it is easy to see that the upper path is preferred to the lower path by defining the following function f from U to L: $f(1, c) = (2, c)$, $f(3, a) = (3, b)$, $f(1, a) = (2, c)$. Moreover, it is *strictly* preferred since we cannot define a function satisfying Definition 5 from L to U.

The next proposition shows in particular that $\precsim_{\max}$ is a partial preorder and that the comparison of two sets of evaluations is equivalent to the comparison of the corresponding maximal evaluations (which are respectively $\{(1, c), (3, a)\}$ and $\{(2, c), (3, b)\}$ in the previous example).

Proposition 1 (elementary properties).
(i) $\precsim_{\max}$ *is a partial preorder on* $\mathcal{P}(E)$*,*
(ii) $A \subseteq B \Longrightarrow A \precsim_{\max} B$*,*
(iii) $\left.\begin{array}{l} A \precsim_{\max} A' \\ B \precsim_{\max} B' \end{array}\right\} \Longrightarrow A \cup B \precsim_{\max} A' \cup B'$*,*
(iv) $A \precsim_{\max} B \iff A_{\max} \precsim_{\max} B_{\max} \iff A \cup B \sim_{\max} B$*,*
(v) $A \sim_{\max} B \iff A_{\max} = B_{\max}$*,*
(vi) $A \precsim_{\max} B \iff (A \cup B)_{\max} = B_{\max}$*,*
for all $A, A', B, B' \subseteq E$*.*

Proof. (i) $\precsim_{\max}$ is reflexive (for $A \subseteq E$, set $f = Id_A$ and then by reflexivity of $\leq_E$) and transitive (for $f : A \to B$ and $g : B \to C$, set $h = g \circ f$ and then by transitivity of $\leq_E$).

(ii) For $A \subseteq B$, set $f = Id_A$ and then by reflexivity of $\leq_E$.

(iii) Define $h : A \cup B \to A' \cup B'$ as follows: For $f : A \to A'$ and $g : B \to B'$, set $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ otherwise.

(iv) Clearly, $A \sim_{\max} A_{\max}$ and $B \sim_{\max} B_{\max}$. By (i), this implies that $A \precsim_{\max} B \iff A_{\max} \precsim_{\max} B_{\max}$. Moreover, $B \subseteq A \cup B$ implies $B \precsim_{\max} A \cup B$ by (ii). Besides, $B \precsim_{\max} B$ and $A \precsim_{\max} B$ implies $A \cup B \precsim_{\max} B$ by (iii). Consequently, $A \precsim_{\max} B \Rightarrow A \cup B \sim_{\max} B$. Conversely, if $A \cup B \sim_{\max} B$ then $A \cup B \precsim_{\max} B$ and therefore $A \precsim_{\max} B$ since $A \precsim_{\max} A \cup B$ by (ii).

(v) If $A_{\max} = B_{\max}$ then we clearly have $A_{\max} \sim_{\max} B_{\max}$ and, by (iv), we deduce $A \sim_{\max} B$. Conversely, assume that $A \sim_{\max} B$ and $A_{\max} \setminus B_{\max} \neq \emptyset$. Let $e \in A_{\max} \setminus B_{\max}$. Since $A \sim_{\max} B$, we have $A_{\max} \sim_{\max} B_{\max}$ and there exists $e' \in B_{\max}$ such that $e \leq_E e'$. Moreover, $e \notin B_{\max} \Rightarrow e \neq e'$. Since $\leq_E$ is a partial order, we deduce $e <_E e'$. Besides, there exists $e'' \in A_{\max}$ such that $e' \leq_E e''$ since $B_{\max} \precsim_{\max} A_{\max}$. Therefore $e <_E e''$ by transitivity of $\leq_E$, which is in contradiction with $e \in A_{\max}$.

(vi) follows from (iv) and (v). □

The problem under consideration in this paper consists in:

$\precsim_{\max}$-SP *(Shortest Paths)* Problem
Instance: $I = (G, s, t, \leq_E)$ where $G = (X, U)$ is a connected digraph with a source vertex s and a destination vertex t included in X, $v : U \to E$ is an evaluation function and $\leq_E$ is a partial order on E, where E denotes the evaluation scale.
Goal: Find the quotient set of minimal paths[2] from s to t with respect to the partial preorder $\precsim_{\max}$, extending $\leq_E$, using mappimax.

5 Complexity considerations

In this section, we study the computational complexity of $\precsim_{\max}$-SP and related problems.

Proposition 2. *The $\precsim_{\max}$-SP problem is intractable.*

Proof. Consider the instance $I = (K_n, 1, n, \leq_E)$ where $K_n = (X, U)$ with $X = \{1, \ldots, n\}$ and $U = \{(i, j) \mid i < j\}$. We are looking for the paths from 1 to n. As long as all arcs have distinct evaluations and the relation $\leq_E$ is empty, there are 2^{n-2} minimal solutions with respect to $\precsim_{\max}$. Indeed, the set of minimal paths is then in bijection with the power set of $\{2, \ldots, n-1\}$. □

[2] Here, we mean that we want one element from each equivalence class.

However, we know that the usual bottleneck shortest path problem (i.e., with a completely ordered scale) is solvable in polynomial time. Thus, it appears that the computational complexity of the problem depends strongly on the cardinality of the ordering relation. An interesting question consists in studying the border line between tractable and intractable problems. We now give an example representative of a special type of partially ordered scale for which the problem is solvable in polynomial time.

Example. Consider a usual bottleneck shortest path problem with missing evaluations, i.e., there are some arcs the evaluations of which are unknown. Moreover, the number k of unknown evaluations is within $O(\log_2 n)$. We wish to determine a set of potentially optimal paths, i.e., a set of paths such that, for any assignments of values to unknown evaluations, there exists a path in this set with the same evaluation as the bottleneck shortest path. This problem reduces to a bottleneck problem on a partially ordered scale E. We can distinguish between two sets of evaluations in E: a set $A \subseteq E$ of completely ordered evaluations and a set $B \subseteq E$ (corresponding to unknown evaluations[3]) such that not($e \lesssim b$) and not($b \lesssim e$) for all $b \in B$ and $e \in E \setminus \{b\}$.

We now presents a polynomial-time algorithm to solve this problem. Without loss of generality, all considered paths are assumed elementary in the sequel. Moreover, for the sake of convenience, we identify every element in B with the corresponding arc in U. We first introduce two preliminary definitions:

Definition 6. *Let $G = (X, U)$ be a graph and $T \subseteq U$. The partial graph $G[T]$ is a graph the vertex set of which is X, and the set of arcs of which is T.*

Definition 7. *Let $G = (X, U)$ be a graph. The contracted graph $G_u = (X_u, U_u)$ with respect to $u = (x, y) \in X^2$ is a graph the vertex set of which is $X_u = X \cup \{z\} \setminus \{x, y\}$ (the pair of vertices $\{x, y\}$ is replaced by a single contracted vertex z), and the set of arcs of which is defined by:*

- $(i, j) \in U_u$ *if* $i, j \in X \setminus \{x, y\}$ *and* $(i, j) \in U$,
- $(i, z) \in U_u$ *if* $i \in X \setminus \{x, y\}$ *and* $(i, x) \in U$,
- $(z, i) \in U_u$ *if* $i \in X \setminus \{x, y\}$ *and* $(y, i) \in U$.

Note that actually the resulting graph might be a multigraph. Moreover, the above definition is not the usual one for a contracted graph. Indeed, our definition allows to conclude that there exists a path P in G containing arc u iff there exists a path P' in G' containing vertex z. Besides, it is easy to get P from P' and P' from P. A quotient set of minimal paths can be computed in polynomial time by Algorithm 1 (*Bottleneck Shortest Path with Missing Evaluations*). At each stage, we denote I (resp. O) the set of arcs from B (resp. not) to be included in the path

[3] There is a distinct element in B for each arc of unknown evaluation.

we are looking for (abbreviations for In and Out). The initial call of the algorithm is $BSPME(G^{(0)}, B, \emptyset, \emptyset)$ with $G^{(0)} = G[A]$.

Algorithm 1 $BSPME(G^{(t)}, B, I, O)$

If $B \setminus (I \cup O) \neq \emptyset$ then
  Find a bottleneck shortest path P in $G^{(t)}$;
  $\mu \leftarrow \max_{u \in P} v(u)$;
  If P includes all vertices in I then $G^{(t+1)} \leftarrow G^{(t)}[\{u \in U : v(u) < \mu\}]$;
    else $G^{(t+1)} \leftarrow G^{(t)}$;
  Choose $b \in B \setminus (I \cup O)$;
  Do $BSPME(G_b^{(t+1)}, B, I \cup \{b\}, O)$;
  Do $BSPME(G^{(t+1)}, B, I, O \cup \{b\})$;
end

end

We now give the recursive principle on which Algorithm 1 relies. Given $b \in B$, the set of minimal paths in G can be separated in two sets:

1. the set $\mathcal{P}_{U \setminus \{b\}}$ of minimal paths in $G[U \setminus \{b\}]$.
2. the set $\mathcal{P}_b$ of minimal paths including b in G (in correspondence with the set of minimal paths including the contracted vertex in G_b),

Moreover, a path in $\mathcal{P}_b$ cannot be optimal if its evaluation on A is not strictly lower than the evaluation of a path in $\mathcal{P}_{U \setminus \{b\}}$ (see the fourth line of the algorithm), as established by the following result:

Proposition 3. *Assume that both paths P and Q are minimal. Then the following implication holds:* $\{u \in P : v(u) \in B\} \subsetneq \{u \in Q : v(u) \in B\} \Longrightarrow \max\{v(u) \in A : u \in P\} > \max\{v(u) \in A : u \in Q\}$.

Proof. By contradiction, assume that $\max\{v(u) \in A : u \in P\} \leq \max\{v(u) \in A : u \in Q\}$. If $\{u \in P : v(u) \in B\} \subsetneq \{u \in Q : v(u) \in B\}$, then $P <_{\max} Q$. This is in contradiction with the minimality of Q. □

The validity of Algorithm 1 is established by the following result:

Proposition 4. *In Algorithm 1, we have*[4]*:*
i) A bottleneck shortest path in $G^{(t)}$ is minimal in G for all t,
ii) For every minimal path in G, there exists t such that an equivalent path is found in $G^{(t)}$.

Proof. (i) By contradiction. Assume that a bottleneck shortest path Q in $G^{(t)}$ is not minimal in G. Then there exists a better path P in the initial graph. Necessarily, by definition of $\lesssim_{\max}$, $\{u \in P : v(u) \in B\} \subseteq \{u \in Q : v(u) \in B\}$ and $\max\{u \in$

[4] For the ease of presentation, we identify a path in the initial graph and the corresponding path after contraction.

$P : v(u) \in B\} \leq \max\{u \in Q : v(u) \in B\}$. Consequently, the path P is also present in $G^{(t)}$. Therefore Q cannot be minimal in $G^{(t)}$.

(ii) Let P be a minimal path in G. It is easy to observe that $1 \leq t \leq 2^r$ where we have assumed that $B = \{b_1, \ldots, b_r\}$. Indeed, t is in bijection with the r-vector where the component i is equal to 1 if the arc of value b_i in G is contracted and 0 otherwise. Thus, there exists a t_0 corresponding to the subset of arcs of P which are in B. In $G^{(t_0)}$, Algorithm 1 finds a path equivalent to P. □

Clearly, the number of recursive calls of Algorithm 1 is within $O(2^k)$. Since $k \in O(\log_2 n)$, Algorithm 1 is processed in polynomial time. The output is the quotient set of minimal paths in G. This result shows well that the computational complexity of the problem depends strongly on the potential number of distinct minimal solutions.

We consider below other indicators to evaluate more precisely the complexity of the problem. Namely, we study two tractable problems (as suggested in Ehrgott (2000) for multicriteria problems):

1. the complexity of the corresponding decision problem,
2. the complexity of the corresponding counting problem.

5.1 Complexity of the corresponding decision problem

The study of the complexity of combinatorial problems is generally realized on decision problems like: "*Does property Pr hold in the graph $G = (X, U)$?*", or "*Given an integer k and a minimization problem (resp. maximization) does there exist a solution of value strictly smaller (resp. strictly greater) than k?*". The latter question is investigated in the framework of a discrete optimization problem π on a completely ordered scale. It is called the *corresponding decision problem* (denoted $D(\pi)$). The complexity of $D(\pi)$ is often a very good indicator of the difficulty to solve π (π is very likely hard to solve as soon as $D(\pi)$ is NP-complete). In our ordinal framework, the decision problem can be written as follows: "*Given a set of evaluations A, does there exist a solution S such that Pr holds and $v(S) <_{\max} A$?*" (where $v(S)$ denotes the set of evaluations represented in S).

The underlying goal in the study of this decision problem is to get an idea about the difficulty determining the quotient set of minimal solutions with respect to $\precsim_{\max}$. Similarly to the classical framework, if the decision problem is NP-complete then the problem of determining the set of minimal solutions is at least as difficult. Moreover, a hardness result (NP-completeness, #P-completeness, etc.) on a particular case remains true in the general case. We now show that the complexity of D($\precsim_{\max}$-Pr SP) is the same as the complexity of the problem consisting in deciding whether a graph $G = (X, U)$ admits a path P from s to t satisfying Pr. More formally, we have:

Proposition 5. *The complexity of D($\precsim_{\max}$-Pr SP) is:*

(i) *polynomial if deciding whether a graph* $G = (X, U)$ *admits a path from* s *to* t *satisfying Pr is polynomial,*
(ii) *NP-complete if deciding whether a graph* $G = (X, U)$ *admits a path from* s *to* t *satisfying Pr is NP-complete.*

Proof. (i) Let A be a set of evaluations, we define a *dominating* set $A_{\text{dom}} = \{e \in E : \exists e' \in A_{\max}, e \leq_E e'\}$. We then apply the following algorithm:

Algorithm 2 *Algorithm solving D($\lesssim_{\max}$-P SP)*

For all $e \in A_{\max}$ do
 $U_e \leftarrow \{u \in U : v(u) \in A_{\text{dom}} \setminus \{e\}\}$ and $G_e = (X, U_e)$;
 If there exists a path from s to t satisfying Pr in G_e then $test_e \leftarrow true$
  else $test_e \leftarrow false$;
end
If there exists $e \in A_{\max}$ such that $test_e = true$ then output "yes"
 else output "no";

end

This algorithm is clearly polynomial. We now prove that it finds the right answer. Assume there exists a path with a set of evaluations B such that $B <_{\max} A$, then there exists $f : B \to A$ such that $e \leq_E f(e)$ for all $e \in B$, and there exists $e' \in A_{\max}$ such that not($e' \leq_E e$) for all $e \in B$. Thus, by contradiction, it implies that $B \subseteq A_{\text{dom}} \setminus \{e'\}$. Indeed, $B \cap (E \setminus A_{\text{dom}}) \neq \emptyset$ would imply not($B \lesssim_{\max} A$), and $e' \in B$ would imply $\exists e \in B$ s.t. $e' \leq_E e$ (by reflexivity). Therefore the previous algorithm answers "yes". Conversely, one checks easily that if the algorithm answers"yes" then there exists a path with a set of evaluations B such that $B <_{\max} A$.

(ii) Assume that for all instances $I = (G, s, t, \leq_E)$ and for any set A of evaluations one can answer in polynomial time to the question: "*Does there exist a path* B *satisfying Pr such that* $B <_{\max} A$ *?*". We now prove that we could then decide in polynomial time whether a graph $G = (X, U)$ admits a path from s to t satisfying Pr. We consider a graph $G = (X, U)$ including two vertices s, t and we build in polynomial time an instance $I = (G, s, t, \leq_E)$ of D($\lesssim_{\max}$-Pr SP) as follows:

- we set $E = \{1, 2\}$ and $1 \leq_E 2$,
- we define $v : U \to E$ as follows: $v(u) = 1$ for all $u \in U$.

Let $A = \{2\}$; we decide whether there exists a path from s to t in G satisfying Pr by answering the question: "*Does there exist a path* B *from* s *to* t *satisfying Pr in* G *and such that:* $B <_{\max} A$ *?*". □

Thus, as long as the property Pr is decidable in polynomial time, we can find one minimal solution in reasonable time but we do not really have an idea about the complexity of finding the quotient set of minimal solutions. In the problem under consideration here, Pr = "true" and therefore the corresponding decision problem is solvable in polynomial time. On the other hand, for a property Pr like "*the path is hamiltonian*", the decision problem becomes clearly NP-complete.

5.2 Complexity of the corresponding counting problem

Another indicator measuring the hardness of that kind of problem is #P-completeness (Valiant, 1979). When a problem belongs to that class, the existence of a polynomial-time algorithm to count the number of optimal solutions is very unlikely. In our problem, we search for the number of minimal solutions. Once more, finding all optimal solutions is clearly at least as difficult as counting them. In an acyclic graph, the complexity of counting minimal paths with respect to $\precsim_{\max}$ remains open. Nevertheless, we now show that this problem is #P-complete when considering only paths without shortcut. A shortcut is an arc e linking two non-consecutive vertices of a path (Gondran and Minoux, 1979). Thus, a path P is without shortcut if and only if the arcs of the subgraph induced by the vertices of this path are exactly the ones along P. Such a path is called an *induced path*.

Proposition 6. *In an acyclic graph, counting the minimal induced paths with respect to $\precsim_{\max}$ (among the set of all induced paths) is #P-complete.*

Proof. We are going to establish a *parsimonious reduction*[5] from a problem already known to be #P-complete. We consider the #P-complete problem consisting in counting the number of perfect matchings in a bipartite graph (Valiant, 1979). A *matching* is a set of edges such that they have no common endpoints. It is said *perfect* if its cardinality is equal to $|X|/2$.

Let $BP = (X_1, X_2; E)$ be a bipartite graph, where $X_1 = \{x_1, ..., x_n\}$ and $X_2 = \{y_1, ..., y_n\}$, we construct n duplicates $BP_1, ..., BP_n$ of that graph. The graph $BP_i = (X_1^i, X_2^i; E_i)$ is determined by $X_1^i = \{x_1^i, ..., x_n^i\}$, $X_2^i = \{y_1^i, ..., y_n^i\}$ and the arcs $U_i = \{(x_p^i, y_q^i) : (x_p, y_q) \in U\}$. Moreover, between two duplicates, all the arcs from X_2^i to X_1^{i+1} do exist. In other words, for any $i \leq n-1$, the arcs $F_i = \{(y_p^i, x_q^{i+1}) : 1 \leq p \leq n, \ 1 \leq q \leq n\}$ do exist. Then, for any p, the arcs $L_p = \{(x_p^i, x_p^j), (y_p^i, y_p^j) : 1 \leq i < j \leq n\} \cup \{(x_p^i, y_p^j) : 1 \leq i < j \leq n\}$ do exist. Finally, we append a source vertex s and the arcs-set $S = \{(s, x_p^1) : 1 \leq p \leq n\}$, as well as a destination vertex t and the arcs-set $T = \{(y_p^n, t) : 1 \leq p \leq n\}$. Let G' denote this graph. We set $E = U$ and $v(u) = u$ for all $u \in U$. We construct the order relation $\leq_E$ on the arcs by: $e \leq_E f$ if and only if $e = f$ or there exists $p \leq n$ such that $e \notin L_p$ and $f \in L_p$. It is easy to check that the graph G' is acyclic and that $\leq_E$ is an order relation. Moreover, the construction is done in polynomial time.

We claim that the number of minimal induced paths from s to t with respect to $\precsim_{\max}$ in G' is equal to $n!$ times the number of perfect matchings in BP. By

5 A parsimonious reduction is a polynomial time reduction that preserves the number of solutions.

construction, a path without shortcut from s to t in G' never links two vertices of the same level p without including an arc from L_p. Thus, there exists two kinds of induced paths: the paths including at least one arc from L_p, and those whose arcs are duplicates of a perfect matching in BP. If A is an induced path from s to t that does not include arcs from the sets $\{L_p \ : \ p \in 1, \ldots, n\}$, then the set $E_A = \{(x_i, y_j) \ : \ \exists p \leq n, \ (x_i^p, y_j^p) \in A\}$ is a perfect matching in BP. Conversely, if E' is a perfect matching in BP then one can construct $n!$ induced paths $A_i, \ i = 1, ..., n!$ from s to t satisfying $E_{A_i} = E'$. Thus, the number of induced paths that do not include arcs from $\{L_p \ : \ p \in 1, \ldots, n\}$ is equal to $n!$ times the number of perfect matchings in BP. To conclude the proof, we show that only those induced paths are minimal with respect to $\precsim_{\max}$. Consider two paths A and B without shortcut. If A and B do not include arcs from $\{L_p \ : \ p \in 1, \ldots, n\}$ then not($A \precsim_{\max} B$) and not($B \precsim_{\max} A$); in other words, the path A is not preferred to the path B and conversely. On the other hand, if A does not include arcs from $\{L_p \ : \ p \in 1, \ldots, n\}$ and if B includes an arc $f \in L_p$ for a given p, then by construction for any arc $e \in A$ we have $e \leq f$ and therefore $A \precsim_{\max} B$. Conversely, we have not($B \precsim_{\max} A$) since for any arc $e \in A$, we have $e \notin (\bigcup_{p \leq n} L_p)$. Thus, the path A is strictly preferred to the path B with respect to $\precsim_{\max}$, which concludes the proof. □

Remark. Finding the quotient set (with respect to the indifference relation) of minimal solutions is also #P-complete since the set of minimal solutions and the quotient set are equal in our reduction. Let us however note that the counting of paths from s to t is solvable in polynomial time.

6 The semiring constructed from *mappimax*

We propose here an adequate semiring to formulate and solve the $\precsim_{\max}$-SP problem. Semiring theory permits to factorize many path algorithms. Indeed, many path algorithms reduce to solving systems of linear equations in an adequate algebraic structure. A semiring is defined as follows (e.g., Rote, 1990):

Definition 8. *A semiring $(S, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ is a set S with two binary operations $\oplus$ and $\otimes$, which fulfills the following axioms:*
(A_1) $(S, \oplus)$ is a commutative semigroup with $\mathbf{0}$ as neutral element:

$$\begin{aligned} a \oplus b &= b \oplus a, \\ (a \oplus b) \oplus c &= a \oplus (b \oplus c), \\ a \oplus \mathbf{0} &= a. \end{aligned}$$

(A_2) $(S, \otimes)$ is a semigroup with $\mathbf{1}$ as neutral element, and for which $\mathbf{0}$ is an absorbing element:

$$\begin{aligned} (a \otimes b) \otimes c &= a \otimes (b \otimes c), \\ a \otimes \mathbf{1} = \mathbf{1} \otimes a &= a, \\ a \otimes \mathbf{0} = \mathbf{0} \otimes a &= \mathbf{0}. \end{aligned}$$

(A_3) $\otimes$ is distributive with respect to $\oplus$:

$$\begin{aligned}(a \oplus b) \otimes c &= (a \otimes c) \oplus (b \otimes c),\\ a \otimes (b \oplus c) &= (a \otimes b) \oplus (a \otimes c).\end{aligned}$$

The following notation will prove useful:

- $\forall \mathcal{X} \subseteq \mathcal{P}(E), \mathcal{P}_{\min}\left(\mathcal{X}, \precsim_{\max}\right) = \{\mathcal{Y} \subseteq \mathcal{X} : \mathcal{Y} = \min\left(\mathcal{Y}, \precsim_{\max}\right)\}$,
- $\forall X \subseteq E, \mathcal{P}_{\max}(X, \leq_E) = \{Y \subseteq X : Y = \max(Y, \leq_E)\}$,
- $\forall \mathcal{A}, \mathcal{B} \in \mathcal{P}(E),\ \mathcal{A} \odot \mathcal{B} = \{\max(A \cup B, \leq_E) : A \in \mathcal{A}, B \in \mathcal{B}\}$.

We have:

Lemma 1. *For any partial order $\leq$ on X, we have the following equalities for $A \subseteq X$ and $B \subseteq X$:*

1. $(A_{\max} \cup B)_{\max} = (A \cup B)_{\max} = (A_{\max} \cup B_{\max})_{\max}$,
2. $(A_{\min} \cup B)_{\min} = (A \cup B)_{\min} = (A_{\min} \cup B_{\min})_{\min}$.

Proof. We only prove part 1. The first equality is established as follows:

- Assume that $x \notin (A \cup B)_{\max}$ and $x \in A \cup B$. Then $\exists y \in A \cup B,\ x < y$. There are two possibilities:
  - $y \in B$. Then, $y \in A_{\max} \cup B$ and therefore $x \notin (A_{\max} \cup B)_{\max}$.
  - $y \in A$. Then $\exists z \in A_{\max},\ y \leq z$. By transitivity, $x < z$ and therefore $x \notin (A_{\max} \cup B)_{\max}$.
- Assume that $x \notin (A_{\max} \cup B)_{\max}$ and $x \in A \cup B$. Then $\exists y \in A_{\max} \cup B,\ x < y$. As $A_{\max} \subseteq A$, there is $y \in A \cup B$ such that $x < y$ and consequently $x \notin (A \cup B)_{\max}$.

The second equality follows from the commutativity of union and the use of the first equality. □

Hence, the following lemma becomes obvious:

Lemma 2. *$\odot$ is commutative and associative.*

Proof. By commutativity of union, $\odot$ is also commutative. By Lemma 1, $\odot$ is associative since $\leq_E$ is a partial order on E:

$$\begin{aligned}(\mathcal{A} \odot \mathcal{B}) \odot \mathcal{C} &= \{((A \cup B)_{\max} \cup C)_{\max} : A \in \mathcal{A}, B \in \mathcal{B}, C \in \mathcal{C}\},\\ &= \{((A \cup B) \cup C)_{\max} : A \in \mathcal{A}, B \in \mathcal{B}, C \in \mathcal{C}\},\\ &= \{(A \cup (B \cup C))_{\max} : A \in \mathcal{A}, B \in \mathcal{B}, C \in \mathcal{C}\},\\ &= \{(A \cup (B \cup C)_{\max})_{\max} : A \in \mathcal{A}, B \in \mathcal{B}, C \in \mathcal{C}\},\\ &= \mathcal{A} \odot (\mathcal{B} \odot \mathcal{C}).\end{aligned}$$

□

We can now introduce the suggested semiring:

$$\begin{aligned} S &= \mathcal{P}_{\min}\left(\mathcal{P}_{\max}(E, \leq_E), \lesssim_{\max}\right), \\ \mathcal{A} \otimes \mathcal{B} &= \min\left(\mathcal{A} \odot \mathcal{B}, \lesssim_{\max}\right), \\ \mathcal{A} \oplus \mathcal{B} &= \min\left(\mathcal{A} \cup \mathcal{B}, \lesssim_{\max}\right), \\ \mathbf{0} &= \max(E, \leq_E), \\ \mathbf{1} &= \emptyset. \end{aligned}$$

We have:

Proposition 7. $(S, \otimes, \oplus, \mathbf{0}, \mathbf{1})$ *is a semiring.*

Proof. Remark first that $\otimes$ and $\oplus$ are internal composition operators. Moreover, $\lesssim_{\max}$ is a partial order on $\mathcal{P}_{\max}(E, \leq_E)$ by items (i), (iv) and (v) of Proposition 1. The axioms of a semiring hold:

(A_1) $(S, \oplus)$ is a commutative semigroup with $\mathbf{0}$ as neutral element:

- $\mathcal{A} \oplus \mathcal{B} = \mathcal{B} \oplus \mathcal{A}$ by commutativity of union.
- $\oplus$ is associative. Indeed, $\lesssim_{\max}$ is a partial order on $\mathcal{P}_{\max}(E, \leq_E)$, and consequently the second part of Lemma 1 applies:

$$\begin{aligned} (\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{C} &= \min\left(\min\left(\mathcal{A} \cup \mathcal{B}, \lesssim_{\max}\right) \cup \mathcal{C}, \lesssim_{\max}\right), \\ &= \min\left((\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C}, \lesssim_{\max}\right), \\ &= \min\left(\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}), \lesssim_{\max}\right), \\ &= \min\left(\mathcal{A} \cup \min\left(\mathcal{B} \cup \mathcal{C}, \lesssim_{\max}\right), \lesssim_{\max}\right), \\ &= \mathcal{A} \oplus (\mathcal{B} \oplus \mathcal{C}). \end{aligned}$$

- $\mathcal{A} \oplus \mathbf{0} = \min\left(\mathcal{A} \cup E_{\max}, \lesssim_{\max}\right) = \min\left(\mathcal{A}, \lesssim_{\max}\right)$ since $\forall A \in \mathcal{A},\ A \lesssim_{\max} E_{\max}$. Moreover, $\min\left(\mathcal{A}, \lesssim_{\max}\right) = \mathcal{A}$ by definition of S. Therefore $\mathcal{A} \oplus \mathbf{0} = \mathcal{A}$.

(A_2) $(S, \otimes)$ is a semigroup with $\mathbf{1}$ as neutral element, and for which $\mathbf{0}$ is an absorbing element:

- $\otimes$ is associative. First of all, $A \lesssim_{\max} A'$ and $B \lesssim_{\max} B'$ imply $A \cup B \lesssim_{\max} A' \cup B'$ by item (iv) of Proposition 1. Hence, we deduce $\min\left(\mathcal{A} \odot \mathcal{B}, \lesssim_{\max}\right) = \min\left(\min\left(\mathcal{A}, \lesssim_{\max}\right) \odot \mathcal{B}, \lesssim_{\max}\right)$. The proof is identical to that of Lemma 1 and is therefore omitted here. Finally, thanks to Lemma 1 and Lemma 2, we deduce:

$$\begin{aligned} (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} &= ((\mathcal{A} \odot \mathcal{B})_{\min} \odot \mathcal{C})_{\min} = ((\mathcal{A} \odot \mathcal{B}) \odot \mathcal{C})_{\min}, \\ &= (\mathcal{A} \odot (\mathcal{B} \odot \mathcal{C}))_{\min} \qquad = ((\mathcal{B} \odot \mathcal{C}) \odot \mathcal{A})_{\min}, \\ &= ((\mathcal{B} \odot \mathcal{C})_{\min} \odot \mathcal{A})_{\min} = (\mathcal{A} \odot (\mathcal{B} \odot \mathcal{C})_{\min})_{\min}, \\ &= \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}). \end{aligned}$$

- $\mathcal{A} \otimes \mathbf{1} = \min\left(\mathcal{A} \odot \emptyset, \lesssim_{\max}\right) = \min\left(\mathcal{A}, \lesssim_{\max}\right) = \mathcal{A}$.
- $\mathcal{A} \otimes \mathbf{0} = \min\left(\mathcal{A} \odot E_{\max}, \lesssim_{\max}\right) = \min\left(E_{\max}, \lesssim_{\max}\right) = \mathbf{0}$.

(A_3) $\otimes$ is distributive with respect to $\oplus$. The proof writes in three stages:
(i) $\mathcal{A} \odot (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \odot \mathcal{B}) \cup (\mathcal{A} \odot \mathcal{C})$

$$\begin{aligned}
\mathcal{A} \odot (\mathcal{B} \cup \mathcal{C}) &= \{\max(A \cup D, \leq_E) : A \in \mathcal{A},\ D \in \mathcal{B} \cup \mathcal{C}\},\\
&= \{\max(A \cup B, \leq_E) : A \in \mathcal{A},\ B \in \mathcal{B}\},\\
&\cup\ \{\max(A \cup C, \leq_E) : A \in \mathcal{A}, C \in \mathcal{C}\},\\
&= (\mathcal{A} \odot \mathcal{B}) \cup (\mathcal{A} \odot \mathcal{C}).
\end{aligned}$$

(ii) $\min\left(\mathcal{A} \cup \mathcal{B}, \lesssim_{\max}\right) = \min\left(\min\left(\mathcal{A}, \lesssim_{\max}\right) \cup \min\left(\mathcal{B}, \lesssim_{\max}\right), \lesssim_{\max}\right)$ by Lemma 1.
(iii) We show that $\otimes$ is distributive.

$$\begin{aligned}
\mathcal{A} \otimes (\mathcal{B} \oplus \mathcal{C}) &= (\mathcal{A}_{\min} \odot (\mathcal{B} \cup \mathcal{C})_{\min})_{\min},\\
&= (\mathcal{A} \odot (\mathcal{B} \cup \mathcal{C}))_{\min},\\
&= ((\mathcal{A} \odot \mathcal{B}) \cup (\mathcal{A} \odot \mathcal{C}))_{\min},\\
&= (\mathcal{A} \otimes \mathcal{B}) \oplus (\mathcal{A} \otimes \mathcal{C}).
\end{aligned}$$

This concludes the proof. □

7 *Jacobi* iterations

The Jacobi algorithm of classical linear algebra is analogous to the well-known Bellman algorithm[6] (Gondran and Minoux, 2000). We briefly recall its principle. Let $M = (m_{ij})$ be the generalized incidence matrix of a graph $G = (X, U)$, defined as follows:

$$m_{ij} = \begin{cases} v(u) & \text{if } u = (i, j) \in U,\\ \mathbf{0} & \text{otherwise,} \end{cases}$$

where $v(u)$ is an evaluation. As indicated in Rote (1990), since we are working in a semiring structure, we can show that:

$$(M^l)_{ij} = \bigoplus_{\substack{\mathcal{P}_{ij} \text{ a path}\\ \text{with } l \text{ arcs}}} v(\mathcal{P}_{ij}),$$

where $\mathcal{P}_{ij}$ denotes a path from i to j with l arcs and $(M^l)_{ij}$ is the element of $M^l = \underbrace{M \otimes \ldots \otimes M}_{l \text{ times}}$ at the intersection of line i and column j.

[6] In the classical framework, the Bellman algorithm writes as follows: Let $x_s^{[0]} = 0$, $x_j^{[0]} = +\infty$ ($\forall j \neq s$); repeat $x_s^{[t]} = 0$ and $x_j^{[t]} = \min_{u=(i,j)\in U}(x_i^{[t-1]} + v(u))$ ($\forall j \neq s$) until no x_j changes during an entire pass (where x denotes the distance vector).

Consequently:

$$(I \oplus M \oplus M^2 \oplus \ldots \oplus M^l)_{ij} = \bigoplus_{\substack{\mathcal{P}_{ij} \text{ a path} \\ \text{with at most } l \text{ arcs}}} v(\mathcal{P}_{ij}),$$

where I denotes the identity matrix with $\mathbf{1}$'s on the main diagonal and $\mathbf{0}$'s otherwise. As soon as $M \oplus M = M$ (which is the case for the considered semiring), we can show that:

$$(I \oplus M)^l = I \oplus M \oplus M^2 \oplus \ldots \oplus M^l.$$

As we are only interested in paths starting from s, we calculate the corresponding line x of $(I \oplus M)^l$. By induction, we could show that it reduces to iterating:

$$\begin{cases} x^{[0]} = (\mathbf{0}, \ldots, \mathbf{0}, \mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}), \\ x^{[l]} = x^{[l-1]} \otimes M \oplus (\mathbf{0}, \ldots, \mathbf{0}, \mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}) \text{ for } l \geq 1, \end{cases}$$

where $\mathbf{1}$ is on the column corresponding to the source s of the graph.

The comparison of two sets of evaluations A and B (induced by operations $\otimes$ and $\oplus$) with respect to the *mappimax* criterion can be realized in polynomial time. A pairwise comparison algorithm could reach this aim. We propose here instead a matrix algorithm. Let R denote the matrix of the binary relation $\leq_E$, whose components are defined as follows:

$$r_{ij} = \begin{cases} 1 \text{ if } e_i \leq_E e_j, \\ 0 \quad \text{otherwise.} \end{cases}$$

In what follows R_{AB} denotes the submatrix whose lines correspond to elements of A and columns to elements of B, and $1_{|A|}$ denotes the unit vector with $|A|$ components. Finally, let $\geq$ denote the partial order corresponding to the usual pointwise ordering relation between vectors ($\forall i,\ x_i \geq y_i$).

Proposition 8. *Let A and B be two sets of evaluations,*

$$A \lesssim_{\max} B \iff R_{AB} 1_{|B|} \geq 1_{|A|}.$$

Proof. $R_{AB} 1_{|B|} \geq 1_{|A|} \iff \forall e_i \in A,\ \exists e_j \in B,\ e_i \leq_E e_j \iff A \lesssim_{\max} B$ $\square$

Remark. Of course, $A <_{\max} B$ iff $R_{AB} 1_{|B|} \geq 1_{|A|}$ and not$(R_{BA} 1_{|A|} \geq 1_{|B|})$.

In our framework, this sequence converges in at most $n - 1$ iterations. As long as the graph is acyclic, we are able to determine $x^{[n-1]}$ in one pass only by sorting the vertices in increasing order of their ranks (so that we get a triangular matrix A). In such a case, the complexity of the algorithm is within $O(n^2 \times |E|^2 \times B^4)$, where $B = |S|^2 - \left|\lesssim_{\max}\right|_S$ and $\left|\lesssim_{\max}\right|_S$ denotes the cardinality of $\lesssim_{\max}$ on S (where S denotes the support of the semiring). Indeed, the number of incomparable minimal paths at each vertex is bounded above by B. Therefore, for each component of the matrix, the number of possible concatenations is bounded above by B^2, the number of comparisons to determine the maximal evaluations of a concatenation is bounded above by $|E|^2$, the number of comparisons to determine the maximal sets of evaluations is bounded above by B^2.

Remark. Note that this result can be interpreted as a kind of “ordinal pseudo-polynomiality". Indeed, the great number of incomparable subsets increases the combinatorial hardness of the problem, just like the great values increase the combinatorial hardness of valued combinatorial optimization problems, which leads to investigate algorithms of polynomial complexity with respect to the size of the instance and to the maximal number of incomparable subsets.

8 Conclusion

We have presented here a semiring that allows to solve the *bottleneck* shortest path problem on a partially ordered scale. Let us underline that the suggested semiring remains adequate whatever the partial order defined on the evaluation space. A promising research prospect would consist in studying other kinds of extension of a binary relation on a set to a binary relation on the corresponding power set. In this respect, we might consider several alternative proposals made in the field of decision theory (Kelly, 1977; MacIntyre and Pattanaik, 1981; Sen, 1991; Bossert et al., 2000; Spiegler, 2001). A characterization result concerning the kind of extensions which allow the use of a semiring structure would be worth investigating.

Acknowledgements. We wish to thank Denis Bouyssou and two anonymous referees for helpful comments on an earlier version of the paper.

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