Combinators for Algebraic Structures (CAS) Version 1

Timothy G. Griffin

July 12, 2016

Abstract

The document presents a small language of combinators for construction semigroups and semirings.

1 Motivation

We assume $2 \leq |S|$ for all carrier sets S.

2 Semigroup Combinators

2.1 Adding an identity

$$\mathrm{AddId}(\alpha,\ (S,\ \bullet)) \equiv (S \uplus \{\alpha\}, \bullet_{\alpha}^{\mathrm{id}})$$

where

$$a \bullet_{\alpha}^{\mathrm{id}} b \quad \equiv \quad \begin{cases} a & (\text{if } b = \mathrm{inr}(\alpha)) \\ b & (\text{if } a = \mathrm{inr}(\alpha)) \\ \mathrm{inl}(x \bullet y) & (\text{if } a = \mathrm{inl}(x),\ b = \mathrm{inl}(y)) \end{cases}$$

2.2 Adding an annihilator

$$\mathrm{AddAn}(\omega,\ (S,\ \bullet)) \equiv (S \uplus \{\omega\}, \bullet_{\omega}^{\mathrm{an}})$$

where

$$a \bullet_{\omega}^{\mathrm{an}} b \quad \equiv \quad \begin{cases} \mathrm{inr}(\omega) & (\text{if } b = \mathrm{inr}(\omega)) \\ \mathrm{inr}(\omega) & (\text{if } a = \mathrm{inr}(\omega)) \\ \mathrm{inl}(x \bullet y) & (\text{if } a = \mathrm{inl}(x),\ b = \mathrm{inl}(y)) \end{cases}$$

2.3 Direct Product

Let $(S,\ \bullet)$ and $(T,\ \diamond)$ be semigroups.

The direct product is denoted

$$(S,\ \bullet) \times (T,\ \diamond) \equiv (S \times T,\ \star)$$

where

$$\star = \bullet \times \diamond$$

is defined as

$$(s_1,\ t_1) \star (s_2,\ t_2) = (s_1 \bullet s_2,\ t_1 \diamond t_2).$$

2.4 Lexicographic Product

Suppose that semigroup $(S,\ \bullet)$ is commutative, idempotent, and selective and that $(T,\ \diamond)$ is a semigroup.

$$(S,\ \bullet)\ \vec{\times}\ (T,\ \diamond) \equiv (S \times T,\ \star)$$

where $\star \equiv \bullet \vec{\times} \diamond$ is defined as

$$(s_1,\ t_1) \star (s_2,\ t_2) = \begin{cases} (s_1 \bullet s_2,\ t_1 \diamond T_2) & s_1 = s_1 \bullet s_2 = s_2 \\ (s_1 \bullet s_2,\ t_1) & s_1 = s_1 \bullet s_2 \neq s_2 \\ (s_1 \bullet s_2,\ t_2) & s_1 \neq s_1 \bullet s_2 = s_2 \end{cases}$$

Examples for $(\mathbb{N},\ \min)\ \vec{\times}\ (\mathbb{N},\ \min)$.

$$\begin{array}{rcl} (1,\ 17) \star (2,3) & = & (1,17) \\ (2,\ 17) \star (2,3) & = & (2,3) \\ (2,\ 3) \star (2,3) & = & (2,3) \end{array}$$

Examples for $(\mathbb{N},\ \min)\ \vec{\times}\ (\mathbb{N},\ \max)$.

$$\begin{array}{rcl} (1,\ 17) \star (2,3) & = & (1,17) \\ (2,\ 17) \star (2,3) & = & (2,17) \\ (2,\ 3) \star (2,3) & = & (2,3) \end{array}$$

Examples for $(\mathbb{N},\ \max)\ \vec{\times}\ (\mathbb{N},\ \min)$.

$$\begin{array}{rcl} (1,\ 17) \star (2,3) & = & (2,3) \\ (2,\ 17) \star (2,3) & = & (2,3) \\ (2,\ 3) \star (2,3) & = & (2,3) \end{array}$$

2.5 Lifted Product

Assume $(S,\ \bullet)$ is a semigroup. Let $\text{lift}(S,\ \bullet) \equiv (\text{fin}(2^S),\ \hat{\bullet})$ where

$$X \hat{\bullet} Y = \{x \bullet y \mid x \in X,\ y \in Y\}.$$

Example.

$$\{1,\ 3,\ 17\}\ \hat{+}\ \{1,\ 3,\ 17\} = \{2,\ 4,\ 6,\ 18,\ 20,\ 34\}$$

2.6 Minimal set union

Let $\leq$ be a partial order on S. For $X \subseteq S$, define

$$\min_{\leq}(X) \equiv \{x \in X \mid \forall y \in X,\ \neg(y < x)\}.$$

Define

$$\mathcal{P}_{\mathrm{fin}}(S,\ \lesssim) \equiv \{X \subseteq S \mid X \text{ finite and } \min_{\leq}(X) = X\}$$

and

$$A\ \cup^{\leq}_{\min} B \equiv \min_{\leq}(A \cup B)$$

$$\mathrm{MinUnion}(S,\ \leq) \quad \equiv \quad (\mathcal{P}_{\mathrm{fin}}(S,\ \leq),\ \cup^{\leq}_{\min})$$

NOTE: perhaps we can always define $\leq$ in terms of a $(S,\ oplus)$, and so get rid of the class of structures $(S,\ \leq)$.

2.7 Minimal set lift

$$A\ \otimes^{\leq}_{\min} B \equiv \min_{\leq}(\{a \otimes b \mid a \in A,\ b \in B\})$$

$$\mathrm{MinLift}(S,\ \leq,\ \otimes) \quad \equiv \quad (\mathcal{P}_{\mathrm{fin}}(S,\ \leq),\ \otimes^{\leq}_{\min})$$

NOTE: perhaps we can always define $\leq$ in terms of a $(S,\ oplus)$, and so get rid of the class of structures $(S,\ \leq)$.

3 Pre-order Combinators

4 Pre-order Semigroup Combinators

5 Bi-Semigroup Combinators

5.1 Adding a zero

$$\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes)) \quad \equiv \quad (S \uplus \{\overline{0}\},\ \oplus^{\mathrm{id}}_{\overline{0}},\ \otimes^{\mathrm{an}}_{\overline{0}})$$

5.2 Adding a one

$$\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes)) \quad \equiv \quad (S \uplus \{\overline{1}\},\ \oplus^{\mathrm{an}}_{\overline{1}},\ \otimes^{\mathrm{id}}_{\overline{1}})$$

5.3 Direct Product

$$(S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T) \equiv (S \times T,\ \oplus_S \times \oplus_T,\ \otimes_S \times \otimes_T)$$

5.4 Lexicographic Product

$$(S,\ \oplus_S,\ \otimes_S) \vec{\times} (T,\ \oplus_T,\ \otimes_T) \equiv (S \times T,\ \oplus_S \vec{\times} \oplus_T,\ \otimes_S \times \otimes_T)$$

5.5 $\mathcal{M}$ and $\mathcal{N}$

Suppose $D \in \{L,\ R\}$. Let $\leq\ \equiv\ \leq_{\oplus}^{D}$ and define

$$\begin{array}{lcl} \mathcal{M}(D,\ (S,\ \oplus,\ \otimes)) & \equiv & (\mathcal{P}_{\text{fin}}(S,\ \leq),\ \otimes_{\text{min}}^{\leq},\ \cup_{\text{min}}^{\leq}) \\ \mathcal{N}(D,\ (S,\ \oplus,\ \otimes)) & \equiv & (\mathcal{P}_{\text{fin}}(S,\ \leq),\ \cup_{\text{min}}^{\leq},\ \otimes_{\text{min}}^{\leq}) \end{array}$$

Recall: $a \leq_{\oplus}^{L} b \equiv a = a \oplus b$ and $a \leq_{\oplus}^{R} b \equiv b = a \oplus b$.

6 Properties

$(S,\ \bullet)$

6.0.1 Initial set of semigroup properties

associative	$\mathbb{AS}$	$\equiv$	$\forall a, b, c \in S,\ a \bullet (b \bullet c) = (a \bullet b) \bullet c$
identity	$\mathbb{ID}$	$\equiv$	$\exists \alpha \in S,\ \forall a \in S,\ a = \alpha \bullet a = a \bullet \alpha$
annihilator	$\mathbb{AN}$	$\equiv$	$\exists \omega \in S,\ \forall a \in S,\ \omega = \omega \bullet a = a \bullet \omega$
commutative	$\mathbb{CM}$	$\equiv$	$\forall a, b \in S,\ a \bullet b = b \bullet a$
selective	$\mathbb{SL}$	$\equiv$	$\forall a, b \in S,\ a \bullet b \in \{a,\ b\}$
idempotent	$\mathbb{IP}$	$\equiv$	$\forall a \in S,\ a \bullet a = a$

6.0.2 Initial set of order properties

$(S,\ \leq)$

reflexive	$\mathbb{RX}$	$\equiv$	$\forall a \in S,\ a \leq a$
transitive	$\mathbb{TR}$	$\equiv$	$\forall a, b, c \in S, a \leq b \wedge b \leq c \implies a \leq c$
anti-symmetric	$\mathbb{AY}$	$\equiv$	$\forall a, b \in S, a \leq b \wedge b \leq a \implies a = b$
total	$\mathbb{TO}$	$\equiv$	$\forall a, b \in S,\ a \leq b \vee b \leq a$
bottom	$\mathbb{BM}$	$\equiv$	$\exists b \in S, \forall a \in S, b \leq a$
top	$\mathbb{TP}$	$\equiv$	$\exists t \in S, \forall a \in S, a \leq t$

6.0.3 Initial set of bi-semigroup properties

$(S,\ \oplus,\ \otimes)$

left distributivity	$\mathbb{LD}$	$\equiv$	$\forall a, b, c \in S,\ a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$
right distributivity	$\mathbb{RD}$	$\equiv$	$\forall a, b, c \in S,\ (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$
zero annihilates	$\mathbb{ZA}$	$\equiv$	$\exists \overline{0} \in S,\ \mathbb{IID}(S,\ \oplus,\ \overline{0}) \wedge \mathbb{IAN}(S,\ \otimes,\ \overline{0})$
one annihilates	$\mathbb{OA}$	$\equiv$	$\exists \overline{1} \in S,\ \mathbb{IID}(S,\ \otimes,\ \overline{1}) \wedge \mathbb{IAN}(S,\ \oplus,\ \overline{1})$

where

$$\mathbb{IID}(S,\ \bullet,\ \alpha) \equiv \forall a \in S,\ a = \alpha \bullet a = a \bullet \alpha$$
$$\mathbb{IAN}(S,\ \bullet,\ \omega) \equiv \forall a \in S,\ \omega = \omega \bullet a = a \bullet \omega$$

6.0.4 Initial set of order semigroup properties

$(S,\ \leq,\ \otimes)$

left monotonicity	$\mathbb{LM}$	$\equiv$	$\forall a,b,c \in S,\ a \leq b \implies (a \otimes c) \leq (b \otimes c)$
right monotonicity	$\mathbb{RM}$	$\equiv$	$\forall a,b,c \in S,\ a \leq b \implies (c \otimes a) \leq (c \otimes b)$

7 Closure

To close we need extra properties.

For semigroups.

				required for
is right	$\mathbb{IR}$	$\equiv$	$\forall s,t \in S, s \bullet t = t$	$\mathbb{SL}(\times)$
is left	$\mathbb{IL}$	$\equiv$	$\forall s,t \in S, s \bullet t = s$	$\mathbb{SL}(\times)$
left cancellative	$\mathbb{LC}$	$\equiv$	$\forall a,b,c \in S,\ c \bullet a = c \bullet b \Rightarrow a = b$	$\mathbb{LD}(\vec{\times})$
right cancellative	$\mathbb{RC}$	$\equiv$	$\forall a,b,c \in S,\ a \bullet c = b \bullet c \Rightarrow a = b$	$\mathbb{RD}(\vec{\times})$
left constant	$\mathbb{LK}$	$\equiv$	$\forall a,b,c \in S,\ c \bullet a = c \bullet b$	$\mathbb{LD}(\vec{\times})$
right constant	$\mathbb{RK}$	$\equiv$	$\forall a,b,c \in S,\ a \bullet c = b \bullet c$	$\mathbb{RD}(\vec{\times})$
anti-left	$\mathbb{AL}$	$\equiv$	$\forall a,b \in S,\ a \bullet b \neq a$	$\mathbb{LC}(\text{AddId}), \mathbb{LAB}(\vec{\times}, \times)$
anti-right	$\mathbb{AR}$	$\equiv$	$\forall a,b \in S,\ b \bullet a \neq a$	$\mathbb{RC}(\text{AddId}), \mathbb{RAB}(\vec{\times}, \times)$

For bi-semigroups $(S,\ \oplus,\ \otimes)$

				required for
left absorption	$\mathbb{LAB}$	$\equiv$	$\forall a,b \in S,\ a = (b \otimes a) \oplus a = a \oplus (b \otimes a)$	$\mathbb{LD}(\text{AddOne})$
right absorption	$\mathbb{RAB}$	$\equiv$	$\forall a,b \in S,\ a = (a \otimes b) \oplus a = a \oplus (a \otimes b)$	$\mathbb{LD}(\text{AddOne})$

7.1 A few usefull implications

$$\begin{aligned}
\mathbb{SL} &\Rightarrow \mathbb{IP} \\
\mathbb{RAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{ID}(S,\ \oplus) &\Rightarrow \mathbb{IP}(S,\ \otimes) \\
\mathbb{LAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{ID}(S,\ \oplus) &\Rightarrow \mathbb{IP}(S,\ \otimes) \\
\mathbb{LD}(S,\ \oplus,\ \otimes) \wedge \mathbb{OA}(S,\ \oplus,\ \otimes) &\Rightarrow \mathbb{RAB}(S,\ \oplus,\ \otimes) \\
\mathbb{RD}(S,\ \oplus,\ \otimes) \wedge \mathbb{OA}(S,\ \oplus,\ \otimes) &\Rightarrow \mathbb{LAB}(S,\ \oplus,\ \otimes)
\end{aligned}$$

7.2 AddId

$$
\begin{aligned}
\mathbb{AS}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{AS}(S,\bullet)\\
\mathbb{ID}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{TRUE}\\
\mathbb{AN}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{AN}(S,\bullet)\\
\mathbb{CM}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{CM}(S,\bullet)\\
\mathbb{IP}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{IP}(S,\bullet)\\
\mathbb{SL}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{SL}(S,\bullet)\\
\mathbb{IR}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{IL}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{LC}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{LC}(S,\ \bullet) \wedge \mathbb{AL}(S,\ \bullet)\\
\mathbb{RC}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{RC}(S,\ \bullet) \wedge \mathbb{AR}(S,\ \bullet)\\
\mathbb{LK}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{RK}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{AL}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{AR}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}
\end{aligned}
$$

7.3 AddAn

$$
\begin{aligned}
\mathbb{AS}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{AS}(S,\bullet)\\
\mathbb{ID}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{ID}(S,\bullet)\\
\mathbb{AN}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{TRUE}\\
\mathbb{CM}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{CM}(S,\bullet)\\
\mathbb{IP}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{IP}(S,\bullet)\\
\mathbb{SL}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{SL}(S,\bullet)\\
\mathbb{IR}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{IL}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{LC}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{RC}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{LK}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{RK}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{AL}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{AR}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) &\Leftrightarrow \mathbb{FALSE}
\end{aligned}
$$

7.4 $\times$

$$
\begin{aligned}
\mathbb{AS}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow \mathbb{AS}(S,\bullet)\wedge\mathbb{AS}(T,\diamond)\\
\mathbb{ID}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow \mathbb{ID}(S,\bullet)\wedge\mathbb{ID}(T,\diamond)\\
\mathbb{AN}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow \mathbb{AN}(S,\bullet)\wedge\mathbb{AN}(T,\diamond)\\
\mathbb{CM}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow \mathbb{CM}(S,\bullet)\wedge\mathbb{CM}(T,\diamond)\\
\mathbb{IP}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow \mathbb{IP}(S,\bullet)\wedge\mathbb{IP}(T,\diamond)\\
\mathbb{SL}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow (\mathbb{IR}(S,\bullet)\wedge\mathbb{IR}(T,\diamond))\vee(\mathbb{IL}(S,\bullet)\wedge\mathbb{IL}(T,\diamond))\\
\mathbb{IR}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow \mathbb{IR}(S,\bullet)\wedge\mathbb{IR}(T,\diamond)\\
\mathbb{IL}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow \mathbb{IL}(S,\bullet)\wedge\mathbb{IL}(T,\diamond)\\
\mathbb{LC}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow \mathbb{LC}(S,\bullet)\wedge\mathbb{LC}(T,\diamond)\\
\mathbb{RC}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow \mathbb{RC}(S,\bullet)\wedge\mathbb{RC}(T,\diamond)\\
\mathbb{LK}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow \mathbb{LK}(S,\bullet)\wedge\mathbb{LK}(T,\diamond)\\
\mathbb{RK}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow \mathbb{RK}(S,\bullet)\wedge\mathbb{RK}(T,\diamond)\\
\mathbb{AL}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow \mathbb{AL}(S,\bullet)\vee\mathbb{AL}(T,\diamond)\\
\mathbb{AR}((S,\bullet)\times(T,\diamond)) &\Leftrightarrow \mathbb{AR}(S,\bullet)\vee\mathbb{AR}(T,\diamond)
\end{aligned}
$$

7.5 $\vec{\times}$ (semigroup)

Assuming $\mathbb{AS}(S,\bullet)$, $\mathbb{CM}(S,\bullet)$, and $\mathbb{SL}(S,\bullet)$.

$$
\begin{aligned}
\mathbb{AS}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{AS}(T,\diamond)\\
\mathbb{ID}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{ID}(S,\bullet)\wedge\mathbb{ID}(T,\diamond)\\
\mathbb{AN}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{AN}(S,\bullet)\wedge\mathbb{AN}(T,\diamond)\\
\mathbb{CM}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{CM}(T,\diamond)\\
\mathbb{IP}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{IP}(T,\diamond)\\
\mathbb{SL}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{SL}(T,\diamond)\\
\mathbb{IR}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{IL}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{LC}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{RC}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{LK}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{RK}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{AL}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{FALSE}\\
\mathbb{AR}((S,\bullet)\vec{\times}(T,\diamond)) &\Leftrightarrow \mathbb{FALSE}
\end{aligned}
$$

7.6 lift

$$
\begin{array}{rcl}
\mathbb{AS}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\
\mathbb{ID}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \;\; (\hat{\alpha} = \{\alpha\}) \\
\mathbb{AN}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{TRUE} \;\; (\omega = \{\}) \\
\mathbb{CM}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\
\mathbb{SL}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{IL}(S, \bullet) \vee \mathbb{IR}(S, \bullet) \vee (\mathbb{IP}(S, \bullet) \;\wedge \mid S \mid = 2) \\
\mathbb{IP}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{SL}((S, \bullet)) \\
\mathbb{IL}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IR}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{LC}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{RC}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{LK}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{RK}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{AL}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{AR}(\text{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE}
\end{array}
$$

7.7 MinUnion

Assume $\mathbb{BM}(S, \leq)$ and $\mathbb{AY}(S, \leq)$. ($\mathbb{AY}$ does not seem to be required for associativity, idempotence... Don't think this is needed?)

$$
\begin{array}{rcl}
\mathbb{AS}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{ID}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{TRUE} \;\; (\alpha = \{\}) \\
\mathbb{AN}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{TRUE} \;\; (\omega = \{\bot\}) \\
\mathbb{CM}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{SL}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{TO}(S, \leq)) \\
\mathbb{IP}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{IL}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IR}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{LC}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{RC}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{LK}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{RK}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{AL}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{AR}(\text{MinUnion}(S, \leq)) & \Leftrightarrow & \mathbb{FALSE}
\end{array}
$$

7.8 MinLift

Assume $\mathbb{LM}(S, \leq, \otimes)$, $\mathbb{RM}(S, \leq, \otimes)$, and $\mathbb{AY}(S, \leq)$. (Needed for associativity) **Perhaps others constraints are needed to make progress on $\mathbb{SL}$ and $\mathbb{IP}$. Lattice Theory to the rescue?**

$$
\begin{array}{rcl}
\mathbb{AS}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & \mathbb{AS}(S,\ \otimes) \\
\mathbb{ID}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & \mathbb{ID}(S,\ \otimes) \\
\mathbb{AN}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & \mathbb{TRUE}\ \ (\omega = \{\}) \\
\mathbb{CM}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & \mathbb{CM}(S,\ \otimes) \\
\mathbb{SL}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & ???\quad (\text{ SOMEDAY !!!}) \\
\mathbb{IP}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & ???\quad (\text{ SOMEDAY !!!}) \\
\mathbb{IL}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IR}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{LC}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{RC}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{LK}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{RK}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{AL}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{AR}(\text{MinLift}(S,\ \leq,\ \otimes)) & \Leftrightarrow & \mathbb{FALSE}
\end{array}
$$

7.9 AddZero

$$
\begin{array}{rcl}
\mathbb{LD}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LD}(S,\ \oplus,\ \otimes) \\
\mathbb{RD}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RD}(S,\ \oplus,\ \otimes) \\
\mathbb{ZA}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{OA}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{OA}(S,\ \oplus,\ \otimes) \\
\mathbb{RAB}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RAB}(S,\ \oplus,\ \otimes) \\
\mathbb{LAB}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LAB}(S,\ \oplus,\ \otimes)
\end{array}
$$

7.10 AddOne

$$
\begin{array}{rcl}
\mathbb{LD}(\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LD}(S,\ \oplus,\ \otimes) \wedge \mathbb{RAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus) \\
\mathbb{RD}(\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RD}(S,\ \oplus,\ \otimes) \wedge \mathbb{LAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus) \\
\mathbb{ZA}(\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{ZA}(S,\ \oplus,\ \otimes) \\
\mathbb{OA}(\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{RAB}(\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus) \\
\mathbb{LAB}(\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus)
\end{array}
$$

7.11 Direct Product, $(\times,\ \times)$

$$
\begin{array}{rcl}
\mathbb{LD}((S,\ \oplus_S,\ \otimes_S) \times (T,\ \oplus_T,\ \otimes_T)) & \Leftrightarrow & \mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \\
\mathbb{RD}((S,\ \oplus_S,\ \otimes_S) \times (T,\ \oplus_T,\ \otimes_T)) & \Leftrightarrow & \mathbb{RD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{RD}(T,\ \oplus_T,\ \otimes_T) \\
\mathbb{ZA}((S,\ \oplus_S,\ \otimes_S) \times (T,\ \oplus_T,\ \otimes_T)) & \Leftrightarrow & \mathbb{ZA}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{ZA}(T,\ \oplus_T,\ \otimes_T) \\
\mathbb{OA}((S,\ \oplus_S,\ \otimes_S) \times (T,\ \oplus_T,\ \otimes_T)) & \Leftrightarrow & \mathbb{OA}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{OA}(T,\ \oplus_T,\ \otimes_T) \\
\mathbb{RAB}((S,\ \oplus_S,\ \otimes_S) \times (T,\ \oplus_T,\ \otimes_T)) & \Leftrightarrow & \mathbb{RAB}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{RAB}(T,\ \oplus_T,\ \otimes_T) \\
\mathbb{LAB}((S,\ \oplus_S,\ \otimes_S) \times (T,\ \oplus_T,\ \otimes_T)) & \Leftrightarrow & \mathbb{LAB}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{LAB}(T,\ \oplus_T,\ \otimes_T)
\end{array}
$$

7.12 Lexicographic Product, $(\vec{\times},\ \times)$

Assume $\mathbb{AS}(S, \oplus_S)$, $\mathbb{CM}(S, \oplus_S)$, and $\mathbb{SL}(S, \oplus_S)$.

$$
\begin{array}{rcl}
\mathbb{LD}((S,\ \oplus_S,\ \otimes_S) \vec{\times} (T,\ \oplus_T,\ \otimes_T)) & \Leftrightarrow & \mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \\
 & & \wedge(\mathbb{LC}(S,\ \otimes_S) \vee \mathbb{LK}(T,\ \otimes_T)) \\
\mathbb{RD}((S,\ \oplus_S,\ \otimes_S) \vec{\times} (T,\ \oplus_T,\ \otimes_T)) & \Leftrightarrow & \mathbb{RD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{RD}(T,\ \oplus_T,\ \otimes_T) \\
 & & \wedge(\mathbb{RC}(S,\ \otimes_S) \vee \mathbb{RK}(T,\ \otimes_T)) \\
\mathbb{ZA}((S,\ \oplus_S,\ \otimes_S) \vec{\times} (T,\ \oplus_T,\ \otimes_T)) & \Leftrightarrow & \mathbb{ZA}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{ZA}(T,\ \oplus_T,\ \otimes_T) \\
\mathbb{OA}((S,\ \oplus_S,\ \otimes_S) \vec{\times} (T,\ \oplus_T,\ \otimes_T)) & \Leftrightarrow & \mathbb{OA}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{OA}(T,\ \oplus_T,\ \otimes_T) \\
\mathbb{RAB}((S,\ \oplus_S,\ \otimes_S) \vec{\times} (T,\ \oplus_T,\ \otimes_T)) & \Leftrightarrow & \mathbb{RAB}(S,\ \oplus_S,\ \otimes_S) \wedge (\mathbb{RAB}(T,\ \oplus_T,\ \otimes_T) \vee \mathbb{AR}(S,\ \otimes_S)) \\
\mathbb{LAB}((S,\ \oplus_S,\ \otimes_S) \vec{\times} (T,\ \oplus_T,\ \otimes_T)) & \Leftrightarrow & \mathbb{LAB}(S,\ \oplus_S,\ \otimes_S) \wedge (\mathbb{LAB}(T,\ \oplus_T,\ \otimes_T) \vee \mathbb{AL}(S,\ \otimes_S))
\end{array}
$$

7.13 $\mathcal{N}$

Which constraints should we impose on $(S,\ \oplus,\ \otimes)$ in order to make this easier, cleaner? And how can we do this without giving up to much in terms of expressive power?

$$
\begin{array}{rcl}
\mathbb{LD}(\mathcal{N}(L,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{RD}(\mathcal{N}(L,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{ZA}(\mathcal{N}(L,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{OA}(\mathcal{N}(L,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{RAB}(\mathcal{N}(L,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{LAB}(\mathcal{N}(L,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ???
\end{array}
$$

$$
\begin{array}{rcl}
\mathbb{LD}(\mathcal{N}(R,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{RD}(\mathcal{N}(R,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{ZA}(\mathcal{N}(R,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{OA}(\mathcal{N}(R,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{RAB}(\mathcal{N}(R,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{LAB}(\mathcal{N}(R,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ???
\end{array}
$$

7.14 $\mathcal{M}$

$$
\begin{array}{rcl}
\mathbb{LD}(\mathcal{M}(L,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{RD}(\mathcal{M}(L,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{ZA}(\mathcal{M}(L,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{OA}(\mathcal{M}(L,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{RAB}(\mathcal{M}(L,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{LAB}(\mathcal{M}(L,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ???
\end{array}
$$

$$
\begin{array}{rcl}
\mathbb{LD}(\mathcal{M}(R,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{RD}(\mathcal{M}(R,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{ZA}(\mathcal{M}(R,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{OA}(\mathcal{M}(R,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{RAB}(\mathcal{M}(R,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ??? \\
\mathbb{LAB}(\mathcal{M}(R,\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & ???
\end{array}
$$

8 Other Problems

8.1 Matrix algebras

8.2 swap

We might want to define

$$\mathrm{swap}(S,\ \oplus,\ \otimes) \equiv (S,\ \otimes,\ \oplus).$$

The disadvantage of this is that we would then have to introduce (and close with respect to) these "dual" properties:

$$\begin{array}{rcl}
\mathbb{LD}^{\delta}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b, c \in S,\ a \oplus (b \otimes c) = (a \oplus b) \otimes (a \oplus c) \\
\mathbb{RD}^{\delta}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b, c \in S,\ (a \otimes b) \oplus c = (a \oplus c) \otimes (b \oplus c) \\
\mathbb{RAB}^{\delta}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b \in S,\ a = (a \oplus b) \otimes a = a \otimes (a \oplus b) \\
\mathbb{LAB}^{\delta}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b \in S,\ a = (b \oplus a) \otimes a = a \otimes (b \oplus a)
\end{array}$$

8.3 k-best paths

8.4 Semi-direct product

8.5 General reductions