MinUnion results

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Abstract

The document presents some theoretical results for the $\cup_{\min}^{\leq}$ operator.

1 Definition

Let $\leq$ be a partial order on some set S with at least two elements. For $X \subseteq S$, define

$$\min_{\leq}(X) \equiv \{x \in X \mid \forall y \in X,\ \neg(y < x)\}.$$

Define

$$\mathcal{P}_{\text{fin}}(S,\ \lesssim) \equiv \{X \subseteq S \mid X \text{ finite and } \min_{\leq}(X) = X\}$$

and

$$X\ \cup_{\min}^{\leq} Y \equiv \min_{\leq}(X \cup Y)$$

then

$$\text{MinUnion}(S,\ \leq) \quad \equiv \quad (\mathcal{P}_{\text{fin}}(S,\ \leq),\ \cup_{\min}^{\leq})$$

2 Summary

The current pre-conditions required for generating necessary and sufficient conditions for each of the currently tracked properties are:

- $\mathbb{TR}(\mathrm{S}, \leq)$: required for associativity
- $\mathbb{BM}(\mathrm{S}, \leq)$: required for annihilator
- $\mathbb{AY}(\mathrm{S}, \leq)$: required for annihilator

Interestingly $\mathbb{RX}(\mathrm{S}, \leq)$ is not required.

3 Lemmas

3.1 Absorbing

$\mathbb{TR}(\mathrm{S}, \leq) \Rightarrow \forall XY : \min_{\leq}(X \cup \min_{\leq}(Y)) = \min_{\leq}(X \cup Y)$

Proof

Assume $z \in \min_{\leq}(X \cup Y)$.

Then we have:

$$z \in X \ \vee \ z \in Y \tag{1}$$

$$\forall x \in X : \neg(x < z) \tag{2}$$

$$\forall y \in Y : \neg(y < z) \tag{3}$$

By Eq 3 and the fact that $\min_{\leq}(Y) \subseteq Y$, we have that

$$\forall y \in \min_{\leq}(Y) : \neg(y < z)$$

and combining this with Eq 2 gives

$$\forall x \in X \cup \min_{\leq}(Y) : \neg(x < z) \tag{4}$$

Case $z \in X$

Clearly $z \in X \cup \min_{\leq}(Y)$ and by combining this with Eq 4 we have that $x \in \min_{\leq}(X \cup \min_{\leq}(Y))$.

Case $z \in Y$

By Eq 3 we have that $z \in \min_{\leq}(Y)$ and hence we have that $z \in X \cup \min_{\leq}(Y)$. By Eq 4 we have that $z \in \min_{\leq}(X \cup \min_{\leq}(Y))$.

Hence $z \in \min_{\leq}(X \cup Y) \Rightarrow z \in \min_{\leq}(X \cup \min_{\leq}(Y))$.

Assume $z \in \min_{\leq}(X \cup \min_{\leq}(Y))$.

Then we have:

$$z \in X \ \vee \ z \in \min_{\leq}(Y) \tag{5}$$

$$\forall x \in X : \neg(x < z) \tag{6}$$

$$\forall y \in \min_{\leq}(Y) : \neg(y < z) \tag{7}$$

Assume $\exists w \in Y : y < w$

Clearly $w \notin \min_{\leq}(Y)$ otherwise Eq 7 would be contradicted. Therefore by definition of $\min_{\leq}$ and $\mathbb{TR}(\mathrm{S}, \leq)$ there exists a $y \in \min_{\leq}(Y)$ such that $y < w$. But again by $\mathbb{TR}(\mathrm{S}, \leq)$ this gives us $y < w < z$ which contradicts Eq 7.

Hence

$$\forall y \in Y : \neg(y < z) \tag{8}$$

Case $z \in X$

Then $z \in X \cup Y$, and by Eq 6 & 8 we have that $\forall x \in X \cup Y : \neg(x < z)$. Therefore $z \in \min_{\leq}(X \cup Y)$.

Case $z \in \min_{\leq}(Y)$

Then $z \in Y$ and hence $z \in X \cup Y$, and by Eq 6 & 8 we have that $\forall x \in X \cup Y : \neg(x < z)$. Therefore $z \in \min_{\leq}(X \cup Y)$.

Hence $z \in \min_{\leq}(X \cup \min_{\leq}(Y)) \Rightarrow z \in \min_{\leq}(X \cup Y)$.

Therefore $\min_{\leq}(X \cup \min_{\leq}(Y)) = \min_{\leq}(X \cup Y)$

4 Properties

4.1 Associative

$\mathbb{TR}(\mathrm{S}, \leq) \Rightarrow (\mathbb{AS}(\mathrm{MinUnion}(\mathrm{S}, \leq)) \Leftrightarrow \mathbb{TRUE})$

Proof

Using the absorbing lemma we have:

$$\begin{aligned}
X \cup^{\leq}_{\min} (Y \cup^{\leq}_{\min} Z) &= \min_{\leq}(X \cup \min_{\leq}(Y \cup Z)) \\
&= \min_{\leq}(X \cup (Y \cup Z)) \\
&= \min_{\leq}((X \cup Y) \cup Z) \\
&= \min_{\leq}(\min_{\leq}(X \cup Y) \cup Z) \\
&= (X \cup^{\leq}_{\min} Y) \cup^{\leq}_{\min} Z
\end{aligned}$$

4.2 Identity

$\mathbb{ID}(\mathrm{MinUnion}(\mathrm{S}, \leq)) \Leftrightarrow \mathbb{TRUE}$

Proof

Witness : $\emptyset$

$$\begin{aligned}
X \cup^{\leq}_{\min} \emptyset &= \min_{\leq}(X \cup \emptyset) \\
&= \min_{\leq}(X) \\
&= X
\end{aligned}$$

$$\begin{aligned}\emptyset \cup^{\leq}_{\min} X &= \min_{\leq}(\emptyset \cup X) \\ &= \min_{\leq}(X) \\ &= X\end{aligned}$$

4.3 Annihilator

$\mathbb{BM}(\mathrm{S}, \leq) \wedge \mathbb{AY}(\mathrm{S}, \leq) \Rightarrow (\mathbb{AN}(\mathrm{MinUnion}(\mathrm{S}, \leq)) \Leftrightarrow \mathbb{TRUE})$

Proof

Witness: $\{\bot\}$

$$\begin{aligned}X \cup^{\leq}_{\min} \{\bot\} &= \min_{\leq}(X \cup \{\bot\}) \\ &= \{\bot\}\end{aligned}$$

$$\begin{aligned}\{\bot\} \cup^{\leq}_{\min} X &= \min_{\leq}(\{\bot\} \cup X) \\ &= \{\bot\}\end{aligned}$$

4.4 Commutative

$\mathbb{CM}(\mathrm{MinUnion}(\mathrm{S}, \leq)) \Leftrightarrow \mathbb{TRUE}$

Proof

$$\begin{aligned}X \cup^{\leq}_{\min} Y &= \min_{\leq}(X \cup Y) \\ &= \min_{\leq}(Y \cup X) \\ &= Y \cup^{\leq}_{\min} X\end{aligned}$$

4.5 Selective

$\mathbb{SL}(\mathrm{MinUnion}(\mathrm{S}, \leq)) \Leftrightarrow \mathbb{TO}((\mathrm{S}, \leq))$

Proof

Assume $\mathbb{TO}((\mathrm{S}, \leq))$.

Then every set can only contain a single element. Hence for all x and y:

$$\begin{aligned}\{x\} \cup^{\leq}_{\min} \{y\} &= \min_{\leq}(\{x\} \cup \{y\}) \\ &= \min_{\leq}(\{x, y\})\end{aligned}$$

and as x and y are totally ordered, the result must either equal $\{x\}$ or $\{y\}$ and therefore we have that $\mathbb{SL}(\text{MinUnion}(\text{S}, \leq))$.

Assume $\neg\mathbb{TO}((\text{S}, \leq))$.

Then there exists x and y such that $x \nleq y$ and $y \nleq x$.

$$\begin{aligned}\{x\} \cup^{\leq}_{\min} \{y\} &= \min_{\leq}(\{x\} \cup \{y\}) \\ &= \min_{\leq}(\{x, y\}) \\ &= \{x, y\}\end{aligned}$$

Hence we have that $\neg\mathbb{SL}(\text{MinUnion}(\text{S}, \leq))$.

4.6 Idempotent

$\mathbb{IP}(\text{MinUnion}(\text{S}, \leq)) \Leftrightarrow \mathbb{TRUE}$

Proof

$$\begin{aligned}X \cup^{\leq}_{\min} X &= \min_{\leq}(X \cup X) \\ &= \min_{\leq}(X) \\ &= X\end{aligned}$$

4.7 IsLeft

$\mathbb{IL}(\text{MinUnion}(\text{S}, \leq)) \Leftrightarrow \mathbb{FALSE}$

Proof

There exists an identity for $\text{MinUnion}(\text{S}, \leq)$, therefore $\neg\mathbb{IL}(\text{MinUnion}(\text{S}, \leq))$.

4.8 IsRight

$\mathbb{IR}(\text{MinUnion}(\text{S}, \leq)) \Leftrightarrow \mathbb{FALSE}$

Proof

There exists an identity for $\text{MinUnion}(\text{S}, \leq)$, therefore $\neg\mathbb{IR}(\text{MinUnion}(\text{S}, \leq))$.

4.9 LeftCancellative

$\mathbb{LC}(\mathrm{MinUnion}(\mathrm{S}, \leq)) \Leftrightarrow \mathbb{FALSE}$

Proof

There exists an identity for $\mathrm{MinUnion}(\mathrm{S}, \leq)$, and at least two elements in S and therefore two non-identity elements in $\mathcal{P}_{\mathrm{fin}}(S, \lesssim)$. Therefore $\neg\mathbb{LC}(\mathrm{MinUnion}(\mathrm{S}, \leq))$.

4.10 RightCancellative

$\mathbb{RC}(\mathrm{MinUnion}(\mathrm{S}, \leq)) \Leftrightarrow \mathbb{FALSE}$

Proof

There exists an identity for $\mathrm{MinUnion}(\mathrm{S}, \leq)$, and at least two elements in S and therefore two non-identity elements in $\mathcal{P}_{\mathrm{fin}}(S, \lesssim)$. Therefore $\neg\mathbb{RC}(\mathrm{MinUnion}(\mathrm{S}, \leq))$.

4.11 LeftConstant

$\mathbb{LK}(\mathrm{MinUnion}(\mathrm{S}, \leq)) \Leftrightarrow \mathbb{FALSE}$

Proof

There exists an identity for $\mathrm{MinUnion}(\mathrm{S}, \leq)$, therefore $\neg\mathbb{LK}(\mathrm{MinUnion}(\mathrm{S}, \leq))$.

4.12 RightConstant

$\mathbb{RK}(\mathrm{MinUnion}(\mathrm{S}, \leq)) \Leftrightarrow \mathbb{FALSE}$

Proof

There exists an identity for $\mathrm{MinUnion}(\mathrm{S}, \leq)$, therefore $\neg\mathbb{RK}(\mathrm{MinUnion}(\mathrm{S}, \leq))$.

4.13 AntiLeft

$\mathbb{AL}(\mathrm{MinUnion}(\mathrm{S}, \leq)) \Leftrightarrow \mathbb{FALSE}$

Proof

There exists an identity for $\mathrm{MinUnion}(\mathrm{S}, \leq)$, therefore $\neg\mathbb{AL}(\mathrm{MinUnion}(\mathrm{S}, \leq))$.

4.14 AntiRight

$\mathbb{AR}(\mathrm{MinUnion}(\mathrm{S}, \leq)) \Leftrightarrow \mathbb{FALSE}$

Proof

There exists an identity for $\mathrm{MinUnion}(\mathrm{S}, \leq)$, therefore $\neg\mathbb{AR}(\mathrm{MinUnion}(\mathrm{S}, \leq))$.