

Chapter 19

Social Choice Theory and Multicriteria Decision Aiding

19.1. Introduction

Many organizations face such complex and important management problems that they sometimes want their decisions to be somehow supported by a ‘scientific approach’, sometimes called a *decision analysis*. The analyst in charge of this preparation faces many diverse tasks: stakeholders identification, problem statement, elaboration of a list of possible actions, definition of one or several criteria for evaluating these actions, information gathering, sensitivity analysis, elaboration of a recommendation (for instance a ranking of the actions or a subset of ‘good’ actions), etc. The desire or necessity to take multiple conflicting viewpoints into account for evaluating the actions often makes this task even more difficult. In that case, we speak of *multicriteria decision aiding* [POM 93, ROY 85, VIN 89]. The expert must then try to synthesize the partial preferences (modeled by each criterion) into a global preference on which a recommendation can be based. This is called *preference aggregation*.

A very similar aggregation problem has been studied for a long time in the framework of *voting theory*. It consists of searching a ‘reasonable’ mechanism (we call it voting system or aggregation method in the sequel) aggregating the opinions expressed by several voters on the candidates in an election, in order to determine a winner or to rank all candidates in order of preference. This problem is of course very old but its modern analysis dates back to the end of the eighteenth century [BOR 81, CON 85].

The diversity of voting systems actually used in the world shows that this problem is still important. In the 1950s, the works of [ARR 63, BLA 58, MAY 52] have initiated a huge literature [KEL 91] forming what is today called *social choice theory*. It analyzes the links that exists

Chapter written by Denis BOUSSOU, Thierry MARCHANT and Patrice PERNY.

(or should exist) between the *individual preferences* of the members of a society and the decisions made by this group when these decisions are supposed to reflect the *collective preference* of the group.

The many results obtained in social choice theory are valuable for multicriteria decision aiding. There are indeed links between these two domains: it is easy to go from one to the other by replacing the words ‘action’, ‘criterion’, ‘partial preference’ and ‘overall preference’ by ‘candidate’, ‘voter’, ‘individual preference’ and ‘collective preference’ [ARR 86].

The aim of this chapter is to present some important results in social choice theory in a simple way and to discuss their relevance for multicriteria decision aiding. Using some classical examples of voting problems (section 19.2), we will show some fundamental difficulties arising when aggregating preferences. We will then present some theoretical results that can help us better understand the nature of these difficulties (section 19.3). We will then try to analyze the consequences of these results for multicriteria decision aiding (section 19.4). A long list of references will help the interested reader to deepen their understanding of these questions.

19.2. Introductory examples

Choices made by a society often impact the individuals making up this society. It therefore seems reasonable to ground these choices on the preferences of the individuals. The choice of a candidate (law, project, social state, etc.) then depends on the outcome of an election in which the individuals (*voters*) express their preferences. A voting system (or aggregation method) uses the information provided by the voters in order to determine the elected candidate or, more generally, the decision made by the group.

In such conditions, how should we conceive a ‘good’ voting system? Common sense tells us that such a system must be democratic, i.e. it must yield collective preferences reflecting the individual preferences as much as possible. In many countries (groups, companies, committees), this is operationalized by the *majority rule* (or some variant of it): candidate a wins against b if the majority of the voters prefer a to b . This simple rule is very intuitive. As we will later see, when there are only two candidates this rule raises almost no problem [MAY 52].

This rule can be adapted in many ways to face situations with more than two candidates. These adaptations can lead to surprising outcomes, which will be illustrated by a few examples in this section. We will begin with *uninominal* voting systems, where each voter expresses their opinion through a ballot that only contains the name of one candidate (section 19.2.1), before moving to other systems where the voters can express their preferences in more complex ways (section 19.2.2).

In all examples, we will assume that each voter is able to rank (possibly with ties) all candidates in order of preference, i.e. can express preferences by means of a weak order. If a voter prefers a to b and b to c (thereby preferring a to c), we write ‘ $a \succ b \succ c$ ’. Except if otherwise stated, we will suppose that the voters are sincere, i.e. they express their ‘true’ preferences. Finally, notice that most examples presented here are classic. Many more examples and the analysis of many voting systems can be found in [DUM 84, FIS 77, MOU 80, MOU 88, NUR 87].

19.2.1. Uninominal systems

Example 19.1. Dictatorship of majority

Let $\{a, b, c, \dots, z\}$ be the set of 26 candidates for an election with 100 voters whose preferences are:

$$\begin{aligned} 51 \text{ voters have preferences } &a \succ b \succ c \succ \dots \succ y \succ z, \\ 49 \text{ voters have preferences } &z \succ b \succ c \succ \dots \succ y \succ a. \end{aligned}$$

It is clear that 51 voters will vote for a while 49 vote for z . Thus a has an absolute majority and, in all uninominal systems we are aware of, a wins. But is a really a good candidate? Almost half of the voters perceive a as the worst one. And candidate b seems to be a good candidate for everyone. Candidate b could be a good compromise. As shown by this example, a uninominal election combined with the majority rule allows a ‘dictatorship of majority’ and doesn’t favor a compromise. A possible way to avoid this problem might be to ask the voters to provide their whole ranking instead of their preferred candidate. We will see some examples in section 19.2.2.

The possibility of a dictatorship of the majority was already acknowledged by classic greek philosophers. The following examples show that many other strange phenomena can occur with uninominal voting systems.

Example 19.2. Respect of majority in the British system

The voting system in the United Kingdom is *plurality voting*, i.e. the election is uninominal and the aggregation method is simple majority. Let $\{a, b, c\}$ be the set of candidates for a 21 voters election (or 21×10^6 voters if one wishes a more realistic example). Suppose that

$$\begin{aligned} 10 \text{ voters have preferences } &a \succ b \succ c, \\ 6 \text{ voters have preferences } &b \succ c \succ a, \\ 5 \text{ voters have preferences } &c \succ b \succ a. \end{aligned}$$

Then a (respectively, b and c) obtains 10 votes (respectively, 6 and 5) so that a is chosen. Nevertheless, this might be different from what a majority of voters wanted. Indeed, an absolute majority of voters prefers any other candidate to a (11 out of 21 voters prefer b and c to a).

Let us see, using the same example, if such a problem could be avoided by the two-stage French system (also called plurality with runoff). After the first stage, as no candidate has an absolute majority, a second stage is run between candidates a and b . We suppose that the voters keep the same preferences on $\{a, b, c\}$. So

$$\begin{aligned} 10 \text{ voters have preferences } &a \succ b, \\ 11 \text{ voters have preferences } &b \succ a. \end{aligned}$$

Thus a obtains 10 votes and b 11 votes so that candidate b is elected. This time, none of the beaten candidates (a and c) are preferred to b by a majority of voters. Nonetheless we cannot conclude that the two-stage French system is superior to the British system from this point of view, as shown by the following example.

Example 19.3. Respect of majority in the two-stage French system

Let $\{a, b, c, d\}$ be the set of candidates for a 21 voters election. Suppose that

- 10 voters have preferences $b \succ a \succ c \succ d$,
- 6 voters have preferences $c \succ a \succ d \succ b$,
- 5 voters have preferences $a \succ d \succ b \succ c$.

After the first stage, as no candidate has absolute majority, a second stage is run between candidates b and c . Candidate b easily wins with 15 out of 21 votes although an absolute majority (11/21) of voters prefer a and d to b .

Because it is not necessary to be a mathematician to figure out such problems, some voters might be tempted not to sincerely report their preferences as shown in the next example.

Example 19.4. Manipulation in the two-stage French system

Let us continue with the example above. Suppose that the six voters having preferences $c \succ a \succ d \succ b$ decide not to be sincere and vote for a instead of c . Then candidate a wins after the first stage because there is an absolute majority for him (11/21). If they had been sincere (as in the previous example), b would have been elected. Thus, casting an insincere vote is useful for those 6 voters as they prefer a to b . Such a system, that may encourage voters to falsely report their preferences, is called manipulable.

This is not the only weakness of the French system, as attested by the following three examples.

Example 19.5. Monotonicity in the two-stage French system

Let $\{a, b, c\}$ be the set of candidates for a 17 voters election. A few days before the election, the results of a survey are as follows:

- 6 voters have preferences $a \succ b \succ c$,
- 5 voters have preferences $c \succ a \succ b$,
- 4 voters have preferences $b \succ c \succ a$,
- 2 voters have preferences $b \succ a \succ c$.

In the French system, a second stage would be run between a and b and a would be chosen obtaining 11 out of 17 votes. Suppose that candidate a , in order to increase his lead over b and to lessen the likelihood of a defeat, decides to strengthen his electoral campaign against b . Suppose that the survey exactly revealed the preferences of the voters and that the campaign has the correct effect on the last two voters. Hence we observe the following preferences.

- 8 voters have preferences $a \succ b \succ c$,
- 5 voters have preferences $c \succ a \succ b$,
- 4 voters have preferences $b \succ c \succ a$.

After the first stage, b is eliminated, due to the campaign of a . The second stage opposes a to c and c wins, obtaining 9 votes. Candidate a thought that his campaign would be beneficial. He was wrong. Such a method is called non-monotonic because an improvement of a candidate's position in some of the voter's preferences can lead to a deterioration of his position after the aggregation.

It is clear with such a system that it is not always interesting or efficient to sincerely report one's preferences. You will note in the next example that some manipulations can be very simple.

Example 19.6. Participation in the two-stage French system

Let $\{a, b, c\}$ be the set of candidates for a 11 voters election. Suppose that

- 4 voters have preferences $a \succ b \succ c$,
- 4 voters have preferences $c \succ b \succ a$,
- 3 voters have preferences $b \succ c \succ a$.

In the French system, a second stage should oppose a to c and c should win the election obtaining 7 out of 11 votes. Suppose that 2 of the first 4 voters (with preferences $a \succ b \succ c$) decide not to vote because c , the worst candidate according to them, is going to win anyway. What will happen? There will only be 9 voters.

- 2 voters have preferences $a \succ b \succ c$,
- 4 voters have preferences $c \succ b \succ a$,
- 3 voters have preferences $b \succ c \succ a$.

Contrary to all expectations, candidate c will loose while b will win, obtaining 5 out of 9 votes. Our two lazy voters can be proud of their abstention since they prefer b to c . Clearly such a method does not encourage participation.

Example 19.7. Separability in the two-stage French system

Let $\{a, b, c\}$ be the set of candidates for a 26 voters election. The voters are located in two different areas: countryside and town. Suppose that the 13 voters located in the town have the following preferences.

- 4 voters have preferences $a \succ b \succ c$,
- 3 voters have preferences $b \succ a \succ c$,
- 3 voters have preferences $c \succ a \succ b$,
- 3 voters have preferences $c \succ b \succ a$.

Suppose that the 13 voters located in the countryside have the following preferences.

- 4 voters have preferences $a \succ b \succ c$,
- 3 voters have preferences $c \succ a \succ b$,
- 3 voters have preferences $b \succ c \succ a$,
- 3 voters have preferences $b \succ a \succ c$.

Suppose now that an election is organized in the town, with 13 voters. Candidates a and c will go to the second stage and a will be chosen, obtaining 7 votes. If an election is organized in the countryside, a will defeat b in the second stage, obtaining 7 votes. Thus a is the winner in both areas. Naturally we expect a to be the winner in a global election. But it is easy to observe that in the global election (26 voters) a is defeated during the first stage. Such a method is called non-separable.

The previous examples showed that, when there are more than 2 candidates, it is not an easy task to imagine a system that would behave as expected. Note that, in the presence of 2 candidates, the British system (uninominal and one-stage) is equivalent to all other systems and it suffers none of the above-mentioned problems [MAY 52]. We might therefore be tempted by a generalization of the British system (restricted to 2 candidates). If there are 2 candidates, we use the British system; if there are more than 2 candidates, we arbitrarily choose 2 of them and we use the British system to select the winner. The winner is opposed (using the British system) to a new arbitrarily chosen candidate, and so on until no more candidates remain. This would require $n - 1$ votes between 2 candidates. Unfortunately, this method suffers severe drawbacks.

Example 19.8. Influence of the agenda in sequential voting

Let $\{a, b, c\}$ be the set of candidates for a 3 voters election. Suppose that

- 1 voter has preferences $a \succ b \succ c$,
- 1 voter has preferences $b \succ c \succ a$,
- 1 voter has preferences $c \succ a \succ b$.

The 3 candidates will be considered two by two in the following order or agenda: a and b first, then c . During the first vote, a is opposed to b and a wins with absolute majority (2 votes against 1). Then a is opposed to c and c defeats a with absolute majority. c is therefore elected.

If the agenda is a and c first, it is easy to see that c defeats a and is then opposed to b . Hence, b wins against c and is elected.

If the agenda is b and c first, it is easy to see that a is finally elected. Consequently, in this example, any candidate can be elected and the outcome depends completely on the agenda, i.e. on an arbitrary decision. Let us note that sequential voting is very common in different parliaments. The different amendments to a bill are considered one by one in a predefined sequence. The first one is opposed to the original bill using the British system; the second one

is opposed to the winner and so on. Finally, the result is opposed to the status quo. Clearly, such a method lacks neutrality. It doesn't treat all candidates in a symmetric way. Candidates (or amendments) appearing at the end of the agenda are more likely to be elected than those at the beginning. We say that such a method is not *neutral*. Notice that the British and French systems are neutral because they do not favor any candidate.

Example 19.9. Violation of unanimity in sequential voting

Let $\{a, b, c, d\}$ be the set of candidates for a 3 voters election. Suppose that

- 1 voter has preferences $b \succ a \succ d \succ c$,
- 1 voter has preferences $c \succ b \succ a \succ d$,
- 1 voter has preferences $a \succ d \succ c \succ b$.

Consider the following agenda: a and b first, then c and finally d . Candidate a is defeated by b during the first vote. Candidate c wins the second vote and d is finally elected although all voters unanimously prefer a to d . Let us remark that this cannot happen with the French and British systems.

Example 19.10. Tie-breaking chairperson

Suppose we use the two-stage French system and, at the second stage, the two candidates have the same number of votes. This is very unlikely in a national election but can often occur in small-scale elections (board of trustees, court jury, Ph.D. jury, etc.). It is then usual to use the chairperson's vote to break the tie. In this case, the opinions of all voters are not treated in the same way. We then say that the voting system is not *anonymous*, unlike all systems we have seen so far. Note that using the chairperson's vote is not the only possibility: we could break the tie by choosing, for instance, the oldest of the two candidates (this would not respect neutrality).

Up until now, we have assumed that the voters are able to rank all candidates from best to worst without ties but the only information that we collected was the best candidate. We could try to palliate the many encountered problems by asking voters to explicitly rank the candidates in order of preference (some systems, like approval voting, use another kind of information; see [BRA 82]). This idea, although interesting, will lead us to many other pitfalls as discussed in the following section.

19.2.2. Systems based on rankings

In this kind of election, each voter provides a ranking without ties of the candidates. Hence the task of the aggregation method is to extract from all these rankings the best candidate or a ranking of the candidates reflecting the preferences of the voters as much as possible. Comparing all candidates pairwise in the following way has been suggested [CON 85].

Condorcet method (or majority method) Candidate a is preferred to b if and only if the number of voters ranking a before b is larger than the number of voters ranking b before a . In case of tie, candidates a and b are indifferent.

Condorcet states the following principle.

Condorcet principle If a candidate is preferred to each other candidate using the majority rule, then he should be chosen. The candidate, the *Condorcet winner*, is necessarily unique.

Note that neither the British or French system respect this principle. Indeed, in example 19.2, the British system leads to the election of a while b is the Condorcet winner and, in example 19.3, the French system elects b while a is the Condorcet winner.

The Condorcet principle seems very sensible and close to the intuitive notion of democracy (yet it can be criticized, as suggested in example 19.1 where candidate a is a Condorcet winner). It is not always operational: in some situations, there is no Condorcet winner; this is the so-called *Condorcet paradox*. Indeed, in example 19.8, a is preferred to b , b is preferred to c and c is preferred to a . No candidate is preferred to all others. In such a case, the Condorcet method fails to elect a candidate. One might think that example 19.8 is very bizarre and unlikely to happen. Unfortunately it isn't. If you consider an election with 25 voters and 11 candidates, the probability of such a paradox is significantly high: approximately $1/2$ [GEH 83]. The more candidates or voters, the higher the probability of such a paradox. Note that, in order to obtain this result, all rankings are supposed to have the same probability. Such an hypothesis is clearly questionable [GEH 83].

We must find how to proceed when there is no Condorcet winner. We may, for example, choose a candidate such that no other candidate defeats him according to the majority rule (weak Condorcet principle), but such a candidate does also not always exist (as in example 19.8). Many methods have been proposed for exploiting the relation constructed using the majority method [FIS 77, LAS 97, NUR 87].

An alternative approach has been proposed by [BOR 81]. He suggests associating a global score to each candidate. This score is the sum of his ranks in the rankings of the voters.

Borda method Candidate a is preferred to b if the sum of the ranks of a in the rankings of the voters is strictly smaller than the corresponding sum for b (we now assume that the rankings are without tie and we assign rank 1 to the best candidate in the ranking, rank 2 to the second best candidate, and so on; as we will see, the method can be easily generalized for handling ties).

Example 19.11. Borda and Condorcet methods

Let $\{a, b, c, d\}$ be the set of candidates for a 3 voters election. Suppose that

$$\begin{aligned} 2 \text{ voters have preferences } &b \succ a \succ c \succ d, \\ 1 \text{ voter has preferences } &a \succ c \succ d \succ b. \end{aligned}$$

The Borda score of a is $5 = 2 \times 2 + 1 \times 1$. For b , it is $6 = 2 \times 1 + 1 \times 4$. Candidates c and d receive 8 and 11. Thus a is the winner and the collective ranking is $a \succ b \succ c \succ d$. Using

the Condorcet method, the conclusion is different: b is the Condorcet winner. Furthermore, the collective preference obtained by the Condorcet method is transitive and yields the ranking $b \succ a \succ c \succ d$. The two methods diverge; the Borda method does not verify the Condorcet principle. Nevertheless, it can be shown that the Borda method never chooses a Condorcet loser, i.e. a candidate that is beaten by all other candidates by an absolute majority (contrary to the British system, see example 19.2).

The Borda method has an important advantage with respect to the Condorcet method. In any situation, it selects one or several winners (those with the lowest sum of ranks). Furthermore, it always yields a ranking of the candidates from best to worse. The Condorcet method, on the contrary, sometimes yields non-transitive preferences and it is then impossible to rank the candidates or even to choose a subset of ‘good’ candidates (see example 19.8). It is easy to verify that the Borda method is neutral, anonymous, separable, monotonic and encourages participation.

The Borda method nevertheless sometimes behaves in a strange way. Indeed, consider example 19.11 and suppose that candidates c and d decide on the eve of the election not to compete because they are almost sure to lose. With the Borda method, the new winner is b . Thus b now defeats a just because c and d dropped out. The fact that a defeats or is defeated by b therefore depends not only on the relative positions of a and b in the rankings of the voters but is also contingent upon the presence of other candidates and on their position with respect to all other candidates. This can be a problem as the set of candidates is not always fixed. It is even more of a problem in decision aiding because the set of actions is seldom given and is, to a large extent, the outcome of a modeling process.

After all these examples, we would like to propose a democratic method with the advantages of the Borda method (transitivity of the collective preferences) and those of the Condorcet method (Condorcet principle and absence of contingency problems). We will see in section 19.3 that it is mainly hopeless.

Let us mention that we limited this discussion to voting systems aimed at choosing a candidate and not a subset of candidates. The reader might then be tempted to conclude that those systems are inferior to systems aimed at choosing a representative body with some ‘proportional’ method. But this is too simple, for at least two reasons. First, the definition of what constitutes a fair or democratic proportional representation is complex and most proportional systems lead to paradoxical situations [BAL 82]. Second, representative bodies must make decisions and, to this end, they need voting systems aimed at choosing a single action.

19.3. Some theoretical results

Based on the preceding examples, we now have the intuition that conceiving ‘good’ preference aggregation methods raises serious problems. This is confirmed by some celebrated results in social choice theory.

19.3.1. Arrow's theorem

Arrow's theorem is central in social choice theory. It is about voting systems aimed at aggregating n ($n \geq 3$) weak orders (rankings possibly with ties) in a collective weak order. Just as in section 19.2.2, each voter ranks all the candidates, possibly with ties.

Formalization 19.1. A binary relation R on a set A is a subset of $A \times A$. We often write aRb instead of $(a, b) \in R$. A weak order on A is a complete (for all $a, b \in A$ we have aRb and/or bRa) and transitive (for all $a, b, c \in A$, aRb and bRc imply aRc) binary relation on A . Let $\mathcal{WO}(A)$ denote the set of all weak orders on the set A . The asymmetric part of R is the binary relation P defined by $aPb \Leftrightarrow [aRb \text{ and Not } bRa]$. The symmetric part of R is the binary relation I defined by $aIb \Leftrightarrow [aRb \text{ and } bRa]$.

Let $N = \{1, 2, \dots, n\}$ represent the set of voters and A the set of candidates. We assume that voter $i \in N$ expresses their preference by means of a weak order $R_i \in \mathcal{WO}(A)$ on the set A . We write P_i (respectively, I_i) for the asymmetric (respectively, symmetric) part of R_i .

Arrow was interested in the aggregation methods satisfying the following conditions.

Universality Every configuration of rankings is admissible.

Formalization 19.2. We want to find an aggregation function F yielding a result (a collective weak order) for every element (R_1, R_2, \dots, R_n) of $\mathcal{WO}(A)^n$.

This condition excludes any constraint on the set of admissible rankings. The examples of the previous section have shown that some problems are caused by some specific rankings or configurations of rankings. A possible way out would then consist of proposing a method that works only with ‘simple’ configurations. Imposing restrictions on the admissible configurations is sometimes reasonable. For instance, one may sometimes assume that all voters and candidates are located on a right-left axis and that each voter ranks the candidates in order of increasing distance between themselves and the candidates. The preferences of the voters are then *single-peaked*; [BLA 58] showed that a Condorcet winner then necessarily exists. However, such restrictions imply e.g. the absence of atypical voters. This cannot be excluded *a priori*. With a non-universal aggregation method, some ballots would be impossible to analyze.

Transitivity The outcome of the aggregation method must always be a complete ranking, possibly with ties.

Formalization 19.3. The aggregation function takes its values in $\mathcal{WO}(A)$.

When there is no ambiguity, we write $R = F(R_1, R_2, \dots, R_n)$ and P (respectively, I) the asymmetric part (respectively, symmetric) of R .

This condition imposes that the outcome is transitive irrespective of the preference of the voters. Whenever the society prefers a to b and b to c , it must therefore prefer a to c . We have seen that the Condorcet method does not satisfy this condition. It is sufficient (but not necessary) to ensure that the method will, in all cases, designate one or several best candidates (those with the best positions in the ranking). We will later see that weakening this condition does not improve the situation formalized by Arrow's theorem.

Unanimity The outcome of the aggregation method may not contradict the voters when they vote unanimously.

Formalization 19.4. *The aggregation function F must be such that, for all $a, b \in A$, if $aP_i b$ for all $i \in N$, then aPb .*

If a is ranked before b in each ranking, then it must be before b in the collective ranking. This condition is very sensible; Example 19.9 nevertheless shows that some methods violate it.

Independence The relative position of two candidates in the collective ranking only depends on their relative position in the individual rankings.

Formalization 19.5. *For all $(R_1, R_2, \dots, R_n), (R'_1, R'_2, \dots, R'_n) \in \mathcal{WO}(A)^n$ and all $a, b \in A$, if $aR_i b \Leftrightarrow aR'_i b$ and $bR_i a \Leftrightarrow bR'_i a$, then $aRb \Leftrightarrow aR'b$.*

This condition is more complex than the previous conditions. When comparing a and b , it forbids

- taking preference intensities into account: the only thing that matters is that a is ranked by the voters before or after b ; and
- taking other candidates into account.

Let us illustrate this condition with an example.

Example 19.12. The Borda method and Independence

Let $\{a, b, c, d\}$ be the set of candidates. Suppose there are three voters with the following preferences:

2 voters have preferences $c \succ a \succ b \succ d$,
1 voter has preferences $a \succ b \succ d \succ c$.

The Borda method yields the ranking: a, c, b, d with the respective scores 5, 6, 8 and 11.

Suppose now that :

2 voters have preferences $c \succ a \succ b \succ d$,
1 voter has preferences $a \succ c \succ b \succ d$.

The Borda method yields the ranking: c, a, b, d with the respective scores 4, 5, 9 and 12.

Note that, in each individual ranking, the relative position of a and c did not vary across ballots: one voter prefers a to c while two voters prefer c to a . Independence then imposes that the position of a and c in the collective ranking be identical. This is not the case with the Borda

method. Indeed, this method uses the fact that the ‘distance’ between a and c seems larger in the ranking $a \succ b \succ d \succ c$ than in the ranking $a \succ c \succ b \succ d$, because b and d lie between a and c in the first case.

The dependence of the relative position of a and c with respect to b and d is ruled out by the Independence condition. It also excludes any method using, in addition to the rankings, some information regarding preference intensities.

The last condition used by Arrow states that no voter can impose, in all circumstances, their preferences to the society. This condition is extremely sensible for anyone willing to use a democratic method.

Non-dictatorship There is no dictator.

Formalization 19.6. *For all $i \in N$ and all $a, b \in A$, there is a profile $(R_1, R_2, \dots, R_n) \in \mathcal{WO}(A)^n$ such that $aP_i b$ and bRa .*

We are now ready to state the following celebrated theorem.

Theorem 19.1. [ARR 63] *If the number of voters is finite and there is at least three candidates, no aggregation method can simultaneously satisfy universality, transitivity, unanimity, independence and non-dictatorship.*

Proof. The proof of Arrow’s theorem uses the following definitions. A subset $I \subseteq N$ of voters is *almost decisive* for the pair of candidates $(a, b) \in A^2$ if, for all $(R_1, R_2, \dots, R_n) \in \mathcal{WO}(A)^n$, $[aP_i b, \forall i \in I \text{ and } bP_j a, \forall j \notin I] \Rightarrow aPb$. Similarly, the subset $I \subseteq N$ of voters is *decisive* for the pair of candidates $(a, b) \in A^2$ if, for all $(R_1, R_2, \dots, R_n) \in \mathcal{WO}(A)^n$, $[aP_i b, \forall i \in I] \Rightarrow aPb$.

We first show that, if I is almost decisive for the pair (a, b) , then I is decisive for all pairs of candidates.

Let c be a candidate distinct from a and b (such a candidate always exists because we assumed $n \geq 3$). Let $(R_1, R_2, \dots, R_n) \in \mathcal{WO}(A)^n$ be a profile such that $aP_i c, \forall i \in I$. Let $(R'_1, R'_2, \dots, R'_n) \in \mathcal{WO}(A)^n$ be a profile such that

- $aP'_i bP'_i c, \forall i \in I$,
- $bP'_j a$ and $bP'_j c, \forall j \notin I$.

Since I is almost decisive for the pair (a, b) , we have $aP'b$. Unanimity imposes $bP'c$. Transitivity then implies $aP'c$. Since the relation between a and c for the voters outside I in the profile $(R'_1, R'_2, \dots, R'_n)$ has not been specified, Independence implies aPc . We have therefore proved that whenever I is almost decisive for the pair (a, b) , then I is decisive for any pair of candidates (a, c) such that $c \neq a, b$. This reasoning is easily generalized to the case where c is not distinct from a or b .

We now show that there is always a voter $i \in N$ almost decisive for some pair of candidates. As shown above, this voter will be decisive for all pairs of candidates and will therefore be a dictator.

By unanimity, N is almost decisive for all pairs of candidates. Since N is finite, there is at least one subset $J \subseteq N$ almost decisive for the pair (a, b) with a minimal cardinality. Suppose $|J| > 1$ and consider a profile $(R_1, R_2, \dots, R_n) \in \mathcal{WO}(A)^n$ such that:

- $aP_i b P_i c$, for $i \in J$,
- $cP_j a P_j b \forall j \in J \setminus \{i\}$,
- $bP_k c P_k a \forall k \notin J$.

Since J is almost decisive for the pair (a, b) we have aPb . It is impossible that cPb . Indeed, by independence, this would imply $J \setminus \{i\}$ is almost decisive for the pair (c, b) and, hence, decisive for all pairs, contrary to our hypothesis. We therefore have bRc and transitivity implies aPc . This implies that $\{i\}$ is almost decisive for the pair (a, c) . \diamond

This negative result applies only when there are at least three candidates. It is easy to verify that the majority method satisfies the five conditions of Arrow's theorem with two candidates. Arrow's theorem explains to a large extent the problems we met in section 19.2 when we were trying to find a 'satisfying' aggregation procedure. Observe, for instance, that the Borda method verifies universality, transitivity, unanimity and non-dictatorship. Hence, it cannot verify independence, as shown in example 19.12. The Condorcet method respects universality, unanimity, independence and non-dictatorship. It cannot therefore be transitive, as shown in example 19.8.

Notice that Arrow's theorem uses only five conditions. In addition to these, we might wish to impose also neutrality, anonymity, monotonicity, non-manipulability, separability or Condorcet's principle. What makes Arrow's theorem so strong is precisely that it uses only five conditions, all seemingly reasonable. This is enough to prove an impossibility.

Arrow's theorem initiated a huge literature, a good overview of which can be found in [CAM 02, FIS 87, KEL 78, SEN 86]. Let us mention that weakening transitivity does not solve the problem revealed by Arrow's theorem. For instance, if we impose quasi-transitivity (i.e. transitivity of the asymmetric part) instead of transitivity, then we can always determine one or several winners. However, it is possible to prove that replacing transitivity by quasi-transitivity in Arrow's theorem leads to an oligarchy instead of a dictatorship. An oligarchy is a subset of voters that can impose their preferences when they are unanimous and such that each of them can veto any strict preference i.e. if a member of the oligarchy strictly prefers a to b , then b cannot be strictly better than a in the collective preference [GIB 69, MAS 72].

Example 19.13. Let us consider six voters numbered from $i = 1$ to 6 and an aggregation method yielding the relation $R = F(R_1, R_2, \dots, R_6)$ by means of:

$$\begin{aligned} xPy &\Leftrightarrow \sum_{\{i:xP_iy\}} w_i > \lambda, \\ xIy &\text{ otherwise,} \end{aligned}$$

with $w_1 = w_2 = 0.4$, $w_3 = w_4 = w_5 = w_6 = 0.05$ and $\lambda = 0.7$. This method is oligarchic. Indeed, consider the set O containing voters 1 and 2. It is easy to verify that, for any profile of

preferences,

$$\begin{aligned}[xP_1y \text{ and } xP_2y] &\Rightarrow xPy, \\ [xP_1y \text{ or } xP_2y] &\Rightarrow \text{Not } yPx.\end{aligned}$$

The existence of an oligarchy is as problematic as the existence of a dictator. Indeed, if the oligarchy contains all voters (this is the only possibility if we want a democratic method) then, because of the veto right of each voter, the collective preference will not be very decisive since it will not discriminate much between candidates. On the contrary, an oligarchy containing only one voter is a dictatorship. Between these two extreme cases, no solution is satisfactory.

We can weaken transitivity even more and impose that there is no circuit in the asymmetric part of the collective preference relation. This condition is necessary and sufficient to guarantee the existence of maximal elements in any finite set of candidates [SEN 70]. However, it is then possible to prove the existence of a voter with an absolute veto [MAS 72] so this does not really help much.

19.3.1.1. Arrow's theorem and fuzzy preferences

Why is it impossible to aggregate voters' preferences in a satisfactory way (i.e. while respecting Arrow's conditions)? There are mainly two reasons:

- The information contained in the weak orders describing the voters' preferences is too poor; it is ordinal. If we use richer structures, we can hope to escape Arrow's theorem. In particular, if we represent the voters' preferences by means of fuzzy relations, we can not only speak of the preference of a over b but also of the intensity of this preference.
- The global preference must be a weak order and this is a strong constraint. If we weaken this condition, we may consider aggregation methods yielding relations with more flexibility, such as fuzzy relations.

Some authors [e.g. BAR 86, BAR 92, LEC 84, PER 92a] have analyzed the consequences of imposing that the outcome of the aggregation is a fuzzy relation, that is a mapping R from A^2 to $[0, 1]$. Their findings are unfortunately largely negative: if we impose that the fuzzy relation has some properties permitting the easy designation of a winner or construction of a ranking, then we find that the only possible aggregation methods give very different powers to the various voters (as in oligarchies or dictatorships). In particular, it is the case if we impose that the collective preference relation verifies min-transitivity, i.e. for all $a, b, c \in A$:

$$R(a, c) \geq \min(R(a, b), R(b, c)).$$

This condition guarantees that the relation R_λ defined by

$$aR_\lambda b \Leftrightarrow R(a, b) \geq \lambda,$$

is transitive for any value of λ . Hence, starting from a min-transitive relation, it is not difficult to designate a winner or to rank the candidates.

However, there are some positive results in the literature which use weaker transitivity conditions [e.g. OVC 91]. It is then tempting to believe that Arrow's theorem does not hold with

fuzzy relations. But these apparently positive results are misleading: the transitivity condition they use is so weak that is not incompatible with Condorcet cycles, as shown in the following example.

Example 19.14. The transitivity condition used by [OVC 91] can be expressed as follows. For all $a, b, c \in A$:

$$R(a, c) \geq R(a, b) + R(b, c) - 1. \quad (19.1)$$

Suppose we want to aggregate the preferences of n voters. We can define the collective fuzzy preference relation by

$$R(a, b) = \frac{1}{n} \# \{i \in A : aR_i b\}.$$

It is easy to show that it satisfies equation (19.1). Let us now consider $3k$ voters with the following preferences:

- k voters have preferences $a \succ b \succ c$,
- k voters have preferences $b \succ c \succ a$,
- k voters have preferences $c \succ a \succ b$.

We obtain: $R(a, b) = 2/3$, $R(b, c) = 2/3$ and $R(c, a) = 2/3$; this is indeed compatible with equation (19.1). However, note that this relation is in some sense cyclic and does not permit us to designate a winner or to rank the candidates. Therefore, this does not solve the problem raised by Arrow's theorem.

In summary, unless we consider a very weak transitivity relation (without any practical interest), aggregation methods yielding fuzzy relations do not escape Arrow's theorem.

19.3.2. Some other results

Arrow's theorem and its many extensions represent only a part of the numerous results in social choice theory. For a comprehensive overview of this field, see [CAM 02, SEN 86]. In this paper, we will roughly group the results into three categories as follows:

1) Impossibility results, as for Arrow's theorem, show that some conditions are incompatible. These results help us to understand better why it is difficult to find a 'good' aggregation method.

2) Characterization results present a set of conditions that a given aggregation method and only this one simultaneously respects. Such results help us understand better the essential characteristics of a method. It is then easier to compare it with other methods.

3) 'Analysis' results: given a set of desirable conditions, these results compare different methods in order to see which satisfies the most axioms. This can help to find a satisfactory method (within the limits revealed by impossibility results).

This distinction is of course to some extent arbitrary, and the three kinds of results are not contradictory. They often use the same conditions.

We will now informally mention some results that we find important or interesting for understanding some phenomena presented in the examples of section 19.2.

19.3.2.1. *Impossibility results*

Among the impossibility results in social choice theory, two are particularly important:

1) Gibbard-Satterthwaite's theorem [GIB 73, SAT 75]. This result shows that there is no aggregation method (for choosing a single candidate) verifying universality, non-dictatorship and non-manipulability when there are at least three candidates. The French electoral system is clearly non-dictatorial and satisfies universality. If we neglect the ties than can occur during the second stage, Gibbard-Satterthwaite's theorem tells us that there is at least one situation where a voter would benefit from voting not sincerely. We have seen such a situation in example 19.4. Note that this result initiated a huge literature analyzing voting problems in terms of non-cooperative games [DUM 84, MOU 80, MOU 88, PEL 84].

2) Sen's theorem of the ‘Paretian liberal’ [SEN 70]. Suppose a society must vote to choose one of several social states. These are defined in such a way that they concern the private sphere of an individual. Clearly, there are conflicts between the majority principle, possibly yielding to a dictatorship of majority (see example 19.1), and the respect of this individual for his private sphere, in which he should decide alone. The theorem of the Paretian liberal tells us much more than this: it proves that the respect of a private sphere is incompatible with universality and unanimity. This result initiated a large literature, a good overview of which can be found in [SEN 83, SEN 92].

19.3.2.2. *Characterizations*

Among the many characterization results (many such results are presented in [SEN 86]), those about the Borda method (section 19.2.2) are particularly interesting. Indeed, this method satisfies most conditions encountered so far and it is very easy to implement.

19.3.2.2.1. A characterization of the Borda method

In this section, we present a characterization of the Borda method proved by [YOU 74]. This method is considered as a choice procedure, i.e. a procedure mapping each profile of weak orders on A to a non-empty subset of A . In this context, the Borda method works as follows: for each candidate a , we calculate a score (Borda score) $B(a)$ equal to the sum of the ranks of candidate a in the weak orders of the voters. In case of tie, we use the mean rank. The choice set then contains the candidate(s) with the smallest score. Example 19.11 illustrates how the scores are computed. Note that, in this example, the Borda method is used to rank and not to choose.

Formalization 19.7. *A choice procedure is a function $f : \mathcal{WO}(A)^n \rightarrow 2^A \setminus \emptyset$. To each n -tuple of weak orders, f associates a non-empty subset of A , interpreted as the set of the best candidates. The Borda method is defined by:*

$$f(R_1, R_2, \dots, R_n) = \{a \in A : B(a) \leq B(b), \forall b \in A\},$$

where $B(a)$ is the Borda score of candidate a and is defined by :

$$B(a) = \sum_{i=1}^n [\#\{b \in A : bR_i a\} - \#\{b \in A : aR_i b\}]. \quad (19.2)$$

This formalization is not exactly the sum of the ranks but the reader will easily check that $B(a)$, defined by equation (19.2), is an affine transformation of the sum of the ranks and therefore, using equation (19.2) or the sum of the ranks always yields the same result. We will use equation (19.2) because it is more convenient than the sum of the ranks.

In order to characterize the Borda method, [YOU 74] uses four conditions.

Neutrality The choice set depends only on the position of the candidates in the preferences of the voters and not, for instance, on the name of the candidates or on their age.

Formalization 19.8. Let \mathcal{P} be the set of all permutations on A , π an element of \mathcal{P} and R a binary relation on A . We write $\pi(T)$ for the binary relation such that $\pi(a) \pi(T) \pi(b) \Leftrightarrow aRb$. A choice method is neutral if and only if $f(R_1, \dots, R_n) = \pi(f(\pi(R_1), \dots, \pi(R_n)))$ for any permutation π in \mathcal{P} .

This condition imposes that all candidates be treated in the same way. It excludes, for instance, methods where the older candidate wins in case of tie. Similarly, sequential voting (example 19.8) is ruled out.

Faithfulness If there is only one voter, then the choice set must contain the best candidates according to this unique voter.

Formalization 19.9. $f(R_1) = \{a \in A : aR_1 b, \forall b \in A\}$.

This condition is extremely intuitive. Indeed, if there is only one voter, why not respect their preferences?

Consistency Suppose, as in example 19.7, that the voters are divided into two groups. We use the same choice method in both groups. If some candidates belong to both choice sets, then these candidates and only these should belong to the choice set which results from applying the same choice method to the whole set of voters.

Formalization 19.10.

$$f(R_1, \dots, R_m) \cap f(R_{m+1}, \dots, R_n) \neq \emptyset \Rightarrow$$

$$f(R_1, \dots, R_n) = f(R_1, \dots, R_m) \cap f(R_{m+1}, \dots, R_n).$$

Consistency is quite sensible. If two groups agree that some candidate, say a , is one of the best, then it is difficult to understand why a would not be a winner when both groups vote together.

Many such conditions, involving two groups of voters, have been used in the literature. They are often called separability. Consistency is one of these conditions.

Cancellation Let us consider two candidates a and b and suppose the number of voters preferring a to b is equal to the number of voters preferring b to a . This is not very particular. Suppose now this is true not only for a and b but for all pairs of candidates, simultaneously. We then face a very particular situation. In such a situation, cancellation requires that the choice set contains all candidates.

Formalization 19.11.

$$\forall a, b \in A, \#\{i \in N : aR_ib\} = \#\{i \in N : bR_ia\} \Rightarrow f(R_1, \dots, R_n) = A.$$

Among the four conditions used by Young, cancellation is probably the most questionable one. In some sense, it is reasonable: when, for each pair a, b of candidates, there are as many voters in favor of a as in favor of b , we can indeed prudently consider that no candidate is better than the other. But there are other situations when prudence recommends considering all candidates tied. For instance, when the majority relation is cyclic (see above, Condorcet paradox). Choosing cancellation rather than another condition imposing a complete tie in case of a cyclic majority relation or in another case is rather arbitrary.

The reader will easily verify that the Borda method verifies neutrality, faithfulness, consistency and cancellation. The following theorem, proved by Young, tells us much more.

Theorem 19.2. [YOU 74] *One and only one choice method verifies neutrality, faithfulness, consistency and cancellation: the Borda method.*

Since the proof of this theorem is quite long, we do not present it in this chapter. Notice that a similar characterization exists for the borda method used to rank [NIT 81]. Moreover, different generalizations of this result have been proved for the Borda method used to aggregate many different kinds of binary relations and even fuzzy binary relations [DEB 87, MAR 96, MAR 98, MAR 00, OUL 00].

19.3.2.3. Generalizations of the Borda method

The Borda method is a particular case of a general family of aggregation methods called *scoring rules*. These rules associate a number (a score) to each position in a binary relation. In order to aggregate n preference relations, we compute, for each candidate, the sum of its scores in the preference relations of the n voters. The winner is the candidate with the smallest total score. The Borda method is a particular scoring rule where the numbers associated to each rank are equally spaced. The British system is also a scoring rule where the best candidate in a preference relation receives 1 point and all the others receive the same score, say 2.

It has been shown that scoring rules are essentially characterized by neutrality, anonymity and separability [SMI 73, YOU 74, YOU 75]. (If we then add cancellation, we obtain a characterization of the Borda method.) For an overview of many results about scoring rules, see [SAA 94]. The French system is not a scoring rule because of the second stage. However, it is neutral and anonymous. It is therefore not separable, as shown in example 13.7. We have noticed in section 13.2 that the British system and the Borda method do not satisfy the Condorcet principle (see examples 19.2 and 19.10). This is not a surprise: indeed, it is possible to prove that no scoring rule can satisfy the Condorcet principle [MOU 88].

The French system can be considered as a scoring rule with iteration: at the first stage, it uses the British system for selecting two candidates. The same system is then used at the second stage. Note that there are many ways to iterate a scoring rule (one could for example use more than two stages). A result by [SMI 73] shows that no iterated scoring rule is monotonic. The violation of monotonicity by the French system (example 19.5) is simply a consequence of this.

19.3.2.4. *A characterization of simple majority*

In this section, we present the characterization of simple majority of [MAY 52] for two candidates. In this case, the distinction between choosing and ranking is no longer meaningful but, in order not to use a new formalism, we adopt here the choice formalism. May considers a choice procedure, i.e. a method designating one or several winners, based on the preferences of the voters. A formal definition of a choice method was presented above, in relation to the Borda method.

A candidate belongs to the choice set with a simple majority if the number of voters supporting them is not smaller than the number of voters supporting their contender.

Formalization 19.12. *The simple majority choice method is defined by: $a \in f(R_1, \dots, R_n)$ if and only if*

$$\#\{i \in N : aR_i b\} \geq \#\{i \in N : bR_i a\}.$$

Note that voters that are indifferent between a and b have no effect on the outcome of the election. Their votes are counted on both sides of the inequality. The outcome would be the same if they did not exist. In order to characterize simple majority, [MAY 52] used three conditions.

Anonymity The choice set depends only on the preferences of the voters and not, for instance, on their name or age.

Formalization 19.13. *Let \mathcal{S} be the set of all permutations on $N = \{1, \dots, n\}$. A choice method is anonymous if and only if $f(R_1, \dots, R_n) = f(R_{\sigma(1)}, \dots, R_{\sigma(n)})$ for any permutation σ in \mathcal{S} .*

This condition rules out, for example, the methods where some voters weigh more than others and methods where a voter (usually the chairperson of the committee) has the power to decide in case of a tie.

Neutrality See above.

Strict monotonicity Given the preferences of the voters, if the candidates a and b are chosen and if one of the voters changes his preferences in favor of a (the other voters do not change anything), then only a is chosen. If, only a was chosen at the beginning, then a stays alone in the choice set.

Formalization 19.14. Consider two weak orders R_i and R'_i identical apart from the fact there is a pair of candidates (a, b) such that:

- Not $aR_i b$ and $aR'_i b$ or
- $bR_i a$ and Not $bR'_i a$.

Strict monotonicity then imposes:

$$f(R_1, \dots, R_i, \dots, R_n) = \{a\} \Rightarrow f(R_1, \dots, R'_i, \dots, R_n) = \{a\},$$

and

$$f(R_1, \dots, R_i, \dots, R_n) = \{a, b\} \Rightarrow f(R_1, \dots, R'_i, \dots, R_n) = \{a\}.$$

A consequence of this condition is that, in case of a tie, a single voter changing their mind is enough to break the tie. Simple majority clearly verifies the three above-mentioned conditions. Moreover, no other method satisfies them all.

Theorem 19.3. [MAY 52] When there are exactly two candidates, the only choice method satisfying neutrality, anonymity and strict monotonicity is simple majority.

To understand why this theorem only applies to the case of two candidates, note that many different choice methods coincide when there are only two candidates. In particular, the Borda method and many scoring methods always yield the same result as simple majority with two candidates. You may then question the interest of this characterization. Actually, Arrow's theorem has shown us that simple majority cannot be extended to more than two candidates (without deeply modifying it). The characterization with two candidates is therefore essential.

19.3.2.5. Analysis

The few aggregation methods presented so far are just a small sample of all the methods proposed in the literature. In particular, we did not mention the methods using the majority relation (constructed by the Condorcet method) to arrive at a choice set or a ranking. Similarly, the properties (such as neutrality or monotonicity) presented so far are also a very small subset of all those studied in the literature. For an overview of methods and properties, see [ARR 63, DED 00, FEL 92, FIS 77, LEV 95, NUR 87, RIC 75, RIC 78a, RIC 78b, RIC 81].

19.4. Multicriteria decision aiding and social choice theory

19.4.1. Relevance and limits of social choice results

We have seen in section 19.1 that aggregation problems in multicriteria decision aiding and social choice are formally very close to each other. The examples of section 19.2 and the results of section 19.3 taught us that conceiving a satisfactory aggregation method is a challenging task. Some authors [e.g. GAR 82] have then concluded that multicriteria decision aiding is doomed to failure. For a detailed answer to this objection, see [ROY 93]. We nonetheless mention the following points:

1) Such a conclusion flows from a biased and too radical interpretation of the available results in social choice theory. There are some impossibility results but this does not mean that resorting to an aggregation method to try to find a collective decision is a futile exercise. It is a demanding task requiring compromises to be made between several exigencies that are in general not compatible.

These results, when combined with characterization and analysis results, provide a good support to motivate the choice of a method. There is no ideal method but some are perhaps more satisfactory than others. See [SAA 94] for a convincing plea in favor of the Borda method or [BRA 82] for approval voting.

2) The formal proximity between both problems does not imply that both problems are identical. In particular:

- The goal of a multicriteria decision aiding process is not always to choose one and only one action. There are many other kinds of outcomes, unlike in social choice theory [ROY 85].

- Some conditions look intuitive in social choice theory but are questionable in multicriteria decision aiding, and conversely. Let us mention, for example, that anonymity is not relevant in multicriteria decision aiding as soon as we wish to take criteria of different importance into account. Conversely, the set of potential actions to be evaluated is seldom given in multicriteria decision aiding (contrary to the set of candidates in social choice theory); it can evolve. The conditions telling us how an aggregation method should behave when this set changes (some actions are added or removed) are therefore more important in multicriteria decision aiding than in social choice theory.

- The preferences to be aggregated in multicriteria decision aiding are the outcome of a long modeling phase along each criterion [BOU 90]. This modeling phase can sometimes lead to incomplete preferences, fuzzy preferences or preferences such that indifference is not transitive [FOD 94, PER 92c, PER 98, ROU 85]. In some circumstances, it is possible to finely model preference intensities or even to compare preference differences on different criteria [KEE 76, VON 86]. Let us mention that handling uncertainty, imprecision or indeterminacy is often necessary to arrive at a recommendation in multicriteria decision aiding [BOU 89], contrary to social choice theory.

- In multicriteria decision aiding, contrary to social choice, it is not always necessary to completely construct the global preference. Indeed, it can occur that the decision maker can express their global preference with respect to some pairs of alternatives. For example, they are able to state that they prefer x to z and y to z but they hesitate between x and y . If they then use an aggregation method, it is in order to construct the preference only between x and y and not on the whole set of alternatives. Of course, these preferences that we construct on some pairs of

alternatives must be based on the single-criterion preferences of the decision maker but also on the global preferences stated.

In multicriteria decision aiding, we therefore have a new element at our disposal: the global preferences. These do not exist in social choice theory. They are of course (very) incomplete but they can nevertheless help construct the global preference relation. In practice, these global preferences are often used by analysts in order to set the value of some parameters of the aggregation method they use. For instance, with the methods based on multi-attribute value theory (MAVT), the decision maker must compare (sometimes fictitious) alternatives in order to assess the value functions. The existence of these global preferences, totally non-existing in social choice theory, breaks the symmetry between multicriteria decision aiding and social choice theory. Few theoretical results have so far taken the global preferences of the decision maker into account. More research is needed [MAR 03].

Even if both domains are formally close to each other and if some conditions used in social choice theory can also be found in multicriteria decision aiding, we must beware of crude transpositions due to the many specificities of multicriteria aggregation.

Conversely, we must not conclude that both domains are unrelated and that the examples and results of sections 19.2 and 19.3 are of no consequence for multicriteria analysis. It has clearly been shown [VAN 86a] that it is possible and useful to consider multicriteria aggregation methods in the light of social choice theory. Let us mention that, for example, the difference between the Condorcet and the Borda method can be found in multicriteria analysis between the ordinal methods [ROY 91, ROY 93] and the cardinal ones where the idea of preference difference is central [KEE 76, VON 86]. In the light of Arrow's theorem, it is not surprising that ordinal methods often lead to global preference relations from which a recommendation is not always easy to derive [VAN 90].

Many results of social choice theory still need to be adapted and/or extended to make them relevant to multicriteria analysis. Among the works in this direction, let us mention:

- impossibility results [ARR 86, BOU 92a, PER 92b],
- characterization results [BOU 92b, BOU 86, BOU 92c, MAR 96, PIR 95, PIR 97] and
- analysis results [BOU 97, LAN 96, LAN 97, PÉR 94, PÉR 95, PIR 97, VIN 92].

However, there is still much to do [BOU 93].

19.4.2. Some results in close relation with multicriteria analysis

So far, we have tried to sketch a global overview of social choice theory and to show the links with multicriteria decision aiding and the limits of this analogy. In this last section, we mention some results of social choice theory that are directly relevant for the analysis of some popular aggregation methods in multicriteria decision aiding.

19.4.2.1. TACTIC [VAN 86b]

The first relevant result is the characterization of a simple majority with two alternatives by [MAY 52], presented higher. This aggregation method can be seen as a particular case of

TACTIC, with a concordance threshold equal to 1, without weights and without discordance. For the case of two alternatives, a result by [FIS 73] characterizes simple majority with weights.

Another article worth mentioning here is by [MAR 03]. It presents two characterizations of weighted simple majority with any number of alternatives. It is therefore slightly more general than the results of May and Fishburn. It corresponds to a particular case of TACTIC with a concordance threshold equal to 1 and no discordance.

19.4.2.2. Multi-attribute value theory (MAVT) [KEE 76, VON 86]

The methods of this family are usually analyzed in the framework of measurement theory [KRA 71, WAK 89]. There are however some relevant results in social choice theory and, in particular, in cardinal social choice theory. In this part of social choice theory, the information to be aggregated is not ordinal (not a binary relation) but cardinal: it consists of utilities, that is, numbers representing preferences [ROB 80]. As far as we know, none of these results have been transposed in multicriteria decision aiding.

19.4.2.3. Weighted sum

The weighted sum is a particular case of MAVT methods. The previous section is therefore relevant for the weighted sum. Let us highlight a particular result: [ROB 80, theorem 2] characterizes the weighted sum. See also [BLA 54, D'A 77].

19.4.2.4. ELECTRE and PROMETHEE [ROY 91, ROY 93, VIN 89]

With ELECTRE and PROMETHEE, each alternative is represented by a vector of \mathbb{R}^n , $x = (x_1, \dots, x_n)$ where x_i represents the performance of x on criterion i (we suppose that all criteria are to be maximized).

The first step in PROMETHEE consists of choosing, for each criterion, a preference function f_i [MAR 88]. This is used to compute, for each pair of alternatives x, y , a number between 0 and 1 representing a preference degree denoted by $P_i(x, y)$ and defined by $P_i(x, y) = f_i(x_i, y_i)$. At the end of the first step, we therefore have a fuzzy preference relation for each criterion, P_i being the fuzzy relation associated to criterion i and $P_i(x, y)$ the value of this relation for the pair x, y .

In the next step, these fuzzy relations are aggregated by means of a generalization of the Borda method. This generalization has been characterized by [MAR 96]. Some variants of this characterization are presented in [MAR 98, MAR 00, OUL 00].

The ELECTRE methods use a somehow similar construction but with veto effects [ROY 91, ROY 93]. The preference relation constructed at the end of the aggregation phase uses some functions f_i and g_i with values in $[0; 1]$ in order to define (1) concordance indices $C_i(x, y) = f_i(x_i, y_i)$ representing to what extent x_i is at least as good as y_i and (2) discordance indices $D_i(x, y) = g_i(x_i, y_i)$ expressing to what extent the difference $y_i - x_i$ is compatible with a global preference of x over y . When $y_i - x_i$ exceeds a certain threshold (*veto threshold*), $D_i(x, y)$ equals 1 and the aggregation method then forbids a preference of x over y [PER 92c].

The ELECTRE and PROMETHEE methods therefore use aggregation procedures based on the construction and aggregation of fuzzy relations. They therefore do not escape the impossibility results mentioned in section 19.3.1.1 or about the aggregation of fuzzy relations [PER 92b]. This is why a last phase (exploitation) is necessary in order to reach a recommendation [ROY 93, VAN 90]. This last phase is often difficult and the problems it raises can also be analyzed in the light of axiomatic results on ordinal aggregation of preferences. For instance, some non-monotonicity phenomena arising with exploitation procedures based on an iterated choice function [FOD 98, PER 92a] can be explained by Smith's theorem presented in section 19.3.2.3 or by more recent axiomatic analyses in the same direction [BOU 04, JUR 03].

Let us finally mention that [BOU 96] has extended the classic results of [MCG 53] regarding simple majority to ELECTRE and PROMETHEE.

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