Discrete Mathematics 26 (1979) 55–78

SEMIRINGS AND PATH SPACES*

Ahnont WONGSEELASHOTE
University of Southampton, England

Received 29 May 1977
Revised 10 April 1978

This paper develops a unified algebraic theory for a class of path problems such as that of finding the shortest or, more generally, the k shortest paths in a network; the enumeration of elemementary or simple paths in a graph. It differs from most earlier work in that the algebraic structure appended to a graph or a network of a path problem is not axiomatically given as a starting point of the theory, but is derived from a novel concept called a "path space". This concept is shown to provide a coherent framework for the analysis of path problems, and hence the development of algebraic methods for solving them.

Introduction

This paper is concerned with the abstract study of path problems. The term "path problems" has, in recent years, been widely used to describe a certain class of mathematical problems which deals with paths in a graph or a network, many of which also have real world applications. These problems had previously been studied separately by a great number of authors in different branches of engineering, Operational Research and Computing Science. Numerous procedures for solving them had also been proposed separately. It was not until the use of algebraic methods for solving these problems became widespread that mathematicians began their search for a unified theory which would be useful for their solution.

Any theory which would adequately describe a path problem must incorporate the two different mathematical aspects of the problem: one algebraic, the other structural. The algebraic aspect of the problem is usually described by a semiring (Section 1.1) which is often called "path algebra", while the structural aspect by a graph (Section 1.2). The roles played by these two mathematical constructs in the abstract study of path problems will be seen in Section 2.3, where we present a general formulation of some of these problems. An important point which emerges from this formulation is that the path algebra of a path problem can be naturally derived from a more basic concept which we shall call a "path space". Section 3 therefore develops this concept. Then, in Section 4, we show how the

* As suggested by one of the referees.

interaction between the two mentioned aspects of a path problem can be fruitfully analysed with the help of a path space. The usefulness of this approach also extends to the development of solution methods akin to classical methods of linear algebra as pioneered by Carré [5].

In order to make this paper self-contained and to prevent possible confusion of terminology, a section on background mathematics is first presented.

1. Preliminaries

1.1. *Monoids and semirings*

As usual, by a *monoid* $(X, \circ)$ we mean a non-empty set X with an associative binary operation $\circ$ for which there is an element $e \in X$ acting as identity. A monoid $(X, \circ)$ is said to be

(i) *commutative* if $x \circ y = y \circ x$ for all $x, y \in X$,

(ii) *non-idempotent* if $x \circ x \neq x$ for all $x \in X \setminus \{e\}$, and

(iii) *locally finite* (cf. Eilenberg [9, p. 170]) if each $x \in X$ admits only a finite number of factorizations of the form $x = x_1 \circ x_2 \circ \cdots \circ x_n$, where each $x_i \in X/\{e\}$.

For example, the usual additive monoid $(N, +)$ of non-negative integers is commutative, non-idempotent and locally finite.

By a *totally ordered monoid* $(X, \leqslant, \circ)$ we mean a monoid $(X, \circ)$ in which the set X is also totally ordered by some relation $\leqslant$ such that from $x \leqslant y$ follow $x \circ u \leqslant y \circ u$ and $u \circ x \leqslant u \circ y$ for all $x, y, u \in X$. This monoid is said to have the *Archimedean property* if from $x > e$, $y > e$, we can find $n \in N \setminus \{0\}$ such that $x^n > y$, where $x > y$ denotes, as usual, $x \geqslant y$ (or $y \leqslant x$) but $x \neq y$.

A *semiring* $(X, +, \circ)$ is a set X on which two binary operations, $+$ and $\circ$, are defined such that

(i) $(X, +)$ forms a commutative monoid with $\mathscr{O}$ as identity,

(ii) $(X, \circ)$ forms a monoid with e as identity, and

(iii) $\circ$ is distributive over $+$, i.e. $x \circ (y + z) = (x \circ y) + (x \circ z)$ and $(x + y) \circ z = (x \circ z) + (y \circ z)$ for all $x, y, z \in X$.

(iv) $x \circ \mathscr{O} = \mathscr{O} = \mathscr{O} \circ x$ for all $x \in X$.

For convenience, we shall here refer to $+$ as *addition*, $\circ$ as *multiplication*, e as *unit* and $\mathscr{O}$ as *zero* of the corresponding semiring. A non-empty subset A of X is said to be a *subsemiring* of the semiring $(X, +, \circ)$ if $(A, +, \circ)$ is itself a semiring. A semiring $(X, +, \circ)$ is said to be *commutative* if $(X, \circ)$ is a commutative monoid, and *idempotent* whenever $e + e = e$ holds. These and other properties of a semiring can also be conveniently expressed via the relation $\prec$ which can be defined on X as follows.

$$x \prec y \quad \text{if and only if} \quad x + y = y \quad \text{for all } x, y \in X.$$

E.g. this relation is anti-symmetric because addition is commutative, transitive because addition is associative. But it is not genernally reflexive, except when X is idempotent. Hence we shall refer to $<$ as the *pseudo-ordering* of the corresponding semiring. The following properties of $<$ will be used subsequently.

P_1: $\mathcal{O} < x$ for all $x \in X$.

P_2: From $x < y$ follows $x \circ u < y \circ u$ and $u \circ x < u \circ y$ for all $u \in X$.

P_3: From $x < y$ and $u < w$ follows $x + u < y + w$ for all $x, y, u, w \in X$.

P_4: From $x < z$ and $y < z$ follow $x + y < z$ for all $x, y, z \in X$.

Here we also need the concept of a *complete semiring* (cf. Eilenberg [9, p. 125]) which we shall define as follows.

Let $(X, \circ)$ be a monoid and consider a *formal sum* $\sum_{i \in I} x_i$ as a well defined element of X which satisfies the following axioms.

(i) If $I = \{i\}$, then $\sum_{i \in I} x_i = x_i$

(ii) If $I = \bigcup_{j \in J} I_j$ is a disjoint partition of I, $\sum_{i \in I} x_i = \sum_{j \in J} (\sum_{i \in I_j} x_i)$

(iii) $z \circ (\sum_{i \in I} x_i) = \sum_{i \in I} (z \circ x_i)$ and $(\sum_{i \in I} x_i) \circ z = \sum_{i \in I} (x_i \circ z)$ for all $z \in X$.

As an example of a complete semiring, consider the set $N_\infty = N \cup \{\infty\}$ which is obtained from the usual semiring $(N, +, \circ)$ of non-negative integers by augmenting it with ∞ in such a way that for all $x \in X$,

$$x + \infty = \infty = \infty + x,$$

and

$$x\infty = \infty x = \begin{cases} 0, & \text{if } x = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

A formal sum $\sum_{i \in I} x_i$ in N_∞ is defined as the sum of all the non-zero x_i if $x_i = 0$ for all but a finite number of $i \in I$, and ∞ otherwise.

Note that a complete semiring is also a semiring if one defines $x_1 + x_2$ as $\sum_{i \in I} x_i$ with $I = \{1, 2\}$, and $\mathcal{O}$ as $\sum_{i \in I} x_i$ with $I = \emptyset$.

Now let $\mathcal{M}_n(X)$ denote the set of all $n \times n$ matrices over X, and for $A \in \mathcal{M}_n(X)$ let A_{ij} denote the (i, j)-entry of A. If X is a semiring (or a complete semiring), then $\mathcal{M}_n(X)$ can also be made into a semiring (or a complete semiring) by defining $A + B$ (or $\sum_{k \in I} A_k$) and $A \circ B$ by

$$(A + B)_{ij} = A_{ij} + B_{ij} \quad (\text{or } (\sum_{k \in I} A_k)_{ij} = \sum_{k \in I} (A_k)_{ij}),$$

and

$$(A \circ B)_{ij} = \sum_{k=1}^{n} A_{ik} \circ B_{kj} \quad \text{for all } i, j \in \{1, 2, \ldots, n\}.$$

Moreover, the unit I and zero θ of $\mathcal{M}_n(X)$ are respectively given by

$$I_{ij} = \begin{cases} e, & \text{if } \quad i = j \\ \mathcal{O}, & \text{otherwise} \end{cases}$$

and

$$\theta_{ij} = \mathbb{O} \quad \text{for all} \quad i, j \in \{1, 2, \ldots, n\}.$$

1.2. *Graphs and stable matrices over semirings*

A *graph* G is an ordered pair (W, V) such that W is a finite set of elements called *nodes* and V is a set of ordered pairs of nodes, called *arcs*. In G we define a path of *length* k to be a sequence of k consecutive arcs. Let

$$p = (x_i, x_1)(x_1, x_2) \cdots (x_{k-1}, x_j),$$

say. Then x_i will be called the *beginning* of the path p, x_j its *end*, and $x_1, x_2, \ldots, x_{k-1}$ its *intermediate nodes*. Moreover, p is said to be *closed* if $x_i = x_j$, *elementary* if from $x_r = x_s$ and $r < s$ follow $r = i$, $s = j$. It is convenient to define a *null path* for each node of G to be a closed path of length zero which begins and ends at that node.

Of particular interest to us here is the concept of a *graph over a monoid* $(X, \circ)$ *or a semiring* $(X, +, \circ)$ which can be defined as a triple (W, V, v) such that (W, V) is a graph and $v: V \to X$ is a function. In such a graph G, v can be extended to the set P of all paths of G by settting

$$v(p) = \begin{cases} e & \text{if } p \text{ is a null path} \\ v(x_i, x_1) \circ v(x_1, x_2) \circ \cdots \circ v(x_{k-1}, x_j) & \text{if } p \text{ is as above.} \end{cases} \tag{1}$$

Note that for economy of notation, we have here used the same letter v for the extended function and that (1) is well defined because $\circ$ is associative. Moreover $v(pq) = v(p) \circ v(q)$ for any two paths p, q such that the end of p is also the beginning of q. Note also that in the case of a graph over a semiring, $v(p)$ is defined in terms of the multiplication of the corresponding semiring.

Let G be a graph over a semiring $(X, +, \circ)$. We define the *arc-value matrix* of G to be the matrix A given by

$$A_{ij} = \begin{cases} v(x_i, x_j), & \text{if} \quad (x_i, x_j) \in V, \\ \mathbb{O}, & \text{otherwise.} \end{cases} \tag{2}$$

Note that A is unique if we choose a fixed numbering for the nodes of G, say $\{x_1, x_2, \ldots, x_n\}$. This set of nodes will be assumed chosen for all the graphs mentioned throughout. Now let $P_{ij}^{(k)}$ be the set of all paths of G which have length k, begin at x_i and end at x_j. Then the matrix A^k which is kth power of A is given by

$$(A^k)_{ij} = \sum_{p \in P_{ij}^{(k)}} v(p). \tag{3}$$

A neat way of proving (3) is to introduce the function $\sigma: 2^{(P)} \to X$ where $2^{(P)}$ denotes the set of all finite subsets of P, as follows.

$$\sigma(Q) = \sum_{p \in Q} v(p) \quad \text{where} \quad \sum_{x \in \emptyset} x = \mathbb{O} \quad \text{by definition.} \tag{4}$$

This function has the following two properties which can be easily verified from its definition.

σ_1: $\sigma(\bigcup_{i=1}^k Q_i)=\sum_{i=1}^k \sigma(Q_i)$ whenever $Q_i \cap Q_j=\emptyset$ for $i\neq j$.

σ_2: $\sigma(Q_{ir_1}Q_{r_1r_2}\cdots Q_{r_kj})=\sigma(Q_{ir_1})\circ\sigma(Q_{r_1r_2})\circ\cdots\circ\sigma(Q_{r_kj})$ where each Q_{rs} denotes a finite subset of paths from x_r to x_s, and $Q_{rs}Q_{st}=\{pq \mid p\in Q_{rs}, q\in Q_{st}\}$.
The proof of (3) now follows easily from σ_1 and σ_2 by noting that

$$P_{ij}^{(k)}=\bigcup_{r_1,r_2,\ldots,r_{k-1}}(P_{ir_1}^{(1)}P_{r_1r_2}^{(1)}\cdots P_{r_{k-1}j}^{(1)}).$$

A consequence of (3) which can be obtained by applying σ_1 is that

$$(A^{[r]})_{ij}=\sigma(P_{ij}^{[r]}) \quad \text{where} \quad A^{[r]}=\sum_{k=0}^{r} A^k \quad \text{and} \quad P_{ij}^{[r]}=\bigcup_{k=0}^{r} P_{ij}^{(k)}. \tag{5}$$

Let A be an $n\times n$ matrix over a semiring X. Then we say that A is *stable* if there is $r\in N=\{0,1,2,\ldots\}$ such that $A^{[r+1]}=A^{[r]}$. The *stability index* of A is the least such integer r if A is stable and ∞ otherwise. Note that in consequence, $A^{[s]}=A^{[r]}$ for all $s\geq r$ if r is the stability index of A. A number of sufficient conditions for a given matrix A to be stable can be conveniently formulated via the graph $G(A)=(W,V,v)$ of the matrix A where

(i) $W=\{A_1,A_2,\ldots,A_n\}$, each A_i denotes ith column of A,
(ii) $V=\{(A_i,A_j) \mid A_{ij}\neq \mathcal{O}\}$, and
(iii) $v(A_i,A_j)=A_{ij}$ for every $i,j\in\{1,2,\ldots,n\}$.

Thus a given matrix A is said to be *absorptive* if $G(A)$ is absorptive, i.e. $v(\omega)<e$ for all elementary non-null closed paths ω in $G(A)$. The definition of absorptive graph has been generalized separately by Gondran [12] and Roy [20] respectively as follows.

(a) A graph G is said to be *q-regular* if there is $q\in N\setminus\{0\}$ such that $v(\omega)^q< e+v(\omega)+v(\omega)^2+\cdots+v(\omega)^{q-1}$ for all elementary non-null closed paths ω in G.

(b) A graph G is said to be *q-absorptive* if

$$v(\omega_1)\circ v(\omega_2)\circ\cdots\circ v(\omega_q) < e+v(\omega_1)+v(\omega_1)\circ v(\omega_2)+\cdots+v(\omega_1)\circ\cdots\circ v(\omega_{q-1})$$

for every q-tuple $(\omega_1,\omega_2,\ldots,\omega_q)$ of elementary non-null closed paths.

Accordingly, we shall say that a matrix A is *q-regular* or *q-absorptive* whenever $G(A)$ is q-regular or q-absorptive.

Stability Theorem. *Let $(X,+,\circ)$ be a semiring. Then the stability index of $A\in\mathcal{M}_n(X)$ is*

(i) *at most $n-1$ if A is absorptive,*

(ii) *at most $nt(q-1)+n-1$ where t denotes the total number of elementary non-null closed paths of $G(A)$ if X is commutative and A is q-regular, and*

(iii) *at most $nq-1$ if X is commutative and A is q-absorptive.*

Our proof of this theorem will be given in the appendix.

2. Path problems in networks. The concept of path spaces

2.1. *The semiring* N_∞^X

Let N_∞ be as before. A *multiset*[1] A with elements from a given set X is a function $A: X \to N_\infty$. Each image $A(x)$ is just the number of occurrences of x in A. We shall call a multiset A *empty*, written $A=\emptyset$, if $A(x)=0$ for all $x\in X$, *non-singular* if $A(x)\neq\infty$ for all $x\in X$, and *finite* if $\sum_{x\in X} A(x)\neq\infty$. For a given multiset A, we can associate a unique set $d(A)=\{x \mid A(x)\neq 0\}$, called the *support* of A. Note that $d(A)$ consists of only distinct elements of A.

By virtue of the fact that multisets are mere generalization of sets, it is natural to make use of set-theoretic notations whenever confusion is not likely. Thus for instance, we shall write $x\in A$ to indicate $A(x)\neq 0$ and sometimes write A in extenso such as $\{1,1,2\}$ or $\{1,1,\ldots,1,2,3\}$ if $d(A)$ is a countable set. However, we find it convenient to reserve the notation $\{x \mid P(x)\}$ exclusively for sets.

Throughout this paper, N_∞^X denotes the set of all multisets with elements from a given monoid $(X,\circ)$, which can be made into a complete semiring as follows[2].

For any $A, B\in N_\infty^X$, we define a *multiproduct* $A\circ B$ by

$$(A\circ B)(x)=\sum_{x=y\circ z} A(y)B(z) \quad \text{for all } x\in X$$

where juxtaposition of $A(y)$ with $B(z)$ denotes the extended multiplication on N_∞. The structure $(N_\infty^X,\circ)$ can be seen to form a monoid with $\{e\}$ as identity. Now a formal sum $\biguplus_{i\in I} A_i$ can be defined in N_∞^X by

$$\Big(\biguplus_{i\in I} A_i\Big)(x)=\sum_{i\in I} A_i(x) \quad \text{for all } x\in X.$$

Since the right-hand side is a formal sum in N_∞, $\biguplus_{i\in I} A_i$ is easily seen to be a formal sum in N_∞^X as claimed. Note that for $I=N$ and $I=\{1,2\}$, we shall respectively write $\biguplus_{i=0}^\infty A_i$ and $A_1\uplus A_2$ in stead of $\biguplus_{i\in I} A_i$. We shall call $A\uplus B$ the *multisum* of A and B. Moreover, the structure $(N_\infty^X, \uplus, \circ)$ forms a semiring with unit $\{e\}$ and zero $\emptyset$.

The set N_∞^X can also be seen to form a complete lattice (see e.g. Birkhoff [3] for definition) with respect to the partial ordering $\subseteq$ which can be defined on N_∞^X as follows.

$$A\subseteq B \quad \text{if and only if} \quad A(x)\leqslant B(x) \quad \text{for all } x\in X.$$

where $\leqslant$ denotes the extension of the usual ordering on N to N_∞ by defining $x\leqslant\infty$ for all $x\in N$.

[1] A kind of K-subsets in Eilenberg [9, p. 126] where $K=N_\infty^X$.

[2] Referee's note: N_∞^X is in fact an instance of a semigroup ring (Rota).

Let $\mathscr{V}$ be a subset of N_∞^X. We say that $\mathscr{V}$ is *hereditary* if from $A \in \mathscr{V}$ and $B \subseteq A$ follow $B \in \mathscr{V}$. Here we are interested in the case where $\mathscr{V}$ is also a subsemiring of N_∞^X such that $\{x\} \in \mathscr{V}$ for all $x \in X$. Such a $\mathscr{V}$ will be called a *hereditary semiring* of N_∞^X. Obviously, N_∞^X is one such candidate. Others include the following.

(i) $\mathscr{F} = \{A \in N_\infty^X \mid \sum_{x \in X} A(x) \neq \infty\}$.

(ii) $\mathscr{Q} = \{A \in N_\infty^X \mid d(A)$ is a countable set$\}$.

(iii) $\mathscr{N} = \{A \in N_\infty^X \mid A(x) \neq \infty$ for all $x \in X\}$, provided that $(X, \circ)$ is a locally finite monoid.

(iv) $\mathscr{W} = \{A \in N_\infty^X \mid d(A)$ is well ordered$\}$, provided that $(X, \leqslant, \circ)$ is a totally ordered monoid.

(v) $\mathscr{W}' = \{A \in N_\infty^X \mid d(A)$ is dually well ordered$\}$, provided that $(X, \leqslant, \circ)$ is a totally ordered monoid.

Now for $A \in N_\infty^X$, we define A^k inductively by $A^0 = \{e\}$, $A^{k+1} = A \circ A^k$ for all $k \in N$. The *star* of A, written A^*, will be defined as $\biguplus_{k=0}^{\infty} A^k$; whereas $A^+ = \biguplus_{k=1}^{\infty} A^k$ will be called the *weak star* of A. Note that $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$ are generally valid but *not* $A^* = (A^*)^*$. A similar observation also applies to A^+. Finally, the star and weak star of $A \in \mathscr{M}_n(N_\infty^X)$ can also be similarly defined.

2.2. *Networks over monoids*

Now using multisets of the previous section, we can generalize the above notion of a graph over a monoid as follows.

By a *network* $\mathscr{A}$ *over a monoid* $(X, \circ)$ we mean an ordered pair (W, U) such that W is a finite set of elements called *nodes* as before and U is a finite multiset with elements from the cartesian product $W \times X \times W$. Each element of U is thus an ordered triple (x, a, x') which we shall call an *arc beginning* at node x, *ending* at node x', and *carrying* the *label* a. All the previous definitions concerning paths of a graph apply to a network as well. Moreover, let p be a path of $\mathscr{A}$ which has length k, say $p = (x_i, a_1, x_1)(x_1, a_2, x_2) \cdots (x_{k-1}, a_k, x_j)$. Then p is said to carry the *label* $a_1 \circ a_2 \circ \cdots \circ a_k$ which is well defined because $\circ$ is associative.

For each network $\mathscr{A} = (W, U)$ over X, we can associate a graph $G(\mathscr{A}) = (W, V, v)$ over N_∞^X by letting $V = \{(x, x') \mid (x, a, x') \in U$ for some $a \in X\}$, and $v(x, x')$ be the multiset of all the labels carried by arcs of U which begin at node x and end at node x'. $G(\mathscr{A})$ will be called the *graph of the network* $\mathscr{A}$.

Here we shall find it convenient to introduce a function $v : 2^P \to N_\infty^X$, where P denotes the set of all paths of $G(\mathscr{A})$, as follows.

$$v(Q) = \biguplus_{p \in Q} v(p) \quad \text{for all } Q \in 2^P. \tag{6}$$

Note that strictly speaking, a different notation should be used in place of v on the left-hand side of (6) above. However, there is no confusion because the distinction can be made from its argument. Let us also note that this function is in fact an extension of the function $\sigma: 2^{(P)} \to N_\infty^X$ as defined by (4) for $G(\mathscr{A})$. Moreover, we have

$$\text{for } Q, Q' \in 2^P,\ v(Q) \subseteq v(Q') \quad \text{whenever } Q \subseteq Q', \tag{7}$$

$$\text{for pairwise disjoint subsets } Q_i \in 2^P\ (i \in I), \quad v\Big(\bigcup_{i\in I} Q_i\Big) = \biguplus_{i \in I} v(Q_i). \tag{8}$$

Now let M be the arc-value marix of $G(\mathscr{A})$, also called the *label matrix* of the network $\mathscr{A}$. Then in terms of the above function we can rewrite (3) and (5) respectively as follows.

$$(M^k)_{ij} = v(P_{ij}^{(k)}) \quad \text{and} \quad (M^{[r]})_{ij} = v(P_{ij}^{[r]}) \quad \text{for all } i, j. \tag{9}$$

Moreover, using (8), we obtain

$$(M^*)_{ij} = v(P_{ij}) \quad \text{and} \quad (M^+)_{ij} = v(P_{ij} \setminus P_{ij}^{(0)}) \quad \text{for all } i, j; \tag{10}$$

where $P_{ij} = \bigcup_{k=0}^{\infty} P_{ij}^{(k)}$ and M^*, M^+ are respectively the star and weak star of M (see Section 2.1).

The following generalization of a result originally due to McNaughton and Yamada [13] will also be useful later.

Let $Q_{ij}^{\{k\}} \subseteq P_{ij} \setminus P_{ij}^{(0)}$ consist of paths which do not use any node x_r of $G(\mathscr{A})$ such that $r > k$ as an intermediate node. Then for all i, j we have

$$\begin{aligned}
&\text{(i) } v(Q_{ij}^{\{0\}}) = v(P_{ij}^{(1)}),\\
&\text{(ii) } v(Q_{ij}^{\{n\}}) = v(P_{ij} \setminus P_{ij}^{(0)}), \text{ and}\\
&\text{(iii) } v(Q_{ij}^{\{k\}}) = v(Q_{ij}^{\{k-1\}}) \uplus (v(Q_{ik}^{\{k-1\}}) \circ v(Q_{kk}^{\{k-1\}})^* \circ v(Q_{kj}^{\{k-1\}}))\\
&\quad\ \text{for all } k \in \{1, 2, \ldots, n\}.
\end{aligned} \tag{11}$$

2.3 *A general formulation of some path problems*

Problem 1 (Shortest Path). Let G be a graph over $(N, +)$. For any two nodes x_i, x_j of G, determine $\min\{v(p) \mid p \in P_{ij}\}$.

Problem 2 (Maximal Capacity Path). Let G be a graph over $(N_\infty, \wedge)$, where $a \wedge b = \min\{a, b\}$ for all $a, b \in N_\infty$. For any two nodes x_i and x_j of G, determine $\max\{v(p) \mid p \in P_{ij}\}$.

Problem 3 (Most Reliable Path). Let G be a graph over the closed interval $[0, 1]$ of real numbers which forms a monoid with respect to ordinary multiplication. For any two nodes x_i, x_j of G, determine $\max\{v(p) \mid p \in P_{ij}\}$.

The above three problems were first seen by Moisil [15] to be equivalent to the determination of $(A^{n-1})_{ij}$ where A is the arc-value matrix of the corrresponding graph G over a suitable semiring such that

$$v(x_i, x_i) = e \quad \text{for each node } x_i \text{ of } G. \tag{12}$$

It became apparent that condition (12) can be omitted if one identifies the above three problems with the determination of $(A^{[n-1]})_{ij}$ rather than $(A^{n-1})_{ij}$. This observation was evident in the subsequent generalization of the above work of Moisil by a number of authors which include Pair [17], Benzaken [2], Peteanu [18, 19], Derniame and Pair [8], Carré [5, 6], Shier [21], Brucker [4], Gondran [12], and Roy [20]. In these references, a number of interesting path problems are formulated and solved as the determination of $(A^{[n_0]})_{ij}$ where A is the arc-value matrix of the corresponding graph over a suitable semiring and n_0 is the stability index of A. The following is one such example.

Problem 4 (k Shortest Paths). Let G be a graph over $(N, +)$ and $k \in N \setminus \{0\}$. For any two nodes x_i, x_j of G, determine $k\text{-min}\{v(p) \mid p \in P_{ij}\}$ which is just the set consisting of the first t smallest elements of $\{v(p) \mid p \in P_{ij}\}$ where t is the largest integer such that $t \leqslant k$.

Note that when $k = 1$, Problem 4 coincides with Problem 1. We chose to illustrate this example because a suitable semiring for solving this problem is not at all obvious. In fact, several analogous proposals were made by a number of authors which include Pair [17], Giffler [11], Derniame and Pair [8], Minieka and Shier [14], Gondran [12], Roy [20], and Shier [22]. The following semiring $(\mathscr{V}_{k\text{-min}}, \oplus, \odot)$ was inspired by their work.

Let $\mathscr{V} = 2^N$ and $n_A = \min\{k, |A|\}$ where $|A|$ denotes the number of elements of A. Define $k\text{-min}(A)$ to be the set of the first n_A elements of A and let $\mathscr{V}_{k\text{-min}} = \{A \in 2^N \mid A = k\text{-min}(A)\}$. Then one can define two binary operations $\oplus$ and $\odot$ on $\mathscr{V}_{k\text{-min}}$ as follows.

$A \oplus B = k\text{-min}(A \cup B)$, and $A \odot B = k\text{-min}(AB)$, where $AB = \{a + b \mid a \in A, b \in B\}$. The structure $(\mathscr{V}_{k\text{-min}}, \oplus, \odot)$ can then be seen to form an idempotent semiring. Now since $k\text{-min}\{x\} = \{x\}$ holds for all $x \in N$, one can identify the elements of N with the singleton subsets of $\mathscr{V}_{k\text{-min}}$. This identification then allows us to consider the graph G of Problem 4 as a graph over the semiring $(\mathscr{V}_{k\text{-min}}, \oplus, \odot)$ rather than the monoid $(N, +)$. Problem 4 is then equivalent to the determination of $(A^{[nk-1]})_{ij}$, since it can be shown (cf. Shier [22]) that

$$k\text{-min}\{v(p) \mid p \in P_{ij}^{[s]}\} = k\text{-min}\{v(p) \mid p \in P_{ij}^{[nk-1]}\} \quad \text{for all } s \geqslant nk - 1.$$

The above construction of the semiring $(\mathscr{V}_{k\text{-min}}, \oplus, \odot)$ from the function k-min suggests a general procedure for constructing a semiring $(\mathscr{V}_r, \oplus, \odot)$ from a certain function $r: \mathscr{V} \to \mathscr{V}$ as follows.

(i) $\mathscr{V}_r = \{A \in \mathscr{V} \mid A = r(A)\}$,

(ii) $A \oplus B = r(A \cup B)$, and

(iii) $A \odot B = r(AB)$, where $AB = \{a \circ b \mid a \in A, b \in B\}$.

The properties that the function r must have are obviously those which will make $(\mathscr{V}_r, \oplus, \odot)$ into a semiring. For this purpose, it can be seen from the definitions of $\oplus$ and $\odot$ above that $\mathscr{V}$ must be closed with respect to set-theoretic union and complex product operations. For such $\mathscr{V}$, the following properties of r can be seen to make $(\mathscr{V}_r, \oplus, \odot)$ into a semiring as required.

(i) $r(\emptyset)=\emptyset$,

(ii) $r(A \cup B) = r(r(A) \cup B)$ for all $A, B \in \mathscr{V}$, and (13)

(iii) $r(AB) = r(r(A)B) = r(Ar(B))$ for all $A, B \in \mathscr{V}$.

In fact, suitable semirings for solving Problems 1, 2 and 3 can also be constructed in this way. Thus in Problem 1, we can set $\mathscr{V} = 2^N$ and $r = \min$. Then the semiring $(\mathscr{V}_{\min}, \oplus, \odot)$ so obtained can also be seen to be isomorphic to the semiring $(N_\infty, \wedge, +)$ where $\wedge = \min$. Also in Problem 2, we can set $\mathscr{V} = 2^{N_\infty}$ and $r = \max$. However, in Problem 3, one must exercise caution. For if we set $\mathscr{V} = 2^{[0,1]}$ and $r = \max$, then it is possible that for some $A \in \mathscr{V}$, $r(A)$ is not defined. For example, let $A = \{1-\frac{1}{2}, 1-\frac{1}{3}, \ldots\}$. Then $\max(A)$ does not exist. Thus to avoid this difficulty, we define $\mathscr{V}$ to be the set of all dually well ordered subsets of $[0, 1]$.

Now observe that in all the above problems, we are required to determine $r(\{v(p) \mid p \in P_{ij}^{[n_0]}\})$ for some $n_0 \in N$. So we must make sure that $\{v(p) \mid p \in P_{ij}^{[n_0]}\} \in \mathscr{V}$. Since this is a finite set, we only need to stipulate that $\mathscr{V}$ contains all the finite subsets, which is so if

$$\mathscr{V} \text{ contains all the singleton subsets of } X, \tag{14}$$

because $\mathscr{V}$ was assumed to be closed with respect to set-theoretic union.

In fact, one can be more general and insist that $\{v(p) \mid p \in P_{ij}\} \in \mathscr{V}$ because all the above problems were originally posed as the determination of $r(\{v(p) \mid p \in P_{ij}\})$ where $r = \min$ or max or k-min. The following example exhibits some of the difficulty which may arise in this case.

Problem 5 (Longest Path). Let G be a graph over $(N, +)$. For any two nodes x_i and x_j of G, determine $\max\{v(p) \mid p \in P_{ij}\}$ if it exists.

Note that $\max\{v(p) \mid p \in P_{ij}\}$ may not exist if G contains a closed path c such that $v(c) > 0$. Thus we see that it is the nature of the graph under consideration which determines whether or not $\{v(p) \mid p \in P_{ij}\} \in \mathscr{V}$. Any graph for which this relation does not hold is in some sense not compatible with $\mathscr{V}$. It turns out that this question of compatibility can be fruitfully analysed if $\mathscr{V}$ has the following property.

$$\text{If} \quad A \in \mathscr{V} \quad \text{and} \quad B \subseteq A, \quad \text{then} \quad B \in \mathscr{V} \quad \text{also.} \tag{15}$$

The detail of this analysis will be given later in the framework of multisets. It suffices to note here that the appropriate $\mathscr{V}$-sets of all the above problems have this property. Our reason for extending the above consideration to multisets is to be able to include the problem of computing all path values in a network as studied by Giffler [10, 11] and Wongseelashote [23] under the same umbrella of path problems. Thus in summary, we shall define a path problem in this paper as follows.

Definition 1. Let $(X, \circ, \mathscr{V}, r)$ be given as in Section 3 below and $\mathscr{A}$ be a given network over $(X, \circ)$ such that $v(P_{ij}) \in \mathscr{V}$ for all i, j. Then by a *path problem* we mean the determination of $r(v(P_{ij}))$ for some (i, j). If we now define $r(A)$ for any $A \in M_n(\mathscr{V})$ by

$$(r(A))_{ij} = r(A_{ij}) \quad \text{for every } i, j,$$

it then follows from (10) that a path problem as specified in Definition 1 is equivalent to the determination of $r((M^*)_{ij})$ for some i, j. But $r(M^*)$ satisfies the matricial equations

$$Y = (r(M) \odot Y) \oplus r(I) \quad \text{or} \quad Y = (Y \odot r(M)) \oplus r(I) \tag{16}$$

because $M^* = (M \circ M^*) \uplus I$ or $M^* = (M^* \circ M) \uplus I$ always.

This suggests that one can determine $r((M^*)_{ij})$ by solving the corresponding system of n equations contained in (16) above. Thus a path problem as defined here can be solved if one knows how to solve a system of n equations over the semiring $(\mathscr{V}_r, \oplus, \odot)$ of the form

$$y = (r(M) \odot y) \oplus b \quad \text{or} \quad y = (y \odot r(M)) \oplus b.$$

Now these systems can be solved quite readily by classical methods of linear algebra such as Gauss or Jordan elimination if the semiring $(\mathscr{V}_r, \oplus, \odot)$ also forms a field or is embeddable in a field (e.g. an integral domain). However, there are relatively few such examples of path problems. It is therefore of significance to have similar methods developed for the more general situation. The interested readers are referred to the pioneer work of Carré [5] and more recently Backhouse and Carré [1], Carré [6], Carré and Wongseelashote [7].

3. Path spaces and their associated semirings

Definition 2. An ordered quadruple $(X, \circ, \mathscr{V}, r)$ is called a *path space* if

(i) $(X, \circ)$ is a monoid,

(ii) $\mathscr{V}$ is a hereditary semiring of N_∞^X (see Section 2.1 above), and

(iii) r is a self-map of $\mathscr{V}$ satisfying

(a) $r(\emptyset) = \emptyset$,

(b) $r(A \uplus B) = r(r(A) \uplus B)$, and

(c) $r(A \circ B) = r(r(A) \circ B) = r(A \circ r(B))$ for all $A, B \in \mathscr{V}$.

For convenience, we shall refer to r as a *reduction*[3] of $\mathscr{V}$ for a given monoid $(X, \circ)$. This reduction can be seen to be an extension of the function r which we defined in Section 2.3. Indeed, let $\mathscr{V} \subseteq 2^X$ be closed with respect to set-theoretic union and complex product operations and satisfies (14) and (15) above, and let r'

[3] This name was inspired by the reduction function studied in Wongseelashote [23]. However, its abstraction was not conceived without the influence of the concept of an extraction function introduced by Roy [20].

be a self-map of $\mathscr{V}' = \{A \in N_\infty^X \mid d(A) \in \mathscr{V}\}$ which is defined by $r'(A) = r(d(A))$ for all $A \in \mathscr{V}'$, where r itself is a self-map of $\mathscr{V}$ satisfying (13) above. Then $\mathscr{V} \subseteq \mathscr{V}'$ and $\mathscr{V}'$ can be seen to form a hereditary semiring of N_∞^X; also $r'(A) = r(A)$ for all $A \in \mathscr{V}$ and r' can be verified to be a reduction of $\mathscr{V}'$. Note that r' is in fact a composition of $r: \mathscr{V} \to \mathscr{V}$ and $d: \mathscr{V}' \to \mathscr{V}$ which is well defined because for each $A \in \mathscr{V}'$, $d(A) \in \mathscr{V}$ by our definition of $\mathscr{V}'$. Thus we see that for each of the path problems given previously, we can obtain a corresponding path space $(X, \circ, \mathscr{V}', r')$ as above. Moreover, $r'(A \uplus A) = r'(A)$ for all $A \in \mathscr{V}'$ can be seen to hold also. Such an r' will be said to be idempotent in view of the next theorem. But first, let us give some other examples of path spaces.

Example 1. $(X, \circ, N_\infty^X, d)$, where $d(A)$ is the support of A.

Example 2.[4] Let $(X, \leqslant, \circ)$ be a totally ordered monoid and $\mathscr{W} = \{A \in N_\infty^X \mid d(A)$ is well ordered$\}$. For a given $A \in \mathscr{W}$, let $a_x = \sum_{y<x} A(y)$ for all $x \in X$ and let k'-min(A) for some given $k \in N \setminus \{0\}$ be defined by

$$(k'\text{-min}(A))(x) = \begin{cases} A(x), & \text{if} \quad a_x + A(x) \leqslant k, \\ k - a_x, & \text{if} \quad a_x < k < a_x + A(x), \\ 0, & \text{if} \quad k \leqslant a_x. \end{cases}$$

It can be verified that $(X, \circ, \mathscr{W}, k'\text{-min})$ forms a path space. Note that $1'$-min$(A) = \min d(A) = 1$-min(A), but k'-min$(A) \neq k$-min(A) in general for $k > 1$, where k-min is defined on $\mathscr{W}$ as we did previously for problem 4.

Example 3. Let $\hat{N} = \{\hat{n} \mid n \in N\}$ be such that $N \cap \hat{N} = \emptyset$. Let $X = N \cup \hat{N}$ and define a binary operation $\circ$ on X by the following rules.

(i) $m \circ n = \hat{m} \circ \hat{n} = m + n$, and

(ii) $\hat{m} \circ n = m \circ \hat{n}$ for all $m, n \in N$.

It can be verified (cf. Wongseelashote [23]) that $(X, \circ)$ forms a commutative monoid with $\mathbb{0}$ as identity. Moreover, this monoid is also locally finite because $(N, +)$ is locally finite. Thus let $\mathscr{N} = \{A \in N_\infty^X \mid A(x) \neq \infty\}(x \in X)$ and for each $A \in \mathscr{N}$, define $r(A)$ by the following rules.

(i) $r(A)(n) = A(n) - \min\{A(n), A(\hat{n})\}$, and

(ii) $r(A)(\hat{n}) = A(\hat{n}) - \min\{A(n), A(\hat{n})\}$ for all $n \in N$.

It can be verified that $(X, \circ, \mathscr{N}, r)$ forms a path space.

Example 4. Let Σ be a finite set of letters, also called an *alphabet*. By a *word* of Σ we mean a finite sequence of letters written one after another in a definite order. A *repetition-free word* is one in which all its letters are pairwise distinct. The word without any letters is called the *empty word*, denoted by λ. Note that λ is repetition-free by definition. The operation $\circ$ which combines two words into one

[4] Its present formulation was suggested by the referee.

is known as *concatenation*, usually denoted by juxtaposition. It is well known that the set Σ^* of all words over Σ (including λ) forms a monoid with respect to concatenation, also known as the *free monoid generated by* Σ. Now for each $A \in N_\infty^{\Sigma^*}$, let $\operatorname{sim}(A)$ be the set of all repetition-free words of A. Then it can be seen that $(\Sigma^*, \circ, N_\infty^{\Sigma^*}, \operatorname{sim})$ forms a path space. This path space is useful for enumerating all the paths in a graph which do not use the same arc more than once, also known as *simple paths*.

Example 5. Let Σ^* be defined as in Example 4 above. A word α is said to be an *abbreviation* of another word β if α can be obtained from β by removing at least one (and possibly all) of the letters of β (Note that every word with at least one letter has the abbreviation λ). For instance, the word "mary" is an abbreviation of the word "elementary". Now for each $A \in N_\infty^{\Sigma^*}$, let $\operatorname{elem}(A)$ be the set of all those words of A which are not abbreviations of any other words of A. Then it can be seen that $(\Sigma^*, \circ, N_\infty^{\Sigma^*}, \operatorname{elem})$ forms a path space. This path space is useful for enumerating all the elementary paths of a graph (see e.g. Murchland [16], Benzaken [2], Backhouse and Carré [1], and Carré [6]).

Example 6. Let $(X, +, \circ)$ be a semiring and $\mathcal{F} = \{A \in N_\infty^X \mid \sum_{x \in X} A(x) \neq \infty\}$. For each $A \in \mathcal{F}$, let $a = \sum_{x \in X} A(x)x$ (sum in X) and define $s(A)$ by

$$s(A) = \begin{cases} \{a\}, & \text{if } a \neq \mathbb{O}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then it can be seen that $(X, \circ, \mathcal{F}, s)$ forms a path space.

This last example shows that each semiring gives rise to at least one path space. We emphasize here that there are more than one way of constructing a path space from a given semiring. For instance, if X is also a complete semiring, then we can choose N_∞^X as the domain of s above. The following theorem shows how one can also obtain a semiring from a path space.

Theorem 1. *Let $(X, \circ, \mathcal{V}, r)$ be a given path space and $\mathcal{V}_r = \{A \in \mathcal{V} \mid A = r(A)\}$. Define $A \oplus B = r(A \uplus B)$ and $A \odot B = r(A \circ B)$ for all $A, B \in \mathcal{V}$. Then the structure $(\mathcal{V}_r, \oplus, \odot)$ forms a semiring with unit $r(\{e\})$ and zero $\emptyset$. Moreover, it is idempotent whenever $r(\{e, e\}) = r(\{e\})$, and commutative if $(X, \circ)$ is a commutative monoid.*

Proof. By direct verification of axioms using definition of a reduction.

For convenience, we shall refer to the semiring $(\mathcal{V}_r, \oplus, \odot)$ as the *reduct*[5] of the path space $(X, \circ, \mathcal{V}, r)$. Let us note in passing that the reduct of the path space $(X, \circ, \mathcal{F}, s)$ of Example 6 above is in fact isomorphic to the semiring $(X, +, \circ)$ which gives rise to this path space, the isomorphism being given by $f(x) = \{x\}$ if $x \neq \mathbb{O}$, otherwise $f(x) = \emptyset$.

[5] Previously called a path algebra by the author; the present name was suggested by the referee.

The rest of this section will be devoted to the study of a reduction r such that $r(\{e, e\}) = r(\{e\})$ holds. Such an r is said to be *idempotent* because of Theorem 1 above. Evidently, r is idempotent if and only if $r(A \uplus A) = r(A)$ for all $A \in \mathcal{V}$. The reductions in Examples 1, 4 and 5 are idempotent. The reduction k'-min in Example 2 is idempotent if $k = 1$ and the reduction s in Example 6 is idempotent if $(X, +, \circ)$ is an idempotent semiring.

Theorem 2. *A reduction r of $\mathcal{V}$ is idempotent if and only if $r(A \uplus B) = r(A \cup B)$ for all $A, B \in \mathcal{V}$, where $(A \cup B)(x) = \max_{x \in X}\{A(x), B(x)\}$.*

Proof. Sufficiency is trivial. For necessity, let $Y = A \cap B$ and $Z = A \cup B$, where $(A \cap B)(x) = \min_{x \in X}\{A(x), B(x)\}$. If $Y = Z$, then $A = B$, and hence

$$r(A \uplus B) = r(A \uplus A) = r(A) = r(A \uplus A) = r(A \cap B)$$

as required. So let $Y = Z$ and define $Z \mathbin{\overset{-}{\cup}} Y$ by $(Z \mathbin{\overset{-}{\cup}} Y)(x) = Z(x) - Y(x)$ for all $x \in X$ if $Z(x) \neq \infty$ and ∞ otherwise. Consequently, $Z = Y \uplus (Z \mathbin{\overset{-}{\cup}} Y)$ always, and hence

$$\begin{aligned} r(A \uplus B) &= r(Y \uplus Z) = r(Y \uplus Y \uplus (Z \mathbin{\overset{-}{\cup}} Y)) = r(r(Y \uplus Y) \uplus (Z \mathbin{\overset{-}{\cup}} Y)) \\ &= r(r(Y) \uplus (Z \mathbin{\overset{-}{\cup}} Y)) = r(Y \uplus (Z \mathbin{\overset{-}{\cup}} Y)) = r(Z) = r(A \cup B). \end{aligned}$$

Corollary. *A reduction r of $\mathcal{V}$ is idempotent if and only if for all $A, B \in \mathcal{V}$ such that $A \subseteq B$, we have $r(A) < r(B)$, where $<$ denotes the pseudo-ordering of $(\mathcal{V}_r, \oplus, \odot)$.*

Let us make a special note here concerning the implication of the above theorems from the view point of Lattice Theory. By Theorem 1, we know that $(\mathcal{V}_r, \oplus, \odot)$ forms an idempotent semiring if r is idempotent, and hence its pseudo-ordering is actually a partial ordering, i.e. $(\mathcal{V}_r, <)$ forms a join-semilattice. Therefore, the addition $\oplus$ of the reduct coincides with the usual join operation. Thus to emphasize that r is idempotent, it is convenient to use the usual join notation $\vee$ in place of $\oplus$ from now on. Also Theorem 2 essentially says that r is a join-morphism from the semilattice $(\mathcal{V}, \subseteq)$ to the semilattice $(\mathcal{V}_r, <)$ if r is idempotent. This is because

$$r(A \cup B) = r(A \uplus B) = r(r(A) \uplus r(B)) = r(A) \vee r(B) \quad \text{for all } A, B \in \mathcal{V}.$$

In fact, if r has an additional property, it can be seen to be a complete join-morphism as in the following theorem. (The readers who are not familiar with the above terminology of Lattice Theory can refer to Birkhoff [3], for instance).

Theorem 3. *Let r be an idempotent reduction of $\mathcal{V}$. Then the following conditions are equivalent.*

(i) $r(\bigcup_{i\in I} A_i) = \bigvee_{i\in I} r(A_i)$ *whenever* $\bigcup_{i\in I} A_i \in \mathcal{V}$.

(ii) $r(\biguplus_{i\in I} A_i) = \bigvee_{i\in I} r(A_i)$ *whenever* $\biguplus_{i\in I} A_i \in \mathcal{V}$.

(iii) Let $A = \biguplus_{i\in I} A_i$ *where* $A_i \in \mathcal{F}$ *for all* $i \in I$. *If* $r(A_i) < r(B)$ *for all* $i \in I$, *then we have* $r(A) < r(B)$.

Proof. (ii) follows from (i) because

$$r\Big(\biguplus_{i\in I} A_i\Big) = r\Big(\biguplus_{J\in 2^{(I)}} \Big(\bigcup_{j\in J} A_j\Big)\Big) = \bigvee_{J\in 2^{(I)}} \Big(\bigcup_{j\in J} A_j\Big) = \bigvee_{J\in 2^{(I)}} \Big(\bigvee_{j\in J} r(A_j)\Big) = \bigvee_{i\in I} r(A_i)$$

(iii) follows from (ii) because

$$r(A \uplus B) = r(\biguplus_{i\in I} A_i \cup B) = \bigvee_{i\in I} r(A_i) \vee r(B) = r(B)$$

since $r(A_i) < r(B)$ $(i \in I)$.

That (i) follows from (iii) can be seen as follows.

Let $Z = \bigcup_{i\in I} A_i$. Then $A_i \subseteq Z$ for all $i \in I$, and hence by the corollary above, $r(A_i) < r(Z)$ for all $i \in I$. We now show that $r(Z)$ is in fact the least upper bound with respect to $<$. Let $r(A_i) < Y$ for all $i \in I$, and let $Z = \biguplus_{j\in J} \{x_j\}$. Then for each $j \in J$, $x_j \in A_i$ for some $i \in I$, and hence $r(\{x_j\}) < r(A_i)$ by the corollary above. But then $r(\{x_j\}) < Y$ for all $j \in J$, and hence $r(Z) < Y$ as required.

In view of this theorem, we shall say that a reduction r of $\mathcal{V}$ is *complete* if it satisfies Theorem 3(iii) above. Note that the reductions of all the above examples are complete. In fact, we are unable to find an example of a reduction which is not complete.

4. Compatibility and stability

Throughout this section, X will denote a monoid, $\mathcal{V}$ a hereditary semiring of N_∞^X, r a reduction of $\mathcal{V}$, and $\mathcal{A}$ a network over X.

Definition 3. Each $A \in \mathcal{V}$ is said to be *closed* in $\mathcal{V}$ if $A^* \in \mathcal{V}$, where $A^* = \biguplus_{k=0}^{\infty} A^k$.

Evidently, A is closed in $\mathcal{V}$ if and only if $A^+ \in \mathcal{V}$ or alternatively, $B^* \in \mathcal{V}$ for all $B \subseteq A$. For the hereditary semirings given in Section 1.2, we obtain the following.

Theorem 4.

(i) $A^* \in N^X_\infty$ *for all* $A \in N^X_\infty$.

(ii) $A^* \in \mathscr{F}$ *if and only if* $A = \emptyset$.

(iii) $A^* \in \mathscr{Q}$ *for all* $A \in \mathscr{Q}$.

(iv) *If* $A^* \in \mathscr{N}$, *then* $e \notin A$ *for all* $A \in \mathscr{N}$. *The converse holds if* X *is locally finite.*

(v) *Let* X *be totally ordered. The condition "*$x \geqslant e$ *(or* $x \leqslant e$*) for all* $x \in A$ *of* $\mathscr{W}$ *(or* $\mathscr{W}'$*)" is sufficient for* $A^* \in \mathscr{W}$ *(or* $\mathscr{W}'$*) if* X *has the Archimedean property. It is also necessary if* X *is non-idempotent.*

Proof. (i) is trivial, (ii) follows from the fact that if $\{x\}^*$ is non-singular, then $d(\{x\}^*)$ cannot be finite. As for (iii), note that $d(A^*) = \bigcup_{k=0}^{\infty} d(A)^k$, and hence countable because each $d(A)^k$ is countable. To prove (iv), let $m = A(e) > 0$. Then $A^k(e) > m^k$ and $A^*(e) > 1 + m + m^2 + \cdots$ cannot be finite. Conversely, if X is locally finite, then each $x \in X$ admits only a finite number of factorizations, and hence $A^k(x) = 0$ for all k big enough. Therefore, $A^*(x) \neq \infty$.

(v) can be proved as follows. Let $A \in \mathscr{W}$ and $a = \min d(A) \geqslant e$. Observe that $A^k \in \mathscr{W}$, $A^{[k]} \in \mathscr{W}$, where $A^{[k]} = e \uplus A \uplus A^2 \uplus \cdots \uplus A^k$, and $a^k = \min d(A^k)$ for every $k \in N$. Suppose $a > e$ and $d(A^*)$ contains a decreasing sequence $b_1 > b_2 > \cdots$. Then $b_i \in d(A^{k_i})$ for some $k_i \in N$. Since $d(A^{[k]})$ is well ordered for every k the sequence (k_i) is not bounded from above by any k. Then $b_i \geqslant a^{k_i}$ which contradicts $b_1 > b_i$ and (ii) above. Thus $A^* \in \mathscr{W}$. So suppose $a = e$ and let $A = B \cup C$, where $d(B) = \{e\}$ and $e \notin d(C)$. Then clearly, $d(A^*) = \{e\} \cup d(C^*)$, and hence $A^* \in \mathscr{W}$ because $C^* \in \mathscr{W}$ by the above argument, thus proving the sufficiency.

Now let X be non-idempotent and assume that $e > a$. Then $a \geqslant a^2$. But $a^2 \neq a$, and hence $e > a > a^2$. Similarly, from $e > a$, we get $a^2 > a^4$, etc. and thus obtaining a decreasing sequence $e > a > a^2 > a^4 > \cdots$ in $d(A^*)$.

The result for $\mathscr{W}'$ can also be shown dually.

Definition 4. Let P be the set of all paths of $G(\mathscr{A})$. Then $\mathscr{A}$ is said to be *compatible* with $\mathscr{V}$ if $v(P) \in \mathscr{V}$.

Evidently, $\mathscr{A}$ is compatible with $\mathscr{V}$ if and only if $v(P_{ij}) \in \mathscr{V}$ for all $i, j \in \{1, 2, \ldots, n\}$. In fact, a stronger result holds.

Lemma 1. $\mathscr{A}$ *is compatible with* $\mathscr{V}$ *if and only if* $v(P_{ii} \setminus P_{ii}^{(0)}) \in \mathscr{V}$ *for all* $i \in \{1, 2, \ldots, n\}$.

Proof. Since $v(P_{ii} \setminus P_{ii}^{(0)}) \subseteq v(P)$, we have $v(P_{ii} \setminus P_{ii}^{(0)}) \in \mathscr{V}$ whenever $v(P) \in \mathscr{V}$. Now suppose $v(P_{ii} \setminus P_{ii}^{(0)}) \in \mathscr{V}$. But $v(Q_{ii}^{[i-1]})^* \subseteq v(P_{ii} \setminus P_{ii}^{(0)})$, and hence $v(Q_{ii}^{[i-1]})^* \in \mathscr{V}$, where $Q_{ii}^{[i-1]}$ is as defined at the end of Section 2.2. Thus by (11), $v(Q_{ij}^{[k]}) \in \mathscr{V}$ for all $k \in \{1, 2, \ldots, n\}$ since $v(Q_{ij}^{(0)}) = v(P_{ij}^{(1)}) \in \mathscr{V}$ also. Therefore, $v(P_{ij}) \in \mathscr{V}$ for every i, j and hence $\mathscr{A}$ is compatible with $\mathscr{V}$.

Theorem 5. (i) *$\mathscr{A}$ is always compatible with $\mathscr{Q}$ and N_∞^X.*

(ii) *$\mathscr{A}$ is compatible with $\mathscr{F}$ if and only if $\mathscr{A}$ is acyclic, i.e. it has no non-null closed paths.*

(iii) *$\mathscr{A}$ is always compatible with $\mathscr{V}$ whenever $\mathscr{A}$ is acyclic.*

(iv) *If $\mathscr{A}$ is compatible with $\mathscr{N}$, then e is not the label of any elementary non-null closed path of $\mathscr{A}$. The converse holds if X is locally finite.*

(v) *Let X be totally ordered. If X is non-idempotent and $\mathscr{A}$ is compatible with $\mathscr{W}$ (or $\mathscr{W}'$), then each label a of any elementary non-null closed path of $\mathscr{A}$ is such that $a \geq e$ (or $a \leq e$). This condition is also sufficient if X has the Archimedean property.*

Proof. (i) Note that $v(P_{ii} \setminus P_{ii}^{(0)}) = \biguplus_{k=1}^{\infty} v(P_{ii}^{(k)}) \in \mathscr{Q}$ because $v(P_{ii}^{(k)}) \in \mathscr{F} \subseteq \mathscr{Q}$ for every k. Thus by Lemma 1, $\mathscr{A}$ is compatible with $\mathscr{Q}$, and hence with N_∞^X because $\mathscr{Q} \subseteq N_\infty^X$.

(ii) By Lemma 1, $\mathscr{A}$ is compatible with $\mathscr{F}$ implies $v(P_{ii} \setminus P_{ii}^{(0)}) \in \mathscr{F}$. Since $\{x\}^* \subseteq v(P_{ii} \setminus P_{ii}^{(0)})$ for any $x \in v(P_{ii} \setminus P_{ii}^{(0)})$, we have $\{x\}^* \in \mathscr{F}$, contradicting Theorem 4(ii). Conversely, if $v(P_{ii} \setminus P_{ii}^{(0)}) = \emptyset$, then $\emptyset^* = \{e\} \in \mathscr{F}$ implies that $\mathscr{A}$ is compatible with $\mathscr{F}$ by Lemma 1.

(iii) follows from (ii) because $\mathscr{F} \subseteq \mathscr{V}$ for any $\mathscr{V}$.

(iv) Note that the condition "e is not the label of any elementary non-null closed path of $\mathscr{A}$" is equivalent to "$e \notin v(P_{ii} \setminus P_{ii}^{(0)})$ for all $i \in \{1, 2, \ldots, n\}$". For let $x \in v(P_{ii} \setminus P_{ii}^{(0)})$ for some i, then $x \in v(p)$ for some $p \in P_{ii} \setminus P_{ii}^{(0)}$. If p is elementary, then x is the label of some elementary closed path of $\mathscr{A}$, and hence $x \neq e$ by assumption. Otherwise, let $p = p'\omega p''$ where ω is an elementary non-null closed path of $G(\mathscr{A})$ and p', p'' may be null, but not both. Then $v(p) = v(p') \circ v(\omega) \circ v(p'')$, and hence $x = y \circ a \circ z$ for some $y \in v(p')$, $a \in v(\omega)$, and $z \in v(p'')$. Since $a \neq e$ and X is locally finite, it follows that $x \neq e$ which proves the equivalence. Now by lemma 1, $\mathscr{A}$ is compatible with $\mathscr{N}$ implies that $v(P_{ii} \setminus P_{ii}^{(0)}) \in \mathscr{N}$, and hence $\{x\}^* \in \mathscr{N}$ for any $x \in v(P_{ii} \setminus P_{ii}^{(0)})$. Thus $x \neq e$ by Theorem 4(iv). Conversely, let $e \notin v(P_{ii} \setminus P_{ii}^{(0)})$. Then by (11), $v(Q_{ij}^{\{1\}}) \in \mathscr{N}$ for every i, j because $v(Q_{11}^{\{0\}}) \in \mathscr{N}$, and $e \notin v(Q_{11}^{\{0\}})$ implies $v(Q_{11}^{\{0\}})^* \in \mathscr{N}$ by Theorem 4(iv). Thus in particular, $v(Q_{22}^{\{1\}}) \in \mathscr{N}$. But again, by Theorem 4(iv), $e \notin v(Q_{22}^{\{1\}})$ implies $v(Q_{22}^{\{1\}})^* \in \mathscr{N}$, and hence $v(Q_{ij}^{\{2\}}) \in \mathscr{N}$ for every i, j. Continuing the argument in this way, we get $v(Q_{ij}^{\{n\}}) \in \mathscr{N}$ for every i, j. Consequently, $v(P_{ij}) \in \mathscr{N}$ for every i, j.

(v) Just as in (iv) above, it can be shown first that the condition stipulated in this case is equivalent to "$x \geq e$ (or $x \leq e$) for all $x \in v(P_{ii} \setminus P_{ii}^{(0)})$ for every i". The remaining argument is also similar to that of (iv), except Theorem 4(v) is now used in place of Theorem 4(iv) above. Its detail will therefore be omitted.

Definition 5. r is said to be *stable* with respect to $\mathscr{A}$ if there is $k \in N$ such that $r(v(P_{ij}^{[k+1]})) = r(v(P_{ij}^{[k]}))$ for all $i, j \in \{1, 2, \ldots, n\}$. The *stability index* of r with respect to $\mathscr{A}$ is the least such integer k if r is stable with respect to $\mathscr{A}$, and ∞ otherwise.

Let M be the label matrix of $\mathscr{A}$ and define $r(M)$ by $(r(M))_{ij} = r(M_{ij})$ for every i, j. Then $r(M)$ can be considered as the arc-value matrix of the graph $G(\mathscr{A})$ over the reduct $(\mathscr{V}_r, \oplus, \odot)$. In this way, one can easily deduce from (5) that the stability index of r with respect to $\mathscr{A}$ is identical with the stability index of $r(M)$. Consequently, if n_0 is the stability index of r with respect to $\mathscr{A}$, then for all $s \geq n_0$, we have $r(v(P_{ij}^{[s]})) = r(v(P_{ij}^{[n_0]}))$ for every i, j. A more important consequence of this observation is that the Stability Theorem of Section 1.2 can be immediately translated into results on the stability index of r with respect to $\mathscr{A}$. For instance, $G(\mathscr{A})$ is absorptive if $r(v(\omega)) \leq r(\{e\})$ for all elementary non-null closed path ω of $G(\mathscr{A})$ which can be seen to hold if $r(\{a\}) \leq r(\{e\})$ for each label a of any elementary non-null closed path of $\mathscr{A}$. Thus if r satisfies this latter condition, then the stability index of r with respect to $\mathscr{A}$ is at most $n-1$ also. More generally, let us make the following

Definition 6. Each $A \in \mathscr{V}$ is said to be *q-absorptive* with respect to r if for each ordered q-tuple $(a_1, a_2, \ldots, a_q)$ of elements of A, we have

$$r(\{a_1 \circ a_2 \circ \cdots \circ a_q\}) \leq r(\{e, a_1, a_1 \circ a_2, \ldots, a_1 \circ a_2 \circ \cdots \circ a_{q-1}\}),$$

where $\{e, a_1, a_1 \circ a_2, \ldots, a_1 \circ a_2 \circ \cdots \circ a_{q-1}\} \subseteq A$.

Let Ω be the set of all elementary non-null closed paths of $\mathscr{A}$. Then we have just argued that *if $v(\Omega)$ is 1-absorptive with respect to r then the stability index of r with respect to $\mathscr{A}$ is at most $n-1$*. In fact, one can also argue similarly (via (ii) and (iii) of the Stability Theorem) that *if X is commutative, then the stability index of r with respect to $\mathscr{A}$ is at most $nq-1$ when $v(\Omega)$ is q-absorptive with respect to r, and at most $n|\Omega|(q-1)+n-1$ when $v(\omega)$ is q-absorptive with respect to r for every $\omega \in \Omega$.*

Definition 7. r is said to be *completely stable* with respect to $\mathscr{A}$ if $\mathscr{A}$ is compatible with $\mathscr{V}$ and there is $k \in N$ such that

$$r(v(P_{ij})) = r(v(P_{ij}^{[k+1]})) = r(v(P_{ij}^{[k]})) \quad \text{for every } i, j.$$

The *complete stability index* of r with respect to $\mathscr{A}$ is the least such integer k and ∞ otherwise.

Lemma 2. *Let $\mathscr{A}$ be compatible with $\mathscr{V}$ and r be complete. Then r is completely stable with respect to $\mathscr{A}$ whenever r is also stable with respect to $\mathscr{A}$.*

Proof. Let us first use mathematical induction to show that

$$r(v(P_{ij}^{(k)}) \uplus v(P_{ij}^{[n_0]})) = r(v(P_{ij}^{[n_0]})) \quad \text{for all } k \geq n_0 + 1$$

if n_0 is the stability index of r with respect to $\mathscr{A}$. For $k = n_0 + 1$, the result is true by assumption. So let us suppose that the result is true for all k such that

$n_0 + 1 \leqslant k \leqslant t$. But then

$$\begin{aligned} r(v(P_{ij}^{(t)}) \uplus v(P_{ij}^{[n_0]})) &= r(v(P_{ij}^{(t)}) \uplus r(v(P_{ij}^{[n_0]}))) \\ &= r(v(P_{ij}^{(t)}) \uplus r(v(P_{ij}^{[t-1]}))), \quad \text{since } t-1 \geqslant n_0 \\ &= r(v(P_{ij}^{(t)}) \uplus v(P_{ij}^{[t-1]})) \\ &= r(v(P_{ij}^{[t]})) \\ &= r(v(P_{ij}^{[n_0]})), \quad \text{since} \quad t \geqslant n_0. \end{aligned}$$

Therefore, the result is true for all $k \geqslant n_0 + 1$.

Now

$$v(P_{ij} \setminus P_{ij}^{[n_0]}) = v\Big(\bigcup_{k=n_0+1}^{*} P_{ij}^{(k)}\Big) = \bigcup_{k=n_0+1}^{*} v(P_{ij}^{(k)})$$

by (8) and since we have just shown that

$$r(v(P_{ij}^{(k)})) \leqslant r(v(P_{ij}^{[n_0]})) \quad \text{for all} \quad k \geqslant n_0 + 1,$$

it follows from the completeness of r that

$$r(v(P_{ij} \setminus P_{ij}^{[n_0]})) \leqslant r(v(P_{ij}^{[n_0]})), \qquad \text{i.e.} \quad r(v(P_{ij})) = r(v(P_{ij}^{[n_0]})).$$

Using this lemma, one can easily extend each of the above results concerning the stability index of r with respect to $\mathscr{A}$ to a corresponding result concerning the complete stability index of r whenever $\mathscr{A}$ is compatible with $\mathscr{V}$ and r is complete. Some other results on complete stability will be given below.

Theorem 6. *The complete stability index of r with respect to an acyclic network $\mathscr{A}$ is at most $n-1$.*

Proof. By Theorem 5(iii), $\mathscr{A}$ is always compatible with $\mathscr{V}$ if $\mathscr{A}$ is acyclic. Also for such $\mathscr{A}$, $P_{ij}^{(k)} = \emptyset$ for all $k \geqslant n$, and hence $r(v(P_{ij})) = r(v(P_{ij}^{[s]}))$ for all $s \geqslant n-1$.

We shall say that r is *intensive* if $r(A) \subseteq A$ for all $A \in \mathscr{V}$, and *stationary* if for some $k \in N$, $r(A^*) = r(\{e\} \uplus A \uplus A^2 \uplus \cdots \uplus A^k)$ whenever $A^* \in \mathscr{V}$. Note that the reductions of all the above examples, except s of Example 6, are intensive. In Example 6, if $X = N$ and $A = \{1, 2, 3\}$, then $s(A) = \{1+2+3\} = \{6\} \nsubseteq A$, and therefore, s is not intensive in this case. The reductions of Examples 2, 4, 5 and 6 are all stationary whereas those of Examples 1 and 3 are not.

Theorem 7. *If r is stationary, idempotent and intensive, then r is completely stable with respect to any $\mathscr{A}$ which is compatible with $\mathscr{V}$.*

Proof. First, let us note that for any $A, B \in \mathscr{V}$ such that $r(A)$, $r(B) \in \mathscr{F}$, both $r(A \uplus B)$ and $r(A \circ B)$ belong to $\mathscr{F}$ because

(i) $r(A \uplus B) = r(r(A) \uplus r(B)) \subseteq r(A) \uplus r(B) \in \mathscr{F}$, and

(ii) $r(A \circ B) = r(r(A) \circ r(B)) \subseteq r(A) \circ r(B) \in \mathscr{F}$.

Next, we claim that $r(v(P_{ij}))\in\mathscr{F}$ *for every* i, j. Since we have $r(v(P_{ij})) = r(v(P_{ij}\setminus P_{ij}^{[n]})\uplus v(P_{ij}^{[n]}))$ and $r(v(P_{ij}^{[n]}))\in\mathscr{F}$, it then suffices to show that $r(v(P_{ij}\setminus P_{ij}^{[n]}))\in\mathscr{F}$. By (ii) of (11), this is equivalent to showing that $r(v(Q_{ij}^{[n]}))\in\mathscr{F}$.

From (i) of (11), we see that $r(v(Q_{ij}^{[0]}))\in\mathscr{F}$ for every i, j because $r(v(Q_{ij}^{[0]})) = r(v(P_{ij}^{[1]}))\subseteq v(P_{ij}^{[1]})\in\mathscr{F}$. Thus in particular, $r(v(Q_{11}^{[0]}))\in\mathscr{F}$. Since r is stationary, it follows that $r(v(Q_{11}^{[0]})^{*})\in\mathscr{F}$, and hence by (iii) of (11), $r(v(Q_{ij}^{[1]}))\in\mathscr{F}$ for every i, j. Continuing the argument in this way, we get $r(v(Q_{ij}^{[n]}))\in\mathscr{F}$ as claimed.

Now since $r(v(P_{ij}))\subseteq v(P_{ij})$, it follows that $r(v(P_{ij})) = v(H)$ for some finite subset H of P_{ij}. Let n_0 be the maximum length of paths in H. Then for each $x\in r(v(P_{ij}))$, $x\in v(P_{ij}^{[s]})$ for all $s\geq n_0$, and hence $r(\{x\})\preccurlyeq r(v(P_{ij}^{[s]}))$ for all $s\geq n_0$ by the corollary of theorem 2. Since this holds for all $x\in r(v(P_{ij}))$ which is finite, it follows that $r(v(P_{ij}))\preccurlyeq r(v(P_{ij}^{[s]}))$ for all $s\geq n_0$. But by the same corollary, $v(P_{ij}^{[s]})\subseteq v(P_{ij})$ implies $r(v(P_{ij}^{[s]}))\preccurlyeq r(v(P_{ij}))$ for all $s\geq n_0$, and hence $r(v(P_{ij})) = r(v(P_{ij}^{[s]})y)$ for all $s\geq n_0$ as required.

Lemma 3. *Let* $A\in\mathscr{F}$ *be closed in* $\mathscr{V}$ *and* q*-absorptive with respect to* r*. If* r *is complete, then* $r(A^{*}) = r(\{e\}\uplus A\uplus A^{2}\uplus\cdots\uplus A^{s})$ *for all* $s\geq q-1$.

Proof. Let $A=\{a_1, a_2, \ldots, a_k\}$ and consider A to be the label matrix of the one-node network $\mathscr{A}'$ below. The required result is then equivalent to showing that the complete stability index of r with respect to $\mathscr{A}'$ is at most $q-1$. But r is complete, and hence by Lemma 2, it suffices to show that $r(v(P^{[q]})) = r(v(P^{[q-1]}))$, where $P^{[q]} = \bigcup_{k=0}^{q} P^{(k)}$, and $P^{(k)}$ is the set of paths of $G(\mathscr{A}')$ which have length k.

Let $a\in v(P^{(q)})$. Then $a\in v(p)$ for some $p\in P^{(q)}$, and hence $a = a_{i_1}\circ a_{i_2}\circ\cdots\circ a_{i_q}$. Since A is q-absorptive with respect to r, we have

$$r(\{a\})\preccurlyeq r(\{e, a_{i_1}, a_{i_1}\circ a_{i_2}, \ldots, a_{i_1}\circ a_{i_2}\circ\cdots\circ a_{i_{q-1}}\}) = r(v(H))$$

for some $H\subseteq P^{[q-1]}$. Since this holds for all $a\in v(P^{(q)})$ which is finite, it follows that $r(v(P^{(q)}))\preccurlyeq r(v(H))$. Therefore, we have

$$r(v(P^{(q)}))\preccurlyeq r(v(P^{[q-1]}\setminus H))\oplus r(v(H)) = r(v(P^{[q-1]})),$$

and hence

$$r(v(P^{[q]})) = r(v(P^{(q)})\uplus v(P^{[q-1]})) = r(v(P^{[q-1]}))$$

as required.

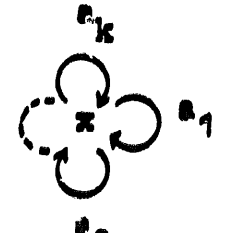

Fig. 1. Network $\mathscr{A}'$.

Theorem 8. *Let r be complete, idempotent and intensive. Let $\mathscr{A}$ be compatible with $\mathscr{V}$ and $v(P_{ii})$ be q-absorptive with respect to r for all $i \in \{1, 2, \ldots, n\}$. Then r is completely stable with respect to $\mathscr{A}$.*

Proof. In view of the end argument of Theorem 7 above, it suffices to show that $r(v(P_{ij}))$ *or, equivalently,* $r(v(Q_{ij}^{(n)}))$ is finite for every i, j.

From (i) of (11), $r(v(Q_{ij}^{(0)})) \in \mathscr{F}$ for every i, j because $r(v(Q_{ij}^{(0)})) = r(v(P_{ij}^{(1)})) \subseteq v(P_{ij}^{(1)}) \in \mathscr{F}$. Thus in particular, we have $r(v(Q_{11}^{(0)})) \in \mathscr{F}$. Since $r(v(Q_{11}^{(0)})) \subseteq v(Q_{11}^{(0)}) \subseteq v(P_{11})$, it follows that $r(v(Q_{11}^{(0)}))$ is q-absorptive with respect to r. Since $\mathscr{A}$ is compatible with $\mathscr{V}$, it follows that $r(v(Q_{11}^{(0)}))$ is also closed in $\mathscr{V}$. Hence, on applying Lemma 3 to $r(A)$ where $A = v(Q_{11}^{(0)})$, we get $r(r(A)^*) \in \mathscr{F}$. But

$$r(r(A)^*) = r(r(A)^{(q-1)}) = r(A^{(q-1)}) = \bigvee_{k=0}^{q-1} r(A^k)$$

$$= r\left(\biguplus_{k=0}^{q-1} A^k\right) = r(A^*) = r(v(Q_{11}^{(0)})^*),$$

and hence by (iii) of (11), we have $r(v(Q_{ij}^{(1)})) \in \mathscr{F}$ for every i, j. Continuing the argument in this way, we have $r(v(Q_{ij}^{(n)})) \in \mathscr{F}$ for every i, j.

Appendix. Proof of Stability Theorem

Let us first obtain the following lemma where we shall say that a subset F of a set of paths Q is *dense* in Q if for any $p \in Q \setminus F$, there is a finite subset H of F such that $v(p) \prec \sigma(H)$.

Lemma 4. *If $F \subseteq B \subseteq Q$ and F is also dense in Q, then so is B. Moreover, if F and B are both finite, then $\sigma(B) = \sigma(F)$.*

Proof. If $p \in Q \setminus B$, then $p \in Q \setminus F$ because $F \subseteq B$. But F is dense in Q and hence there is a finite subset H of F and hence of B such that $v(p) \prec \sigma(H)$ which proves the first part. Now by P_1, P_3 and σ_1, we get

$$v(p) = v(p) + \mathscr{O} \prec \sigma(H) + \sigma(F \setminus H) = \sigma(F).$$

Consequently, by P_4, $\sigma(B \setminus F) \prec \sigma(F)$. Therefore, by σ_1, we have

$$\sigma(B) = \sigma(B \setminus F) + \sigma(F) = \sigma(F).$$

We can now prove (i), (ii) and (iii) of the theorem as follows.

(i) From (4), this is equivalent to $\sigma(P_{ij}^{[s]}) = \sigma(P_{ij}^{[n-1]})$ for all $s \geqslant n-1$. But by the lemma above, it suffices to show that $\Omega_{ij}^{(1)} \subseteq P_{ij}^{[s]} \subseteq P_{ij}$ for all $s \geqslant n-1$ and that $\Omega_{ij}^{(1)}$ is dense in P_{ij}, where $\Omega_{ij}^{(1)}$ denotes the set of all paths of $G(A)$ which do not traverse any elementary non-null closed paths. The former is obvious while the latter can be proved as follows.

Let $p \in P_{ij} \setminus \Omega_{ij}^{(1)}$. Then in traversing p, we must come across at least one elementary non-null closed path say ω_1. Accordingly, $p = p_1\omega_1 q_1$ and therefore, $v(p) < v(p_1 q_1)$ since $v(\omega_1) < e$ by assumption. If $p_1 q_1 \in \Omega_{ij}^{(1)}$, we are done. Otherwise, repeat the above argument using $p_1 q_1$ in place of p until we eventually obtain an elementary open path $\bar{p}$ such that $v(p) < v(\bar{p})$.

(ii) Like (i) above, it suffices to show that $\Omega_{ij}^{(q)} \subseteq P_{ij}^{[s]} \subseteq P_{ij}$ for all $s \geq nt(q-1)+n-1$ and that $\Omega_{ij}^{(q)}$ is dense in P_{ij} where $\Omega_{ij}^{(q)}$ denotes the set of all paths of $G(A)$ which do not traverse any elementary non-null closed path more than $q-1$ times. The former follows because the maximum order of any path in $\Omega_{ij}^{(q)}$ corresponds to that which traverses each of the elementary non-null closed path exactly $q-1$ times and an elementary open path which amounts to $nt(q-1)+n-1$ if there are exactly t elementary non-null closed paths in $G(A)$. The latter can be proved as follows.

First let us note that if $G(A)$ is q-regular, then for every elementary closed path ω of G,

$$v(\omega)^s < e + v(\omega) + v(\omega)^2 + \cdots + v(\omega)^{q-1} \quad \text{for all} \quad s \geq q. \tag{17}$$

Now let $p \in P_{ij} \setminus \Omega_{ij}^{(q)}$. Then $p \notin \Omega_{ij}^{(q)}$ and hence p must traverse at least one elementary non-null closed path, say ω_1, exactly s_1 times where $s_1 \geq q$. Let us express p by the manner which it traverses ω_1 as follows.

$$p = p_1\omega_1 p_2 \omega_1 p_3 \cdots \omega_1 p_{s_1+1},$$

where the p_i's may be null. But then by the commutativity assumption,

$$v(p) = v(p_1) \circ v(p_2) \circ \cdots \circ v(p_{s_1+1}) \circ v(\omega_1)^{s_1}.$$

Therefore, by P_2 and (17) above, we obtain

$$v(p) < v(p_1) \circ v(p_2) \circ \cdots \circ v(p_{s_1+1}) \circ (e + v(\omega_1) + v(\omega_1)^2 + \cdots + v(\omega_1)^{q-1}). \tag{18}$$

Let us now write

$$p^{(1)} = p_1 p_2 \cdots p_{s+1},$$
$$p^{(2)} = p_1 \omega_1 p_2 \cdots p_{s+1},$$
$$\cdots$$
$$p^{(q)} = p_1\omega_1 p_2 \omega_1 p_3 \cdots \omega_1 p_q p_{q+1} \cdots p_{s+1}.$$

Then it follows from (18) and the commutativity assumption that

$$v(p) < v(p^{(1)}) + v(p^{(2)}) + \cdots + v(p^{(q)}) = \sigma(\{p^{(1)}, p^{(2)}, \ldots, p^{(q)}\}).$$

Moreover, each of the $p^{(i)}$ is a subpath of p which does not traverse ω_1 for more than $q-1$ times. If any $p^{(i)}$ also traverses another elementary non-null closed path for more than $q-1$ times, then we can apply the above argument to it and so on. Since there are only a finite number of elementary closed paths of G, this process must terminate. Hence we must eventually obtain $v(p) < \sigma(H)$ where $H \subseteq \Omega_{ij}^{(q)}$.

(iii) Again, it suffices to show that $\Omega_{ij}^{[q]} \subseteq P_{ij}^{[s]} \subseteq P_{ij}$ for all $s \geq nq-1$ and that $\Omega_{ij}^{[q]}$ is dense in P_{ij} where $\Omega_{ij}^{[q]}$ denotes the set of all paths of $G(A)$ which do not traverse more than $q-1$ elementary non-null closed paths. The former follows because the maximum order of any path in $\Omega_{ij}^{[q]}$ corresponds to that which traverses exactly $q-1$ elementary non-null closed paths and an elementary open path which amounts to $n(q-1)+n-1=nq-1$. The latter can be proved as follows.

First let us note that if $G(A)$ is q-absorptive, then for every s-tuple $(\omega_1, \omega_2, \ldots, \omega_s)$ of elementary closed paths of G,

$$v(\omega_1)\circ v(\omega_2)\circ\cdots\circ v(\omega_s) \tag{19}$$
$$\prec e+v(\omega_1)+v(\omega_1)\circ v(\omega_2)+\cdots+v(\omega_1)\circ v(\omega_2)\circ\cdots\circ v(\omega_{q-1})$$
$$\text{for all } s \geqslant q.$$

Now let $p \in P_{ij} \setminus \Omega_{ij}^{[q]}$. Then $p \notin \Omega_{ij}^{[q]}$ and hence p must traverse exactly s elementary non-null closed paths where $s \geqslant q$, say $\omega_1, \omega_2, \ldots, \omega_s$. Thus we may express p as follows.

$$p=p_1\omega_1q_1,$$
$$p_1q_1=p_2\omega_2q_2,$$
$$\ldots$$
$$p_{s-1}q_{s-1}=p_s\omega_sq_s,$$

where p_sq_s is an elementary open path or a null path.

But then by the commutativity assumption, we have

$$v(p)=v(p_sq_s)\circ v(\omega_s)\circ v(\omega_{s-1})\circ\cdots\circ v(\omega_1).$$

Thus by P_2, and (19), we obtain

$$v(p)\prec v(p_sq_s)\circ(e+v(\omega_s)+v(\omega_s)\circ v(\omega_{s-1})+\cdots+v(\omega_s)\circ v(\omega_{s-1}) \circ\cdots\circ v(\omega_{s-q+2}) \tag{20}$$

If we now set $H=\{p_sq_s, p_{s-1}q_{s-1}, \ldots, p_{s-q+1}q_{s-q+1}\}$, then it follows from (20) and the commutativity assumption that $v(p)\prec\sigma(H)$ and $H\subseteq\Omega_{ij}^{[q]}$.

Acknowledgement

The author is indebted to his supervisor, Dr. B.A. Carré, for his help and encouragement throughout this work. He is also grateful to Dr. E.K. Lloyd and the referees for their helpful comments on the earlier versions of this paper. Financial support for this work was kindly provided by the Bank of Thailand.

References

[1] R.C. Backhouse and B.A. Carré, Regular algebra applied to path-finding problems, J. Inst. Math. Applics. 15 (1975) 161–186.

[2] C. Benzaken, *Structures algébriques des cheminements*: Pseudo-treillis, gerbiers de carré nul, in: G. Biorci, ed., Network and Switching Theory (Academic Press, New York, 1968) 40–57.

[3] G. Birkhoff, Lattice Theory, AMS Colloquim Publ., 1967.
[4] P. Brucker, Theory of matrix Algorithms, Mathematical Systems in Economics No. 13 (Verlag Anton Hain, Germany, 1974).
[5] B.A. Carré, An algebra for network routing problems, J. Inst. Math. Applics. 7 (1971) 273–294.
[6] B.A. Carré, a book to be published by Oxford University Press, 1978.
[7] B.A. Carré and A. Wongseelashote, A scheme of iterative methods of solving certain problems in connection with graphs, in preparation.
[8] J.C. Derniame and C. Pair, Problèmes de Cheminement dans les Graphes (Dunod, Paris, 1971).
[9] S. Eilenberg, Automata, Languages, and Machines: Volume A, (Academic Press, New York, 1974).
[10] B. Giffler, Scheduling general production system using schedule algebra, Nav. Res. Log. Quart. 10 (1963) 237–255.
[11] B. Giffler, Schedule algebra: A progress report, Nav. Res. Log. Quart. 15 (1968) 255–280.
[12] M. Gondran, Algébre linéaire et cheminement dans un graphe, Rev. Fr. Aut. Inf. Rech. Opér. 9 (1975) V-1, 77–99.
[13] R. McNaughton and H. Yamada, Regular expressions and state graphs for automata, IRE Trans. Electron. Comput. 9 (1960) 39–47.
[14] E. Minieka and D.R. Shier, Note on an algebra for the K- best routes in a network, J. Inst. Math. Applics. 11 (1973) 145–149.
[15] G.C. Moisil, Asupra unor reprezentări ale grafurilor ce intervin în probleme de economia transporturilor, Com. Acad. R. P. Romîne 10 (1960) 647–652.
[16] J.D. Murchland, A new method for finding all elementary paths in a complete directed graph, Report LSE-TNT-22, London School of Economics, 1965.
[17] C. Pair, Sur des algorithmes pour les problèmes de cheminement dans les graphes finis, in: P. Rosenstiehl, ed., Theory of Graphs (Dunod, Paris, Gordon and Breach, New York, 1967) 271–300.
[18] V. Peteanu, Optimal paths in network and generalisations (I), Mathematica (Cluj) 11 (1969) 311–327.
[19] V. Peteanu, Optimal paths in network and generalisations (II), Mathematica (Cluj) 12 (1970) 159–186.
[20] B. Roy, Chemins et circuits: Enumération et optimisation, in: B. Roy et al. eds., Combinatorial Programming: Methods and Applications (D. Reidel, Boston, 1975) 105–136.
[21] D.R. Shier, Network optimality problems: Theory and algorithms, Ph.D. Thesis (London), 1973.
[22] D.R. Shier, Iterative methods for determining the K shortest paths in a network, Networks 6 (1976) 205–229.
[23] A. Wongseelashote, An algebra for determining all path values in a network with applications to K-shortest-paths problems, Networks 6 (1976) 307–334.