TOPOLOGICAL FIELD THEORIES AND FACTORIZATION HOMOLOGY

– Project –

for

Mathematical Physics - Quantum Relativistic Theories

submitted by

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Preface

A large part of this project is inspired by the work of Jacob Lurie and lectures of Claudia Scheimbauer and John Baez. Topological (quantum) field theories are functors from the category of bordisms to the category of vector spaces that preserve their monoidal structure. Such functors arose in Physics but have proven to be useful in various fields of Mathematics. They give topological and geometric invariants of manifolds, and thus may help in understanding and classifying them.

In the first chapter we give all the background in manifold theory and symmetric monoidal categories required to understand the various variants of definitions of bordisms and topological field theory. In the second chapter we looks at higher categorical tools to generalize the concept of topological field theories and bordisms to $(\infty, n)-$ versions of them. In the third chapter, we realize how factorization homology gives a topological field theory. And in the final chapter we see how the mathematical concept of topological quantum field theories was reconciles with Quantum Physics and General Relativity.

I would like to thank Prof. Valter Moretti for his helpful suggestion. Lastly, thank you for choosing to read my note. I hope you will enjoy it :)

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1 Topological Field Theory

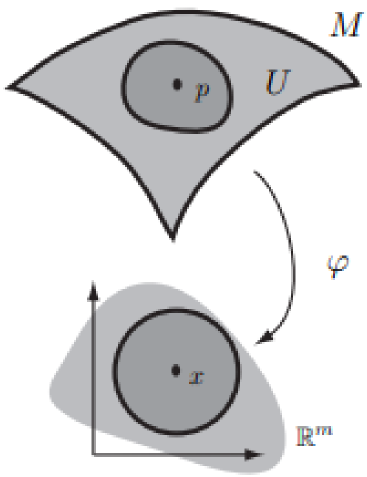
1.1 Smooth Manifolds

Definition 1.1. A *topological manifold with boundary* of dimension n is a topological space $\mathcal{M}$ which:

1. is locally homeomorphic to $\mathbb{R}^n_\pm$ i.e. $\forall p \in \mathcal{M}$, $\exists U(p)$ open and $p \in U(p)$ such that $U(p) \cong V \underset{open}{\subset} \mathbb{R}^n_\pm$
2. is Hausdorff
3. is connected
4. is second countable

NOTE: The symbol $\mathbb{R}^n_\pm$ means: $\mathbb{R}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n\}$, $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$, $\mathbb{R}^n_- = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \leq 0\}$.

Definition 1.2. A *chart* (U, φ) on $\mathcal{M}$ is given by an open set $U \subset \mathcal{M}$ and an homeomorphism $\varphi: U \to D$, onto an open set D of $\mathbb{R}^m_\pm$.



Example 1.3. Every open set of $\mathbb{R}^n_\pm$ is a topological manifold with boundary. We can choose an atlas which consists of only one coordinate chart $U = \mathbb{R}^n_\pm$, and $\varphi = Id$.

While using a paper map, we have to move from one map to another. To follow our path we need to find the coordinates, in both maps, of the same point of our position in that moment. The way to do so, is the following:

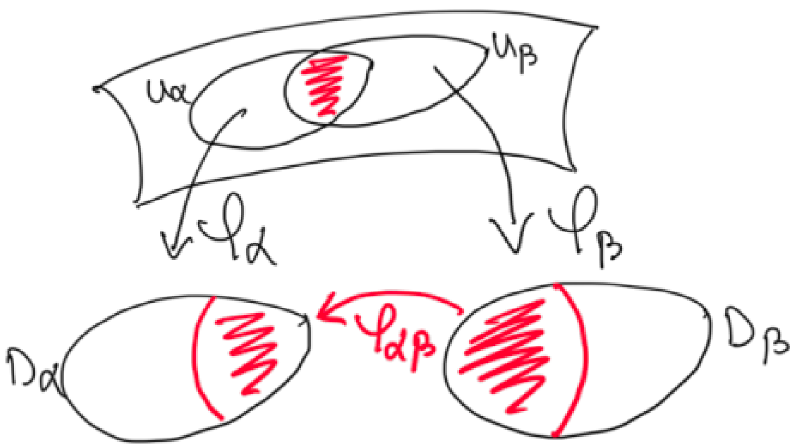
Definition 1.4. For every ordered pair of charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), the set of $\{\varphi_{\alpha\beta}\}$ is the set of *transition functions*, $\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}$,

$$\varphi_{\alpha\beta} = \varphi_\alpha |_{U\alpha \cap U_\beta} \circ \varphi_\beta^{-1}|_{\varphi_\beta(U_\alpha \cap U_\beta)}$$
$$\varphi_{\alpha\beta}: \varphi_\beta(U\alpha \cap U_\beta) \to \varphi_\alpha(U\alpha \cap U_\beta)$$

Definition 1.5. An *atlas* for a topological space $\mathcal{M}$ is a family of charts $\{U_\alpha, \varphi_\alpha\}_{\alpha \in \mathcal{I}}$ on $\mathcal{M}$ such that $\bigcup_{\alpha \in \mathcal{I}} U_\alpha = \mathcal{M}$ and all transition functions $\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}$ are smooth.

The transition function satisfies the following *cocylce conditions*:

1. $\forall \alpha \in \mathcal{I}$, $\varphi_{\alpha\alpha} = Id$
2. $\forall \alpha, \beta \in \mathcal{I}$, $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$
3. $\forall \alpha, \beta, \gamma \in \mathcal{I}$, $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$



Example 1.6. Let $\mathcal{M} = \mathbb{R}^n$ and take $U = \mathcal{M}$ with $\varphi = Id$. We could also take $\mathcal{M}$ to be any open set in $\mathbb{R}^n$.

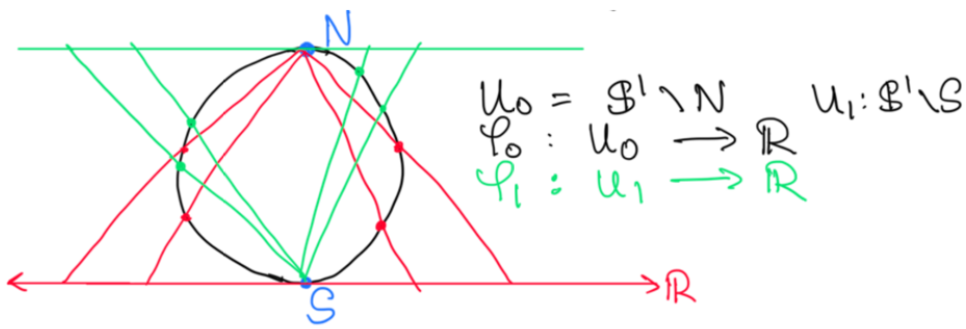
A chart allows to use the coordinates of $\mathbb{R}^n$ to identify a point of the mapped object U of the manifold $\mathcal{M}$.
From now on we will denote by u_i the i-th coordinate function on $\mathbb{R}^n$

$$u_i : \mathbb{R}^n \to \mathbb{R}$$
$$(w_1, \dots, w_n) \to w_i$$

Each chart (U, φ) induces *local coordinates* $(x_1, \dots, x_n)$ defined by

$$x_i := u_i \circ \varphi : U \to \mathbb{R}$$

Example 1.7. $\mathbb{S}^1 = \{(x, y) \in \mathbb{R} \mid x^2 + y^2 = 1\}$
$U_0 = \mathbb{S}^1 \setminus N$ and $\varphi_0 : U_0 \to \mathbb{R}$
$U_1 = \mathbb{S}^1 \setminus N$ and $\varphi_1 : U_1 \to \mathbb{R}$
transition function φ_{01} is $\mathcal{C}^\infty$.

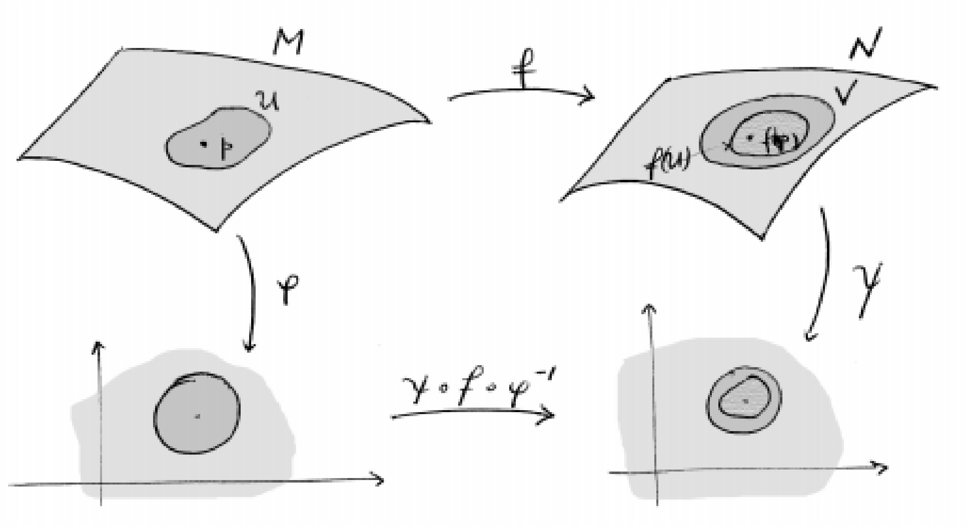


Example 1.8. $\mathbb{S}^n = \{(x_0, \dots, x_n) \mid \Sigma_{i=0}^{i=n} x_i = 1\}$
where north pole $N = (1, \dots, 0)$ and south pole $S = (-1, \dots, 0)$
charts $U_0 = \mathbb{S}^n \setminus N$ and $U_1 = \mathbb{S}^n \setminus S$
and stereographic projections $\varphi_0 : U_0 \to \mathbb{R}^n$ is an isomorphism and $\varphi_1 : U_1 \to \mathbb{R}^n$ is an isomorphism.

Definition 1.9. Let U be an open set of $\mathbb{R}^n_\pm$. A function $F : U \to \mathbb{R}^m$ is *smooth* if there is an open set $V \subset \mathbb{R}^n$ with $V \cap \mathbb{R}^n_\pm = U$ and a smooth function $G : V \to \mathbb{R}^m$ which extends F, i.e. such that $G|_U = F$.

Definition 1.10. Let $\mathcal{M}$ be a manifold with atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{I}}$ and $\mathcal{N}$ a manifold with atlas $\{(V_\beta, \psi_\beta)\}_{\beta \in \mathcal{I}'}$. A function $f : \mathcal{M} \to \mathcal{N}$ is *smooth (or $\mathcal{C}^\infty$) in a point*

$p \in \mathcal{M}$ if, given a chart $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{I}}$ with $p \in U_\alpha$, and a chart $\{(V_\beta, \psi_\beta)\}_{\beta\in\mathcal{I}'}$ with $f(p) \in V_\beta$,the function $\psi_\beta \circ f \circ \varphi_\beta^{-1}$ is smooth.



Example 1.11. An important example of $\mathcal{C}^\infty$ functions is a *bump function* on a manifold $\mathcal{M}$. More precisely, for any open sets $U, V \subset \mathcal{M}$ with U compact and $U \subset V$, there exists some $f \in C^\infty(\mathcal{M})$, such that

$$f(x) = \begin{cases} 1 & x \in \overline{U}, \\ 0 & q \notin V. \end{cases}$$

Example 1.12. The natural projection $\pi_1 \colon \mathcal{M} \times \mathcal{N} \to \mathcal{M}$ and $\pi_2 \colon \mathcal{M} \times \mathcal{N} \to \mathcal{N}$ given by $\pi_1(x,y) \mapsto x$ and $\pi_2 \colon (x,y) \mapsto y$ are smooth maps.

Example 1.13. Another important example of $\mathcal{C}^\infty$ maps is a *smooth curve* on a manifold $\mathcal{M}$ a $\mathcal{C}^\infty$ map from some open interval $I \subset \mathbb{R}$ to $\mathcal{M}$.

A rather but obvious theorem about the boundaries of a topological manifold is,

Theorem 1.14. *If M is a topological n–manifold with boundary, then the boundary of M, ∂M is a topological $(n–1)$–manifold without boundary.*

This is true because if x is in the boundary and U is an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n_+$, then the intersection of U and the boundary is homeomorphic to an open subset of $\mathbb{R}^{n-1}$.

Definition 1.15. A *diffeomorphism* is a smooth function which is invertible and whose inverse function is also smooth.

M. H. Brown stated the Collar Neighborhood Theorem for compact topological manifolds [Bro62], which can be modified for smooth manifolds with boundaries as,

Theorem 1.16 (Collar Neighborhood Theorem)**.** *Let M be a smooth manifold with boundary. Then the boundary ∂M has an open neighborhood which is diffeomorphic to the product $\partial M \times [0,1)$.*

This theorem is central notably for the definition and behaviour of categories of cobordisms, which we will define in the next section.

Theorem 1.17. *The boundary ∂M of a manifold M with boundary has a collar.*

If M and N are two topological manifolds with boundaries, then their product $M \times N$ is also a topological manifold with boundary, which is defined as $\partial M \times N \cup M \times \partial N$. However, if both manifolds with boundaries have smooth structures and both boundaries are nonempty, then there is no "natural" smooth structure on the product; instead, one obtains objects known as *manifolds with corners.*

Definition 1.18. A manifold X is called *closed* if it is compact but does not have a boundary.

1.2 Symmetric Monoidal Categories

In order to define monoidal categories we first need to define Cartesian categories.

Definition 1.19. A *cartesian product* of two categories C and D is defined as $\mathcal{C} \times \mathcal{D}$ such that,
Objects: $\mathcal{O}(\mathcal{C} \times \mathcal{D}) \coloneqq \mathcal{O}(\mathcal{C}) \times \mathcal{O}(\mathcal{D})$
Morhisms: $Hom_{\mathcal{C}\times\mathcal{D}}(A \times B, A' \times B') \coloneqq Hom_{\mathcal{C}}(A, A') \times Hom_{\mathcal{D}}(B, B')$ for all objects $A, A' \in \mathcal{O}(\mathcal{C})$ and $B, B' \in \mathcal{O}(\mathcal{D})$
Composition: is defined componentwise, as $(f, g) \circ (f', g') = (f \circ f', g \circ g')$
Identity: is defined levelwise, as $1_{(f,g)} = (1_f, 1_g)$.

We define the empty product category, **1**, to be the category with only an object and only its identity morphism.
Thus, we can write,

$$(\mathcal{C} \times \mathcal{D}) \times \mathcal{E} = \mathcal{C} \times \mathcal{D} \times \mathcal{E} = \mathcal{C} \times (\mathcal{D} \times \mathcal{E})$$

and,

$$\mathcal{C} \times \mathbf{1} = \mathcal{C} = \mathbf{1} \times \mathcal{C}$$

Definition 1.20. A *monoidal category* is a category M equipped with,

1. a functor $\otimes: M \times M \to M$ called the *tensor product* (we write $\otimes(A, B) = A \otimes B$ and $\otimes(f, g) = f \otimes g$)
2. a functor $I: \mathbf{1} \to M$ called the *unit* (we write the image of the only object in $\mathbf{1}$ simply as $I \in \mathcal{O}(M)$)
3. a natural isomorphism, α, called an *associator*,

$$\begin{tikzcd} & M \times M \times M \arrow[dl, "\otimes \times 1_M"'] \arrow[dr, "1_M \times \otimes"] & \\ M \times M \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, "\otimes"'] & & M \times M \arrow[dl, "\otimes"] \\ & M & \end{tikzcd}$$

(i.e. with components $\alpha_{A,B,C}: (A \otimes B) \otimes C \to A \otimes (B \otimes C)$)

4. and two natural isomorphisms, λ and ρ, called *unit isomorphisms*,

$$\begin{tikzcd} & M \times M \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\lambda"] & \\ \mathbf{1} \times M \arrow[ur, "I \times 1_M"] \arrow[rr, equal] & {} & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & M \times M \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\rho"] & \\ \mathbf{1} \times M \arrow[ur, "1_M \times I"] \arrow[rr, equal] & {} & M \end{tikzcd}$$

(i.e. with components $\lambda_A: I \otimes A \to A$ and $\rho_A: A \otimes I \to A$ respectively)

such that the following diagrams commute for any $A, B, C, D \in M$:

1. The *pentagon diagram*:

$$\begin{tikzcd} & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C \otimes D}"] & \\ ((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A \otimes B, C, D}"] \arrow[d, "\alpha_{A,B,C} \otimes 1_D"'] & & A \otimes (B \otimes (C \otimes D)) \\ (A \otimes (B \otimes C) \otimes D \arrow[rr, "\alpha_{A, B \otimes C, D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "1_A \otimes \alpha_{B,C,D}"'] \end{tikzcd}$$

2. and the *triangle diagram*:

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes 1_B"'] & & A \otimes (I \otimes B) \arrow[dl, "1_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

These two diagrams are known as *coherence conditions* and are to ensure that all diagrams involving α, λ and ρ commute.

Now we would like to define monoidal categories with commutative tensor products. To do so first we define the *twist functor* $T : \mathcal{C} \times \mathcal{D} \to \mathcal{D} \times \mathcal{C}$ as follows:

1. $T(A \times B) = B \times A$ for any objects A, B in $\mathcal{C}$ and $\mathcal{D}$ respectively,
2. $T(f \times g) = g \times f$ for any morphisms f, g in $\mathcal{C}$ and $\mathcal{D}$ respectively

Definition 1.21. A *symmetric monoidal category* is a monoidal category M equipped with a natural isomorphism, β, called *symmetric braiding*,

$$\begin{tikzcd}
M \times M \arrow[rr, "\otimes"] \arrow[dr, "T"'] & \arrow[d, phantom, "\Downarrow\beta"] & M \\
& M \times M \arrow[ur, "\otimes"'] &
\end{tikzcd}$$

(i.e. with components $\beta_{A,B} : A \otimes B \to B \otimes A$), such that,

1. it satisfies the *symmetric condition*: β is its own inverse, i.e. its components satisfy $\beta_{B,A} \circ \beta_{A,B} = 1_{A \otimes B}$
2. and makes the *hexagon diagram* commute for any objects $A, B, C \in M$:

$$\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha_{B,C,A}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{A,B,C}"] \arrow[dr, "\beta_{A,B} \otimes 1_C"'] & & & B \otimes (C \otimes A) \\
& (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[ur, "1_B \otimes \beta_{A,C}"'] &
\end{tikzcd}$$

Example 1.22. Let k be a field. Then the category $Vect_k$ of vector spaces over k can be regarded as a symmetric monoidal category with respect to the usual tensor product functor

$$\otimes : Vect_k \times Vect_k \to Vect_k$$

The unit object of $Vect_k$ is the vector space k itself.
This symmetric monoidal category of vector spaces is denoted by $\mathbf{Vect}_{\mathbf{k}}^{\otimes}$.

Definition 1.23. A *monoidal functor* between two monoidal categories $(M_1, \otimes_1, I_1)$ and $(M_2, \otimes_2, I_2)$ is a functor $F : M_1 \to M_2$ equipped with:

1. a natural isomorphism, ϕ,

$$\begin{tikzcd}
& M_1 \times M_1 \arrow[dl, "F\times F"'] \arrow[dr, "\otimes_1"] & \\
M_2 \times M_2 \arrow[rr, Rightarrow, "\phi"] \arrow[dr, "\otimes_2"'] & & M_1 \arrow[dl, "F"] \\
& M_2 &
\end{tikzcd}$$

   (i.e. with components $\phi_{A,B} : F(A) \otimes_2 F(B) \to F(A \otimes_1 B)$)

2. and an isomorphism $\omega : I_2 \to F(I_1)$ in M_2

such that,

1. the following diagram commutes for any objects $A, B, C \in M_1$,

$$\begin{tikzcd}
F(A \otimes_1 B) \otimes_2 F(C) \arrow[r, "\phi_{A\otimes_1 B,C}"] & F((A \otimes_1 B) \otimes_1 C) \arrow[d, "\alpha_{A,B,C}"] \\
(F(A) \otimes_2 F(B)) \otimes_2 F(C) \arrow[u, "\phi_{A,B}\otimes_2 1_{F(C)}"] \arrow[d, "\alpha_{F(A),F(B),F(C)}"'] & F(A \otimes_1 (B \otimes_1 C)) \\
F(A) \otimes_2 (F(B) \otimes_2 F(C)) \arrow[r, "1_{F(A)}\otimes_2\phi_{B,C}"] & F(A) \otimes (B \otimes_1 C) \arrow[u, "\phi_{A,B\otimes_1 C}"']
\end{tikzcd}$$

2. and the following diagrams commute for any object $A \in M_1$,

$$\begin{tikzcd} I_2 \otimes_2 F(A) \arrow[r, "\lambda_{F(A)}"] \arrow[d, "\omega \otimes_2 F(A)"'] & F(A) \\ F(I_1) \otimes_2 F(A) \arrow[r, "\phi_{I_1,A}"] & F(I_1 \otimes_1 A) \arrow[u, "F(\lambda_A)"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F(A) \otimes_2 F I_2 \arrow[r, "\rho_{F(A)}"] \arrow[d] & F(A) \\ F(A) \otimes_2 F(I_1) \arrow[r, "\phi_{A,I_1}"] & F(A \otimes_1 I_1) \arrow[u, "F(\rho_A)"'] \end{tikzcd}$$

We can define a symmetric monoidal functor in a similar fashion:

Definition 1.24. A *symmetric monoidal functor* is a monoidal functor, F, between two symmetric monoidal categories $(M_1, \otimes_1, I_1)$ and $(M_2, \otimes_2, I_2)$ that makes the following diagram commute for any $A, B \in M_1$:

$$\begin{tikzcd} & F(B) \otimes_2 F(A) \arrow[dr, "\phi_{B,A}"] & \\ F(A) \otimes_2 F(B) \arrow[ur, "\beta_{F(A),F(B)}"] \arrow[dr, "\phi_{A,B}"'] & & F(B \otimes_1 A) \\ & F(A \otimes_1 B) \arrow[ur, "F(\beta_{A,B})"'] & \end{tikzcd}$$

Finally we can extend the notion of monoidality to natural transformations:

Definition 1.25. A *monoidal natural transformation* between two functors $(F: M_1 \to M_2, \Phi, \phi)$ and $(G: M_1 \to M_2, \Gamma, \gamma)$ is a natural transformation $\mu: F \implies G$ that makes the two following diagrams commute for any objects $A, B \in M_1$:

$$\begin{tikzcd} F(A) \otimes_2 F(B) \arrow[r, "\Phi_{A,B}"] \arrow[d, "\mu_A \otimes_2 \mu_B"'] & F(A \otimes_1 B) \arrow[d, "\mu_{A \otimes_1 B}"] \\ G(A) \otimes_2 G(B) \arrow[r, "\Gamma_{A,B}"] & G(A \otimes_1 B) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I_2 \arrow[r, "\phi"] \arrow[d, equal] & F(I_1) \arrow[d, "\mu I_1"] \\ I_2 \arrow[r, "\gamma"] & G(I_1) \end{tikzcd}$$

Definition 1.26. A *symmetric monoidal natural transformation* is a monoidal natural transformation between two symmetric monoidal functors.

1.3 Topological Field Theory

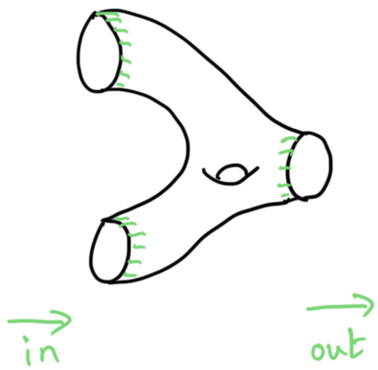
A bordism (or cobordism) between two manifolds is essentially a manifold with boundary of one higher dimension.

Definition 1.27 ([Fre12])**.** Let, M and N be closed $n-$manifolds. A *bordism* $(\Sigma, p, \theta_0, \theta_1): M \to N$ is a compact $(n+1)-$manifold with boundary, together with a map

$$p: \partial\Sigma \to \{0, 1\}$$

which partitions the boundary (into incoming boundary ∂_{in} and outgoing boundary ∂_{out}),

$$\begin{aligned} \partial\Sigma &= p^{-1}(0) \coprod p^{-1}(1) \\ &= \partial_{in}\Sigma \coprod \partial_{out}\Sigma \end{aligned}$$



and embeddings (choice of collars),

$$\begin{aligned} \theta_0&: [0, 1) \times M \to \Sigma \\ \theta_1&: (-1, 0] \times N \to \Sigma \end{aligned}$$

such that,

$$\begin{aligned} \theta_0(0, M) &= \partial_{in}\Sigma \\ \theta_1(0, N) &= \partial_{out}\Sigma \end{aligned}$$

Remark 1.28. The choice of these embeddings is contractible.

If we have two embeddings,

$$\Sigma\colon M_0 \to M_1$$
$$\Sigma'\colon M_1 \to M_2$$

then their composition is defined by gluing them together along M_1, $\Sigma \coprod_{M_1} \Sigma'$.

Remark 1.29. For each $n > 0$, the category **nCob** can be endowed with the structure of a symmetric monoidal category, where the tensor product operation

$$\otimes\colon \mathbf{nCob} \times \mathbf{nCob} \to \mathbf{nCob}$$

is given by the disjoint union of manifolds.
The unit object of **nCob** is the empty set (regarded as a manifold of dimension $(n-1)$).

Notation: We denote $(\Sigma, p, \theta_0, \theta_1)$ by just Σ for convenience when there is no confusion.

Definition 1.30. A *diffeomorphism* between bordisms $(\Sigma, p, \theta_0, \theta_1), (\Sigma', p', \theta_0', \theta_1')\colon M \to N$ is a diffeomorphism of smooth manifolds $F\colon M \to N$ with boundary, which commutes with p, θ_0, θ_1.

Example 1.31. If we take $\partial\Sigma$, then $F\mid_\partial$ gives a diffeomorphism, i.e. we have the following commutative diagram,

$$\begin{tikzcd}
\partial\Sigma \arrow[rr, "F|_\partial"] \arrow[dr, "p"'] & & \partial\Sigma' \arrow[dl, "p'"] \\
& \{0,1\} &
\end{tikzcd}$$

and the same for θ_0 and θ_1.

A topological (quantum) field theory is a functor from the symmetric monoidal category of bordisms **nCob** to the symmetric monoidal category of vector spaces, $\mathbf{Vect}_{\mathbf{k}}^{\otimes}$.

Definition 1.32. An *n-dimensional topological field theory* is a symmetric monoidal functor of categories,

$$Z\colon \mathbf{nCob} \to \mathbf{Vect}_{\mathbf{k}}^{\otimes}$$

Objects: closed $(n-1)$-dimensional smooth manifolds.
Morphisms: from M to N are diffeomorphic classes of bordisms from M to N.

Definition 1.33. Let M be an $n-$manifold. A *framing* of M is trivialization of the tangent bundle of M, i.e. an isomorphism $TM \simeq \underline{\mathbb{R}}^n$ of vector bundles over M; here $\underline{\mathbb{R}}^n$ denotes the trivial bundle with fiber $\mathbb{R}^n$.

Definition 1.34. The *framed bordism is denoted as $n-$category* $\mathbf{nCob^{fr}}$, and is defined in the same way as **nCob**, except that we require that all manifolds be equipped with an $n-$framing.

We can add different structure, like orientation, framing etc on the topological field theory and the definition changes accordingly, for example,

Definition 1.35 (Alternative Definition)**.** An *n-dimensional (oriented/framed) topological field theory* is a symmetric monoidal functor of categories,

$$Z\colon \mathbf{nCob^{or/fr}} \to \mathbf{Vect}_{\mathbf{k}}^{\otimes}$$

Objects: closed $(n-1)$-dimensional (oriented/framed) smooth manifolds.
Morphisms: from M to N are diffeomorphic classes of (oriented/framed) bordisms from M to N.

An $n-$framing on bordism Σ is an identificaton of tangent of Σ with the trivial bundle, i.e.

$$T\Sigma \simeq \Sigma \times \mathbb{R}^n$$

And an $n-$framing on an object M (Note: $(n-1)$-dimensional) is also an identification of the form,

$$T(M \times \mathbb{R}) \simeq (M \times \mathbb{R}) \times \mathbb{R}^n$$

Remark 1.36*.* We would like to have an $n-$framing on the objects, so that we are able to restrict the morphisms on the boundaries and extract the source and target objects.

Our aim is to understand Cobordism Hypothesis, which gives us information about classification of topological (quantum) field theories.

Example 1.37. Let us look at the classification of 1−dimensional oriented topological field theories for the time being.
Given,

$$Z\colon \mathbf{1Cob^{or}} \to \mathbf{Vect_k^{\otimes}}$$

let us explicity see examples of some 0−dimensional oriented objects and observe what happens to the incoming and outgoing boundaries for each of them.
The empty set ϕ is the symmetric monoidal unit, so it goes to the ground field k,

$$\phi \mapsto k$$

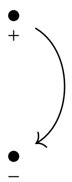
The incoming boundary $\partial_{in}\phi$ and outgoing boundary $\partial_{out}\phi$ of the empty set are also empty sets.
We can think of the orientation of a point by thinking of the point as an "interval". Hence, a point has a positive and a negative orientation.

$$\underset{+}{\bullet} \longrightarrow \underset{+}{\bullet} \;\; \mapsto V$$

for some vector space V.

$$\underset{-}{\bullet} \longleftarrow \underset{-}{\bullet} \;\; \mapsto W$$

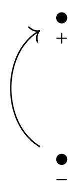
for some vector space W.
The incoming and outgoing boundaries for both of the above two figures have a single point.
Similarly, there are other nice cobordisms,



$$\to \phi \mapsto W \otimes V \xrightarrow{ev_V} k$$

There are two points in the incoming boundary and the outgoing boundary is an empty set.

$$\phi \to$$



$$\mapsto k \xrightarrow{coev_V} V \otimes W$$

Finally, there are two points in the outgoing boundary and the incoming boundary is empty.

Further, we can even replace the $\mathbf{Vect}_{\mathbf{k}}^{\otimes}$ by any symmetric monoidal categories to obtain a general definition of topological (quantum) field theories.

Definition 1.38 (Alternative Definition)**.** An *n-dimensional (oriented/framed) topological field theory* is a symmetric monoidal functor of categories,

$$Z\colon \mathbf{nCob} \to \mathbf{C}$$

where, $\mathbf{C}$ is any symmetric monoidal category.
Objects: closed $(n-1)$-dimensional (oriented/framed) smooth manifolds.
Morphisms: from M to N are diffeomorphic classes of (oriented/framed) bordisms from M to N.

Remark 1.39. We will explore in depth later what a correct $\mathbf{C}$ is!

Following this line of analogy the above classification of 1−dimensional oriented topological field theories then becomes,

Example 1.40. Given,

$$Z\colon \mathbf{1Cob^{or}} \to \mathbf{C}$$

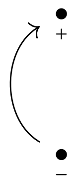
If the unit object of the arbitary symmetric monoidal category is denoted by 1, then the 0−dimensional oriented objects are defined as,

$$\phi \mapsto \mathbf{1}$$

$$\underset{+}{\bullet} \longrightarrow \underset{+}{\bullet} \;\mapsto V$$

$$\underset{-}{\bullet} \longleftarrow \underset{-}{\bullet} \;\mapsto W$$

$$\begin{matrix} \underset{+}{\bullet} \\ \\ \underset{-}{\bullet} \end{matrix} \to \phi \mapsto W \otimes V \xrightarrow{ev_V} \mathbf{1}$$

$\phi \to$  $\mapsto \mathbf{1} \xrightarrow{coev_V} V \otimes W$

Remark 1.41. The above evaluation map $ev_V: W \otimes V \to \mathbf{1}$ is called a *perfect pairing*, i.e. it induces an isomorphism between V and dual of W, and W and dual of V,

$$V \cong W^*$$
$$W \cong V^*$$

Remark 1.42. For $\mathbf{Vect}_{\mathbf{k}}^{\otimes}$, the evaluation map $ev_V: W \otimes V \to k$ is a perfect pairing $\implies V$ is finite dimensional.

Remark 1.43. The above two remarks are true even for $n-$dimensional oriented topological field theory $Z: \mathbf{nCob^{or}} \to \mathbf{C}$.

Remark 1.44. The converse of Remark 1.41 is also true, i.e. given a finite dimensional vector space, we can define an 1−dimensional oriented topological field theory.

Definition 1.45. An *dual of an object* $X \in \mathbf{C}$ is an object $Y \in \mathbf{C}$ together with an evaluation

$$ev_X: Y \otimes X \to \mathbf{1}$$

and coevaluation

$$coev_X: \mathbf{1} \to X \otimes Y$$

such that,

$$X \xrightarrow{coev_X \otimes Id} X \otimes Y \otimes X \xrightarrow{Id \otimes ev_X} X$$

is the identity Id_X and,

$$Y \xrightarrow{Id \otimes coev_X} Y \otimes X \otimes Y \xrightarrow{ev_X \otimes Id} Y$$

is the identity Id_Y.

2 Higher Categories

Idea: We would like to use enriched categories.
Let, $(\mathbf{S}, \otimes)$ be a monoidal category which is an enriched category.

Definition 2.1. An $\mathbf{S}-$*enriched category* $\mathbf{C}$ consists of,

1. a set of *objects*, which we denote by $\mathcal{O}(\mathbf{C})$
2. *morphisms*, $\forall X, Y \in \mathcal{O}(\mathbf{C})$, we have a hom object $Hom_{\mathbf{C}}(X,Y) \in \mathbf{S}$
3. *composition*, $\forall X, Y, Z \in \mathcal{O}(\mathbf{C})$ we have,

$$\circ_{XYZ} \colon Hom_{\mathbf{C}}(X,Y) \otimes Hom_{\mathbf{C}}(Y,Z) \to Hom_{\mathbf{C}}(X,Z)$$

4. *identity*, $\forall X \in \mathcal{O}(\mathbf{C})$, is a morphism

$$1 \xrightarrow{Id_X} Hom_{\mathbf{C}}(X,X)$$

5. *associativity*, $\forall X, Y, Z, W \in \mathcal{O}(\mathbf{C})$, we have the following commutative diagram,

$$\begin{tikzcd}
Hom_{\mathbf{C}}(X,Y) \otimes (Hom_{\mathbf{C}}(Y,Z) \otimes Hom_{\mathbf{C}}(Z,W)) \arrow[r, "1 \otimes \circ_{YZW}"] \arrow[dd, equal] & Hom_{\mathbf{C}}(X,Y) \otimes Hom_{\mathbf{C}}(Y,W) \arrow[d, "\circ_{XYW}"] \\
 & Hom_{\mathbf{C}}(X,W) \\
Hom_{\mathbf{C}}(X,Y) \otimes (Hom_{\mathbf{C}}(Y,Z)) \otimes Hom_{\mathbf{C}}(Z,W) \arrow[r, "\circ_{XYZ} \otimes 1"] & Hom_{\mathbf{C}}(X,Z) \otimes Hom_{\mathbf{C}}(Z,W) \arrow[u, "\circ_{XZW}"']
\end{tikzcd}$$

Remark 2.2. In composition we use monoidal product instead of the usual Cartesian product because $Hom_{\mathbf{C}}(X,Y)$ and $Hom_{\mathbf{C}}(Y,Z)$ are objects in $\mathbf{S}$, which may or may not be sets.

Example 2.3. Let us look at various categories we can obtain by changing $\mathbf{S}$,

1. If $(\mathbf{S}, \otimes) \coloneqq (\mathbf{Set}, \times)$, then we obtain small categories. Small categories are enriched over the category of sets, **Set** with Cartesian product as monoidal product.

2. If $(\mathbf{S}, \otimes) \coloneqq (\mathbf{Vect_k}, \otimes)$, then we obtain linear categories. Linear categories are enriched over the category of vector spaces over the field k, $\mathbf{Vect_k}$ with tensor product as monoidal product.

3. If $(\mathbf{S}, \otimes) \coloneqq (\mathbf{Ch}, \otimes)$, then we obtain dg-categories. Dg-categories are enriched over the category of chain complexes, $\mathbf{Ch}$ with tensor product as monoidal product.

4. If $(\mathbf{S}, \otimes) \coloneqq (\mathbf{Cat}, \times)$, then we obtain $2-$categories. 2-categories are enriched over the category of small categories, **Cat** with Cartesian product as monoidal product.

5. If $(\mathbf{S}, \otimes) \coloneqq (\mathbf{Top}, \times)$, then we obtain topological categories. Topological categories are enriched over the category of topological spaces, **Top** with Cartesian product as monoidal product.

6. If $(\mathbf{S}, \otimes) \coloneqq (\mathbf{sSet}, \times)$, then we obtain simplicial categories. Simplicial categories are enriched over the category of simplicial sets, **sSet** with Cartesian product as monoidal product.

Definition 2.4. The above Example 4 can be used as a definition for $2-$*categories*.

The notion of a $2-$category is a generalization of a category. For a standard definition of $2-$categories readers are advised to refer to [Tat18].

Example 2.5. The simplest example of a $2-$category is **Cat** where,
Objects: are categories
Morphisms: are functors
2-morphisms: are natural transformations.

Example 2.6. A symmetric monoidal category is a $2-$category, where,
Objects: are symmetric monoidal categories
Morphisms: are symmetric monoidal functors
2-morphisms: are symmetric monoidal natural transformations.

Definition 2.7. A *bicategory* has the same data as a $2-$category but with associativity and unitality hold upto $2-$isomorphisms.

Example 2.8. **Cat** is a bicategory in which the associators and unitors are identities.

Example 2.9. The category of bimodules over algebra, $\mathbf{Alg^{bi}}$ is a bicategory, where,
Objects: are algebras X, Y
Morphisms: are (X, Y)–bimodules, $Hom_{\mathbf{Alg^{bi}}}(X, Y)$
2-morphisms: are homorphisms of (X, Y)–bimodules
And a *composition* between an (B, C)–bimodule ${}_BN_C$ and (A, B)–bimodule ${}_AM_B$ is defined as a relative tensor product as,

$$ {}_BN_C \circ_A M_B \coloneqq_A (M \underset{B}{\otimes} N)_C $$

Example 2.10. The category of span of sets, $\mathbf{Span^{bi}}$ is a bicategory, where,
Objects: are sets X, Y
Morphisms: between objects X, Y is a pair of maps arising from some common source S,

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

2-morphisms: between two morphisms $\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$ and $\begin{tikzcd} & S' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ X & & Y \end{tikzcd}$
is an isomorphism class of maps that makes both halves of the following diagram commute,

$$\begin{tikzcd} & S' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ X & \Sigma \arrow[u] \arrow[d] & Y \\ & S \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

And a *composition* between two morphisms $\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$ and $\begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ Y & & Z \end{tikzcd}$
is defined by taking a pullback in the following way,

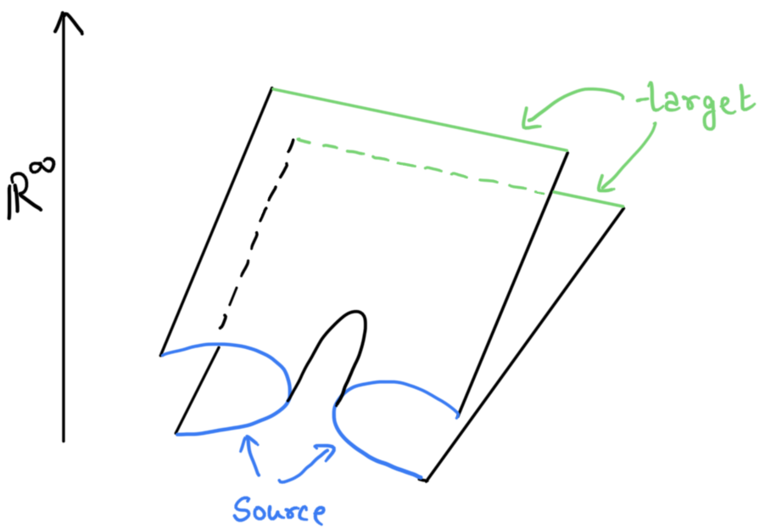
$$\begin{tikzcd}
& & S \underset{Y}{\times} T \arrow[dl] \arrow[dr] & & \\
& S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

Remark 2.11. In the last example for 2−morphisms we need to take isomorphism clases of maps and not pairs of maps, because we will need the morphisms to compose.

Example 2.12 ([SP14])**.** The category of 2-cobordisms, **2Cob** is a bicategory, where,
Objects: finite set of points, i.e. closed 0−dimensional compact manifolds
Morphisms: is a bordism $\Sigma\colon M \to N$, where M and N are closed 0−manifolds, such that,

$$\partial\Sigma = \underbrace{\partial_{in}\Sigma}_{source} \coprod \underbrace{\partial_{out}\Sigma}_{target}$$

2-morphisms: are isomorphism classes of 2−dimensional bordisms between bordisms, i.e. 2−dimensional manifold with corners, which can be embedded in $[0,1]^2 \times \mathbb{R}^\infty$



Remark 2.13. Example 2.12 is a symmetric monodial bicategory.

In principle we can replace 2 by n in Example 2.12 and obtain an $n-$Category of bordisms but then we will have $(n+1)-$layers instead of $3-$layers as we have in case of bicategories. This is basically, the origin of trying to build an **nCob** and has a very long history. But it is already complicated to write down how a symmetric monoidal $3-$category looks like.
To overcome this difficulty, we will talk about higher category in a different flavor, i.e. instead of adding $k-$morphisms upto n and getting an $n-$Category, we will add all $k-$morphisms simultaneously.

Question 2.14. How can we add all $k-$morphisms $\forall k \geq 1$?

We start with the situation, where all $k-$morphisms are invertible. This is called an $\infty-$groupoid.

Remark 2.15. The "∞" in $\infty-$groupoid refers to the fact that we have $k-$morphisms for all k.

We follow Grothendieck's idea [GpGM10]: given a topological space X, we can associate to it the fundamental groupoid $\pi_{\leq 1}X$, whose,
Objects: are points in X
Morphisms: are homotopy classes of paths in X.

Similarly, given a topological space X we can associate to it the fundamental $2-$groupoid $\pi_{\leq 2}X$, whose,
Objects: are points in X
Morphisms: are paths in X
2-morphisms: are homotopy classes of homotopies between paths.

Remark 2.16. The fundamental $2-$groupoid $\pi_{\leq 2}X$ is a bicategory.

$\vdots$

Continuing in this fashion, given a topological space X we can associate to it the fundamental $\infty-$groupoid $\pi_{\leq \infty}X$.

Homotopy Hypothesis: An $\infty-$groupoid is a space, i.e. it is a space upto weak equivalence.

Remark 2.17. Since we do not have a concrete definition of $\infty-$groupoid, we use the homotopy hypothesis as a definition! But, if we obtain an equivalence of some definition with the homotopy hypothesis, then we can use that as a defintion of

$\infty-$groupoid, and the the homotopy hypothesis will then become a theorem in regard to this definition.

We can use this knowledge to define an $(\infty,1)-$category $\mathcal{C}$.

Remark 2.18. The "∞" and "1" in $(\infty,1)-$category refers to the fact that we have $k-$morphisms for all k and this $k-$morphisms are invertible for all $k > 1$ respectively.

For fixed objects $X, Y \in \mathcal{O}(\mathcal{C})$, we have $Hom_{\mathcal{C}}(X,Y)$, whose,
Objects: are morphisms in $\mathcal{C}$, $X \to Y$
Morphisms: are 2−morphisms in $\mathcal{C}$, which are invertible by definition of $(\infty,1)-$category.
Hence, they are $\infty-$groupoids, i.e. by homotopy hypothesis they are spaces.

Definition 2.19. An $(\infty,1)-$*category* is a category enriched in spaces.

Remark 2.20. By spaces, we would generally mean compactly generated Hausdorff spaces, **CGHaus** or simplicial sets, **sSet**.
We are able to take either **CGHaus** or **sSet** because their model categories are equivalent, i.e. the geometric realization functor $|-|$ is left adjoint to the singular complex functor $\mathcal{S}$,

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \mathbf{CGHaus} \arrow[l, bend left, "\mathcal{S}"] \end{tikzcd}.$$

Example 2.21. CGHaus is an $(\infty,1)-$category.

Example 2.22. sSet is an $(\infty,1)-$category.

Example 2.23 ([Lur09a])**.** Any combinatorial simplicial model category is a presentation of $(\infty,1)-$category.

Example 2.24. Cobordism category are $(\infty,1)-$category.

Theorem 2.25 (Whitney's Embedding Theorem)**.** *Any smooth $n-$manifold can be smoothly embedded in* $\mathbb{R}^{\infty} = \varinjlim_{k\to\infty} \mathbb{R}^k$.

Since, we are interested in understanding cobordism, we first modify the above theorem for manifolds with boundary and corners [Lau00] in the following way,

Theorem 2.26 (Whitney's Embedding Theorem)**.** *Any smooth $n-$manifold with boundary can be smoothly embedded in* $\mathbb{R}_{\geq 0} \times \mathbb{R}^{\infty}$.

Theorem 2.27 (Whitney's Embedding Theorem)**.** *Any smooth n−manifold with corners of dimension k can be smoothly embedded in* $(\mathbb{R}_{\geq 0})^k \times \mathbb{R}^\infty$.

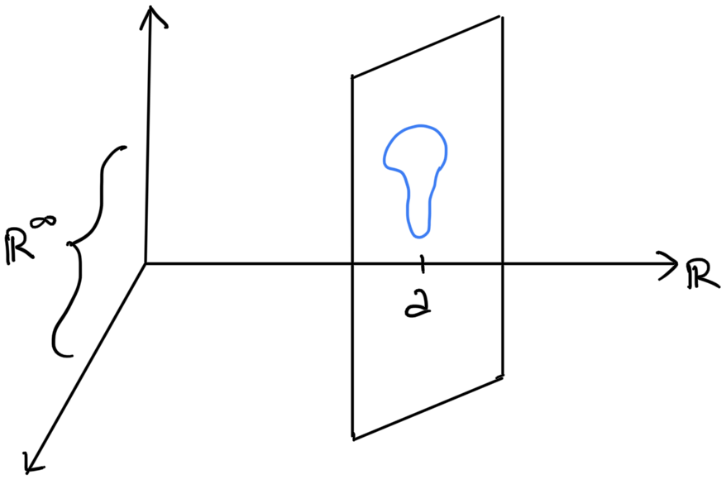
We can finally modify this theorem for bordisms in the following way,

Theorem 2.28 (Whitney's Embedding Theorem)**.** *Any bordism can be smoothly embedded in* $[0,1] \times \mathbb{R}^\infty$.

Remark 2.29. The smooth embedding of a bordism Σ in $[0,1] \times \mathbb{R}^\infty$ is such that the incoming and outgoing boundary of Σ is over 0 and 1 respectively of $[0,1] \times \mathbb{R}^\infty$.

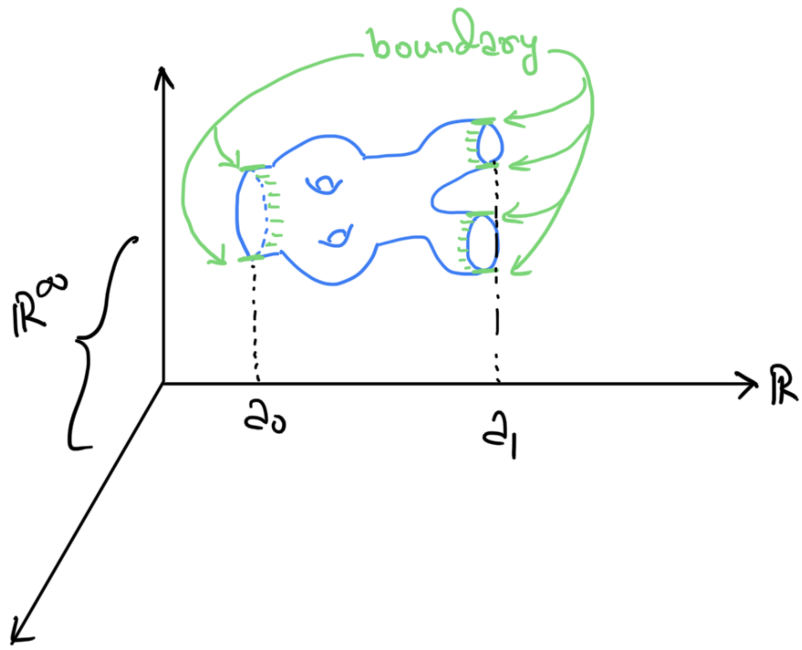
Remark 2.30. The space of these embeddings is contractible.

Definition 2.31 ([GMTW09])**.** The category of n−bordisms, **nCob** is defined as,
Objects: are pairs (M,a), where M is closed $(n-1)$−dimensional sub-manifold of $\mathbb{R}^\infty$ and $a \in \mathbb{R}$



Morphisms: from (M_0,a_0) to (M_1,a_1) is a triple (W,a_0,a_1), where $a_0 \leq a_1$ and W is a compact sub-manifold of $[a_0,a_1] \times \mathbb{R}^\infty$ with boundary such that,

1. $\partial W = W \cap (\{a_0,a_1\} \times \mathbb{R}^\infty)$
2. $\exists \epsilon > 0$ such that, $W \cap ([a_0,a_0+\epsilon) \times \mathbb{R}^\infty) = [a_0,a_0+\epsilon) \times M_0$
3. $\exists \epsilon > 0$ such that, $W \cap ((a_1-\epsilon,a_1] \times \mathbb{R}^\infty) = (a_1-\epsilon,a_1] \times M_1$



Composition: is done by gluing cylinders.

Remark 2.32. Points (2) and (3) gives the cylindricity of the embedding.

We would like to topologize **nCob** for which we use the following topology, for a manifold M the embedding of M into $\mathbb{R}^\infty$ is a colimit,

$$Emb(M, \mathbb{R}^\infty) = \varinjlim_{k\to\infty} Emb(M, \mathbb{R}^k)$$

Now we can endow the right had side with the Whitney C^∞−topology. But, we would like to have sub-manifold and not embeddings, for which we act freely on M by diffeomorphism of M, $Diff(M)$. So, we have the following quotient, which we denote by $B_\infty(M)$,

$$Emb(M, \mathbb{R}^\infty) \to Emb(M, \mathbb{R}^\infty)/Diff(M) =: B_\infty(M)$$

This will actually be a principle $Diff(M)$−bundle.
By the Whitney's embedding theorem, we know that the space of embeddings is contractible, so $Emb(M, \mathbb{R}^\infty)$ is contractible.
This means, that $B_\infty(M)$ is a model for the classifying space of diffeomorphisms of M, $BDiff(M)$.
And this is exactly what we would want to use to topologize **nCob**. Hence, we have now,

Objects:= $\mathbb{R} \times \coprod_{Diff(M)} B_\infty(M)$
Morphisms:= $\{a_0, a_1\} \times \coprod_W B_\infty(W) = \{a_0, a_1\} \times \coprod_W Emb(W, \mathbb{R}^\infty)/Diff(W)$.

Remark 2.33. This is not a category enriched in spaces, but rather a category internal to spaces.

Remark 2.34. It is still an $(\infty, 1)$−category, because category internal to spaces are also a model of $(\infty, 1)$−category [Hor15].

Let us look at another a very important model of $(\infty, 1)$−categories due to Charles Rezk [Rez01].

2.1 Complete Segal Spaces

Complete Segal spaces is introduced by Rezk [Rez01] and and was later shown by Joyal and Tierney [JT06] to be a model for $(\infty, 1)$−categories.

Definition 2.35. Let $\mathbf{\Delta}$ be the *category of finite ordinal numbers*, with objects as finite, non-empty, totally ordered sets,

$$[0] \coloneqq \{0\}, [1] \coloneqq \{0 \to 1\}, \dots, [n] \coloneqq \{0 \to 1 \to \dots \to n\}, \dots$$

and morphisms are order preserving maps (set functions) between them.

Simplicial sets, introduced by Eilenberg and Zilber [EZ50] are a generalization of geometric simplicial complexes that appears in algebraic topology, and provide a combinatorial model for the (weak) homotopy type of topological spaces.

Definition 2.36. A *simplicial set* X is a presheaf on the category of finite ordinal numbers $\mathbf{\Delta}$

$$\begin{aligned} X \colon \mathbf{\Delta}^{op} &\to \mathbf{Set} \\ [n] &\mapsto X([n]) \\ ([n] \xrightarrow{f} [m]) &\mapsto (X([m]) \xrightarrow{X(f)} X([n])) \end{aligned}$$

Notation: We denote $X([n])$ as X_n for convenience and call its elements n−*simplices*. Simplicial sets form a category, which we will denote by **sSet**, where the objects are simplicial sets and morphisms are natural transformation between the corresponding functors.

Example 2.37. Let, $\mathcal{C}$ be a small category. The *nerve* of $\mathcal{C}$ is a **sSet** and is denoted as $\mathcal{N}\mathcal{C}$ and is defined levelwise as $\mathcal{N}\mathcal{C}_n \coloneqq Hom_{Cat}([n], \mathcal{C})$, explicitly:

- $\mathcal{N}\mathcal{C}_0 \coloneqq$ objects of $\mathcal{C} = \{\bullet\} \eqqcolon \mathcal{O}$
- $\mathcal{N}\mathcal{C}_1 \coloneqq$ morphisms of $\mathcal{C} = \{\bullet \to \bullet\} \eqqcolon \mathcal{M}$
- $\mathcal{N}\mathcal{C}_2 \coloneqq$ pairs of composable morphisms of $\mathcal{C} = \{\bullet \to \bullet \to \bullet\} \eqqcolon \mathcal{M} \underset{\mathcal{O}}{\times} \mathcal{M}$

$\vdots$

- $\mathcal{N}\mathcal{C}_n \coloneqq$ strings of $n-$composable morphisms of $\mathcal{C} = \{\bullet \to \ldots \to \bullet\} \eqqcolon \underbrace{\mathcal{M} \underset{\mathcal{O}}{\times} \ldots \underset{\mathcal{O}}{\times} \mathcal{M}}_{(n+1)-\ factors}$.

Example 2.38. The *standard n−simplex* Δ^n are the simplest example of **sSet**. For all $n \geq 0$, it is a representable functor for each object $[n]$ of $\mathbf{\Delta}$.
By the Yoneda embedding,

$$\mathbf{\Delta} \hookrightarrow \mathbf{Set}^{\mathbf{\Delta}^{op}} \eqqcolon \mathbf{sSet}$$
$$[n] \to \mathbf{\Delta}(-, [n]) \eqqcolon \Delta^n$$
$$([n] \to [m]) \mapsto (\Delta^n \to \Delta^m)$$

It is defined levelwise as, $\Delta^n_k = \mathbf{\Delta}([k], [n])$.

By the Yoneda lemma, we can classify $n-$simplices X_n of a simplicial set X, since there is a natural bijection between X_n and the natural transformation between Δ^n and X,

$$X_n \cong Hom_{\mathbf{sSet}}(\Delta^n, X).$$

Definition 2.39. The k^{th} *horn* Λ^n_k $\forall 0 \leq k \leq n$ is a sub-simplicial set, which is defined as,

$$\Lambda^n_k \coloneqq \bigcup_{i \neq k} \partial_i \Delta^n.$$

Definition 2.40. A simplicial set X is called a *Kan complex* if every map $\Lambda^n_k \to X$ from a horn has an extension to Δ^n, i.e. there is a lift $\Delta^n \to X$ such that

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"] & \end{tikzcd}$$

Definition 2.41. A *simplicial space* (or *bisimplicial set*) X is a contravariant functor from $\mathbf{\Delta}$ to **sSet**

$$\begin{aligned} X\colon \mathbf{\Delta}^{op} &\to \mathbf{sSet} \\ [n] &\mapsto X([n]) \\ ([n] \xrightarrow{f} [m]) &\mapsto (X([m]) \xrightarrow{X(f)} X([n])). \end{aligned}$$

Remark 2.42. Here, we treat simplicial sets **sSet** as spaces and not category, more precisely as Kan complexes.

Notation: We denote $X([n])$ as X_n for convenience and call its elements $n-$*simplicial object*.
Simplicial spaces form a category, which we will denote by **sS**, where the objects are simplicial spaces and morphisms are natural transformation between the corresponding functors.

Remark 2.43. Simplcial spaces are also called *bisimplicial sets* on a categorical level! It can be observed from the following adjunction,

$$Fun(\Delta^{op} \times \Delta^{op}, \mathbf{Set}) \cong Fun(\Delta^{op}, Fun(\Delta^{op}, \mathbf{Set})) =: Fun(\Delta^{op}, \mathbf{sSet}).$$

Example 2.44. The $k^{\text{th}}-$*space functor* $F(k)$ is a discrete simplcial space and is defined levelwise as,

$$(F(k))_{nl} \coloneqq Hom_{\mathbf{\Delta}}([n],[k])$$

while, the *standard* $k-$*simplex* Δ^k is not a discrete simplicial space and is defined levelwise as,

$$(\Delta^k)_{nl} \coloneqq Hom_{\mathbf{\Delta}}([l],[k])$$

Remark 2.45. $F(n) \times \Delta^l$ is the generator of the simplicial space. Explicitly, $F(n)$ generated the columns and Δ^l generates the rows of the simplicial space, and together $F(n) \times \Delta^l$ generate the entire simplicial space.

By the Yoneda lemma, we can classify a simplicial space X into simplicial sets X_n, since there is a natural isomorphism between X_n and the natural transformation between $F(n)$ and X,

$$Map_{\mathbf{sS}}(F(n), X) \cong X_n$$

Remark 2.46. The category of simplicial spaces is *Cartesian closed*, i.e. for $X, Y, Z \in \mathbf{sS}$,

$$Map_{\mathbf{sS}}(X \times Y, Z) \cong Map_{\mathbf{sS}}(X, Z^Y)$$

In particular the simplicial space $Y^X \in \mathbf{sS}$ is defined as,

$$(Y^X)_{nl} \coloneqq Hom_{\mathbf{sS}}(F(n) \times \Delta^l \times X, Y)$$

Definition 2.47. Let, X, Y be $\mathbf{sS}$, then, the simplicial morphism $f: X \to Y$ on the *Reedy model structure on* $\mathbf{sS}$ [Ree74] are defined as,

- fibrations are map $f: X \to Y$ such that $\forall k \geq 0$, the induced map

  $$Map_{\mathbf{sS}}(F(k), Y) \to Map_{\mathbf{sS}}(F(k), X) \underset{Map_{\mathbf{sS}}(\partial F(k), X)}{\times} Map_{\mathbf{sS}}(\partial F(k), Y)$$

  is a *fibration of* $\mathbf{sSet}$.
- cofibrations are *monomorphisms*, i.e. $f_n: X_n \to Y_n$ are levelwise injective maps $\forall n \geq 0$ of $\mathbf{sSet}$
- weak equivalences are *levelwise weak equivalence of* $\mathbf{sSet}$.

Definition 2.48. A simplicial space X is called *Reedy fibrant* if $\forall n, l \geq 0$ and $0 \leq i \leq n$, the following diagram,

$$\begin{tikzcd}
\partial F(n) \times \Delta^l \coprod_{\partial F(n) \times \Lambda^l_i} F(n) \times \Lambda^l_i \arrow[r] \arrow[d] & X \\
F(n) \times \Delta^l \arrow[ur, dashed, "\exists"] &
\end{tikzcd}$$

has the right lifting property.

Example 2.49. $F(n)$ is a Reedy fibrant simplicial space $\forall n \geq 0$.

The definition of Segal spaces originally appeared in Section 4 in [Rez01] observing the Segal condition in [Seg68] following Grothendieck [Gro61].

Definition 2.50. A Reedy fibrant simplicial space X is a *Segal space* if the map,

$$X_n \xrightarrow{\simeq} \underbrace{X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1}_{n-\ factors}$$

is a Kan equivalence $\forall n \geq 2$.

The above definition can be represented as a limit diagram $\forall n \geq 2$

$$\begin{tikzcd}
 & & & & X_n \arrow[dlll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drrr] & & & & \\
 & X_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & & X_1 \arrow[dl, "d_0"'] & \dots & X_1 \arrow[dr, "d_1"] & & X_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\
X_0 & & X_0 & \dots & \dots & \dots & X_0 & & X_0
\end{tikzcd}$$

Segal spaces classify a simplicial space X as follows,

- X_0 represents the *space of objects* of X
- X_1 represents the *space of maps* of X
- X_2 represents the *space of pairs of composable maps* of X, i.e. $X_2 \xrightarrow{\simeq} X_1 \underset{X_0}{\times} X_1$

$\vdots$

- X_n represents the *space of $n-$composable morpsisms* of X, i.e. $X_n \xrightarrow{\simeq} \underbrace{X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1}_{n-\ factors}.$

Definition 2.51 (Informally)**.** X_0 encodes the underlying $\infty-$groupoid.

Having, defined Reedy fibrancy condition, Segal condition and completeness condition, we are ready to define what complete Segal spaces are.

Definition 2.52. A complete Segal space X is a simplicial space which satisfies the following condition:

- Reedy fibrancy condition
- Segal condition
- Completeness condition

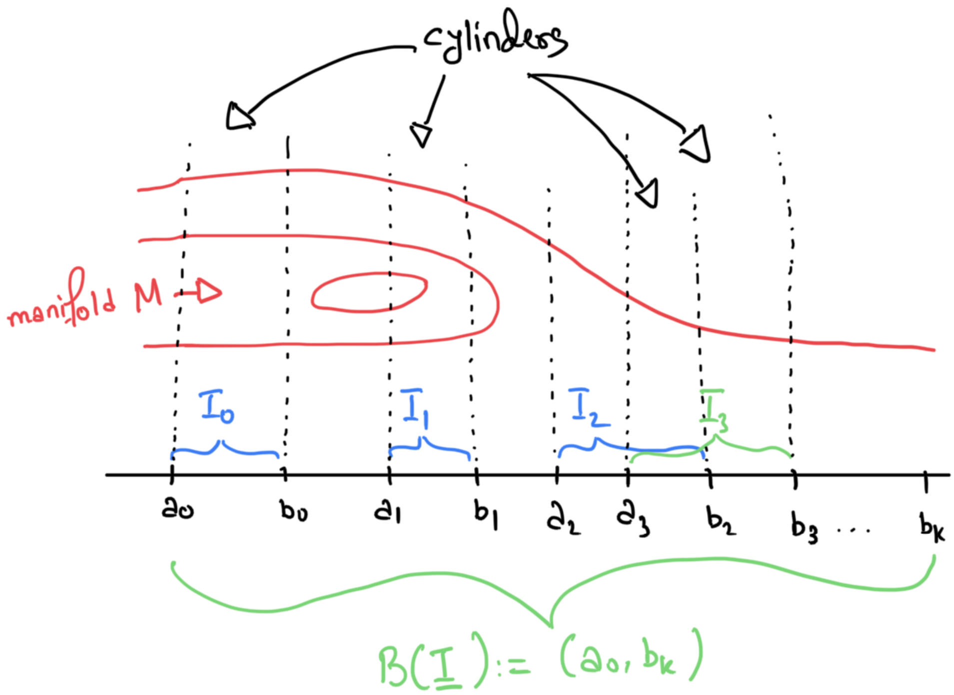
This is a very brief and non-technical introduction to the complete Segal space. The treatise of complete Segal spaces is interesting but not what we want to develop in this project. Interested readers are advised to refer to the original paper by Rezk [Rez01] or to my Master Thesis titled "Complete Segal Space as a model for Higher Categories".

3 Factorization Homology

3.1 Bordisms as a Segal space

Idea: nCob is a category internal to spaces, and acting on it by the nerve functor we get a simplical space, i.e. $\mathcal{N}(\mathbf{nCob})$ is a simplicial space. Infact, we will see that it will be a Segal space.
But, we will not follow this path! Instead, we will modify this definition a little bit, because our goal is to understand how factorization homology gives a topological (quantum) field theory. We do so by first defining the $(\infty,1)$−bordism category, which is a generalization of **1Cob**.

Definition 3.1. The $(\infty,1)$−*bordism category*, $X = Bord_n^{(\infty,1)}$ is defined levelwise as,
The k^{th}−space is defined as a pair of manifold M and $\underline{I} = (I_0 \leq I_1 \leq \ldots \leq I_k)$, where, I_i are intervals $[a_i, b_i]$ and, $a_i \leq a_j$ and $b_i \leq b_j$ $\forall i,j$ and $a_i, b_i \in \mathbb{R}$,



$$X_k \coloneqq (M, \underline{I})$$

where $M \subset B(\underline{I}) \times \mathbb{R}^\infty$, for $B(\underline{I}) = (a_0, b_k)$ and p_1 is the first projection map, so we have,

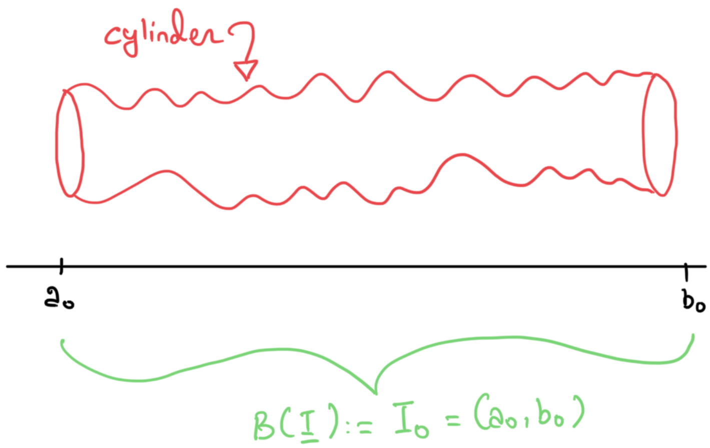
$$\begin{tikzcd} M \arrow[r, hook] \arrow[dr, "\pi"'] & B(\underline{I}) \times \mathbb{R}^\infty \arrow[d, "p_1"] \\ & B(\underline{I}) \end{tikzcd}$$

such that,

1. π is a proper map
2. over $(I_i \coloneqq [a_i, b_i]) \cap B(\underline{I})$, π is submersive

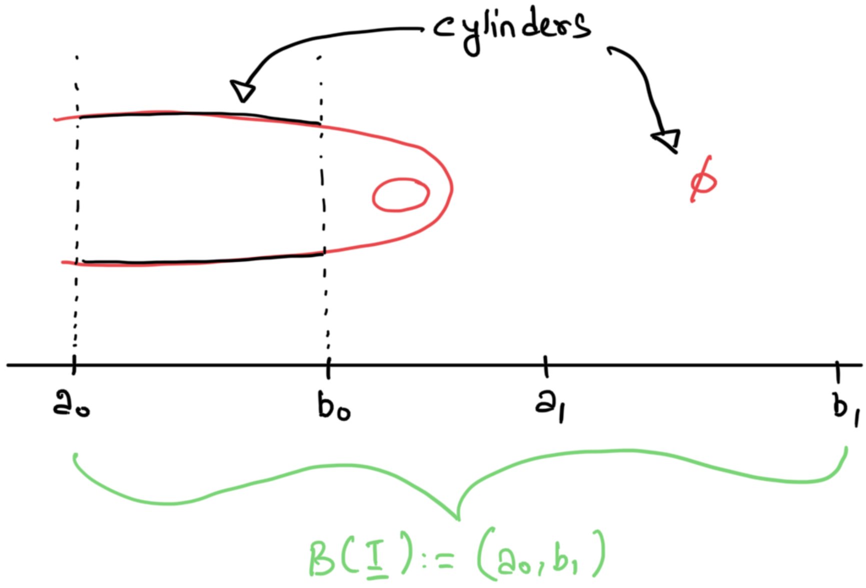
Remark 3.2. As opposed to Definition 2.31, the interval I_i replaces choosing a point $a \in \mathbb{R}$ and the manifold M now sits inside $B(\underline{I}) \times \mathbb{R}^\infty$ instead of $\{a\} \times \mathbb{R}^\infty \cong \mathbb{R}^\infty$. Also, point (2) above replaces cylindricity in Definition 2.31 given by points (2) and (3).

In particular, the 0^{th}–level X_0, which denotes the *space of objects* of the bordism category $Bord_n^{(\infty,1)}$ is the pair $(M, \underline{I} \coloneqq I_0)$, where M is a manifold and I_0 is the interval $I_0 = (a_0, b_0)$ for $a_0, b_0 \in \mathbb{R}$ and $B(I_0) = (a_0, b_0)$,



such that, $\pi^{-1}(B(I_0)) \cong$ cylinder over $\pi^{-1}(x)$ for any $x \in I_0$.

The 1^{th}–level X_1, which denotes the *space of morphisms* of the bordism category $Bord_n^{(\infty,1)}$ is the pair $(M, \underline{I} \coloneqq (I_0 \leq I_1))$, where M is a manifold and I_i is the interval $I_i = (a_i, b_i)$ for $a_i, b_i \in \mathbb{R}$ $\forall i = \{0, 1\}$ and $B(\underline{I}) = (a_0, b_1)$,



For more similar construction interested readers are advised to refer to [CS19].

Remark 3.3. For the purpose of building a Segal space, we could have just used points instead of intervals like in Definition 2.31, but this intervals I_i will be useful in the construction of topological (quantum) field theories.

There are in general many model of (∞, n)−categories and we are choosing one of them which will be suitable for bordisms. n−fold Segal spaces were introduced by Clark Barwick in his thesis [Bar05].

Definition 3.4. An (∞, n)−category is a complete n−fold Segal space, i.e. an n−fold simplicial space $X\colon (\mathbf{\Delta}^{\mathrm{op}})^n \to \mathit{Space}$, such that,

1. If we fix all but one component then the simplicial space $X_{k_1,\ldots,k_{i-1},-,k_{i+1},\ldots,k_n}\colon \mathbf{\Delta}^{op} \to \mathit{Space}$ is a complete Segal space
2. If one of the component is 0 then the simplicial space $X_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}\colon \mathbf{\Delta}^{op} \to \mathit{Space}$ is levelwise equivalent to a constant $(n-i)$−fold Segal space, i.e.

$$X_{k_1,\ldots,k_{i-1},0,0,\ldots,0} \xrightarrow[\simeq]{\mathit{degeneracy}} X_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$$

In particular, let us look at the case when $n = 2$.

Definition 3.5. An $(\infty, 2)$−category is a complete 2−fold Segal space such that,

1. $X_{-,k}$ and $X_{k,-}$ is a complete Segal space.

2. $X_{0,-}$ is essentially constant, i.e.,

$$X_{0,0} \xrightarrow{\simeq} X_{0,k}$$

In particular, in the $(\infty, 2)$−categories,
Objects: are given by $X_{0,0}$
Morphisms: are given by $X_{1,0}$
2-morphisms: are given by $X_{1,1}$

Remark 3.6. We have 2−morphisms as paths in $X_{1,0}$ and as points in $X_{1,1}$.

Remark 3.7. The completeness condition would imply that $X_{1,0}$ is equivalent to groupoid of invertible 2−morphisms from $X_{1,1}$.

Lurie had the idea [Lur09b] of using complete n−fold Segal spaces to define the bordism categories. We will essentially add a bunch of sets of intervals instead of adding one set of interval as in Definition 3.8.

Definition 3.8. The (∞, n)−*bordism category*, $X = Bord_n^{(\infty,n)}$ is defined levelwise as, a pair of manifold M and bunch of sets of intervals $(\underline{I}^i)_{1\leq\ldots\leq n}$, defined as earlier as $X_{k_1,\ldots,k_n} := (M, (\underline{I}^i)_{1\leq\ldots\leq n})$, where, $M \subset B((\underline{I}^i)_{1\leq\ldots\leq n}) \times \mathbb{R}^\infty$, where, $B((\underline{I}^i)_{1\leq\ldots\leq n}) := (a_0, b_{k_1}) \times \ldots \times (a_0, b_{k_n})$ and p_1 is the first projection map, so we have,

$$\begin{tikzcd} M \arrow[r, hook] \arrow[dr, "\pi"'] & B((\underline{I}^i)_{1\leq\ldots\leq n}) \times \mathbb{R}^\infty \arrow[d, "p_1"] \\ & B((\underline{I}^i)_{1\leq\ldots\leq n}) \end{tikzcd}$$

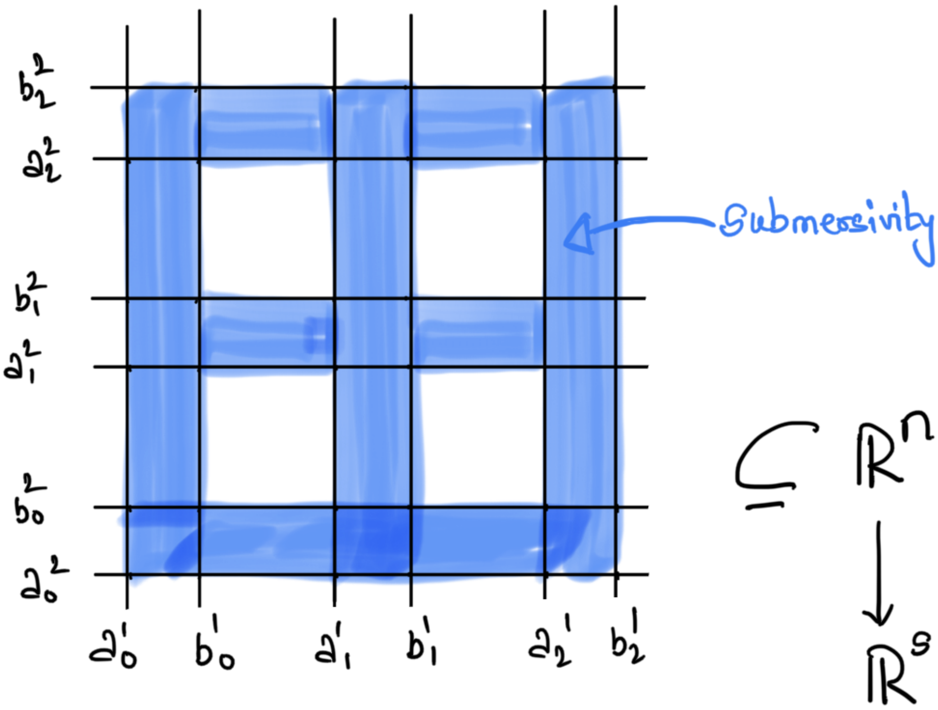
such that,

1. π is a proper map

2. for $s \in \{1, \ldots n\}$ then $p_s \colon M \xrightarrow{\pi} B((\underline{I}^i)_{1\leq\ldots\leq n}) \to \mathbb{R}^s$ at $x \in p_i^{-1}(I_j^i)$, then $p_{i_{1,\ldots,n}}$ is submersive

Remark 3.9. The (∞, n)−*bordism category*, $Bord_n^{(\infty,n)}$ is not complete.

In particular, for the case $n = 2$, we have a diagram looking like the following,



3.2 Factorization Homology

Definition 3.10. The topologically enriched category or ((∞, 1)−category) of *framed $n-$manifold $Mfld_n^{fr}$* has,

Objects: are smooth $n-$manifolds with a framing ($TM \simeq M \times \mathbb{R}^n$), such that we have the following non-commutative diagram,

$$\begin{tikzcd} & * \arrow[d, "p"] \\ M \arrow[ur, dashed, "\phi"] \arrow[r, "\tau_M"'] & BGL_n \end{tikzcd}$$

such that there is a homotopy, $p\phi \simeq \tau_M$

Morphisms: are space of embeddings compatible with framing, i.e. we have a pull-back of $(\infty, 1)-$categories,

$$\begin{tikzcd}
Mfld_n^{fr} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Top/* \arrow[d] \\
Mfld_n \arrow[r] & Top/BGL_n
\end{tikzcd}$$

$$M \longmapsto \tau_M$$

Idea: Manifolds $Mfld$ are build out of disks $Disk$, so if we want to associate something to a manifolds, we need to know what it does on the disks.

If we take *framed* $n-$*disks* $Disk_n^{fr} \subset Mfld_n^{dr}$ then,

Objects: are finite disjoint unions of framed disks, i.e. they are diffeomorphic to $\coprod_k \mathbb{R}^n$.

Remark 3.11. Framed $n-$manifold $Mfld_n^{fr}$ and framed $n-$disks $Disk_n^{fr}$ are both symmetric monoidal categories and we denote them as $Mfld_n^{fr,\amalg}$ and $Disk_n^{fr,\amalg}$ respectively to emphasis on the symmetric monoidality.

Definition 3.12. An E_n-*algebra* in a symmetric monoidal $(\infty,1)-$categories $\mathcal{S}$ is a symmetric monoidal functor $\mathcal{A}\colon Disk_n^{fr,\amalg} \to (\mathcal{S},\otimes)$.

Example 3.13. If $n = 1$ and $\mathcal{S} = \mathbf{Vect}$, then the E_n-algebra $\mathcal{A}$ are just associative algebras.

Example 3.14. If $n = 1$ and $\mathcal{S} = \mathbf{Ch}$ then the E_n-algebra $\mathcal{A}$ are $A_\infty-$algebras.

Example 3.15. If $n = 2$ and $\mathcal{S} = \mathbf{Cat}$, then the E_n-algebra $\mathcal{A}$ are braided monoidal categories.

Example 3.16. If n is any and $\mathcal{S} = \mathbf{Top}$ then the E_n-algebra $\mathcal{A}$ are $n-$fold loop spaces.

Now we are ready to give our main definition.

Definition 3.17 ([AFT16])**.** Given $\mathcal{A}$ as above, a *factorization homology* is the left Kan extension of the colimit, $\varinjlim_{Disk_n^{fr}/M} A\colon Mfld_n^{fr,\amalg} \to (\mathcal{S},\otimes)$, such that,

$$\begin{tikzcd}
Disk_n^{fr,\amalg} \arrow[r, "\mathcal{A}"] \arrow[d, hook] & (\mathcal{S},\otimes) \\
Mfld_n^{fr,\amalg} \arrow[ur, dashed, "\varinjlim_{Disk_n^{fr}/M} A"']
\end{tikzcd}$$

Remark 3.18 ([GTZ12]). Factorization homology is a generalization of higher Horschild homology for E_∞–algebras $\mathcal{R}$ as,

$$\begin{tikzcd}
Fin^{\amalg} \arrow[r, "\mathcal{R}"] \arrow[d, hook] & (\mathcal{S}, \otimes) \\
\mathbf{sSet} \arrow[ur, dashed] &
\end{tikzcd}$$

where, $Fin^{\amalg}$ denotes the symmetric monoidal category of finite sets and $\mathcal{R}\colon Fin^{\amalg} \to (\mathcal{S}, \otimes)$ is the symmetric monoidal functor.

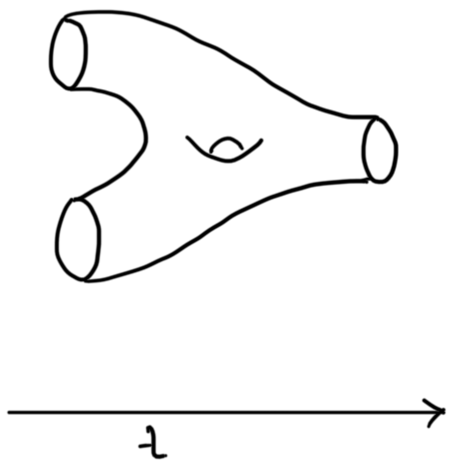
Remark 3.19. Factorization homology is a generalized theory for manifolds

4 Relationship to Physics

We will see how the mathematical concept of topological (quantum) field theories is related to Quantum Physics and General Relativity. Interested readers are advised to refer to the Master Thesis of Jaume Baixas Estradé [QTEC20] or to the paper by Baez [Bae06] for further details.

The key to grasp the importance of topological (quantum) field theories as proposed by Witten and Atiyah is to understand that both quantum processes and spacetime can be described categorically in a similar way. At first glance, general relativity and quantum theory "use different sorts of mathematics", says Baez: "one is based on objects such as manifolds, the other on objects such as Hilbert spaces".

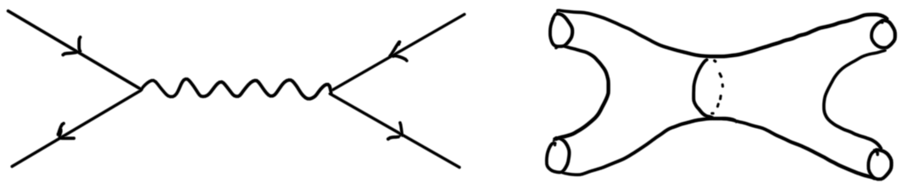
Under the assumption that neither space nor spacetime have a fixed topology- an idea that general relativity put on the table- manifolds of dimension n are used to model spacetime, and submanifolds of dimension $(n-1)$, space at a given instant. Thus, a bordism can represent an evolution of the space along the time.



Spacetime as a bordism

On the other hand, we want to establish a connection between quantum states and vector spaces. In fact, Quantum Mechanics is mathematically formulated by means of complex Hilbert spaces, a $\mathbb{C}$-vector spaces with further structure: the possible states of a quantum system are associated with the unit vectors of a Hilbert space. Moreover, the processes that occur between such sets of states are described by bounded linear maps (or more commonly operators). Therefore topological (quantum) field theories, in this case, will be understood as functors from $Bord_n$ to **Hilb**, the category of the aforementioned vector spaces.

The resemblances between these two categories become more apparent when we considering the *Feynman diagrams*, which are used to visualize operators: intuitively these diagrams exhibit the analogy between Quantum Physics and Topology.



Feynman diagram understood as a 2-bordism

Later in the 1970s, "Penrose realized that generalizations of Feynman diagrams arise throughout quantum theory, and might even lead to revisions in our understanding of spacetime" [BS10]. Furthermore, in string theory, Feynman diagrams are substituted by *worldsheets*, which are 2−dimensional bordisms that describe the embedding of a string in spacetime; and similarly in the loop quantum gravity (LQG) theory, etc.

4.1 Hilbert Spaces

Definition 4.1. A *Hilbert space* is a $\mathbb{C}$−vector space, H equipped with a positive definite inner product whose induced norm makes H a complete metric space.

To understand this definition completely, let us look at what do we mean by a positive definite inner product space.

Definition 4.2. An *inner product space* is a sesquilinear, conjugate-symmetric map $< -|- >: H \times H \to \mathbb{C}$, which satisfies,

1. $< \psi | a\phi + b\omega > = a < \psi|\phi > + b < \psi|\omega >$
2. $< \psi|\phi > = \overline{< \phi|\psi >}$

for all vectors $\psi, \phi, \omega \in H$ and all scalars $a, b \in \mathbb{C}$.

Remark 4.3. Combining the above two properties, we have,

$$< a\psi + b\phi|\omega >= \overline{a} < \psi|\omega > + \overline{b} < \phi|\omega >$$

Remark 4.4. Antilinearity on the first variable and linearity on the second is the usual convention in Physics, but it is sometimes defined conversely.

Definition 4.5. The inner product is said to be *positive definite* if $< \psi|\psi > \geq 0$.

Remark 4.6. The *induced norm* is $||\psi|| = \sqrt{< \psi|\psi >}$.

Definition 4.7. *Unitary operators* are linear operators $T \colon H_1 \to H_2$ that preserves inner products, i.e.,

$$< T(\psi)|T(\phi) >_{H_2} = < \psi|\phi >_{H_1}$$

where, $\psi, \phi \in H_1$ and H_1 and H_2 are Hilbert spaces.

Definition 4.8. The category of Hilbert spaces is denoted as **Hilb** and is defined as,
Objects: are Hilbert spaces
Morphisms: are unitary operators

Remark 4.9. However, many of the operators that arise in Quantum Physics are not unitary.

We require the morphisms only to be bounded linear operators, i.e. $\exists M \geq 0$, a constant such that $||T(\psi)|| \leq M||\psi||$ for every $\psi \in H_1$.
Since, it can be shown that in finite dimensions, every linear map is bounded, the category of finite-dimensional Hilbert spaces is equivalent to the category of finite dimensional vector space over $\mathbb{C}$, $\mathbf{FinVect}_{\mathbb{C}}$, and hence the inner product is irrelevant. If we allow Hilbert spaces to have any dimension, possibly infinite, the category **Hilb** is not equivalent to $\mathbf{Vect}_{\mathbb{C}}$, but to the subcategory of Hilbertizable vector spaces, those that can be equipped with the topology of a certain Hilbert space. Bounded linear maps do not preserve inner products, though, only the topology they provide. The fact that we require the objects to be Hilbert spaces, and not just vector spaces with a specific kind of topology, is that inner products allow us to induce **Hilb** with some new structure, namely it makes a †−category. it makes it.

4.2 †−Category

Definition 4.10. The *†−category* is a category C equipped with a functor,

$$\dagger\colon C^{op} \to C$$

which is,

1. the identity on objects
2. an involution, i.e. $\dagger \circ \dagger = 1_C$

Remark 4.11. The image $f^\dagger\colon B \to A$ is said to be the *†−adjoint* of $f\colon A \to B$.

The common operation in Quantum Physics that makes **Hilb** into a †−category is the *Hermitian conjugate* or *adjoint*, i.e. given a bounded linear operator $T\colon H_1 \to H_2$ we define its Hermitian conjugate to be unique bounded linear operator $T^\dagger\colon H_2 \to H_1$ satisfying,

$$< T^\dagger(\phi)|\psi >_{H_1} =< \phi|T(\psi) >_{H_2}$$

for all $\psi \in H_1, \phi \in H_2$.

Therefore, the inner product serves to regard **Hilb** as a †−category, and in fact, conversely, one can recover the inner product solely form the †−structure of **Hilb**. By means of the canonical correspondence between elements of a Hilbert space H and linear maps $\mathbb{C} \to H$,

$$\begin{aligned} \psi &\mapsto T_\psi\colon 1 \mapsto \psi \\ T(1) &\leftarrow\!\shortmid T \end{aligned}$$

the inner product of H can be expressed in terms of morphisms, $\mathbb{C} \to H$

$$\begin{aligned} T_\psi^\dagger \circ T_\psi\colon \mathbb{C} &\to \mathbb{C} \\ 1 &\mapsto T_\psi^\dagger(\phi) \\ &\coloneqq< 1|T_\psi^\dagger(\phi) >_{\mathbb{C}} \\ &=< (T_\psi^\dagger)_\psi^\dagger(1)|\phi >_H \\ &=< T_\psi(1)|\phi >_H \\ &=< \psi|\phi >_H \end{aligned}$$

Dirac's *bra-ket notation* is commonly used to denote such morphisms,

$$T_\psi^\dagger = < \psi|$$
$$T_\phi = |\phi >$$

The category of bordisms is a $\dagger-$category. The definition of the $\dagger-$adjoint morphisms is simpler in this case. Given a bordism $M\colon \Sigma_1 \to \Sigma_2$, we can set its adjoint to be $M^\dagger \coloneqq \overline{M}\colon \Sigma_2 \to \Sigma_1$, i.e. the bordism M with opposite orientation. Notice that, since the orientations of Σ_1 and Σ_2 remain unchanged, the incoming boundary ∂_{in} becomes the outgoing boundary ∂_{out} and vice versa.

If bordisms represent spacetime, the $\dagger-$functor can be understood as a time reversal operation. If the bordism, $M\colon \Sigma_1 \to \Sigma_2$ describes a process in time where the space Σ_1 is transformed into the space Σ_2, then $\overline{M}$ is the reverse process from Σ_2 to Σ_1, switching past and future.

Topological (quantum) field theories can be enriched by requiring it to preserve the $\dagger-$structure.

Definition 4.12. A *$\dagger-$functor* is a functor between two $\dagger-$categories, $F\colon (C, \dagger) \to (D, \ddagger)$, such that it commutes with the $\dagger-$structures, i.e.

$$F \circ \dagger = \ddagger \circ F^{op}$$

where, $F^{op}\colon C^{op} \to D^{op}$ is the opposite functor induced by F.

Topological (quantum) field theories, that satisfy this condition are called *unitary topological (quantum) field theories*. In our case that would mean that, if Z is a topological quantum field theory,

$$Z(\overline{M}) = Z(M)^\dagger$$

which is the *Hermitian axiom* in Atiyah's paper [Ati88]. This establishes an analogy between time reversal in general relativity and taking the adjoint of an operator between Hilbert spaces.

In Quantum Physics, operators that describe the time evolution of a quantum system are usually assumed to be unitary operators T, such that $< T(\psi)|T(\phi) >$ or equivalently, such that $T^\dagger = T^{-1}$, a common hypothesis known as *unitarity*. However, if one accepts the possibility of topology changes in space along time, then, as Baez points out, other operators should be considered: It can be shown that a bordism $M\colon \Sigma \to \Sigma$ is unitary (i.e. $M = M$) if it involves no topology changes on Σ, i.e. if it is a cylinder.

Remark 4.13. The converse is true only for $n \leq 3$.

Therefore, an absence of topology change implies unitary time evolution. This fact reinforces a point already well-known from quantum field theory on curved spacetime, namely that unitary time evolution is not a built in feature of quantum theory but rather the consequence of specific assumptions about the nature of spacetime.

4.3 Quantum Entanglement

In this section we will see how spacetime and Hilbert spaces fit into the monoidality of their associated categories.

We already know that the tensor product in $Bord_n$ is the disjoint union. This corresponds in our analogy with stating that a disjoint union of two spacetimes is to be understood simply by letting them evolve in parallel, being independent from one another.

In the other hand, the fact that the monoidality of **Hilb** comes from the usual tensor product of vector spaces (instead of the Cartesian product). This leads to many paradoxes in Quantum Physics, in particular *quantum entanglement*. Intuitively, the state of a joint system, a system consisting on two separate parts, is uniquely determined by the state of each part. However, one of the most lurid discoveries in the twentieth century discussed by Einstein, Podolsky and Rosen in their famous paper [EPR35] and by Schrődinger shorty after is that, whereas that is true in Classical Physics, in the quantum scenario there exist *entangled states*: systems that cannot be described by the sum (or, more rightly, the product) of its constituent parts, but by the *superposition* of such. Put in other words, tensor product best describes quantum systems, while in the classical context Cartesian product is enough.

This is another evidence for the similarity between **Hilb** and $Bord_n$. To name another one, consider the Wootters–Zurek argument [WZ82], which states that no quantum system can be cloned. If joint system could be expressed with Cartesian product, there would be the canonical diagonal map $\Delta\colon H \to H \times H$ that duplicated information. However, as **Hilb** is monoidal with the tensor product, the possibility of cloning would imply the existence of an operator $H \to H \otimes H$; but it can be proven that there is no canonical way to define such map.

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