UNIVERSITÀ DEGLI STUDI DI TRENTO

DEPARTMENT OF MATHEMATICS

MASTER DEGREE IN MATHEMATICS (LM-40 - MATHEMATICS)
ADVANCED MATHEMATICS CURRICULUM

Complete Segal Spaces as a model of Higher Categories

Supervisor:
Dr. Nima Rasekh

Signature N. Rasekh

Prof. Edoardo Ballico

Signature Edoardo Ballico

Candidate:
Chirantan Mukherjee

Signature Mukherjee

DATE OF DISCUSSION: 18/03/2022

Page left intentionally blank.

COMPLETE SEGAL SPACES AS A MODEL OF HIGHER CATEGORIES

A dissertation presented

by

CHIRANTAN MUKHERJEE

to

The Department of Mathematics

in partial fulfillment of the requirements

for the degree of

Master of Sciences

in the subject of

Mathematics (LM-40 - Mathematics)

in the track of Advanced Mathematics

University of Trento

Italy

March 2022

COMPLETE SEGAL SPACES AS A MODEL OF HIGHER CATEGORIES

CHIRANTAN MUKHERJEE

Abstract

This thesis aims to define an important model of $(\infty, 1)-$categories, namely the complete Segal spaces, and understand the necessary foundations needed to define it. We achieve this in several steps. First, we review categorical homotopy theory. Then we study the theory of model categories and give a detailed characterization of the Kan model structure on simplicial sets. We then focus on understanding Reedy fibrant simplicial spaces, the Segal condition and its relation to the composition of maps, and how it is used to define Segal spaces. This is followed by appending the importance of the completeness condition via Dwyer-Kan equivalences. Next we define the twisted arrow construction for simplicial spaces, and show that the twisted arrow construction of a complete Segal space X, $Tw(X)$ is a complete Segal space. Further, we show that the projection map $Tw(X) \to X^{op} \times X$ is a left fibration of the complete Segal space.

Contents

Abstract iii

Acknowledgments vii

Dedications viii

1 Introduction 1

2 Two Versions Of Simplicial Sets 5

2.1 Category Theory 5

2.2 Homotopy Theory 21

2.3 Simplicial Sets 23

2.4 Kan Complexes 28

2.5 Model Categories 35

3 Simplicial Spaces 39

3.1 Introduction 39

3.2 Simplicial Spaces 40

4 Segal Spaces 45

4.1 Reedy Model Structure 45

4.2 Segal Spaces 51

4.3 Composition Of Maps 56

4.4 Homotopy Equivalence 62

5 Complete Segal Spaces **67**

5.1 Introduction 67

5.2 Completenes Condition 69

6 Twisted Arrow Construction **77**

6.1 Twisted Arrow Category 77

6.2 Twisted Arrow Segal Space 81

6.3 Left Fibration 95

References **103**

Acknowledgments

I want to express my gratitude to my advisor, Nima Rasekh, for introducing me to homotopy and higher category theories. I will miss our weekly meetings and intensive mathematical discussions the most. He showed me alternative approaches to the same problem and taught me to write down my ideas with mathematical rigor. Above all, he pushed me to discover them on my own. Apart from being an exceptional mathematician, he is a humble human being who assisted me in selecting universities for my Ph.D., writing papers, and whatnot. I aspire to become a mathematician and a person like him.

I would also like to thank my supervisor, Edoardo Ballico, for being one of the kindest people I have ever met. I am not sure what I would have done if he had not always encouraged and guided me. I am also grateful to Riccardo Ghiloni for his course on algebraic topology, where I first learned about homotopy theory.

I am indebted to Goutam Mukherjee for always supporting me and inspiring me to become a better person and mathematician. I want to express my thanks for his guidance during my undergraduate years. His passion for mathematics has left an indelible impression on me.

I will always be grateful to my parents for their unwavering support and instilling a passion for mathematics in me. I owe all my achievements to them.

Last but most importantly I would like to thank my friends, especially Jahnabi and Nisha. You were always my constant Jahnabi, whether we were going on excursions, sampling delicious foods, or discovering new areas. I owe it all to you that I enjoyed some of the finest years of my life.

Finally, I would like to thank the Erasmus+ Grant from the Italian Erasmus+ Agency, which allowed me to spend a year at the University of Warsaw, where I learned a lot about higher mathematics and met some incredible people, as well as the Thesis Research Abroad scholarship from the University of Trento, which allowed me to write my thesis in the research group of Prof. Kathryn Hess at EPFL.

To the kind individuals, who will ever read my thesis!

&

To Nima, this document exists because of you.

Chapter 1

Introduction

The theory of higher categories has become prominent in modern mathematics in recent years. As such, there are different models to study them. The most popular being quasi-categories, which was introduced by Michael Boardman and Rainer Vogt [BV73], and has been extensively studied first by André Joyal [Joy02], [Joy08], [Joy09], and later by Jacob Lurie [Lur09a]. Other higher-order category models include simplicially enriched categories, topologically enriched categories, Segal categories, $A_\infty-$categories et. al.. There is, however, a model-independent approach to study higher category theory, which is being developed by Emily Riehl and Dominic Verity [RV22].

Another such model is the model of complete Segal spaces, which was introduced by Charles Rezk [Rez01] and later shown to be a model of $(\infty, 1)-$categories, independently by André Joyal and Myles Tierney [JT07], and Bertrand Toën [To5]. The comparison between quasi-categories, simplicially enriched categories, Segal categories, and complete Segal spaces have been studied in detail by Julia Bergner [Ber10], [Ber07]. Whereas, the comparison between topologically enriched categories and quasi-categories has been studied by Jacob Lurie [Lur09b].

The category theory of quasi-categories and simplicially enriched categories have been explored extensively. In contrast, the category theory of complete Segal spaces has not been studied as much and can, for example, be found in [Ras18]. This thesis aims to fill this void and develop a complete categorical introduction to complete Segal spaces. No prior knowledge is expected from the readers, since we start from scratch by reviewing category theory [Mac71], [Rie16] and homotopy theory [Rot88] in Chapter 2. We then shift our focus to the foundational concepts of categorical homotopy theory. In particular, we review the concept of simplicial sets, its relation to topological spaces via geometric realizations, and its relation to categories via nerves [GJ99], [Rie11]. We then review the theory of model categories and how it is understood via fibrations, cofibrations, and weak equivalences [DS95], [Hov99]. In particular, we give a detailed characterization of the Kan model structure on simplicial sets.

Chapter 2 introduces two different ways of constructing simplicial sets: from any general category using the nerve functor and the other from any topological space using the singular functor. A higher category should generalize both these ideas together in a larger setting. In Chapter 3 we fit both versions of simplicial sets into one and obtain simplicial spaces.

Chapter 3 combines the categorical properties and homotopical properties of simplicial sets into one and introduces the notion of simplicial spaces. This leads to the study of higher categories. However, we can not consider any simplicial space. We need some additional conditions to develop the theory of higher categories. We achieve this in several steps. In Chapter 4, we understand Reedy fibrant simplicial spaces, the Segal condition and its relation to the composition of maps, and how it is used to define Segal spaces.

Chapter 4 discusses the homotopical and categorical properties of a Segal space. However, the homotopy theory and category theory of a Segal space are not compatible with each other. This is resolved in Chapter 5 by understanding the importance of the completeness condition via Dwyer-Kan equivalences [Rez01]. Finally, we study how to relate classical categories to complete Segal spaces via the classifying diagram, which generalizes nerves.

The twisted arrow category has been studied for quasi-categories by Lurie in [Lur11], and by Barwick, Glasman and Nardin in [BGN18] and for Segal spaces by Bergner, Osorno, Ozornova, Rovelli and Scheimbauer in [BOO+20], and by Martini in [Mar21]. As a last step we want to study the twisted arrow construction for complete Segal spaces.

Chapter 4 discusses how composition can be uniquely defined upto a contractible space of choices (4.4). A similar notion follows when we want to generalize functors in higher categories, which leads to the study of Grothendieck fibration [Gro95a]. A left fibration is homotopical anlogue of Grothendieck fibrations, and model functors valued in spaces. It plays an essential role in all of higher category theory, particularly in Lurie's work. Left fibrations were first studied for quasi-categories by Joyal in [Joy08], [Joy09], and later by Jacob Lurie in [Lur09a]. They were first introduced for Segal spaces independently by de Brito in [BdB18] and, Kazhdan and Varshavsky in [VK14], and for simplicial spaces by Rasekh in [Ras17]. Although the idea of generalizing the left fibrations to complete Segal spaces is due to Charles Rezk yet he did not publish any papers related to it.

Chapter 5 defines the complete Segal space as a model of higher categories. Chapter 6 aims to generalize the twisted arrow construction to complete Segal spaces and has been adapted from [Muk22] for our purpose with permission from the author. We then prove that for a given complete Segal space X, the twisted arrow simplicial space $Tw(X)$ is a complete Segal space (Theorem 6.2.9). And extend this result further by showing that the projection map $p\colon Tw(X) \to X^{op} \times X$ is a left fibration of complete Segal spaces (Theorem 6.3.9).

Chapter 2

Two Versions Of Simplicial Sets

2.1 Category Theory

Samuel Eilenberg and Saunders Mac Lane invented category theory to study homology theory in an algebraic setting. It was first introduced in the paper [EM45] in 1945. In this section we will recall some basic facts about category theory that will be essential to us later.

Definition 2.1.1. A *category* $\mathcal{C}$ consists of a class of *objects* $\mathcal{O}$ and class of *morphisms* $\mathcal{M}$ between them, such that:

- each object has a designated *identity map*, $Id\colon \mathcal{O} \to \mathcal{M}$
- each morphism has a specified *domain* (also called *source*, s) and *codomain* (also called *target*, t), $(s,t)\colon \mathcal{M} \to \mathcal{O} \times \mathcal{O}$
- if the codomain of the first map is equal to the domain of the second map then, *composition* of map is defined, $m\colon \mathcal{M} \, {}^{s}\!\underset{\mathcal{O}}{\times}{}^{t} \, \mathcal{M} \to \mathcal{M}$

making the following diagrams commute,

- source-target preservation:

$$\begin{tikzcd}
\mathcal{M} \arrow[d, "s"] & \mathcal{M} \underset{\mathcal{O}}{\times} \mathcal{M} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "m"] & \mathcal{M} \arrow[d, "t"] \\
\mathcal{O} & \mathcal{M} \arrow[l, "s"'] \arrow[r, "t"] & \mathcal{O}
\end{tikzcd}$$

- identity relation:

$$\begin{tikzcd} \mathcal{O} \arrow[r, "Id"] \arrow[dr, "(Id_{\mathcal{O}}, Id_{\mathcal{O}})"'] & \mathcal{M} \arrow[d, "(s,t)"] \\ & \mathcal{O} \times \mathcal{O} \end{tikzcd}$$

- identity composition:

$$\begin{tikzcd} \mathcal{O} \times \mathcal{M} \arrow[r, "Id \times Id_{\mathcal{M}}"] \arrow[dr, "\pi_2"'] & \mathcal{M} \times \mathcal{M} \arrow[d, "m"] & \mathcal{M} \times \mathcal{O} \arrow[l, "Id_{\mathcal{M}} \times Id"'] \arrow[dl, "\pi_1"] \\ & \mathcal{M} & \end{tikzcd}$$

- associativity:

$$\begin{tikzcd} \mathcal{M} \underset{\mathcal{O}}{\times} \mathcal{M} \underset{\mathcal{O}}{\times} \mathcal{M} \arrow[r, "Id_{\mathcal{M}} \times m"] \arrow[d, "m \times \mathcal{M}"'] & \mathcal{M} \underset{\mathcal{O}}{\times} \mathcal{M} \arrow[d, "m"] \\ \mathcal{M} \underset{\mathcal{O}}{\times} \mathcal{M} \arrow[r, "m"] & \mathcal{M}. \end{tikzcd}$$

Example 2.1.2. The category of sets is represented as **Set**, whose objects are sets and morphisms are set-functions.

Example 2.1.3. The category of topological spaces is represented as **Top**, whose objects are topological spaces and morphisms are continuous functions.

Definition 2.1.4. If $\mathcal{C}$ is a category, then the *opposite category* $\mathcal{C}^{op}$ is obtained by reversing the morphisms of $\mathcal{C}$.

We would like to build new categories from existing categories, for example,

Example 2.1.5. If $\mathcal{C}$ and $\mathcal{D}$ are two categories, then we can construct their *product category* denoted by $\mathcal{C} \times \mathcal{D}$, whose objects are ordered pairs (C, D), where C is an object of $\mathcal{C}$ and D is an object of $\mathcal{D}$, and morphisms are also ordered pairs $(f, g)\colon \mathcal{C} \times \mathcal{C}' \to \mathcal{D} \times \mathcal{D}'$, where f and g are morphisms from $\mathcal{C} \xrightarrow{f} \mathcal{D}$ and $\mathcal{C}' \xrightarrow{g} \mathcal{D}'$ respectively.

Example 2.1.6. For a category $\mathcal{C}$ and an object $C \in \mathcal{C}$, we can construct the *slice category of* $\mathcal{C}$ *under* C denoted by $C/\mathcal{C}$, where D is an object of $\mathcal{C}$, whose objects are are morphisms $C \xrightarrow{f} D$, and morphisms from $C \xrightarrow{f} D'$ and $C \xrightarrow{g} D''$ form commutative triangle,

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "g"] & \\ D' \arrow[rr, "h"] & & D'' \end{tikzcd}$$

such that $g = hf$.

Example 2.1.7. For a category $\mathcal{C}$ and an object $C \in \mathcal{C}$, we can construct the *slice category of* $\mathcal{C}$ *over* C denoted by $\mathcal{C}/C$, whose objects are are morphisms $D \xrightarrow{f} C$, where D is an object of $\mathcal{C}$, and morphisms from $D' \xrightarrow{f} C$ and $D'' \xrightarrow{g} C$ form commutative triangle,

$$\begin{tikzcd} & C & \\ D' \arrow[ur, "f"] \arrow[rr, "h"] & & D'' \arrow[ul, "g"'] \end{tikzcd}$$

such that $f = gh$.

If we treat the categories as objects, then the morphisms between categories can be defined as functors.

Definition 2.1.8. A *functor* $\mathcal{F}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of maps $\mathcal{F} = (\mathcal{F}_{\mathcal{O}}, \mathcal{F}_{\mathcal{M}})$ such that:

- $\mathcal{F}_{\mathcal{O}} \colon \mathcal{O}(\mathcal{C}) \to \mathcal{O}(\mathcal{D})$
- $\mathcal{F}_{\mathcal{M}} \colon \mathcal{M}(\mathcal{C}) \to \mathcal{M}(\mathcal{D})$

satisfying the following conditions,

- preserving identities: $Id_{\mathcal{D}}\mathcal{F}_{\mathcal{O}} = \mathcal{F}_{\mathcal{M}}Id_{\mathcal{C}}$
- preserving source-target: $s_{\mathcal{D}}\mathcal{F}_{\mathcal{M}} = \mathcal{F}_{\mathcal{O}}s_{\mathcal{C}}$ and $t_{\mathcal{D}}\mathcal{F}_{\mathcal{M}} = \mathcal{F}_{\mathcal{O}}t_{\mathcal{C}}$
- preserving composition: $\mathcal{F}_{\mathcal{M}}m_{\mathcal{C}} = m_{\mathcal{D}}(\mathcal{F}_{\mathcal{M}} \times \mathcal{F}_{\mathcal{M}})$.

Definition 2.1.9. A category $\mathcal{C}$ is called a *small category* if the objects and morphisms of $\mathcal{C}$ are sets and not a proper classes. Otherwise it is called a *large category.*

Example 2.1.10. The category of small categories is represented as **Cat**, whose objects are small categories and morphisms are functors.

Definition 2.1.11. A category $\mathcal{C}$ is called a *locally small category* if the morphism between any two objects of $\mathcal{C}$ is a set and not a proper class.

Example 2.1.12. The category of large categories is represented as **CAT**, whose objects are large categories and morphisms are functors.

Remark 2.1.13. Cat is actually an element of **CAT**, therefore we can not define **Cat** of all **Cat**s without running into existential inconsistencies like the Russell's paradox in set theory.

Example 2.1.14. Let $\mathcal{C}$ be a locally small category, then the *hom-functor* is a functor from the product category to the category of sets,

$$\begin{aligned} Hom_{\mathcal{C}} \colon \mathcal{C}^{op} \times \mathcal{C} &\to \mathbf{Set} \\ (C, C') &\mapsto Hom_{\mathcal{C}}(C, C') \\ ((f,g) \colon (C, C') \to (D, D') &\mapsto (Hom_{\mathcal{C}}(f,g) \coloneqq Hom_{\mathcal{C}}(C, C') \to Hom_{\mathcal{C}}(D, D')) \end{aligned}$$

where, $Hom_{\mathcal{C}}(f,g)$ takes $h \in Hom_{\mathcal{C}}(C, C')$ to a morphism $ghf \in Hom_{\mathcal{C}}(D, D')$,

$$\begin{tikzcd} C \arrow[r, "h"] & C' \arrow[d, "g"] \\ D \arrow[u, "f"] \arrow[r, dashed, "ghf"'] & D' \end{tikzcd}$$

Repeating the previous analogy on functors we can define natural transformations.

Definition 2.1.15. A *natural transformation* between two functors $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathcal{D}$, represented as $\mathcal{F} \implies \mathcal{G}$, where $\mathcal{C}$ and $\mathcal{D}$ are two categories, is defined as a collection of maps

$$\alpha_C \colon \mathcal{F}(C) \to \mathcal{G}(C)$$

for every object C in $\mathcal{C}$ such that for every morphism $f \colon C \to D$, where $D \in \mathcal{D}$, such that the following diagram commutes,

$$\begin{tikzcd} \mathcal{F}(C) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\alpha_C"'] & \mathcal{F}(D) \arrow[d, "\alpha_D"] \\ \mathcal{G}(C) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(D). \end{tikzcd}$$

Definition 2.1.16. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, then the category of functors is called the *functor category* and is represented as $[\mathcal{C}, \mathcal{D}]$, whose objects are functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and morphisms are natural transformations.

An important example of the functor category is the category of presheaves.

Example 2.1.17. A *presheaf* on a small category $\mathcal{C}$ is a functor

$$\mathcal{F}\colon \mathcal{C}^{op} \to \mathbf{Set}.$$

The *category of presheaves* on $\mathcal{C}$ is represented as $[\mathcal{C}^{op}, \mathbf{Set}]$, whose objects are presheaves and morphisms are natural transformation.

Remark 2.1.18. For a locally small category $\mathcal{C}$, the category of presheaves can also be represented as $[\mathcal{C}^{op}, \mathbf{Set}] \coloneqq Hom_{\mathcal{C}}(\mathcal{C}^{op}, \mathbf{Set}) \coloneqq \mathbf{Set}^{\mathcal{C}^{op}}$.

An important result in category theory is the Yoneda lemma, which emphasizes on understanding mathematical objects by their relationships to other mathematical objects, rather than defining them on their own. Saunders Mac Lane coined the term "Yoneda lemma" after an interview with Nobuo Yoneda, but the origins of the lemma are unknown. Before stating the Yoneda lemma we have to define the Yoneda embedding functor.

Definition 2.1.19. The *Yoneda embedding functor* $\mathcal{Y}$ is a functor from a locally small category $\mathcal{C}$ to the category of presheaves $[\mathcal{C}^{op}, \mathbf{Set}]$, which maps each object C in $\mathcal{C}$ to the representable functor $Hom_{\mathcal{C}}(-, C)$ and each morphism f to the natural transformation $f_*\colon \mathcal{Y}(C) \to \mathcal{Y}(D)$,

$$\begin{aligned} \mathcal{Y}\colon \mathcal{C} &\to [\mathcal{C}^{op}, \mathbf{Set}] \\ C &\mapsto \mathcal{Y}(C) \coloneqq Hom_{\mathcal{C}}(-, C) \\ (f\colon C \to D) &\mapsto (f_*\colon \mathcal{Y}(C) \to \mathcal{Y}(D)) \end{aligned}$$

The *Yoneda lemma* states that:

Lemma 2.1.20. If $\mathcal{C}$ is a locally small category and $F \in [\mathcal{C}^{op}, \mathbf{Set}]$, then there is a canonical isomorphism between the hom-set of presheaf homomorphisms and the elements of the set $F(C)$,

$$Hom_{[\mathcal{C}^{op}, \mathbf{Set}]}(\mathcal{Y}(C), F) \cong F(C).$$

An isomorphism is a two-sided-invertible morphism. Two categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic if there exists an isomorphism from $\mathcal{C}$ to $\mathcal{D}$.

Definition 2.1.21. Two categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic* if there exist functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ such that,

$$\mathcal{F}\mathcal{G} = Id_{\mathcal{D}} \text{ and } \mathcal{G}\mathcal{F} = Id_{\mathcal{C}}$$

where, $Id_{\mathcal{C}}$ and $Id_{\mathcal{D}}$ are the identity functors defined on category $\mathcal{C}$ and $\mathcal{D}$ respectively.

The notion of isomorphism is too strict to be useful in mathematics, and most categories do not even satisfy it. However, they do satisfy the notion of equivalence of categories, which is a generalization of isomorphism of categories.

Definition 2.1.22. Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exist functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ such that there are natural isomorphisms,

$$\mathcal{F}\mathcal{G} \cong Id_{\mathcal{D}} \text{ and } \mathcal{G}\mathcal{F} \cong Id_{\mathcal{C}}$$

where, $Id_{\mathcal{C}}$ and $Id_{\mathcal{D}}$ are identity functors defined on category $\mathcal{C}$ and $\mathcal{D}$ respectively.

Repeating the previous analogy on equivalence of categories, we can define adjunctions.

Definition 2.1.23. An *adjunction* between two categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$, together with natural transformations,

- $\eta\colon Id_{\mathcal{C}} \implies \mathcal{G}\mathcal{F}$ called *unit* and,
- $\epsilon\colon \mathcal{F}\mathcal{G} \implies Id_{\mathcal{D}}$ called *counit*

where, $Id_{\mathcal{C}}$ and $Id_{\mathcal{D}}$ are identity functors defined on category $\mathcal{C}$ and $\mathcal{D}$ respectively, such that for all object X in $\mathcal{C}$ and for all object Y in $\mathcal{D}$ the two triangles commute,

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "\mathcal{F}\eta_X"] \arrow[dr, "Id_X"'] & \mathcal{F}\mathcal{G}\mathcal{F}(X) \arrow[d, "\epsilon_{\mathcal{F}X}"] \\ & \mathcal{F}(X) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{G}(Y) \arrow[r, "\eta_{\mathcal{G}Y}"] \arrow[dr, "Id_{\mathcal{G}Y}"'] & \mathcal{G}\mathcal{F}\mathcal{G}(Y) \arrow[d, "\mathcal{G}\epsilon_Y"] \\ & \mathcal{G}(Y) \end{tikzcd}$$

$\mathcal{F}$ and $\mathcal{G}$ are called *adjoint functors* and is represented as $\mathcal{F} \dashv \mathcal{G}$, where $\mathcal{F}$ is called the *left adjoint* of $\mathcal{G}$ and $\mathcal{G}$ is called the *right adjoint* of $\mathcal{F}$.

Example 2.1.24. Let $\mathcal{D}$: **Set** $\to$ **Top** be the discrete space functor, which endows each set S with the discrete topology. Let U: **Top** $\to$ **Set** be the forgetful functor that sends each topological space T and each continuous map to its underlying set and underlying function respectively.
Then, there is a natural bijection,

$$Hom_{\mathbf{Top}}(\mathcal{D}(S), T) \cong Hom_{\mathbf{Set}}(S, \mathcal{U}(T))$$

and the discrete space functor is left adjoint to the forgetful functor, i.e.,

$$\mathcal{D} \dashv \mathcal{U}.$$

Let $\mathcal{I}$: **Set** $\to$ **Top** be the indiscrete space functor, which endows each set X with the indiscrete topology.
Then, there is a natural bijection,

$$Hom_{\mathbf{Top}}(T, \mathcal{I}(X)) \cong Hom_{\mathbf{Set}}(\mathcal{U}(T), X)$$

and the indiscrete space functor is right adjoint to the forgetful functor, i.e.,

$$\mathcal{U} \dashv \mathcal{I}.$$

We can combine them together and obtain,

$$\mathcal{D} \dashv \mathcal{U} \dashv \mathcal{I}.$$

Universal properties are present all over mathematics. Initial and terminal objects are universal elements, which satisfy the universal properties in some locally small category.

Definition 2.1.25. Let $\mathcal{C}$ be a category. An object T in $\mathcal{C}$ is called *terminal* if it satisfies the universal property, i.e., for every object X in $\mathcal{C}$ there is a unique morphism $X \to T$.

Example 2.1.26. In **Set**, every $1-$point set is terminal.

Example 2.1.27. In **Top**, every $1-$point space is terminal.

Example 2.1.28. In **Grp**, the trivial group is terminal.

Lemma 2.1.29. Terminal objects are unique upto unique isomophism.

Proof. Let T and T' be two terminal objects in a category $\mathcal{C}$.
Since, T is a terminal object, by definition there is a unique morphism $T' \xrightarrow{f} T$.

Similarly, since, T' is a terminal object, by definition there is a unique morphism $T \xrightarrow{g} T'$.
Claim: T and T' are inverse to each other.
Since, T and T' are terminal objects, there is a unique morphism $T \xrightarrow{Id_T} T$ and $T' \xrightarrow{Id_{T'}} T'$ respectively.
In particular,

$$fg = Id_{T'}$$
$$gf = Id_T.$$

Hence, we get a unique isomorphism $T \xrightarrow{\cong} T'$. □

Remark 2.1.30. Terminal objects need not exist, but if they do, they are unique.

Initial objects are dual to terminal objects, i.e., an initial object in $\mathcal{C}$ is a terminal object in $\mathcal{C}^{op}$.

Definition 2.1.31. Let $\mathcal{C}$ be a category. An object I in $\mathcal{C}$ is called *initial* if it satisfies the universal property, i.e., for every object X in $\mathcal{C}$ there is a unique morphism $I \to X$.

Example 2.1.32. In **Set**, the empty set is initial.

Example 2.1.33. In **Top**, the empty space is initial.

Example 2.1.34. In **Grp**, the trivial group is initial.

Lemma 2.1.35. Initial objects are unique upto unique isomophism.

Proof. An initial object in $\mathcal{C}$ is a terminal object in $\mathcal{C}^{op}$. And we know that terminal objects are unique upto unique isomorphism. □

Remark 2.1.36. Initial objects need not exist, but if they do, they are unique.

Like we defined the product of categories, we can also define the product of objects in a category. This is another kind of universal property.

Definition 2.1.37. Given objects X, Y and V in a category $\mathcal{C}$. A *product* of X and Y is represented as $X \times Y$ and is given by a diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

satisfying the universal property, i.e., for every diagram

$$\begin{tikzcd} & V \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

there exists a unique factorization h, making the entire diagam commutative,

$$\begin{tikzcd} & V \arrow[d, dashed, "\exists! h"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

where π_1 and π_2 are projections onto the first and second coordinate respectively.

Example 2.1.38. Cartesian products are products in **Set**.

Example 2.1.39. Product topologies are products in **Top**.

Example 2.1.40. Direct products are products in **Grp**.

Lemma 2.1.41. Products are unique upto unique isomorphism.

Proof. Let $X \times Y$ and $X \times' Y$ be two products in a category $\mathcal{C}$.
Since, $X \times Y$ is a product, by definition there is a unique morphism h, such that the following diagram commutes,

$$\begin{tikzcd} & X \times' Y \arrow[d, dashed, "\exists! h"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

Similarly, since, $X \times' Y$ is a product, by definition there is a unique morphism h', such that the following diagram commutes,

$$\begin{tikzcd}
& X \times Y \arrow[d, dashed, "\exists ! h'"] \arrow[ddl, bend right, "\pi_1"'] \arrow[ddr, bend left, "\pi_2"] & \\
& X \times' Y \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X & & Y
\end{tikzcd}$$

Claim: h and h' are inverse to each other.
Since, $X \times Y$ and $X \times' Y$ are products, there are unique morphisms $Id_{X\times Y}$ and $Id_{X\times' Y}$ respectively, such that the following diagrams

$$\begin{tikzcd}
& X \times Y \arrow[d, dashed, "\exists ! Id_{X\times Y}"] \arrow[ddl, bend right, "\pi_1"'] \arrow[ddr, bend left, "\pi_2"] & \\
& X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X & & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
& X \times' Y \arrow[d, dashed, "\exists ! Id_{X\times' Y}"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\
& X \times' Y \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X & & Y
\end{tikzcd}$$

commutes respecitvely.
In particular,

$$hh' \cong Id_{X\times Y}$$
$$h'h \cong Id_{X\times' Y}.$$

Hence, we get a unique isomorphism between the products $X \times Y \xrightarrow{\cong} X \times' Y$. $\square$

Coproducts are dual to products, i.e., a coproduct in $\mathcal{C}$ is a product in $\mathcal{C}^{op}$.

Definition 2.1.42. Given objects X, Y and V in a category $\mathcal{C}$. A *coproduct* of X and Y is represented as $X \sqcup Y$ and is given by a diagram

$$\begin{tikzcd}
& X \sqcup Y & \\
X \arrow[ur, "i_1"] & & Y \arrow[ul, "i_2"']
\end{tikzcd}$$

satisfying the universal property, i.e., for every diagram
$$\begin{tikzcd} & V & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$
there exists a unique factorization h, making the entire diagam commutative,

$$\begin{tikzcd} & V & \\ & X \sqcup Y \arrow[u, dashed, "h"] & \\ X \arrow[ur, "f"] \arrow[uur, bend left, "i_1"] & & Y \arrow[ul, "g"'] \arrow[uul, bend right, "i_2"'] \end{tikzcd}$$

where i_1 and i_2 are inclusions.

Example 2.1.43. Disjoint unions are coproducts in **Set**.

Example 2.1.44. Disjoint unions with their disjoint union topologies are coproducts in **Top**.

Example 2.1.45. Free products are coproducts in **Grp**.

Lemma 2.1.46. Coproducts are unique upto unique isomorphism.

Proof. A coproduct in $\mathcal{C}$ is a product in $\mathcal{C}^{op}$. And we know that products are unique upto unique isomorphism. $\square$

Definition 2.1.47. Let X, Y, Z and V be objects in a category $\mathcal{C}$. A *pullback* of f and g is represented as $X \underset{Z}{\times} Y$ and is given by a commutative square,
$$\begin{tikzcd} X \underset{Z}{\times} Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$
satisfying the universal property, i.e., for every other square
$$\begin{tikzcd} V \arrow[r, "t"] \arrow[d, "s"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$
there exists a unique factorization h, making the entire diagram commutative,

$$\begin{tikzcd}
V \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\
& X \underset{Z}{\times} Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

where, π_1 and π_2 are projections onto the first and second coordinate respectively.

Example 2.1.48. In **Set**, the pullback of $f\colon X \to Z$ and $g\colon Y \to Z$ is given by $X \underset{Z}{\times} Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$.

Lemma 2.1.49. Pullbacks are unique upto unique isomorphism.

Proof. Let $X \underset{Z}{\times} Y$ and $X \underset{Z}{\times}' Y$ be two pullbacks in a category $\mathcal{C}$.
Since, $X \underset{Z}{\times} Y$is a pullback, by definition there is a unique morphism h, such that the following diagram commutes,

$$\begin{tikzcd}
X \times'_Z Y \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\
& X \underset{Z}{\times} Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

Similarly, since, $X \underset{Z}{\times} Y$is a pullback, by definition there is a unique morphism h, such that the following diagram commutes,

$$\begin{tikzcd}
X \underset{Z}{\times} Y \arrow[dr, dashed, "\exists! h'"] \arrow[drr, bend left, "\pi_2"] \arrow[ddr, bend right, "\pi_1"'] & & \\
& X \times'_Z Y \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

Claim: h and h' are inverse to each other.
Since, $X \underset{Z}{\times} Y$ and $X \underset{Z}{\times}' Y$ are pullbacks, there are unique morphisms $Id_{X \underset{Z}{\times} Y}$ and $Id_{X \underset{Z}{\times}' Y}$ respectively, such that the following diagrams

$$
\begin{tikzcd}
X \underset{Z}{\times} Y \arrow[drr, bend left, "\pi_2"] \arrow[ddr, bend right, "\pi_1"'] \arrow[dr, dashed, "\exists! Id_{X \underset{Z}{\times} Y}"] & & \\
& X \underset{Z}{\times} Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
X \underset{Z}{\times}' Y \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] \arrow[dr, dashed, "\exists! Id_{X \underset{Z}{\times}' Y}"] & & \\
& X \times'_Z Y \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}
$$

commutes respectively.

In particular,

$$hh' \cong Id_{X \underset{Z}{\times} Y}$$
$$h'h \cong Id_{X \underset{Z}{\times}' Y}.$$

Hence, we get a unique isomorphism between the products $X \underset{Z}{\times} Y \xrightarrow{\cong} X \underset{Z}{\times}' Y$. $\square$

Pushouts are dual to pullbacks, i.e., a pushout in $\mathcal{C}$ is a pullback in $\mathcal{C}^{op}$.

Definition 2.1.50. Let X, Y, Z and V be objects in a category $\mathcal{C}$. A *pushout* of f and g is represented as $X \underset{Z}{\sqcup} Y$ and is given by a commuting square,

$$
\begin{tikzcd}
X \underset{Z}{\sqcup} Y & Y \arrow[l, "i_2"'] \\
X \arrow[u, "i_1"] & Z \arrow[l, "f"] \arrow[u, "g"'] \arrow[ul, phantom, "\lrcorner", very near end]
\end{tikzcd}
$$

satisfying the universal property, i.e., for every other square

$$
\begin{tikzcd}
V & Y \arrow[l, "t"'] \\
X \arrow[u, "s"] & Z \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
$$

there exists a unique factorization h, making the entire diagram commutative,

$$\begin{tikzcd}
V & & \\
& X \underset{Z}{\sqcup} Y \arrow[ul, dashed, "\exists! h"'] & Y \arrow[l, "i_2"'] \arrow[ull, bend right, "t"'] \\
& X \arrow[u, "i_1"] \arrow[uul, bend left, "s"] & Z \arrow[l, "f"] \arrow[u, "g"'] \arrow[ul, phantom, "\lrcorner", very near end]
\end{tikzcd}$$

where, i_1 and i_2 are inclusions.

Example 2.1.51. In **Set**, the pushout of $f\colon X \to Z$ and $g\colon Y \to Z$ is given by $X \underset{Z}{\sqcup} Y = (X \sqcup Y)/\sim$, where $f(z) \sim g(z)$, $\forall z \in Z$.

Lemma 2.1.52. Pushouts are unique upto unique isomorphism.

Proof. A pushout in $\mathcal{C}$ is a pullback in $\mathcal{C}^{op}$. And we know that pullback are unique upto unique isomorphism. $\square$

Remark 2.1.53. Products, coproducts, pullbacks and pushouts need not exist, but if they do, they are unique.

All these abstract notions of universal constructions are reflected through the concept of limits and colimits. They are defined over diagrams, which are functors. More formally,

Definition 2.1.54. Let $\mathcal{I}$ and $\mathcal{C}$ be two categories. An $\mathcal{I}-$shaped *diagram* of $\mathcal{C}$ is the image of a functor $\mathcal{F}\colon \mathcal{I} \to \mathcal{C}$. $\mathcal{I}$ is called the *indexing category.*

Definition 2.1.55. Let $\mathcal{I}$ be the indexing category, and X, Y and V be objects in the category $\mathcal{C}$. A *limit* of a diagram $\mathcal{F}\colon \mathcal{I} \to \mathcal{C}$ is represented as $\varprojlim \mathcal{F}$ and is given by a commuting triangle

$$\begin{tikzcd}
& \varprojlim \mathcal{F} \arrow[dl, "\eta_X"'] \arrow[dr, "\eta_Y"] & \\
X \arrow[rr, "\phi_{XY}"] & & Y
\end{tikzcd}$$

satisfying the universal property, i.e., for every other triangle

$$\begin{tikzcd}
& V \arrow[dl, "\alpha_X"'] \arrow[dr, "\alpha_Y"] & \\
X \arrow[rr, "\phi_{XY}"] & & Y
\end{tikzcd}$$

there exists a unique factorization h, making the entire diagram commutative, where $\eta_X\colon \varprojlim \mathcal{F}\colon \to X$ for all X in the diagram satisfying $\eta_Y = \phi_{XY}\eta_X$ for every morphism $\phi_{XY}\colon X \to Y$,

$$\begin{tikzcd} & V \arrow[dl, bend right, "\alpha_X"'] \arrow[d, dashed, "\exists! h"] \arrow[dr, bend left, "\alpha_Y"] & \\ & \varprojlim \mathcal{F} \arrow[dl, "\eta_X"'] \arrow[dr, "\eta_Y"] & \\ X \arrow[rr, "\phi_{XY}"] & & Y \end{tikzcd}$$

Example 2.1.56. Terminal objects are limits of an empty diagram.

Example 2.1.57. Products are limits of a discrete diagram.

Example 2.1.58. If X, Y, Z are objects of **Set**, the pullback

$$X \underset{Z}{\times} Y \coloneqq \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

is a limit of the functions $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$

$$\begin{tikzcd} X \underset{Z}{\times} Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Example 2.1.59. Let X, Y be two objects of a category $\mathcal{C}$, and f, g are two morphisms from X to Y, then the *equalizer*, $Eq(f, g)$ is a limit of a diagram $\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$, which is represented as,

$$\begin{tikzcd} Eq(f, g) \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

such that $fe = ge$.

Definition 2.1.60. A category $\mathcal{C}$ is called a *complete category* if it contains all its small limits.

Colimits are dual to limits, i.e., a colimit in $\mathcal{C}$ is a limit in $\mathcal{C}^{op}$.

Definition 2.1.61. Let $\mathcal{I}$ be the indexing category, and X, Y and V be objects in the category $\mathcal{C}$. A *colimit* of a diagram $\mathcal{F} \colon \mathcal{I} \to \mathcal{C}$ is denoted by $\varinjlim \mathcal{F}$ and is given by a commutative triangle

$$\begin{tikzcd} & \varinjlim \mathcal{F} & \\ X \arrow[ur, "\epsilon_X"] \arrow[rr, "\phi_{XY}"] & & Y \arrow[ul, "\epsilon_Y"'] \end{tikzcd}$$

satisfying the universal property, i.e., for every other triangle

$$\begin{tikzcd} & V & \\ X \arrow[ur, "\alpha_X"] \arrow[rr, "\phi_{XY}"] & & Y \arrow[ul, "\alpha_Y"'] \end{tikzcd}$$

there exists a unique factorization h, making the entire diagram commutative, where $\epsilon_X \colon X \to \varinjlim \mathcal{F}$ for all X in the diagram satisfying $\epsilon_X = \epsilon_Y \phi_{XY}$ for every morphism $\phi_{XY} \colon X \to Y$,

$$\begin{tikzcd} & V & \\ & \varinjlim \mathcal{F} \arrow[u, dashed, "h"] & \\ X \arrow[ur, "\epsilon_X"] \arrow[rr, "\phi_{XY}"] \arrow[uur, bend left, "\alpha_X"] & & Y \arrow[ul, "\epsilon_Y"'] \arrow[uul, bend right, "\alpha_Y"'] \end{tikzcd}$$

Example 2.1.62. Initial objects are colimits of an empty diagram.

Example 2.1.63. Coproducts are colimits of a discrete diagram.

Example 2.1.64. In **Set** pushouts are colimit of a pair of morphisms with equal domain.

Example 2.1.65. Let X, Y be two objects of a category $\mathcal{C}$, and f, g are two morphisms from X to Y, then the *coequalizer*, $Coeq(f, g)$ is a colimit of a diagram $\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$, which is represented as,

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "c"] & Coeq(f, g) \end{tikzcd}$$

such that $cf = cg$.

Definition 2.1.66. A category $\mathcal{C}$ is called a *cocomplete category* if it contains all its small colimits.

2.2 Homotopy Theory

Definition 2.2.1. Let X and Y be two topological spaces. Two continuous functions $f, g\colon X \to Y$ are called (left) *homotopic*, which is represented as $f \simeq g$ if there exists a continuous function $H\colon X \times I \to Y$ making the following diagram commutative,

$$\begin{tikzcd}
X \times \{0\} \arrow[d, hook] \arrow[dr, "f"] & \\
X \times I \arrow[r, dashed, "H"] & Y \\
X \times \{1\} \arrow[u, hook] \arrow[ur, "g"'] &
\end{tikzcd}$$

i.e., for every $x \in X$,

$$H(x, 0) = f(x)$$
$$H(x, 1) = g(x)$$

where, I denotes the unit interval $I \coloneqq [0, 1]$.

Definition 2.2.2. Let X and Y be two topological spaces. A continuous map $f\colon X \to Y$ is called a *homotopy equivalence* if there exists another continuous map $g\colon Y \to X$, such that,

$$fg \simeq Id_Y$$
$$gf \simeq Id_X.$$

Remark 2.2.3. If there is a homotopy equivalence $f\colon X \to Y$, then X and Y are said to have the *same homotopy type*.

Definition 2.2.4. Let X be a topological space. X is called *contractible* if the identity map $Id_X\colon X \to X$ defined on X is nullhomotopic, i.e. it is homotopic to a constant map $\{x_0\}$ defined on X, i.e.,

$$Id_X \simeq \{x_0\}.$$

Theorem 2.2.5. A topological space X is contractible if and only if it has the same homotopy type as a single point space.

Proof. Assume X is contractible. Then, by definition $Id_X \simeq c$, where $Id_X \colon X \to X$ denotes the identity map on X and $c \colon X \to \{x_0\}$ denotes the constant map on X. Let $g \colon \{x_0\} \to X$ is defined as $g(x_0) = x_0$. Then,

$$\begin{aligned} cg &\simeq Id_{x_0} \\ gc &\simeq Id_X. \end{aligned}$$

Hence, X and $\{x_0\}$ have the same homotopy type.
Conversely, assume X and $Y = \{*\}$ have the same homotopy type. Then, by definition there are continuous maps $g \colon Y \to X$ and $f \colon X \to Y$ with,

$$\begin{aligned} fg &\simeq Id_Y \\ gf &\simeq Id_X. \end{aligned}$$

Let $x_0 = g(*)$ and $c \colon X \to \{x_0\}$. Then, $c = gf \simeq Id_X$.
Hence, X is contractible. $\square$

Theorem 2.2.6. If a topological space Y is contractible, then every morphisms $X \to Y$ are homotopic to each other.

Proof. We will show that if Y is contractible, then any two morphisms $f, g \colon X \to Y$ are homotopic.
Let us assume $Id_Y \simeq c$, where c is a constant map, i.e., $c(y) = y_0$.
Define $g \colon X \to Y$ as the constant map $g(x) = y_0$ for all x in X. If $f \colon X \to Y$ is a continuous map, then consider the diagram,

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "c"] \arrow[r, shift right, "Id_Y"'] & Y \end{tikzcd}$$

we have, $Id_Y \simeq c \implies f = fId_Y \simeq cf = g$.
Hence, $f \simeq g$. $\square$

Definition 2.2.7. A *path* from x to y in X is a continuous function, $f \colon I \to X$, such that,

$$\begin{aligned} f(0) &= x \\ f(1) &= y. \end{aligned}$$

Definition 2.2.8. The *set of path components* of X is denoted by $\pi_0(X)$ is an equivalence class of X, where the equivalence relation $x \sim y$ is given if and only if there is a path from x to y in X.

2.3 Simplicial Sets

Simplicial sets were introduced by Eilenberg and Zilber [EZ50], and are a generalization of the geometric simplicial complex that appears in algebraic topology and provide a combinatorial model for the (weak) homotopy type of topological spaces.

Definition 2.3.1. Let $\mathbf{\Delta}$ be the *category of finite ordinal numbers*, where objects are finite, non-empty, totally ordered sets,

$$[0] \coloneqq \{0\}, [1] \coloneqq \{0 \to 1\}, \dots, [n] \coloneqq \{0 \to 1 \to \dots \to n\}, \dots$$

and morphisms are order preserving maps (set functions) between them.

$\mathbf{\Delta}$ is a full subcategory of the category of small categories, **Cat**.
There are two special morphisms in the category of finite ordinal numbers, $\mathbf{\Delta}$, namely,

- there is a unique injective map, called *coface map*,

$$d^i\colon [n-1] \to [n], \forall i \text{ such that } 0 \leq i \leq n$$
$$d^i(0 \to 1 \to \dots \to n-1) \coloneqq (0 \to 1 \to i-1 \to i+1 \dots \to n).$$

explicitly, $d^i(j) = \begin{cases} j & \text{for } j < i \\ j+1 & \text{for } j \geq i \end{cases}$ and,

- there is a unique surjective map, called *codegeneracy map*,

$$s^i\colon [n+1] \to [n], \forall i \text{ such that } 0 \leq i \leq n$$
$$s^i(0 \to 1 \to \dots \to n+1) \coloneqq (0 \to 1 \to i \xrightarrow{Id} i \dots \to n).$$

explicitly, $s^i(j) = \begin{cases} j & \text{for } j \leq i \\ j-1 & \text{for } j > i \end{cases}$ and,

Remark 2.3.2. We have the following identity of the codegeneracy maps, $s^i(i) = i = s^i(i+1)$.

Every morphism in $\mathbf{\Delta}$ can be written as a composition of the above two maps [GJ99]. They also satisfy *cosimplicial identities*:

- $d^j d^i = d^i d^{j-1}, \forall i < j$
- $s^j d^i = d^i s^{j-1}, \forall i < j$

- $s^j d^j = 1 = s^j d^{j+1}$
- $s^j d^i = d^{i-1} s^j, \forall i > j+1$
- $s^j s^i = s^i s^{j+1}, \forall i \leq j$

There are $n+1$ coface maps, d^i from $[n-1]$ to $[n]$ and n codegeneracy maps, s^i from $[n]$ to $[n-1]$, i.e.,

$$\begin{tikzcd}
{[0]} \arrow[r, shift left=3, "d^0"] \arrow[r, shift right=3, "d^1"'] & {[1]} \arrow[l, "s^0"'] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & {[2]} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=6] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift right=6] & \cdots \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4]
\end{tikzcd}$$

Definition 2.3.3. A *simplicial set* X is a presheaf on the category of finite ordinal numbers $\mathbf{\Delta}$,

$$\begin{aligned}
X\colon \mathbf{\Delta}^{op} &\to \mathbf{Set} \\
[n] &\mapsto X([n]) \\
([n] \xrightarrow{f} [m]) &\mapsto (X([m]) \xrightarrow{X(f)} X([n])).
\end{aligned}$$

Notation: We represent $X([n])$ as X_n for convenience and call its elements $n-$*simplices*.

Remark 2.3.4. Simplicial sets form a category, which is represented as **sSet**, where objects are simplicial sets and morphisms are natural transformation between the corresponding functors.

There are two special morphisms in **sSet**, namely,

- there are unique injective maps, called *face maps*,

$$d_i\colon X_n \to X_{n-1}, \forall i \text{ such that } 0 \leq i \leq n.$$

- there are unique surjective maps, called *degeneracy maps*,

$$s_i\colon X_n \to X_{n+1}, \forall i \text{ such that } 0 \leq i \leq n.$$

Every morphism in **sSet** can be written as a composition of the above two maps [GJ99]. They also satisfy *simplicial identities*, which are dual to the cosimplicial identities:

- $d_i d_j = d_{j-1} d_i, \ \forall i < j$

- $d_i s_j = s_{j-1} d_i$, $\forall i < j$
- $d_j s_j = 1 = d_{j+1} s_j$
- $d_i s_j = s_j d_{i-1}$, $\forall i > j+1$
- $s_i s_j = s_{j+1} s_i$, $\forall i \leq j$

There are $n+1$ face maps, d_i from X_n to X_{n-1} and n degeneracy maps, s_i from X_n to X_{n+1}, i.e.,

$$\begin{tikzcd}
X_0 \arrow[r, "s_0" description] & X_1 \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_1"] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] & X_2 \arrow[l, shift right=3] \arrow[l] \arrow[l, shift left=3] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & \cdots \arrow[l, shift right=4.5] \arrow[l, shift right=1.5] \arrow[l, shift left=1.5] \arrow[l, shift left=4.5]
\end{tikzcd}$$

The notion of the nerve of a category is attributed to Grothendieck [Gro95b], which is based on the work of Aleksandrov's nerve covering [Ale29]. And has been intensively studied by Graeme Segal in [Seg68].

Example 2.3.5. Let $\mathcal{C}$ be a small category. The *nerve* of $\mathcal{C}$ is a **sSet** and is represented as $\mathcal{NC}$, and is defined levelwise as $\mathcal{NC}_n \coloneqq Hom_{Cat}([n], \mathcal{C})$, explicitly:

- $\mathcal{NC}_0 \coloneqq$ objects of $\mathcal{C} = \{\bullet\} \eqqcolon \mathcal{O}$
- $\mathcal{NC}_1 \coloneqq$ morphisms of $\mathcal{C} = \{\bullet \to \bullet\} \eqqcolon \mathcal{M}$
- $\mathcal{NC}_2 \coloneqq$ pairs of composable morphisms of $\mathcal{C} = \{\bullet \to \bullet \to \bullet\} \eqqcolon \mathcal{M} \underset{\mathcal{O}}{\times} \mathcal{M}$

$\vdots$

- $\mathcal{NC}_n \coloneqq$ strings of $n-$composable morphisms of $\mathcal{C} = \{\bullet \to \ldots \to \bullet\} \eqqcolon \underbrace{\mathcal{M} \underset{\mathcal{O}}{\times} \ldots \underset{\mathcal{O}}{\times} \mathcal{M}}_{n-\ factors}$.

Lemma 2.3.6. The nerve, $\mathcal{N}$ is a functor between,

$$\begin{aligned}
\mathcal{N} : \mathbf{Cat} &\to \mathbf{sSet} \\
\mathcal{C} &\to \mathcal{NC} \\
(\mathcal{C} \xrightarrow{f} \mathcal{D}) &\mapsto (\mathcal{NC} \xrightarrow{\mathcal{N}f} \mathcal{ND})
\end{aligned}$$

Example 2.3.7. The *standard $n-$simplex*, Δ^n is the simplest example of a **sSet**. For all $n \geq 0$, it is a representable functor for each object $[n]$ of $\mathbf{\Delta}$.
By the Yoneda embedding,

$$\begin{aligned} \mathbf{\Delta} &\hookrightarrow \mathbf{Set}^{\mathbf{\Delta}^{op}} =: \mathbf{sSet} \\ [n] &\to \mathbf{\Delta}(-, [n]) =: \Delta^n \\ ([n] \to [m]) &\mapsto (\Delta^n \to \Delta^m) \end{aligned}$$

It is defined levelwise as, $\Delta^n_k = \mathbf{\Delta}([k], [n])$.

By the Yoneda lemma, we can classify $n-$simplices, X_n of a simplicial set, X, since there is a natural bijection between X_n and the natural transformation between Δ^n and X,

$$X_n \cong Hom_{\mathbf{sSet}}(\Delta^n, X).$$

Let X be a simplicial set, then the *category of simplices*, $\mathbf{\Delta} \downarrow X$, with objects as maps $Hom_{\mathbf{sSet}}(\Delta^n, X)$ and morphisms from $\Delta^n \to \Delta^m$ over X, which gives rise to the following commutative diagram,

$$\begin{tikzcd} \Delta^n \arrow[rr] \arrow[dr] & & \Delta^m \arrow[dl] \\ & X & \end{tikzcd}$$

By the density theorem [Mac71], we can conclude that any simplicial set, X, is a colimit of the standard $n-$simplex,

$$X \cong \varinjlim_{\Delta^n \to X} \Delta^n.$$

Remark 2.3.8. Given any small category we can transform it into a **sSet** using the nerve functor. But this list is not exhaustive, i.e., we can not obtain every **sSet** using this construction!

Question 2.3.9. What kind of **sSet** can we build using the nerve functor?

The nerve theorem due to Segal helps us resolve the problem, but first we must define what it means to satisfy the Segal condition.

Definition 2.3.10. A simplicial set, X, satisfies the *Segal condition* if the following maps,

$$X_n \xrightarrow{\cong} \underbrace{X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1}_{n-\ factors}$$

are bijections $\forall n \geq 2$.

The above definition can be represented as a limit diagram $\forall n \geq 2$,

$$\begin{tikzcd}
 & & & & X_n \arrow[dlll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drrr] & & & & \\
 & X_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & & X_1 \arrow[dl, "d_0"'] & \ldots & X_1 \arrow[dr, "d_1"] & & X_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\
X_0 & & X_0 & \ldots & \ldots & \ldots & X_0 & & X_0
\end{tikzcd}$$

Example 2.3.11. The nerve of any small category, $\mathcal{C}$, satisfies the Segal condition. It follows from the simple observation that if we replace the X_i's in the Segal condition by $\mathcal{NC}_i$, it yields the nerve construction,

$$\mathcal{NC}_n \xrightarrow{\cong} \underbrace{\mathcal{NC}_1 \underset{\mathcal{NC}_0}{\times} \cdots \underset{\mathcal{NC}_0}{\times} \mathcal{NC}_1}_{n-\ factors}$$

The following observation due to Segal [Seg68] following Grothendieck [Gro95b] gives the *Nerve Theorem*:

Theorem 2.3.12 (Nerve Theorem)**.** A simplicial set, X, satisfies the Segal condition if and only if it is the nerve of some category $\mathcal{C}$.

Following the above theorem, we observe that levelwise simplicial sets can be represented as,

$$X_n \cong \mathcal{NC}_n \xrightarrow{\cong} \underbrace{\mathcal{NC}_1 \underset{\mathcal{NC}_0}{\times} \cdots \underset{\mathcal{NC}_0}{\times} \mathcal{NC}_1}_{n-\ factors} \cong \underbrace{X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1}_{n-\ factors}$$

2.4 Kan Complexes

Definition 2.4.1. The (geometric) *realization* transforms any simplicial set into a topological space. The realization functor $|-|$ is defined from **sSet** to the category of compactly generated Hausdorff spaces **CGHaus**,

$$\begin{aligned} |-|\colon \mathbf{sSet} &\to \mathbf{CGHaus} \\ X &\mapsto |X|. \end{aligned}$$

To define the realization of a simplicial set X, we start by defining the realization of the standard $n-$simplex, known as the *standard topological $n-$simplex* $|\Delta^n|$,

$$|-|\colon \mathbf{sSet} \to \mathbf{CGHaus}$$

$$\Delta^n \mapsto |\Delta^n| \coloneqq \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i = 1, 0 \le x_i \le 1\}.$$

The realization of the standard $n-$simplex, $|X|$ is the colimit of the standard topological $n-$simplex, $|\Delta^n|$, i.e.,

$$|X| \cong \varinjlim_{\Delta^n \to X} |\Delta^n|.$$

We can also build the standard topological $n-$simplex from the category of finite ordinal numbers using the functor,

$$\begin{aligned} \boldsymbol{\Delta} &\to \mathbf{CGHaus} \\ [n] &\mapsto |\Delta^n|. \end{aligned}$$

Definition 2.4.2. Let T be a topological space (compactly generated Hausdorff space), then the *singular complex* functor, $\mathcal{S}(T)$ is a **sSet** and can be defined levelwise as

$$\mathcal{S}(T_n) = Hom_{\mathbf{CGHaus}}(|\Delta^n|, T).$$

Theorem 2.4.3. The geometric realization functor is left adjoint to the singular complex functor,

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \mathbf{CGHaus} \arrow[l, bend left, "\mathcal{S}"] \end{tikzcd}.$$

Observation: The proof follows essentially from the observation that, for any locally small category $\mathcal{C}$ and any category $\mathcal{D}$, and any object $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, the functor $\mathcal{C}(-, X) \colon \mathcal{C}^{op} \to$**Set** sends colimits to limits,

$$Hom_{\mathcal{C}}(\varinjlim Y, X) \cong \varprojlim Hom_{\mathcal{C}}(Y, X).$$

Proof. Let X be a simplicial set and Y be a compactly generated Hausdorff space.

$$\begin{aligned}
Hom_{\mathbf{CGHaus}}(|X|, Y) &= Hom_{\mathbf{CGHaus}}(\varinjlim_{\Delta^n \to X} |\Delta^n|, Y) \\
&\cong \varprojlim_{\Delta^n \to X} Hom_{\mathbf{CGHaus}}(|\Delta^n|, Y) \\
&\cong \varprojlim_{\Delta^n \to X} Hom_{\mathbf{sSet}}(\Delta^n, \mathcal{S}(Y)) \\
&\cong Hom_{\mathbf{sSet}}(\varinjlim_{\Delta^n \to X} \Delta^n, \mathcal{S}(Y)) \\
&= Hom_{\mathbf{sSet}}(X, \mathcal{S}(Y)).
\end{aligned}$$

□

Remark 2.4.4. The realization functor preserves all colimits, i.e. it is cocomplete.

Notation: We will distinguish between the simplicial sets obtained from **Cat** and from **CGHaus**. We will denote the former by **sSet** and the later by **S**.

Remark 2.4.5. Given any topological space, we can transform it into a **S** using the singular functor. But this list is also not exhaustive, i.e., we can not obtain every **S** using this construction!

Question 2.4.6. What kind of simplicial set can we build using the singular functor?

The simplicial set we obtain are Kan complexes, but first we must define sub-simplicial sets.

Definition 2.4.7. We say Y is a *sub-simplicial* set of a simplicial set X whenever $Y_n \subset X_n$ $\forall n \geq 0$ and the face and degeneracy maps of Y are restrictions of the face and degeneracy maps of X.

Definition 2.4.8. The i^{th} *face of the standard $n-$simplex*, $\partial_i \Delta^n$ is a sub-simplicial set generated by the coface maps, i.e.,

$$\partial_i \Delta^n \cong [\Delta^{n-1} \xrightarrow{d^i} \Delta^n]$$

Definition 2.4.9. The *boundary of the standard* $n-$*simplex*, $\partial\Delta^n$ is the union of all the faces of the standard $n-$simplex, i.e.,

$$\partial\Delta^n \coloneqq \bigcup_{i=0}^{n} \partial_i \Delta^n.$$

Definition 2.4.10 (Alternative Definition)**.** The *boundary of the standard* $n-$*simplex*, $\partial\Delta^n$ is expressed as a coequalizer,

$$\coprod_{0\leq i\leq j\leq n} \Delta^{n-2} \rightrightarrows \coprod_{0\leq i\leq n} \Delta^{n-1} \longrightarrow \partial\Delta^n$$

which is given by the identity $d^j d^i = d^i d^{j-1}$ for all $i < j$.

Remark 2.4.11. Hence, the boundary of the $n^{\text{th}}-$level of a simplicial set, X_n is expressed as an equalizer,

$$\prod_{0\leq i\leq j\leq n} Hom(\Delta^{n-2}, X) \leftleftarrows \prod_{0\leq i\leq n} Hom(\Delta^{n-1}, X) \longleftarrow Hom(\partial\Delta^n, X)$$

or,

$$\prod_{0\leq i\leq j\leq n} X_{n-2} \leftleftarrows \prod_{0\leq i\leq n} X_{n-1} \longleftarrow Hom(\partial\Delta^n, X)$$

Definition 2.4.12. The k^{th} *horn* is a sub-simplicial set, which is the union of all but the k^{th} face of the standard n-simplex Δ^n, i.e., $\Lambda^n_k \subset \Delta^n$ is generated by the set $\{d_0, d_1, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n\}$, i.e., $\forall 0 \leq k \leq n$,

$$\Lambda^n_k \coloneqq \bigcup_{i\neq k} \partial_i \Delta^n.$$

Remark 2.4.13. We distinguish between outer horns and inner horns,
$$\Lambda^n_k = \begin{cases} \textit{outer horn} & \text{for k=0 or n} \\ \textit{inner horn} & \text{otherwise} \end{cases}$$

Definition 2.4.14. Let X, X', Y and Y' be objects of some category $\mathcal{C}$ and i, p, f and g be morphisms of $\mathcal{C}$. Then we say that i has the *left lifting property* with respect to p and that p has the *right lifting property* with respect to i if for every commutative diagram,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ X' \arrow[r, "g"] \arrow[ur, dashed, "h"] & Y' \end{tikzcd}$$

there exists a *lift* $h \in \mathcal{M}(\mathcal{C})$ (dotted arrow) making the above diagram commute, such that $hi = f$ and $ph = g$.

Definition 2.4.15. A morphism $f\colon X \to Y$ between two simplicial sets X and Y is called a *Kan fibration* if it has the right lifting property with respect to all horn inclusions, i.e., for every commutative diagram of the following form,

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

there exists a lift $\Delta^n \to X$ (also called filler) making the above diagram commute.

Serre fibrations are topological analogs of Kan fibrations.

Definition 2.4.16. A continuous map $f\colon X \to Y$ between two topological spaces X and Y is called a *Serre fibration* if it has the right lifting property with respect to realization of all horn inclusions, i.e., for every commutative diagram of the following form

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "f"] \\ {|\Delta^n|} \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

there exists a lift $\Delta^n \to X$ (also called filler) making the above diagram commute.

Remark 2.4.17. The action of a singular functor on a Serre fibration is a Kan fibration.

However, an interesting paper [Qui68] by Quillen shows that the non-trivial converse result is also true.

Remark 2.4.18. The geometric realization of a Kan fibration is a Serre fibration.

Definition 2.4.19. A simplicial set X is called a *Kan complex* if the unique morphism $f\colon X \to \Delta^0$ is a Kan fibration, i.e., f as the right lifting property with respect to all horn inclusions,

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & \Delta^0 \end{tikzcd}$$

Observation: Since, the singleton set Δ^0 is a terminal object in simplicial set, the lower part of the triangle trivially commutes for every filler $\Delta^n \to X$. Thus, we can rewrite the definition of Kan complex.

Definition 2.4.20 (Equivalent Definition)**.** A simplicial set X is called a *Kan complex* if every horn in X has a filler, i.e., every morphism $\Lambda^n_k \to X$ can be lifted to a morphism $\Delta^n \to X$

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"] & \end{tikzcd}$$

Remark 2.4.21. The fillers may not be unique!

Definition 2.4.22 (Equivalent Definition)**.** A simplicial set X is a *Kan complex* if $\forall n \geq 0$ and $0 \leq i \leq n$, the maps,

$$Hom_{\mathbf{S}}(\Delta^n, X) \twoheadrightarrow Hom_{\mathbf{S}}(\Lambda^n_k, X)$$

are surjective.

Theorem 2.4.23. Let X be a topological space. Then the $S(X)$ is a Kan complex.

For a proof of the above theorem readers are advised to refer to in [Lur21, Proposition 1.1.9.8].

Definition 2.4.24. A simplicial set X is called a *quasi-category* if every inner horn in X has a filler.

Hence, every Kan complex is a quasi-category.

Example 2.4.25. Δ^0 is a Kan complex. It follows from the fact that there is only one map from any set to a singleton,

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f"] \arrow[d, hook] & \Delta^0 = [0] \\ \Delta^n \arrow[ur, dashed, "g"] & \end{tikzcd}$$

All of the n $(n-1)-$simplices of Λ^n_k gets mapped to $[0,\ldots 0]$, degenerate of $[0]$ via the map f. And same for Δ^n via g.

Let us look at an important counter-example.

Lemma 2.4.26. The standard $n-$simplex is not a Kan complex for $n>0$.

Proof. Let us prove the case for $n=1$, others follow similarly.
If $\Delta^1=[0,1]$ is a Kan complex the following diagram must commutes.

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "f"] \arrow[d, hook, "i"] & \Delta^1 \\ \Delta^2 \arrow[ur, dashed, "g"] & \end{tikzcd}$$

But it is not the case, since, the non-degenerate simplex of Λ^2_0, $[0,1]$ and $[0,2]$ gets send via f to Δ^1 by $f([0,1])=[0,1]$ and $f([0,2])=[0,0]$.
If the triangle has to commutes g would send the non-degenerate of Δ^2, $[0,1]$, $[0,2]$ and $[1,2]$ to $g([0,1])=[0,1]$, $g([0,2])=[0,0]$ and $g([1,2])=[1,0]$, since $g([0])=[0]$, $g([1])=1$ and $g([2])=0$.
But, $g([1,2])=[1,0]\notin\Delta^1$. $\square$

Definition 2.4.27. Let X and Y be two Kan complexes and let two morphisms $f,g\colon X\to Y$ are called *homotopic*, which is denoted by $f\simeq g$ if there exists a morphism $H\colon X\times\Delta^1\to Y$ such that the following diagram commutes,

$$\begin{tikzcd} X\times\Delta^0 \arrow[d, hook] \arrow[dr, "f"] & \\ X\times\Delta^1 \arrow[r, dashed, "H"] & Y \\ X\times\Delta^0 \arrow[u, hook] \arrow[ur, "g"'] & \end{tikzcd}$$

Remark 2.4.28. This definition is also true for any general simplicial set, but we are interested in Kan complex because in this case it is an equivalence relation.

We are interested in two kinds of homotopy equivalences, *strong homotopy equivalence*, which we will simply call as homotopy equivalence and *weak homotopy equivalence.*

Definition 2.4.29. Let X and Y be two Kan complexes and a morphism $f\colon X \to Y$ is called a *homotopy equivalence* if there exists a morphism $g\colon Y \to X$ such that,

$$fg \simeq Id_Y$$
$$gf \simeq Id_X$$

Definition 2.4.30. Let X and Y be two Kan complex and a morphism $f\colon X \to Y$ is called a *weak homotopy equivalences* if $\forall x \in X$, $\pi_n(f,x)\colon \pi_n(X,x) \to \pi_n(Y, f(x))$ are isomorphism $\forall n > 0$ and a bijection of sets for $n = 0$.

The next lemma states that studying homotopy equivalence of topological spaces is the same as studying homotopoy equivalence of analogous Kan complexes. For a proof, readers are requested to refer to Theorem 11.4 in [GJ99].

Lemma 2.4.31. Let X and Y be two topological spaces, then $f\colon X \to Y$ is a homotopy equivalence if and only if $\mathcal{S}f\colon \mathcal{S}X \to \mathcal{S}Y$ is a weak homotopy equivalence of Kan complexes.

The notion of contractible Kan complexes are of utmost importance because they are the homotopical analogs of uniqueness.

Definition 2.4.32. A Kan complex X is called *contractible* if the map $K \to \Delta^0$ is a homotopy equivalence.

Definition 2.4.33. A *groupoid* is a category in which every morphism is invertible.

Kan complexes have a lot of topological properties, in particular homotopical properties.

Remark 2.4.34. A *group* is a special case of a groupoid with only one object.

We know that the nerve functor transforms any category into **sSet** that satisfies the Segal condition. Furthermore, in general, the fillers in a Kan complex (are **sSet**) may not be unique! However, if $\mathcal{C}$ is a groupoid, then the classical theorem due to Lee [Lee72] states that the fillers are unique. Or that Segal condition implies unique fillers.

Theorem 2.4.35. For a small category $\mathcal{C}$, $\mathcal{NC}$ is a Kan complex $\iff$ $\mathcal{C}$ is a groupoid.

For proof of this classical theorem readers are advised to refer to Theorem 3 of [Lee72] or Proposition 1.2.4.2 in [Lur21].

2.5 Model Categories

Model Category (or Quillen Model Category) was developed by Quillen [Qui67] to formalize the similarities between homotopy theory and homological algebra. The axioms of a model category provide a natural setting for homotopy theory.

Definition 2.5.1. Let X be a subspace of a topological space Y. X is called a *retract* of Y if there exists a continuous map $r\colon Y \to X$ such that, the restriction of r to X is the identity map on X, i.e., $r(x) = x$ for all $x \in X$.

The concept of retraction in category theory comes from the concept of retraction in topological spaces, which was developed by Borsuk [Bor31].

Definition 2.5.2. Let X, X', Y and Y' be objects of some category $\mathcal{C}$, and f, f', g and g' be morphisms of $\mathcal{C}$, then f is a *retract* of g if there is a commutative diagram,

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"] \arrow[rr, bend left, "Id_X"] & Y \arrow[r, "j"] \arrow[d, "g"] & X \arrow[d, "f"] \\
X' \arrow[r, "i'"] \arrow[rr, bend right, "Id_{X'}"'] & Y' \arrow[r, "j'"] & X'
\end{tikzcd}$$

such that $ji = Id_X$ and $j'i = Id_{X'}$.

Having defined retract and lift, we are now ready to define model category.

Definition 2.5.3. A *model category* is a category $\mathcal{C}$, with three distinct subcategories,

- fibrations ($\twoheadrightarrow$)
- cofibrations ($\hookrightarrow$)
- weak equivalences ($\xrightarrow{\sim}$)

satisfying the following axioms,

- *Limit axiom*: $\mathcal{C}$ is complete and cocomplete
- $2-$*out-of*-3 *axiom*: if f and g are morphisms in $\mathcal{C}$ such that gf is defined and any two out of f, g and gf are weak equivalences, so is the third

- *Retract axiom*: if f and g are morphisms in $\mathcal{C}$ and f is a retract of g, and g is a fibration, cofibration or weak equivalence, then so is f
- *Lifting axiom*: a map which is both a fibration (similarly cofibration) and a weak equivalence is called trivial fibration (similarly trivial cofibration) or acyclic fibration (similarly acyclic cofibration). Trivial fibration, p has right lifting property with respect to cofibration, i and trivial cofibration, i have left lifting property with respect to fibration, p,

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook, "i"] & Y \arrow[d, two heads, "p"] \\ X' \arrow[r] \arrow[ur, dashed, "\exists"] & Y' \end{tikzcd}$$

- *Factorization axiom*: every map f admits functorial factorizations, in two ways, $f = pi$, where
  - p is a trivial fibration and i is a cofibration, and,
  - p is a fibration and i is a trivial fibration.

The above definition of a model category differs from the original definition [Qui67] of model category due to Quillen. Quillen distinguished between a closed model category and a model category, which we do not and refer to both kinds as just model category. Quillen also assumed the existence of finite limits and finite colimts, but all the examples he considered with this property are full subcategories of model categories, which are complete and cocomplete. So it is convenient to assume (model) categories which are complete and cocomplete. These changes are due to Kan [DHK97].

Remark 2.5.4. Suppose $\mathcal{C}$ is a model category,then we can define a model category on $\mathcal{C}^{op}$ where the fibrations and cofibrations if $\mathcal{C}$ are the cofibrations and fibrations in $\mathcal{C}^{op}$ respectively.

We define the classical model structure on $\mathbf{S}$, namely the *Kan-Quillen model structure on* $\mathbf{S}$, which is represented as $\mathbf{S_{Quillen}}$.

Definition 2.5.5. Let X, Y be $\mathbf{S}$, then, the simplicial morphism $f\colon X \to Y$ on the *Kan-Quillen model structure on* $\mathbf{S}$ are defined as,

- fibrations are the *Kan fibrations*, i.e., it has right lifting property with respect to all horn inclusions $\forall n \geq 1$ and $0 \leq k \leq n$,

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

- cofibrations are *monomorphisms*, i.e., $f_n \colon X_n \to Y_n$ are levelwise injective maps $\forall n \geq 0$

- weak equivalences are *weak homotopy equivalences* $|f| \colon |X| \to |Y|$ in **Top**, i.e., $\forall x \in X$, $\pi_n(f, x) = \pi_n(X, x) \to \pi_n(Y, f(x))$ are isomorphism $\forall n > 0$ and a bijection of sets for $n = 0$.

The classical model category on **Top** due to Quillen [Qui67] is called the *Serre-Quillen model structure on* **Top** and is represented as $\mathbf{Top_{Quillen}}$.

Definition 2.5.6. Let X, Y be **Top**, then the morphisms, which are continuous functions $f : X \to Y$ on the *Serre-Quillen model structure on* **Top** are defined as,

- fibrations are *Serre fibrations* i.e., it has right lifting property with respect to all horn inclusions $\forall n \geq 1$ and $0 \leq k \leq n$,

$$\begin{tikzcd} |\Lambda^n_k| \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ |\Delta^n| \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

- cofibrations are retract of a *relative cell-complexes*, i.e., (possibly infinite) composition of maps of the form

$$X_0 \to X_1 \to ... \to X_k \to ...$$

  where, each $X_k \to X_{k+1}$ is a pushout of the form,

$$\begin{tikzcd} \coprod |\partial \Delta^n| \arrow[r] \arrow[d, hook] & X_k \arrow[d] \\ \coprod |\Delta^n| \arrow[r] & X_{k+1} \end{tikzcd}$$

- weak equivalences are *weak homotopy equivalences*, i.e., $\forall x \in X$, $\pi_n(f, x) \colon \pi_n(X, x) \to \pi_n(Y, f(x))$ are isomorphism $\forall n > 0$ and a bijection of sets for $n = 0$.

Definition 2.5.7. An object X in a model category is called a *fibrant object* if the unique morphism from X to the terminal object 1 is a fibration, i.e., $X \twoheadrightarrow 1$.

Example 2.5.8. Kan complexes are fibrant objects in the $\mathbf{S}_{\mathbf{Quillen}}$.

Cofibrant objects are defined in a similar way but using initial objects, which are dual to terminal objects.

Definition 2.5.9. An object X in a model category is a *cofibrant object* if the unique morphism from the initial object 0 to X is a cofibration, i.e., $0 \hookrightarrow X$.

Example 2.5.10. CW complexes are cofibrant objects in the $\mathbf{Top}_{\mathbf{Quillen}}$.

Before describing the Quillen equivalence between $\mathbf{S}_{\mathbf{Quillen}}$ and $\mathbf{Top}_{\mathbf{Quillen}}$, let us define Quillen adjunction.

Definition 2.5.11. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, then an adjoint pair of functors F and G defined as $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ is called a *Quillen adjunction* if G preserves fibrations and trivial fibrations.

Equivalently, G implies a symmetric condition on F, because F and G are adjoint pair of functors i.e.,

Definition 2.5.12 (Equivalent Definition)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be model categories then an adjoint pair of functors F and G defined as $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ is called a *Quillen adjunction* if F preserves cofibrations and trivial cofibrations.

Having defined Quillen adjunction, we are not ready to give the definition of Quillen equivalence.

Definition 2.5.13. A Quillen adjunction $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ is *Quillen equivalence* if every cofibrant object $C \in \mathcal{C}$ and every fibrant object $D \in \mathcal{D}$, a map $C \to G(D)$ is a weak equivalence in $\mathcal{C}$ when the adjunct $F(C) \to D$ is a weak equivalence in $\mathcal{D}$.

The Quillen equivalence [Qui67] is a weaker condition than an equivalence. Hence, the Quillen equivalence $\begin{tikzcd} \mathbf{S} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, bend left, "\mathcal{S}"] \end{tikzcd}$ between $\mathbf{S}_{\mathbf{Quillen}}$ and $\mathbf{Top}_{\mathbf{Quillen}}$ follows from the adjunction in Theorem 2.4.3. For a proof, readers are requested to refer to Proposition 10.10 in [GJ99].

Chapter 3

Simplicial Spaces

3.1 Introduction

In the last chapter we have seen two different ways of constructing simplicial sets: from any general category using the nerve functor, and from any topological space using the singular functor. A higher category should generalize both these ideas together in a larger setting. In this section, we will fit both versions of simplicial sets into one and obtain simplicial spaces.

We can build a simplicial set **sSet** X (satisfying the Segal condition) from any given category using the nerve $\mathcal{N}$ functor. This simplicial sets behave as categories and have a notion of 'direction'. We represent them as,

$$\begin{tikzcd} X_0 \arrow[r, "s_0" description] & X_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \arrow[r, shift left=1] \arrow[r, shift right=1] & X_2 \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r] \arrow[r, shift left=2] \arrow[r, shift right=2] & \cdots \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift left=3] \arrow[l, shift right=3] \end{tikzcd} \tag{3.1}$$

We can also build a simplicial set, $\mathbf{S}$ (with the Kan-Quillen model structure)Y from any given topological space using the singular functor $\mathcal{S}$. This simplicial sets, which we will call as *space* from now on, has homotopical properties (Kan complexes) but do not have a notion of 'direction'. We represent them as,

$$
\begin{tikzcd}
Y_0 \arrow[d, "s_0"] \\
Y_1 \arrow[u, shift left=3, "d_0"] \arrow[u, shift right=3, "d_1"'] \arrow[d, shift left=1.5] \arrow[d, shift right=1.5] \\
Y_2 \arrow[u] \arrow[u, shift left=3] \arrow[u, shift right=3] \arrow[d] \arrow[d, shift left=3] \arrow[d, shift right=3] \\
\vdots \arrow[u, shift left=1.5] \arrow[u, shift right=1.5] \arrow[u, shift left=4.5] \arrow[u, shift right=4.5]
\end{tikzcd}
\tag{3.2}
$$

Our goal is to combine both this diagrams into one, such that the horizontal part of the (new) diagram (4.2) will represent categorical properties and the vertical part of the (new) diagram (3.2) will represent homotopical properties. But we should merge this two diagrams in such a way that the horizontal and vertical part are independent of each other.

3.2 Simplicial Spaces

Definition 3.2.1. A *simplicial space* (or *bisimplicial set*) X is a contravariant functor from $\mathbf{\Delta}$ to $\mathbf{S}$,

$$
\begin{aligned}
X \colon \mathbf{\Delta}^{op} &\to \mathbf{S} \\
[n] &\mapsto X([n]) \\
([n] \xrightarrow{f} [m]) &\mapsto (X([m]) \xrightarrow{X(f)} X([n])).
\end{aligned}
$$

Notation: We denote $X([n])$ as X_n for convenience and call its elements $n-$*simplicial object*.

Remark 3.2.2. Simplicial spaces form a category, which is denoted as $\mathbf{sS}$, where the objects are simplicial spaces and morphisms are natural transformations between the corresponding functors.

Just like simplicial sets there are two special morphisms in $\mathbf{sS}$, namely,

- there is a unique injective map, called *face map*,

$$d_i \colon X_n \to X_{n-1}, \forall i \text{ such that } 0 \leq i \leq n$$

- there is a unique surjective map, called *degeneracy map*,

$$s_i \colon X_n \to X_{n+1}, \forall i \text{ such that } 0 \leq i \leq n.$$

Every morphism in **sS** can be written as a composition of the above two maps [Rez01]. They also satisfy the *simplicial identities*, which are dual to the cosimplicial identities :

- $d_i d_j = d_{j-1} d_i, \forall i < j$
- $d_i s_j = s_{j-1} d_i, \forall i < j$
- $d_j s_j = 1 = d_{j+1} s_j$
- $d_i s_j = s_j d_{i-1}, \forall i > j + 1$
- $s_i s_j = s_{j+1} s_i, \forall i \leq j$

There are $n + 1$ face maps, d_i from X_n to X_{n-1} and n degeneracy maps, s_i from X_n to X_{n+1}, i.e.,

$$\begin{tikzcd}
X_0 \arrow[r, "s_0" description] & X_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \arrow[r, shift left=1] \arrow[r, shift right=1] & X_2 \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r] \arrow[r, shift left=2] \arrow[r, shift right=2] & \cdots \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift left=3] \arrow[l, shift right=3]
\end{tikzcd}$$

Remark 3.2.3. Simplicial spaces are also called *bisimplicial sets* on a categorical level! It can be observed from the following adjunction,

$$Fun(\Delta^{op} \times \Delta^{op}, \mathbf{Set}) \cong Fun(\Delta^{op}, Fun(\Delta^{op}, \mathbf{Set})) =: Fun(\Delta^{op}, \mathbf{S}).$$

Thus, levelwise a **sS**, X can be diagrammatically represented as,

$$\begin{array}{ccccccccc}
X_{00} & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} & X_{10} & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} & X_{20} & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} & \cdots & = & X_{-0} \\
\upuparrows\downarrow & & \upuparrows\downarrow & & \upuparrows\downarrow & & & & \upuparrows\downarrow \\
X_{01} & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} & X_{11} & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} & X_{21} & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} & \cdots & = & X_{-1} \\
\upuparrows\downdownarrows\uparrow & & \upuparrows\downdownarrows\uparrow & & \upuparrows\downdownarrows\uparrow & & & & \upuparrows\downdownarrows\uparrow \\
X_{02} & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} & X_{12} & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} & X_{22} & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} & \cdots & = & X_{-2} \\
\upuparrows\downdownarrows\upuparrows\downarrow & & \upuparrows\downdownarrows\upuparrows\downarrow & & \upuparrows\downdownarrows\upuparrows\downarrow & & & & \upuparrows\downdownarrows\upuparrows\downarrow \\
\vdots & & \vdots & & \vdots & & \ddots & = & \vdots \\
\| & & \| & & \| & & \| & & \\
X_{0-} & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} & X_{1-} & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} & X_{2-} & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} & \cdots & &
\end{array}$$

where,

- each element X_{ij} represents **Set**
- each column X_{i-} represents **S**
- each row X_{-i} represents **sSet**

It is important to understand that the columns and rows both represent the same simplicial sets, i.e. the objects and morphism of the category of the simplicial sets are the same. However, they differ in properties, i.e. the columns have homotopical properties, while the rows have categorical properties, just as we wanted. For an explicit construction, we define two embedding functors, using which we embed **sSet** and **S** into **sS**. They are defined as follows:

- Given a functor i_F defined as,

$$\begin{aligned} i_F\colon \mathbf{\Delta} \times \mathbf{\Delta} &\to \mathbf{\Delta} \\ ([n],[m]) &\mapsto [n] \end{aligned}$$

  induces the *vertical embedding functor* as,

$$\begin{aligned} i_F^*\colon \mathbf{S} &\to \mathbf{sS} \\ X_{ij} &\mapsto X_i \end{aligned}$$

- Given a functor i_Δ defined as,

$$\begin{aligned} i_\Delta \colon \mathbf{\Delta} \times \mathbf{\Delta} &\to \mathbf{\Delta} \\ ([n],[m]) &\mapsto [m] \end{aligned}$$

  induces the *horizontal embedding functor* as,

$$\begin{aligned} i_\Delta^* \colon \mathbf{sSet} &\to \mathbf{sS} \\ X_{ij} &\mapsto X_j \end{aligned}$$

We know that $\Delta^k_l =: Hom_{\mathbf{\Delta}}([l],[k])$. We want to embed it using the embedding functors as,

Definition 3.2.4. We define levelwise the $k^{\text{th}}-$*space functor* $F(k)$ as,

$$(F(k))_{nl} \coloneqq Hom_{\mathbf{\Delta}}([n],[k]) = i_F^*(\Delta^k)$$

which is a discrete simplicial space and levelwise the *standard* $k-$*simplex* Δ^k as,

$$(\Delta^k)_{nl} \coloneqq Hom_{\mathbf{\Delta}}([l],[k]) = i_\Delta^*(\Delta^k)$$

which is not a discrete simplicial space.
In the same way, we can define the *boundary of the* $k^{th}-$*space functor* $\partial F(k) = i_F^*(\partial\Delta^k)$.

Definition 3.2.5. The *boundary of the* $k^{th}-$*space functor*, $\partial F(k)$ is defined as,

$$\partial F(k) = i_F^*(\partial\Delta^k).$$

Definition 3.2.6 (Alternative Definition)**.** The *boundary of the* $k^{\text{th}}-$*space functor*, $\partial F(k)$ is expressed as a coequalizer,

$$\coprod_{0\leq i\leq j\leq k} F(k-2) \rightrightarrows \coprod_{0\leq i\leq k} F(k-1) \longrightarrow \partial F(k)$$

which is given by the identity $d^j d^i = d^i d^{j-1}$ for all $i<j$.

Remark 3.2.7. $F(n)\times\Delta^l$ is the generator of the simplicial space. Explicitly, $F(n)$ generated the columns and Δ^l generates the rows of the simplicial space, and together $F(n)\times\Delta^l$ generate the entire simplicial space.

Notation: Where there is no confusion, we will denote $X_{n-}\in\mathbf{S}$ as just X_n for notational convenience.

By the Yoneda lemma, we can classify a simplicial space, X into simplicial sets, X_n, since there are natural isomorphisms between X_n and the natural transformation between $F(n)$ and X,

$$Map_{\mathbf{sS}}(F(n), X) \cong X_n$$

Remark 3.2.8. We introduce two isomorphisms which will be useful to us later,

$$Map_{\mathbf{sS}}(F(0), X) \cong X_0$$

and

$$\begin{aligned} Map_{\mathbf{sS}}(\partial F(0), X) &= Map_{\mathbf{sS}}(\phi, X) \\ &\cong \Delta^0 =: \{*\} \end{aligned}$$

Definition 3.2.9. A category $\mathcal{C}$ with finite products is called *Cartesian closed* if the functor $(-)^Y : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is right adjoint to $- \times Y : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, i.e. $- \times Y \dashv (-)^Y$,

$$Hom_{\mathcal{C}}(X \times Y, Z) \cong Hom_{\mathcal{C}}(X, Z^Y)$$

for $X, Y, Z \in \mathcal{C}$.

Example 3.2.10. The category of sets, **Set** is Cartesian closed. For $X, Y \in$ **Set**, the product $X \times Y$ is the cartesian product of X and Y, and Z^Y is the defined as the set of all functions from $Y \to Z$. The adjointness follows from the fact that a function $f : X \times Y \to Z$ is naturally identified with the function $g : X \to Z^Y$, which is defined by $g(x)(y) = f(x, y)$ for all $x \in X$ and $y \in Y$.

Example 3.2.11. The category of small categories, **Cat** is Cartesian closed.

Example 3.2.12. The category of simplicial sets, **sSet** is Cartesian closed.

Example 3.2.13. The category of topological spaces, **Top** is not Cartesian closed.

Remark 3.2.14. However the category of compactly generated Hausdorff space, **CGHaus** is Cartesian closed. Hence, we prefer working on **CGHaus** over **Top**.

Example 3.2.15. The category of simplicial spaces is Cartesian closed, i.e. for $X, Y, Z \in \mathbf{sS}$,

$$Map_{\mathbf{sS}}(X \times Y, Z) \cong Map_{\mathbf{sS}}(X, Z^Y)$$

In particular the simplicial space $Y^X \in \mathbf{sS}$ is defined as,

$$(Y^X)_{nl} := Hom_{\mathbf{sS}}(F(n) \times \Delta^l \times X, Y)$$

Chapter 4

Segal Spaces

We introduced the notion of simplicial spaces to combine the categorical properties and homotopical properties of simplicial sets into one. This leads to the study of higher categories. However, we can not consider any simplicial space. We need some additional conditions to develop the theory of higher categories. We achieve this in several steps.

4.1 Reedy Model Structure

Let us first define the Reedy model structure on the category of simplicial spaces **sS**. Charles Reedy introduced the Reedy model structure in [Ree74].

Definition 4.1.1. Let, X, Y be **sS**, then, the simplicial morphism $f\colon X \to Y$ on the *Reedy model structure on* **sS** are defined as,

- fibrations are map $f\colon X \to Y$ such that $\forall k \geq 0$, the induced maps

  $$Map_{\mathbf{sS}}(F(k), Y) \to Map_{\mathbf{sS}}(F(k), X) \underset{Map_{\mathbf{sS}}(\partial F(k), X)}{\times} Map_{\mathbf{sS}}(\partial F(k), Y)$$

  are *Kan fibrations of* **S**

- cofibrations are *monomorphisms*, i.e. $f_n\colon X_n \to Y_n$ are levelwise injective maps $\forall n \geq 0$ of **S**

- weak equivalences are *levelwise weak equivalence of* **S**.

Notation: We will call the Reedy model structure on **sS** as **sS** for short.
We would like the vertical axis of the simplicial space (3.2) to behave like a space as

we had discussed before. This is achieved using the Reedy fibrancy condition denoted in the Reedy model structure on **sS**.

We will later see in Theorem 4.1.8 that the vertical axis of the **sS** X is a Kan complex $X_n (= X_{n-})$. Hence, the Reedy fibrancy condition can be expressed as a generalization of the Kan fibration condition. The Reedy model structure on **sS** is cofibrantly generated [DHK97] and can be defined as,

Definition 4.1.2 (Equivalent Definition)**.** A morphism $f\colon X \to Y$ between two simplicial spaces, X and Y is called a *Reedy fibration* if it has the right lifting property $\forall n, l \geq 0$ and $0 \leq i \leq n$, i.e. for every commutative diagram of the following form,

$$\begin{tikzcd} \partial F(n) \times \Delta^l \coprod_{\partial F(n) \times \Lambda^l_i} F(n) \times \Lambda^l_i \arrow[r] \arrow[d] & X \arrow[d, two heads, "f"] \\ F(n) \times \Delta^l \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

there exists a lift $F(n) \times \Delta^l \to X$ making the above diagram commute.

Definition 4.1.3. A simplicial space X is called *Reedy fibrant* if the unique morphism, $f\colon X \to \Delta^0$ between two simplicial spaces, X and Δ^0 is a Reedy fibration $\forall n, l \geq 0$ and $0 \leq i \leq n$,

$$\begin{tikzcd} \partial F(n) \times \Delta^l \coprod_{\partial F(n) \times \Lambda^l_i} F(n) \times \Lambda^l_i \arrow[r] \arrow[d] & X \arrow[d, two heads, "f"] \\ F(n) \times \Delta^l \arrow[r] \arrow[ur, dashed, "\exists"] & \Delta^0 \end{tikzcd}$$

Just as we did for the definition of Kan complex, we can rewrite the above definition as follows,

Definition 4.1.4 (Equivalent Definition)**.** A simplicial space X is called *Reedy fibrant* if $\forall n, l \geq 0$ and $0 \leq i \leq n$, the following diagram,

$$\begin{tikzcd} \partial F(n) \times \Delta^l \coprod_{\partial F(n) \times \Lambda^l_i} F(n) \times \Lambda^l_i \arrow[r] \arrow[d] & X \\ F(n) \times \Delta^l \arrow[ur, dashed, "\exists"] & \end{tikzcd}$$

has the right lifting property.

Definition 4.1.5 (Equivalent Definition)**.** A simplicial space X is called *Reedy fibrant* if $\forall n, l \geq 0$ and $0 \leq i \leq n$, the maps,

$$Map_{\mathbf{sS}}(F(n), X) \twoheadrightarrow Map_{\mathbf{sS}}(\partial F(n), X)$$

are Kan fibrations.

The above definition says that we can express the Reedy fibrancy condition using the Kan fibrancy condition. This is exactly what we wanted: that X_n behaves as Kan complexes so that we can do homotopy theory.

Theorem 4.1.6. *$F(n)$ is a Reedy fibrant simplicial space $\forall n \geq 0$.*

Proof. To show that $F(n)$ is a Reedy fibrant simplicial space, we need to show that the lift α exists,

$$\begin{tikzcd} \partial F(k) \times \Delta^l \coprod\limits_{\partial F(k)\times \Lambda^l_i} F(k) \times \Lambda^l_i \arrow[r] \arrow[d] & F(n) \\ F(k) \times \Delta^l \arrow[ur, dashed, "\alpha"] & \end{tikzcd}$$

Claim: $Map_{\mathbf{sS}}(F(k) \times \Delta^l, F(n)) \cong Map_{\mathbf{sS}}(F(k), F(n))$
By the Yoneda Lemma,

$$\begin{aligned} Map_{\mathbf{sS}}(F(k) \times \Delta^l, F(n)) &\cong Map_{\mathbf{\Delta}\times\mathbf{\Delta}}([k] \times [l], [n] \times [0]) \\ &= Map_{\mathbf{\Delta}\times\mathbf{\Delta}}([k], [n]) \\ &\cong Map_{\mathbf{sS}}(F(k), F(n)) \end{aligned}$$

$$\begin{tikzcd} F(k) \times \Lambda^l_i \arrow[r, hook] \arrow[rr, bend left, "\theta"] & \partial F(k) \times \Delta^l \coprod\limits_{\partial F(k)\times \Lambda^l_i} F(k) \times \Lambda^l_i \arrow[r] \arrow[d] & F(n) \\ & F(k) \times \Delta^l \arrow[ur, dashed, "\alpha"] \arrow[r, "\pi_1"] & F(k) \arrow[u, "\theta"'] \end{tikzcd}$$

$\theta\colon F(k)\times\Lambda^l_i$ is the same as $\theta\colon F(k)\to F(n)$, since Λ^l is constant.
Hence, the lift α can be written as a composition $\alpha=\theta\circ\pi_1$. □

The Reedy fibrancy condition is suitable for simplicial spaces because the vertical generator, $F(n)$ of simplicial spaces are Reedy fibrant and, hence by last definition they are levelwise Kan complexes. Thus, Reedy fibrant simplicial space are Kan fibrant, which we are going to prove in Theorem 4.1.8.

Observation: For a simplicial space X there is a map,

$$i\colon [0]\to[n]$$

that takes the unique point in $[0]$ to $i\in[n]$, induces a map,

$$i^*\colon X_n\to X_0.$$

We can combine all these various maps (for different i) to a map,

$$\begin{aligned}(0^*,1^*,\dots,n^*)\colon X_n\to X_0&\times X_0\times\dots\times X_0\\ &\cong(X_0)^{n+1}.\end{aligned}$$

Lemma 4.1.7. If X is a Reedy fibrant simplicial space, then $X_n\to(X_0)^{n+1}$ is a Kan fibration.

Proof. That is, we have to show that the following diagram,

$$\begin{tikzcd}
\Lambda^k_i \arrow[r] \arrow[d, hook] & X_n\cong Map_{\mathbf{sS}}(F(n),X) \arrow[d] \\
\Delta^k \arrow[r] \arrow[ur, dashed] & (X_0)^{n+1}\cong Map_{\mathbf{sS}}(\coprod_{n+1}F(0),X)
\end{tikzcd}$$

has a lift.
Which is equivalent to showing that,

$$\begin{array}{c} Hom_{\mathbf{S}}(\Delta^k, Map_{\mathbf{sS}}(F(n),X)) \\ \big\downarrow\!\!\downarrow \\ Hom_{\mathbf{S}}(\Lambda^k_i, Map_{\mathbf{sS}}(F(n),X)) \underset{Map_{\mathbf{sS}}(\Lambda^k_i, Map_{\mathbf{sS}}(\coprod\limits_{n+1} F(0,X)))}{\times} Map_{\mathbf{S}}(\Delta^k, Map_{\mathbf{sS}}(\coprod\limits_{n+1} F(0,X))) \end{array}$$

is a surjection.
Which is equivalent to showing that,

$$\begin{array}{c} Hom_{\mathbf{sS}}(F(n)\times\Delta^k, X) \\ \big\downarrow\!\!\downarrow \\ Hom_{\mathbf{sS}}(F(n)\times\Lambda^k_i, X) \underset{Map_{\mathbf{sS}}(\coprod\limits_{n+1} F(0)\times\Lambda^k_i, X)}{\times} Map_{\mathbf{sS}}(\coprod\limits_{n+1} F(0)\times\Delta^k, X) \end{array}$$

is a surjection.
Which is equivalent to showing that the following diagram,

$$\begin{tikzcd} F(n)\times\Lambda^k_i \coprod\limits_{(\coprod\limits_{n+1} F(0)\times\Lambda^k_i)} \times\Delta^k \arrow[r] \arrow[d] & X \\ F(n)\times\Delta^k \arrow[ur, dashed] & \end{tikzcd}$$

has a lift.
By assumption X is a Reedy fibrant, then,

$$(F(n))\times(\Lambda^k_i) \coprod_{(\coprod\limits_{n+1} F(0)\times\Lambda^k_i)} (\coprod_{n+1} F(0)\times\Delta^k) \twoheadrightarrow F(n)\times\Delta^k$$

is a trivial cofibration in the Reedy model structure.
Which is equivalent to showing that, $\forall l\geq 0$, the maps of spaces,

$$(F(n))_l\times(\Lambda^k_i)_l \coprod_{(\coprod\limits_{n+1} F(0)_l\times(\Lambda^k_i)_l)} (\coprod_{n+1} F(0)_l\times(\Delta^k)_l) \twoheadrightarrow F(n)_l\times(\Delta^k)_l$$

is a trivial Kan cofibration.

Which is equivalent to showing that,

$$Hom_{\mathbf{\Delta}}([l],[n]) \times \Lambda^k_i \coprod_{(\coprod_{n+1} \Delta^0 \times \Lambda^k_i)} (\coprod_{n+1} \Delta^0 \times \Delta^k) \twoheadrightarrow Hom_{\mathbf{\Delta}}([l],[n]) \times \Delta^k$$

is a Kan equivalence, which can be seen from the following diagram,

$$\begin{tikzcd}
\coprod_{n+1} \Lambda^k_i \arrow[r] \arrow[d, hook, "\simeq"] & Hom_{\mathbf{\Delta}}([l],[n]) \times \Lambda^k_i \arrow[d, hook] \arrow[ddr, bend left, hook, "\simeq"] & \\
\coprod_{n+1} \Delta^k \arrow[r] \arrow[drr, bend right, hook] & Hom_{\mathbf{\Delta}}([l],[n]) \times \Lambda^k_i \coprod_{(\coprod_{n+1} \Lambda^k_i)} (\coprod_{n+1} \Delta^k) \arrow[dr, dashed] & \\
 & & Hom_{\mathbf{\Delta}}([l],[n]) \times \Delta^k
\end{tikzcd}$$

The squared diagram is a pushout, and Kan equivalences are preserved under pushout, hence by the 2−out−of−3 axiom the dashed line is a Kan equivalence. □

Theorem 4.1.8. If X is a Reedy fibrant simplicial space, then X_n is Kan fibrant $\forall n \geq 0$.

Proof. The simplicial space X is Reedy fibrant if $\forall n \geq 0$,

$$Map_{\mathbf{sS}}(F(n), X) \twoheadrightarrow Map_{\mathbf{sS}}(\partial F(n), X)$$

is a Kan fibration.
For, $n = 0$, we have $\forall l \geq 0$

$$\begin{aligned}
Map_{\mathbf{sS}}(F(0), X)_l &= Map_{\mathbf{sS}}(\Delta^l, X) \cong X_{0l} \\
Map_{\mathbf{sS}}(F(0), X) &\cong X_0 \\
\text{And, } Map_{\mathbf{sS}}(\partial F(0), X) &= Map_{\mathbf{sS}}(\phi, X) \\
&\cong \Delta^0
\end{aligned}$$

Thus,

$$(Map_{\mathbf{sS}}(F(0), X) \twoheadrightarrow Map_{\mathbf{sS}}(\partial F(0), X)) \cong (X_0 \twoheadrightarrow \Delta^0)$$

is a Kan fibration.
Hence, X_0 is Kan fibrant when X is a Reedy fibration. Also, by Lemma 4.1.7, we know that $X_n \to (X_0)^{n+1}$ is a Kan fibration when, X is a Reedy fibrant simplicial space.
Thus, X_n is Kan fibrant $\forall n \geq 0$. $\square$

Definition 4.1.9. A model structure is *Cartesian* if $i\colon A \to B$, $j\colon C \to D$ are two cofibrations then the map $A \times D \coprod\limits_{A \times C} B \times C \to B \times D$ is a cofibration, which is a trivial cofibration if either i or j are trivial cofibrations.

Theorem 4.1.10. The Reedy model structure of simplicial spaces is Cartesian.

Observation: The proof essentially follows from the observation that the Kan-Quillen model structure is Cartesian. For a proof readers are advised to take a look at Proposition 11.5 in [GJ99]

Proof. $i\colon A \to B$ is a cofibration (or trivial cofibration) in the Reedy model structure then $i_n\colon A_n \to B_n$ is a cofibration (or trivial cofibration) in the Kan model structure.
$j\colon C \to D$ is a cofibration (or trivial cofibration) in the Reedy model structure then $j_n\colon C_n \to D_n$ is a cofibration (or trivial cofibration) in the Kan model structure.
Since, $A_n \times D_n \coprod\limits_{A_n \times C_n} B_n \times C_n \to B_n \times D_n$ is a Kan cofibration (or tivial Kan cofibration).
$\implies$ $A \times D \coprod\limits_{A \times C} B \times C \to B \times D$ is a Reedy cofibration (or Reedy trivial cofibration). $\square$

Remark 4.1.11. Being Cartesian closed is a property of categories, while a model structure is Cartesian if its underlying category is Cartesian closed. However, the converse is not true, i.e. a category being Cartesian closed does not imply that its model structure is going to be Cartesian.

4.2 Segal Spaces

Having ensured that by adding the Reedy fibrancy condition to simplicial spaces, the vertical axis behaves like space, we would like to focus on the horizontal axis in this section. As discussed, we would like the horizontal axis to have categorical properties. This leads to the definition of Segal spaces, which originally appeared in Section 4 in [Rez01].

Segal spaces are a generalization of Segal condition on **sSet** defined in Definition 2.3.10.

Definition 4.2.1. A Reedy fibrant simplicial space X is a *Segal space* if the maps,

$$X_n \xrightarrow{\simeq} \underbrace{X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1}_{n-\ factors}$$

are Kan equivalences $\forall n \geq 2$.

The above definition can be represented as a limit diagram $\forall n \geq 2$,

$$\begin{tikzcd}
 & & & & X_n \arrow[dlll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drrr] & & & & \\
 & X_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & & X_1 \arrow[dl, "d_0"'] & \dots & X_1 \arrow[dr, "d_1"] & & X_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\
X_0 & & X_0 & \dots & \dots & \dots & X_0 & & X_0
\end{tikzcd}$$

Remark 4.2.2. By Reedy fibrancy condition, the above map is actually a Kan fibration. Hence, for Segal spaces this map is actually a trivial Kan fibration.

Definition 4.2.3. A simplicial sub-space $G(n) \subset F(n)$ is called *spine* of $F(n)$ if,

$$G(n) = \underbrace{F(1) \coprod_{F(0)} \cdots \coprod_{F(0)} F(1)}_{n-\ factors}$$

Remark 4.2.4. This gives us,

$$Map_{\mathbf{sS}}(G(n), X) \cong \underbrace{X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1}_{n-\ factors}$$

The right hand side above can be represented as a limit diagram $\forall n \geq 2$,

$$\begin{tikzcd}
 & X_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & & X_1 \arrow[dl, "d_0"'] & \dots & X_1 \arrow[dr, "d_1"] & & X_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\
X_0 & & X_0 & \dots & \dots & \dots & X_0 & & X_0
\end{tikzcd}$$

Remark 4.2.5. Hence, a Segal space X can be represented as a Kan equivalence $\forall n \geq 2$,

$$X_n \cong Map_{\mathbf{sS}}(F(n), X) \xrightarrow{\simeq} Map_{\mathbf{sS}}(G(n), X) \cong \underbrace{X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1}_{n-\ factors}$$

Just as with Segal condition on $\mathbf{S}$, Segal spaces classify a simplicial space X as follows,

- X_0 represents the *space of objects* of X

- X_1 represents the *space of maps* of X

- X_2 represents the *space of pairs of composable maps* of X, i.e. $X_2 \xrightarrow{\simeq} X_1 \underset{X_0}{\times} X_1$

$\vdots$

- X_n represents the *space of* $n-$*composable morpsisms* of X, i.e. $X_n \xrightarrow{\simeq} \underbrace{X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1}_{n-\ factors}$.

The above classification looks similar to that of a nerve of a category (defined in Example 2.3.5) and that is no coincidence. The next theorem shows that any category can be transformed into a Segal space using the nerve functor.

The nerve is a discrete simplicial space and can be represented as $\mathcal{NC}_{nl} = \mathcal{NC}_n$. Hence, it is Reedy fibrant. The simplicial set NC_n is constant, i.e. the arrows in the vertical directions are equalities,

$$\begin{array}{ccccccc}
\mathcal{NC}_{00} & \substack{\longleftarrow\\\longrightarrow\\\longleftarrow} & \mathcal{NC}_{10} & \substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow} & \mathcal{NC}_{20} & \substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow} & \cdots \\
\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \\
\mathcal{NC}_{01} & \substack{\longleftarrow\\\longrightarrow\\\longleftarrow} & \mathcal{NC}_{11} & \substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow} & \mathcal{NC}_{21} & \substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow} & \cdots \\
\uparrow\downarrow\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow\downarrow\uparrow & & \\
\mathcal{NC}_{02} & \substack{\longleftarrow\\\longrightarrow\\\longleftarrow} & \mathcal{NC}_{12} & \substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow} & \mathcal{NC}_{22} & \substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow} & \cdots \\
\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow & & \\
\vdots & & \vdots & & \vdots & & \ddots \\
\| & & \| & & \| & & \| \\
\mathcal{NC}_{0} & \substack{\longleftarrow\\\longrightarrow\\\longleftarrow} & \mathcal{NC}_{1} & \substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow} & \mathcal{NC}_{2} & \substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow} & \cdots
\end{array}$$

Hence, the first vertical column is represented as $\mathcal{NC}_0$, and the second vertical column is represented as $\mathcal{NC}_1$ and so on are all discrete simplicial sets.

Lemma 4.2.6. $\mathcal{NC}$ is a Segal space for every category $\mathcal{C}$.

Proof. A Segal space is a simplicial space that satisfies the Segal condition. And a nerve satisfies the Segal condition (Example 2.3.11), so $\mathcal{NC}$ is a Segal space, $\mathcal{N}C_n \xrightarrow{\simeq} \underbrace{\mathcal{N}C_1 \underset{\mathcal{N}C_0}{\times} \cdots \underset{\mathcal{N}C_0}{\times} \mathcal{N}C_1}_{n-\ factors}$, where $\mathcal{NC}_0$ denotes the set of object of a $\mathcal{C}$ and $\mathcal{NC}_1$ denotes the set of morphisms of $\mathcal{C}$, and $\mathcal{NC}_n$ denotes the set of $n-$composable morphisms of $\mathcal{C}$. $\square$

Remark 4.2.7. Given any small category, we can transform it into a **sS** using the nerve functor. However, the nerve functor gives **sS** that satisfy a strict Segal condition, i.e., much more restrictive than then a general Segal space. A better understanding can be found later in Remark 4.3.5.

Theorem 4.2.8. If the Reedy fibrant simplicial space X is homotopically constant, then X is a Segal space.

Proof. We know that a Reedy fibrant simplicial space is a Segal space if it satisfies the Segal condition, i.e.,

$$X_n \xrightarrow{\simeq} \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n-\ factors}$$

is a Kan equivalence.
Let us prove the case for $n = 2$, the higher cases follows by induction,

$$\begin{tikzcd}
X_2 \arrow[dr, dashed, "\simeq"] \arrow[ddr, bend right, "\simeq"'] & & \\
& X_1 \underset{X_0}{\times} X_1 \arrow[r] \arrow[d, "\simeq"] \arrow[dr, phantom, "\lrcorner"] & X_1 \arrow[d, "\simeq"] \\
& X_1 \arrow[r, "\simeq"] & X_0
\end{tikzcd}$$

Since, $X_1 \to X_0$ are Kan fibration $\implies$ Kan equivalences. Homotopy pullback of Kan equivalences are Kan equivalences $\implies X_1 \underset{X_0}{\times} X_1 \to X_1$ is a Kan equivalence. Also, $X_2 \to X_1$ is a Kan equivalence, hence by $2-out-of-3$ axiom $X_2 \xrightarrow{\simeq} X_1 \underset{X_0}{\times} X_1$ is a Kan equivalence, as required. Hence, X is a Segal space. $\square$

Next, we define the Segal space model structure on **sS**.

Definition 4.2.9. Let X, Y be **sS**, then, the simplicial morphism, $f\colon X \to Y$ on the *Segal space model structure on* **sS** are defined as,

- fibrant objects, Z are Reedy fibrant such that,

$$Map(F(k), Z)) \to Map(G(k), Z)$$

  are Kan equivalences $\forall k \geq 2$

- cofibrations are *monomorphisms*, i.e. $f_n\colon X_n \to Y_n$ are levelwise injective maps $\forall n \geq 0$ of **S**

- weak equivalences are maps, $f\colon X \to Y$ such that for every Segal space, Z, the induced map,

$$Map(X, Z) \to Map(Y, Z)$$

  is a weak equivalence.

Segal spaces do not form a model of $(\infty, 1)-$categories. However, Ayala and Francis give an alternative, model-independent way of defining Segal spaces in [AF18].

4.3 Composition Of Maps

Let us explore these spaces in more details to gain a better understanding of how Segal spaces generalizes concepts arising in categorical setting into homotopical setting.

Observation: For a Segal space, X and x, y, z objects of X, the space of composable map, X_2 is expressed as a Kan equivalence, $X_2 \xrightarrow{\simeq} X_1 \underset{X_0}{\times} X_1$,

$$\begin{tikzcd}
X_2 \arrow[r] \arrow[d] & X_1 \arrow[d, "d_1"] \\
X_1 \arrow[r, "d_0"] & X_0
\end{tikzcd}$$

The left hand side of the above Kan equivalence, X_2 is a $2-$cell and can be represented as,

$$\begin{tikzcd}
& y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "gf"] & & z
\end{tikzcd}$$

Similarly, the right hand side of the above Kan equivalence, $X_1 \underset{X_0}{\times} X_1$ are two $1-$cells and can be represented as,

$$\begin{tikzcd}
& y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] & & z
\end{tikzcd}$$

But the Segal condition guarantees that the triangle can be completed to a $2-$cell,

$$\begin{tikzcd}
& y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, dashed, "h"] & & z
\end{tikzcd} \tag{4.1}$$

Notice that the Segal condition just says that the dashed arrow, h and thus the $2-$cell exists, but it does not confirm that they are unique. Higher category is fundamentally different than the usual categorical setting where h is expressed uniquely as the composition of $h = gf$. Later in Remark 4.3.4, we will justify our reasoning for calling

$h = gf$, despite the fact that the composition of map is not unique.

Observation: For a Segal space, X and x, y, z, w objects of X, the space of composable map, X_3 can be expressed as a Kan equivalence $X_3 \xrightarrow{\simeq} X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1$.
Just like the last observation, the left hand side of the above Kan equivalence, X_3 is a 3−cell and can be expressed as,

$$\begin{tikzcd}
 & w & \\
x \arrow[ur] \arrow[rr, "f"] \arrow[dr] & & y \arrow[ul] \arrow[dl, "g"] \\
 & z \arrow[uu, "h"] &
\end{tikzcd}$$

While the right hand side of the above Kan equivalence, $X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1$ are three 1−cells and can be expressed as,

$$\begin{tikzcd}
 & w & \\
x \arrow[rr, "f"] & & y \arrow[dl, "g"] \\
 & z \arrow[uu, "h"] &
\end{tikzcd} \tag{4.2}$$

But the Segal condition guarantees that the tetrahedron can be completed to a 3−cell,

$$\begin{tikzcd}
 & w & \\
x \arrow[ur, dashed, "j"] \arrow[rr, "f"] \arrow[dr, dashed, "i"'] & & y \arrow[ul, dashed, "k"'] \arrow[dl, "g"] \\
 & z \arrow[uu, "h"] &
\end{tikzcd} \tag{4.3}$$

As we stated in the last observation, the Segal condition only guarantees the existence of the 3−cell and the compositions, but not uniqueness. Again, we will denote $i = gf$ and $k = hg$ and justify the reasoning in Remark 4.4.5. However, something even more interesting happens: j can be depicted as both $j = h(gf)$ and $j = (hg)f$, by our earlier reasoning. We will show that these two (associative) ways of depicting

j are actually equivalent in Theorem 4.3.9.

Segal spaces are not a category, so we can not define the composition of morphism in the usual way. Earlier we stated that we would call the morphism h as the composition $h = gf$ in Diagram (4.1) even though the composition is not uniquely defined. Let us justify our reasoning now.

Definition 4.3.1. For $x_0, \ldots, x_n$ objects of a Segal space X, the *mapping space*, $map_X(x_0, \ldots, x_n)$ is defined as the following homotopy pullback,

$$\begin{tikzcd} map_X(x_0,\ldots,x_n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d] \\ \{*\} = \Delta^0 \arrow[r, "(x_0,\ldots,x_n)"] & (X_0)^{n+1} = \underbrace{X_0 \times \ldots \times X_0}_{(n+1)-\ factors} \end{tikzcd}$$

Since Kan fibrations are preserved under pullback, and the Reedy fibrancy condition states $X_n \to (X_0)^{n+1}$ is a Kan fibration, then $map_X(x_0, \ldots, x_n) \to \Delta^0$ is a Kan fibration. Hence, $map_X(x_0, \ldots, x_n)$ is a Kan complex.

Remark 4.3.2. A point in the mapping space, $map_X(x_0, \ldots, x_n)$ is a $(n+1)-$simplex with vertices $(x_0, \ldots, x_n)$.

Since, X is a Segal space, $X_n \xrightarrow{\simeq} \underbrace{X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1}_{n-\ factors}$ is a Kan equivalence and the mapping space can be extended to,

$$\begin{tikzcd} map_X(x_0,\ldots,x_n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[r, "\simeq"] \arrow[d] & \underbrace{X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1}_{n-\ factors} \arrow[dl] \\ \Delta^0 \arrow[r, "(x_0,\ldots,x_n)"] & (X_0)^{n+1} & \end{tikzcd}$$

Remark 4.3.3. The commutative triangle in the above diagram induces a trivial Kan fibration,

$$map_X(x_0, \ldots, x_n) \xrightarrow{\simeq} map_X(x_0, x_1) \times \ldots \times map_X(x_{n-1}, x_n)$$

Observation: The following diagram is obtained from the above diagram where x, y, z are objects of a Segal space X,

$$\begin{tikzcd}
map_X(x,y,z) \arrow[r, "d_1"] \arrow[d, "\simeq"] & map_X(x,z) \\
map_X(x,y) \times map_X(y,z) &
\end{tikzcd}$$

Let, $f \in map_X(x,y)$ and $g \in map_X(y,z)$ be two morphisms, then we can define the space of all composition of f and g as $Comp(f,g) \subset map_X(x,y,z)$ as the following pullback diagram,

$$\begin{tikzcd}
Comp(f,g) \arrow[r, hook] \arrow[d, "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] & map_X(x,y,z) \arrow[r, "d_1"] \arrow[d, "\simeq"] & map_X(x,z) \\
\Delta^0 \arrow[r] & map_X(x,y) \times map_X(y,z) &
\end{tikzcd} \tag{4.4}$$

Explicitly, $Comp(f,g)$ can be expressed in the following way,

$$Comp(f,g) = \left\{ \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr] & & z \end{tikzcd} \in X_2 \right\}$$

$$Comp(f,g) = \{\sigma \in X_2 \mid d_0\sigma = g, d_2\sigma = f\}$$

$$Comp(f,g) = (d_0, d_2)^{-1}(f,g).$$

Remark 4.3.4. $Comp(f,g) \xrightarrow{\simeq} \Delta^0$ is a Kan equivalence, i.e. $Comp(f,g)$ is contractible. Thus, any two points in $Comp(f,g)$ are equivalent. This justifies calling the morphism h as the composition $h = gf$ in Diagram (4.1). Thus, compositions are uniquely defined using the $Comp(f,g)$ upto contractible space of such choices. Hence, it justifies to depict the Diagram (4.1) as,

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "gf"] & & z \end{tikzcd}$$

Remark 4.3.5. In the special case, when the Segal space X is a nerve of some category $\mathcal{C}$, $Comp(f,g)$ is a unique point, i.e. $Comp(f,g) \cong \Delta^0$, because $\mathcal{NC}_2 \xrightarrow{\cong} \mathcal{NC}_1 \underset{\mathcal{NC}}{\times} \mathcal{NC}_1$,

$$\begin{tikzcd}
Comp(f,g) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{NC}_2 \arrow[d, "\cong"] \\
\Delta^0 \arrow[r] & \mathcal{NC}_1 \underset{\mathcal{NC}_0}{\times} \mathcal{NC}_1
\end{tikzcd}$$

where $f, g \in \mathcal{NC}_1$.
Hence, if the Segal space is the nerve of some category, then two maps are homotopic to each other if and only if they are equal to each other. Moreover, a map is a homotopy equivalence if and only if it is an isomorphism.

Definition 4.3.6. In a Segal space X, for every object $x \in X$ we can define the *identity map* of x denoted as Id_x in $map_X(x,x)$.

Remark 4.3.7. Explicitly the identity map of an object x in the Segal space X is constructed using the degeneracy map s_0 as,

$$\begin{aligned} s_0 \colon X_0 &\to X_1 \\ x &\mapsto map_x(x,x) = Id_x \end{aligned}$$

Definition 4.3.8. In a Segal space X, two morphisms $f \in map_X(x,y)$, $g \in map_X(x,y)$ are called *homotopic* if $f, g \colon \Delta^0 \to map_X(x,y)$ are homotopic in the sense of Definition 2.2.1.

Theorem 4.3.9. In a Segal space X, let $f \in map_X(x,y)$, $g \in map_X(y,z)$ and $h \in map_X(z,w)$ be three composable morphisms for objects x, y, z, w in X then $h(gf) \sim (hg)f$ and $fId_x \sim f \sim Id_y f$.

The above theorem shows that composition is associative and unital upto homotopy. For a proof readers are advised to refer to Proposition 5.4 in [Rez01]. Hence, the two ways of expressing j in Diagram (4.3), which are equivalent upto homotopy.

Using this last theorem we are able to construct the homotopy category HoX out of a Segal space X, which establishes a relation between Segal spaces and category theory. But first let us define what a homotopy category is,

Definition 4.3.10. For a Segal space X, the *homotopy category* of X, denoted as HoX is defined as follows:

- The *objects* of HoX are the objects of X

- For x, y objects of X, the *morphism* of HoX is defined as the space of homotopy classes of morphisms from x to y as follows,

$$Hom_{HoX}(x, y) = \pi_0(map_X(x, y))$$

- For x, y, z objects of X, the *composition* of morphisms between $Hom_{HoX}(x, y)$ and $Hom_{HoX}(y, z)$ is defined as,

$$\begin{aligned} Hom_{HoX}(x, y) \times Hom_{HoX}(y, z) &\to Hom_{HoX}(x, z) \\ ([f], [g]) &\mapsto ([f \circ g]) \end{aligned}$$

Remark 4.3.11. It is possible to define the homotopic morphism because we have mapping spaces rather than sets.

Theorem 4.3.12. *HoX is a category for every Segal space X.*

Proof. We have already defined objects and morphims in Definition 4.3.10 and the associativity and identity follows from Theorem 4.3.9. We only have to define composition,

$$Hom_{HoX}(x, y) \times Hom_{HoX}(y, z) \to Hom_{HoX}(x, z)$$

for objects x, y, z in a Segal space X, which is induced by the composition map,

$$map_X(x, y) \times map_X(y, z) \to map_X(x, z).$$

□

Theorem 4.3.13. *Every category $\mathcal{C}$ is isomorphic to $Ho\mathcal{N}\mathcal{C}$.*

Proof. The objects and morphisms of $Ho\mathcal{N}\mathcal{C}$ are defined as,

$$\begin{aligned} \mathcal{O}(Ho\mathcal{N}\mathcal{C}) &= \mathcal{O}(\mathcal{N}\mathcal{C}) \\ &= \mathcal{N}\mathcal{C}_0 \\ &= \mathcal{O}(\mathcal{C}) \end{aligned}$$

Let, $x, y \in Ho\mathcal{N}\mathcal{C}$, then,

$$\begin{aligned} Hom_{Ho\mathcal{N}\mathcal{C}}(x, y) &= \pi_0(Map_{\mathcal{N}\mathcal{C}}(x, y)) \\ &= \pi_0(\mathcal{N}\mathcal{C}_1 \underset{\mathcal{N}\mathcal{C}_0 \times \mathcal{N}\mathcal{C}_0}{\times} *) \end{aligned}$$

The observation that $\mathcal{NC}_1$ and $\mathcal{NC}_0 \times \mathcal{NC}_0$ are discrete simplicial space yields,

$$\begin{aligned}\pi_0(\mathcal{NC}_1 \underset{\mathcal{NC}_0 \times \mathcal{NC}_0}{\times} *) &\cong \mathcal{NC}_1 \underset{\mathcal{NC}_0 \times \mathcal{NC}_0}{\times} * \\ &= Hom_{\mathcal{C}}(x, y)\end{aligned}$$

Hence, $\mathcal{C}$ is isomorphic to $Ho\mathcal{NC}$. $\square$

4.4 Homotopy Equivalence

As we had pointed out before, a Segal space should be a juxtaposition of category theory and homotopy theory. However, we have mostly looked at how we can endow categorical properties to a Segal space in the last section. In this section, we will focus on the homotopical aspects of it.

Having defined homotopy category and homotopic morphism, we are ready to define homotopy equivalence of Segal spaces which follows from homotopy equivalence of topological spaces in Definition 2.2.2.

Definition 4.4.1. For a Segal space X, a map $f \in map_X(x, y)$ is called a *homotopy equivalence* if there exist maps $g, h \in map_X(y, x)$ for every object $x, y \in X$ such that:

$$\begin{aligned} gf &\sim Id_x \\ fh &\sim Id_y. \end{aligned}$$

Remark 4.4.2. This can be diagrammatically represented as,

$$\begin{tikzcd} & x & \\ x \arrow[ur, "Id_x"] \arrow[rr, "f"] & & y \arrow[ul, "g"'] \\ & y \arrow[ul, "h"] \arrow[ur, "Id_y"'] & \end{tikzcd}$$

where, the composition and identities are unique upto homotopy by Theorem 4.3.9. Hence, the inverses are also unique upto homotopy.

Observation: Let us denote the following homotopy colimit,

$$\begin{tikzcd}
F(1) & & F(1) & & F(1) \\
& F(0) \arrow[ul, "1"] \arrow[ur, "1"'] & & F(0) \arrow[ul, "0"] \arrow[ur, "0"'] &
\end{tikzcd}$$

as the simplicial space $Z(3) \subset F(3)$. Hence, the space $Map_{\mathbf{sS}}(Z(3), X)$ can be represented as the homotopy limit,

$$Map_{\mathbf{sS}}(Z(3), X) \cong X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1,$$

which can be diagrammatically represented as,

$$\begin{tikzcd}
X_1 \arrow[dr, "d_1"'] & & X_1 \arrow[dl, "d_1"] \arrow[dr, "d_0"'] & & X_1 \arrow[dl, "d_0"] \\
& X_0 & & X_0 &
\end{tikzcd}$$

Thus, we have the map,

$$(d_1 d_3, d_0 d_3, d_1 d_0) \colon X_3 \cong Map_{\mathbf{sS}}(F(3), X) \to Map_{\mathbf{sS}}(Z(3), X) \cong X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1.$$

This map is well-defined by the properties of simplicial identities, i.e.

$$d_1 d_1 d_3 = d_1 d_0 d_3$$
$$d_0 d_0 d_3 = d_0 d_1 d_0$$

Definition 4.4.3. (Alternative Definition) For a Segal space X, a map $f \in map_X(x, y)$ is a *homotopy equivalence* if $(Id_x, f, Id_y) \in Map_{\mathbf{sS}}(Z(3), X)$ admits a lift to an element $Y \in X_3$.

Remark 4.4.4. If $f \in map_X(x, y)$ as above then,

$$s_0 d_1 f = Id_x$$
$$s_0 d_0 f = Id_y$$

Remark 4.4.5. It is crucial to point out at this moment that although $Z(3)$ and $G(3)$ (defined in Definition 4.2.3) are both sub-simplicial spaces of $F(3)$, whose 1−simplices are glued in certain order, they are not the same sub-objects.
For a Segal space X and x, y objects of X, the element $(Id_x, f, Id_y) \in Map_{\mathbf{sS}}(Z(3), X) \cong X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1$ are three 1−cells and can be represented as,

$$\begin{tikzcd}
& x & \\
y \arrow[dr, "Id_y"'] & & x \arrow[ul, "Id_x"'] \arrow[dl, "f"] \\
& y &
\end{tikzcd}$$

while, $Map_{\mathbf{sS}}(G(3), X) \cong X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1$ are three $1-$cells and can be represented as in Diagram (4.2).
But by the last definition, the tetrahedron can be completed to a $3-$cell in $Map_{\mathbf{sS}}(F(3), X) \cong X_3$,

$$\begin{tikzcd}
& x & \\
y \arrow[ur, dashed, "g"] \arrow[rr, dashed, "g"] \arrow[dr, "Id_y"'] & & x \arrow[ul, "Id_x"'] \arrow[dl, "f"] \\
& y \arrow[uu, dashed, "h"] &
\end{tikzcd}$$

Thus, for a map $f \in map_{(}x, y)$ there are two maps $g, h \in map_X(y, x)$ as constructed in the above diagram such that g is left inverse and h is right inverse, i.e.

$$gf = Id_x$$
$$fh = Id_y$$

The readers are encouraged to compare the above diagram with the Diagram (4.3), so as understand the essential difference between $Z(3)$ and $G(3)$.

Using this definition, we can easily conclude that the notion of homotopy equivalence is actually homotopy invariant. A proof can be found in Lemma 5.8 in [Rez01].

Lemma 4.4.6. If there is a path between two maps, $f, g \in X_1$ in a Segal space X and if f is a homotopy equivalence, then so is g.

Lemma 4.4.7. If Theorem 4.2.8 holds, then HoX is precisely the fundamental groupoid of the Kan complex X_0

Proof. If X_0 is a Kan complex, we can define its fundamental groupoid, $\pi_1(X_0)$ as,

$$\begin{aligned} \mathcal{O}_{\pi_1(X_0)} &:= X_{00} \\ Hom_{\pi_1(X_0)}(x, y) &:= \pi_0(Hom_{X_0}(x, y)) \\ &= \{f \in X_{01} \mid f : x \to y\}/\sim \\ &= \pi_0(\Delta^0 \underset{X_0 \times X_0}{\times} X_0^{\Delta^1}) \end{aligned}$$

This is precisely the definition of HoX.
Hence, HoX is the fundamental groupoid of the Kan complex X_0. □

Definition 4.4.8. For a Segal space X, we define the *space of homotopy equivalences* of X by $X_{hoequiv}$ to be a subspace of X_1, such that every map in $X_{hoequiv} \subset X_1$ is a homotopy equivalence.

Remark 4.4.9. This means a point in $X_{hoequiv}$ is just a morphism in our Segal space X. In fact, $X_{hoequiv}$ is the full subspace of X_1 that consists of the homotopy equivalences, i.e. morphisms, f in X such that there exists left inverses, g and right inverses, h.

Remark 4.4.10. There is another equivalent space $X_{hoeqchoice}$ of $X_{hoequiv}$, such that every point in the space $X_{hoeqchoice}$ is a choice of equivalence f along with a choice of left inverse, g and right inverse, h, defined as,

$$\begin{tikzcd} X_{hoequiv} \arrow[dr, hook] & X_{hoeqchoice} \arrow[l, "\simeq"'] \arrow[d] \\ & X_1 \end{tikzcd}$$

Since, by definition $X_{hoeqchoice}$ is not a subspace of X_1 like $X_{hoequiv}$, rather a subspace of X_3, the above diagram can be extended to the following pullback diagram,

$$\begin{tikzcd} X_{hoequiv} \arrow[dr, hook] & X_{hoeqchoice} \arrow[l, "\simeq"'] \arrow[d] \arrow[r, hook] \arrow[dr, phantom, "\lrcorner", very near start] & X_3 \arrow[d, "(d_1d_3, d_0d_3, d_1d_0)"] \\ & X_1 \arrow[r, "(s_0d_0, Id_{X_1}, s_0d_1)"'] & X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1 \end{tikzcd}$$

Observation: $X_{hoeqchoice}$ carries more information that $X_{hoequiv}$. Since, a point in $X_{hoequiv}$ is a morphism such that there exists inverses. Whereas, a point in $X_{hoeqchoice}$ is a morphism and their (left and right) inverses. The crucial fact is that, given a morphism, f in a Segal space that has an inverse, the space of inverses is contractible, meaning that up to homotopy the choice of inverse is unique!

Definition 4.4.11. A Segal space X is called a *Segal space groupoid* if every map of X is a homotopy equivalence.

The following homotopy pullback on a mapping space of a Segal space, X,

$$\begin{tikzcd} map_X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "{(d_0,d_1)}"] \\ \Delta^0 \arrow[r, "{(x,y)}"] & X_0 \times X_0 \end{tikzcd}$$

induces a local definition of homotopy equivalence,

Definition 4.4.12. For a Segal space X, *the space of homotopy equivalences between two objects x and y of X* is denoted as $hoequiv_X(x,y)$ and is defined as the following homotopy pullback,

$$\begin{tikzcd} hoequiv_X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{hoequiv} \arrow[d, "{(d_0,d_1)}"] \\ \Delta^0 \arrow[r, "{(x,y)}"] & X_0 \times X_0 \end{tikzcd}$$

Example 4.4.13. From Lemma 4.2.6 we know that if $\mathcal{C}$ is a category, then $\mathcal{N}(\mathcal{C})$ is a Segal space. Further, the homotopy equivalences in the Segal space $\mathcal{N}(\mathcal{C})$ are precisely the isomorphisms in $\mathcal{C}$.
We need to show $[\sigma] \in \mathcal{C}$ is an isomorphism if $\sigma \in \mathcal{N}(\mathcal{C})$ is a equivalence.
We know $\sigma \in \mathcal{N}(\mathcal{C})$ is an equivalence if and only if $[\sigma] \in Ho\mathcal{N}(\mathcal{C})$ is an isomorphism.
By Theorem 4.3.13, $\mathcal{C}$ is isomorphic to $Ho\mathcal{N}(\mathcal{C})$, then $[\sigma] \in \mathcal{C}$ is an isomorphism.

Example 4.4.14. From Theorem 4.2.8, we know that if X is a constant simplicial space, then it is a Segal space. Further, every morphism in X_1 is an homotopy equivalence because in a constant simplicial space X, the morphisms are invertible.

Chapter 5

Complete Segal Spaces

5.1 Introduction

In the last chapter, we have discussed the homotopical and categorical properties of a Segal space. But the homotopy theory and category theory of a Segal space are not compatible with each other.

To observe this, let us define a category $I(1)$, which has two objects $\{x, y\}$ and one invertible morphism between them,

$$x \overset{\cong}{\leftrightarrow} y.$$

We would like to understand the Segal space, $\mathcal{N}(I(1))$, which we would denote by $E(1) \coloneqq \mathcal{N}(I(1))$. Since, we have applied the nerve functor on $I(1)$, $\mathcal{N}(I(1))$ is clearly a discrete simplicial space and can be levelwise denoted as,

$$E(1)_n = \{x, y\}^{[n]}.$$

That is $E(1)_n$ has 2^{n+1} elements.
Explicitly, at the 0^{th} level, the two elements of $E(1)_0$ represent the two objects $\{x, y\}$ of the category $I(1)$.
While, $E(1)_1$ has four elements represented as $\{xx, xy, yx, yy\}$, where xx and yy denote the identity morphisms on objects x and y respectively, and xy and yx represent the morphism $x \to y$ with inverse $y \to x$ and vice-versa. This analogy can be carried over to higher n similarly.

Example 5.1.1. The category $I(1)$ and $[0]$ are equivalent, i.e. there exists two functors $F\colon [0] \to I(1)$ and $G\colon I(1) \to [0]$ such that,

$$FG\colon I(1) \to I(1)$$
$$GF\colon [0] \to [0]$$

where, FG is not the identity map but, $FG \cong Id_{I(1)}$, while GF is the identity map $Id_{[0]}$.
However, the Segal spaces $i_F^*(I(1)) \coloneqq E(1)$ and $i_\Delta^*(I(1)) \coloneqq F(1)$ are not equivalent, since $E(1)$ is not levelwise contractible.

Remark 5.1.2. The underlying homotopy theory of the groupoid, $I(1)$ is overlooked by the Segal space.

Example 5.1.3. We know that $E(1) \coloneqq \{x, y\}$ is a discrete Segal space and has two invertible morphisms, which is denoted by xy and yx. Two objects are equivalent if there is a homotopy equivalence between them. However, the objects in $E(1)_0$ are not equivalent, since there is no path between them.

Remark 5.1.4. Here the distinction between the notion of category theory and homotopy theory is very subtle. Categorically, the two points $\{x, y\}$ are equivalent, but homotopically, they are not!

Definition 5.1.5. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is an *equivalence* if and only if the following two conditions hold,

- F is fully faithful, i.e. for objects $x, y \in \mathcal{C}$,

  $$Hom_{\mathcal{C}}(x, y) \to Hom_{\mathcal{D}}(Fx, Fy)$$

  is a bijection

- F is essentially surjective, i.e. for every object $y \in \mathcal{D}$ there exists an object $x \in \mathcal{C}$ such that Fx is equivalent to y.

Remark 5.1.6. The functor, $F\colon [0] \to I(1)$ is fully faithful and essentially surjective, hence is equivalent.

Remark 5.1.7. It is similar to what happens in Example 5.1.3, the two points, $\{x, y\}$ are equivalent in the Segal space $E(1)$, but homotopically they are not in the space $E(1)_0$.

Example 5.1.8. Equivalence for categories is not well-defined for Segal spaces. $E(1)$ and $F(0)$ are equivalence of categories satisfying the above definition for a functor $F(0) \to E(1)$. However they are not equivalence of Segal spaces.

Observation: We know that the nerve functor transforms any small category into simplicial space,

$$\begin{aligned} \mathcal{N}\colon \mathbf{Cat} &\to \mathbf{sS} \\ \mathcal{C} &\to \mathcal{N}(\mathcal{C}) \end{aligned}$$

and by Lemma 4.2.6, this transformation gives us a Segal space, i.e. $\mathcal{N}(\mathcal{C})$ is a Segal space.
However, the equivalence of categories does not imply the equivalence of Segal spaces even by the action of nerves, i.e. if $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence, then it does not imply an equivalence of Segal spaces $\mathcal{N}F\colon \mathcal{N}\mathcal{C} \to \mathcal{N}\mathcal{D}$.

Example 5.1.9. $F\colon [0] \to I(1)$ is an equivalence of categories. However, $\mathcal{N}F\colon \mathcal{N}([0]) \to \mathcal{N}(I(1))$ is not an equivalence of Segal space because,

$$\begin{aligned} \mathcal{N}([0]) &\coloneqq F(0) \\ \mathcal{N}(I(1)) &\coloneqq E(1) \end{aligned}$$

and the Kan equivalence of $\mathbf{S}$ implies the equivalence of $\mathbf{sS}$, i.e. $f\colon X \to Y$ is an equivalence of $\mathbf{sS}$ if $\forall n \geq 0$m $f_n\colon X_n \to Y_n$ is levelwise a Kan equivalence of $\mathbf{sSet}$. Observe that,

$$\begin{aligned} F(0)_0 &\cong \{x\} \\ E(1)_0 &\cong \{x, y\} \end{aligned}$$

are not Kan equivalence of $\mathbf{sSet}$.
Hence, the nerve functor does not take an equivalence of small categories $\mathbf{Cat}$ to an equivalence of $\mathbf{sS}$.

5.2 Completenes Condition

Observation: In the last Example 5.1.9 there are two points $\{x, y\}$ in $E(1)$. We saw that these two points $\{x, y\}$ are homotopy equivalent in the Segal space $E(1)$. However, they are not in $E(1)_0$. We want to rectify it, i.e., we need a condition that guarantees us that whenever two points are $\{x, y\}$ are homotopy equivalent in the Segal space X, they are also equivalent in X_0.
This observation leads to the notion of *completeness*.

Remark 5.2.1. From Definition 4.4.8, we know that for a Segal space, X, the space of homotopy equivalences of X, $X_{hoequiv} \subset X_1$. The identity map is defined as,

$$\begin{aligned} s_0 \colon X_0 &\to X_1 \\ x &\mapsto Id_x \end{aligned}$$

which, takes every object of the Segal space X to the identity map. Since, an identity map is a homotopy equivalence, hence factors through $X_{hoequiv}$ and we obtain the following diagram,

$$\begin{tikzcd} X_0 \arrow[r, "s_0"] \arrow[dr] & X_1 \\ & X_{hoequiv} \arrow[u, hook] \end{tikzcd}$$

The proof that in the above diagram $X_{hoequiv} \hookrightarrow X_1$ is the inclusion of spaces can be found in Lemma 5.8 in [Rez01].

Lemma 5.2.2. For a Segal space X, $X_{hoequiv} \hookrightarrow X_1$ is an inclusion of path components.

Remark 5.2.3. In simpler words the above Lemma 5.2.2 states that for two morphisms, f, g in X_1, if $f \in X_{hoequiv}$ and there is a path from $f \to g$ in X_1, then $g \in X_{hoequiv}$.

Remark 5.2.4. The map $s_0 \colon X_0 \to X_1$ is always injective, but is also a surjection when every isomophism in X_1 is the identity.
This leads to the next definition, that of a complete Segal space.

Definition 5.2.5. A Segal space, X is called a *complete Segal space* if the map,

$$s_0 \colon X_0 \to X_{hoequiv}$$

is an equivalence of spaces.

Lemma 5.2.6. The space $E(1)$ is not a complete Segal space.

Proof. We know that $E(1)$ is a Segal space, so we will just chek for the completeness condition.

From the last Subsection 5.1 we know that,

$$\begin{aligned} E(1)_0 &\coloneqq \{x, y\} \\ E(1)_1 &\coloneqq \{xx, xy, yx, yy\} \\ \text{Also, } E(1)_{hoequiv} &\coloneqq \{xx, xy, yx, yy\} \\ &= E(1)_1 \end{aligned}$$

However,

$$\begin{aligned} E(1)_0 &\to E(1)_{hoequiv} \\ \{x, y\} &\mapsto \{xx, xy, yx, yy\} \end{aligned}$$

is not an equivalence.
Hence, the space $E(1)$ is not a complete Segal space. □

There are several equivalent ways of defining a complete Segal space X, which we state in the following theorem. The proof that these definitions are equivalent to each other can be found in Proposition 6.4 in [Rez01].

Theorem 5.2.7. For a Segal space X, the following are equivalent:

- X is a *complete Segal space*
- The following diagram is a homotopy pullback,

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_3 \arrow[d] \\ X_1 \arrow[r] & X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1 \end{tikzcd}$$

- The following map of spaces is a weak equivalence,

$$Map_{\mathbf{sS}}(E(1), X) \to Map_{\mathbf{sS}}(F(0), X)$$

- For two objects, $x, y \in X_0$, the space of homotopy equivalence between x and y in X, $hoequiv_X(x, y)$ is weakly equivalent to the space of paths in X_0 from x to y.

Summing up, everything we have done so far makes Complete Segal spaces a bisimplicial set.

Definition 5.2.8 (Alternative Definition)**.** A *complete Segal space* is a simplicial space (as defined in Section 3.2), where,

- the vertical axis (Diagram 3.2), endowed with the *Reedy fibrancy condition* (as defined in Section 4.1) has homotopical properties
- the horizontal axis (Diagram 4.2), endowed with the *Segal condition* (as defined in Section 4.3) has categorical properties
- and the two interact with each other using the *completeness condition* (as defined in Definition 5.2.18).

This is the definition that we were chasing from the beginning.

The definition of a complete Segal space groupoid is inspired from the definition of a Segal space groupoid in Definition 4.4.11.

Definition 5.2.9. A complete Segal space X is called a *complete Segal space groupoid* if every morphism of X is an equivalence.

We observed that in Example 5.1.9 that since, $I(1)$ had non-trivial isomorphism $\mathcal{N}(I(1))$ behaves terribly. In particular, if a category $\mathcal{C}$ has non-trivial isomorphism, then $\mathcal{NC}$ behaves terribly.

Theorem 5.2.10. For a category $\mathcal{C}$, $\mathcal{NC}$ is a complete Segal space if and only if $\mathcal{C}$ has no non-trivial isomorphisms.

Proof. $(\Rightarrow)$ Since, $\mathcal{NC}$ is a complete Segal space, then

$$(\mathcal{NC})_0 \xrightarrow{\simeq} (\mathcal{NC})_{hoequiv}$$

is an equivalence of spaces. But, $(\mathcal{NC})_0$ and $(\mathcal{NC})_{hoequiv}$ are both discrete sets, then,

$$(\mathcal{NC})_0 \xrightarrow{\cong} (\mathcal{NC})_{hoequiv}$$

is a bijection of sets. Hence, every homotopy equivalence is the identity map.

$(\Leftarrow)$ $\mathcal{C}$ has no non-trivial isomorphism then,

$$\mathcal{NC}_0 \xrightarrow{\cong} \mathcal{NC}_{hoequiv} = Iso(\mathcal{C})$$
$$X \mapsto Id_X$$

is a bijection of sets. Hence, is also an equivalence (because bijection is stronger than equivalence). Thus, the Segal space $\mathcal{N}\mathcal{C}$ is complete.

□

Remark 5.2.11. If the category $\mathcal{C}$ has non-trivial isomorphism, we can not work with our usual nerve. We need a better nerve. Charles Rezk introduced the classifying diagram of a category as an improvement of the usual nerve of a category in Section 3 in [Rez01].

But before introducing the classifying diagram of a category, we need another definition.

Definition 5.2.12. For a category $\mathcal{C}$, we define $\mathcal{C}^{core}$ as the sub-category of $\mathcal{C}$ with same objects as that of $\mathcal{C}$, and invertible morphisms between any two objects.

Remark 5.2.13. $\mathcal{C}^{core}$ is the maximal sub-groupoid of the category $\mathcal{C}$.

Definition 5.2.14. For a category $\mathcal{C}$, the *classifying diagram of* $\mathcal{C}$ is a **sS**, denoted by $\mathcal{NC}$, and $\forall n \geq 0$ are denoted levelwise as,

$$(\mathcal{NC})_n := \mathcal{N}(Fun([n], \mathcal{C}))^{core}.$$

Remark 5.2.15. At the 0^{th} level,

$$\begin{aligned}(\mathcal{NC})_0 &:= \mathcal{N}(Fun[0], \mathcal{C})^{core}\\ &= \mathcal{N}(\mathcal{C})^{core}\end{aligned}$$

represents the space of objects.

Concretely each cell of the classifying diagram can be denoted as,

$$(\mathcal{NC})_{n;} := Hom_{\mathbf{Cat}}([n] \times I(l), \mathcal{C})$$

Thus, levelwise a $\mathcal{NC}$ can be diagrammatically represented as,

$$\begin{tikzcd}
(\mathcal{N}\mathcal{C})_{00} \arrow[r, <-, shift left=2] \arrow[r] \arrow[r, <-, shift right=2] \arrow[d, <-, shift left=2] \arrow[d] \arrow[d, <-, shift right=2] & (\mathcal{N}\mathcal{C})_{10} \arrow[r, <-, shift left=4] \arrow[r, shift left=2] \arrow[r, <-] \arrow[r, shift right=2] \arrow[r, <-, shift right=4] \arrow[d, <-, shift left=2] \arrow[d] \arrow[d, <-, shift right=2] & (\mathcal{N}\mathcal{C})_{20} \arrow[r, <-, shift left=6] \arrow[r, shift left=4] \arrow[r, <-, shift left=2] \arrow[r] \arrow[r, <-, shift right=2] \arrow[r, shift right=4] \arrow[r, <-, shift right=6] \arrow[d, <-, shift left=2] \arrow[d] \arrow[d, <-, shift right=2] & \cdots \\
(\mathcal{N}\mathcal{C})_{01} \arrow[r, <-, shift left=2] \arrow[r] \arrow[r, <-, shift right=2] \arrow[d, <-, shift left=4] \arrow[d, shift left=2] \arrow[d, <-] \arrow[d, shift right=2] \arrow[d, <-, shift right=4] & (\mathcal{N}\mathcal{C})_{11} \arrow[r, <-, shift left=4] \arrow[r, shift left=2] \arrow[r, <-] \arrow[r, shift right=2] \arrow[r, <-, shift right=4] \arrow[d, <-, shift left=4] \arrow[d, shift left=2] \arrow[d, <-] \arrow[d, shift right=2] \arrow[d, <-, shift right=4] & (\mathcal{N}\mathcal{C})_{21} \arrow[r, <-, shift left=6] \arrow[r, shift left=4] \arrow[r, <-, shift left=2] \arrow[r] \arrow[r, <-, shift right=2] \arrow[r, shift right=4] \arrow[r, <-, shift right=6] \arrow[d, <-, shift left=4] \arrow[d, shift left=2] \arrow[d, <-] \arrow[d, shift right=2] \arrow[d, <-, shift right=4] & \cdots \\
(\mathcal{N}\mathcal{C})_{02} \arrow[r, <-, shift left=2] \arrow[r] \arrow[r, <-, shift right=2] \arrow[d, <-, shift left=6] \arrow[d, shift left=4] \arrow[d, <-, shift left=2] \arrow[d] \arrow[d, <-, shift right=2] \arrow[d, shift right=4] \arrow[d, <-, shift right=6] & (\mathcal{N}\mathcal{C})_{12} \arrow[r, <-, shift left=4] \arrow[r, shift left=2] \arrow[r, <-] \arrow[r, shift right=2] \arrow[r, <-, shift right=4] \arrow[d, <-, shift left=6] \arrow[d, shift left=4] \arrow[d, <-, shift left=2] \arrow[d] \arrow[d, <-, shift right=2] \arrow[d, shift right=4] \arrow[d, <-, shift right=6] & (\mathcal{N}\mathcal{C})_{22} \arrow[r, <-, shift left=6] \arrow[r, shift left=4] \arrow[r, <-, shift left=2] \arrow[r] \arrow[r, <-, shift right=2] \arrow[r, shift right=4] \arrow[r, <-, shift right=6] \arrow[d, <-, shift left=6] \arrow[d, shift left=4] \arrow[d, <-, shift left=2] \arrow[d] \arrow[d, <-, shift right=2] \arrow[d, shift right=4] \arrow[d, <-, shift right=6] & \cdots \\
\vdots \arrow[d, equal] & \vdots \arrow[d, equal] & \vdots \arrow[d, equal] & \\
\mathcal{C}^{core} \arrow[r, <-, shift left=2] \arrow[r] \arrow[r, <-, shift right=2] & Fun([1],\mathcal{C})^{core} \arrow[r, <-, shift left=4] \arrow[r, shift left=2] \arrow[r, <-] \arrow[r, shift right=2] \arrow[r, <-, shift right=4] & Fun([2],\mathcal{C})^{core} \arrow[r, <-, shift left=6] \arrow[r, shift left=4] \arrow[r, <-, shift left=2] \arrow[r] \arrow[r, <-, shift right=2] \arrow[r, shift right=4] \arrow[r, <-, shift right=6] & \cdots
\end{tikzcd}$$

Lemma 5.2.16. For any category $\mathcal{C}$, $\mathcal{N}\mathcal{C}$ is a Reedy fibrant simplicial space.

For a proof of the above lemma readers are advised to refer to Lemma 3.9 in [Rez01].

Lemma 5.2.17. For any category $\mathcal{C}$, $\mathcal{N}\mathcal{C}$ is a Segal space.

Proof. To show $\mathcal{N}\mathcal{C}$ is a Segal space, we need to show that,

$$\begin{tikzcd}
\mathcal{N}\mathcal{C}_2 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{N}\mathcal{C}_1 \arrow[d] \\
\mathcal{N}\mathcal{C}_1 \arrow[r] & \mathcal{N}\mathcal{C}_0
\end{tikzcd}$$

for which we can show that,

$$\begin{tikzcd}
Fun([2],\mathcal{C}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Fun([1],\mathcal{C}) \arrow[d] \\
Fun([1],\mathcal{C}) \arrow[r] & Fun([0],\mathcal{C})
\end{tikzcd}$$

and then action with *core* and $\mathcal{N}$ because, they preserve pullbacks where $\mathcal{N}\mathcal{C}_i \coloneqq \mathcal{N}(Fun([i],\mathcal{C})^{core})$.
But showing the second diagram is the same as showing,

$$\begin{tikzcd} \mathcal{NC}_2 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{NC}_1 \arrow[d] \\ \mathcal{NC}_1 \arrow[r] & \mathcal{NC}_0 \end{tikzcd}$$

because, $Fun([i], \mathcal{C}) =: \mathcal{NC}_i$.
We already know that $\mathcal{NC}$ is a Segal space, i.e.

$$\mathcal{N}C_n \xrightarrow{\simeq} \underbrace{\mathcal{N}C_1 \times_{\mathcal{N}C_0} \cdots \times_{\mathcal{N}C_0} \mathcal{N}C_1}_{n\ factors}.$$

Checking for $n = 2$, yields the above pullback. □

Lemma 5.2.18. For any category $\mathcal{C}$, $\mathcal{NC}$ satisfies the completeness condition.

Proof. We need to show that $\mathcal{NC}_0 \to \mathcal{NC}_{hoequiv} =: \mathcal{N}Iso(\mathcal{C}^{core})$ is an equivalence.
We know that if $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence then $F^{core}\colon \mathcal{C}^{core} \to \mathcal{D}^{core}$ is an equivalence of groupoids and then $\mathcal{N}F^{core}\colon \mathcal{NC}^{core} \to \mathcal{ND}^{core}$ is a Kan equivalence. Substituting, $\mathcal{D}$ with $Iso(\mathcal{C})$ we are done. □

Theorem 5.2.19. $\mathcal{NC}$ is a complete Segal space for any category $\mathcal{C}$.

Proof. By Lemma 5.2.16, Lemma 5.2.17 and Lemma 5.2.18, we can conclude that for any category $\mathcal{C}$, $\mathcal{NC}$ is a complete Segal space. □

Lemma 5.2.20. Let, $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. If F is an equivalence, then $\mathcal{N}F\colon \mathcal{NC} \to \mathcal{ND}$ is a levelwise equivalence of complete Segal spaces.

Proof. If $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence if $\exists G\colon \mathcal{D} \to \mathcal{C}$ such that $FG \cong Id_{\mathcal{D}}$ and $GF \cong Id_{\mathcal{C}}$. Hence, $(\mathcal{N}F)_{(n,l)}\colon (\mathcal{NC}) := Hom_{Cat}([n] \times I(l), \mathcal{C}) \to Hom_{Cat}([n] \times I(l), \mathcal{D}) =: (\mathcal{ND})_{n,l}$ are levelwise simplicical spaces and equivalence, since $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories.
By, Theorem 5.2.19 for any category $\mathcal{C}$, $\mathcal{NC}$ is a complete Segal space, then $\mathcal{N}F\colon \mathcal{NC} \to \mathcal{ND}$ is a levelwise equivalence of complete Segal spaces, when $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories. □

Example 5.2.21. By Example 5.1.1 above, $I(1)$ and $[0]$ are equivalent as categories. Hence, $\mathcal{N}I(1)$ is equivalent to $F(0)$

We can define the complete Segal space model structure on **sS**.

Definition 5.2.22. Let, X, Y be $\mathbf{sS}$, then the simplicial morphism, $f\colon X \to Y$ on the *complete Segal space model structure on* $\mathbf{sS}$ are defined as,

- fibrant objects Z are Segal space such that,

  $$Map(E(1), Z) \to Map(F(0), Z)$$

  are Kan equivalences $\forall k \geq 2$

- cofibrations are *monomorphisms*, i.e. $f_n\colon X_n \to Y_n$ are levelwise injective maps $\forall n \geq 0$ of $\mathbf{S}$

- weak equivalences are map $f\colon X \to Y$ such that for every complete Segal space, Z, the induced map,

  $$Map(X, Z) \to Map(Y, Z)$$

  is a Kan equivalence.

Joyal and Tierney [JT07], and Toën [To5] independently proved that the complete Segal space is equivalent to other models of $(\infty, 1)-$categories and hence is a model of $(\infty, 1)$-categories.

Chapter 6

Twisted Arrow Construction

This thesis chapter is adapted from the "Twisted Arrow Construction for Segal Spaces" by Chirantan Mukherjee in [Muk22].

Having extensively defined complete Segal spaces, we would now like to see an example of the construction of complete Segal space in this chapter.

The twisted arrow category originated from the construction of comma category, which was introduced by Lawvere in [Law63]. It was been shown to be a model of quasi-category in [Lur11] and [BGN18].

6.1 Twisted Arrow Category

Definition 6.1.1. For any category $\mathcal{C}$ the twisted arrow category of $\mathcal{C}$, denoted as $Tw(\mathcal{C})$ is defined as follows:

- The *objects* of $Tw(\mathcal{C})$ are morphisms $C \xrightarrow{f} D$ in $\mathcal{C}$ for $C, D \in \mathcal{O}(\mathcal{C})$
- For f, g objects in $Tw(\mathcal{C})$, the *morphism* of $Tw(\mathcal{C})$ is defined as a commutative diagram,

$$\begin{tikzcd}
C \arrow[r, "k"] \arrow[d, "f"'] & C' \arrow[d, "g"] \\
D & D' \arrow[l, "h"]
\end{tikzcd}$$

which satisfies $f = hgk$, where h, k are morphisms of $\mathcal{C}$.

- For f, f', f'' objects in $Tw(\mathcal{C})$, the *composition* of mophisms between

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, "f"'] & C' \arrow[d, "f'"] \\ D & D' \arrow[l, "h"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C' \arrow[r, "k'"] \arrow[d, "f'"'] & C'' \arrow[d, "f''"] \\ D' & D'' \arrow[l, "h'"] \end{tikzcd}$$

in $Tw(\mathcal{C})$ is defined as a commutative diagram of the following form,

$$\begin{tikzcd} C \arrow[r, "k''"] \arrow[d, "f"'] & C'' \arrow[d, "f''"] \\ D & D'' \arrow[l, "h''"] \end{tikzcd}$$

which satisfies $f = h''f''k''$, and $h'' \coloneqq h'h$ and $k'' \coloneqq k'k$, where h, k, h', k', h'', k'' are all morphisms of $\mathcal{C}$.

Remark 6.1.2. The source and target of the morphisms in $Tw(\mathcal{C})$ above are given by f and g respectively.

Thus, there is a forgetful functor, which takes an object in $Tw(\mathcal{C})$ to its source and target in $\mathcal{C}$ as follows,

$$Tw(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C}$$
$$(C \xrightarrow{f} D) \mapsto (C, D)$$

The twisted arrow category has been studied for quasi-categories by Lurie in [Lur11], by Barwick, Glasman and Nardin in [BGN18], and for Segal spaces by Bergner, Osorno, Ozornova, Rovelli and Scheimbauer in [BOO$^+$20]. Our approach are similar to Proposition A.2.3 in [HMS20], though we use model categorical methods.

Definition 6.1.3. Let, I and J be two totally ordered set, then the *concatenation* of I and J is denoted by $I * J$ and is a coproduct $I \coprod J$ equipped with a total ordering given by $i \leq j, \forall i \in I, j \in J$.

Remark 6.1.4. Since, $[n]$ is a totally ordered in $\mathbf{\Delta}$, we get a functor

$$*\colon \mathbf{\Delta} \times \mathbf{\Delta} \to \mathbf{\Delta}$$
$$([n], [m]) \mapsto [n] * [m]$$

where, the concatenation $[n] * [m]$ is defined as in Definition 6.1.3.

Remark 6.1.5. Using Remark 6.1.4 and Example 2.3.7, the concatenation $\Delta^n \star \Delta^m$ is defined as,

$$\star \colon \mathbf{sSet} \times \mathbf{sSet} \to \mathbf{sSet}$$
$$(\Delta^n, \Delta^m) \coloneqq (\mathbf{\Delta}(-,[n]), \mathbf{\Delta}(-,[m])) \mapsto \mathbf{\Delta}(-,[n] * [m]) =: \Delta^n \star \Delta^m$$

Definition 6.1.6. For a simplicial set, X, the *twisted arrow simplicial set* of X, denoted by $Tw(X)$ is a simplicial set and is defined levelwise as,

$$\begin{aligned} Tw(X)_n &= Hom_{\mathbf{sSet}}((\Delta^n)^{op} \star \Delta^n, X) \\ &\cong X_{2n+1} \end{aligned}$$

Remark 6.1.7. $Tw(X) \cong Tw(X^{op})$ because $(\Delta^n)^{op} \star \Delta^n = \Delta^n \star (\Delta^n)^{op}$, i.e. the source and target maps are interchanged.

Example 6.1.8. The twisted arrow category of **Set** is denoted by $Tw(\mathbf{Set})$, whose objects are the morphisms of **Set**, and morphisms are given by the following commutative diagram,

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & X' \arrow[d, "g"] \\ Y & Y' \arrow[l, "\beta"'] \end{tikzcd}$$

where, $f \coloneqq \beta g \alpha$.

Example 6.1.9. Let us now give a detailed treatment to what happens to the twisted arrow simplicial set $X \in \mathbf{sSet}$.
Levelwise it is defined as,

- $Tw(X)_0 \cong Hom_{Tw(\mathbf{sSet})}((\Delta^0)^{op} \star \Delta^0, X) \cong X_1$
- $Tw(X)_1 \cong Hom_{Tw(\mathbf{sSet})}((\Delta^1)^{op} \star \Delta^1, X) \cong X_3$
- $Tw(X)_2 \cong Hom_{Tw(\mathbf{sSet})}((\Delta^2)^{op} \star \Delta^2, X) \cong X_5$

To understand the face and degeneracy maps of $Tw(X)_n$, it is enough to identify the coface and codegeneracy maps of Δ^{2n+1}, because by Yoneda lemma,

$$X_{2n+1} \cong Hom(\Delta^{2n+1}, X).$$

Since, the morphisms in Δ^n, are objects in Tw, we can label,

- Δ^1 as $(01) :=\ 0 \longrightarrow 1$
- $\Delta^3 :=$
$$\begin{tikzcd} & 2 \arrow[rr] & & 3 \\ 0 \arrow[ur] \arrow[urrr] \arrow[rr] & & 1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$
as $(12) \longrightarrow (03)$

Then, the degeneracy and face maps are defined as follows,

- For $Tw(X)_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} \xrightarrow{s_0} Tw(X)_1$, the degeneracy map is as follows,

$$\begin{aligned} s_0 \colon Tw(X)_0 &\to Tw(X)_1 \\ X_{(01)} &\mapsto \{X_{(01)} \xrightarrow{Id} X_{(01)}\} \end{aligned}$$

  and the face maps are as follows,

$$\begin{aligned} d_0 \colon Tw(X)_1 &\to Tw(X)_0 \\ \{X_{(12)} \to X_{(03)}\} &\mapsto X_{(12)} \end{aligned}$$

$$\begin{aligned} d_1 \colon Tw(X)_1 &\to Tw(X)_0 \\ \{X_{(12)} \to X_{(03)}\} &\mapsto X_{(03)} \end{aligned}$$

- For $Tw(X)_1 \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} Tw(X)_2$, the degeneracy maps are as follows,

$$s_0, s_1 \colon Tw(X)_1 \to Tw(X)_2$$
$$\{X_{(12)} \to X_{(03)}\} \mapsto \left\{ \begin{tikzcd} & X_{(14)} \arrow[d] \\ X_{(23)} \arrow[ur] \arrow[r] & X_{(05)} \end{tikzcd} \right\}$$

  and the face maps are as follows,

$$d_0 \colon Tw(X)_2 \to Tw(X)_1$$
$$\left\{ \begin{tikzcd} & X_{(14)} \arrow[d] \\ X_{(23)} \arrow[ur] \arrow[r] & X_{(05)} \end{tikzcd} \right\} \mapsto \{X_{(14)} \to X_{(05)}\}$$

$$d_1 \colon Tw(X)_2 \to Tw(X)_1$$

$$\left\{ \begin{tikzcd} & X_{(14)} \arrow[d] \\ X_{(23)} \arrow[ur] \arrow[r] & X_{(05)} \end{tikzcd} \right\} \mapsto \{X_{(23)} \to X_{(05)}\}$$

$$d_2 \colon Tw(X)_2 \to Tw(X)_1$$

$$\left\{ \begin{tikzcd} & X_{(14)} \arrow[d] \\ X_{(23)} \arrow[ur] \arrow[r] & X_{(05)} \end{tikzcd} \right\} \mapsto \{X_{(23)} \to X_{(14)}\}$$

6.2 Twisted Arrow Segal Space

Our aim in this section is to prove the twisted arrow complete Segal space is a complete Segal space. This breaks down into proving the following:

- Twisted arrow Reedy fibrant simplicial space is a Reedy fibrant simplicial space (as proven in Theorem 6.2.4)
- Twisted arrow Segal space is a Segal space (as proven in Theorem 6.2.5)
- Twisted arrow complete Segal space satisfies the completeness condition (as proven in Theorem 6.2.9)

To show that the twisted arrow Reedy fibrant simplicial space is a Reedy fibrant simplicial space, we need to first define the twisted arrow construction of a simplicial space. Simplicial spaces are bismiplicial sets, meaning they have two simplicial directions. Hence, there are three plausible choices for the twisted arrow construction for a simplicial space X, $Tw(X)$.

1. Applying the twisted arrow construction in both the directions at the same time, we obtain,

$$Tw(X)_{mn} = X_{2m+1,2n+1}$$

i.e. concretely,

$$Tw(F(m)) = F(2m+1)$$
$$Tw(\Delta^n) = \Delta^{2n+1}$$

2. Applying the twisted arrow construction only in the categorical direction, we obtain,

$$Tw(X)_{mn} = X_{2m+1,n}$$

i.e. concretely,

$$Tw(F(m)) = F(2m+1)$$
$$Tw(\Delta^n) = \Delta^n$$

3. Applying the twisted arrow construction only in the spatial direction, we obtain,

$$Tw(X)_{mn} = X_{m,2n+1}$$

i.e. concretely,

$$Tw(F(m)) = F(m)$$
$$Tw(\Delta^n) = \Delta^{2n+1}$$

We are going to use the construction (2) as our definition, since it is easier to work with.

Lemma 6.2.1. The construction (1) and (2) are equivalent to each other, while (3) is equivalent to the identity.

Proof. To show: (1) and (2) are equivalent to each other.
Concretely, we need to show that there is a natural equivalence between $F(2m+1) \times \Delta^{2n+1}$ and $F(2m+1) \times \Delta^n$, which is same as showing that there is a natural equivalence between Δ^{2n+1} and Δ^n. This is trivially true, hence (1) and (2) are equivalent to each other.
To show: (3) is equivalent to identity.
Again, concretely we need to show that there is a natural equivalence between $F(m)\times \Delta^{2n+1}$ and $F(m) \times \Delta^n$, which is same as showing that there is a natural equivalence between Δ^{2n+1} and Δ^n. This is trivially true beacuse Δ^{2n+1} and Δ^n are equivalent, hence (3) is equivalent to identity. $\square$

Thus, we obtain the following levelwise isomorphism,

$$Hom(F(n), Tw(X)) \cong Hom(F(2n+1), X)$$
$$Tw(X)_n \cong X_{2n+1}$$

Next, we would like to see how the boundary maps behave, for which we have the following lemma,

Lemma 6.2.2. The boundary of the twisted arrow simplicial space X can be expressed as the equalizer

$$Hom(\partial F(n), Tw(X)) \longrightarrow \prod_{0\leq i\leq n} X_{2n-1} \rightrightarrows \prod_{0\leq i\leq j\leq n} X_{2n-3}$$

Proof. From the definition of the twisted arrow category, we have,

$$Hom(F(n-2), Tw(X)) \cong Hom(F(2n-3), X)$$
$$Hom(F(n-1), Tw(X)) \cong Hom(F(2n-1), X)$$

Then, we have the following coequalizer,

$$\coprod_{0\leq i\leq j\leq n} F(n-2) \overset{f}{\underset{g}{\rightrightarrows}} \coprod_{0\leq i\leq n} F(n-1) \longrightarrow \partial F(n)$$

Hence, we have the following equalizer,

$$\prod_{0\leq i\leq j\leq n} Hom(F(n-2), Tw(X)) \overset{f'}{\underset{g'}{\leftleftarrows}} \prod_{0\leq i\leq n} Hom(F(n-1), Tw(X))) \longleftarrow Hom(\partial F(n), Tw(X))$$

By definition,

$$\prod_{0\leq i\leq j\leq n} Hom(F(2n-3), X) \overset{f'}{\underset{g'}{\leftleftarrows}} \prod_{0\leq i\leq n} Hom(F(2n-1, X)) \longleftarrow Hom(\partial F(n), Tw(X))$$

Thus, the boundary of twisted arrow category is an equalizer of the following form,

$$\prod_{0\leq i\leq j\leq n} X_{2n-3} \overset{f'}{\underset{g'}{\leftleftarrows}} \prod_{0\leq i\leq n} X_{2n-1} \longleftarrow Hom(\partial F(n), Tw(X))$$

□

Lemma 6.2.3. If $p\colon C \twoheadrightarrow A$ is a Kan fibration and $i\colon E \hookrightarrow A$ be a sub-simplicial set such that p factors through i, then $C \to E$ is also a Kan fibration.

Proof. We have the following pullback diagram,

$$\begin{tikzcd}
C \arrow[r, "Id_C"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \\
E \arrow[r, "Id_E"] \arrow[d, "Id_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, hook, "i"] \\
E \arrow[r, hook, "i"'] & A
\end{tikzcd}$$

Using, the fact that Kan fibrations are closed under pullback,

$$\begin{tikzcd}
C \arrow[r, "Id_C"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, dashed, bend right=60] & C \arrow[d] \arrow[dd, two heads, bend left=60, "p"] \\
E \arrow[r, "Id_E"] \arrow[d, "Id_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, hook, "i"] \\
E \arrow[r, hook, "i"'] & A
\end{tikzcd}$$

we can conclude that $C \to E$ is a Kan fibration, because p is. □

Theorem 6.2.4. If X is a Reedy fibrant simplicial space, then $Tw(X)$ is also a Reedy fibrant simplicial space.

Proof. We need to show that,

$$Map(F(n), Tw(X)) \twoheadrightarrow Map(\partial F(n), Tw(X))$$

is a Kan fibration.
By Lemma 6.2.2, we know that the $Map(\partial F(n), X)$ is an equalizer, i.e.

$$Eq(f', g') \coloneqq Hom(\partial F(n), Tw(X)) \longrightarrow \prod_{0 \leq i \leq n} X_{2n-1} \overset{f'}{\underset{g'}{\rightrightarrows}} \prod_{0 \leq i \leq j \leq n} X_{2n-3}$$

Substituting, $C = Map(F(n), Tw(X))$, $E = Eq(f', g')$ and $A = \prod_{0 \leq i \leq n} X_{2n-1}$ in Lemma 6.2.3, we have

$$Map(F(n), Tw(X)) \twoheadrightarrow Map(\partial F(n), Tw(X))$$

is a Kan fibration. $\square$

Having shown the twisted arrow Reedy fibrant simplicial space is a Reedy fibrant simplicial space, we are now ready to show that the twisted arrow Segal space is a Segal space.

Theorem 6.2.5. If X is a Segal space, then $Tw(X)$ is also a Segal space.

Proof. To show that $Tw(X)$ is a Segal space we need to show that $Tw(X)$ is a Reedy fibrant simplicial space for which the following map,

$$Tw(X)_n \xrightarrow{\simeq} \underbrace{Tw(X)_1 \underset{Tw(X)_0}{\times} \cdots \underset{Tw(X)_0}{\times} Tw(X)_1}_{n-\ factors} \tag{6.1}$$

is a Kan equivalence $\forall n \geq 2$.
By Theorem 6.2.4, we know that $Tw(X)$ is a Reedy fibrant simplicial space if X is a Reedy fibrant simplicial space. We already know from Definition 4.2.1, that a Segal space is a Reedy fibrant simplicial space. Hence, we just have to show that the map defined in Equation (6.1) is a Kan equivalence.
By property of Twisted arrow construction we can write $Tw(X)_n \cong X_{2n+1}$. Hence, proving that Equation (6.1) is a map of Kan equivalence is the same as proving the following map is a Kan equivalence,

$$X_{2n+1} \xrightarrow{\simeq} \underbrace{X_3 \underset{X_1}{\times} \cdots \underset{X_1}{\times} X_3}_{n-\ factors}. \tag{6.2}$$

We prove Equation (6.2) using induction.
For $n = 2$, we have to show that the following map is a Kan equivalence,

$$X_5 \xrightarrow{\simeq} X_3 \underset{X_1}{\times} X_3$$

The right hand side can be written as a pullback diagram and using the fact that X

is a Segal space, we have,

$$X_3 \xrightarrow{\simeq} X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1$$

is a Kan equivalence. Thus, we want to show that the following is a pullback square,

$$\begin{tikzcd}
X_3 \underset{X_1}{\times} X_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_3 \arrow[r, "\simeq"] \arrow[d] & X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1 \\
X_3 \arrow[r] \arrow[d, "\simeq"'] & X_1 & \\
X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1 & &
\end{tikzcd}$$

We want to show that the dashed arrows denoted by π_1 and π_3 are projections of the 1^{st} and 3^{rd} components respectively,

$$\begin{tikzcd}
X_5 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_3 \arrow[r, "\simeq"] \arrow[d, "Tw(d_1)"] & X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1 \arrow[dl, dashed, "\pi_1"] \\
X_3 \arrow[r, "Tw(d_0)"] \arrow[d, "\simeq"'] & X_1 & \\
X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1 \arrow[ur, dashed, "\pi_3"'] & &
\end{tikzcd}$$

Using, the fact that,

$$Tw(d_0)\colon Tw(X_1) \cong X_3 \to Tw(X_0) \cong X_1$$

we conclude, $X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1 \to X_1$ is the 3^{rd} projection map.
Similarly, using, the fact that,

$$Tw(d_1)\colon Tw(X_1) \cong X_3 \to Tw(X_0) \cong X_1$$

we conclude, $X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1 \to X_1$ is the 1^{st} projection map.

From the above diagram we have,

$$X_5 \xrightarrow{\simeq} (X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1) \underset{X_1}{\times} (X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1)$$

Using the fact that X is a Segal space, we have,

$$X_3 \xrightarrow{\simeq} X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1$$

is a Kan equivalence.
Hence, we can conclude that,

$$X_5 \xrightarrow{\simeq} X_3 \underset{X_1}{\times} X_3$$

$$Tw(X)_2 \xrightarrow{\simeq} Tw(X)_1 \underset{Tw(X)_0}{\times} Tw(X)_1$$

is a Kan equivalence.
The higher inductive steps follows similarly.
Hence, $Tw(X)$ is a Segal space, if X is a Segal space. □

Having shown the twisted arrow Segal space is a Segal space, we are ready to show the twisted arrow complete Segal space is a complete Segal space. But before that we prove the following useful lemmas which will be useful to us.

Lemma 6.2.6. If X is a Segal space then there is an equivalence of categories $Tw(HoX) \simeq HoTw(X)$.

Proof. The objects and morphisms of $Tw(HoX)$ are defined as,

$$\begin{aligned}\mathcal{O}(Tw(HoX)) &= \mathcal{M}(HoX) \\ &= \pi_0(Map_W(x,y)) \\ &= X_{10}/\sim\end{aligned}$$

where, $x, y \in X_{00}$ and $x \sim y$ if there is a path from x to y in X_{10}.
Let, $[f], [g] \in \mathcal{O}(Tw(HoX))$, then,

$$Hom_{Tw(HoX)}([f],[g]) = \{([k],[h] \mid [h][g][k] = [f])\}$$

The objects of $HoTw(X)$ are defined as,

$$\begin{aligned}\mathcal{O}(HoTw(X)) &= \mathcal{O}(HoX)\\ &= \mathcal{M}(W)\\ &= X_{10}\end{aligned}$$

Let, $f, g \in \mathcal{O}(HoTw(X))$ then,

$$\begin{aligned}Hom_{HoTw(X)}(f,g) &= \pi_0(Map_{Tw(X)}(f,g))\\ &= \pi_0(X_3 \underset{X_1 \times X_1}{\times} *)\\ &= \{\sigma \in X_{30} \mid d_1 d_1 \sigma = f, d_0 d_3 \sigma = g\}/\sim\end{aligned}$$

where, $f, g \in X_{10}$ and $f \sim g$ if there is a path from f to g in X_{30}.
Let us define a functor,

$$F\colon HoTw(X) \to Tw(HoX)$$

which acts on objects in the following way,

$$\begin{aligned}F\colon X_{10} &\to X_{10}/\sim\\ f &\mapsto [f]\end{aligned}$$

and acts on morphisms as,

$$\begin{aligned}F\colon\ \{\sigma \in X_{30} \mid d_1 d_1 \sigma = f, d_0 d_3 \sigma = g\}/\sim &\to \{([k],[h] \mid [h][g][k] = [f])\}\\ [\sigma] &\mapsto ([d_2 d_2 \sigma], [d_0 d_0 \sigma])\end{aligned}$$

We further show that F preserves identities. Fix an object $f\colon x \to y$ in X_{10}, where, $x, y \in X_{00}$. Then,

$$\begin{aligned}F([s_1 s_1 f]) &= ([d_2 d_2 s_1 s_1 f], [d_0 d_0 s_1 s_1 f])\\ &\coloneqq ([Id_x], [Id_y])\end{aligned}$$

Finally, we show that F also preserves compositions.
Let, $[\sigma] \in Hom_{HoTw(X)}(f,g)$ and $[\sigma'] \in Hom_{HoTw(X)}(g,p)$ then the composition $Comp([\sigma],[\sigma'])$ is defined by the pullback,

$$\begin{tikzcd}
Comp([\sigma],[\sigma']) \arrow[r, hook] \arrow[d, "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] & map(f,g,p) \arrow[d, "\simeq"] \\
\Delta^0 \arrow[r] & map(f,g) \times map(g,p)
\end{tikzcd}$$

Since, $Comp([\sigma],[\sigma'])$ is contractible, then for $\sigma'' \in Comp([\sigma],[\sigma'])$, we have $[\sigma''] = [\sigma'] \circ [\sigma]$.
Hence,

$$\begin{aligned}
F([\sigma']) \circ F([\sigma]) &= ([d_2 d_2 \sigma'], [d_0 d_0 \sigma']) \circ ([d_2 d_2 \sigma], [d_0 d_0 \sigma]) \\
&= ([d_2 d_2 \sigma''], [d_0 d_0 \sigma'']) \\
&= F([\sigma''])
\end{aligned}$$

We are just left to show that the functor F is an equivalence.
By definition F is essentially surjective.
To show, that F is full, we need to show that for any $([k],[h]) \in Hom_{Tw(HoX)}([f],[g])$, $\exists [\sigma] \in Hom_{HoTw(X)}(f,g)$ such that $F([\sigma]) = ([k],[h])$.
By Segal condition, we have,

$$X_3 \xrightarrow{\simeq} X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1$$

Thus, we have the following equivalence of spaces,

$$\pi_0(X_3) \xrightarrow{\simeq} \pi_0(X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1)$$

where, $\pi_0(X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1) \coloneqq \{[(\alpha,\beta,\gamma)] | d_0\alpha = d_1\beta, d_0\beta = d_1\gamma\}$.
Given $([k],[h]) \in Hom_{Tw(HoX)}([f],[g])$, we pick representatives $h, k \in X_1$ such that,

$$hgk \sim f$$

and

$$\begin{aligned}
[(k,g,h)] &\in \pi_0(X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1) \\
&\simeq \pi_0(X_3)
\end{aligned}$$

Hence, there exists a unique $[\sigma] \in \pi_0(X_3)$ such that it corresponds to $[(k,g,h)] \in \pi_0(X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1)$ by the Segal condition.

We have the following commutative triangles by Theorem 6.2.5, such that

$$\pi_0(d_0d_0)([\sigma]) = [h]$$
$$\pi_0(d_2d_2)([\sigma]) = [k]$$

$$\begin{tikzcd}
 & \pi_0(X_3) \arrow[dl, "\pi_0(d_0d_0)"'] \arrow[d, "\simeq"'] \arrow[dr, "\pi_0(d_2d_2)"] & \\
\pi_0(X_1) & \pi_0(X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1) \arrow[l, "\pi_0(\pi_1)"] \arrow[r, "\pi_0(\pi_3)"'] & \pi_0(X_1)
\end{tikzcd}$$

where, π_1 and π_3 are the 1$^{\text{st}}$ and 3$^{\text{rd}}$ projection map.
From which we obtain the following commutative triangle by picking representatives, such that

$$d_0d_0(\sigma) = k$$
$$d_2d_2(\sigma) = h$$

$$\begin{tikzcd}
 & X_3 \arrow[dl, "d_0d_0"'] \arrow[d, "\simeq"'] \arrow[dr, "d_2d_2"] & \\
X_1 & X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1 \arrow[l, "\pi_1"] \arrow[r, "\pi_3"'] & X_1
\end{tikzcd}$$

Hence, $F([\sigma]) = ([k], [h])$.
To show, that F is faithful, we need to show that for any $[\sigma], [\sigma'] \in Hom_{HoTw(X)}(f, g)$, $F([\sigma]) = F([\sigma']) \implies [\sigma] = [\sigma']$.
Since,

$$F([\sigma]) = F([\sigma'])$$
$$\implies ([k], [h]) = ([k'], [h'])$$

we have,

$$[k] = [k']$$
$$[h] = [h']$$

So, there exists paths $\gamma_k \colon \Delta^1 \to X_1$, such that

$$\gamma_k(0) = k$$
$$\gamma_k(1) = k'$$

$\gamma_h \colon \Delta^1 \to X_1$, such that,

$$\gamma_h(0) = h$$
$$\gamma_h(1) = h'$$

and $\gamma_g \colon \Delta^1 \to X_1$ is the identity, i.e.,

$$\gamma_g(0) = g$$
$$\gamma_g(1) = g$$

This implies the existence of a lift

$$\begin{tikzcd}
\partial\Delta^1 \arrow[r] \arrow[d, hook] & X_3 \arrow[d, two heads, "\simeq"] \\
\Delta^1 \arrow[r] \arrow[ur, dashed] & X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1
\end{tikzcd}$$

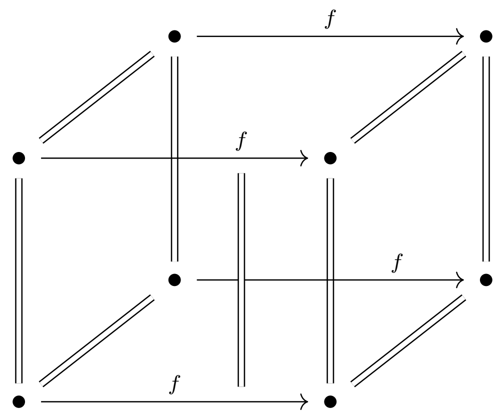
We need to construct a path $\gamma \colon \Delta^1 \to map_{Tw(X)}(f, g) \coloneqq X_3 \underset{X_1 \times X_1}{\times} *$, such that,

$$\gamma(0) = \sigma$$
$$\gamma(1) = \sigma'$$

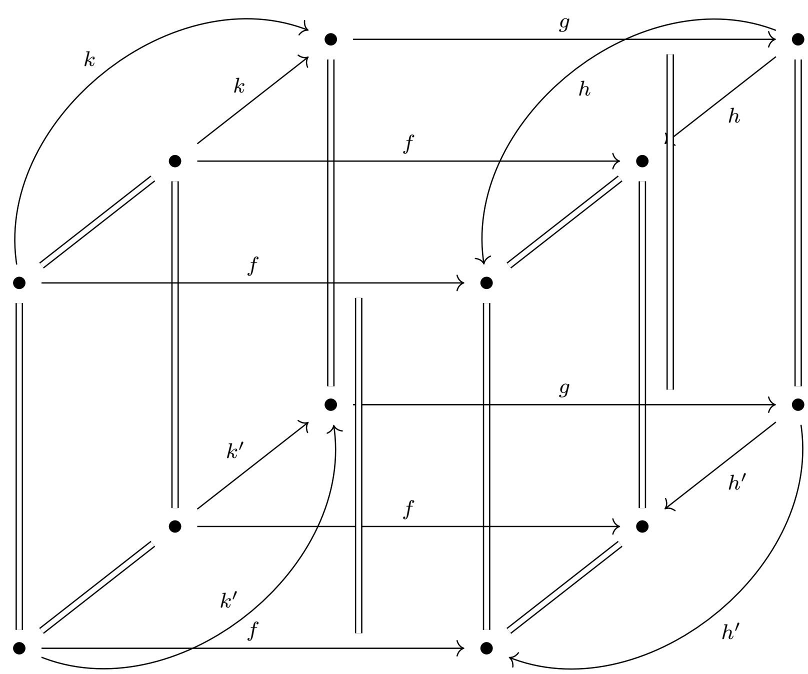
Observe, the path $\gamma_1 \colon \Delta^1 \to map_{Tw(X)_1}(f, g) \coloneqq X_3 \underset{(f,g)}{\times} *$, has the form,

$$\begin{tikzcd}
& \bullet \arrow[rr, "g"] \arrow[dd, equal] & & \bullet \arrow[dl, "h"] \arrow[dd, equal] \\
\bullet \arrow[ur, "k"] \arrow[rr, "f"] \arrow[dd, equal] & & \bullet \arrow[dd, equal] & \\
& \bullet \arrow[rr, "g"] & & \bullet \arrow[dl, "h'"] \\
\bullet \arrow[ur, "k'"] \arrow[rr, "f"] & & \bullet &
\end{tikzcd}$$

But, f and f might not have trivial homotopy, hence, γ_1 might not be constant! This, can be rectified by gluing a path $\gamma_2\colon \Delta^1 \to map_{Tw(X)_1}(f, f)$ with the above diagram,



This diagram exists, because every homotopy from f to itself is homotopic to the identity.
Hence, the path γ can be represented by gluing the paths γ_1 and γ_2 as $\gamma \coloneqq \gamma_1 * \gamma_2$ has the form,



Since, all the maps but f in path γ_2 are identities, then gluing path γ_1 and γ_2 preserves k, h, k' and h' as can be seen in the bend arrows in the above diagram.
Thus, the path γ is constant and can be represented as,

$$\begin{tikzcd}
 & \bullet \arrow[rr, "g"] \arrow[dd, equal] & & \bullet \arrow[dl, "h"] \arrow[dd, equal] \\
\bullet \arrow[ur, "k"] \arrow[rr, "f"] \arrow[dd, equal] & & \bullet \arrow[dd, equal] & \\
 & \bullet \arrow[rr, "g"] & & \bullet \arrow[dl, "h'"] \\
\bullet \arrow[ur, "k'"] \arrow[rr, "f"] & & \bullet &
\end{tikzcd}$$

Thus, $\sigma \sim \sigma \implies [\sigma] = [\sigma']$. Hence, F is faithful.
Thus, F is an equivalence. $\square$

Lemma 6.2.7. If X is a Segal space, then a morphism σ

$$\begin{tikzcd}
\bullet \arrow[r, "k"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\
\bullet & \bullet \arrow[l, "h"]
\end{tikzcd}$$

in the Segal space $Tw(X)$ is a homotopy equivalence if k and h are homotopy equivalence in X.

Proof. We need to show that $\sigma \in Tw(X)$ is an equivalence iff $[\sigma] \in HoX$ is an isomorphism.
By definition, we have σ is an equivalence in $Tw(X)$ iff $[\sigma]$ is an isomorphism in $HoTw(X)$.
By Lemma 6.2.6, $HoTw(X) \simeq Tw(HoX)$, we have $[\sigma]$ is an isomorphism in $HoTw(X)$ iff $[\sigma]$ is an isomorphism in $Tw(HoX)$.
Further, we have $[\sigma]$ is an isomorphism in $Tw(HoX)$ iff $[k], [h]$ are isomorphism in HoX, where $[k], [h] \in X_{10}/\sim$ and $x \sim y$ if there is a path from x to y in X_{10}.
By definition, we have $[k], [h]$ are isomorphism in HoX iff k, h are equivalences in X. $\square$

Lemma 6.2.8. $Tw(X)_{hoequiv}$ can be written as the following pullback square,

$$\begin{tikzcd}
Tw(X)_{hoequiv} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Tw(X)_1 \arrow[d] \\
X^{op}_{hoequiv} \times X_{hoequiv} \arrow[r, hook] & X_1^{op} \times X_1
\end{tikzcd}$$

Proof. We denote the pullback by Pb and conclude that $Pb = Tw(X)_{hoequiv}$.
So, we have the following pullback square,

$$\begin{tikzcd}
Pb \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Tw(X)_1 \arrow[d] \\
X^{op}_{hoequiv} \times X_{hoequiv} \arrow[r, hook] & X^{op}_1 \times X_1
\end{tikzcd}$$

Since, $X^{op}_{hoequiv} \times X_{hoequiv} \to X^{op}_1 \times X_1$ is an inclusion of path components, then $Pb \to Tw(X)_1$ is an inclusion of path components because inclusion of path components are stable under pullback.
From Section 5.7 of [Rez01], we have, $Tw(X)_{hoequiv} \to Tw(X)_1$ is an inclusion of path components. Thus, it suffices to show that $Tw(X)_{hoequiv}$ and Pb include the same path components of $Tw(X)_1$.
For, $f \in Tw(X)_{10}$, we need to show $f \in Pb_0 \iff f \in (Tw(X)_{hoequiv})_0$.
If $f \in Pb_0$, then $\alpha(f) \in X^{op}_{hoequiv} \times X_{hoequiv}$. Then by Lemma 6.2.7, $f \in (Tw(X)_{hoequiv})_0$.
Conversely, if $f \in (Tw(X)_{hoequiv})_0$, then we have the following commutative diagram

$$\begin{tikzcd}
Tw(X)_{hoequiv} \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Pb \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Tw(X)_1 \arrow[d] \\
& X^{op}_{hoequiv} \times X_{hoequiv} \arrow[r] & X^{op}_1 \times X_1
\end{tikzcd}$$

then by universal property of pullback, $f \in Pb_0$.
Thus, $Pb_0 = (Tw(X)_{hoequiv})_0 \implies Pb = Tw(X)_{hoequiv}$. $\square$

Theorem 6.2.9. If X is a complete Segal space, then $Tw(X)$ is a complete Segal space.

Proof. We have the following pullback diagram,

$$\begin{tikzcd} Tw(X)_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Tw(X)_1 \arrow[d] \\ X_0^{op} \times X_0 \arrow[r] & X_1^{op} \times X_1 \end{tikzcd} \tag{6.3}$$

because we have the following pullback diagram,

$$\begin{tikzcd} Pb \arrow[r] \arrow[d, two heads, "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] & Tw(X)_1 \arrow[d, two heads, "\simeq"] \\ Tw(X)_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1 \arrow[d, "(\pi_1, \pi_3)"] \\ X_0^{op} \times X_0 \arrow[r] & X_1^{op} \times X_1 \end{tikzcd}$$

The pullback diagram (6.3) factors through,

$$\begin{tikzcd} Tw(X)_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Tw(X)_{hoequiv} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Tw(X)_1 \arrow[d] \\ X_0^{op} \times X_0 \arrow[r, "\simeq"] & X_{hoequiv}^{op} \times X_{hoequiv} \arrow[r, hook] & X_1^{op} \times X_1 \end{tikzcd}$$

Since, X is a complete Segal space, then, $X_0^{op} \times X_0 \xrightarrow{\simeq} X_{hoequiv}^{op} \times X_{hoequiv}$ is an equivalence of spaces, this implies that $Tw(X)_0 \xrightarrow{\simeq} Tw(X)_{hoequiv}$ is an equivalence of spaces.
Hence, $Tw(X)$ is a complete Segal space. □

6.3 Left Fibration

Left fibrations were first studied for quasi-categories by Joyal in [Joy08], [Joy09], and later by Jacob Lurie in [Lur09a]. They were first introduced for Segal spaces independently by de Brito in [BdB18] and, Kazhdan and Varshavsky in [VK14], and for simplicial spaces by Rasekh in [Ras17]. Although the idea of generalizing the left fibrations to complete Segal spaces is due to Charles Rezk yet he did not publish any papers related to it.

Composition in a higher category can not be defined uniquely (4.1) but they can be uniquely defined upto a contractible space of choices (4.4). Similarly, functors can not be defined uniquely in a higher category. As a result, the Yoneda lemma (Lemma 2.1.20) cannot easily be generalized in higher categories since it is dependent on the Yoneda embedding functor (Definition 2.1.19). This leads to the concept of Grothendieck fibration, which was introduced by Grothedieck in the context of descent theory in [Gro95a].

Definition 6.3.1. For a category $\mathcal{C}$, a *discrete Grothendieck fibration* is a functor $p\colon \mathcal{D} \to \mathcal{C}$ such that for each morphism $f\colon c \to c'$ and choice of lift d' of c', there exists a unique lift of f, $\hat{f}$

$$\begin{tikzcd} d \arrow[r, dashed, "\exists! \hat{f}"] \arrow[d, dashed] & d' \arrow[d] \\ c \arrow[r, "f"'] & c' \end{tikzcd}$$

Definition 6.3.2. If $\mathcal{C}$ is a category and $F\colon \mathcal{C} \to \mathbf{Set}$ is a functor, then the category of *Grothendieck construction* is represented as $\int_{\mathcal{C}} F$, is defined as follows:

1. The *objects* are ordered pairs (c, x), where $c \in \mathcal{C}$ and $x \in F(c)$

2. For (c, x) and (c', x') objects in $\int_{\mathcal{C}} F$, the *morphisms* of $\int_{\mathcal{C}} F$ is defined as a set,

$$Hom_{\int_{\mathcal{C}} F}((c, x), (c', x')) = \{f \in Hom(c, c') \mid F(f)(x) = x'\}$$

Remark 6.3.3. There is a forgetful functor, which takes an object in $\int_{\mathcal{C}} F$ to its underlying category $\mathcal{C}$ using the first projection map,

$$\int_{\mathcal{C}} F \to \mathcal{C}$$
$$(c, x) \mapsto c$$

Theorem 6.3.4. If $\mathcal{C}$ is a category and $F\colon \mathcal{C} \to \mathbf{Set}$ is a functor, then the Grothendieck construction, $\int_{\mathcal{C}} F$ is a Grothendieck fibration,

$$p\colon \int_{\mathcal{C}} F \to \mathcal{C}$$

Proof. Let, $f\colon c \to c'$ be a morphism in $\mathcal{C}$ and an element $x' \in F(c')$. Letting $x = F(f)(x')$ we hence have a unique lift of f, $f\colon (c,x) \to (c',x')$. $\square$

This leads to the notion of Yoneda lemma for Grothendieck fibration, a proof of which can be found in Theorem 3.3.1 in [LR20].

Lemma 6.3.5. For any Grothendieck fibration $p\colon \mathcal{D} \to \mathcal{C}$, there is an equivalence of functors,

$$Fun_{\mathcal{C}}(\mathcal{C}/c, \mathcal{D}) \simeq Fun_{\mathcal{C}}([0], \mathcal{D})$$

The concept of functoriality depends on the existence of unique lifts. However, we know exactly how to alter the concept of uniqueness in order to acquire a working definition for higher categories, specifically by substituting contractibility for uniqueness. This leads to the notion of left fibration, which are homotopical analogue of Grothendieck fibrations, and model functors valued in spaces.

The definition of left fibration can be found in Definition 3.2 in [Ras17].

Definition 6.3.6. A *left fibration* $p\colon X \to Y$ is a Reedy fibration such that the following is a pullback diagram for all $n \geq 0$,

$$\begin{tikzcd} X_n \arrow[r, "<0>^\star"] \arrow[d, "p_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \arrow[d, "p_0"] \\ Y_n \arrow[r, "<0>^\star"'] & Y_0 \end{tikzcd}$$

We would like to show in Theorem 6.3.9 that if X is a Segal space then the projection map $p\colon Tw(X) \to X^{op} \times X$ is a left fibration from the Segal space $Tw(X)$. Following Definition 6.3.6 we need to show two things, for a Segal space X,

- $p\colon Tw(X) \to X^{op} \times X$ is a Reedy fibration (Lemma 6.3.7)
- The following diagram is a homotopy pullback square (Lemma 6.3.8),

$$\begin{tikzcd} Tw(X)_n \arrow[r, "<0>^\star"] \arrow[d, "p_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & Tw(X)_0 \arrow[d, "p_0"] \\ X_n^{op} \times X_n \arrow[r, "<0>^\star"'] & X_0^{op} \times X_0 \end{tikzcd}$$

Lemma 6.3.7. The map $Tw(X) \to X^{op} \times X$ is a Reedy fibration.

Proof. We need to show that,

$$Tw(X)_k \twoheadrightarrow (X_k^{op} \times X_k) \underset{Map(\partial F(k), X^{op}\times X)}{\times} Map(\partial F(k), Tw(X))$$

is a Kan fibration.
Hence, we need to show,

$$\begin{tikzcd}
Tw(X)_k \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& (X_k^{op} \times X_k) \underset{Map(\partial F(k), X^{op}\times X)}{\times} Map(\partial F(k), Tw(X)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Map(\partial F(k), Tw(X)) \arrow[d] \\
& X_k^{op} \times X_k \arrow[r] & Map(\partial F(k), X^{op} \times X)
\end{tikzcd}$$

Since, $Tw(X)$ and X are a Reedy fibrant simplicial space, then

$$Tw(X)_k \twoheadrightarrow Map(\partial F(k), Tw(X))$$
$$X_k^{op} \times X_k \twoheadrightarrow Map(\partial F(k), X^{op} \times X)$$

are Kan fibrations respectively.
Next, we show that the following map

$$Map(\partial F(k), Tw(X)) \twoheadrightarrow Map(\partial F(k), X^{op} \times X) \tag{6.4}$$

is a Kan fibration, by writing

$$\coprod_{0 \le i \le j \le k} F(k-2) \rightrightarrows \coprod_{0 \le i \le k} F(k-1) \to \partial F(k)$$

as a coequalizer as in Theorem 6.2.5, so we have,

$$\begin{tikzcd}
\prod\limits_{0 \le i \le j \le k} Tw(X)_{k-2} \arrow[d, two heads] & \prod\limits_{0 \le i \le k} Tw(X)_{k-1} \arrow[l, shift left] \arrow[l, shift right] \arrow[d, two heads] & Hom(\partial F(k), Tw(X)) \arrow[l] \arrow[d, two heads] \\
\prod\limits_{0 \le i \le j \le k} X_{k-2}^{op} \times X_{k-2} & \prod\limits_{0 \le i \le k} X_{k-1}^{op} \times X_{k-1} \arrow[l, shift left] \arrow[l, shift right] & Hom(\partial F(k), X^{op} \times X) \arrow[l]
\end{tikzcd}$$

Hence it suffices to show that $Tw(X)_0 \twoheadrightarrow X_0^{op} \times X_0$ is a Kan fibration.
But, $X_0^{op} = X_0$, and because, X is a Segal space, $Tw(X)_0 \twoheadrightarrow X_0 \times X_0$ is a Kan fibration.
Finally, we show the following map is a Kan fibration,

$$Tw(X)_k \twoheadrightarrow X_k^{op} \times X_k$$

because the Reedy model structure is simplicial and

$$F(k)^{op} \coprod F(k) \to F(2k+1)$$

is a cofibration. Then, X is Reedy fibrant, implies,

$$Map(F(k)^{op} \coprod F(k), X) \leftarrow Map(F(2k+1), X)$$

is a Kan fibration. Hence, $Tw(X)_k \twoheadrightarrow X_k^{op} \times X_k$ is a Kan fibration.
Since, composition of Kan fibrations are Kan fibration, we have the following diagram,

$$\begin{tikzcd}
Tw(X)_k \arrow[r] \arrow[d, two heads] & Tw(X)_k \arrow[d, two heads] & Tw(X)_k \arrow[l] \arrow[d, two heads] \\
X_k^{op} \times X_k \arrow[r] & Map(\partial F(k), X^{op} \times X) & Map(\partial F(k), Tw(X)) \arrow[l]
\end{tikzcd}$$

Hence, we can conclude that

$$Tw(X)_k \twoheadrightarrow (X_k^{op} \times X_k) \underset{Map(\partial F(k), X^{op} \times X)}{\times} Map(\partial F(k), Tw(X))$$

is a Kan fibration. □

Lemma 6.3.8. If X is a Segal space then the following diagram is a homotopy pullback square,

$$\begin{tikzcd}
Tw(X)_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Tw(X)_0 \arrow[d] \\
X_1^{op} \times X_1 \arrow[r] & X_0^{op} \times X_0
\end{tikzcd}$$

Proof. We have the following pullback diagram,

$$\begin{tikzcd}
X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1 \arrow[r, "\pi_2"] \arrow[d, "(\pi_1,\pi_3)"] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "(d_1,d_0)"] \\
X_1 \times X_1 \arrow[r, "(d_0,d_1)"] & X_0 \times X_0
\end{tikzcd}$$

But, by the Segal condition, $Tw(X)_1 \simeq X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1$, so we have the following commutative diagram,

$$\begin{tikzcd}
Tw(X)_1 \arrow[dr, twoheadrightarrow, "\simeq"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} X_1 \arrow[r, "\pi_2"] \arrow[d, "(\pi_1,\pi_3)"] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "(d_1,d_0)"] \\
& X_1 \times X_1 \arrow[r, "(d_0,d_1)"] & X_0 \times X_0
\end{tikzcd}$$

Hence,

$$\begin{tikzcd}
Tw(X)_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Tw(X)_0 \arrow[d] \\
X_1^{op} \times X_1 \arrow[r] & X_0^{op} \times X_0
\end{tikzcd}$$

is a homotopy pullback square. ☐

It might create an impression that if X is a Segal space then the projection map $p\colon Tw(X) \to X^{op} \times X$ is a not left fibration from the Segal space $Tw(X)$ because we only state and prove Lemma 6.3.8 for $n = 1$ and not a general n as we had stated to do earlier. However, that is not the case. It is correct, specifically because X is a Segal space. We have implicitly used Lemma 3.29 in [Ras17].

Theorem 6.3.9. If X is a Segal space then $Tw(X) \to X^{op} \times X$ is a left fibration from the Segal space $Tw(X)$. Moreover, if X is complete, then $Tw(X)$ is a complete Segal space.

Proof. Because X is a complete Segal space, $Tw(X)$ is a complete Segal space according to Theorem 6.2.9. Now that the map is a Reedy fibration according to Lemma

6.3.7, it is sufficient to examine the necessary homotopy pullbacks as described in Definition 6.3.6. However, because X is a Segal space, Lemma 3.29 in [Ras17] asserts that it suffices to check the homotopy pullback condition for $n = 1$, which we did in Lemma 6.3.8. $\square$

References

[AF18] David Ayala and John Francis. Flagged higher categories. In *Topology and quantum theory in interaction*, volume 718 of *Contemp. Math.*, pages 137–173. Amer. Math. Soc., [Providence], RI, 2018.

[Ale29] Paul Alexandroff. Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension. *Ann. of Math. (2)*, 30(1-4):101–187, 1928/29.

[BdB18] Pedro Boavida de Brito. Segal objects and the Grothendieck construction. In *An alpine bouquet of algebraic topology*, volume 708 of *Contemp. Math.*, pages 19–44. Amer. Math. Soc., [Providence], RI, 2018.

[Ber07] Julia E. Bergner. Three models for the homotopy theory of homotopy theories. *Topology*, 46(4):397–436, 2007.

[Ber10] Julia E. Bergner. A survey of $(\infty, 1)$-categories. In *Towards higher categories*, volume 152 of *IMA Vol. Math. Appl.*, pages 69–83. Springer, New York, 2010.

[BGN18] Clark Barwick, Saul Glasman, and Denis Nardin. Dualizing cartesian and cocartesian fibrations. *Theory Appl. Categ.*, 33:Paper No. 4, 67–94, 2018.

[BOO$^+$20] Julia E. Bergner, Angélica M. Osorno, Viktoriya Ozornova, Martina Rovelli, and Claudia I. Scheimbauer. The edgewise subdivision criterion for 2-Segal objects. *Proc. Amer. Math. Soc.*, 148(1):71–82, 2020.

[Bor31] Karol Borsuk. Sur les rétractes. *Fundamenta Mathematicae*, 17(1):152–170, 1931.

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973.

[DHK97] W. Dwyer, Philip Hirschhorn, and Daniel Kan. Model categories and more general abstract homotopy theory: a work in what we like to think of as progresss. https://people.math.rochester.edu/faculty/doug/otherpapers/dhk.pdf, 1997.

[DS95] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.

[EM45] Samuel Eilenberg and Saunders MacLane. General theory of natural equivalences. *Trans. Amer. Math. Soc.*, 58:231–294, 1945.

[EZ50] Samuel Eilenberg and J. A. Zilber. Semi-simplicial complexes and singular homology. *Ann. of Math. (2)*, 51:499–513, 1950.

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[Gro95a] Alexander Grothendieck. Technique de descente et théorèmes d'existence en géométrie algébrique. VI. Les schémas de Picard: propriétés générales. In *Séminaire Bourbaki, Vol. 7*, pages Exp. No. 236, 221–243. Soc. Math. France, Paris, 1995.

[Gro95b] Alexander Grothendieck. Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert. In *Séminaire Bourbaki, Vol. 6*, pages Exp. No. 221, 249–276. Soc. Math. France, Paris, 1995.

[HMS20] Rune Haugseng, Valerio Melani, and Pavel Safronov. Shifted coisotropic correspondences. pages 1–65. Journal of the Institute of Mathematics of Jussieu, 2020.

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[Joy02] A. Joyal. Quasi-categories and Kan complexes. volume 175, pages 207–222. 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.

[Joy08] André Joyal. Notes on quasi-categories. http://www.math.uchicago.edu/~may/IMA/Joyal.pdf, 2008.

[Joy09] André Joyal. The theory of quasi-categories and its applications. https://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf, 2009.

[JT07] André Joyal and Myles Tierney. Quasi-categories vs Segal spaces. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 277–326. Amer. Math. Soc., Providence, RI, 2007.

[Law63] F. William Lawvere. Functorial semantics of algebraic theories. *Proc. Nat. Acad. Sci. U.S.A.*, 50:869–872, 1963.

[Lee72] Ming Jung Lee. Homotopy for functors. *Proc. Amer. Math. Soc.*, 36:571–577; erratum, ibid. 42 (1973), 648–650, 1972.

[LR20] Fosco Loregian and Emily Riehl. Categorical notions of fibration. *Expo. Math.*, 38(4):496–514, 2020.

[Lur09a] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur09b] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.

[Lur11] Jacob Lurie. Derived algebraic geometry x: Formal moduli problems. https://www.math.ias.edu/~lurie/papers/DAG-X.pdf, 2011.

[Lur21] Jacob Lurie. Kerodon. https://kerodon.net, 2021.

[Mac71] Saunders MacLane. *Categories for the working mathematician*. Graduate Texts in Mathematics, Vol. 5. Springer-Verlag, New York-Berlin, 1971.

[Mar21] Louis Martini. Yoneda's lemma for internal higher categories. *arXiv preprint*, 2021. arXiv:2103.17141v2.

[Muk22] Chirantan Mukherjee. Twisted arrow construction for segal spaces. *arXiv preprint*, 2022. arXiv:2203.01788v1.

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.

[Qui68] Daniel G. Quillen. The geometric realization of a Kan fibration is a Serre fibration. *Proc. Amer. Math. Soc.*, 19:1499–1500, 1968.

[Ras17] Nima Rasekh. Yoneda lemma for simplicial spaces. *arXiv preprint*, 2017. arXiv:1711.03160v3.

[Ras18] Nima Rasekh. Introduction to complete segal spaces. *arXiv preprint*, 2018. arXiv:1805.03131v1.

[Ree74] C. L. Reedy. Homotopy theory of model categories. https://klein.mit.edu/~psh/reedy.pdf, 1974.

[Rez01] Charles Rezk. A model for the homotopy theory of homotopy theory. *Trans. Amer. Math. Soc.*, 353(3):973–1007, 2001.

[Rie11] Emily Riehl. A leisurely introduction to simplicial sets. https://math.jhu.edu/~eriehl/ssets.pdf, 2011.

[Rie16] Emily Riehl. *Category Theory in Context*. Aurora: Modern Math Originals. Dover Publications, 2016.

[Rot88] Joseph J. Rotman. *An introduction to algebraic topology*, volume 119 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988.

[RV22] Emily Riehl and Dominic Verity. *Elements of ∞-Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2022.

[Seg68] Graeme Segal. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.*, (34):105–112, 1968.

[To5] Bertrand Toën. Vers une axiomatisation de la théorie des catégories supérieures. *K-Theory*, 34(3):233–263, 2005.

[VK14] Ya. Varshavskiĭ and D. Kazhdan. The Yoneda lemma for complete Segal spaces. *Funktsional. Anal. i Prilozhen.*, 48(2):3–38, 2014.