MODEL STRUCTURE ON SIMPLICIAL CATEGORIES

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Abstract. In the first part of the talk, we review the axioms of model categories and define the Kan-Quillen model structure on simplicial sets. We then move on to define a model structure on the category enriched over simplicial sets. This, further forms a model of cofibrantly $(\infty, 1)-$categories. We also define the generating cofibrations for this model structure.

0. Introduction

(1) Simplicial categories are categories enriched over **sSet** and are used in the study of homotopy theories.
(2) Dwyer, Hirschon and Kan presented a cofibrantly generated model structure on **sSet** in an earlier version of their online notes in [DHKS04].
(3) Toën and Vezzosi pointed out in [TV05] that this model structure is incorrect. In particular some of the trivial cofibrations are not actually weak equivalences.
(4) Bergner in [Ber07a] in 2004 considers a different set of generating trivial cofibrations which along with cofibrations in [DHKS04] gives the required model structure on simplicial categories, which we will call as the **Bergner model structure**.

1. Motivation

(1) Quasi-category, which is a model for $(\infty, 1)-$categories is a full subcategory of **sSet**.
(2) Another model is the Bergner model on simplicial categories, whose Hom-sets are **sSet**. An $(\infty, 1)-$category on the simplicial categories is a category over quasi-categories. This implies that $\mathcal{C}(x, y) \in$quasi-categories.
(3) There is a stronger relation between them, the Bergner model structure on $\mathcal{SC}$ is left adjoint to the Joyal model structure on **sSet**,

$$\begin{tikzcd} \mathbf{sSet}(Joyal) \arrow[r, bend left, "N^{hc}"] \arrow[r, phantom, "\perp"] & SC(Bergner) \arrow[l, bend left, "S"] \end{tikzcd}$$

(4) The fibrant-cofibrant objects in the Joyal model structure on **sSet** are quasi-categories. This implies that the fibrant-cofibrant objects in the Bergner model structure on simplicial categories are the Kan-complexes ($\infty-$groupoid).

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2. Background

A morphism $f\colon \mathcal{C} \to \mathcal{D}$ on simplicial categories is a simplicially enriched functor between $f\colon \mathcal{C}(x,y) \to \mathcal{D}(fx, fy)$ for all $x, y \in \mathcal{C}$.

Definition 2.1. For a category $\mathcal{C}$, the **homotopy category** of $\mathcal{C}$, denoted as $\pi_0\mathcal{C}$ is defined as follows:

- The *objects* of $\pi_0\mathcal{C}$ are the objects of $\pi_0\mathcal{C}$
- For x, y objects of $\pi_0\mathcal{C}$, the **morphism** of $\pi_0\mathcal{C}$ is defined as the space of homotopy class of morphisms from x to y as follows,

$$Hom_{\pi_0\mathcal{C}}(x,y) = \pi_0(map_{\pi_0\mathcal{C}}(x,y))$$

Remark 2.2. By naturality, any functor $f\colon \mathcal{C} \to \mathcal{D}$ induces the homotopy functor $\pi_0(f)\colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$. Hence, π_0 is a functor from $\mathcal{SC}$ to **Cat**.

If $x, y \in \mathcal{C}$, where $\mathcal{C}$ is a simplicial category, $f \in \mathcal{C}(x,y)_0$ then a morphism $f\colon x \to y$ is a homotopy equivalence if f is an isomorphism in $\pi_0\mathcal{C}$, i.e., $f \in \mathcal{C}(x,y)_0$ induces $[f]\colon x \to y$.

Theorem 2.3. *$\mathcal{SC}$ is complete and co-complete.*

Remark 2.4. Every model category is complete and co-complete by definition.

3. Model Structure

We begin by defining a model category. Note that we have implemented the changes suggested by Kan [DHK97] for the definition of model categories, hence our definition of model category is different from the original definition by Quillen [Qui67].

Definition 3.1. A **model category** is a category $\mathcal{C}$, with three distinct subcategories,

- fibrations ($\twoheadrightarrow$)
- cofibrations ($\hookrightarrow$)
- weak equivalences ($\xrightarrow{\sim}$)

satisfying the following axioms,

- **Limit axiom**: $\mathcal{C}$ is complete and cocomplete.
- $2-$**out-of**-3 **axiom**: If f and g are morphisms in $\mathcal{C}$ such that gf is defined and any two out of f, g and gf are weak equivalences, so is the third.
- **Retract axiom**: if f and g are morphisms in $\mathcal{C}$ and f is a retract of g, and g is a fibration, cofibration or weak equivalence, then so is f.
- *Lifting axiom*: A map which is both a fibration (similarly cofibration) and a weak equivalence is called trivial fibration (similarly trivial cofibration). Trivial fibration, p has right lifting property with respect to cofibration, i and trivial cofibration, i have left lifting property with respect to fibration, p,

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook, "i"'] & Y \arrow[d, two heads, "p"] \\ X' \arrow[r] \arrow[ur, dashed, "\exists"] & Y' \end{tikzcd}$$

- **Factorization axiom**: every map f admits functorial factorizations, in two ways, $f = pi$, where
  - p is a trivial fibration and i is a cofibration, and,
  - p is a fibration and i is a trivial fibration.

We define the classical model structure on **sSet**, namely the *Kan-Quillen model structure on* **sSet**.

Definition 3.2. Let X, Y be **sSet**, then, the simplicial morphism $f\colon X \to Y$ on the *Kan-Quillen model structure on* **sSet** are defined as,

[F1] Fibrations are the *Kan fibrations*, i.e., it has right lifting property with respect to all horn inclusions $\forall n \geq 1$ and $0 \leq k \leq n$.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

[C] Cofibrations are *monomorphisms*, i.e., $f_n\colon X_n \to Y_n$ are levelwise injective maps $\forall n \geq 0$.

[W1] Weak equivalences are *weak homotopy equivalences* $|f|\colon |X| \to |Y|$ in **Top**, i.e., $\forall x \in X$, $\pi_n(f, x) = \pi_n(X, x) \to \pi_n(Y, f(x))$ are isomorphism $\forall n > 0$ and a bijection of sets for $n = 0$.

Now we are ready to define the **Bergner model structure** on $\mathcal{SC}$.

Definition 3.3. Let, $\mathcal{C}, \mathcal{D}$ be $\mathcal{SC}$, then a morphism $f\colon \mathcal{C} \to \mathcal{D}$ is defined as,

[W1] for all $x, y \in \mathcal{C}$ the morphism, $f\colon \mathcal{C}(x, y) \to \mathcal{D}(fx, fy)$ are weak equivalence of **sSet**.

Remark 3.4. Weak equivalence in **sSet**.

[W2] The induced functor $\pi_0 f\colon \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ are essentially surjective.

Remark 3.5. Weak equivalence in **Cat**.

Remark 3.6. From [W1] and [W2] we see that $\pi_0 f$ is an equivalence of categories, since a morphism $\mathcal{C}(x, y) \to \mathcal{D}(fx, fy)$ is weak equivalence and there is a bijection $\pi_0\mathcal{C}(x, y) \to \mathcal{D}(fx, fy)$.

[F1] For all $x, y \in \mathcal{C}$ the morphism, $f\colon \mathcal{C}(x, y) \to \mathcal{D}(fx, fy)$ is a fibration of **sSet**.

Remark 3.7. Fibration in **sSet**.

[F2] For all $x \in \mathcal{C}, y \in \mathcal{D}$ and homotopy equivalence $e\colon fx \to b$ in $\mathcal{D}$, there exists homotopy equivalence $d\colon x \to y$ in $\mathcal{C}$ such that $fd = e$, and $fy = b$.

Remark 3.8. Equivalently, $\pi_0 f\colon \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is an isofibration.

Remark 3.9. Fibration in **Cat**.

[C] Are maps which have the left lifting properties with respect to trivial fibrations.

Theorem 3.10. *The $\mathcal{SC}$ with the above conditions is a model structure and it is cofibrantly generated.*

4. Generating Cofibrations

The idea of generating cofibrations for $\mathcal{SC}$ is based on the generating cofibration on **sSet**, which is also a cofibrantly generated model category.

Before defining the generating cofibrations, let us define a functor,

$$U\colon \mathbf{sSet} \to \mathcal{SC} \tag{4.1}$$

$$X \mapsto \mathcal{C} \tag{4.2}$$

such that,

$$\mathcal{O}(\mathcal{C}) := x, y$$

$$\mathcal{C}(x, y) = X \text{ and } \mathcal{C}(x, x) = * = \mathcal{C}(y, y)$$

Remark 4.1. The generating cofibration of $\mathcal{SC}$ will be images of U.

Definition 4.2. The generating fibration and cofibration of $\mathcal{SC}$ are as follows,

[GC1] $U(\partial\Delta^n \to \Delta^n)$ for all $n \geq 0$.

Remark 4.3. Generating cofibration in **sSet**.

[GC2] $\phi \to *$, where ϕ is the empty simplicial category, and $*$ is the terminal category.

[GTC1] $U(\Lambda_i^n \to \Delta^n)$ for all $n \geq 1$.

Remark 4.4. Generating trivial cofibration in **sSet**.

GTC2 $* \to \mathcal{H}$, where $\{\mathcal{H}\}$ is a set of representatives for the isomorphism classes of simplicial categories with two objects x and y, weakly contractible function complexes, and only countably many simplices in each function complex.

Finally we want to prove that the Bergner model structure on simplicial category is cofibrantly generated for which we use the following theorem 11.3.1 from [Hir03].

Theorem 4.5. *Let $\mathcal{C}$ be a complete and co-complete category with specified weak equivalence and fibrations. Define a map to be cofibration if it has the left lifting property with respect to trivial fibrations. Suppose that the class of weak equivalences are closed under retracts and satisfy the 2-out-of-3 property. Suppose there exists sets I and J of maps in $\mathcal{C}$ satisfying the following:*

1. *I and J satisfy the small object argument.*
2. *A map is a fibration if and only if has the right lifting property with respect to J.*
3. *A map is a trivial fibration if and only if it has the right lifting property with respect to I.*
4. *A map is a trivial cofibration if and only if it has the left lifting property with respect to the fibrations.*

Then $\mathcal{C}$ is said to be a **cofibrantly generated model structure** *with the generating cofibrations I and generating trivial cofibrations J.*

5. Comments

1. The Begner model structure [Ber07a] on simplicial category is **right proper**. Further Lurie has shown that this model structure is also **left proper** in A.3.2.4 in [Lur09].

(2) The fibrant-cofibrant objects of $\mathcal{SC}$ with the Bergner model structure are the Kan-complex enriched categories.
(3) Bergner showed in [Ber07b] that this model structure is Quillen equivalent to three other models of $(\infty, 1)-$categories, namely Segal categories, quasi-categories and complete Segal spaces.
(4) Bergner mentioned in [Ber07a] that it is possible to put up a similar model structure on the **Top**, and it has been verified by Ilias in [Amr11].

Further work: Does it have the additional structure of a simplicial model category?

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