The Delinearization of *C* programs

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Content

Linearized multi-dimensional array

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1. Can we parallelize the two `for-loops`?
2. Is there data dependence between two different iterations of the nest?
3. Are there integer solutions to the following system of linear inequalities?

$$\begin{cases} 0 \leq i_1 < n \\ i_1 + 1 \leq j_1 < n \\ 0 \leq i_2 < n \\ i_2 + 1 \leq j_2 < n \\ i_1 \times n + j_1 = n \times j_2 - n + j_2 - i_2 - 1 \end{cases}$$

Delinearize the array accesses

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There is no integer solution, therefore, no dependence.

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such that:

$\boxed{R = f_1 M_2 \cdots M_e \; + \; \cdots \; + \; f_{e-1} M_2 + f_e}$ for each $(i_1, \ldots, i_d)$ in the iteration domain we have the validity conditions:

$\boxed{0 \leq f_1 < M_1, \quad \ldots, 0 \leq f_e < M_e.}$

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4. QE over $\mathbb{Z}$.

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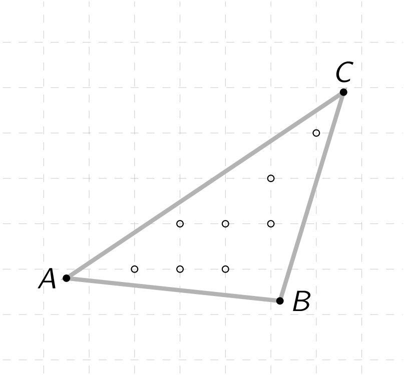

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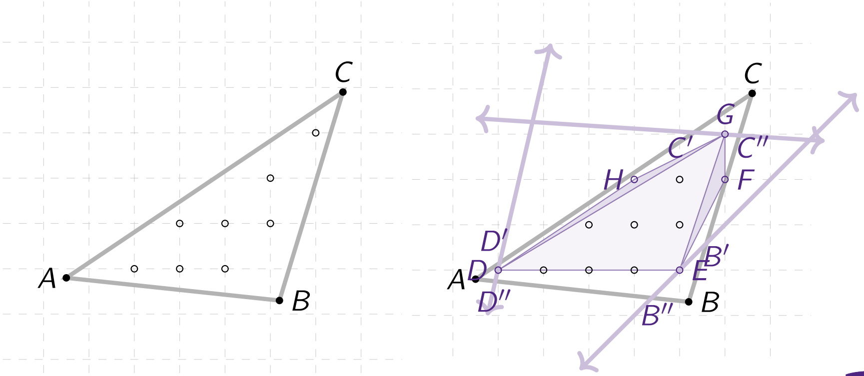

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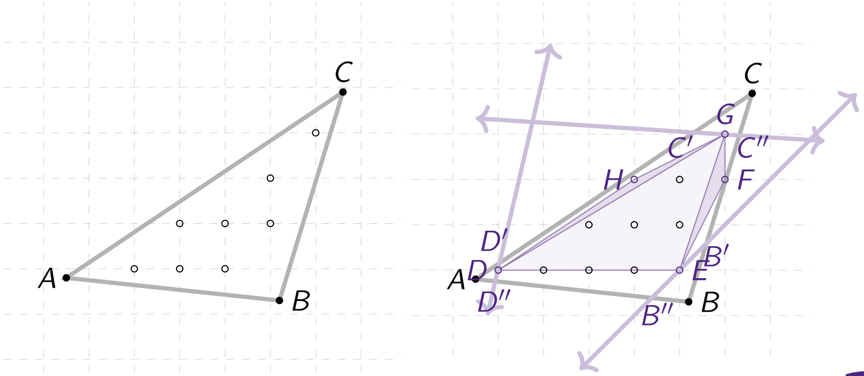

The integer hull $\{D, E, F, G, H\}$ is formed by $\{D, E, G\}$ and searching integer points $\{F, H\}$ in quadrilaterals $DD''B''E$, $EB'C''G$ and $D'DGC'$.

Parametric Integer Linear Programming

$$
\begin{array}{ll}
\max\limits_{(i_1,\ldots,i_d)} & f_k \\
\text{subject to} & (i_1, \ldots, i_d) \in \text{iteration domain} \\
 & i_1, \ldots, i_d \in \mathbb{Z}
\end{array}
$$

For which, $0 \leq \max\limits_{(i_1,\ldots,i_d)} f_k < M_k$ for all $k \in [1, \ldots, e]$.

1. `PIP/PipLib` by Paul Feautrier
2. `isl` by Sven Verdoolaege
3. `barvinok` by Sven Verdoolaege

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The reference $A[R]$ to A which encodes a reference $B[f1][f2]$ to B, where:

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- $\begin{cases} f1 = f_{11} i_1 + f_{12} i_2 + f_{10} \\ f2 = f_{21} i_1 + f_{22} i_2 + f_{20} \end{cases}$ and $R = f1 M_2 + f2$

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   A[R(i_1, ..., i_d, m_1, ..., m_e)] <- ...
```

Output

```
for(i_1, ..., i_1 ++)
 ...
  for(i_d, ..., i_d ++)
   B[f_1], ..., B[f_e] <- ...
```

$$\left.\begin{array}{ll} d = 2 \rightsquigarrow i_1, i_2 & \text{loop counters} \\ e = 2 \rightsquigarrow m_1, m_2 & \text{program parameters} \end{array}\right\} \text{known at compile time.}$$

$$\left.\begin{array}{l} M_1 = a_1 m_1 + b_1 \\ M_2 = a_2 m_2 + b_2 \end{array}\right\}, \text{ where } a_1, b_1, a_2, b_2 \in \mathbb{Z} \text{ TBD}.$$

The reference $A[R]$ to A which encodes a reference $B[f1][f2]$ to B, where:

- $\begin{cases} f1 = f_{11} i_1 + f_{12} i_2 + f_{10} \\ f2 = f_{21} i_1 + f_{22} i_2 + f_{20} \end{cases}$ and $R = f1 M_2 + f2$
- for each (i_1, i_2), we have, $\begin{cases} 0 \leq f1 < M_1 \\ 0 \leq f2 < M_2 \end{cases}$ and $\begin{cases} 0 \leq \max f1 < M_1 \\ 0 \leq \max f2 < M_2 \end{cases}$.

2D-2D polynomial system solving

Substituting $f1$ and $f2$ in R, we obtain, $R = \underbrace{a_2 f_{11}}_{T_1} i_1 m_2 + \underbrace{a_2 f_{12}}_{T_2} i_2 m_2 + \underbrace{a_2 f_{10}}_{T_3} m_2 + (\underbrace{b_2 f_{11} + f_{21}}_{T_4}) i_1 + (\underbrace{b_2 f_{12} + f_{22}}_{T_5}) i_2 + (\underbrace{b_2 f_{10} + f_{20}}_{T_6})$.

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$$\begin{cases} T_1 = a_2 f_{11} \\ T_2 = a_2 f_{12} \\ T_3 = a_2 f_{10} \\ T_4 = b_2 f_{11} + f_{21} \\ T_5 = b_2 f_{12} + f_{22} \\ T_6 = b_2 f_{10} + f_{20} \end{cases} \implies \begin{cases} f_{11} = \dfrac{T_1}{a_2} \\ f_{12} = \dfrac{T_2}{a_2} \\ f_{10} = \dfrac{T_3}{a_2} \\ f_{21} = T_4 - b_2 f_{11} \\ f_{22} = T_5 - b_2 f_{12} \\ f_{20} = T_6 - b_2 f_{10} \end{cases}$$

2D-2D polynomial system solving

Substituting $f1$ and $f2$ in R, we obtain, $R = \underbrace{a_2 f_{11}}_{T_1} i_1 m_2 + \underbrace{a_2 f_{12}}_{T_2} i_2 m_2 + \underbrace{a_2 f_{10}}_{T_3} m_2 + (\underbrace{b_2 f_{11} + f_{21}}_{T_4}) i_1 + (\underbrace{b_2 f_{12} + f_{22}}_{T_5}) i_2 + (\underbrace{b_2 f_{10} + f_{20}}_{T_6})$.

$$\begin{cases} T_1 = a_2 f_{11} \\ T_2 = a_2 f_{12} \\ T_3 = a_2 f_{10} \\ T_4 = b_2 f_{11} + f_{21} \\ T_5 = b_2 f_{12} + f_{22} \\ T_6 = b_2 f_{10} + f_{20} \end{cases} \implies \begin{cases} f_{11} = \dfrac{T_1}{a_2} \\ f_{12} = \dfrac{T_2}{a_2} \\ f_{10} = \dfrac{T_3}{a_2} \\ f_{21} = T_4 - b_2 f_{11} \\ f_{22} = T_5 - b_2 f_{12} \\ f_{20} = T_6 - b_2 f_{10} \end{cases}$$

a_2, b_2 can NOT be uniquely determined, but $a_2 \mid gcd(T_1, T_2, T_3)$.

2D-2D quantifier elimination (I/II)

For each (i_1, i_2), we have, $0 \leq f2 < M_2$, thus $0 \leq \max f2 < M_2$,

$$\begin{cases} \max\limits_{(i_1,i_2)} & f_2 \\ \text{subject to} & (i_1, i_2) \in \text{iteration domain} \\ & i_1, i_2 \in \mathbb{Z} \end{cases}$$

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Depending on the shape of the iteration domain, we solve on a case to case basis.

2D-2D quantifier elimination (I/II) - Rectangular domain

For a `for loop` of the form,

```
for (int i_1 = 0; i_1 < r_1; i_1 ++)
  for (int i_2 = 0; i_2 < r_2; i_2 ++)
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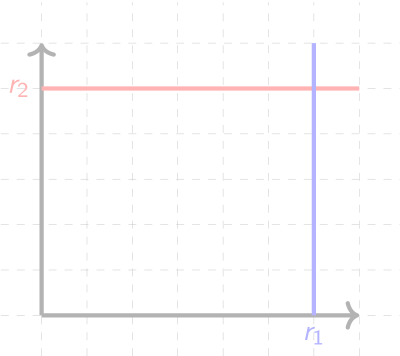
the iteration domain is of the shape,



2D-2D quantifier elimination (I/II) - Triangular domain

For a `for loop` of the form,

```
for (int i_1 = 0; i_1 < r_1; i_1 ++)
  for (int i_2 = 0; m_1 * i_1 + m_2 * i_2 < r_2; i_2 ++)
```

2D-2D quantifier elimination (I/II) - Triangular domain

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for (int i_1 = 0; i_1 < r_1; i_1 ++)
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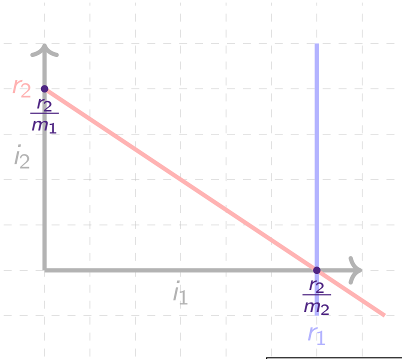

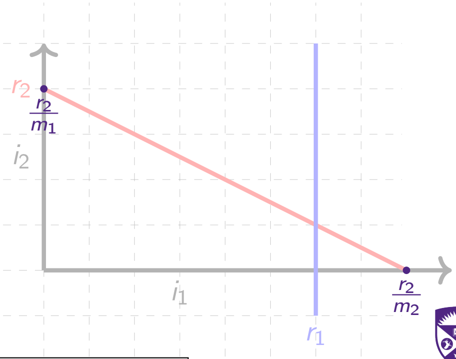

Notice, that $i_2 = \frac{-i_1 m_2 + r_2}{m_1}$

2D-2D quantifier elimination (I/II)

The parametric integer linear problem:

$$\begin{cases} \max\limits_{(i_1,i_2)} & f_2 = f_{21} i_1 + f_{22} i_2 + f_{20} \\ \text{subject to} & (i_1, i_2) \in \text{iteration domain} \\ & i_1, i_2 \in \mathbb{Z} \end{cases}$$

can be solved for,

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can be solved for,

1. Rectangular domain by case inspection

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can be solved for,

1. Rectangular domain by case inspection
2. Triangular domain by case inspection except when `f_21, f_22 > 0`, in which case the problem becomes,

$$\begin{array}{lll} \max\limits_{i_1} & f_{21} i_1 + f_{22} \lfloor \frac{-i_1 m_2 + r_2}{m_1} \rfloor + f_{20} & \\ \text{subject to} & 0 \le i_1 < r_1, & i_1 \in \mathbb{Z} \end{array}$$

2D-2D quantifier elimination (II/II)

QE over $\mathbb{R}$,

2D-2D quantifier elimination (II/II)

QE over $\mathbb{R}$, uses the `QuantifierElimination` function from the `RegularChains:-SemiAlgebraicSetTools` package.

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f := &A([i_1, i_2]), ((0 < i_1) &and (i_1 < r_1) &and
                      (0 < i_2) &and (i_2 < r_2) &and
                      (0 < r_1) &and (0 < r_2) &and
                      (0 < f_{21}) &and (0 < f_{22}) &and (0 < B))
                      \\ B = M_2 - f_{20}
            &implies (f_{21} * i_1 + f_{22} * i_2 < B);
```

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            &implies (f_{21} * i_1 + f_{22} * i_2 < B);
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After simplification, `r_1 * f_{21} + r_2 * f_{22} + f_{20} < M_2`.

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                      (0 < r_1) &and (0 < r_2) &and
                      (0 < f_{21}) &and (0 < f_{22}) &and (0 < B))
                      \\ B = M_2 - f_{20}
           &implies (f_{21} * i_1 + f_{22} * i_2 < B);
```

After simplification, `r_1 * f_{21} + r_2 * f_{22} + f_{20} < M_2`. Substituting, $f_{11} = \frac{T_1}{a_2}, f_{12} = \frac{T_2}{a_2}, f_{10} = \frac{T_3}{a_2}, f_{21} = T_4 - b_2 f_{11}, f_{22} = T_5 - b_2 f_{12}, f_{20} = T_6 - b_2 f_{10}, M_2 = a_2 m_2 + b_2$, we obtain,

$$\boxed{r_1 \left(T_4 - b_2 \frac{T_1}{a_2}\right) + r_2 \left(T_5 - b_2 \frac{T_2}{a_2}\right) + T_6 - b_2 \frac{T_3}{a_2} < a_2 m_2 + b_2.}$$

Examples (I/II) - Rectangular domain

```
for (int i_1 = 0; i_1 <= r_1; i_1 ++)
  for (int i_2 = 0; i_2 <= r_2; i_2 ++)
    A[2 * i_1 * m_2 + m_2 + 3 * i_2 + 2] = ...;
```

1. $max\ i_1 = r_1, max\ i_2 = r_2$

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6. assuming $m_2 = 10$, i.e. $B[...][10]$, we get,

   delinearization valid when $r_1 = r_2 = 1, max\ f2 = 5 < 10$

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   delinearization valid when $r_1 = r_2 = 2, max\ f2 = 8 < 10$

Examples (I/II) - Rectangular domain

```
for (int i_1 = 0; i_1 <= r_1; i_1 ++)
  for (int i_2 = 0; i_2 <= r_2; i_2 ++)
    A[2 * i_1 * m_2 + m_2 + 3 * i_2 + 2] = ...;
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   delinearization invalid when $r_1 = r_2 = 3, max\ f2 = 11 \not< 10$

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Examples (II/II) - Triangular domain

```
for (int i_1 = 0; i_1 <= r_1; i_1 ++)
  for (int i_2 = 0; i_1 + 2 * i_2 <= r_2; i_2 ++)
    A[2 * i_1 * m_2 + m_2 + 3 * i_2 + 2] = ...;
```

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Examples (II/II) - Triangular domain

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3. $a_2 =?, b_2 =?, f_{11} = 2, f_{12} = 0, f_{10} = 1, f_{21} = -2b_2, f_{22} = 3, f_{20} = 2 - b_2$
4. the validity condition $-r_1 b_2 \frac{2}{a_2} + 3r_2 + 2 - b_2 \frac{1}{a_2} < a_2 m_2 + b_2$
5. evaluating at $a_2 = 1, b_2 = 0$, we obtain, $f1 = 2i_1 + 1, f2 = 3i_2 + 2$
6. assuming $m_2 = 10$, i.e. $B[...][10]$, we get,
   delinearization valid when $r_1 = r_2 = 1, max\ f2 = 2 < 10$
   delinearization valid when $r_1 = r_2 = 2, max\ f2 = 5 < 10$

Examples (II/II) - Triangular domain

```
for (int i_1 = 0; i_1 <= r_1; i_1 ++)
  for (int i_2 = 0; i_1 + 2 * i_2 <= r_2; i_2 ++)
    A[2 * i_1 * m_2 + m_2 + 3 * i_2 + 2] = ...;
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   delinearization valid when $r_1 = r_2 = 4, max\ f2 = 8 < 10$

Examples (II/II) - Triangular domain

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for (int i_1 = 0; i_1 <= r_1; i_1 ++)
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   delinearization valid when $r_1 = r_2 = 3, max\ f2 = 5 < 10$
   delinearization valid when $r_1 = r_2 = 4, max\ f2 = 8 < 10$
   delinearization valid when $r_1 = r_2 = 5, max\ f2 = 8 < 10$

Examples (II/II) - Triangular domain

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   delinearization valid when $r_1 = r_2 = 3, max\ f2 = 5 < 10$
   delinearization valid when $r_1 = r_2 = 4, max\ f2 = 8 < 10$
   delinearization valid when $r_1 = r_2 = 5, max\ f2 = 8 < 10$
   delinearization invalid when $r_1 = r_2 = 6, max\ f2 = 11 \not< 10$

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