Pricing Algorithm and Optimal exercise time for American Derivative Securities

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Abstract

A pricing strategy for American Options is derived by applying a Binomial Model framework and using topics from the Probability Theory, Martingale Theory, and Stochastic Processes, by establishing a backwards recursive algorithm. Furthermore, the concept of Stopping time is applied, to arrive at the optimal exercise time for any such American derivative security.

1 Introduction

This report intends to come up with a pricing strategy for an American Derivative Security, as described by Steven E. Shreve[1]. We first define the Financial terms and model them with certain justified assumptions. For our purpose, we shall assume a Multi-period Binomial model, in an efficient market, such that, there exists no possibility of an arbitrage. Then, we shall recall a few of the elementary topics from the Probability theory, such as martingales, Markov processes, stopping time, etc. We then arrive at a pricing strategy for the European options which can by modified to give the corresponding result for the American derivatives. We shall explore both cases: firstly, when the Stock price is independent of the path of the previous coin flips, and them the general case, where it depends on the path. Finally we apply the theory of Stopping times and prove some theorems regarding then, in order to arrive at an optimal exercise time for such an American option.

2 Financial Concepts and their Mathematical Interpretation

In this section, we briefly define and describe the concepts used from Financial Theory, that are used in this report. We then interpret them mathematically, by modeling them using basic Probability Theory, specifically Martingale Theory and Stochastic Processes.

2.1 Financial Terminology

1. A *financial market* (or simply, market) is a platform for people to trade financial securities and derivatives at low transaction costs. For our purpose, we assume zero-transaction cost.

2. An *interest rate* r is the excess yield, when a unit currency is invested in the money-market (which is assumed risk-free).

3. An *arbitrage* is any trading strategy in a market, which requires zero initial wealth and has a positive probability to yield non-negative final wealth, thereby making money without any loss of risk. An arbitrage-free market is said to be *efficient*.

4. An *option* is a financial instrument whose value is based on another stock. A *Call* (*put*) option contract provides buyer the right, but not the obligation to buy (sell) the underlying stock at a predetermined time (T) in the future, at a predetermined strike price K.

5. In case, of a *European option*, there is a single date for exercise, known as the expiration date. In contrast, an *American option* can be exercised any time before and up to the expiration date ($t \leq T$).

Mathematically we define the payoff of such an option as follows.

Definition 2.1. A *European call*, option with a specified expiration time T, and exercise/strike price K, has the payoff, $\max(S_T - K, 0)$.

For a put option, this payoff becomes $\max(K - S_T, 0)$. Modifying definition 2.1 for all time periods before the expiration date T, we get the payoff for an American option.

Definition 2.2. An *American call*, option with a specified expiration time T, and exercise/strike price K, has the payoff:

$$\max(S_t - K, 0),\ \forall t \in \mathbb{N}_0 : 0 \leq t \leq T$$

2.2 Binomial Model

In order to price a given derivative security (American Options in our case), we employ a Binomial Model, where we assume that in a single time period, the stock price S_0 either increases by an up-factor u, or decreases by a down-factor d, determined by a coin-flip (Head or Tail):

$$S_1(H) = uS_0$$
$$S_1(T) = dS_0$$

Lemma 2.1 (No-arbitrage Condition)**.** *For a Binomial Model, in order to have an arbitrage-free (efficient) market, the following inequality must hold:*

$$0 < d < (1+r) < u \tag{1}$$

Proof. Since stocks can only take positive values, we get $d > 0$.
"$d < (1+r)$":
Assume on the contrary that $d \geq (1+r)$. Then at time 0, an investor can buy one share of this stock by borrowing S_0 amount from the money market at interest rate r. At time T, both cases, T and H, will yield a higher return that the money market ($S_1(T) = dS_0 \geq S_0(1+r)$, and $u > d$ by assumption of the model). Thus, at T, the investor can sell back this share of stock and make a minimum risk-less positive profit ($S_0(d-(1+r))$), with zero initial wealth thereby resulting arbitrage.
"$(1+r) < u$":
Using a similar argument, we can see that assuming the contrary inequality gives us the possibility to short sell the stock and invest this S_0 amount in the money market. This gives a positive risk-less profit ($S_0((1+r)-u)$) at time T, with zero initial wealth (arbitrage-condition) □

From here onward, we will assume the No-arbitrage condition (2.1), to derive fair pricing for any given instrument.

The probability of head and tail (say p and q) are random in nature. In reality, they are such that the tendency for a stock price to go up or down depends on the growth of the stock and can't be determined beforehand, hence random. But, if we neglect any trend specific to the stock, thereby neglecting any bias for going up or down, and only consider the growth due to risk-free rate, we derive the *risk-neutral probabilities*, $\tilde{p}$ and $\tilde{q}$ as follows.

Let V_1 be the value of the derivative security at $T = 1$, depending on coin-toss and V_0 be its associated value at $T = 0$, thus the price. We shall determine this value by hedging against an equivalent portfolio with an initial wealth X_0 to match the final wealth. Let us by Δ_0 shares at time 0. Then

$$\text{Cash at time } 0 = X_0 - \Delta_0 S_0 \tag{2}$$

$$\implies \text{Final Wealth}, X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \tag{3}$$

Based on the coin-toss, we obtain the two equations:

$$X_0 + \Delta_0\left(\frac{S_1(H)}{1+r} - S_0\right) = \frac{V_1(H)}{1+r}, \tag{4}$$

$$X_0 + \Delta_0\left(\frac{S_1(T)}{1+r} - S_0\right) = \frac{V_1(T)}{1+r}. \tag{5}$$

To solve these two equations (4) and (5), we multiply them by $\tilde{p}$ and $\tilde{q} = 1-\tilde{p}$ and add them to get

$$X_0 + \Delta_0\left(\frac{\tilde{p}S_1(H) + \tilde{q}S_1(T)}{1+r} - S_0\right) = \frac{1}{1+r}\big[\tilde{p}V_1(H) + \tilde{q}V_1(T)\big]$$

Also, since $\tilde{p}$ and $\tilde{q}$ are the risk-neutral probabilities as described before, we can set the coefficient of Δ_0 to 0 and have

$$S_0 = \frac{1}{1+r}\big[\tilde{p}S_1(H) + \tilde{q}S_1(T)\big]$$

Solving this, by using $S_1(H) = uS_0, S_1(T) = dS_0$, gives us the values for the risk-neutral probabilities as

$$\tilde{p} = \frac{1+r-d}{u-d}, \quad \tilde{q} = \frac{u-1-r}{u-d}$$

Upon solving the above equations for the initial wealth X_0 (and hence, V_0) and initial number of shares Δ_0, we obtain

$$V_0 = X_0 = \frac{1}{1+r}\big[\tilde{p}V_1(H) + \tilde{q}V_1(T)\big], \quad \Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$

Remark. As pointed before, these are the risk-neutral probabilities for head and tail and not the true probabilities, p and q of the trend of the stock. In general, they grow more than risk-less asset, on average, due to the inherent risk associated.

$$S_0 < \left(\frac{pS_1(H) + qS_1(T)}{1+r}\right)$$

2.3 Prerequisites from Probability Theory

For our purpose, we are only concerned with the *Finite Probability Spaces.* Let us now recall the following definition:

Definition 2.3. Given a non-empty finite sample space Ω and a probability measure $\mathbb{P} : \Omega \to [0,1]$, so that $\mathbb{P}(\Omega) = 1$. Then, the tuple $(\Omega, \mathbb{P})$ is called a *finite probability space.*

Definition 2.4. Given a finite probability space $(\Omega, \mathbb{P})$, a *random variable* is defined as any real-valued function on Ω:

$$X : \Omega \to \mathbb{R}$$

Definition 2.5. Given a finite probability space $(\Omega, \mathbb{P})$, and a random variable X defined on it. Then *Expectation* of X is defined as:

$$\mathbb{E}[X] \coloneqq \sum_{\omega\in\Omega} X(\omega)\mathbb{P}(\omega)$$

Theorem 2.2 (Conditional Expectation)**.** *Given the risk-neutral probability space* $(\Omega, \widetilde{\mathbb{P}})$*, and a random variable* S_n *denoting the stock price at time* n*. Then Conditional Expectation of X, given the first* n *coin tosses satisfies:*

$$S_n = \frac{\widetilde{\mathbb{E}}_n[S_{n+1}]}{1+r}, \quad \forall n = 0, 1, ..., N-1$$

Proof. Recalling the risk-neutral probabilities from above, we have

$$\tilde{p} = \frac{1+r-d}{u-d}, \quad \tilde{q} = \frac{u-1-r}{u-d}$$

$$\begin{aligned}
\implies \frac{\tilde{p}u + \tilde{q}d}{1+r} &= \frac{1}{1+r}\left(\frac{1}{u-d}\right)\Big[\big(1+r-d\big)u + \big(u-1-r\big)d\Big] \\
&= \frac{1}{(1+r)(u-d)}\big[u + ru - du + u - d - rd\big] \\
&= \frac{1}{(1+r)(u-d)}\big[(u-d) + r(u-d)\big] \\
&= \frac{1}{(1+r)(u-d)}\big[(u-d)(1+r)\big] = 1
\end{aligned}$$

$\therefore \forall n \in \{0, 1, ..., N-1\}$ $\forall \omega_1 ... \omega_n$ coin tosses :

$$\begin{aligned}
\frac{\widetilde{\mathbb{E}}_n[S_{n+1}]}{1+r} &= \frac{1}{1+r}\Big[\tilde{p}S_{n+1}(\omega_1...\omega_n H) + \tilde{q}S_{n+1}(\omega_1...\omega_n T)\Big] \\
&= \frac{1}{1+r}\Big[\tilde{p}uS_n(\omega_1...\omega_n) + \tilde{q}dS_n(\omega_1...\omega_n)\Big] \\
&= S_n(\omega_1...\omega_n)\left[\frac{\tilde{p}u + \tilde{q}d}{1+r}\right] \\
&= S_n(\omega_1...\omega_n)
\end{aligned}$$

□

Definition 2.6. Consider a Binomial Model and let $(M_n)_{0 \leq n \leq N}$ be a sequence of random variables such that M_0 is a constant and each M_n depends only on the first n coin tosses. Such a sequence is known as an *adapted sequence*.
Furthermore
(i) $(M_n)_{0 \leq n \leq N}$ is said to be a *martingale*, if

$$M_n = \mathbb{E}_n[M_{n+1}], \quad n = 0, 1, ..., N-1$$

(ii) $(M_n)_{0 \leq n \leq N}$ is said to be a *submartingale*, if

$$M_n \leq \mathbb{E}_n[M_{n+1}], \quad n = 0, 1, ..., N-1$$

(iii) $(M_n)_{0 \leq n \leq N}$ is said to be a *supermartingale*, if

$$M_n \geq \mathbb{E}_n[M_{n+1}], \quad n = 0, 1, ..., N-1.$$

Theorem 2.3. *Consider an N-period Binomial Model with the no-arbitrage condition $(0 < d < (1+r) < u)$. Then, under the risk-neutral probability measure,*
(i) the discounted stock price is a martingale:

$$\frac{S_n}{(1+r)^n} = \widetilde{\mathbb{E}}\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right],\ n = 0, 1, ..., N-1,$$

(ii) the discounted wealth process is a martingale:

$$\frac{X_n}{(1+r)^n} = \widetilde{\mathbb{E}}\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right],\ n = 0, 1, ..., N-1,$$

(iii) the discounted derivative price is a martingale:

$$\frac{V_n}{(1+r)^n} = \widetilde{\mathbb{E}}\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right],\ n = 0, 1, ..., N-1.$$

Proof. For any $n \in \{0, 1, ...N-1\}$, let the n coin-tosses $\omega_1...\omega_n$ be given. Then (i): Conditional expectation of the discounted stock price, given the n coin tosses is

$$\begin{aligned}
\widetilde{\mathbb{E}}_n\left[\frac{S_{n+1}(\omega_1...\omega_n)}{(1+r)^{n+1}}\right] &= \frac{1}{(1+r)^n}\left[\frac{\tilde{p}S_{n+1}(\omega_1...\omega_n H) + \tilde{q}S_{n+1}(\omega_1...\omega_n T)}{1+r}\right] \\
&= \frac{1}{(1+r)^n}\left[\frac{\tilde{p}uS_n(\omega_1...\omega_n) + \tilde{q}dS_n(\omega_1...\omega_n)}{1+r}\right] \\
&= \frac{S_n(\omega_1...\omega_n)}{(1+r)^n}\left[\frac{\tilde{p}u + \tilde{q}d}{1+r}\right] = \frac{S_n(\omega_1...\omega_n)}{(1+r)^n}
\end{aligned}$$

(ii): Recalling the Wealth equation (3) from above:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

Hence, the conditional expectation of discounted wealth process is

$$\begin{aligned}
\implies \widetilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \widetilde{\mathbb{E}}_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r)^n}\right] \\
&= \widetilde{\mathbb{E}}_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}}\right] + \widetilde{\mathbb{E}}_n\left[\frac{X_n - \Delta_n S_n}{(1+r)^n}\right] \\
&= \Delta_n \widetilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] + \left(\frac{X_n - \Delta_n S_n}{(1+r)^n}\right) \\
&= \left(\Delta_n \frac{S_n}{(1+r)^n} + \frac{X_n - \Delta_n S_n}{(1+r)^n}\right) \\
&= \frac{X_n}{(1+r)^n}
\end{aligned}$$

(iii): Conditional expectation of the discounted derivative price, given the n coin tosses is

$$\begin{aligned}
\widetilde{\mathbb{E}}_n\left[\frac{V_{n+1}(\omega_1...\omega_n)}{(1+r)^{n+1}}\right] &= \frac{1}{(1+r)^n}\left[\frac{\tilde{p}V_{n+1}(\omega_1...\omega_n H)+\tilde{q}V_{n+1}(\omega_1...\omega_n T)}{1+r}\right] \\
&= \frac{1}{(1+r)^n}\left[\frac{(1+r)V_n(\omega_1...\omega_n)}{1+r}\right] \\
&= \frac{V_n(\omega_1...\omega_n)}{(1+r)^n}
\end{aligned}$$

□

Our discussion till now, assumed path-dependence, i.e., the conditional expectations depended on the sequence of the coin toss path $\omega_1...\omega_n$. However, as we will see now, that we only need the values at time n and not the whole path to calculate the expectations for the next time period. This requires us with the definition of a Markov Process.

Definition 2.7 (Markov Process)**.** Consider an N-period Binomial Model. Let $(X_n)_{0\leq n\leq N}$ be an adapted process. We say that $(X_n)_{0\leq n\leq N}$ is a *Markov Process*, if it holds that:

$$\forall n \in \{0, 1, ..., N-1\}\ \forall f\ \exists g : \mathbb{E}[f(X_{n+1})] = g(X_n)$$

This merely suggests that the conditional expected value of a random variable here is dependent only on the current value and not on the past values. This helps us model the situation of stock prices in the Binomial Model, as it eases our calculation and provides with a much simpler theory for deriving the pricing algorithm.

Theorem 2.4. *Consider a Binomial model. Let the Stock price process be $(S_n)_{0\leq n\leq N}$. This process is Markov under risk-neutral probability measure.*

Proof. We have, for any n before N, and with the first n coin tosses

$$\begin{aligned}
S_{n+1}(\omega_1...\omega_n H) &= uS_n(\omega_1...\omega_n) \\
S_{n+1}(\omega_1...\omega_n T) &= dS_n(\omega_1...\omega_n) \\
\implies \widetilde{\mathbb{E}}_n[f(S_{n+1})] &= \tilde{p}f(uS_n) + \tilde{q}f(dS_n)
\end{aligned}$$

On setting a function $g(x) = \tilde{p}f(ux)+\tilde{q}f(dx)$, we see that the stock price process is Markov. □

Theorem 2.5. *Consider a Binomial Model with an asset-price process $(S_n)_{0\leq n\leq N}$, that is Markov under the risk-neutral probability measure $\widetilde{\mathbb{P}}$. Let the payoff $V_N = v_N(S_N)$. Then for all n in the set $\{0,1,...,N\}$, we get*

$$V_n = v_n(S_n)$$

Proof. From Theorem 2.3, we know that the discounted price of derivative is a martingale under the risk-neutral probability measure:

$$V_n = \frac{\widetilde{\mathbb{E}}_n[V_{n+1}]}{1+r},\ n = 0,1,...,N-1$$

But $V_N = v_N(S_N)$, and the stock price is Markov, thereby giving us

$$V_{N-1} = \frac{\widetilde{\mathbb{E}}_{N-1}[v_N(S_N)]}{1+r} = v_{N-1}(S_{N-1})$$

Thus on applying Mathematical Induction, we see that this equality holds true for any $n \leq N$, thereby giving us $V_n = v_n(S_n)$ for all n. □

Definition 2.8. Consider an N-period Binomial Model. A *stopping time* is a random variable, $\tau : \Omega \to \{0,1,...N,\infty\}$, such that for any n and for all future coin tosses $\omega'_{n+1}...\omega'_N$, if $\tau(\omega_1...\omega_n\omega_{n+1}...\omega_n) = n$, then $\tau(\omega_1...\omega_n\omega'_{n+1}...\omega'_n) = n$.

Theorem 2.6 (Optional Sampling Theorem)**.** *A martingale (resp., supermartingale and submarginale) stopped at stopping time is a martingale (respectively, supermartingale and submartingale).*

Proof. Let us first show this statement for supermartingales. Let $(M_n)_{0\leq n\leq N}$ be a supermartingale.

$$\implies \mathbb{E}_n[M_{n+1}] \leq M_n,\ \forall n \in \{0,1,...,N-1\}$$

Let τ be the supermartingale corresponding to this supermartingale and define a stopped process $Y_n \coloneqq M_{\min\{n,\tau\}}$. For any n:

$$\begin{aligned}\implies \mathbb{E}_n[Y_{n+1}] &= \mathbb{E}_n[M_{\min\{n+1,\tau\}}] \\ &\leq M_{\min\{n,\tau\}} \\ &\leq Y_n\end{aligned}$$

Hence, a supermartingale stopped at a stopping time is a supermartingale itself. Similarly, it can be shown that a submartingale stopped at a stopping time is a submartingale (all the inequalities signs are reversed). Now, we know that a martingale is both a submartingale, as well as a supermartingale. Therefore, a martingale stopped at stopping time is both a submartingale and a supermartingale, therefore, a martingale. □

3 Pricing American Derivatives

Let us now discuss about a method to price an American derivative security, by using the previous topics built so far. Till now we have studied about the European options, where the exercise of the contract is only possible at a single date. This gets modified for the American options, in order to accommodate for all possible times before and including the expiry. Sub-section (3.1) will develop on the theory of Markov Processes, by considering path-independent random variables. Then, we generalize this in the next sub-section by using the concept of Stopping time, earlier developed.

3.1 Path-independent Derivatives

Consider an N-period Binomial model, with up-factor u, down-factor d, satisfying the no-arbitrage condition (2.1). Let us now consider a derivative security that pays off $g(S_N)$ at time N. From Theorem 2.5, we have $V_n = v_n(S_n)$, $n = 0, 1, ..., N$. Thus we arrive at the following European Algorithm:

$$v_N(S_N) = \max\{g(S_N), 0\}$$

$$v_n(S_n) = \frac{1}{1+r}\left[\tilde{p}v_{n+1}(uS_n) + \tilde{q}v_{n+1}(dS_n)\right],\ n \in \{0, 1, ..., N-1\}$$

The key difference between an American option from its counterpart European option is that the holder of the contract can exercise the contract, any time before expiry and get a payoff of $g(S_n)$.

$$X_n \geq g(S_n),\ \forall n \in \{0, 1, ..., N\}$$

Hence, we arrive at the following path-independent American algorithm:

$$v_N(S_N) = \max\{g(S_N), 0\}$$

$$v_n(S_n) = \max\left\{g(S_n), \frac{1}{1+r}\left[\tilde{p}v_{n+1}(uS_n) + \tilde{q}v_{n+1}(dS_n)\right]\right\},\ n \in \{0, 1, ..., N-1\}$$

3.2 Path-dependent Derivatives

Now we shall develop a general theory, by assuming path-dependence, which will help us in determining an optimal exercise time.
Consider an N-period Binomial model (up-factor u, down-factor d) satisfying the no-arbitrage condition (2.1). Let S_n be the set of all stopping times τ in the set $\{n, n+1, ..., N, \infty\}$.

Definition 3.1. Consider a Binomial Model and let the random variable G_n be the intrinsic value process of an American derivative security, depending on first n coin tosses. The American risk-neutral price process is defined as follows

$$V_n = \max_{\tau \in S_n} \widetilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\tau \leq N\}} \frac{G_\tau}{(1+r)^{\tau-n}}\right],\ \ n = 0, 1, ..., N$$

Theorem 3.1. *Let the American derivative security price process* $(V_n)_{0\le n\le N}$ *be given by definition 3.1. Then*
(i) $V_n \ge \max\{G_n, 0\},\ \forall n \in \{0,1,...,N\}$
(ii) $\dfrac{V_n}{(1+r)^n}$ *is a supermartingale*

Proof. (i):
Let $n \in \{0,1,...,N\}$ and let $\hat{\tau} \in S_n$, such that $\hat{\tau} = n$, regardless of the coin toss.

$$\implies \widetilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\hat{\tau}\le N\}}\frac{G_{\hat{\tau}}}{(1+r)^{\hat{\tau}-n}}\right] = G_n$$

From definition 3.1, it is clear that V_n is the largest of all such expectations.

$$\implies V_n \ge G_n$$

Now, consider a stopping time $\bar{\tau} \in S_n$ such that $\hat{\tau} = \infty$, regardless of coin tosses.

$$\implies \widetilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\bar{\tau}\le N\}}\frac{G_{\bar{\tau}}}{(1+r)^{\bar{\tau}-n}}\right] = 0$$

Again, from definition 3.1, V_n is the largest of all such expectations.

$$\implies V_n \ge 0$$

$$\therefore V_n \ge \max\{G_n, 0\},\ \forall n \in \{0,1,...,N\}$$

(ii): Let $n \in \{0,1,...,N\}$ and let $\tau^* \in S_{n+1}$ be the expectation maximizing stopping time for V_{n+1} in definition 3.1.

$$\implies V_{n+1} = \widetilde{\mathbb{E}}_{n+1}\left[\mathbb{1}_{\{\tau^*\le N\}}\frac{G_{\tau^*}}{(1+r)^{\tau^*-n-1}}\right]$$

$$\begin{aligned}
\implies V_n &\ge \widetilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\tau^*\le N\}}\frac{G_{\tau^*}}{(1+r)^{\tau^*-n}}\right]\\
&= \widetilde{\mathbb{E}}_n\left[\widetilde{\mathbb{E}}_{n+1}\left[\mathbb{1}_{\{\tau^*\le N\}}\frac{G_{\tau^*}}{(1+r)^{\tau^*-n}}\right]\right]\\
&= \widetilde{\mathbb{E}}_n\left[\frac{1}{1+r}\widetilde{\mathbb{E}}_{n+1}\left[\mathbb{1}_{\{\tau^*\le N\}}\frac{G_{\tau^*}}{(1+r)^{\tau^*-n-1}}\right]\right]\\
&= \widetilde{\mathbb{E}}\left[\frac{1}{1+r}V_{n+1}\right]
\end{aligned}$$

$$\therefore \frac{V_n}{(1+r)^n} \ge \widetilde{\mathbb{E}}\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right]$$

Hence, the discounted price process is a super martingale. $\square$

Theorem 3.2. *Let the American derivative security price process $(V_n)_{0\leq n\leq N}$ be given by definition 3.1. Let $(Y_n)_{0\leq n\leq N}$ be another process satisfying both conditions of theorem 3.1, i.e.*
(i) $Y_n \geq \max\{G_n, 0\}$, $\forall n \in \{0, 1, ..., N\}$, and
(ii) $\dfrac{Y_n}{(1+r)^n}$ is a supermartingale.
Then, $Y_n \geq V_n$, for all n in the set $\{0, 1, ..., N\}$.

Proof. Let Y_n satisfy the conditions (i) and (ii) from the theorem. Let $n \leq N$ be given and let τ be a stopping time in S_n. From (i):

$$\begin{aligned}
\mathbb{1}_{\{\tau\leq N\}} G_\tau &\leq \mathbb{1}_{\{\tau\leq N\}} \max\{G_\tau, 0\} \\
&\leq \mathbb{1}_{\{\tau\leq N\}} \max\{G_{\min\{N,\tau\}}, 0\} \\
&\leq (\mathbb{1}_{\{\tau\leq N\}} + \mathbb{1}_{\{\tau=\infty\}}) \max\{G_{\min\{N,\tau\}}, 0\} \\
&= \max\{G_{\min\{N,\tau\}}, 0\} \\
&\leq Y_{\min\{N,\tau\}}
\end{aligned}$$

From the Optional Sampling theorem 2.6 and (ii), we obtain:

$$\begin{aligned}
\widetilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\tau\leq N\}} \frac{G_\tau}{(1+r)^\tau}\right] &= \widetilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\tau\leq N\}} \frac{G_\tau}{(1+r)^{\min\{N,\tau\}}}\right] \\
&\leq \widetilde{\mathbb{E}}_n\left[\frac{Y_{\min\{N,\tau\}}}{(1+r)^{\min\{N,\tau\}}}\right] \\
&\leq \frac{Y_{\min\{n,\tau\}}}{(1+r)^{\min\{n,\tau\}}}
\end{aligned}$$

Since, τ is in S_n, we have $n \geq \tau$.

$$\therefore \frac{Y_{\min\{n,\tau\}}}{(1+r)^{\min\{n,\tau\}}} = \frac{Y_n}{(1+r)^n}$$

$$\implies \widetilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\tau\leq N\}} \frac{G_\tau}{(1+r)^\tau}\right] \leq \frac{Y_n}{(1+r)^n}$$

$$\implies \widetilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\tau\leq N\}} \frac{G_n}{(1+r)^{\tau-n}}\right] \leq Y_n$$

But, $V_n = \max_{\tau\in S_n} \widetilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\tau\leq N\}} \dfrac{G_\tau}{(1+r)^{\tau-n}}\right] \leq Y_n$. Hence, $V_n \leq Y_n$. $\square$

We are now ready to come up with the pricing algorithm for the generalized path-dependent American derivative securities.

Theorem 3.3. *Consider an N-period Binomial Model. Let $(V_n)_{0\leq n\leq N}$ be the American risk-neutral price process as given by the definition 3.1. Then this price process follows the American pricing algorithm, for all n:*

$$V_N(\omega_1...\omega_N) = \max\{G_N(\omega_1...\omega_N), 0\},$$

$$V_n(\omega_1...\omega_n) = \max\left\{G_n(\omega_1...\omega_n), \frac{1}{1+r}\left[\tilde{p}V_{n+1}(\omega_1...\omega_n H) + \tilde{q}V_{n+1}(\omega_1...\omega_n T)\right]\right\}$$

Proof. Let V_n be defined as per the above theorem. Let us first verify whether V_n satisfies the conditions of (i) and (ii) of theorem 3.1.
We see that for $n = N$, it holds true. Applying Mathematical Induction backwards from N to 0, let $V_{n+1} \geq \max\{G_{n+1}, 0\}$. Then from the definition of V_n as per the theorem, we get $V_n \geq \max\{G_n, 0\}$. Also,

$$\begin{aligned}\frac{V_n(\omega_1...\omega_n)}{(1+r)^n} &\geq \frac{1}{(1+r)^{n+1}}\left[\tilde{p}V_{n+1}(\omega_1...\omega_n H) + \tilde{q}V_{n+1}(\omega_1...\omega_n T)\right] \\ &= \widetilde{\mathbb{E}}_n\left[\frac{V_{n+1}(\omega_1...\omega_n)}{(1+r)^{n+1}}\right]\end{aligned}$$

Hence, V_n satisfies both conditions of the theorem 3.1. From the definition, $V_N = \max\{G_N, 0\}$ is the smallest random variable satisfying conditions (i) and (ii). Applying Mathematical Induction backwards.
Let V_{n+1} be the smallest random variable satisfying. From the supermartingale property shown above for V_n, and due to the fact that $V_n \geq G_n$, we get

$$V_n(\omega_1...\omega_n) \geq \max\left\{G_n(\omega_1...\omega_n), \frac{1}{1+r}\left[\tilde{p}V_{n+1}(\omega_1...\omega_n H) + \tilde{q}V_{n+1}(\omega_1...\omega_n T)\right]\right\}$$

But, the definition of V_n gives the equality of both sides. Hence, from the principle of mathematical induction, V_n is the smallest random variable, satisfying both conditions (i) and (ii). Hence, from theorem 3.2, we see that V_n defined by this theorem is the same as the V_n from the definition 3.1. □

Theorem 3.4 (Optimal Exercise Time). *The stopping τ^* that maximizes the right-hand side of the expectation expression of the definition 3.1 at $n = 0$, and thus gives the optimal time to exercise an American option, is given by*

$$\tau^* = \min\{n : V_n = G_n\}$$

Proof. We need to show that

$$V_0 = \widetilde{\mathbb{E}}\left[\mathbb{1}_{\tau^* \leq N}\frac{G_{\tau^*}}{(1+r)^{\tau^*}}\right]$$

Consider the first n coin tosses $\omega_1...\omega_n$.
Let $\tau^* \geq n+1$.

$$\implies V_n(\omega_1...\omega_n) > G_n(\omega_1...\omega_n)$$

$$\begin{aligned}\therefore V_{\min\{n,\tau^*\}}(\omega_1...\omega_n) &= V_n(\omega_1...\omega_n) \\ &= \frac{1}{1+r}\left[\tilde{p}V_{n+1}(\omega_1...\omega_n H) + \tilde{q}V_{n+1}(\omega_1...\omega_n T)\right] \\ &= \frac{1}{1+r}\left[\tilde{p}V_{\min\{n+1,\tau^*\}}(\omega_1...\omega_n H) + \tilde{q}V_{\min\{n+1,\tau^*\}}(\omega_1...\omega_n T)\right]\end{aligned}$$

Now, let $\tau^* \leq n+1$.

$$\begin{aligned}\implies V_{\min\{n,\tau^*\}}(\omega_1...\omega_n) &= V_{\tau^*}(\omega_1...\omega_{\tau^*})\\ &= \tilde{p}V_{\tau^*}(\omega_1...\omega_{\tau^*}) + \tilde{q}V_{\tau^*}(\omega_1...\omega_{\tau^*})\\ &= \tilde{p}V_{\min\{n+1,\tau^*\}}(\omega_1...\omega_n H) + \tilde{q}V_{\min\{n+1,\tau^*\}}(\omega_1...\omega_n T)\end{aligned}$$

Therefore, the stopped process, $\left(\frac{V_{\min\{n,\tau^*\}}}{(1+r)^{\min\{n,\tau^*\}}}\right)_{0\leq n\leq N}$ is a martingale, under the risk-neutral probability measure.

$$\begin{aligned}\implies V_0 &= \widetilde{\mathbb{E}}\left[\frac{V_{\min\{N,\tau^*\}}}{(1+r)^{\min\{N,\tau^*\}}}\right]\\ &= \widetilde{\mathbb{E}}\left[\mathbb{1}_{\{\tau^*\leq N\}}\frac{G_{\tau^*}}{(1+r)^{\tau^*}}\right] + \widetilde{\mathbb{E}}\left[\mathbb{1}_{\{\tau^*=\infty\}}\frac{V_N}{(1+r)^N}\right]\end{aligned}$$

Notice that when $\tau^* = \infty$, we have $V_n \geq G_n$ for all n.

$$\implies V_N \geq G_N, \text{when } \tau^* = \infty.$$

From theorem 3.3, this occurs when $G_N < 0$ and $V_N = 0$.

$$\therefore \mathbb{1}_{\{\tau^*=\infty\}}V_N = 0$$

$$\implies V_0 = \widetilde{\mathbb{E}}\left[\mathbb{1}_{\tau^*\leq N}\frac{G_{\tau^*}}{(1+r)^{\tau^*}}\right]$$

Hence, τ^* is the optimal exercise time. □

4 Conclusion

A recursive algorithm was constructed for pricing any given American derivative security, when its payoff can be mathematically described. An application of this is to directly calculate American *Call* and *Put* options. We were also able to find out the optimal time to exercise such contracts, using the concept of stopping time.

References

[1] S. Shreve, *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*. Springer Finance, New York, 2004.