Solving the scalar wave equation using spacetime discretization

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A. Discretizing the Action

Consider the Lagrangian density $\mathcal{L}$ for a free scalar wave equation, that is used to extremize the action $\mathcal{S}$

$$S = \int \mathrm{d}x\,\mathrm{d}t\,\mathcal{L} = \int \left[\frac{1}{2}\eta^{\alpha\beta}\,\partial_\alpha u\,\partial_\beta u\right]\,\mathrm{d}x\,\mathrm{d}t. \tag{1}$$

We wish to find a solution vector $\mathbf{u}$ over a $N \times N$ spacetime grid. Thus $\mathbf{u}$ is a vector of length $N \times N$. Now, we write the discretized version of Eq. (1), by first replacing the continous integral with a summation, and replacing the derivatives by derivative operators. Note, that the summation is over spacetime points.

$$S_d = \frac{1}{2}\sum_{p=0}^{N\times N}\sum_{q=0}^{N\times N} w_{pq}\,\eta^{ab}\,D^p_{ak}u^k\,D^q_{bl}u^l \tag{2}$$

The matrix w^{pq} (an outer product of one-dimensional Curtis-Clenshaw weights) corresponds to the integration weights at each point on the spacetime grid, arranged along the diagonal of a $(N \times N)$ matrix. Further, the derivative operator D^a_{pk} corresponds to the derivative of the vector u^k evaluated at p. Concretly, the operator can be expressed as

$$\eta^{ab}\,D^p_{ak}\,D^q_{bl} = -\mathrm{I}\otimes\mathrm{D}^2_\mathrm{t} + \mathrm{D}^2_\mathrm{x}\otimes\mathrm{I} \tag{3}$$

for a (1+1) dimensionsal case, which gives a square matrix of dimension $(N \times N)$. Eq. (2), therefore, can be written in terms of a quadratic form as

$$S_d = \frac{1}{2}\mathbf{u}^T\mathbf{C}\,\mathbf{u}; \qquad \mathbf{C} = w^{pq}\,\eta^{ab}\,D^p_{ak}\,D^q_{bl}. \tag{4}$$

Now, given the method of integration, and the choice of basis functions we can construct the matrix C. It's a quadratic form, i.e. S_d is a scalar. Therefore, our problem reduces to finding the vector u^l, such that the scalar S_d is extremum. This is done by requiring

$$\frac{\partial S_d}{d\mathbf{u}} = 0 = \frac{1}{2}\mathbf{u}^T(\mathbf{C}+\mathbf{C}^T) \tag{5}$$

where the last equality is a standard result applicable to quadratic forms. Therefore, to get the solution, we solve the homogenous system of linear equations given by Eq. (5). For us, the form of Eq. (5) isn't the most intuitive. We can also write the result of the minization as

$$\frac{1}{2}(\mathbf{C}+\mathbf{C}^T)\,\mathbf{u} = \mathbf{b} \tag{6}$$

by manipulating the second term of the derivative instead of the first term.[1] Note, however, the matrix $\mathbf{C}$ is symmetric[2], we can just solve the system $\mathbf{Cu} = \mathbf{b}$, where $\mathbf{b}$ contains the conditions imposed by the evolution equation and the boundary conditions.

The above formalism has been succesfully implemented in Python for the 1D scalar wave equation in Cartesian coordinates, and extending it to higher dimensions seems straightforward at the moment. We know wish to evolve the scalar wave equation in null coordinates. To begin, we define two new coordinates (u, v) such that

$$u = t - r, \qquad v = t + r, \tag{7}$$

which immediately leads us to the inverse transformation

$$t = \frac{u+v}{2}, \qquad r = \frac{v-u}{2}. \tag{8}$$

[1] For a quick reference, see Eq. (46) in this document.

[2] This needs checking. In our implementation, the matrix A is not symmetric. However, symmetrizing the matrix leads to the wrong solution.

Since we'd still be working in 1D, the notation is consistent with 1D scalar wave equation in Cartesian coordinates. We now calculate how the action (see Eq. 1) looks, in these new coordinates. We get

$$S = \frac{1}{2}\int \eta^{uv}\partial_u\phi\,\partial_v\phi\sqrt{-\eta}\,du\,dv, \tag{9}$$

where η^{uv} is the Minkowski metric in (u, v) coordinates. However, if we further introduce a conformal compactification, we get an additional term since the Ricci scalar is no longer zero. Thus, we'd now have

$$S = \frac{1}{2}\int \left[\tilde{\eta}^{uv}\partial_u\phi\,\partial_v\phi + \tilde{R}\right]\sqrt{-\tilde{\eta}}\,du\,dv, \tag{10}$$

where $\tilde{\eta}^{uv}$ and $\tilde{R}$ are the metric tensor, and the Ricci scalar for the conformally compactified Minkowski spacetime. Further, if we consider a massive scalar field with a time-varying potential, we can add two additional terms to get such an action for the system:

$$S = \frac{1}{2}\int \left[-\tilde{\eta}^{uv}\partial_u\phi\,\partial_v\phi + m^2\phi + V(\phi) + \tilde{R}\right]\sqrt{-\tilde{\eta}}\,du\,dv. \tag{11}$$

If we adopt the hyperboloidal scri-fixing coordinates the authors of [1] adopt, then we have

[1] A. Zenginoğlu and L. E. Kidder, Phys. Rev. D **81**, 124010 (2010), arXiv:1004.0760 [gr-qc].