CHAPTER 5

Finite Fields

LEARNING OBJECTIVES

After studying this chapter, you should be able to:

- Distinguish among groups, rings, and fields.
- Define finite fields of the form GF(p).
- Explain the differences among ordinary polynomial arithmetic, polynomial arithmetic with coefficients in Z_p, and modular polynomial arithmetic in GF(2^n).
- Define finite fields of the form GF(2^n).
- Explain the two different uses of the mod operator.

Finite fields have become increasingly important in cryptography. A number of cryptographic algorithms rely heavily on properties of finite fields, notably the Advanced Encryption Standard (AES) and elliptic curve cryptography. Other examples include the message authentication code CMAC and the authenticated encryption scheme GCM.

This chapter provides the reader with sufficient background on the concepts of finite fields to be able to understand the design of AES and other cryptographic algorithms that use finite fields. Because students unfamiliar with abstract algebra may find the concepts behind finite fields somewhat difficult to grasp, we approach the topic in a way designed to enhance understanding. Our plan of attack is as follows:

1. Fields are a subset of a larger class of algebraic structures called rings, which are in turn a subset of the larger class of groups. In fact, as shown in Figure 5.1, both groups and rings can be further differentiated. Groups are defined by a simple set of properties and are easily understood. Each successive subset (abelian group, ring, commutative ring, and so on) adds additional properties and is thus more complex. Sections 5.1 through 5.3 will examine groups, rings, and fields, successively.
2. **Finite fields** are a subset of fields, consisting of those fields with a finite number of elements. These are the class of fields that are found in cryptographic algorithms. With the concepts of fields in hand, we turn in Section 5.4 to a specific class of finite fields, namely those with p elements, where p is prime. Certain asymmetric cryptographic algorithms make use of such fields.
3. A more important class of finite fields, for cryptography, comprises those with 2^n elements depicted as fields of the form GF(2^n). These are used in a wide variety of cryptographic algorithms. However, before discussing these fields, we need to analyze the topic of polynomial arithmetic, which is done in Section 5.5.
4. With all of this preliminary work done, we are able at last, in Section 5.6, to discuss finite fields of the form GF(2^n).

Before proceeding, the reader may wish to review Sections 2.1 through 2.3, which cover relevant topics in number theory.

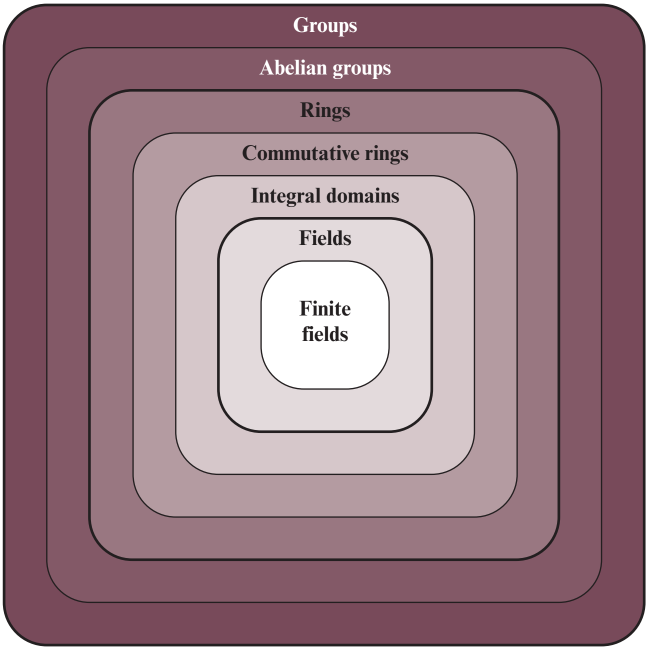


Figure 5.1 Groups, Rings, and Fields

5.1 GROUPS

Groups, rings, and fields are the fundamental elements of a branch of mathematics known as abstract algebra, or modern algebra. In abstract algebra, we are concerned with sets on whose elements we can operate algebraically; that is, we can combine two elements of the set, perhaps in several ways, to obtain a third element of the set. These operations are subject to specific rules, which define the nature of the set. By convention, the notation for the two principal classes of operations on set elements is usually the same as the notation for addition and multiplication on ordinary numbers. However, it is important to note that, in abstract algebra, we are not limited to basic arithmetical operations. All this should become clear as we proceed.

Groups

A group G, sometimes denoted by $\{G, \cdot\}$, is a set of elements with a binary operation denoted by $\cdot$ that associates to each ordered pair (a, b) of elements in G an element $(a \cdot b)$ in $G \times G$, such that the following axioms are obeyed:[1]

(A1) Closure: If a and b belong to G, then $a \cdot b$ is also in G.

(A2) Associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all a, b, c in G.

[1] The operator $\cdot$ is generic and can refer to addition, multiplication, or some other mathematical operation.

(A3) Identity element: There is an element e in G such that $a \cdot e = e \cdot a = a$ for all a in G.

(A4) Inverse element: For each a in G, there is an element a' in G such that $a \cdot a' = a' \cdot a = e$.

Let N_n denote a set of n distinct symbols that, for convenience, we represent as $\{1, 2, \dots, n\}$. A **permutation** of n distinct symbols is a one-to-one mapping from N_n on to N_n.[2] Define S_n to be the set of all permutations of n distinct symbols. Each element of S_n is represented by a permutation π of the integers in $1, 2, \dots, n$. It is easy to demonstrate that S_n is a group:

A1: If $(\pi, \rho \in S_n)$, then the composite mapping $\pi \cdot \rho$ is formed by permuting the elements of ρ according to the permutation π. For example, $\{3, 2, 1\} \cdot \{1, 3, 2\} = \{2, 3, 1\}$. The notation for this mapping is explained as follows: The value of the first element of π indicates which element of ρ is to be in the first position in $\pi \cdot \rho$; the value of the second element of π indicates which element of ρ is to be in the second position in $\pi \cdot \rho$; and so on. Clearly, $\pi \cdot \rho \in S_n$.

A2: The composition of mappings is also easily seen to be associative.

A3: The identity mapping is the permutation that does not alter the order of the n elements. For S_n, the identity element is $\{1, 2, \dots, n\}$.

A4: For any $\pi \in S_n$, the mapping that undoes the permutation defined by π is the inverse element for π. There will always be such an inverse. For example $\{2, 3, 1\} \cdot \{3, 1, 2\} = \{1, 2, 3\}$.

If a group has a finite number of elements, it is referred to as a finite group, and the **order** of the group is equal to the number of elements in the group. Otherwise, the group is an infinite group.

Abelian Group

A group is said to be abelian if it satisfies the following additional condition:

(A5) Commutative: $a \cdot b = b \cdot a$ for all a, b in G.

The set of integers (positive, negative, and 0) under addition is an abelian group. The set of nonzero real numbers under multiplication is an abelian group. The set S_n from the preceding example is a group but not an abelian group for $n > 2$.

[2] This is equivalent to the definition of permutation in Chapter 2, which stated that a permutation of a finite set of elements S is an ordered sequence of all the elements of S, with each element appearing exactly once.

When the group operation is addition, the identity element is 0; the inverse element of a is $-a$; and subtraction is defined with the following rule: $a - b = a + (-b)$.

Cyclic Group

We define exponentiation within a group as a repeated application of the group operator, so that $a^3 = a \cdot a \cdot a$. Furthermore, we define $a^0 = \text{e}$ as the identity element, and $a^{-n} = (a')^n$, where a' is the inverse element of a within the group. A group G is cyclic if every element of G is a power a^k (k is an integer) of a fixed element $a \in G$. The element a is said to **generate** the group G or to be a generator of G. A cyclic group is always abelian and may be finite or infinite.

> The additive group of integers is an infinite cyclic group generated by the element 1. In this case, powers are interpreted additively, so that n is the nth power of 1.

5.2 RINGS

A ring R, sometimes denoted by $\{R, +, \times\}$, is a set of elements with two binary operations, called *addition* and *multiplication*,[3] such that for all a, b, c in R the following axioms are obeyed.

(A1–A5) R is an abelian group with respect to addition; that is, R satisfies axioms A1 through A5. For the case of an additive group, we denote the identity element as 0 and the inverse of a as $-a$.

(M1) Closure under multiplication:	If a and b belong to R, then ab is also in R.
(M2) Associativity of multiplication:	$a(bc) = (ab)c$ for all a, b, c in R.
(M3) Distributive laws:	$a(b + c) = ab + ac$ for all a, b, c in R. $(a + b)c = ac + bc$ for all a, b, c in R.

In essence, a ring is a set of elements in which we can do addition, subtraction $[a - b = a + (-b)]$, and multiplication without leaving the set.

> With respect to addition and multiplication, the set of all n-square matrices over the real numbers is a ring.

A ring is said to be **commutative** if it satisfies the following additional condition:

(M4) Commutativity of multiplication: $ab = ba$ for all a, b in R.

[3] Generally, we do not use the multiplication symbol, ×, but denote multiplication by the concatenation of two elements. Thus, $a \times b$ is written as ab.

Let S be the set of even integers (positive, negative, and 0) under the usual operations of addition and multiplication. S is a commutative ring. The set of all n-square matrices defined in the preceding example is not a commutative ring.

The set Z_n of integers $\{0, 1, \ldots, n - 1\}$, together with the arithmetic operations modulo n, is a commutative ring (Table 4.3).

Next, we define an integral domain, which is a commutative ring that obeys the following axioms.

(M5) Multiplicative identity: There is an element 1 in R such that $a1 = 1a = a$ for all a in R.

(M6) No zero divisors: If a, b in R and $ab = 0$, then either $a = 0$ or $b = 0$.

Let S be the set of integers (positive, negative, and 0) under the usual operations of addition and multiplication. S is an integral domain.

5.3 FIELDS

A **field** F, sometimes denoted by $\{F, +, \times\}$, is a set of elements with two binary operations, called *addition* and *multiplication*, such that for all a, b, c in F the following axioms are obeyed.

(A1–M6) F is an integral domain; that is, F satisfies axioms A1 through A5 and M1 through M6.

(M7) Multiplicative inverse: For each a in F, except 0, there is an element a^{-1} in F such that $aa^{-1} = (a^{-1})a = 1$.

In essence, a field is a set of elements in which we can do addition, subtraction, multiplication, and division without leaving the set. Division is defined with the following rule: $a/b = a(b^{-1})$.

Familiar examples of fields are the rational numbers, the real numbers, and the complex numbers. Note that the set of all integers is not a field, because not every element of the set has a multiplicative inverse; in fact, only the elements 1 and -1 have multiplicative inverses in the integers.

In gaining insight into fields, the following alternate characterization may be useful. A **field** F, denoted by $\{F, +, \times\}$, is a set of elements with two binary operations, called *addition* and *multiplication,* such that the following conditions hold:

1. F forms an abelian group with respect to addition.
2. The nonzero elements of F form an abelian group with respect to multiplication.

3. The distributive law holds. That is, for all a, b, c in F,

$$a(b + c) = ab + ac.$$

$$(a + b)c = ac + bc.$$

4. Figure 5.2 summarizes the axioms that define groups, rings, and fields.

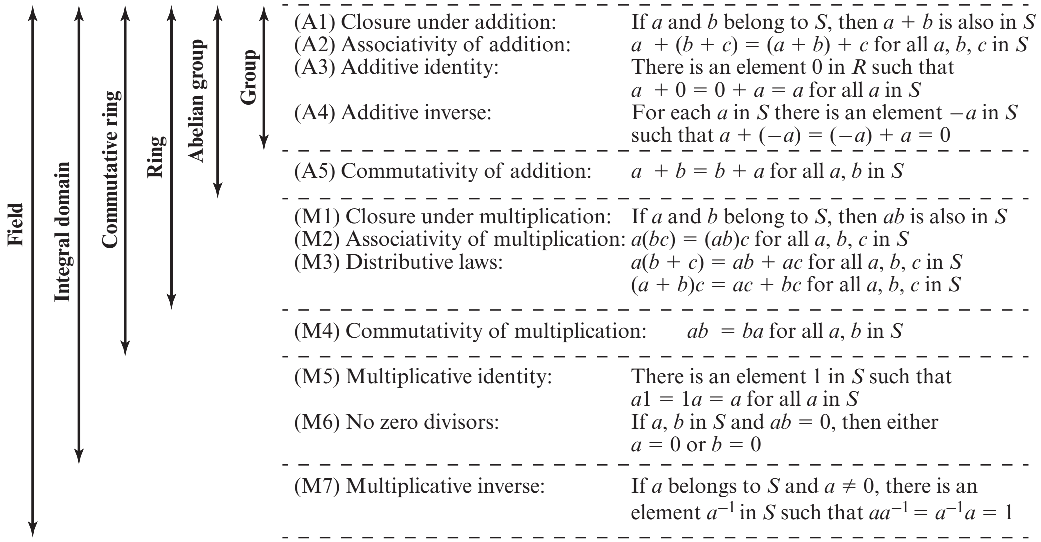
5.4 FINITE FIELDS OF THE FORM GF(p)

In Section 5.3, we defined a field as a set that obeys all of the axioms of Figure 5.2 and gave some examples of infinite fields. Infinite fields are not of particular interest in the context of cryptography. However, in addition to infinite fields, there are two types of finite fields, as illustrated in Figure 5.3. Finite fields play a crucial role in many cryptographic algorithms.

It can be shown that the order of a finite field (number of elements in the field) must be a power of a prime p^n, where n is a positive integer. The finite field of order p^n is generally written GF(p^n); GF stands for Galois field, in honor of the mathematician Galois who first studied finite fields. Two special cases are of interest for our purposes. For $n = 1$, we have the finite field GF(p); this finite field has a different structure than that for finite fields with $n > 1$ and is studied in this section. For finite fields of the form GF(p^n), GF(2^n) fields are of particular cryptographic interest, and these are covered in Section 5.6.

Finite Fields of Order p

For a given prime, p, we define the finite field of order p, GF(p), as the set Z_p of integers $\{0, 1, \ldots, p - 1\}$ together with the arithmetic operations modulo p. Note therefore that we are using ordinary modular arithmetic to define the operations over these fields.



(A1) Closure under addition:	If a and b belong to S, then $a + b$ is also in S
(A2) Associativity of addition:	$a + (b + c) = (a + b) + c$ for all a, b, c in S
(A3) Additive identity:	There is an element 0 in R such that $a + 0 = 0 + a = a$ for all a in S
(A4) Additive inverse:	For each a in S there is an element $-a$ in S such that $a + (-a) = (-a) + a = 0$
(A5) Commutativity of addition:	$a + b = b + a$ for all a, b in S
(M1) Closure under multiplication:	If a and b belong to S, then ab is also in S
(M2) Associativity of multiplication:	$a(bc) = (ab)c$ for all a, b, c in S
(M3) Distributive laws:	$a(b + c) = ab + ac$ for all a, b, c in S $(a + b)c = ac + bc$ for all a, b, c in S
(M4) Commutativity of multiplication:	$ab = ba$ for all a, b in S
(M5) Multiplicative identity:	There is an element 1 in S such that $a1 = 1a = a$ for all a in S
(M6) No zero divisors:	If a, b in S and $ab = 0$, then either $a = 0$ or $b = 0$
(M7) Multiplicative inverse:	If a belongs to S and $a \neq 0$, there is an element a^{-1} in S such that $aa^{-1} = a^{-1}a = 1$

Figure 5.2 Properties of Groups, Rings, and Fields

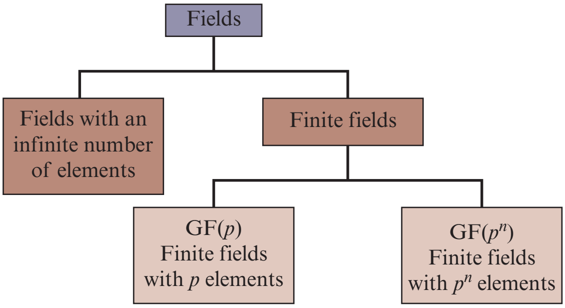

Fields
Fields with an infinite number of elements
Finite fields
GF(p) Finite fields with p elements
GF(p^n) Finite fields with p^n elements

Figure 5.3 Types of Fields

Recall that we showed in Section 5.2 that the set Z_n of integers $\{0, 1, \ldots, n - 1\}$, together with the arithmetic operations modulo n, is a commutative ring (Figure 5.2). We further observed that any integer in Z_n has a multiplicative inverse if and only if that integer is relatively prime to n [see discussion of Equation (2.5)].[4] If n is prime, then all of the nonzero integers in Z_n are relatively prime to n, and therefore there exists a multiplicative inverse for all of the nonzero integers in Z_n. Thus, for Z_p we can add the following properties to those listed in Table 5.2:

Multiplicative inverse (w^{-1})	For each $w \in Z_p$, $w \neq 0$, there exists a $z \in Z_p$ such that $w \times z \equiv 1 \pmod{p}$

Because w is relatively prime to p, if we multiply all the elements of Z_p by w, the resulting residues are all of the elements of Z_p permuted. Thus, exactly one of the residues has the value 1. Therefore, there is some integer in Z_p that, when multiplied by w, yields the residue 1. That integer is the multiplicative inverse of w, designated w^{-1}. Therefore, Z_p is in fact a finite field. Furthermore, Equation (2.5) is consistent with the existence of a multiplicative inverse and can be rewritten without the condition that a is relatively prime to n. So, for a and b in Z_p, with $a \neq 0$:

$$\textbf{if } (a \times b) \equiv (a \times c)(\text{mod } p) \textbf{ then } b \equiv c(\text{mod } p) \qquad \textbf{(5.1)}$$

Multiplying both sides of Equation (5.1) by the multiplicative inverse of a, we have

$$((a^{-1}) \times a \times b) \equiv ((a^{-1}) \times a \times c)(\text{mod } p)$$
$$b \equiv c \; (\text{mod } p)$$

The simplest finite field is GF(2). Its arithmetic operations are easily summarized:

+	0	1
0	0	1
1	1	0

Addition

×	0	1
0	0	0
1	0	1

Multiplication

w	$-w$	w^{-1}
0	0	–
1	1	1

Inverses

In this case, addition is equivalent to the exclusive-OR (XOR) operation, and multiplication is equivalent to the logical AND operation.

[4] As stated in the discussion of Equation (2.5), two integers are **relatively prime** if their only common positive integer factor is 1.

The right-hand side of Table 5.1 shows arithmetic operations in GF(7). This is a field of order 7 using modular arithmetic modulo 7. As can be seen, it satisfies all of the properties required of a field (Figure 5.2). Compare with the left-hand side of Table 5.1, which reproduces Table 2.2. In the latter case, we see that the set Z_8, using modular arithmetic modulo 8, is not a field. Later in this chapter, we show how to define addition and multiplication operations on Z_8 in such a way as to form a finite field.

Finding the Multiplicative Inverse in GF(p)

It is easy to find the multiplicative inverse of an element in GF(p) for small values of p. You simply construct a multiplication table, such as shown in Table 5.1e, and the desired result can be read directly. However, for large values of p, this approach is not practical.

If a and b are relatively prime, then b has a multiplicative inverse modulo a. That is, if $\gcd(a, b) = 1$, then b has a multiplicative inverse modulo a. Thus, for positive integer $b < a$, there exists a $b^{-1} < a$ such that $bb^{-1} \equiv 1 \bmod a$. If a is a prime number and $0 < b < a$, then clearly a and b are relatively prime and have a greatest common divisor of 1. We now show that we can easily compute b^{-1} using the extended Euclidean algorithm.

We repeat here Equation (2.7), which we showed can be solved with the extended Euclidean algorithm:

$$ax + by = d = \gcd(a, b)$$

Now, if $\gcd(a, b) = 1$, then we have $ax + by = 1$. Using the basic equalities of **modular arithmetic**, defined in Section 2.3, we can say

$$[(ax \bmod a) + (by \bmod a)] \bmod a = 1 \bmod a$$
$$0 + (by \bmod a) = 1$$

But if $by \bmod a = 1$, then $y = b^{-1}$. Thus, applying the extended Euclidean algorithm to Equation (2.7) yields the value of the multiplicative inverse of b if $\gcd(a, b) = 1$.

Consider the example that was shown in Table 2.4. Here we have $a = 1759$, which is a prime number, and $b = 550$. The solution of the equation $1759x + 550y = d$ yields a value of $y = 355$. Thus, $b^{-1} = 355$. To verify, we calculate $550 \times 355 \bmod 1759 = 195250 \bmod 1759 = 1$.

More generally, the extended Euclidean algorithm can be used to find a multiplicative inverse in Z_n for any n. If we apply the extended Euclidean algorithm to the equation $nx + by = d$, and the algorithm yields $d = 1$, then $y = b^{-1}$ in Z_n.

Table 5.1 Arithmetic Modulo 8 and Modulo 7

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

(a) Addition modulo 8

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

(b) Multiplication modulo 8

w	0	1	2	3	4	5	6	7
$-w$	0	7	6	5	4	3	2	1
w^{-1}	—	1	—	3	—	5	—	7

(c) Additive and multiplicative inverses modulo 8

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

(d) Addition modulo 7

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

(e) Multiplication modulo 7

w	0	1	2	3	4	5	6
$-w$	0	6	5	4	3	2	1
w^{-1}	—	1	4	5	2	3	6

(f) Additive and multiplicative inverses modulo 7

Summary

In this section, we have shown how to construct a finite field of order p, where p is prime. Specifically, we defined GF(p) with the following properties.

1. GF(p) consists of p elements.
2. The binary operations + and × are defined over the set. The operations of addition, subtraction, multiplication, and division can be performed without leaving the set. Each element of the set other than 0 has a multiplicative inverse, and division is performed by multiplication by the multiplicative inverse.

We have shown that the elements of GF(p) are the integers $\{0, 1, \ldots, p - 1\}$ and that the arithmetic operations are addition and multiplication mod p.

5.5 POLYNOMIAL ARITHMETIC

Before continuing our discussion of finite fields, we need to introduce the interesting subject of polynomial arithmetic. We are concerned with polynomials in a single variable x, and we can distinguish three classes of polynomial arithmetic (Figure 5.4).

- Ordinary polynomial arithmetic, using the basic rules of algebra.
- Polynomial arithmetic in which the arithmetic on the coefficients is performed modulo p; that is, the coefficients are in GF(p).
- Polynomial arithmetic in which the coefficients are in GF(p), and the polynomials are defined modulo a polynomial $m(x)$ whose highest power is some integer n.

This section examines the first two classes, and the next section covers the last class.

Ordinary Polynomial Arithmetic

A polynomial of degree n (integer $n \geq 0$) is an expression of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i$$

where the a_i are elements of some designated set of numbers S, called the coefficient set, and $a_n \neq 0$. We say that such polynomials are defined over the coefficient set S.

A zero-degree polynomial is called a constant polynomial and is simply an element of the set of coefficients. An nth-degree polynomial is said to be a monic polynomial if $a_n = 1$.

In the context of abstract algebra, we are usually not interested in evaluating a polynomial for a particular value of x [e.g., $f(7)$]. To emphasize this point, the variable x is sometimes referred to as the **indeterminate**.

Polynomial arithmetic includes the operations of addition, subtraction, multiplication, and division. These operations are defined in a natural way as though the

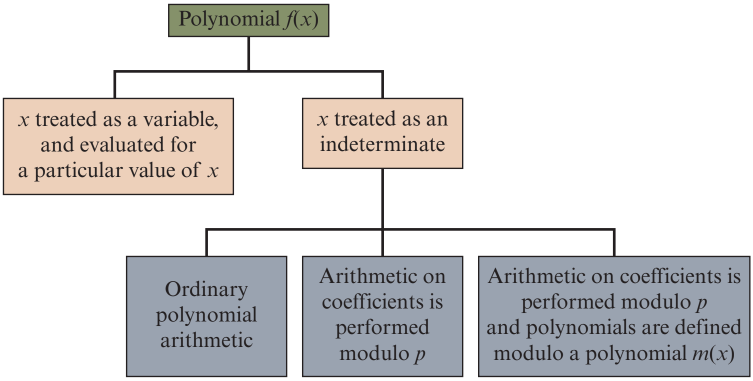

Polynomial $f(x)$
x treated as a variable, and evaluated for a particular value of x
x treated as an indeterminate
Ordinary polynomial arithmetic
Arithmetic on coefficients is performed modulo p
Arithmetic on coefficients is performed modulo p and polynomials are defined modulo a polynomial $m(x)$

Figure 5.4 Treatment of Polynomials

variable x was an element of S. Division is similarly defined, but requires that S be a field. Examples of fields include the real numbers, rational numbers, and Z_p for p prime. Note that the set of all integers is not a field and does not support polynomial division.

Addition and subtraction are performed by adding or subtracting corresponding coefficients. Thus, if

$$f(x) = \sum_{i=0}^{n} a_i x^i; \quad g(x) = \sum_{i=0}^{m} b_i x^i; \quad n \geq m$$

then addition is defined as

$$f(x) + g(x) = \sum_{i=0}^{m} (a_i + b_i)x^i + \sum_{i=m+1}^{n} a_i x^i$$

and multiplication is defined as

$$f(x) \times g(x) = \sum_{i=0}^{n+m} c_i x^i$$

where

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0$$

In the last formula, we treat a_i as zero for $i > n$ and b_i as zero for $i > m$. Note that the degree of the product is equal to the sum of the degrees of the two polynomials.

As an example, let $f(x) = x^3 + x^2 + 2$ and $g(x) = x^2 - x + 1$, where S is the set of integers. Then

$$\begin{aligned} f(x) + g(x) &= x^3 + 2x^2 - x + 3 \\ f(x) - g(x) &= x^3 + x + 1 \\ f(x) \times g(x) &= x^5 + 3x^2 - 2x + 2 \end{aligned}$$

Figures 5.5a through 5.5c show the manual calculations. We comment on division subsequently.

Polynomial Arithmetic with Coefficients in *Zp*

Let us now consider polynomials in which the coefficients are elements of some field F; we refer to this as a polynomial over the field F. In this case, it is easy to show that the set of such polynomials is a ring, referred to as a polynomial ring. That is, if we consider each distinct polynomial to be an element of the set, then that set is a ring.[5]

When polynomial arithmetic is performed on polynomials over a field, then division is possible. Note that this does not mean that *exact division* is possible. Let

[5] In fact, the set of polynomials whose coefficients are elements of a commutative ring forms a polynomial ring, but that is of no interest in the present context.

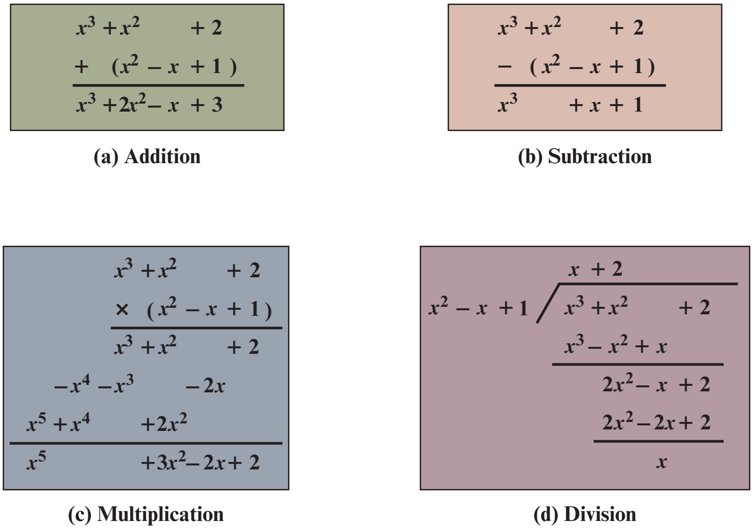


Figure 5.5 Examples of Polynomial Arithmetic

us clarify this distinction. Within a field, given two elements a and b, the quotient a/b is also an element of the field. However, given a ring R that is not a field, in general, division will result in both a quotient and a remainder; this is not exact division.

Consider the division 5/3 within a set S. If S is the set of rational numbers, which is a field, then the result is simply expressed as 5/3 and is an element of S. Now suppose that S is the field Z_7. In this case, we calculate (using Table 5.1f)

$$5/3 = (5 \times 3^{-1}) \bmod 7 = (5 \times 5) \bmod 7 = 4$$

which is an exact solution. Finally, suppose that S is the set of integers, which is a ring but not a field. Then 5/3 produces a quotient of 1 and a remainder of 2:

$$5/3 = 1 + 2/3$$
$$5 = 1 \times 3 + 2$$

Thus, division is not exact over the set of integers.

Now, if we attempt to perform polynomial division over a coefficient set that is not a field, we find that division is not always defined.

If the coefficient set is the integers, then $(5x^2)/(3x)$ does not have a solution, because it would require a coefficient with a value of 5/3, which is not in the coefficient set. Suppose that we perform the same polynomial division over Z_7. Then we have $(5x^2)/(3x) = 4x$, which is a valid polynomial over Z_7.

However, as we demonstrate presently, even if the coefficient set is a field, polynomial division is not necessarily exact. In general, division will produce a quotient and a remainder. We can restate the division algorithm of Equation (2.1) for polynomials over a field as follows. Given polynomials $f(x)$ of degree n and $g(x)$

of degree (m), $(n \geq m)$, if we divide $f(x)$ by $g(x)$, we get a quotient $q(x)$ and a remainder $r(x)$ that obey the relationship

$$f(x) = q(x)g(x) + r(x) \tag{5.2}$$

with polynomial degrees:

Degree $f(x) = n$
Degree $g(x) = m$
Degree $q(x) = n - m$
$0 \leq$ Degree $r(x) \leq m - 1$

With the understanding that remainders are allowed, we can say that polynomial division is possible if the coefficient set is a field. One common technique used for polynomial division is polynomial long division, similar to long division for integers. Examples of this are shown subsequently.

In an analogy to integer arithmetic, we can write $f(x)$ mod $g(x)$ for the remainder $r(x)$ in Equation (5.2). That is, $r(x) = f(x) \bmod g(x)$. If there is no remainder [i.e., $r(x) = 0$], then we can say $g(x)$ **divides** $f(x)$, written as $g(x)|f(x)$. Equivalently, we can say that $g(x)$ is a **factor** of $f(x)$ or $g(x)$ is a **divisor** of $f(x)$.

For the preceding example [$f(x) = x^3 + x^2 + 2$ and $g(x) = x^2 - x + 1$], $f(x)/g(x)$ produces a quotient of $q(x) = x + 2$ and a remainder $r(x) = x$, as shown in Figure 5.5d. This is easily verified by noting that

$$\begin{aligned} q(x)g(x) + r(x) &= (x + 2)(x^2 - x + 1) + x = (x^3 + x^2 - x + 2) + x \\ &= x^3 + x^2 + 2 = f(x) \end{aligned}$$

For our purposes, polynomials over GF(2) are of most interest. Recall from Section 5.4 that in GF(2), addition is equivalent to the XOR operation, and multiplication is equivalent to the logical AND operation. Further, addition and subtraction are equivalent mod 2:

$$\begin{aligned} 1 + 1 = 1 - 1 = 0 \\ 1 + 0 = 1 - 0 = 1 \\ 0 + 1 = 0 - 1 = 1 \end{aligned}$$

Figure 5.6 shows an example of polynomial arithmetic over GF(2). For $f(x) = (x^7 + x^5 + x^4 + x^3 + x + 1)$ and $g(x) = (x^3 + x + 1)$, the figure shows $f(x) + g(x)$; $f(x) - g(x)$; $f(x) \times g(x)$; and $f(x)/g(x)$. Note that $g(x)|f(x)$.

A polynomial $f(x)$ over a field F is called irreducible if and only if $f(x)$ cannot be expressed as a product of two polynomials, both over F, and both of degree greater than 0 and lower than that of $f(x)$. By analogy to integers, an irreducible polynomial is also called a prime polynomial.:

The polynomial[6] $f(x) = x^4 + 1$ over GF(2) is reducible, because

$$x^4 + 1 = (x + 1)(x^3 + x^2 + x + 1).$$

[6] In the reminder of this chapter, unless otherwise noted, all examples are of polynomials over GF(2).

Consider the polynomial $f(x) = x^3 + x + 1$. It is clear by inspection that x is not a factor of $f(x)$. We easily show that $x + 1$ is not a factor of $f(x)$:

$$\begin{array}{r|llll}
\multicolumn{1}{r}{} & x^2 & + x & & \\ \cline{2-5}
x + 1 & x^3 & & + x & + 1 \\
 & x^3 & + x^2 & & \\ \cline{2-3}
 & & x^2 & + x & \\
 & & x^2 & + x & \\ \cline{3-4}
 & & & & 1
\end{array}$$

Thus, $f(x)$ has no factors of degree 1. But it is clear by inspection that if $f(x)$ is reducible, it must have one factor of degree 2 and one factor of degree 1. Therefore, $f(x)$ is irreducible.

$$\begin{array}{llllll}
x^7 & +x^5 & +x^4 & +x^3 & +x & +1 \\
 & & & +(x^3 & +x & +1\,) \\ \hline
x^7 & +x^5 & +x^4 & & &
\end{array}$$

(a) Addition

$$\begin{array}{llllll}
x^7 & +x^5 & +x^4 & +x^3 & +x & +1 \\
 & & & -(x^3 & +x & +1\,) \\ \hline
x^7 & +x^5 & +x^4 & & &
\end{array}$$

(b) Subtraction

$$\begin{array}{lllllllllll}
 & & & x^7 & & +x^5 & +x^4 & +x^3 & & +x & +1 \\
 & & & & & & & \times(x^3 & & +x & +1\,) \\ \cline{4-11}
 & & & x^7 & & +x^5 & +x^4 & +x^3 & & +x & +1 \\
 & & x^8 & & +x^6 & +x^5 & +x^4 & & +x^2 & +x & \\
x^{10} & & +x^8 & +x^7 & +x^6 & & +x^4 & +x^3 & & & \\ \hline
x^{10} & & & & & & +x^4 & & +x^2 & & +1
\end{array}$$

(c) Multiplication

$$\begin{array}{r|llllll}
\multicolumn{1}{r}{} & x^4 & +1 & & & & \\ \cline{2-7}
x^3 + x + 1 & x^7 & +x^5 & +x^4 & +x^3 & +x & +1 \\
 & x^7 & +x^5 & +x^4 & & & \\ \cline{2-4}
 & & & & x^3 & +x & +1 \\
 & & & & x^3 & +x & +1 \\ \cline{5-7}
\end{array}$$

(d) Division

Figure 5.6 Examples of Polynomial Arithmetic over GF(2)

Finding the Greatest Common Divisor

We can extend the analogy between polynomial arithmetic over a field and integer arithmetic by defining the **greatest common divisor** as follows. The polynomial $c(x)$ is said to be the greatest common divisor of $a(x)$ and $b(x)$ if the following are true.

1. $c(x)$ divides both $a(x)$ and $b(x)$.
2. Any divisor of $a(x)$ and $b(x)$ is a divisor of $c(x)$.

An equivalent definition is the following: $\gcd[a(x), b(x)]$ is the polynomial of maximum degree that divides both $a(x)$ and $b(x)$.

We can adapt the Euclidean algorithm to compute the greatest common divisor of two polynomials. Recall Equation (2.6), from Chapter 2, which is the basis of the Euclidean algorithm: $\gcd(a, b) = \gcd(b, a \bmod b)$ assuming $a > b$. This equality can be rewritten as the following equation:

$$\gcd[a(x), b(x)] = \gcd[b(x), a(x) \bmod b(x)] \qquad \textbf{(5.3)}$$

The equation assumes that the degree of $a(x)$ is greater than the degree of $b(x)$. Equation (5.3) can be used repetitively to determine the greatest common divisor. Compare the following scheme to the definition of the Euclidean algorithm for integers.

Euclidean Algorithm for Polynomials	
Calculate	**Which satisfies**
$r_1(x) = a(x) \bmod b(x)$	$a(x) = q_1(x)b(x) + r_1(x)$
$r_2(x) = b(x) \bmod r_1(x)$	$b(x) = q_2(x)r_1(x) + r_2(x)$
$r_3(x) = r_1(x) \bmod r_2(x)$	$r_1(x) = q_3(x)r_2(x) + r_3(x)$
• • •	• • •
$r_n(x) = r_{n-2}(x) \bmod r_{n-1}(x)$	$r_{n-2}(x) = q_n(x)r_{n-1}(x) + r_n(x)$
$r_{n+1}(x) = r_{n-1}(x) \bmod r_n(x) = 0$	$r_{n-1}(x) = q_{n+1}(x)r_n(x) + 0$ $d(x) = \gcd(a(x), b(x)) = r_n(x)$

At each iteration, we have $d(x) = \gcd(r_{i+1}(x), r_i(x))$ until finally $d(x) = \gcd(r_n(x), 0) = r_n(x)$. Thus, we can find the greatest common divisor of two polynomials by repetitive application of the division algorithm. This is the Euclidean algorithm for polynomials.

Find $\gcd[a(x), b(x)]$ for $a(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ and $b(x) = x^4 + x^2 + x + 1$. First, we divide $a(x)$ by $b(x)$:

$$\begin{array}{rrrrrrrr}
 & x^2 & +x & & & & & \\ \cline{2-8}
x^4 + x^2 + x + 1 \big) & x^6 & +x^5 & +x^4 & +x^3 & +x^2 & +x & +1 \\
 & x^6 & & +x^4 & +x^3 & +x^2 & & \\ \cline{2-6}
 & & x^5 & & & & +x & +1 \\
 & & x^5 & & +x^3 & +x^2 & +x & \\ \cline{3-8}
 & & & & x^3 & +x^2 & & +1
\end{array}$$

This yields $r_1(x) = x^3 + x^2 + 1$ and $q_1(x) = x^2 + x$.

Then, we divide $b(x)$ by $r_1(x)$.

$$
\begin{array}{rlllll}
 & & x & +\ 1 & & \\ \cline{2-6}
x^3 + x^2 + 1 \big) & x^4 & & +\ x^2 & +\ x & +\ 1 \\
 & x^4 & +\ x^3 & & +\ x & \\ \cline{2-5}
 & & x^3 & +\ x^2 & & +\ 1 \\
 & & x^3 & +\ x^2 & & +\ 1 \\ \cline{3-6}
\end{array}
$$

This yields $r_2(x) = 0$ and $q_2(x) = x + 1$.

Therefore, $\gcd[a(x), b(x)] = r_1(x) = x^3 + x^2 + 1$.

Summary

We began this section with a discussion of arithmetic with ordinary polynomials. In ordinary polynomial arithmetic, the variable is not evaluated; that is, we do not plug a value in for the variable of the polynomials. Instead, arithmetic operations are performed on polynomials (addition, subtraction, multiplication, division) using the ordinary rules of algebra. Polynomial division is not allowed unless the coefficients are elements of a field.

Next, we discussed polynomial arithmetic in which the coefficients are elements of GF(p). In this case, polynomial addition, subtraction, multiplication, and division are allowed. However, division is not exact; that is, in general division results in a quotient and a remainder.

Finally, we showed that the Euclidean algorithm can be extended to find the greatest common divisor of two polynomials whose coefficients are elements of a field.

All of the material in this section provides a foundation for the following section, in which polynomials are used to define finite fields of order p^n.

5.6 FINITE FIELDS OF THE FORM GF(2^n)

Earlier in this chapter, we mentioned that the order of a finite field must be of the form p^n, where p is a prime and n is a positive integer. In Section 5.4, we looked at the special case of finite fields with order p. We found that, using modular arithmetic in Z_p, all of the axioms for a field (Figure 5.2) are satisfied. For polynomials over p^n, with $n > 1$, operations modulo p^n do not produce a field. In this section, we show what structure satisfies the axioms for a field in a set with p^n elements and concentrate on GF(2^n).

Motivation

Virtually all encryption algorithms, both symmetric and asymmetric, involve arithmetic operations on integers. If one of the operations that is used in the algorithm is division, then we need to work in arithmetic defined over a field. For convenience and for implementation efficiency, we would also like to work with integers that fit exactly into a given number of bits with no wasted bit patterns. That is, we wish to work with integers in the range 0 through $2^n - 1$, which fit into an n-bit word.

Suppose we wish to define a conventional encryption algorithm that operates on data 8 bits at a time, and we wish to perform division. With 8 bits, we can represent integers in the range 0 through 255. However, 256 is not a prime number, so that if arithmetic is performed in Z_{256} (arithmetic modulo 256), this set of integers will not be a field. The closest prime number less than 256 is 251. Thus, the set Z_{251}, using arithmetic modulo 251, is a field. However, in this case the 8-bit patterns representing the integers 251 through 255 would not be used, resulting in inefficient use of storage.

As the preceding example points out, if all arithmetic operations are to be used and we wish to represent a full range of integers in n bits, then arithmetic modulo 2^n will not work. Equivalently, the set of integers modulo 2^n for $n > 1$, is not a field. Furthermore, even if the encryption algorithm uses only addition and multiplication, but not division, the use of the set Z_{2^n} is questionable, as the following example illustrates.

Suppose we wish to use 3-bit blocks in our encryption algorithm and use only the operations of addition and multiplication. Then arithmetic modulo 8 is well defined, as shown in Table 5.1. However, note that in the multiplication table, the nonzero integers do not appear an equal number of times. For example, there are only four occurrences of 3, but twelve occurrences of 4. On the other hand, as was mentioned, there are finite fields of the form GF(2^n), so there is in particular a finite field of order $2^3 = 8$. Arithmetic for this field is shown in Table 5.2. In this case, the number of occurrences of the nonzero integers is uniform for multiplication. To summarize,

Integer	1	2	3	4	5	6	7
Occurrences in Z_8	4	8	4	12	4	8	4
Occurrences in GF(2^3)	7	7	7	7	7	7	7

For the moment, let us set aside the question of how the matrices of Table 5.2 were constructed and instead make some observations.

1. The addition and multiplication tables are symmetric about the main diagonal, in conformance to the commutative property of addition and multiplication. This property is also exhibited in Table 5.1, which uses mod 8 arithmetic.
2. All the nonzero elements defined by Table 5.2 have a multiplicative inverse, unlike the case with Table 5.1.
3. The scheme defined by Table 5.2 satisfies all the requirements for a finite field. Thus, we can refer to this scheme as GF(2^3).
4. For convenience, we show the 3-bit assignment used for each of the elements of GF(2^3).

Intuitively, it would seem that an algorithm that maps the integers unevenly onto themselves might be cryptographically weaker than one that provides a uniform mapping. That is, a cryptanalytic technique might be able to exploit the fact that some integers occur more frequently and some less frequently in the ciphertext. Thus, the finite fields of the form GF(2^n) are attractive for cryptographic algorithms.

To summarize, we are looking for a set consisting of 2^n elements, together with a definition of addition and multiplication over the set that define a field. We can assign a unique integer in the range 0 through $2^n - 1$ to each element of the set. Keep in mind that we will not use modular arithmetic, as we have seen that this does not result in a field. Instead, we will show how polynomial arithmetic provides a means for constructing the desired field.

Modular Polynomial Arithmetic

Consider the set S of all polynomials of degree $n - 1$ or less over the field Z_p. Thus, each polynomial has the form

$$f(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0 = \sum_{i=0}^{n-1} a_i x^i$$

Table 5.2 Arithmetic in GF(2^3)

		000	001	010	011	100	101	110	111
	+	0	1	2	3	4	5	6	7
000	0	0	1	2	3	4	5	6	7
001	1	1	0	3	2	5	4	7	6
010	2	2	3	0	1	6	7	4	5
011	3	3	2	1	0	7	6	5	4
100	4	4	5	6	7	0	1	2	3
101	5	5	4	7	6	1	0	3	2
110	6	6	7	4	5	2	3	0	1
111	7	7	6	5	4	3	2	1	0

(a) Addition

		000	001	010	011	100	101	110	111
	×	0	1	2	3	4	5	6	7
000	0	0	0	0	0	0	0	0	0
001	1	0	1	2	3	4	5	6	7
010	2	0	2	4	6	3	1	7	5
011	3	0	3	6	5	7	4	1	2
100	4	0	4	3	7	6	2	5	1
101	5	0	5	1	4	2	7	3	6
110	6	0	6	7	1	5	3	2	4
111	7	0	7	5	2	1	6	4	3

(b) Multiplication

w	$-w$	w^{-1}
0	0	–
1	1	1
2	2	5
3	3	6
4	4	7
5	5	2
6	6	3
7	7	4

(c) Additive and multiplicative inverses where each a_i takes on a value in the set $\{0, 1, \ldots, p - 1\}$. There are a total of p^n different polynomials in S.

For $p = 3$ and $n = 2$, the $3^2 = 9$ polynomials in the set are

$$0, 1, 2, x, x + 1, x + 2, 2x, 2x + 1, 2x + 2$$

For $p = 2$ and $n = 3$, the $2^3 = 8$ polynomials in the set are

$$0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1$$

With the appropriate definition of arithmetic operations, each such set S is a finite field. The definition consists of the following elements.

1. Arithmetic follows the ordinary rules of polynomial arithmetic using the basic rules of algebra, with the following two refinements.
2. Arithmetic on the coefficients is performed modulo p. That is, we use the rules of arithmetic for the finite field Z_p.
3. If multiplication results in a polynomial of degree greater than $n - 1$, then the polynomial is reduced modulo some irreducible polynomial $m(x)$ of degree n. That is, we divide by $m(x)$ and keep the remainder. For a polynomial $f(x)$, the remainder is expressed as $r(x) = f(x) \bmod m(x)$.

The Advanced Encryption Standard (AES) uses arithmetic in the finite field GF(2^8), with the irreducible polynomial $m(x) = x^8 + x^4 + x^3 + x + 1$. Consider the two polynomials $f(x) = x^6 + x^4 + x^2 + x + 1$ and $g(x) = x^7 + x + 1$. Then

$$\begin{aligned} f(x) + g(x) &= x^6 + x^4 + x^2 + x + 1 + x^7 + x + 1 \\ &= x^7 + x^6 + x^4 + x^2 \end{aligned}$$

$$\begin{aligned} f(x) \times g(x) &= x^{13} + x^{11} + x^9 + x^8 + x^7 \\ &\quad + x^7 + x^5 + x^3 + x^2 + x \\ &\quad + x^6 + x^4 + x^2 + x + 1 \\ &= x^{13} + x^{11} + x^9 + x^8 + x^6 + x^5 + x^4 + x^3 + 1 \end{aligned}$$

$$\begin{array}{r|llllllllll}
\multicolumn{1}{r}{} & x^5 & + x^3 & & & & & & & & \\ \cline{2-11}
x^8 + x^4 + x^3 + x + 1 & x^{13} & + x^{11} & + x^9 & + x^8 & & + x^6 & + x^5 & + x^4 & + x^3 & + 1 \\
\multicolumn{1}{r}{} & x^{13} & & + x^9 & + x^8 & & + x^6 & + x^5 & & & \\ \cline{2-8}
\multicolumn{1}{r}{} & & x^{11} & & & & & & + x^4 & + x^3 & \\
\multicolumn{1}{r}{} & & x^{11} & & & + x^7 & + x^6 & & + x^4 & + x^3 & \\ \cline{3-11}
\multicolumn{1}{r}{} & & & & & x^7 & + x^6 & & & & + 1
\end{array}$$

Therefore, $f(x) \times g(x) \bmod m(x) = x^7 + x^6 + 1$.

As with ordinary modular arithmetic, we have the notion of a set of residues in **modular polynomial arithmetic**. The set of residues modulo $m(x)$, an nth-degree polynomial, consists of p^n elements. Each of these elements is represented by one of the p^n polynomials of degree $m < n$.

The residue class $[x + 1]$, (mod $m(x)$), consists of all polynomials $a(x)$ such that $a(x) \equiv (x + 1)(\text{mod } m(x))$. Equivalently, the residue class $[x + 1]$ consists of all polynomials $a(x)$ that satisfy the equality $a(x) \text{ mod } m(x) = x + 1$.

It can be shown that the set of all polynomials modulo an irreducible nth-degree polynomial $m(x)$ satisfies the axioms in Figure 5.2, and thus forms a finite field. Furthermore, all finite fields of a given order are isomorphic; that is, any two finite-field structures of a given order have the same structure, but the representation or labels of the elements may be different.

To construct the finite field GF(2^3), we need to choose an irreducible polynomial of degree 3. There are only two such polynomials: $(x^3 + x^2 + 1)$ and $(x^3 + x + 1)$. Using the latter, Table 5.3 shows the addition and multiplication tables for GF(2^3). Note that this set of tables has the identical structure to those of Table 5.2. Thus, we have succeeded in finding a way to define a field of order 2^3.

We can now read additions and multiplications from the table easily. For example, consider binary $100 + 010 = 110$. This is equivalent to $x^2 + x$. Also consider $100 \times 010 = 011$, which is equivalent to $x^2 \times x = x^3$ and reduces to $x + 1$. That is, $x^3 \text{ mod } (x^3 + x + 1) = x + 1$, which is equivalent to 011.

Finding the Multiplicative Inverse

Just as the Euclidean algorithm can be adapted to find the greatest common divisor of two polynomials, the extended Euclidean algorithm can be adapted to find the multiplicative inverse of a polynomial. Specifically, the algorithm will find the multiplicative inverse of $b(x)$ modulo $a(x)$ if the degree of $b(x)$ is less than the degree of $a(x)$ and $\gcd[a(x), b(x)] = 1$. If $a(x)$ is an irreducible polynomial, then it has no factor other than itself or 1, so that $\gcd[a(x), b(x)] = 1$. The algorithm can be characterized in the same way as we did for the extended Euclidean algorithm for integers. Given polynomials $a(x)$ and $b(x)$ with the degree of $a(x)$ greater than the degree of $b(x)$, we wish to solve the following equation for the values $v(x)$, $w(x)$, and $d(x)$, where $d(x) = \gcd[a(x), b(x)]$:

$$a(x)v(x) + b(x)w(x) = d(x)$$

If $d(x) = 1$, then $w(x)$ is the multiplicative inverse of $b(x)$ modulo $a(x)$. The calculations are as follows.

Table 5.3 Polynomial Arithmetic Modulo $(x^3 + x + 1)$

		000	001	010	011	100	101	110	111
	+	0	1	x	$x + 1$	x^2	$x^2 + 1$	$x^2 + x$	$x^2 + x + 1$
000	0	0	1	x	$x + 1$	x^2	$x^2 + 1$	$x^2 + x$	$x^2 + x + 1$
001	1	1	0	$x + 1$	x	$x^2 + 1$	x^2	$x^2 + x + 1$	$x^2 + x$
010	x	x	$x + 1$	0	1	$x^2 + x$	$x^2 + x + 1$	x^2	$x^2 + 1$
011	$x + 1$	$x + 1$	x	1	0	$x^2 + x + 1$	$x^2 + x$	$x^2 + 1$	x^2
100	x^2	x^2	$x^2 + 1$	$x^2 + x$	$x^2 + x + 1$	0	1	x	$x + 1$
101	$x^2 + 1$	$x^2 + 1$	x^2	$x^2 + x + 1$	$x^2 + x$	1	0	$x + 1$	x
110	$x^2 + x$	$x^2 + x$	$x^2 + x + 1$	x^2	$x^2 + 1$	x	$x + 1$	0	1
111	$x^2 + x + 1$	$x^2 + x + 1$	$x^2 + x$	$x^2 + 1$	x^2	$x + 1$	x	1	0

(a) Addition

		000	001	010	011	100	101	110	111
	×	0	1	x	$x + 1$	x^2	$x^2 + 1$	$x^2 + x$	$x^2 + x + 1$
000	0	0	0	0	0	0	0	0	0
001	1	0	1	x	$x + 1$	x^2	$x^2 + 1$	$x^2 + x$	$x^2 + x + 1$
010	x	0	x	x^2	$x^2 + x$	$x + 1$	1	$x^2 + x + 1$	$x^2 + 1$
011	$x + 1$	0	$x + 1$	$x^2 + x$	$x^2 + 1$	$x^2 + x + 1$	x^2	1	x
100	x^2	0	x^2	$x + 1$	$x^2 + x + 1$	$x^2 + x$	x	$x^2 + 1$	1
101	$x^2 + 1$	0	$x^2 + 1$	1	x^2	x	$x^2 + x + 1$	$x + 1$	$x^2 + x$
110	$x^2 + x$	0	$x^2 + x$	$x^2 + x + 1$	1	$x^2 + 1$	$x + 1$	x	x^2
111	$x^2 + x + 1$	0	$x^2 + x + 1$	$x^2 + 1$	x	1	$x^2 + 1$	x^2	$x + 1$

(b) Multiplication

Extended Euclidean Algorithm for Polynomials			
Calculate	**Which satisfies**	**Calculate**	**Which satisfies**
$r_{-1}(x) = a(x)$		$v_{-1}(x) = 1; w_{-1}(x) = 0$	$a(x) = a(x)v_{-1}(x) + bw_{-1}(x)$
$r_0(x) = b(x)$		$v_0(x) = 0; w_0(x) = 1$	$b(x) = a(x)v_0(x) + b(x)w_0(x)$
$r_1(x) = a(x) \bmod b(x)$ $q_1(x) =$ quotient of $a(x)/b(x)$	$a(x) = q_1(x)b(x) + r_1(x)$	$v_1(x) = v_{-1}(x) - q_1(x)v_0(x) = 1$ $w_1(x) = w_{-1}(x) - q_1(x)w_0(x) = -q_1(x)$	$r_1(x) = a(x)v_1(x) + b(x)w_1(x)$
$r_2(x) = b(x) \bmod r_1(x)$ $q_2(x) =$ quotient of $b(x)/r_1(x)$	$b(x) = q_2(x)r_1(x) + r_2(x)$	$v_2(x) = v_0(x) - q_2(x)v_1(x)$ $w_2(x) = w_0(x) - q_2(x)w_1(x)$	$r_2(x) = a(x)v_2(x) + b(x)w_2(x)$
$r_3(x) = r_1(x) \bmod r_2(x)$ $q_3(x) =$ quotient of $r_1(x)/r_2(x)$	$r_1(x) = q_3(x)r_2(x) + r_3(x)$	$v_3(x) = v_1(x) - q_3(x)v_2(x)$ $w_3(x) = w_1(x) - q_3(x)w_2(x)$	$r_3(x) = a(x)v_3(x) + b(x)w_3(x)$
• • •	• • •	• • ⋮	• • •
$r_n(x) = r_{n-2}(x) \bmod r_{n-1}(x)$ $q_n(x) =$ quotient of $r_{n-2}(x)/r_{n-2}(x)$	$r_{n-2}(x) = q_n(x)r_{n-1}(x) + r_n(x)$	$v_n(x) = v_{n-2}(x) - q_n(x)v_{n-1}(x)$ $w_n(x) = w_{n-2}(x) - q_n(x)w_{n-1}(x)$	$r_n(x) = a(x)v_n(x) + b(x)w_n(x)$
$r_{n+1}(x) = r_{n-1}(x) \bmod r_n(x) = 0$ $q_{n+1}(x) =$ quotient of $r_{n-1}(x)/r_n(x)$	$r_{n-1}(x) = q_{n+1}(x)r_n(x) + 0$		$d(x) = \gcd(a(x), b(x)) = r_n(x)$ $v(x) = v_n(x); w(x) = w_n(x)$

Table 5.4 shows the calculation of the multiplicative inverse of $(x^7 + x + 1)$ mod $(x^8 + x^4 + x^3 + x + 1)$. The result is that $(x^7 + x + 1)^{-1} = (x^7)$. That is, $(x^7 + x + 1)(x^7) \equiv 1(\bmod\,(x^8 + x^4 + x^3 + x + 1))$.

Computational Considerations

A polynomial $f(x)$ in GF(2^n)

$$f(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0 = \sum_{i=0}^{n-1} a_i x^i$$

can be uniquely represented by the sequence of its n binary coefficients $(a_{n-1}, a_{n-2}, \ldots, a_0)$. Thus, every polynomial in GF(2^n) can be represented by an n-bit number.

Table 5.4 Extended Euclid $[(x^8 + x^4 + x^3 + x + 1), (x^7 + x + 1)]$

Initialization	$a(x) = x^8 + x^4 + x^3 + x + 1; v_{-1}(x) = 1; w_{-1}(x) = 0$ $b(x) = x^7 + x + 1; v_0(x) = 0; w_0(x) = 1$
Iteration 1	$q_1(x) = x; r_1(x) = x^4 + x^3 + x^2 + 1$ $v_1(x) = 1; w_1(x) = x$
Iteration 2	$q_2(x) = x^3 + x^2 + 1; r_2(x) = x$ $v_2(x) = x^3 + x^2 + 1; w_2(x) = x^4 + x^3 + x + 1$
Iteration 3	$q_3(x) = x^3 + x^2 + x; r_3(x) = 1$ $v_3(x) = x^6 + x^2 + x + 1; w_3(x) = x^7$
Iteration 4	$q_4(x) = x; r_4(x) = 0$ $v_4(x) = x^7 + x + 1; w_4(x) = x^8 + x^4 + x^3 + x + 1$
Result	$d(x) = r_3(x) = \gcd(a(x), b(x)) = 1$ $w(x) = w_3(x) = (x^7 + x + 1)^{-1} \bmod (x^8 + x^4 + x^3 + x + 1) = x^7$

Tables 5.2 and 5.3 show the addition and multiplication tables for GF(2^3) modulo $m(x) = (x^3 + x + 1)$. Table 5.2 uses the binary representation, and Table 5.3 uses the polynomial representation.

Addition We have seen that addition of polynomials is performed by adding corresponding coefficients, and, in the case of polynomials over Z_2, addition is just the XOR operation. So, addition of two polynomials in GF(2^n) corresponds to a bitwise XOR operation.

Consider the two polynomials in GF(2^8) from our earlier example:

$$f(x) = x^6 + x^4 + x^2 + x + 1 \text{ and } g(x) = x^7 + x + 1.$$

$(x^6 + x^4 + x^2 + x + 1) + (x^7 + x + 1) = x^7 + x^6 + x^4 + x^2$ (polynomial notation)
$(01010111) \oplus (10000011) = (11010100)$ (binary notation)
$\{57\} \oplus \{83\} = \{D4\}$ (hexadecimal notation)[7]

Multiplication There is no simple XOR operation that will accomplish multiplication in GF(2^n). However, a reasonably straightforward, easily implemented technique is available. We will discuss the technique with reference to GF(2^8) using $m(x) = x^8 + x^4 + x^3 + x + 1$, which is the finite field used in AES. The technique readily generalizes to GF(2^n).

The technique is based on the observation that

$$x^8 \bmod m(x) = [m(x) - x^8] = (x^4 + x^3 + x + 1) \tag{5.4}$$

[7] A basic refresher on number systems (decimal, binary, hexadecimal) can be found at the Computer Science Student Resource Site at WilliamStallings.com/StudentSupport.html. Here each of two groups of 4 bits in a byte is denoted by a single hexadecimal character, and the two characters are enclosed in brackets.

A moment's thought should convince you that Equation (5.4) is true; if you are not sure, divide it out. In general, in GF(2^n) with an nth-degree polynomial $p(x)$, we have $x^n \bmod p(x) = [p(x) - x^n]$.

Now, consider a polynomial in GF(2^8), which has the form $f(x) = b_7x^7 + b_6x^6 + b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$. If we multiply by x, we have

$$x \times f(x) = (b_7x^8 + b_6x^7 + b_5x^6 + b_4x^5 + b_3x^4 + b_2x^3 + b_1x^2 + b_0x) \bmod m(x) \tag{5.5}$$

If $b_7 = 0$ in Equation (5.5), then the result is a polynomial of degree less than 8, which is already in reduced form, and no further computation is necessary. If $b_7 = 1$, then reduction modulo $m(x)$ is achieved using Equation (5.4):

$$x \times f(x) = (b_6x^7 + b_5x^6 + b_4x^5 + b_3x^4 + b_2x^3 + b_1x^2 + b_0x) + (x^4 + x^3 + x + 1)$$

It follows that multiplication by x (i.e., 00000010) can be implemented as a 1-bit left shift followed by a conditional bitwise XOR with (00011011), which represents $(x^4 + x^3 + x + 1)$. To summarize,

$$x \times f(x) = \begin{cases} (b_6b_5b_4b_3b_2b_1b_00) & \text{if } b_7 = 0 \\ (b_6b_5b_4b_3b_2b_1b_00) \oplus (00011011) & \text{if } b_7 = 1 \end{cases} \tag{5.6}$$

Multiplication by a higher power of x can be achieved by repeated application of Equation (5.6). By adding intermediate results, multiplication by any constant in GF(2^8) can be achieved.

In an earlier example, we showed that for $f(x) = x^6 + x^4 + x^2 + x + 1$, $g(x) = x^7 + x + 1$, and $m(x) = x^8 + x^4 + x^3 + x + 1$, we have $f(x) \times g(x) \bmod \mathrm{m}(x) = x^7 + x^6 + 1$. Redoing this in binary arithmetic, we need to compute (01010111) × (10000011). First, we determine the results of multiplication by powers of x:

$$\begin{aligned}
(01010111) \times (00000010) &= (10101110) \\
(01010111) \times (00000100) &= (01011100) \oplus (00011011) = (01000111) \\
(01010111) \times (00001000) &= (10001110) \\
(01010111) \times (00010000) &= (00011100) \oplus (00011011) = (00000111) \\
(01010111) \times (00100000) &= (00001110) \\
(01010111) \times (01000000) &= (00011100) \\
(01010111) \times (10000000) &= (00111000)
\end{aligned}$$

So,

$$\begin{aligned}
(01010111) \times (10000011) &= (01010111) \times [(00000001) \oplus (00000010) \oplus (10000000)] \\
&= (01010111) \oplus (10101110) \oplus (00111000) = (11000001)
\end{aligned}$$

which is equivalent to $x^7 + x^6 + 1$.

Using a Generator

An equivalent technique for defining a finite field of the form GF(2^n), using a primitive polynomial, is sometimes more convenient. To begin, we need several new definitions. A **generator** g of a finite field F of order q (contains q elements) is an element whose first $q - 1$ powers generate all the nonzero elements of F. That is, the elements of F consist of 0, $g^0, g^1, \ldots, g^{q-2}$.

Recall from the discussion in Chapter 2 that if a is a primitive root of n, then its powers $a, a^2, \ldots, a^{\phi(n)}$ are distinct (mod n) and are all relatively prime to n. In particular, for a prime number p, if a is a primitive root of p, then $a, a^2, \ldots, a^{p-1}$ are distinct (mod p). Consider a field F defined by a polynomial $f(x)$. An element b contained in F is called a **root** of the polynomial if $f(b) = 0$.

A monic polynomial $f(x)$ is a **primitive polynomial** of degree n over a finite field GF(p) if and only if all of its roots are generators of the nonzero elements of the finite field GF(p^n). In particular, it can be shown that $f(x)$ satisfies the following equation:

$$x^{p^n-1} \equiv 1(\text{mod}(f(x))$$

Moreover, (p^{n-1}) is the least positive integer for which the preceding equation is true. That is, there is no integer $m < (p^{n-1})$ for which $f(x)$ divides $(x^m - 1)$. For example, for GF(2^3), $f(x) = x^3 + x + 1$ is a primitive polynomial. Thus,

$$x^{2^3-1} = x^7 \equiv 1(\text{mod}\, x^3 + x + 1)$$

which is easily shown.

All primitive polynomials are also irreducible, but the reverse is not true. For an irreducible polynomial that is not a primitive polynomial, we can find a positive integer $m < (p^{n-1})$. For example, the irreducible polynomial used to define the GF(2^8) finite field for AES is $f(x) = x^8 + x^4 + x^3 + x + 1$. In this case, it can be easily calculated that $f(x)$ divides $(x^{51} - 1)$. But, because $51 \leq (2^8 - 1)$, $f(x)$ is not a primitive polynomial. A root of this polynomial can generate only 51 nonzero elements of GF(2^8).

Let us consider the finite field GF(2^3), defined over the primitive polynomial $x^3 + x + 1$, discussed previously. Thus, the generator g must satisfy $f(g) = g^3 + g + 1 = 0$. Keep in mind, as discussed previously, that we need not find a numerical solution to this equality. Rather, we deal with polynomial arithmetic in which arithmetic on the coefficients is performed modulo 2. Therefore, the solution to the preceding equality is $g^3 = -g - 1 = g + 1$. We now show that g in fact generates all of the polynomials of degree less than 3. We have the following.

$$g^4 = g(g^3) = g(g + 1) = g^2 + g$$
$$g^5 = g(g^4) = g(g^2 + g) = g^3 + g^2 = g^2 + g + 1$$
$$g^6 = g(g^5) = g(g^2 + g + 1) = g^3 + g^2 + g = g^2 + g + g + 1 = g^2 + 1$$
$$g^7 = g(g^6) = g(g^2 + 1) = g^3 + g = g + g + 1 = 1 = g^0$$

We see that the powers of g generate all the nonzero polynomials in GF(2^3). Also, it should be clear that $g^k = g^{k \bmod 7}$ for any integer k. Table 5.5 shows the power representation, as well as the polynomial and binary representations.

This power representation makes multiplication easy. To multiply in the power notation, add exponents modulo 7. For example, $g^4 \times g^6 = g^{(10 \bmod 7)} = g^3 = g + 1$. The same result is achieved using polynomial arithmetic: We have $g^4 = g^2 + g$ and $g^6 = g^2 + 1$. Then, $(g^2 + g) \times (g^2 + 1) = g^4 + g^3 + g^2 + g$. Next, we need to determine $(g^4 + g^3 + g^2 + g) \bmod (\mathrm{g}^3 + \mathrm{g} + 1)$ by division:

$$
\begin{array}{r l}
 & g \quad + 1 \\
\cline{2-2}
g^3 + g + 1 \big) & g^4 + g^3 + g^2 + g \\
 & \underline{g^4 + \qquad\quad g^2 + g} \\
 & \qquad g^3 \\
 & \qquad \underline{g^3 + \qquad\quad g + 1} \\
 & \qquad\qquad\qquad\quad g + 1
\end{array}
$$

We get a result of $g + 1$, which agrees with the result obtained using the power representation.

Table 5.6 shows the addition and multiplication tables for GF(2^3) using the power representation. Note that this yields the identical results to the polynomial representation (Table 5.3) with some of the rows and columns interchanged.

Table 5.5 Generator for $GF(2^3)$ using $x^3 + x + 1$

Power Representation	Polynomial Representation	Binary Representation	Decimal (Hex) Representation
0	0	000	0
$g^0 (= g^7)$	1	001	1
g^1	g	010	2
g^2	g^2	100	4
g^3	$g + 1$	011	3
g^4	$g^2 + g$	110	6
g^5	$g^2 + g + 1$	111	7
g^6	$g^2 + 1$	101	5

In general, for GF(2^n) with primitive polynomial $f(x)$, determine $g^n = \mathrm{f}(g) - g^n$. Then calculate all of the powers of g from g^{n+1} through $g^{2^n - 2}$. The elements of the field correspond to the powers of g from g^0 through $g^{2^n - 2}$ plus the value 0. For multiplication of two elements in the field, use the equality $g^k = g^{k \bmod (2^n - 1)}$ for any integer k.

Table 5.6 GF(2^3) Arithmetic Using Generator for the Polynomial ($x^3 + x + 1$)

		000	001	010	100	011	110	111	101
	+	0	1	g	g^2	g^3	g^4	g^5	g^6
000	0	0	1	g	g^2	$g + 1$	$g^2 + g$	$g^2 + g + 1$	$g^2 + 1$
001	1	1	0	$g + 1$	$g^2 + 1$	g	$g^2 + g + 1$	$g^2 + g$	g^2
010	g	g	$g + 1$	0	$g^2 + g$	1	g^2	$g^2 + 1$	$g^2 + g + 1$
100	g^2	g^2	$g^2 + 1$	$g^2 + g$	0	$g^2 + g + 1$	g	$g + 1$	1
011	g^3	$g + 1$	g	1	$g^2 + g + 1$	0	$g^2 + 1$	g^2	$g^2 + g$
110	g^4	$g^2 + g$	$g^2 + g + 1$	g^2	g	$g^2 + 1$	0	1	$g + 1$
111	g^5	$g^2 + g + 1$	$g^2 + g$	$g^2 + 1$	$g + 1$	g^2	1	0	g
101	g^6	$g^2 + 1$	g^2	$g^2 + g + 1$	1	$g^2 + g$	$g + 1$	g	0

(a) Addition

		000	001	010	100	011	110	111	101
	×	0	1	g	g^2	g^3	g^4	g^5	g^6
000	0	0	0	0	0	0	0	0	0
001	1	0	1	g	g^2	$g + 1$	$g^2 + g$	$g^2 + g + 1$	$g^2 + 1$
010	g	0	g	g^2	$g + 1$	$g^2 + g$	$g^2 + g + 1$	$g^2 + 1$	1
100	g^2	0	g^2	$g + 1$	$g^2 + g$	$g^2 + g + 1$	$g^2 + 1$	1	g
011	g^3	0	$g + 1$	$g^2 + g$	$g^2 + g + 1$	$g^2 + 1$	1	g	g^2
110	g^4	0	$g^2 + g$	$g^2 + g + 1$	$g^2 + 1$	1	g	g^2	$g + 1$
111	g^5	0	$g^2 + g + 1$	$g^2 + 1$	1	g	g^2	$g + 1$	$g^2 + g$
101	g^6	0	$g^2 + 1$	1	g	g^2	$g + 1$	$g^2 + g$	$g^2 + g + 1$

(b) Multiplication

Summary

In this section, we have shown how to construct a finite field of order 2^n. Specifically, we defined GF(2^n) with the following properties.

1. GF(2^n) consists of 2^n elements.
2. The binary operations + and × are defined over the set. The operations of addition, subtraction, multiplication, and division can be performed without leaving the set. Each element of the set other than 0 has a multiplicative inverse.

We have shown that the elements of GF(2^n) can be defined as the set of all polynomials of degree $n - 1$ or less with binary coefficients. Each such polynomial can be represented by a unique n-bit value. Arithmetic is defined as polynomial arithmetic modulo some irreducible polynomial of degree n. We have also seen that an equivalent definition of a finite field GF(2^n) makes use of a generator and that arithmetic is defined using powers of the generator.

5.7 KEY TERMS, REVIEW QUESTIONS, AND PROBLEMS

Key Terms

commutative divisor field	finite field greatest common divisor identity element	modular arithmetic order relatively prime

Review Questions

5.1 Briefly define a group.

5.2 Briefly define a ring.

5.3 Briefly define a field.

5.4 Briefly define an irreducible polynomial.

Problems

5.1 Consider the group S_7 of all permutations of 7 distinct symbols.

a. Let $x = (1, 2, 3)\ (4, 6)$ and $y = (2, 3, 4, 5, 6)$ in S_7 be two permutations that are written in disjoint cycle notation. Compute $x \cdot y$ and $y \cdot x$.

b. Is S_7 abelian?

5.2 Does the set of residue classes (mod3) form a group

a. with respect to modular addition?

b. with respect to modular multiplication?

5.3 Let $S = \{0, a, b, c\}$ The addition and multiplication on the set S is defined in the following tables:

+	0	a	B	C
0	0	a	B	C
A	a	0	c	B
B	b	c	0	A
C	c	b	a	0

×	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

Is S a noncommutative ring? Justify your answer.

5.4 Develop a set of tables similar to Table 5.1 for GF(5).

5.5 Demonstrate that the set of polynomials whose coefficients form a field is a ring.

5.6 Let R be the field of real numbers. Let $R[x]$ be the ring of polynomials with coefficients in field R. State whether each of the following statements is true or false.

a. $R[x]$ is a commutative ring with unity, with multiplicative identity being the constant polynomial 1.

b. $f \in R[x]$ has a multiplicative inverse if and only if f is a non-zero constant.

c. $R[x]$ is also a field.

5.7 For polynomial arithmetic with coefficients in Z_{11}, perform the following calculations.

a. $(x^2 + 2x + 9)(x^3 + 11x^2 + x + 7)$

b. $(8x^2 + 3x + 2)(5x^2 + 6)$

5.8 Determine which of the following polynomials are reducible over GF(2).

a. $x^2 + 1$

b. $x^2 + x + 1$

c. $x^4 + x + 1$

5.9 Determine the gcd of the following pairs of polynomials.

a. $(x^3 + 1)$ and $(x^2 + x + 1)$ over GF(2)

b. $(x^3 + x + 1)$ and $(x^2 + 1)$ over GF(3)

c. $(x^3 - 2x + 1)$ and $(x^2 - x - 2)$ over GF(5)

d. $(x^4 + 8x^3 + 7x + 8)$ and $(2x^3 + 9x^2 + 10x + 1)$ over GF(11)

5.10 Develop a set of tables similar to Table 5.3 for GF(3) with $m(x) = x^2 + x + 1$.

5.11 Determine the multiplicative inverse of $x^2 + 1$ in GF(2^3) with $m(x) = x^3 + x - 1$.

5.12 Develop a table similar to Table 5.5 for GF(2^5) with $m(x) = x^5 + x^4 + x^3 + x + 1$.

Programming Problems

5.1 Write a simple four-function calculator in GF(2^4). You may use table lookups for the multiplicative inverses.

5.2 Write a simple four-function calculator in GF(2^8). You should compute the multiplicative inverses on the fly.