Direct method for 1D steady heat conduction equation

Consider the cooling of a circular solid cylinder by means of convective heat transfer along its length. The solid cylinder is a thin rod of length 10 cm and constant cross section with diameter 1 mm. The left end of the cylinder is held at a constant temperature, $T_L = 200°\mathrm{C}$, and the right end is insulated. The ambient temperature, T_∞, is $15°\mathrm{C}$.

The heat transfer along the cylinder is governed by the 1D steady heat equation:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + S(T) = 0$$

where k is the thermal conductivity constant and $S(T)$ is a source term given as follows

$$S(T) = -h\frac{P}{A_C}(T - T_\infty)$$

h is the convective heat transfer coefficient, and P and A_C are the perimeter and the area,

respectively, of the cross-section of the cylinder.

Discretization: we use centered finite difference for the second order derivative

$$\frac{d^2T}{dx^2}|_j \approx \frac{T_{j+1} - 2T_j + T_{j-1}}{\Delta x^2}$$

Substituting the discretized conduction term and the source term into the governing equation leads to a linear algebraic equation of the form

$$a_P T_j = a_W T_{j-1} + a_E T_{j+1} + S_u$$

The coefficients:

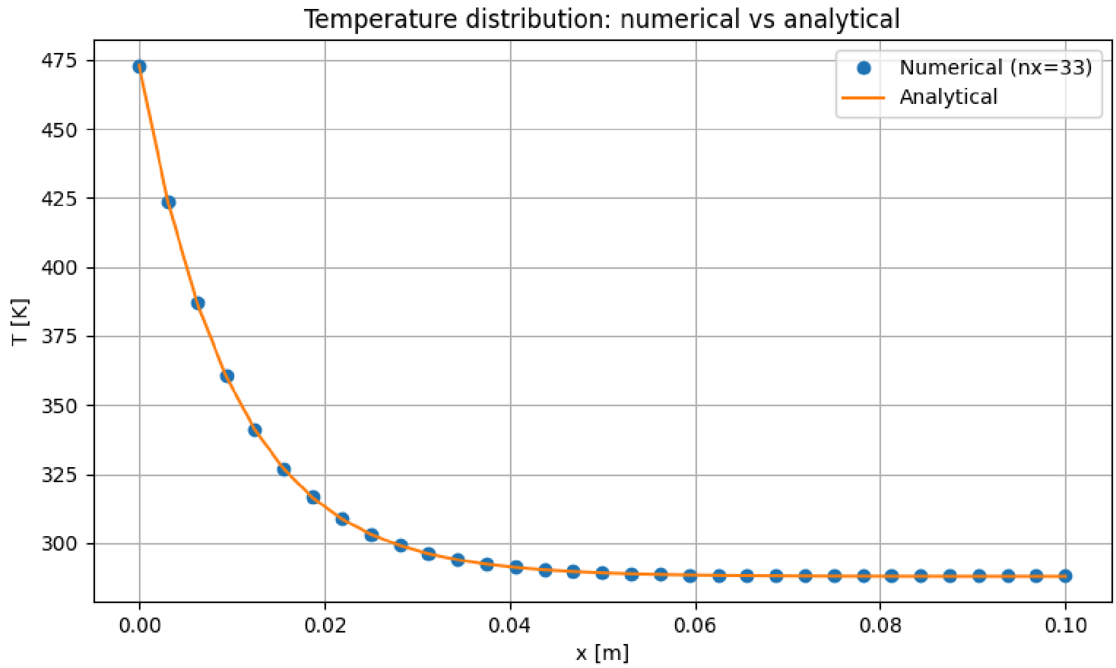
- West and east coefficients: $a_E = a_W = k/\Delta x^2$
- Central coefficients: $a_P = a_E + a_W - S_P$

The source term is rewritten in linear form: $S(T) = S_P T + S_U$

$$S_P = -h\frac{P}{A_C}, \qquad S_U = h\,\frac{P}{A_C}\,T_\infty.$$

Results

The following figure compares the **numerical finite-difference solution** (blue circles) with the **analytical insulated-tip fin solution** (orange line). The curves lie almost on top of each other. Physically, the temperature drops rapidly near the heated base $x = 0$, because the fin loses heat strongly to convection, and then levels off toward the ambient temperature as $x \to L$; the insulated tip enforces zero heat flux, so the profile becomes nearly flat near the end.



Error

We compute the numerical temperature $T_n(x)$ on a uniform grid with n_x points, so the spacing is

$$\Delta x = \frac{L}{n_x - 1}.$$

For each grid, we compare the numerical solution with the analytical insulated-tip fin solution

$$T_{\text{exact}}(x) = T_\infty + (T_L - T_\infty)\frac{\cosh\big(m(L-x)\big)}{\cosh(mL)}, \qquad m = \sqrt{\frac{hP}{kA_c}}.$$

The error vector at the grid points is: $e_j = T_{j,\text{num}} - T_{j,\text{exact}}$. The reported "$(L^2)$" error in the table is a discrete approximation of the continuous (L^2) norm over the domain:

$$|e|_{L^2} \approx \left(\sum_{j=1}^{N_J} e_j^2, \Delta x\right)^{1/2} = \sqrt{\Delta x}\left(\sum_{j=1}^{N_J} e_j^2\right)^{1/2}.$$

Finally, the observed order of accuracy (p) between two successive grids is computed from

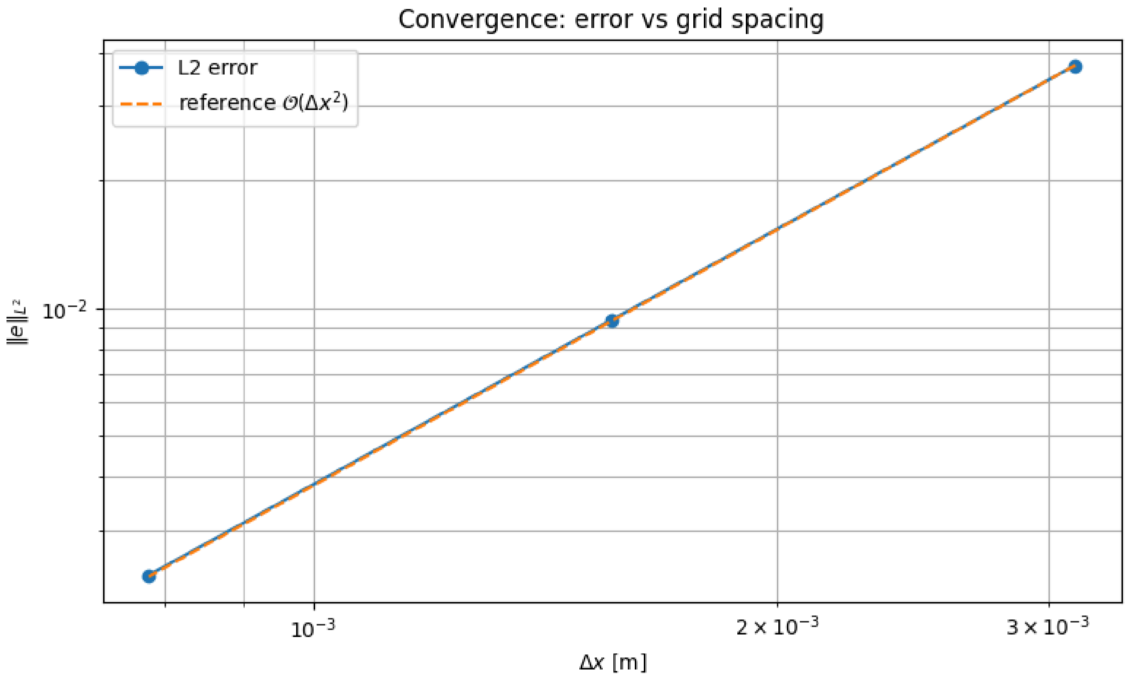
$$p = \frac{\ln(E_1/E_2)}{\ln(\Delta x_1/\Delta x_2)},$$

where E_1, E_2 are the two L^2 errors and $\Delta x_1, \Delta x_2$ the corresponding grid spacings. The values $p \approx 2$ indicate second-order convergence as the mesh is refined.

Error + convergence table

nx	Δx [m]	$\|e\|_{L2}$	p
33	3.125000e-03	0.037332	1.9911
65	1.562500e-03	0.009391	1.9980
129	7.812500e-04	0.002351	-

The second figure shows a **grid-convergence study**, plotting the L^2 error norm versus grid spacing Δx on log–log axes. The error decreases as the mesh is refined, and the dashed reference line proportional to Δx^2 matches the measured error trend closely. This indicates **second-order accuracy** overall, consistent with using central differences for the second derivative.



We see that the error is not zero even though a direct method is used to solve the linear system. The reason is that the error comes from the **finite-difference discretization**. In the finite-difference model, the second derivative $\mathrm{d}^2T/\mathrm{d}x^2$ is replaced by an approximation based on values at discrete grid points, which introduces a truncation (discretization) error. This error is inherent to the method and only decreases as the grid is refined (smaller Δx).