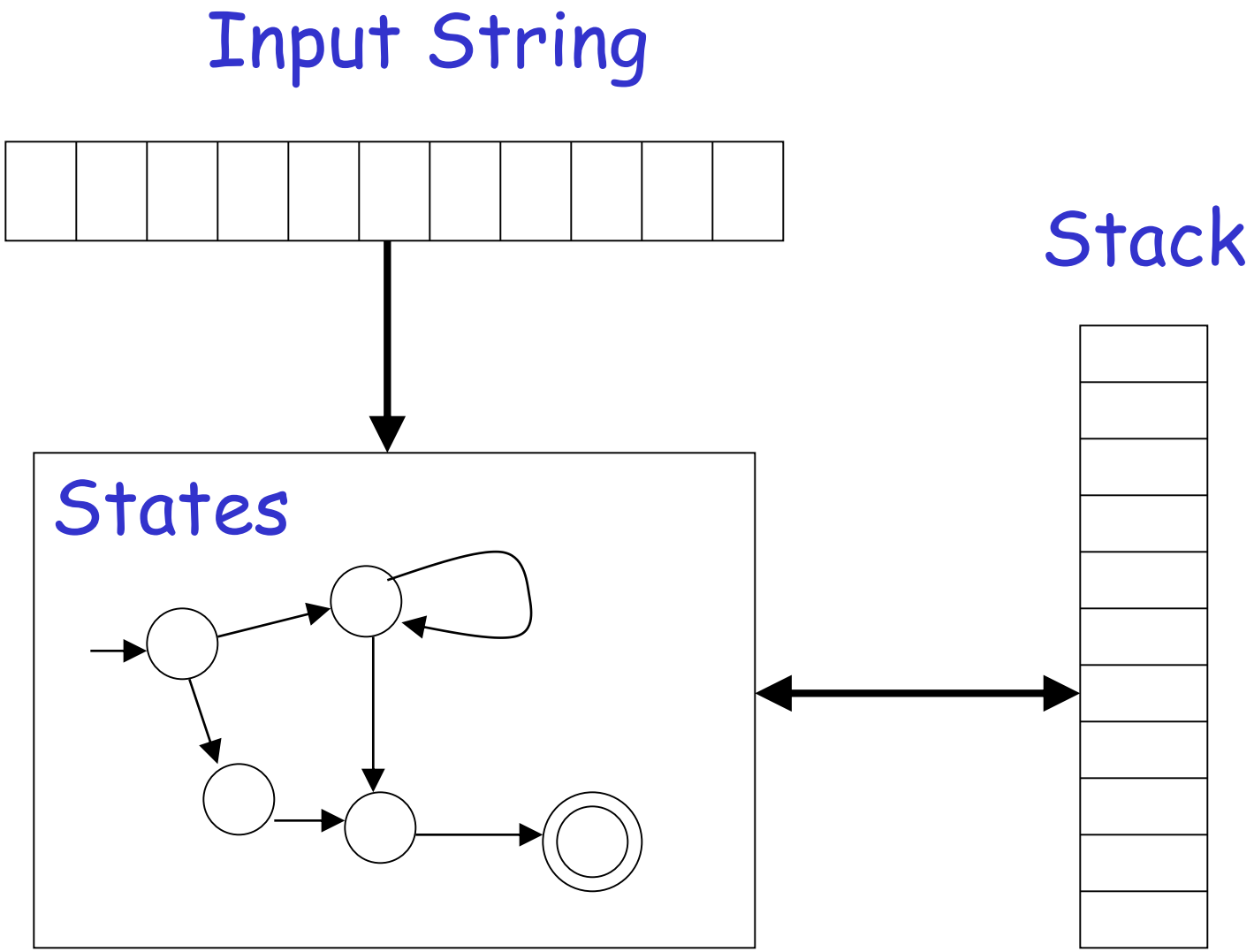
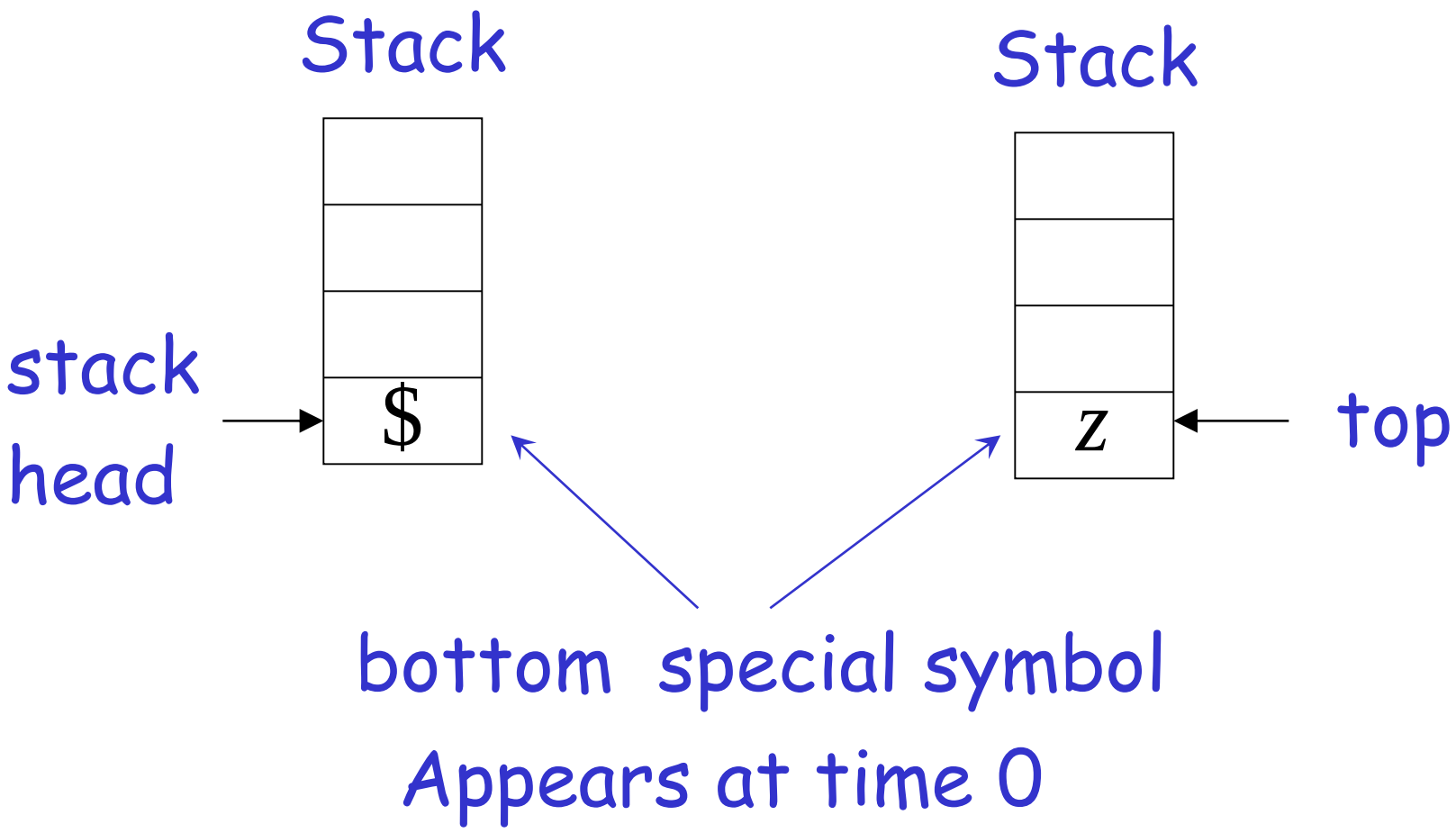
Pushdown Automata
PDAs

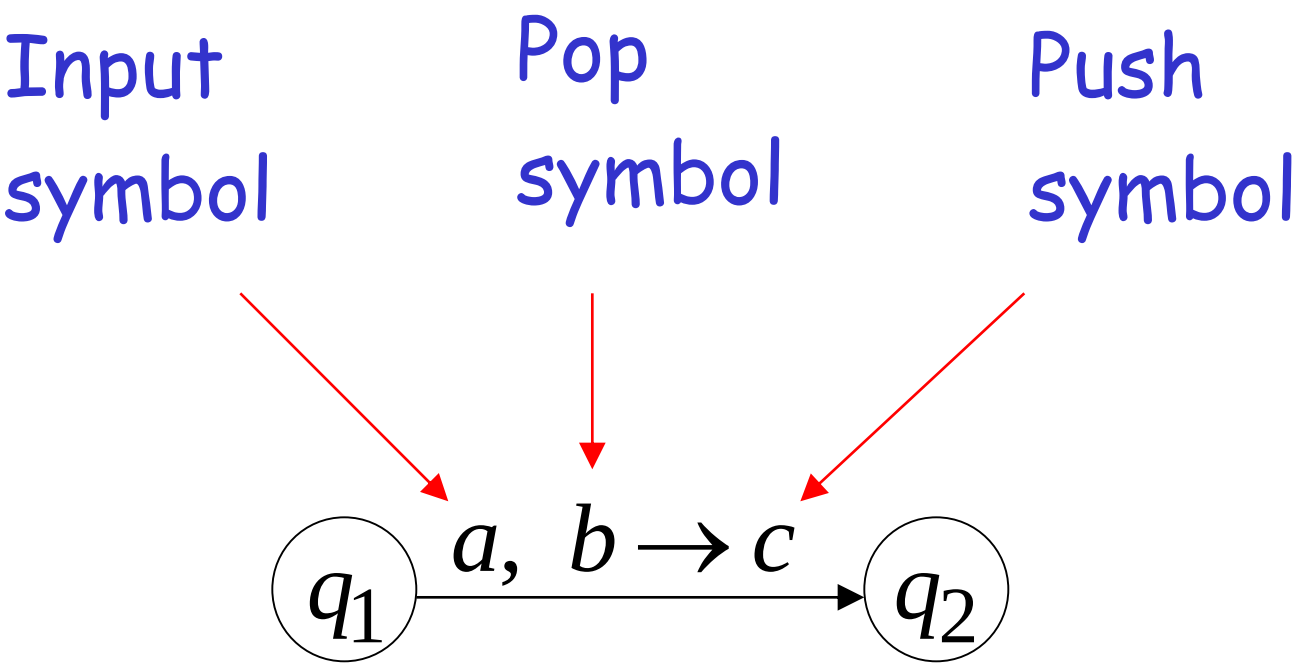
Pushdown Automaton -- PDA

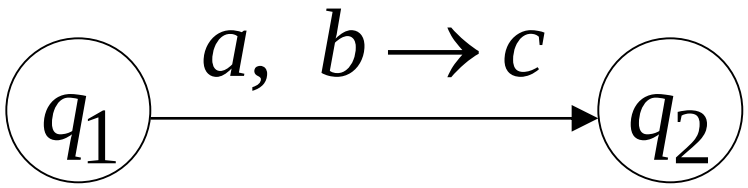


Initial Stack Symbol

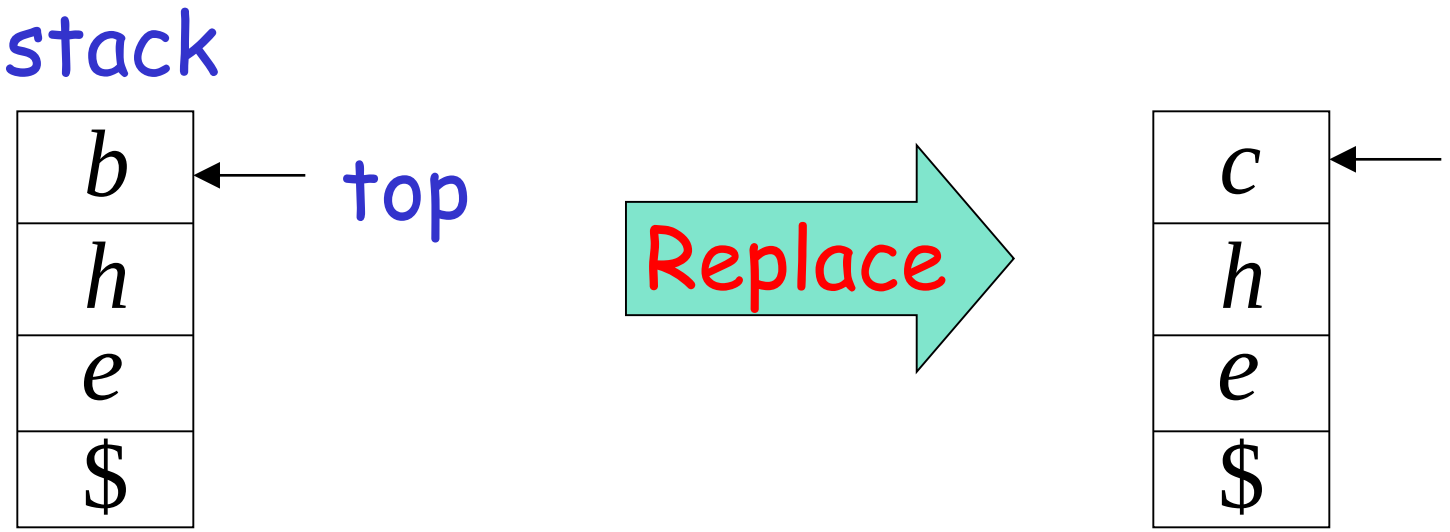


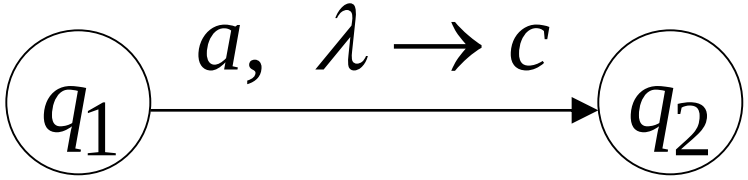
The States



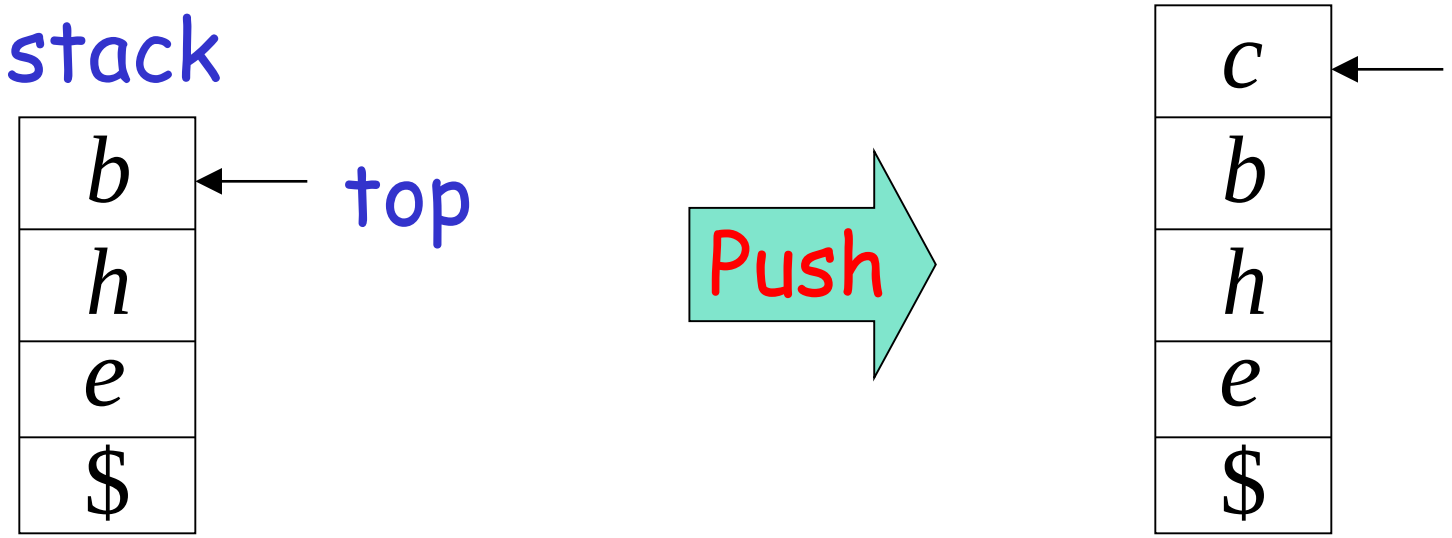

$a,\ b \to c$
q_1
q_2

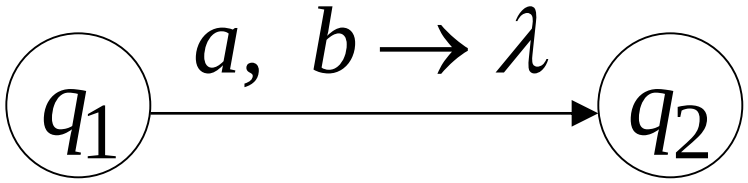

input
…
a
…
…
a
…


stack
b
h
e
$
top
Replace
c
h
e
$

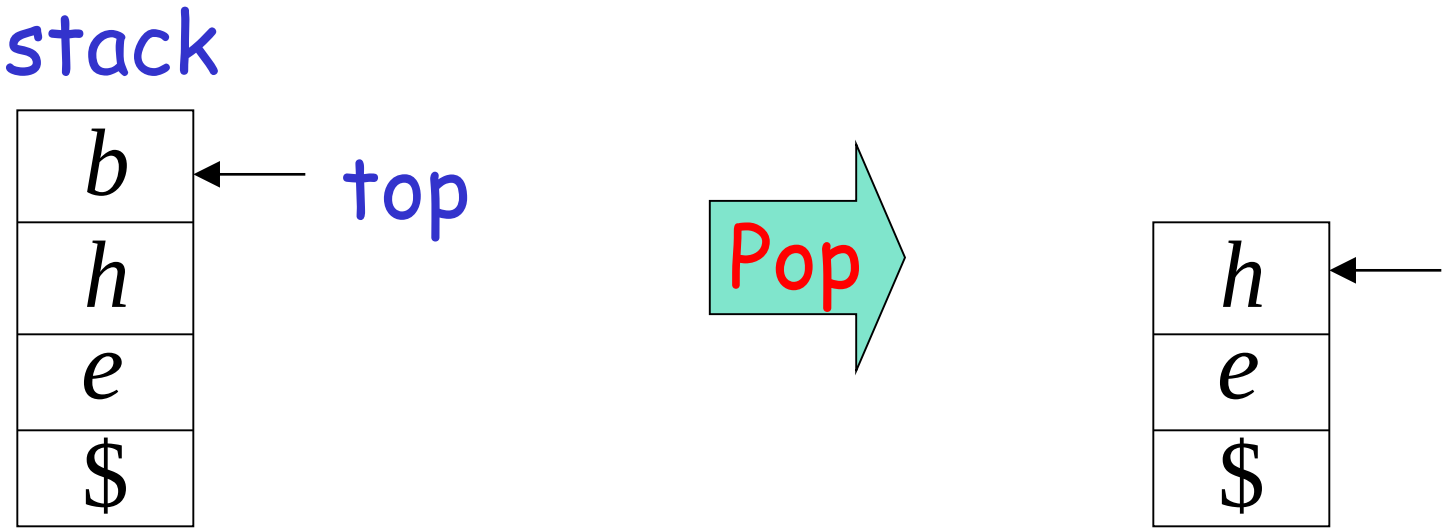

$a,\ \lambda \rightarrow c$
q_1
q_2

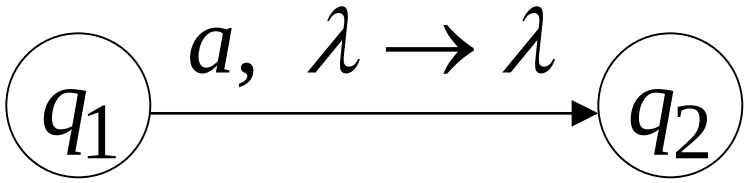

input
…
a
…
…
a
…


stack
b
top
h
e
$
Push
c
b
h
e
$

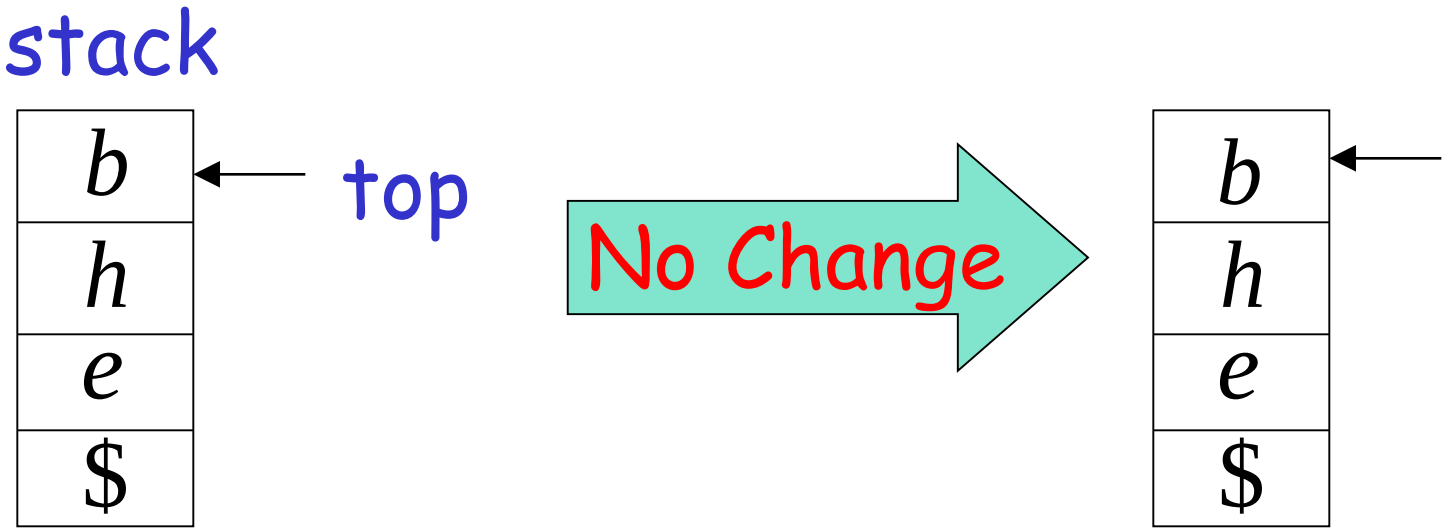

$a,\ b \to \lambda$
q_1
q_2


input
…
a
…
…
a
…

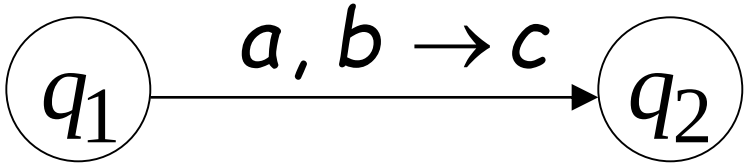

stack
b
top
h
e
$
Pop
h
e
$

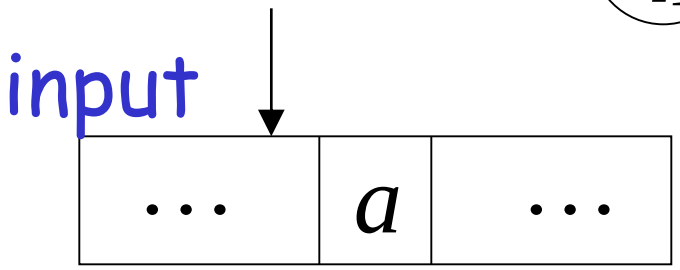

$a,\ \lambda \to \lambda$
q_1
q_2


input
…
a
…
…
a
…

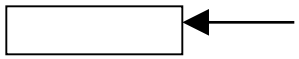

stack
b
h
e
$
top
No Change
b
h
e
$

Pop from Empty Stack





stack



Pop

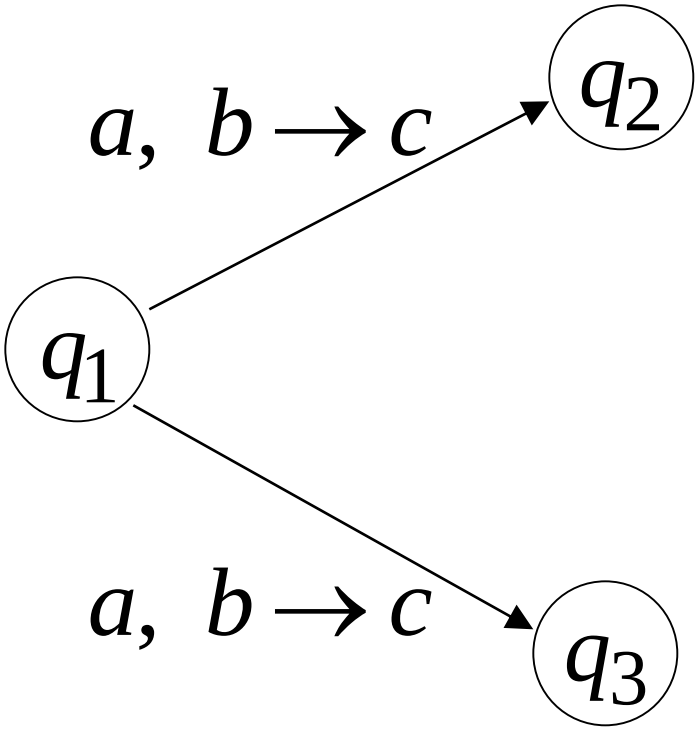
Automaton halts!

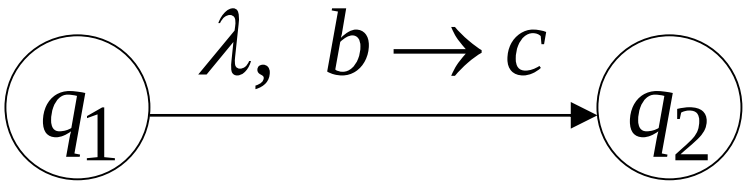
If the automaton attempts to pop from empty stack then it halts and rejects input

Non-Determinism

PDAs are non-deterministic

Allowed non-deterministic transitions

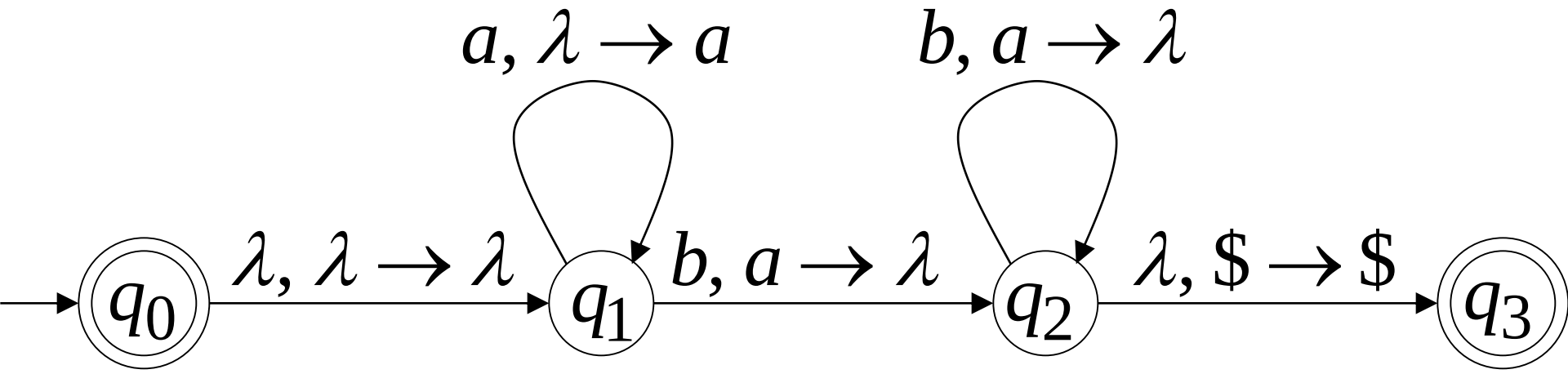




λ – transition

Example PDA

PDA M: $L(M) = \{a^n b^n : n \geq 0\}$



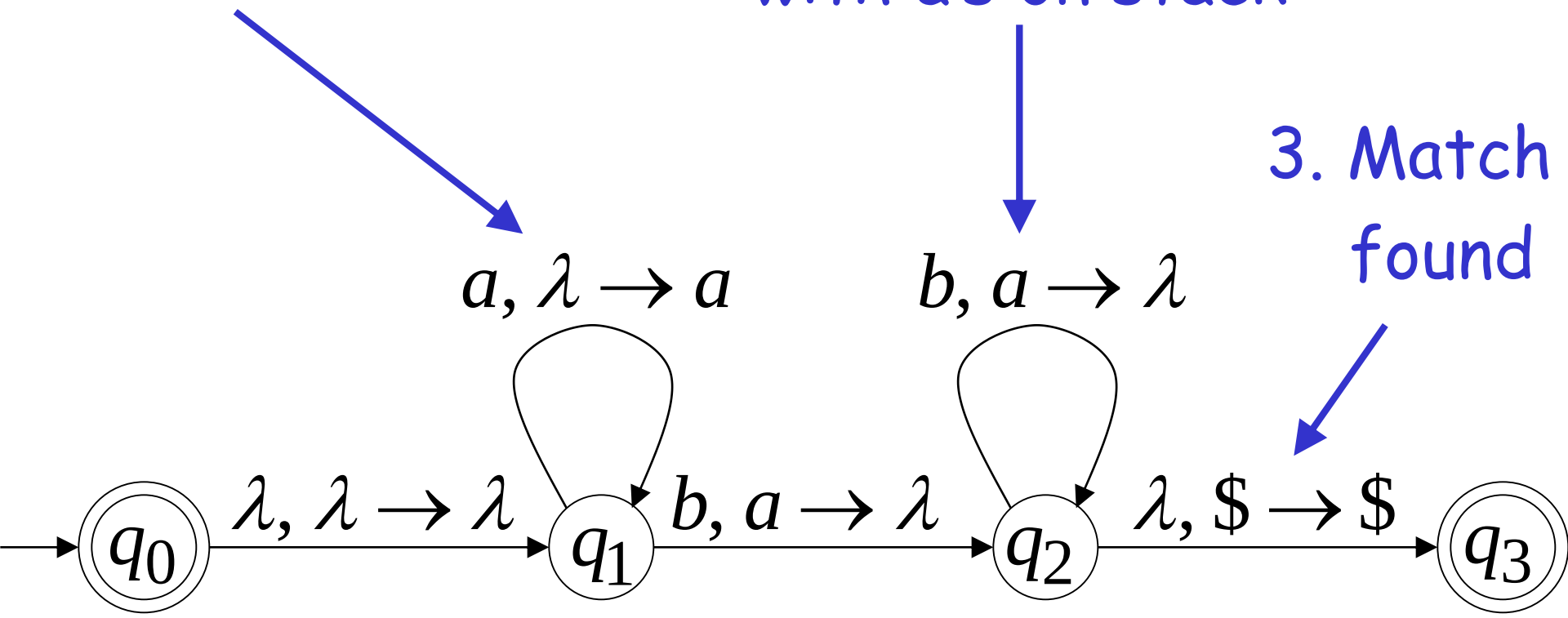
$$L(M) = \{a^n b^n : n \geq 0\}$$

Basic Idea:

1. Push the a's on the stack

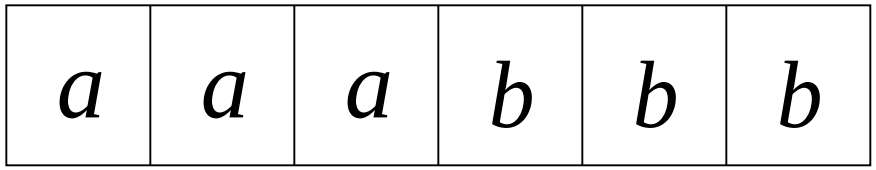
2. Match the b's on input with a's on stack

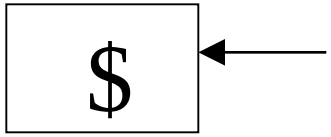
3. Match found



Execution Example: Time 0

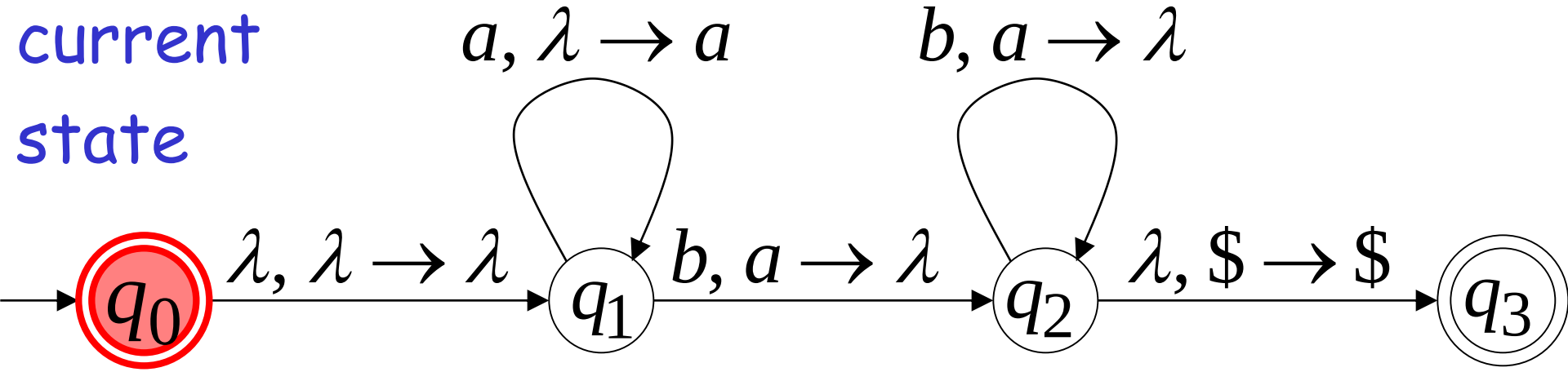
Input





Stack

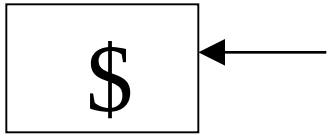
current state



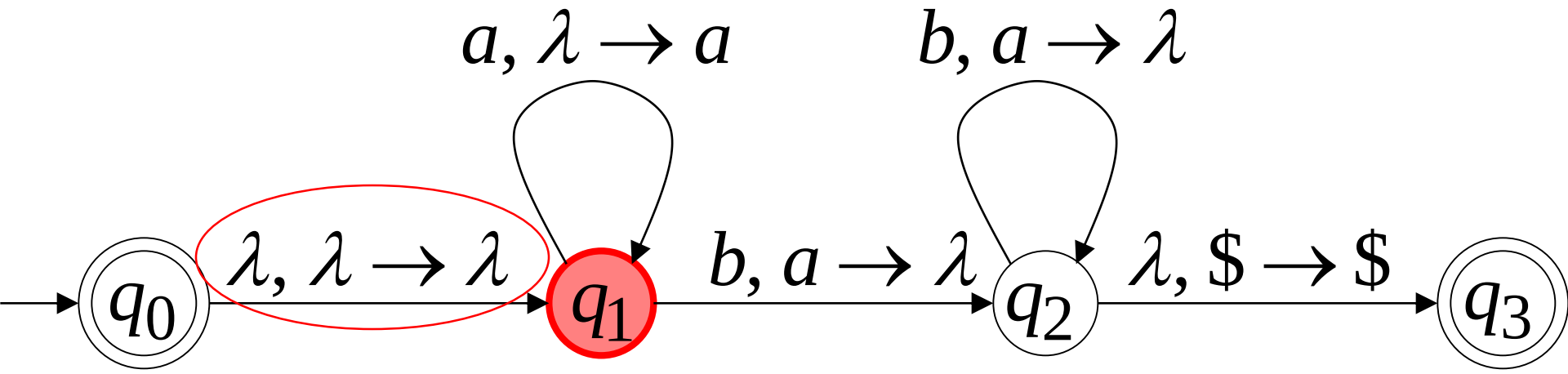
Time 1

Input

a	a	a	b	b	b

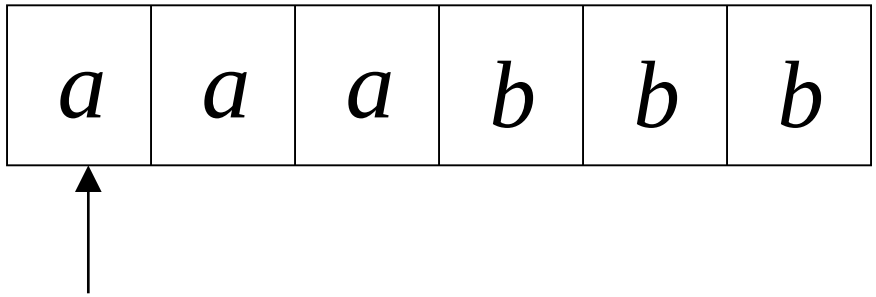


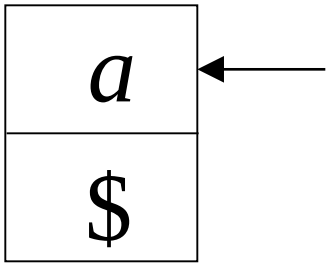
Stack



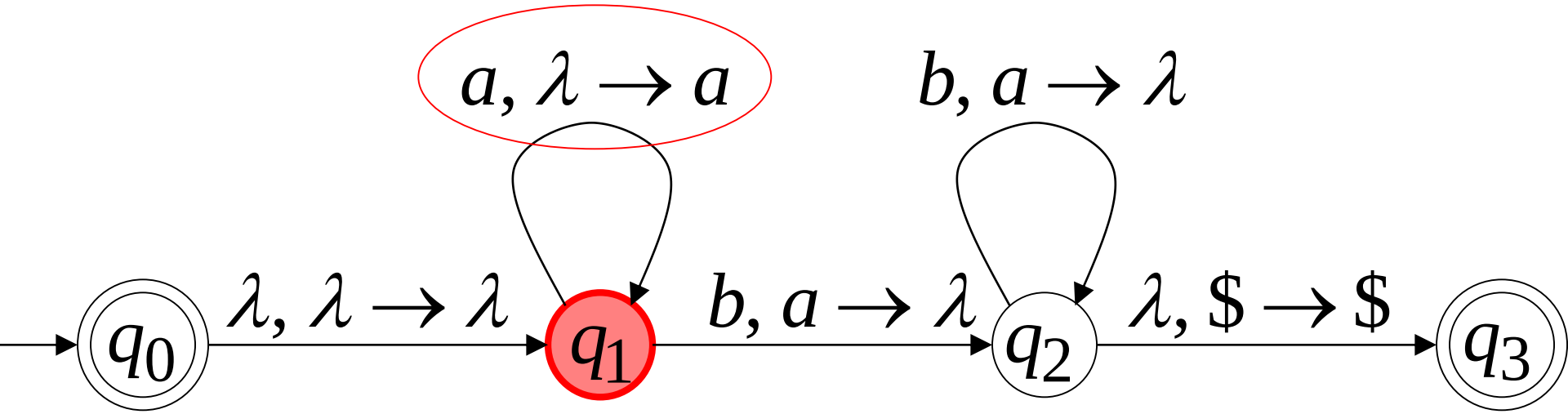
Time 2

Input



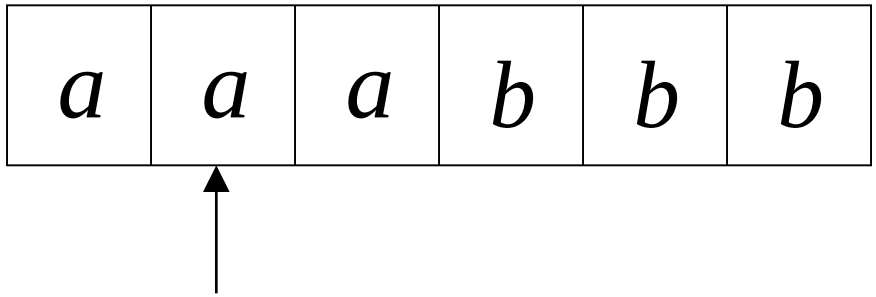


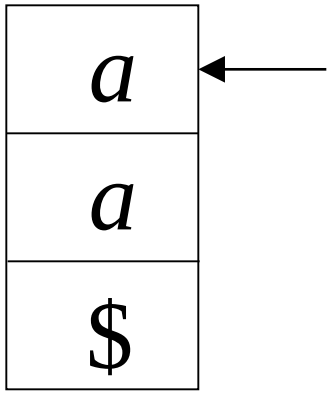
Stack



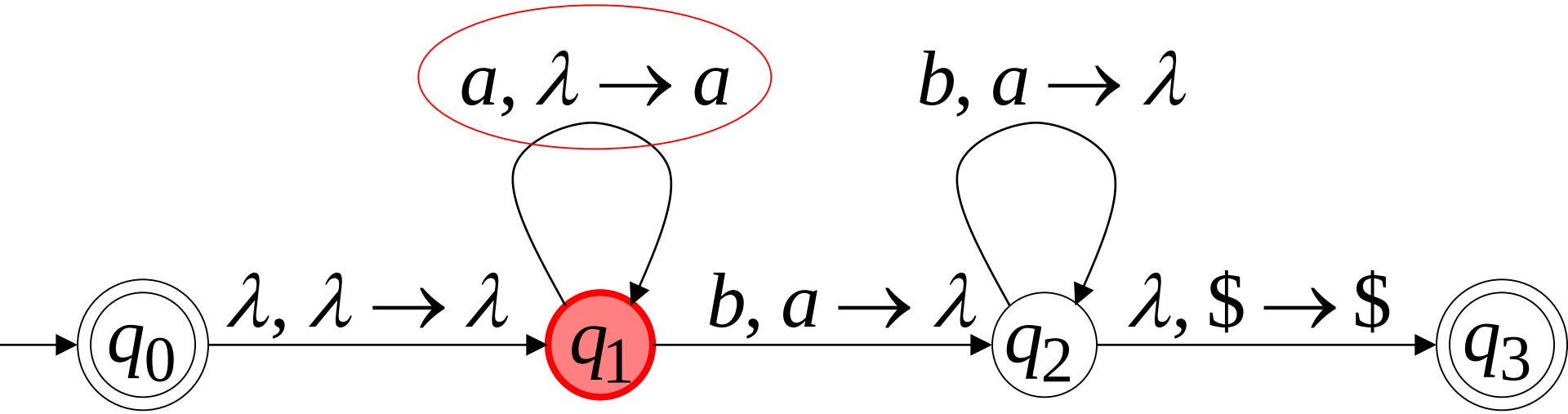
Time 3

Input



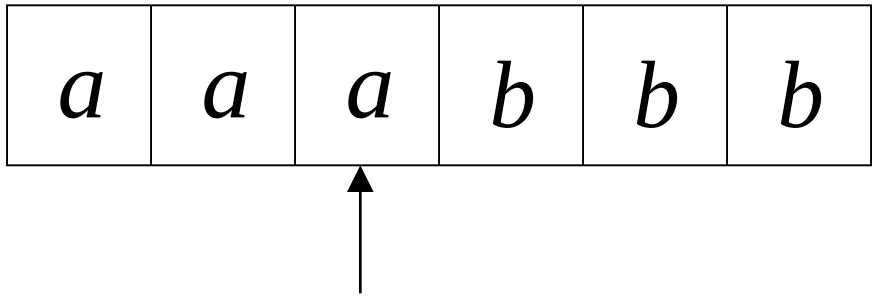


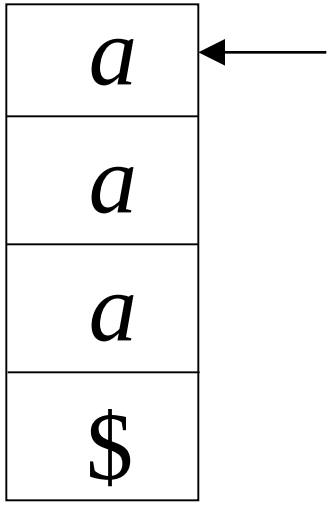
Stack



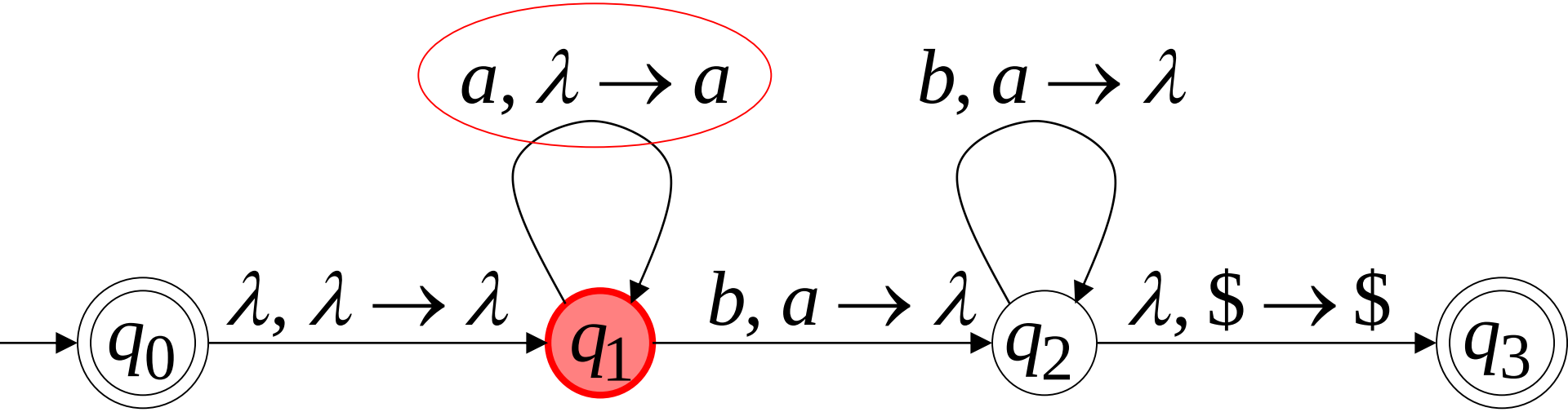
Time 4

Input



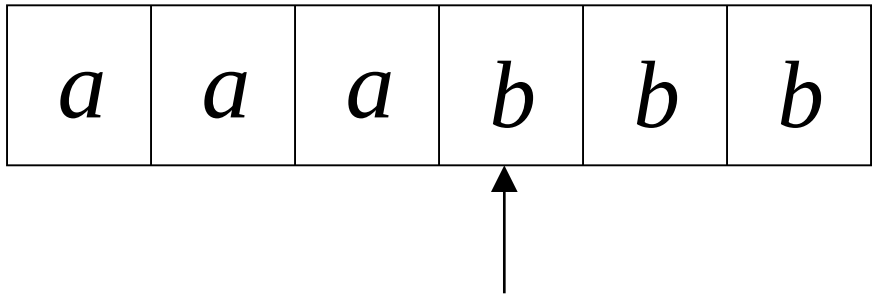


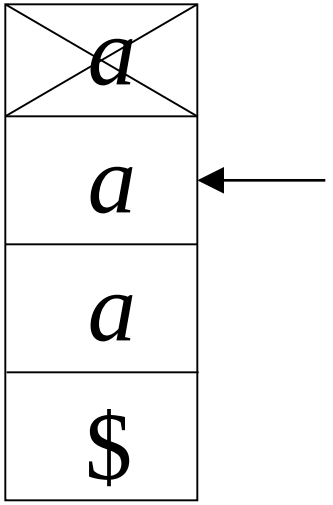
Stack



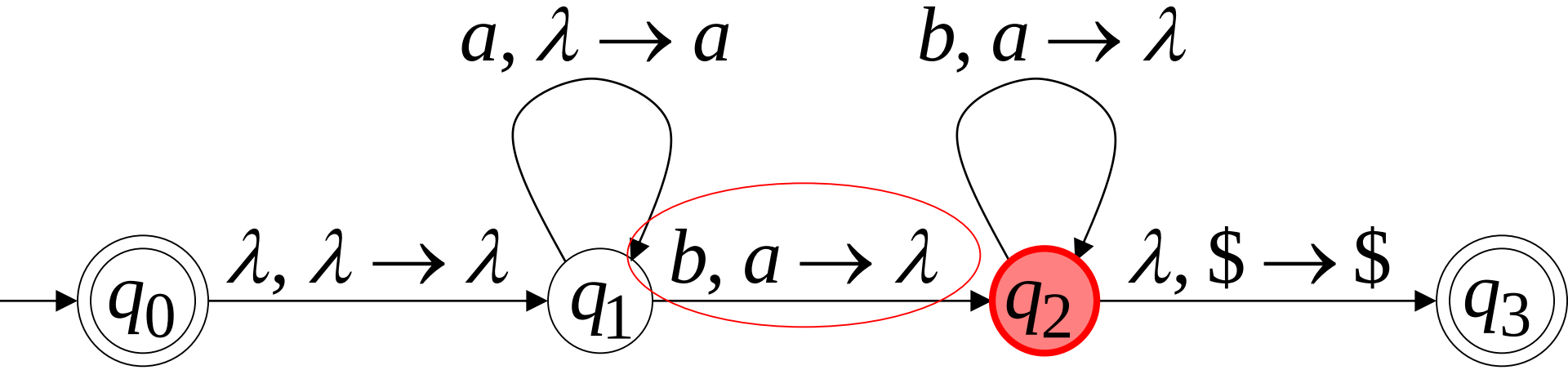
Time 5

Input



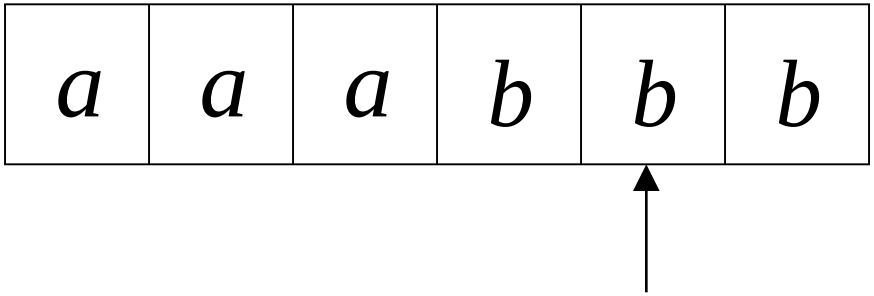


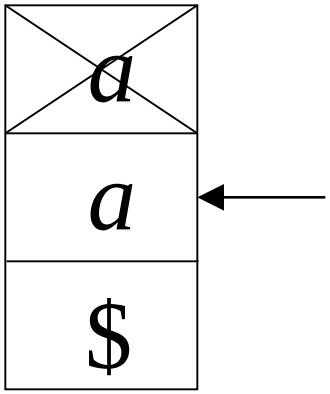
Stack



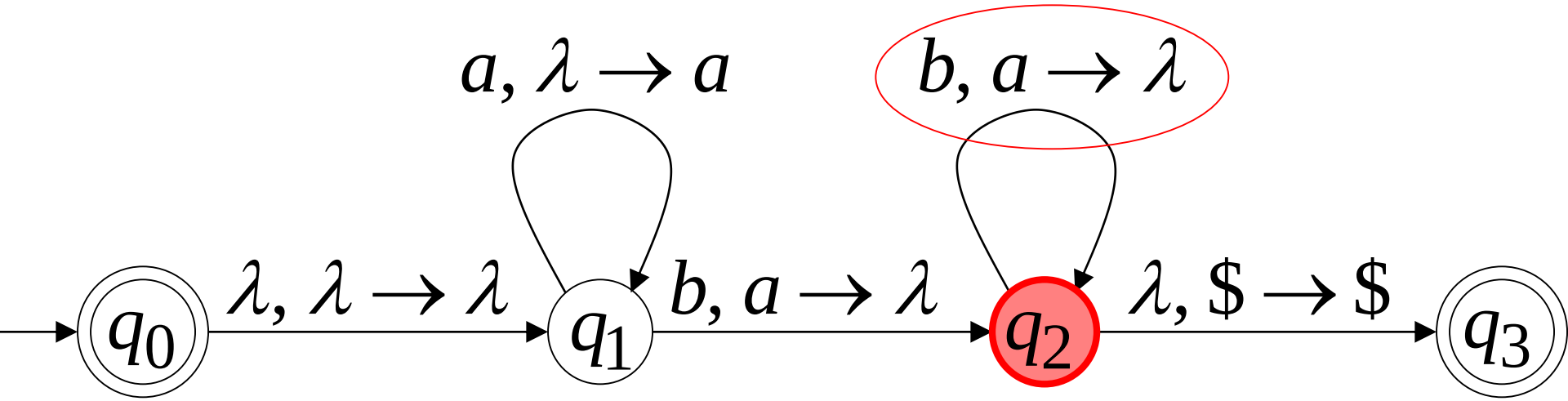
Time 6

Input



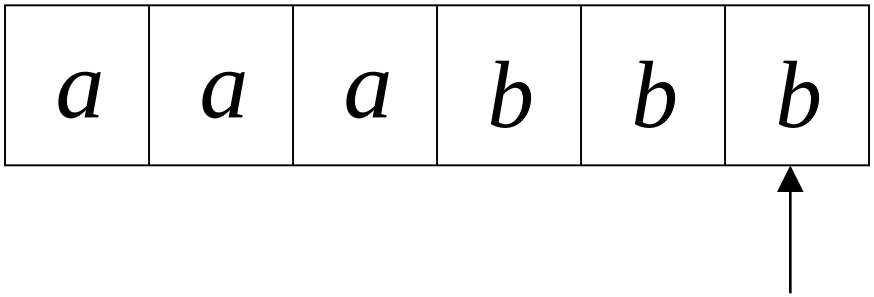


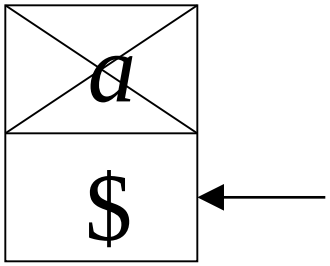
Stack



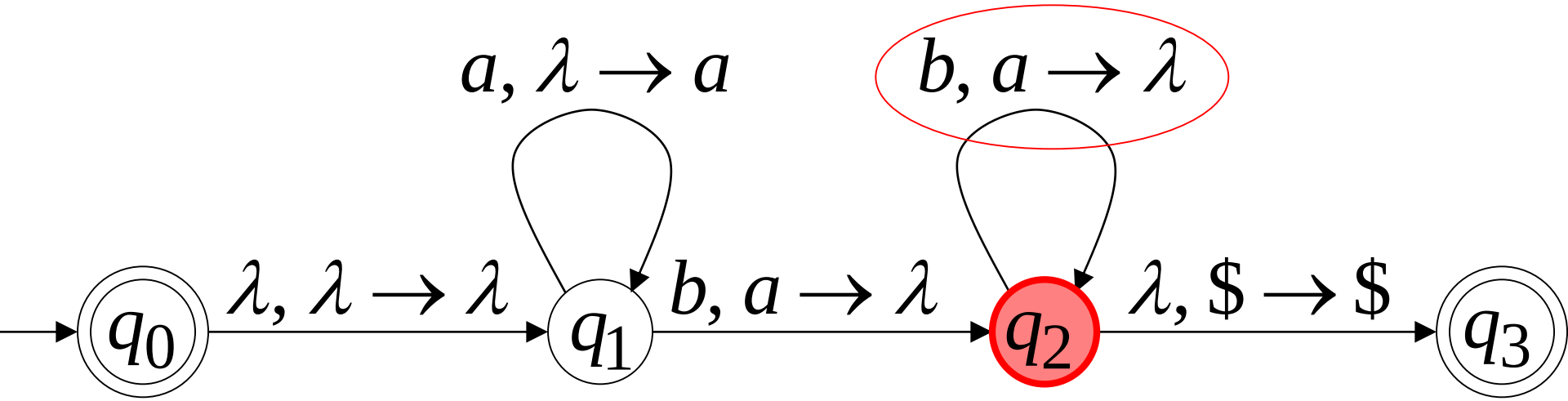
Time 7

Input



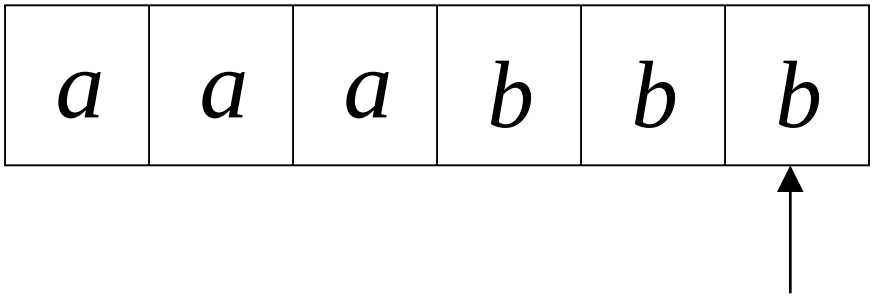


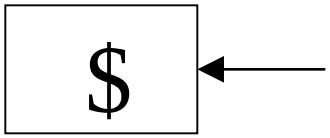
Stack



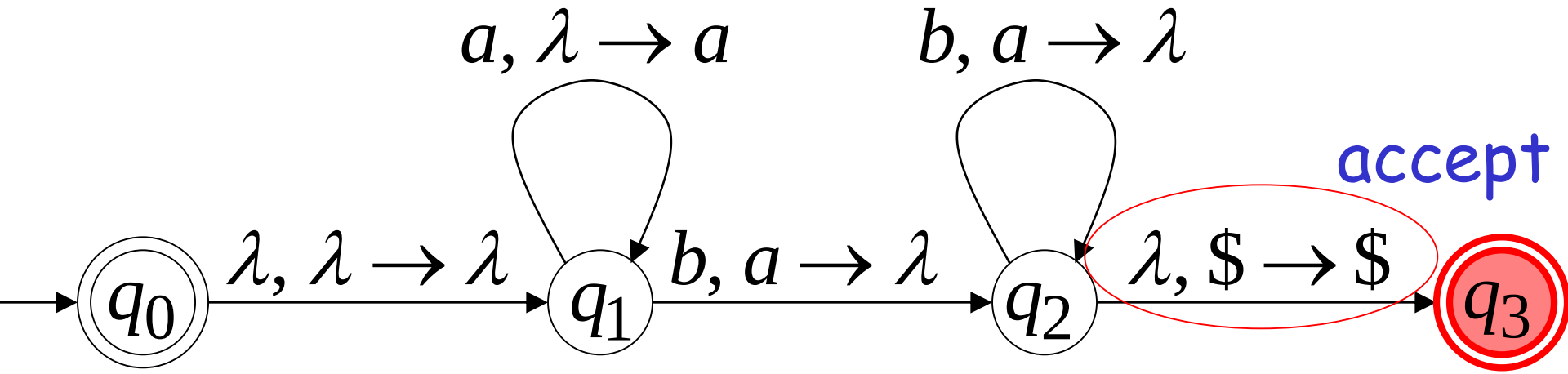
Time 8

Input





Stack



A string is accepted if there is a computation such that:

All the input is consumed

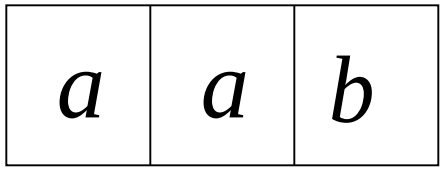
AND

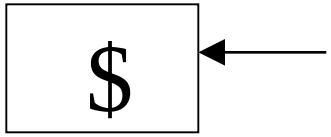
The last state is an accepting state

we do not care about the stack contents at the end of the accepting computation

Rejection Example: Time 0

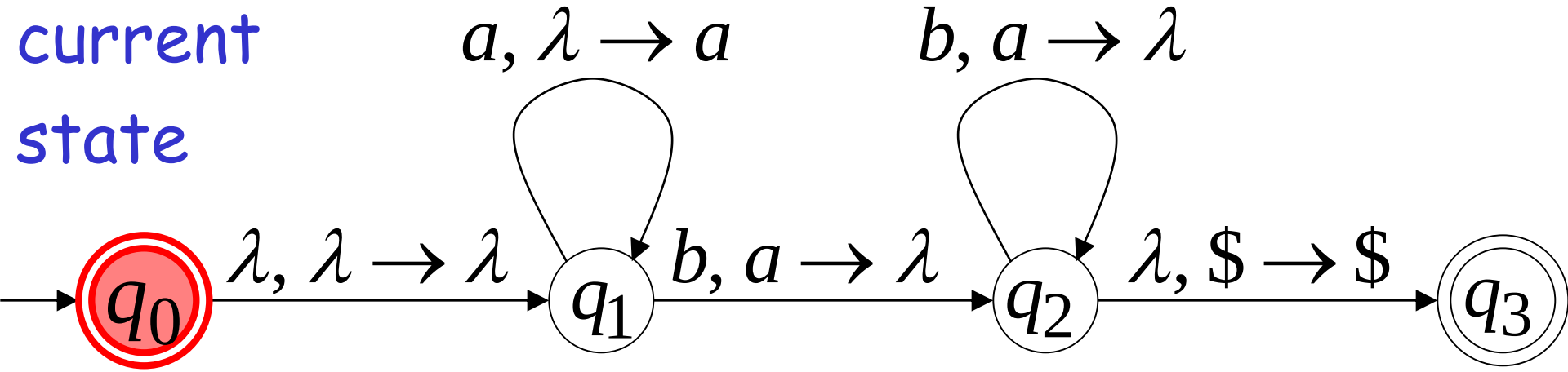
Input





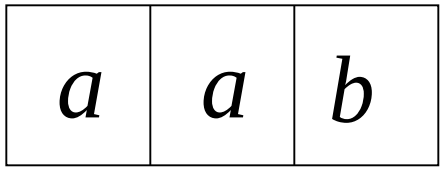
Stack

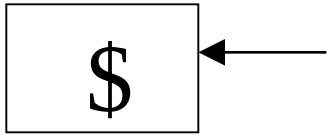
current state



Rejection Example: Time 1

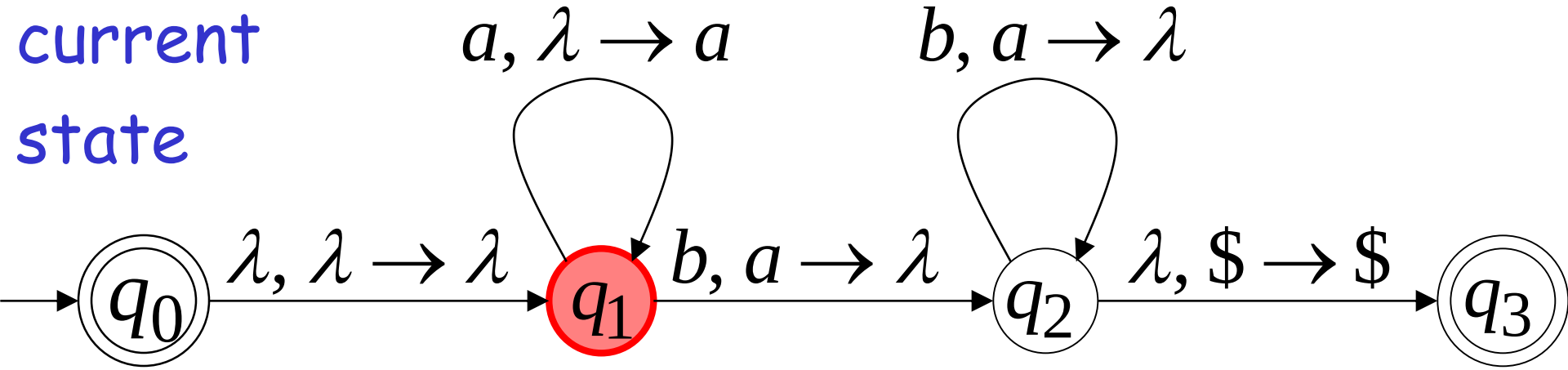
Input





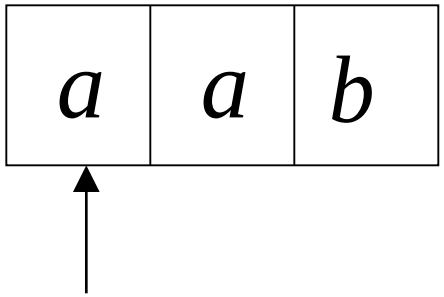
Stack

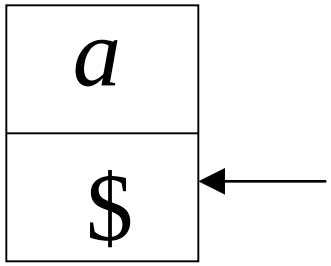
current state



Rejection Example: Time 2

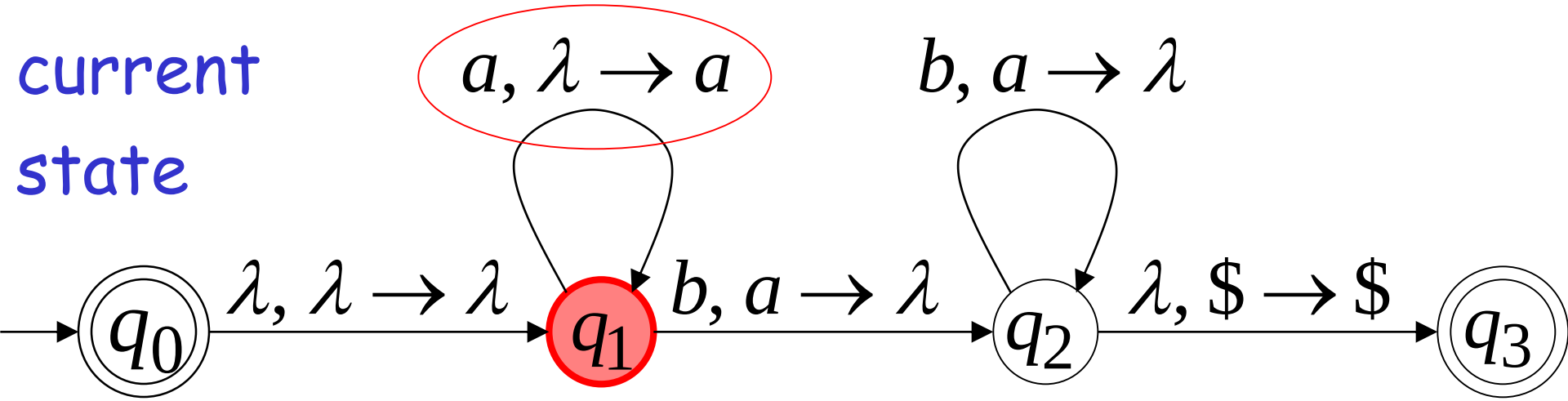
Input





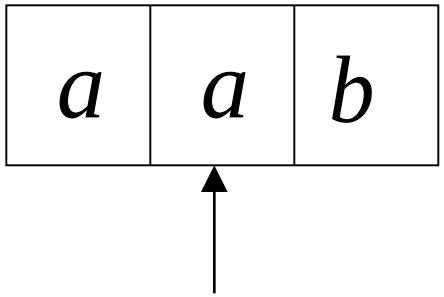
Stack

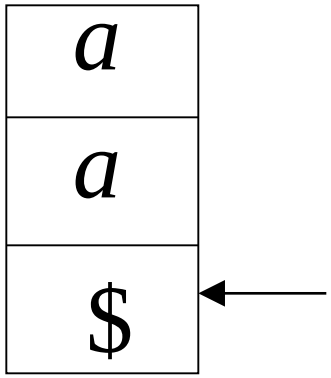
current state



Rejection Example: Time 3

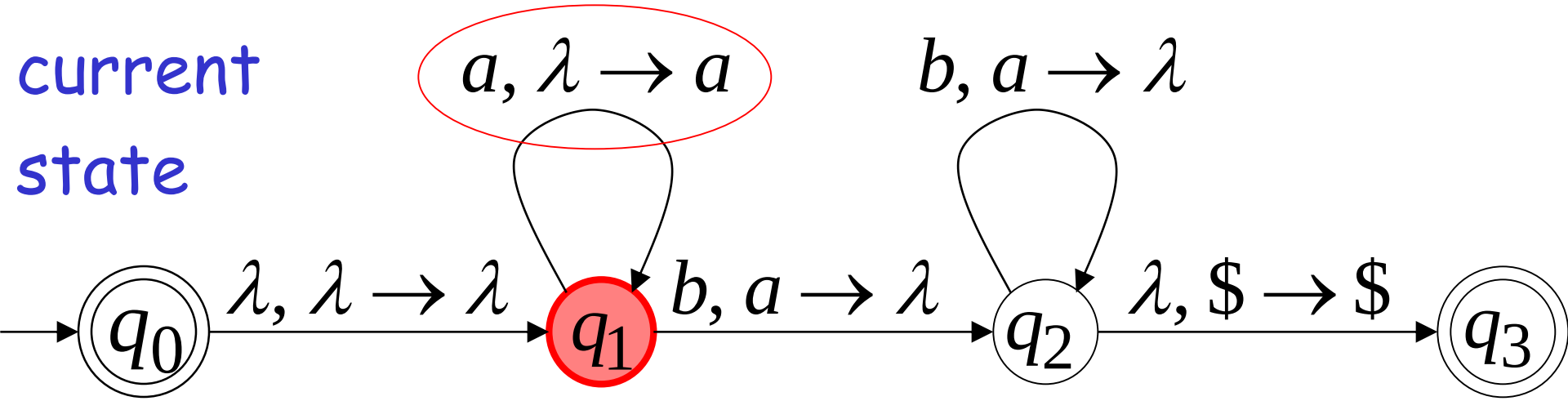
Input





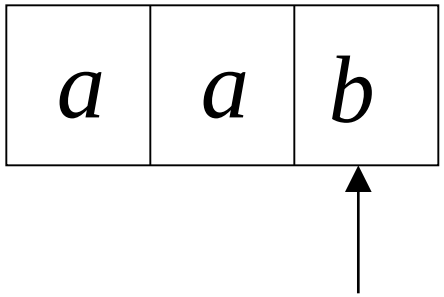
Stack

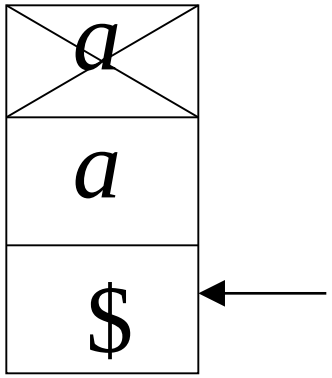
current state



Rejection Example: Time 4

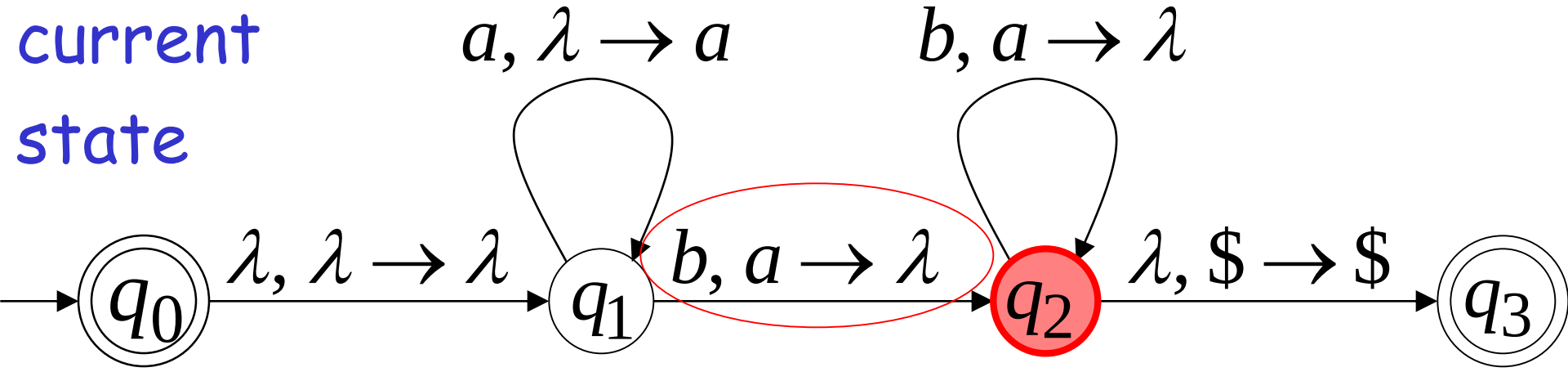
Input





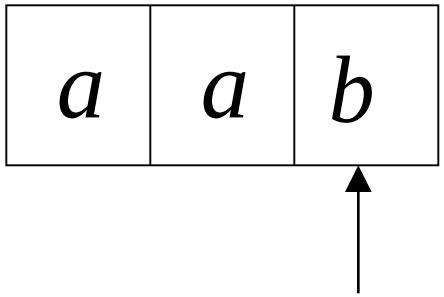
Stack

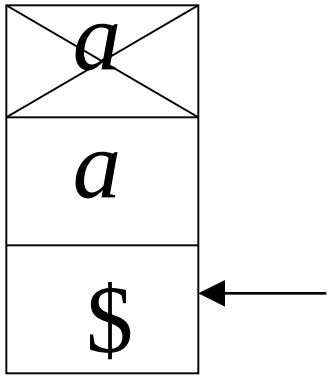
current state



Rejection Example: Time 4

Input

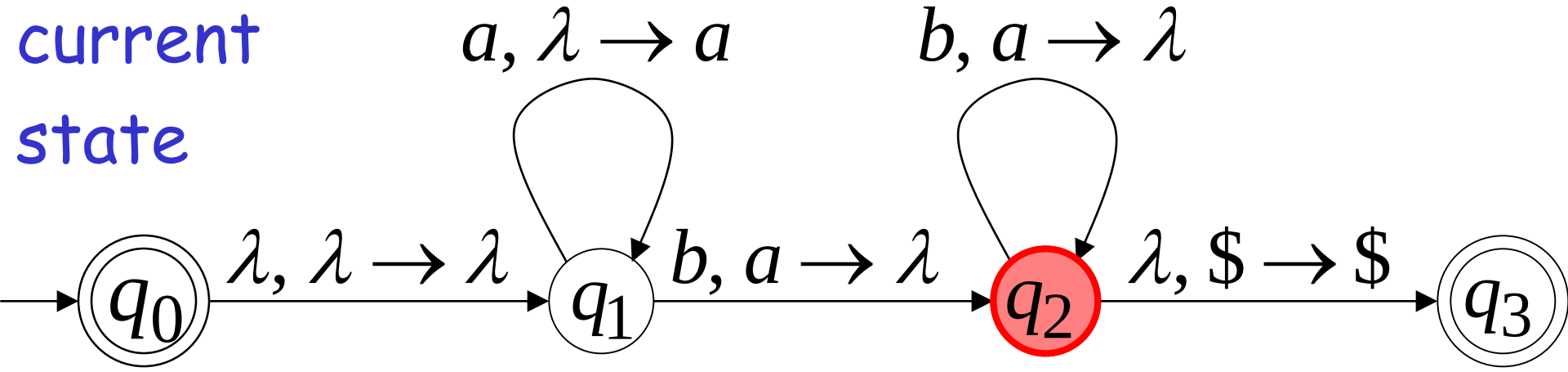




Stack

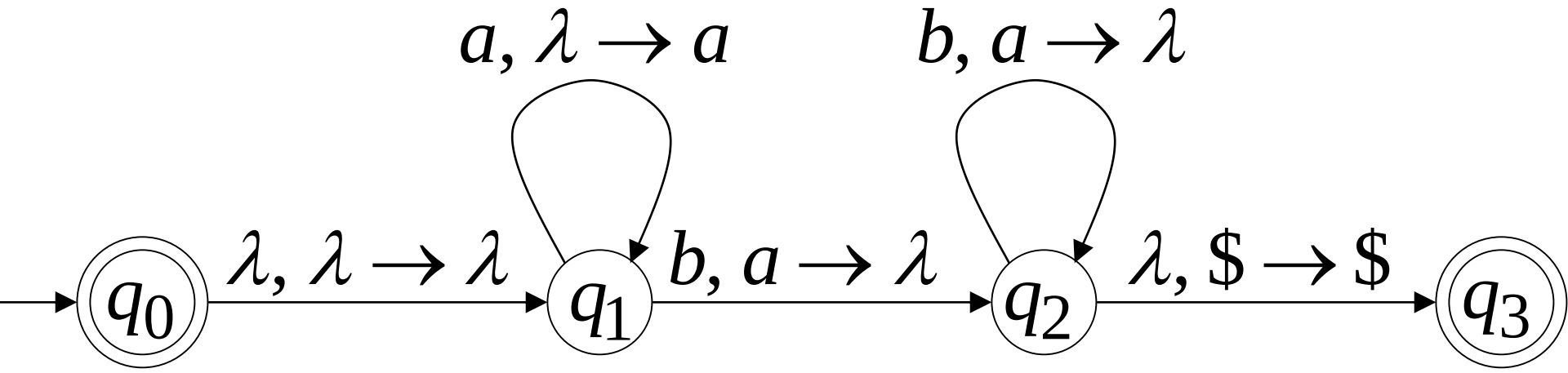
reject

current state



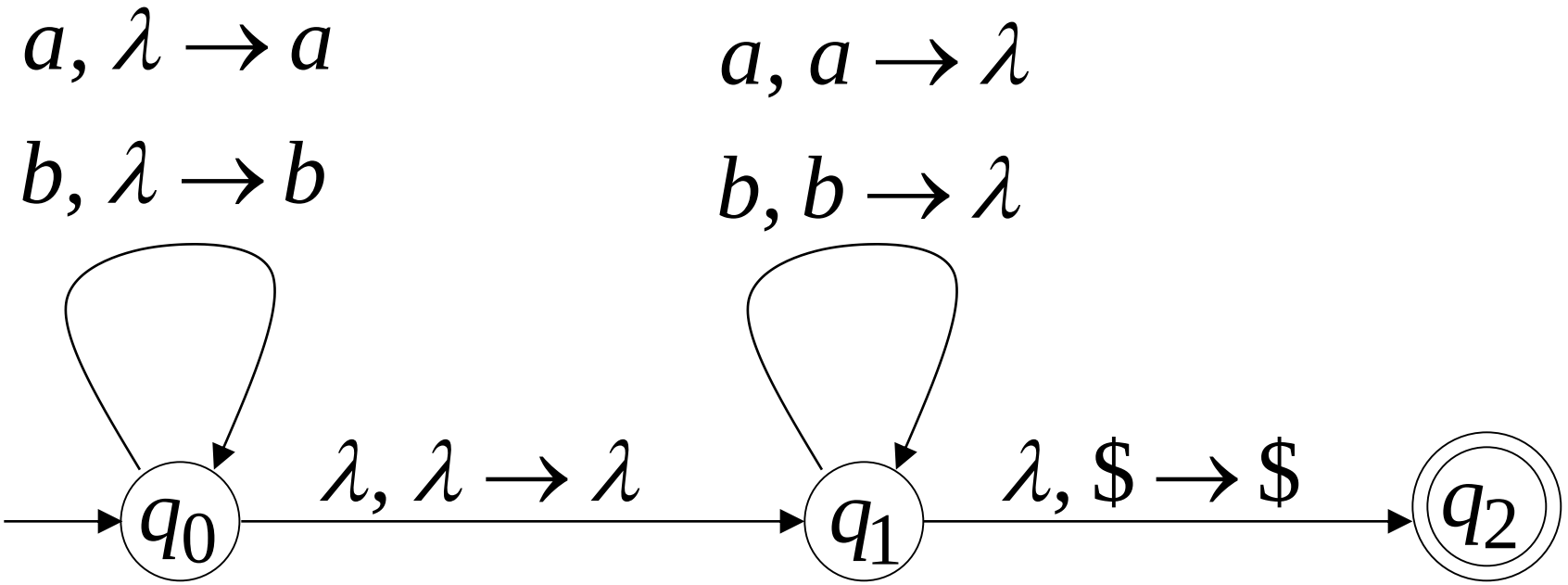
There is no accepting computation for aab

The string aab is rejected by the PDA



Another PDA example

PDA M: $L(M) = \{vv^R : v \in \{a,b\}^*\}$



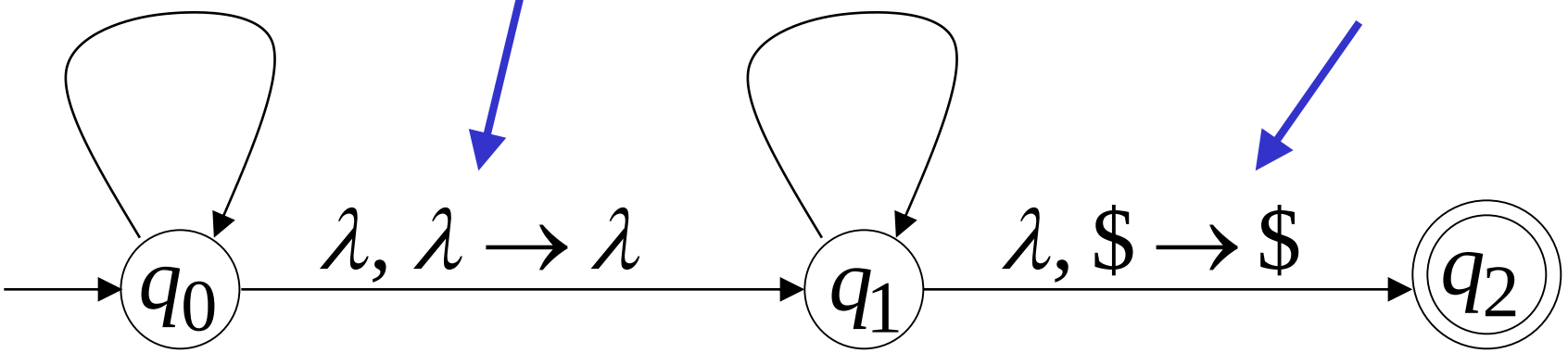
Basic Idea: $L(M) = \{vv^R : v \in \{a,b\}^*\}$

1. Push v on stack

2. Guess middle of input

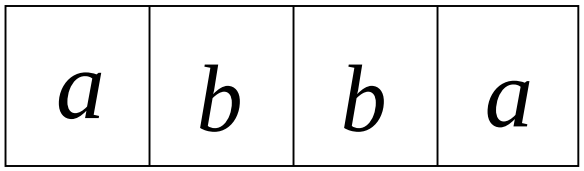
3. Match v^R on input with v on stack

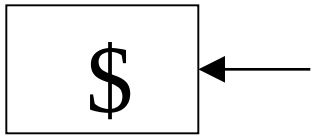
4. Match found



Execution Example: Time 0

Input





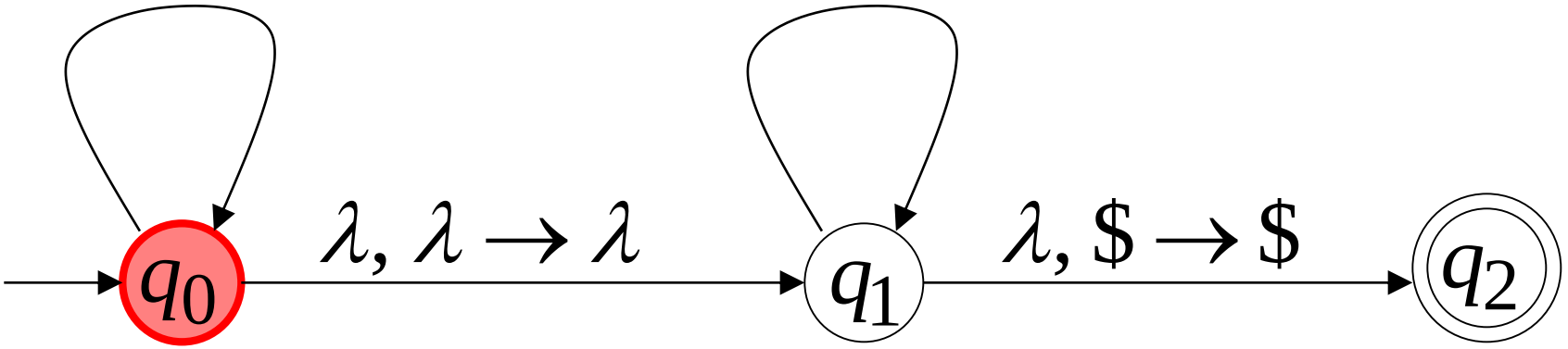
Stack

$a, \lambda \to a$

$b, \lambda \to b$

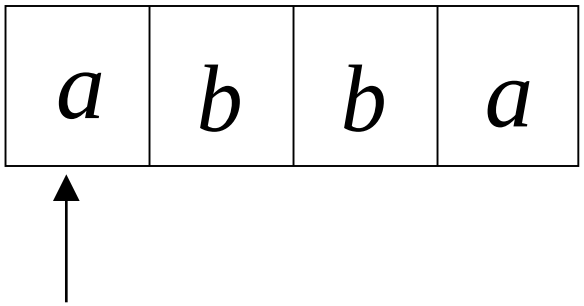
$a, a \to \lambda$

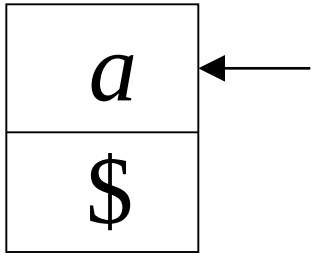
$b, b \to \lambda$



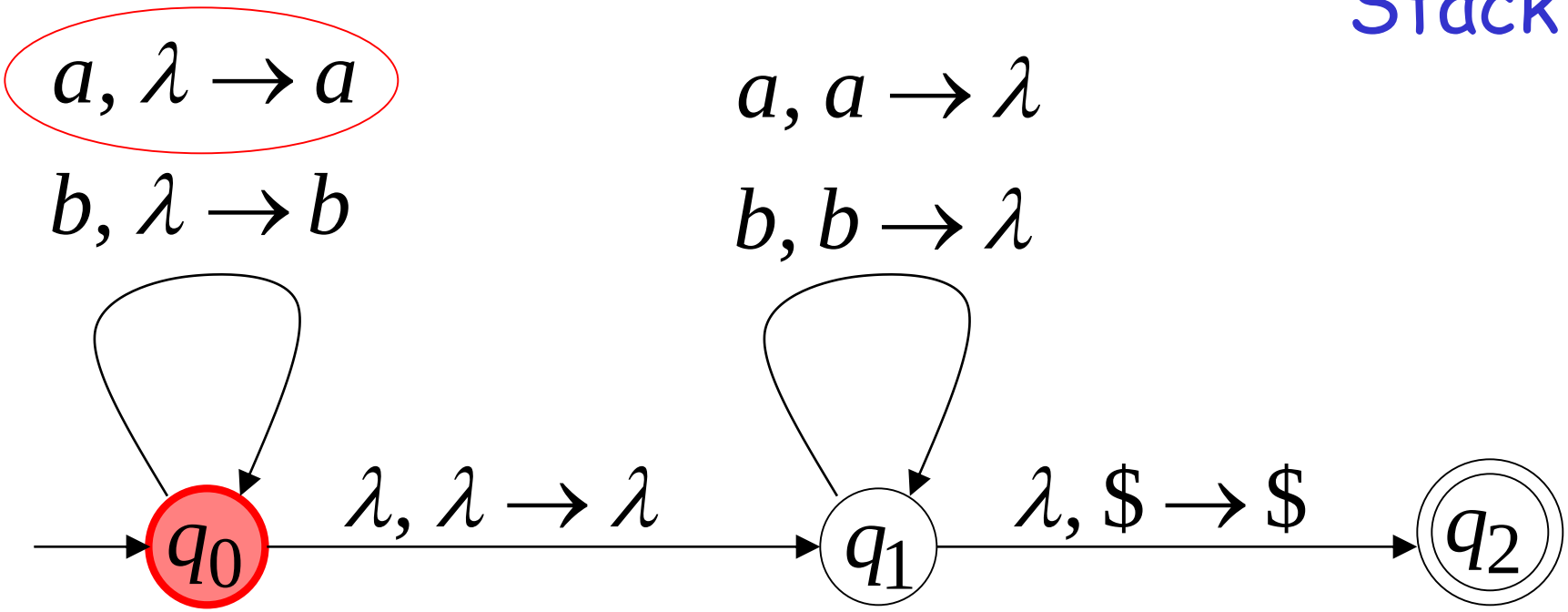
Time 1

Input



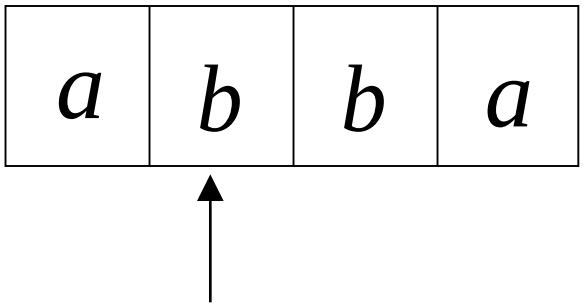


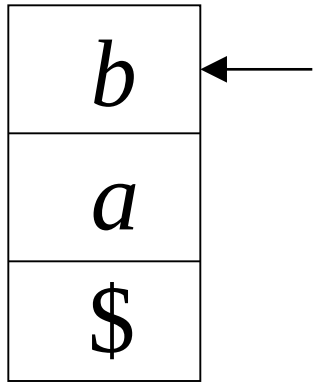
Stack



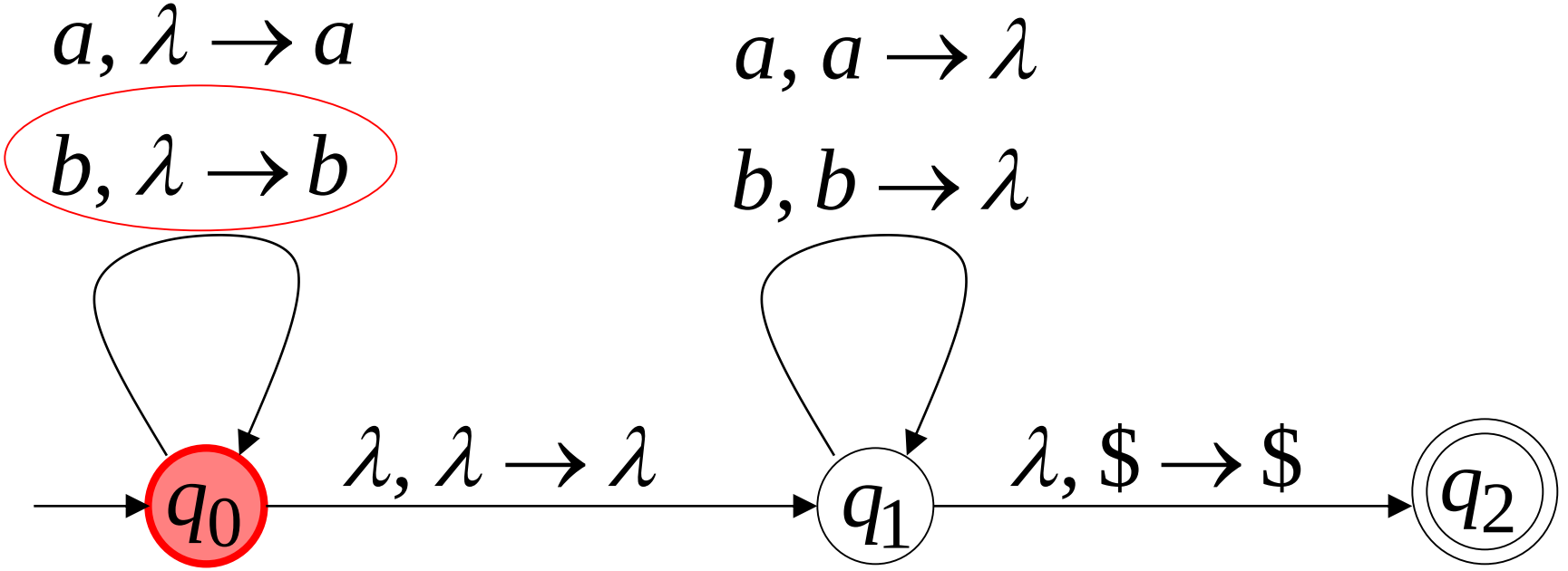
Time 2

Input

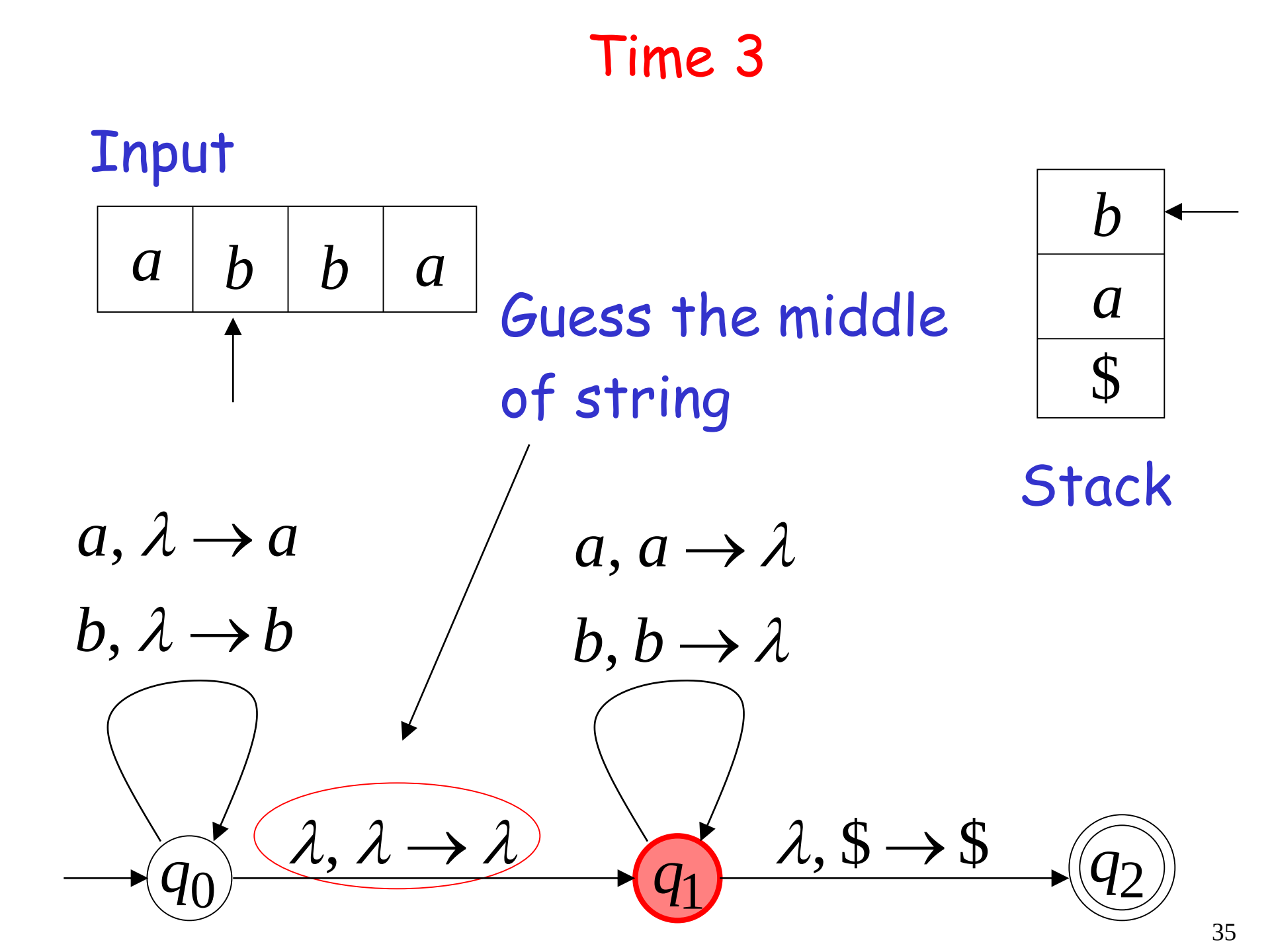




Stack

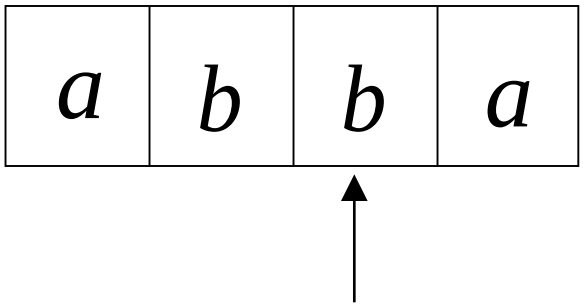


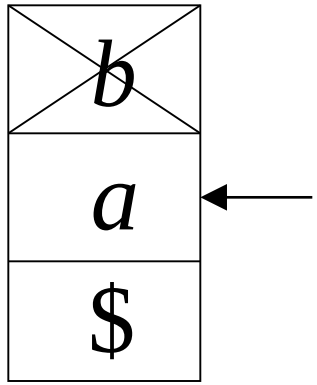
Time 3


Input
a b b a
Guess the middle of string
b
a
$
Stack
a, λ → a
b, λ → b
a, a → λ
b, b → λ
λ, λ → λ
λ, $ → $
q0
q1
q2

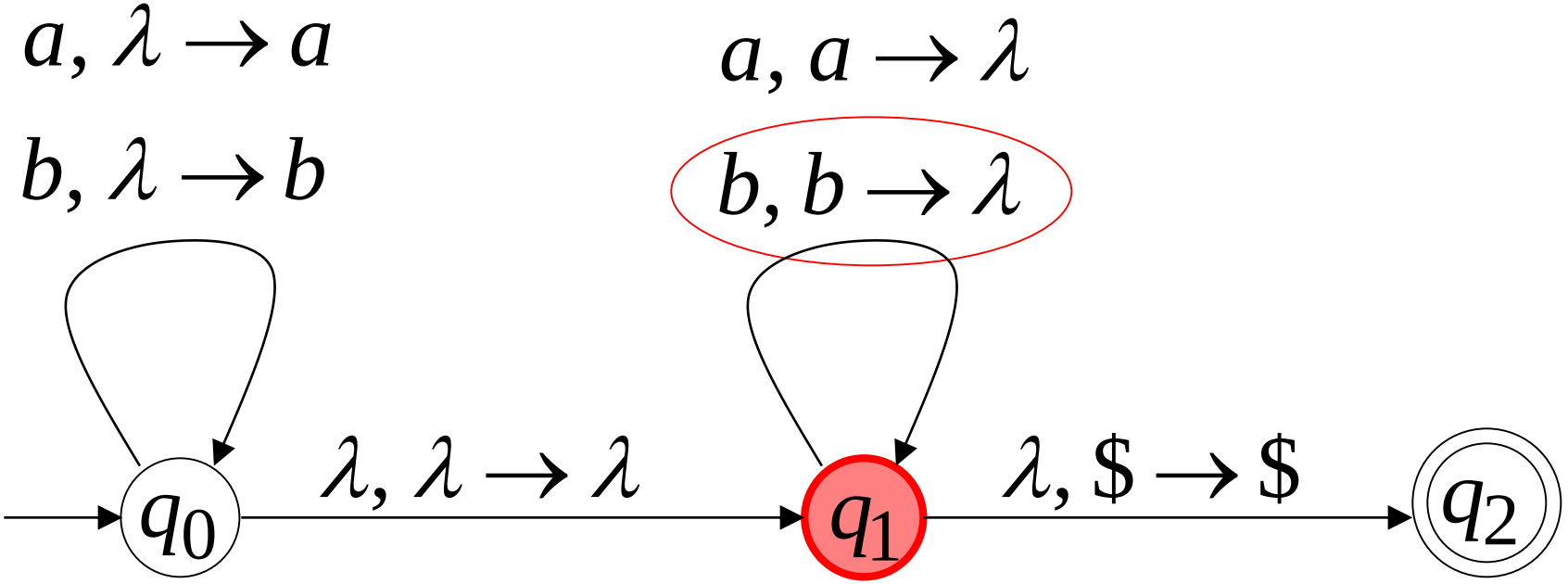
Time 4

Input



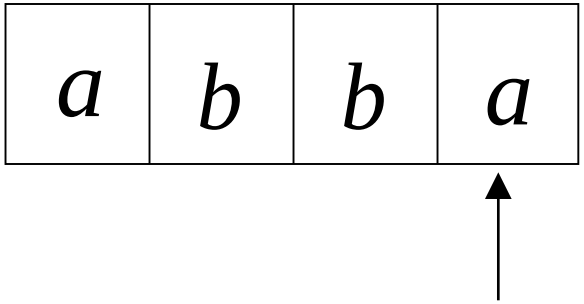


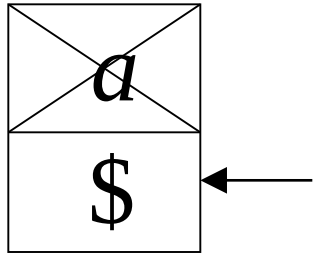
Stack



Time 5

Input





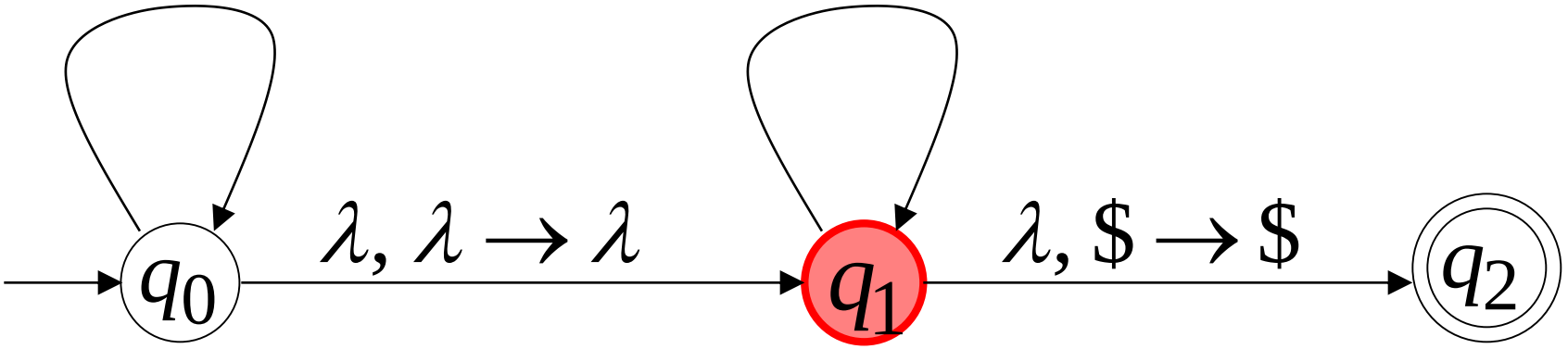
Stack

$a, \lambda \to a$

$b, \lambda \to b$

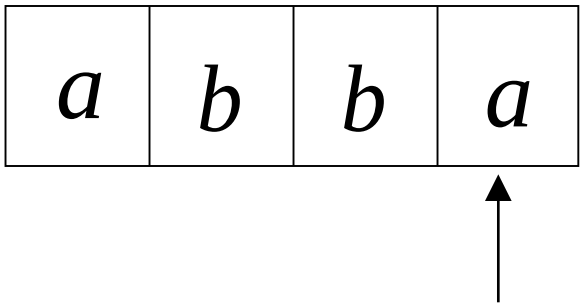
$a, a \to \lambda$

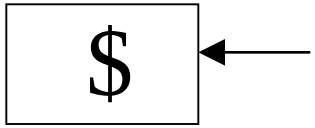
$b, b \to \lambda$



Time 6

Input





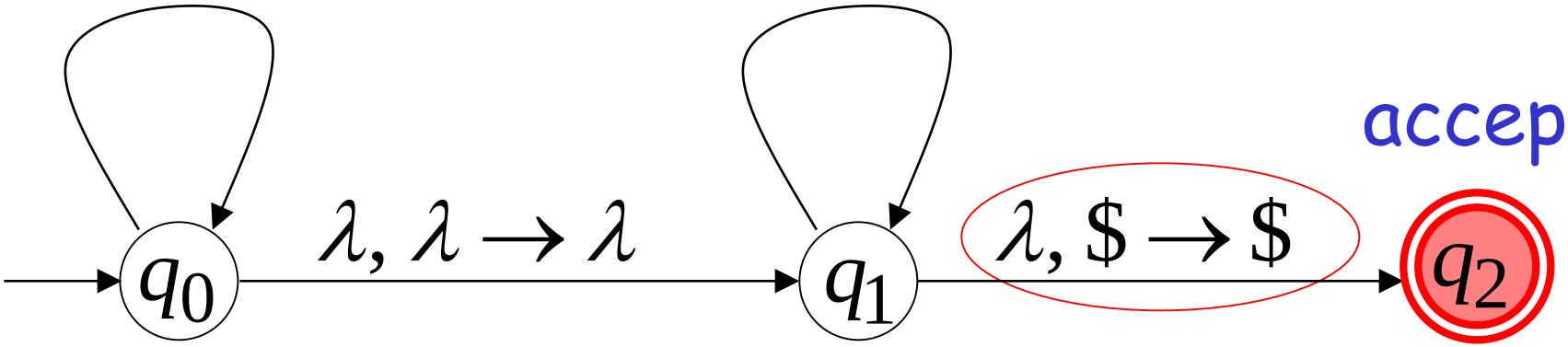
Stack

$a, \lambda \to a$

$b, \lambda \to b$

$a, a \to \lambda$

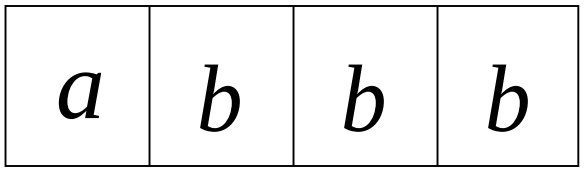
$b, b \to \lambda$

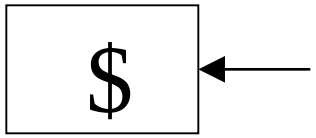


accept

Rejection Example: Time 0

Input





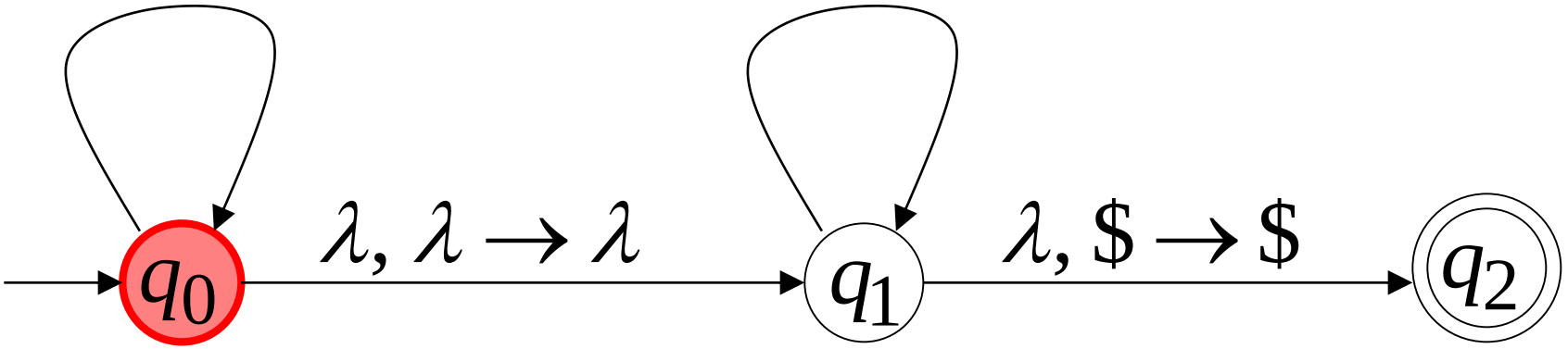
Stack

$a, \lambda \to a$

$b, \lambda \to b$

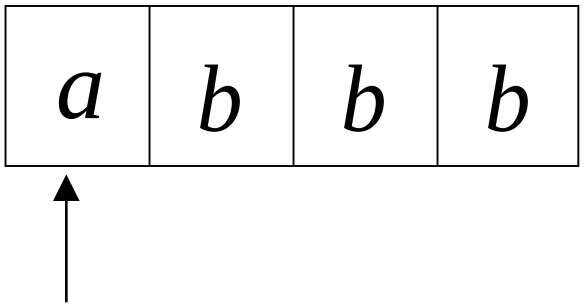
$a, a \to \lambda$

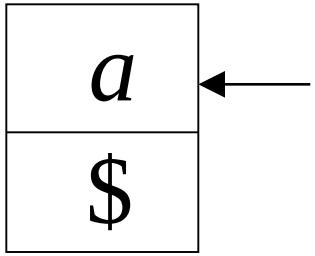
$b, b \to \lambda$



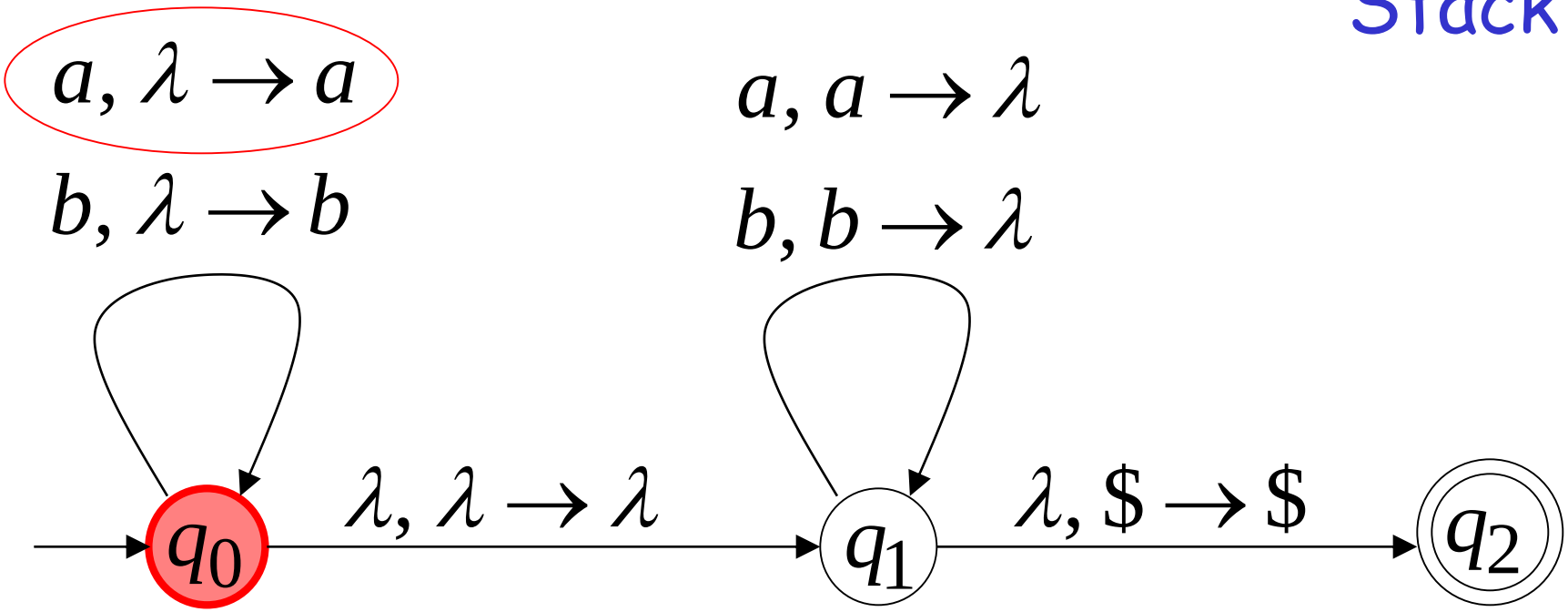
Time 1

Input



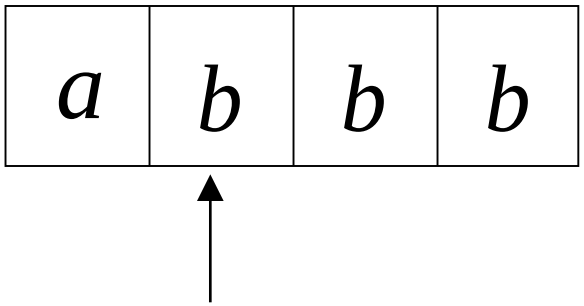


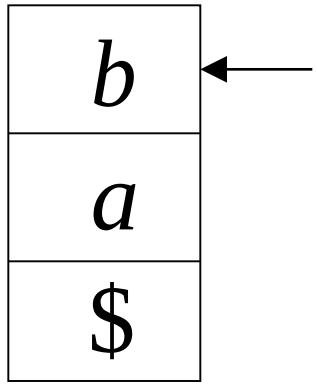
Stack



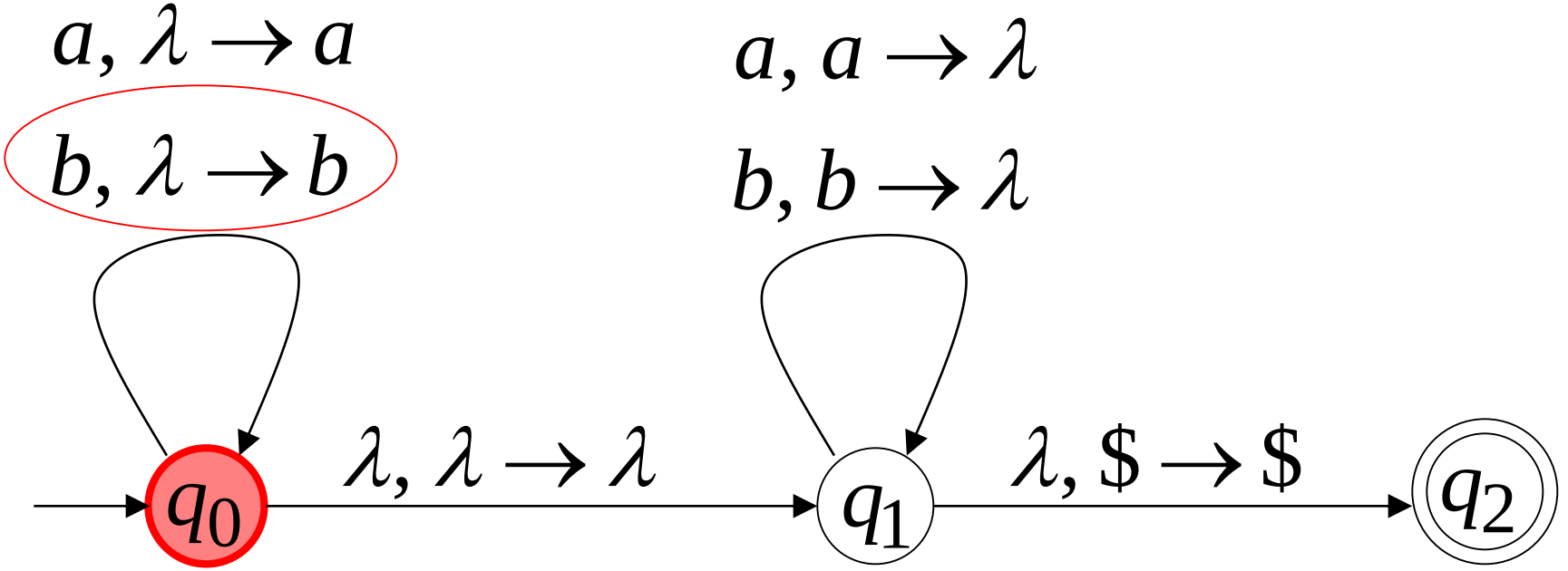
Time 2

Input



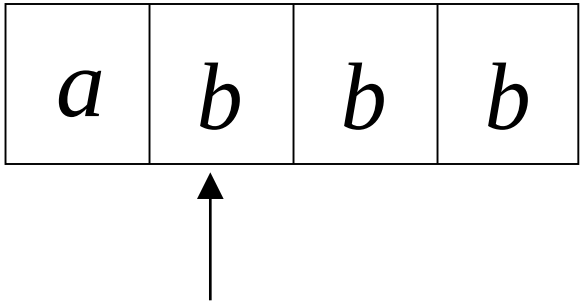


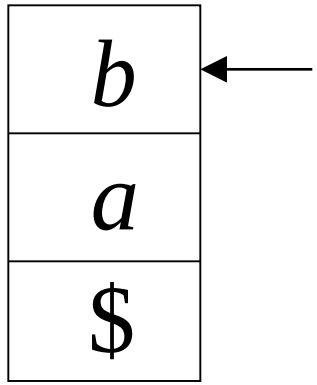
Stack



Time 3

Input





Stack

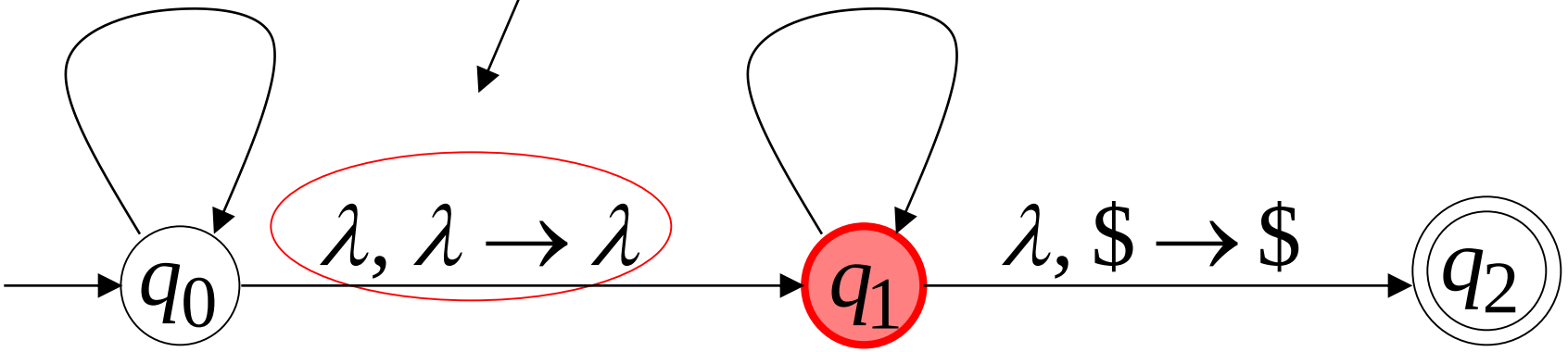
Guess the middle of string

$a, \lambda \to a$

$b, \lambda \to b$

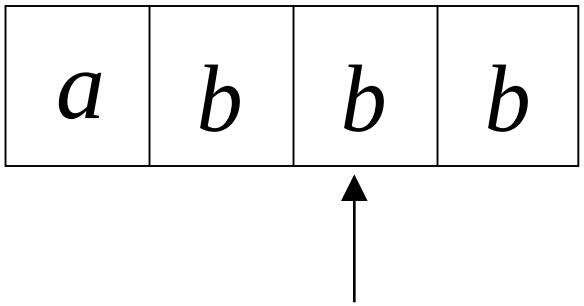
$a, a \to \lambda$

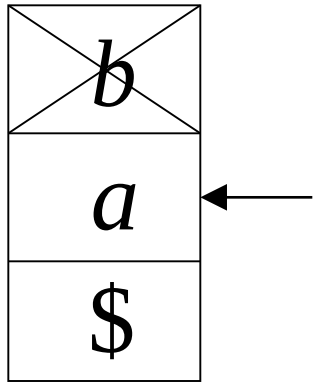
$b, b \to \lambda$



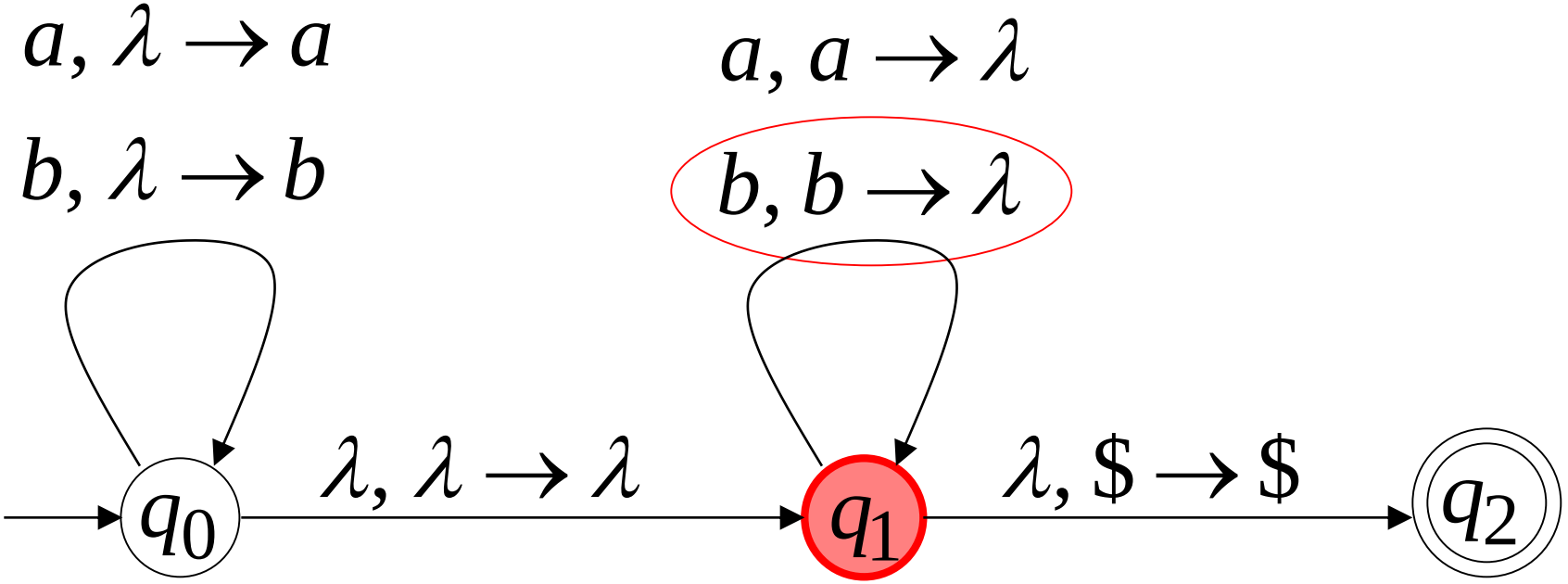
Time 4

Input





Stack



Time 5

There is no possible transition.

Input

a	b	b	b

Input is not consumed

a
$

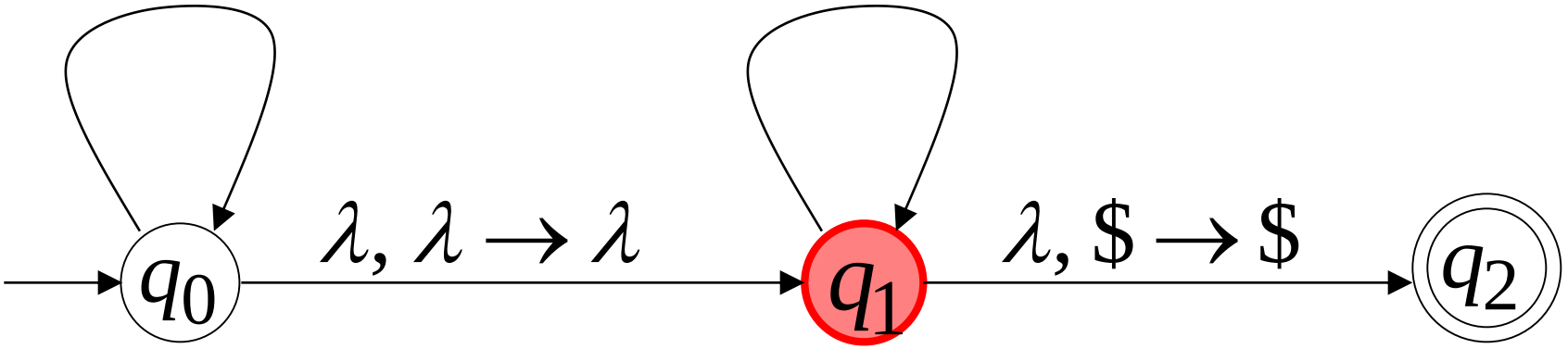
Stack

$a, \lambda \to a$

$b, \lambda \to b$

$a, a \to \lambda$

$b, b \to \lambda$

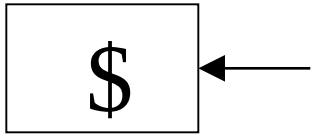


Another computation on same string:

Input

Time 0

a	b	b	b



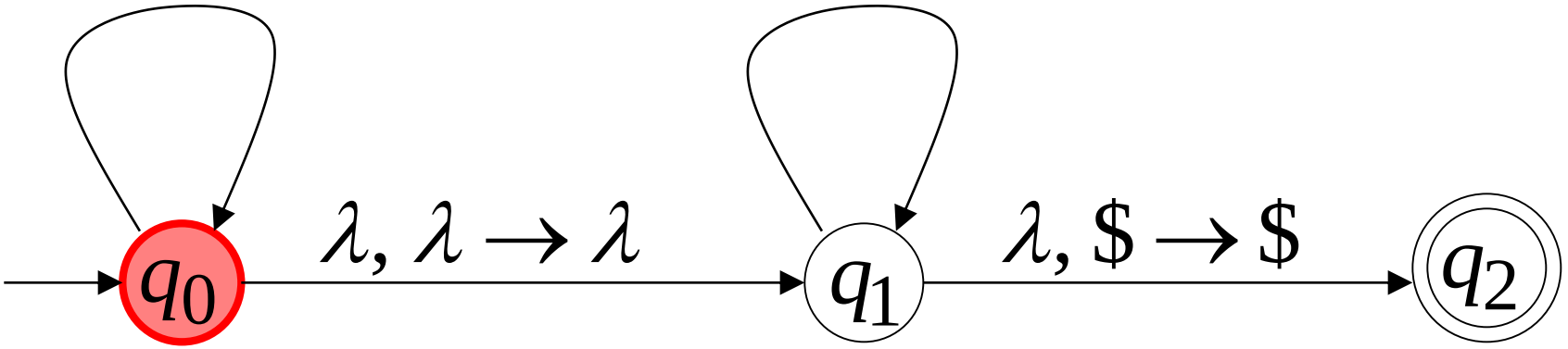
Stack

$a, \lambda \to a$

$b, \lambda \to b$

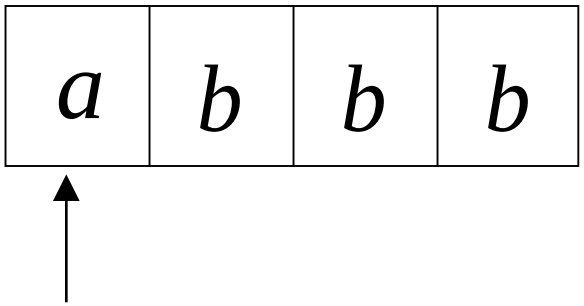
$a, a \to \lambda$

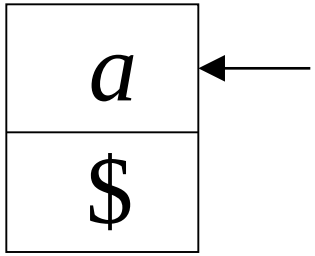
$b, b \to \lambda$



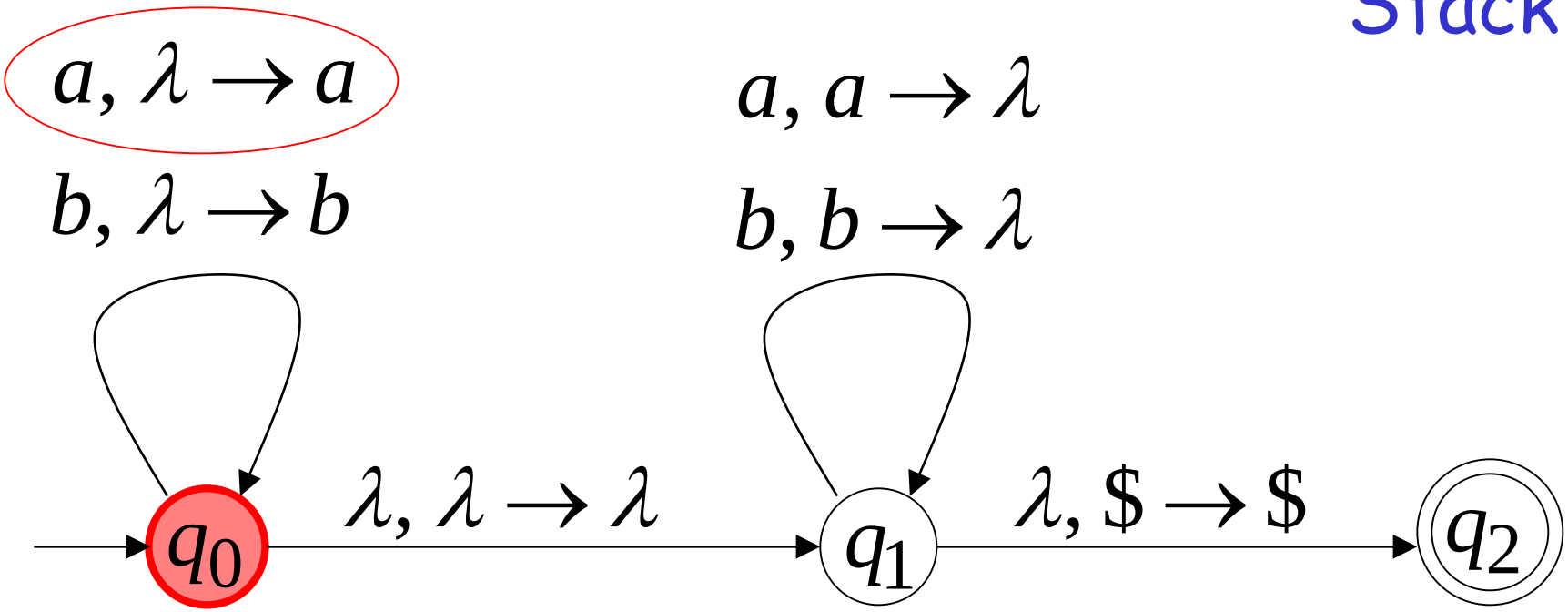
Time 1

Input



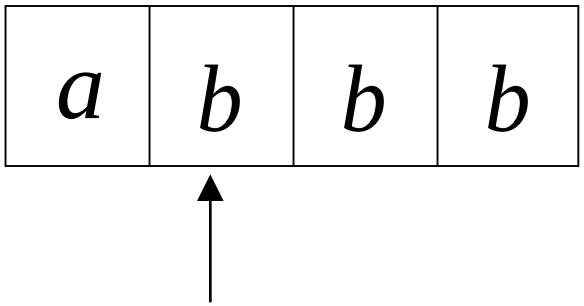


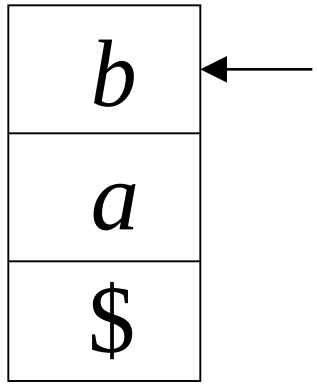
Stack



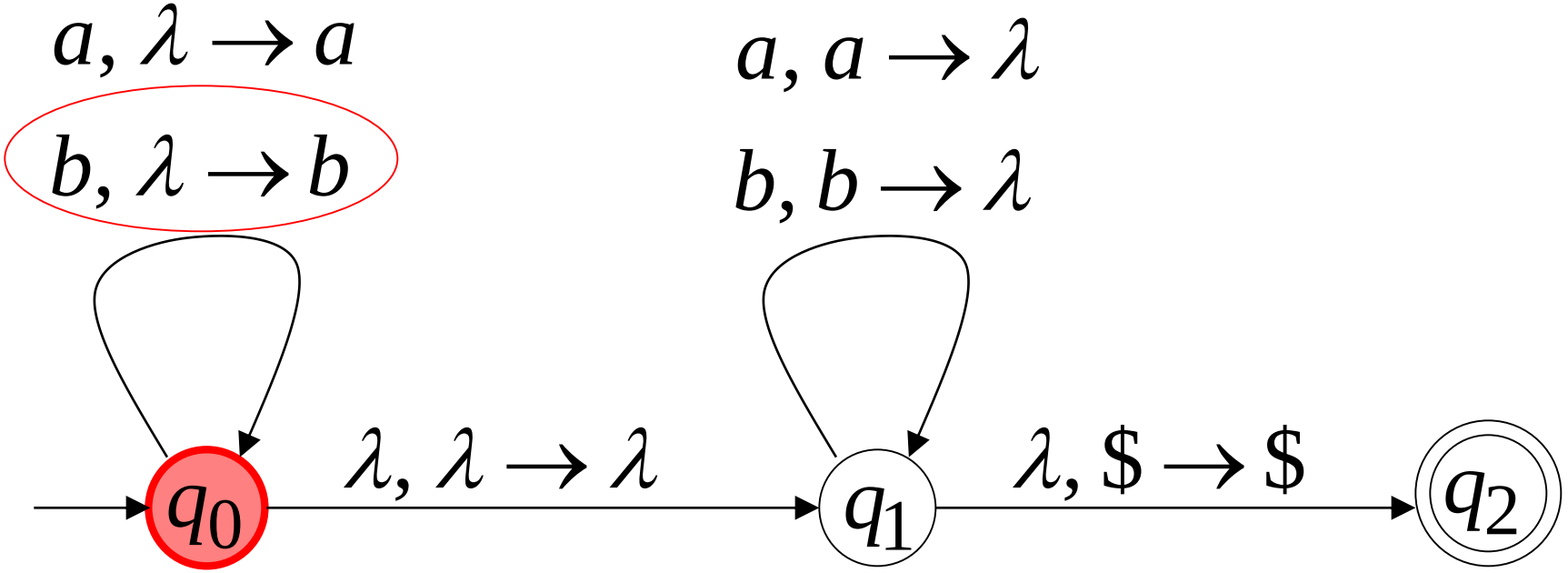
Time 2

Input



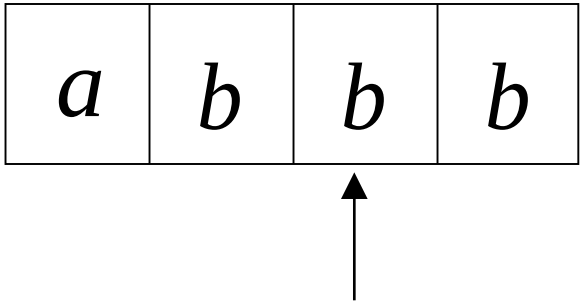


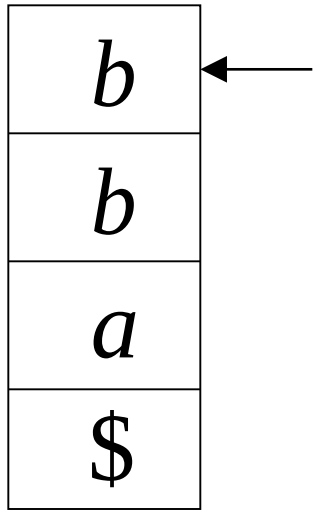
Stack



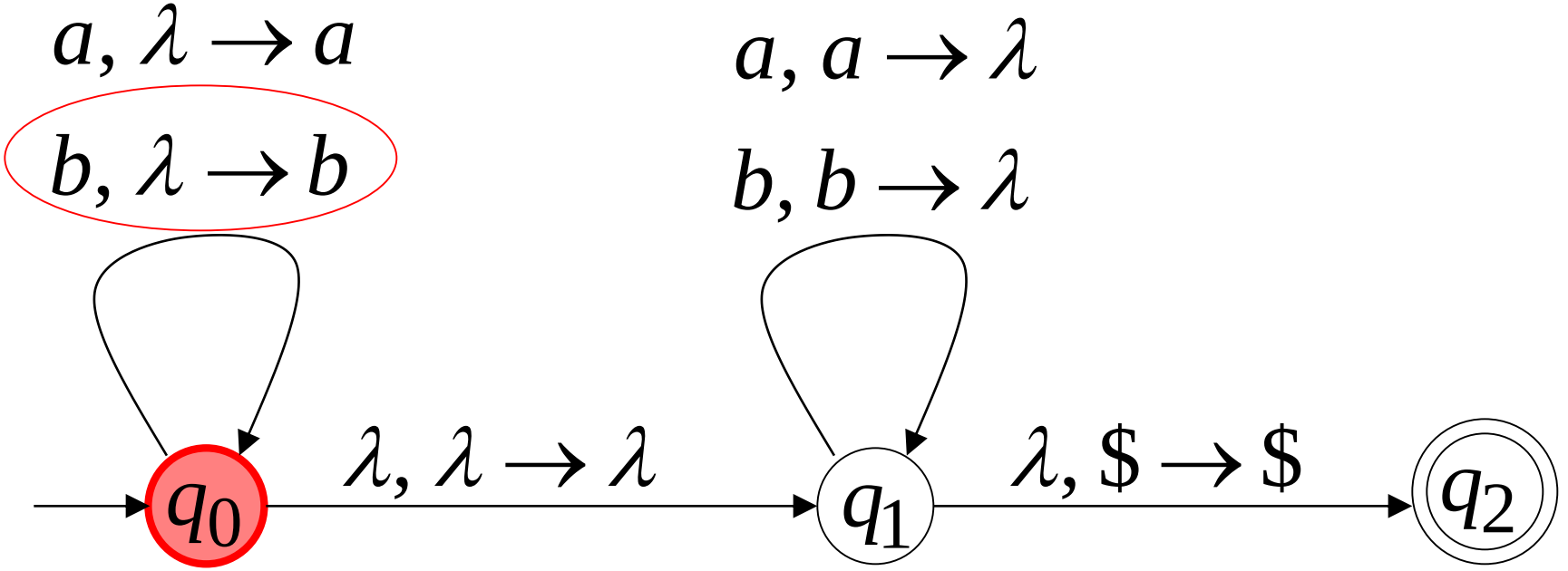
Time 3

Input



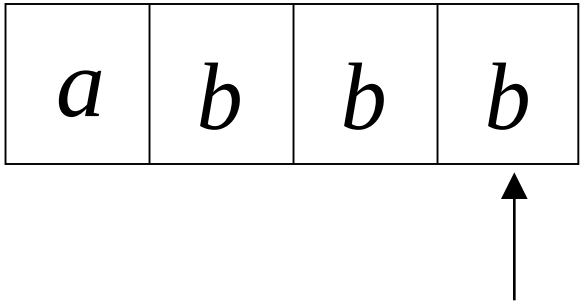


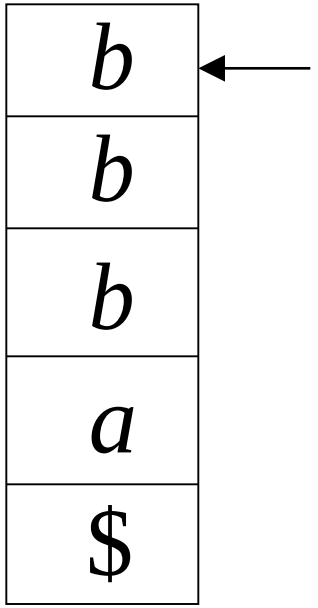
Stack



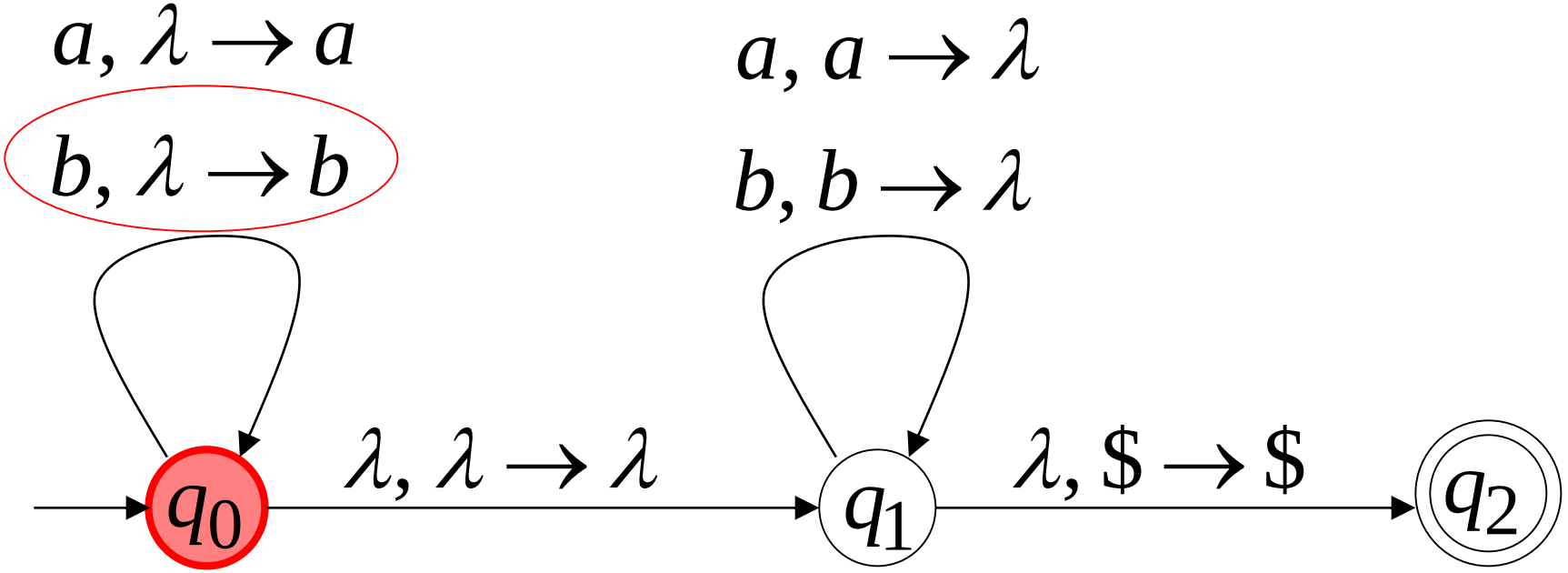
Time 4

Input





Stack



Time 5

Input

a	b	b	b

No accept state is reached

b
b
b
a
$

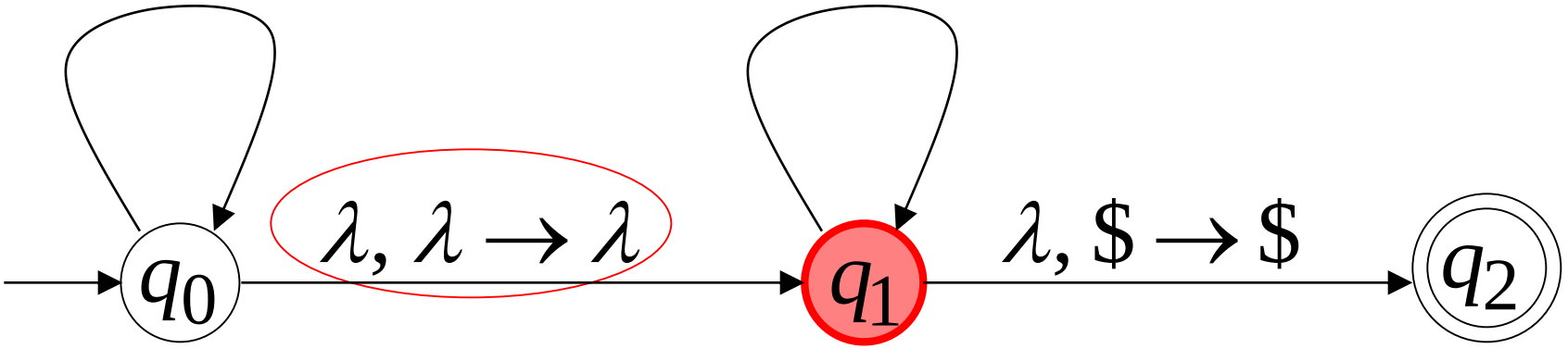
Stack

$a, \lambda \to a$

$b, \lambda \to b$

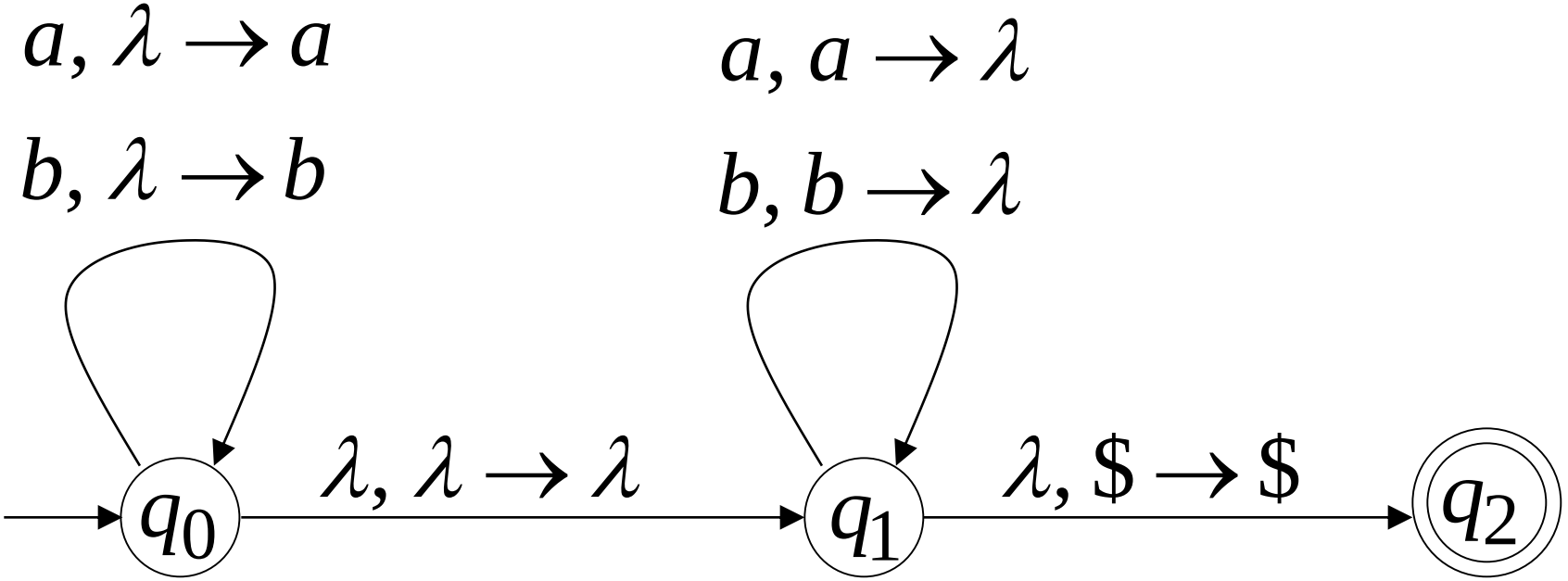
$a, a \to \lambda$

$b, b \to \lambda$

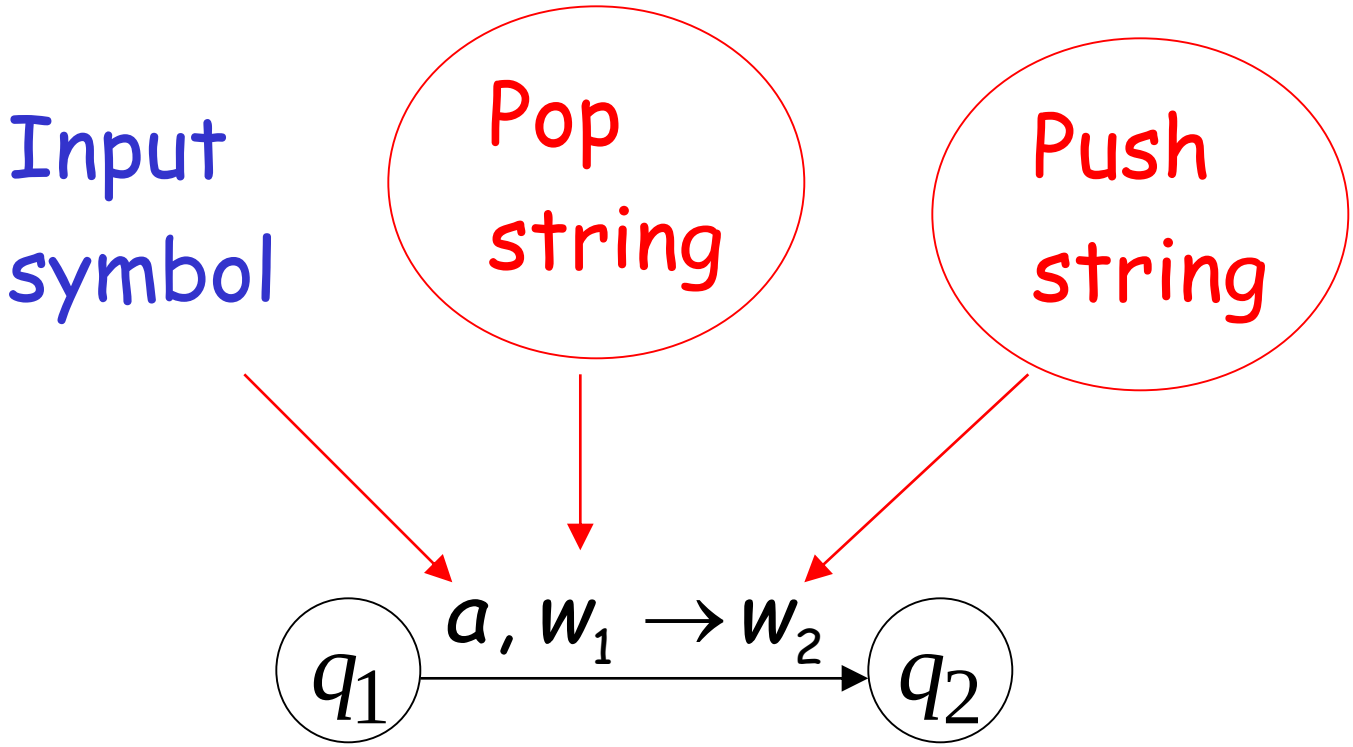


There is no computation that accepts string $abbb$

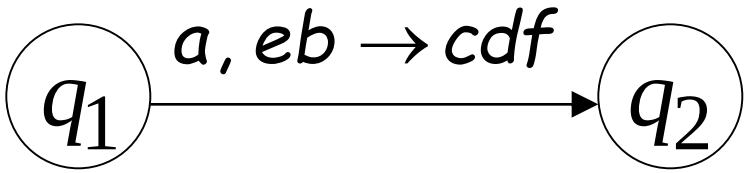
$$abbb \notin L(M)$$



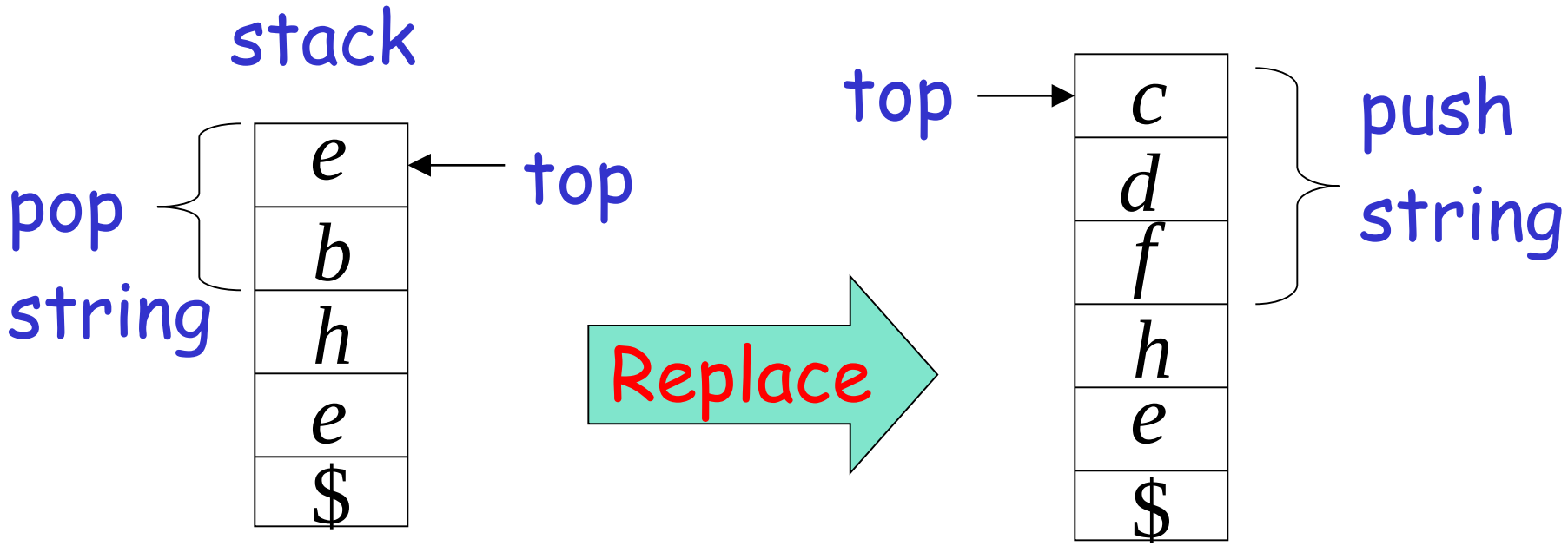
Pushing & Popping Strings

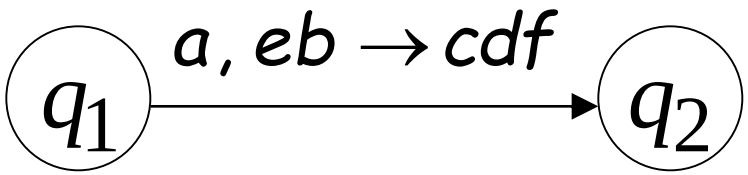


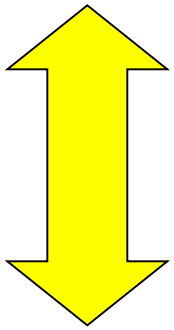
Example:


$a, eb \rightarrow cdf$
q_1
q_2


input
…
a
…
…
a
…

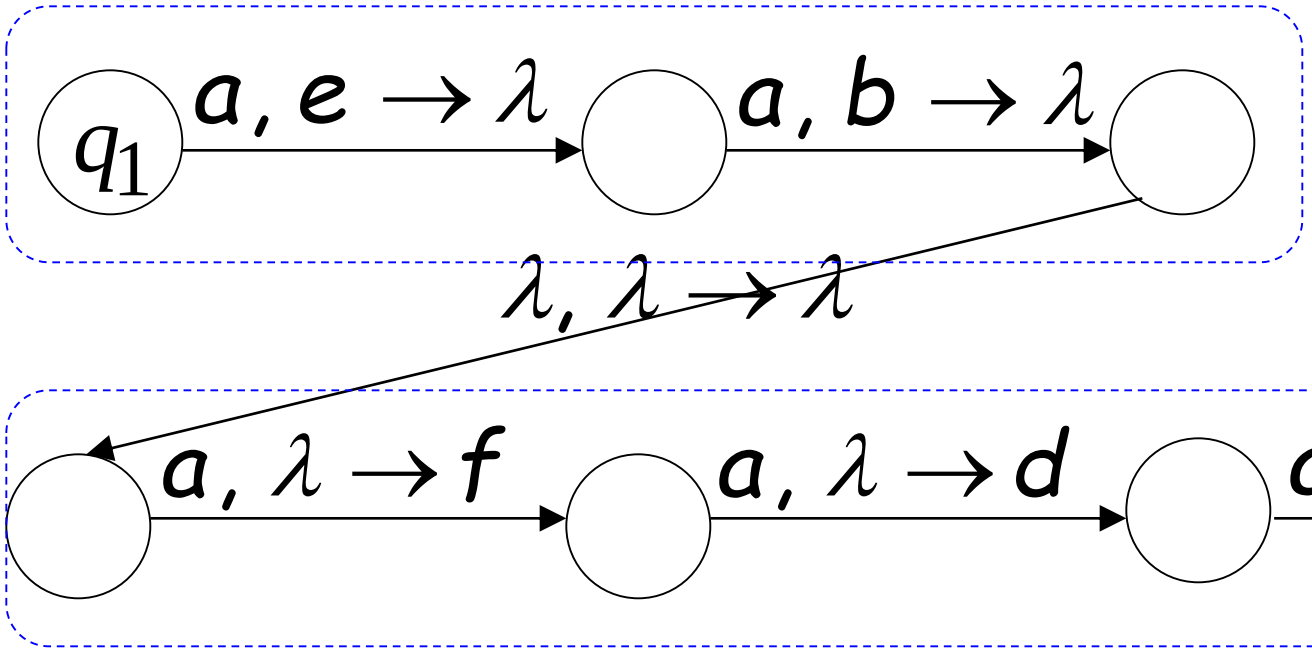

stack
pop string
e
b
h
e
$
top
Replace
top
c
d
f
h
e
$
push string





Equivalent transitions

pop



Another PDA example

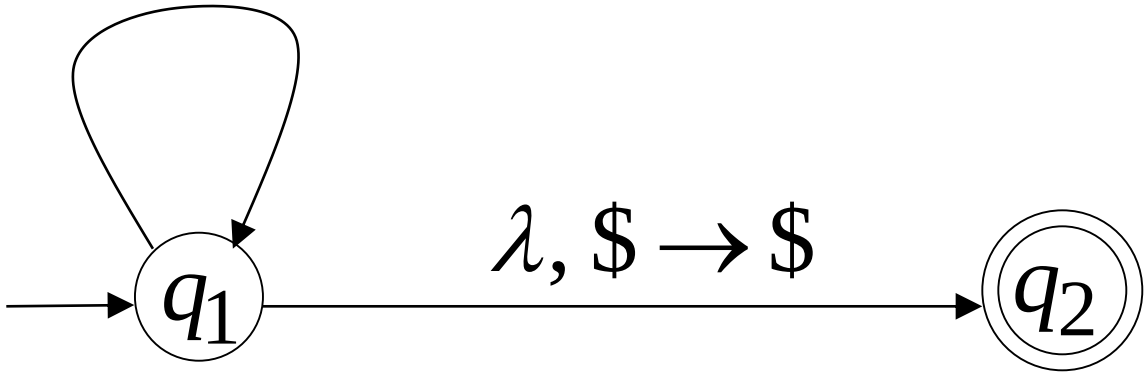
$$L(M) = \{w \in \{a,b\}^* : \quad n_a(w) = n_b(w)\}$$

PDA M

$a, \$ \to 0\$$ $\quad b, \$ \to 1\$$

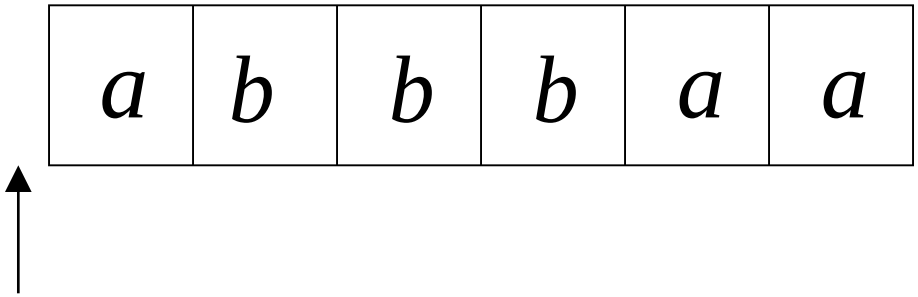
$a, 0 \to 00$ $\quad b, 1 \to 11$

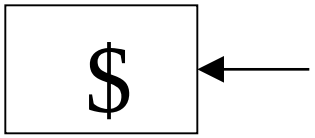
$a, 1 \to \lambda$ $\quad b, 0 \to \lambda$



Execution Example: Time 0

Input





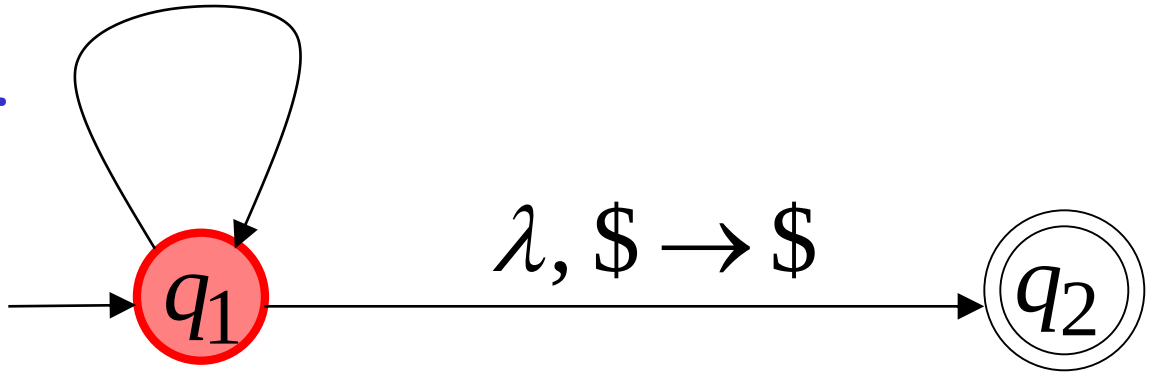
Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

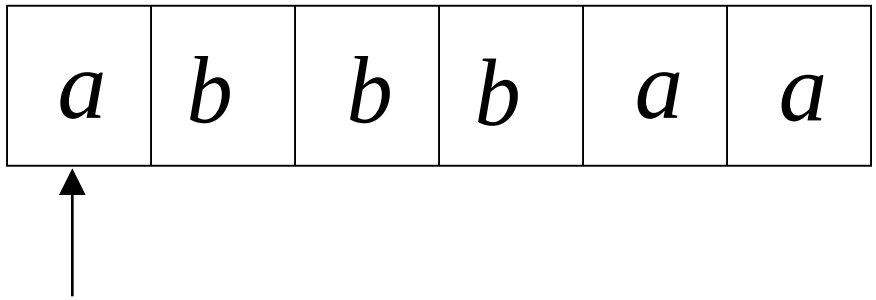
$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$

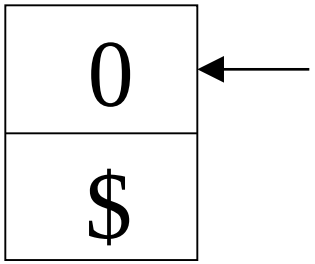
current state



Time 1

Input



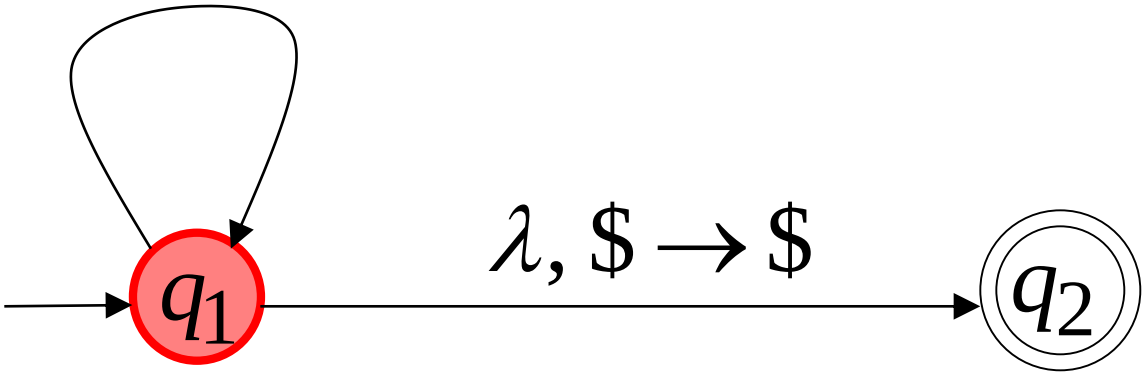


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

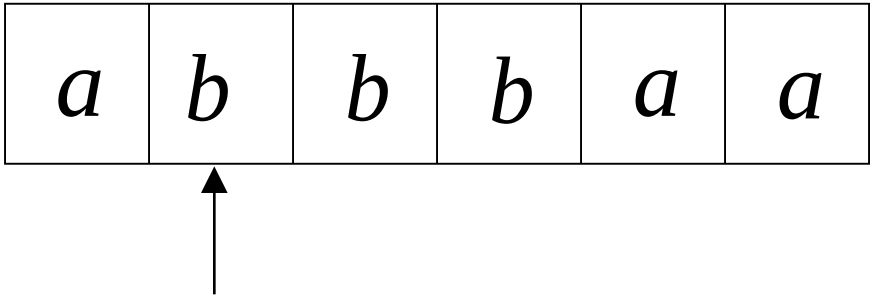
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

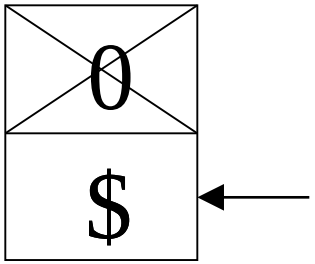
$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



Time 3

Input



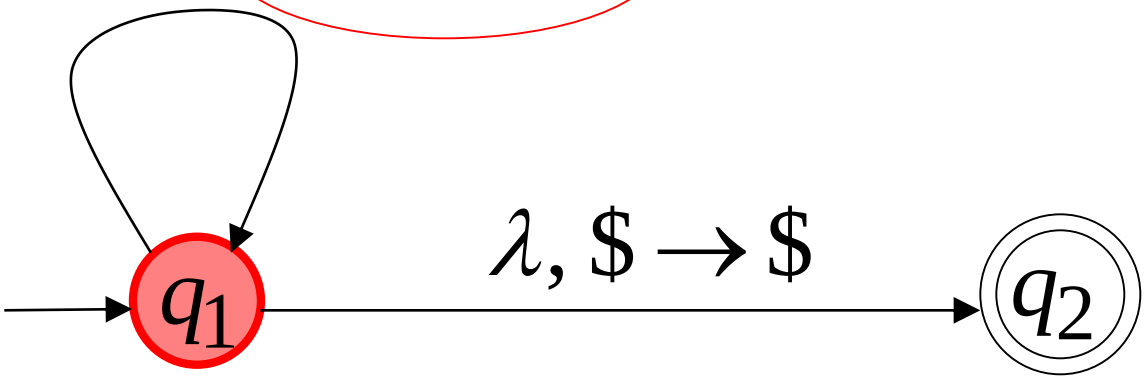


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

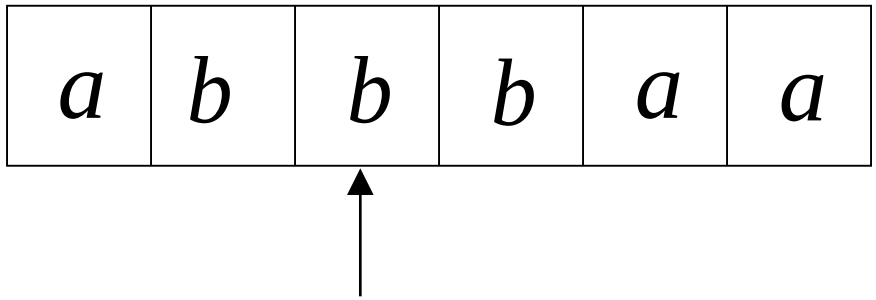
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

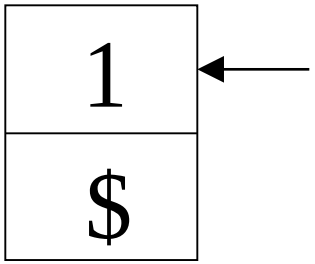
$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



Time 4

Input



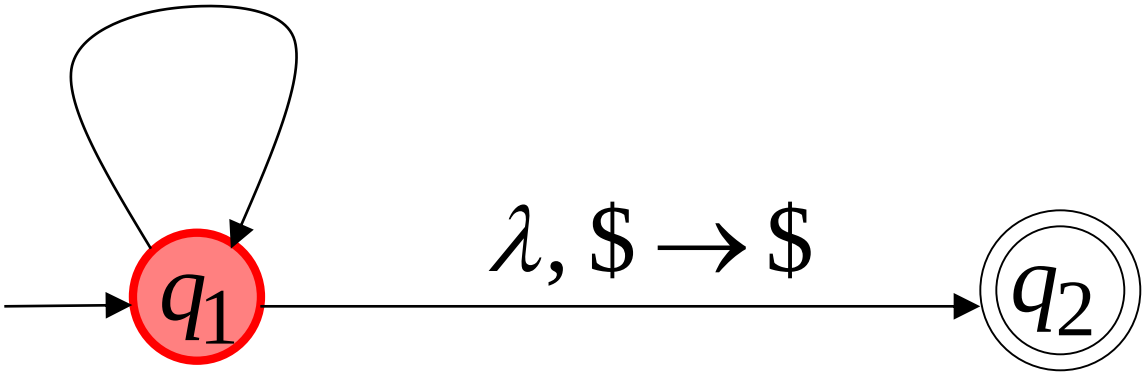


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

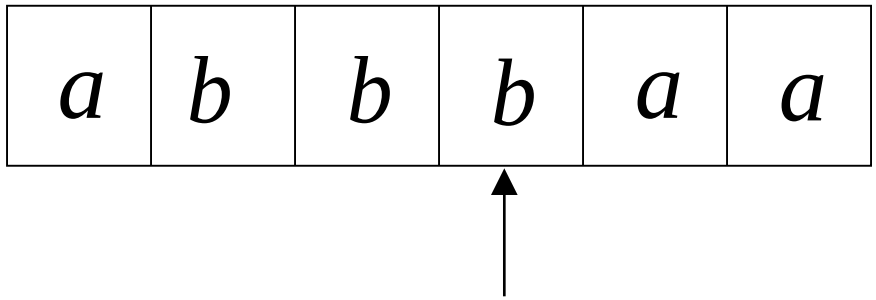
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

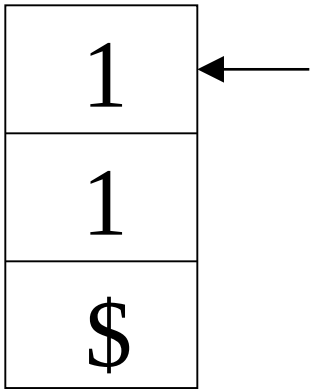
$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



Time 5

Input



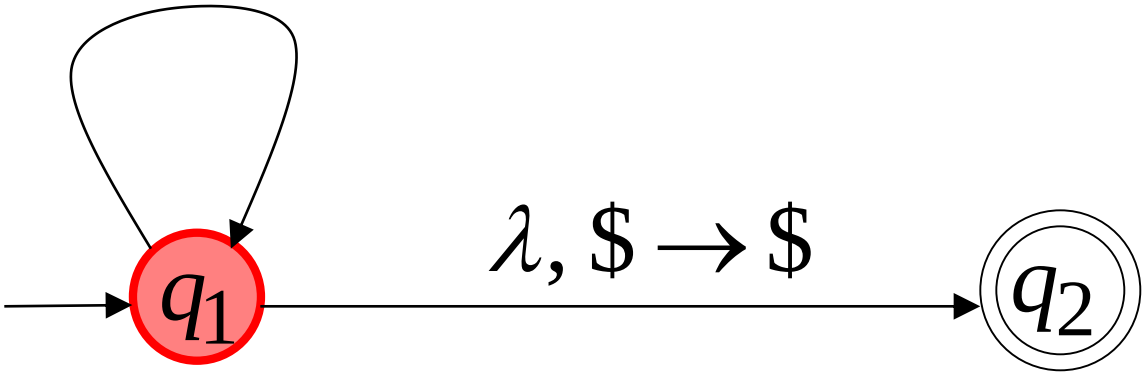


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

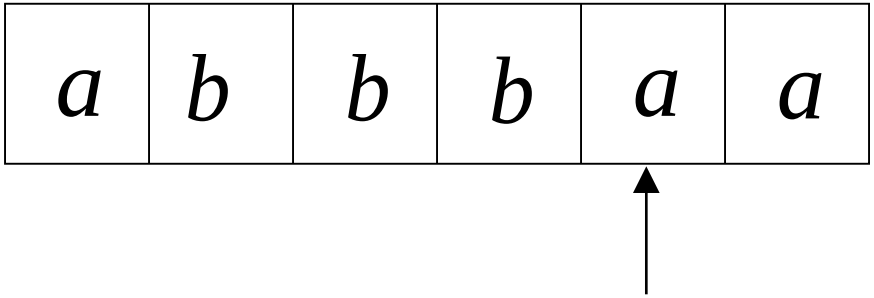
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

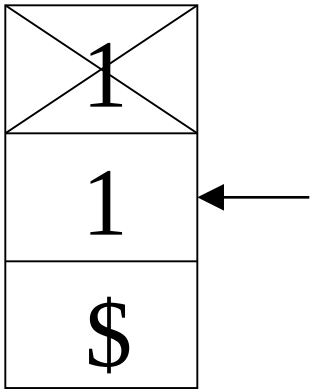
$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



Time 6

Input



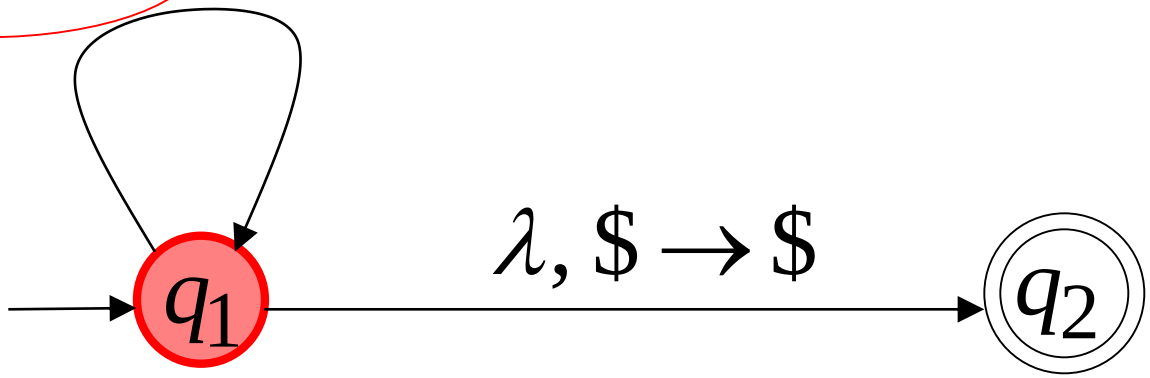


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

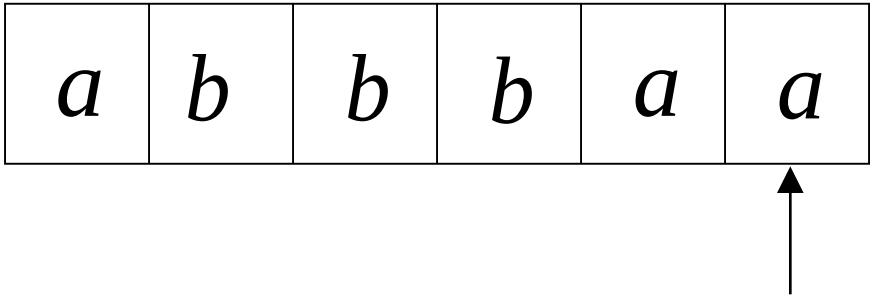
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

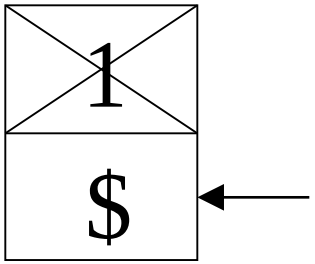
$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



Time 7

Input



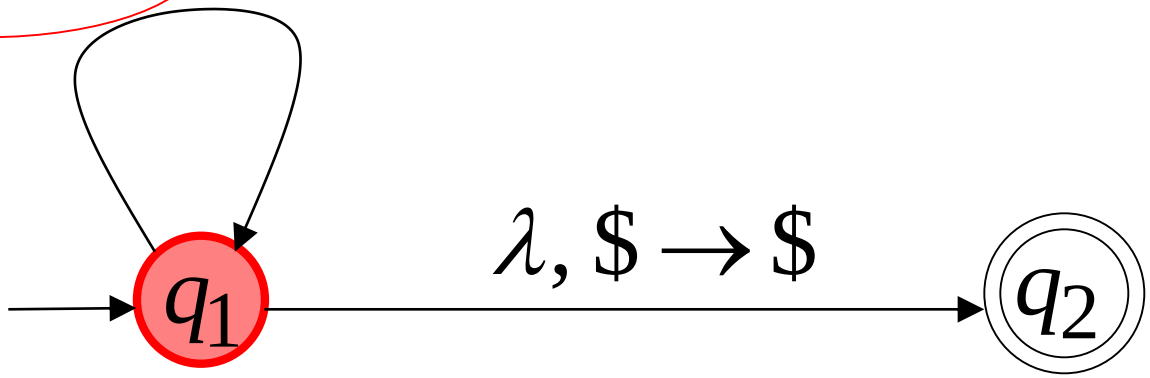


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

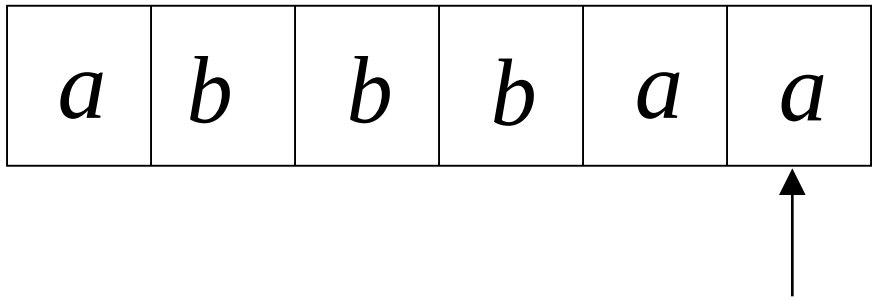
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

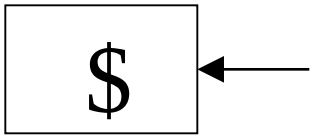
$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



Time 8

Input



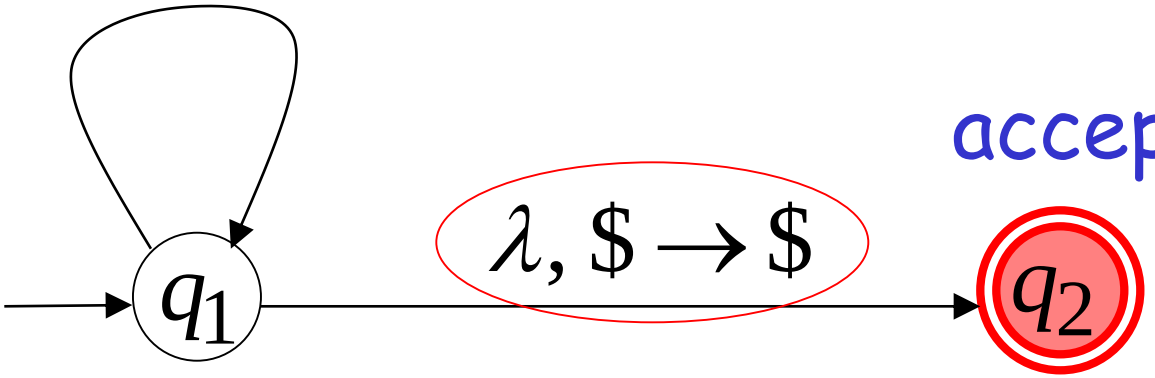


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

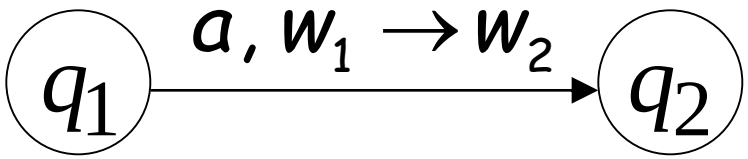
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



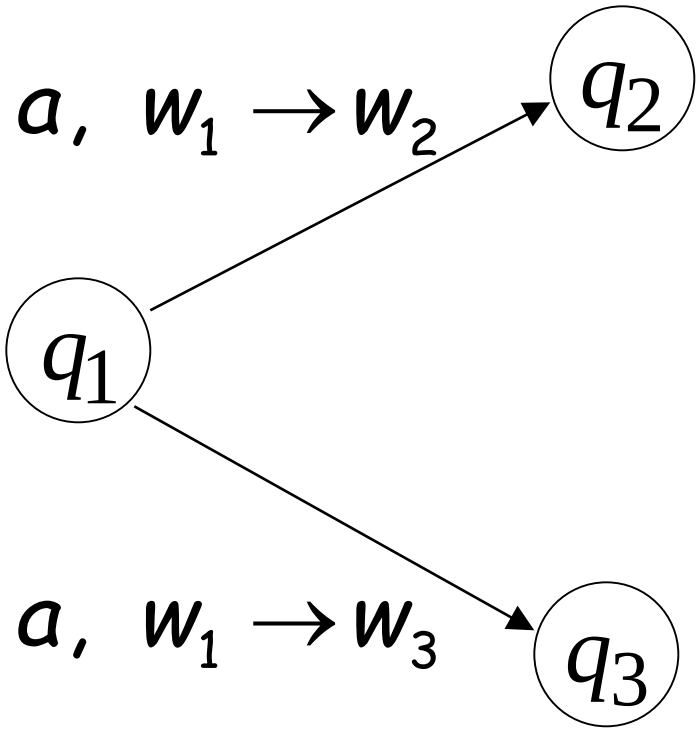
accept

Formalities for PDAs



Transition function:

$$\delta(q_1, a, w_1) = \{(q_2, w_2)\}$$



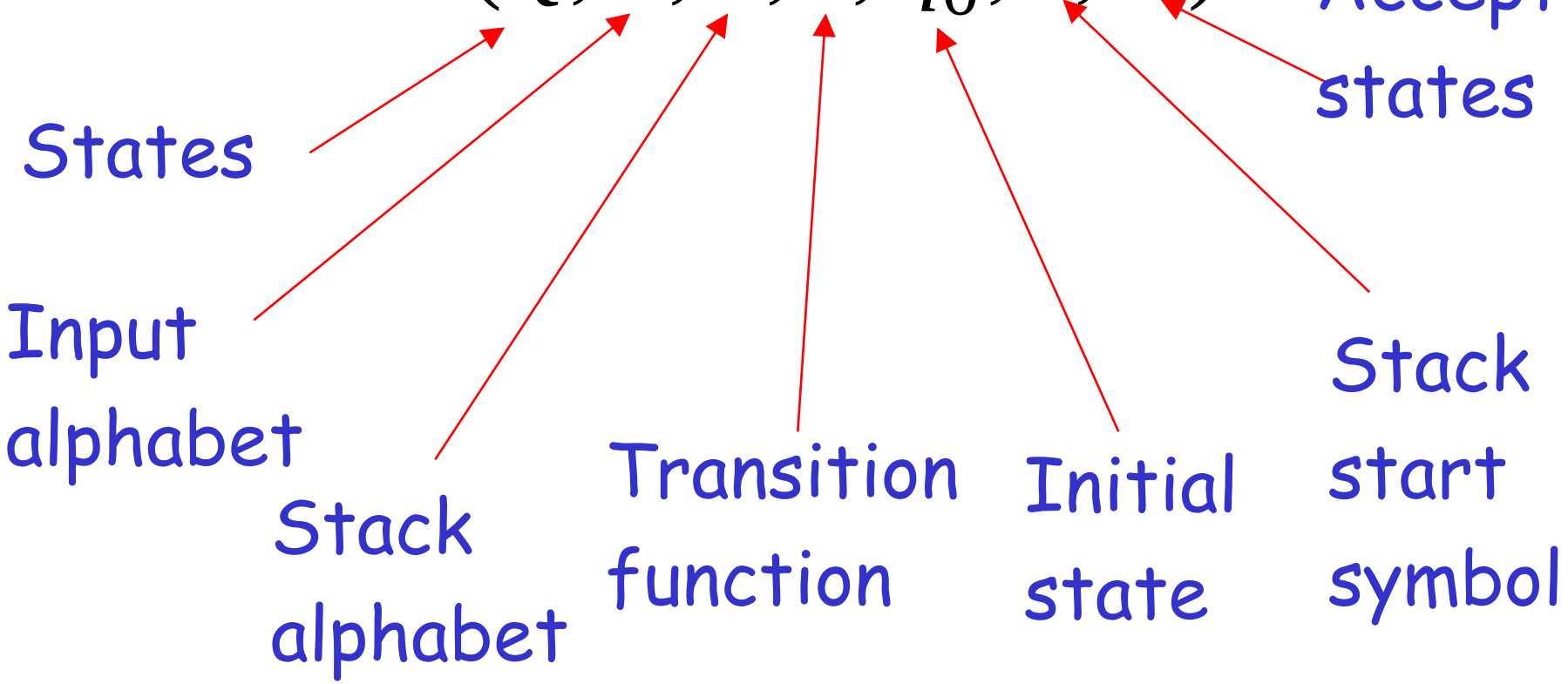
Transition function:

$$\delta(q_1, a, w_1) = \{(q_2, w_2),\ (q_3, w_3)\}$$

Formal Definition

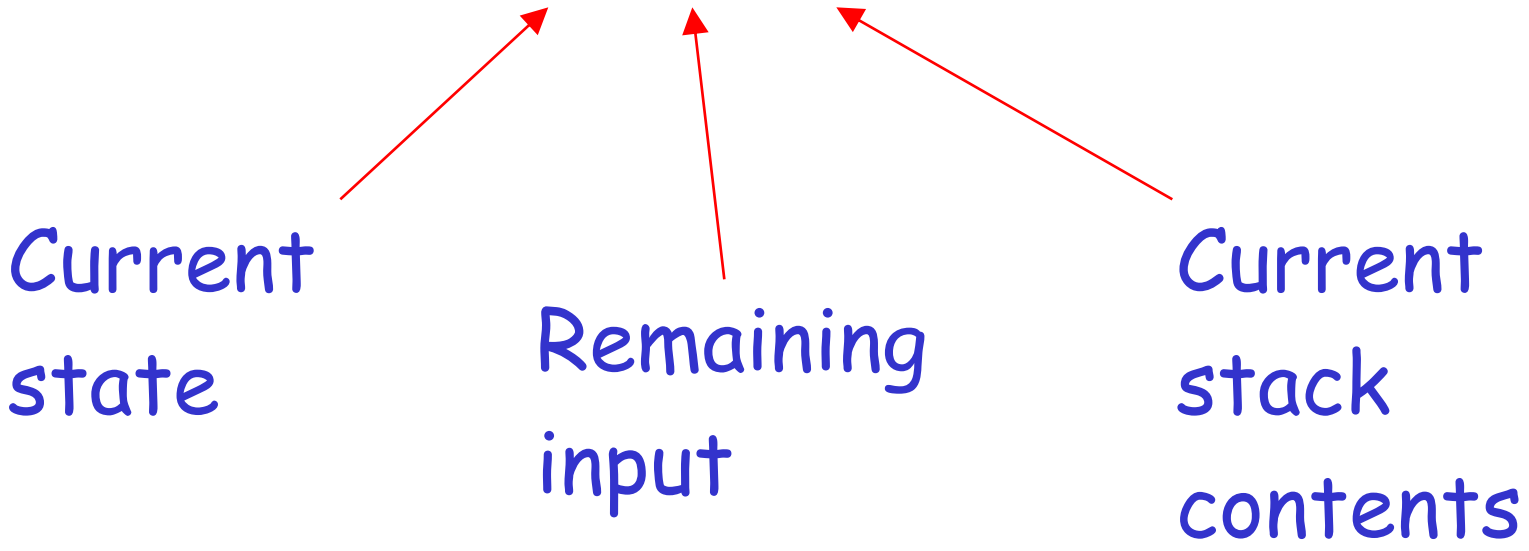
Pushdown Automaton (PDA)

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$



Instantaneous Description

$$(q,\ u,\ s)$$

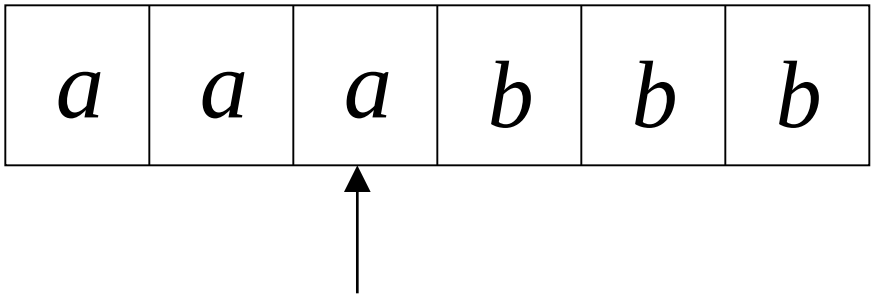


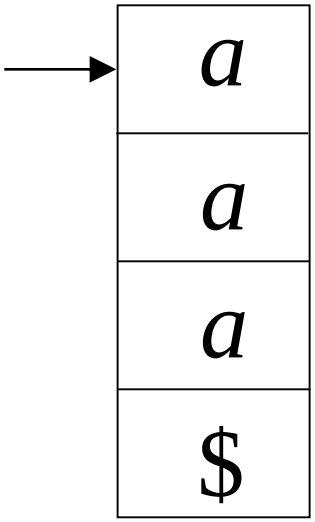
Example: **Instantaneous Description**

$$(q_1, bbb, aaa\$)$$

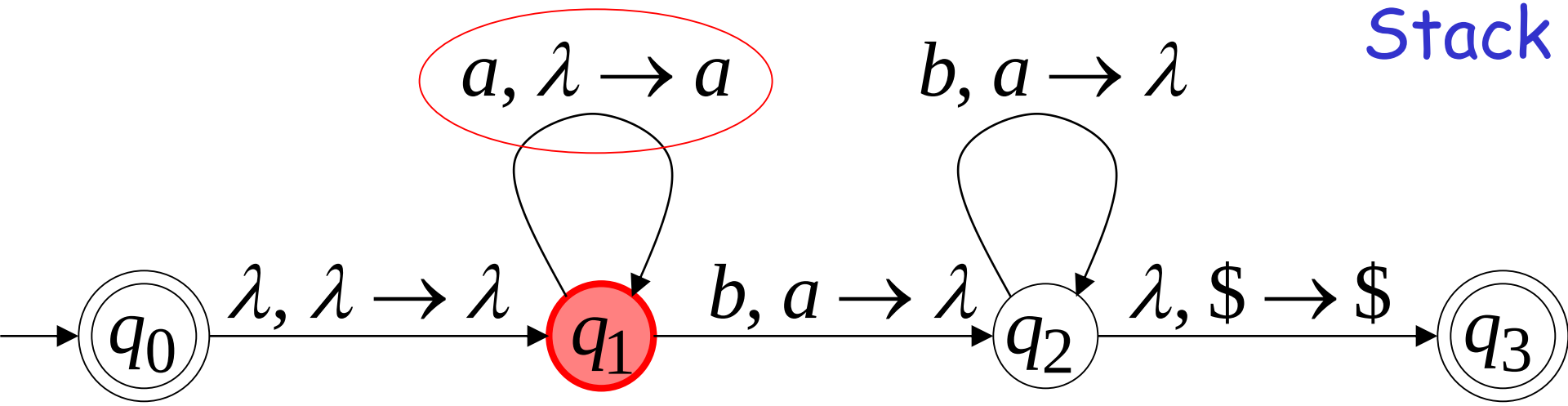
Time 4:

Input





Stack

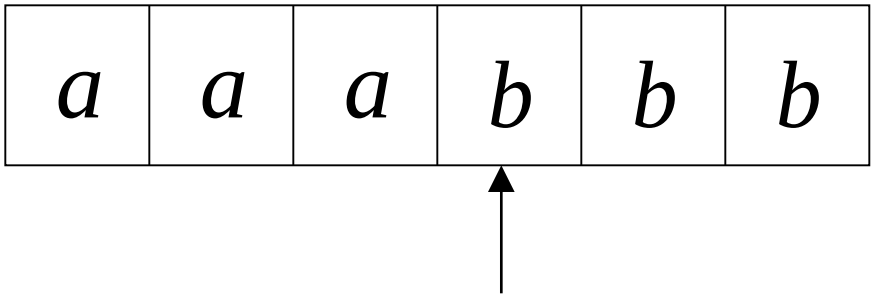


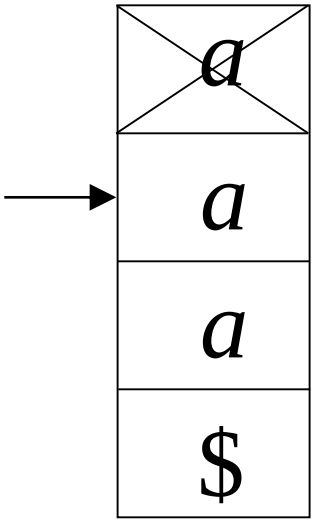
Example: Instantaneous Description

$$(q_2, bb, aa\$)$$

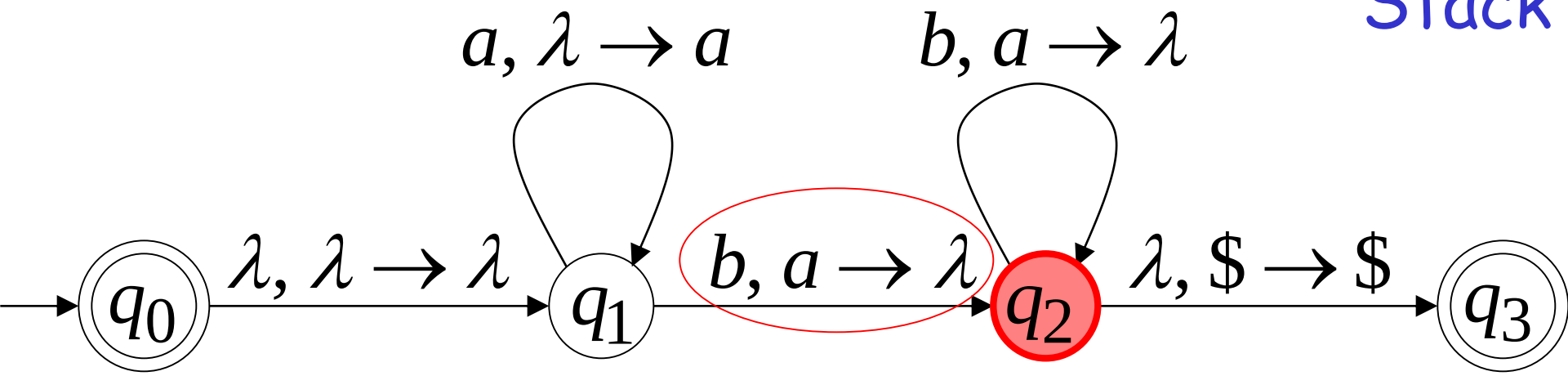
Time 5:

Input





Stack



We write:

$$(q_1, bbb, aaa\$) \succ (q_2, bb, aa\$)$$

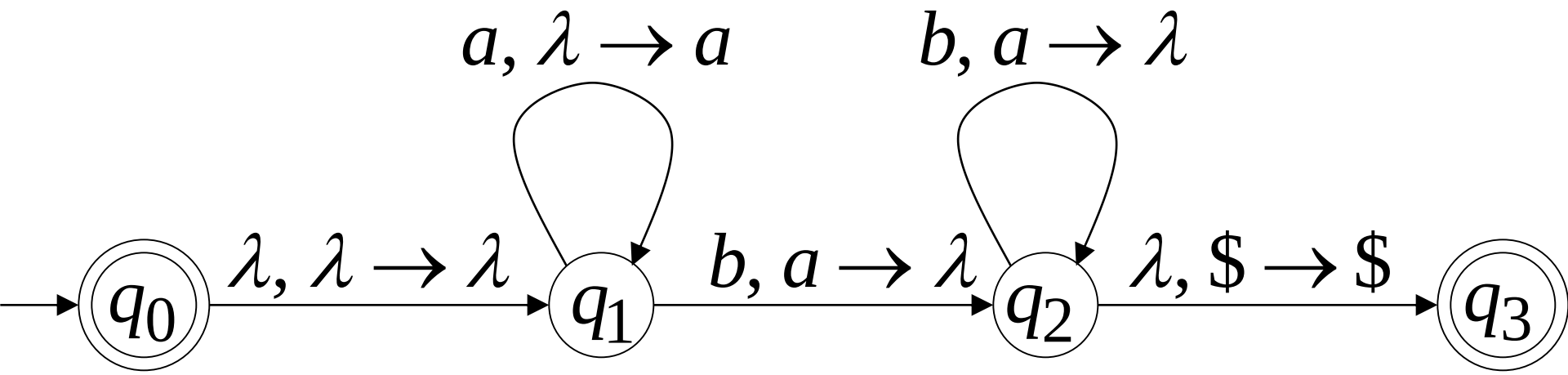
Time 4 Time 5

A computation:

$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$

$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$

$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \lambda, \$) \succ (q_3, \lambda, \$)$$



$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$

$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$

$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \lambda, \$) \succ (q_3, \lambda, \$)$$

For convenience we write:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

Language of PDA

Language $L(M)$ accepted by PDA M:

$$L(M) = \{w : (q_0, w, z) \overset{*}{\vdash} (q_f, \lambda, s)\}$$

Initial state

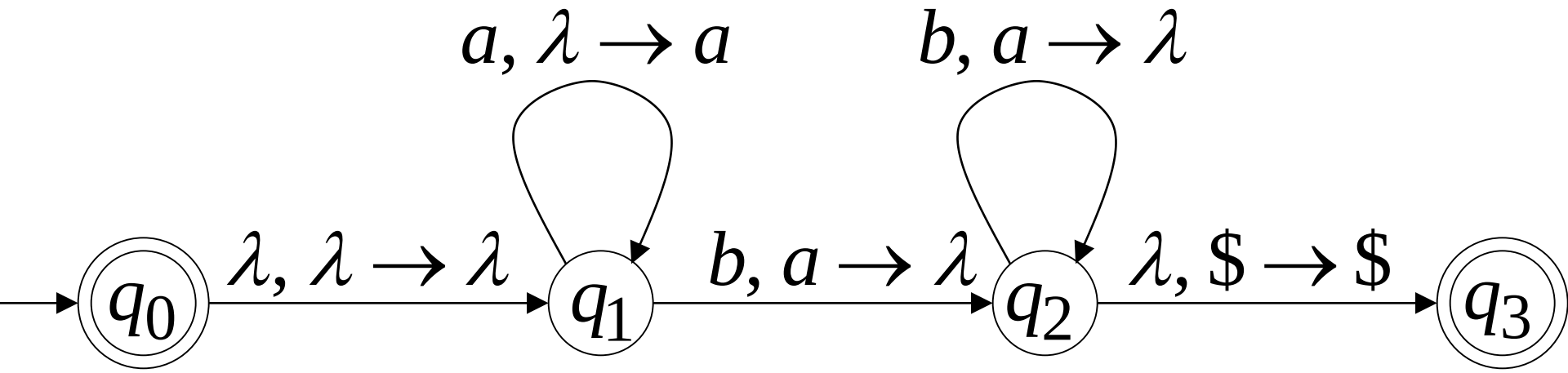
Accept state

Example:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

$$aaabbb \in L(M)$$

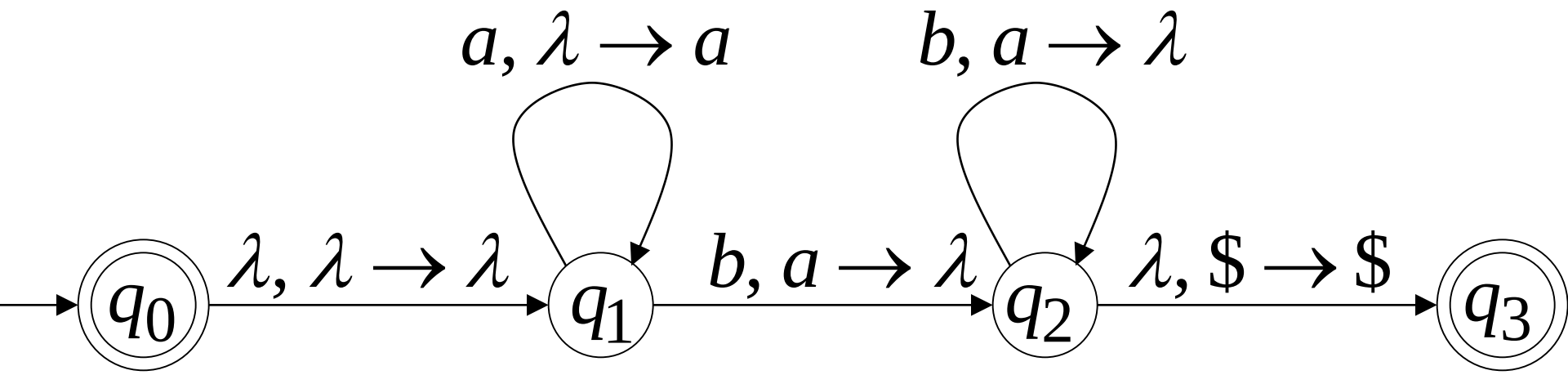
PDA M:



$$(q_0, a^n b^n, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

$$a^n b^n \in L(M)$$

PDA M :



Therefore: $L(M) = \{a^n b^n : n \geq 0\}$

PDA M :