Nondeterministic Finite Automata

Nondeterminism

Subset Construction

Nondeterminism

- A *nondeterministic finite automaton* has the ability to be in several states at once.
- Transitions from a state on an input symbol can be to any set of states.

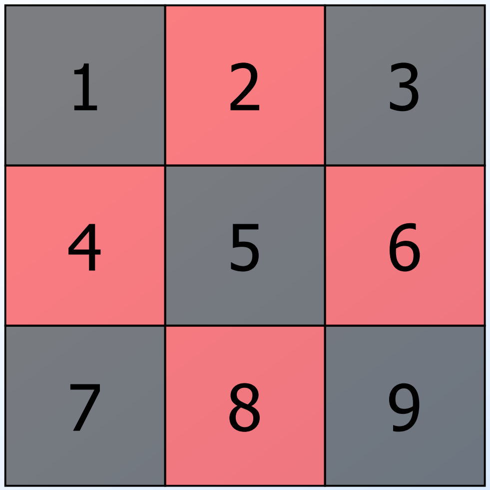
Nondeterminism – (2)

- Start in one start state.
- Accept if any sequence of choices leads to a final state.
- Intuitively: the NFA always "guesses right."

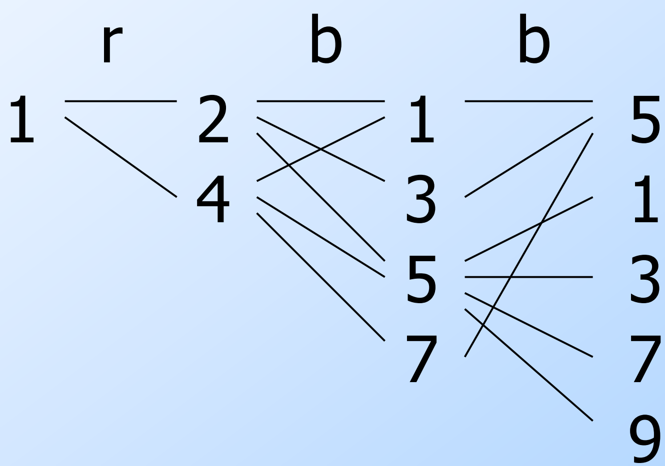
Example: Moves on a Chessboard

- States = squares.
- Inputs = r (move to an adjacent red square) and b (move to an adjacent black square).
- Start state, final state are in opposite corners.

Example: Chessboard – (2)



		r	b
→	1	2,4	5
	2	4,6	1,3,5
	3	2,6	5
	4	2,8	1,5,7
	5	2,4,6,8	1,3,7,9
	6	2,8	3,5,9
	7	4,8	5
	8	4,6	5,7,9
*	9	6,8	5



← Accept, since final state reached

Formal NFA

- A finite set of states, typically Q.
- An input alphabet, typically Σ.
- A transition function, typically δ.
- A start state in Q, typically q_0.
- A set of final states $F \subseteq Q$.

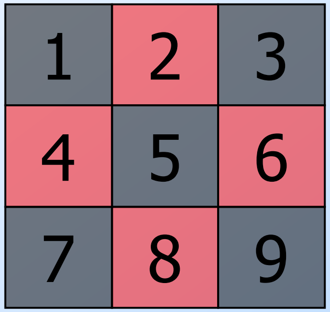
Transition Function of an NFA

- $\delta(q, a)$ is a set of states.
- Extend to strings as follows:
- Basis: $\delta(q, \epsilon) = \{q\}$
- Induction: $\delta(q, wa)$ = the union over all states p in $\delta(q, w)$ of $\delta(p, a)$

Language of an NFA

- A string w is accepted by an NFA if $\delta(q_0, w)$ contains at least one final state.
- The language of the NFA is the set of strings it accepts.

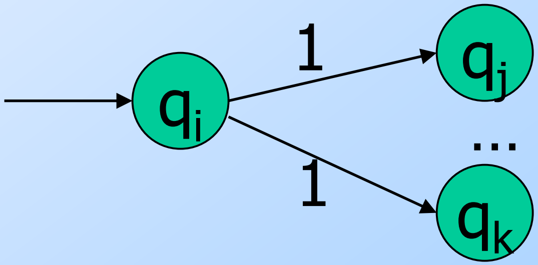
Example: Language of an NFA



- For our chessboard NFA we saw that rbb is accepted.
- If the input consists of only b's, the set of accessible states alternates between {5} and {1,3,7,9}, so only even-length, nonempty strings of b's are accepted.
- What about strings with at least one r?

Non-deterministic Finite Automata (NFA)

- A Non-deterministic Finite Automaton (NFA)
  - is of course "non-deterministic"
    - Implying that the machine can exist in more than one state at the same time
    - Transitions could be non-deterministic



- Each transition function therefore maps to a <u>set</u> of states

Non-deterministic Finite Automata (NFA)

- A Non-deterministic Finite Automaton (NFA) consists of:
  - Q ==> a finite set of states
  - ∑ ==> a finite set of input symbols (alphabet)
  - q_0 ==> a start state
  - F ==> set of accepting states
  - δ ==> a transition function, which is a mapping between Q x ∑ ==> subset of Q
- An NFA is also defined by the 5-tuple:
  - {Q, ∑ , q_0,F, δ }

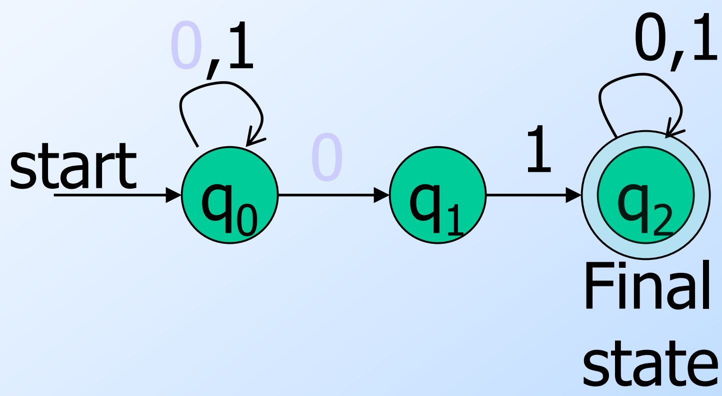
How to use an NFA?

- Input: a word w in Σ^*
- Question: Is w acceptable by the NFA?
- Steps:
  - Start at the "start state" q_0
  - For every input symbol in the sequence w do
    - Determine all possible next states from all current states, given the current input symbol in w and the transition function
  - If after all symbols in w are consumed and if at least one of the current states is a final state then *accept w;*
  - Otherwise, *reject w.*

Regular expression: (0+1)*01(0+1)*

NFA for strings containing 01

Why is this non-deterministic?



What will happen if at state q_1 an input of 0 is received?

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{0,1\}$
- start state = q_0
- $F = \{q_2\}$
- Transition table

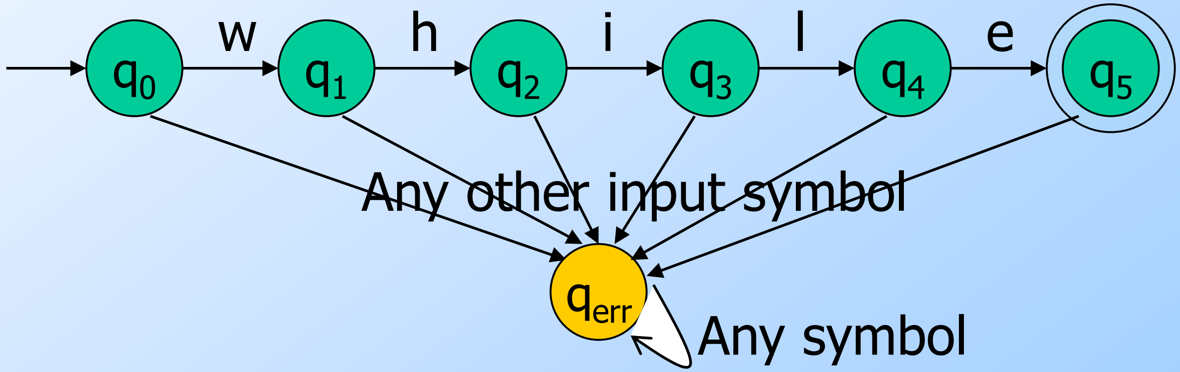
symbols

states: δ	0	1
→ q_0	$\{q_0, q_1\}$	$\{q_0\}$
q_1	Φ	$\{q_2\}$
$*q_2$	$\{q_2\}$	$\{q_2\}$

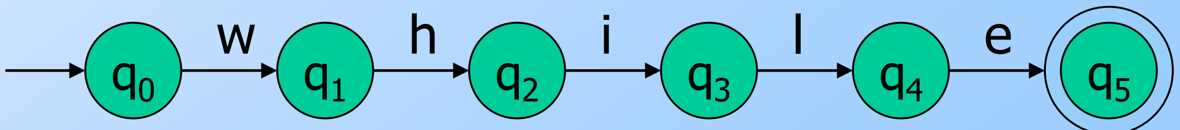
Note: Omitting to explicitly show error states is just a matter of design convenience
(one that is generally followed for NFAs), and
i.e., this feature should not be confused with the notion of non-determinism.

What is an "error state"?

- A DFA for recognizing the key word "*while*"



- An NFA for the same purpose:



Transitions into a dead state are implicit

Extension of δ to NFA Paths

- <u>Basis:</u> $\hat{\delta}(q,\varepsilon) = \{q\}$
- <u>Induction:</u>
  - Let $\hat{\delta}(q_0,w) = \{p_1,p_2,\dots,p_k\}$
  - $\delta(p_i,a) = S_i$ *for* $i=1,2,\dots,k$
  - *Then,* $\hat{\delta}(q_0,wa) = S_1 \cup S_2 \cup \dots \cup S_k$

Language of an NFA

- An NFA accepts *w* if *there exists at least one* path from the start state to an accepting (or final) state that is labeled by *w*
- $L(N) = \{ w \mid \hat{\delta}(q_0, w) \cap F \neq \Phi \}$

Example #2

- Build an NFA for the following language:
  L = { w | w ends in 01}
- ?
- Other examples
  - Keyword recognizer (e.g., if, then, else, while, for, include, etc.)
  - Strings where the first symbol is present somewhere later on at least once

Example #3

- Build an NFA which accepts exactly those strings that have the symbol 1 in the second position

Example #4

- Build an NFA which accepts a^n b^m where n,m>=1

Example #5

- Build an NFA for the following

L={ w|w $\varepsilon\ 0101^n\ or\ \ 010^n\ where\ n \geq 0$}

Exercises

1. Build an NFA for the following

L={x|x is in Σ^* *and the third last symbol in x is b*}

2. Build an NFA for 2 (or) more c's where $\Sigma = \{a, b, c\}$

3. The set of all strings such that containing either 101 or 110 as substring

4. The set of all strings such that every 1 is followed by 00

Advantages & Caveats for NFA

- Great for modeling regular expressions
  - String processing - e.g., grep, lexical analyzer
- Could a non-deterministic state machine be implemented in practice?
  - Probabilistic models could be viewed as extensions of non-deterministic state machines
    (e.g., toss of a coin, a roll of dice)
    - They are not the same though
  - A parallel computer could exist in multiple “states” at the same time

But, DFAs and NFAs are equivalent in their power to capture langauges !!

Differences: DFA vs. NFA

- **DFA**

1. All transitions are deterministic
   - Each transition leads to exactly one state
2. For each state, transition on all possible symbols (alphabet) should be defined
3. Accepts input if the last state visited is in F
4. Sometimes harder to construct because of the number of states
5. Practical implementation is feasible

- **NFA**

1. Some transitions could be non-deterministic
   - A transition could lead to a subset of states
2. Not all symbol transitions need to be defined explicitly (if undefined will go to an error state – this is just a design convenience, not to be confused with "non-determinism")
3. Accepts input if *one of* the last states is in F
4. Generally easier than a DFA to construct
5. Practical implementations limited but emerging (e.g., Micron automata processor)

Equivalence of DFA's, NFA's

- A DFA can be turned into an NFA that accepts the same language.
- If $\delta_D(q, a) = p$, let the NFA have $\delta_N(q, a) = \{p\}$.
- Then the NFA is always in a set containing exactly one state – the state the DFA is in after reading the same input.

Equivalence – (2)

- Surprisingly, for any NFA there is a DFA that accepts the same language.
- Proof is the *subset construction*.
- The number of states of the DFA can be exponential in the number of states of the NFA.
- Thus, NFA's accept exactly the regular languages.

Subset Construction

- Given an NFA with states Q, inputs Σ, transition function δ_N, state state q_0, and final states F, construct equivalent DFA with:
  - States 2^Q (Set of subsets of Q).
  - Inputs Σ.
  - Start state $\{q_0\}$.
  - Final states = all those with a member of F.

Critical Point

- The DFA states have *names* that are sets of NFA states.
- But as a DFA state, an expression like {p,q} must be read as a single symbol, not as a set.
- Analogy: a class of objects whose values are sets of objects of another class.

Subset Construction – (2)

- The transition function δ_D is defined by:

$\delta_D(\{q_1,\ldots,q_k\}, a)$ is the union over all $i = 1,\ldots,k$ of $\delta_N(q_i, a)$.

- Example: We'll construct the DFA equivalent of our "chessboard" NFA.

Example: Subset Construction

		r	b
→	1	2,4	5
	2	4,6	1,3,5
	3	2,6	5
	4	2,8	1,5,7
	5	2,4,6,8	1,3,7,9
	6	2,8	3,5,9
	7	4,8	5
	8	4,6	5,7,9
*	9	6,8	5

		r	b
→	{1}	{2,4}	{5}
	{2,4}		
	{5}		

Alert: What we're doing here is the *lazy* form of DFA construction, where we only construct a state if we are forced to.

Example: Subset Construction

		r	b
→	1	2,4	5
	2	4,6	1,3,5
	3	2,6	5
	4	2,8	1,5,7
	5	2,4,6,8	1,3,7,9
	6	2,8	3,5,9
	7	4,8	5
	8	4,6	5,7,9
*	9	6,8	5

	r	b
→ {1}	{2,4}	{5}
{2,4}	{2,4,6,8}	{1,3,5,7}
{5}		
{2,4,6,8}		
{1,3,5,7}		

Example: Subset Construction

		r	b
→	1	2,4	5
	2	4,6	1,3,5
	3	2,6	5
	4	2,8	1,5,7
	5	2,4,6,8	1,3,7,9
	6	2,8	3,5,9
	7	4,8	5
	8	4,6	5,7,9
*	9	6,8	5

		r	b
→	{1}	{2,4}	{5}
	{2,4}	{2,4,6,8}	{1,3,5,7}
	{5}	{2,4,6,8}	{1,3,7,9}
	{2,4,6,8}		
	{1,3,5,7}		
*	{1,3,7,9}		

Example: Subset Construction

		r	b
→	1	2,4	5
	2	4,6	1,3,5
	3	2,6	5
	4	2,8	1,5,7
	5	2,4,6,8	1,3,7,9
	6	2,8	3,5,9
	7	4,8	5
	8	4,6	5,7,9
*	9	6,8	5

		r	b
→	{1}	{2,4}	{5}
	{2,4}	{2,4,6,8}	{1,3,5,7}
	{5}	{2,4,6,8}	{1,3,7,9}
	{2,4,6,8}	{2,4,6,8}	{1,3,5,7,9}
	{1,3,5,7}		
*	{1,3,7,9}		
*	{1,3,5,7,9}		

Example: Subset Construction

		r	b
→	1	2,4	5
	2	4,6	1,3,5
	3	2,6	5
	4	2,8	1,5,7
	5	2,4,6,8	1,3,7,9
	6	2,8	3,5,9
	7	4,8	5
	8	4,6	5,7,9
*	9	6,8	5

		r	b
→	{1}	{2,4}	{5}
	{2,4}	{2,4,6,8}	{1,3,5,7}
	{5}	{2,4,6,8}	{1,3,7,9}
	{2,4,6,8}	{2,4,6,8}	{1,3,5,7,9}
	{1,3,5,7}	{2,4,6,8}	{1,3,5,7,9}
*	{1,3,7,9}		
*	{1,3,5,7,9}		

Example: Subset Construction

		r	b
→	1	2,4	5
	2	4,6	1,3,5
	3	2,6	5
	4	2,8	1,5,7
	5	2,4,6,8	1,3,7,9
	6	2,8	3,5,9
	7	4,8	5
	8	4,6	5,7,9
*	9	6,8	5

		r	b
→	{1}	{2,4}	{5}
	{2,4}	{2,4,6,8}	{1,3,5,7}
	{5}	{2,4,6,8}	{1,3,7,9}
	{2,4,6,8}	{2,4,6,8}	{1,3,5,7,9}
	{1,3,5,7}	{2,4,6,8}	{1,3,5,7,9}
*	{1,3,7,9}	{2,4,6,8}	{5}
*	{1,3,5,7,9}		

Example: Subset Construction

		r	b
→	1	2,4	5
	2	4,6	1,3,5
	3	2,6	5
	4	2,8	1,5,7
	5	2,4,6,8	1,3,7,9
	6	2,8	3,5,9
	7	4,8	5
	8	4,6	5,7,9
*	9	6,8	5

		r	b
→	{1}	{2,4}	{5}
	{2,4}	{2,4,6,8}	{1,3,5,7}
	{5}	{2,4,6,8}	{1,3,7,9}
	{2,4,6,8}	{2,4,6,8}	{1,3,5,7,9}
	{1,3,5,7}	{2,4,6,8}	{1,3,5,7,9}
*	{1,3,7,9}	{2,4,6,8}	{5}
*	{1,3,5,7,9}	{2,4,6,8}	{1,3,5,7,9}

Proof of Equivalence: Subset Construction

- The proof is almost a pun.
- Show by induction on |w| that

  $\delta_N(q_0, w) = \delta_D(\{q_0\}, w)$
- Basis: $w = \epsilon$: $\delta_N(q_0, \epsilon) = \delta_D(\{q_0\}, \epsilon) = \{q_0\}$.

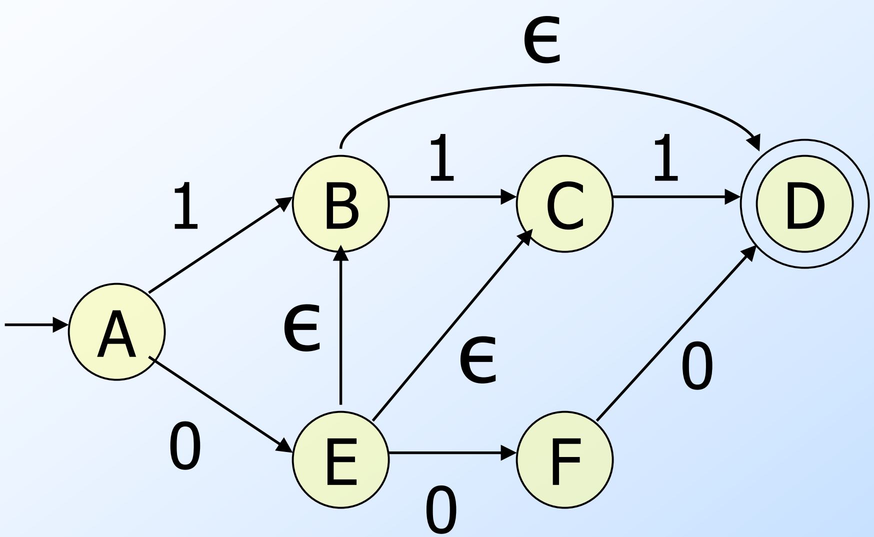
Induction

- Assume IH for strings shorter than w.
- Let w = xa; IH holds for x.
- Let $\delta_N(q_0, x) = \delta_D(\{q_0\}, x) = S$.
- Let T = the union over all states p in S of $\delta_N(p, a)$.
- Then $\delta_N(q_0, w) = \delta_D(\{q_0\}, w) = T$.
  - For NFA: the extension of δ_N.
  - For DFA: definition of δ_D plus extension of δ_D.
    - That is, $\delta_D(S, a) = T$; then extend δ_D to w = xa.

NFA's With ϵ-Transitions

- We can allow state-to-state transitions on ϵ input.
- These transitions are done spontaneously, without looking at the input string.
- A convenience at times, but still only regular languages are accepted.

Example: ϵ-NFA

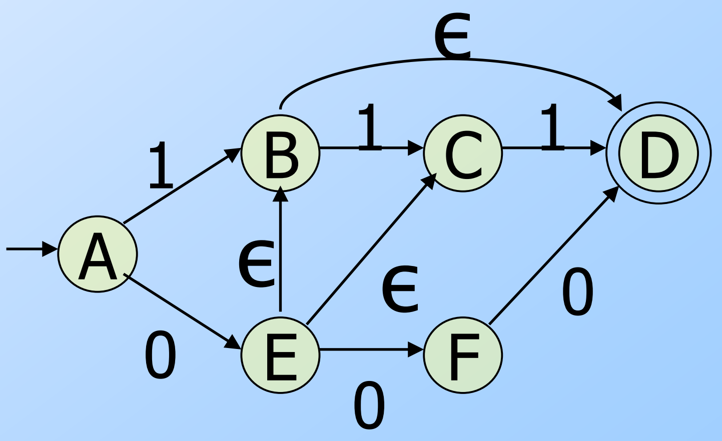


		0	1	ϵ
→	A	{E}	{B}	∅
	B	∅	{C}	{D}
	C	∅	{D}	∅
*	D	∅	∅	∅
	E	{F}	∅	{B, C}
	F	{D}	∅	∅

Closure of States

- CL(q) = set of states you can reach from state q following only arcs labeled ϵ.
- Example: CL(A) = {A};

  CL(E) = {B, C, D, E}.



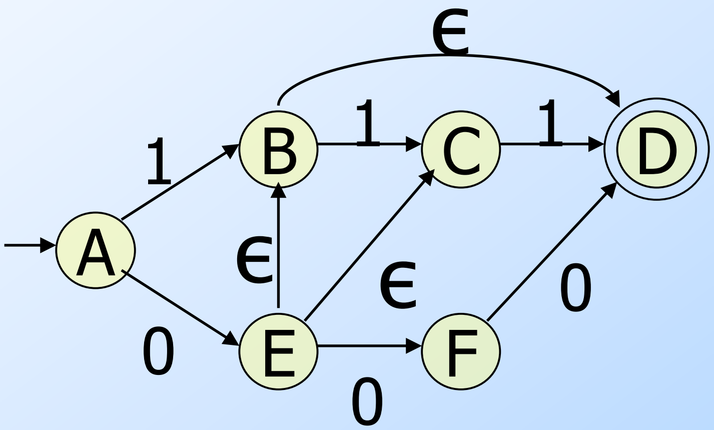
- Closure of a set of states = union of the closure of each state.

Extended Delta

- Basis: $\hat{\delta}(q, \epsilon) = CL(q)$.
- Induction: $\hat{\delta}(q, xa)$ is computed as follows:
  1. Start with $\hat{\delta}(q, x) = S$.
  2. Take the union of $CL(\delta(p, a))$ for all p in S.
- Intuition: $\hat{\delta}(q, w)$ is the set of states you can reach from q following a path labeled w.

And notice that $\delta(q, a)$ is *not* that set of states, for symbol a.

Example: Extended Delta



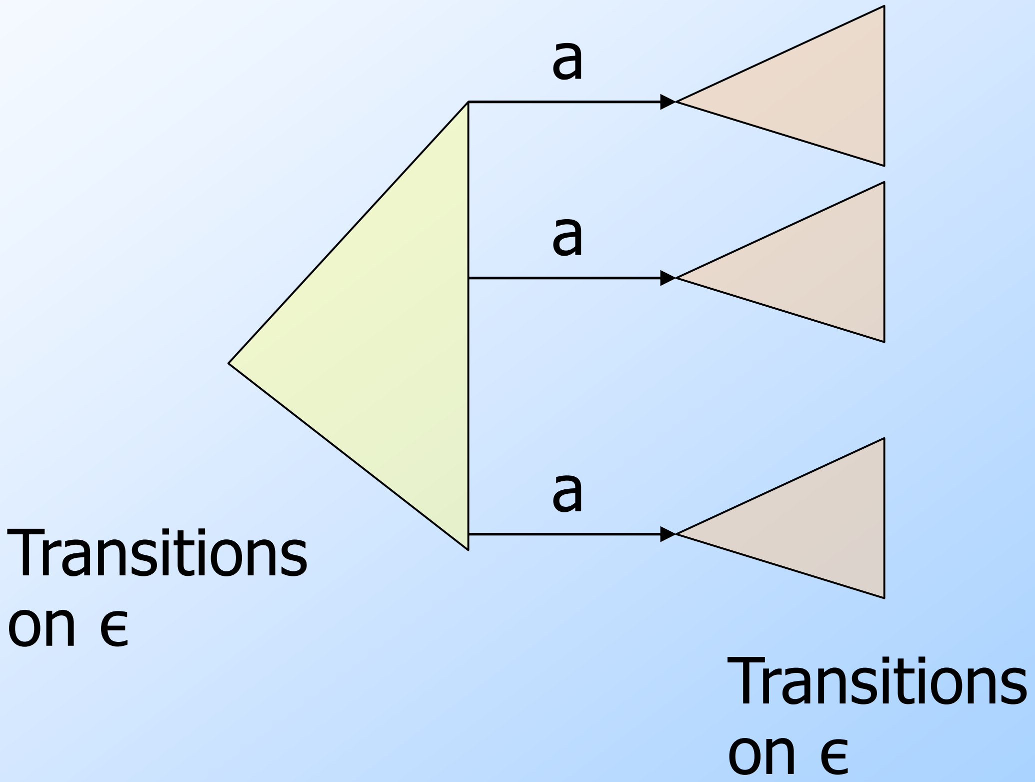
- $\hat{\delta}(A, \epsilon) = CL(A) = \{A\}$.
- $\hat{\delta}(A, 0) = CL(\{E\}) = \{B, C, D, E\}$.
- $\hat{\delta}(A, 01) = CL(\{C, D\}) = \{C, D\}$.
- *Language* of an ϵ-NFA is the set of strings w such that $\hat{\delta}(q_0, w)$ contains a final state.

Equivalence of NFA, ϵ-NFA

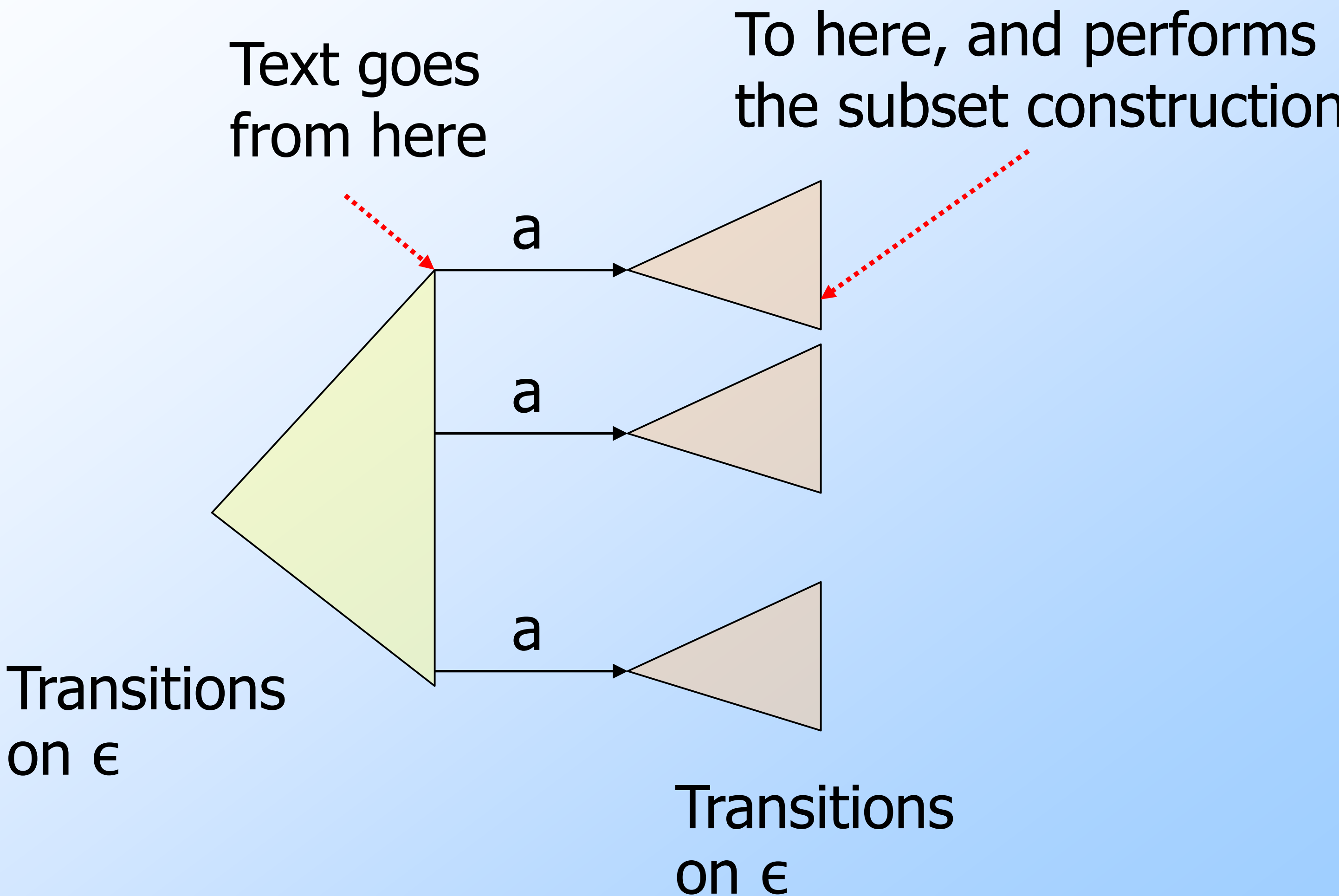
- Every NFA is an ϵ-NFA.
  - It just has no transitions on ϵ.
- Converse requires us to take an ϵ-NFA and construct an NFA that accepts the same language.
- We do so by combining ϵ–transitions with the next transition on a real input.

Warning: This treatment is a bit different from that in the text.

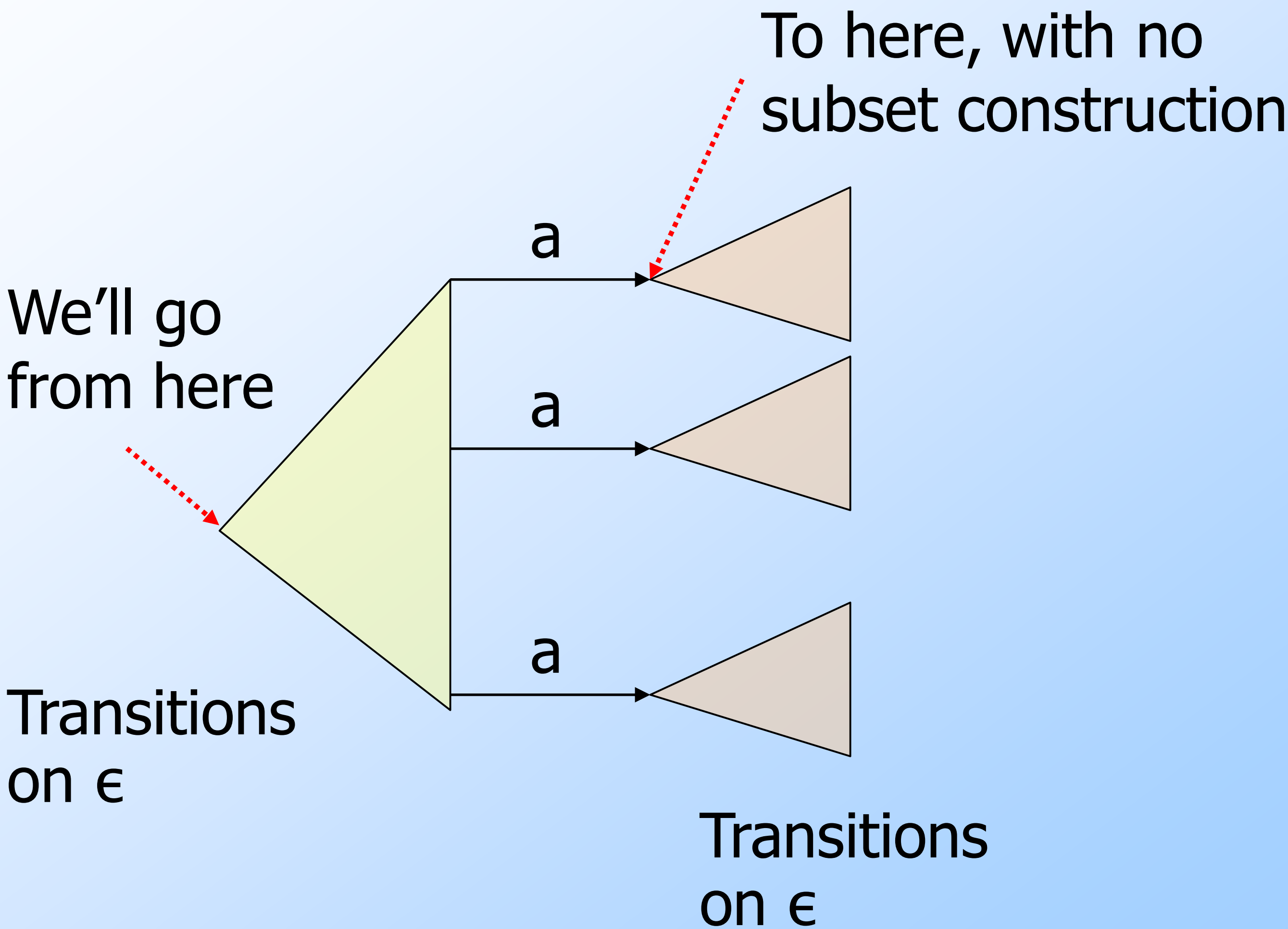
Picture of ϵ-Transition Removal



Picture of ϵ-Transition Removal



Picture of ϵ-Transition Removal



Equivalence – (2)

- Start with an ϵ-NFA with states Q, inputs Σ, start state q_0, final states F, and transition function δ_E.
- Construct an "ordinary" NFA with states Q, inputs Σ, start state q_0, final states F', and transition function δ_N.

Equivalence – (3)

- Compute $\delta_N(q, a)$ as follows:
  1. Let S = CL(q).
  2. $\delta_N(q, a)$ is the union over all p in S of $\delta_E(p, a)$.
- F' = the set of states q such that CL(q) contains a state of F.
- Intuition: δ_N incorporates ϵ–transitions before using *a* but not after.

Equivalence – (4)

- Prove by induction on |w| that

$$CL(\delta_N(q_0, w)) = \hat{\delta}_E(q_0, w).$$

- Thus, the ε-NFA accepts w if and only if the "ordinary" NFA does.

Example: ϵ-NFA-to-NFA

Interesting closures: CL(B) = {B,D}; CL(E) = {B,C,D,E}

	0	1	ϵ
→ A	{E}	{B}	∅
B	∅	{C}	{D}
C	∅	{D}	∅
* D	∅	∅	∅
E	{F}	∅	{B, C}
F	{D}	∅	∅

ϵ-NFA

	0	1
→ A	{E}	{B}
* B	∅	{C}
C	∅	{D}
* D	∅	∅
* E	{F}	{C, D}
F	{D}	∅

Since closures of B and E include final state D.

Since closure of E includes B and C; which have transitions on 1 to C and D.

Summary

- DFA's, NFA's, and ϵ–NFA's all accept exactly the same set of languages: the regular languages.
- The NFA types are easier to design and may have exponentially fewer states than a DFA.
- But only a DFA can be implemented!