Regular Expressions

Definitions

Equivalence to Finite Automata

Regular Expressions

- Notation to specify a language
  - Declarative
  - Sort of like a programming language.
    - Fundamental in some languages like perl and applications like grep or lex
  - Capable of describing the same thing as a NFA
    - The two are actually equivalent, so RE = NFA = DFA
  - We can define an algebra for regular expressions

Introduction

- The Language accepted by finite automata are easily described by simple expressions called regular expressions.

- The regular expression is the most effective way to represent any language.

- The language accepted by some regular expression is known as a regular language.

RE's: Introduction

- *Regular expressions* are an algebraic way to describe languages.
- They describe exactly the regular languages.
- If E is a regular expression, then L(E) is the language it defines.
- We'll describe RE's and their languages recursively.

Definition of a Regular Expression

- R is a regular expression if it is:
  1. **a** for some *a* in the alphabet $\sum$, standing for the language {a}
  2. ε, standing for the language {ε}
  3. Ø, standing for the empty language
  4. R_1+R_2 where R_1 and R_2 are regular expressions, and + signifies union (sometimes | is used)
  5. R_1R_2 where R_1 and R_2 are regular expressions and this signifies concatenation
  6. R* where R is a regular expression and signifies closure
  7. (R) where R is a regular expression, then a parenthesized R is also a regular expression

This definition may seem circular, but 1-3 form the basis

Precedence: Parentheses have the highest precedence, followed by *, concatenation, and then union.

RE's: Definition

- Basis 1: If *a* is any symbol, then **a** is a RE, and L(**a**) = {a}.
  - Note: {a} is the language containing one string, and that string is of length 1.
- Basis 2: ϵ is a RE, and L(ϵ) = {ϵ}.
- Basis 3: $\emptyset$ is a RE, and L($\emptyset$) = $\emptyset$.

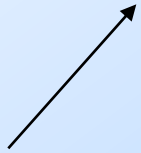
RE's: Definition – (2)

- Induction 1: If E_1 and E_2 are regular expressions, then E_1+E_2 is a regular expression, and $L(E_1+E_2) = L(E_1) \cup L(E_2)$.
- Induction 2: If E_1 and E_2 are regular expressions, then E_1E_2 is a regular expression, and $L(E_1E_2) = L(E_1)L(E_2)$.

Concatenation : the set of strings wx such that w Is in $L(E_1)$ and x is in $L(E_2)$.

RE's: Definition – (3)

- Induction 3: If E is a RE, then E* is a RE, and L(E*) = (L(E))*.



Closure, or "Kleene closure" = set of strings $w_1w_2...w_n$, for some $n \geq 0$, where each w_i is in L(E).
Note: when n=0, the string is ϵ.

Precedence of Operators

- Parentheses may be used wherever needed to influence the grouping of operators.
- Order of precedence is * (highest), then concatenation, then + (lowest).

Algebra for Languages

- Previously we discussed these operators:
  - Union
  - Concatenation
  - Kleene Star

Examples: RE's

- L(**01**) = {01}.
- L(**01**+**0**) = {01, 0}.
- L(**0**(**1**+**0**)) = {01, 00}.
  - Note order of precedence of operators.
- L(**0***) = {ϵ, 0, 00, 000,… }.
- L((**0**+**10**)*(ϵ+**1**)) = all strings of 0's and 1's without two consecutive 1's.

Identity Rules :

Let P,Q and R are the regular expressions then the identity r
are as follows:

1.φ +R = R

2.φR = Rφ = φ

3.€R = R € = R

4.€ * = € and φ *= €

5. R + R = R

6. R*R* = R*

7. RR* = R*R 8. (R*)*=R*

9. € + RR*= R* = € +R*R

10. (PQ)*P = P(QP)*

11. (P+Q)* = (P*Q*)* = (P* + Q*)

12. (P+Q)R = PR + QR and R(P+Q) = RP + RQ

- Exercise 1: Construct the regular expression for the language which accepts all the strings which begin or end with either 0 or 11.

Solution:

The R.E. can be categorized into two subparts.

$R = L_1 + L_2$

L1 = The String which begin with 00 or 11. L2 = The String wh end with 00 or 11.

Let us find out L_1 and L_2.

L_1 = (00+11)(any number of 0's and 1's) $L_1 = (00+11)(0+1)^*$

L_2 = (any number of 0's and 1's) (00+11) $L_2 = (0+1)^*(00+11)$

Hence

$R = [(00+11)(0+1)^*] + [(0+1)^*(00+11)]$

Set of strings which consists of event length where input alphabet Is {a,b}
Strings{ab,ba,aa,bb}

RE Examples

- L(**001**) = {001}
- L(**0+10***) = { 0, 1, 10, 100, 1000, 10000, … }
- L(**0*10***) = {1, 01, 10, 010, 0010, …} i.e. {w | w has exactly a single 1}
- L(∑∑)* = {w | w is a string of even length}
- L((**0(0+1)**)*) = { ε, 00, 01, 0000, 0001, 0100, 0101, …}
- L((**0+ε)(1+ ε**)) = {ε, 0, 1, 01}
- L(1Ø) = Ø ; concatenating the empty set to any set yields the empty set.
- Rε = R
- R+Ø = R

- Note that R+ε may or may not equal R (we are adding ε to the language)
- Note that RØ will only equal R if R itself is the empty set.

RE Exercise

- Exercise: Write a regular expression for the set of strings that contains an even number of 1's over $\Sigma=\{0,1\}$. Treat zero 1's as an even number.

Equivalence of FA and RE

- Finite Automata and Regular Expressions are equivalent. To show this:
  - Show we can express a DFA as an equivalent RE
  - Show we can express a RE as an ε-NFA. Since the ε-NFA can be converted to a DFA and the DFA to an NFA, then RE will be equivalent to all the automata we have described.

Turning a DFA into a RE

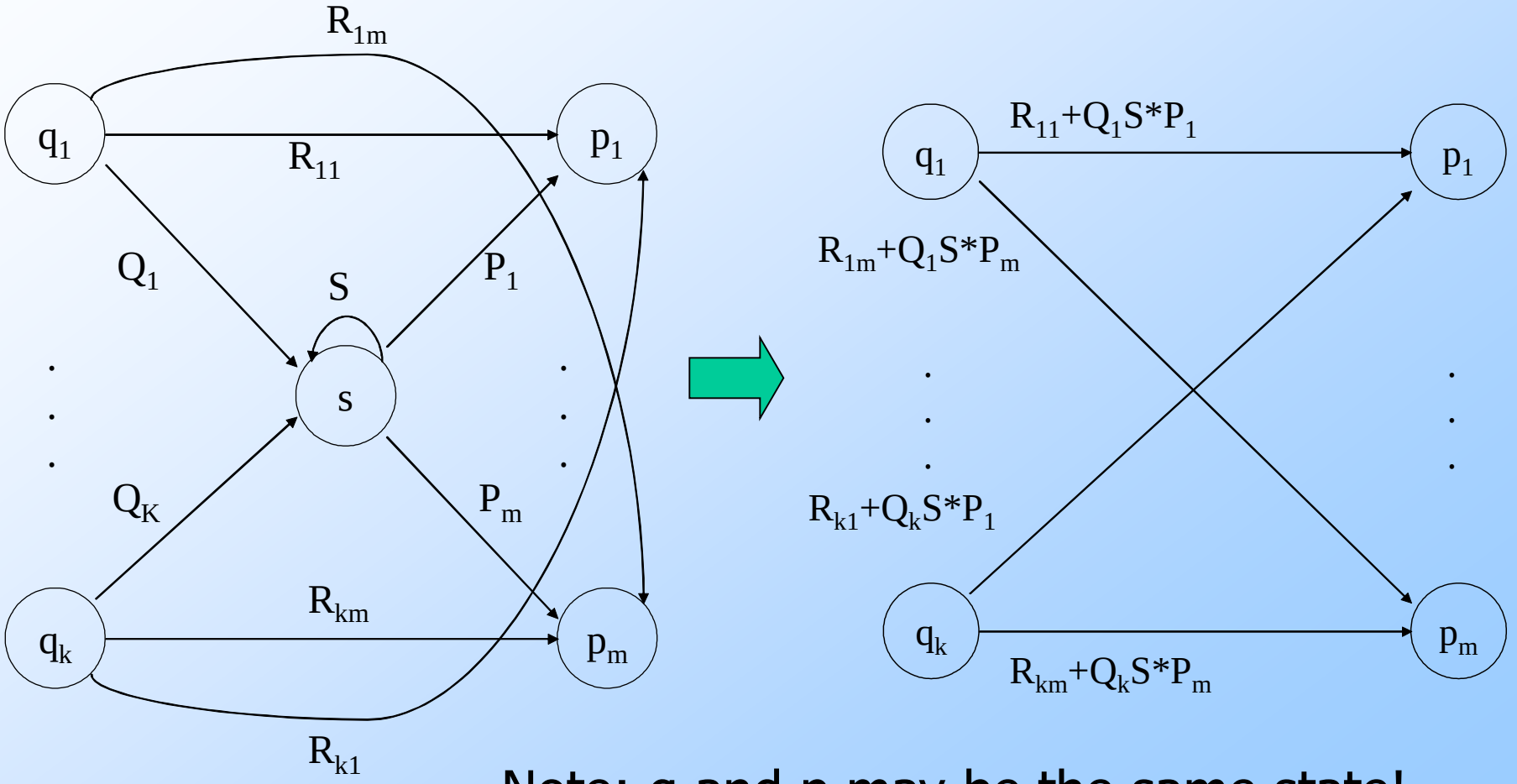
- Theorem: If L=L(A) for some DFA A, then there is a regular expression R such that L=L(R).
- Proof
  - Construct GNFA, Generalized NFA
    - We'll skip this in class, but see the textbook for details
  - State Elimination
    - We'll see how to do this next, easier than inductive construction, there is no exponential number of expressions

DFA to RE: State Elimination

- Eliminates states of the automaton and replaces the edges with regular expressions that includes the behavior of the eliminated states.
- Eventually we get down to the situation with just a start and final node, and this is easy to express as a RE

State Elimination

- Consider the figure below, which shows a generic state s about to be eliminated. The labels on all edges are regular expressions.
- To remove s, we must make labels from each q_i to p_1 up to p_m that include the paths we could have made through s.



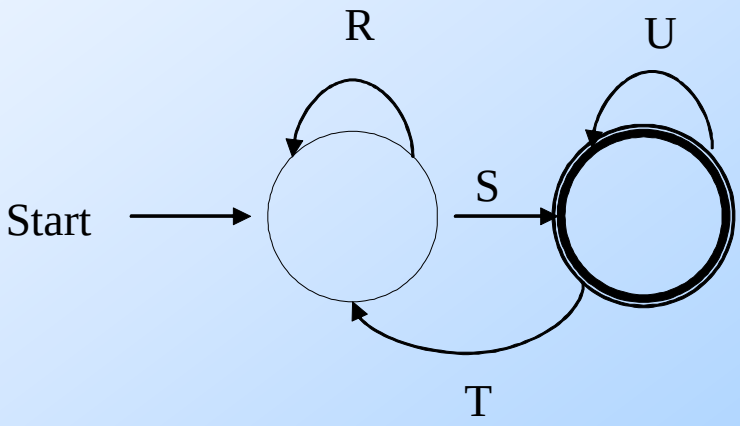
Note: q and p may be the same state!

DFA to RE via State Elimination (1)

1. Starting with intermediate states and then moving to accepting states, apply the state elimination process to produce an equivalent automaton with regular expression labels on the edges.
   - The result will be a one or two state automaton with a start state and accepting state.

DFA to RE State Elimination (2)

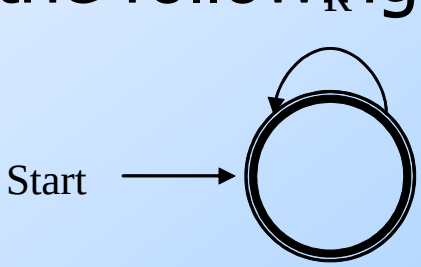
2. If the two states are different, we will have an automaton that looks like the following:



We can describe this automaton as: (R+SU*T)*SU*

DFA to RE State Elimination (3)

3. If the start state is also an accepting state, then we must also perform a state elimination from the original automaton that gets rid of every state but the start state. This leaves the following:



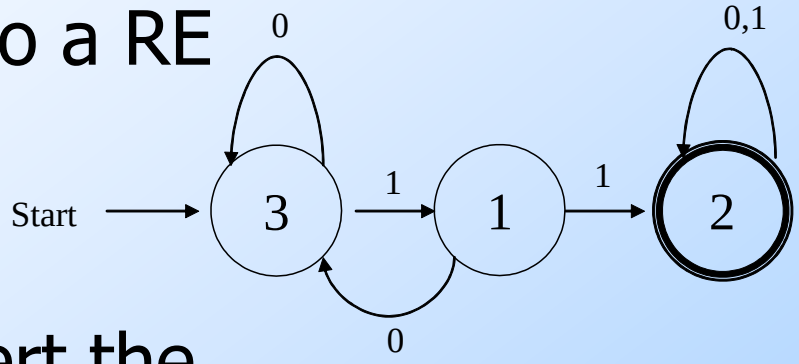
We can describe this automaton as simply R*.

DFA to RE State Elimination (4)

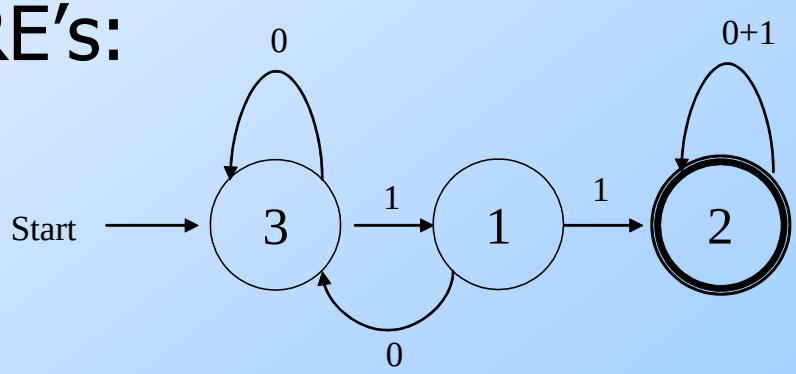
4. If there are n accepting states, we must repeat the above steps for each accepting states to get n different regular expressions, R_1, R_2, … R_n. For each repeat we turn any other accepting state to non-accepting. The desired regular expression for the automaton is then the union of each of the n regular expressions: $R_1 \cup R_2 \ldots \cup R_N$

DFA→RE Example

- Convert the following to a RE

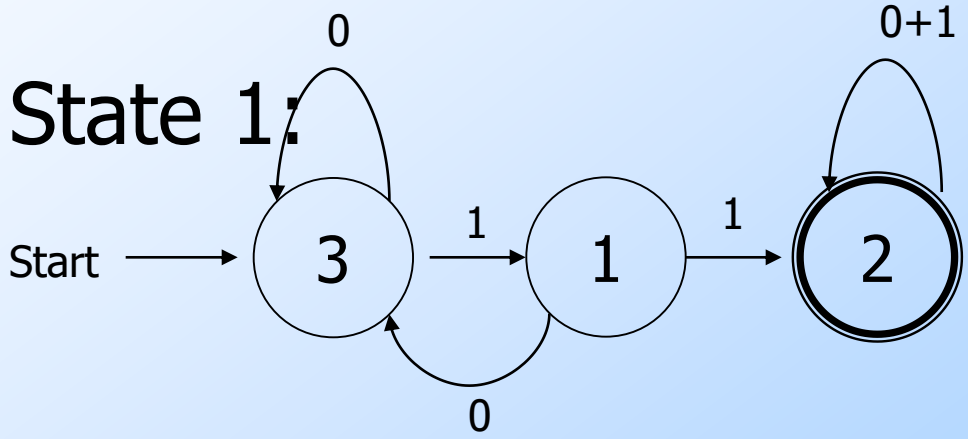


- First convert the edges to RE's:



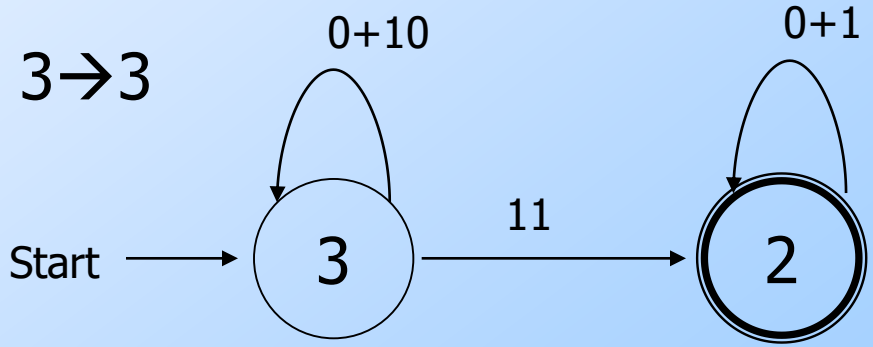
DFA → RE Example (2)

- Eliminate State 1:



- To:

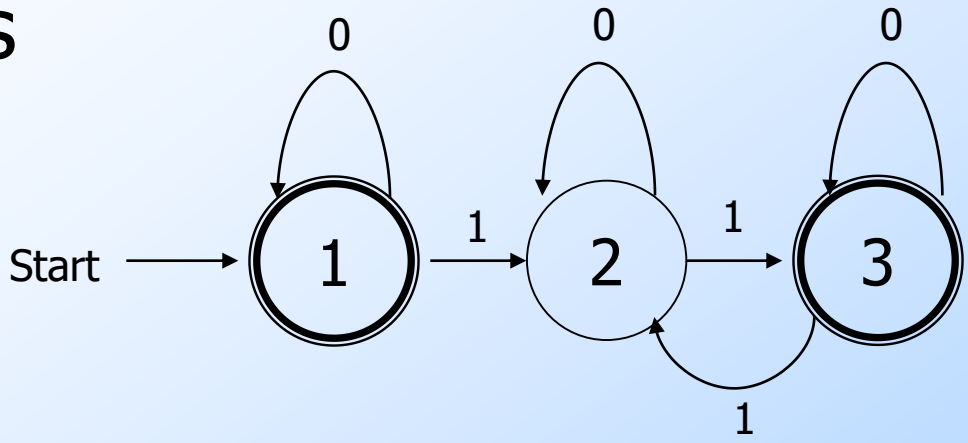
Note edge from 3→3



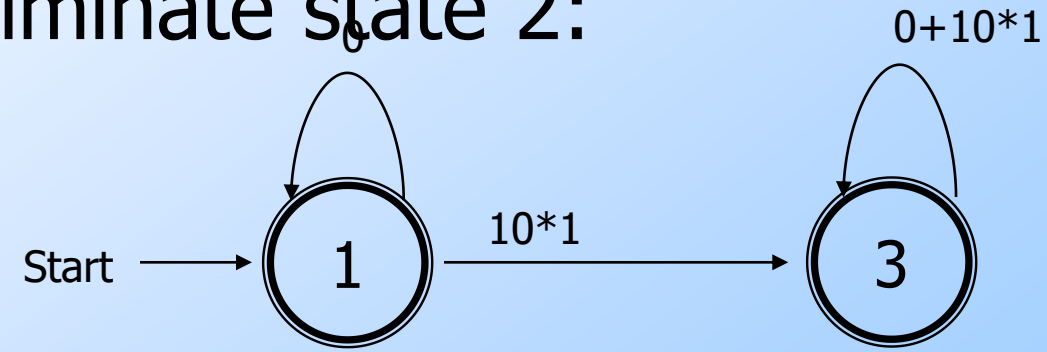
Answer: (0+10)*11(0+1)*

Second Example

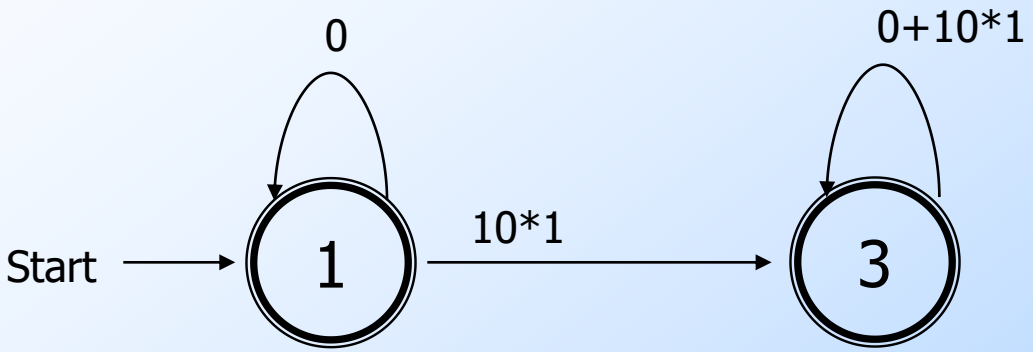
- Automata that accepts even number of 1's



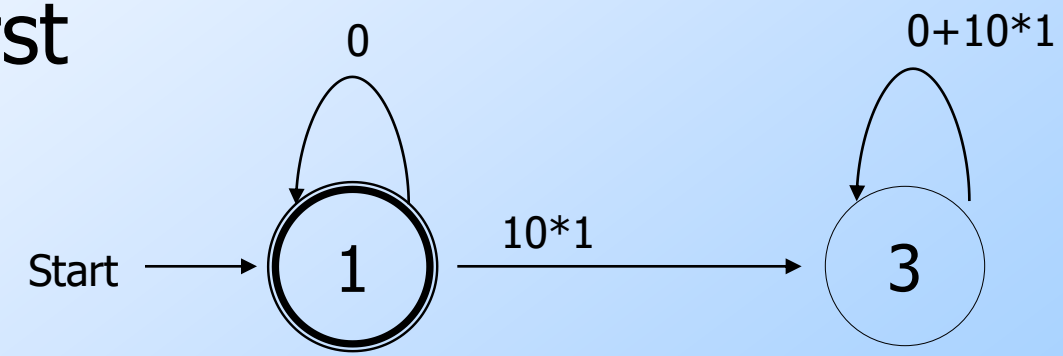
- Eliminate state 2:



Second Example (2)

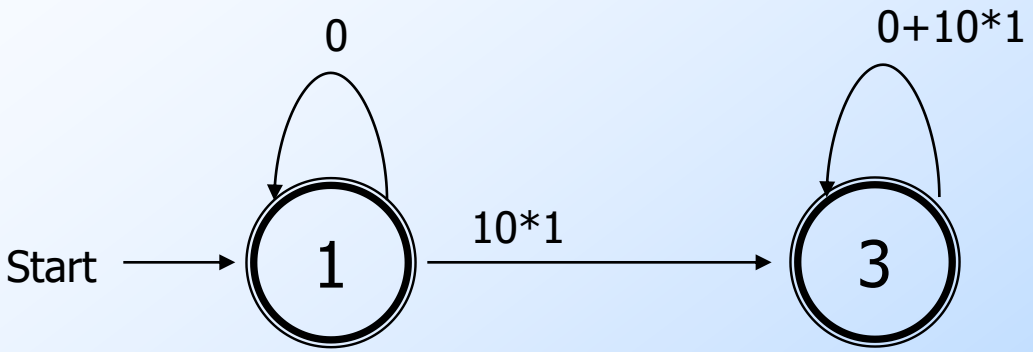


- Two accepting states, turn off state 3 first

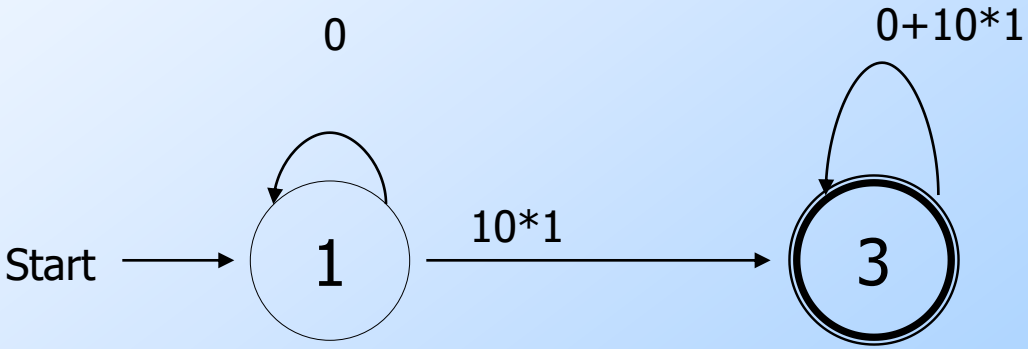


This is just 0*; can ignore going to state 3 since we would "die

Second Example (3)



- Turn off state 1 second:



This is just 0*10*1(0+10*1)*

Combine from previous slide to get 0* + 0*10*1(0+10*1)*

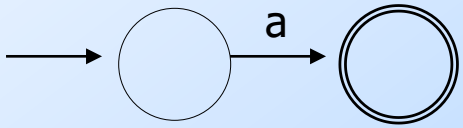
Converting a RE to an Automata

- We have shown we can convert an automata to a RE. To show equivalence we must also go the other direction, convert a RE to an automaton.
- We can do this easiest by converting a RE to an ε-NFA
  - Inductive construction
  - Start with a simple basis, use that to build more complex parts of the NFA

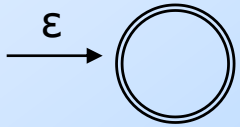
RE to ε-NFA

- Basis:

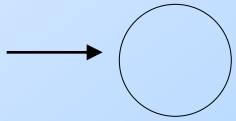
R=a



R=ε

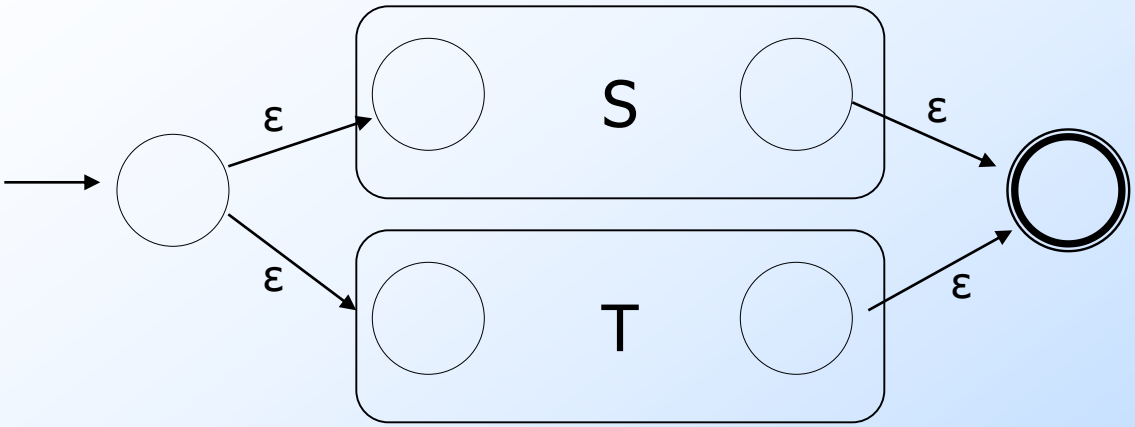


R=Ø

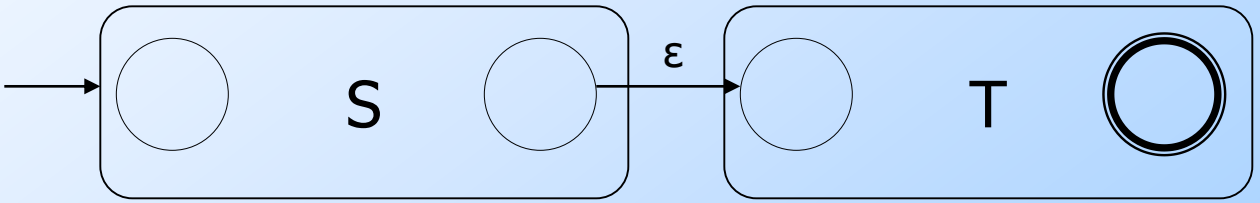


Next slide: More complex RE's

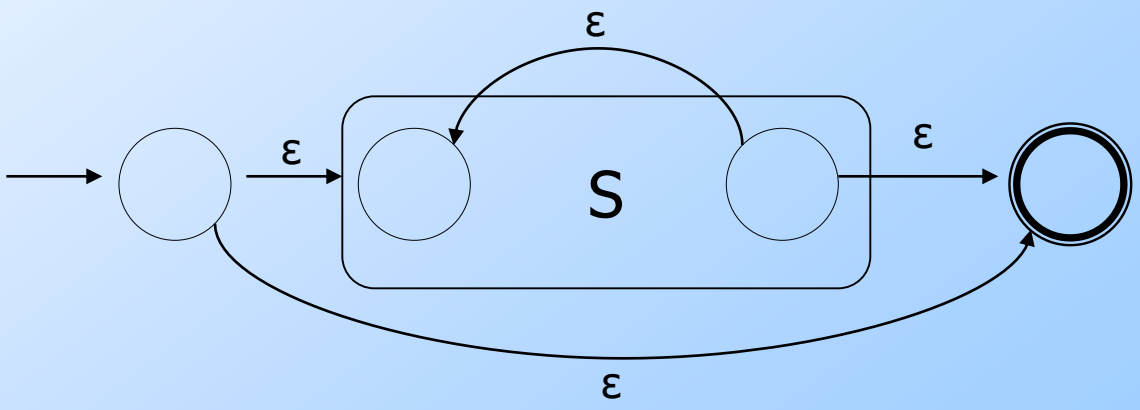
R=S+T



R=ST



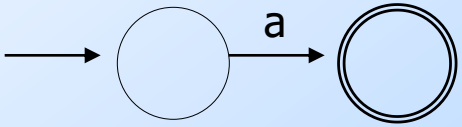
R=S*



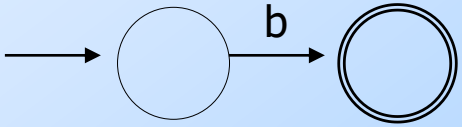
RE to ε-NFA Example

- Convert R= (ab+a)* to an NFA
  - We proceed in stages, starting from simple elements and working our way up

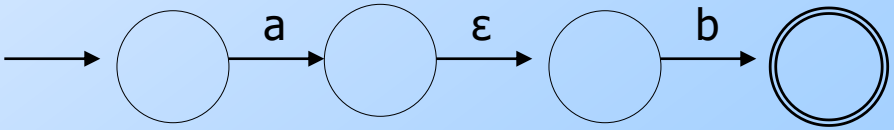
a



b

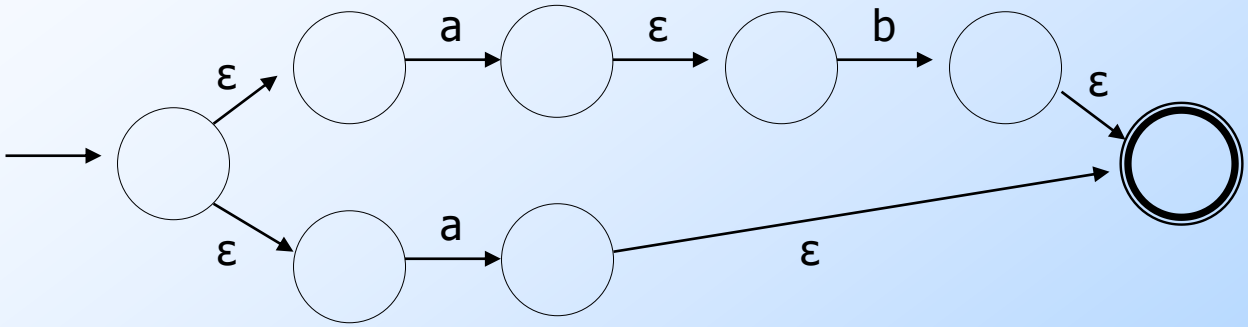


ab

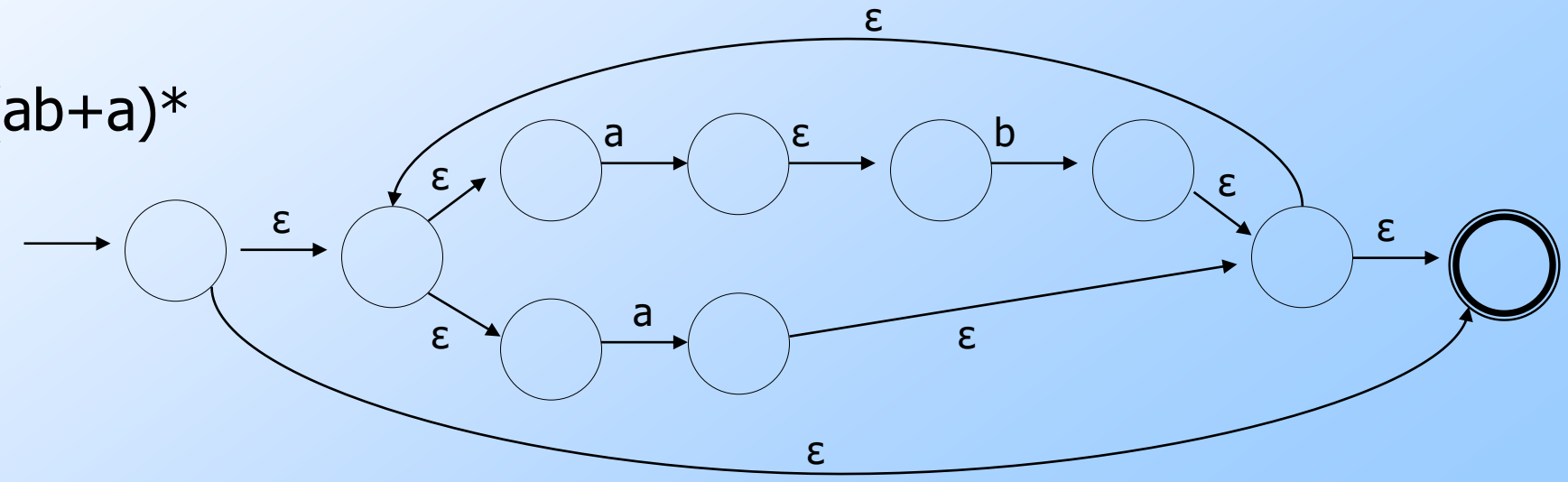


RE to ε-NFA Example (2)

ab+a



(ab+a)*



What have we shown?

- Regular expressions and finite state automata are really two different ways of expressing the same thing.
- In some cases you may find it easier to start with one and move to the other
  - E.g., the language of an even number of one's is typically easier to design as a NFA or DFA and then convert it to a RE

Algebraic Laws for RE's

- Just like we have an algebra for arithmetic, we also have an algebra for regular expressions.
  - While there are some similarities to arithmetic algebra, it is a bit different with regular expressions.

Algebra for RE's

- Commutative law for union:
  - $L + M = M + L$
- Associative law for union:
  - $(L + M) + N = L + (M + N)$
- Associative law for concatenation:
  - $(LM)N = L(MN)$
- Note that there is no commutative law for concatenation, i.e. $LM \neq ML$

Algebra for RE's (2)

- The identity for union is:
  - $L + \emptyset = \emptyset + L = L$
- The identity for concatenation is:
  - $L\varepsilon = \varepsilon L = L$
- The annihilator for concatenation is:
  - $\emptyset L = L\emptyset = \emptyset$
- Left distributive law:
  - $L(M + N) = LM + LN$
- Right distributive law:
  - $(M + N)L = LM + LN$
- Idempotent law:
  - $L + L = L$

Laws Involving Closure

- $(L^*)^* = L^*$
  - i.e. closing an already closed expression does not change the language
- $\emptyset^* = \varepsilon$
- $\varepsilon^* = \varepsilon$
- $L^+ = LL^* = L^*L$
  - more of a definition than a law
- $L^* = L^+ + \varepsilon$
- $L? = \varepsilon + L$
  - more of a definition than a law

Checking a Law

- Suppose we are told that the law

  $(R + S)^* = (R^*S^*)^*$

  holds for regular expressions. How would we check that this claim is true?

1. Convert the RE's to DFA's and minimize the DFA's to see if they are equivalent (we'll cover minimization later)
2. We can use the "concretization" test:
   - Think of R and S as if they were single symbols, rather than placeholders for languages, i.e., R = {0} and S = {1}.
   - Test whether the law holds under the concrete symbols. If so, then this is a true law, and if not then the law is false.

Concretization Test

- For our example

  $(R + S)^* = (R^*S^*)^*$

  We can substitute 0 for R and 1 for S.

  The left side is clearly any sequence of 0's and 1's. The right side also denotes any string of 0's and 1's, since 0 and 1 are each in L(0*1*).

Concretization Test

- NOTE: extensions of the test beyond regular expressions may fail.
- Consider the "law" $L \cap M \cap N = L \cap M$.
- This is clearly false
  - Let L=M={a} and N=Ø. {a} ≠ Ø.
  - But if L={a} and M = {b} and N={c} then
  - L∩M does equal $L \cap M \cap N$ which is empty.
  - The test would say this law is true, but it is not because we are applying the test beyond regular expressions.
- We'll see soon various languages that do not have corresponding regular expressions.

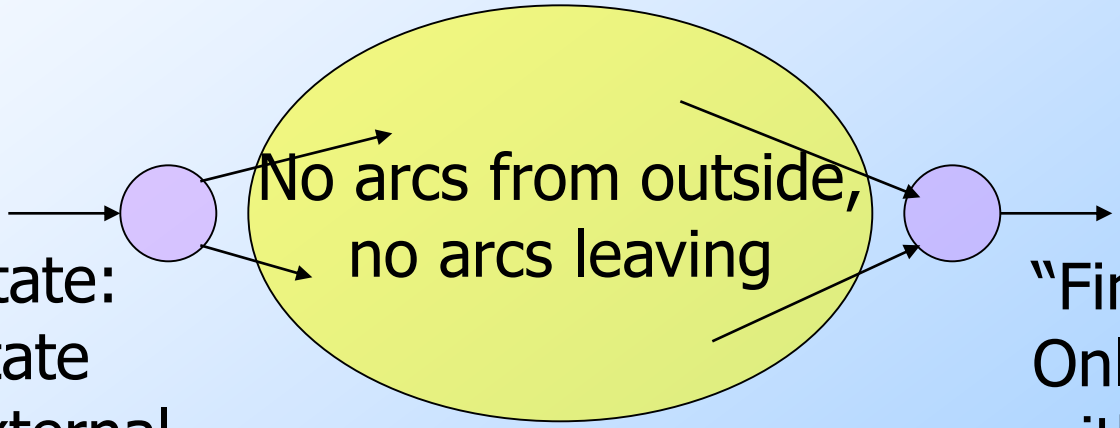
Equivalence of RE's and Automata

- We need to show that for every RE, there is an automaton that accepts the same language.
  - Pick the most powerful automaton type: the ϵ-NFA.
- And we need to show that for every automaton, there is a RE defining its language.
  - Pick the most restrictive type: the DFA.

Converting a RE to an ϵ-NFA

- Proof is an induction on the number of operators (+, concatenation, *) in the RE.
- We always construct an automaton of a special form (next slide).

Form of ϵ-NFA's Constructed

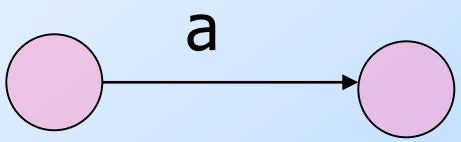


Start state:
Only state
with external
predecessors

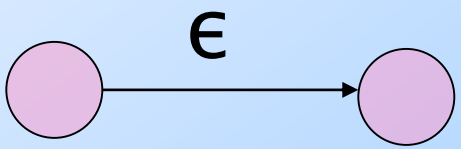
"Final" state:
Only state
with external
successors

RE to ϵ-NFA: Basis

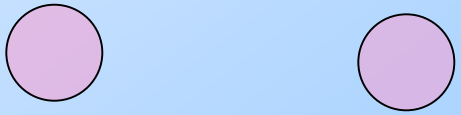
- Symbol **a**:



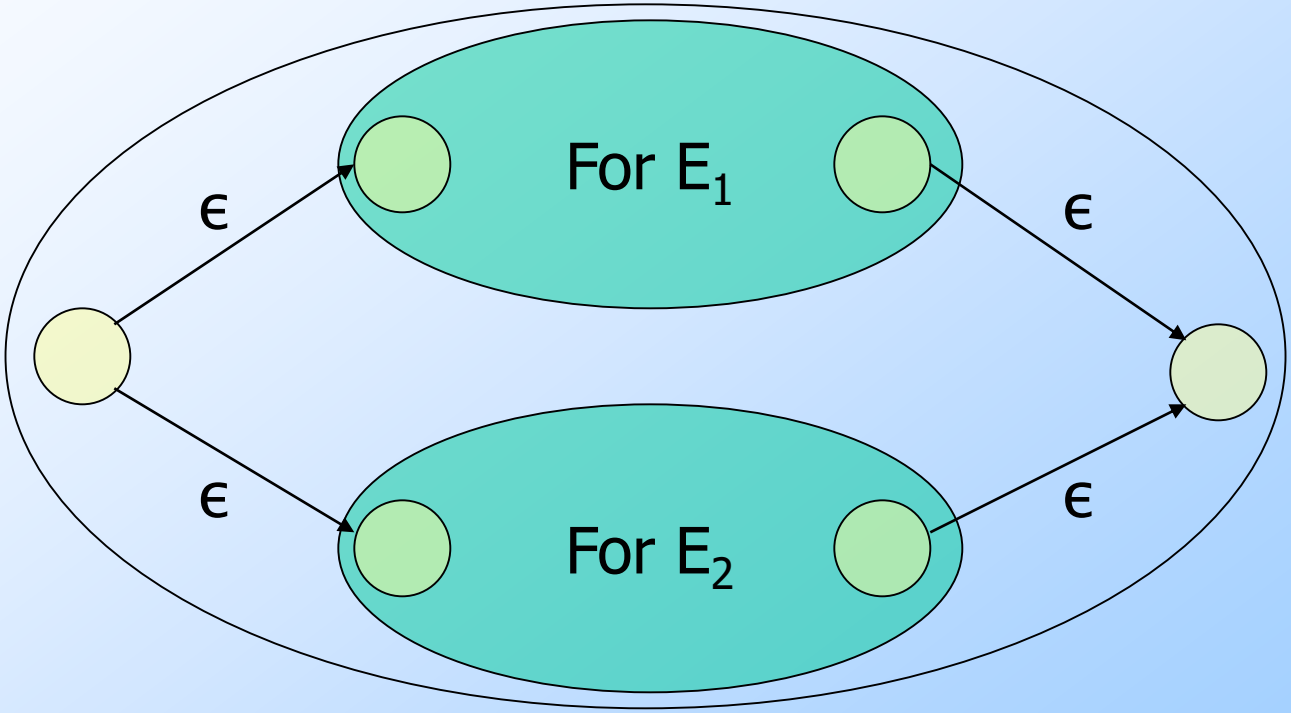
- ϵ:



- ∅:

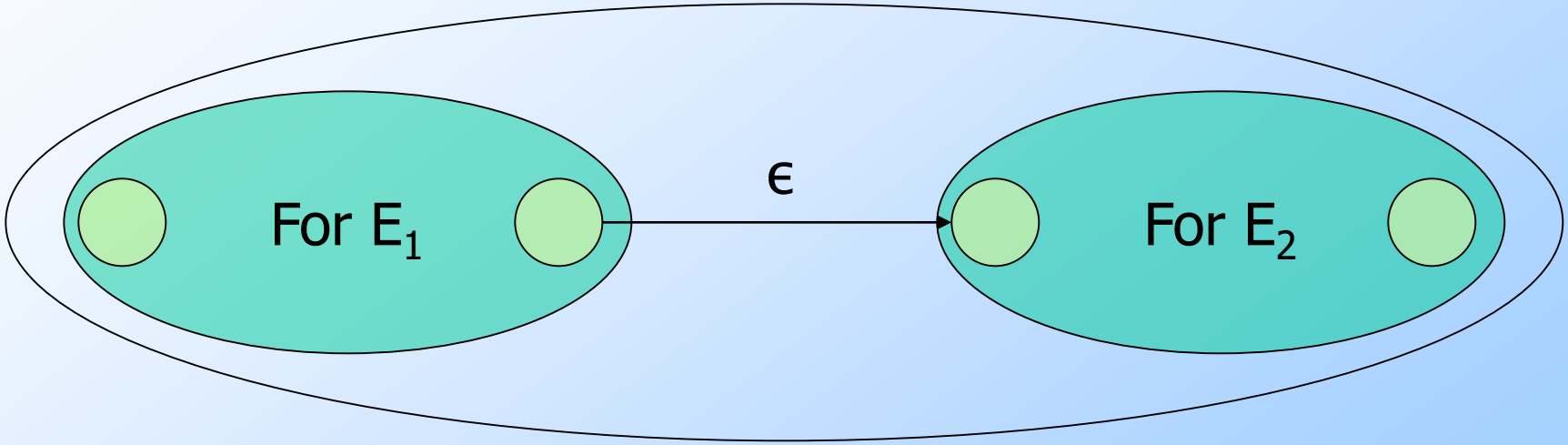


RE to ϵ-NFA: Induction 1 – Union



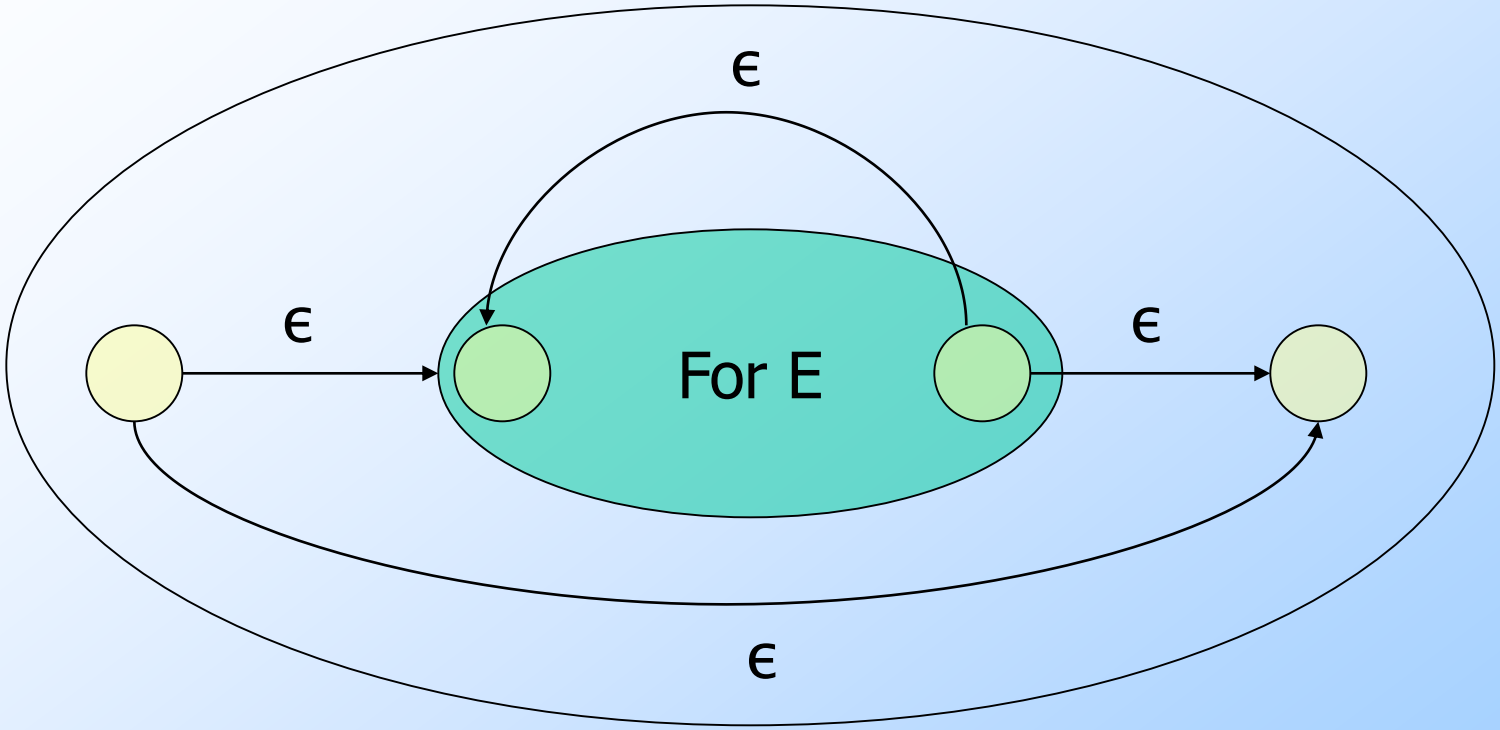
For $E_1 \cup E_2$

RE to ϵ-NFA: Induction 2 – Concatenation



For E_1E_2

RE to ϵ-NFA: Induction 3 – Closure



For E*

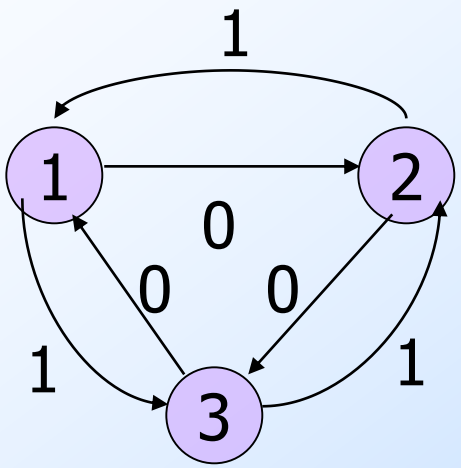
DFA-to-RE

- A strange sort of induction.
- States of the DFA are assumed to be 1,2,…,n.
- We construct RE's for the labels of restricted sets of paths.
  - Basis: single arcs or no arc at all.
  - Induction: paths that are allowed to traverse next state in order.

k-Paths

- A k-path is a path through the graph of the DFA that goes though no state numbered higher than k.
- Endpoints are not restricted; they can be any state.

Example: k-Paths



0-paths from 2 to 3:
RE for labels = **0**.

1-paths from 2 to 3:
RE for labels = **0**+**11**.

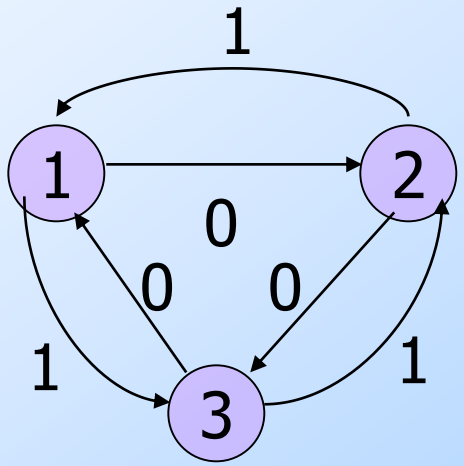
2-paths from 2 to 3:
RE for labels =
(**10**)*__0__+**1**(**01**)*__1__

3-paths from 2 to 3:
RE for labels = ??

k-Path Induction

- Let R_{ij}^k be the regular expression for the set of labels of k-paths from state i to state j.
- Basis: k=0. R_{ij}^0 = sum of labels of arc from i to j.
  - $\emptyset$ if no such arc.
  - But add ϵ if i=j.

Example: Basis



- $R_{12}^{0} = \mathbf{0}$.
- $R_{11}^{0} = \emptyset + \epsilon = \epsilon$.

k-Path Inductive Case

- A k-path from i to j either:
  1. Never goes through state k, or
  2. Goes through k one or more times.

$$R_{ij}^{k} = R_{ij}^{k-1} + R_{ik}^{k-1}(R_{kk}^{k-1})^{*} R_{kj}^{k-1}.$$

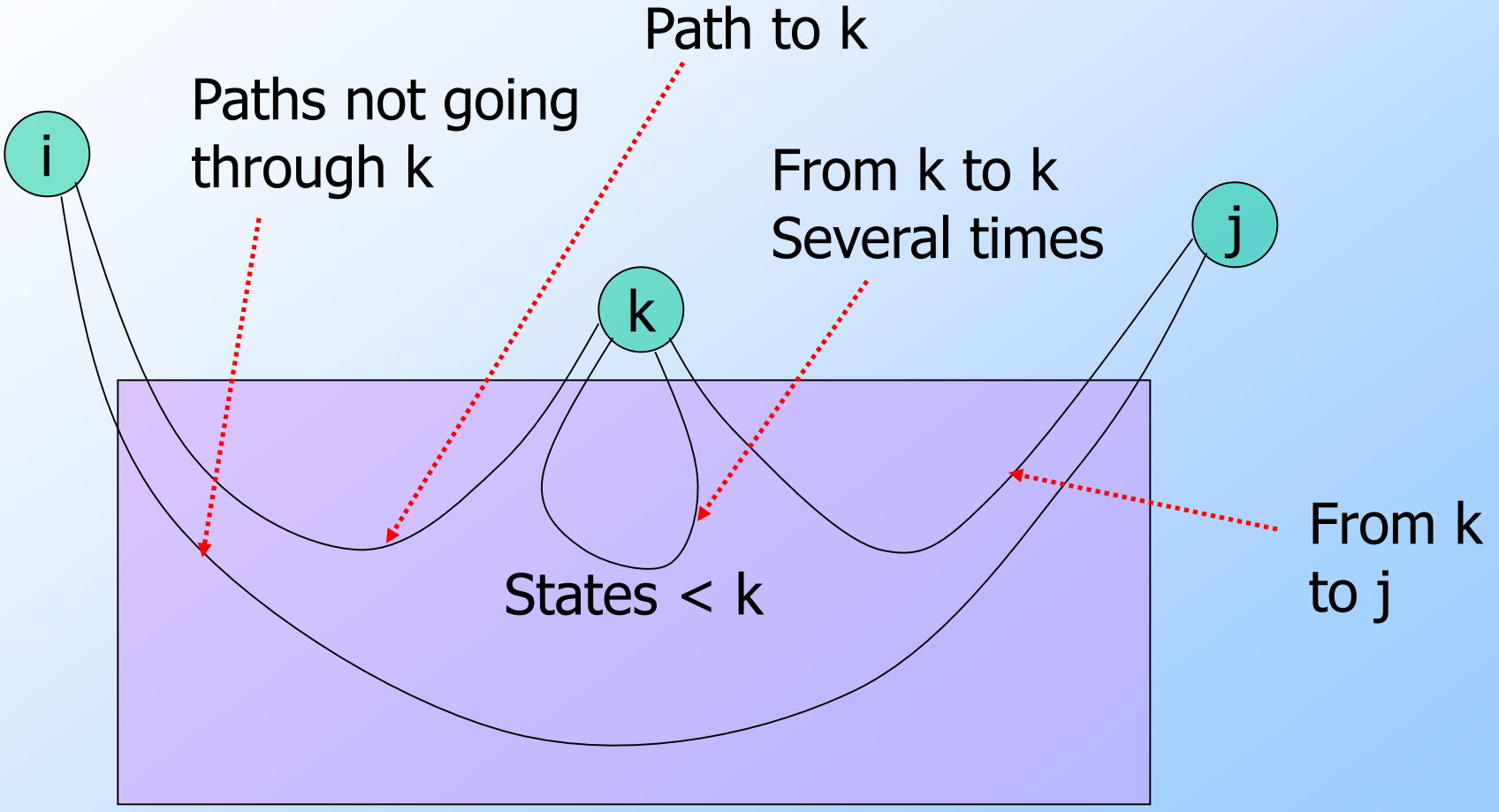
Doesn't go through k

Goes from i to k the first time

Zero or more times from k to k

Then, from k to j

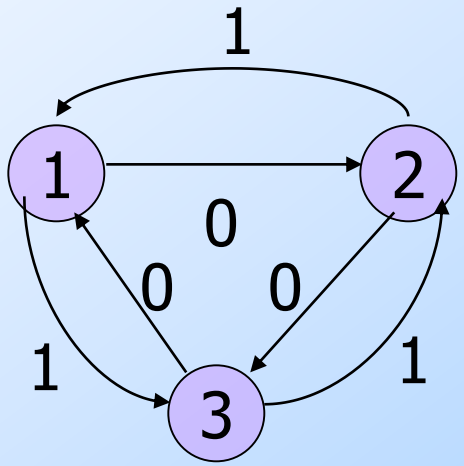
Illustration of Induction



Final Step

- The RE with the same language as the DFA is the sum (union) of R_{ij}^{n}, where:
  1. n is the number of states; i.e., paths are unconstrained.
  2. i is the start state.
  3. j is one of the final states.

Example



- $R_{23}^3 = R_{23}^2 + R_{23}^2(R_{33}^2)^*R_{33}^2 = R_{23}^2(R_{33}^2)^*$
- $R_{23}^2 = (\mathbf{10})^*\mathbf{0}+\mathbf{1}(\mathbf{01})^*\mathbf{1}$
- $R_{33}^2 = \mathbf{0}(\mathbf{01})^*(\mathbf{1}+\mathbf{00}) + \mathbf{1}(\mathbf{10})^*(\mathbf{0}+\mathbf{11})$
- $R_{23}^3 = [(\mathbf{10})^*\mathbf{0}+\mathbf{1}(\mathbf{01})^*\mathbf{1}]$ $[\mathbf{0}(\mathbf{01})^*(\mathbf{1}+\mathbf{00}) + \mathbf{1}(\mathbf{10})^*(\mathbf{0}+\mathbf{11}))]^*$

Summary

- Each of the three types of automata (DFA, NFA, ϵ-NFA) we discussed, and regular expressions as well, define exactly the same set of languages: the regular languages.

Algebraic Laws for RE's

- Union and concatenation behave sort of like addition and multiplication.
  - + is commutative and associative; concatenation is associative.
  - Concatenation distributes over +.
  - Exception: Concatenation is not commutative.

Identities and Annihilators

- $\emptyset$ is the identity for +.
  - $R + \emptyset = R$.
- ϵ is the identity for concatenation.
  - $\epsilon R = R\epsilon = R$.
- $\emptyset$ is the annihilator for concatenation.
  - $\emptyset R = R\emptyset = \emptyset$.