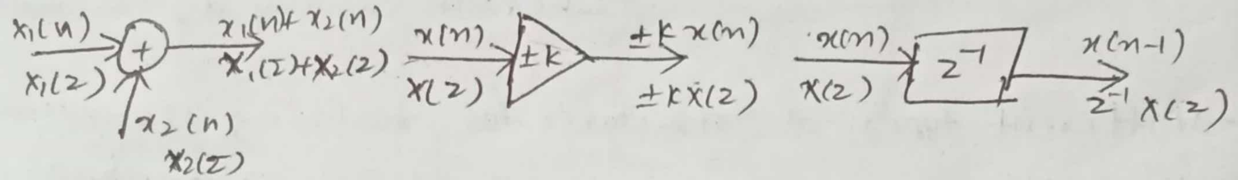
UNIT-2

* Realization of Discrete-Time Systems:-

To realize a discrete-time system, the given difference eqn in time domain is to be converted to an algebraic eqn in z-domain and each term of eqn is to be represented by a suitable element.

Then using adders, all elements representing various terms of eqn are to be connected to obtain o/p. The symbols of the basic elements used for constructing the block diagram of DT system (adder, constant multiplier and unit delay element) are:-



* Realization of IIR SYSTEMS:

IIR systems are Infinite impulse response systems.

- Convolution formula for IIR systems is given by

$$\boxed{y(n) = \sum_{k=0}^{\infty} h(k)\cdot x(n-k)}$$

which involves present and past i/p samples, we can say that IIR system has an infinite memory.

→ It is also called as "Recursive system".

* TF of IIR SYSTEM:-

- IIR system is described by difference equation:

$$y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=0}^{M} b_k \cdot x(n-k)$$

- Taking Z-Transform, we get

$$Y(z) = -\sum_{k=1}^{N} a_k \cdot z^{-k}\, Y(z) + \sum_{k=0}^{M} b_k \cdot z^{-k} \cdot X(z)$$

$$\Rightarrow Y(z) + \sum_{k=1}^{N} a_k \cdot z^{-k}\, Y(z) = \sum_{k=0}^{M} b_k \cdot z^{-k}\, X(z)$$

⇒ TF of IIR system is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{1+\sum_{k=1}^{N} a_k \cdot z^{-k}} = \frac{b_0+b_1 z^{-1}+b_2 z^{-2}+\dots+b_M z^{-M}}{1+a_1 z^{-1}+a_2 z^{-2}+\dots+a_N z^{-N}}$$

- Factors that influence the choice of structure for realization of LTI system are:
  - Computational complexity (No. of arithmetic operations)
  - Memory requirements (No. of memory locations required)
  - Finite word length effects in computation. (quantization effect that are inherent in any digital implementation)

⇒ Different types of structures for realizing IIR systems are
- Direct form-I structure
- Direct form-II structure
- Cascade form structure
- Parallel form structure

① DIRECT FORM-I STRUCTURE: It is the direct implementation of difference eqn/TF. It is simplest & most straight forward realization structure available.

- Difference eqn. of an IIR system is

$$y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=0}^{M} b_k \cdot x(n-k)$$

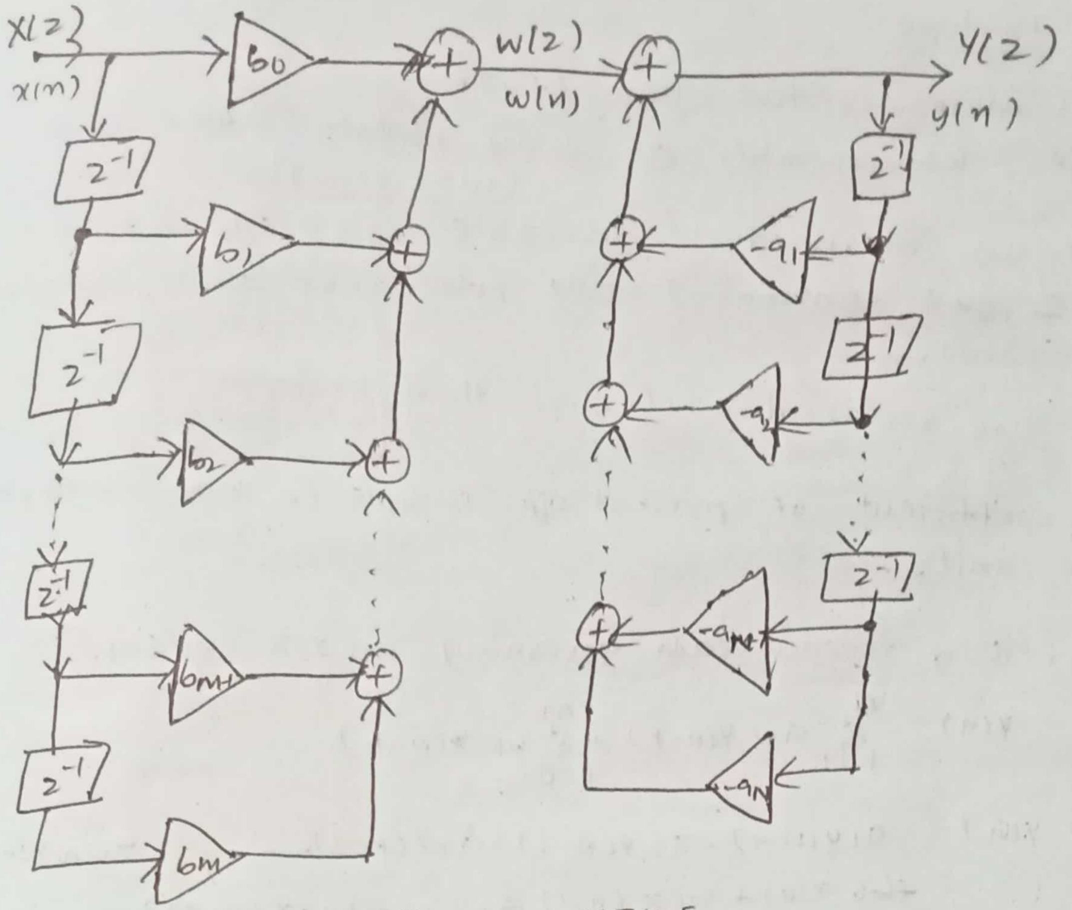
$$\Rightarrow y(n) = -a_1 y(n-1) - a_2 y(n-2) - \dots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \dots + b_M x(n-M)$$

Taking ZT, we get:

$$\Rightarrow \boxed{Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \dots - a_N z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \dots + b_M z^{-M} X(z)}$$

This eqn. can be directly represented by a block diagram as shown below, which is called DF-I structure.

- More memory is required
- This form provides a direct relation b/w time domain and z-domain eqns.



DF-I STRUCTURE

- It is called non-canonical structure, bcz no. of delay elements $(M+N)$ is more than order of difference eqn.
- No. of Adders = $M+N$
- No. of multipliers = $M+N+1$

- If IIR system is more complex, that is higher order, then introduce new variable $W(z)$, so that

$$W(z) = \sum_{k=0}^{M} b_k z^{-1} X(z) = b_0 X(z) + b_1 z^{-1} X(z) + \dots + b_M z^{-M} X(z)$$

$$\therefore \quad Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \dots + W(z)$$

$\Rightarrow$ DF-I structure is in 2 parts. The first part contains only zeros and 2nd part contains only poles.
In DF-I, the zeros are realised first and poles are realised second.

Limitations:-

- Not effective
- Lacks Hardware flexibility
- Chances of instability due to quantization noise.

② DF-II structure:

- It is more effective than DF-I.
- It uses less number of delay elements than DF-I. $(\max(M,N))$
- Here, an intermediate variable is used and TF is split into 2, one containing only poles and other containing only zeros.
- Poles are realized first and then zeros.

If coefficient of present o/p sample is non unity, transform it to unity.

. Consider difference eqn governing an IIR system.

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$$

$$\Rightarrow y(n) = -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) - \dots - a_N y(n-N)$$
$$+ b_0 x(n) + b_1 x(n-1) + \dots + b_M x(n-M).$$

ZT $\Rightarrow$ $$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \dots - a_N z^{-N} Y(z) + b_0 X(z)$$
$$+ b_1 z^{-1} X(z) + \dots + b_M z^{-M} X(z)$$

$$\Rightarrow Y(z)\left[1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}\right] = X(z)\left[b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots b_M z^{-M}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

$$\frac{Y(z)}{X(z)} = \underbrace{\frac{Y(z)}{W(z)}}_{\text{zeros}} \cdot \underbrace{\frac{W(z)}{X(z)}}_{\text{poles}}$$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

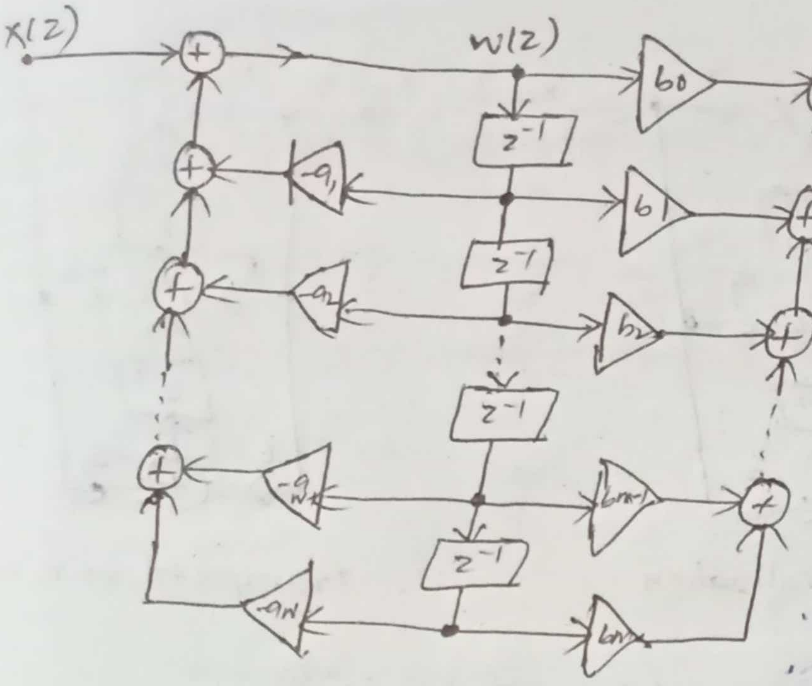
$$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + \dots + b_M z^{-M}$$

On cross multiplying, we get

$$W(z) + a_1 z^{-1} W(z) + \dots + a_N z^{-N} W(z) = X(z)$$

$$\therefore W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - \dots - a_N z^{-N} W(z)$$

and $$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + \dots + b_M z^{-M} W(z)$$



DF-II structure

Advantages:

No. of delay elements is less than that of DF-I.

LIMITATIONS:

- Lacks Hardware flexibility
- Chances of instability due to quantization noise.

- No. of delay elements used in DF-II is same as order of difference eqn. Hence also called as "Canonical structure".

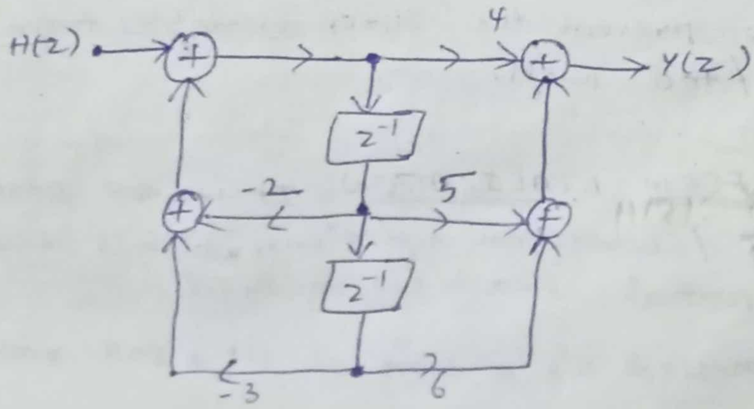
③ CASCADE FORM REALIZATION:

It is a cascaded or series interconnection of the sub TFs (or) sub system functions which are realized by using DF structures.

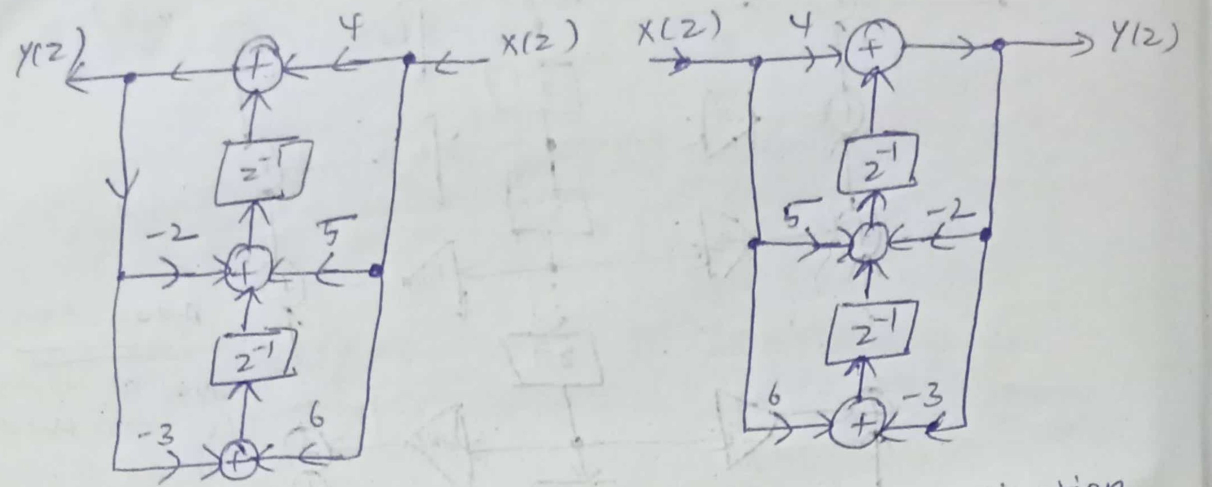
Hence, in cascade form realization, the given TF $H(z)$ is expressed as a product of a number of second order (or) first order sections as indicated below:

$$H(z) = \frac{Y(z)}{X(z)} = \prod_{i=1}^{K} H_i(z)$$

[Reduces range of coefficients]



DF-II realization

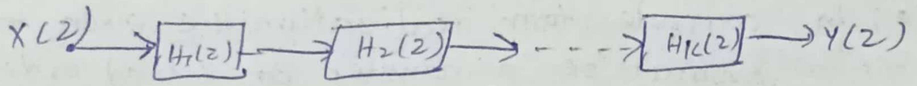


Recovered realization   Transposed realization

where $H_i(z) = \dfrac{C_{0i} + C_{1i}z^{-1} + C_{2i}z^{-2}}{d_{0i} + d_{1i}z^{-1} + d_{2i}z^{-2}}$ (second order section)

(or) $H_i(z) = \dfrac{C_{0i} + C_{1i}z^{-1}}{d_{0i} + d_{1i}z^{-1}}$ (first order section)

Each of these sections is realized seperately & all of them are connected in cascade. Therefore, cascade form is also called a series structure in which one sub TF is i/p to other TF and so on.



Difficulties:

1. Decision of pairing poles and zeros.
2. Deciding the order of cascading the 1st & 2nd order section
3. Scaling multipliers should be provided b/w individual sections to prevent the filter variables from becoming too large / too small.

④ PARALLEL FORM REALIZATION: It is the parallel connection of sub-TFs / subsystem functions, which is decomposed by using the partial fraction method.

• TF is expressed as a sum of 1st & 2nd order sections

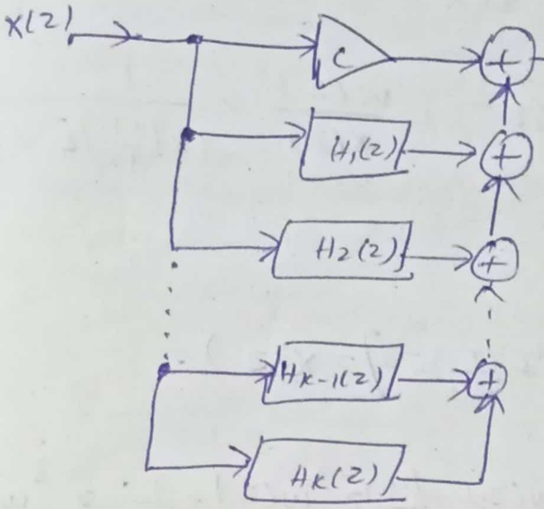
$$H(z) = \frac{Y(z)}{X(z)} = C + \sum_{i=1}^{K} H_i(z)$$

where,

$$H_i(z) = \frac{C_{0i} + C_{1i}z^{-1}}{d_{0i} + d_{1i}z^{-1} + d_{2i}z^{-2}} \quad \text{(2nd order section)}$$

$$H_i(z) = \frac{C_{0i}}{d_{0i} + d_{1i} z^{-1}}$$ (first order section]

- Each order section is realized either in DF-I structure or DF-II structure and individual sections are connected in parallel to obtain overall system as shown:



[Increases speed of realization]

- Difficulty
- expressing TF in partial fraction is not easy for higher order system.

Parallel form realization

Ex: Obtain DF-I, DF-II, cascade and parallel form realizations of LTI system governed by equation

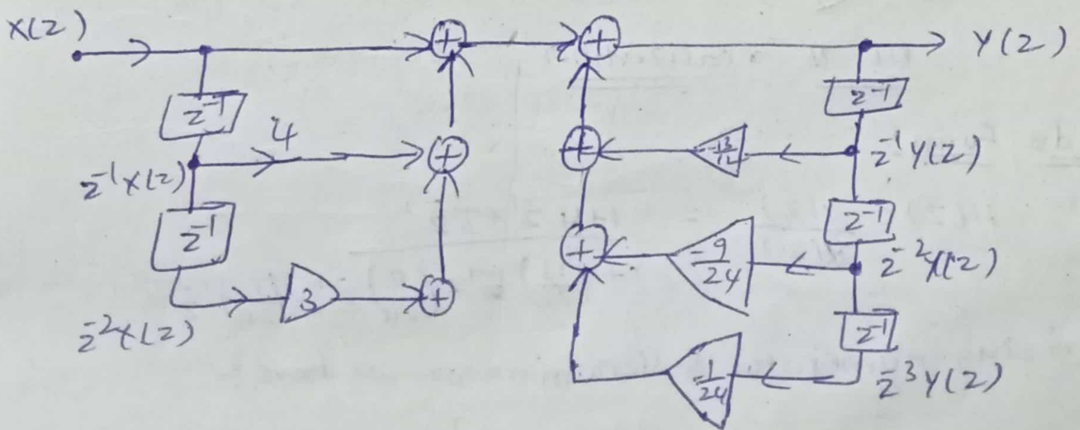
$$y(n) = -\frac{13}{12} y(n-1) - \frac{9}{24} y(n-2) - \frac{1}{24} y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$$

sol:

DF-I:

Given $y(n) = -\frac{13}{12} y(n-1) - \frac{9}{24} y(n-2) - \frac{1}{24} y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$

$$ZT \Rightarrow Y(z) = -\frac{13}{12} z^{-1} Y(z) - \frac{9}{24} z^{-2} Y(z) - \frac{1}{24} z^{-3} Y(z) + X(z) + 4z^{-1}X(z) + 3z^{-2}X(z) \quad \text{--- (1)}$$



DF-II: rewriting (1), we get

$$Y(z) + \frac{13}{12} z^{-1} Y(z) + \frac{9}{24} z^{-2} Y(z) + \frac{1}{24} z^{-3} Y(z) = X(z) + 4z^{-1}X(z) + 3z^{-2}X(z)$$

$$Y(z)\left[1 + \frac{13}{12}z^{-1} + \frac{9}{24}z^{-2} + \frac{1}{24}z^{-3}\right] = X(z)\left[1 + 4z^{-1} + 3z^{-2}\right]$$

TF $\Rightarrow$ $\dfrac{Y(z)}{X(z)} = \dfrac{1+4z^{-1}+3z^{-2}}{1+\left(\frac{13}{12}\right)z^{-1}+\left(\frac{9}{24}\right)z^{-2}+\left(\frac{1}{24}\right)z^{-3}}$

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

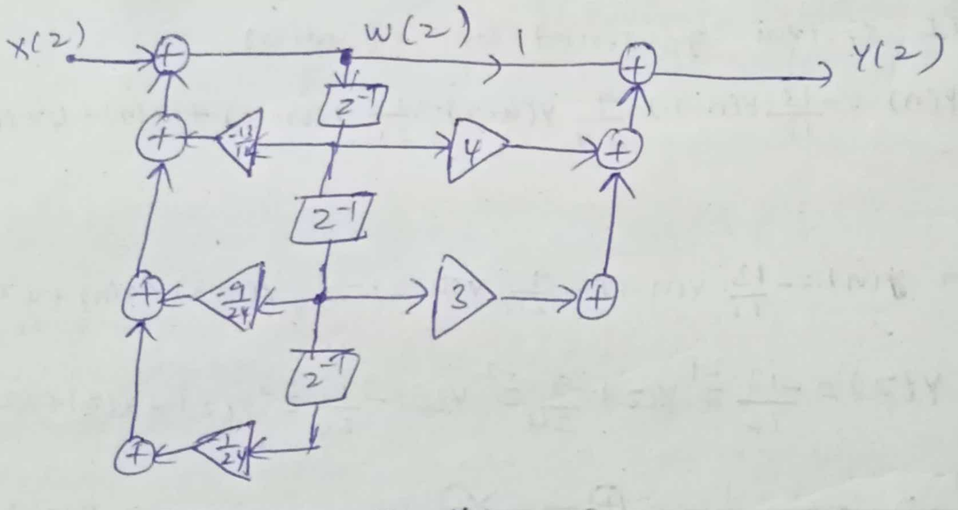
$\Rightarrow$ $\dfrac{Y(z)}{W(z)} = 1+4z^{-1}+3z^{-2}$; $\dfrac{W(z)}{X(z)} = \dfrac{1}{1+\left(\frac{13}{12}\right)z^{-1}+\left(\frac{9}{24}\right)z^{-2}+\left(\frac{1}{24}\right)z^{-3}}$

cross multiplying, we get:

$$W(z)\left[1+\left(\frac{13}{12}\right)z^{-1}+\left(\frac{9}{24}\right)z^{-2}+\left(\frac{1}{24}\right)z^{-3}\right] = X(z)$$

$\Rightarrow$ $W(z) = X(z) - \left(\frac{13}{12}\right)z^{-1}W(z) - \left(\frac{9}{24}\right)z^{-2}W(z) - \frac{1}{24}z^{-3}W(z)$

and $Y(z) = W(z) + 4z^{-1}W(z) + 3z^{-2}W(z)$



DF-II realization

Cascade Form:

TF is $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1+4z^{-1}+3z^{-2}}{1+\left(\frac{13}{12}\right)z^{-1}+\left(\frac{9}{24}\right)z^{-2}+\left(\frac{1}{24}\right)z^{-3}}$

Factorizing numerator & denominator, we have:

$$H(z) = \frac{(1+z^{-1})(1+3z^{-1})}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

since, there are 3 1st order factors in denominator of H(z), it can be expressed as:

$$H(z) = H_1(z) \cdot H_2(z) \cdot H_3(z)$$

where, $H_1(z) = \dfrac{Y_1(z)}{X(z)} = \dfrac{1+z^{-1}}{1+\frac{1}{2}z^{-1}}$

$H_2(z) = \dfrac{1+3z^{-1}}{1+\frac{1}{3}z^{-1}}$ and $H_3(z) = \dfrac{1}{1+\frac{1}{4}z^{-1}}$

* $H_1(z)$ can be realized in DF-II as:

$$H_1(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \cdot (1+z^{-1}) = \frac{Y_1(z)}{W_1(z)} \cdot \frac{W(z)}{X(z)}$$

$\Rightarrow \dfrac{W_1(z)}{X(z)} = \dfrac{1}{1+\frac{1}{2}z^{-1}} \Rightarrow X(z) = W_1(z)\left[1+\frac{1}{2}z^{-1}\right]$

$\Rightarrow W_1(z) = X(z) - \frac{1}{2}z^{-1}W_1(z)$ — (1)

and $\dfrac{Y_1(z)}{W_1(z)} = 1+z^{-1} \Rightarrow Y_1(z) = W_1(z) + z^{-1}W_1(z)$ — (2)

* $H_2(z)$ can also realized in DF-II as:

$$H_2(z) = \frac{Y_2(z)}{W_2(z)} \cdot \frac{W_2(z)}{Y_1(z)} = \frac{1+3z^{-1}}{1+\frac{1}{3}z^{-1}}$$

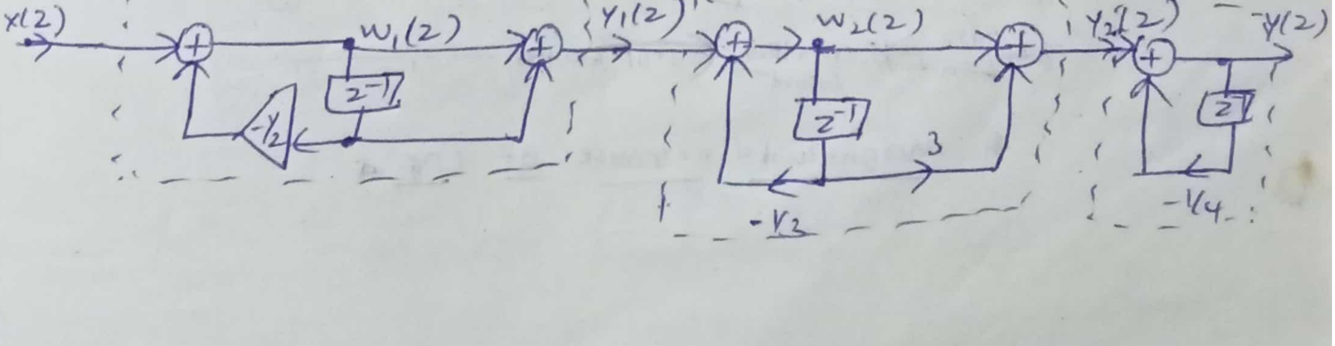
$\Rightarrow \dfrac{W_2(z)}{Y_1(z)} = \dfrac{1}{1+\frac{1}{3}z^{-1}} \Rightarrow W_2(z) = Y_1(z) - \frac{1}{3}z^{-1}W_2(z)$ — (3)

and $\dfrac{Y_2(z)}{W_2(z)} = 1+3z^{-1} \Rightarrow Y_2(z) = W_2(z) + 3z^{-1}W_2(z)$ — (4)

* $H_3(z)$ can be realized using DF-II as:

$$H_3(z) = \frac{Y(z)}{Y_2(z)} = \frac{1}{1+\frac{1}{4}z^{-1}}$$

$$\Rightarrow Y(z) = Y_2(z) - \frac{1}{4}z^{-1}Y(z)$$



(TGP)

- ## IIR Response Filters:-

-> Comparision of digital and analog filters:-

DIGITAL FILTER	ANALOG FILTER
1. It operates on digital samples of signal.	1. It operates on analog signal
2. It is governed by linear difference equations.	2. It is governed by linear differential eq
3. It consists of adders, multiplier & delay elements in digital logic.	3. It consists of electrical components like resistors, capacitors, and inductors
4. In digital filters, filter coefficients are designed to satisfy the desired frequency response.	4. Here, approximation problem is solved to satisfy desired FR.

- Transfer function of filter is given as; H(s)

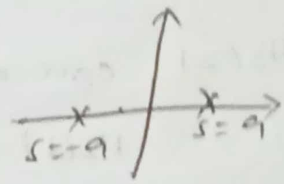
$$H(j\Omega) = H(s)|_{s=j\Omega}$$

-> Magnitude response is,

$$|H(j\Omega)|^2 = H(j\Omega).H(-j\Omega) = H(s)(-s)|_{s=j\Omega}$$

-> For stable system transfer function:

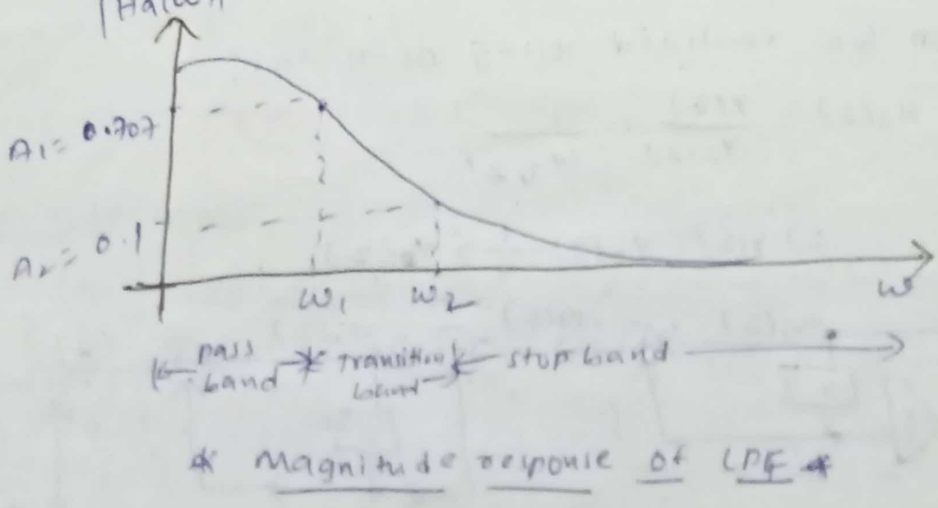
->
$$|H(j\Omega)|^2 = \frac{1}{a^2+\Omega^2} \quad ; \quad s = j\Omega, \quad \Omega = s/j$$



$$\Rightarrow H(s)H(-s) = \frac{1}{(a+s)(a-s)} = \frac{-1}{(s+a)(s-a)}$$

- Let us consider pole on LHS, $\Rightarrow H(s) = \frac{1}{s+a}$.

-> Frequency response of LPF is given as:



* Magnitude response of LPF *

596

* Design of Low-pass digital butterworth filter:-

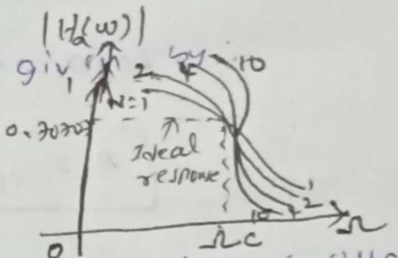
· Designing IIR digital filter involves the design of equivalent analog filter and then converting the analog filter to digital filter. Hence to design a Butterworth IIR digital filter, first analog Butterworth filter TF is determined using given specification, then it is converted to a digital filter TF.

→ **Analog Butterworth filter:-**

It is designed by approximating the ideal frequency response using an error function. The error function is selected such that the magnitude is maximally flat in passband and monotonically decreasing in stopband.

∴ The magnitude response of LPF is given

$$|H_a(\omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$



→ where, Ω_c is 3dB cutoff frequency & N is order of filter.

→ **Order of filter:-** (No need, written in next pages)

To determine the frequency response of filter, 'N' has to estimated to satisfy the specifications which are given interms of 'A' (or) 'gain' attenuation 'α' at passband (or) stopband frequency as:

$$A_1 \le |H(\omega)| \le 1 \qquad 0 \le \omega \le \omega_1$$

$$|H(\omega)| \le A_2, \qquad \omega_2 \le \omega \le \pi$$

→ Order of filter is determined as below:

→ let Ω_1, Ω_2 be analog filter edge frequencies corresponding to digital frequencies ω_1, ω_2.

$$\therefore\ A_1^2 \le \frac{1}{1+\left(\frac{\Omega_1}{\Omega_c}\right)^{2N}} \le 1 \ ; \quad \frac{1}{1+\left(\frac{\Omega_2}{\Omega_c}\right)^{2N}} \le A_2^2$$

$$\Rightarrow \left(\frac{\Omega_1}{\Omega_c}\right)^{2N} \le \frac{1}{A_1^2} - 1 \quad \text{—(1)} \ ; \quad \left(\frac{\Omega_2}{\Omega_c}\right)^{2N} \ge \frac{1}{A_2^2} - 1 \quad \text{—(2)}$$

$$\frac{①}{②} \Rightarrow \left(\frac{\Omega_1}{\Omega_2}\right)^{2N} = \frac{\frac{1}{A_1^2} - 1}{\frac{1}{A_2^2} - 1}$$

$$\Rightarrow N = \frac{\frac{1}{2}\log\left\{\left(\frac{1}{A_2^2} - 1\right) / \left(\frac{1}{A_1^2} - 1\right)\right\}}{\log \frac{\Omega_2}{\Omega_1}} \quad — (A)$$

and $\Omega_c = \dfrac{\Omega_1}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}}$ — (B) ; A_1, A_2 are in dB.

$\rightarrow A_1\,dB = -20\log A_1$

$\therefore \log A_1 = -\dfrac{A_1 dB}{20}$

$$\boxed{A_1 = 10^{-(A_1 dB)/20}}$$

$$\therefore \frac{1}{A_1^2} - 1 = \frac{1}{\left[10^{\frac{-A_1 dB}{20}}\right]^2} - 1$$

i.e. $$\frac{1}{A_1^2} - 1 = 10^{0.1 A_1 dB} - 1 \quad — ③$$

lly $$\frac{1}{A_2^2} - 1 = 10^{0.1 A_2 dB} - 1 \quad — ④$$

③, ④ in (A)

$$\Rightarrow N = \frac{\frac{1}{2}\cdot\log\left[\frac{10^{0.1 A_2 dB} - 1}{10^{0.1 A_1 dB} - 1}\right]}{\log\left(\frac{\Omega_2}{\Omega_1}\right)}$$

$$(B) \Rightarrow \Omega_c = \frac{1}{2}\left[\frac{\Omega_1}{\left(10^{0.1 A_1 dB} - 1\right)^{1/2N}} + \frac{\Omega_2}{\left[10^{0.1 A_2 dB} - 1\right]^{1/2N}}\right]$$

Transfer function:-

- Unnormarlized TF is given by:

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_k^2} \quad (N \text{ is even})$$

$$H_a(s) = \frac{\Omega_c}{s+\Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c + \Omega_k^2} \quad (N \text{ is odd})$$

$$b_k = 2\sin\left(\frac{(2k-1)\pi}{2N}\right)$$

→ If $\left(\frac{s}{\Omega_c}\right)$ is replaced by 's_n', then the normalised Butterworth filter TF is given by:

* $$H_a(s) = \prod_{k=1}^{N/2} \frac{1}{s_n^2 + b_k s_n + 1} \quad (N \text{ is even})$$

* $$H_a(s) = \frac{1}{s_n+1} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s_n^2 + b_k s_n + 1} \quad (N \text{ is odd})$$

→ Design procedure for LP digital Butterworth IIR Filter.

① Choose type of transformation (Bilinear / Impulse Invariant)

② calculate ratio of analog edge frequencies Ω_2/Ω_1.

- For Bilinear transformation,

* $\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2}$; $\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2}$ $\Rightarrow \frac{\Omega_2}{\Omega_1} = \frac{\tan \omega_2/2}{\tan \omega_1/2}$

- For impulse invariant,

* $\Omega_1 = \frac{\omega_1}{T}$; $\Omega_2 = \frac{\omega_2}{T} \Rightarrow \frac{\Omega_2}{\Omega_1} = \frac{\omega_2}{\omega_1}$

③ Decide order 'N' of filter, such that

$$N \geq \frac{1}{2} \frac{\log\left\{\left(\frac{1}{A_2^2}-1\right) \Big/ \left(\frac{1}{A_1^2}-1\right)\right\}}{\log\frac{\Omega_2}{\Omega_1}}$$

④ Calculate anolog cutoff frequency, $\Omega_c = \frac{\Omega_1}{\left(\frac{1}{A_1^2}-1\right)^{1/2N}}$

→ substitute Ω_1 according to transformation selected.

⑤ Determine TF of analog filter:

* If 'N' is even, $H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$

* If 'N' is odd, $H_a(s) = \frac{\Omega_c}{s+\Omega_c} \cdot \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$

where, $b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$

• For normalised case, $\Omega_c = 1$ rad/sec.

⑥ Using the chosen transformation, transform analog filter TF $H_a(s)$ to digital filter TF $H(z)$.

⑦ Realize the digital filter TF $H(z)$ by suitable stru

→ Poles of normalised Butterworth filter:—

w.k.T, ~~$|H_a(\Omega)|^2$~~ $= |H_a(\omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$

• N = odd ⇒ $S_k = 1 \angle \frac{k\pi}{\dots}$

N = even ⇒ $S_k = 1 \angle \frac{\pi}{2n} \dots$

⇒ $H_n(s) = \frac{1}{\prod(s-s_k)}$.

→ Frequency response is obtained by substituting $s = j\Omega$ in the analog TF $H_a(s)$.

⇒ System TF is obtained by replacing 'Ω' by '$\frac{s}{j}$' in eqn

$$\Rightarrow H_a(s)H_a(-s) = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}} = \frac{1}{1+\left(\frac{s^2}{j^2\Omega_c^2}\right)^N}$$

→ In above eqn, when 's/Ω_c' is replaced by 's_n', the TF is called normalized TF.

$$H_a(s_n) \cdot H_a(-s_n) = \frac{1}{1+(-s_n^2)^N} \,//$$

→ The TF of above eqn will have '2N' poles which are given by the roots of denominator polynomial.

The poles of a Butterworth ~~filter~~ polynomial lie on a circle, whose radius is ω_c. To determine the number of poles of Butterworth filter and the angle b/w them we use following rules.

. Number of Butterworth poles = 2N

Angle b/w 2 poles, $\theta = 360^\circ/(2N)$

→ If order of filter 'N' is even, location of first pole is at $\frac{\theta}{2}$. with angle measured in counter clock wise direction, and subsequent poles are at,

$$\left(\frac{\theta}{2}+\theta\right), \left(\frac{\theta}{2}+2\theta\right), \left(\frac{\theta}{2}+3\theta\right) - - - - - - \left(360-\frac{\theta}{2}\right)$$

→ If order is odd, location of first pole is on x-axis, and remaining are at $\theta, 2\theta, \ldots 360-\theta$.

→ If ϕ is angle of a pole w.r.to x-axis, then poles & its conjugate are at $[\omega_c(\cos\phi \pm j\sin\phi)]$.

Properties:–

1. Butterworth filters are all pole designs.
2. Filter order 'N' completely specifies the filter.
3. MR approaches ideal response as 'N' (↑).
4. Magnitude is maximally flat at origin.
5. Magnitude is monotonically decreasing function of Ω.
6. At Ω_c, magnitude $= \frac{1}{\sqrt{2}}$.

Q: Design a Butterworth digital filter using bilinear transformation. The specifications of the desired LPF are

$$0.9 \le |H(\omega)| \le 1 \; ; \quad 0 \le \omega \le \pi/2$$

$$|H(\omega)| \le 0.2 \; ; \quad \frac{3\pi}{4} \le \omega \le \pi \quad \text{with } T = 1s.$$

Sol: we have,

$$A_1 = 0.9, \; \omega_1 = \pi/2$$

$$A_2 = 0.2, \; \omega_2 = \frac{3\pi}{4} \text{ and } T = 1s.$$

① Here bilinear Transformation is specified.

② $\frac{\Omega_2}{\Omega_1}$:–

$$* \; \Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = \frac{2}{1}\tan\frac{\frac{\pi}{2}}{2} = 2$$

$$* \; \Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = \frac{2}{1}\tan\frac{3\pi}{8} = 4.828$$

$$\Rightarrow \frac{\Omega_2}{\Omega_1} = \frac{4.828}{2} = 2.414$$

③ $\underline{\underline{N}}$:

$$N \geq \frac{1}{2} \cdot \frac{\log\left\{\left[\frac{1}{A_2^2}-1\right] / \left[\frac{1}{A_1^2}-1\right]\right\}}{\log \frac{\Omega_2}{\Omega_1}}$$

$$\geq \frac{1}{2} \frac{\log\left\{\left[\frac{1}{(0.2)^2}-1\right] / \left[\frac{1}{(0.9)^2}-1\right]\right\}}{\log(1.207)}$$

$$\geq \frac{1}{2} \frac{\log(24/0.2345)}{\log 2.4}$$

$$N \geq 2.626$$

$$\Rightarrow \boxed{N=3} \quad \text{(Assume)}$$

④ $\underline{\underline{\Omega_c}}$:

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}} = \frac{2}{\left[\frac{1}{(0.9)^2}-1\right]^{1/2 \cdot 3}} = 2.5467$$

⑤ $\underline{\underline{H_a(s)}}$:

* $N=3 \Rightarrow$ odd

$$\therefore H_a(s) = \frac{\Omega_c}{s+\Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2+b_k\Omega_c s+\Omega_c^2}$$

$$, \quad \frac{N-1}{2} = \frac{3-1}{2} = 2$$

$$= \frac{\Omega_c}{s+\Omega_c} \cdot \frac{\Omega_c^2}{s^2+b_1\Omega_c s+\Omega_c^2}$$

$$, b_1 = 2\sin\left(\frac{(2k-1)\pi}{2N}\right)$$

$$= \left(\frac{2.54}{s+2.54}\right)\left(\frac{(2.54)^2}{s^2+1(2.54)s+(2.54)^2}\right)$$

$$= 2\sin\left(\frac{\pi}{6}\right)$$

$$\boxed{b_1 = 1}$$

⑥ $\underline{\underline{H_a(s) \rightarrow H_a(z)}}$:

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$[\because T=1]$$

$$\Rightarrow H(z) = \left(\frac{2.54}{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2.54}\right)\left[\frac{(2.54)^2}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2+2.54\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]+(2.54)^2}\right]$$

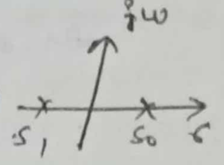
$$= \frac{0.2332(1+z^{-1})^3}{1+0.439z^{-1}+0.384z^{-2}+0.0416z^{-3}} //$$

* ~~DIGITAL FILTERS FROM ANALOG FILTERS~~ :-

→ TF from poles & order :

→ If $n = 1$, ⇒ $S_K = 1 \angle K\pi/n$ ⇒ $S_0 = 1\angle 0 = 1 + j \cdot 0$

$S_1 = 1 \angle \pi = -1 + j \cdot 0$



→ considering only left half splane, we get:

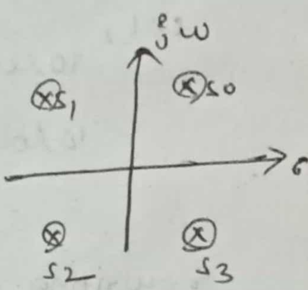
$$H_1(s) = \frac{1}{s - s_1} = \frac{1}{s - (-1)} = \frac{1}{s+1}$$

which is the TF of normalised LPF of order 1.

→ If $n = 2$, ⇒ $S_K = 1 \angle \frac{\pi}{2n} + \frac{K\pi}{n}$; $K = 0 \to 2n-1$

$= 0 \to 3$

* $S_0 = 1 \angle \frac{\pi}{4} = 0.707 + j0.707$
* $S_1 = 1 \angle \frac{\pi}{4} + \frac{\pi}{2} = -0.707 + j0.707$
* $S_2 = 1 \angle \frac{\pi}{4} + \pi = -0.707 - j0.707$
* $S_3 = 1 \angle \frac{\pi}{4} + \frac{3\pi}{2} = 0.707 - j0.707$



considering left half of s-plane, we get:

$$H_2(s) = \frac{1}{\pi(s - s_K)}$$

$$= \frac{1}{(s - s_1)(s - s_2)}$$

$$\boxed{H_2(s) = \frac{1}{s^2 + \sqrt{2}s + 1}}$$

for stability purpose, we consider poles on left half only

which is TF of normalised LPF of order 2.

→ If $n = 3$ ⇒ $S_K = 1 \angle \frac{K\pi}{n}$; $K = 0 \to 5$

after solving, we get:

$$\boxed{H_3(s) = \frac{1}{(s^2+s+1)(s+1)}}$$

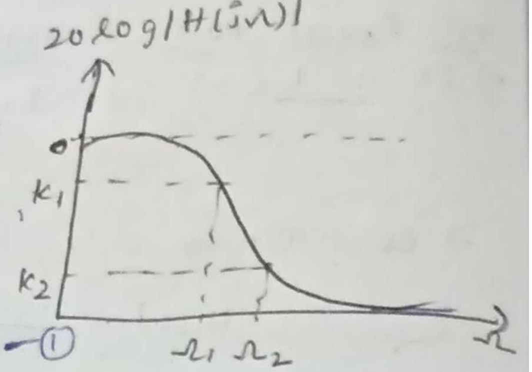
Ily, $$H_4(s) = \frac{1}{(s^2 + 0.765s + 1)(s^2 + 1.85s + 1)}$$

which is TF of normalised LPF of order 4.

Evaluation of order of LPF:

→ K_1, K_2 = Gains

→ Ω_1, Ω_2 = passband & stop band frequencies



MR of LPF

From the graph,

$$0 \geq 20\log|H(j\Omega)| \geq K_1 \quad ; \Omega \leq \Omega_1 \quad -①$$

and $$20\log|H(j\Omega)| \leq K_2 \quad ; \Omega \geq \Omega_2 \quad -②$$

→ Magnitude response is given by, $|H_n(j\Omega)|^2 = \dfrac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2n}}$; Ω_1, Ω_2

let, $10\log|H(j\Omega)|^2 = K_1$ at $\Omega = \Omega_1$ ⇒ $10\log\left[\dfrac{1}{1+\left(\frac{\Omega_1}{\Omega_c}\right)^{2n}}\right] = K_1$ — Ⓐ

$10\log|H(j\Omega)|^2 = K_2$ at $\Omega = \Omega_2$ ⇒ $10\log\left[\dfrac{1}{1+\left(\frac{\Omega_2}{\Omega_c}\right)^{2n}}\right] = K_2$ — Ⓑ

rewriting above equations, we get:

Ⓐ ⇒ $\log\left[\dfrac{1}{1+\left(\frac{\Omega_1}{\Omega_c}\right)^{2n}}\right] = \dfrac{K_1}{10} = \log_{10} 10^{\frac{K_1}{10}}$; Ⓑ ⇒ $\log\left[\dfrac{1}{1+\left(\frac{\Omega_2}{\Omega_c}\right)^{2n}}\right] = \dfrac{K_2}{10} = \log 10^{\frac{K_2}{10}}$

⇒ $$\frac{1}{1+\left(\frac{\Omega_1}{\Omega_c}\right)^{2n}} = 10^{K_1/10} \quad ; \quad \frac{1}{1+\left(\frac{\Omega_2}{\Omega_c}\right)^{2n}} = 10^{K_2/10}$$

⇒ $$1+\left(\frac{\Omega_1}{\Omega_c}\right)^{2n} = 10^{-K_1/10} \quad -③ \quad ; \quad 1+\left(\frac{\Omega_2}{\Omega_c}\right)^{2n} = 10^{-K_2/10} \quad -④$$

$\dfrac{③}{④}$ ⇒ $$\left(\frac{\Omega_1}{\Omega_2}\right)^{2n} = \frac{10^{-K_1/10} - 1}{10^{-K_2/10} - 1}$$

Taking log on both sides, then:

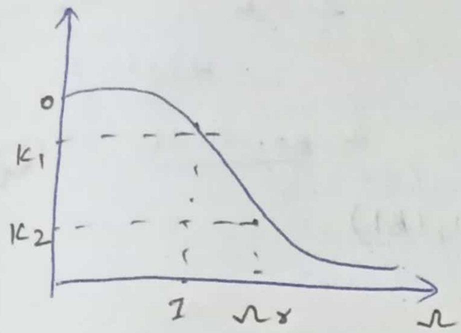
⇒ $$2n\log\left(\frac{\Omega_1}{\Omega_2}\right) = \log(10^{-K_1/10} - 1) - \log(10^{-K_2/10} - 1)$$

⇒ $$\boxed{n = \frac{\log\left(\dfrac{10^{-K_1/10} - 1}{10^{-K_2/10} - 1}\right)}{2\log\left(\dfrac{\Omega_1}{\Omega_2}\right)}}$$

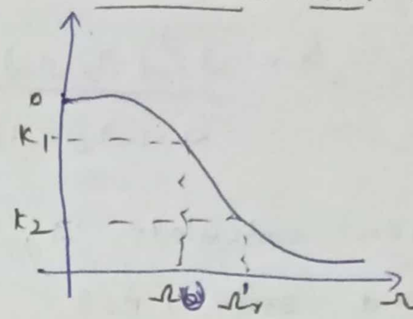
* ANALOG TO ANALOG TRANSFORMATION:

- To design any type of filter, first we need to find unnormalised filter and then convert it to normalised.

Prototype Response



Transformed filter response



- Normalised low pass $H(s)$
- Unnormalised low pass $H'(s)$

- To find TF of unnormalised filter, find TF of normalised filter and then replace $s \to \frac{s}{\Omega_0}$ in $H(s)$, We get $\underline{H'(s)}$

Ex: If $n=1 \Rightarrow H(s) = \frac{1}{s+1}$

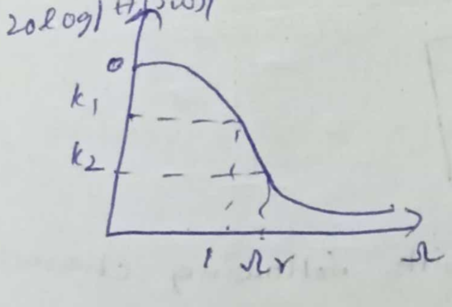
$$\Rightarrow H'(s) = \frac{1}{\frac{s}{\Omega_0}+1} = \frac{\Omega_0}{s+\Omega_0}$$

$$\boxed{\Omega_r = \frac{\Omega_r'}{\Omega_0}}$$

which is TF of unnormalised LPF.

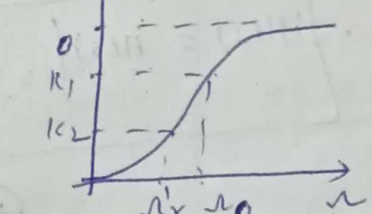
- LOWPASS $\rightarrow$ HIGH PASS:

Prototype Response



TF = $H(s)$

HP Response



TF = $H'(s)$

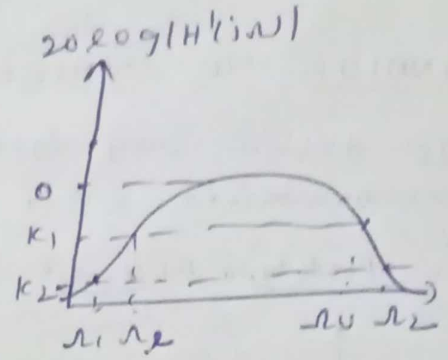
- we generally be given $\underline{H'(s)}$, to find TF of HPF,

Find $\Omega_r = \frac{\Omega_c}{\Omega_r'}$, Find n :) then $H(s)$

and $\boxed{H'(s) = H(s)\big|_{s=\frac{\Omega_c}{s}}}$

• **Lowpass - Band Pass:**

→ Find $A = \dfrac{(-\Omega_1^2 + \Omega_l \Omega_u)}{[\Omega_1(\Omega_u - \Omega_l)]}$



$B = \dfrac{\Omega_2^2 - (\Omega_l \Omega_u)}{\Omega_2(\Omega_u - \Omega_l)}$

$H'(s) = ?$

* Bandpass response *

→ then calculate, $\Omega_r = \min(|A|, |B|)$

→ Find order, $n = ?$

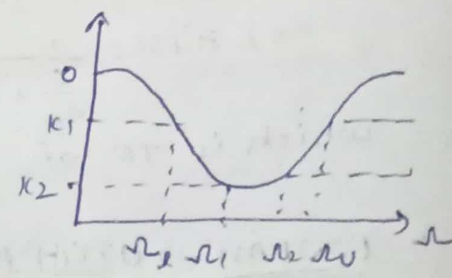
→ Find $H(s)$.

then TF of BPF is obtained by replacing 's' by $\dfrac{s^2 + \Omega_l\Omega_u}{s(\Omega_u - \Omega_l)}$

$$\Rightarrow \boxed{H'(s) = H(s)\Big|_{s = \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}}}$$

. **Lowpass - BandReject / Bandstop:**

→ Calculate, $A = \dfrac{\Omega_1(\Omega_u - \Omega_l)}{(-\Omega_1^2 + \Omega_l \Omega_u)}$



$B = \dfrac{\Omega_2(\Omega_u - \Omega_l)}{(-\Omega_2^2 + \Omega_l \Omega_u)}$

→ Then, Ω_r = minimum of $(|A|, |B|)$

$$\cdot \boxed{H'(s) = H(s)\Big|_{s = \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l\Omega_u}}}$$

Q: Design an analog $H'(s)$ BPF with following characteristics

(a) -3.0103 dB, upper & lower cutoff frequency of 20 & 50 Hz respectively.

(b) A stop band attenuation of atleast 20dB at 20Hz and 45KHz

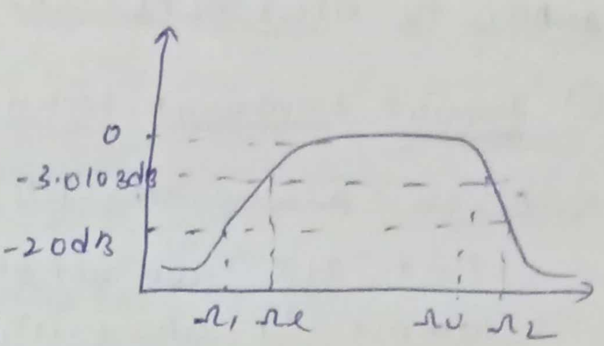
(c) A monotonic frequency response.
↓ Use Butterworth Technique.

sol:

Given,

$K_1 = -3.0103\ dB$

$K_2 = -20\ dB$



$\Rightarrow \Omega_1 = 2\pi f_1 = 2\pi(20) = 40\pi$

$\Omega_2 = 2\pi f_2 = 2\pi(45\times10^3) = 90000\pi$

$\Omega_l = 2\pi f_l = 2\pi(50) = 100\pi$

$\Omega_u = 2\pi f_u = 2\pi(20\times10^3) = 40000\pi$

$$\Rightarrow A = \frac{-\Omega_1^2 + \Omega_l\Omega_u}{\Omega_1(\Omega_u - \Omega_l)} = \frac{-(40\pi)^2 + (100\pi)(40000\pi)}{40\pi[39900\pi]} = \frac{-1600\pi^2 + 4000000\pi^2}{40\pi\times39900\pi}$$

$A = 2.505$

$B = 2.25$

$$\Rightarrow B = \frac{\Omega_2^2 - (\Omega_l\Omega_u)}{\Omega_2(\Omega_u - \Omega_l)} = \frac{(90000\pi)^2 - (100\pi)(40000\pi)}{90000\pi(39900\pi)} \Rightarrow \Omega_r = 2.25$$

$\Rightarrow \Omega_r = \min(|A|, |B|) = 2.25.$

$$\Rightarrow \text{calculate, order of filter, } n = \frac{\log\left(\frac{10^{-K_1/10} - 1}{10^{-K_2/10} - 1}\right)}{2\log\left(\frac{\Omega_1}{\Omega_r}\right)}$$

~~$K_1 = 10\log(-3.0103) = 4.2$~~

~~$K_2 = 10\log(-20) = -13.0$~~

$$= \frac{\log\left(\frac{10^{+3.0103/10} - 1}{10^{+20/10} - 1}\right)}{2\log\left(\frac{400\pi}{90000\pi}\right)}$$

$n = 2.83$

Assume, $\boxed{n = 3}$

$\Rightarrow N$ is odd.

w.k.T, TF of 3^{rd} order LPF is,

$$\boxed{H(s) = \frac{1}{(s+1)(s^2+s+1)}}$$

$$s = \frac{s^2 + (40000\pi)(100\pi)}{s(40000\pi - 100\pi)} = \frac{s^2 + 39478417.36}{s(39900\pi)}$$

$= 125349.54$

$\Rightarrow$ TF of BPF is, $H'(s) = H(s)\Big|_{s = \frac{s^2 + \Omega_u\Omega_l}{s(\Omega_u - \Omega_l)}}$

$$= \frac{1}{\left(\frac{s^2 + 394784.709}{s(125349.54)} + 1\right)\left(\left(\frac{s^2 + 3947884.36}{s(39900\pi)}\right)^2 + \cdots + 1\right)}$$

- **ANALOG TO DIGITAL FILTERS:**

① **Impulse Invariant Technique:**

For Given Analog TF, $H(s)$

- Find ILT, we get $h(t)$
- sample it at $t = nT_s$
- For discrete transfer function, compute ZT.
- we get $H(z)$.

which is TF of digital filter.

→ **Limitations:**

Can't be applied for BP, BS filters.

② **Bilinear Transformation technique:**

- $x(t) \to \boxed{\int} \to y(t)$

- $y(t) = \int x(t)$; $\frac{dy(t)}{dt} = x(t)$ → Take LT, we get: $\frac{Y(s)}{X(s)} = \frac{1}{s}$ — ①

- $$\int_{(n-1)T_s}^{nT_s} \frac{d}{dt} y(t)\, dt = \int_{(n-1)T_s}^{nT_s} x(t)\, dt$$

solve using Trapezoidal technique
↓
Area can be approximated by mean height of $x(t)$ b/w 2 event & multiplied by T_s

$$\Rightarrow y(t)\Big|_{(n-1)T_s}^{nT_s} = \int_{(n-1)T_s}^{nT_s} x(t)\, dt$$

$$\Rightarrow y[nT_s] - y[(n-1)T_s] = \left[\frac{x(nT_s) + x[(n-1)T_s]}{2}\right] T_s$$

→ T_s is very small, Hence neglect it, we get

$$y[n] - y[n-1] = \left[\frac{x[n] + x[n-1]}{2}\right] T_s$$

Take Z-T on b.s, we get:

$$y[z] - z^{-1} y[z] = \frac{T_s}{2}\left[x[z] + z^{-1} x[z]\right]$$

$$Y[z][1 - z^{-1}] = \frac{T_s}{2}[1 + z^{-1}] X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{T_s}{2} \frac{[1 + z^{-1}]}{[1 - z^{-1}]} \quad — ②$$

comparing ① & ②, we get

$$\frac{1}{s} = \frac{T_s}{2}\left[\frac{1+z^{-1}}{1-z^{-1}}\right]$$

$$\Rightarrow s = \frac{2}{T_s}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] = \frac{2}{T_s}\left[\frac{z-1}{z+1}\right]$$

⇒ To convert analog to digital filter, replace 's' by " $s = \frac{2}{T_s}\left[\frac{z-1}{z+1}\right]$ " in $H(s)$.

* w.r.t, $s = \sigma + j\Omega$; $z = re^{j\omega}$

$$\Rightarrow \sigma + j\Omega = \frac{2}{T_s}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right]\left[\frac{re^{-j\omega}+1}{re^{-j\omega}+1}\right]$$

on solving, we get:

$$\sigma = \frac{2}{T_s}\left[\frac{r^2-1}{r^2+2r\cos\omega+1}\right] ; \Omega = \frac{2}{T_s}\left[\frac{2r\sin\omega}{r^2+2r\cos\omega+1}\right]$$

⇒ $r<1$, σ is $-ve$

⇒ pole lies inside unit circle in z-plane & left half of s-plane.

⇒ $r>1$, σ is $+ve$

⇒ poles lies outside unit circle in z-plane and right half of s-plane.

⇒ $r=1$, $\sigma = 0$, poles are imaginary $\left[\Omega = \frac{2}{T_s}\tan\left(\frac{\omega}{2}\right)\right]$ (Analog freq, Digital freq)

⇒ poles lies on unit circle in z-plane and on Imaginary axis in s-plane.

- while designing digital filter, when 'ω' is low, we have $\boxed{\omega = \Omega T_s}$ and relation b/w analog & digital frequencies is linear.

- At higher frequencies, the relation is not linear, hence we may get distortion, it is called as "WARPING EFFECT". This effect can be eliminated by using 'PREWARPING' specifications, i.e., $\Omega = \frac{2}{T_s}\tan\left(\frac{\omega}{2}\right)$.

→ If digital specifications are given for designing the digital filter i.e., $\omega_1, \omega_2 \ldots \omega_N$ & $k_1, k_2 \ldots k_N$,

- get PREWRAPPED analog specifications i.e. $\Omega_1', \Omega_2', \ldots \Omega_N'$
$k_1, k_2 \ldots k_N$
using $\Omega = \frac{2}{T_s} \tan\left(\frac{\omega}{2}\right)$.

- Then design analog filter.
- Then convert it to digital filter using B.T.

$$\boxed{\begin{array}{c}\text{Digital}\\ \text{specifications}\end{array}} \rightarrow \boxed{\begin{array}{c}\text{prewrapped}\\ \text{Analog specification}\end{array}} \rightarrow \boxed{H_a(s)} \xrightarrow{BT} \boxed{H(z)}$$

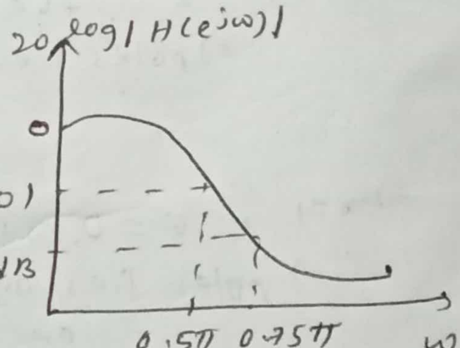
$\omega_1, \omega_2 \ldots \omega_N$ $\Omega_1', \Omega_2', \ldots \Omega_N$
$k_1, k_2, \ldots k_N$ $k_1, k_2, \ldots k_N$

Q: Design and realize LPF (digital) using BT method to satisfy following characteristics:

(a) Monotonic stopband & passband
(b) −3.01dB cutoff frequency of 0.5π rad
(c) magnitude down atleast 15dB at 0.75π rad.

} Digital specification (ω)

Required frequency response is shown below:



sol: Assume, $T_s = 1s$

→ $\Omega_1' = 2\tan\left(\frac{0.5\pi}{2}\right) = 2\,\text{rad/sec}$

∴ $\Omega_2' = 2\tan\left(\frac{0.75\pi}{2}\right) = 4.82\,\text{rad/sec}$

and we have, $k_1 = -3.01\,dB$
$k_2 = -15\,dB$

→ Find Ω_r: $\frac{4.82}{2} = 2.41\,\text{rad/sec}$

→ Find $n = \dfrac{\log\left(10^{-(-3.01)/10} - 1\right) - \log\left(10^{-(-15)/10} - 1\right)}{2\log\left(\frac{2}{4.82}\right)}$

$= 2$ //

$n = 1.91 \approx 2$

$\Rightarrow H_a(s) = \dfrac{1}{s^2+\sqrt{2}s+1}$

$\Rightarrow H_a'(s) = H_a(s)\Big|_{s=\frac{s}{\Omega_1'}}$

$$H_a'(s) = \frac{1}{\left(\frac{s}{2}\right)^2+\sqrt{2}\left(\frac{s}{2}\right)+1} = \frac{4}{s^2+2\sqrt{2}s+4}$$

which is TF of unnormalised LPF.

$\rightarrow$ To find TF of digital LPF, replace $s=\frac{2}{T_s}\left(\frac{z-1}{z+1}\right)$

$$\Rightarrow H(z) = \frac{4}{\left\{\frac{2}{T_s}\left[\frac{z-1}{z+1}\right]\right\}^2 + 2\sqrt{2}\left[\frac{2}{T_s}\left[\frac{z-1}{z+1}\right]\right]+4}$$

$$= \frac{4}{4\left[\frac{(z-1)^2}{(z+1)^2}\right]+4\sqrt{2}\left[\frac{z-1}{z+1}\right]+4}$$

$$= \frac{4(z+1)^2}{4(z-1)^2+4\sqrt{2}(z^2-1)+4(z+1)^2} = \frac{4z^2+4+8z}{4z^2+4-\cancel{8z}+4\sqrt{2}z^2-4\sqrt{2}+4z^2+4+\cancel{8z}}$$

$$= \frac{4z^2+8z+4}{(8+4\sqrt{2})z^2+(8-4\sqrt{2})}$$

$$\boxed{H(z) = \frac{1+2z^{-1}+z^{-2}}{3.414+0.58z^{-2}}}$$

* CHEBYSHEV FILTER :-

There are 2 types of chebyshev filter :

→ One containing "Ripple in pass Band" (type-1) and the
→ another containing "Ripple in stopBand" (type-2).

→ A type-1 LP normalised chebychev's filter with a Ripple in the pass Band is characterised by the "Magnitude squared frequency response".

$$\boxed{|H_N(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega)}}$$

$$|H(j\Omega)|^2 = \frac{r}{1+\epsilon^2 T_N^2(\Omega)}$$

where,

$T_N(\Omega)$ - n^{th} order chebychev's polynomial.

→ the chebychev's polynomial can be generated by Recursive formula.

$$\boxed{T_n(x) = 2x T_{n-1}(x) - T_{n-2}(x), \; n \geq 2}$$

with $T_0(x) = 1$ & $T_1(x) = x$

n	$T_n(x)$
0	1
1	x
2	$2x(x) - 1 = 2x^2 - 1$
3	$2x(2x^2-1) - x = 4x^3 - 3x$
4	$2x(4x^3-3x) - (2x^2-1) = 8x^4 - 8x^2 + 1$
5	$2x(8x^4-8x^2+1) - (4x^3-3x) = 16x^5 - 20x^3 + 5x$
6	$2x(16x^5-20x^3+5x) - (8x^4-8x^2+1) = 32x^6 - 48x^4 + 18x^2 - 1$

→ For $n=5$; $T_5(x) = 16x^5 - 20x^3 + 5x$

Take, $x = 0 \Rightarrow T_5(x) = 0$

$x = 0.3124 \Rightarrow 1$

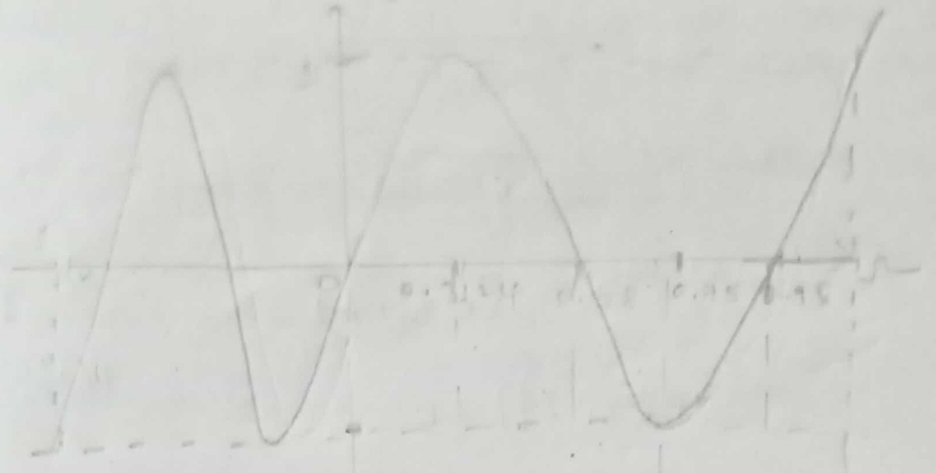
$x = 0.58 \Rightarrow 16(0.58)^2 - 20(0.58)^3 + 5(0.58) = 0$

$x = 0.75 \Rightarrow -1$

$x = 0.95 \Rightarrow 0$

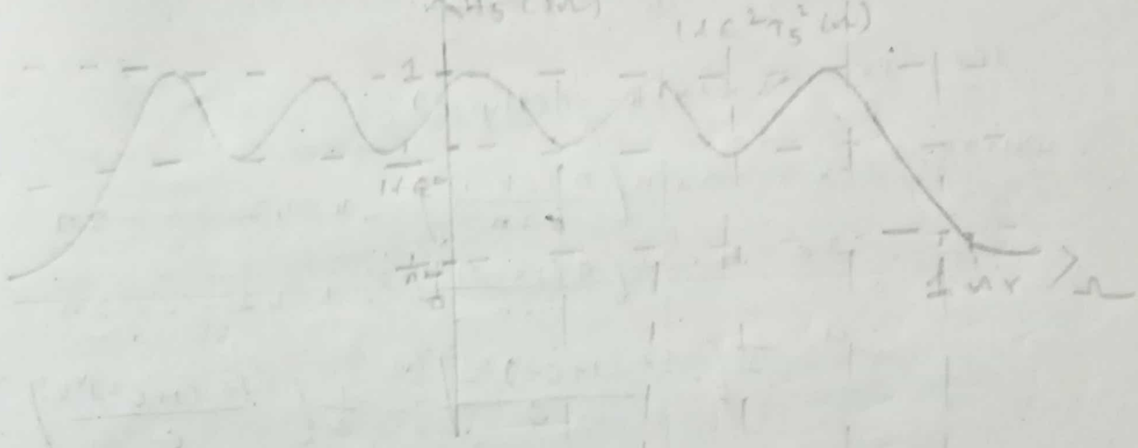
$x = 1 \Rightarrow 1$

The plot of $T_5(\Omega)$ is shown in figure. From that, it is observed for $n=5$, the cheychev polynomial oscillates between +1 & -1 within the interval $\Omega = -1 \to 1$ while outside that interval it moves towards +∞ & -∞



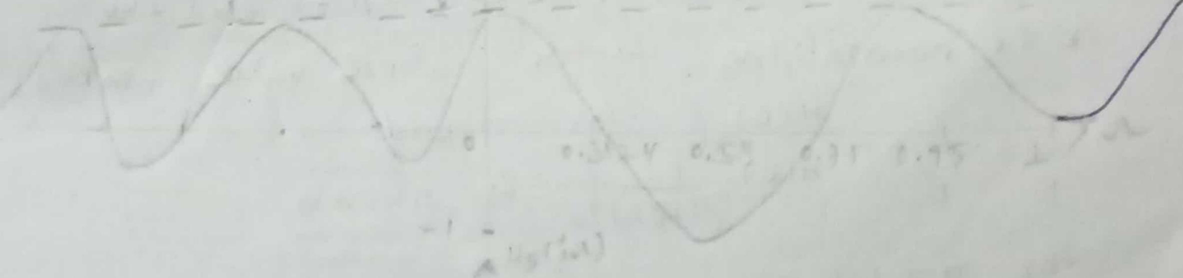
The plot of $T_5(\Omega)$ is shown in figure. From that, it is observed for $n=5$, The cheby

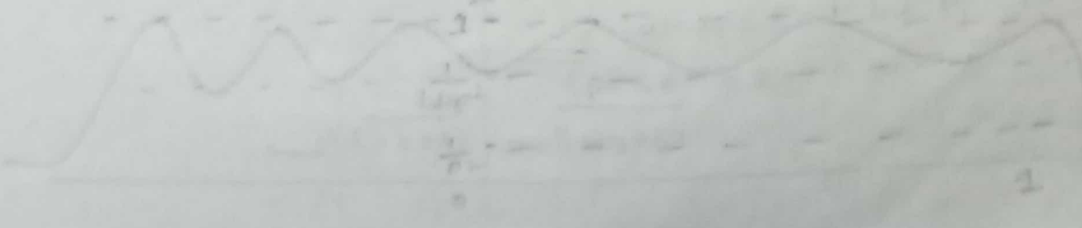
Normalised frequency Response of chebychev polynomial for $n=5$; $T_5(\Omega)$:



The same type of oscillations take place for every chebychev polynomial and causes the equal magnitude ripple in the $|H_n(j\Omega)|^2$, shown in figure.

The amplitude of $|H_n(j\Omega)|^2$ oscillates between 1 and $\frac{1}{1+\epsilon^2}$ and Ω goes from -1 to $+1$. The $|H_n(j\Omega)|^2$ is moving towards '0' outside $\Omega=-1$ to 1. Because the magnitude of chebychev's polynomial moving towards '∞' as Ω goes away from $-1/+1$.





→ To obtain the causal stable Transfer function, $H_n(s)$, start from the chebychev magnitude squared function

$$|H_n(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_n^2(\Omega)}$$

$$H_n(s) \cdot H_n(-s) = \frac{1}{1+\epsilon^2 T_n^2(\frac{s}{j})} \quad ; \quad \Omega = \frac{s}{j}$$

$$H(j)(-j) = |H(j)|^2$$

$$1+\epsilon^2 T_n^2\left(\frac{s}{j}\right) = 0$$

$$T_n^2\left(\frac{s}{j}\right) = \frac{-1}{\epsilon^2}$$

$$T_n^2(-js) = \frac{j^2}{\epsilon^2}$$

$$T_n(-js) = \pm\frac{j}{\epsilon}$$

let $s_k = \sigma_k + j\Omega_k$ (ellipse)

where;

$$\sigma_k = -a\sin\left[\frac{\pi(2k-1)}{2n}\right] \quad ; k = 1,2,\ldots 2n$$

$$\Omega_k = b\cos\left(\frac{\pi(2k-1)}{2n}\right] \quad ; k = 1,2,\ldots 2n$$

$$a = \frac{1}{2}\left[\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right]^{1/n} - \frac{1}{2}\left(\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right)^{-1/n}$$

$$b = \frac{1}{2}\left[\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right]^{1/n} + \frac{1}{2}\left[\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right]^{-1/n}$$

→ Use the left half of the s-plane poles only. $H_n(s)$ can be written in the form;

TF: $$H_n(s) = \frac{K}{\prod(s-s_k)} = \frac{k}{V_n(s)}$$

where, $V_n(s)$ is polynomial in 's',

$$V_n(s) = s^n + b_{n-1}s^{n-1} + \cdots + b_2 s^2 + b_1 s + b_0$$

K is normalising factor whose value makes,

$H(0) = 1$ for n = odd

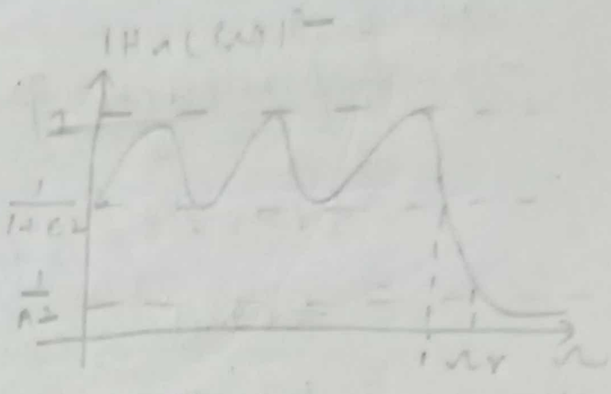
$H(0) = \frac{1}{(1+\epsilon^2)^{1/2}}$ for n = even

for n = odd: $k = V_n(0) = b_0$

n = even: $k = \frac{V_n(0)}{(1+\epsilon^2)^{1/2}} = \frac{b_0}{(1+\epsilon^2)^{1/2}}$

Evaluation of order of the filter:

$$n = \frac{\log_{10}\left[g + (g^2-1)^{1/2}\right]}{\log_{10}\left[\Omega_r + (\Omega_r^2-1)^{1/2}\right]}$$



Normalized Chebyshev filter

$$g = \left[\frac{A^2-1}{\epsilon^2}\right]^{1/2}$$

$\epsilon \to$ Passband Ripple.

Ex: $\frac{1}{1+\epsilon^2} \to$ passband gain $-0.5\,dB$; $\underline{\underline{n=1}}$; poles $= \underline{\underline{1}}$

$\rightarrow 10\log\left(\frac{1}{1+\epsilon^2}\right) = -0.5\,dB \Rightarrow \boxed{\epsilon = 0.349}$

$\rightarrow a = \frac{1}{2}\left[\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right]^{1/n} - \frac{1}{2}\left[\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right]^{-1/n}$

$= \frac{1}{2}\left[\frac{1+(1+0.3492^2)^{1/2}}{0.349}\right] - \frac{1}{2}\left[\frac{1+1.05}{0.349}\right]^{-1}$

$= \frac{1}{2}\left[\frac{1+1.05}{0.349}\right] - \frac{1}{2}\left[5.8\right]^{-1}$

$= \frac{1}{2}[5.8] - \frac{1}{2}[5.8]^{-1}$

$= 2.9 - \frac{1}{2}\times\frac{1}{5.8} = 2.9 - 0.08$

$\boxed{a = 2.82}$

$\rightarrow b = \frac{1}{2}\left[\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right]^{1/n} + \frac{1}{2}\left[\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right]^{-1/n}$

$\boxed{b = 3.03}$

$s_k = \sigma_k + j\Omega_k$

$\sigma_k = -a\sin\left[\frac{\pi}{2n}(2k-1)\right]$; $k = 1, 2, \ldots 2n$

$\Omega_k = b\cos\left[\frac{\pi(2k-1)}{2n}\right]$; $k = 1, 2, \ldots 2n$

$s_1 = \sigma_1 + j\Omega_1$

$$\boxed{H_1(s) = \frac{k}{\prod_{k=1}^{2n}(s-s_k)}}$$

product

$\Omega_1 = 0,\ \sigma_1 = -2.86$ $\quad\therefore V_n(s) = s + 2.86$

$\Omega_2 = 0,\ \sigma_2 = 2.86$ $\quad\therefore V_n(0) = 2.86 = K$

$\Rightarrow H_1(s) = \frac{K}{\pi(s+2.86)}$; $\boxed{H_1(s) = \frac{2.86}{\pi(s+2.86)}}$

→ $n=2$; poles $=4$

→ $\epsilon = 0.349$;

→ $a = \frac{1}{2}\left[\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right]^{1/2} - \frac{1}{2}\left[\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right]^{-1/2}$

$= \frac{1}{2}[5.8]^{1/2} - \frac{1}{2}[5.8]^{-1/2}$

$= \frac{1}{2}(2.4) - \frac{1}{2}(2.4)^{-1}$

$= 1.2 - 0.20$

$\boxed{a = 1.008}$

→ $b = \frac{1}{2}\left[1+\frac{(1+\epsilon^2)^{1/2}}{\epsilon}\right]^{1/2} + \frac{1}{2}\left[\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right]^{-1/2}$

$= 1.2 + 0.20$

$\boxed{b = 1.41}$

$\therefore$ $s_1 = \sigma_1 + j\Omega_1$; $\sigma_1 = -a\sin\left(\frac{\pi}{4}\right) = -\frac{a}{\sqrt{2}}$, $\Omega_1 = b\cos\left(\frac{\pi}{4}\right) = \frac{b}{\sqrt{2}}$

$s_2 = \sigma_2 + j\Omega_2$; $\sigma_2 = -a\sin\left(\frac{3\pi}{4}\right) = -\frac{a}{\sqrt{2}}$, $\Omega_2 = b\cos\left(\frac{3\pi}{4}\right) = \frac{-b}{\sqrt{2}}$

$s_3 = \sigma_3 + j\Omega_3$; $\sigma_3 = -a\sin\left(\frac{5\pi}{4}\right) = \frac{a}{\sqrt{2}}$, $\Omega_3 = b\cos\left(\frac{5\pi}{4}\right) = \frac{-b}{\sqrt{2}}$

$s_4 = \sigma_4 + j\Omega_4$; $\sigma_4 = -a\sin\left(\frac{7\pi}{4}\right) = \frac{a}{\sqrt{2}}$, $\Omega_4 = b\cos\left(\frac{7\pi}{4}\right) = \frac{b}{\sqrt{2}}$

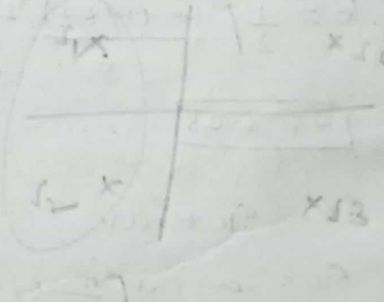
⇒ $s_1 = -\frac{a}{\sqrt{2}} + j\left(\frac{b}{\sqrt{2}}\right) = -(0.008)(0.707) + j(1.41)(0.707)$

$s_1 = -0.713 + j1.004$

lly, $s_2 = -0.713 - j1.004$

$s_3 = 0.713 - j1.004$

$s_4 = 0.713 + j1.004$



$H_2(s) = \frac{k}{(s-s_1)(s-s_2)}$

$H_2(s) = \frac{k}{s^2 + 1.43s + 1.52}$

$b_0 = v_N(0)$

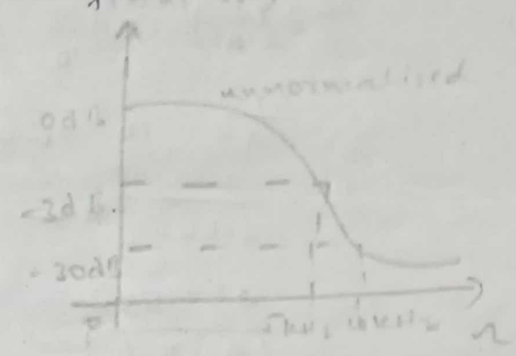
w.k.t; $k = \frac{b_0}{(1+\epsilon^2)^{1/2}} = \frac{1.52}{[1+(0.349)^2]^{1/2}}$

$k = 1.43$

⇒ $H_2(s) = \frac{1.43}{s^2 + 1.43s + 1.52}$ //

Determine the order of the Butterworth and chebychev's filter that will realise a LPF to satisfy a response flat within 3 dB from DC ('0' freq) to 5KHz and attenuation ≥ 30dB for frequencies ≥ 10 KHz.

Butterworth :-



$$n = \frac{\log(10^{-K_1/10} - 1) - \log(10^{-K_2/10} - 1)}{2\log\left(\frac{\Omega_1}{\Omega_2}\right)}$$

$$= \frac{\log(10^{-K_1/10} - 1) - \log(10^{-\frac{K_2}{10}} - 1)}{2\log\left(\frac{1}{\Omega_r}\right)}$$

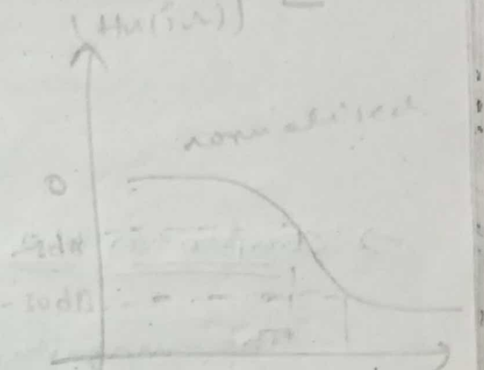
$\rightarrow f_p = 5000$ Hz, $\quad \Omega = 2\pi f$, $\omega = \Omega_p T$

$\Omega_p = 5000 \times 2\pi = 10000\pi$ rad/sec

$f_s = 10$ KHz

$\Omega_s = 10000 \times 2\pi = 20000\pi$ rad/sec

$\rightarrow w_p = \Omega_p T$



$$\Rightarrow n = \frac{\log(10^{-(-3/10)} - 1) - \log(10^{-(-30/10)} - 1)}{2\log\left(\frac{1}{\Omega_r}\right)}$$

$$= \frac{\log(10^{0.3} - 1) - \log(10^3 - 1)}{2\log\left(\frac{1}{2}\right)} = \frac{\log(1.99 - 1) - \log(999)}{2(-0.3010)}$$

$$= \frac{\log(0.99) - \log(999)}{-0.6010} = \frac{-0.0043 - 2.999}{-0.6010} = \frac{-3.033}{-0.6010}$$

$= 4.92$

$\boxed{n = 5}$ //

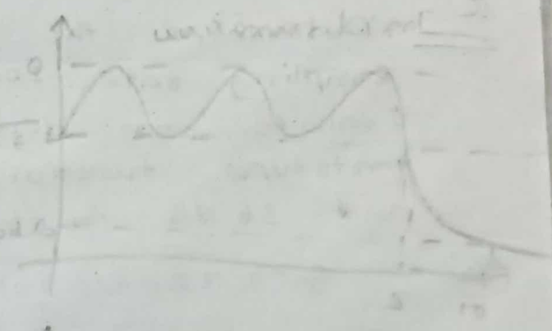
Chebychev's :-

$10\log\left(\frac{1}{1+\epsilon^2}\right) = -3$dB

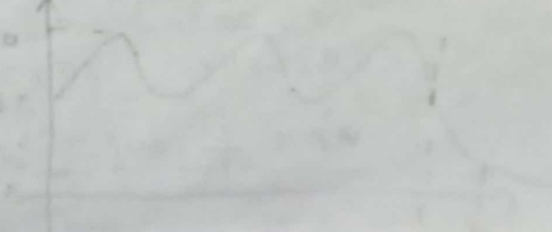
$\log\left(\frac{1}{1+\epsilon^2}\right) = \frac{-3}{10}$

$\log\left(\frac{1}{1+\epsilon^2}\right) = \log_{10}^{-0.3}$

$\frac{1}{1+\epsilon^2} = 10^{-0.3} = 0.5$



$\frac{1}{0.5} = 1 + \epsilon^2$



$2 - 1 = \epsilon^2$

$\epsilon^2 = 1$

$\boxed{\epsilon = 1}$

$\rightarrow 10\log\left(\frac{1}{A^2}\right) = -30$; $g = \left[\frac{A^2-1}{\epsilon^2}\right]^{1/2} = 3.15$

$\log\left(\frac{1}{A^2}\right) = -\frac{30}{10}$

$\log\left(\frac{1}{A^2}\right) = \log_{10}^{-3}$

$\frac{1}{A^2} = 10^{-3} = 0.001$

$\frac{1}{0.001} = A^2$

$A^2 = 1000$.

$\boxed{A = 31.6}$

; $n = \frac{\log_{10}[g + (g^2-1)^{1/2}]}{\log_{10}[\Omega_r + (\Omega_r^2 - 1)^{1/2}]}$

$= \frac{\log_{10}(31.5 + (398.56-1)^{1/2})}{\log_{10}(2 + (4-1)^{1/2})}$

$= \frac{\log_{10}(3.15+3.15)}{\log_{10}(2+1.732)}$

$= \frac{\log_{10}(63)}{\log_{10}(3.732)}$

$= \frac{1.799}{0.571}$

$= 3.15$

$\simeq n = 4$

→ Comparision b/w Butterworth & Chebychev's:-

The magnitude Response of Butter filter decreases monotonically as 'Ω' increases from 0 to α., whereas the magnitude response in chebychev's technique exhibits ripple for passband /stopband accordingly to the type of filter.

→ The transmission Band is more in B.W compared to chebychev's.

→ The poles of B.W filter lies on circle, where as the poles of a chebychev's filter lies on a Ellipse.

→ For The same specifications, the number of poles in B.W filter is more when compared to the chebychev's filter. i.e., the order of chebychev's filter is less than that of Butterworth filter.

Q:- Determine the TF H(z) of a Highpass filter having sampling rate 200 Hz ⇒ $T_s = \frac{1}{200}$, that satisfies a Bilinear Transformation design based on Butterworth analog prototype function with attenuation 3 dB for f>20 Hz and 36 dB for f = 5 Hz.

sol:- $f_p = 5$ Hz, $f_s = 20$ Hz, $T_s = \frac{1}{200\text{ Hz}}$

→ $\Omega_p = 2\pi f_p = 2\pi(5) = 10\pi$ rad/sec

→ $\Omega_s = 2\pi f_s = 2\pi(20) = 40\pi$ rad/sec

$\Omega_p' = \frac{2}{T_s}\tan\left(\frac{\omega_p}{2}\right)$, $\Omega_s' = \frac{2}{T_s}\tan\left(\frac{\omega_s}{2}\right)$

$\rightarrow \omega_p = \Omega_p T \Rightarrow 10\pi\left(\frac{1}{200}\right) = \frac{\pi}{20}$

$\rightarrow \omega_s = \Omega_s T \Rightarrow 40\pi\left(\frac{1}{200}\right) = \frac{\pi}{5}$

$\Omega_p' = \frac{2}{\frac{1}{200}} \tan\left(\frac{\pi}{40}\right) = 400 \tan\left(\frac{\pi}{40}\right) = 31.48$

$\Omega_s' = \frac{2}{\frac{1}{200}} \tan\left(\frac{\pi}{10}\right) = 129.96$

$\rightarrow n = \frac{\log(10^{-K_1/10} - 1) - \log(10^{-K_2/10} - 1)}{2\log\left(\frac{\Omega_p'}{\Omega_s'}\right)} = \frac{\log(10^{0.3} - 1) - \log(10^{3.6} - 1)}{2\log\left(\frac{31.48}{129.96}\right)}$

$= \frac{-0.043 - 3.599}{-1.22} = \frac{3.6}{1.22} = 2.95$

$\Rightarrow n = 3$

	BW	CH ripple
• MR		
• poles position	circle	ellipse
• BW	more	less
• No. of poles	more	less
• order	More	less
• $\frac{1}{\sqrt{2}}$ at $\omega_c \rightarrow$	$\frac{1}{\sqrt{1+\epsilon^2}} \rightarrow \omega_c$	
• few parameters to determine TF	• large no. of parameters to determine TF.	