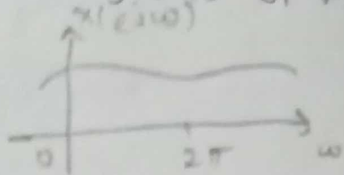
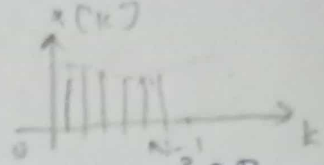
* DISCRETE FOURIER TRANSFORM:

- It is applicable only for periodic signals of finite length.



$\rightarrow$ $x[n] \Rightarrow X(e^{j\omega}) \xrightarrow{\text{SAMPLE}} X[K] = \text{DFT}.$



W.K.T. $\quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \quad$ —(1)

Let $\omega = \frac{2\pi}{N} k$; $\Rightarrow X\left(\frac{2\pi}{N}k\right) = X[k] = \sum_{n=-\infty}^{\infty} x[n] . e^{-j\frac{2\pi}{N}k.n}$

$[0 \le k \le N-1]$

$$X[k] = \dots + \sum_{n=-N}^{-1} x[n] . e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}k.n} + \sum_{n=N}^{2N-1} x[n]\, e^{-j\frac{2\pi}{N}} + \dots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x[n]\, e^{-j\frac{2\pi}{N}.k.n} \quad ; 0 \le k \le N-1$$

$[n = n - ln]$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x[n-lN] . e^{-j\frac{2\pi}{N}kn} . \underbrace{e^{j\frac{2\pi}{N}klN}}_{[1]}$$

$$= \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x[n+lN] . e^{-j\frac{2\pi}{N}.k.n} \quad ; 0 \le k \le N-1$$

$$\boxed{X[k] = \sum_{n=0}^{N-1} x_p[n] . e^{-j\frac{2\pi}{N}.k.n}} \quad ; 0 \le k \le N-1 \quad —(2)$$

* IDFT:

* Discrete Fourier series of a periodic signal with period 'N' is:

$$x_p[n] = \sum_{k=0}^{N-1} a_k . e^{j.\frac{2\pi}{N}.k.n.} \quad —(3)$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n]\, e^{-j\frac{2\pi}{N}kn.} \quad —(4)$$

$a_k = \frac{1}{N} X[k]$ [From ② & ④]

③ :- $$\boxed{x_p[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] . e^{j\frac{2\pi}{N}kn}} \quad 0 \le n \le N-1$$

which is Inverse Discrete Fourier Transform (DFT)

→ DFT of $x[n]$ with length "$0 \le n \le N-1$",

$$X[k] = \sum_{n=0}^{N-1} x_p[n] e^{-j\frac{2\pi}{N}kn} \quad ; \quad 0 \le k \le N-1 .$$

and $$\boxed{X[k] = \sum_{n=0}^{N-1} x_p[n] . W_N^{kn}}$$

* PHASE FACTOR / TWIDDLE FACTOR

$$\left[W_N = e^{\frac{-j2\pi}{N}} \right]$$

and IDFT of $X[k]$ is:

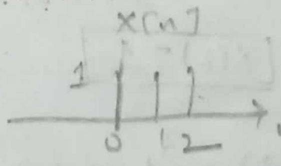
$$x_p[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{\frac{j2\pi}{N}kn} \quad ; \quad 0 \le n \le N-1$$

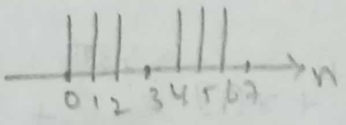
$$\boxed{x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}}$$

0,1,2 = 1,1,1,0
0,1,2,3

Q: Find DFT of $x[n] = 1 \quad 0 \le n \le 2$, for $N = 4$. Sketch Magnitude and Phase spectrum.

sol: Given, $x[n] = 1, \quad ; 0 \le n \le 2$



$x_p[n] =$ 

* $W_4^0 = 1$

* $W_4^1 = -j$

* $W_4^2 = -1$

w.k.T., $$X[k] = \sum_{n=0}^{N-1} x_p[n] e^{\frac{-j2\pi}{N}kn} \quad ; \quad 0 \le k \le N-1$$

$$= \sum_{n=0}^{3} x[n] e^{-j2\pi/4 . kn}$$

$$= 1. e^{-\frac{j2\pi}{4}k(0)} + 1. e^{-\frac{j2\pi}{4}.2.k} + 1.e^{\frac{j2\pi}{4}.3k} + 0.$$

we have, $x(0) = 1, x[1] = 1, x(2) = 1, x[3] \ldots = 0$
and $N=4 \Rightarrow k = 0, 1, 2, 3$

$$\therefore X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\left(\frac{2\pi}{N}\right)nk}$$

Now, $\boxed{X(k) = \sum_{n=0}^{3} x_p[n] \cdot e^{-j\left(\frac{2\pi}{4}\right)nk}}$ $, k = 0, 1, 2, 3$

$$\Rightarrow X(0) = \sum_{n=0}^{3} x(n) \cdot e^{0} = x(0) + x(1) + x(2) + x(3) = 1+1+1+0 = 3$$

$$* \; X(1) = \sum_{n=0}^{3} x(n) e^{-j\left(\frac{\pi}{2}\right)n \cdot 1}$$
$$= x(0) + x(1) e^{-j(\pi/2)} + x(2) e^{-j(\pi/2) \cdot 2} + x(3) e^{-j(\frac{\pi}{2})3}$$
$$= 1 + 1 \cdot (\cancel{+1}) + 1 \cdot (-1) + 0.$$
$$= \cancel{1 + j) - 1}$$
$$\boxed{X(1) = \cancel{(1)}} \Rightarrow \boxed{X(1) = -j}$$

$$* \; X(2) = \sum_{n=0}^{3} x(n) \cdot e^{-j\left(\frac{\pi}{2}\right) \cdot 2 \cdot n} = \sum_{n=0}^{3} x(n) e^{-j\pi \cdot n}$$
$$= x(0) e^{-j\pi \cdot 0} + x(1) \cdot e^{-j\pi \cdot 1} + x(2) e^{-j\pi \cdot 2} + x(3) \cdot e^{-j\pi \cdot 3}$$
$$= 1 \cdot 1 + 1 \cdot (-1) + 1 \cdot (1) + 0.$$
$$= 1 + (-1) + 1$$
$$\boxed{X(2) = 1}$$

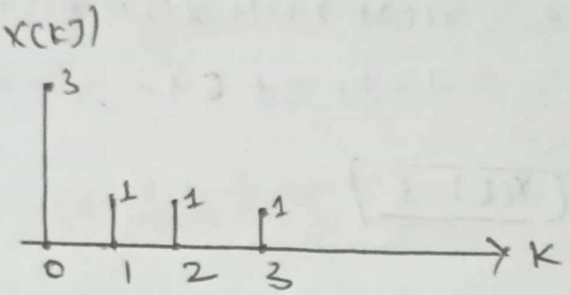
$$* \; X(3) = \sum_{n=0}^{3} x(n) e^{-j\left(\frac{\pi}{2}\right) \cdot 3 \cdot n}$$
$$= x[0] e^{-j\frac{3\pi}{2} \cdot 0} + x[1] \cdot e^{-j\frac{3\pi}{2}} + x[2] \cdot e^{-j\frac{3\pi}{2} \cdot 2} + x[3] e^{-j\frac{3\pi}{2} 3}$$
$$= 1 \cdot 1 + 1 \cdot (j) + 1 \cdot (-1) + 0$$
$$\boxed{X(3) = j}$$

$\Rightarrow X[k] = \{3, -j, 1, j\}$

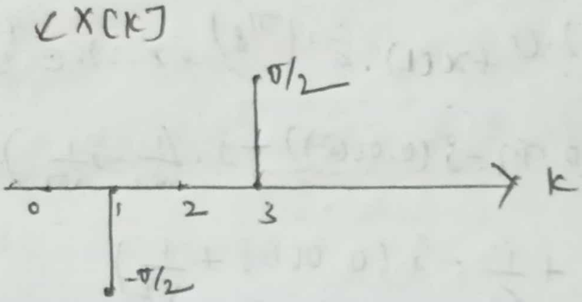
$|X[k]| = \{3, 1, 1, 1\}$

$\angle X[k] = \{0, -\frac{\pi}{2}, 0, \frac{\pi}{2}\}$

MAGNITUDE SPECTRUM



PHASE SPETRUM



* IDFT:

$X(k) = \{3, -j, 1, j\}$

$\Rightarrow x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{j\frac{2\pi}{N} n \cdot k} \quad ; \; 0 \le n \le N-1$

$= \frac{1}{4} \sum_{k=0}^{3} X[k] e^{j\frac{2\pi}{N} nk} \quad ; \; 0 \le n \le 3$

$x(n) = \frac{1}{4}\left[X[0] + X[1] e^{j\frac{2\pi}{N} k} + X[2] e^{j\frac{2\pi}{N} \cdot 2k} + X[3] e^{j\frac{2\pi}{N} 3k}\right]$

$x(n) = \frac{1}{4}\left[3 - j e^{j\frac{2\pi}{4} k} + 1 e^{j\frac{4\pi}{4} k} + j e^{j(\frac{2\pi}{4} \cdot 3k)}\right]$

$x(n) = \frac{1}{4}\left[3 - j e^{j\frac{\pi}{2} k} + e^{j\pi k} + j e^{j\frac{3\pi}{2} k}\right]$

* $x[0] = \frac{1}{4}[3 - j + 1 + j] = \frac{4}{4} = 1$

* $x[1] = \frac{1}{4}\left[3 - j e^{j\pi/2} + e^{j\pi} + j e^{j 3\pi/2}\right] = \frac{1}{4}\left[3 - j(j) + (-1) + j(-j)\right]$

* $x[2] = \frac{1}{4}\left[3 - j e^{j\pi} + e^{j\pi 2} + j e^{j3\pi}\right] = \frac{1}{4}(3 - j(-1) + (1) + j(-1)) = 1$

$x(3) = \frac{1}{4}(3 - j e^{j\frac{3\pi}{2}} + e^{j3\pi} + j e^{\dots})$
$= \frac{1}{4}(3 - 3) = 0$

* MATRIX REPRESENTATION OF DFT:

w.k.t $X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_N^{nk}$

$\Rightarrow X[k] = x(0) + x(1) W_N^{k} + x(2) W_N^{2k} + \cdots x(N-1) W_N^{(N-1)k}$

$\cdot\ X(0) = x(0) + x(1) + x(2) + \cdots + x(N-1)$

$\cdot\ X(1) = x(0) + W_N x(1) + W_N^{2} x(2) + \cdots + W_N^{(N-1)(1)} x(N-1)$

$\vdots$

$\cdot\ X(N-1) = x(0) + W_N^{N-1} x(1) + W_N^{2(N-1)} x(2) + \cdots + W_N^{(N-1)(N-1)} x(N-1)$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}_{N\times 1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^{1} & \cdots & W_N^{N-1} \\ \vdots & & & \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)^2} \end{bmatrix}_{N\times N} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}_{N\times 1}$$

$$\boxed{X_N = W_N x_N} \quad \text{①}$$

$\rightarrow$ to find IDFT:

w.k.T. $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot W_N^{-nk}$

$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(W_N^{nk}\right)^*$

$x[n] = \frac{1}{N} W_N^{*} X_N$

$$\boxed{x_N = \frac{1}{N} W_N^{*} X_N} \quad \text{②}$$

① $\times W_N^{-1} \Rightarrow \quad W_N^{-1} X_N = W_N^{-1} W_N x_N$

$$x_N = W_N^{-1} X_N. \quad —③$$

from ② & ③, $\boxed{W_N^{-1} = \frac{1}{N} W_N^{*}}$

CIRCULAR SHIFT OF SEQUENCE:-

In the time domain circular time shitting in frequency domain circular frequency shifting.

→ Consider a sequence $x(n)$ of length N defined for $0 \le n \le N-1$.

⇒ sequence is '0', for $n<0$; $n>N$

→ If $x(n)$ is such a sequence, then for any arbitary integer 'n_0', the shifted sequence $x(n-n_0)$ is no longer defined for range "$0 \le n \le N-1$". Therefore, define another type of shift that will always keep the shifted sequence in the range $0 \le n \le N-1$. This is achieved by using the modulo operation.

i.e., " n modulo. N = $\langle n \rangle_N$ "

→ Using modulo operation circular shift of a length 'N' sequence $x(n)$ is defined by equation

$$x_c(n) = x[\langle n-n_0 \rangle_N] \quad —①$$

where $x_c(n)$ is also a length 'N'.

* If $n_0 > 0$, it is a right circular shift.

* If $n_0 < 0$, it is a left circular shift.

→ For $n_0 > 0$:

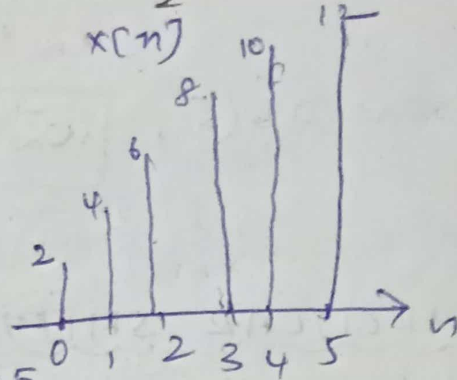
① ⇒ $x_c[n] = x[\langle n-n_0 \rangle_N] = x[n-n_0]$; $n_0 \le n < N$

$= x[n-n_0+N]$; $0 \le n < n_0$

Ex:-

Calculate $x_c[n]$ for $x[n] = \{2,4,6,8,10,12\}$

$0 \le n \le 5$,



Sol: Let $n_0 = 2$,

$N = 6$

$\Rightarrow x[\langle n-2 \rangle_6] = x[n-n_0]$; $2 \le n \le 5$

$x[n-n_0+N]$; $0 \le n < 2$

$n=0$, $x(0) = x(4) = 10$

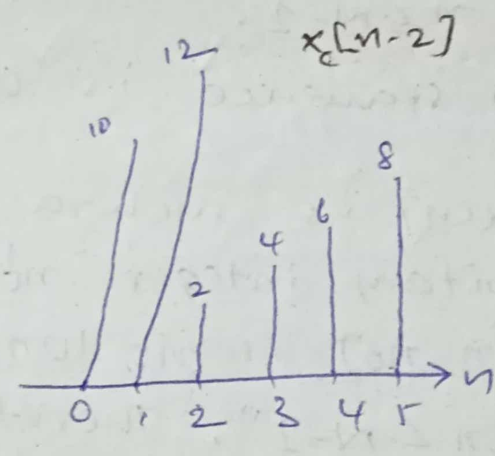
$n=1$, $x(1) = x(5) = 12$

$n=2$, $x(2) = x(0) = 2$

$n=3$, $x(3) = x(1) = 4$

$n=4$, $x(4) = x(2) = 6$

$n=5$, $x(5) = x(3) = 8$



$\rightarrow n_0 = 4$

- $x_c(n) = x(\langle n-4 \rangle_6]$

$= x[n-4]$; $4 \le n \le 6$

$x[n-4+6]$; $0 \le n < 4$

$\therefore x(0) = x(2) = 6$

$x(1) = x(3) = 8$

$x(2) = x(4) = 10$

$x(3) = x(5) = 12$

$x(4) = x(0) = 2$

$x(5) = x(1) = 4$

$\rightarrow n_0 = 6$

- $x_c(n) = x(\langle n-6 \rangle_6]$

$= x[n-6]$; $6 \le n \le 5$

$x[n-6+6]$; $0 \le n < 6$

- $x(0) = x(0)$
- $x(1) = x(1)$
- $x(2) = x(2)$
- $x(3) = x(3)$
- $x(4) = x(4)$
- $x(5) = x(5)$ //

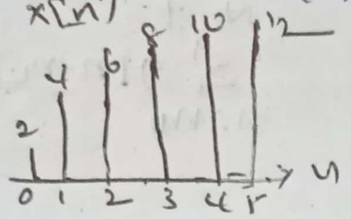
* CIRCULARY TIME-REVERSAL SEQUENCE:

The circularly time-reversal sequence is also of length 'N' and given by:

$$x[\langle -n \rangle_N] = x[\langle N-n \rangle_N]$$

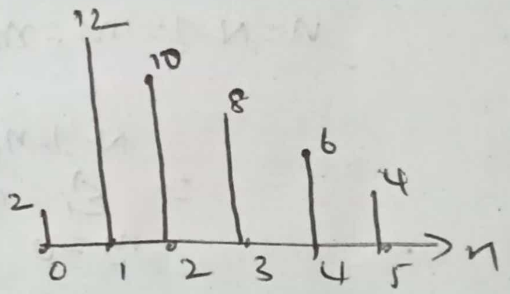
$$x[\langle -n \rangle_N] = x[N-n] \quad ; \; 1 \le n \le N-1$$
$$= x[n] \quad ; \; n=0$$

Ex:- $x[n] =$



$x[\langle -n \rangle_N] = x[6-n] \; ; \; 1 \le n \le N-1 \, (5)$
$x[n] \; ; \; n=0$

$n=0$, $x(0) = x(0) = 2$
$n=1$, $x(1) = x(5) = 12$
$n=2$, $x(2) = x(4) = 10$
$n=3$, $x(3) = x(3) = 8$
$n=4$, $x(4) = x(2) = 6$
$n=5$, $x(5) = x(1) = 4$



→ **Properties:**

1. LINEARITY :- if DFT of $g_1[n] = G_1[k]$, $g_2(n) = G_2[k]$ then DFT of $a g_1(n) + b g_2(n) = aG_1[k] + bG_2[k]$

Proof: DFT of $a g_1[n] + b g_2[n]$

$$= \sum_{n=0}^{N-1} \left(a g_1[n] + b g_2[n]\right) W_N^{nk}$$

$$= \sum_{n=0}^{N-1} a g_1[n] W_N^{nk} + \sum_{n=0}^{N-1} b g_2[n] W_N^{nk}$$

$$= a G_1[k] + b G_2[k] \; //$$

2. TIME SHIFTING: If DFT of $g[n] = G[k]$ then

DFT of $g[\langle n-n_0\rangle_N] = W_N^{kn_0} G[k]$

$= e^{-j\frac{2\pi}{N}kn_0} G[k]$

<u>proof:</u> DFT of $g[\langle n-n_0\rangle_N] = \sum_{n=0}^{N-1} g[\langle n-n_0\rangle_N] W_N^{nk}$

$= \sum_{n=n_0}^{N-1} g[n-n_0] W_N^{nk} + \sum_{n=0}^{n_0-1} g[n-n_0+N] W_N^{nk}$

let, $n-n_0=m$; $n-n_0+N=m$

$n=m+n_0$; $n=m+n_0-N$

$n=n_0 \Rightarrow m=0$; $n=0 \Rightarrow m=-n_0+N$

$n=N-1 \Rightarrow m=(N-1)-n_0$; $n=n_0-1 \Rightarrow m=N-1$

$= \sum_{m=0}^{N-1-n_0} g[m] \cdot W_N^{k(n_0+m)} + \sum_{m=N-n_0}^{N-1} g[m] W_N^{k(m+n_0-N)}$

$= \sum_{m=0}^{N-1-n_0} g[m] W_N^{kn_0} \cdot W_N^{km} + \sum_{m=N-n_0}^{N-1} g[m] W_N^{km} \cdot W_N^{kn_0}$

$= \sum_{n=0}^{N-1-n_0} g[n] \cdot W_N^{kn} \cdot W_N^{kn_0} + \sum_{m=N-n_0}^{N-1} g[n] W_N^{kn} W_N^{kn_0}$

$= \sum_{n=0}^{N-1-n_0} g[n] W_N^{nk} \cdot W_N^{kn_0} + \sum_{n=N-n_0}^{N-1} g[n] W_N^{nk} W_N^{kn_0} (1)$

$= W_N^{kn_0} \cdot \sum_{n=0}^{N-1} g[n] W_N^{nk}$ //

$$\boxed{g[\langle n-n_0\rangle_N] = W_N^{kn_0} \cdot G[k]}$$

3. CIRCULAR FREQUENCY SHIFTING PROPERTY:

If DFT of $g[n] = G[k]$ then, DFT of

DFT $[W_N^{-k_0 n} g[n]] = G[\langle k-k_0\rangle_N]$

proof:

LHS: $\text{DFT of } W_N^{-k_0 n} \cdot g[n] = \sum_{n=0}^{N-1} W_N^{-k_0 n}\, g[n] \cdot W_N^{kn}$

$$= \sum_{n=0}^{N-1} g[n]\, W_N^{n(k-k_0)}$$

$$\boxed{W_N^{-k_0 n}\, g[n] = G[\langle k-k_0 \rangle_N]}$$

(or)

RHS: $G[\langle k-k_0 \rangle_N] = G[k-k_0] \;;\; k_0 \le k \le N-1$

$G[k-k_0+N] \;;\; 0 \le k \le k_0-1$

$\therefore$ w.k.T., $G[k] = \sum_{n=0}^{N-1} g[n]\, W_N^{kn} \;:\; 0 \le k \le N-1$

① — $G[k-k_0] = \sum_{n=0}^{N-1} g[n] \cdot W_N^{kn}\, W_N^{-k_0 n} \;;\; k_0 \le k \le N-1$

② — $G[k-k_0+N] = \sum_{n=0}^{N-1} g[n]\, W_N^{kn}\, W_N^{-k_0 N}\, W_N^{N} \;;\; 0 \le k \le k_0$

①+② $\Rightarrow$ $G[\langle k-k_0 \rangle_N] = \sum_{n=0}^{N-1} g[n]\, W_N^{-k_0 N} \cdot W_N^{kN}$

$$\boxed{G[\langle k-k_0 \rangle_N] = \text{DFT}\{g[n]\, W_N^{-k_0 N}\}}$$ //

4. DUALITY PROPERTY: If DFT of $x[n]$ is $X[k]$, then DFT of $X[n] = N \cdot x[\langle -k \rangle_N]$

proof: w.k.T, $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N} kn}$

replace $n \to k$, $k \to n$

$$x[k] = \frac{1}{N} \sum_{k=0}^{N-1} X[n]\, e^{j(\frac{2\pi}{N})kn}$$

replace $k = -k$

$$x[\langle -k \rangle_N] = \frac{1}{N} \sum_{k=0}^{N-1} X[n]\, e^{-j(\frac{2\pi}{N})kn}$$

$\Rightarrow$ $\boxed{\text{DFT}[X[n]] = N x[\langle -k \rangle_N]}$

⑤ CIRCULAR CONVOLUTION THEOREM:

* statement: $g[n] \circledast h[n] = G[k]H[k]$

proof:

let $y[n] = g[n] \circledast h[n]$

$\therefore$ DFT of $\{y[n]\} = \sum_{n=0}^{N-1} y[n] W_N^{kn}$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} g[m] \cdot h[\langle k-m \rangle_N] \cdot W_N^{kn}$$

$$= \sum_{m=0}^{N-1} g[m] \sum_{n=0}^{N-1} h[\langle k-m \rangle_N] \cdot W_N^{kn}$$

$$= \sum_{m=0}^{N-1} g[m] \cdot e^{-j\left(\frac{2\pi}{N}\right)km} \cdot H[k]$$

$$\boxed{\text{DFT of } g[n] \circledast h[n] = G[k] \cdot H[k]}$$

⑥ MODULATION THEOREM / MULTIPLICATION IN TIME DOMAIN

· Statement: $g[n] \cdot h[n] \rightleftharpoons \frac{1}{N} \sum_{l=0}^{N-1} G[l] \cdot H[\langle k-l \rangle_N]$

· Proof: W.K.T, $g[n] \rightleftharpoons \frac{1}{N} \sum_{l=0}^{N-1} G[l] \cdot W_N^{-ln}$

$\Rightarrow$ DFT of $g[n]h[n] = \frac{1}{N} \sum_{l=0}^{N-1} G[l] \underbrace{\sum_{n=0}^{N-1} h(n) W_N^{-ln} \cdot W_N^{kn}}_{h[\langle k-l \rangle_N]}$

$$\boxed{g[n]h[n] \rightleftharpoons \frac{1}{N} \sum_{l=0}^{N-1} G[l] H[\langle k-l \rangle_N]}$$ //

* PARSEVAL'S RELATION: $\sum_{n=0}^{N-1} |g[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |G[k]|^2$

$$\sum_{n=0}^{N-1} g[n] \cdot h^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} G[k] \cdot H^*[k]$$

proof:

LHS: $\sum_{n=0}^{N-1} g[n] \cdot h^*[n]$

$$= \frac{1}{N} \sum_{k=0}^{N-1} G[k] \sum_{n=0}^{N-1} h^*[n] W_N^{-kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} G[k]\left[\sum_{n=0}^{N-1} h[n]\, W_N^{kn}\right]^*$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} G[k]\,.\, H^*[k]$$

$= RHS$ //

* Symmetry properties of DFT of a real sequence of length N:

- $x[n] = x^*[n]$ Hence no need to take conjugate for real sequence.

(i) $x[n] — X[k]$

(ii) $x[\langle -n\rangle_N] — X^*[k]$

(iii) $x_{ev}[n] — X_{re}[k]$

(iv) $x_{od}[n] — j X_{im}[k]$

(v) $X[k] = X^*[\langle -k\rangle_N]$

(vi) $X_{re}[k] = X_{re}[\langle -k\rangle_N]$

(vii) $X_{im}[k] = -X_{im}[\langle -k\rangle_N]$

(viii) $|X[k]| = |X[\langle -k\rangle_N]|$

(ix) $\operatorname{Arg} X[k] = -\operatorname{Arg} X[\langle -k\rangle_N]$

(3) $x_{ev}[n] - X_{re}[k]$

(4) $x_{od}[n] - j X_{im}[k]$

w.k.t $x_{ev}[n] = [x[n] + x[\langle -n\rangle_N]]/2$

$x_{od}[n] = [x[n] - x[\langle -n\rangle_N]]/2$

$\Rightarrow x_{ev}[n] \rightleftharpoons \frac{X[k] + X^*[k]}{2}$ — (1)

and $x_{od}[n] \rightleftharpoons \frac{X[k] - X^*[k]}{2}$ — (2)

w.k.t, $X[k] = X_{re}[k] + j X_{im}[k]$ — (A)

$X^*[k] = X_{re}[k] - j X_{im}[k]$ — (B)

A + B in (1) $\Rightarrow$

$x_{ev}[n] \rightleftharpoons \frac{2 X_{re}[k]}{2}$

$$\boxed{x_{ev}[n] \rightleftharpoons X_{re}[k]}$$

(A) - (B) in (2) $\Rightarrow$

$x_{od}[n] \rightleftharpoons \frac{2 j X_{im}[k]}{2}$

$$\boxed{x_{od}[n] \rightleftharpoons j X_{im}[k]}$$

⑤

$$X[k] = X^{*}[\langle -k \rangle_N]$$

w.k.t $X[k] = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$ —①

, $k = -k$

$$X[\langle -k \rangle_N] = \sum_{n=0}^{N-1} x(n) e^{j(\frac{2\pi}{N})kn}$$

$$X^{*}[\langle -k \rangle_N] = \sum_{n=0}^{N-1} x(n) e^{-j(\frac{2\pi}{N})kn} \quad [\text{From ①}]$$

$$= X[k]$$

$$\Rightarrow \boxed{X[k] = X^{*}[\langle -k \rangle_N]}$$

⑥ $X_{re}[k] = X_{re}[\langle -k \rangle_N]$

⑦ $X_{im}[k] = -X_{im}[\langle -k \rangle_N]$

w.k.t, $X[k] = X_{re}[k] + X_{im}[k]$

$$\sum_{n=0}^{N-1} x(n) e^{-j(\frac{2\pi}{N})kn} = X_{re}[k] + X_{im}[k]$$

$$\sum_{n=0}^{N-1} [x(n) \cdot \cos(\frac{2\pi}{N})kn - j x(n) \sin(\frac{2\pi}{N})kn] = X_{re}[k] + X_{im}[k]$$

$$\Rightarrow X_{re}[k] = \sum_{n=0}^{N-2} x[n] \cos(\frac{2\pi}{N})kn \quad —①$$

$$X_{im}[k] = -j \sum_{n=0}^{N-1} x(n) \sin(\frac{2\pi}{N})kn \quad —②$$

similarly, $X[\langle -k \rangle_N] = X_{re}[\langle -k \rangle_N] - X_{im}[\langle -k \rangle_N]$

$$\sum_{n=0}^{N-1} x(n) e^{j(\frac{2\pi}{N})kn} = X_{re}[\langle -k \rangle_N] - X_{im}[\langle -k \rangle_N]$$

$$\Rightarrow X_{re}[\langle -k \rangle_N] = \sum_{n=0}^{N-1} x(n) \cos(\frac{2\pi}{N})kn \quad —③$$

$$X_{im}[\langle -k \rangle_N] = j \sum_{n=0}^{N-1} x(n) \sin(\frac{2\pi}{N})kn \quad —④$$

From ①, ③ & ②, ④

$$X_{re}[k] = X_{re}[\langle -k \rangle_N]$$

$$X_{im}[k] = -X_{im}[\langle -k \rangle_N]$$

(7) $x(\langle -n\rangle_N) \rightleftharpoons X^*[k]$

proof: w.k.T $x(n) \rightleftharpoons \sum_{n=0}^{N-1} x(n) W_N^{nk}$

$\Rightarrow x(\langle -n\rangle_N) \rightleftharpoons \sum_{n=0}^{N-1} x(-n) W_N^{nk}$

$\rightleftharpoons \left[\sum_{n=0}^{N-1} x(n).W_N^{nk}\right]^*$

$x(n); n=0$
$x(N-n); 1\le n\le N-1$

$$\boxed{x(\langle -n\rangle_N) \rightleftharpoons X^*[k]}$$

(or)

$\therefore DFT[x(\langle -n\rangle_N)] = \sum_{n=0}^{N-1} x(\langle -n\rangle_N) W_N^{kn}$

$x(\langle -n\rangle_N) = x(\langle N-n\rangle_N)$
$= x(N-n); 1\le n\le N-1$
$= x(n), n=0$

$= \sum_{n=0}^{N-1} x(\langle N-n\rangle_N) W_N^{kn}$

$= \sum_{n=0} x(n).W_N^{kn} + \sum_{n=1}^{N-1} x[N-n] W_N^{kn}$

put $N-n=m$
$n=N-m$

$= \sum_{n=0} x(n) W_N^{kn} + \sum_{m=1}^{N-1} x(m) W_N^{k(N-m)}$ if $m=1 \Rightarrow n=N-1$

$= \sum_{n=0} x(n) W_N^{kn} + \sum_{n=1}^{N-1} x(n) W_N^{-kn}$

$= x(0) + \sum_{n=1}^{N-1} x(n) W_N^{-kn}$

$= \sum_{n=0}^{N-1} x(n) W_N^{-kn}$

$= \left[\sum_{n=0}^{N-1} x(n).W_N^{kn}\right]^*$

of $x[\langle -n\rangle_N] = x(N-n); 1\le n\le N-1$
$x(n), n=0$

$= X^*(k)$ //

(8) $|X[k]| = |X[\langle -k\rangle_N]|$

w.k.T $|X[k]|^2 = X[k].X^*[k]$

$= X^*[\langle -k\rangle_N].X[\langle -k\rangle_N]$

$|X[k]|^2 = |X[\langle -k\rangle_N]|^2$

$|X[k]| = |X[\langle -k\rangle_N]|$ //

(9) $\text{Arg}[X[k]] = -\text{Arg}[X[\langle -k \rangle_N]]$

w.k.t $\text{Arg}\, X[k] = \tan^{-1}\left(\frac{X_{im}[k]}{X_{re}[k]}\right)$

$= \tan^{-1}\left(-\frac{X_{im}[\langle -k\rangle_N]}{X_{re}[\langle -k\rangle_N]}\right)$

$= -\tan^{-1}\left(\frac{X_{im}[\langle -k\rangle_N]}{X_{re}[\langle -k\rangle_N]}\right)$

$= -\text{Arg}\{X[\langle -k\rangle_N]\}$ //

* Symmetry properties of DFT of complex sequence

$x(n) = x_{re}(n] + j x_{im}[n]$

$X[k] = X_{re}[k] + j X_{im}[k]$

(i) $x^*[n] - X^*[\langle -k\rangle_N]$ (ii) $x^*[\langle -n\rangle_N] - X^*[k]$

(iii) $x_{re}[n] - X_{cs}[k] = \frac{1}{2}[X[k] + X^*[\langle -k\rangle_N]]$

(iv) $j x_{im}[n] - X_{cas}[k] = \frac{1}{2}[X[k] - X^*[\langle -k\rangle_N]]$

(v) $x_{cs}[n] - X_{re}[k]$

(vi) $x_{cas}[n] - j X_{im}[k]$

(i) $x^*[n] - X^*[\langle -k\rangle_N]$,

soln: IDFT of $X^*[\langle -k\rangle_N] = \frac{1}{N}\sum_{k=0}^{N-1} X^*[\langle -k\rangle_N] W_N^{-kn}$

$= \frac{1}{N}\left[\sum_{k=0} X^*[+k] + \sum_{k=1}^{N-1} X^*[N-k] W_N^{-kn}\right]$

$N-k = m$
$k = N-m$

$= \frac{1}{N}\left[\sum_{k=0} X^*[k] + \sum_{m=1}^{N-1} X^*[m] \cdot W_N^{mn}\right]$

$= \frac{1}{N}\left[\sum_{k=0} X^*[k] + \sum_{k=1}^{N-1} X^*[k] W_N^{+kn}\right]$

$= \frac{1}{N}\sum_{k=0}^{N-1} X^*[k] W_N^{+kn}$

$$= \left[\frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot W_N^{-kn}\right]^*$$

$$= [x[n]]^*$$

$$\boxed{X^*[\langle -k \rangle_N] \rightleftharpoons x^*[n]}$$

ii) $x[\langle -n \rangle_N] - X[\langle -k \rangle_N]$

DFT of $x[\langle -n \rangle_N] = \sum_{n=0}^{N-1} x[\langle -n \rangle_N] W_N^{kn}$

$$= \sum_{n=0} x[n] W_N^{kn} + \sum_{n=1}^{N-1} x[N-n] W_N^{kn}$$

$N-n = M$

$n = N-M$

$$= \sum_{n=0} x[n] W_N^{kn} + \sum_{M=1}^{N-1} x[M] W_N^{k(N-M)}$$

$$= x(0) + \sum_{n=1}^{N-1} x[m] W_N^{-kn}$$

$$= \left[\sum_{n=0}^{N-1} x^*[n] W_N^{(+k)n}\right]^*$$

$$= [X^*[\langle -k \rangle_N]]^*$$

$$= X[\langle -k \rangle_N] \;//$$

iii) $x_{re}[n] = X_{cs}[k]$

Proof: w.k.T. $x(n) = x_{re}[n] + j x_{im}[n]$

$X(k) = X_{cs}[k] + X_{cas}[k]$

w.k.T $X_{cs}[k] = \frac{X[k] + X^*[\langle -k \rangle_N]}{2}$ — (1)

$$X[k] \rightleftharpoons \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$X[k] \rightleftharpoons \sum_{n=0}^{N-1} [x_{re}[n] + j x_{im}[n]] W_N^{kn} \quad — (2)$$

$$X^*[\langle -k \rangle_N] \rightleftharpoons \sum_{n=0}^{N-1} [x_{re}[n] - j x_{im}[n]] W_N^{kn} \quad — (3)$$

(2) + (3) ⇒ $X[k] + X^*[\langle -k \rangle_N] \rightleftharpoons 2\sum_{n=0}^{N-1} x_{re}[n] W_N^{kn}$

$$X_{cs}[k] \rightleftharpoons x_{re}[n] \;//$$

ii) $x^*[\langle -n \rangle_N] - X^*[k]$

w.k.t

$$x[n] \rightleftharpoons \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

$$x^*[\langle -n \rangle_N] \rightleftharpoons \sum_{n=0}^{N-1} x^*[\langle -n \rangle_N] W_N^{kn}$$

$$\rightleftharpoons \sum_{n=0} x^*[n] W_N^{kn} + \sum_{n=1}^{N-1} x^*[N-n] W_N^{kn}$$

$N-n=m$

$n=N-m$

$$\rightleftharpoons \sum_{n=0} x^*[n] + \sum_{m=1}^{N-1} x^*[m] W_N^{k(N-m)}$$

$$\rightleftharpoons \sum_{n=0} x^*[n] + \sum_{n=1}^{N-1} x^*[n] W_N^{-kn}$$

$$\rightleftharpoons \sum_{n=1}^{N-1} x^*[n] W_N^{-kn}$$

$$\rightleftharpoons \sum_{n=1}^{N-1} x[n] W_N^{kn}]^*$$

$$\rightleftharpoons X^*[k]$$ //

iv) $j x_{im}[n] - X_{cas}[k] = \frac{1}{2}[X[k] - X^*[\langle -k \rangle_N]]$

Proof:

$$X[k] \rightleftharpoons \sum_{n=0}^{N-1} x[n] W_N^{nk} \quad -① $$

$$X^*[\langle -k \rangle_N] \rightleftharpoons \sum_{n=0}^{N-1} x^*[n] W_N^{nk} \quad -②$$

① - ② => $X[k] - X^*[\langle -k \rangle_N] \rightleftharpoons 2j \sum_{n=0} x_{im}[n] W_N^{kn}$

$$X_{cas}[k] \rightleftharpoons j x_{im}[n]$$ //

v) $x_{cs}[n] - X_{re}[k]$

sol: DFT of $x_{cs}[n] = \sum_{n=0}^{N-1} \frac{1}{2}(x[n] + x^*[\langle -n \rangle_N]) W_N^{nk}$

$$= \frac{1}{2}\left[\sum_{n=0}^{N-1} x(n) W_N^{kn} + \sum_{n=0}^{N-1} x^*[\langle -n \rangle_N] W_N^{kn}\right]$$

$$= \frac{1}{2}\left(2X_{re}[k]\right)$$

$$x_{cs} \rightleftharpoons X_{re}[k] \;//$$

(vi) $x_{cas}[n] - j X_{im}[k]$

$$\text{DFT of } x_{cas}[n] = \frac{1}{2}\left[\sum_{n=0}^{N-1} \left(x[n] - x^*[\langle -n \rangle_N]\right) W_N^{nk}\right]$$

$$\rightleftharpoons \frac{1}{2}\left[X[k] - X^*[\langle -k \rangle_N]\right]$$

$$\rightleftharpoons \frac{2 j X_{im}[k]}{2}$$

$$x_{cas}[n] \rightleftharpoons j X_{im}[k] \;//$$

*

Q: The first 5 points of 8 point DFT of $x[k]$ are $\{0.25, 0.125 - j0.3018, 0, 0.125 - j0.518, 0\}$. Determine remaining 3 points. Also estimate the value of $x(0)$.

sol: Given, 5 points of 8 point DFT are:

$$\{0.25, 0.125 - j0.3018, 0, 0.125 - j0.518, 0\}$$

w.k.T $\boxed{X[k] = X^*[\langle -k \rangle_N]}$

- $X[k] = X^*[N-k]$; 1 to N-1

$$X[5] = X^*[3] = 0.125 + j0.518$$

$$X[6] = X^*[2] = 0$$

$$X[7] = X^*[1] = 0.125 + j0.3018$$

- w.k.T $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}$

$$\Rightarrow x[0] = \frac{1}{8}\sum_{k=0}^{7} X[k] W_N^{-k(0)}$$

$$= \frac{1}{8}[0.25 + 0.125 - j0.3018 + 0.125 - \cancel{j0.518} + 0.125 + \cancel{j0.518} + 0.125 + j0.318$$

$$= \frac{1}{8}[0.75] = 0.093 \;//$$

* CIRCULAR CONVOLUTION: $x_1(n) \circledast x_2(n) = X_1(K) \cdot X_2(K)$

· METHOD-1: CONCENTRIC CIRCULAR METHOD / CIRCULAR ARRAYS FOR FINDING CIRCULAR CONVOLUTION OF 2 SEQUENCES.

1. First graph each sequence as a point on a circle in a counter clockwise direction, then fold one sequence that is representing as a point on circle in a clockwise direction.

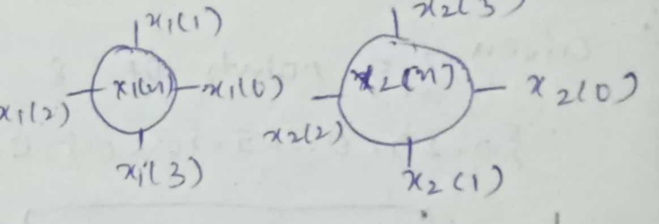
2. Multiply corresponding samples on 2 circles & sum the product to produce o/p.

3. Rotate folded sequence one sample at a time in anticlockwise direction and go to step-2 to obtain next value of o/p.

4. Repeat step-3, untill folded sequence circle first sample lines up with first sample of other circle once again.

Ex:- $x_1(n) = [2\ 2\ 2\ 1]$; $x_2(n) = [1\ 2\ 2\ 2]$

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2(n-m)$$

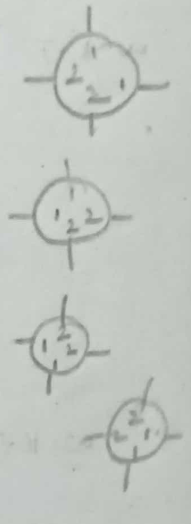


$$x_3(0) = \sum_{m=0}^{N-1} x_1(m)\, x_2(-m) = 1 + 2 + 4 + 2 = 9$$

$$x_3(1) = \sum_{m=1} x_1(1)\, x_2(1-m) = 2 + 2 + 2 + 2 = 8$$

$$x_3(2) = x_1(2)\, x_2(2-m) = 2 + 4 + 2 + 1 = 9$$

$$x_3(3) = x_1(3) + x_2(3) = 1 + 4 + 4 + 1 = 10$$



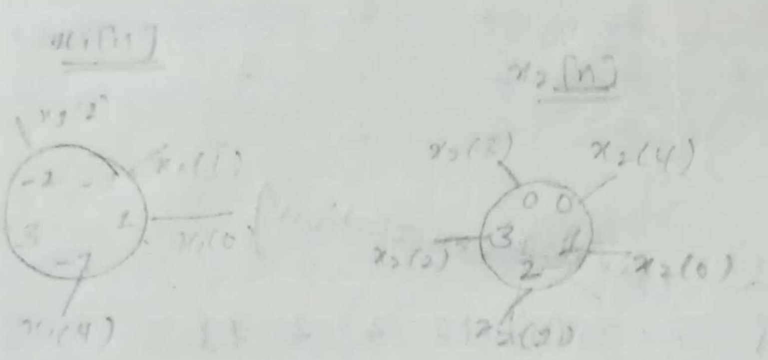
*** matrix method:**

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} : [9, 8, 9, 10]$$

⑧ Find the circular convolution of the sequence
$x_1[n] = \{1, -1, -2, 3, -1\}$; $x_2[n] = \{1, 2, 3\}$

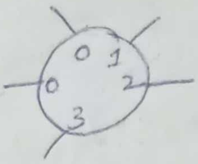
Soll: $x_1[n] = \{1, -1, -2, 3, -1\}$ $L = 5$

$x_2[n] = \{1, 2, 3, 0, 0\}$ $L = 5$



$Y(0) = \sum_{n=0} x_1[n]\, x_2[-n]$

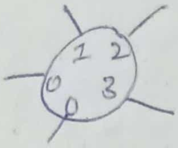
$= 1 + 0 + 0 + 9 - 2 = 8$



Matrix method

$Y(1) = 2 - 1 - 3 = -2$

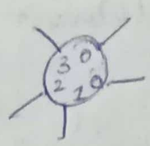
$$\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$$



$Y(2) = 3 - 2 - 2 = -2$

$$= \begin{bmatrix} 1 + 0 + 0 + 9 - 2 \\ 2 - 1 + 0 + 0 - 3 \\ 3 - 2 - 2 + 0 + 0 \\ 0 - 3 - 4 + 3 + 0 \\ 0 + 0 - 6 + 6 - 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}$$

$Y(3) = 0 - 3 - 4 + 3 = -4$



$Y(4) = 0 + 0 - 6 + 6 - 1 = -1$

* TABULAR ARRAYS :- (METHOD-2)

$x_1(n) = \{1 \quad 2 \quad 2 \quad 1\}$; $x_2(n) = \{1 \quad 2 \quad 2 \quad 1\}$

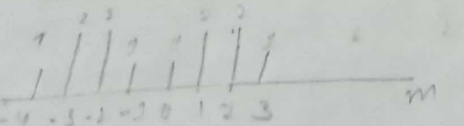
$x_1[m]$ $-(N-1)$ to $(N-1)$

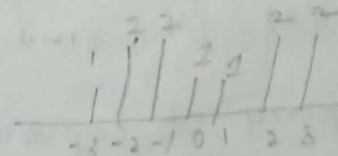
$x_2(n-m)$ $n = 0, 1, \ldots N-1$

$x_2(1-m)$

$x_1[m] = x_2[m]$

* $x_2[-m]$





(2 2 1)	1	1	2	2	9
(1 2 2)	2	1	1	2	8
(2 2 1)	2	2	1	1	9
(2 2 1)	1	2	2	1	10

m	-3	-2	-1	0	1	2	3
$x_1(m)$				1	2	2	1
$x_2(-m)$	2	2	1	1			
$x_2(1-m)$		2	2	1	1		
$x_2(2-m)$			2	2	1	1	
$x_2(3-m)$				1	2	2	1

- $y(0) = 9$
- $y(1) = 8$
- $y(2) = 9$
- $y(3) = 10$

STOCKHAM'S METHOD: (~~linear convolution~~)

① $x_1[n] = \{1\ 2\ 2\ 1\}$; $x_2[n] = \{1\ 2\ 2\ 1\}$

Sol:

$x_1(k) = DFT\{x_1(n)\}$

$x_2[k] = DFT\{x_2(n)\}$

$x_1(n) * x_2(n) \longrightarrow x_1(k) \cdot x_2(k)$

- $x_1[k] \cdot x_2(k) = Y(k)$
- $Y(n) = IDFT[Y(k)]$
- $x_1(n) = \{1, 2, 2, 1\}$; $x_1(k) = \sum_{n=0}^{N-1} x_1(n) W_N^{nk}$

$x_1[k] = \sum_{n=0}^{3} x_1(n) W_4^{nk}$

- $x_1[k] = \sum_{n=0}^{3} x_1(n) W_4^{nk} = x_1(0) W_4^0 + x_1(1) W_4^k + x_1(2) W_4^{2k} + x_1(3) W_4^{3k}$

* $x_1(k) = 1(1) + 2W_4^k + 2W_4^{2k} + 1 W_4^{3k}$

- $x_1(0) = 1 + 2 + 2 + 1 = 6$
- $x_1(1) = 1 + 2W_4^1 + 2W_4^2 + W_4^3 = -1 - j$
- $x_1(2) = 1 + 2W_4^2 + 2W_4^4 + W_4^6 = 1 - 2 + 2 - 1 = 0$
- $x_1(3) = 1 + 2W_4^3 + 2W_4^6 + W_4^9 = 1 + 2j - 2 - j = -1 + j$

$\Rightarrow x_1(k) = \{6, -1-j, 0, -1+j\}$

* property of convolution:

$x_1(k) \cdot x_2(k) = Y(k)$

- $Y(0) = x_1(0) \cdot x_2(0) = 36$
- $Y(1) = x_1(1) \cdot x_2(1) = 2j$
- $Y(2) = x_1(2) \cdot x_2(2) = 0$
- $Y(3) = x_1(3) \cdot x_2(3) = -2j$

IDFT.

$Y(k) = \{36, 2j, 0, -2j\}$

* $y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot W_N^{-nk} = \frac{1}{4}\left(Y(0) + Y(1)W_4^{-n(1)} + Y(2)W_4^{-n(2)} + Y(3)W_4^{-n(3)}\right]$

- $y(0) = \frac{36}{4} = 9$
- $y(1) = \frac{1}{4}(36 + 2j(j) + 0 + (-2j)(-j)) = \frac{1}{4}(36 - 2 - 2) = 8$
- $y(2) = \frac{1}{4}(36 + 2j(-1) + (-2j)(-1)) = \frac{36}{4} = 9$
- $y(3) = \frac{1}{4}(36 + 2 + 2) = \frac{40}{4} = 10$

$\Rightarrow y[n] = \{9, 8, 9, 10\}$ //

* FAST FOURIER TRANSFORM (FFT)

- To reduce computation time, memory space:

$4N^2 \rightarrow$ no. of real multiplications

$N^2 \rightarrow$ no. of complex multiplications

$N(4N-2) \rightarrow$ no. of real additions

$N(N-1) \rightarrow$ no. of complex ~~multiplications~~ additions

- If $N = 128$; $N^2 \rightarrow (128)^2 \rightarrow$ large space required and memory space to store them in digital computers.

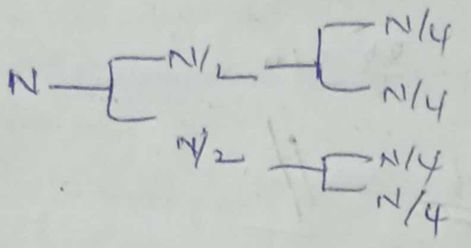
- $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \cdot W_N^{-nk}$; $0 \leq k \leq N-1$ —①

from eq① , it is clear that for each value of k the direct computation of $X[k]$ requires $4N$ (real multi) $4N-2$ (real additions). Since $X(k)$ must be computed for N, different values of k, the direct computation of DFT of a sequence $x[n]$ requires $4N^2$ real multiplications and $N(4N-2)$ real additions (or) approximately N^2 complex multiplications and $N(N-1)$ complex additions.

The implementation of the computation of ~~comp~~ DFT on a general purpose digital computer requires very large memory for storage of data and more time for computation. In order to reduce the above things we will go for the FFT. It is a set of computational Algorithms.

FFT Algorithms exploids the symmetry property and periodicity property. Therefore, it is an efficient algorithm.

DIT-FFT:-



- $x(n) - N \quad x(0), x(1) \dots x(N-1)$
- $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad ; \quad 0 \le k \le N-1$ — (1)

$x(0), x(2) \dots, x(N-2)$; Even
$x(1), x(3) \dots, x(N-1)$; odd

$$X(k) = \sum_{n=0}^{N-2} x(n) W_N^{kn} + \sum_{n=1}^{N-1} x(n) W_N^{kn} \qquad n = 2r \qquad \boxed{n = 2r+1}$$

$$\boxed{W_N^{k2r} = W_{N/2}^{kr}}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_{N/2}^{kr} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) \cdot W_{N/2}^{kr} \cdot W_N^{k} \quad \text{— (2)}$$

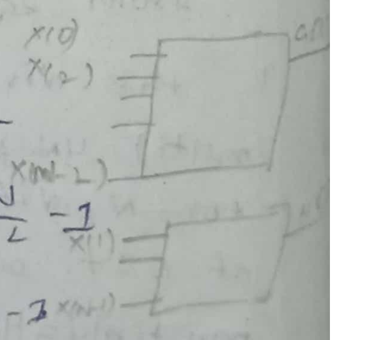
$\frac{N}{2}$ PDFT $\quad \left(\frac{N}{2}\right)^2$

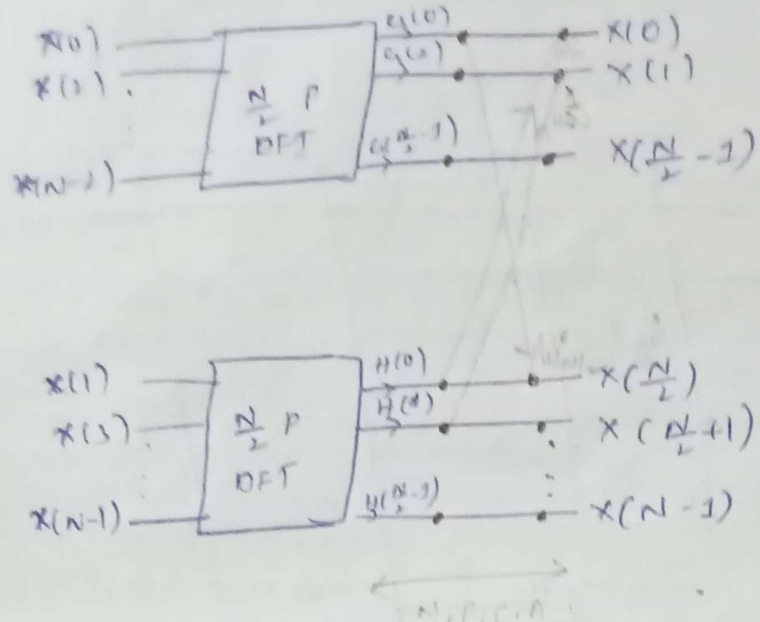
$\frac{N}{2}$ PDFT $\quad \left(\frac{N}{2}\right)^2$

- $G(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_{N/2}^{kr}$ — (3) $\quad ; 0 \le k \le \frac{N}{2} - 1$
- $H(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_{N/2}^{kr}$ — (4) $\quad ; 0 \le k \le \frac{N}{2} - 1$

$X(k) = G(k) + W_N^{k} H(k)$ — (5) $\quad 0 \le k \le \frac{N}{2} - 1$

$X(k) = G\left(k - \frac{N}{2}\right) + W_N^{k} H\left(k - \frac{N}{2}\right)$ (6)





$\cdot\ \frac{N}{2} \rightarrow 2\,\frac{N}{4}$.

put $2r = r$ in ③

$$\Rightarrow G[k] = \sum_{r=0}^{N/2-1} x(r)\cdot W_N^{kr}$$

$$= \sum_{r=0}^{N-4} x(r)\cdot W_N^{kr} + \sum_{r=2}^{N-2} x(r)\, W_N^{kr}$$

$r = 4l$ $\qquad$ $r = 4l+2$

$$= \sum_{l=0}^{\frac{N}{4}-1} x(4l)\, W_N^{k\cdot 4l} + \sum_{l=0}^{\frac{N}{4}-1} x(4l+2)\, W_N^{k\cdot 4l}\cdot W_N^{k\cdot 2}$$

$$= \underbrace{\sum_{l=0}^{\frac{N}{4}-1} x(4l)\cdot W_{N/4}^{kl}}_{G'[k]} + W_N^{2k}\underbrace{\sum_{l=0}^{\frac{N}{4}-1} x(4l+2)\cdot W_{\frac{N}{4}}^{kl}}_{H'[k]}$$

$\frac{N}{4}$ PDFT $\qquad$ $\frac{N}{4}$ PDFT

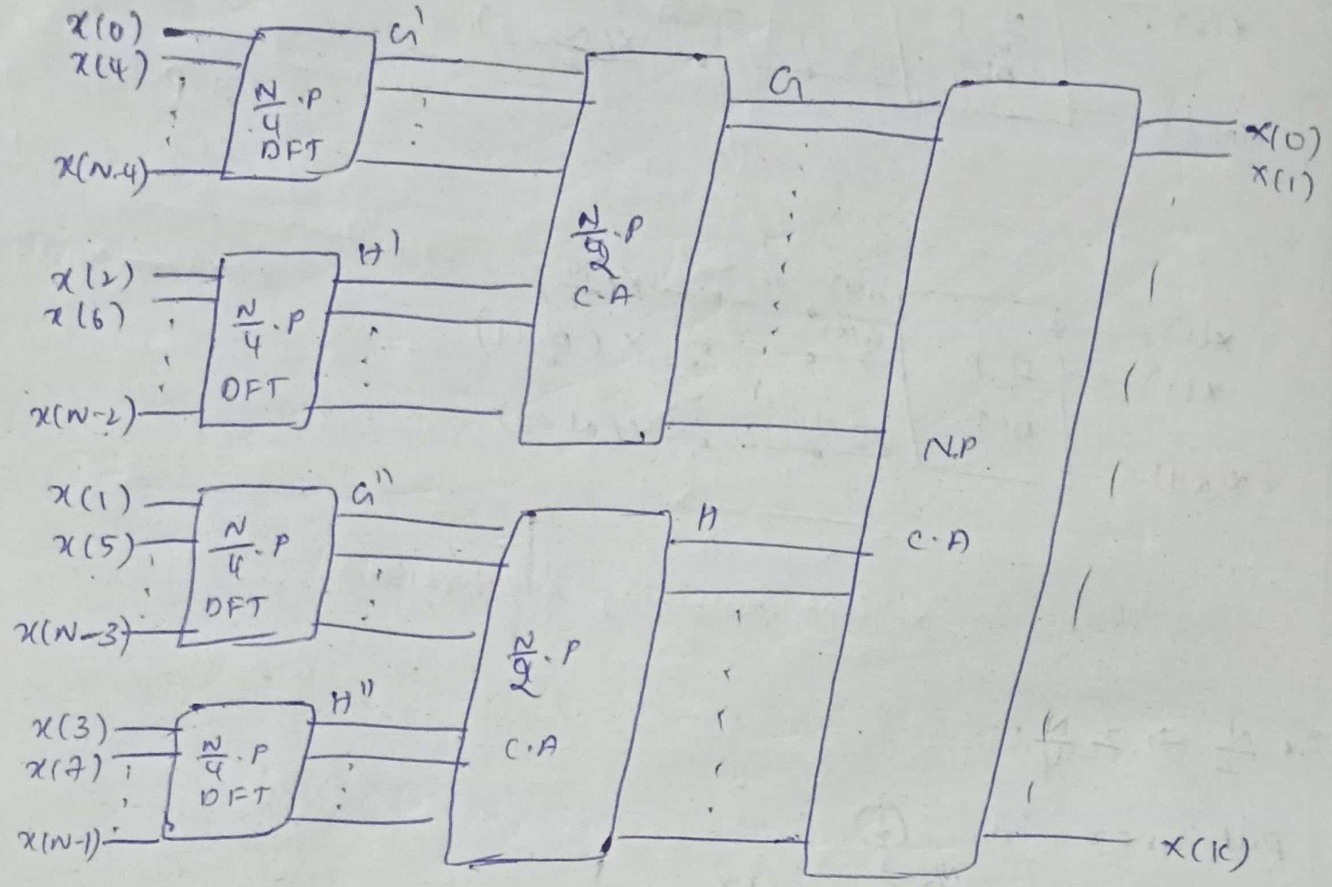
* $\Rightarrow G[k] = G'[k] + W_N^{2k} H'[k]$; $0 \le k \le \frac{N}{4}-1$ — ⑦

* and $G[k] = G'\left[k-\frac{N}{4}\right] + W_N^{2k}\cdot H'\left(k-\frac{N}{4}\right]$; $\frac{N}{4} \le k \le \frac{N}{2}-1$ — ⑧

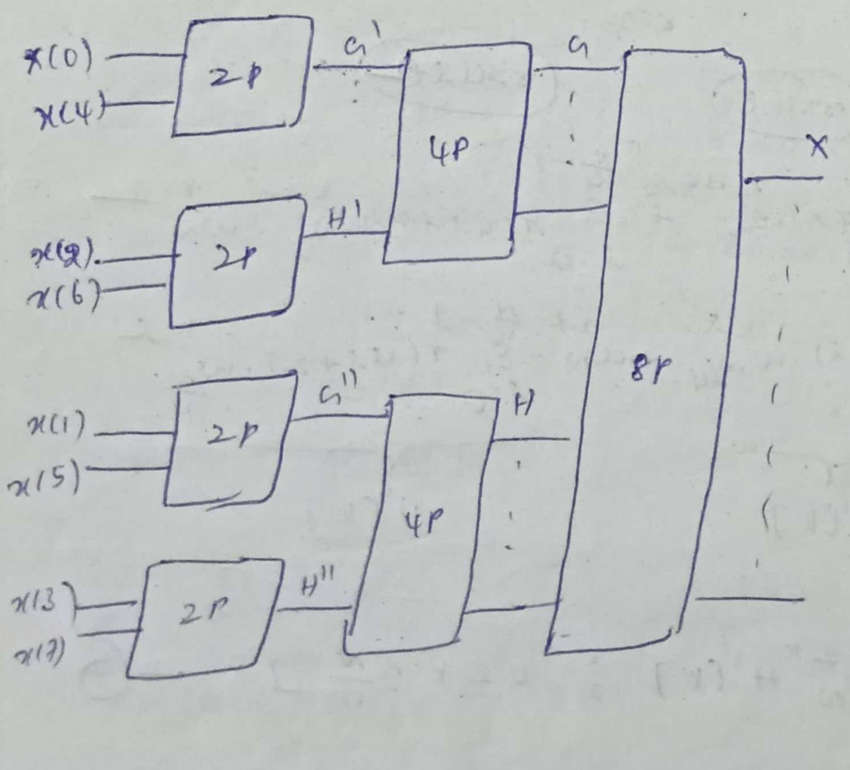
* lly $H[k] = G''[k] + W_N^{2k} H''[k]$; $0 \le k \le \frac{N}{4}-1$ — ⑨

* $H[k] = G''\left[k-\frac{N}{4}\right] + W_N^{2k} H''\left[k-\frac{N}{4}\right]$; $\frac{N}{4} \le k \le \frac{N}{2}-1$ — ⑩

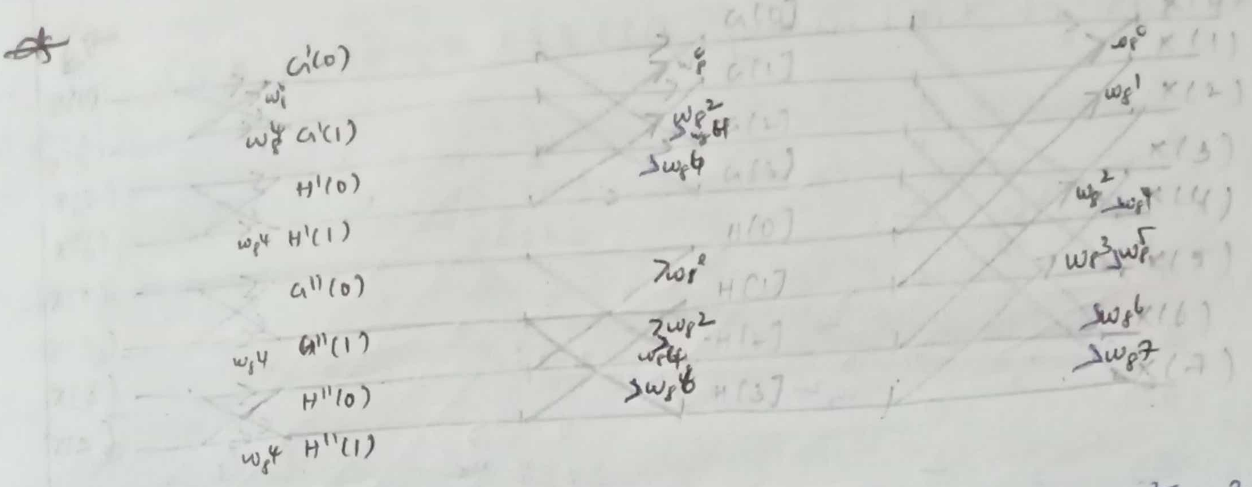
Decomposition of 'N' P DFT interms of '$\frac{N}{4}$' P DFTs :



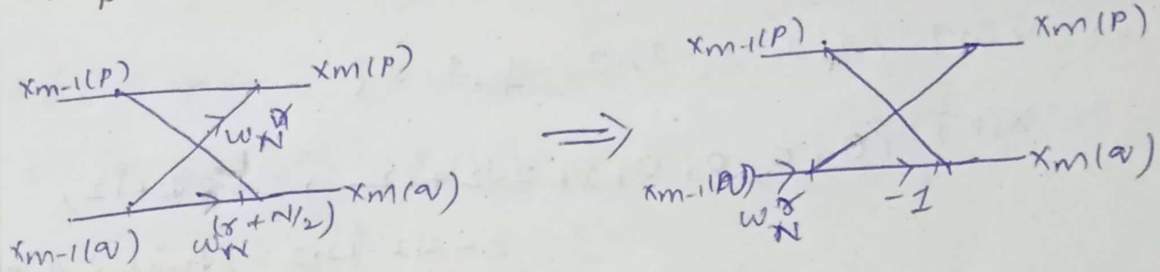
• Take N=8 and show decomposition :



*



- In Direct computation, no. of multiplications $= N^2 = 8^2 = 64$
- Using FFT, no. of multiplications $= \underline{\underline{N \log_2 N}} = 8 \log_2 2^3 = 24$
- Simplified Butterfly Diagram:-



* $X_m(P) = X_{m-1}(P) + W_N^r \, X_{m-1}(q)$
* $X_m(q) = X_{m-1}(P) + \underline{W_N^{(r+N/2)}} \, X_{m-1}(q)$
* $W_N^{r+N/2} = W_N^r \cdot W_N^{N/2}$

$= W_N^r \cdot e^{-j\left(\frac{2\pi}{N}\right)\cdot\frac{N}{2}}$

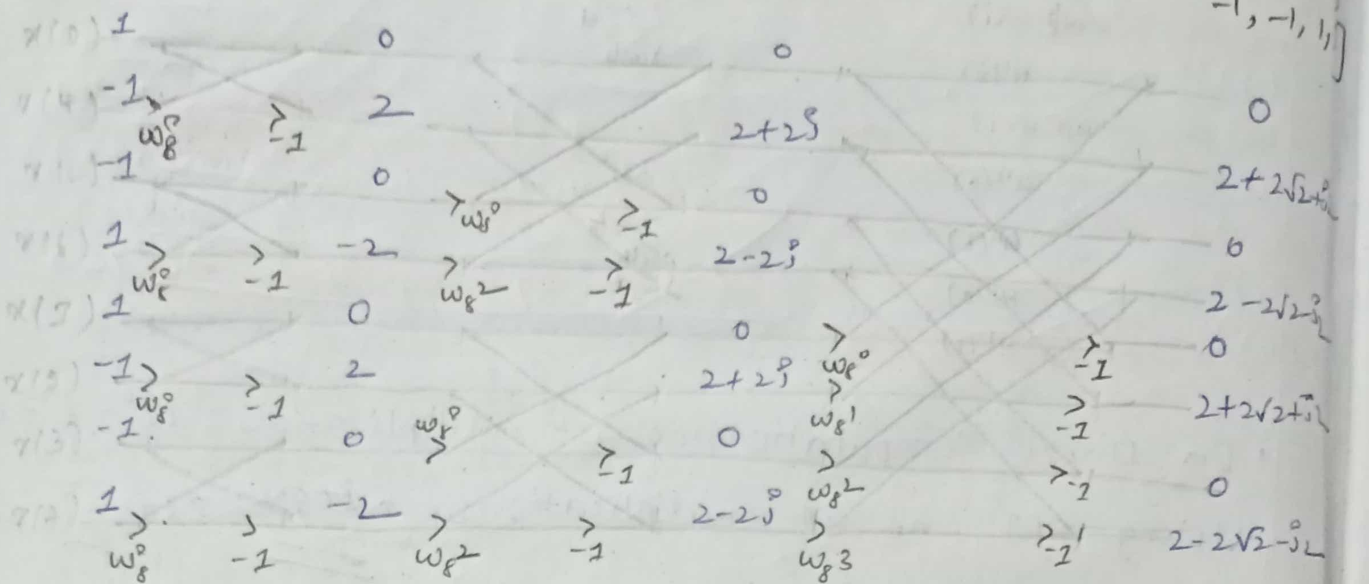
$= -W_N^r$

$\Rightarrow X_m(q) = X_{m-1}(P) + [-W_N^r] \, X_{m-1}(q)$

$N \log N = 4 \times 3 = 12$ //

Q: Find DFT of $x[n]$ using DIT FFT Algorithm, $x(n) = \{1, 1, -1, -1, -1, -1, 1, 1\}$

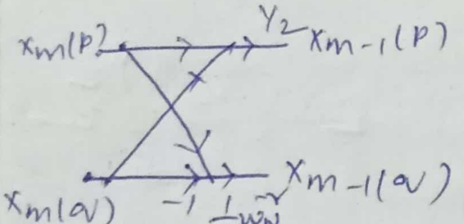
Sol:



$\therefore \quad x[n] = \{1, 1, -1, -1, -1, -1, 1, 1\}$

$X[k] = \{0, 0, 0, 0, 2+2\sqrt{2}+j2, 2+2\sqrt{2}+j2, 2-2\sqrt{2}-j2, 2-2\sqrt{2}-j2\}$ B.RO

*** CALCULATION OF IDFT USING DITFFT ALGORITHM:**



W.K.T.,

$$X_m(p) = X_{m-1}(p) + W_N^r \cdot X_{m-1}(q) \quad \text{--- (1)}$$

$$X_m(q) = X_{m-1}(p) - W_N^r \cdot X_{m-1}(q) \quad \text{--- (2)}$$

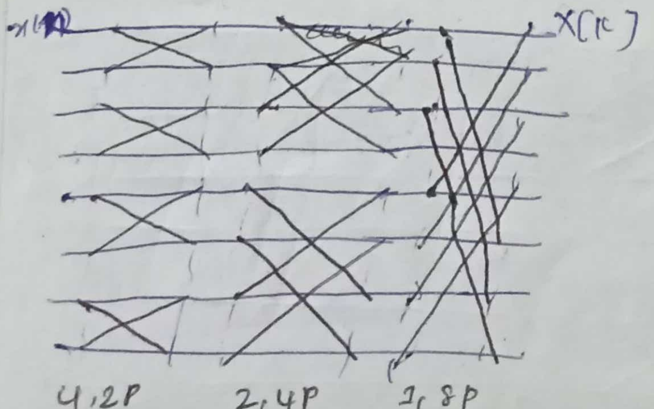
$\Rightarrow$ (1)+(2)

$$X_{m-1}(p) = \frac{1}{2}[X_m(p) + X_m(q)] \quad \text{--- (3)}$$

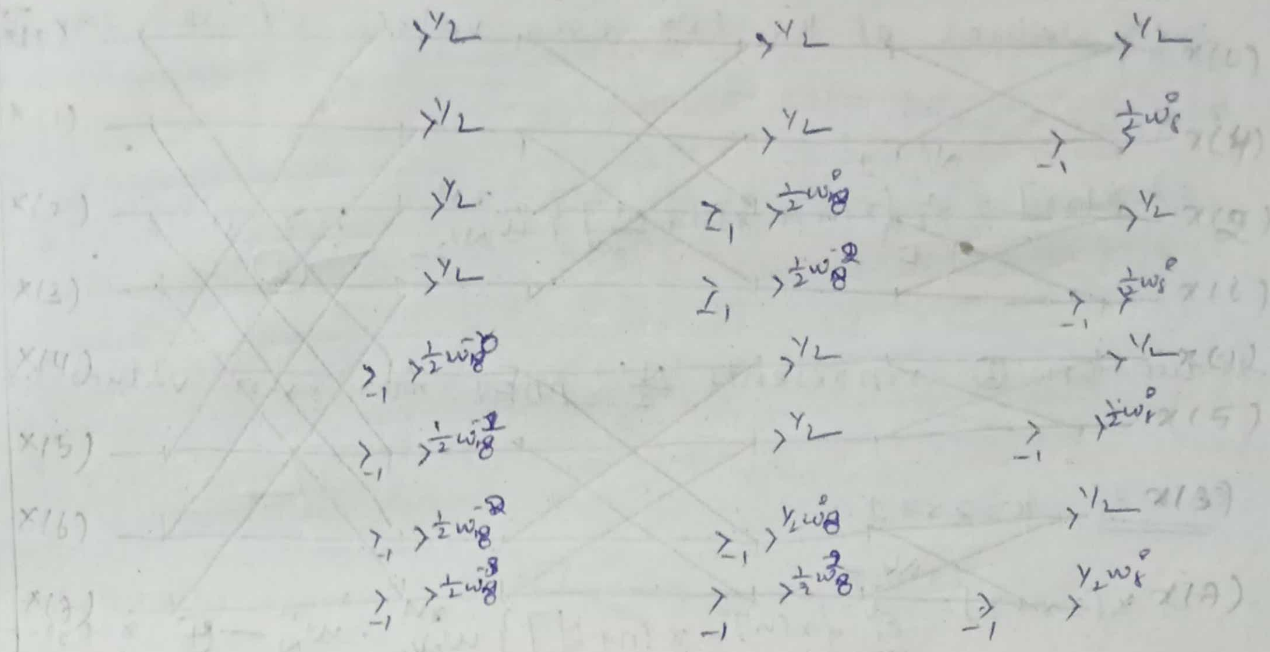
and (1)-(2)

$$\Rightarrow X_{m-1}(q) = \frac{1}{2} W_N^{-r} [X_m(p) - X_m(q)] \quad \text{--- (4)}$$

- For DFT, if N=8



For IDFT, if N=8 the signal flow graph is:-



DECIMATION IN FREQUENCY FAST FOURIER TRANSFORM:

W.K.T, $X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_N^{kn}$ — ① ; $0 \le k \le N-1$

$$X[k] = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn}$$

$$\boxed{n = r + \frac{N}{2}}$$

$$= \sum_{n=0}^{N/2-1} x(n) \cdot W_N^{kn} + \sum_{\cancel{r=0}}^{\frac{N}{2}-1} x[r + \tfrac{N}{2}] \cdot W_N^{rk} \cdot \underbrace{W_N^{k \cdot N/2}}_{\to (-1)^k}$$

$$X[k] = \sum_{n=0}^{N/2-1} \left\{ x(n) + (-1)^k x(n + \tfrac{N}{2}) \right\} W_N^{nk} \quad \text{—②}$$

~~CASE-i: put k=2r~~

$$\cancel{\Rightarrow X(2r) = \sum_{n=0}^{N/2-1}}$$

Now decimation is obtained by taking odd and even values of 'k'. For even values of 'k', say $\boxed{k=2r}$ $r = 0, 1, 2 \ldots$

$$\Rightarrow X[2r] = \sum_{n=0}^{N/2-1} \{x(n) + x(n + \tfrac{N}{2})\} W_{N/2}^{rn} \quad ; r = 0, 1, 2 \ldots \tfrac{N}{2}-1 \quad — (3)$$

→ Equation (3) represents '$\frac{N}{2}$' pDFT for even values of k.

<u>case-2:</u> $k = 2r+1$.

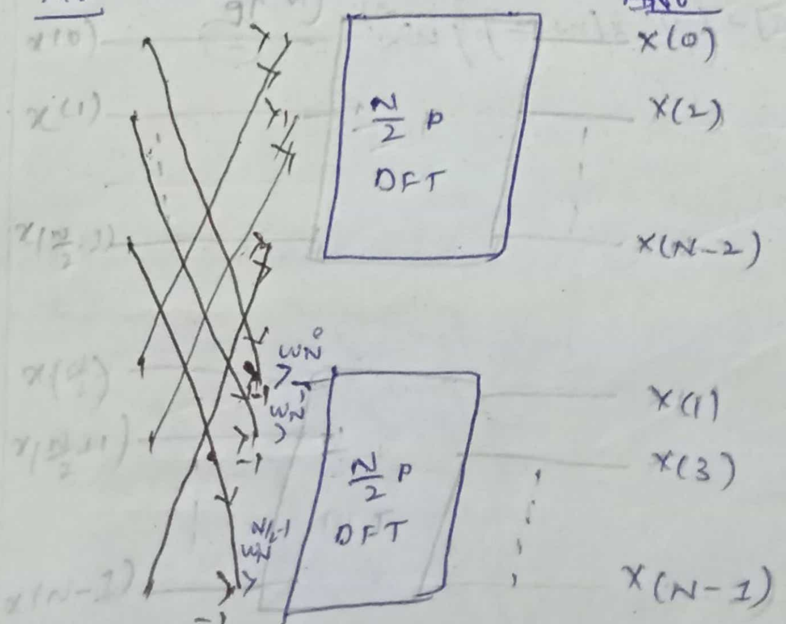
$$\Rightarrow X(2r+1) = \sum_{n=0}^{N/2-1} \{x(n) - x(n + \tfrac{N}{2})\} W_{N/2}^{rn} \cdot W_N^n \quad — (4) \; ; r = 0, 1, \ldots \tfrac{N}{2}-1$$

- $$X(2r) = \sum_{n=0}^{N/2-1} g(n) W_{N/2}^{rn} \quad ; r = 0, 1, \ldots \tfrac{N}{2}-1$$

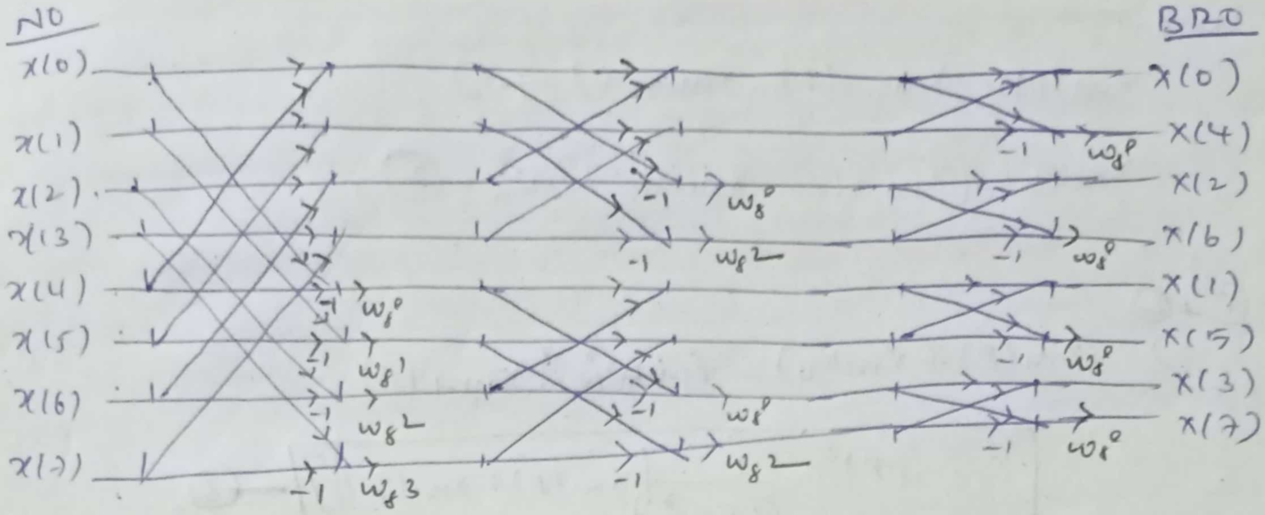
- $$X(2r+1) = \sum_{n=0}^{N/2-1} h(n) \cdot W_{N/2}^{rn}.$$

$\Rightarrow g(n) = x(n) + x(n + N/2)$ and $h(n) = \{x(n) - x(n + \tfrac{N}{2})\} W_N^n$

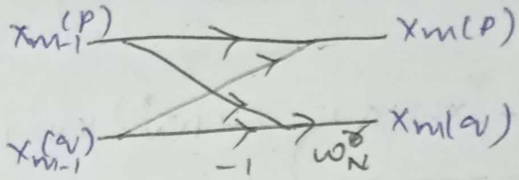
- Signal flow graph is shown below:-



Ex: For $N=8$, calculate DFT using DIFFFT.



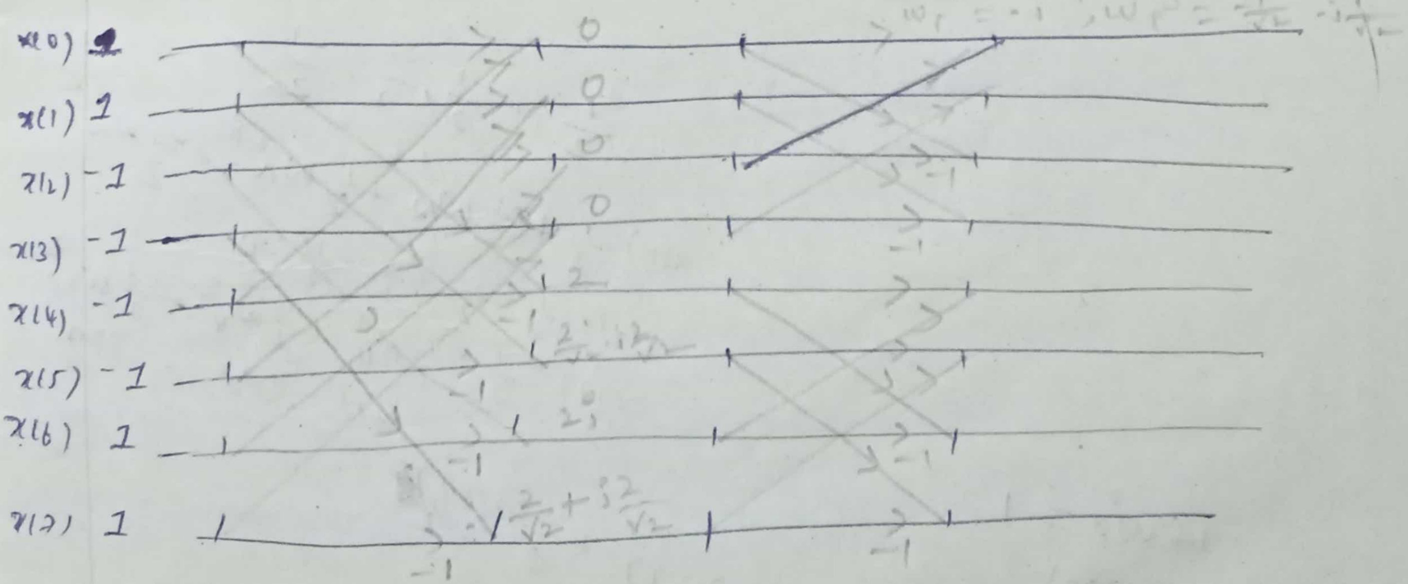
→ Butterfly block is:



* $X_m(p) = X_{m-1}(p) + X_{m-1}(q)$ — (a)

* $X_m(q) = [X_{m-1}(p) - X_{m-1}(q)]\, W_N^r$ — (b)

Ex: If $x(n) = \{1, 1, -1, -1, -1, -1, 1, 1\}$, calculate DFT using DIFFFT.



$\frac{1}{2} W_N^{-r}$

IDFT using DIF-FFT Algorithm:

W.K.T,

$$X_m(p) = X_{m-1}(p) + X_{m-1}(q) \quad — (1)$$

$$X_m(q) = [X_{m-1}(p) - X_{m-1}(q)]\, W_N^r \quad — (2)$$

(1) + (2)

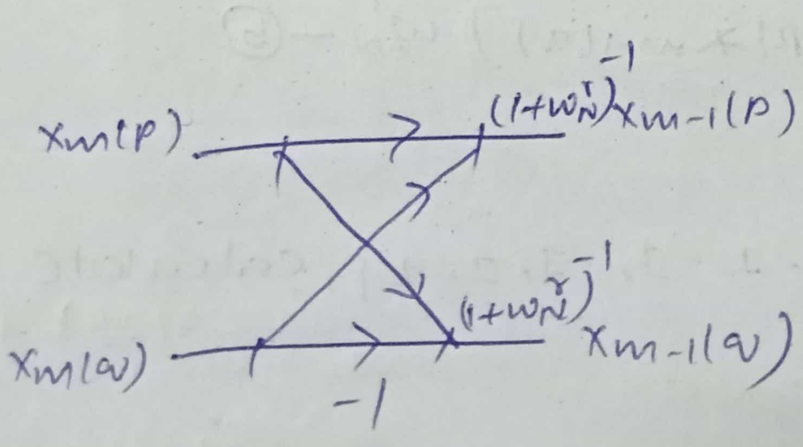
$$\Rightarrow X_m(p) + X_m(q) = (1+W_N^r)\, X_{m-1}(p)$$

$$\Rightarrow \boxed{X_{m-1}(p) = \frac{1}{(1+W_N^r)}\left[X_m(p) + X_m(q)\right]} \quad — (3)$$

(1) − (2)

$$\Rightarrow X_m(p) - X_m(q) = (1+W_N^r)\, X_{m-1}(q)$$

$$\Rightarrow \boxed{X_{m-1}(q) = \frac{1}{(1+W_N^r)}\left\{X_m(p) - X_m(q)\right\}} \quad — (4)$$



$\frac{1}{2}$, $\frac{-1}{2}$, $\frac{1}{2}$, $\frac{-1}{2}$