17/11/23

3. SIGNAL SPACE ANALYSIS:

Gram-Schmdit Procedure:-

The task of transforming an incoming msg (m_i), $i = 0, 1, 2 \ldots M$ into a modulated wave $\{s_i(t)\}$ may be divided into seperate discrete time and continuous time operations. The justification for this seperation lies in Gram-Schmdit Orthogonalization procedure which permitts the representation of any set of M energy signals represented by $\{s_i(t)\}$ as linear combination of 'N' orthonormal basis functions where "$N \leq M$".

We represent the given set of real valued signals i.e., $s_1(t), s_2(t) \ldots s_M(t)$ each of energy duration T-seconds in the form: $s_i(t) = \sum_{j=1}^{N} s_{ij} \cdot \phi_j(t)$.

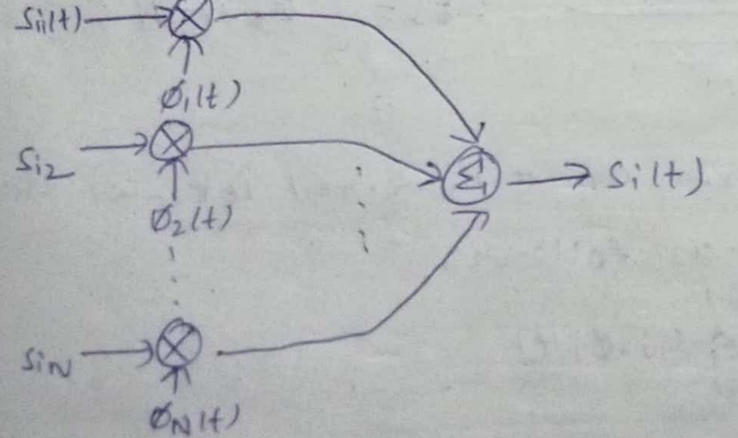
; $0 \leq t \leq T$. , $i = 1, 2, \ldots M$.

where, $s_{ij} = \int_0^T s_i(t) \cdot \phi_j(t)\, dt$; $i = 1, 2, \ldots, M$; $j = 1, 2, \ldots, N$

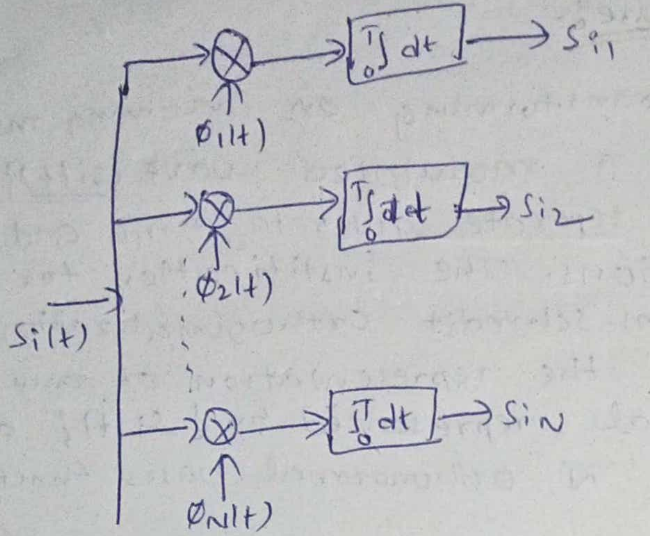
→ The real valued basis functions $\phi_1(t), \phi_2(t) \ldots \phi_N(t)$ are orthonormal (If any 2 signals are integrated over that interval, i.e., $\int_0^T \phi_i(t) \cdot \phi_j(t)\, dt = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$), Each basis function is normalised to have unit energy as specified by first condition.

The second condition states that, the basis functions $\phi_1(t) \ldots \phi_N(t)$ are orthogonal w.r.to each other over the interval over the interval $0 \leq t \leq T$.

i) Generating Signals "$s_i(t)$":-



ii) scheme for generating Coefficients:



The second scheme consists of a bank of 'N' product integrators/correlators with a common i/p and with each one supplied with it's own basis function. Such a bank of correlators may be used as a first stage (or) detector in receivers.

It is possible to construct a set of 'N' orthonormal basis functions $\phi_1(t), \phi_2(t) \ldots \phi_N(t)$ from the linearly independent signals $s_1(t), s_2(t) \ldots s_N(t)$. As a starting point defines the first basis function as:

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} \quad \text{—} (1)$$

where, E_1 = Energy of $\phi_1(t)$

and $s_1(t) = \sqrt{E_1} \cdot \phi_1(t) = s_{11} \cdot \phi_1(t)$ — (2) ; $s_{11} = \sqrt{E_1}$ = unit Energy

$$s_{21}(t) = \int_0^T s_2(t)\, \phi_1(t)\, dt \quad \text{—} (3)$$

We define a new intermediate function $g_2(t)$,

$$g_2(t) = s_2(t) - s_{21}\phi_1(t) \quad \text{—} (4)$$

which is orthogonal to $\phi_1(t)$ over the interval $0 \le t \le T$. Now, we define the second basis function as:

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)\, dt}} \quad \text{—} (5)$$

$$\phi_2(t) = \frac{s_2(t) - s_{21} \cdot \phi_1(t)}{\sqrt{E_2 - s_{21}^2}} \quad ; \quad E_2 = \int_0^T \phi_2^2(t)\, dt$$

where, $\phi_1(t)$ & $\phi_2(t)$ form an orthonormal set, we have to generalise $g(t)$ as follows:

$$\Rightarrow g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij} \cdot \phi_j(t)$$

where, $s_{ij} = \int_0^T s_i(t) \cdot \phi_j(t)\, dt$

and $\phi_i(t) = \dfrac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)\, dt}}$

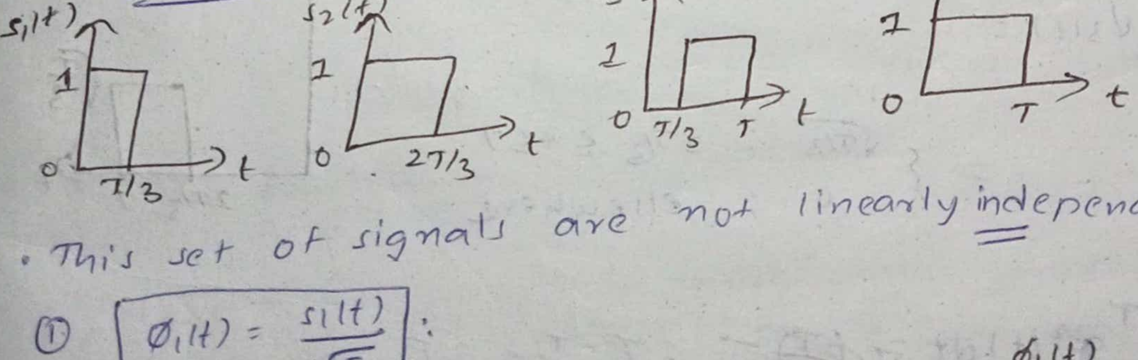
→ Each one of the derived subset of linearly independent signals $s_1(t), \ldots, s_N(t)$ may be expressed as a linear combination of orthonormal basis functions $\phi_1(t), \phi_2(t), \ldots \phi_N(t)$. It follows that each one of the original set of signals may be expressed as a linear combination of this set of functions. This completes the proof of Gram-Schmidt orthogonalisation procedure.

Conventional fourier series of expansion of a periodic signal is similar to this type of expansion. However, 2 important distinctions must be made.

1, The form of the basis functions $\phi_1(t) - - \phi_N(t)$ has not been specified. This clearly indicates that unlike the fourier series expansion of a periodic signal GSO procedure will not restrict interms of sinusoidal functions.

2, The expansion of "$s_i(t)$" into a finite number of terms is not an approximation where in only first 'k' terms are significant but rather an exact expression, where 'N' terms are significant.

Q: Signals s_1, s_2, s_3, s_4 are shown in figure. Find out the basis functions ϕ_1 to ϕ_4.

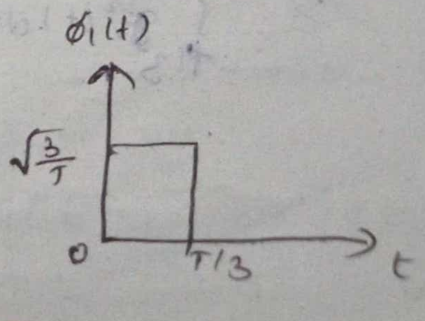


• This set of signals are not linearly independent.

① $\boxed{\phi_1(t) = \dfrac{s_1(t)}{\sqrt{E_1}}}$:

* $E_1 = \int_0^T s_1^2(t)\,dt = \int_0^{T/3} (1)\,dt = \frac{T}{3}$.

$\Rightarrow \phi_1(t) = \dfrac{s_1(t)}{\sqrt{T/3}} = \dfrac{1}{\sqrt{\frac{T}{3}}} = \sqrt{\dfrac{3}{T}}$ — ①



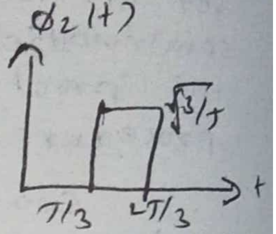
(2) $$\phi_2(t) = \frac{s_2(t) - s_{21}\cdot\phi_1(t)}{\sqrt{E_2 - s_{21}^2}} \quad — (2)$$

$$\cdot E_2 = \int_0^T s_2^2(t)\,dt = \int_0^{2T/3} 1\cdot dt = \frac{2T}{3}$$

$$\cdot s_{21} = \int_0^T s_2(t)\cdot\phi_1(t)\,dt = \int_0^{T/3} 1/\left(\sqrt{\frac{T}{3}}\right) dt$$

$$= \sqrt{\frac{3}{T}}\cdot\left(\frac{T}{3}\right) = \sqrt{\frac{T}{3}}$$

$$* \quad \phi_2(t) = \begin{cases} \sqrt{\frac{3}{T}} & ; \quad T/3 \le t \le 2T/3 \\ 0 & ; \quad \text{else where.} \end{cases}$$



(3) $$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)\,dt}} \quad ; \quad g_3(t) = s_3(t) - s_{31}\cdot\phi_1(t) - s_{32}\phi_2(t)$$

$$g_3(t) = \begin{cases} 1 & ; \quad 2T/3 \le t \le T \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

$$* \quad s_{31} = \int_0^T s_3(t)\cdot\phi_1(t)\,dt$$

$$= \int_0^{T/3} 1\cdot\sqrt{\frac{3}{T}}\,dt = \sqrt{\frac{3}{T}}\cdot\frac{T}{3} = \sqrt{\frac{T}{3}}$$

$$* \quad s_{32} = \int_{T/3}^{2T/3} 1\cdot\phi_2(t)\,dt = \int_{T/3}^{2T/3} 1\cdot\sqrt{\frac{3}{T}}\cdot dt = \sqrt{\frac{3}{T}}\left[\frac{2T}{3} - \frac{T}{3}\right] = \sqrt{\frac{3}{T}}\cdot\frac{T}{3}$$

$$* \quad g_3(t) = 1 - \sqrt{\frac{T}{3}}\cdot\cancel{\sqrt{\frac{3}{T}}} - \sqrt{\frac{T}{3}}\sqrt{\frac{T}{3}} = \frac{-T}{3}$$

$$\Rightarrow \quad \phi_3(t) = \begin{cases} \sqrt{T/3} & ; \quad 2T/3 \le t \le T \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

$$\int_{T/3}^{T} g_3^2(t)\,dt = T - \cancel{\left(\frac{T}{3}\right)} = T - \frac{T}{3} = \frac{2T}{3}$$

* N-dimensional Representation of signals:-

w.k.T, $s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t)$; $0 \le t \le T$, $i = 1, 2, \ldots M$

, $s_{ij} = \int_0^T s_i(t) \cdot \phi_j(t)\,dt$; $i = 1, \ldots M$, $j = 1, \ldots N$

$\rightarrow \mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}$; $i = 1, \ldots M$

↓

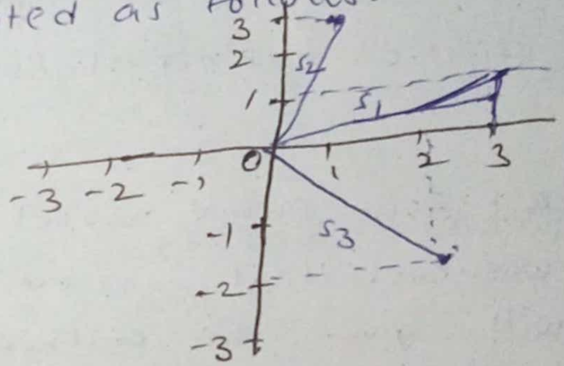
signal vector.

→ To represent signals $\{s_i\}$, $i = 1, \ldots M$, in N-dimensional Euclidean phase, signals will be represented as set of 'M' points in N-dimensional space. Hence, this space is called as "signal space".

→ It contains N-mutually perpendicular axis represented as: $\phi_1, \phi_2 \ldots \phi_N$.

ex:- let $N = 2$ with 3-dimensional signals. ⇒ $M = 3$, signal space is represented as follows:

- s_1, s_2, s_3 are vectors.



- length of signals is also called as Norm of signal vector i.e, $\|s_i\|$.

- The squared length of any signal vector s_i is defined as inner product/dot product of s_i with itself.

- Cosine of angle b/w s_i & s_j vectors is given by

$$\text{Angle} = \frac{(s_i, s_j)}{\|s_i\| \cdot \|s_j\|}$$

- s_i, s_j are orthogonal, if their inner product is zero,
- and Energy of vector E_i is given by, $E_i = \sum_{j=1}^{N} s_{ij}^2$

- Incase of pair of signals $s_i(t), s_k(t)$, we can write $\|s_i - s_k\|^2 = \sum_{j=1}^{N}(s_{ij} - s_{kj})^2 = \int_0^T [s_i(t) - s_k(t)]\,dt$.

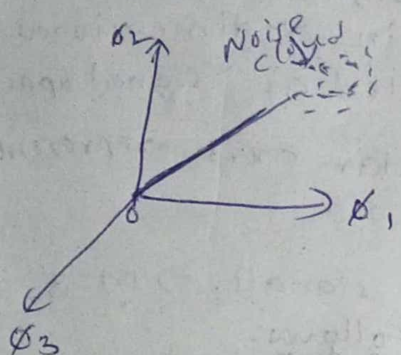
It is called "Eucledian Distance".

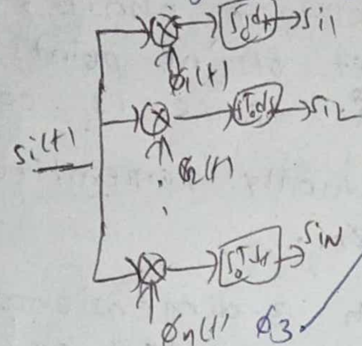
M-2

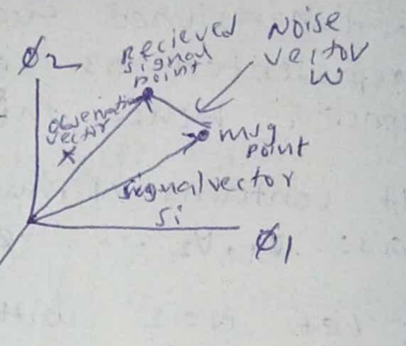
Coherent detection of signals in presence of noise

→ suppose one of 'M' possible signals $s_1(t), s_2(t) \ldots s_M(t)$ is transmitted with equal probability. Each symbol duration is T.

→ In geometric representation the signal $s_i(t)$; $i=1,2,\ldots M$ is given to Bank of correlators, it will generate signal vector s_i. we may represent $s_i(t)$ by a point in Euclidean space of dimension $N \le M$. this point is referred as transmitted signal point (or) message point. "The set of message points corresponding to the set of transmitted signals is called a signal constellation".







Effect of noise perturbation

effect of noise on location of recieved signal point

- But from AWGN model channel, the recieved signal is $x(t)$ whenever it is passed through N bank of correlators it will gives the observation vector x. It is differ from s_i by noise vector w.

- we prepresent recieved signal $x(t)$ by a point in 'N' Euclidian space to represent transmitted signal. This point is reffered as recieved signal point.

- suppose, given the observation vector x, we make a decision as $\hat{m} = m_i$, probability of error is

$$P_e(m_i|x) = P(m_i \text{ not sent} | x) = 1 - P(m_i \text{ sent} | x) \quad (1)$$

- From (1), the optimum decision rule is stated as:

set $\hat{m} = m_i$ if $P(m_i \text{ sent} | x) \geq P(m_k \text{ sent} | x)$ for all $k \neq i$ — (2)

and set $\hat{m} = m_i$ if $\dfrac{(p_k f_x(x|m_k))}{f_x(x)}$ is maximum for $k = i$ — (3)

→ (2) interms of likelihood conditions are: set $\hat{m} = m_i$, if $l(m_k)$ is max for $k=i$

* Graphical representation of maximum likelihood decision rule:

- Let z denote the N-dimensional space of all possible observations vectors x. This is reffered as observation space, we assume that $\hat{m} = m$. So the total observation space 'z' is portioned into M-decision regions.

- Observation vector(s) lies in z_i region if $l(m_k)$ is max for $k = i$ — (5)

- From log likelihood function $l(m_k)$ gets minimum value by minimizing summation term for $k = i$

$$\sum_{j=1}^{N} (x_j - s_{kj})^2$$

$\Rightarrow$ observation vector lies in 'z_i' if

$$\sum_{j=1}^{N} (x_j - s_{kj})^2 \text{ is max for } k = i \quad — (6)$$

W.k.T,

$$\sum_{j=1}^{N} (x_j - s_{kj})^2 = \| x - s_k \|^2 \quad — (7)$$

From (7), observation vector 'x' lies in z_i if Euclidean distance $\|x - s_k\|$ is max for $k = i$. — (8)

→ From (8), we can say that the max likelihood decision rule is simply to choose msg point closest to received signal point.

From (8), we can write, observation vector 'x' lies in z_i if

$$\sum_{j=1}^{N} x_j s_{kj} - \frac{1}{2} E_k \text{ is max for } k = i \;//$$

Advantages:
- Better sensitivity
- More tolerance
- High data rate

Disadvantages:
- Synchronization
- Coping with noise & interference
- Complex hardware.

* Coherent detection involves the matched filter detection of a single pulse. In this situation, the matched filter performs a coherent integration and a decision regarding presence/absence of pulse is based upon SNR of optimally sampled matched filter o/p.

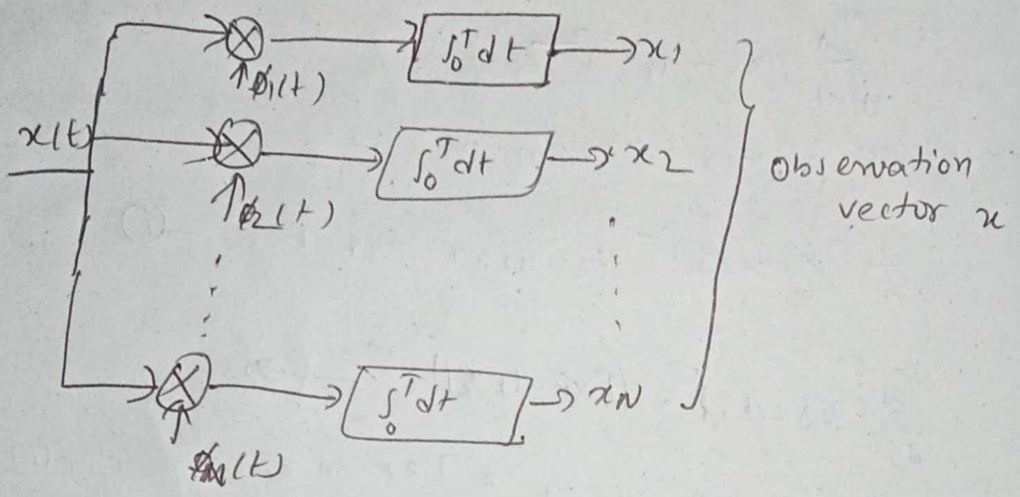
M-2

* correlation Receiver:-

* This reciever consists of two sub systems:
  1. Detector / demodulator
  2. signal transmission decoder

① ⇒ Detector is shown below. It consists of N-bank of correlators, each one having one orthogonal basis function that are locally generated.

-> This bank of correlators operates on recieved signal $x(t)$ and produce observation vector 'x'

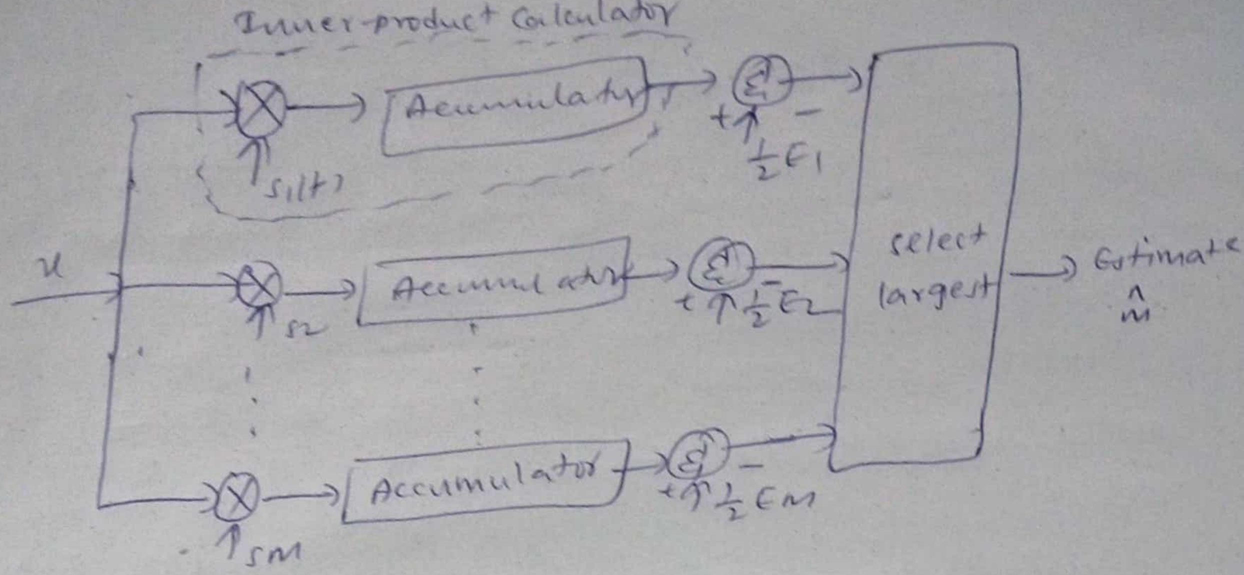


* Demodulator *

② - The second part of receiver is signal transmission decoder, it is shown below.

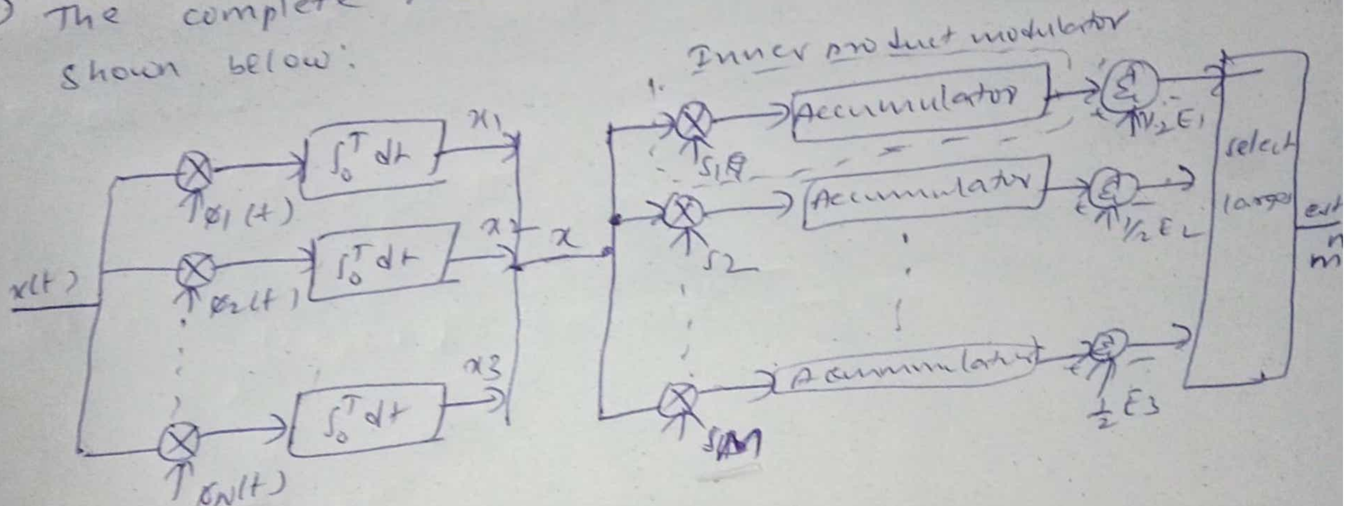
- It is implemented in the form of maximum likelihood decoder that operates on observation vector x to produce an estimate $\hat{m}$ of transmitted symbol mi in a way to minimize the average probability error.

- First, the observation vector 'x' are multiplied by the individual signal vectors s_1, s_2 -- s_m and the resulting products are successively summed by accumulators to produce corresponding set of inner products.



* Signal transmission decoder *

-> The complete structure of correlation reciever is shown below:



* correlation reciever *

- Advantages
  - Simultaneous detection of inphase & quadrature phase components
  - Optimum Noise figure
  - Improved interference rejection
  - self contained EMAT Flaw detector

- Disadvantages
  - same IF filter must be used in all recievers
  - Limited flexibility in design of the reciever
  - predistortion by imparting undesired characteristics to transmitted signal like splatter, high peak amplitude

- Application
  - sample radio interferometers
  - scientific Researches
  - Medical fields.