20/12/23

4. MULTIRATE SIGNAL PROCESSING

→ The sampling rate of digital signal can be changed by using a multirate digital signal processing system. which uses a down sampler & Up sampler, which are the 2 basic sampling rate alteration devices. In additional to conventional elements such as an adder, a multiplier & a delay.

Discrete time systems with unequal sampling rates at various parts of the system are called "Multirate systems".

Sampling rate alteration is employed to generate a new sequence with a sampling rate higher / lower than that of a given sequence. Let '$x(n)$' is a sequence with sampling rate of "F_T" and it is used to generate another sequence "$y(n)$" with a desired sampling rate of "F'_T", then the sampling rate alteration ratio is given by: $\boxed{R = \frac{F'_T}{F_T}}$

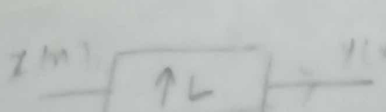
- If $R>1$, the process is called "interpolation" & results in a sequence with higher sampling rate. The discrete time system implementing interpolation process is called an "Interpolator / Expandor".

- On the otherhand, if '$R<1$', the process is called "Decimation" and results in a sequence with lower sampling rate. The discrete time system implementing decimation process is called a "Decimator / Compressior" (sampling rate compressor).

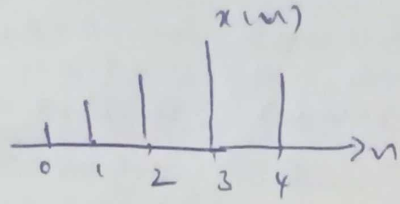
* UPSAMPLER / Sampling Rate Expander:

Upsampler is a multirate operation, in which no. of of samples are increased. If $x(n)$ is the i/p, it is upsampled by a factor of 'L', the o/p $y(n)$ is defined as:

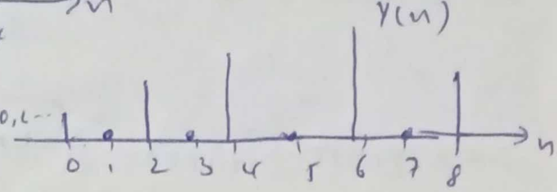
$$y(n) = \begin{cases} x\left[\frac{n}{L}\right] & ; \quad n = 0, \pm L, \pm 2L - - - \\ 0 & ; \quad \text{otherwise} \end{cases}$$



Ex:- (e) $x(n) =$, $L = 2$



$\Rightarrow y(n) = x\left[\frac{n}{L}\right] = x\left(\frac{n}{2}\right)$; $n = 0, L, \ldots$
0 , ow



$\rightarrow Y(z) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{L}\right) z^{-n}$

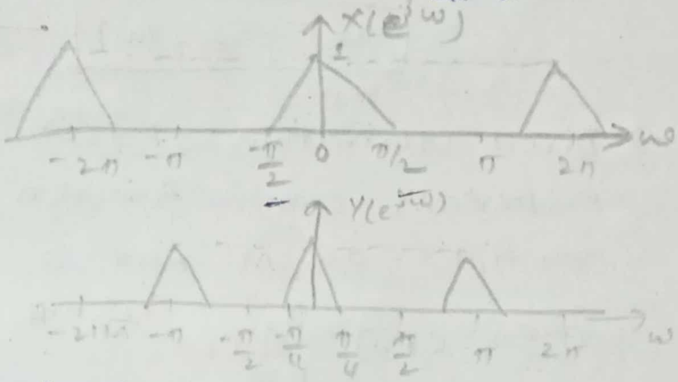
let, $m = \frac{n}{L} \Rightarrow n = mL$

$\Rightarrow Y(z) = \sum_{m=-\infty}^{\infty} x(m) (z^{L})^{-m}$ w.r.t. $\left[X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}\right]$

$$\boxed{Y(z) = X(z^{L})}$$

$\Rightarrow Y(e^{j\omega}) = X(e^{j\omega L})$

$$\boxed{Y(\omega) = X(\omega L)}$$



NOTE:- Upsampled by 'L' i.e, (L > 1), L-1 equidistant zero valued sampled are inserted by the upsampler b/w each 2 consecutive samples of i/p sequence $x(n)$ to develop o/p sequence $y(n)$.

* DOWN SAMPLER :-

$x(n) \rightarrow \boxed{\downarrow M} \rightarrow y(n)$

The down sampling operation by an integer factor $M > 1$, on a sequence $x(n)$ consists of keeping every Mth sample of $x(n)$ and removing $M-1$ b/w samples generating an o/p sequence $y(n)$.

$Y(z) = \sum_{n=-\infty}^{\infty} x(nM) z^{-n}$

$p = nM \Rightarrow n = \frac{p}{M}$

$= \sum_{p=-\infty}^{\infty} x(p) z^{-\left(\frac{p}{M}\right)}$

$= \sum_{p=-\infty}^{\infty} x(p) z^{\left(\frac{1}{M}\right)^{-p}}$

$Y(z) = X(z^{1/M})$

$\rightarrow$ NOT CORRECT, since $y(n)$ is defined for multiple integers of M.

NOTE:- The samples b/w $0 \rightarrow M$, $M \rightarrow 2M$, $2M \rightarrow 3M$ ... are knocked off.

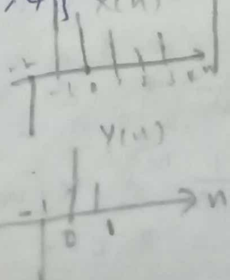
$$y(n) = \begin{cases} x(nM), & n = 0, \pm M, \pm 2M \ldots \\ \text{discarded}, & \text{ow} \end{cases}$$

Ex:-

let $x(n) = \{-8, 7, \underset{\uparrow\, n=0}{6}, 5, 3, 4\}$

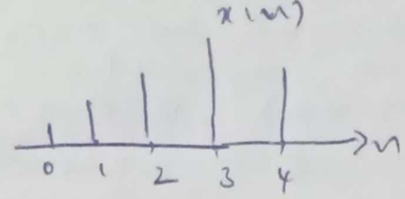
$M = 2$

$\Rightarrow y(n) = \{-8, 6, 3\}$

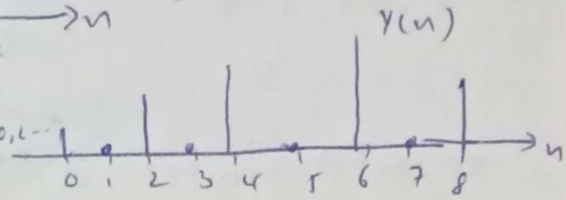


$\Rightarrow \left| Y(z) = \sum_{n=-\infty}^{\infty} x(nM) z^{-n} \right.$

Ex:- (c) $x(n) =$



$, L = 2$



$\Rightarrow y(n) = x\left[\frac{n}{L}\right] = x\left(\frac{n}{2}\right); n = 0, L, \ldots$
$\quad 0, \text{ o.w}$

$\rightarrow Y(z) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{L}\right) z^{-n}$

let, $m = \frac{n}{L} \Rightarrow n = mL$

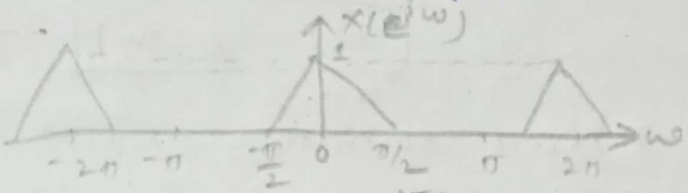
$\Rightarrow Y(z) = \sum_{m=-\infty}^{\infty} x(m) (z^{L})^{-m}$

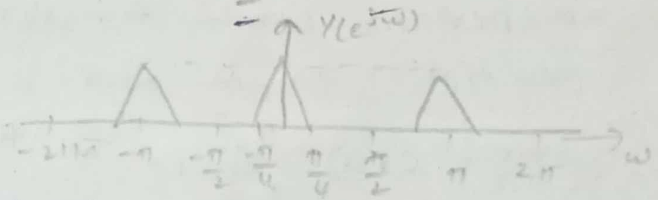
w.r.t, $\left[X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}\right]$

$$\boxed{Y(z) = X(z^{L})}$$

$\Rightarrow Y(e^{j\omega}) = X(e^{j\omega L})$

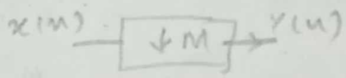
$$\boxed{Y(\omega) = X(\omega L)}$$





NOTE:- Upsampled by 'L' i.e.,(L > 1), $L-1$ equidistant zero valued sampled are inserted by the upsampler b/w each 2 consecutive samples of i/p sequence $x(n)$ to develop o/p sequence $y(n)$.

* DOWN SAMPLER:-



The down sampling operation by an integer factor $M > 1$, on a sequence $x(n)$ consists of keeping every M^{th} sample of $x(n)$ and removing $M-1$ b/w samples generating an o/p sequence $y(n)$.

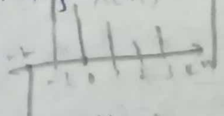
NOTE:- The samples b/w $0 \rightarrow M, M \rightarrow 2M, 2M \rightarrow 3M \ldots$ are knocked off.

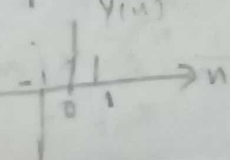
$$y(n) = \begin{cases} x(nM), & n = 0, \pm M, \pm 2M \ldots \\ \text{discarded}, & \text{ow} \end{cases}$$

Ex:- let $x(n) = \{-8, 7, 6, 5, 3, 4\}$
$\uparrow$ n=0

$M = 2$

$\Rightarrow y(n) : \{-8, 6, 3\}$





$$\Rightarrow \boxed{Y(z) = \sum_{n=-\infty}^{\infty} x(nM) z^{-n}}$$

$Y(z) = \sum_{n=-\infty}^{\infty} x(nM) z^{-n}$

$p = nM \Rightarrow n = \frac{p}{M}$

$= \sum_{p=-\infty}^{\infty} x(p) z^{-\left(\frac{p}{M}\right)}$

$= \sum_{p=-\infty}^{\infty} x(p) z^{\left(\frac{1}{M}\right)^{-p}}$

$Y(z) = X(z^{1/M})$

→ NOT CORRECT, since $y(n)$ is defined for multiple integers of M.

→ The above equ. can't be expressed directly interms of $X(z)$. Define an intermediate sequence.

→ Let, $x_{int}(n) = \begin{cases} x(nM) & ; n = 0, \pm M, \pm 2M \dots \\ 0 & ; \text{otherwise} \end{cases}$

$$\Rightarrow Y(z) = X_{int}(z) = \sum_{n=-\infty}^{\infty} x(nM) \cdot z^{-n}$$

$$= \sum_{p=-\infty}^{\infty} x(p) \cdot z^{-\frac{p}{M}}$$

Since, $nM = p$

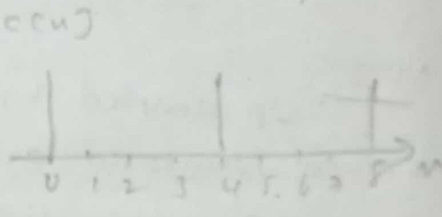
$n = \frac{p}{M}$

$$= \sum_{p=-\infty}^{\infty} x(p) \cdot [z^{1/M}]^{-p}$$

$$\boxed{Y(z) = X[z^{1/M}]} \quad \text{—} (1)$$

→ Direct derivation of $y(z)$ is difficult, hence consider a periodic comb function which is '1' at integer multiplies of 'M' and '0' otherwise.

i.e., $c[n] = \sum_{k=0}^{M-1} c_k e^{j\frac{2\pi}{M}nk}$



$$c_k = \frac{1}{M} \underbrace{\sum_{n=0}^{M-1} c[n] e^{-j\frac{2\pi}{M}nk}}_{1}$$

$$\Rightarrow c_k = \frac{1}{M}$$

$$\Rightarrow c[n] = \sum_{k=0}^{M-1} \frac{1}{M} e^{j\left(\frac{2\pi}{M}\right)nk}$$

$$\boxed{c[n] = \sum_{k=0}^{M-1} \frac{1}{M} W_M^{-nk}}$$

→ '$x_{int}[n]$' can be formally related to $x[n]$ through

$x_{int}[n] = x[n]\, c[n]$

Taking ZT, we get

$$X_{int}[z] = \sum_{n=-\infty}^{\infty} x[n]\, c[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot \sum_{k=0}^{M-1} \frac{1}{M} W_M^{-nk} z^{-n}$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} \sum_{n=-\infty}^{\infty} x[n] (W_M^{k} z)^{-n}$$

$$\boxed{X_{int}(z) = \frac{1}{M} \sum_{k=0}^{M-1} X[W_M^{k} z]} \quad \text{—} (2)$$

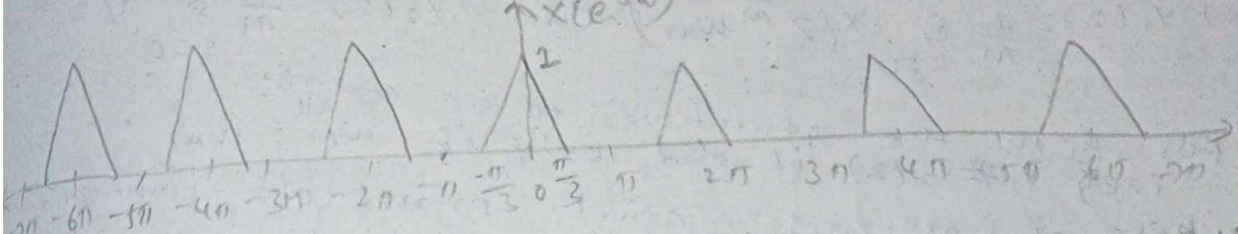
Equating ①, ② we get:

$$Y(z) = X[z^{1/M}] = \frac{1}{M}\sum_{k=0}^{M-1} X[W_M^k z^{1/M}]$$

$$\therefore \boxed{Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} X[W_M^k z^{1/M}]}$$

⇒ Frequency domain representation is:-

$$Y(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j\left(\frac{\omega-2\pi k}{M}\right)}\right)$$



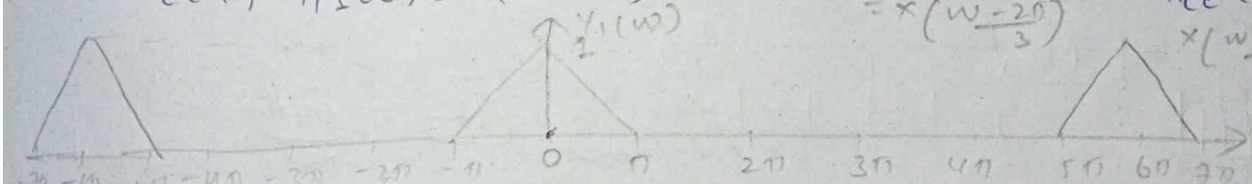
→ let M=3

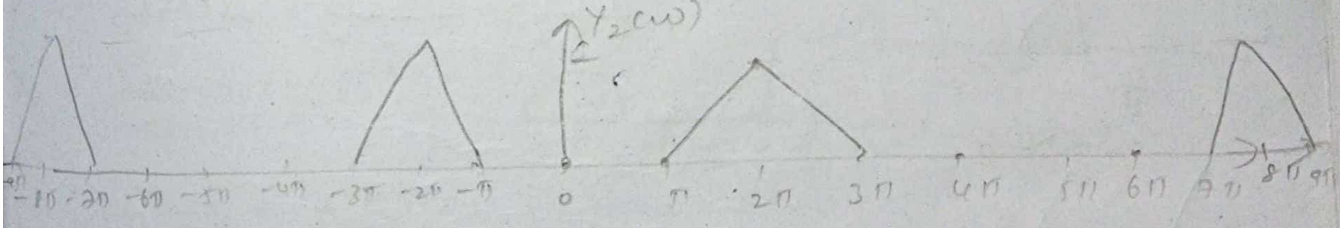
$$\Rightarrow Y(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{M-1} X\left(e^{j\left(\frac{\omega-2\pi k}{3}\right)}\right)$$

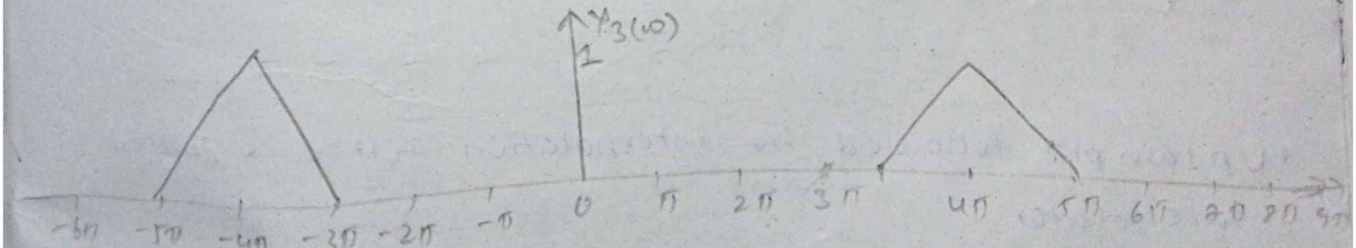
$\omega=0$, $-\frac{2\pi}{3} = -\pi+\frac{\pi}{3}$; $\omega=\pi$, $X(\frac{-\pi}{3})$; $\omega=2\pi$, $X(0)$; $\omega=3\pi$, $X(\frac{\pi}{3})$; $\omega=4\pi$, $X(\frac{2\pi}{3})$; $\omega=5\pi=0$; $\omega=7\pi$, $X(\frac{4\pi}{3})$

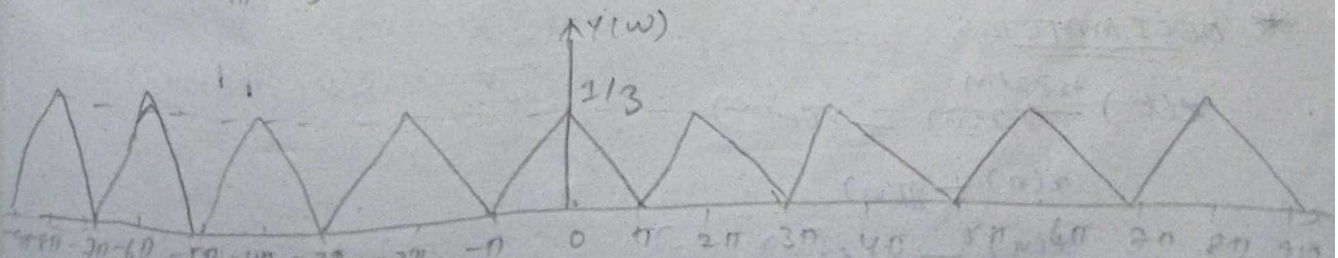
$$Y(e^{j\omega}) = \frac{1}{3}\left[X\left(e^{\frac{j\omega}{3}}\right) + X\left(e^{j\left(\frac{\omega-2\pi}{3}\right)}\right) + X\left(e^{j\left(\frac{\omega-4\pi}{3}\right)}\right)\right]$$

* let, $Y_1(\omega) = X(e^{j\omega/3})$; $Y_2(\omega) = X(e^{j(\frac{\omega-2\pi}{3})}) = X(\frac{\omega-2\pi}{3})$; $Y_3(\omega) = X(e^{j(\frac{\omega-4\pi}{3})}) = X(\frac{\omega-4\pi}{3})$









* CASCADE EQUIVALENCE OF UP SAMPLER & DOWN SAMPLER

$x(n) \rightarrow \boxed{\uparrow L} \xrightarrow{v(n)} \boxed{\downarrow M} \rightarrow y_1(n)$

$x(n) \rightarrow \boxed{\downarrow M} \xrightarrow{v(n)} \boxed{\uparrow L} \rightarrow y_2(n)$

- $V(z) = X(z^L)$
- $Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} V(z^{\frac{1}{M}} W_M^k)$

$\Rightarrow \boxed{Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{\frac{L}{M}} W_M^{kL})}$ — ①

- $V(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{\frac{1}{M}} W_M^k)$
- $Y_2(z) = V(z^L)$

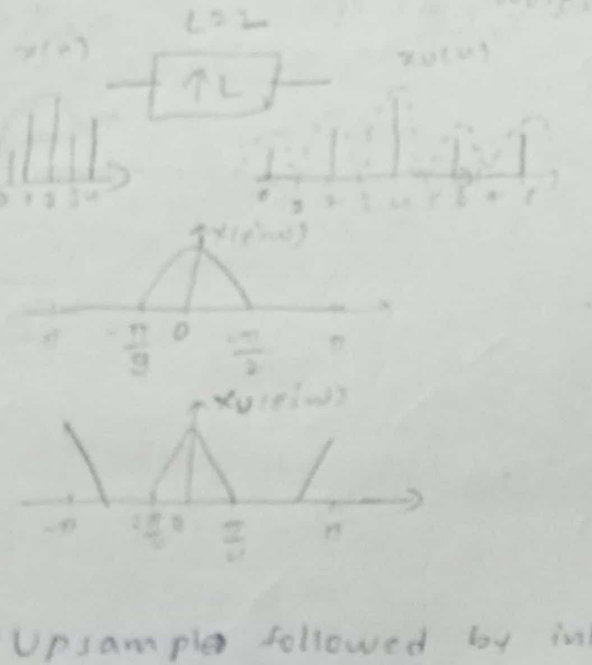
$\Rightarrow \boxed{Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{\frac{L}{M}} W_M^k)}$ — ②

→ For both the structures to be equal:
i.e. $Y_1(z) = Y_2(z)$ if and only if $W_M^{kL} = W_M^k$; $k = 0, 1, 2, \ldots$
where L & M are relatively prime (No common factor)

Ex: $(L, M) = (2, 3)$; $(5, 7)$; $(11, 15)$

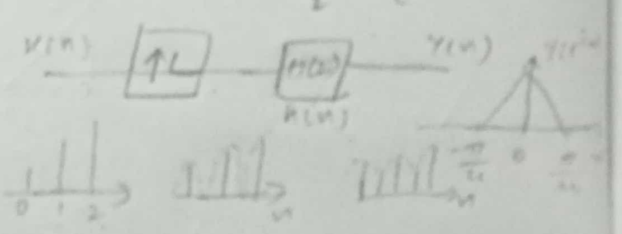
Then up sampler and down sampler can be interchanged.

* INTERPOLATOR :- (MITRA WORDS)



INTERPOLATOR

$-\frac{\pi}{L}$ to $\frac{\pi}{L}$



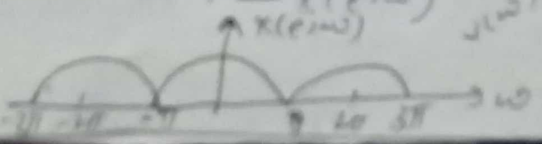
- Upsampler followed by interpolation filter is called Interpolator

* DECIMATOR :

$x(t) \xrightarrow{f_s \geq 2f_m} x(n) — X(e^{j\omega})$

$x(n) — y(n)$

$x(n) — X(e^{j\omega})$



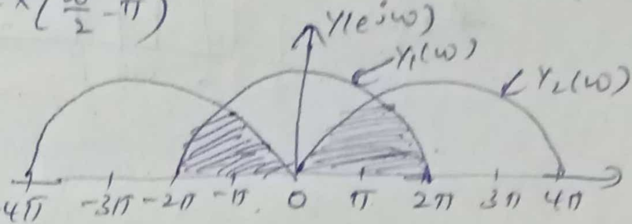
$x(n) \longrightarrow \boxed{\downarrow 2} \longrightarrow y(n)$

$\Rightarrow Y(z) = \frac{1}{2} \sum_{k=0}^{1} X(z^{1/2} W_2^{k})$

$Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\frac{\omega}{2}}) + X(e^{j(\frac{\omega}{2}-\pi)})\right]$

$Y_1(\omega) = X(\frac{\omega}{2})$; $Y_2(\omega) = X(\frac{\omega}{2}-\pi)$

- $x(n) \longrightarrow \boxed{\downarrow M} \longrightarrow \boxed{H(z)} \longrightarrow y(n)$ $Y(z)$



$\to Y_1(\omega) = X(\frac{\omega}{2})$

$\omega = 0 \Rightarrow Y_1(0) = X(0)$; $\omega = -\pi \Rightarrow Y_1(-\pi) = X(-\pi/2)$

$\omega = \pi \Rightarrow Y_1(\pi) = X(\pi/2)$; $\omega = -2\pi \Rightarrow Y_1(-2\pi) = X(-\pi)$

$\omega = 2\pi \Rightarrow Y(2\pi) = X(\pi) = 0$

$\to Y_2(\omega) = X(\frac{\omega}{2} - \pi)$

$\omega = 0 \Rightarrow Y_2(0) = X(-\pi)$; $\omega = -\pi \Rightarrow Y_2(-\pi) = X(\frac{-3\pi}{2})$

$\omega = \pi \Rightarrow Y_2(\pi) = X(\frac{-\pi}{2})$; $\omega = -2\pi \Rightarrow Y_2(-2\pi) = X(\frac{-5\pi}{2})$

$\omega = 2\pi \Rightarrow Y_2(2\pi) = X(-\pi)$

$\to$ we come across aliasing, if we use downsampling filter after down sampler.

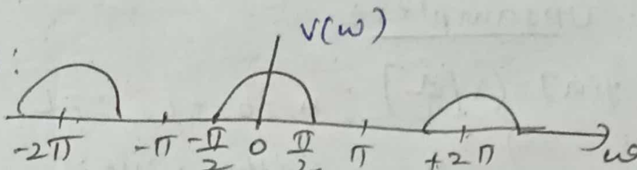
- $x(n) \longrightarrow \boxed{H(z)} \xrightarrow{v(n)} \boxed{\downarrow M} \longrightarrow y(n)$

$\to$ cutoff frequency of prealising filter is, "$\frac{\pi}{M}$".

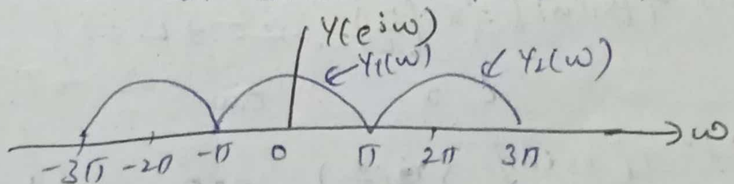
$\Rightarrow$ Let $M = 2$,

$\Rightarrow$ output of filter is:

- cutoff freq $= \pm\frac{\pi}{2}$



$\to$ Now, the output of down sampler is,



- $Y_1(\omega) = V(\frac{\omega}{2})$
- $\omega = 0 \Rightarrow Y_1(0) = X_1(0)$
- $\omega = \pi \Rightarrow Y_1(\pi) = V(\frac{\pi}{2}) = 0$
- $\omega = 2\pi \Rightarrow Y_1(2\pi) = V(\pi) = 0$

- $Y_2(\omega) = V(\frac{\omega}{2} - \pi)$

$Y_2(0) = V(-\pi) = 0$

$Y_2(\pi) = V(-\frac{\pi}{2}) = 0$

$Y_2(2\pi) = V(0) = 1$ //

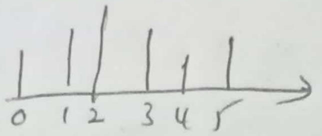
Fractional Sampling Rate Alteration:

$x(n) \rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z)} \xrightarrow{v(n)} \boxed{\downarrow M} \rightarrow y(n) \rightarrow \boxed{\dfrac{f_s}{M/L} = \dfrac{f_s \times L}{M}}$

f_s

$M/L - \uparrow L - \text{Min}\left(\frac{\pi}{L}, \frac{\pi}{M}\right) - \downarrow M$

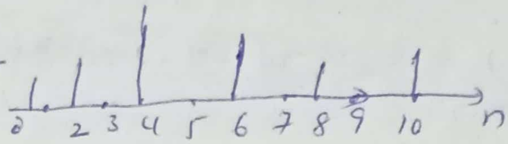
$x(n) =$



- No. of samples $= 6 = f_s$
- $M = 3$
- $L = 2$

$\Rightarrow y(n) = \dfrac{6 \times 2}{3} = 4$ samples.

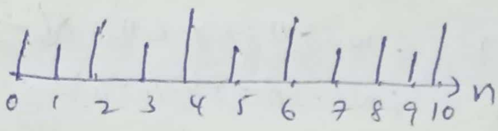
- $x(n)$: $\uparrow 2$

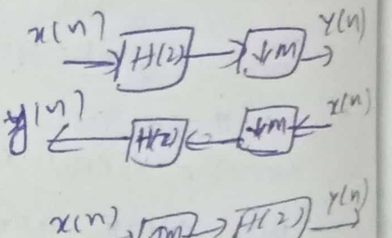


$\left(\frac{\pi}{2}, \frac{\pi}{3}\right)$

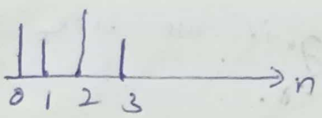
$H(z) = \pi/3$

$v(n) =$





- $\downarrow 3$



Q: Find whether upsampler & Down sampler are Linear / Non-linear systems.

Sol: 1. Upsampler:

$$y[n] = \begin{cases} x\left[\frac{n}{L}\right] & ; n = 0, \pm L, \pm 2L \\ 0 & ; \text{otherwise} \end{cases}$$

$$\rightarrow y_1[n] = T[x_1[n]] = \begin{cases} x_1\left(\frac{n}{L}\right) & ; n = 0, \pm L, \pm 2L \\ 0 & ; \text{o.w} \end{cases}$$

$$\rightarrow y_2[n] = T[x_2[n]] = \begin{cases} x_2\left(\frac{n}{L}\right) & ; n = 0, \pm L, \pm 2L \\ 0 & ; \text{o.w} \end{cases}$$

$$\rightarrow y_3[n] = T[a x_1[n] + b x_2[n]] = \begin{cases} a x_1\left[\frac{n}{L}\right] + b x_2\left(\frac{n}{L}\right) \\ 0 \end{cases} \quad -(1)$$

$$\rightarrow y_4[n] = a y_1[n] + b y_2[n] = \begin{cases} a x_1\left(\frac{n}{L}\right) + b x_2\left(\frac{n}{L}\right) & ; n = 0, \pm L, \pm 2L \\ 0 & ; \text{o w} \end{cases} \quad -(2)$$

$\therefore$ $\boxed{y_3[n] = y_4[n]}$ $\Rightarrow$ A Linear system.

② Down sampler :- $y[n] = x[nm]$

(1) $y_1[n] = x_1[nm]$

(2) $y_2[n] = x_2[nm]$

(3) $y_3[n] = a x_1[nm] + b x_2[nm]$ — ①

(4) $y_4[n] = a y_1(n) + b y_2(n)$

$y_4[n] = a x_1[nm] + b x_2[nm]$ — ④

$\therefore \boxed{y_3[n] = y_4[n]}$

Hence, a Linear system.

Q Prove that upsampler and down sampler are Time variant.

Sol: ① upsampler:- $y[n] = \begin{cases} x\left[\frac{n}{L}\right] & ; n = 0, \pm L, \pm 2L \\ 0 & ; o.w \end{cases}$

- $y[n,k] = T[x[n-k]] = x\left[\frac{n}{L} - k\right]$; $n = 1, \pm L, \pm 2L, \pm \dots$ — ①
- $y[n-k] = y[n]|_{n = n-k}$

$= x\left[\frac{n-k}{L}\right]$; — ②

$\boxed{y[n,k] \neq y[n-k]}$

$\therefore$ Up sampler is a Time variant system.

② Down sampler:

$y[n] = x[nm]$

- $y[n,k] = x[nm-k]$ — ①
- $y[n-k] = y[n]|_{n = n-k}$

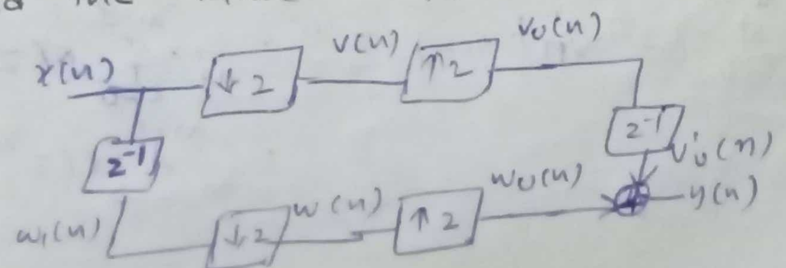
$= x[(n-k)m]$

$y[n-k] = x[nm-km]$ — ②

$\therefore y[n,k] \neq y[n-k]$

$\therefore$ Down sampler is a Time variant system.

Q:- Find the value of "y(n)" in the following system.



Sol:

· $w_1(z) = z^{-1}x(z)$

· $w(z) = \frac{1}{2}\sum_{k=0}^{1} w_1(z^{1/2} W_2^k)$

$= \frac{1}{2}\left[\boxed{w_1(z^{1/2})} + w_1(-z^{1/2})\right]$

$w(z) = \frac{1}{2}\left[z^{-1/2}x(z^{1/2}) - z^{-1/2}x(-z^{1/2})\right]$

· $w_U(z) = w(z^2) = \frac{1}{2}\left[z^{-1}x(z) - z^{-1}x(-z)\right]$ — (1)

· $v(z) = \frac{1}{2}\sum_{k=0}^{1} x(z^{1/2} W_2^k)$

$\Rightarrow v(z) = \frac{1}{2}\left[x(z^{1/2}) + x(z^{1/2}W_2)\right]$

$= \frac{1}{2}\left[x(z^{1/2}) + x(-z^{1/2})\right]$

$W_2^1 = e^{-j\left(\frac{2\pi}{2}\right)(1)}$
$= e^{-j\pi}$
$\boxed{W_2 = -1}$

· $v_U(z) = v(z^2)$

$= \frac{1}{2}\left[x(z) + x(-z)\right]$

· $v_U'(z) = \frac{1}{2}\left[z^{-1}x(z) + z^{-1}x(-z)\right]$ — (2)

· From figure, $y(z) = w_U(z) + v_U'(z)$

$= \frac{1}{2}\left[z^{-1}x(z) - \cancel{z^{-1}x(-z)} + z^{-1}x(z) + \cancel{z^{-1}x(-z)}\right]$

$= \frac{2z^{-1}x(z)}{2}$

$\boxed{Y(z) = z^{-1}X(z)}$

· prove that $y_1(n)$ & $y_2(n)$ are equal from the following structures.

$x(n) \rightarrow \boxed{\downarrow M} \xrightarrow{v_1(n)} \boxed{H(z)} \rightarrow y_1(n)$; $x(n) \rightarrow \boxed{H(z^M)} \xrightarrow{v_2(n)} \boxed{\downarrow M} \rightarrow y_2(n)$

Sol:

· $v_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} x(z^{1/M} W_M^k)$

· $y_1(z) = v_1(z)H(z)$

$y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} x(z^{\frac{1}{M}} W_M^k)\cdot H(z)$

· $v_2(z) = x(z)\cdot H(z^M)$

· $y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} v_2(z^{\frac{1}{M}}, W_M^k)$

$= \frac{1}{M}\cdot\sum_{k=0}^{M-1} x(z^{\frac{1}{M}} W_M^k) H(z^{\frac{M}{M}} W_M^{kM})$

$= \frac{1}{M}\sum_{k=0}^{M-1} x(z^{1/M} W_M^k) H(z)$

• NOBEL IDENTITIES:

(1) $x(n) \rightarrow \boxed{\downarrow M} \xrightarrow{v_1(n)} \boxed{H(z)} \rightarrow y_1(n) \quad \equiv \quad x(n) \rightarrow \boxed{H(z^M)} \xrightarrow{v_2(n)} \boxed{\downarrow M} \rightarrow y_2(n)$

(2) $x(n) \rightarrow \boxed{\uparrow L} \xrightarrow{v(n)} \boxed{H(z^L)} \rightarrow y_1(n) \quad \equiv \quad x(n) \rightarrow \boxed{H(z)} \xrightarrow{v(n)} \boxed{\uparrow L} \rightarrow y_2(n)$

(2) $v_1(z) = X(z^L)$ $\qquad V_2(z) = X(z)\,H(z)$

$Y_1(z) = H(z^L)\,X(z^L)$ $\qquad Y_2(z) = X(z^L)\cdot H(z^L)$

Q Develop an expression as a function of i/p $x(n)$ for a multirate structure shown in figure.

$x(n) \rightarrow \boxed{z^{-6}} \rightarrow \boxed{\downarrow 2} \rightarrow \underbrace{\boxed{\uparrow 4} \rightarrow \boxed{\downarrow 3}}_{\text{prime lateral}} \rightarrow y(n)$

Sol: Given, can be rewritten as

(1) $x(n) \rightarrow \boxed{z^{-6}} \rightarrow \boxed{\downarrow 2} \rightarrow \boxed{\downarrow 3} \rightarrow \boxed{\uparrow 4} \rightarrow y(n)$

(2) $x(n) \rightarrow \boxed{z^{-6}} \rightarrow \boxed{\downarrow 6} \rightarrow \boxed{\uparrow 4} \rightarrow y(n)$

(3) $x(n) \rightarrow \boxed{\downarrow 6} \xrightarrow{v_1(n)} \boxed{z^{-1}} \xrightarrow{v_2(n)} \boxed{\uparrow 4} \rightarrow y(n)$

$v_1(n) = x(n-6)$

$v_2(n) = v_1(6n)$

$= x(6n-6)$

$y(n) = v_2\left(\frac{n}{4}\right)$

$= x\left(6\left(\frac{n}{4}\right) - 6\right)$

$\rightarrow v_1(n) = x(6n)$

$\rightarrow v_2(n) = v_1(n-1) = x(6n-6)$

$\rightarrow y(n) = v_2\left(\frac{n}{4}\right) = x\left[\frac{6n}{4} - 6\right]$; $n = 0, \pm 4, \pm 8 \ldots$

$$\therefore\ y(n) = \begin{cases} x\left[\frac{6n}{4} - 6\right] & ; n = 0, \pm 4, \pm 8 \ldots \\ 0 & ; \text{ow} \end{cases}$$

Q: Develop an expression for o/p $y(n)$ as the function of $x(n)$ for the multirate function shown in figure.

$x(n) \rightarrow \boxed{\uparrow 3} \xrightarrow{x_1(n)} \boxed{\downarrow 15} \xrightarrow{x_2(n)} \boxed{\uparrow 5} \rightarrow y(n)$

sol:

→ $x_1(n) = \begin{cases} x\left(\frac{n}{3}\right) & ; n = 0, \pm 3, \pm 6, \pm 9 \ldots \\ 0 & ; \text{otherwise} \end{cases}$

→ $x_2(n) = x_1(15n)$

$= \begin{cases} x\left(\frac{15n}{3}\right) & ; n = 0, \pm 3, \pm 6, \pm 9 \ldots \\ 0 & ; \text{otherwise} \end{cases}$

→ $y(n) = \begin{cases} x_2\left(\frac{n}{5}\right) & ; n = 0, \pm 5, \pm 10, \ldots \\ 0 & ; \text{otherwise} \end{cases}$

$= \begin{cases} x\left(\frac{15}{3}\left(\frac{n}{5}\right)\right) & ; n = 0, \pm 15, \pm 60, \ldots \\ 0 & ; \text{otherwise} \end{cases}$

→ POLYPHASE REALIZATION:-

→ $n = 9$

- $H(z) = \sum_{n=0}^{8} h(n) z^{-n}$

①

- $H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}$

$= h(0) + h(2)z^{-2} + h(4)z^{-4} + h(6)z^{-6} + h(8)z^{-8} + z^{-1}[h(1) + h(3)z^{-2} + h(5)z^{-4} + h(7)z^{-6}]$

$$\boxed{H(z) = [E_0(z^2) + z^{-1}E_1(z^2)]}$$

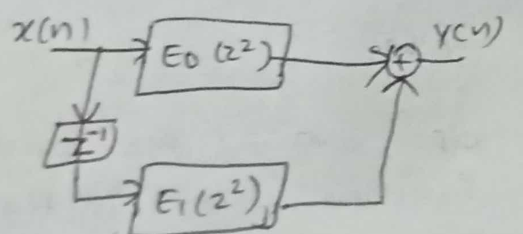
where,

- $E_0(z^2) = h(0) + h(2)z^{-2} + h(4)z^{-4} + h(6)z^{-6} + h(8)z^{-8}$
- $E_1(z^2) = h(1) + h(3)z^{-2} + h(5)z^{-4} + h(7)z^{-6}$

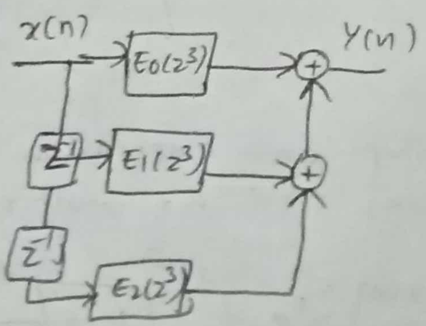
and

- $E_0(z) = h(0) + h(2)z^{-1} + h(4)z^{-2} + h(6)z^{-3} + h(8)z^{-4}$
- $E_1(z) = h(1) + h(3)z^{-1} + h(5)z^{-2} + h(7)z^{-3}$

①



②



② $H(z) = E_0(z^3) + z^{-1}E_1(z^3) + z^{-2}E_2(z^3)$

where,

$E_0(z) = h(0) + h(3)z^{-1} + h(6)z^{-2}$

$E_1(z) = h(1) + h(4)z^{-1} + h(7)z^{-2}$

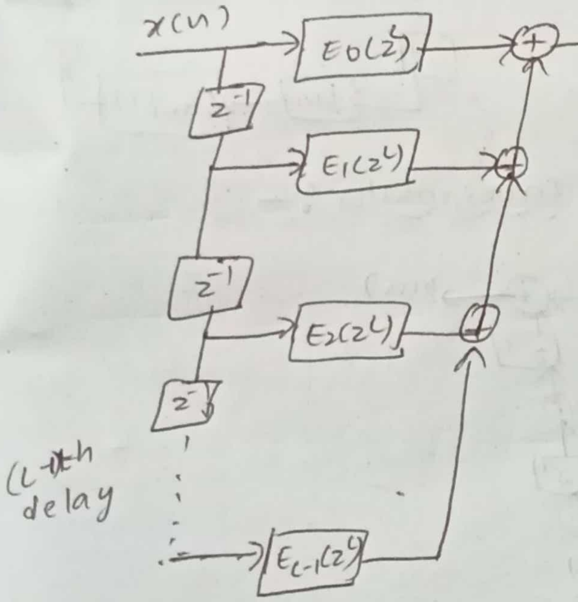
$E_2(z) = h(2) + h(5)z^{-1} + h(8)z^{-2}$

$\therefore H(z) = h(0) + h(3)z^{-3} + h(6)z^{-6} + h(1)z^{-1} + h(4)z^{-4} + h(7)z^{-7} + h(2)z^{-2} + h(5)z^{-5} + h(8)z^{-8}.$

→ For 'L' branch polyphase decomposition the transfer function $H(z)$ and polyphase components are obtained by:

* $\boxed{H(z) = \sum_{m=0}^{L-1} z^{-m} E_m(z^L)}$

* $\boxed{E_m(z^L) = \sum_{n=0}^{int\left(\frac{N+1}{L}\right)} h[Ln+m]z^{-n} \quad ; 0 \le m \le L-1}$



$H(z) = \sum_{m=0}^{L-1} z^{-m} E_m(z^L)$

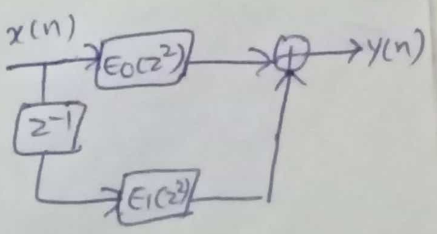
$E_m(z) = \sum_{n=0}^{int\left(\frac{N+1}{L}\right)} h(nL+m)z^{-n}$

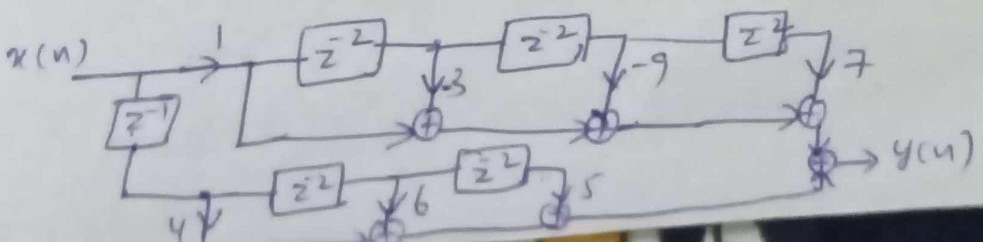
Q: $H(z) = 1 + 4z^{-1} - 3z^{-2} + 6z^{-3} - 9z^{-4} + 5z^{-5} + 7z^{-6}$, realize in two branch polyphase realization.

$L = 2$

Sol:

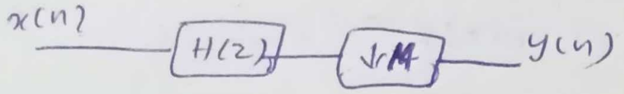
* $H(z) = E_0(z^2) + E_1(z^2).z^{-1}$
* $E_0(z^2) = 1 - 3z^{-2} - 9z^{-4} + 7z^{-6}$
* $E_1(z^2) = 4 + 6z^{-2} + 5z^{-4}$





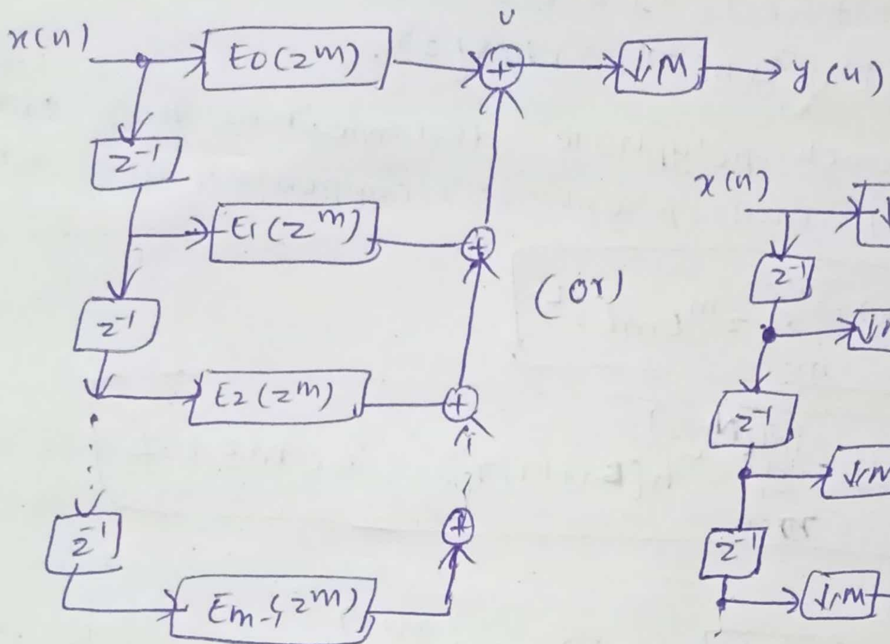
- **POLYPHASE STRUCTURE FOR DECIMATOR:-**

→ A decimator is given as:-

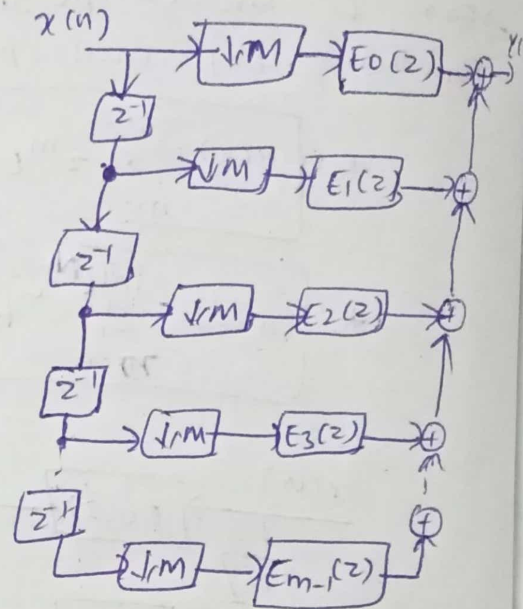


$\cdot\ H(z) = \sum_{m=0}^{M-1} z^{-m} E_m(z^M)$

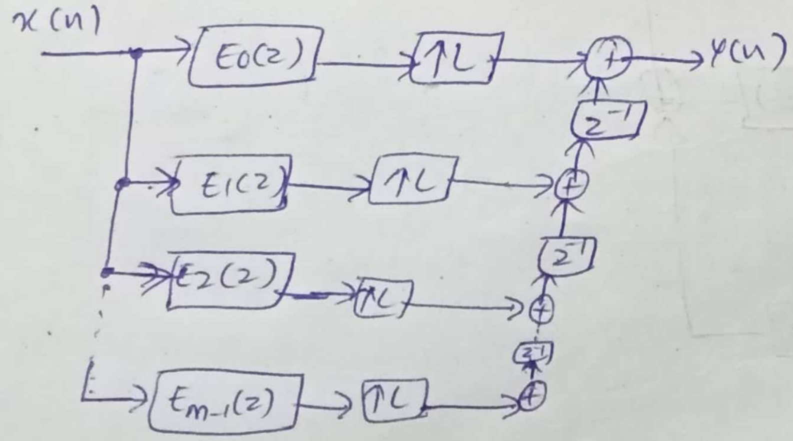
→ polyphase structure:-



(or)



- **Polyphase structure for Interpolator:-**



Q: