The irrationality of π

Theorem.

The number π is irrational.

Proof.

Assume $\pi = \frac{a}{b}$ is rational, where *a*, *b* are positive integers. We define the polynomials:

$$f(x) = \frac{x^n (a-bx)^n}{n!}$$
$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \ldots + (-1)^n f^{(2n)}(x)$$

where *n* is a positive integer which will be specified later. Note that $F(0)$ is integer and $f(x) = f(\frac{a}{b} - x)$, follows that $F(\pi)$ is also an integer.

Now, consider the derivative of $F'(x)\sin x - F(x)\cos x$, by simple calculus we have

$$\begin{aligned} \frac{d}{dx}\left[F'(x)\sin x - F(x)\cos x\right] &= F''(x)\sin x + F(x)\sin x \\ &= f(x)\sin x \end{aligned}$$

and

$$\begin{aligned} \int_0^\pi f(x)\sin x &= \left[F'(x)\sin x - F(x)\cos x\right]_0^\pi \\ &= F(\pi) + F(0) \end{aligned}$$

We conclude that $\int_0^\pi f(x)\sin x$ is an integer. But for $0 < x < \pi$,

$$0 < f(x)\sin x < \frac{\pi^n a^n}{n!},$$

so the integral is positive and arbitrarily small for sufficient large *n*. This is a contradiction, and disproved our assumption that π is rational.

Q.E.D.