The irrationality of e

Theorem.

The number $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \dots$ is irrational.

Proof.

For convenience, we denote $s_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$ and $e = \lim_{n \to \infty} s_n$.

Assume that $e = \frac{p}{q}$ is rational, here *p*, *q* are positive integers having no common prime factor. Then the number $q!(e - s_q)$ should be a positive integer since $q!e$ and $q!s_q$ are both integers and $e > s_q$. On the other hand, note that

$$
\begin{aligned}
e - s_q &= \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \dots \\
&< \frac{1}{(q+1)!}\left(1 + \frac{1}{(q+1)} + \frac{1}{(q+1)^2} + \dots\right) \\
&= \frac{1}{q!} \times \frac{1}{q}
\end{aligned}
$$

It follows that $q!(e - s_q) < \frac{1}{q} < 1$, which is a contradiction.

Q.E.D.